PRODUCT TYPE ACTIONS OF $G_q$

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Abstract. We will study a faithful product type action of $G_q$, the $q$-deformation of a connected semisimple compact Lie group $G$, and prove that such an action is induced from a minimal action of the maximal torus $T$ of $G_q$. This enables us to classify product type actions of $SU_q(2)$ up to conjugacy. We also compute the intrinsic group of $G_{q, \Omega}$, the 2-cocycle deformation of $G_q$ that is naturally associated with the quantum flag manifold $L^\infty(T\backslash G_q)$.

1. Introduction

In this paper, we will study a product type action of a $q$-deformed compact quantum group. Theory of a quantum group was initiated by Drinfel’d and Jimbo [13, 21]. They have introduced the quantum group $U_q(g)$, the $q$-deformation of the enveloping algebra of a Kac–Moody Lie algebra $g$. In the operator algebraic approach, Woronowicz has defined $SU_q(N)$ and introduced the concept of a compact quantum group [56, 57, 58, 60] by deforming a function algebra. We can construct the $q$-deformed compact quantum group $G_q$ for a connected semisimple compact Lie group $G$ using $U_q(g)$ for a finite dimensional simple $g$ (see [27, 15]).

Let us consider a product type action of $G_q$ on a uniformly hyperfinite C$^*$-algebra. Such an action has been studied by Konishi–Nagisa–Watatani [26]. They have shown that the fixed point algebra of the product type action of $SU_q(2)$ with respect to the spin-1/2 irreducible representation is generated by certain Jones projections. In particular, if we take its weak closure, then the product type action is never minimal.

In [17], Izumi has elucidated this interesting phenomenon by introducing the concept of a (non-commutative) Poisson boundary. Namely, he has constructed a von Neumann algebra from a random walk on the dual discrete quantum group of a compact quantum group, and shown that it is isomorphic to the relative commutant of a fixed point algebra inside an infinite tensor product factor of matrix algebras. Since this pioneering work, the program of realization of Poisson boundaries has been carried out in several papers [17, 20, 48, 53, 54]. In particular, it is known that the Poisson boundary of $G_q$ is isomorphic to the quantum flag manifold $T \backslash G_q$ [17, 20, 48].

On the center of a Poisson boundary as a von Neumann algebra, it has been conjectured in [50] that the center could coincide with the classical part of the Poisson boundary, that is, the center could come exactly from the random walk on the von Neumann algebraic center of the dual discrete quantum group. Indeed,
it is well-known for experts that this is the case to $SU_q(2)$. Also for universal quantum groups $A_o(F)$ and $A_u(F)$, the conjecture has been affirmatively solved [51, 53, 54]. We will show the following result which states that the conjecture also holds for every $G_q$ (Theorem 3.1).

**Theorem 1.** The von Neumann algebra $L^\infty(T\setminus G_q)$ is a factor of type I.

Let us again consider a product type action of $G_q$ on a factor $M$. From Izumi's result and Theorem 1, it turns out that the relative commutant $\mathcal{Q} := (\mathcal{M}^{G_q})' \cap \mathcal{M}$ is the infinite dimensional type I factor, where $\mathcal{M}^{G_q}$ denotes the fixed point algebra. Thus we obtain the tensor product splitting $M \cong \mathcal{R} \otimes \mathcal{Q}$, where $\mathcal{R} := \mathcal{Q}' \cap M$. It is then shown that the inclusion $\mathcal{M}^{G_q} \subset \mathcal{R}$ is irreducible and of depth 2 (Lemma 5.1). Hence it arises from a minimal action of a unique compact quantum group on $\mathcal{R}$. Actually, we will show that the compact quantum group is nothing but the maximal torus $T$ (Theorem 5.6). Hence we have a $T$-equivariant copy of $L^\infty(T)$ inside $\mathcal{R}$. Then it is natural to ask whether this copy and $\mathcal{Q}$ generate a von Neumann algebra that is $G_q$-isomorphic to $L^\infty(G_q)$.

To solve this question, we need to show the triviality of the $G_q$-equivariant automorphism on $L^\infty(T\setminus G_q)$. In [40], Soibelman has classified irreducible representations of $C(G_q)$ as a $C^*$-algebra. Namely, it has become clear that irreducible representations of $C(G_q)$ are parametrized by the maximal torus $T$ and the Weyl group $W$ of $\mathfrak{g}$ as $\{\pi_{t,w}\}_{t \in T, w \in W}$. Then Dijkhuizen–Stokman’s result [12, Theorem 5.9] (Theorem 3.6) states that any irreducible representation of $C(T\setminus G_q)$ actually comes from that of $C(G_q)$. This, in particular, implies that the counit gives a unique character on $C(T\setminus G_q)$, and we obtain the triviality of the $G_q$-equivariant automorphism group of $C(T\setminus G_q)$ (Corollary 3.11).

By using these results, we study an $L^\infty(T\setminus G_q)$-valued invariant cocycle. Actually, it is shown that any such cocycle is a coboundary with a unique solution in $Z(L^\infty(G_q))$ up to a scalar multiple (Theorem 4.1). As an application, we can show the following main result of this paper (Theorem 5.14).

**Theorem 2.** A faithful product type action of $G_q$ is induced from a minimal action of $T$ on a type III factor. Moreover, such a minimal action is unique.

Another application of Theorem 1 concerns theory of 2-cocycle deformation of locally compact quantum groups. In [9], De Commer has shown that a 2-cocycle twisted von Neumann bi-algebra of a locally compact quantum group again has a locally compact quantum group structure. In our setting, we will encounter with a 2-cocycle $\Omega$ that is canonically associated with an irreducible projective unitary representation of $G_q$ coming from $L^\infty(T\setminus G_q)$. We can determine the intrinsic group of $G_q,\Omega$, the deformation of $G_q$ by $\Omega$ (Theorem 4.5) as follows.

**Theorem 3.** The intrinsic group of $G_q,\Omega$ is isomorphic to $\hat{T}$.

When $G_q = SU_q(2)$, it has been proved by De Commer that $G_q,\Omega$ is isomorphic to $\tilde{E}_q(2)$, Woronowicz’s quantum $E(2)$ group [11]. The above theorem generalizes a partial result of [59].

This paper is organized as follows.
In Section 2, we will give a brief summary of theory of a compact quantum group and a $q$-deformed Lie group $G_q$.

In Section 3, we will show the factoriality of $L^\infty(T \backslash G_q)$, and present an alternative proof of Dijkhuizen–Stokman’s classification result (Theorem 3.6). We especially emphasize that we use the von Neumann algebra $L^\infty(T \backslash G_q)$ to classify all irreducible representations of the $C^*$-algebra $C(T \backslash G_q)$. As an application, we will compute a density operator of the Haar state (Theorem 3.13) and derive the well-known quantum Weyl dimension formula (Proposition 3.15).

In Section 4, the notion of an invariant cocycle is introduced. We will prove that all invariant cocycles evaluated in $L^\infty(T \backslash G_q)$ come from the canonical generators of $Z(L^\infty(G_q))$. As an application, we compute the intrinsic group of a 2-cocycle deformation $G_q, \Omega$.

In Section 5, we will discuss a product type action. First, we will deduce from sector theory that the canonical inclusion of factors stated before corresponds to a minimal action of the maximal torus $T$. Next, by using invariant cocycles and the triviality of $G_q$-equivariant automorphism of $L^\infty(T \backslash G_q)$, we will show that a faithful product type action is actually induced from a minimal action of $T$. Then the classification of product type actions is studied. Especially, we will present a complete classification of product type actions of $SU_q(2)$. Uncountably many non-product type and mutually non-cocycle conjugate actions of $SU_q(2)$ on the injective type $III_1$ factors are also constructed.

In the last section, we will pose a problem concerning the main results in a more general situation.

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2. Preliminary

2.1. Notations and terminology. In this paper, $\mathbb{Z}_+$ denotes the set of non-negative integers, that is, $\mathbb{Z}_+ = \{0, 1, \ldots\}$.

The tensor symbol $\otimes$ denotes the minimal tensor product for $C^*$-algebras and the von Neumann algebra tensor product for von Neumann algebras.

We denote by $\text{span} S$ and $\text{span}^w S$, the linear span of a set $S$ and the weak closure of span $S$, respectively.

For a von Neumann algebra $\mathcal{M}$, we will denote by $Z(\mathcal{M})$ its center. By $\text{End}(\mathcal{M})$, we will denote the set of normal endomorphisms on $\mathcal{M}$. For $\rho, \sigma \in \text{End}(\mathcal{M})$, $(\rho, \sigma)$ denotes the set of intertwiners. Namely, an element $a \in (\rho, \sigma)$ satisfies $a\rho(x) = \sigma(x)a$ for all $x \in \mathcal{M}$. If $(\rho, \rho) = \mathbb{C}$, then we will say that $\rho$ is irreducible.

Two endomorphisms $\rho, \sigma$ on $\mathcal{M}$ are said to be equivalent if there exists a unitary $u \in \mathcal{M}$ such that $\rho = \text{Ad} u \circ \sigma$. By $\text{Sect}(\mathcal{M})$, we denote the quotient space of $\text{End}(\mathcal{M})$. The equivalence class of $\rho$ is denoted by $[\rho]$ which is called a sector. For sector theory, reader’s are referred to [16, 29, 30].
Recall the notion of a Hilbert space in a von Neumann algebra \[38\]. A weakly closed linear space \( \mathcal{H} \) in a von Neumann algebra \( \mathcal{M} \) is called a \textit{Hilbert space in} \( \mathcal{M} \) if \( W^*W \in \mathcal{C} \) for all \( V, W \in \mathcal{H} \). Then \( \mathcal{H} \) is a Hilbert space with the inner product \( \langle V, W \rangle := W^*V \). The support of \( \mathcal{H} \), which we denote by \( s(\mathcal{H}) \), is the infimum of projections \( p \in \mathcal{M} \) such that \( pV = V \) for all \( V \in \mathcal{H} \). If \( \{ V_i \}_{i \in I} \) is an orthonormal base of \( \mathcal{H} \), then we have \( s(\mathcal{H}) = \sum_{i \in I} V_i V_i^* \).

If \( \rho, \sigma \in \text{End}(\mathcal{M}) \) and \( \rho \) is irreducible, then \((\rho, \sigma)\) is a Hilbert space in \( \mathcal{M} \) by the inner product \( \langle V, W \rangle := W^*V \) for \( V, W \in (\rho, \sigma) \).

Let \( \mathcal{N} \subset \mathcal{M} \) be an inclusion of properly infinite von Neumann algebras. Then \( L^2(\mathcal{M}) \) also has the structure of the standard form for \( \mathcal{N} \). Let \( J_\mathcal{M} \) and \( J_\mathcal{N} \) be the modular conjugations of \( \mathcal{M} \) and \( \mathcal{N} \), respectively. Then \( \gamma^M_N(x) := J_\mathcal{N} J_\mathcal{M} x J_\mathcal{M} J_\mathcal{N}, \ x \in \mathcal{M} \), is called the \textit{canonical endomorphism} from \( \mathcal{M} \) into \( \mathcal{N} \). It is known that the sector \([\gamma^M_N]\) in Sect(\( \mathcal{N} \)) does not depend on the choice of the structure of the standard forms of \( \mathcal{N} \) and \( \mathcal{M} \).

2.2. \textbf{Compact quantum group}. We will quickly review theory of compact quantum groups introduced by Woronowicz. Our references are \[41\] [60].

\textbf{Definition 2.1} (Woronowicz). We will say that a pair \((A, \delta)\) of a separable unital \( C^* \)-algebra \( A \) and a faithful unital \( * \)-homomorphism \( \delta : A \to A \otimes A \) is a \textit{compact quantum group} when the following conditions hold:

- \( \delta \) is a coproduct, that is, \((\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta) \circ \delta\);
- \( \delta(A)(\mathcal{C} \otimes A) \) and \( \delta(A)(A \otimes \mathcal{C}) \) are norm dense subspaces in \( A \otimes A \).

If \( \mathcal{G} := (A, \delta) \) is a compact quantum group, we write \( C(\mathcal{G}) := A \). It is known that there exists a unique state \( h \) called the \textit{Haar state} such that

\[
(id \otimes h)(\delta(x)) = h(x)1 = (h \otimes \text{id})(\delta(x)) \quad \text{for all} \ x \in C(\mathcal{G}).
\]

We always assume that \( h \) is faithful in what follows. Let \( \{ L^2(\mathcal{G}), 1_h \} \) be the GNS representation with respect to \( h \). We will regard \( C(\mathcal{G}) \) as a \( C^* \)-subalgebra of \( B(L^2(\mathcal{G})) \) from now on. By \( L^\infty(\mathcal{G}) \), we denote the weak closure of \( C(\mathcal{G}) \).

The \textit{multiplicative unitary} \( V \) is a unitary on \( L^2(\mathcal{G}) \otimes L^2(\mathcal{G}) \) such that

\[
V(x_1 \otimes \xi) = \delta(x)(1_h \otimes \xi) \quad \text{for} \ x \in C(\mathcal{G}), \ \xi \in L^2(\mathcal{G}).
\]

Then \( V \) satisfies the pentagon equation \( V_{12}V_{13}V_{23} = V_{23}V_{12} \). The coproduct \( \delta \) extends to the normal coproduct \( \delta : L^\infty(\mathcal{G}) \to L^\infty(\mathcal{G}) \otimes L^\infty(\mathcal{G}) \) by

\[
\delta(x) = V(x \otimes 1)V^* \quad \text{for} \ x \in L^\infty(\mathcal{G}).
\]

Note \( V \) belongs to \( B(L^2(\mathcal{G})) \otimes L^\infty(\mathcal{G}) \). The Haar state \( h \) also extends to a faithful normal invariant state on \( L^\infty(\mathcal{G}) \) by putting \( h(x) := \langle x_1h, h \rangle \) for \( x \in L^\infty(\mathcal{G}) \).

Let \( H \) be a Hilbert space. A unitary \( v \in B(H) \otimes L^\infty(\mathcal{G}) \) is called a \textit{unitary representation} on \( H \) when \( (id \otimes \delta)(v) = v_{12}v_{13} \). If \( v = (v_{ij})_{i,j} \) is the matrix representation with \( v_{ij} \in L^\infty(\mathcal{G}) \), then we have \( \delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \).

Let \( v \) and \( w \) be unitary representations on Hilbert spaces \( H \) and \( K \), respectively. The \textit{intertwiner space} \((v, w)\) is the set of bounded linear operators \( a : H \to K \) such that \((a \otimes 1)v = w(a \otimes 1)\). If \((v, v) = C1_H\), then \( v \) is said to be \textit{irreducible}. If this is the case, \( H \) must be finite dimensional.
By $\text{Rep}_f(G)$, we denote the set of finite dimensional unitary representations. We set
\[
A(G) := \text{span}\{ (\omega \otimes \text{id})(v) \mid \omega \in B(H)_*, \ v \in \text{Rep}_f(G) \}.
\]
Then $A(G)$ is a dense unital $*$-subalgebra of $C(G)$. We can define an anti-
multiplicative linear map $\kappa$ on $A(G)$, which is called the antipode, such that
$(\text{id} \otimes \kappa)(v) = v^*$ for $v \in \text{Rep}_f(G)$. The counit is the character $\varepsilon : A(G) \to \mathbb{C}$ such
that $(\text{id} \otimes \varepsilon)(v) = 1$ for $v \in \text{Rep}_f(G)$. We only treat a co-amenable $G$ in this
paper, and $\varepsilon$ extends to the character on $C(G)$. See [3, 4, 5, 17] for details of amenability.

We will introduce the Woronowicz characters $\{f_z\}_{z \in \mathbb{C}}$. They are multiplicative linear functionals on $A(G)$ uniquely determined by the following properties:

(1) $f_0 = \varepsilon$;
(2) For any $a \in A(G)$, the function $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is entirely holomor-
phic;
(3) $(f_{z_1} \otimes f_{z_2}) \circ \delta = f_{z_1 + z_2}$ for all $z_1, z_2 \in \mathbb{C}$;
(4) $f_z(\kappa(a)) = f_{-z}(a)$, $f_z(a^*) = f_{-z}(a)$ for all $z \in \mathbb{C}$ and $a \in A(G)$;
(5) $\kappa^2 = (f_1 \otimes \text{id} \otimes f_{-1}) \circ \delta^{(2)}$;
(6) $h(ab) = h(b(f_1 \otimes \text{id} \otimes f_1)(\delta^{(2)}(a)))$ for all $a, b \in A(G)$,
where $\delta^{(2)} := (\delta \otimes \text{id}) \circ \delta$. In general, for $k \in \mathbb{N}$, we let $\delta^{(k)} := (\delta^{(k-1)} \otimes \text{id}) \circ \delta$.

The modular automorphism group $\sigma^h$ is given by
\[
\sigma^h_t(x) = (f_{it} \otimes \text{id} \otimes f_{-it})(\delta^{(2)}(x)) \quad \text{for all } t \in \mathbb{R}, \ x \in A(G).
\]
Define the scaling automorphism group $\tau$ by
\[
\tau_t(x) = (f_{it} \otimes \text{id} \otimes f_{-it})(\delta^{(2)}(x)) \quad \text{for all } t \in \mathbb{R}, \ x \in A(G).
\]

Let $v \in B(H) \otimes L^\infty(G)$ be a finite dimensional unitary representation. Then
it is known that $v$ in fact belongs to $B(H) \otimes A(G)$. We let $F_v := (\text{id} \otimes f_1)(v)$,
which is non-singular and positive. Then $F_v^z = (\text{id} \otimes f_z)(v)$ for all $z \in \mathbb{C}$. It is
known that $\text{Tr}(F_v) = \text{Tr}(F_v^{-1})$, which is called the quantum dimension of $v$, and
denoted by $\dim_q(v)$ or $\dim_H H$.

Let $\text{Irr}(G)$ be the complete set of unitary equivalence classes of irreducible
unitary representations. For $s \in \text{Irr}(G)$, we fix a section $v(s) = (v(s)_{ij})_{i,j \in I_s}$.
Then we have the following orthogonal equalities: for all $s, t \in \text{Irr}(G)$, $i, j \in I_s$
and $k, \ell \in I_t$,
\[
\begin{align*}
&h(v(s)_{ij}v(t)_{k\ell}) = \dim_q(v(s))^{-1}(F_{v_s})_{\ell,j} \delta_{s,t} \delta_{i,k}, \\
&h(v(s)_{ij}v(t)_{k\ell}) = \dim_q(v(s))^{-1}(F_{v_s})_{k,i} \delta_{s,t} \delta_{j,\ell}.
\end{align*}
\]

2.3. **Action.** Let $A$ be a unital C*-algebra. We will say that a faithful unital
$*$-homomorphism $\alpha : A \to A \otimes C(G)$ is a (right) action of $G$ on $A$ if $(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \delta) \circ \alpha$, and $\alpha(A)(C \otimes C(G))$ is a dense subspace of $A \otimes C(G)$. Similarly, we
can define a left action.

By $A^\alpha$, we denote the fixed point algebra $\{ x \in A \mid \alpha(x) = x \otimes 1 \}$. If $A^\alpha = C$, then $\alpha$ is said to be ergodic.
For a von Neumann algebra $\mathcal{M}$, a (right) action means a faithful normal unital $*$-homomorphism $\alpha: \mathcal{M} \to \mathcal{M} \otimes L^\infty(G)$ satisfying $(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \delta) \circ \alpha$. It is known that the condition of the density like the above automatically holds. The fixed point algebra $\mathcal{M}^\alpha$ is similarly defined.

We will say that a state $\varphi \in \mathcal{M}_*$ is invariant when $(\varphi \otimes \text{id})(\alpha(x)) = \varphi(x)1$ for all $x \in \mathcal{M}$.

The crossed product is defined by

$$\mathcal{M} \rtimes \alpha \mathbb{G} := \overline{\text{span}}^w \{\alpha(M)(C \otimes R(G))\} \subset \mathcal{M} \otimes B(L^2(G)),$$

where $R(G)$ denotes the right quantum group algebra, that is,

$$R(G) := \overline{\text{span}}^w \{(\text{id} \otimes \omega)(V) \mid \omega \in L^\infty(G)_+\}.$$

Let $e_1 := (\text{id} \otimes h)(V) \in R(G)$. Then $e_1$ is a minimal projection of $B(L^2(G))$, and $(1 \otimes e_1)(\mathcal{M} \rtimes \alpha \mathbb{G})(1 \otimes e_1) = \mathcal{M}^\alpha \otimes Cc_1$. Thus $\mathcal{M}^\alpha$ is a corner of $\mathcal{M} \rtimes \alpha \mathbb{G}$. In particular, if $\mathcal{M} \rtimes \alpha \mathbb{G}$ is a factor, then so is $\mathcal{M}^\alpha$.

2.4. Quantum subgroup. Let $\mathbb{H}$ and $\mathbb{G}$ be compact quantum groups. We will say that $\mathbb{H}$ is a quantum subgroup of $\mathbb{G}$ when there exists a unital surjective $*$-homomorphism $r_\mathbb{H}: C(\mathbb{G}) \to C(\mathbb{H})$, which we will call a restriction map, such that $\delta_\mathbb{H} \circ r_\mathbb{H} = (r_\mathbb{H} \otimes r_\mathbb{H}) \circ \delta_\mathbb{G}$, where $\delta_\mathbb{H}$ and $\delta_\mathbb{G}$ denote the coproducts of $\mathbb{H}$ and $\mathbb{G}$, respectively.

Then $\mathbb{H}$ acts on $C(\mathbb{G})$ from the both sides. Namely, let $\gamma^\ell_\mathbb{H} := (r_\mathbb{H} \otimes \text{id}) \circ \delta_\mathbb{G}$ and $\gamma^r_\mathbb{H} := (\text{id} \otimes r_\mathbb{H}) \circ \delta_\mathbb{G}$. Then they are left and right actions of $\mathbb{H}$. Moreover, they are commuting, that is, $(\text{id} \otimes \gamma^r_\mathbb{H}) \circ \gamma^\ell_\mathbb{H} = (\gamma^\ell_\mathbb{H} \otimes \text{id}) \circ \gamma^r_\mathbb{H}$.

Let us introduce the function algebras on the homogeneous spaces as follows:

$$C(\mathbb{H}\backslash \mathbb{G}) := \{x \in C(\mathbb{G}) \mid \gamma^\ell_\mathbb{H}(x) = 1 \otimes x\},$$

$$C(\mathbb{G}/\mathbb{H}) := \{x \in C(\mathbb{G}) \mid \gamma^r_\mathbb{H}(x) = x \otimes 1\}.$$

Then the restrictions of $\delta_\mathbb{G}$ on $C(\mathbb{H}\backslash \mathbb{G})$ and $C(\mathbb{G}/\mathbb{H})$ yield actions from the right and left, respectively. The weak closures of $C(\mathbb{H}\backslash \mathbb{G})$ and $C(\mathbb{G}/\mathbb{H})$ are denoted by $L^\infty(\mathbb{H}\backslash \mathbb{G})$ and $L^\infty(\mathbb{G}/\mathbb{H})$, respectively.

Let $\varepsilon_\mathbb{H}$ and $\varepsilon_\mathbb{G}$ be the counits of $\mathbb{H}$ and $\mathbb{G}$, respectively. Then $\varepsilon_\mathbb{H} \circ r_\mathbb{H} = \varepsilon_\mathbb{G}$. So we will denote simply by $\varepsilon$ the counits of $\mathbb{H}$ and $\mathbb{G}$.

**Lemma 2.2.** Let $\alpha$ be an action of $\mathbb{G}$ on a von Neumann algebra $\mathcal{M}$. Then there uniquely exists a unital $C^*$-subalgebra $A$ of $\mathcal{M}$ such that

- $\alpha(A) \subset A \otimes C(\mathbb{G})$;
- If a $C^*$-subalgebra $B \subset \mathcal{M}$ satisfies $\alpha(B) \subset B \otimes C(\mathbb{G})$, then $B \subset A$;
- $A$ is weakly dense in $\mathcal{M}$.

**Proof.** Let $A$ be a $C^*$-algebra generated by all $C^*$-subalgebras $B \subset \mathcal{M}$ which satisfy $\alpha(B) \subset B \otimes C(\mathbb{G})$. Then it is clear that (1) and (2) hold. We will check (3) as follows.

Let $s \in \text{Irr}(\mathbb{G})$ and $H_s$ the corresponding irreducible module. Let $\text{Hom}_\mathbb{G}(H_s, \mathcal{M})$ be the set of $\mathbb{G}$-equivariant linear maps. Then $\mathcal{M}$ is weakly spanned by $T(H_s)$ for $T \in \text{Hom}_\mathbb{G}(H_s, \mathcal{M})$ and $s \in \text{Irr}(\mathbb{G})$. It is clear that $T(H_s) \subset A$, and we are done. Note that $A$ is in fact generated by such $T(H_s)$’s.

□
Let \( \alpha \) and \( A \) be as above and \( \mathbb{H} \) a quantum subgroup of \( \mathbb{G} \) with a restriction map \( r_{\mathbb{H}} \). Then we can restrict \( \alpha \) on \( \mathbb{H} \), that is, \( \alpha_{\mathbb{H}} := (\text{id} \otimes r_{\mathbb{H}}) \circ \alpha \) gives an action of \( \mathbb{H} \) on \( A \). It is clear that \( \alpha_{\mathbb{H}} \) preserves any \( \alpha \)-invariant normal state on \( \mathbb{M} \). Thus \( \alpha_{\mathbb{H}} \) extends to \( \mathbb{M} \) as the action of \( \mathbb{H} \). We will call \( \alpha_{\mathbb{H}} \) the restriction of \( \alpha \) by \( \mathbb{H} \).

2.5. \( U_q(\mathfrak{g}) \). We will review the definition of \( U_q(\mathfrak{g}) \) introduced by Drinfel’d and Jimbo \([13, 21]\), and the highest weight theory. Our references are \([6, 22, 23, 25, 27]\).

Let \( A = (a_{ij})_{i,j \in I} \) be an irreducible Cartan matrix of finite type (\( I := \{1, \ldots, n\} \)), and \( (\mathfrak{h}, \{h_i\}_{i \in I}, \{\alpha_i\}_{i \in I}) \) the root data (the realization of \( A \)), that is,

- \( \mathfrak{h} \) is an \( n \)-dimensional vector space over \( \mathbb{C} \), and \( \{h_i\}_i \) is a base of \( \mathfrak{h} \);
- \( \{\alpha_i\}_i \) is a base of \( \mathfrak{h}^* \), the space of linear functionals on \( \mathfrak{h} \);
- \( \alpha_i(h_j) = a_{ij} \) for all \( i, j \in I \).

Each \( \alpha_i \) is called a simple root. The simple reflection \( s_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \) is defined by \( s_i(\lambda) := \lambda - \lambda(h_i)\alpha_i \). Note that \( s_i^2 = 1 \). The Weyl group \( W \) is the finite group generated by \( s_i \)'s. The word length of \( w \in W \) with respect to \( \{s_1, \ldots, s_n\} \) is denoted by \( \ell(w) \). We denote by \( w_0 \) an element of maximal length. It is known that any \( w \in W \) is contained in \( w_0 \), that is, \( \ell(w_0) = \ell(w) + \ell(w_0^{-1}) \). It follows from this equality that \( w_0 \) is unique and \( w_0^2 = 1 \).

Take positive integers \( \{d_i\}_{i \in I} \) such that \( d_ia_{ij} = d_ja_{ji} \) for all \( i, j \in I \). A standard form is an inner product \( (\cdot, \cdot) \) on \( \mathfrak{h}^* \) such that \( (\alpha_i, \alpha_j) = d_ia_{ij} \) for \( i, j \in I \). It is known that \( (\alpha_i, \alpha_i) = 2d_i \) can attain at most two values. We normalize \( \{d_i\}_{i \in I} \) so that the smallest value of \( (\alpha_i, \alpha_i) \) is equal to 2. Then the normalized standard form satisfies the following properties:

- \( W \)-invariance, that is, \( (w\lambda, w\mu) = (\lambda, \mu) \) for \( w \in W \), \( \lambda, \mu \in \mathfrak{h}^* \);
- \( (\alpha_i, \alpha_i)/2 \in \mathbb{Z}_+ \) for \( i \in I \);
- \( \lambda(h_i) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \) for \( i \in I \).

We associate \( A \) with a finite dimensional simple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \) whose Cartan subalgebra is \( \mathfrak{h} \).

**Definition 2.3** (Drinfel’d, Jimbo). Let \( 0 < q < 1 \). The *quantum universal enveloping algebra* \( U_q(\mathfrak{g}) \) is the unital \( \mathbb{C} \)-algebra generated by \( \{K_i, X_i^+, X_i^-\}_{i \in I} \) such that \( K_i \) is invertible for all \( i \in I \), and the following relations hold for all \( i, j \in I \):

\[
K_iK_j = K_jK_i, \quad K_iX_j^\pm = q^{(\alpha_i, \alpha_j)/2}X_j^\pm K_i;
\]

\[
X_i^+X_j^- - X_j^-X_i^+ = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}};
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_i^{-k(1-a_{ij}-k)}(X_i^\pm)^kX_j^\pm(X_i^\pm)^{-1-a_{ij}-k} = 0 \quad \text{if } i \neq j,
\]

where \( q_i := q^{(\alpha_i, \alpha_i)/2} \), and the \( t \)-binomial is defined by

\[
\binom{m}{n}_t := \frac{(t; t)_m}{(t; t)_n(t; t)_{m-n}}, \quad (a; t)_m = (1 - a)(1 - at) \cdots (1 - at^{m-1})
\]

for \( m, n \in \mathbb{Z}_+ \) and \( a \in \mathbb{C} \).
Then $U_q(\mathfrak{g})$ has the Hopf*-algebra structure defined as follows: For $i \in I$,

- **(Coproduct)**
  \[ \Delta(K_i^\pm) = K_i^\pm \otimes K_i^\pm, \quad \Delta(X_i^\pm) = X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm, \]

- **(Antipode)**
  \[ S(K_i^\pm) = K_i^\mp, \quad S(X_i^\pm) = -q_i^\pm X_i^\pm, \]

- **(Counit)**
  \[ \varepsilon(K_i) = 1, \quad \varepsilon(X_i^\pm) = 0, \]

- **(Involution)**
  \[ K_i^* = K_i, \quad (X_i^\pm)^* = X_i^\mp. \]

(2.5)

2.6. Representation theory of $U_q(\mathfrak{g})$. For a finite dimensional Hilbert space $H$, we call a $*$-homomorphism $\pi$ from $U_q(\mathfrak{g})$ into $B(H)$ a representation. If the commutant of $\text{Im} \pi$ is trivial, $\pi$ is said to be irreducible. In general, $\text{Im} \pi$ is a finite dimensional C*-algebra, and $\pi$ is the direct sum of irreducibles.

Let $\pi: U_q(\mathfrak{g}) \to B(H)$ be an irreducible representation. From (2.5), we obtain non-singular self-adjoint operators $\pi(K_i)$. Let $p_i^\pm$ be the projections onto the positive and negative spectral subspaces of $\pi(K_i)$, respectively. Then the relations (2.2) and (2.3) imply $p_i^\pm H$ are $U_q(\mathfrak{g})$-invariant. Hence each $K_i$ has either only positive eigenvalues or only negative ones. We will say that $\pi$ is admissible when $\pi(K_i)$ is a positive operator for all $i \in I$.

For $g = (g_k)_k \in \mathbb{Z}_2^n = \prod_{k=1}^n \{1, -1\}$, we will define an automorphism $\sigma_g$ on $U_q(\mathfrak{g})$ as a $*$-algebra (not as a Hopf*-algebra) by

\[ \sigma_g(K_i^\pm) = g_i K_i^\pm, \quad \sigma_g(X_i^\pm) = X_i^\mp \quad \text{for } i \in I. \]

Then for any irreducible $\pi$, we can find $g \in \mathbb{Z}_2^n$ so that $\pi \circ \sigma_g$ is admissible. Hence the classification of irreducible admissible representations is essential.

Let $\pi: U_q(\mathfrak{g}) \to B(H)$ be a finite dimensional admissible representation. Then $\pi(K_i)$’s are generating a commutative C*-subalgebra. Hence we obtain the decomposition of $H$ into the common eigenspaces. For $\lambda \in \mathfrak{h}^*$, we define

\[ H_\lambda := \{ \xi \in H \mid \pi(K_i)\xi = q_i^{(\lambda, \alpha_i)/2} \xi \}. \]

Note that $q_i^{(\lambda, \alpha_i)/2} = q_i^{(\lambda, h_i)/2}$. Then the positivity of each $\pi(K_i)$ implies the direct sum decomposition of $H$:

\[ H = \bigoplus_{\lambda \in \mathfrak{h}^*} H_\lambda. \]

If $H_\lambda \neq \{0\}$, then $\lambda$ and $H_\lambda$ are called a weight and a weight space of $\pi$ (or of the $U_q(\mathfrak{g})$-module $H$), respectively. By $\text{Wt}(H)$, we denote the collection of the weights of $\pi$. It is known that $\dim H_\lambda = \dim H_{w\lambda}$ for all $\lambda \in \text{Wt}(H)$ and $w \in W$.

If a non-zero vector $\xi \in H_\lambda$ is cyclic and $\pi(X_i^\pm)\xi = 0$ for all $i \in I$, then $\lambda$ and $\xi$ are called a highest weight and a highest weight vector of $H$, respectively. A lowest weight and a lowest vector are similarly introduced by using $X_i^-$ instead of $X_i^+$. It is known that if $\lambda$ is a highest weight, then $w_0 \lambda$ is a lowest weight.

By $\text{Irr}^+(U_q(\mathfrak{g}))$, we denote the set of the unitary equivalence classes of irreducible admissible representations of finite dimension.
Let us introduce the root lattice \( Q := \sum_{i \in I} \mathbb{Z} \alpha_i \subset \mathfrak{h}^* \), and set \( Q_+ := \sum_{i \in I} \mathbb{Z}^+ \alpha_i \). We will equip \( \mathfrak{h}^* \) with the partial order \( \leq \) such that \( \lambda \leq \mu \) if and only if \( \mu - \lambda \in Q_+ \). Then it turns out that a finite dimensional irreducible admissible representation has a unique maximal weight that is in fact the highest weight. Moreover, the weight space of the highest weight is one-dimensional.

Let
\[
P := \{ \lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I \},
\]
and
\[
P_+ := \{ \lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_+ \text{ for all } i \in I \}.
\]
We will call an element of \( P \) and \( P_+ \) an integral weight and a dominant integral weight, respectively. When \( \lambda \in P_+ \) satisfies \( \lambda(h_i) > 0 \) for all \( i \in I \), \( \lambda \) is said to be regular. We will denote by \( P_{++} \) the set of regular dominant weights. Note that \( Q \subset P \subset \sum_{i \in I} \mathbb{Q} \alpha_i \) since \( A \) is an invertible matrix.

From \( \lambda \in P_+ \), we can construct an irreducible module \( L(\lambda) \) called the Verma module. The highest weight theory says that there exists a one-to-one correspondence \( P_+ \ni \lambda \leftrightarrow L(\lambda) \in \text{Irr}^+(U_q(\mathfrak{g})) \). Let \( \pi_\lambda : U_q(\mathfrak{g}) \rightarrow B(L(\lambda)) \) be the corresponding representation.

Let \( \omega_i \in P_+ \) be the fundamental weight that is defined by \( \omega_i(h_j) = \delta_{i,j} \) for \( i, j \in I \). Then \( P = \sum_{i \in I} \mathbb{Z} \omega_i \) and \( P_+ = \sum_{i \in I} \mathbb{Z}_+ \omega_i \).

The Weyl vector is defined by \( \rho := (1/2) \sum_{\alpha \in \Delta} \alpha \), where \( \Delta := \{ w\alpha_i \mid w \in W, i \in I \} \) and \( \Delta_+ := \Delta \cap Q_+ \). It is known that \( \rho(h_i) = 1 \) for all \( i \in I \), that is, \( \rho = \sum_{i \in I} \omega_i \). In particular, \( \rho \) is a dominant integral weight.

2.7. \( C(G_q) \). Let \( \pi : U_q(\mathfrak{g}) \rightarrow B(H_\pi) \) be a finite dimensional representation. For vectors \( \xi, \eta \in H_\pi \), we will define a linear functional \( C^\pi_{\xi,\eta} \) on \( U_q(\mathfrak{g}) \) by
\[
C^\pi_{\xi,\eta}(x) := \langle \pi(x) \eta, \xi \rangle \quad \text{for } x \in U_q(\mathfrak{g}).
\]

For \( \lambda \in P_+ \) and \( \mu \in \text{Wt}(L(\lambda)) \), we will fix an orthonormal base \( \{ \xi_i^\lambda \mid i \in I^\lambda_\mu \} \) of \( L(\lambda)_\mu \), where \( I^\lambda_\mu := \{ 1, \ldots, \dim L(\lambda)_\mu \} \). We often write \( C^\lambda_{\xi_i^\mu,\xi_j^\nu} \) or, more simply, \( C^\lambda_{\mu,\nu} \) for \( C^\lambda_{\xi_i^\mu,\xi_j^\nu} \).

Since \( \dim L(\lambda)_w = \dim L(\lambda)_\lambda = 1 \) for \( w \in W \), we simply denote by \( C^\lambda_{\xi,\xi'} \) for \( C^\lambda_{\xi_w,\xi_{w'}} \) where \( \xi, \xi' \in L(\lambda)_w \) and \( \xi, \xi' \in L(\lambda)_w \) are fixed unit vectors for \( w, w' \in W \). Note that there exists an ambiguity of a constant factor of modulus one about this expression.

Then we will introduce the following subspace of \( U_q(\mathfrak{g})^* \):
\[
A(G_q) := \text{span}\{ C^\pi_{\xi,\eta} \mid \xi, \eta \in H_\pi, \pi \text{ is an admissible representation} \} = \text{span}\{ C^\lambda_{\mu,\nu} \mid \mu, \nu \in \text{Wt}(L(\lambda)), i \in I^\lambda_\mu, j \in I^\lambda_\nu, \lambda \in P_+ \}.
\]

We put the Hopf-algebra structure on \( A(G_q) \) as follows: for \( \phi, \psi \in A(G_q) \) and \( x, y \in U_q(\mathfrak{g}) \),

- (Product)
\[
(\phi \psi)(x) := (\phi \otimes \psi)(\Delta(x)),
\]
• (Coproduct) \( \delta(\phi)(x \otimes y) := \phi(xy) \),
• (Antipode) \( \kappa(\phi)(x) := \phi(S(x)) \),
• (Counit) \( \varepsilon(\phi) := \phi(1) \),
• (Involution) \( \phi^*(x) = \overline{\phi(S(x))^*} \).

The following equality is frequently used:
\[
\delta(C^\lambda_{\mu,\nu}) = \sum_{\zeta \in \text{Wt}(\lambda), k \in I^\lambda_k} C^\lambda_{\mu,\zeta} \otimes C^\lambda_{\zeta,\nu},
\]
where \( \text{Wt}(\lambda) \) denotes \( \text{Wt}(L(\lambda)) \). The \( C^* \)-completion of \( A(G_q) \) is denoted by \( C(G_q) \). It is known that \( G_q \) is co-amenable. Hence the Haar state \( h \) is faithful, and the counit \( \varepsilon \) is norm bounded on \( C(G_q) \) (see, for example, [2 Corollary 5.1]). We will describe the Woronowicz character of \( A(G_q) \), which is well-known for experts (see [23 Example 9, p.425]).

**Lemma 2.4.** For \( z \in \mathbb{C} \), the Woronowicz character \( f_z : A(G_q) \to \mathbb{C} \) is given by
\[
f_z(C^\lambda_{\xi,\nu}) = \langle \xi, \nu \rangle \left(q^{2(\mu,\rho)}z\right) \quad \text{for all } \xi, \nu \in L(\lambda)_\mu, \nu.
\]

**Proof.** Let \( \phi \in A(G_q) \). Then \( \tau_{-i}(\phi)(x) = \kappa^2(\phi)(x) = \phi(S^2(x)) \) for \( x \in U_q(\mathfrak{g}) \). If \( x = X_{i_1} \cdots X_{i_k} K X_{j_1} \cdots X_{j_l} \) for some \( i_1, \ldots, i_k, j_1, \ldots, j_l \in I \) and \( K \in U_q(\mathfrak{h}) \),
\[
S^2(x) = a_{i_1}^2 \cdots a_{i_k}^2 q_{j_1}^2 \cdots q_{j_l}^2 x,
\]
where \( U_q(\mathfrak{h}) \) denotes the Hopf-algebra generated by \( K_i \)'s. Since \( C^\lambda_{\xi,\nu}(x) \) can be non-zero only if \( \mu = \nu + \alpha_{i_1} + \cdots + \alpha_{j_l} - \alpha_{i_1} - \cdots - \alpha_{j_l} \), we obtain \( C^\lambda_{\xi,\nu}(S^2(x)) = q^{2(\mu,\nu)}C^\lambda_{\xi,\nu}(x) \), where we have used the equality \( 2(\mu,\nu) = (\alpha_{i_1}, \alpha_{i_1}) \). The positivity of \( C^\lambda_{\xi,\nu}(x) \) is of exponential type (see, for example, [12 Proposition 11.11]), we obtain the following equality:
\[
\tau_t(C^\lambda_{\xi,\nu}) = q^{2(\mu,\nu)t}C^\lambda_{\xi,\nu} \quad \text{for all } t \in \mathbb{R}.
\]

We let \( F_\lambda := (\text{id} \otimes f_1)(C^\lambda) \). Since \( (\text{id} \otimes \tau_t)(C^\lambda) = (F^t_\lambda \otimes 1)C^\lambda(F^-t_\lambda \otimes 1) \) and \( C^\lambda \) is irreducible, we have \( f_{it}(C^\lambda_{\xi,\nu}) = \langle \xi, \nu \rangle q^{2(\mu,\nu)t}g^\lambda(it) \) for some entire function \( g^\lambda \). Thus \( f_z : A(G_q) \to \mathbb{C} \) is given by
\[
f_z(C^\lambda_{\xi,\nu}) = \langle \xi, \nu \rangle q^{2(\mu,\rho)}g^\lambda(z) \quad \text{for all } \xi, \nu \in L(\lambda)_\mu, \nu.
\]

Since \( f_{z_1} \cdot f_{z_2} = f_{z_1 \otimes f_{z_2}} \circ \delta \) for \( z_1, z_2 \in \mathbb{C} \), we obtain \( g^\lambda(z_1 + z_2) = g^\lambda(z_1)g^\lambda(z_2) \). Hence there exists \( \theta_\lambda \in \mathbb{C} \) such that \( g^\lambda(z) = e^{\theta_\lambda z} \) for all \( z \in \mathbb{C} \). The positivity of \( F^t_\lambda = (\text{id} \otimes f_1)(C^\lambda) \) for \( t \in \mathbb{R} \) implies that \( \theta_\lambda \) belongs to \( \mathbb{R} \). By definition of \( f_z \), we must have \( \text{Tr}(F_\lambda) = \text{Tr}(F^{-1}_\lambda) \). Hence
\[
g^\lambda(1) \sum_{\mu \in \text{Wt}(\lambda)} q^{2(\mu,\rho)} \dim L(\lambda)_\mu = g^\lambda(-1) \sum_{\mu \in \text{Wt}(\lambda)} q^{-2(\mu,\rho)} \dim L(\lambda)_\mu.
\]
However, by $w_0^2 = 1$, $w_0 \rho = -\rho$ and $\dim L(\lambda)_{\mu} = \dim L(\lambda)_{w_0 \mu}$,

$$\sum_\mu q^{-2(\mu, \rho)} \dim L(\lambda)_\mu = \sum_\mu q^{2(\mu, w_0 \rho)} \dim L(\lambda)_\mu = \sum_\mu q^{2(\mu, \rho)} \dim L(\lambda)_{w_0 \mu} = \sum_\mu q^{2(\mu, \rho)} \dim L(\lambda)_\mu.$$ 

Thus $g^\lambda(1) = g^\lambda(-1)$. Hence $g^\lambda(1)^2 = g^\lambda(1) g^\lambda(-1) = g^\lambda(0) = 1$. Since $g^\lambda(1) > 0$, $e^\theta = g^\lambda(1) = 1$. This shows $\theta_\lambda = 0$ and $g^\lambda(z) = 1$ for all $z \in \mathbb{C}$.

Hence for $t \in \mathbb{R}$, $\lambda \in P_+$ and $\mu, \nu \in \text{Wt}(\lambda)$, we have

$$\sigma_t^h(C_{\xi_\mu, \xi_\nu}^\lambda) = q^{2(\mu + \nu, \rho)_{t \cdot h}} C_{\xi_\mu, \xi_\nu}^\lambda$$

for all $\xi_\mu \in L(\lambda)_\mu$, $\xi_\nu \in L(\lambda)_\nu$, (2.6)

and the quantum dimension $\dim_q L(\lambda)$ is given by

$$\dim_q L(\lambda) = \sum_{\mu \in \text{Wt}(\lambda)} q^{2(\mu, \rho)} \dim L(\lambda)_\mu.$$  

(2.7)

2.8. Quantum subgroups of $G_q$. Let us denote by $T$ the $n$-torus, that is, $T = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid |t_i| = 1, i = 1, \ldots, n\}$. Recall that $n = |I|$. By the coupling $\langle \cdot, \cdot \rangle: T \times P \to \mathbb{C}$ defined by

$$\langle (t_1, \ldots, t_n), (\mu) \rangle := \mu_1^{t_1 h_1} \cdots \mu_n^{t_n h_n}$$

for $(t_1, \ldots, t_n) \in T$, $\mu \in P$, we will regard $P$ as the dual group of $T$. Let $\text{ev}_t: C(T) \to \mathbb{C}$ be the evaluation map at $t \in T$. Then the restriction map $r_T: C(G_q) \to C(T)$ is defined as follows:

$$\text{ev}_t \circ r_T(C_{\xi_\mu, \xi_\nu}^\lambda) = \langle t, \mu \rangle \langle \xi_\mu, \xi_\nu \rangle$$

for all $t \in T$, $\xi_\mu \in L(\lambda)_\mu$, $\xi_\nu \in L(\lambda)_\nu$. (2.8)

With this map, $T$ is a quantum subgroup of $G_q$ and called the maximal torus. Note that $r_T$ actually comes from the inclusion map $U_q(\mathfrak{h})$ into $U_q(\mathfrak{g})$.

Let $\gamma_t := (\text{ev}_t \circ r_T \otimes \text{id}) \circ \delta$ be the left action of $T$ on $C(G_q)$. Then we have

$$\gamma_t(C_{\xi_\mu, \xi_\nu}^\lambda) = \langle t, \mu \rangle C_{\xi_\mu, \xi_\nu}^\lambda$$

for all $t \in T$, $\xi_\mu \in L(\lambda)_\mu$, $\xi_\nu \in L(\lambda)_\nu$. (2.9)

Let $U_q(su(2))$ be the Hopf-$*$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{1, K_i^{\pm 1}, X_i^{\pm}\}$. Then the canonical embedding of $U_q(su(2))$ into $U_q(\mathfrak{g})$ gives a surjective $*$-homomorphism $r_i: C(G_q) \to C(SU_q(2))$. It turns out that $r_i$ is a restriction map. So, $SU_q(2)$ is a quantum subgroup of $G_q$.

2.9. Classification of irreducible representations of $C(G_q)$. In this subsection, we recall theory of irreducible representations of $C(G_q)$ as a $C^*$-algebra.

For $i \in I$, set $H_{s_i} := \ell^2(\mathbb{Z}_\pm)$ with $\{e_k\}_{k \in \mathbb{Z}_\pm}$, the standard orthonormal base. Let us introduce an irreducible representation $\pi_i: C(SU_q(2)) \to B(H_{s_i})$ as follows:

$$\pi_i(x_i) e_k = \sqrt{1 - q_i^{2k+2}} e_{k+1}, \quad \pi_i(u_i) e_k = q_i^k e_k,$$

(2.10)

$$\pi_i(v_i) e_k = -q_i^{k+1} e_k, \quad \pi_i(y_i) e_k = \sqrt{1 - q_i^{2k}} e_{k-1}$$

for all $k \geq 0$, where

$$(x_i \quad u_i \quad v_i \quad y_i) := \begin{pmatrix} r_i(C_{\omega_i, \omega_i}^\omega) & r_i(C_{\omega_i, s_1(\omega_i)}^\omega) \\ r_i(C_{s_1(\omega_i), \omega_i}^\omega) & r_i(C_{s_1(\omega_i), s_1(\omega_i)}^\omega) \end{pmatrix}.$$
Since $\pi_i(u_i)$ generates an atomic maximal abelian subalgebra of $B(H_{s_i})$ and $\pi_i(x_i)$ is acting on $H_{s_i}$ as a weighted unilateral shift, it turns out that $\pi_i$ is irreducible.

Note that $\pi(u_i)$ and $\pi(v_i)$ are compact operators. Let $p_i$ be the quotient map from $B(H_{s_i})$ onto the Calkin algebra $B(H_{s_i})/K(H_{s_i})$. Then we obtain $p_i(\pi_i(x_i)) = p_i(S)$ and $p_i(\pi_i(u_i)) = 0$, where $S$ denotes the unilateral shift on $H_{s_i}$. This shows that $\text{Im} p_i \circ \pi_i$ coincides with the commutative $C^*$-algebra $C^*(p_i(S)) \cong C(S^1)$.

Let $\omega_i : \text{Im} p_i \circ \pi_i \to \mathbb{C}$ be the character defined by $\omega_i(p_i(S)) = 1$. Then the composition $\omega_i \circ p_i \circ \pi_i$ is nothing but the counit of $C(SU_q(2))$. Let us denote by $\eta_i$ the map $\omega_i \circ p_i$. Then we obtain $\varepsilon = \eta_i \circ \pi_i$.

Since $\pi_i$ is irreducible, $\pi_{s_i} := \pi_i \circ r_i$ is an irreducible representation of $C(G_q)$.

The following theorem due to Soibel’man says that those $\pi_{s_i}$ play a role of building blocks of any irreducible representation of $C(G_q)$. See [40] Theorem 3.4, 5.7 or [27] Theorem 5.3.3, 6.2.7, Chapter 3 for its proof. Also see [39] Theorem 4 and [53] Theorem 3.1 for the statement in the case of $SU_q(N)$.

For $w \in W$, $I_w$ denotes the norm-closed ideal in $C(G_q)$ that is generated by $C^\lambda_{\xi,\xi}$ for $\lambda \in P_\pm$ such that $\langle U_q(b_+) \lambda \rangle \xi = 0$, where $U_q(b_+)$ is the subalgebra of $U_q(g)$ generated by $\{1, K^\pm, X^\pm_i \}_{i \in I}$.

**Theorem 2.5** (Soibel’man). Let $G_q$ be as before.

1. For any irreducible representation $\pi$ of $C(G_q)$, there exists a unique $w \in \text{W}$ such that $I_w \subseteq \ker \pi$;

2. Let $\pi$ be an irreducible representation of $C(G_q)$ corresponding to $w \in \text{W}$. Then there exists $t \in T$ such that $\pi$ is unitarily equivalent to the following representation:

$$\pi_{t,w} := (\pi_t \otimes \pi_{s_{i_1}} \otimes \cdots \otimes \pi_{s_{i_k}}) \circ \delta^{(k)},$$

where $\pi_t := ev_t \circ r_t$, and $w = s_{i_1} \cdots s_{i_k}$ is a minimal expression of $w$. The representation space is $H_w := H_{s_{i_1}} \otimes \cdots \otimes H_{s_{i_k}}$.

If $t = e$ (the neutral element of $T$), then we will denote by $\pi_w$ for $\pi_{e,w}$. By definition, we obtain $\pi_{t,w} = \pi_w \circ \gamma_t$.

For a minimal expression $w = s_{i_1} \cdots s_{i_k}$, we let $\eta_w := \eta_{i_1} \otimes \cdots \otimes \eta_{i_k}$. Then $\eta_w \circ \pi_w = \varepsilon$. By definition of $\delta^{(n)}$ for $n \in \mathbb{N}$, we have $\pi_w = (\pi_{w_1} \otimes \pi_{w_2}) \circ \delta$ for $w = w_1 w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$.

3. **Quantum Flag Manifolds**

In this section, we will study the quantum flag manifold $C(T\setminus G_q)$ and its measure theoretic analogue $L^\infty(T\setminus G_q)$. In particular, we will prove the factoriality of $L^\infty(T\setminus G_q)$. Then we will present an alternative proof of Dijkhuizen–Stokman’s result for $T \subset G_q$. Our proof involves knowledge of Poisson boundary, but it is relatively short. On recent development of this subject in a more general setting, readers are referred to [35] and references therein. As applications, we will derive the quantum Weyl dimension formula of $L(\lambda)$ and also determine the intrinsic group of $G_q,\Omega$, the 2-cocycle deformation associated with $L^\infty(T\setminus G_q)$.
3.1. Classification of irreducible representations of $C(T\backslash G_q)$. Let $G$ be a compact quantum group. In [50], it is conjectured that the classical Poisson boundary $H_{\mathrm{class}}^\infty(\hat{G}, \mu)$ could coincide with the center of the Poisson boundary $Z(H^\infty(\hat{G}, \mu))$. (See [17] [20] [48] for a detail of theory of a Poisson boundary.) In particular, if the fusion rule of $G$ is commutative, the conjecture asks the factoriality of the Poisson boundary. It has been verified that the conjecture is true for $SU_q(2)$, $A_q(2)$ [27], $A_q(2)$ [27], and $A_q(2)$ [27]. To these examples, we can show that a stronger property that the canonical $G$-action on a Poisson boundary is “approximately inner.” We will show the factoriality of $L^\infty(T\backslash G_q)$ as follows.

By (1), $Z(L^\infty(G_q))$ is a faithful action of $\hat{G}$ on $L^\infty(T\backslash G_q)$. Hence $\gamma_t(v_\lambda) = (t, \lambda)v_\lambda$ for $t \in T$. Thus $\gamma$ is faithful on $Z(L^\infty(G_q))$.

Let $a_\lambda = v_\lambda|^a_\lambda$ be the polar decomposition in $L^\infty(G_q)$. Then $v_\lambda C^\lambda(\xi, \nu) = C^\lambda(\xi, \nu) v_\lambda$. Hence $v_\lambda$ belongs to the center of $L^\infty(G_q)$. Then $\gamma_t(v_\lambda) = (t, \lambda)v_\lambda$ for $t \in T$. Thus $\gamma$ is faithful on $Z(L^\infty(G_q))$.

(2). This is a direct consequence of [50] Theorem 4.7]. We will sketch the proof for readers’ convenience. Let $H^\infty(\hat{G}_q)$ be the Poisson boundary of $\hat{G}_q$ and $\Theta: L^\infty(G_q) \to R(G_q)$ the Poisson integral defined by $\Theta(x) := (\text{id} \otimes h)(V^*(1 \otimes x)V)$ for $x \in L^\infty(G_q)$. Then $\Theta$ is a $G_q$-equivariant isomorphism from $L^\infty(T\backslash G_q)$ onto $H^\infty(\hat{G}_q)$. See [17] Theorem 5.10], [20] Theorem A,B, and [48] Corollary 4.11] for the proof.

Let $x \in Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q)$. Then $\Theta(x)$ is fixed by the coproduct of $R(G_q)$, and $\Theta(x)$ is a scalar. Hence $Z(L^\infty(G_q))^\gamma = Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q) = \mathbb{C}$ which shows the central ergodicity of $\gamma$.

By (1), $Z(L^\infty(G_q))$ is generated by unitaries $v_\lambda$, $\lambda \in P_+$, such that $\gamma_t(v_\lambda) = (t, \lambda)v_\lambda$ for $t \in T$. From [2.9], it turns out that $L^\infty(G_q) = Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q)$. Hence $Z(L^\infty(T\backslash G_q)) \subset Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q) = \mathbb{C}$. 

Note that by ergodicity, $v_\lambda$ in the above proof must be a unitary. As a corollary, we have the following.

**Corollary 3.2.** The Poisson boundary $H^\infty(\hat{G}_q)$ is the type $I_\infty$ factor.
Now let us present our approach to Dijkhuizen–Stokman’s result.

**Lemma 3.3.** The representation $\pi_{w_0}$ is faithful on $C(T \backslash G_q)$.

**Proof.** Let $x \in C(T \backslash G_q)$ with $\pi_{w_0}(x) = 0$. Let $w \in W \setminus \{w_0\}$. Then $w' := w^{-1}w_0$ satisfies $\ell(w') = \ell(w) + \ell(w')$. Hence we may assume that $\pi_{w_0} = (\pi_w \otimes \pi_{w'}) \circ \delta$. Using the character $\eta_{w'}$: $\text{Im} \pi_{w'} \to \mathbb{C}$ with $\eta_{w'} \circ \pi_{w'} = \varepsilon$, we have $(\text{id} \otimes \eta_{w'}) \circ \pi_{w_0} = \pi_w$. Thus $\pi_w(x) = 0$ for any $w \in W$. Since the family $\{\pi_{t,w}\}_{t \in T, w \in W}$ separates the elements of $C(G_q)$ and $\pi_{t,w} = \pi_w$ on $C(T \backslash G_q)$, we have $x = 0$. □

**Lemma 3.4.** For any $\lambda \in P_{++}$, $|a_\lambda|^n$ uniformly converges to the same minimal projection $p_0$ in $C(T \backslash G_q)$ as $n \to \infty$.

**Proof.** With a minimal expression $w_0 = s_{i_1} \cdots s_{i_k}$, we have the following (up to a factor of modulus one):

$$\pi_{w_0}(a_\lambda) = \pi_{s_{i_1}}(u_{i_1})^{s_{i_2} \cdots s_{i_k} \lambda(h_{i_1})} \otimes \ldots \otimes \pi_{s_{i_k}}(u_{i_k})^{\lambda(h_{i_k})}.$$

(See [27], p.120, (6.2.4)). Since $\lambda$ is regular, $s_{i_{\ell+1}} \cdots s_{i_k} \lambda(h_{i_\ell})$ is strictly positive for each $\ell = 1, \ldots, k$. Thus $\pi_{w_0}(|a_\lambda|)$ is a non-singular compact operator with spectrum $\{0\} \cup \{q^n\}_{m \in S} \cup \{1\}$ for some $S \subset \mathbb{N}$. Note that the eigenspace with respect to the eigenvalue 1 is the one-dimensional space $\mathbb{C} \xi_0 \otimes \ell(w_0)$. Hence we obtain

$$\|\pi_{w_0}(|a_\lambda|^m) - \pi_{w_0}(|a_\lambda|^n)\| \leq q^m \quad \text{if} \quad m \leq n.$$

It follows from Lemma 3.3 that $||a_\lambda|^n - |a_\lambda|^n\| \leq q^m$. In particular, $|a_\lambda|^n$ uniformly converges to an element $p_\lambda$ in $C(T \backslash G_q)$. Then $\pi_{w_0}(p_\lambda)$ is nothing but the one rank projection onto $\mathbb{C} \xi_0 \otimes \ell(w_0)$. Since $\pi_{w_0}$ is faithful, $p_\lambda$ does not depend on $\lambda$, and we denote it by $p_0$.

Moreover, the following equality shows the minimality of $p_0$ in $C(T \backslash G_q)$.

$$\pi_{w_0}(p_0C(T \backslash G_q)p_0) = \pi_{w_0}(p_0)\pi_{w_0}(C(T \backslash G_q))\pi_{w_0}(p_0) = C\pi_{w_0}(p_0).$$

□

Now we consider a commutative $C^*$-algebra $A := \pi_{w_0}(C^*(|a_\lambda| \mid \lambda \in P_{++}))$ that is contained in $K(H_{w_0})$. Since $A$ is non-degenerately acting on $H_{w_0}$, there exists a sequence of projections $q_\ell \in A$ for $\ell$ in some index set $J$ such that $q_0 = \pi_{w_0}(p_0)$, $\sum_{\ell \in J} q_\ell = 1$ (strongly), $\dim q_\ell H_{w_0} < \infty$ and $\pi_{w_0}(|a_\lambda|)q_\ell = \nu_{\lambda,\ell}q_\ell$ for some $\nu_{\lambda,\ell} > 0$. Using the faithfulness of $\pi_{w_0}$ on $C(T \backslash G_q)$, we take a family of orthogonal projections $\{r_\ell\}_{\ell \in J}$ from $C(T \backslash G_q)$ such that $\pi_{w_0}(r_\ell) = q_\ell$. Also we must have $|a_\lambda|r_\ell = \nu_{\lambda,\ell}r_\ell$.

Since $\pi_{w_0}$ is faithful, $r_\ell C(T \backslash G_q)r_\ell$ embeds into $q_\ell B(H_{w_0})q_\ell$, which shows the finite dimensionality of $r_\ell C(T \backslash G_q)r_\ell$ for all $\ell \in J$.

**Lemma 3.5.** The restriction of $\pi_{w_0}$ on $C(T \backslash G_q)$ is irreducible.

**Proof.** Since we know the minimality of $p_0$, it suffices to show that $\xi_0 := e_0 \otimes \ell(w_0)$ is cyclic for $\pi_{w_0}(C(T \backslash G_q))$. Take a vector $\xi \in H_{w_0}$ that is orthogonal to $\pi_{w_0}(C(T \backslash G_q))\xi_0$. Thus we have $\langle \xi, \pi_{w_0}(x)\xi_0 \rangle = 0$ for any $x \in C(T \backslash G_q)$. This implies that $\pi_{w_0}(p_0x)\xi = 0$ for any $x \in C(T \backslash G_q)$. □
Now recall that \( L^\infty(T \setminus G_q) \) is a type I factor. So, we have a faithful irreducible representation \( \psi: C(T \setminus G_q) \to B(\ell^2) \). Since each \( r_I C(T \setminus G_q)r_I \) is finite dimensional, we know that \( \text{Im} \psi \) has a non-zero intersection with \( K(\ell^2) \). Hence \( K(\ell^2) \subset \text{Im} \psi \). This enables us to take partial isometries \( v_i, i \in I_\ell \) in \( C(T \setminus G_q) \) such that \( v_i = r_i v_i'q_0 \) and \( r_i = \sum_{i \in I_\ell} v_i' q_0 v_i \) (finite sum). Thus we have \( q_0 \xi = \sum_i \pi_{w_0}(v_i' q_0 v_i^*) \xi = 0 \). This shows \( \xi = 0 \) because \( \sum_{i \in I_\ell} q_\ell = 1 \). \( \square \)

We will state Dijkhuizen–Stokman’s result for \( T \subset G_q \) \cite[Theorem 5.9]{DS}.

Theorem 3.6 (Dijkhuizen–Stokman). The following statements hold:

1. For any \( w \in W \), the restriction of \( \pi_w \) on \( C(T \setminus G_q) \) gives an irreducible representation;
2. Any irreducible representation of \( C(T \setminus G_q) \) is unitarily equivalent to the restriction of \( \pi_w \) with unique \( w \in W \).

Proof. (1) By the previous lemma, it suffices to show the statement when \( w \neq w_0 \). Let \( w' = w^{-1} w_0 \). Then \( \ell(w_0) = \ell(w) + \ell(w') \). Let \( L \) be a non-zero \( \pi_w(C(T \setminus G_q)) \)-invariant closed subspace. Since \( \pi_{w_0}(C(T \setminus G_q)) \subset \pi_w(C(T \setminus G_q)) \otimes \pi_w(C(G_q)) \), \( L \otimes H_{w'} \) is invariant for \( \pi_{w_0}(C(T \setminus G_q)) \). The irreducibility of \( \pi_{w_0}(C(T \setminus G_q)) \) shows that \( L \otimes H_{w'} = \pi_{w_0} = H_w \otimes H_{w'} \), and hence \( L = H_w \).

(2) Let \( \varphi \) be a pure state on \( C(T \setminus G_q) \) and \( \psi \) a pure state extension on \( C(G_q) \). Then by Theorem 2.5 (2) due to Soibel’man, there exists \( t \in T, w \in W \) and a unit vector \( \xi \in H_w \) such that \( \psi(x) = \langle \pi_{t,w}(x)\xi, \xi \rangle \) for all \( x \in C(G_q) \). By (1), the restriction of \( \pi_{t,w} \) on \( C(T \setminus G_q) \), which coincides with \( \pi_w \) on \( C(T \setminus G_q) \), is irreducible. Hence \( \xi \) is cyclic for \( \pi_w(C(T \setminus G_q)) \), that is, \( \varphi \) comes from \( \pi_w \).

Suppose that the restrictions of \( \pi_w \) and \( \pi_{w'} \) on \( C(T \setminus G_q) \) are unitarily equivalent. Then \( \ker \pi_w \cap C(T \setminus G_q) = \ker \pi_{w'} \cap C(T \setminus G_q) \). Thanks to Theorem 2.5 (1), it suffices to show that \( I_w \subset \ker \pi_{w'} \). Take any \( C_{\xi^w,\xi^w}^\lambda \) in \( I_w \) with \( \xi^w \in L(\lambda)_\mu \) and \( \mu \in \text{Wt}(\lambda) \). Then \( \pi_w(\langle C_{\xi^w,\xi^w}^\lambda \rangle) = 0 \). Since \( C_{\xi^w,\xi^w}^\lambda \in C(T \setminus G_q) \), we have \( \pi_w(\langle C_{\xi^w,\xi^w}^\lambda \rangle) = 0 \), that is, \( C_{\xi^w,\xi^w}^\lambda \in \ker \pi_{w'} \). Hence \( I_w \subset \ker \pi_{w'} \), and we are done.

Remark 3.7. In the above proof, we in fact have shown that if \( \pi_w \) and \( \pi_{w'} \) are unitarily equivalent as representations of \( C(T \setminus G_q)/T \), then \( w = w' \).

Remark 3.8. We can show \( \pi_w \) with \( w \neq w_0 \) has the non-trivial kernel in \( C(T \setminus G_q) \) as follows. Recall that for any \( \lambda \in P_+ \), the lowest weight of \( L(\lambda) \) is given by \( w_0 \lambda \). In particular, if \( \mu \in \text{Wt}(\lambda) \), then \( \mu \geq w_0 \lambda \). Take \( \lambda \in P_+ \) such that \( w_0 \lambda \neq w_0 \lambda \). Then \( \pi_w(\langle C_{w_0 \lambda, w_0 \lambda}^\lambda \rangle) = 0 \), which shows the non-triviality of \( I_w \cap C(T \setminus G_q) \). In particular, the restriction of \( \pi_{w_0} \) is conjugate to the embedding \( \nu: C(T \setminus G_q) \to L^\infty(T \setminus G_q) \cong B(\ell^2) \).

The previous result immediately implies the following.

Corollary 3.9. The counit is the unique character of \( C(T \setminus G_q) \).
For an action $\alpha$ of $G_q$ on a unital $C^*$-algebra (or von Neumann algebra) $A$, we denote by $\text{Aut}_{G_q}(A)$ the set of $G_q$-equivariant automorphisms on $A$, that is,

$$ \text{Aut}_{G_q}(A) := \{ \theta \in \text{Aut}(A) \mid \alpha \circ \theta = (\theta \otimes \text{id}) \circ \alpha \}. $$

**Theorem 3.10.** The embedding of $C(T\backslash G_q)$ into $C(G_q)$ is unique.

**Proof.** Suppose that $\psi : C(T\backslash G_q) \to C(G_q)$ is a $G_q$-equivariant embedding. By Corollary 3.9, we have $\varepsilon \circ \psi = \varepsilon$ on $C(T\backslash G_q)$. Then for $x \in C(T\backslash G_q)$,

$$ x = (\varepsilon \otimes \text{id})(\delta(x)) = (\varepsilon \circ \psi \otimes \text{id})(\delta(x)) = (\varepsilon \otimes \text{id})(\delta(\psi(x))) = \psi(x). $$

$\square$

From the previous result, we have the following.

**Corollary 3.11.** One has $\text{Aut}_{G_q}(C(T\backslash G_q)) = \{ \text{id} \}$.

This also implies $\text{Aut}_{G_q}(L^\infty(T\backslash G_q)) = \{ \text{id} \}$ because an equivariant map on $L^\infty(T\backslash G_q)$ preserves each finite dimensional spectral subspace and also $C(T\backslash G_q)$.

### 3.2. Some formulae concerning the Haar state.

We will close this section computing the Haar state $h$ on $C(G_q)$ in terms of a density operator. The results obtained in this subsection is not used in the subsequent sections. Recall that $|a_\rho| = |C^\rho_{\rho,\nu_0 \rho}|$ is a non-singular positive operator.

**Lemma 3.12.** For $t \in \mathbb{R}$, one has $\sigma^h_t = \text{Ad} |a_\rho|^{2it}$ on $L^\infty(G_q)$.

**Proof.** Put $\Lambda = \rho$ in (3.1). Then we have $|a_\rho|^2 C^\Lambda_{\xi_\mu,\xi_\nu} = q^{2(\mu + \nu \rho)} C^\Lambda_{\xi_\mu,\xi_\nu} |a_\rho|^2$, that is,

$$ (1 \otimes |a_\rho|^2) C^\Lambda = (F^\Lambda \otimes 1) C^\Lambda = (F^\Lambda \otimes |a_\rho|^2) $$

for all $\lambda \in P_+$. Since $C^\Lambda \in B(L(\Lambda)) \otimes C(G_q)$ is a unitary, we have

$$ (\text{id} \otimes |a_\rho|^{2it}) C^\Lambda = (F^\Lambda \otimes 1) C^\Lambda = (F^\Lambda \otimes |a_\rho|^{2it}) $$

for all $\lambda \in P_+$, $t \in \mathbb{R}$.

Thus $\sigma^h_t = \text{Ad} |a_\rho|^{2it}$.

$\square$

Let $\varphi$ be the restriction of $h$ on $L^\infty(T\backslash G_q)$. Let $E_\gamma : L^\infty(G_q) \to L^\infty(T\backslash G_q)$ be the conditional expectation defined by $E_\gamma(x) = \int_T \gamma_t(x) dt$, where $dt$ denotes the normalized Haar measure on $T$. Then $h \circ E_\gamma = h$, and we have $\sigma^h_t = \sigma^h_t|_{L^\infty(T\backslash G_q)} = \text{Ad} |a_\rho|^{2it}$ for $t \in \mathbb{R}$. Since $L^\infty(T\backslash G_q)$ is a type I factor, we obtain $\varphi = c \text{Tr} |a_\rho|^2$ for some $c > 0$, where $\text{Tr}$ denotes the canonical trace of $L^\infty(T\backslash G_q)$.

Recall the distinguished minimal projection $p_0$ in $L^\infty(T\backslash G_q)$. Using $|a_\rho|p_0 = p_0$, we have $c = \varphi(p_0)$. Hence we get the following result which generalizes [37, Lemma 2].

**Theorem 3.13.** One has $h(x) = h(p_0) \text{Tr} |a_\rho|^2 E_\gamma(x)$ for all $x \in L^\infty(G_q)$. In other words, with the identification $L^\infty(G_q) = L^\infty(T) \otimes L^\infty(T\backslash G_q)$, one has

$$ h = h(p_0) \int_T dt \otimes \text{Tr} |a_\rho|^2. $$
We will compute \( h(p_0) \). Let \( w_0 = w_{i_1} \cdots w_{i_k} \) be a minimal expression. We have known that \( \pi_{w_0} \colon C(T \setminus G_q) \to B(H_{w_0}) \) is conjugate to the embedding \( \iota \colon C(T \setminus G_q) \to L^\infty(T \setminus G_q) \). Let us identify \( L^\infty(T \setminus G_q) \) with \( B(H_{w_0}) \) in what follows. Let \( \lambda \in P_+ \). By (3.1), we have the following:

\[
h(|a_\lambda|^2) = h(p_0) \cdot \prod_{\alpha \in \Delta_+} (1 - q^{2(\alpha, \rho)})
\]

where each \( \text{Tr} \) denotes the canonical trace of \( B(\ell^2) \). Using (2.10) and \( q_j = q^{(\alpha_j, \alpha_j)/2} \) for \( j \in I \), we obtain the following for \( \ell = 1, \ldots, k \):

\[
\text{Tr} \left( \pi_i (u_{i_k})^{2s_{i_{\ell+1}} \cdots s_{i_k} (\lambda + \rho)(h_{i_1})} \cdots \text{Tr} \left( \pi_i (u_{i_k})^{2(\lambda + \rho)(h_{i_k})} \right) \right) = \left( 1 - q^{2(\lambda + \rho, s_{i_{\ell+1}} \cdots s_{i_k} \alpha_\ell)} \right)^{-1}.
\]

Since \( \Delta_+ = \{ s_{i_k} \cdots s_{i_{\ell+1}} \alpha_\ell \mid \ell = 1, \ldots, k \} \), we get

\[
h(|a_\lambda|^2) = h(p_0) \prod_{\alpha \in \Delta_+} (1 - q^{2(\lambda + \rho, \alpha)})^{-1}.
\]

(3.2)

Putting \( \lambda = 0 \) in the above, we obtain \( a_\lambda = 1 \) and the following result.

**Lemma 3.14.** One has the following equality:

\[
h(p_0) = \prod_{\alpha \in \Delta_+} (1 - q^{2(\alpha, \rho)}).
\]

For instance, if \( G_q = SU_q(n) \), we obtain \( h(p_0) = (q^2; q^2)_1 \cdots (q^2; q^2)_{n-1} \).

Now we can derive the well-known quantum Weyl dimension formula. From (2.11) and Lemma 2.4 we obtain

\[
h(|a_\lambda|^2) = h(C_{\lambda, w_0 \lambda} (C_{\lambda, w_0 \lambda}^*) = (\dim_q L(\lambda))^{-1} (F_{\lambda, w_0 \lambda, w_0 \lambda} = (\dim_q L(\lambda))^{-1} q^{-2(\lambda, \rho)}.
\]

(3.3)

Employing (3.2) and Lemma 3.14 we get the following.

**Proposition 3.15 (Quantum Weyl dimension formula).** For \( \lambda \in P_+ \), one has

\[
\dim_q L(\lambda) = q^{-2(\lambda, \rho)} \prod_{\alpha \in \Delta_+} \frac{(1 - q^{2(\lambda + \rho, \alpha)})}{(1 - q^{2(\rho, \alpha)})} = \prod_{\alpha \in \Delta_+} \frac{[(\lambda + \rho, \alpha)]_q}{[(\rho, \alpha)]_q},
\]

where \( [n]_q \) denotes the q-integer \( (q^n - q^{-n})/(q^{-1} - q) \) for \( n \in \mathbb{Z} \).

**Remark 3.16.** We can compute the value of \( h(p_0) \) by using the Weyl character formula and the fact that \( |a_\lambda|^{2n} \) converges to \( p_0 \) as \( n \to \infty \) in the norm topology.

Let \( \text{ch}(L(\Lambda)) \) be the character of \( L(\Lambda) \), that is,

\[
\text{ch} L(\Lambda) = \sum_{\mu \in Wt(\Lambda)} \dim L(\Lambda)_{\mu} e(\mu),
\]

(3.4)
where \( e(\cdot) \) denotes the formal exponential (see [23, \S 10.2, p.172]). By the Weyl character formula (see, for example, [24, Theorem 10.4, p.173] or [25, Proposition 12, p.203]), we obtain

\[
\text{ch} \, L(\Lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w(\Lambda + \rho) - \rho)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))}.
\]

If we put \( e(\mu) = q^{2(\mu, \rho)} \) in (3.4), then it follows from (2.7) that \( \text{ch} \, L(\Lambda) \) is equal to \( \dim_q L(\Lambda) \). Thus

\[
\dim_q L(\Lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} q^{2(w(\Lambda + \rho) - \rho, \rho)}}{\prod_{\alpha \in \Delta_+} (1 - q^{-2(\alpha, \rho)})}.
\]

Now we put \( \Lambda = n\rho \) in the equality above. Then

\[
\dim_q L(n\rho) = q^{-2(\rho, \rho)} \frac{\sum_{w \in W} (-1)^{\ell(w)} q^{2(n+1)(w\rho, \rho) - 2n\rho, \rho}}{\prod_{\alpha \in \Delta_+} (1 - q^{-2(\alpha, \rho)})}.
\]

Recall that \( w_0\rho \) is the lowest weight of \( L(\rho) \). In particular, \( w_0\rho \leq \rho \) for any \( w \in W \) and the equality holds if and only if \( w = w_0 \). Hence if \( w \neq w_0 \), then \( (w\rho, \rho) > (w_0\rho, \rho) = -(\rho, \rho) \), and \( q^{2(n+1)(w\rho, \rho)} q^{2n\rho, \rho} \to 0 \) as \( n \to \infty \). From \( \varphi(p_0) = \lim_n h(|a_{\rho}|^{2n}) = \lim_n h(|a_{\rho}|^2) \) and (3.3), we have

\[
h(p_0) = q^{2(\rho, \rho)} \lim_{n \to \infty} \frac{\prod_{\alpha \in \Delta_+} (1 - q^{-2(\alpha, \rho)})}{\sum_{w \in W} (-1)^{\ell(w)} q^{2(n+1)(w\rho, \rho) + 2n\rho, \rho}}
= q^{2(\rho, \rho)} \lim_{n \to \infty} \prod_{\alpha \in \Delta_+} (1 - q^{-2(\alpha, \rho)})
= q^{4(\rho, \rho)} (-1)^{\ell(w_0)} \prod_{\alpha \in \Delta_+} (1 - q^{-2(\alpha, \rho)}).
\]

Using the fact that \( \ell(w_0) \) equals \( |\Delta_+| \), the last term above is equal to

\[
q^{4(\rho, \rho)} \prod_{\alpha \in \Delta_+} (q^{-2(\alpha, \rho)} - 1) = q^{4(\rho, \rho)} \prod_{\alpha \in \Delta_+} q^{-2(\alpha, \rho)} \prod_{\alpha \in \Delta_+} (1 - q^{2(\alpha, \rho)})
= \prod_{\alpha \in \Delta_+} (1 - q^{2(\alpha, \rho)})
\]
since \( 2\rho = \sum_{\alpha \in \Delta_+} \alpha \). Hence we are done.

It is also possible to compute the value of the Haar state at each diagonal minimal projection. Let \( m \in \mathbb{Z}_+ \) and \( p_m : \ell^2(\mathbb{Z}_+) \to \mathbb{C} \) the minimal projection. Then for \( \ell = 1, \ldots, k \), we have

\[
\pi_{\ell} (u_{\ell, k})^{2m_{\ell+1,s_{\ell}k}} p_m = q^{2m_{\ell,s_{\ell}-s_{\ell+1,1}a_{\ell}}(\rho, s_{\ell}k \cdots s_{\ell+1,1}a_{\ell})} p_m.
\]

Therefore, we obtain the following result.

**Proposition 3.17.** For \( m_1, \ldots, m_k \in \mathbb{Z}_+ \), one has

\[
h(p_{m_1} \otimes \cdots \otimes p_{m_k}) = h(p_0) \prod_{\ell=1}^k q^{2m_{\ell}(\rho, s_{\ell}k \cdots s_{\ell+1,1}a_{\ell})}.
\]
4. Invariant cocycles and 2-cocycle deformations

In this section, we will introduce the notion of invariant cocycles evaluated in $L^\infty(T\backslash G_q)$ and determine them. As an application, we will describe the intrinsic group of a twisted quantum group $G_{q,q}$.

4.1. Invariant cocycles evaluated in $L^\infty(T\backslash G_q)$. Recall that the left $T$-action $\gamma$ is faithful and ergodic on $Z(L^\infty(G_q))$. For each fundamental weight $\omega_i \in P$, $i \in I$, we take a unitary $v_{\omega_i}$ in $Z(L^\infty(G_q))$ such that $\gamma_t(v_{\omega_i}) = (t, \omega_i)v_{\omega_i}$ for $t \in T$. Next, for $\lambda = \sum_{i=1}^n a_i\omega_i \in P$, $a_i \in \mathbb{Z}$, we set $v_\lambda := v_{\omega_1}^{a_1} \cdots v_{\omega_n}^{a_n}$. Then we have

$$\gamma_t(v_\lambda) = \langle t, \lambda \rangle v_\lambda, \quad v_\lambda v_\mu = v_{\lambda+\mu} \quad \text{for } t \in T, \ \lambda, \mu \in P.$$ \hfill (4.1)

Note that $v_\lambda$'s are generating $Z(L^\infty(G_q))$ as a von Neumann algebra, and $L^\infty(G_q) = Z(L^\infty(G_q)) \vee L^\infty(T\backslash G_q)$.

Now for each $\lambda \in P$, we introduce the unitary $w_\lambda$ defined by:

$$\delta(v_\lambda) = (v_\lambda \otimes 1)w_\lambda.$$ \hfill (4.2)

Since $\delta$ and $\gamma$ are commuting, $w_\lambda$ belongs to $L^\infty(T\backslash G_q) \otimes L^\infty(G_q)$. From the above equation, the cocycle relation $(w_\lambda \otimes 1)(\delta \otimes \text{id})(w_\lambda) = (\text{id} \otimes \delta)(w_\lambda)$ holds. Moreover, setting $\delta^{w_\lambda} := \text{Ad} w_\lambda \circ \delta$, we have $\delta^{w_\lambda} = \delta$ on $L^\infty(G_q)$ because $v_\lambda$ is a central unitary.

Let us denote by $Z^1_{\text{inv}}(\delta, L^\infty(T\backslash G_q))$ the collection of $L^\infty(T\backslash G_q)$-valued cocycles $w$ such that $\delta^w = \delta$ on $L^\infty(T\backslash G_q)$. In general, when $\alpha$ is an action of $G_q$ on a von Neumann algebra $N$, we denote by $Z^1_{\text{inv}}(\alpha, N)$ the set of $\alpha$-cocycles $w$ with $\alpha^w = \alpha$. We call such $w$ an invariant cocycle.

We know $w_\lambda$'s are invariant cocycles of the action of $\delta$ on $L^\infty(T\backslash G_q)$, but in fact, they are all.

**Theorem 4.1.** In the above setting, the following statements hold:

1. $Z^1_{\text{inv}}(\delta, L^\infty(T\backslash G_q)) = \{w_\lambda \mid \lambda \in P\}$;
2. The map $P \ni \lambda \mapsto w_\lambda \in Z^1_{\text{inv}}(\delta, L^\infty(T\backslash G_q))$ is a group isomorphism.

**Proof.** (1). Let $w \in Z^1_{\text{inv}}(\delta, L^\infty(T\backslash G_q))$. First, we will observe that the twisted right action $\delta^w$ is ergodic on $L^\infty(G_q)$. Let $x \in L^\infty(G_q)\delta^w$. Since $w \in L^\infty(T\backslash G_q) \otimes L^\infty(G_q)$, the actions $\gamma$ and $\delta^w$ are commuting. Thus we may and do assume that $\gamma_t(x) = (t, \mu)x$ for some $\mu \in P$ and all $t \in T$. Then $\gamma$ fixes $y := v_\mu^*x$, and $y$ belongs to $L^\infty(T\backslash G_q)$. Using $w^\delta(x)w^* = \delta^w(x) = x \otimes 1$ and $\delta^w = \delta$ on $L^\infty(T\backslash G_q)$, we have

$$ww_\mu w^\delta(y) = w(v_\mu^* \otimes 1)\delta(v_\mu) \cdot (y)w^* = (v_\mu^* \otimes 1)w\delta(x)w^* = y \otimes 1.$$ \hfill (4.3)

In particular, $y^*y$ and $yy^*$ are fixed by $\delta$ since $ww_\mu w^*$ commutes with $\delta(yy^*)$. By ergodicity of $\delta$ on $L^\infty(T\backslash G_q)$, we may and do assume that $y$ is a unitary.

Set the inner automorphism $\psi := \text{Ad} y^*$ on $L^\infty(T\backslash G_q)$. Then for $z \in L^\infty(T\backslash G_q)$,

$$\delta(\psi(z)) = \delta(y^*)\delta(z)\delta(y) = (y^* \otimes 1)ww_\mu w^* \cdot \delta(z) \cdot ww_\mu w^*(y \otimes 1) \quad \text{by (4.3)}$$

$$= (y^* \otimes 1)\delta(z)(y \otimes 1) = (\psi \otimes \text{id})(\delta(z)).$$
Namely, $\psi \in \text{Aut}_{G_q}(L^\infty(T\setminus G_q))$. However, we must have $\psi = \text{id}$ by Corollary 3.11. It turns out from Theorem 3.1 that $y \in \mathbb{C}$. From (4.3), we have $w_\mu = 1$. This shows $\mu = 0$. (See the proof of the second statement.) Hence $x$ is a scalar, and $\delta^w$ is ergodic on $L^\infty(G_q)$.

Second, we will use Connes’ $2 \times 2$ matrix trick. Let $N := M_2(\mathbb{C}) \otimes L^\infty(G_q)$ and $\{e_{ij}\}_{i,j=1}^2$ be a system of matrix units of $M_2(\mathbb{C})$. We set $\alpha := \text{id} \otimes \delta$ and $\overline{w} := e_{11} \otimes 1 + e_{22} \otimes w$. Then $\overline{w}$ is an $\alpha$-cocycle. We will show that the projections $p_1 := e_{11} \otimes 1$ and $p_2 := e_{22} \otimes 1$ are Murray–von Neumann equivalent in $N^\alpha$.

Consider the crossed product $N \rtimes_{\alpha} G_q$. Since $\alpha$ is a cocycle perturbation of $\text{id} \otimes \delta$, $N \rtimes_{\alpha} G_q$ is canonically isomorphic to $M_2(\mathbb{C}) \otimes L^\infty(G_q) \rtimes_{\delta} G_q$, and also to $M_2(\mathbb{C}) \otimes B(L^2(G_q))$. Thus $N \rtimes_{\alpha} G_q$ is the type I factor. The fixed point algebra $N^\alpha$ is a corner of $N \rtimes_{\alpha} G_q$, and $N^\alpha$ is a type I factor. Since $p_1 N^\alpha p_1 = L^\infty(G_q)^\delta = \mathbb{C}$ and $p_2 N^\alpha p_2 = L^\infty(G_q)^{\delta^w} = \mathbb{C}$, $p_1$ and $p_2$ are minimal projections in $N^\alpha$. Hence they are equivalent.

Let us take a unitary $v \in L^\infty(G_q)$ such that $e_{12} \otimes v \in N^\alpha$, that is, $\delta(v) = (v \otimes 1)w$. The ergodicity of $\delta$ shows that $v$ is the unique solution of this equation up to a scalar multiple. Indeed, if $v'$ is a (not necessarily unitary) another solution, then $\delta(v'v^*) = (v' \otimes 1)w \cdot w^*(v^* \otimes 1) = v'v^* \otimes 1$, and $v'v^* \in \mathbb{C}$. Then by the commutativity of $\gamma$ and $\delta$ and also the equality $(\gamma_t \otimes \text{id})(w) = w$, we can find a unique $\lambda \in P$ such that $v = v_\lambda v_1$ for some unitary $v_1 \in L^\infty(T\setminus G_q)$. However, the invariance $\delta^w = \delta$ on $L^\infty(T\setminus G_q)$ deduces $\text{Ad} v_1 \in \text{Aut}_{G_q}(L^\infty(T\setminus G_q))$. Hence $v_1$ is a scalar as before, and $w = w_\lambda$.

(2). Let $\lambda, \mu \in P$. Since $v_\lambda$ is central and $v_\lambda v_\mu = v_{\lambda+\mu}$, we have
\[
 w_\lambda w_\mu = (v_\lambda^* \otimes 1)\delta(v_\lambda) \cdot (v_\mu^* \otimes 1)\delta(v_\mu) = (v_\lambda^* \otimes 1)(v_\mu^* \otimes 1)\delta(v_\lambda) \cdot \delta(v_\mu) = w_{\lambda+\mu}.
\]

To show the injectivity, let $w_\lambda = 1$. Then $v_\lambda$ is fixed by $\delta$, and $v_\lambda$ is a scalar. Hence $\langle t, \lambda \rangle = 1$ for all $t \in T$, and $\lambda = 0$.

4.2. 2-cocycle deformations. As we have shown, the quantum flag manifold $L^\infty(T\setminus G_q)$ is a type I factor. So, we would like to find a unitary which implements the right action $\delta$ on $L^\infty(T\setminus G_q)$.

**Lemma 4.2.** There exists a unitary $U \in L^\infty(T\setminus G_q) \otimes L^\infty(G_q)$ such that $\delta(x) = U(x \otimes 1)U^*$ for all $x \in L^\infty(T\setminus G_q)$.

**Proof.** Let $p_0$ be the distinguished minimal projection of $C(T\setminus G_q)$ as before. Actually, $p_0$ is also contained in $C(G_q/T)$, and $\delta(p_0) \in C(T\setminus G_q) \otimes C(G_q/T)$. Recall that $L^\infty(T\setminus G_q) \otimes L^\infty(G_q/T)$ is a type I factor. We will show that $\delta(p_0)$ is an infinite projection.

Suppose that $\delta(p_0)$ is of finite rank. Then $\delta(p_0)(C(T\setminus G_q) \otimes C(G_q/T))\delta(p_0)$ is finite dimensional. Since the restrictions of $\pi_{w_0}$ on $C(T\setminus G_q)$ and $C(G_q/T)$ are irreducible, $e := (\pi_{w_0} \otimes \pi_{w_0})(\delta(p_0))$ is a finite rank projection of $B(H_{w_0}) \otimes B(H_{w_0})$. In particular, $e$ is a compact operator on $H_{w_0} \otimes H_{w_0}$. Then we obtain $(\text{id} \otimes \eta_{w_0})(e) = 0$ since $K(H_{w_0}) = K(H_{s_{i_1}}) \otimes \cdots \otimes K(H_{s_{i_k}})$ for a minimal expression $w_0 = s_{i_1} \cdots s_{i_k}$. However, since $\eta_{w_0} \otimes \pi_{w_0} = \varepsilon$, we obtain $\pi_{w_0}(p_0) = 0$. It follows that $p_0 = 0$, a contradiction. Hence $\delta(p_0)$ is an infinite projection of $L^\infty(T\setminus G_q) \otimes L^\infty(G_q/T)$. 

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Take a partial isometry $v \in L^\infty(T\setminus G_q) \otimes L^\infty(G_q/T)$ such that $v^*v = p_0 \otimes 1$ and $vv^* = \delta(p_0)$. Let $\{e_{ij}\}_{i,j=0}^\infty$ be a system of matrix units of $L^\infty(T\setminus G_q)$ such that $e_{00} = p_0$. Then a unitary $U := \sum_{i=0}^\infty \delta(e_{i0})v(e_{0i} \otimes 1)$ implements $\delta$.

**Remark 4.3.** By the same proof as the above, we can show that $(\pi_{w_0} \otimes \pi_w)\delta(p_0)$ is a projection of infinite rank on $H_{w_0} \otimes H_w$ for any $w \in W \neq \{e\}$.

The coassociativity of $\delta$, implies that $(\id \otimes \delta)(U)^*U_{12}U_{13}$ commutes with $x \otimes 1 \otimes 1$ for all $x \in L^\infty(T\setminus G_q)$. By factoriality of $L^\infty(T\setminus G_q)$, we obtain a unitary $\Omega \in L^\infty(G_q) \otimes L^\infty(G_q)$ such that

$$U_{12}U_{13} = (\id \otimes \delta)(U)(1 \otimes \Omega^*).$$

Then $\Omega$ satisfies the following 2-cocycle relation:

$$(\Omega \otimes 1)(\delta \otimes \id)(\Omega) = (1 \otimes \Omega)(\id \otimes \delta)(\Omega).$$

Denote by $\delta_\Omega$ the twisted coproduct $\Ad \Omega \circ \delta$. Then thanks to [9, Theorem 6.2], the pair $(L^\infty(G_q), \delta_\Omega)$ becomes a new, in general non-compact, locally compact quantum group in the sense of [28], which we will denote by $G_{q,\Omega}$. Set $L^\infty(G_{q,\Omega}) := L^\infty(G_q)$. Readers are referred to [10] for a general treatment of projective representations.

We will not study a concrete description of $G_{q,\Omega}$, but describe a group-like elements. A unitary $u \in L^\infty(G_{q,\Omega})$ is called a group-like element when $\delta_\Omega(u) = u \otimes u$. Denote by $\mathcal{U}(G_{q,\Omega})$ the collection of all group-like elements of $G_{q,\Omega}$, which is called the intrinsic group of $G_{q,\Omega}$.

For a unitary $u \in L^\infty(G_{q,\Omega})$, we will put

$$w^u := U(1 \otimes u)U^* \in L^\infty(T\setminus G_q) \otimes L^\infty(G_q). \quad (4.4)$$

**Lemma 4.4.** The unitary $w^u$ is a $\delta$-cocycle if and only if $u \in \mathcal{U}(G_{q,\Omega})$.

**Proof.** We have

$$(w^u \otimes 1)(\delta \otimes \id)(w^u) = U_{12}w_2U_{12}^* \cdot U_{12}(w^u)_{13}U_{13}^*$$

$$= U_{12}w_2U_{13}u_3U_{13}^*U_{12}^*$$

$$= (\id \otimes \delta)(U)\Omega_{23}(1 \otimes u \otimes u)\Omega_{23}(\id \otimes \delta)(U^*),$$

and

$$(\id \otimes \delta)(w^u) = (\id \otimes \delta)(U)(1 \otimes \delta(u))(\id \otimes \delta)(U^*).$$

Hence we are done. \qed

By Lemma 4.2 and (4.4), $w^u\delta(x) = \delta(x)w^u$ for all $x \in L^\infty(T\setminus G_q)$. Thus $w^u$ belongs to $Z^{\text{inv}}_\delta(L^\infty(T\setminus G_q))$ if $u \in \mathcal{U}(G_{q,\Omega})$.

**Theorem 4.5.** The map $\mathcal{U}(G_{q,\Omega}) \ni u \mapsto w^u \in Z^{\text{inv}}_\delta(L^\infty(T\setminus G_q))$ is a group isomorphism. In particular, $\mathcal{U}(G_{q,\Omega})$ is isomorphic to $P = \hat{T}$.

**Proof.** It is trivial that this map is a injective group homomorphism. To show the surjectivity, let $w \in Z^{\text{inv}}_\delta(L^\infty(T\setminus G_q))$. Then for $x \in L^\infty(T\setminus G_q)$, we have $w\delta(x)w^* = \delta(x)$, and so $U^*wU$ commutes with $x \otimes 1$. By factoriality of $L^\infty(T\setminus G_q)$,
there exists a unitary \( u \in L^\infty(G_q) \) such that \( w = U(1 \otimes u)U^* \). Since \( w \) is a \( \delta \)-cocycle, \( u \) is group-like with respect to \( \delta_{\Omega} \) by the previous lemma. The remaining statement follows from Theorem 4.11.

This result shows that we have a Hopf algebra embedding of \( L^\infty(T) \) into \( L^\infty(G_q,\Omega) \), that is, the \( n \)-dimensional torus \( T \) is a “quotient quantum group” of \( G_q,\Omega \).

When \( G_q = SU_q(2) \), it has been proved that \( G_q,\Omega \) is isomorphic to \( \tilde{E}_q(2) \), Woronowicz’s quantum \( E(2) \) group in [11 Theorem 4.5]. In [59 Theorem 2.1], Woronowicz has classified unitary representations of \( \tilde{E}_q(2) \). In particular, the intrinsic group of \( \tilde{E}_q(2) \) is generated by the canonical unitary representation \( v \) (see [59 p.254, (1)]), and is indeed isomorphic to \( \mathbb{Z} \cong \tilde{T} \).

5. Product type actions

In this section, we will study a product type action of \( G_q \). We will fix our notations.

Let \( v \) be a unitary representation of \( G_q \) on a Hilbert space \( H_v \) with \( 2 \leq \dim H_v \leq \infty \). Take a faithful invariant state \( \phi \in B(H_v)_* \), that is, \( \phi \) satisfies \( (\phi \otimes \text{id})(v(x \otimes 1)v^*) = \phi(x)1 \) for all \( x \in B(H_v) \). Note that \( \phi \) is never tracial since \( (\text{id} \otimes f_1)(C^\lambda) \neq 1 \) for any non-zero \( \lambda \in P_+ \).

Consider the infinite tensor product \( (M, \varphi) := \bigotimes_{m=1}^\infty (B(H_v), \phi)^{\prime\prime} \) that is a factor of type III. The Connes’ \( S \)-invariant is computed from the period of \( \varphi \). Let \( \alpha: M \to M \otimes L^\infty(G_q) \) be the product type action with respect to \( v \) [17 20 26]. Let \( E_{\alpha}: M \to M^\alpha \) be the conditional expectation defined by \( E_{\alpha} := (\text{id} \otimes h) \circ \alpha \). Then \( \varphi \circ E_{\alpha} = \varphi \).

5.1. Depth 2 inclusions. In what follows, we always assume that \( \alpha \) is faithful, that is, the subspace \( \alpha(M)(M \otimes \mathbb{C}) \) is dense in \( M \otimes L^\infty(G_q) \). This is the case when each irreducible representation of \( G_q \) is contained in a product unitary representations \( (v \otimes \overline{v})^\otimes m \) for some \( m \in \mathbb{N} \). Then as remarked in [20 p.509], each irreducible is contained in \( v^\otimes m \) for some \( m \in \mathbb{N} \). Therefore, the faithfulness of \( \alpha \) implies the generating property of the corresponding probability measure on \( \text{Irr}(G_q) \).

Thanks to [17 Corollary 3.9], the relative commutant \( \Omega := (M^\alpha)' \cap M \) is non-trivial. Moreover, we know by [17 Theorem 5.10], [20 Theorem A] and [48 Corollary 4.11] that there exists a \( G_q \)-equivariant isomorphism from \( L^\infty(T\backslash G_q) \) onto \( \Omega \). In particular, \( \Omega \) is a type I factor by Theorem 3.1. Then we have the following tensor product decomposition:

\[ M = \mathcal{R} \vee \Omega \cong \mathcal{R} \otimes \Omega, \tag{5.1} \]

where \( \mathcal{R} := \Omega' \cap M \). Note that the invariant state \( \varphi \) is of the form \( \varphi|_{\mathcal{R} \otimes \varphi}|_{\Omega} \). Indeed, the modular automorphism group \( \sigma^\varphi \) preserves \( M^\alpha \), and it does \( \Omega \). Hence, by Takesaki’s theorem [45 p.309], there exists a unique conditional expectation \( F: M \to \Omega \) with \( \varphi \circ F = \varphi \). Then \( F \) maps \( \mathcal{R} \) into the center \( Z(\Omega) = \mathbb{C} \), and

\[ \varphi(xy) = \varphi(F(x)y) = \varphi(F(x))\varphi(y) = \varphi(x)\varphi(y) \quad \text{for } x \in \mathcal{R}, \ y \in \Omega. \]
We will study the inclusion $\mathcal{M}^\alpha \subset \mathcal{R}$.

**Lemma 5.1.** The inclusion $\mathcal{M}^\alpha \subset \mathcal{R}$ is irreducible and of depth 2.

**Proof.** First we will show the irreducibility. Let $x \in (\mathcal{M}^\alpha)' \cap \mathcal{R}$. Then $x \in (\mathcal{M}^\alpha)' \cap \mathcal{M} = \mathcal{Q}$, but $x \in \mathcal{R} = \mathcal{Q}' \cap \mathcal{M}$. Hence $x \in Z(\mathcal{Q}) = \mathbb{C}$.

Next we let $\mathcal{M}^\alpha \subset \mathcal{R} \subset \mathcal{R}_1 \subset \mathcal{R}_2$ be the Jones tower. We will show that $(\mathcal{M}^\alpha)' \cap \mathcal{R}_2$ is a type I factor. Let $\mathcal{M}^\alpha \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2$ be the Jones tower. By (5.1), this is isomorphic to the following:

$$\mathcal{M}^\alpha \otimes \mathbb{C} \subset \mathcal{R} \otimes \mathcal{Q} \subset \mathcal{R}_1 \otimes \mathcal{Q}_1 \subset \mathcal{R}_2 \otimes \mathcal{Q}_2,$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are type I factors. Thus it suffices to show that $(\mathcal{M}^\alpha)' \cap \mathcal{M}_2$ is a type I factor.

**Claim.** The Jones tower $\mathcal{M}^\alpha \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2$ is isomorphic to $\mathcal{M}^\alpha \otimes \mathbb{C} \subset \alpha(\mathcal{M}) \subset \mathcal{M} \rtimes_\alpha \mathbb{G}_q \subset \mathcal{M} \otimes B(L^2(\mathbb{G}_q))$.

**Proof of Claim.** Since $\alpha$ is integrable, there exists a canonical surjection from $\mathcal{M} \rtimes_\alpha \mathbb{G}_q$ onto $\mathcal{M}_1$ (see, for example, [20, p.510] or [52, Theorem 5.3]). Recall that $\alpha$ is faithful. Thanks to [20, Corollary 1.5], we have an isomorphism from $\alpha(\mathcal{M})' \cap (\mathcal{M} \rtimes_\alpha \mathbb{G}_q)$ onto $\mathcal{M}^\prime \cap \mathcal{M}_1$. In particular, the canonical surjection from $\mathcal{M} \rtimes_\alpha \mathbb{G}_q$ onto $\mathcal{M}_1$ is an isomorphism. Hence the tower $\mathcal{M}^\alpha \subset \mathcal{M} \subset \mathcal{M}_1$ is isomorphic to $\mathcal{M}^\alpha \otimes \mathbb{C} \subset \alpha(\mathcal{M}) \subset \mathcal{M} \rtimes_\alpha \mathbb{G}_q$. The basic extension of $\alpha(\mathcal{M}) \subset \mathcal{M} \rtimes_\alpha \mathbb{G}_q$ is realized as $\mathcal{M} \otimes B(L^2(\mathbb{G}_q))$ through the computation of the modular conjugation of $\mathcal{M} \rtimes_\alpha \mathbb{G}_q$. See [52, Proof of Proposition 5.9] or [18, Lemma 5.7] for its proof.

Hence $(\mathcal{M}^\alpha)' \cap \mathcal{M}_2$ is isomorphic to $((\mathcal{M}^\alpha)' \cap \mathcal{M}) \otimes B(L^2(\mathbb{G}_q)) = \mathcal{Q} \otimes B(L^2(\mathbb{G}_q))$, which is a type I factor. $\square$

The restriction of $E_\alpha$ on $\mathcal{R}$ gives a conditional expectation from $\mathcal{R}$ onto $\mathcal{M}^\alpha$. Therefore, there exists a compact quantum group and its minimal action $\beta$ on $\mathcal{R}$ with the fixed point algebra $\mathcal{M}^\alpha$. (See [14], [31], [44]. It is worth mentioning that Longo’s sector-approach still works in our case.) We will show that the quantum group is nothing but the maximal torus $T$.

Note that $\mathcal{R}^\beta$ can be of type $\Pi_1$ though $\mathcal{R}$ is of type III. Thus $\beta$ is not dual in general (see [18, Proposition 5.2 (5)]). However, the action is semidual (see, for example, [34, Theorem 4.4], [52, Proposition 6.4] and [61, Theorem 2.2]). So let us take the tensor product by $B(\ell^2)$ as follows:

$$\overline{\mathcal{M}} := B(\ell^2) \otimes \mathcal{M}, \quad \overline{\alpha} := \text{id} \otimes \alpha.$$

Also set $\overline{\mathcal{R}} := B(\ell^2) \otimes \mathcal{R}$ and $\overline{\mathcal{Q}} := \mathbb{C} \otimes \mathcal{Q}$. Then we have

$$\overline{\mathcal{M}} = \overline{\mathcal{R}} \vee \overline{\mathcal{Q}} \cong \overline{\mathcal{R}} \otimes \overline{\mathcal{Q}}, \quad \overline{\mathcal{M}}^\prime = B(\ell^2) \otimes \mathcal{M}^\alpha, \quad (\overline{\mathcal{M}}^\prime)' \cap \overline{\mathcal{M}} = \overline{\mathcal{Q}}.$$

Let $\pi$ be a $G_q$-equivariant isomorphism from $L^\infty(T \setminus \mathbb{G}_q)$ onto $\overline{\mathcal{Q}}$, which is unique by Corollary 3.11.

Recall $w_\lambda \in Z^1_{\text{inv}}(\delta, L^\infty(T \setminus \mathbb{G}_q))$ introduced in (4.2). Set an $\overline{\alpha}$-cocycle $w_\lambda^\alpha$ defined by

$$w_\lambda^\alpha := (\pi \otimes \text{id})(w_\lambda). \quad (5.2)$$
Then \( w^o_{\lambda} \in Z_{\text{inv}}^1(\overline{\alpha}, \overline{Q}) \) for all \( \lambda \in P \).

Note that \( \overline{M} \rtimes_\theta G_q \) is an infinite factor. Since \( \overline{M}^\tau \) and \( \overline{M}^{\tau^\theta} \) contain \( B(\ell^2) \otimes \mathbb{C} \), they are also infinite factors. Using the \( 2 \times 2 \) matrix trick as before, we obtain a unitary \( u_\lambda \in \overline{M} \) for \( \lambda \in P \) such that \( \overline{\alpha}(u_\lambda) = (u_\lambda \otimes 1)w^o_{\lambda} \). For \( x \in \overline{M}^\tau \), we have

\[
\overline{\alpha}(u_\lambda xu_\lambda^*) = (u_\lambda \otimes 1)w^o_{\lambda}(x \otimes 1)(w^o_{\lambda})^*(u_\lambda^* \otimes 1) = u_\lambda xu_\lambda^* \otimes 1.
\]

Hence \( \theta_\lambda := \text{Ad} u_\lambda \) gives an endomorphism on \( \overline{M}^\tau \). We will show that \( \theta_\lambda \) is in fact an automorphism.

**Lemma 5.2.** For any \( \lambda \in P \), \( u_\lambda \) belongs to \( \overline{\mathcal{R}} \).

**Proof.** Let \( x \in \overline{Q} \) and \( y \in \overline{M}^\tau \). Then

\[
u^*x_\lambda y = u^*_\lambda x_\lambda y = u^*_\lambda \theta_\lambda(y) xu_\lambda = yu^*_\lambda xu_\lambda.
\]

Hence \( \rho_\lambda := \text{Ad} u^*_\lambda \) defines an endomorphism on \( \overline{Q} \). Moreover for \( x \in \overline{Q} \), we have

\[
\overline{\alpha}(\rho_\lambda(x)) = \overline{\alpha}(u^*_\lambda \overline{\alpha}(x)) = (w^o_{\lambda})^*(w^o_{\lambda} \otimes 1)\overline{\alpha}(x)(u_\lambda \otimes 1)w^o_{\lambda} = (w^o_{\lambda})^*(\rho_\lambda \otimes \text{id})(\overline{\alpha}(x))w^o_{\lambda}.
\]

Thus

\[
\overline{\alpha}(\rho_\lambda(x)) = (w^o_{\lambda})^* \overline{\alpha}(\rho_\lambda(x))(w^o_{\lambda})^* \text{ because } w^o_{\lambda} \in Z_{\text{inv}}^1(\overline{\alpha}, \overline{Q})
\]

\[
= w^o_{\lambda} \cdot (w^o_{\lambda})^*(\rho_\lambda \otimes \text{id})(\overline{\alpha}(x))w^o_{\lambda} \cdot (w^o_{\lambda})^*
\]

\[
= (\rho_\lambda \otimes \text{id})(\overline{\alpha}(x)).
\]

Namely, \( \rho_\lambda \) is a \( G_q \)-equivariant embedding of \( \overline{Q} \) into itself. The injectivity of \( \rho_\lambda \) implies that spectral multiplicities of \( \rho_\lambda(\overline{Q}) \) and \( \overline{Q} \) must coincide, and \( \rho_\lambda(\overline{Q}) = \overline{Q} \). Hence \( \rho_\lambda \) is a \( G_q \)-equivariant automorphism on \( \overline{Q} \cong L^\infty(T \setminus G_q) \). By Corollary 3.11, we obtain \( \rho_\lambda = \text{id} \), that is, \( u_\lambda \in \overline{Q} \cap \overline{M} = \overline{\mathcal{R}} \). \( \square \)

Let \( \lambda, \mu \in P \). Since \( u_\mu \) belongs to \( \overline{\mathcal{R}} \), we see \( w^o_{\lambda}(u_\mu \otimes 1) = (u_\mu \otimes 1)w^o_{\lambda} \), and

\[
\overline{\alpha}(u_\lambda u_\mu) = (u_\lambda \otimes 1)w^o_{\lambda}(u_\mu \otimes 1)w^o_{\mu} = (u_\lambda u_\mu \otimes 1)w^o_{\lambda}w^o_{\mu} = (u_\lambda u_\mu \otimes 1)w^o_{\lambda+\mu} = (u_\lambda u_\mu u^*_\lambda+\mu \otimes 1)\overline{\alpha}(u_{\lambda+\mu}).
\]

It follows that \( c_{\lambda, \mu} := u_\lambda u_\mu u^*_\lambda+\mu \) is contained in \( \overline{M}^\tau \), and

\[
\theta_\lambda \circ \theta_\mu = \text{Ad} c_{\lambda, \mu} \circ \theta_\lambda + \theta_\mu \text{ for all } \lambda, \mu \in P.
\]

(5.3)

We will show that \((\theta, c)\) is a cocycle action of \( P \) on \( \overline{M}^\tau \) as below. (See 36 for the definition of a cocycle action.) If we put \( \mu = -\lambda \), we have \( u_{\lambda+\mu} = u_0 = 1 \), and \( \theta_\lambda \circ \theta_{-\lambda} = \text{Ad} c_{\lambda, -\lambda} \). This shows the surjectivity of \( \theta_\lambda \), that is, \( \theta_\lambda \in \text{Aut}(\overline{M}^\tau) \).

The 2-cocycle identity of \( c \) is verified as follows: for \( \lambda, \mu, \nu \in P \),

\[
c_{\lambda, \mu}c_{\lambda+\mu, \nu} = u_\lambda u_\mu u^*_\lambda+\mu \cdot u_{\lambda+\mu} u_{\nu} u^*_\lambda+\mu+\nu = u_{\lambda+\mu} u_{\nu} u^*_\lambda+\mu+\nu,
\]

\[24\]
and
\[
\theta_{\lambda}(c_{\mu,\nu})u_{\lambda,\mu+\nu} = u_{\lambda} \cdot u_{\mu}u_{\nu}u_{\mu+\nu}^* \cdot u_{\lambda}^*u_{\mu+\nu}u_{\lambda+\mu+\nu}^* \cdot u_{\lambda}^*u_{\mu+\nu}u_{\lambda+\mu+\nu}^* = u_{\lambda}u_{\mu}u_{\nu}u_{\lambda+\mu+\nu}^*.
\]

From (5.3), it turns out that \((\theta, c)\) gives a cocycle action on an infinite factor \(\overline{M}\). Then \(c\) is in fact a coboundary by [13] Proposition 2.1.3. Take unitaries \(u_\lambda\) in \(\overline{M}\) for \(\lambda \in P\) so that \(u_\lambda^*\theta_\lambda(u_\lambda^*)c_{\lambda,\mu}(u_{\lambda+\mu}^*)^* = 1\) for \(\lambda, \mu \in P\). By replacing \(u_\lambda\) with \(u_\lambda'\) if necessary, we may and do assume that our \(u_\lambda\)'s are satisfying
\[
u_\lambda \in \overline{\mathcal{R}}, \quad u_\lambda u_\mu = u_{\lambda+\mu}, \quad \overline{\alpha}(u_\lambda) = (u_\lambda \otimes 1)u_\lambda^* \quad \text{for all } \lambda, \mu \in P. \quad (5.4)
\]

Then we have an outer action \(\theta\) of \(P\) on \(\overline{M}\). Indeed, if for some \(\lambda \in P\), \(a \in \overline{\mathcal{M}}\) satisfies \(ax = \theta_\lambda(x)a\) for all \(x \in \overline{\mathcal{M}}\), then \(u_\lambda^*a \in (\overline{\mathcal{M}}')' \cap \overline{\mathcal{R}} = \mathbb{C}\). This, however, implies that \(u_\lambda \in \overline{\mathcal{M}}\) and \(w_\lambda^* = 1\), that is, \(\lambda = 0\). We will prove that \(\overline{\mathcal{R}}\) is actually generated by \(\overline{\mathcal{M}}\) and \(\{u_\lambda\}_{\lambda \in P}\) by sector technique developed in [10].

Since the inclusion \(\mathcal{N} := \overline{\mathcal{M}} \subset \overline{\mathcal{R}}\) comes from a compact quantum group action, \(\overline{\mathcal{R}}\) is the crossed product of \(\mathcal{N}\) by the dual discrete quantum group action. We would like to determine this action. For the sake of this, we should study the sector \([\gamma_{\overline{\mathcal{R}}}[\mathcal{N}]\) in Sect(\(\mathcal{N}\)), where \(\gamma_{\overline{\mathcal{R}}}[\mathcal{N}]\) denotes the canonical endomorphism from \(\overline{\mathcal{R}}\) into \(\mathcal{N}\). Since there exists a conditional expectation from \(\overline{\mathcal{M}}\) onto \(\overline{\mathcal{R}}\), we have a canonical embedding \(\gamma\mathcal{L}^2(\mathcal{R}) \subset \gamma\mathcal{L}^2(\mathcal{M})\). Hence \([\gamma_{\overline{\mathcal{R}}}[\mathcal{N}]\) is contained in \([\gamma_{\overline{\mathcal{M}}}[\mathcal{N}]\). So let us study \(\gamma\mathcal{L}^2(\mathcal{M})\) first. The author thanks Izumi for suggesting this method.

Lemma 5.3. For any \(\lambda \in P_+\), there exists a Hilbert space \(\mathcal{H}_\lambda\) with \(s(\mathcal{H}_\lambda) = 1\) in \(\overline{\mathcal{M}}\) which admits an isometric \(G_q\)-isomorphism from \(L(\lambda)\) onto \(\mathcal{H}_\lambda\).

Proof. Recall that \(\mathcal{M}^\alpha\) is not a type I factor. Otherwise, it follows that \(\mathcal{M} = \mathcal{M}^\alpha \lor \mathcal{Q}\), and \(\mathcal{M}\) would be of type I. Thus we can take a von Neumann algebra embedding \(\psi\) of \(B(L(\lambda))\) into \(\mathbb{C}1_{\mathcal{F}} \otimes \mathcal{M}^\alpha \subset \overline{\mathcal{M}}\). Let \(w := (\psi \otimes \text{id})(C^\lambda)\). Then \(w\) is an \(\overline{\mathcal{R}}\)-cocycle, and \(B(\mathcal{F}^2) \otimes \mathbb{C}\) is contained in the fixed point algebra of \(\overline{\mathcal{M}}\) by \(\overline{\alpha}\). Hence we can employ the \(2 \times 2\)-matrix trick as usual, we obtain a unitary \(u \in \overline{\mathcal{M}}\) such that \(\overline{\alpha}(u) = (u \otimes 1)w\).

Let \(\{e_{ij}\}_{i,j \in I}\) be a system of matrix units of \(\text{Im } \psi\) such that each \(e_{ii}\) is minimal. Fix an element \(i_0\) in \(I\). Since \(1_{\mathcal{F}} \otimes e_{i_0i_0}\) is infinite projection, there exists an isometry in \(V_{i_0} \in B(\mathcal{F}^2) \otimes \mathcal{M}^\alpha\) such that \(V_{i_0}V_{i_0}^* = 1_{\mathcal{F}} \otimes e_{i_0i_0}\). For \(i \in I\), we set \(V_i := (1 \otimes e_{i_0})V_{i_0}\). Then it turns out that \(V_i^*V_j = \delta_{ij}1\) and \(V_iV_j^* = 1 \otimes e_{ij}\) for \(i, j \in I\). We let \(W_i := uV_i\). Then
\[
\overline{\alpha}(W_i) = \overline{\alpha}(u)\overline{\alpha}(V_i) = (u \otimes 1)w(V_i \otimes 1)
\]
\[
= \sum_{j \in I}(W_j \otimes 1)(V_j^* \otimes 1)w(V_i \otimes 1).
\]

Therefore, the statement follows by setting \(\mathcal{H}_\lambda := \text{span}\{W_i \mid i \in I\}\). \(\square\)
For $\lambda \in P_+$, let $T_\lambda$ be an isometric $G_q$-isomorphism from $L(\lambda)$ onto a Hilbert space $H_\lambda$ in $\mathcal{M}$. We let $V^\lambda_{\mu_i} := T_\lambda(\xi_{\mu_i})$ for $\mu \in Wt(\lambda)$ and $i \in I^\lambda_\mu$. Then we obtain

$$\pi(V^\lambda_{\mu_i}) = \sum_{\nu \in Wt(\lambda), j \in I^\lambda_\nu} V^\lambda_{\nu_j} \otimes C^\lambda_{\nu_j,\mu_i}.$$

For $\lambda, \Lambda \in P_+$, $\mu \in Wt(\lambda)$, $i \in I^\lambda_\mu$, $\nu \in Wt(\Lambda)$ and $j \in I^\lambda_\nu$, we have

$$E_{\pi}(V^\lambda_{\mu_i}(V^\Lambda_{\nu_j})^*) = (\text{id} \otimes h)(\overline{\pi}(V^\lambda_{\mu_i}(V^\Lambda_{\nu_j})^*))$$

$$= \sum_{\eta, \zeta, k, \ell} V^\lambda_{\eta_k}(V^\Lambda_{\zeta_l})^* \cdot h(C^\lambda_{\eta_k,\mu_i}(C^\Lambda_{\zeta_l,\nu_j})^*)$$

$$= \delta_{\lambda,\Lambda} \delta_{\mu,\nu} \delta_{i,j}(\dim_q L(\lambda))^{-1} F^\lambda_{\mu_i,\mu} \sum_{\eta, \zeta, k, \ell} \delta_{\eta,\zeta} \delta_{k,\ell} V^\lambda_{\eta_k}(V^\Lambda_{\zeta_l})^*$$

$$= \delta_{\lambda,\Lambda} \delta_{\mu,\nu} \delta_{i,j}(\dim_q L(\lambda))^{-1} F^\lambda_{\mu_i,\mu} \sum_{\eta} V^\lambda_{\eta_k}(V^\Lambda_{\zeta_l})^*$$

$$= \delta_{\lambda,\Lambda} \delta_{\mu,\nu} \delta_{i,j}(\dim_q L(\lambda))^{-1} \sum_{\eta, \zeta} V^\lambda_{\eta_k}(V^\Lambda_{\zeta_l})^*$$

$$= \delta_{\lambda,\Lambda} \delta_{\mu,\nu} \delta_{i,j}(\dim_q L(\lambda))^{-1} q^{2(\mu,\nu)}$$

by Lemma 2.4.

So if we put $W^\lambda_{\mu_i} := (\dim_q L(\lambda))^{1/2} q^{-(\mu,\nu)} V^\lambda_{\mu_i}$, then $\{W^\lambda_{\mu_i}\}_{\mu, i}$ is a linear base of $H_\lambda$ such that

$$E_{\pi}(W^\lambda_{\mu_i}(W^\Lambda_{\nu_j})^*) = \delta_{\lambda,\Lambda} \delta_{\mu,\nu} \delta_{i,j}. \quad (5.5)$$

Now we let

$$\sigma_\lambda(x) := \sum_{\mu \in Wt(\lambda), i \in I^\lambda_\mu} V^\lambda_{\mu_i} x(V^\Lambda_{\nu_j})^* \quad \text{for } x \in \mathcal{N}.$$

Then $\sigma_\lambda$ is an endomorphism on $\mathcal{N}$. We will determine the intertwiner space $(\sigma_\lambda, \sigma_\lambda)$. Recall $v_\mu \in Z(L^\infty(G_q))$ and $w_\mu \in Z^\text{inv}_{\text{inv}}(\delta, L^\infty(T \setminus G_q))$ introduced in (4.1) and (4.2). Then we set

$$a^\lambda_{\mu_i} := \sum_{\nu \in Wt(\lambda), j \in I^\lambda_\nu} V^\lambda_{\nu_j} \pi((C^\lambda_{\mu_i,\nu_j})^* v_\mu) u^*_\mu$$

for $\lambda \in P_+$, $\mu \in Wt(\lambda)$, $i \in I^\lambda_\mu$,

where we recall that $u_\mu$ is satisfying (5.4). Note that $(C^\lambda_{\mu_i,\nu_j})^* v_\mu$ is contained in $L^\infty(T \setminus G_q)$, and $\pi((C^\lambda_{\mu_i,\nu_j})^* v_\mu)$ is well-defined.

**Lemma 5.4.** Let $\lambda \in P_+$. Then the following statements hold:

1. For all $\mu \in Wt(\lambda)$, $\{a^\lambda_{\mu_i}\}_{i \in I^\lambda_\mu}$ is an orthonormal base of $(\theta_\mu, \sigma_\lambda)$;
2. $(\sigma_\lambda, \sigma_\lambda)$ is linearly spanned by $a^\lambda_{\mu_i} (a^\lambda_{\mu_j})^*$ for $\mu \in Wt(\lambda)$ and $i, j \in I^\lambda_\mu$. 

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Proof. (1). We will check that \( a^\lambda_{\mu_i} \) is contained in \( \overline{M} \). Indeed,

\[
\overline{\alpha}(a^\lambda_{\mu_i}) = \sum_{\nu, j} \overline{\alpha}(V^\lambda_{\nu_j} \cdot (\pi \otimes \text{id}) (\delta((C^\lambda_{\mu_i, \nu_j})^* v_{\mu_j})) \cdot \overline{\alpha}(u^*_{\mu})
\]

\[
= \sum_{\nu, \eta, \zeta, j, k, \ell} (V^\lambda_{\eta_k} \otimes C^\lambda_{\eta_k, \nu_j}) \cdot (\pi \otimes \text{id}) (((C^\lambda_{\mu_i, \zeta})^* \otimes (C^\lambda_{\zeta, \nu_j})^*) (v_{\mu} \otimes 1) w_{\mu})
\]

\[
= \sum_{\nu, \eta, \zeta, j, k, \ell} \delta_{\eta, \zeta} \delta_{k, \ell} (V^\lambda_{\eta_k} \otimes 1) \cdot (\pi \otimes \text{id}) (((C^\lambda_{\mu_i, \zeta})^* \otimes 1) (v_{\mu} \otimes 1)) \cdot (u^*_{\mu} \otimes 1)
\]

\[
= a^\lambda_{\mu_i} \otimes 1.
\]

Next we have the following for \( x \in \overline{M} \):

\[
a^\lambda_{\mu_i} \theta_{\mu_j} (x) = \sum_{\nu, j} V^\lambda_{\nu_j} \pi((C^\lambda_{\mu_i, \nu_j})^* v_{\mu_j}) u^*_{\mu} \theta_{\mu}(x)
\]

\[
= \sum_{\nu, j} V^\lambda_{\nu_j} \pi((C^\lambda_{\mu_i, \nu_j})^* v_{\mu}) x u^*_{\mu}
\]

\[
= \sum_{\nu, j} V^\lambda_{\nu_j} x \pi((C^\lambda_{\mu_i, \nu_j})^* v_{\mu}) u^*_{\mu} = \sigma_{\lambda}(x) a^\lambda_{\mu_i}.
\]

Thus \( a^\lambda_{\mu_i} \in (\theta_{\mu}, \sigma_{\lambda}) \). We have

\[
(a^\lambda_{\mu_i})^* a^\lambda_{\nu_j} = \sum_{\eta, \zeta, k, \ell} u_{\mu} \pi(v^*_{\mu} C^\lambda_{\mu_i, \eta_k}) (V^\lambda_{\eta_k})^* \cdot V^\lambda_{\zeta, \nu_j} \pi((C^\lambda_{\nu_j, \zeta})^* v_{\nu}) u^*_{\nu}
\]

\[
= \sum_{\eta, \zeta} u_{\mu} \pi(v^*_{\mu} C^\lambda_{\mu_i, \eta_k} (C^\lambda_{\nu_j, \eta_k})^* v_{\nu}) u^*_{\nu}
\]

\[
= \delta_{\mu, \nu} \delta_{\nu_j, \nu},
\]

and

\[
\sum_{\mu, i} a^\lambda_{\mu_i} (a^\lambda_{\mu_i})^* = \sum_{\mu, \eta, \zeta, i, k, \ell} V^\lambda_{\eta_k} \pi((C^\lambda_{\mu_i, \eta_k})^* v_{\mu}) u^*_{\mu} \cdot \mu \pi(v^*_{\mu} C^\lambda_{\mu_i, \zeta}) (V^\lambda_{\zeta, \nu_j})^* u_{\nu}
\]

\[
= \sum_{\eta, \zeta, k, \ell} \delta_{\eta, \zeta} \delta_{k, \ell} V^\lambda_{\eta_k} (V^\lambda_{\zeta, \nu_j})^* = 1.
\]

Hence for \( x \in \overline{M} \), we obtain

\[
\sigma_{\lambda}(x) = \sum_{\mu, i} a^\lambda_{\mu_i} \theta_{\mu}(x) (a^\lambda_{\mu_i})^*,
\]

and we are done. \( \square \)

By the previous lemma, we get the following equality in Sect(\( N \)):

\[
[\sigma_{\lambda}] = \bigoplus_{\mu \in \text{Wt}(\lambda)} \dim L(\lambda)_\mu [\theta_{\mu}].
\] (5.6)
Lemma 5.5. In $\text{Sect}(\mathcal{N})$, one has

$$[\gamma_{\mathcal{N}}^{\mathcal{M}}]_{\mathcal{N}} = \bigoplus_{\lambda \in P_{+}} \dim L(\lambda)[\sigma_{\lambda}].$$

Proof. We will decompose the $\mathcal{N}$-$\mathcal{N}$ bimodule $\mathcal{N} L^{2}(\mathcal{M})_{\mathcal{N}}$ as follows. First we observe that the linear span of $\mathcal{H}^{*}_{\lambda} \mathcal{N}$, $\lambda \in P_{+}$ is weakly dense in $\mathcal{M}$. Indeed, for any $\lambda \in P$ and any equivariant map $T: L(\lambda) \to \mathcal{M}$, it turns out that $a := \sum_{\mu,i} V_{\mu i}^{\lambda} T(\xi_{\mu i})^{*}$ belongs to $\mathcal{M}^{*}$. It follows that $T(\xi_{\mu i})^{*} (V_{\mu i}^{\lambda})^{*} a \in \mathcal{H}^{*}_{\lambda} \mathcal{N}$. Since the linear span of $T(\xi_{\mu i})^{*}$'s for $T$ and $\lambda, \mu, i$ is weakly dense in $\mathcal{M}$, we are done.

Next recall the $\alpha$-invariant state $\varphi$ on $\mathcal{M}$. Take a faithful normal state $\psi$ on $\mathcal{B}(\ell^{2})$ and put $\varphi := \psi \otimes \varphi$. Then $\varphi$ is an $\alpha$-invariant state on $\mathcal{M}$. For $\lambda \in P_{+}$, we let $e_{\lambda}: L^{2}(\mathcal{M}) \to \mathcal{N}_{\varphi}$ be the Jones projection. Then from (5.5), $z_{\lambda} := \sum_{\mu,i} (W_{\mu i}^{\lambda})^{*} e_{\lambda} W_{\mu i}^{\lambda}$ is a projection onto the subspace $X_{\lambda} := \mathcal{H}^{*}_{\lambda} \mathcal{N}_{\varphi}$. Since each $(W_{\mu i}^{\lambda})^{*} e_{\lambda} W_{\mu i}^{\lambda}$ belongs to $\mathcal{N}' \cap \mathcal{J}_{\mathcal{N}} \mathcal{N}'$, the subspace $(W_{\mu i}^{\lambda})^{*} \mathcal{N}_{\varphi}$ is an $\mathcal{N}$-$\mathcal{N}$-bimodule. Thus we have the following decomposition as $\mathcal{N}$-$\mathcal{N}$-bimodules:

$$\mathcal{N} L^{2}(\mathcal{M})_{\mathcal{N}} = \bigoplus_{\lambda \in P_{+}} \bigoplus_{\mu \in W(\lambda), i \in I_{\mu}} (W_{\mu i}^{\lambda})^{*} \mathcal{N}_{\varphi}. \quad (5.7)$$

Let us consider the map $\mathcal{N}_{\varphi} \ni x_{\varphi} \mapsto (W_{\mu i}^{\lambda})^{*} x_{\varphi}$. Again by (5.5), it turns out that this map extends to the unitary map $U$ from $\mathcal{N}_{\varphi}$ onto $(W_{\mu i}^{\lambda})^{*} \mathcal{N}_{\varphi}$. Then for $a, b \in \mathcal{N}$ and $\xi \in \mathcal{N}_{\varphi}$, we have

$$U(\sigma_{\lambda}(a) \xi b) = (W_{\mu i}^{\lambda})^{*} \sigma_{\lambda}(a) \xi b = (W_{\mu i}^{\lambda})^{*} \sigma_{\lambda}(a) \xi b = a((W_{\mu i}^{\lambda})^{*} \xi) b = a(U \xi) b.$$ 

Hence as $\mathcal{N}$-$\mathcal{N}$-bimodules, $(W_{\mu i}^{\lambda})^{*} \mathcal{N}_{\varphi}$ and $\mathcal{N}_{\sigma_{\lambda}} L^{2}(\mathcal{N})_{\mathcal{N}}$ are isomorphic. Then the statement follows from (5.7). \qed

By (5.6) and the previous lemma, we obtain

$$[\gamma_{\mathcal{N}}^{\mathcal{M}}]_{\mathcal{N}} = \bigoplus_{\mu \in P} \infty[\theta_{\mu}].$$

Since $[\gamma_{\mathcal{N}}^{\mathcal{M}}]_{\mathcal{N}}$ is contained in $[\gamma_{\mathcal{N}}^{\mathcal{M}}]_{\mathcal{N}}$, $[\gamma_{\mathcal{N}}^{\mathcal{M}}]_{\mathcal{N}}$ is a direct sum of multiples of $[\theta_{\mu}]$'s. Now we know $\mathcal{N} \subset \mathcal{R}$ comes from a minimal action of a compact quantum group. Since every $\theta_{\mu}$ is an automorphism, each irreducible representation of the compact quantum group is one-dimensional. Thanks to [19, p.49] or [49, Lemma 3.5], we have

$$[\gamma_{\mathcal{N}}^{\mathcal{M}}]_{\mathcal{N}} = \bigoplus_{\mu \in S} \{\theta_{\mu}\} \text{ for some } S \subset P.$$
However, each $u_\mu$ is actually an element of $\overline{\mathcal{R}}$ which implements $\theta_\mu$, $[\theta_\mu]$ must be contained in $[\gamma_N^\mathcal{R}]$. Thus we obtain $S = P$, that is,

$$[\gamma_N^\mathcal{R}] = \bigoplus_{\mu \in P} [\theta_\mu].$$

Then by \cite{[19], Theorem 3.9}, it turns out that $\overline{\mathcal{R}}$ is generated by $\mathcal{N} = \overline{\mathcal{M}}^\mathcal{R}$ and $u_\mu$, $\mu \in P$. For $\mu \in P$, $E_\mathcal{N}(u_\mu)$ is an element in $(\text{id}, \theta_\mu)$ The outerness of $\theta$ implies that $E_\mathcal{N}(u_\mu) = 0$ for $\mu \neq 0$. Hence we obtain the following result.

**Theorem 5.6.** The inclusion $\overline{\mathcal{M}}^\mathcal{R} \subset \overline{\mathcal{R}}$ is isomorphic to $\overline{\mathcal{M}}^\mathcal{R} \rtimes_\theta \hat{T}$, where $\hat{T} = \mathcal{P}$ as usual.

**Remark 5.7.** Recall the unitary $U$ introduced in Lemma \ref{lem:42}. Let $\Gamma(x) := (\pi \otimes \text{id})(U^*)\alpha(x)(\pi \otimes \text{id})(U)$ for $x \in \mathcal{M}$. Then $(\Gamma \otimes \text{id}) \circ \Gamma = (\text{id} \otimes \delta_\mathcal{N}) \circ \Gamma$, that is, $\Gamma$ is an action of $G_{\mathcal{Q},\Omega}$ on $\mathcal{R}$. However, $\Gamma$ is not faithful. Indeed, $\mathcal{R}\Gamma = \mathcal{M}^\mathcal{R}$ and $\Gamma(u_\lambda) = u_\lambda \otimes \nu_\lambda$ for $\lambda \in \mathcal{P}$, where $\nu_\lambda$ is a group-like unitary of $G_{\mathcal{Q},\Omega}$ such that $U^*\nu_\lambda U = 1 \otimes \nu_\lambda$. Hence $\Gamma$ is nothing but the dual action of $\theta$.

### 5.2. Induced actions.

Let us introduce the action $\beta$ on $\overline{\mathcal{R}}$, that is,

$$\beta_t(u_\mu) = (t, \mu)u_\mu \quad \text{for all } t \in T, \mu \in P.$$ 

By definition, $\beta_t = \hat{\theta}_{t^{-1}}$, where $\hat{\theta}$ denotes the dual action of $\theta$. Then $\beta$ extends to $\overline{\mathcal{M}}$ by putting $\beta = \text{id}$ on $\overline{\mathcal{Q}}$. Then $\overline{\varphi} \circ \beta_t = \overline{\varphi}$ for all $t \in T$ since $E_\mathcal{N}(u_\mu) = 0$ if $\mu \neq 0$. We will show that $W^*(u_\mu \mid \mu \in \mathcal{P}) \vee \mathcal{Q}$ is naturally isomorphic to $L^\infty(G_q)$.

**Lemma 5.8.** There exists a von Neumann algebra isomorphism $\pi: L^\infty(G_q) \to W^*(u_\mu \mid \mu \in \mathcal{P}) \vee \mathcal{Q}$ such that

- $\overline{\varphi} \circ \pi = (\pi \otimes \text{id}) \circ \delta$;
- $\beta_t \circ \pi = \pi \circ \gamma_t$ for all $t \in T$;
- $\pi(v_\mu) = u_\mu$ for $\mu \in \mathcal{P}$;
- $\pi(L^\infty(T\setminus G_q)) = \mathcal{Q}$.

**Proof.** Let $\pi: L^\infty(T\setminus G_q) \to \mathcal{Q}$ be a $G_q$-equivariant isomorphism as before. Let $w_\mu, w_\mu^\alpha$ be the invariant cocycles defined in \ref{lem:42} and \ref{lem:52}. They are satisfying the following equalities:

$$\delta(v_\mu) = (v_\mu \otimes 1)w_\mu, \quad \alpha(u_\mu) = (u_\mu \otimes 1)w_\mu^\alpha \quad \text{for } \mu \in \mathcal{P}.$$

Put $\mathcal{P} := W^*(u_\mu \mid \mu \in \mathcal{P}) \vee \mathcal{Q}$. Let us introduce a unitary map $U: L^2(G_q) \to L^2(\mathcal{P})$ such that $U(v_\mu a_1^h) = u_\mu \pi(a)1_{\overline{\mathcal{Q}}}$ for $\mu \in \mathcal{P}$ and $a \in L^\infty(T\setminus G_q)$. Then we have $Uv_\mu U^* = u_\mu$ and $UaU^* = \pi(a)$ for $\mu \in \mathcal{P}$ and $a \in L^\infty(T\setminus G_q)$. The map $\pi$ extends to a map, which we also denote by $\pi$, from $L^\infty(G_q)$ into $\mathcal{M}$. The $G_q$-equivariance of $\pi$ is verified as

$$\overline{\pi}(\text{Ad} U(v_\mu)) = \overline{\pi}(u_\mu) = (u_\mu \otimes 1)w_\mu^\alpha$$

$$= (u_\mu \otimes 1)(\pi \otimes \text{id})(w_\mu) = (\text{Ad} U \otimes \text{id})((v_\mu \otimes 1)w_\mu)$$

$$= (\text{Ad} U \otimes \text{id})(\delta(v_\mu)).$$

□
Remark 5.9. It turns out from the previous lemma that \( \alpha \) is semidual. Hence there exists an action \( \sigma \) of \( \hat{G}_q \) on \( N = B(\ell^2) \otimes M^\alpha \) such that \( \overline{M} = N \rtimes_\sigma \hat{G}_q \).

Recall the restriction of an action by a quantum subgroup (see Section 2.4). In the following lemma, we will show that the minimal action \( \beta \) actually comes from the restriction of \( \alpha \) by the maximal torus \( T \) though it seems not so clear at first.

Let \( \alpha_T \) be the restriction of \( \alpha \) on \( T \). We denote by \( \alpha_t \) the restriction of \( \alpha \) on \( t \in T \), that is, \( \alpha_t := (\text{id} \otimes ev_t) \circ \alpha \) for \( t \in T \). Let \( w_0t \) be the element satisfying \( \langle w_0t, \mu \rangle = \langle t, w_0\mu \rangle \) for all \( \mu \in P \).

Lemma 5.10. The minimal action \( \beta_t \) of \( T \) on \( \mathcal{R} \) is given by \( \alpha_{w_0t} \) on \( \mathcal{R} \).

Proof. To see this, we may assume that \( M^\alpha \) is infinite. Then \( \mathcal{R} \) is generated by \( M^\alpha \) and \( u_\lambda \)'s as before.

By the above equivariant embedding \( \pi \), \( \alpha_T \) on \( \{u_\lambda\}_\Lambda \cap \Omega \) is conjugate to the right torus action \( \gamma^R \) on \( L^\infty(G_q) \), where \( \gamma^R_t := (\text{id} \otimes ev_t \circ r_T) \circ \delta \) for \( t \in T \). Using \( \pi(v_\gamma) = u_\lambda \) and the polar decomposition of \( C^\Lambda_{\Lambda_0, w_0\Lambda} \) with \( \Lambda \in I_+ \), we have \( \alpha_t(u_\lambda) = \pi(\gamma^R_t(v_\lambda)) = \langle t, w_0\lambda \rangle u_\lambda = \beta_{w_0t}(u_\lambda) \). \( \square \)

Remark 5.11. Let \( x \in \mathbb{R}^n \) and \( y := A^{-1}x \), where \( A \) denotes the Cartan matrix. Then we put \( t = (t_j)_j \) with \( t_j = q^{iy_j} \) for \( j = 1, \ldots, n \), and we get \( (w_0t, \nu) = \prod_j q^{(w_0\omega_j, \nu)x_j} \). By the commutation relation in the proof of Theorem 3.1, we obtain
\[
\gamma^R_{w_0t} = \text{Ad}|a_{\omega_1}|^{ix_1} \cdots |a_{\omega_n}|^{ix_n} \quad \text{on} \quad L^\infty(T\backslash G_q).
\]
This shows the right action \( \gamma^R \) on \( L^\infty(T\backslash G_q) \) is implemented by a unitary representation.

Lemma 5.12. The map
\[
\Xi : (\overline{M^\tau} \otimes \mathbb{C}) \cup W^*(u_\mu \otimes v_\mu | \mu \in P) \cup (\mathbb{C} \otimes L^\infty(T\backslash G_q)) \to \overline{M}
\]
with \( \Xi((a \otimes 1)(u_\mu \otimes v_\mu)(1 \otimes b)) = au_\mu \pi(b) \) for \( a \in \overline{M^\tau} \), \( \lambda \in P \) and \( b \in \Omega \) is a well-defined \( G_q \)-equivariant isomorphism.

Proof. Let \( \mathcal{L} := (\overline{M^\tau} \otimes \mathbb{C}) \cup W^*(u_\mu \otimes v_\mu | \lambda \in P) \cup (\mathbb{C} \otimes L^\infty(T\backslash G_q)) \). Then \( \mathcal{L} \subset \overline{\mathcal{R}} \otimes L^\infty(G_q) \).

Claim. The following map \( U : L^2(\mathcal{L}) \to L^2(\overline{M}) \) is a well-defined unitary:
\[
U((a \otimes 1)(u_\mu \otimes v_\mu)(1 \otimes b)(1_\pi \otimes 1_h)) := au_\mu \pi(b)1_\overline{\pi}
\]
for \( a \in \overline{M^\tau} \), \( \mu \in P \) and \( b \in L^\infty(T\backslash G_q) \), where \( \pi \) is the one defined in the previous lemma.

Proof of Claim. Recall that \( \overline{M} \cong \overline{\mathcal{R}} \otimes \mathbb{Q} \) and \( \overline{\pi} \) is splitted to \( \overline{\pi}|_{\overline{\pi}} \otimes \varphi_0 \). Then the well-definedness follows from \( \overline{\pi}(au_\mu) = 0 \) for \( a \in \overline{M^\tau} \) and a non-zero \( \mu \in P \). \( \square \)
Using this map, we obtain an isomorphism \( \Xi : \mathcal{L} \to \overline{\mathcal{M}} \) as in the statement. We will check the \( G_q \)-equivariance. Let \( a \in \overline{\mathcal{M}}^r \), \( \mu \in P \) and \( b \in L^\infty(T \setminus G_q) \). Then
\[
\Xi(a) = a \text{ and } \Xi(1 \otimes b) = \pi(b).
\]
Next,
\[
(\Xi \otimes \text{id}_{L^\infty(G_q)}) ((\text{id}_{\mathcal{L}} \otimes \delta)(u_\mu \otimes v_\mu)) = (\Xi \otimes \text{id}_{L^\infty(G_q)}) ((u_\mu \otimes v_\mu \otimes 1)(1 \otimes w_\mu))
\]
\[
= (u_\mu \otimes 1)(\pi \otimes \text{id})(w_\mu)
\]
\[
= (u_\mu \otimes 1)w_\mu = \pi(u_\mu)
\]
\[
= \overline{\pi}(\Xi(u_\mu \otimes v_\mu)).
\]
Therefore, \( \Xi \) is \( G_q \)-equivariant. \( \square \)

We will recall the notion of the induction of actions.

**Definition 5.13.** Let \( \mathbb{H} \) be a quantum subgroup of \( \mathcal{G} \) and \( \Gamma : \mathcal{A} \to \mathcal{A} \otimes L^\infty(\mathbb{H}) \) an action of \( \mathbb{H} \) on a von Neumann algebra \( \mathcal{A} \). Let \( \gamma_\mathbb{H} := (\gamma_\mathbb{H} \otimes \text{id}) \circ \delta \) be the left action of \( \mathbb{H} \) on \( L^\infty(\mathcal{G}) \). Set
\[
\text{Ind}^\mathcal{G}_\mathbb{H} \mathcal{A} := \mathcal{A} \otimes_{\mathbb{H}} L^\infty(\mathcal{G}) = \{ x \in \mathcal{A} \otimes L^\infty(\mathcal{G}) \mid (\Gamma \otimes \text{id})(x) = (\text{id} \otimes \gamma_\mathbb{H})(x) \}.
\]
Then the restriction of \( \text{id} \otimes \delta \) on \( \text{Ind}^\mathcal{G}_\mathbb{H} \mathcal{A} \), which we will denote by \( \text{Ind}^\mathcal{G}_\mathbb{H} \Gamma \), gives an action of \( \mathcal{G} \), and we will call it the *induction of \( \Gamma \)* from \( \mathbb{H} \) to \( \mathcal{G} \).

Note that the fixed point algebra of \( \text{Ind}^\mathcal{G}_\mathbb{H} \Gamma \) is equal to \( \mathcal{A}^\Gamma \). Now we will prove the following main result of this paper.

**Theorem 5.14.** A faithful product type action of \( G_q \) is induced from a minimal action of \( T \) on a type III factor. Moreover, such minimal action is unique in the following sense: If there exists a minimal action \( \chi \) of \( T \) on a factor \( \mathcal{N} \) such that \( \text{Ind}^\mathcal{G}_T \mathcal{N} \) is \( G_q \)-equivariantly isomorphic to \( \mathcal{M} \), then there exist a \(*\)-isomorphism \( \zeta \) from \( \mathcal{R} \) onto \( \mathcal{N} \) and a topological group isomorphism \( f \) on \( T \) such that \( \chi_t = \zeta \circ \beta_{f(t)} \circ \zeta^{-1} \).

**Proof.** We let \( \mathcal{A} := W^*(u_\mu \mid \lambda \in P) \subseteq \overline{\mathcal{R}} \). Since \( \beta_t(u_\mu) = \langle t, \mu \rangle u_\mu \) and \( \gamma_t(v_\mu) = \langle t, \mu \rangle v_\mu \), we have
\[
\mathcal{A} \otimes_T Z(L^\infty(T \setminus G_q)) = W^*(u_\mu \otimes v_\mu \mid \lambda \in P).
\]
Therefore,
\[
B(\ell^2) \otimes \text{Ind}^\mathcal{G}_T \mathcal{R} = \overline{\mathcal{R}} \otimes_T L^\infty(G_q) = (\overline{\mathcal{M}}^r \vee \mathcal{A}) \otimes_T (Z(L^\infty(G_q)) \vee L^\infty(T \setminus G_q))
\]
\[
= (\overline{\mathcal{M}}^r \otimes \mathbb{C}) \vee W^*(u_\mu \otimes v_\mu \mid \lambda \in P) \vee (\mathbb{C} \otimes L^\infty(T \setminus G_q)),
\]
which is isomorphic to \( B(\ell^2) \otimes \mathcal{M} \) through \( \Xi \), the map constructed in the previous lemma. By definition, \( \Xi \) maps the fixed point algebra \( (\overline{\mathcal{R}} \otimes_T L^\infty(G_q))^\mathcal{G}_q = \overline{\mathcal{R}} = \overline{\mathcal{M}}^r \) onto \( \overline{\mathcal{M}}^r \) identically. Thus we can remove the contribution of \( B(\ell^2) \).

Next suppose that we have a minimal action \( \chi \) of \( T \) on a factor \( \mathcal{N} \) such that \( \mathcal{P} := \text{Ind}^\mathcal{G}_T \mathcal{N} \) is \( G_q \)-equivariantly isomorphic to \( \mathcal{M} \). Then the inclusion \( \mathcal{N}^\alpha \subseteq (\mathbb{C} \otimes L^\infty(T \setminus G_q)) \cap \mathcal{P} \) is isomorphic to \( \mathcal{M}^\alpha \subseteq \mathcal{R} \). Thus there exist a \(*\)-isomorphism \( \zeta \) from \( \mathcal{R} \) onto \( \mathcal{N} \) and a topological group isomorphism \( f \) on \( T \) such that \( \chi_t = \zeta \circ \beta_{f(t)} \circ \zeta^{-1} \) since every automorphism \( \psi \) on \( \mathcal{R} \) which fixes \( \mathcal{M}^\alpha \) is of the form \( \beta_t \).
for some $t \in T$. Indeed, $u^*_\lambda \psi(u_\lambda)$ commutes with $M^\alpha$, and it is a scalar (see [1] p.131 for a more general situation). □

5.3. **Classification of product type actions.** As mentioned in Lemma 5.10, the minimal action $\beta$ on $R$ comes from the restriction of $\alpha$ on $T$, which we denote by $\alpha_T$ as usual. Let $\alpha_t := (id \otimes ev_t) \circ \alpha_T$ for $t \in T$. Readers are referred to [33, 36] for the notion of conjugacy and cocycle conjugacy. We will say that a (quantum) group action is stable when every cocycle is a coboundary.

Recall that $\alpha_T$ on $Q$ is implemented by a unitary representation (see Remark 5.11). Then we have

$$\alpha_{w_0 t} \approx \alpha_{w_0 t}|_R \otimes \alpha_{w_0 t}|_Q = \beta_t \otimes \alpha_{w_0 t}|_Q \sim \beta_t \otimes id|_Q \sim \beta_t$$ for $t \in T$,

where we have used the infiniteness of $R$ at the last cocycle conjugacy. The notations $\approx$ and $\sim$ denote the conjugacy and the cocycle conjugacy, respectively. We will summarize this observation in the following (cf. Lemma 5.10). For the notion of invariant approximate innerness, readers are referred to [33, Definition 4.5, Lemma 4.7].

**Theorem 5.15.** The minimal action $\beta_t$ of the maximal torus $T$ on $R$ is cocycle conjugate to $\alpha_{w_0 t}$. In particular, $\beta$ is invariantly approximately inner.

Note that $M$ is the completion of the infinite tensor product of $B(H)$ by a product state, $M$ is of type III$_\lambda$ with $0 < \lambda \leq 1$. To compute the type of $M^\alpha$, the following result is useful. Note that $M^\alpha$ is not of type I as remarked in the proof of Lemma 5.3.

**Corollary 5.16.** The following statements hold:

1. The fixed point algebra $M^{\alpha_T}$ is not of type III$_0$;
2. If $M^{\alpha_T}$ is of type III$_\lambda$ with $0 < \lambda \leq 1$, then so is $M^\alpha$. In this case, $\alpha$ is stable;
3. If $M^{\alpha_T}$ is of type II, then so is $M^\alpha$.

**Proof.** (1). It is clear that the canonical action of the infinite symmetric group $S_\infty$ is commuting not only $\alpha_T$ but $\sigma^\varphi$, where $\varphi$ is the product state with respect to $\phi$. Therefore, $(M^{\alpha_T})^\varphi \cap M^{\alpha_T} = C$, and $\Gamma(\sigma^\varphi|_{M^{\alpha_T}}) = \text{Sp}(\sigma^\varphi|_{M^{\alpha_T}})$. This shows that $M^{\alpha_T}$ is not of type III$_0$.

(2). By [18, Proposition 5.2 (4)], $\alpha_T$ is stable. This implies that $\alpha_t$ is conjugate to $\beta_{w_0 t}$, and $M^{\alpha_T} \cong R^\beta = M^\alpha$. The stability of $\alpha$ is shown by using $2 \times 2$-matrix trick.

(2). If $M^\alpha = R^\beta$ were of type III, then so would $M^{\alpha_T}$ since there exists a normal conditional expectation from $M^{\alpha_T}$ onto $M^\alpha$. This is a contradiction. □

Theorem 5.15 enables us to classify some product type actions of $G_q$.

**Corollary 5.17.** A product type action $\alpha$ is unique up to conjugacy if $M^\alpha$ is of type III$_1$. More precisely, such $\alpha$ is conjugate to $\text{Ind}_{\mathbb{G}_q}^{G_q}(id_{R_\infty} \otimes m)$, where $R_\infty$
denotes the injective type III\textsubscript{1} factor and \(m\) the minimal action of \(T\) on the type II\textsubscript{1} injective factor \(\mathcal{R}_0\).

**Proof.** Let \(\beta\) be the associated minimal action on \(\mathcal{R}\). Then \(\mathcal{R}^\beta = \mathcal{M}^\alpha\) is of type III\textsubscript{1}. It follows that \(\beta\) is a dual action of an outer action \(\theta^{-1}\) on \(\mathcal{R}^\beta\). Then \(\theta_\mu\) for each \(\mu \in \hat{T}\) is approximately inner since \(\text{Aut}(\mathcal{R}_\infty) = \text{Int}(\mathcal{R}_\infty)\) [24, Theorem 1]. By [33, Theorem 4.11], \(\theta\) has the Rohlin property, that is, the central freeness. Thus \(\theta\) is unique up to cocycle conjugacy [36, Theorem 1.4, p.7]. This implies the uniqueness of \(\beta\) up to conjugacy. \(\square\)

**Example 5.18.** We will construct a model of a product type action whose fixed point algebra is of type III\textsubscript{1}. As a result, it turns out that \(\text{Ind}^G_T(id_{\mathcal{R}_\infty} \otimes m)\) is indeed of product type.

Take an \(n\)-dimensional unitary representation \(v\) of \(G_q\) such that the matrix elements \(v_{ij}\) generate \(C(G_q)\). Then we set the \((n + 3)\)-dimensional representation \(w := 1 \oplus 3 \oplus v\). Let \(\lambda, \mu > 0\) such that \(\lambda/\mu \notin \mathbb{Q}\). We introduce \(\text{Ad}w\)-invariant state \(\phi\) defined by the normalisation of \(\text{Tr}_k\), where \(k\) denotes the diagonal matrix \(\text{diag}(1, \lambda, \mu, F_v)\).

By Corollary 5.16, it suffices to show that \(\mathcal{M}^{\alpha T}\) is of type III\textsubscript{1}. It follows from the proof of Corollary 5.16 that \(\Gamma(\sigma^e|_{\mathcal{M}^{\alpha T}}) = \text{Sp}(\sigma^e|_{\mathcal{M}^{\alpha T}})\). By construction of \(w\), it turns out that \(\log \lambda, \log \mu \in \text{Sp}(\sigma^e|_{\mathcal{M}^{\alpha T}})\). Thus \(\mathcal{M}^{\alpha T}\) is of type III\textsubscript{1}.

When the fixed point algebra is of another type, it seems that the general classification is complicated. So, let us treat \(SU_q(2)\) in what follows. Our main ingredient is the complete invariant treated in [33, Theorem 6.28]. Note that two actions of the torus \(\mathbb{R}/2\pi\mathbb{Z}\) are cocycle conjugate if and only if so are they as \(\mathbb{R}\)-actions.

We now suppose that \(\alpha\) is a product type action of \(SU_q(2)\) and \(v\) a finite dimensional representation. To compute the invariant, we give a parametrization of \(v\) and \(\phi\) as follows. The irreducible representations of \(SU_q(2)\) are parametrized by \(\mathbb{Z}_+\omega_1\), or equivalently, the half spins \((1/2)\mathbb{Z}_+\). Let us decompose \(v\) into the direct sum of irreducible representations as follows:

\[
v = \bigoplus_{\nu \in (1/2)\mathbb{Z}_+} \bigoplus_{k=1}^{m_\nu} C^\nu,
\]

where \(m_\nu\) denotes the multiplicity of \(C^\nu\) in \(v\). Under identification of \(T = \mathbb{R}/2\pi\mathbb{Z}\), we have

\[
v_t = \bigoplus_{\nu \in (1/2)\mathbb{Z}_+} \bigoplus_{k=1}^{m_\nu} \text{diag}(e^{2\nu it}, e^{(2\nu - 2)it}, \ldots, e^{-2\nu it}) \quad \text{for } t \in \mathbb{R}.
\]

Changing the orthonormal base of each intertwiner space if necessary, we may and do assume that \(\phi\) is the normalization of \(\text{Tr}_{k_\phi}\), where \(k_\phi\) is defined as

\[
k_\phi = \bigoplus_{\nu \in (1/2)\mathbb{Z}_+} \bigoplus_{k=1}^{m_\nu} c_k^\nu \text{diag}(q^{2\nu}, q^{2\nu - 2}, \ldots, q^{-2\nu}), \quad \text{for some } c_k^\nu > 0.
\]
From the faithfulness of $\alpha$, $v$ has at least one non-integer-spin representation and at least one integer. Thus we may assume that $c_k^\nu = 1$ for a fixed even $\nu$ and $k$. Note that the density matrix $\text{diag}(q^{2\nu}, q^{2\nu-2}, \ldots, q^{-2\nu})$ contains $1$ as its spectrum for any integer-spin $\nu$.

Then the invariant $G_{\lambda, \mu}$ stated in \cite[Theorem 6.28]{33} is computed as follows:

$$G_{\alpha_T} := \langle (\log(c_k^\nu q^\ell), \ell) \mid \ell = 2\nu, 2\nu - 2, \ldots, -2\nu, \ k = 1, \ldots, m_\nu, \ \nu \in (1/2)\mathbb{Z}_+ \rangle,$$

which is a closed subgroup of $\mathbb{R}^2$. Since there exists $\nu \in (1/2) + \mathbb{Z}_+$ with $m_\nu > 0$, $G_{\alpha_T}$ can be written as the following form:

$$G_{\alpha_T} = \langle (\log(c_k^\nu, 0), (2\log(c_k^\nu, 0), (\log(c_k^\nu q^k), 1) \mid k, \nu \in \mathbb{Z}_+, \nu_0 \in 1/2 + \mathbb{Z}_+ \rangle \quad (5.8)$$

**Theorem 5.19.** If $G_q = SU_q(2)$, and $\mathcal{M}_\alpha$ is of type II, then $\mathcal{M}_\alpha$ and $\mathcal{M}$ must be of type II$_1$ and III$_q$, respectively. Moreover, $\alpha$ is conjugate to the induction of the torus action $\sigma^q_{t/\log q}$, where $\varphi_q$ denotes the Powers state on the Powers factor $\mathcal{R}_q$ of type III$_q$.

**Proof.** Let $tr$ be the tracial weight on $\mathcal{R}^\beta$. Then by \cite[Proposition 5.2 (5)]{18}, $\{\sigma_{t, E}^{\tau_0\alpha_1|\lambda}\}_{t \in \mathbb{R}}$ is contained in $\{\beta_t\}_{t \in \mathbb{R}/2\pi \mathbb{Z}}$. In particular, $\sigma_{t, E}^{\tau_0\alpha_1|\lambda}$ is periodic, and $\mathcal{R}$ is of type III$_\lambda$ for some $0 < \lambda < 1$. We will show that $\lambda$ must be equal to $q$.

By \cite[Proposition 6.34]{33}, $\beta_t$ is cocycle conjugate to $\sigma^{}_{t/\log \lambda}$ or $\sigma^{}_{-t/\log \lambda}$, where $\psi$ denotes the Powers state on the Powers factor $\mathcal{R}_\lambda$. From Theorem 5.15, we have $\alpha_t \sim \beta_t \sim \sigma^{}_{t/\log \lambda}$, and their invariants introduced in \cite[Section 6.5]{33} coincide. The invariant of $\sigma^{}_{t/\log \lambda}$ equals $G_\lambda = \mathbb{Z}(\log \lambda, \mp 1)$. It follows immediately from (5.8) that $c_k^\nu = 1$ for all $\nu$ and $k$, and $\lambda = q$. Hence we have $\beta_t \sim \sigma^{}_{t/\log \lambda}$ and $\alpha_t = \sigma^{}_{t/\log q}$ for $t \in \mathbb{R}$. So, $\mathcal{M}_\alpha = \mathcal{M}_\varphi$ is of type II$_1$.

We will show that $\beta_t$ is in fact conjugate to $\sigma^{}_{-t/\log q}$. Employing Lemma 5.10, we have $\beta_t = \alpha_{-t} = \sigma^{}_{-t/\log q} = \sigma^{}_{t/\log q}$ on $\mathcal{R}$. Note that $\mathcal{R} \cong \mathcal{R}_q$ and $\varphi$ and $\psi_q$ are periodic states. Then by adjusting a Connes–Takesaki module, there exists an isomorphism $\zeta : \mathcal{R} \to \mathcal{R}_q$ such that $\varphi^{}_{\mathcal{R}} = \psi_q \circ \zeta$. Thus $\beta_t \approx \sigma^{}_{-t/\log q}$. It is not so difficult to show that the induction of an action is stable with respect to an automorphism of $T$, and we have $\alpha \approx \text{Ind}^G q \sigma^{}_{-t/\log q} \approx \text{Ind}^G q \sigma^{}_{t/\log q}$. \hfill $\square$

**Example 5.20.** Let $v$ be the direct sum of the spin-0 and 1/2 irreducible representations. Namely, a unitary $v$ has the following form:

$$v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & u \\ 0 & v & y \end{pmatrix} \in M_3(\mathbb{C}) \otimes C(SU_q(2)) = M_3(C(SU_q(2))),$$

where $x, u, v$ and $y$ are the canonical generators of $C(SU_q(2))$ as a $C^*$-algebra (see \cite[Section 2.3]{32}).

Now we set the following density matrix:

$$k_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}.$$
where \( \text{Tr} \) denotes the canonical non-normalized trace of \( M_3(\mathbb{C}) \). Let \( \varphi \) be the product state of \( \phi \) as usual. Then \( \alpha_t = \sigma_{-t/\log q}^\varphi \) for \( t \in \mathbb{R}/2\pi\mathbb{Z} \), and \( M^\alpha \) is of type \( \text{II}_1 \). Thus so is \( M^\alpha \).

The remaining case is when \( M^\alpha \) is of type III\( \lambda \) with \( 0 < \lambda < 1 \). The infiniteness of \( \mathcal{R}^\beta \) implies that the crossed product decomposition of \( \mathcal{R} \), that is, \( \mathcal{R} = \mathcal{R}^\beta \rtimes q\mathbb{Z} \). Recall that \( \beta_t = \hat{\vartheta}_{-t} \) for \( t \in \mathbb{R}/2\pi\mathbb{Z} \). Since \( \beta \) is invariantly approximately inner, \( \theta \) is centrally free.

Let \( \text{mod}(\theta) \) be the Connes–Takesaki module of \( \theta \). Identifying the flow space of \( \mathcal{R}^\beta \) with \( (\lambda, 1] = \mathbb{R}_{>0}/\lambda\mathbb{Z} \), we may assume that \( \lambda \leq \text{mod}(\theta) < 1 \). Let \( \mu := \text{mod}(\theta) \). Thanks to the classification of \( \mathbb{Z} \)-actions, (see \cite[Theorem 1, Corollary 6, p.385]{7} or \cite[Theorem 1.13, p.311]{16}), \( \theta \) is cocycle conjugate to \( \text{id}_\mathcal{R} \otimes \theta^\mu \), where \( \theta^\mu \) denotes the automorphism on the injective type II\( _\infty \) factor \( \mathcal{R}_{0,1} \) with \( \text{tr} \circ \theta^\mu = \mu \text{ tr} \). Thus \( \mathcal{R} \cong \mathcal{R}_\lambda \otimes \mathcal{R}_\mu \) and \( \beta_t \approx \text{id}_\mathcal{R} \otimes \sigma_{t/\log \mu}^\varphi \). So, the invariant of \( \beta \) is computed as follows:

\[
G_{\lambda, \mu} := Z(\log \lambda, 0) + Z(\log \mu, 1). \tag{5.9}
\]

Note that we can replace \( \mu \) with \( \lambda \mu \), that is, \( G_{\lambda, \lambda \mu} = G_{\lambda, \mu} \). This shows that \( \text{id}_\mathcal{R} \otimes \sigma_{t/\log \mu}^\varphi \) is (cocycle) conjugate to \( \text{id}_\mathcal{R} \otimes \sigma_{t/\log (\lambda \mu)}^\varphi \).

**Theorem 5.21.** If \( G_q = SU_q(2) \), and \( M^\alpha \) is of type III\( \lambda \) with \( 0 < \lambda < 1 \), then \( \text{mod}(\theta) = q \) or \( \lambda^{1/2} q \) in \( \mathbb{R}_{>0}/\lambda\mathbb{Z} \). In each case, \( \alpha \) is unique up to conjugacy.

**Proof.** In this case, we have \( G_{\alpha_t} = G_{\lambda, \mu} \). By (5.8) and (5.9), we can use the fact that \( c_k^{\mu} \in \lambda^{m_k} \) with \( m_k \in (1/2)\mathbb{Z}_+ \) and \( k = q c_k^{\mu} \lambda^{Z_k} \). Hence \( \mu = q \lambda^n \) for some \( n \in (1/2)\mathbb{Z}_+ \), and \( \text{mod}(\theta) = q \) or \( \lambda^{1/2} q \) in \( \mathbb{R}_{>0}/\lambda\mathbb{Z} \). \( \square \)

**Example 5.22.** Let \( 0 < \lambda < 1 \) and \( \epsilon \in \{0, 1/2\} \). Set \( v \) and \( k_\phi \) as follows:

\[
v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & u \\ 0 & 0 & v & y \end{pmatrix}, \quad k_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^q & 0 \\ 0 & 0 & 0 & \lambda^{q-1} \end{pmatrix}.
\]

Then we can see \( G_{\alpha_t} = G_{\lambda, \lambda \epsilon \lambda^q} \) by direct calculation.

So, if \( \mu = \lambda^k q < 1 \) with \( k \) a half integer, then \( \text{Ind}_T^G (\text{id}_\mathcal{R} \otimes \sigma_{t/\log \mu}^\varphi) \) is of product type and falls into two categories.

The following result is a direct consequence of the previous theorem.

**Corollary 5.23.** Let \( G_q = SU_q(2) \) and \( 0 < \lambda < 1 \). Suppose that \( \mu \) satisfies \( 0 < \mu < 1 \) and \( \mu/q \notin (\lambda^{1/2})\mathbb{Z}_+ \). Then the induced action \( \text{Ind}_T^G (\text{id}_\mathcal{R} \otimes \sigma_{t/\log \mu}^\varphi) \) is not of product type. In particular, for any \( 0 < \lambda < 1 \), there exist uncountably many, non-product type, mutually non-cocycle conjugate actions of \( SU_q(2) \) on the injective type III\( _1 \) factor with fixed point factor of type III\( \lambda \).

6. Related problems

Let \( G \) be a compact quantum group and \( K \) the maximal quantum subgroup of Kac type introduced in \cite[App. A]{1} and \cite[Definition 4.6]{8}. We would like to generalize Dijkhuizen-Stokman’s result stated in Section 3.1.
Problem 6.1. Does the following equality hold?
\[ \text{Irr}(C(\mathbb{K}\setminus \mathbb{G})) = \{ \pi |_{C(\mathbb{K}\setminus \mathbb{G})} | \pi \in \text{Irr}(C(\mathbb{G})) \}, \]
where \( \text{Irr}(A) \) denotes the equivalence classes of irreducible representation of a C*-algebra \( A \).

Problem 6.2. Is the counit a unique character on \( C(\mathbb{K}\setminus \mathbb{G}) \)?

We will remark on this problem. Let \( \Gamma \) be the set of characters on \( C(\mathbb{G}) \). Then it is probably well-known for experts that \( \Gamma \) is a compact group that is regarded as a quantum subgroup of \( \mathbb{G} \). The maximality of \( \mathbb{K} \) implies that \( C(\mathbb{K}\setminus \mathbb{G}) \subset C(\Gamma\setminus \mathbb{G}) \). In particular, the restriction of every element of \( \Gamma \) on \( C(\mathbb{K}\setminus \mathbb{G}) \) gives a counit. Thus if Problem 6.1 is solved, this problem holds.

Problem 6.3. \( \text{Aut}_G(C(\mathbb{K}\setminus \mathbb{G})) = \{ \text{id} \} ? \)

If \( \mathbb{G} \) is a compact group, then \( \mathbb{K} = \mathbb{G} \). So these problems are trivial. We will explain why the last problem seems plausible. Let \( G \) be a compact group and \( H \) a closed subgroup of \( G \). Then \( \text{Aut}_G(C(H\setminus \mathbb{G})) \) is isomorphic to \( N_G(H)/H \), where \( N_G(H) \) denotes the normalizer group of \( H \). If there exists a non-trivial \( g \in N_G(H) \), then \( H \) and \( g \) generate a closed subgroup larger than \( H \). Hence the maximality of \( \mathbb{K} \) would imply the triviality of \( \text{Aut}_G(C(\mathbb{K}\setminus \mathbb{G})) \).

In the last section, in order to show that a faithful product type action of \( G_q \) is induced from a minimal action of the maximal torus \( T \), we have exploited the representation theory of \( G_q \) and \( C(G_q) \) to a full. We would like to obtain this in a more conceptual way.

Problem 6.4. Let \( \mathbb{G} \) be a co-amenable compact quantum group with commutative fusion rules and \( \mathbb{K} \) the maximal quantum subgroup of Kac type. Then is any faithful product type action of \( \mathbb{G} \) induced from a minimal action of \( \mathbb{K} \)?

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