The Optimal Quantile Estimator for Compressed Counting

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Abstract
Compressed Counting (CC) was recently proposed for very efficiently computing the (approximate) αth frequency moments of data streams, where $0 < \alpha \leq 2$. Several estimators were reported including the geometric mean estimator, the harmonic mean estimator, the optimal power estimator, etc. The geometric mean estimator is particularly interesting for theoretical purposes. For example, when $\alpha \to 1$, the complexity of CC (using the geometric mean estimator) is $O(1/\epsilon)$, breaking the well-known large-deviation bound $O(1/\epsilon^2)$. The case $\alpha \approx 1$ has important applications, for example, computing entropy of data streams.

For practical purposes, this study proposes the optimal quantile estimator. Compared with previous estimators, this estimator is computationally more efficient and is also more accurate when $\alpha > 1$.

1 Introduction

Compressed Counting (CC) was very recently proposed for efficiently computing the $\alpha$th frequency moments, where $0 < \alpha \leq 2$, in data streams. The underlying technique of CC is maximally skewed stable random projections, which significantly improves the well-known algorithm based on symmetric stable random projections, especially when $\alpha \to 1$. CC boils down to a statistical estimation problem and various estimators have been proposed. In this study, we present an estimator based on the optimal quantiles, which is computationally more efficient and significantly more accurate when $\alpha > 1$, as long as the sample size is not too small.

One direct application of CC is to estimate entropy of data streams. A recent trend is to approximate entropy using frequency moments and estimate frequency moments using symmetric stable random projections. applied CC to estimate entropy and demonstrated huge improvement (e.g., 50-fold) over previous studies.

CC was recently presented at MMDS 2008: Workshop on Algorithms for Modern Massive Data Sets. Slides are available at [http://www.stanford.edu/group/mmds/slides2008/li.pdf](http://www.stanford.edu/group/mmds/slides2008/li.pdf).

1.1 The Relaxed Strict Turnstile Data Stream Model

Compressed Counting (CC) assumes a relaxed strict Turnstile data stream model. In the Turnstile model, the input stream $a_t = (i_t, I_t)$, $i_t \in [1, D]$ arriving sequentially describes the underlying signal $A$, meaning

$$A_t[i_t] = A_{t-1}[i_t] + I_t,$$

where the increment $I_t$ can be either positive (insertion) or negative (deletion). Restricting $A_t[i] \geq 0$ at all $t$ results in the strict Turnstile model, which suffices for describing most natural phenomena. CC constrains $A_t[i] \geq 0$ only at the $t$ we care about; however, when at $s \neq t$, CC allows $A_s[i]$ to be arbitrary.

Under the relaxed strict Turnstile model, the $\alpha$th frequency moment of a data stream $A_t$ is defined as

$$F(\alpha) = \sum_{i=1}^{D} A_t[i]^\alpha.$$  

When $\alpha = 1$, it is obvious that one can compute $F(1) = \sum_{i=1}^{D} A_t[i] = \sum_{s=1}^{t} I_s$ trivially, using a simple counter. When $\alpha \neq 1$, however, computing $F(\alpha)$ exactly requires $D$ counters.

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The results were initially drafted in Jan 2008, as part of a report for private communications with several theorists. That report was later filed to arXiv, which, for shortening the presentation, excluded the content of the optimal quantile estimator.
1.2 Maximally-skewed Stable Random Projections

Based on maximally skewed stable random projections, CC provides an very efficient mechanism for approximating $F(\alpha)$. One first generates a random matrix $R \in \mathbb{R}^D$, whose entries are i.i.d. samples of a $\beta$-skewed $\alpha$-stable distribution with scale parameter 1, denoted by $r_{ij} \sim S(\alpha, \beta, 1)$.

By property of stable distributions[12, 10], entries of the resultant projected vector $X = R^T A_t \in \mathbb{R}^k$ are i.i.d. samples of a $\beta$-skewed $\alpha$-stable distribution whose scale parameter is the $\alpha$ frequency moment of $A_t$ we are after:

$$x_j = [R^T A_t]_j = \sum_{i=1}^D r_{ij} A_t[i] \sim S(\alpha, \beta, F(\alpha) = \sum_{i=1}^D A_t[i]^\alpha).$$

The skewness parameter $\beta \in [-1, 1]$. CC recommends $\beta = 1$, i.e., maximally-skewed, for the best performance.

In real implementation, the linear projection $X = R^T A_t$ is conducted incrementally, using the fact that the Turnstile model is also linear. That is, for every incoming $a_t = (i_t, I_t)$, we update $x_j \leftarrow x_j + r_{i_t,j} I_t$ for $j = 1$ to $k$. This procedure is similar to that of symmetric stable random projections[3, 6]; the difference is the distribution of the elements in $R$.

2 The Statistical Estimation Problem and Previous Estimators

CC boils down to a statistical estimation problem. Given $k$ i.i.d. samples, $x_j \sim S(\alpha, \beta = 1, F(\alpha))$, estimate the scale parameter $F(\alpha)$.

Assume $k$ i.i.d. samples $x_j \sim S(\alpha, \beta = 1, F(\alpha))$. Various estimators were proposed in [4, 7], including the geometric mean estimator, the harmonic mean estimator, the maximum likelihood estimator, the optimal quantile estimator. Figure 1 compares their asymptotic variances along with the asymptotic variance of the geometric mean estimator for symmetric stable random projections[6].

![Figure 1](image.png)

Figure 1: Let $\hat{F}$ be an estimator of $F$ with asymptotic variance $\text{Var}(\hat{F}) = V F^2 + O(1/k^2)$. We plot the $V$ values for the geometric mean estimator, the harmonic mean estimator (for $\alpha < 1$), the optimal power estimator (the lower dashed curve), and the optimal quantile estimator, along with the $V$ values for the geometric mean estimator for symmetric stable random projections in [6] (“symmetric GM”, the upper dashed curve). When $\alpha \rightarrow 1$, CC achieves an “infinite improvement” in terms of the asymptotic variances.
2.1 The geometric mean estimator, \( \hat{F}_{(\alpha),gm} \), for \( 0 < \alpha \leq 2, (\alpha \neq 1) \)

\[
\hat{F}_{(\alpha),gm} = \left( \cos^k \left( \frac{\alpha \pi}{2k} \right) / \cos \left( \frac{\pi \alpha}{2} \right) \right) \frac{1}{k} \sum_{j=1}^{k} |x_j|^\alpha/k
\]

\[
\text{Var} \left( \hat{F}_{(\alpha),gm} \right) = \frac{\alpha^2}{k} \left( \alpha^2 + 3\kappa^2(\alpha) \right) + O \left( \frac{1}{k^2} \right),
\]

\( \kappa(\alpha) = \alpha \), if \( \alpha < 1 \), \( \kappa(\alpha) = 2 - \alpha \), if \( \alpha > 1 \).

\( \hat{F}_{(\alpha),gm} \) is unbiased and has exponential tail bounds for all \( 0 < \alpha \leq 2 \).

2.1.1 The harmonic estimator, \( \hat{F}_{(\alpha),hm,c} \), for \( 0 < \alpha < 1 \)

\[
\hat{F}_{(\alpha),hm,c} = \frac{k}{\sum_{j=1}^{k} |x_j|^\alpha} \left( 1 - \frac{1}{k} \left( \frac{2\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} - 1 \right) \right),
\]

\[
\text{E} \left( \hat{F}_{(\alpha),hm,c} \right) = F_{(\alpha)} + O \left( \frac{1}{k^2} \right),
\text{Var} \left( \hat{F}_{(\alpha),hm,c} \right) = \frac{F_{(\alpha)}^2}{k} \left( \frac{2\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} - 1 \right) + O \left( \frac{1}{k^2} \right).
\]

\( \hat{F}_{(\alpha),hm,c} \) has exponential tail bounds.

2.2 The maximum likelihood estimator, \( \hat{F}_{(0.5),mle,c} \), for \( \alpha = 0.5 \) only

\[
\hat{F}_{(0.5),mle,c} = \left( 1 - \frac{31}{4k} \right) \sqrt{\frac{k}{\sum_{j=1}^{k} \frac{1}{x_j^2}}}
\]

\[
\text{E} \left( \hat{F}_{(0.5),mle,c} \right) = F_{(0.5)} + O \left( \frac{1}{k^2} \right),
\text{Var} \left( \hat{F}_{(0.5),mle,c} \right) = \frac{F_{(0.5)}^2}{2} + \frac{9F_{(0.5)}^2}{8k^2} + O \left( \frac{1}{k^3} \right).
\]

\( \hat{F}_{(0.5),mle,c} \) has exponential tail bounds.

2.3 The optimal power estimator, \( \hat{F}_{(\alpha),op,c} \), for \( 0 < \alpha \leq 2, (\alpha \neq 1) \)

\[
\hat{F}_{(\alpha),op,c} = \left( \frac{1}{k} \left( \frac{\cos (\kappa(\alpha) \lambda^* \pi)}{\cos^{\lambda^*} \left( \frac{\lambda^* \pi}{2} \right)} \frac{2}{\pi} \Gamma(1 - \lambda^*) \Gamma(\lambda^* \sin \left( \frac{\pi \lambda^*}{2} \right)) \right) \right)^{1/\lambda^*}
\]

\[
\times \left( 1 - \frac{1}{k} \left[ \frac{1}{\lambda^*} \left( \frac{1}{\lambda^*} - 1 \right) \left( \frac{\cos (\kappa(\alpha) \lambda^* \pi)}{\cos \left( \frac{\lambda^* \pi}{2} \right)} \frac{2}{\pi} \Gamma(1 - 2\lambda^*) \Gamma(2\lambda^* \sin \left( \frac{\pi \lambda^*}{2} \right)) \right) - 1 \right) \right),
\]

\[
\text{E} \left( \hat{F}_{(\alpha),op,c} \right) = F_{(\alpha)} + O \left( \frac{1}{k^2} \right)
\]

\[
\text{Var} \left( \hat{F}_{(\alpha),op,c} \right) = F_{(\alpha)}^2 \frac{1}{\lambda^{*2k}} \left( \frac{\cos (\kappa(\alpha) \lambda^* \pi)}{\cos \left( \frac{\lambda^* \pi}{2} \right)} \frac{2}{\pi} \Gamma(1 - 2\lambda^*) \Gamma(2\lambda^* \sin \left( \frac{\pi \lambda^*}{2} \right)) - 1 \right) + O \left( \frac{1}{k^2} \right).
\]

\[ \lambda^* = \arg \min g (\lambda; \alpha), \quad g (\lambda; \alpha) = \frac{1}{\lambda^2} \left( \frac{\cos (\kappa(\alpha) \lambda \pi)}{\cos \left( \frac{\lambda \pi}{2} \right)} \frac{2}{\pi} \Gamma(1 - 2\lambda) \Gamma(2\lambda \sin \left( \frac{\pi \lambda}{2} \right)) - 1 \right). \]

When \( 0 < \alpha < 1, \lambda^* < 0 \) and \( \hat{F}_{(\alpha),op,c} \) has exponential tail bounds.

\( \hat{F}_{(\alpha),op,c} \) becomes the harmonic mean estimator when \( \alpha = 0+ \), the arithmetic mean estimator when \( \alpha = 2 \), and the maximum likelihood estimator when \( \alpha = 0.5 \).
3 The Optimal Quantile Estimator

Because \( X \sim S(\alpha, \beta = 1, F(\alpha)) \) belongs to the location-scale family (location is zero always), one can estimate the scale parameter \( F(\alpha) \) simply from the sample quantiles.

3.1 A General Quantile Estimator

Assume \( x_j \sim S(\alpha, 1, F(\alpha)) \), \( j = 1 \) to \( k \). One possibility is to use the \( q \)-quantile of the absolute values, i.e.,

\[
\hat{F}_{(\alpha), q} = \left( q\text{-Quantile}\{|x_j|, j = 1, 2, \ldots, k\} \right)^\alpha.
\]

where

\[
W_q = q\text{-Quantile}\{|S(\alpha, \beta = 1, 1)|\}.
\]

Denote \( Z = |X| \), where \( X \sim S(\alpha, 1, F(\alpha)) \). Note that when \( \alpha < 1 \), \( Z = X \). Denote the probability density function of \( Z \) by \( f_Z(z; \alpha, F(\alpha)) \), the probability cumulative function by \( F_Z(z; \alpha, F(\alpha)) \), and the inverse cumulative function by \( F_Z^{-1}(q; \alpha, F(\alpha)) \).

We can analyze the asymptotic (as \( k \rightarrow \infty \)) variance of \( \hat{F}_{(\alpha), q} \), presented in Lemma 1.

Lemma 1

\[
\text{Var}\left(\hat{F}_{(\alpha), q}\right) = \frac{1}{k} \frac{(q - q^2)\alpha^2}{f_Z^2(F_Z^{-1}(q; \alpha, F(\alpha)); \alpha, F(\alpha))(F_Z^{-1}(q; \alpha, F(\alpha)))^2} F_{(\alpha)}^2 + O\left(\frac{1}{k^2}\right).
\]

Proof: The proof directly follows from known statistical results on sample quantiles, e.g., [1, Theorem 9.2], and the “delta” method.

\[
\text{Var}\left(\hat{F}_{(\alpha), q}\right) = \frac{1}{k} \frac{q - q^2}{f_Z^2(F_Z^{-1}(q; \alpha, F(\alpha)); \alpha, F(\alpha))(F_Z^{-1}(q; \alpha, F(\alpha)))^2} (F(\alpha))^{((\alpha - 1)/\alpha)^2}\alpha^2 + O\left(\frac{1}{k^2}\right)
\]

\[
= \frac{1}{k} \frac{(q - q^2)\alpha^2}{f_Z^2(F_Z^{-1}(q; \alpha, F(\alpha)); \alpha, F(\alpha))(F_Z^{-1}(q; \alpha, F(\alpha)))^2} F_{(\alpha)}^2 + O\left(\frac{1}{k^2}\right),
\]

using the fact that

\[
F_Z^{-1}(q; \alpha, F(\alpha)) = F_{(\alpha)}^1 F_Z^{-1}(q; \alpha, 1), \quad f_Z(z; \alpha, F(\alpha)) = F_{(\alpha)}^{-1}\alpha f_Z(z; \alpha^{-1}/\alpha, 1).
\]

We can choose \( q = q^* \) to minimize the asymptotic variance factor, \( f_Z^2(F_Z^{-1}(q; \alpha, 1); \alpha, F(\alpha)))^2 \), which is apparently a convex function of \( q \), although there appears no simple algebraic method to prove it (except when \( \alpha = 0^+ \)).

We denote the optimal quantile estimator as \( \hat{F}_{(\alpha), q^*} = \hat{F}_{(\alpha), q^*} \).

3.2 The Optimal Quantiles

The optimal quantiles, denoted by \( q^* = q^*(\alpha) \), has to be determined by numerical procedures, using the simulated probability density functions for stable distributions. We used the \texttt{fBasics} package in \texttt{R}. We, however, found those functions had numerical problems when \( 1 < \alpha < 1.011 \) and \( 0.989 < \alpha < 1 \).

For all other estimators, we have not noticed any numerical issues even when \( \alpha = 1 - 10^{-4} \) or \( 1 + 10^{-4} \). Therefore, we do not consider there is any numerical instability for CC, as far as the method itself is concerned.

Table 1 presents the numerical results, including \( q^* \), \( W_{q^*} = q^*\text{-Quantile}\{|S(\alpha, \beta = 1, 1)|\} \), and the variance of \( \hat{F}_{(\alpha), q^*} \) (without the \( \frac{1}{k} \) term). The variance factor is also plotted in Figure 1, indicating significant improvement over the geometric mean estimator when \( \alpha > 1 \).
### 3.3 Comments on the Optimal Quantile Estimator

The optimal quantile estimator has at least two advantages:

- When the sample size $k$ is not too small (e.g., $k \geq 50$), $\hat{F}_{(\alpha),oq}$ is more accurate than $\hat{F}_{(\alpha),gm}$, especially for $\alpha > 1$.
- $\hat{F}_{(\alpha),oq}$ is computationally more efficient.

The disadvantages are:

- For small samples (e.g., $k \leq 20$), $\hat{F}_{(\alpha),oq}$ exhibits bad behaviors when $\alpha > 1$.
- Its theoretical analysis, e.g., variances and tail bounds, is based on the density function of skewed stable distributions, which do not have closed-forms. The tail bound bounds can be obtained similarly using the method developed in [5].
- The important parameters, $q^*$ and $W_{q^*}$, are obtained from the numerically-computed density functions. Due to the numerical difficulty in those functions, we can only obtain $q^*$ and $W_{q^*}$ values for $\alpha \geq 1.011$ and $\alpha \leq 0.989$.

### 4 Conclusion

Compressed Counting (CC) dramatically improves symmetric stable random projections, especially when $\alpha \approx 1$, and has important applications in data streams computations such as entropy estimation.

CC boils down to a statistical estimation problem. We propose the optimal quantile estimator, which considerably improves the previously proposed geometric mean estimator when $\alpha > 1$, at least asymptotically. For practical purposes, this estimator should be very useful. However, for theoretical purposes, it can not replace the geometric mean estimator.
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