Dynamics of Certain Smooth One-dimensional Mappings

I. The $C^{1+\alpha}$-Denjoy-Koebe distortion lemma

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Abstract

We prove a technical lemma, the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma, estimating the distortion of a long composition of a $C^{1+\alpha}$ one-dimensional mapping $f : M \to M$ with finitely many, non-recurrent, power law critical points. The proof of this lemma combines the ideas of the distortion lemmas of Denjoy and Koebe.

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§1 Introduction

There are two techniques in studying the distortion of a long composition of a one-dimensional smooth mapping.

“Denjoy Principle”: One technique goes back to Denjoy. Many people have contributed to this technique [D], [S], [N1], [N2], [N3], [H], [M], etc. We call one of the formulations of this technique the naive distortion lemma because its proof is straightforward – any one, who has been trained in Calculus, will understand the proof ([J1], p25–26, or Lemma 3 in this paper). The naive distortion lemma is one of the key lemmas in studying a long composition of a one-dimensional $C^{1+\alpha}$-endomorphism.

“Koebe principle”: The second technique was found in recent years in studying a long composition of a mapping with critical points from a one-dimensional manifold to itself. Many people formulated this principle in different ways, [GS], [Su1], [Su2], [WS], [Sw], etc. We call one version the $C^{3}$-Koebe distortion lemma (see also [J1, p26–27] for a complete proof). I learned this from Sullivan, who invented the name “Koebe principle” in analogy with the Koebe lemma in one variable complex analytic functions. We consider the nonlinearity of a $C^{2}$-function $f$ on an interval $I$ as a one-form

$$n(f) = \frac{f''}{f'} dx.$$

If the nonlinearity of the function $f$ is integrable on $I$, then the distortion $|f'(x)/f'(y)|$ of $f$ at any pair $x$ and $y$ in $I$ is bounded. The problem is that the nonlinearity of $f$ may be non-integrable if $f$ has a critical point. The $C^{3}$-Koebe distortion lemma estimates the nonlinearity of a one-dimensional $C^{3}$-mapping $f$ with nonnegative Schwarzian derivative. This property, nonnegative Schwarzian derivative, is preserved under composition, which makes the $C^{3}$-Koebe distortion lemma a very useful technique in studying a long composition of a one-dimensional $C^{3}$-mapping with nonpositive Schwarzian derivative (its inverse branches have nonnegative Schwarzian derivatives). However, the assumption of nonpositive Schwarzian derivative is a very strong one.

What we would like to say in this paper. We prove a technical lemma, the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma, estimating the dis-
tortion of a long composition of a one-dimensional $C^{1+\alpha}$-mapping with finitely many non-recurrent critical points of certain types. The formulation and the proof of this lemma combine the ideas of the distortion lemmas of Denjoy and Koebe.

Suppose $M$ is an oriented connected one-dimensional $C^2$-Riemannian manifold with Riemannian metric $dx^2$ and associated length element $dx$. Suppose $f: M \to M$ is a continuous mapping. A critical point $c$ of $f$ is a point in $M$ such that either $f$ is not differentiable at this point or $f$ is differentiable at this point with zero derivative. We always assume that $f$ is $C^1$ at a noncritical point $p$, namely there is a neighborhood $U_p$ of $p$ such that the restriction of $f$ to $U_p$ is differentiable and the derivative $(f|_{U_p})'$ is continuous. We call the image of a critical point under $f$ a critical value of $f$. We say a critical point $c$ of $f$ is a power law critical point if it is an isolated critical point and there is a number $\gamma \geq 1$ such that the limits of ratio, $f'(x)/|x-c|^{-1}$, exist and equal nonzero numbers as $x$ goes to $c$ from below and from above. We call the number $\gamma$ the exponent of $f$ at the power law critical point $c$. We will assume that $f: M \to \tilde{M}$ is a $C^1$-mapping for we are only interested in a smooth critical point of $f$.

For a $C^1$-mapping $f: M \to M$ with only power law critical points such that the set of critical points and the set of critical values of $f$ are disjoint, we define a new differentiable structure on the underlying space $M$. This new differentiable structure associated with the mapping $f$ has the local parameter $\int dx/|x|^{\tau}$, where $\tau = 1 - 1/\gamma$, on a neighborhood of a critical value of $f$ if the corresponding critical point has the exponent $\gamma$. On a neighborhood of any other point, the new differentiable structure has the local parameter $\int \rho(x) dx$ where $\rho(x)$ is a positive $C^2$-function. With respect to the new differentiable structure, the left and the right derivatives of $f$ at any critical point exist and equal nonzero numbers (see Figure 1). We call the original differentiable structure the old one.

We use the oriented connected one-dimensional smooth manifolds $M$ and $\tilde{M}$, which are the same topological space but with the old and the new differentiable structures, respectively, to study the dynamics of the mapping $f: M \to M$: the distortions of a long composition of a one-dimensional $C^{1+\alpha}$-mapping $f: M \to M$ with only finitely many,
non-recurrent, power law critical points has an estimate like that in the naive distortion lemma and that in the $C^3$-Koebe distortion lemma.

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§2 A Very Good Mappings

Suppose $M$ is an oriented connected one-dimensional $C^2$-Riemannian manifold with Riemannian metric $dx^2$ and associated length element $dx$. Suppose $f : M \mapsto M$ is a continuous mapping. A critical point $c$ of $f$ is a point in $M$ such that

(a) $f$ is not differentiable at this point or

(b) $f$ is differentiable at this point but the derivative of $f$ at this point is zero.

We always assume that $f$ is $C^1$ at a noncritical point $p$, namely there is a neighborhood $U_p$ of $p$ such that the restriction of $f$ to $U_p$ is differentiable and the derivative $(f|U_p)'$ is continuous. We call the
image of a critical point under $f$ a critical value of $f$.

§2.1 A power law critical point.

We give a definition of a power law critical point for the one-dimensional mapping $f : M \mapsto M$ as follows.

**Definition 1.** Suppose $c$ is an isolated critical point of $f$ and suppose there are $\gamma^-, \gamma^+ \geq 1$ such that

$$\lim_{x \to c^-} f'(x)/|x - c|^{\gamma^- - 1} = A$$

and

$$\lim_{x \to c^+} f'(x)/|x - c|^{\gamma^+ - 1} = B$$

exist and equal nonzero numbers. Then we say that $c$ is a power law critical point with the left and right exponents $\gamma^-$ and $\gamma^+$.

![Examples of power law critical points](image)

Figure 2

The following is essentially proved in [J4] (see [J5], too).

**Preliminary Lemma.** Suppose $f : M \mapsto M$ is a continuous mapping and $c$ is a power law critical point with the left and right exponents $\gamma^-$ and $\gamma^+$. Then there is a continuous mapping $\tilde{f} : M \mapsto M$ and a real number $\sigma \neq 0$ such that

(a) the mapping $\tilde{f}$ either has the form

$$\tilde{f} = \begin{cases} -\sigma|x - c|^{\gamma^-} + f(c) & x \leq c, \\ |x - c|^{\gamma^+} + f(c) & x \geq c \end{cases}$$

or

$$\tilde{f} = \begin{cases} \sigma|x - c|^{\gamma^-} + f(c) & x \leq c, \\ -|x - c|^{\gamma^+} + f(c) & x \geq c \end{cases}$$

where $x$ is in a small neighborhood of $c$,

(b) the mapping $f$ is semi-conjugate to the mapping $\tilde{f}$. This means that there is a monotone and continuous mapping $h$ from $M$ onto $M$ such that

$$h \circ f = \tilde{f} \circ h$$

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and $h$ is differentiable at $c$ with $h'(c) > 0$.

Moreover,

(i) if $f$ is $C^{1+\alpha}$ on $x \leq c$ and on $x \geq c$ for some $0 < \alpha \leq 1$, and $r_-(x) = f'(x)/|x-c|^{\gamma-1}$, $x \leq c$, and $r_+(x) = f'(x)/|x-c|^{\gamma+1}$, $x \geq c$, are $C^\beta$ for some $0 < \beta \leq 1$, where $x$ is in a small neighborhood of $c$, then the mapping $h$ can be an orientation-preserving $C^1$-diffeomorphism.

(ii) The mapping $h$ can be an orientation-preserving $C^{1,1}$ or $C^2$-diffeomorphism if and only if $f$ is $C^{1,1}$ or $C^2$ on $x \leq c$ and $x \geq c$, and $r_-(x)$ and $r_+(x)$ are Lipschitz or $C^1$, where $x$ is in a small neighborhood of $c$.

Remark. For a power law critical point of $f$, the left and right exponents are $C^1$-invariants. By this we mean that they are the same numbers for $f$ and for $h \circ f \circ h^{-1}$ whenever $h$ is an orientation-preserving $C^1$-diffeomorphism. When the left and right exponents are the same, we then have an important $C^1$-invariant

$$\sigma = \lim_{x \rightarrow c} \frac{f'(x)}{f'(-x + 2c)}$$

which we call the asymmetry of $f$ at $c$. The number $\sigma$ in Preliminary Lemma is the asymmetry. In the paper [J1], we showed that the asymmetry is an independent $C^1$-invariant.

§2.2 The new differentiable structure associated with a semi-good mapping.

Although the results in the rest of the paper hold for a mapping $f$ with both smooth and non-smooth critical points, but we are only interested in a smooth critical point of $f$. Henceforth we will assume that $f : M \mapsto M$ is a $C^1$-mapping. Moreover we will assume that the left and right exponents of $f$ at a power law critical point are the same.

Definition 2. We say $f$ is a semi-good mapping if

(I) the mapping $f$ has only finitely many power law critical points, 
(II) the set of critical points and the set of critical values of $f$ are disjoint, and if
(III) the exponents of $f$ at two critical points are the same whenever the images of these two points under $f$ are the same.
Suppose \( f : M \mapsto M \) is a semi-good mapping. Let \( CP = \{c_1, \ldots, c_d\} \) be the set of critical points of \( f \) and \( \Gamma = \{\gamma_1, \ldots, \gamma_d\} \) be the set of corresponding exponents. We define a new differentiable structure associated with \( f \) as follows.

Suppose \( \Phi = \{(w_j, W_j)\}_{j \in \Lambda} \) is a \( C^2 \)-atlas of \( M \), this means that \( \{W_j\}_{j \in \Lambda} \) is a cover of open sets of \( M \) and \( \{w_j : W_j \mapsto \mathbb{R}^1\}_{j \in \Lambda} \) is a set of homeomorphisms such that every \( w_{jk} = w_j \circ w_k^{-1} \) is a \( C^2 \)-function whenever \( W_j \) and \( W_k \) are overlap. Suppose every critical value \( v_i = f(c_i) \) is in one and only one chart \((w_i, W_i)\) and \( w_i \) maps the critical value \( v_i \) to 0. For every critical value \( v_i = f(c_i) \), we use \( k_i(x) \) to denote the homeomorphism \( \int_0^x dx/|x|^{\tau_i} : \mathbb{R}^1 \mapsto \mathbb{R}^1 \) where \( \tau_i = 1 - 1/\gamma_i \). Let \( \tilde{w}_j = k_i \circ w_i \) if \( W_j = W_i \) contains a critical value \( v_i = f(c_i) \) and \( \tilde{w}_j = w_j \) if \( W_j \) does not contain any critical values. The set \( \tilde{\Phi} = \{((\tilde{w}_j, W_j))\} \) is another \( C^2 \)-atlas of \( M \). We call the maximal \( C^2 \)-atlas of \( M \) which contains the set \( \tilde{\Phi} = \{((\tilde{w}_j, W_j))\} \) the new differentiable structure associated with \( f \) on \( M \). We denote the topological space \( M \) equipped with this new differentiable structure as a differentiable manifold \( \tilde{M} \).

It is often convenient to think the new differentiable structure associated with \( f \) as a singular metric \( \rho(x)dx \) with respect to \( dx \) and the mapping \( h = \int_0^x \rho(x)dx : M \mapsto M \) as the corresponding change of coordinate on \( M \). The mapping \( \tilde{f} = h \circ f \circ h^{-1} : M \mapsto M \) is the representation of the mapping \( f : \tilde{M} \mapsto \tilde{M} \).

**Lemma 1.** Suppose \( f : M \mapsto M \) is a semi-good mapping and \( CP \) is the set of critical points of \( f \). Then the mapping \( f : \tilde{M} \mapsto \tilde{M} \) is a continuous mapping and at every point \( c_i \in CP \), the left and right derivatives of \( f : \tilde{M} \mapsto \tilde{M} \) exist and equal nonzero numbers.

**Proof.** The proof of this lemma is easy. The reader may do it as an exercise or refer to the proof in [J1, p21].

§2.3 The definition of a very good mapping.

We define a **very good mapping**. Before to give the definition of a very good mapping, we define the term \( C^{1+\alpha} \) for a real number \( 0 < \alpha \leq 1 \) and a semi-good mapping.

Suppose \( f : M \mapsto M \) is a semi-good mapping. Let \( CP \) be the set of critical points of \( f \). Suppose \( \eta_0 \) is the set of the closures of the intervals
of the complement of $CP$. We say a homeomorphism $g : I \mapsto J$ is a $C^{1+\alpha}$-embedding for some $0 < \alpha \leq 1$ if $g$ and $g^{-1}$ are both differentiable with $\alpha$-Hölder continuous derivatives.

**Definition 3.** We say $f$ is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if

(1) the restriction of $f$ to every interval in $\eta_0$ is differentiable with $\alpha$-Hölder continuous derivative,

(2) for every critical point $c_i$, there is a neighborhood $U_i$ of $c_i$ such that the restrictions of $f : \tilde{M} \mapsto \tilde{M}$ to the intersection of $U_i$ and $\{x \leq c_i\}$ and the intersection of $U_i$ and $\{x \geq c_i\}$ are $C^{1+\alpha}$-embeddings.

We will assume that $U_i$ is a closed interval for every $i = 1, \ldots, d$. Suppose $U$ be the union $\bigcup_{i=1}^d U_i$ and $V$ be the closure of the complement of $U$ in $M$.

**Definition 4.** A $C^1$-mapping $f : M \mapsto M$ is a very good $C^{1+\alpha}$-mapping (or a very good mapping) for some $0 < \alpha \leq 1$ if it is a semi-good mapping and satisfies

$(IV)$ $f$ is $C^{1+\alpha}$,

$(V)$ the set $CP$ of critical points and the closure of the post-critical orbits $\bigcup_{n=1}^{\infty} f^n(CP)$ are disjoint and

$(VI)$ there are two constants $K > 0$ and $\nu > 1$ such that for any $O_{x,n} = \{x, f(x), \ldots, f^{(n-1)}(x)\}$ with $O_{x,n} \cap U = \emptyset$, $|(f^k)'(x)| \geq K\nu^k$ for any $1 \leq k \leq n$.

The space of good mappings is a quite large one, for example, it contains all $C^3$ semi-good mappings with nonpositive Schwarzian derivative and finitely many non-recurrent critical points (see, for example, [Mi], [MS] and [J2]).

§ 3 The Distortion Of A Long Composition Of A Very Good Mapping

Suppose $f : M \mapsto M$ is a very good $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$. We always assume that $U$, the union of all $U_i$ in Definition 3, is disjoint with the closure of the post-critical orbits $\bigcup_{n=1}^{\infty} f^n(CP)$. We use $U_i-$ to denote the subset consisting of all points $x$ in $U_i$ with $x \leq c_i$ and use $U_i+$ to denote the subset consisting of all points $x$ in $U_i$ with $x \geq c_i$. Let $W$ be the collection of all $U_i-$ and $U_i+$. Remember that $V$ is the closure of the complement of $U$ in $M$. We say a sequence
\[ I = \{ I_j \}_{j=0}^n \] of intervals of \( M \) is suitable if

(i) \( I_j \) is the image of \( I_{j+1} \) under \( f \) for \( j = 0, \ldots, n-1 \) and

(ii) either \( I_j \) is in \( V \) or \( I_j \) is in some interval in \( W \) for every \( j = 0, \ldots, n \).

For a suitable sequence \( I = \{ I_j \}_{j=0}^n \) of intervals of \( M \), we use \( g_j \) to denote the inverse of the restriction of \( f \circ j \) to \( I_j \). For a pair of points \( x \) and \( y \) in \( I_0 \), we use \( x_j \) and \( y_j \) to denote the images of \( x \) and \( y \) under \( g_j \) and call the ratio \( |g'_n(x)|/|g'_n(y)| \) the distortion of \( f \) at \( x \) and \( y \) along \( I \). We use \( D_{xy} \) to denote the distance between \( \{ x, y \} \) and post-critical orbits \( \cup_{j=1}^\infty f^j(CP) \).

The main result of this paper is the following:

**Lemma 2** (the \( C^{1+\alpha} \)-Denjoy-Koebe distortion lemma). Suppose \( f : M \mapsto M \) is a very good \( C^{1+\alpha} \)-mapping for some \( 0 < \alpha \leq 1 \). There are two positive constants \( A \) and \( B \) such that for any suitable sequence \( I = \{ I_j \}_{j=0}^n \) of intervals of \( M \) and any pair \( x \) and \( y \) in \( I_0 \), the distortion of \( f \) at \( x \) and \( y \) along \( I \) satisfies

\[
|g'_n(x)|/|g'_n(y)| \leq \exp \left( A \sum_{i=0}^n |x_i - y_i|^\alpha + B |x - y|/D_{xy} \right).
\]

**§3.1. The naive distortion lemma.**

Before to prove Lemma 2, we state the naive distortion lemma. Suppose \( g : U \mapsto M \) is a \( C^{1+\alpha} \)-mapping for some \( 0 < \alpha \leq 1 \) where \( U \) is an interval of \( M \). Let \( K \) be the \( \alpha \)-Hölder constant of the derivative of \( g \), this means that \( K \) is the smallest positive constant such that

\[
|g'(x) - g'(y)| \leq K|x - y|^\alpha
\]

for all \( x \) and \( y \) in \( U \). Suppose \( \{ I_j \}_{j=1}^n \) is a sequence of intervals of \( U \) and \( x_i \) and \( y_i \) are two points in \( I_j \) for \( 1 \leq j \leq n \). We also call the product of ratios \( \prod_{j=1}^n |g'(x_j)|/|g'(y_j)| \) the distortion of \( g \) at \( \{ x_j \}_{j=1}^n \) and \( \{ y_j \}_{j=1}^n \). Let \( \beta \) be the minimum of \( |g'| \) on \( \cup_{j=0}^n I_j \).

**Lemma 3** (the naive distortion lemma). The distortion of \( g \) at \( \{ x_j \}_{j=1}^n \) and \( \{ y_j \}_{j=1}^n \) satisfies

\[
\prod_{j=1}^n |g'(x_j)|/|g'(y_j)| \leq \exp \left( \frac{K}{\beta} \sum_{j=0}^n |x_j - y_j|^\alpha \right).
\]
Proof. Take the function $\log x$ at $\prod_{j=1}^{n} |g'(x_j)|/|g'(y_j)|$, we have

$$\log \left( \prod_{j=1}^{n} \frac{|g'(x_j)|}{|g'(y_j)|} \right) = \sum_{j=1}^{n} \left( \log |g'(x_j)| - \log |g'(y_j)| \right).$$

Because $\log x$ is Lipschitz continuous with the Lipschitz constant $1/\beta$ on the interval $[\beta, +\infty)$ and the $\alpha$-Hölder constant of $g'$ on $U$ is $K$, we have that

$$\left| \sum_{j=0}^{n} \left( \log |g'(x_j)| - \log |g'(y_j)| \right) \right| \leq \frac{1}{\beta} \sum_{j=0}^{n} |g'(x_j) - g'(y_j)|$$

which is bounded above by $(K/\beta) \sum_{j=0}^{n} |x_j - y_j|^\alpha$.

§3.2 The proof of $C^{1+\alpha}$-Denjoy-Koebe distortion lemma.

We call $U$, the union of $U_i$ for $i = 1, \cdots, d$, the critical set and $\mathcal{V}$, the closure of the complement of $U$ in $M$, the noncritical set (see Figure 3). Let $\eta_0$ be the set of the closures of the intervals of the complement of the set $CP$ of critical points of $f$ in $M$. Let $\tilde{f} = h \circ f \circ h^{-1} : M \mapsto \tilde{M}$ be the representation of $f : M \mapsto \tilde{M}$, where $h$ is the corresponding change of coordinate. Remember that $U_{i-} = U_i \cap \{x : x \leq c_i\}$ and $U_{i+} = U_i \cap \{x : x \geq c_i\}$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \draw[step=1cm,black,very thin] (0,0) grid (4,2);
  \draw[very thick] (0,1) -- (1,1) -- (1,2) -- (2,2) -- (2,1) -- (3,1) -- (3,2) -- (4,2) -- (4,1);
  \node at (0.5,1.5) {$c_1$}; \node at (1.5,1.5) {$c_2$}; \node at (2.5,1.5) {$c_3$}; \node at (3.5,1.5) {$c_d$};
  \node at (0,0) {$M$}; \node at (0.5,0) {$U_1$}; \node at (1.5,0) {$U_2$}; \node at (2.5,0) {$U_3$}; \node at (3.5,0) {$U_{d+1}$};
\end{tikzpicture}
\caption{Figure 3}
\end{figure}

Let $K_1 > 0$ be the maximum of the $\alpha$-Hölder constants of the derivatives of the restrictions of $f$ to the intervals in $\eta_0$ and $\beta_1 > 0$ be the minimum of the absolute value of the restriction of the derivative $f'$ of $f$ to $\mathcal{V}$.

The restrictions of $\tilde{f}$ to the sets $U_{i-}$ and $U_{i+}$ are $C^{1+\alpha}$-embeddings for $i = 1, \cdots, d$. Let $K_2 > 0$ be the maximum of the $\alpha$-Hölder constants of the derivatives of these restrictions and $\beta_2 > 0$ be the minimum of the absolute value of the derivatives of these restrictions.
The restrictions of $h$ to the intervals of $\mathcal{U}$ are $C^{1,1}$. Let $K_3 > 0$ be the maximum of Lipschitz constants of the derivatives of these restrictions and $\beta_3 > 0$ be the minimum of the absolute value of the derivatives of these restrictions.

The distortion of $f$ along $I$ at $x$ and $y$ satisfies
\[
\frac{|g_n'(x)|}{|g_n'(y)|} = \left| \frac{(f^{\circ n})'(y_n)}{(f^{\circ n})'(x_n)} \right|.
\]
By the chain rule, the ratio $|\frac{(f^{\circ n})'(y_n)}{(f^{\circ n})'(x_n)}|$ equals the product of ratios $|\frac{f'(y_{n-i})}{f'(x_{n-i})}|$ where $i$ runs from 0 to $n-1$. This product can be factored into two products,
\[
\prod_{x_i,y_i \in V} \frac{|f'(y_i)|}{|f'(x_i)|} \quad \text{and} \quad \prod_{x_i,y_i \in U} \frac{|f'(y_i)|}{|f'(x_i)|}.
\]
We note that the subscript $i$ in the products are integers in the range $[1, n]$.

Using Lemma 3 (the naive distortion lemma), we can show that the first product
\[
\prod_{x_i,y_i \in V} \frac{|f'(y_i)|}{|f'(x_i)|} \leq \exp \left( \frac{K_1}{\beta_1} \sum_{i=0}^{n} |x_i - y_i|^\alpha \right).
\]

The second product
\[
\prod_{x_i,y_i \in U} \frac{|f'(y_i)|}{|f'(x_i)|}
\]
can be factored into three products
\[
\prod_{x_i,y_i \in U} \frac{|h'(y_i)|}{|h'(x_i)|} \cdot \prod_{x_i,y_i \in U} \frac{|\tilde{f}'(h(y_i))|}{|\tilde{f}'(h(x_i))|} \cdot \prod_{x_i,y_i \in U} \frac{|h'(f)(x_i))|}{|h'(f)(y_i)|},
\]
by using the formula
\[
f'(x) = \frac{h'(x)\tilde{f}'(h(x))}{h'(f(x))}.
\]
By using Lemma 3 again, the first product
\[ \prod_{x_i, y_i \in \mathcal{U}} \left| \frac{h'(y_i)}{h'(x_i)} \right| \leq \exp \left( \frac{K_3}{\beta_3} \sum_{i=0}^{n} |x_i - y_i| \right) \]
and the second product
\[ \prod_{x_i, y_i \in \mathcal{U}} \left| \frac{\tilde{f}'(h(y_i))}{f'(h(x_i))} \right| \leq \exp \left( \frac{K_2}{\beta_2} \sum_{i=0}^{n} |x_i - y_i|^{\alpha} \right). \]

Suppose \( x_i, y_i \) and \( c_{k(i)} \) are in the same set \( U_{k(i)} \) and \( v_{k(i)} = f(c_{k(i)}) \) is the critical value. Because \( h'(x) = 1/|x - v_{k(i)}|^{\tau_{k(i)}} \) on a neighborhood of \( v_{k(i)} \), where \( \tau_{k(i)} = 1 - 1/\gamma_{k(i)} \) and \( \gamma_{k(i)} \) is the exponent of \( f \) at \( c_{k(i)} \), the third product has the form
\[ \prod_{x_i, y_i \in \mathcal{U}} \left( \frac{|y_i - v_{k(i)}|}{|x_i - v_{k(i)}|} \right)^{\tau_{k(i)}}. \]

We note that \( x_{i-1} = f(x_i) \) and \( y_{i-1} = f(y_i) \) are the points near the critical value \( v_{k(i)} \) for \( x_i \) and \( y_i \) are in the set \( U_{k(i)} \).

To control the third product we write
\[ \frac{|y_{i-1} - v_{k(i)}|}{|x_{i-1} - v_{k(i)}|} = |1 + \frac{y_{i-1} - x_{i-1}}{x_{i-1} - v_{k(i)}}|, \]
which is less than or equal to \( 1 + |x_{i-1} - y_{i-1}|/|x_{i-1} - v_{k(i)}| \), for every pair \( x_i \) and \( y_i \) in \( \mathcal{U} \).

Suppose \( l \) is the smallest positive integer such that \( x_l \) and \( y_l \) are in \( \mathcal{U} \). We consider \( l \) in the two cases. The first case is that \( l = 1 \) and the second case is that \( l > 1 \).

In the first case, the images of \( x_l \) and \( y_l \) under \( f \) are \( x \) and \( y \). We have that
\[ \frac{|x - y|}{|x - v_{k(l)}|} \leq \frac{|x - y|}{D_{xy}}. \]

In the second case, suppose \( I_l \) is the smallest interval containing \( x_l \), \( y_l \) and \( c_{k(l)} \) and \( I_{l-i} = f^{(i)}(I_l) \) for \( i = 0, \ldots, l \). Because the intervals \( I_{l-i} \) are contained in \( \mathcal{V} \) for \( i = 1, \ldots, l - 1 \) (we can always reduce to this
case), by using \((VI)\) of Definition 4 and Lemma 3, there is a constant \(K_4 > 1\) such that

\[
\frac{|x_{l-1} - y_{l-1}|}{|x_{l-1} - v_{k(l)}|} \leq K_4 \frac{|x - y|}{|x - f^{\circ (l-1)}(v_{k(l)})|}.
\]

We note that \(f^{\circ l}(x_l) = x\) and \(f^{\circ l}(y_l) = y\) (see Figure 4-a). This implies that

\[
\frac{|y_{l-1} - x_{l-1}|}{|x_{l-1} - v_{k(l)}|} \leq K_4 \frac{|x - y|}{|x - f^{\circ (l-1)}(v_{k(l)})|} \leq K_4 \frac{|x - y|}{D_{xy}}.
\]

\[\text{Figure 4}\]

For any \(q > l\) with \(x_q\) and \(y_q\) in \(U\), let \(m_q\) be the smallest positive integer such that \(x_{q-m_q}\) and \(y_{q-m_q}\) are in \(U\) (see Figure 4-b).

Suppose \(I_q\) is the smallest interval containing \(x_q\), \(y_q\) and \(c_{k(q)}\) and \(I_{q-i} = f^{\circ i}(I_q)\) for \(i = 0, \cdots, m_q\). The intervals \(I_{q-i}\) for \(i = 1, \cdots, m_q - 1\) are contained in \(V\) (we always can reduce to this case). By using \((VI)\) of Definition 4 and Lemma 3, there is a positive constant, we still denote it as \(K_4\), such that
\[
\frac{|y_{q-1} - x_{q-1}|}{|x_{q-1} - v_{k(q)}|} \leq K_4 \frac{|y_{q-m_q} - x_{q-m_q}|}{|x_{q-m_q} - f^{\circ(q-m_q)}(c_{k(q)})|}.
\]

Because \( x_{q-m_q} \) is in \( \mathcal{U} \) and \( f^{\circ(q-m_q)}(c_{k(q)}) \) is not in \( \mathcal{U} \), the number \( |x_{q-m_q} - f^{\circ(q-m_q)}(c_{k(q)})| \) is bigger than or equal to \( L \), the distance between the set \( \mathcal{U} \) and the closure of the post-critical orbits \( \cup_{n=1}^{\infty} f^{\circ n}(CP) \). Hence we get

\[
\frac{|x_{q-1} - y_{q-1}|}{|x_{q-1} - v_{k(q)}|} \leq K_4 \frac{|x_{q-m_q} - y_{q-m_q}|}{L}.
\]

Now the third product satisfies that

\[
\prod_{x_i, y_i \in \mathcal{U}} \left( \frac{|y_i - v_{k(i)}|}{|x_i - v_{k(i)}|} \right)^{\tau_{k(i)}} \leq \exp \left( K_4 |x - y| + \frac{K_4}{L\tau} \sum_{i=1}^{n} |x_i - y_i| \right),
\]

where \( \tau \) is the maximum of \( \tau_j = 1 - 1/\gamma_j \) for \( j = 1, \ldots, d \).

We now prove Lemma 2 by putting all the estimates together and

\[
A = K_1/c_1 + (K_3^\alpha K_2)/c_2 + K_3/c_3 + K_4/(L\tau) \quad \text{and} \quad B = K_4/\tau.
\]

§3.3 A larger class of one-dimensional mappings.

We can actually prove Lemma 2 for a wider class of one-dimensional mappings as follows.

Suppose \( f : M \mapsto M \) is a \( C^1 \)-mapping with only power law critical points. Let \( CP = \{c_1, \ldots, c_d\} \) be the set of critical points of \( f \) and \( \Gamma = \{\gamma_1, \ldots, \gamma_d\} \) be the set of corresponding exponents. Suppose \( \eta_0 \) be the set of the closures of the intervals of the complement of the set \( CP \) of critical points of \( f \) in \( M \).

**Definition 5.** We say \( f \) is \( C^{1+\alpha} \) for some \( 0 < \alpha \leq 1 \) if

1. the restriction of \( f \) to every interval in \( \eta_0 \) is differentiable with \( \alpha \)-Hölder continuous derivative,

2. for every critical point \( c_i \), there is a neighborhood \( U_i \) of \( c_i \) such that the functions \( r_{i,-}(x) = f'(x)/|x - c_i|^{\gamma_i - 1} \) for \( x < c_i \) in \( U_i \) and \( r_{i,+}(x) = f'(x)/|x - c_i|^{\gamma_i - 1} \) for \( x > c_i \) in \( U_i \) are \( \alpha \)-Hölder continuous.
Suppose $U$ is the union of $U_i$ for $i = 1, \cdots, d$ and $V$ is the closure of the complement of $U$ in $M$. Let $U_{i-}$ be the subset consisting of all points $x$ in $U_i$ with $x \leq c_i$ and $U_{i+}$ be the subset consisting of all points $x$ in $U_i$ with $x \geq c_i$, for $i = 1, \cdots, d$. Suppose $W$ be the collection of all $U_{i-}$ and $U_{i+}$.

**Definition 6.** Suppose $f : M \mapsto M$ is a $C^1$-mapping. We say $f$ is a good $C^{1+\alpha}$-mapping (or good mapping) for some $0 < \alpha \leq 1$ if

(I) $f$ has only finitely many power law critical points,

(II) $f$ is $C^{1+\alpha}$,

(III) there is a positive integer $N$ such that the set $CP$ of critical points and the closure of the set $\bigcup_{n=N}^\infty f^n(CP)$ are disjoint,

(IV) there are two constants $K > 0$ and $\nu > 1$ such that for any $O_{x,n} = \{x, f(x), \cdots, f^{c(n-1)}(x)\}$ with $O_{x,n} \cap U = \emptyset$, $|f^k(x)| \geq K\nu^k$ for any $1 \leq k \leq n$.

We say a sequence $I = \{I_j\}_{j=0}^n$ of intervals of $M$ is suitable if

(i) $I_j$ is the image of $I_{j+1}$ under $f$ for $j = 0, \cdots, n - 1$ and

(ii) either $I_j$ is in $V$ or $I_j$ is in some interval in $W$, for every $j = 0, \cdots, n$.

**Lemma 4 (C$^{1+\alpha}$-Denjoy-Koebe distortion lemma).** Suppose $f$ is a good $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$. There are positive constants $A$ and $B$ such that for any suitable sequence $I = \{I_j\}_{j=0}^n$ of intervals of $M$ and any pair $x$ and $y$ in $I_0$, the distortion of $f$ at $x$ and $y$ along $I$ satisfies

$$\frac{|g'_n(x)|}{|g'_n(y)|} \leq \exp\left(A \sum_{i=0}^n |x_i - y_i|^{\alpha} + \frac{B|x - y|}{D_{xy}}\right)$$

where $D_{xy}$ is the distance between the set $\{x, y\}$ and the post-critical orbit $\bigcup_{n=1}^\infty f^n(CP)$.

The idea of the proof of this lemma is the same as that of Lemma 2. Details will be omitted.

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Figure 1