ANOSOV FLOWS AND DYNAMICAL ZETA FUNCTIONS
(ERRATA)

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Abstract. This errata fixes a mistake in the part of [1] which proves a spectral
gap for contact Anosov flows with respect to the measure of maximal entropy
([1, Section 7]). However, the first part of [1], in which it is proved that the
Ruelle zeta function is meromorphic, is unaffected.

1. The Mistake

Equation [1, Equation (7.14)] is wrong, since it does not take into account the
factor $e^{z\tau_W \circ H_{\beta,W}}$ from [1, Equation (7.10)] and estimates incorrectly the norm of
$\varphi_{k,\beta,i} \circ H_{\beta,i,W}$. The correct version of [1, Equation (7.14)] is (see (6) below):

\begin{equation}
(1) \| e^{z\tau_W \circ H_{\beta,i,W}} \hat{g}_{k,\beta,i,W} \|_{E_{\epsilon} \circ \mathcal{W}_{\beta,i}} \leq C_{\#}(r^{-1} + |z|) \frac{(kr)^n e^{-akr}}{(n-1)!} \|g\|_{E_{\epsilon} \circ \mathcal{W}_{\beta,i}}.
\end{equation}

Unfortunately, this weaker estimate does not suffice to carry out the proof of [1, Proposition 7.5] as presented in [1] (i.e. using [1, Equation (7.16)]).

2. Correction

Nevertheless, [1, Theorem 2.4] holds under a stronger assumption (namely some
homogeneity), as we shall show here. In particular, it applies to small perturbations
of constant curvature geodesic flows in any dimension. To simplify the argument we
did not try to optimise the estimate of the size of the perturbation. Before stating
the correct results we need to recall and introduce some notation.

Let $C_0, c_0 > 0$, and $\lambda_+(x,t), \lambda_-(x,t) > 0$ be such that, for each $x \in M$ and $t > 0$,
\begin{align*}
\sup_{v \in E^s(x)} \| D_x \varphi_t \|_{E^s(x)} &\leq C_0 e^{\lambda_+(x,t)} \quad \text{and} \\
\inf_{v \in E^s(x)} \| D_x \varphi_t \|_{E^s(x)} &\geq c_0 e^{\lambda_-(x,t)}.
\end{align*}

Also, let
\begin{align*}
\lambda_+(t) &= \sup_{x \in M} \lambda_+(x,t) \quad ; \quad \lambda_-(t) = \inf_{x \in M} \lambda_-(x,t).
\end{align*}

and, for some $n_0$ large enough, $\lambda_+ = \sup_{t \geq n_0} \frac{\lambda_+(t)}{t}$, $\lambda_- = \inf_{t \geq n_0} \frac{\lambda_-(t)}{t}$. In [1] we use the notation $\tilde{\omega} = \frac{2\lambda_-}{\lambda_+}$, $\omega' = \min\{1, \tilde{\omega}\}$.

Next, we introduce a parameter $\vartheta > 0$ which measures the homogeneity. Let $J^s \varphi_t$ be the stable Jacobian of the flow. Given $n_0 \in \mathbb{N}$, we assume, for all $t \geq n_0$,
\begin{equation}
(1) \left[ \sup_{x \in M} \ln J^s \varphi_t(x) - \inf_{x \in M} \ln J^s \varphi_t(x) \right] \leq \vartheta \lambda_-(t) d_s.
\end{equation}

With this notation we can state a correct version of [1, Theorem 2.4].
Theorem 1. For any $C^r$, $r > 2$, contact flow with

$$\sqrt{5} - 1 < \hat{\omega}; \quad \vartheta < \frac{(\omega')^2 + \omega' - 1}{2d_a(1 + \omega')},$$

there exists $\tau_*>0$ such that the Ruelle zeta function is analytic in $\{z \in \mathbb{C} : \Re(z) \geq h_{\text{top}}(\phi_1) - \tau_*\}$ apart from a simple pole at $z = h_{\text{top}}(\phi_1)$.

The rest of this note contains the proof of Theorem 1.

First, it suffices to prove [1, Proposition 7.5] under the hypotheses of Theorem 1, since the derivation of [1, Theorem 2.4] from [1, Proposition 7.5], holds unchanged.

By [1, Remark 7.1] we can restrict the discussion to $d_\alpha = (d - 1)/2$ forms. Since the first inequality in [1, Proposition 7.5] is correct, we need only prove the second. Also [1, equation (7.6)] is correct, hence it suffices to estimate the $\| \cdot \|_{1+\eta}$ norm of some power of $\hat{R}_n(z)$. Indeed, by [1, equation (7.7) and the previous displayed equation], for each $z = a + ib$, $a \geq \sigma d_\alpha$, and $\eta \in [0,1]$,

$$\left\| \hat{R}_n(z)^3 h \right\|_{1+\eta} \leq \frac{C_\eta}{(a - h_{\text{top}}(\phi_1) + \lambda \eta)^n (a - h_{\text{top}}(\phi_1))^2 n} \| h \|_{\eta}^3 + \frac{C_\eta}{(a - h_{\text{top}}(\phi_1))} \left\| \hat{R}_n(z)^2 h \right\|_{1+\eta}.$$

We will use the above equation instead of [1, equation (7.7)].

Remark 2. The estimate (2) can be restricted to forms proportional to the volume on the stable manifold. More precisely, given a stable manifold $W$, if $\{v_i^u\}_{i=1}^{d_u}$, $\{v_i^s\}_{i=1}^{d_s}$ are a base for the tangent space of $W$ and the unstable foliation, respectively, and $\{dx_i\}_{i=1}^{2d_s+1} = \{dx_i^s, dx_i^u\}_{i=1}^{d_s} \cup \{dx_0\}$ the dual base ($dx_0$ being the flow direction), then for all $g$ not proportional to $w^s := dx_1^s \wedge \cdots \wedge dx_{d_s}^s$, we have

$$\left| \int_W \langle g, \hat{R}_n(z)^3 h \rangle \right| \leq \frac{C_\eta}{(a - h_{\text{top}}(\phi_1) + \lambda \eta)^{3n}} \| h \|_{\eta}^3,$$

which yields already the required estimate. Hence, from now on, by $\Gamma^{d_s,\alpha}_{c}(\hat{W}_+)$, defined in [1, Section 3.2], we mean the subset of forms proportional to $w^s$.

Remark 3. If $v \in \mathcal{V}^n$, then the Lie derivative $L_v$ acting on the above $d_s$ forms is well defined even for Hölder vector fields. Indeed, the pushforward by the flow generated by $v$ yields a quantity proportional to the Jacobian of the unstable holonomy which is well defined, together with its derivative along the unstable direction.

Next, we must estimate the right hand side of (2) let $g \in \Gamma^{d_s,1+\eta}_{c}$ and $h \in \Omega_{0,1}^{d_s}$,

$$\int_{W_n,g} \langle g, \hat{R}_n(z)^2 h \rangle = \sum_{k,\beta,i} \sum_{W \in \mathcal{W}_{k,\beta,i}} \int_{\tilde{W}} \langle \hat{g}_{k,\beta,i}, \hat{R}_n(z)h \rangle$$

$$= \hat{g}_{k,\beta,i} = \varphi_{k,\beta,i} \frac{s_{kr}^{n-1} J_{kr} \phi_{kr} \circ \phi_{kr}^* \phi_{kr} \circ \phi_{kr}}{e^{c^2 (kr + \gamma_*) (n - 1)!}} g$$

$$\varphi_{\beta,\gamma,i}(x) = \psi_{\beta}(x) \Phi_{\beta,i}(\Theta_{\beta}(x)) p(r^{-1} \gamma_*(x)) |V(x)|^{-1}.$$
Recall that $\tau_W : \tilde{W} \cong \cup_{l \in [-r,2r]} \tilde{\phi}_l W \to \mathbb{R}$ is defined by $\phi_{-\tau_W(x)}(x) \in W$.3

Next, as in [1, Equation (7.13)], we want to “project” $\hat{g}_{k,\beta,i}$ from $\tilde{W}$ to some preferred manifold $\tilde{W}_{k,\beta,i}$. To this end, we need a refinement of [1, Lemma 7.3].

**Lemma 4.** For each $\alpha \in \mathcal{A}$, $W, W' \in \Sigma_\alpha$ such that $H_{W,W'}(\tilde{W}) \subset \tilde{W}_+, \varphi \in \Gamma_{e^{d_{\varphi}}(\tilde{W})}$, $\varphi \in \Gamma_{e^{d_{\varphi}}(\tilde{W})}$, $q \in [0,1]$, supported in a ball of size $r$, there exists $\hat{\varphi} \in \Gamma_{e^{d_{\varphi}}(\tilde{W})}$, such that for all $h \in \Omega_{d_s}$ we have

$$\left| \int_{\tilde{W}} \langle \varphi, h \rangle - \int_{\tilde{W}_+} \langle \hat{\varphi}, h \rangle \right| \leq C_{\#} \| \hat{\varphi} \|_{\Gamma_{e^{d_{\varphi}}(\tilde{W})}}.$$  

**Proof.** Working in appropriate coordinates we can write $\tilde{W}_+$ as $\{(\xi, 0, \tau)\}_{(\xi, \tau) \in \mathbb{R}^d + 1}$, and $\tilde{W}$ as $\{(\xi, \varphi, \tau)\}_{(\xi, \varphi, \tau) \in \mathbb{R}^d + 3}$ for $\varphi = d(W, W') e_1$.

We can describe the unstable foliation by $U(\xi, \eta, \tau) = (U(\xi, \eta), \eta, \Psi(\xi, \eta) + \tau)$, $U(\xi, 0, \tau) = (\xi, 0, \tau)$. Then the intersection between $\tilde{W}_+ = \{(\xi, \varphi, \tau)\}_{(\xi, \varphi, \tau) \in \mathbb{R}^d + 1}$ and the fiber $U(\xi, \tau)$ gives the holonomy $\tilde{H}_U(\xi, \tau) = (U(\xi, \eta), \varphi, \Psi(\xi, \varphi, \tau) + \tau)$.

As mentioned in Remark 2, $\varphi = \hat{\varphi} d\xi_1 \wedge \cdots \wedge d\xi_d$, hence we can assume w.l.o.g. that $\hat{h} = \hat{h} d\xi_1 \wedge \cdots \wedge d\xi_d$, for some function $\hat{h}$. It is then natural to define, for each $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$ and $\eta \in \mathbb{R}^d$,

$$\tilde{H}_s(\xi, \eta, \tau) = U(U^{-1}(\xi, \eta, \tau) + (0, s \varphi, 0)).$$

Since, $\tilde{H}_0(\xi, \eta, \tau) = (\xi, \eta, \tau)$ and $\tilde{H}_s \circ \tilde{H}_r = \tilde{H}_{s+r}$, we have just defined a flow, let $\tilde{\varphi} = \langle \varphi, \varphi, \tau \rangle$ be the associated vector field. Note that, by construction, $\tilde{\varphi}$ is a vector field in the unstable direction. By the regularity of the holonomy (see the discussion at the beginning of [1, Appendix E]), we have $\| \tilde{\varphi} \| \leq C_{\#} d(W, W')$. Hence, $\tilde{\varphi} = d(W, W')^{-1} \tilde{\varphi} \in \mathcal{V}^u$.

Since $\tilde{H}_s h = \hat{h} \circ \tilde{H}_s J_{\tilde{H}_s} d\xi_1 \wedge \cdots \wedge d\xi_d$, $J_{\tilde{H}_s}$ is the Jacobian of $\tilde{H}_s$, we have

$$\int_{\tilde{W}} \langle \varphi, h \rangle = \int_{\tilde{W}_+} \langle \tilde{\varphi}, h \rangle \circ \tilde{H}_s \cdot J_{\tilde{H}_s} = \int_{\tilde{W}_+} \tilde{\varphi} \circ \tilde{H}_1 \hat{h} \circ \tilde{H}_1 J_{\tilde{H}_s}$$

$$= \int_{\tilde{W}_+} \int_0^1 \tilde{\varphi} \circ \tilde{H}_1 \frac{d}{ds} (\hat{h} \circ \tilde{H}_s J_{\tilde{H}_s}) \, ds + \int_{\tilde{W}_+} \tilde{\varphi} \circ \tilde{H}_1 \hat{h}.$$  

Since

$$\frac{d}{ds} (\hat{h} \circ \tilde{H}_s J_{\tilde{H}_s}) = \frac{d}{ds} (d\xi_1 \wedge \cdots \wedge d\xi_d, \tilde{H}_s h) = (d\xi_1 \wedge \cdots \wedge d\xi_d, \tilde{H}_s h)$$

$$= (d\xi_1 \wedge \cdots \wedge d\xi_d, L_{\tilde{H}_s} h) \circ \tilde{H}_s J_{\tilde{H}_s},$$

it is convenient to define,

$$\hat{\varphi} = \tilde{\varphi} \circ \tilde{H}_1 (\xi, \tau) d\xi_1 \wedge \cdots \wedge d\xi_d$$

$$\psi_{s} = \tilde{\varphi} \circ \tilde{H}_1 \circ \tilde{H}_{-s}^{-1} d\xi_1 \wedge \cdots \wedge d\xi_d,$$

3The point of the above equation is that it allows one to go from an integral over a strong stable manifold to integrals over weak stable manifolds. See [1, Section 3] for the necessary definitions. To compare the formulae below with [1, equations (7.9, 7.10, 7.11)] recall that the flow is contact, hence $\int \psi = 1$, and $(-1)^{d_{\psi}(d-\xi_d)} = (-1)^{d_{\psi}(d-\xi_d+1)} = 1$. Also, recall that $\sum_{k \in \mathbb{Z}} p(k + \epsilon) = 1$ and $\text{supp}(\rho) C \subset (-1, 1)$. Finally, the minus sign in front of $z$ in [1, equation (7.10)] is a misprint and, just before [1, Equation (7.9)], the definition of $\tau_W$ has a minus sign missing due to a misprint.
which, setting \( \hat{W}_s = \{(\xi, s, \tau)\} \) with \( s \in \mathbb{R}, \tau \in \mathbb{R}_{d+1} \), allows to write
\[
\int_{\hat{W}_s} \langle \varphi, h \rangle = \int_{\hat{W}_s} \int_0^{1} \langle \psi_s, L\hat{w}h \rangle \circ \mathbb{H}_s J\mathbb{H}_s ds + \int_{\hat{W}_s} \langle \hat{\varphi}, h \rangle \\
= \int_{\hat{W}_s} \int_0^{1} \langle \psi_s, L\hat{w}h \rangle \circ \mathbb{H}_s J\mathbb{H}_s ds + \int_{\hat{W}_s} \langle \hat{\varphi}, h \rangle \\
= d(W, W') \int_0^{1} ds \int_{\hat{W}_s} \langle \psi_s, L\hat{w}h \rangle + \int_{\hat{W}_s} \langle \hat{\varphi}, h \rangle.
\]
From the above equation the Lemma follows, since \( \|\psi_s\|_{\Gamma^{\frac{\alpha}{2}, \frac{\alpha}{2}+\eta}(\hat{W}_s)} \leq C_\# \|\varphi\|_{\Gamma^{\frac{\alpha}{2}, \frac{\alpha}{2}+\eta}(\hat{W})} \) and \( \|\hat{\varphi}\|_{\Gamma^{\frac{\alpha}{2}, \frac{\alpha}{2}+\eta}(\hat{W})} \leq C_\# \|\varphi\|_{\Gamma^{\frac{\alpha}{2}, \frac{\alpha}{2}+\eta}(\hat{W})} \). The extra \( r \) comes from the size of the support of \( \varphi \), and hence of \( \psi_s \), in the flow direction.

Next, following verbatim \cite{1}, and using Lemma 4 (with \( q = 1 \)), we obtain the equivalent of \cite[Equation (7.13)]{1}: for each \( g \in \Gamma^{d+1+\eta}(W_\alpha, \mathcal{G}) \),
\[
\int_{W_\alpha, \mathcal{G}} \langle g, \hat{R}_n(z)^2 \rangle h = \sum_{k, \beta, \iota \in W} \sum_{\hat{W}_{\beta, \iota}} \int_{\hat{W}_{\beta, \iota}} \langle \hat{g}_{k, \beta, \iota, W}, \hat{R}_n(z) \rangle h \\
+ \sum_k \mathcal{O} \left( \frac{r^2 (kr)^{n-1} \| \hat{R}_n(z) h \|_{\Gamma^{d+1+\eta}} \| g \|_{\Gamma^{d+1+\eta}}}{(n-1)! C_\#^2 \kappa kr} \right),
\]
where \( \hat{g}_{k, \beta, \iota, W} \) is given by (4), with \( \mathbb{H}_1 = \mathbb{H}_{\beta, \iota, W} \) being the holonomy between \( \hat{W}_{\beta, \iota} \) and \( W \). Note that equation (4) was implicitly used (but missing) in \cite{1}.

Next, we slightly depart from \cite{1} insofar as we simplify immediately the expression of \( \hat{g}_{k, \beta, \iota, W} \) instead of doing it during the proof of \cite[Lemma 7.10]{1}.

By (3) we can write
\[
\hat{g}_{k, \beta, \iota} = \varphi_{k, \beta, \iota} e^{-\tau \hat{w}} \hat{g}_{k, \beta, \iota}.
\]
Hence, setting \( \varphi_{k, \beta, \iota, W} = \varphi_{k, \beta, \iota} \circ \mathbb{H}_{\beta, \iota, W} \), by the first line of (4) we have
\[
\hat{g}_{k, \beta, \iota, W} = \varphi_{k, \beta, \iota, W} e^{-\tau \hat{w} \circ \mathbb{H}_{\beta, \iota, W}} \hat{g}_{k, \beta, \iota, W},
\]
where, recalling \cite[Equation (7.12)]{1},
\[
\| \hat{g}_{k, \beta, \iota, W} \|_{\Gamma^{d+1+\eta}(\hat{W}_{\beta, \iota})} \leq C_\# (kr)^{n-1} e^{-akr} \| g \|_{\Gamma^{d+1+\eta}(W)}
\]
\[
\| \varphi_{k, \beta, \iota, W} \|_{\Gamma^{d+1+\eta}(\hat{W}_{\beta, \iota})} \leq C_\# r^{1-1}.
\]
Hence, setting \( \varepsilon_{k, \beta, \iota, W} = \hat{g}_{k, \beta, \iota, W}(x^\iota) \),
\[
\| \hat{g}_{k, \beta, \iota, W} - \varepsilon_{k, \beta, \iota, W} \|_{\Gamma^{d+1+\eta}(\hat{W}_{\beta, \iota})} \leq C_\# r^{1-1} (kr)^{n-1} e^{-akr} \| g \|_{\Gamma^{d+1+\eta}(W)}.
\]
Next, setting \( \Delta^*_W(\xi) = \tau_W \circ \mathbb{H}_{\beta, \iota, W}(\xi) - \xi_{2d+1} \) and \( w_W(\xi) = \mathbb{H}_{\beta, \iota, W}(\xi) - \xi_{2d+1} \), we have that \cite[Equations (7.27) and (7.30)]{1} implies, for all \( \zeta = (\zeta, 0) \) with \( \| \zeta \| \leq r \),
\[
\| \Delta^*_W(\xi + \zeta) - \Delta^*_W(\xi) - d\sigma_0(w_W(x^\iota), \zeta) \| \leq C_\# r^{2-1} \| w_W(x^\iota) \|^\# \| \zeta \| \| \zeta \|^\# \\
\leq C_\# r^{2+1}.
\]
\[\text{4Here, again, we are computing using some appropriate coordinates.}\]
We impose, for \( \varsigma \) small enough,
\[
|b| \leq r^{-2-\varsigma} + \varsigma,
\]
and define
\[
\mathcal{G}_{k,\beta,i} = \sum_{W \in \mathcal{W}_{k,\beta,i}} R_{k,\beta,i}(W)
\]
and
\[
\mathcal{G}_{k,\beta,i,W}(\xi) = \varphi_{k,\beta,i,W}(\xi)e^{-dz_{\beta_0}(W_{k,\beta,i},Z_{k,\beta,i})}G_{k,\beta,i,W}(\xi).
\]

Letting \( D_{k,\beta,i} \) and recalling (8), (9) and [1, Equation (7.30)], we can write, for \( \varsigma \leq \nu' \),
\[
\|g_{k,\beta,i} - g^*_{k,\beta,i}\|_{\Gamma^2(\tilde{W}_{k,\beta,i})} \leq C\#^{\nu} D_{k,\beta,i}\|g\|_{\Gamma^{d_1+\nu}}
\]
\[
\|g_{k,\beta,i}\|_{\Gamma^2(\tilde{W}_{k,\beta,i})} + \|g^*_{k,\beta,i}\|_{\Gamma^2(\tilde{W}_{k,\beta,i})} \leq C\# \left\{ \frac{1}{r} + \frac{|b|}{r^{-2+\nu'}} \right\} D_{k,\beta,i}\|g\|_{\Gamma^{d_1+\nu}}.
\]

Note that, by the definition of \( R(z) \) in [1, Section 7.1], [1, Equation (4.11)], [1, Equation (4.17)] and the related notation, for all \( n \geq c_*\log r^{-1} \), with \( c_* \) large enough,

\[
\sum_{k,\beta,i} \left| \int_{\tilde{W}_{k,\beta,i}} \langle [g_{k,\beta,i} - g^*_{k,\beta,i}], R_n(z)h \rangle \right| = \int_{c_*n}^{\infty} dt e^{-zt} \frac{t^{n-1}}{(n-1)!} 
\]
\[
\times \sum_{k,\beta,i} \sum_{k' \in K_\beta} \int_{\tilde{W}_{k',\beta,i}} J_{W_{k',\beta,i}}(\phi_i) \langle \phi_i^*, [g_{k,\beta,i} - g^*_{k,\beta,i}], h \rangle 
\]
\[
\leq C\# \int_{c_*n}^{\infty} dt e^{-at} \frac{t^{n-1}}{(n-1)!} \sum_{k,\beta,i} \sum_{k' \in K_\beta} r^{1+\varsigma} D_{k,\beta,i}\|g\|_{\Gamma^{d_1+\nu}}\|h\|_{\Gamma^2}^p 
\]
\[
\leq C\# \int_{c_*n}^{\infty} dt e^{-(a+h_{\text{top}}(\phi))(t+s)} \frac{t^{n-1}}{(n-1)!} \frac{\theta^{n-1} s^n}{(a-h_{\text{top}}(\phi))^n} \|h\|_{\Gamma^{d_1+\nu}}^p 
\]
\[
\leq C\#(a-h_{\text{top}}(\phi))^{-2n\nu'}\|g\|_{\Gamma^{d_1+\nu}}^p \|h\|_{\Gamma^2}^p.
\]

Hence, by (5), (12) and [1, Equation (7.6)], we can write
\[
\int_{W_{k,\beta,i}} \langle g, R_n(z)h \rangle = \sum_{k,\beta,i} \int_{\tilde{W}_{k,\beta,i}} \langle g_{k,\beta,i}, R_n(z)h \rangle + O \left( \frac{r^{n}\|g\|_{\Gamma^{d_1+\nu}}}{(a-h_{\text{top}}(\phi))2n\|h\|_{\Gamma^2}} \right) 
\]
\[
+ O \left( \frac{\|h\|_{\Gamma^2}\|g\|_{\Gamma^{d_1+\nu}}}{(a-h_{\text{top}}(\phi)+\lambda)^n(a-h_{\text{top}}(\phi))^{\nu'}} \right).
\]

To estimate the integral on the right hand side of (13) we define, similarly to [1]:
\[
\mathcal{G}^*_{k,\beta,i,A} = \sum_{W \in \mathcal{W}_{k,\beta,i}} \sum_{W' \in \mathcal{W}_{k,\beta,i}(W)} \langle g_{k,\beta,i,W}, g_{k,\beta,i,W'} \rangle
\]
\[
\mathcal{G}^*_{k,\beta,i,B} = \sum_{W \in \mathcal{W}_{k,\beta,i}} \sum_{W' \in \mathcal{W}_{k,\beta,i}(W)} \langle g_{k,\beta,i,W}, g_{k,\beta,i,W'} \rangle.
\]
To conclude, we need Lemmata 5 and 6 which are refinements of [1, Lemma 7.9] and [1, Lemma 7.10], respectively. The proof of Lemma 6 follows closely [1, Lemma 7.10], but it applies the same logic to different objects. Conversely, Lemma 5 differs from [1, Lemma 7.9] as we take advantage of our new homogeneity hypothesis (1).

**Lemma 5.** If \( c_\alpha \geq n_0 \) and \( C_\# |b|^{-\frac{\lambda(C_\alpha)}{n_0}} \leq \vartheta \leq C_\# r^{\frac{1+\ell d_s}{1+\gamma(W)}} \), for some \( \gamma > 0 \), and \( \vartheta \in (0,1) \), we have

\[
\| \Theta_{k,\beta,i,A} \|_\infty \leq C_\# D_{k,\beta,i}^{d_s} \| g \|^{2}_{[r^{d_s + 1 + \gamma(W)}].}
\]

**Proof.** The lower bound and the fact that the upper bound is bounded by the ratio between the volume of \( \phi_t(D^u(W)) \) and \( \phi_t(D^s(W)) \) is proven exactly as in [1, Lemma 7.9]. The novelty here consists in a different estimate of such a ratio.

Let \( t_0 \in \mathbb{N} \) be such that \( e^{\lambda(-t_0)} g = 1 \). Then, for each \( x \in D^s_\phi(W) \), let \( B(x) \) be an unstable disc of diameter 1 and centred at \( \phi_{t_0}(x) \), clearly \( B(x) \subset \phi_{t_0}(D^s_\phi(W)) \).

Thus we can cover \( \phi_{t_0}(D^u_\phi(W)) \) with \( N_\vartheta = C_\# |\phi_{t_0}(D^u_\phi(W))| \) discs. On the other hand, arguing analogously, we can find \( N_r = C_\# |\phi_{t_0}(D^s_\phi(W))| \) disjoint unstable discs of diameter 1 contained in \( \phi_{t_0}(D^s_\phi(W)) \).

By (1) and since the flow is contact, setting \( J^u := \inf_{x \in M} J^u_{\phi, t_0}(x) \), we have

\[
\frac{N_r}{N_\vartheta} \geq C_\# \frac{|\phi_{t_0}(D^u_\phi(W))|}{|\phi_{t_0}(D^s_\phi(W))|} \geq C_\# \frac{J^u_{r \vartheta^{d_s}}}{J^u_{\vartheta^{d_s}}} = C_\# (r \vartheta^{d_s - 1} \vartheta^{d_s}) \geq C_\# r^{-\gamma}.
\]

On the other hand by [1, Lemmata C.1, C.3] we have that all the discs of radius one grow under the dynamics at the same rate (given by the topological entropy), hence for all \( t \geq t_0 \), we have the required estimate

\[
\frac{|\phi_t(D^u_\phi(W))|}{|\phi_t(D^s_\phi(W))|} \geq C_\# r^{-\gamma}.
\]

□

**Lemma 6.** For \( |b| \leq r^{-2-\gamma + \epsilon} \) we have

\[
\left| \int_{\hat{W}_{\beta,i}} \Theta_{k,\beta,i,B}^* \right| \leq C_\# |b|^{-\gamma} \vartheta^{-d_s} D_{k,\beta,i}^{2} \| g \|^{2}_{[r^{d_s + 1 + \gamma(W)}].}
\]

**Proof.** For future convenience let us set \( (\eta^+_{W,W',i}, \eta^-_{W,W',i}, \eta^0_{W,W',i}) = (\eta_W, \eta_W', i) = w_W(x^i) - w_W(x^i), w_W(x^i) + w_W(x^i) \). By assumption \( \|\eta_W, \eta_W'\| \geq \vartheta \).

Also, it is convenient to work in coordinates \( (\xi, \eta, \tau) \), \( \xi, \eta \in \mathbb{R}^{d_x} \), in which \( x_i = 0 \) and \( W_{\beta,i} \subset \{(\xi, 0) : \xi \in \mathbb{R}^{d_x} \} \) and \( d\alpha_0 = \sum_{i=1}^{d_x} d\xi_i \wedge d\eta_i \). We must estimate

\[
\int_{\hat{W}_{\beta,i}} \langle \Theta_{k,\beta,i,\hat{W}}^{*}, \Theta_{k,\beta,i,\hat{W'}}^{*} \rangle = \int_{\hat{W}_{\beta,i}} \varphi_{k,\beta,i,\hat{W}}(\xi) \varphi_{k,\beta,i,\hat{W'}}(\xi)
\]

\[
\times \varphi_{k,\beta,i,\hat{W}}(\tau) \hat{c}_{k,\beta,i,\hat{W}}(\tau) e^{-i\alpha_0(\eta_W, \eta_W', i, \xi - x^i)} e^{-i\alpha_0(\eta^0_{W, W', i}, \xi - x^i)}.
\]

As in [1, Section 7.2] we choose \( y_{W,W'} = (-\eta^+_{W,W',i}, 1, 1, -\eta^-_{W,W',i}) \) which implies that \( \hat{d}\alpha_0(\eta_W, \eta_W', i, y_{W,W'}) = \hat{d}\alpha_0(\eta^0_{W, W', i}, x_{2d_x + 1}) = 0 \). Also, let \( \Sigma_W = \{(\xi, \tau) \in \mathbb{R}^{d_x + 1} \mid (\xi, \eta_W, \eta_W', i) = 0 \} \) and

\[
A(\xi, s, \tau) = \varphi_{k,\beta,i,W}(\xi, 0, \tau) + s y_{W,W'} \varphi_{k,\beta,i,W'}((\xi, 0, \tau) + s y_{W,W'})
\]

\[
\times e^{-i\alpha_0(\eta^0_{W, W', i}, (\xi, s, \tau) + s y_{W,W'})}.
\]
Then, we can write
\[
\int_{W_{\beta,i}} \langle \mathfrak{g}_{k,\beta,i}, \mathfrak{g}_{k,\beta,i}, W, W' \rangle = \int_{\mathbb{R}^d} d\tau d\pi \mathfrak{g}_{k,\beta,i} W(\tau) \mathfrak{g}_{k,\beta,i} W'(\tau) \int_{-c_{\#}r}^{c_{\#}r} ds A(\xi, s, \tau) e^{-ibb_{\#}^w W', v s}. \]

Note that, by (6), \(\|A(\xi, \cdot, \tau)\|_{C^\infty} \leq C_{\#}r^{-1}\), hence (as in [1, Lemma 7.10])
\[
\int_{-c_{\#}r}^{c_{\#}r} ds A(\xi, s, \tau) e^{-ibb_{\#}^w W', v s} \leq C_{\#} |b|^{-\omega'} \rho^{-\omega}. \qedhere
\]

Here our strategy departs from [1] as we control directly where \(\mathfrak{g}_{k,\beta,i}^w\) is large.

Let \(\Omega = \{ x \in \mathbb{W}_{\beta,i} : \| \mathfrak{g}_{k,\beta,i}^w(x) \| \geq 4C_{\#} r'^{1/2} D_{k,\beta,i} \} \) and \(\Omega_{1} = \{ x \in \mathbb{W}_{\beta,i} : \mathfrak{g}_{k,\beta,i}^{*, B} \geq C_{\#} r'^{1/2} D_{k,\beta,i} \} \), while \(\Omega\) and \(\Omega_{1}\) are \(r\) thickenings in the flow direction. Note that if \(x \in \Omega\), then \(\phi_{t}(x) \in \Omega_{1}\) for all \(t \leq C_{\#} r\). By Lemma 5, choosing \(C_{\#}\) large, we have \(\Omega \subset \Omega_{1}\). By Chebychev inequality, Lemma 6 implies
\[
|\Omega| = \int_{\mathbb{W}_{\beta,i}} 1_{\Omega_{1}} \leq C_{\#} \int_{\mathbb{W}_{\beta,i}} \mathfrak{g}_{k,\beta,i}^{*, B} r'^{-\varsigma} D_{k,\beta,i}^{2} \leq C_{\#} (|b|^{2} r'^{-\omega'}) r^{d_{s} - \varsigma}.
\]
Thus
\[
|\Omega| \leq C_{\#} |b|^{-\omega'} \rho^{-\omega'} r^{d_{s} - 1 - \varsigma}.
\]

If \(x \in \Omega\), then, by (10), (11), we have that \(\| \mathfrak{g}_{k,\beta,i}^w(y) \| \geq C_{\#} r'^{1/2} D_{k,\beta,i}\) provided
\[
|y - x|^{\omega'} r^{-1} + |y - x| |b| r \leq C_{\#} r^{\epsilon/2}.
\]
The above holds for
\[
|y - x| \leq C_{\#} \min \{r^{1+\epsilon/2} |b|^{-1} r^{-1+\epsilon/2}\} =: \rho.
\]

We are finally ready to prove the stated Theorem.

**Proof of Theorem 1.** Let \(t_{0} > 0\) be such that \(e^{\lambda(0)} \rho = 1\). Then, recalling (1),
\[
|\phi_{-t_{0}}(\Omega)| \leq e^{(J_{-}^{*}(0_{+}) + \lambda_{-}(0_{+}) d_{s} \theta)} |\Omega|
\]
\[
|\phi_{-t_{0}}(W_{\beta,i})| \geq e^{J_{-}^{*}(0_{+}) d_{s}}.
\]
It follows that if we cover \(\phi_{-t_{0}}(W_{\beta,i})\) by discs of radius 1, then, recalling (17), for each disc that intersects \(\phi_{-t_{0}}(\Omega)\) there are at least
\[
K = \frac{r^{d_{s}}}{e^{\lambda_{-}(0_{+}) d_{s} \theta} |\Omega|} \geq C_{\#} \rho^{d_{s}} b^{2\omega'} r^{1+ \varsigma}
\]
disks that are disjoint from \(\phi_{-t_{0}}(\Omega)\). Indeed, if a disc intersects \(\phi_{-t_{0}}(\Omega)\), then a disc twice its radius must have a fixed proportion of its volume belonging to \(\phi_{-t_{0}}(\Omega)\).

We chose \(\rho = C_{\#} r^{1+\epsilon/2} \) (so Lemma 5 applies), \(|b| = r^{2-\omega'+\epsilon}\) (so Lemma 6 applies). Accordingly, \(K\) can be larger than one only if \((\omega')^{2} + \omega' - 1 > 0\), but then, choosing \(\epsilon\) small enough, (18) implies \(\rho = C_{\#} r^{1+\omega'-\epsilon}\), which implies
\[
K \geq r^{-\varsigma}
\]
provided \(\theta < \frac{(\omega')^{2} + \omega' - 1}{2d_{s}(1+\omega')}\) and \(\varsigma\) is small enough. Again by [1, Lemmata C.1, C.3] this ratio persists under iteration. Hence, for each \(t \geq t_{0}\),
\[
|\phi_{-t}(\Omega)| \leq C_{\#} r^{\epsilon} |\phi_{-t}(W_{\beta,i})|.
\]

If \( n \geq C_1 \ln |b| \), for \( C_1 \) and \( b \) large enough, by [1, Equation (7.12)], we have
\[
\| J_W \phi \* \phi_{t_0} \* g_{k,\beta,i}^{\phi_{t_0}} \|_{L^2(W)} \leq C_\# \| g_{k,\beta,i}^{\phi_{t_0}}(W) \|_{L^2(W)}.
\]
Hence, for each \( k' \geq c_\alpha n \), we have
\[
\| J_W \phi \* \phi_{t_0} \* g_{k,\beta,i}^{\phi_{t_0}} \|_{L^2(W)} \leq C_\# \| g_{k,\beta,i}^{\phi_{t_0}}(W) \|_{L^2(W)}.
\]

Using the above inequality in (7.13), we write
\[
\sum_{k,\beta,i} \int \langle g_{k,\beta,i}^{\phi_{t_0}}, r \rangle_W h \leq C_\# \sum_{k,\beta,i} \int \langle g_{k,\beta,i}^{\phi_{t_0}}, r \rangle_W h \leq C_\# \sum_{k,\beta,i} \int \langle g_{k,\beta,i}^{\phi_{t_0}}, r \rangle_W h.
\]

With \( \| \phi_{k',i} \|_{L^2(W)} \leq C_\# \| g_{k,\beta,i}^{\phi_{t_0}}(W) \|_{L^2(W)} \), it follows,
\[
\sum_{k,\beta,i} \int \langle g_{k,\beta,i}^{\phi_{t_0}}, r \rangle_W h \leq C_\# \sum_{k,\beta,i} \int \langle g_{k,\beta,i}^{\phi_{t_0}}, r \rangle_W h.
\]

Using the above inequality in (13) provides an estimate of \( \| \hat{R}_n(z)^2 h \|^{s}_{s+\eta} \), which, substituted into (2), yields [1, Proposition 7.5].

Theorem 1 follows then as in [1, Theorem 2.4].

References

[1] Giulietti, P.; Liverani, C.; Pollicott, M. Anosov flows and dynamical zeta functions. Ann. of Math. (2) 178 (2013), no. 2, 687–773.