Regularized maximum pure-state input-output fidelity of a quantum channel

Moritz F. Ernst and Rochus Klesse
Universität zu Köln, Institut für Theoretische Physik, Zülpicher Straße 77, 50937 Köln, Germany
(Dated: December 12, 2017)

As a toy model for the capacity problem in quantum information theory we investigate finite and asymptotic regularizations of the maximum pure-state input-output fidelity \( F(\mathcal{N}) \) of a general quantum channel \( \mathcal{N} \). We show that the asymptotic regularization \( \tilde{F}(\mathcal{N}) \) is lower bounded by the maximum output \( \infty \)-norm \( \nu_\infty(\mathcal{N}) \) of the channel. For \( \mathcal{N} \) being a Pauli channel we find that both quantities are equal.

I. INTRODUCTION

An open problem of quantum information theory is finding an efficient method to compute certain information capacities of a general quantum channel \( \mathcal{N} \), for instance, its capacity for transmission of classical, private-classical or quantum information [1]. The problem arises because, according to the present state of the theory, determining such a capacity \( C(\mathcal{N}) \) requires regularizing a corresponding single-shot capacity \( C^{(1)}(\mathcal{N}) \) as

\[
C(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} C^{(1)}(\mathcal{N}^\otimes n). \tag{1}
\]

The computation of \( C^{(1)}(\mathcal{N}^\otimes n) \) involves typically maximization of an entropic expression over the input quantum states of the \( n \)-times replicated channel \( \mathcal{N}^\otimes n \). This renders determining \( C(\mathcal{N}) \) in general an analytically as well as numerically formidable problem, to which presently no good solution is available.

We do not attempt to solve any of the above capacity problems. Instead, here we study as a toy problem a structurally similar but technically far less demanding problem, namely regularizing the maximum pure-state input-output fidelity \( F(\mathcal{N}) \) of a quantum channel \( \mathcal{N} \) [2]. We find that in general the \( n \)-th regularization \( F^{(n)}(\mathcal{N}) = F(\mathcal{N}^\otimes n)^{1/n} \) shows a non-trivial \( n \)-dependence, as it is seen e.g. for certain Pauli channels (cf. Fig. 1, 2 or 3). Determining the asymptotic regularization \( \tilde{F}(\mathcal{N}) = \lim_{n \to \infty} F^{(n)}(\mathcal{N}) \) therefore represents a problem that is structurally similar to determining capacities of a quantum channel. By employing symmetric trial states we show that the maximum output \( \infty \)-norm \( \nu_\infty(\mathcal{N}) \) is an easily-computable single-letter lower bound of the asymptotic regularization \( \tilde{F}(\mathcal{N}) \) for a general channel \( \mathcal{N} \). Moreover, from a result of King [3], stating that the maximum output \( \infty \)-norm is multiplicative for unital qubit channels, it follows that for a general Pauli channel \( \nu_\infty \) actually coincides with \( \tilde{F} \). This establishes \( F \) on Pauli channels as a simple toy model within which non-trivial \( n \)-th regularizations are observed while at the same time the asymptotic regularization \( \tilde{F} \) is available.

II. NOTATIONS, DEFINITIONS, RELATIONS

We consider a quantum channel \( \mathcal{N} \) with identical input and output Hilbert space \( \mathcal{H} \) of finite dimension \( d \). I.e. \( \mathcal{N} \) is a completely positive and trace preserving endomorphism on the linear operators on \( \mathcal{H} \). The Hermitian conjugate of the channel \( \mathcal{N} \) with respect to the Hilbert-Schmidt inner product \( (A, B) = \text{tr} A^\dagger B \) will be denoted by \( \mathcal{N}^\dagger \). For a pure-state (i.e. rank-1) density operator \( \psi \) on \( \mathcal{H} \) let

\[
F(\mathcal{N}, \psi) = \text{tr} \psi \mathcal{N}(\psi)
\]

be the input-output fidelity of the channel \( \mathcal{N} \) on \( \psi \). As usual, we denote a state vector of \( \mathcal{H} \) and its dual by \( |\psi\rangle \) and \( \langle \psi | \), respectively, and the rank-1 density operator of the associated pure state by \( \psi \).

For the channel \( \mathcal{N} \) we define the maximum input-output fidelity \( F(\mathcal{N}) \) on pure states, its \( n \)-th regularization \( F^{(n)}(\mathcal{N}) \), and its asymptotic regularization \( \tilde{F}(\mathcal{N}) \) as

\[
F(\mathcal{N}) = \max_{\psi} F(\mathcal{N}, \psi), \quad F^{(n)}(\mathcal{N}) = F(\mathcal{N}^\otimes n)^{1/n}, \quad \tilde{F}(\mathcal{N}) = \lim_{n \to \infty} F^{(n)}(\mathcal{N}),
\]

where the maximum is taken with respect to rank-1 density operators \( \psi \). We will also need the maximum output \( q \)-norm \( \nu_q(\mathcal{N}) \) of \( \mathcal{N} \), its \( n \)-th regularization \( \nu_q^{(n)}(\mathcal{N}) \), and its asymptotic regularization \( \tilde{\nu}_q(\mathcal{N}) \), defined by

\[
\nu_q(\mathcal{N}) = \max_{\psi} ||\mathcal{N}(\psi)||_q, \quad \nu_q^{(n)}(\mathcal{N}) = \nu_q(\mathcal{N}^\otimes n)^{1/n}, \quad \tilde{\nu}_q(\mathcal{N}) = \lim_{n \to \infty} \nu_q^{(n)}(\mathcal{N}).
\]

Note that \( \nu_2(\mathcal{N}) \) can be expressed by the maximum input-output fidelity as

\[
\nu_2(\mathcal{N}) = F(\mathcal{N}^\dagger \mathcal{N})^{1/2},
\]

since

\[
||\mathcal{N}(\psi)||_2^2 = (\mathcal{N}(\psi), \mathcal{N}(\psi)) = (\psi, \mathcal{N}^\dagger \mathcal{N}(\psi)) = \text{tr} \psi \mathcal{N}^\dagger \mathcal{N}(\psi).
\]
Another obvious relation is
\[ F \leq \nu_{\infty}, \tag{2} \]
following from the fact that the maximum norm \(|A|_{\infty}\) of an operator \(A\) can be expressed as \(|A|_{\infty} = \max_{\phi} \text{tr} \, \phi A\), where the maximum is taken w.r.t. rank-1 density operators \(\phi\), and so
\[
F(\mathcal{N}) = \max_{\psi} \text{tr} \, \psi \mathcal{N}(\psi) \\
\leq \max_{\phi} \text{tr} \, \phi \mathcal{N}(\psi) \\
= \max_{\psi} ||\mathcal{N}(\psi)||_{\infty} = \nu_{\infty}(\mathcal{N}).
\]

Despite being an upper bound of the single-shot maximum fidelity, \(\nu_{\infty}\) is also a lower bound of the regularized maximum fidelity,
\[ \tilde{F} \geq \nu_{\infty}, \tag{3} \]
as we will be prove below in Sec. III. This inequality cannot be an equality in general because \(\tilde{F}\) is weakly multiplicative [4] whereas \(\nu_{\infty}\) is known to be not weakly multiplicative [5]. Nevertheless, after regularization we obtain
\[ \tilde{F} \geq \tilde{\nu}_{\infty}, \tag{4} \]
which together with the regularized version of relation [2] eventually proves
\[ \tilde{F} = \tilde{\nu}_{\infty} \tag{5} \]
(cf. Sec. III). This equality of the regularized quantities does not look very promising. However, with the aforementioned result of King [3], we can use it to compute the regularized maximum fidelity for Pauli channels in Sec. IV.

### III. \( \tilde{F} \geq \tilde{\nu}_{\infty} \)

First, we will prove
\[ \tilde{F}(\mathcal{N}) \geq \nu_{\infty}(\mathcal{N}) \tag{6} \]
for a fixed but arbitrary channel \(\mathcal{N}\). For this specific \(\mathcal{N}\) let \(|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}\) such that \(\nu_{\infty}(\mathcal{N}) = \langle \phi_1 | \mathcal{N}(\phi_2) | \phi_1 \rangle\). It may happen that \(|\phi_1\rangle\) and \(|\phi_2\rangle\) are linearly dependent. In this case \(\nu_{\infty}(\mathcal{N}) = \langle \phi_1 | \mathcal{N}(\phi_1) | \phi_1 \rangle\) and the inequality [6] holds trivially since
\[ \tilde{F}(\mathcal{N}) \geq F(\mathcal{N} \otimes \mathcal{N}, \phi_1 \otimes \phi_2) = \text{tr} \, \phi_1 \mathcal{N}(\phi_1) = \nu_{\infty}(\mathcal{N}) \]
for arbitrary \(n\). For the rest of the proof we can therefore assume without loss of generality that \(|\phi_1\rangle\) and \(|\phi_2\rangle\) are linearly independent, and that
\[ \nu_{\infty}(\mathcal{N}) > \langle \psi | \mathcal{N}(\psi) | \psi \rangle \]
for all \(|\psi\rangle \in \mathcal{H}\). Beyond that, we will also assume that \(\langle \phi_1 | \mathcal{N}(\phi_2) | \phi_2 \rangle\) is a non-negative real number, which can be always achieved by multiplying \(|\phi_1\rangle\) with an appropriate phase.

Based on \(|\phi_1\rangle\) and \(|\phi_2\rangle\) let a sequence \(|\psi_n\rangle\) of state vectors be given by
\[ |\psi_n\rangle = \frac{c_n}{\sqrt{2}} (|\phi_1\rangle \otimes |\phi_2\rangle), \quad n \in \mathbb{N}, \]
where \(c_n = (1 + \mathbb{R} \langle \phi_1 | \phi_2 \rangle)^{-1/2}\) ensures normalization of \(|\psi_n\rangle\). Note that by assumption \(|\langle \phi_1 | \phi_2 \rangle| < 1\) and hence \(\lim_n c_n = 1\). In the following we will show that
\[ \lim_{n} F(\mathcal{N} \otimes \mathcal{N}, |\psi_n\rangle) = \nu_{\infty}(\mathcal{N}), \tag{7} \]
which suffices to prove the inequality [6]. To do so, we start with expanding \(F(\mathcal{N} \otimes \mathcal{N}, |\psi_n\rangle)\) in terms of \(n\)-th powers of coefficients
\[ N_{ijkl} = \langle \phi_1 | \mathcal{N}(\langle \phi_j | \phi_k \rangle | \phi_1 \rangle, \quad i,j,k,l \in \{1,2\}, \]
as
\[ F(\mathcal{N} \otimes \mathcal{N}, |\psi_n\rangle) = \frac{c_n^4}{4} \sum_{ijkl} (N_{ijkl})^n. \]

Clearly, for large \(n\) the sum is dominated by those coefficients \(N_{ijkl}\) which have maximal absolute value, namely, \(N_{1221} = \nu_{\infty}(\mathcal{N})\) and potentially \(N_{2112}, N_{1212},\) and \(N_{2121}\). The sequence \(F(\mathcal{N} \otimes \mathcal{N}, |\psi_n\rangle)\) will thus converge to \(\nu_{\infty}(\mathcal{N})\), provided that there will be no accidental cancellation among the terms. To show that this is actually the case, we represent \(\mathcal{N}\) with suitable Kraus operators \(A_1, \ldots, A_K\) as
\[ \rho \mapsto \mathcal{N}(\rho) = \sum_{\nu=1}^{K} A_{\nu} \rho A_{\nu}^\dagger, \]
and define four \(K\)-dimensional complex vectors
\[ w \equiv r_{11}, \quad x \equiv r_{12}, \quad y \equiv r_{21}, \quad z \equiv r_{22} \]
by
\[ (r_{ij})_{\nu} = \langle \phi_i | A_{\nu} | \phi_j \rangle. \]
This allows us to write
\[ N_{ijkl} = r_{ik}^\dagger r_{ij} \]
and hence
\[ F(\mathcal{N} \otimes \mathcal{N}, |\psi_n\rangle) = \frac{c_n^4}{4} \sum_{a,b \in \{w,x,y,z\}} (a^\dagger b)^n. \tag{8} \]
By the properties of $|\phi_1\rangle$ and $|\phi_2\rangle$ with respect to $\mathcal{N}$ and the Cauchy-Schwarz inequality we find that
\[
|x|^2 = N_{121} = \nu_\infty(\mathcal{N}),
\]
\[
|y|^2 = N_{211} \leq |x|^2,
\]
\[
|w|^2 = N_{111} \leq |x|^2,
\]
\[
|z|^2 = N_{222} \leq |x|^2,
\]
\[
0 \leq N_{121} = y^*x \leq |y||x| \leq |x|^2,
\]
\[
0 \leq N_{211} = x^*y \leq |x||y| \leq |x|^2,
\]
\[
|a^\dagger b| < |x|^2 \text{ if } a \text{ or } b \text{ in } \{w, z\}.
\]

This means that all terms in $\{a^\dagger b\}_{a,b \in \{w, x, y, z\}}$ that are of maximal absolute value $|x|^2 = \nu_\infty(\mathcal{N})$ are real positive numbers, which with Eq. (3) immediately shows (7) and so concludes the proof of the inequality (4).

The inequality (1) follows then by regularization: To this end we employ the weak multiplicativity of $F$ (2), by which
\[
\tilde{F}(\mathcal{N}) = \tilde{F}(\mathcal{N}^{\otimes m})^{1/m} \geq \nu_\infty(\mathcal{N}^{\otimes m})^{1/m}.
\]

This holds for all positive, integer $m$ and thus proves $\tilde{F}(\mathcal{N}) \geq \tilde{\nu}_\infty(\mathcal{N})$, which is inequality (1). From inequality (2) it is clear that also $\tilde{F}(\mathcal{N}) \leq \tilde{\nu}_\infty(\mathcal{N})$, and therefore actually $\tilde{F}(\mathcal{N}) = \tilde{\nu}_\infty(\mathcal{N})$ for any channel $\mathcal{N}$, which shows Eq. (5).

IV. PAULI CHANNEL

As an example we study the maximum input-output fidelity of a general Pauli channel $\mathcal{P}$ on a qubit ($d = 2$), defined as
\[
\mathcal{P}(\rho) = \sum_{\alpha=0}^{3} p_\alpha \sigma_\alpha \rho \sigma_\alpha,
\]
where $\sigma_0$ is the identity, $\sigma_1, \sigma_2, \sigma_3$ are the standard Pauli operators, and the non-negative coefficients $p_\alpha$ sum up to unity. Without loosing generality we demand that $p_1 \leq p_2 \leq p_3$.

It is not difficult to show that
\[
\nu_\infty(\mathcal{P}) = \begin{cases} 
 p_0 + p_3 : & \text{for } p_0 \geq p_2 \quad (a) \\
 p_2 + p_3 : & \text{for } p_0 < p_2 \quad (b) 
\end{cases}
\]

In the first case, (a), we find
\[
\nu_\infty(\mathcal{P}) = \langle \phi_3 | \mathcal{P}(\phi_3) | \phi_3 \rangle ,
\]
where $|\phi_3\rangle$ is an eigenstate of $\sigma_3$ (either for eigenvalue $+1$ or $-1$), while for the second case, (b),
\[
\nu_\infty(\mathcal{P}) = \langle \phi_+ | \mathcal{P}(\phi_-) | \phi_+ \rangle
\]
with $|\phi_+\rangle$ and $|\phi_-\rangle$ eigenstates of $\sigma_1$ for eigenvalues $+1$ and $-1$, respectively.

According to King [3], the maximum $q$-norm $\nu_q$ of any unital qubit channel $\Phi$ is multiplicative for any $q \geq 1$. This means that for any other qubit channel $\Omega$
\[
\nu_q(\Omega \otimes \Phi) = \nu_q(\Omega) \nu_q(\Phi).
\]

Since $\mathcal{P}$ is unital this result particularly implies that its maximum output $\infty$-norm does not change under regularization, i.e. for any $n$
\[
\nu_\infty(\mathcal{P}) = \nu^{(n)}(\mathcal{P}) ,
\]
and so clearly
\[
\nu_\infty(\mathcal{P}) = \tilde{\nu}_\infty(\mathcal{P}) .
\]

Thanks to this fortunate situation we can actually use our result Eq. (5) to determine the asymptotic regularized input-output fidelity of a Pauli channel as
\[
\tilde{F}(\mathcal{P}) = \begin{cases} 
 p_0 + p_3 : & \text{for } p_0 \geq p_2 \quad (a) \\
 p_2 + p_3 : & \text{for } p_0 < p_2 \quad (b) 
\end{cases}
\]

In the first case, (a), $F(\mathcal{P}, \phi_3) = p_0 + p_3 = F(\mathcal{P}^{\otimes n}, \phi_3^{\otimes n})^{1/n}$, and thus regularization has no effect: $F(\mathcal{P}) = F^{(n)}(\mathcal{P}) = \tilde{F}(\mathcal{P})$. In the second case, (b), we observe that
\[
F(\mathcal{P}) = F(\mathcal{P}, \phi_3) = p_0 + p_3 ,
\]
showing that here regularization increases the fidelity as
\[
F(\mathcal{P}) = p_0 + p_3 < p_2 + p_3 = \tilde{F}(\mathcal{P}) .
\]

V. NUMERICAL RESULTS FOR PAULI CHANNELS

Taking the limit $n$ to infinity is essential in deriving the lower bound $\nu_\infty$ of $\tilde{F}$. For this reason we do not have analytic results for the $n$-th regularization for finite $n$. To obtain some insight into the $n$-dependence of the regularized fidelity, we determined $F^{(n)}$ for finite $n$ for three exemplary Pauli channels by numerical maximization with a variant of the Barzilai-Borwein gradient method [6]. Up to $n = 6$ we maximized over the entire $n$-qubit Hilbert space. Beyond that, up to $n = 26$, we restricted the maximization to symmetric state vectors, i.e. to state vectors that do not change under permutation of qubits. For all three investigated channels we found that for $n \leq 6$ maximization over the symmetric state vectors and maximization over all states gave identical values within numerical precision of $10^{-6}$.

The first Pauli channel is given by probabilities $p = (p_0, p_1, p_2, p_3) = (0.1, 0.2, 0.3, 0.4)$. Since here $p_0 < p_2$, according to Eq. (4) the asymptotic regularization of the input-output fidelity is $\tilde{F}(\mathcal{P}) = p_2 + p_3 = 0.7$, indicated by the dashed line in Fig. [1]. The numerically determined maximum $n$-fidelities $F^{(n)}(\mathcal{P})$ increase strictly monotonically and evidently approach $\tilde{F}(\mathcal{P})$ as $n$ increases from
FIG. 1. Regularized maximum input-output fidelity for a Pauli channel $\mathcal{P}$ with probabilities $p = (0.1, 0.2, 0.3, 0.4)$. $\circ : F^{(n)}(\mathcal{P})$ determined by numerical maximization, $+ : F(\mathcal{P}^{\otimes n}, \psi_n)^{1/n}$ for trial states $\psi_n$, dashed line : $\tilde{F}(\mathcal{P})$.

1 to 20 (cf. $\circ$-symbols in Fig. 1). These $n$-fidelities agree within numerical precision with $F(\mathcal{P}^{\otimes n}, \psi_n)^{1/n}$ on trial states

$$|\psi_n\rangle = \frac{1}{\sqrt{2}}(|\phi_+\rangle^{\otimes n} + |\phi_-\rangle^{\otimes n}),$$

which can be easily computed to be

$$F(\mathcal{P}^{\otimes n}, \psi_n)^{1/n} = \frac{1}{2^{\frac{1}{n}}}[ (p_0 + p_1)^n + (p_0 - p_1)^n + (p_3 - p_2)^n + (p_3 + p_2)^n ]^{\frac{1}{n}},$$

(11)

(cf. $+$-symbols in Fig. 1). We emphasize that this agreement is coincidental, since we only proved $\tilde{F}(\mathcal{P}) = \lim_{n \to \infty} F(\mathcal{P}^{\otimes n}, \psi_n)^{1/n}$ (cf. Eq. 7).

We studied a second Pauli channel with probabilities

$$p = \left(\frac{1}{3} - \epsilon, 0, \frac{1}{3}, \frac{1}{3} + \epsilon\right)$$

where $\epsilon = 1/21 = 0.04762$. As shown in Fig. 2 here the numerically determined fidelities $F^{(n)}$ are constant of value $F^{(1)}$ for $n \leq 10$ and increase only for larger $n$ in order to approach asymptotically $F(\mathcal{P}) = p_3 = \frac{1}{2} + \epsilon = 0.714$. Fittingly, the regularized fidelities $F(\mathcal{P}, \psi_n)^{1/n}$ for the trial states Eq. 10 are submaximal for $1 < n < 10$, but appear again to be maximal for $n \geq 10$.

Closer inspection of Eq. 11 reveals that for sufficiently small $\epsilon$ the fidelity $F(\mathcal{P}^{\otimes n}, \psi_n)^{1/n}$ exceeds $F^{(1)}(\mathcal{P})$ at

$$n \approx n_0 = \frac{\ln 4}{\frac{1}{3} - \epsilon}. \quad (13)$$

If we take it for granted that the behavior shown in Fig. 2 is representative for sufficiently small $\epsilon$ it is clear that for arbitrarily large $n_0$ one can always find a Pauli channel $\mathcal{P}_0$ such that $F^{(n)}(\mathcal{P}_0) = F^{(1)}(\mathcal{P}_0)$ for $n \leq n_0$ while $F^{(n)}(\mathcal{P}_0) > F^{(1)}(\mathcal{P}_0)$ for $n > n_0$.

This is confirmed by our last numerical example presented in Fig. 3. Here we investigated a Pauli channel again with probabilities as in Eq. 12, but with $\epsilon = 0.025$, leading to $[n_0] = 19$.

VI. SUMMARY

We addressed the problem of determining finite and asymptotic regularizations of the maximum input-output fidelity of a general quantum channel. Using symmetric trial states we showed that the maximum output $\infty$-norm $\nu_\infty(\mathcal{N})$ is a lower bound of the asymptotically regularized maximum input-output fidelity $\tilde{F}(\mathcal{N})$ for a general quantum channel $\mathcal{N}$. Moreover, for $\mathcal{N}$ being a Pauli channel we found that a result of King already implies equality of $\nu_\infty(\mathcal{E})$ and $\tilde{F}(\mathcal{N})$. Numerically determined finite regularizations $F^{(n)}(\mathcal{P})$ for Pauli channels show a non-trivial
\( n \)-dependence and confirm the results for the asymptotic
regularization.

Financial support from DFG Grant No. ZI-513/1-2, from the center for Quantum Matter and Materials of the University of Cologne, and from the SFB/TR 12 is gratefully acknowledged.

[1] M. M. Wilde, *Quantum Information Theory*, Cambridge University Press, 2013 [arXiv:1106.1445]; and references therein.

[2] Average and minimum input-output fidelity have been investigated as average and minimum gate fidelity in: M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999); M. Nielsen, Phys. Lett. A 303, 249 (2002); M. D. Bowdrey, D. K. L. Oi, A. J. Short, K. Banaszek, J. A. Jones, Phys. Lett. A 294, 258 (2002); A. Gilchrist, N. K. Langford, and M. A. Nielsen, Phys. Rev. A 71, 062310 (2005); J. Emerson, R. Alicki, K. Zyczkowski, J. Opt. B: Quantum Semiclass. Opt. 7, 347 (2005); N. Johnston and D. W. Kribs, J. Phys. A: Math. Theor. 44, 495303 (2011).

[3] C. King, J. Math. Phys. 43, 4641 (2002).

[4] Immediate by definition: \( \tilde{F}(\mathcal{N}^\otimes m) = \lim_n F^{(n)}(\mathcal{N}^\otimes m) = \left( \lim_n F(\mathcal{N}^\otimes mn)^{1/nm} \right)^m = \tilde{F}(\mathcal{N})^m \).

[5] R. F. Werner, A. S. Holevo, J. Math. Phys. 43, 4353 (2002).

[6] J. Barzilai, J. M. Borwein, IMA J. Numer. Anal., 8, 141 (1988).