Proof of the renormalizability of the gradient flow

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We give an alternative perturbative proof of the renormalizability of the system defined by the gradient flow and the fermion flow in vector-like gauge theories.

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1. Introduction

The non-abelian gauge theory is renormalizable in the sense that one can send the momentum cutoff infinity while keeping the strength of the interaction finite. In such a theory with an infinite cutoff, any observable must be renormalized one and hence in gauge theory how to construct renormalized quantities is a fundamental question. The gradient or Wilson flow [1, 2] provides a surprisingly versatile method to define renormalized quantities in gauge theory, without explicitly referring to the perturbative renormalization. See also Ref. [3]. Since renormalized quantities have the meaning being independent of regularization, this method is especially useful in the context of lattice regularization with which one clearly wants to define renormalized quantities without referring to perturbation theory as much as possible.

That the bare gauge field and its composite operators evolved by the gradient flow possess finite correlation functions without any wave function renormalization was proven in Ref. [4] to all orders of perturbation theory. The gradient flow has been generalized to the flow of the fermion field [5] and it is known that wave function renormalization of the flowed fermion elementary field makes any correlation functions and composite operators finite. Backed by these theoretical understandings, the gradient flow has been applied to many problems in lattice gauge theory; Refs. [6] and [7] are recent reviews. More recently, as the reference list [8-44] shows, the number of related works is rapidly growing (see also proceedings of the recent lattice conferences). In most of these works, the gradient flow is used to the scale setting, definition of a non-perturbative running coupling, and to define the topological charge. Another interesting branch of application is to define a wide class of renormalized composite operators using the flow, such as the energy–momentum tensor [45, 46] and the axial-vector current [53]. In Refs. [54-57], applications of the gradient flow for the construction of the energy–momentum tensor are studied from a somewhat different perspective from Refs. [45, 46]. See also Refs. [62-68] for related works.

In the present paper, we present an alternative perturbative proof of the renormalizability of the system defined by the gradient flow and the fermion flow. Although the proof of the renormalizability has already been given in Refs. [4, 5], at some stages of the proof, the explanations in Refs. [4, 5] seem rather concise. In the present paper, we tried to improve the presentations in the following points. As Refs. [4, 5], we use a (D + 1)-dimensional field theoretical representation of the system defined by the flow. However, to show the renormalizability, we employ the Ward–Takahashi (WT) relation associated with the Becchi–Rouet–Stora (BRS) symmetry [69] expressed in terms of the effective action (the generating functional of the 1PI correlation functions), the Zinn-Justin equation [70, 71] (see Ref [72] for example for the exposition). This type of proof, that was adopted also in Ref. [62] to prove the renormalizability of the gradient flow in the 2D $O(N)$ non-linear sigma model, is easy to follow step by step, although tends to become somewhat lengthy. Also, in the present paper, as in Ref. [62], we provide a precise definition of the flow-time derivative by the

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1 The validity of these constructions has been tested numerically [47–52].

2 For recent developments in the problem of the construction of the energy–momentum tensor in lattice gauge theory, see Refs. [58-61].

3 In our proof, one can find expressions being analogue to those in the proof of the renormalizability of the stochastic quantization [73, 74].
forward difference prescription. We show that this prescription is consistent with the above BRS symmetry on which the derivation of the WT relation relies. This precise definition resolves a certain subtle issue concerning the so-called “flow-line loop”; the vanishing-ness of the flow-line loop is crucially important for the renormalizability proof. The prescription also clearly illustrates the origin of the “boundary counterterm”. We believe that our proof is quite accessible because of these aspects.

This paper is organized as follows. In Sect. 2 we recapitulate the basic definition of the gradient flow and the fermion flow in vector-like gauge theories. We then elucidate the perturbative expansion of the flowed system. In Sect. 3 we introduce a \((D + 1)\)-dimensional local field theory \([4, 5]\) whose Feynman rules reproduce the perturbative expansion of the flowed system in Sect. 2.2. We lay some emphasis on the usefulness of discretized flow time; this resolves a certain subtle issue concerning a “flow-line loop”, that is crucial for the perturbative equivalence of the \((D + 1)\)-dimensional field theory and the flowed system. Section 4 is the main part of the present paper, the proof of the renormalizability. First, we study the case of the pure Yang–Mills theory. In Sect. 4.1 we derive the WT relation in the \((D + 1)\)-dimensional field theory. After stating our way of multiplicative renormalization (Eqs. \((4.15)-(4.19)\)) and expressing the WT relation in terms of renormalized quantities (Sect. 4.2), we argue the most general form of the divergent part in the effective action (Sect. 4.3). Finally, by examining the constraint imposed by the WT relation on the divergent part, we find the validity of our renormalization \((4.15)-(4.19)\) (Sect. 4.4) to all orders of perturbation theory. In Sect. 4.5 the proof is generalized to include the fermion fields. The renormalization factors are defined by Eqs. \((4.51)-(4.57)\). Here, we find that the flowed fermion fields require wave function renormalization \([5]\), in contrast to the flowed gauge field. In Sect. 4.6, we argue the finiteness of composite operators of the flowed fields. Section 5 is devoted for the conclusion.

Here is our notation: The spacetime dimension is set to \(D \equiv 4 - 2\varepsilon\) because in what follows we implicitly assume dimensional regularization that preserves the gauge symmetry in perturbation theory. The signature of the metric is euclidean and all Dirac matrices are hermitian. The summation over repeated Lorentz and gauge indices is understood. The generators of the gauge group are normalized as
\[
\text{tr}(T^a T^b) = -\frac{1}{2} \delta^{ab}.
\]
We use the following abbreviation for the momentum integral:
\[
\int_p \equiv \int \frac{d^D p}{(2\pi)^D}.
\]

2. Gradient flow, fermion flow and the perturbative expansion

2.1. The \(D\)-dimensional system

We consider the gradient flow and the fermion flow in the vector-like gauge theory defined by
\[
S = -\frac{1}{2g_0} \int d^D x \text{tr} [F_{\mu\nu}(x) F_{\mu\nu}(x)] + \int d^D x \bar{\psi}(x) (\slash D + m_0) \psi(x),
\]
where \(g_0\) and \(m_0\) are the bare gauge coupling and the bare mass parameter, respectively. The field strength is given by
\[
F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]
\]
from the gauge potential $A_\mu(x) = A_\mu^a(x) T^a$ and the covariant derivative on the fermion field $\psi(x)$ in $\mathcal{D} = \gamma_\mu D_\mu$ is defined by

$$D_\mu = \partial_\mu + A_\mu.$$  \hfill (2.3)

To define perturbation theory, we introduce the gauge fixing term and the Faddeev–Popov ghost term by

$$S_{gf} + S_{\bar{c}c} \equiv \delta \frac{-2}{g_0^2} \int d^D x \text{ tr} \left\{ \bar{c}(x) \left[ \partial_\mu A_\mu(x) - \frac{1}{2\lambda_0} B(x) \right] \right\},$$  \hfill (2.4)

where $\delta$ is the nilpotent BRS transformation on the gauge field, the ghost field ($c(x) = c^a(x) T^a$), the anti-ghost field ($\bar{c}(x) = \bar{c}^a(x) T^a$), and the Nakanishi–Lautrup field ($B(x) = B^a(x) T^a$):

$$\delta A_\mu(x) = D_\mu c(x), \quad \delta c(x) = -c(x)^2,$$  \hfill (2.5)

$$\delta \psi(x) = -\bar{c}(x) \psi(x), \quad \delta \bar{\psi}(x) = -\bar{\psi}(x)c(x),$$  \hfill (2.6)

$$\delta \bar{c}(x) = B(x), \quad \delta B(x) = 0.$$  \hfill (2.7)

From these constructions, the above $D$-dimensional field theory is manifestly BRS invariant:

$$\delta S = 0, \quad \delta (S_{gf} + S_{\bar{c}c}) = 0.$$  \hfill (2.8)

### 2.2. Flow equations

The Yang–Mills gradient flow is an evolution of a gauge field configuration $A_\mu(x)$ along a fictitious time (the flow time) $t \in [0, \infty)$, according to the flow equation \cite{1, 2}

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x), \quad B_\mu(t = 0, x) = A_\mu(x),$$  \hfill (2.9)

where $G_{\mu\nu}(t, x)$ is the field strength of the flowed field,

$$G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)],$$  \hfill (2.10)

and $D_\mu$ is the covariant derivative on the gauge field,

$$D_\mu = \partial_\mu + [B_\mu,].$$  \hfill (2.11)

The first term on the right-hand side of the flow equation (2.9) is the “gradient” in the functional space, $-g_0^2 \delta S / \delta B_\mu(t, x)$, where $S$ is the Yang–Mills action integral for the flowed field. The flow equation (2.9) has a form of the diffusion equation and the evolution for $t > 0$ effectively damps high-frequency modes in the configuration. The second term on the right-hand side of the flow equation (2.9) is a “gauge fixing term” and it makes the integrand of the Feynman integral well behaved. It can be shown \cite{2} that, however, any gauge-invariant quantity which does not contain the flow-time derivative $\partial_t$ is independent of the “gauge fixing parameter” $\alpha_0$.  

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An evolution of fermion fields being similar to the above can be introduced. A possible choice is
\[ \partial_t \chi(t, x) = [\Delta - \alpha_0 \partial_\mu B_\mu(t, x)] \chi(t, x), \quad \chi(t = 0, x) = \psi(x), \] (2.12)
\[ \partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) \left[ \Delta + \alpha_0 \partial_\mu B_\mu(t, x) \right], \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x), \] (2.13)
where
\[ \Delta = D_\mu D_\mu, \quad D_\mu = \partial_\mu + B_\mu, \] (2.14)
\[ \Delta = \bar{D}_\mu \bar{D}_\mu, \quad \bar{D}_\mu = \bar{\partial}_\mu - B_\mu. \] (2.15)

These evolutions of the fermion fields are referred to as the fermion flow in what follows.

2.3. Perturbative solutions to the flow equations

The flow equation for the gauge field, Eq. (2.9), can be formally solved as
\[ B_\mu(t, x) = \int d^D y \left[ K_t(x - y)_{\mu\nu} A_\nu(y) + \int_0^t ds K_{t-s}(x - y)_{\mu\nu} R_\nu(s, y) \right], \] (2.16)
where
\[ K_t(x)_{\mu\nu} \equiv \int_p \frac{e^{ipx}}{p^2} \left[ (\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2} \right] \] (2.17)
is termed the heat kernel in what follows, and
\[ R_\mu = 2[B_\nu, \partial_\mu B_\nu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]] \] (2.18)
arises from the non-linear terms in the flow equation. Then by iteratively solving Eq. (2.10), we have a perturbative expansion for the flowed field \( B_\mu(t, x) \) in terms of the initial value \( A_\nu(y) \). Note the retarded nature of the solution (2.16); the heat kernels connect a sequence of points, \((t, x), (s_1, y_1), (s_2, y_2), \ldots\), in the temporal order \( t \geq s_1 \geq s_2 \geq s_3 \geq \cdots \geq 0 \). Note also that the Gaussian damping factor in the heat kernel \( K_{t-s}(x)_{\mu\nu} \) behaves as \( \sim e^{-(t-s)p^2} \) or \( \sim e^{-\alpha_0 (t-s)p^2} \), where \( t \) and \( s \) are flow times at end points of the heat kernel, so the damping factors become inactive when the two flow times \( t \) and \( s \) coincide.

For the fermion flow (2.12) and (2.13), we have
\[ \chi(t, x) = \int d^D y \left[ K_t(x - y)\psi(y) + \int_0^t ds K_{t-s}(x - y)\Delta'\chi(s, y) \right], \] (2.19)
\[ \bar{\chi}(t, x) = \int d^D y \left[ \bar{\psi}(y)K_t(x - y) + \int_0^t ds \bar{\chi}(s, y)\bar{\Delta}'K_{t-s}(x - y) \right], \] (2.20)
where
\[ K_t(x) \equiv \int_p e^{ipx}e^{-tp^2} \] (2.21)
is also referred to as the heat kernel, and
\[ \Delta' \equiv (1 - \alpha_0)\partial_\mu B_\mu + 2B_\mu \partial_\mu + B_\mu B_\mu, \] (2.22)
\[ \bar{\Delta}' \equiv -(1 - \alpha_0)\partial_\mu B_\mu - 2\bar{\partial}_\mu B_\mu + B_\mu B_\mu. \] (2.23)

By iteratively solving Eqs. (2.19) and (2.20), we again have a perturbative expansion of the fermion flow. Again note the retarded nature of the solutions and the fact that the Gaussian factor behaves as \( \sim e^{-(t-s)p^2} \).
The initial values for the above flows, $A_\mu(x)$, $\psi(x)$, and $\bar{\psi}(x)$, are quantum fields being subject to the functional integral. Correlation functions of the flowed fields are thus obtained by expressing the flowed fields in terms of the initial values following the above procedures and then carrying out the functional integral over the latter. For example, in the tree-level approximation, we have

$$
\langle B^a_\mu(t, x) B^b_\nu(s, y) \rangle_0 = \delta^{ab} \gamma_0^2 \int_p \frac{e^{ip(x-y)}}{(p^2)^2} \left[ (\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0(t+s)p^2} \right]
$$

and, similarly for the fermion field,

$$
\langle \chi(t, x) \bar{\chi}(s, y) \rangle_0 = \int_p \frac{e^{ip(x-y)}}{ip + m_0} e^{-(t+s)p^2},
$$

in the tree-level approximation. Eqs. (2.24) and (2.25) are referred to as propagators in what follows.

In a diagrammatic representation of the above perturbative expansion of the flowed system (see Refs. [2, 4, 46] for details), we thus have two types of “lines”, one represents the heat kernels (2.17) and (2.21), and another is for the propagators (2.24) and (2.25). The former line is called the flow line (or heat kernel), while the latter is simply called the propagator. Note that the propagators carry the Gaussian damping factor of the form $\sim e^{-(t+s)p^2}$ or $\sim e^{-\alpha_0(t+s)p^2}$, where $t$ and $s$ are flow times at end points of the propagator, so the damping factor is always active even when the two flow times $t$ and $s$ coincide as far as $t > 0$ or $s > 0$; this is a crucial difference from the heat kernel or the flow line. Also we have two types of vertices, one arises from the non-linear terms in the flow equations, Eqs. (2.18), (2.22), and (2.23), and another represents the conventional interaction vertices in Eq. (2.1). An important feature of this diagrammatic representation of the flowed system is that there is no closed loop totally consisting of the flow lines (or heat kernels). This property, which turns out to be crucial for the renormalizability of the flowed system, follows from the fact that to form a loop in a diagram, we have to Wick contract the initial values and then the contraction results in one of the propagators (2.24) and (2.25), but not heat kernels.

The statement of the renormalizability of the flowed system, first proved in Refs. [4, 5] to all orders of perturbation theory, is that any correlation functions and composite operators of flowed fields are made finite by the conventional renormalization in the gauge theory and wave function renormalization of the flowed fermion fields. In particular, no wave function renormalization of the flowed gauge field is required. The purpose of the present paper is to give an another proof of this statement.

3. A $(D + 1)$-dimensional field theoretical representation of the flowed system

3.1. $(D + 1)$-dimensional field theory and the BRS invariance

The total action of a $(D + 1)$-dimensional local field theory we will consider is

$$
S_{\text{tot}} = S + S_{gf} + S_{\text{cc}} + S_{\bar{H}} + S_{\bar{d}d},
$$

(3.1)
where $D$-dimensional actions, $S$, $S^g$, and $S^{\chi\chi}$, are already introduced in Sect. 2.1. The $(D + 1)$-dimensional actions are defined by [4, 5]

$$
S^g = -2\int_0^\infty dt \int d^Dx \text{tr} \left\{ L_\mu(t, x) \left[ \partial_t B_\mu(t, x) - D_\nu G_{\nu\mu}(t, x) - \alpha_0 D_\mu \partial_\nu B_\nu(t, x) \right] \right\}
$$

$$
+ \int_0^\infty dt \int d^Dx \left\{ \lambda(t, x) \left[ \partial_t - \Delta + \alpha_0 \partial_\mu B_\mu(t, x) \right] \chi(t, x)
+ \chi(t, x) \left[ \partial_t - \Delta - \alpha_0 \partial_\mu B_\mu(t, x) \right] \lambda(t, x) \right\}
$$

(3.2)

and

$$
S_{\bar{a}\bar{d}} = -2\int_0^\infty dt \int d^Dx \text{tr} \left\{ \bar{d}(t, x) \left[ \partial_t d(t, x) - \alpha_0 D_\mu \partial_\nu d(t, x) \right] \right\}.
$$

(3.3)

In Eqs. (3.1) and (3.2), $(D + 1)$-dimensional fields, $B_\mu(t, x)$, $\chi(t, x)$, and $\bar{\chi}(t, x)$, are subject to the boundary conditions,

$$
B_\mu(t = 0, x) = A_\mu(x), \quad \chi(t = 0, x) = \psi(x), \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x),
$$

(3.4)

corresponding to the initial conditions of the flow equations in Eqs. (2.9), (2.12), and (2.13).

The Lagrange multiplier fields, $L_\mu(t, x) = L^a_\mu(t, x)T^a$, $\lambda(t, x)$, and $\bar{\lambda}(t, x)$, obey free boundary conditions, that is, $L_\mu(t = 0, x)$, $\lambda(t = 0, x)$, and $\bar{\lambda}(t = 0, x)$ are integrated as independent variables in the functional integral.

In Eq. (3.3), $d(t, x)$ is a $(D + 1)$-dimensional ghost field which obeys the boundary condition,

$$
d(t = 0, x) = c(x),
$$

(3.5)

and $\bar{d}(t, x) = \bar{d}^a(t, x)T^a$ is a ghost analogue of the Lagrange multiplier field; it also obeys the free boundary condition at $t = 0$.

In addition to Eqs. (2.5)–(2.7), we introduce the BRS transformation on the above $(D + 1)$-dimensional fields by

$$
\delta B_\mu(t, x) = D_\mu d(t, x), \quad \delta d(t, x) = -d(t, x)^2,
$$

(3.6)

$$
\delta L_\mu(t, x) = [L_\mu(t, x), d(t, x)],
$$

(3.7)

$$
\delta \chi(t, x) = -d(t, x)\chi(t, x), \quad \delta \bar{\chi}(t, x) = -\bar{\chi}(t, x)d(t, x),
$$

(3.8)

$$
\delta \lambda(t, x) = -d(t, x)\lambda(t, x), \quad \delta \bar{\lambda}(t, x) = -\bar{\lambda}(t, x)d(t, x).
$$

(3.9)

These are simply a standard form of the BRS transformation in which the parameter of the gauge transformation is replaced by the $(D + 1)$-dimensional ghost field $d(t, x)$; here $L_\mu(t, x)$ is regarded as a field in the adjoint representation and $\lambda(t, x)$ ($\bar{\lambda}(t, x)$) is regarded as a field in the same representation as $\chi(t, x)$ ($\bar{\chi}(t, x)$). By this construction, the nilpotency $\delta^2 = 0$ on these fields immediately follows as for the conventional $D$-dimensional BRS transformation.

On the other hand, for the “ghost Lagrange multiplier” $\tilde{d}(t, x)$, we define the BRS transformation as

$$
\delta \tilde{d}(t, x) = D_\mu L_\mu(t, x) - \left\{ d(t, x), \tilde{d}(t, x) \right\} + \bar{\lambda}(t, x)T^a\chi(t, x)T^a - \bar{\chi}(t, x)T^a\lambda(t, x)T^a.
$$

(3.10)

By a somewhat troublesome calculation, one can confirm that this transformation on $\tilde{d}(t, x)$ is nilpotent $\delta^2 = 0$ without using any equation of motion. One may wonder how this intricate structure of nilpotent $\delta \tilde{d}(t, x)$ arises. In Appendix A we show that this structure of $\delta \tilde{d}(t, x)$
can be naturally understood if one considers an enlarged field space in which the covariance under a \((D+1)\)-dimensional gauge transformation is manifest.

One can also confirm by a direct calculation that the \((D+1)\)-dimensional system is BRS invariant:

\[ \delta (S_{\text{fl}} + S_{\text{dd}}) = 0. \] (3.11)

This can be seen by writing

\[
S_{\text{fl}} + S_{\text{dd}} = -2 \int_0^\infty dt \int d^D x \left( \operatorname{tr} L_\mu(t, x) E_\mu(t, x) + \int_0^\infty dt \int d^D x \left[ \bar{\lambda}(t, x) f(t, x) + \bar{f}(t, x) \lambda(t, x) \right] - 2 \int_0^\infty dt \int d^D x \operatorname{tr} \bar{d}(t, x)e(t, x), \right. \]

where

\[
E_\mu(t, x) \equiv \partial_t B_\mu(t, x) - D_\nu G_{\nu\mu}(t, x) - \alpha_0 D_\mu \partial_\nu B_\nu(t, x), \] (3.13)

\[
e(t, x) \equiv \partial_t d(t, x) - \alpha_0 D_\mu \partial_\mu d(t, x), \] (3.14)

\[
f(t, x) \equiv [\partial_t - \Delta + \alpha_0 \partial_\mu B_\mu(t, x)] \chi(t, x), \] (3.15)

\[
\bar{f}(t, x) \equiv \bar{\chi}(t, x) \left[ \bar{\partial}_t - \bar{\Delta} - \alpha_0 \partial_\mu B_\mu(t, x) \right], \] (3.16)

and then using the relations \[4\]

\[
\delta E_\mu(t, x) = [E_\mu(t, x), d(t, x)] + D_\mu e(t, x), \] (3.17)

\[
\delta e(t, x) = -\{e(t, x), d(t, x)\}, \] (3.18)

\[
\delta f(t, x) = -d(t, x)f(t, x) - e(t, x)\chi(t, x), \] (3.19)

\[
\delta \bar{f}(t, x) = -\bar{f}(t, x)d(t, x) - \bar{\chi}(t, x)e(t, x). \] (3.20)

Again, this (apparently miraculous) BRS invariance of the action can be naturally understood in the enlarged field space (see Appendix \[A\]).

### 3.2. Derivation of the Feynman rules

In this subsection, we show that perturbation theory in the above \((D+1)\)-dimensional field theory precisely reproduces the perturbative expansion of the gradient flow we have considered in Sect. \[2.3\]. This fact then allows us to use the \((D+1)\)-dimensional field theory to prove the perturbative renormalizability of the flowed system.

For this, it is quite helpful to specify the precise meaning of the flow-time derivative \(\partial_t\). In what follows, we understand that the flow time is discretized in the step \(\epsilon\). The integral is thus given by the sum,

\[ \int_0^\infty dt \equiv \epsilon \sum_{t=0}^\infty. \] (3.21)

For the flow-time derivative, we adopt the forward difference prescription,

\[ \partial_t f(t, x) \equiv \frac{1}{\epsilon} [f(t + \epsilon, x) - f(t, x)] \] (3.22)
for any field \( f(t, x) \). In particular, this prescription implies that for \( t = 0 \),

\[
\partial_t B_\mu(t = 0, x) \equiv \frac{1}{\epsilon} [B_\mu(t = \epsilon, x) - A_\mu(x)],
\]

\[
\partial_t \chi(t = 0, x) \equiv \frac{1}{\epsilon} [\chi(t = \epsilon, x) - \psi(x)],
\]

\[
\partial_t d(t = 0, x) \equiv \frac{1}{\epsilon} [d(t = \epsilon, x) - c(x)],
\]

because of the boundary conditions (3.4) and (3.5).

The ordering of removing regularizations is also quite important: After applying the Feynman rules to a given set of diagrams, we understand that we first take the continuous flow-time limit \( \epsilon \to 0 \) and then, after the renormalization, remove the dimensional regularization, \( D \to 4 \) (or other 4-dimensional regularization such as the lattice).

Under the above prescription, to obtain the tree-level propagators, we introduce the heat kernels with the discretized flow time [62] by

\[
K_\epsilon^f(x)_{\mu\nu} \equiv \int \frac{e^{ipx}}{p^2} \left[ (\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1 - \epsilon p^2)^{t/\epsilon} + p_\mu p_\nu(1 - \epsilon \alpha_0 p^2)^{t/\epsilon} \right],
\]

\[
K_\epsilon^f(x; \alpha_0) \equiv \int \frac{e^{ipx}(1 - \epsilon \alpha_0 p^2)^{t/\epsilon}}{p^2}.
\]

These quantities satisfy, by construction,

\[
\frac{1}{\epsilon} \left[ K_\epsilon^f(x)_{\mu\nu} - K_\epsilon^f(0)_{\mu\nu} \right] = \partial_\mu \partial_\rho K_\epsilon^f(x)_{\rho\nu} + (\alpha_0 - 1) \partial_\mu \partial_\nu K_\epsilon^f(x)_{\mu\rho},
\]

\[
K_\epsilon^f(0)_{\mu\nu} = \delta_{\mu\nu} \delta(x),
\]

and

\[
\frac{1}{\epsilon} \left[ K_\epsilon^f(x; \alpha_0) - K_\epsilon^f(0; \alpha_0) \right] = \alpha_0 \partial_\mu \partial_\nu K_\epsilon^f(x; \alpha_0),
\]

\[
K_\epsilon^f(0; \alpha_0) = \delta(x).
\]

First, we consider the propagators containing the flowed gauge field \( B_\mu(t, x) \). We change the variable from \( B_\mu \) to \( b_\mu \) as [4]

\[
B_\mu(t, x) = \int d^D y \ K_\epsilon^f(x - y)_{\mu\nu} A_\nu(y) + b_\mu(t, x).
\]

Then, because of Eq. (3.29), the boundary condition (3.4) becomes simply

\[
b_\mu(t = 0, x) = 0
\]

and, because of Eq. (3.28), the kinetic term of \( b_\mu \) is given by

\[
S_\phi = \epsilon \sum_{\alpha = 0}^\infty \int d^D x \ L^a_\mu(t, x)
\]

\[
\times \left\{ \frac{1}{\epsilon} \left[ b^a_\mu(t + \epsilon, x) - b^a_\mu(t, x) \right] - \partial_\nu \partial_\rho b^a_\mu(t, x) + (1 - \alpha_0) \partial_\mu \partial_\nu b^a_\mu(t, x) \right\} + \cdots.
\]
The corresponding Schwinger–Dyson equation in the tree-level yields,
\[
\left\langle \frac{1}{\epsilon} \left[ b^a_{\mu}(t + \epsilon, x) - b^a_{\mu}(t, x) \right] L^b_\nu(s, y) + \left[ -\delta_{\mu\rho} \partial_\sigma + (1 - \alpha_0)\partial_\mu \partial_\rho \right] b^a_{\mu}(t, x) L^b_\nu(s, y) \right\rangle_0 = \frac{1}{\epsilon} \delta^{ab} \delta_{\mu\nu} \delta(t, s) \delta(x - y).
\] (3.35)

This equation can then be solved step by step in the discretized \( t \), starting from \( t = 0 \) at which
\[
\left\langle b^a_{\mu}(t = 0, x) L^b_\nu(s, y) \right\rangle_0 = 0, \quad s \geq 0,
\] (3.36)

because of the boundary condition \( (3.33) \). In particular, we find
\[
\left\langle b^a_{\mu}(s, x) L^b_\nu(s, y) \right\rangle_0 = 0,
\] (3.37)
\[
\left\langle b^a_{\mu}(s + \epsilon, x) L^b_\nu(s, y) \right\rangle_0 = \delta^{ab} \delta_{\mu\nu} \delta(x - y),
\] (3.38)

and generally,
\[
\left\langle b^a_{\mu}(t, x) L^b_\nu(s, y) \right\rangle_0 = \delta^{ab} \vartheta(t - s) K^\epsilon_{t-s-\epsilon}(x - y)_{\mu\nu},
\] (3.39)

where \( \vartheta(t) \) is a “regularized step function”,
\[
\vartheta(t) \equiv \begin{cases} 1, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \\ 0, & \text{for } t < 0. \end{cases}
\] (3.40)

We then note that, from the structure of the action, the \( AL \) and \( bb \) propagators in the tree-level vanish. Thus, by using Eq. \( (3.32) \), we find
\[
\left\langle B^a_{\mu}(t, x) L^b_\nu(s, y) \right\rangle_0 = \delta^{ab} \vartheta(t - s) K^\epsilon_{t-s-\epsilon}(x - y)_{\mu\nu},
\] (3.41)
\[
\left\langle B^a_{\mu}(t, x) B^b_\nu(s, y) \right\rangle_0 = g_0^2 \delta^{ab} \int \frac{e^{ip(x-y)}}{(p^2)^2} \left[ (\delta_{\mu\rho}p^2 - p_\mu p_\nu)(1 - \epsilon p^2)^{(t+s)/\epsilon} + \frac{1}{\lambda_0} p_\mu p_\nu(1 - \epsilon \alpha_0 p^2)^{(t+s)/\epsilon} \right].
\] (3.42)

These results show that, in the \( \epsilon \to 0 \) limit, the former \( BL \) propagator becomes the heat kernel (or flow line) in the perturbative expansion of the gradient flow, Eq. \( (2.17) \), and the latter \( BB \) propagator reproduces Eq. \( (2.21) \).

In a similar way, we find for other flowed fields,
\[
\left\langle \chi(t, x) \lambda(s, y) \right\rangle_0 = \vartheta(t - s) K^\epsilon_{t-s-\epsilon}(x - y; 1),
\] (3.43)
\[
\left\langle \chi(t, x) \bar{\lambda}(s, y) \right\rangle_0 = \vartheta(s - t) K^\epsilon_{s-t-\epsilon}(x - y; 1),
\] (3.44)
\[
\left\langle \chi(t, x) \bar{\lambda}(s, y) \right\rangle_0 = \int \frac{e^{ip(x-y)}}{p^2 + m_0^2} (1 - \epsilon p^2)^{(t+s)/\epsilon},
\] (3.45)
\[
\left\langle \delta^a(t, x) \delta^b(s, y) \right\rangle_0 = \delta^{ab} \vartheta(t - s) K^\epsilon_{t-s-\epsilon}(x - y; \alpha_0),
\] (3.46)

which, for \( \epsilon \to 0 \), reproduce heat kernels and propagators in Sect. \( 2.3 \).

From the structure of \( S_0 \) in Eq. \( (3.2) \), it is obvious that the interaction vertices derived from Eq. \( (3.2) \) are identical to the non-linear terms in the flow equations, \( R_{\mu} \) in Eq. \( (2.18) \).
and $\Delta'$ and $\bar{\Delta}'$ in Eqs. (2.22) and (2.23). Thus, we see that the Feynman rules of the above $(D + 1)$-dimensional field theory basically coincide with the perturbative expansion of the flowed system in Sect. 2.3.

A possible subtlety arises, however, for a Feynman loop consisting only of the $BL$ propagators (and similarly for a loop consisting only of the $\chi\bar{\lambda}$ and $\lambda\bar{\chi}$ propagators), because although we can write such a Feynman diagram in the $(D + 1)$-dimensional field theory, we have no such loop diagram of the flow lines (heat kernels) in the perturbative expansion of the flowed system, as we have noted at the end of Sect. 2.3. In reality, this possible discrepancy does not arise according to the above forward difference prescription (3.22), because we have for example $\langle B_{\mu}(t, x) L_{\mu}(s, y) \rangle_0 = 0$ for $t \leq s$. This implies that such a loop consisting only of the $BL$ propagators (and a similar loop of the $\chi\bar{\lambda}$ or $\lambda\bar{\chi}$ propagators) identically vanishes; this is the role of the forward difference prescription (3.22).

A similar remark applies to a loop consisting of the $d\bar{d}$ propagators. The interaction vertex in $S_{d\bar{d}}$ in Eq. (3.3) does not contribute to correlation function without containing $d$ and $\bar{d}$, because any loop consisting of the $d\bar{d}$ propagator vanishes because of the property $\langle d(t, x)\bar{d}(s, y) \rangle_0 = 0$ for $t \leq s$. Note that we are interested only in correlation functions without containing $d$ and $\bar{d}$; these fields are auxiliary tools simply to derive the WT relation associated with the BRS symmetry.

This completes our argument for the perturbative equivalence of the $(D + 1)$-dimensional field theory and the system defined by the flow equations. We can now study the structure of the renormalization in the flowed system, by employing the $(D + 1)$-dimensional local field theory.

4. Proof of the renormalizability

In this section, we give a proof of the renormalizability of the flowed system on the basis of the $(D + 1)$-dimensional local field theory introduced in the previous section. To avoid our basic reasoning from being made obscure by too complicated mathematical expressions, we first extensively study the case of the pure Yang–Mills theory. Then, in Sect. 4.5 we explain how the proof is modified by the inclusion of fermion fields.

4.1. Ward–Takahashi relation or the Zinn-Justin equation

In this subsection, we derive the WT relation associated with the BRS symmetry (the Zinn-Justin equation) of the $(D + 1)$-dimensional field theory. The argument is standard, except a caution associated with non-trivial relations among field variables, Eqs. (3.4) and (3.5).

First, we introduce the external sources for each field by

$$ S_J = 2 \int d^Dx \text{ tr} \left[ J_\mu^A(x) A_\mu(x) + J^c(x)c(x) + J^{\bar{c}}(x)\bar{c}(x) + J^B(x)B(x) \right] $$

$$ + 2 \int_0^\infty dt \int d^Dx \text{ tr} \left[ J_\mu^B(t, x) B_\mu(t, x) + J^d(t, x)d(t, x) \right] $$

$$ + 2 \int_0^\infty dt \int d^Dx \text{ tr} \left[ J_\mu^L(t, x) L_\mu(t, x) + J^{\bar{d}}(t, x)\bar{d}(t, x) \right]. \quad (4.1) $$

In writing this, we have noted that the field $B_\mu(t = 0, x)$ is not an independent dynamical variable but rather $B_\mu(t = 0, x) = A_\mu(x)$. Thus, we should not include the external
source \( J^B t, x \) for \( B t, x \) at \( t = 0 \), \( J^B t, 0, x \) cannot be distinguished from \( J^A t, x \). In order to take this point into account, we have introduced a flow-time integration with the \( t = 0 \) time-slice being excluded:

\[
\int_0^\infty dt \equiv \epsilon \sum_\tau = \epsilon.
\]

(4.2)

This remark is applied also to the external source \( J^d t, x \) for \( d t, x \), because \( d t, 0, x \) is not an independent variable.

We next introduce another source term for the BRS transformations of each field:

\[
S_K = 2 \int d^D x \text{tr} \left[ K^A t, x D^x c(x) - K^c x^c c(x)^2 \right]
+ 2 \int_0^\infty dt \int d^D x \text{tr} \left[ K^B t, x D^x d t, x - K^d x^d d x^2 \right]
+ 2 \int_0^\infty dt \int d^D x \text{tr} \left\{ K^L t, x [L^x t, x, d t, x] \right. \\
\left. + K^d x^d \left[ D^x L^x t, x - \{ d t, x \}, \bar{d}(t, x) \} \right] \right\}. \quad (4.3)
\]

The symbol \( \int_0^\infty dt \) is used for the same reason as for Eq. (4.1). Note that because of the nilpotency of the BRS transformation \( \delta^2 = 0 \), \( \delta S_K = 0 \).

Then, as usual, we consider the BRS transformation \( \delta \) of integration variables in the functional integral

\[
e^{-W[J,K]} \equiv \int d\mu e^{-S_{\text{tot}} - S_J - S_K}, \quad (4.4)
\]

where \( d\mu \) is the functional integration measure. Then, since \( S_{\text{tot}} (3.1) \) and \( S_K (4.3) \) are invariant under \( \delta \) while \( S_J (4.1) \) is not, we have the following identity among correlation functions:

\[
\left\langle -2 \int d^D x \text{tr} \left[ J^A t, x D^x c(x) + J^c x^c c(x)^2 - J^c x^c c(x)^2 \right] \right\rangle
+ \left\langle -2 \int_0^\infty dt \int d^D x \text{tr} \left[ J^B t, x D^x d t, x + J^d x^d d x^2 \right] \right\rangle
+ \left\langle -2 \int_0^\infty dt \int d^D x \text{tr} \left\{ J^L t, x [L^x t, x, d t, x] \right. \\
\left. - J^d \left[ D^x L^x t, x - \{ d t, x \}, \bar{d}(t, x) \} \right] \right\} \right\rangle = 0. \quad (4.5)
\]

In reality, this identity is broken by \( O(\epsilon) \) terms under the discrete flow-time prescription that we are assuming at present. To explicitly see those \( O(\epsilon) \) terms safely vanish for the \( \epsilon \to 0 \) limit requires a little work and Appendix B is devoted to this task. In what follows, we neglect those \( O(\epsilon) \) terms by assuming that the limit \( \epsilon \to 0 \) is taken in an appropriate stage.

Another possible approach is to consider a lattice regularization of the \((D + 1)\)-dimensional field theory [5] in which the BRS symmetry is manifest and then study the perturbative renormalizability by using this regularization.
We then introduce the effective action $\Gamma$, the generating functional of one-particle irreducible (1PI) diagrams in the presence of $K$ external sources, by the Legendre transformation from external sources to the expectation values by,

$$
\Gamma[A_\mu, c, \bar{c}, B, \tilde{K}^A_\mu, K^c; B_\mu, d, L_\mu, \bar{d}, K^B_\mu, K^d, K^L_\mu, K^d] \equiv W[J, K]
$$

$$
+ 2 \int d^Dx \, \text{tr} \left[ J^A_\mu(x)A_\mu(x) + J^c(x)c(x) + J^c(x)\bar{c}(x) + J^B(x)B(x) \right]
$$

$$
+ 2 \int_0^{\infty} dt \int d^Dx \, \text{tr} \left[ J^B_\mu(t, x)B_\mu(t, x) + J^d(t, x)d(t, x) \right]
$$

$$
+ 2 \int_0^{\infty} dt \int d^Dx \, \text{tr} \left[ J^L_\mu(t, x)L_\mu(t, x) + J^\bar{d}(t, x)d(t, x) \right], \quad (4.6)
$$

where field variables in this expression, such as $A_\mu(x)$, stand for expectation values rather than dynamical variables. Note that the $K$ external sources are not Legendre transformed; the dependence on $K$ is thus taken over from $W$ to $\Gamma$. Then, noting the relation such as $\delta \Gamma/\delta A_\mu^a(x) = J_\mu^a(x)$, the WT relation $4.3$ is expressed in terms of the effective action $\Gamma$ as

$$
\int d^Dx \, \left[ \frac{\delta \Gamma}{\delta A_\mu^a(x)} \frac{\delta \Gamma}{\delta K^A_\mu(x)} + \frac{\delta \Gamma}{\delta c^a(x)} \frac{\delta \Gamma}{\delta K^c(x)} - \frac{\delta \Gamma}{\delta \bar{c}^a(x)} B^a(x) \right]
$$

$$
+ \int_0^{\infty} dt \int d^Dx \, \left[ \frac{\delta \Gamma}{\delta B_\mu^a(t, x)} \frac{\delta \Gamma}{\delta K^B_\mu(t, x)} + \frac{\delta \Gamma}{\delta d^a(t, x)} \frac{\delta \Gamma}{\delta K^d(t, x)} \right]
$$

$$
+ \int_0^{\infty} dt \int d^Dx \, \left[ \frac{\delta \Gamma}{\delta L_\mu^a(t, x)} \frac{\delta \Gamma}{\delta K^L_\mu(t, x)} + \frac{\delta \Gamma}{\delta \bar{d}^a(t, x)} \frac{\delta \Gamma}{\delta K^\bar{d}(t, x)} \right] = 0. \quad (4.7)
$$

Next, we note that the equations of motion (the Schwinger–Dyson equations),

$$
\left\langle \frac{1}{g_0^2} \left[ \partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0} B^a(x) \right] \right\rangle - J^{B\mu}(x) = 0,
$$

$$
\left\langle -\frac{1}{g_0^2} \partial_\mu D_\mu c^a(x) \right\rangle + J^{\bar{c}^a}(x) = 0, \quad (4.8)
$$

are expressed in terms of the effective action as

$$
\frac{\delta \Gamma}{\delta B^a(x)} = \frac{1}{g_0^2} \left[ \partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0} B^a(x) \right], \quad (4.10)
$$

$$
\frac{\delta \Gamma}{\delta c^a(x)} - \frac{1}{g_0^2} \partial_\mu \frac{\delta \Gamma}{\delta K^a_\mu(x)} = 0. \quad (4.11)
$$

These relations show that, by defining

$$
\tilde{K}^A_\mu \equiv K^A_\mu - \frac{1}{g_0^2} \partial_\mu c^a \quad (4.12)
$$

and

$$
\tilde{\Gamma}[A_\mu, c, \tilde{K}^A_\mu, K^c; B_\mu, d, L_\mu, \bar{d}, K^B_\mu, K^d, K^L_\mu, K^d]
$$

$$
\equiv \Gamma + \frac{2}{g_0^2} \int d^Dx \, \text{tr} \left\{ B(x) \left[ \partial_\mu A_\mu(x) - \frac{1}{2\lambda_0} B(x) \right] \right\}, \quad (4.13)
$$
the dependence on \( B \) and \( c \) are completely eliminated; \( \tilde{\Gamma} \) does not depend on neither \( B \) nor \( c \). The reduced effective action \( \tilde{\Gamma} \) then obeys

\[
\int d^D x \left[ \frac{\delta \tilde{\Gamma}}{\delta A^a_{\mu}(x)} \frac{\delta \tilde{\Gamma}}{\delta K_{\mu}^{Aa}(x)} + \frac{\delta \tilde{\Gamma}}{\delta c^a(x)} \frac{\delta \tilde{\Gamma}}{\delta K^{ca}(x)} \right] + \int_0^\infty dt \int d^D x \left[ \frac{\delta \tilde{\Gamma}}{\delta B^a_{\mu}(t,x)} \frac{\delta \tilde{\Gamma}}{\delta K_{\mu}^{Ba}(t,x)} + \frac{\delta \tilde{\Gamma}}{\delta d^a(t,x)} \frac{\delta \tilde{\Gamma}}{\delta K^{da}(t,x)} \right] = 0. \tag{4.14}
\]

This is the WT relation following from the BRS invariance of the \((D+1)\)-dimensional field theory.

### 4.2. Statement of the renormalizability

Now, our statement of renormalizability is that the generating functional of the 1PI correlation functions \( \Gamma \) can be made finite in terms of renormalized quantities, by choosing three renormalization constants \( Z \), \( Z_3 \), and \( \tilde{Z}_3 \) in

\[
\begin{align*}
    g^2_0 &= \mu^{2\varepsilon} g^3 Z, \tag{4.15} \\
    A^a_{\mu} &= Z^{1/2} \tilde{Z}_3^{1/2}(A_R)_{\mu}^a, \quad K_{\mu}^{Aa} = Z^{-1/2} \tilde{Z}_3^{-1/2}(K^{Aa}_R)_{\mu}, \tag{4.16} \\
    c^a &= \tilde{Z}_3 Z^{1/2} \tilde{Z}_3^{1/2}(c_R)^a, \quad K^{ca} = Z^{-1/2} \tilde{Z}_3^{-1/2}(K^{ca}_R) \tag{4.17}
\end{align*}
\]

and

\[
\begin{align*}
    \lambda_0 &= \lambda \tilde{Z}_3^{-1}, \\
    B^a &= Z^{1/2} \tilde{Z}_3^{-1/2} B^{a}_R, \quad \bar{c}^a = Z^{1/2} \tilde{Z}_3^{-1/2} \bar{c}^a_R. \tag{4.19}
\end{align*}
\]

appropriately order by order in the loop expansion of perturbation theory. Note that the above statement in particular claims that the flowed gauge field \( B_{\mu}(t,x) \) does not require any wave function renormalization. From Eq. (4.13), it suffices to show the finiteness of the reduced effective action \( \tilde{\Gamma} \), because

\[
- \frac{2}{g^2_0} \text{tr} \left[ B \left( \partial_\mu A_\mu - \frac{1}{2\lambda_0} B \right) \right] = - \frac{2}{\mu^{2\varepsilon} g^2} \text{tr} \left\{ B_R \left[ \partial_\mu (A_R)_\mu - \frac{1}{2\lambda} B_R \right] \right\}, \tag{4.20}
\]

is finite with the above definitions. For \( \tilde{\Gamma} \), defining

\[
\tilde{K}_{\mu}^{Aa} = K_{\mu}^{Aa} - \frac{1}{g^2_0} \partial_\mu \bar{c}^a \equiv Z^{-1/2} \tilde{Z}_3^{-1/2}(\tilde{K}_{R}^{Aa})_{\mu}, \tag{4.21}
\]

5 When fermion fields are included, we need additional renormalization constants, \( Z_\psi \), \( Z_m \), and \( Z_\chi \); see Sect. 4.5.
6 It also claims that the parameter \( \alpha_0 \) is not renormalized.
the WT relation \((1.14)\) in terms of the renormalized quantities yields,

\[
\int d^Dx \left[ \frac{\delta \tilde{\Gamma}}{\delta (A_R)^a_\mu (x)} \frac{\delta \tilde{\Gamma}}{\delta (\hat{K}_{R}^{a\mu}) (x)} + \frac{\delta \tilde{\Gamma}}{\delta \hat{c}_{R}^{a} (x)} \frac{\delta \tilde{\Gamma}}{\delta \hat{K}_{R}^{a\mu} (x)} \right] + \int_0^\infty dt \int d^Dx \left[ \frac{\delta \tilde{\Gamma}}{\delta L^a_\mu (t, x) \delta \hat{K}_{R}^{a\mu} (t, x)} + \frac{\delta \tilde{\Gamma}}{\delta d^a(t, x) \delta \hat{K}_{R}^{a\mu} (t, x)} \right] = 0. \tag{4.22}
\]

Let us now consider the loop expansion of \(\tilde{\Gamma}\) in the renormalized perturbation theory,

\[
\tilde{\Gamma} = \sum_{\ell=0}^\infty \tilde{\Gamma}^{(\ell)}, \tag{4.23}
\]

where \(\ell\) denotes the order of the loop expansion; the tree-level effective action \(\tilde{\Gamma}^{(0)}\) is given by the tree-level action in renormalized quantities:

\[
\tilde{\Gamma}^{(0)} = (S + S_{\text{fl}} + S_{\text{dd}} + S_{\text{K}})|_{Z=Z_3=Z_3^1=\hat{K}_{R}^{a\mu} \to (\hat{K}_{R}^{a\mu})}. \tag{4.24}
\]

Corresponding to the tree-level action \((4.24)\), the counterterm in the renormalized perturbation theory is given by

\[
\Delta S = S + S_{\text{fl}} + S_{\text{dd}} + S_{\text{K}} - (S + S_{\text{fl}} + S_{\text{dd}} + S_{\text{K}})|_{Z=Z_3=Z_3^1=\hat{K}_{R}^{a\mu} \to (\hat{K}_{R}^{a\mu})}. \tag{4.25}
\]

Note that this counterterm contains in particular the “boundary counterterm” \(\Delta S_{\text{bc}}\),

\[
\Delta S_{\text{bc}} = 2 \int d^Dx \text{tr} \left[ L_\mu(0, x)(Z^{1/2}Z_3^{1/2} - 1)(A_R)_\mu(x) \right] + 2 \int d^Dx \text{tr} \left[ \tilde{d}(0, x)(\bar{Z}_3Z^{1/2}Z_3^{1/2} - 1)c_R(x) \right], \tag{4.26}
\]

because in Eqs. \((3.2)\) and \((3.3)\), the \(t\)-derivatives at \(t = 0\) are defined by Eqs. \((3.23)\) and \((3.25)\) and \(A_\mu(x)\) and \(c(x)\) are renormalized as \(Z^{1/2}Z_3^{1/2}(A_R)_\mu(x)\) and \(\bar{Z}_3Z^{1/2}Z_3^{1/2}c_R(x)\), while \(B_\mu(t = \epsilon, x)\) and \(d(t = \epsilon, x)\) are not renormalized. We also note that the tree-level action \((4.24)\) yields the tree-level propagators

\[
\langle (A_R)^a_\mu(x)(A_R)^b_\nu(y) \rangle_0 = \mu^{2g_2} \delta^{ab} \frac{e^{i(p\cdot(x-y))}}{(p^2)^2} \left( \delta_{\mu\nu}p^2 - p_\mu p_\nu + \frac{1}{\lambda} p_\mu p_\nu \right), \tag{4.27}
\]

and (in the \(\epsilon \to 0\) limit)

\[
\langle B^a_\mu(t, x)B^b_\nu(s, y) \rangle_0 = \mu^{2g_2} \delta^{ab} \frac{e^{i(p\cdot(x-y))}}{(p^2)^2} \left[ (\delta_{\mu\nu}p^2 - p_\mu p_\nu) e^{-(t+s)p^2} + \frac{1}{\lambda} p_\mu p_\nu e^{-a_0(t+s)p^2} \right]. \tag{4.28}
\]

In the renormalized perturbation theory, these propagators are used.

Our proof for the renormalizability proceeds by induction: Suppose that, to the \(\ell\)-th order of the loop expansion, \(Z, Z_3,\) and \(\bar{Z}_3\) can be chosen so that \(\tilde{\Gamma}^{(\ell)}\) is finite. Then, considering
the divergent part of the WT relation (4.22) in the \((\ell + 1)\)-th loop order, we have
\[ \tilde{\Gamma}(0) * \tilde{\Gamma}^{(\ell+1)\text{div}} = 0, \] (4.29)
where \(\tilde{\Gamma}^{(\ell+1)\text{div}}\) denotes the divergent part of \(\tilde{\Gamma}^{(\ell+1)}\) and
\[ \tilde{\Gamma}(0) = -\int d^D x \left[ \frac{\delta \tilde{\Gamma}(0)}{\delta (A_R)_\mu(x)} \frac{\delta}{\delta (K_R^a)_\mu(x)} + \frac{\delta \tilde{\Gamma}(0)}{\delta (K_R^a)_\mu(x)} \frac{\delta}{\delta (A_R)_\mu(x)} \right] \]
\[ - \epsilon \int d^D x \left[ \frac{\delta \tilde{\Gamma}(0)}{\delta \bar{d}^a(0,x)} \frac{\delta}{\delta K_{L_a}^a(0,x)} + \frac{\delta \tilde{\Gamma}(0)}{\delta K_{L_a}^a(0,x)} \frac{\delta}{\delta \bar{d}^a(0,x)} \right] \]
\[ + (\text{derivatives w.r.t. field variables at } t \neq 0). \] (4.30)

In this expression, we have explicitly used the discrete time prescription (3.21).

One can confirm that because of the BRS invariance of the action \(S + S_{fl} + S_{d \bar{d}} + S_K\), \(\tilde{\Gamma}(0)\) is nilpotent:
\[ \tilde{\Gamma}(0) * \tilde{\Gamma}(0) = 0. \] (4.31)

Eqs. (4.29) and (4.31) are basic relations for the proof. Next, we have to clarify the most general form of the divergent part \(\tilde{\Gamma}^{(\ell+1)\text{div}}\).

### 4.3. General form of the divergent part

First of all, we note that there is no “bulk” divergence that is given by a \((D + 1)\)-dimensional integral \(\int_0^\infty dt \int d^D x\) of a local polynomial of fields. This follows from the fact that a Feynman loop that consists only of the flow lines (the heat kernels) identically vanishes as we have observed. Thus the integrand of the loop integral for a loop diagram whose vertices reside in the “bulk” of the \((D + 1)\)-dimensional spacetime, \(t > 0\), contains at least one propagator which provides the Gaussian damping factor \(\sim e^{-tp^2}\). Then the loop-momentum integral is absolutely convergent; there is no “bulk” divergence.

Thus \(\tilde{\Gamma}^{(\ell+1)\text{div}}\) consists of the \(D\)-dimensional integral \(\int d^D x\) of a local polynomial of fields, the “boundary” divergence. The mass dimension of the local polynomial is less or equal to 4 for \(D = 4\) and the ghost number is 0.

Let us next show that
\[ -2 \int d^D x \text{ tr } [\partial_t B_\mu(0,x)(A_R)_\mu(x)], \quad -2 \int d^D x \text{ tr } [\partial_t d(0,x)\bar{e}_R(x)], \] (4.32)
cannot appear in \(\tilde{\Gamma}^{(\ell+1)\text{div}}\), although the mass dimension and the ghost number perfectly match. Note that two combinations in Eq. (4.32) contain, according to our prescription,
$B_\mu(t = \epsilon, x)$ and $d(t = \epsilon, x)$, respectively. That the terms in Eq. (4.32) cannot appear in the divergent part follows again from the fact that a Feynman loop that consists only of the flow lines, the $BL$ and $d\bar{d}$ propagators, identically vanishes. Because of this fact and from the structure of the interaction vertices containing $B_\mu$ or $d$ in $S_{tot} + S_K$, we see that any divergent term that contains $B_\mu$ or $d$ must involve $L_\mu$, $\bar{d}$, or the external source $K$. The combinations in Eq. (4.32) do not satisfy this criterion.

Next, consider terms being proportional to the flow-time derivative at $t = 0$, $\partial_t B_\mu^a(0, x)$ or $\partial_t d^a(0, x)$ in the WT relation $\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)}_{\text{div}} = 0$. In $\tilde{\Gamma}^{(0)}$ in Eq. (4.30), only the coefficient functions,

$$\frac{\delta \tilde{\Gamma}^{(0)}}{\delta L_\mu^a(0, x)} = \partial_t B_\mu^a(0, x) + \cdots, \quad \frac{\delta \tilde{\Gamma}^{(0)}}{\delta d^a(0, x)} = \partial_t d^a(0, x) + \cdots,$$

contain the flow-time derivative at $t = 0$. Since $\tilde{\Gamma}^{(0)} *$ has the structure

$$\tilde{\Gamma}^{(0)} * \sim \frac{\delta \tilde{\Gamma}^{(0)}}{\delta L_\mu^a(0, x)} \delta K_L^a(0, x) + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta d^a(0, x)} \delta K^d(0, x) + \cdots,$$

taking into account the fact that $\tilde{\Gamma}^{(\ell+1)}_{\text{div}}$ contains neither $\partial_t B_\mu^a(0, x)$ nor $\partial_t d^a(0, x)$ as we have observed above, we infer that $\tilde{\Gamma}^{(\ell+1)}_{\text{div}}$ cannot contain external fields, $K_L^a(0, x)$ and $K^d(0, x)$.

Finally, the mass dimension for $D = 4$ and the ghost number of $K_L^a(B)$ and $K^d$ are 4 and $-1$ (4 and $-2$), respectively. For this reason, these sources cannot appear in $\tilde{\Gamma}^{(\ell+1)}_{\text{div}}$.

From these considerations, we find that the most general form of the divergent part is given by

$$\tilde{\Gamma}^{(\ell+1)}_{\text{div}} = \int d^D x \mathcal{L} \left( (A_R)_\mu, c_R, (\tilde{K}_R^A)_\mu, K^{c}_R; L_\mu(t = 0), \bar{d}(t = 0) \right),$$

where $\mathcal{L}$ is a local polynomial of fields; its mass dimension is at most 4 for $D = 4$ and the ghost number is 0.

### 4.4. Final steps

Our strategy to analyze the WT relation (4.29) is to decompose it into a $D$-dimensional part that is well-understood in the renormalization of the Yang–Mills theory and an exotic part that originates from the $(D + 1)$-dimensional fields.

Thus, in Eq. (4.35), we set $\tilde{\Gamma}^{(\ell+1)}_{\text{div}} = \tilde{\Gamma}^{(\ell+1)}_{D} + \tilde{\Gamma}^{(\ell+1)}_{D+1}$, where

$$\tilde{\Gamma}^{(\ell+1)}_{D} \equiv \tilde{\Gamma}^{(\ell+1)}_{\text{div}} \bigg|_{L_\mu = \bar{d} = 0},$$
$$\tilde{\Gamma}^{(\ell+1)}_{D+1} \equiv \tilde{\Gamma}^{(\ell+1)}_{\text{div}} - \tilde{\Gamma}^{(\ell+1)}_{D}.$$

Note that $\tilde{\Gamma}^{(\ell+1)}_{D+1}$ contains at least one $L_\mu$ or one $\bar{d}$.

---

8One has to note this property of the divergence part also in the proof of Ref. [4]. We would like to thank Martin Lüscher for explanation on this point.
We also decompose $\tilde{\Gamma}^{(0)}$ into $\tilde{\Gamma}^{(0)} = \tilde{\Gamma}_D^{(0)} + \tilde{\Gamma}_{D+1}^{(0)}$, where $\tilde{\Gamma}_D^{(0)}$ is the “BRS operator” in the original Yang–Mills theory

\[
\tilde{\Gamma}_D^{(0)} \equiv - \int d^D x \left[ \frac{\delta \tilde{\Gamma}^{(0)}}{\delta (A_R)^{a\mu}(x)} \frac{\delta A_R^{\mu}(x)}{\delta (K_R)^{a\mu}(x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta (A_R)^{a\mu}(x)} \frac{\delta (K_R)^{a\mu}(x)}{\delta (A_R)^{a\mu}(x)} \right] + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta c_R^a(x)} \frac{\delta c_R^a(x)}{\delta c_R^a(x)}
\]

and $\tilde{\Gamma}_{D+1}^{(0)}$ is the remaining

\[
\tilde{\Gamma}_{D+1}^{(0)} \equiv \tilde{\Gamma}^{(0)} - \tilde{\Gamma}_D^{(0)}.
\]
This shows that the counterterm in Eq. (4.25) precisely cancels the flowed gauge field does not require wave function renormalization. In particular, the renormalization constants in Eqs. (4.15)–(4.19) makes the generating functional of the 1PI gauge invariant part in the first line of Eq. (4.44) is also invariant under \( \tilde{\Gamma}^{(0)} \). Also, because of the nilpotency (4.31), the WT relation (4.29) for Eq. (4.44) requires (the gauge invariant part in the first line of Eq. (4.44) is also invariant under \( \tilde{\Gamma}^{(0)} \)).

This shows that \( z_1 = z_2 = 0 \) and, by Eq. (4.46), \( \tilde{\Gamma}^{(0)} \) diverges. Thus, going back to Eq. (4.44), we conclude that the most general form of \( \tilde{\Gamma}^{(0)} \) is

\[
\tilde{\Gamma}^{(0)} = \sum_{\ell} (z_1 - z_2) L_\mu(0, x) \partial_\mu c_R(x)
\]

with three constants \( x_1, y_1, \) and \( y_2 \).

Then, it is straightforward to see that if one chooses the \((\ell+1)\)-th loop order coefficients in \( Z = \sum_{\ell'=0}^{\ell+1} Z^{(\ell')}, \quad Z_3 = \sum_{\ell'=0}^{\ell+1} Z_3^{(\ell')}, \quad Z_3^{(\ell+1)} = x_1, \quad Z_3^{(\ell+1)} = -x_1 + 2y_1, \quad Z_3^{(\ell+1)} = -y_1 - y_2, \)

the counterterm in Eq. (4.25) precisely cancels the \((\ell+1)\)-th order divergence (4.48). In particular, the boundary counterterm (4.25) cancels the last line of Eq. (4.48). This completes the mathematical induction: We have shown that an appropriate order by order choice of renormalization constants in Eqs. (4.15)–(4.19) makes the generating functional of the 1PI diagrams—thus all correlation functions of renormalized fields—finite. In particular, the flowed gauge field does not require wave function renormalization.

Note that \( Z^{(0)} = Z_3^{(0)} = \hat{Z}_3^{(0)} = 1 \).
4.5. Renormalization of fermion fields

One can include the fermion fields into the above argument. Corresponding to the BRS transformations in Eqs. (2.6), (3.8), and (3.9), the action $S_K$ has additional terms,

$$S_K = \cdots + \int d^D x \left[ \bar{K}^\psi(x)c(x)\psi(x) + \bar{\psi}(x)c(x)K^\psi(x) \right]$$

and

$$+ \int_0^\infty dt \int d^D x \left[ \bar{K}^\chi(t,x)d(t,x)\chi(t,x) + \bar{\chi}(t,x)d(t,x)K^\chi(t,x) \right]$$

and

$$+ \int_0^\infty dt \int d^D x \left[ \bar{K}^\lambda(t,x)d(t,x)\lambda(t,x) + \bar{\lambda}(t,x)d(t,x)K^\lambda(t,x) \right]. \quad (4.50)$$

The renormalization constants are introduced as

$$m_0 = Z_m^{-1}m_R, \quad (4.51)$$

$$\psi = Z_\psi^{-1/2}\psi_R, \quad \bar{\psi} = Z_\psi^{-1/2}\bar{\psi}_R, \quad (4.52)$$

and

$$\bar{\lambda} = Z_\chi^{1/2}\bar{\lambda}_R, \quad (4.53)$$

$$\lambda = Z_\chi^{1/2}\lambda_R, \quad (4.54)$$

$$K^\chi = Z_\chi^{-1/2}K_R^\chi, \quad (4.55)$$

and

$$\bar{\lambda} = Z_\chi^{1/2}\bar{\lambda}_R, \quad (4.56)$$

$$\lambda = Z_\chi^{1/2}\lambda_R, \quad (4.57)$$

The derivative operator $\hat{\Gamma}^{(0)*}$ has additional terms,

$$\hat{\Gamma}^{(0)*} = \cdots - \int d^D x \left[ \frac{\delta \hat{\Gamma}^{(0)}}{\delta \bar{\psi}_R(x)\delta K^\psi_R(x)} + \frac{\delta \hat{\Gamma}^{(0)}}{\delta \bar{\lambda}_R(x)\delta K^\chi_R(x)} \delta \psi_R(x) \right]$$

and

$$- \epsilon \int d^D x \left[ \frac{\delta \hat{\Gamma}^{(0)}}{\delta \lambda_R(0,x)\delta K^\chi_R(0,x)} + \frac{\delta \hat{\Gamma}^{(0)}}{\delta \lambda_R(0,x)\delta K^\lambda_R(0,x)} \right]$$

+ (derivatives w.r.t. field variables at $t \neq 0$). \quad (4.58)$$

Following the same line of argument as the pure Yang–Mills case, the representation of the possible divergent term, Eq. (4.44), is modified to

$$\hat{\Gamma}^{(\ell+1)\text{div}} = \cdots + \int d^D x \left\{ w_1 \bar{\psi}_R(x)\gamma_\mu [\partial_\mu + (A_R)_\mu(x)] \psi_R(x) + w_2 m_R \bar{\psi}_R(x)\psi_R(x) \right\}$$

and

$$+ \hat{\Gamma}^{(0)*} \int d^D x w_3 \left[ K^\psi_R(x)\psi_R(x) + \bar{\psi}_R(x)K^\psi_R(x) \right]$$

and

$$+ \int d^D x \xi_1 \left[ \bar{\lambda}_R(0,x)\psi_R(x) + \bar{\psi}_R(x)\lambda_R(0,x) \right]. \quad (4.59)$$
where \( w_1, w_2, w_3, \) and \( \xi_1 \) are (divergent) constants. This time, unlike the pure Yang–Mills case \( 4.44 \), the last term of Eq. \( 4.59 \) is manifestly BRS (or gauge) invariant and it is annihilated by \( \tilde{I}^{(0)*} \). Thus we cannot conclude \( \xi_1 = 0 \) from the WT relation \( \tilde{I}^{(0)*} \tilde{I}^{(\ell+1)\div} = 0 \) \(^{10}\).

Eq. \( 4.59 \) then yields \( \tilde{I}^{(\ell+1)\div} \)

\[
= \cdots + \int d^D x \left\{ (w_1 + 2w_3) \bar{\psi}_R(x) \gamma_\mu \partial_\mu \psi_R(x) \right.
+ (w_1 + 2w_3 - y_1) \bar{\psi}_R(x) \gamma_\mu (A_R)_\mu \psi_R(x) \\
+ (w_2 + 2w_3) m_R \bar{\psi}_R(x) \psi_R(x) \\
+ y_2 \left[ \bar{K}^\psi_R(x)c_R(x) \psi_R(x) + \bar{\psi}_R(x) c_R(x) K^\psi_R(x) \right] \\
+ \left( -w_3 + \xi_1 \right) \left[ \bar{\lambda}_R(0, x) \psi_R(x) + \bar{\psi}_R(x) \lambda_R(0, x) \right] \right\}.
\]

These divergences can be canceled by setting the \( (\ell + 1) \)-th order renormalization constants as

\[
Z^{(\ell+1)}_{\psi} = w_1 + 2w_3, \quad Z^{(\ell+1)}_m = -w_1 + w_2, \quad Z^{(\ell+1)}_\chi = w_1 + 4w_3 - 2\xi_1.
\]

In particular, the last line of Eq. \( 4.60 \) is canceled by the boundary counterterm arising from the flow time derivative in Eq. \( 3.2 \). Note that there is no particular reason for \( w_1 + 4w_3 - 2\xi_1 = 0 \) (i.e., \( Z_\chi = 1 \)) to hold and consequently the Lagrange multiplier fields for the flowed fermion fields, \( \bar{\lambda}(t, x) \) and \( \lambda(t, x) \), must be renormalized. This is a crucial difference from the gauge field for which \( L_\mu(t, x) \) is not renormalized.

Finally, the renormalization of the flowed fermion fields \( \chi(t, x) \) and \( \bar{\chi}(t, x) \) in Eq. \( 4.56 \) being reciprocal to that of \( \lambda(t, x) \) and \( \bar{\lambda}(t, x) \) is required, not to generate a “bulk counterterm” from \( S_\Pi \) \( 3.2 \). As we emphasized, there is no divergence written as a \((D + 1)\)-dimensional integral of a local polynomial of fields. Thus we should not have any \((D + 1)\)-dimensional counterterm. This requirement fixes the renormalization factors in Eq. \( 4.56 \).

Thus, unlike the flowed gauge field, the flowed fermion field \( \chi(t, x) \) and \( \bar{\chi}(t, x) \) requires wave function renormalization. As Eq. \( 4.61 \) indicates, the required renormalization constant \( Z_\chi \) is generally different from that for the original \( D \)-dimensional fermion field, \( Z_\psi \); an explicit perturbative calculation confirms that this is actually the case. From this observation, it is expected that this wave function renormalization of a flowed field persists also for generic matter fields.

### 4.6. Composite operators

Finally, we mention on the finiteness of local products of flowed fields (i.e., composite operators). In the above, we have shown that a correlation function of flowed fields

\[
Z^{(n+1)/2}_\chi (B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \chi(s_1, y_1) \cdots \chi(s_m, y_m) \bar{\chi}(u_1, z_1) \cdots \bar{\chi}(u_l, z_l)),
\]

\(^{10}\)Recall the argument around Eq. \( 4.47 \) that eliminated the possibility of \( z_1 \) and \( z_2 \).
is finite under the conventional renormalization in the original 4-dimensional gauge theory and an appropriate choice of the wave function renormalization factor $Z_\chi$. We can further see that this finiteness persists, even if some of spacetime points (say, $x_1$ and $x_2$ in Eq. (4.62)) coincide, as far as the corresponding flow times ($t_1$ and $t_2$ in this example) are strictly positive. This implies that any local products of flowed elementary fields (with the factor $Z_\chi^{1/2}$ for flowed fermion fields) is finite. Because of this remarkable property, the flow provides a versatile method to define renormalized composite operators.

This finiteness follows again from the fact that a Feynman loop that consists only of the flow lines (the heat kernels) identically vanishes. Let us take the above situation in which two spacetime points $x_1$ and $x_2$ coincide. In the loop that contains the points $(t_1, x_1)$ and $(t_2, x_2 = x_1)$, there must exist at least one propagator. The propagator carries the Gaussian damping factor $\sim e^{-v+wp^2}$, where $p_\mu$ denotes the loop momentum and $v$ and $w$ are flow times at end points of the propagator. In this situation, if $v > 0$ or $w > 0$, then the loop integral for the loop absolutely converges because of the damping factor. If $v = w = 0$, on the other hand, one of the propagators or the heat kernels contained in the loop gives rise to the factor that behaves like $\sim e^{-t_1p^2}$ or $\sim e^{-t_2p^2}$. Thus the loop integral again converges. This shows that the coincidence $x_1 = x_2$ does not produces new divergence as far as the corresponding flow times are strictly positive.

5. Conclusion

In the present paper, we have given another proof of the renormalizability of the system defined by the gradient flow and the fermion flow in vector-like gauge theories, the theorem first proven in Ref. [4, 5]: Any correlation functions and composite operators of flowed fields are made finite by the conventional renormalization of gauge theory and wave function renormalization of the flowed fermion fields. We believe that our proof is quite accessible if not elegant. We hope that our proof will be helpful to understand the essential feature of the flow that underlies its various applications in (lattice) gauge theory.

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A. The $(D + 1)$-dimensional field theory in an enlarged field space

In this Appendix, we show that the BRS transformation of the ghost Lagrange multiplier $\bar{d}$, Eq. (3.10), and the BRS invariance of the $(D + 1)$-dimensional action $S_\text{fl} + S_{\bar{d}d}$, Eq. (3.11), can naturally be understood if one enlarges the field space so that the gauge covariance in a $(D + 1)$-dimensional sense manifest. The idea of this enlarged field space can be found in Ref. [76] in the context of the stochastic quantization and also in Ref. [54] in the context of the gradient flow.
For this, we introduce the gauge potential in the flow-time direction, \( B_t(t, x) \), and set

\[
G_{t\mu}(t, x) \equiv \partial_t B_\mu(t, x) - \partial_\mu B_t(t, x) + [B_t(t, x), B_\mu(t, x)],
\]

(A1)

\[
D_t \equiv \partial_t + B_t,
\]

(A2)

\[
\tilde{D}_t \equiv \partial_t - B_t,
\]

(A3)

and consider a \((D + 1)\)-dimensional action

\[
\hat{S}_{fl} = -2 \int_0^\infty dt \int d^D x \ tr \{ L_\mu(t, x) [G_{t\mu}(t, x) - D_\nu G_{\nu\mu}(t, x)] \}
\]

\[
+ \int_0^\infty dt \int d^D x \left[ \bar{\lambda}(t, x) (D_t - \Delta) \chi(t, x) + \bar{\chi}(t, x) \left( \tilde{D}_t - \tilde{\Delta} \right) \lambda(t, x) \right].
\]

(A4)

We regard the new gauge potential at \( t = 0 \), \( B_t(t = 0, x) \), as an independent integration variable.

The above \((D + 1)\)-dimensional action \( \hat{S}_{fl} \) is manifestly invariant under the gauge transformation in the \((D + 1)\)-dimensional sense. Correspondingly, \( \hat{S}_{fl} \) is invariant under the following \((D + 1)\)-dimensional BRS transformation:

\[
\hat{\delta} B_\mu(t, x) = D_\mu d(t, x),
\]

(A5)

\[
\hat{\delta} B_t(t, x) = D_t d(t, x),
\]

(A6)

\[
\hat{\delta} L_\mu(t, x) = [L_\mu(t, x), d(t, x)],
\]

(A7)

\[
\hat{\delta} \chi(t, x) = -d(t, x) \chi(t, x),
\]

\[
\hat{\delta} \bar{\chi}(t, x) = -\bar{\chi}(t, x) d(t, x),
\]

(A8)

\[
\hat{\delta} \lambda(t, x) = -d(t, x) \lambda(t, x),
\]

\[
\hat{\delta} \bar{\lambda}(t, x) = -\bar{\lambda}(t, x) d(t, x),
\]

(A9)

\[
\hat{\delta} d(t, x) = -d(t, x)^2,
\]

(A10)

where \( d(t, x) \) is a \((D + 1)\)-dimensional ghost field. As usual, the above transformation is nilpotent by construction, \( \hat{\delta}^2 = 0 \).

Next, we introduce a \((D + 1)\)-dimensional “gauge fixing term” \( \hat{S}_{GF} \) and a corresponding “ghost-anti-ghost term” \( \hat{S}_{d\bar{d}} \) by

\[
\hat{S}_{GF} + \hat{S}_{d\bar{d}} \equiv \hat{\delta}(-2) \int_0^\infty dt \int d^D x \ tr \{ -\bar{d}(t, x) [B_t(t, x) - \alpha_0 \partial_\mu B_\mu(t, x)] \}.
\]

(A11)

In this expression, \( \bar{d}(t, x) \) is a \((D + 1)\)-dimensional analogue of the anti-ghost and its BRS transformation is defined by

\[
\hat{\delta} \bar{d}(t, x) = C(t, x),
\]

\[
\hat{\delta} C(t, x) = 0,
\]

(A12)

where \( C(t, x) = C^a(t, x) T^a \) is a \((D + 1)\)-dimensional analogue of the Nakanishi–Lautrup field. Clearly, the nilpotency \( \hat{\delta}^2 = 0 \) is preserved with this definition. Thus, by construction,

\footnote{Throughout this appendix, the flow time is assumed to be continuous. We may discretize the flow time while keeping the gauge invariance by using lattice regularization as in Ref. [5].}
the above action whose explicit form is given by

\[
\dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}} = -2 \int_0^\infty dt \int d^Dx \text{tr} \left\{-C(t,x) [B_t(t,x) - \alpha_0 \partial_\mu B_\mu(t,x)]\right\} \\
- 2 \int_0^\infty dt \int d^Dx \text{tr} \left\{\tilde{d}(t,x) [D_t\tilde{d}(t,x) - \alpha_0 \partial_\mu D_\mu \tilde{d}(t,x)]\right\},
\]

(A13)
is manifestly BRS invariant:

\[
\delta \left(\dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}}\right) = 0.
\]

(A14)

So far, the nilpotency and the BRS invariance of the action are manifest. We will now consider a reduced field space in which \(B_t\) and \(C\) are eliminated by using the equations of motion (we denote them EOM in what follows):

\[
B_t(t,x) = \alpha_0 \partial_\mu B_\mu(t,x),
\]

(A15)

\[
\Leftrightarrow \frac{\delta}{\delta C^a(t,x)} \left(\dot{S}_{\tilde{h}} + \dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}}\right) = 0,
\]

(A16)

\[\delta \tilde{d}(t,x) = C(t,x)|_{EOM}
= D_\mu L_\mu(t,x) - \left\{d(t,x), \bar{d}(t,x)\right\} + \lambda(t,x) T^a \chi(t,x) T^a - \bar{\chi}(t,x) T^a \lambda(t,x) T^a.
\]

(A17)

Interestingly, even after this elimination of \(B_t\) and \(C\), the BRS transformation remains nilpotent, \(\delta^2 = 0\). The only non-trivial relation for this is

\[
\delta^2 \tilde{d}(t,x) = \delta \left[D_\mu L_\mu(t,x) - \left\{d(t,x), \bar{d}(t,x)\right\} + \lambda(t,x) T^a \chi(t,x) T^a - \bar{\chi}(t,x) T^a \lambda(t,x) T^a\right]
= 0.
\]

(A18)

This relation may directly be confirmed (cf. Ref [4]). Instead, one can see this by noting the commutation relation,

\[
\left[\delta, \frac{\delta}{\delta B_t^a(t,x)}\right] = -f^{abc} d^b(t,x) \frac{\delta}{\delta B_t^c(t,x)},
\]

(A19)

which immediately follows from the definition of \(\delta\). By applying this to \(\dot{S}_{\tilde{h}} + \dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}}\) and using \(\delta \left(\dot{S}_{\tilde{h}} + \dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}}\right) = 0\), we have

\[
\frac{\delta}{\delta B_t^a(t,x)} \left(\dot{S}_{\tilde{h}} + \dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}}\right) = -f^{abc} d^b(t,x) \frac{\delta}{\delta B_t^c(t,x)} \left(\dot{S}_{\tilde{h}} + \dot{S}_{GF} + \dot{S}_{\tilde{d}\tilde{d}}\right).
\]

(A20)

Since the right-hand side vanishes under the EOM,

\[
\left\{\delta \left[D_\mu L_\mu^a(t,x) - \left\{d(t,x), \bar{d}(t,x)\right\}\right] + \lambda(t,x) T^a \chi(t,x) - \bar{\chi}(t,x) T^a \lambda(t,x) \right\}|_{EOM} = 0,
\]

(A21)

where we have used \(\delta C^a(t,x) = 0\). Finally, since \(\left(\delta \tilde{d}\right)|_{EOM} = \delta \tilde{d}\), we have

\[
\delta \left[D_\mu L_\mu^a(t,x) - \left\{d(t,x), \bar{d}(t,x)\right\}\right] + \lambda(t,x) T^a \chi(t,x) - \bar{\chi}(t,x) T^a \lambda(t,x) = 0.
\]

(A22)

This is the relation (A18).
If we integrate over \( C(t,x) \), \( B_t(t,x) \) is eliminated from the system and we have the action in the reduced field space \( S_{\bar{n}} + S_{\bar{d}d} \) in Eqs. (3.2) and (3.3). We can see that the reduced action \( S_{\bar{n}} + S_{\bar{d}d} \) is invariant under the reduced BRS transformation \( \delta \). The argument proceeds as follows:

\[
\delta (S_{\bar{n}} + S_{\bar{d}d}) = \delta \left[ \left( S_{\bar{n}} + \hat{S}_{GF} + \hat{S}_{\bar{d}d} \right) \right]_{\text{EOM}} = \left\{ \int_0^\infty dt \int d^Dx \alpha_0 \partial_\mu D_\mu d^a(t,x) \frac{\delta}{\delta B_t^a(t,x)} \right. \\
\left. + \delta - \int_0^\infty dt \int d^Dx D_t d^a(t,x) \frac{\delta}{\delta B_t^a(t,x)} \right\} \left( S_{\bar{n}} + \hat{S}_{GF} + \hat{S}_{\bar{d}d} \right) \right|_{\text{EOM}} = \left[ \delta \left( S_{\bar{n}} + \hat{S}_{GF} + \hat{S}_{\bar{d}d} \right) \right]_{\text{EOM}} = 0. \quad (A23)
\]

In the second equality, we have noted

\[
\delta [B_t(t,x)]_{\text{EOM}} - \left[ \hat{\delta}_t B_t(t,x) \right]_{\text{EOM}} = \alpha_0 \partial_\mu D_\mu d(t,x) - D_t d(t,x) |_{\text{EOM}}, \quad (A24)
\]

\[
\delta \left( \text{other fields} \right)_{\text{EOM}} - \left[ \hat{\delta} \left( \text{other fields} \right) \right]_{\text{EOM}} = 0, \quad (A25)
\]

and, in the third equality, we have used the fact that \( \frac{\delta}{\delta B_t^a(t,x)} (\hat{S}_{\bar{n}} + \hat{S}_{GF} + \hat{S}_{\bar{d}d}) = 0 \) under the EOM.

The bottom line of the above argument is

\[
\delta^2 = 0, \quad \delta (S_{\bar{n}} + S_{\bar{d}d}) = 0, \quad (A26)
\]

without using any equation of motion in the reduced field space; this is the property that we employed in the main text.

**B. The WT relation with the discretized flow time**

Throughout this appendix, the symbols \( \partial_t \) and \( \int dt \) always stand for the difference operator (3.22) and the sum (3.21), respectively, and do not refer to their continuum counterparts.

When the flow-time derivative is replaced by the difference operator (3.22), the Leibniz rule does not hold and the BRS transformations on quantities containing \( \partial_t \) in Eqs. (3.13)–(3.16) are modified by \( O(\epsilon) \) terms as \(^{12}\)

\[
\delta E_\mu(t,x) = [E_\mu(t,x), d(t,x)] + D_\mu e(t,x) + \epsilon [\partial_t B_\mu(t,x), \partial_t d(t,x)], \quad (B1)
\]

\[
\delta e(t,x) = - \{ e(t,x), d(t,x) \} - \epsilon \partial_t d(t,x) \partial_t d(t,x), \quad (B2)
\]

\[
\delta f(t,x) = - f(t,x) d(t,x) - e(t,x) \chi(t,x) - \epsilon \partial_t d(t,x) \partial_t \chi(t,x), \quad (B3)
\]

\[
\delta \bar{f}(t,x) = - \bar{f}(t,x) d(t,x) - \bar{\chi}(t,x) e(t,x) - \epsilon \partial_t \bar{\chi}(t,x) \partial_t d(t,x). \quad (B4)
\]

\(^{12}\)The BRS transformations on elementary fields, Eqs. (3.6)–(3.9), do not contain the flow-time derivative and they remain nilpotent \( \delta^2 = 0 \) even if the flow time is discretized. As a consequence, the BRS invariance of the source term \( S_K \) (4.3) is preserved.
As a consequence, the BRS invariance of the action, Eq. (3.11), and the resultant WT relation (4.5) are broken by $O(\epsilon)$ terms. To be definite, we have

$$\delta (S_{\text{fl}} + S_{\text{di}}) = -2 \int_0^\infty dt \int d^Dx \text{tr} [L_\mu(t,x)\epsilon [\partial_t B_\mu(t,x), \partial_t d(t,x)]]$$

$$+ \int_0^\infty dt \int d^Dx \left[ \lambda(t,x)\epsilon \partial_t \chi(t,x) - \epsilon \partial_t \bar{\chi}(t,x) \partial_t d(t,x) \lambda(t,x) \right]$$

$$- 2 \int_0^\infty dt \int d^Dx \text{tr} \left[ \bar{d}(t,x)\epsilon \partial_t d(t,x) \partial_t d(t,x) \right]. \quad (B5)$$

The WT relation (4.5) is then modified by the expectation value of this combination. In what follows, we show that the above $O(\epsilon)$ terms do not contribute in the $\epsilon \to 0$ limit and the WT relation assumes the present form (4.5); this point requires careful examination because some $O(1/\epsilon)$ quantities are hidden in our Feynman rules.

First of all, Eq. (B5) is an $O(\epsilon)$ quantity and thus its contribution basically vanishes as $\epsilon \to 0$ unless some element of $O(1/\epsilon)$ emerges. The unique place in our Feynman rules where such an $O(1/\epsilon)$ quantity can appear is the $t = s$ case of

$$\left\langle \partial_t B^a_\mu(t,x) L^b_\nu(s,y) \right\rangle_0 = \delta^{ab} \delta(t-s) K^\epsilon_{\epsilon-s}(x-y)_{\mu\nu} + \delta^{ab} \vartheta(t-s+\epsilon) \partial_t K^\epsilon_{\epsilon-s}(x-y)_{\mu\nu}, \quad (B6)$$

$$\left\langle \partial_t \chi(t,x) \bar{\lambda}(s,y) \right\rangle_0 = \delta(t-s) K^\epsilon_{\epsilon-s}(x-y;1) + \vartheta(t-s+\epsilon) \partial_t K^\epsilon_{\epsilon-s}(x-y;1), \quad (B7)$$

$$\left\langle \lambda(t,x) \partial_s \bar{\chi}(s,y) \right\rangle_0 = \delta(s-t) K^\epsilon_{\epsilon-t}(x-y;1) + \vartheta(s-t+\epsilon) \partial_s K^\epsilon_{\epsilon-t}(x-y;1), \quad (B8)$$

$$\left\langle \partial_t d^a(t,x) \bar{d}^b(s,y) \right\rangle_0 = \delta^{ab} \delta(t-s) K^\epsilon_{\epsilon-s}(x-y;a_0) + \delta^{ab} \vartheta(t-s+\epsilon) \partial_t K^\epsilon_{\epsilon-s}(x-y;a_0), \quad (B9)$$

where the “delta function” has been introduced by

$$\delta(t) \equiv \partial_t \vartheta(t) = \begin{cases} 0, & \text{for } t > 0, \\ \frac{1}{\epsilon}, & \text{for } t = 0, \\ 0, & \text{for } t < 0. \end{cases} \quad (B10)$$

Note that $\delta(0) = 1/\epsilon$ and this factor can potentially cancel the factor $\epsilon$.

Let us consider the expectation value of the right-hand side of Eq. (B5) under the functional integral (4.4). For example, for the first line of Eq. (B5), we consider a flow line starting from the factor $\partial_t B_\mu(t,x)$. The two-point function (B6) is attached to this flow line. If this flow line ends at a certain interaction vertex contained in the action $S_{\text{tot}} + S_J + S_K$, since those interaction vertices do not contain $\partial_s$, the factor $\delta(0) = 1/\epsilon$ does not arise because of the flow-time integration at the vertex, $\int ds \delta(t-s) = 1$; the contribution remains $O(\epsilon)$. The cancellation $\epsilon \cdot 1/\epsilon = 1$ can occur only when the flow line goes back to Eq. (B5) itself, i.e., only when Eq. (B5) is self-contracted by Eq. (B6). A similar argument applies to other lines of Eq. (B5). However, the self-contractions are proportional to the factor $f^{abc} \delta^{ab}$ or $\text{tr} T^a$.

\textsuperscript{13} In fact, in the proof of the renormalizability of the gradient flow in the 2D $O(N)$ non-linear sigma model \cite{62}, similar seemingly $O(\epsilon)$ terms survive even in the $\epsilon \to 0$ limit and play an important role.
which identically vanishes.  This shows that the WT relation restores the desired form safely in the $\epsilon \to 0$ limit.

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\[14\] With dimensional regularization, the self-contractions vanish also because $\lim_{\epsilon \to 0} K^\epsilon_{\epsilon}(0) = 0$. 

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