New Hadamard-type inequality for new class of geodesic convex functions

Wedad Saleh

Department of Mathematics, Taibah University, Al-Medina 20012, Saudi Arabia.

Abstract

This paper aims to introduce the concept of \((E,\mu,\kappa)\)-convex function by using special inequality. Hadamard integral inequality for this new class of geodesic convex function in the case of Lebesgue and Sugeno integrals is given.

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1. Introduction

It is commonly known that convexity is used in modern analysis, either directly or indirectly [18]. The idea of convexity has been developed and generalized in numerous directions due to its uses and significance, see [7, 8, 15, 16]. E-convexity of sets and functions, which is a broader function than invexity, was introduced in 1999 [28]. However, Young [27] claims that some of the results in [28] are inaccurate. In [4], the E-convexity was expanded to a semi-E-convexity. See [5, 6, 23] for further information on the E-convex or semi-E-convex functions. Furthermore, Youness and Emam in [29] discuss a novel class of functions called as strongly E-convex functions. In particular, semi-strong E-convexity as well as quasi and pseudo semi-strong E-convexity was added to this class of functions [30].

A manifold is not a linear space, and extensions of concepts and techniques from linear spaces to Riemannian manifolds are natural. Many authors, including Udrist [24] and Rapcsak [20], have studied generalized convex functions in Riemannian manifolds Geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds are investigated in 2012 [10]. Moreover, geodesic semi E-convex functions are introduced in [9]. Recently, geodesic strongly E-convex functions have been introduced, and some of their properties [11].

Based on these ideas, a new class of functions, which are called geodesic semi strongly E-convex functions, are defined and some of their properties are presented in [12]. A class of functions on Riemannian manifolds, which are called geodesic semilocal E-preinvex functions, as a generalization of geodesic semilocal E-convex and geodesic semi E-preinvex functions, are given in [13]. In [21], geodesic E-b-vex
sets and geodesic E-b-vex functions on a Riemannian manifold are extended to geodesic strongly E-b-vex sets and geodesic strongly E-b-vex functions.

Sugeno integrals are a type of nonlinear integral invented by Sugeno [22] to capture and integrate interactions between criteria of various phenomena. The most well-known integral inequalities for Sugeno integral have been proven, see [1, 25].

2. Preliminaries

In this section, we present some definitions and properties that can be found in many books on differential geometry, such as [24].

Suppose that \( X \) is a \( C^\infty \) \( n \)-dimensional Riemannian manifold, and \( T_tX \) is the tangent space to \( X \) at \( t \). Also, assume that \( \mu_t(y_1, y_2) \) is a positive inner product on the tangent space \( T_tX (y_1, y_2 \in T_tX) \), which is given for each point of \( X \). Then, a \( C^\infty \) map \( \mu: t \mapsto \mu_t \), which assigns a positive inner product \( \mu_t \) to \( T_tX \), \( \forall t \in X \) is called a Riemannian metric.

The length of a piecewise \( C^1 \) curve \( \eta: [a_1, a_2] \rightarrow X \) which is defined as follows:

\[
L(\eta) = \int_{a_1}^{a_2} \|\dot{\eta}(t)\| dt.
\]

We define \( d(t_1, t_2) = \inf\{L(\eta): \eta \) is a piecewise \( C^1 \) curve joining \( t_1 \) to \( t_2 \} \) for any points \( t_1, t_2 \in X \). Furthermore, a smooth path \( \eta \) is a geodesic if and only if its tangent vector is a parallel vector field along the path \( \eta \), i.e., \( \eta \) satisfies the equation \( \nabla_{\dot{\eta}(t)} \dot{\eta}(t) = 0 \). Every path \( \eta \) is joining \( t_1, t_2 \in X \), where \( L(\eta) = d(t_1, t_2) \) is a minimal geodesic.

Finally, assume that \( (X, \mu) \) is a complete \( n \)-dimensional Riemannian manifold with Riemannian connection \( \nabla \). Let \( y_1, y_2 \in X \) and \( \eta: [0, 1] \rightarrow X \) be a geodesic joining the points \( y_1 \) and \( y_2 \), which means that \( \eta(y_1, y_2)(0) = x_2 \) and \( \eta(y_1, y_2)(1) = y_1 \).

A set \( A \) in a Riemannian manifold \( X \) is called t-convex if \( A \) contains every geodesic \( \eta(y_1, y_2) \) of \( N \) whose endpoints \( y_1 \) and \( y_2 \) belong to \( A \).

Note that the whole of the manifold \( X \) is t-convex, and conventionally, so is the empty set. The minimal circle in a hyperboloid is t-convex, but a single point is not. Also, any proper subset of a sphere is not necessarily t-convex.

The following theorem was proved in [24].

**Theorem 2.1** ([24]). The intersection of any number of t-convex sets is t-convex.

**Remark 2.2.** In general, the union of a t-convex set is not necessarily t-convex.

**Definition 2.3** ([24]). A function \( g: A \rightarrow \mathbb{R} \) is called g-convex function on a t-convex set \( A \subset X \) if for every geodesic \( \eta(y_1, y_2) \), then

\[
g(\eta(y_1, y_2)(\gamma)) \leq \gamma g(y_1) + (1 - \gamma)g(y_2)
\]

holds \( \forall y_1, y_2 \in A \) and \( \gamma \in [0, 1] \).

Now let \( M \) be a non-empty set and \( \xi \) be a \( \sigma \)-algebra of subsets of \( M \).

**Definition 2.4** ([19]). Let \( N: \xi \rightarrow [0, \infty) \) be a set function, then \( N \) is called a Sugeno measure if it satisfies

1. \( N(\emptyset) = 0 \);
2. if \( A, B \in \xi \) and \( A \subset B \), then \( \xi(A) \leq \xi(B) \);
3. \( A_i \in N \), where \( i \in \mathbb{N}, A_{i-1} \subset A_i \), then \( \lim_{i \rightarrow \infty} \xi(A_i) = \xi(\bigcup_{i=1}^{\infty} E_i) \);
4. \( A_n \in \xi \), where \( i \in \mathbb{N}, A_{i-1} \subset A_i \), then \( \lim_{i \rightarrow \infty} \xi(A_i) = \xi(\bigcap_{i=1}^{\infty} E_i) \).

Assume that \( (M, \xi, N) \), which is said to be a sugeno measure space, is a fuzzy measure space. By \( H_\xi(M) \), then

\[
\chi_\xi(M) = \{h: M, \xi, \rightarrow [0, \infty) : h \text{ is measurable with respect to } \xi\}.
\]
Definition 2.5 ([17, 22]). Assuming \((M, \mathcal{E}, N)\) is a fuzzy measure space, \(h \in \chi_{\mathcal{E}}(M)\) and \(X \in \mathcal{E}\), then the Sugeno integral of \(h\) on \(A\) w.r.t. the \(N\) is defined by

\[
\int_X h dN = \bigvee_{\alpha \geq 0} (\alpha \wedge N(X \cap H_\alpha)),
\]

where \(H_\alpha = u \in M : h(u) \geq \alpha\), \(\wedge\) is the prototypical t-normal minimum and \(\bigvee\) the prototypical t-conorm maximum. If \(X = M\), then

\[
\int_X h dN = \bigvee_{\alpha \geq 0} (\alpha \wedge N(H_\alpha)).
\]

Some properties of the Sugeno integral can be found in [17, 26] such as following.

Theorem 2.6. Assume that \((M, \mathcal{E}, N)\) is a fuzzy measure space, \(X, Y \in \mathcal{E}\), and \(h_1, h_2 \in \chi_N(M)\), then

1. \(\int_X h_1 dN \leq N(X)\);
2. \(\int_X adN = a \wedge N(X)\), where \(a\) is non-negative constant;
3. If \(h_1 \leq h_2\) on \(X\), then \(\int_X h_1 dN \leq \int_X h_2 dN\);
4. If \(X \subset Y\), then \(\int_X h_1 dN \leq \int_Y h_1 dN\).

3. The main results

In this part of the paper, let us take \((M, \mathcal{E})\) be a fuzzy measure space for a given \(h \in H^N(M)\) and \(X \in \mathcal{E}\), then

\[\Gamma = \{ \alpha : \alpha \geq 0, N(X \cap h_\alpha) > N(X \cap h_\beta) \text{ for any } \beta > \alpha \}\].

Moreover, \(\int_X h dN = \bigvee_{\alpha \in \Gamma} (\alpha \wedge N(X \cap h_\alpha)) \) [2].

In the next definition, the concept of \((E, \mu, \kappa)\)-convexity is given.

Definition 3.1. Considering \(Y_1\) and \(Y_2\) are two \(E\)-convex sets, where \(E : \mathbb{R}^+ \rightarrow \mathbb{R}^+\). Assume that \(\mu_{E(u_1),E(u_2)} : [0,1] \rightarrow Y_1\) is a geodesic arc joining the points \(u_1, u_2 \in Y_1\) and \(\kappa_{E(v_1),E(v_2)} : [0,1] \rightarrow Y_2\) is a geodesic arc joining the points \(v_1, v_2 \in Y_2\). A real valued function \(h : Y_1 \rightarrow Y_2\) is called a \((E, \mu, \kappa)\)-convex if

\[h(\mu_{E(u_1),E(u_2)}(\lambda)) \leq \kappa_{h(E(v_1)),h(E(v_2))}(\lambda), \forall u_1, u_2 \in Y_1, \lambda \in [0,1].\]

Remark 3.2.

1. For a \((E, \mu, \kappa)\)-convex function \(h : [x_1, x_2] \rightarrow [y_1, y_2]\), then

\[h(u) = h \left( \mu_{E(x_1),E(x_2)} \left( \mu_{E(x_1),E(x_2)}^{-1}(u) \right) \right) \leq \kappa_{h(E(x_1)),h(E(x_2))} \left( \mu_{E(x_1),E(x_2)}^{-1}(u) \right) \] (3.1)

for all \(u \in [x_1, x_2]\), then the inequality is sharp for all \(u \in [x_1, x_2]\).
2. If \(E = I\), where \(I\) is the indentity mapping, then the inequality (3.1) becomes the inequality (2) in [2].

Next, some generalizations of Hadamard in inequality for different geodesic convex functions are given.

Theorem 3.3. Assume that \(Y_1\) and \(Y_2\) are two \(E\)-convex subsets of \(\mathbb{R}\), \(x_1, x_2 \in Y_1^0\) with \(x_1 < x_2\) and \(y_1, y_2 \in Y_2^0\) with \(y_1 < y_2\). For the particular geodesic arcs \(\mu : [0,1] \rightarrow Y_1\) and \(\kappa : [0,1] \rightarrow Y_2\) defined by \(\mu_{E(u_1),E(u_2)}(\lambda) = (1-\lambda)E(u_1) + \lambda E(u_2)\) and \(\kappa_{v_1,v_2}(\lambda) = E(v_1)^{1-\lambda}E(v_2)^\lambda\), then the next inequalities hold.

1. If \(h : Y_1 \rightarrow Y_2\) is a \((E, \mu, \mu)\)- convex function, then

\[
\frac{1}{x_2 - E(x_1)} \int_{E(x_1)}^{E(x_2)} h(E(u)) dE(u) \leq \frac{h(E(x_1)) + h(E(x_2))}{2}, \forall u \in [x_1, x_2].
\]
2. If \( h: Y_1 \rightarrow Y_2 \subseteq (0, \infty) \) is a \((E, \mu, \kappa)\)-convex function with \( h(E(x_1)) \neq h(E(x_2)) \), then
\[
\frac{1}{E(x_2) - E(x_1)} \int_{E(x_1)}^{E(x_2)} h(E(u))dE(u) \leq \frac{h(E(x_1))}{\ln \left( \frac{h(E(x_2))}{h(E(x_1))} \right)} \left( \frac{h(E(x_2))}{h(E(x_1))} - 1 \right).
\]

3. If \( h: Y_2 \subseteq (0, \infty) \rightarrow Y_1 \) is a \((E, \kappa, \mu)\)-convex function, then
\[
\frac{1}{E(y_1) - E(y_2)} \int_{E(y_1)}^{E(y_2)} h(E(u))dE(u) \leq h(E(y_1)) + \frac{h(E(y_2)) - h(E(y_1))}{\ln \left( \frac{h(E(y_2))}{h(E(y_1))} \right)} \left( \frac{E(y_2)}{E(y_1)} - 1 \right).
\]

4. If \( h: Y_2 : (0, \infty) \rightarrow Y_2 : (0, \infty) \) is a \((E, \kappa, \kappa)\)-convex function with \( h(E(y_1)) \neq h(E(y_2)) \), then
\[
\frac{1}{E(y_1) - E(y_2)} \int_{E(y_1)}^{E(y_2)} h(E(u))dE(u) \leq \frac{E(y_1)h(E(y_1))}{E(y_2) - E(y_1)} \left( \frac{E(y_2)}{E(y_1)} \frac{\log \left( \frac{h(E(y_2))}{h(E(y_1))} + 1 \right)}{\ln \left( \frac{h(E(y_2))}{h(E(y_1))} \right)} - 1 \right).
\]

Proof. The first inequality is the well-known Hadamard’s inequality for \( E \)-convex functions. If we use the inequality \((3.1)\), then we have the following inequalities.

1. The function \( h: [x_1, x_2] \rightarrow [x_1, x_2] \) is \((E, \mu, \mu)\)-convex iff
\[
h(u) \leq h(E(x_1)) + \frac{u - E(x_1)}{E(x_2) - E(x_1)}(h(E(x_2)) - h(E(x_1))), \quad \forall u \in [x_1, x_2].
\]

2. The function \( h: [x_1, x_2] \rightarrow [y_1, y_2] \) is \((E, \mu, \kappa)\)-convex iff
\[
h(u) \leq h(E(x_1)) \left( \frac{h(E(x_2))}{h(E(x_1))} \right)^{\frac{u - E(x_1)}{E(x_2) - E(x_1)}}, \quad \forall u \in [x_1, x_2].
\]

3. The function \( h: [y_1, y_2] \rightarrow [x_1, x_2] \) is \((E, \kappa, \mu)\)-convex iff
\[
h(u) \leq h(E(y_1)) + \log \left( \frac{E(y_2)}{E(y_1)} \right) \frac{u}{h(E(y_1))}(h(E(y_2)) - h(E(y_1))), \quad \forall u \in [y_1, y_2].
\]

4. The function \( h: [x_1, x_2] \rightarrow [y_1, y_2] \) is \((E, \mu, \kappa)\)-convex iff
\[
h(u) \leq h(E(y_1)) \left( \frac{h(E(y_2))}{h(E(y_1))} \right)^{\log \left( \frac{E(y_2)}{E(y_1)} \right) \frac{u}{h(E(y_1))}}, \quad \forall u \in [y_1, y_2].
\]

If we integrate the inequalities \((3.2), (3.3), (3.4), \) and \((3.5)\) from both sides over \([x_1, x_2]\) or \([y_1, y_2]\), we obtain the results in the theorem. \(\square\)

**Theorem 3.4.** Let \( (\mathbb{R}, E, N) \) be the fuzzy measurement space. Assume that \( \mu : [0, 1] \rightarrow [x_1, x_2] \) and \( \kappa : [0, 1] \rightarrow [y_1, y_2] \) are two invertible geodesic arcs. If \( h: [x_1, x_2] \rightarrow [y_1, y_2] \) is a \((E, \mu, \kappa)\)-convex function, then
\[
\int_{x_1}^{x_2} h dN \leq \begin{cases} 
\bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))]} \left( \alpha \land N \left( \left[ \mu(E(x_1), E(x_2))(\kappa^{-1}(h(E(x_1)), h(E(x_2))(\alpha))) \right], E(x_2) \right) \right), & \text{if } \mu, \kappa \text{ are comonotone,} \\
\bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))]} \left( \alpha \land N \left( \left[ E(x_1), \mu(E(x_1), E(x_2))(\kappa^{-1}(h(E(x_1)), h(E(x_2))(\alpha))) \right] \right), & \text{if } \mu, \kappa \text{ are countermonotone.}
\end{cases}
\]
Proof. Since $h$ is a $(E, \mu, \kappa)$-convex function and by using the property (3 in Theorem 2.6) of fuzzy measure, we get
\[ \int_{x_1}^{x_2} h dN = \int_{x_1}^{x_2} h(\mu_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(u))) dN \]
\[ \leq \int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(u)) dN. \] \hfill (3.6)

If $\mu$ and $\kappa$ are comonotone, then $\kappa \circ \mu^{-1}$ is an increasing function, then by Definition 2.5
\[ \int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(u)) dN \]
\[ = \bigvee_{\alpha \geq 0} \left( \alpha \wedge N([E(x_1), E(x_2)] \cap \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha))) \right) \]
\[ = \bigvee_{\alpha \geq 0} \left( \alpha \wedge N(\mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha))) \right) \]
\[ = \bigvee_{\alpha \geq 0} \left( \alpha \wedge N \left( \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)), E(x_2) \right) \right) \]. \hfill (3.7)

Since $\kappa \circ \mu^{-1}$ is increasing, we get
\[ E(x_1) \leq \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)) < E(x_2) \]
\[ \Rightarrow \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(E(x_1))) \leq \alpha < \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(E(x_2))) \]
\[ \Rightarrow \kappa_{E(x_1),E(x_2)}(0) \leq \alpha < \kappa_{E(x_1),E(x_2)}(1) \]
\[ \Rightarrow h(E(x_1)) \leq \alpha < h(E(x_2)). \] \hfill (3.8)

Thus, $\Gamma = [h(E(x_1)), h(E(x_2))]$ and we only need to consider $\alpha \in [h(E(x_1)), h(E(x_2))]$. It follows from (3.6), (3.7), and (3.8), that
\[ \int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(u)) dN \]
\[ \leq \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))]} \left( \alpha \wedge N \left( \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)), E(x_2) \right) \right) \).

If $\mu$ and $\kappa$ are countermonotone, then $\kappa \circ \mu^{-1}$ is a decreasing function. Then, by Definition 2.5, we get
\[ \int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(u)) dN \]
\[ = \bigvee_{\alpha \geq 0} \left( \alpha \wedge N([E(x_1), E(x_2)] \cap \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha))) \right) \]
\[ = \bigvee_{\alpha \geq 0} \left( \alpha \wedge N(u \leq \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha))) \right) \]
\[ = \bigvee_{\alpha \geq 0} \left( \alpha \wedge N \left( \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)), E(x_2) \right) \right) \). \hfill (3.9)

Since $\kappa \circ \mu^{-1}$ is decreasing, we get
\[ E(x_1) \leq \mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)) < E(x_2) \]
Corollary 3.6. \[ \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(E(x_1))) \leq \alpha < \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(E(x_2))) \] (3.10)\\
\[ \Rightarrow \kappa_{E(x_1),E(x_2)}(0) \leq \alpha < \kappa_{E(x_1),E(x_2)}(1) \]
\[ \Rightarrow h(E(x_1)) \leq \alpha < h(E(x_2)). \]

Thus, \( \Gamma = [h(E(x_2)), h(E(x_1))] \) and we only need to consider \( \alpha \in [h(E(x_2)), h(E(x_1))] \). It follows from (3.6), (3.9), and (3.10) that
\[
\int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(u))dN \leq \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))] } \left( \alpha \wedge N \left( \left[ \mu_{E(x_1),E(x_2)}(E(x_2)), \kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha) \right] \right) \right).
\]

\[ \square \]

Remark 3.5. Consider \( h : [x_1, x_2] \rightarrow [y_1, y_2] \) is a \((E, \mu, \kappa)\)-convex function, \( \xi \) is the Borel field, and \( N \) is the Lebesgue measure on \( \mathbb{R} \). Then
\[
\int_{x_1}^{x_2} hdN \leq \begin{cases} \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))] } \left( \alpha \wedge \left[ E(x_2) - \mu_{E(x_1),E(x_2)}(h(E(x_1)),h(E(x_2)))(\alpha) \right] \right), & \text{if } \mu, \kappa \text{ are comonotone,} \\ \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))] } \left( \alpha \wedge \left[ \mu_{E(x_1),E(x_2)}(h(E(x_1)),h(E(x_2)))(\alpha) - E(x_1) \right] \right), & \text{if } \mu, \kappa \text{ are countermonotone.} \end{cases}
\]

In the following corollaries, consider that \(((\mathbb{R}, \mathbb{R}, \xi, N))\) is the fuzzy measure space.

Corollary 3.6. Let \( h : [x_1, x_2] \rightarrow [x_1, x_2] \) be a \((E, \mu, \mu)\)-convex function, hence
\[
\int_{x_1}^{x_2} hdN \leq \begin{cases} \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))] } \left( \alpha \wedge \left[ \left[ E(x_1) + (E(x_2) - E(x_1)) \frac{\alpha - h(E(x_1))}{h(E(x_2)) - h(E(x_1))} \right] \right] \right), & \text{if } h(E(x_1)) < h(E(x_2)), \\ h(E(x_1)) \wedge N(E(x_1), E(x_2)), & \text{if } h(E(x_1)) = h(E(x_2)), \\ \bigvee_{\alpha \in [h(E(x_2)), h(E(x_1))] } \left( \alpha \wedge \left[ \left[ E(x_1), E(x_1) + (E(x_2) - E(x_1)) \frac{\alpha - h(E(x_1))}{h(E(x_2)) - h(E(x_1))} \right] \right] \right), & \text{if } h(E(x_1)) > h(E(x_2)). \end{cases}
\]

If we take
\[
\mu_{E(x_1),E(x_2)}(\mu_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)) = E(x_1) + (E(x_2) - E(x_1)) \frac{\alpha - h(E(x_1))}{h(E(x_2)) - h(E(x_1))}
\]
in Theorem 2.6, then the Corollary 3.6 can be proved.

Corollary 3.7. Let \( h : [x_1, x_2] \rightarrow [y_1, y_2] \) be a \((E, \mu, \kappa)\)-convex function. Then
\[
\int_{x_1}^{x_2} hdN \leq \begin{cases} \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))] } \left( \alpha \wedge \left[ \left[ E(x_1) + (E(x_2) - E(x_1)) \log_{h(E(x_2))} \frac{\alpha}{h(E(x_1))} \right] \right] \right), & \text{if } h(E(x_1)) < h(E(x_2)), \\ h(E(x_1)) \wedge N(E(x_1), E(x_2)), & \text{if } h(E(x_1)) = h(E(x_2)), \\ \bigvee_{\alpha \in [h(E(x_2)), h(E(x_1))] } \left( \alpha \wedge \left[ \left[ E(x_1), E(x_1) + (E(x_2) - E(x_1)) \log_{h(E(x_2))} \frac{\alpha}{h(E(x_1))} \right] \right] \right), & \text{if } h(E(x_1)) > h(E(x_2)). \end{cases}
\]
If we take
$$\mu_{E(x_1),E(x_2)}(\kappa_{h(E(x_1)),h(E(x_2))}^{-1}(\alpha)) = E(x_1) + (E(x_2) - E(x_1)) \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha}{h(E(x_1))}$$
in Theorem 2.6, then the Corollary 3.7 can be proved.

**Corollary 3.8.** Let \( h : [y_1, y_2] \rightarrow [x_1, x_2] \) be an \((E, \kappa, \mu)\)-convex function. Then
$$\int_{x_1}^{x_2} h dN \leq \begin{cases} \forall \alpha \in [h(E(x_1)), h(E(x_2))] \left( \alpha \land \mathcal{N} \left( \left[ E(x_1) + \left( \frac{E(x_2)}{E(x_1)} \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha - h(E(x_1))}{h(E(x_1))} \right), E(x_2) \right] \right) \right), \\
\text{if } h(E(x_1)) < h(E(x_2)), \\
\forall \alpha \in [h(E(x_2)), h(E(x_1))] \left( \alpha \land \mathcal{N} \left( \left[ E(x_1), E(x_1) \left( \frac{E(x_2)}{E(x_1)} \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha - h(E(x_1))}{h(E(x_1))} \right) \right), E(x_2) \right) \right), \\
\text{if } h(E(x_1)) > h(E(x_2)). \end{cases}$$

If we take
$$\kappa_{E(x_1), E(x_2)}(\mu_{h(E(x_1)), h(E(x_2))}^{-1}(\alpha)) = E(x_1) \left( \frac{E(x_2)}{E(x_1)} \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha}{h(E(x_1))} \right)$$
in Theorem 2.6, then the Corollary 3.8 can be proved.

**Corollary 3.9.** Let \( h : [y_1, y_2] \rightarrow [y_1, y_2] \) be an \((E, \kappa, \mu)\)-convex function. Then
$$\int_{x_1}^{x_2} dN \leq \begin{cases} \forall \alpha \in [h(E(x_1)), h(E(x_2))] \left( \alpha \land \mathcal{N} \left( \left[ E(x_1), E(x_1) \left( \frac{E(x_2)}{E(x_1)} \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha - h(E(x_1))}{h(E(x_1))} \right) \right), E(x_2) \right) \right), \\
\text{if } h(E(x_1)) < h(E(x_2)), \\
\forall \alpha \in [h(E(x_2)), h(E(x_1))] \left( \alpha \land \mathcal{N} \left( \left[ E(x_1), E(x_1) \left( \frac{E(x_2)}{E(x_1)} \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha - h(E(x_1))}{h(E(x_1))} \right) \right), E(x_2) \right) \right), \\
\text{if } h(E(x_1)) > h(E(x_2)). \end{cases}$$

If we take
$$\kappa_{E(x_1), E(x_2)}(\kappa_{h(E(x_1)), h(E(x_2))}^{-1}(\alpha)) = E(x_1) \left( \frac{E(x_2)}{E(x_1)} \log \frac{h(E(x_2))}{h(E(x_1))} \frac{\alpha}{h(E(x_1))} \right)$$
in Theorem 2.6, then Corollary 3.9 can be proved.

**Example 3.10.**
1. If \( E : \mathbb{R} \rightarrow \mathbb{R} \) is \( E(x) = x^2 \), then the function \( h : [1, 3] \rightarrow [0, +\infty) \), which is defined as \( h(x) = \frac{1}{\ln^2(x+1)} \), is \((E, \mu, \mu)\)-convex function.
2. If \( E : \mathbb{R} \rightarrow \mathbb{R} \) is \( E(x) = x^x \), then the function \( h : [1, 2] \rightarrow [0, +\infty) \), which is defined as \( h(x) = x \), is \((E, \kappa, \mu)\)-convex function.
3. If \( E : \mathbb{R} \rightarrow \mathbb{R} \) is \( E(x) = x \), then the function \( h : [\frac{\pi}{4}, \frac{\pi}{2}] \), which is defined as \( h(x) = x^2 \sin^2 x \), is \((E, \kappa, \mu)\)-convex function.
4. If \( E : \mathbb{R} \rightarrow \mathbb{R} \) is \( E(x) = 2x \), then the function \( h : [1, 2] \rightarrow [0, +\infty) \), which is defined as \( h(x) = (\cosh(x))^\frac{1}{2} \), is \((E, \kappa, \mu)\)-convex function.

**4. Conclusion**

This paper introduces a new class of geodesic convex functions, namely \((E, \mu, \kappa)\)-convex function, and considers and generalizes the Hadamard inequality for \((E, \mu, \kappa)\)-convex function.
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