World-Line Path Integral for the Propagator Expressed as an Ordinary Integral: Concept and Applications

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Abstract
The (Feynman) propagator $G(x_2, x_1)$ encodes the entire dynamics of a massive, free scalar field propagating in an arbitrary curved spacetime. The usual procedures for computing the propagator—either as a time ordered correlator or from a partition function defined through a path integral—requires introduction of a field $\phi(x)$ and its action functional $A[\phi(x)]$. An alternative, more geometrical, procedure is to define a propagator in terms of the world-line path integral which only uses curves, $x^i(s)$, defined on the manifold. I show how the world-line path integral can be reinterpreted as an ordinary integral by introducing the concept of effective number of quantum paths of a given length. Several manipulations of the world-line path integral becomes algebraically tractable in this approach. In particular I derive an explicit expression for the propagator $G_{QG}(x_2, x_1)$, which incorporates the quantum structure of spacetime through a zero-point-length, in terms of the standard propagator $G_{\text{std}}(x_2, x_1)$, in an arbitrary curved spacetime. This approach also helps to clarify the interplay between the path integral amplitude and the path integral measure in determining the form of the propagator. This is illustrated with several explicit examples.

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1 Relativistic Propagator from a Geometrical Perspective

Consider a free scalar field of mass $m$ which is propagating in a spacetime with metric $g_{ik}$ and is treated within the context of quantum field theory in curved spacetime. I take the (generally accepted) point-of-view that the dynamics of such a field is completely contained in the standard relativistic (Feynman) propagator $G_{\text{std}}(x,y;m^2)$ or, equivalently, in the rescaled\(^1\) propagator $\tilde{G}_{\text{std}}(x,y;m) \equiv m G_{\text{std}}(x,y;m^2)$. So if we have a prescription for computing $\tilde{G}_{\text{std}}(x,y;m)$, we can completely determine the dynamics of the field.

The usual procedure to determine $G_{\text{std}}(x,y;m)$ is to start with a Lagrangian for the scalar field, quantize the field and obtain the propagator from it. This itself can be done in two different ways. (i) One can identify the canonical momentum $\pi$ for $\phi$ and impose equal-time-commutation rules (ETCR) between them. If $\phi$ is expanded in terms a set of mode functions, the ETCR will lead to the identification of creation/annihilation operators. One can then construct a vacuum state, Fock basis etc.\(^2\) The propagator is then identified as the time-ordered vacuum expectation value $\langle 0 | T[\phi(x_2)\phi(x_1)] | 0 \rangle$. (ii) Alternatively one can find the partition function $Z[J]$ by evaluating a path integral over $\phi$ of $\exp iA[\phi,J]$ after adding a source term $J(x)\phi(x)$ to the Lagrangian. The propagator can then be obtained as the second functional derivative of of $Z[J]$ evaluated at $J = 0$.

Both these, standard, procedures use a field $\phi(x)$ and its action functional $A[\phi]$ as tools to arrive at the propagator $G_{\text{std}}$, which, ultimately, encodes the entire dynamics. It is therefore useful to inquire whether we can determine $G_{\text{std}}$ directly (and geometrically) without using the crutch of a field or its action functional. One motivation for this inquiry is the following: A purely geometrical definition of the propagator may be robust enough to survive (and be useful) at scales close to—but somewhat larger than—Planck scales. (I call this regime mesoscopic; I will say more about it later on).

Such a geometric approach is indeed possible because we know the differential equation and boundary conditions which $G_{\text{std}}$ satisfies. The relevant solution can be expressed in terms of the zero-mass-Schwinger-kernel (ZMSK) of the spacetime in the Schwinger’s propertime representation as follows:\(^3\)

\[
\tilde{G}_{\text{std}}(x,y;m) \equiv m G_{\text{std}}(x,y;m^2) = \int_0^\infty m \, ds \, e^{-m^2 s} K_{\text{std}}(x,y;s).
\]  

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\(^1\) As we shall see, there is some algebraic advantage in using $G_{\text{std}}$ rather than $G_{\text{std}}$. Of course, both contain the same amount of information in the case of a massive field, which I will be focusing on; the massless case can be treated by a limiting procedure and I will comment on it when relevant.

\(^2\) In a time dependent background, there is the usual ambiguity of choice of positive/negative frequency mode functions, inequivalent vacuua etc. These are not relevant to the main thrust of the current discussion. Any one choice of mode functions and vacuum state is good enough for my purpose.

\(^3\) I will work in the Euclidean space(time) for mathematical convenience and will assume that the results in the pseudo-Riemannian spacetime arise through analytic continuation. This is particularly useful in the path integral representation discussed below.
where $K_{\text{std}}$ is the standard ZMSK. This can be specified as a solution to a differential equation or as $K_{\text{std}}(x,y;s) \equiv \langle x|e^{\Box_g} |y\rangle$ where $\Box_g$ is the Laplacian in the background space(time). This kernel is a purely geometric object, entirely determined by the background geometry; the information about the scalar field is contained in the single parameter $m$. The $K_{\text{std}}(x,y;s)$ has the structure (in $D = 4$):

$$K_{\text{std}}(x,y;s) \propto e^{-\bar{\sigma}^2(x,y)/4s} \left[1 + \text{curvature corrections}\right] \tag{2}$$

where $\bar{\sigma}^2(x,y)$ is the geodesic distance between the two events. The curvature corrections, encoded in the Schwinger–Dewitt expansion, will involve powers of $(s/L_{\text{curv}}^2)$.

An equivalent, more intuitive but formal, definition of $G_{\text{std}}(x_1,x_2;m)$ is through a world-line path integral for a relativistic particle:

$$G_{\text{std}}(x_1,x_2;m) = \sum_{\text{paths } \sigma} M \left[\exp[-m\sigma(x_1,x_2)] \right] \tag{3}$$

where $\sigma(x_1,x_2)$ is the geometrical length of a path connecting the two points $x_1,x_2$ and the sum is over all paths connecting these two events. This is appealing because it uses the lengths of paths in space(time), which are purely geometrical entities, to give meaning to the propagator. Of course, the result of the path integral sum depends both on the summand $e^{-m\sigma}$ and on the measure chosen for the path integral, indicated by $M$ on top of the summation symbol in Eq. (3). In flat spacetime, there is a straightforward procedure to define $M$: This is to define the sum over paths on a lattice and compute it—with a suitable measure—in the limit of zero lattice spacing [1–3]. The lattice measure can then be chosen so that the sum will lead to the same propagator as in Eq. (1). But, as is evident, defining and manipulating the measure, in a general curved background, is a nontrivial task. I stress that, the definition of the propagator through the path integral in Eq. (3) is valid in arbitrary curved spacetime. In fact it is usually taken to be the definition of the propagator, in the world-line approach, and is often used with a quadratic action and a gauge function to ensure reparametrisation invariance.

I will now describe how this path integral can be converted into an ordinary integral with a suitable integration measure which can be interpreted as the effective number density of paths in space(time). This conceptual advance—which will provide a completely geometrical and useful description of the propagator — is one of the key results of this paper. As we will see, this definition, for example, will allow us to explore how the propagator gets modified when the quantum nature of the spacetime geometry is taken into account.

To convert the path integral to an ordinary integral, I will introduce a Dirac delta function into the path integral sum in Eq. (3) and use the fact that both $\ell'$ and $\sigma$ are positive definite, to obtain:

$$G_{\text{std}}(x_1,x_2;m) = \sum_{\text{paths } \sigma} M \left[\delta(x_1-x_2) \exp[-m\sigma(x_1,x_2)] \right] \tag{3}$$

This works best in Euclidean sector because path lengths $d\sigma = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$ are real. In an earlier work I have tried to do it with Lorentzian signature but it leads to ambiguities.
\[
G_{\text{std}}(x_1, x_2; m) = \int_0^\infty d\ell \ e^{-m\ell} \sum_{\text{paths}} \sigma^{*M} \delta_D(\ell - \sigma(x_2, x_1))
\]

\[
\equiv \int_0^\infty d\ell \ e^{-m\ell} N_{\text{std}}(\ell; x_2, x_1)
\]

where we have defined the function \(N_{\text{std}}(\ell; x_2, x_1)\) to be:

\[
N_{\text{std}}(\ell; x_2, x_1) \equiv \sum_{\text{paths}} \sigma^{*M} \delta_D(\ell - \sigma(x_2, x_1))
\]

The last equality in Eq. (4) describes the path integral as an ordinary integral with an integration measure \(N_{\text{std}}(\ell; x_2, x_1)\). This measure—according to Eq. (5)—can be thought of as counting the effective number of paths of length \(\ell\) connecting \(x_1\) and \(x_2\). This is a purely geometrical quantity defined in the space(time). I will hereafter just write \(N_{\text{std}}(\ell)\) for \(N_{\text{std}}(\ell; x_2, x_1)\) etc., without explicitly displaying the dependence on the spacetime coordinates, for notational simplicity. The propagator \(G_{\text{std}}(m)\) is just the Laplace transform of \(N_{\text{std}}(\ell)\) from the variable \(\ell\) to \(m\); the measure \(N_{\text{std}}(\ell)\) is the inverse Laplace transform of \(G_{\text{std}}(m)\) from the variable \(m\) to \(\ell\).

To conclude this section, let me illustrate the explicit form of \(N_{\text{free}}(\ell)\) in the case of a free field in flat space.\(^6\) Translation invariance demands that both \(G_{\text{free}}(x_1, x_2; m)\) and \(N_{\text{free}}(\ell; x_2, x_1)\) will only depend on the difference \(x \equiv (x_2 - x_1)\). Fourier transforming the last equality in Eq. (4) with respect to \(x\) we obtain a similar relation between \(G_{\text{free}}(p; m)\) and \(N_{\text{free}}(p, \ell)\) in the momentum space. The \(N_{\text{free}}(p, \ell)\) in the momentum space (in any dimension \(D\)) is given by the very simple expression:\(^7\)

\[
N_{\text{free}}(p, \ell) = \cos(p\ell).
\]

Direct computation of this result will require lattice regularization of the sum in Eq. (5). However, since the sum in Eq. (3) can indeed be computed by lattice regularization\(^1\),\(^2\),\(^3\), we can also compute \(N_{\text{free}}(\ell)\) by an additional Laplace transform to arrive at Eq. (6). The result can be verified by taking the inverse Laplace transform of \(G_{\text{free}}(p, m) = m(p^2 + m^2)^{-1}\) or, more directly, by observing that:

\[
\int_0^\infty d\ell \ e^{-m\ell} \cos p\ell = \frac{m}{m^2 + p^2} = mG_{\text{free}}(p^2, m^2) = G_{\text{free}}(p^2, m)
\]

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\(^5\) Of course, the actual number of geometrical paths, of a given length connecting any two points in the Euclidean space, is either zero or infinity. But the effective number of paths \(N_{\text{std}}(\ell; x_2, x_1)\), formally defined as the inverse Laplace transform of \(G_{\text{std}}(x_1, x_2; m)\) (see Eq. (4)), will be a finite quantity.

\(^6\) Notation: I will use the subscript ‘std’ for functions pertaining to a classical gravitational background, not necessarily a flat spacetime; for corresponding expressions evaluated in the flat spacetime, I will use the subscript ‘free’. In the later discussion the subscript ‘QG’ will give the corresponding functions with quantum gravitational corrections.

\(^7\) From our definition, it follows that \(N(x, \ell)\) has the dimensions of \(L^{-D}\) in a \(D\) dimensional space(time) which is the same as that of space(time) number density. Its Fourier transform \(N(p, \ell)\) is dimensionless in all \(D\).
Given \( N_{\text{free}}(p, \ell) \) in the momentum space, \( N_{\text{free}}(\ell, x) \) in real space can be computed by evaluating the Fourier transform of \( \cos p \ell \). The result is again quite simple. In \( D = 4 \), we find that (see Appendix A.1 for calculational details):

\[
N_{\text{free}}(\ell, x) = \frac{3}{4\pi^2} \frac{\ell \Theta[\ell^2 - x^2]}{(\ell^2 - x^2)^{5/2}}
\]

where \( \Theta \) is the Heaviside theta function. It is amusing to note that \( N_{\text{free}}(\ell, x) \) vanishes for paths with length \( \ell < x \) so that no such path will contribute to the Euclidean integration measure. This is reminiscent of the fact that there are no geometrical paths with \( \ell < x \) connecting the two events in the Euclidean space.

2 Corrections to the Propagator Due to the Quantum Structure of Spacetime

I have now defined the propagator entirely in terms of a geometric object \( N_{\text{std}}(\ell; x_2, x_1) \) which could be thought of as the density of effective number of paths between \( x_2 \) and \( x_1 \). As an application of this formalism, let us consider the following context.

The description based on Eq. (1)—which describes the quantum field theory in a classical background spacetime—is expected to breakdown when we probe the spacetime at length scales \( \lambda \lesssim L_p \) where \( L_p \equiv (G\hbar/c^3)^{1/2} \) is the Planck length. I will call this regime microscopic and the regime of QFT in curved spacetime (CST), with \( \lambda \gg L_p \), macroscopic. We need the full formalism of QG to study microscopic scales while QFT in CST is adequate for macroscopic scales.

I am interested in the intermediate, mesoscopic scales, at which one can describe the spacetime in the usual continuum language and incorporate the prominent effects of QG by modifying the propagator \( G_{\text{std}} \) to a quantum gravity corrected propagator \( G_{\text{QG}} \). Such a description is expected to be valid at length scales \( \lambda \gtrsim C L_p \), with \( C = 10^3 \), say, for definiteness. A factor of \( 10^3 \) could allow for the continuum description to emerge, but—at the same time—be sensitive to the microscopic physics through a non-zero \( L_p \). One cannot calculate \( G_{\text{QG}} \) from first principles, without the full theory of quantum gravity. In the absence of such a luxury, I will use the following working hypothesis to go forward.

It may be possible to capture the key effects of quantum gravity by introducing a zero-point-length to the spacetime [2–14]. This is based on the idea that the dominant effect of quantum gravity at mesoscopic scales can be described by replacing the (squared) path length \( \sigma^2(x_2, x_1) \) by \( \sigma^2(x_2, x_1) \rightarrow \sigma^2(x_2, x_1) + L^2 \) where \( L^2 \) is of the order of Planck area \( L_p^2 \). Such an idea is decades old and has been explored extensively in the previous literature [2–14]. All the same, let me stress some aspects of

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8 I want to work with a descriptor of the field dynamics which is robust enough to survive (and be useful) at mesoscopic scales. The propagator, described in terms of \( N(\ell) \), is a good choice for such a description.
this approach for the sake of conceptual completeness. This will be useful to readers who are not sufficiently familiar with the earlier work on this approach.

1. The idea of bringing in the zero-point-length by the modification \( \sigma^2(x_2, x_1) \to \sigma^2(x_2, x_1) + L^2 \) should be thought of as a working hypothesis which is postulated to make progress, in the absence of of a complete theory of quantum gravity. This postulate is assumed to hold in an arbitrary curved spacetime. idea has been introduced and explored extensively in the past two decades or so in the literature (for some early work, see [2–16]; for more recent work, see [17–23]) and I am describing some further consequences of this approach in this paper. In principle, one should be able to derive this form of the propagator from a more complete theory of quantum gravity. For example, it can be obtained from the string theory [24] in a specific approximation; but for the purpose of this work, it is enough to consider it as a postulate.

2. By working directly with the propagator, we bypass several nuances of standard QFT which may all require some revision at mesoscopic scales. However, we know that both the dynamics and the symmetries of a free quantum field, propagating in a curved geometry, is encoded in the Feynman propagator. So, if we understand how QG effects modify the propagator, we can obtain a direct handle on both the dynamics and the symmetries of the theory at mesoscopic scales. This is the motivation for working directly with the propagator containing QG corrections, without worrying about the (unknown) modifications to the standard formalism of QFT at mesoscopic scales.

3. As an example of the economy gained by this approach, let me stress the notion of diffeomorphism invariance in a curved geometry and—as a special case—the Lorentz symmetry in flat spacetime. The prescription \( \sigma^2(x_2, x_1) \to \sigma^2(x_2, x_1) + L^2 \) is generally covariant because \( L \) is a, constant, (scalar) number. In flat spacetime, our ansatz will replace \((x_2 - x_1)^2\) by \((x_2 - x_1)^2 + L^2\) which is clearly Lorentz invariant. The mere introduction of a constant, scalar, length scale into the propagator will not violate Lorentz invariance; this should be obvious from the fact that the propagator for the massive scalar field does depend on the length scale \( m^{-1} \) and is still perfectly Lorentz invariant. The results of detailed computations (see for example, the extensive set of computations in [15, 16]) explicitly demonstrate the general covariance and Lorentz invariance of the procedure. This result is somewhat similar to that in, for example, LQG which also contains a length scale but does not violate Lorentz invariance. Moreover, in our approach, the general covariance (and Lorentz invariance) is manifestly apparent in the prescription \( \sigma^2(x_2, x_1) \to \sigma^2(x_2, x_1) + L^2; \) so no special demonstration of this fact is required unlike, in the case of, for example, LQG [25]. Some other prescriptions in the literature, for introducing a ‘minimal length’, do create issues with Lorentz invariance but our prescription is (manifestly) generally covariant.

4. The action for the relativistic particle possesses a simple — though not well-appreciated feature—which plays a key role in this approach. The action, a priori, is expected to be a functional of the form \( A[x^a(\tau); x_1, x_2] \); that is, it is the functional of the trajectory \( x^a(\tau) \) and a function of the end points \( x_2 \) and \( x_1 \). But, it can be expressed purely as a function \( A = A(\ell) \) of the length of the path \( \ell [x^a(\tau); x_2, x_1], \)
which carries the functional dependence on $x^a(\tau)$. This geometrical structure of relativistic action is rather special; in fact, the usual action for the non-relativistic particle cannot be expressed as a function of the length of the path.

As we shall see, it is this property which will allow us to translate the modification of path lengths in spacetime (by the addition of the zero-point-length) to the modification of the relativistic action. This, in turn, helps us to maintain all the relevant symmetries of the theory and directly compute the corrections to the propagator at mesoscopic scale, using the world-line path integral (bypassing e.g., the standard canonical quantization etc).

After this rather long aside, let me come back to the main theme. It is easy to see how the introduction of zero-point-length into the geometry changes the form of path integral Eq. (3) for the propagator. The existence of the zero-point-length changes each path length $\sigma$ appearing in the amplitude to $(\sigma^2 + L^2)^{1/2}$. The quantum corrected propagator $G_{QG}$ will then be given by the path integral sum:

$$G_{QG}(x_1, x_2; m) = \sum_{\text{paths}} \sigma^M \exp[-m\sqrt{\sigma^2 + L^2}]$$

(9)

with the same measure $M$. Once again, introducing the Dirac delta function and carrying out the steps which led to Eq. (4), we will get:

$$G_{QG}(x_1, x_2; m) = \int_0^\infty d\ell \, N_{\text{std}}(\ell; x_1, x_2) \exp\left(-m\sqrt{\ell^2 + L^2}\right)$$

(10)

where $N_{\text{std}}(\ell; x_1, x_2)$ is again defined through Eq. (5) and counts the effective number of paths. This is the expression for the propagator in an (effective) quantum geometry with a zero-point-length. The modification $\ell \to (\ell^2 + L^2)^{1/2}$ ensures that all path lengths are bounded from below by the zero-point-length as expected.9

Before proceeding further, let me again illustrate this result in the case of flat spacetime. Since $N_{\text{std}}(\ell; x_1, x_2)$ depends only on $x = x_2 - x_1$, so does $G_{QG}(x_1, x_2; m) = G_{QG}(x_2 - x_1; m)$. Fourier transforming both sides of Eq. (10) we get a similar relation in the momentum space. Now using the result that, in flat spacetime, $N_{\text{free}}(\ell', p) = \cos p\ell$ in the momentum space, we get:

$$G_{QG}(p^2) = \int_0^\infty d\ell \, e^{-m\sqrt{\ell^2 + p^2}} \cos(p\ell) = \frac{mL}{\sqrt{p^2 + m^2}} K_1[L\sqrt{p^2 + m^2}]$$

(11)

where $K_1(z)$ is the Bessel function of second kind and we have used a standard cosine transform (see p.16(26) of [27]). As to be expected, the $L \to 0$ limit leads

9 What happens to the classical relativistic particle if the action is modified from $m\ell$ to another function $A(\ell)$ monotonic in $\ell$? Since $\delta A = A'(\ell)\delta\ell$, the equations of motion does not change. This implies that, at least in the classical case, the dispersion relation $\omega^2 = p^2 + m^2$ does not change by the addition of zero-point-length. In fact, it turns out that the dispersion relations for the excitations does not change even when the propagator $G_{QG}$ is obtained from a QFT but the discussion of this feature goes beyond the scope of this paper [26].
to the standard expression $G_{\text{std}}(p^2) = m(p^2 + m^2)^{-1}$ when we use the fact that in this limit $K_1(z) \to 1/z$. The result in real space is obtained by a Fourier transform using standard integrals and we get:

$$G_{QG}(x) = \frac{1}{4\pi^2} \frac{m}{\sqrt{x^2 + L^2}} K_1[m\sqrt{x^2 + L^2}] = G_{\text{std}}(\sqrt{x^2 + L^2})$$  \hspace{1cm} (12)

The first equality comes from explicit Fourier transform of the result in Eq. (11). The second equality, however, tells us that the result could have been “guessed”. Since $G_{\text{std}}(x)$ in flat spacetime depends only on $x^2$, we could have obtained $G_{QG}(x)$ by simply replacing $x^2$ by $x^2 + L^2$. As we shall see later, such simplicity does not occur in an arbitrary curved spacetime.

Similar results hold for the zero-mass-Schwinger-kernel (ZMSK). One can rewrite the result in Eq. (11) in a different form, using another standard integral representation of the $K_1(z)$ function (see 3.324 (1) of [28]). We can write:

$$G_{QG}(p^2) = \frac{mL}{\sqrt{p^2 + m^2}} K_1[L\sqrt{p^2 + m^2}] = \int_0^\infty ds \, m \, \exp \left[ -s(p^2 + m^2) - \frac{L^2}{4s} \right]$$

$$= \int_0^\infty ds \, m \, e^{-m^2s} [K_{\text{free}}(s;p) e^{-L^2/4s}]$$  \hspace{1cm} (13)

First equality is just Eq. (11), the second equality uses an integral representation of the $K_1(z)$ function and in the third equality we have used the flat space expression $K_{\text{free}}(s;p) = \exp(-sp^2)$ for the ZMSK in the momentum space. This suggests that, at least in this case of flat space/time, the introduction of zero-point-length modifies the ZMSK by the replacement $K_{\text{std}} \to K_{QG} = K_{\text{std}} e^{-L^2/4s}$. Again, in flat spacetime we could have “guessed” this result; the ZMSK in flat spacetime, $K_{\text{free}}(s;x)$, depends only on $x^2$ through a factor $\exp(-x^2/4s)$; so replacing $x^2$ by $x^2 + L^2$ gives $K_{QG}(s;x) = K_{\text{std}} e^{-L^2/4s}$. Very surprisingly, as we shall see soon, this result $K_{\text{std}} \to K_{QG} = K_{\text{std}} e^{-L^2/4s}$ holds even in arbitrary curved spacetime, even when $K_{\text{std}}$ is not just a function of $x^2$. That fact cannot be “guessed”.

Let us now get back to the path integral expression in Eq. (9) for $G_{QG}(x_1,x_2;m)$, which is valid in an arbitrary, curved spacetime. It is a nontrivial problem to define a proper measure and do this summation in a curved spacetime even to obtain $G_{\text{std}}(x_1,x_2;m)$, let alone obtain $G_{QG}(x_1,x_2;m)$. Given the two path integrals:

$$G_{\text{std}} = \sum_{\text{paths}} \sigma^M \exp[-m\sigma]; \quad G_{QG} = \sum_{\text{paths}} \sigma^M \exp\left[ -m\sqrt{\sigma^2 + L^2} \right]$$  \hspace{1cm} (14)

in a curved spacetime, the best we can hope is a relation between the two. Even this is very difficult to achieve in terms of path integrals. Here is where the introduction of $N(\ell)$ comes to our rescue. We can now rewrite the propagators in Eq. (14) in terms of integrals with measure $N(\ell)$ instead of as path integrals. This leads to Eq. (4) for $G_{\text{std}}(x_1,x_2;m)$ and Eq. (10) for $G_{QG}(x_1,x_2;m)$:
If we eliminate the measure $N_{std}(\ell' : x_1, x_2)$ between these two equations, we can express $G_{QG}$ directly in terms of $G_{std}$.

While we do not know the form of $N_{std}(\ell' : x_1, x_2)$ in an arbitrary curved spacetime, we will often know—in practical contexts—the form of standard propagator $G_{std}(x_1, x_2)$ [or the standard ZMSK $K_{std}(s;x_1, x_2)$] in a given spacetime. So, to actually compute the QG effects at mesoscopic scales, in a given curved spacetime, it will be useful if we can (a) express $G_{QG}(x_1, x_2)$ in terms of $G_{std}(x_1, x_2)$ and (b) express the quantum corrected ZMSK $K_{QG}(x_1, x_2;s)$ in terms of the standard ZMSK $K_{std}(x_1, x_2;s)$. Remarkably enough this can be done, without us knowing the explicit form of $N_{std}(\ell' ; x_1, x_2)$.

This is a key practical result of our approach, which allows explicit computations without worrying about the measure for path integral etc. I will first quote the result, then describe some elementary consequences. The proof is given in the Appendix A.2.

The modification of the Schwinger kernel, due to introduction of zero-point-length, in arbitrary curved spacetime, is extremely simple: the QG effects involve the replacement:

$$K_{std}(s;x_1, x_2) \rightarrow K_{QG} = K_{std}(s;x_1, x_2)e^{-L^2/4s}$$

so that Eq. (1) gets replaced by:

$$G_{QG}(x, y;m) = \int_0^\infty m \ ds \ e^{-m^2s-L^2/4s}K_{std}(s;x_1, x_2)$$

Recalling that the leading order behaviour of the ZMSK is $K_{std} \sim s^{-2}\exp[-\tilde{\sigma}^2(x, y)/4s]$ (where $\tilde{\sigma}^2$ is the geodesic distance between the two events) we see that the modification in Eq. (17) amounts to the replacement $\tilde{\sigma}^2 \rightarrow \tilde{\sigma}^2 + L^2$ in the exponential factor. We have already verified in Eq. (13) that this is indeed the case in flat spacetime.

This result, valid in arbitrary curved spacetime, is highly non-trivial and could not be “guessed”. This is because, in a general spacetime $K_{std}(s;x, y)$ will have complicated dependences on $x$, $y$ and $K_{std}(s;x, y)$ cannot be expressed as function of $\tilde{\sigma}^2(x, y)$ alone; that is $K_{std}(s;x, y) \neq K_{std}(s;\tilde{\sigma}^2(x, y))$. (Such a simplification arises only in highly symmetric spacetimes like e.g., flat spacetime.) All the same, the replacement $\tilde{\sigma}^2 \rightarrow \tilde{\sigma}^2 + L^2$ in the leading exponential correctly captures all the effects of the zero-point-length.

The relationship between $G_{QG}$ and $G_{std}$ is in the form of a convolution over the mass parameter. One can show that:
This result is more useful in practical computations than Eq. (16) because one more often knows the form of $G_{\text{std}}(x_2, x_1; m_0)$ than the form of $K_{\text{std}}(x_2, x_1; s)$.

In particular, if the background spacetime has certain symmetries, they constrain the structure of both $G_{\text{std}}$ and $G_{\text{QG}}$ in a similar manner. For example, consider a class of homogeneous spacetimes (like e.g., the Friedmann universe) in which $G_{\text{std}}(x_1, x_2, m^2) = G_{\text{std}}(x_1 - x_2, t_1, t_2; m^2)$. It follows from Eq. (18) that we will also have $G_{\text{QG}}(x_1, x_2, m^2) = G_{\text{QG}}(x_1 - x_2, t_1, t_2; m^2)$. Therefore, one can Fourier transform both propagators with respect to spatial coordinates and obtain a relation identical to Eq. (18) in the momentum space:

$$G_{\text{QG}}(p, t_1, t_2; m) = \int_0^{\infty} dm_0 \mathcal{P}[m_0; m, L] G_{\text{std}}(p, t_1, t_2; m_0)$$  \hspace{1cm} (20)

Similarly, if the background spacetime is static, then one can Fourier transform both $G_{\text{QG}}$ and $G_{\text{std}}$ with respect to the difference in the time coordinates and obtain

$$G_{\text{QG}}(\omega, x_1, x_2; m) = \int_0^{\infty} dm_0 \mathcal{P}[m_0; m, L] G_{\text{std}}(\omega, x_1, x_2; m_0)$$  \hspace{1cm} (21)

More complicated symmetries of $G_{\text{std}}$ can be handled in a similar manner. For example, in any maximally symmetric spacetimes, both $G_{\text{std}}$ and $G_{\text{QG}}$ will (essentially) depend on the geodesic distance and one can often deal with the Fourier transform with respect to the geodesic distance. This opens up several further avenues for concrete computation of QG effects in curved spacetime. I hope to explore these in a future work.

These expression in Eq. (18) is given as a convolution over the mass parameter in the propagator. It is also possible obtain other forms of relations from this result which could lead to algebraic simplifications or better intuitive understanding. I mention two alternative ways of relating $G_{\text{QG}} = G_{\text{QG}}/m$ with $G_{\text{std}} = G_{\text{std}}/m$; the proofs are in the Appendix A.2.

1. By changing the integration variable in Eq. (18) to $\mu \equiv (m_0^2 - m^2)^{1/2}$ we can show that:

\begin{equation}
G_{\text{QG}}(x_2, x_1; m) = \int_0^{\infty} dm_0 \mathcal{P}[m_0; m, L] G_{\text{std}}(x_2, x_1; m_0)  \hspace{1cm} (18)
\end{equation}

where

\begin{equation}
\mathcal{P}[m_0; m, L] = -\frac{\partial}{\partial m} \left\{ \Theta(m_0 - m) J_0 \left[ L \sqrt{m_0^2 - m^2} \right] \right\} \hspace{1cm} (19)
\end{equation}

One can carry out the differentiation in this expression and obtain two terms, one involving a Dirac delta function $\delta(m_0 - m)$ and the other involving a product $\Theta(m_0 - m) J_0 [L(m_0^2 - m^2)^{1/2}]$. It turns out to be often more convenient not to carry out this differentiation and work with the expression in Eq. (19).
(2) It is also provide a higher dimensional interpretation of this relation. Consider a fictitious \( N = D + 2 \), Euclidean curved space(time) with the metric

\[
dS_N^2 = (g_{ab}(x)dx^a dx^b)_D + \delta_{AB} dX^A dX^B \quad (A, B = 1, 2)
\]

where we have added two “flat” directions, \( X^A \) with \( A = 1, 2 \). Let \( G_{\text{std}}^N(x, L; y, 0) \) be the standard propagator in the \( N \) dimensional space with propagation in the fictitious directions being from origin to a point \( L \) with \( |L| \) being the zero-point-length. Then, one can show that the propagator \( G_{\text{QG}}^D(x, y) \) in \( D \) dimensions with zero-point-length \( L \) is related to \( G_{\text{std}}^N(x, L; y, 0) \) (both for same mass \( m \) which is not explicitly displayed) by:

\[
G_{\text{QG}}^D(x, y) = -4\pi \frac{d}{dm^2} G_{\text{std}}^N(x, L; y, 0) \bigg|_{L^2 = L^2}
\]

The \( N = D + 2 \) dimensional propagator, \( G_{\text{std}}^N(x, L; y, 0) \), of course has a standard QFT interpretation in the curved spacetime. The zero-point-length in \( D \) dimensions arises as the magnitude of the (fictitious) propagation distance in the extra dimensions.

(3) One can also convert these relations into differential equations and show that:

\[
(-\Box^N + m^2)^2 G_{\text{QG}}^D = 4\pi \delta(x, y) \delta(L)
\]

and

\[
(-\Box^N + m^2) G_{\text{QG}}^D = 4\pi G_{\text{std}}^N(x, L; y, 0)
\]

These relations could form a basis for alternative interpretations of \( G_{\text{QG}}^D \). (Note that, in Eq. (25), the \( G^D \) refers to the propagator in \( D \)-dimensions while the Laplacian \( \Box^N \) is a \( N = D + 2 \) dimensional one. The result is valid in arbitrary curved spacetime. In flat spacetime, if you take the Fourier transform of Eq. (26), the left hand side will lead to the square of the conventional momentum space propagator but the integration over the extra dimensions will lead to the correct result. The derivation of Eq. (25) in arbitrary curved spacetime and the explicit computation, in terms of the square of the momentum space propagator in flat spacetime, are given in Appendix A.2.)

I conclude this section with two technical comments.

The procedure we have used, viz. the introduction of the effective number of paths and its QG generalization, certainly works for the test scalar field propagating in an arbitrary curved space(time). For interacting fields (e.g. a scalar field with \( \lambda \phi^4 \) coupling) it is not easy to represent the propagator in terms of a world-line path integral; therefore, the generalization of the current idea to interacting field theories is not straightforward. One possibly could capture some of the leading corrections to the perturbation theory by using the \( G_{\text{QG}} \) in the standard Feynman rules but it will miss non-perturbative effects. To tackle this problem, we first need to represent the
interacting field theories, in curved spacetime, entirely in terms of the world-line approach—which is a nontrivial task. But as long as we are only interested in probing the mesoscopic structure of spacetime using a simple quantum field, which is our primary goal here, this approach is adequate.

Let me conclude this section with a technical comment on the analytic continuation. Our approach starts with a curved space of Euclidean signature and obtains the results for the Lorentzian (pseudo-Riemannian) curved spacetime by an analytic continuation. It is well-known that there is no rigorous, unique, correspondence between the set of all Euclidean metrics and the Lorentzian (pseudo-Riemannian) metrics. So, our procedure should be thought of as a general prescription and ambiguous cases need to be handled on a case-by-case basis. In this context, the following two points need to be kept in mind: (a) This problem with analytic continuation arises whenever one uses Euclidean methods in curved spacetime and is not specific to the discussion in this work. (b) As long as one is concerned with the short-distance behaviour at mesoscopic scales, this problem can be circumvented.

To make this notion precise, consider a non-singular spacetime with curvatures nowhere close to Planck values. That is, we assume, $L_p \ll L_{cur}$ where the curvature length scale defined by, say, $L_{cur}^{-4} = R_{abcd}R^{abcd}$. Around any event $P$, one can then introduce a locally inertial frame and use flat spacetime notion of analytic continuation at scales $L_p \lesssim x \ll L_{cur}$ which includes the mesoscopic scales we are interested in. This provides a way around the problem of analytic continuation in the regime we are interested in but, of course, it is not a solution to the broader issue of correspondence between Euclidean and pseudo-Riemannian spaces.

### 3 Additional Comments on the Path Measure

It should be obvious from the above discussion that manipulating $N_{std}(\ell, x)$ is far easier than working with path integral measures and limiting processes. This has allowed us to obtain rather general results in the previous section in a concrete and simple manner. I will now discuss some further possible applications of this approach.

The standard propagator $G_{std}(x_1, x_2, m)$ was obtained by a path integral sum in Eq. (3) with a specific measure $M$ indicated as a superscript on the summation symbol. The same result translates to the ordinary integral in Eq. (4) in terms of the integration measure $N_{std}(\ell, x_1, x_2)$. The definition of $N_{std}(\ell, x_1, x_2)$ in Eq. (5) shows that it is defined using the path integral measure $M$ and thus maintains a one-to-one correspondence with the choice of path integral measure. If we change the path integral measure, the functional form of $N_{std}(\ell, x_1, x_2)$ will change and vice-versa. But for calculational purposes it is easy to change the form of the integration measure $N_{std}(\ell, x_1, x_2)$ rather than the more abstractly defined path integral measure $M$, with the implicit understanding that different choices of $N_{std}(\ell, x_1, x_2)$ corresponds to different choices of the path integral measure $M$.

This algebraic fact becomes significant when we realize that, in any physical situation, we are only concerned with the propagator and not individually on the form $G_{std}(x_1, x_2, m)$.
of the integration measure \( N(\ell', x_1, x_2) \) (or \( \mathcal{M} \)) and the form of the action used in the amplitude \( \exp[-A(\ell')] \). That is, in the expressions:

\[
\mathcal{G}(x_1, x_2; m) = \sum_{\text{paths}} \sigma^\mathcal{M} \exp -A(\sigma) = \int_0^\infty d\ell' \ N(\ell'; x_2, x_1) e^{-A(\ell')}
\] (27)

what matters for physics is the propagator \( \mathcal{G}(x_1, x_2; m) \) in the left-hand-side. It depends on both the form of \( A(\sigma) \) as well as the measure \( \mathcal{M} \) in the path integral or, equivalently, on the form of \( A(\ell') \) and the path density \( N(\ell'; x_2, x_1) \) in the ordinary integral. The pair \( \{\mathcal{M}, A(\sigma)\} \) or the pair \( \{N(\ell'), A(\ell')\} \), determines \( \mathcal{G}(x_1, x_2; m) \) and different pairs can lead to the same propagator. This fact is difficult to visualize or manipulate in terms of \( \mathcal{M} \) but completely straightforward when we use \( N(\ell') \)

Consider, as an example, the expressions for \( \mathcal{G}_{\text{std}} \) and \( \mathcal{G}_{\text{QG}} \) in Eqs. (4) and (10). In proceeding from Eqs. (4) to (10), we postulated that QG effects modify the world-line action by the replacement \( m\ell' \to m\sqrt{\ell'^2 + L^2} \). We then evaluated the sum over paths with the same original measure \( \mathcal{M} \), which is equivalent to using the original integration measure \( N_{\text{std}}(x_1, x_2, \ell') \) in Eq. (10). But, as I said, the physics only depends on the form of \( \mathcal{G}_{\text{QG}} \) and not individually on the form of integration measure or the action. For example, we could have obtained the same result (i.e., the same \( \mathcal{G}_{\text{QG}} \)) by keeping the path integral amplitude to be the same (i.e., keeping the amplitude as \( \exp[-m\sigma(x, x')] \)) but introducing all the quantum gravity corrections on the path measure, by replacing \( N_{\text{std}} \) by another measure \( N_{\text{QG}} \). These two measures are related by the condition that we should get the same \( \mathcal{G}_{\text{QG}}(m) \). This requires:

\[
\mathcal{G}_{\text{QG}}(x_1, x_2; m) = \int_0^\infty d\ell' \ N_{\text{std}}(\ell'; x_2, x_1) e^{-m\sqrt{\ell'^2 + L^2}} = \int_0^\infty d\ell' \ N_{\text{QG}}(\ell'; x_2, x_1) e^{-m\ell'}
\] (28)

It is easy to determine \( N_{\text{QG}}(\ell') \) by changing the integration variable in the first integral in Eq. (28) from \( \ell' \) to \( \mu \) through \( \mu^2 = \ell'^2 + L^2 \) and rewrite the integral as:

\[
\mathcal{G}_{\text{QG}}(x_1, x_2; m) = \int_0^\infty \frac{\mu \ d\mu}{\sqrt{\mu^2 - L^2}} \Theta(\mu - L) \ N_{\text{std}}(x_1, x_2; \ell' = \sqrt{\mu^2 - L^2}) e^{-m\mu}
\] (29)

This is the form of the second integral Eq. (28) in which we keep the standard form of the amplitude \( \exp(-m\ell') \) but introduce the zero-point-length in the path measure. The quantum corrected path measure \( N_{\text{QG}}(\mu) \) is then related to the standard path measure \( N_{\text{std}}(\ell') \) by the simple relation

\[
N_{\text{QG}}(x_1, x_2; \mu) = \frac{\mu}{\sqrt{\mu^2 - L^2}} N_{\text{std}} \left[ x_1, x_2; \ell' = \sqrt{\mu^2 - L^2} \right] \Theta(\mu - L)
\] (30)

\[\text{[1] Of course if you change } \ell' \text{ to some other function } f(\ell') \text{ both in the measure and in the amplitude, you will change nothing in a definite integral.}\]
The $N_{\text{QG}}(\sigma, x_2, x_1)$, vanishes for paths with length less than the zero-point-length; they are treated as irrelevant to physics and does not contribute. I stress that (a) this interpretation was possible only because $N_{\text{QG}}$ etc. are purely geometrical constructs, independent of the mass $m$ of the field and (b) the result is valid in arbitrary curved spacetime, in which defining and manipulating path integral measures are difficult tasks. Again let me quote the specific expression for flat space(time) for illustration. Using the momentum space expressions for $N_{\text{free}}$ and $N_{\text{QG}}$, and noting that $N_{\text{free}}(\sigma, \ell) = \cos (\sigma \ell)$, Eq. (30) gives the corresponding $N_{\text{QG}}(\sigma, \ell)$ to be:

$$N_{\text{QG}}(\sigma, \ell) = \Theta(\ell - L) \frac{\ell \cos p \sqrt{\ell^2 - L^2}}{\sqrt{\ell^2 - L^2}} \tag{31}$$

As a final application, let me consider a particular form of the QG modified action, which has been used in previous literature. This is given by:

$$A(\sigma) = -m \left[ \sigma + \frac{L_0^2}{\sigma} \right] \tag{32}$$

which has the nice property that it is invariant under $\sigma \to L_0^2/\sigma$ which was called ‘duality’ in previous literature. It turns out that one can actually compute the path integral sum with this action (see [2, 3]), using a non-standard measure $\mathcal{M}'$ and arrive at the same QG corrected propagator $G_{\text{QG}}$. Our conversion of the path integral into an ordinary integral, in terms of an integration measure $N(\ell', x_2, x_1)$, allows us to see more transparently how this result comes about. By changing the variable from $\ell$ to $\tilde{\ell}$, with $\ell = \tilde{\ell} - (L_0^2/\tilde{\ell})$—so that the integration range $\ell' = (0, \infty)$ can be mapped to $\tilde{\ell} = (\ell_0, \infty)$—we can re-express the integral for $G_{\text{QG}}$ in the desired form. Elementary algebra gives

$$G_{\text{QG}}(x_2, x_1, m) \equiv \int_{\ell_0}^{\infty} d\ell' N_{\text{std}}(\ell', x_2, x_1) e^{-m \sqrt{\ell'^2 + L^2}} = \int_{0}^{\infty} d\sigma N_{\text{QG}}(\sigma, x_2, x_1) e^{-m \left( \sigma + \frac{L_0^2}{\sigma} \right)} \tag{33}$$

with $L_0 = L/2$ and

$$N_{\text{QG}}(\sigma, x_2, x_1) = \Theta(\sigma - L_0) \left( 1 + \frac{L_0^2}{\sigma^2} \right) N_{\text{std}} \left( \ell' = \sigma - \frac{L_0^2}{\sigma}, x_2, x_1 \right) \tag{34}$$

This result explicitly relates the two integration measures used in the two cases and remains valid in arbitrary curved spacetime. We see that $N_{\text{QG}}(\sigma)$ restricts the paths to those with length $\sigma > \ell'$. Such manipulations are not so transparent when one works with the path integral measure $\mathcal{M}$ and $\mathcal{M}'$. 

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4 Summary of Results

I summarize below the key new results and their significance.

- The propagator for a spinless particle of mass $m$ in an arbitrary curved background with metric $g_{ab}(x)$ encodes the full quantum dynamics of the system. It can be represented as a purely geometric object in terms of a world-line path integral. The measure for such a path integral is difficult to define and manipulate in an arbitrary curved spacetime. I introduce the concept of effective number of paths $N(\ell; x_1, x_2)$ and re-write the world-line path integral as an ordinary integral:

$$G(m; x_1, x_2) = \sum_{\text{paths } \sigma}^M e^{-A[\sigma]} = \int_0^\infty d\ell \ N(\ell; x_1, x_2) e^{-A[\ell]} \quad (35)$$

where $A[\sigma] = m\sigma(x_1, x_2)$ is the standard action for the relativistic particle. As it turns out, this conversion of a path integral measure $\mathcal{M}$ to an ordinary integral measure $N(\ell; x_1, x_2)$ is a key technical and conceptual advance introduced in this paper. This idea works only because the action for a relativistic particle is a geometrical entity and can be expressed entirely in terms of the path length $\sigma(x_1, x_2)$.

- I illustrate this concept in terms of flat spacetime and show that the effective number of paths has an extremely simple expression in momentum space and is given by $N(\ell, p) = \cos p\ell$.

- This approach really comes alive when one considers the propagator $G_{\text{QG}}(m; x_1, x_2)$ at mesoscopic scales, incorporating the effects of zero-point-length by the modification $\sigma \rightarrow \sqrt{\sigma^2 + L^2}$. This propagator $G_{\text{QG}}(m; x_1, x_2)$ is given by exactly the same expression as in Eq. (35) with $A(\sigma) = m\sigma$ replaced by $A(\sigma) = m\sqrt{\sigma^2 + L^2}$. The path integral sum is now intractable but the ordinary integral in terms of $N(\ell; x_1, x_2)$ comes to our rescue. For example, one can explicitly compute the QG corrected propagator when the background metric is flat and obtain, in momentum space, the result:

$$G_{\text{QG}}(p^2) = \int_0^\infty d\ell \ e^{-m\sqrt{L^2+\ell^2}} \cos(p\ell) = \frac{mL}{\sqrt{p^2+m^2}} K_1[L\sqrt{p^2+m^2}] \quad (36)$$

It is non-trivially difficult to do this explicit computation, even in flat spacetime, using path integral measure.

- The concept of effective number of paths turns out to be much more useful in curved spacetime. In an arbitrary curved spacetime we can express both the standard propagator and the QG corrected one in terms of $N(\ell; x_1, x_2)$ by converting the respective path integrals into ordinary integrals and thereby obtaining:
It is now possible to eliminate $N(\ell' x_1 x_2)$ between these two relations, by some algebraic gymnastics, and relate $G_{QG}(x_1, x_2; m)$ directly to $G_{std}(x_1, x_2; m)$. These relations can be expressed in many different forms; for e.g., one can show that

$$G_{QG}(x_1, x_2; m^2) = -4\pi \frac{\partial}{\partial m^2} \int \frac{d^2 k}{(2\pi)^2} e^{i k \cdot L} G_{std}(x_1, x_2; k^2 + m^2)$$

where $L$ is a 2-dimensional vector with magnitude equal to the zero-point-length.

- One can also relate $G_{QG}^D(x, y)$ in $D$-dimensions with the standard propagator in a fictitious space of $N = D + 2$ dimensions with metric $ds^2 = g_{ab}(x)dx^a dx^b + \delta_{AB} dX^A dX^B$, where we have added two “flat directions“ $X^A$ with $A = 1, 2$. We can prove that:

$$G_{QG}^D(x, y) = -4\pi \frac{\partial}{\partial m^2} G_{std}^N(x, L; y, 0) \bigg|_{L^2 = L^2}$$

The $N = D + 2$ dimensional propagator, $G_{std}^N(x, L; y, 0)$, of course has a standard QFT interpretation in the curved spacetime. The zero-point-length in $D$ dimensions arises as the magnitude of the (fictitious) propagation distance in the extra dimensions. We also have the differential relations:

$$(-\Box^N + m^2) G_{QG}^D = 4\pi \delta(x, y) \delta(L)$$

and

$$(-\Box^N + m^2) G_{QG}^D = 4\pi G_{std}^N(x, L; y, 0)$$

between the two propagators.

**Appendix: Calculational Details**

**Path Measure in Real Space**

Given the path measure in the momentum space, $N_{\text{free}}(\ell', p) = \cos p \ell'$, we can find the measure $N_{\text{free}}(\ell', x)$ in real space by evaluating the D-dimensional Fourier transform of $\cos p \ell'$. To do this, we start with the standard result for the Fourier transform of spherically symmetric function. If

$$F(k) = \int d^D x f(|x|) e^{-i k \cdot x}$$

then we can write
This allows us to write the relevant Fourier transform as:

\[ N_{\text{free}}(\ell', x) = \int_0^\infty \frac{d^D p}{(2\pi)^D} e^{ip.x} \cos p\ell' = \frac{1}{(2\pi)^{D/2}} \frac{1}{x^\alpha} \int_0^\infty dp \, p^{\alpha+1} J_\alpha(px) \cos \ell p \]

where \( \alpha = (D/2) - 1 \). We next evaluate this integral using the standard cosine transform (see, for e.g., page 45, 1.12 (12) of [27])

\[ \int_0^\infty dx \, (\cos xy) x^{\nu+1} J_\nu(ax) = \frac{2^{\nu+1} \sqrt{\pi} a^{\nu} \Theta[y - a]}{\Gamma\left(-\frac{1}{2} - \nu\right)(y^2 - a^2)^{\nu+\frac{1}{2}}} \]

This gives the result

\[ N_{\text{free}}(\ell', x) = \frac{1}{\pi^{\frac{D-1}{2}}} \frac{\Gamma\left(\nu + 1\right)}{\Gamma\left(-\frac{D-1}{2}\right)} \frac{\ell'}{\left[\ell'^2 - x^2\right]^{\frac{D+1}{2}}} \]

which reduces to the expression quoted in the text when \( D = 4 \). (Strictly speaking the integral in Eq. (45) is defined only for \(-1 < \text{Re} \nu < -1/2\) but can be analytically continues for other \( \nu \), as often done in dimensional regularization.) Note that \( N_{\text{free}}(\ell', x) \) vanishes for \( \ell' < x \); paths, with lengths less than the geometrical length between the two points, do not contribute to the path integral which is rather nice feature.

One can directly verify that this expression leads to the correct massive propagator in real space which, of course, is obvious from the fact that \( N_{\text{free}}(\ell', x) \) is the Fourier transform of \( N_{\text{free}}(\ell', p) \) and we know that the latter gives the correct propagator in the momentum space. To verify this directly we need the integral

\[ \int_1^\infty dt \, \frac{t \, e^{-zt}}{(t^2 - 1)^{\frac{1}{2} - \nu}} = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}} \left(\frac{2}{z}\right) K_{\nu+1}(z) \]

which can be obtained from a standard result (see, 8.432 (3) of [28]) by differentiating with respect to \( z \) and using the recursion relation for \( K_\nu(z) \) (see page 929 (13) of [28]). Using this integral and the expression for \( N_{\text{free}}(\ell', x) \) in Eq. (46), we find that (with \( \mu \equiv (1/2)(D + 1) \))

\[ \int_0^\infty d\ell' \, N_{\text{free}}(\ell', x) \, e^{-m\ell'} = \int_x^\infty d\ell' \frac{1}{\pi^{\mu-1}} \frac{1}{\Gamma(1 - \mu)} \frac{\ell'}{(\ell'^2 - x^2)^\mu} \]

\[ = \frac{x^{2(1 - \mu)}}{\pi^{\mu-1} \sqrt{\pi}} \left(\frac{2}{mx}\right)^{\frac{1}{2} - \mu} K_{\mu - \frac{3}{2}}(mx) = G_\mu(mx) \]
which is indeed $m$ times the standard massive propagator $G_D(mx) = mG_D(mx)$ in D-dimensions. In the case of $D = 4$, this reduces to

$$
G(x) = m \, G(x) = \frac{m}{4\pi^2} \left( \frac{m}{x} \right) K_1(mx)
$$

which is the familiar expression.

**Relation Between $G_{QG}$ and $G_{std}$**

I will now sketch the proof of Eq. (18) and Eq. (16) which involves slightly nontrivial manipulations of integrals over Bessel functions. I begin with two easily provable identities:

$$
e^{-m\sqrt{\epsilon^2 + L^2}} \sqrt{\epsilon^2 + L^2} = \int_m^\infty dm_0 \, e^{-m_0 \epsilon} J_0 \left[ L\sqrt{m_0^2 - m^2} \right]
$$

and

$$
\frac{1}{2s} e^{-m^2 x \frac{L^2}{4s}} = \int_m^\infty dm_0 \, m_0 e^{-m_0^2 \epsilon} J_0 \left[ L\sqrt{m_0^2 - m^2} \right]
$$

The first identity in Eq. (50) can be proved by changing the integration variable on the right hand side to $x \equiv m_0/m$ and using a standard integral (see, 6.616 (2) of [28])

$$
\int_1^\infty dx \, e^{-ax} J_0 \left[ \beta \sqrt{x^2 - 1} \right] = \frac{e^{-\sqrt{a^2 + \beta^2}}}{\sqrt{a^2 + \beta^2}}
$$

The second identity in Eq. (51) can again be proved by changing the integration variable on the right hand side to $x = m_0/m$ and using a result derivable from 6.614 (1) of [28]:

$$
\int_0^\infty 2kdk \, J_0(kL)e^{-sk^2} = \frac{1}{s} \exp \left( -\frac{L^2}{4s} \right)
$$

Differentiating both sides of Eq. (50) with respect to $m$, we obtain

$$
e^{-m\sqrt{\epsilon^2 + L^2}} = -\frac{\partial}{\partial m} \int_m^\infty dm_0 \, e^{-m_0 \epsilon} J_0 \left[ L\sqrt{m_0^2 - m^2} \right]
$$

$$
= \int_0^\infty dm_0 \, e^{-m_0 \epsilon} \left( -1 \right) \frac{\partial}{\partial m} \left\{ \Theta(m_0 - m) J_0 \left[ L\sqrt{m_0^2 - m^2} \right] \right\}
$$

This gives

$$
e^{-m\sqrt{\epsilon^2 + L^2}} = \int_0^\infty dm_0 \, e^{-m_0 \epsilon} \mathcal{P}[m_0;m,L]
$$
with \( \mathcal{P} \) defined by Eq. (19). Similarly, differentiating both sides of Eq. (51) with respect to \( m \) and manipulating as before we get

\[
me^{-m^2s-\frac{L^2}{4}} = \int_0^\infty dm_0 \ m_0 e^{-m_0^2s} \mathcal{P}[m_0; m, L]
\]  

(56)

Notice that the right hand sides of Eqs. (56) and (55) have very similar structures with \( e^{-m_0^2s} \) replaced by \( m_0 e^{-m_0^2s} \).

The results in Eqs. (18) and (17), which we need to prove, can now be obtained in a straightforward manner as follows: Multiply both sides of Eq. (55) by \( N(\ell) \) and integrate over \( \ell \) to get:

\[
\mathcal{G}_{\text{QG}} = \int_0^\infty d\ell' \ N(\ell') e^{-m\sqrt{\ell'^2+L^2}} = \int_0^\infty dm_0 \ \mathcal{P}[m_0; m, L] \int_0^\infty d\ell' \ N(\ell') e^{-m_0\ell'}
\]

\[
= \int_0^\infty dm_0 \ \mathcal{P}[m_0; m, L] \mathcal{G}_{\text{std}}(m_0)
\]

(57)

where \( \mathcal{P}[m_0; m, L] \) is defined by Eq. (19), reproduced here for ready reference:

\[
\mathcal{P}[m_0; m, L] = -\frac{\partial}{\partial m} \left\{ \Theta(m_0 - m)J_0 \left[ L\sqrt{m_0^2 - m^2}\right] \right\}
\]

(58)

This gives Eq. (18). To obtain Eq. (17), we write \( \mathcal{G}_{\text{std}} \) in terms of \( K_{\text{std}}(s) \). This leads to

\[
\mathcal{G}_{\text{QG}} = \int_0^\infty dm_0 \ \mathcal{P}[m_0; m, L] \int_0^\infty ds \ e^{-m_0^2s}K_{\text{std}}(s)m_0
\]

\[
= \int_0^\infty ds \ K_{\text{std}}(s) \int_0^\infty dm_0 \ m_0 e^{-m_0^2s} \mathcal{P}[m_0; m, L]
\]

\[
= \int_0^\infty ds \ K_{\text{std}}(s)me^{-m^2s-\frac{L^2}{4}}
\]

(59)

In arriving at the last equality we have used the result in Eq. (56). This proves Eq. (17).

To obtain the relation between \( \mathcal{G}_{\text{QG}} = \mathcal{G}_{\text{QG}}/m \) and \( \mathcal{G}_{\text{std}} = \mathcal{G}_{\text{std}}/m \) we can proceed as follows. We pull the derivative with respect to \( m \) (arising from the expression in Eq. (58)) out of the integral sign in Eq. (57) and change the variable of integration to \( k \) with \( k^2 \equiv m_0^2 - m^2 \). This leads to the relation:

\[
\mathcal{G}_{\text{QG}}(m) = -\frac{\partial}{\partial m} \int_0^\infty \frac{kdk}{\sqrt{k^2 + m^2}} \ J_0(Lk) \ \mathcal{G}_{\text{std}}(m^2 + k^2)
\]

(60)

where \( \mathcal{G}_{\text{std}}(m^2 + k^2) \) is the standard propagator with \( m^2 \) replaced by \( m^2 + k^2 \). Using the fact that \( \mathcal{G}(M) = MG(M) \) for any mass parameter \( M \), this is equivalent to
This is the relation quoted in the main text; see Eq. (22). Further, for any function \( f(k) \) which depends only on the magnitude of the 2-dimensional vector \( k \), we have the identity

\[
\int_0^\infty \frac{dk}{k^2 + m^2} J_0(Lk) \sqrt{k^2 + m^2} G_{\text{std}}(m^2 + k^2) = \frac{\partial}{\partial m^2} \int_0^\infty 2k \, dk \, J_0(Lk) \, G_{\text{std}}(k^2 + m^2)
\]

(61)

Using this Eq. (61) can be expressed in the form

\[
G_{\text{QG}}(m) = -4\pi \frac{\partial}{\partial m^2} \int d^2k (2\pi)^2 e^{ik \cdot L} f(k) = \frac{1}{2\pi} \int_0^\infty k \, dk \, J_0(kL) f(k)
\]

(62)

where \( L \) is a 2-dimensional vector with magnitude equal to the zero-point-length. From this it is possible to obtain a higher dimensional interpretation of \( G_{\text{QG}} \). However, the result is probably more transparent when obtained from first principles and I will provide such a derivation:

I will now work in \( D \) dimensional space(time) and express \( G_{\text{QG}} \) in \( D \)-dimensions in terms of the standard propagator \( G_{\text{std}}^N \) in \( N = D + 2 \) dimensions. To do this, let us consider a fictitious \( N = D + 2 \), Euclidean curved space(time) with the metric

\[
dS_N^2 = (g_{ab} dx^a dx^b)_D + \delta_{AB} dX^A dX^B
\]

(64)

where we have added two “flat” directions, \( X^A \) with \( A = 1, 2 \). (The metric \( g_{ab} \) in \( D \) dimensions, of course depends only on the \( D \) coordinates \( x^\mu \).) The \( N \)-dimensional Schwinger kernel for \( m = 0 \) (ZMSK) now factorizes and we can write

\[
K_{m=0}^N \equiv \langle x, L | e^{\square_N} | y, 0 \rangle = \left( \frac{1}{4\pi s} \right) e^{-L^2/4s} K_{m=0}^D(x, y; s); \quad L^2 \equiv L^A L^A
\]

(65)

Therefore, the corresponding \( N \)-dimensional, massive, Schwinger kernel becomes

\[
K_N(x, L; y, 0; s) = \left( \frac{1}{4\pi s} \right) e^{-(L^2/4s) - m^2 s} K_{m=0}^D(x, y; s)
\]

(66)

The corresponding \( N \)-dimensional massive propagator is obtained by integrating this kernel over \( s \) in the range 0 to \( \infty \). This is almost the same as \( G_{\text{QG}} \) but for the extra factor \( (1/4\pi s) \). This factor can be taken care of by differentiating the propagator with respect to \( m^2 \). We then find that

\[
-4\pi \frac{\partial}{\partial m^2} G_{\text{std}}^N(x, L; y, 0) = \int_0^\infty ds \, e^{-(L^2/4s) - m^2 s} K_{m=0}^D(x, y; s) = G_{\text{QG}}^D(x, y)
\]

(67)
So, we have related the quantum corrected propagator $G_{QG}^D(x, y)$ in $D$ dimensions to the standard Klein-Gordon propagator in the fictitious $N = D + 2$ space with the metric in Eq. (64), through the relation:

$$G_{QG}^D(x, y) = -4\pi \left. \frac{\partial}{\partial m^2} G_{std}^N(x, L; y, 0) \right|_{L^2=L^2}$$

(68)

The $N = D + 2$ dimensional propagator, $G_{std}^N(x, L; y, 0)$, of course has a standard QFT interpretation in the curved spacetime. The zero-point-length in $D$ dimensions arises as the magnitude of the (fictitious) propagation distance in the extra dimensions. This proves Eq. (24) in the main text.

The derivative of standard Klein-Gordon propagator with respect to $m^2$—which appears in the right hand side of Eq. (68)—can be related to the ‘transitivity integral’ for the propagator $G_{std}^N$. To see this, consider the following integral (with the notation $d^N\tilde{z} \equiv d^Dz \, d^2Z$):

$$\int d^N\tilde{z} \, G_{std}^N(x, L; z, Z) \, G_{std}^N(z, Z; y, 0)$$

$$= \int d^N\tilde{z} \, \langle x, L|(-\Box^N + m^2)^{-1}|z, Z\rangle \langle z, Z|(-\Box^N + m^2)^{-1}|y, 0\rangle$$

(69)

$$= \langle x, L|(-\Box^N + m^2)^{-2}|y, 0\rangle$$

We can, however, write:

$$\langle x, L|(-\Box^N + m^2)^{-2}|y, 0\rangle = -\left. \frac{\partial}{\partial m^2} G_{std}^N(x, L; y, 0) \right|_{L^2=L^2}$$

(70)

Combining this result with Eq. (68) we find that

$$G_{QG}^D = \left. (4\pi) \langle x, L|(-\Box^N + m^2)^{-2}|y, 0\rangle \right|_{L^2=L^2}$$

(71)

This result, in turn, can expressed in the form of either of the following differential equations for $G_{QG}^D$, viz.:

$$(-\Box^N + m^2)^2 G_{QG}^D = 4\pi \delta(x, y) \, \delta(L)$$

(72)

and

$$(-\Box^N + m^2) G_{QG}^D = 4\pi G_{std}^N(x, L; y, 0)$$

(73)

These are valid any curved space(time). We can now write $G_{std}^N(x, L; y, 0)$ as the standard vacuum expectation value of time ordered products of the KG field operators in $N = D + 2$ space(time); then the QG corrected propagator in $D$ dimensional space is given by a solution to Eq. (73). This proves Eqs. (25) and (26).

We can, of course, verify that, in flat space(time) either of these equations lead to the correct $G_{QG}$. For example, if we Fourier transform either Eqs. (72) or (73), we will find that $G_{QG}^D$ can be expressed as the integral...
The square appearing in the denominator can be taken care of by the usual trick of differentiating the expression with respect to \( m^2 \). Performing the 2-dimensional integral over the measure \( d^2 K = K dK d\theta \) we get the result in terms of the Bessel function \( J_0(\alpha) \):

\[
G_{QG}^D(x, L; 0, 0) = 4\pi \iint \frac{d^D k \, d^2 K}{(2\pi)^N} \frac{e^{ikx} e^{iK \cdot L}}{(k^2 + m^2 + K^2)^2} \tag{74}
\]

Therefore the \( D \)-dimensional Fourier transform \( G_{QG}^D(k, L) \) of \( G_{QG}^D(x, L; 0, 0) \) is given by

\[
G_{QG}^D(k, L) = -\frac{\partial}{\partial m^2} \int_0^\infty 2K dK \frac{J_0(KL)}{k^2 + m^2 + K^2} \tag{76}
\]

To perform the integral over \( K \) we write the denominator in the exponential form, leading to:

\[
-\frac{\partial}{\partial m^2} \int_0^\infty 2K dK J_0(KL) \int_0^\infty ds \ e^{-s(k^2 + m^2)} \ e^{-sk^2} \tag{77}
\]

\[= \int_0^\infty ds \ s e^{-s(k^2 + m^2)} \times \int_0^\infty 2K dK J_0(KL) e^{-sk^2} \]

Finally, we use the identity

\[
\int_0^\infty 2K dK J_0(KL) e^{-sk^2} = \frac{1}{s} \exp \left( -\frac{L^2}{4s} \right) \tag{78}
\]

to recover the standard result

\[
G_{QG}^D(k, L) = \int_0^\infty ds \ e^{-s(k^2 + m^2) - (L^2/4s)} \tag{79}
\]

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