VARIANCE REDUCTION FOR DISCRETISED DIFFUSIONS VIA REGRESSION

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Abstract. In this paper we present a novel approach towards variance reduction for discretised diffusion processes. The proposed approach involves specially constructed control variates and allows for a significant reduction in the variance for the terminal functionals. In this way the complexity order of the standard Monte Carlo algorithm ($\varepsilon^{-3}$ in the case of a first order scheme and $\varepsilon^{-2.5}$ in the case of a second order scheme) can be reduced down to $\varepsilon^{-2+\delta}$ for any $\delta \in [0, 0.25)$ with $\varepsilon$ being the precision to be achieved. These theoretical results are illustrated by several numerical examples.

Keywords. Control variates; Monte Carlo methods; regression methods; stochastic differential equations; weak schemes.

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1. Introduction

Let $T > 0$ be a fixed time horizon. Consider a $d$-dimensional diffusion process $(X_t)_{t \in [0,T]}$ defined by the Itô stochastic differential equation

\[ dx_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = x_0 \in \mathbb{R}^d, \]

for Lipschitz continuous functions $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, where $(W_t)_{t \in [0,T]}$ is a standard $m$-dimensional Brownian motion. Recall that, since $\mu$ and $\sigma$ are Lipschitz, the stochastic differential equation (1.1) has a strong solution, and pathwise uniqueness holds. Suppose we want to find a continuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, which has a continuous first derivative with respect to the time argument and continuous first and second derivatives with respect to the components of the space argument on $[0,T] \times \mathbb{R}^d$ such that it solves the partial differential equation

\[ \frac{\partial u}{\partial t} + Lu = 0 \quad \text{on } [0,T) \times \mathbb{R}^d, \]

\[ u(T,x) = f(x) \quad \text{for } x \in \mathbb{R}^d, \]

where $f$ is a given continuous function on $\mathbb{R}^d$. Here and in what follows, $t$ denotes the time argument, $x$ denotes the space argument of $u$, and $L$ is the differential operator associated with the equation (1.1):

\[ (Lu)(t,x) := \sum_{i=1}^d \mu_i(x) \frac{\partial u}{\partial x_i}(t,x) + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^\top)_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t,x), \]

where $\sigma^\top$ denotes the transpose of $\sigma$, and the components of $\mu$ and $\sigma \sigma^\top$ (and later the ones of $\sigma$) are denoted by superscripts. Under appropriate conditions on $\mu$, $\sigma$ and $f$, there is a solution of the Cauchy problem (1.2)–(1.3), which is unique in the class of solutions satisfying certain growth conditions, and it has the following Feynman-Kac stochastic representation

\[ u(t,x) = \mathbb{E}[f(X_T^{t,x})] \]

(see Section 5.7 in [4]), where $X_T^{t,x}$ denotes the solution started at time $t$ in point $x$. Moreover it holds (see e.g. Newton [12])

\[ \mathbb{E}[f(X_T^{0,x}) | X_t^{0,x}] = u(t, X_t^{0,x}), \quad \text{a.s.} \]

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for \( t \in [0, T] \) and
\[
(1.4) \quad f(X_{T}^{0,x}) = \mathbb{E}[f(X_{T}^{0,x})] + M_{T}^{x}, \quad \text{a.s.}
\]
with
\[
(1.5) \quad M_{T}^{x} := \int_{0}^{T} \nabla_{x} u(t, X_{t}^{0,x}) \sigma(X_{t}^{0,x}) \, dW_{t} = \int_{0}^{T} \sum_{i=1}^{d} \frac{\partial u}{\partial x_{i}}(t, X_{t}^{0,x}) \sum_{j=1}^{m} \sigma^{ij}(X_{t}^{0,x}) \, dW_{t}^{j}.
\]

The standard Monte Carlo (SMC) approach for computing \( u(0, x) \) at a fixed point \( x \in \mathbb{R}^{d} \) consists of three steps. First an approximation \( \overline{X}_{T} \) for \( X_{T}^{0,x} \) is constructed via a time discretisation of the equation \([1.1]\) (we refer to \([5]\) for a nice overview of various discretisation schemes). Next \( N_{0} \) independent copies of the approximation \( \overline{X}_{T} \) are generated and finally a Monte Carlo estimate \( V_{N_{0}} \) is defined as an average of the values of \( f \) at simulated points:
\[
V_{N_{0}} := \frac{1}{N_{0}} \sum_{i=1}^{N_{0}} f(\overline{X}_{T}^{(i)}).
\]

In the computation of \( u(0, x) = \mathbb{E}[f(X_{T}^{0,x})] \) by the SMC approach there are two types of error inherent: a discretisation error \( \mathbb{E}[f(X_{T}^{0,x})] - \mathbb{E}[f(\overline{X}_{T})] \) and a Monte Carlo (statistical) error, which results from the substitution of \( \mathbb{E}[f(\overline{X}_{T})] \) with the sample average \( V_{N_{0}} \). The aim of variance reduction methods is to reduce the statistical error. For example, in the so-called control variate variance reduction approach one looks for a random variable \( \xi \) with \( \mathbb{E}\xi = 0 \) such that the variance of the difference \( f(\overline{X}_{T}) - \xi \) is minimised, i.e.,
\[
\text{Var}[f(\overline{X}_{T}) - \xi] \to \min \text{ under } \mathbb{E}\xi = 0.
\]

The use of control variates for solving \([1.1]\) via Monte Carlo path simulation approach was initiated by Newton \([12]\) and further developed in Milstein and Tretyakov \([10]\). In fact, the construction of the appropriate control variates in the above two papers essentially relies on the identity \([1.4]\) implying that the zero-mean random variable \( M_{T}^{x} \) can be viewed as an optimal control variate, since
\[
\text{Var}[f(X_{T}^{0,x}) - M_{T}^{x}] = \text{Var}[\mathbb{E}[f(X_{T}^{0,x})]] = 0.
\]

Let us note that it would be desirable to have a control variate reducing the variance of \( f(\overline{X}_{T}) \) rather than the one of \( f(X_{T}^{0,x}) \) because we simulate from the distribution of \( f(\overline{X}_{T}) \) and not from the one of \( f(X_{T}^{0,x}) \). Moreover, the control variate \( M_{T}^{x} \) cannot be directly computed, since the function \( u(t, x) \) is unknown. This is why Milstein and Tretyakov \([10]\) proposed to use regression for getting a preliminary approximation for \( u(t, x) \) in a first step.

The contribution of our work is threefold. First, we propose an approach for the construction of control variates which reduce the variance of \( f(\overline{X}_{T}) \). As a by-product our control variates can be computed in a rather simple way. More importantly, we are able to achieve a higher order convergence of the resulting variance to zero, which in turn leads to a significant complexity reduction as compared to the SMC algorithm. Other prominent examples of Monte Carlo algorithms with this property are multilevel Monte Carlo (MLMC) algorithm of \([2]\), a quadrature-based algorithm of \([11]\), and a variance reduction algorithm of \([16]\) based on importance sampling. Our approach becomes especially simple in the case of the so-called weak approximation schemes, i.e. the schemes, where simple random variables are used in place of Brownian increments. In recent years weak approximation schemes became quite popular. The weak Euler scheme is a first order scheme with weak order of convergence \( \alpha = 1 \), and has been studied by many researchers. Milstein \([8]\) showed the first order convergence of the weak Euler scheme. The Itô-Taylor (weak Taylor) high-order scheme is a natural extension of the weak Euler scheme. In the diffusion case, some new discretization schemes (also called Kusuoka type schemes) which are of order \( \alpha \geq 2 \) without the Romberg extrapolation have been introduced by Kusuoka \([6]\), Lyons and Victoir \([17]\), Ninomiya and Victoir \([14]\), and Ninomiya and Ninomiya \([13]\). A general class of weak approximation methods, comprising many well-known discretisation schemes, was constructed in Tanaka and Kohatsu-Higa \([17]\). The main advantage of the weak approximation schemes is that simple discrete random variables can be used to approximate multiple Wiener integrals arising in higher order schemes.
Summing up, we propose a new regression-type approach for the construction of higher order control variates. It takes advantage of the smoothness in $\mu, \sigma$ and $f$ (which is needed for nice convergence properties of regression methods) in order to significantly reduce the variance of the random variable $f(X_T)$.

The paper is organised as follows. Section 2 contains a discrete-time analogue of the Clark-Ocone formula for schemes with Gaussian innovations, which provides the basis for constructing control variates via regression methods. The corresponding formulas for weak approximation schemes are discussed in Section 3 where the schemes of first and second order are analysed in detail. Section 4 describes a generic regression algorithm for the construction of control variates. Error bounds in the generic algorithm depend on a particular implementation, i.e. on the choice of basis functions for regression. For the specific choice of piecewise polynomial regression, the error bound is presented in Section 5 and the complexity analysis of the algorithm in Section 6. Section 7 is devoted to a simulation study. Finally, all proofs are collected in Section 8.

2. Control variates for schemes with Gaussian increments

To begin with, we introduce some notations, which will be frequently used in the sequel. Throughout the paper $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $J \in \mathbb{N}$ denotes the time discretisation parameter, we set $\Delta := T/J$ and consider discretisation schemes defined on the grid $\{j\Delta : j = 0, \ldots, J\}$. We recall that $X$ in (1.1) is $d$-dimensional and $W$ in (1.1) is $m$-dimensional for some fixed $d, m \in \mathbb{N}$. For $j \in \{1, \ldots, J\}$, we define $\Delta_jW := W_{j\Delta} - W_{(j-1)\Delta}$, and by $W^i$ we denote the $i$-th component of the vector $W$. Finally, for $k \in \mathbb{N}_0$, $H_k : \mathbb{R} \to \mathbb{R}$ stands for the (normalised) $k$-th Hermite polynomial, i.e.

$$H_k(x) := \frac{(-1)^k}{\sqrt{k!}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$ 

Notice that $H_0 \equiv 1$. To motivate a general construction of optimal control variates, let us first look at an example.

2.1. Motivating example. Consider a simple one-dimensional SDE

$$dX_t = \sigma X_t dW_t, \quad t \in [0, T],$$

with $X_0 = x_0$, and its Euler discretisation $(X_{\Delta,j})_{j=0,\ldots,J}$, where $X_{\Delta,0} = x_0$ and

$$X_{\Delta,j} = X_{\Delta,(j-1)}(1 + \sigma \Delta_jW), \quad j = 1, \ldots, J.$$ 

Suppose that we would like to approximate the quantity $V := \mathbb{E}[X_T^2]$. It is easy to see that $\mathbb{E}[X_{\Delta,j\Delta}^2] = x_0^2 (1 + \sigma^2 \Delta)^j$ and using a telescopic sum trick, we derive

$$X^2_{\Delta,j\Delta} - \mathbb{E}[X^2_{\Delta,j\Delta}] = \sum_{j=1}^{J} \left( X^2_{\Delta,j\Delta}(1 + \sigma^2 \Delta)^{j-j} - X^2_{\Delta,(j-1)\Delta}(1 + \sigma^2 \Delta)^{j-j+1} \right).$$

Since $\Delta_jW = \frac{X_{\Delta,j\Delta} - X_{\Delta,(j-1)\Delta}}{\sigma X_{\Delta,(j-1)\Delta}}$, we get

$$X^2_{\Delta,j\Delta} - X^2_{\Delta,(j-1)\Delta}(1 + \Delta \sigma^2) = 2\sigma X^2_{\Delta,(j-1)\Delta} \Delta_jW + \sigma^2 X^2_{\Delta,(j-1)\Delta} (\Delta_jW^2 - \Delta).$$

As a result

$$(2.1) \quad X^2_{\Delta,j\Delta} - \mathbb{E}[X^2_{\Delta,j\Delta}] = \sum_{j=1}^{J} \left( a_{j,1}(X_{\Delta,(j-1)\Delta})H_1 \left( \frac{\Delta_jW}{\sqrt{\Delta}} \right) + a_{j,2}(X_{\Delta,(j-1)\Delta})H_2 \left( \frac{\Delta_jW}{\sqrt{\Delta}} \right) \right)$$

with $a_{j,1}(y) = 2\sigma \sqrt{\Delta} y^2 (1 + \sigma^2 \Delta)^{j-j}$ and $a_{j,2}(y) = \sqrt{2} \sigma^2 \Delta y^2 (1 + \sigma^2 \Delta)^{j-j}$. Notice that representation (2.1) has a very simple form. Furthermore, the coefficients $a_{j,1}$ and $a_{j,2}$ can be represented as conditional expectations

$$a_{j,k}(X_{\Delta,(j-1)\Delta}) = \mathbb{E} \left[ X^2_{\Delta,j\Delta} H_k \left( \frac{\Delta_jW}{\sqrt{\Delta}} \right) \bigg| X_{\Delta,(j-1)\Delta} \right], \quad k = 1, 2.$$
Thus, the control variate
\begin{equation}
M_{\Delta,j}\Delta := \sum_{j=1}^{J} \sum_{k=1}^{2} a_{j,k}(X_{\Delta,(j-1)\Delta}) H_k(\Delta_j W/\sqrt{\Delta}),
\end{equation}
is a perfect control variate, as it satisfies \( \text{Var}[X_{\Delta,j}\Delta - M_{\Delta,j}\Delta] = 0 \). The above example encourages us to look for control variates in the form (2.2), where the coefficients \( a_{k,j}(x) \) have the form of conditional expectations, which in turn can be computed by regression methods. As we will see in the next sections, such perfect control variates can be constructed in the general case.

**Discussion.** The control variate in (2.2) is a sum over all time steps. At this point it is, therefore, unclear whether the variance reduction achieved in the proposed method outweighs the additional computational work required to implement such a control variate. After the detailed description of our algorithm we will present the complexity analysis, which shows that, given the precision \( \varepsilon \) to be achieved, implementing such a control variate results in less total computational work, provided several parameters are chosen a proper way.

### 2.2. Control variate construction.

Let us consider a scheme, where \( d \)-dimensional approximations \( X_{\Delta,j}\Delta, j = 0, \ldots, J \), satisfy \( X_{\Delta,0} = x_0 \) and
\begin{equation}
X_{\Delta,j}\Delta = \Phi_j \left( X_{\Delta,(j-1)\Delta}, \sqrt{\Delta_j W'} \right)
\end{equation}
for some Borel measurable functions \( \Phi_j : \mathbb{R}^{d-m} \to \mathbb{R} \) (clearly, the Euler scheme is a special case of this setting).

**Theorem 2.1.** Let \( f : \mathbb{R}^{d} \to \mathbb{R} \) be a Borel measurable function such that \( \mathbb{E}|f(X_{\Delta,T})|^2 < \infty \). Then the following representation holds
\begin{equation}
f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \sum_{k=1}^{n} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{r=1}^{m} H_k \left( \frac{\Delta_j W^{(r)}}{\sqrt{\Delta}} \right),
\end{equation}
where \( k = (k_1, \ldots, k_m) \) and \( 0 = (0, \ldots, 0) \) (in the second summation), and the coefficients \( a_{j,k} : \mathbb{R}^{d} \to \mathbb{R} \) are given by the formula
\begin{equation}
a_{j,k}(x) = \mathbb{E} \left[ f(X_{\Delta,T}) \prod_{r=1}^{m} H_k \left( \frac{\Delta_j W^{(r)}}{\sqrt{\Delta}} \right) \bigg| X_{\Delta,(j-1)\Delta} = x \right],
\end{equation}
for all \( j \in \{1, \ldots, J\} \) and \( k \in \mathbb{N}_0^m \setminus \{0\} \).

**Remark 2.2.** (i) Representation (2.4) can be viewed as a discrete-time analogue of the Clark-Ocone formula. See e.g. [11] (Gaussian increments), [15] (Bernoulli increments) and the references therein for representations of similar types. Our form (2.4) is aimed at constructing control variates via regression methods.

(ii) A comparison of (2.2) and (2.4) gives rise to the question whether our motivating example fits the framework (2.4). The answer is affirmative: a straightforward calculation using the facts that \( f(x) = x^2 \) in the motivating example and that, for \( k \geq 3 \), \( H_k(\Delta_j W/\sqrt{\Delta}) \) is orthogonal to all polynomials of \( \Delta_j W \) of degree two reveals that \( a_{j,k} \equiv 0 \) whenever \( k \geq 3 \) in the situation of our motivating example.

**Discussion.** Representation (2.4) shows that the random variable
\begin{equation}
M_{\Delta,T} := \sum_{j=1}^{J} \sum_{k=1}^{n} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{r=1}^{m} H_k \left( \frac{\Delta_j W^{(r)}}{\sqrt{\Delta}} \right)
\end{equation}
is a perfect control variate for the functional \( f(X_{\Delta,T}) \), i.e. \( \text{Var}[f(X_{\Delta,T}) - M_{\Delta,T}] = 0 \). In order to be able to use this control variate, we need to truncate the summation (over \( k \)) in (2.5) and study the order of the corresponding truncation error. However, as we will see in the next section, we can avoid this problem by using the so-called weak approximation schemes, where the Brownian motion increments in (2.3) are replaced by simple discrete-valued random variables.
3. Schemes with discrete random variables in the increments

In this section we derive the analogue of representation \((2.4)\) for the case of weak approximation schemes. In order to be more concise, we focus on the weak schemes of first and second order.

3.1. First order schemes. In this subsection we treat weak schemes of order 1. Let us consider a scheme, where \(d\)-dimensional approximations \(X_{\Delta,j}\Delta, j = 0, \ldots, J\), satisfy \(X_{\Delta,0} = x_0\) and
\[
X_{\Delta,j}\Delta = \Phi_{\Delta}(X_{\Delta,(j-1)\Delta}\Delta, \xi_j), \quad j = 1, \ldots, J,
\]
for some functions \(\Phi_{\Delta}: \mathbb{R}^{d+m} \to \mathbb{R}^d\), with \(\xi_j = (\xi_{j,1}, \ldots, \xi_{j,m})\), \(j = 1, \ldots, J\), being \(m\)-dimensional iid random vectors with iid coordinates such that
\[
\mathbb{P}(\xi_{k}^i = \pm 1) = \frac{1}{2}, \quad k = 1, \ldots, m.
\]
A particular case is the Euler weak scheme (also called the simplified weak Euler scheme in \([5]\) Section 14.1]) of order 1, which is given by
\[
\Phi_{\Delta}(x, y) = x + \mu(x) \Delta + \sigma(x) y \sqrt{\Delta}.
\]

**Theorem 3.1.** The following representation holds
\[
f(X_{\Delta,T}) = \mathbb{E}f(X_{\Delta,T}) + \sum_{j=1}^{J} \sum_{r=1}^{m} \sum_{1 \leq s_1 < \ldots < s_r \leq m} a_{j,r,s}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{r} \xi_{j,i}^s,
\]
where we use the notation \(s = (s_1, \ldots, s_r)\). Moreover, the coefficients \(a_{j,r,s} : \mathbb{R}^d \to \mathbb{R}\) can be computed by the formula
\[
a_{j,r,s}(x) = \mathbb{E} \left[ f(X_{\Delta,T}) \prod_{i=1}^{r} \xi_{j,i}^s \mid X_{\Delta,(j-1)\Delta} = x \right]
\]
for all \(j, r, s\) as in \((3.3)\).

The next proposition shows the properties of the simplified Euler scheme combined with the control variate
\[
M^{(1)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{r=1}^{m} \sum_{1 \leq s_1 < \ldots < s_r \leq m} a_{j,r,s}(X_{\Delta,(j-1)\Delta}) \prod_{i=1}^{r} \xi_{j,i}^s,
\]
where the coefficients \(a_{j,r,s}(x)\) are given by \((3.4)\). It is a combination of the above Theorem 3.1 together with Theorem 2.1 in \([9]\).

**Proposition 3.2.** Assume that \(\mu\) and \(\sigma\) in \((1.1)\) are Lipschitz continuous with components \(\mu^i, \sigma^{i,r}: \mathbb{R}^d \to \mathbb{R}, \ i = 1, \ldots, d, \ r = 1, \ldots, m\), being 4 times continuously differentiable with their partial derivatives of order up to 4 having polynomial growth. Let \(f : \mathbb{R}^d \to \mathbb{R}\) be 4 times continuously differentiable with partial derivatives of order up to 4 having polynomial growth. Provided that \((3.2)\) holds and that, for sufficiently large \(p \in \mathbb{N}\), the expectations \(\mathbb{E}|X_{\Delta,j}\Delta|^{2p}\) are uniformly bounded in \(J\) and \(j = 0, \ldots, J\), we have for this “simplified weak Euler scheme”
\[
|\mathbb{E}[f(X_{T}) - f(X_{\Delta,T})]| \leq c \Delta,
\]
where the constant \(c\) does not depend on \(\Delta\). Moreover, it holds \(\text{Var} \left[ f(X_{\Delta,T}) - M^{(1)}_{\Delta,T} \right] = 0\).

**Discussion.** In order to use the control variate \(M^{(1)}_{\Delta,T}\) in practice, we need to estimate the unknown coefficients \(a_{j,r,s}\). Thus, practically implementable control variates \(M^{(1)}_{\Delta,T}\) have the form \((3.5)\) with some estimated functions \(\tilde{a}_{j,r,s} : \mathbb{R}^d \to \mathbb{R}\). Notice that they remain valid control variates, i.e. we still have \(\mathbb{E}[\tilde{M}^{(1)}_{\Delta,T}] = 0\), which is due to the martingale transform structure\(^1\) in \((3.5)\).

\(^1\)This phrase means that the discrete-time process \(\tilde{M} = (\tilde{M}_t)_{t=0,\ldots,J}\), where \(\tilde{M}_0 = 0\) and \(\tilde{M}_t\) is defined like the right-hand side of \((3.5)\) but with \(\sum_{j=1}^{t}\) being replaced by \(\sum_{j=1}^{t}\) and \(a_{j,r,s}\) by \(\tilde{a}_{j,r,s}\), is a martingale, which is a straightforward calculation.
3.1.1. Computation of coefficients. Coefficients \( (3.4) \) can be directly computed using various regression algorithms as discussed in Section 3. From a computational point of view it is sometimes advantageous to look for another representation which only involves a regression over one time step (note that in \( (3.4) \) regression should be performed over \( J - j + 1 \) time steps). To this end, we introduce the functions
\[
q_j(x) := \mathbb{E}[f(X_{\Delta,T})|X_{\Delta,j\Delta} = x].
\]
The next proposition contains backward recursion formulas for the functions \( q_j \) as well as the expressions for the coefficients \( (3.4) \) in terms of \( q_j, j = 1, \ldots, J \).

**Proposition 3.3.** We have \( q_j \equiv f \) and for each \( j \in \{1, \ldots, J\} \),
\[
q_{j-1}(x) = \mathbb{E}[q_j(X_{\Delta,j\Delta})|X_{\Delta,(j-1)\Delta} = x] = \frac{1}{2^m} \sum_{y=(y', \ldots, y^m) \in \{-1, 1\}^m} q_j(\Phi \Delta(x, y)).
\]
Moreover, the coefficients \( (3.4) \) can be expressed in terms of the functions \( q_j, j = 1, \ldots, J, \) as
\[
a_{j,r,s}(x) = \frac{1}{2^m} \sum_{y=(y', \ldots, y^m) \in \{-1, 1\}^m} \left[ \prod_{i=1}^r y_i \right] q_j(\Phi \Delta(x, y))
\]
for all \( j, r \) and \( s = (s_1, \ldots, s_r) \) as in \( (3.3) \).

**Discussion.** The advantage of the representation \( (3.8) \) over the original one consists in the fact that all functions \( q_j, j = 1, \ldots, J \), can be recursively computed using regressions over one time step (based on the first equality in \( (3.7) \)) and without involvement of the independent of \( X_{\Delta,(j-1)\Delta} \) centred random variables \( \xi^t \) (cf. \( (3.4) \)), rendering the estimates for \( q_j \) more stable. If \( q_j \) is approximated as a linear combination of \( n \) basis functions, then the cost of computing the coefficients in this linear combination by least squares regression on \( N \) paths is of order \( N \times n^2 \). Once \( q_j \) is approximated, the cost of estimating \( a_{j,r,s}(x) \) in a given point \( x \) via \( (3.8) \) is of order \( 2^m \times (c_1 + c_2 \times n) \), where the constant \( c_1 \) describes the cost of computing \( \Phi \Delta(x, y) \) for given points \( x \) and \( y \) (this is \( d \times m \) in case of \( (3.2) \)), and the constant \( c_2 \) describes the cost of computing the value of a basis function at a point in \( \mathbb{R}^d \) (this is typically \( d \)).

3.2. Second order schemes. Now we treat weak schemes of order 2. We consider a scheme, where \( d \)-dimensional approximations \( X_{\Delta,j\Delta}, j = 0, \ldots, J, \) satisfy \( X_{\Delta,0} = x_0 \) and
\[
X_{\Delta,j\Delta} = \Phi \Delta(X_{\Delta,(j-1)\Delta}, \xi_j, V_j), \quad j = 1, \ldots, J,
\]
for some functions \( \Phi \Delta: \mathbb{R}^{d+m+m \times m} \to \mathbb{R}^d \). Here,
\begin{itemize}
  \item [(S1)] \( \xi_j = (\xi_j^k)_{k=1}^m \) are \( m \)-dimensional random vectors,
  \item [(S2)] \( V_j = (V_j^{kl})_{k,l=1}^m \) are random \( m \times m \)-matrices,
  \item [(S3)] the pairs \( (\xi_j, V_j), j = 1, \ldots, J, \) are i.i.d.,
  \item [(S4)] for each \( j \), the random elements \( \xi_j \) and \( V_j \) are independent,
  \item [(S5)] for each \( j \), the random variables \( \xi_j^k, k = 1, \ldots, m, \) are i.i.d. with
    \[
    \mathbb{P}\left( \xi_j^k = \pm \sqrt{3} \right) = \frac{1}{6}, \quad \mathbb{P}\left( \xi_j^k = 0 \right) = \frac{2}{3},
    \]
  \item [(S6)] for each \( j \), the random variables \( V_j^{kl}, 1 \leq k < l \leq m, \) are i.i.d. with
    \[
    \mathbb{P}\left( V_j^{kl} = \pm 1 \right) = \frac{1}{2},
    \]
  \item [(S7)] \( V_j^{kl} = -V_j^{lk}, 1 \leq k < l \leq m, \) \( j = 1, \ldots, J, \)
  \item [(S8)] \( V_j^{kk} = -1, k = 1, \ldots, m, \) \( j = 1, \ldots, J, \).
\end{itemize}

**Remark 3.4.** In order to obtain an order 2 weak scheme in the multidimensional case, we need to incorporate additional random elements \( V_j \) into the structure of the scheme. This is the reason why we now consider \( (3.9) \) instead of \( (3.1) \). For instance, to get the simplified order 2 weak Taylor scheme of \( \Phi \) Section 14.2 in the multidimensional case, we need to define the functions \( \Phi \Delta(x, y, z), x \in \mathbb{R}^d, y \in \mathbb{R}^m, z \in \mathbb{R}^m, z \) as explained below. First we define the function \( \Sigma: \mathbb{R}^d \to \mathbb{R}^{d \times d} \) by the formula
\[
\Sigma(x) = \sigma(x)\sigma(x)^\top
\]
and recall that the coordinates of vectors and matrices are denoted by superscripts, e.g. \( \Sigma(x) = (\Sigma^k_l(x))_{k,l=1}^d \), \( \Phi_\Delta(x, y, z) = (\Phi^k_\Delta(x, y, z))_{k=1}^d \). Let us introduce the operators \( \mathcal{L}^r, r = 0, \ldots, m \), that act on sufficiently smooth functions \( g : \mathbb{R}^d \to \mathbb{R} \) as follows:

\[
\mathcal{L}^0 g(x) := \sum_{k=1}^d \mu^k(x) \frac{\partial g}{\partial x^k}(x) + \frac{1}{2} \sum_{k,l=1}^d \Sigma^{kl}(x) \frac{\partial^2 g}{\partial x^k \partial x^l}(x),
\]

\[
\mathcal{L}^r g(x) := \sum_{k=1}^d \sigma^{kr}(x) \frac{\partial g}{\partial x^k}(x), \quad r = 1, \ldots, m.
\]

The r-th coordinate \( \Phi^r_\Delta, r = 1, \ldots, d \), in the simplified order 2 weak Taylor scheme of [5, Section 14.2] is now given by the formula

\[
(3.10) \quad \Phi^r_\Delta(x, y, z) = x^r + \sum_{k=1}^m \sigma^{kr}(x) y^k \Delta + \mu^r(x) y^k \Delta^2 + \frac{1}{2} \sum_{k,l=1}^m \mathcal{L}^k \sigma^{rl}(x)(y^k y^l + z^{kl}) \Delta^2 + \frac{1}{2} \mu^r(x) \Delta^2, \tag{3.12}
\]

provided the coefficients \( \mu \) and \( \sigma \) of (1.1) are sufficiently smooth. We will need to work explicitly with (3.10) at some point, but all results in this subsection assume structure (3.9) only.

Let us define the index sets

\[
\mathcal{I}_1 = \{1, \ldots, m\}, \quad \mathcal{I}_2 = \{(k, l) : k < l\}
\]

and the system

\[
\mathcal{A} = \{(U_1, U_2) \in \mathcal{P}(\mathcal{I}_1) \times \mathcal{P}(\mathcal{I}_2) : U_1 \cup U_2 \neq \emptyset\},
\]

where \( \mathcal{P}(\mathcal{I}) \) denotes the set of all subsets of a set \( \mathcal{I} \). For any \( U_1 \subseteq \mathcal{I}_1 \) and \( o \in \{1, 2\}^{U_1} \), we write \( o \) as \( o = (o_r)_{r \in U_1} \). Below we use the convention that a product over the empty set is always one.

**Theorem 3.5.** It holds

\[
(3.11) \quad f(X_{\Delta, T}) = \mathbb{E} f(X_{\Delta, T}) + \sum_{j=1}^J \sum_{(U_1, U_2) \in \mathcal{A}} \sum_{o \in \{1, 2\}^{U_1}} \sum_{r \in U_1} a_{j, o, U_1, U_2} f(X_{\Delta, (j-1)\Delta}) \prod_{r \in U_1} H_{o_r}(\xi^j_r) \prod_{(k, l) \in U_2} V_{j}^{kl},
\]

where the coefficients \( a_{j, o, U_1, U_2} : \mathbb{R}^d \to \mathbb{R} \) can be computed by the formula

\[
(3.12) \quad a_{j, o, U_1, U_2}(x) = \mathbb{E} \left[ f(X_{\Delta, T}) \prod_{r \in U_1} H_{o_r}(\xi^j_r) \prod_{(k, l) \in U_2} V_{j}^{kl} \middle| X_{\Delta, (j-1)\Delta} = x \right].
\]

Combining Theorem 3.5 with Theorem 2.1 in [9] we obtain the following result, which provides a bound for the discretisation error and a perfect control variate for the discretised quantity.

**Proposition 3.6.** Assume, that \( \mu \) and \( \sigma \) in (1.1) are Lipschitz continuous with components \( \mu^i, \sigma^{i,r} : \mathbb{R}^d \to \mathbb{R}, \quad i = 1, \ldots, d, \quad r = 1, \ldots, m \), being 6 times continuously differentiable with their partial derivatives of order up to 6 having polynomial growth. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be 6 times continuously differentiable with partial derivatives of order up to 6 having polynomial growth. Provided that (3.10) holds and that, for sufficiently large \( p \in \mathbb{N} \), the expectations \( \mathbb{E} |X_{\Delta, j}\Delta^{2p}| \) are uniformly bounded in \( J \) and \( j = 0, \ldots, J \), we have for this “simplified second order weak Taylor scheme”

\[
\|\mathbb{E} [f(X_T) - f(X_{\Delta, T})]\| \leq c \Delta^2,
\]
where the constant $c$ does not depend on $\Delta$. Moreover, we have $\text{Var} \left[f(X_{\Delta,T}) - M^{(2)}_{\Delta,T}\right] = 0$ for the control variate

\begin{equation}
M^{(2)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(U_1,U_2) \in \mathcal{A} \cap [1,2]^U_1} a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \prod_{r \in U_1} H_{o_r}(\xi^r_j) \prod_{(k,l) \in U_2} V^{kl}_j,
\end{equation}

where the coefficients $a_{j,o,U_1,U_2}(x)$ are defined in (3.12).

3.2.1. Computation of coefficients. Similarly to the case of first order schemes, one can derive an alternative representation for the coefficients (3.12) making their computation more stable. The next result contains backward recursions for the functions $q_j$ of (3.6) and for $a_{j,o,U_1,U_2}$ of (3.12).

**Proposition 3.7.** We have $q_{j} \equiv f$ and, for each $j \in \{1,\ldots,J\},$

\begin{equation}
q_{j-1}(x) = \mathbb{E}[q_j(X_{\Delta,j\Delta})|X_{\Delta,(j-1)\Delta} = x] = \frac{1}{2^{m(m-1)}} \sum_{(y^1,\ldots,y^m) \in \{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{(z^uv) \in \{-1,0,1\}^{m(m-1)}} 4\sum_{i=1}^{m} I(y^{(i)} = 0) q_j(\Phi_\Delta(x,y,z)),
\end{equation}

and, for all $j \in \{1,\ldots,J\}, (U_1,U_2) \in \mathcal{A}$ and $o \in \{1,2\}^{U_1}$, it holds

\begin{equation}
a_{j,o,U_1,U_2}(x) = \frac{1}{2^{m(m-1)}} \sum_{(y^1,\ldots,y^m) \in \{-\sqrt{3},0,\sqrt{3}\}^m} \sum_{(z^uv) \in \{-1,0,1\}^{m(m-1)}} 4\sum_{i=1}^{m} I(y^{(i)} = 0) \prod_{r \in U_1} H_{o_r}(y^r) \prod_{(k,l) \in U_2} z^{kl} q_j(\Phi_\Delta(x,y,z)),
\end{equation}

where $y = (y^1,\ldots,y^m)$ and $z = (z^uv)$ is the $m \times m$-matrix with $z^uv = -z^{uv}, u < v$, $z^{uu} = -1$.

4. Generic regression algorithm

In the previous sections we have given several representations for perfect control variates. Now we discuss how to compute the coefficients in these representations via regression. For the sake of clarity, we focus on second order schemes and representation (3.11) with coefficients given by (3.12).

4.1. Monte Carlo regression. Fix a $n$-dimensional vector of real-valued functions $\psi = (\psi^1,\ldots,\psi^n)$ on $\mathbb{R}^d$. Simulate a big number of independent “training paths” of the discretised diffusion $X_{\Delta,j\Delta}$, $j = 0,\ldots,J$. In what follows these $N$ training paths are denoted by $D^i_N$:

\begin{equation}
D^i_N := \left\{(X^{tr,(i)}_{\Delta,j\Delta})_{j=0,\ldots,J} : i = 1,\ldots,N\right\}.
\end{equation}

Let $\alpha_{j,o,U_1,U_2} = (\alpha_{j,o,U_1,U_2}^1,\ldots,\alpha_{j,o,U_1,U_2}^n)$, where $j \in \{1,\ldots,J\}$, $(U_1,U_2) \in \mathcal{A}$, $o \in \{1,2\}^{U_1}$, be a solution of the following least squares optimisation problem:

\begin{equation}
\arg \min_{\alpha \in \mathbb{R}^n} \sum_{i=1}^{N} \left[ a_{j,o,U_1,U_2}(X^{tr,(i)}_{\Delta,j\Delta}) - \alpha^1 \psi^1(X^{tr,(i)}_{\Delta,(j-1)\Delta}) - \cdots - \alpha^n \psi^n(X^{tr,(i)}_{\Delta,(j-1)\Delta})\right]^2
\end{equation}

with

\begin{equation}
a_{j,o,U_1,U_2}(x) := f(X^{tr,(i)}_{\Delta,j\Delta}) \prod_{r \in U_1} H_{o_r}(\xi^{tr,(i)}_r) \prod_{(k,l) \in U_2} (V^{tr,(i)}_j)^{kl}.
\end{equation}

Define an estimate for the coefficient function $a_{j,o,U_1,U_2}$ via

\begin{equation}
\hat{a}_{j,o,U_1,U_2}(x) := \hat{a}_{j,o,U_1,U_2}(x, D^i_N) := \alpha_{j,o,U_1,U_2}^1 \psi^1(x) + \cdots + \alpha_{j,o,U_1,U_2}^n \psi^n(x), \quad x \in \mathbb{R}^d.
\end{equation}

The intermediate expression $\hat{a}_{j,o,U_1,U_2}(x, D^i_N)$ in the above formula emphasises that the estimates $\hat{a}_{j,o,U_1,U_2}$ of the functions $a_{j,o,U_1,U_2}$ are random in that they depend on the simulated training paths.

\footnote{In the complexity analysis below we show how large $N$ is required to be in order to provide an estimate within some given tolerance.}
The cost of computing $\alpha_{j,o,U_1,U_2}$ is of order $O(Nn^2)$, since each $\alpha_{j,o,U_1,U_2}$ is of the form $\alpha_{j,o,U_1,U_2} = B^{-1}b$ with
\begin{equation}
B_{k,l} := \frac{1}{N} \sum_{i=1}^{N} \psi^k(X_{\Delta,j-1}\Delta) \psi^l(X_{\Delta,j-1}\Delta)
\end{equation}
and
\begin{equation}
b_k := \frac{1}{N} \sum_{i=1}^{N} \psi^k(X_{\Delta,j-1}\Delta) \Sigma_{j,o,U_1,U_2},
\end{equation}
k, $l \in \{1, \ldots, n\}$. The cost of approximating the family of the coefficient functions $a_{j,o,U_1,U_2}$, $j \in \{1, \ldots, J\}$, $(U_1, U_2) \in \mathcal{A}$, $o \in \{1, 2\}^{|U_1|}$, is of order $O(J(3n^{2} - 1)n^2)$.

4.2. Summary of the algorithm. The algorithm consists of two phases: training phase and testing phase. In the training phase, we simulate $N$ independent training paths $D_{N}^{r}$ and construct regression estimates $\hat{\alpha}_{j,o,U_1,U_2}(\cdot, D_{N}^{r})$ for the coefficients $a_{j,o,U_1,U_2}(\cdot)$. In the testing phase, independently from $D_{N}^{r}$ we simulate $N_0$ independent testing paths $(X_{\Delta,j}^{(i)})_{j=0,\ldots,J}$, $i = 1, \ldots, N_0$, and build the Monte Carlo estimator for $\mathbb{E}[f(X_T)]$ as
\begin{equation}
\mathcal{E} = \frac{1}{N_0} \sum_{i=1}^{N_0} \left( f(X_{\Delta,T}^{(i)}) - \hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} \right),
\end{equation}
where
\begin{equation}
\hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(U_1, U_2) \in \mathcal{A}} \sum_{o \in \{1, 2\}^{|U_1|}} \hat{\alpha}_{j,o,U_1,U_2}(X_{\Delta,j-1}\Delta) \alpha_{j,o,U_1,U_2} \prod_{r \in U_1} H_{\alpha_{j,o,U_1,U_2}}(\xi_{j,o,U_1,U_2}^{(r)}) \prod_{(k,l) \in U_2} V_{j,k,l}^{(i)}
\end{equation}
(cf. with (3.13)). Due to the martingale transform structure in (4.3) (recall footnote 1 on page 5), we have $\mathbb{E}[\hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} | D_{N}^{r}] = 0$, hence $\mathbb{E}[\mathcal{E} | D_{N}^{r}] = \mathbb{E}[f(X_{\Delta,T}^{(i)}) - \hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} | D_{N}^{r}] = \mathbb{E}[f(X_{\Delta,T}^{(i)})]$, and we obtain
\begin{align*}
\text{Var}[\mathcal{E}] &= \mathbb{E}[\text{Var}(\mathcal{E} | D_{N}^{r})] + \mathbb{E}[\text{Var}(\mathcal{E} | D_{N}^{r})] = \mathbb{E}[\text{Var}(\mathcal{E} | D_{N}^{r})] \\
&= \frac{1}{N_0} \mathbb{E} \left[ \text{Var} \left( f(X_{\Delta,T}^{(i)}) - \hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} | D_{N}^{r} \right) \right] = \frac{1}{N_0} \text{Var} \left[ f(X_{\Delta,T}^{(i)}) - \hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} \right].
\end{align*}
Summarising, we have
\begin{equation}
\mathbb{E}[\mathcal{E}] = \mathbb{E}[f(X_{\Delta,T})],
\end{equation}
\begin{equation}
\text{Var}[\mathcal{E}] = \frac{1}{N_0} \text{Var} \left[ f(X_{\Delta,T}^{(i)}) - \hat{\mathcal{M}}^{(2),(i)}_{\Delta,T} \right].
\end{equation}
Notice that the result of (4.5) indeed requires the computations above and cannot be stated right from the outset because the summands in (4.2) are dependent (through $D_{N}^{r}$).

This concludes the description of the generic regression algorithm for constructing the control variate. Further details, such as bounds for the right-hand side of (4.5), depend on a particular implementation, i.e. on the quality of the chosen basis functions. In what follows, we perform a detailed analysis for the specific choice of the basis functions, which leads to the so-called piecewise polynomial partitioning estimates.

5. Error bounds for piecewise polynomial regression

We fix some $p \in \mathbb{N}$, which will denote the maximal degree of polynomials involved in our basis functions. The piecewise polynomial partitioning estimate of $a_{j,o,U_1,U_2}$ works as follows: consider some $R > 0$ and an equidistant partition of $[-R, R]^d$ in $Q^d$ cubes $K^1, \ldots, K^{Q^d}$. Further, consider the basis functions $\psi^{k,1}, \ldots, \psi^{k,n}$ with $k \in \{1, \ldots, Q^d\}$ and $n = \binom{p+d}{d}$ such that $\psi^{k,1}(x), \ldots, \psi^{k,n}(x)$ are polynomials with degree less than or equal to $p$ for $x \in K^k$ and $\psi^{k,1}(x) = \ldots = \psi^{k,n}(x) = 0$ for $x \notin K^k$. Then we obtain the least squares regression estimate $\hat{a}_{j,o,U_1,U_2}(x)$ for $x \in \mathbb{R}^d$ as described in Section 4 based on $Q^d = O(Q^d p^d)$ basis functions. In particular, we have $\hat{a}_{j,o,U_1,U_2}(x) = 0$ for any $x \notin [-R, R]^d$. We note that the cost of computing $\hat{a}_{j,o,U_1,U_2}$ for all $j, o, U_1, U_2$ is of order $O(JNQ^d p^{2d})$.
rather than $O(JNQ^{2d}p^{2d})$ due to a block diagonal matrix structure of $B$ in (4.1). An equivalent approach, which leads to the same estimator $\hat{a}_{j,o,U_1,U_2}(x)$, is to perform separate regressions for each cube $K^1, \ldots, K^{Q^d}$. Here, the number of basis functions at each regression is of order $O(p^d)$ so that the overall cost is of order $O(JNQ^dp^{2d})$, too. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $h \in [1, \infty)$, we will use the notations

$$|x|_h := \left( \sum_{i=1}^{d} |x_i|^h \right)^{1/h}, \quad |x|_{\infty} := \max_{i=1,\ldots,d} |x_i|.$$ 

For $s \in \mathbb{N}_0$, $C > 0$ and $h \in [1, \infty]$, we say that a function $f: \mathbb{R}^d \to \mathbb{R}$ is $(s+1,C)$-smooth w.r.t. the norm $| \cdot |_h$ whenever, for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with $\sum_{i=1}^{d} \alpha_i = s$, we have

$$|\partial^{\alpha} f(x) - \partial^{\alpha} f(y)| \leq C|x-y|_h, \quad x,y \in \mathbb{R}^d,$$

i.e. the function $\partial^{\alpha} f$ is globally Lipschitz with the Lipschitz constant $C$ with respect to the norm $| \cdot |_h$ on $\mathbb{R}^d$ (cf. Definition 3.3 in [3]). In what follows, we use the notation $P_{\Delta,j-1}$ for the distribution of $X_{\Delta,(j-1)\Delta}$. In particular, we will work with the corresponding $L^2$-norm:

$$\|g\|_{L^2(P_{\Delta,j-1})} := \int_{\mathbb{R}^d} g^2(x) P_{\Delta,j-1}(dx) = \mathbb{E} \left[ g^2(X_{\Delta,(j-1)\Delta}) \right].$$

Let us now fix some $j \in \{1, \ldots, J\}$, $(U_1, U_2) \in \mathcal{A}$, $o \in \{1,2\}^{U_1}$, set

$$\zeta_{j,o,U_1,U_2} := f(X_{\Delta,T}) \prod_{r \in U_1} H_{\nu} (\xi_r^j) \prod_{(k,l) \in U_2} V_{j,l}^{kl}$$

and remark that $a_{j,o,U_1,U_2}(x) = \mathbb{E}[\zeta_{j,o,U_1,U_2} X_{\Delta,(j-1)\Delta} = x]$. We assume that for some constant $h \in [1, \infty]$ and some positive constants $\Sigma, A, C_h, \nu, B_\nu$, it holds:

- $(A1)$ $\sup_{x \in \mathbb{R}^d} \text{Var}[\zeta_{j,o,U_1,U_2} X_{\Delta,(j-1)\Delta} = x] \leq \Sigma^2 < \infty$,
- $(A2)$ $\sup_{x \in \mathbb{R}^d} |a_{j,o,U_1,U_2}(x)| \leq A \Delta^{1/2} < \infty$,
- $(A3)$ $a_{j,o,U_1,U_2}$ can be extended to $\mathbb{R}^d$ in a $(p+1,C_h)$-smooth way w.r.t. the norm $| \cdot |_h$,
- $(A4)$ $\mathbb{P}(|X_{\Delta,(j-1)\Delta}|_{\infty} > R) \leq B_\nu R^{-\nu}$ for all $R > 0$.

**Remark 5.1.** Due to representation (3.15), the smoothness of the coefficients $a_{j,o,U_1,U_2}$ is related to the smoothness of the one step conditional distribution of $X_{\Delta,j\Delta}$, given $X_{\Delta,(j-1)\Delta} = x$, for any $j = 1, \ldots, J$ (recall the first equality in (3.14)), and to the smoothness in $x$ of the mapping $\Phi_\Delta$ from (3.9). In the case when the mapping $\Phi_\Delta$ is given by (3.10), its smoothness in $x$ is related to the smoothness of the coefficients $\mu$ and $\sigma$. Let us also notice that it is only a matter of convenience which $h$ to choose in (A3) because all norms $| \cdot |_h$ are equivalent.

Let $\hat{a}_{j,o,U_1,U_2}$ be the piecewise polynomial partitioning estimate of $a_{j,o,U_1,U_2}$ described in the beginning of this section. By $\tilde{a}_{j,o,U_1,U_2}$ we denote the truncated estimate, which is defined as follows:

$$(5.1) \quad \tilde{a}_{j,o,U_1,U_2}(x) := T_{A \Delta^{1/2}} \hat{a}_{j,o,U_1,U_2}(x) := \begin{cases} a_{j,o,U_1,U_2}(x) & \text{if } |\hat{a}_{j,o,U_1,U_2}(x)| \leq A \Delta^{1/2}, \\ A \Delta^{1/2} \text{sgn} \hat{a}_{j,o,U_1,U_2}(x) & \text{otherwise}. \end{cases}$$

We again emphasise that, in fact, $\hat{a}_{j,o,U_1,U_2}(x) = \tilde{a}_{j,o,U_1,U_2}(x, D_{\hat{N}}^r)$, that is, the estimates $\hat{a}_{j,o,U_1,U_2}$ of the functions $\tilde{a}_{j,o,U_1,U_2}$ depend on the simulated training paths.

**Theorem 5.2.** Under $(A1)$–$(A4)$, we have

$$(5.2) \quad \mathbb{E} \| \tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2} \|_{L^2(P_{\Delta,j-1})}^2 \leq \tilde{c} \left( \Sigma^2 + A^2 \Delta (\log N + 1) \right) \frac{(p+d)Q^d}{N} + \frac{8 C_h^2}{(p+1)!^2 d^{2-2/n}} \left( \frac{Rd}{Q} \right)^{2p+2} + 8A^2 \Delta B_\nu R^{-\nu},$$

where $\tilde{c}$ is a universal constant.
It is worth noting that the expectation in the left-hand side of (5.2) accounts for the averaging over the randomness in \( D^N_N \). To explain this in more detail, let \((X_{\Delta,j})_{j=0,\ldots,J}\) be a “testing path" which is independent of the training paths \( D^N_t \). Then it holds

\[
\| \tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2} \|^2_{L^2(P_{\Delta,j-1})} = \mathbb{E} \left[ \left( \tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, D^N_N) - a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \right)^2 \right],
\]

hence,

\[
(5.3) \quad \mathbb{E} \| \tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2} \|^2_{L^2(P_{\Delta,j-1})} = \mathbb{E} \left[ \left( \tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, D^N_N) - a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) \right)^2 \right],
\]

which provides an alternative form for the expression in the left-hand side of (5.2).

We now estimate the variance of the random variable \( f(X_{\Delta,T}) - \tilde{M}^{(2)}_{\Delta,T} \), where

\[
(5.4) \quad \tilde{M}^{(2)}_{\Delta,T} := \sum_{j=1}^{J} \sum_{(U_1,U_2) \in A} \sum_{o \in \{1,2\}^{U_1}} \tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, D^N_N) \prod_{r \in U_1} H_{o_r}(\xi^r_{j,o,U}) \prod_{(k,l) \in U_2} V_{kl}^{j}. \]

Using the martingale transform structure in (5.4) and (3.13) (recall footnote 1 on page 5) together with the orthonormality (in \( L^2 \)) of the system \( \prod_{r \in U_1} H_{o_r}(\xi^r_{j,o,U}) \prod_{(k,l) \in U_2} V_{kl}^{j} \), we get by Theorem 5.2

\[
(5.5) \quad \text{Var}[f(X_{\Delta,T}) - \tilde{M}^{(2)}_{\Delta,T}] = \text{Var}[M^{(2)}_{\Delta,T} - \tilde{M}^{(2)}_{\Delta,T}]
\]

\[
= \sum_{j=1}^{J} \sum_{(U_1,U_2) \in A} \sum_{o \in \{1,2\}^{U_1}} \mathbb{E} \| \tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2} \|^2_{L^2(P_{\Delta,j-1})}
\]

\[
\leq J \left( 3^m - \frac{\delta(m-1)}{2} - 1 \right) \left\{ \tilde{c} \left( \Sigma^2 + A^2 \Delta (\log N + 1) \right) \left( \frac{p+d}{D} \right) \frac{Q^d}{N} 
\]

\[
+ \frac{8 C_h^2}{(p+1)^2 d^{p-2}/h} \left( \frac{Rd}{Q} \right)^{2p+2} + 8A^2 \Delta B_p R^{-\nu} \right\}.
\]

In the case of piecewise polynomial regression, the estimator \( \mathcal{E} \) given in (4.2) with “hat" replaced by “tilde" is an unbiased estimator of \( \mathbb{E}[f(X_{\Delta,T})] \), and, by (4.5), the upper bound for its variance is \( \frac{1}{N_0} \) times the last expression in (5.3).

6. Complexity analysis for piecewise polynomial regression

Below we present a complexity analysis, which explains how we can go beyond the complexity order \( \varepsilon^{-2} \) with \( \varepsilon \) being the precision to be achieved\footnote{Notice that the multilevel Monte Carlo (MLMC) algorithm can at best achieve the complexity of order \( \varepsilon^{-2} \).}.

We will consider two variants of the Monte Carlo approach with regression-based control variate. The first algorithm, which is abbreviated below as RCV approach (“RCV” stands for “Regression-based Control Variate”), is the algorithm described in detail in Section 4. Here the estimates \( \tilde{a}_{j,o,U_1,U_2} \) needed in (5.4) are constructed via regressions based on (3.12). In the second algorithm, which we call recursive RCV (RCV) approach, we construct in the training phase regression-based estimates \( \tilde{q}_j \) of the functions \( q_j \) backwards in time via regressions based on the first equality in (3.14). Given the approximations \( \tilde{q}_j(\cdot, D^N_N) \) of the functions \( q_j(\cdot) \), we construct in the testing phase the approximations of the values \( \tilde{a}_{j,o,U_1,U_2}(X^{(i)}_{\Delta,(j-1)\Delta}, D^N_N) \) on the testing paths via (3.15) with \( q_j(\cdot) \) replaced by \( \tilde{q}_j(\cdot, D^N_N) \).

Then, again, the values of the control variate on the testing paths are computed via (5.4), and the Monte Carlo estimator for \( \mathbb{E}[f(X_T)] \) is computed as in (4.2).
6.1. Complexity analysis of the RCV approach. The overall cost of the algorithm (training and testing phase) is of order

\[ C \asymp JQ^d \max \{ N, N_0 \}, \]

provided that we only track the parameters \( J, N, N_0, Q \) that tend to infinity when \( \epsilon \downarrow 0 \). Further, we have the following constraints

\[ \max \left\{ \frac{1}{J^2}, \frac{JQ^d}{N^2}, \frac{J}{N_0} \left( \frac{R}{Q} \right)^{2(p+1)}, \frac{1}{R^2N_0} \right\} \lesssim \epsilon^2, \]

provided that we, in addition to \( J, N, N_0, Q \), track the parameter \( R \), which also tends to infinity when \( \epsilon \downarrow 0 \). Note that the first term in (6.2) comes from the squared bias of the estimator and the remaining three ones come from the variance of the estimator (see (5.5) and (4.2)).

**Theorem 6.1.** We obtain the following parameter values

\[ J \asymp \epsilon^{-1/2}, \quad Q \asymp \epsilon^{\nu - \frac{5(p+1)}{2d+4(p+1)}}, \quad R \asymp \epsilon^{\nu - \frac{6(p+1)-d}{2d+4(p+1)}} \]

provided that \( p > \frac{d-2}{2} \) and \( \nu > \frac{2d(p+1)}{2(p+1)-2} \). As a result the complexity order is given by

\[ C_{RCV} \asymp JQ^d N \asymp JQ^d N_0 \asymp \epsilon^{-\frac{11d+2(p+1)(7\nu+8d)}{2d+4(p+1)(2p+d)}}. \]

6.2. Complexity of the RRCV approach. In the training phase, the cost of approximating all functions \( q_j \) is of order \( NJQ^d \). In the testing phase, the coefficients \( \tilde{a}_{j,o,U_1,U_2} \) are computed via direct summation in (3.15) (with \( q_j \) replaced by their approximations \( \tilde{q}_j \)) at a cost of order \( N_0JQ^d \), and, finally, the control variate is computed via (5.4) on all testing paths at a cost of order \( N_0J \). Therefore, the overall cost is of order \( JQ^d \max \{ N, N_0 \} \), which is the same as for the RCV approach. (In the latter formula we ignore the cost constituents of smaller orders.)

We now establish the constraints that are pertinent to the RRCV approach. The regressions are now performed for the functions \( q_j \). Pertinent assumptions are in the spirit of (A1)–(A4) with different bounds in (A1) and (A2): the conditional variance in such regressions over one time step is typically of order \( \Delta \), hence we require the bound \( \Sigma^2 \Delta \) in the analogue of (A1); while in the analogue of (A2) and in formula (5.1) for the truncated estimate we require only the constant bound \( A \). For the regression error, instead of (5.2) we get

\[ \mathbb{E}\| \tilde{q}_j - q_j \|^2_{L^2(\mathbb{P}_{\Delta,j})} \leq \tilde{c} \left( \Sigma^2 \Delta + A^2 \log N + 1 \right) \left( \frac{p+d}{N} \right)Q^d + \frac{8C_h^2}{(p+1)^2d^2-2/h} \left( \frac{Rd}{Q} \right)^{2p+2} + 8A^2BpR^{-\nu}. \]

It turns out that

\[ \mathbb{E}\| \tilde{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2} \|^2_{L^2(\mathbb{P}_{\Delta,j-1})} \leq \mathbb{E}\| \tilde{q}_j - q_j \|^2_{L^2(\mathbb{P}_{\Delta,j})}, \]

for all \( j, o, U_1 \) and \( U_2 \). To prove (6.6), we use (5.3) and the similar formula involving \( q_j \) and \( \tilde{q}_j \). As in (5.3), we consider a testing path \( (X_{\Delta,j})_{j=0,...,J} \) which is independent of \( D_N^H \). Since \( \tilde{a}_{j,o,U_1,U_2}(\cdot, D_N^H) \) is given by (3.15) with \( q_j(\cdot) \) replaced by \( \tilde{q}_j(\cdot, D_N^H) \), it holds

\[ \tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, D_N^H) = \mathbb{E} \left[ \tilde{q}_j(X_{\Delta,j\Delta}, D_N^H) \prod_{r \in U_1} H_{\alpha_r}(\xi_{jk}^r) \prod_{(k,l) \in U_2} V_{jk} V_{lk} X_{\Delta,(j-1)\Delta}, D_N^H \right]. \]

\[ \text{When deriving the solution via Lagrange multipliers (cf. proof of Theorem 6.1) one can see that these parameter values are not optimal if } p \leq \frac{d-2}{2} \text{ or } \nu \leq \frac{2d(p+1)}{2(p+1)+d} \text{ (a Lagrange multiplier corresponding to a } \leq 0 \text{ constraint is negative). Therefore, the recommendation is to choose } p \in \mathbb{N}_0 \text{ and } \nu > 0 \text{ according to } p > \frac{d-2}{2} \text{ and } \nu > \frac{2d(p+1)}{2(p+1)-d}. \]
Furthermore, we have
\[
a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}) = \mathbb{E} \left[ q_j(X_{\Delta,j\Delta}) \prod_{r \in U_1} H_{\nu_r}(\xi_{j,r}^r) \prod_{(k,l) \in U_2} V_{j,k}^{l_k} \right].
\]
The latter formula remains true also without conditioning on \( D_{N_t}^p \), but this (seemingly superfluous) conditioning is helpful in the following calculation:
\[
(6.7) \quad (\tilde{a}_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}, D_{N_t}^p) - a_{j,o,U_1,U_2}(X_{\Delta,(j-1)\Delta}))^2
\]
\[
\leq \mathbb{E} \left[ \left( \tilde{q}_j(X_{\Delta,j\Delta}, D_{N_t}^p) - q_j(X_{\Delta,j\Delta}) \right)^2 \right] + \mathbb{E} \left[ \prod_{r \in U_1} H_{\nu_r}(\xi_{j,r}^r) \prod_{(k,l) \in U_2} V_{j,k}^{l_k} \right]
\]
\[
= \mathbb{E} \left[ (\tilde{q}_j(X_{\Delta,j\Delta}, D_{N_t}^p) - q_j(X_{\Delta,j\Delta}))^2 \right] \times \mathbb{E} \left[ \prod_{r \in U_1} H_{\nu_r}(\xi_{j,r}^r) \prod_{(k,l) \in U_2} V_{j,k}^{l_k} \right].
\]
We arrive at (6.6) by taking expectations in (6.7) and using (5.3) together with the similar formula for \( q_j \) and \( \tilde{q}_j \). Finally, we get an upper bound for the variance in the RRCV approach by the same calculation as in (5.5) using (6.5) and (6.6) instead of (5.3), and the resulting upper bound is the same as in (5.5) except that \( A^2 \Delta \) is replaced by \( A^2 \), while \( \Sigma^2 \) is replaced by \( \Sigma^2 \Delta \). Thus, in the case of the RRCV approach, our constraints are
\[
(6.8) \quad \max \left\{ \frac{1}{J^4}, JQd \log N, \frac{J}{N_0}, \frac{R}{Q}, \frac{J}{R^\nu N_0} \right\} \leq \varepsilon^2,
\]
where we again only track the parameters \( J, N, N_0, Q, R \).

**Theorem 6.2.** We obtain the following parameter values
\[
J \asymp \varepsilon^{-\frac{1}{2}}, \quad Q \asymp \varepsilon^{-\frac{5(p+1)+10}{2d+4(p+1)(d+1)}}, \quad R \asymp \varepsilon^{-\frac{5(p+1)}{2d+4(p+1)(d+1)}},
\]
\[
N \asymp N_0 \asymp \varepsilon^{-\frac{5(p+1)+10(p+1)+(d+1)}{2d+4(p+1)(d+1)}} \sqrt{\log \varepsilon},
\]
provided that \( p > \frac{d-2}{2} \) and \( \nu > \frac{2d(p+1)}{2(p+1)-2} \). Thus, we have for the complexity
\[
(6.9) \quad \mathcal{C}_{RRCV} \asymp JQ^d N \asymp JQ^d N_0 \asymp \varepsilon^{-\frac{11d(p+1)(7\nu+14d)}{2d+4(p+1)(d+1)} \sqrt{\log \varepsilon}}.
\]

**6.3. Discussion.** For the sake of comparison with the SMC and MLMC approaches, we recall at this point that their complexities are
\[
\mathcal{C}_{SMC} \asymp \varepsilon^{-2.5} \quad \text{and} \quad \mathcal{C}_{MLMC} \asymp \varepsilon^{-2}
\]
at best (we are considering the second order scheme). Complexity estimates (6.4) and (6.9) show that one can go beyond the complexity order \( \varepsilon^{-2} \), provided that
\[
p > \frac{7d-2}{2}, \quad \nu > \frac{8d(p+1)}{2(p+1) - 7d}
\]
in case of the RCV approach and
\[
p > \frac{7d-2}{2}, \quad \nu > \frac{14d(p+1)}{2(p+1) - 7d}
\]
in case of the RRCV approach. Both in (6.4) and (6.9) the power of \( \varepsilon \) converges to \(-1.75\) as \( p, \nu \to \infty \) (the log-term is ignored). Notice that, while \( d \) and \( m \) are fixed, \( p \) and \( \nu \) are free parameters in our algorithms, which can be chosen large, provided the smoothness in \( \mu, \sigma \) and \( f \) allows that. Therefore, whenever it is possible to take arbitrarily large \( p \) and \( \nu \), the complexity of our scheme can be reduced to \( \varepsilon^{-1.75-\delta} \) for arbitrarily small \( \delta > 0 \).

Notice that we obtain such a complexity for piecewise polynomial regression with the second order weak scheme. A natural question is to perform a similar complexity analysis also for the weak Euler

\footnote{Footnote 1 on page 12 applies.}
scheme. We then get the complexity \( \varepsilon^{-2.5} \) in the limit as \( p, \nu \to \infty \), that is, both the RCV and the RRCV approaches with the weak Euler scheme cannot outperform the MLMC approach as well as the SMC approach with the second order scheme (but they still outperform the SMC approach with the Euler or the weak Euler scheme because the complexity of the latter is \( \varepsilon^{-3} \)). Still, both the RCV and the RRCV approaches might be useful also with the weak Euler scheme, provided we choose basis functions other than those in piecewise polynomial regression (recall the last paragraph in Section 4).

Obviously, the complexity estimate \([6.4]\) of the RCV approach gives us a better order compared to the one of the RRCV approach \([6.9]\) (due to the factor \( J \) which arises in the last expression of the maximum term \([6.8]\) but not in \([6.2]\)). However, the larger is the parameter \( \nu \), the closer are both complexities to each other (provided that we ignore the log-term). As we mentioned in Sections 3.1.1 and 3.2.1, from the computational point of view it is preferable to consider the RRCV approach rather than the RCV one, since we perform regressions over only one time step in RRCV. In addition, in case of the RCV approach, there are destabilising factors \( \prod_{r \in U_1} H_{\nu, (c r_j)} \prod_{(k,l) \in U_2} V_{\nu}^{kl} \) in the estimation of \( a_{j,o,U_1,U_2} \), which are independent of \( X_{\Delta,(j-1)\Delta} \) and have zero expectation and thus may lead to poor regression results. Regarding the RRCV approach, such destabilising factors are not present in the regression for \( q_1 \).

7. Numerical results

In this section, we consider weak schemes of second order and compare the numerical performance of the SMC, MLMC, RCV and RRCV approaches. For simplicity we implemented a global regression (i.e. the one without truncation and partitioning, as a part of the general description in Section 4). In what follows it is convenient to have notations for the following constants

\[
c_m := 3^m 2^{\frac{m(m-1)}{2}}, \quad c_{p,d} := \left( p + \frac{d}{d} \right) + 1.
\]

Regarding the choice of basis functions, we use in both RCV and RRCV approaches the same polynomials \( \psi(x) = \prod_{i=1}^d x_i^k \), where \( l_1, \ldots, l_d \in \{0, 1, \ldots, p\} \) and \( \sum_{i=1}^d l_i \leq p \). In addition to the polynomials, we consider the function \( f \) as a basis function. Hence, we have overall \( c_{p,d} \) basis functions in each regression.

The following results are based on program codes written and vectorised in MATLAB and running on a Linux 64-bit operating system.

7.1. One-dimensional example. Here \( d = m = 1 \). We consider the following SDE

\[
dx_t = -\frac{1}{2} \tanh (X_t) \text{sech}^2 (X_t) \, dt + \text{sech} (X_t) \, dW_t, \quad X_0 = 0,
\]

for \( t \in [0, 1] \), where \( \text{sech}(x) := \frac{1}{\cosh(x)} \). This SDE has an exact solution \( X_t = \text{arsinh} (W_t) \). Furthermore, we consider the functional \( f(x) = \text{sech}(x) + 15 \arctan(x) \), that is, we have

\[
E[f(X_1)] = E[\text{sech}(\text{arsinh}(W_1))] = E\left[ \frac{1}{\sqrt{1 + W_1^2}} \right] \approx 0.789640.
\]

We choose \( p = 3 \) (that is, 5 basis functions) and, for each \( \varepsilon = 2^{-i} \), \( i \in \{2, 3, 4, 5, 6\} \), we set the parameters \( J \), \( N \) and \( N_0 \) as follows (compare with the formulas in Section 5 for the “limiting” case \( \nu \to \infty \) and ignore the log-terms for the RCV approach):

\[
J = \left[ \varepsilon^{-0.5} \right], \quad N = c_N \cdot \left[ \varepsilon^{-1.3235} \right], \quad c_N = \begin{cases} \frac{64}{32} & \text{RRCV} \\ \frac{128}{32} & \text{RCV} \end{cases}, \quad N_0 = 128 \cdot \left[ \varepsilon^{-1.3235} \right].
\]

Regarding the SMC approach, the number of paths is set \( N_0 = 32 \cdot \varepsilon^{-2} \). The factors 32, 64 and 128 are here for stability purposes. We use different constants for the training and testing paths due the fact that, if we also track the constants \( c_{p,d} \) and \( c_m \), we will have the cost of order \( O(J c_{p,d} (c_m - 1) \max \{N c_{p,d}, N_0 c_m\}) \) for the RCV approach and \( O(J c_{p,d} \max \{N c_{p,d}, N_0 c_m\}) \) for the RRCV approach (cf. \([6.4]\)). Since we get from Theorems 6.1 and 6.2 that both components in the maximum term are of the same order in the optimal solution, we choose the constants such that \( N c_{p,d} \approx N_0 \) in case of the RCV approach and \( N c_{p,d} \approx N_0 c_m \) in case of the RRCV approach. As for the MLMC approach, we set the initial number of paths for the first level \( (l = 0) \) equal to \( 10^3 \) as well as the “discretisation
parameter” $M = 4$ (leading to timesteps of size $\frac{1}{4}$ at level $l$). Next we compute the numerical RMSE (the exact value is known, see (7.2)) by means of 100 independent repetitions of the algorithm. As can be seen from the left-hand side of Figure 1, the estimated numerical complexity is about $\text{RMSE}^{-1.41}$ for the RRCV approach, $\text{RMSE}^{-1.66}$ for the RCV approach, $\text{RMSE}^{-1.99}$ for the MLMC approach and $\text{RMSE}^{-2.53}$ for the SMC approach, which we get by regressing the log-time (logarithmic computing time of the whole algorithm in seconds) vs. log-RMSE. Thus, the complexity reduction works best with the RRCV approach.

7.2. Five-dimensional example. Here $d = m = 5$. We consider the SDE

$$dX^i_t = -\sin(X^i_t) \cos^3(X^i_t) \ dt + \cos^2(X^i_t) \ dW^i_t, \quad X^i_0 = 0, \quad i \in \{1, 2, 3, 4\},$$

(7.3)$$dX^5_t = \sum_{i=1}^4 \left[ -\frac{1}{2} \sin(X^i_t) \cos^2(X^i_t) \ dt + \cos(X^i_t) \ dW^i_t \right] + dW^5_t, \quad X^5_0 = 0.$$

The solution of (7.3) is given by

$$X^i_t = \arctan(W^i_t), \quad i \in \{1, 2, 3, 4\},$$

$$X^5_t = \sum_{i=1}^4 \text{arsinh}(W^i_t) + W^5_t.$$

for $t \in [0, 1]$. Further, we consider the functional

$$f(x) = \cos\left(\sum_{i=1}^5 x^i\right) - 20 \sum_{i=1}^4 \sin(x^i),$$

that is, we have

$$\mathbb{E}[f(X_1)] = \left(\mathbb{E}\left[\cos(\arctan(W^1_t) + \text{arsinh}(W^1_t))\right]\right)^4 \mathbb{E}[\cos(W^5_t)] \approx 0.002069.$$

Note that we do not need to consider random variables $V^{kl}_j$ in the second order weak scheme, since $L^k\sigma^l(x) = 0$ for $k \neq l$ (see (3.10)). This gives us a smaller constant $\tilde{c}_m := 3^m = 243$ compared to $c_m = 248832$ and hence a smaller number of terms for the control variate (the factor $2^{m(m-1)/2} = 1024$ is no longer present). We again choose $p = 3$ (this now results in 57 basis functions), consider the same values of $\varepsilon$ as above (and, in addition, consider the value $\varepsilon = 2^{-7}$ for the SMC approach to obtain a similar computing time as for the RCV, RRCV and MLMC approaches). Moreover, we set

$$J = \lfloor \varepsilon^{-0.5} \rfloor, \quad N = c_N \lfloor \varepsilon^{-1.5476} \rfloor, \quad c_N = \begin{cases} 512 & \text{RRCV} \\ 32 & \text{RCV} \end{cases},$$

$$N_0 = c_{N_0} \lfloor \varepsilon^{-1.5476} \rfloor, \quad c_{N_0} = \begin{cases} 128 & \text{RRCV} \\ 1024 & \text{RCV} \end{cases}$$
(similar to the previous example we consider the limiting case \( \nu \to \infty \), ignore the log-terms for the RRCV approach and consider the relations \( N_{cp,d} \approx N_0 \) in case of the RCV approach and \( N_{cp,d} \approx N_0 \) in case of the RRCV approach). The number of paths for the SMC approach is set \( N_0 = 512 \cdot \varepsilon^{-2} \). Since the estimated variance of \( f(X_{\Delta,T}) \) is much higher than in the previous example, we use a higher constant here for the SMC approach. This is due to the fact that we get \( N \) in the last equality we use independence between \( G \).

8.1. Proof of Theorem 2.1

The proof uses the well-known fact that the system

\[
\left\{ \prod_{j=1}^{J} \prod_{r=1}^{m} H_{k_{j,r}} \left( \frac{\Delta_j^W}{\sqrt{\Delta}} \right) : k = (k_{j,r}) \in N_0^{J \times m} \right\}
\]

is an orthonormal basis in \( L^2(G_j) \), where the \( \sigma \)-field \( G_j \) is generated by the Brownian increments, \( G_j = \sigma(\Delta_j W : j = 1, \ldots, J) \), and goes along the lines of the proof of Theorem 3.5.

8.2. Proof of Theorem 3.1

The proof is similar to the one of Theorem 3.5.

8.3. Proof of Proposition 3.3

Let \( G_0 \) be the trivial \( \sigma \)-field and \( G_j = \sigma(\xi_1, \ldots, \xi_j), j = 1, \ldots, J \). It follows from (3.1) that the process \( (X_{\Delta,j})_{j=0}^J \) is Markov with respect to \( (G_j)_{j=0}^J \). By the Markov property, we have

\[
q_j(X_{\Delta,j,\Delta}) \equiv E[f(X_{\Delta,T})|X_{\Delta,j,\Delta}] = E[f(X_{\Delta,T})|G_j],
\]

hence, by the tower property of conditional expectation,

\[
q_{j-1}(x) = E[q_j(\Phi_{\Delta}(X_{\Delta,(j-1),\Delta}, \xi_j))|X_{\Delta,(j-1),\Delta} = x] = \frac{1}{2^{m}} \sum_{y=(y_1, \ldots, y_m) \in \{\pm 1\}^m} q_j(\Phi_{\Delta}(x, y)),
\]

where in the last equality we use independence between \( X_{\Delta,(j-1),\Delta} \) and \( \xi_j \). This proves (3.7). We now apply intermediate conditioning with respect to \( G_j \) in (3.4) and arrive at

\[
a_{j,r,s}(x) = E\left[ q_j(\Phi_{\Delta}(X_{\Delta,(j-1),\Delta}, \xi_j)) \prod_{i=1}^{r} \xi_j^s \left| X_{\Delta,(j-1),\Delta} = x \right. \right]
\]

which implies (3.8) due to the independence between \( X_{\Delta,(j-1),\Delta} \) and \( \xi_j \).

8.4. Proof of Theorem 3.5

Let \( G_0 \) denote trivial \( \sigma \)-field, and, for \( j = 1, \ldots, J \), define the \( \sigma \)-field \( G_j = \sigma(\xi_1, V_1, \ldots, \xi_j, V_j) \). Since each of the random variables \( \xi_j^r, j = 1, \ldots, J, r \in I_1 \) can take 3 different values, each of the random variables \( V_j^{kl}, (k, l) \in I_2 \), can take 2 different values and \( |I_1| = m, |I_2| = \frac{m(m-1)}{2} \), where \( |\cdot| \) means the cardinality of a set, \( L^2(G_j) \) is a \( (3^m \cdot \frac{m(m-1)}{2})^J \)-dimensional vector space. A simple calculation reveals that, for any fixed \( j = 1, \ldots, J \), the system

\[
(\prod_{r \in I_1} H_{o_j^r}(\xi_j^r)) \prod_{(k, l) \in I_2} (V_j^{kl})^{s_j^{kl}} : o_j^r \in \{0, 1, 2\}, s_j^{kl} \in \{0, 1\} \}
\]

different and \( |I_1| = m, |I_2| = \frac{m(m-1)}{2} \), where \( |\cdot| \) means the cardinality of a set, \( L^2(G_j) \) is a \( (3^m \cdot \frac{m(m-1)}{2})^J \)-dimensional vector space. A simple calculation reveals that, for any fixed \( j = 1, \ldots, J \), the system

\[
(\prod_{r \in I_1} H_{o_j^r}(\xi_j^r)) \prod_{(k, l) \in I_2} (V_j^{kl})^{s_j^{kl}} : o_j^r \in \{0, 1, 2\}, s_j^{kl} \in \{0, 1\} \}
\]

Due to independence of \( \xi_1, V_1, \ldots, \xi_j, V_j \), the system

\[
(8.1) \quad \left\{ \prod_{j=1}^{J} r \in I_1 \prod_{(k, l) \in I_2} (V_j^{kl})^{s_j^{kl}} : o_j^r \in \{0, 1, 2\}, s_j^{kl} \in \{0, 1\} \right\}
\]
is orthonormal in $L^2(\mathcal{G}_j)$, and therefore, linear independent. The cardinality of system \((8.1)\) is \((3m^2 m^{-1})^J\), i.e. equals the dimension of $L^2(\mathcal{G}_j)$. Hence, linear independent system \((8.1)\) is an orthonormal basis in $L^2(\mathcal{G}_j)$. We have $E|f(X_{\Delta,T})|^2 < \infty$ because $X_{\Delta,T}$ takes finitely many values. Therefore, $f(X_{\Delta,T})$ belongs to $L^2(\mathcal{G}_j)$ and can be written

$$f(X_{\Delta,T}) = \sum_{\delta \in \{0,1,2\}^m J} \sum_{s \in \{0,1\}^m J} c_{\delta s} \prod_{j=1}^J \prod_{r \in I_1} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in I_2} (V_{kl})^{s_{kl}},$$

where \(\delta = (o_1^j, \ldots, o_j^L, \ldots, o_m^J), \bar{s} = (s_1^2, \ldots, s_{j-1}^2, \ldots, s_j^2, \ldots, s_{(m-1)}^J, \ldots, s_{(m-1)}^J, \ldots, s_{j}^{(m-1)} \ldots, s_j^{(m-1)} m). \) Note that $c_{\delta s} = E[f(X_{\Delta,T})] \prod_{j=1}^J H_{\delta j}(\xi_j^r) \prod_{(k,l) \in I_2} (V_{kl})^{s_{kl}}$, in particular, $c_{00} = E[f(X_{\Delta,T})]$. Rearranging the terms in the expression for $f(X_{\Delta,T})$ we rewrite it as

$$f(X_{\Delta,T}) = E[f(X_{\Delta,T})] + \sum_{j=1}^J \sum_{(U_1, U_2) \in A_{p \in \{1,2\}^J}} A_{j,p,U_1,U_2} \prod_{r \in U_1} H_{p \delta}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} \prod_{j=1}^J H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} \prod_{j=1}^J H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} \prod_{j=1}^J H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j},$$

with $\mathcal{G}_{j-1}$-measurable random variables $A_{j,p,U_1,U_2}$. Let us now multiply both sides of the last equality by $\prod_{r \in U_0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j}$, with some $j^0 \in \{1, \ldots, J\}, (U_0^0, U_2^0) \in A, p^0 \in \{1,2\} U_0^0$ and calculate conditional expectations of the resulting expressions given $\mathcal{G}_{j^0-1}$. Notice that, with $j^h < j^0$ and $j^g > j^0$, we have

$$E[\prod_{r \in U_0^0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2^0} V_{kl}^{j} | \mathcal{G}_{j^0-1}] = E[\prod_{r \in U_0^0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2^0} V_{kl}^{j}] = 0,$$

$$E[A_{j^h,p,U_1, U_2} \prod_{r \in U_1} H_{p \delta}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} \prod_{r \in U_0^0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2^0} V_{kl}^{j} \prod_{j=1}^J \prod_{r \in U_0^0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2^0} V_{kl}^{j} | \mathcal{G}_{j^0-1}] = 0,$$

$$E[A_{j^g,p,U_1, U_2} \prod_{r \in U_1} H_{p \delta}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} \prod_{r \in U_0^0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2^0} V_{kl}^{j} \prod_{j=1}^J \prod_{r \in U_0^0} H_{\delta j}(\xi_j^r) \prod_{(k,l) \in U_2^0} V_{kl}^{j} | \mathcal{G}_{j^0-1}] = 0,$$

$$\delta_{\cdot} \text{ is the Kronecker delta. Thus, the coefficients } A_{j,p,U_1, U_2} \text{ in } (8.2) \text{ are given by}\n
\begin{align*}
(8.3) \quad A_{j,p,U_1, U_2} &= E[f(X_{\Delta,T})] \prod_{r \in U_1} H_{p \delta}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} | \mathcal{G}_{j-1}.
\end{align*}

Let us now prove that

$$E[f(X_{\Delta,T})] \prod_{r \in U_1} H_{p \delta}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} | \mathcal{G}_{j-1} = E[f(X_{\Delta,T})] \prod_{r \in U_1} H_{p \delta}(\xi_j^r) \prod_{(k,l) \in U_2} V_{kl}^{j} | X_{\Delta,(j-1)\Delta}].$$

In what follows we use the functions $q_j$ from (3.6) and notice that, by the Markov property of $(X_{\Delta, 1\Delta})_{t=0, \ldots, J}$ with respect to $(\mathcal{G}_t)$, which is due to (3.9), we also have

$$q_j(X_{\Delta, J\Delta}) = E[f(X_{\Delta,T}) | \mathcal{G}_J].$$
Let us set
\[
\tag{8.6}
    h(X_{\Delta,(j-1)\Delta}, \xi_j, V_j) = \prod_{r \in U_1} H_{pr}(\xi_r^j) \prod_{(k,l) \in U_2} V_k^j q_j(X_{\Delta,j\Delta})
\]
and notice that, due to (3.9), this is indeed a function of \(X_{\Delta,(j-1)\Delta}, \xi_j\) and \(V_j\) only. Further, let us set
\[
\tag{8.7}
g(x) = E[h(x, \xi_j, V_j)].
\]

Using the tower property of conditional expectations together with (8.5), (8.6) and (8.7), we get
\[
\tag{8.8}
    E[f(X_{\Delta,T}) \prod_{r \in U_1} H_{pr}(\xi_r^j) \prod_{(k,l) \in U_2} V_k^j |G_{j-1}] = E[\prod_{r \in U_1} H_{pr}(\xi_r^j) \prod_{(k,l) \in U_2} V_k^j E[f(X_{\Delta,T}) |G_j] |G_{j-1}]
\]
where the last equality is due to the facts that \(X_{\Delta,(j-1)\Delta}\) is \(G_{j-1}\)-measurable and the pair \((\xi_j, V_j)\) is independent of \(G_{j-1}\). Moreover, applying (8.8), we also obtain
\[
\tag{8.9}
    E[f(X_{\Delta,T}) \prod_{r \in U_1} H_{pr}(\xi_r^j) \prod_{(k,l) \in U_2} V_k^j |X_{\Delta,(j-1)\Delta}] = E[g(X_{\Delta,(j-1)\Delta}) |X_{\Delta,(j-1)\Delta}] = g(X_{\Delta,(j-1)\Delta}).
\]
Comparing (8.8) and (8.9), we arrive at (8.4). Together with (8.3) and (8.2), this proves (3.11) and (3.12).

8.5. **Proof of Proposition 3.7** The proof is similar to the one of Proposition 3.3.

8.6. **Proof of Theorem 5.2** For the proof, we need the following multivariate generalisation of Lemma 11.1 in [3].

**Lemma 8.1.** Let \(a : [0, 1]^d \to \mathbb{R}\) be a \((p + 1, C)\)-smooth function w.r.t. the norm \(|.|_h\), where \(d \in \mathbb{N}\), \(h \in [1, \infty]\) and \(p \in \mathbb{N}_0\). Further, let \(g\) be a piecewise polynomial of degree less than or equal to \(p\) w.r.t. an equidistant partition of \([0, 1]^d\) in \(Q^d\) cubes. Then it holds
\[
\tag{8.10}
    \sup_{x \in [0, 1]^d} |a(x) - g(x)| \leq \frac{C}{d^{1/2} (p + 1)!} \left( \frac{d}{2Q} \right)^{p+1}.
\]

**Proof.** Consider the Taylor expansion of the function \(a\) up to the degree \(p\) around \(z \in (0, 1)^d\):
\[
    a_p(x) = \sum_{n=0}^{p} \frac{1}{n!} \sum_{l_1+\ldots+l_d=n} \left( \begin{array}{c} n \\ l_1, \ldots, l_d \end{array} \right) \frac{\partial^n m(z)}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} \prod_{i=1}^{d} (x_i - z_i)^{l_i}.
\]
The remainder term has the form
\[
a(x) - a_p(x) = \frac{1}{p!} \int_0^1 (1 - t)^p \sum_{l_1+\ldots+l_d=p+1} \left( \begin{array}{c} p+1 \\ l_1, \ldots, l_d \end{array} \right) \frac{\partial^{p+1} a(z + t (x - z))}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} \prod_{i=1}^{d} (x_i - z_i)^{l_i} dt.
\]
At first, we will focus on the case \(p > 0\). For \(g = a_p\) we have
\[
a(x) - g(x) = a(x) - a_p(x) - \frac{1}{p!} \sum_{l_1+\ldots+l_d=p} \left( \begin{array}{c} p \\ l_1, \ldots, l_d \end{array} \right) \frac{\partial^p a(z)}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} \prod_{i=1}^{d} (x_i - z_i)^{l_i}
\]
\[
= \frac{1}{(p-1)!} \int_0^1 (1 - t)^{p-1} \sum_{l_1+\ldots+l_d=p} \left( \begin{array}{c} p \\ l_1, \ldots, l_d \end{array} \right) \left( \frac{\partial^p a(z + t (x - z))}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} - \frac{\partial^p a(z)}{\partial x_1^{l_1} \ldots \partial x_d^{l_d}} \right)
\]
\[
\times \prod_{i=1}^{d} (x_i - z_i)^{l_i} dt.
\]
Since \( a \) is \((p + 1, C)\)-smooth, we obtain
\[
|a(x) - g(x)| \leq \frac{C}{(p - 1)!} |x - z|_h \int_0^1 t(1-t)^{p-1} dt \sum_{l_1 + \ldots + l_d = p} \left( \prod_{i=1}^d |x_i - z_i|^{l_i} \right) \\
= \frac{C}{(p + 1)!} |x - z|_h \left( \sum_{i=1}^d |x_i - z_i| \right)^p \leq \frac{C}{(p + 1)!} |x - z|^p_1 d^p(1/h).
\]
In the case \( p = 0 \) this inequality holds, too. This follows directly from the \((p + 1, C)\)-smoothness assumption.

Next, we consider the equidistant partitioning of \([0, 1]^d\) into \(Q^d\) cubes \(K^1, \ldots, K^{Q^d}\) with \(\bigcup_{k=1}^{Q^d} K^k = [0, 1]^d\). Let \(z^k\) be the midpoint of \(K^k\). We then have \(\sup_{x \in K^k} |x - z^k|_h = \frac{d}{2Q}\) for all \(k \in \{1, \ldots, Q^d\}\).

This finally yields (8.10)

\[\boxed{(8.10)}\]

We now proceed with the proof of Theorem 5.2. Define the set
\[
\Psi_{Q,p} := \text{span} \left\{ \psi^{k,1}, \ldots, \psi^{k,n} : k \in \{1, \ldots, Q^d\}, n = \binom{p + d}{d} \right\}.
\]

We would like to apply Theorem 11.3 in [3], which gives us
\[
\mathbb{E}\|\hat{a}_{j,o,U_1,U_2} - a_{j,o,U_1,U_2}\|_{L^2(P_{\Delta,j-1})}^2 \leq \tilde{c} \max \left\{ \Sigma^2, A^2 \Delta \right\} \left( \log N + 1 \right) \frac{(p+d)Q^d}{N} \frac{N}{p+d} + 8 \inf_{g \in \Psi_{Q,p}} \int_{\mathbb{R}^d} (a_{j,o,U_1,U_2}(x) - g(x))^2 P_{\Delta,j-1}(dx).
\]

However, the maximum in (8.11) is in fact a sum of two terms \(A^2 \Delta (\log N + 1)\) and \(\Sigma^2\) so that the logarithm is only included in one term (see proof of Theorem 11.3 in [3]). Next, we split the integral in (8.11) into two parts:
\[
\int_{\mathbb{R}^d} (a_{j,o,U_1,U_2}(x) - g(x))^2 P_{\Delta,j-1}(dx) = \int_{[-R,R]^d} (a_{j,o,U_1,U_2}(x) - g(x))^2 P_{\Delta,j-1}(dx) + \int_{\mathbb{R}^d \setminus [-R,R]^d} a_{j,o,U_1,U_2}^2(x) P_{\Delta,j-1}(dx),
\]
since \(g(x) = 0\) for \(x \not\in [-R,R]^d\) for \(g \in \Psi_{Q,p}\). The second integral in (8.12) refers to the case \(|X_{\Delta,(j-1)\Delta}|_\infty > R\), where we simply use Assumptions (A2) and (A4) to get
\[
\int_{\mathbb{R}^d \setminus [-R,R]^d} a_{j,o,U_1,U_2}^2(x) P_{\Delta,j-1}(dx) \leq \sup_{x \in \mathbb{R}^d} |a_{j,o,U_1,U_2}(x)|^2 P(|X_{\Delta,(j-1)\Delta}|_\infty > R) \leq A^2 \Delta B \nu R^{-\nu}.
\]

Regarding the first integral in (8.12), we obtain by Lemma 8.1
\[
\inf_{g \in \Psi_{Q,p}} \int_{[-R,R]^d} (a_{j,o,U_1,U_2}(x) - g(x))^2 P_{\Delta,j-1}(dx) \leq \inf_{g \in \Psi_{Q,p}} \sup_{x \in [-R,R]^d} |a_{j,o,U_1,U_2}(x) - g(x)|^2 \leq \frac{C_h^2}{d^{2-2/h}(p + 1)!} \left( \frac{Rd}{Q} \right)^{2p+2}
\]

(\text{notice that, since we consider \([-R,R]^d\) instead of \([0,1]^d\), the expression \(\frac{d}{2Q}\) in (8.10) is replaced by \(\frac{Rd}{Q}\) because \(\sup_{x \in K^k} |x - z^k|_h = \frac{Rd^{1/h}}{Q}\) with \(z^k\) being the midpoint of \(K^k\)).

8.7. Proof of Theorem 6.1. Let us consider the log-cost and log-constraints rather than (6.1) and (6.2). Further, let us subdivide the optimisation problem into two cases:
(1) \( N \lesssim N_0 \). This gives us the Lagrange function

\[
L_{\lambda_1, \ldots, \lambda_5}(J, N, N_0, Q, R) := \log(J) + \log(N) + d \log(Q) + \lambda_1(-4 \log(J) - 2 \log(\epsilon)) \\
+ \lambda_2(\log(J) + d \log(Q) - \log(N) - \log(N_0) - 2 \log(\epsilon)) \\
+ \lambda_3(\log(J) + (p + 1)\log(R) - \log(Q)) - \log(N_0) - 2 \log(\epsilon)) \\
+ \lambda_4(-\nu \log(R) - \log(N_0) - 2 \log(\epsilon)) + \lambda_5(\log(N) - \log(N_0)),
\]

where \( \lambda_1, \ldots, \lambda_5 \geq 0 \). Thus, considering the conditions \( \frac{\partial L}{\partial J} = \frac{\partial L}{\partial N} = \frac{\partial L}{\partial N_0} = \frac{\partial L}{\partial Q} = \frac{\partial L}{\partial R} = 0 \) gives us the following Lagrange parameters

\[
\lambda_1 = \frac{3d\nu + 6\nu(p + 1)}{4(d\nu + 2(p + 1)(2\nu + d))}, \\
\lambda_2 = \frac{2(p + 1)(\nu - d) - d\nu}{d\nu + 2(p + 1)(2\nu + d)} = \lambda_5, \\
\lambda_3 = \frac{3d\nu}{d\nu + 2(p + 1)(2\nu + d)}, \\
\lambda_4 = \frac{6d(p + 1)}{d\nu + 2(p + 1)(2\nu + d)}.
\]

Obviously it holds \( \lambda_1, \lambda_3, \lambda_4 > 0 \), so that we can deduce

\[
J \gtrsim \epsilon^{-\frac{1}{2}}, \quad R \gtrsim \left( Q^{4(p+1)} \epsilon \right)^{\frac{3}{2p+4(p+1)}}, \quad N_0 \gtrsim \left( Q^{4\nu(p+1)} \epsilon^{5\nu+8(p+1)} \right)^{-\frac{1}{2p+4(p+1)}}.
\]

Regarding \( \lambda_2 (\equiv \lambda_5) \), we have to consider two cases again.

- Case (1a): \( \lambda_2 = \lambda_5 = 0 \). From this condition, we get \( \nu = \frac{2d(p+1)}{2(p+1)-d} \) and \( p > \frac{d-2}{2} \). (The latter guarantees that \( \nu \) is positive.) Thus,

\[
R \gtrsim \left( Q^{4(p+1)} \epsilon \right)^{\frac{1}{(p+1)}} \gtrsim \epsilon^{-\frac{8(p+1)+d}{4(p+1)}} Q^{-d}.
\]

Hence, the complexity \( JQ^d N_0 \gtrsim \epsilon^{-\frac{10(p+1)+d}{4(p+1)}} \gtrsim \epsilon^{-2.5} \) is worse than that of the SMC in this case.

- Case (1b): \( \lambda_2 = \lambda_5 > 0 \). This implies \( \nu > \frac{2d(p+1)}{2(p+1)-d} \) and \( p > \frac{d-2}{2} \). (Again, the second condition guarantees that \( \nu \) is positive.) We can deduce

\[
Q \gtrsim \epsilon^{-\frac{5\nu+6(p+1)}{2d\nu+4(p+1)(2\nu+d)}}, \quad R \gtrsim \epsilon^{-\frac{6(p+1)+d}{2d\nu+4(p+1)(2\nu+d)}}, \quad N_0 \gtrsim \epsilon^{-\frac{5d\nu+2(p+1)(5\nu+4d)}{2d\nu+4(p+1)(2\nu+d)}} \gtrsim N,
\]

so that the complexity \( JQ^d N_0 \gtrsim \epsilon^{-\frac{11d\nu+2(p+1)(7\nu+8d)}{2d\nu+4(p+1)(2\nu+d)}} \gtrsim N \)

is a better solution than that in case (1a).

(2) \( N \gtrsim N_0 \). This gives us the Lagrange function

\[
\tilde{L}_{\lambda_1, \ldots, \lambda_5}(J, N, N_0, Q, R) := \log(J) + \log(N) + d \log(Q) + \lambda_1(-4 \log(J) - 2 \log(\epsilon)) \\
+ \lambda_2(\log(J) + d \log(Q) - \log(N) - \log(N_0) - 2 \log(\epsilon)) \\
+ \lambda_3(\log(J) + (p + 1)\log(R) - \log(Q)) - \log(N_0) - 2 \log(\epsilon)) \\
+ \lambda_4(-\nu \log(R) - \log(N_0) - 2 \log(\epsilon)) + \lambda_5(\log(N) - \log(N_0)).
\]

Analogously to the procedure above we get the same optimal solution (8.14).

Thus, we arrive at (6.3) and (6.4), provided that \( p > \frac{d-2}{2} \) and \( \nu > \frac{2d(p+1)}{2(p+1)-d} \). Let us finally prove the statement in footnote 4 on page 12 i.e. that the complexity of the RCV approach would be worse than that of the SMC whenever at least one of the above inequalities is violated. More precisely, the statement we are going to prove sounds as follows. If either \( p \leq \frac{d-2}{2} \) (recall that \( p \in N_0 \)) or \( \nu \leq \frac{2(p+1)d}{2(p+1)-d} \) (recall that \( \nu > 0 \)), then the cost \( C \) of the RCV algorithm given in (6.1) is worse than \( \epsilon^{-2.5} \) regardless of the choice of \( J, Q, R, N \) and \( N_0 \) such that (6.2) holds true.
We first remark that any choice of $J$, $Q$, $R$, $N$, $N_0$ such that $R$ does not tend to infinity as $\varepsilon \downarrow 0$ results in $C \gtrsim \varepsilon^{-2.5}$. Indeed, in this case we see from the first and the fourth terms in (6.2) that $J \gtrsim \varepsilon^{-0.5}$ and $N_0 \gtrsim \varepsilon^{-2}$, hence $C \gtrsim JN_0 \gtrsim \varepsilon^{-2.5}$. Therefore, below we consider without loss of generality only such choices of $J$, $Q$, $R$, $N$, $N_0$, where $R$ tends to infinity as $\varepsilon \downarrow 0$, and discuss the following two cases.

Let $p < \frac{d-2}{2}$, that is, $2(p+1) \leq d$. Then we obtain from the third term in (6.2)

$$Q^dN_0 \gtrsim Q^{2(p+1)}N_0 \gtrsim \varepsilon^{-2}JR^{2(p+1)} \gtrsim \varepsilon^{-2}J$$

and hence, together with $J \gtrsim \varepsilon^{-0.5}$ (see the first term in (6.2)), we have for the cost

$$C \gtrsim JQ^dN_0 \gtrsim \varepsilon^{-2}J^2 \gtrsim \varepsilon^{-3},$$

which is even worse than $\varepsilon^{-2.5}$.

Finally, let $p > \frac{d-2}{2}$, that is, $2(p+1) > d$, and $0 < \nu \leq \frac{2(p+1)d}{2(p+1)-d}$. Then we get from the third and the fourth terms in (6.2)

$$R^{2(p+1)} \lesssim J^{-1}Q^{2(p+1)}N_0\varepsilon^2,$$

$$R^{2(p+1)d} \gtrsim R^\nu \gtrsim N_0^{-1}\varepsilon^{-2}.$$

Therefore,

$$J^{-\frac{d}{2(p+1)-d}}Q^{\frac{2(p+1)d}{2(p+1)-d}}N_0^{\frac{d}{2(p+1)-d}} \gtrsim \varepsilon^{-\frac{2d}{2(p+1)-d}}N_0^{-1}\varepsilon^{-2}.$$  

This yields

$$J^{-\frac{d}{2(p+1)-d}}Q^{\frac{2(p+1)d}{2(p+1)-d}}N_0^{\frac{d}{2(p+1)-d}} \gtrsim \varepsilon^{-\frac{4(p+1)}{2(p+1)-d}},$$

and we deduce

$$J^{-\frac{d}{2(p+1)-d}}Q^dN_0 \gtrsim \varepsilon^{-2}.$$  

Together with $J \gtrsim \varepsilon^{-0.5}$, we obtain for the cost

$$C \gtrsim JQ^dN_0 \gtrsim J^{1+\frac{d}{2(p+1)}}\varepsilon^{-2} \gtrsim \varepsilon^{-2.5},$$

which concludes the proof.

8.8. **Proof of Theorem 6.2.** The proof is similar to the one of Theorem 6.1.

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