Wigner-Moyal Operator in Loop Quantum Cosmology

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Abstract

In this paper we derive the Wigner-Moyal operator and the characteristic function for the Loop Quantum Cosmology homogeneous and isotropic FLRW model. In our derivation we use the Wigner-Moyal-Groenewold phase space formalism applied to the LQC holonomy-flux algebra.

1 Introduction

We derive the Wigner-Moyal operator in Loop Quantum Gravity in case of homogeneous and isotropic space by using the Wigner-Moyal-Groenewold phase space approach. The Wigner transform is the inverse of the Weyl transform from the space of functions to the space of operators. The inverse Fourier transform of the characteristic function becomes a phase-space quasi distribution of the two non-commuting observables. The Wigner-Moyal-Groenewold approach can be applied to derive the Wigner-Moyal operator and a characteristic function for the Loop Quantum Cosmology in the homogeneous and isotropic case with the holonomy-flux algebra: $\{\hat{p}, \hat{N}\} = \frac{8\pi G}{\hbar} \mu \hat{N}$, where $\hat{N} = e^{i\mu c}$ is an LQC holonomy operator, $c$ - is the configuration variable corresponding to the connection, $\mu$ - the number of the fiducial cell repetition, $\mu \in R$, $c \in R_b$ - Bohr real line compactification.

The LQC Wigner-Moyal operator is an operator with the following property:

$$M(\tau, \theta) = \langle \psi^*, \hat{M} \psi \rangle$$  \hspace{1cm} (1)

where $\psi$ are the LQC cylindrical functions, $M(\tau, \theta)$ is the LQC characteristic function, which is the Fourier transform of the the quasi probability density function with respect to the group characters.

As a result we obtain the following expressions for the LQC Wigner-Moyal operator:

$$\hat{M}(\tau, \theta) = e^{i\hat{p}\hat{\theta}} e^{i\hat{N}\phi} e^{i\hat{p}\phi}$$  \hspace{1cm} (2)
and the LQC characteristic function:

\[
M(\tau, \theta) = \int \psi^*(c - \frac{\tau}{2}) e^{i\theta c} \psi(c + \frac{\tau}{2}) \, dc \tag{3}
\]

In homogeneous isotropic space a connection \(c\) is gauge and diffeomorphism invariant \([12], [5], [11]\), and therefore the operator \(-i \frac{d}{dc}\) is well defined.

The paper is organized as follows. In section 2 we derive the holonomy-flux characteristic function. In section 3 we obtain the LQC Wigner-Moyal operator. The discussion section 4 concludes the paper.

## 2 Holonomy-Flux Characteristic Function

Before we derive a Wigner-Moyal operator for the Loop Quantum Cosmology holonomy-flux algebra, we need to derive the LQC version of the Wigner’s characteristic function. We need to prove that the function we derive has the following property: when integrated by one variable it becomes the distribution density of the other variable. In addition we will prove that the first momentum integral used in Moyal’s formalism \([1]\) still exists when integration is performed with respect to the holonomy-flux measures. The Wigner function for LQC was first derived in \([6]\), however only in the dual space variables. We derive it first time in the original variables in order to find the Wigner-Moyal operator.

The holonomy-flux algebra for homogeneous isotropic space FLWR model is \([4] [5] [11]\)

\[
\left[ \hat{N}(\mu), \hat{p} \right] = -\frac{8\pi\gamma G\hbar}{3} \mu \hat{N}(\mu) \tag{4}
\]

The holonomy and flux operators act as follows:

\[
\hat{N}(\mu) \Psi(c) = e^{i\mu c} \Psi(c), \quad \hat{p} \Psi(c) = -i \frac{8\pi\gamma G\hbar}{3} \frac{d\Psi}{dc} \tag{5}
\]

The basis of the physical Hilbert space is given by LQC analogs of LQG spin-networks: \(\hat{N}(\mu) = e^{i\mu c}\), where \(c\) - is the configuration variable corresponding to the connection, \(\mu\) - the number of the fiducial cell repetition, \(c \in R_b\) - Bohr compactified real line, \(\mu \in R\).

The basis functions satisfy the relation:

\[
\langle N(\mu)|N(\mu') \rangle = \langle e^{i\mu c} e^{i\mu' c} \rangle = \delta_{\mu,\mu'} \tag{6}
\]

The FLWR holonomy-flux algebra commutator is of the form:

\[
[\hat{p}, \hat{N}] = a\mu \hat{N} \tag{7}
\]

where \(a\) is a constant:

\[
a = \frac{4\pi\gamma G\hbar}{3} \tag{8}
\]
We are going to obtain the LQC characteristic function $M(\tau, \theta)$ and its inverse Fourier transform, which is the Wigner function.

Let us first formally define the LQC Winger function and then prove that it satisfies the main property: when integrated by one variable it becomes the distribution density of the other variable. We define it as:

$$F(\mu, c) = \int \psi^*(c - a\tau)e^{-2ia\tau\mu}\psi(c + a\tau)d\tau \quad (9)$$

where $\psi(c)$ are the cylindrical functions of the $c \in R_b$, compactified real line. The cylindrical functions are of the form:

$$\psi(c) = \sum_{n=0}^{N} \hat{\Psi}_{\mu_n} e^{i\mu_n c}, \; \mu_n \in R \quad (10)$$

In order for $F(\mu, c)$ to have the meaning of the mutual quasi distribution function of $\mu$ and $c$ the following two equalities should be true. When integrating with respect to one variable it becomes the distribution density of the other one:

$$\rho_c = \int F(\mu, c)d\mu = |\psi(c)|^2 \quad (11)$$

and

$$\rho_\mu = \int F(\mu, c)dc = |\hat{\Psi}_\mu|^2 \quad (12)$$

In order to prove both equalities we use measures $dc$ and $d\mu$ as in [6] [9] [10].

$$\int_{\hat{R}_b} f_{\mu}d\mu = \sum_{\mu \in R} f_{\mu} \quad (13)$$

$$\int_{\hat{R}_b} e^{i\mu c}dc = \delta_{\mu,0} \quad (14)$$

where $\hat{R}_b$ is Bohr’s dual space, $\delta_{\mu,0}$ - a Kronecker delta.

The characters of the compactified line $R_b$ are the functions $h_{\mu}(c) = e^{i\mu c}$ [9]. The Fourier transform of the function on $R_b$ is given by:

$$\hat{f}_{\mu} = \int f(c)h_{-\mu}(c)dc \quad (15)$$

This is an isomorphism of $L^2(R_b, c) \rightarrow L^2(\hat{R}_b, \mu)$. $e^{i\mu c}$ comprise the basis of $H = L^2(R_b, dc)$. Let us prove the equalities (11) and (12). We begin with the first one. We substitute the expression (9) of $F(\mu, c)$ and the expression (10) for $\psi(c)$ into (11).

$$\int F(\mu, c)d\mu = \int \left( \sum_{n=0}^{N} \sum_{k=0}^{K} \hat{\Psi}_{\mu_n}^* e^{-ia\mu_n c}e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c}e^{ia\mu_k \tau} e^{-2ia\tau\mu}d\tau \right) d\mu \quad (16)$$
where $\tau \in R_b$, $\mu \in R$. The integration with respect to $\mu$ is just a sum as $\mu$ is discrete.

$$
\int F(\mu, c) d\mu = \sum_{\mu \in R} \sum_{n=0}^{N} \sum_{k=0}^{K} \hat{\Psi}_{\mu_n}^{*} e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau
$$

By collecting the terms containing $\tau$ and integrating with respect to $\tau$, by using (14) we obtain:

$$
\int e^{ia\mu_n \tau} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau = \delta_{2\mu_n, \mu_k + \mu_n}
$$

Since $\mu \in R$, summation by $\mu$ makes the terms with $2\mu \neq \mu_k + \mu_n$ equal zero and the terms with $\mu = \mu_k + \mu_n$ equal one and all terms with $\tau$ zero out in the integration and by using (10) we obtain:

$$
\int F(\mu, c) d\mu = \sum_{n=0}^{N} \sum_{k=0}^{K} \hat{\Psi}_{\mu_n}^{*} e^{-ia\mu_n c} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} = \psi^{*}(c)\psi(c) = |\psi(c)|^2
$$

In order to prove the equality (12), we substitute the expression (9) of $F(\mu, c)$ and the expression (10) for $\psi(c)$ into it.

$$
\int F(\mu, c) dc = \int \int \hat{\Psi}_{\mu_n}^{*} e^{-ia\mu_n c} e^{ia\mu_n \tau} \hat{\Psi}_{\mu_k} e^{ia\mu_k c} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau dc
$$

By substituting the expression (20) provides:

$$
\int e^{-ia\mu_n c} e^{ia\mu_k c} dc = \delta_{\mu_k - \mu_n, 0}
$$

Therefore only the terms with $\mu_k = \mu_n$ remain in (20). The integration with respect to $d\tau$ in turn gives

$$
\int e^{ia\mu_n \tau} e^{ia\mu_k \tau} e^{-2ia\tau \mu} d\tau = \delta_{2\mu_n, \mu_k + \mu_n}
$$

From (21) and (22) it follows that:

$$
\mu_n = \mu_k = \mu
$$

after substituting it into (20) we obtain:

$$
\int F(\mu, c) dc = \int \hat{\Psi}_{\mu_n}^{*} e^{-ia\mu c} e^{ia\mu \tau} \hat{\Psi}_{\mu_k} e^{ia\mu c} e^{ia\mu \tau} e^{-2ia\tau \mu} d\tau dc
$$

the integrals with respect to $d\tau$ and $dc$ are equal to one according to (14), so (24) becomes:

$$
\int F(\mu, c) dc = |\hat{\Psi}_{\mu}|^2
$$
Thus (19) and (25) imply that \( F(\mu, c) \) is an LQC Wigner function in \((c, \mu)\) variables. This completes the proof.

In addition we would like to consider the first momentum and prove the following property:

\[
\int F(\mu, c)e^{2i\alpha\tau_0 \mu}d\mu = \psi^*(c - a\tau_0)\psi(c + a\tau_0)
\]  \( (26) \)

,where \( c, \tau_0 \in \mathbb{R}_b \)

We begin by substituting the expression of \( F(\mu, c) \) (9) into the l.h.s. of (26)

\[
\int F(\mu, c)e^{2i\alpha\tau_0 \mu}d\mu = \int \int \psi^*(c - a\tau)e^{-2i\alpha\tau \mu}e^{2i\alpha\tau_0 \mu}\psi(c + a\tau)d\tau d\mu
\]  \( (27) \)

By using the expression (10) for the \( \psi(c) \) function, we obtain:

\[
\int F(\mu, c)e^{2i\alpha\tau_0 \mu}d\mu = \sum_{n=0}^{N} \sum_{k=0}^{K} \hat{\Psi}^*_n e^{-i\alpha\mu_n c} e^{i\alpha\mu_n \tau} \hat{\Psi}_{\mu_k} e^{i\alpha\mu_k c} e^{i\alpha\mu_k \tau} e^{-2i\alpha\tau \mu} e^{2i\alpha\tau_0 \mu} d\tau d\mu
\]  \( (28) \)

again, the integration by \( \mu \) can be replaced with the sum over \( \mu \)

\[
\int F(\mu, c)e^{2i\alpha\tau_0 \mu}d\mu = \sum_{\mu \in \mathbb{R}} \sum_{n=0}^{N} \sum_{k=0}^{K} \hat{\Psi}^*_n e^{-i\alpha\mu_n c} e^{i\alpha\mu_n \tau} \hat{\Psi}_{\mu_k} e^{i\alpha\mu_k c} e^{i\alpha\mu_k \tau} e^{-2i\alpha\tau \mu} e^{2i\alpha\tau_0 \mu} d\tau
\]  \( (29) \)

The integration by \( \tau \) gives us as before:

\[
\int e^{i\alpha\mu_n \tau} e^{i\alpha\mu_k \tau} e^{-2i\alpha\tau \mu} e^{2i\alpha\tau_0 \mu} d\tau = \delta_{2\mu, \mu_k + \mu_n}
\]  \( (30) \)

Which means that only those \( \mu \) satisfying \( 2\mu = \mu_k + \mu_n \) are equal to one after the integration, all the rest are zeros and we obtain:

\[
\int F(\mu, c)e^{2i\alpha\tau_0 \mu}d\mu = \sum_{n} \sum_{k} \hat{\Psi}^*_n e^{-i\alpha\mu_n c} \hat{\Psi}_{\mu_k} e^{i\alpha\mu_k c} e^{2i\alpha\tau_0 \mu}
\]  \( (31) \)

We substitute \( 2\mu = \mu_k + \mu_n \) into (31) and use the definition of the LQC cylindrical functions (10):

\[
\int F(\mu, c)e^{2i\alpha\tau_0 \mu}d\mu = \sum_{n} \sum_{k} \hat{\Psi}^*_n e^{-i\alpha\mu_n c} \hat{\Psi}_{\mu_k} e^{i\alpha\mu_k c} e^{i\alpha\tau_0 (\mu_n + \mu_k)} = \psi^* (c - a\tau_0) \psi(c + a\tau_0)
\]  \( (32) \)

or by taking \( \tau_0/2 \) instead of \( \tau_0 \) it can be rewritten in the form:

\[
\int F(\mu, c)e^{i\alpha\tau_0 \mu}d\mu = \sum_{n} \sum_{k} \hat{\Psi}^*_n e^{-i\alpha\mu_n c} \hat{\Psi}_{\mu_k} e^{i\alpha\mu_k c} e^{i\alpha\tau_0 \frac{\mu_n + \mu_k}{2}} = \psi^* (c - \frac{a\tau_0}{2}) \psi(c + \frac{a\tau_0}{2})
\]  \( (33) \)

This completes the proof of the equality (20).
3 LQC Wigner-Moyal Operator

In this section we would like to find the LQC Wigner-Moyal operator, i.e the operator with the following property:

\[ M(\tau, \theta) = \langle \psi^*, \hat{M} \psi \rangle \]  \hspace{1cm} (34)

where \( \psi \) are the LQC cylindrical functions \( (10) \). \( M(\tau, \theta) \) is the LQC characteristic function, which is the Fourier transform of the quasi probability density function \( F(\mu, c) \) with respect to the group characters:

\[ M(\tau, \theta) = \int \int F(\mu, c) e^{i\tau \mu} e^{i\theta c} d\mu \, dc \]  \hspace{1cm} (35)

By substituting \( F(\mu, c) \) expression from \( (9) \) into \( (35) \) we obtain:

\[ M(\tau, \theta) = \int \int \psi^*(c - a\tau_0) e^{-2ia\tau_0 \mu} \psi(c + a\tau_0) \, d\tau_0 \, e^{i\tau \mu} e^{i\theta c} d\mu \, dc \]  \hspace{1cm} (36)

after using the \( \psi(c) \) function definition \( (10) \) it becomes:

\[ M(\tau, \theta) = \int \int \sum_n \sum_k \hat{\psi}_{\mu_n} e^{-i\mu_n c} \hat{\psi}_{\mu_k} e^{i\mu_k c} e^{i\tau (\mu_k + \mu_n)} e^{i\theta c} d\tau_0 \, e^{i\tau \mu} e^{i\theta c} d\mu \, dc \]  \hspace{1cm} (37)

As in the previous section the integration with regard to \( \tau_0 \) provides the condition by which all terms in the sum zero out except those \( \mu_n \) and \( \mu_k \) satisfying:

\[ 2\mu = \mu_k + \mu_n \]  \hspace{1cm} (38)

The integration over \( \tau_0 \) and \( \mu \) provides one as a result. After substituting instead of \( \mu \) its expression from \( (38) \) into \( (37) \) we obtain:

\[ M(\tau, \theta) = \int \int \sum_n \sum_k \hat{\psi}_{\mu_n} e^{-i\mu_n c} \hat{\psi}_{\mu_k} e^{i\mu_k c} e^{i\tau (\mu_k + \mu_n)} e^{i\theta c} d\mu \, dc \]  \hspace{1cm} (39)

Finally we can combine the terms in the exponents and by using the LQC cylindrical function definition \( (10) \), write the formula as \( \psi(c - \frac{\tau}{2}) \) and \( \psi(c + \frac{\tau}{2}) \):

\[ M(\tau, \theta) = \int \psi^*(c - \frac{\tau}{2}) e^{i\theta c} \psi(c + \frac{\tau}{2}) \, dc \]  \hspace{1cm} (40)

Thus we have found the LQC characteristic function \( M(\tau, \theta) \) as a Fourier transform of \( F(\mu, c) \). Now let us prove that the operator

\[ \hat{M}(\tau, \theta) = e^{i\tilde{\tau} \tilde{\theta}} \hat{h}_\theta e^{i\tilde{\tau} \tilde{\theta}} = e^{i\tilde{\tau} \tilde{\theta}} e^{i\theta c} e^{i\tau \tilde{\theta}} \]  \hspace{1cm} (41)

is an LQC Wigner-Moyal operator, i.e it satisfies the property \( (34) \) for the found LQC characteristic function \( (40) \). We can prove \( (34) \) directly by substituting the operator \( (41) \) into \( (34) \)

\[ \langle \psi^*, \hat{M} \psi \rangle = \int \psi^*(c) e^{i\tilde{\tau} \tilde{\theta}} e^{i\theta c} e^{i\tau \tilde{\theta}} \psi(c) \, dc \]  \hspace{1cm} (42)
and expanding the exponents into the Taylor series. We obtain:

\[ \langle \psi^*, \hat{M} \psi \rangle = \int \psi^* (c) (1 + \frac{i \tau}{2a} (-ia \frac{d}{dc} + \ldots) e^{i \theta c} (1 + \frac{i \tau}{2a} (-ia \frac{d}{dc} + \ldots) \psi(c) \, dc \]

\[ = \int \psi^* (c - \frac{\tau}{2}) e^{i \theta c} \psi(c + \frac{\tau}{2}) \, dc = M(\tau, \theta) \quad (43) \]

this is exactly what we aimed to prove:

\[ \langle \psi^*, \hat{M} \psi \rangle = M(\tau, \theta) \quad (44) \]

Thus \( \hat{M}(\tau, \theta) = e^{i \hat{\mathcal{H}}_\theta} \hat{\hat{\mathcal{H}}}_\theta = e^{i \hat{\mathcal{H}}_\theta} e^{i \theta c} e^{i \hat{\mathcal{H}}_\theta} \) is an LQC Wigner operator.

4 Discussion

In this paper we have found the expression of the LQC Wigner-Moyal operator in case of homogeneous and isotropic space, i.e the operator with the property:

\[ M(\tau, \theta) = \langle \psi^*, \hat{M} \psi \rangle \quad (45) \]

where \( \psi \) are the LQC cylindrical functions \((10)\), \( M(\tau, \theta) \) is the LQC characteristic function, which is the Fourier transform of the the quasi probability density function \( F(\mu, c) \) with respect to the group characters:

\[ F(\mu, c) = \int \psi^* (c - a \tau) e^{-2ia \tau \mu} \psi(c + a \tau) d\tau \quad (46) \]

The found Wigner-Moyal operator has the form:

\[ M(\tau, \theta) = e^{i \hat{\mathcal{H}}_\theta} \hat{\hat{\mathcal{H}}}_\theta = e^{i \hat{\mathcal{H}}_\theta} e^{i \theta c} e^{i \hat{\mathcal{H}}_\theta} \], where \( \hat{\hat{\mathcal{H}}}_\theta = -ia \frac{d}{dc} \quad (47) \)

In homogeneous isotropic space a connection \( c \) is gauge and diffeomorphism invariant \([12, 5, 11]\), and therefore the operator \(-i \frac{d}{dc}\) is well defined.

The found LQC characteristic function is of the form:

\[ M(\tau, \theta) = \int \psi^* (c - \frac{\tau}{2}) e^{i \theta c} \psi(c + \frac{\tau}{2}) \, dc \quad (48) \]

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