Bargmann representations for deformed harmonic oscillators

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Abstract

Generalizing the case of the usual harmonic oscillator, we look for Bargmann representations corresponding to deformed harmonic oscillators. Deformed harmonic oscillator algebras are generated by four operators $a, a^\dagger, N$ and the unity $1$ such as $[a, N] = a, \ [a^\dagger, N] = -a^\dagger, \ a^\dagger a = \psi(N)$ and $aa^\dagger = \psi(N+1)$. We discuss the conditions of existence of a scalar product expressed with a true integral on the space spanned by the eigenstates of $a$ (or $a^\dagger$). We give various examples, in particular we consider functions $\psi$ that are linear combinations of $q^N, q^{-N}$ and unity and that correspond to $q$-oscillators with Fock-representations or with non-Fock-representations.

1 Introduction

The harmonic oscillator Lie-algebra is defined by four operators : the annihilation operator $a$, the creation operator $a^\dagger$, the energy operator $N$ and the unity $1$ satisfying the following commutation relations :

$$[a, N] = a, \ [a^\dagger, N] = -a^\dagger$$

and
\[ [a, a^\dagger] = 1 \]  

where \( a^\dagger \) is the adjoint of \( a \) and \( N \) is self-adjoint.

This algebra has been deformed in many different ways (see in particular \[1\], \[2\], \[3\], \[4\], \[5\], \[6\]) and the representations of the deformed algebras were widely studied. In this paper, the deformed harmonic oscillator is defined by the relations \([1]\) and by the following relations between the three operators \( a, a^\dagger \) and \( N \):

\[ a^\dagger a = \psi(N), \quad aa^\dagger = \psi(N + 1) \]  

where \( \psi \) is a real analytical function.

In the other formulations encountered in the literature, \([3]\) is replaced by:

\[ [a, a^\dagger] = f(N, q) \]  

or

\[ [a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = f_q(N, q) \]

In these approach, the function \( \psi \) is not given but results of solving the equations:

\[ f(N, q) = \psi(N + 1) - \psi(N) \text{ or } f_q(N, q) = \psi(N + 1) - q\psi(N) \]

The resolution of these equations leads to some arbitrariness that is eliminated in our formulation, \( f \) and \( f_q \) being uniquely determined in terms of \( \psi \).

Let us give some examples:
- the usual harmonic oscillator defined by \( f(N) = 1 \) corresponds to \( \psi_{\text{usual}}(N) = N + \sigma \).
- the \( q \)-oscillator \([1], [2]\) defined by \( f_q(N, q) = q^{-N} \) corresponds to

\[ \psi_{\text{qosc}}(N) = -q^{-N}/(q - q^{-1}) + \sigma q^N/(q - q^{-1}), \forall \sigma. \]

- the \( q \)-oscillator defined by \( f_q(N, q) = 1 \) corresponds to

\[ \psi'_{\text{qosc}}(N) = (1 - q)^{-1} + \sigma q^N, \forall \sigma. \]
- with the usual notation:

\[ [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{9} \]

the function \( \psi_{suq(2)}(N) = \sigma - [N - 1/2]^2, \forall \sigma \), corresponds to \( f(N) = -[2N] \); that is to the deformation \( su_q(2) \) of the Lie-algebra \( su(2) \) after the identification \( a = L_-, a^\dagger = L_+ \) and \( L_z = N \).

- \( su_q(1, 1) \) is obtained for the \( \psi_{suq(1,1)}(N) = -\psi_{suq(2)}(N) \).

Generalizing the pioneer work of Bargmann for the usual harmonic oscillator, the purpose of this paper is to study if the deformed harmonic oscillator defined by (9) and (3) admits representations on one space of complex variable functions. In [8], we restricted to the case where the function \( \psi \) does not vanish. The scalar product of the representations we are looking for, is written with a true integral as in ([7], [8], [10], [11], [12]) and contrarily to the works of ([13], [14], [15], [16], [17], [18], [19]) where a \( q \)-integration occurs.

In section 2, we recall how to build the irreducible representations on the basis of the eigenvectors of \( N \). They are determined by the spectrum of \( N \) which is depending on the zeros of \( \psi \). Then, in section 3, we discuss the existence of the coherent states that are defined as the eigenstates of the operators \( a \) (or \( a^\dagger \)). In section 4, we study the possibility of Bargmann representations. The formulation of the problem is done in a general framework. We show on various examples how the construction works: section 5 is devoted to strictly positive function \( \psi \), other cases are considered in section 6.

## 2 Representations

Let \( |0> \) be the eigenvector of \( N \) with eigenvalue \( \mu \). We built the normalized vectors \( |n> \)

\[
| n > = \begin{cases} 
\lambda_n a^n |0>, & n \in N^+ \\
\lambda_n a^{-n} |0>, & n \in N^- 
\end{cases} \tag{10}
\]

with

\[
\lambda_n^{-2} = \psi(\mu + n)! = \begin{cases} 
\prod_{i=1}^{n} \psi(\mu + i), & n \in N^+ \\
\prod_{i=0}^{n+1} \psi(\mu + i), & n \in N^- 
\end{cases} \tag{11}
\]

3
\( N^+ \) and \( N^- \) are the set of integers \( \geq 0 \) and \( < 0 \).
The vectors \( | n > \) are the eigenvectors of \( N \) with eigenvalue \( \mu + n \) and span the Hilbert space \( \mathcal{H} \). As \( < n | a a^\dagger | n > \) is necessarily positive or zero, the construction of the increasing states stops if it exists an integer \( \nu_+ \) such that
\[
\psi(\mu + \nu_+ + 1) = 0
\]
in which case the representation labelled by \( \mu \) and \( \nu_+ \) admits a highest weight state \( | \nu_+ > \). We have an analogous situation for the decreasing states built with \( a \), when it exists an integer \( \nu_- \) such as \( \psi(\mu + \nu_-) = 0 \). The representation labelled by \( \mu \) and \( \nu_- \) then admits a lowest weight state \( | \nu_- > \).

We get different types of representations [3], [4], [5], [6]:

1) \( \psi \) has no zero.
The inequivalent representations are labelled by the decimal part of \( \mu \) and are defined by:
\[
\begin{align*}
| a^\dagger | n > &= (\psi(\mu + n + 1))^{1/2} | n + 1 > \\
| a | n > &= (\psi(\mu + n))^{1/2} | n - 1 >, \quad n \in \mathbb{Z} \\
| N | n > &= (\mu + n) | n >
\end{align*}
\]
The spectrum of \( N \), \( SpN \), is \( \mu + Z \). The operator \( N \) has no lowest and no highest eigenstates. These representations, thus, are non equivalent to Fock-representations and are called non-Fock-representations [20], [21]. It is the case when \( \psi \) is equal to \( \psi_{\text{qosc}} \) with \( q \in [0,1] \) and \( \sigma \leq 0 \).

An interpretation of this case [12] is obtained by identifying the states \( | n > \) to the functions on a circle.

2) \( \psi \) has zeros.
We are interested in the intervals where \( \psi \) is positive:

a) finite intervals
We can associate a representation to the intervals that have a length equal to an integer.
The spectrum of \( N \) is \( [\mu + \nu_-, \mu + \nu_+] \cap Z + \mu \).

For example, in the case \( \psi_{\text{suq}(2)} \), when \( \sigma = [l + 1/2]^2 \), \( l \) being a positive integer, the dimension of the representation is \( 2l + 1 \) and verifies
\[
\begin{align*}
| a^\dagger | l, m > &= ([l + 1/2]^2 - [m + 1 - \frac{1}{2}])^{1/2} | l, m + 1 > \\
| a | l, m > &= ([l + 1/2]^2 - [m - \frac{1}{2}])^{1/2} | l, m - 1 > \\
| N | l, m > &= m | l, m >
\end{align*}
\]
b) infinite intervals
The representations are similar to the Fock-representation of the usual harmonic oscillator.

The spectrum of $N$ is $\mu + \nu_- + N^+$ or $\mu + \nu_+ + N^-$. Let us give an example:
when $\psi$ is equal to $\psi_{qosc}$ with $\sigma = 1$, we recover the usual q-oscillator case such as

$$
\begin{align*}
\left\{ a^\dagger \mid n > = \right. & \left. [n + 1]^{1/2} \mid n + 1 > \\
a \mid n > = & \left. [n]^{1/2} \mid n - 1 >, \quad n \in \mathbb{Z}^+ \right.
\end{align*}
(15)
$$

The first step to build a Bargmann representation requires to study the coherent vectors.

3 Coherent states
We call coherent states [22], the eigenvectors of the operator $a$ or $a^\dagger$.

The state $\mid z > = \sum_p c_p \mid p >$ is an eigenvector of $a$ if the coefficients $c_p$ verify the recursive relation

$$
z c_p = \psi(\mu + p + 1)^{1/2} c_{p+1}
(16)
$$

- When the spectrum of $N$ is upper bounded, (16) implies that all the $c_p$ vanish and then that $a$ has no eigenvectors.

If we look for the eigenvectors of $a^\dagger$, the situation is analogous : $a^\dagger$ has no eigenvectors if the spectrum of $N$ is lower bounded. Therefore, in the case (2.a) of the previous section as the spectrum of $N$ is finite, $a$ and $a^\dagger$ have no eigenvectors, hence no Bargmann representation exists.

- When the spectrum of $N$ is no upper bounded, the eigenvectors $\mid z >$ of $a$ take the form :
\[
\begin{aligned}
\text{when } & SpN = Z + \mu, \\
|z> = & \sum_{n=-1}^{\infty} z^n (\psi(\mu + n)!)^{1/2} |n> + \sum_{n=0}^{\infty} \frac{z^n}{\psi(\mu + n)!^{1/2}} |n>, \\
\text{when } & SpN = \nu_- + \mu + N^+, \\
|z> = & \sum_{n=-1}^{\nu_-} z^n (\psi(\mu + n)!)^{1/2} |n> + \sum_{n=0}^{\infty} \frac{z^n}{\psi(\mu + n)!^{1/2}} |n>, \quad \nu_- < 0 \\
|z> = & \sum_{n=\nu_-}^{\infty} z^n (\psi(\mu + n)!)^{-1/2} |n>, \quad \nu_- \geq 0
\end{aligned}
\]

(17)

with the convention $\psi(\mu)! = 1$. 

The domain $D$ of existence of the coherent states depends on the function $\psi$. Indeed, $|z>$ belongs to the Hilbert space spanned by the basis $|n>$ only if the series in the left hand side of (17) are convergent in norm.

- When $SpN = Z + \mu$, this implies that:

\[
|z| < \lim_{p \to \infty} \psi(p)^{1/2} = r_2
\]

(18)

and

\[
|z| > \lim_{p \to -\infty} \psi(p)^{1/2} = r_1
\]

(19)

Thus when $r_2 = 0$, the annihilation operator has no eigenvectors. When $r_1$ is smaller than $r_2$, the eigenstates of $a$ exist and their domain is $r_1 < |z| < r_2$. When $r_1$ is larger than $r_2$, the annihilation operator $a$ has no eigenstates, but then we can establish by analogous reasoning that the creation operator $a^\dagger$ has eigenvectors if $r_1 \neq 0$.

- When the spectrum of $N$ is lower bounded, $SpN = \mu + \nu_- + N^+$, the second condition (19) does not exist and the eigenstates of $a$ always exist provided $r_2 \neq 0$; their domain is defined by $|z| < r_2$.

When the spectrum of $N$ is upper bounded, $SpN = \mu + \nu_+ + N^-$, the eigenvectors of $a^\dagger$ exist only if $|z| < r_1$.

To summarize, the eigenvectors of $a$ exist if:
- $SpN = \mu + Z$ and $r_2^2 \equiv \psi(+\infty) > r_1^2 \equiv \psi(-\infty)$, the domain of existence $D$ of the coherent states is $D = \{z; r_1 < |z| < r_2\}$ or
- $SpN = \mu + \nu_- + N^+$, then $D = \{z; |z| < r_2\}$.

The eigenvectors of $a^\dagger$ exist if:
- $SpN = \mu + Z$ and $r_2 < r_1$, then $D = \{z; r_2 < |z| < r_1\}$ or
- $SpN = \mu + \nu_+ + N^-$, then $D = \{z; |z| < r_1\}$.

We do not study here the case where $r_1 = r_2$.

Although $\mu$ is a significant quantity as labelling inequivalent representations, it does not play a part in the present problem. So we simplify the notation in assuming $\mu = 0$ from now on. Indeed, this is equivalent to substitute $N - \mu$ to $N$ and $\psi_\mu(N) = \psi(\mu + N)$ to $\psi(N)$.

4 Bargmann representation

4.1 Representation space

Following the construction [7], in the Bargmann representation any state $|f>$ of $\mathcal{H}$:

$$|f> = \sum_{n \in SpN} f_n |n>, \quad \sum_{n \in SpN} |f_n|^2 < \infty$$

is represented as the function of a complex variable $z$, $f(z) = <\overline{z}|f>$, with a Laurent expansion:

$$f(z) = \begin{cases} 
\sum_{n \geq 0} \frac{z^n}{\psi(n)!} + \sum_{n < 0} z^n \frac{f_n(\psi(n)!)}{1/2}, & SpN = Z, \\
\sum_{n \geq 0} \frac{z^n}{\psi(n)!} + \sum_{n < 0} z^n \frac{f_n(\psi(n)!)}{1/2}, & SpN = \nu_- + N^+, \quad \nu_- < 0, \\
\sum_{n \geq \nu_-} z^n \frac{f_n(\psi(n)!)}{-1/2}, & SpN = \nu_- + N^+, \quad \nu_- \geq 0,
\end{cases}$$

(21)
on the domain $D$ of definition of the eigenvectors of $a$, $r_1 < |z| < r_2$. The
space of the representation $S$ is constituted with holomorphic functions in
$D$, strongly depending on $\psi$ ( cf. subsection (4.3)).
In particular, to the basis vectors $|n>, n \in SpN$, correspond the monomials :
\[
< \varpi | n > = \begin{cases} 
z^n(\psi(n))^{-1/2}, & n \geq 0 \\
z^n(\psi(n))^{1/2}, & n < 0 
\end{cases}
\tag{22}
\]

### 4.2 Resolution of unity

A Bargmann representation exists if we can exhibit a positive real function
$F(x)$ such as
\[
\int F(z\varpi) | \varpi > < \varpi | dzd\varpi = 1 
\tag{23}
\]
where the integration is extended to the whole complex plane and where
$F(|z|^2)$ contains the characteristic function of the domain $D$ of existence of
the coherent states. Denoting $z = \rho e^{i\theta}$, Equation (23) reads :
\[
\int_0^{2\pi} d\theta \int_0^{r_2^2} F(\rho^2) d\rho^2 | \rho e^{-i\theta} > < \rho e^{i\theta} | = 1 
\tag{24}
\]

From the resolution of unity (23), we obtain the scalar product in $S$ :
\[
(g | f) = \int F(z\varpi)f(z)\overline{g(z)}dzd\varpi 
\tag{25}
\]
and the representation of any linear operator $K$ of $H$ by the kernel $K(\zeta, \varpi) =
< \varpi | K | \varpi >$ such as :
\[
(Kf)(\zeta) = \int F(z\varpi)K(\zeta, \varpi)f(z)dzd\varpi 
\tag{26}
\]

### 4.3 Reproducing Kernel

Let $G(x)$ be the function :
\[
G(x) = \begin{cases} 
\sum_{n \geq 0} x^n(\psi(n))^{-1} + \sum_{n < 0} x^n\psi(n)!, & SpN = Z, \\
\sum_{n \geq 0} x^n(\psi(n))^{-1} + \sum_{n < 0} x^n\psi(n)!, & SpN = \nu_- + N^+, \ \nu_- < 0 \\
\sum_{n \geq \nu_-} x^n(\psi(n))^{-1}, & SpN = \nu_- + N^+, \ \nu_- \geq 0 
\end{cases}
\tag{27}
\]
We easily verify that \( G(\zeta z) = \langle z \mid \zeta > \), so that \( G(\zeta z) \) is the function of \( S \) representing the coherent state \( \mid \zeta > \). The function \( G(x) \) plays a prominent part for, if (23) should be true, we will have

\[
\int F(z \bar{z}) \langle z \mid \bar{z} > < \bar{z} \mid f > dz d\bar{z} = \langle \zeta \mid f >
\]

(28)

and \( G(z \bar{z}) \) appears to be one reproducing kernel. Moreover, we get from the Schwarz inequality:

\[
|f(z)| \leq |G(z \bar{z})|^{1/2}
\]

(29)

Thus, \( S \) is the set of holomorphic functions the growth of which on the edge of \( D \), is controlled by the growth of \(|G(x)|^{1/2}\).

We easily prove that, in the Bargmann representation, \( a^\dagger \) is the multiplication by \( z \) and \( N \) is the operator \( zd/dz \), as in the usual case, while \( a \) is the operator \( z^{-1}\psi(zd/dz) \). As \( G(\zeta z) \) corresponds to the coherent state \( \mid \zeta > \), we obtain for \( G \) the following equation

\[
xG(x) = \psi(x \frac{d}{dx})G(x)
\]

(30)

which could be obtained directly from the expansion (27).

### 4.4 Weight Function

Let us introduce the Mellin transform \( \hat{F}(\rho) \) of the weight function \( F(x) \):

\[
\hat{F}(\rho) = \int_0^\infty F(x)x^{\rho-1}dx = \int_{r_1^2} F(x)x^{\rho-1}dx
\]

(31)

From (22) and (23), we deduce that \( F(x) \) must verify the following condition:

\[
\hat{F}(n+1) = \begin{cases} 
\psi(n)!, & n \geq 0 \\
(\psi(n)!)^{-1}, & n < 0, n \in SpN
\end{cases}
\]

(32)

Let us remark that

\[
\hat{F}(\rho) \leq \hat{F}(n) + \hat{F}(n+1), \quad n \leq Re\rho < n + 1
\]

(33)

as \( F(x) \) is a positive function. As the Mellin transform of \( F \) exists for all the integers belonging to the spectrum of \( N \), then, due to (33), it exists for any
\[ \rho \text{ such as } \text{Re}\rho \text{ is greater than the lowest bound } \nu_\cdot \text{ of } SpN, \nu_\cdot \text{ can be finite or } -\infty. \]

Formula (32) is equivalent to

\[ \hat{F}(n + 1) = \psi(n)\hat{F}(n), \text{ with } \hat{F}(1) = 1 \]  

(34)

which ensures that the operators \( a^\dagger = z, a = z^{-1}\psi(zd/dz) \) be adjoint on the basis \(|n>\).

The function \( \psi \) being given, \( \hat{F} \) verifying (34) must be interpolated in order to get \( F \) as the Mellin inverse of (31).

Equation (34) can be obviously interpolated by:

\[ \hat{F}(\rho + 1) = \psi(\rho)\hat{F}(\rho) \]  

(35)

Any other interpolation of \( \hat{F}(n) \) is obtained by:

\[ \hat{F}_h(\rho) = \hat{F}(\rho)\hat{h}(\rho) \]  

(36)

where \( \hat{F}(\rho) \) is some solution of (35) and where \( \hat{h}(\rho) \) is equal to 1 on the spectrum of \( N \).

In order to tackle the discussion on the existence of a Bargmann representation, we first state the following lemmas:

**Lemma 1**: If there exists one particular solution of (35) which does not admit a Mellin inverse, the same holds for every solution of (35).

**Proof**: Two solutions of (35) differ by a multiplicative factor \( \hat{k}(\rho) \) which is a periodic function of period 1:

\[ \hat{k}(\rho + 1) = \hat{k}(\rho) \]  

(37)

Let \( \hat{F}(\rho) \) be one particular solution of (35), without Mellin inverse. We cannot find \( \hat{k}(\rho) \) such as \( \hat{k}(\rho)\hat{F}(\rho) \) has a Mellin inverse. Indeed, the behavior of \( \hat{k}(\rho) \) at infinities on a parallel to the imaginary axis must compensate the corresponding bad behavior of \( \hat{F}(\rho) \). This implies that \( \hat{k}(\rho) \) has a Mellin inverse \( k(x) \). Then from Equation (37), we obtain that \( k(x) \) must verify

\[ x\hat{k}(x) = k(x) \]  

(38)

and then \( k(x) \) is equal to \( k\delta(x - 1) \), so that \( \hat{k}(\rho) \) is a constant in contradiction to the assumption that \( \hat{k}(\rho)\hat{F}(\rho) \) have a Mellin inverse.
Lemma 2: If \( \hat{h}(\rho) \) is equal to 1 on \( SpN \) and admits a Mellin inverse \( h(x) \), \( \hat{h}(\rho) \) is the identity.

Proof: Let us assume that \( SpN \) contains \( N^+ \). Then we have:

\[
\hat{h}(n+1) = \int_0^\infty h(x)x^n dx = 1
\]  

(39)

From this equation, we easily deduce:

\[
\int_0^\infty h(x)e^{\rho x} dx = e^\rho
\]  

(40)

so that \( h(x) = \delta(x-1) \) and \( \hat{h}(\rho) = 1 \).

The same method works when \( SpN = \nu_+ + N^+, \nu_+ > 0 \).

Discussion:

1) If there exists a solution of (35) with a positive Mellin inverse, our problem is solved.

2) If we exhibit a solution of (35) without Mellin inverse, we know from Lemma 1 that no solution of (35) with Mellin inverse exists and from Lemma 2 that these solutions cannot be corrected by a suitable multiplicative factor.

3) If there exists a solution of (35) with a Mellin inverse not strictly positive on \( D \), we cannot conclude, for we cannot ensure this solution cannot be improved by a suitable multiplicative factor leading to positivity.

Therefore, a Bargmann representation can be obtained only if there exist solution of (35) with Mellin inverse, so that the interpolation problem (34) is completely solved by (35) precisely.

Concerning the remaining positivity condition, let us point out that it will be satisfied if \( \psi(\rho) \) is such that \( \psi(\rho + i \sigma) \) is a definite positive function of \( \sigma \) for any \( \rho \in SpN \). This can be proved by an adaptation to the Mellin transform of the well-known Bochner theorem for Fourier transform. But this condition on \( \psi(\rho) \) is only a sufficient condition as it is readily seen from the counter-example of the usual oscillator: \( \psi(\rho) = \rho \).

Therefore, the only interpolation of (34) to be considered in the following is precisely the simplest one, namely (35).

Remains one case where we cannot conclude, namely when the Mellin inverse of the solution of (35) exists but is not positive on \( D \), we cannot ensure that the result can be improved by a suitable factor leading to the positivity.

To summarize, a Bargmann representation can be defined on a deformed harmonic oscillator algebra if the coherent states exist and if it exists at least
one solution of (35) with a positive Mellin inverse on D. Furthermore, let us remark that (35) can be written:

\[ \int F(x)x^{\rho}dx = \int F(x)\psi(x\partial_x + 1)x^{\rho-1}dx \]  

(41)

The right member is equal to \( \int (\psi(-x\partial_x)F(x))x^{\rho-1}dx \) if the integration by parts (or the change of variables) can be done without extra terms and this gives:

\[ xF(x) = \psi(-x\frac{d}{dx})F(x) \]  

(42)

This equation, when it holds, can be used to study the positivity of \( F(x) \), when we cannot obtain an explicit expression for this function.

In the next section [8], we restrict to strictly positive \( \psi \) and to \( F \) solution of (42), fast decreasing at infinity and at the origin.

In section 6, we illustrate the construction of Bargmann representation with specific vanishing functions \( \psi \), in particular we discuss the cases of the q-oscillators, looking for a resolution of the identity involving a true integral.

5 \( \psi \) strictly positive

In this section, the eigenvalues of \( N \) are all the integers. The coherent states lie in the whole complex plane, except the origin, in the two first subsections and in a ring in the last one.

5.1 the q-oscillator

The function \( \psi \) associated to the q-oscillator (7) is strictly positive if \( q < 1 \) and \( \sigma \leq 0 \). We easily see that when \( \sigma \neq 0 \), the coherent states do not exist. Let \( \psi_{\lambda,q}(N) \) be equal to \( \lambda q^{-N}, q \leq 1, \lambda > 0 \). When \( \lambda = (q^{-1} - q) \), we get the function \( \psi_{qosc} \) for \( \sigma = 0 \) previously introduced and corresponding to the q-oscillator algebra (4) with non-Fock representation and with coherent states.

Let us remark that this function also corresponds to \( aa^\dagger = q^{-1}a^\dagger a \).

In this case, the functional equations (35):

\[ \hat{F}_{\lambda,q}(\rho + 1) = \lambda q^{-\rho}\hat{F}_{\lambda,q}(\rho) \]  

(43)
and

\[ xF_{\lambda,q}(x) = \lambda F_{\lambda,q}(qx) \] (44)

are equivalent for \( F_{\lambda,q}(x) \) has to be fast decreasing at infinity and at the origin.

Introducing \( \hat{F}_{\lambda,q}(\rho) \), we solve the resulting equation in terms of first and second Bernoulli polynomials so that we get the particular solution:

\[ \hat{F}_{\lambda,q}^0(\rho) = \exp \left( \rho \ln \lambda - \frac{1}{2}(\rho^2 - \rho) \ln q \right) \] (45)

Taking the inverse Mellin transform, we obtain the following particular solution of (44) [12] [8]:

\[ F_{\lambda,q}^0(x) = \exp \left( \frac{\ln^2(x\lambda^{-1})}{2\ln(q)} - \frac{\ln(x\lambda^{-1})}{2} \right) \] (46)

The general solution of (44) is given by:

\[ F_{\lambda,q}(x) = F_{\lambda,q}^0(x) h_{\lambda,q}(x) \] (47)

where \( h_{\lambda,q}(x) \) is a function satisfying

\[ h_{\lambda,q}(x) = h_{\lambda,q}(qx) \] (48)

that is a periodic function of \( \ln x / \ln q \) of period 1. We verify directly that up to a constant the momentums \( M(n) \) of \( F_{\lambda,q}^0(x) \) are well equal to \( \lambda^{-n} q^{-n(n+1)/2} \) for \( n \in \mathbb{Z} \), as wanted, and this is yet true for \( F_{\lambda,q}(x) \) given in (47) as implied by the definition of \( h_{\lambda,q}(x) \).

As the absolute value of a solution of equation (48) is also a solution, the function \( h_{\lambda,q}(x) \) can always be taken strictly positive. Nevertheless, there exist an infinite number of \( h_{\lambda,q}(x) \), in particular all the powers of any solution, so that we get a lot of candidates to define the norm we are looking for. But mutatis mutandis, the situation is analogous to that initially studied by Bargmann and we can extend the largest part of its proof.

Particularly we can prove that the set \( S \) of holomorphic functions under consideration is the set of functions \( f(z) \) such that:

\[ |f(z)| \leq C \exp \left( -\frac{\ln^2(x\lambda^{-1})}{4\ln q} - \frac{\ln(x\lambda^{-1})}{4} \right) \times \sum_{-\infty}^{+\infty} \exp \left( \frac{\ln q}{2} \left( \frac{\ln(x\lambda^{-1})}{\ln q} + n + \frac{1}{2} \right)^2 \right) \] (49)
The representation Hilbert space is the Hilbert space defined on $S$ by

$$(f,g)_F = \int F_{\lambda,q}(z\bar{z}) f(z) \overline{g(z)} dz d\bar{z}$$

(50)

where $F_{\lambda,q}(z\bar{z})$ is a positive function of the form given in equation (47). The various norms such obtained are proportional as verified on the basis elements.

Furthermore, following always Bargmann’s procedure, we can prove the closeness of the operator $z$ and $z^{-1}q^z \frac{d}{dz}$. Moreover we have:

$$|zf(z)|^2 = \sum_{-\infty}^{\infty} |f_n|^2 \psi(n+1)$$

(51)

and

$$|z^{-1}\psi(z \frac{d}{dz} f(z)|^2 = \sum_{-\infty}^{\infty} |f_n|^2 \psi(n)$$

(52)

Using the two previous equations, we obtain:

$$|zf(z)|^2 = q |z^{-1}q^z \frac{d}{dz} f(z)|^2$$

(53)

This proves that the operators $a$ and $a^\dagger$ have the same domain of definition and are mutually adjoint [7].

We have proved the existence of Bargmann representations for the deformed oscillator defined by (1) and (3) when $\psi_{q^{-1}-q,q}(N) = (q^{-1} - q)q^{-N}$. The resolution of identity is not unique and is given explicitly by (23), (46) and (47).

### 5.2 Generalization of the previous example

In this section, we point out some directions to extend the results of the previous section when $\psi(x)$ is of the form:

$$\psi(x) = \exp \left( \sum_{0}^{2p+1} a_n x^n \right), \quad a_{2p+1} > 0$$

(54)

A solution of Equation (55) is:
\[
\hat{F}(\rho) = \exp \left( \sum_{0}^{2p+1} \frac{a_n}{n+1} B_{n+1}(\rho) \right)
\]  
(55)

where \( B_{n+1}(\rho) \) are the Bernoulli polynomials \(^{26}\) defined by the difference equation:

\[
B_n(x + 1) - B_n(x) = nx^{n-1}
\]  
(56)

The term of highest degree in (55) is \( a_{2p+1} \rho^{2p+2}/(2p + 1) \). When \( \rho \) is pure imaginary \( \rho = i\sigma, \sigma \in \mathbb{R} \), this term is \( a_{2p+1}(-1)^{p+1} \sigma^{2p+2}/(2p + 1) \), therefore the inverse Mellin transform of \( \hat{F}(\rho) \) exists only if \( p \) is an even number. The function \( F(x) \) thus obtained is always real, but not necessarily positive. Nevertheless, in specific cases, for example when the exponent in (55) contains only the term of highest degree, \( F(x) \) is actually strictly positive and the Bargmann procedure works as before.

Since \( \psi(n + 1)/\psi(n) \) grows indefinitely as \( n \to \pm\infty \), Equations (51) and (52) imply that the domain of \( z^{-1}\psi(z \frac{d}{dz}) \) is included in the domain of \( z \) but cannot be identical. Nevertheless, the mutual adjointness can be proved on the basis, using equation (34).

It is worthwhile to underline that this generalization gives an example where the Bargmann representation does not exist, namely when \( p \) is odd.

### 5.3 The ring case

Let us consider the simple case \( \psi(x) = a + qx, \ q > 1, \ a > 0 \), that is mainly involved in the study of the q-oscillator \(^8\) with non-Fock representations. The domain of existence of the coherent states is \( a \leq |z|^2 \). The momentum \( \hat{F}(n) \) reads:

\[
\hat{F}(n) = \int_{a}^{+\infty} F(x)x^{n-1}dx
\]  
(57)

We first prove that equations (34) and (42) are not equivalent if \( F(x) \) is positive on the whole positive axis. Indeed, let us start with a solution of (42), Equation (34) reads:

\[
\int_{q^{-1}a}^{a} F(x)x^{n-1}dx = 0
\]  
(58)
that is obviously impossible if \( F(x) \) is positive on \([q^{-1}a, a]\). Therefore in this case, the momentums deduced from the weight function solution of (12) are not the expected ones (solutions of (14)). Moreover, in [8], we proved that the solution of (12) is identically zero.

Let us look for a solution of (35) that cannot have poles, due to (33) :

\[
\hat{F}(\rho + 1) = (q^\rho + a)\hat{F}(\rho)
\]

(59)

We have as a convenient particular solution the following entire function :

\[
\hat{F}(\rho) = a^\rho \prod_{p \geq 0} (1 + a^{-1}q^\rho - p - 1)
\]

(60)

but it is not a Mellin transform of a true function \( F(x) \). Indeed if it has an inverse Mellin transform, it can be calculated on any parallel to the imaginary, for instance on \( \text{Re} \rho = \ln a/\ln q \). On this axis, \( |\hat{F}(iy)| \geq \prod_{p \geq 0} (1 - q^{-p}) \), so that (60) is not the Mellin transform of a true function.

Nevertheless, we can write (60) in the form [24]:

\[
\hat{F}(\rho) = a^\rho \left( 1 + \sum_{n \geq 1} \frac{a^{-n}q^{n\rho}}{(q - 1) \cdots (q^n - 1)} \right)
\]

(61)

The series is absolutely convergent as \( q > 1 \). It is easily verified that this expression can be seen as the Mellin transform of the following measure :

\[
F(x) = \sum_{n \geq 0} \frac{a^{-n}}{(q - 1) \cdots (q^n - 1)} \delta(\ln a + \ln q - \ln x)
\]

(62)

Therefore, in this case we obtain a Bargmann representation if we accept the weight function to be a true measure.

The same is true when we consider \( \psi(x) = 1/(q^x + a), q < 1 \). Equation (33) reads :

\[
\hat{F}(\rho + 1) = \frac{1}{q^\rho + a} \hat{F}(\rho)
\]

(63)

The domain of existence of the coherent states is the disc of radius \( 1/a \).

We then obtain :

\[
F(x) = \sum_{n \geq 0} \frac{q^{-n}a^{-n}}{(q^{-1} - 1) \cdots (q^{-n} - 1)} \delta(- \ln a + n \ln q - \ln x)
\]

(64)
In this subsection, we gave examples where the Bargmann representations only exist if we admit that the scalar product be expressed by means of true measures.

6 \( \psi \) vanishes

In this section, we consider two cases where the spectrum of \( N \) is the set \( N^+ \) of the positive integers and where the coherent states are defined in the whole complex plane.

6.1 q-oscillators

The first example corresponds to (7) with \( \sigma = 1 \) and the second one to (8) with \( \sigma = (q - 1)^{-1} \).

a) \( \psi(x) = [x] \equiv (q^x - q^{-x})/(q - q^{-1}) \)

A resolution of the identity was shown to be obtained with a q-integration \[13\]. The q-integration \( \int_0^x d_q x \) is the inverse operator of the q-derivative \( D_q = x \frac{q^x \partial_x - q^{-x} \partial_x}{q-q^{-1}} \) that vanishes at the origin:

\[
\int_0^x d_q x = \frac{q - q^{-1}}{q^x \partial_x - q^{-x} \partial_x} x = (q^{-1} - q) \sum_{n \geq 0} q^{(2n+1)x} \partial_x x, \text{ when } q < 1 \tag{65}
\]

The q-exponential is defined by:

\[
\text{Exp}_q(x) = D_q \text{Exp}_q(x) = \frac{\text{Exp}_q(qx) - \text{Exp}_q(q^{-1}x)}{x(q - q^{-1})} \tag{66}
\]

with the condition that it is equal to one when \( x \) is zero. This function reads:

\[
\text{Exp}_q(x) = \sum_{n \geq 0} \frac{x^n}{n!} \tag{67}
\]

and vanishes on the negative axis \([13]\). Denoting by \( -\zeta \) the first zero at the left of the origin the resolution of identity then reads:
\[
\int \frac{d\theta}{2} \int_{0}^{\zeta^2} d\rho \rho^2 \text{Exp}(\rho^2) \cdot \text{Exp}(-\rho^2) \cdot \text{Exp}(\rho e^{-i\theta}) \cdot \text{Exp}(\rho e^{i\theta}) \cdot 2\rho d\rho = 1
\]  

(68)

Here we look for a Bargmann representation where the scalar product involves a true integral.

First, it is easy to verify that in this case as in the following, if \( F \) verifies (42), its Mellin transform is solution of (35) and the moments are the expected ones. In both cases, we choose to define the weight function, not through its Mellin transform but directly as solution of (42).

The equation (42) for this particular case reads:

\[
x F(x) = q^{-x} \frac{q^{x} - q}{q - q^{-1}} F(x)
\]

(69)

The obvious solution of this equation

\[
F(x) = \text{Exp}(x)
\]

(70)

is not positive for all positive values of \( x \) and the Bargmann representation as defined in section 4 does not exist.

Following the trick used to get (68), we can try to limit the integration to the domain where \( \text{Exp}(x) \) is positive. Let us see if

\[
\hat{F}(n) = \int_{0}^{\zeta} \text{Exp}(x) x^{n-1} dx
\]

(71)

could work. Equation (34) gives:

\[
\int_{\zeta} \text{Exp}(x) x^{n-1} dx - \int_{\zeta^{-1}} \text{Exp}(x) x^{n-1} dx = 0
\]

(72)

The problem is symmetric under the change \( q \) into \( q^{-1} \). Let us choose \( q > 1 \), (72) takes the form:

\[
\int_{\zeta^{-1}} (\text{Exp}(x) q^n + \text{Exp}(q^{-1}x)) x^{n-1} dx = 0
\]

(73)

The integrand of (73) reads \((\text{Exp}(x)(q^n - x(q^{-1})) + \text{Exp}(q^{-1}x))x^{n-1} \) and is positive for \( n \) enough large; this leads to \( \text{Exp}(x)(q^n - x(q^{-1})) + \text{Exp}(q^{-1}x) = 0 \), that is impossible.
Therefore in order to obtain a Bargmann representation, we must look for another solution of (42) that will be positive. As already noticed, the problem being symmetric under the change $q \rightarrow q^{-1}$, we assume $q > 1$. Let us start with (35) that reads:

$$\hat{F}(\rho + 1) = \frac{q^\rho - q^{-\rho}}{q - q^{-1}} = \frac{q^\rho}{q - q^{-1}} (1 - q^{-2}\rho) \hat{F}(\rho) \quad (74)$$

Let us write $\hat{F}$ on the form:

$$\hat{F}(\rho) = \phi q^{\hat{f}(\rho-1)} (q - q^{-1})^{-\rho} \hat{f}(\rho) \quad (75)$$

The function $\hat{f}(\rho)$ must verify

$$\hat{f}(\rho + 1) = (1 - q^{-2}\rho) \hat{f}(\rho) \quad (76)$$

and is given by:

$$\hat{f}(\rho) = \sum_{n \geq 0} q^{-2n\rho} \frac{1}{(1 - q^{-2}) \cdots (1 - q^{-2n})} \quad (77)$$

The condition $\hat{F}(0) = 1$, furnishes the normalization factor:

$$\phi = (q - q^{-1}) \left( 1 + \sum_{n > 0} q^{-2n} \frac{1}{(1 - q^{-2}) \cdots (1 - q^{-2n})} \right)^{-1} \quad (78)$$

Putting (74) and (78) in (75), we obtain $\hat{F}(\rho)$, and then we can calculate its inverse Mellin transform:

$$F(x) = \exp\left(-\frac{1}{2\ln q} (\ln x + \ln(q - q^{-1}) + \frac{1}{2} \ln q)^2\right) \frac{1}{1 + \sum_{n > 0} \frac{q^{-2n}}{(1-q^{-2}) \cdots (1-q^{-2n})}} \sum_{n \geq 0} \frac{q^{-n(2n+1)} ((q - q^{-1})x)^{-2n}}{(1-q^{-2}) \cdots (1-q^{-2n})} \quad (79)$$

This function being positive, we have obtained a Bargmann representation where the scalar product is written with a true integral. Let us stress that $F(-x)$ is solution of (12) and is thus a possible candidate to write the resolution of identity with a q-integration and a positive function on the whole positive axis.

The same is true for in the next example where two resolutions of the identity coexist.
b) \( \psi(x) = (x) \equiv (q^x - 1)/(q - 1) \), with \( q > 1 \)

First we show that the resolution of the identity can be obtained with a q-integral as in [13].

The q-integration [23], [24], [25] is defined to be the inverse of the q-derivative
\[
D_q \equiv \frac{1}{x} \frac{q^x - 1}{q - 1}:
\]
\[
\int_0^x d_q x \equiv \frac{q - 1}{q^x - 1} x = (q - 1) \sum_{n \geq 0} q^{-(n+1)x} x
\]
(80)

The q-exponential, solution of the equation :
\[
Exp_q(x) = D_q Exp_q(x) = \frac{Exp_q(qx) - Exp_q(x)}{x(q - 1)}
\]
is given by :
\[
Exp_q(x) = \prod_{p \geq 0} (1 + x(1 - q^{-1})q^{-p}) = \sum_{n \geq 0} x^n / (n)!
\]
(82)

and vanishes for \( x = -q^p(1 - q^{-1})^{-1} \). The nearest zero on the left of the origin is \( -\zeta = -(1 - q^{-1})^{-1} \). Therefore the resolution of the identity takes the same form as in (68) with the new expressions for \( \int_0^x d_q x \), \( Exp_q \) and \( \zeta \).

Let us now look for a Bargmann representation as defined in section 4. We see that the equation (42) can be written :
\[
F(q^{-1}x) = (x(q - 1) + 1)F(x)
\]
(83)

We easily prove that the weight function is given by
\[
F(x) = \frac{1}{Exp_q(qx)}
\]
(84)

It is a positive function when \( x > 0 \) and its Mellin transform fulfills :
\[
\hat{F} (\rho + 1) = \frac{q^\rho - 1}{q - 1} \hat{F} (\rho)
\]
(85)

This ensures that the momentum \( \hat{F} (n) \) are the expected one (32). Thus, in this case, coexist two resolutions of the identity, one involving a true integral and a weight function \( F(x) = (Exp_q(qx))^{-1} \) and one with a q-integral, the weight function being \( Exp_q(-x) \).

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6.2 $\psi(x) = x^n, n > 0$

The Mellin transform of the weight function is solution of the equation deduced from (35):

$$\hat{F}(\rho + 1) = \rho^n \hat{F}(\rho)$$  \hspace{1cm} (86)

and can be expressed with the gamma-function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$:

$$\hat{F}(\rho) = (\Gamma(\rho))^n$$  \hspace{1cm} (87)

When $n$ is an integer, the inverse Mellin transform gives $F(x)$

$$F(x) = \int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots + t_n)} dt_1 \cdots dt_n \delta(x - t_1 \times \cdots \times t_n)$$  \hspace{1cm} (88)

On this expression, we see that $F(x)$ is a positive function so that the Bargmann representation exists. In the case $n = 1$, we recover the usual harmonic oscillator where $F(x) = e^{-x}$.

7 Conclusion

We have studied the possibility of Bargmann representations for any deformed oscillator algebra characterized by a function $\psi$. We gave the conditions to be verified by this function for admitting representations with coherent states. We get the unique functional equation to be satisfied by the Mellin transform of the weight function defining the scalar product. We were able to get definite and positive answer in many cases including in particular some types of q-oscillators. Although we cannot succeed to obtain a general characterization of the function $\psi$ leading to Bargmann representations, we underline two points:

- We exhibit cases where the Bargmann representations do not exist even when coherent states do (subsection (5.2));
- The analysis of subsection (5.3) shows that the scope of our study have to be extended up to include true measures for writing the scalar product.

Finally let us remark that we have obtained scalar products for the Bargmann representations of the usual q-oscillators, involving true integrals instead of q-integrations as previously proposed in literature.
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