STUDY OF A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM VIA THE DICHOTOMY–TYPE TECHNIQUE

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Abstract

We present a new view onto the successive approximations' approach in study of the two–point nonlinear fractional boundary value problems. In order to reduce the original problem and further construct its approximate solution we use the so–called 'freezing' technique and the dichotomy–type approach. These lead to improvement of the sufficient conditions for application of the aforementioned method and sharpen the obtained estimates.

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1. Introduction

The fractional differential equations have been waking a high interest during the last decades. The variety of their applications in biology, physics, engineering and economics lead to development of the proper and precise techniques to study behavior of solutions of the aforementioned equations and their systems.

Particular attention is paid to the class of nonlinear fractional boundary value problems (FBVPs), since construction of their exact solutions may be impossible or one may face computational difficulties trying to find their analytical representation. However, the high precise constructive methods of approximation of solutions may help to simplify and even solve this task.

In the current paper we give a new view on the successive approximations approach, recently used in study of the FBVPs for periodic and
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Cauchy–Nicoletti type boundary conditions (see [2]–[5]). An original ‘freezing’ technique, initially suggested for the nonlinear systems of ordinary differential equations (see discussions [8], [7]), and a dichotomy–type approach (see [9], [10]) lead to investigation of solutions of two ’model’–type FBVPs, containing some artificially introduced parameters. The approximate solutions of these problems are constructed analytically, while the numerical values of parameters are determined as solutions of the so–called ’bifurcation’ equations.

It should be emphasized that the suggested in this paper technique for study of the FBVPs allows us to improve the applicability conditions of the successive approximations approach and to essentially sharpen the estimates, obtained in the earlier papers (see [2]–[5]).

2. Problem setting

Consider a two–point nonlinear boundary–value problem for a system of fractional differential equations (FDEs)

\[ C_a^p D_t^x = f(t, x(t)), \quad t \in [a, b], \quad x, f \in \mathbb{R}^n \]  \hfill (2.1)

for some \( p \in (0, 1) \), where \( C_a^p D_t^x \) is the generalized Caputo fractional derivative with lower limit at \( a \) (see [12, Definition 1.8], [11, Definition 2.3]) \( f : \mathcal{G}_f \to \mathbb{R}^n \) is a continuous vector–function and \( \mathcal{G}_f := [a, b] \times \mathcal{D}, \mathcal{D} \subset \mathbb{R}^n \) is a closed and bounded domain, subjected to the two–point boundary constraint

\[ g(x(a), x(b)) = 0, \]  \hfill (2.2)

where \( g : \mathcal{D} \times \mathcal{D} \to \mathbb{R}^n \) is a continuous function.

Together with the FBVP (2.1), (2.2) we study two ’model’–type FBVPs with separated two–point linear boundary conditions

\[ C_a^p D_t^x = f(t, u(t)), \quad t \in \left[a, \frac{a+b}{2}\right], \quad u, f \in \mathbb{R}^n, \]  \hfill (2.3)

\[ u(a) = z, \quad u\left(\frac{a+b}{2}\right) = \lambda \]  \hfill (2.4)

and

\[ C_{a+b}^p D_t^x = f(t, v(t)), \quad t \in \left[\frac{a+b}{2}, b\right], \quad v, f \in \mathbb{R}^n, \]  \hfill (2.5)

\[ v\left(\frac{a+b}{2}\right) = \lambda, \quad v(b) = \eta, \]  \hfill (2.6)

where \( z, \lambda, \eta \in \mathbb{R}^n \) are considered as parameters.

The problem is to find a continuous function \( x : [a, b] \to \mathcal{D} \) satisfying the system of FDEs (2.1) and the nonlinear boundary conditions (2.2).
Remark 2.1. Note that in the FBVPs (2.3), (2.4) and (2.5), (2.6) the length of the interval of definition of the independent variable $t$ is equal to $I := \frac{b-a}{2}$, that is a half of the interval in the original problem (2.1), (2.2). The results, presented in the upcoming sections show, that this approach enables us to reduce some values in the qualitative analysis of the given FBVP and to essentially improve the estimates of the constructed iteration schemes.

3. Reduction of the original FBVP and some subsidiary statements

Let $D_a, D_{a+b}, D_b \subset \mathbb{R}^n$ be some convex domains containing boundary values of the continuous solution $x$ of the FBVP (2.1), (2.2):

$$x(a) \in D_a, \quad x\left(\frac{a+b}{2}\right) \in D_{a+b}, \quad x(b) \in D_b,$$

and let us introduce a set

$$D_{a,b} := (1 - \theta)z + \theta \lambda, \quad z \in D_a, \lambda \in D_{a+b}, \theta \in [0, 1] \quad (3.2)$$

with its $\rho^u$ neighborhood of the form:

$$D^u := B(D_{a,b}, \rho^u).$$

Here

$$B(\Omega, r) := \bigcup_{y \in \Omega} B(y, r)$$

is a componentwise $r$–neighborhood of a bounded connected set $\Omega \subset \mathbb{R}^n$, where under

$$B(y, r) := \{\xi \in \mathbb{R}^n : |\xi - y| \leq r\}$$

we understand the componentwise $\rho$–neighborhood of a point $y \in \mathbb{R}^n$ with $r$ to be some non–negative real vector (see [10], Definition 1).

Similarly, based on the sets $D_{a+b}, D_b$ we introduce a set

$$D_{a+b,b} := (1 - \theta)\lambda + \theta \eta, \quad \lambda \in D_{a+b}, \eta \in D_b, \theta \in [0, 1] \quad (3.3)$$

and its $\rho^v$ neighborhood

$$D^v := B(D_{a+b,b}, \rho^v).$$

Using the aforementioned 'freezing' technique (see discussions [8], [7], [10]) we introduce the following vector parameters

$$z = col(z_1, z_2, \ldots, z_n), \quad \lambda = col(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \eta = col(\eta_1, \eta_2, \ldots, \eta_n)$$
assigning them values of solution $x$ of the FBVP (2.1), (2.2) at the points $t = a$, $t = \frac{a + b}{2}$, $t = b$:

$$z := x(a), \quad \lambda := x \left( \frac{a + b}{2} \right), \quad \eta := x(b).$$

(3.4)

The parametrization (3.4) reduces study of the original FBVP (2.1), (2.2) with nonlinear boundary conditions at the full interval $[a, b]$ to investigation of solutions of two 'model–type' problems (2.3), (2.4) and (2.5), (2.6) with parametrized linear boundary conditions, defined on the half intervals $[a, \frac{a + b}{2}]$ and $[\frac{a + b}{2}, b]$ respectively.

It is worth to emphasize that the set of solutions of the FBVP (2.1), (2.2) coincides with the set of solutions of the modified problems (2.3), (2.4) and (2.5), (2.6) under additional conditions (3.4).

The following lemmas hold.

**Lemma 3.1.** Let $f(t)$ be a continuous function for $t \in [a, b]$. Then for all $t \in [a, b]$ the following estimate is true:

$$
\frac{1}{\Gamma(p)} \left| \int_{\tau}^{t} (t - s)^{p-1} f(s) ds - \left( \frac{t - \tau}{I} \right)^{p} \int_{\tau}^{\tau + I} (\tau + I - s)^{p-1} f(s) ds \right| 
\leq \alpha_1(t, \tau, I) \max_{t \in [\tau, \tau + I]} |f(t)|,
$$

(3.5)

where

$$
\alpha_1(t, \tau, I) = \left( \frac{t - \tau}{I} \right)^{p} \Gamma(p + 1) \left[ 1 - I^{p-1} + 2 \left( 1 - \frac{t - \tau}{I} \right)^{p} \right]^{-1}
$$

(3.6)

and $\Gamma(\cdot)$ is the Gamma–function.

**Proof.** The direct calculations show that

$$
\frac{1}{\Gamma(p)} \left| \int_{\tau}^{t} (t - s)^{p-1} f(s) ds - \left( \frac{t - \tau}{I} \right)^{p} \int_{\tau}^{\tau + I} (\tau + I - s)^{p-1} f(s) ds \right|
\leq \frac{1}{\Gamma(p)} \left[ \int_{\tau}^{t} \left( (t - s)^{p-1} - \left( \frac{t - \tau}{I} \right)^{p} (\tau + I - s)^{p-1} \right) |f(s)| ds 
+ \left( \frac{t - \tau}{I} \right)^{p} \int_{\tau}^{\tau + I} (\tau + I - s)^{p-1} |f(s)| ds \right]
\leq \frac{1}{\Gamma(p)} \left[ \int_{\tau}^{t} \left( (t - s)^{p-1} - \left( \frac{t - \tau}{I} \right)^{p} (\tau + I - s)^{p-1} \right) ds 
+ \left( \frac{t - \tau}{I} \right)^{p} \int_{\tau}^{\tau + I} (\tau + I - s)^{p-1} ds \right]
\leq \frac{1}{\Gamma(p)} \left[ \int_{\tau}^{t} \left( (t - s)^{p-1} - \left( \frac{t - \tau}{I} \right)^{p} (\tau + I - s)^{p-1} \right) ds 
+ \left( \frac{t - \tau}{I} \right)^{p} \frac{1}{p} \left[ (\tau + I)^{p} - (\tau + I - s)^{p} \right] \right]
\leq \frac{1}{\Gamma(p)} \left[ \int_{\tau}^{t} \left( (t - s)^{p-1} - \left( \frac{t - \tau}{I} \right)^{p} (\tau + I - s)^{p-1} \right) ds 
+ \left( \frac{t - \tau}{I} \right)^{p} \frac{1}{p} \left[ (\tau + I)^{p} - (\tau + I - s)^{p} \right] \right]
$$

(3.7)
+ \left( \frac{t - \tau}{I} \right)^p \int_t^{\tau+I} (\tau + s - I)^{p-1} ds \right]_{\max_{t \in [\tau, \tau+I]} |f(t)|} = \alpha_1(t, \tau, I) \max_{t \in [\tau, \tau+I]} |f(t)|.

Note, that we could omit the absolute value for some terms under the integrals, since

\[(t - s)^{p-1} - \left( \frac{t - \tau}{I} \right)^p (\tau + s - I)^{p-1} = (t - s)^{p-1} \left[ 1 - \left( \frac{t - \tau}{I} \right)^p \left( \frac{t - s}{\tau + s - I} \right)^{p-1} \right] \geq (t - s)^{p-1} \left[ 1 - \left( \frac{t - \tau}{I} \right)^p \left( \frac{t - s}{\tau} \right)^{p-1} \right] = (t - s)^{p-1} \left[ 1 - \left( \frac{t - \tau}{I} \right)^p \right] = (t - s)^{p-1} \frac{\tau + s - t}{I} \geq 0, \quad t \in [\tau, \tau - I].\]

\[\square\]

**Lemma 3.2.** Let \( \{\alpha_m(\cdot, \tau, I)\}_{m \in \mathbb{N}} \) be a sequence of continuous functions at the interval \([a, b]\) given by

\[
\alpha_m(t, \tau, I) := \frac{1}{\Gamma(p)} \int_{\tau}^{t} \left( (t - s)^{p-1} - \left( \frac{t - \tau}{I} \right)^p (\tau + s - I)^{p-1} \right) \alpha_{m-1}(s, \tau, I) ds
+ \left( \frac{t - \tau}{I} \right)^p \int_{t}^{\tau+I} (\tau + s - I)^{p-1} \alpha_{m-1}(s, \tau, I) ds, \quad m \in \mathbb{N},
\]

(3.7)

where \( \alpha_0(\cdot, \tau, I) = 1 \) and \( \alpha_1(\cdot, \tau, I) \) defined by formula (3.6).

Then the following estimate holds:

\[
\alpha_m(t, \tau, I) \leq \frac{I^{(m-1)p}}{2^{(m-1)(2p-1)}\Gamma^{m-1}(p+1)} \alpha_1(t, \tau, I)
\]

\[
\leq \frac{I^{mp}}{2^m(2p-1)!^{m}(p+1)}
\]

(3.8)

for all \( m \in \mathbb{N} \).
Proof. Let us first estimate $\alpha_1(t, \tau, I)$. Indeed, using its explicit form \((3.6)\) we get:

$$\alpha_1(t, \tau, I) = \frac{(t-\tau)^p}{\Gamma(p+1)} \left(1 - \frac{t-\tau}{I}\right)^p \leq \frac{2(t-\tau)^p}{\Gamma(p+1)} \left(1 - \frac{t-\tau}{I}\right)^p \leq \frac{I^p}{2^{2p-1}\Gamma(p+1)}.$$

For $m = 2$ from the recurrent formula \((3.8)\) we obtain:

$$\alpha_2(t, \tau, I) = \frac{1}{\Gamma(p)} \left[ \int_{t}^{t} \left( (t-s)^{p-1} - \left( \frac{t-\tau}{I} \right)^p \left( \tau + I - s \right)^{p-1} \right) \alpha_1(s, \tau, I) ds + \left( \frac{t-\tau}{I} \right)^p \int_{t}^{\tau + I} \alpha_1(s, \tau, I) ds \right] \leq \frac{1}{\Gamma(p)} \left[ \int_{t}^{t} \left( (t-s)^{p-1} - \left( \frac{t-\tau}{I} \right)^p \left( \tau + I - s \right)^{p-1} \right) ds \right] \frac{I^p}{2^{2p-1}\Gamma(p+1)} \leq \frac{I^2}{2^{2(2p-1)}\Gamma^2(p+1)}.$$

Suppose that for $(m-1)$ the estimate

$$\alpha_{m-1}(t, \tau, I) \leq \frac{I^{(m-2)p}}{2^{(m-2)(2p-1)}\Gamma^{m-1}(p+1)} \alpha_1(t, \tau, I) \leq \frac{I^{(m-1)p}}{2^{(m-1)(2p-1)}\Gamma^{m-1}(p+1)}$$

is true, and let us prove it in the case of $m$. The direct calculations show that

$$\alpha_m(t, \tau, I) = \frac{1}{\Gamma(p)} \left[ \int_{t}^{t} \left( (t-s)^{p-1} - \left( \frac{t-\tau}{I} \right)^p \left( \tau + I - s \right)^{p-1} \right) \alpha_{m-1}(s, \tau, I) ds + \left( \frac{t-\tau}{I} \right)^p \int_{t}^{\tau + I} \alpha_{m-1}(s, \tau, I) ds \right] \leq \frac{1}{\Gamma(p)} \left[ \int_{t}^{t} \left( (t-s)^{p-1} - \left( \frac{t-\tau}{I} \right)^p \left( \tau + I - s \right)^{p-1} \right) ds \right] \frac{I^p}{2^{2p-1}\Gamma(p+1)} \leq \frac{I^{2p}}{2^{2(2p-1)}\Gamma^2(p+1)}.$$
\[ + \left( \frac{t - \tau}{I} \right)^p \int_{\tau}^{\tau + I} (\tau + I - s)^{p-1} ds \frac{I^{(m-1)p}}{2^{(m-1)(2p-1)} \Gamma^{m-1}(p+1)} \leq \frac{I^{mp}}{2^{m(2p-1)} \Gamma^m(p+1)} \]

The last inequality proves lemma. \(\square\)

4. Successive approximation techniques on the half intervals

Suppose that \(f \in \text{Lip}(K_u, D^u)\) with \(\rho^u\) satisfying an inequality

\[ \rho^u \geq \frac{(b - a)^p M_u}{2^{3p-1} \Gamma(p+1)}. \quad (3.1) \]

Let us connect with the first 'model'-type FBVP (2.3), (2.4) the following sequence of functions

\[ u_m(t, z, \lambda) := z + \frac{1}{\Gamma(p)} \int_a^t (t - s)^{p-1} f(s, u_{m-1}(s, z, \lambda)) ds \]

\[ - \frac{1}{\Gamma(p)} \left( \frac{2(t - a)}{b - a} \right)^p \int_{a+b/2}^t \left( \frac{a + b}{2} - s \right)^{p-1} f(s, u_{m-1}(s, z, \lambda)) ds \]

\[ + \left( \frac{2(t - a)}{b - a} \right)^p \left[ \lambda - z \right], \quad (3.2) \]

with the zero–approximation to the exact solution given by

\[ u_0(t, z, \lambda) := \left[ 1 - \left( \frac{2(t - a)}{b - a} \right)^p \right] z + \left( \frac{2(t - a)}{b - a} \right)^p \lambda, \quad (3.3) \]

for all \(m \in \mathbb{N}\) and \((t, z, \lambda) \in \mathcal{G}_u, \mathcal{G}_u := [a, \frac{a+b}{2}] \times \mathcal{D}_a \times \mathcal{D}_{a+b, \frac{a+b}{2}}\).

Similarly to the FBVP (2.3), (2.4) we assume that for the problem (2.5), (2.6) function \(f \in \text{Lip}(K_v, D^v)\) with \(\rho^v\) satisfying an inequality

\[ \rho^v \geq \frac{(b - a)^p M_v}{2^{3p-1} \Gamma(p+1)} \quad (3.4) \]

and construct an appropriate sequence of functions

\[ v_m(t, \lambda, \eta) := \lambda + \frac{1}{\Gamma(p)} \int_{a+b}^t (t - s)^{p-1} f(s, v_{m-1}(s, \lambda, \eta)) ds \]

\[ - \frac{1}{\Gamma(p)} \left( \frac{2(t - b)}{b - a} + 1 \right)^p \int_{a+b}^t (b - s)^{p-1} f(s, v_{m-1}(s, \lambda, \eta)) ds \]

\[ + \left( \frac{2(t - b)}{b - a} + 1 \right)^p \left[ \eta - \lambda \right], \quad (3.5) \]
where the zero–approximation to the exact solution is

\[ v_0(t, \lambda, \eta) := \left[ 1 - \left( \frac{2(t-b)}{b-a} + 1 \right)^p \right] \lambda + \left( \frac{2(t-b)}{b-a} + 1 \right)^p \eta, \] (3.6)

for all \( m \in \mathbb{N} \) and \( (t, \lambda, \eta) \in \mathcal{G}_v, \mathcal{G}_v := [\frac{a+b}{2}, b] \times \mathcal{D}_{a+b} \times \mathcal{D}_b. \)

**Theorem 3.1.** Let for a parametrized FBVP (2.3), (2.4) there exists a non–negative vector \( \rho \) satisfying an inequality (3.1) such that \( f \in \text{Lip}(K_u, D^u) \) on an interval \( t \in [a, \frac{a+b}{2}] \) and for the matrix

\[ Q_u := \frac{(b-a)K_u}{2^{3p-1}\Gamma(p+1)} \] (3.7)

an inequality holds

\[ r(Q_u) < 1. \] (3.8)

Then for arbitrary pair of vector parameters \( (z, \lambda) \in \mathcal{D}_a \times \mathcal{D}_{a+b} : \)

1. All functions of the sequence are continuous on the interval \( [a, \frac{a+b}{2}] \) and satisfy the linear boundary conditions (2.4).
2. The sequence of functions (3.2) for \( t \in [a, \frac{a+b}{2}] \) converges uniformly as \( m \to \infty \) to its limit function

\[ u_\infty(t, z, \lambda) = \lim_{m \to \infty} u_m(t, z, \lambda). \] (3.9)

3. The limit function (3.9) satisfies boundary conditions

\[ u_\infty(a, z, \lambda) = z, u_\infty\left(\frac{a+b}{2}, z, \lambda\right) = \lambda \] (3.10)

and is a unique solution of an integral equation

\[ u(t) := z + \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s, u(s))ds \]

\[ -\frac{1}{\Gamma(p)} \left( \frac{2(t-a)}{b-a} \right)^p \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - s \right)^{p-1} f(s, u(s))ds \]

\[ + \left( \frac{2(t-a)}{b-a} \right)^p [\lambda - z], \ t \in \left[ a, \frac{a+b}{2} \right] \] (3.11)

in the domain \( D^u, \text{i.e. it is a solution of the corresponding Cauchy problem for a perturbed system of FDEs:} \)

\[ C^a_D^p t u = f(t, u(t)) + \left( \frac{2}{b-a} \right)^p \Delta(z, \lambda), \ t \in \left[ a, \frac{a+b}{2} \right], \] (3.12)

\[ u(a) = z. \] (3.13)
where $\Delta : D_a \times D_{a+b/2} \to \mathbb{R}^n$ is a mapping given by formula:

$$\Delta(z, \lambda) := \Gamma(p+1)|\lambda - z| - p \int_a^{a+b/2} \left(\frac{a + b}{2} - s\right)^{p-1} f(s, u(s))ds.$$  \tag{3.14}

(4) The following error estimation holds:

$$|u_\infty(t, z, \lambda) - u_m(t, z, \lambda)| \leq \frac{(b-a)^p}{2^{3p-1}\Gamma(p+1)}Q_u^m(\mathbb{I}_n - Q_u)^{-1}M_u,$$  \tag{3.15}

where $\mathbb{I}_n$ is the $n$–dimensional unit matrix.

Proof. Simple calculations show that the first statement of the theorem holds, i.e. all functions of the sequence \ref{3.2} are continuous and satisfy the parametrized boundary restrictions \ref{2.4}.

Now we prove that for all $m \in \mathbb{N}$ functions $u_m$ of the sequence \ref{3.2} will remain in its domain of definition, i.e. the iteration process can last infinitely long. For this purpose let us estimate the differences $d_m^0(t, z, \lambda) := |u_m(t, z, \lambda) - u_0(t, z, \lambda)|$, $m \in \mathbb{N}$, where functions $u_m(\cdot, z, \lambda)$ and $u_0(\cdot, z, \lambda)$ are defined by formulas \ref{3.2}, \ref{3.3}. We get

$$d_m^0(t, z, \lambda) = \frac{1}{\Gamma(p)} \int_a^t (t - s)^{p-1} f(s, u_{m-1}(s, z, \lambda))ds - \frac{2(t-a)}{b-a} \int_a^{a+b/2} \left(\frac{a + b}{2} - s\right)^{p-1} f(s, u_{m-1}(s, z, \lambda))ds$$

$$\leq \frac{1}{\Gamma(p)} \left[ \int_a^t \left\{ (t-s)^{p-1} - \left(\frac{2(t-a)}{b-a}\right)^p \left(\frac{a + b}{2} - s\right)^{p-1} \right\} ds \right]$$

$$\leq \frac{2(t-a)}{b-a} \int_t^{a+b/2} \left(\frac{a + b}{2} - s\right)^{p-1} ds$$

$$\times \max_{(t,z,\lambda) \in G_u} |f(t, u_{m-1}(t, z, \lambda))| = M_u\alpha_1 \left( t, a, \frac{b-a}{2} \right),$$  \tag{3.16}

where $\alpha_1(t, a, \frac{b-a}{2})$ is defined by \ref{3.6} and

$$M_u := \max_{(t,z,\lambda) \in G_u} |f(t, u_{m-1}(t, z, \lambda))|, \ m \in \mathbb{N}.$$  

Let us now analyse the difference $d_m^{m+1}(t, z, \lambda) := |u_{m+1}(t, z, \lambda) - u_m(t, z, \lambda)|$, $\forall m \in \mathbb{N}$, where $u_m(\cdot, z, \lambda)$ are functions of the sequence \ref{3.2}.

From the inequality \ref{3.16} for $m = 0$ we already obtained an estimate

$$d_1^0(t, z, \lambda) \leq M_u\alpha_1 \left( t, a, \frac{b-a}{2} \right).$$
Computations show that for the general case of the iteration step $m$ one gets

$$d_{m+1}^m(t, z, \lambda) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} |f(s, u_m(s, z, \lambda)) - f(s, u_{m-1}(s, z, \lambda))| ds$$

$$- \left( \frac{2(t-a)}{b-a} \right)^p \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - s \right)^{p-1} |f(s, u_m(s, z, \lambda)) - f(s, u_{m-1}(s, z, \lambda))| ds$$

$$\leq K_u \Gamma(p) \left[ \int_a^t \left\{ (t-s)^{p-1} - \left( \frac{2(t-a)}{b-a} \right)^p \left( \frac{a+b}{2} - s \right)^{p-1} \right\} |u_m(s, z, \lambda) - u_{m-1}(s, z, \lambda)| ds \right]$$

$$+ \left( \frac{2(t-a)}{b-a} \right)^p \int_t^{\frac{a+b}{2}} \left( \frac{a+b}{2} - s \right)^{p-1} |u_m(s, z, \lambda) - u_{m-1}(s, z, \lambda)| ds$$

$$\leq \left( \frac{T^p K_u}{2^{3p-1} \Gamma(p+1)} \right)^m M_u \alpha_1 \left( t, a, \frac{b-a}{2} \right) = Q^m M_u \alpha_1 \left( t, a, \frac{b-a}{2} \right)$$

$$\leq \frac{(b-a)^p}{2^{3p-1} \Gamma(p+1)} Q^m M.$$  \hfill (3.17)

In view of the inequality (3.17) \hfill (3.18)

$$d_{m+j}^m(t, z, \lambda) \leq \sum_{k=1}^j d_{m+k}^{m+k-1}(t, z, \lambda) \leq \sum_{k=1}^j K_u^{m+k-1} M_u \alpha_{m+k}(t)$$

$$\leq \sum_{k=1}^j K_u^{m+k-1} (b-a)^{(m+k-1)p} \frac{1}{2^{(m+k-1)(3p-1)} \Gamma^{m+k-1}(p+1)} M_u \alpha_1 \left( t, a, \frac{b-a}{2} \right)$$

$$= \sum_{k=0}^{j-1} Q_u^{m+k} M_u \alpha_1 \left( t, a, \frac{b-a}{2} \right) = Q_u^m \sum_{k=0}^{j-1} Q_u^k M_u \alpha_1 \left( t, a, \frac{b-a}{2} \right)$$

$$\leq \frac{(b-a)^p}{2^{3p-1} \Gamma(p+1)} Q_u^m \sum_{k=0}^{j-1} Q_u^k M_u.$$  \hfill (3.18)

Due to the condition (3.18) the spectrum radius of the matrix $Q_u$ does not exceed 1.
This means that
\[
\sum_{k=0}^{j-1} Q_k^u \leq (I_n - Q_u)^{-1}, \quad \lim_{m \to \infty} Q_u^m = \emptyset_n,
\]
where \(\emptyset_n\) is the zero \(n\)-dimension matrix.

Passing in (3.18) to the limit for \(j \to \infty\), we get the estimate (3.15). Moreover, according to the Cauchy criteria the sequence of functions \(\{u_m(\cdot, z, \lambda)\}\), defined by the iterative formula (3.2), is uniformly convergent in the domain \(G_u\) to the limit function \(u_\infty(\cdot, z, \lambda)\).

Since all functions of the sequence (3.15) satisfy two-point parametrized boundary conditions (2.4), the limit function \(u_\infty(\cdot, z, \lambda)\) also satisfied them.

Analogically to Theorem 1 in [2] it is easy to show, that letting \(m \to \infty\) in the relation (3.2), the limit function (3.9) is the solution of the integral equation (3.11), i.e. it is a unique solution of the Cauchy problem (3.12) with the perturbation term \(\Delta(z, \lambda)\) defined by (3.14).

Under similar to Theorem 3.1 conditions one can prove convergence of the sequence of functions \(v_m(\cdot, \lambda, \eta)\), i.e. theorem holds.

**Theorem 3.2.** Let for a parametrized FBVP (2.5), (2.6) there exists a non-negative vector \(\rho^v\) satisfying an inequality (3.4) such that
\(f \in \text{Lip}(K_v, D_v)\), \(\forall t \in [a+b^2, b]\) and for the matrix
\[
Q_v := \frac{(b-a)^p K_v}{2^{p-1}(p+1)}
\]
an inequality holds
\[
r(Q_v) < 1.
\]

Then for arbitrary pair of vector parameters \((\lambda, \eta) \in D_{a+b^2} \times D_b:\)

(1) All functions of the sequence are continuous on the interval \([a+b^2, b]\) and satisfy the separated boundary conditions (2.6).
(2) The sequence of functions (3.5) for \(t \in [a+b^2, b]\) converges uniformly as \(m \to \infty\) to its limit function
\[
v_\infty(t, \lambda, \eta) = \lim_{m \to \infty} v_m(t, \lambda, \eta).
\]
(3) The limit function (3.21) satisfies boundary conditions
\[
v_\infty\left(\frac{a+b}{2}, \lambda, \eta\right) = \lambda, u_\infty(b, z, \lambda) = \eta
\]
and is a unique solution of an integral equation
\[
v(t) := \lambda + \frac{1}{\Gamma(p)} \int_{\frac{a+b}{2}}^{t} (t-s)^{p-1} f(s, v(s)) ds
\]
\[
- \frac{1}{\Gamma(p)} \left( \frac{2(t-b)}{b-a} + 1 \right) \int_{\frac{a+b}{2}}^{b} (b-s)^{p-1} f(s, v(s)) ds
\]
\[
+ \left( \frac{2(t-b)}{b-a} + 1 \right)^p [\eta - \lambda]
\]
in the domain \(D^p\), i.e. it is a solution of the corresponding Cauchy problem for a perturbed system of FDEs:
\[
\frac{C_{\frac{a+b}{2}}}{D_t^p} v = f(t, v(t)) + \left( \frac{2}{b-a} \right)^p \Theta(\lambda, \eta), \ t \in \left[ \frac{a+b}{2}, b \right],
\]
\[
v \left( \frac{a+b}{2} \right) = \lambda,
\]
where \(\Theta : D_{\frac{a+b}{2}} \times D_b \to \mathbb{R}^n\) is a mapping, given by formula:
\[
\Theta(\lambda, \eta) := \Gamma(p+1)[\eta - \lambda] - p \int_{\frac{a+b}{2}}^{b} (b-s)^{p-1} f(s, v(s)) ds.
\]

(4) The following error estimation holds:
\[
|v_\infty(t, z, \lambda) - v_m(t, z, \lambda)| \leq \frac{(b-a)^p}{2^{3p-1} \Gamma(p+1)} Q_m^{m}(I - Q_v)^{-1} M_v
\]

Proof. The proof is similar to the aforementioned Theorem 3.1. \(\square\)

REMARK 3.1. Theorem 3.1 and 3.2 guarantee that under assumed conditions functions
\[
u_\infty(t, z, \lambda) : \left[ a, \frac{a+b}{2} \right] \times D_a \times D_{\frac{a+b}{2}} \to \mathbb{R}^n,
\]
\[
v_\infty(t, \lambda, \eta) : \left[ a, \frac{a+b}{2} \right] \times D_{\frac{a+b}{2}} \times D_b \to \mathbb{R}^n
\]
are well defined for all pairs of artificially introduced parameters \((z, \lambda) \in \times D_a \times D_{\frac{a+b}{2}}\) and \((\lambda, \eta) \in D_{\frac{a+b}{2}} \times D_b.\)

Then by putting
\[
x_\infty(t, z, \lambda) := \begin{cases} 
    u_\infty(t, z, \lambda), & t \in \left[ a, \frac{a+b}{2} \right], \\
    v_\infty(t, \lambda, \eta), & t \in \left[ \frac{a+b}{2}, b \right]
\end{cases}
\]
we obtain a well defined continuous function \( x_\infty (\cdot, z, \lambda, \eta) \), which at the point \( t = \frac{a+b}{2} \) attains the value
\[
  x_\infty \left( \frac{a+b}{2}, z, \lambda, \eta \right) = u_\infty \left( \frac{a+b}{2}, z, \lambda \right) = v_\infty \left( \frac{a+b}{2}, z, \eta \right) = \lambda. \quad (3.30)
\]

4. Main result

Let us now study two fractional initial value problems (FIVP) with some constant perturbation vector terms:

\[
  C_a D_t^p u = f(t, u(t)) + \left( \frac{2}{b-a} \right)^p \mu^u, \quad t \in \left[ a, \frac{a+b}{2} \right], \quad u(a) = z \quad (4.1)
\]

and

\[
  C_{a+b} D_t^p v = f(t, v(t)) + \left( \frac{2}{b-a} \right)^p \mu^v, \quad t \in \left[ \frac{a+b}{2}, b \right], \quad v \left( \frac{a+b}{2} \right) = \lambda, \quad (4.4)
\]

where \( \mu^u = \text{col}(\mu_1^u, \mu_2^u, \ldots, \mu_n^u) \), \( \mu^v = \text{col}(\mu_1^v, \mu_2^v, \ldots, \mu_n^v) \in \mathbb{R}^n \) we will call 'control parameters'.

**Theorem 4.1.** Let \( z \in \mathcal{D}_a, \lambda \in \mathcal{D}_{a+b} \) and \( \eta \in \mathcal{D}_b \) are fixed values of parameters. Assume that conditions of Theorem 3.1, Theorem 3.2 hold.

Then the solutions \( u(\cdot, z, \lambda) \) and \( v(\cdot, \lambda, \eta) \) of the FIVPs (4.1), (4.2) and (4.3), (4.4) respectively will satisfy conditions
\[
uu a+b/2 z, \lambda uu = \lambda, \quad (4.5)\]

and
\[
vv b, \lambda, \eta vv = \eta, \quad (4.6)\]
i.e. they will be solutions of the 'model-type' FBVPs with separated two-point parametrized boundary conditions if and only if the control parameters \( \mu^u, \mu^v \) in (4.1), (4.3) have the form:
\[
  \mu^u = \Gamma(p+1)[\lambda - z] - p \int_{a}^{a+b/2} \left( \frac{a+b}{2} - s \right)^{p-1} f(s, u_\infty(s, z, \lambda)) ds \quad (4.7)
\]
and
\[
  \mu^v = \Gamma(p+1)[\eta - \lambda] - p \int_{a+b/2}^{b} (b - s)^{p-1} f(s, v_\infty(s, \lambda, \eta)) ds \quad (4.8)
\]
respectively, where \( u_\infty(\cdot, z, \lambda), v_\infty(\cdot, \lambda, \eta) \) are the limit functions (3.9), (3.21).

**Proof.** The proof can be carried out using a similar approach described in Theorem 2 (see discussion [2]).

**Theorem 4.2.** Assume that conditions of Theorem 3.1 and Theorem 3.2 are true. Then

1. Function \( x_\infty(\cdot, z, \lambda, \eta) : [a, b] \times \mathcal{D}_a \times \mathcal{D}_{a+b} \times \mathcal{D}_b \to \mathbb{R}^n \) is a continuous solution of the original nonlinear FBVP (2.1), (2.2) if and only if the triplet \((z, \lambda, \eta)\) satisfies the system of determining equations

\[
\Delta(z, \lambda) = 0, \quad (4.9)
\]

\[
\Theta(\lambda, \eta) = 0, \quad (4.10)
\]

\[
\Xi(z, \lambda, \eta) = 0, \quad (4.11)
\]

where \( \Delta \) and \( \Theta \) are the mappings defined by formulas (3.14), (3.26) respectively, and \( \Xi : D^u \times D^v \to \mathbb{R}^n \), given by

\[
\Xi(z, \lambda, \eta) := g\left( u_\infty(a, z, \lambda), u_\infty\left( \frac{a+b}{2}, z, \lambda \right) \right) + g\left( v_\infty\left( \frac{a+b}{2}, \lambda, \eta \right), v_\infty(b, \lambda, \eta) \right).
\]

2. For every function \( X(\cdot) \) of the FBVP (2.1), (2.2) with values \((X(a), X(\frac{a+b}{2}), X(b)) \in \mathcal{D}_a \times \mathcal{D}_{\frac{a+b}{2}} \times \mathcal{D}_b\), there exists a triplet \((z^0, \lambda^0, \eta^0)\) such that \( X(\cdot) = x_\infty(t, z, \lambda, \eta) \), where function \( x_\infty \) is defined by (3.29).

**Proof.** We refer to proofs of Theorem 3 (see discussion in [2]) and Theorem 3 (see [3]) and note that the equations (3.30), (3.12), (3.24), (4.9), (4.10) lead straightforward to the continuity of function \( x_\infty(\cdot, z, \lambda, \eta) \) at the point \( t = \frac{a+b}{2} \). Moreover, according to the definition (3.29) of the aforementioned function, its continuity at all other points of the interval \([a, b]\) holds as well. \( \square \)

5. Some remarks

In practice it is more reasonable to consider an approximate determining system

\[
\Delta_m(z, \lambda) := \Gamma(p+1)[\lambda - z]
\]

\[
-p \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - s \right)^{p-1} f(s, u_m(s, z, \lambda)) ds = 0, \quad (5.1)
\]
\[ \Theta_m(\lambda, \eta) := \Gamma(p+1)[\eta - \lambda] - p \int_{\frac{a+b}{2}}^{b} (b-s)^{p-1} f(s, v_m(s, \lambda, \eta)) ds = 0, \quad (5.2) \]

\[ \Xi_m(z, \lambda, \eta) := g \left( u_m(a, z, \lambda), u_m \left( \frac{a+b}{2}, z, \lambda \right) \right) + g \left( v_m \left( \frac{a+b}{2}, \lambda, \eta \right), v_m(b, \lambda, \eta) \right) = 0 \quad (5.3) \]

instead of the exact one. Here \( \Delta_m : D_a \times D_{a+b/2} \to \mathbb{R}^n \), \( \Theta_m : D_{a+b/2} \times D_b \to \mathbb{R}^n \) and \( \Xi_m : D^u \times D^v \to \mathbb{R}^n \) are continuous mappings.

Using our conclusions about function \( x_\infty(\cdot, z, \lambda, \eta) \) given by (3.29), it is natural that its \( m \)-th approximation will be defined as

\[ x_m(t, z, \lambda, \eta) := \begin{cases} u_m(t, z, \lambda), & t \in [a, \frac{a+b}{2}], \\ v_m(t, \lambda, \eta), & t \in \left[ \frac{a+b}{2}, b \right], \end{cases} \quad (5.4) \]

where the sequences of function \( u_m(\cdot, z, \lambda), v_m(\cdot, \lambda, \eta) \) have the form (3.2), (3.5) accordingly.

**Theorem 5.1.** If the values of parameters \( z, \lambda, \eta \) satisfy the \( m \)-approximate system of determining equations (5.1)–(5.3), then the function \( x_m(\cdot, z, \lambda, \eta) \) in (5.4) is continuous on \([a, b] \).

**Proof.** Since the functions \( u_m(\cdot, z, \lambda) \) and \( v_m(\cdot, \lambda, \eta) \), defined by the successive approximations (3.2), (3.5) satisfy the consistency condition

\[ u_m \left( \frac{a+b}{2}, z, \lambda \right) = v_m \left( \frac{a+b}{2}, \lambda, \eta \right) = \lambda, \quad (5.5) \]

it follows that

\[ \frac{C_a}{a} D_t^p u_m \left( \frac{a+b}{2}, z, \lambda \right) = f \left( \frac{a+b}{2}, u_m \left( \frac{a+b}{2}, z, \lambda \right) \right) \\
- \left( \frac{2}{b-a} \right)^p \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - s \right)^{p-1} f \left( \frac{a+b}{2}, u_m \left( \frac{a+b}{2}, z, \lambda \right) \right) ds \\
+ \left( \frac{2}{b-a} \right)^p \Gamma(p+1)[\lambda - z] \quad (5.6) \]
and
\[
\frac{C}{a+b}D^p_t v_m\left(\frac{a+b}{2}, \lambda, \eta\right) = f\left(\frac{a+b}{2}, v_m\left(\frac{a+b}{2}, \lambda, \eta\right)\right)
\]
\[
- \left(\frac{2}{b-a}\right)^p p \int_{\frac{a+b}{2}}^{b} (b-s)^{p-1} f\left(\frac{a+b}{2}, v_m\left(\frac{a+b}{2}, \lambda, \eta\right)\right) ds \quad (5.7)
\]
\[
+ \left(\frac{2}{b-a}\right)^p \Gamma(p+1)[\eta - \lambda].
\]

Due to assumptions of the theorem, parameters $z, \lambda, \eta$ satisfy the so-called 'bifurcation equations' (5.1), (5.2). This means that (5.6), (5.7) may be simplified to the form
\[
\frac{C}{a}D^p_t u_m\left(\frac{a+b}{2}, z, \lambda\right) = f\left(\frac{a+b}{2}, u_m\left(\frac{a+b}{2}, z, \lambda\right)\right) \quad (5.8)
\]
and
\[
\frac{C}{a+b}D^p_t v_m\left(\frac{a+b}{2}, \lambda, \eta\right) = f\left(\frac{a+b}{2}, v_m\left(\frac{a+b}{2}, \lambda, \eta\right)\right) \quad (5.9)
\]
respectively.

Since (5.5) holds, from (5.8), (5.9) we come to the conclusion that
\[
\frac{C}{a}D^p_t u_m\left(\frac{a+b}{2}, z, \lambda\right) = \frac{C}{a+b}D^p_t v_m\left(\frac{a+b}{2}, \lambda, \eta\right),
\]
which under relation (5.4) prove the continuity of function $x_m(\cdot, z, \lambda, \eta)$ at the point $t = \frac{a+b}{2}$. The fact, that this function is also continuous at all the other points follows straightforward from its definition. □

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