More on $U_q(su(1,1))$ with $q$ a Root of Unity

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Abstract

Highest weight representations of $U_q(su(1,1))$ with $q = \exp \pi i/N$ are investigated. The structures of the irreducible highest weight modules are discussed in detail. The Clebsch-Gordan decomposition for the tensor product of two irreducible representations is discussed. By using the results, a representation of $SL(2,\mathbb{R}) \otimes U_q(su(2))$ is also presented in terms of holomorphic sections which also have $U_q(su(2))$ index. Furthermore we realise $Z_N$-graded supersymmetry in terms of the representation. An explicit realization of $Osp(1|2)$ via the highest weight representation of $U_q(su(1,1))$ with $q^2 = -1$ is given.
1 Introduction

Many works on quantum groups or \( q \)-deformations of universal enveloping algebras (qUEA) have revealed a variety of fascinating features in both the fields of mathematics and physics. In particular, from the physical viewpoint, they are of great significance with their connection to exactly solvable systems and 2D conformal field theories. The connections between qUEA of compact Lie algebra, e.g., \( U_q(su(2)) \), with \( q \) a root of unity and rational conformal field theories (RCFT)[19] whose central charges are given by rational numbers have been discussed extensively [2]-[6]. In RCFT quantum group structures appear essentially in the monodromy properties of conformal blocks. The authors of Refs.[8, 9] have shown more transparent connections between quantum groups and RCFT by constructing good representation spaces of \( U_q(su(2)) \) in terms of the Coulomb gas representations of RCFT. An important feature of the construction is that the highest weight module of \( U_q(su(2)) \) emerges as the the family of screened vertex operators, that is, each highest weight vector \( e^j_m \) corresponds to the screened vertex operator with \( j - m \) screening operators, and the generators of \( U_q(su(2)) \), \( X_+ \) and \( X_- \), are represented as contour creation and annihilation operators. Thus, the quantum groups associated with the compact Lie algebras when \( q \) is a root of unity act as a relevant symmetry of RCFT.

In contrast, the qUEA, \( U_q(su(1,1)) \), of the non-compact Lie algebra, \( su(1,1) \), has not been well discussed. Several dynamical models where \( U_q(su(1,1)) \) appears as a symmetry or a dynamical algebra are known [14]-[18], and it is likely that \( U_q(su(1,1)) \) will play a significant role in the models with non-compact spaces. The representation theories of \( U_q(su(1,1)) \) were given in [13]-[17] for generic \( q \), i.e., \( q \) not a root of unity. As in the compact case, however, we can expect to extract new and illuminating features from \( U_q(su(1,1)) \) with \( q \) a root of unity. In Ref.[18], Matsuzaki and the author have investigated highest weight representations of \( U_q(su(1,1)) \) when \( q \) is a root of unity and revealed a remarkable feature: for unitary representation space, \( U_q(su(1,1)) \) has the structure

\[
U_q(su(1,1)) = U(su(1,1)) \otimes U_q(su(2)).
\]  

(1)

The important point is that the non-compact nature appears through the ‘classical’ Lie algebra \( su(1,1) \) and the \( q \)-deformed effects are only in \( U_q(su(2)) \). This relation means that \( U_q(su(1,1)) \) with \( q \) a root of unity is the unified algebra of the quantum algebra \( U_q(su(2)) \) and the non-compact \( su(1,1) \simeq sl(2, \mathbb{R}) \). The importance of this observation is recognized by noticing that the Lie group \( SL(2, \mathbb{R}) \) has deep connection with the theory of topological 2D gravity [19]-[21]. The spirit of this is that \( SL(2, \mathbb{R}) \) plays the role of the gauge group on the Riemann surface \( \Sigma_g \) with constant negative curvature, i.e., with genus \( g \geq 2 \) and the moduli space of the complex structure of \( \Sigma_g \) is identified with the moduli space of \( SL(2, \mathbb{R}) \) flat connections. Along this line, we can expect that \( U_q(su(1,1)) \) is regarded as the symmetry of the theory of topological gravity coupled to RCFT as a matter. Here the \( sl(2, \mathbb{R}) \) sector of \( U_q(su(1,1)) \) gives rise to the deformation of the complex structure of the
base manifold $\Sigma_g$. With this perspective in mind, the aim of this paper is to further investigate $U_q(su(1,1))$ when $q = \exp \pi i/N$ as the first step toward this goal. We will obtain a holomorphic representation of $SL(2, \mathbb{R}) \otimes U_q(su(2))$ in terms of holomorphic vectors, which are holomorphic sections of a line bundle over the homogeneous space $SL(2, \mathbb{R})/U(1)$ and that have an index with respect to $U_q(su(2))$.

Another novel aspect of $q$UEA at a root of unity appears in the connection with generalized ($\mathbb{Z}_N$-graded) supersymmetry or paragrassmann algebras [22] as discussed in Ref.[23]-[26]. Here we will present a different and concrete realization of this connection as another remarkable consequence of the relation (1). By using the highest weight representations, we will explicitly construct $N$ holomorphic functions on the homogeneous space. The $N$-th powers of the generators of $U_q(su(1,1))$ act on them and give rise to infinitesimal transformations of these functions under the transformation of the homogeneous space. Furthermore we will see that these $N$ functions can be classified into two sets according to their dimensions or, so called, $sl(2, \mathbb{R})$-spins. One of them is the set of functions which have dimensions $\zeta$, and the other is the set of functions with dimensions $\zeta + \frac{1}{2}$.

The organization of this paper is as follows: In the next chapter, we briefly review the highest weight representations of $U_q(su(1,1))$ in order to make this article self-contained although it has already been done in Ref.[18]. Chapter 3, which is the main part of this paper, looks in detail at the structure of $U_q(su(1,1))$ highest weight module when $q = \exp \pi i/N$. We also give the Clebsch-Goldan decomposition for the tensor product of two highest weight representations. In addition, representations of $SL(2, \mathbb{R}) \otimes U_q(su(2))$ are discussed in terms of holomorphic sections which have $U_q(su(2))$ index over the homogeneous space $SL(2, \mathbb{R})/U(1)$ and discuss $\mathbb{Z}_N$-graded supersymmetry. In chapter 4, we explicitly derive $Osp(1|2)$ via the highest weight representation of $U_q(su(1,1))$ with $q^2 = -1$. We conclude with some discussion.

It is convenient to summarize here the conventions and notation we will make use in this paper; $\mathbb{Z}_+$ stands for the non-negative integers, i.e., $\mathbb{Z}_+ = \{0, 1, 2, \cdots \}$. $[n]$ for $n \in \mathbb{Z}$ describes a $q$-integer defined by $[n] = (q^n - q^{-n})/(q - q^{-1})$. This convention is useful in our later discussion because, for $\forall n \in \mathbb{Z}$, $[n] \in \mathbb{R}$ if $q \in \mathbb{R}$ or $|q| = 1$. Finally $\left[ \begin{array}{c} n \\ r \end{array} \right]_q = [n]!/[n-r]!/[r]!$ is a $q$-binomial coefficient.

2 Highest Weight Representations of $U_q(su(1,1))$

To begin with, it is helpful for our discussions to give a brief review of unitary representations of the classical Lie algebra $su(1,1)$. This appears as a non-compact real form of the Lie algebra $sl(2, \mathbb{C})$ generated by $E_+, E_-$ and $H$. The relations among them are

\[ [E_+, E_-] = 2H, \quad [H, E_\pm] = \pm E_\pm \]  

The difference between the compact real form $su(2)$ and the non-compact one $su(1,1)$ will appear only through the definitions of Hermitian conjugations. They
are defined by,
\[
E^\dagger_\pm = E_\mp, \quad H^\dagger = H, \quad \text{for } su(2),
\]
\[
E^\dagger_\pm = -E_\mp, \quad H^\dagger = H, \quad \text{for } su(1,1).
\]

Via the substitution \( E_\pm \to \mp G_\mp, E_0 \to G_0 \) we can get another formulation of \( su(1,1) \simeq sl(2, \mathbb{R}) \). Now the relations and Hermitian conjugations are,
\[
[G_n, G_m] = (n - m)G_{n+m}, \quad n, m = 0, \pm 1,
\]
\[
G^\dagger_\pm = G_\mp, \quad G^\dagger_0 = G_0, \quad \text{for } su(1,1),
\]
\[
G^\dagger_\pm = -G_\mp, \quad G^\dagger_0 = G_0, \quad \text{for } su(2)
\]

We will use the latter formulation in chapter 3.

Representations are classified by means of eigenvalues of Cartan operator \( H \) and the second Casimir operator. It is well known that there are four classes of unitary irreducible representations: (a) Identity \( I \); The trivial representation of the form \( I = \{|0\rangle\} \). (b) Discrete series \( D^+_n \); \( D^+_n = \{|k + \phi \rangle \ | \ k = n, n+1, \cdots \} \) with \( n \in \mathbb{Z}_+ \), that is, each representation \( D^+_n \) is bounded below such that \( E_-|n + \phi \rangle = 0 \). The state \( |n + \phi \rangle \) is called a highest weight state. \( D^+_n \) is referred to as the highest weight representation and we will concentrate only on this type. (c) Discrete series \( D^-_n \); \( D^-_n = \{|k - \phi \rangle \ | \ k = -n, -n-1, \cdots \} \) for \( n \in \mathbb{Z}_+ \), that is, each representation \( D^-_n \) has the upper bound state such that \( E_+| -n - \phi \rangle = 0 \). This type of representation is called the lowest weight representation. (d) Continuous series \( B \): Representations of the form \( B = \{|k + \phi \rangle \ | \ k \in \mathbb{R} \} \). In the cases (b)~(d), \( \phi \) takes its value in \( 0 < \phi \leq 1 \) and it cannot be further determined. In particular, representations \( D^\pm_n \) are not really discrete in this sense. Only by a consideration of the Lie group action of \( SU(1,1) \) on the highest or lowest weight representations, does the discreteness arise. Then \( \phi \) is determined to be 1/2, or 1. It will turn out that the highest weight representation of \( U_q(su(1,1)) \) with \( q \) a root of unity is actually a discrete series without any other considerations. Let us proceed to the quantum cases.

### 2.1 The case when \( q \) is not a root of unity

We briefly summarize the highest weight representation of the non-compact real form of \( U_q(sl(2, \mathbb{C})) \) when \( q \) is not a root of unity in order to make the differences from the case \( q \) when \( q \) is a root of unity clear. Generators and relations of \( U_q(sl(2, \mathbb{C})) \) are as follows.

**Definition 2.1** \( U_q(sl(2, \mathbb{C})) \) is generated by \( X_+, X_-, K \) with relations among them given by
\[
[X_+, X_-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad KX_\pm = q^{\pm 1}X_\pm K.
\]
As in the classical case, the Hermitian conjugation rule determines whether its real form is compact or non-compact. Conjugations consistent with the relations exist if and only if $q$ is real or $|q| = 1$. The non-compact real form $U_q(su(1,1))$ can be obtained by defining the conjugation as follows:

$$
\begin{align*}
K^\dagger &= K, & X_\pm^\dagger &= -X_\mp, & \text{when } q \in \mathbb{R} \\
K^\dagger &= K^{-1}, & X_\pm^\dagger &= -X_\mp, & \text{when } |q| = 1
\end{align*}
$$

If we take the conjugation of $X_\pm$ to be $X_\pm^\dagger = X_\mp$ instead of the above, we obtain compact real form, $U_q(su(2))$. A Hopf algebraic structure results upon defining a coproduct $\Delta$, a counit $\epsilon$ and an antipode $\gamma$ by

$$
\begin{align*}
\Delta(K) &= K \otimes K, \\
\Delta(X_\pm) &= X_\pm \otimes K^{-1} + K \otimes X_\mp, \\
\epsilon(K) &= 1, \\
\epsilon(X_\pm) &= 0, \\
\gamma(K) &= K^{-1}, \\
\gamma(X_\pm) &= -q^{\pm 1}X_\mp.
\end{align*}
$$

One can represent $U_q(su(1,1))$ by constructing a highest weight module as in the classical case. Each module is characterized by a positive parameter $h$ which is the highest weight. The highest weight $h$ is specified by the second order Casimir operator. The highest weight module over a highest weight vector $|h; 0\rangle$ is given by

$$
V_h = \{|h; r\rangle | |h; r\rangle := \frac{(X_\pm)^r}{[r]!}|h; 0\rangle, \ r \in \mathbb{Z}_+\}
$$

where the highest weight vector is characterized by

$$
X_-|h; 0\rangle = 0, \ K|h; 0\rangle = q^h|h; 0\rangle.
$$

Using the definition of weight vectors $|h; r\rangle$ and the relations (3), the action of $U_q(su(1,1))$ on the highest weight module $V_h$ is given as follows,

$$
\begin{align*}
X_+|h; r\rangle &= [r + 1]|h; r + 1\rangle, \\
X_-|h; r\rangle &= -[2h + r - 1]|h; r - 1\rangle, \\
K|h; r\rangle &= q^{h+r}|h; r\rangle.
\end{align*}
$$

With the Hermitian conjugation (6), the norm of the state $|h; r\rangle$ is given as the inner product $\| |h; r\rangle \|^2 := \langle h; r | h; r \rangle$ and is

$$
\| |h; r\rangle \|^2 = \left[ \begin{array}{c} 2h + r - 1 \\ r \end{array} \right]_q,
$$

where we have normalized the norm of the highest weight vector as $\| |h; 0\rangle \|^2 = 1$. Since the $q$-integer $[2h + r - 1]$ in the r.h.s. of eq.(11) is always non-zero for $r \geq 1$, we see that no other submodules appear, that is, the highest weight module $V_h$ is irreducible. Furthermore, if $q \in \mathbb{R}$, we can also see that all weight vectors have positive definite norm, i.e., $V_h$ is unitary. Notice that every highest weight module $V_h$ is isomorphic to the highest weight module $V_0^{cl}$ of the classical $su(1,1)$. 

4
2.2 The case when $q$ is a root of unity

In the following we set the deformation parameter $q$ to be $q = \exp\pi i/N$. Then $q^N = -1$ and $[N] = 0$. In this case the situation is drastically different from both the classical case and the case with generic $q$ and some difficulties appear.

The first problem is that the norm of the $N$-th state diverges due to the $[N]$ in the denominator of eq.(12). The same problem occurs in the $U_q(su(2))$ case as well. Fortunately in that case the problem could be resolved utilizing the fact that highest and lowest weight states exist for each module. Namely, we can always choose a value of the highest weight such that the module does not have the state whose norm diverges. This is the origin of the finiteness of the number of the highest weight states in the unitary representation of $U_q(su(2))$. However in the non-compact case, we cannot remedy this problem in such a manner because the unitary representation is always of infinite dimension. Instead, in order for the $N$-th state to have finite norm, we have to impose the condition that there exists an integer $\mu$ satisfying

$$[2h + \mu - 1] = 0.$$  \hfill (13)

This factor appearing in the numerator cancels the factor $[N]$ at the $N$-th state. Of course, it is necessary that $\mu \leq N$. Thus we are led to the following proposition:

**Proposition 2.2** A highest weight module of $U_q(su(1,1))$ with $q = \exp\pi i/N$ is well-defined if and only if the highest weight is labelled by two integers $\mu$ and $\nu$ as

$$h_{\mu\nu} = \frac{1}{2}(N\nu - \mu + 1), \quad \mu = 1, 2, \ldots, N, \quad \nu \in \mathbb{N}.$$  \hfill (14)

The restriction $\nu \in \mathbb{N}$ (equivalently $h_{\mu\nu} > 0$) follows because we are considering the highest weight representations. The important point coming from the proposition is that the highest weight representations of $U_q(su(1,1))$ with $q = \exp\pi i/N$ are actually a discrete series with the highest weights taking values in $\{1/2, 1, 3/2, \ldots\}$. Unlike the classical case, no consideration about a group action is needed to show this discreteness. As we will see in section 3.2, these values of $h_{\mu\nu}$ are compatible with representations of $SL(2,\mathbb{R}) \otimes U_q(su(2))$.

In the construction of a highest weight module we come upon further difficulties. First, noticing (9), (10), the generators $X_\pm$ are nilpotent on the module due to $[N] = 0$ and $[2h_{\mu\nu} + \mu - 1] = 0$, i.e.,

$$(X_\pm)^N|\psi\rangle = 0,$$  \hfill (15)

where $|\psi\rangle$ is an arbitrary state in the highest weight module on $|h_{\mu\nu}; 0\rangle$. Therefore one cannot move from a state to another state by acting $X_+$ or $X_-$ successively. However $(X_\pm)^N/[N!]$ is well-defined on the module, and one can reach every state with the set $\{X_\pm, (X_\pm)^N/[N!]\}$. Secondly, the operator $K$ is not sufficient to specify the weight of a state due to the relation $K^{2N} = 1$. Indeed, two states $|h; r\rangle$ and
\(|h; r + 2N\rangle\) have the same eigenvalue \(q^{h_{\mu + r}}\) with respect to the operator \(K\). This fact means that we need other operator in addition to \(K\) to specify weight completely. These problems have already appeared in the compact case and Lusztig resolved them by adding generators and redefining \(U_q(\mathfrak{sl}(2, \mathbb{C}))\) \[[27]\]. His method is applicable to our non-compact case as well, and so we add new generators

\[
L_1 := -\frac{(-X_-)^N}{[N]!}, \quad L_{-1} := \frac{(X_+)^N}{[N]!}, \quad L_0 := \frac{1}{2} \left[ \frac{2H + N - 1}{N} \right]_q,
\]

(16)

where \(K = q^H\). In order to obtain non-compact representations, Hermitian conjugations are given by the second line of (6) for the operators \(X_\pm, K\) and, therefore, those for the new operators are

\[
L_{\pm 1}^\dagger = -L_{\mp 1}, \quad L_0^\dagger = L_0.
\]

(17)

Now we are ready to investigate the highest weight representations of \(U_q(\mathfrak{su}(1, 1))\) with \(q\) a root of unity. Let us construct the highest weight module, \(V_{\mu, \nu}\), on the highest weight vector \(|h_{\mu \nu}; 0\rangle\),

\[
V_{\mu, \nu} = \{|h_{\mu \nu}; r\rangle \mid r \in \mathbb{Z}_+\}.
\]

(18)

The highest weight vector is characterized by

\[
X_-|h_{\mu \nu}; 0\rangle = L_{-1}|h_{\mu \nu}; 0\rangle = 0,
K|h_{\mu \nu}; 0\rangle = q^{-h_{\mu \nu}}|h_{\mu \nu}; 0\rangle, \quad L_0|h_{\mu \nu}; 0\rangle = \ell |h_{\mu \nu}; 0\rangle,
\]

(19)

(20)

where \(\ell := \frac{1}{2} \left[ \frac{2h_{\mu \nu} + N - 1}{N} \right]_q\). Although this \(q\)-binomial coefficient includes a factor \(0^0\), we can estimate it by using \([kN]/[N] = (-)^{k-1}k\) and obtain \(\ell = (-)^{N\nu + \mu + \frac{1}{2}}\). In the discussions of the highest weight module, it is convenient to consider two cases (I) \(1 \leq \mu \leq N - 1\) and (II) \(\mu = N\) separately, since zero-norm states appear only in the case (I).

(I) \(1 \leq \mu \leq N - 1\): The drastic difference from both the classical and generic cases is in the fact that the highest weight module is not of itself irreducible owing to the state \(|h_{\mu \nu}; \mu\rangle\). By the definition of the parameter \(\mu\), this state has zero norm. The appearance of the zero-norm state is indispensable for obtaining a well defined representation. The state is not only a zero-norm state but also a highest weight state (we will call such a state a null state) due to the relations,

\[
X_-|h_{\mu \nu}; \mu\rangle = 0, \quad L_{-1}|h_{\mu \nu}; \mu\rangle = 0.
\]

(21)

The first equation comes from the definition of \(\mu\), and the second equation follows from the fact that there is not the corresponding state in \(V_{\mu, \nu}\) because \(\mu \leq N - 1\). The weight of \(|h_{\mu \nu}; \mu\rangle\) is \(h_{\mu \nu} + \mu = h_{-\mu \nu}\) and therefore the state \(|h_{\mu \nu}; \mu\rangle\) can be regarded as the highest weight state \(|h_{-\mu \nu}; 0\rangle\). Thus the original highest weight
module $V_{\mu,\nu}$ has the submodule $V_{-\mu,\nu}$ on the null state $|h_{\mu,\nu};\mu\rangle$ and, therefore, the module $V_{\mu,\nu}$ is not irreducible. Further, we can easily show that $V_{-\mu,\nu}$ has again the submodule $V_{-\mu,\nu+2}$ which has the submodule $V_{-\mu,\nu+2}$ and so on. Finally we obtain the following embedding of the submodules in the original highest weight module $V_{\mu,\nu}$:

$$V_{\mu,\nu} \rightarrow V_{-\mu,\nu} \rightarrow V_{\mu,\nu+2} \rightarrow V_{-\mu,\nu+2} \rightarrow \cdots \rightarrow V_{\mu,\nu+2k} \rightarrow V_{-\mu,\nu+2k} \rightarrow \cdots$$  \hspace{1cm} (22)

An irreducible highest weight module $V^{\text{irr}}_{\mu,\nu}$ on the highest weight state $|h_{\mu,\nu};0\rangle$ is constructed by subtracting all the submodules, and finally we obtain

$$V^{\text{irr}}_{\mu,\nu} = \sum_{k \in \mathbb{Z}_+} V^{(k)}_{\mu,\nu},$$  \hspace{1cm} (23)

where $V^{(k)}_{\mu,\nu} := V_{\mu,\nu+2k} - V_{-\mu,\nu+2k}$. Notice that all the zero-norm states that lie on the levels from $(kN + \mu)$ to $((k + 1)N - 1)$ disappear by the subtraction. The remarkable point is that, the irreducible highest weight module has block structure, that is to say, $V^{\text{irr}}_{\mu,\nu}$ consists of infinite series of blocks $V^{(k)}_{\mu,\nu}$, $k = 0, 1, \cdots$, and each block has finite number of states, $|h_{\mu,\nu};kN + r\rangle$, $r = 0, 1, \cdots, \mu - 1$. The operators $X_\pm$ move states in each block according to $|h_{\mu,\nu};kN + r\rangle$ to $|h_{\mu,\nu};(k \mp 1)N + r\rangle$.

(II) $\mu = N$: In this case, no null state appears in $V_{N,\nu}$. The module is, therefore, irreducible of itself. The only difference from the classical and generic cases is that the actions of $X_+$ and $X_-$, respectively, on the $(kN - 1)$-th state and the $kN$-th state vanish, i.e., $X_+|h_{N,\nu};kN - 1\rangle = 0$ due to $[N] = 0$ and $X_-|h_{N,\nu};kN\rangle = 0$ due to $[2h_{N,\nu} + N - 1] = 0$. Instead, the state $|h_{N,\nu};kN\rangle$ can be generated by the operation of $L_{-1}$ on the state $|h_{N,\nu};(k - 1)N\rangle$. Thus the irreducible highest weight module also has the block structure as in the case (I),

$$V^{\text{irr}}_{N,\nu} = \sum_{k \in \mathbb{Z}_+} V^{(k)}_{N,\nu},$$  \hspace{1cm} (24)

where $V^{(k)}_{N,\nu}$ is the $k$-th block which consists of $|h_{N,\nu};kN + r\rangle$, $r = 0, 1, \cdots, N - 1$.

Now we have obtained irreducible highest weight modules of $U_q(su(1,1))$ when $q = \exp \pi i/N$. The crucial point is that unlike the classical or generic cases, every irreducible module has the block structure (23) or (24) which can be written as $V_k \otimes V_r$, where $V_k$ is the space which consists of infinite number of blocks and $V_r$ stands for a block having finite number of states $|h_{\mu,\nu};kN + r\rangle$, $r = 0 \sim \mu - 1$. It will turn out that the block structure is the very origin of novel features of $U_q(su(1,1))$ at a root of unity. Furthermore we should notice that the irreducible module $V^{\text{irr}}_{\mu,\nu}$ is not necessarily unitary, because $[x]$ for a positive integer $x$ is not always positive.

Finally we present character formula of the representation.

$$\chi_{\mu,\nu}(x) := \text{Tr}_{V^{\text{irr}}_{\mu,\nu}} x^H = \sum_{k=0}^{\infty} \left( \frac{x^{h_{\mu,\nu+2k}}}{1-x} - \frac{x^{h_{-\mu,\nu+2k}}}{1-x} \right)$$
This holds for $1 \leq \mu \leq N$. In the $\mu = N$ case, the character $\chi_{N\nu}(x)$ is the same as that of classical highest weight representation of $su(1,1)$.

3 Structure of $U_q(su(1,1))$ with $q = \exp \pi i/N$

The aim of this chapter is to elaborate on the irreducible highest weight module $V_{irr}^{\mu,\nu}$, $\mu = 1, 2, \cdots N$ and its attendant the block structure. It is convenient to write the level of a state in $V_{\mu,\nu}^{irr}$ by $kN + r$ with $r = 0, 1, \cdots, \mu - 1$ rather than $r = 0, 1, \cdots$, and we always adopt this notation hereafter.

3.1 Main Theorem

In this section we prove the following theorem which states the structure of the irreducible highest weight module of $U_q(su(1,1))$.

**Theorem 3.1** When $q = \exp \pi i/N$, the irreducible highest weight $U_q(su(1,1))$-module is isomorphic to a tensor product of two spaces as follows;

$$V_{\mu,\nu}^{irr} \simeq V_{\zeta}^{cl} \otimes \mathcal{U}_j;$$

where $\zeta = \frac{1}{2}\nu$ and $j = \frac{1}{2}(\mu - 1)$. These two spaces $V_{\zeta}^{cl}$ and $\mathcal{U}_j$ are understood as follows;

(1) if $V_{\mu,\nu}^{irr}$ is unitary (see proposition 3.2 below), then

- $V_{\zeta}^{cl}$: unitary irreducible infinite dimensional $su(1,1)$-module
- $\mathcal{U}_j$: unitary irreducible finite dimensional $U_q(su(2))$-module.

(2) if $V_{\mu,\nu}^{irr}$ is not unitary, there are three cases as follows;

- (2-1) $\nu \in 2N + 1$ and $\nu N + \mu \in 2N + 1$
  - $V_{\zeta}^{cl}$: non-unitary irreducible infinite dimensional $su(2)$-module
  - $\mathcal{U}_j$: unitary irreducible $U_q(su(2))$-module.

- (2-2) $\nu \in 2N$ and $\nu N + \mu \in 2N$
  - $V_{\zeta}^{cl}$: unitary irreducible infinite dimensional $su(1,1)$-module
  - $\mathcal{U}_j$: non-unitary irreducible finite dimensional $U_q(su(1,1))$-module.

- (2-3) $\nu \in 2N$ and $\nu N + \mu \in 2N + 1$
  - $V_{\zeta}^{cl}$: non-unitary irreducible infinite dimensional $su(2)$-module
  - $\mathcal{U}_j$: non-unitary irreducible finite dimensional $U_q(su(1,1))$-module.

To prove this theorem we first find an isomorphism $\rho$ between $V_{\mu,\nu}^{irr}$ and $V_k \otimes V_r$, and then discuss what $V_k$ and $V_r$ are. (For the time being, we will denote the two spaces
$V^d_\xi$ and $O_j$ in the theorem respectively by $V_k$ and $V_r$ in accordance with the previous chapter.) We begin with the observation that the operator $(X_\pm)^{kN+r}/[kN+r]!$ can be rewritten in terms of $X_\pm$ and $L_{\pm 1}$. By the straightforward calculation, one finds

$$\frac{X_\pm^{kN+r}}{[kN+r]!} = (-)^{\frac{1}{2}k(k-1)N+kr} \frac{X_\pm (L_{\pm 1})^k}{[r]!}.$$  \hspace{1cm} (27)

Furthermore one should notice that the generators $L_n$, $n = 0, \pm 1$ (anti-)commute with the generators $X_\pm, K$ on the module $V^{\text{irr}}_{\mu,\nu}$. That is, for $\psi \in V^{\text{irr}}_{\mu,\nu}$,

$$[X_\pm, L_n] \psi = 0, \quad K^{-1} L_n K \psi = -L_n \psi, \quad n = 0, \pm 1.$$  \hspace{1cm} (28)

These equations mean that up to a sign we can reach the $(kN+r)$-th state by letting $L_{-1}/k!$ and $X_\pm/[r]!$ operate separately on the highest weight state. These facts indicate that there exits a map $\rho : V^{\text{irr}}_{\mu,\nu} \to V_k \otimes V_r$ and the map $\rho$ induces another map $\hat{\rho} : U_q(su(1,1)) \to U_k \otimes U_r$, where $U_r$ and $U_k$ are the universal enveloping algebras generated by $X_\pm, K$ and $L_n$, $n = 0, \pm 1$, respectively. In order to define the maps $\rho$ and $\hat{\rho}$, we observe the actions of $X_\pm, K$ and $L_n$ on $V^{\text{irr}}_{\mu,\nu}$. The actions of $U_r$ are generated by

$$X_+ |h_{\mu \nu}; kN + r\rangle = (-)^k [r+1] |h_{\mu \nu}; kN + r + 1\rangle,$$  \hspace{1cm} (29)

$$X_- |h_{\mu \nu}; kN + r\rangle = -(-)^{\nu+1} (-)^k [\mu - r] |h_{\mu \nu}; kN + r - 1\rangle,$$  \hspace{1cm} (30)

$$K |h_{\mu \nu}; kN + r\rangle = (-)^{\frac{k}{2} + kr} q^{r+k} |h_{\mu \nu}; kN + r\rangle,$$  \hspace{1cm} (31)

and as far as the operators $L_n$ are concerned, it is sufficient to consider the actions on $|h_{\mu \nu}; kN\rangle$. They are as follows:

$$L_1 |h_{\mu \nu}; kN\rangle = (-)^k (k+1)|h_{\mu \nu}; (k+1)N\rangle,$$  \hspace{1cm} (32)

$$L_{-1} |h_{\mu \nu}; kN\rangle = (-)^{\nu + 1} (-)^k (\nu + k - 1)|h_{\mu \nu}; (k-1)N\rangle,$$  \hspace{1cm} (33)

$$L_0 |h_{\mu \nu}; kN\rangle = (-)^{\nu + 1} \mu \left(\nu + \frac{1}{2}\right) |h_{\mu \nu}; kN\rangle.$$  \hspace{1cm} (34)

The map $\rho$ can be defined as follows,

$$\rho(|h_{\mu \nu}; kN + r\rangle) = (-)^{\frac{1}{2}k(k-1)N+kr} |\zeta; k\rangle \otimes |j; -j + r\rangle,$$  \hspace{1cm} (35)

where $\zeta = \frac{1}{2} \nu$ and $j = \frac{1}{2} (\mu - 1)$. This is an isomorphism between $V^{\text{irr}}_{\mu,\nu}$ and $V_k \otimes V_r$. From (29)-(34) together with (33) we obtain the map $\hat{\rho} : U_q(su(1,1)) \to U_k \otimes U_r$, such that $\rho(O|\psi\rangle) = \hat{\rho}(O) \rho(|\psi\rangle)$ for $|\psi\rangle \in V^{\text{irr}}_{\mu,\nu}$, $O \in \{X_\pm, K, L_{0,\pm 1}\}$. Explicitly we find

$$\hat{\rho}(X_+) = 1 \otimes (-)^{\nu+1} J_+,$$  \hspace{1cm} (36)

$$\text{where} \quad J_+ |j; -j + r\rangle = (-)^{\nu+1} [r+1]|j; -j + r + 1\rangle,$$

$$\hat{\rho}(X_-) = 1 \otimes (-J_-),$$  \hspace{1cm} (37)

$$\text{where} \quad J_- |j; -j + r\rangle = (-)^{\nu+1} [2j-r+1]|j; -j + r - 1\rangle,$$
\[
\hat{\rho}(K) = (-)^{L_0 + 1} \hat{1} \otimes \hat{K}, \\
\text{where } \quad \hat{K}|j; -j + r\rangle = q^{-j + r}|j; -j + r\rangle
\]

\[
\hat{\rho}(L_1) = (-G_1) \otimes \hat{1}, \\
\text{where } \quad G_1|\zeta; k\rangle = (-)^{\mu N + \mu}(2\zeta + k - 1)|\zeta; k - 1\rangle,
\]

\[
\hat{\rho}(L_{-1}) = (-)^{\mu N + \mu}G_{-1} \otimes \hat{1}, \\
\text{where } \quad G_{-1}|\zeta; k\rangle = (-)^{\mu N + \mu}(k + 1)|\zeta; k + 1\rangle,
\]

\[
\hat{\rho}(L_0) = (-)^{\mu N + \mu}G_0 \otimes \hat{1}, \\
\text{where } \quad G_0|\zeta; k\rangle = (\zeta + k)|\zeta; k\rangle,
\]

Here we have included some sign factors \((-)^{\nu + 1}, (-)^{\nu N + \mu}\) for later convenience. The next step to complete the proof of the theorem is to examine in detail the modules \(V_k\) and \(V_r\) which are spanned by \(|\zeta; k\rangle\) and \(|j; -j + r\rangle\), respectively. We calculate the commutation relations among \(J_\pm, J_r, K\) and among \(G_1, G_{-1}, G_0\). As for \(V_k\) and \(U_k\), one finds from the equations (39)-(41) that,

\[
[G_n, G_m] = (n - m)G_{n + m}, \quad n, m = 0, \pm 1.
\]

They are just the relations in (4) of the classical \(sl(2, \mathbb{C})\) and we set the Hermitian conjugations as

\[
G_{\pm 1}^\dagger = (-)^{\nu N + \mu}G_{\mp 1}, \quad G_0^\dagger = G_0.
\]

Upon using the conjugations, the norm of the state \(|\zeta; k\rangle := ((-)^{\nu N + \mu}G_{-1})^k|\zeta; 0\rangle/k!\) is

\[
\| |\zeta; k\rangle \|^2 = ((-)^{\nu N + \mu})^k \left( \frac{2\zeta + k - 1}{k} \right).
\]

The Hermitian conjugations and the signature of the norm depend on the value of \(\nu N + \mu\). When \(\nu N + \mu\) is even, eqs. (39)-(41) say \(U_k\) is the classical \(su(1, 1)\) and \(V_k\) is the unitary infinite dimensional \(su(1, 1)\)-module. On the other hand, when \(\nu N + \mu\) is odd, \(U_k\) is the classical \(su(2)\) and \(V_k\) is a non-unitary infinite dimensional \(su(2)\)-module. Hereafter we write \(V_k\) as \(V_k^{cl}\).

We turn to \(U_r\) and \(V_r\). Let us define \(\mathcal{U}_j\) to be the finite dimensional module by rewriting the level \(r = 0, 1, \cdots, 2j\) in \(V_r\) by means of \(m = -j + r\), that is

\[
\mathcal{U}_j = \{|j; m\rangle | j; m\rangle := \frac{((-)^{\nu + 1}J_\pm)^{j - m}}{[j - m]!}|j; j\rangle, \quad m = -j, \cdots, j\}
\]

Instead of (36)-(38), the actions of \(J_\pm, K\) on the module \(\mathcal{U}_j\) are written as

\[
J_{\pm}|j; m\rangle = (-)^{\nu + 1}[j \pm m + 1]|j; m \pm 1\rangle, \quad K|j; m\rangle = q^m|j; m\rangle
\]

with \(J_+|j; j\rangle = J_-|j; -j\rangle = 0\). Upon using (46), one finds the relations among \(J_\pm, K\) to be

\[
[J_+, J_-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad KJ_\pm = q^{\pm 1}J_\pm K,
\]

and Hermitian conjugations are defined by
\[ J^\dagger_\pm = (-)^{\nu+1} J_\mp, \quad \mathcal{K}^\dagger = \mathcal{K}^{-1}. \]  

With the Hermitian conjugations and the definition of \(|j;m\rangle\) given in (45), the norm of the state \(|j;m\rangle\) can be calculated as

\[ \| |j;m\rangle \|^2 = ((-)^{\nu+1})^{j-m} \left[ \begin{array}{c} 2j \\ j-m \end{array} \right]_q \]  

(49)

Notice that the \(q\)-binomial coefficient in eq.(49) is always positive because \([x] = \frac{\sin(\pi x/N)}{\sin(\pi/N)} > 0\) for \(x < N\). Relation (47) and the conjugation (48) together with the norm (49) say that when \(\nu+1\) is even, \(U_r\) is \(U_q\(su(2)\))-unitary and \(\mathbf{\Psi}\) is a \((2j+1)\)-dimensional \(U_q\(su(2)\))-module, whereas when \(\nu+1\) is odd \(U_r\) is \(U_q\(su(1,1)\)) and \(\mathbf{\Psi}\) is a \((2j+1)\)-dimensional non-unitary \(U_q\(su(1,1)\))-module.

Now we have understood the two spaces \(V_{\zeta}^{cl}\) and \(\mathbf{\Psi}\) together with the unitarity conditions of them. We then wish to ask how the unitarity of the origin al module \(V_{\mu,\nu}^{irr}\) relates to the unitarity of \(V_{\zeta}^{cl}\) and \(\mathbf{\Psi}\). It is easy to answer the question by noticing that the norm of the state \(|h_{\mu,\nu}; kN + r\rangle\) is written as

\[ \| |h_{\mu,\nu}; kN + r\rangle \|^2 = \| |j;m\rangle \|^2 \cdot \| |\zeta;k\rangle \|^2. \]  

(50)

Therefore \(V_{\mu,\nu}^{irr}\) is unitary if and only if both \(V_{\zeta}^{cl}\) and \(\mathbf{\Psi}\) are unitary and we have shown the following proposition:

**Proposition 3.2** The irreducible highest weight module \(V_{\mu,\nu}^{irr}\) is unitary if and only if

\[ \nu \in 2N - 1, \quad \text{and} \quad \mu \in \begin{cases} 2N - 1 & \text{if} \quad N \in 2N - 1 \\ 2N & \text{if} \quad N \in 2N \end{cases} \]  

(51)

Now the proof of Theorem 3.1 has been completed.

We end this section with presenting irreducible decomposition of the completely reducible \(U_q(su(1,1))\)-module \(V^{su_q(1,1)}\). Basically \(V^{su_q(1,1)}\) is a direct sum of \(V_{\mu,\nu}^{irr}\) \(\otimes \mathbf{\Psi}\) and the values \(\zeta\) and \(j\) take \(\zeta \in \mathbb{N}/2\) and \(j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{N-1}{2}\}\). However the maximal value of \(j\), i.e., \(j = (N - 1)/2\) is problematic as in the \(U_q(su(2))\) case. In \(U_q(su(2))\) with \(q = \exp \pi i/N\), the highest weight state is restricted such as \(|j;j\rangle \in \text{Ker} J_+ / \text{Im} J_+^{N-1} \) \[4\]. Hence the value \(j = (N - 1)/2\) is excluded. The quantum dimension defined by \(\text{Tr}_{R_j} \mathcal{K}^2\) for the highest weight representation \(R_j\) of \(U_q(su(2))\) is zero for \(R_{N-1/2}\). In our case, using the character formula (25), the quantum dimension is calculated as follows;

\[ d_q := \text{Tr}_{V_{\mu,\nu}^{irr}} \mathcal{K}^2 = \chi_\zeta \cdot \chi_j, \]  

(52)

where

\[ \chi_\zeta = \frac{q^{2N\zeta}}{1 - q^{2N}}, \quad \chi_j = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}}. \]  

(53)
When \( j = (N - 1)/2 \), \( \chi_j = 0 \) as in the \( U_q(\mathfrak{su}(2)) \) case, but \( d_q \) is finite. In contrast, when \( j \leq (N - 2)/2 \), \( \chi_j = [2j + 1] \) and the quantum dimension \( d_q \) diverges as expected. Therefore we have shown that

\[
V^{\mathfrak{su}(1,1)} = \left( \bigoplus_{\zeta \in \mathbb{N}/2} V^\zeta \right) \otimes \left( \bigoplus_{j \in \mathcal{A}} \bar{\mathcal{U}}_j \right),
\]

where \( \mathcal{A} = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{N-2}{2}\} \).

### 3.2 Clebsch-Gordan Decomposition

Let us study the Clebsch-Gordan (CG) decomposition for the tensor product of two irreducible highest weight representations of \( U_q(\mathfrak{su}(1,1)) \),

\[
V_1 \otimes V_2 \rightarrow V_3,
\]

where \( V_i := V^\zeta_{\mu_i, \nu_i} \simeq V^\zeta_\zeta \otimes \bar{\mathcal{U}}_j \). The quantum CG coefficient, known as the \( q \)-3\( j \) symbol, for \( U_q(\mathfrak{su}(1,1)) \) has been obtained in Refs.\[28, 29\] when \( q \) is generic. In this section, in order to derive the CG decomposition rule for \( U_q(\mathfrak{su}(1,1)) \) with \( q = \exp \pi i/N \) we will make full use of the result obtained by Liskova and Kirillov\[28\]. The CG coefficient they have given is,

\[
\begin{bmatrix}
h_1 & h_2 & h_3 \\
M_1 & M_2 & M_3
\end{bmatrix}^{\mathfrak{su}(1,1)} = C(q) \delta_{M_1 + M_2, M_3} \tilde{\Delta}(h_1, h_2, h_3)
\times \begin{cases}
[2j - 1][M_3 - h_3][M_1 - h_1][M_2 - h_2][M_1 + h_1 - 1][M_2 + h_2 - 1] \\
[M_3 + h_3 - 1]!
\end{cases}^{1/2}
\times \sum_{R \geq 0} (-1)^R q^{R(M_3 + h_3 - 1)} 
\frac{1}{[R]! [M_3 - h_3 - R]! [M_1 - h_1 - R]! [M_2 + h_2 - 1]!} 
\frac{1}{[h_3 - h_1 - M_3 + R]! [h_3 + h_2 - M_1 + R - 1]!} 
\]

where \( C(q) \) is a factor which is not important for our analysis below and

\[
\tilde{\Delta}(h_1, h_2, h_3) = \left\{ [h_3 - h_1 - h_2][h_3 - h_1 + h_2 - 1][h_3 + h_1 - h_2 - 1][h_1 + h_2 + h_3 - 2] \right\}^{1/2}.
\]

The notations \( h_i = h_{\mu_i, \nu_i} = \zeta_i N - j_i, M_i = h_i + k_i N + r_i = (\zeta_i + k_i)N - m_i \) have been made. Of course for our case, \( i.e., q = \exp \pi i/N \), the CG coefficient is not necessarily well-defined due to the factor \( |N| = 0 \). What we will do is to count the number of \( [N] \) and look for the condition such that the numbers of \( [N] \) in the numerator and in the denominator should be equal. Lengthy examination derives the following result: Finite CG coefficients exist if and only if

\[
|j_1 - j_2| - 1 < j_3 \leq \min (j_1 + j_2, N - 2 - j_1 - j_2).
\]
It should be noticed that, on the module $V_{\mu,\nu}^{irr}$, the coproduct of the operators $K$ and $L_0$ are $\Delta(K) = K \otimes K$ and $\Delta(L_0) = L_0 \otimes 1 + 1 \otimes L_0$. The coproduct of the Cartan operator yields the conservation law of the highest weights, physically speaking, the conservation of the spin or angular-momentum along the $z$-axis. Now, from the above coproducts, we have the conservation laws $(\zeta_1 + k_1) + (\zeta_2 + k_2) = (\zeta_3 + k_3)$ and $m_1 + m_2 = m_3$ (mod $N$). Therefore the minimum value of $j_3$ is just $|j_1 - j_2|$ because the difference between $|j_1 - j_2|$ and $j_3$ is always integer. We then obtain the following decomposition rule of the tensor product of two representations of $U_q(sl(1,1))$,

$$
(V_{\zeta_1}^{cl} \otimes \mathcal{O}_{j_1}) \otimes (V_{\zeta_2}^{cl} \otimes \mathcal{O}_{j_2}) = \left( \bigoplus_{\zeta_1 + \zeta_2 \leq \zeta_3} V_{\zeta_3}^{cl} \right) \otimes \left( \bigoplus_{j_3 = |j_1 - j_2|} \mathcal{O}_{j_3} \right).
$$

The decomposition rules for the tensor products of $V_\zeta^{cl}$ and of $\mathcal{O}_j$ are the same as those for the tensor products of the non-compact representations of $sl(2, \mathbb{C})$ and of the compact representation of $U_q(sl(2, \mathbb{C}))$, respectively. However, taking the unitarity conditions for $V_\zeta^{cl}$ and $\mathcal{O}_j$ and the conservation laws of the highest weights into account, we find that all the classical modules $V_{\zeta_i}^{cl}$, $(i = 1, 2, 3)$ cannot be simultaneously unitary, and similarly for all the modules $\mathcal{O}_{j_i}$, $(i = 1, 2, 3)$.

### 3.3 Representation of $SL(2, \mathbb{R}) \otimes U_q(su(2))$

Through this section we suppose that the $U_q(su(1,1))$-module $V_{\mu,\nu}^{irr}$ is unitary. Namely, the integers $\mu$ and $\nu$ take their values in accordance with proposition 3.2. Now we have found the classical sector $sl(2, \mathbb{R}) \simeq su(1,1)$ in $U_q(su(1,1))$ and, therefore, we can construct a representation of $SL(2, \mathbb{R}) \otimes U_q(su(2))$ via the representations we have obtained.

The basic strategy we will follow is to represent the $SL(2, \mathbb{R})$ sector, roughly speaking, as follows: Let $G$ be a semi-simple Lie group, and $T$ a maximal torus. The homogeneous space $G/T$ has a complex homogeneous structure, i.e., the group $G$ acts on $G/T$ by means of holomorphic transformation. Then we can interpret the unitary irreducible representations of $G$ as spaces of holomorphic sections of holomorphic line bundles over $G/T$. In our case, the group is $G = SL(2, \mathbb{R})$ and $T = U(1)$. The homogeneous space $D = SL(2, \mathbb{R})/U(1)$ can be identified with the complex upper half plane or alternatively with the Poincaré disk $|w| < 1$. Now we wish to obtain a representation not only of $SL(2, \mathbb{R})$ but also of $U_q(su(2))$, that is to say, the holomorphic sections are also the representations of the quantum Lie algebra $U_q(su(2))$. This means that each section should have the index with respect to $U_q(su(2))$ as well.

Let $L$ be a line bundle over $SL(2, \mathbb{R})/U(1)$, and define $L_j := L \otimes \mathcal{O}_j$. We then construct $\Psi_{j,m}^\zeta \in L_j$, that is, $\Psi_{j,m}^\zeta$ is a holomorphic section of $L$ and also a vector in $\mathcal{O}_j$. Let $\mathcal{H}$ be the Hilbert space spanned by $|\zeta;k\rangle$. Having an element $\langle \Psi | \in \mathcal{H}^t$, we
give the section by,

\[ \Psi_{j,m}^\xi(w) := \langle \Psi \mid \sum_{k=0}^{\infty} w^k |\zeta;k \rangle \otimes |j;m \rangle, \]  

(60)

Under the action of \( SL(2,\mathbb{R}) \), \( w \to w' = (aw + b)/(cw + d) \), \( \Psi_{j,m}^\xi(w) \) transforms as

\[ \Psi_{j,m}^\xi(w) \to \Psi_{j,m}^\xi(w') = \left( \frac{1}{cw + d} \right)^{2\zeta} \Psi_{j,m}^\xi \left( \frac{aw + b}{cw + d} \right) \]  

(61)

Note that the action of \( t \in U(1) \) is \( t \cdot \Psi_{j,m}^\xi(w) = \pi(t) \Psi_{j,m}^\xi(w) \) with \( \pi(t) \in \mathbb{C} \). It is well known that, on \( L \), we can pick a hermitian metric \( \langle f, g \rangle_\zeta = e^\eta f^* \cdot g \) by choosing the symplectic (Kähler) potential \( \eta = 2\zeta \log(1 - |w|^2) \), i.e., the curvature form on \( L \) is given by \( w = -i\partial\bar{\partial}\eta(w^*,w) \). With this hermitian metric, the inner product on the space of sections of \( L \) is given by

\[ \langle f, g \rangle_\zeta = \int_{|w| < 1} d\mu e^{\eta(w^*,w)} f^* \cdot g. \]  

(62)

where \( d\mu = (2\zeta - 1)/\pi dw^*dw(1 - |w|^2)^{-2} \) is the \( SL(2,\mathbb{R}) \) invariant measure. Hence the inner product on \( L_i \) is

\[ \langle \Psi_{j,m}^\xi, \Psi_{j,m'}^\xi \rangle_\zeta = \delta_{m,m'} \left[ \frac{2j}{\pi} \int_{|w| < 1} dw^*dw(1 - |w|^2)^{2\zeta - 2} \psi^*_\zeta(w) \cdot \psi_\zeta(w) \right], \]  

(63)

where \( \psi_\zeta(w) = \langle \Psi | \sum_{k=0}^{\infty} w^k |\zeta;k \rangle \). It is worthwhile to notice that from equation (11), the group \( SL(2,\mathbb{R}) \) now acts in a single valued way because \( 2\zeta \in \mathbb{N} \). In the classical theory of representation, the condition that the \( SL(2,\mathbb{R}) \)-spin \( \zeta \) should be a half-integer stems from the requirement of the group action to be single-valued, while in the representation theory of \( U_q(su(1,1)) \) at a root of unity, the condition originates from the requirement that all states have definite norms. The action of the Lie algebra \( sl(2,\mathbb{R}) \) on \( \Psi_{j,m}^\xi(w) \) is obtained upon defining the action of \( g \in sl(2,\mathbb{R}) \) on the sections of \( L \) as \( g \cdot \psi^\xi(w) = \langle \Psi | \sum_{k=0}^{\infty} w^k g |\zeta;k \rangle \). By using \( G_n \) actions (39)-(11), we obtain

\[ \hat{G}_n \Psi_{j,m}^\xi(w) = \left( w^{n+1} \partial_w + \zeta(n+1)w^n \right) \Psi_{j,m}^\xi(w), \quad n = 0, \pm 1, \]  

(64)

where we have denoted the \( sl(2,\mathbb{R}) \)-actions on the space of holomorphic vectors as \( \hat{G}_n \). The right hand side of (64) spans infinitesimal transformations of \( SL(2,\mathbb{R}) \), \( w \to w + \epsilon(w) \) with \( \epsilon(w) = \alpha w^2 + \beta w + \gamma \).

On the other hand, the holomorphic vector \( \Psi_{j,m}^\xi \) transforms under \( U_q(su(2)) \) as follows;

\[ J_{\pm} \Psi_{j,m}^\xi(w) = [j \pm m + 1] \Psi_{j,m \pm 1}^\xi(w), \]  

\[ \mathcal{K} \Psi_{j,m}^\xi(w) = q^m \Psi_{j,m}^\xi(w), \]  

(65)
with \( J_\pm \Psi^\zeta_{j,j}(w) = J_\pm \Psi^\zeta_{j-j}(w) = 0 \). In the above we have given only the highest weight representations for the \( U_q(su(2)) \) sector. However, it is possible to represent this sector in other ways which are more useful for physical applications. In particular, the construction given in [8, 9] of the representation space of \( U_q(su(2)) \) is important for the further investigations, especially the connections with 2D conformal field theories.

Let us turn our attention to another remarkable feature of \( U_q(su(1,1)) \) at a root of unity, that is, the connection with the generalized supersymmetry. Of course we can easily guess this connection from (16) together with the fact that \( L_0 \) generate \( sl(2, \mathbb{R}) \) when the representation is unitary. In the following we will explicitly show that the operators \( L_{0, \pm 1} \) in (14) can be written as the infinitesimal transformations of the Poincaré disk and therefore the operators \( X_\pm \) can be interpreted as the \( N \)-th roots of the transformations. Our discussion proceeds as follows. First of all, let us find suitable functions on the disk so that the operators \( L_{0, \pm 1} \) act on them as the infinitesimal transformations. In the previous part of this section, we constructed the holomorphic vector \( \Psi^\zeta_{j,m}(w) \) by summing \( |\zeta; k\rangle \otimes |j; m\rangle \) over the level \( k \) in the \( sl(2, \mathbb{R}) \) sector. Noticing that on \( \Psi^\zeta_{j,m}(w) \), \( \hat{G}_n \) rather than \( L_n \) played the roles of such transformations, we should change our standing point and obtain other holomorphic vectors on which \( L_n \) naturally act. Therefore, in this case, we have to construct the vectors by means of the original states \( |h_{\mu\nu}; kN + r\rangle \in V_{\mu,\nu} \) instead of \( |\zeta; k\rangle \otimes |j; m\rangle \). Notice here that we do not restrict the highest weight modules to the irreducible modules and so the level \( r \) runs from 0 to \( N - 1 \). Let us define the holomorphic vectors as follows:

\[
\Phi^\zeta_r(w) := \langle \Phi | \sum_{k \in \mathbb{Z}_+} (-)^{\frac{1}{2}k(k-1)+kr} w^k |h_{\mu\nu}; kN + r\rangle,
\]

where \(|w| < 1\). \( \Phi^\zeta_r(w) \) also behaves as a holomorphic section of the line bundle over \( SL(2, \mathbb{R})/U(1) \). Now we have \( N \) holomorphic functions \( \Phi^\zeta_0(w), \Phi^\zeta_1(w), \ldots, \Phi^\zeta_{N-1}(w) \).

From eq.(29) one can easily calculate the actions of \( X_+ \), denoted as \( \hat{X}_+ \), on these functions as follows:

\[
\hat{X}_+^\zeta \Phi^\zeta_r(w) = \begin{cases} 
[r + 1] \Phi^\zeta_{r+1}(w), & \text{for } 0 \leq r \leq N - 2 \\
[N] \partial_w \Phi^\zeta_0(w), & \text{for } r = N - 1.
\end{cases}
\]

Similarly upon using eq.(31) we obtain

\[
- \hat{X}_-^\zeta \Phi^\zeta_r(w) = \begin{cases} 
[w] \mu \Phi^\zeta_{N-1}(w), & \text{for } r = 0 \\
[\mu - r] \Phi^\zeta_{r-1}(w), & \text{for } 1 \leq r \leq \mu - 1 \\
[N] \partial_w + 2\zeta \Phi^\zeta_{\mu-1}(w), & \text{for } r = \mu \\
[N + \mu - r] \Phi^\zeta_{r-1}(w), & \text{for } \mu + 1 \leq r \leq N - 1.
\end{cases}
\]
Here we have introduced the symbols $[N]_\varepsilon$ and $\hat{X}_\pm^\varepsilon$ such that $[N]_\varepsilon \neq 0$ and $\lim_{\varepsilon \to 0}[N]_\varepsilon = [N] = 0$, and $\lim_{\varepsilon \to 0} \hat{X}_\pm^\varepsilon = \hat{X}_\pm$. Now let us calculate the $N$-th powers of $\hat{X}_\pm^\varepsilon$. We obtain

\[
\lim_{\varepsilon \to 0} \frac{(\hat{X}_\varepsilon^0)^N}{[N]_\varepsilon!} \Phi^\varepsilon_r(w) = \partial_w \Phi^\varepsilon_r(w) \quad (69)
\]

\[
\lim_{\varepsilon \to 0} \frac{(-\hat{X}_\varepsilon^0)^N}{[N]_\varepsilon!} \Phi^\varepsilon_r(w) = \begin{cases} (w^2 \partial_w + 2\zeta w) \Phi^\varepsilon_r(w), & 0 \leq r \leq \mu - 1 \\ (w^2 \partial_w + 2(\zeta + \frac{1}{2})w) \Phi^\varepsilon_r(w), & \mu \leq r \leq N - 1 \end{cases} \quad (70)
\]

Thus we have shown that $\hat{X}_\pm^\varepsilon$, which move the function $\Phi^\varepsilon_r(w)$ to $\Phi^\varepsilon_{r \pm 1}(w)$ according to (69), are related to the $N$-th roots of infinitesimal transformations of the Poincaré disk. Furthermore, eq.(70) tells us that the functions $\Phi^\varepsilon_r(w)$ for $0 \leq r \leq \mu - 1$ have dimensions $\zeta$ and the functions $\Phi^\varepsilon_r(w)$ for $\mu \leq r \leq N - 1$ have dimensions $\zeta + \frac{1}{2}$. Let $\tilde{\Phi}^\varepsilon_r(w)$ be the former functions, i.e., $\Phi^\varepsilon_r$ with dimensions $\zeta$ and $\Xi_r^{\zeta+\frac{1}{2}}(w)$ be the latter with dimensions $\zeta + \frac{1}{2}$. In the above, we examined $\hat{X}_\pm$ and the $N$-powers of them only. Of course we can also obtain

\[
\lim_{\varepsilon \to 0} \frac{1}{2} \left[ \frac{2\hat{K} + N - 1}{N} \right]_q = (w \partial_w + h). \quad (71)
\]

where $\hat{K} = q^{H}$ and $h$ is the dimension, i.e. $h = \zeta$ for the functions $\tilde{\Phi}^\varepsilon_r(w)$ and $h = \zeta + \frac{1}{2}$ for the functions $\Xi_r^{\zeta+\frac{1}{2}}(w)$. Thus we have obtained all the generators which span the infinitesimal holomorphic transformations of the homogeneous space $SL(2, \mathbb{R})/U(1)$ and obtained two kinds of functions. One is the set of functions whose dimensions are $\zeta$ and the other is the set of functions whose dimensions are $\zeta + \frac{1}{2}$. Moreover the generators $\hat{X}_\pm$ mix between them. Now we can conclude that $U_q(su(1,1))$ with the deformation parameter $q = \exp \pi i/N$ may be viewed in terms of a $Z_N$-graded supersymmetry with the upper half plane or Poincaré disk interpreted as an external space.

We have constructed holomorphic vectors over the Poincaré disk in two ways and obtained two sets, $\Psi^\varepsilon_{j,m}(w)$ and $\Phi^\varepsilon_r(w) = \{ \tilde{\Phi}^\varepsilon_r(w), \Xi_r^{\zeta+\frac{1}{2}}(w) \}$. We end this section with the discussion of the connection between them. Upon the substitution $m = -j + r$, the actions of $\hat{X}_\pm$ on $\tilde{\Phi}^\varepsilon_r(w)$ coincide with those of $J_{\pm}$ on $\Psi^\varepsilon_{j,m}(w)$ except the actions $\hat{X}_\mu \tilde{\Phi}^\varepsilon_r$ and $\hat{X}_- \tilde{\Phi}^\varepsilon_0$ corresponding to $J_+ \Psi^\varepsilon_j$ and $J_- \Psi^\varepsilon_{-j}$, respectively. The latter vanish because $\Psi^\varepsilon_{j,j}$ is the highest weight vector and $\Psi^\varepsilon_{j,-j}$ is the lowest weight vector with respect to $U_q(su(2))$. However $\hat{X}_+ \tilde{\Phi}^\varepsilon_\mu$ and $\hat{X}_- \tilde{\Phi}^\varepsilon_0$ do not vanish but yield the functions $\Xi^{\zeta+\frac{1}{2}}_{\mu+1}$ and $\Xi^{\zeta+\frac{1}{2}}_{N-1}$. In other words, through these two actions the two classes of functions, $\tilde{\Phi}^\varepsilon_r$ and $\Xi_r^{\zeta+\frac{1}{2}}$, mix with each other after taking the limit $\varepsilon \to 0$ in eqs.(67),(68). Noticing that the functions $\Xi_r^{\zeta+\frac{1}{2}}$ have zero norms because they correspond to the states lying between the $(kN + r)$-th level to the $(kN + N - 1)$-th
level in the original module \( V_{\mu,\nu} \), we see that they are proportional to \( \sqrt{[N]_\varepsilon} \) and may rescale them as \( \Xi^{\zeta+\frac{1}{2}}_r = \sqrt{[N]_\varepsilon} \Xi^{\zeta+\frac{1}{2}}_r \). By the rescaling, the set of functions \( \Xi^{\zeta+\frac{1}{2}}_r \) completely decouples from the set \( \Phi^{\zeta}_r \) after taking the limit \( \varepsilon \to 0 \). We can then identify \( \Phi^{\zeta}_r(w) \) with \( \Psi^{\zeta}_{j,m}(w) \), and we have another set of functions \( \Xi^{\zeta+\frac{1}{2}}_r \) whose dimensions differ by 1/2 from those of \( \Phi^{\zeta}_r \).

4 \( Osp(1|2) \) and \( U_q(su(1,1)) \) with \( q^2 = -1 \)

In this section we devote ourselves only to the case \( N = 2 \), that is, \( [2] = 0 \) and find that \( Osp(1|2) \) can be represented in terms of the representation of \( U_q(su(1,1)) \). To this end it is convenient to introduce operators \( L_1 \) and \( L_{-1} \), which are related to \( X_\pm \) and \( K \) by the relations

\[
L_1 = iq^{-\frac{1}{2}}KX_-, \quad L_{-1} = iq^{-\frac{1}{2}}X_+K.
\]

Further we define vectors \( \phi^{\zeta}_r(w) \), \( r = 0, 1 \) by means of the highest weight representations of \( U_q(su(1,1)) \) by

\[
\phi^{\zeta}_r(w) = \sum_{k=0}^{\infty} w^k |h_{\mu\nu}; 2k + r\rangle, \quad r = 0, 1
\]

where we have introduced new weight vectors

\[
|h_{\mu\nu}; r\rangle := \frac{L_r}{(r)!} |h_{\mu\nu}; 0\rangle,
\]

with \( \langle r \rangle = q^{-1}[r] \). The new weight vector \( |h_{\mu\nu}; r\rangle \) coincides with the original one \( |h_{\mu\nu}; r\rangle \) up to a phase factor. We can, therefore, deal with the highest weight modules spanned by the new weight vectors in the same fashion as \( V_{\mu,\nu} \). In particular, the operator \( L_1 \) and \( L_{-1} \) act as

\[
L_1|h_{\mu\nu}; r\rangle = \langle 2h_{\mu\nu} + r - 1 |h_{\mu\nu}; r - 1\rangle, \\
L_{-1}|h_{\mu\nu}; r\rangle = \langle r + 1 |h_{\mu\nu}; r + 1\rangle.
\]

From these actions it is easily seen that the \( \mu \)-th state has zero norm as in the previous case. Indeed, in \( N = 2 \) case, the highest weight is given by (see eq.(14)),

\[
h_{\mu\nu} = \frac{1}{2}(2\nu - \mu + 1).
\]

Because \( 1 \leq \mu \leq N \), there are two cases, \( \mu = 1 \) and \( \mu = 2 \); in the case when \( \mu = 1 \), \( \phi^{\zeta}_1(w) \) has zero norm, while neither \( \phi^{\zeta}_0(w) \) nor \( \phi^{\zeta}_1(w) \) has zero norm in the case when \( \mu = 2 \). We treat these cases separately.
We first examine the case when $\mu = 1$. The highest weight is given by $h_{1\nu} = \nu$. Upon using the relation $\langle 2n \rangle = \langle 2 \rangle n$ for an integer $n$, the actions of $\mathcal{L}_{-1}$ on the vectors $\phi_0^L(w)$, $\phi_1^L(w)$ are easily obtained,

$$
\mathcal{L}_{-1}\phi_0^L(w) = \phi_1^L(w), \quad \mathcal{L}_{-1}\phi_1^L(w) = \langle 2 \rangle \partial_w \phi_0^L(w), \quad (77)
$$

and $\mathcal{L}_1$ acts on them as

$$
\mathcal{L}_1\phi_0^L(w) = w\phi_1^L(w), \quad \mathcal{L}_1\phi_1^L(w) = \langle 2 \rangle (\nu + w\partial_w) \phi_0^L(w). \quad (78)
$$

At first sight of eqs. (77) and (78) we might suspect that $\phi_0^L(w)$ and $\phi_1^L(w)$ are super-partner with each other and $\mathcal{L}_{\pm 1}$ are the generators of supersymmetry transformations. However we must be more cautious because the actions of $\mathcal{L}_{\pm 1}$ on $\phi_1^L(w)$ are zero-actions due to the factor $\langle 2 \rangle$. Note that the vector $\phi_1^L(w)$ must be proportional to $\sqrt{\langle 2 \rangle}$ since the norm of the vector is proportional to $\langle 2 \rangle$. Fortunately we can remedy the situation by scaling the operators and the vector $\phi_1^L(w)$ as follows;

$$
\sqrt{\langle 2 \rangle} \mathcal{G}_{-1} = \mathcal{L}_{-1}, \quad \sqrt{\langle 2 \rangle} \mathcal{G}_1 = \mathcal{L}_1. \quad (79)
$$

The actions of $\mathcal{G}_{\pm}$ on $\phi^L$ and $\psi^L$ are now non-vanishing and $\psi^L$ has definite norm.

We are ready to discuss the connection between two-dimensional supersymmetry and $U_q(su(1, 1))$. Let us define an infinitesimal transformation $\delta_\varepsilon$ as

$$
\delta_\varepsilon := a \mathcal{G}_1 + b \mathcal{G}_{-1}, \quad (80)
$$

where $a$, $b$ are infinitesimal Grassmann numbers. Under the transformation the fields $\phi^L(w)$ and $\psi^L(w)$ transform into each other according to

$$
\delta_\varepsilon \phi^L(w) = \varepsilon(w) \psi^L(w), \quad \delta_\varepsilon \psi^L(w) = (\nu(\partial_w \varepsilon(w)) + \varepsilon(w)\partial_w) \phi^L(w), \quad (81)
$$

where $\varepsilon(w) = aw + b$ is an anticommuting analytic function which parametrises infinitesimal holomorphic transformation. The commutation relations of two transformations $\delta_{\varepsilon_1}$ and $\delta_{\varepsilon_2}$ are

$$
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \phi^L(w) = (\zeta(\partial_w \xi(w)) + \xi(w)\partial_w) \phi^L(w), \quad [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \psi^L(w) = ((\zeta + \frac{1}{2})(\partial_w \xi(w)) + \xi(w)\partial_w) \psi^L(w), \quad (82)
$$

with $\xi(w) = 2\varepsilon_1(w)\varepsilon_2(w)$. The right hand sides of equations (82) are just the transformations of the fields $\phi^L(w)$ and $\psi^L(w)$ having dimensions $\zeta$ and $\zeta + \frac{1}{2}$, respectively, under the the infinitesimal transformation of $SL(2, \mathbb{R})$, $w \to w + \xi(w)$. We can therefore conclude that the infinitesimal transformation $\delta_\varepsilon$ which is written in terms of generators of $U_q(su(1, 1))$ is just the 'square root' of infinitesimal
$SL(2, \mathbb{R})$ transformation, that is to say, $\delta_c$ is an infinitesimal supersymmetry transformation. Further the fields $\phi^c(w)$ and $\psi^c(w)$ which are constructed in terms of the highest weight representations of $U_q(sl(1, 1))$ can be regarded as superpartners with each other. Finally, $Osp(1|2)$ algebra is obtained as follows: Let $L_{\pm 1} = (\mathcal{G}_{\pm 1})^2$ and $F_{\pm \frac{1}{2}} = \mathcal{G}_{\pm 1}$, then the following commutation relation are easily checked on the fields $\phi^c(w)$ and $\psi^c(w)$ to be

\[
\begin{align*}
[L_n, L_m] &= (n - m) L_{n+m}, \quad n, m = 0, \pm 1, \\
[L_n, F_r] &= \left(\frac{1}{2} n - r \right) F_{n+r}, \quad r = \frac{1}{2}, -\frac{1}{2}, \\
\{F_r, F_s\} &= 2L_{r+s}, \quad r, s = \frac{1}{2}, -\frac{1}{2}.
\end{align*}
\] (83)

Thus we have succeeded in building the super-algebra $Osp(1|2)$ and its representation in terms of the representation of $U_q(sl(1, 1))$ when $q^2 = -1$ and $\mu = 1$, i.e., all states at the $(2N - 1)$-th levels are zero-norm states.

Next we turn to the $\mu = 2$ case. By eq. (20), the highest weight is given by $\zeta - \frac{1}{2}$. On the contrary to the case when $\mu = 1$, no zero-norm state appears. The actions of $L_1$ and $L_{-1}$ on $\phi_0 1$ are as follows;

\[
\begin{align*}
L_1 \phi_0^c(w) &= \langle 2 \rangle (\nu w + w^2 \partial_w) \phi_0^c(w), \quad L_1 \phi_1^c(w) = \phi_0^c(w), \\
L_{-1} \phi_0^c(w) &= \phi_1^c(w), \quad L_{-1} \phi_1^c(w) = \langle 2 \rangle \partial_w \phi_0^c(w).
\end{align*}
\] (84)

Unfortunately, we cannot find any consistent ways to remove the factor $\langle 2 \rangle$ as in the previous case (73).

## 5 Discussion

In this article the highest weight representations of $U_q(sl(1, 1))$ when $q = \exp \pi i/N$ has been investigated in detail. We have shown that the highest weight module $V_{\mu, \nu}^{\text{irr}}$ is isomorphic to the tensor product of two highest weight modules $V_{\zeta}^{\text{cl}}$ and $\mathcal{U}_j$. This fact played a key role of this work, and novel features of $U_q(sl(1, 1))$ originated from this structure of $V_{\mu, \nu}^{\text{irr}}$. The module $V_{\zeta}^{\text{cl}}$ is a classical non-compact $sl(2, \mathbb{C})$-module, while $\mathcal{U}_j$ is a $(2j + 1)$-dimensional module of the quantum universal envelopping algebra $U_q(sl(2, \mathbb{C}))$. Theorem 3.1 states what $V_{\zeta}^{\text{cl}}$ and $\mathcal{U}_j$ are. In particular, when the original $U_q(sl(1, 1))$-module $V_{\mu, \nu}^{\text{irr}}$ is unitary, $V_{\zeta}^{\text{cl}}$ is the unitary highest weight module of $su(1, 1) \simeq sl(2, \mathbb{R})$ and $\mathcal{U}_j$ is the unitary highest weight $U_q(sl(2))$-module. In the following we restrict our discussions to this case, i.e., $V_{\mu, \nu}^{\text{irr}}$ is unitary.

We summarize here the novel features of $U_q(sl(1, 1))$ when $q = \exp \pi i/N$:

First we should notice that the non-compact nature appears only through the classical module $V_{\zeta}^{\text{cl}}$, and the effects of $q$-deformation arise only from the compact sector $\mathcal{U}_j$. Since the non-compact sector $V_{\zeta}^{\text{cl}}$ is classical, a representation of the Lie group $SL(2, \mathbb{R})$ is naturally induced. Indeed, we gave a representation of $SL(2, \mathbb{R}) \otimes U_q(sl(2))$ by means of the holomorphic vector $\Psi_j^m(w) = \psi_\zeta(w) \otimes |j; m\rangle$. 


Here we used holomorphic sections $\psi_\zeta(w)$ of a line bundle over the homogeneous space $SL(2,\mathbb{R})/U(1)$ in order to represent the $SL(2,\mathbb{R})$ sector and $|j;m\rangle \in \mathcal{O}_j$ is a weight vector with respect to $U_q(su(2))$. With our deformation parameter $q$, i.e., $q = \exp \pi i/N$, we have shown that the value of the highest weight $j$ lies in $\mathcal{A} = \{0, 1, 1/2, 1, \cdots, N/2\}$. Notice that this finiteness of the number of the highest weight states for the $U_q(su(2))$ sector comes from the condition that the original highest weight representations $V^\text{irr}_{\mu,K}$ of $U_q(su(1,1))$ be well-defined, that is, every state in them has finite norm. The representation $\Psi^\zeta_{j,m}(w)$ says that every point on the homogeneous space, (i.e., the upper half plane or the Poincaré disk) has the representation space of $U_q(su(2))$. In this sense, we suggest that the non-compact homogeneous space can be viewed as a base space or an external space and the representation space of $U_q(su(2))$ as an internal space.

We have also discussed the connection between $U_q(su(1,1))$ with $q = \exp \pi i/N$ and $\mathbb{Z}_{N}$-graded supersymmetry by presenting $N$ holomorphic vectors, denoted as $\Phi^\zeta_r(w)$, $r = 0, 1, \cdots, N - 1$, in another way. The generators, $X_{\pm}, K$ of $U_q(su(1,1))$ act on them and map $\Phi^\zeta_r$ to $\Phi^\zeta_{r+1}$. On the other hand, the operator $L_n$ which are related to the $N$-th powers of $X_{\pm}, K$ by the relations (14) generate the holomorphic transformations of the functions under the infinitesimal transformations of the homogeneous space $SL(2,\mathbb{R})/U(1)$. In this sense, we may say that generators of $U_q(su(1,1))$ give rise to $\mathbb{Z}_{N}$-graded supersymmetry transformations and the $N$-th powers of them are related to the infinitesimal transformations with respect to the external space. Furthermore, by observing the transformations under $L_n$, we have shown that these $N$ functions separate into two classes. One of them is the set of functions, $\Phi^\zeta_r(w)$, $r = 0 \sim 2j$, with dimensions $\zeta$ and the other is the set of functions, $\Xi^\zeta_r^{+1/2}(w), r = 2j + 1 \sim N - 1$, which have dimensions $\zeta + 1/2$ and have zero norms. That is to say, the functions $\bar{\Phi}^\zeta_r$ and $\Xi^\zeta_r^{+1/2}$ behave as the covariant vectors with dimensions $\zeta$ and $\zeta + 1/2$, respectively, under $sl(2,\mathbb{R})$. In particular, we have shown the explicit realization of two-dimensional supersymmetry $Osp(1|2)$ via the representation of $U_q(su(1,1))$ when the deformation parameter satisfies $q^2 = -1$.

We have also discussed the Clebsch-Gordan decomposition for the tensor product of two irreducible highest weight modules and found that the decomposition rules for the two sectors $V^\text{cl}_\zeta$ and $\mathcal{O}_j$ coincide with those for the classical non-compact representations of $sl(2,\mathbb{R})$ and the representations of $U_q(su(2))$.

Finally, we would like to future issues to be investigated. As mentioned in chapter 1, it is quite interesting to expect the relationship between $U_q(su(1,1))$ and topological $2D$ gravity coupled with RCFT. In order to make this expectation come true, we have to find a good representation space of $U_q(su(1,1))$ for which such a physical theory is associated. The $\mathbb{Z}_{N}$ graded supersymmetry implies that the internal symmetry, $U_q(su(2))$, is not independent of the base manifold $\Sigma_g$ but yields the deformation of the metric of $\Sigma_g$ through the $N$-th powers of the action.

Second, it is also interesting to investigate geometrical aspect of our result. As
for the geometrical viewpoint of quantum groups, it is widely expected that quantum
groups will shed light on the concept of “quantum” space-time. In particular, quantum
groups in the sense of $A_q(G)$, the $q$-deformation of the functional ring over the
group $G$, rather than $U_q(g)$ play the central role in the noncommutative geometry
initiated by Manin \[31\], and Wess and Zumino \[32\]. Further Wess and Zumino have
studied a $q$-deformed quantum mechanics in terms of the noncommutative differential geometry based on $A_q(G)$. The phenomena observed in this paper suggest that
by the quantization of the Poincaré disk, a certain “$q$-deformed space” appears as a
($q$-deformed) fiber at each point on the disk which remains classical. This observation is reminiscent of the result obtained in Ref.\[33\]. Actually, in order to construct
$q$-deformed mechanics, a $q$-deformed phase space was introduced in \[33\] by attaching
an internal space at each point on the phase space of the classical mechanics and all
effects of $q$-deformation stemmed only from the internal space.

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