A Gedanken Experiment For Gravitomagnetism

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Abstract

A gedanken experiment implies the existence of gravitomagnetism and raises a question about what we know about the weak-field limit of the gravitomagnetic field of General Relativity.

1 The Experiment

Imagine positive charge +Q essentially fixed to the origin of coordinates of system $S'$ by virtue of having a large rest mass, $M_0$. Let a positive charge, $q$, of mass $m_o (<< M_o)$ be placed distance $y'$ from Q in $S'$ and experience the influence of a Coulomb force, $F' = kQqj/y'^2$. The amounts of charge are chosen such that their repulsion just balances their gravitational attraction $F' = -GM_0m_0j'/y'^2$. Having set up this nice balance, you depart along the $-x'$-axis at speed $-v$.

Obviously, the masses will remain balanced from your new perspective in system S. But in your now moving frame it appears that both electric and magnetic forces act on $q$ such that a magnetic field $B = v \times E/c^2$ is created by the motion of Q relative to you, where $E = \gamma kQj/y'^2$ is the (foreshortened) electric field at q, due to Q. Here $\gamma = 1/\sqrt{1 - v^2/c^2}$. The sum of electric and magnetic forces as calculated in S is $F = q(E + u \times B)$, where $u$ is the velocity of $q$ relative to S and, in this case $u = v$. This leads to combined electric and magnetic forces of

$$F = \frac{\gamma kQqj}{y'^2}(1 - v^2/c^2)$$

Clearly this is a weaker force than that observed in the original system $S'$ and if gravity were not present it would lead to a slower separation of $Q$ and $q$ as they repelled each other. From S, you would see the separation occur in a time dilated way. But now consider the gravitational attraction between masses $M$ and $m$. The fact that you chose to go away and leave this balanced arrangement of electric...
Figure 1: Here $M >> m$ is located at the origin of coordinates of the $S'$ system, which moves at speed $v$ down the common $x,x'$ axes relative to system $S$. Masses $M$ and $m$ are positively electrically charged with charges $Q$ and $q$, respectively. Electric repulsion and gravitational attraction are balanced, leaving both objects at rest with no net force between them in $S'$. In system $S$, electric, magnetic, gravitational and gravitomagnetic forces act.

and gravitational forces can surely have no bearing on its subsequent behavior. To remain motionless as observed in system $S$, the total force of gravitational origin between the objects in system $S$ must surely be

$$
F = \gamma GM_0 m_0 \frac{j}{y^2} (1 - v^2/c^2)
$$

(2)

And as originally balanced in $S'$, $GM_0 m_0 = kQq$. The part of the force in Eq. (2) that is velocity dependent can be considered to be a gravitomagnetic force. In the $S$ frame it must exist and must, and does, balance the ordinary magnetic force. One would expect that Eq. (2) could be obtained by analogy with the electromagnetic case. $M$ would be the source of a $g$ field as observed in $S$ to be $g = -\gamma GM_0 j/y^2$ and also a source of a gravitomagnetic field $B_g = v \times g/c^2$ in which $m$ moves. Then if the sum of gravitostatic and gravitomagnetic forces would be

$$
F = m_0 (g + u \times B_g)
$$

(3)

we would obtain Eq. (2) without further ado.

But the weak-field limit of General Relativity produces a gravitomagnetic field that is

$$
B_g = 4v \times g/c^2
$$

(4)

(see, e.g., ref [1]) which would lead to gravitomagnetic forces four times stronger than required to balance the ordinary magnetic effect. The force of gravitational
origin in Eq. (3) would become
\[ F = -\frac{\gamma GM_0 m_0}{y^2}(1 - 4v^2/c^2) \] (5)

I leave the reader to ponder where this goes wrong.

2 The Free-Fall Problem in General Relativity

To see what gravitomagnetic effects we should expect, let us dispense with the electric charges on M and m (Q = q = 0) and just consider the free fall of m toward M down the y'-axis starting from rest. In either frame, S or S', the small test mass, m, would follow a geodesic. In the weak field limit of S' and in isotropic coordinates the metric is
\[ ds^2 = (1 - 2\phi)c^2 dt'^2 - (1 + 2\phi)(dx'^2 + dy'^2 + dz'^2) \] (6)

where \( \phi = GM_0/c^2r' \). A Lorentz transform to frame S \([x = \gamma(x' + vt'), \ y = y', \ z = z']\) yields
\[ ds^2 = (1 - 2\phi\gamma^2(1+v^2/c^2))c^2 dt'^2 - (1 + 2\phi\gamma^2(1+v^2/c^2))(dx'^2 + dy'^2 + dz'^2) + (8\gamma^2\phi v/c) dx dt \] (7)

where, at the instant depicted in Figure 1, \( r' = y' = y \). The last terms with \( dx dt \) arise from the Lorentz transformation. The coefficient of 8 is exactly 4X larger than expected from Newtonian theory combined with special relativity. It partially describes the “gravitomagnetic” effect. Of particular interest is that \( g_{01} = 4\gamma^2\phi v/c \) can be considered as arising from a vector potential \( A_g = (4\gamma^2 G/c^2) \int (J/r')dV \)
where \( J = \rho v \) is the mass current density and the gravitomagnetic field is taken to be \( B_g = \nabla \times A_g \). But this term with its extra factor of 4 cannot completely describe all of the gravitomagnetic effects, for we have seen that that the assumption that it does leads to an unacceptable result for our gedanken experiment.

In the weak-field limit, we have \( \det g \approx -1 \), neglecting terms involving \( \phi \). We can use the inverse of the Minkowski metric \( \eta^{ij} \) to raise indexes. The geodesic equation for the y-component of motion of m is
\[ \frac{d^2y}{c^2dt^2} \approx -(\Gamma^2_{00} + \Gamma^2_{11}v^2/c^2 + 2\Gamma^2_{01}v_x/c) \] (8)

Taking \( g^{22} = \eta^{22} = -1 \), there follows \( \Gamma^2_{00} = \Gamma^2_{11} = -\gamma^2(1+v^2/c^2)(\partial\phi/\partial y) \) and \( \Gamma^2_{01} = (2\gamma^2v/c)(\partial\phi/\partial y) \). Then with a little algebra and using \( v_x = v \gg 0, v_z = 0, v_y \approx 0, \) we obtain
\[ \frac{d^2y}{c^2dt^2} = \gamma^2\frac{\partial\phi}{\partial y}(1 + v^2/c^2)^2 - 4v^2/c^2 \] = \( (1 - v^2/c^2)\frac{\partial\phi}{\partial y} \) (9)

which is exactly the same result that we get from Eq. (2) if the inertial mass on which F acts is \( \gamma m_0 \). Eq. (9) is a geodesic equation that is correct to order \( v^2/c^2 \).
What happens in Eq. (9) is that the term $\gamma^2(1 + v^2/c^2)^2$ manages to cancel $3/4$ of the effect of the factor of 4 in the gravitomagnetic field term. It would have been very surprising to have obtained any different result, for that would have left the general relativistic approach in conflict with special relativity. From the standpoint of special relativity, the freely falling mass could serve as an interval timer, with the interval corresponding to the time to fall some particular small distance down the $y'$-axis. As observed in $S$, the interval would be dilated and that is exactly what is described by Eqs. (2) or (9). Apparently magnetic and gravitomagnetic forces provide mechanism for enforcing time dilation.

The reason that the combination of Eqs. (3) and (4) fails to yield Eq. (9) and correctly describe the free-fall problem is that they are only correct to order $v/c$. They have been combined to produce terms of order $v^2/c^2$, but apparently not all of them. Under these circumstances, it would also be expected that they would not correctly predict all gravitomagnetic effects to order $v^2/c^2$.

3 Special Relativity

The gedanken experiment can be made more interesting by allowing $m$ to orbit $M$ in a circular orbit of radius $b$ as observed in $S'$. As observed in $S$, the orbital time period will be dilated and it can be shown from the kinematic equations of special relativity that the orbit diameter will be the usual $2b$ along $y$-axis, but be contracted to $2b/\gamma$ along the $x$-axis. Further, a special case in agreement with special relativity can be obtained from Eq. (8) using $v_x = v \pm u$, where $u$ would be the speed of $m$ relative to $M$, as observed in $S$, when $m$ is on the $y$-axis. Neglecting terms in $u^2$, the resulting acceleration of $m$ for these positions would be

$$\frac{d^2y}{c^2dt^2} \approx \frac{\partial \phi}{\partial y}(1 - \frac{v^2}{c^2} \pm 2uv/c^2) \quad (10)$$

The last term of the equation accounts for the small difference of speeds of $m$ and $M$ relative to reference frame $S$.

4 Laser Lunar Ranging

The foregoing ruminations have been motivated by recent discussions of laser lunar ranging. Murphy, Nordtvedt and Turyshev [2] have recently published new results that claim to have verified Eq. (4) as a source of perturbing acceleration acting on the lunar orbit. It is claimed that they have established this weak-field limit prediction of General Relativity to an accuracy of about 0.1%, which is far better confirmation than expected from the Gravity Probe B experiment. Ciufolini [3] has pointed out that the result of Murphy et al. is merely a coordinate dependent effect. Their results were obtained relative to a solar system barycenter frame (hereafter
SSB) and thus could be transformed away. Kopeiken, on the other hand, says [4] that there should be no observable effect at all relative to the SSB.

While the gravitomagnetic field due to the sun's rotation cannot be transformed away, it is much too small to have a measurable effect on the lunar orbit. If a gravitomagnetic effect exists relative to the SSB it must be associated with the motions of earth and moon relative to the SSB. For analysis of the lunar motions, the orbital motion of earth would be regarded as the cause of a $B_g$ field that is some four orders of magnitude larger than that due to solar rotation at the position of the moon.

At the time of a new moon, the situation would correspond to the masses $m$ and $M$ positioned as shown in Figure 1, with the sun located a long way up the y-axis. System $S'$ would be comoving with the earth while system $S$ would remain at rest relative to the SSB. Thus whatever would be observed in S should be the same as what is measured relative to the SSB for this particular configuration. Considering the outcome of the previous gedanken experiment, the major effects that we should observe would be time dilation and length contraction. For a lunar orbit radius of $3.8 \times 10^8$ m and lunar velocity of $\sim 30$ km/s, the contraction of the lunar orbit parallel to the nearly joint earth-moon orbital motion about the sun would be about 1.9 m. This would be observed in frame S of Fig. 1, but not in the SSB frame.

In view of the gedanken experiment and Eqs. (2), (9) and (10), we should not expect the combination of Eqs. (3) and (4) to be confirmed in the SSB frame, notwithstanding the apparent confirmation by Murphy, Nordtvedt and Turyshev [1]. They transformed the LLR to the SSB frame and then fitted the lunar trajectory using multiple adjustable parameters. In recent years the amplitude of the dominant gravitomagnetic oscillation has been reported as 6.5 $m$ [1], 9.3 $m$ [5] and 5.3 $m$ [6]. Although the residuals of these curve fits to the lunar range observations are typically less than one cm, the oscillation amplitudes are obviously much more variable. Until the statistical errors of determination of the oscillation amplitude exclude the alternatives considered here, the claim that Eq. (4) has been confirmed is unwarranted.

Finally, we should note that calculating gravitomagnetic effects relative to a geocentric frame removes several opportunities for error. In this frame the moon moves in a gravitomagnetic field due to the apparent motion of the sun. Shahid-Saless has calculated the expected amplitude of gravitomagnetic lunar orbit oscillations to be about 3 cm in the geocentric frame.

References

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