GENERIC TRANSVERSALITY OF MINIMAL SUBMANIFOLDS
AND GENERIC REGULARITY OF TWO-DIMENSIONAL
AREA-MINIMIZING INTEGRAL CURRENTS

BRIAN WHITE

Abstract. Suppose that $N$ is a smooth manifold with a smooth Riemannian metric $g_0$, and that $\Gamma$ is a smooth submanifold of $N$. This paper proves that for a generic (in the sense of Baire category) smooth metric $g$ conformal to $g_0$, if $F$ is any simple $g$-minimal immersion of a closed manifold into $N$, then $F$ is transverse to $\Gamma$, and $F$ is self-transverse.

The paper also proves that for a generic ambient metric, every 2-dimensional surface (integral current or flat chain mod 2) without boundary that minimizes area in its homology class has support equal to a smoothly embedded minimal surface.

1. Introduction

In this paper, we prove:

**Theorem 1.** Suppose that $N$ is a smooth manifold with a smooth Riemannian metric $g_0$, and that $\Gamma$ is a smooth submanifold of $N$. For a generic (in the sense of Baire category) smooth metric $g$ conformal to $g_0$, if $F$ is any simple $g$-minimal immersion of a closed manifold into $N$, then

1. $F$ is transverse to $\Gamma$, and
2. $F$ is self-transverse.

An immersion $F : M \to N$ is called **self-transverse** provided the following holds: if $U$ and $W$ are disjoint open sets in $M$ and if the restrictions $F|_U$ and $F|_W$ are embeddings, then $F(U)$ and $F(W)$ are transverse.

An immersion $F : M \to N$ is called **simple** if each connected component of $M$ contains a point $q$ such that $F(q)$ and $F(M \setminus \{q\})$ are disjoint. In case $F$ is $g$-minimal for a smooth metric $g$, unique continuation implies that if $F$ is simple, $F(q)$ and $F(M \setminus q)$ are disjoint except for a closed, nowhere dense, measure-0 set of $q \in M$. Thus a $g$-minimal immersion is simple if and only if the image has multiplicity 1 almost everywhere.

Theorem 1 is false without the word “simple”, even in the case of 1-dimensional minimal submanifolds (i.e., geodesics). For there is a nonempty open set of metrics on $N$ for which there is a closed geodesic. If we traverse the geodesic multiple times, the result is a closed geodesic with non-transverse self-intersections.

We also prove a stronger version of Theorem 1:
Theorem 2. Theorem 1 remains true with “strongly transverse” and “strongly-self-transverse” in place of “transverse” and “self-transverse.”

Strong transversality and strong self-transversality are defined in Section 7. Theorem 21 in Section 8 gives a more geometrically intuitive characterization of those terms. The terminology is easiest to understand when $M$ and $\Gamma$ are hypersurfaces in $N$ (with $M$ immersed and $\Gamma$ embedded). In that case,

1. $M$ is strongly self-transverse if for each point $p \in N$, the unit normals to the sheets of $M$ passing through $p$ are linearly independent.
2. $M$ is strongly transverse to $\Gamma$ if for each point $p \in \Gamma$, the unit normal to $\Gamma$ at $p$ and the unit normals to the sheets of $M$ passing through $p$ are linearly independent.

See Theorem 21 in §8.

Theorems 1 and 2 also hold for constant mean curvature immersions and more generally for prescribed mean curvature immersions. See §11.

The proofs of Theorems 1 and 2 are based on the Bumpy Metrics Theorem (see §2) together with a very general theorem (Theorem 29) about linear elliptic partial differential equations. The flavor of the PDE Theorem is indicated by the following (which is equivalent to a special case of that theorem):

Theorem 3. Let $M$ be a smooth, compact, connected Riemannian manifold with smooth, nonempty boundary. Let $S$ be a finite subset of the interior of $M$ and let $f : S \to \mathbb{R}$ be any function. Then there is a harmonic function $h$ on $M$ such that $h(x) = f(x)$ for each $x \in S$.

The PDE Theorem is a rather direct consequence of a theorem of Peter Lax.

The transversality results of this paper play a key role in Xin Zhou’s proof of the Marques-Neves multiplicity-one conjecture. Indeed, this paper grew out of a conversation in which Professor Zhou explained to me how better knowledge of generic behavior of prescribed mean curvature surfaces could be very useful in min-max theory.

In §12 we apply the results in the preceding sections to show that for a generic smooth Riemannian metric on a manifold $N$, every 2-dimensional locally area-minimizing integral current without boundary has support equal to a smoothly embedded minimal surface. The same is true for flat chains mod 2. In particular, for generic ambient metrics, 2-dimensional varieties that minimize in their homology classes (integral or mod 2) are smoothly embedded minimal surfaces, possibly (in the integral case) with multiplicity.

2. The Bumpy Metrics Theorem

Let $N$ be a smooth manifold with a smooth Riemannian metric $g_0$.

Two smooth immersions $F_i : M_i \to N$ of closed manifolds into $N$ are called equivalent if there is a smooth diffeomorphism $u : M_1 \to M_2$ such that $F_2 = F_1 \circ u$. If $F$ is a smooth immersion into $N$, we let $[F]$ denote its equivalence class. If $F_i$ and $F$ are smooth immersions, we say that $[F_i]$ converges smoothly to $[F]$ if there are immersions $F'_i \in [F_i]$ such that $F'_i$ converges smoothly to $F$.

Let $\mathcal{M}$ be the space of all pairs $(\gamma, [F])$ such that $\gamma \in C^\infty(N)$ and $F$ is a smooth, simple, $c^\gamma g_0$-minimal immersion of a closed manifold into $N$. 

Define a projection $\Pi$ by

$$\Pi : \mathcal{M} \to C^\infty(N),$$

$$\Pi(\gamma, [F]) = \gamma.$$

Let $\mathcal{M}_{\text{reg}}$ be the union of open sets $U \subset \mathcal{M}$ such that $\Pi$ maps $U$ homeomorphically onto an open subset of $C^\infty(N)$. It follows from the implicit function theorem that $(\gamma, [F]) \in \mathcal{M}_{\text{reg}}$ if and only if $[F]$ has no nonzero Jacobi fields (for the metric $e^\gamma g_0$).

Let $\mathcal{M}_{\text{sing}} = \mathcal{M} \setminus \mathcal{M}_{\text{reg}}$.

**Theorem 4** (Bumpy Metrics Theorem). The set $\Pi(\mathcal{M}_{\text{sing}})$ is a meager subset of $C^\infty(N)$.

For proof, see [Whi17].

**Corollary 5** (Bumpy Metrics Corollary). Suppose that $K$ is a closed subset of $\mathcal{M}$, or, more generally, a relatively closed subset of an open subset $\mathcal{U}$ of $\mathcal{M}$. Suppose also that every nonempty open subset of $\mathcal{M}_{\text{reg}}$ contains a point of $\mathcal{M} \setminus K$.

Then $\Pi(K)$ is nowhere dense in $C^\infty(N)$.

**Proof.** Because $\mathcal{M}$ is second countable (see Remark 6), $\mathcal{M}_{\text{reg}} \cap \mathcal{U}$ is a countable union of open sets $U_i$ such that $\Pi$ maps $U_i$ homeomorphically onto an open subset $\Pi(U_i)$ of $C^\infty(N)$. Now

$$\Pi(K) \subset \Pi(\mathcal{M}_{\text{sing}}) \cup (\bigcup_i \Pi(K \cap U_i)).$$

Since $K \cap U_i$ is a relatively closed subset of $U_i$, the hypothesis of the Corollary implies that $K \cap U_i$ is nowhere dense in $U_i$. Hence $\Pi(K \cap U_i)$ is nowhere dense in $\Pi(U_i)$ and therefore is nowhere dense in $C^\infty(N)$. Thus by (1) and the Bumpy Metrics Theorem, $\Pi(K)$ is meager in $C^\infty(N)$. □

**Remark 6.** Second countability of $\mathcal{M}$ may be proved as follows. Up to smooth diffeomorphism, there are only countably many smooth, closed $m$-manifolds. Let $M_1, M_2, \ldots$ be an enumeration of them. Give each $M_i$ a smooth Riemannian metric. For each $i$, let $\mathcal{C}_i$ be a countable dense subset of $C^\infty(N) \times C^\infty(M_i, N)$. For $(\gamma, F) \in \mathcal{C}_i$, $j \in \mathbb{N}$, and $k \in \mathbb{N}$, let $U_{(\gamma, F), j, k}$ be the set of $(\gamma, [F])$ in $\mathcal{M}$ such that:

- $\text{domain}(F') = M_i$,
- $\|F' - F\|_{C^j} < \frac{1}{k}$,
- $\|\gamma' - \gamma\|_{C^j} < \frac{1}{k}$,

(where the $C^j$ norms are with respect to the background metric $g_0$ on $N$.) Then the $U_{(\gamma, F), j, k}$ form a countable basis for topology of $\mathcal{M}$. (To make sense of $F' - F$, we isometrically embed $(N, g_0)$ in some Euclidean space.)

**3. The Mean Curvature Operator**

Let $N$ be a smooth $n$-dimensional manifold with a smooth Riemannian metric $g$. Let $M$ be a smooth, closed manifold, and $F : M \to N$ be a smooth, $g$-minimal immersion. Let $\mathcal{V}_F$ be the space of all smooth normal vectorfields to $F$. Thus $f \in \mathcal{V}_F$ if and only if $f$ is a smooth function that assigns to each $x \in M$ a vector $f(x)$ in $\text{Tan}_{F(x)}N$ that is perpendicular to the image of $DF(x)$.
If \( u \in \mathcal{V}_F \) is sufficiently small (in \( C^1 \) norm), then
\[
(2) \quad x \in M \mapsto F(x) + u(x)
\]
is also an immersion.

(The expression \( (2) \) makes sense if \( N = \mathbb{R}^n \). In a general ambient manifold \( N \), the right hand side of \( (2) \) should be replaced by the image of \( u(x) \) under the exponential map.)

If \( \gamma \in C^\infty(N) \), the immersion \( (2) \) is minimal with respect to the Riemannian metric \( e^\gamma g \) if and only if \( u \) satisfies the relevant Euler-Lagrange system:
\[
H(\gamma, u) = 0.
\]

Here \( H(\gamma, \cdot) : \mathcal{V}_F \to \mathcal{V}_F \) is a second-order quasilinear elliptic operator.

Let \( G = D_1 H(0, 0) \) and \( J = D_2 H(0, 0) \). Then \( J \) is the Jacobi operator, a second-order, self-adjoint, linear elliptic operator; it is the sum of the Laplace operator and a zero-order operator. The Jacobi operator reflects how the mean curvature changes (to first order) as we move the surface while keeping the metric fixed.

The operator \( G : C^\infty(N) \to \mathcal{V}_F \) is a linear differential operator that reflects how the mean curvature changes (to first order) as we vary the metric while keeping the surface fixed. In fact, one easily calculates that
\[
(3) \quad G(\gamma)(p) = -\frac{m}{2}(\langle \nabla u(F(p)) \rangle)^\perp,
\]
where \( m = \dim(M) \) and the \( \perp \) indicates the projection onto the orthogonal complement of \( \text{Tan}(F, p) \).

In general, the map \( G : \mathcal{V} \to C^\infty(N) \) need not be surjective. For example, if \( p_1 \) and \( p_2 \) are distinct points in \( M \) with \( F(p_1) = F(p_2) \) and \( \text{Tan}(F, p_1) = \text{Tan}(F, p_2) \), then for any \( \gamma \), we have \( G(\gamma)(p_1) = G(\gamma)(p_2) \) by \( (3) \).

On the other hand, if \( F \) is an embedding, then \( G \) is surjective by \( (4) \).

4. Notation

In the remainder of the paper, except where otherwise stated, \( M \) and \( N \) are smooth manifolds with \( \dim(M) < \dim(N) \), \( g \) is a smooth Riemannian metric on \( N \), and \( F : M \to N \) is simple, smooth, \( g \)-minimal immersion.

We let \( W \) be an open subset of \( N \) such that \( U := F^{-1}(W) \) contains a point from each component of \( N \) and such that \( F|_U \) is an embedding. (Such a \( W \) exists since \( F \) is simple.) If \( M \) is connected, one can choose \( W \) to be a small neighborhood of a point \( p \in N \) such that exactly one sheet of \( F(M) \) passes through \( p \).

As in \( (3) \) we let \( \mathcal{V} = \mathcal{V}_F \) be the space of smooth normal vectorfields on \( F \). We let \( V_0 \) be the set of \( f \in \mathcal{V} \) such that \( Jf \) is supported in \( U \). It may be helpful to think of \( f \in V_0 \) as “almost” a Jacobi field: \( Jf = 0 \) outside of the very small set \( U \).

5. Families of Immersions

Proposition 7. If \( f \in V_0 \), then there is a \( \gamma \in C^\infty(N) \) such that \( G \gamma = f \).

Proof. This follows immediately from \( (3) \). \( \square \)

Theorem 8. Suppose that \( F : M \to N \) is a simple, smooth, \( g \)-minimal immersion with no nontrivial Jacobi fields (i.e., no nonzero solutions \( v \in \mathcal{V}_F \) of \( Jv = 0 \)). Let \( U, W \) and \( V_0 \) be as in \( (4) \). Let \( f_1, \ldots, f_k \) be vectorfields in \( V_0 \).
There exist $\epsilon > 0$ and smooth maps
\[
\gamma : B^k(0, \epsilon) \to C^\infty(N),
\]
\[
\mathcal{F} : B^k(0, \epsilon) \times M \to N
\]
with the following properties:

1. $\gamma(0, \cdot) = 0$.
2. $\mathcal{F}(0, \cdot) = F(\cdot)$.
3. For each $\tau \in B^k(0, \epsilon)$, the map $\mathcal{F}(\tau, \cdot) : M \to N$ is an immersion that is minimal with respect to the metric $e^{\gamma(\tau)}g$.
4. For each $\tau$, $\gamma(\tau)$ is supported in $W$.
5. For each $i$,
\[
(d/dt)_{t=0} \mathcal{F}(t\mathbf{e}_i, \cdot) = f_i(\cdot),
\]
or, in other notation,
\[
D_1 \mathcal{F}(0, \cdot) \mathbf{e}_i = f_i(\cdot).
\]

Remark 9. Theorem 8 can be restated as follows. Given a $k$-dimensional linear subspace $V$ of $V_0$, there exist smooth maps $\gamma$ and $F$ such that (1) – (4) hold, and such that $v \mapsto D_1 \mathcal{F}(0, \cdot) v$ is a surjective linear map from $\mathbb{R}^d$ onto $V$.

Proof. By Proposition 7, for each $i$, we can find a $\gamma_i \in C^\infty(N)$ supported in $W$ such that $G\gamma_i = - Jv_i$. Define $\gamma : R^k \times N \to R$ by
\[
\gamma(\tau, \cdot) = \sum_{i=1}^k \tau_i \gamma_i(\cdot).
\]
By the implicit function theorem, there is an $\epsilon > 0$ and a smooth map
\[
\mathcal{F} : B^k(0, \epsilon) \times M \to N
\]
such that (1) – (3) hold. Note also that (4) holds by our choice of the $\gamma_i$. Thus it remains only to show (5).

Since $\mathcal{F}(t\mathbf{e}_i, \cdot)$ is $e^{t\gamma_i}g$-minimal,
\[
0 = H(t\gamma_i, \mathcal{F}(t\mathbf{e}_i, \cdot)).
\]
Taking the derivative at $t = 0$ gives
\[
0 = D_1 H(0, 0) \gamma_i + D_2 H(0, 0)(d/dt)_{t=0} \mathcal{F}(t\mathbf{e}_i, \cdot) = G\gamma_i + J(d/dt)_{t=0} \mathcal{F}(t\mathbf{e}_i, \cdot) = - Jv_i + J(d/dt)_{t=0} \mathcal{F}(t\mathbf{e}_i, \cdot).
\]
Since $J$ has no nontrivial kernel,
\[
v_i = (d/dt)_{t=0} \mathcal{F}(t\mathbf{e}_i, \cdot).
\]

6. First Transversality Theorem

Theorem 10 (Submersion Theorem). Suppose that $F : M \to N$ is a smooth, simple, $g$-minimal immersion with no nontrivial Jacobi fields (i.e., no nonzero solutions $v \in V_F$ of $Jv = 0$.)

Then for some finite $k$ and some $\epsilon > 0$, there is a smooth function $\gamma : B^k(0, \epsilon) \to C^\infty(N)$ and a smooth map $\mathcal{F} : B^k(0, \epsilon) \times M \to N$ such that

1. $\gamma(0) = 0$ and $\mathcal{F}(0, \cdot) = F$. \hfill $\square$
(2) For each \( \tau \in \mathbb{B}^k(0, \epsilon) \), \( \mathcal{F}(\tau, \cdot) : M \to N \) is a smooth immersion that is minimal with respect to the metric \( \varepsilon^\gamma(t) g \).

(3) \( \mathcal{F} \) is a submersion.

Proof. Since \( \mathcal{F} : M \to N \) is simple, there is an open subset \( W \) of \( N \) such that that \( \mathcal{F} \) is an embedding of \( U := \mathcal{F}^{-1}(W) \) into \( W \) and such that \( U \) has points in each connected component of \( M \).

Let \( V_0 = \{ f \in V_{\mathcal{F}} : Jf \text{ is supported in } U \} \).

By a very general fact about solutions of linear PDEs on compact manifolds (see Theorem 29 below), \( V_0 \) has a finite-dimensional subspace \( V \) with the following property:

(4) For every point \( p \in M \) and normal vector \( v \in \text{Tan}(\mathcal{F}, p) \perp \), there is an \( f \in V \) such that \( f(p) = v \).

By Theorem 8 and Remark 9, there exist \( \epsilon > 0 \) and smooth maps
\[
\gamma : \mathbb{B}^k(0, \epsilon) \to C^\infty(N),
\]
\[
\mathcal{F} : \mathbb{B}^k(0, \epsilon) \times M \to N
\]
such that Assertions (1) and (2) hold and such that

(5) \( D\mathcal{F}(0, \cdot)_{\tau} \) is a surjection from \( \mathbb{R}^k \) onto \( V \).

To complete the proof, we show that by replacing \( \epsilon > 0 \) by a smaller positive number \( \epsilon' \) (and by replacing \( \gamma \) and \( \mathcal{F} \) by their restrictions to \( \mathbb{B}^k(0, \epsilon') \) and \( \mathbb{B}^k(0, \epsilon') \times \Omega^k M \)) we can make \( \mathcal{F} \) be a submersion.

Let \( x \in M \), let \( p = \mathcal{F}(x) = \mathcal{F}(0, x) \), and let \( v \in \text{Tan}_p N \). Write \( v = v' + v'' \) where \( v' \in \text{Tan}(\mathcal{F}, x)\perp \) and \( v'' \in \text{Tan}(\mathcal{F}, x) \).

By (4), there is an \( f \in V \) such that \( f(x) = v' \).

By (3), there is a vector \( \tau \in \mathbb{R}^k \) such that
\[
D_1\mathcal{F}(0, \cdot)\tau = f.
\]

Thus
\[
D_1\mathcal{F}(0, x)\tau = f(x) = v'.
\]

Since \( v'' \in \text{Tan}(\mathcal{F}, x) \), there is a vector \( \xi \) in \( \text{Tan}(M, x) \) such that
\[
D\mathcal{F}(x)\xi = v''.
\]

Since \( \mathcal{F}(0, \cdot) = \mathcal{F}(\cdot) \), we see that \( D_2\mathcal{F}(0, x) = D\mathcal{F}(x) \) and therefore
\[
D_2\mathcal{F}(0, x)\xi = v''.
\]

Thus
\[
D\mathcal{F}(0, x)(\tau, \xi) = D_1\mathcal{F}(0, x)\tau + D_2\mathcal{F}(0, x)\xi
\]
\[
= v' + v''
\]
\[
= v.
\]

We have shown that \( D\mathcal{F}(0, x) \) is surjective for every \( x \in M \). Hence by replacing \( \epsilon \) by a smaller \( \epsilon > 0 \), we can guarantee that \( D\mathcal{F} \) is surjective at all points of \( \mathbb{B}^k(0, \epsilon) \times M \), i.e., that \( \mathcal{F} \) is a submersion. \( \square \)
Theorem 11. Suppose that $N$ is a smooth manifold with a smooth Riemannian metric $g_0$, and that $\Gamma$ is a smooth submanifold of $N$. For a generic smooth metric $g$ conformal to $g_0$, the following holds: if $F : M \to N$ is a simple, $g$-minimal immersion of a closed manifold $M$ into $N$, then $F$ is transverse to $\Gamma$.

Proof. Let $F$ be a $g$-minimal immersion of a closed manifold $M$ into $N$ with no nontrivial Jacobi fields. Let

$$\gamma : B^k(0, \epsilon) \to C^\infty(N),$$

and

$$F : B^k(0, \epsilon) \times M \to N$$

be as in the Submersion Theorem (Theorem 10).

Since $F$ is a submersion, it is transverse to $\Gamma$. Therefore (by the Parametric Transversality Theorem), $F(\tau, \cdot) : M \to N$ is transverse to $\Gamma$ for almost all $\tau$.

In particular, there is a sequence $\tau_i \to 0$ such that $F(\tau_i, \cdot) : M \to N$ is transverse to $\Gamma$. Theorem 11 now follows from the Bumpy Metrics Corollary 5. (To see Corollary 5 applies, let $Q$ be the set of points in $M$ corresponding to immersions that are transverse to $\Gamma$. Note that transversality is an open condition, so $Q$ is an open subset of $M$. Thus we can apply Corollary 5 to the set $K := M \setminus Q$.)

Remark 12. Let $g$ be a metric satisfying the conclusion of Theorem 11. Then the conclusion also holds for non-simple immersions. For let $F : M \to N$ be any $g$-minimal immersion of a closed manifold $M$ into $N$. Using unique continuation, it is not hard to prove that $F$ factors through a simple immersion. To be precise, there is a closed manifold $M'$, a simple immersion $F' : M' \to N$, and a covering map $\pi : M \to M'$ such that $F = F' \circ \pi$. By hypothesis, $F'$ is transverse to $\Gamma$. But then (trivially) $F$ is also transverse to $\Gamma$.

7. Strong Transversality

If $S$ is a set, we let $\Delta^k S$ be the diagonal in $S^k$:

$$\Delta^k S = \{(x_1, x_2, \ldots, x_k) \in S^k : x_1 = x_2 = \cdots = x_k\},$$

and we let

$$\Omega^k S = \{(x_1, \ldots, x_k) \in S^k : x_i \neq x_j \text{ for every } i \neq j\}.$$

Definition 13. Let $F : M \to N$ be a smooth map between smooth manifolds $M$ and $N$, let $\Gamma$ be a smooth submanifold of $N$, and let $k \geq 1$ be an integer. We say that $F$ is $k$-transverse to $\Gamma$ provided the map

$$\tilde{F} : \Omega^k M \to N^k,$$

$$\tilde{F}(x_1, \ldots, x_k) = (F(x_1), \ldots, F(x_k))$$

is transverse to $\Delta^k \Gamma$. We say that $F$ is strongly transverse to $\Gamma$ if it is $k$-transverse for every $k \geq 1$.

Note that 1-transversality is the same as transversality. Theorem 21 in Section 8 gives a more geometrically intuitive description of strong transversality.

Theorem 14. Given $F$ as in §4 with no nontrivial Jacobi fields and a positive integer $k$, there exist $\epsilon > 0$, $d < \infty$, and smooth maps

$$\gamma : B^d(0, \epsilon) \to C^\infty(N),$$

$$F : B^d(0, \epsilon) \times M \to N$$
with the following properties:

1. \( \gamma(0) = 0 \) and \( F(0, \cdot) = F(\cdot) \).
2. For each \( \tau \), the map \( F(\tau, \cdot) : M \to N \) is a smooth immersion that is minimal with respect to the metric \( e^{\gamma(\tau)} g \).
3. The map \( \tilde{F} : B^d(0, \epsilon) \times \Omega^k M \to N^k \),
   \[ \tilde{F}(\tau, x_1, \ldots, x_k) = (F(\tau, x_1), \ldots, F(\tau, x_k)) \]
   is a submersion at each point of \( \mathcal{C} := \tilde{F}^{-1}(\Delta^k N) \).

Proof. Since \( F \) is an immersion, the set \( \mathcal{C} := \{(x_1, \ldots, x_k) \in \Omega^k M : F(x_1) = F(x_2) = \cdots = F(x_k)\} \)

is a compact subset of \( \Omega^k M \).

Hence, by a general PDE theorem (Theorem 29) there is a finite-dimensional subspace \( V \) of \( V_0 \) with the following property:

6. Given \( (x_1, \ldots, x_k) \in C \) and \( v_i \in \text{Tan}(F, x_i) \), there is a section \( f \in V \) such that \( f(x_i) = v_i \) for each \( i \in \{1, \ldots, k\} \).

By Theorem 8 there exist \( \gamma \) and \( F \) such that (1) and (2) hold and such that

7. \( D_1 F(0, \cdot) \) is a surjection from \( R^d \) onto \( V \).

It remains only to verify (3).

Define

\[ \rho : R^k \times \Omega^k M \to R, \]
\[ \rho(\tau, x_1, \ldots, x_k) = |\tau|, \]

and let

\[ \mathcal{C}_0 = \rho^{-1}(0) \cap \mathcal{C} = \{(\tau, x_1, \ldots, x_k) : \tau = 0 \text{ and } (x_1, \ldots, x_k) \in C\}. \]

Claim 1. \( \tilde{F} \) is a submersion at each point \((0, x_1, \ldots, x_k)\) of \( \mathcal{C}_0 \).

Proof of claim. Suppose \((0, x_1, \ldots, x_k) \in \mathcal{C}_0 \). By definition of \( \mathcal{C}_0 \),
\[ F(x_1) = F(x_2) = \cdots = F(x_k) = q \]

for some \( q \in N \). Let
\[ (v_1, \ldots, v_k) \in \text{Tan}(N^k, (q, \ldots, q)) = (\text{Tan}(N, q))^k. \]

Write \( v_i = v'_i + v''_i \) where \( v'_i \in \text{Tan}(F, x_i)^\perp \) and \( v''_i \in \text{Tan}(F, x_i) \).

By (10), there is an \( f \in V \) such that \( f(x_i) = v'_i \) for each \( i \). By (7), there is a \( \tau \in R^d \) such that
\[ D_1 F(0, \cdot) \tau = f. \]

Thus
\[ D_1 F(0, x_i) \tau = f(x_i) = v'_i. \]

Since \( v''_i \in \text{Tan}(F, x_i) \), there is a vector \( \xi_i \in \text{Tan}(M, x_i) \) such that
\[ DF(x_i) \xi_i = v''_i. \]
Since $F(0, \cdot) = F(\cdot)$,

\begin{equation}
D_2 F(0, x_i) x_i = D F(x_i) x_i = v''_i.
\end{equation}

Now

\[ D F(0, x_i)(\tau, x_i) = D_1 F(0, x_i)\tau + D_2 F(0, x_i) x_i = v'_i + v''_i = v_i. \]

Consequently,

\[ D \tilde{F}(0, x_1, \ldots, x_k)(\tau, x_1, \ldots, x_k) = (v_1, \ldots, v_k). \]

This completes the proof of Claim 1. \qed

Now let $K$ be the set of points in $C$ where $\tilde{F}$ is not a submersion. If $K$ is empty, we are done. Otherwise, note that $K$ is compact (it is a closed subset of the compact set $C$). By Claim 1, $\rho(\cdot) > 0$ at each point in $K$. Hence

\[ \epsilon' := \min_K \rho(\cdot)/2 > 0. \]

Now replace $\epsilon$ by $\epsilon'$ (and therefore $B^d(0, \epsilon)$ by $B^d(0, \epsilon')$). \qed

**Theorem 15.** Suppose that $N$ is a smooth manifold with a smooth Riemannian metric $g_0$, and that $\Gamma$ is a smooth submanifold of $N$. For a generic set of smooth metrics conformal to $g_0$, the following holds: if $F : M \to N$ is a simple, $g$-minimal immersion of a closed manifold $M$ into $N$, then $F$ is strongly transverse to $\Gamma$.

**Proof.** It suffices to show for each $k$ that the theorem holds with “$k$-transversality” in place of “strong transversality”.

Let $F$ be a $g$-minimal immersion of a closed manifold $M$ into $N$ with no nontrivial Jacobi fields. Let

\begin{align*}
\gamma : B^d(0, \epsilon) &\to C^\infty(N), \\
F : B^d(0, \epsilon) \times M &\to N, \text{ and} \\
\tilde{F} : B^d(0, \epsilon) \times \Omega^k M &\to N^k
\end{align*}

be as in the Theorem 13.

Since $\tilde{F}$ is a submersion at all points of $\tilde{F}^{-1}(\Delta^k N)$, it is transverse to $\Delta^k \Gamma$. By the Parametric Transversality Theorem, $\tilde{F}(\tau, \cdot) : M \to N$ is transverse to $\Delta^k \Gamma$ for almost all $\tau$. Therefore $F(\tau, \cdot) : M \to N$ is $k$-transverse to $\Gamma$ for almost all $\tau$.

In particular, there is a sequence $\tau_i \to 0$ such that $F(\tau_i, \cdot) : M \to N$ is $k$-transverse to $\Gamma$. Theorem 13 (with “$k$-transverse” in place of “strongly transverse”) now follows from the Bumpy Metrics Corollary 5. (Let $Q$ be the set of points in $M$ corresponding to immersions that are $k$-transverse to $\Gamma$. Since $k$-transversality is an open condition, $Q$ is an open subset of $M$. Thus we can apply Corollary 5 to the set $K := M \setminus Q$. \qed

**Definition 16.** A smooth immersion $F : M \to N$ is called “$k$-self-transverse” provided $F$ is $k$-transverse to $N$. We say that $F$ is strongly self-transverse if it is $k$-self-transverse for every $k \geq 1$. 
The immersion $F$ is 2-self-transverse if and only if it is self-transverse as defined in the introduction.

As the special case $\Gamma = N$ of Theorem 15 we have

**Theorem 17.** Suppose that $N$ is a smooth manifold with a smooth Riemannian metric $g_0$. For a generic set of smooth metrics $g$ conformal to $g_0$, the following holds: if $F : M \to N$ is a simple, $g$-minimal immersion of a closed manifold $M$ into $N$, then $F$ is strongly self-transverse.

**Corollary 18.** In Theorem 17, the following holds for a generic set of smooth metrics $g$ conformal to $g_0$: if $F : M \to N$ is a simple, $g$-minimal immersion of a closed manifold $M$ into $N$ and if $\dim(M) < \frac{1}{2} \dim(N)$, then $F$ is an embedding.

8. The Geometry of Strong Transversality

In this section, we give a more geometrically intuitive characterization of strong transversality of maps. Nothing in the section is required for the rest of the paper.

**Definition 19.** Let $V_1, \ldots, V_k$ be linear subspaces of a finite-dimensional Euclidean space $V$. We say that $V_1, \ldots, V_k$ are strongly transverse provided the following holds: if $v_i \in V_i^\perp$ and $\sum v_i = 0$, then $v_1 = \cdots = v_k = 0$.

The following theorem gives equivalent characterizations of strong transversality of linear subspaces. In particular, it shows that whether $V_1, \ldots, V_k$ are strongly transverse does not depend on choice of the inner product on $V$.

**Theorem 20.** Let $V_1, \ldots, V_{k+1}$ be linear subspaces of a finite-dimensional Euclidean space $V$. Let $\Pi_i : V \to V_i^{\perp}$ be the orthogonal projection, and let

$$L : (\bigcap_i V_i)^\perp \to V_1^{\perp} \times \cdots \times V_{k+1}^{\perp},$$

$$L(v) = (\Pi_1 v, \ldots, \Pi_{k+1} v).$$

The following are equivalent:

1. The map $L$ is an isomorphism.
2. The map $L$ is surjective.
3. $\dim((\bigcap_i V_i)^\perp) = \sum \dim(V_i^{\perp})$.
4. $\dim((\bigcap_i V_i)^\perp) \geq \sum \dim(V_i^{\perp})$.
5. $V_1 \times \cdots \times V_k$ is transverse to $\Delta^k V_{k+1}$. That is, $V^k = (V_1 \times V_2 \times \cdots \times V_k) + \Delta^k V_{k+1}$.
6. The map

$$T : V_1^{\perp} \times \cdots \times V_{k+1}^{\perp} \to (\bigcap_i V_i)^\perp,$$

$$T(v_1, \ldots, v_{k+1}) = v_1 + \cdots + v_{k+1}$$

is an isomorphism.
7. The map $T$ in (6) is injective.
8. The spaces $V_1, \ldots, V_{k+1}$ are strongly transverse in $V$.

From Condition (3), it is easy to check that $V_1, V_2$ are strongly transverse if and only if $V_1 + V_2 = V$. Thus in the case of a pair of subspaces, strong transversality and transversality are the same.
Proof of Theorem 20. That $L$ is injective follows immediately from its definition. Thus the following are equivalent: (i) $L$ is an isomorphism, (ii) $L$ is surjective, (iii) the dimensions of the domain and range are equal, (iv) the dimension of the domain is $\geq$ the dimension of the target. Now the dimensions of the domain and target are the right and left sides of the equation in (3). This proves the equivalence of (1)–(4).

To show that (3) implies (5), assume that (3) holds and let $(v_1, \ldots, v_k) \in V^k$. Then $(v_1, \ldots, v_k, 0) \in V_1 \times \cdots \times V_{k+1}$, so by (2), there is a vector $v \in (\cap_i V_i) \perp$ such that

$$v_i = \Pi_i v_i \quad \text{for } i \leq k,$$

and

$$\Pi_{k+1} v = 0.$$  
Thus $v \in V_{k+1}$, and, for $i \leq k$, $\Pi_i (v_i - v) = 0$, so $v_i - v \in V_i$. Consequently,

$$(v_1, \ldots, v_k) = (v_1 - v, v_2 - v, \ldots, v_k - v) + (v, v, \ldots, v) \in (V_1 \times V_2 \times \cdots \times V_k) + \Delta^k V_{k+1}.$$

This completes the proof that (2) implies (5).

Now suppose that (5) holds and let $(v_1, \ldots, v_k) \in V^k$. Then $(v_1, \ldots, v_k) \in V^k$, so by (3), there exists $u_i$ in $V_i$ (for $i \leq k$) and $u \in V_{k+1}$ such that

$$(v_1, \ldots, v_k) = (u_1, \ldots, u_k) + (u, \ldots, u).$$
Thus $v_1 = u_1 + u$ for $i \leq k$, so, applying $\Pi_i$ gives

$$v_i = \Pi_i u \quad (i \leq k)$$
since $v_i \in V_i \perp$ and $u_i \in V_i$. Also by (5), there exist $w_i \in V_i$ (for $i \leq k$) and $w \in V_{k+1}$ such that

$$(v_{k+1}, \ldots, v_k) = (w_1, \ldots, w_k) + (w, \ldots, w).$$
Thus $v_{k+1} = w_i + w$ for $i \leq k$, so applying $\Pi_i$ gives

$$\Pi_i v_{k+1} = \Pi_i w \quad (\text{for } i \leq k).$$

Now let

$$\sigma = u + v_{k+1} - w.$$  
Applying $\Pi_i$ gives

$$\Pi_i \sigma = v_i \quad (\text{for } i \leq k),$$

by (9) and (10). Applying $\Pi_{k+1}$ to (11) gives

$$\Pi_{k+1} \sigma = v_{k+1}$$
since $u$ and $w$ are in $V_{k+1}$.

Now let $v = \sigma - \Pi \sigma$, where $\Pi : V \to \cap_i V_i$ is the orthogonal projection. Then $v \in (\cap_i V_i) \perp$ and $\Pi_i v = v_i$ for all $i \leq k + 1$ by (12) and (13). Thus (2) holds. This completes the proof that (5) implies (2).

Note that (3) and (7) are equivalent to (6) and (9), respectively, because $T$ is the adjoint of the map $L$. Finally, (7) and (8) are trivially equivalent. □

Theorem 21. Suppose that $F : M \to N$ is a smooth immersion, that $\Gamma$ is a smooth submanifold of $N$, and that $p$ is a point in $\Gamma$. Let $V_1, V_2, \ldots, V_k$ be the tangent planes to the sheets of $F : M \to N$ that pass through $p$. Then
Theorem 23. Suppose that $X \subset M$, that $V$ and $V'$ are linear subspaces of $K(X)$, and that $S$ is a finite subset of $X$. If $V$ is ample for $S$ and if $V'$ is dense in $V$, then $V'$ is ample for $S$.

Proof. Since $V'$ is dense in $V$ and since $V|_S = K(S)$, we see that $V'|_S$ is dense in $K(S)$. But since $K(S)$ is finite-dimensional, a subspace of $K(S)$ that is dense in $K(S)$ must be all of $K(S)$.

9. Ample Spaces

In the next section, we prove the general theorems about linear PDE that were needed in the proof of the main theorem (Theorem 15). In this section, we present a few preliminaries results about spaces of sections of vector bundles.

Throughout this section, we fix a $d$-dimensional vector bundle over a Hausdorff space $M$. If $S \subset M$, we let $K(S)$ denote the space of continuous sections over $S$, i.e., the set of continuous maps that assign to each $x \in S$ a vector in the fiber at $x$.

Of course if the bundle is trivial over $S$, then $K(S)$ may be identified with $C^0(S, \mathbb{R}^d)$. Though we need the results of this section for vector bundles that may not be trivial, the proofs are the same whether or not the bundle is trivial. Thus there is no real loss of generality if this section is read with a trivial bundle in mind. That is, wherever we write $K(S)$, the reader may think $C^0(S, \mathbb{R}^d)$.

Note that if $S \subset M$ is finite set with $k$ elements, then $K(S)$ is a finite-dimensional vector space, since $K(S) \cong C^0(S, \mathbb{R}^d) \cong \mathbb{R}^{kd}$.

If $X \subset Y \subset M$ and if $V$ is a linear subspace of $K(Y)$, we let

$$V|_X = \{f|_X : f \in V\},$$

where $f|_X$ denotes the restriction of $f$ to $X$.

Definition 22. Suppose that $X \subset M$ and that $V$ is a linear subspace of $K(X)$. Let $S$ be a finite subset of $X$. We say that $V$ is ample for $S$ if

$$V|_S = K(S).$$

In other words, $V$ is ample for $\{p_1, \ldots, p_k\}$ provided the following holds: given vectors $v_i$ at $p_i$, there is an $f \in V$ such that $f(p_i) = v_i$ for each $i \in \{1, 2, \ldots, k\}$.

We will also write “$V$ is ample for the point $p$” to mean “$V$ is ample for $\{p\}$”.

Note that if $S$ is a finite set with more than one point, then being ample for $S$ is much stronger than being ample for each point of $S$. For example, suppose that the bundle is trivial, and let $V$ be the space of constant functions in $K(M)$. Then $V$ is ample for each point of $M$, but $V$ is not ample for any set with more than one point.

Theorem 23. Suppose that $X \subset M$, that $V$ and $V'$ are linear subspaces of $K(X)$, and that $S$ is a finite subset of $X$. If $V$ is ample for $S$ and if $V'$ is dense in $V$, then $V'$ is ample for $S$. 

Proof. Since $V'$ is dense in $V$ and since $V|_S = K(S)$, we see that $V'|_S$ is dense in $K(S)$. But since $K(S)$ is finite-dimensional, a subspace of $K(S)$ that is dense in $K(S)$ must be all of $K(S)$.

□
Theorem 24. Suppose that $C \subset M$ is compact, that $V_0$ is a linear subspace of $K(M)$, and that $V_0$ is ample for every $p \in C$. Then $V_0$ has a finite-dimensional subspace $V$ such that $V$ is ample for every $p \in C$.

Proof. Let $\mathcal{V}$ be the collection of all finite-dimensional subspaces of $V_0$. For each $V \in \mathcal{V}$, let $U(V)$ be the set of points $x$ such that $V$ is ample at $x$. Note that $U(V)$ is an open set.

If $p \in C$, then $V_0$ is ample for $p$, from which it trivially follows that $V_0$ has a finite-dimensional subspace that is ample at $p$. Consequently,

$$C \subset \bigcup_{V \in \mathcal{V}} U(V).$$

Since $C$ is compact, this open cover has a finite subcover:

$$C \subset U(V_1) \cup \cdots \cup U(V_k).$$

Now let $V$ be $V_1 + \cdots + V_k$. □

Theorem 25. Let $k$ be a positive integer and $C$ be a compact collection of $k$-element subsets of $M$. Suppose $V_0$ is a linear subspace of $K(M)$ such that $V_0$ is ample for every $S \in C$. Then $V_0$ contains a finite-dimensional subspace $V$ such that $V$ is ample for every $S \in C$.

Here the topology on $C$ is the obvious one. (It is the topology that comes from identifying the space of $k$-element subsets of $M$ with a subset of the quotient of $M^k$ by the action of the permutation group $S_k$.)

Note that Theorem 24 is the special case $k = 1$ of Theorem 25.

Proof. Theorem 25 can be proved almost exactly as Theorem 24 was proved. Alternatively, we can deduce Theorem 25 from Theorem 24 as follows. For notational simplicity, assume that the vector bundle over $M$ is trivial, so $K(M) = C^0(M, \mathbb{R}^d)$.

Let $\hat{C}$ be the set of $(p_1, \ldots, p_k) \in M^k$ such that $\{p_1, \ldots, p_k\} \in C$.

If $f \in C^0(M, \mathbb{R}^d)$, let $\hat{f} \in C^0(M^k, (\mathbb{R}^d)^k)$ be given by

$$\{\hat{f}(p_1, \ldots, p_k) = (f(p_1), f(p_2), \ldots, f(p_k))\}.$$

Let

$$\hat{V} = \{\hat{f} : f \in V\}.$$

Note that if $S = \{p_1, \ldots, p_k\}$ is a $k$-element subset of $M$, then $f$ is ample for $S$ if and only if $\hat{f}$ is ample at the point $(p_1, \ldots, p_k)$.

Now apply Theorem 24 to the linear space $\hat{V}$ and the set $\hat{C}$. □

10. PDE

In this section, we assume that $M$ is a smooth, closed, connected Riemannian manifold. We consider some fixed smooth vector bundle over $M$ endowed with a smooth inner product on the fibers. Let $\mathcal{V}$ be the space of all smooth sections of the vector bundle. If $D$ is a subdomain of $M$, we let $\mathcal{V}(D)$ be the space of smooth sections whose domain is $D$. In the notation of Section 9,

$$\mathcal{V}(D) = \{f \in K(D) : f \text{ is smooth}\}.$$

Let $J : \mathcal{V} \to \mathcal{V}$ be a second order, linear elliptic differential operator. We assume that $J$ has the unique continuation property:
Definition 26. We say that $J$ has the **unique continuation property** provided the following holds. If $f \in \mathcal{V}$, if $Jf = 0$ on a connected open set $U \subset M$, and if $Jf$ vanishes to infinite order at a point $p \in U$, then $f$ vanishes everywhere in $U$.

For the $J$ that arise in this paper, $J$ is the Laplacian plus lower order terms. In that case, it is well-known that $J$ has the unique continuation property. For example, it may be proved using Almgren’s frequency function (as in [GL86]), or by a theorem of Calderon [Cal62, Theorem 11].

In [Lax56, p. 760], Peter Lax proved that the unique continuation property is equivalent to what he called the Runge Property:

Definition 27. We say that $J$ has the **Runge Property** provided the following holds. If $D_1 \subset D_2$ are smooth closed domains in $M$ with nonempty boundaries such that $D_1$ is in the interior of $D_2$ and such that $D_2 \setminus D_1$ is connected, then

$$\{ f|_{D_1} : f \in \mathcal{V}(D_2) \text{ and } Jf = 0 \}$$

is dense (in the $L^2$ norm) in

$$\{ f \in \mathcal{V}(D_1) : Jf = 0 \}.$$

Of course this $L^2$ denseness implies by elliptic regularity that if $C$ is a compact subset of the interior of $D_1$, then the space

$$\{ f|_C : f \in \mathcal{V}(D_2) \text{ and } Jf = 0 \}$$

is dense with respect to smooth convergence in

$$\{ f|_C : f \in \mathcal{V}(D_1) \text{ and } Jf = 0 \}.$$

In particular,

$$\text{if } C \subset D_1 \setminus \partial D_1 \text{ is finite, then the spaces } (14) \text{ and } (15) \text{ are the same.}$$

Theorem 28 (Runge-Type Theorem). Let $U$ be a nonempty open subset of $M$ (e.g., a small open ball around a point) and let

$$\mathcal{V} = \{ f \in \mathcal{V} : Jf \text{ is supported in } U \}.$$

Suppose that $S$ is a finite subset of $M$ and that $\phi \in \mathcal{V}$. Then there is an $f \in \mathcal{V}$ such that $f(p) = \phi(p)$ and $Df(p) = D\phi(p)$ for each $p \in S$.

In other words, there exists an $f \in \mathcal{V}$ with any prescribed values and prescribed first derivatives at the points of $S$.

**Proof.** Let us assume for notational convenience that the bundle is trivial, and thus that $\mathcal{V} = \mathcal{C}^\infty(M, \mathbb{R}^d)$ for some $d$. Let $A$ be the finite dimensional vector space

$$A = \oplus_{p \in S}(\text{Tan}(M, p) \oplus L(\text{Tan}(M, p), \mathbb{R}^d)).$$

If $W$ is an open subset of $M$ containing $S$ and if $f \in \mathcal{C}^\infty(W, \mathbb{R}^d)$, we define $\lambda(f) \in A$ by

$$\lambda(f) = (f(p), Df(p))_{p \in S}.$$

Thus the assertion of the Lemma is that $\lambda$ maps $\mathcal{V}$ surjectively to $A$.

Let $q$ be a point in $U \setminus S$. Choose $r > 0$ small enough that the closed geodesic balls of radius $r$ around the points in $S \cup \{q\}$ are disjoint and diffeomorphic to closed balls in $\mathbb{R}^m$, and so that $\overline{B(q, r)}$ is contained in $U$. 


By standard PDE (see Lemma 29 below), we can choose \( r > 0 \) sufficiently small that for each \( p \in S \), there exists an \( f \in C^\infty(B(p, r), \mathbb{R}^d) \) with \( Jf = 0 \) having any prescribed values of \( f(p) \) and \( Df(p) \). Thus if we let \( D_1 = \cup_{p \in S} B(p, r) \), then
\[
\lambda : \{ f \in C^\infty(D_1, \mathbb{R}^d) : Jf = 0 \} \to A \text{ is a surjective linear map.}
\]
Let \( D_2 = M \setminus B(q, r/4) \). By the Runge Property,
\[
\{ f|D_1 : f \in C^\infty(D_2, \mathbb{R}^d), Jf = 0 \}
\]
is dense in
\[
\{ f \in C^\infty(D_1, \mathbb{R}^d) : Jf = 0 \}.
\]
Thus since \( A \) is finite dimensional, it follows from (17) that
\[
\lambda : \{ f \in C^\infty(D_2, \mathbb{R}^d) : Jf = 0 \} \to A \text{ is a surjective linear map.}
\]
Let \( \alpha \in A \). Then there exists an \( f \in C^\infty(D_2, \mathbb{R}^d) \) such that \( Jf = 0 \) and such that \( \lambda(f) = \alpha \). Let \( \tilde{f} \in C^\infty(M, \mathbb{R}^d) \) be a smooth function such that \( \tilde{f} - f \) vanishes outside of \( B(q, r/2) \). Then \( \tilde{f} \in V \) and \( \lambda(\tilde{f}) = \lambda(f) = \alpha \).

**Theorem 29.** Let \( U \) be a nonempty open subset of \( M \) (e.g., a small open ball around a point) and let
\[
V = \{ f \in \mathcal{V} : Jf \text{ is supported in } U \}.
\]
Then
1. For every finite subset \( S \) of \( M \), \( V \) is ample for \( S \).
2. If \( k \) is positive integer and \( C \) is a compact collection of \( k \)-element subsets of \( M \), then there is a finite-dimensional subspace \( \tilde{V} \) of \( V \) such that \( \tilde{V} \) is ample for every \( S \in C \).
3. There is a finite-dimensional subspace \( \tilde{V} \) of \( V \) such that \( \tilde{V} \) is ample for every point in \( M \).

**Proof.** Assertion (1) follows from the Runge-Type Theorem 28. (In fact, the Runge-Type Theorem is much stronger: it asserts that we can prescribe the values and the first derivatives of \( f \in V \) at the points in \( S \), whereas Assertion (1) only asserts that we can prescribe the values.) Assertion (2) follows from Assertion (1) by Theorem 25. Assertion (3) is the special case of Assertion (2) when \( k = 1 \) and \( C = M \).

**Lemma 30.** Suppose that \( J : C^\infty(\Omega, \mathbb{R}^d) \to C^\infty(\Omega, \mathbb{R}^d) \) is a 2nd-order linear elliptic operator, where \( \Omega \subset \mathbb{R}^m \) is an open set containing the origin. For all sufficiently small \( r > 0 \), the following holds. For every vector \( v \in \mathbb{R}^d \) and for every linear map \( L : \mathbb{R}^m \to \mathbb{R}^d \), there is a solution \( f \) of \( Jf = 0 \) on \( B(0, r) \) such that \( f(0) = v \) and such that \( Df(0) = L \).

**Proof.** Choose \( R > 0 \) so that for every \( r \leq R \) and for every affine function \( \phi : \mathbb{R}^m \to \mathbb{R}^d \), there is a unique solution \( f_{r, \phi} : B(0, r) \to \mathbb{R}^d \) to the boundary value problem
\[
Jf_{r, \phi} = 0,
\]
\[
f_{r, \phi}|_{\partial B(0, r)} = \phi|_{\partial B(0, r)}.
\]
Let \( \tilde{f}_{r, \phi} \) be the rescaled function
\[
\tilde{f}_{r, \phi} : x \in B(0, 1) \mapsto f_{r, \phi}(0) + r^{-1}(f_{r, \phi}(rx) - f_{r, \phi}(0)).
\]
Note that
\[ \tilde{f}_{r,\phi}(0) = f_{r,\phi}(0) \text{ and } D\tilde{f}_{r,\phi}(0) = Df_{r,\phi}(0), \]
and that \( \tilde{f}_{r,\phi} = \phi \) on \( \partial B(0,1) \).

As \( r \to 0 \), \( \tilde{f}_{r,\phi} \) converges smoothly to the solution \( \tilde{f}_\phi : B(0,1) \to \mathbb{R}^m \)
of
\[ J_0\tilde{f}_\phi = 0, \]
\[ \tilde{f}_\phi|_{\partial B(0,1)} = \phi|_{\partial B(0,1)}. \]
where \( J_0 \) is a constant-coefficient, homogeneous, 2nd order linear elliptic operator. Thus \( \tilde{f}_\phi = \phi \).

Let \( A \) be the space of affine maps from \( \mathbb{R}^m \) to \( \mathbb{R}^d \). Since the linear map
\[ \phi \in A \mapsto (\tilde{f}_{r,\phi}(0), D\tilde{f}_{r,\phi}(0)) \in \mathbb{R}^d \times L(\mathbb{R}^m, \mathbb{R}^d) \]
converges as \( r \to 0 \) to the linear bijection
\[ \phi \in A \mapsto (\phi(0), D\phi(0)) \in \mathbb{R}^d \times L(\mathbb{R}^m, \mathbb{R}^d), \]
it follows that the map (20) must be a bijection for all sufficiently small \( r \). Thus we are done by (19). \( \square \)

**Remark 31.** (Not needed in this paper.) A slight modification of the proof of Lemma 30 shows that for each positive integer \( k \), the following holds for all sufficiently small \( r > 0 \). If \( \phi : \mathbb{R}^m \to \mathbb{R}^d \) is a polynomial of degree \( k \) such that \( J_0\phi = 0 \), then there is an \( f : B^m(0,r) \to \mathbb{R}^d \) such that \( Jf = 0 \) and such that \( \phi \) is the degree \( k \) Taylor polynomial for \( f \) at \( 0 \).

11. Prescribed Mean Curvature Hypersurfaces

The theorems in this paper easily extend to hypersurfaces with constant mean curvature or, more generally, with prescribed mean curvature. In those settings, one works with oriented surfaces. We say that two immersions \( F_i : M_i \to N \) of smooth, oriented manifolds \( M_1 \) and \( M_2 \) are equivalent if there is an orientation-preserving diffeomorphism \( u : M_1 \to M_2 \) such that \( F_1 = F_2 \circ u \). We let \([F]\) denote the equivalence class of \( F \).

Now suppose that \( N \) is an oriented smooth \((m+1)\)-dimensional manifold with Riemannian metric \( g \), and that \( h \) is a smooth function on \( N \). Suppose that \( F : M \to N \) is an immersion of an oriented \( m \)-manifold into \( N \). We say that \( F \) has prescribed mean curvature \( h \) with respect to the metric \( g \) provided the mean curvature vector at \( x \in M \) is given by \(-h(F(x))\nu_F(x)\) where \( \nu_F(x) \) is the unit normal to \( \text{Tan}(F,x) \) corresponding to the orientations of \( N \) and of \( M \).

If we linearize the prescribed mean curvature equation about a critical point, we get the \((g,h)\)-Jacobi operator \( J \). As in the minimal case, if we restrict \( J \) to the normal bundle, it is a self-adjoint, second-order, linear elliptic operator whose leading term is the Laplacian. As in the minimal case, solutions of \( Ju = 0 \) are called \((g,h)\)-Jacobi fields, or just Jacobi fields if the \( g \) and \( h \) are understood.

**Theorem 32.** Let \( M \) be a smooth, closed, oriented \( m \)-dimensional manifold. Let \( N \) be a smooth, oriented, \((m+1)\)-dimensional manifold with a smooth Riemannian metric \( g \). Let \( h \) be a smooth function on \( N \). Let \( q \) and \( j \) be positive integers with \( q > j \geq 2 \) and let \( \alpha \in (0,1) \). Let \( M^q(\gamma, j, \alpha) \) be the set of pairs \((\gamma, [F])\) where \( \gamma \) is a \( C^q \) function on \( N \) and \( F : M \to N \) is a simple, \( C^{j+\alpha} \) immersion that has prescribed
mean curvature \(h\) with respect to the metric \(e^\gamma g\). Then \(\mathcal{M}^h(q,j,\alpha)\) is a separable, \(C^{q-j}\) Banach manifold and the map

\[
\Pi: \mathcal{M}^h(q,j,\alpha) \rightarrow C^q(N),
\]

\(\Pi(\gamma, [F]) = \gamma\)

is a \(C^{q-j}\) Fredholm map of Fredholm index 0. The kernel of \(D\Pi(\gamma, [F])\) is naturally isomorphic to the space of normal \((e^\gamma g, h)\)-Jacobi fields to \([F]\).

In case \(h\) is constant, this is proved in [Whi91, §7]. For functions \(h\), the same proof works, except that in the first equation on page 198, one replaces

\[
h \int u^\# \omega_\gamma,
\]

by

\[
\int u^\# (h \omega_\gamma),
\]

and similarly for the other formulas on that page.

**Corollary 33.** In Theorem 32, the set of critical values of \(\Pi\) is meager in \(C^q(N)\).

The Corollary follows from Theorem 32 and the Sard-Smale Theorem [Sma65, 1.3].

Let \(N, g,\) and \(h\) be as in Theorem 32. Let \(\mathcal{M}^h\) be the space of all pairs \((\gamma, [F])\) such that \(\gamma \in C^\infty(N)\) and \(F\) is a smooth, simple, immersion with prescribed mean curvature \(h\) with respect to the metric \(e^\gamma g\) from some closed, oriented \(m\)-manifold into \(N\). Let \(\Pi: \mathcal{M}^h \rightarrow C^\infty(N)\) be the projection onto the first factor:

\[
\Pi(\gamma, [F]) = \gamma.
\]

Let \(\mathcal{M}^h_{reg}\) be the union of open sets \(U \in \mathcal{M}^h\) such that \(\Pi\) maps \(U\) homeomorphically onto an open subset of \(C^\infty(N)\). It follows from the implicit function theorem that \((\gamma, [F]) \in \mathcal{M}^h_{reg}\) if and only if \([F]\) has no nonzero normal \((e^\gamma, h)\)-Jacobi fields.

Let \(\mathcal{M}^h_{sing} = \mathcal{M}^h \setminus \mathcal{M}^h_{reg}\).

**Theorem 34.** The set \(\Pi(\mathcal{M}^h_{sing})\) is a meager subset of \(C^\infty(N)\).

The paper [Whi17] proves that Theorem 34 follows from Corollary 33 in the case \(h = 0\), but the proof given there works equal well for arbitrary \(h\).

**Theorem 35.** Suppose that \(N\) is a smooth, oriented, \((m+1)\)-dimensional manifold with a smooth Riemannian metric \(g_0\), that \(h\) is a smooth function on \(N\), and that \(\Gamma\) is a smooth submanifold of \(N\). For a generic (in the sense of Baire category) smooth metric \(g\) conformal to \(g_0\), if \(F\) is any simple immersion of a closed, oriented \(m\)-manifold into \(N\) that has prescribed mean curvature \(h\) with respect to \(g\), then

1. \(F\) is strongly transverse to \(\Gamma\), and
2. \(F\) is strongly self-transverse.

Given Theorem 34, the proof of Theorem 35 is exactly as in the minimal case.

(Theorem 34 is about a given closed \(m\)-manifold, whereas Theorem 35 is an assertion about all closed \(m\)-manifolds. Note that for assertions about Baire Category, it does not matter whether or not one fixes the domain manifold, since there are only countably many diffeomorphism types of smooth, closed \(m\)-manifolds.)

The following special case of Theorem 35 is important in Xin Zhou’s proof [Zho19] of the multiplicity-one conjecture:
Corollary 36. Suppose in Theorem 35 that $h^{-1}(0)$ is smoothly embedded $m$-manifold in $N$. For a generic (in the sense of Baire category) smooth metric $g$ conformal to $g_0$, if $F$ is any simple immersion of a closed, oriented $m$-manifold into $N$ that has prescribed mean curvature $h$ with respect to $g$, then $F$ is transverse to $h^{-1}(0)$.

Remark 37. In this section, we have been assuming that $N$ and $M$ are orientable. Actually, such orientations are not necessary: it suffices for the immersions one works with to have oriented normal bundles. That is, we work with immersions $F : M \rightarrow N$ that are equipped with nowhere vanishing sections of the normal bundle. With minor changes to the definitions, all the results in this section remain true (with the same proofs) in that slightly more general setting.

12. Generic Regularity of 2-Dimensional Locally Area-Minimizing Cycles

In this section, we prove

Theorem 38. For a generic smooth Riemannian metric $g$ on a manifold $N$, if $T$ is a 2-dimensional locally $g$-area-minimizing integral cycle in $N$, then the support of $T$ is a smoothly embedded submanifold.

At the end of this section, we prove the analogous result for flat chains mod 2.

Here, “integral cycle” means “integral current with boundary 0”, and “$T$ is locally area-minimizing” means that each point of $N$ has a neighborhood $B$ such that the area (i.e., mass) of $T \cap B$ is less than or equal to the area of $T'$ for any integral current $T'$ in $B$ such that $\partial T' = \partial(T \cap B)$. In particular, if $T$ minimizes area in its homology class, then it is locally area-minimizing. Thus we have

Corollary 39. For a generic smooth Riemannian metric $g$ on a manifold $N$, if $T$ is a 2-dimensional integral cycle in $N$ that minimizes $g$-area in its integral homology class, then the support of $T$ is a smoothly embedded submanifold.

If the dimension of $N$ is less than 4, then the support of any locally area minimizing integral cycle (for an arbitrary smooth ambient metric) is a smoothly embedded submanifold. Thus we will assume throughout this section that $\dim(N) \geq 4$.

De Lellis, Spadaro, and Spolaor [DLSS18,DLSS17,DLSS18b], building on earlier work of Sheldon Chang [Cha88], proved that if $T$ is a 2-dimensional locally area-minimizing integral cycle, then the support of $T$ is a branched minimal surface. Thus there is a closed (not necessarily connected) 2-manifold $M$ and a branched minimal immersion $F : M \rightarrow N$ such that $F$ is simple and such that $F(M)$ is the support of $T$.

(If $M$ has connected components $M_1, M_2, \ldots, M_k$, then $T$ will be the current $\sum_i n_i F_\# [M_i]$ for some positive integers $n_1, n_2, \ldots, n_k$.)

According to a theorem of J. D. Moore [Moo06] and [Moo07], for a generic smooth metric on $N$, every simple branched minimal immersion into $N$ is in fact an immersion (that is, free of branch points). Consequently, for such a metric, every area-minimizing integral cycle $T$ has support equal to $F(M)$ for a simple minimal immersion $F : M \rightarrow N$ of a closed 2-manifold $M$ into $N$.

Thus Theorem follows from
**Theorem 40.** A generic smooth Riemannian metric $g$ on $N$ has the following property. If $F$ is a simple $g$-minimal immersion of a closed 2-manifold $M$ into $N$ and if $F(M)$ is the support of a locally area-minimizing integral cycle, then $F$ is an embedding.

In fact, we will prove a somewhat stronger result:

**Theorem 41.** Let $g_0$ be a smooth Riemannian metric on $N$. A generic smooth metric $g$ conformal to $g_0$ has the following property. If $F : M \to N$ is a simple $g$-minimal immersion of a closed 2-manifold $M$ into $N$ and if $F(M)$ is the support of a locally $g$-area-minimizing integral cycle, then $F$ is an embedding.

**Proof.** If $\dim(N) > 4$, the theorem is true even without the condition “and if $F(\Sigma)$ is the support of an area-minimizing integral cycle”; see Corollary 18. Thus we may assume that $N$ is a 4-manifold.

If $\Sigma$ is the support of a 2-dimensional locally area-minimizing integral cycle in the 4-manifold $N$ and if $P$ and $P'$ are planes in $\Tan(N, p)$ that are tangent to $\Sigma$ at a point $p \in N$, then the pair $(P, P')$ has the following property [Mor82]:

There is a complex structure $J$ on $\Tan(N, p)$ compatible with the conformal structure on $N$ such that $JP = P$ and $JP' = P'$. Equivalently, $\dist(x, P')$ is independent of $x$ for $x \in P$ with $|x| = 1$.

As in [2] we let $\mathcal{M}$ be the space of pairs $(\gamma, [F])$ where $\gamma \in C^\infty(N)$ and where $F$ is a smooth, simple, $e^\gamma g_0$-minimal immersion of a closed 2-dimensional manifold into $N$.

Let $\mathcal{K}$ be the set of $(\gamma, [F])$ in $\mathcal{M}$ such that $F$ is not strongly self-transverse.

If $(\gamma, [F]) \in \mathcal{M}$ and if $F$ is strongly transverse, then $F$ has double points but no triple points. Let $\mathcal{K}'$ be the set of $(\gamma, [F'])$ in $\mathcal{M} \setminus \mathcal{K}$ such that

1. $F$ is strongly self-transverse,
2. $F$ is not an embedding, and
3. at each self-intersection, the two tangent planes have the property [4],

Note that $\mathcal{K}$ is a closed subset of $\mathcal{M}$ and that $\mathcal{K}'$ is a relatively closed subset of the open set $\mathcal{U} := \mathcal{M} \setminus \mathcal{K}$.

It suffices to show that $\Pi(\mathcal{K} \cup \mathcal{K}')$ is meager in $C^\infty(N)$. Since $\Pi(\mathcal{K})$ is meager in $C^\infty(N)$ (by Theorem 17), it suffices to show that $\Pi(\mathcal{K}')$ is meager in $C^\infty(N)$. By Corollary 5 it suffices to show that if $O$ is an open subset of $\mathcal{M}_{\text{reg}}$ that contains a point $(\gamma, [F])$ in $\mathcal{K}'$, then $O$ also contains a point not in $\mathcal{K}'$.

By replacing the background metric $g_0$ by $e^\gamma g_0$, we can assume that $\gamma = 0$.

By definition of $\mathcal{K}'$, there are distinct points $p$ and $q$ in the domain $M$ of $F$ such that $F(p) = F(q)$ and such that the two tangent planes to $F(M)$ at $F(p)$ belong to $\mathcal{P}$. Since $(\gamma, [F]) \notin \mathcal{K}$, the two planes cross transversely.

Let $f$ be any smooth normal vectorfield on $F$ such that $f(p) = f(q) = 0$, $Df(p) = 0$, $Df(q) \neq 0$, and $Df(q)v = 0$ for some nonzero vector $v$.

Now suppose that $t \mapsto F_t$ is a one-parameter family of immersions with $F_0 = F$ and with $(d/dt)_{t=0}F_t = f$. By transversality, there are one-parameter families $p_t$ and $q_t$ with $p_0 = p$ and $q_t = q$ such that $F_t(p_t) = F_t(q_t)$ for $t$ near 0. For
all sufficiently small $t \neq 0$, (21) implies that the two tangent planes to $F_t(M)$ at $F_t(p_t) = F_t(q_t)$ do not have the property (4).

It remains only to show that we can choose $f$ and the family $F_t$ so that $(\gamma_t, [F_t]) \in M$ for some smooth 1-parameter family $\gamma_t \in C^\infty(N)$ with $\gamma_0 = 0$. For then we will have for all sufficiently small $t \neq 0$ that $(\gamma_t, [F_t])$ is in $\mathcal{O}$ but not in $\mathcal{K}'$. (It is not in $\mathcal{K}'$ because $F_t(M)$ has pair of tangent planes that violate the property (4).)

As in [21] we let $W$ be an open subset of $N$ such that $U := F^{-1}(W)$ contains a point from each component of $N$ and such that $F|_U$ is an embedding. (Such a $W$ exists since $F$ is simple.) By the Runge-Type Theorem [25] there exists a smooth normal vectorfield $f$ to $F$ such that $Jf$ is supported in a compact subset of $U$ and such that (21) holds. By Theorem 8 there exist $\epsilon > 0$ and smooth one-parameter families

$$t \in (-\epsilon, \epsilon) \mapsto \gamma_t \in C^\infty(M),$$

$$t \in (-\epsilon, \epsilon) \mapsto F_t \in C^\infty(M, N)$$

such that $(\gamma_t, [F_t]) \in M$, $\gamma_0 = 0$, $F_0 = F$, and $(d/dt)|_{t=0}F_t = f$. □

**Theorem 42.** A generic smooth Riemannian metric $g$ on a manifold $N$ has the following property. If $T$ is a 2-dimensional, mod 2 cycle that minimizes $g$-area in its mod 2 homology class, or, more generally, that is locally $g$-area-minimizing, then $T$ is a smooth embedded minimal surface with multiplicity 1.

Here “mod 2 cycle” means “flat chain mod 2 with boundary 0” and “$T$ is locally area-minimizing” means that every point in $N$ has a neighborhood $B$ such that the area of $T \cap B$ is less than equal to the area of any mod 2 flat chain $T'$ in $B$ with $\partial T' = \partial T$.

Every 2-dimensional locally area-minimizing cycle mod 2 is a smoothly immersed minimal surface, and if $P$ and $P'$ are two distinct tangent planes at a self-intersection point, the $P$ and $P'$ are orthogonal: $P$ lies in the orthogonal complement of $P'$. Thus for mod 2 cycles, neither the Chang-De-Lellis-Spadaro-Spoliar Theorem nor Moore’s Theorem about generic absence of branch points is needed. Otherwise, the proof is identical to the proof of Theorem 42.

**References**

[Cal62] A.-P. Calderón, *Existence and uniqueness theorems for systems of partial differential equations*, Fluid Dynamics and Applied Mathematics (Proc. Sympos., Univ. of Maryland, 1961), Gordon and Breach, New York, 1962, pp. 147–195. MR0166685

[Cha88] Sheldon Xu-Dong Chang, *Two-dimensional area minimizing integral currents are classical minimal surfaces*, J. Amer. Math. Soc. 1 (1988), no. 4, 699–778, DOI 10.2307/1990991. MR946554

[DLSS17] Camillo De Lellis, Emanuele Spadaro, and Luca Spoliar, *Regularity theory for 2-dimensional almost minimal currents II: Branched center manifold*, Ann. PDE 3 (2017), no. 2, Art. 85, DOI 10.1007/s40817-017-0035-7. MR3671256

[DLSS18a] Camillo De Lellis, Emanuele Spadaro, and Luca Spoliar, *Regularity theory for 2-dimensional almost minimal currents I: Lipschitz approximation*, Trans. Amer. Math. Soc. 370 (2018), no. 3, 1783–1801, DOI 10.1090/tran/6995. MR3739191

[DLSS18b] Camillo De Lellis, Emanuele Spadaro, and Luca Spoliar, *Regularity theory for 2-dimensional almost minimal currents III: blowup*, Preprint arXiv:1508.05510 (2018).

[GL86] Nicola Garofalo and Fang-Hua Lin, *Monotonicity properties of variational integrals, $A_p$ weights and unique continuation*, Indiana Univ. Math. J. 35 (1986), no. 2, 245–268, DOI 10.1512/iumj.1986.35.35015. MR833393
[Lax56] P. D. Lax, *A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations*, Comm. Pure Appl. Math. 9 (1956), 747–766, DOI 10.1002/cpa.3160090407. MR0086991

[Moo06] John Douglas Moore, *Bumpy metrics and closed parametrized minimal surfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. 358 (2006), no. 12, 5193–5256, DOI 10.1090/S0002-9947-06-04317-0. MR2238914

[Moo07] John Douglas Moore, *Correction for: “Bumpy metrics and closed parametrized minimal surfaces in Riemannian manifolds” [Trans. Amer. Math. Soc. 358 (2006), no. 12, 5193–5256 (electronic); MR2238914]*, Trans. Amer. Math. Soc. 359 (2007), no. 10, 5117–5123, DOI 10.1090/S0002-9947-07-04438-8. MR2329662

[Mor82] Frank Morgan, *On the singular structure of two-dimensional area minimizing surfaces in $R^n$*, Math. Ann. 261 (1982), no. 1, 101–110, DOI 10.1007/BF01456413. MR675210

[Sma65] S. Smale, *An infinite dimensional version of Sard’s theorem*, Amer. J. Math. 87 (1965), 861–866. MR0185664

[Whi91] Brian White, *The space of minimal submanifolds for varying Riemannian metrics*, Indiana Univ. Math. J. 40 (1991), no. 1, 161–200, DOI 10.1512/iumj.1991.40.40008. MR1101226

[Whi17] Brian White, *On the bumpy metrics theorem for minimal submanifolds*, Amer. J. Math. 139 (2017), no. 4, 1149–1155, DOI 10.1353/ajm.2017.0029. MR3683255

[Zho19] Xin Zhou, *On the multiplicity one conjecture in min-max theory*, arXiv:1901.01173 (2019), 1–40. MR3689325

Department of Mathematics, Stanford University, Stanford, CA 94305

E-mail address: bcwhite@stanford.edu