Ordering dynamics of the driven lattice gas model

E. Levine, Y. Kafri, and D. Mukamel

Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

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The evolution of a two-dimensional driven lattice-gas model is studied on an $L_x \times L_y$ lattice. Scaling arguments and extensive numerical simulations are used to show that starting from random initial configuration the model evolves via two stages: (a) an early stage in which alternating stripes of particles and vacancies are formed along the direction $y$ of the driving field, and (b) a stripe coarsening stage, in which the number of stripes is reduced and their average width increases. The number of stripes formed at the end of the first stage is shown to be a function of $L_x/L_y^2$, with $\phi \simeq 0.2$. Thus, depending on this parameter, the resulting state could be either single or multi-striped. In the second, stripe coarsening stage, the coarsening time is found to be proportional to $L_y$, becoming infinitely long in the thermodynamic limit. This implies that the multi-striped state is thermodynamically stable. The results put previous studies of the model in a more general framework.

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I. INTRODUCTION

Driven diffusive systems have been extensively studied in recent years. They serve as a fruitful framework for studying the statistical mechanics of systems far from thermal equilibrium. Driven by an external field these systems reach a steady state with a non-vanishing current and as such do not satisfy detailed balance. Studies of these models have revealed many differences between such systems and systems in thermal equilibrium. For example, several one-dimensional driven diffusive systems with local dynamics exhibit long range order and spontaneous symmetry breaking. Such phenomena can not occur in thermal equilibrium when the interactions are short ranged.

Many studies of driven diffusive systems have focused on a driven lattice gas (Ising) model. The model was introduced by Katz, Lebowitz, and Spohn and is often referred to as the “standard model”. In $d = 2$ dimensions the model is defined on an $L_x \times L_y$ lattice. Each of the lattice sites $i$, is either occupied by a particle or is vacant. A macroscopic configuration is characterized by a set of occupation numbers $\{n_i\}$ where $n_i = 0, 1$ represents a vacant or an occupied site, respectively. Usually the model is studied with an equal number of occupied and vacant sites. An energy $H = -\sum_{\langle ij \rangle} n_in_j$ is associated with each configuration. Here the sum is over $\langle ij \rangle$ nearest neighbor sites. The energy represents an attractive interaction between the particles. An external drive is introduced through a field $E$ which biases the motion of the particles in the $-y$ direction. Imposing periodic boundary conditions in this direction results in a current of particles through the system along the field direction. Specifically the dynamics of the model is defined through the exchange of nearest neighbor particles with a rate

$$W = \min \{1, \exp(-\beta\Delta H - E\Delta y)\}. \quad (1)$$

Here $\beta$ is an inverse temperature-like parameter, and $\Delta y = (-1, 0, 1)$ for a particle attempting to hop along, orthogonal to, or against the direction of the driving field. The energy difference between the two configurations after and before the particle exchange is denoted by $\Delta H$.

The model has been studied extensively for nearly two decades. Monte-Carlo simulations suggest that the $(T, E)$ phase diagram of the model is composed of two phases: A high temperature disordered phase in which the particle density is homogeneous, and a low temperature phase in which the system orders and phase separates into high density and low density regimes. It was found that in this phase the particles evolve towards a striped structure parallel to the direction of the driving field. Numerical studies indicate that a slow coarsening takes place in this state. As the magnitude of the driving field is increased, the transition temperature between the two phases increases and saturates at about $1.41 T_O$, where $T_O$ is the Onsager temperature corresponding to $E = 0$.

Recent Monte-Carlo simulations of this model suggest that the evolution of the striped phase is rather complex. For a square system the stripes are found not to coarsen in the thermodynamic limit, yielding a multi-stripped ordered state. This phase was termed extraordinary or “stringy”. On the other hand systems with large aspect ration, $L_y \gg L_x$, were found to evolve toward a single stripe phase.

In order to get a better understanding of the nature of the ordered phase of the driven lattice gas model we carry out in this paper a finite size scaling analysis of the evolution process starting from a fully disordered state. We find that the model evolves via two stages: (a) an early stripe formation stage in which stripes are formed from the initially disordered state; and (b) a stripe coarsening stage in which the multi-stripe configuration formed in the early stage coarsen by reducing the number of stripes and increasing their average width. A typical evolution of such a system is shown in...
The number of stripes which are formed at the end of Fig. 1.

Simple arguments are presented to show that durations from times (a) 50, (b) 2000 and (c) 500,000 Monte-Carlo sweeps are shown. Here $\beta = 2$ and $E = \infty$. One can clearly observe the two steps of the coarsening process described in the text.

Our studies yield two main results:

1. The number of stripes which are formed at the end of the initial stripe formation stage strongly depends on the aspect ratio of the system. In particular we find that the number of stripes $m$ scales as $m \sim L_x/L_y^\phi$, with $\phi \simeq 0.2$. This implies that for narrow systems ($L_x/L_y^\phi \lesssim 1$) a single stripe is formed at the end of the first stage, while for wide systems ($L_x/L_y^\phi \gg 1$) the resulting structure is multi-striped.

2. Simple arguments are presented to show that during the stripe coarsening stage the average width of the stripes grows with time as $(t/L_y)^{1/3}$. This behavior is verified by extensive numerical simulations. Therefore, the coarsening of the stripes becomes slower as the system size in the direction of the drive is increased. This implies that in the thermodynamic limit a multiple striped configuration is in fact stable. We note that similar phenomena of arrested striped configurations have been observed in previous studies of coarsening of other models with striped structures perpendicular to the direction of the drive [3].

The paper is organized as follows: In Section II the stripe formation stage is discussed. Section III considers the stripe coarsening stage. We end with a summary and discussion of the implications of our results to other related works in Section IV.

II. THE STRIPE FORMATION STAGE

The evolution of the driven lattice gas model in the early stripe formation stage has received some attention [3, 4]. Numerical simulations indicate that the domain growth process which takes place in this stage is highly anisotropic. The typical domain size in the direction of the drive and the direction perpendicular to it grow differently. In particular it has been observed [3, 4] that the typical domain size parallel to the drive grows roughly as $\ell_x \sim t^{\varphi_\perp}$ with $\varphi_\perp \simeq 1$, while the typical domain size perpendicular to the drive grows roughly as $\ell_y \sim t^{\varphi_\parallel}$, $\varphi_\parallel \simeq 0.2$. This behavior is very different from that of a non-driven system evolving towards equilibrium. It is well known that in such a system, when the dynamics is conserving, as is the case here, the average linear domain size $\xi$ grows as $t^{1/3}$ [2]. The difference in behavior is due to the inherent anisotropy induced by the drive.

The number of stripes formed in the system at the end of the stripe formation stage can be estimated using the results described above. For a stripe to form in the system the size of a domain along the direction of the drive $\ell_x(t)$ must be of the order of the system size $L_y$.

Since $\ell_y(t) \sim t^{\varphi_\parallel}$ the time for this to occur $t_s$ scales as $t_s \sim L_y^{1/\varphi_\parallel}$. At this time the typical domain size perpendicular to the drive is

$$\ell_x(t_s) \sim t_s^{\varphi_\perp} \sim L_y^{\varphi_\perp/\varphi_\parallel}.$$  \hspace{1cm} (2)$$

Thus, the number of stripes formed, $m$, scales as

$$m \sim \frac{L_x}{\ell_x(t_s)} \sim \frac{L_x}{L_y^{\phi}},$$  \hspace{1cm} (3)$$

where $\phi = \varphi_\perp/\varphi_\parallel$. Using the estimates for the exponents $\varphi_\perp$ and $\varphi_\parallel$ one has $\phi \simeq 0.2$.

Specifically, for a square system, where $L_x = L_y \equiv L$, Eq. (2) implies $m \sim L^{1-\phi}$. Since $\phi < 1$ we find that the number of stripes grows as the system size is increased, and one always reaches a multi-striped state. The stripe density, $m/L_x$, vanishes in the thermodynamic limit. In fact, this result is independent of the exact value of the exponent $\phi$ as long as $\varphi_\parallel > \varphi_\perp$.

To verify these results Monte-Carlo simulations are performed for various system sizes, starting from a random initial condition. The Monte-Carlo procedure we use is standard: At each time step a pair of neighboring sites is chosen randomly and updated according to the rate $W$ given in [4]. Throughout the paper we use $E = \infty$ and $\beta = 2$, for which the system is ordered. We have checked that the main features of this study are unchanged for other values of the parameters as long as the system is in the ordered phase. We first verify the growth law of $t_s$ with $L_y$. In order to evaluate $t_s$ the equal-time correlation of two sites at a distance $L_y/2$ in the drive direction is measured and averaged over the sample. The time $t_s$ is estimated by the time at which the average measured correlation reaches the value of 0.4. The results are shown in Fig. 2(a). One can see that the behavior of $t_s$ with $L_y$ is consistent with $\varphi_\parallel \simeq 1$.

The number of stripes initially formed in the system is estimated by performing a Fourier transform of the density in the $x$ direction and locating its first peak at a non-zero wave length. This procedure is repeated 40 times for each system size. For simplicity we consider only square systems. In Fig. 2(b) we plot the location of the peak for square systems as a function of $L$. The fitted exponent for slightly over a decade of system sizes

\[\frac{\ell_y(t)}{L} \sim t^{\varphi_\parallel}, \quad \ell_x(t) \sim t^{\varphi_\perp}.\]
main features of the driven system: (i) the fact that the
ordered domains, namely the stripes, are of the size of
the system, and (ii) the smoothness of the domain walls
bounding the stripes. This last feature has been shown
to be a result of the drive [13–14]. In contrast to the
non-driven two-dimensional Ising model, where the
domain walls are rough, here the driving field makes
the domain walls smooth.

We proceed by considering a striped state composed
of alternating stripes of particles and vacancies with aver-
age width \( \ell \). Neighboring stripes of particles interact
with each other by an exchange of particles. Since the bound-
aries of the stripes are smooth, the lateral distance that
particles have to travel in order to move from one stripe
to the other is of the order of \( \ell \). To estimate the coarsen-
ing time we assume that within a stripe of vacancies the
density of particles is low enough so that the particles
may be considered as non-interacting. This assumption
is qualitatively supported by the configurations observed
in simulations (see, e.g., Fig. 1). When a particle reaches
the boundary it is absorbed in the neighboring particle
stripe. Thus the lateral motion of the particles within a
stripe of vacancies can be considered as a one-dimensional
random walk in the \( x \) direction with two absorbing walls
located at \( x = 0 \) and \( x = \ell \). This problem is known as
the gambler’s ruin problem [17]. The probability of such
a particle to move from 0 to \( \ell \) is given by \( p(\ell) \sim 1/\ell \).

For the width of a stripe to decrease by one lattice
spacing it has to lose \( L_y \) particles. Due to the right-left
symmetry of the problem, the particle currents from one
stripe to the other are balanced on average. Therefore a
net transfer of particles from one stripe to another is only
due to fluctuations in the lateral current. The net excess
in the number of particles transferred at a time interval
\( t \) is then proportional to \( \sqrt{L_y p(\ell) t} \). For one stripe to
shrink and disappear \( \ell L_y \) particles must be transferred
so that

\[
\sqrt{L_y p(\ell) t} \sim \ell L_y .
\]

Combining this result with \( p(\ell) \sim 1/\ell \) one finds that the
average stripe width in the system grows as

\[
\ell(t) \sim \left( \frac{t}{L_y} \right)^{1/3}.
\]

This suggests that the coarsening time scales with \( L_y \),
yielding a stable striped structure in the thermodynamic
limit.

The scaling form [13] may be verified numerically by
studying the two point particle-particle correlation func-
tion. To carry out this analysis we note that in an
isotropic system without a driving field, the coarsen-
ing process is characterized by a single length scale \( \xi(t) \),
which could be the linear size of the growing domains. In
this case the two point particle-particle correlation func-
tion obeys a scaling form [12].
\[ C(r, t) = g \left( \frac{r}{\xi(t)} \right), \tag{6} \]

where \( r \) is the distance between two points. Driven systems, on the other hand, are non-isotropic, and correlations along the drive and perpendicular to it behave differently. The typical length scale perpendicular to the drive is given by \( \xi(t) \). Thus we expect the correlation function in the \( x \) direction to be of the form

\[ C_\perp(x, t) = g_\perp \left( \frac{x}{(t/L_y)^{1/3}} \right). \tag{7} \]

The asymptotic behavior of \( g_\perp(z) \) for \( z \to 0 \) is expected to obey Porod’s law, which states that \( g_\perp(z) = 1/2 - \eta z \) with some constant \( \eta \). For \( z \to \infty \) one should have \( g_\perp(z) \to 1/4 \).

Next, we verify the full scaling form \( \xi(t) \). In Fig. 3 correlation functions for the three different system sizes are plotted. For each system size the correlation function is evaluated for arbitrarily chosen \( t \) and the data is then plotted as a function of the scaling variable \( x/(t/L_y)^{1/3} \). Again, the quality of the data collapse supports our main result. Although computation time limits us to a relatively small systems, we believe the quality of the data backs our scaling argument.

![Image](image_url)

**FIG. 3.** The two-point particle-particle correlation function \( C_\perp \) is plotted as a function of the scaling variable \( x/(t/L_y)^{1/3} \), for systems of size \( 960 \times 8 \) (marked by \( \times \)), \( 800 \times 16 \) (\( \sigma \)), \( 960 \times 32 \) (\( \dagger \)). The times of measurement are chosen arbitrarily \( t = 1, 3, 8 \times 10^6 \) Monte-Carlo sweeps, respectively. In the inset, \( C_\perp \) as a function of \( x/t^{1/3} \) is shown for a system of size \( 960 \times 8 \) and times \( t = 0.2(\times), 1(\sigma), 2(\dagger) \times 10^6 \) Monte-Carlo sweeps.

**IV. DISCUSSION**

The evolution of the driven lattice gas model was considered starting from a random initial configuration. We have shown, using simple scaling arguments and extensive numerical simulations, that the evolution proceeds via two stages: an early, stripe formation stage in which stripes of the size of the system are formed, followed by a second stage in which the stripes coarsen. While the first stage lasts \( t_s \sim L_y \), the system evolves towards a single stripe configuration in the second stage at a time of order \( \sim L_y^3 L_y \). This is a result of the fact that the typical width of stripes in the coarsening stage scales with time as \( \ell(t) \sim (t/L_y)^{1/3} \). This result indicates that the coarsening time of multi-striped configurations scales with the system length \( L_y \), suggesting that these configurations exist as stable states in the thermodynamic limit \( L_y \to \infty \).

Thus, starting from a random initial configuration, the system evolves to one of two types of states, depending on its aspect ratio. For \( L_x/L_y^6 \lesssim 1 \) (\( \phi \approx 0.2 \)) the stripe formation stage leads directly to a single stripe state, while for \( L_x/L_y^6 \gg 1 \) multi-striped states are reached. The coarsening process of these states proceeds with a time scale proportional to \( L_y \).

These results put in a more general framework previous studies of this model which considered either the early stages \( \xi(t) \) of the evolution or the nature of the steady state \( \xi(t) \). A recent study of a square system has shown \( \xi(t) \) that a multi striped state (termed "stringy") is
reached from a random initial condition. It was suggested that this state is stable. Our studies indicate that this is indeed the case for an infinitely large system. However we expect a finite system to coarsen to a single striped state at a time of the order of $L_x^3 L_y$. The fact that the steady state of a system with a small aspect ration, $L_x/L_y$, was found to be composed of a single stripe is consistent with our scaling picture.

Finally, note that the slow coarsening of the stripes is a direct consequence of stripes spanning the entire system. This is a result of the existence of the drive, and is expected to be valid also in higher dimensions. It would be interesting to study such processes in high-dimensional systems. We note, however, that already in two dimensions the computational effort was considerable.

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