Erratic birational behavior of mappings in positive characteristic

Steven Dale Cutkosky

Abstract
Birational properties of generically finite morphisms $X \to Y$ of algebraic varieties can be understood locally by a valuation of the function field of $X$. In finite extensions of algebraic local rings in characteristic zero algebraic function fields which are dominated by a valuation, there are nice monomial forms of the mapping after blowing up enough, which reflect classical invariants of the valuation. Further, these forms are stable upon suitable further blowing up. In positive characteristic algebraic function fields, it is not always possible to find a monomial form after blowing up along a valuation, even in dimension two. In dimension two and positive characteristic, after enough blowing up, there are stable forms of the mapping which hold upon suitable sequences of blowing up. We give examples showing that even within these stable forms, the forms can vary dramatically (erratically) upon further blowing up. We construct these examples in defect Artin–Schreier extensions which can have any prescribed distance.

KEYWORDS
morphism, positive characteristic, valuation

1 | INTRODUCTION

Suppose that $\phi : X \to Y$ is a morphism of algebraic varieties over a field $k$. We would like to find resolutions of singularities $X_1$ of $X$ and $Y_1$ of $Y$ such that there is an induced morphism $X_1 \to Y_1$ which has the simplest possible local structure. In [4, Problem 6.2.1], it is asked if, with the assumption that $k$ has characteristic zero, there always exists such an $X_1 \to Y_1$ which is toroidal (the problem of toroidalization). Certainly no simpler global structure can always be found. This toroidal form can be found if the varieties have characteristic zero and are of dimension $\leq 3$ ([9, 12] and a simplified proof in [14]). Stronger local forms than toroidalization are true in complete generality in characteristic zero. To formulate this result, we use valuations.

The use of valuations converts the study of birational properties of morphisms into a problem in local commutative algebra. This approach was initiated and developed by Zariski. Suppose that $\phi : Y \to X$ is a dominant morphism of projective (or proper) $k$-varieties and we have a commutative diagram of morphisms

$$
\begin{array}{ccc}
Y_1 & \to & X_1 \\
\downarrow & & \downarrow \\
Y & \to & X
\end{array}
$$
where $Y_1 \to Y$ and $X_1 \to X$ are birational and projective (or proper). A valuation $\omega$ of the function field $k(Y)$ (which is trivial on $k$) restricts to a valuation $\nu$ on $k(X)$. The valuations have centers at points $y$ and $x$ on $Y$ and $X$, respectively, and have centers $y_1$ and $x_1$ on $Y_1$ and $X_1$, respectively. The behavior of the morphisms in the diagram near the centers of the valuations is determined completely by the commutative diagram

$$
\begin{array}{ccc}
R_1 & \to & S_1 \\
\uparrow & & \uparrow \\
R & \to & S \\
\end{array}
$$

where $R, S, R_1, S_1$ are the respective local rings $O_{X,x}, O_{Y,y}, O_{X_1,x_1}, O_{Y_1,y_1}$. Thus, we are able to formulate locally problems in the birational geometry in terms of valuation theory and local commutative algebra.

A classical example of this approach is the problem of local uniformization as formulated by Zariski. Suppose that $R$ is a local domain which is essentially of the finite type over a field $k$ with quotient field $K$. Let $\nu$ be a valuation of $K$ which dominates $R$. If $R \to S$ is an extension of local rings such that the maximal ideal of $S$ contracts to the maximal ideal of $R$ then we say that $S$ dominates $R$. If $S$ is dominated by the valuation ring $\mathcal{O}_{\omega}$ of a valuation $\omega$ we say that $\omega$ dominates $S$. A local uniformization of $R$ along $\nu$ is a birational extension $R \to R_1$, where $R_1$ is a regular local ring which is essentially of finite type over $R$ and is dominated by $\nu$. Zariski proved local uniformization in all dimensions and characteristic zero in [38] and established resolution of singularities in characteristic zero from local uniformization in dimension three [39]. Hironaka later proved resolution of singularities in all dimensions and characteristic zero in [22], using a different method. All current proofs of resolution of singularities of 3-folds in positive characteristic are accomplished by first proving local uniformization ([3], a simplification of this proof in [13], Cossart and Piltant [7]). As of this time, the existence of resolution of singularities in dimension $\geq 4$ is unknown in positive characteristic. The problem of local uniformization (and resolution of singularities) is closely related to the problem of finding good local forms of mappings along a valuation, as the construction of a resolution of singularities generally starts with a generically finite projection onto a nonsingular variety.

In characteristic zero, there is a very nice local form for morphisms, called local monomialization. This result is a little stronger than what comes immediately from the assumption that toroidalization is possible.

**Theorem 1** Local monomialization [8, 11]. Suppose that $k$ is a field of characteristic zero and $R \to S$ is an extension of regular local rings such that $R$ and $S$ are essentially of finite type over $k$ and $\omega$ is a valuation of the quotient field of $S$ which dominates $S$ and $S$ dominates $R$. Then, there is a commutative diagram

$$
\begin{array}{ccc}
R_1 & \to & S_1 \\
\uparrow & & \uparrow \\
R & \to & S \\
\end{array}
$$

such that $\omega$ dominates $S_1$, $S_1$ dominates $R_1$, and the vertical arrows are products of monoidal transforms; that is, these arrows are factored by the local rings of blowups of prime ideals whose quotients are regular local rings. In particular, $R_1$ and $S_1$ are regular local rings. Further, $R_1 \to S_1$ has a locally monomial form; that is, there exist regular parameters $u_1, \ldots, u_m$ in $R_1$ and $x_1, \ldots, x_n$ in $S_1$, an $m \times n$ matrix $A = (a_{ij})$ with integral coefficients such that $\text{rank}(A) = m$ and units $\delta_i \in S_1$ such that

$$u_i = \delta_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

for $1 \leq i \leq m$.

The difficulty in the proof is to obtain the condition that $\text{rank}(A) = m$. To do this, it is necessary to blow up above both $R$ and $S$.

In the case when the extension of quotient fields $K \to L$ of the extension $R \to S$ is a finite extension and $k$ has characteristic zero, it is possible to find a local monomialization such that the structure of the matrix of coefficients recovers classical invariants of the extension of valuations in $K \to L$, and this form holds stably along suitable sequences of birational morphisms which generate the respective valuation rings. This form is called strong local monomialization.
It is established for rank 1 valuations in [8] and for general valuations in [19]. The case which has the simplest form and will be of interest to us in this paper is when the valuation has rational rank 1. In this case, if \( R_1 \rightarrow S_1 \) is a strong local monomialization, then there exist regular parameters \( u_1, \ldots, u_m \) in \( R_1 \) and \( v_1, \ldots, v_m \) in \( S_1 \), a positive integer \( a \) and a unit \( \delta \in S_1 \) such that

\[
    u_1 = \delta v_1^a, \quad u_2 = v_2, \ldots, \quad u_m = v_m.
\]

The stable forms of mappings in positive characteristic and dimension \( \geq 2 \) are much more complicated. For instance, local monomialization does not always hold. An example is given in [15] where \( R \rightarrow S \) are local rings of points on nonsingular algebraic surfaces over an algebraically closed field \( k \) of positive characteristic \( p \) and \( k(X) \rightarrow k(Y) \) is finite and separable.

This leads to the question of determining the best local forms that are possible along a valuation for a generically finite morphism of surfaces in positive characteristic. In this paper, we will consider this situation, assuming that \( k \) is algebraically closed of positive characteristic \( p \).

The obstruction to local monomialization is the defect. The defect \( \delta(\omega/\nu) \), which is a power of the residue characteristic \( p \) of \( \omega \), is defined and its basic properties developed in [33, Chapter VI, Section 11], [19, 25, Section 7.1]. The defect is discussed in Section 2.1. We have the following theorem, showing that the defect is the only obstruction to strong local monomialization for maps of surfaces.

**Theorem 2.** ([19, Theorem 7.35]) Suppose that \( K \rightarrow L \) is a finite, separable extension of algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \), \( R \rightarrow S \) is an extension of local domains such that \( R \) and \( S \) are essentially of finite type over \( k \) and the quotient fields of \( R \) and \( S \) are \( K \) and \( L \) respectively such that \( S \) dominates \( R \). Suppose that \( \omega \) is valuation of \( L \) which dominates \( S \). Let \( \nu \) be the restriction of \( \omega \) to \( K \). Suppose that the extension is defectless (\( \delta(\omega/\nu) = 1 \)). Then, the conclusions of Theorem 1 hold. In particular, \( R \rightarrow S \) has a local monomialization (and a strong local monomialization) along \( \omega \).

Suppose that \( K \rightarrow L \) is a Galois extension of fields of characteristic \( p > 0 \) and \( \omega \) is a valuation of \( L \), \( \nu \) is the restriction of \( \omega \) to \( K \). Then, there is a classical tower of fields ([21, p. 171])

\[
    K \rightarrow K^s \rightarrow K^l \rightarrow K^v \rightarrow L,
\]

where \( K^v \) is the ramification field and the extension \( K \rightarrow K^v \) has no defect. Thus, the essential difficulty comes from the extension from \( K^v \) to \( L \) which could have defect. The extension \( K^v \rightarrow L \) is a tower of Artin–Schreier extensions, so the Artin–Schreier extension is of fundamental importance in this theory.

Kuhlmann has extensively studied defect in Artin–Schreier extensions in [26]. He separated these extensions into dependent and independent defect Artin–Schreier extensions. This definition is reproduced in Section 2.5. Kuhlmann also defined an invariant called the distance to distinguish the natures of Artin–Schreier extensions. This definition is given in Sections 2.3 and 2.5.

We now specialize to the case of a finite separable extension \( K \rightarrow L \) of two-dimensional algebraic function fields over an algebraically closed field \( k \) of characteristic \( p > 0 \), and suppose that \( \omega \) is a valuation of \( L \) which is trivial on \( k \) and \( \nu \) is the restriction of \( \omega \) to \( K \). If \( L/K \) has defect then \( \omega \) must have rational rank 1 and be nondiscrete. We will assume that \( \omega \) has rational rank 1 and is nondiscrete for the remainder of the introduction.

With these restrictions, the distance \( \delta \) of an Artin–Schreier extension is \( \leq 0^- \) when the extension has defect. We will define \( a^- \) in Section 2.2. If the Artin–Schreier extension is a defect extension with \( \delta = 0^- \) it is an independent defect extension. If it is a defect extension and the distance is less than \( 0^- \), then the extension is a dependent defect extension.

A quadratic transform along a valuation is the center of the valuation at the blow up of a maximal ideal of a regular local ring. There is the sequence of quadratic transforms along \( \nu \) and \( \omega \)

\[
    R \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots \quad \text{and} \quad S \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots.
\]

We have that \( \nu_{i=1}^\infty R_i = \mathcal{O}_\nu \), the valuation ring of \( \nu \), and \( \nu_{i=1}^\infty S_i = \mathcal{O}_\omega \), the valuation ring of \( \omega \). These sequences can be factored by standard quadratic transform sequences (defined in Section 3). It is shown in [19] that given positive integers
Consider $r_0$ and $s_0$, there exists $r \geq r_0$ and $s \geq s_0$ such that $R_r \to S_s$ has the following form:

$$u = \delta x^a, \quad v = x^b(y^d\gamma + x\Omega),$$

(3)

where $u, v$ are regular parameters in $R_r$, $x, y$ are regular parameters in $S_s$, $\gamma$ and $\tau$ are units in $S_s$, $\Omega \in S_s$, $a$ and $d$ are positive integers and $b$ is a non-negative integer. If we choose $r_0$ sufficiently large, then we have that the complexity $ad$ of the extension $R_r \to S_s$ is a constant which depends on the extension of valuations, which we call the stable complexity of Equation (2). When $R_r \to S_s$ has this stable complexity, we call the forms (3) stable forms.

The strongly monomial form is the case when $b = 0$ and $d = 1$; that is, after making a change of variables in $y$,

$$u = \delta x^a, \quad v = y.$$

As we observed earlier (Theorem 2) if the extension $K \to L$ has no defect, then the stable form is the strongly monomial form. If there is defect, then it is possible for the $a$ and $d$ in stable forms along a valuation to vary wildly, even though their product $ad$ is fixed by the extension, as we will see in Theorem 5. An interesting open question is if we can always find stable local forms (3) with $b = 0$.

We make an extensive study of the local forms which can occur in an Artin–Schreier extension under a sequence of quadratic transforms in Section 4. In an Artin–Schreier extension, the stable complexity is either 1 or $p$. If the stable complexity is 1, then a stable form $R_r \to S_s$ is unramified. If the stable complexity is $p$, then a stable form $R_r \to S_s$ is either of type 1 or of type 2, as defined below, and the type can vary within the sequences (2). The two types are defined as follows. There are regular parameters $u_r, v_r$ in $R_r$ and $x_s, y_s$ in $S_s$ such that if $R_r \to S_s$ has type 1, then

$$u_r = x_s, \quad v_r = y_s^p\gamma + x_s\Sigma$$

(4)

where $\Sigma$ is a unit in $S_s$ and $\Omega \in S_s$. If $R_r \to S_s$ is of type 2, then

$$u_r = \gamma x_s^p, \quad v_r = y_s,$$

(5)

where $\gamma, \tau$ are units in $S_s$ and $\Omega \in S_s$. The Artin–Schreier extension $L/K$ has no defect if and only if the stable forms $R_r \to S_s$ are of type 2 for $r \gg 0$ (Proposition 6). Observe that type 2 is the condition of being strongly monomial.

In Theorem 5, it is shown that we can construct defect Artin–Schreier extensions with any prescribed (nonpositive) distance and any prescribed switching between types. Since we are constructing defect extensions, we must impose the condition that the stable forms are not eventually always of type 2.

The construction is such that if

$$R = R_0 \to R_1 \to R_2 \to \cdots \quad \text{and} \quad S = S_0 \to S_1 \to S_2 \to \cdots$$

(6)

are the sequences of standard quadratic transform sequences along the valuation, then $R_i \to S_i$ is a stable form for all $i$, and if $\Phi : \mathbb{N} \to \{1, 2\}$ is any function (the prescribed switching) which is not equal to 2 for all sufficiently large integers, then $R_i \to S_i$ is of type $\Phi(i)$ for all $i$.

We use a formula derived in Proposition A.2 to compute distances from sequences (2) in Artin–Schreier extensions by Piltant and Kuhlmann [27]. For the reader’s convenience, we give a proof of this formula in the Appendix to this paper. A more general form of the theorem is proven in [28].

We analyze in Section 6 an example in [19], showing failure of strong local monomialization. This example is of the type of the constructions in Theorem 5. It is a tower of two defect Artin–Schreier extensions, each of the type of Theorem 5. The first extension is of type 1 for even integers and of type 2 for odd integers. The second extension is of type 2 for even integers and of type 1 for odd integers. The composite gives a sequence of extensions of regular local rings $R_i \to S_i$, where $R_i$ has regular parameters $u_i, v_i$ and $S_i$ has regular parameters $x_i, y_i$ such that the stable form is

$$u_i = \gamma x_i^p, \quad v_i = y_i^p\tau + x_i\Omega$$

(7)

for all $i$. 
Using the formula of Proposition A.2, we compute the distances of these two Artin–Schreier extensions, concluding that both extensions have dependent defect. We show that the first Artin–Schreier extension has distance \((-\frac{p^4-2}{p^4-1})\) and the second Artin–Schreier extension has distance \((-\frac{cp^3+(c-1)p^2+cp+c}{p^4-1})\), where \(c\) is a number occurring in the equation defining the second extension. The first of these distances was computed in [23] by a different method.

Suppose that \(K \to L\) is a finite extension of fields of positive characteristic and \(\nu\) is a valuation of \(L\) with restriction \(\nu\) to \(K\). It is known that there is no defect in the extension if and only if there is a finite generating sequence in \(L\) for the valuation \(\omega\) over \(K\) [30, 36]. The calculation of generating sequences for extensions of Noetherian local rings which are dominated by a valuation is extremely difficult. This has been accomplished for two-dimensional regular local rings in [20, 35] and for many hypersurface singularities above a regular local ring of the arbitrary dimension in [18].

The nature of a generating sequence in an extension of \(S\) over \(R\) determines the nature of the mappings in the stable forms. It is shown in [16, Theorem 1] that if \(R \to S\) is an extension of two-dimensional excellent regular local rings, whose quotient fields give a finite extension \(K \to L\) and \(\omega\) is a valuation of \(L\) which dominates \(S\) then the extension is without defect if and only if there exist sequences of quadratic transform \(R \to R_1\) and \(S \to S_1\) along \(\nu\) such that \(\omega\) has a finite generating sequence in \(S_1\) over \(R_1\). This shows us that we can expect good stable forms (as do hold by Theorem 2) if there is no defect, but not otherwise.

I thank Franz-Viktor Kuhlmann and Olivier Piltant for telling me about the beautiful formulas from [27], comparing distance and ramification cuts in an Artin–Schreier extension, and sharing their manuscript [27] with me. For the readers convenience, I give proofs of these formulas in the Appendix.

In [5], it is shown that quite generally there are distance bounds for Artin–Schreier extensions, and in particular in the case considered in this paper of two dimensional algebraic function fields over an algebraically closed field, the bound is 4.

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2 | PRELIMINARIES

2.1 | Some notation

Let \(K\) be a field with a valuation \(\nu\). The valuation ring of \(\nu\) will be denoted by \(\mathcal{O}_\nu\), \(\nu K\) will denote the value group of \(\nu\) and \(K\nu\) will denote the residue field of \(\mathcal{O}_\nu\).

The maximal ideal of a local ring \(A\) will be denoted by \(m_A\). If \(A \to B\) is an extension (inclusion) of local rings such that \(m_B \cap A = m_A\) we will say that \(B\) dominates \(A\). If a valuation ring \(\mathcal{O}_\nu\) dominates \(A\) we will say that the valuation \(\nu\) dominates \(A\).

Suppose that \(K\) is an algebraic function field over a field \(k\). An algebraic local ring \(A\) of \(K\) is a local domain which is a localization of a finite type \(k\)-algebra whose quotient field is \(K\). A \(k\)-valuation of \(K\) is a valuation of \(K\) which is trivial on \(k\).

Suppose that \(K \to L\) is a finite algebraic extension of fields, \(\nu\) is a valuation of \(K\) and \(\omega\) is an extension of \(\nu\) to \(L\). Then, the reduced ramification index of the extension is \(e(\omega/\nu) = [\omega L : \nu K]\) and the residue degree of the extension is \(f(\omega/\nu) = [L \omega : K \nu]\).

The defect \(\delta(\omega/\nu)\), which is a power of the residue characteristic \(p\) of \(\mathcal{O}_\omega\), is defined and its basic properties developed in [33, Chapter VI, Section 11], [25] and [19, Section 7.1]. In the case that \(L\) is Galois over \(K\), we have the formula

\[ [L : K] = e(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu)g \]

where \(g\) is the number of extensions of \(\nu\) to \(L\). In fact, we have the equation (cf. [26] or Section 7.1 [19])

\[ |G^i(\omega/\nu)| = e(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu), \]

where \(G^i(\omega/\nu)\) is the decomposition group of \(L/K\).

If \(K \to L\) is a finite Galois extension, then we will denote the Galois group of \(L/K\) by \(\text{Gal}(L/K)\).
2.2 | Initial and final segments and cuts

We review some basic material about cuts in totally ordered sets from [26]. Let \((S, <)\) be a totally ordered set. An initial segment of \(S\) is a subset \(\Lambda \subseteq S\) such that if \(\alpha \in \Lambda\) and \(\beta < \alpha\) then \(\beta \in \Lambda\). A final segment of \(S\) is a subset \(\Lambda \subseteq S\) such that if \(\alpha \in \Lambda\) and \(\beta > \alpha\) then \(\beta \in \Lambda\). A cut in \(S\) is a pair of sets \((\Lambda_L, \Lambda_R)\) such that \(\Lambda_L\) is an initial segment of \(S\) and \(\Lambda_R\) is a final segment of \(S\) satisfying \(\Lambda_L \cup \Lambda_R = S\) and \(\Lambda_L \cap \Lambda_R = \emptyset\). If \(\Lambda_1\) and \(\Lambda_2\) are two cuts in \(S\), write \(\Lambda_1 < \Lambda_2\) if \(\Lambda_L^1 \subseteq \Lambda_L^2\). Suppose that \(S \subseteq T\) is an order preserving inclusion of ordered sets and \(\Lambda = (\Lambda_L, \Lambda_R)\) is a cut in \(S\). Then, define the cut induced by \(\Lambda = (\Lambda_L, \Lambda_R)\) in \(T\) to be the cut \(\Lambda \uparrow T = (\Lambda_L \uparrow T, T \setminus (\Lambda_L \uparrow T))\) where \(\Lambda_L \uparrow T\) is the least initial segment of \(T\) in which \(\Lambda_L\) forms a cofinal subset.

We embed \(S\) in the set of all cuts of \(S\) by sending \(s \in S\) to \(s^+ = \{t \in S \mid t \leq s\}, \{t \in S \mid t > s\}\). we may identify \(s\) with the cut \(s^+\). Define
\[s^- = \{t \in S \mid t < s\}, \{t \in S \mid t \geq s\} \text{.}\]

Given a cut \(\Lambda = (\Lambda_L, \Lambda_R)\) of an ordered Abelian group \(S\), we define \(-\Lambda = (-\Lambda_R, -\Lambda_L)\), where \(-\Lambda_L = \{-s \mid s \in \Lambda_L\}\) and \(-\Lambda_R = \{-s \mid s \in \Lambda_R\}\). We have that if \(\Lambda_1\) and \(\Lambda_2\) are cuts, then \(\Lambda_1 < \Lambda_2\) if and only if \(-\Lambda_2 < -\Lambda_1\).

2.3 | Distances

Let \(K \to L\) be an extension of fields and \(\omega\) be a valuation of \(L\) with restriction \(\nu\) to \(K\). Let \(\nu K\) be the divisible hull of \(\nu K\). Suppose that \(z \in L\). Then, the distance of \(z\) from \(K\) is defined in [26, Section 2.3] to be the cut \(\text{dist}(z, K)\) of \(\nu K\) in which the initial segment of \(\text{dist}(z, K)\) is the least initial segment of \(\nu K\) in which \(\omega(z - K)\) is cofinal. That is,
\[\text{dist}(z, K) = (\Lambda^L(z, K), \Lambda^R(z, K)) \uparrow \nu K\]
where
\[\Lambda^L(z, K) = \{\omega(z - c) \mid c \in K \text{ and } \omega(z - c) \in \nu K\}\].

The following notion of equivalence is defined in [26, Section 2.3]. If \(y, z \in L\), then \(z \sim_K y\) if \(\omega(z - y) > \text{dist}(z, K)\).

2.4 | Higher ramification groups

We recall material from pp. 78 and 79 of [33]. Suppose that \(K \to L\) is a finite Galois extension with Galois group \(G\) and \(\omega\) is a valuation of \(L\). Suppose that \(I \subseteq O_\omega\) is an ideal. Associate with \(I\) higher ramification subgroups of \(G\) by
\[G_I = \{s \in G \mid s(x) - x \in I \text{ for every } x \in O_\omega\}\],
\[H_I = \{s \in G \mid s(x) - x \in Ix \text{ for every } x \in L\}\].

The group \(H_I\) is denoted by \(G_I^r\) in [27]. We always have that \(H_I \subseteq G_I\). If \(I \subseteq J\), then \(G_I \subseteq G_J\) and \(H_I \subseteq H_J\).

2.5 | Artin–Schreier extensions

Let \(K \to L\) be an Artin-Schreier extension of fields of characteristic \(p > 0\) and \(\omega\) be a valuation of \(L\) with restriction \(\nu\) to \(K\). The field \(L\) is Galois over \(K\) with Galois group \(G \cong \mathbb{Z}_p\), where \(p\) is the characteristic of \(K\).
Let $\Theta \in L$ be an Artin–Schreier generator of $K$; that is, there is an expression

$$\Theta^p - \Theta = a$$

for some $a \in K$. We have that

$$\text{Gal}(L/K) \cong \mathbb{Z}_p = \{\text{id}, \sigma_1, \ldots, \sigma_{p-1}\}$$

where $\sigma_i(\Theta) = \Theta + i$.

Since $L$ is Galois over $K$, we have that $ge(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu) = p$, where $g$ is the number of extensions of $\nu$ to $L$. So, we either have that $g = 1$ or $g = p$. If $g = 1$, then $\omega$ is the unique extension of $\nu$ to $L$ and either $e(\omega/\nu)f(\omega/\nu) = 1$ and $\delta(\omega/\nu) = p$. In particular, the extension is defect if and only if it is an immediate extension ($e = f = 1$) and $\omega$ is the unique extension of $\nu$ to $L$.

For $\alpha \in \omega L$, we have an associated ideal $I_\alpha = \{f \in \mathcal{O}_\omega \mid \omega(f) \geq \alpha\}$. We have that $\beta < \alpha$ implies $I_\alpha \subset I_\beta$. Thus, $\{\alpha \in \omega L \mid G_{I_\alpha} = 1\}$ is a final segment of $\omega L$. We define the ramification cut $\text{Ram}(\omega/\nu)$ of the valued field extension $L/K$ to be the cut in $\nu K$ induced by this final segment; that is, $\text{Ram}(\omega/\nu) = (\text{Ram}(\omega/\nu)^L, \text{Ram}(\omega/\nu)^R)$, where $\text{Ram}(\omega/\nu)^R$ is the smallest final segment of $\nu K$ in which $\{\alpha \in \omega L \mid G_{I_\alpha} = 1\}$ is coinital.

From now on in this subsection, suppose that $L$ is a defect extension of $K$. By [26, Lemma 4.1], the distance $\delta = \text{dist}(\Theta, K)$ does not depend on the choice of Artin–Schreier generator $\Theta$, so $\delta$ can be called the distance of the Artin–Schreier extension. Since $L/K$ is an immediate extension, the set $\omega(\Theta - K)$ is an initial segment in $\nu K$ which has no maximal element by [26, Theorem 2.19].

We have, since the extension is defect, that

$$\delta = \text{dist}(\Theta, K) \leq 0^-$$

by [26, Corollary 2.30].

A defect Artin–Schreier extension $L$ is defined in [26, Section 4] to be a dependent defect Artin–Schreier extension if there exists an immediate purely inseparable extension $K(\eta)$ of $K$ of degree $p$ such that $\eta \sim_K \Theta$. Otherwise, $L/K$ is defined to be an independent defect Artin–Schreier defect extension. We have by [26, Proposition 4.2] that for a defect Artin–Schreier extension,

$$L/K$$

is independent if and only if the distance $\delta = \text{dist}(\Theta, K)$ satisfies $\delta = p\delta$.

### 2.6 Extensions of rank 1 valuations in an Artin–Schreier extension

In this subsection, we suppose that $L$ is an Artin–Schreier extension of a field $K$ of characteristic $p$, $\omega$ is a rank 1 valuation of $L$ and $\nu$ is the restriction of $\omega$ to $K$. We suppose that $L$ is a defect extension of $K$. To simplify notation, we suppose that we have an embedding of $\omega L$ in $\mathbb{R}$. Since $L$ has defect over $K$ and $L$ is separable over $K$, $\omega L$ is nondiscrete by the corollary on p. 287 of [32], so that $\omega L$ is dense in $\mathbb{R}$.

We define a cut in $\mathbb{R}$ by extending the cut dist$(\Theta, K)$ in $\nu K$ to a cut of $\mathbb{R}$ by taking the initial segment of the extended cut to be the least initial segment of $\mathbb{R}$ in which the cut dist$(\Theta, K)$ is cofinal. This cut is then dist$(\Theta, K) \uparrow \mathbb{R}$. This cut is either $s$ or $s^-$ for some $s \in \mathbb{R}$. If $L$ is a defect extension of $K$ then dist$(\Theta, K) \uparrow \mathbb{R} = s^-$, where $s$ is a nonpositive real number by [26, Theorem 2.19] and [26, Corollary 2.30]. We will set dist$(\omega/\nu)$ to be this real number $s$, so that

$$\text{dist}(\Theta, K) \uparrow \mathbb{R} = s^- = (\text{dist}(\omega/\nu))^-$$. 

The real number dist$(\omega/\nu)$ is well defined since it is independent of choice of Artin–Schreier generator of $L/K$ by Lemma 4.1 [26].

With the assumptions of this subsection, by Equations (9) and (10), the distance $\delta = \text{dist}(\Theta, K)$ of an Artin–Schreier extension is $\leq 0^-$ when the extension has defect. If it is a defect extension with distance equal to $0^-$ then it is an independent defect extension. If it is a defect extension and the distance is less than $0^-$ then the extension is a dependent defect.
extension. Thus, if $L/K$ is a defect extension, we have that $\text{dist}(\omega/\nu) \leq 0$ and the defect extension $L/K$ is independent if and only if $\text{dist}(\omega/\nu) = 0$.

3 | EXTENSIONS OF TWO-DIMENSIONAL REGULAR LOCAL RINGS

Suppose that $M$ is a two-dimensional algebraic function field over an algebraically closed field $k$ of characteristic $p > 0$ and $\mu$ is a nondiscrete rational rank 1 valuation of $M$. Suppose that $A$ is an algebraic regular local ring of $M$ such that $\mu$ dominates $A$. A quadratic transform of $A$ is an extension $A \to A_1$, where $A_1$ is a local ring of the blowup of the maximal ideal of $A$ such that $A_1$ dominates $A$ and $A_1$ has dimension two. A quadratic transform $A \to A_1$ is said to be along the valuation $\mu$ if $\mu$ dominates $A_1$.

Let

$$A = A_0 \to A_1 \to A_2 \to \cdots$$

be the sequence of quadratic transforms along $\mu$ so that the valuation ring $\mathcal{O}_A = \bigcup A_i$ (by [2, Lemma 12]). Let $P$ be a height one prime ideal in $A$ such that $A/P$ is a regular local ring. $A_i$ is said to be free if the radical of $PA_i$ is a height one prime ideal (so that $A_i/\sqrt{PA_i}$ is a regular local ring). In particular, $A_0$ is free.

Now, there exist (as explained in more detail in [19, Definition 7.11]) positive integers $r'_i$ and $\tilde{r}_i$ such that $A_0 = A_{r'_1}$ and for all $i \geq 1$, $r'_i \leq \tilde{r}_i < r'_{i+1} - 1$ such that if $r'_i \leq j \leq \tilde{r}_i$ then $A_j$ is free and if $\tilde{r}_i < j < r'_{i+1}$ then $A_j$ is not free. Here, we modify [19, Definition 7.5] slightly by defining $E_0$ on Spec($A_0$) to be $Z(P)$ and $E_i$ on Spec($A_i$) to be $Z(PA_i)$.

Suppose that $A_j$ is free. Then, there exists $i$ such that $r'_i \leq j \leq \tilde{r}_i$. Let $u, v$ be regular parameters in $A_j$ such that $u = 0$ is a local equation of $Z(PA_j)$; that is, $(u) = \sqrt{PA_j}$. We will say that $u, v$ are admissible parameters in $A_i$. Let $\tilde{u}, \tilde{v}$ be defined by

$$u = \tilde{u}^m (\tilde{v} + \alpha)^{a'}, v = \tilde{u}^d (\tilde{v} + \alpha)^{b'}$$

where

$$\frac{\nu(u)}{\nu(v)} = \frac{m}{q}$$

with $\gcd(m, q) = 1$ and $0 \neq \alpha \in k$ is such that $\nu(\tilde{v}) > 0$.

Then, there exists $k > j$ such that $\tilde{u}$ and $\tilde{v}$ are regular parameters in $A_k$. Further, $A_k$ is free, and $\tilde{u} = 0$ is a local equation of the reduced exceptional divisor of Spec($A_k$) $\to$ Spec($A$), so that $\tilde{u}, \tilde{v}$ are admissible parameters in $A_k$. If $\mu(v) \not\in \mu(u)Z$, then $m > 1$ and $k = r'_{i+1}$. If $\mu(v) \in \mu(u)Z$ then $m = 1$ and $k \leq \tilde{r}_i$.

A regular system of parameters with $\mu(v) \not\in \mu(u)Z$ can always be found from a given regular system of parameters $u, v$ by possibly replacing $v$ with the difference of $v$ and a suitable polynomial $g(u) \in k[u]$ (which necessarily has no constant term).

We will call the sequence $A_0 = A_{r'_0} \to A_{r'_1} \to A_{r'_2} \to \cdots$ the sequence of standard sequences of quadratic transforms along $\mu$. Observe that this sequence depends on the choice of $P$ in $A$.

Let $K \to L$ be a finite separable extension of two-dimensional algebraic function fields over an algebraically closed field $k$. Suppose that $R$ is a two-dimensional regular algebraic local ring of $K$ and $S$ is a two-dimensional regular algebraic local ring of $L$ such that $S$ dominates $R$. Let $P$ be a height one prime ideal in $R$ such that $P/R$ is a regular local ring and let $Q$ be a height one prime ideal in $S$ such that $S/Q$ is a regular local ring. We do not insist in this definition that the good condition that $Q \cap R = P$ holds.

Let $R \to R_1 \to \cdots \to R_r$ and $S \to S_1 \to \cdots \to S_s$ be sequences of quadratic transforms such that $S_i$ dominates $R_i$. Let $E_i$ be the divisor $Z(PR_i)$ and $F_i$ be the divisor $Z(QS_i)$.

**Definition 1** [19, Definition 7.5]. Suppose that $S_i$ dominates $R_i$. The map $R_r \to S_s$ is said to be prepared if both $R_r$ and $S_s$ are free, the critical locus of Spec($S_s$) $\to$ Spec($R_r$) is contained in $F_s$ and we have an expression $u = \gamma x^a$, where $u$ is part of a regular system of parameters of $R_r$ such that $u = 0$ is a local equation of $E_r$, $x$ is part of a regular system of parameters of $S_s$ such that $x = 0$ is a local equation of $F_s$ and $\gamma$ is a unit in $S_s$. 
Suppose that $R_r \to S_s$ is prepared. Let $(u, v)$ and $(x, y)$ be admissible parameters in $R_r$ and $S_s$, respectively; that is, $u$ is part of a regular system of parameters of $R_r$ such that $u = 0$ is a local equation of $E_r$, $x$ is part of a regular system of parameters of $S_s$ such that $x = 0$ is a local equation of $F_s$. Further, $u = \gamma x^a$, where $\gamma$ is a unit in $S_s$. Then, there is an expression

$$u = \gamma x^a, v = x^b f,$$

where $\gamma \in S_s$ is a unit, $f \in S_s$ and $x$ does not divide $f$ in $S_s$.

**Definition 2.** We will say that $R_r \to S_s$ is well prepared if $f$ is not a unit in $S_s$.

Suppose that $R_r \to S_s$ is well prepared. The complexity of $R_r \to S_s$ is $ad$ where $d$ is the order of the residue of $f$ in the one-dimensional regular local ring $S_s/(x)$.

The complexity is defined in [19, Definition 7.9]. It is shown there that the complexity depends only on the extension $R_r \to S_s$.

**Proposition 1** [19, Proposition 7.2]. Suppose that $S_s$ dominates $R_r$, $R_r \to S_s$ is well prepared and $R_r$ and $S_s$ have admissible parameters $(u, v)$ and $(x, y)$ satisfying equation (11). Let $S^*$ be the local ring which is the localization of the normalization of $R_r$ in $L$ which is dominated by $S_s$. Then, there exists a commutative diagram

$$
\begin{array}{ccc}
S^* & \to & S_s \\
\uparrow & & \uparrow \\
R_r & \to & R^*
\end{array}
$$

where all arrows are dominating maps such that $R^*$ is a two-dimensional algebraic normal local ring of $K$ such that $S_s$ is a local ring of the integral closure of $R^*$ in $L$. We have

$$\left[\text{QF}(\hat{S}_s) : \text{QF}(\hat{R}^*)\right] = ad$$

where $d$ is the order of the residue of $f$ in the one-dimensional regular local ring $S_s/(x)$.

**Remark 1.** Suppose that assumptions are as in Proposition 1 with the additional assumption that $K \to L$ is Galois. Then, the complexity $ad$ of $R_r \to S_s$ divides the degree $[L : K]$.

**Proof.** Let $\hat{S}$ be the integral closure of $R^*$ in $L$. Let $m_1, \ldots, m_g$ be the maximal ideals of $\hat{S}$. The local ring $S_s$ is one of the localizations $\hat{S}_{m_i}$. The Galois group $G(L/K)$ acts transitively on the local rings $\hat{S}_{m_i}$ so these rings are all isomorphic $k$-algebras. The $m_{R^*}$-adic completion of $\hat{S}$ is the direct sum of the complete local rings $\hat{S}_{m_i}$, and one of these local rings is $\hat{S}_s$. We have that

$$[L : K] = \sum_{i=1}^g [\text{QF}(\hat{S}_{m_i}) : \text{QF}(\hat{R}^*)]$$

by [1, Proposition 1]. Thus, $[L : K] = g[\text{QF}(\hat{S}_s) : \text{QF}(\hat{R}^*)] = gad$. □

The following remark follows from [2, Theorem 2].

**Remark 2.** Suppose that $\omega$ is a rational rank 1 nondiscrete valuation of the quotient field of $K^*$ with restriction $\nu$ to $K$ such that $\omega$ dominates $S_j$ for all $j$ and $\nu$ dominates $R_i$ for all $i$. Then given $r_0 > 0$ and $s_0 > 0$, there exist $r \geq r_0$ and $s \geq s_0$ such that $S_s$ dominates $R_r$ and $R_r \to S_s$ is well prepared. This result is true for any initial choice of $P$ in $R$ and $Q$ in $S$. 
**Proposition 2.** Let $K \rightarrow L$ be a finite separable extension of two-dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$, $\omega$ be a rational rank 1 nondiscrete valuation of $L$ (with residue field $k$) and $\nu$ be the restriction of $\omega$ to $K$.

Suppose that $A$ is an algebraic local ring of $K$ which is dominated by $\nu$. Then, there exists an algebraic regular local ring $R'$ of $K$ which is dominated by $\nu$ and dominates $A$ with the following property.

Suppose that $R$ is a regular algebraic local ring of $K$ which dominates $R'$ and $S$ is a regular algebraic local ring of $L$ which is dominated by $\omega$ and dominates $R$ such that

1) There exist regular parameters $x, y$ in $S$ and $u, v$ in $R$ such that $u = \gamma x^a$ and $v = x^b f$ with $\gamma$ a unit in $S$ and $f \in S$ such that $f$ is not a unit in $S$, $x$ does not divide $f$.
2) The critical locus of $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is contained in $Z(x_S)$.

Then, letting $d = \dim_k S/(f, x)$, we have that the complexity $ad = e(\omega/\nu) \delta(\omega/\nu)$.

This is proved in [19, Section 7.9] and [17, Proposition 3.4].

Let $K \rightarrow L$ be a finite separable extension of two-dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$, $\omega$ be a rational rank 1 nondiscrete valuation of $L$ (with residue field $k$) and $\nu$ be the restriction of $\omega$ to $K$. Let $R$ be a regular algebraic local ring of $K$ and $S$ be a regular algebraic local ring of $L$ which is dominated by $\omega$ and dominates $R$. Let $P$ be a height one prime ideal in $R$ such that $R/P$ is a regular local ring and let $Q$ be a height one prime ideal in $S$ such that $S/Q$ is a regular local ring. Let

$$R \rightarrow R_1 \rightarrow \cdots \rightarrow \cdots \text{ and } S \rightarrow S_1 \rightarrow \cdots \rightarrow \cdots$$

be the infinite sequences of quadratic transforms along $\nu$ and $\omega$, respectively. By Remark 2 and Proposition 2, there exists a positive integer $r_0$ such that whenever $r \geq r_0$ and $R_r \rightarrow S_r$ is well prepared, we have that the complexity $ad$ of this extension is equal to $e(\omega/\nu) \delta(\omega/\nu)$. We will call this the stable complexity of the sequences (12).

Suppose that $K \rightarrow L$ is a finite extension of two-dimensional algebraic function fields, $R$ is an algebraic regular local ring of $K$ which is dominated by a regular algebraic local ring $S$ of $L$ such that $\dim R = \dim S = 2$. Let $x, y$ be regular parameters in $S$ and $u, v$ be regular parameters in $R$. Then, we can form the Jacobian ideal

\[ J(S/R) = \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right). \]

This ideal is independent of choice of regular parameters.

The following proposition is established in [19].

**Proposition 3.** Suppose that $A$ is an algebraic local ring of $K$ and $B$ is an algebraic local ring of $L$ which is dominated by a rational rank 1 nondiscrete valuation $\omega$ of $L$ such that $B$ dominates $A$. Then, there exists a commutative diagram of homomorphisms

$$R \rightarrow S \quad \text{ and } \quad A \rightarrow B$$

such that $R$ is a regular algebraic local ring of $K$ with regular parameters $u, v$, $S$ is a regular algebraic local ring of $L$ with regular parameters $x, y$ such that $S$ is dominated by $\omega$, $S$ dominates $R$, $J(S/R) = (x \tilde{c})$ for some non-negative integer $\tilde{c}$ and there is an expression

$$u = \gamma x^a, \quad v = x^b (y^n \tau + x\Omega)$$

where $\tau, \gamma$ are units in $S, \Omega \in S$ and $n > 0, 0 \leq b < a$. Thus, the quadratic transform of $R$ along $\nu$ is not dominated by $S$. 

Proof. By two-dimensional local uniformization (or resolution of singularities), \[1, 29\] or \[6\], there exists a commutative diagram
\[
\begin{array}{ccc}
R_0 & \rightarrow & S_0 \\
\uparrow & & \uparrow \\
A & \rightarrow & B
\end{array}
\]
such that \(S_0\) is an algebraic local ring of \(L\) which dominates \(B\), \(R_0\) is an algebraic regular local ring of \(K\) which dominates \(A\), \(\omega\) dominates \(S_0\) and \(S_0\) dominates \(R_0\). Fix regular parameters \(u_0, v_0\) in \(R_0\).

Let \(S_0 \rightarrow S\) be a sequence of quadratic transforms along \(\omega\) such that \(S\) is free and \(S\) dominates \(R_1\), and the support of \(u_0J(S_0/R_0)S\) is the last exceptional divisor \(F\) of the sequence of quadratic transforms factoring \(S_0 \rightarrow S\) (such an \(S\) exists since \(\omega\) has rank 1 and \(\omega\) is nondiscrete). Let \(x, y\) be regular parameters in \(S\) such that \(x = 0\) is a local equation of \(F\) in \(\text{Spec}(S)\). Thus, \(J(S/R_0) = J(S_0/R_0)J(S/S_0) = (x^f)\) for some positive integer \(f\). Further, \(u_0 = \gamma x^l\) for some unit \(\gamma\) in \(S\) and positive integer \(l\). Thus, there exist \(b, n \in \mathbb{N}\), a unit \(\tau\) in \(S\) and \(\Omega \in S\) such that \(v_0 = x^b(\tau y^n + x \Omega)\). There exists a sequence of quadratic transforms \(R_0 \rightarrow R_1\) along \(\nu\) such that \(S\) dominates \(R_1\), and after replacing \(R_0\) with \(R_1\) we have that \(n > 0\).

If \(\left\lfloor \frac{b}{l} \right\rfloor = 0\) then the quadratic transform of \(R_0\) along \(\nu\) is not dominated by \(S\) and we set \(R = R_0\). If \(\left\lfloor \frac{b}{l} \right\rfloor > 0\), then set \(e = \left\lfloor \frac{b}{l} \right\rfloor\), and let \(R_0 \rightarrow R\) be the sequence of \(e\) quadratic transforms along \(\nu\). Then, \(R \rightarrow S\) satisfies the conclusions of the proposition. \(\square\)

Remark 3. Let \(K \rightarrow L\) be an Artin–Schreier extension of two-dimensional algebraic function fields over an algebraically closed field \(k\) of characteristic \(p > 0\). Let \(\omega\) be a rational rank 1 nondiscrete valuation of \(L\) (with residue field \(k\)) and \(\nu\) be the restriction of \(\omega\) to \(K\). Since \(L\) is Galois over \(K\), we have that \(g(e(\omega/\nu)\delta(\omega/\nu)) = p\), where \(g\) is the number of extensions of \(\nu\) to \(L\). So, we either have that \(g = 1\) or \(g = p\). If \(g = 1\), then \(\omega\) is the unique extension of \(\nu\) to \(L\) and either \(e(\omega/\nu) = p\) and \(\delta(\omega/\nu) = 1\) or \(e(\omega/\nu) = 1\) and \(\delta(\omega/\nu) = p\). If \(g = 1\), we have by Proposition 2 that the stable complexity of the sequences (12) is \(ad = p\). If \(g = p\), then \(e(\omega/\nu) = 1\) and \(\delta(\omega/\nu) = 1\) and the stable complexity of the sequences (12) is \(ad = 1\).

The following proposition is proven in \[31\].

**Proposition 4.** Suppose that \(K \rightarrow L\) is an Artin–Schreier extension of two-dimensional algebraic function fields over an algebraically closed field \(k\) of characteristic \(p > 0\), \(\omega\) is a rational rank 1 nondiscrete valuation of \(L\) with restriction \(\nu = \omega|K\). Further suppose that \(A\) is an algebraic local ring of \(K\) and \(B\) is an algebraic local ring of \(L\) which is dominated by \(\omega\) such that \(B\) dominates \(A\). Then, there exists a commutative diagram of homomorphisms
\[
\begin{array}{ccc}
R & \rightarrow & S \\
\uparrow & & \uparrow \\
A & \rightarrow & B
\end{array}
\]
such that \(R\) is a regular algebraic local ring of \(K\) and \(S\) is a regular algebraic local ring of \(L\) such that \(S\) is dominated by \(\omega\), \(S\) dominates \(R\), \(R \rightarrow S\) is well prepared with admissible parameters \(u, v\) in \(R\) and \(x, y\) in \(S\) (with respect to the prime ideals \(P = uR\) in \(R\) and \(Q = xS\) in \(S\)). We further have that \(R \rightarrow S\) is quasi-finite, \(J(S/R) = (x^c)\) for some non-negative integer \(c\) and one of the following three cases holds:

0) \(u = x, v = y\) (\(R \rightarrow S\) is unramified).
1) \(u = x, v = y^p \gamma + x \Sigma\) where \(\gamma\) is a unit in \(S\) and \(\Sigma \in S\).
2) \(u = \gamma x^p, v = y\) where \(\gamma\) is a unit in \(S\) and \(\Omega \in S\).

**Proof.** By Proposition 3 and Remark 1, we may construct a diagram
\[
\begin{array}{ccc}
\bar{R} & \rightarrow & \bar{S} \\
\uparrow & & \uparrow \\
A & \rightarrow & B
\end{array}
\]
such that all the conclusions of the proposition hold, except possibly $\overline{R} \to \overline{S}$ is not quasi finite, and we have a form

$$u = \delta x^p, v = x^b y$$  \hspace{1cm} (13)$$

where $b$ is an integer with $0 < b < p$ and $\delta$ is a unit in $\overline{S}$. We will show that after one more sequence of blowups, we obtain a map of the form of (0), (1), or (2).

There exists a sequence of quadratic transforms $S \to S'$ of regular local rings along $\omega$ such that $S'$ has regular parameters $\tilde{x}_1, \tilde{y}_1$ defined by

$$x = \tilde{x}_1^{\tilde{y}} (\tilde{y} + \alpha)^{\tilde{y}'}, y = \tilde{x}_1^{\tilde{y}} (\tilde{y} + \alpha)^{\tilde{y}'}$$  \hspace{1cm} (14)$$

where $0 \neq \alpha \in k$ and $\tilde{a}' = \tilde{a} = 1$. Then, we have an expression

$$u = \delta \tilde{x}_1^{d_1}(\tilde{y}_1 + \alpha)^{e_1}, v = \tilde{x}_1^{f_1}(\tilde{y}_1 + \alpha)^{g_1},$$

where $d_1 g_1 - e_1 f_1 = p$. Let $l = \gcd(d_1, f_1)$. The number $l$ must either be $1$ or $p$. We have that

$$\delta = \delta_0 + \tilde{x}_1 \Omega$$

for some $0 \neq \delta_0 \in k$ and $\Omega \in S'$.

We have a sequence of quadratic transforms $R \to R'$ of regular local rings such that $S'$ dominates $R'$ and $R'$ has regular parameters $\tilde{u}$ and $\tilde{v}$ such that

$$\tilde{u} = \tilde{x}_1^{\tilde{y}_1}(\tilde{y}_1 + \alpha)^{\tilde{e}_1}, \tilde{v} = (\tilde{y}_1 + \alpha)^{\tilde{g}_1} - \alpha^{\tilde{g}_0}$$

We have that $lg_2 = p$ as

$$p = \begin{vmatrix} d_1 & e_1 \\ f_1 & g_1 \end{vmatrix} = \begin{vmatrix} l & e_2 \\ 0 & g_2 \end{vmatrix}.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \\

First, suppose that $l = p$. Then, $g_2 = 1$ and we have an expression of the form of (2). Now suppose that $l \neq p$. Then, $l = 1$ and we have an expression of the form of (1).

\section{SOME CALCULATIONS IN TWO-DIMENSIONAL ARTIN–SCHREIER EXTENSIONS}

Let $K \to L$ be an Artin–Schreier extension of two-dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$. Let $R \to S$ be an extension from a regular algebraic local ring of $K$ to a regular algebraic local ring of $L$ such that $S$ dominates $R$.

Let $u, v$ be regular parameters in $R$ and $x, y$ be regular parameters in $S$. We will say that $R \to S$ is of type 0 with respect to these parameters if

**Type 0:** $u = \gamma x, v = y \tau + x \Omega$

where $\gamma, \tau$ are units in $S$ and $\Omega \in S$, so that $R \to S$ is unramified. We will say that $R \to S$ is of type 1 with respect to these parameters if

**Type 1:** $u = \gamma x, v = y^p \tau + x \Omega,$

where $\gamma, \tau$ are units in $S$ and $\Omega \in S$. We will say that $R \to S$ is of type 2 with respect to these parameters if

**Type 2:** $u = \gamma x^p, v = y \tau + x \Omega,$

where $\gamma, \tau$ are units in $S$ and $\Omega \in S$. 
These definitions are such that if one of these types holds, and \( \overline{u}, \overline{v} \) are regular parameters in \( R, \overline{x}, \overline{y} \) are regular parameters in \( S \) such that \( \overline{u} \) is a unit in \( R \) times \( u \) and \( \overline{x} \) is a unit in \( S \) times \( x \), then \( R \to S \) is of the same type for the new parameters \( \overline{u}, \overline{v} \) and \( \overline{x}, \overline{y} \).

If we replace \( R \to S \) with a well-prepared map \( R_i \to S_j \) in the sequences (12), we will insist that the above parameters be admissible. We see that the three types are preserved by allowable changes of variables (changes of variables which preserve the condition that the regular parameters are admissible). In particular, we may obtain the respective forms of Proposition 4 by an allowable change of variables.

**Theorem 3.** Suppose that \( R \to S \) is of type 1 with respect to regular parameters \( x, y \) in \( S \) and \( u, v \) in \( R \) and that \( J(S/R) = (x^c) \). Let \( \overline{x} = u, \overline{y} = y - g(\overline{x}) \), where \( g(\overline{x}) \in k[\overline{x}] \) is a polynomial with zero constant term, so that \( \overline{x}, \overline{y} \) are regular parameters in \( S \). Computing the Jacobian determinate \( J(S/R) \), we see that

\[
u = \overline{x}, v = \overline{y}^p \gamma + \overline{x} \overline{y} \tau + f(\overline{x}) \quad (15)
\]

where \( \gamma, \tau \) are unit series in \( \hat{S} \) and \( f(\overline{x}) = \sum e_i \overline{x}^i \in k[[\overline{x}]] \). Make the change of variables \( \overline{v} = v - \sum u_i^j \), where the sum is over \( i \) such that \( i \leq \frac{pq}{m} \) so that \( u, \overline{v} \) are regular parameters in \( R \).

Suppose that \( m, q \) are positive integers with \( m > 1 \) and \( \gcd(m, q) = 1 \). Let \( \alpha \) be a nonzero element of \( k \). Let \( a', b' \in \mathbb{N} \) be such that \( mb' - qa' = 1 \). Then, define the sequence of quadratic transforms \( S \to S_1 \) so that the two-dimensional algebraic local ring \( S_1 \) has regular parameters \( x_1, y_1 \) defined by

\[
x_1 = x_1^{m}(y_1 + \alpha)^{a'}, y_1 = x_1^{q}(y_1 + \alpha)^{b'}.
\]

We have that \( R \to S \) is of type 1 with respect to the regular parameters \( \overline{x}, \overline{y} \) and \( u, v \). Let \( \sigma = \gcd(m, pq) \) which is 1 or \( p \). There exists a unique sequence of quadratic transforms \( R \to R_1 \) such that \( R_1 \) has regular parameters \( u_1, v_1 \) defined by

\[
u_1 = u_1^{\overline{m}}(v_1 + \beta)^{c'}, v_1 = u_1^{\overline{q}}(v_1 + \beta)^{d'}
\]

with \( 0 \neq \beta \in k \) giving a commutative diagram of homomorphisms

\[
\begin{array}{ccc}
R_1 & \to & S_1 \\
\uparrow & & \uparrow \\
R & \to & S
\end{array}
\]

such that \( R_1 \to S_1 \) is quasi finite. We have that \( J(S_1/R_1) = (x_1^{c_1}) \) for some positive integer \( c_1 \). Further:

0) If \( \frac{q}{m} \geq \frac{p-1}{p-1} \) then \( R_1 \to S_1 \) is of type 0.

1) If \( \frac{q}{m} < \frac{p-1}{p-1} \) and \( \sigma = 1 \) then \( R_1 \to S_1 \) is of type 1 and

\[
\left( \frac{c_1}{p-1} \right) = \left( \frac{c}{p-1} \right) m - q.
\]

2) If \( \frac{q}{m} < \frac{p-1}{p-1} \) and \( \sigma = p \) then \( R_1 \to S_1 \) is of type 2 and

\[
\left( \frac{c_1}{p-1} \right) = \left( \frac{c}{p-1} \right) m - q + 1.
\]

In cases 1) and 2), \( m = \sigma \overline{m}, pq = \sigma \overline{q} \) and \( \overline{mc'} - \overline{qd'} = 1 \).

**Proof.** Define a monomial valuation \( \mu \) dominating \( \hat{S} \) by prescribing that \( \mu(\overline{x}) = m, \mu(\overline{y}) = q \) and for \( 0 \neq \sum a_{ij} \overline{x}^i \overline{y}^j \in \hat{S}, \nu(\sum a_{ij} \overline{x}^i \overline{y}^j) = \min\{im + jq \mid a_{ij} \neq 0\}. \)
Expand

\[
\bar{v} = \sum_{i=1}^{d} \gamma_i \bar{x}^{\alpha_i} \bar{y}^{\beta_i} + \sum_{i>d} \gamma_i \bar{x}^{\alpha_i} \bar{y}^{\beta_i}
\]  

(16)

where all \( \gamma_i \in k \) are nonzero, \( \bar{x}^{\alpha_i} \bar{y}^{\beta_i} \) have minimal \( \mu \) value \( \rho \) for \( 1 \leq i \leq d \) and \( \bar{x}^{\alpha_i} \bar{y}^{\beta_i} \) have value larger than \( \rho \) for \( i > d \) and \( \beta_1 < \cdots < \beta_d \). By our choice of \( \bar{v} \) and Equation (15), we have that \( d \leq 2 \) and the initial form of \( \bar{v} \) with respect to \( \mu \) has one of the three special forms (to be enumerated in the list of three cases in the “finer analysis” later in the proof). Further, the substitution defining \( \bar{v} \) ensures that

\[
\beta_1 > 0.
\]  

(17)

Substitute

\[
\bar{x} = x^m_1(y_1 + \alpha)^{a'}, \bar{y} = x^q_1(y_1 + \alpha)^{b'}.
\]  

(18)

into \( u \) and expression (16) of \( \bar{v} \) to obtain

\[
u = x^m_1(y_1 + \alpha)^{a'}, \quad \bar{v} = x^{\alpha_1 m + \beta_1 q}_1 (y_1 + \alpha)^{a' \alpha_1 + b' \beta_1} \Lambda
\]  

where

\[
\Lambda = \left( \sum_{i=1}^{d} \gamma_i (y_1 + \alpha)^{\beta_i - \beta_1} m + x_1 \Omega \right)
\]  

(19)

with \( \frac{\beta_i - \beta_1}{m} \in \mathbb{N} \) for all \( i \). This expression for \( \bar{v} \) is shown in [37] and in the proof of [10, Theorem 8.4].

Assume that \( \Lambda \) is a unit. We will show later in this proof that with our choice of variables, if \( \Lambda \) is not a unit then we will reach the case where \( R_1 \to S_1 \) is of type 0 in the conclusions of the theorem.

Define

\[
\bar{\sigma} = \gcd(m, \alpha_1 m + \beta_1 q) = \gcd(m, \beta_1 q).
\]  

(20)

Let

\[
\tau = \text{Det} \left( \begin{array}{cc} m & a' \\ \alpha_1 m + \beta_1 q & \alpha_1 a' + \beta_1 b' \end{array} \right) = \beta_1 (mb' - a'q) = \beta_1 > 0.
\]  

(21)

Let

\[
\phi = \text{Det} \left( \begin{array}{cc} m & 0 \\ \alpha_1 m + \beta_1 q & 1 \end{array} \right) = m > 0.
\]  

(22)

Let \( R \to R^* \) be defined by \( u = u_1^h v_1^{g'}, \bar{v} = u_1^h \bar{v}_1^{g'} \) where \( g, g', h, h' \in \mathbb{N}, gh' - hg' = \pm 1 \) and

\[
u_1 = x^c_1(y_1 + \alpha)^{\bar{\alpha}} \Lambda^c, \quad \bar{v}_1 = x^d_1(y_1 + \alpha)^{\bar{\beta}} \Lambda^d
\]  

where

\[
\text{Det} \left( \begin{array}{cc} \bar{\sigma} & c \\ \bar{\sigma} & d \end{array} \right) = \bar{\sigma}(d - c) = \tau = \beta_1
\]  

(23)
and

$\text{Det} \begin{pmatrix} \sigma & e \\ \sigma & f \end{pmatrix} = \sigma(f - e) = \phi = m.$ \hspace{1cm} (24)

Now, perform a single quadratic transform $R^x \to R_1$ so that $R_1$ is dominated by $S_1$ and $R_1$ has regular parameters $u_1, v_1$ satisfying

$$u_1 = x_1^x(y_1 + \alpha)^\gamma \Lambda^e, v_1 = \frac{v_1}{u_1} - \Lambda(0,0)^{f-e} \alpha^{d-c} = (y_1 + \alpha)^{d-c} \Lambda^{f-e} - \Lambda(0,0)^{f-e} \alpha^{d-c}. \hspace{1cm} (25)$$

We have an expression for $R \to R_1$ of the form

$$u = u_1^{\overline{m}}(v_1 + \beta)^{\overline{q}}, \overline{v} = u_1^{\overline{q}}(v_1 + \beta)^{\overline{p}}$$

where $\overline{m} = \frac{m}{\sigma}, \overline{q} = \frac{\alpha m + \beta q}{\sigma}$ and $\beta = \psi \alpha \sigma$. We have that $\left( \frac{v_1}{u_1}, 0, y_1 \right)$ is a polynomial in $y_1$ since $d - c > 0$ and $f - e > 0$.

We now make a finer analysis. We have three cases:

1) $\overline{x} \overline{y}$ is the unique minimal value term in expansion (16).
2) $\overline{x} \overline{y}$ and $\overline{y}^\gamma$ are the two minimal value terms in expansion (16).
3) $\overline{y}^\gamma$ is the unique minimal value term in expansion (16).

Suppose that we are in Case 1), so that $\overline{x} \overline{y}$ is the unique minimal value term in expansion (16), so $\beta_1 = 1$. Further, $\Lambda = \gamma_1 + x_1 \Omega$ (with $\gamma_1 \in k \neq 0$). Now $\overline{\sigma}(d - c) = \tau = \beta_1 = 1$, and $\overline{\sigma}(f - e) = m$. Thus, $\overline{\sigma} = 1, d - c = 1$ and $f - e = m$. Thus,

$$u_1 = x_1(y_1 + \alpha)^\gamma \Lambda^e, \quad v_1 = (y_1 + \alpha)(\gamma_1 + x_1 \Omega)^m - \alpha \gamma^m = \gamma_1 y_1 + x_1 \Omega'.$$

Thus, $R_1 \to S_1$ is unramified and we are in Case 0) of the conclusions of the theorem.

Suppose that we are in Case 2), so that $\overline{x} \overline{y}$ and $\overline{y}^\gamma$ are the two minimal value terms in the expansion (16). The expansion of Equation (16) is then

$$\overline{v} = \gamma_1 x_1 \overline{y} + \gamma_2 \overline{y}^\gamma + \text{higher value terms}.$$ 

Since $\mu(\overline{y}^\gamma) = \mu(\overline{x} \overline{y})$ we have that $(p - 1)\mu(\overline{y}) = \overline{\sigma}(\overline{x})$. Thus, char $k \neq 2$ since $\mu(\overline{y}) \notin \mu(\overline{x}) \mathbb{Z}$. Let $\psi = \gcd(p - 1, \overline{c})$. Then in the substitution

$$\overline{x} = x_1^m(y_1 + \alpha)^a, \overline{y} = x_1^q(y_1 + \alpha)^b$$

with $mb' - a'd' = 1$ of Equation (18), we have $\psi m = p - 1$ and $\psi q = \overline{c}$. Substituting Equation (18) in $u$ and $\overline{v}$, we obtain

$$u = x_1^{\psi m}(y_1 + \alpha)^{a'}$$

and

$$\overline{v} = \gamma_1 x_1^{\psi m + q}(y_1 + \alpha)^{a' + b'} + \gamma_2 x_1^{\psi p}(y_1 + \alpha)^{b'} + \cdots = x_1^{\psi p}(y_1 + \alpha)^{a' + b'} \Lambda$$

where $\Lambda = \gamma_2(y_1 + \alpha)^{\psi} + \gamma_1 + x_1 \Omega$. 
Suppose that $\Lambda$ is not a unit. Let $d_1 = \text{ord}_{y_1} [(y_1 + \alpha)^{d-c+b} \Lambda(0, y_1)]$. We have that $0 < d_1 < \infty$ since $\Lambda$ is not a unit. By [17, Proposition 3.1] and since our extension is Galois, we have that the complexity $md_1$ of $R \rightarrow S_1$ divides $p = [L:K]$, which is a contradiction to our assumption that $m > 1$ and the fact that $m$ divides $p - 1$.

Suppose that $\Lambda$ is a unit. Following the analysis of the case when $\Lambda$ is a unit above, we have that $\tau = \beta_1 = 1$ and $\sigma = \gcd(m, \tilde{m} + q) = 1$. Thus, from Equation (23), we have $d - c = 1$ and from Equation (24), we have $f - e = m$. From Equation (25), we obtain

$$u_1 = x_1(y_1 + \alpha)^c \Lambda^e$$

and

$$v_1 = (y_1 + \alpha)\Lambda^m - \alpha(y_2\Lambda^\psi + \gamma_1)^m.$$ 

We compute

$$u_1(0, y_1) = (y_1 + \alpha)(y_2(y_1 + \alpha)^\psi + \gamma_1)^m - \alpha(y_2\Lambda^\psi + \gamma_1)^m.$$ 

We have

$$\frac{\partial}{\partial y_1} u_1(0, y_1) = \frac{\partial}{\partial y_1} (y_2(y_1 + \alpha)^\psi + \gamma_1)^m = \frac{\partial}{\partial y_2} (y_2(y_1 + \alpha)^\psi + \gamma_1)^m - m(y_2(y_1 + \alpha)^\psi + \gamma_1 + \alpha \gamma_2 \psi^2 + \gamma_1)^m$$

$$= (y_2(y_1 + \alpha)^\psi + \gamma_1)^m - m(y_2(y_1 + \alpha)^\psi + \gamma_1 + \alpha \gamma_2 \psi^2 + \gamma_1)^m - 1$$

$$= (y_2(y_1 + \alpha)^\psi + \gamma_1)^m - m(y_2(y_1 + \alpha)^\psi + \gamma_1 + \alpha \gamma_2 \psi^2 + \gamma_1)^m - 1$$

$$= (y_2(y_1 + \alpha)^\psi + \gamma_1)^m - m(y_2(y_1 + \alpha)^\psi + \gamma_1 + \alpha \gamma_2 \psi^2 + \gamma_1)^m - 1$$

is a unit so $\text{ord}_{y_1} u_1(0, y_1) = 1$. Thus, the complexity of $R_1 \rightarrow S_1$ is $m < p$, so the complexity must be one, so that $R_1 \rightarrow S_1$ is unramified and we are in Case 0) of the conclusions of the theorem.

Now, suppose that we are in Case 3) so that $\tilde{y}^\beta$ is the unique minimal value term of the expansion (16) of $\tilde{v}$. Then, $\beta_1 = p$ in Equation (16) and $\Lambda = \gamma_1 + x_1 \Omega$ is a unit in Equation (19). In the analysis of the case when $\Lambda$ is a unit following Equation (19), we have that $d - c = 1$ if $\overline{\sigma} = p$ and $d - c = p$ if $\overline{\sigma} = 1$, so that

$$\text{ord}_{y_1} u_1(0, y_1) = \text{ord}_{y_1} (y_1 + \alpha)^{d-c} \gamma_1^{f-e} - \alpha^{d-c} \gamma_1^{f-e} = \begin{cases} 1 & \text{if } \overline{\sigma} = p \\ p & \text{if } \overline{\sigma} = 1. \end{cases}$$

Thus, from Equation (25), we see that $R_1 \rightarrow S_1$ is of type 1 if $\overline{\sigma} = p$ and is of type 2 if $\overline{\sigma} = 1$.

We now establish the forminals for the Jacobian ideal $J(S_1/R_1)$. We have that

$$J(R_1/R)J(S_1/R_1) = J(S/R)J(S_1/S),$$

where $J(S_1/R_1) = x_1^c S_1$, $J(S/R) = \tilde{x}^c S$ and $J(S_1/S) = x_1^{m+q-1} S_1$.

We have that $\sigma = \gcd(m, pq)$ which is 1 or $p$. Thus, $m = \sigma m$ and $pq = \sigma q$. Further, we have shown that $\sigma = 1$ implies $R_1 \rightarrow S_1$ is of type 1 and $\sigma = p$ implies $R_1 \rightarrow S_1$ is of type 2.
Now, $\sigma = 1$ implies $p$ divides $\overline{q}$ and $\sigma = p$ implies $p$ does not divide $\overline{q}$, since $p$ divides $m$ implies $q \not| \overline{q}$. We have that $J(R_1/R) = u_1^{m+q-1}R_1$ and thus

$$J(R_1/R)S_1 = \begin{cases} x_1^{m+pq-1}S_1 & \text{if } \sigma = 1 \\ x_1^{m+pq-p}S_1 & \text{if } \sigma = p. \end{cases}$$

In the case $\sigma = 1$ we have $(x_1^{m+pq-1})(x_1^{c_1})S_1 = (x_1^{c_1})S_1$ so $(x_1^{c_1})S_1 = (x_1^{m+p-q})S_1$ and we obtain the formula of Case 1) of the conclusions of the theorem. In the case $\sigma = p$, we have $(x_1^{m+p-q})(x_1^{c_1})S_1 = (x_1^{c_1})S_1 = (x_1^{m+p-q})S_1$ and we obtain the formula of Case 2) of the conclusions of the theorem.

□

**Remark 4.** Suppose that $\omega$ is a rational rank 1 nondiscrete valuation of $L$ dominating $S$ and $R \to S$ is of type 1. Let $\nu$ be the restriction of $\omega$ to $K$. Let $\overline{x} = u$ and $\overline{y}$ be the difference of $y$ and a nonzero polynomial in $\overline{x}$ so that $\omega(\overline{y}) \not\in \omega(\overline{x}) \mathbb{Z}$. Let $\overline{v}$ be the change of variables in Theorem 3.

Define $m$ and $q$ to be the unique relatively prime positive integers such that $mq(\overline{y}) = q(\overline{x})$. We have that $m > 0$. There exist $0 \neq \alpha \in k$ and $a', b' \in \mathbb{N}$ such that $mb' - qa' = 1$ and if $S \to S_1$ is the sequence of quadratic transforms defined by

$$\overline{x} = x_1^m(y_1 + \alpha)^{a'}, \overline{y} = x_1^q(y_1 + \alpha)^{b'}$$

then $\omega$ dominates $S_1$.

Let $\nu$ be the restriction of $\omega$ to $K$. The formulas of Cases 0), 1) and 2) of Theorem 3 can then be stated in terms of the valuation $\omega$. They are:

0) If $\frac{q}{m} \geq \frac{c}{p-1}$ then $R_1 \to S_1$ is unramified.

1) If $\frac{q}{m} < \frac{c}{p-1}$ and $\sigma = 1$ then $R_1 \to S_1$ is of type 1 and

$$\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{c}{p-1} \right) \omega(x) - \omega(y).$$

2) If $\frac{q}{m} < \frac{c}{p-1}$ and $\sigma = p$ then $R_1 \to S_1$ is of type 2 and

$$\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{c}{p-1} \right) \omega(x) - \omega(y) + \omega(x_1).$$

In the conclusions of the theorem, suppose that $R_1 \to S_1$ is of type 1. Then, we necessarily have that $\nu(\overline{v}) \not\in \nu(\overline{u}) \mathbb{Z}$ since $\sigma = 1$ and thus $\overline{m} = m > 1$.

**Theorem 4.** Suppose that $R \to S$ is of type 2 with respect to regular parameters $x, y$ in $S$ and $u, v$ in $R$ and that $J(S/R) = (x^c)$. Let $g(u) \in k[u]$ be a polynomial with no constant term. Make the change of variables, letting $\overline{v} = v - g(u)$ and $\overline{y} = \overline{v}$, so that $x, \overline{y}$ are regular parameters in $S$ and $u, \overline{v}$ are regular parameters in $R$.

Suppose that $m, q$ are positive integers with gcd$(m, q) = 1$. Let $\alpha$ be a nonzero element of $k$. Consider the sequence of quadratic transforms $S \to S_1$ so that $S_1$ has regular parameters $x_1, y_1$ defined by

$$x = x_1^m(y_1 + \alpha)^{a'}, \overline{y} = x_1^q(y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

Let $\sigma = \text{gcd}(pm, q)$ which is 1 or $p$. There exists a unique sequence of quadratic transforms $R \to R_1$ such that $R_1$ has regular parameters $u_1, v_1$ defined by

$$u = u_1^{\overline{m}}(v_1 + \beta)^c, \overline{v} = u_1^{\overline{q}}(v_1 + \beta)^{d'}$$
where \( pm = \sigma m, q = \sigma q, \overline{md'} - c'\overline{q} = 1 \) and \( 0 \neq \beta \in k \), giving a commutative diagram of homomorphisms

\[
\begin{array}{c}
R_1 \rightarrow S_1 \\
\uparrow \quad \uparrow \\
R \rightarrow S
\end{array}
\]

such that \( R_1 \rightarrow S_1 \) is quasi finite. We have that \( J(S_1/R_1) = (x_1^{c_1}) \) for some positive integer \( c_1 \). Further:

1) If \( \sigma = 1 \) then \( R_1 \rightarrow S_1 \) is of type 1 and

\[
\left( \frac{c_1}{p-1} \right) = \left( \frac{c'}{p-1} \right) m - m.
\]

2) If \( \sigma = p \) then \( R_1 \rightarrow S_1 \) is of type 2 and

\[
\left( \frac{c_1}{p-1} \right) = \left( \frac{c'}{p-1} \right) m - m + 1.
\]

**Proof.** Substitute

\[
x = x_1^m(y_1 + \alpha)^{a'}, \overline{y} = x_1^q(y_1 + \alpha)^{b'}.
\]

into \( u \) and \( v \) to obtain

\[
\begin{align*}
u &= x_1^{mp}(y_1 + \alpha)^{a'p}(\lambda + x_1\Omega) \\
v &= x_1^q(y_1 + \alpha)^{b'}
\end{align*}
\]

where \( 0 \neq \lambda \in k \). Let

\[
\sigma = \gcd(mp, q).
\]

Let

\[
\tau = \Det\begin{pmatrix} mp & a'p \\
q & b'
\end{pmatrix} = p(mb' - a'q) = p.
\]

Let \( R \rightarrow R^* \) be the sequence of quadratic transforms defined by \( u = u_1^g\overline{v}_1^h, \overline{v} = u_1^h\overline{v}_1^g \) where \( g, g', h, h' \in \mathbb{N}, gh' - hg' = \pm 1 \) and

\[
u_1 = x_1^c(y_1 + \alpha)^c(\lambda + x_1\Omega)^e, \overline{v}_1 = x_1^d(y_1 + \alpha)^d(\lambda + x_1\Omega)^f
\]

where

\[
\Det\begin{pmatrix} c \\ d
\end{pmatrix} = \sigma(d - c) = \tau = p.
\]

Now, perform a single quadratic transform \( R^* \rightarrow R_1 \) so that \( R_1 \) is dominated by \( S_1 \) and \( R_1 \) has regular parameters \( u_1, v_1 \) satisfying

\[
u_1 = x_1^c(y_1 + \alpha)^c(\lambda + x_1\Omega)^e, v_1 = \frac{\overline{v}_1}{u_1} - \alpha^{d-c} \lambda^{f-e} = (y_1 + \alpha)^{d-c}(\lambda + x_1\Omega)^{f-e} - \alpha^{d-c} \lambda^{f-e}.
\]

We have an expression for \( R \rightarrow R_1 \) of the form

\[
u = u_1^{m}(v_1 + \beta)^{a'}, \overline{v} = u_1^{q}(v_1 + \beta)^{b'}
\]
where $\overline{m} = \frac{mp}{\sigma}, \overline{q} = \frac{q}{\sigma}$ and $\beta = \frac{\beta}{\sigma} \gamma(0, 0) - \frac{q}{\sigma}$. We have that $\nu_1(0, y_1) = (y_1 + \alpha)^{d-c} \lambda^{f-e} - \alpha^{d-c} \lambda^{f-e}$ with $d - c > 0$. Hence,

$$0 < d_1 = \text{ord}_{y_1} \nu_1(0, y_1) < \infty.$$

If $\sigma = p$ then $d - c = 1$ so that $d_1 = 1$ and the complexity of $R_1 \to S_1 = p$. We then have that $R_1 \to S_1$ is of type 2, so that it is in Case 2 of the conclusions of the theorem. If $\sigma = 1$ then $d - c = p$ and $d_1 = p$, so that the complexity of $R_1 \to S_1$ is $p$ and it is in Case 1 of the conclusions of the theorem.

If $\sigma = 1$ then $p$ divides $\overline{m}$ and $\sigma = p$ implies $p$ does not divide $\overline{m}$, since $p$ then divides $q$ which implies $p$ does not divide $m = \overline{m}$.

We have that $J(R_1/R) = (u_{1\overline{m}+q-1})$ and thus

$$J(R_1/R)S_1 = \begin{cases} x_1^{pm+q-1}S_1 & \text{if } \sigma = 1 \\ x_1^{pm+q-p}S_1 & \text{if } \sigma = p. \end{cases}$$

In the case $\sigma = 1$, we have $(x_1^{pm+q-1})(x_1^{c_1})S_1 = (x^{\overline{c}})(x_1^{m+q-1})S_1$ so $(x_1^{c_1})S_1 = (x_1^{mc-p(m-1)})S_1$ and we obtain the formula of Case 1) of the statement of the theorem.

In the case $\sigma = p$, we have

$$(x_1^{pm+q-p})(x_1^{c_1})S_1 = (x^{\overline{c}})(x_1^{m+q-1})S_1$$

so

$$(x_1^{c_1})S_1 = (x_1^{mc-(p-1)m+(p-1)})S_1$$

and we obtain the formula of Case 2) of the statement of the theorem.

Remark 5. Suppose that $\omega$ is a nondiscrete rational rank 1 valuation of $L$ dominating $S$ and $R \to S$ is of type 2. Let $\nu$ be the restriction of $\omega$ to $K$. Make the change of variables, letting $\overline{\nu}$ be the difference of $\nu$ and a polynomial in $u$ so that $\omega(\overline{\nu}) \notin \omega(u)\mathbb{Z}$ and letting $\overline{\nu} = \overline{\nu}$.

Define $m$ and $q$ to be the unique relatively prime positive integers such that $m\omega(\overline{\nu}) = q\omega(x)$. There exist $0 \neq \alpha \in k$ and $a', b' \in \mathbb{N}$ such that $mb' - qa' = 1$ and if $S \to S_1$ is the sequence of quadratic transforms defined by

$$x = x_1^m(y_1 + \alpha)^{a'}, \overline{\nu} = x_1^q(y_1 + \alpha)^{b'}$$

then $\omega$ dominates $S_1$.

The formulas of Cases 1) and 2) of Theorem 4 can then be stated in terms of the valuation $\omega$. They are:

1) If $\sigma = 1$, then $R_1 \to S_1$ is of type 1 and

$$\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{\overline{c}}{p-1} \right) \omega(x) - \omega(x).$$

2) If $\sigma = p$, then $R_1 \to S_1$ is of type 2 and

$$\left( \frac{c_1}{p-1} \right) \omega(x_1) = \left( \frac{\overline{c}}{p-1} \right) \omega(x) - \omega(x) + \omega(x_1).$$

If $R_1 \to S_1$ is of type 2, then we have that $\nu(\overline{\nu}) \notin \nu(x)\mathbb{Z}$, since $\text{gcd}(pm, q) = p$.

We will show that $\nu(\overline{\nu}) \notin \nu(x)\mathbb{Z}$ if $R_1 \to S_1$ is of type 2. We have that $\text{gcd}(pm, q) = p$. If $\nu(\overline{\nu}) \in \nu(x)\mathbb{Z}$, then $\nu(\overline{\nu}) = q\nu(x)$ and since $p \mid q$, we have that $\nu(\overline{\nu}) = q\nu(u\overline{\nu})$ with $\overline{q} = \frac{q}{p}$, a contradiction to the assumption that $\nu(\overline{\nu}) \notin \nu(u)\mathbb{Z}$.
5  |  SWITCHING OF TYPES OF EXTENSIONS UNDER BLOWING UP

Suppose that $K$ and $L$ are two-dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$ and $K \rightarrow L$ is an Artin–Schreier extension. Suppose that $R$ is a regular algebraic local ring of $K$ and $S$ is a regular algebraic local ring of $L$ such that $S$ dominates $R$ and $R \rightarrow S$ is of type 1 or 2 as defined at the beginning of Section 4. Further assume that the Jacobian ideal $J(S/R)$ satisfies $\sqrt{J(S/R)} = xS$. Let $P = uR$ and $Q = xS$. Then, $R \rightarrow S$ is well prepared and the regular parameters $u, v$ and $x, y$ are admissible parameters. Such an extension $R \rightarrow S$ exists by Remark 2 and Proposition 4.

Inductively applying Theorems 3 and 4, and making choices for the construction of $S_i \rightarrow S_{i+1}$ consistent with the assumptions of Theorems 3 and 4, we construct a diagram, where the horizontal sequences are birational extensions of regular local rings (sequences of quadratic transforms)

$$
\begin{align*}
S &= S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \\
R &= R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots
\end{align*}
$$

(30)

Every $R_i \rightarrow S_i$ is well prepared of type 0, 1, or 2. Each $R_i$ has admissible regular parameters $(u_i, v_i)$ and each $S_i$ has admissible regular parameters $(x_i, y_i)$ and $(\overline{x}_i, \overline{y}_i)$. The map $S_i \rightarrow S_{i+1}$ is defined by

$$
\overline{x}_i = x_{i+1}^{m_{i+1}}(y_{i+1} + \alpha_{i+1})^{a'_{i+1}}, \overline{y}_i = x_{i+1}^{q_{i+1}}(y_{i+1} + \alpha_{i+1})^{b'_{i+1}}
$$

(31)

and the map $R_i \rightarrow R_{i+1}$ is defined by

$$
u_i = u_{i+1}^{m_{i+1}}(v_{i+1} + \beta_{i+1})^{c'_{i+1}}, \overline{v}_i = u_{i+1}^{q_{i+1}}(v_{i+1} + \beta_{i+1})^{d'_{i+1}}
$$

(32)

where $0 \neq \alpha_{i+1}, 0 \neq \beta_{i+1} \in k$. If $R_i \rightarrow S_i$ is of type 1 or of type 2 then $\overline{x}_i, \overline{y}_i$ and $\overline{v}_i$ are defined by our changes of variables in Theorem 3 or 4. If $R_i \rightarrow S_i$ is of type 0, then we take $\overline{x}_i = u_i$ and $\overline{y}_i = \overline{v}_i = v_i$. If $R_i \rightarrow S_i$ is of type 2, we will impose the extra condition that

$$
m_{i+1} = \frac{pm_{i+1}}{\gcd(pm_{i+1}, q_{i+1})} > 1.
$$

(33)

We will say that the sequence (30) switches infinitely often if there are infinitely many $i$ such that $R_i \rightarrow S_i$ is of type 1 and there are infinitely many $i$ such that $R_i \rightarrow S_i$ is of type 2.

Since $\text{trdeg}_k L = 2$, we have that $\cup_{i=1}^{\infty} R_i$ and $\cup_{i=1}^{\infty} S_i$ are valuation rings (by [2, Lemma 12]). Further, given $f \in R_i$ (or $f \in S_i$), there exists $j \geq i$ such that there is an expression $f = u_i^t \gamma_i$, where $t_i \in \mathbb{N}$ and $\gamma_i$ is a unit in $R_j$ (or $f = x_i^r \gamma_i$ where $r_i \in \mathbb{N}$ and $\gamma_i$ is a unit in $S_j$), as shown, for instance, in [2].

Let $\nu$ and $\omega$ be valuations which have these respective valuation rings and such that $\omega|K = \nu$. These valuations are uniquely determined up to equivalence of valuations. We have that $\omega$ and $\nu$ are nondiscrete rational rank 1 valuations, with value groups

$$
\nu K = \cup_{i=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_i} \mathbb{Z} \nu(u) \quad \text{and} \quad \omega L = \cup_{i=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_i} \mathbb{Z} \omega(x).
$$

Equation (33) is just the statement that $\nu(\overline{v}_i) \notin \mathbb{Z} \nu(u_i)$. The condition that all $m_{i+1} > 1$ in Theorem 3 is just the statement that $\omega(\overline{y}_i) \notin \mathbb{Z} \omega(\overline{x}_i)$.

Suppose that $\overline{w}$ is a valuation ring of $L$ which dominates $S$ which is nondiscrete of rational rank 1 and $\overline{\nu} = \overline{w}|K$. Then, we can inductively construct a sequence (30) so that $\overline{w}$ dominates $S_i$ for all $i$, and so $\overline{w} S_i = \cup_{i=1}^{\infty} R_i$ and $\overline{w} S_i = \cup_{i=1}^{\infty} S_i$, so that the valuations $\omega$ and $\nu$ determined by the sequence are $\overline{w}$ and $\overline{\nu}$, respectively (up to equivalence of valuations).

The complexity of the maps $R_i \rightarrow S_i$ in the diagram (30) must either be $p$ for all $i$, or will be $p$ until some $i_0$ and then the complexity will be $1$ for all $j \geq i_0$, so that $R_j \rightarrow S_j$ is of type 1 or 2 for $j < i_0$ and $R_j \rightarrow S_j$ is unramified (has type 0) for all $j \geq i_0$. 
We will say that a sequence \((30)\) has stable complexity \(p\) if the complexity of \(R_i \to S_i\) is \(p\) for all \(i \geq 0\). With this assumption, each map \(R_i \to S_i\) in \((30)\) is either of type 1 or of type 2. We draw the following conclusions from Theorems 3 and 4.

Assume that the stable complexity of a sequence \((30)\) is \(p\). If \(R_i \to S_i\) is of type 1, then \(S_i \to S_{i+1}\) is the standard sequence of quadratic transforms along \(\omega\). Further, \(R_i \to R_{i+1}\) is the standard sequence of quadratic transforms along \(\nu\) unless \(m_{r+1} = p\). In this case, \(m_{r+1} = 1\), so that the standard sequence of quadratic transforms of \(R_i\) along \(\nu\) dominates \(R_{r+1}\), and \(R_{r+1} \to S_{r+1}\) is of type 2.

If \(R_i \to S_i\) is of type 2, then \(R_i \to R_{i+1}\) is the standard sequence of quadratic transforms along \(\nu\). Further, \(S_i \to S_{i+1}\) is the standard sequence of quadratic transforms along \(\omega\) unless \(m_{r+1} = p\). In this case, \(m_{r+1} = 1\), so that the standard sequence of quadratic transforms of \(S_i\) along \(\omega\) dominates \(S_{r+1}\), and \(R_{r+1} \to S_{r+1}\) is of type 1.

**Proposition 5.** Suppose that a sequence \((30)\) has stable complexity \(p\). Then, the sequence \((30)\) switches infinitely often if and only if \(vK\) is \(p\)-divisible.

**Proof.** We have that \(\omega L\) is \(p\)-divisible if and only if \(vK\) is \(p\)-divisible (for instance, by (3) of [19, Lemma 7.32]).

Suppose that \(vK\) is not \(p\)-divisible. Then, there exists \(r_0\) such that for \(r > r_0\) we have that \(p \nmid m_r\) and \(p \nmid m_r\). Now, if \(r > r_0\) and \(R_r \to S_r\) is of type 1 then \(R_{r+1} \to S_{r+1}\) must be of type 1 since \(\sigma_{r+1} = p\) in Theorem 3 implies \(p\) divides \(m_{r+1}\). Further, if \(R_r \to S_r\) is of type 2 then \(R_{r+1} \to S_{r+1}\) must be of type 2 since \(\sigma_{r+1} = 1\) in Theorem 4 implies \(p\) divides \(m_{r+1}\). Thus for \(r > r_0\) there can be no switching.

Suppose that \(vK\) is \(p\)-divisible. Suppose that Equation \((30)\) does not switch infinitely often. Then, there exists \(r_0\) such that for \(r \geq r_0\), \(R_r \to S_r\) is of the same type as \(R_{r_0} \to S_{r_0}\). Suppose that \(R_{r_0} \to S_{r_0}\) is of type 1. Since \(\omega L\) is \(p\)-divisible, there exists \(r \geq r_0\) such that \(p\) divides \(m_{r+1}\). But then \(R_{r+1} \to S_{r+1}\) must be of type 2 since \(\sigma_{r+1} = p\) in Theorem 3. Suppose that \(R_{r_0} \to S_{r_0}\) is of type 2. Since \(\nu L\) is \(p\)-divisible, there exists \(r \geq r_0\) such that \(p\) divides \(m_{r+1}\). Now, \(p\) divides \(m_{r+1}\) implies \(p \nmid \gamma_{r+1}\) which implies \(\sigma_{r+1} = \gcd(pm_{r+1}, q_{r+1}) = 1\) in Theorem 4. But then \(R_{r+1} \to S_{r+1}\) must be of type 1. We have arrived at a contradiction. This completes the proof that Equation \((30)\) switches infinitely often.

**Remark 6.** Suppose that a sequence \((30)\) has stable complexity \(p\). If \(vK\) is not \(p\)-divisible then \(m_r = \overline{m_r}\) for \(r \gg 0\) and we have that \(m_r > 1\) and \(\overline{m_r} > 1\) for \(r \gg 0\) in \((30)\).

As the following Proposition shows, the nicest form that a sequence \((30)\) can take is when \(R_r \to S_r\) is of type 2 for all \(r \gg 0\). This is the strongly monomial form (defined in the introduction).

**Proposition 6.** The following are equivalent for a sequence \((30)\) with stable complexity \(p\), and valuations \(\nu\) and \(\omega\) which it determines.

1) There exists an \(r_0\) such that \(R_r \to S_r\) is of type 2 in Equation \((30)\) for \(r \geq r_0\).
2) \([\omega L : vK] = p\).
3) The valued extension \(L/K\) is defectless.

**Proof.** Since the stable complexity of the sequence \((30)\) is \(p\), we have that

\[ p = [\omega L : vK]\delta(\omega/\nu) \]

by Remark 3 and Proposition 2. Thus, statement 2) is equivalent to statement 3). We now prove that statement 1) is equivalent to statement 2). Suppose that there exists \(r_0\) such that \(R_r \to S_r\) is of type 2 in Equation \((30)\) for \(r \geq r_0\). Then, \(u_i = \gamma_i\overline{m_i}\) for all \(i \geq r_0\) and \(m_i = \overline{m_i}\) for \(i \geq r_0\) by Theorem 4. Thus,

\[ vK = \bigcup_{i=r_0+1}^{\infty} \frac{1}{\overline{m_{i+1}} \cdots \overline{m_i}} Z(v(u_{r_0})) = \bigcup_{i=r_0+1}^{\infty} \frac{1}{m_{i+1} \cdots m_i} Z(p\omega(x_{r_0})) = p\omega L. \]

If \(vK = \omega L\) then \(p\omega L = \omega L\), which implies that \(\omega L\) is \(p\)-divisible, a contradiction to Proposition 5. Thus, \([\omega L : vK] = p\).
Suppose that there exists \( r_0 \) such that \( R_r \to S_r \) is of type 1 in Equation (30) for \( r \geq r_0 \). Then, \( u_i = x_i \) for \( i \geq r_0 \). Thus,

\[
\nu K = \bigcup_{i=r_0}^{\infty} \mathbb{Z} \nu(u_i) = \bigcup_{i=r_0}^{\infty} \mathbb{Z} \omega(x_i) = \omega L.
\]

Finally, suppose that Equation (30) switches infinitely often. Then, \( \nu K \) and \( \nu L \) are \( p \)-divisible by Proposition 5. Since \([L : K] = p\), \( \omega L = p \omega L \subset \nu K \subset \omega L \) which implies that \( \nu K = \omega L \).

We see that if the sequence (30) switches infinitely often then the extension must be a defect extension. Any configuration of switching is possible. A sequence with prescribed switching can be created by iterating the constructions of Theorems 3 and 4.

If a sequence stabilizes with \( R_r \to S_r \) of type 2 for all \( r \geq r_0 \), then from iteration of formula (2) of Theorem 4, for all \( s > 0 \) we have that

\[
\left( \frac{c_{r_0+s}}{p-1} \right) \frac{1}{m_1 \cdots m_{r_0+s}} = \left( \frac{c_{r_0}}{p-1} \right) \frac{1}{m_1 \cdots m_{r_0}} - \frac{1}{m_1 \cdots m_{r_0}} + \frac{1}{m_1 \cdots + m_{r_0+s}}.
\]

By Remarks 4 and 5, \( \omega(J(S_i/R_i)) = c_i \omega(x_i) \) is monotonically decreasing with \( i \). We calculate that for \( s \geq 1 \),

\[
\omega(J(S_{r_0+s}/R_{r_0+s})) = \left[ \frac{c_{r_0}}{m_1 \cdots m_{r_0}} - \frac{(p-1)}{m_1 \cdots m_{r_0}} + \frac{p-1}{m_1 \cdots + m_{r_0+s}} \right] \omega(x_0).
\]

Thus, since infinitely many \( m_i \) are greater than 1,

\[
\inf \{ \omega(J(S_i/R_i)) \} = \left[ \frac{c_{r_0}}{m_1 \cdots m_{r_0}} - \frac{(p-1)}{m_1 \cdots m_{r_0}} \right] \omega(x_0) \in \omega(L).
\]

Thus by Proposition A.2,

\[
\text{dist}(\omega/\nu) = -\frac{1}{p-1} \inf \{ \omega(J(S_i/R_i)) \} \leq \frac{1}{p-1} \omega(L).
\]

In contrast, we can get any nonpositive real number as the distance \( \text{dist}(\omega/\nu) \) on \( K \) if we allow a sequence which does not stabilize to type 2, as is shown in the following theorem.

**Theorem 5.** Suppose that \( K \) is an algebraic function field of transcendence degree 2 over an algebraically closed field \( k \) of characteristic \( p > 0 \), and that \( A \) is an algebraic regular local ring of \( K \) with regular parameters \( z \) and \( w \). Let \( \alpha \in \mathbb{R} \geq 0 \) and let \( \Phi : \mathbb{N} \to \{1, 2\} \) be a function such that \( \Phi(n) \) is not identically equal to 2 for \( n \gg 0 \). Then, there exists an Artin–Schreier extension \( K \to L \) and a sequence (30) such that \( R_0 = A \), \( R_r \to S_r \) is of type 1 if \( \Phi(r) = 1 \) and of type 2 if \( \Phi(r) = 2 \) and the induced defect extension of valuations satisfies

\[
\text{dist}(\omega/\nu) = -\alpha,
\]

where the valuation \( \nu \) of \( L \) is normalized so that \( \nu(z) = 1 \). We will further have that \( m_i > 1 \) and \( \overline{m}_1 > 1 \) for all \( i \) in the sequence (30).

**Proof.** First, assume that \( \Phi(0) = 1 \). Let \( R_0 = A, u_0 = z \) and \( v_0 = w \). Let \( e \) be a positive integer such that \( e > \alpha \). Let \( c_0 = (p - 1)e \). Let \( \Theta \) be a root of the Artin–Schreier polynomial \( X^p - X - v_0 u_0^{p-1} \). Let \( L = K(\Theta) \). Set \( x_0 = u_0, y_0 = u_0^e \Theta \). Let \( S_0 = R[y_0/(x_0,y_0)] \), which is an algebraic regular local ring of \( L \) which dominates \( R_0 \). The regular parameters \( x_0, y_0 \) in \( S_0 \) satisfy \( u_0 = x_0, v_0 = y_0^p - x_0^{p-1} y_0 \), so that the extension \( R \to S \) is of type 1. We have that \( J(S_0/R_0) = (x_0^{p-1}) \), with \( \frac{c_0}{p-1} > \alpha \).

Suppose that we have a sequence

\[
\begin{align*}
S_r & \to S_{r+1} \\
\uparrow & \quad \uparrow \\
R_r & \to R_{r+1}
\end{align*}
\]
where \( R_r \rightarrow S_r \) and \( R_{r+1} \rightarrow S_{r+1} \) are both of type 1. Then from Theorem 3, we have that

\[
\left( \frac{c_{r+1}}{p-1} \right) \frac{1}{m_1 \cdots m_{r+1}} = \left( \frac{c_r}{p-1} \right) \frac{1}{m_1 \cdots m_r} - \frac{q_{r+1}}{m_{r+1}} \left( \frac{1}{m_1 \cdots m_r} \right). \tag{34}
\]

Suppose that we have a sequence

\[
\begin{align*}
S_r & \rightarrow S_{r+1} \rightarrow \cdots \rightarrow S_{r+s} \rightarrow S_{r+s+1} \\
\uparrow & \quad \uparrow \quad \uparrow \\
R_r & \rightarrow R_{r+1} \rightarrow \cdots \rightarrow R_{r+s} \rightarrow R_{r+s+1}
\end{align*}
\]

where \( s \geq 1 \), \( R_i \rightarrow S_i \) is of type 1 if \( i = r \) or \( i = r + s + 1 \) and \( R_i \rightarrow S_i \) is of type 2 if \( r + 1 \leq i \leq r + s \). Then from Theorems 3 and 4, we have that

\[
\left( \frac{c_{r+s+1}}{p-1} \right) \frac{1}{m_1 \cdots m_{r+s+1}} = \left( \frac{c_r}{p-1} \right) \frac{1}{m_1 \cdots m_r} - \frac{q_{r+1}}{m_{r+1}} \left( \frac{1}{m_1 \cdots m_r} \right). \tag{35}
\]

Let \( p' \) be a prime distinct from \( p \).

We now inductively construct the sequence (30), so that \( m_i > 1 \) and \( \overline{m_i} > 1 \) for all \( i \). Suppose that the sequence has been constructed up to \( R_r \rightarrow S_r \), \( \Phi(r) = 1 \) and we have that for all \( t \leq r \) such that \( \Phi(t) = 1 \),

\[
\alpha < \left( \frac{c_t}{p-1} \right) \frac{1}{m_1 \cdots m_t} \quad \text{and} \quad \alpha + \frac{1}{2t} \quad \text{if } t > 0. \tag{36}
\]

First, suppose that \( \Phi(r+1) = 1 \). There exists \( \lambda(r+1) \in \mathbb{Z}^+ \) such that there exists \( q_{r+1} \in \mathbb{Z}^+ \) such that \( \gcd(q_{r+1}, p') = 1 \) and

\[
\frac{c_r}{p-1} - \alpha m_1 \cdots m_r > \frac{q_{r+1}}{(p')^{\lambda(r+1)}} > \frac{c_r}{p-1} - \left( \alpha + \frac{1}{2(r+1)} \right) m_1 \cdots m_r. \tag{37}
\]

Set \( m_{r+1} = (p')^{\lambda(r+1)} \). Then

\[
\alpha + \frac{1}{2(r+1)} > \left( \frac{c_r}{p-1} \right) \frac{1}{m_1 \cdots m_r} - \frac{q_{r+1}}{m_{r+1}} \left( \frac{1}{m_1 \cdots m_r} \right) > \alpha. \tag{38}
\]

Now, we have that \( \frac{q_{r+1}}{m_{r+1}} < \frac{c_r}{p-1} \) with \( \gcd(m_{r+1}, pq_{r+1}) = 1 \) so we may define from Theorem 3 and the above values of \( q_{r+1} \) and \( m_{r+1} \) a commutative diagram

\[
\begin{align*}
S_r & \rightarrow S_{r+1} \\
\uparrow & \quad \uparrow \\
R_r & \rightarrow R_{r+1}
\end{align*}
\]

such that \( R_{r+1} \rightarrow S_{r+1} \) is of type 1. Further, Equation (36) holds for \( t = r + 1 \) by Equations (34) and (38).

Now, suppose that \( \Phi(r+1) = 2 \). Let \( s \geq 1 \) be the smallest integer such that \( \Phi(r + s + 1) = 1 \). There exists an integer \( \lambda(r + 1) > 1 \) such that there exists \( q_{r+1} \in \mathbb{Z}^+ \) such that \( \gcd(q_{r+1}, p) = 1 \) and

\[
\frac{c_r}{p-1} - \alpha m_1 \cdots m_r > \frac{q_{r+1}}{(p')^{\lambda(r+1)}} > \frac{c_r}{p-1} - \left( \alpha + \frac{1}{2(r+1)} \right) m_1 \cdots m_r. \tag{39}
\]

Set \( m_{r+1} = p^{\lambda(r+1)} \). Then

\[
\alpha + \frac{1}{2(r+1)} > \left( \frac{c_r}{p-1} \right) \frac{1}{m_1 \cdots m_r} - \frac{q_{r+1}}{m_{r+1}} \left( \frac{1}{m_1 \cdots m_r} \right) > \alpha. \tag{40}
\]
We have that \( \frac{q_{r+1}}{m_{r+1}} < \frac{c_r}{p-1} \) with \( \gcd(m_{r+1}, pq_{r+1}) = p \) so we may define from Theorem 3 and the above values of \( q_{r+1} \) and \( m_{r+1} \) a commutative diagram

\[
\begin{array}{ccc}
S_r & \rightarrow & S_{r+1} \\
\uparrow & & \uparrow \\
R_r & \rightarrow & R_{r+1}
\end{array}
\]

such that \( R_{r+1} \rightarrow S_{r+1} \) is of type 2. We have \( \sigma = p \) in Theorem 3 and \( \overline{m}_{r+1} = \frac{m_{r+1}}{\sigma} > 1 \). For \( r+1 \leq i \leq r + s \), define

\[
\begin{array}{ccc}
S_i & \rightarrow & S_{i+1} \\
\uparrow & & \uparrow \\
R_i & \rightarrow & R_{i+1}
\end{array}
\]

from Theorem 4 by taking \( m_{i+1} = (p')^2 \) and \( q_{i+1} = p^2 \) if \( i < s \) and taking \( m_{i+1} = p^2 \) and \( q_{i+1} = (p')^2 \) if \( i = s \). From Theorem 4, we have a commutative diagram

\[
\begin{array}{ccc}
S_r & \rightarrow & S_{r+1} \rightarrow \cdots \rightarrow S_{r+s} \rightarrow S_{r+s+1} \\
\uparrow & & \uparrow \uparrow \uparrow \uparrow \\
R_r & \rightarrow & R_{r+1} \rightarrow \cdots \rightarrow R_{r+s} \rightarrow R_{r+s+1}
\end{array}
\]

such that \( R_i \rightarrow S_i \) is of type 1 if \( i = r \) or \( i = r + s + 1 \) and \( R_i \rightarrow S_i \) is of type 2 if \( r + 1 \leq i \leq r + s \). Further, Equation (36) is satisfied with \( t = r + s + 1 \) by Equations (35) and (40). We saw above that \( m_{r+1} > 1 \) and \( \overline{m}_{r+1} > 1 \). If \( r + 1 < i \leq r + s \), we have \( \sigma = p \) in Theorem 4 so that \( \overline{m}_i = \frac{pm_i}{\sigma} = m_i > 1 \). If \( i = r + s + 1 \), then \( \sigma = 1 \) in Theorem 4 and \( \overline{m}_{r+s+1} = pm_{r+s+1} > 1 \). Thus, \( m_i > 1 \) and \( \overline{m}_i > 1 \) for \( r \leq i \leq r + s + 1 \).

Now by Proposition A.2, we have that

\[
\text{dist}(\omega/\nu) = \frac{1}{p-1} \inf_i \{\alpha(J(S_i/R_i))\} = \alpha(x_0) = \alpha.
\]

Now, suppose that \( \Phi(0) = 2 \). Using the construction of the above case (when \( \Phi(0) = 1 \)), we can in this case construct an augmented sequence

\[
\begin{array}{ccc}
B & \rightarrow & S_{-1} \rightarrow S_0 \rightarrow S_1 \rightarrow \cdots \\
\uparrow & & \uparrow \uparrow \uparrow \\
A & \rightarrow & R_{-1} \rightarrow R_0 \rightarrow R_1 \rightarrow \cdots
\end{array}
\] (41)

where \( \Phi \) is extended to the set \( \{-1, 0, 1, 2, \ldots\} \) by defining \( \Phi(-1) = 1 \), and such that the conclusions of the theorem hold for this augmented sequence. We then get the statement of the theorem by forgetting the map \( R_{-1} \rightarrow S_{-1} \).

Remark 7. In the construction of the sequence (30) in Theorem 5, we have \( m_i > 1 \) and \( \overline{m}_i > 1 \) for all \( i \), so that \( \omega(\overline{x}_i) \notin \omega(x_i)Z \) for all \( i \) and \( \omega(x_i) \notin \omega(\overline{x}_i)Z \) for all \( i \). Thus, \( R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots \) is the sequence of sequences of standard quadratic transforms along \( \nu \) and \( S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \) is the sequence of sequences of standard quadratic transforms along \( \omega \).

6 | CALCULATION OF DISTANCE IN SOME EXAMPLES

We give an analysis of the tower of two Artin–Schreier extensions constructed in [19, Theorem 7.38]. The example gives a diagram

\[
\begin{array}{ccc}
R_1 & \rightarrow & A_1 \rightarrow S_1 \\
\downarrow & & \downarrow \downarrow \\
R_2 & \rightarrow & A_2 \rightarrow S_2 \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
\]
where the first two columns and the last two columns are diagrams of the type of Equation (30). The union of the $S_i$ is the valuation ring of a rational rank 1 nondiscrete valuation $\omega$. The vertical arrows are all standard sequences of quadratic transforms.

The rows are such that $R_i \to A_i$ is of type 2 if $i$ is odd and of type 1 if $i$ is even. The extension $A_i \to S_i$ is of type 1 if $i$ is odd and of type 2 if $i$ is even. We have that $R_i$ has regular parameters $u_i, v_i$ and $S_i$ has regular parameters $x_i, y_i$ such that

$$u_i = \gamma_i x_i^p, v_i = \tau_i y_i^p + x_i g_i$$

for all $i$, where $\gamma_i, \tau_i$ are units in $S_i$ and $g_i \in S_i$. A further analysis in [19] shows that strong local monomialization fails for this example.

An example is given in the later paper [15], where the condition of local monomialization itself fails. The example of Section 7 of [19] is a composite of two defect Artin–Schreier extensions,

$$K = k(u, v) \to K_1 = k(x, v) \to K^* = k(x, y)$$

where

$$u = \frac{x^p}{1 - x^p}, v = y^p - x^c y$$

(42)

with $c$ being a positive integer which is divisible by $p - 1$. A rational rank 1 valuation $\omega$ is given of $K^*$, which is trivial on $k$. Let $\nu_1$ be the restriction of $\omega$ to $K_1$ and $\nu$ be the restriction of $\omega$ to $K$.

We will determine the distances of these extensions, illustrating an application of Proposition A.2, showing that the extension $K \to K_1 \to K^*$ is a tower of two dependent defect Artin–Schreier extensions. The first of these extensions was computed by a different method in [23].

We have a sequence of algebraic regular local rings of $K, K_1$, and $K^*$,

$$R_1 = k[u, v]_{(u, v)} \to A_1 = k[x, v]_{(x, v)} \to S_1 = k[x, y]_{(x, y)}$$

such that $\omega$ dominates $S_1$. We normalize $\omega$ by setting $\omega(x) = 1$. There are sequences of homomorphisms

$$\begin{array}{ccc}
R_{r+1} & \to & A_{r+1} \\
\uparrow & \uparrow & \uparrow \\
R_r & \to & A_r \\
\uparrow & \uparrow & \uparrow \\
R_2 & \to & A_2 \\
\uparrow & \uparrow & \uparrow \\
R_1 & \to & A_1 \\
\end{array}$$

where the vertical arrows are sequences of quadratic transforms which are dominated by $\omega$ and $S_r$ dominates $A_r$ and $A_r$ dominates $R_r$. These sequences are calculated above $R_1$ and $S_1$ in [19] and above $A_1$ in [17]. The homomorphism $R_k \to A_k$ is of type 1 if $k$ is even, and of type 2 if $k$ is odd. The homomorphism $A_k \to S_k$ is of type 1 if $k$ is odd and of type 2 if $k$ is even.

The local ring $A_r$ has regular parameters $(x_{A_k}, u_{A_k})$ and $(x_{A_k}, \bar{v}_k)$ such that $A_k \to A_{k+1}$ is defined (Equations (70) and (71) of [17]) by

$$x_{A_k} = x_{A_{k+1}}^p, (u_{A_{k+1}} + 1), \bar{v}_A = x_{A_{k+1}}$$

(43) if $k$ is odd
and
\[ x_{A_k} = x_{A_{k+1}}^{p^3}(v_{A_{k+1}} + 1), \quad \overline{u}_{A_k} = x_{A_{k+1}} \text{ if } k \text{ is even.} \] (44)

The local ring \( S_r \) has regular parameters \((x_{S_k}, y_{S_k})\) and \((x_{S_k}, \overline{y}_{S_k})\) such that \( S_k \to S_{k+1} \) is defined for all \( k \geq 1 \) (Equation (45) of [17]) by
\[ x_{S_k} = x_{S_{k+1}}^{p^2}(\overline{y}_{k+1} + 1), \quad \overline{y}_k = x_{S_{k+1}}. \] (45)

By [19, Theorem 7.38], the local ring \( R_r \) has regular parameters \((x_{R_k}, y_{R_k})\) such that the homomorphism \( R_k \to S_k \) has a stable form
\[ u_{R_k} = y_k x_{S_k}^p, \quad v_{R_k} = \alpha_k y_{S_k}^p + x_{S_k} g_k \]
for all \( k \geq 1 \), where \( \gamma_k \) and \( \alpha_k \) are units in \( S_k \) and \( g_k \in S_k \). We have that \( \omega(x_{A_1}) = 1 \) and for \( k \geq 2 \), we deduce from Equations (43) and (44) that
\[
\omega(x_{A_k}) = \begin{cases} \frac{1}{p^{2k-2}} & \text{if } k \text{ is odd,} \\ \frac{1}{p^{2k-3}} & \text{if } k \text{ is even.} \end{cases}
\]

Letting \( J(A_k/R_k) = (x_{A_k}^{c_k}) \), we calculate from 2) of Remark 4 that if \( k \) is even, then
\[
\left( \frac{c_{k+1}}{p-1} \right) \omega(x_{A_{k+1}}) = \left( \frac{c_k}{p-1} \right) \omega(x_{A_k}) - \omega(\overline{v}_{A_k}) + \omega(x_{A_{k+1}}) = \left( \frac{c_k}{p-1} \right) \omega(x_{A_k})
\]
and if \( k \) is odd, we calculate from 1) of Remark 5 that
\[
\left( \frac{c_{k+1}}{p-1} \right) \omega(x_{A_{k+1}}) = \left( \frac{c_k}{p-1} \right) \omega(x_{A_k}) - \omega(x_{A_k}) = \left( \frac{c_k}{p-1} \right) \omega(x_{A_k}) - \frac{1}{p^{2k-2}}.
\]

From
\[ u_{R_1} = \frac{x_{A_1}^p}{1-x_{A_1}^{p-1}}, \quad v_{R_1} = u_{A_1}, \]

we compute \( J(A_1/R_1) = (x_{A_1}^{2p-2}) \) so \( \omega(J(A_1/R_1)) = 2(p-1) \) and \( c_1 = 2p - 2 \). Now, we compute from Equations (46) and (47) and Proposition A.2 that
\[
-\text{dist}(\nu_1/\nu) = \frac{1}{p-1} \inf_k \omega(J(A_k/R_k)) = 1 - \left( \sum_{i=1}^{\infty} \frac{1}{p^{4i}} \right) = 1 - \left( \frac{1}{p^4 - 1} \right) = \frac{p^4 - 2}{p^4 - 1}.
\]

Since dist(\( \nu_1/\nu \)) is less than zero, the extension is dependent. This distance is computed using a different method in [23].

From Equation (45), we compute
\[
\omega(x_{S_k}) = \frac{1}{p^{2k-2}}
\]
for \( k \geq 1 \). If \( k \) is odd, we compute from 2) of Remark 4 that
\[
\left( \frac{c_{k+1}}{p-1} \right) \omega(x_{S_{k+1}}) = \left( \frac{c_k}{p-1} \right) \omega(x_{S_k}) - \omega(\overline{y}_{S_k}) + \omega(x_{S_{k+1}}) = \left( \frac{c_k}{p-1} \right) \omega(x_{S_k}).
\]

(49)
If \( k \) is even, we compute from 1) of Remark 5 that
\[
\left( \frac{c_k + 1}{p - 1} \right) \omega(x_{S_{k+1}}) = \left( \frac{c_k}{p - 1} \right) \omega(x_{S_k}) - \frac{1}{p^{2k-2}}. \tag{50}
\]
Now, we compute from Equation (42) that \( J(S_1/A_1) = (x_1^c) \) so \( c_1 = c \). Next, we compute from Equations (49) and (50) and Proposition A.2 that
\[
-\text{dist}(\omega/\nu_1) = \frac{1}{p-1} \inf_k \omega(J(S_k/A_k))
= \frac{c}{p-1} - \frac{1}{p-1} \left( \sum_{i=0}^{\infty} \frac{1}{p^i} \right)
= \frac{c}{p-1} - \frac{1}{p^2} \left( \frac{1}{p^3} \right) = \frac{c p^3 + (c-1)p^2 + cp + c}{p^4-1}.
\]
Now, \( c p^3 + (c-1)p^2 + cp + c \) is positive for all positive integers \( c \), so dist(\( \omega/\nu_1 \)) is less than zero for all positive integral \( c \) and thus the extension is dependent.

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**ORCID**
Steven Dale Cutkosky ♦ https://orcid.org/0000-0002-9319-0717

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APPENDIX A

In this Appendix, we give proofs of the results on defect cuts and ramification cuts of Artin–Schreier extensions from [26]. These results are stated in [26] and their proofs are outlined there. A much more general statement than Theorem A.1 is proven in [27, Theorem 3.5].

We suppose throughout this section that $L$ is an Artin–Schreier extension of a field $K$ of characteristic $p$, $\omega$ is a rank 1 valuation of $L$ and $\nu$ is the restriction of $\omega$ to $K$. We suppose that $L$ is a defect extension of $K$. We will use the notation of Sections 2.5 and 2.6.

Let $\Theta$ be an Artin–Schreier generator of $K$. We have that

$$\text{Gal}(L/K) \cong \mathbb{Z}_p = \{\text{id}, \sigma_1, \ldots, \sigma_{p-1}\},$$

where $\sigma_i(\Theta) = \Theta + i$. Since $L/K$ is an immediate extension, the set $\omega(\Theta - K)$ is an initial segment in $\nu K$ which has no maximal element. Further, $\omega(\Theta) < 0$ by [25, Lemma 2.28].

Let $s = \text{dist}(\omega/\nu) \in \mathbb{R}$, so that

$$\text{dist}(\Theta, K) \uparrow \mathbb{R} = s^- = \text{dist}(\omega/\nu)^- \leq 0^-.$$

There exists a sequence $\{c_i\}_{i \in \mathbb{N}}$ in $K$ such that

$$\omega(\Theta - c_i) < \omega(\Theta - c_{i+1})$$

for all $i$, and

$$\lim_{i \to \infty} \omega(\Theta - c_i) = \text{dist}(\omega/\nu).$$

Let $r$ and $t$ be positive integers with $r < t$. Then

$$\nu(c_i - c_r) = \omega((\Theta - c_r) - (\Theta - c_i)) = \omega(\Theta - c_r).$$

Thus for $r < t < u$,

$$\nu(c_i - c_r) = \omega(\Theta - c_r) < \omega(\Theta - c_i) = \nu(c_u - c_i),$$

so $\{c_i\}$ is a pseudo-convergent sequence in $K$ ([23] or p. 39 [33]). By the above, for $r \in \mathbb{N}$, $\nu(c_i = c_r)$ has a common value $\gamma_r$ for all $t > r$. Further, the above shows that

$$\gamma_r = \omega(\Theta - c_r).$$

The Artin–Schreier generator $\Theta$ is a pseudo-limit of $\{c_i\}$ ([23] or p. 47 [33]).
**Lemma A.1.** The pseudo-convergent sequence \( \{c_i\} \) does not have a pseudo-limit in \( K \).

**Proof.** Suppose \( c \in K \) is a pseudo-limit of \( \{c_i\} \) in \( K \). Then

\[ \nu(c - c_r) = \gamma_r = \omega(\Theta - c_r) \text{ for all } r. \]

Thus for all \( r \),

\[ \omega(\Theta - c) = \omega((\Theta - c_r) + (c_r - c)) \geq \omega(\Theta - c_r) \]

so \( \omega(\Theta - c) \geq \text{dist}(\omega/\nu) \), a contradiction. \( \square \)

**Lemma A.2.** Suppose that \( f(\mathbf{x}) \in \mathbb{K}[\mathbf{x}] \) is a polynomial such that \( \deg(f) < p \). Then, there exists \( t_0 \in \mathbb{N} \) such that \( \nu(f(c_t)) = \nu(f(c_{t_0})) \) for \( t \geq t_0 \).

**Proof.** Since \( \nu \) has rank 1, by [23, Lemma 10], the smallest degree of a polynomial \( g(\mathbf{x}) \in \mathbb{K}[\mathbf{x}] \) such that \( \nu(g(c_t)) \) does not stabilize for large \( t \) is a power of \( p \). Since \( \{c_t\} \) does not have a pseudo-limit in \( K \), we have that this degree is \( \geq p \). \( \square \)

**Proposition A.1.** Suppose that \( f(\mathbf{x}) \in \mathbb{K}[\mathbf{x}] \) is a polynomial of degree \( r < p \). Let \( \Theta_i = \Theta - c_i \). Then, there are polynomials \( g_j(\mathbf{Y}) \in \mathbb{K}[\mathbf{Y}] \) of degree \( \leq j \) such that

\[ f(\Theta) = g_0(c_i)\Theta^r_i + g_1(c_i)\Theta^{r-1}_i + \cdots + g_{r-1}(c_i)\Theta_1 + g_r(c_i) \]  

(A.1)

for all \( i \), with \( g_0(c_i) = g_0 \) a nonzero element of \( K \). Further, there exists \( i_0 \) and \( \lambda_j \in \nu \mathbb{K} \) (depending on \( f \)) such that

\[ \nu(g_j(c_i)) = \lambda_j \]

for \( i \geq i_0 \) and \( 0 \leq j \leq r \), and

\[ \omega(g_j(c_i)\Theta^{r-j}_i) \neq \omega(g_k(c_i)\Theta^{r-k}_i) \]

(A.2)

for all \( 0 \leq j < k \leq r \) and \( i \geq i_0 \).

**Proof.** We have a factorization \( f(x) = f_0f_1(\mathbf{x}) \cdots f_l(\mathbf{x}) \), where \( f_0 \in \mathbb{K} \) and \( f_i(\mathbf{x}) \) are monic and irreducible for \( 1 \leq i \leq l \). Let \( r_j = \deg(f_j(\mathbf{x})) \). Let \( \Omega \) be an algebraic closure of \( \mathbb{K} \) containing \( L \). We have factorizations

\[ f_j(\mathbf{x}) = (x - a_{j1})(x - a_{j2}) \cdots (x - a_{jr}) \]

and

\[ f_j(x+y) = x^{r_j} - S_1(a_{j1}, \ldots, a_{jr})x^{r_j-1} + \cdots + (-1)^{r_j}S_{r_j}(a_{j1}, \ldots, a_{jr}) \]

where \( S_i \) is the elementary symmetric function of degree \( i \). Let \( y \) be an indeterminate. Then

\[ f_j(x + y) = x^{r_j} - S_1(a_{j1} - y, \ldots, a_{jr} - y)x^{r_j-1} + \cdots + (-1)^{r_j}S_{r_j}(a_{j1} - y, \ldots, a_{jr} - y). \]

Let \( L_j = \mathbb{K}(a_{j1}, \ldots, a_{jr}) \) and set

\[ h_i = S_i(a_{j1} - y, \ldots, a_{jr} - y) \in L_j[\mathbf{y}] \]

for \( 1 \leq i \leq r_j \). \( h_i \) is a polynomial of degree \( i \). \( h_i \) is invariant under permutation of the \( a_{jk} \) and \( L_j \) is Galois over \( \mathbb{K} \) (since it is a normal extension of \( K \) and \( r_j < p \)). Thus, \( h_i \in L_j[\mathbf{y}] \) for \( i \leq r_j \), and we have an expression

\[ f(x + y) = g_0x^r + g_1(\mathbf{y})x^{r-1} + \cdots + g_{r-1}(\mathbf{y})x + g_r(\mathbf{y}) \]  

(A.3)
where \( g_i(y) \in K[y] \) is a polynomial of degree \( \leq i \) \( (g_0 = f_0) \). For \( i \in \mathbb{N} \), we have an expression

\[
f(\Theta) = f(\Theta_i + c_i) = g_0\Theta_i^r + g_1(c_i)\Theta_i^{r-1} + \cdots + g_{r-1}(c_i)\Theta_i + g_r(c_i).
\]

By Lemma A.2, there exists \( i_0 \) such that \( \nu(g_j(c_i)) \) is a constant value \( \lambda_j \) for \( i \geq i_0 \) and \( 0 \leq j \leq r \). Now, \( \omega(\Theta_{i+1}) > \omega(\Theta_i) \) for all \( i \), so for all \( j, k \) with \( 0 \leq j < k \leq r \), there exists \( i(j,k) \) such that

\[
\omega(\Theta_i) \neq \frac{\lambda_k - \lambda_j}{t}
\]

for \( t \) any integer with \( 1 \leq t \leq r \), whenever \( i > i_0 \).

Thus for \( i \) such that \( i > i_0 \) and \( i > \max\{i(j,k) \mid 0 \leq j < k \leq r\} \), \( 0 \leq j < k \leq r \),

\[
\omega(g_j(c_i)\Theta_i^{r-j}) \neq \omega(g_k(c_i)\Theta_i^{r-k}). \quad \square
\]

**Corollary A.1.** The valuation ring \( O_\omega \) is generated as an \( O_\nu \)-module by

\[
\{ g\Theta_i^j \mid g \in K, 0 \leq j \leq p-1, i \in \mathbb{N} \text{ and } \omega(g\Theta_i^j) \geq 0 \}. \quad (A.4)
\]

**Proof.** Let \( M \) be the \( O_\nu \)-module generated by the set \( (A.4) \). The module \( M \) is certainly contained in \( O_\omega \). Suppose that \( h \in O_\omega \). Then, \( \omega(h) \geq 0 \) and \( h = f(\Theta) \) for some polynomial \( f \in K[x] \) of degree < \( p \). By Proposition A.1, we have expression \((A.1)\) of \( h \). Taking \( i \) sufficiently large (so that Equation \((A.2)\) holds) we have that

\[
0 \leq \omega(h) = \min\{\omega(g_j(c_i)\Theta_i^{r-j}) \mid 0 \leq j \leq r\}. \quad (A.5)
\]

Thus, \( h \in M \). \( \square \)

**Corollary A.2.** Suppose that \( h \in L \). Then

\[
\omega(\sigma(h) - h) = \omega(\tau(h) - h)
\]

for \( \sigma, \tau \in Gal(L/K) \) which are both not the identity.

**Proof.** \( h = h(\Theta) \) has an expression of the form \((A.1)\) of Proposition A.1 such that Equation \((A.2)\) holds. We compute using Equation \((A.1)\) for \( 0 < j < p \),

\[
\sigma_j(h) - h = h(\Theta + j) - h(\Theta) = h(\Theta_i + j + c_i) - h(\Theta_i + c_i)
\]

\[
= g_0(c_i)(\Theta_i + j)^r + \cdots + g_{r-1}(c_i)(\Theta_i + j) + g_r(c_i)
\]

\[
- [g_0(c_i)\Theta_i^r + \cdots + g_{r-1}(c_i)\Theta_i + g_r(c_i)]
\]

\[
= g_0(c_i)\left( \sum_{k=0}^{r-1} \binom{r}{k} j^k \Theta_i^{r-k} \right) + g_1(c_i)\left( \sum_{k=0}^{r-2} \binom{r-1}{k} j^{r-1-k} \Theta_i^{r-k} \right) + \cdots + g_{r-1}(c_i)j.
\]

Since \( \omega(\Theta_i) < 0 \), we have that

\[
\omega\left( g_0(c_i)\left( \sum_{k=0}^{r-1} \binom{r}{k} j^k \Theta_i^{r-k} \right) \right) = \omega(g_0(c_i)j\Theta_i^{r-1}) = \omega(g_0(c_i)\Theta_i^{r-1})
\]

for \( 0 \leq l \leq r - 1 \). Thus,

\[
\omega(\sigma_j(h) - h) = \min\{\omega(g_0(c_i)\Theta_i^{r-1-l}) \mid 0 \leq l \leq r - 1\} \quad (A.6)
\]

for \( i \gg 0 \) by Equation \((A.2)\). \( \square \)
For $\alpha \in \omega L$, define the ideal $I_\alpha = \{ f \in \mathcal{O}_L \mid \omega(f) \geq \alpha \}$ in $\mathcal{O}_\omega$. Then, $G_{I_\alpha}$ (defined in Section 2.4) is the subgroup

$$G_{I_\alpha} = \{ s \in \text{Gal}(L/K) \mid \omega(s(x) - x) \geq \alpha \text{ for all } x \in \mathcal{O}_\omega \}$$

of $G = \text{Gal}(L/K)$.

**Corollary A.3.** Continuing with our assumption that $L/K$ is a defect Artin–Schreier extension and $\omega$ has rank 1, suppose that $\alpha \in \mathbb{R}$. Then,

$$G_{I_\alpha} = \begin{cases} \{1\} & \text{if } \alpha > -\text{dist}(\omega/\nu) \\ \text{Gal}(L/K) & \text{if } \alpha \leq -\text{dist}(\omega/\nu). \end{cases}$$

**Proof.** Suppose that $h \in \mathcal{O}_\omega$. Then, $h$ has an expression of the form (A.1) of Proposition A.1 such that Equation (A.2) holds. By the calculation of Equation (A.5) of the proof of Corollary A.1 we have that for $i \gg 0$,

$$0 \leq \omega(h) = \min\{\omega(g_j(c_i)\Theta_{r-j}^i) \mid 0 \leq j \leq r\}. \tag{A.7}$$

By the calculation of Equation (A.6) of the proof of Corollary A.2 and Equation (A.7), we have that for $i \gg 0$,

$$\omega(\sigma(h) - h) = \min\{\omega(g_l(c_i)\Theta_{r-l-1}^i) \mid 0 \leq l \leq r - 1\} \geq -\omega(\Theta_i) > -\text{dist}(\omega/\nu).$$

Thus, $G_{\alpha} = \text{Gal}(L/K)$ for $\alpha \leq -\text{dist}(\omega/\nu)$.

Given $\epsilon > 0$ there exists $c \in K$ such that $\omega(\Theta - c) > \text{dist}(\omega/\nu) - \frac{\epsilon}{2}$. Let $\Theta = \Theta - c$. The group $\omega K$ is dense in $\mathbb{R}$, so there exists $g \in K$ such that

$$0 \leq \omega(g\Theta) < \frac{\epsilon}{2}.$$

We have that $\sigma_1(g\overline{\Theta}) - g\overline{\Theta} = g$, so

$$\omega(\sigma_1(g\overline{\Theta}) - g\overline{\Theta}) = \omega(g) < -\text{dist}(\omega/\nu) + \epsilon.$$

Thus, $G_{\alpha} = \{1\}$ for $\alpha > -\text{dist}(\omega/\nu)$. \qed

As a consequence of the above corollary, we obtain the following theorem. A more general version of this theorem is proven in [27].

**Theorem A.1** Kuhlmann and Piltant [26]. Continuing with our assumption that $L/K$ is a defect Artin–Schreier extension and $\omega$ has rank 1, let $\text{Ram}(\omega/\nu)$ be the ramification cut of $L/K$ defined in Section 2.5. Then

$$\text{dist}(\omega/\nu)^- \cap \nu K = \text{dist}(\Theta, K) \cap \nu K = -\text{Ram}(\omega/\nu) \cap \nu K.$$

**Proof.** By Corollary A.3

$$(\text{Ram}(\omega/\nu)^\mathbb{R}) = \{\alpha \in \mathbb{R} \mid G_{I_\alpha} = 1\} = \cup\{\alpha \in \mathbb{R} \mid \alpha > -\text{dist}(\omega/\nu)\}.$$

Thus,

$$(-\text{Ram}(\omega/\nu)^\mathbb{R}) = \{\alpha \in \mathbb{R} \mid \alpha < \text{dist}(\omega/\nu)\}$$

and so $\text{dist}(\Theta, K) \cap \nu K = -\text{Ram}(\omega/\nu) \cap \nu K$. \qed
Lemma A.3 Kuhlmann and Piltant, [26]. Suppose that $K$ and $L$ are two-dimensional algebraic function fields over an algebraically closed field $k$ of characteristic $p > 0$ and $K \to L$ is an Artin–Schreier extension. Let $\omega$ be a rational rank one non-discrete valuation of $L$ and let $\nu$ be the restriction of $\omega$ to $K$. Suppose that $L$ is a defect extension of $K$.

Suppose that $R$ is a regular algebraic local ring of $K$ and $S$ is a regular algebraic local ring of $L$ such that $\omega$ dominates $S$, $S$ dominates $R$ and $R \to S$ is of type 1 or 2. Inductively applying Theorems 3 and 4, we construct a diagram where the horizontal sequences are birational extensions of regular local rings

\[
\begin{array}{cccc}
S &=& S_0 & \to & S_1 & \to & S_2 & \to & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
R &=& R_0 & \to & R_1 & \to & R_2 & \to & \cdots
\end{array}
\] (A.8)

with $\bigcup_{i=1}^{\infty} S_i = \mathcal{O}_L$. Further assume that for each map $R_i \to S_i$, there are regular parameters $u, v$ in $R_i$ and $x, y$ in $S_i$ such that one of the following forms hold:

1) $u = x, v = f$ \hspace{1cm} (A.9)

where $\dim_k S_i/(x, f) = p$, or

2) $u = \delta x^p, v = y$ \hspace{1cm} (A.10)

where $\delta$ is a unit in $S_i$ and in both cases that $x = 0$ is a local equation of the critical locus of $\text{Spec}(S_i) \to \text{Spec}(R_i)$.

Let

\[ J_i = J(S_i/R_i) = \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) \]

be the Jacobian ideal of the map $R_i \to S_i$.

Then, there exists $c > 0$ such that $J_i = x^c S_i$ (since the critical locus is supported on $x = 0$). Let $\sigma$ be a generator of $\text{Gal}(L/K)$. Then

1) $\omega(\sigma(y) - y) = \frac{c}{p-1} \omega(x) = \frac{1}{p-1} \omega(J_i)$ if Equation (A.9) holds,

2) $\omega(\sigma(x) - x) = \frac{c}{p-1} \nu(x) = \frac{1}{p-1} \omega(J_i)$ if Equation (A.10) holds.

Proof. We prove the first statement 1). The proof of the second statement is similar. Suppose that a form (A.9) holds. Let $N$ be the $S_i$-ideal $N = \text{Ann}_{S_i}(\Omega^1_{S_i/R_i})$.

Since $\omega$ is the unique extension of $\nu$ to $L$ and $R_i \to S_i$ is quasi finite with complexity $p = [L : K]$, we have that $S_i$ is the integral closure of $R_i$ in $L$ and is thus a finite $R_i$-module. There exists a unit $\delta \in S_i$ and $\gamma \in S_i$ such that

\[ u = x, v = \delta y^p + xy. \]

Let $M$ be the $R_i$-module $M = R_i + R_i y + \cdots + R_i y^{p-1}$. We have that

\[ y^p = \delta^{-1} v - u \delta^{-1} \gamma \in (u, v)S_i \]

and $x = u \in (u, v)S_i$ so $S_i = M + (u, v)S_i$. Thus, $S_i = M$ by Nakayama’s lemma. Let $f(t) \in K[t]$ be the minimal polynomial of $y$ over $K$. The polynomial $f(t)$ has degree $p$ since $[L : K] = p$. Since $R_i$ is normal and $y$ is integral over $R_i$, by Theorem 4 on p. 260 of [31], the coefficients of $f(t)$ are in $R_i$, and thus $S_i \cong R_i[t]/(f(t))$. We have an isomorphism of $S_i$-modules

\[ \Omega^1_{S_i/R_i} \cong S_i/f'(y)S_i \]
where \( f'(t) = \frac{df}{dt} \). Thus, \( N = (f'(y)) \). We compute \( N \) in another way, from the right exact sequence

\[
Ω^1_{R_i/k} \otimes_{R_i} S_i \to Ω^1_{S_i/k} \to Ω^1_{S_i/R_i} \to 0,
\]

showing that we have a presentation

\[
S_i^2 \xrightarrow{A} S_i \to Ω^1_{S_i/R_i} \to 0,
\]

where

\[
A = \begin{pmatrix}
\frac{∂u}{∂x} & \frac{∂u}{∂y} \\
\frac{∂v}{∂x} & \frac{∂v}{∂y}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & \frac{∂v}{∂x} & \frac{∂v}{∂y}
\end{pmatrix}.
\]

From the fact that

\[
\left\{ \begin{pmatrix} 1 \\
\frac{∂u}{∂x} \\
\frac{∂v}{∂y}
\end{pmatrix}, \begin{pmatrix} 0 \\
1
\end{pmatrix} \right\}
\]

is an \( S_i \)-basis of \( S_i^2 \), we see that \( Ω^1_{S_i/R_i} \cong S_i/\frac{∂v}{∂y}S_i \), and so \( N = \begin{pmatrix} \frac{∂v}{∂y} \end{pmatrix} = J_i = (x^c) \). Factoring

\[
f(t) = \prod_{τ \in \text{Gal}(L/K)} (t - τ(y))
\]

in \( L[t] \), we see that

\[
f'(y) = \prod_{id≠τ \in \text{Gal}(L/K)} (y - τ(y)).
\]

Thus,

\[
cω(x) = \sum_{id≠τ \in \text{Gal}(L/K)} ω(τ(y) - y) = (p - 1)ω(σ(y) - y)
\]

by Corollary A.2.

**Proposition A.2** Kuhlmann and Piltant, [26]. Let assumptions be as in Lemma A.3. Then, the distance \( \text{dist}(ω/ν) \) is computed by the formula

\[
-\text{dist}(ω/ν) = \frac{1}{p - 1} \inf_{i} \{ω(J(S_i/R_i))\}
\]

where the infimum is over the \( R_i → S_i \) in the sequence (A.8).

**Proof.** Let \( σ \) be a generator of \( \text{Gal}(L/K) \). Since \( \cup S_i = \mathcal{O}_ω \), we have that

\[
-\text{dist}(ω/ν) = \inf \{ω(σ(h) - h) \mid h ∈ S_i \text{ for some}\}
\]

by Corollaries A.3 and A.2. By the proof of Lemma A.3, \( h ∈ S_i \) implies there exists a polynomial \( f(t) ∈ R_i[t] \) of degree < \( p \) such that \( h = f(z_i) \), where \( z_i = y \) if \( R_i → S_i \) is in case (A.9), \( z_i = x \) if \( R_i → S_i \) is in case (A.10). Thus, we have an expression

\[
h = a_0z_i^{p-1} + a_1z_i^{p-2} + \cdots + a_{p-1}
\]
with \(a_0, \ldots, a_{p-1} \in R_i\). For \(s \geq 1\), we have a factorization

\[
\sigma(z_i)^s - z_i^s = (\sigma(z_i) - z_i)(\sigma(z_i)^{s-1} + z_i\sigma(z_i)^{s-2} + \cdots + z_i^{s-1}).
\]

The valuation \(\omega\) is the unique extension of \(\nu\) to \(L\), so \(\omega(\sigma(z_i)) = \omega(z_i) \geq 0\). Thus, \(\omega(\sigma(z_i)^s - z_i^s) \geq \omega(\sigma(z_i) - z_i)\) for all \(s \geq 1\).

We have that

\[
\sigma(h) - h = a_0(\sigma(z_i)^{p-1} - z_i^{p-1}) + \cdots + a_{p-2}(\sigma(z_i) - z_i)
\]

so \(\omega(\sigma(h) - h) \geq \omega(\sigma(z_i) - z_i)\). Thus

\[
-\text{dist}(\omega/\nu) = \inf\{\omega(\sigma(z_i) - z_i)\}.
\]

The proposition now follows from Lemma A.3. \(\square\)