Supersymmetry and PT-Symmetric Spectral Bifurcation

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Abstract:
Dynamical systems exhibiting both PT and Supersymmetry are analyzed in a general scenario. It is found that, in an appropriate parameter domain, the ground state may or may not respect PT-symmetry. Interestingly, in the domain where PT-symmetry is not respected, two superpotentials give rise to one potential; whereas when the ground state respects PT, this correspondence is unique. In both scenarios, supersymmetry and shape-invariance are intact, through which one can obtain eigenfunctions and eigenstates exactly. Our procedure enables one to generate a host of complex potentials which are not PT-symmetric, and can be exactly solved.

Introduction

Complex potentials symmetric under simultaneous Parity (P) and Time-reversal (T) transformations are known to yield real energy eigenvalues in suitable parameter domain, sharing common eigenfunctions with the PT operator [1-3]. Beyond this range, these PT-
symmetric potentials correspond to complex-conjugate spectra with eigenfunctions connected through PT operation. This **spontaneous breaking** of PT-symmetry is expected as PT is a non-linear operator. Various models have been studied, both numerically and analytically, to illustrate the above structure [4, 5]. When these systems are analyzed under Supersymmetric quantum mechanics (SUSY-QM) [6], in the real domain, they are found to be isospectral to a real potential [7]. We here put forward an approach to arrive at both real and complex-conjugate domains starting from a class of superpotentials, with complex parameters. For the real case, it is possible to have a unique superpotential. For the complex-conjugate domain we have multiple superpotentials, each representing a **separate** section of the complete Hilbert space, but possessing the same potential. Moreover, these two classes of superpotentials are related through simple parametric variation, showing a bifurcation in the complex energy plane.

In the following, we will discuss SUSY-QM briefly and then directly proceed to discuss a particular example of PT-symmetric potential [8]. Analysis of other complex potentials which are not PT-symmetric will be discussed, followed by some recent experimental findings.

1. **SUSY-QM: A Brief Introduction**

Following the factorization approach to analytically solvable Hamiltonians [9], in SUSY-QM [6, 10, 11, 12], the Hamiltonian can be written as $H_-(x) = A^\dagger A$, where $A^\dagger = -\partial/\partial x + W(x)$ and $A = \partial/\partial x + W(x)$; with the **superpotential** $W(x)$ being defined as:

$$V_\pm(x) = W^2(x) \pm \partial W/\partial x.$$ 

Here $V_-$ is the potential corresponding to $H_-(x)$ and $V_+$ corresponds to another Hamiltonian $H_+(x) = AA^\dagger$, which is the **superpartner** of $H_-(x)$. The eigenstates of $H_\pm(x)$ have one-to-one correspondence:
\[ \psi^+_n(x) = [E^{-}_{n+1}]^{-1/2} A \psi^-_{n+1}(x), \]

and,

\[ \psi^-_{n+1}(x) = [E^+_{n}]^{-1/2} A^\dagger \psi^+_n(x); \]  

except for the ground state of \( H_-(x) \), which is defined as \( A \psi_0(x) = 0 \), and is expressible as

\[ \psi_0(x) = e^{-\int^x w(x') dx'}. \]

Also, from (1.1), the energies are related as \( E^+_n = E^-_{n+1} \), whence the potentials \( V_\pm(x) \) are called *isospectral*. Further, Gendenshtein [13] showed that, if two isospectral potentials are related as

\[ V_+(x; a_0) = V_-(x; a_1) + R(a_1), \]

where \( a_0 \) is a parameter in \( V_\pm \), \( a_1 = f(a_0) \) and \( R(a_1) \) is independent of \( x \); then the potentials are called *shape-invariant*. From such potentials, one can construct a hierarchy of Hamiltonians:

\[ H^s = -\partial^2 / \partial x^2 + V_-(x; a_s) + \sum_{k=1}^s R(a_k) = -\partial^2 / \partial x^2 + V_+(x; a_{s-1}) + \sum_{k=1}^{s-1} R(a_k), \]

with ground state energy \( E^s_0 = \sum_{k=1}^s R(a_k) \). On identifying, \( H^1 = H_+ \) and \( H^0 = H_- \), the energy of the \( n \)th level of \( H_- \) is found to be \( E^+_0 = \sum_{k=1}^n R(a_k) \). Further, the excited states of the original Hamiltonian can be found from the ground state of \( H^s \) as:

\[ \psi^-_s(x; a_s) \propto A^\dagger(x; a_0) A^\dagger(x; a_1) \ldots \ldots \ldots A^\dagger(x; a_{n-1}) \psi^-_0(x; a_n). \]  

Therefore if the potential is shape-invariant, the eigenvalue problem can be solved completely by this algebraic method and the spectrum with corresponding eigenfunctions can be obtained. The application of the SUSY-QM approach to PT-symmetric pseudo-Hermitian Hamiltonians has been carried out in detail [14, 15], including the construction of the appropriate norm. Recently [16], the procedure to construct the norm for general pseudo-Hermitian Hamiltonians has been shown.
2. **Construction of Complex PT-symmetric Potentials**

Non-Hermitian, complex PT-symmetric potentials are known to have real eigenvalues over a definite range of parameters appearing in them, beyond which they show complex-conjugate (CC) spectra, with corresponding wavefunctions being PT-related. If such potentials are shape-invariant then the eigenvalue problem can be solved through SUSY-QM.

We start with general superpotentials with complex parameters, leading to both real and CC spectra for different range of the parameter values. In the real domain, for a given potential, the superpotential is *unique*, for a given parameter range. For the CC domain, two *different* superpotentials yield the *same* potential, for a different range of parameterization. Thus, we arrive at the SUSY condition for *phase-transition* of the spectrum from real to CC values owing to spontaneous PT-symmetry breaking.

For example, we will consider generalizations of PT-symmetric shape-invariant complex potentials, \( V(x; \alpha) = -V_1 \text{sech}^2(\alpha x) - iV_2 \text{sech}(\alpha x) \tan(\alpha x) \), which was analytically solved by Z. Ahmed [8]. Here \( V_{1,2} \) and \( \alpha \) are real constant parameters. We propose a superpotential,

\[
W^\pm_{PT} = (A \pm iC^{PT}) \tanh(\alpha x) + (\pm C^{PT} + iB) \text{sech}(\alpha x),
\]

with \( A, B, \alpha \) are real parameters. The corresponding potentials are,

\[
V^\pm(x) = -[(A \pm iC^{PT})(A \pm iC^{PT} + \alpha) - (\pm C^{PT} + iB)^2] \text{sech}^2(\alpha x) - i(\pm C^{PT} - B)[2(A \pm iC^{PT}) + \alpha] \text{sech}(\alpha x) \tan(\alpha x).
\]

This potential is not PT-symmetric, for which the co-efficient of the even-P term has to be real and that of the odd-P part has to be imaginary. Invoking this condition, we arrive at the *bifurcation condition*,

\[
C^{PT}[2(A - B) + \alpha] = 0.
\]
From (2.3), there can be two possibilities. For $C^{PT} = 0$, we have from (2.1),

$$W_{PT}(x) \equiv W_{real}(x) = A \tanh(ax) + iB \text{sech}(ax),$$

(2.4)
giving, from (2.2),

$$V_{\pm}(x) \equiv V_{\pm}(x) = -[A(A + \alpha) + B^2] \text{sech}^2(ax) + iB(2A + \alpha) \text{sech}(ax) \tanh(ax).$$

(2.5)
Then, from (2.4), we get the real spectrum of the above potential through shape-invariance as,

$$E = -(n\alpha - A)^2,$$

(2.6)
modulo a constant term, and from (1.3), the eigenfunctions as [9],

$$\psi_n(x) \propto [\sec(ax)]^{-1/2} \exp\left[-i \frac{B}{\alpha} \tan^{-1}\{\sinh(ax)\}\right] \mathcal{P}_n^{A + B - \frac{1}{2} A - \frac{1}{2} B \pm \frac{1}{2} \left[i \sinh(ax)\right]},$$

(2.7)
Again, if $C^{PT} \neq 0$, from (2.3), one gets $A = B - \frac{\alpha}{2}$, which when substituted in (2.1) and (2.2), yields,

$$W_{PT}^{\pm}(x) \equiv W_{\pm}(x) = (A \pm iC^{PT}) \tanh(ax) + \left[\pm C^{PT} + i\left(A + \frac{\alpha}{2}\right)\right] \text{sech}(ax),$$

(2.8)
and,

$$V_{\pm}^{\pm}(x) \equiv V_{\pm}(x) = -\left[2A(A + \alpha) - 2(C^{PT})^2 + \frac{\alpha^2}{4}\right] \text{sech}^2(ax) +$$

$$+ i \left[2A(A + \alpha) + 2(C^{PT})^2 + \frac{\alpha^2}{2}\right] \text{sech}(ax) \tanh(ax).$$

(2.9)
Corresponding to two different superpotentials, we arrive at the CC spectrum,

$$E_{n}^{\pm} = 2n(A \pm iC^{PT})\alpha + (n\alpha)^2,$$

(2.10)
modulo a constant term again, and the eigenfunctions as,

$$\psi_n^{\pm}(x) \propto [\text{sech}(ax)]^{-1/2(A \pm iC^{PT})} \exp\left[-\frac{1}{\alpha} \left(A + \frac{\alpha}{2}\right) \mp \frac{C^{PT}}{\alpha}\right] \tan^{-1}\{\sinh(ax)\} \times$$

$$\mathcal{P}_n^{\pm 2C^{PT} - 2\alpha A + \frac{1}{2} \left[i \sinh(ax)\right]},$$

(2.11)
The specific parameterization condition $C^{PT} \neq 0$ is the SUSY criterion for spontaneously broken PT, which is different from the analytic parameter criterion with PT being unbroken over a range of parameters. Further, for broken PT, two superpotentials corresponding to the
**Conclusion**

One can construct non-PT-symmetric shape-invariant complex potential through *minimal complexification* of the superpotential corresponding to the real counterparts [1]. For example, the Pöschl-Teller potential [18] \( U(x) = U_a \text{sech}^2(\alpha x) + U_b \text{csch}^2(\alpha x) \), where \( U_{a,b} \) and \( \alpha \) are constant parameters, can be complexified by considering the superpotentials

\[
W_1 = \tanh(\alpha x) + i \coth(\alpha x) \quad \text{or} \quad W_2 = i \tanh(\alpha x) + \coth(\alpha x),
\]

where \( A, B \) and \( \alpha \) being constant parameters. In both cases, the spectra are complex. The non-uniqueness of the real spectra for the real potential is lifted by the complexification, with the imaginary part being equispaced. Further, the wave-functions become normalizable over a greater parameter range. A complex radial Coulomb potential can be constructed from the superpotential, \( W(r) = \frac{i\alpha}{r} + \beta \), \( \alpha \) and \( \beta \) being independent of \( r \), which results into complex principal quantum numbers, with the wave-function being normalizable over a greater parameter range than those corresponding to the real counterpart [19]. Recently, spontaneous PT breaking has been observed experimentally in optical fibers, where the gain-loss profile showed bifurcation in the complex plane beyond a
critical value of the \textit{optical loss co-efficient} [20]. We propose similar observation in terms of our parameters, if they are identified properly for an optical system.

Recently it has been shown for unbroken PT-symmetry, the potential in Eq.\,(2.5) corresponds to not one, but two superpotentials, under the additional $\text{sl}(2)$ symmetry of the system [21,22]. But each of them is shown to be independently mapped to the same pair of superpotentials in the broken PT sector under the SUSY parameterization [23].

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\textbf{References:}

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. \textbf{80} (1998) 5243-5246; C. M. Bender, S. Boettcher and P.N. Meisinger, J. Math. Phys. \textbf{40} (1999) 2201-2229; C. M. Bender, S. Boettcher, H. F. Jones and V. M. Savage, J. Math. Phys. A: Math. Gen. \textbf{32} (1999) 6771-6781.

[2] C. M. Bender, Rept. Prog. Phys., \textbf{70} (2007) 947-1018; A. Mostafazadeh, [quant-ph:0810.5643], (revised version to appear in Int. J. Geom. Meth. Mod. Phys).

[3] Pramana, Special issue, \textit{Non-Hermitian Hamiltonians in Quantum Physics}-Part I (08- 2009).

[4] A. Khare and B. P. Mandal, Phys. Lett. A \textbf{272} (2000) 53-56.
[5] S. Sree Ranjani, A. K. Kapoor and P. K. Panigrahi, IJMPA 20 (2005) 4067-4077.

[6] *Super Symmetry in Quantum Mechanics*, F. Cooper, A. Khare, U. P. Sukhatme, World Scientific, Singapore (2001) and references therein.

[7] B. Bagchi and R. Roychoudhury, J. Phys. A: Math. Gen. 33 (2000) L1-L3.

[8] Z. Ahmed, Phys. Lett. A 282 (2001) 343-348.

[9] G. Darboux, C. R. Acad. Sci. (Paris) 94 (1882) 1456-1459.

[10] E. Witten, Nucl. Phys. B 188 (1981) 513-554.

[11] R. Dutt, A. Khare and U. P. Sukhatme, Phys. Lett B 181 (1986) 295-298; R. Dutt, A. Khare, and U. P. Sukhatme, Am. J. Phys. 56(2) (1988) 163-168; R. Dutt, A. Gangopadhyaya, C. Rasinariu, and U. P. Sukhatme, J. Phys. A 34 (2001) 4129-4142.

[12] C. V. Sukumar, J. Phys. A 18 (1985) 2917-2939; F. Cooper and B. Freedman, Ann. Phys. (NY) 146 (1983) 262-288.

[13] L. Gendenshtein, Pis'ma Zh. Eksp. Teor. Fiz. 38 (1983) 299 [JETP Lett. 38 (1983) 356].

[14] A. Mostafazadeh, Nucl. Phys. B, 640 (2002) 419-434.

[15] F. Correa and M. S. Plyushchay. Ann. Phys. 322 (2007) 2493-2500; [hep-th/0605104]; F. Correa and M. S. Plyushchay, J. Phys. A 40 (2007) 14403-14412; [arXiv:0706.1114]; F. Correa, V. Jakubsky and M. S. Plyushchay, Ann. Phys. 324 (2009) 1078-1094; nb [arXiv:0809.2854]; F. Correa, V. Jakubsky, L. Nieto and M. S. Plyushchay, Phys. Rev. Lett. 101 (2008) 030403; [arXiv:0801.1671].

[16] A. Das and L. Greenwood, Phys. Lett. B, 678 (2009) 504.

[17] J. W. Dabrowwaska, A. Khare and U. P. Sukhatme, J. Phys. A: Math. Gen. 21 (1988) L195

[18] P. M. Morse, Phys. Rev. 34 (1928) 57-64; G. Pöschl and E. Teller, Z. Phys. 21 (1949) 488.

[19] A. Gangopadhyaya, P. K. Panigrahi and U. P. Sukhatme, J. Phys. A: Math. Gen. 27 (1994)
[20] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou and D. N. Christodoulides, Phys. Rev. Lett. 103 (2009) 093902.

[21] B. Bagchi and C. Quesne, Phys. Lett. A 273 (2000) 285.

[22] B. Bagchi and C. Quesne, [quant-ph:1007.3870], to be published in Annals of Physics (N.Y.).

[23] K. Abhinav and P. K. Panigrahi, DOI:10.1016/j.aop.2010.10.012, to be published in Annals of Physics (N.Y.).