INFLATION, OUTPUT, AND WELFARE*

BY RICARDO LAGOS AND GUILLAUME ROCHETEAU1

Federal Reserve Bank of Minneapolis and New York University; Federal Reserve Bank of Cleveland and Australian National University

We study the effects of anticipated inflation on aggregate output and welfare within a search-theoretic framework. We consider two pricing mechanisms: ex post bargaining and a notion of competitive pricing. Under bargaining, the equilibrium is generically inefficient and an increase in inflation reduces buyers’ search intensities, output, and welfare. If prices are posted and buyers can direct their search, search intensities are increasing with inflation for low inflation rates and decreasing for high inflation rates. The Friedman rule achieves the efficient allocation, and inflation always reduces welfare, although it can have a positive effect on output for low inflation rates.

In a monetary economy, it is in everyone’s private interest to try to get someone else to hold non-interest-bearing cash and reserves. But someone has to hold it all, so all of these efforts must simply cancel out. All of us spend several hours per year in this effort, and we employ thousands of talented and highly trained people to help us. These person-hours are simply thrown away, wasted on a task that should not have to be performed at all.

Robert E. Lucas, Jr., 2000

1. INTRODUCTION

It is a commonly held view that inflation induces economic agents to undertake costly actions in order to reduce their exposure to the inflation tax. The costs associated with these actions are a social waste, part of the welfare cost of inflation. This conventional wisdom is succinctly articulated by Lucas (2000). In this article we formalize this argument and study its implications for the effects that anticipated inflation has on aggregate output and welfare.

The search-theoretic framework pioneered by Kiyotaki and Wright (1989, 1991, 1993) explicitly models the frictions that make money essential and relates them to the decentralized nature of trade. This makes the search-based approach a natural

* Manuscript received June 2004; revised September 2004.
1 We are grateful to Randall Wright for his input. We also thank Paul Chen, Patrick Kehoe, Ellen McGrattan, Flavio Menezes, Paulo Monteiro, Graeme Wells, and seminar participants at the Australian National University, Indiana University, and University of Sydney. Lagos thanks the C.V. Starr Center for Applied Economics at NYU for financial support. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Cleveland, the Federal Reserve Bank of Minneapolis, or the Federal Reserve System. Please address correspondence to: R. Lagos, Department of Economics, NYU, 269 Mercer Street, New York, NY 10003. E-mail: ricardo.lagos@nyu.edu.
setup for our analysis. We let agents choose costly search intensities to determine the frequency with which they trade and study how changes in the rate of inflation affect their search effort decisions, as well as the number of trades, the quantity of output produced in any trade, and welfare in the equilibrium. We view these search efforts as a natural way of formalizing the “efforts” to avoid the inflation tax alluded to in the quote from Lucas (2000).

The particular economic environment we consider has the structure introduced by Lagos and Wright (2003, 2005). Agents periodically participate in centralized and decentralized markets. Trade in the centralized market involves all agents and occurs at market-clearing prices. The centralized market allows agents to rebalance their cash holdings and at the same time keeps the model tractable. In the decentralized market agents are matched pairwise and trade is bilateral. Lagos and Wright (2003, 2005) follow the previous literature and assume that agents in these bilateral situations bargain over the terms of trade. Instead, here we follow Rocheteau and Wright (2005) and generalize the model to allow for different pricing mechanisms.

We first consider bargaining since it is often regarded as the standard pricing mechanism for search environments with bilateral meetings. In this case, buyers respond to increases in inflation by reducing their real money holdings and search intensities. This complementarity between real balances and search intensity can be explained as follows. If agents hold smaller real balances, they enjoy smaller gains from trade when matched. Since each trade yields less utility, agents exert less effort to generate trades. As a result fewer matches occur (extensive margin effect) and less output is produced in every match (intensive margin effect), so aggregate output falls with inflation. Under the Friedman rule, buyers’ search effort choices are generically inefficient because of a congestion externality characteristic of search environments. For example, if buyers have all the bargaining power, their search effort is too high, and as inflation goes up they reduce their search intensities, which tends to mitigate this inefficiency. Nevertheless, the negative effect of inflation on real money balances dominates and higher inflation always leads to lower welfare. Note that the prototypical search model with bargaining predicts that individual search effort decreases with inflation and hence fails to rationalize the conventional wisdom discussed above.

We then study the model under price posting with partially directed search—one of the pricing mechanisms considered by Rocheteau and Wright (2005)—as this is a natural notion of competitive pricing for search models. This version of the model has some agents posting prices and other agents directing their search toward a particular price. For this case the Friedman rule is the optimal monetary policy and it generates the first best allocation: Buyers choose the socially

---

2 A model similar to our setup with bargaining but based on Shi (1997) instead of on Lagos and Wright (2005) has been developed independently by Peterson and Shi (2003). They investigate the effect of inflation on price dispersion by combining both a stochastic match-specific component and buyers’ endogenous search intensities. A key distinction between our model and theirs is that we allow individual agents to choose their money holdings. Our results differ in terms of both the basic properties of the equilibrium (for example, existence, uniqueness) as well as comparative static properties.

3 This is the competitive price posting or directed search construct introduced in the labor literature by Moen (1997) and Shimer (1996).
efficient search intensity and real balances. For high inflation rates, search intensity decreases with inflation, just as in the model with bargaining. But for low inflation rates, an increase in inflation raises buyers’ search intensities. Therefore with the competitive notion of price posting, the model is able to generate predictions that are in line with the conventional wisdom discussed above: Agents increase their search intensities when the inflation rate is higher, and the additional search costs are socially wasteful.

To see the intuition for this result, suppose that prices are set by a group of agents we call sellers and that another group of agents—buyers—direct their search toward the sellers. Since sellers compete against each other to attract buyers, they internalize the effect of inflation on the buyers’ willingness to carry real balances. In equilibrium, competition forces them to partially compensate buyers for the inflation tax by raising the buyer’s share in the total surplus of a match. For low inflation rates, the gains from trade that accrue to the buyers increase, so they search harder to generate trades. Furthermore, when this happens, the effect of a change in inflation on aggregate output is ambiguous. For some parameterizations the level of output increases with the rate of inflation. Interestingly, we find that the effect of inflation on output can be nonmonotonic, just as suggested by the recent empirical evidence surveyed by Bullard (1999). Inflating in excess of the Friedman rule may help raise output, but it always reduces welfare.

The rest of the article is organized as follows. Section 2 lays out the environment. Section 3 analyzes the model under ex post bargaining with undirected search. The model with ex ante price posting and directed search is studied in Section 4. Section 5 concludes. The Appendix contains all the proofs.

2. THE MODEL

Time is discrete and the horizon infinite. Each period is divided into two sub-periods. There are two types of nonstorable consumption goods: search goods (produced and consumed in the first subperiod) and general goods (produced and consumed in the second subperiod). The economy is populated by a set $A_b \subseteq [0, 1]$ with mass $\mu_b$ of agents we call buyers and a set $A_s \subseteq [0, 1]$ with mass $\mu_s$ of sellers. All agents are infinitely lived. Buyers and sellers differ in their preferences and production possibilities. During the second subperiod both have the ability to produce and wish to consume. But in the first subperiod, buyers want to consume but cannot produce, whereas sellers are able to produce but do not wish to consume. This double coincidence of wants problem in the first subperiod is what generates an essential role for money. We describe the preferences of buyers and sellers in detail below.

There is an intrinsically useless, perfectly divisible, and storable asset called money. We use $M_t$ to denote the quantity of money in the first subperiod of period $t$. Let the distributions of money across buyers and sellers at the beginning of the first subperiod be $F_b^s$ and $F_s^s$, respectively. The gross growth rate of the money supply is constant over time and equal to $\gamma$; that is, $M_{t+1} = \gamma M_t$. New money is injected, or withdrawn if $\gamma < 1$, by lump-sum transfers, or taxes. These transfers take place during the second subperiod, and for simplicity we specify that they go only to buyers.
The market structure differs across subperiods. In the first subperiod, trade takes place in a decentralized market where agents are matched and trade bilaterally. In the second subperiod there is a centralized market where agents can trade general goods and money. All agents in this market are price takers, and the relative price of money in terms of the general good, \( \phi_t \), adjusts to clear the market.

Search frictions are modeled by an aggregate matching function \( \zeta(\bar{\mu}_b, \mu_s) \), where

\[
\bar{e} = \frac{\int_{A_b} e_i \, di}{\mu_b}
\]

is the average search intensity of buyers, and \( e_i \) denotes the search intensity of buyer \( i \). Below, when no confusion may arise, we will often use \( e \) to denote \( e_i \).

Assume \( \zeta \) is homogeneous of degree one, twice continuously differentiable, strictly increasing, and strictly concave with respect to each argument. Also, suppose \( \zeta(0, \mu_s) = \zeta(\bar{\mu}_b, 0) = 0 \), and \( \zeta(\bar{\mu}_b, \mu_s) \leq \min(\mu_b, \mu_s) \) for any \( \bar{e} \geq 0 \). When we let \( \theta \equiv \frac{\mu_s}{\bar{\mu}_b} \), an individual buyer’s meeting probability is \( \alpha_b = \frac{\epsilon(\bar{\mu}_b, \mu_s)}{\epsilon(\bar{\mu}_b, \bar{\mu}_b)} = \epsilon(1, \theta) \).

To make the notation more compact, we will let \( \alpha \equiv \zeta(1, \theta) \) and write an individual buyer’s meeting probability as \( \alpha_b = e \alpha \). We assume that \( \alpha \in [0, 1] \) for any \( \theta \geq 0 \) and that \( \lim_{\theta \to \infty} \alpha = 1 \). Similarly, the meeting probability of a seller is \( \alpha_s = \alpha/\theta \), and we assume that \( \lim_{\theta \to 0} \alpha_s = 1 \). The dependence of \( \alpha_b \) and \( \alpha_s \) on \( \theta \) reflects standard search externalities. In what follows we normalize the population sizes to one: \( \mu_b = \mu_s = 1 \).

The instantaneous utility function of a buyer is

\[
U_b(x, y, q, e, \varepsilon) = \varepsilon u(q) - \psi(e) + x - y
\]

where \( q \) is consumption in the first subperiod, \( x \) is the quantity consumed, and \( y \) the quantity produced in the second subperiod. Given the linear preferences over \( x \) and \( y \), it is not worth producing general goods for oneself. In one of the formulations buyers receive a match-idiosyncratic utility shock \( \varepsilon \) in the first subperiod. Shocks \( \varepsilon_i \) are independent and identically distributed with cumulative distribution \( G(\varepsilon) \) on \([0, 1]\). The realization of the preference shock is observed by both the buyer and the seller. We assume \( u(0) = 0 \), \( u'(0) = \infty \), \( u'(q) > 0 \), and \( u''(q) < 0 \) for \( q > 0 \). Finally, the utility cost for a buyer to search with intensity \( e \) is \( \psi(e) \). We assume that \( \psi \) satisfies \( \psi(e) \in [0, +\infty) \) for all \( e \in [0, 1] \).

---

4 We will explore two alternative specifications of the search market. In Section 3 we assume buyers and sellers are randomly matched (that is, search is undirected). In Section 4 we consider a search market in which search can be at least partially directed: Each seller can locate herself in a distinct “submarket” by credibly posting terms of trade, and each buyer can direct his search toward a particular submarket but contacts potential trading partners at random within the submarket.

5 For example, if the matching function is \( \zeta(\bar{\mu}_b, \mu_s) = \bar{\mu}_b[1 - \exp(-\eta \theta)] \) for some \( \eta > 0 \), then \( \alpha = 1 - \exp(-\eta \theta) \). It is easy to check that \( \alpha \in [0, 1] \) for all \( \theta \), as well as that \( \lim_{\theta \to \infty} \alpha = \lim_{\theta \to 0} \alpha_s = 1 \).

6 One can make the model more general by assuming quasi-linear preferences \( v(x) = x \). Agents would then consume \( x^* \) in the second subperiod, where \( x^* \) satisfies \( v'(x^*) = 1 \).
for \( e > 1, \psi' > 0, \psi'' > 0, \psi(0) = \psi'(0) = 0, \) and \( \lim_{e \to 1} \psi'(e) = +\infty. \) A buyer’s lifetime expected utility is \( E_0 \sum_{t=0}^{\infty} \beta^t U^b(x_t, y_t, q_t, e_t, \epsilon_t) \), where \( E_0 \) is the expectation operator conditional on all information available at \( t = 0. \) The discount factor \( \beta \in (0, 1) \) is the same for all agents and assumed to be smaller than \( \gamma \) throughout the analysis. The instantaneous utility function of a seller is

\[
U^s(x, y, q) = -c(q) + x - y
\]

We let \( c(0) = c'(0) = 0, c'(q) > 0, \) and \( c''(q) \geq 0 \) for \( q > 0, \) and we let \( c(q) = u(q) \) for some \( q > 0. \) Lifetime utility for a seller is given by \( E_0 \sum_{t=0}^{\infty} \beta^t U^s(x_t, y_t, q_t) \). We assume that agents are anonymous and that there are no forms of commitment or public memory that would render money inessential.

In the centralized market of a given period \( t, \) buyers will want to acquire cash to be able to buy their consumption good in the decentralized market of the following period \( t + 1. \) Since agents are anonymous and they cannot commit, trade must be quid pro quo in the decentralized market. Thus buyers produce during the second subperiod in order to carry some money into the decentralized market. Sellers, on the other hand, never wish to consume in the first subperiod, so they will have no use for cash in the decentralized market. Thus, in the centralized market of period \( t, \) they will sell off any cash they may have accumulated. Sellers accept money in the decentralized market because they anticipate that they will be able to trade it for goods in the second subperiod, in the next meeting of the centralized market. These observations imply that at the beginning of every period, all the cash is being held by buyers. After the round of decentralized trading, some of the money will end up being held by sellers. These sellers will then sell those balances off in the following centralized trading session, and so on.

Throughout the article we take production in the decentralized market to be the relevant measure of aggregate output:

\[
Y = \zeta (\bar{\epsilon} \mu_b, \mu_s) \int_0^{\bar{\epsilon}} q(\epsilon) dG(\epsilon)
\]

where \( q(\epsilon) \) is the quantity traded in a meeting when the buyer receives an \( \epsilon \) preference shock. We define velocity as the nominal value of transactions in the decentralized market divided by the money stock:

\[
V = \frac{\zeta (\bar{\epsilon} \mu_b, \mu_s) \int_0^{\bar{\epsilon}} d(\epsilon) dG(\epsilon)}{M}
\]

where \( d(\epsilon) \) is the amount of money a buyer spends in a decentralized trade when he receives a preference shock \( \epsilon. \)

In the following sections we will be introducing several notions of equilibrium. To assess the efficiency properties of the equilibrium allocations,

---

7 This last condition guarantees that the buyer’s optimal choice of intensity will be in \( [0, 1] \). Together with the assumption \( \sigma (\theta) \in [0, 1] \) for any \( \theta \geq 0, \) this guarantees that \( \sigma_b(\epsilon) \in [0, 1] \) in any equilibrium.
here we consider the social planner’s problem. The planner chooses a non-
negative sequence \( \{e_t, (x^t_i, y^t_i, q^t_i)\}_{t=0}^{\infty} \) in order to maximize \( \sum_{t=0}^{\infty} \beta^t \times \left[ U^b(x^b_t, y^b_t, q^b_t, e_t, e_t) + U^s(x^s_t, y^s_t, q^s_t) \right] \), subject to \( x^b_t + x^s_t \leq y^b_t + y^s_t \) and \( q^b_t \leq q^s_t \). Using (1), (2), and the feasibility constraints at equality and writing out the ex-
pectations operator explicitly, the planner’s objective can be rewritten as

\[
\sum_{t=0}^{\infty} \beta^t \left\{ -\psi(e_t) + e_t \alpha(1/\bar{\varepsilon}_t) \int_0^1 \left[ \varepsilon u[q_t(\varepsilon)] - c[q_t(\varepsilon)] \right] dG(\varepsilon) \right\}
\]

Optimality requires \( q_t(\varepsilon) = q(\varepsilon) \) for all \( t \), where \( q(\varepsilon) \) satisfies

\[
\frac{\varepsilon u'[q(\varepsilon)]}{c'[q(\varepsilon)]} = 1, \quad \forall \varepsilon \in [0, 1]
\]

For given \( \varepsilon \), let \( q^* \) denote the socially efficient quantity traded that satisfies (5). Note that \( q^*_0 = 0 \) and \( \frac{\partial q^*}{\partial \varepsilon} = \frac{-u'}{u'' - c} > 0 \). The socially efficient choice of search intensity for buyer \( i \), namely, \( \bar{\varepsilon}^* \) satisfies

\[
\psi'(\bar{\varepsilon}^*) = \alpha(1/\bar{\varepsilon}^*) \eta(1/\bar{\varepsilon}^*) S^*
\]

where \( \eta(1/\bar{\varepsilon}) = 1 - \frac{(1/\bar{\varepsilon})u'(1/\bar{\varepsilon})}{u(1/\bar{\varepsilon})} \) and \( S^* = \int_0^1 [\varepsilon u(q^*_\varepsilon) - c(q^*_\varepsilon)] dG(\varepsilon) \). Note that
\( \eta(1/\bar{\varepsilon}) \in (0, 1) \) is the elasticity of the matching function with respect to \( \bar{\varepsilon} \). When we let \( Q^* = \int_0^1 q^*_\varepsilon dG(\varepsilon) \), aggregate output (in the decentralized market) along the optimal path is

\[
Y^* = \zeta(\bar{\varepsilon}^* \mu_b, \mu_s) Q^*
\]

3. BARGAINING

This section investigates the output and welfare effects of inflation when prices are determined according to a simple bargaining protocol, that is, buyers make take-it-or-leave-it offers. This pricing mechanism is the standard benchmark in the search-theoretic literature on monetary exchange since Shi (1995) and Trejos and Wright (1995). At the end of the section we also discuss the implications of giving sellers some bargaining power.

First consider a buyer’s problem in the first subperiod. Given his choice of search intensity, \( e \), he contacts a seller with probability \( \alpha e \). Once matched, the buyer receives \( q(m_b, m_s, e) \) in exchange for \( d(m_b, m_s, e) \) dollars. The notation reflects that, in general, the terms of trade \( (q, d) \) may depend on the money holdings of the buyer and the seller, \( m_b \) and \( m_s \), as well as on the buyer’s preference shock, \( \varepsilon \). For convenience, we suppress the time subscript \( t \) and shorten the subscript \( t + 1 \) to +1, \( t - 1 \) to -1, and so on. Let \( V^b(m, \phi) \) be the value function of a buyer with \( m \) dollars upon entering the decentralized market when the price of money in terms of general goods in the following centralized market is \( \phi \). Similarly, let \( W^b(m, \phi) \) be the value function of a buyer with \( m \) dollars upon entering the centralized
market in a period where the price of money in terms of general goods is \( \phi \). The value functions satisfy

\[
V^b(m, \phi) = \max_e \left\{ -\psi(e) + (1 - \alpha e) W^b(m, \phi) \\
+ \alpha e \int \{ \varepsilon u[q(m, m_s, \varepsilon)] + W^b[m - d(m, m_s, \varepsilon), \phi] \} dF^s(m_s) dG(\varepsilon) \right\}
\]

and

\[
W^b(m, \phi) = \max_{\hat{m} \geq 0} \left[ x - y + \beta V^b(\hat{m}, \phi + 1) \right]
\]

s.t. \( x + \phi \hat{m} = y + \phi(m + T) \)

where \( T = M_{t+1} - M \) is the monetary transfer the buyer receives and \( \hat{m} \) is the money he chooses to take into the decentralized market of the following subperiod. Using the constraint to substitute for \( x - y \), we have

\[
W^b(m, \phi) = \phi(m + T) + \max_{\hat{m} \geq 0} \left[ \beta V^b(\hat{m}, \phi + 1) - \phi \hat{m} \right]
\]

From (9), we see that the maximizing choice of \( \hat{m} \) is independent of \( m \) and that \( W^b(m + d, \phi) - W^b(m, \phi) = \phi d \). This second observation implies

\[
V^b(m, \phi) = W^b(m, \phi) + \max_{\varepsilon \geq 0} \left\{ -\psi(e) + \alpha e \int \{ \varepsilon u[q(m, m_s, \varepsilon)] \} \right\}
- \phi d(m, m_s, \varepsilon) \right\} dH^s(m_s, \varepsilon)
\]

Let \( V^s(m, \phi) \) and \( W^s(m, \phi) \) be the corresponding value functions for sellers. The value function of a seller entering the first subperiod with \( m \) dollars satisfies

\[
V^s(m, \phi) = (1 - \alpha \bar{e}) W^s(m, \phi)
+ \alpha \bar{e} \int \{-c[q(m_b, m, \varepsilon)] + W^s[m + d(m_b, m, \varepsilon), \phi] \} dF^b(m_b) dG(\varepsilon)
\]

The value function of a seller entering the centralized market with \( m \) dollars satisfies

\[
W^s(m, \phi) = \phi m + \max_{\hat{m} \geq 0} \left[ \beta V^s(\hat{m}, \phi + 1) - \phi \hat{m} \right]
\]

Again, note that \( \hat{m} \) is independent of \( m \) and that \( W^s(m, \phi) = \phi m + W^s(0, \phi) \), so
(12) \[ V^*(m, \phi) = W^*(m, \phi) + \alpha \tilde{e} \int \{ -c(q(m_b, m, \varepsilon)) + \phi d(m_b, m, \varepsilon) \} dH_b(m_b, \varepsilon) \]

3.1. Prices. Prices in the decentralized market are determined by take-it-or-leave-it offers by buyers. Consider a match between a buyer and a seller where the buyer holds \( m \) units of money and his preference shock is \( \varepsilon \). The terms of trade \((q, d)\) satisfy

\[
\max_{q, d \leq m} \left[ \varepsilon u(q) - \phi d \right] \quad \text{s.t.} \quad -c(q) + \phi d \geq 0
\]

The solution is \( q = q^*_\varepsilon \) and \( d = \frac{c(q^*_\varepsilon)}{\phi} \) if \( c(q^*_\varepsilon) \leq \phi m \), or \( c(q) = \phi m \) and \( d = m \) otherwise. In either case the pair \((q, d)\) is independent of the seller’s money holdings. Moreover, the solution depends only on the buyer’s real money balances \( z = \phi m \), so we write it as

\[
q_\varepsilon(z) = \begin{cases} 
q^*_\varepsilon & \text{if } z \geq c(q^*_\varepsilon) \\
\hat{q}(z) & \text{otherwise}
\end{cases}
\]

where \( \hat{q}(z) \) is the \( q \) that solves \( c(q) = z \). Since \( q^*_\varepsilon \) is an increasing function of \( \varepsilon \), there is a threshold \( R(z) \) such that \( q_\varepsilon(z) = q^*_\varepsilon \) if \( \varepsilon \leq R(z) \) and \( q_\varepsilon(z) = \hat{q}(z) \) for all \( \varepsilon > R(z) \). In other words, \( R(z) \) is the \( \varepsilon \)-draw that renders the buyer’s cash constraint in the bargaining just binding. The threshold \( R \) satisfies \( c(q^*_R) = z \) or equivalently \( Ru'[\hat{q}(z)] = c'[\hat{q}(z)] \). For future reference, note that \( R(0) = 0 \), \( R[c(q^*_1)] = 1 \), and \( R'(z) > 0 \) for \( z \in [0, c(q^*_1)) \). Also,

\[
q'_\varepsilon(z) = \begin{cases} 
\frac{-1}{c'(q)} & \text{if } \varepsilon > R(z) \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
q''_\varepsilon(z) = \begin{cases} 
\frac{-c'(q)}{[c'(q)]^2} & \text{if } \varepsilon > R(z) \\
0 & \text{otherwise}
\end{cases}
\]

3.2. Equilibrium. Let \( Z_t = \phi_t M_t \) denote aggregate real balances. Hereafter we specialize the analysis to stationary equilibria with constant aggregate real balances, that is, with \( Z_{t+1} = Z \) for all \( t \), which implies \( \phi/\phi_{t+1} = \gamma \). Define \( \tau \equiv \phi T \), the real transfer received by the buyer, and

\[
S(z) = \int_0^1 \{ \varepsilon u[q_\varepsilon(z)] - c[q_\varepsilon(z)] \} dG(\varepsilon)
\]

the expected surplus of a match. The following lemma gives the closed form expressions for the value functions rewritten in terms of real balances.
Lemma 1.

(a) Sellers do not carry cash into the decentralized market. Moreover,

\[ V^s(z) = z \quad \text{and} \quad W^s(z) = z \]

(b) Let \( g(z) \equiv \tau + \max_e [\alpha e S(z) - \psi(e)] \). The value functions for the buyer are

\[ V^b(z) = \frac{B}{1 - \beta} + g(z) + z \]

and

\[ W^b(z) = \tau + z + \max_z [\beta g(z) + (\beta - \gamma) z] + \frac{\beta}{1 - \beta} B \]

where \( B \equiv \max_z [\beta g(z) + (\beta - \gamma) z] \).

Buyers choose real balances in the centralized market and search intensity upon entering the decentralized market. From (17), their decision problem is

\[ \max_{e, z} \left[ \alpha e S(z) - \psi(e) + \left( \frac{\beta - \gamma}{\beta} \right) z \right] \]

The following lemma establishes some properties of the individual decision problem faced by buyers in a stationary equilibrium.

Lemma 2.

(a) The buyer’s optimal choice of search intensity as a function of his real balances, \( e(z) \), is characterized by

\[ \psi'(e) = \alpha S(z) \]

Moreover, \( e(z) \in [0, 1) \), \( e'(z) > 0 \) for all \( z \in [0, c(q^*_1)) \), and \( e'(z) = 0 \) for all \( z \geq c(q^*_1) \)

(b) The (set of) optimal choice(s) of real balances \( D(\bar{e}, \gamma) = \arg \max_z [\beta g(z) + (\beta - \gamma) z] \) is nonempty, compact-valued, and upper-hemi continuous. In addition, \( D(\bar{e}, \gamma) \subseteq [0, c(q^*_1)) \) is decreasing in \( (\bar{e}, \gamma) \).

Part (a) shows that the search effort is increasing in the buyer’s expected surplus from trade, \( S(z) \). Since the trade surplus is increasing in real balances, there is a complementarity between money demand and search intensity and the buyer’s maximization problem need not be concave. Part (b) provides a complete characterization of the set of money demands that maximize the buyer’s problem. The largest solution will be strictly below \( c(q^*_1) \), namely, the amount of cash that renders
the buyer’s cash constraint slack for every possible realization of the preference shock. As inflation (γ) or congestion in the decentralized market (\(\bar{e}\)) increases, buyers reduce both their real balances and their search effort. Even though the buyer’s problem may have multiple solutions, our comparative static results do not rely on an arbitrary selection rule: According to part (b), the buyer’s cash constraint slack for every possible realization of the preference shock. As inflation (γ) or congestion in the decentralized market (\(\bar{e}\)) increases, buyers reduce both their real balances and their search effort. Even though the buyer’s problem may have multiple solutions, our comparative static results do not rely on an arbitrary selection rule: According to part (b), any selection from among multiple solutions for \(\bar{e}\) is used in Lemma 2 only to establish the strict monotonicity of the maximizers in the very last step of the proof of part (b).

**Definition 1.** Given a money supply process \(M_t+1/M_t = \gamma\), a stationary monetary equilibrium is a collection \(((z_i, e_i)_{i \in [0,1]}, \bar{e})\) and a sequence \(\{\phi_t, Z_t\}\) with \(Z_t = Z\) for all \(t\), such that

(B1) given \(\bar{e}, z_i \in D(\bar{e}, \gamma)\), and \(e_i = e(z_i)\) is given in (19), for every buyer \(i \in [0, 1]

(B2) \(\int_{[0,1]} e_i \, di = \bar{e}\)

(B3) \(\int_{[0,1]} z_i \, di = Z\)

(B4) \(\phi_t = \frac{Z}{M_t}\).

**Proposition 1.** There exists a stationary monetary equilibrium if

\[
\max_{\bar{e}, \bar{z}} \{\beta [eS(z) - \psi(e)] + (\beta - \gamma)z\} > 0
\]

The average equilibrium search intensity \(\bar{e}\) is uniquely determined, and it is decreasing in \(\gamma\).

In order to study the effects of inflation on output and welfare, we focus on the case where the solution to the buyer’s problem is unique.\(^8\) Using (3) and (7), the output is

\[
\frac{\int_{[0,1]} \{\epsilon u'[q(z)] / c[q(z)] - 1\} \, dG(e) = \frac{\gamma - \beta}{e\alpha\beta}}
\]

is used in Lemma 2 only to establish the strict monotonicity of the maximizers in the very last step of the proof of part (b).

\(^8\) The correspondence \(D(\bar{e}, \gamma)\) is decreasing in \((\bar{e}, \gamma)\) if \((\bar{e}', \gamma') \succ (\bar{e}, \gamma)\), \(\bar{e}' \in D(\bar{e}, \gamma)\), and \(\bar{e}' \in D(\bar{e}', \gamma')\), then \(\bar{z}' \leq \bar{z}'\) if \(\bar{z}'' > 0\). Here \(\succeq\) denotes the pairwise ordering relation on \(\mathbb{R}\). That is, for any \((x', y'), (x'', y'') \in \mathbb{R}\), we write \((x', y') \succeq (x'', y'')\) if \(x' \geq x''\) and \(y' \geq y''\). We write \((y'', \bar{e}'') \prec (\bar{e}', \bar{e}')\) if \((y'', \bar{e}'') \preceq (\bar{e}', \bar{e}')\) and \((y'', \bar{e}'') \neq (\bar{e}', \bar{e}')\).

\(^9\) In general, the buyer’s problem (18) need not be strictly concave, allowing for the possibility of multiple solutions. When this is the case, there may be equilibria differing only in the fractions of buyers choosing each solution. There are several ways to ensure that the buyer’s problem has a unique solution. We could focus on inflation rates close to the Friedman rule (\(\gamma = \beta\)). One can also make assumptions on primitives so that there is a unique positive solution to the first-order necessary conditions. For example, it is sufficient to assume \(\psi'' > 0\) and \(S''(z)S'(z) - 2S'(z)^2 < 0\).
\begin{equation}
Y = \zeta(\bar{e} \mu_b, \mu_s) \left\{ Q^* - \int_{\bar{R}(z)}^{1} \left[ q^*_e - \hat{q}(z) \right] dG(\varepsilon) \right\}
\end{equation}

where \( \hat{q}(z) = c^{-1}(z) \) and \( R(z) = c'[\hat{q}(z)]/u'[\hat{q}(z)] \). We study the effects of inflation on welfare from the perspective of a buyer upon entering the decentralized market. We ignore sellers since their expected utility is zero in equilibrium. Along the equilibrium path, we have

\[(1 - \beta) V^b(z) = \alpha e(z) S(z) - \psi[e(z)]\]

where \( z \in \arg \max \beta g(\hat{z}) + (\beta - \gamma) \hat{z} \) and \( e(z) \) is implicitly defined by (19). Note that this welfare criterion is essentially the same one used by the planner.

**Proposition 2.** Equilibrium is socially inefficient. Furthermore, aggregate output and welfare are decreasing with inflation.

3.3. **Discussion.** A surprising feature of the model with ex post bargaining is that buyers do not increase their search intensity to spend their cash faster as the rate of anticipated inflation increases. In fact, what happens is that real balances fall, and since real balances and search effort are complements (recall part (a) of Lemma 2), buyers reduce their intensity of search. Since they can get less for their cash, buyers choose not to search as hard. A direct consequence of this is that output declines as the inflation rate increases: Agents produce less in each trade (intensive margin) due to the lower real balances, and fewer trades take place (extensive margin) due to the reduction in search intensity. All this has interesting implications for the velocity of money, an observable variable, which is closely related to search intensity according to the theory.

Using (4) and the bargaining solution, we can write velocity as

\[
\bar{e}\alpha(1/\bar{\varepsilon}) \frac{\int_0^{R(z)} \frac{c(q^*_e)}{\phi} dG(\varepsilon) + \int_{R(z)}^{1} M dG(\varepsilon)}{M}
\]

Letting \( v^* = \int_0^{R} \frac{c(q^*_e)}{\phi} dG(\varepsilon) \), we can conveniently rewrite this expression as

\[
V = \bar{e}\alpha(1/\bar{\varepsilon}) [G(R)v^* + [1 - G(R)]]
\]

This shows aggregate velocity as a weighted average of one, velocity in all trades for which the buyer’s cash constraint binds, and \( v^* \), the average velocity of money in those trades with slack cash constraints. Since \( v^* < 1 \), there are high and low velocity trades, namely, those with binding and slack constraints, respectively. The effect of inflation on velocity cannot be signed in general. To this end, the first thing to note is that if there were no idiosyncratic preference shock, that is, if \( \varepsilon = 1 \) with probability one, then the cash constraint would be binding in every trade, implying \( V = \bar{e}\alpha(1/\bar{\varepsilon}) = \zeta(\bar{e}, 1) \). In this case velocity would be increasing
in average search intensity and hence (by Proposition 1) always decreasing in the rate of inflation. On the other hand, for fixed search intensity, note that velocity rises when real balances fall.

This happens for two reasons. First, when real balances fall, velocity increases in those trades with slack cash constraints: Since real balances are lower, a buyer needs to spend a larger fraction of his cash holdings to buy the first best quantity. And second, \( R \) falls when \( z \) falls, and this increases the fraction of high velocity trades (that is, the fraction of trades with binding constraints). On the one hand, inflation reduces real balances, and this makes buyers search less intensively. This is the (negative) extensive-margin effect of inflation on velocity. On the other hand, higher inflation causes velocity to rise in low velocity trades, and the proportion of these trades fall. This is the (positive) intensive-margin effect of inflation on velocity. The extensive and intensive margins move in opposite directions, and the net effect is ambiguous in general.

Comparing (5) to (20) and (6) to (19), we see that the equilibrium with ex post bargaining is inefficient. Conditions (5) and (20) coincide if and only if \( \gamma = \beta \). That is, the equilibrium achieves efficiency along the intensive margin under the Friedman rule (agents exchange the efficient quantities in all trades). For the equilibrium to also achieve efficiency along the extensive margin (that is, for search intensities to coincide), in addition we would need to have \( \eta(1/e) = 1 \). But since \( \eta(1/e) \in (0, 1) \) for all \( e \), buyers search excessively in the equilibrium.\(^{10}\)

The normative and positive implications of the model are not in line with the conventional wisdom articulated by Lucas (2000). Lucas' view is that agents invest additional resources to get away from cash as the inflation rate increases and that these resources are part of the welfare cost of inflation. But in the model with bargaining, agents always reduce their search effort if the inflation rate increases. If the buyer’s bargaining power is large, then search costs are inefficiently high under the Friedman rule, and higher inflation reduces those search costs, bringing the extensive margin in the equilibrium closer to the efficient benchmark. All else constant, this effect has a positive impact on welfare. However, in equilibrium, this partial effect is outweighed by the distortion in the intensive margin caused by the reduction in real money balances. As a result, welfare unambiguously decreases with the rate of inflation.

4. COMPETITIVE PRICE POSTING

In this section, we depart from the random matching assumption of the previous section and set up a decentralized market in which search can be partially directed. The sequencing is as follows. First, each buyer locates himself in one of possibly many distinct submarkets by credibly posting terms of trade at which he stands ready to trade with any seller he contacts.\(^{11}\) Each seller then directs her search

\(^{10}\) In Lagos and Rocheteau (2004) we verify that the main results generalize to the case of an arbitrary bargaining power.

\(^{11}\) This notion of equilibrium requires that the agents who announce the terms of trade be able to commit to their announcement. As is well known, money is inessential in environments in which
toward a particular submarket, and once there, meets potential trading partners at random within the submarket. As before, the extent of the meeting frictions will depend on the number of sellers posting the same terms of trade and the number of buyers searching for these sellers. Both buyers and sellers form rational expectations about the market tightness on the different submarkets.

We define a submarket as a subset of buyers who post the same contract and randomly search for sellers with whom to trade at the terms of trade specified by this contract. Formally, a submarket is a triple $S = (q_{ms}, d_{ms}, \theta)$ where $[q_{ms}, d_{ms}]$ is the menu posted by buyers and at which both buyers and sellers commit to trade, if they choose to trade, and $\theta$ is the tightness of the submarket $S$ that is generated by $(q, d)$. Note that terms of trade $(q, d)$ may in principle be contingent on sellers’ money balances, although sellers will not hold money in equilibrium. The choice of menu also depends on the buyer’s own money holdings $mb$. To make the notation consistent with that of the previous section, we use $q_{mb, ms}$ to denote the quantity produced by the seller and consumed by the buyer and $d_{mb, ms}$ to denote the money transfer from the buyer to the seller when their money balances are $mb$ and $ms$, respectively. We use $\theta(S)$ to denote the ratio of the measure of sellers to the measure of effective buyers searching in submarket $S$. Through the matching probabilities $\alpha_b = e^\zeta[1, \theta(s)]$ and $\alpha_s = \zeta[1/\theta(s), 1]$, the “(sub)market tightness” $\theta(s)$ determines the expected waiting times to complete a trade for buyers and sellers in submarket $s$. If there is a subset $A_b(s) \subseteq A_b$ of buyers and a subset $A_s(s) \subseteq A_s$ of sellers in submarket $s$, with masses $\mu_b(s)$ and $\mu_s(s)$, respectively, then $\theta(s) = \mu_s(s)/\mu_b(s)$. In this section we let $\alpha[\theta(s)] \equiv \zeta[1, \theta(s)]$, so we write $\alpha_b = e\alpha[\theta(s)]$ and $\alpha_s = \alpha[\theta(s)]/\theta(s)$. (We will omit the argument of $\alpha$ and $\alpha_s$ when no confusion may result.)

The problem of a seller holding $m$ dollars and searching for a buyer in submarket $s = (q, d, \theta)$ when the price of money in terms of general goods in the next centralized market is $\phi$ is summarized by the following Bellman equation:

$$V^s(m, s, \phi) = \alpha_s \left\{ -c[q_{mb, m}] + W^s[m + d_{mb, m}, \phi] + (1 - \alpha_s)W^s(m, \phi) \right\}$$

where $m_b$ is the money holdings of buyers in submarket $s$. The value of a seller entering the centralized market with $m$ dollars when the real price of a dollar is $\phi$ is
where $S$ is the set of all submarkets that are active, $\hat{m} \geq 0$, and $\hat{W}^s(m, s, \phi)$ satisfies

$$\hat{W}^s(m, s, \phi) = \phi m + \max_{\hat{m}} \left[ \beta V^b(\hat{m}, s, \phi+1) - \phi \hat{m} \right]$$

Equations (22)–(23) state that sellers choose their money holdings and direct their search toward the submarket that maximizes their expected utility.

The value of search to a buyer holding $m$ units of money who has chosen to post terms of trade $(q, d)$ and hence to be in submarket $s = (q, d, \theta)$ is given by

$$V^b(m, s, \phi) = \max_e \left\{ -\psi(e) + (1 - \alpha e) W^b(m, \phi) + \alpha e \int \{u[q(m, \tilde{m})] + W^b[m - d(m, \tilde{m}), \phi]\} dF^s(\tilde{m}) \right\}$$

where $F^s$ is the distribution of money holdings among sellers searching in submarket $s$. The value of a buyer entering the centralized market with $m$ dollars is

$$W^b(m, \phi) = \phi (m + T) + \max_{\hat{m}, s} \left[ \beta V^b(\hat{m}, s, \phi+1) - \phi \hat{m} \right]$$

subject to $\theta \geq 0$, $\hat{m} \geq 0$, $d(\hat{m}, m') \leq \hat{m}$ for all $m'$, and

$$\hat{W}^s(0, s, \phi) \geq W^s(0, \phi)$$

The left-hand side of (25) is the expected utility of a seller who chooses to search in submarket $s$, and the right-hand side is the expected utility from searching in the submarket that maximizes her expected utility. Note that condition (25) ensures that sellers who carry no money are just as well off searching in submarket $s$ as elsewhere. But $\hat{W}^s(m', s, \phi) - W^s(m', \phi)$ is independent of $m'$, so if (25) holds for $m' = 0$ (which will be the case in equilibrium), then it holds for any $m'$. This condition defines an implicit relationship between $\theta$, the terms of trade $(q, d)$, and $W^s(0, \phi)$, and says that a submarket receives sellers only if it offers them the maximum level of expected utility they can achieve by searching in another submarket. Submarkets for which (25) does not hold do not attract sellers and therefore are inactive.

4.1. Prices. As in the previous sections, we specialize the analysis to stationary equilibria in which $M_{+1}/M = \phi/\phi_{+1} = \gamma > \beta$. The terms of trade $(q, d)$ are posted by buyers, so to see how they are determined we take a closer look at their maximization problem in the decentralized market. First note that since $W^b(m, \phi) = W^b(0, \phi) + \phi m$, the value function $V^b$ can be written as
\[ V^b(m, s, \phi) = \max_e \left[ \alpha e S_b(m, F^s, \phi) - \psi(e) + W^b(0, \phi) + \phi m \right] \]

where \( S_b(m, F^s, \phi) \equiv \int [u(q(m, \hat{m})) - \phi d(m, \hat{m})] dF^s(\hat{m}) \) is the buyer’s expected surplus from trade, and \( e \) is constrained to be nonnegative. Substituting this expression into (24), we get

\[ W^b(m, \phi) = \phi(m + T) + \beta W^b(0, \phi_{+1}) \]
\[ + \beta \max_{\hat{m}, s, e} \left[ \alpha e S_b(\hat{m}, F^s, \phi_{+1}) - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) \phi_{+1} \hat{m} \right] \]

where the maximization is subject to \( \theta \geq 0, \hat{m} \geq 0, e \geq 0, d(\hat{m}, \tilde{m}) \leq \hat{m} \) for all \( \hat{m} \), and (25). Similarly for sellers, \( W^s(m, \phi) = W^s(0, \phi) + \phi m \), and therefore

\[ V^s(m, s, \phi) = \alpha_s S_s(m_b, m, \phi) + \phi m + W^s(0, \phi) \]

where \( S_s(m_b, m, \phi) \equiv \phi d(m_b, m) - c[q(m_b, m)] \) is the seller’s surplus. Substituting this expression in (23) yields

\[ \hat{W}^s(m, \phi) = \phi m + \beta W^s(0, \phi_{+1}) + \beta \max_{\hat{m}} \left[ \alpha_s S_s(m_b, \hat{m}, \phi_{+1}) - \left( \frac{\gamma}{\beta} - 1 \right) \phi_{+1} \hat{m} \right] \]

Therefore the buyer’s problem in the centralized market is

\[ \max_{\hat{m}, s, e} \left[ \alpha e S_b(\hat{m}, F^s, \phi_{+1}) - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) \phi_{+1} \hat{m} \right] \]

with \( s \equiv [q(\hat{m}, \bar{m}), d(\hat{m}, \bar{m}), \theta] \) and subject to \( \theta \geq 0, \hat{m} \geq 0, e \geq 0, d(\hat{m}, \tilde{m}) \leq \hat{m} \) for all \( \hat{m} \), and

\[ \max_{\hat{m}} \left[ \alpha_s S_s(m_b, \hat{m}, \phi_{+1}) - \left( \frac{\gamma}{\beta} - 1 \right) \phi_{+1} \hat{m} \right] \geq U^s \]

where

\[ U^s \equiv \max_{s \in S, \hat{m}} \left[ \alpha_s S_s(m_b, \hat{m}, \phi_{+1}) - \left( \frac{\gamma}{\beta} - 1 \right) \phi_{+1} \hat{m} \right] \]

So, the buyer maximizes his expected surplus in the decentralized market net of the search cost and the cost of carrying real balances and subject to the constraint that the contract he chooses satisfies sellers’ participation constraint. The following lemma summarizes the main properties of the solutions to the individual decision problems.

**Lemma 3.**

(a) *Sellers carry no cash into the decentralized market.*
(b) The buyer’s optimal choices are described by the following correspondence:

\[ \Upsilon(U^s) = \arg \max_{q, z, e, \theta} \left\{ e\alpha(\theta) \left[ u(q) - z \right] - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) z \right\} \]

subject to \( q, z, e \geq 0 \), and \( \theta \geq 0 \) if \( \alpha_s(\theta) \left[ z - c(q) \right] \geq U^s \) or \( \theta = 0 \) otherwise.

(c) \( \Upsilon(U^s) \) is nonempty and upper-hemi continuous.

As before, \( m^s = 0 \). The buyer’s objective function in part (b) is identical to (18), that of a buyer under bargaining. We can think of the buyer’s problem as one of choosing search intensity and a simple contract \((q, z)\) specifying a level of real balances \( z \) that he will carry into the submarket and will hand over to any seller he meets in exchange for \( q \) units of output. The participation condition for sellers can be used to solve for the submarket tightness, \( \theta \), implied by any combination of terms of trade \((q, z)\) offered in the submarket and any market-wide utility level \( U^s \). Note that if the submarket is active, then the participation condition for sellers will bind at the optimum, so \( \alpha_s(\theta) \left[ z - c(q) \right] = U^s \).

4.2. Equilibrium.

**Definition 2.** Given a money supply process \( M_{t+1}/M = \gamma \), a stationary competitive price posting monetary equilibrium is a sequence \( \{ \phi_t, Z_t \} \) with \( Z_t = Z \) for all \( t \) and a collection \( \{ U^b, U^s, (q_i, z_i, e_i, \theta_i)_{i \in [0,1]} \} \) such that

1. \( \phi_t = \frac{Z}{M} \)
2. \( U^b = e_i a(\theta) \left[ u(q_i) - z_i \right] - \psi(e_i) - \left( \frac{\gamma}{\beta} - 1 \right) z_i \), for \((q_i, z_i, \theta_i, e_i) \in \Upsilon(U^s) \).

(C1) says that the allocation \((q_i, z_i, \theta_i, e_i)\) must be an equilibrium of the price posting game. Condition (C2) is an aggregate consistency condition: It states that the numbers of buyers and sellers each add up to one. We can also think of this condition as stating that the aggregate demand for sellers (the left-hand side of (C2)) must equal the aggregate supply, that is, one. Under this interpretation (C2) determines \( U^s \), the “price of a seller” that clears the market for sellers. (C3) is the clearing condition for the money market. (C4) maps real balances and the money supply into the (reciprocal of the) price level. (C5) simply defines \( U^b \) as the maximum value of the buyer’s program. The following proposition states that a monetary equilibrium exists as long as the inflation rate \( \gamma \) is not too high.

**Proposition 3.** A stationary competitive price posting monetary equilibrium exists if

\[ \max_{q, z, e} \left\{ e[u(q) - c(q)] - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) c(q) \right\} > 0 \]
Given \( U^s \), in any active submarket \( s \) the contract \((q, z)\), the tightness \( \theta \), and the buyer’s search intensity \( e \) must satisfy

\[
\frac{u'(q)}{c'(q)} = 1 + \frac{\gamma - \beta}{e\alpha(\theta)\beta}
\]

(26)

\[
\psi'(e) = \alpha(\theta)[u(q) - z]
\]

(27)

\[
u(q) - z = \frac{\eta(\theta)u'(q)}{\eta(\theta)u'(q) + [1 - \eta(\theta)]c'(q)}[u(q) - c(q)]
\]

(28)

\[
U^s = \alpha_s(\theta)[z - c(q)]
\]

(29)

Equations (26), (27), and (28) are the first-order conditions of the buyer’s problem for \( q, e, \) and \( \theta \), respectively. Just as in the previous section, (26) defines a relationship between \( q \) and the buyer’s probability of trade, \( e \). The higher the probability of trade, the larger the real balances the buyer will carry. And more real balances translate into larger quantities traded of the search good. Condition (27) defines the equilibrium level of search intensity as the one that equates the marginal cost of search to the marginal increase in the buyer’s expected gain from trade. Condition (27) comes from the choice of market tightness and defines the gain from trade that accrues to the buyer:

\[
u(q) - z = \omega(\theta, q) [u(q) - c(q)]
\]

where

\[
\omega(\theta, q) = \frac{\eta(\theta)u'(q)}{\eta(\theta)u'(q) + [1 - \eta(\theta)]c'(q)}
\]

is the buyer’s share of the total surplus.

In what follows we focus on equilibria, where the buyer’s problem has a unique solution and hence a single submarket is active. This, for example, is the case for inflation rates close to the Friedman rule (see Lemma 4 in the Appendix). When the buyer’s problem has a unique solution, \( (C2) \) simplifies to \( \theta = 1/e \), and \( (C3) \) to \( z = Z \). Conditions (26)–(28) can be solved for \((q, z, e)\): Given \( e \) we know \( \theta \), and (29) yields \( U^s \). This allows us to rewrite (26) and (27) as

\[
\frac{u'(q)}{c'(q)} = 1 + \frac{\gamma - \beta}{e\alpha(1/e)\beta}
\]

(30)

\[
\psi'(e) = \alpha(1/e)\frac{\eta(1/e)u'(q)}{\eta(1/e)u'(q) + [1 - \eta(1/e)]c'(q)}[u(q) - c(q)]
\]

(31)

From (30) it is immediate that the Friedman rule implies \( q = q^* \) and hence achieves efficiency along the intensive margin. Also, when \( \gamma = \beta \), (31) is identical to (6) for the case with \( \epsilon = 1 \) with probability one. Thus under competitive price posting, the Friedman rule also achieves efficiency along the extensive margin: It induces efficient search intensity. The following proposition summarizes the effects of inflation on the intensive margin, welfare, and search intensity under competitive price posting.
Proposition 4. The equilibrium is efficient under the Friedman rule. Inflation reduces \( q \), and deviations from the Friedman rule reduce welfare. For \( \gamma \) close to \( \beta \) an increase in \( \gamma \) increases search intensity, \( e \).

4.3. Discussion. The determination of the equilibrium is illustrated in Figure 1. The dotted lines represent the planner’s first-order conditions. The \((q, e)\) combinations that satisfy (30) lie on the curve labeled \( q(e; \gamma) \). The \((q, e)\) pairs that satisfy (30) lie on the curve labeled \( e(q) \). Note that \( q(e; \beta) = q^* \) and \( e(q^*) = e^*(q^*) \), so the Friedman rule induces efficient production and search intensity decisions. Inflation shifts up the \( q(e; \gamma) \) locus and always reduces \( q \). The effect of inflation on search intensity depends on the level of \( \gamma \). For high \( \gamma \), an increase in inflation induces buyers to reduce search intensity \( e \). For low \( \gamma \), buyers raise their search intensity in response to an increase in inflation. The key to a positive extensive margin effect lies in the nonmonotonic relationship between \( e \) and \( q \) defined by (27), namely,

\[
\psi'(e) = \alpha(1/e)\omega(1/e, q)[u(q) - c(q)]
\]

The right-hand side represents the buyer’s expected gains from trade in the decentralized market and depends on the product of two factors: the total surplus, \( u(q) - c(q) \), and the buyer’s share of this surplus, \( \omega \). The total surplus is increasing, but the buyer’s share is decreasing in the size of the trade, \( q \). Intuitively, as \( q \) increases, the marginal utility of a buyer falls relative to the marginal disutility from production incurred by the seller, and therefore equating their marginal utilities from trade requires the buyer to get a smaller share of the total surplus. The net effect on the buyer’s expected gains from trade can go either way, and this causes the relationship between \( e \) and \( q \) to be nonmonotonic. For small \( q \) the surplus is steep so the “total surplus effect” dominates and the buyer’s gain from trade is
increasing in \( q \). The “share effect” dominates for large \( q \), and consequently the buyer’s gain from trade is decreasing in the size of the trade, \( q \), beyond some threshold \( \bar{q} < q^* \). Suppose, for example, that the inflation rate increases slightly from \( \gamma = \beta \). The quantity \( q \) falls slightly below \( q^* \), but since \( u(q) - c(q) \) is maximized at \( q^* \), the total surplus suffers only a second-order reduction. The buyer’s share \( \omega(1/e, q) \) will experience a first-order increase, and as a result, the buyer’s expected gain from trade will rise, and this will induce him to increase his search intensity.

The economic reasoning behind the nonmonotonic relationship between \( e \) and \( \gamma \) deserves further attention. The distinctive feature of competitive price posting is that price setters compete for trading probabilities. They internalize both congestion and thick market effects, which is precisely why the equilibrium allocation is efficient at \( \gamma = \beta \). As \( \gamma \) increases, buyers reduce their real balances and this reduces the match surplus \( u(q) - c(q) \). If the buyer’s share of this surplus remains constant, the buyer’s expected gains from trade fall and so does his willingness to pay for a contact with a seller. For the market for contact probabilities to clear, the “price” \( U^* \) needs to adjust downward. The reduction in \( U^* \) occurs in equilibrium through a fall in the seller’s share. A similar story can be told for the case in which sellers are the ones who post prices. As inflation increases, buyers will tend to carry lower real balances into the decentralized market. To encourage buyers to bring more cash (in order to increase the gains of trade \( u(q) - c(q) \) of the buyer–seller match), sellers offer them a better deal, that is, a larger share of the surplus.

An increase in inflation always has a negative intensive margin effect on output: Less output is traded in each match. Furthermore, for high inflation, the number of trades falls so that the extensive margin effect of inflation on output is also negative. Thus for high \( \gamma \), aggregate output unambiguously falls with inflation. At low inflation rates, however, the frequency of trades is increasing in the rate of inflation. Therefore, if the positive extensive margin effect dominates the negative intensive margin effect, then total output will rise. We parameterized the model, computed several examples, and found that the positive effects of inflation on output can be large. For example, suppose \( \alpha(1/e) = [1 - \exp(-\epsilon/e)] \), \( u(q) = \frac{(q + b)^{1-\sigma} - b^{1-\sigma}}{1-\sigma} \), \( \psi(s) = Ae^\rho(1-e)^{-\epsilon} \), and \( c(q) = q^\rho / \kappa \). When \( \beta = 0.99 \), \( \epsilon = 0.01 \), \( \sigma = 0.7 \), \( b \approx 0 \), \( \rho = 1.2 \), \( A = 1 \), and \( \kappa = 25 \), going from the Friedman rule (which corresponds to a 1 percent deflation rate) to price stability entails a 5 percent increase in aggregate output. Increasing inflation further from 0 percent to 1 percent implies an additional output gain of 1.4 percent. Output peaks at around 6 percent inflation, and at that point aggregate output is about 7.6 percent higher than it would be under the Friedman rule.\(^{13} \)

Interestingly, this model is consistent with the common wisdom on the welfare effects of anticipated inflation we discussed in the introduction. An increase

\(^{13} \)The intensive margin is less responsive to inflation if \( c(q) \) is very convex, so the size of the output effect is increasing in \( \kappa \). For instance if \( \kappa = 15 \), output is 2.7% higher at 0% inflation than at the Friedman rule. The extensive margin is more responsive if \( \rho \) is small (\( A \) big), so the output effect is larger for smaller (bigger) \( \rho \) (\( A \)). For example, if we set \( A = 1.5 \), then going from the Friedman rule to price stability yields a 6.8% increase in aggregate output.
inflation can induce agents to spend additional resources to try to get rid of their cash, and these additional resources spent are a social waste. The model is also broadly consistent with the recent empirical evidence on the long-run output effects of permanent changes in inflation of Bullard and Keating (1995).

5. CONCLUSION

It is only natural for economic agents to devote more effort to trading away their cash holdings at higher inflation rates. But ultimately, money has to be held by someone, so these efforts are bound to be socially wasteful. The actions taken in response to the inflation tax are also likely to have an impact on macroeconomic outcomes. Indeed, recent evidence seems to indicate that permanent increases in inflation are associated with permanent increases (reductions) in the level of output in countries with low (high) average inflation. We constructed two models of monetary exchange and used them to study the effects of permanent changes in the rate of inflation on the level of output and welfare.

We first considered the canonical search model of money in which prices are set according to ex post bilateral Nash bargaining between a buyer and a seller. We found that higher inflation always decreases search effort, the frequency of trades, and aggregate output. From a normative point of view, ex post bargaining generically implements inefficient equilibria, and any deviation from the Friedman rule reduces welfare. We then analyzed the model under competitive price posting and found that search intensity and the frequency of trades increase with inflation at low inflation rates. When this extensive margin effect is strong enough, it is possible to have aggregate output increasing with anticipated inflation at low inflation rates but decreasing at high inflation rates. Thus, this version of the model can rationalize the conventional wisdom regarding the changes in agents’ trading behavior in response to inflation and at the same time help explain the nonlinear relationship between inflation and the level of output. Although it may be possible to increase output by running a mild inflation, inflating in excess of the Friedman rule always reduces welfare.

APPENDIX

PROOF OF LEMMA 1. (a) Using the bargaining solution (14) together with the fact that \( c[q_t(z)] = z \) in all matches, we rewrite the value functions in terms of real balances:

\[
V^b(z) = \max_{e \geq 0} \left[ \alpha e S(z) - \psi(e) + W^b(z) \right]
\]

(A.1)

\[
W^b(z) = z + \tau + \max_{\tilde{z} \geq 0} \left[ \beta V^b(\tilde{z}) - \gamma \tilde{z} \right]
\]

(A.2)

\[
V^s(z) = W^s(z)
\]

(A.3)
It can easily be checked from (A.3) and (A.4) that $\dot{z}_s = 0$ for all $\gamma > \beta$. This gives the first part of the lemma. (b) Combine (A.1) and (A.2) and write

\begin{equation}
V^b(z) = g(z) + z + \max_{\tilde{z} \geq 0} [\beta V^b(\tilde{z}) - \gamma \tilde{z}] \tag{A.5}
\end{equation}

Since $g$ is continuous and bounded (see proof of Lemma 2), we can use standard dynamic programming to establish the existence and uniqueness of a $V^b(z)$ that solves (A.5). By substituting the expressions in the statement of the lemma into (A.1) and (A.2), it is easy to see that they indeed solve the Bellman equations.

**Proof of Lemma 2.** (a) Our assumptions on primitives imply that $aeS(z) - \psi(e)$ is strictly concave in $e$, so the first-order condition (19) is necessary and sufficient. It is immediate from this that $e(z) = S(z) = 0$ if $z = 0$. The fact that $e(z) < 1$ for all $z$ follows from our assumption that $\lim_{e \to 1} \psi'(e) = +\infty$. Let $z^*_1 \equiv c(q_1^*)$. Note that $S'(z) = \int_{R_+} \frac{e^{\psi(q)} - 1 - e^{\psi(q)}}{e^{\psi(q)}} \, dG(e)$, so $S'(z) > 0$ for all $z \in [0, z^*_1]$ and $S'(z) = 0$ for all $z \geq z^*_1$, since $S(z) = S^*$ for all $z \geq z^*_1$. Therefore, $e'(z) = \frac{aS(z)}{\psi'(e)} > 0$ for all $z \in [0, z^*_1)$, and $e'(z) = 0$ for all $z \geq z^*_1$, so the buyer’s optimal choice of search intensity is increasing in his real money balances. (b) Let $\Delta(z; \tilde{e}, \gamma) = \beta ae(z)S'(z) - (\gamma - \beta)$, so from part (a) we have $\Delta(z; \tilde{e}, \gamma) = -(\gamma - \beta) < 0$ for all $z \geq z^*_1$. Thus, we can write $\max_{z \geq 0} \Delta(z; \tilde{e}, \gamma) = \max_{z \in [0, z^*_1]} \Delta(z; \tilde{e}, \gamma)$. The buyer is maximizing a continuous function over a compact set. By the theorem of the maximum (for example, in Stokey et al., 1989, p. 62) the correspondence $D : [0, \infty) \times [\beta, \infty) \to [0, z^*_1]$ defined by $D(\tilde{e}, \gamma) = \arg \max_{z \geq 0} \Delta(z; \tilde{e}, \gamma)$ is nonempty, compact-valued, and upper-hemi continuous. We know that $D(\tilde{e}, \gamma) \subseteq [0, z^*_1)$ because $\Delta(z^*_1; \tilde{e}, \gamma) < 0$. Next, we establish the comparative static results. The correspondence $D(\tilde{e}, \gamma)$ is decreasing in $(\tilde{e}, \gamma)$ if $(\tilde{e}'', \gamma'') > (\tilde{e}', \gamma')$, $\tilde{z} \in D(\tilde{e}', \gamma')$, and $\tilde{z}'' \in D(\tilde{e}'', \gamma'')$, then $\tilde{z}' \leq \tilde{z}''$ (and $\tilde{z}' < \tilde{z}''$ if $\tilde{z}' > 0$), where “$\leq$” denotes the pairwise ordering relation on $\mathbb{R}^2$. The choice set $[0, z^*_1]$ is independent of $(\tilde{e}, \gamma)$. Note that $\Delta_{13}(z; \tilde{e}, \gamma) = -1 < 0$ and that $\Delta_{12}(z; \tilde{e}, \gamma) = \beta [e + \frac{aS(z)}{\psi'(e)}]S'(z) = 0$. Thus the objective function has strictly decreasing differences in $(\tilde{e}, \gamma)$; that is, for all $(\tilde{e}'', \gamma'') > (\tilde{e}', \gamma')$, we have $\Delta(z; \tilde{e}'', \gamma'') - \Delta(z; \tilde{e}', \gamma') < \Delta(z; \tilde{e}'', \gamma'') - \Delta(z; \tilde{e}', \gamma')$ if $z > \tilde{z}$ (see Topkis, 1998). We now show that $(\tilde{e}'', \gamma'') > (\tilde{e}', \gamma')$ implies $\tilde{z}' < \tilde{z}$. Suppose to the contrary that $\tilde{z}' > \tilde{z}'$, then

$$0 \leq \Delta(z''; \tilde{e}'', \gamma'') - \Delta(z'; \tilde{e}'', \gamma'') < \Delta(z''; \tilde{e}'', \gamma'') - \Delta(z'; \tilde{e}', \gamma') \leq 0$$

a contradiction. So we conclude that $\tilde{z}' \leq \tilde{z}'$. Next, we show that if $\tilde{z}'' > 0$, then $(\tilde{e}'', \gamma'') > (\tilde{e}', \gamma')$ implies $\tilde{z}'' < \tilde{z}'$. Since $\tilde{z}'' \leq \tilde{z}'$, $\tilde{z}' > 0$, implies $\tilde{z}'' > 0$, so $\tilde{z}''$ and $\tilde{z}'$
must satisfy the first-order necessary conditions \( \Delta_1 (z'''; \bar{e}'', \gamma '') = \Delta_1 (\bar{z}''', \bar{e}'', \gamma') = 0 \). Suppose that \( z''' = \bar{z}' \). Then since \( \Delta (z; \bar{e}, \gamma) \) has strictly decreasing differences in \( (\bar{e}, \gamma) \), we have \( 0 = \Delta_1 (\bar{z}''', \bar{e}'', \gamma') > \Delta_1 (\bar{z}''', \bar{e}'', \gamma'') = \Delta_1 (\bar{z}'', \bar{e}'', \gamma'') = 0 \), a contradiction. Therefore \( \bar{z}' < \bar{z}''. \)

**Proof of Proposition 1.** By Corollary 2.7.1 in Topkis (1998), we know that \( D(\bar{e}, \gamma) \) has a greatest and a least element. Denote them \( z_H(\bar{e}) \) and \( z_L(\bar{e}) \), respectively. Then define a correspondence \( E: [0, \infty) \times [\beta, \infty) \to [0, \infty) \) by

\[
E(\bar{e}; \gamma) = \{ e \in \mathbb{R} : e = \sigma h[z_H(\bar{e}), \bar{e}] + (1 - \sigma) h[z_L(\bar{e}), \bar{e}] \text{ for some } \sigma \in [0, 1] \}
\]

So \( E(\bar{e}; \gamma) \) is the set of average search intensities resulting from all convex combinations of the elements of \( D(\bar{e}, \gamma) \). The equilibrium condition \( \int_{[0,1]} e(i) di = \bar{e} \) can be reformulated as \( \bar{e} \in E(\bar{e}; \gamma) \); that is, for given \( \gamma, \bar{e} \) is a fixed point of the correspondence \( E(\bar{e}; \gamma) \). The correspondence \( E(\bar{e}; \gamma) \) is upper-hemi continuous and convex valued. From (19), we see that \( \bar{e} \to \infty \) implies \( \alpha(\bar{e}) \to 0 \), and hence \( h[z_i(\bar{e}), \bar{e}] \to 0 \) for \( i = H, L \). (To see this, note that \( S(z) \leq S^* < \infty \).) As \( \bar{e} \to 0 \), \( \alpha(\bar{e}) \to 1 \) and the buyer’s maximization problem in the centralized market becomes \( \max \{ \beta [eS(z) - \psi(e)] + (\beta - \gamma) z \} \) subject to \( e \geq 0 \) and \( z \geq 0 \). Since the value of setting \( z = e = 0 \) is 0, the condition \( \max \{ \beta [eS(z) - \psi(e)] + (\beta - \gamma) z \} > 0 \) implies \( z_L(0) > 0 \), and then from (19) we have \( h[z_L(0), 0] > 0 \). Thus a monetary equilibrium exists if \( \max_{e \geq 0, z \geq 0} \{ \beta [eS(z) - \psi(e)] + (\beta - \gamma) z \} > 0 \). (The only if part is immediate from the monotonicity of the equilibrium correspondence in \( \bar{e} \), which we establish below.) From (19) and the comparative static results in parts (a) and (c) of Lemma 2, we know that \( E(\bar{e}; \gamma) \) is strictly decreasing in \( \bar{e} \), so there is a unique \( \bar{e} \) satisfying \( \bar{e} \in E(\bar{e}; \gamma) \) (see Figure A.1).

To show the comparative static result on the equilibrium, consider \( \gamma' \) and \( \gamma'' \) such that \( \gamma'' > \gamma' \). From part (c) of Lemma 2 we know that for all \( e' \in E(\bar{e}; \gamma') \) and

---

**Figure A.1**

**Equilibrium with Bargaining**
all \( e'' \in E(\bar{e}; \gamma') \), \( e' \geq e'' \) with a strict inequality if \( e'' > 0 \). As a consequence, the fixed point \( \bar{e} = E(\bar{e}; \gamma) \) is strictly decreasing in \( \gamma \).

**Proof of Proposition 2.** According to (20), \( \varepsilon u'(q) = c'(q) \) in all trades if and only if \( \gamma = \beta \). From (19), \( e \) satisfies \( \psi'(e) = \alpha(1/e)S^* \). It is then easy to check that \( e > \bar{e}' \).

To show the second part of the proposition, take two inflation rates \( \gamma'' \) and \( \gamma' \) with \( \gamma'' > \gamma' > \beta \). We first establish that real balances are decreasing with inflation. Let \( \bar{e}'' \in E(\bar{e}'', \gamma'') \) and \( \bar{e}' \in E(\bar{e}', \gamma') \). From Proposition 1 we know that \( \bar{e}'' \leq \bar{e}' \), with strict inequality if \( \bar{e}'' > 0 \). From (19), \( \bar{e}'(\bar{e}'') \) implies a unique \( z'(z'') \) and \( \bar{e}'' < \bar{e}' \) implies \( z'' < z' \). Now consider the effect of inflation on output. Recall that \( \hat{q}'(z) \geq 0 \); also from (19), \( e'(z) = \frac{uS}{\varphi - \sigma S} \geq 0 \). Differentiating (21), we find that

\[
(A.6) \quad \frac{\partial Y}{\partial z} = \frac{\zeta_1(e, 1)e'(z)Y}{\zeta(e, 1)} + \frac{\zeta(e, 1)\hat{q}'(z)[1 - G(R)]}{1 - \beta} \geq 0
\]

with strict inequality if \( z < c(q_1^*) \). For the welfare effect of inflation, use (19) to write \( (1 - \beta)V^b(z) = e\psi'(e) - \psi(e) \) and get

\[
(A.7) \quad \frac{\partial V^b(z)}{\partial z} = \frac{e(z)e'(z)\psi''[e(z)]}{1 - \beta} \geq 0
\]

with strict inequality if \( z < c(q_1^*) \). ■

**Proof of Lemma 3.** We first show that the following problem,

\[
\max_{S \in S, \hat{m}, e} \left[ \alpha eS_b(\hat{m}, F^s, \phi) - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) \phi\hat{m} \right]
\]

with \( S = [q(\hat{m}, \hat{m}), d(\hat{m}, \hat{m}), \theta] \) and subject to \( \theta \geq 0, \hat{m} \geq 0, e \geq 0, d(\hat{m}, \hat{m}) \leq \hat{m}, \) and

\[
(A.8) \quad \max_{\hat{m} \geq 0} \left[ \alpha_S S_S(m_b, \hat{m}, \phi) - \left( \frac{\gamma}{\beta} - 1 \right) \phi\hat{m} \right] \geq U^s
\]

reduces to the one in the statement of the lemma. We proceed in several steps. Let \( [q(\cdot, \cdot), d(\cdot, \cdot), \theta] \) be part of the solution to the buyer’s problem. Then

(i) If the submarket \( S = [q(\cdot, \cdot), d(\cdot, \cdot), \theta] \) is active, then \( d(m_b, m_s) > 0 \).

Assume not; then \( S_S(m_b, m_s, \phi) = -c[q(m_b, m_s)] \), and therefore the left-hand side of the seller’s participation constraint (A.8) is strictly negative. But in any period sellers can achieve \( 0 \leq U^s \), meaning that (A.8) is violated, and hence the submarket \( S = [q(\cdot, \cdot), d(\cdot, \cdot), \theta] \) is inactive.
(ii) If the submarket \( s \equiv [q(\cdot, \cdot), d(\cdot, \cdot), \theta] \) is active, then the contract \([q(m_b, m_s), d(m_b, m_s)]\) is independent of \( m_s \), and sellers carry no cash into submarket \( s \).

Assume not; suppose that
\[
\phi d(m_b, m_s) - c[q(m_b, m_s)] > \phi d(m_b, 0) - c[q(m_b, 0)]
\]
and that the seller chooses to carry \( m'_s > 0 \) into submarket \( s \). (If the above inequality does not hold, then it immediately follows that a seller would choose \( m_s = 0 \).

Then a seller’s payoff in submarket \( s \) is
\[
\alpha_s (\phi d(m_b, m'_s) - c[q(m_b, m'_s)]) - \left( \frac{\gamma}{\beta} - 1 \right) \phi m'_s
\]
Consider the contract \([\hat{q}(m_b), \hat{d}(m_b)]\) where terms of trade are independent of the seller’s money holdings, that is, \( \hat{q}(m_b) = q(m_b, m'_s) \) and \( \hat{d}(m_b) = d(m_b, m'_s) \). It yields the seller
\[
\alpha_s (\phi d(m_b) - c[q(m_b)]) = \alpha_s (\phi d(m_b, m'_s) - c[q(m_b, m'_s)])
\]
\[
> \alpha_s (\phi d(m_b, m'_s) - c[q(m_b, m'_s)]) - \left( \frac{\gamma}{\beta} - 1 \right) \phi m'_s
\]
meaning that the seller’s participation constraint is violated, and hence the submarket is inactive.

Parts (i) and (ii) and the fact that \( \gamma > \beta \) imply that any active submarket \( s \equiv [q(\cdot, \cdot), d(\cdot, \cdot), \theta] \) will have \( q(m_b, m_s) = q(m_b) \) and \( d(m_b, m_s) = m_b \). (The terms of trade are independent of the seller’s money holdings, sellers carry no cash, and buyers hand over all their cash to sellers in every trade.) This establishes part (a) and allows us to rewrite the buyer’s problem as

\[
\max_{m, \theta, e} \left[ \alpha(\theta)e[u(q) - \phi m] - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) \phi m \right]
\]
subject to \( q \geq 0, m \geq 0, \theta \geq 0, e \geq 0, \) and \( \alpha_s(\theta) [\phi m - c(q)] \geq U^s \). Letting \( z = \phi m \), we get the expressions in part (b) of the statement of the lemma.

For part (c), we first show that the maximizers \((q, z, e)\) lie in a compact set. Suppose the buyer chooses \((q', z')\) with \( q' > q^* \). Then the seller’s surplus is \( z' - c(q') \) and the buyer’s is \( u(q') - z' \). But note that the contract \((q^*, z^*)\) with \( z^* = c(q^*) + z' - c(q') \) keeps the seller’s expected surplus unchanged but gives the buyer surplus \( u(q^*) - z^* = u(q^*) - c(q^*) - z + c(q') > u(q') - z' \). So we conclude that \( q \in [0, q^*] \). Since \( \psi'(e) \rightarrow +\infty \) as \( e \rightarrow 1 \), we know that at the optimum, \( e \in [0, 1] \). Finally, we can restrict \( z \in [c(q), u(q)] \) without affecting the value of the program, since choosing \( u(q) - z < 0 \) yields value zero (and this can be achieved by setting \( e = z = 0 \), and choosing \( z - c(q) < 0 \) implies the submarket is inactive.

Thus, the value of the program is not affected by assuming \( z \in [c(q), u(q)] \). We can invert \( \alpha_s = \zeta(1/\theta, 1) \) and use it to define \( \theta(\alpha_s) \), so we can think of \( \alpha_s \) as the
choice variable. Then, our assumptions on the matching function imply that \( \alpha_s \in [0, 1] \). Now the problem in part (b) of the lemma can be restated as

\[
\max_{q, z, \alpha_s, e} \left[ e \alpha(\theta(\alpha_s))[u(q) - z] - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) z \right]
\]

subject to \((q, z, \alpha_s, e) \in \Gamma(U^s)\), where

\[
\Gamma(U^s) = \{ (q, z, \alpha_s, e) \in \mathbb{R}^4 : q \in [0, q^s], z \in [c(q), u(q)], \alpha_s, e \in [0, 1], \text{ and } \alpha_s[z - c(q)] \geq U^s \}
\]

Since the objective function is continuous and the constraint correspondence \( \Gamma \) is continuous and compact-valued, the correspondence \( \Upsilon(U^s) \) is nonempty and upper-hemi continuous.

**Proof of Proposition 3.** (a) Suppose \((q(U^s), z(U^s), e(U^s), \theta(U^s)) \in \Upsilon(U^s)\), and let \( r(U^s) = e(U^s)\theta(U^s) \). We denote \( r_H(U^s) \) and \( r_L(U^s) \) the greatest and the least element of the set \( \{ r \in \mathbb{R} : r = e(U^s)\theta(U^s) \} \). Then, we define the correspondence \( \hat{E}(U^s) \) as follows:

\[
\hat{E}(U^s) = \{ r \in \mathbb{R} : r = \sigma r_H(U^s) + (1 - \sigma)r_L(U^s) \text{ for some } \sigma \in [0, 1] \}.
\]

This is the set of all convex combinations of \( r_H(U^s) \) and \( r_L(U^s) \). With this notation, condition (C2) can be written as \( 1 \in \hat{E}(U^s) \). All we need to do to establish that an equilibrium exists is show that there exists a \( U^s \) that satisfies this condition. (i) From the buyer’s objective function we know that if \( U^s > u(q^s) - c(q^s) \) implies \( \hat{E}(U^s) = \{0\} \). (ii) Now suppose \( U^s = 0 \). We want to show that in this case we have \( e(U^s) > 0 \) and \( \theta(U^s) = +\infty \) for all elements in \( \Upsilon(U^s) \). Suppose \( \theta < +\infty \); then the seller’s participation constraint implies the buyer should set \( z = c(q) \). If \( \max_{q, z, e}[e[u(q) - c(q)] - \psi(e) - (\frac{\gamma}{\beta} - 1)c(q)] > 0 \), then the buyer chooses \( e > 0 \) and wants to set \( \theta = +\infty \) to maximize his chances to trade. This means that \( r(U^s) = e(U^s)\theta(U^s) \to +\infty \) as \( U^s \to 0 \), and therefore all elements in \( \hat{E}(U^s) \) approach \(+\infty\) as \( U^s \to 0 \). Given that \( E(U^s) \) is upper-hemi continuous, parts (i) and (ii) imply that there exists a \( U^s \) such that \( 1 \in \hat{E}(U^s) \). See Figure A.2.

**Lemma 4.** The buyer’s problem in part (b) of Lemma 3 has a unique solution for \( \gamma \) close to \( \beta \).

**Proof of Lemma 4.** In what follows we assume that \( U^s < u(q^s) - c(q^s) \), which has to hold in equilibrium. The seller’s participation constraint can be rewritten as \( z = c(q) + U^s/\alpha_s(\theta) \). Substituting \( z \) into the buyer’s objective allows us to rewrite the buyer’s problem as \( \max_{(q, \theta, e) \in \mathbb{R}^3} \Psi(q, \theta, e) \), where

\[
\Psi(q, \theta, e) = e\alpha(\theta) \left[ u(q) - c(q) - \frac{U^s}{\alpha_s(\theta)} \right] - \psi(e) - \left( \frac{\gamma}{\beta} - 1 \right) \left[ c(q) + \frac{U^s}{\alpha_s(\theta)} \right]
\]

If \( \gamma = \beta \), the first-order conditions give
EXISTENCE OF EQUILIBRIUM UNDER PRICE POSTING

\[ q = q^* \quad (A.9) \]

\[ \psi'(e) = \alpha'(\theta) \left[ u(q^*) - c(q^*) - \frac{U^s}{\alpha_s(\theta)} \right] \quad (A.10) \]

\[ U^s = \alpha' \left[ u(q^*) - c(q^*) \right] \quad (A.11) \]

From our assumptions, \( \alpha' \) is strictly decreasing, \( \alpha'(0) = 1 \), and \( \alpha'(\infty) = 0 \). Given that \( \frac{U^s}{u(q^*) - c(q^*)} \in [0, 1] \), there is a unique \( \theta \) that solves (A.11). Given this \( \theta \), since \( \psi'' > 0 \), there is a unique \( e \) that solves (A.10). Given that \( U^s < u(q^*) - c(q^*) \), it can be checked that \( \theta > 0 \) and \( \max_{q, \theta, e} \psi(q, \theta, e) > 0 \). To show that \( \psi(q, \theta, e) \) is strictly concave in the neighborhood of the solution for \( \gamma = \beta \), we compute

\[ \Psi_{qq} = e\alpha'(\theta)[u''(q) - c''(q)] - \left( \frac{\gamma}{\beta} - 1 \right) c''(q) \]
\[ \Psi_{ee} = -\psi''(e) \]

\[ \Psi_{q\theta} = e\alpha''(\theta)[u(q) - c(q)] + \left( \frac{\gamma}{\beta} - 1 \right) \left( \frac{\alpha''(\theta)\alpha_s(\theta) - 2[\alpha'_s(\theta)]^2}{[\alpha_s(\theta)]^3} \right) U^s \]
\[ \Psi_{qc} = \alpha'(\theta)[u'(q) - c'(q)] \]
\[ \Psi_{q\theta} = e\alpha'(\theta)[u'(q) - c'(q)] \]
\[ \Psi_{e\theta} = \alpha'(\theta)[u(q) - c(q)] - U^s \]

The fact that \( \alpha'(0) = 1 \) comes from our assumption that \( \lim_{\theta \to 0} \frac{\alpha(\theta)}{\theta} = 1 \). The fact that \( \alpha'(\infty) = 0 \) comes from our assumption that \( \lim_{\theta \to \infty} \alpha(\theta) = 1 \).
At $\gamma = \beta$ and $q = q^*$, we have $\Psi_{q_e} = \Psi_{q_\theta} = \Psi_{e_\theta} = 0$ and $\Psi_{q_\theta} < 0$, $\Psi_{e_\theta} < 0$ and $\Psi_{q_\theta} < 0$. Since the Hessian is continuous, the function $\Psi(q, \theta, e)$ is strictly concave in the neighborhood of the solution under the Friedman rule and hence for $\gamma$ in the neighborhood of $\beta$. Thus, $\Psi(q, \theta, e)$ admits a unique local maximum in the neighborhood of $q = q^*$. The first-order necessary condition with respect to $q$, is $e\alpha(\theta)[u'(q) - c'(q)] = (\gamma/\beta - 1)c'(q)$. So any other candidate for an optimum must be such that either $\theta$ or $e$ is in the neighborhood of zero, which would imply that $\max_{q, \theta, e} \Psi(q, \theta, e)$ is close to zero (and it can be made arbitrarily close to zero by choosing $\gamma$ arbitrarily close to $\beta$). Given that $\max_{(q, \theta, e) \in \mathbb{R}_+^3} \Psi(q, \theta, e) > 0$ at $\gamma = \beta$, such candidates can be eliminated.

**Proof of Proposition 4.** As argued above the statement of the proposition, the Friedman rule implements the planner’s solution, so welfare decreases as we move away from $\gamma = \beta$. For the second part, combine (26)–(28) to get the following two equations in $e$ and $q$:

\[
\begin{align*}
(A.12) & \quad \frac{u'(q)}{c'(q)} = 1 + \frac{\gamma - \beta}{ae\beta} \\
(A.13) & \quad \psi'(e) = \xi_1(e, 1) \frac{u'(q)}{\eta \left(\frac{1}{2}\right) u'(q) + \left[1 - \eta \left(\frac{1}{2}\right)\right] c'(q)}[u(q) - c(q)]
\end{align*}
\]

where $\xi_1(e, 1)$ is the partial derivative of the matching function with respect to its first argument. By Lemma 4, for $\gamma$ close to $\beta$ (A.12) and (A.13) characterize the unique equilibrium. Condition (A.12) defines a monotonic relationship between $q$ and $e$ with slope $\frac{ae}{\partial q} = \frac{\beta(c''u' - u'c')\xi_1(e, 1)c'}{(\gamma - \beta)\xi_1(e, 1)c'} > 0$. Similarly, if we think of (A.13) as a curve in $(q, e)$ space, its slope at $q^*$ has the same sign as

\[
\frac{\partial e}{\partial q} = \frac{\xi_1(e^*, 1)}{u'(q^*)} \left[1 - \eta \left(\frac{1}{2}\right)\right] \left[u''(q^*) - c''(q^*)\right][u(q^*) - c(q^*)] \leq 0
\]

Therefore, $e$ as a function of $q$ as defined by (A.13) is downward sloping at $q = q^*$. A small deviation from the Friedman rule shifts the curve defined by (A.12) up in $(q, e)$ space although (A.13) is not affected. This implies that $e$ increases and $q$ falls in response to the increase in $\gamma$. See Figure 1.

**REFERENCES**

Bullard, J., “Testing Long-Run Monetary Neutrality Propositions: Lessons from the Recent Research,” *Federal Reserve Bank of St. Louis Review* 81 (1999), 57–78.

——, AND J. W. Keating, “The Long-Run Relationship between Inflation and Output in Postwar Economies,” *Journal of Monetary Economics* 36 (1995), 477–96.

Kiyotaki, N., AND R. Wright, “On Money as a Medium of Exchange,” *Journal of Political Economy* 97 (1989), 927–54.

——, AND ———, “A Contribution to the Pure Theory of Money,” *Journal of Economic Theory* 53 (1991), 215–35.
LAGOS, R., AND G. ROCHETEAU, “Inflation, Output, and Welfare,” Working Paper 04-07, Federal Reserve Bank of Cleveland, and Staff Report 342, Federal Reserve Bank of Minneapolis Staff Report 342, 2004.

——, AND ——, “Dynamics, Cycles, and Sunspot Equilibria in ‘Genuinely Dynamic, Fundamentally Disaggregative’ Models of Money,” Journal of Economic Theory 109 (2003), 156–71.

——, AND R. WRIGHT, “A Unified Framework for Monetary Theory and Policy Analysis,” Federal Reserve Bank of Minneapolis Staff Report 346, 2004.

——, AND ——, “A Unified Theory and Policy Analysis,” Journal of Political Economy 113 (2005), forthcoming.

LUCAS, R. E., JR., “Inflation and Welfare,” Econometrica 68 (2000), 247–74.

MOEN, E. R., “Competitive Search Equilibrium,” Journal of Political Economy 105 (1997), 385–411.

PETERSON, B., AND S. SHI, “Search, Price Dispersion and Welfare,” Mimeo, 2003.

ROCHETEAU, G., AND R. WRIGHT, “Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium,” Econometrica 73 (2005), 175–202.

SHI, S., “Money and Prices: A Model of Search and Bargaining,” Journal of Economic Theory 67 (1995), 467–96.

——, “A Divisible Search Model of Fiat Money,” Econometrica 65 (1997), 75–102.

SHIMER, R., “Essays in Search Theory,” dissertation, Massachusetts Institute of Technology, 1996.

STOKEY, N. L., R. E. LUCAS, JR., AND E. C. PRESCOTT, Recursive Methods in Economic Dynamics (Cambridge, MA: Harvard University Press, 1989).

TOPKIS, D. M., Supermodularity and Complementarity (Princeton, NJ: Princeton University Press, 1998).

TREJOS, A., AND R. WRIGHT, “Search, Bargaining, Money, and Prices,” Journal of Political Economy 103 (1995), 118–41.