Characterizations of plurisubharmonic functions

Fusheng Deng¹, Jiafu Ning²,∗ & Zhiwei Wang³

¹School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China; ²School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, China; ³Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

Email: fshdeng@ucas.ac.cn, jfning@csu.edu.cn, zhiwei@bnu.edu.cn

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Abstract We give characterizations of (quasi-)plurisubharmonic functions in terms of $L^p$-estimates of $\bar{\partial}$ and $L^p$-extensions of holomorphic functions.

Keywords plurisubharmonic function, Hörmander’s $L^2$-estimate, Ohsawa-Takegoshi extension theorem, characterization of plurisubharmonicity

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1 Introduction

In the fundamental work [13], Hörmander established a systematic $L^2$-theory for the $\bar{\partial}$ operator. In [15], Ohsawa and Takegoshi proved an extension theorem for $L^2$ holomorphic functions, now known as the Ohsawa-Takegoshi extension theorem. Hörmander’s $L^2$-theory for $\bar{\partial}$ and the Ohsawa-Takegoshi extension theorem are of fundamental importance in several complex variables and have terrific applications in algebraic geometry.

In both Hörmander’s $L^2$-theory for $\bar{\partial}$ and the Ohsawa-Takegoshi extension theorem, plurisubharmonic functions are used as weights and play a key role. It is natural to ask if plurisubharmonic functions are the only choice for weights in both theories. The aim of the present paper is to answer this question affirmatively in a certain sense and give characterizations of (quasi-)plurisubharmonic functions in terms of $L^p$-estimates of $\bar{\partial}$ and $L^p$-extensions of holomorphic functions, which can be roughly understood as converses of Hörmander’s $L^2$-estimate for $\bar{\partial}$ and the Ohsawa-Takegoshi extension theorem. To state the results, we first introduce some notions.

Definition 1.1. Let $\phi : D \to [-\infty, +\infty)$ be an upper semi-continuous function on a domain $D$ in $\mathbb{C}^n$. We have

(1) $\phi$ satisfies the optimal $L^p$-estimate property if for any $\bar{\partial}$-closed smooth $(0,1)$-form $f$ on $D$ with compact support and any smooth strictly plurisubharmonic function $\psi$ on $D$, the equation $\bar{\partial}u = f$ can be solved on $D$ with the estimate

$$\int_D |u|^p e^{-\phi - \psi} \leq \int_D |f|^p |e^{-\phi - \psi}|_{\partial D}$$

*Corresponding author
provided that the right-hand side is finite.

(2) $\phi$ satisfies the multiple coarse $L^p$-estimate property if for any $m \geq 1$, any $\bar{\partial}$-closed smooth $(0,1)$-form $f$ on $D$ with compact support and any smooth strictly plurisubharmonic function $\psi$ on $D$, the equation $\partial u = f$ can be solved on $D$ with the estimate
\[
\int_D |u|^p e^{-m\phi - \psi} \leq C_m \int_D |f|^p |\partial\partial \psi| e^{-m\phi - \psi}
\]
provided that the right-hand side is finite, where $C_m$'s are constants such that $\lim_{m \to \infty} \log \frac{C_m}{m} = 0$.

(3) $\phi$ satisfies the optimal $L^p$-extension property if for any $z \in D$ with $\phi(z) \neq -\infty$ and any holomorphic cylinder $P$ with $z + P \subset D$, there is a holomorphic function $f$ on $z + P$ such that $f(z) = 1$ and
\[
\frac{1}{\mu(P)} \int_{z+P} |f|^p e^{-\phi(z)} \leq e^{-\phi(z)},
\]
where $\mu(P)$ is the volume of $P$ with respect to the Lebesgue measure. (Here by a holomorphic cylinder we mean a domain of the form $A(P_{r,s})$ for some $A \in U(n)$ and $r, s > 0$ with $P_{r,s} = \{(z_1, z_2, \ldots, z_n) : |z_1|^2 < r^2, |z_2|^2 + \cdots + |z_n|^2 < s^2\}$.)

(4) $\phi$ satisfies the multiple coarse $L^p$-extension property if for any $m \geq 1$ and any $z \in D$ with $\phi(z) \neq -\infty$, there is a holomorphic function $f$ on $D$ such that $f(z) = 1$ and
\[
\int_D |f|^p e^{-m\phi} \leq C_m e^{-m\phi(z)},
\]
where $C_m$'s are constants such that $\lim_{m \to \infty} \log \frac{C_m}{m} = 0$.

A related but different form of the optimal $L^p$-extension property for $p = 2$ in terms of analytic discs was introduced in [12], the multiple coarse $L^p$-extension property was introduced in [9], and the multiple coarse $L^p$-estimate property was introduced and named as the twisted Hörmander condition in [14] in the case where $p = 2$ with a different form in the dimension one case introduced in [2].

If $\phi$ is plurisubharmonic, $p = 2$, and $D$ is a bounded pseudoconvex domain, then $\phi$ satisfies:

- the optimal $L^2$-estimate property and the multiple coarse $L^2$-estimate property by Hörmander’s $L^2$-estimate of $\bar{\partial}$ (see [1, Theorem 1.6.4] for an appropriate formulation);
- the multiple coarse $L^2$-extension property by Ohsawa and Takegoshi [15]; and
- the optimal $L^2$-extension property by Blocki [5] and Guan and Zhou [10, 11].

In the present paper, we will discuss the converses of the above results. We will see that $\phi$ must be plurisubharmonic if it satisfies one of the properties in Definition 1.1. The precise results are formulated as several theorems as follows.

**Theorem 1.2.** Let $D$ be a domain in $\mathbb{C}^n$ and $\phi \in C^2(D)$. If $\phi$ satisfies the optimal $L^2$-estimate property, then $\phi$ is plurisubharmonic on $D$.

In fact, Theorem 1.2 can be strengthened to characterize quasi-plurisubharmonicity as follows.

**Theorem 1.3.** Let $D$ be a domain in $\mathbb{C}^n$, $\phi \in C^2(D)$ and $\omega$ be a continuous real $(1,1)$-form on $D$. If for any $\bar{\partial}$-closed smooth $(0,1)$-form $f$ on $D$ with compact support and any smooth strictly plurisubharmonic function $\psi$ on $D$ with $i\partial \partial \psi + \omega > 0$ on supp $f$, the equation $\partial u = f$ can be solved on $D$ with the estimate
\[
\int_D |u|^2 e^{-\phi - \psi} \leq \int_D |f|^2 i\partial \partial \psi + \omega e^{-\phi + \psi}
\]
provided that the right-hand side is finite, then $i\partial \partial \phi \geq \omega$ on $D$.

We prove Theorem 1.3 by connecting $\partial \partial \phi$ with the optimal $L^2$-estimate property via a Bochner type identity, and then using a localization technique to produce a contradiction if $i\partial \partial \phi \geq \omega$ is assumed to be not true.

**Remark 1.4.** Note that $\phi$ is plurisubharmonic if $\phi + \psi$ is plurisubharmonic for any strictly plurisubharmonic function $\psi$. However, the role played by $\psi$ in Theorem 1.3 is very different. Indeed, there $\psi$ is used to do localization and only those $\psi$’s such that $i\partial \partial \psi$ is very large are used in the proof of Theorem 1.3.
Theorem 1.5. Let $D$ be a domain in $\mathbb{C}^n$ and $\phi : D \to \mathbb{R}$ be a continuous function. If $\phi$ satisfies the multiple coarse $L^p$-estimate property for some $p > 1$, then $\phi$ is plurisubharmonic on $D$.

In connection with Theorem 1.5, we remark that Berndtsson [1, Proposition 2.2] has proved the following result: let $D$ be a domain in $\mathbb{C}$ and $\phi$ be a continuous function on $D$, such that for any $m \geq 1$ and any $f \in C^\infty_c(D)$ we can solve the equation $\bar{\partial}u = fd\bar{z}$ with the estimate

$$
\int_D |u|^2 e^{-m\phi} \leq C \int_D |f|^2 e^{-m\phi}
$$

with $C$ a uniform constant independent of $m$, and then $\phi$ is subharmonic. The method of Berndtsson depends on the fact that the $(0,1)$-Dolbeault cohomology of $\mathbb{C}$ with compact support does not vanish, and seems difficult to be generalized to higher dimensions. The case of Theorem 1.5 where $\phi$ depends on the fact that the $(0,1)$-Dolbeault cohomology of $\mathbb{C}$ with compact support does not vanish, and seems difficult to be generalized to higher dimensions. The case of Theorem 1.5 where $\phi$ is locally Hölder continuous for general dimensions was proved in [14], by showing that the multiple coarse $L^2$-estimate property implies the multiple coarse $L^2$-extension property, and then applying Theorem 1.7 in the following. Theorem 1.5 is proved by modifying the idea in [14].

Theorem 1.6. Let $\phi : D \to [-\infty, +\infty)$ be an upper semi-continuous function on a domain $D$ in $\mathbb{C}^n$. If $\phi$ satisfies the optimal $L^p$-extension property for some $p > 0$, then $\phi$ is plurisubharmonic on $D$.

Theorem 1.6 for $n = 1$ is essentially contained in [11], where it is shown that the optimal estimate of the Ohsawa-Takegoshi extension theorem implies Berndtsson’s plurisubharmonic variation of relative Bergman kernels [3, 4]. The general case is proved by modifying Guan-Zhou’s method in [11].

Theorem 1.7. Let $\phi : D \to [-\infty, +\infty)$ be an upper semi-continuous function on a domain $D$ in $\mathbb{C}^n$. If $\phi$ satisfies the multiple coarse $L^p$-extension property for some $p > 0$, then $\phi$ is plurisubharmonic on $D$.

Theorem 1.7 was originally proved in [9], where the method is motivated by Demailly’s idea on the regularization of plurisubharmonic functions [6]. In the present paper, we give a new proof of it based on Guan-Zhou’s method mentioned above.

Remark 1.8. Note that plurisubharmonicity involves exact inequalities. It is more or less reasonable to expect that sharp estimates for $\bar{\partial}$ (namely the optimal $L^p$-estimate property and the optimal $L^p$-extension property) could imply the plurisubharmonicity of $\phi$. However, it is difficult to expect that coarse estimates can encode plurisubharmonicity. The point in Theorems 1.5 and 1.7 is that we have to consider powers of the trivial line bundle with product metrics $e^{-m\phi}$, and then there is a procedure similar to taking $m$-th roots of $C_m$ and finally we get the exact number $1 = \lim \sqrt[m]{C_m}$ in the limit. This is the main observation in [9].

Remark 1.9. The continuity assumption in Theorem 1.5 is in some sense optimal since the weight $\phi$ appears in integrations on both sides of the involved estimates. This condition can be weakened to semi-continuity in Theorems 1.6 and 1.7 since the related estimates involve pointwise evaluations of $\phi$. Therefore it is natural to ask whether the regularity condition on $\phi$ in Theorem 1.3 can be weakened to being continuous.

The above theorems, combined with the $L^2$-theory of $\bar{\partial}$, imply that the four properties in Definition 1.1 are essentially equivalent for $\phi$ at least for the case where $p = 2$. As this conclusion is built on some heavy theories, it seems interesting to find more straightforward methods to establish the equivalence of these properties.

We now recall some background and geometric meaning of plurisubharmonic functions.

A plurisubharmonic function on an open set in $\mathbb{C}^n$ is an upper semi-continuous function with values in $[-\infty, +\infty)$ whose restriction to any complex line is subharmonic. It turns out that the definition does not depend on the linear structure on $\mathbb{C}^n$ and is invariant under holomorphic coordinate changes, and hence plurisubharmonic functions can be defined on complex manifolds (or even complex spaces).

Plurisubharmonic functions play very important roles in several complex variables and complex geometry. Geometrically, a basic fact is that plurisubharmonic functions are connected to positivity of curvatures of Hermitian holomorphic vector bundles.
For the line bundle case, let $L$ be a holomorphic line bundle over a complex manifold $X$ and $h$ be a Hermitian metric on $L$. Let $e$ be a holomorphic local frame of $L$ on some open set $U \subset X$. Then $\|e\|^2$ is a positive function on $U$ and thus can be written as $e^{-\phi}$ for some smooth function $\phi$ on $U$. Then the curvature of $(L, h)$ on $U$ can be written as $i\partial\bar{\partial} \phi$. Therefore $(L, h)$ is positive (resp. semi-positive) if and only if its local weights $\phi$'s are strictly plurisubharmonic (resp. plurisubharmonic). In applications, it is also very important to allow $\phi$ to have singularities, and then we get the so-called singular Hermitian metrics on $L$ (see [7]).

The above characterization of positivity of Hermitian line bundles in terms of plurisubharmonicity can be generalized to Hermitian holomorphic vector bundles of higher rank. Let $(E, h)$ be a Hermitian holomorphic vector bundle over $X$. Then we can define some positivity—called the Griffiths positivity—of the curvature of the Chern connection on $E$. It turns out that $(E, h)$ is Griffiths negative (resp. semi-negative) if for any local nonvanishing holomorphic section $s$ of $E$, $\log \|s\|$ is a strictly plurisubharmonic (resp. plurisubharmonic) function, and $(E, h)$ is Griffiths positive (resp. semi-positive) if and only if its dual bundle $E^\ast$ with the dual metric $h^\ast$ is Griffiths negative (resp. semi-negative).

From the above discussion, we can get the conclusion that geometrically plurisubharmonicity is in some sense equivalent to Griffiths positivity of Hermitian holomorphic vector bundles.

Theorems 1.2, 1.3 and 1.5–1.7 can be viewed as local characterizations of positive Hermitian line bundles. These results can be generalized to Hermitian holomorphic vector bundles of higher rank. However, to highlight the key ideas in our method, we will not discuss vector bundles of higher rank in the present paper. A development of these ideas in the setting of vector bundles and its geometric applications will appear in the forthcoming paper [8].

2 Characterizations of plurisubharmonic functions in terms of $L^p$-estimates of $\bar{\partial}$

2.1 Optimal $L^2$-estimate property

The aim of this subsection is to prove Theorem 1.3. For convenience, we restate it here.

**Theorem 2.1** (= Theorem 1.3). Let $D$ be a domain in $\mathbb{C}^n$, $\phi \in C^2(D)$ and $\omega$ be a continuous real $(1,1)$-form on $D$. If for any $\bar{\partial}$-closed smooth $(0,1)$-form $f$ on $D$ with compact support and any smooth strictly plurisubharmonic function $\psi$ on $D$ with $i\partial\bar{\partial}\psi + \omega > 0$ on supp$f$, the equation $\bar{\partial}u = f$ can be solved on $D$ with the estimate

$$\int_D \|u\|^2 e^{-\phi-\psi} \leq \int_D \|f\|^2_0 e^{-\phi+\psi}$$

provided that the right-hand side is finite, then $i\partial\bar{\partial}\phi \geq \omega$ on $D$.

We need the following lemma.

**Lemma 2.2** (See [13, Proposition 2.1.2]). Let $D$ be a domain in $\mathbb{C}^n$ and $\phi \in C^2(D)$. For any $\alpha = \sum_{j=1}^n \alpha_j dz_j \in D_{0,1}(D)$, we have

$$\int_D \sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-\phi} + \int_D \sum_{j,k=1}^n \left| \frac{\partial \alpha_j}{\partial \bar{z}_k} \right|^2 e^{-\phi} = \int_D |\bar{\partial} \alpha|^2 e^{-\phi} + \int_D |\bar{\partial}^* \alpha|^2 e^{-\phi}.$$  \hspace{1cm} (2.1)

We now give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We give the proof of Theorem 2.1 in the case where $\omega$ is $C^1$, and the general case follows from the proof by an approximation argument.

Let $\psi$ be any smooth strictly plurisubharmonic function on $D$, and

$$\omega = \frac{1}{n} \sum_{j,k=1}^n g_{jk} dz_j \wedge d\bar{z}_k.$$
By assumption, we can solve the equation $\bar{\partial}u = f$ for any $\bar{\partial}$-closed $f \in \mathcal{D}_{0,1}(D)$ with the estimate
\[
\int_D |u|^2 e^{-(\phi + \psi)} \leq \int_D |f|^2 e^{-(\phi + \psi)}.
\] (2.2)

For any $\alpha \in \mathcal{D}_{0,1}(D)$, we have
\[
|\langle \alpha, f \rangle_{\phi + \psi}| = |\langle \alpha, \bar{\partial}u \rangle_{\phi + \psi}|
= |\langle \bar{\partial} \alpha, u \rangle_{\phi + \psi}|
\leq \|u\|_{\phi + \psi} \|\bar{\partial} \alpha\|_{\phi + \psi}.
\]

From (2.2) and Lemma 2.2, we obtain
\[
|\langle \alpha, f \rangle_{\phi + \psi}|^2
\leq \int_D |f|^2 e^{-(\phi + \psi)} \times \left( \int_D \sum_{j,k} \frac{\partial^2 (\phi + \psi)}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} + \int_D \sum_{j,k} \frac{\partial^2 (\phi + \psi)}{\partial \bar{z}_j \partial z_k} \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} \right)
\leq \int_D |f|^2 e^{-(\phi + \psi)} \times \left( \int_D \sum_{j,k} \frac{\partial^2 (\phi + \psi)}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} + \int_D \sum_{j,k} \frac{\partial^2 (\phi + \psi)}{\partial \bar{z}_j \partial z_k} \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} \right).
\] (2.3)

Let $f = \sum_{j=1}^n f_j dz_j$, and set
\[
(\alpha_1, \alpha_2, \ldots, \alpha_n) = (f_1, f_2, \ldots, f_n) \left( \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} + g_{jk} \right)^{-1}_{n \times n},
\]
i.e.,
\[
(f_1, f_2, \ldots, f_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \left( \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} + g_{jk} \right)^{-1}_{n \times n}.
\]

Then the inequality (2.3) becomes
\[
\left( \int_D \sum_{j,k=1}^n \left( \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} + g_{jk} \right) \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} \right)^2
\leq \int_D \sum_{j,k=1}^n \left( \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} + g_{jk} \right) \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} \left( \int_D \sum_{j,k=1}^n \frac{\partial^2 (\phi + \psi)}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} + \int_D \sum_{j,k=1}^n \frac{\partial^2 (\phi + \psi)}{\partial \bar{z}_j \partial z_k} \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} \right).
\]

Therefore, we can get
\[
\int_D \sum_{j,k=1}^n \left( \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k} - g_{jk} \right) \alpha_j \bar{\alpha}_k e^{-(\phi + \psi)} + \int_D \sum_{j,k=1}^n \frac{\partial \alpha_j}{\partial \bar{z}_k} \frac{\partial \alpha_k}{\partial z_j} e^{-(\phi + \psi)} \geq 0.
\] (2.4)

We argue by contradiction. Suppose that $i\partial \bar{\partial} \phi - \omega$ is not a semipositive $(1,1)$-form on $D$. Then there are $z_0 \in D$, $r > 0$, and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$ with $|\xi| = 1$, such that
\[
\sum_{j,k=1}^n \left( \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} - g_{jk} \right) \xi_j \bar{\xi}_k < -c
\]
holds for any
\[
z \in B(z_0, r) := \{ z \in \mathbb{C}^n : |z - z_0| < r \} \subset D.
\]

We may assume that $z_0 = 0$, and write $B(0, r)$ as $B_r$. 
Choose \( \chi \in C_0^\infty(B_r) \) satisfying \( \chi(z) = 1 \) for \( z \in B_{r/2} \). Let \( f = \bar{\partial} \nu \) with
\[
\nu(z) = \left( \sum_{j=1}^{n} \xi_j \bar{z}_j \right) \chi(z).
\]
Then
\[
f(z) = \sum_{j=1}^{n} \xi_j d\bar{z}_j
\]
for \( z \in B_{r/2} \). For \( s > 0 \), set
\[
\psi_s(z) = s \left( |z|^2 - \frac{r^2}{4} \right).
\]
It is obvious that \( i \partial \bar{\partial} \psi_s + \omega > 0 \) on \( \text{supp} f \) for \( s \gg 1 \). As before, set
\[
(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s) = (f_1, f_2, \ldots, f_n) \left( \frac{\partial^2 \psi_s}{\partial \bar{z}_j \partial z_k} + g_{jk} \right)^{-1} = (f_1, f_2, \ldots, f_n) \cdot \frac{1}{s} \left( \delta_{jk} + \frac{g_{jk}}{s} \right)^{-1}.
\]
We now estimate the integrations on the left-hand side of (2.4) on \( B_{r/2} \) and \( D \setminus B_{r/2} \) separately with \( \alpha \) and \( \psi_s \) replaced by \( \alpha^s \) and \( \psi_s \), respectively.

- On \( B_{r/2} \), as \( s \to +\infty \), we have
  \[
  \alpha^s(z) = \frac{1}{s} \sum_{j=1}^{n} \xi_j d\bar{z}_j + o \left( \frac{1}{s} \right)
  \]
  and
  \[
  \frac{\partial \alpha^s_j}{\partial \bar{z}_k}(z) = o \left( \frac{1}{s} \right), \quad j, k = 1, 2, \ldots, n.
  \]
  Hence we have
  \[
s^2 \int_{B_{r/2}} \sum_{j,k=1}^{n} \left( \frac{\partial^2 \phi}{\partial \bar{z}_j \partial z_k} - g_{jk} \right) \alpha^s_j \bar{\alpha}^s_k e^{-(\phi + \psi_s)} + s^2 \int_{B_{r/2}} \sum_{j,k=1}^{n} \left| \frac{\partial \alpha^s_j}{\partial \bar{z}_k} \right|^2 e^{-(\phi + \psi_s)}
  \leq \int_{B_{r/2}} (-c + o(1)) e^{-(\phi + \psi_s)}. \tag{2.5}
\]

- On \( D \setminus B_{r/2} \), since \( f \) has compact support, there is a constant \( C > 0 \), such that \( |\alpha^s_j| \leq \frac{C}{s} \) and \( \left| \frac{\partial \alpha^s_j}{\partial \bar{z}_k} \right| \leq \frac{C}{s} \) hold for \( j, k = 1, 2, \ldots, n \) and \( s > 0 \). Note also that
  \[
  \lim_{s \to +\infty} \psi_s(z) = +\infty
  \]
  for \( z \in D \setminus \overline{B}_{r/2} \). We get
  \[
  \lim_{s \to +\infty} \left( s^2 \int_{D \setminus B_{r/2}} \sum_{j,k=1}^{n} \left( \frac{\partial^2 \phi}{\partial \bar{z}_j \partial z_k} - g_{jk} \right) \alpha^s_j \bar{\alpha}^s_k e^{-(\phi + \psi_s)} + s^2 \int_{D \setminus B_{r/2}} \sum_{j,k=1}^{n} \left| \frac{\partial \alpha^s_j}{\partial \bar{z}_k} \right|^2 e^{-(\phi + \psi_s)} \right) = 0. \tag{2.6}
\]

Noting that \( \psi_s \leq 0 \) on \( B_{r/2} \), and combining the equalities (2.5) and (2.6), we have
\[
\int_{D} \sum_{j,k=1}^{n} \left( \frac{\partial^2 \phi}{\partial \bar{z}_j \partial z_k} - g_{jk} \right) \alpha^s_j \bar{\alpha}^s_k e^{-(\phi + \psi_s)} + \int_{D} \sum_{j,k=1}^{n} \left| \frac{\partial \alpha^s_j}{\partial \bar{z}_k} \right|^2 e^{-(\phi + \psi_s)} < 0
\]
for \( s \gg 1 \), which contradicts the inequality (2.4). \( \square \)

**Remark 2.3.** From the proof, we can see that in Theorem 2.1, \( \psi \) can be taken to be of the form \( a|z - w|^2 + b \), with \( a \gg 1 \) and \( b \in \mathbb{R} \).

**Remark 2.4.** In Theorem 1.2, we can allow \( \phi \) to have poles with the condition that \( \text{Pol}(\phi) := \phi^{-1}(-\infty) \) is closed in \( D \), and \( \phi \) is upper semi-continuous on \( D \), and is \( C^2 \)-smooth and satisfies the optimal \( L^2 \)-estimate property on \( D \setminus \text{Pol}(\phi) \).
2.2 Multiple coarse $L^p$-estimate property

The purpose of this subsection is to prove Theorem 1.5, which is the following theorem.

**Theorem 2.5** (= Theorem 1.5). Let $D$ be a domain in $\mathbb{C}^n$ and $\phi : D \to \mathbb{R}$ be a continuous function. If $\phi$ satisfies the multiple coarse $L^p$-estimate property for some $p > 1$, then $\phi$ is plurisubharmonic on $D$.

Proof. We prove the theorem by modifying the idea in [14]. We will show that $(D, \phi)$ satisfies the multiple coarse $L^p$-extension property. Noting that $\phi |_{D'}$ also satisfies the multiple coarse $L^p$-estimate property for any open set $D' \subset D$, replacing $D$ by a relatively compact open subset of it, we may assume that $D$ is bounded and $\phi$ is uniformly continuous on $D$.

Fix an integer $m > 0$ and $w \in D$. We will construct a holomorphic function $f \in \mathcal{O}(D)$ such that $f(w) = 1$ and

$$
\int_D |f|^p e^{-m\phi(w)} \leq C_m e^{-m\phi(w)},
$$

where $C_m'$ is a constant independent of the choice of $w \in D$ satisfying $\lim_{m \to \infty} \frac{\log C_m'}{m} = 0$.

Let $\chi = \chi(t)$ be a smooth function on $\mathbb{R}$, such that

- $\chi(t) = 1$ for $t \leq 1/4$,
- $\chi(t) = 0$ for $t \geq 1$, and
- $|\chi'(t)| \leq 2$ on $\mathbb{R}$.

Define a $(0, 1)$-form $\alpha_\epsilon$ by

$$
\alpha_\epsilon := \partial \chi \left( \frac{|z - w|^2}{\epsilon^2} \right) = \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \sum_j \frac{z_j - w_j}{\epsilon} \, dz_j
$$

and set

$$
\psi_\delta := |z|^2 + n \log(|z - w|^2 + \delta^2),
$$

where $0 < \epsilon \leq 1$ and $\delta \geq 0$. From the multiple coarse $L^p$-estimate property, we obtain a smooth function $u_{\epsilon, \delta}$ on $D$ such that $\partial u_{\epsilon, \delta} = \alpha_\epsilon$ and

$$
\int_D |u_{\epsilon, \delta}|^p e^{-(m\phi + \psi_\delta)} \leq C_m \int_D |\alpha_\epsilon|^p e^{-(m\phi + \psi_\delta)}.
$$

Since

$$
|\alpha_\epsilon|_{\partial \bar{\partial} \psi_\delta} = \left| \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \right| \cdot \frac{1}{\epsilon^2} \left| \sum_j (z_j - w_j) \, dz_j \right|_{\partial \bar{\partial} \psi_\delta}
$$

$$
\leq \left| \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \right| \cdot \frac{1}{\epsilon^2} \left| \sum_j (z_j - w_j) \, dz_j \right|_{\partial \bar{\partial} |z|^2}
$$

$$
= \left| \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \right| \cdot \frac{1}{\epsilon^2} |z - w|,
$$

the support of $\chi' \left( \frac{|z - w|^2}{\epsilon^2} \right)$ is in $\{1/4 \leq |z - w|^2/\epsilon^2 \leq 1\}$, and $\psi_\delta \geq 2n \log |z - w|$, we have

$$
C_m \int_D |\alpha_\epsilon|^p e^{-(m\phi + \psi_\delta)} \leq C_m \int_{\{\epsilon^2/4 \leq |z - w|^2 \leq \epsilon^2\}} \left| \chi' \left( \frac{|z - w|^2}{\epsilon^2} \right) \right|^p \frac{1}{\epsilon^{2p}} |z - w|^p e^{-(m\phi + \psi_\delta)}
$$

$$
\leq C_m \frac{2p}{\epsilon^{2p}} \int_{\{\epsilon^2/4 \leq |z - w|^2 \leq \epsilon^2\}} |z - w|^p e^{-(m\phi + \psi_\delta)}
$$

$$
\leq C_m \frac{2p}{\epsilon^{2p}} \int_{\{\epsilon^2/4 \leq |z - w|^2 \leq \epsilon^2\}} e^{p e^{-m \inf_{B(w, \epsilon)} \phi} e^{-2n \log |z - w|}}
$$

$$
\leq CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \frac{e^{-2n \log |z - w|}}{\epsilon^p},
$$

where $C = 2^{p+2n} \mu(B_1)$.

To summarize, we have obtained a smooth function $u_{\epsilon, \delta}$ on $D$ such that
• \(\partial u_{\epsilon, j} = \alpha_{\epsilon} \), and 
• the following estimate holds:

\[
\int_{D} |u_{\epsilon, j}|^p e^{-(m\phi + \psi_{\epsilon})} \leq CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p}.
\]  

(2.9)

Note that the weight function \(\psi_{\delta}\) is decreasing when \(\delta \searrow 0\), and \(e^{-\psi_{\delta}}\) is increasing when \(\delta \searrow 0\). Fix \(\delta_0 > 0\). Then for \(\delta < \delta_0\), we have 

\[
\int_{D} |u_{\epsilon, j}|^p e^{-(m\phi + \psi_{\epsilon})} \leq \int_{D} |u_{\epsilon, j}|^p e^{-(m\phi + \psi_{\delta})} \leq CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p}.
\]

Thus \(\{u_{\epsilon, j}\}_{\delta < \delta_0}\) forms a bounded sequence in \(L^p(D, e^{-(m\phi + \psi_{\delta})})\). Noting that \(p > 1\), we can choose a sequence \(\{u_{\epsilon, j}\}_k\) in \(L^p(D, e^{-(m\phi + \psi_{\delta})})\) which weakly converges to some \(u_{\epsilon} \in L^p(D, e^{-(m\phi + \psi_{\delta})})\), satisfying

\[
\int_{D} |u_{\epsilon}|^p e^{-(m\phi + \psi_{\epsilon})} \leq CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p}.
\]

By repeating this argument for a sequence \(\{\delta_j\}\) decreasing to 0, and by the diagonal argument, we can select a sequence \(\{u_{\epsilon, j}\}_k\) which weakly converges to \(u_{\epsilon} \in L^p(D, e^{-(m\phi + \psi_{\epsilon})})\) and where \(u_{\epsilon}\) satisfies the estimates

\[
\int_{D} |u_{\epsilon}|^p e^{-(m\phi + \psi_{\epsilon})} \leq CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p}.
\]

for all \(j\). By the monotone convergence theorem,

\[
\int_{D} |u_{\epsilon}|^p e^{-(m\phi + \psi_{\epsilon})} \leq CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p}.
\]

Since \(\bar{\partial} e\) is weakly continuous, we also have \(\partial u_{\epsilon} = \alpha_{\epsilon}\).

Since \(\frac{1}{|z-w|^2}\) is not integrable near \(w\), \(u_{\epsilon}(w)\) must be 0. Let \(f_{\epsilon} := \chi(|z-w|^2/\epsilon^2) - u_{\epsilon}\).

Then \(f_{\epsilon} \in \mathcal{O}(D), \ f_{\epsilon}(0) = 1\) and

\[
\int_{D} |f_{\epsilon}|^p e^{-m\phi} \leq \left( \left( \int_{D} \chi \left( \frac{|z-w|^2}{\epsilon^2} \right) \frac{e^{-m\phi}}{e^2} \right)^{1/p} + \left( \int_{D} |u_{\epsilon}|^p e^{-m\phi} \right)^{1/p} \right)^p \leq 2^p \left( \int_{D} \chi \left( \frac{|z-w|^2}{\epsilon^2} \right) \frac{e^{-m\phi}}{e^2} + \int_{D} |u_{\epsilon}|^p e^{-m\phi} \right).
\]

(2.10)

Since \(\chi \leq 1\) and the support of \(\chi(|z-w|^2/\epsilon^2)\) is contained in \(\{|z-w|^2 \leq \epsilon^2\}\) and \(0 < \epsilon \leq 1\), we have

\[
\int_{D} \chi \left( \frac{|z-w|^2}{\epsilon^2} \right) \frac{e^{-m\phi}}{e^2} \leq \mu(B_{1}) e^{-m \inf_{B(w, \epsilon)} \phi}.
\]

We also have

\[
\int_{D} |u_{\epsilon}|^p e^{-m\phi} \leq \sup_{z \in D} e^{\psi_{\delta}(z)} \cdot \int_{D} |u_{\epsilon}|^p e^{-(m\phi + \psi_{\epsilon})} \leq \sup_{z \in D} e^{\psi_{\delta}(z)} \cdot CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p} \leq C' CC_m e^{-m \inf_{B(w, \epsilon)} \phi} \cdot \frac{\epsilon^p}{e^p},
\]

where \(C'\) is a constant depending only on the diameter of \(D\). We may assume \(C_m \geq 1\). Combining these estimates with (2.10), we obtain

\[
\int_{D} |f_{\epsilon}|^p e^{-m\phi} \leq C'' C_m \frac{1}{\epsilon^p} e^{-m \inf_{B(w, \epsilon)} \phi}.
\]
where $C''$ is a constant independent of $m$ and $w$.

Let 

$$O_\epsilon = \sup_{z,w \in D, |z-w| \leq \epsilon} |\phi(z) - \phi(w)|.$$ 

By the uniform continuity of $\phi$, $O_\epsilon$ is finite and goes to 0 as $\epsilon \to 0$. Let $\epsilon := 1/m$. We have 

$$|m\phi(z) - m\phi(w)| \leq mO_{1/m}$$ 

for $|z - w| \leq 1/m$. Then 

$$\int_D |f_{1/m}|^p e^{-m\phi} \leq C''C_m m^p e^{-m\phi(w) + mO_{1/m}} = C''C_m m^p e^{mO_{1/m}} e^{-m\phi(w)}.$$ 

(2.11)

Let $C'_m = C''C_m m^p e^{mO_{1/m}}$. We have 

$$\log \frac{C'_m}{m} = \frac{\log(C''C_m m^p)}{m} + O_{1/m} \to 0,$$ 

which implies that $\phi$ satisfies the multiple coarse $L^p$-extension property on $D$, and hence $\phi$ is plurisubharmonic by Theorem 1.7.

Remark 2.6. In Theorem 2.5, we can allow $\phi$ to have poles with the condition that 

$$\text{Pol}(\phi) := \phi^{-1}(-\infty)$$ 

is closed in $D$, and $\phi$ is upper semi-continuous on $D$, and is continuous and satisfies the multiple coarse $L^p$-estimate property on $D \setminus \text{Pol}(\phi)$.

3 Characterizations of plurisubharmonic functions in terms of $L^p$-extensions of holomorphic functions

In this section, we discuss characterizations of plurisubharmonic functions in terms of $L^p$-extensions of holomorphic functions. The aim is to prove Theorems 1.6 and 1.7. The idea is inspired by Guan-Zhou’s method in [11].

We first prove a lemma as follows.

Lemma 3.1. Let $D \subset \mathbb{C}^n$ be a domain, and $\phi$ be an upper semi-continuous function on $D$. Then $\phi$ is plurisubharmonic if and only if for any $z_0 \in D$ and any holomorphic cylinder $P$ with $z_0 + P \subset \subset D$,

$$\phi(z_0) \leq \frac{1}{\mu(P)} \int_{z_0 + P} \phi.$$ 

Proof. The “only if” part follows easily from the mean value inequality for plurisubharmonic functions. We now give the proof of the “if” part.

For any point $z_0 \in D$, any $\xi \in \mathbb{C}^n$ with $|\xi| = 1$, and $r > 0$ such that $\{z_0 + \zeta_1 \xi : |\zeta_1| \leq r\} \subset D$, choose an orthonormal basis $f_1, f_2, \ldots, f_n$ of $\mathbb{C}^n$ with $f_1 = \xi$. There is an $s_0 > 0$ such that 

$$z_0 + \zeta_1 \xi + \sum_{j=2}^n \zeta_j f_j \in D$$ 

for all $|\zeta_1| \leq r$ and $\sum_{j=2}^n |\zeta_j|^2 \leq s_0^2$. Let $A = (f_1, f_2, \ldots, f_n)$. Then $z_0 + A(P_{r,s}) \subset \subset D$ for $s < s_0$. By assumption, we have 

$$\phi(z_0) \leq \frac{1}{\mu(A(P_{r,s}))} \int_{z_0 + A(P_{r,s})} \phi.$$
$$= \frac{1}{\mu(P_{r,1})} \int_{P_{r,1}} \phi \left( z_0 + \zeta_1 \xi + s \sum_{j=2}^{n} \zeta_j f_j \right). \tag{3.1}$$

As \( \phi \) is upper semi-continuous on \( D \) and \( \{ z_0 + \zeta_1 \xi : |\zeta_1| \leq r \} \) is compact, we may assume \( \phi \leq 0 \) on \( z_0 + A(P_{r,s_0}) \). Then we have

$$\phi(z_0) \leq \frac{1}{\mu(P_{r,1})} \int_{P_{r,1}} \limsup_{s \to 0} \phi \left( z_0 + \zeta_1 \xi + s \sum_{j=2}^{n} \zeta_j f_j \right)$$

$$\leq \frac{1}{\pi r^2} \int_{|\zeta_1|<r} \phi(z_0 + \zeta_1 \xi),$$

where the first inequality follows from (3.1) and Fatou’s lemma, the second inequality follows from the fact that \( \phi \) is upper semi-continuous, and the last equality holds since \( \phi(z_0 + \zeta_1 \xi) \) is independent of \( \zeta_2, \ldots, \zeta_n \).

\[\square\]

**Remark 3.2.** The condition “holomorphic cylinder” in Lemma 3.1 cannot be replaced by “polydisc”. For example, let \( \phi(z_1, z_2) = z_1 \bar{z}_2 + \bar{z}_1 z_2 \). One can get that for any polydisc \( P = \{(z_1, z_2) : |z_1| < r_1, |z_2| < r_2\} \),

where \( r_1 > 0, r_2 > 0 \) with \( z_0 + P \subset \subset \mathbb{C}^2 \),

$$\phi(z_0) = \frac{1}{\mu(P)} \int_{z_0 + P} \phi.$$

However, \( \phi \) is not a plurisubharmonic function.

**Theorem 3.3** (\( \equiv \) Theorem 1.6). Let \( \phi : D \to [-\infty, +\infty) \) be an upper semi-continuous function on a domain \( D \) in \( \mathbb{C}^n \). If \( \phi \) satisfies the optimal \( L^p \)-extension property for some \( p > 0 \), then \( \phi \) is plurisubharmonic on \( D \).

**Proof.** Take arbitrary \( z_0 \in D \), such that \( \phi(z_0) \neq -\infty \). For any holomorphic cylinder \( P \) with \( z_0 + P \subset D \), by the optimal \( L^p \)-extension property and taking log, we get

$$\phi(z_0) \leq - \log \left( \frac{1}{\mu(P)} \int_{z_0 + P} |f|^p e^{-\phi} \right).$$

By Jensen’s inequality, we have

$$\phi(z_0) \leq \frac{1}{\mu(P)} \int_{z_0 + P} - \log(|f|^p e^{-\phi})$$

$$= \frac{1}{\mu(P)} \int_{z_0 + P} \log \phi - \frac{1}{\mu(P)} \int_{z_0 + P} p \log |f|.$$

Noting that \( p \log |f| \) is a plurisubharmonic function and \( f(z_0) = 1 \), by Fubini’s theorem, we have

$$- \frac{1}{\mu(P)} \int_{z_0 + P} p \log |f| \leq 0.$$

Therefore, we get the desired mean-value inequality

$$\phi(z_0) \leq \frac{1}{\mu(P)} \int_{z_0 + P} \phi.$$

By Lemma 3.1, \( \phi \) is plurisubharmonic on \( D \).  \( \square \)
Theorem 3.4 (\(=\) Theorem 1.7). Let \(\phi: D \to [-\infty, +\infty)\) be an upper semi-continuous function on a domain \(D\) in \(\mathbb{C}^n\). If \(\phi\) satisfies the multiple coarse \(L^p\)-extension property for some \(p > 0\), then \(\phi\) is plurisubharmonic on \(D\).

Proof. Let \(z_0 \in D\) such that \(\phi(z_0) \neq -\infty\), and let \(P\) be a holomorphic cylinder with \(z_0 + P \subseteq D\). By assumption, for any \(m \geq 1\), there is a holomorphic function \(f_m\) on \(D\) with \(f_m(z_0) = 1\) and

\[
\int_{z_0 + P} |f_m|^p e^{-m\phi} \leq \int_D |f_m|^p e^{-m\phi} \leq C_m e^{-m\phi(z_0)}.
\]

Dividing both sides by \(\mu(P)\) and taking log, we get

\[
\log \frac{1}{\mu(P)} \int_{z_0 + P} |f_m|^p e^{-m\phi} \leq \log \frac{C_m}{\mu(P)} - m\phi(z_0).
\]

Then we have

\[
\phi(z_0) \leq \frac{\log C_m}{m} - \frac{\log \mu(P)}{m} - \frac{1}{m} \log \frac{1}{\mu(P)} \int_{z_0 + P} |f_m|^p e^{-m\phi}.
\]

By Jensen’s inequality, we have

\[
- \frac{1}{m} \log \left( \frac{1}{\mu(P)} \int_{z_0 + P} |f_m|^p e^{-m\phi} \right)
\leq - \frac{1}{m} \frac{1}{\mu(P)} \int_{z_0 + P} \log(|f_m|^p e^{-m\phi})
= - \frac{1}{m} \frac{1}{\mu(P)} \int_{z_0 + P} p \log |f_m| + \frac{1}{\mu(P)} \int_{z_0 + P} \phi.
\]

By noting that \(p \log |f_m|\) is a plurisubharmonic function and \(f_m(z_0) = 1\), the first term in the above equality is nonpositive. Therefore,

\[
- \frac{1}{m} \log \left( \frac{1}{\mu(P)} \int_{z_0 + P} |f_m|^p e^{-m\phi} \right) \leq \frac{1}{\mu(P)} \int_{z_0 + P} \phi.
\]

Combining the inequalities (3.2) and (3.3), we get

\[
\phi(z_0) \leq \frac{\log C_m}{m} - \frac{\log \mu(P)}{m} + \frac{1}{\mu(P)} \int_{z_0 + P} \phi.
\]

As \(\lim_{m \to \infty} \frac{\log C_m}{m} = 0\) and \(\lim_{m \to \infty} \frac{\log \mu(P)}{m} = 0\), taking limit, we have

\[
\phi(z_0) \leq \frac{1}{\mu(P)} \int_{z_0 + P} \phi.
\]

From Lemma 3.1, \(\phi\) is plurisubharmonic on \(D\).

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