2-CLUB is NP-hard for distance to 2-club cluster graphs

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Abstract. We show that 2-CLUB is NP-hard for distance to 2-club cluster graphs.

1 Introduction

A complete graph or clique is a graph that contains an edge for every pair of distinct vertices. Diameter of a graph is the length of a longest shortest path in the graph. Any clique has diameter 1. A generalization of this notion is s-club, a graph of diameter s. In general graphs, finding a set of vertices that induces a subgraph of diameter s is NP-hard. For s = 2, Hartung et al [HKN13] have studied the problem with many structural restrictions on the input graph. This paper answers one of the open problems mentioned in [HKN13].

Given a class of graphs with some property \( \Pi \), we can define another class of graphs by the parameter distance to \( \Pi \), namely the number of vertices that needs to be deleted from the graph to make the resultant graph have property \( \Pi \). For example, distance 2 to bipartiteness defines a class of graphs that become bipartite after deleting at most 2 vertices. A graph where each connected component is an s-club is called s-club cluster graph. In this paper, we show that finding 2-club in distance d to 2-club cluster graphs is NP-hard for \( d \geq 2 \).

2 Constant Distance to 2-club cluster

We define the 2-CLUB problem as follows: Given an undirected graph \( G = (V,E) \) and \( k \in \mathbb{N} \), is there a vertex set \( S \subseteq V \) of size at least \( k \) such that \( G[S] \) has diameter at most 2?

**Theorem 1.** 2-CLUB is NP-hard even on graphs with distance two to 2-club cluster.

**Proof.** We reduce from the NP-hard CLIQUE problem: Given a positive integer \( k \) and a graph \( H \), the question is whether there is a clique of size at least \( k \).

Given an instance \((H,k)\) of CLIQUE, we construct an undirected graph \( G = (V,E) \).

Let \( |V(H)| = n \). Define the vertex set

\[
V(G) := V(H) \cup A \cup \{a,b,u\} \cup X_1 \cup X_2
\]

where \( a, b, u \) are vertices and \( A, B, X_1, X_2 \) are sets of vertices with sizes \( |X_1| = n^3, |X_2| = n^2 - n \) and \( |A| = n^2 \).

For every vertex \( v_i \in V(H) \), we label \( n \) vertices of \( A \) as \( V_i = \{v_{i,1}, ..., v_{i,n}\} \). The edge set \( E(G) \) is defined as

\[
E(G) = E(H) \cup a \times \{b\} \cup X_1 \cup A \cup X_2 \cup V(H) \times X_2 \cup u \times \{A \cup V(H) \cup X_2\} \bigcup_{v_i \in [n]} v_i \times V_i
\]

Note that all the edges are undirected. See Figure below.

**Claim.** \( H \) has a clique of size \( k \) if and only if \( G \) has a 2-club of size \( n^3 + n^2 + (k-1)n + k + 2 \).
Proof. Let $S$ be a clique of size $k$ in $H$. Then, $X_1 \cup X_2 \cup S \cup (N(S) \cap A) \cup \{a, b\}$ is a 2-club of size $n^3 + n^2 + (k - 1)n + k + 2$.

Let $Y$ be a 2-club of size $n^3 + n^2 + (k - 1)n + k + 2$ in $G$.

By size consideration $X_1 \subset Y$. If $b \notin Y$, then none of $B$ and $X_2$ can be in $Y$. Consequently, the size of any 2-club in $G$ can be $n^3 + n^2 + 1$. Hence, we must have that $b \in Y$. By similar reasoning, we have that $a \in Y$.

If $A \cap Y = \emptyset$, then the size of the largest 2-club can be at most $n^3 + n^2 + 2$ implying that $Y$ must intersect with $A$. Moreover, $|A \cap Y|$ must be a multiple of $n$ as for $v_i \in A$ contained in $Y$, the whole subset $V_i \subset A$ can be included in $Y$ preserving the 2-club property. If $|A \cap Y| < (k - 1)n$, then size of the maximum 2-club can be at most $n^3 + n^2 + (k - 1)n + 2$, the size of $X_1 \cup X_2 \cup V(H) \cup \{a, b\} \cup (A \cap Y)$ which is less than $n^3 + n^2 + (k - 1)n + k + 2$. Hence at least $k$ vertices in $V(H) \cap Y$ have neighbors in $A \cap Y$. This also implies that $V(H) \cap Y$ forms a clique in $H$. If $\{x, y\} \in V(H) \cap Y$ are not adjacent and have neighbors $\{x', y'\} \in A \cap Y$. Then, there is no path of length $\leq 2$ between $x$ and $y'$. Hence, $H$ has a clique of size $k$.

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References

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