Weakly singular symmetric Galerkin boundary element method for fracture analysis of three-dimensional structures considering rotational inertia and gravitational forces

Shuangxin He¹, Xuan Zhou¹, Leiting Dong¹*, Satya N. Atluri²

¹ School of Aeronautic Science and Engineering, Beihang University, China
² Department of Mechanical Engineering, Texas Tech University, USA

Abstract: The Symmetric Galerkin Boundary Element Method is advantageous for the linear elastic fracture and crack-growth analysis of solid structures, because only boundary and crack-face elements are needed. However, for engineering structures subjected to body forces such as rotational inertia and gravitational loads, additional domain integral terms in the Galerkin boundary integral equation will necessitate meshing of the interior of the domain. In this study, weakly-singular SGBEM for fracture analysis of three-dimensional structures considering rotational inertia and gravitational forces are developed. By using divergence theorem or alternatively the radial integration method, the domain integral terms caused by body forces are transformed into boundary integrals. And due to the weak singularity of the formulated boundary integral equations, a simple Gauss-Legendre quadrature with a few integral points is sufficient for numerically evaluating the SGBEM equations. Some numerical examples are presented to verify this approach and results are compared with benchmark solutions.

Keywords: Symmetric Galerkin boundary element method, Rotational inertia, Gravitational force, Weak singularity, Stress intensity factor

1. Introduction

The Symmetric Galerkin Boundary Element Method (SGBEM)[1-3] has gained increasing popularity in fracture and crack-growth analysis of solid structures due to its attractive features of symmetric coefficient matrices, weak-singularity, and that only boundary & crack-face elements are needed. The papers by Bonnet, Mair, Frangi et al.[4-7] are devoted to the formulation, numerical evaluation and implementation of SGBEM. Atluri, Okada, Han and Dong [8-11] utilized a simple and straightforward methodology to develop regularized traction Boundary Integral Equations (tBIE) two and three-dimensional linear-elastic solids containing cracks, and also developed weakly-singular SGBEMs for the fracture and fatigue analysis of various complex structures. However, for the fracture mechanics problems such as of turbine discs and turbine blades of aircraft engines, concrete gravity dam, etc., SGBEM may lose it advantages, because evaluation of domain integral terms resulting from body forces such as rotational inertia and gravitational loads leads to the meshing of the interior of the domain. For this reason, is a method to evaluate such domain integral terms using only boundary meshes, is desired to efficiently analyze cracked structures considering body forces with SGBEM.

For the conventional collocation boundary element method based on Somigliana’s identity

* Corresponding author: ltdong@buaa.edu.cn (L. Dong). Address (L.D.): School of Aeronautic Science and Engineering, Beihang University, Beijing, 100191, CHINA.
for the displacement vector, a few methods were developed for this purpose. Considering centrifugal loads present in rotating gas turbines, Cruse [12] transformed domain integrals to boundary integrals by utilizing the divergence theorem. By making use of the Galerkin vector or the Green’s second identity, Danson [13] transformed the volume integral terms to boundary integral terms, for three kind of body forces, i.e. gravitational loads, the rotational inertia and steady state thermal loads. Gao [14] also developed a radial integration technique and applied it to deal with various body forces. Nardini and Brebbia [15] developed the dual reciprocity method [16] which converts the associated domain integrals into boundary integrals by using a series of basis functions to approximate the body force fields. Nowak and Brebbia extended the idea of dual reciprocity and proposed another approach, multiple reciprocity method [17].

Different from the conventional collocation boundary element method [18] based on the Somigliana’s identity, formulations of SGBEM [5, 8, 19] result in weak-form displacement Boundary Integral Equations (dBIE) and weak-form traction Boundary Integral Equations (tBIE). As a matter of fact, the domain integrals caused by body forces appear both in dBIE and tBIE. Moreover, it is beneficial to use tBIE to derive weak-form equations on crack-faces, where displacement discontinuities are the to be solved unknowns [19]. Thus, if SGBEM is utilized for linear fracture analysis of cracked structures, while for domain integrals appearing in dBIE one may refer to the above-mentioned transformation techniques, the treatment for domain integral terms appearing in tBIE needs further study.

This paper presents the weakly singular traction boundary integral equation for solids undergoing rotational inertia and gravitational Loads. By using the divergence theorem (div) or a radial integration method (RIM), rotational inertia or gravitational forces induced domain integrals are transformed into boundary integrals correspondingly. The thus derived formulas show that these transformed boundary integral terms have no influence upon the coefficient matrix of SGBEM, but only affect the right-hand-side vector. The transformed boundary integral terms derived by the divergence theorem and radial integration method, possessing $1/r$ singularity, is weakly singular. Numerical examples demonstrate that only a few Gauss points are sufficient to evaluate to boundary integrals. The developed SGBEM with only weakly-singular boundary integrals are thus applied to simulate various examples of 3D solids with/without considering rotational inertia and gravitational loads.

This paper is organized as follows. In Section 2, transformation from domain integrals induced by gravitational and rotational inertia forces to the boundary by div or RIM respectively is carried out. Some numerical examples for solids undergoing rotational inertia or gravitational loads are presented in section 3 and 4 with and without cracks correspondingly. In section 5, we complete this paper with some concluding remarks.

2. Weakly-singular Galerkin boundary integral equations and boundary element method with rotational inertia and gravitational loads

The symmetric Galerkin formulations of displacement and traction boundary Integral Equations (d&tBIE) for linear elastic solids can be found in [8]. Here, the domain integrals containing body forces are added in the formulations.
\[-\int_{\Omega} \frac{1}{2} \delta t_p(x) u_p(x) dS_x = -\int_{\Omega} \delta t_p(x) dS_x \int_{\Omega} f_j(\xi) u^{*p}_j(\xi - x) d\Omega_\xi \]

\[-\int_{\Omega} \delta t_p(x) dS_x \int_{\Omega} t_j(\xi) u^{*p}_j(\xi - x) d\Gamma_\xi \]

\[-\int_{\Omega} \delta t_p(x) dS_x \int_{\Omega} D_t(\xi) u_j(\xi) G_{ij}^{*p}(\xi - x) d\Gamma_\xi \]

\[-\int_{\Omega} \delta t_p(x) dS_x \int_{\Omega} \frac{1}{2} \delta u_b(\xi) t_b(\xi) dS_x = -\int_{\Omega} \delta u_b(\xi) n_a(\xi) dS_x \int_{\Omega} f_j(\xi) \sigma_{ab}^{*j}(\xi - x) d\Omega_\xi \]

\[-\int_{\Omega} \delta t_p(x) dS_x \int_{\Omega} \frac{1}{2} \delta u_b(\xi) t_b(\xi) dS_x \int_{\Omega} D_t(\xi) u_j(\xi) G_{tbpq}^{*p}(\xi - x) d\Gamma_\xi \]

\[-\int_{\Omega} D_t(\xi) \delta u_b(\xi) dS_x \int_{\Omega} D_p(\xi) u_q(\xi) H_{tbpq}^{*p}(\xi - x) d\Gamma_\xi \]

\[\frac{1}{2} \int_{\Omega} \delta u_b(\xi) t_b(\xi) dS_x = -\int_{\Omega} \delta u_b(\xi) n_a(\xi) dS_x \int_{\Omega} f_j(\xi) \sigma_{ab}^{*j}(\xi - x) d\Omega_\xi \]

\[\frac{1}{2} \int_{\Omega} \delta t_p(x) u_p(x) dS_x = -\int_{\Omega} \delta t_p(x) dS_x \int_{\Omega} f_j(\xi) u^{*p}_j(\xi - x) d\Omega_\xi \]

\[\frac{1}{2} \int_{\Omega} \delta u_b(\xi) t_b(\xi) dS_x = -\int_{\Omega} \delta u_b(\xi) n_a(\xi) dS_x \int_{\Omega} f_j(\xi) \sigma_{ab}^{*j}(\xi - x) d\Omega_\xi \]

In the above two equations, \( x \) represents the source point and \( \xi \) represents the field point. \( r \) is the distance between source point and field point. \( \Omega \) is the domain of the problem. If the domain integral or boundary integral is with respect to the field point, the integral domain is denoted by \( \Omega_\xi \) or \( \Gamma_\xi \) respectively; otherwise, the integral domain is denoted by \( \Omega_x \) or \( \Gamma_x \) respectively. \( u_p(x) \) and \( t_p(x) \) are the displacement and traction at the source point respectively. \( f_j(\xi) \) is the body force per unit volume. \( n_i(\xi) \) is the component of outward unit normal at a field point on the boundary. \( D_t \) is a surface tangential operator defined in Eq. (3)

\[D_t(\xi) = n_r(\xi) e_{rst} \frac{\partial}{\partial \xi_s}\]

where \( e_{rst} \) is the permutation coefficient defined by \( e_{123} = e_{231} = e_{312} = 1; e_{321} = e_{213} = e_{132} = -1; e_{rst} = 0 \) if any two of the indices are identical.

Several kernel functions \( u_j^{*p}(\xi - x) \), \( G_{ij}^{*p} \), \( \phi^{*p}_{ij} \), \( \sigma_{ab}^{*j} \), \( H_{tbpq}^{*p} \) appear in Eqs. (1) and (2). \( u_j^{*p}(\xi - x) \) is the fundamental solution of three dimensional linear elasticity. Consider a unit load applied in an arbitrary direction \( p \), at point \( x \) in a linear elastic isotropic homogeneous infinite medium as shown in Fig. 1, \( u_j^{*p}(\xi - x) \) is the displacement in the \( j \) direction at field point \( \xi \).

Analytical expression of other kernel functions utilized in this paper is listed in the Appendix. One may also refer to other forms of these kernel functions in [5, 19].
The radial integration method is introduced here briefly. For further details, one may refer to [14]. Domain integral on the left-hand side of Eq. (5) with a general function $f(\xi)$ may be written in Cartesian coordinate system $(x_1, x_2, x_3)$ or in spherical coordinate system $(r, \theta, \phi)$ with the origin at the source point $P$ shown in Fig. 2.

In the Cartesian coordinate system
\[
\int_{\Omega} f(\xi) d\Omega = \int_0^{2\pi} \int_0^\pi \int_0^{r(\theta, \phi)} f(r, \theta, \phi) r^2 dr \sin \theta d\theta d\phi = \int_0^{2\pi} \int_0^\pi F(\theta, \phi) \sin \theta d\theta d\phi \quad (5)
\]
where
\[
F(\theta, \phi) = \int_0^{r(\theta, \phi)} f(r, \theta, \phi) r^2 dr \quad (6)
\]

In the spherical coordinate system, the area of element $dS$ on the spherical surface can be expressed as
\[
dS = r^2 \sin \theta d\theta d\phi \quad (7)
\]

Fig. 2: Cartesian and spherical coordinate systems.

Fig. 3: Spherical surface $dS$ and real boundary $d\Gamma$

If the field point is on the boundary $\Gamma$ of domain $\Omega$, geometric projective transformation can be established between the spherical surface element $dS$ and the real boundary surface element $d\Gamma$. 
where \( n_i \) is the component of outward unit normal of field on the real boundary surface \( d\Gamma \), \( r_i \) is the Cartesian component of \( r \), i.e.

\[ n_i = \xi_i - x_i \]  

By some derivations, the domain integral may be rewritten as

\[ \int_{\Omega} f(\xi) dV = \int_{\partial\Omega} \frac{1}{r^2} \frac{\partial r}{\partial n} F(r) d\Gamma \]  

where \( F(r) \) is evaluated by a radial integration of \( R^2 f(\xi) \) on the segment linking the source point and the field point, i.e.

\[ F(r) = \int_0^r R^2 f(\xi) dR. \]

\( \frac{\partial r}{\partial n} \) is the directional derivative at the field point on the boundary, which may be expressed as

\[ \frac{\partial r}{\partial n} = r_i n_i \]

where \( (\cdot)_j \) denotes partial differentiation with respect to the Cartesian component of field point if not otherwise stated. By this definition, we have

\[ r_i = \frac{\partial r}{\partial \xi_i} = r_i = -\frac{\partial r}{\partial x_i}. \]

Some useful equations related to \( r \) are listed as follows.

\[ r = \sqrt{r_i r_i} \]

\[ r_i r_j = 1 \]

\[ r_{ij} = \frac{1}{r} \left( \delta_{ij} - r_i r_j \right) \]

\[ r_{ii} = \frac{2}{r} \]

\[ r_{ij} r_{ij} = 0 \]

\[ r_{ijk} = -\frac{1}{r^2} \left( r_i \delta_{jk} + r_j \delta_{ik} + r_k \delta_{ij} - 3r_i r_j r_k \right) \]

\[ r_{iij} = -2 \frac{1}{r^2} r_{ij} \]

\[ r_{kij} = 2 \frac{1}{r^3} \left( 3r_i r_j - \delta_{ij} \right) \]

\[ \left( \frac{1}{r} \right)_i = -\frac{1}{r^2} r_i \]

\[ \left( \frac{1}{r} \right)_{ij} = \frac{1}{r} \left( 3r_i r_j - \delta_{ij} \right) \]

In section 2.1 and 2.2, the domain integral terms with rotational inertia and gravitational loads in tBIE are transformed into weakly singular boundary integral terms by the two methods of divergence theorem and radial integration method respectively.

2.1. Transformation of domain integrals with Gravitational loads to boundary integrals

Consider a solid body with a constant mass density \( \rho \), and a constant gravitational field
\( g_i = \text{const.} \) The body force will also be constant, where
\[ f_i = \rho g_i = \text{const.} \] (24)

\( \sigma_{ab}^{ij}(\xi - x) \) is the stress field of Kelvin’s solution:
\[ \sigma_{ab}^{ij}(\xi - x) = \frac{1}{8\pi(1-\nu)^2} \left[ (1 - 2\nu)(\delta_{ab} r_{ij} - \delta_{aj} r_{ib} - \delta_{bj} r_{ia}) - 3r_{aj} R_{ib} r_{ij} \right] \] (25)

where \( \nu \) is the Poisson’s ratio; \( \delta_{ab} \) is the Kronecker Delta.

Thus, the constant gravity force \( f_i \) can be taken outside the integral in Eq. (4). Then we get
\[ \int \rho g j \sigma_{ab}^{ij}(\xi - x) d\Omega_{\xi} \]
\[ = \rho g j \int_{\Omega} \frac{1}{8\pi(1-\nu)^2} \left[ (1 - 2\nu)(\delta_{ab} r_{ij} - \delta_{aj} r_{ib} - \delta_{bj} r_{ia}) - 3r_{aj} R_{ib} r_{ij} \right] d\Omega_{\xi} \] (26)

2.1.1. Using divergence theorem to transform domain integrals with gravitational forces

Substitution of Eq. (19) into Eq. (26), we have
\[ \int \rho g j \sigma_{ab}^{ij}(\xi - x) d\Omega_{\xi} = \rho g j \frac{1}{8\pi(1-\nu)} \int_{\Omega} 2\nu \delta_{ab} \left( -\frac{1}{r^2} r_a + 2(1 - \nu) \delta_{aj} \left( -\frac{1}{r^2} r_a \right) \right) + 2(1 - \nu) \delta_{bj} \left( \frac{1}{r^2} r_b \right) \] (27)

Substituting Eq. (22) into Eq. (27), we have
\[ \int \rho g j \sigma_{ab}^{ij}(\xi - x) d\Omega_{\xi} = \rho g j \frac{1}{8\pi(1-\nu)} \int_{\Omega} 2\nu \delta_{ab} \left( \frac{1}{r^2} r_a + 2(1 - \nu) \delta_{aj} \left( \frac{1}{r^2} r_a \right) \right) + 2(1 - \nu) \delta_{bj} \left( \frac{1}{r^2} r_b \right) \] (28)

Using divergence theorem and Eq. (16), we can get that
\[ \int \rho g j \sigma_{ab}^{ij}(\xi - x) d\Omega_{\xi} = \rho g j \frac{1}{8\pi(1-\nu)} \int_{\partial\Omega} \left( 2\nu - 1 \right) \delta_{ab} n_{a}(\xi) + 2(1 - \nu) \delta_{aj} n_{b}(\xi) + 2(1 - \nu) \delta_{bj} n_{a}(\xi) \] (29)

Note that, a singularity of \( 1/r \) appears in the boundary integral of Eq. (29). This integral is weakly-singular [8], thus Cauchy principal value integral does not need to be taken into account.

The numerical integration method to evaluate this weakly-singular integral is stated briefly in section 2.3.3.

2.1.2. Using the radial integration method to transform domain integrals with gravitational forces

Using radial integration method, Eq. (26) can be rewritten as
\[ \int \rho g j \sigma_{ab}^{ij}(\xi - x) d\Omega_{\xi} = \rho g j \int_{\partial\Omega} \frac{\partial r}{\partial n} F(r) d\Gamma_{\xi} \] (30)

where
\[ F(r) = \int_{0}^{R} \frac{R^2}{8\pi(1-\nu)^2} \left[ (1 - 2\nu)(\delta_{ab} R_{ij} - \delta_{ij} R_{a} - \delta_{bj} R_{b}) - 3R_{a} R_{b} R_{ij} \right] dR \] (31)

From Eq. (13) and Fig. 2, one can find that \( R_{i} \) is the cosine between \( r \) and coordinate axis \( i \), i.e. \( R_{i} = r_i \). Thus, \( R_{i} \) can be taken out of this radial integral Eq. (31) directly. Substitution of Equation (31) into Equation (30) gives
\[ \int_{\Omega} \rho g_j \sigma_{a b}^j (\xi - x) d\Omega_{\xi} = \rho g_j \int_{\Omega} \frac{1}{8\pi} \frac{1}{r} \frac{\partial r}{\partial n} \left[ (1 - 2\nu) (\delta_{ab} r_j - \delta_{aj} r_b - \delta_{bij} r_a) - 3 r_a r_b r_j \right] d\Gamma_{\xi} \] (32)

Note that, when the field point approaches the source point, \( \partial r / \partial n \to 0 \). Singularity of the boundary integral in Eq. (32) may be weaker than that in Eq. (29).

2.2. Transform domain integrals with Rotational inertia to boundary integrals

About an analytical expression of the rotational inertial force in detail, one may refer to [18]. Here we introduce it briefly. Consider a solid body of uniform mass density \( \rho \) rotating about one axis with angular velocity \( \omega_i \). For simplicity and without loss of the generality, we consider that the axis of rotation passes thorough the origin of Cartesian coordinate system shown in Fig. 4.

Fig. 4: The rotational axis passing thorough the origin of Cartesian coordinate system

By the d’Alembert’s principle, body force resulting from the rotational inertia is

\[ f(\xi) = -\rho \omega \times (\omega \times \xi) \] (33)

Eq. (33) may be written in index notation as

\[ f_i(\xi) = -\rho e_{ijk} \omega_j e_{kpq} \omega_p \xi_q = h_{iq} \xi_q \] (34)

where

\[ h_{iq} = -\rho e_{ijk} \omega_j e_{kpq} \omega_p \] (35)

Note that \( h_{iq} \) is constant and may be described in a more straightforward way:

\[ [h_{iq}] = \rho \begin{bmatrix} \omega_2^2 + \omega_3^2 & -\omega_2 \omega_3 & -\omega_3 \omega_1 \\ -\omega_2 \omega_3 & \omega_1^2 + \omega_3^2 & -\omega_1 \omega_3 \\ -\omega_3 \omega_1 & -\omega_1 \omega_3 & \omega_1^2 + \omega_2^2 \end{bmatrix} \] (36)

Then this dynamic problem can be treated as a elasto-static problem. Using Eq. (4), Eq. (25) and Eq. (34), we get

\[ \int_{\Omega} h_{ij} \xi_i \sigma_{a b}^j (\xi - x) d\Omega_{\xi} = \int_{\Omega} h_{ij} \xi_i \frac{1}{8\pi(1-\nu)r^2} \left[ (1 - 2\nu) (\delta_{ab} r_j - \delta_{aj} r_b - \delta_{bij} r_a) - 3 r_a r_b r_j \right] d\Omega_{\xi} \] (37)

2.2.1. Using divergence theorem to transform domain integrals with inertial force

Similar to the derivation of Eq. (28), the inertial force domain integrals with the rotational inertia can be written as
\[ \int_\Omega h_{ij} \xi_i \sigma_{ab} (\xi - x) \, d\Omega_\xi = \int_\Omega \frac{1}{8\pi(1-\nu)r^2} h_{ij} \left[ 2\nu \delta_{ab} \xi_i \left( \frac{1}{r} \right)_j + 2(1-\nu)\delta_{aj} \xi_i \left( \frac{1}{r} \right)_b + 2(1-\nu)\delta_{bj} \xi_i \left( \frac{1}{r} \right)_a - \xi_i r_{ab} \right] \, d\Omega_\xi \]  

(38)

Substituting Eq. (39) and Eq. (40) into Equation (38) and using Eq. (17)

\[
\left( \xi_i \left( \frac{1}{r} \right)_j \right)_j = \delta_{ij} \left( \frac{1}{r} \right)_j + \xi_i \left( \frac{1}{r} \right)_j
\]

(39)

\[
\left( \xi_i r_{ab} \right)_j = \delta_{ij} r_{ab} + \xi_i r_{ab}
\]

(40)

we get

\[
\int_\Omega h_{ij} \xi_i \sigma_{ab} (\xi - x) \, d\Omega_\xi = \int_\Omega \frac{1}{8\pi(1-\nu)} \frac{1}{r^2} \left[ 2\nu \delta_{ab} n_j (\xi) h_{ij} \xi_i - \nu \delta_{ab} g_{ii} r_r m_n (\xi) + 2(1-\nu) n_b (\xi) h_{ai} \xi_i + 2(1-\nu) n_a (\xi) h_{bi} \xi_i - (1-\nu) r_r m_n (\xi) (h_{ab} + h_{ba}) + r h_{ii} n_a (\xi) - (\delta_{ab} - r_{ai} r_{ab}) n_j (\xi) h_{ij} \xi_i \right] \, d\Omega_\xi
\]

(41)

Then using the divergence theorem, we get

\[
\int_\Omega h_{ij} \xi_i \sigma_{ab} (\xi - x) \, d\Omega_\xi = \int_{\partial\Omega} \frac{1}{8\pi(1-\nu)} \frac{1}{r^2} \left[ 2\nu \delta_{ab} n_j (\xi) h_{ij} \xi_i - \nu \delta_{ab} g_{ii} r_r m_n (\xi) + 2(1-\nu) n_b (\xi) h_{ai} \xi_i + 2(1-\nu) n_a (\xi) h_{bi} \xi_i - (1-\nu) r_r m_n (\xi) (h_{ab} + h_{ba}) + r h_{ii} n_a (\xi) - (\delta_{ab} - r_{ai} r_{ab}) n_j (\xi) h_{ij} \xi_i \right] \, d\Gamma_\xi
\]

(42)

Note that, as is mentioned above, boundary integral terms in Eq. (42) have the property of 1/r weak-singularity.

2.2.2. Using radial integration method to transform domain integrals with inertial force

Using radial integration Method, Eq. (37) may be rewritten as

\[
\int_\Omega h_{ij} \xi_i \sigma_{ab} (\xi - x) \, d\Omega_\xi = \int_{\partial\Omega} \frac{1}{r^2} \frac{1}{r} F(r) \, d\Gamma_\xi
\]

(43)

\[
F(r) = \int_0^r R^2 h_{ij} \xi_i \xi_j \left( \frac{1}{8\pi(1-\nu)r^2} \right) \left[ (1-2\nu) (\delta_{ab} R_j - \delta_{aj} R_b - \delta_{bj} R_a) - 3R_a R_b R_r \right] \, dR
\]

(44)

As is mentioned above, \( R_{,j} \) can be taken outside the integral directly. Note that, \( F(r) \) is the radial integral about the field point \( \xi \). Substitution of Equation (9) into Equation (44) gives

\[
F(r) = h_{ij} \left( \frac{1}{8\pi(1-\nu)} \right) \left[ (1-2\nu) (\delta_{ab} R_j - \delta_{aj} R_b - \delta_{bj} R_a) - 3R_a R_b R_r \right] \int_0^r R (R_i + \frac{x_i}{r}) \, dR
\]

(45)

Note that, for radial integral \( F(r) \), source point \( x \) is constant. We can directly compute this radial integral. Substitution of Equation (45) into Equation (43) gives

\[
\int_\Omega h_{ij} \xi_i \sigma_{ab} (\xi - x) \, d\Omega_\xi = \int_{\partial\Omega} \left( \frac{1}{r^2} \frac{1}{r} h_{ij} (\xi_i + x_i) \right) \left[ (1-2\nu) (\delta_{ab} R_j - \delta_{aj} R_b - \delta_{bj} R_a) - 3R_a R_b R_r \right] \, d\Gamma_\xi
\]

(46)

Eq. (46) is the boundary integral form with the rotational inertia force obtained by the radial integration method.
2.3 Weakly-singular SGBEM with numerical implementation

We have obtained weakly singular boundary integrals transformed from domain integrals considering rotational inertia and gravitational loads by the divergence theorem or radial integration method respectively. In this section, the displacement and traction boundary integral equations (t&dBIE) considering crack surfaces and rotational inertia and gravitational loads are given. Then numerical evaluation of weakly singular double surface integrals by using quadrilateral elements is introduced briefly.

2.3.1. Traction and displacement BIEs considering rotational inertia and gravitational loads by divergence theorem

Consider a crack embedded in the domain $\Omega$, the crack surfaces are denoted as $S_C^+$ and $S_C^-$ which are geometrically coincident. The outward normal direction of $S_C^+$ is opposite to that of $S_C^-$. With the assumption that the traction acting on crack surfaces satisfies that $t_j^+ + t_j^- = 0$, the boundary of the domain $\Omega$ can be defined as

$$\partial \Omega = S_u + S_t + S_C \quad (47)$$

where $S_u$ is the part of boundary where displacement is known and $S_t$ is the part of boundary where traction is known. The displacement discontinuity on crack surfaces may be defined as

$$\Delta u = u^+(x^+) - u^-(x^-) \quad (48)$$

where $u^+(x^+)$ is the displacement of point $x^+$ on $S_C^+$; $u^-(x^-)$ is the displacement of point $x^-$ on $S_C^-$; $\Delta u$ must be zero around the crack front. Points $x^+$ and $x^-$ are geometrically coincident. In this paper, quarter-point singular elements are used at the crack front.

If the weak-form traction boundary integral equation is applied on $S_t$, we may get that

$$\frac{1}{2} \int_{S_t} \delta u_b(x) t_b(x) dS_x + \int_{S_t} \delta u_b(x) n_a(x) dS_x \int_{S_u + S_t} \rho_j \frac{1}{8\pi(1-v)} \frac{1}{r} [2(1-v) \delta_{ab} n_j(\xi) + 2(1-v) \delta_{aj} n_b(\xi) + r_a r_b \xi_j(\xi)] d\Gamma_\xi +$$

$$\int_{S_t} \delta u_b(x) n_a(x) dS_x \int_{S_u + S_t} \frac{1}{8\pi(1-v)} \frac{1}{r^2} [2\nu \delta_{ab} n_j(\xi) h_{ii} h_{ij} \xi_i - \nu \delta_{ab} h_{ii} r_m n_m(\xi) + 2(1-v) n_a(\xi) h_{ii} h_{ij} \xi_i + (1-v) r_m n_m(\xi) h_{ab} + h_{ba}] +$$

$$r h_{ii} r_a n_b(\xi) - (\delta_{ab} - r_a r_b) n_j(\xi) h_{ij} \xi_j d\Gamma_\xi = - \int_{S_t} \delta u_b(x) d\Gamma_x \int_{S_u + S_t} t_j(\xi) G_{tb}^{ij}(\xi - \xi)$$

Fig. 5: Displacement discontinuity in domain $\Omega$
functions can be written in terms of nodal shape functions as $u_i$ at the discretized traction and displacement SGBEM equations are obtained, and we denote this method as SGBEM-div in this paper.

Then we may discretize boundary surfaces considering rotational inertia and gravitational loads obtained by using divergence theorem. Then we may discretize boundary surfaces

$$\int_S \delta u_b(x) d\Gamma_x + \int_{S_u+\Gamma} t_j(\xi) \varphi_{ab}^* \left( \xi - x \right) d\Gamma_x -$$

$$\int_{S_u+\Gamma} D_t \delta u_b(x) d\Gamma_x \int_{S_u+\Gamma} D_p u_q(\xi) H_{tbpq}(\xi - x) d\Gamma_x - \int_{S_u+\Gamma} D_t \delta u_b(x) d\Gamma_x \int_{S_u+\Gamma} D_p \Delta u_q(\xi) H_{tbpq}(\xi - x) d\Gamma_x =$$

$$\int_{S_u+\Gamma} D_t \delta u_b(x) d\Gamma_x \int_{S_u+\Gamma} D_p u_q(\xi) H_{tbpq}(\xi - x) d\Gamma_x -$$

And if the weak-form traction boundary integral equation is applied on the crack surfaces $S_c$, we may get that

$$\int_S \delta u_b(x) t_b(x) d\Gamma_x + \int_S \delta u_b(x)n_a(x) dS_x \int_{S_u+\Gamma} \rho \frac{\partial}{\partial n} \rho g \varphi_{ab}^* \left( \xi - x \right) d\Gamma_x -$$

Finally, if the weak-form displacement boundary integral equation is applied on the prescribed displacement boundary surfaces $S_u$, we may get that

$$- \frac{1}{2} \int_{S_u} \delta t_p(\xi) u_p(\xi) d\Gamma_x + \int_{S_u} \delta t_p(\xi) d\Gamma_x \int_{S_u+\Gamma} \left( \frac{1}{2(1-v)} \rho \frac{\partial}{\partial n} \rho g \varphi_{ab}^* \left( \xi - x \right) d\Gamma_x -$$

Eq. (49), (50), (51) are the weakly-singular traction and displacement boundary integral equations considering rotational inertia and gravitational loads obtained by using divergence theorem. Then we may discretize boundary surfaces $\partial \Omega$ into boundary elements. Traction field functions can be written in terms of nodal shape functions as $t_j = t_{j|i} N^{m}_{i}$ at $S_t$, $t_j = \bar{t}_{j|i} N^{m}_{i}$ at $S_e$; similarly displacement field functions can be written as $u_i = \bar{u}_{i|i} N^{m}_{i}$ at $S_u$, $u_i = \bar{u}_{i|i} N^{m}_{i}$ at $S_e$, where the hat symbol denotes that the nodal variables are known. In this way, the discretized traction and displacement SGBEM equations are obtained, and we denote this method as SGBEM-div in this paper.
2.3.2. Traction and displacement BIEs considering rotational inertia and gravitational loads by the radial integration method

Similar to Eq. (49), (50), (51), the weakly-singular traction and displacement BIE considering rotational inertia and gravitational loads by radial integration method can be written as follows.

\[
\frac{1}{2} \int_{S_t} \delta u_b(x) t_b(x) \, dS_x + \int_{S_t} \delta u_b(x) n_a(x) \, dS_x \int_{S_u + S_t} \rho g_j \frac{1}{8 \pi (1-\nu) r} \frac{1}{r} \frac{\partial r_j}{\partial n} (1 - 2\nu) (\delta a_r + \delta a \delta r_j - \delta b_j \delta r_j - 3r_a \delta r_j) \, d\Gamma_x + \int_{S_t} \delta u_b(x) n_a(x) \, dS_x \int_{S_u + S_t} \frac{1}{16 \pi (1-\nu) r} \frac{1}{r} \frac{\partial r_j}{\partial n} h_j (\xi_l + x_i) \big( (1 - 2\nu) (\delta a_r - \delta b_j \delta r_j - 3r_a \delta r_j) \big) \, d\Gamma_x = \int_{S_t} D_t \delta u_b(x) \, dS_x \int_{S_u + S_t} D_p u_q(\xi) H^*_{tbpq}(\xi - x) \, d\Gamma_x - \int_{S_t} D_t \delta u_b(x) \, dS_x \int_{S_u + S_t} D_p u_q(\xi) H^*_{tbpq}(\xi - x) \, d\Gamma_x
\]

\[(52)\]

\[
\frac{1}{2} \int_{S_u} \delta t_p(x) u_p(x) \, dS_x + \int_{S_u} \delta t_p(x) \, dS_x \int_{S_u + S_t} \frac{1}{16 \pi (1-\nu) r} \frac{1}{r} \frac{\partial r_j}{\partial n} (3 - 4\nu) \rho g_p(\xi) + r_p r_j \rho g_p(\xi) \big( (1 - 2\nu) (\delta a_r - \delta b_j \delta r_j - 3r_a \delta r_j) \big) \, d\Gamma_x = \int_{S_u} \delta t_p(x) \, dS_x \int_{S_u + S_t} \frac{1}{16 \pi (1-\nu) r} \frac{1}{r} \frac{\partial r_j}{\partial n} n_m(\xi) \big[ (3 - 4\nu) \delta p_j + r_p r_j \big] h_j (\xi_l + x_i) \big( (1 - 2\nu) (\delta a_r - \delta b_j \delta r_j - 3r_a \delta r_j) \big) \, d\Gamma_x - \int_{S_u} \delta t_p(x) \, dS_x \int_{S_u + S_t} D_t u_j(\xi) G^*_{ij}(x, \xi) \, d\Gamma_x - \int_{S_u} \delta t_p(x) \, dS_x \int_{S_u + S_t} u_j(\xi) n_i(\xi) \psi^*_i(\xi, \xi) \, d\Gamma_x - \int_{S_u} \delta t_p(x) \, dS_x \int_{S_c} \Delta u_j(\xi) n_i(\xi) \psi^*_i(\xi, \xi) \, d\Gamma_x
\]

\[(53)\]

\[
\frac{1}{2} \int_{S_u} \delta t_p(x) \, dS_x \int_{S_u + S_t} D_t u_j(\xi) G^*_{ij}(x, \xi) \, d\Gamma_x - \int_{S_u} \delta t_p(x) \, dS_x \int_{S_u + S_t} u_j(\xi) n_i(\xi) \psi^*_i(\xi, \xi) \, d\Gamma_x - \int_{S_u} \delta t_p(x) \, dS_x \int_{S_c} \Delta u_j(\xi) n_i(\xi) \psi^*_i(\xi, \xi) \, d\Gamma_x
\]

\[(54)\]

By the same discretization procedure mentioned above for Eq. (52), (53), (54), the SGBEM equations obtained by radial integration method can be obtained, and we denote this as SGBEM-RIM in this paper.
It can be seen that, for Eq. (49), (50) using the divergence theorem, there exists $1/r$ singularity in boundary integral terms containing rotational inertia and gravitational loads; while for Eq. (52), (53) using the radial integration method, there exists $1/r \cdot \partial r / \partial n$ in boundary integral terms containing rotational inertia and gravitational loads. As is mentioned above, when the field point approaches the source point, $\partial r / \partial n \to 0$. In other words, by the radial integration method, the obtained boundary integral terms may have weaker singularity compared with those obtained by the divergence theorem.

2.3.3. Numerical evaluation of weakly-singular double surface integrals using quadrilateral elements

In this paper, 8-noded quadrilateral isoparametric elements are selected for the numerical implementation, and quarter-point singular quadrilateral elements with two mid-side nodes shifted towards the crack front as shown in Fig. 6 are adopted at the crack front. For the numerical evaluation of double surface integrals by quadrilateral isoparametric elements in detail, one may refer to [20], here it is introduced briefly.

As shown in Fig. 7, there are four quadrilateral elements $A, B, C, D$. In the computation of the double surface ($S_x \& \Gamma_\xi$) integrals, two elements will form a pair. One appears in the $S_x$, while the other appears in $\Gamma_\xi$. There exist four kind of cases: coincident elements, e.g. $A_x \& A_\xi$; adjacent elements sharing one edge, e.g. $A_x \& B_\xi$ sharing edge $pq$; adjacent elements sharing one vertex, e.g. $A_x \& C_\xi$ sharing vertex $p$; distinct elements, e.g. $A_\xi \& D_\xi$. Numerical integral for a pair of distinct elements do not need special treatment. But for the first three cases, a coordinate transformation is used for numerical integration, which can introduce a Jacobian exploited to cancel singularity of the boundary integral.

For a pair of distinct elements, standard isoparametric coordinate transformation is used.
together with the standard Gauss-Legendre quadrature. As an example, the double surface integral containing gravitational loads obtained by the divergence theorem in Eq. (49) is considered at here.

\[
I = \int_{S_t} \delta u_b(x) n_a(x) dS_x \int_{S_t+S_s} \rho g_j \frac{1}{8\pi(1-v)} \left[(2v-1)\delta_{ab}n_j(\xi) + 2(1-v)\delta_{aj}n_b(\xi) + 2(1-v)\delta_{aj}n_b(\xi) + r_a r_b n_j(\xi)\right] d\Gamma_{\xi}
\]

(55)

For simplicity, we rewrite it as

\[
I = \int_{S_x} dS_x \int_{S_t} B(x, \xi) d\Gamma_{\xi} = \int_0^1 \int_0^1 \int_0^1 B'[x(x_1, x_2), \xi(\xi_1, \xi_2)] dx_1 \cdot dx_2 \cdot d\xi_1 \cdot d\xi_2
\]

(56)

where \(x_1', x_2', \xi_1', \xi_2'\) are isoparametric coordinates corresponding to Cartesian coordinates \(x_1, x_2, \xi_1, \xi_2\). It should be noted that, in Eq. (56), \(B'[x(x_1', x_2'), \xi(\xi_1', \xi_2')\}] = B(x, \xi)/J_xJ_\xi\) due to the transformation integral variables.

For cases of coincident elements, adjacent elements sharing one edge, adjacent elements sharing one vertex, further coordinate transformations are given follows to cancel the singularity caused by \(1/r\) appearing in Eq. (56).

Fig. 8: Isoparametric coordinates for a pair of coincident elements

For a pair of coincident elements, local isoparametric coordinates are shown in Fig. 8. The boundary integral domain is partitioned into 8 subdomains. For each case we may implement a further transformation of variables listed in Table 1.

| case | \(x_1\) | \(x_2\) | \(\xi_1\) | \(\xi_2\) |
|------|--------|--------|--------|--------|
| 1    | \(v_3\) | \(v_4\) | \(v_1 + v_3\) | \(v_2 + v_4\) |
| 2    | \(v_3\) | \(v_2 + v_4\) | \(v_1 + v_3\) | \(v_3\) |
| 3    | \(v_1 + v_3\) | \(v_2 + v_4\) | \(v_3\) | \(v_4\) |
| 4    | \(v_1 + v_3\) | \(v_3\) | \(v_2 + v_4\) | \(v_3\) |
| 5    | \(v_3\) | \(v_2 + v_4\) | \(v_3\) | \(v_1 + v_3\) |
| 6    | \(v_2 + v_4\) | \(v_3\) | \(v_4\) | \(v_3\) |
| 7    | \(v_2 + v_4\) | \(v_1 + v_3\) | \(v_4\) | \(v_3\) |
| 8    | \(v_4\) | \(v_1 + v_3\) | \(v_2 + v_4\) | \(v_3\) |

In Table 1, \(v_1, v_2, v_3, v_4\) are defined as follows.

\[
\begin{align*}
v_1 &= w_1 \\
v_2 &= w_1 w_2 \\
v_3 &= w_3(1-w_1) \\
v_4 &= w_4(1-w_1 w_2) \quad \text{with} \quad 0 \leq w_4 \leq 1
\end{align*}
\]

(57)
The Jacobian for such a variable transformation can be used to cancel the singularity in Eq. (56):

\[ J = w_1(1 - w_1)(1 - w_1 w_2) \]  

(58)

For a pair of coincident elements, Eq. (56) can be rewritten as:

\[ I = \sum_{\text{case}=1}^{8} \int_0^1 \int_0^1 \int_0^1 B'[\mathbf{x}(x'_1, x'_2), \xi(\xi'_1, \xi'_2)] w_1(1 - w_1)(1 - w_1 w_2) dw_1 dw_2 dw_3 dw_4 \]  

(59)

For a pair of common-edge elements, local isoparametric coordinates are shown in Fig. 9.

![Diagram of local isoparametric coordinates for a pair of common-edge elements](image)

**Fig. 9:** Local isoparametric coordinates for a pair of common-edge elements

This boundary integral domain is partitioned into 6 subdomains. For each case we may implement a transformation of variables listed in Table 2.

| case | \( x'_1 \) | \( x'_2 \) | \( \xi'_1 \) | \( \xi'_2 \) | Jacobians |
|------|-------------|-------------|-------------|-------------|-----------|
| 1    | \( v_4 \)   | \( v_2 \)   | \( v_1 + v_4 \) | \( v_3 \)   | \( J_1 \)  |
| 2    | \( v_5 \)   | \( v_1 \)   | \( v_5 + v_2 \) | \( v_3 \)   | \( J_2 \)  |
| 3    | \( v_5 \)   | \( v_3 \)   | \( v_5 + v_2 \) | \( v_1 \)   | \( J_2 \)  |
| 4    | \( v_1 + v_4 \) | \( v_2 \)   | \( v_4 \)   | \( v_3 \)   | \( J_1 \)  |
| 5    | \( v_5 + v_2 \) | \( v_1 \)   | \( v_5 \)   | \( v_3 \)   | \( J_2 \)  |
| 6    | \( v_5 + v_2 \) | \( v_3 \)   | \( v_5 \)   | \( v_1 \)   | \( J_2 \)  |

In Table 2, \( v_1, v_2, v_3, v_4, v_5 \) and \( J_1, J_2 \) are defined as follows:

\[
\begin{align*}
  v_1 &= w_1 \\
  v_2 &= w_1 w_2 \\
  v_3 &= w_1 w_3 \\
  &\text{with} \quad 0 \leq w_1 \leq 1 \\
  v_4 &= w_4(1 - w_1) \\
  v_5 &= w_4(1 - w_1 w_2) \\
  &\text{with} \quad 0 \leq w_2 \leq 1 \\
  &\quad 0 \leq w_3 \leq 1 \\
  &\quad 0 \leq w_4 \leq 1
\end{align*}
\]  

(60)

Jacobs of the variable transformation are:

\[
J_1 = w_1^2(1 - w_1) \\
J_2 = w_1^2(1 - w_1 w_2)
\]  

(61)

For a pair of elements with a common vertex, local isoparametric coordinates is shown in Fig. 10.
This boundary integral domain is partitioned into 4 subdomains. For each case, a transformation of variables listed in Table 3 is implemented.

Table 3: The transformation of variables for a pair of elements with a common vertex

| case | $x'_1$ | $x'_2$ | $\xi'_1$ | $\xi'_2$ |
|------|--------|--------|--------|--------|
| 1    | $v_1$  | $v_2$  | $v_3$  | $v_4$  |
| 2    | $v_2$  | $v_1$  | $v_3$  | $v_4$  |
| 3    | $v_2$  | $v_3$  | $v_1$  | $v_4$  |
| 4    | $v_2$  | $v_3$  | $v_4$  | $v_1$  |

Variables $v_1, v_2, v_3, v_4$ are defined as follows

\[
\begin{align*}
  v_1 &= w_1 & 0 \leq w_1 \leq 1 \\
  v_2 &= w_1w_2 & 0 \leq w_2 \leq 1 \\
  v_3 &= w_1w_3 & \text{with } 0 \leq w_3 \leq 1 \\
  v_4 &= w_1w_4 & 0 \leq w_4 \leq 1
\end{align*}
\]  

(62)

The Jacobian of the variable transformation can be used to cancel the singularity in Eq. (56):

\[
J = w_1^3
\]  

(63)

3 Numerical examples without cracks

In this section and the next sections some examples without or with crack are implemented respectively to verify SGBEM-div or SGBEM-RIM developed in section 2.

3.1 Numerical test of the effect of the number of integration points

In this section, the double surface integral term in Eq. (55), for a pair of coincident square elements, is evaluated using the quadrature method given in section 2.3.3, considering the problem of a cube undergoing gravity given in section 3.2. The effect of the number of integration points are is shown in Fig. 11, where the relative error is defined as follows:

\[
\text{relative error} = \frac{I(n) - I(48)}{I(48)} \times 100\%
\]

where $I(n)$ is evaluated double surface integral with $n$ Gauss points.
Fig. 11: Relative errors for the evaluated weakly-singular boundary integral

Fig. 11 shows that the error for the numerical integration is very small when the number of integration points is larger than 6. Thus, 8 gauss points are used for the evaluation of double surface integrals in the following examples.

3.2. A cube undergoing gravitational loads

We consider a cube with dimensions of $10\text{mm} \times 10\text{mm} \times 10\text{mm}$ [14], which is discretized into 96 quadratic boundary elements with 290 boundary nodes (Fig. 2). The surface $z = 0$ is completely fixed. The elastic constants are chosen to be $E = 1000\text{Mpa}$ and $\nu = 0$.

![Mesh of a cube](image)

The gravitational force $\rho g = -10\text{Mpa/mm}$ is considered. And the exact solution for the vertical displacement is:

$$u_z = \frac{\rho g}{E} z \left( L - \frac{z}{2} \right)$$  \hspace{1cm} (64)

Table 4 shows vertical displacements along the direction $z$. The computational results are in excellent agreement with the exact solution.

| $z$/mm | 2.5     | 5       | 7.5     | 10      |
|--------|---------|---------|---------|---------|
| Exact  | -0.218750 | -0.375000 | -0.468750 | -0.500000 |
| SGBEM-div | -0.218708 | -0.374989 | -0.468747 | -0.499985 |
3.3. A rotating disk

In the second example, a disk with inner radius of 0.1m and outer radius of 0.2m, rotating at a constant angular speed \( \omega = 10000 \text{rpm} \) (Fig. 13), is considered. The thickness of this disk is \( t = 0.02 \text{m} \). The elastic constants are chosen to be \( E = 7000 \text{Mpa} \) and \( v = 0.3 \); density \( \rho = 2800 \text{kg/m}^3 \). All the boundary surface of this disk is free from traction. The boundary of the disk is discretized with 5 elements in radial direction, 32 elements in circumferential direction, and 4 elements in axial direction (Fig. 14).

Table 5 and 6 shows the computed radial displacements with the mesh in Fig. 14. “FEM” denotes numerical solutions by the Finite Element Method. For each point, “Error” is computed with FEM solution as the reference. As can be seen, computational results by SGBEM-div and SGBEM-RIM are in excellent agreement with the FEM solutions.

| \( R/\text{m} \) | 0.1   | 0.13  | 0.16  | 0.2   |
|----------------|-------|-------|-------|-------|
| SGBEM-div      | 1.49121| 1.40577| 1.36749| 1.31153|
| FEM            | 1.49479| 1.40857| 1.37257| 1.31922|
| Error          | −0.24% | −0.20% | −0.37% | −0.58% |

Table 6: Radial displacements by SGBEM-RIM (10\(^{-3}\)m)

| \( R/\text{m} \) | 0.1   | 0.13  | 0.16  | 0.2   |
|----------------|-------|-------|-------|-------|
| SGBEM-RIM      | 1.49122| 1.40578| 1.36750| 1.31155|
| FEM            | 1.49479| 1.40857| 1.37257| 1.31922|
| Error          | −0.24% | −0.20% | −0.37% | −0.58% |
4 Numerical examples with cracks

In this section, numerical examples with cracked solids considering body forces are given. In each example, after obtaining the displacement discontinuities for the quarter-point node using the developed SGBEM method, displacement extrapolation is used to calculate the stress intensity factors.

4.1 A cuboid hanging under its own weight with a through-thickness crack

Consider a solid cuboid with a crack of length $2a$ (see Fig. 15) under gravitational loads[21], where $l = 4$, $b = 1$, $h = 0.5l$, $a = 0.1$, $t = 0.2$, $\rho g = -10$. The elastic constants are chosen to be $E = 1000$ and $\nu = 0$.

Computed stress intensity factors are presented in Table 7, in which “Error” means the relative error between SGBEM with div and FEM solution. For this through-thickness crack, $K_I$ computed by SGBEM with RIM are in better agreement with the FEM solution.

Table 7 : $K_I$ for the problem shown in Fig. 15

| $y/t$  | $x = 0.1$ |          |          | $x = -0.1$ |          |        |
|--------|-----------|----------|----------|-----------|----------|--------|
|        |          |          |          |           |          |        |
| SGBEM-div | 11.4655  | 11.4744  | 11.4654  | 11.4655   | 11.4744  | 11.4655 |
| Error  | 1.64%     | 1.72%    | 1.64%    | 1.64%     | 1.72%    | 1.64%  |
| SGBEM-RIM | 11.2705  | 11.2658  | 11.2705  | 11.2705   | 11.2658  | 11.2705 |
| FEM    | 11.28     | 11.28    | 11.28    | 11.28     | 11.28    | 11.28  |

4.2. A rotating disk with a through-thickness crack

A rotating disk with a through-thickness crack ($a = 0.03$m) is computed. The rotating disk is identical to the disk in section 3.3. Again, excellent agreement between the computed SGBEM results and FEM results are shown in Table 8 and Table 9.
Fig. 16: SGBEM mesh of a cracked rotating disk

Table 8: KI of through-thickness crack on rotating disk (MPa√m)

| z/t | SGBEM-div | FEM | Error |
|-----|-----------|-----|-------|
| 0.25 | 36.497 | 36.150 | 0.96% |
| 0.5  | 36.540 | 36.303 | 0.65% |
| 0.75 | 36.494 | 36.150 | 0.95% |

Table 9: KI of through-thickness crack on rotating disks (MPa√m)

| z/t | SGBEM-RIM | FEM | Error |
|-----|-----------|-----|-------|
| 0.25 | 36.498 | 36.150 | 0.96% |
| 0.5  | 36.541 | 36.303 | 0.66% |
| 0.75 | 36.495 | 36.150 | 0.95% |

4.3. A rotating disk with semi-elliptic surface cracks

This section gives a series of results for a cracked disk in Fig. 17 with various semi-elliptic surface cracks, shown in Fig. 18. All the parameters of this disk are identical to that of disk in section 3, except for the semi-elliptic cracks. Various semi-elliptic cracks with a fixed depth \( a=0.004 \text{m} \), and various semi-elliptic cracks with a fixed length/depth ratio \( b/a=2 \), are computed using both SGBEM-div and SGBEM-RIM.
For simplicity, we give stress intensity factor $K_I$ at point P, i.e. the deepest point of various semi-elliptic cracks, as shown in Fig. 19, and Fig. 20. These results can be used for the benchmark solutions for future studies.

5 Conclusions

In this paper, weakly-singular SGBEM for fracture analysis of three-dimensional structures considering rotational inertia and gravitational forces are developed. By using the divergence theorem (div) or a radial integration method (RIM), rotational inertia or gravitational forces
induced domain integrals are transformed into boundary integrals correspondingly. The derived boundary integral terms with the gravitational and inertial forces are weakly-singular, which only influence the SGBEM right-hand-side vector.

Several numerical examples of solids with and without cracks undergoing body forces are studied. The calculated stress intensity factors and displacements show high accuracy compared with reference solutions. The test of numerical integration also shows that only a small number of quadrature points are needs.

The symmetric Galerkin boundary element method considering gravity and inertia loads presented in this paper appears promising in the fracture analysis of structural components with body forces, such as dams and rotating machineries. Furthermore, with some effort, the methodology given in this study can also be extended to deal with domain integrals for SGBEM with thermoelastic problems, which will be given in a subsequent work.

Acknowledgements

The first four authors acknowledge the support of the National Natural Science Foundation of China (12072011).

Appendix

Kernel functions listed here are utilized in the numerical implementation of the SGBEM. Kernel functions (A 1), (A 2), (A 3) appear in the displacement boundary integral equation; kernel functions (A 2), (A 3), (A 4) appear in the traction boundary integral equation.

\[
\begin{align*}
\mathbf{u}_i^p(x, \xi) &= \frac{1}{16\pi\mu(1-\nu)r} \left[ (3-4\nu)\delta_{ip} + r_i r_p \right] \quad \text{(A 1)} \\
G_{ij}^p(x, \xi) &= \frac{1}{8\pi(1-\nu)r} \left[ (1-2\nu)e_{ij} + e_{lk} r_k r_p \right] \quad \text{(A 2)} \\
\phi_{ij}^p(x, \xi) &= \delta_{pj} \frac{1}{4\pi r^2} r_i \quad \text{(A 3)} \\
H_{ijpq}^r(x, \xi) &= \frac{\mu}{8\pi(1-\nu)r} \left[ 4\nu \delta_{iq} \delta_{jp} - \delta_{ip} \delta_{jq} - 2\nu \delta_{ij} \delta_{pq} + \delta_{ij} r_p r_q + \delta_{pq} r_i r_j - 2\delta_{ip} r_j r_q - \delta_{jq} r_i r_p \right] \quad \text{(A 4)}
\end{align*}
\]

References

1. Sutradhar, A., G.H. Paulino, and L.J. Gray, Symmetric Galerkin Boundary Element Methods. Applied Mechanics Reviews, 2008. 51(11): p. 669.
2. Giorgio, N. and F. Attilio, Symmetric Galerkin BEM in 3D Elasticity: Computational Aspects and Applications to Fracture Mechanics. 2002: Springer Vienna.
3. Bonnet, M., G. Maier, and C. Polizzotto, Symmetric Galerkin Boundary Element Methods. Applied Mechanics Reviews, 1998.
4. Bonnet, M., Boundary Integral Equation Methods for Solids and Fluids. Meccanica, 1999.
5. Li, S. and M.E. Mear, Singularity-reduced integral equations for displacement discontinuities in three-dimensional linear elastic media. International Journal of Fracture, 1998. 93(1-4): p. 87-114.
6. Nishimura, N. and S. Kobayashi, A regularized boundary integral equation
method for elastodynamic crack problems. Comp. Mech, 1989. 4(4): p. 319-328.

7. Sladek, V. and J. Sladek, Advances in Boundary Element Method Series: Singular Integrals in Boundary Element Methods. Computational Mechanics, 1998.

8. Han, Z.D. and S.N. Atluri, On Simple Formulations of Weakly-Singular Traction & Displacement BIE, and Their Solutions through Petrov-Galerkin Approaches. Computer Modeling in Engineering and Sciences, 2003. 4(1): p. 5-20.

9. Dong, L.T. and S.N. Atluri, SGBEM (Using Non-hyper-singular Traction BIE), and Super Elements, for Non-Collinear Fatigue-growth Analyses of Cracks in Stiffened Panels with Composite-Patch Repairs. Computer Modeling in Engineering and Sciences, 2012. 89: p. 417-458.

10. Okada, H., H. Rajiyah, and S.N. Atluri, A Novel Displacement Gradient Boundary Element Method for Elastic Stress Analysis With High Accuracy. Journal of Applied Mechanics, 1988. 55(4): p. 786-794.

11. Okada, H., H. Rajiyah, and S.N. Atluri, Non-hyper-singular integral-representations for velocity (displacement) gradients in elastic/plastic solids (small or finite deformations). Computational Mechanics, 1989. 4(3): p. 165-175.

12. Cruse, T.A., Boundary-Integral Equation Method for Three-Dimensional Elastic Fracture Mechanics Analysis. 1975.

13. Danson, D.J., A Boundary Element Formulation of Problems in Linear Isotropic Elasticity with Body Forces. 1981: A Boundary Element Formulation of Problems in Linear Isotropic Elasticity with Body Forces.

14. Gao, X.W., The radial integration method for evaluation of domain integrals with boundary-only discretization. Engineering Analysis with Boundary Elements, 2002. 26(10): p. 905-916.

15. Nardini, D. and C.A. Brebbia, A new approach to free vibration analysis using boundary elements. Applied Mathematical Modelling, 1983. 7(3): p. 157-162.

16. Partridge, P.W., C. Brebbia, and L.C. Wrobel, The dual reciprocity boundary element method. 1992: The dual reciprocity boundary element method.

17. Nowak, A.J. and C.A. Brebbia, The multiple-reciprocity method. A new approach for transforming BEM domain integrals to the boundary. Engineering Analysis with Boundary Elements, 1989. 6(3): p. 164-167.

18. Brebbia, C.A. and H. Saun De Rs, Progress in Boundary Element Methods (Vol. 1). London Pentech Press P, 1983. 106(1): p. 153-155.

19. Frangi, A., et al., 3D fracture analysis by the symmetric Galerkin BEM. Computational Mechanics, 2002. 28(3-4): p. 220-232.

20. Frangi, A., et al., Symmetric Galerkin Boundary Element Analysis in Three-Dimensional Linear-Elastic Fracture Mechanics. 2004: Boundary Element Methods for Soil-Structure Interaction.

21. Ostanin, I.A., et al., Complex variables boundary element method for elasticity problems with constant body force. Engineering Analysis with Boundary Elements, 2011. 35(4): p. 623-630.