A DEEP LOOK INTO THE DAGUM FAMILY OF ISOTROPIC COVARIANCE FUNCTIONS

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Abstract

The Dagum family of isotropic covariance functions has two parameters that allow for decoupling of the fractal dimension and the Hurst effect for Gaussian random fields that are stationary and isotropic over Euclidean spaces. Sufficient conditions that allow for positive definiteness in \( \mathbb{R}^d \) of the Dagum family have been proposed on the basis of the fact that the Dagum family allows for complete monotonicity under some parameter restrictions. The spectral properties of the Dagum family have been inspected to a very limited extent only, and this paper gives insight into this direction. Specifically, we study finite and asymptotic properties of the isotropic spectral density (intended as the Hankel transform) of the Dagum model. Also, we establish some closed-form expressions for the Dagum spectral density in terms of the Fox–Wright functions. Finally, we provide asymptotic properties for such a class of spectral densities.

Keywords: Hankel transforms; Mellin–Barnes transforms; spectral theory; positive definite

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1. Introduction

Isotropic covariance functions have a long history that can be traced back to Matheron [16] and Yaglom [29]. The mathematical machinery that allows us to implement isotropic covariance functions is based on positive definite functions that are radially symmetric over \( d \)-dimensional Euclidean spaces. In particular, the characterization of the radial part of radially symmetric positive definite functions was provided in the tour de force by Schoenberg [23]. There is a rich catalogue of isotropic covariance functions that are obtained by composing a given parametric family of functions defined on the positive real line with the Euclidean norm.
The Dagum family of isotropic covariance functions in a \(d\)-dimensional Euclidean space [6]. What makes them interesting is that some parameters have a corresponding interpretation in terms of geometric properties of Gaussian random fields. For instance, the Matérn family has a parameter that allows us to verify the mean square differentiability of its corresponding Gaussian random field, in concert with its fractal dimension.

The Dagum parametric family of functions was originally proposed by Porcu [18] as a new family of isotropic covariance functions associated with Gaussian random fields that are weakly stationary and isotropic over \(d\)-dimensional Euclidean spaces. Porcu, Mateu, Zini, and Pini [21] provided sufficient conditions for positive definiteness in \(\mathbb{R}^3\) on the basis of a criterion of Pólya type [9]. Later, Mateu, Porcu, and Nicolis [15] showed that the Dagum family allows for decoupling fractal dimension and the Hurst effect, allowing us to avoid self-similar random fields and consequently all the issues that are related to the estimation of fractal dimension and long memory parameters under self-similarity [12].

Positive definiteness of a given radial function over all \(d\)-dimensional Euclidean spaces is equivalent to complete monotonicity of its radial part [23]. Berg, Mateu, and Porcu [4] have proved sufficient conditions for complete monotonicity of the Dagum class. They also provided some necessary conditions, but unfortunately these do not match the sufficient ones. Hence a complete characterization for the Dagum function is, to date, still elusive.

A wealth of applications in applied branches of science has shown how the Dagum family can be used to model temporal or spatial phenomena where local properties (fractal dimension) and global ones (the Hurst effect) are decoupled, and we refer the reader to [24] and the references therein.

Positive definiteness of a radially symmetric function in a \(d\)-dimensional Euclidean space, for a given dimension \(d\), is equivalent to its Hankel transform, called the radial spectral density, being non-negative and integrable [6]. The radial spectral density is often not available in closed form, with the notable exception of the Matérn model [25]. A big effort in this direction was provided by Lim and Teo [14] with the generalized Cauchy model, being also a decoupler of fractal dimension and the Hurst effect.

Radial spectral densities are fundamental to spatial statistics. On the one hand, knowing at least local and global properties of a radial spectral density allows us, by application of Tauberian theorems [25], to inspect the properties of the associated Gaussian field in terms of mean square differentiability, fractal dimension, the Hurst effect, and reproducing kernel Hilbert spaces [19]. On the other hand, the radial spectral density covers a fundamental part of statistical inference for Gaussian fields under infill asymptotics [5, 25]. Finally, radial spectral densities are fundamental to inspecting the so-called screening effect, which in turn plays an important role in spatial prediction, and we refer the reader to [27] as well as to the recent paper [22].

An expression for the radial spectral density associated with the Dagum family has been elusive so far. A first attempt was made by Laudani et al. [13], who showed that such a spectral density admits a series expansion that is absolutely convergent.

This paper provides some insights in this direction. After background material in Section 2, Section 3 provides the main results, which are classified into three parts. We start by deriving series expansions associated with the isotropic spectral density of the Dagum class. We then provide a closed-form expression, in terms of the Fox–Wright functions, for such a class of isotropic spectral densities. Finally we provide local and global asymptotic identities. The proofs are lengthy and technical; for a neater exposition we have deferred them to the Appendix. Section 4 concludes the paper with a short discussion.
2. Background material

2.1. Positive definite radial functions

We let \( \{Z(s), s \in \mathbb{R}^d\} \) denote a centred Gaussian random field in \( \mathbb{R}^d \), with the stationary covariance function \( C: \mathbb{R}^d \to \mathbb{R} \). We consider the class \( \Phi_d \) of continuous mappings \( \phi: [0, \infty) \to \mathbb{R} \) with \( \phi(0) = 1 \), such that

\[
\text{cov}(Z(s), Z(s')) = C(s' - s) = \phi(||s' - s||),
\]

with \( s, s' \in \mathbb{R}^d \), and \( || \cdot || \) denoting the Euclidean norm. Gaussian fields with such covariance functions are called weakly stationary and isotropic. The function \( C \) is called isotropic or radially symmetric, and the function \( \phi \) its radial part.

Schoenberg [23] characterized the class \( \Phi_d \) as being scale mixtures of the characteristic functions of random vectors uniformly distributed on the spherical shell of \( \mathbb{R}^d \):

\[
\phi(r) = \int_0^\infty \Omega_d(r \xi) F(d \xi), \quad r \geq 0,
\]

with \( \Omega_d(r) = r^{-(d-2)/2} J_{d-2}/2(r) \) and \( J_v \) a Bessel function of order \( v \). Here \( F \) is a probability measure. The function \( \phi \) is the uniquely determined characteristic function of a random vector, \( X \), such that \( X = U \cdot R \), where equality is intended in the distributional sense, where \( U \) is uniformly distributed over the spherical shell of \( \mathbb{R}^d \), \( R \) is a non-negative random variable with probability distribution \( F \), and where \( U \) and \( R \) are independent.

Daley and Porcu [6] have described the properties of the measures, \( F \), which they term the Schoenberg measures, and they have shown the existence of projection operators that map the elements of \( \Phi_d \) onto the elements of \( \Phi_{d'} \), for \( d' \neq d \). Throughout, we adopt their illustrative name and will call the function \( F \) associated with \( \phi \) a Schoenberg measure. The derivative of \( F \) is called the isotropic spectral density. If \( \phi \) is absolutely integrable, then the Fourier inversion (the Hankel transform) becomes possible. The Fourier transforms of radial versions of the members of \( \Phi_d \), for a given \( d \), have a simple expression, as reported in [26] and [30]. For a member \( \phi \) of the family \( \Phi_d \), we define its isotropic spectral density as

\[
\hat{\phi}(z) = \frac{z^{1-d/2}}{(2\pi)^{d/2}} \int_0^\infty u^{d/2} J_{d/2-1}(uz) \phi(u) \, du, \quad z \geq 0.
\]

The classes \( \Phi_d \) are nested, with the inclusion relation \( \Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_\infty \) being strict, and where \( \Phi_\infty := \bigcap_{d \geq 1} \Phi_d \) is the class of mappings \( \phi \) whose radial version is positive definite on all \( d \)-dimensional Euclidean spaces.

2.2. Parametric families of isotropic covariance functions

The generalized Cauchy family of members of \( \Phi_\infty \) [10] is defined as

\[
C_{\delta, \lambda}(r) = (1 + r^\delta)^{-\lambda/\delta}, \quad r \geq 0,
\]

where the conditions \( \delta \in (0, 2] \) and \( \lambda > 0 \) are necessary and sufficient for \( C_{\delta, \lambda} \) to belong to the class \( \Phi_\infty \). The parameter \( \delta \) is crucial to the differentiability at the origin and, as a consequence, for the degree of differentiability of the associated sample paths. Specifically, for \( \delta = 2 \) they are infinitely differentiable, and they are not differentiable for \( \delta \in (0, 2) \).
The Dagum family of isotropic covariance functions

Faouzi et al. [7] and Lim and Teo [14] have shown that the isotropic spectral density, $\widehat{C}_{\delta, \lambda}(\omega)$ of the generalized Cauchy covariance function is identically equal to

$$\widehat{C}_{\delta, \lambda}(\omega) = -\frac{z^{-d}}{2d/2-1} \frac{\pi^{d/2}}{\Gamma(d/2)} K_{d-2}(\omega),$$

for $\delta > 0$ and $\lambda \in (0, 2)$. Here $K_\nu$ denotes the modified Bessel function of the second kind with order $\nu > 0$.

The Dagum family $D_{\delta, \lambda} : [0, \infty) \to \mathbb{R}$ is defined as

$$D_{\delta, \lambda}(r) = 1 - \left( \frac{\rho^\delta}{1 + \rho^\delta} \right)^{\lambda}, \quad r \geq 0. \quad (2)$$

Porcu [18] and subsequently Porcu et al. [21] have shown that $D_{\delta, \lambda}$ belongs to the class $\Phi_3$ provided $\delta < (7 - \lambda)/(1 + 5\lambda)$ and $\lambda < 7$. Berg et al. [4] have shown that $D_{\delta, \lambda} \in \Phi_\infty$ if and only if the function $A_{\delta, \lambda}$, defined as

$$A_{\delta, \lambda}(r) = \frac{\rho^{\delta\lambda - 1}}{(1 + \rho^\delta)^{\lambda + 1}}, \quad r \geq 0,$$

belongs to $\Phi_\infty$. In particular, sufficient conditions for $D_{\delta, \lambda} \in \Phi_\infty$ become $\delta\lambda \leq 2$ and $\delta < 2$. Also, for $\delta = 1/\lambda$ we have $D_{\lambda, 1/\lambda} \in \Phi_\infty$ if and only if $\lambda \geq 1/2$.

To simplify notation, throughout we shall write $D_{\delta, \lambda}(r)$ for $D_{\delta, \lambda}(\|r\|)$, $r \in \mathbb{R}^d$, with $\|$ denoting composition. Similar notation will be used for $C_{\delta, \lambda}(r)$, $r \in \mathbb{R}^d$. Analogously, we use $\widehat{D}_{\delta, \lambda}(z)$ for $\widehat{D}_{\delta, \lambda}(|z|)$, $z \in \mathbb{R}^d$, and sometimes we shall make use of the notation $z$ for $\|z\|$. Similar notation will be used for the isotropic spectral density $\widehat{C}_{\delta, \lambda}$.

2.3. Fractal dimensions and the Hurst effect

The local properties of a time series or a surface of $\mathbb{R}^d$ are identified through the so-called fractal dimension $D$, which is a roughness measure with range $[d, d + 1)$, and with higher values indicating rougher surfaces. The long memory in time series or spatial data is associated with power law correlations, and is often referred to as the Hurst effect. Long memory dependence is characterized by the $H$ parameter. Local and global properties of a Gaussian random field have an intimate connection with its associated isotropic covariance function. In particular, if, for some $\alpha \in (0, 2]$, the radial part $\phi \in \Phi_d$ satisfies

$$\lim_{r \to 0} \frac{\phi(r)}{r^\alpha} = 1, \quad (3)$$

then the realizations of the Gaussian random field have fractal dimension $D = d + 1 - \alpha/2$, with probability 1. Thus estimation of $\alpha$ is linked to that of the fractal dimension $D$. Conversely, if for some $\beta \in (0, 1)$

$$\lim_{r \to \infty} \phi(r)r^\beta = 1, \quad (4)$$

then the Gaussian random field is said to have long memory, with Hurst coefficient $H = 1 - \beta/2$. For $H \in (1/2, 1)$ or $H \in (0, 1/2)$ the correlation is said to be persistent or anti-persistent, respectively. In general, $D$ and $H$ are independent of each other, but under the assumption of self-similarity they find an intimate connection in the well-known linear relationship $D + H = d + 1$. The Cauchy model behaves like (3) for $\alpha = \delta \in (0, 2]$ and like (4) for $\beta = \lambda \in (0, 1]$. For
the reparametrized version $D_{\lambda, \delta/\lambda}$, we have exactly the same result. For both models the local and global behaviour parameters may be estimated independently.

For $\delta \in (0, 2)$, the Dagum covariance function can be rewritten as

$$D_{\lambda, \delta}(r) = 1 - (1 + 1/\|r\|^\delta)^{-\lambda}, \quad r \in \mathbb{R}^d.$$  

When $\|r\|$ is large, the Dagum covariance function has the following asymptotic behaviour:

$$D_{\lambda, \delta}(r) \sim \lambda \|r\|^{-\delta} \quad \text{for } \delta \in (0, 1).$$

Hence, under these parameter restrictions, a Gaussian random field has long memory with Hurst coefficient $H = 1 - \delta/2$ with $\delta \in (0, 1)$.

A notable fact is the following:

$$\int_{[0, \infty)^d} DR_{\lambda, \delta}(r) \, dr = \frac{\pi^{d/2}}{2^{d-1} \Gamma(d/2)} \int_0^\infty r^{d-1} \left(1 - \frac{1}{1/r^\delta + 1}\right) \, dr. \quad \text{(5)}$$

Furthermore,

$$r^{d-1}D_{\lambda, \delta}(r) \sim r^{d-\delta-1}, \quad r \to \infty, \quad \text{with } \|r\| = r.$$  

The above implies that the integral (5) is finite if $\delta > d$. An alternative way to see this is to notice that equation (18) of this paper proves that

$$\int_{\mathbb{R}^d} DR_{\lambda, \delta}(r) \, dr = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \int_0^\infty r^{d-1} \left(1 - \frac{1}{1/r^\delta + 1}\right) \, dr$$

$$= \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \frac{B(d/\delta + \lambda, -d/\delta)}{\delta}.$$

3. Theoretical results

3.1. Isotropic spectral density of the Dagum covariance function

This subsection aims to compute the isotropic spectral density associated with the Dagum class in $\mathbb{R}^d$, for a given positive integer $d$, when $\delta > d$. The spectral density of the Dagum class can be written as

$$\hat{D}_{\delta, \lambda}(z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iz^T r} D_{\lambda, \delta}(r) \, dr,$$

with $i$ denoting the imaginary unit. Laudani et al. [13] showed that when $\lambda = k \in \mathbb{Z}_+$ is an integer, the Dagum spectral density can be written as

$$\hat{D}_{\delta, k}(z) = -\sum_{h=0}^{k-1} (-1)^{k-h} C_h^k \hat{C}_{\delta, k-h}(z), \quad z \in \mathbb{R}^d,$$

where $\hat{C}_{\delta, k-h}$ is a generalized Cauchy isotropic spectral density associated with the generalized Cauchy function, $\hat{C}_{\delta, k-h}$ as defined in (1).

We start by extending this result for any $\lambda > 0$ and for $d = 1$.  

Theorem 1. For $d = 1$ and $\delta > 1$, the Dagum isotropic spectral density $\hat{D}_{\delta, \lambda}$ has the following explicit form:

$$
\hat{D}_{\delta, \lambda}(z) = -\frac{1}{\pi} \text{Im} \int_0^\infty \left[ 1 - \frac{e^{\delta \lambda \pi/2} r^{\delta \lambda}}{(1 + e^{\pi \delta/2} r^\delta)^{\lambda}} \right] e^{-zr} \, dr.
$$

To extend this result to $\mathbb{R}^d$, for $d > 1$, we start by considering the case $\delta \leq d$, $\delta \lambda \in (0, 2)$. We can show the following.

Theorem 2. For $d > 1$, $\delta \in (0, 2)$, and $\delta \lambda \in (0, 2)$, the Dagum isotropic spectral density $\hat{D}_{\delta, \lambda}$ is given by

$$
\hat{D}_{\delta, \lambda}(z) = -\frac{z^{1-d/2}}{2^{d/2-1} \pi^{d/2+1}} \text{Im} \int_0^\infty K_{d/2-1}(zt) \left( 1 - \frac{t^{\delta \lambda} e^{i\pi \delta/2}}{(1 + e^{\pi \delta/2} t^\delta)^{\lambda}} \right) t^{d/2} \, dt,
$$

with $z = \|z\|$.

Although Theorems 1 and 2 provide some insight into understanding the isotropic spectral density associated with the Dagum covariance function, it might be desirable to have an explicit expression for such spectral densities. The following subsection tackles this problem.

3.2. Dagum spectral density expressed as Fox–Wright function

We start by defining the Fox–Wright function \([8, 28]\) $p\Psi_q$, via the identity

$$
p\Psi_q \left[ (a_1, A_1), \ldots, (a_p, A_p); (b_1, B_1), \ldots, (b_q, B_q); -z \right] = \sum_{k=0}^\infty \frac{(-1)^k \Gamma(a_1 + kA_1) \cdots \Gamma(a_p + kA_p)}{k! \Gamma(b_1 + kB_1) \cdots \Gamma(b_q + kB_q)} z^k.
$$

It turns out that this class of special functions is intimately related to the Dagum spectral density. We formally state this fact below.

Theorem 3. Let $d$ be a positive integer, $z \in \mathbb{R}^d$, and $z = \|z\|$. Let $D_{\lambda, \delta}$ be the Dagum class as defined in (2). For $\delta \in (0, 2)$ and $\delta \lambda \in (0, 2)$, the isotropic spectral density $D_{\lambda, \delta}$ in $\mathbb{R}^d$ is given by

$$
\hat{D}_{\delta, \lambda}(z) = \delta^{(d)}(z) - \frac{\pi^{-d/2} z^{-d}}{\Gamma(\lambda)} 2\Psi_1 \left[ (\lambda, 1); (d/2, -\delta/2) \right] - \frac{1}{2^d \pi^{d/2}} \frac{1}{\Gamma(\lambda)} 2\Psi_1 \left[ (\lambda + d/\delta, 2/\delta); (d/2, -\delta/2) \right] - \left( \frac{z}{\sqrt{2}} \right)^2,
$$

where

$$
\delta^{(d)}(z) = (2\pi)^{-d} \lim_{\epsilon \to 0} \prod_{k=1}^d \left( \frac{1}{\epsilon} \xi_{\epsilon}(z_k) \right),
$$

with $\xi_{\epsilon}$ being the unit impulse function, and $2\Psi_1$ is the Fox–Wright function given by (7).

3.3. Asymptotic properties of Dagum spectral density

We finish with some theoretical results relating to the asymptotic behaviour of the Dagum isotropic spectral density.
Theorem 4. For all $\delta \in (0, 2)$ and $\delta\lambda \in (0, 2]$, the low frequency limit of the spectral density $\tilde{D}_{\delta,\lambda}$ is given by

$$
\tilde{D}_{\delta,\lambda}(z) \sim \begin{cases} 
2^{-\delta\lambda} \frac{\Gamma(d/2 - \delta/2)}{\pi^{d/2}\Gamma(\delta/2)} z^{-d} & \text{if } \delta \in \left( \frac{d - 2}{2}, d \right), z \to 0,
\end{cases}
$$

(i) $\tilde{D}_{\delta,\lambda}(z) \sim \frac{1}{\delta^{d/2}2^{d-1}\Gamma(d/2)} \frac{\Gamma(-d/\delta)\Gamma(d/\delta + \lambda)}{\Gamma(\lambda)}$ if $\delta \in (d, 2), z \to 0$.

Theorem 5. For $0 < \delta < 2$, and $0 < \delta\lambda \leq 2$ when $z \to \infty$,

$$
\tilde{D}_{\delta,\lambda}(z) \sim \frac{2^{\delta\lambda}z^{-d-\delta\lambda}}{\pi^{d/2}\Gamma(\lambda)} \Psi_2\left[ \left( \lambda, 1 \right), \left( \delta\lambda/2 + 1, \delta/2 \right), \left( \delta\lambda/2, \delta/2 \right), \left( 1 - \delta\lambda/2, -\delta/2 \right) ; - \left( \frac{2}{z} \right)^\delta \right] 
$$

$$
\sim \frac{2^{\delta\lambda-1}\lambda\delta z^{-d-\delta\lambda}}{\pi^{d/2}} \frac{\Gamma(d/2 + \lambda\delta/2)}{\Gamma(1 - \lambda\delta/2)}. 
$$

4. Conclusion

We have obtained the expressions for the isotropic spectral density related to the Dagum family. Our results can now be used in research related to (a) best optimal unbiased linear prediction (kriging) under infill asymptotics when using the Dagum family. This in turn relies on equivalence of Gaussian measures and on the ratio between the correct and the mis-specified spectral density [5]. While the Matérn covariance function has already been studied in this setting [31], the characterization of equivalence of Gaussian measures under the Dagum family has been elusive so far. Also, (b) knowing the form of the spectral density will be crucial to obtaining the space–time spectral densities associated with covariance functions having a dynamical support depending on a Dagum radius, as detailed by Porcu, Bevilacqua, and Genton [20].

Appendix A. Proofs

Proof of Theorem 1. We provide a constructive proof. We start by considering the Fourier transform, $\tilde{D}_{\delta,\lambda}$, of the function $D_{\delta,\lambda}$, defined as

$$
\tilde{D}_{\delta,\lambda}(z) = \frac{1}{\pi} \Re \int_0^\infty e^{i|t|} \left( 1 - \frac{t^{\delta\lambda}}{(1 + t^{\delta})^{\lambda}} \right) dt.
$$

We sketch the main arguments as follows.

1. We define the integrand in the expression above via

$$
f(t; z) := e^{i|t|} \left( 1 - \frac{t^{\delta\lambda}}{(1 + t^{\delta})^{\lambda}} \right).$$

2. We consider $D_\xi = \{ t \in \mathbb{C} ; |t| \leq \xi, \Re(t) > 0, \Im(t) > 0 \}$, and note that for $\delta \in (0, 2)$ and $\delta\lambda \in (0, 2), f$ is an analytic function on $D_\xi$.

3. Hence we can use the Cauchy integral formula, that is,

$$
\oint_{\partial D_\xi} f(t; z) dt = 0,
$$

where $\partial D_\xi$ is the boundary of $D_\xi$. 

4. We now note that such a boundary can actually be split into the union of three components: the line segment $L_1$ along the real axis from 0 to $\xi$, the arc $C_\xi$ of the circle $|t| = \xi$ from $\xi$ to $i\xi$, and the line segment $L_2$ along the imaginary axis from $i\xi$ to 0.

5. The rest of the proof comes from direct inspection in concert with standard arguments from harmonic analysis. In fact, for any $t$ belonging to the arc $C_\xi = \{ t \in \mathbb{C}; |t| = \xi \}$, there exists a phase $\varphi \in [0, \pi/2]$ such that $t = \xi e^{i\varphi}$. Thus

$$f(t; z) = f(\xi e^{i\varphi}; z) = e^{i|z|\xi} e^{i\varphi} \left( 1 - \left[ \frac{\xi^\delta e^{i\varphi}}{1 + \xi^\delta e^{i\varphi}} \right]^\lambda \right).$$

The last term of the above equality can be expressed as

$$1 - \left[ \frac{\xi^\delta e^{i\varphi}}{1 + \xi^\delta e^{i\varphi}} \right]^\lambda = 1 - \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\lambda + j)}{\Gamma(\lambda)} (\xi^{-\delta} e^{-i\varphi})^j.$$

6. Hence we have proved that

$$\int_{C_\xi} f(t, z) \, dt = -\sum_{j=1}^{\infty} (-1)^j \frac{\Gamma(\lambda + j)}{\Gamma(\lambda)} \int_0^{\pi/2} e^{-i\varphi j} e^{i|z|\xi} e^{i\varphi} \, d\varphi. \tag{9}$$

Swapping integral with series is licit, owing to the fact that it is a proper definite integral and that the series is absolutely convergent.

7. The rest of the proof follows from manipulation of (9), which can be written as

$$\int_0^{\pi/2} e^{-i\varphi j} e^{i|z|\xi} e^{i\varphi} \, d\varphi = \frac{1}{i} \int_0^{\pi/2} e^{-i(j\delta+1)\varphi} e^{i|z|\xi} e^{i\varphi} \, d\varphi = \frac{1}{i} \int_{C_1} e^{i|z|\xi e^{i\varphi} 0} e^{-(j\delta+1)} \, d\varphi$$

$$= \frac{(|z|\xi)^j}{i} \int_{C_{|z|\xi}} e^{iu} u^{-(j\delta+1)} \, du,$$

where $C_1$ is an arc of the circle $|\omega| = 1$ from 1 to i and $C_{|z|\xi}$ is an arc of the circle $|u| = |z|\xi$ from $|z|\xi$ to $i|z|\xi$. Then

$$\lim_{\xi \to \infty} \oint_{\partial D_\xi} f(t; z) \, dt = \lim_{\xi \to \infty} \int_{L_1} f(t; x) \, dt + \lim_{\xi \to \infty} \int_{L_2} f(t; z) \, dt$$

$$+ i \sum_{j=1}^{\infty} (-1)^j \frac{\Gamma(\lambda + j)}{\Gamma(\lambda)} |z|^\delta \lim_{\xi \to \infty} \int_{C_{|z|\xi}} e^{iu} u^{-(j\delta+1)} \, du = 0.$$

However, we may state that

$$\lim_{\xi \to \infty} \int_{C_{|z|\xi}} e^{iu} u^{-(j\delta+1)} \, du = 0.$$
Finally,
\[
\lim_{\xi \to \infty} \int_{L_1} f(t; x) \, dt = - \lim_{\xi \to \infty} \int_{L_2} f(t; z) \, dt.
\]

The last result implies
\[
\hat{D}_{\delta, \lambda}(z) = \frac{1}{\pi} \text{Re} \int_0^\infty \left[ 1 - \frac{r^{\delta \lambda}}{(1 + r^{\delta \lambda})} \right] e^{i|z|r} \, dr = -\frac{1}{\pi} \text{Im} \int_0^\infty \left[ 1 - \frac{e^{\delta \lambda \pi/2} r^{\delta \lambda}}{(1 + e^{i\pi/2} r^{\delta \lambda})} \right] e^{-|z|r} \, dr.
\]

\textbf{Proof of Theorem 2.} The idea of this proof is based on the inverse Fourier transform associated with the Dagum covariance function. With \( \|z\| = z \), we write
\[
\int_{\mathbb{R}^d} e^{i r z} \left( -\frac{z^{1-d/2}}{2^{d/2-1} \pi^{d/2-1}} \text{Im} \int_0^\infty K_{d/2-1}(z t) \left( 1 - \frac{r^{\delta \lambda} e^{i \pi \delta t/2}}{(1 + e^{i \pi/2} r^{\delta t})} \right) r^{d/2} \, dt \right) \, dz
\]
\[
= (2\pi)^{d/2} r^{1-d/2} \int_0^\infty J_{d/2-1}(r z) z^{d/2} \left( -\frac{z^{1-d/2}}{2^{d/2-1} \pi^{d/2-1}} \text{Im} \int_0^\infty K_{d/2-1}(z t) \right)
\times \left( 1 - \frac{r^{\delta \lambda} e^{i \pi \delta t/2}}{(1 + e^{i \pi/2} r^{\delta t})} \right) r^{d/2} \, dt \, dz
\]
\[
= -\frac{2 r^{1-d/2}}{\pi} \text{Im} \int_0^\infty \left( 1 - \frac{e^{i \pi \delta t/2} r^{\delta \lambda}}{(1 + e^{i \pi/2} r^{\delta t})} \right) \frac{t^{d/2}}{r^{d/2+t^2}} \, dt \int_0^\infty z K_{d/2-1}(z t) J_{d/2-1}(z r) \, dz
\]
\[
= -\frac{1}{i\pi} \int_{-\infty}^\infty \left( 1 - \frac{r^{\delta \lambda} e^{i \pi \delta t/2}}{(1 + e^{i \pi/2} r^{\delta t})} \right) \frac{t}{r^{2+t^2}} \, dt.
\]

Both improper integrals in this proof are convergent for \( d > 1 \), and for arbitrary values of \( \lambda \) and \( \delta \). Hence the order of integration can be swapped. When \( \delta \in (0, 2) \) and \( \delta \lambda \in (0, 2) \), the last integral is expressed as
\[
-\frac{1}{i\pi} \int_{-\infty}^\infty \left( 1 - \frac{r^{\delta \lambda} e^{i \pi \delta t/2}}{(1 + e^{i \pi/2} r^{\delta t})} \right) \frac{t}{r^{2+t^2}} \, dt = 1 - \frac{r^{\delta \lambda}}{(1 + r^{\delta \lambda})}.
\]

\textbf{Proof of Theorem 3.} To find the explicit form of the Dagum spectral density, we use the Mellin–Barnes transform [3] defined by the identity
\[
\frac{1}{(1 + x)^\delta} = \frac{1}{2i\pi} \frac{1}{\Gamma(\delta)} \int_{\Lambda} x^{u} \Gamma(-u) \Gamma(\delta + u) \, du,
\]
where \( \Gamma(\cdot) \) denotes the Gamma function. This representation is valid for any \( x \in \mathbb{R} \). The contour \( \Lambda \) contains the vertical line which passes between left and right poles in the complex plane \( u \) from negative to positive imaginary infinity, and should be closed to the left if \( x > 1 \) and to the right complex infinity if \( 0 < x < 1 \).

We start by proving equation (10). We have
\[
\frac{1}{(1 + x)^\delta} = \frac{1}{2i\pi} \frac{1}{\Gamma(\delta)} \int_{c-i\infty}^{c+i\infty} x^{-u} \Gamma(u) \Gamma(\delta - u) \, du,
\]
\[0 < c < \text{Re} \delta, \ x > 0.\]
The beta function is defined by

\[ B(u, \delta) = \int_0^1 y^{u-1}(1 - y)^{\delta-1} \, dy, \quad \text{Re } u, \text{ Re } \delta > 0. \]

The change of variable \( x = y/(1 - y) \) gives

\[ B(u, \delta) = \int_0^\infty \frac{x^{u-1}}{(1 + x)^{u+\delta}} \, dx. \]

We now note that the beta function \( B(u, \delta - u) \) is the Mellin transform of the function \( 1/(1 + x)^\delta \). The inversion formula gives

\[
\frac{1}{(1 + x)^\delta} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-u} B(u, \delta - u) \, du
\]

\[
= \frac{1}{2\pi i} \frac{1}{\Gamma(\delta)} \int_{c-i\infty}^{c+i\infty} x^{-u} \Gamma(u) \Gamma(\delta - u) \, du.
\]

By construction, \( c \) should be in the strip \( 0 < c < \text{Re } \delta \) due to the integrability conditions of the Mellin transformation (see [2]).

We may of course change the sign of the integration variable \( u \) and obtain the identity

\[
\frac{1}{(1 + x)^\delta} = \frac{1}{2\pi i} \frac{1}{\Gamma(\delta)} \int_{-c-i\infty}^{-c+i\infty} x^{-u} \Gamma(-u) \Gamma(\delta + u) \, du.
\]

Equation (10) is established.

Applying (10), we obtain

\[
\hat{D}_{\delta,\lambda}(z) = \frac{1}{(2\pi)^d} \left[ \int_{\mathbb{R}^d} e^{iz\cdot r} \, dr - \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}^d} e^{iz\cdot r} \Gamma(-u) \Gamma(u + \lambda) r^{-u\delta} \, du \, dr \right]
\]

\[
= \frac{1}{(2\pi)^d} \left[ \int_{\mathbb{R}^d} e^{iz\cdot r} \, dr - \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}^d} \Gamma(-u) \Gamma(u + \lambda) \int_{\mathbb{R}^d} e^{-z\cdot r} r^{-u\delta} \, dr \, du \right].
\]

We now invoke the well-known relationship [1]

\[
\int_{\mathbb{R}^d} e^{iz\cdot r} \, dr = \frac{2^{d-\delta} \pi^{d/2} \Gamma(d/2 - u\delta/2)}{\Gamma(u\delta/2) \|z\|^{d-u\delta}}.
\]

Hence

\[
\hat{D}_{\delta,\lambda}(z) = \delta^{(d)}(z) - \frac{1}{(2\pi)^d} \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{\mathbb{R}^d} \frac{2^{d-\delta} \pi^{d/2} \Gamma(d/2 - u\delta/2) \Gamma(-u) \Gamma(u + \lambda)}{\Gamma(u\delta/2)} \frac{1}{z^{d-u\delta}} \, du
\]

\[
= \delta^{(d)}(z) - \frac{z^{-d}}{\pi^{d/2}} \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}^d} \frac{\Gamma(-u) \Gamma(u + \lambda) \Gamma(d/2 - u\delta/2)}{\Gamma(u\delta/2)} \left( \frac{z}{2} \right)^{u\delta} \, du. \quad (11)
\]

For any given value of \( |z/2| \), it is not relevant whether it is smaller or greater than 1. In fact, the contour might be closed to the right complex infinity for any value of \( z \) because in the
integrand the number of Euler gamma functions with the negative signs of its arguments \( u \) is bigger than the number of Euler gamma functions with the positive signs of its arguments \( u \). Thus the integrand is asymptotically decreasing on the right complex infinity and the arc does not contribute. The series resulting from the residue calculus is convergent for any values of the variable \( z \). The functions \( u \mapsto \Gamma(-u) \) and \( u \mapsto \Gamma(d/2 - u\delta/2) \) contain poles in the complex plane, respectively when \( -u = -n \), and when \( d/2 - u\delta/2 = -n, n \in \mathbb{Z}_+ \). Using this fact and through direct inspection, we obtain that the right-hand side in (11) matches

\[
\hat{D}_{\delta,\lambda}(z) = \delta^{(d)}(z) - \frac{z^{-d}}{\pi d/2} \frac{1}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + n)\Gamma(d/2 - n\delta/2)}{n!\Gamma(n\delta/2)} \left( \frac{z}{2} \right)^{n\delta} - \frac{1}{2^d \pi d/2} \frac{1}{\Gamma(\lambda)} \frac{2}{\delta} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + (d + 2n)/\delta)\Gamma(-(d + 2n)/\delta)}{n!\Gamma(n + d/2)} \left( \frac{z}{2} \right)^{2n}.
\]

(12)

Next we invoke the expression of the Fox–Wright function as in (7). In particular, using (7) and (8), we obtain a new form of the Dagum spectral density:

\[
\hat{D}_{\delta,\lambda}(z) = \delta^{(d)}(z) - \frac{\pi^{-d/2} z^{-d}}{\Gamma(\lambda)} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + n)\Gamma(d/2 - n\delta/2)}{n!\Gamma(n\delta/2)} \left( \frac{z}{2} \right)^{n\delta} - \frac{1}{2^d \pi d/2} \frac{1}{\Gamma(\lambda)} \frac{2}{\delta} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + (d + 2n)/\delta)\Gamma(-(d + 2n)/\delta)}{n!\Gamma(n + d/2)} \left( \frac{z}{2} \right)^{2n} \right].
\]

(13)

This is the same constant that stands in the formulation of Theorem 4 that states that the limit \( z \to 0 \) is smooth for \( \delta > d \) and the function \( \hat{D}_{\delta,\lambda}(z) \) is continuous at \( z = 0 \) if \( \delta > d \). This means the Dirac \( \delta \)-function should be cancelled with the \( n = 0 \) term in the first sum of (12), and for any \( z \) the final form of the spectral density is

\[
\hat{D}_{\delta,\lambda}(z) = -\frac{z^{-d}}{\pi d/2} \frac{1}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(\lambda + n)\Gamma(d/2 - n\delta/2)}{n!\Gamma(n\delta/2)} \left( \frac{z}{2} \right)^{n\delta} - \frac{1}{2^d \pi d/2} \frac{1}{\Gamma(\lambda)} \frac{2}{\delta} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda + (d + 2n)/\delta)\Gamma(-(d + 2n)/\delta)}{n!\Gamma(n + d/2)} \left( \frac{z}{2} \right)^{2n}.
\]

(14)

This may be only if the \( n = 0 \) term in the first sum of (12) is interpreted as the Dirac \( \delta \)-function in the sense of distributions.

\textbf{Proof of Theorem 4.} The first point can be proved by direct construction. We have

\[
\hat{D}_{\delta,\lambda}(z) = \frac{z^{1-d/2}}{(2\pi)^{d/2}} \int_{0}^{\infty} J_{d/2-1}(rz) r^{d/2} \left( 1 - \frac{r^{\delta \lambda}}{(1 + r^{\delta \lambda})} \right) dr.
\]

(15)
To find the low frequency behaviour of $\hat{D}_{\delta,\lambda}(z)$, we make use of (15). A change of variable of the type $y = rz$ shows that
\[
\hat{D}_{\delta,\lambda}(z) = \frac{z^{-d/2}}{(2\pi)^{d/2}} \int_0^\infty J_{d/2-1}(y) \left(1 - \frac{(y/z)^{\delta\lambda}}{(1 + (y/z)^{\delta})^\lambda}\right)(y/z)^{d/2} dy \\
= \frac{z^{-d}}{(2\pi)^{d/2}} \int_0^\infty J_{d/2-1}(y) \left(1 - \frac{(y/z)^{\delta\lambda}}{(1 + (y/z)^{\delta})^\lambda}\right)y^{d/2} dy.
\]

We now invoke the identity
\[
1 - \frac{1}{(1 + (z/y)^{\delta})^\lambda} = -\sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(\lambda + j)}{\Gamma(\lambda)} (z/y)^{\lambda j} \sim \lambda(z/y)^\delta \quad \text{as } z \to 0^+,
\]
to obtain
\[
\hat{D}_{\delta,\lambda}(z) \sim \frac{\lambda z^{-d}}{(2\pi)^{d/2}} \int_0^\infty J_{d/2-1}(y) \left(\frac{z}{y}\right)^{\delta} y^{d/2} dy = \frac{\lambda z^{-d}}{(2\pi)^{d/2}} \int_0^\infty J_{d/2-1}(y) y^{d/2-\delta} dy.
\]
The improper integral of each term of this absolutely convergent series is finite, so we may exchange summation and improper integration. In the limit $z \to 0^+$ we leave only the first (most singular) term of this series.

Using [11, 6.651 (14)], we find that if $d/2 - 1/2 < \delta < d$,
\[
\hat{D}_{\delta,\lambda}(z) \sim \frac{2^{d/2-\delta} \lambda \Gamma(d/2 - \delta/2)}{(2\pi)^{d/2} \Gamma(\delta/2)} \zeta^{\delta-d} = \frac{2^{\delta-d} \lambda \Gamma(d/2 - \delta/2)}{\pi^{d/2} \Gamma(\delta/2)} \zeta^{\delta-d}.
\]

This result coincides with the $n = 1$ term of the first sum of (12) and (14). We may conclude that this limit $z \to 0$ is singular for the Dagum spectral density if $\delta < d$. The Dagum spectral density is not continuous at the point $z = 0$ under the condition $\delta < d$. The limit $z \to 0$ corresponds to the behaviour of the Dagum correlation function at $r \to \infty$ and to integrability of its Fourier transformation. In this case the Dagum spectral density is singular at $z = 0$ because the integral of the Fourier transformation is not convergent for $z = 0$ under the condition $\delta < d$.

We now prove the second point. When $z \to 0^+$, the Bessel function of the second kind can be expressed asymptotically as [17]
\[
J_\nu(rz) \sim \frac{(rz)^\nu}{2^\nu \Gamma(\nu + 1)}, \quad (16)
\]
that is, an upper bound on the Bessel function in this case. Since this improper integral is finite when $\delta > d$, we may replace the Bessel function in this improper integral with its upper bound in order to estimate the asymptotic behaviour of this integral. Thus we may write in this limit
\[
\hat{D}_{\delta,\lambda}(z) = \frac{z^{1-d/2}}{(2\pi)^{d/2}} \int_0^\infty J_{d/2-1}(rz)r^{d/2} \left(1 - \frac{r^{\delta\lambda}}{(1 + r^{\delta})^\lambda}\right) dr \\
\sim \frac{z^{1-d/2}}{(2\pi)^{d/2}} \int_0^\infty \frac{(rz)^{d/2-1}}{2^{d/2-1} \Gamma(d/2)} r^{d/2} \left(1 - \frac{r^{\delta\lambda}}{(1 + r^{\delta})^\lambda}\right) dr \\
= \frac{1}{\pi^{d/2} 2^{d-1} \Gamma(d/2)} \int_0^\infty r^{d-1} \left(1 - \frac{r^{\delta\lambda}}{(1 + r^{\delta})^\lambda}\right) dr.
\]
We make a change of variable of the type $r^\delta = u$, and we find that if $\delta > d$,
\[
\hat{D}_{\delta,\lambda}(z) \sim \frac{1}{\delta \pi^{d/2} 2^{d-1} \Gamma(d/2)} \int_0^\infty u^{(d-\delta)/\delta} \left(1 - \frac{u^\delta}{(1+u)^\lambda}\right) du.
\] (17)

This integral is a finite constant under the condition $\delta > d$ and may be found by the change of variables
\[
\tau = \frac{u}{1+u} \Rightarrow u = \frac{\tau}{1-\tau},
\]
that is,
\[
\int_0^\infty u^{(d-\delta)/\delta} \left(1 - \frac{u^\delta}{(1+u)^\lambda}\right) du = -\int_0^1 \frac{\tau^{d-1}}{(1-\tau)^{d/\delta+1}} (1 - \tau^\lambda) d\tau
\]
\[
= - \lim_{\epsilon \to 0} \int_0^1 \frac{\tau^{d-1}}{(1-\tau)^{d/\delta+1-\epsilon}} (1 - \tau^\lambda) d\tau
\]
\[
= - \lim_{\epsilon \to 0} \left[ B\left(\frac{d}{\delta}, \frac{d}{\delta} + \epsilon\right) - B\left(\frac{d}{\delta} + \lambda, \frac{d}{\delta} + \epsilon\right) \right]
\]
\[
= - \lim_{\epsilon \to 0} \left[ \frac{\Gamma(d/\delta)}{\Gamma(\epsilon)} - \frac{\Gamma(d/\delta + \lambda)}{\Gamma(\epsilon + \lambda)} \right] \Gamma\left(\frac{d}{\delta} + \epsilon\right)
\]
\[
= \frac{\Gamma(d/\delta + \lambda)}{\Gamma(\lambda)} \frac{\Gamma(-d/\delta)}{\Gamma(d/\delta + \lambda)}.
\] (18)

Thus we have
\[
\hat{D}_{\delta,\lambda}(z) \sim \frac{1}{\delta \pi^{d/2} 2^{d-1} \Gamma(d/2)} \frac{\Gamma(-d/\delta)}{\Gamma(\lambda)} \frac{\Gamma(d/\delta + \lambda)}{\Gamma(\lambda)}.
\]

This result coincides with the $n = 0$ term (13) of the second sum of (12) and (14). We may conclude that this limit $z \to 0$ is smooth for the Dagum spectral density if $\delta > d$. The Dagum spectral density is continuous at the point $z = 0$ under the condition $\delta > d$. The limit $z \to 0$ corresponds to the behaviour of the Dagum correlation function at $r \to \infty$ and to integrability of its Fourier transformation. In this case the Dagum spectral density is not singular at $z = 0$ because the integral of the Fourier transformation is convergent for $z = 0$ under the condition $\delta > d$. \qed

**Proof of Theorem 5.** To find the high frequency behaviour of the Dagum spectral density, we need to use (6). Indeed, as $z \to \infty$, we have
\[
\hat{D}_{\delta,\lambda}(z)
\]
\[
= - \frac{z^{1-d/2}}{2^{d/2-1} \pi^{d/2+1}} \text{Im} \int_0^\infty K_{d/2-1}(z t) \left(1 - \frac{z^{\delta t}}{(1 + z^{\delta t})^{\lambda}}\right) t^{\delta/2} dt
\]
\[
= - \frac{z^{-d}}{2^{d/2-1} \pi^{d/2+1}} \text{Im} \int_0^\infty K_{d/2-1}(t) \left(1 - \frac{z^{-\delta t}}{(1 + z^{-\delta t})^{\lambda}}\right) t^{\delta/2} dt
\]
\[
= \frac{z^{-d}}{2^{d/2-1} \pi^{d/2+1}} \text{Im} \int_0^\infty K_{d/2-1}(t) (1 + z^{\delta} e^{-i\pi \delta/2} t^{-\delta})^{-\lambda} t^{\delta/2} dt
\]
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Asymptotic result (21).

Asymptotic result (21). Studying the asymptotic of the spectral density at z would satisfy the standard criteria of convergence. Such a form of the integrand is useful for the contour of the integral to the left complex infinity becomes obvious, because this series the convergence of the series that arises in the result of the residue calculus when we close the contour of the integral to the left complex infinity becomes obvious, because this series would satisfy the standard criteria of convergence. Such a form of the integrand is useful for studying the asymptotic of the spectral density at z → ∞ and would allow us to reproduce the asymptotic result (21).

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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