Limit Distributions and Sensitivity Analysis for Entropic
Optimal Transport on Countable Spaces

Shayan Hundrieser * † Marcel Klatt * Axel Munk * † ‡

April 30, 2021

Abstract

For probability measures supported on countable spaces we derive limit distributions for empirical entropic optimal transport quantities. In particular, we prove that the corresponding plan converges weakly to a centered Gaussian process. Furthermore, its optimal value is shown to be asymptotically normal. The results are valid for a large class of ground cost functions and generalize recently obtained limit laws for empirical entropic optimal transport quantities on finite spaces. Our proofs are based on a sensitivity analysis with respect to a weighted $\ell^1$-norm relying on the dual formulation of entropic optimal transport as well as necessary and sufficient optimality conditions for the entropic transport plan. This can be used to derive weak convergence of the empirical entropic optimal transport plan and value that results in weighted Borisov-Dudley-Durst conditions on the underlying probability measures. The weights are linked to an exponential penalty term for dual entropic optimal transport and the underlying ground cost function under consideration. Finally, statistical applications, such as bootstrap, are discussed.

Keywords: Optimal transport, entropy regularization, central limit theorem, bootstrap, sensitivity analysis
MSC 2020 subject classification Primary: 60B12, 60F05, 62E20
Secondary: 90C06, 90C25, 90C31

1 Introduction

Over the last decades, the theory of optimal transport (OT), originating in the seminal work by Monge (1781) and later by Kantorovich (1958), has gradually established itself as an active area of modern mathematical research and related areas (see Rachev & Rüschendorf (1998a,b), Villani (2008), Santambrogio (2015), or Galichon (2016) for comprehensive monographs). Recently, OT and variants thereof have also been recognized as an important tool for statistical data analysis, e.g., in genetics (Evans & Matsen, 2012), fingerprint identification (Sommerfeld & Munk, 2018), computational biology (Schiebinger et al., 2019; Tameling et al., 2021), deformation analysis (Zemel & Panaretos, 2019), and medical imaging (Chen, 2020), among others. However, for routine data analysis the computational speed to solve the underlying linear program is still a bottleneck and the development of algorithms for fast computation is a highly active area of research. Common linear OT solvers (for bounded costs) such as the Auction algorithm (Bertsekas, 1981;
Bertsekas & Castanon, 1989) or Orlin’s algorithm (Orlin, 1988) have a worst case complexity \( \tilde{O}(N^3) \) for \( N \) denoting the size of the data and where \( \tilde{O} \) suppresses polylogarithmic terms. The best known (theoretical) worst case complexity to solve OT as a linear program is given by \( \tilde{O}(N^{2.5}) \) (Lee & Sidford, 2014) for which, however, no practical implementation is known. As an alternative approach Cuturi (2013) suggests to replace the original OT optimization problem by an entropy regularized surrogate. The proposed algorithm based on the work by Sinkhorn (1964, 1967) solves the corresponding optimization problem in about \( O(N^2) \) elementary operations (Altschuler et al., 2017; Dvurechensky et al., 2018). Since then, entropy regularized OT (EROT) has become a frequently used computational scheme for the approximation of OT (Peyré & Cuturi, 2019; Amari et al., 2019; Clason et al., 2021; Tong & Kobayashi, 2021).

In this paper we are concerned with EROT on countable spaces \( \mathcal{X} = \{x_1, x_2, \ldots\} \) and \( \mathcal{Y} = \{y_1, y_2, \ldots\} \) (possibly \( \mathcal{X} = \mathcal{Y} \)). A probability measure \( r \) on \( \mathcal{X} \) (\( s \) on \( \mathcal{Y} \)) is represented as an element in \( \ell^1(\mathcal{X}) \) (resp. \( \ell^1(\mathcal{Y}) \)), the space of absolutely summable sequences indexed over \( \mathcal{X} \) (resp. \( \mathcal{Y} \)), such that \( \sum_{x \in \mathcal{X}} r_x = 1 \) and \( r \geq 0 \). The set of couplings between \( r \) and \( s \), also known as transport plans, on the product space \( \mathcal{X} \times \mathcal{Y} \) is defined by

\[
\Pi(r, s) := \left\{ \pi \in \ell^1(\mathcal{X} \times \mathcal{Y}) : A(\pi) = \begin{pmatrix} r \\ s \end{pmatrix}, \pi \geq 0 \right\},
\]

where \( A \) is the marginalization operator

\[
A : \ell^1(\mathcal{X} \times \mathcal{Y}) \to \ell^1(\mathcal{X}) \times \ell^1(\mathcal{Y}), \quad \pi \mapsto \left( \frac{\sum_{y \in \mathcal{Y}} \pi_{xy}}{\sum_{x \in \mathcal{X}} \pi_{xy}}, \frac{\sum_{x \in \mathcal{X}} \pi_{xy}}{\sum_{y \in \mathcal{Y}} \pi_{xy}} \right).
\]

For a cost function \( c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) and a regularization parameter \( \lambda > 0 \) the entropic optimal transport value between probability measures \( r \) and \( s \) is defined as

\[
EROT^\lambda(r, s) := \inf_{\pi \in \Pi(r, s)} \langle c, \pi \rangle + \lambda M(\pi). \tag{EROT}
\]

The quantity \( \langle c, \pi \rangle = \sum_{(x, y) \in \mathcal{X} \times \mathcal{Y}} c(x, y) \pi_{xy} \) denotes the total costs associated to a transport plan \( \pi \in \Pi(r, s) \) and \( M(\pi) \) represents the mutual information

\[
M(\pi) := \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \pi_{xy} \log \left( \frac{\pi_{xy}}{\sum_{y' \in \mathcal{Y}} \pi_{xy'}} \right) \leq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \pi_{xy} \log \left( \frac{\pi_{xy}}{\pi_{x'y}} \right) \in [0, \infty],
\]

where by convention \( 0 \log(0) = 0 \). If there exists \( \pi \in \Pi(r, s) \) such that \( \langle c, \pi \rangle + \lambda M(\pi) < \infty \) and \( EROT^\lambda(r, s) > -\infty \), i.e., if (EROT) is feasible, then there exists a unique minimizer

\[
\pi^\lambda(r, s) := \arg\min_{\pi \in \Pi(r, s)} \langle c, \pi \rangle + \lambda M(\pi), \tag{1.1}
\]

known as entropic optimal transport plan (Proposition 2.1). Plugging \( \pi^\lambda(r, s) \) into the functional for total costs \( \langle c, \cdot \rangle \) yields the Sinkhorn costs

\[
S^\lambda(r, s) := \langle c, \pi^\lambda(r, s) \rangle.
\]

Moreover, (EROT) is a convex optimization problem and hence exhibits a dual formulation

\[
\sup_{\alpha \in \ell^1(\mathcal{X})} \langle \alpha, r \rangle + \sup_{\beta \in \ell^1(\mathcal{Y})} \langle \beta, s \rangle - \lambda \left[ \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \exp \left( \frac{\alpha_x + \beta_y - c(x, y)}{\lambda} \right) r_x s_y - r_x s_y \right], \tag{DEROT}
\]
where $\ell^r_s(\mathcal{X})$ and $\ell^s_r(\mathcal{Y})$ denote the spaces of functions on $\mathcal{X}$, $\mathcal{Y}$ with finite expectation under $r$ and $s$, respectively (for a general dual formulation on Polish spaces see Chizat et al. (2016)). Elements $\alpha = (\alpha_x)_{x \in \mathcal{X}} \in \ell^r_s(\mathcal{X})$, $\beta = (\beta_y)_{y \in \mathcal{Y}} \in \ell^s_r(\mathcal{Y})$ attaining the supremum in (DEROT) are called optimal entropic dual potentials and the quantities $(\alpha, r) = \sum_{x \in \mathcal{X}} \alpha_x r_x$, $(\beta, s) = \sum_{y \in \mathcal{Y}} \beta_y s_y$ are equal to their expectation with respect to $r$ and $s$.

Statistical questions arise as soon as the probability measures $r$ and $s$ are estimated by (discrete) empirical measures

$$\hat{r}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \quad \text{and} \quad \hat{s}_m = \frac{1}{m} \sum_{j=1}^{m} \delta_{Y_j} \quad \quad (1.2)$$

for samples $X_1, \ldots, X_n \overset{i.i.d.}{\sim} r$ and independent $Y_1, \ldots, Y_m \overset{i.i.d.}{\sim} s$ where $\delta_x$ is the Dirac measure at $x$. This scenario occurs, e.g., if the underlying probability measures are unknown (Genevay et al., 2019), when subsampling methods are applied for randomized approximations (Sommerfeld et al., 2019), or if statistical inference based on EROT is aimed for (Bigot et al., 2019; Klatt et al., 2020b). Fundamental to such tasks are asymptotic limit laws of the empirical EROT value and its corresponding plan. So far, existing results are limited to certain restrictions on the ground space. For probability measures supported on finitely many points Bigot et al. (2019) proved that the limit law for the empirical EROT value centered by its population counterpart is normal and Klatt et al. (2020b) showed asymptotic normality for the empirical EROT plan and empirical Sinkhorn costs. For the Euclidean space $\mathbb{R}^d$ and squared Euclidean costs Mena & Niles-Weed (2019) derived a normal limit law of the empirical EROT value when sampling from sub-Gaussian probability measures. In contrast to the finite case, the centering constant is given by the expected value of the empirical estimator rather than the population quantity.

Herein, we extend such results to countable spaces. Under suitable assumptions (to be discussed below) on the ground costs and probability measures $r$, $s$ we prove that the empirical EROT value is asymptotically normal

$$\sqrt{n}(\text{EROT}^\lambda(\hat{r}_n, s) - \text{EROT}^\lambda(r, s)) \xrightarrow{D} \mathcal{N}(0, \sigma^2_{\lambda}(r|s)), \quad (1.3)$$

as $n \to \infty$ (Theorem 4.1). Throughout this paper $\xrightarrow{D}$ denotes weak convergence (Billingsley, 1999). The asymptotic variance $\sigma^2_{\lambda}(r|s)$ depends on the variance of the optimal entropic dual potential $\alpha^\lambda$ for (DEROT) with respect to $r$ and is equal to

$$\sigma^2_{\lambda}(r|s) = \sum_{x \in \mathcal{X}} (\alpha^\lambda_x)^2 r_x - \left( \sum_{x \in \mathcal{X}} \alpha^\lambda_x r_x \right)^2.$$

Concerning the EROT plan, we show for the empirical version centered by its population counterpart, as $n \to \infty$, that

$$\sqrt{n}(\pi^\lambda(\hat{r}_n, s) - \pi^\lambda(r, s)) \xrightarrow{D} \mathcal{G}(0, \Sigma_{\lambda,\pi^\lambda}(r|s)), \quad (1.4)$$

for a centered Gaussian process $G$ with covariance $\Sigma_{\lambda,\pi^\lambda}(r|s)$ (Theorem 4.2). The covariance $\Sigma_{\lambda,\pi^\lambda}(r|s)$ can be stated explicitly and depends on the regularization parameter $\lambda$ and the EROT plan $\pi^\lambda$ between $r$ and $s$. Notably, weak convergence in (1.4) takes place in a suitable weighted $\ell^2$-space over $\mathcal{X} \times \mathcal{Y}$. This allows to characterize the limit distribution of empirical Sinkhorn costs (Corollary 4.4)

$$\sqrt{n}(S^\lambda(\hat{r}_n, s) - S^\lambda(r, s)) \xrightarrow{D} \mathcal{N}(0, \sigma^2_{\lambda,\pi^\lambda}(r|s)), \quad (1.5)$$

3
as \( n \to \infty \). The asymptotic variance \( \tilde{\sigma}_{\lambda,n}^2 (r|s) \) is given by
\[
\tilde{\sigma}_{\lambda,n}^2 (r|s) = \sum_{x,x' \in \mathcal{X}, y,y' \in \mathcal{Y}} c(x,y) c(x',y') \left( \sum_{\lambda,n} (r|s) \right)_{(x,y),(x',y')}.
\]

Our limit laws are generically Gaussian for \( r \neq s \) and \( r = s \). This is in strict contrast to limit results obtained for the empirical non-regularized \((\lambda = 0)\) OT value (Tameling et al., 2019) and for the OT plan (Klatt et al., 2020a) on discrete spaces. Heuristically speaking, whereas the asymptotic law of the non-regularized OT quantities depends on the geometry of the boundary of the underlying transport simplex, the entropy regularization smooths such quantities (unique solutions are attained in the interior of the simplex). Consequently, Gaussian fluctuations in the marginals translate to Gaussian fluctuations of EROT quantities.

Additionally, our results extend to the two-samples case and our method of proof implies consistency of the naive \( n\)-out-of-\( n \) bootstrap (Theorem 4.9), which is also in contrast to non-regularized OT (Sommerfeld & Munk, 2018). The latter being useful as the limit quantities in (1.4) and (1.5) are in general not accessible and its estimation from data is computationally cumbersome.

Our analysis reveals an interesting interplay between the cost function and the probability measures \( r \) and \( s \) in order to guarantee weak convergence of empirical quantities, an issue which does not arise for finite ground spaces (Bigot et al., 2019; Klatt et al., 2020b). In analogy to asymptotic results on finite spaces, for uniformly bounded ground costs
\[
\sup_{(x,y) \in \mathcal{X} \times \mathcal{Y}} |c(x,y)| < \infty
\]
we find that our convergence results are valid if the Borisov-Dudley-Durst condition
\[
\sum_{x \in \mathcal{X}} \sqrt{r_x} < \infty \tag{1.6}
\]
(Durst & Dudley, 1980; Borisov, 1981, 1983) is fulfilled. This condition is known to be necessary and sufficient for the function class \( \mathcal{H} := \{ \mathbf{1}_K: K \subseteq \mathcal{X} \} \) to be \( r \)-Donsker\(^1\) which is equivalent to weak convergence of the empirical process \( \sqrt{n}(\tilde{r}_n - r) \) in \( \ell^1(\mathcal{X}) \) towards a centered tight Gaussian process. For unbounded ground costs a careful modification of the usual \( \ell^1 \)-norm to a weighted version is required. Given a strictly positive function \( \Phi_X: \mathcal{X} \to [1, \infty) \) we introduce the weighted \( \ell^1 \)-norm
\[
\|a\|_{\ell^1_{\Phi_X}(\mathcal{X})} := \sum_{x \in \mathcal{X}} \Phi_X(x) |a_x|.
\]
A weak limit of the empirical process \( \sqrt{n}(\tilde{r}_n - r) \) to a centered Gaussian process in the corresponding weighted \( \ell^1 \)-space is then achieved by the weighted Borisov-Dudley-Durst condition (Yukich, 1986; Tameling et al., 2019)
\[
\sum_{x \in \mathcal{X}} \Phi_X(x) \sqrt{r_x} < \infty. \tag{1.7}
\]
In analogy to (1.6), weak convergence of the empirical process in \( \ell^1_{\Phi_X}(\mathcal{X}) \) is equivalent to the (weighted) function class \( \mathcal{H}_{\Phi_X} := \{ \Phi_X \mathbf{1}_K: K \subseteq \mathcal{X} \} \) being \( r \)-Donsker. The function

\(^1\)A class of real functions \( \mathcal{G} \) on \( \mathcal{X} \) is called \( r \)-Donsker if the empirical process \( \sqrt{n}(\tilde{r}_n - r) \) converges weakly in the Banach space \( \ell^\infty(\mathcal{G}) \) of uniformly bounded real functions on \( \mathcal{G} \), equipped with supremum norm \( \|a\|_{\ell^\infty(\mathcal{G})} := \sup_{g \in \mathcal{G}} |a(g)| \), towards a tight Borel-measurable random element (Van der Vaart & Wellner, 1996).
\[ \Phi_X \text{ is linked to the ground cost and the exponential penalty term of the dual formulation (DEROT) in an intricate fashion. More precisely, we assume throughout that there exist (possibly unbounded) functions } c_X^+, c_X^-, c_Y^+, c_Y^- : X \rightarrow \mathbb{R}, \ c_Y^+, c_Y^- : Y \rightarrow \mathbb{R}\text{ with } c_X^+ \leq c_X^- \text{ and } c_Y^+ \leq c_Y^-\] such that
\[ c_X^+(x) + c_Y^-(y) \leq c(x, y) \leq c_X^+(x) + c_Y^+(y) \tag{1.8} \]
for all \((x, y) \in X \times Y\). Then, the sum of absolute values of the functions \(c_X^+\) and \(c_X^-(c_Y^+\) and \(c_Y^-)\) multiplied with an exponential term of their respective difference (see Section 2.2) yields the function \(\Phi_X : X \rightarrow [1, \infty)\) (resp. \(\Phi_Y : Y \rightarrow [1, \infty)\)) of the weighted \(\ell^1\)-norm for \(\ell^1_\Phi(X)\) (resp. \(\ell^1_\Phi(Y)\)), i.e.,
\[
\Phi_X(x) := (1 + |c_X^+(x)| + |c_X^-(x)|) \exp \left( \frac{c_X^+(x) - c_X^-(x)}{\lambda} \right) \quad \text{for all } x \in X,
\]
\[
\Phi_Y(y) := (1 + |c_Y^+(y)| + |c_Y^-(y)|) \exp \left( \frac{c_Y^+(y) - c_Y^-(y)}{\lambda} \right) \quad \text{for all } y \in Y.
\]

As a consequence, we require for our limit laws on the empirical EROT value (1.3) in addition to the weighted Borisov-Dudley-Durst condition (1.7) for \(r\) a moment condition for \(s\), given by
\[
\sum_{y \in Y} \Phi_Y(x)s_y < \infty. \tag{1.9}
\]

Note, that the different nature of the conditions (1.7) and (1.9) results from the fact that the measure \(r\) is randomly perturbed (sampled) while \(s\) is assumed to be fixed. For the related two-sample results for which both \(r\) and \(s\) are randomly perturbed we also need to consider the square-root of \(s_y\) in the sum, hence the conditions become symmetric. Our results on the empirical EROT plan and Sinkhorn costs additionally require the dominating functions \(c_X^+, c_X^-\) to satisfy
\[
\sup_{x \in X} |c_X^+(x) - c_X^-(x)| < \infty.
\]

This condition is met, e.g., for squared Euclidean costs \(c(x, y) = ||x - y||^2\) when the ground space \(X\) is bounded whereas \(Y\) may be selected to be unbounded (Section 4.1.1), or for costs induced by a metric if the ground spaces \(X\) and \(Y\) fulfill certain geometrical properties (Section 4.1.2). The respective summability constraints for our limit laws of the empirical EROT plan (1.4) and Sinkhorn costs (1.5) are similar to (1.9). We distinguish between three types of cost functions and refer to Table 1 for an overview. Let us emphasize that our conditions on the probability measures \(r\) and \(s\) to ensure weak convergence of the empirical EROT plan are sharp under uniformly bounded ground costs or ground costs with bounded variation (Remark 4.5).

Our proof technique is based on a general functional delta method with respect to Hadamard differentiable functionals in a (weighted) \(\ell^1\)-space. To verify this notion of differentiability for the EROT value we use the dual formulation (DEROT), exploit strong duality and explicitly use the (weighted) \(\ell^1\)-structure. Concerning the EROT plan we employ an optimality criterion for primal and dual optimizers of (EROT) and perform a sensitivity analysis that is based on an implicit function approach. Additional mathematical challenges come into play as a certain operator is only invertible on a rather small domain which avoids an ad-hoc application of a standard implicit function theorem for Hadamard differentiable functionals (Römisch, 2004, Proposition 4). We refer to Remark 3.7 for further details. Instead, we carefully assess the individual error terms that are caused by the perturbation and exploit explicit bounds for primal and dual optimizers.
Table 1: Interplay between cost functions and weighted Borisov-Dudley-Durst conditions on the probability measure \(r\) for weak limits of empirical EROT plan and Sinkhorn costs.

| Type                              | Cost                             | Weighted Borisov-Dudley-Durst Condition |
|-----------------------------------|----------------------------------|----------------------------------------|
| Bounded costs \(\|c\|_{\ell^\infty(X \times Y)} < \infty\) | \(\sum_{x \in X} \sqrt{T_x} < \infty\) | \(\sum_{y \in Y} \left(1 + \|c_X(x)\| + \|c_Y(y)\|\right) s_y < \infty\) |
| Costs with bounded variation \(\|c'_X - c_X\|_{\ell^\infty(X)} < \infty\) | \(\sum_{y \in Y} \left(1 + \|c'_X(x)\|\right) s_y < \infty\) | \(\sum_{x \in X} \left(1 + \|c_Y(y)\|\right) s_y < \infty\) |
| Costs with bounded variation in \(X\)-component \(\|c'_X - c_X\|_{\ell^\infty(X)} < \infty\) | \(\sum_{y \in Y} \left(1 + \|c'_X(x)\|\right) s_y < \infty\) | \(\sum_{x \in X} \left(1 + \|c_Y(y)\|\right) s_y < \infty\) |

The paper is structured as follows. Section 2 introduces basic notation, states an optimality criterion for primal and dual optimizers of (EROT), and shows existence and uniqueness of optimizers. Furthermore, we derive explicit bounds for optimal solutions and prove continuity of primal and dual optimizers. Section 3 proves that the EROT value and plan are Hadamard differentiable in a weighted \(\ell^1\)-space with respect to their marginal probability measures (Theorems 3.2 and 3.4). Herein, we also motivate our candidates for the respective derivative of EROT value and plan. Section 4 presents the main results on limit distributions for the empirical EROT value and plan (Theorems 4.1 and 4.2). As particular cases, we investigate the setting of squared Euclidean costs as well as cost functions which are induced by a metric. Furthermore, we study the behavior of the limit distributions as the regularization parameter tends to zero (Section 4.2) and verify that the naïve \(n\)-out-of-\(n\) bootstrap is consistent for the empirical EROT value and plan (Theorem 4.9). Section 5 contains the proofs for our sensitivity analysis of the EROT value and plan. In Section 6, we discuss our results and conclude with open questions for future research. The Appendix contains proofs for results of Section 2 and technical details that are employed in the proof of the sensitivity for the EROT plan.

We finally stress that our results have immediate statistical applications, which will be detailed in subsequent work. We briefly mention here biological colocalization of protein networks recorded with super-resolution microscopy which can be defined as a certain functional of the regularized OT plan. This has been done by Klatt et al. (2020b) in the context of finite ground spaces, i.e., for the setting of a finite number of pixels. Our conditions in Table 1 can be viewed as a guarantee for the stability of this method for large scale images with many pixels (corresponding to the support of the intensity profile of protein structures).

2 Preliminaries

In this section, we state some basic results for (EROT) most of which are well known and the purpose here is to generalize them to a broader class of cost functions. For the sake of readability the technical details are deferred to Appendix A.

Throughout this work we denote by \(\ell^1(\mathcal{X})\) the space of summable sequences indexed over \(\mathcal{X}\) and equipped with total variation norm \(\|a\|_{\ell^1(\mathcal{X})} = \sum_{x \in \mathcal{X}} |a_x|\). Note that \(\ell^1(\mathcal{X})\) can be interpreted as the space of finite signed measures on \(\mathcal{X}\). Its dual space can be identified by \(\ell^\infty(\mathcal{X})\) with norm \(\|b\|_{\ell^\infty(\mathcal{X})} = \sup_{x \in \mathcal{X}} |b_x|\). Given a positive function \(f : \mathcal{X} \rightarrow (0, \infty)\) we introduce the weighted \(\ell^1\)-space \(\ell^1_f(\mathcal{X})\) and its corresponding dual space \(\ell^\infty_f(\mathcal{X})\) with
For $a \in \ell^1_1(\mathcal{X})$ and $b \in \ell^\infty_1(\mathcal{X})$ we denote their dual pairing by $\langle a, b \rangle := \sum_{x \in \mathcal{X}} b_x a_x$ and let $\mathcal{P}(\mathcal{X}) = \{ r \in \ell^1(\mathcal{X}) : \sum_{x \in \mathcal{X}} r_x = 1, r \geq 0 \}$ be the set of probability measures on $\mathcal{X}$. We emphasize that we equip $\mathcal{X}$ with the discrete topology and do not embed it, e.g., in $\mathbb{R}^d$. Hence, for any probability measure $r \in \mathcal{P}(\mathcal{X})$ its (topological) support is equal to $\text{supp}(r) = \{ x \in \mathcal{X} : r_x > 0 \}$ and $r$ is of full support if and only if $r_x > 0$ for all $x \in \mathcal{X}$.

For a probability measure $r \in \mathcal{P}(\mathcal{X})$ we define with slight abuse of notation $\ell^1_1(\mathcal{X})$ as the space of functions on $\mathcal{X}$ with finite expectation with respect to $r$. For the weighted $\ell^1$- and $\ell^\infty$-spaces on $\mathcal{Y}$ and $\mathcal{X} \times \mathcal{Y}$ we adapt the same notation.

## 2.1 Primal and Dual Optimizers

Without loss of generality, we assume that the cost function $c$ is non-negative. Otherwise, according to (1.8) define the non-negative function

$$
\tilde{c} : \mathcal{X} \times \mathcal{Y} \to [0, \infty), \quad \tilde{c}(x, y) := c(x, y) - c^\mathcal{X}_x \oplus c^\mathcal{Y}_y (x, y)
$$

with $c^\mathcal{X}_x \oplus c^\mathcal{Y}_y (x, y) := c^\mathcal{X}_x (x) + c^\mathcal{Y}_y (y)$ which is bounded by $\tilde{c}(x, y) \leq (c^\mathcal{X}_x - c^\mathcal{X}_x^\lambda)(x) + (c^\mathcal{Y}_y - c^\mathcal{Y}_y^\lambda)(y)$ and satisfies for any $\pi \in \Pi (r, s)$ that $\langle c, \pi \rangle = \langle \tilde{c}, \pi \rangle + \langle c^\mathcal{X}_x, r \rangle + \langle c^\mathcal{Y}_y, s \rangle$. Hence, in case $c^\mathcal{X}_x \in \ell^1_1(\mathcal{X})$ and $c^\mathcal{Y}_y \in \ell^1_1(\mathcal{Y})$ it follows that the objective functions of (EROT) with cost function $c$ and with the shifted non-negative cost function $\tilde{c}$ only differ by a constant which does not affect the set of minimizers.

We start with a general result for existence of optimizers for (EROT) and formulate a necessary and sufficient optimality criterion. Recall that the primal problem (EROT) and its dual (DEROT) are said to satisfy strong duality if their respective optimal values are equal.

**Proposition 2.1.** If the primal problem (EROT) is feasible, then there exists a unique EROT plan $\pi^\lambda \in \Pi (r, s)$ and a pair $(\alpha^\lambda, \beta^\lambda) \in \ell^1_1(\mathcal{X}) \times \ell^1_1(\mathcal{Y})$ of optimal entropic dual potentials for (DEROT) unique up to a constant shift, i.e., for any pair of optimal entropic dual potentials $(\tilde{\alpha}^\lambda, \tilde{\beta}^\lambda) \in \ell^1_1(\mathcal{X}) \times \ell^1_1(\mathcal{Y})$ there exists a constant $\eta \in \mathbb{R}$ with

$$
\alpha^\lambda - \tilde{\alpha}^\lambda = \beta^\lambda - \tilde{\beta}^\lambda = \eta
$$

for all $x \in \text{supp}(r), y \in \text{supp}(s)$. Moreover, strong duality holds and the elements $\pi \in \ell^1(\mathcal{X} \times \mathcal{Y})$ and $(\alpha, \beta) \in \ell^1_1(\mathcal{X}) \times \ell^1_1(\mathcal{Y})$ are optimal for (EROT) and (DEROT), respectively, if and only if

$$
\sum_{y \in \mathcal{Y}} \pi_{xy} = r_x \quad \text{and} \quad \sum_{x \in \mathcal{X}} \pi_{xy} = s_y \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}. \tag{2.2}
$$

The assertion on strong duality follows by (Chizat et al., 2016, Theorem 3.2), which also contains an optimality criterion that is however limited to $(\alpha, \beta) \in \ell^\infty(\mathcal{X}) \times \ell^\infty(\mathcal{Y})$. For an extension to potentials in $\ell^1_1(\mathcal{X}) \times \ell^1_1(\mathcal{Y})$ we follow a technique by Braunsmann (2018). Details can be found in Appendix A.
Remark 2.2. According to Proposition 2.1 optimal entropic dual potentials \((\alpha^\lambda, \beta^\lambda)\) are uniquely characterized by the relation
\[
\alpha^\lambda_x = -\lambda \log \left[ \sum_{y \in Y} \exp \left( \frac{\beta^\lambda_y - c(x, y)}{\lambda} \right) s_y \right] \quad \text{for all } x \in \text{supp}(r),
\]
\[
\beta^\lambda_y = -\lambda \log \left[ \sum_{x \in X} \exp \left( \frac{\alpha^\lambda_x - c(x, y)}{\lambda} \right) r_x \right] \quad \text{for all } y \in \text{supp}(s).
\]

2.2 Bounds and Continuity for Entropic Optimizers

Bounds for optimal entropic dual potentials turn out to be of crucial importance as they will be of particular use for proving Hadamard differentiability of the EROT value and plan. For squared Euclidean costs such bounds are readily available by Mena & Niles-Weed (2019). It is the purpose of this section to generalize their argument to a broader class of functions. To simplify notation, we define for \(\lambda > 0\) and \(\delta \in \mathbb{R}\) the functions
\[
C_\lambda: X \to [1, \infty), \quad x \mapsto 1 + |c^+_X(x)| + |c^-_X(x)|,
\]
\[
\phi^\lambda_X: X \to [1, \infty), \quad x \mapsto \exp \left( \delta \frac{c^+_X(x) - c^-_X(x)}{\lambda} \right),
\]
\[
\Phi^\lambda_X: X \to [1, \infty), \quad x \mapsto C_\lambda(x) \exp \left( \delta \frac{c^+_X(x) - c^-_X(x)}{\lambda} \right),
\]
where we note by \(c^+_X \geq c^-_X\) and \(C_\lambda \geq 1\) for \(\delta \geq 0\) that \(1 \leq \phi^\lambda_X \leq \Phi^\lambda_X\). Further, we set \(\phi_X := \phi^1_X\), \(\Phi_X := \Phi^1_X\) and define likewise the functions \(C_Y\), \(\phi^\lambda_Y\), \(\Phi^\lambda_Y\) on \(Y\).

Proposition 2.3. Let \(r \in \ell^1(X), s \in \ell^1(Y)\) be two probability measures and suppose that EROT\(^\lambda\)(\(r, s\)) is feasible. Then there exists an \((r, s)\)-almost surely unique pair of optimal entropic dual potentials \((\alpha^\lambda, \beta^\lambda) \in \ell^1(X) \times \ell^1(Y)\) with \((\alpha^\lambda, r) = (\beta^\lambda, s) = \frac{1}{\lambda} \text{EROT}^\lambda(r, s) \geq 0\) and
\[
c^-_X(x) - (c^+_X, r) - \lambda \log(\phi_Y, s) \leq \alpha^\lambda_x \leq c^+_X(x) + (c^-_Y, s),
\]
\[
c^-_Y(y) - (c^+_Y, s) - \lambda \log(\phi_X, r) \leq \beta^\lambda_y \leq c^+_Y(y) + (c^-_X, r)
\]
for all \(x \in \text{supp}(r), y \in \text{supp}(s)\). Further, for the EROT plan \(\pi^\lambda\) in (1.1) it holds for all \((x, y) \in X \times Y\) that
\[
r_{x} s_{y} \phi^\lambda_X(x) \phi^{-\lambda}_Y(y) \exp \left( \frac{- (c^+_X, r) - (c^+_Y, s)}{\lambda} \right) \pi^\lambda_{xy} \leq \phi^\lambda_X(x) \phi^\lambda_Y(y) \exp \left( \frac{(c^+_X, r) + (c^+_Y, s)}{\lambda} \right).
\]
In particular, we note that the upper bounds for the optimal entropic dual potentials \(\alpha^\lambda, \beta^\lambda\) are finite if \(c^+_X \in \ell^1(X)\) and \(c^+_Y \in \ell^1(Y)\), whereas the lower bounds are finite if additionally the condition \(\phi_X \in \ell^1(X)\) and \(\phi_Y \in \ell^1(Y)\) is fulfilled. These bounds imply a variety of convergence results for EROT\(^\lambda\) as well as for optimizers of (EROT) and (DEROT). We fix an element \(y_1 \in Y\) and assume that \(s_{y_1} > 0\).
Proposition 2.4. Let \((r_k, s_k)_{k \in \mathbb{N}} \subseteq \ell_\Phi^1(\mathcal{X}) \times \ell_\Psi^1(\mathcal{Y})\) be pairs of probability measures on \(\mathcal{X}\) and \(\mathcal{Y}\) that converge to \((r, s)\) and consider corresponding dual optimizers \((\alpha_k^\lambda, \beta_k^\lambda)\) \(\in \mathbb{R}^\mathcal{X} \times \mathbb{R}^\mathcal{Y}\) of (DEROT) for \((r_k, s_k)_{k \in \mathbb{N}}, (r, s)\), respectively, such that \(\beta_{k,y}^\lambda = \beta_y^\lambda = 0\) for all \(k \in \mathbb{N}\). For each \(x \in \text{supp}(r)\) and \(y \in \text{supp}(s)\) it then follows as \(k\) tends to infinity that

\[
\alpha_{k,x}^\lambda \to \alpha_x^\lambda \quad \text{and} \quad \beta_{k,y}^\lambda \to \beta_y^\lambda.
\]

As a consequence, we obtain for \(k \to \infty\) that

\[
\text{EROT}^\lambda(r_k, s_k) \to \text{EROT}^\lambda(r, s) \quad \text{and} \quad \pi^\lambda(r_k, s_k) \to \pi^\lambda(r, s) \quad \text{in} \quad \ell_{\mathcal{C}_x \otimes \mathcal{C}_y}^1(\mathcal{X} \times \mathcal{Y}).
\]

Proposition 2.4 states that the EROT value (resp. plan) is a continuous mapping from a subset of \(\ell_\Phi^1(\mathcal{X}) \times \ell_\Psi^1(\mathcal{Y})\) into \(\mathbb{R}\) (resp. \(\ell_{\mathcal{C}_x \otimes \mathcal{C}_y}^1(\mathcal{X} \times \mathcal{Y})\)). This observation will be strengthened by a refined sensitivity analysis which is the focus of the next section.

3 Sensitivity Analysis

In this section, we prove Hadamard differentiability of the EROT value and plan with respect to the marginal probability measures for a suitable weighted \(\ell^1\)-norm.

Definition 3.1. A mapping \(\Psi: U \to V\) between normed spaces \(U, V\) is said to be Hadamard differentiable at \(u \in U\) if there exists a continuous linear map \(\mathcal{D}_u^H \Psi: U \to V\) such that for any sequence \(h_n \in U\) converging to \(h\) and any positive sequence \((t_n)_{n \in \mathbb{N}}\) with \(t_n \searrow 0\) such that \(u + t_nh_n \in U\) it holds

\[
\left\| \frac{\Psi(u + t_nh_n) - \Psi(u)}{t_n} - \mathcal{D}_u^H \Psi(h) \right\|_V \to 0 \quad (3.1)
\]

for \(n \to \infty\). Further, let \(U_0 \subseteq U\) then \(\Psi\) is Hadamard differentiable tangentially to \(U_0\) at \(u\) if the limit (3.1) exists for all sequences \(h_n = t_n^{-1}(k_n - u)\) converging to \(h\) where \(k_n \in U_0\) and \(t_n \searrow 0\). The Hadamard derivative is then defined on the contingent (Bouligand) cone to \(U_0\) at \(u\)

\[
T_u(U_0) = \left\{ h \in U : h = \lim_{n \to \infty} t_n^{-1}(k_n - u), (k_n)_{n \in \mathbb{N}} \subseteq U_0, (t_n)_{n \in \mathbb{N}} \in \mathbb{R}_+; t_n \searrow 0 \right\}.
\]

Concerning the derivative of the EROT value (defined in (EROT)) we recall by the relation between primal and dual optimizers \(\pi^\lambda, \alpha^\lambda, \beta^\lambda\) for \(r, s\) (Proposition 2.1) that

\[
\text{EROT}^\lambda(r, s) = \langle \alpha^\lambda, r \rangle + \langle \beta^\lambda, s \rangle - \lambda \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \pi_{xy}^\lambda - r_xs_y = \langle \alpha^\lambda, r \rangle + \langle \beta^\lambda, s \rangle.
\]

This indicates that the Hadamard derivative of \(\text{EROT}^\lambda\) at \((r, s)\) is characterized by \(\langle \alpha^\lambda, \cdot \rangle + \langle \beta^\lambda, \cdot \rangle\), an observation that is in line with findings on finite ground spaces by Bigot et al. (2019) and Klatt et al. (2020b). On countable spaces this is also valid under suitable conditions on the cost functional and the probability measures \(r, s\).

Theorem 3.2. Assume the cost function satisfies (1.8) and let \(r \in \ell_\Phi^1(\mathcal{X}), s \in \ell_\Psi^1(\mathcal{Y})\) be
probability measures with full support. Then for $\lambda > 0$ the function

$$\text{EROT}^\lambda : \big( \mathcal{P}(\mathcal{X}) \cap \ell_1^{\Phi_X}(\mathcal{X}) \big) \times \big( \mathcal{P}(\mathcal{Y}) \cap \ell_1^{\Phi_Y}(\mathcal{Y}) \big) \to \mathbb{R},$$

is Hadamard differentiable at $(r, s)$ tangentially to $\big( \mathcal{P}(\mathcal{X}) \cap \ell_1^{\Phi_X}(\mathcal{X}) \big) \times \big( \mathcal{P}(\mathcal{Y}) \cap \ell_1^{\Phi_Y}(\mathcal{Y}) \big)$ with Hadamard derivative

$$\mathcal{D}^H_{(r, s)} \text{EROT}^\lambda : T_{(r, s)} \left( \big( \mathcal{P}(\mathcal{X}) \cap \ell_1^{\Phi_X}(\mathcal{X}) \big) \times \big( \mathcal{P}(\mathcal{Y}) \cap \ell_1^{\Phi_Y}(\mathcal{Y}) \big) \right) \to \mathbb{R},$$

$$(h^X, h^Y) \mapsto \langle \alpha^\lambda, h^X \rangle + \langle \beta^\lambda, h^Y \rangle,$$

where $(\alpha^\lambda, \beta^\lambda)$ are optimal entropic dual potentials for (DEROT) with marginals $(r, s)$. Herein, the set

$$T_{(r, s)} \left( \big( \mathcal{P}(\mathcal{X}) \cap \ell_1^{\Phi_X}(\mathcal{X}) \big) \times \big( \mathcal{P}(\mathcal{Y}) \cap \ell_1^{\Phi_Y}(\mathcal{Y}) \big) \right)$$

represents the contingent cone at $(r, s)$ with respect to $\ell_1^{\Phi_X}(\mathcal{X}) \times \ell_1^{\Phi_Y}(\mathcal{Y})$.

The proof is deferred to Section 5.1. At its heart is the strong duality statement (Proposition 2.1) in conjunction with point-wise convergence of optimal entropic dual potentials (Proposition 2.4). We explicitly exploit the weighted $\ell^1$-convergence for Hadamard differentiability of the EROT value on countable spaces (Remark 5.1).

Deriving the Hadamard derivative of the EROT plan is more challenging and requires sophisticated considerations. Before we state this in a formal way, we give a heuristic derivation. For this purpose, we consider the fixed element $y_1 \in \mathcal{Y}$ from previous section and adapt the notation that for any element $b \in \mathbb{R}^\mathcal{Y}$ we set $b_* := (b_{y_2}, b_{y_1}, \ldots) \in \mathbb{R}^{\mathcal{Y}\setminus\{y_1\}}$, i.e., the element where we omit the entry at $y_1$. In particular, we define

$$\mathcal{P}(\mathcal{Y})_* = \left\{ s_* \in \ell^1(\mathcal{Y}\setminus\{y_1\}) : \sum_{y \in \mathcal{Y}\setminus\{y_1\}} s_{*,y} \in [0, 1], \ s_{*,y} \geq 0 \right\}$$

as the set of probability vectors on $\mathcal{Y}$, where we omit the entry for $y_1$. Note that for a given element $s_* \in \mathcal{P}(\mathcal{Y})_*$ we obtain the associated probability measure by $s = (1 - \sum_{y \in \mathcal{Y}\setminus\{y_1\}} s_{*,y}, s_{*,y_1}, s_{*,y_2}, s_{*,y_3}, \ldots)$. At the core of our approach is the reformulation of the optimality criterion (Proposition 2.1) as an operator defined on suitable spaces. We introduce the marginalization operator omitting the entry at $y_1$, i.e.,

$$A_* : \ell^1(\mathcal{X} \times \mathcal{Y}) \to \ell^1(\mathcal{X}) \times \ell^1(\mathcal{Y}\setminus\{y_1\}), \ \pi \mapsto \left( \left( \sum_{x \in \mathcal{X}} \pi_{x,y} \right)_{x \in \mathcal{X}}, \left( \sum_{x \in \mathcal{X}} \pi_{x,y} \right)_{y \in \mathcal{Y}\setminus\{y_1\}} \right).$$

Its dual operator, a mapping $\ell^\infty(\mathcal{X}) \times \ell^\infty(\mathcal{Y}\setminus\{y_1\}) \to \ell^\infty(\mathcal{X} \times \mathcal{Y})$, will be extended as follows

$$A^T_* : \mathbb{R}^\mathcal{X} \times \mathbb{R}^{\mathcal{Y}\setminus\{y_1\}} \to \mathbb{R}^{\mathcal{X} \times \mathcal{Y}}, \ (a, b_*) \mapsto \begin{pmatrix} a_{x_1} & a_{x_1} + b_{*,y_2} & a_{x_1} + b_{*,y_3} & \cdots \\ a_{x_2} & a_{x_2} + b_{*,y_2} & a_{x_2} + b_{*,y_3} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$
These operators enable us to define
\[
\mathcal{F}: \left( \ell^1(\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^X \times \mathbb{R}^{\mathcal{Y}(y_1)} \right) \times \left( \ell^1(\mathcal{X}) \times \ell^1(\mathcal{Y}\{y_1\}) \right) \to \mathbb{R}^{X \times Y} \times \mathbb{R}^X \times \mathbb{R}^{\mathcal{Y}(y_1)},
\]
\[
\left( (\pi, \alpha, \beta), (r, s) \right) \mapsto \left( \pi - \exp \left( \frac{1}{c} \left[ A^T_\alpha(\alpha, \beta) - c \right] \right) \odot (r \otimes s), \right) \quad \left( A_\pi(\pi) - \left( \begin{array}{c} r \\ s \end{array} \right) \right),
\]
where $\odot$ denotes the component-wise product and $\otimes$ is defined as the tensor product of the probability measures $(r \otimes s)_{xy} = r_x s_y$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. In terms of $\mathcal{F}$ the optimality criterion from Proposition 2.1 is restated as follows.

**Corollary 3.3.** For given probability measures $(r, s) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$ and $\lambda > 0$ the element $\pi \in \ell^1(\mathcal{X} \times \mathcal{Y})$ and the potentials $(\alpha, (0, \beta)) \in \ell^1_\alpha(\mathcal{X}) \times \ell^1_\beta(\mathcal{Y}) \subseteq \mathbb{R}^X \times \mathbb{R}^{\mathcal{Y}}$ are optimal for (EROT) and (DEROT), respectively, if and only if
\[
\mathcal{F}(\pi, \alpha, \beta, (r, s)) = 0. \tag{3.2}
\]
Equation (3.2) intuitively contains a proposal for the Hadamard derivative of $\pi^\lambda$. Indeed, a naïve application of the usual calculus of partial derivatives (denoted by $\mathcal{D}$) from finite-dimensional spaces with respect to $r, s$ yields the partial derivative of $\mathcal{F}$ at $\tilde{r}, \tilde{s} \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$, for fixed elements $\bar{\pi} \in \ell^1(\mathcal{X} \times \mathcal{Y})$, $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^X \times \mathbb{R}^{\mathcal{Y}(y_1)}$, given by
\[
\mathcal{D}_{r,s}((\bar{\pi}, \bar{\alpha}, \bar{\beta}, \tilde{r}, \tilde{s})): \ell^1(\mathcal{X} \times \mathcal{Y}) \times \ell^1_\alpha(\mathcal{X}) \times \ell^1_\beta(\mathcal{Y}) \to \mathbb{R}^{X \times Y} \times \mathbb{R}^X \times \mathbb{R}^{\mathcal{Y}(y_1)},
\]
\[
(h^X, h^Y) \mapsto \left( - \exp \left( \frac{1}{c} \left[ A^T_\alpha(\bar{\alpha}, \bar{\beta}) - c \right] \right) \odot (\tilde{r} \otimes h^Y + h^X \otimes \tilde{s}), \right) \quad \left( - \left( \begin{array}{c} h^X \\ h^Y \end{array} \right) \right),
\]
where $h^Y := (-\sum_{y \in \mathcal{Y}(y_1)} h^Y_{y, y}, h^Y_{y, y_2}, h^Y_{y, y_3}, \ldots)$. Likewise, the naïve partial derivative of $\mathcal{F}$ with respect to $\pi, \alpha, \beta$ for the elements $(\bar{\pi}, \bar{\alpha}, \bar{\beta}, \tilde{r}, \tilde{s})$ is equal to
\[
\mathcal{D}_{\pi,\alpha,\beta}((\bar{\pi}, \bar{\alpha}, \bar{\beta}, \tilde{r}, \tilde{s})): \ell^1(\mathcal{X} \times \mathcal{Y}) \times \ell^1_\alpha(\mathcal{X}) \times \ell^1_\beta(\mathcal{Y}) \to \mathbb{R}^{X \times Y} \times \mathbb{R}^X \times \mathbb{R}^{\mathcal{Y}(y_1)},
\]
\[
(h^X, h^Y, h^X, h^Y) \mapsto \left( h^X \otimes A^T_\alpha(\bar{\alpha}, \bar{\beta}) - c \right) \odot (\tilde{r} \otimes \tilde{s}) \odot A^T_\beta(h^X, h^Y, \tilde{r}, \tilde{s}), \right) \quad \left( A_\pi(h^X \otimes Y) \right).
\]
Motivated by an implicit function approach a proposal for the Hadamard derivative of $\pi^\lambda$ at the pair of probability measures $(r, s) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$ is
\[
- \left[ \mathcal{D}_{\pi,\alpha,\beta}((\pi^\lambda, \alpha^\lambda, \beta^\lambda, r, s)) \mathcal{F} \right]^{-1} \circ \mathcal{D}_{r,s}(\pi^\lambda, \alpha^\lambda, \beta^\lambda, r, s) \mathcal{F}. \tag{3.3}
\]
Let us stress that the operator in (3.3) is neither obtained by rigorous mathematical considerations nor is it clear that such an operator is bounded or even exists. It is the content of the next statement to make (3.3) mathematically precise. This requires assumptions on the cost function and the probability measures $r$ and $s$. For their formulations we remind the reader of the functions $C_X$ and $C_Y$ from Section 2.2.

**Theorem 3.4.** Suppose $\|c_X^r - c_X^r\|_{\ell^\infty(\mathcal{X})} < \infty$, $\|c_Y^s - c_Y^s\|_{\ell^\infty(\mathcal{Y})} < \infty$, and let $r \in \ell^1_{C_X}(\mathcal{X})$, $s \in \ell^1_{C_Y}(\mathcal{Y})$ be probability measures with full support. Further, consider the EROT plan
\( \pi^\lambda(r, s) \) for \( \lambda > 0 \) as a mapping

\[
\pi^\lambda: (\mathcal{P}(\mathcal{X}) \cap \ell^1_{\mathcal{CX}}(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{CY}}(\mathcal{Y})) \rightarrow \ell^1_{\mathcal{CX} \oplus \mathcal{CY}}(\mathcal{X} \times \mathcal{Y}).
\]

Then \( \pi^\lambda \) is Hadamard differentiable at \( (r, s) \) tangentially to \((\mathcal{P}(\mathcal{X}) \cap \ell^1_{\mathcal{CX}}(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{CY}}(\mathcal{Y}))\) with Hadamard derivative given by

\[
D^H_{\pi^\lambda(r, s)}(h^x, h^y) = \left( \left[ D_{r, s}((\pi^\lambda(x, y), \ell^1_{\mathcal{CY}}(\mathcal{Y}))) \right]^{-1} \circ D_{r, s}((\pi^\lambda(x, y), \ell^1_{\mathcal{CY}}(\mathcal{Y}))) \right)(h^x, h^y).
\]

The proof is deferred to Section 5.2 but we like to sketch its main arguments here.

**Sketch of Proof.** We first verify that the proposed operator for the derivative (3.3) is well-defined and bounded (Proposition 5.2). Herein, we require that \( \|c^*_X - c^*_X\|_{\ell^\infty(\mathcal{X})} < \infty \) to ensure the validity of the Neumann series-calculus. We then proceed with the proof of Hadamard differentiability of the EROT plan. By definition, this requires to consider a sequence \((t_n)_{n \in \mathbb{N}}\) such that \(t_n \searrow 0\) and a converging sequence \((h^x_n, h^y_n)_{n \in \mathbb{N}} \in \ell^1_{\mathcal{CX}}(\mathcal{X}) \times \ell^1_{\mathcal{CY}}(\mathcal{Y})\) with limit \((h^x, h^y)\) and \((r + t_n h^x_n, s + t_n h^y_n) \in (\mathcal{P}(\mathcal{X}) \cap \ell^1_{\mathcal{CX}}(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{CY}}(\mathcal{Y}))\) for each \(n \in \mathbb{N}\). To this end, we verify that EROT plan is locally Lipschitz continuous (Proposition 5.3) and show that the difference quotient for finitely supported perturbations can be approximated by the proposed derivative (Proposition 5.4). These results, in conjunction with a notion of finite support approximation (Appendix C), allow us to find for any \( \varepsilon > 0 \) an integer \( N \in \mathbb{N} \) such that for all \( n \geq N \) holds

\[
\left\| \frac{\pi^\lambda(r + t_n h^x_n, s + t_n h^y_n) - \pi(r, s)}{t_n} - D^H_{\pi^\lambda(r, s)}(h^x, h^y) \right\|_{\ell^1_{\mathcal{CX} \oplus \mathcal{CY}}(\mathcal{X} \times \mathcal{Y})} < \varepsilon. \tag*{\Box}
\]

**Remark 3.5.** For any \( r, r' \in \mathcal{P}(\mathcal{X}) \cap \ell^1_{\mathcal{CX}}(\mathcal{X}), s, s' \in \mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{CY}}(\mathcal{Y}) \) it holds that

\[
\left\| \pi^\lambda(r, s) - \pi^\lambda(r', s') \right\|_{\ell^1_{\mathcal{CX} \oplus \mathcal{CY}}(\mathcal{X} \times \mathcal{Y})} \geq \max \left\{ \left\| r - r' \right\|_{\ell^1_{\mathcal{CX}}(\mathcal{X})}, \left\| s - s' \right\|_{\ell^1_{\mathcal{CY}}(\mathcal{Y})} \right\}.
\]

This implies under \( \|c^*_X - c^*_X\|_{\ell^\infty(\mathcal{X})} < \infty \) and \( \|c^*_Y - c^*_Y\|_{\ell^\infty(\mathcal{Y})} < \infty \) for a sequence \((t_n)_{n \in \mathbb{N}} \subseteq (0, \infty)\) such that \(t_n \searrow 0\) and a converging sequence \((h^x_n, h^y_n)_{n \in \mathbb{N}} \in \ell^1_{\mathcal{CX}}(\mathcal{X}) \times \ell^1_{\mathcal{CY}}(\mathcal{Y})\) with limit \((h^x, h^y)\) and \((r + t_n h^x_n, s + t_n h^y_n) \in (\mathcal{P}(\mathcal{X}) \cap \ell^1_{\mathcal{CX}}(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{CY}}(\mathcal{Y}))\) for all \(n \in \mathbb{N}\)
that
\[
\|D_{(r,s)}^H \pi^\lambda(h^X, h^Y)\|_{\ell^1_{\mathcal{C}_X\otimes\mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y})} = \lim_{n \to \infty} \frac{\|\pi^\lambda(r + t_n h^X_n, s + t_n h^Y_n) - \pi^\lambda(r,s)\|_{\ell^1_{\mathcal{C}_X\otimes\mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y})}}{t_n} \geq \lim_{n \to \infty} \max \left( \|h^X_n\|_{\ell^1_{\mathcal{C}_X}(\mathcal{X})}, \|h^Y_n\|_{\ell^1_{\mathcal{C}_Y}(\mathcal{Y})} \right) = \max \left( \|h^X\|_{\ell^1_{\mathcal{C}_X}(\mathcal{X})}, \|h^Y\|_{\ell^1_{\mathcal{C}_Y}(\mathcal{Y})} \right),
\]
which asserts \(D_{(r,s)}^H \pi^\lambda \neq 0\).

**Remark 3.6 (Generalization to unbounded ground costs).** Our proof technique generalizes to an asymmetric setting where we assume \(\|c^+ - c^-\|_{\ell^\infty(\mathcal{X})} < \infty\) and allow for \(\|c^r - c^s\|_{\ell^\infty(\mathcal{Y})} = \infty\). For this extension, we employ the space \(\ell_{\mathcal{C}_Y}^1(\mathcal{Y})\) which exhibits a strictly stronger norm than \(\ell_{\mathcal{C}_Y}^1(\mathcal{Y})\) when \(\|c^r - c^s\|_{\ell^\infty(\mathcal{Y})} = \infty\). In fact, we show in our proof in Section 5.2 that
\[
\pi^\lambda: (\mathcal{P}(\mathcal{X}) \cap \ell_{\mathcal{C}_X}^1(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell_{\mathcal{C}_Y}^1(\mathcal{Y})) \to \ell^1_{\mathcal{C}_X\otimes\mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y})
\]
is Hadamard-differentiable with an analogous derivative as in Theorem 3.4. In the proof, we specifically make use of the fact that at least one component of the cost function has bounded variation. It remains an open problem if this can be relaxed to account for general unbounded ground costs (Remark 5.5).

**Remark 3.7.** Our proof for the sensitivity of the EROT plan does not rely on a standard implicit function theorem for Hadamard differentiable functions (Römisch, 2004, Proposition 4). The main issue in employing this result lies in the selection of suitable normed spaces for the domain and range of \(\mathcal{F}\). To ensure that the mapping \(\mathcal{F}\) is well-defined, the range space has to be chosen sufficiently large while at the same time the range space has to be sufficiently small such that the operator \([D_{\pi,\alpha,\beta,\lambda}(\mathcal{P},\mathcal{Y}),\mathcal{F}]^{-1}\) is well-defined on a neighborhood around the origin in the range space. As it turns out by Lemma B.1 in Appendix B, the operator \([D_{\pi,\alpha,\beta,\lambda}(\mathcal{P},\mathcal{Y}),\mathcal{F}]^{-1}\) only has a fairly small domain (Remark B.2). Hence, to prove the claim on Hadamard differentiability we instead perform a careful analysis of the individual perturbation errors and show that they tend towards zero.

## 4 Limit Distributions

We derive in this section the limit distributions of the empirical EROT value and plan. More precisely, we estimate the EROT quantities \(\text{EROT}^\lambda(r, s)\) and \(\pi^\lambda(r, s)\) by plug-in estimators \(\text{EROT}^\lambda(\tilde{r}_n, \tilde{s}_m)\) and \(\pi^\lambda(\tilde{r}_n, \tilde{s}_m)\) based on empirical counterparts \(\tilde{r}_n\) and \(\tilde{s}_m\) (1.2) of the probability measures \(r\) and \(s\). We then characterize the statistical fluctuation \(\text{EROT}^\lambda(\tilde{r}_n, \tilde{s}_m)\) and \(\pi^\lambda(\tilde{r}_n, \tilde{s}_m)\) around their respective population version \(\text{EROT}^\lambda(r, s)\) and \(\pi^\lambda(r, s)\) by limit distributions. Herein, weak convergence of measures is denoted by \(\overset{\mathcal{D}}{\to}\) and we refer to Van der Vaart & Wellner (1996) for a general introduction. The underlying metric space in which weak convergence of the empirical EROT plan occurs is a suitable weighted \(\ell^1\)-space.

Our results are based on an application of the functional delta method for tangentially Hadamard differentiable functionals (Van der Vaart & Wellner, 1996). Hence, as our sensitivity analysis (Section 3) is based on a weighted \(\ell^1\)-space \(\ell^1(\mathcal{X})\) for a suitable choice of \(f: \mathcal{X} \to (0, \infty)\) we emphasize that the empirical process \(\sqrt{n}(\tilde{r}_n - r)\) has to converge...
weakly in this respective space $\ell^1_f(Y)$. To guarantee such weak convergence it is necessary and sufficient that the probability measure $r$ satisfies the weighted Borisov-Dudley-Durst condition $\sum_{x \in X} f(x) \sqrt{r_x} < \infty$ (Yukich, 1986, Theorem 6), (Tameling et al., 2019, Lemma 2.6). We note that weak convergence of the empirical process in $\ell^1_f(X)$ is equivalent to the function class $H_f := \{ f \mathbb{1}_K: K \in \mathcal{X} \}$ being $r$-Donsker. Overall, this leads to conditions as summarized in (1.9) and Table 1. The limit law is characterized by a zero mean Gaussian process with covariance $\Sigma(r) \in \mathbb{R}^{X \times X}$, defined by

$$\Sigma(r)_{xx'} = \begin{cases} r_x(1 - r_x) & \text{if } x = x', \\ -r_xr_{x'} & \text{if } x \neq x'. \end{cases} \quad \text{(4.1)}$$

Analogous assertions hold for the empirical process $\sqrt{m}(\hat{s}_m - s)$ with covariance $\Sigma(s) \in \mathbb{R}^{Y \times Y}$.

### 4.1 Empirical Entropic Optimal Transport

Without loss of generality, we assume that $r$ and $s$ have full support. Else we consider $\check{X} := \text{supp}(r)$ and $\check{Y} := \text{supp}(s)$ as the respective ground spaces. We first state our main results on limit laws of EROT quantities for general cost functions. Implications to more popular ground costs such as squared Euclidean costs are derived in further subsections. Recall the definition of $\Phi_X$ in (2.4).

**Theorem 4.1.** Suppose that the cost functions $c$ satisfies (1.8) and let $r \in \mathcal{P}(X) \cap \ell^1_{\Phi_X}(X)$ and $s \in \mathcal{P}(Y) \cap \ell^1_{\Phi_Y}(Y)$ be two probability measures. For $\lambda > 0$ denote by $(\alpha^\lambda, \beta^\lambda)$ corresponding optimal entropic dual potentials of (DEROT).

(i) (One sample) Suppose $X_1, \ldots, X_n \stackrel{i.i.d.}{\sim} r$ and that $\sum_{x \in X} \Phi_X(x) \sqrt{r_x} < \infty$. Then, as $n$ tends to infinity, weak convergence holds

$$\sqrt{n}(EROT^\lambda(\hat{r}_n, s) - EROT^\lambda(r, s)) \xrightarrow{D} (G_r, \alpha^\lambda) \overset{\text{D}}{=} \mathcal{N}(0, \sigma^2_{\lambda}(r|s)),$$

where $G_r$ is a tight, centered Gaussian process with covariance $\Sigma(r)$ defined in (4.1) and

$$\sigma^2_{\lambda}(r|s) = \sum_{x \in X} (\alpha^\lambda_x)^2 r_x - \left( \sum_{x \in X} \alpha^\lambda_x r_x \right)^2.$$

(ii) (Two samples) Suppose in addition that $Y_1, \ldots, Y_m \sim s$, independently of the $X_i$’s, and that $\sum_{y \in Y} \Phi_Y(y) \sqrt{s_y} < \infty$ and $\sum_{y \in Y} \Phi_Y(y) \sqrt{s_y} < \infty$. Then, for $m(n, n) \to \infty$ with $\frac{m}{n(n+1)} \to \delta \in (0, 1)$, it follows that

$$\sqrt{\frac{n+m}{n+1}}(EROT^\lambda(\hat{r}_n, \hat{s}_m) - EROT^\lambda(r, s)) \xrightarrow{D} \sqrt{\delta}(G_r, \alpha^\lambda) + \sqrt{1-\delta}(G_s, \beta^\lambda) \overset{\text{D}}{=} \mathcal{N}(0, \sigma^2_{\lambda, \delta}(r, s)).$$

where $G_r, G_s$ are tight, centered, independent Gaussian processes with associated covariances $\Sigma(r)$ and $\Sigma(s)$, respectively, and

$$\sigma^2_{\lambda, \delta}(r, s) = \delta \left[ \sum_{x \in X} (\alpha^\lambda_x)^2 r_x - \left( \sum_{x \in X} \alpha^\lambda_x r_x \right)^2 \right] + (1 - \delta) \left[ \sum_{y \in Y} (\beta^\lambda_y)^2 s_y - \left( \sum_{y \in Y} \beta^\lambda_y s_y \right)^2 \right].$$

For our main results on the empirical EROT plan (1.1) we require additional assumptions on the cost function.
Theorem 4.2. Assume the cost function satisfies (1.8) such that $\|c_X^* - c_X\|_{\ell^\infty(X)} < \infty$ and $\|c_Y^* - c_Y\|_{\ell^\infty(Y)} < \infty$. Further, let $\lambda > 0$ and consider probability measures $r \in P(X) \cap \ell_1^1(X)$ and $s \in P(Y) \cap \ell_1^1(Y)$.

(i) (One sample) Suppose $X_1, \ldots, X_n$ are i.i.d. $r$ and that $\sum_{x \in X} C_X(x) \sqrt{r_x} < \infty$. Then as $n$ tends to infinity, weak convergence holds

$$\sqrt{n}(\pi^\lambda(\hat r_n, s) - \pi^\lambda(r, s)) \xrightarrow{D} D_{(r,s)}^H \pi^\lambda(G_r, 0) \text{ in } \ell_1^1(\mathcal{X} \times \mathcal{Y}),$$

where $G_r$ is a tight, centered Gaussian process with covariance $\Sigma(r)$ from (4.1) and $D^H$ the Hadamard derivative in Theorem 3.4.

(ii) (Two samples) Suppose in addition that $Y_1, \ldots, Y_m \sim s$, independently of the $X_i$’s, and that $\sum_{x \in X} C_X(x) \sqrt{r_x} < \infty$ and $\sum_{y \in Y} C_Y(y) \sqrt{\sigma_y} < \infty$. Further, let $\min(m, n) \to \infty$ with $\frac{m}{m+n} \to \delta \in (0, 1)$, then it follows that

$$\sqrt{\frac{nm}{n + m}} \left(\pi^\lambda(\hat r_n, \hat s_m) - \pi^\lambda(r, s)\right) \xrightarrow{D} D_{(r,s)}^H \pi^\lambda(\sqrt{\delta}G_r, \sqrt{1 - \delta}G_s)$$

in $\ell_1^1(\mathcal{X} \times \mathcal{Y})$, with independent Gaussian processes $G_r, G_s$ as in Theorem 4.1.

Remark 4.3. By linearity of the Hadamard derivative $D_{(r,s)}^H \pi^\lambda$ it follows that the weak limits for the one-sample and two-samples case are both centered Gaussian processes.

Proofs of Theorems 4.1 and 4.2. We start with the assertions on the EROT value in Theorem 4.1. By our summability constraints (1.7) and (1.9) for assertion (i) we obtain using (Tameling et al., 2019, Lemma 2.6) for $n \to \infty$ the weak convergence

$$\sqrt{n}(\hat r_n - r) \xrightarrow{D} G_r \text{ in } \ell_1^1(X).$$

Further, by Portmanteau’s characterization of weak convergence in terms of closed sets (Van der Vaart & Wellner, 1996, Theorem 1.3.4(ii)) it follows that

$$1 = \limsup_{n \to \infty} P\left(\sqrt{n}(\hat r_n - r) \in \left\{ h^X \in \ell_1^1(X) : \sum_{x \in X} h^X_x = 0 \right\} \right)$$

$$\leq P\left(G_r \in \left\{ h^X \in \ell_1^1(X) : \sum_{x \in X} h^X_x = 0 \right\} \right),$$

i.e., the weak limit $G_r$ takes values in $\{h^X \in \ell_1^1(X) : \sum_{x \in X} h^X_x = 0\}$. For assertion (ii) we conclude by the weighted Borisov-Dudley-Durst conditions for $r$ and $s$ combined with independence of the samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ using (Van der Vaart & Wellner, 1996, Corollary 1.4.5) for $\min\{m, n\} \to \infty$, $\frac{m}{m+n} \to \delta$ that

$$\sqrt{\frac{nm}{n + m}} \left((\hat r_n, \hat s_m) - (r, s)\right) \xrightarrow{D} (\sqrt{\delta}G_r, \sqrt{1 - \delta}G_s) \text{ in } \ell_1^1(\mathcal{X}) \times \ell_1^1(\mathcal{Y}),$$

where $G_r$ is independent from $G_s$ and the pair $(G_r, G_s)$ takes values in

$$\{h^X \in \ell_1^1(X) : \sum_{x \in X} h^X_x = 0\} \times \{h^Y \in \ell_1^1(Y) : \sum_{y \in Y} h^Y_y = 0\}.$$
Concluding, by Theorem 3.4 the EROT value is Hadamard differentiable tangentially to\( (\mathcal{P}(\mathcal{X}) \cap \ell^2_{\Phi X}(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{\Phi Y}(\mathcal{Y})).\) Note that the underlying topology for Hadamard differentiability and for the weak convergence coincide. Hence, all assertions from Theorem 4.1 on the limit law follow as an application of the functional delta method for tangentially Hadamard differentiable functions (Van der Vaart & Wellner, 1996, Theorem 3.9.4). In particular, by linearity of the tangential Hadamard derivative, the asymptotic distribution is characterized by a zero mean Gaussian distribution where the limiting variance for the one-sample case is given by

\[
\sigma^2_n(r|s) = \sum_{x,x^\prime \in \mathcal{X}} \alpha_x^r \alpha_x^s \Sigma(r)_{x,x^\prime} = \sum_{x \in \mathcal{X}} (\alpha_x^r)^2 r_x - \left( \sum_{x \in \mathcal{X}} \alpha_x^r r_x \right)^2.
\]

The calculation for two samples is analogous. For the assertions on the EROT plan we use the same proof strategy and apply the functional delta method in conjunction with Hadamard differentiability (Theorem 3.4).

As a simple corollary we derive the limit laws of empirical Sinkhorn costs. We only present the one sample case, the two-samples case is analogous.

**Corollary 4.4.** Assume the same setting as for assertion (i) of Theorem 4.2. Then, as \(n\) tends to infinity, it follows that

\[
\sqrt{n}(S^\lambda(\hat{r}_n,s) - S^\lambda(r,s)) \overset{D}{\rightarrow} \{(c, D^H_{(r,s)} \pi^\lambda(G_r,0)) \}.
\]

Notably, by continuity and linearity of \((c, \cdot) : l^1_{\mathcal{C}_X \oplus \mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y}) \rightarrow \mathbb{R}\) the weak limit is also centered gaussian.

**Remark 4.5 (Sharpness of conditions).** For cost functions with bounded variation \(\|c^\lambda_X - c_X\|_{\ell^\infty(\mathcal{X})} < \infty, \|c^\lambda_Y - c_Y\|_{\ell^\infty(\mathcal{Y})} < \infty\) and in particular uniformly bounded ground costs \(\|c\|_{\ell^\infty(\mathcal{X} \times \mathcal{Y})} < \infty\) our stated assumptions for the validity of our limit laws for the empirical EROT plan are sharp. This is a simple consequence of the fact that weak convergence of the empirical EROT plan \(\sqrt{n}(\pi^\lambda(\hat{r}_n,s_n) - \pi^\lambda(r,s))\) in \(l^1_{\mathcal{C}_X \oplus \mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y})\) requires that the marginal empirical processes \(\sqrt{n}(\hat{r}_n - r)\) and \(\sqrt{n}(\hat{s}_n - s)\) converge weakly in \(l^1_{\mathcal{C}_X}(\mathcal{X})\) and \(l^1_{\mathcal{C}_Y}(\mathcal{Y})\), respectively. Such weak convergence holds if and only if the respective weighted Borisov-Dudley-Durst conditions \(\sum_{x \in \mathcal{X}} (\mathcal{C}_X \sqrt{r_x} < \infty, \sum_{y \in \mathcal{Y}} (\mathcal{C}_Y \sqrt{s_y} < \infty\) are fulfilled (Tameling et al., 2019).

**Remark 4.6 (Generalizations and degeneracy of limit laws).** (i) As discussed in Section 2.1, our theory generalizes to cost functions which might attain negative values.

(ii) Our sensitivity analysis on the EROT plan can be extended to the asymmetric case where \(\|c^\lambda_X - c_X\|_{\ell^\infty(\mathcal{X})} < \infty\) and \(\|c^\lambda_Y - c_Y\|_{\ell^\infty(\mathcal{Y})} = \infty\) (Remark 3.6). This covers limit results for the empirical EROT plan and Sinkhorn costs to this setting. For these limit laws we require that \(r \in l^1_{\mathcal{C}_X}(\mathcal{X})\) and \(s \in l^1_{\mathcal{C}_Y}(\mathcal{Y})\) as well as a weighted Borisov-Dudley-Durst condition (1.7) for the probability measures from which samples are taken. A summary of the different settings for cost function and associated summability constraints for the one-sample case from \(r\) is detailed in Table 1.

(iii) For a common countable metric ground space \((\mathcal{X}, d)\) i.e., \(\mathcal{X} = \mathcal{Y}\) our results for the cost function \(c(x,y) = d^p(x,y), p \geq 1\) show no substantial difference in the limit.
law for empirical EROT quantities between the cases \( r = s \) and \( r \neq s \). Additionally, our derived limit distributions generally do not degenerate as optimal entropic dual potentials are typically non-constant due to their relation with each other (Remark 2.2) and since the Hadamard derivative \( D^H_{(r,s)} \pi^L \) does not vanish for any \( r, s \) (Remark 3.5). These observations are in line with previous findings for finite spaces (Bigot et al., 2019; Klatt et al., 2020b) and for continuous spaces with squared Euclidean costs (Mena & Niles-Weed, 2019). In fact, for probability measures with finite support our required summability constraints are trivial and our findings coincide with results by Bigot et al. (2019) and Klatt et al. (2020b).

(vi) In contrast, limit results for the empirical (non-regularized) OT value obtained by Tameling et al. (2019) show a clear distinction in the limit behavior between the cases \( r = s \) and \( r \neq s \). Further, for \( \mathcal{X} = \mathcal{Y} \) and \( c(x, y) = d^p(x, y) \) with \( p > 1 \) the obtained limits by Tameling et al. (2019) degenerate for \( r = s \) with \( \text{supp}(r) = \mathcal{X} \) if and only if \( \mathcal{X} \) has no isolated point. This illustrates again that limit laws for empirical non-regularized OT differ fundamentally from their entropy regularized counterparts.

### 4.1.1 Squared Euclidean Costs

For squared Euclidean costs the theory of non-regularized optimal transport is well developed, e.g., existence of an optimal map to the Monge problem (Brenier, 1987, 1991) or for gradient flows in the 2-Wasserstein space on \( \mathbb{R}^d \) (Santambrogio, 2015). We like to contribute to this theory by stating explicit results for the framework of squared Euclidean costs \( c(x, y) = \| x - y \|_2^2 \) with ground spaces \( \mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d \) for some \( d \in \mathbb{N} \). By Young’s inequality it holds for \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \gamma > 0 \) for all \( (x, y) \in \mathcal{X} \times \mathcal{Y} \) that

\[
\| x \|_2^2 - \frac{2 \gamma \| x \|_p^p}{p} + \| y \|_2^2 - \frac{2 \| y \|_q^q}{q \gamma} \leq \| x - y \|_2^2 \leq \| x \|_2^2 + \frac{2 \gamma \| x \|_p^p}{p} + \| y \|_2^2 + \frac{2 \| y \|_q^q}{q \gamma}.
\]

Hence, we see that the dominating functions from (1.8) can be chosen as

\[
c^\pi_{\mathcal{X}}(x) := \| x \|_2^2 - \frac{2 \gamma \| x \|_p^p}{p}, \quad c^\pi_{\mathcal{Y}}(y) := \| y \|_2^2 - \frac{2 \| y \|_q^q}{q \gamma},
\]

\[
\Phi_{\mathcal{X}}(x) \asymp (1 + \| x \|_2^{2p}) \exp \left( \frac{4 \gamma}{p \lambda} \| x \|_p^p \right), \quad \Phi_{\mathcal{Y}}(y) \asymp (1 + \| y \|_2^{2q}) \exp \left( \frac{4}{q \gamma \lambda} \| y \|_q^q \right),
\]

where \( \vee \) denotes the maximum between two numbers and \( f \asymp g \) states the existence of two constants \( k, K > 0 \) such that \( k g \leq f \leq K g \).

Notably, the assertions of Theorem 4.1 for the empirical EROT value hold if \( r \in \ell_{\Phi_{\mathcal{X}}}^d(\mathcal{X}), s \in \ell_{\Phi_{\mathcal{Y}}}^d(\mathcal{Y}) \) and an associated weighted Borisov-Dudley-Durst condition (1.7) is satisfied. In case of \( p = q = 2 \) these conditions require that the probability measures \( r, s \) are sub-Gaussian up to a certain order. These findings complement results by Mena & Niles-Weed (2019) on the limit law of the empirical EROT value for sub-Gaussian probability measures on \( \mathbb{R}^d \) of any order. However, in contrast to our findings the centering constant in their central limit theorem is given by the expected value of the empirical estimator instead of the population version. Moreover, our limit laws remain valid if one probability measure is not sub-Gaussian provided that the other one is sufficiently concentrated.

In case both ground spaces \( \mathcal{X}, \mathcal{Y} \) are bounded subsets in \( \mathbb{R}^d \), we note that the cost function is uniformly bounded. Hence, our results on the empirical EROT value (Theorem 4.1) as well as the empirical EROT plan (Theorem 4.2) and empirical Sinkhorn costs
(Corollary 4.4) are valid as long as the probability measures satisfy the usual (unweighted) Borisov-Dudley-Durst condition. If only the ground space $\mathcal{X}$ is contained in a closed ball $B_R(0) = \{x \in \mathbb{R}^d : \|x\|_2 \leq R\}$ whereas $\mathcal{Y}$ is an unbounded set, we obtain by Cauchy-Schwarz inequality for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ that

$$\|x\|_2^2 - 2R \|y\|_2 + \|y\|_2^2 \leq \|x - y\|_2^2 \leq 2R \|y\|_2 + \|y\|_2^2.$$  

Upon defining the functions $c^X\_M(x) := \|x\|_2^2$ and $c^Y(y) := \|y\|_2^2 + 2R \|y\|_2$ we see that $\|c^X - c^X\|_{\ell^\infty(\mathcal{X})} = 0$ and $\|c^Y - c^Y\|_{\ell^\infty(\mathcal{Y})} = \infty$. As a consequence, it follows that the weight functions for our results on the EROT value (Theorem 4.1) are characterized by

$$\Phi_X(x) \asymp 1, \quad \Phi_Y(y) \asymp (1 + \|y\|_2^2) \exp\left(\frac{4}{\gamma \lambda \|y\|_2}\right).$$

For our limit results on the EROT plan (Theorem 4.2) and Sinkhorn costs (Corollary 4.4) we require that $r \in \ell^1_{\mathcal{X}\_M}(\mathcal{X})$, $s \in \ell^1_{\mathcal{Y}\_M}(\mathcal{Y})$ as well as associated Borisov-Dudley-Durst conditions (1.7) on the measure from which sampling occurs (Remark 4.6 (ii)). The underlying weight functions are chosen as

$$c^X\_M(x) \asymp 1, \quad \Phi^S\_M(y) \asymp (1 + \|y\|_2^2) \exp\left(\frac{16}{\gamma \lambda \|y\|_2}\right).$$

### 4.1.2 General Metric Costs

Cost functions in (regularized) optimal transport on general metric spaces are typically defined by the underlying metric. Assuming both ground spaces $\mathcal{X}, \mathcal{Y}$ to be subsets of a common metric space $(M, d)$ let the cost function be $c(x, y) = d(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Given some fixed element $z \in M$ we see by triangle inequality that

$$c^X\_M(x) := d(x, z), \quad c^X\_M(x) := 0, \quad c^Y\_M(y) := d(y, z), \quad c^Y\_M(y) := 0$$

are suitable dominating functions for the cost function satisfying (1.8). Hence, the weight functions for our limit laws on the empirical EROT value (Theorem 4.1) are equal to

$$\Phi_X(x) \asymp (1 + d(x, z)) \exp\left(\frac{d(x, z)}{\lambda}\right), \quad \Phi_Y(y) \asymp (1 + d(y, z)) \exp\left(\frac{d(y, z)}{\lambda}\right).$$

In order to obtain weight functions without the exponential term, we assume that the ground metric $d$ fulfills

$$\kappa := \sup_{(x, y) \in \mathcal{X} \times \mathcal{Y}} \left( d(x, z) + d(z, y) - d(x, y) \right) < \infty. \quad (4.2)$$

This condition is satisfied if $\mathcal{X}$ is contained in a bounded set, i.e., in case $\sup_{x \in \mathcal{X}} d(x, z) < \infty$. It also holds for $(M, d) = (\mathbb{R}^d, \|\cdot\|_1)$ with $\mathcal{X} \subseteq (-\infty, a], \mathcal{Y} \subseteq [b, \infty)$ for some $a, b \in \mathbb{R}^d$. In fact, condition (4.2) is a separability constraint, i.e., the larger the distances $d(x, z)$ and $d(y, z)$, the better the separation between $x$ and $y$ meaning the larger the distance $d(x, y)$. Most notably, both ground spaces $\mathcal{X}$ and $\mathcal{Y}$ can be unbounded sets. Under condition (4.2) we can select the functions $c^X\_M, c^Y\_M$ as

$$c^X\_M(x) := d(x, z) - \kappa/2 \quad \text{and} \quad c^Y\_M(y) := d(y, z) - \kappa/2$$

18
which yields that \( \|c_X^+ - c_X^-\|_\infty(\mathcal{X}) < \infty \) and \( \|c_Y^+ - c_Y^-\|_\infty(\mathcal{Y}) < \infty \). Hence, it follows that the exponential terms in the weight functions \( \Phi_X \) and \( \Phi_Y \) disappear

\[
\Phi_X(x) \times C_X \propto 1 + d(x, z) \quad \text{and} \quad \Phi_Y(y) \times C_Y \propto 1 + d(y, z).
\]

In particular, we note that our results on the limit laws of the empirical EROT value and plan (Theorems and 4.1 and 4.2) are both valid under identical assumptions, i.e., \( r \in \ell_1^\lambda(\mathcal{X}) \) and \( s \in \ell_1^\lambda(\mathcal{Y}) \) as well as a weighted Borisov-Dudley-Durst condition (1.7).

### 4.2 Relation to Non-regularized Optimal Transport

We assess the limit behavior of the empirical EROT value and Sinkhorn costs in the regime \( \lambda \searrow 0 \). For this purpose, we introduce the (non-regularized) optimal transport value between \( r \) and \( s \) as

\[
\text{OT}(r, s) := \inf_{\pi \in \Pi(r,s)} \{c, \pi\}.
\]

(OT)

Provided that \( C_X \in \ell_1^\lambda(\mathcal{X}) \), \( C_Y \in \ell_1^\lambda(\mathcal{Y}) \) it follows by (Villani, 2008, Theorem 5.9) that \( \text{OT}(r, s) \) is finite and that there exists a (possibly non-unique) optimal solution \( \pi^0 \in \Pi(r, s) \) such that \( \text{OT}(r, s) = (c, \pi^0) \). For our analysis we require an upper bound for the quantities \( |\text{EROT}^\lambda(r, s) - \text{OT}(r, s)| \) and \( |S^\lambda(r, s) - \text{OT}(r, s)| \). Hence, let \( \pi^0 \) be an OT plan for the non-regularized problem \( (\text{OT}) \) for \( r, s \) and denote for \( \lambda > 0 \) its entropy regularized counterpart by \( \pi^\lambda \). By optimality for their respective optimization problem it follows that

\[
(c, \pi^0) \leq (c, \pi^\lambda) \leq (c, \pi^\lambda) + \lambda M(\pi^\lambda) \leq (c, \pi^0) + \lambda M(\pi^0),
\]

where \( M(\cdot) \) represents the mutual information. This yields

\[
0 \leq S^\lambda(r, s) - \text{OT}(r, s) \leq \text{EROT}^\lambda(r, s) - \text{OT}(r, s) \leq \lambda M(\pi^0).
\]

Most notably, if one of the probability measures has finite entropy, i.e.,

\[
H(r, s) := \min \left( \sum_{x \in \mathcal{X}} r_x \log \frac{1}{r_x}, \sum_{y \in \mathcal{Y}} s_y \log \frac{1}{s_y} \right) < \infty,
\]

we obtain by the bound \( 0 \leq M(\pi) \leq H(r, s) \) for any \( \pi \in \Pi(r, s) \) (Cover & Thomas, 1991, Theorem 2.4.1) that

\[
|S^\lambda(r, s) - \text{OT}(r, s)| \leq |\text{EROT}^\lambda(r, s) - \text{OT}(r, s)| \leq \lambda H(r, s) = O(\lambda).
\]

Note that if a probability measure fulfills the Borisov-Dudley-Durst condition, then it necessarily has finite entropy.

**Remark 4.7.** For finitely supported probability measures it was shown by Cominetti & San Martín (1994) that \( |S^\lambda(r, s) - \text{OT}(r, s)| = o(\exp(-\kappa/\lambda)) \) for some constant \( \kappa > 0 \) depending on \( r \) and \( s \) as \( \lambda \) tends to zero. However, as noted by Weed (2018) the upper bound \( o(\exp(-\kappa/\lambda)) \) appears to fail in general for countably supported probability measures where instead the rate \( |S^\lambda(r, s) - \text{OT}(r, s)| = O(\lambda) \) seems to be tight.

For a sequence of i.i.d. samples \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} r \) and its associated empirical measure \( \hat{r}_n \) we obtain by (Antos & Kontoyiannis, 2001, Corollary 1) that \( \lim_{n \to \infty} H(\hat{r}_n, s) = H(r, s) \), hence \( \limsup_{n} H(\hat{r}_n, s) < \infty \). Performing a decomposition gives

\[
\sqrt{n} \left( \text{EROT}^\lambda(n)(\hat{r}_n, s) - \text{EROT}^\lambda(n)(r, s) \right) + \sqrt{n} \left( \text{OT}(\hat{r}_n, s) - \text{OT}(r, s) \right) + \sqrt{n} \left( \text{EROT}^\lambda(n)(\hat{r}_n, s) - \text{EROT}^\lambda(n)(r, s) \right).
\]
For $\lambda(n) = o(1/\sqrt{n})$ the first and second term converge to zero whereas the third term converges weakly as described by Tameling et al. (2019). In our notation the weak limit of the empirical non-regularized OT value as $n$ tends to infinity is characterized by

$$
\sqrt{n}(OT(\hat{r}_n, s) - OT(r, s)) \xrightarrow{D} \max_{\alpha^* \in S^*} \langle \alpha^*, G_r \rangle,
$$

(4.3)

where $S^*$ denotes the set of dual optimizers of $(OT)$ and $G_r$ represents the Gaussian process with covariance $\Sigma(r)$. These considerations are valid if $r \in \ell^1_{\mathcal{C}_r}(\mathcal{X}), s \in \ell^1_{\mathcal{C}_s}(\mathcal{Y})$, and $r$ additionally satisfies an associated weighted Borisov-Dudley-Durst condition (1.7). Both limit laws, for non-regularized OT value and its entropy regularized counterpart, are essentially governed by the respective set of dual optimizers. However, unlike the entropy variant where the limit law is always characterized by a Gaussian due to uniqueness of dual optimizers, the set of dual optimizers for the non-regularized OT problem may not be unique resulting in a maximum of Gaussian distributions which in general fails to be Gaussian. In case of $\lambda(n) = o(1/\sqrt{n})$ we also obtain a similar limit law for $\sqrt{n}(S^{(n)}(\hat{r}_n, s) - S^{(n)}(r, s))$ as $n$ tends to infinity.

**Remark 4.8 (Degeneracy of limit law).** As noted by Tameling et al. (2019) the limit distributions provided in (4.3) may degenerate in certain cases, namely when the set of dual optimal solutions $S^*$ for the non-regularized OT problem only contains constant elements. This occurs on a common metric space $(\mathcal{X}, d)$, i.e., $\mathcal{X} = \mathcal{Y}$ under $r = s$ with $\text{supp}(r) = \mathcal{X}$ for a cost function $c(x, y) = d^p(x, y)$ with $p > 1$ if and only if $\mathcal{X}$ has no isolated points. Hence, for $\lambda(n) = o(1/\sqrt{n})$ the limit distribution of the empirical EROT value and Sinkhorn costs may degenerate. In contrast, for fixed $\lambda > 0$ the limit law generally does not degenerate since optimal entropic dual potentials are typically non-constant (Remark 2.2) and as the Hadamard derivative of the EROT plan differs from zero (Remark 3.5).

### 4.3 Bootstrap

Our findings from Theorems 4.1 and 4.2 on the distributions of the empirical EROT value and plan are asymptotic results. In order to estimate the respective non-asymptotic distribution typically bootstrap methods are applied. On finite and countable spaces Sommerfeld & Munk (2018) and Tameling et al. (2019) showed that the non-regularized OT value is only directionally Hadamard differentiable, i.e., that the Hadamard derivative with respect to $r$ and $s$ is non-linear. As a consequence, the (naïve) $n$-out-of-$n$ bootstrap for the approximation of the distribution of the empirical non-regularized OT value fails. However, the EROT plan on countable spaces is Hadamard differentiable with a linear derivative. Therefore, it follows that the naïve $n$-out-of-$n$ bootstrap appears to be a consistent estimation method.

To make this statement precise we follow Van der Vaart & Wellner (1996). Denote the notion of convergence in outer probability by $P_{\rightarrow}$ and consider for a given Banach space $B$ the set of bounded Lipschitz functions with Lipschitz modulus at most one

$$
\text{BL}_1(B) := \left\{ g : B \rightarrow \mathbb{R} : \sup_{x \in B} |g(x)| \leq 1, |g(x_1) - g(x_2)| \leq \|x_1 - x_2\|_B \forall x_1, x_2 \in B \right\}.
$$

With this notation we will prove the consistency of the bootstrap for the EROT value and plan as an application of the functional delta method in conjunction with consistency of the bootstrap empirical process.
Theorem 4.9. Consider an empirical measure \( \hat{\rho}_n \) derived by a sample \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} r \) and denote by \( \hat{\mu}^* = \frac{1}{n} \sum_{i=1}^{n} \delta X_i^* \) the empirical bootstrap estimator for \( \hat{\mu}_n \) based on a sample \( X_1^*, \ldots, X_n^* \overset{i.i.d.}{\sim} r_n \). Under the same setting as in assertion (i) of Theorem 4.1 the (naive) bootstrap is consistent for the EROT value, i.e., as \( n \) tends to infinity it holds that

\[
\sup_{h \in \text{BL}(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n}(\text{ERO}\lambda(\hat{\mu}^*_n, s) - \text{ERO}\lambda(\hat{\mu}_n, s)) \right) \right]_{X_1, \ldots, X_n} \right| - \mathbb{E} \left[ h \left( \sqrt{n}(\text{ERO}\lambda(\hat{\mu}_n, s) - \text{ERO}\lambda(r, s)) \right) \right] \xrightarrow{p} 0.
\]

Likewise, under the same assumptions as for assertion (i) of Theorem 4.2 the EROT plan is also consistent for the (naive) bootstrap, i.e., as \( n \) tends to infinity it holds that

\[
\sup_{h \in \text{BL}(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n}(\pi\lambda(\hat{\mu}^*_n, s) - \pi\lambda(\hat{\mu}_n, s)) \right) \right]_{X_1, \ldots, X_n} \right| - \mathbb{E} \left[ h \left( \sqrt{n}(\pi\lambda(\hat{\mu}_n, s) - \pi\lambda(r, s)) \right) \right] \xrightarrow{p} 0.
\]

Proof. For a positive function \( f: \mathcal{X} \to (0, \infty) \) we prove that if \( \sum_{x \in \mathcal{X}} f(x) \sqrt{\pi_x} < \infty \) then

\[
\sup_{h \in \text{BL}(\ell_1'(\mathcal{X}))} \left| \mathbb{E} \left[ h \left( \sqrt{n}(\hat{\mu}^*_n - \hat{\mu}_n) \right) \right]_{X_1, \ldots, X_n} \right| - \mathbb{E} \left[ h \left( \sqrt{n}(\hat{\mu}_n - r) \right) \right] \xrightarrow{p} 0
\]

as \( n \) tends towards infinity. The assertion then follows by tangential Hadamard differentiability of the EROT value and plan (Theorems 3.2 and 3.4) in conjunction with the functional delta method for the bootstrap (Van der Vaart & Wellner, 1996, Theorem 3.9.11). For the function class \( \mathcal{H}_f := \{ f \mathds{1}_{K}: K \subset \mathcal{X} \} \) we define the Banach space \( \ell^\infty(\mathcal{H}_f) \) as the space of uniformly bounded real functions on \( \mathcal{H}_f \) equipped with supremum norm \( \| a \|_{\ell^\infty(\mathcal{H}_f)} := \sup_{g \in \mathcal{H}_f} | a(g) | \). Then it holds by (Van der Vaart & Wellner, 1996, Theorem 3.6.1) for \( n \to \infty \) that

\[
\sup_{h \in \text{BL}(\ell^\infty(\mathcal{H}_f))} \left| \mathbb{E} \left[ h \left( \sqrt{n}(\hat{\mu}^*_n - \hat{\mu}_n) \right) \right]_{X_1, \ldots, X_n} \right| - \mathbb{E} \left[ h \left( \sqrt{n}(\hat{\mu}_n - r) \right) \right] \xrightarrow{p} 0
\]

if \( \mathcal{H}_f \) is \( r \)-Donsker. Indeed, by (Yukich, 1986, Theorem 6) the function class \( \mathcal{H}_f \) is \( r \)-Donsker if and only if \( \sum_{x \in \mathcal{X}} f(x) \sqrt{\pi_x} < \infty \). As \( \ell_1'(\mathcal{X}) \) can be continuously embedded into \( \ell^\infty(\mathcal{H}_f) \) and since any bounded Lipschitz function on \( \ell_1'(\mathcal{X}) \) whose modulus is bounded by one can be extended to a bounded Lipschitz function on \( \ell^\infty(\mathcal{H}_f) \) (McShane, 1934, Theorem 1) we conclude that the condition \( \sum_{x \in \mathcal{X}} f(x) \sqrt{\pi_x} < \infty \) implies (4.4).

5 Proofs for Sensitivity Analysis

We recall that the cost function \( c \) is dominated by functions \( c_X^*, c_X^\gamma, c_Y^*, c_Y^\gamma \) as in (1.8). Additionally, let us recall the functions \( C_X = 1 + |c_X^\gamma| + |c_X^\gamma|, c_X^\delta := \exp \left( \frac{\delta}{c_X^\gamma} - c_X^\gamma \right), \) and \( \Phi_X^\delta := C_X \phi_X^\delta \) for given parameters \( \delta \in \mathbb{R} \) and \( \lambda > 0 \). Since \( c_X^\gamma \geq c_Y^\gamma \) as well as \( C_X \geq 1 \) it holds for any \( \delta \geq 0 \) that \( 1 \leq \phi_X^\delta \leq \Phi_X^\delta \). Further, given another real number \( \delta' \in \mathbb{R} \) it holds that \( \phi_X^\delta \Phi_X^\delta' = \Phi_X^\delta' \). The functions \( \phi_Y^\gamma \) and \( \Phi_Y^\gamma \) are defined analogously and feature similar properties.
5.1 Sensitivity of EROT Value

Proof of Theorem 5.2. The claim on the contingent cone is a simple consequence of (Aubin & Frankowska, 1990, Proposition 4.2.1) in conjunction with \( r \) and \( s \) having full support. For given probability measures \( \tilde{r} \in (\mathcal{P}(\mathcal{X}) \cap \ell^1_{\phi_X}(\mathcal{X})) \) and \( \tilde{s} \in (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{\phi_Y}(\mathcal{Y})) \) denote the objective of (DEROT) by

\[
\mathbb{D}^{\tilde{r}, \tilde{s}}(\ell^0_{\phi_X}(\mathcal{X}) \times \ell^0_{\phi_Y}(\mathcal{Y}) \to [-\infty, \infty], \quad (\alpha, \beta) \mapsto (\alpha, \tilde{r}) + (\beta, \tilde{s}) - \lambda \left( \sum_{x \in \mathcal{X}} \exp \left( \frac{\alpha_x + \beta_y - c(x,y)}{\lambda} \right) \tilde{r}_x \tilde{s}_y - \tilde{r}_x \tilde{s}_y \right). \]

By definition of Hadamard differentiability, consider \((t_n)_{n \in \mathbb{N}} \) with \( t_n \to 0 \) and a converging sequence \((h_{\alpha_n}^X, h_{\beta_n}^Y)_{n \in \mathbb{N}} \in \ell^1_{\phi_X}(\mathcal{X}) \times \ell^1_{\phi_Y}(\mathcal{Y}) \) with limit \((h^X, h^Y)\) such that \((r + t_nh_{\alpha_n}^X, s + t_nh_{\beta_n}^Y) \in (\mathcal{P}(\mathcal{X}) \times \ell^1_{\phi_X}(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \times \ell^1_{\phi_Y}(\mathcal{Y})) \) for all \( n \in \mathbb{N}. \) Denote by \((\alpha^\lambda_n, \beta^\lambda_n)\) and \((\alpha^\lambda, \beta^\lambda)\) the optimal entropic dual potentials for \((r + t_nh_{\alpha_n}^X, s + t_nh_{\beta_n}^Y)\) and \((r, s)\), respectively, as in Proposition 2.4. Notably, we consider those potentials that are equal to zero outside of the support of the underlying probability measure. By definition of these optimal potentials we see that

\[
\begin{align*}
\mathbb{D}^{r+tnh_n^X,sh_nh_n^Y}(\alpha^\lambda_n, \beta^\lambda_n) &- \mathbb{D}^{r,s}(\alpha^\lambda, \beta^\lambda) \\
\leq \mathbb{D}^{r+tnh_n^X,sh_nh_n^Y}(\alpha^\lambda_n, \beta^\lambda_n) &- \mathbb{D}^{r,s}(\alpha^\lambda, \beta^\lambda) \\
\leq \mathbb{D}^{r+tnh_n^X,sh_nh_n^Y}(\alpha^\lambda_n, \beta^\lambda_n) &- \mathbb{D}^{r,s}(\alpha^\lambda, \beta^\lambda). 
\end{align*}
\]

The term from the second line is equal to \( \text{EROT}^\lambda(r + t_nh_n^X, s + t_nh_n^Y) - \text{EROT}^\lambda(r, s). \) Hence, it follows that

\[
\left| \frac{1}{t_n} \left( \text{EROT}^\lambda(r + t_nh_n^X, s + t_nh_n^Y) - \text{EROT}^\lambda(r, s) - (\alpha^\lambda, h^X) - (\beta^\lambda, h^Y) \right) \right| \\
\leq \frac{1}{t_n} \left| \left( \mathbb{D}^{r+tnh_n^X,sh_nh_n^Y}(\alpha^\lambda_n, \beta^\lambda_n) - \mathbb{D}^{r,s}(\alpha^\lambda, \beta^\lambda) \right) - (\alpha^\lambda_n, h^X_n) - (\beta^\lambda_n, h^Y_n) \right| \\
+ \frac{1}{t_n} \left| \left( \mathbb{D}^{r+tnh_n^X,sh_nh_n^Y}(\alpha^\lambda_n, \beta^\lambda_n) - \mathbb{D}^{r,s}(\alpha^\lambda, \beta^\lambda) \right) - (\alpha^\lambda, h^X) - (\beta^\lambda, h^Y) \right|. \quad (5.1)
\]

Once we show that the terms in (5.1) and (5.2) converge to zero as \( n \) tends to infinity our proof is finished. Denoting the EROT plan for \((r, s)\) by \( \pi^\lambda \) we obtain using the relation between primal and dual optimizers (Proposition 2.1) that

\[
\begin{align*}
\frac{1}{t_n} \left( \mathbb{D}^{r+tnh_n^X,sh_nh_n^Y}(\alpha^\lambda_n, \beta^\lambda_n) - \mathbb{D}^{r,s}(\alpha^\lambda, \beta^\lambda) \right) - (\alpha^\lambda_n, h^X_n) - (\beta^\lambda_n, h^Y_n) \\
= (\alpha^\lambda_n, h^X_n - h^X) + (\beta^\lambda_n, h^Y_n - h^Y) - \lambda t_n \sum_{x \in \mathcal{X}} \pi^\lambda_{xy} \left( (r_x + t_nh_{\alpha_n}^X)(s_y + t_nh_{\beta_n}^Y) - r_x s_y \right) \\
= (\alpha^\lambda_n, h^X_n - h^X) + (\beta^\lambda_n, h^Y_n - h^Y) - \lambda t_n \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \exp \left( \frac{\alpha_x + \beta_y - c(x,y)}{\lambda} \right) h_{\alpha_n}^X h_{\beta_n}^Y. \quad (5.3)
\end{align*}
\]

Herein, we exploit the fact that \( \sum_{x \in \mathcal{X}} h_{\alpha_n}^X = \sum_{y \in \mathcal{Y}} h_{\beta_n}^Y = 0 \) for all \( n \in \mathbb{N}. \) For the first term of (5.3) it follows by Hölder’s inequality for \( n \to \infty \) that

\[
|\langle \alpha^\lambda_n, h^X_n - h^X \rangle| \leq \sum_{x \in \mathcal{X}} |\alpha^\lambda_n| |h_{\alpha_n}^X - h_x^X| = \sum_{x \in \mathcal{X}} \frac{|\alpha^\lambda_n|}{C^X(x)} |h_{\alpha_n}^X - h_x^X| C^X(x)
\]
where we explicitly made use of the weighted $\ell^1$-convergence of $h_n$ towards $h$ and our bounds for $\alpha^\lambda$ from Proposition 2.3 to ensure that $\|\alpha^\lambda\|_{c(x)} < \infty$. Likewise, it follows for $n \to \infty$ that $(\beta^\lambda, h^\lambda_n - h^\lambda) \to 0$. For the third term of (5.3) we note by our bounds from Proposition 2.3 and convergence of $(h^\lambda_n, h^\mu_n)$ in $\ell^1(\phi^\lambda_n) \times \ell^1(\phi^\mu_n)$ that

$$\sup_{n \in \mathbb{N}} \left| \sum_{x \in X} \sum_{y \in Y} \exp \left( \frac{\alpha^\lambda_n + \beta^\lambda_n - c(x, y)}{\lambda} \right) h^\lambda_n(x) h^\mu_n(y) \right| \leq \exp \left( \frac{(c^\lambda_n, r) + (c^\mu_n, s)}{\lambda} \right) \left| \sum_{x \in X} \phi^\lambda_n(x) \phi^\mu_n(y) h^\lambda_n(x) h^\mu_n(y) \right| < \infty,$$  

(5.5)

which implies by $t_n \to 0$ that all terms of (5.3) tend towards zero. For the terms from (5.2) we obtain

$$\frac{1}{t_n} \left( \mathbb{D}^r + t_n h^\lambda_n \right) = (\alpha^\lambda, h^\lambda) + (\alpha^\lambda, h^\lambda_n - h^\lambda) + (\beta^\lambda, h^\lambda_n - h^\lambda) + (\beta^\lambda, h^\lambda_n - h^\lambda) + \frac{\lambda}{t_n} \sum_{x \in X} \sum_{y \in Y} \exp \left( \frac{\alpha^\lambda_n + \beta^\lambda_n - c(x, y)}{\lambda} \right) \left[ (r + t_n h^\lambda_n)(s + t_n h^\lambda_n) - r_s \right].$$  

(5.6)

Recall that we already showed that the second and the last term in (5.6) converge to zero. For the first and third term of (5.6) we use point-wise convergence of the optimal entropic dual potentials (Proposition 2.4) in conjunction with our bounds from Proposition 2.3 to apply Lebesgue’s dominated convergence theorem. At last, it remains to show that the series in (5.7) converges to zero. To simplify notation we write $\tilde{X}_n$ and $\tilde{Y}_n$ instead of $\text{supp}(r + t_n h^\lambda_n)$ and $\text{supp}(s + t_n h^\lambda_n)$, respectively. Exploiting the relation between primal and dual optimizers (Proposition 2.1) we see by a direct computation as in (5.3) and due to our convention of $\alpha^\lambda_n, \beta^\lambda_n$ vanishing on $(\tilde{X}_n)^c, (\tilde{Y}_n)^c$ for the term in (5.7) that

$$\frac{\lambda}{t_n} \sum_{x \in X} \sum_{y \in Y} \exp \left( \frac{\alpha^\lambda_n + \beta^\lambda_n - c(x, y)}{\lambda} \right) \left[ (r + t_n h^\lambda_n)(s + t_n h^\lambda_n) - r_s \right]$$

(5.8)

$$= \lambda \left[ \sum_{y \in \tilde{Y}_n} h^\mu_n(y) + \sum_{x \in \tilde{X}_n} h^\lambda_n(x) + t_n \sum_{x \in \tilde{X}_n} \sum_{y \in \tilde{Y}_n} \exp \left( \frac{\alpha^\lambda_n + \beta^\lambda_n - c(x, y)}{\lambda} \right) h^\lambda_n(x) h^\mu_n(y) \right]$$

$$+ \lambda \sum_{x \in \tilde{X}_n} \sum_{y \in \tilde{Y}_n} \exp \left( \frac{\alpha^\lambda_n - c(x, y)}{\lambda} \right) h^\lambda_n(x) h^\mu_n(y) + \lambda \sum_{x \in \tilde{X}_n} \sum_{y \in \tilde{Y}_n} \exp \left( \frac{\beta^\lambda_n - c(x, y)}{\lambda} \right) h^\lambda_n(x) h^\mu_n(y)$$

$$- t_n \lambda \sum_{x \in \tilde{X}_n} \sum_{y \in \tilde{Y}_n} \exp \left( \frac{-c(x, y)}{\lambda} \right) h^\lambda_n(x) h^\mu_n(y).$$

(5.9)

(5.10)

Herein, we used for the terms in (5.9) and (5.10) that $r + t_n h^\lambda_n = 0$ holds if and only if $r = -t_n h^\lambda_n$ and likewise for $s + t_n h^\mu_n = 0$. In the following we show that all these terms
tend to zero. We start with the terms from (5.8). By convergence of $h_n^y$ towards the element $h^y$, which satisfies $\sum_{y \in \mathcal{Y}} h_n^y = 0$, and since $\mathcal{Y}_n = \text{supp}(r + t_n h_n^X)$ converges in a set-theoretical sense to $\mathcal{Y} = \text{supp}(r)$ it follows that the first sum in (5.8) tends to zero for $n \to \infty$. Similarly, the second sum also tends to zero for increasing $n$. For the third term of (5.8) we recall by our bounds on optimal entropic dual potentials and the type of convergence for $(h_n^X, h_n^Y)$ that the sum stays uniformly bounded over all $n \in \mathbb{N}$, whereas $t_n \searrow 0$. Hence, all terms from (5.8) converge to zero. For the first term in (5.9) the bounds for $\alpha_n^\lambda$ (Proposition 2.3) imply

$$
\lambda \sum_{x \in \mathcal{X}_n \atop y \not\in \mathcal{Y}_n} \exp \left( \frac{\alpha_n^\lambda x - c(x, y)}{\lambda} \right) r_x h_n^y \leq \lambda \sum_{x \in \mathcal{X}_n \atop y \not\in \mathcal{Y}_n} \exp \left( \frac{\alpha_n^\lambda x + c_y^\lambda(y) - c(x, y)}{\lambda} \right) |r_x h_n^y|
$$

$$
\leq \lambda K \sum_{x \in \mathcal{X}_n} \phi_\lambda(x) |r_x| \sum_{y \not\in \mathcal{Y}_n} \phi_y(y) |h_n^y| \leq \lambda K \sum_{x \in \mathcal{X}_n} \phi_\lambda(x) |r_x| \sum_{y \not\in \mathcal{Y}_n} \phi_y(y) |h_n^y|
$$

for some constant $K > 0$. By the type of convergence of $h_n^\lambda$ and since $\mathcal{Y}_n \to \mathcal{Y}$ we obtain that the quantity $\sum_{y \not\in \mathcal{Y}_n} \phi_y(y) |h_n^y|$ tends to zero for $n \to \infty$. Likewise, the second term from (5.9) also converges to zero. Lastly, by non-negativity of the cost function we obtain for the term in (5.10) that

$$
t_n^\lambda \sum_{x \not\in \mathcal{X}_n \atop y \not\in \mathcal{Y}_n} \exp \left( \frac{-c(x, y)}{\lambda} \right) h_n^X h_n^Y \leq t_n^\lambda \sum_{x \not\in \mathcal{X}_n} |h_{n,x}^X| \sum_{y \not\in \mathcal{Y}_n} |h_{n,y}^Y|,
$$

which also converges to zero for $n \to \infty$ and finishes the proof. 

\[\square\]

**Remark 5.1.** For Hadamard differentiability of $\text{EROT}^\lambda$ at $(r, s)$ we need to show for $n \to \infty$ that

$$
\left| \frac{1}{t_n} \left( \text{EROT}^\lambda(r + t_n h_n^X, s + t_n h_n^Y) - \text{EROT}^\lambda(r, s) \right) - (\alpha^\lambda, h^X) - (\beta^\lambda, h^Y) \right| \to 0.
$$

Our proof technique relies on the weighted $\ell^1$-convergence of $(h_n^X, h_n^Y)$ towards $(h^X, h^Y)$ in combination with Hölder’s inequality (Equation (5.4) and (5.5)). In particular, the weighted $\ell^1$-norm appears to be a necessary condition for Hadamard differentiability of the $\text{EROT}$ value functional on countable spaces.

### 5.2 Sensitivity of EROT Plan

The proof of Theorem 3.4 for the Hadamard differentiability of the EROT plan requires three key results: well-definedness and boundedness of the proposed derivative, a local Lipschitz-continuity property for the mapping $\pi^\lambda$, and convergence of the difference quotient of $\pi^\lambda$ for finitely supported perturbations towards the proposed derivative. For the sake of readability these three statements are proven separately in Propositions 5.2, 5.3, and 5.4.

Furthermore, the proof employs various approximation results. For an element $h^X \in \ell^1(\mathcal{X})$ we define its finite support approximation of order $l \geq 2$, denoted by $\tilde{h}_l^X$, for $x \in \mathcal{X} = \{x_1, x_2, \ldots\}$ as

$$
\tilde{h}_l^X := \begin{cases} 
  h_{x_1}^X + \sum_{i=0}^{l-1} h_{x_{i+1}}^X & \text{if } x = x_1, \\
  h_x^X & \text{if } x \in \{x_2, \ldots, x_l\}, \\
  0 & \text{else}.
\end{cases}
$$
Similarly, we define for \( h^\gamma \in \ell^1(\mathcal{Y}) \) its finite support approximation by \( \hat{h}_n^\gamma \). The relevant properties of this type of approximation are shown in Appendix C (Lemmas C.1, C.2, and C.3) and will play an important role in the following.

**Proof of Theorem 3.4.** According to the definition of Hadamard differentiability, consider a sequence \((t_n)_{n \in \mathbb{N}}\) such that \( t_n \to 0 \) and \((h^X_n, h^\gamma_n) \in \ell^1_{C^X}(\mathcal{X}) \times \ell^1_{\mathcal{Y}}(\mathcal{Y})\) converging to \((h^X, h^\gamma)\) such that it holds for each \( n \in \mathbb{N} \)

\[
(r + t_n h^X_n, s + t_n h^\gamma_n) \in \left( \mathcal{P}(\mathcal{X}) \cap \ell^1_{C^X}(\mathcal{X}) \right) \times \left( \mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{Y}}(\mathcal{Y}) \right).
\]

The assertion on the contingent cone follows by (Aubin & Frankowska, 1990, Proposition 4.2.1) in conjunction with \( r \) and \( s \) having full support. The quantity of interest is

\[
\left\| \pi^\lambda(r + t_n h^X_n, s + t_n h^\gamma_n) - \pi^\lambda(r, s) - \mathcal{D}^H_{(r, s)} \pi^\lambda(h^X, h^\gamma) \right\|_{\ell^1_{C^X} \oplus \ell^1_{\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})}
\]

for which we need to prove that it converges to zero as \( n \) tends to infinity. Let \( \varepsilon > 0 \), then there exists by Lemma C.1 and by boundedness of \( \mathcal{D}^H_{(r, s)} \pi^\lambda \) (Proposition 5.2) an integer \( l \in \mathbb{N} \) such that it follows for the finite support approximations \( \hat{h}_n^X \) of \( h^X \) and \( \hat{h}_n^\gamma \) of \( h^\gamma \), respectively, that

\[
\left\| h^X - \hat{h}_n^X \right\|_{\ell^1_{C^X}(\mathcal{X})} + \left\| h^\gamma - \hat{h}_n^\gamma \right\|_{\ell^1_{\mathcal{Y}}(\mathcal{Y})} < \varepsilon 4^{-1} \left\| \mathcal{D}^H_{(r, s)} \pi^\lambda \right\|_{OP}.
\]

Further, denote by \( \rho_0 > 0 \) the radius from Proposition 5.3 such that \( \pi^\lambda \) is Lipschitz with modulus \( A > 0 \) on the set

\[
B_{\rho_0}(r, s) := \left\{ (\tilde{r}, \tilde{s}) \in \left( \mathcal{P}(\mathcal{X}) \cap \ell^1_{C^X}(\mathcal{X}) \right) \times \left( \mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{Y}}(\mathcal{Y}) \right) : \right. \]

\[
\left. \| r - \tilde{r} \|_{\ell^1_{C^X}(\mathcal{X})} + \| s - \tilde{s} \|_{\ell^1_{\mathcal{Y}}(\mathcal{Y})} \leq \rho_0 \right\}.
\]

Moreover, by Lemma C.3 there exists \( N_1 \in \mathbb{N} \) such that it follows for all \( n \geq N_1 \) that

\[
(r + t_n \hat{h}_n^X, r + t_n \hat{h}_n^\gamma) \in \mathcal{P}(\mathcal{X}) \text{ with } \text{supp}(r + t_n \hat{h}_n^X) = \text{supp}(r + t_n \hat{h}_n^X) = \mathcal{X}, \quad \text{and likewise}
\]

\[
s + t_n \hat{h}_n^\gamma \in \mathcal{P}(\mathcal{Y}) \text{ with } \text{supp}(s + t_n \hat{h}_n^\gamma) = \text{supp}(s + t_n \hat{h}_n^\gamma) = \mathcal{Y}.
\]

Additionally, Lemma C.2 asserts existence of \( N_2 \in \mathbb{N} \) such that it holds for all \( n \geq N_2 \) that

\[
(r + t_n h^X_n, s + t_n h^\gamma_n), (r + t_n \hat{h}^X_n, s + t_n \hat{h}^\gamma_n), (r + t_n \hat{h}^X_n, s + t_n \hat{h}^\gamma_n) \in B_{\rho_0}(r, s).
\]

Using Lemma C.1 there also exists \( N_3 \in \mathbb{N} \) such that it follows for all \( n \geq N_3 \)

\[
\left\| h^X_n - \hat{h}^X_n \right\|_{\ell^1_{C^X}(\mathcal{X})} < \varepsilon 4^{-1} \quad \text{and}
\]

\[
\left\| \hat{h}^X_n - \hat{h}^X \right\|_{\ell^1_{C^X}(\mathcal{X})} < \varepsilon 4^{-1}.
\]

Finally, by Proposition 5.4 there exists \( N_4 \in \mathbb{N} \) such that it follows for \( n \geq N_4 \)

\[
\left\| \pi^\lambda(r + t_n \hat{h}_n^X, s + t_n \hat{h}_n^\gamma) - \pi^\lambda(r, s) - \mathcal{D}^H_{(r, s)} \pi^\lambda(\hat{h}_n^X, \hat{h}_n^\gamma) \right\|_{\ell^1_{C^X} \oplus \ell^1_{\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})} < \varepsilon 4^{-1}.
\]
Summarizing, for all \( n \geq \max\{N_1, N_2, N_3, N_4\} \) we obtain that

\[
\left\| \pi^\lambda(r + t_n h^{X,s}_n, s + t_n h^{Y}_n) - \pi^\lambda(r, s) - \mathcal{D}^H_{(r,s)} \pi^\lambda(h^{X,Y}) \right\|_{\ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(X \times Y)} \\
\leq \left\| \frac{\pi^\lambda(r + t_n h^{X,s}_n, s + t_n h^{Y}_n) - \pi^\lambda(r, s) - \mathcal{D}^H_{(r,s)} \pi^\lambda(h^{X,Y})}{t_n} \right\|_{\ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(X \times Y)} + \frac{\pi^\lambda(r + t_n \tilde{h}^{X,s}_n, s + t_n \tilde{h}^{Y}_n) - \pi^\lambda(r, s) - \mathcal{D}^H_{(r,s)} \pi^\lambda(h^{X,Y})}{t_n} \right\|_{\ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(X \times Y)} + \frac{\pi^\lambda(r + t_n \tilde{h}^{X,Y}_n) - \pi^\lambda(r, s) - \mathcal{D}^H_{(r,s)} \pi^\lambda(h^{X,Y})}{t_n} \right\|_{\ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(X \times Y)} + \frac{\mathcal{D}^H_{(r,s)} \pi^\lambda(h^{X,Y}) - \mathcal{D}^H_{(r,s)} \pi^\lambda(h^{X,Y})}{t_n} \right\|_{\ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(X \times Y)}
\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,
\]

which proves the claim. \(\Box\)

We continue with our assertions on the well-definedness and boundedness of the proposed derivative, the local Lipschitz property for the EROT plan as well as our result concerning the convergence of the difference quotient for finitely supported perturbations. For the sake of readability, we first state these results and prove them afterwards. Notably, for all of these results we assume that \(\|c_X - c_Y\|_{L^\infty(\mathcal{X}, \mathcal{Y})} < \infty\).

**Proposition 5.2.** Let \( r \in \ell^1_{\mathcal{C}_X}(\mathcal{X}), s \in \ell^1_{\mathcal{C}_Y}(\mathcal{Y}) \) be two probability measures with full support, i.e., \(\text{supp}(r) = \mathcal{X}\) and \(\text{supp}(s) = \mathcal{Y}\), then the linear mapping given by

\[
-\left[\mathcal{D}_{\pi,\alpha,\beta, \varphi} \circ \mathcal{D}_{\rho,\sigma, (\varphi(r,s_r), \varphi(s,s_s))} \mathcal{F}\right]^{-1}\mathcal{F} : \ell^1_{\mathcal{C}_X}(\mathcal{X}) \times \ell^1_{\mathcal{C}_Y}(\mathcal{Y}) \to \ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y}) \times \ell^\infty_{\mathcal{C}_X}(\mathcal{X}) \times \ell^\infty_{\mathcal{C}_Y}(\mathcal{Y})
\]

is well-defined and bounded.

**Proposition 5.3.** Let \( (r, s) \in \ell^1_{\mathcal{C}_X}(\mathcal{X}) \times \ell^1_{\mathcal{C}_Y}(\mathcal{Y}) \) be probability measures with full support. Denote for \( \rho > 0 \) the set

\[
B_\rho(r, s) := \left\{ (\tilde{r}, \tilde{s}) \in \left( \mathcal{P}(\mathcal{X}) \cap \ell^1_{\mathcal{C}_X}(\mathcal{X}) \right) \times \left( \mathcal{P}(\mathcal{Y}) \cap \ell^1_{\mathcal{C}_Y}(\mathcal{Y}) \right) : \left\| r - \tilde{r} \right\|_{\ell^1_{\mathcal{C}_X}(\mathcal{X})} + \left\| s - \tilde{s} \right\|_{\ell^1_{\mathcal{C}_Y}(\mathcal{Y})} \leq \rho \right\}.
\]

Then there exist \( \rho_0, N, N' > 0 \) such that for any \( (\tilde{r}, \tilde{s}), (r', s') \in B_{\rho_0}(r, s) \) with \( \text{supp}(\tilde{r}') \subseteq \text{supp}(\tilde{r}) \) and \( \text{supp}(\tilde{s}') \subseteq \text{supp}(\tilde{s}) \) it follows that

\[
\left\| \pi^\lambda(\tilde{r}, \tilde{s}) - \pi^\lambda(r', s') \right\|_{\ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(X \times Y)} \leq \Lambda \left\| (\tilde{r}, \tilde{s}) - (r', s') \right\|_{\ell^1_{\mathcal{C}_X}(X) \times \ell^1_{\mathcal{C}_Y}(Y)}, \tag{5.11}
\]

\[
\left\| (\pi^\lambda(\tilde{r}, \tilde{s}) - \alpha^\lambda(\tilde{r}', \tilde{s}')) 1_{\text{supp}(\tilde{r})} \right\|_{L^\infty(\mathcal{X})} \leq \Lambda' \left\| (\tilde{r}, \tilde{s}) - (r', s') \right\|_{\ell^1_{\mathcal{C}_X}(X) \times \ell^1_{\mathcal{C}_Y}(Y)}, \tag{5.12}
\]

26
\[
\left\| (\beta^\lambda(\hat{r}, \hat{s}) - \beta^\lambda(\hat{r}', \hat{s}')) \mathbb{1}_{\text{supp}(\hat{s}')} \right\|_{L^\infty(\mathcal{Y})} \leq A' \left\| (\hat{r}, \hat{s}) - (\hat{r}', \hat{s}') \right\|_{\ell_{\mathcal{X}}^1(\mathcal{X}) \times \ell_{\mathcal{Y}}^1(\mathcal{Y})},
\]  

(5.13)

where \((\alpha^\lambda, \beta^\lambda)\) represent optimal entropic dual potentials as in Proposition 2.4, i.e., \(\beta_{y_1}^\lambda = 0\).

**Proposition 5.4.** Let \((r, s) \in \ell_{\mathcal{X}}^1 (\mathcal{X}) \times \ell_{\mathcal{Y}}^1 (\mathcal{Y})\) be probability measures with full support. Given \(l \in \mathbb{N}\) consider \(N \in \mathbb{N}\) as in Lemma \(\mathcal{C}.3\) such that for all \(n \geq N\) holds

\[
(r + t_n \hat{h}_1^X, s + t_n \hat{h}_1^Y) \in (\mathcal{P}(\mathcal{X}) \times \ell_{\mathcal{X}}^1(\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell_{\mathcal{Y}}^1(\mathcal{Y}))
\]

with \(\text{supp}(r + t_n \hat{h}_1^X) = \mathcal{X}\) and \(\text{supp}(s + t_n \hat{h}_1^Y) = \mathcal{Y}\). Then it follows as \(n\) tends to infinity that

\[
\frac{\left\| \pi^\lambda (r + t_n \hat{h}_1^X, s + t_n \hat{h}_1^Y) - \pi^\lambda (r, s) \right\|_{t_n}}{t_n} - \mathcal{D}_{r,s}^H \pi^\lambda (\hat{h}_1^X, \hat{h}_1^Y) \to 0.
\]

Proof of Proposition 5.2. Given a pair \((h^X, h^Y) \in \ell_{\mathcal{X}}^1 (\mathcal{X}) \times \ell_{\mathcal{Y}}^1 (\mathcal{Y})\) we need to show that there exists a unique element \((h^X, h^\infty)\) \(\in \ell_{\mathcal{X}}^1(\mathcal{X}) \times \ell_{\mathcal{Y}}^{\infty}(\mathcal{Y})\) such that

\[
\mathcal{D}_{\pi,\alpha,\beta,\sigma}(\theta(r,s),r,s)\mathcal{F}(h^X, h^\infty, h^Y) = \mathcal{D}_{r,s}(\theta(r,s),r,s)\mathcal{F}(h^X, h^Y).
\]

(5.14)

Denoting the EROT plan for \(r, s\) as \(\pi^\lambda\) and by the relation between optimizers of (EROT) and (DEROT) (Proposition 2.1) Equation (5.14) can be rewritten as

\[
\begin{pmatrix}
\dot{h}^X - \frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} \

\end{pmatrix} = \begin{pmatrix}
-\frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} \
-\frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} 
\end{pmatrix} \circ \begin{pmatrix}
r \circ h^Y + h^X \otimes s \

-h^X
\end{pmatrix}.
\]

(5.15)

In order to solve this system of countably many equations we set

\[
\dot{h}^X = \frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} \circ \begin{pmatrix}
-\frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} \
-\frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda}
\end{pmatrix} \circ \begin{pmatrix}
r \circ h^Y + h^X \otimes s \

-h^X
\end{pmatrix}.
\]

(5.16)

which reduces (5.15) to

\[
\dot{h}^X = \frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} \circ \begin{pmatrix}
-\frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda} \
-\frac{\pi^\lambda}{\pi_r^\kappa \circ \alpha^\lambda}
\end{pmatrix} \circ \begin{pmatrix}
r \circ h^Y + h^X \otimes s \

-h^X
\end{pmatrix}.
\]

(5.17)

We now have to show that a unique solution \((h^X, h^\infty)\) for (5.17) exists. For this purpose, we evaluate for \(x \in \mathcal{X}\) the corresponding component-wise equation of (5.17) and obtain

\[
\frac{\pi^\lambda}{\lambda} h^X + \sum_{y \in \mathcal{Y} \setminus \{y_1\}} \frac{\pi^\lambda}{\lambda} (h^X + h^Y) - \sum_{y \in \mathcal{Y} \setminus \{y_1\}} \left[ \frac{\pi^\lambda}{\lambda} r x h^Y + h^X s y \right] = -h^Y_x,
\]

(5.18)
\[ \frac{r_x}{\lambda} h^\infty_x + \left[ \sum_{y \in \mathcal{Y}\setminus\{y_1\}} \frac{\pi^\lambda_{xy}}{r_x} h^\infty_y \right] - \left[ \sum_{y \in \mathcal{Y}} \frac{\pi^\lambda_{xy}}{s_y} h^\infty_y \right] - h^\lambda_x \left[ \sum_{y \in \mathcal{Y}} \frac{\pi^\lambda_y}{r_x} \right] = -h^\lambda_y. \]

Note, for all \( x \in \mathcal{X} \) it holds that \( \sum_{y \in \mathcal{Y}} \frac{\pi^\lambda_{xy}}{r_x} = 1 \), this leads to

\[ h^\infty_x + \left[ \sum_{y \in \mathcal{Y}\setminus\{y_1\}} \frac{\pi^\lambda_{xy}}{r_x} h^\infty_y \right] = \lambda \left[ \sum_{y \in \mathcal{Y}} \frac{\pi^\lambda_{xy}}{r_x s_y} h^\infty_y \right]. \]

Similarly, since for any \( y \in \mathcal{Y}\setminus\{y_1\} \) holds \( \sum_{x \in \mathcal{X}} \frac{\pi^\lambda_{xy}}{s_y} = 1 \) we deduce that

\[ \left[ \sum_{x \in \mathcal{X}} \frac{\pi^\lambda_{xy}}{s_y} h^\infty_x \right] + h^\infty_y = \lambda \left[ \sum_{x \in \mathcal{X}} \frac{\pi^\lambda_{xy}}{r_x s_y} h^\infty_x \right]. \]

Summarizing, we have to solve a system of countably many linear equations. For suitable matrices of countable dimension this can be rewritten as

\[
\begin{pmatrix}
1 & 0 & \cdots & \frac{\pi^\lambda_{1y_2}}{r_{x_1}} & \frac{\pi^\lambda_{1y_3}}{r_{x_2}} & \cdots \\
0 & 1 & \cdots & \frac{\pi^\lambda_{2y_2}}{r_{x_1}} & \frac{\pi^\lambda_{2y_3}}{r_{x_2}} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\frac{\pi^\lambda_{1y_2}}{s_{y_2}} & \frac{\pi^\lambda_{2y_2}}{s_{y_2}} & \cdots & 1 & 0 & \cdots \\
\frac{\pi^\lambda_{1y_3}}{s_{y_3}} & \frac{\pi^\lambda_{2y_3}}{s_{y_3}} & \cdots & 0 & 1 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
h^\infty_{x_1} \\
h^\infty_{x_2} \\
\vdots \\
h^\infty_{x_1} h^\infty_{x_2} h^\infty_{x_3} \\
\vdots \\
\end{pmatrix}
= \lambda
\begin{pmatrix}
\text{Id}_{\mathcal{X}} \\
-1_{\mathcal{Y}\setminus\{y_1\}} \\
\vdots \\
\end{pmatrix}
\begin{pmatrix}
h^\infty_{x_1} \\
h^\infty_{x_2} \\
\vdots \\
\end{pmatrix},
\]

where we denote the operator of the l.h.s. by \( M = M(r, s, \pi^\lambda) \) and the composition of both operators on the r.h.s. with the factor \( \lambda \) by \( Q = Q(\lambda, r, s, \pi^\lambda) \). As we will show, these operators are defined on the following spaces

\[
M = M(r, s, \pi^\lambda) : \ell^\infty(\mathcal{X}) \times \ell^\infty(\mathcal{Y}\setminus\{y_1\}) \rightarrow \ell^\infty(\mathcal{X}) \times \ell^\infty(\mathcal{Y}\setminus\{y_1\}),
Q = Q(\lambda, r, s, \pi^\lambda) : \ell^1_{\mathcal{X}}(\mathcal{X}) \times \ell^1_{\mathcal{Y}\setminus\{y_1\}}(\mathcal{Y}\setminus\{y_1\}) \rightarrow \ell^1_{\mathcal{X}}(\mathcal{X}) \times \ell^1_{\mathcal{Y}\setminus\{y_1\}}(\mathcal{Y}).
\]

Herein, we equip the product of \( \ell^1 \)-spaces with sum of the norms of each subspace and for the \( \ell^\infty \)-spaces we consider the maximum of the norms of the subspaces. We now derive a bound for the operator norm of \( Q \) and afterwards show that \( M \) is invertible. The operator norm of the mapping

\[
\begin{pmatrix}
\text{Id}_{\mathcal{X}} \\
-1_{\mathcal{Y}\setminus\{y_1\}} \\
\vdots \\
\end{pmatrix}
: \ell^1_{\mathcal{X}}(\mathcal{X}) \times \ell^1_{\mathcal{Y}\setminus\{y_1\}}(\mathcal{Y}\setminus\{y_1\}) \rightarrow \ell^1_{\mathcal{X}}(\mathcal{X}) \times \ell^1_{\mathcal{Y}\setminus\{y_1\}}(\mathcal{Y})
\]

28
can be computed as follows. Denote the closed ball of radius 1 by
\[ B_1 := \{(h^x, h^y) \in \ell^1_{\phi_2}(\mathcal{X}) \times \ell^1_{\phi_2}(\mathcal{Y}); \|h^x\|_{\ell^1_{\phi_2}(\mathcal{X})} + \|h^y\|_{\ell^1_{\phi_2}(\mathcal{Y}\setminus\{y_1\})} \leq 1\}. \]

Then, the operator norm is given by
\[
\sup_{(h^x, h^y) \in B_1} \left| h^x \phi_{\mathcal{X}}(x) + h^y \phi_{\mathcal{Y}}(y) \right| = 1 + \sup_{(h^x, h^y) \in B_1} \left( \sum_{y \neq y_1} h^y \y \phi_{\mathcal{Y}}(y) \right) = 1 + \sup_{y \neq y_1} \frac{\sum_{y \neq y_1} h^y \y \phi_{\mathcal{Y}}(y)}{\sum_{y \neq y_1} h^y \y \phi_{\mathcal{Y}}(y)} \leq 1 + \phi^2_{\mathcal{Y}}(y_1).
\]

Moreover, based on our bounds for \( \pi^\lambda \) (Proposition 2.3) it follows for each \( x \in \mathcal{X} \) and \( h^y \in \ell^1_{\phi_2}(\mathcal{Y}) \) that
\[
- \sum_{y \neq y_1} \frac{\pi_{xy} h^y}{r_x s_y} \leq \exp \left( \frac{(c^*_x, r) + (c^*_y, s) + \|c^*_x - c^*_y\|_{\ell^\infty(\mathcal{X})}}{\lambda} \right) \|h^y\|_{\ell^1_{\phi_2}(\mathcal{Y})} \leq \exp \left( \frac{(c^*_x, r) + (c^*_y, s) + \|c^*_x - c^*_y\|_{\ell^\infty(\mathcal{X})}}{\lambda} \right) \|h^y\|_{\ell^1_{\phi_2}(\mathcal{Y})}.
\]

Likewise, it follows for each \( y \in \mathcal{Y}\setminus\{y_1\} \) and \( h^x \in \ell^1_{\phi_2}(\mathcal{X}) \) that
\[
\phi_{\mathcal{Y}}(y) - \frac{1}{\phi_{\mathcal{Y}}(y)} \sum_{x \in \mathcal{X}} \frac{\pi_{xy} h^x}{r_x s_y} \leq \exp \left( \frac{(c^*_x, r) + (c^*_y, s) + \|c^*_x - c^*_y\|_{\ell^\infty(\mathcal{X})}}{\lambda} \right) \|h^y\|_{\ell^1_{\phi_2}(\mathcal{X})}.
\]

Hence, we obtain for the operator norm of \( Q \) that
\[
\|Q\|_{Q^2} \leq \lambda(1 + \phi^2_{\mathcal{Y}}(y_1)) \exp \left( \frac{(c^*_x, r) + (c^*_y, s) + \|c^*_x - c^*_y\|_{\ell^\infty(\mathcal{X})}}{\lambda} \right).
\]

Next, we show that \( M \) is invertible. Note, that \( M \) can be represented by
\[
M = \begin{pmatrix} \text{Id}_{\ell^\infty(\mathcal{X})} & P \\ R & \text{Id}_{\ell^1_{\phi_2}(\mathcal{Y}\setminus\{y_1\})} \end{pmatrix}
\]

for some suitable linear operators \( P = P(r, \pi^\lambda): \ell^\infty_{\phi_2}(\mathcal{Y}\setminus\{y_1\}) \to \ell^\infty_{\phi_2}(\mathcal{X}) \) and \( R = R(s, \pi^\lambda): \ell^\infty_{\phi_2}(\mathcal{X}) \to \ell^1_{\phi_2}(\mathcal{Y}\setminus\{y_1\}) \). We prove existence of the inverse of \( M \) by applying the Neumann-series calculus for \( (\text{Id} - M) \) (Sasane, 2017, Theorem 2.9). In particular, this requires that the series \( \sum_{k=0}^{\infty} (\text{Id} - M)^k \) converges in operator norm. To this end, we perform a change to the norm \( \ell^\infty_{\phi_2}(\mathcal{Y}\setminus\{y_1\}) \) which does not change the topology. We first label \( \mathcal{Y} = \{y_1, y_2, \ldots\} \) and set \( \eta := \inf_{x \in \mathcal{X}} \frac{\pi^\lambda_{xy}}{r_x} > 0 \) which is strictly positive by the lower bounds for \( \pi^\lambda \) (Proposition 2.3) and since . Using the upper bound for \( \pi^\lambda \) (Proposition 2.3) we obtain that
\[
\phi_{\mathcal{Y}}(y) \frac{\pi^\lambda_{xy}}{r_x} \leq \exp \left( \frac{(c^*_x - c^*_y)_{\ell^\infty(\mathcal{X})} + (c^*_y, s)}{\lambda} \right) \phi_{\mathcal{Y}}(y) s_y,
\]

which shows that \( \phi_{\mathcal{Y}}(y) \frac{\pi^\lambda_{xy}}{r_x} \) is summable over \( y \in \mathcal{Y} \) as \( s \in \ell^1_{\phi_2}(\mathcal{Y}) \in \ell^1_{\phi_2}(\mathcal{Y}) \). Notably, the dominating function is independent of \( x \in \mathcal{X} \). Hence, there exists some \( N \in \mathbb{N} \) such that
\[
\sum_{i=N+1}^{\infty} (\phi_{\mathcal{Y}}(y_i) - 1) \frac{\pi^\lambda_{xy}}{r_x} \leq \frac{\eta}{2}.
\]
For such \( N \in \mathbb{N} \) it follows for all \( x \in \mathcal{X} \) that
\[
\sum_{i=2}^{\infty} \frac{\pi^\lambda_{xy}}{r_x} + \sum_{i=N+1}^{\infty} \left( \phi_y(y_i) - 1 \right) \frac{\pi^\lambda_{xy}}{r_x} \leq 1 - \frac{\pi^\lambda_{xy}}{r_x} + \frac{\eta}{2} \leq 1 - \frac{\eta}{2} \tag{5.21}
\]

To change the norm of \( \ell^\infty_{\phi_{xy}}(\mathcal{Y}\setminus \{y_1\}) \) we define the weight function
\[
\tilde{\phi}_y : \mathcal{Y} \to [1, \infty), \quad \tilde{\phi}_y(y) = \begin{cases} 1 & \text{if } y \in \{y_1, \ldots, y_N\}, \\ \phi_y(y) & \text{else}, \end{cases}
\]
where we note that \( L_\mathcal{Y}^{-1} \phi_y(y) \leq \tilde{\phi}_y(y) \leq \phi_y(y) \) for \( L_\mathcal{Y} = \max_{i=1,\ldots,N} \phi_y(y) \). Consequently, it follows that \( \| \cdot \|_{\ell^\infty_{\tilde{\phi}_y}(\mathcal{Y})} \leq \| \cdot \|_{\ell^\infty_{\phi_y}(\mathcal{Y})} \leq L_\mathcal{Y} \| \cdot \|_{\ell^\infty_{\phi_y}(\mathcal{Y})} \). Hence, we can consider the operators \( P, R \) as mappings
\[
\tilde{P} : \ell^\infty_{\tilde{\phi}_y}(\mathcal{Y}\setminus \{y_1\}) \to \ell^\infty(\mathcal{X}), \quad \tilde{R} : \ell^\infty(\mathcal{X}) \to \ell^\infty_{\tilde{\phi}_y}(\mathcal{Y}\setminus \{y_1\}),
\]
respectively, and introduce \( \tilde{M} \) likewise. Based on (5.21) we then note that \( \| \tilde{P} \|_{OP} \leq 1 - \eta/2 \) and since \( \sum_{x\in\mathcal{X}} \frac{\pi^\lambda_{xy}}{s_y} = 1 \) for all \( y \in \mathcal{Y}\setminus \{y_1\} \), it follows that \( \| \tilde{R} \|_{OP} \leq 1 \). Thus, we assert that
\[
\|(Id - \tilde{M})^2\|_{OP} \leq \max\left\{\|\tilde{R}\tilde{P}\|_{OP}, \|\tilde{P}\tilde{R}\|_{OP}\right\} \leq 1 - \frac{\eta}{2},
\]
which implies that the Neumann-series \( \sum_{k=0}^{\infty} (Id - \tilde{M})^k \) converges in operator norm with
\[
\|\tilde{M}^{-1}\|_{OP} \leq \sum_{k=0}^{\infty} \|(Id - \tilde{M})^k\|_{OP} \leq \left(\|Id\|_{OP} + \|(Id - \tilde{M})\|_{OP}\right) \sum_{k=0}^{\infty} \|(Id - \tilde{M})^2\|_{OP}^k
\]
\[
\leq \frac{2}{1 - (1 - \eta/2)} = \frac{4}{\eta} < \infty.
\]
This also yields that \( \|\tilde{M}^{-1}\|_{OP} \leq 4L_\mathcal{Y}/\eta \). Concluding, there exists a unique pair of elements \((h^{X,\infty}, h^\mathcal{Y}_{\infty,\infty}) \in \ell^\infty(\mathcal{X}) \times \ell^\infty_{\tilde{\phi}_y}(\mathcal{Y}\setminus \{y_1\})\) that solves equation (5.18) and \( M^{-1}Q \) is a bounded operator.

Finally, we prove that \( h^{X\times \mathcal{Y}} \) from (5.16) is contained in \( \ell^1_{\ell_\mathcal{X}\otimes \ell_\mathcal{Y}}(\mathcal{X} \times \mathcal{Y}) \). This follows by the following calculation
\[
\left\| h^{X\times \mathcal{Y}} \right\|_{\ell^1_{\ell_\mathcal{X}\otimes \ell_\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})} \leq \left\| \frac{\pi^\lambda}{\lambda} \otimes A^T_t \left( h^{X,\infty}, h^\mathcal{Y}_{\infty,\infty} \right) \right\|_{\ell^1_{\ell_\mathcal{X}\otimes \ell_\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})}
\]
\[
+ \left\| \frac{\pi^\lambda}{r \otimes s} \otimes \left( r \otimes h^\mathcal{Y} + h^X \otimes s \right) \right\|_{\ell^1_{\ell_\mathcal{X}\otimes \ell_\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})}
\]
\[
\leq 2 \left( \frac{\pi^\lambda}{\lambda} \right)_{\ell^1_{\ell_\mathcal{X}\otimes \ell_\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})} \left( \left\| h^{X,\infty} \right\|_{\ell^\infty(\mathcal{X})} + \left\| h^\mathcal{Y}_{\infty,\infty} \right\|_{\ell^\infty_{\tilde{\phi}_y}(\mathcal{Y}\setminus \{y_1\})} \right)
\]
\[
+ \left( \frac{\pi^\lambda}{r \otimes s} \right)_{\ell^\infty_{\ell_\mathcal{X}\otimes \ell_\mathcal{Y}}(\mathcal{X} \times \mathcal{Y})} 2 \left( \left\| r \right\|_{\ell^1_{\ell_\mathcal{X}}(\mathcal{X})} \left\| h^\mathcal{Y} \right\|_{\ell^1_{\tilde{\phi}_y}(\mathcal{Y})} + \left\| h^X \right\|_{\ell^1_{\ell_\mathcal{X}}(\mathcal{X})} \left\| s \right\|_{\ell^1_{\tilde{\phi}_y}(\mathcal{Y})} \right), \tag{5.22}
\]
where we used in the second inequality for \( (x, y) \in \mathcal{X} \times \mathcal{Y} \) the following bound
\[
(\mathcal{C}_\mathcal{X}(x) + \mathcal{C}_\mathcal{Y}(x)) \phi_y(y) = \mathcal{C}_\mathcal{X}(x) \phi_y(y) + \Phi_y(y) \leq 2\mathcal{C}_\mathcal{X}(x) \Phi_y(y).
\]
In particular, it holds by the upper bound for $\pi^\lambda$ (Proposition 2.3) that
\[
\|\pi^\lambda\|_{1_{X}⊗\phi_\lambda}(X\times Y) = \sum_{(x,y)\in X\times Y} C_X(x)\Phi_Y(y)\pi^\lambda_{xy} \\
\leq \sum_{(x,y)\in X\times Y} C_X(x)\Phi_Y(y)\phi_\lambda(y)\exp\left(\frac{(c^*_X,r) + (c^*_Y,s) + \|c^*_X - c^*_X\|_{l^\infty}(X)}{\lambda}\right)_r x y \\
= \exp\left(\frac{(c^*_X,r) + (c^*_Y,s) + \|c^*_X - c^*_X\|_{l^\infty}(X)}{\lambda}\right) \sum_{(x,y)\in X\times Y} C_X(x)\Phi^2_Y(y)r_x s_y < \infty,
\]
which is finite by $(r,s)\in l^1(X)\times l^1(\mathcal{Y})$. By the same upper bound we see that the term $\|\pi^\lambda/\otimes s\|_{\ell^1_{\otimes\phi_\lambda}}(X\times Y)$ is bounded by
\[
\sup_{(x,y)\in X\times Y} \frac{\pi^\lambda_{xy}/(r_x s_y)}{\phi_\lambda(y)} \leq \exp\left(\frac{(c^*_X,r) + (c^*_Y,s) + \|c^*_X - c^*_X\|_{l^\infty}(X)}{\lambda}\right) < \infty.
\]
This also shows that $h^{X\times Y}$ continuously depends on $(h^X, h^Y) \in \ell^{\infty}(X) \times \ell^{\infty}(\mathcal{Y})$ and $(h^X, h^Y) \in l^1(X) \times l^1(\mathcal{Y})$ and thus concludes the proof on well-definedness and boundedness of the operator of the claim.

**Remark 5.5.** Crucial for the well-definedness of the proposed derivative for the EROT plan (Proposition 5.2) for the setting $\|c^*_X - c^*_X\|_{l^\infty}(X) < \infty$ is that the operator $M: \ell^{\infty}(X) \times \ell^{\infty}(\mathcal{Y}) \to \ell^{\infty}(X) \times \ell^{\infty}(\mathcal{Y})$ in (5.20) has a bounded inverse for which we employ the Neumann-series calculus. To this end, we verify that there exists $\varepsilon > 0$ and construct a function $\tilde{\phi}_\lambda: \mathcal{Y} \to [1,\infty]$ with $\tilde{\phi}_\lambda \geq k\phi_\lambda$ for some $k > 0$ such that
\[
\sum_{y\in\mathcal{Y}\setminus\{y_1\}} \tilde{\phi}_\lambda(y) \frac{\pi^\lambda_{xy}}{r_x} \leq 1 - \varepsilon \quad \text{for all } x \in \mathcal{X}, \\
\frac{1}{\tilde{\phi}_\lambda(y)} \sum_{x\in\mathcal{X}} \frac{\pi^\lambda_{xy}}{s_y} \leq 1 \quad \text{for all } y \in \mathcal{Y}.
\]
Generalizing this approach to ground costs with unbounded variation in both components $\|c^*_X - c^*_X\|_{l^\infty}(X) = \infty$ and $\|c^*_Y - c^*_Y\|_{l^\infty}(\mathcal{Y}) = \infty$ would require existence of $\varepsilon > 0$ and suitable functions $\psi_X: \to [1,\infty), \psi_Y: \to [1,\infty)$ with $\psi_X \geq k\phi_X$ and $\psi_Y \geq k\phi_Y$ for some $k > 0$ such that
\[
\frac{1}{\psi_X(x)} \sum_{y\in\mathcal{Y}\setminus\{y_1\}} \psi_Y(y) \frac{\pi^\lambda_{xy}}{r_x} \leq 1 - \varepsilon \quad \text{for all } x \in \mathcal{X}, \\
\frac{1}{\psi_Y(y)} \sum_{x\in\mathcal{X}} \psi_X(x) \frac{\pi^\lambda_{xy}}{s_y} \leq 1 \quad \text{for all } y \in \mathcal{Y}.
\]
The explicit construction of such functions $\psi_X, \psi_Y$ remains challenging is and is left for future research.

**Proof of Proposition 5.3.** The proof consists of two steps. We first show the claims for finitely supported probability measures and extend them afterwards to probability measures with countable support.

The Lipschitz bound will be derived by showing that the operator norm of the derivative of primal and dual optimizers for a given pair of probability measures with respect to
perturbations on the same support can be uniformly bounded. For the proof, we first define the quantity
\[
\kappa := \sup_{(\tilde{r}, \tilde{s}) \in B_1} \exp \left( \frac{(c_X^+, \tilde{r}) + (c_Y^+, \tilde{s}) + \|c_X^+ - c_Y^+\|_{\ell^\infty}}{\lambda} \right) \epsilon [1, \infty).
\]

We further introduce the quantity \( \eta \in (0, 1) \) by
\[
\eta := \inf_{(\tilde{r}, \tilde{s}) \in B_{\rho_0}} \frac{s_{yi}}{2} \exp \left( \frac{-\epsilon (c_X^+, \tilde{r}) - (c_Y^+, \tilde{s}) - 2 \|c_X^+ - c_Y^+\|_{\ell^\infty}}{\lambda} \right) (\phi_{\tilde{r}}, \tilde{s})^{-1},
\]
and define \( \rho_0 := \eta / 4\kappa \). Next, we consider \( (\tilde{r}, \tilde{s}) \in B_{\rho_0} \) with finite support. Note that the inequality \( s_{yi} \geq s_{yi}/2 > 0 \) holds. Furthermore, we define \( \hat{X} := \text{supp}(\tilde{r}) \), \( \hat{Y} := \text{supp}(\tilde{s}) \) and introduce for given positive function \( f: X \to (0, \infty) \) the spaces \( \ell^f_f(X) := \ell^f_f(\hat{X}) \) and \( \ell^\infty_f(\hat{X}) := \ell^\infty_{f,(\hat{X})} \). With this notation, we define the operator \( \hat{A}_* \) as \( A_* \) from Section 3 restricted to \( \ell^f_f(\hat{X}) \times \ell^f_f(\hat{Y}) \setminus \{y1\} \) and, similarly, introduce \( \hat{F} \) as \( F \) from Section 3 with a modified domain and range space
\[
\hat{F}: (\ell^f_f(\hat{X}) \times \ell^\infty(\hat{X}) \times \ell^\infty_f(\hat{Y}) \setminus \{y1\}) \times (\ell^f_f(\hat{X}) \times \ell^f_f(\hat{Y}) \setminus \{y1\}) \to \mathbb{R}^{[\hat{X} \times \hat{Y}] \setminus \{x1\} \times [\hat{X} \times \hat{Y}] \setminus \{y1\}}^{-1}.
\]

By Corollary 3.3 the triplet \((\hat{\pi}^\lambda, \hat{\alpha}^\lambda, \hat{\beta}^\lambda) \in (\ell^1_{\ell^f_f(\hat{X}) \times \ell^f_f(\hat{Y}) \setminus \{y1\}) \times \ell^\infty_f(\hat{X}) \times \ell^\infty_f(\hat{Y}))\) are optimizers of \((\text{EROT})\) and \((\text{DEROT})\) for the probability measures \((\tilde{r}, \tilde{s})\) if and only if \( \hat{F}((\hat{\pi}^\lambda, \hat{\alpha}^\lambda, \hat{\beta}^\lambda), (\tilde{r}, \tilde{s})) = 0 \). Furthermore, the function \( \hat{F} \) is Fréchet differentiable (Averbukh & Smolyanov, 1967), the derivative in this notion will be denoted by \( D^F \). Following the arguments by Klatt et al. (2020b) the partial derivative of \( \hat{F} \) with respect to \((\hat{\pi}, \hat{\alpha}, \hat{\beta})\) at optimizers \((\hat{\pi}^\lambda, \hat{\alpha}^\lambda, \hat{\beta}^\lambda)\) for \((\tilde{r}, \tilde{s})\) in matrix representation is then given by
\[
\left[ D^F_{\hat{\pi}, \hat{\alpha}, \hat{\beta}, s}(\hat{\pi}^\lambda, \hat{\alpha}^\lambda, \hat{\beta}^\lambda, \hat{r}, \hat{s}) \hat{F} \right] = \left[ \begin{array}{c}\text{Id}_{\ell^1_{\ell^f_f(\hat{X}) \times \ell^f_f(\hat{Y}) \setminus \{y1\}}} \\
\frac{1}{\kappa} \exp \left( \frac{1}{\kappa} \left[ \tilde{A}_T^f((\hat{\alpha}^\lambda, \hat{\beta}^\lambda) - c) \odot (\hat{r} \odot \hat{s}) \odot \hat{A}_2^f \right] \right) \end{array} \right],
\]
which is an invertible operator since the identity operator is invertible in conjunction with \( A_* \) having full rank of order \( |X| + |Y| - 1 \), and because \( \frac{1}{\kappa} \exp \left( \frac{1}{\kappa} \left[ \tilde{A}_T^f((\hat{\alpha}^\lambda, \hat{\beta}^\lambda) - c) \odot (\hat{r} \odot \hat{s}) \odot \hat{A}_2^f \right] \right) \) is component-wise strictly positive. By the implicit function theorem this induces a mapping on an open set \( \mathcal{U} \in \ell^1_{\ell^f_f(\hat{X}) \times \ell^f_f(\hat{Y}) \setminus \{y1\}} \) with \((\hat{r}, \hat{s}) \in \mathcal{U}\)
\[
\hat{\vartheta}: \mathcal{U} \to \left( \ell^1_{\ell^f_f(\hat{X}) \times \ell^f_f(\hat{Y}) \setminus \{y1\}} \times \ell^\infty_f(\hat{X}) \times \ell^\infty_f(\hat{Y}) \setminus \{y1\}) \right)
\]
such that for any \((\sigma, \tau, \pi) \in \mathcal{U}\) the relation \( \hat{F}(\hat{\vartheta}(\sigma, \tau, \pi), (\sigma, \tau)) = 0 \) holds. In particular, if \((\sigma, \tau, \pi) \in \mathcal{P}(\hat{X}) \times \mathcal{P}(\hat{Y}) \) it follows that \( \hat{\vartheta}(\sigma, \tau) \) coincides with the triplet of optimizers of \((\hat{\pi}^\lambda, \hat{\alpha}^\lambda, \hat{\beta}^\lambda)\) for these respective probability measures.

Moreover, the implicit function theorem yields that \( \hat{\vartheta} \) is Fréchet differentiable at \((\hat{r}, \hat{s})\) with derivative
\[
D^F_{\hat{\pi}, \hat{\alpha}, \hat{\beta}, s}(\hat{\vartheta}(\hat{r}, \hat{s}), \hat{r}, \hat{s}) \hat{F} = -D^F_{\hat{\pi}, \hat{\alpha}, \hat{\beta}, s}(\hat{\vartheta}(\hat{r}, \hat{s}), \hat{r}, \hat{s}) \hat{F}^{-1} \odot D^F_{\hat{\pi}, \hat{\alpha}, \hat{\beta}, s}(\hat{\vartheta}(\hat{r}, \hat{s}), \hat{r}, \hat{s}) \hat{F}.
\]

32
Hence, it remains to bound this operator. Adapting the notation of the proof for Proposition 5.2 we know that there exist suitable operators \( M, \hat{Q} \) such that the derivative for the component of \( \hat{\vartheta} \) in \( \ell^\infty(\mathcal{X}) \times \ell^\infty(\mathcal{Y}) \), i.e., the component for optimal entropic dual potentials, is given by \( \hat{M}^{-1}\hat{Q} \). For the operator \( \hat{Q} \) we know that

\[
\| \hat{Q} \|_{OP} \leq (1 + \Phi_2^2(y_1)) \lambda \exp \left( \frac{(c^r, \tilde{r}) + (c^\prime, \tilde{s}) + \| c^1_x - c^1_x \|_{\ell^\infty(\mathcal{X})}}{\lambda} \right)
\]

\[
\leq (1 + \Phi_2^2(y_1)) \lambda \zeta = A_1 < \infty.
\]

For a bound on the operator norm of \( M \) we obtain by Proposition 2.3 the lower bound

\[
\min_{x \in \mathcal{X}} \pi^\lambda_{xy}(\tilde{r}, \tilde{s})/r_x \geq \eta.\]

Moreover, we choose \( N \in \mathbb{N} \) such that

\[
\sum_{i=N+1}^{\infty} \Phi_2^2(y_i) \kappa s_{yi} \leq \frac{\eta}{4}.
\]

By definition of \( \rho_0 \) we obtain for all \((\tilde{r}, \tilde{s}) \in B_{\rho_0}\) that

\[
\sum_{i=1}^{\infty} \Phi_2^2(y_i)(|\pi_{yi} - s_{yi}|) \leq \rho_0 = \frac{\eta}{4\kappa'},
\]

which yields by our choice on \( N \) for each \((\tilde{r}, \tilde{s}) \in B_{\rho_0}\) and \( x \in \mathcal{X} \) that

\[
\sum_{i=N+1}^{\infty} (\phi_2(y_i) - 1) \frac{\pi^\lambda_{xy}(\tilde{r}, \tilde{s})}{r_x} \leq \sum_{i=N+1}^{\infty} (\phi_2(y_i) - 1) \frac{\| c^1_x - c^1_x \|_{\ell^\infty(\mathcal{X})} + c^2_y(y_i) - c^2_y(\tilde{y}) + (c^1_x, \tilde{r}) + (c^2_y, \tilde{s})}{\lambda} \frac{\pi_{yi}}{r_x} \leq \sum_{i=N+1}^{\infty} \Phi_2^2(y_i) \kappa \frac{\pi_{yi} - s_{yi} + s_{yi}}{2} \leq \frac{\eta}{2}.
\]

In particular, it follows that \( \sum_{i=N+1}^{\infty} (\phi_2(y_i) - 1) \frac{\pi^\lambda_{xy}(\tilde{r}, \tilde{s})/r_x}{\eta} \leq \frac{4}{\eta} \) for all \( x \in \mathcal{X} \) and the quantity \( L_Y = \sup_{i=1,...,N} \phi_2(y) \) is finite. Hence, by the Neumann-series calculus we obtain that the operator norm of \( \hat{M}^{-1} \) can be bounded by

\[
\| \hat{M}^{-1} \|_{OP} \leq L_Y \frac{4}{\eta} = A_2,
\]

which yields \( \| \hat{M}^{-1}\hat{Q} \|_{OP} \leq A_1 A_2 = A' \). By definition, this bound is independent from \((\tilde{r}, \tilde{s}), \) i.e., for any two pairs of probability measures \((\tilde{r}, \tilde{s}), (\tilde{r}', \tilde{s}') \in B_{\rho_0}\) with finite, coinciding support it follows that \((5.13)\) is valid for the Lipschitz-constant \( \tilde{A}' \). Moreover, by Proposition 2.4 we note that \((5.13)\) generalizes to the setting of \( \text{supp}(\tilde{r}') \subseteq \text{supp}(\tilde{r}) \) and \( \text{supp}(\tilde{s}') \subseteq \text{supp}(\tilde{s}) \).

Next, we derive the Lipschitz property for the EROT plan \( \pi^\lambda \) in case of finitely supported probability measures. To this end, we again consider the pair \((\tilde{r}, \tilde{s})\) and note by \((5.16)\) from the proof of Proposition 5.2 that the derivative for the component of \( \vartheta \) in \( \ell^1_{\mathcal{X} \times \mathcal{Y}}(\mathcal{X} \times \mathcal{Y}) \), denoted by \( \vartheta_{\tilde{r}, \tilde{s}} \), is given by

\[
D^F_{\tilde{r}, \tilde{s}} \vartheta_{\tilde{r}, \tilde{s}} \pi^\lambda_{\mathcal{X}}(\mathcal{X}) \times \pi^\lambda_{\mathcal{Y}}(\mathcal{Y}\setminus\{y_1\}) \rightarrow \ell^1_{\mathcal{X} \times \mathcal{Y}}(\mathcal{X} \times \mathcal{Y}),
\]

\[
(\hat{h}^\lambda, \hat{h}^\lambda_\prime) \mapsto \frac{\pi^\lambda(\tilde{r}, \tilde{s})}{\lambda} \odot \tilde{A}'^{-1} \hat{M}^{-1}\hat{Q}(\tilde{r}, \hat{h}^\lambda_\prime) + \frac{\pi^\lambda(\tilde{r}, \tilde{s})}{\tilde{r} \odot \tilde{s}} \odot \left[ \hat{h}^\lambda \odot \hat{h}^\lambda_\prime + \hat{h}^\lambda \odot \tilde{s} \right].
\]
Similar to the upper bound for (5.22) we see that

\[
\left\| D^F_{\tilde{r}, \tilde{s}} \right\|_{OP} \leq \sup_{(\tilde{r}, \tilde{s}) \in B_{\rho_0}} \left( 2 \lambda^1 \left\| \pi^\lambda(\tilde{r}, \tilde{s}) \right\|_{l^1_\ell(x) \otimes l^1_\ell(y)} A' \right)
+ 2 \left\| \pi^\lambda(\tilde{r}, \tilde{s}) \right\|_{l^1_\ell(x) \otimes l^1_\ell(y)} \left( \left\| \tilde{r} \right\|_{l^1_\ell(x)} + \left\| \tilde{s} \right\|_{l^1_\ell(y)} \right)
\]

is finite. Since the upper bound is independent from \((\tilde{r}, \tilde{s})\) it follows for any two pairs of probability measures \((\tilde{r}, \tilde{s}), (\tilde{r}', \tilde{s}') \in B_{\rho_0}\) with coinciding support that the following inequality is valid

\[
\left\| \pi^\lambda(\tilde{r}, \tilde{s}) - \pi^\lambda(\tilde{r}', \tilde{s}') \right\|_{C^\lambda_x \otimes C^\lambda_y(\chi \times \gamma)} \leq \tilde{A} \left\| (\tilde{r}, \tilde{s}) - (\tilde{r}', \tilde{s}') \right\|_{l^1_\ell(x) \otimes l^1_\ell(y)}.
\]

By Proposition 2.4 this inequality also holds in case of \(\text{supp}(\tilde{r}) \subseteq \text{supp}(r)\) and \(\text{supp}(\tilde{s}) \subseteq \text{supp}(s)\) and thus finishes the first step of the proof.

For step two of this proof it remains to show that these Lipschitz bounds extend to probability measures \((\tilde{r}, \tilde{s}), (\tilde{r}', \tilde{s}') \in B_{\rho_0}\) with \(\text{supp}(\tilde{r}') \subseteq \text{supp}(r)\), \(\text{supp}(\tilde{s}') \subseteq \text{supp}(s)\) where at least one probability measure has infinite support. Concerning the Lipschitz property for the EROT plan we consider finite support approximations of the probability measures. By Proposition 2.4 it follows for given \(\varepsilon > 0\) that there exists \(l \in \mathbb{N}\) such that \((\tilde{r}_l, \tilde{s}_l), (\tilde{r}'_l, \tilde{s}'_l) \in B_{\rho_0}\) with

\[
\left\| \pi^\lambda(\tilde{r}, \tilde{s}) - \pi^\lambda(\tilde{r}_l, \tilde{s}_l) \right\|_{C^\lambda_x \otimes C^\lambda_y(\chi \times \gamma)} \leq \frac{\varepsilon}{2}, \quad \left\| \pi^\lambda(\tilde{r}', \tilde{s}') - \pi^\lambda(\tilde{r}'_l, \tilde{s}'_l) \right\|_{C^\lambda_x \otimes C^\lambda_y(\chi \times \gamma)} \leq \frac{\varepsilon}{2}.
\]

By our Lipschitz bounds for finitely supported probability measures it then follows that

\[
\left\| \pi^\lambda(\tilde{r}, \tilde{s}) - \pi^\lambda(\tilde{r}', \tilde{s}') \right\|_{C^\lambda_x \otimes C^\lambda_y(\chi \times \gamma)} \leq \left\| \pi^\lambda(\tilde{r}_l, \tilde{s}_l) - \pi^\lambda(\tilde{r}'_l, \tilde{s}'_l) \right\|_{C^\lambda_x \otimes C^\lambda_y(\chi \times \gamma)} + \varepsilon
\]

As \(\varepsilon > 0\) can be chosen arbitrarily small we deduce the local Lipschitz property with modulus \(A := \tilde{A}(C^\lambda(x_1) + \Phi^2_\gamma(y_1))\).

For the dual solutions, we only prove the claim for \(\alpha^\lambda\), for \(\beta^\lambda\) the proof is analogous. Consider \(x \in \text{supp}(\tilde{r}')\), then it follows by Proposition 2.4 for given \(\varepsilon > 0\) that there exists \(l \in \mathbb{N}\) such that \(|\alpha^\lambda_x(\tilde{r}, \tilde{s}) - \alpha^\lambda_x(\tilde{r}_l, \tilde{s}_l)| < \varepsilon/2\) and \(|\alpha^\lambda_x(\tilde{r}', \tilde{s}') - \alpha^\lambda_x(\tilde{r}'_l, \tilde{s}'_l)| < \varepsilon/2\) as well as \((\tilde{r}_l, \tilde{s}_l), (\tilde{r}'_l, \tilde{s}'_l) \in B_{\rho_0}\). Applying our Lipschitz bound then yields

\[
|\alpha^\lambda_x(\tilde{r}, \tilde{s}) - \alpha^\lambda_x(\tilde{r}', \tilde{s}')| \leq |\alpha^\lambda_x(\tilde{r}_l, \tilde{s}_l) - \alpha^\lambda_x(\tilde{r}'_l, \tilde{s}'_l)| + \varepsilon
\]

Choosing \(\varepsilon\) arbitrarily small and setting \(A' := \tilde{A}(C^\lambda(x_1) + \Phi^2_\gamma(y_1))\) gives

\[
|\alpha^\lambda_x(\tilde{r}, \tilde{s}) - \alpha^\lambda_x(\tilde{r}', \tilde{s}')| \leq A' \left\| (\tilde{r}, \tilde{s}) - (\tilde{r}', \tilde{s}') \right\|_{l^1_\ell(x) \otimes l^1_\ell(y)}.
\]

taking the supremum over all \(x \in \text{supp}(\tilde{r}')\) then proves the claim.
Proof of Proposition 5.4. Define the mapping from probability measures to optimizers of (EROT) and (DEROT)

\[ \psi: (\mathcal{P}(\mathcal{X}) \cap \ell^1_{C_{\mathcal{X}}} (\mathcal{X})) \times (\mathcal{P}(\mathcal{Y}) \cap \ell^1_{C_{\mathcal{Y}}} (\mathcal{Y}\setminus\{y_1\})) \to \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^X \times \mathbb{R}^Y \setminus \{y_1\}, \]

\[ (r, s) \mapsto (\pi^\lambda(r, s), \alpha^\lambda(r, s), \beta^\lambda(r, s)), \]

where we select \((\alpha^\lambda, \beta^\lambda)\) according to Proposition 2.4, i.e., such that the element \((0, \beta^\lambda) \in \mathbb{R}^Y\) represents a dual optimizer. Recalling the function \(\mathcal{F}\) from Section 3, it holds by Corollary 3.3 for each \(n \geq N\) that

\[ 0 = \mathcal{F}(\vartheta(r, s), r, s) = \mathcal{F}(\vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y). \]

which yields

\[ 0 = \left( \mathcal{F}(\vartheta(r, s), r, s) - \mathcal{F}(\vartheta(r, s), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right) \]

\[ - \left( \mathcal{F}(\vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right), \]

Adding another three terms of non-trivial zeros leads to the following equation

\[ \left[ D_{\pi, \alpha, \beta, \vartheta}(\vartheta(r, s), r, s) \right] \mathcal{F} \left( \vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) - \vartheta(r, s) \right) \]

\[ = -t_n \left[ D_{\vartheta}(\vartheta(r, s), r, s) \right] \mathcal{F} \left( \hat{h}_1^X, \hat{h}_s^Y \right) + \left( \mathcal{F}(\vartheta(r, s), r, s) - \mathcal{F}(\vartheta(r, s), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right) + t_n \left[ D_{\vartheta}(\vartheta(r, s), r, s) \right] \mathcal{F} \left( \hat{h}_1^X, \hat{h}_s^Y \right) \]

\[ + \left( \mathcal{F}(\vartheta(r, s), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right. \]

\[ - \mathcal{F}(\vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right) + \left[ D_{\vartheta}(\vartheta(r, s), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right] \mathcal{F} \left( \vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) - \vartheta(r, s) \right) \]

\[ + \left[ D_{\vartheta}(\vartheta(r, s), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right] \mathcal{F} \left( \vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) - \vartheta(r, s) \right) \]

\[ - \left[ D_{\vartheta}(\vartheta(r, s), r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) \right] \mathcal{F} \left( \vartheta(r + t_n \hat{h}_1^X, s + t_n \hat{h}_s^Y) - \vartheta(r, s) \right), \]

where \(D_{\vartheta}, D_{\vartheta} \mathcal{F}\) represent the naïve component-wise derivatives as employed in Section 3. For the term in (5.23) we already know by Proposition 5.2 that applying \(\left[ D_{\pi, \alpha, \beta, \vartheta}(\vartheta(r, s), r, s) \right]^{-1}\) is well-defined.

We need to show that applying \(\left[ D_{\pi, \alpha, \beta, \vartheta}(\vartheta(r, s), r, s) \right]^{-1}\) on each of the summands (5.24), (5.25), and (5.26) is also well-defined and that the \(\ell^1_{C_{\mathcal{X}@C_{\mathcal{Y}}}}(\mathcal{X} \times \mathcal{Y})\)-norm of the resulting component in \(\ell^1_{C_{\mathcal{X}@C_{\mathcal{Y}}}}(\mathcal{X} \times \mathcal{Y})\), i.e., the \(\pi\)-component decreases with order \(o(t_n)\) for \(n \to \infty\). This part of the proof is technical and deferred to Lemma B.3 in Appendix.
B. Most notably, for this purpose we require \( s \in \ell_1^\delta (\mathcal{X}) \). As a consequence, we obtain for \( n \to \infty \) that

\[
\left\| \frac{\pi^\lambda (r + t_n \hat{h}^X_t, s + t_n \hat{h}^Y_t) - \pi^\lambda (r, s)}{t_n} - \mathcal{D}^H_{r,s} \pi^\lambda (\hat{h}^X_t, \hat{h}^Y_t) \right\|_{\ell_1^\delta (\mathcal{X} \times \mathcal{Y})} = \frac{\| \varphi_t (r + t_n \hat{h}^X_t, s + t_n \hat{h}^Y_t) - \varphi_t (r, s) \|}{t_n} \\
+ \left( \mathcal{D}^-_{\varphi (r, s)} \mathcal{F} \right)^{-1} \circ \mathcal{D}^-_{\varphi (r, s)} \mathcal{F} (\hat{h}^X_t, \hat{h}^Y_t) \right\|_{\ell_1^\delta (\mathcal{X} \times \mathcal{Y})} = o(1),
\]
which proves the assertion.

\[ \square \]

6 Discussion

It remains an open question if the conditions on limit laws for empirical EROT plan and value for partially bounded ground costs or generally unbounded ground costs can be weakened omitting the exponential term in (1.7), (1.9), and Table 1 by a refined sensitivity analysis. In particular, it would be interesting to investigate the sensitivity of the EROT plan in case of generally unbounded ground costs. Notably, in our results the condition \( \| c^+_{\mathcal{X}} - c^-_{\mathcal{X}} \|_{\ell^\infty (\mathcal{X})} < \infty \) is of particular use for the Neumann series-calculus of bounded operators. In case of ground costs that are generally unbounded this proof technique does not generalize well. Therefore, limit distributions for the empirical EROT plan remain unknown (Remark 5.5) although a similar structure is reasonable to conjecture.

In addition to our limit results for fixed regularization parameter \( \lambda > 0 \), we characterize in Section 4.2 the asymptotic behavior of the empirical EROT value and Sinkhorn costs for the regime of a decreasing regularization parameter \( \lambda (n) = o(1/\sqrt{n}) \). We see that the resulting limit law is given by the respective limit law of the empirical non-regularized OT value which is fundamentally different Tameling et al. (2019). Naturally, for \( \lambda (n) \) of slower order than \( O(1/\sqrt{n}) \) the question arises whether the empirical EROT value still converges weakly towards a suitable limit distribution and when a phase transition to the obtained Gaussian limit occurs. Recent results demonstrate that the sample complexity of the EROT value \( \mathbb{E} \left[ | \text{EROT}^\lambda (\hat{r}_n, \hat{s}_n) - \text{EROT}^\lambda (r, s) | \right] \) decreases of order \( O(\exp(\kappa/\lambda)\lambda^{-d/2}n^{-1/2}) \) for some \( \kappa > 0 \) in certain settings in \( \mathbb{R}^d \) as \( n \) tends to infinity (Genevay et al., 2019). Extensions are provided by Mena & Niles-Weed (2019) and Chizat et al. (2020) who refine this to \( O((1 + \lambda^{-d/2})n^{-1/2}) \). Consequently, when \( \lambda (n) \) decreases sufficiently slow such that the sample complexity rate stays bounded from above it follows by Markov’s inequality that \( \sqrt{n}(\text{EROT}^\lambda (\hat{r}_n, \hat{s}_n) - \text{EROT}^\lambda (r, s)) \) is a tight sequence of random variables. By Prokhorov’s Theorem there exists a subsequence which converges weakly towards a tight limit.

The analysis of the limit behavior of the empirical EROT plan on countable spaces for the regime \( \lambda < 0 \) is even more involved. Mimicking Section 4.2 two aspects appear to us as crucial. First, it is necessary to obtain suitable bounds between EROT plans and non-regularized OT plans. These bounds are available on finite spaces (Weed, 2018) but unknown for countable ground spaces. Second, the limit distribution of the empirical non-regularized OT plan on countable spaces has to be characterized. For finitely supported probability measures with a unique non-regularized OT plan Klatt et al. (2020a) recently
obtained such a limit law explicitly relying on finite-dimensional linear programming. This
approach does not generalize well to infinite-dimensional linear programming and hence
similar statements for countable spaces remain to be investigated in further studies.

Acknowledgements
S. Hundrieser and A. Munk acknowledge funding by the Deutsche Forschungsgemeinschaft
(DFG, German Research Foundation) under Germany’s Excellence Strategy - EXC
2067/1-390729940. Further, M. Klatt acknowledges support from the DFG Research
Training Group 2088 Discovering structure in complex data: Statistics meets Optimization
and Inverse Problems.

References
Altschuler, J., Niles-Weed, J., & Rigollet, P. (2017). Near-linear time approximation
algorithms for optimal transport via sinkhorn iteration. In I. Guyon, U. V. Luxburg,
& others (Eds.), Advances in Neural Information Processing Systems, volume 30:
Curran Associates, Inc.

Amari, S.-i., Karakida, R., Oizumi, M., & Cuturi, M. (2019). Information geometry
for regularized optimal transport and barycenters of patterns. Neural Computation,
31(5), 827–848.

Antos, A. & Kontoyiannis, I. (2001). Convergence properties of functional estimates for
discrete distributions. Random Structures & Algorithms, 19(3-4), 163–193.

Aubin, J. & Frankowska, H. (1990). Set-valued analysis. Modern Birkhäuser Classics.
Springer.

Averbukh, V. I. & Smolyanov, O. G. (1967). The theory of differentiation in linear topo-
logical spaces. Russian Mathematical Surveys, 22(6), 201–258.

Bertsekas, D. P. (1981). A new algorithm for the assignment problem. Mathematical
Programming, 21(1), 152–171.

Bertsekas, D. P. & Castanon, D. A. (1989). The auction algorithm for the transportation
problem. Annals of Operations Research, 20(1), 67–96.

Bigot, J., Cazelles, E., & Papadakis, N. (2019). Central limit theorems for entropy-
regularized optimal transport on finite spaces and statistical applications. Electronic
Journal of Statistics, 13(2), 5120–5150.

Billingsley, P. (1999). Convergence of probability measures. Wiley Series in Probability
and Statistics. Wiley.

Borisov, I. S. (1981). Some limit theorems for empirical distributions. In Abstracts of
Reports. Third Vilnius Conference on Probability Theory and Mathematical Statistics,
volume 1, pages 71–72.

Borisov, I. S. (1983). Problem of accuracy of approximation in the central limit theorem
for empirical measures. Siberian Mathematical Journal, 24(6), 833–843.
Braunsmann, J. (2018). The entropy-regularized Wasserstein distance as a metric for machine learning based post-processing of structural MR images of the brain. Master’s thesis, University Münster.

Brenier, Y. (1987). Decomposition polaire et rearrangement monotone des champs de vecteurs. *Comptes Rendus de l’Académie des Sciences - Série I - Mathematics*, 305, 805–808.

Brenier, Y. (1991). Polar factorization and monotone rearrangement of vector-valued functions. *Communications on Pure and Applied Mathematics*, 44(4), 375–417.

Chen, C. (2020). Spatiotemporal Imaging with Diffeomorphic Optimal Transportation. *arXiv e-prints*, page 2011.11906.

Chizat, L., Peyré, G., Schmitzer, B., & Vialard, F.-X. (2016). Scaling algorithms for unbalanced transport problems. *Mathematics of Computation*, 87(314), 2563–2609.

Chizat, L., Roussillon, P., Léger, F., Vialard, F.-X., & Peyré, G. (2020). Faster wasserstein distance estimation with the sinkhorn divergence. In H. Larochelle, M. Ranzato, & others (Eds.), *Advances in Neural Information Processing Systems*, volume 33, pages 2257–2269: Curran Associates, Inc.

Clason, C., Lorenz, D. A., Mahler, H., & Wirth, B. (2021). Entropic regularization of continuous optimal transport problems. *Journal of Mathematical Analysis and Applications*, 494(1), 124432.

Cominetti, R. & San Martín, J. (1994). Asymptotic analysis of the exponential penalty trajectory in linear programming. *Mathematical Programming*, 67(1-3), 169–187.

Cover, T. & Thomas, J. (1991). *Elements of information theory*. Wiley series in telecommunications. Wiley.

Csiszár, I. (1975). *I*-divergence geometry of probability distributions and minimization problems. *The Annals of Probability*, 3(1), 146–158.

Cuturi, M. (2013). Sinkhorn distances: Lightspeed computation of optimal transport. In C. J. C. Burges, L. Bottou, & others (Eds.), *Advances in Neural Information Processing Systems*, volume 26: Curran Associates, Inc.

Dessein, A., Papadakis, N., & Rouas, J.-L. (2018). Regularized optimal transport and the ROT mover’s distance. *Journal of Machine Learning Research*, 19(1), 590–642.

Durst, M. & Dudley, R. M. (1980). Empirical processes, Vapnik-Chervonenkis classes and Poisson processes. *Probability and Mathematical Statistics*, 1(2), 109–115.

Dvurechensky, P., Gasnikov, A., & Kroshnin, A. (2018). Computational optimal transport: Complexity by accelerated gradient descent is better than by Sinkhorn’s algorithm. In J. Dy & A. Krause (Eds.), *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 1367–1376: PMLR.

Evans, S. N. & Matsen, F. A. (2012). The phylogenetic Kantorovich–Rubinstein metric for environmental sequence samples. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 74(3), 569–592.
Feydy, J., Séjourné, T., Vialard, F.-X., Amari, S.-i., Trouve, A., & Peyré, G. (2019). Interpolating between optimal transport and MMD using Sinkhorn divergences. In K. Chaudhuri & M. Sugiyama (Eds.), Proceedings of Machine Learning Research, volume 89 of Proceedings of Machine Learning Research, pages 2681–2690. PMLR.

Galichon, A. (2016). Optimal transport methods in economics. Princeton University Press.

Genevay, A., Chizat, L., Bach, F., Cuturi, M., & Peyré, G. (2019). Sample complexity of sinkhorn divergences. In K. Chaudhuri & M. Sugiyama (Eds.), Proceedings of Machine Learning Research, volume 89 of Proceedings of Machine Learning Research, pages 1574–1583. PMLR.

Kantorovich, L. (1958). On the translocation of masses. Management Science, 5(1), 1–4.

Klatt, M., Munk, A., & Zemel, Y. (2020a). Limit laws for empirical optimal solutions in stochastic linear programs. arXiv preprint 2007.13473.

Klatt, M., Tameling, C., & Munk, A. (2020b). Empirical regularized optimal transport: Statistical theory and applications. SIAM Journal on Mathematics of Data Science, 2(2), 419–443.

Lee, Y. T. & Sidford, A. (2014). Path finding methods for linear programming: Solving linear programs in $O(\sqrt{\text{rank}})$-iterations and faster algorithms for maximum flow. In 2014 IEEE 55th Annual Symposium on Foundations of Computer Science, pages 424–433.

McShane, E. J. (1934). Extension of range of functions. Bulletin of the American Mathematical Society, 40(12), 837 – 842.

Mena, G. & Niles-Weed, J. (2019). Statistical bounds for entropic optimal transport: Sample complexity and the central limit theorem. In H. Wallach, H. Larochelle, & others (Eds.), Advances in Neural Information Processing Systems, volume 32, pages 4541–4551.

Monge, G. (1781). Mémoire sur la théorie des déblais et des remblais. In Histoire de l’Académie Royale des Sciences de Paris, pages 666–704.

Orlin, J. (1988). A faster strongly polynomial minimum cost flow algorithm. In Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing, STOC ’88, pages 377–387. Association for Computing Machinery.

Peyré, G. & Cuturi, M. (2019). Computational optimal transport: With applications to data science. Foundations and Trends in Machine Learning, 11(5-6), 355–607.

Rachev, S. & Rüschendorf, L. (1998a). Mass transportation problems: Volume I: Theory. Probability and Its Applications. Springer.

Rachev, S. & Rüschendorf, L. (1998b). Mass transportation problems: Volume II: Applications. Probability and Its Applications. Springer.

Römisch, W. (2004). Delta method, infinite dimensional. In S. Kotz, N. Balakrishnan, & others (Eds.), Encyclopedia of Statistical Sciences. Wiley.

Santambrogio, F. (2015). Optimal transport for applied mathematicians: Calculus of variations, PDEs, and modeling. Progress in Nonlinear Differential Equations and Their Applications. Springer.
Sasane, A. (2017). A friendly approach to functional analysis. Essential Textbooks In Mathematics. World Scientific Publishing Company.

Schiebinger, G., Shu, J., Tabaka, M., Cleary, B., Subramanian, V., Solomon, A., Gould, J., Liu, S., Lin, S., Berube, P., Lee, L., Chen, J., Brumbaugh, J., Rigollet, P., Hochedligner, K., Jaenisch, R., Regev, A., & Lander, E. S. (2019). Optimal-transport analysis of single-cell gene expression identifies developmental trajectories in reprogramming. Cell, 176(4), 928 – 943.e22.

Sinkhorn, R. (1964). A relationship between arbitrary positive matrices and doubly stochastic matrices. The Annals of Mathematical Statistics, 35(2), 876–879.

Sinkhorn, R. (1967). Diagonal equivalence to matrices with prescribed row and column sums. The American Mathematical Monthly, 74(4), 402–405.

Sommerfeld, M. & Munk, A. (2018). Inference for empirical Wasserstein distances on finite spaces. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 80(1), 219–238.

Sommerfeld, M., Schrieber, J., Zemel, Y., & Munk, A. (2019). Optimal transport: Fast probabilistic approximation with exact solvers. Journal of Machine Learning Research, 20, 1–23.

Tameling, C., Sommerfeld, M., & Munk, A. (2019). Empirical optimal transport on countable metric spaces: Distributional limits and statistical applications. The Annals of Applied Probability, 29(5), 2744–2781.

Tameling, C., Stoldt, S., Stephan, T., Naas, J., Jakobs, S., & Munk, A. (2021). Colocalization for super-resolution microscopy via optimal transport. Nature Computational Science, 1, 199–211.

Tong, Q. & Kobayashi, K. (2021). Entropy-regularized optimal transport on multivariate normal and q-normal distributions. Entropy, 23(3), 302.

Van der Vaart, A. & Wellner, J. (1996). Weak convergence and empirical processes: With applications to statistics. Springer Series in Statistics. Springer.

Villani, C. (2008). Optimal transport: old and new. A Series of Comprehensive Studies in Mathematics. Springer.

Weed, J. (2018). An explicit analysis of the entropic penalty in linear programming. In S. Bubeck, V. Perchet, & P. Rigollet (Eds.), Conference On Learning Theory, COLT 2018, Stockholm, Sweden, 6-9 July 2018, volume 75 of Proceedings of Machine Learning Research, pages 1841–1855.: PMLR.

Yukich, J. E. (1986). Metric entropy and the central limit theorem in banach spaces. In X. Fernique, B. Heinzel, & others (Eds.), Geometrical and Statistical Aspects of Probability in Banach Spaces, pages 113–128.: Springer.

Zemel, Y. & Panaretos, V. M. (2019). Fréchet means and procrustes analysis in Wasserstein space. Bernoulli, 25(2), 932–976.
A Proofs for Preliminary Results

Proof of Proposition 2.1. Existence of a unique EROT plan $\pi^\lambda$ and strong duality follow by (Chizat et al., 2016, Theorem 3.2). For the existence of dual optimizers we first prove the optimality criterion with an approach that is inspired by (Braunsmann, 2018, Proposition 3.3.18 and Theorem 3.3.19). Assume that $\pi^\lambda, \alpha^\lambda, \beta^\lambda$ are optimal and define $\pi \in \mathbb{R}^{X \times Y}$ as

$$\pi_{xy} := \exp \left( \frac{\alpha^\lambda + \beta^\lambda - c(x,y)}{\lambda} \right) r_x s_y \geq 0$$

for all $(x,y) \in X \times Y$. Note that $\pi \in \ell^1(X \times Y)$ because otherwise $(\alpha^\lambda, \beta^\lambda)$ could not be optimizers. Further, denote the objective of (DEROT) by

$$\mathbb{D}^r,s(\alpha, \beta) := (\alpha, r) + (\beta, s) - \lambda \left[ \sum_{x \in X} \sum_{y \in Y} \exp \left( \frac{\alpha^\lambda + \beta^\lambda - c(x,y)}{\lambda} \right) r_x s_y - r_x s_y \right].$$

Then it follows by optimality of $\alpha^\lambda$ for any $\varphi^x \in \ell^\infty(X)$ that $\mathbb{D}(\alpha^\lambda + t\varphi^x, \beta^\lambda)$ is finite for each $t \in \mathbb{R}$ and differentiable at $t = 0$ with

$$0 = \frac{d}{dt}{|}_{t=0} \mathbb{D}^r,s(\alpha^\lambda + t\varphi^x, \beta^\lambda) = \sum_{x \in X} \varphi^x r_x - \sum_{y \in Y} \varphi^y s_y + \lambda \left[ \sum_{x \in X} \sum_{y \in Y} \exp \left( \frac{\alpha^\lambda + \beta^\lambda - c(x,y)}{\lambda} \right) r_x s_y - r_x s_y \right].$$

Likewise, it follows for any $\varphi^y \in \ell^\infty(Y)$ that

$$0 = \frac{d}{dt}{|}_{t=0} \mathbb{D}^r,s(\alpha^\lambda, \beta^\lambda + t\varphi^y) = \sum_{y \in Y} \varphi^y \left( s_y - \sum_{x \in X} \pi_{xy} \right).$$

Since $\varphi^x$ and $\varphi^y$ were arbitrary we see that $\pi$ is contained in $\Pi(r,s)$. Next, we show that $\pi$ optimizes (EROT) and thus coincides with $\pi^\lambda$. To this end, we note that $t \mapsto t \log(t/a) - t$ with $a > 0$ is strictly convex and differentiable on $(0, \infty)$ with derivative $\log(t/a)$. Since $\supp(\pi) = \supp(r \otimes s)$ we obtain for any other feasible transport plan $\tilde{\pi} \in \Pi(r,s)$ and $(x,y) \in \supp(r \otimes s)$ that

$$c(x,y)\tilde{\pi}_{xy} + \lambda \left( \tilde{\pi}_{xy} \log \left( \frac{\tilde{\pi}_{xy}}{r_x s_y} \right) - \tilde{\pi}_{xy} \right) - c(x,y)\pi_{xy} - \lambda \left( \pi_{xy} \log \left( \frac{\pi_{xy}}{r_x s_y} \right) - \pi_{xy} \right) \geq 0$$

Note that the terms from (A.1) for $(x,y) \notin \supp(r \otimes s)$ sum up to zero. Due to $\alpha^\lambda \in \ell^1_p(X)$ and $\beta^\lambda \in \ell^1_p(Y)$, the terms on the r.h.s. of (A.2) are absolutely summable over $X \times Y$ and also sum up to zero. Consequently, summing up both sides of (A.1) and (A.2) over $X \times Y$ yields

$$\langle c, \tilde{\pi} \rangle + \lambda M(\tilde{\pi}) - \langle c, \pi \rangle - \lambda M(\pi) \geq 0,$$

which proves optimality of $\pi$. For the converse, suppose that $\pi, \alpha, \beta$ satisfy conditions (2.1) and (2.2). Condition (2.1) implies $\pi_{xy} \geq 0$ for each $(x,y) \in X \times Y$, by (2.2) we then
obtain that \( \pi \in \Pi(r,s) \). An analogous argument as in (A.1), (A.2), yields that \( \pi \) is an EROT plan. For the optimality of \( \alpha \) we note for arbitrary \( \varphi^X \in \ell^\infty(\mathcal{X}) \) that
\[
\frac{d}{dt}_{|t=0} \mathbb{D}^r,s(\alpha + t\varphi^X, \beta) = \sum_{x \in \mathcal{X}} \varphi^X_x \left( r_x - \sum_{y \in \mathcal{Y}} \pi_{xy} \right) = 0,
\]
where the last equality holds due to \( \pi \in \Pi(r,s) \). Likewise, it follows for any \( \varphi^Y \in \ell^\infty(\mathcal{Y}) \) that \( \frac{d}{dt}_{|t=0} \mathbb{D}^r,s(\alpha, \beta + t\varphi^Y) = 0 \), which yields by strict concavity of \( \mathbb{D}^r,s(\cdot, \cdot) \) that \( (\alpha, \beta) \) are optimal for (DEROT). For our assertions on existence and uniqueness up to a constant of optimal entropic dual potentials we perform a similar calculation as in (Dessein et al., 2018, Proposition 8) and obtain
\[
\langle c, \pi \rangle + \lambda M(\pi) - \langle 1_{\mathcal{X} \times \mathcal{Y}}, (r \otimes s) - K^\lambda \rangle = \text{KL}(\pi || K^\lambda),
\]
where \( K^\lambda_{xy} := \exp(-c(x,y)/\lambda) r_x s_y \) for all \((x,y) \in \mathcal{X} \times \mathcal{Y} \) and \( 1_{\mathcal{X} \times \mathcal{Y}} \) represents the constant function on \( \mathcal{X} \times \mathcal{Y} \) with value 1. By non-negativity of the cost function we note that \( K^\lambda \in \ell^1(\mathcal{X} \times \mathcal{Y}) \) and thus \( \langle 1_{\mathcal{X} \times \mathcal{Y}}, K^\lambda \rangle < \infty \). Hence, since (ERO) is feasible it follows that \( \min_{\pi \in \Pi(r,s)} \text{KL}(\pi || K^\lambda) \) is also feasible. Most notably, the sets of optimizers coincide and existence of a unique EROT plan \( \pi^\lambda \) yields \( \{ \pi^\lambda \} = \text{argmin}_{\pi \in \Pi(r,s)} \text{KL}(\pi || K^\lambda) \). Moreover, since the derivative of \( \log(\cdot) \) diverges to \( +\infty \) near 0 and by lower semi-continuity of \( \text{KL}(-||K^\lambda) \) there exists an element \( \hat{\pi} \in \Pi(r,s) \) with \( \text{supp}(\hat{\pi}) = \text{supp}(r \otimes s) \) and \( \text{KL}(\hat{\pi} || K^\lambda) < \infty \). Consequently, by (Csiszár, 1975, Corollary 3.2) there exist functions \( a : \mathcal{X} \to [0, \infty), b : \mathcal{Y} \to [0, \infty) \) with \( \log(a) \in \ell^1_r(\mathcal{X}) \) and \( \log(b) \in \ell^1_s(\mathcal{Y}) \) such that
\[
\pi^\lambda_{xy} = a_x b_y K^\lambda_{xy} = \exp \left( \frac{\lambda \log(a_x) + \lambda \log(b_y) - c(x,y)}{\lambda} \right) r_x s_y \quad \text{for all } (x,y) \in \mathcal{X} \times \mathcal{Y}.
\]
By optimality criterion (2.1) we deduce that \( \alpha^\lambda := \lambda \log(a) \) and \( \beta^\lambda := \lambda \log(b) \) are optimal entropic dual potentials. Additionally, given another pair of optimal entropic dual potentials \((\tilde{\alpha}^\lambda, \tilde{\beta}^\lambda) \in \ell^1_r(\mathcal{X}) \times \ell^1_s(\mathcal{Y}) \) we see by uniqueness of the EROT plan for all \((x,y) \in \mathcal{X} \times \mathcal{Y} \) that
\[
\exp \left( \frac{\alpha^\lambda_x + \beta^\lambda_y - c(x,y)}{\lambda} \right) r_x s_y = \pi^\lambda_{xy} = \exp \left( \frac{\tilde{\alpha}^\lambda_x + \tilde{\beta}^\lambda_y - c(x,y)}{\lambda} \right) r_x s_y.
\]
This can be rewritten to the following system of equations
\[
\alpha^\lambda_x - \tilde{\alpha}^\lambda_x = \tilde{\beta}^\lambda_y - \beta^\lambda_y \quad \text{for all } x \in \text{supp}(r), y \in \text{supp}(s).
\]
Setting \( \eta := \alpha^\lambda_x - \tilde{\alpha}^\lambda_x \) for some \( x \in \text{supp}(r) \) then yields the claim. \( \square \)

**Proof of Proposition 2.3.** The proof is based on recent findings by Mena & Niles-Weed (2019). By relation (2.1) between primal and dual optimizers for (ERO) any pair of optimal entropic dual potentials \((\alpha^\lambda, \beta^\lambda) \in \ell^1_r(\mathcal{X}) \times \ell^1_s(\mathcal{Y}) \) satisfies \( \text{ERO}^\lambda(r,s) = \langle \alpha^\lambda, r \rangle + \langle \beta^\lambda, s \rangle \). Almost sure uniqueness of dual optimizers up to a constant (Proposition 2.1) then allows us to select potentials such that \( \langle \alpha^\lambda, r \rangle = \langle \tilde{\beta}^\lambda, s \rangle = \text{ERO}^\lambda(r,s)/2 \geq 0 \) (recall by Section 2.1 that \( c \geq 0 \)). Optimality of \((\alpha^\lambda, \beta^\lambda) \) implies by Proposition 2.1 for all \( x \in \text{supp}(r) \) and \( y \in \text{supp}(s) \) that
\[
\alpha^\lambda_x = -\lambda \log \left[ \sum_{y \in \mathcal{Y}} \exp \left( \frac{\tilde{\beta}^\lambda_y - c(x,y)}{\lambda} \right) s_y \right],
\]
\[
\beta^\lambda_y = -\lambda \log \left[ \sum_{x \in \mathcal{X}} \exp \left( \frac{\alpha^\lambda_x - c(x,y)}{\lambda} \right) r_x \right].
\]
As noted in Remark 2.2 this condition is not only necessary but also sufficient for optimality of potentials. Applying Jensen’s inequality for the convex function \(-\log(\cdot)\) and by our choice of \((\alpha^\lambda, \beta^\lambda)\) it then follows for each \(x \in \mathcal{X}\) that

\[
\alpha^\lambda_x = -\lambda \log \left[ \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\lambda_y - c(x, y)}{\lambda} \right) s_y \right] \leq -\langle \beta^\lambda, s \rangle + \sum_{y \in \mathcal{Y}} c(x, y) s_y \leq c^\lambda_X(x) + \langle c^\lambda_Y, s \rangle.
\]

Note, that the bound holds trivially if \(\langle c^\lambda_Y, s \rangle = \infty\). Likewise, it follows for all \(y \in \mathcal{Y}\) that \(\beta^\lambda_y \leq c^\lambda_Y(y) + \langle c^\lambda_X, r \rangle\). For the lower bound of \(\alpha^\lambda_x\) consider the upper bound of \(\beta^\lambda_y\) as well as the lower bound on \(c\) and see

\[
\alpha^\lambda_x = -\lambda \log \left[ \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\lambda_y - c(x, y)}{\lambda} \right) s_y \right] \\
\geq -\lambda \log \left[ \sum_{y \in \mathcal{Y}} \exp \left( \frac{c^\lambda_Y(y) + \langle c^\lambda_X, r \rangle - c^\lambda_X(x) - c^\lambda_Y(y)}{\lambda} \right) s_y \right] \\
\geq c^\lambda_X(x) - \langle c^\lambda_X, r \rangle - \lambda \log(\phi_X, s).
\]

Indeed, the lower bound for \(\alpha^\lambda_x\) is valid if \(\langle c^\lambda_X, r \rangle = \infty\) or \(\langle \phi_X, s \rangle = \infty\). The lower bound for \(\beta^\lambda_y\) follows analogously. Lastly, the bounds for the EROT plan follow from the correspondence to the optimal entropic dual potentials (Proposition 2.1) and their bounds. \(\square\)

**Proof of Proposition 2.4.** For the pointwise convergence of optimal entropic dual potentials we follow the approach by Mena & Niles-Weed (2019), who were inspired by Feydy et al. (2019), and afterwards exploit for the convergence of the EROT plan the relation between primal and dual optimizers (Proposition 2.1).

As a first step we state a bound for \((\alpha^\lambda_k, \beta^\lambda_k)\) that is uniform over all \(k \in \mathbb{N}\). Recall that we consider optimal entropic dual potentials such that \(\beta^\lambda_{k, y_1} = 0\). By convergence of \((r_k)_{k \in \mathbb{N}}\) to \(r\) with respect to \(\ell^1_{\Phi_X}(\mathcal{X})\) and similar for \((s_k)_{k \in \mathbb{N}}\) to \(s\) in \(\ell^1_{\Phi_Y}(\mathcal{Y})\) it follows that

\[
K_X := \sup_{k \in \mathbb{N}}(\Phi_X, r_k) < \infty \quad \text{and} \quad K_Y := \sup_{k \in \mathbb{N}}(\Phi_Y, s_k) < \infty
\]

Let us denote by \((\tilde{\alpha}^\lambda_k, \tilde{\beta}^\lambda_k)_{k \in \mathbb{N}}\) the optimal entropic dual potentials for probability measures \((r_k, s_k)_{k \in \mathbb{N}}\) which satisfy \(\langle \tilde{\alpha}^\lambda_k, r_k \rangle = \langle \tilde{\beta}^\lambda_k, s_k \rangle\). By Proposition 2.3 we then infer for all \(k \in \mathbb{N}\) and \(x \in \text{supp}(r_k), y \in \text{supp}(s_k)\) that

\[
c^\lambda_X(x) - K_X - K_Y \leq \tilde{\alpha}^\lambda_{k,x} \leq c^\lambda_X(x) + K_Y, \\
c^\lambda_Y(y) - K_X - K_Y \leq \tilde{\beta}^\lambda_{k,y} \leq c^\lambda_Y(y) + K_X.
\]

Note for all \(k \in \mathbb{N}\) that \((\tilde{\alpha}^\lambda_k + \beta^\lambda_{k,y_1} - \beta^\lambda_{k,y_1})\) is \((r_k, s_k)\)-almost surely equal to \((\alpha^\lambda_k, \beta^\lambda_k)\). Moreover, since \(K_\beta := \sup_{k \in \mathbb{N}}(\beta^\lambda_{k,y_1}) < \infty\), this leads for all \(k \in \mathbb{N}, x \in \text{supp}(r_k), y \in \text{supp}(s_k)\) to the following bounds

\[
c^\lambda_X(x) - K_X - K_Y - K_\beta \leq \alpha^\lambda_{k,x} \leq c^\lambda_X(x) + K_Y + K_\beta, \\
c^\lambda_Y(y) - K_X - K_Y - K_\beta \leq \beta^\lambda_{k,y} \leq c^\lambda_Y(y) + K_X + K_\beta. \tag{A.5}
\]

A diagonalization argument proves existence of a subsequence \((\alpha^\lambda_{k_m}, \beta^\lambda_{k_m})_{m \in \mathbb{N}}\) converging pointwise for each \(x \in \text{supp}(r)\) and \(y \in \text{supp}(s)\) to a limit \((\alpha^\lambda_\infty, \beta^\lambda_\infty) \in \mathbb{R}^X \times \mathbb{R}^Y\). It remains to show that \((\alpha^\lambda_\infty, \beta^\lambda_\infty)\) is an optimizer for (DEROT) for probability measures \(r\) and \(s\). Uniqueness up to constant for each \((x, y) \in \text{supp}(r \otimes s)\) and \(\beta^\lambda_\infty_{y_1} = 0\) then implies
\((\alpha^\alpha, \beta^\alpha) = (\alpha^\beta, \beta^\beta)\). Relabelling, we may assume that the sequence \((\alpha^\alpha, \beta^\alpha)\) already converges pointwise. This leads for each \(x \in \text{supp}(r), y \in \text{supp}(s)\) to

\[
\exp \left( \frac{-\alpha^\alpha}{\lambda} \right) = \lim_{k \to \infty} \exp \left( \frac{-\alpha^\alpha_k}{\lambda} \right) = \lim_{k \to \infty} \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) s_{k,y},
\]

\[
\exp \left( \frac{-\beta^\beta}{\lambda} \right) = \lim_{k \to \infty} \exp \left( \frac{-\beta^\beta_k}{\lambda} \right) = \lim_{k \to \infty} \sum_{x \in \mathcal{X}} \exp \left( \frac{\alpha^\beta_k - c(x, y)}{\lambda} \right) r_{k,x}.
\]

Once we show that the limit expression and the sum on the r.h.s. can be interchanged optimality of \((\alpha^\alpha, \beta^\alpha)\) follows by condition (A.4). Based on bound (A.5) it follows for all \(y \in \text{supp}(s)\) and \(k \in \mathbb{N}\) that

\[
\exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) \leq \exp \left( \frac{c^\alpha_k(y) - c^\beta_k(y)}{\lambda} + K_x + K_{\beta} \right) . \tag{A.6}
\]

Convergence of \((s_k)_{k \in \mathbb{N}}\) in \(\ell^1_{f,\mathcal{X}}(\mathcal{Y})\) implies that there exists for all \(\varepsilon > 0\) a finite set \(S_\varepsilon \subset \mathcal{Y}\) such that for all \(k \in \mathbb{N}\) holds

\[
\sum_{y \in S_\varepsilon} \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) s_{k,y} \leq \frac{\varepsilon}{4}, \quad \sum_{y \notin S_\varepsilon} \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) s_{k,y} \leq \frac{\varepsilon}{4}.
\]

Further, by pointwise convergence of \(\beta^\alpha_k\) there exists \(N_1 \in \mathbb{N}\) such that it holds for all \(k \geq N_1\) and \(y \in S_\varepsilon\) that

\[
\left| \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) - \exp \left( \frac{\beta^\alpha_{\infty} - c(x, y)}{\lambda} \right) \right| \leq \frac{\varepsilon}{4}.
\]

Moreover, by (A.6) there exists \(N_2 \in \mathbb{N}\) such that for all \(k \geq N_2\) follows

\[
\left| \sum_{y \notin S_\varepsilon} \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) (s_{k,y} - s_y) \right| \leq \frac{\varepsilon}{4}.
\]

These previous four inequalities yield for \(k \geq \max\{N_1, N_2\}\) that

\[
\left| \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) s_{k,y} - \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\alpha_{\infty} - c(x, y)}{\lambda} \right) s_y \right| \leq \varepsilon.
\]

As \(\varepsilon > 0\) can be chosen arbitrarily small, it holds for all \(x \in \text{supp}(r)\) that

\[
\exp \left( \frac{-\alpha^\alpha}{\lambda} \right) = \lim_{k \to \infty} \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\alpha_k - c(x, y)}{\lambda} \right) s_{k,y} = \sum_{y \in \mathcal{Y}} \exp \left( \frac{\beta^\alpha_{\infty} - c(x, y)}{\lambda} \right) s_y.
\]

Likewise, it follows by an analogous argument for all \(y \in \text{supp}(s)\) that

\[
\exp \left( \frac{-\beta^\beta}{\lambda} \right) = \lim_{k \to \infty} \sum_{x \in \mathcal{X}} \exp \left( \frac{\alpha^\beta_k - c(x, y)}{\lambda} \right) r_{k,x} = \sum_{x \in \mathcal{X}} \exp \left( \frac{\alpha^\beta_{\infty} - c(x, y)}{\lambda} \right) r_x,
\]

which finishes the proof for pointwise convergence of dual optimizers. Hence, we obtain in conjunction with our optimality criterion (2.1) the asserted convergence result for the EROT value as \(k\) tends to infinity, i.e.,

\[
|\text{EROT}^\lambda(r_k, s_k) - \text{EROT}^\lambda(r, s)| = |\langle \alpha^\lambda_k, r_k \rangle + \langle \beta^\lambda_k, s_k \rangle - \langle \alpha^\lambda, r \rangle - \langle \beta^\lambda, r \rangle|
\]

\[
\leq |\langle \alpha^\lambda_k, r_k - r \rangle + \langle \beta^\lambda_k, s_k - s \rangle| + |\langle \alpha^\lambda - \alpha^\lambda, r \rangle + \langle \beta^\lambda - \beta^\lambda, r \rangle| \to 0.
\]
Herein, the first term of the second line tends to zero by our bounds on $(\alpha_k^1, \beta_k^1)_{k \in \mathbb{N}}$ from (A.5) in conjunction with the type of convergence of $(r_k, s_k)$ towards $(r, s)$, whereas the second term converges to zero by dominated convergence. The assertion on the convergence for the EROT plan follows by an analogous argument and explicitly uses that $\Phi_X = C_X \phi_X$ as well as $\Phi_Y = C_Y \phi_Y$.

## B Lemmas for Sensitivity Analysis

For the proof of Lemma B.3 we employ the following result on invertibility of the operator $\mathcal{F}_{\pi, \alpha, \beta}((\varrho(r,s),r,s)) \mathcal{F}_{\pi, \alpha, \beta}^{-1}$ and the norm of the resulting element.

**Lemma B.1.** Let $r \in \ell_{C_X}^1(\mathcal{X})$ and $s \in \ell_{\psi}^1(\mathcal{Y})$ be two probability measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively, with full support and consider a monotone, possibly unbounded, function $\psi : \mathbb{N} \to [1, \infty)$ such that $\sum_{i=1}^{\infty} \psi(i) \Phi^i_y(y_i) s_{y_i} < \infty$. Further, define the function

$$
\Gamma : \mathbb{R}^{\mathcal{X} \times \mathcal{Y}} \to [0, \infty], \quad \xi \mapsto \max \left( \sup_{x \in \mathcal{X}, y \in \mathcal{Y}} \frac{|\xi_{xy}|}{s_{y_i} \psi(i) \Phi^i_y(y_i)}, \sup_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{|\xi_{xy}|}{s_{y_i} \psi(i) \Phi^i_y(y_i)} \right)
$$

and consider an element $\xi \in \ell_{C_X \otimes C_Y}^1(\mathcal{X} \times \mathcal{Y})$. Then applying the operator $\mathcal{F}_{\pi, \alpha, \beta}((\varrho(r,s),r,s)) \mathcal{F}_{\pi, \alpha, \beta}^{-1}$ onto $(\xi, 0, 0) \in \mathbb{R}^{\mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \setminus \{y_1\}}$ is well-defined and gives an element in $\ell_{C_X \otimes C_Y}^1(\mathcal{X} \times \mathcal{Y})$. In particular, there exists $\kappa > 0$ which is independent from $\xi$ such that

$$
\left\| \mathcal{F}_{\pi, \alpha, \beta}((\varrho(r,s),r,s)) \mathcal{F}_{\pi, \alpha, \beta}^{-1}(\xi, 0, 0) \right\|_{\ell_{C_X \otimes C_Y}^1(\mathcal{X} \times \mathcal{Y})} \leq \kappa \Gamma(\xi).
$$

**Proof.** We will show that there exists a unique triplet $(\zeta_{\mathcal{X} \times \mathcal{Y}}, \zeta_{\mathcal{X} \times \mathcal{Y}}^{\infty}, \zeta_{\mathcal{Y} \times \mathcal{Y}}^{\infty}) \in \ell_{C_X \otimes C_Y}^1(\mathcal{X} \times \mathcal{Y}) \times \ell_{\psi}^1(\mathcal{Y}) \times \ell_{\psi}^1(\mathcal{Y} \setminus \{y_1\})$ such that

$$
\begin{pmatrix}
\zeta_{\mathcal{X} \times \mathcal{Y}} - \frac{\pi^1}{\lambda} \otimes A^T \left(\zeta_{\mathcal{X} \times \mathcal{Y}}^{\infty}, \zeta_{\mathcal{Y} \times \mathcal{Y}}^{\infty}\right)
\end{pmatrix}
= \begin{pmatrix}
\xi \\
0 \\
0
\end{pmatrix},
$$

where we exploited on the l.h.s. the relation between primal and dual optimizers of (EROT) (Proposition 2.1). By setting

$$
\zeta_{\mathcal{X} \times \mathcal{Y}} := \xi + \frac{\pi^1}{\lambda} \otimes A^T \left(\zeta_{\mathcal{X} \times \mathcal{Y}}^{\infty}, \zeta_{\mathcal{Y} \times \mathcal{Y}}^{\infty}\right)
$$

we reduce the system of countably many equalities. Then it remains to find $(\zeta_{\mathcal{X} \times \mathcal{Y}}^{\infty}, \zeta_{\mathcal{Y} \times \mathcal{Y}}^{\infty}) \in \ell_{\psi}^1(\mathcal{X}) \times \ell_{\psi}^1(\mathcal{Y} \setminus \{y_1\})$ such that

$$
A^\ast \left(\frac{\pi^1}{\lambda} \otimes A^T \left(\zeta_{\mathcal{X} \times \mathcal{Y}}^{\infty}, \zeta_{\mathcal{Y} \times \mathcal{Y}}^{\infty}\right)\right) = -A^\ast(\xi).
$$

This relation means that for all $x \in \mathcal{X}$ the following equation is valid

$$
\frac{\pi^1_{x y_1}}{\lambda} \xi_{x, y_1} + \sum_{y \in \mathcal{Y} \setminus \{y_1\}} \frac{\pi^1_{x y}}{\lambda} (\zeta_{x, y}^{\mathcal{X} \times \mathcal{Y}} + \zeta_{x, y}^{\mathcal{Y} \times \mathcal{Y}}) = -\sum_{y \in \mathcal{Y}} \xi_{x y}.
$$
with different domain and range. Notably, by our assumption that

\[ \pi \text{ solution} \]

which is equivalent to

\[ \zeta_{s_{x},y}^{X,\infty} + \sum_{y \in \mathcal{Y} \setminus \{y_1\}} \frac{\pi_{xy}^{Y}}{\lambda} \zeta_{s_{y},y}^{Y,\infty} = -\sum_{y \in \mathcal{Y}} \xi_{xy}, \]  

(B.2)

Similarly, we obtain for each \( y \in \mathcal{Y} \setminus \{y_1\} \) that

\[ \sum_{x \in \mathcal{X}} \frac{\pi_{xy}^{Y}}{\lambda} \zeta_{s_{x},y}^{X,\infty} = \sum_{x \in \mathcal{X}} \xi_{xy}, \]

which implies that

\[ \left( \sum_{x \in \mathcal{X}} \frac{\pi_{xy}^{Y}}{s_{y}} \right) \zeta_{s_{y},y}^{Y,\infty} = -\frac{\lambda}{s_{y}} \sum_{x \in \mathcal{X}} \xi_{xy}, \]  

(B.3)

The equalities (B.2) and (B.3) can therefore be represented by

\[ M(\zeta_{s_{x},y}^{X,\infty}, \zeta_{s_{y},y}^{Y,\infty}) = -\lambda \bar{\xi}, \]  

(B.4)

where \( M: \ell^{\infty}(\mathcal{X}) \times \ell^{\infty}_{\psi \phi \mathcal{Y}}(\mathcal{Y} \setminus \{y_1\}) \rightarrow \ell^{\infty}(\mathcal{X}) \times \ell^{\infty}_{\psi \phi \mathcal{Y}}(\mathcal{Y} \setminus \{y_1\}) \) denotes the operator from the proof of Proposition 5.2 with different domain and range. Notably, by our assumption \( \Gamma(\xi) < \infty \) it follows that

\[ \left( \sum_{y \in \mathcal{Y}} \frac{\xi_{xy}}{r_{x}}, \sum_{y \in \mathcal{Y}} \frac{\xi_{xy}}{r_{x}}, \ldots, \sum_{x \in \mathcal{X}} \frac{\xi_{xy}}{s_{y}}, \sum_{x \in \mathcal{X}} \frac{\xi_{xy}}{s_{y}}, \ldots \right) \in \ell^{\infty}(\mathcal{X}) \times \ell^{\infty}_{\psi \phi \mathcal{Y}}(\mathcal{Y} \setminus \{y_1\}). \]

Moreover, since \( \sum_{i=1}^{\infty} \psi(i) \phi_{3}^{3}(y_i) s_{y_i} < \infty \), it follows by a similar argument as in the proof of Proposition 5.2 that \( M \) has a bounded inverse operator and thus there exists a unique solution \( (\zeta_{s_{x},y}^{X,\infty}, \zeta_{s_{y},y}^{Y,\infty}) \in \ell^{\infty}(\mathcal{X}) \times \ell^{\infty}_{\psi \phi \mathcal{Y}}(\mathcal{Y} \setminus \{y_1\}) \) for equation (B.4). Hence, we obtain that

\[ \left\| (\zeta_{s_{x},y}^{X,\infty}, \zeta_{s_{y},y}^{Y,\infty}) \right\|_{\ell^{\infty}(\mathcal{X}) \times \ell^{\infty}_{\psi \phi \mathcal{Y}}(\mathcal{Y} \setminus \{y_1\})} \leq \left\| M^{-1} \right\|_{OP} \lambda \Gamma(\xi). \]

Finally, it remains to show that \( \psi_{\times \mathcal{Y}} \), as defined in (B.1), is contained in \( \ell_{C_{X} \oplus C_{Y}}^{1}(\mathcal{X} \times \mathcal{Y}) \). Exploiting the upper bound for \( \pi^{Y} \) (Proposition 2.3) yields that

\[ \left\| \psi_{\times \mathcal{Y}} \right\|_{\ell_{C_{X} \oplus C_{Y}}^{1}(\mathcal{X} \times \mathcal{Y})} \leq \left\| \psi_{\times \mathcal{Y}} \right\|_{\ell_{C_{X} \oplus C_{Y}}^{1}(\mathcal{X} \times \mathcal{Y})} + \left\| \pi^{Y} \circ A_{1}^{*} \left( (\zeta_{s_{x},y}^{X,\infty}, \zeta_{s_{y},y}^{Y,\infty}) \right) \right\|_{\ell_{C_{X} \oplus C_{Y}}^{1}(\mathcal{X} \times \mathcal{Y})} \]

\[ \leq \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} C_{X}(x) |\psi_{xy}^{X} x| + \sum_{(y,y) \in \mathcal{Y} \times \mathcal{Y}} C_{Y}(y) |\psi_{xy}^{Y} y| \]

\[ + \left\| \pi^{Y} \right\|_{\ell_{C_{X} \oplus C_{Y}}^{1}(\mathcal{X} \times \mathcal{Y})} \cdot 2 \left\| (\zeta_{s_{x},y}^{X,\infty}, \zeta_{s_{y},y}^{Y,\infty}) \right\|_{\ell^{\infty}(\mathcal{X}) \times \ell^{\infty}_{\psi \phi \mathcal{Y}}(\mathcal{Y} \setminus \{y_1\})} \]

\[ \leq \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} C_{X}(x) \frac{|\psi_{xy}^{X} x|}{r_{x}} + \sum_{(y,y) \in \mathcal{Y} \times \mathcal{Y}} \psi(i) C_{Y}(y_i) \phi_{3}^{3}(y_i) s_{y_i} \frac{|\xi_{xy} y|}{\psi(i) \phi_{3}^{3}(y_i) s_{y_i}}. \]
\begin{align*}
+ \|r\|_{\ell^2_{\mathcal{C}}(\chi)} \cdot \|s\|_{\ell^2_{\mathcal{Y}}(\gamma)} \exp\left(\frac{(c^*_X, r) + (c^*_Y, s) + \|c^*_X - c^*_Y\|_{\ell^\infty(\chi)}}{\lambda} \right) 2\|M^{-1}\|_{\mathcal{O}_P} \lambda \Gamma(\xi)
\leq \Gamma(\xi) \left(\|r\|_{\ell^2_{\mathcal{C}}(\chi)} + \|s\|_{\ell^2_{\mathcal{Y}}(\gamma)} + \|r\|_{\ell^2_{\mathcal{C}}(\chi)} \cdot \|s\|_{\ell^2_{\mathcal{Y}}(\gamma)} \cdot \exp\left(\frac{(c^*_X, r) + (c^*_Y, s) + \|c^*_X - c^*_Y\|_{\ell^\infty(\chi)}}{\lambda} \right) 2\|M^{-1}\|_{\mathcal{O}_P} \lambda \right) =: \kappa \Gamma(\xi) < \infty,
\end{align*}

which proves that the claim.

\hfill \Box

**Remark B.2.** We like to discuss the domain of the operator \([\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}]^{-1} - \mathcal{F}(r,s)\). For an element \(\xi \in \ell^4_{\mathcal{C}}(\chi \times Y)\) the previous proof shows that the element \((\xi,0,0)\) \(\in \ell^4_{\mathcal{C}}(\chi \times Y) \times \ell^\infty(\chi) \times \ell^\infty(\gamma)\) has a well-defined image under the mapping \([\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}]^{-1}\) in \(\ell^4_{\mathcal{C}}(\chi \times Y) \times \ell^\infty(\chi) \times \ell^\infty(\gamma)\) if and only if

\[
\left(\sum_{y \in Y} \xi_{xy_1} r_{y_1}, \sum_{y \in Y} \xi_{xy_2} r_{y_2}, \ldots, \sum_{x \in \chi} \xi_{x\gamma_1} s_{\gamma_1}, \sum_{x \in \chi} \xi_{x\gamma_2} s_{\gamma_2}, \ldots\right) \in \ell^\infty(\chi) \times \ell^\infty(\gamma)\). \tag{B.5}
\]

In particular, by Lemma B.4 one can construct for any \(\varepsilon > 0\) and \(\xi\) satisfying (B.5) an element \(\xi' \in \ell^4_{\mathcal{C}}(\chi \times Y)\) such that \(\|\xi - \xi'\|_{\ell^4_{\mathcal{C}}(\chi \times Y)} < \varepsilon\) but where \(\xi'\) does not fulfill (B.5). Hence, the domain of \([\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}]^{-1}\) does not even contain an open ball in \(\ell^4_{\mathcal{C}}(\chi \times Y)\).

We now proceed with the proof of the error bounds from Proposition 5.4.

**Lemma B.3.** Assume the same setting as in Proposition 5.4 and denote by \(\theta\) the mapping as defined in its proof. Then it follows for \(n \to \infty\) that

\[
\left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r,s)\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
+ \left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
+ \left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
\] \tag{B.6}

\[
- \mathcal{F}(\theta(r,s),r + \gamma h_1^X, s + \gamma h_1^Y) \right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
+ \left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
+ \left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
\] \tag{B.7}

\[
\left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
+ \left\|\left[\mathcal{D}_{\pi,\alpha,\beta,\theta}(r,s),r,s)\mathcal{F}\right]^{-1}_n \left(\mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right) - \mathcal{F}(r,s),r + \gamma h_1^X, s + \gamma h_1^Y\right\|_{\ell^4_{\mathcal{C}}(\chi \times Y)}
\] \tag{B.8}

47
Further, consider the closed ball \( B \) such that optimal entropic dual potentials for optimizers are Lipschitz (Proposition 5.3) and let \( n \) be sufficiently large such that \((r + t_n \hat{h}_l^X, s + t_n \hat{h}_l^Y) \in B_{\rho_0}(r, s)\) (Lemma C.2). This guarantees that dual optimizers are Lipschitz (Proposition 5.3). A straightforward calculation yields

\[
\left(\mathcal{D}_{\pi,\alpha,\beta,|\varrho(r(s), r(s))\mathcal{F}}\right)^{-1}(\xi_1(n), 0, 0) \bigg|_{\ell^1_{\mathcal{X} \oplus \mathcal{Y}}(X \times Y)} = o(t_n).
\]

\[
\left(\mathcal{D}_{\pi,\alpha,\beta,|\varrho(r(s), r(s))\mathcal{F}}\right)^{-1}(\xi_2(n), 0, 0) \bigg|_{\ell^1_{\mathcal{X} \oplus \mathcal{Y}}(X \times Y)} = o(t_n).
\]

where \( \Delta \) is given by \( \Delta: \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto \exp(x) - 1 - x \) and is evaluated component-wise for each entry of \( A^2_\lambda(\alpha_n^\lambda - \alpha^\lambda, \beta_n^\lambda - \beta^\lambda) \). The function \( \Delta \) is smooth, non-negative and its derivative vanishes at zero, hence, for a positive null-sequence \( x_n \to 0 \) it follows that \( \Delta(x_n) = o(x_n) \). In addition, for each \( x \geq 0 \) the inequality \( \Delta(-x) \leq \Delta(x) \) is satisfied. Before we apply Lemma B.1 let us state a component-wise bound for \( \xi_2(n) \). For this purpose, let

\[
\kappa := \exp\left(\lambda^{-1}\sup_{n \in \mathbb{N}}(c_\mathcal{X}^+, r + t_n \hat{h}_l^X) + \lambda^{-1}\sup_{n \in \mathbb{N}}(c_\mathcal{Y}^+, s + t_n \hat{h}_l^Y)\right) < \infty
\]
and recall by our bounds from Proposition 2.3 that \( \exp \left( \lambda^{-1} \left[ A^T_x(\alpha^\lambda, \beta^\lambda) - c \right] \right) \leq \kappa(\phi_X \otimes \phi_Y) \). Hence, for all \( n \in \mathbb{N} \) it holds that

\[
|\xi_2(n)| = \left| (r + t_n \hat{h}^X_t) \otimes (s + t_n \hat{h}^Y_t) \right| \exp \left( \frac{A^T_x(\alpha^\lambda, \beta^\lambda) - c}{\lambda} \right)
\]

\[
\otimes \left| \Delta \left( \frac{A^T_x(\alpha^\lambda_n - \alpha^\lambda, \beta^\lambda_n - \beta^\lambda)}{\lambda} \right) \right|
\]

\[
\leq \left| (r + t_n \hat{h}^X_t) \otimes (s + t_n \hat{h}^Y_t) \right| \otimes \kappa(\phi_X \otimes \phi_Y)
\]

\[
\otimes \Delta \left( \frac{A^T_x(\alpha^\lambda_n - \alpha^\lambda, |\beta^\lambda_n - \beta^\lambda|)}{\lambda} \right)
\]

\[
\leq \left| (r + t_n \hat{h}^X_t) \otimes (s + t_n \hat{h}^Y_t) \right| \otimes \kappa(\phi_X \otimes \phi_Y)
\]

\[
\otimes A^T_x \left( \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda}{\lambda} \right), \kappa' \phi_Y |\beta^\lambda_n - \beta^\lambda| \right)
\]

(B.9)

(B.10)

for some constant \( \kappa' > 0 \). Herein, we use for the second inequality that \( \Delta \) is Lipschitz with modulus \( \exp(t) \) on the bounded domain \([-t, t]\) for \( t \geq 0 \) in conjunction with our bounds for \(|\alpha^\lambda_n - \alpha^\lambda|\), \(|\beta^\lambda_n - \beta^\lambda|\) (Proposition 2.3), which assert for \((x, y) \in \mathcal{X} \times \mathcal{Y} \{y_1\})\) that

\[
\left| \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda}{\lambda} + |\beta^\lambda_n - \beta^\lambda| \right) - \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda}{\lambda} \right) \right| \leq \kappa' \phi_Y |\beta^\lambda_n - \beta^\lambda|.
\]

(B.11)

Notably, based on (B.10) we infer by \((r, s) \in \ell^1_{\mathcal{C}_X}(\mathcal{X}) \otimes \ell^1_{\mathcal{C}_Y}(\mathcal{Y})\) that \(\xi_2(n) \in \ell^1_{\mathcal{C}_X \otimes \mathcal{C}_Y}(\mathcal{X} \times \mathcal{Y})\).

In order to apply Lemma B.1 let us consider a monotonous unbounded function \(\psi : \mathbb{N} \to [1, \infty)\) such that \(\sum_{i=1}^\infty \psi(i) \phi^i (y_i) s_{y_i} < \infty\) (Lemma B.4). For this function \(\psi\) we now show that \(\Gamma(\xi_2(n)) = o(t_n)\), so let \(I \in \mathbb{N}\) and consider \(x \in \mathcal{X}\). Then it holds that

\[
\sum_{y \in Y} \frac{|r_x + t_n \hat{h}^X_t(x) \cdot s_y + t_n \hat{h}^Y_t(y)|}{r_x} \exp \left( \frac{\alpha^\lambda_n + \beta^\lambda y - c(x, y)}{\lambda} \right) \left| \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda + \beta^\lambda y - \beta^\lambda}{\lambda} \right) \right|
\]

\[
\leq \sum_{i=1}^I \frac{|r_x + t_n \hat{h}^X_t(x) \cdot s_y + t_n \hat{h}^Y_t(y)|}{r_x} \kappa \Phi_X(x) \Phi_Y(y) \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda + \beta^\lambda y - \beta^\lambda}{\lambda} \right)
\]

\[
+ \sum_{i=I+1}^\infty \frac{|r_x + t_n \hat{h}^X_t(x) \cdot s_y + t_n \hat{h}^Y_t(y)|}{r_x} \kappa \Phi_X(x) \Phi_Y(y) \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda + \beta^\lambda y - \beta^\lambda}{\lambda} \right)
\]

\[
\cdot \left( \left| \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda}{\lambda} \right) \right| + \left| \frac{\beta^\lambda_n - \beta^\lambda}{\psi(y_i) \Phi_Y(y_i) \lambda} \right| \right)
\]

\[
\leq \left( 1 + t_n \max_{x \in \{x_1, \ldots, x_I\}} \left| \hat{h}^X_t \right| \right) \kappa \| \Phi_X \|_{\ell^\infty(\mathcal{X})} \left\| s + t_n \hat{h}^Y_t \right\|_{\ell^1_{\psi^\infty(\mathcal{Y})}}
\]

\[
\cdot \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda}{\lambda} \right) + \max_{i=1}^I \left| \beta^\lambda_n - \beta^\lambda \right|
\]

\[
+ \left( 1 + t_n \max_{x \in \{x_1, \ldots, x_I\}} \left| \hat{h}^X_t \right| \right) \kappa \kappa' \Phi_X \|_{\ell^\infty(\mathcal{X})} \left\| s + t_n \hat{h}^Y_t \right\|_{\ell^1_{\psi^\infty(\mathcal{Y})}} \Delta \left( \frac{\alpha^\lambda_n - \alpha^\lambda}{\lambda} \right)
\]

\[
+ \left( 1 + t_n \max_{x \in \{x_1, \ldots, x_I\}} \left| \hat{h}^X_t \right| \right) \kappa \kappa' \Phi_X \|_{\ell^\infty(\mathcal{X})} \left\| s + t_n \hat{h}^Y_t \right\|_{\ell^1_{\psi^\infty(\mathcal{Y})}} \psi(I) \lambda
\]
\[\kappa'' \left(\lambda^{-1} \left( \|\alpha_n^\lambda - \alpha^\lambda\|_{\ell^\infty(x)} + \max_{i=1}^I |\beta_{n,y_i}^\lambda - \beta_{y_i}^\lambda| \right) + \psi(I)^{-1} \|\beta_n^\lambda - \beta^\lambda\|_{\ell^\infty(Y)} \right)\]

for some \(\kappa'' > 0\) that is independent of \(I\) and \(x \in X\). By the local Lipschitz property (Proposition 5.3) we know that

\[
\left\|\alpha_n^\lambda - \alpha^\lambda\right\|_{\ell^\infty(x)} \leq t_n A' \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)},
\]

\[
\left\|\beta_n^\lambda - \beta^\lambda\right\|_{\ell^\infty(Y)} \leq t_n A' \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)}.
\]

Hence, for \(\varepsilon > 0\) select \(I \in \mathbb{N}\) large enough such that

\[
\kappa'' \frac{\left\|\beta_n^\lambda - \beta^\lambda\right\|_{\ell^\infty(Y)}}{\psi(I)} \leq \psi(I)^{-1} t_n \kappa'' A' \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)} < \frac{\varepsilon t_n}{2},
\]

and choose \(N \in \mathbb{N}\) sufficiently large such that for all \(n \geq N\) holds

\[
\kappa'' \Delta \left(\lambda^{-1} \left( \|\alpha_n^\lambda - \alpha^\lambda\|_{\ell^\infty(x)} + \max_{i=1}^I |\beta_{n,y_i}^\lambda - \beta_{y_i}^\lambda| \right) \right)
\]

\[
\leq \kappa'' \Delta \left( t_n \lambda^{-1} \left( 1 + \max_{i=1}^I \phi \chi(i) \right) A' \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)} \right) < \frac{\varepsilon t_n}{2}.
\]

Consequently, we obtain that \(\sup_{x \in X} \sum_{y \in Y} |\xi_2(n)_{xy}|/r_x < \varepsilon t_n\) and thus it holds that \(\sup_{x \in X} \sum_{y \in Y} |\xi_2(n)_{xy}|/r_x = o(t_n)\). To show that \(I(\xi_2(n)) = o(t_n)\) let us again consider some integer \(I \in \mathbb{N}\). For \(y_i \notin \{y_1, \ldots, y_I\}\) we employ the upper bound (B.10) and see for some \(\kappa'''' > 0\) independent from \(y_i\) and \(I\) that

\[
\sum_{x \in X} \frac{|r_x + t_n \hat{h}_{1,i,x}^X \cdot s_{y_i} + t_n \hat{h}_{1,i,y_i}^Y|}{s_{y_i} \psi(i) \phi^\lambda \phi^\lambda(y_i)} \exp \left( \frac{\alpha_n^\lambda + \beta_n^\lambda - c(x, y_i)}{\lambda} \right) \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)} \lambda < \varepsilon t_n.
\]

Note that the first four factors can be bounded by a constant independent of \(I\) and \(y \in Y\). For the last factor in (B.12) we employ the local Lipschitz property of dual optimizers with modulus \(\Lambda'\) (Proposition 5.3) and obtain

\[
\frac{|\beta_{y_i}^\lambda - \beta_{y_i}^\lambda|}{\psi(i) \phi^\lambda \phi^\lambda(y_i)} \leq \psi(i)^{-1} t_n A' \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)}.
\]

Hence, for \(\varepsilon > 0\) we may choose \(I\) sufficiently large such that

\[
\psi(i)^{-1} t_n A' \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)} < \varepsilon t_n \quad \text{for all } i > I.
\]

For each \(y_i \in \{y_1, \ldots, y_I\}\) we then use the upper bound from (B.9) in conjunction with a similar bound as in (B.11) and obtain for some \(\kappa'''''' > 0\) independent from \(y_i\) and \(I\) that

\[
\sum_{x \in X} \frac{|r_x + t_n \hat{h}_{1,i,x}^X \cdot s_{y_i} + t_n \hat{h}_{1,i,y_i}^Y|}{s_{y_i} \psi(i) \phi^\lambda \phi^\lambda(y_i)} \exp \left( \frac{\alpha_n^\lambda + \beta_n^\lambda - c(x, y_i)}{\lambda} \right) \left\|\left(\hat{h}_{1,i}^X, \hat{h}_{1,i}^Y\right)\right\|_{\ell_{\Omega}^1(x) \times \ell_{\Omega}^2(Y)} \lambda < \varepsilon t_n.
\]
Here we exploited the local Lipschitz property of $\beta^\lambda$ for the components $y_i \in \{y_1, \ldots, y_l\}$. As a result we obtain by (B.13) and (B.14) that $\Gamma(\xi_2(n)) = o(t_n)$. Hence, Lemma B.1 implies that

$$
\left\| \left[ D_{\pi,\beta_i(\theta(r_{s*}),r_{s*})} \right] \right\|_{C_X \oplus C_Y} = o(t_n).
$$

For the last term, i.e., for (B.8) we obtain

$$
\left[ D_{\pi,\beta_i(\theta(r_{s*}),r_{s*})} \right] \left( \theta(r + t_n h^X_i, s_{s*} + t_n h^Y_i) - \theta(r, s_{s*}) \right)
$$

$$
- \left[ D_{\pi,\beta_i(\theta(r_{s*}),r + t_n h^Y_i, s_{s*} + t_n h^Y_i)} \right] \left( \theta(r + t_n h^X_i, s_{s*} + t_n h^Y_i) - \theta(r, s_{s*}) \right)
$$

$$
= \begin{pmatrix}
-t_n h^X_i \otimes s + t_n r \otimes h^Y_i + t_n h^X_i \otimes h^Y_i \otimes \exp \left( \frac{A_T(\alpha^\lambda - \beta^\lambda)}{\lambda} \right) \otimes \frac{A_T(\alpha^\lambda - \alpha^\lambda, \beta^\lambda - \beta^\lambda)}{\lambda} \\
0 \\
0
\end{pmatrix}
$$

Since $(r, s) \in \ell^1_{C_X}(\mathcal{X}) \otimes \ell^1_{C_Y}(\mathcal{Y})$ and as $h^X_i \otimes h^Y_i$ both have finite support, we obtain that $\xi_3(n) \in \ell^1_{C_X \oplus C_Y}(\mathcal{X} \times \mathcal{Y})$. To finish the proof, we show that $\Gamma(\xi_3(n)) = o(t_n)$, where we select $\psi \equiv 1$. For any $x \in \mathcal{X}$ it then follows using the Lipschitz property for dual optimizers (Proposition 5.3) that

$$
\frac{1}{r_n} \sum_{y \in \mathcal{Y}} \left| h^X_{i,x} s_y + r_x h^Y_i + t_n h^X_i h^Y_i \right| 
\cdot \exp \left( \frac{\alpha^\lambda - \beta^\lambda}{\lambda} \cdot c(x, y) \right) \left| \alpha^\lambda - \alpha^\lambda \right| \left| \beta^\lambda - \beta^\lambda \right|
\leq t_n \left( \|\alpha^\lambda - \alpha^\lambda\|_{\ell^\infty(\mathcal{X})} + \|\beta^\lambda - \beta^\lambda\|_{\ell^\infty(\mathcal{Y})} \right) \exp \left( \frac{\|c^\lambda - c^\lambda\|_{\ell^\infty(\mathcal{X})}}{\lambda} \right)
\cdot \left| \frac{1}{r_n} \sum_{x \in \mathcal{X}} \left| h^X_{i,x} \right| \sum_{y \in \mathcal{Y}} (s_y + (1 + t_n) |h^Y_i|) \phi^\lambda_\pi(y) \right|
\leq \mathcal{O} \left( t_n \left( \|\alpha^\lambda - \alpha^\lambda\|_{\ell^\infty(\mathcal{X})} + t_n \|\beta^\lambda - \beta^\lambda\|_{\ell^\infty(\mathcal{Y})} \right) \right) \leq \mathcal{O}(t_n^2) \leq o(t_n).
$$

Likewise, we see for all $y \in \mathcal{Y}$ that

$$
\frac{1}{s_y \phi^\lambda_\pi(y)} t_n \left( \sum_{y \in \mathcal{Y}} \left| h^X_{i,x} s_y + r_x h^Y_i + t_n h^X_i h^Y_i \right| 
\cdot \exp \left( \frac{\alpha^\lambda + \beta^\lambda - c(x, y)}{\lambda} \right) \left| \alpha^\lambda - \alpha^\lambda \right| \left| \beta^\lambda - \beta^\lambda \right| \right)
$$

$$
\leq \mathcal{O} \left( t_n \left( \|\alpha^\lambda - \alpha^\lambda\|_{\ell^\infty(\mathcal{X})} + t_n \|\beta^\lambda - \beta^\lambda\|_{\ell^\infty(\mathcal{Y})} \right) \right) \leq \mathcal{O}(t_n^2) \leq o(t_n).
$$

51
We define the function \( \Phi \). We then define the monotone function \( \sum \). In this section we show a variety of useful characteristics of finite support approximations. For Lemma C.1.

### Properties of Finite Support Approximation

In this section we show a variety of useful characteristics of finite support approximations. Based on the notation in the proof of Theorem 3.4, we consider a probability measure \( r \in \ell_{cX}(\mathcal{X}) \) on \( \mathcal{X} \) with full support. Further, let \( (t_n)_{n \in \mathbb{N}} \) be a sequence such that \( t_n \to 0 \) and \( (h_n^X)_{n \in \mathbb{N}} \in \ell_{cX}(\mathcal{X}) \) with limit \( h^X \) and that \( r + t_n h_n^X \in \mathcal{P}(\mathcal{X}) \cap \ell_{cY}(\mathcal{X}) \) for each \( n \in \mathbb{N} \). Additionally, for an integer \( l \geq 2 \) we denote the finite support approximations for \( h^X \) by \( \hat{h}_l^X \), i.e., labelling

\[
\hat{h}_l^X := \begin{cases} 
  h_{X,x}^X + \sum_{i=l+1}^{\infty} h_{x,i}^X & \text{if } x = x_1, \\
  h_x^X & \text{if } x \in \{x_2, \ldots, x_l\}, \\
  0 & \text{else.}
\end{cases}
\]

**Lemma C.1.** For \( \delta > 0 \) there exist \( l \in \mathbb{N} \) such that

\[
\| h^X - \hat{h}_l^X \|_{\ell_{cY}(\mathcal{X})} \leq \delta.
\]
Additionally, there is $N \in \mathbb{N}$ such that for all $n \geq N$ holds
\[
\| \hat{h}_n^X - h_n^X \|_{l^1_C(X)}(\alpha) \leq \delta, \quad \| h_n^X - \hat{h}_n^X \|_{l^1_C(X)}(\alpha) \leq \delta.
\]

Proof. Choose $l$ large enough such that $C_X(x_1) \sum_{i=l+1}^{\infty} C_X(x_i) |h_{x_i}^X| \leq \delta/6$. Then we see
\[
\| h^X - \hat{h}_l^X \|_{l^1_C(X)}(\alpha) \leq C_X(x_1) \sum_{i=l+1}^{\infty} |h_{x_i}^X| + \sum_{i=2}^{\infty} C_X(x_i) |h_{x_i}^X - h_{\hat{x}_i}^X| \leq \frac{\delta}{3},
\]
Let $N$ be large enough such that $C_X(x_1) \| h^X - h_n^X \|_{l^1_C(X)}(\alpha) \leq \delta/3$ for all $n \geq N$. This yields
\[
\| \hat{h}_l^X - \hat{h}_n^X \|_{l^1_C(X)}(\alpha) = C_X(x_1) \| h_{x_1}^X - h_{\hat{x}_1}^X + \sum_{i=2}^{\infty} (h_{x_i}^X - h_{\hat{x}_i}^X) \| + \sum_{i=2}^{l} C_X(x_i) |h_{x_i}^X - h_{\hat{x}_i}^X| \leq C_X(x_1) \sum_{x \in X} C_X(x)|h_x^X - h_{\hat{x}_i}^X| = C_X(x_1) \| h^X - h_n^X \|_{l^1_C(X)}(\alpha) \leq \frac{\delta}{3},
\]
which implies by triangle inequality
\[
\| \hat{h}_n^X - \hat{h}_n^X \|_{l^1_C(X)}(\alpha) \leq \| h_n^X - h^X \|_{l^1_C(X)}(\alpha) + \| h^X - \hat{h}_l^X \|_{l^1_C(X)}(\alpha) + \| \hat{h}_l^X - \hat{h}_n^X \|_{l^1_C(X)}(\alpha) \leq \delta,
\]
and thus finishes the proof.

Lemma C.2. It holds that
\[
K := \| h^X \|_{l^1_C(X)} + \sup_{n \in \mathbb{N}} \| h_n^X \|_{l^1_C(X)} + \sup_{n,l \in \mathbb{N}} \| \hat{h}_n^X \|_{l^1_C(X)} < \infty.
\]

Further, for $\rho > 0$ there is $N \in \mathbb{N}$ such that it follows for all $n \geq N$ and $l \in \mathbb{N}$ that
\[
r + t_n h_n^X, r + t_n \hat{h}_n^X, r + t_n \hat{h}_l^X \in \{ \tilde{r} \in \ell^1_C(X) : \| \tilde{r} - r \|_{l^1_C(X)} \leq \rho \}.
\]

Proof. By convergence of $h_n^X$ towards $h^X$ with respect to $\ell^1_C(X)$ we see that
\[
\| h^X \|_{l^1_C(X)} + \sup_{n \in \mathbb{N}} \| h_n^X \|_{l^1_C(X)} < \infty.
\]

Further, for given $l,n \in \mathbb{N}$ and it follows that $\| \hat{h}_n^X \|_{l^1_C(X)}(\alpha) \leq C_X(x_1) \| h_n^X \|_{l^1_C(X)}(\alpha)$, which yields that $K < \infty$. For $N \in \mathbb{N}$ large enough such that $t_n < \rho K^{-1}$ we conclude the second claim.

Lemma C.3. For given $l \in \mathbb{N}$ and a probability measure $r$ with $\text{supp}(r) = X$ there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$ it holds $r + t_n \hat{h}_l^X \in \mathcal{P}(X)$ and $r + t_n \hat{h}_n^X \in \mathcal{P}(X)$ with $\text{supp}(r + t_n \hat{h}_n^X) = \text{supp}(r + t_n \hat{h}_l^X) = X$.

Proof. First, we note that $\sum_{x \in X} h_{n,x}^X = 0$ for all $n \in \mathbb{N}$ implies $\sum_{x \in X} h_x^X = 0$. By construction of the finite support approximation, it also follows for any $l,n \in \mathbb{N}$ that $\sum_{x \in X} \hat{h}_{n,l,x}^X = \sum_{x \in X} h_{l,x}^X = 0$. Hence, it remains to show for given $l \in \mathbb{N}$ that there exists $N \in \mathbb{N}$ such that the elements $r + t_n \hat{h}_l^X, r + t_n \hat{h}_n^X$ are strictly positive in each entry. To this end, consider
$K > 0$ as in Lemma C.2 and choose $N \in \mathbb{N}$ large enough such that it holds for all $n \geq N$ that $t_n < (2K)^{-1} \min_{i=1,\ldots,l} \min(r_{x_i}, 1 - r_{x_i})$. This yields for any $i \in \{1, \ldots, l\}$ that

$$r_{x_i} + t_n \hat{h}_{i,x_i} = r_{x_i} + t_n K \cdot \frac{\hat{h}_{i,x_i}^X}{K} \begin{cases} \leq r_{x_i} + \frac{1}{2} \min(r_{x_i}, 1 - r_{x_i}) \leq \frac{1}{2} r_{x_i} + \frac{1}{2} < 1, \\ \geq r_{x_i} - \frac{1}{2} \min(r_{x_i}, 1 - r_{x_i}) \geq \frac{1}{2} r_{x_i} > 0. \end{cases}$$

Further, since $\hat{h}_x^X = 0$ for all $x \notin \{x_1, \ldots, x_l\}$ it follows for all $n \geq N$ that $r_x + t_n \hat{h}_x^X = r_x \in (0, 1)$, which shows that $r + t_n \hat{h}_x^X \in \mathcal{P}(\mathcal{X})$ and $\text{supp}(r + t_n \hat{h}_x^X) = \mathcal{X}$. The same argument implies for all $n \geq N$ that $r + t_n \hat{h}_{n,i}^X \in \mathcal{P}(\mathcal{X})$ and $\text{supp}(r + t_n \hat{h}_{n,i}^X) = \mathcal{X}$. \qed