THE DIMENSION OF DECOMPACTIFIED SPACETIME FROM STRING THEORY

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ABSTRACT
The implications of string theory for understanding the dimension of decompactified spacetime are discussed. Results from a computer model designed to simulate expansion of the early universe during the string dominated phase are presented. This model focuses on the effects of string winding modes on inflation and is based on the theory of random walks. We demonstrate that our decompactified four-dimensional spacetime can be explained by the proper choice of initial conditions.

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1. Introduction

In spite of extraordinary successes, traditional cosmology has left unanswered a number of fundamental questions and been plagued by potential inconsistencies. Arguably the most troubling problem is the pointlike initial singularity at the time of the big bang, \( t = 0 \). Almost equally distressing is the related prediction of infinite initial temperature. Frequently, the preceding problems are sidesteped by appeals to some future theory of quantum gravity. A classical theory, general relativity is expected to break down at small scales where quantum effects should dominate. Thus, the divergences predicted as \( t \to 0 \) by the standard Friedman-Robertson-Walker cosmology are expected to be artifacts of using a classical theory in a quantum regime. As this limit is approached, it is hoped that the proper theory of gravity would predict a small but non-singular universe, which would have no divergent physical quantities. Indeed, such an outcome can be seen as a test for any candidate theory of quantum gravity.

An equally compelling, albeit less common, open question in traditional cosmology is why we live in a four-dimensional universe. While many are content to insert the dimension of spacetime by hand, it would be more satisfying to explain its value.

We need no longer talk about quantum gravity as a distant dream; with the advent of string theory, we have a candidate theory of quantum gravity today and therefore an unrivaled potential tool for understanding cosmology. Conversely, cosmology provides a unique arena for testing string theory’s performance as a theory of quantum gravity. Since string theory may make qualitatively different predictions than point particle theories, we can hope that some of the consequences are observable and will lead to the first experimental (or at least observational) tests of string theory.

Indeed, using arguments based on the duality symmetry of string theory, the potential problem of an initial pointlike singularity is eliminated. This duality implies that the smallest possible radius of the universe has some non-zero value. This minimum radius correspond to the fixed-point of the duality transformation.

Duality is most easily demonstrated by considering the mode expansion\(^1\) of a compactified bosonic string coordinate:

\[
X = x + \left( \frac{m}{2R} + nR \right) (\tau + \sigma) + \left( \frac{m}{2R} - nR \right) (\tau - \sigma) + \text{oscillators},
\]

where \( m, n \in \mathbb{Z} \). (Note we have chosen \( \sqrt{\alpha'} = \sqrt{\frac{1}{2}} \) in units of the Planck length, \( l_{Pl} = \sqrt{\hbar G_N / c^3} \).) The left- and right-moving total momenta are,

\[
(p_L, p_R) = \left( \frac{m}{2R} + nR, \frac{m}{2R} - nR \right).
\]

The first term, \( \frac{m}{2R} \), is interpreted as one half the center of mass momentum of the string, while the second term, \( \pm nR \) is the winding mode “momentum.” The corresponding string mass spectrum is,

\[
\frac{1}{4} M^2 = N + \frac{1}{2} (\frac{m}{2R} - nR)^2 - 1 + \tilde{N} + \frac{1}{2} (\frac{m}{2R} + nR)^2 - 1.
\]

If we let \( R \to \frac{\alpha'}{R} \), while simultaneously \( m \leftrightarrow n \), the spectrum is preserved. Indeed, the scattering amplitudes also respect “\( R \leftrightarrow \frac{\alpha'}{R} \) duality,” and it has been shown that replacing

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\(^1\) The superscript denotes that this equation is used to introduce a variable for the expansion of a string coordinate.
with \( R' \) produces an isomorphic conformal field theory.\cite{2,3} Thus, by duality, a pointlike universe is equivalent to an infinitely big universe. Thus, the “smallest” universe possible has \( R = 1 \) in units of \( \sqrt{\alpha'} \approx \) the Planck length, which is of the same order of the minimum size expected to result from quantum gravity. (Unless otherwise noted, we express \( R \) in units of \( \sqrt{\alpha'} \).)

2. The Original Paradigm of Brandenberger and Vafa

The basic framework of string cosmology rests on a reversal of the usual compactification scenario. Rather than begin with \( D_{\text{crit}} \) uncompactified (i.e., large) dimensions and posit the spontaneous compactification of \( D_{\text{crit}} - 4 \) of them, it has been suggested\cite{4} we adopt a view more compatible with the cosmological idea of a small universe that expands i.e., that the universe began with all \( D_{\text{crit}} - 1 \) of its spatial dimensions compactified near the Planck radius. We can ask why precisely three of these spatial dimensions became very large (“decompactified”) in a stringy big bang. Since the universe has not expanded infinitely since the big bang, we anticipate that all of its spatial dimensions are still compactified today. Some simply have a larger radius of compactification than others. This view allows strings to have winding modes about all spatial dimensions.

In their seminal paper,\cite{4} Brandenberger and Vafa suggested a tantalizing scenario in which string winding modes can severely hinder (or halt) the expansion of spatial dimensions. They argued heuristically that winding modes exert a negative pressure on the universe, thereby slowing and ultimately reversing the expansion. Since winding mode energy is linear with the radius of compactification (i.e., the scale factor of the universe), there is a large energy cost to expanding with winding modes present. The question becomes “in how many dimensions can winding modes be expected to interact frequently enough to annihilate?” If the universe expands in a direction where annihilation is incomplete, the windings will eventually force a recollapse of the universe along that direction to and possibly past \( R = 1 \) (which by duality can be interpreted as another attempt at expansion). Through their study of the low energy expansion of the tree level gravitational-dilaton effective action, Tseytlin and Vafa\cite{5} showed that the general form of the equations of motion do indeed imply that winding modes inhibit expansion.

Tseytlin and Vasa’s equations of motion can be solved analytically once initial conditions are chosen, if the total string energy, \( E(R_i) \), as a function of the nine radii of compactification, \( R_i=1 \) to 9, are known. Unfortunately, this function is not well understood. However, a universe with all its energy in windings will have \( E \sim R_i \) so it is reasonable to assume that for relatively large \( R_i \)

\[
E(R_i) \sim R_i^{\alpha_i},
\]

where the \( \alpha_i \) are of order unity. The solution for the \( R_i \) is easiest when \( \alpha_i = 0 \), for all \( i \), and the nine radii are all equivalent, \( R_i = R \). In this case the radius varies by

\[
R = R_0 \left( \frac{t - 2\sqrt{dA/E}}{t + 2\sqrt{dA/E}} \right),
\]

where \( A \) is an integration constant. Even in this extreme case, the expansion slows and ultimately is halted. When \( \alpha > 0 \) and \( R_i = R \), not only is the expansion stopped in finite
time, but it is also reversed. Thus we see that winding strings must collide and annihilate for significant and continued expansion to occur.

Our computer model was designed to explore qualitatively the implications of this new string scenario. The goal was to test the viability of the Brandenberger-Vafa-Tseytlin paradigm by asking if it could yield sensible results. Although detailed predictions must await a fuller understanding of string interactions and string thermodynamics at extreme temperatures, our findings indicate that the preferred dimension of spacetime need not be two, as might have been inferred from the original model of Brandenberger and Vafa. Additionally, our computer simulations yielded approximate limits on the magnitude of the Hubble expansion rate that are consistent with theoretical estimates.

3. Dimensional Predictions and Random Walks of Winding Modes

In what number of spacetime dimensions can annihilation be expected? Clearly, annihilation is easier in fewer dimensions. For example, in 1+1 dimensions, the windings must lie on top of each other. In 2+1 dimensions, they can be separated by one coordinate, but would be expected to interact frequently. In a very large number of dimensions, we would expect the equilibrium between winding modes to most likely be lost, so that their number density need not fall drastically as their energy increases. What is the maximum spacetime dimension which would allow thermal equilibrium between winding modes and thus lead to their total annihilation during expansion? Many have argued that a pair of 2-dimensional world sheets should generically intersect in four or fewer spacetime dimensions because $2 + 2 = 4$. Even if four were a rigorously proven maximum, the question would remain why four dimensions are favored over a smaller number, which seems much more likely from the point of view of ease of interaction. One suggestion has been that entropy considerations might favor four dimensions.

We probably do exist in a large four-dimensional universe because $2 + 2 = 4$, but due to more involved physics than this simple numerical argument implies. The world sheets of strings with winding modes (and string world sheets as a whole) are not simple planes, especially at high energies. Instead they may have many complex bends and twists, at least at high energies. At or near the Planck scale it is probably more accurate to model strings as random surfaces. In point particle field theory, correlation functions can be bounded by the intersection properties of two random walks. Whether or not two random walks will intersect in a $D$-dimensional embedding space depends of their Hausdorff dimension, $d_H$. For our purposes, we use the common definition of $d_H$ which is that $<X^2> \sim N^{2/d_H}$ for a walk of $N$ steps. This definition suffices for most occasions, and agrees with the more rigorous mathematical definitions in all but singular cases. For random surfaces, $<X^2>$ is the average square size of the object, the hypersurface, that is formed by the walk and $N$ is the number of faces of dimension $d_{\text{geo}}$ forming the hypersurface. The Hausdorff dimension indicates how the size of the walk (surface) scales, which relates to how “space filling” the walk (surface) is. The higher the Hausdorff dimension, the slower the size grows and the better the embedding space is filled. If two random surfaces of geometrical dimension $d_{\text{geo}}$ and Hausdorff dimension $d_H$ are moving in an embedding space $\mathbb{R}^D$, there will be a non-trivial intersection of the two surfaces if $2d_H > D$. If $2d_H < D$

*For a review of work in this area, see the forthcoming ref. [7].
the probability of intersection is low. In the latter case, the two random walks (surfaces) will each fill their own $d_H$ dimensional subspaces and not likely overlap with each other. For the boundary case, $2d_H = D$, the outcome is not so certain \textit{a priori}. Note that for random walks of points, the Hausdorff dimension is two, independent of the embedding dimension.\textsuperscript{[13]}

Thus, we suggest string interactions can be studied by considering the intersection properties of random surfaces. Whether two free strings will interact in $D$-dimensions depends on the sum of the Hausdorff dimensions of two random surfaces. Surprisingly, the Hausdorff dimension of a random two-dimensional surface is not less than eight (and possibly infinite) for $D \geq 1$,\textsuperscript{[13,14]} suggesting strings should interact with high probability in \textit{at least} up to 16 dimensions and are as likely to interact in ten dimensions as in four.

We must ask, though, if this result is valid for winding strings in particular? For high energy non-winding strings with wild fluctuations and twists in spacetime it is, indeed, reasonable. However, for strings with winding modes around a dimension with growing radius, the Hausdorff dimension should quickly reduce to two. Winding strings will not move freely in all directions; they will lie basically parallel to the direction they are wound about. Granted, if winding strings also have high energy oscillations, they may also bend far away from parallel. But transverse fluctuations of a winding string quickly die down, through emission of low energy gravitons, resulting from self-intersection of the string. We can understand this most simply from the qualitative argument that, since the central difficulty for expansion is the energy cost of expanding with windings present, we would expect the oscillation energy to be minimized. Although a typical non-winding string of length $l$ will have fluctuations of the order $\sqrt{l}$, a winding string of the same length, deprived of high energy oscillations, will only have fluctuations of the order $\langle \omega \rangle$, where

$$\langle \omega \rangle^2 = c_0 + \frac{1}{4\pi\sigma} \ln \left(\frac{l^2}{\alpha'}\right). \quad (6)$$

Here $\sigma$ is the string tension and $(c_0)^{\frac{1}{2}}$ is a constant of order $1 l_{\text{Pl}}$.\textsuperscript{[15]} Like $R$, $\langle \omega \rangle$ is expressed in units of $\sqrt{\alpha'}$.

This allows us to treat winding modes not as random surfaces, but rather as random walks in the $D - 1$ spacetime dimensions orthogonal to the winding direction. This would suggest that interaction (self-cancellation) of winding modes is possible only if $D - 1 < 4$, which is quite in agreement with phenomenological results! Be that as it may, there are still problems with this approach. First, it could be suggested that a string with winding modes can rap around both decompactifying dimensions and smaller non-decompactifying and intersect in the latter subspace. Second, random walks are generally done in static Euclidean embedding space, quite unlike the early (stringy) universe. The approach of Brandenberger, Vafa, and Tseytlin has been to view the early ten-dimensional spacetime universe as a nine-dimensional spacetime universe as a nine-dimensional torus tensored with time, \textit{i.e.} a topologically non-trivial $\mathbb{T}^9 \times \mathbb{R}$. Compactifying embedding space from $\mathbb{R}^{D-1}$ to $\mathbb{T}^{D-1}$ should significantly enhance the interaction rate and perhaps increase the dimension of embedding space in which random walks will intersect. On the other hand, the early universe underwent significant expansion, giving the opposite effect. Our computer generated toy model of the early universe considers these factors and additional ones.
4. The Computer Simulation

Treating all these effects analytically is prohibitively difficult, since we are forced to consider the universe near or even at the Planck scale. A proper treatment of the creation rate of windings requires a knowledge of string thermodynamics well beyond the current level of understanding. Near the Hagedorn temperature, $T_H$, the microcanonical ensemble must be used, which ostensibly requires counting all the states in the universe. Progress has been made in limiting regimes, but general results for independent radii of varying size have yet to be exhibited and are sure to be unwieldy at best. Furthermore, the inclusion of gravitational effects, appropriate for the early universe, could lead us to question the validity of using thermodynamics. Even if a careful thermodynamic treatment of winding creation in an equilibrium ensemble were possible, it would leave unanswered the most interesting question: how does the winding creation drop as equilibrium is lost? This would require understanding non-equilibrium statistical mechanics in the early universe. Another problem is that the stability of the very topology of spacetime has come into question at extreme temperatures. Above the Hagedorn temperature, the conservation of winding number cannot be guaranteed.[16,17]

In spite of the preceding difficulties, we believe it is feasible to test whether this model for the expansion of the universe can work. In other words, we can ask whether we can make reasonable phenomenological assumptions about various processes in the early universe which in this model would lead to a strong prediction that three dimensions expand. Furthermore, we can turn the problem around, asking what must the early universe have been like in order to produce our four-dimensional spacetime. Ultimately, useful constraints may be placed on the expansion rate, the radius at which equilibrium is effectively lost, the number of windings surviving at that radius, interaction rates, temperature and other quantities in the early universe by this procedure.

In this spirit, our computer model was developed to simulate winding string collisions in the early universe. While we would like to follow the model from $t = 0$, it is expected to be much more reliable below $T_H$, where we can more confidently use a string description and assume that oscillations are suppressed. Thus, we begin evolving the model after the temperature has dropped slightly below $T_H$, in an inflationary era. The primary goal is to better understand the evolution of the universe just after the equilibrium of winding strings is lost.

The windings about each dimension are represented by points in $D - 2$ spatial dimensions, where $D$ is the total number of spacetime coordinates in the theory, including both the compactified and the “decompactified” coordinates. Our study focuses on $D = 10$, which is the dimension of Type II and heterotic superstrings, although other values of $D$ are also considered to study the effect of dimension on various processes. Since the radius and corresponding temperature at which the windings drop out of equilibrium, is not known, the initial radii of the spatial dimensions is left variable, but is typically chosen to be a few Planck lengths. The radii of compactification, which truly are quantum mechanical objects, can be allowed to fluctuate, typically up to $l_{Pl}$ per time step. Since our results indicate this effect is not very significant, it is only incorporated into some of the trials.

A certain number of windings are presumed to remain in this epoch, but the precise number is unknown, so the initial number of windings is also a free parameter. Since
the total number of high energy strings in the early universe roughly equals the log of the energy, the number of windings about each direction should not be huge. If the primordial universe contained precisely the energy in our observable universe, assuming critical density, there would only be 135 energetic strings in the entire universe! Of course, this is an extreme lower bound. Nevertheless, since we do not expect all the strings in the universe to be winding strings, the number of windings about each direction can reasonably be assumed to be at most of order ten when equilibrium is lost. Only windings of ±1 about a single direction are considered. By the time temperatures below \( T_H \) are reached, we expect strings with higher winding excitations about a given direction will have decayed to strings with unit winding number, as required by Boltzmann suppression. Perhaps more important is the possibility of strings with single windings about more than one direction. While these may have significance, they are not tracked in our model, since although they may increase the overall interaction rate, they should not change the kinds of qualitative effects we seek to study.

Also considered was the issue of creation of pairs of winding modes after winding modes have fallen out of equilibrium. A naive argument shows that creation of windings must cease at a very small radius. If the expansion process is roughly adiabatic, then \( T \sim \frac{1}{R} \). Furthermore, the energy of windings is linear with \( R \). Thus, the Boltzmann suppression factor goes as \( e^{-R^2} \). Even though deviations from adiabaticity may occur, the suppression is strong enough that we believe the creation rate is negligible in the relevant regime. Thus, we assumed creation of winding modes ceases by the time the winding modes have effectively fallen out of equilibrium and we begin our numerical trials.

The net winding number is set to zero, in order to satisfy observational isotropy constraints.\(^{18,19}\) The windings then execute a random walk, stepping up to \( l_{Pl} \) in each time step of \( \delta t = 1 \) Planck time unit \( \equiv t_{Pl} \). During each step, the computer checks for annihilations of pairs of winding modes with opposite winding number. Unlike most work in this field that assumes an ideal gas, this analysis explicitly allows interactions. Naively, if two windings come within \( l_{Pl} \), they can be expected to annihilate.\(^{4,9,15}\) However, as the length of the string and thus its energy grow, oscillations cost less and less energy, compared to the total energy of the string so that the effective thickness of the string increases. Owing to quantum correlations, interactions are expected for windings that come within \( \langle \omega \rangle \) of each other.\(^{15}\) (Having the extra oscillations in no way violates our assumption regarding the straightness of the winding strings, since the scale of the oscillations grows slowly compared to the size of the string.)

The probability of interaction, given a collision, may also vary with radius. However, we can show that the coupling remains fairly constant for small radii in the limit of \( \alpha_i = 0 \) and all the radii are equal. More precisely, in that limiting case,

\[
g^2 = e^{2\phi} = K \left( 1 - \left( \frac{R}{R_{\text{max}}} \right)^{\sqrt{D-1}} \right)^2. \tag{7}
\]

\( R_{\text{max}} \) is the radius at which expansion stops and \( K \) is an unknown constant. We see that for \( R \ll R_{\text{max}} \) the coupling remains constant to lowest order. For larger \( R \), the coupling \( e^{2\phi} \) decreases with radius, ultimately dropping to zero. We can use eq. (7) to get an indication of the importance the variation of the dilaton, even in the current context where the radii
are all independent, if we replace \((R_{\text{max}})^{\sqrt{D-1}}\) by \((V_{\text{max}})^{\sqrt{D-1}}\). At \(R = 1\), the probability of annihilation, given a collision, is taken to be one, providing the normalization to the coupling constant. Computer runs were conducted assuming either a constant dilaton or a coupling varying by eq. (7). In any case, the decrease in the coupling is not expected to be significant, at least in ten dimensions, since the dramatic drop in the collision rate with increasing radius will dominate over any effect of the dilaton.

The universe is allowed to expand during each time step. The proper expansion equation can be found analytically provided that we knew how the energy of the matter varied with independent radii. The assumptions made in [5] that all the radii are equal and that \(E \sim R^a\) clearly do not hold here, so their solutions are also inappropriate since they show all radii tending to a fixed value. No dimensions effectively “decompactify.” Even if \(E(R_i)\) were well known, we would be forced to solve a system of 19 coupled differential equations to get a rigorous result. However, in order to understand the qualitative implications of the model, we only need to use an expansion equation that has the correct features. The Brandenberger-Vafa framework, verified in special cases by [5], requires that the windings slow the expansion as the radius increases and can ultimately stop or reverse it. These essential features are captured by the following procedure: During each time step, each radius \(R_i\) is rescaled by a function of the number of windings, \(n_i\), around the direction \(i\) and the radius itself,

\[
R_i(t + 1) = R_i(t)(1 + \epsilon(n_i(t), R_i(t))).
\]

(8)

If \(\epsilon\) were independent of time, in the limit of \(\delta t \to 0\) (8) would approach exponential expansion with constant Hubble parameter, \(H = \epsilon = \dot{R}/R\).

This is, indeed, the form predicted by some authors to result from string driven inflation,[18–20] In light of recent discussions of a possible phase transition at the Hagedorn temperature,[16,21–23,18,19] we assume, in the absence of windings, that \(\epsilon\) is constant, whereas in the presence of windings, that \(\epsilon\) decreases linearly with increasing radius and with increasing number of windings. Some suggest this phase transition corresponds, as the universe cools below \(T_H\), to individual strings “condensing” out of a single string or out of a string “soup”. At or above \(T_H\), this single string (“soup”) fills all of spacetime and carries all, or nearly all, the energy. Thus, in our model we combine the ideas of exponential inflation with the expansion hindering effects of winding modes. However, we do not expect our results to alter significantly if we were to replace exponential expansion with power-law expansions. Our results indicate that if winding modes are to annihilate, they must do so very early, early enough that exponential expansion is still subluminal and is closely approximated by a (low) power expansion.

The following form satisfies our requirements for \(\epsilon\):

\[
\epsilon(n_i(t), R_i(t)) = H(1 - \frac{n_i(t)R_i(t)}{2R_{\text{max}}}) ,
\]

(9)

where \(H\) is the maximum expansion rate, as well as the Hubble constant for de Sitter inflation, and \(R_{\text{max}}\) is the radius at which two windings will halt the expansion. \(R_{\text{max}}\) appeared previously in eq. (7). Clearly, \(R_{\text{max}}\) must be less than the radius at which GUT physics takes place, but is not otherwise well constrained. The importance and reasonable ranges of both parameters were determined by studying how they effect the prediction...
for the dimension of spacetime. In our program, we take the limit of $\delta t \to 0$ and replace $1 + \epsilon(t)$ with $\exp(\epsilon(t))$ in eq. (8).

This prescription yields an expansion rate that decreases to zero as $R_i$ increases along directions with corresponding windings present, but results in constant exponential expansion about any direction for which all the corresponding windings have been annihilated. Dimensions do not recontract if their windings remain, instead staying compactified at $R_i = 2R_{\text{max}}/n_i$. Though an expansion equation that allows contraction could be constructed, it would not be useful in the model. It is of course possible in a given expansion attempt for no dimensions to lose all their windings. In that case, the dimensions are expected to recontract and ultimately begin expansion again. However, for inflation to occur a second time, the universe must first recontract back into the (topological) phase at or above the Hagedorn temperature. We cannot follow the windings as they enter this phase. When the universe begins expansion again, we begin modeling below the Hagedorn temperature as before, essentially treated this as an independent attempt at expansion. Thus, our model should be interpreted as following the evolution of the universe during its final and only successful attempt at expansion. The above reasoning also implies that if some dimensions lose all their windings, then these will not stay forever at the Planck scale. With some dimensions large, the temperature can no longer grow high enough to restore the compactified space back into its original vacuum, so that the inflation of the small dimensions cannot be repeated.

Causality raises some questions about how to implement the preceding prescription. These difficulties result from trying to incorporate the effects of a purely global concept like winding number into local physical effects like expansion. The number of windings about a given direction is globally defined, irrespective of position. However, the effect of those windings on local physics cannot change everywhere instantaneously. By causality, if a winding is annihilated at some spacetime point, $X^\mu$, we would expect the expansion rate far away to be unaffected initially. Strictly, winding annihilations would lead to growing bubbles of spacetime, which are expanding at a faster rate then the rest of space. After a number of annihilations, each spacetime point would have expanded a different amount. This is very difficult to model. Instead, a retardation is introduced into our model so that $n_i$ in eq. (9) counts windings that either have not annihilated or have annihilated too recently for most of the universe to know about it. Specifically, an annihilation is “counted” after a time equal to the effective radius of the universe, $R = \sqrt{\sum_{i=1}^{d} R_i^2}$, at the time of the annihilation. Runs were conducted both with and without the retardation to determine its importance.

5. Results of the Simulation and Predictions of the Model

The central result of the computer simulation is that a two-dimensional decompactified universe need not be the most probable outcome of the model just presented. If we consider the full parameter space described in the last section, the vast majority of it corresponds to either a two- or a ten-dimensional spacetime. However, appropriate choices of parameters can be found to make any dimension, from two to ten, the most probable. Unfortunately, in cases when the most likely dimension is neither two nor ten, it generally becomes impossible to predict the outcome with reasonable certainty.

Two dimensions result when annihilations are extremely unlikely. All dimensions then
have \( n_i \) windings about them that survive so that each dimension can at most expand to \( 2R_{\text{max}}/n_i \). The simulation would then show a result of zero large spatial dimensions or a one-dimensional spacetime. However, we know that given sufficient time, annihilation must ultimately occur, since the space is no longer rapidly expanding. This time may have to be integrated over several expansion attempts if it is more likely that the universe will recollapse before such annihilations occur! (See figure 2.) In this case the entire scenario would be repeated, presumably with the same choice of initial parameters, since they are determined by the poorly understood physics of the Planck scale. Once the rare annihilation occurs that leaves a dimension without windings, this dimension will expand without bound, forever suppressing annihilations along other dimensions. More than one large dimension would require two rare annihilations to occur almost simultaneously. If the annihilations along different dimensions occurred at significantly different times, then the large spatial dimensions we observe today would have undergone vastly different amounts of expansion. This is probably ruled out by the isotropy of our universe.

A ten dimensional spacetime is achieved when the parameters are such that annihilation is extremely efficient once equilibrium is lost. Then all winding strings are destroyed almost immediately and all dimensions expand without constraint.

The more interesting situation occurs for a relatively narrow band of parameter space in which winding annihilation is moderately likely. (See figures 4 and 5.) The most important variable is the radius at which equilibrium is lost and the simulation begins. The importance of radius can be seen by examining how the collision rate falls with the radius of compactification in various dimensions. For walks in one spatial dimension, one would expect the number of steps required for collision to scale as the square of the radius of compactification. This generally holds, especially at large radii. Deviations result from the logarithmic growth of the size of the string with radius. At small radii, \( \sqrt{\ln R} \) is not negligible, accounting for the greater deviation in small spaces. Figure 2 shows how the number of steps required for a pair of windings to annihilate (in just three dimensional compactified space) with at least 98 percent probability varies with the radius of compactification. As expected, in more dimensions the collision rate drops dramatically. In nine dimensions, roughly 180,000 steps (e.g., 180,000 Planck time units) are required to get a collision with 98 percent probability at a constant radius of only three. As a result, we see that the winding creation rate must drop effectively to zero at a radius not much larger than \( R = 1 \), or the expansion rate must be small enough so that hundreds of thousands of time steps lead to negligible expansion. (An expansion rate of \( 10^{-4} \) would increase the radius by a factor of \( 6 \times 10^7 \) in 180,000 time steps.) If not, windings would be created at a radius where they had little chance of annihilating, leading to a two dimensional universe.

This raises the question about how fast the universe can expand without preventing winding collision and annihilation. To answer this question, trials were conducted with the expansion rate taken to be a constant, independent of the number of windings present. We then checked to see how large the expansion rate could be such that two windings would collide with 98 percent probability before the radii of compactification were clearly too large for annihilation to occur (\( R > 500 \)). In three spatial dimensions, a very large Hubble parameter, \( H \), (of order one) is allowed if equilibrium is lost at the rather improbable value, \( R = 1 \). However, if the proper initial radius for the model is \( R = 4 \) then the maximum Hubble parameter is about \( 10^{-4} \) in Planck units. (See figure 3.) With nine spatial dimen-
sions, as is appropriate for the superstring, the largest possible Hubble parameter is around $10^{-5}$ for $R$ starting slightly above 1. These constraints are not precise limits, since the actual expansion rate is not constant, as assumed above, but gets reduced in the presence of windings. Thus, the maximum expansion rate without windings could be larger. More complicated string processes than those treated here could also increase the annihilation rate and allow greater expansion. Nevertheless, the preceding analysis indicates that the expected magnitude of the maximum expansion is very small. It is also pleasing that our bound on $H$ for $D = 10$ is in agreement with the phenomenological bound of Abbott and Wise,\cite{Abbott:85} which is based on the anisotropy in the cosmic microwave background radiation.

Of course, the most direct and revealing way to determine the effect of the radius of compactification and Hubble parameter on the expected dimension of spacetime is to simply run many (50) trials for various values of these parameters and compute the average dimension obtained. Typically, we find $\langle D \rangle = 10$ up to some $R_1$ and then falls rapidly as a function of $R$ up until $R_2$, beyond which the expected dimension is two. Unless otherwise specified, all following runs use 10 windings about each of nine compactified directions, an $\hat{R}_{\text{max}}$ (the maximum radius obtainable with two windings present) of 50, and an effective string width $\langle \omega \rangle$ chosen to equal $2\pi$ at $R = 1$. If the Hubble parameter, $H$, is between .1 and .01, $R_1$ is equal to one and $R_2$ is a very small 1.5. Four-dimensional spacetime is then the most probable only in the narrow range of $1.18 \leq R_0 \leq 1.21$ for $H = .1$ and of $1.22 \leq R_0 \leq 1.24$ for $H = .01$. Since it is hard to believe that the temperature could have dropped sufficiently below the Hagedorn temperature for the windings to have fallen out of equilibrium so near to the dual radius, we again conclude that a small expansion rate is necessary. For $H = .001$, the interesting range for $R_0$ has only increases to between 1.2 and 1.6, with four-dimensions most-probable between 1.5 and 1.58. If $H = 10^{-4}$, $R_1$ and $R_2$ are 1.5 and 2.5 respectively. The range for four dimensions is now 1.99 to 2.10. We estimate the uncertainties on $R_{1,2}$ to be about .05. For $H = 10^{-5}$, $R_1$ and $R_2$ are extrapolated to be $2.0 \pm .1$ and $3.3 \pm .1$, respectively, with four dimensions most probable in the range of 2.8 to 2.93. Thus, for the dimension of spacetime considered solely as a function of $H$ and $R_0$, only a very narrow range of the parameter space predicts “four” as the outcome.

Another parameter upon which the final dimension of spacetime sensitively depends is the effective width of a string, determined by $c_0$ in the expression above eq. (7). (See figure 6.) Even with $R = 1$ and $H = .01$, we find that $\langle D \rangle = 1$ (i.e., that only time is uncompactified) for $c_0$ ranging from zero to 15. The expected dimension rises rapidly as $c_0$ increases from 15 to 30. Of course, a wider string should act equivalently to a narrower string in a smaller space, so this behavior is not surprising. Clearly, if a change in phase occurs as $R$ approaches 1, then the interaction width of strings should be large at $R = 1$, especially if this change in phase corresponds to all strings merging into a single string filling all of compactified space. Thus, unless otherwise specified, we select $c = 37.64$, giving $\langle \omega \rangle = 2\pi R$ at $R = 1$.

The number of windings surviving when equilibrium is lost has a variable effect. If the initial radius is small, it has almost no effect. For example, with $R = 1.4$, and $H = .1$, 500 trials were conducted with either 2, 10, 50 or 100 windings about each dimension. Even with a sensitivity of .07 in the average dimension, no statistically significant change in the average dimension was observed when the number of windings ranged from 10 to
We can argue that since the initial volume of the transverse space was only about 15, ten or more windings completely filled the space. This implies that for sufficiently small radii the total number is irrelevant and, further, might lead us to expect that almost all of the windings would annihilate, as they are guaranteed to be in close proximity. In reality, most of the time no dimensions got large. This is because adjacent windings often do not have opposite winding number. For larger initial radii, the effect of the number of windings is, however, very significant. When the initial radius is two and \( H = 0.0001 \), the average dimension of spacetime drops by over four when the number of windings increases from 10 to 25.

Other parameters are less significant. The radius at which two windings stop expansion, \( R_{\text{max}} \), does not greatly affect the results. In many cases, varying \( R_{\text{max}} \) from 5 to 100 has no effect, above error. If the initial radius is close enough to \( R_{\text{max}} \), then this parameter can reduce the expected dimension of spacetime by about one. (See figure 7.) This is to be expected since a larger \( R_{\text{max}} \) allows faster expansion for a given number of windings, resulting in less collisions and a smaller dimension. The effect of the evolution of the dilaton was also considered. While it can sometimes drop the expected dimension by one or two sigma, the effect is insignificant compared to other uncertainties, so many trials are conducted with a constant dilaton. This result gives us confidence that deviations from the approximate dilaton evolution equation being used (7), will not significantly affect the results. The effect of radii fluctuations was also studied. When radii fluctuations were allowed, they had no effect whatsoever if \( R_i > 1.3 \), so fluctuations were subsequently ignored for most runs.

Finally, the importance of a time delay to enforce causality was determined. In almost all cases the time delay had almost no statistically significant impact on the results. For some trials the time delay reduced the expected dimension by up to two sigma (i.e., by 4) for \( \langle D \rangle \) near five. The minimal effect of the time delay indicates that we need not be concerned with constructing a more realistic time delay algorithm.

The average dimension of spacetime is by no means the only quantity that should be studied. The width of the expected distribution of dimensions is also critically important. When the average dimension is one (ultimately two) or ten, the width can be arbitrarily small. However, for intermediate values, \( \sigma \) is roughly 1 - 1.5. (See figure 5.) Thus, while it is possible to have an average dimension of spacetime of four, we cannot rule out other alternatives based on the gross initial conditions of the universe. This lack of determinism is not pleasing.

The above analysis shows that there are a number of parameters that can be tuned to produce any desired average dimension of spacetime. The maximum expansion rate, the radius at which equilibrium is lost, the number of strings remaining at this radius and the effective width of those strings are certainly the most important. Unfortunately, a firm prediction for the most probable dimension of spacetime is not possible from this model because of the number of free parameters and the omission of possibly important physical effects. Nevertheless, this work does demonstrate how string theory can be used to make such a prediction. A more complete model, properly incorporating as yet poorly understood physics, is clearly called for. Lastly, the narrow range of parameters that give a four-dimensional universe should be seen as a blessing in disguise, rather than a fine tuning disaster. Once our knowledge of some of the relevant parameters improves, we can
use the fact that we live in a four-dimensional world in analysis as done above to determine
the values of the remaining parameters to good accuracy.

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