The Cauchy problem in general relativity: an algebraic characterization

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Abstract

In this paper we shall analyse the structure of the Cauchy problem for general relativity by applying the theory of first-order symmetric hyperbolic systems. The role of harmonic coordinates will be discussed.

Keywords: general relativity, Cauchy problem, harmonic gauge

1. Introduction

The Cauchy problem (CP) for standard general relativity (GR) has been studied in several papers, from a numerical point of view (see, for example, [1–4] and the pioneering works [5–9] and references therein) as well as from an analytical point of view [10–12]. It is at the basis of all applications of numerical gravity as well as at the basis of the physical interpretation of GR theory.

The analysis of the CP for GR is essentially based on the seminal paper of Arnowitt, Deser and Misner (ADM); see [13] and also [14–16]. The ADM decomposition is introduced and particular coordinate systems, called harmonic coordinates, are used to study the conditions under which one has a well-posed CP.

In this paper we shall review such a procedure, aiming to clarify the role of the choice of harmonic coordinates. In fact, in GR one is analysing covariant equations, the Cauchy theorem is stated in terms of the principal symbol of the differential operator which is defined intrinsically (as we shall show also in GR in which the equations are non-linear but quasi-linear and the principal symbol depends on fields) and hence it does not depend on the choice of coordinates. This is a first step in a project aiming to characterise integrability and constraints (as done in a Hamiltonian framework) though in a Lagrangian setting.

The ADM decompositions transforms the original (vacuum) Einstein field equations (which are ten PDEs in $\dim(M) = 4$) into a new system formed by six equations, which will be a hyperbolic system, and four constraint equations. Only the first system of six equations is
necessary to define a CP but we have to keep in mind that the second system is as important as the first one for the physical problem and it constrains the allowed initial conditions.

ADM decomposition also allows us to define a parameter $\tau$ that represents the evolution parameter of the system. In other words, from a mathematical point of view, ADM decomposition is a choice of a bundle structure $(M, \mathbb{R}, \tau, \Sigma)$ over the spacetime $M$, namely:

$$\tau: M \to \mathbb{R}.$$  

(1.1)

The standard fiber $\Sigma$ is a model for the isochronous space submanifolds $\Sigma_{t_0} := \tau^{-1}(t_0) \subset M$.

The ADM transformations are the transformations that preserve the bundle structure:

$$\begin{cases} 
    x'^0 = x^0(t^0) \\
    x'^i = x^i(t^0, x^0)
\end{cases}$$

(1.2)

where $(x^0, x^i)$ are fibered coordinates over $M$ (see [16] for further details).

On the spacetime $M$, one can restrict to Lorentzian metrics $g$ for which the fibers $\Sigma_{t_0}$ are space-like submanifolds. One can then decompose the metric tensor $g_{\mu\nu}$ (or its inverse $g^{\mu\nu}$) in the following way:

$$g_{\mu\nu} = -N^2 + \tilde{N}^2 \left( N^i \gamma_{ij} \right), \\
g^{\mu\nu} = \begin{pmatrix} 
- N^{-2} & N^{-2} N^i \\
N^{-2} N^i & \gamma_{ij} - N^{-2} N^i N^j \end{pmatrix}.$$  

(1.3)

where $N$ is a spatial scalar field called the lapse, $N^i$ a spatial vector called the shift and $\gamma_{ij}$ a 3-Euclidean metric called the induced metric, defined on $\Sigma$, with respect to transformations on $\Sigma$ (see [16]).

Let $\vec{n}$ be the (future directed) unit vector $g$-orthogonal to $\Sigma$ and $e_i$ a basis for vectors tangent to the fibers $\Sigma_{t_0}$. Then the lapse and shift are defined by the relation

$$\partial_0 = N \vec{n} + N^i e_i.$$  

(1.4)

The Ricci tensor, written in these new fields and in the frame $(n, e_i)$, is

$$\begin{cases} 
    R_{00} = -\frac{1}{N} \left( \delta_0 K - D_i D^i N \right) - K^i N_i \\
    R_{0i} = D_i \left( K_{j}^i - \delta_j^i K \right) = R_{i0} \\
    R_{ij} = \frac{1}{N} \left( \gamma_{ij} \delta_0 K_{k}^k - D_i D_j N \right) + \gamma^2 R_{ij} + K K_{ij},
\end{cases}$$

(1.5)

where $\delta_0 = \partial_0 - E \delta_i K_i$ is the extrinsic curvature (namely $K_{ij} = \frac{1}{2N^i} \delta_0 \gamma_{ij}$) and $D_i$ is the covariant derivative with respect to the affine connection $\gamma^2$ induced by $\gamma_{ij}$.

The Ricci scalar is

$$R = \frac{2}{N} \left( \delta_0 K - D_i D^i N \right) + \gamma^2 R + K^2 + K_{ij} K^{ij}.$$  

(1.6)

The Einstein equations do not determine the evolution of $N$ and $N^i$, i.e. they are not dynamical fields, so they can be arbitrarily chosen. We will set hereafter, for sake of the simplicity, $N = 1$ and $N^i = 0$. 


With these choices we have that the metric tensor and its inverse become
\[
g_{\mu\nu} = \left( \begin{array}{cc} -1 & \gamma^i j \\ 0 & \gamma^j i \end{array} \right) \quad g^{\mu\nu} = \left( \begin{array}{cc} -1 & 0 \\ 0 & \gamma^j i \end{array} \right)
\]  
and the ADM evolution Einstein equations \( R_{\mu} = 0 \) become
\[
A_{(ij)(lm)} \partial_{(i} \gamma_{lm)} + B_{(ij)(mn)} \partial_{i} \gamma_{mn} \approx 0,
\]
where we set
\[
A_{(ij)(lm)} = \gamma(i \gamma_{m)j}
\]
and
\[
B_{(ij)(mn)} = -\gamma_{kl} \gamma_{r(s)l} - \gamma_{lm} \delta(l \delta_{r j} + \gamma_{j(n} \delta_{r m)} \delta^l) + \gamma_{i(m} \delta_{r n)l},
\]
where the symbol \( \approx \) means modulo lower-order terms with respect to the derivative degree. See [16] for further details.

Let us stress that the other Einstein equations \( R_{\mu} = 0 \) and \( R_{\mu00} = 0 \) do not contain second-order time derivatives (one can eliminate the term \( \delta_{0}K \) by summing with the trace of the equation \( R_{\mu} = 0 \) and are hence to be interpreted as constraints on the allowed initial conditions. We shall not consider these constraints here (which contain information about the canonical analysis of the system; see [17]).

The next section is devoted to stating the Cauchy problem for the evolution PDE (1.8).

### 2. The Cauchy problem for a PDE

First of all, let us recall that we are not analysing a generic PDE but quasi-linear systems that come from a Lagrangian. This means that we have a Lagrangian \( L(x^i, y^j, \dot{y}^j) \, dS \) and we obtain the Euler-Lagrangian equation by standard action variation. After the action
\[
A_{D} = \int_{D} L(y(x), \dot{y}(x)) \, dS
\]
has been varied (with fixed boundary conditions) we obtain
\[
\delta A_{D} = \int_{\Sigma} \alpha_{f}(x, y) \dot{y}^j \, dS = 0
\]
and the relative equations are
\[
\alpha_{f}(x, y) = 0.
\]
It is clear that if equations of motion come from a Lagrangian (by using the principle of least action) they will live in the dual space of the field deformations \( \delta y^j \). In general, using the geometrical framework for variational calculus (see, e.g., [18]), one can show that the Euler–Lagrange equations are described by a (vertical) bundle morphism
\[
E : J^2 \text{Lor}(M) \rightarrow V^\bullet (\text{Lor}(M)) \otimes \mathcal{A}_{m}(M)
\]
which exactly expresses this remark.

Let us briefly review the CP for a first-order PDE. Although GR has second-order equations let us introduce it for first order and then we shall extend the results to the second-order system. Let us also stress that the Einstein equation are quasi-linear, i.e., they are linear in the highest derivative terms. From this viewpoint, a first-order PDE analogous to the one for standard GR for us is written as follows:
In the same way, in order to define a CP, we have to define an initial condition, namely

\[ \alpha_{IJ}(x, y) \partial_0 y^J + \alpha^I_{IJ}(x, y) \partial_0 y^I + \gamma_I(x, y) = 0. \] (2.5)

where \( f^I(x') \) is called initial conditions or initial data.

We are ready to state the existence and uniqueness theorem:

**Theorem 2.1.** Let \( \alpha_{IJ} \) be a non-degenerate positive-definite bilinear form and let \( \partial_0 \) be symmetric in indices \( IJ \) for all \( i \), then under these hypotheses, given the initial data in \( H^k(M) \), with \( k > \frac{n}{2} + 1 \), the existence and uniqueness are ensured in an open interval \( I \subset \mathbb{R} \) and in a suitable Sobolev space, which depends on the regularity of the initial data (see [12]).

We stress that, under the above conditions, if we have \( f^I \) smooth, we will have a smooth solution defined on subset \( I \) of the real line that contains \( t = 0 \). In other words, we imposed initial conditions and the equations determine a unique solution in a neighbourhood of the Cauchy surface \( \Sigma_0 \).

Overviewing technical details, we can notice that two aspects are involved in the theorem above: an analytical condition and an algebraic one. Although they are both important we shall focus on the algebraic one, also in view of the fact that in most contexts physical fields are chosen to be smooth.

Let us remark that the theorem above states that the well-posedness of the CP is subject to some algebraic conditions of the coefficients appearing in the differential operator.

Now, we can consider the case of second-order quasi-linear systems. As done in (2.5) we define a second-order system as follows:

\[ \alpha_{IJ}(x, y) \partial_0 y^J - \alpha^I_{IJ}(x, y) \partial_0 y^I - \alpha^I_{IJ}(x, y) \partial_0 y^I + \gamma_I(x, y, dy) = 0. \] (2.7)

and its associated CP (hereafter we drop the coefficients’ dependence):

\[ \alpha_{IJ} \partial_0 y^J - \alpha^I_{IJ} \partial_0 y^I - \alpha^I_{IJ} \partial_0 y^I \approx 0. \]
\[ y^I(0, x^i) = f^I(x^i), \quad \partial_0 y^I(0, x^i) = g^I(x^i). \] (2.8)

which will be called hereafter CP2.

Our goal is to transform a second-order PDE into a first-order system by introducing auxiliary fields. Inspired by the method used for ordinary differential equations we can define the following new fields:

\[
\begin{align*}
\nu^I &= \partial_0 y^I \\
\nu^I &= \partial_0 y^I \Rightarrow \partial_0 \nu^I = \partial_I \nu^I.
\end{align*}
\] (2.9)

However, if we wish to consider these equations as part of the original system, then one should notice that they do not correspond to differential operators with values in the dual space of field variations, as was the case for the original equation. Then one should introduce some suitable bilinear forms to write them equivalently in the form.
\[
\begin{align*}
\beta_{IJ} \left( \partial_{0} y^{I} - v^{I} \right) &= 0 \\
\beta_{IJ}^{\mu} \left( \partial_{0} y^{I} - \partial_{\mu} y^{I} \right) &= 0
\end{align*}
\]

(2.10)

for some invertible coefficients \( \beta_{IJ} \) and \( \beta_{IJ}^{\mu} \).

Let us remark that the equation \( v_{I}^{f} = \partial_{f} y^{I} \) contains no time derivative and as such is a constraint on initial conditions and will not contribute to the CP.

Then equation (2.7) can be written in terms of the new fields \((y^{I}, v^{I}, v_{I}^{f})\), so that the CP can be recast in the following form:

\[
\begin{align*}
\beta_{IJ} \partial_{0} y^{I} &\approx 0 \\
\alpha_{IJ} \partial_{0} v^{I} - \alpha_{IJ}^{\mu} \partial_{\mu} v^{I} - \alpha_{IJ}^{0} \partial_{0} v^{I} &\approx 0 \\
\beta_{IJ}^{\mu} \partial_{0} v^{I} - \beta_{IJ}^{\mu} \partial_{j} v^{I} &= 0
\end{align*}
\]

(2.11)

together with the constraint \( \partial_{0} y^{I} = v^{I} \). We can write this system in the block-matrix form as follows:

\[
\begin{pmatrix}
\beta_{IJ} & 0 & 0 \\
0 & \alpha_{IJ} & 0 \\
0 & 0 & \beta_{IJ}^{\mu}
\end{pmatrix}
\begin{pmatrix}
y^{I} \\
v^{I} \\
v_{I}^{f}
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 \\
0 & \alpha_{IJ}^{\mu} & \alpha_{IJ}^{0} \\
0 & \beta_{IJ}^{\mu} & 0
\end{pmatrix}
\begin{pmatrix}
y^{I} \\
v^{I} \\
v_{I}^{f}
\end{pmatrix}
= 0.
\]

(2.12)

This is a first-order CP so theorem 2.1 applies to it. One has the existence and uniqueness of solutions if the first matrix is symmetric, non-degenerate and positive-definite and the second is symmetric.

We already know that \( \alpha_{IJ} \) is non-degenerate and positive-definite. If also \( \beta_{IJ} \) and \( \beta_{IJ}^{\mu} \) are non-degenerate and positive-definite then the whole matrix is. Since we are free to choose \( \beta_{IJ} \) as we wish (provided that the choice is non-degenerate and positive-definite) we can fix it as \( \beta_{IJ} = \alpha_{IJ} \), which is automatically a good choice.

For the second matrix to be symmetric, \( \alpha_{IJ}^{\mu} \) must be symmetric and one must have (see appendix B)

\[
\beta_{IJ}^{\mu} = \alpha_{IJ}^{\mu}.
\]

(2.13)

Thus the block \( \beta_{IJ}^{\mu} \) (and as a consequence of this choice the coefficient \( \alpha_{IJ}^{\mu} \)) must be symmetric in \((IJ)\) and non-degenerate positive-definite.

We can rewrite the original system as

\[
\begin{align*}
\alpha_{IJ} &\partial_{0} y^{I} - \alpha_{IJ}^{0} \partial_{0} y^{I} - \alpha_{IJ}^{\mu} \partial_{\mu} y^{I} &\approx 0 \\
\alpha_{IJ}^{\mu} &\partial_{0} v^{I} - \alpha_{IJ}^{0} \partial_{0} v^{I} - \alpha_{IJ}^{\mu} \partial_{\mu} v^{I} &\approx 0 \\
0 &\partial_{0} v_{I}^{f} - \alpha_{IJ}^{0} \partial_{0} v_{I}^{f} - \alpha_{IJ}^{\mu} \partial_{\mu} v_{I}^{f} &\approx 0
\end{align*}
\]

(2.14)

which, together with the constraint \( \partial_{0} y^{I} = v^{I} \), is called the reduced CP or CP1 for short.

Let us remark that once again, also for second-order operators, the well-posedness of the CP is subject to algebraic requirements. Unlike for first-order operators, symmetry is no longer enough and one needs to require that the coefficient \( \alpha_{IJ}^{\mu} \) is also positive-definite.

Now we have to show that the original CP2 (namely (2.8)) is dynamically equivalent to the reduced CP (2.14), namely CP1.
Obviously, if we have a solution $y^J(t, x^i)$ of $CP2$, then
\[ \begin{align*}
(y^J, \nu^J) := \partial_0 y^J, \nu^J := \partial_0 \nu^J
\end{align*} \]  
(2.15)
is a solution of the reduced $CP1$. In fact, one immediately has that
\[ \begin{align*}
\alpha_U(y^J - \partial_0 \nu^J) &= 0 \\
\alpha_U^J(\partial_0 \nu^J - \partial_j \nu^J) &= 0
\end{align*} \]  
(2.16)
while the second-order equation can be recast as
\[ \begin{align*}
\alpha_U \partial_0 \nu^J - \alpha_U^J \partial_j \nu^J - \alpha_U^J \partial_0 \partial_j \nu^J &\approx 0.
\end{align*} \]  
(2.17)

Thus the equations of $CP1$ are satisfied, the constraint is satisfied (since we defined $\nu^J := \partial_0 y^J$) and the constraint, evaluated at $t = 0$, shows that $h_J^J(x) = \partial_0 f^J(x)$ are the only initial conditions compatible with it. Thus $CP1$ holds true.

Vice versa, we need to prove that, given a solution $(y^J(t, x), \nu^J(t, x))$ of $CP1$ that satisfies the constraint, then $y^J(t, x)$ is also a solution of $CP2$. One has that $\nu^J = \partial_0 y^J$, by the first equation in $CP1$, and that $\partial_0 \nu^J = \partial_0 = \partial_0 \nu^J$, by the third equations. Thus, one has $\nu^J = \partial_0 y^J + k(\nu)$ though, because of the constraint, $k(\nu) = 0$. Accordingly, one also has
\[ \begin{align*}
\nu^J &= \partial_0 y^J \\
y^J &= \partial_0 y^J
\end{align*} \]  
(2.18)
with which the second equation of $CP1$ implies the second-order equation.

One obviously has that $y^J(0, x) = f^J(x)$. Now, the first equation of (2.14) tells us that
\[ \begin{align*}
\partial_0 y^J(t, x) &= \partial_0 y^J(0, x) = \nu^J(0, x) = \nu^J(0, x)
\end{align*} \]  
(2.19)
which is the second initial condition and $CP2$ holds true.

Obviously the correspondence
\[ \begin{align*}
y^J(t, x) \\ \leftrightarrow \quad (y^J, \nu^J := \partial_0 y^J, \nu^J := \partial_0 \nu^J)
\end{align*} \]  
(2.20)
transforms solutions of $CP2$ into solutions of $CP1$ and is a bijection, proving dynamical equivalence.

We eventually have to prove that if the constraint is satisfied at the initial time, then it will be satisfied at all time. For this, let us define the quantity
\[ \begin{align*}
k^J_i &= \nu^J_i - \partial_0 y^J_i.
\end{align*} \]  
(2.21)
Since we know that $\partial_0 \nu^J = \nu^J$, then
\[ \begin{align*}
\partial_0 k^J_i &= \partial_0 \nu^J_i - \partial_0 y^J_i = \partial_0 \nu^J_i - \partial_0 \nu^J_i = 0,
\end{align*} \]  
(2.22)
where the last equality holds true by the last equation of $CP1$.

In particular, by imposing the constraint at $t = 0$, we obtain $k^J_i(0, x) = 0$ and then we have the following Cauchy problem:
\[ \begin{align*}
\alpha_U \partial_0 k^J_i &= 0 \\
k^J_i(0, x) &= 0
\end{align*} \]  
(2.23)
which has a solution $k^J_i(t, x) = 0$, which is unique since the system is symmetric hyperbolic. This means $y^J_i(t, x) = \partial_0 y^J_i(t, x)$ at any time and the constraint is satisfied at all times.

The next section is devoted to applying the theory of PDEs developed above to the GR case.
3. Harmonic coordinates

We can start to reduce the system (1.8) of second order to a first-order one. We have to define some new fields, as was done in the previous section:

\[
\begin{align*}
\Sigma_{ij}^j &:= \partial_0 \gamma_{ij}^j \\
\Sigma_{ij}^j &:= \partial_i \gamma_{ij}^j
\end{align*}
\] (3.1)

so that the system can be rewritten in the following way (as in the previous section):

\[
\begin{align*}
A_{ij}^{(\text{lm})} \partial_0 \gamma_{ij}^{\text{lm}} - A_{ij}^{(\text{lm})} \Sigma_{\text{lm}}^j &= 0 \\
A_{ij}^{(\text{lm})} \partial_0 \Sigma_{\text{lm}}^j + B_{ij}^{(kl)} \partial_k \Sigma_{mn}^m &\approx 0 \\
B_{ij}^{(kl)} \partial_0 \Sigma_{mn}^m - B_{ij}^{(kl)} \partial_k \Sigma_{mn}^m &= 0
\end{align*}
\] (3.2)

with the constraint

\[
\Sigma_{ij}^j = \partial_i \gamma_{ij}^j.
\] (3.3)

In matrix form the system above can be written as follows:

\[
\begin{pmatrix}
A_{ij}^{(\text{lm})} & 0 & 0 \\
0 & A_{ij}^{(\text{lm})} & 0 \\
0 & 0 & B_{ij}^{(kl)}^{(mn)}
\end{pmatrix}
\begin{pmatrix}
\gamma_{ij}^{\text{lm}} \\
\Sigma_{\text{lm}}^j \\
\Sigma_{mn}^m
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & B_{ij}^{(kl)}^{(mn)} \\
0 & B_{ij}^{(kl)}^{(mn)} & 0
\end{pmatrix}
\begin{pmatrix}
\gamma_{ij}^{\text{lm}} \\
\Sigma_{\text{lm}}^j \\
\Sigma_{mn}^m
\end{pmatrix}
\approx 0.
\] (3.4)

We want to verify whether the first-order system (3.4) is symmetric hyperbolic. We have two conditions that have to be satisfied: the first one is that the matrix

\[
B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & B_{ij}^{(kl)}^{(mn)} \\
0 & B_{ij}^{(kl)}^{(mn)} & 0
\end{pmatrix}
\] (3.5)

has to be symmetric, which is automatically satisfied by construction. The second one is that the matrix

\[
A = \begin{pmatrix}
A_{ij}^{(\text{lm})} & 0 & 0 \\
0 & A_{ij}^{(\text{lm})} & 0 \\
0 & 0 & B_{ij}^{(kl)}^{(mn)}
\end{pmatrix}
\] (3.6)

has to be symmetric and positive-definite. One can easily show that the block \(A_{ij}^{(\text{lm})}\) is positive-definite and symmetric. In contrast, \(B_{ij}^{(kl)}^{(mn)}\) is not symmetric with respect to the exchange of the pairs \(\{ij\}\{lm\}\).

In fact, it is easy to see that its antisymmetric part is

\[
B_{ij}^{(kl)}^{(mn)} = \frac{1}{2} \left( \gamma_{mn}^{(i)} \delta_{j}^{(k)} \delta_{l}^{(l)} - \gamma_{ij}^{(i)} \delta_{mn}^{(k)} \delta_{l}^{(l)} \right)
\] (3.7)

and it does not generically vanish. Let us also remark that it is a tensor on the space manifold \(\Sigma\) so that one cannot hope it will vanish in any (spatial) coordinate system.

Let us introduce a new coordinates system, called (spatial) harmonic coordinates and defined by the following conditions:
\[ 3 \Gamma^l = \gamma^j \Gamma^l_{ij} = 0, \]  
(3.8)

where \( \Gamma^l_{ij} \) are the Christoffel symbols of \( \gamma^j \). It is easy to prove that harmonic coordinates always exist. The condition (3.8) is equivalent to

\[ \frac{1}{2} \gamma^{kl} \partial_l \gamma_{kl} = \gamma_{ij} \partial_l \gamma_{ij} \]  
(3.9)

which in turn implies

\[ \frac{1}{2} \gamma_{ij} \partial_l \gamma_{ij} \approx \gamma_{ij} \partial_l \partial_i \gamma_{ij}. \]  
(3.10)

Then we have

\[ \gamma_{im} \partial_m \gamma_{mn} + \gamma_{im} \partial_m \gamma^{mn} \approx \gamma_{im} \partial_m \gamma^{mn} \]  
(3.11)

so that, in harmonic coordinates, the coefficient \( B_{ij\{mn\}}^{kl} \) takes the form

\[ B_{ij\{mn\}}^{kl} = \gamma^{kl} \gamma_{mn} = \gamma^{kl} A_{ij\{mn\}} \]  
(3.12)

which is symmetric (as well as automatically non-degenerate positive-definite since \( A_{ij\{mn\}} \) is).

In other words, in harmonic coordinates the antisymmetric part of the full operator becomes lower order, it contributes to the lower-order tail and consequently the operator becomes symmetric hyperbolic and CP is well posed.

Let us remark that there is no contradiction between \( B_{ij\{mn\}}^{kl} \) being a tensor and it becoming symmetric in harmonic coordinates since what we are really saying is that in harmonic coordinates one has

\[ B_{ij\{mn\}}^{kl} \partial_l \gamma^{mn} \approx 0 \]  
(3.13)

not that the coefficients \( B_{ij\{mn\}}^{kl} \) become zero. In fact, the equations (3.13) are not covariant and they can be satisfied in particular coordinate systems (e.g. harmonic coordinates) without being satisfied in others.

Now that we have seen that harmonic coordinates make the evolution equation symmetric hyperbolic (though of course they spoil general covariance), we shall show that coordinates more general than harmonic coordinates exist for which the evolution equations are still symmetric hyperbolic.

Since we have already found the antisymmetric part (3.7), we can directly impose that

\[ B_{ij\{mn\}}^{kl} \partial_l \gamma^{mn} \approx 0 \]  
(3.14)

without imposing conditions on first derivatives as in (3.9).

This condition is weaker than the conditions for harmonic coordinates. Obviously, harmonic coordinates imply (3.14). Vice versa, we shall see that there exists a coordinate system in which the evolution equations become symmetric hyperbolic and this system is not harmonic; see [19]. In fact, the condition (3.14) is also satisfied if one simply has

\[ \partial_k \partial_l \gamma^{mn} = 0. \]  
(3.15)

Indeed, if the inverse metric \( \gamma^{ij} \) takes a linear form in coordinates, e.g.

\[ \gamma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + t \end{pmatrix}, \]  
(3.16)

then (3.15) vanishes, while \( \Gamma^l = \gamma^j \Gamma^l_{ij} \) does not.
4. Bianchi identities and constraints

In this section we shall see how one can use projectors (see appendix A) to prove that evolution preserves the constraints

\[
\begin{align*}
H & := -\frac{1}{N}(\delta_0 K - D_i D^i N) - K^{ij} K_{ij} = 0 \\
M_i & := D_i \left( K^{j} \downarrow - \delta^j_i K \right) = 0
\end{align*}
\]

(4.1)
due to (contracted) Bianchi identities. In fact, Bianchi identities are

\[
\nabla_j G^{ij} = 0
\]

(4.2)
which by projection provide the following conditions

\[
\begin{align*}
\nabla_j G^{ij} n_\nu &= 0 \iff \delta_0 H = -D_i \left( N M^i \right) - M^i D_i N - 2 N K H \\
\nabla_j G^{ij} \sigma_\nu &= 0 \iff \delta_0 M^i = -D^i \left( N H \right) - 2 N M K^j \downarrow - N K M^j - H D^i N.
\end{align*}
\]

(4.3)

By setting, as done above, \( N = 1 \) and \( \bar{N} = 0 \) and we obtain

\[
\begin{align*}
\delta_0 H & \approx -\delta_i^j \partial_0 M^i \\
\gamma_m \partial_0 M^n & \approx -\delta_i^j \partial_0 H \\[10pt]
\implies \begin{pmatrix} 1 & 0 \\ 0 & \gamma_m \end{pmatrix} \partial_0 \begin{pmatrix} H \\ M^n \end{pmatrix} + \begin{pmatrix} 0 & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix} \partial_0 \begin{pmatrix} H \\ M^i \end{pmatrix} \approx 0
\end{align*}
\]

(4.4)
which is symmetric hyperbolic. Then, a unique solution exists for any initial condition. If we set \( H_{t=0} = 0 \) and \( M^i_{t=0} = 0 \) for the initial condition, then \( H = M^i = 0 \) at all \( t \in I \) is a solution, and since the equations are symmetric hyperbolic it is the only solution. Thus the constraints are preserved by Bianchi identities.

5. Conclusions and perspectives

We have seen that (vacuum) Einstein equations can be split into an (elliptic) system of constraints (four equations in \( \dim(M) = 4 \)) and an evolutionary system (six equations in \( \dim(M) = 4 \)).

The evolutionary part is not symmetric hyperbolic in general. However, one can split the evolutionary part into a symmetric hyperbolic equation and a further constraint, which is non-covariant with respect to change of coordinates on the spatial manifold \( \Sigma \). Hence one can find spatial coordinates for which the antisymmetric part vanishes and, in those coordinates, solve the symmetric evolutionary part of the equations.

The evolutionary equation is covariant with respect to change of coordinates on \( \Sigma \), thus if a unique solution is found in a coordinate system, then a solution is found in any coordinate system. This is not strange after all, since being symmetric hyperbolic is a sufficient (not necessary) condition for solving CP.

Thus there exists a solution to the evolutionary part for any initial condition. However, not all initial conditions are the same: there are initial conditions that satisfy the elliptic constraints as well as initial conditions that do not. For any initial condition that satisfies the elliptic constraints one can find a spatial metric \( \gamma_j(t, x) \), which together with a choice of the lapse \( N \) and shift fields \( \bar{N} \) (which in fact fixes the ADM foliation) defines a global Lorentzian metric \( g \) that solves the original Einstein equations.
This is more or less well known since it is the basis for numerical gravity (see [1–3]). However, a detailed analysis allows us to clarify some of the details that are relatively less well known and to draw some conclusions which, to the best of our knowledge, are new.

First we clarify the role of (spatial) harmonic coordinates. The coefficients \( A_{ij|mn} \) and \( B_{ij|mn}^{kl} \) of ADM splitting of the Einstein equations (i.e. their principal symbols) are spatial tensors. Hence, one can prove that changing the coordinates will not make them symmetric if they are not in the first place.

This originally appeared as an issue to us: how can covariant CP be well posed in one coordinate system and not well posed in another? The solution is quite simple: the evolutionary system is well posed in any coordinate system, but in some coordinate system one can use the theorem about symmetric hyperbolic systems while in other coordinates the same system is not symmetric hyperbolic (though the CP is well posed anyway). In other words, it is the condition of being symmetric hyperbolic that is not covariant. Nevertheless, the Cauchy theorem for symmetric hyperbolic systems, despite not being covariant, seems to be enough to deal with standard GR in full generality.

As a first issue then, one should define an integrable evolutionary system as one that is symmetric hyperbolic in at least some coordinate system.

Changing coordinates does not make the system symmetric hyperbolic. It simply makes their antisymmetric part identically satisfied. In other words, the antisymmetric part of the coefficient \( B_{ij|mn}^{kl} \) is and remains non-zero. However, the equation from it,

\[
B_{ij|mn}^{kl} \partial_{kl} \gamma^{mn} \approx 0
\]

becomes identically satisfied. This is possible precisely because this equation is non-covariant.

Moreover, in view of a generalisation, we can stress that the fact that the original Einstein equations come from a variational principle plays a fundamental role, in particular in GR when a metric on fields (namely, \( A_{ij|mn} \)) is unknown since it depends on the unknown field to be determined, \( \gamma_{ij} \). In particular, one has as many equations as fields. If \( k \) equations are constraints then it is reasonable to expect that \( k \) fields will be left undetermined, eventually spoiling uniqueness as we know must be so in view of the hole argument (see [20–22]). Moreover, constraints are responsible for the fact that the system is overdetermined. This simply accounts for the fact that equations of physics are at the same time overdetermined and underdetermined, due to gauge symmetries, see [23–26]. The variational origin of equations also leads us to assume that equations live in the dual space of field deformations, which leads us to a framework in which the metric on fields is not needed to define symmetric hyperbolic systems.

Further investigations are needed to generalise this to more general models of interest for (fundamental) physics. It seems possible that integrability is generically equivalent to Hamiltonian formulation, though in a completely Lagrangian setting.

We also found a more general class of coordinate systems than harmonic coordinates for which the system becomes symmetric. This is not that important in vacuum gravity since harmonic coordinates always exist and they are sufficient to solve the CP. However, when gravity is coupled to matter fields, matter field equations may need to be symmetrised as well. Unfortunately, the choice of spatial coordinates is a game that can be played only once since matter equations are coupled to gravity equations. One would need a coordinate system in which the whole system is symmetric, while the symmetrisation of matter equations depends on the matter–gravity coupling. Having more coordinate systems in which gravity becomes symmetric hyperbolic may help when coupled with some matter fields. We still do not have
examples of matter fields that can be solved in this way, though it is clear that harmonic coordinates play no distinguished role within the class of coordinates that turn the system into a symmetric hyperbolic system.

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Appendix A. Projectors

Let us define the projectors for the ADM decomposition. Due to the immersion of the space-like hypersurface $\Sigma (i:\Sigma \hookrightarrow M)$, we have a normal covector:

$$u = \frac{1}{3!} \varepsilon^{ABC} \partial_\alpha x^\mu \partial_\beta x^\nu \partial_\gamma x^\rho \epsilon_{\mu\nu\rho\sigma} dx^\sigma$$

and, due to the metric structure, its normal vector:

$$\bar{u} = g^{\mu\nu} u_\mu \partial_\nu.$$  (A.2)

Since $\bar{n}$ is not light-like, we have its normal unit vector $n$, obtained by normalization. With $n$ we can define a basis adapted to the foliation: let us take $x^i (k) \in M$ and we can take a quadruple of vectors $(n, e_A)$ where $e_A$ is a basis in $T_k \Sigma$.

This set has the property that each $e_A$ is orthogonal to $n$, so it is a basis for $T_k \Sigma$. Also we have that the norm of $n$ is $-1$, i.e.

$$g(n, n) = -1.$$  (A.3)

Now, we can define some maps that allow us to decompose each geometrical object (like tensors, metrics and so on). These are defined as follows:

$$\sigma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu,$$  (A.4)

and it is easy to see that they are idempotents, so they are projectors. Namely, we have

$$\sigma^\mu_\nu \sigma^\nu_\rho = \sigma^\rho_\mu.$$  (A.5)

Once projectors are defined we can decompose vectors and covectors (and, in general, tensors) into tangent and normal parts.

**Theorem.** For all $v = v^\mu \partial_\mu \in T_i \Sigma$ there exists a decomposition

$$v = v_\parallel + v_\perp$$  (A.6)

with $v_\parallel \in T_k \Sigma$. This decomposition is unique.

**Proof.**

$$v = v^\mu \partial_\mu = v^\mu \delta^\mu_\mu = v^\mu \delta^\nu_\mu \partial_\nu.$$
We can obtain (by using (A.4))
\[
\delta_\mu^\nu = \sigma_\mu^\nu - n^\nu n_\nu
\]
so that
\[
v = v^\nu \left( \sigma_\mu^\nu - n^\nu n_\nu \right) \partial_\nu = v^\nu \sigma_\mu^\nu \partial_\nu - v^\nu n^\nu n_\nu \partial_\nu = v^\nu \sigma_\mu^\nu \partial_\nu + \left( - v^\nu n_\nu \right) n^\nu \partial_\nu,
\]
and we have
\[
v = v\parallel + v\perp \tag{A.7}
\]
with \(v\parallel = v^\nu \sigma_\mu^\nu \partial_\nu\) and \(v\perp = - v^\nu n_\nu n^\nu \partial_\nu\). It is easy to see that \(g(v\perp, e_\lambda) = g(n, v\parallel) = 0\).

A similar result holds for the covectors. Let \(\alpha\) be a covector on \(\iota(\Sigma)\), namely
\[
\alpha = \alpha_\mu dx^\mu \in T^*_{\iota(k)}(M) \tag{A.8}
\]
Then there exist a decomposition of \(\alpha\) (parallel and perpendicular parts) and a covector \(\beta\) on \(\Sigma\) such that
\[
\iota^* \left( \alpha_\parallel \right) = \beta \in T^*_{\iota(k)}(\Sigma) \tag{A.9}
\]
Let now \(g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu\) be a metric on \(\iota(\Sigma)\) and we can write its tangent part as
\[
g_{\parallel} = g_{\alpha\beta}(x)\sigma^\alpha_\mu \sigma^\beta_\nu dx^\mu dx^\nu \tag{A.10}
\]
Now, we can take the pullback along the immersion:
\[
\iota^* \left( g_{\parallel} \right) = g_{\alpha\beta}(x(k))\sigma^\alpha_\mu \sigma^\beta_\nu \partial_\lambda x^\mu \partial_\beta x^\nu dk^A \otimes dk^B = g_{\mu\nu}\partial_\lambda x^\mu \partial_\beta x^\nu dk^A \otimes dk^B \tag{A.11}
\]
This means that the parallel part of the metric is restricted to the induced metric over \(\Sigma\).

**Appendix B**

Let us consider here an example of the situation discussed in section 2. Fix two fields \(y^1\) and \(y^2\) and one spatial coordinate on the space \(\Sigma\) (i.e. \(\dim(M) = 2\)). In this simple case equation (2.12) reads as
\[
\begin{pmatrix}
\beta_{11} & \beta_{12} & 0 & 0 & 0 & 0 \\
\beta_{21} & \beta_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_{11} & \alpha_{12} & 0 & 0 \\
0 & 0 & \alpha_{21} & \alpha_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & \beta_{11}^{(1)} & \beta_{12}^{(1)} \\
0 & 0 & 0 & 0 & \beta_{11}^{(2)} & \beta_{12}^{(2)} \\
0 & 0 & 0 & 0 & \beta_{21}^{(1)} & \beta_{22}^{(1)} \\
0 & 0 & 0 & 0 & \beta_{21}^{(2)} & \beta_{22}^{(2)} \\
\end{pmatrix}
\begin{pmatrix}
\partial_0 \\
\partial_1 \\
\partial_2 \\
\partial_3 \\
\partial_4 \\
\partial_5 \\
\partial_6 \\
\partial_7 \\
\end{pmatrix}
\begin{pmatrix}
y^1 \\
y^2 \\
y^3 \\
y^4 \\
y^5 \\
y^6 \\
y^7 \\
y^8 \\
\end{pmatrix}
\approx 0
\]
from which one sees that in order to have a symmetric hyperbolic system one needs to verify condition (2.13).
Let us stress that the symmetry of the system relies on a suitable ordering of fields and equations. This is acceptable since symmetric hyperbolic form is a sufficient, not a necessary, condition for having existence and uniqueness of solutions.

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