Gradient Estimates for the Perfect Conductivity Problem

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0 Introduction

Let Ω be a bounded open set in \( \mathbb{R}^n \) with \( C^{2,\alpha} \) boundary, \( n \geq 2, 0 < \alpha < 1 \), \( D_1 \) and \( D_2 \) be two bounded strictly convex open subsets in \( \Omega \) with \( C^{2,\alpha} \) boundaries which are \( \varepsilon \) apart and far away from \( \partial \Omega \), i.e.

\[
D_1, D_2 \subset \Omega, \quad \text{the principal curvature of } \partial D_1, \partial D_2 \geq \kappa_0 \\
\varepsilon := \text{dist}(D_1, D_2) > 0, \quad \text{dist}(D_1 \cup D_2, \partial \Omega) > r_0, \quad \text{diam}(\Omega) < \frac{1}{r_0},
\]

(0.1)

where \( \kappa_0, r_0 > 0 \) are universal constants independent of \( \varepsilon \).

We denote

\[
\tilde{\Omega} := \Omega \setminus (D_1 \cup D_2).
\]

Given \( \varphi \in C^2(\partial \Omega) \), consider the following scalar equation with Dirichlet boundary condition:

\[
\begin{cases}
\text{div}(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\
u_k = \varphi & \text{on } \partial \Omega,
\end{cases}
\]

(0.2)

where

\[
a_k(x) = \begin{cases}
k \in (0, \infty) & \text{in } D_1 \cup D_2, \\
1 & \text{on } \Omega \setminus (D_1 \cup D_2).
\end{cases}
\]

(0.3)

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It is well known that there exists a unique solution $u^k \in H^1(\Omega)$ of the above equation, which is also the minimizer of $I^k$ on $H^1_\varphi(\Omega)$, where

$$H^1_\varphi(\Omega) := \{ u \in H^1(\Omega) \mid u = \varphi \text{ on } \partial\Omega \}, \quad I^k[v] := \frac{1}{2} \int_\Omega a_k|\nabla v|^2.$$

As explained in the introduction of [9], the above equation in dimension $n = 2$ can be used as a simple model in the study of composite media with closely spaced interfacial boundaries. For this purpose, the domain $\Omega$ would model the cross-section of a fiber-reinforced composite, $D_1$ and $D_2$ would represent the cross-sections of the fibers, $\tilde{\Omega}$ would represent the matrix surrounding the fibers, and the shear modulus of the fibers would be $k$ and that of the matrix would be 1. Equation (0.2) is then obtained by using a standard model of anti-plane shear, and the solution $u^k$ represents the out of plane elastic displacement. The most important quantities from an engineering point of view are the stresses, in this case represented by $\nabla u^k$.

It is well known that the solution $u^k$ satisfies $\|u^k\|_{C_{\alpha}^2(D_i)} < \infty$. In fact, if $\partial D_1$ and $\partial D_2$ are $C_{\alpha}^{m,\alpha}$, we have $\|u^k\|_{C_{\alpha}^{m,\alpha}(D_i)} < \infty$. Such results do not require $D_i$ to be convex and hold for general elliptic systems with piecewise smooth coefficients; see e.g. theorem 9.1 in [9] and proposition 1.6 in [8]. For a fixed $0 < k < \infty$, the $C_{\alpha}^{m,\alpha}(D_i)$-norm of the solution might tend to infinity as $\varepsilon \to 0$. Babuska, Anderson, Smith and Levin [4] were interested in linear elliptic systems of elasticity arising from the study of composite material. They observed numerically that, for solution $u$ to certain homogeneous isotropic linear systems of elasticity, $\|\nabla u\|_{L^\infty}$ is bounded independently of the distance $\varepsilon$ between $D_1$ and $D_1$. Bonnetier and Vogelius [5] proved this in dimension $n = 2$ for the solution $u^k$ of (0.2) when $D_1$ and $D_2$ are two unit balls touching at a point. This result was extended by Li and Vogelius in [9] to general second order elliptic equations with piecewise smooth coefficients, where stronger $C^{1,\beta}$ estimates were established. The $C^{1,\beta}$ estimates were further extended by Li and Nirenberg in [8] to general second order elliptic systems including systems of elasticity. For higher derivative estimates, e.g. an $\varepsilon$-independent $L^\infty$-estimate of second derivatives of $u^k$ in $D_1$, we draw attention of readers to the open problem on page 894 of [8]. In [9] and [8], the ellipticity constants are assumed to be away from 0 and $\infty$. If we allow ellipticity constants to deteriorate, the situation is different. It has been shown in various papers, see e.g. [6] and [10], that when $k = \infty$ the $L^\infty$-norm of $\nabla u^k$ for the solution $u^k$ of equation (0.2) generally becomes unbounded as $\varepsilon$ tends to zero. The rate at which the $L^\infty$ norm of the gradient of a special solution has been shown in [6] to be $\varepsilon^{-1/2}$.

In this paper, we consider the perfect conductivity problem, where $k = +\infty$. It was proved by Ammari, Kang and Lim in [3] and Ammari, Kang, H. Lee, J. Lee and Lim in [2] that, when $D_1$ and $D_2$ are balls of comparable radii embedded in $\Omega = \mathbb{R}^2$, the blow-up rate of the gradient of the solution to the perfect conductivity problem is $\varepsilon^{-1/2}$ as $\varepsilon$ goes to zero; with the lower bound given in [3] and the upper bound given in [2]. Yun in [11] generalized the above mentioned result in [3] by establishing the same lower bound, $\varepsilon^{-1/2}$, for two strictly convex subdomains in $\mathbb{R}^2$. 

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In this paper, we give both lower and upper bounds to blow-up rate of the gradient for the solution to the perfect conductivity problem in a bounded matrix, where two strictly convex subdomains are embedded. Our methods apply to dimension $n \geq 3$ as well. One might reasonably suspect that the blow-up rate in dimension $n \geq 3$ should be smaller than that in dimension $n = 2$. However we prove the opposite: As $\varepsilon$ goes to zero, the blow-up rate is $\varepsilon^{-1/2}$, $(\varepsilon |\ln \varepsilon|)^{-1}$ and $\varepsilon^{-1}$ for $n = 2$, $3$ and $n \geq 4$, respectively. We also give a criteria, in terms of a linear functional of the boundary data $\varphi$, for the situation where the rate of blow-up is realized. Note that [3] and [2] contain also results for $k < \infty$.

The perfect conductivity problem is described as follows:

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \tilde{\Omega}, \\
|u|_+ &= |u|_- \quad \text{on } \partial D_1 \cup \partial D_2, \\
\nabla u &\equiv 0 \quad \text{in } D_1 \cup D_2, \\
\int_{\partial D_i} \frac{\partial u}{\partial \nu}^+ &= 0 \quad (i = 1, 2), \\
u &= \varphi \quad \text{on } \partial \Omega.
\end{aligned}
\]

where

\[\frac{\partial u}{\partial \nu}^+ := \lim_{t \to 0^+} \frac{u(x + t\nu) - u(x)}{t}.\]

Here and throughout this paper $\nu$ is the outward unit normal to the domain and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively.

The existence and uniqueness of solutions to equation (0.4) are well known, see the Appendix. Moreover, the solution $u \in H^1(\Omega)$ is the weak limit of the solutions $u_k$ to equations (0.2) as $k \to +\infty$. It can be also described as the unique function which has the “least energy” in appropriate functional space, defined as $I_\infty[u] = \min_{v \in A} I_\infty[v]$, where

\[I_\infty[v] := \frac{1}{2} \int_{\tilde{\Omega}} |\nabla v|^2, \quad v \in A,\]

\[A := \{v \in H^1_\varphi(\Omega) | \nabla v \equiv 0 \text{ in } D_1 \cup D_2\}.\]

The readers can refer to the Appendix for the proofs of the above statements.

We now state more precisely what it means by saying that the boundary of a domain, say $\Omega$, is $C^{2,\alpha}$ for $0 < \alpha < 1$: In a neighborhood of every point of $\partial \Omega$, $\partial \Omega$ is the graph of some $C^{2,\alpha}$ functions of $n - 1$ variables. We define the $C^{2,\alpha}$ norm of $\partial \Omega$, denoted as $\|\partial \Omega\|_{C^{2,\alpha}}$, as the smallest positive number $\frac{1}{a}$ such that in the $2a$–neighborhood of every point of $\partial \Omega$, identified as 0 after a possible translation and rotation of the coordinates so that $x_n = 0$ is the tangent to $\partial \Omega$ at 0, $\partial \Omega$ is given by the graph of a $C^{2,\alpha}$ function, denoted as $f$, which is defined as $|x'| < a$, the $a$–neighborhood of 0 in the tangent plane. Moreover, $\|f\|_{C^{2,\alpha}(|x'| < a)} \leq \frac{1}{a}$. 3
**Theorem 0.1** Let $\Omega, D_1, D_2 \subset \mathbb{R}^n$, $\varepsilon$ be defined as in (0.1), $\varphi \in C^2(\partial \Omega)$. Let $u \in H^1(\Omega) \cap C^1(\overline{\Omega})$ be the solution to equation (0.4). For $\varepsilon$ sufficiently small, there is a positive constant $C$ which depends only on $n, \kappa_0, r_0, \|\partial \Omega\|_{C^{2,\alpha}}, \|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$, but independent of $\varepsilon$ such that

$$
\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}} \|\varphi\|_{C^2(\partial \Omega)} \quad \text{for } n = 2,
$$

$$
\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon |\ln \varepsilon|} \|\varphi\|_{C^2(\partial \Omega)} \quad \text{for } n = 3,
$$

$$
\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} \|\varphi\|_{C^2(\partial \Omega)} \quad \text{for } n \geq 4.
$$

**Remark 0.1** We draw attention of readers to the independent work of Yun [12] where he has also established the upper bound, $\varepsilon^{-1/2}$, in $\mathbb{R}^2$. The methods are very different. Results in this paper and those in [11] and [12] do not really need $D_1$ and $D_2$ to be strictly convex, the strict convexity is only needed for the portions in a fixed neighborhood (the size of the neighborhood is independent of $\varepsilon$) of a pair of points on $\partial D_1$ and $\partial D_2$ which realize minimal distance $\varepsilon$. In fact, our proofs of Theorem 0.1-0.2 also apply, with minor modification, to more general situations where two inclusions, $D_1$ and $D_2$, are not necessarily convex near points on the boundaries where minimal distance $\varepsilon$ is realized; see discussions after the proofs of Theorem 0.1-0.2 in Section 1.3.

To prove Theorem 0.1 we first decompose the solution $u$ of equation (0.4) as follows:

$$
u = C_1 v_1 + C_2 v_2 + v_3 \quad (0.6)$$

where $C_i := C_i(\varepsilon) \ (i = 1, 2)$ be the boundary value of $u$ on $\partial D_i \ (i = 1, 2)$ respectively, and $v_i \in C^2(\overline{\Omega}) \ (i = 1, 2, 3)$ satisfies

$$
\begin{align*}
\Delta v_1 &= 0 & \text{in } \tilde{\Omega}, \\
v_1 &= 1 & \text{on } \partial D_1, \quad v_1 = 0 & \text{on } \partial D_2 \cup \partial \Omega, \\
\end{align*}
$$

$$
\begin{align*}
\Delta v_2 &= 0 & \text{in } \tilde{\Omega}, \\
v_2 &= 1 & \text{on } \partial D_2, \quad v_2 = 0 & \text{on } \partial D_1 \cup \partial \Omega, \\
\end{align*}
$$

$$
\begin{align*}
\Delta v_3 &= 0 & \text{in } \tilde{\Omega}, \\
v_3 &= 0 & \text{on } \partial D_1 \cup \partial D_2, \quad v_3 = \varphi & \text{on } \partial \Omega. \\
\end{align*}
$$

Define

$$Q_\varepsilon[\varphi] := \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_3}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu}, \quad (0.10)$$

then $Q_\varepsilon : C^2(\partial \Omega) \rightarrow \mathbb{R}$ is a linear functional.
Theorem 0.2 With the same conditions in Theorem \[\text{0.1}\] let \( u \in H^1(\Omega) \cap C^1(\bar{\Omega}) \) be the solution to equation \((0.4)\). For \( \varepsilon \) sufficiently small, there exists a positive constant \( C \) which depends on \( n, \kappa_0, \rho_0, \|\partial\Omega\|_{C^{2,\alpha}}, \|\partial D_1\|_{C^{2,\alpha}}, \|\partial D_2\|_{C^{2,\alpha}} \) and \( \|\varphi\|_{C^2(\partial\Omega)} \), but is independent of \( \varepsilon \) such that

\[
\|\nabla u\|_{L^\infty(e\Omega)} \geq \frac{|Q_\varepsilon[\varphi]|}{C} \cdot \frac{1}{\varepsilon^\alpha} \quad \text{for } n = 2,
\]

\[
\|\nabla u\|_{L^\infty(e\Omega)} \geq \frac{|Q_\varepsilon[\varphi]|}{C} \cdot \frac{1}{\varepsilon |\ln\varepsilon|} \quad \text{for } n = 3,
\]

\[
\|\nabla u\|_{L^\infty(e\Omega)} \geq \frac{|Q_\varepsilon[\varphi]|}{C} \cdot \frac{1}{\varepsilon} \quad \text{for } n \geq 4.
\]

(0.11)

Remark 0.2 If \( \varphi \equiv 0 \), then the solution to equation \((0.4)\) is \( u \equiv 0 \). Theorem \[\text{0.1}\] and Theorem \[\text{0.2}\] are obvious in this case. So we only need to prove them for \( \|\varphi\|_{C^2(\partial\Omega)} = 1 \), by considering \( u/\|\varphi\|_{C^2(\partial\Omega)} \).

Remark 0.3 It is interesting to know when \( |Q_\varepsilon[\varphi]| \geq \frac{1}{C} \) for some positive constant \( C \) independent of \( \varepsilon \). Roughly speaking \( Q_\varepsilon[\varphi] \to Q^*[\varphi] \) as \( \varepsilon \to 0 \), and this amounts to \( Q^*[\varphi] \neq 0 \). For details, see Section 2.

Theorem \[\text{0.1}-\text{0.2}\] can be extended to equations with more general coefficients as follows: Let \( n, \Omega, D_1, D_2, \varepsilon \) and \( \varphi \) be same as in Theorem \[\text{0.1}\] and let

\[ A_2(x) := (a_{ij}^2(x)) \in C^2(\bar{\Omega}) \]

be \( n \times n \) symmetric matrix functions in \( \bar{\Omega} \) satisfying for some constants \( 0 < \lambda \leq \Lambda < \infty \),

\[ \lambda|\xi|^2 \leq a_{ij}^2(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n, \]

and \( a_{ij}^2(x) \in C^2(\Omega\setminus\omega) \).

We consider

\[
\begin{align*}
\partial_{x_i} \left( a_{ij}^2(x) \partial_{x_j} u \right) &= 0 \quad \text{in } \bar{\Omega}, \\
u \big|_+ &= u \big|_- \quad \text{on } \partial D_1 \cup \partial D_2, \\
\nabla u &= 0 \quad \text{in } D_1 \cup D_2, \\
\int_{\partial D_i} a_{ij}^2(x) \partial_{x_i} u \nu_j \big|_+ &= 0 \quad (i = 1, 2), \\
u &= \varphi \quad \text{on } \partial\Omega.
\end{align*}
\]

(0.12)

where repeated indices denote as usual summations.

Here is an extension of Theorem \[\text{0.1}\]
Theorem 0.3  With the above assumptions, let $u \in H^1(\Omega) \cap C^1(\overline{\Omega})$ be the solution to equation (0.12). For $\varepsilon$ sufficient small, there is a positive constant $C$ which depends only on $n$, $\kappa_0$, $r_0$, $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$, $\lambda$, $\Lambda$ and $\|A_2\|_{C^2(\overline{\Omega})}$, but independent of $\varepsilon$ such that estimate (0.5) holds.

Similar to the decomposition formula (0.6), we decompose the solution $u$ of equation (0.12) as follows:

$$u = C_1 V_1 + C_2 V_2 + V_3$$

where $C_i := C_i(\varepsilon)$ $(i = 1, 2)$ be the boundary value of $u$ on $\partial D_i$ $(i = 1, 2)$ respectively, and $V_i \in C^2(\overline{\Omega})$ $(i = 1, 2, 3)$ satisfies

$$\begin{align*}
\partial_{x_j} \left( a_{i,j}^2(x) \partial_{x_i} V_1 \right) &= 0 \quad \text{in } \tilde{\Omega}, \\
V_1 &= 1 \quad \text{on } \partial D_1, \quad V_1 = 0 \quad \text{on } \partial D_2 \cup \partial \Omega, \\
\partial_{x_j} \left( a_{i,j}^2(x) \partial_{x_i} V_2 \right) &= 0 \quad \text{in } \tilde{\Omega}, \\
V_2 &= 1 \quad \text{on } \partial D_2, \quad V_2 = 0 \quad \text{on } \partial D_1 \cup \partial \Omega, \\
\partial_{x_j} \left( a_{i,j}^2(x) \partial_{x_i} V_3 \right) &= 0 \quad \text{in } \tilde{\Omega}, \\
V_3 &= 0 \quad \text{on } \partial D_1 \cup \partial D_2, \quad V_3 = \varphi \quad \text{on } \partial \Omega.
\end{align*}$$

(0.13)

Define

$$Q_\varepsilon[\varphi] := \int_{\partial D_1} a_{i,j}^2(x) \partial_{x_i} V_3 \nu_j \int_{\partial \Omega} a_{i,j}^2(x) \partial_{x_i} V_2 \nu_j - \int_{\partial D_2} a_{i,j}^2(x) \partial_{x_i} V_3 \nu_j \int_{\partial \Omega} a_{i,j}^2(x) \partial_{x_i} V_1 \nu_j,$$

(0.14)

then $Q_\varepsilon : C^2(\partial \Omega) \rightarrow \mathbb{R}$ is a linear functional.

Theorem 0.4  With the same conditions in Theorem 0.3 let $u \in H^1(\Omega) \cap C^1(\overline{\Omega})$ be the solution to equation (0.12). For $\varepsilon$ sufficiently small and $Q_\varepsilon[\varphi]$ defined by (0.14), there is a positive constant $C$ which depends only on $n$, $\kappa_0$, $r_0$, $\|\partial D_1\|_{C^{2,\alpha}}$, $\|\partial D_2\|_{C^{2,\alpha}}$, $\lambda$, $\Lambda$ and $\|A_2\|_{C^2(\overline{\Omega})}$, but independent of $\varepsilon$ such that estimate (0.11) holds.

The paper is organized as follows. In Section 1 we prove Theorem 0.1–0.2. In Section 2 we give a criteria for $|Q_\varepsilon[\varphi]|$ to be bounded below by a positive constant independent of $\varepsilon$. Theorem 0.3–0.4 are proved in Section 3. In the Appendix we present some elementary results for the conductivity problem.
Proof of Theorem 0.1 and 0.2

In the introduction, we write \( u = C_1 v_1 + C_2 v_2 + v_3 \) as in (0.6). To prove our main theorems, we first estimate \( \|\nabla u\|_{L^\infty(\tilde{\Omega})} \) in terms of \( |C_1 - C_2| \), and then estimate \( |C_1 - C_2| \).

In this section we use, unless otherwise stated, \( C \) to denote various positive constants whose values may change from line to line and which depend only on \( n, \kappa_0, r_0, \|\partial\Omega\|_{C^{2,\alpha}}, \|\partial D_1\|_{C^{2,\alpha}} \) and \( \|\partial D_2\|_{C^{2,\alpha}} \).

Proposition 1.1 Under the hypotheses of Theorem 0.1, let \( u \) be the solution of equation (0.4). There exists a positive constants \( C \), such that, for sufficiently small \( \varepsilon > 0 \),

\[
\frac{1}{\varepsilon} |C_1 - C_2| \leq \|\nabla u\|_{L^\infty(\tilde{\Omega})} \leq \frac{C}{\varepsilon} |C_1 - C_2| + C \|\varphi\|_{C^2(\partial\Omega)}. \tag{1.18}
\]

To prove this proposition, we first estimate the gradients of \( v_1, v_2 \) and \( v_3 \). Without loss of generality, we may assume throughout the proof of the proposition that \( \|\varphi\|_{C^2(\partial\Omega)} = 1 \); see Remark 0.2.

Lemma 1.1 Let \( v_1, v_2 \) be defined by equations (0.7) and (0.8), then for \( n \geq 2 \), we have

\[
\|\nabla v_1\|_{L^\infty(\tilde{\Omega})} + \|\nabla v_2\|_{L^\infty(\tilde{\Omega})} \leq \frac{C}{\varepsilon}, \quad \|\frac{\partial v_1}{\partial \nu}\|_{L^\infty(\partial\Omega)} + \|\frac{\partial v_2}{\partial \nu}\|_{L^\infty(\partial\Omega)} \leq C.
\]

Proof: By the maximum principle, \( \|v_1\|_{L^\infty(\tilde{\Omega})} \leq 1 \), and since \( v_1 \) achieves constants on each connected component of \( \partial\tilde{\Omega} \), and each connected component of \( \partial\tilde{\Omega} \) is \( C^{2,\alpha} \) then the gradient estimates for harmonic functions implies that

\[
\|\nabla v_1\|_{L^\infty(\tilde{\Omega})} \leq \frac{C \|v_1\|_{L^\infty}}{\text{dist}(\partial D_1, \partial D_2)} = \frac{C}{\varepsilon}.
\]

Similarly, we can prove \( \|\nabla v_2\|_{L^\infty(\tilde{\Omega})} \leq C/\varepsilon \). The second inequality follows from the boundary estimates for harmonic functions.

Before estimating \( |\nabla v_3| \), we first prove:

Lemma 1.2 Let \( \rho \in C^2(\tilde{\Omega}) \) be the solution to:

\[
\begin{cases}
\Delta \rho = 0 & \text{in } \tilde{\Omega}, \\
\rho = 0 & \text{on } \partial D_1 \cup \partial D_2, \\
\rho = 1 & \text{on } \partial\Omega. 
\end{cases}
\tag{1.19}
\]

Then \( \|\nabla \rho\|_{L^\infty(\tilde{\Omega})} \leq C \).

Proof: Let \( \rho_i (i = 1, 2) \in C^2(\Omega \setminus \overline{D_i}) \cap C^1(\overline{\Omega \setminus D_i}) \) be the solution to:

\[
\begin{cases}
\Delta \rho_i = 0 & \text{in } \Omega \setminus \overline{D_i}, \\
\rho_i = 0 & \text{on } \partial D_i, \\
\rho_i = 1 & \text{on } \partial\Omega.
\end{cases}
\]
Again by the maximum principle and the strong maximum principle, we obtain $0 < \rho_1 < 1$ in $\Omega \setminus D_1$. Since $D_2 \subset \Omega \setminus D_1$, we have $\rho_1 > 0 = \rho$ on $\partial D_2$. And since $\rho_1 = \rho$ on $\partial D_1$ and $\partial \Omega$, therefore $\rho_1 > \rho$ on $\Omega$. Now because $\rho_1 = \rho = 0$ on $\partial D_1$ and $\partial \Omega$, therefore $\rho_1 > \rho > 0$ on $\Omega$, so

$$\|\nabla \rho\|_{L^\infty(\partial D_1)} \leq \|\nabla \rho_1\|_{L^\infty(\partial D_1)} \leq C.$$  

Similarly,

$$\|\nabla \rho\|_{L^\infty(\partial D_2)} \leq \|\nabla \rho_2\|_{L^\infty(\partial D_2)} \leq C.$$  

By the boundary estimate of harmonic functions, we know that $\|\nabla \rho\|_{L^\infty(\partial \Omega)} \leq C$.

Since $\Delta \rho = 0$ in $\tilde{\Omega}$, $\partial x_i \rho$ is also harmonic, by the maximum principle,

$$\|\nabla \rho\|_{L^\infty(\tilde{\Omega})} \leq \max \left( \|\nabla \rho\|_{L^\infty(\partial D_1)}, \|\nabla \rho\|_{L^\infty(\partial D_2)}, \|\nabla \rho\|_{L^\infty(\partial \Omega)} \right) \leq C.$$

□

Now, we estimate $|\nabla v_3|$:  

**Lemma 1.3** Let $v_3$ be defined by equation (0.9), for $n \geq 2$, we have

$$\|\nabla v_3\|_{L^\infty(\tilde{\Omega})} \leq C.$$  

**Proof:** Since $v_3 = -\rho = \rho = 0$ on $\partial D_i (i = 1, 2)$, and $-\rho \leq v_3 = \varphi \leq \rho$ on $\partial \Omega$, we have, by the maximum principle,

$$-\rho \leq v_3 \leq \rho \quad \text{in} \quad \tilde{\Omega}.$$  

It follows, for $i = 1, 2$, that

$$\|\nabla v_3\|_{L^\infty(\partial D_i)} \leq \|\nabla \rho\|_{L^\infty(\partial D_i)} \leq C.$$  

By the boundary estimate,

$$\|\nabla v_3\|_{L^\infty(\partial \Omega)} \leq C.$$  

By the harmonicity of $\partial x_i v_3$ and the maximum principle,

$$\|\nabla v_3\|_{L^\infty(\tilde{\Omega})} \leq C.$$  

□

**Remark 1.1** Without assuming $\|\varphi\|_{C^2(\partial \Omega)} = 1$, we have

$$\|\nabla v_3\|_{L^\infty(\partial D_1 \cup \partial D_2)} \leq C \|\varphi\|_{L^\infty(\partial \Omega)},$$

where $C$ has the dependence specified at the beginning of this section, except that it does not depend on $\|\partial \Omega\|_{C^{2,\alpha}}$. This is easy to see from the proof of Lemma 1.3.

The above lemma yields the main result of [1].
Corollary 1.1 ([1]) Let $B_1$ and $B_2$ be two spheres with radius $R$ and centered at $(\pm R \pm \frac{\varepsilon}{2}, 0, \cdots, 0)$, respectively. Let $H$ be a harmonic function in $\mathbb{R}^3$. Define $u$ to be the solution to

\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^3 \setminus (B_1 \cup B_2), \\
u = 0 & \text{on } \partial B_1 \cup \partial B_2, \\
u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to +\infty.
\end{cases}
\]

Then there is a constant $C$ independent of $\varepsilon$ such that

$$
\|\nabla (u - H)\|_{L^\infty(\mathbb{R}^3 \setminus (B_1 \cup B_2))} \leq C.
$$

Proof: By the maximum principle and interior estimates of harmonic functions, the $C^3$ norm of $\nu|_{B_{2R}(0)}$ is bounded by a constant independent of $\varepsilon$. Apply Lemma 1.3 with $\Omega = B_{2R}(0)$ and $\varphi = u|_{B_{2R}(0)}$, we immediately obtain the above corollary. \(\square\)

With the above lemmas, we give the

Proof of Proposition 1.1 Since $u = C_1$ on $\partial D_1$, $u = C_2$ on $\partial D_2$, dist($\partial D_1, \partial D_2$) = $\varepsilon$, by the mean value theorem, $\exists \xi \in \tilde{\Omega}$ such that

$$
\|\nabla \nu\|_{L^\infty(\tilde{\Omega})} \geq |\nabla \nu(\xi)| \geq \frac{|C_1 - C_2|}{\varepsilon}.
$$

By the decomposition formula (0.6),

$$
\nabla \nu = C_1 \nabla v_1 + C_2 \nabla v_2 + \nabla v_3 = (C_1 - C_2) \nabla v_1 + C_2 \nabla (v_1 + v_2) + \nabla v_3.
$$

Hence,

$$
\|\nabla \nu\|_{L^\infty(\tilde{\Omega})} \leq |C_1 - C_2| \|\nabla v_1\|_{L^\infty(\tilde{\Omega})} + |C_2| \|\nabla (v_1 + v_2)\|_{L^\infty(\tilde{\Omega})} + \|\nabla v_3\|_{L^\infty(\tilde{\Omega})}.
$$

By Lemma 1.2 since $v_1 + v_2 = 1 - \rho$ in $\tilde{\Omega}$, we have

$$
\|\nabla (v_1 + v_2)\|_{L^\infty(\tilde{\Omega})} = \|\nabla (1 - \rho)\|_{L^\infty(\tilde{\Omega})} = \|\nabla \rho\|_{L^\infty(\tilde{\Omega})} \leq C.
$$

Using the fact we showed in the Appendix, $\|u\|_{H^1(\Omega)} \leq C$, so $|C_1| + |C_2| \leq C$. Therefore using also Lemma 1.1 we obtain,

$$
\|\nabla \nu\|_{L^\infty(\tilde{\Omega})} \leq \frac{C}{\varepsilon} |C_1 - C_2| + C.
$$

This proof is now completed. \(\square\)

Later we will give an estimate of $|C_1 - C_2|$, which, together with Proposition 1.1 yields the lower and upper bounds of $\|\nabla \nu\|_{L^\infty(\tilde{\Omega})}$ for strictly convex subdomains $D_1$ and $D_2$. 9
1.1 Estimate of $|C_1 - C_2|$ 

Back to the decomposition formula (0.6), denote

\[ a_{ij} = \int_{\partial D_i} \frac{\partial v_j}{\partial \nu} \quad (i, j = 1, 2), \quad b_i = \int_{\partial D_i} \frac{\partial v_3}{\partial \nu} \quad (i = 1, 2). \tag{1.20} \]

We first give some basic lemmas:

**Lemma 1.4** Let $a_{ij}$ and $b_i$ be defined as in (1.20), then they satisfy the following:

1. $a_{12} = a_{21} > 0$, $a_{11} < 0$, $a_{22} < 0$,
2. $-C \leq a_{11} + a_{21} \leq -\frac{1}{C}$, $-C \leq a_{22} + a_{12} \leq -\frac{1}{C}$,
3. $|b_1| \leq C$, $|b_2| \leq C$.

By the fourth line of equation (0.4), $C_1$ and $C_2$ satisfy

\[
\begin{align*}
   a_{11}C_1 + a_{12}C_2 + b_1 &= 0, \\
   a_{21}C_1 + a_{22}C_2 + b_2 &= 0.
\end{align*} \tag{1.21}
\]

By solving the above linear system, using $a_{12} = a_{21}$ and $a_{11}a_{22} - a_{12}a_{21} > 0$ which follows from Lemma 1.4 we obtain

\[
C_1 = \frac{-b_1a_{22} + b_2a_{12}}{a_{11}a_{22} - a_{12}^2}, \quad C_2 = \frac{-b_2a_{11} + b_1a_{12}}{a_{11}a_{22} - a_{12}^2}, \tag{1.22}
\]

and therefore,

\[
|C_1 - C_2| = \frac{|b_1 - \alpha b_2|}{|a_{11} - \alpha a_{12}|}, \quad \text{where} \quad \alpha = \frac{a_{11} + a_{12}}{a_{22} + a_{12}} > 0. \tag{1.23}
\]

Based on this formula, we will give the estimates for $|a_{11} - \alpha a_{12}|$ and $|b_1 - \alpha b_2|$, then the estimate for $|C_1 - C_2|$ follows immediately.

**Proof of Lemma 1.4** (1) By the maximum principle and the strong maximum principle,

\[ 0 < v_1 < 1 \quad \text{in} \quad \Omega. \]

By the Hopf Lemma, we know that

\[
\frac{\partial v_1}{\partial \nu} \bigg|_{\partial D_1} < 0, \quad \frac{\partial v_1}{\partial \nu} \bigg|_{\partial D_2} > 0, \quad \frac{\partial v_1}{\partial \nu} \bigg|_{\partial \Omega} < 0.
\]

Similarly,

\[
\frac{\partial v_2}{\partial \nu} \bigg|_{\partial D_1} > 0, \quad \frac{\partial v_2}{\partial \nu} \bigg|_{\partial D_2} < 0, \quad \frac{\partial v_2}{\partial \nu} \bigg|_{\partial \Omega} < 0.
\]

Thus $a_{11} < 0$, $a_{12} > 0$, $a_{21} > 0$ and $a_{22} < 0$. 

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Also, since \( v_1 \) and \( v_2 \) are the solutions of equations (0.7) and equations (0.8), respectively, we have

\[
0 = \int_{\tilde{\Omega}} \Delta v_1 \cdot v_2 - \int_{\tilde{\Omega}} \Delta v_2 \cdot v_1 = -\int_{\partial D_2} \frac{\partial v_1}{\partial \nu} \cdot 1 + \int_{\partial D_1} \frac{\partial v_2}{\partial \nu} \cdot 1 = -a_{21} + a_{12},
\]

i.e. \( a_{21} = a_{12} \).

(2) We will prove the first inequality, the second one stands with the same reason.

By the harmonicity of \( v_1 \) in \( \tilde{\Omega} \),

\[
a_{11} + a_{21} = -\int_{\tilde{\Omega}} \Delta v_1 + \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} < 0.
\]

By Lemma 1.1

\[
a_{11} + a_{21} = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \geq -C.
\]

On the other hand, since \( 0 < v_1 < 1 \) in \( \tilde{\Omega} \) and \( v_1 = 1 \) on \( \partial D_1 \), by the boundary gradient estimates of a harmonic function, \( \exists B(\bar{x}, 2\bar{r}) \subset \tilde{\Omega} \), such that \( v_1 > 1/2 \) in \( B(\bar{x}, \bar{r}) \), where \( \bar{r} \) is independent of \( \varepsilon \). Let \( \rho \in C^2(\Omega \setminus D_2 \cup B(\bar{x}, \bar{r})) \cup C^1(\partial \Omega \cup \partial D_2 \cup \partial B(\bar{x}, \bar{r})) \) be the solution of the following equation:

\[
\begin{aligned}
  \Delta \rho &= 0 \quad \text{in} \quad \Omega \setminus D_2 \cup B(\bar{x}, \bar{r}), \\
  \rho &= 1/2 \quad \text{on} \quad \partial B(\bar{x}, \bar{r}) \\
  \rho &= 0 \quad \text{on} \quad \partial D_2 \cup \partial \Omega.
\end{aligned}
\]

By the maximum principle and the strong maximum principle, \( 0 < \rho < 1/2 \) in \( \Omega \setminus D_2 \cup B(\bar{x}, \bar{r}) \). A contradiction argument based on the Hopf Lemma yields,

\[
-\frac{\partial \rho}{\partial \nu} \geq \frac{1}{C} \quad \text{on} \quad \partial \Omega.
\]

On the other hand, since \( \rho \leq v_1 \) on the boundary of \( \Omega \setminus D_1 \cup D_2 \cup B(\bar{x}, \bar{r}) \), we obtain, via the maximum principle, \( 0 < \rho \leq v_1 \) in \( \Omega \setminus D_1 \cup D_2 \cup B(\bar{x}, \bar{r}) \). It follows, using \( \rho = v_1 = 0 \) on \( \partial \Omega \), that

\[
\frac{\partial v_1}{\partial \nu} \leq \frac{\partial \rho}{\partial \nu} \quad \text{on} \quad \partial \Omega.
\]

Thus,

\[
a_{11} + a_{21} = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \leq \int_{\partial \Omega} \frac{\partial \rho}{\partial \nu} \leq -\frac{1}{C}.
\]

(3) Clearly,

\[
0 = \int_{\tilde{\Omega}} \Delta v_1 \cdot v_3 - \int_{\tilde{\Omega}} \Delta v_3 \cdot v_1 = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \cdot \varphi + \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \cdot 1 = \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \cdot \varphi + b_1.
\]
Thus,
\[ |b_1| = \left| \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \cdot \varphi \right| \leq \int_{\partial\Omega} \left| \frac{\partial v_1}{\partial \nu} \right| \leq C. \]
Thus, we finished the proof. \[\Box\]

### 1.2 Estimate of $|a_{11} - \alpha a_{12}|$

By a translation and rotation of the axis, we may assume without loss of generality that $D_1$, $D_2$ are two strictly convex subdomains in $\Omega \subset \mathbb{R}^n$ which satisfy the following:

\[ (-\varepsilon/2, 0') \in \partial D_1, \ (\varepsilon/2, 0') \in \partial D_2, \ \varepsilon = \text{dist}(\partial D_1, \partial D_2) = \text{dist}(D_1, D_2). \quad (1.25) \]

Near the origin, we can find a ball $B(0, r)$ such that the portion of $\partial D_i$ ($i = 1, 2$) in $B(0, r)$ is strictly convex, where $r > 0$ is independent of $\varepsilon$. Then $\partial D_1 \cap B(0, r)$ and $\partial D_2 \cap B(0, r)$ can be represented by the graph of $x_1 = f(x') - \varepsilon/2$ and $x_1 = g(x') + \varepsilon/2$ respectively, where $x' = (x_2, \ldots, x_n)$. Thus $f(0') = g(0') = 0$, $\nabla f(0') = \nabla g(0') = 0$, and $-CI \leq (D^2 f(0')) \leq -\frac{1}{C} I$, $\frac{1}{C} I \leq (D^2 g(0')) \leq CI$.

With these notations, we first estimate $a_{ii}$ for $i = 1, 2$.

**Lemma 1.5** Let $a_{ii}$ be defined by (1.20), then

\[ \frac{1}{C \sqrt{\varepsilon}} \leq -a_{ii} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \text{for } n = 2, \ i = 1, 2. \]

**Proof:** It suffices to prove it for $a_{11}$. By the harmonicity of $v_1$, we have

\[ 0 = \int_\Omega \Delta v_1 \cdot v_1 = -\int_\Omega |\nabla v_1|^2 - \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} = -\int_\Omega |\nabla v_1|^2 - a_{11}, \]

i.e.

\[ a_{11} = -\int_\Omega |\nabla v_1|^2. \]

Now we construct a function (here in $\mathbb{R}^2$, we let $x = x_1$, $y = x_2$)

\[ \overline{w}(x, y) = -\frac{x - g(y) - \frac{\varepsilon}{2}}{g(y) - f(y) + \varepsilon} \quad (1.26) \]

on $O_r := \Omega \cap \{(x, y) \mid |y| < r\}$. It is clear that $\overline{w}(x, y)$ is linear in $x$ for fixed $y$ and

\[ \overline{w} \big|_{B(0, r) \cap \partial D_1} = 1; \quad \overline{w} \big|_{B(0, r) \cap \partial D_2} = 0, \]

so we have

\[ \int_{f(y) - \frac{\varepsilon}{2}}^{g(y) + \frac{\varepsilon}{2}} |\partial_x \overline{w}(x, y)|^2 dx \leq \int_{f(y) - \frac{\varepsilon}{2}}^{g(y) + \frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2 dx, \]
Let \( \psi \). On the other hand, we can find \( v \)

Indeed,

\[
\int_0^{r/2} \int_0^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2 dxdy \geq \int_0^{r/2} \frac{1}{g(y) - f(y) + \varepsilon} dy
\]

\[
\geq \frac{1}{C} \int_0^{r/2} \frac{1}{y^2 + \varepsilon} dy = \frac{1}{C \sqrt{\varepsilon}}.
\]

Thus

\[
-a_{11} \geq \int_0^{r/2} \int_0^{g(y)+\frac{\varepsilon}{2}} |\partial_x v_1(x, y)|^2 dxdy \geq \frac{1}{C \sqrt{\varepsilon}}.
\]

On the other hand, we can find \( \psi \in C^2(\overline{\Omega}) \) such that

\[
\psi = 0 \text{ on } \overline{O}_{r/8}, \quad \psi = 1 \text{ on } \partial D_1 \setminus \overline{O}_{r/4}, \quad \psi = 0 \text{ on } \partial D_2 \setminus \overline{O}_{r/4}, \quad
\]

\[
\psi = 0 \text{ on } \partial \Omega, \quad \text{and} \quad \|\nabla \psi\|_{L^\infty(\Omega)} \leq C.
\]

We can also find \( \rho \in C^2(\overline{\Omega}) \) such that

\[
0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } \overline{O}_{r/2}, \quad \rho = 0 \text{ on } \overline{O} \setminus O_r \text{ and } |\nabla \rho| \leq C.
\]

Let \( w = \rho \overline{w} + (1 - \rho)\psi \), then \( w = 1 = v_1 \) on \( \partial D_1 \setminus \overline{O}_{r/4} \); \( w = 0 = v_1 \) on \( \partial D_2 \setminus \overline{O}_{r/4} \); \( w = 0 = v_1 \) on \( \partial \Omega \) and \( w = \overline{w} \) on \( \overline{O}_{r/2} \). Then by the properties of \( \psi, \rho \) and the harmonicity of \( v_1 \), we have

\[
\int_{\overline{\Omega}} |\nabla v_1|^2 \leq \int_{\overline{\Omega}} |\nabla w|^2 \leq \int_{\overline{\Omega} \cap \overline{O}_{r/2}} |\nabla \overline{w}|^2 + C.
\]

A calculation gives

\[
\partial_y \overline{w} = \frac{g'(y)(g(y) - f(y) + \varepsilon) - (g(y) - x + \frac{\varepsilon}{2})(g'(y) - f'(y))}{(g(y) - f(y) + \varepsilon)^2}.
\]

We will show \( \int_{\overline{\Omega} \cap \overline{O}_{r/2}} |\partial_y \overline{w}|^2 \leq C. \)

Indeed,

\[
\int_0^{r/2} \int_0^{g(y)+\frac{\varepsilon}{2}} |\partial_y \overline{w}(x, y)|^2 dxdy
\]

\[
\leq 2 \int_0^{r/2} \int_0^{g(y)+\frac{\varepsilon}{2}} \left( \frac{g'(y)^2}{(g(y) - f(y) + \varepsilon)^2} + \frac{(g(y) - x + \frac{\varepsilon}{2})(g'(y) - f'(y))^2}{(g(y) - f(y) + \varepsilon)^4} \right) dxdy
\]

\[
= 2 \int_0^{r/2} \frac{g'(y)^2}{g(y) - f(y) + \varepsilon} dy + 2 \int_0^{r/2} \frac{(g'(y) - f'(y))^2}{g(y) - f(y) + \varepsilon} dy
\]

\[
\leq C \int_0^{r/2} \frac{y^2}{y^2 + \varepsilon} dy + C \int_0^{r/2} \frac{y^2}{y^2 + \varepsilon} dy
\]

\[
\leq C.
\]

(1.29)
Then by (1.28) and (1.29)

\[
|a_{11}| = \int_{\tilde{\Omega}} |\nabla v_1|^2 \leq \int_{\tilde{\Omega} \cap O_{r/2}} |\nabla w|^2 + C \\
\leq C \int_0^{r/2} \int_{f(y) - \frac{\varepsilon}{2}}^{g(y) + \frac{\varepsilon}{2}} |D_x \overline{w}(x, y)|^2 dy dx + C \\
= C \int_0^{r/2} \frac{1}{g(y) - f(y) + \varepsilon} dy + C \leq C \int_0^{r/2} \frac{1}{y^2 + \varepsilon} dy + C \\
\leq \frac{C}{\sqrt{\varepsilon}}.
\]

The proof is completed. \(\square\)

Similarly, we have

**Lemma 1.6** Let \(a_{ii}\) be defined by (1.20),

\[
\frac{1}{C} |\ln \varepsilon| \leq -a_{ii} \leq C |\ln \varepsilon|, \quad \text{for } n = 3, \ i = 1, 2.
\]

*Proof:* We consider

\[
\overline{w}(x_1, x') = -\frac{x - g(x') - \frac{\varepsilon}{2}}{g(x') - f(x') + \varepsilon}
\]

on \(O_{r/2} := \tilde{\Omega} \cap \{(x_1, x') \mid |x'| < \frac{r}{2}\}\). Use the same proof in Lemma 1.5 we have

\[
\int_0^{r/2} \int_{f(x') - \frac{\varepsilon}{2}}^{g(x') + \frac{\varepsilon}{2}} |\partial_{x'} \overline{w}(x_1, x')|^2 dx_1 dx' \leq C.
\]

Therefore, it suffices to verify that

\[
\int_{\tilde{\Omega} \cap O_{r/2}} |\partial_{x_1} \overline{w}(x_1, x')|^2 \sim |\ln \varepsilon|.
\]

Indeed,

\[
\int_{\tilde{\Omega} \cap O_{r/2}} |\partial_{x_1} \overline{w}(x_1, x')|^2 = \int_{|x'| < r/2} \frac{1}{g(x') - f(x') + \varepsilon} dx' \sim \int_0^{r/2} \frac{t}{Ct^2 + \varepsilon} dt \sim |\ln \varepsilon|.
\]

This completes the proof. \(\square\)

**Lemma 1.7** Let \(a_{ii}\) be defined by (1.20),

\[
\frac{1}{C} \leq -a_{ii} \leq C \quad \text{for } n \geq 4, \ i = 1, 2.
\]
Proof: We only need
\[
\int_{|x'|<r/2} \frac{1}{g(x') - f(x') + \varepsilon} dx' \sim \int_{0}^{r/2} \frac{t^{n-2}}{Ct^2 + \varepsilon} dt \sim C.
\]
The proof is completed. □

Lemma 1.8 Let \( \alpha \) be defined by (1.23), we have
\[
\frac{1}{C} \leq \alpha \leq C.
\]
Proof: By the definition of \( \alpha \) and using the second statement in Lemma 1.4, we are done. □

To summarize, we have

Proposition 1.2 Let \( a_{ij} \) and \( \alpha \) be defined by (1.20) and (1.23), we have
1. \( \frac{1}{C\sqrt{\varepsilon}} \leq |a_{11} - \alpha a_{12}| \leq \frac{C}{\sqrt{\varepsilon}} \) for \( n = 2 \),
2. \( \frac{1}{C} |\ln \varepsilon| \leq |a_{11} - \alpha a_{12}| \leq C |\ln \varepsilon| \) for \( n = 3 \),
3. \( \frac{1}{C} \leq |a_{11} - \alpha a_{12}| \leq C \) for \( n \geq 4 \).

Proof: Since \( a_{11} < 0, a_{12} > 0, a_{11} + a_{12} < 0 \) and \( \alpha > 0 \), we have
\[
|a_{11}| < |a_{11} - \alpha a_{12}| < (1 + \alpha)|a_{11}|.
\]
Combining the results of Lemma 1.5, Lemma 1.6, Lemma 1.7 and Lemma 1.8, the proof is completed. □

1.3 Estimate of \( |b_1 - \alpha b_2| \)

Proposition 1.3 Let \( b_1, b_2, \alpha \) and \( Q_{\varepsilon}[\varphi] \) be defined by (1.20), (1.23) and (0.10), we have
\[
\frac{|Q_{\varepsilon}[\varphi]|}{C} \leq |b_1 - \alpha b_2| \leq C\|\varphi\|_{C^2(\partial \Omega)}.
\]

Proof: Combining the third result in Lemma 1.4 and Lemma 1.8 we have
\[
|b_1 - \alpha b_2| \leq |b_1| + |\alpha||b_2| \leq C\|\varphi\|_{C^2(\partial \Omega)}.
\]
On the other hand, by the definition and the harmonicity of \( v_1 \) and \( v_2 \) and using Lemma 1.4, we obtain
\[
|b_1 - \alpha b_2| = \frac{|a_{22} + a_{12}|}{|a_{22} + a_{12}|} \geq \frac{1}{C} \cdot \left| \int_{\partial D_1} \frac{\partial v_3}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_3}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \right| = \frac{|Q_{\varepsilon}[\varphi]|}{C}.
\]

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This completes the proof. □

Now we are ready to prove our two main theorems:

**Proof of Theorem 0.1-0.2** By Proposition 1.1 and (1.23), then using Proposition 1.2, 1.3 we are done. □

As we mentioned in Remark 0.1 the strict convexity assumption of the two inclusions can be weakened. In fact, our proofs of Theorem 0.1-0.2 apply, with minor modification, to more general situations:

In $\mathbb{R}^n$, $n \geq 2$, under the same assumptions in the beginning of Section 1.2 except for the strict convexity condition, $\partial D_1 \cap B(0, r)$ and $\partial D_2 \cap B(0, r)$ can be represented by the graph of $x_1 = f(x') - \frac{r}{2}$ and $x_1 = g(x') + \frac{r}{2}$, then $f(0') = g(0') = 0$, $\nabla (g - f)(0') = 0$. Assume further that

$$\lambda_0 |x'|^{2m} \leq g(x') - f(x') \leq \lambda_1 |x'|^{2m}, \quad \forall |x'| \leq r/2,$$

for some $\varepsilon$-independent $\lambda_0, \lambda_1 > 0, m \geq 1 \in \mathbb{Z}$.

Under the above assumption, let $u \in H^1(\Omega) \cap C^1(\overline{\Omega})$ be the solution to equation (0.4). For $\varepsilon$ sufficiently small, there exist positive constants $C$ and $C'$, such that

$$\frac{|Q_\varepsilon[\varphi]|}{C'} \cdot |\varphi| \cdot \varepsilon^{- \frac{n+1}{2m}} \leq \|\nabla u\|_{L^\infty(\Omega)} \leq C \|\varphi\|_{C^2(\partial \Omega)} \cdot \varepsilon^{- \frac{n-1}{2m}}, \quad \text{if } n - 1 < 2m,$$

$$\frac{|Q_\varepsilon[\varphi]|}{C'} \cdot \frac{1}{\varepsilon} \leq \|\nabla u\|_{L^{\infty}(\Omega)} \leq C \|\varphi\|_{C^2(\partial \Omega)} \cdot \frac{1}{\varepsilon \ln \varepsilon}, \quad \text{if } n - 1 = 2m,$$

$$\frac{|Q_\varepsilon[\varphi]|}{C'} \cdot \frac{1}{\varepsilon} \leq \|\nabla u\|_{L^{\infty}(\Omega)} \leq C \|\varphi\|_{C^2(\partial \Omega)} \cdot \frac{1}{\varepsilon}, \quad \text{if } n - 1 > 2m,$$

where $Q_\varepsilon[\varphi]$ is defined by (0.10), and $C$ depends on $n, m, \lambda_0, \lambda_1, r_0, \|\partial \Omega\|_{C^{2,\alpha}}$, $\|\partial D_1\|_{C^{2,\alpha}}$ and $\|\partial D_2\|_{C^{2,\alpha}}$, $C'$ depends on the same as $C$ and also $\|\varphi\|_{C^2(\partial \Omega)}$, but both are independent of $\varepsilon$.

The proof is essentially the same except for the computation of $\int_{\Omega} \|\nabla v\|$. In fact,

$$\int_0^{r/2} \int_{f(x') - \frac{r}{2}}^{g(x') + \frac{r}{2}} \left| \partial_x \varphi(x_1, x') \right|^2 dx_1 dx' \leq C,$$

still holds. Then by (1.27) and (1.30) we only need to calculate

$$\int_{|x'| < r/2} \frac{1}{g(x') - f(x') + \varepsilon} dx' \sim \int_0^{r/2} \frac{\rho^{n-2}}{\rho^{2m} + \varepsilon} d\rho.$$

Indeed, if $n - 1 < 2m$,

$$\int_0^{r/2} \frac{\rho^{n-2}}{\rho^{2m} + \varepsilon} d\rho = \varepsilon^{- \frac{n-1}{2m}} \int_0^{r/2} \frac{\rho^{n-2}}{s^{2m}} ds \sim C\varepsilon^{- \frac{n-1}{2m}},$$
if $n - 1 = 2m$,
\[ \int_0^{r/2} \frac{\rho^{n-2}}{\rho^{2m} + \varepsilon} d\rho = \frac{1}{2m} \int_0^{r/2} \frac{1}{\rho^{2m} + \varepsilon} d\rho^{2m} \sim C|\ln \varepsilon|, \]
if $n - 1 > 2m$,
\[ \int_0^{r/2} \frac{\rho^{n-2}}{\rho^{2m} + \varepsilon} d\rho \sim C. \]

Therefore, we obtain (1.33) by using the same arguments in the proofs of Theorem 0.1 and Theorem 0.2.

Actually, we can replace $2m$ by any real number $\beta > 0$, the results still hold.

## 2 Estimate of $|Q_{\varepsilon}[\varphi]|$

In order to identify situations when $\|\nabla u\|_{L^\infty}$ behaves exactly as the upper bound established in Theorem 0.1, we estimate in this section $|Q_{\varepsilon}[\varphi]|$. To emphasize the dependence on $\varepsilon$, we denote $D_1, D_2$ by $D_{1\varepsilon}, D_{2\varepsilon}$, denote $\varphi$ by $\varphi_{\varepsilon}$, and denote $v_1, v_2, v_3$ defined by equation (0.7), (0.8), (0.9) as $v_{1\varepsilon}, v_{2\varepsilon}, v_{3\varepsilon}$. In this section we assume, in addition to the hypotheses in Theorem 0.1, that along a sequence $\varepsilon \to 0$ (we still denote it as $\varepsilon$), $D_{1\varepsilon} \to D_1^*, D_{2\varepsilon} \to D_2^*$ in $C^2, \alpha$ norm, $\varphi_{\varepsilon} \to \varphi^*$ in $C^{1,\alpha}(\partial \Omega)$.

We use notation $\tilde{\Omega}^* = \Omega \setminus \overline{D_1^* \cup D_2^*}$, and assume, without loss of generality, that $D_1^* \cap D_2^* = \{0\}$. We will show that as $\varepsilon \to 0$, $v_{1\varepsilon}$ converges, in appropriate sense, to $v_{1*}$ which satisfies

\[
\begin{aligned}
\Delta v_{1*} &= 0 \quad \text{in } \tilde{\Omega}^*, \\
v_{1*} &= 1 \quad \text{on } \partial D_1^* \setminus \{0\}, \quad v_{1*} = 0 \quad \text{on } \partial \Omega \cup \partial D_2^* \setminus \{0\},
\end{aligned}
\]

\[
\begin{aligned}
\Delta v_{2*} &= 0 \quad \text{in } \tilde{\Omega}^*, \\
v_{2*} &= 1 \quad \text{on } \partial D_2^* \setminus \{0\}, \quad v_{2*} = 0 \quad \text{on } \partial \Omega \cup \partial D_1^* \setminus \{0\},
\end{aligned}
\]

\[
\begin{aligned}
\Delta v_{3*} &= 0 \quad \text{in } \tilde{\Omega}^*, \\
v_{3*} &= 0 \quad \text{on } \partial D_1^* \cup \partial D_2^*, \quad v_{3*} = \varphi^* \quad \text{on } \partial \Omega.
\end{aligned}
\]

First we prove

**Lemma 2.1** There exist unique $v_{i*} \in L^\infty(\tilde{\Omega}^*) \cap C^0(\tilde{\Omega}^* \setminus \{0\}) \cap C^2(\tilde{\Omega}^*), i = 1, 2, 3$, which solve equations (2.34), (2.35) and (2.36) respectively. Moreover, $v_{i*} \in C^1(\tilde{\Omega}^* \setminus \{0\})$.

**Proof:** The existence of solutions to the above equations can easily be obtained by Perron’s method, see theorem 2.12 and lemma 2.13 in [7]. For reader’s convenience, we give below a simple proof of the uniqueness. We only need to prove that 0 is the only solution in $L^\infty(\tilde{\Omega}^*) \cap C^0(\tilde{\Omega}^* \setminus \{0\}) \cap C^2(\tilde{\Omega}^*)$ to the following equation:

\[
\begin{aligned}
\Delta w &= 0 \quad \text{in } \tilde{\Omega}^*, \\
w &= 0 \quad \text{on } \partial \tilde{\Omega}^* \setminus \{0\}.
\end{aligned}
\]
Indeed, $\forall \varepsilon > 0$, we have
\[
|w(x)| \leq \frac{\varepsilon^{n-2} \|w\|_{L^\infty(\tilde{\Omega}^*)}}{|x|^{n-2}}, \quad \text{on } \partial(\tilde{\Omega}^* \setminus B_\varepsilon)(0).
\]

By the maximum principle,
\[
|w(x)| \leq \frac{\varepsilon^{n-2} \|w\|_{L^\infty(\tilde{\Omega}^*)}}{|x|^{n-2}}, \quad \forall x \in \tilde{\Omega}^* \setminus B_\varepsilon(0).
\]

Thus $w \equiv 0$ in $\tilde{\Omega}^*$. The additional regularity $v_i^* \in C^1(\tilde{\Omega}^* \setminus \{0\})$ follows from standard elliptic estimates and the regularity of the $\partial D_i$ and $\partial \Omega$.

**Lemma 2.2** For $i = 1, 2, 3$,
\[
v_i \varepsilon \longrightarrow v_i^* \quad \text{in } C^2_\text{loc}(\tilde{\Omega}^*), \quad \text{as } \varepsilon \to 0,
\]
\[
\int_{\partial \Omega} \frac{\partial v_i \varepsilon}{\partial \nu} \rightarrow \int_{\partial \Omega} \frac{\partial v_i^*}{\partial \nu}, \quad \text{as } \varepsilon \to 0, \quad i = 1, 2,
\]
\[
\int_{\partial D_i \varepsilon} \frac{\partial v_3 \varepsilon}{\partial \nu} \rightarrow \int_{\partial D_i^*} \frac{\partial v_3^*}{\partial \nu}, \quad \text{as } \varepsilon \to 0.
\]

**Proof:** By the maximum principle, $\{\|v_i \varepsilon\|_{L^\infty}\}$ is bounded by a constant independent of $\varepsilon$. By the uniqueness part of Lemma 2.1, we obtain (2.38) using standard elliptic estimates. By Lemma 1.3, $\{\|\nabla v_3 \varepsilon\|_{L^\infty}\}$ is bounded by some constant independent of $\varepsilon$, so $\|\nabla v_3^*\|_{L^\infty} < \infty$. Estimate (2.39) and (2.40) follow from standard elliptic estimates. The proof is completed.

Similar to $Q_\varepsilon[\varphi \varepsilon]$, we define
\[
Q^*[\varphi^*] := \int_{\partial D_1^*} \frac{\partial v_1^*}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_2^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_2^*}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_3^*}{\partial \nu} - \int_{\partial D_3^*} \frac{\partial v_3^*}{\partial \nu} \int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu},
\]
then $Q^* : C^2(\partial \Omega) \to \mathbb{R}$ is a linear functional. Let $Q_\varepsilon[\varphi \varepsilon]$ and $Q^*[\varphi^*]$ be defined by equation (0.10), (2.41), then, by the above lemmas,
\[
Q_\varepsilon[\varphi \varepsilon] \longrightarrow Q^*[\varphi^*], \quad \text{as } \varepsilon \to 0.
\]

**Corollary 2.1** If $\varphi^* \in C^2(\partial \Omega)$ satisfies $Q^*[\varphi^*] \neq 0$, then $|Q_\varepsilon[\varphi \varepsilon]| \geq \frac{1}{C}$, for some positive constant $C$ which is independent of $\varepsilon$.

In the following we give some examples to show that, in general, the rates of the lower bounds established in Theorem 0.2 are optimal.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, which is symmetric with respect to $x_1$-variable, i.e., $(x_1, x') \in \Omega$ if and only if $(-x_1, x') \in \Omega$, where $x' = (x_2, \cdots, x_n)$.
Let $D^*_1$ be a strictly convex bounded open set in $\{(x_1, x') \in \mathbb{R}^n | x_1 < 0\}$ with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, satisfying $0 \in \partial D^*_1$ and $\overline{D}^*_1 \subset \Omega$. Set $D^*_2 = \{(x_1, x') \in \mathbb{R}^n | (-x_1, x') \in D^*_1\}$.

Let $\varphi \in C^2(\partial \Omega) \{0\}$ satisfy
\[
\varphi_{\text{odd}}(x_1, x') := \frac{1}{2} [\varphi(x_1, x') - \varphi(-x_1, x')] \leq 0 \text{ (or } \geq 0),
\] on $(\partial \Omega)^+ := \{(x_1, x') \in \partial \Omega | x_1 > 0\}$.

For $\varepsilon > 0$ sufficiently small, let
\[D_1^\varepsilon := \{(x_1, x') \in \Omega | (x_1 + \varepsilon, x') \in D^*_1\},\]
\[D_2^\varepsilon := \{(x_1, x') \in \Omega | (x_1 - \varepsilon, x') \in D^*_2\},\]
\[\varphi^\varepsilon := \varphi.
\]

**Proposition 2.1** Under the above assumptions, we have $|Q_\varepsilon[\varphi]| \geq \frac{1}{C}$, for some positive constant $C$ independent of $\varepsilon$. Consequently,
\[
\|\nabla u^\varepsilon\|_{L^\infty(\Omega)} \geq \frac{1}{C\sqrt{\varepsilon}} \quad \text{for } n = 2,
\]
\[
\|\nabla u^\varepsilon\|_{L^\infty(\Omega)} \geq \frac{1}{C\varepsilon |\ln \varepsilon|} \quad \text{for } n = 3,
\]
\[
\|\nabla u^\varepsilon\|_{L^\infty(\Omega)} \geq \frac{1}{C\varepsilon} \quad \text{for } n \geq 4,
\]
where $u^\varepsilon$ is the solution to equation (0.4).

The above proposition can be easily obtained by the following lemma which gives a necessary and sufficient condition instead of condition (2.42) on $\varphi$ for the lower bounds (2.43) to hold.

Let
\[\left(v^*_3\right)_{\text{odd}}(x_1, x') := \frac{1}{2} [v^*_3(x_1, x') - v^*_3(-x_1, x')],\]
we have

**Lemma 2.3** Under the same hypotheses in Proposition 2.1 except for the condition (2.42), let $Q_\varepsilon[\varphi]$ and $(v^*_3)_{\text{odd}}(x)$ be defined by equation (0.10) and (2.44), then the following statements are equivalent:

1. For some positive constant $C$ independent of $\varepsilon$, we have $|Q_\varepsilon[\varphi]| \geq \frac{1}{C}$,
2. $\int_{\partial D^*_2} \frac{\partial (v^*_3)_{\text{odd}}}{\partial \nu} \neq 0$. 

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Proof: By symmetry, the strong maximum principle and the Hopf Lemma, we can easily obtain

$$\int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu} = \int_{\partial \Omega} \frac{\partial v_2^*}{\partial \nu} < 0.$$ 

Then

$$Q^*[\varphi] = \int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu} \left( \int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu} \right)$$

$$= \int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu} \left( \int_{\partial D_1^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} \right)$$

$$= -2 \int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu} \int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu}.$$ 

Hence, $Q^*[\varphi] \neq 0$ if and only if $\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} \neq 0$. Then by Corollary 2.1 we complete the proof. 

Proof of Proposition 2.1. Note that $(v_3^*)_{\text{odd}}(0, x') = 0$ by symmetry, and $(v_3^*)_{\text{odd}}$ is harmonic with $(v_3^*)_{\text{odd}} = \varphi_{\text{odd}} \leq 0$ (or $\geq 0$) but not identically 0 on $(\partial \Omega)^+$. Now by using the strong maximum principle and the Hopf Lemma, it is clear that $\int_{\partial D_2^*} \frac{\partial (v_3^*)_{\text{odd}}}{\partial \nu} \neq 0$. Hence, by Lemma 2.3 and Theorem 0.2 we are done.

Remark 2.1 If $\varphi = \sum_{i=1}^{n} b_i x_i$ with $b_i \in \mathbb{R}$ and $b_1 \neq 0$, then by Proposition 2.1 we have $|Q_\varepsilon[\varphi]| \geq \frac{1}{\varepsilon}$. Therefore, by Theorem 0.1 and 0.2, the blow-up rates of $\|\nabla u\|_{L^\infty(\tilde{\Omega})}$ are $\varepsilon^{-1/2}$ in dimension $n = 2$, $(\varepsilon|\ln \varepsilon|)^{-1}$ in dimension $n = 3$ and $\varepsilon^{-1}$ in dimension $n \geq 4$.

Now instead of in a bounded set $\Omega$, we consider in $\mathbb{R}^n$:

$$\begin{cases}
\Delta u_\varepsilon = 0 & \text{in } \mathbb{R}^n \setminus D_{1\varepsilon} \cup D_{2\varepsilon}, \\
u_\varepsilon \big|_+ = u_\varepsilon \big|_- & \text{on } \partial D_{1\varepsilon} \cup \partial D_{2\varepsilon}, \\
 u_\varepsilon \big|_+ = 0 & \text{in } D_{1\varepsilon} \cup D_{2\varepsilon}, \\
\int_{\partial D_{i\varepsilon}} \frac{\partial u_\varepsilon}{\partial \nu} \big|_+ = 0 & (i = 1, 2), \\
\limsup_{|x| \to \infty} |x|^{n-1}|u_\varepsilon(x) - H(x)| < \infty,
\end{cases}
\quad (2.45)$$

where $H(x)$ is a given entire harmonic function in $\mathbb{R}^n$.

we have the following result regarding the lower bound for $|\nabla u_\varepsilon|$:

Proposition 2.2 With the same assumptions on $D_{1\varepsilon}$ and $D_{2\varepsilon}$ as in Proposition 2.1 and let $H(x)$ be an entire harmonic function in $\mathbb{R}^n$ satisfying $H_{\text{odd}}(x_1, x') :=$
\[
\frac{1}{2}[H(x_1, x') - H(-x_1, x')] < 0 \text{ (or > 0) on } \mathbb{R}^n_+ := \{(x_1, x') \in \mathbb{R}^n | x_1 > 0\}, \text{ then for some positive constant } C \text{ independent of } \varepsilon, \text{ we have}
\]
\[
\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \geq \frac{1}{C \sqrt{\varepsilon}} \text{ for } n = 2,
\]
\[
\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \geq \frac{1}{C \varepsilon|\ln \varepsilon|} \text{ for } n = 3,
\]
\[
\|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \geq \frac{1}{C \varepsilon} \text{ for } n \geq 4,
\]
where \(u_\varepsilon\) is the solution to equation (2.45).

\textbf{Proof:} Step 1. First, we show that there exists a positive constant \(C\) independent of \(\varepsilon\), such that for any small \(\varepsilon > 0\),
\[
|x|^{n-1}|u_\varepsilon(x) - H(x)| \leq C, \quad \forall x \in \mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}. \tag{2.47}
\]

(i) For any bounded open set \(U \subset \mathbb{R}^n\) with \(C^1\) boundary \(\partial U\) satisfying \(\partial U \cap \overline{D_{1\varepsilon} \cup D_{2\varepsilon}} = \emptyset\), we have, in view of the first and the fourth lines in (2.45),
\[
\int_{\partial U} \frac{\partial u_\varepsilon}{\partial \nu} = \int_{U \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}}} \Delta u_\varepsilon = 0. \tag{2.48}
\]

(ii) We show that there exists a positive constant \(M\) independent of \(\varepsilon\), such that
\[
\|u_\varepsilon - H\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \leq M, \quad \forall \text{ small } \varepsilon > 0.
\]

We only need to prove
\[
\|u_\varepsilon - H\|_{L^\infty(\mathbb{R}^n \setminus \overline{D_{1\varepsilon} \cup D_{2\varepsilon}})} \leq \sum_{i=1}^2 (\max_{\overline{D_{1\varepsilon}}} H - \min_{\overline{D_{1\varepsilon}}} H). \tag{2.49}
\]

Since \(\nabla u_\varepsilon = 0\) in \(D_{1\varepsilon} \cup D_{2\varepsilon}\), \(u_\varepsilon\) is constant on each \(D_{i\varepsilon}\), denoted as \(C_i(\varepsilon)\). We know that
\[
\lim_{|x| \to \infty} (u_\varepsilon(x) - H(x)) = 0, \tag{2.50}
\]
and
\[
C_i(\varepsilon) - \max_{\overline{D_{i\varepsilon}}} H \leq u_\varepsilon - H \leq C_i(\varepsilon) - \min_{\overline{D_{i\varepsilon}}} H, \quad \text{on } D_{i\varepsilon}, \quad i = 1, 2. \tag{2.51}
\]
If (2.49) did not hold, say,
\[
\sup_{\mathbb{R}^n} (u_\varepsilon - H) > \sum_{i=1}^2 (\max_{\overline{D_{i\varepsilon}}} H - \min_{\overline{D_{i\varepsilon}}} H),
\]
then, because of (2.50) and (2.51), there would exist \(0 < a < \sup_{\mathbb{R}^n} (u_\varepsilon - H)\) such that \(U := \{x \in \mathbb{R}^n \mid (u_\varepsilon - H)(x) > a\} \neq \emptyset\) satisfies \(\partial U \cap \overline{D_{1\varepsilon} \cup D_{2\varepsilon}} = \emptyset\). We may assume,
by the Sard theorem, that $a$ is a regular value of $u_\varepsilon - H$, and therefore $\partial U$ is $C^1$. By the Hopf lemma, $\frac{\partial (u_\varepsilon - H)}{\partial \nu} < 0$ on $\partial U$, and therefore

$$\int_{\partial U} \frac{\partial (u_\varepsilon - H)}{\partial \nu} < 0.$$ 

On the other hand, using (2.48) and the harmonicity of $H$ in $U$, we have

$$\int_{\partial U} \frac{\partial (u_\varepsilon - H)}{\partial \nu} = - \int_{\partial U} \frac{\partial H}{\partial \nu} = - \int_U \Delta H = 0.$$ 

A contradiction.

(iii) Consider $w_\varepsilon(x) := u_\varepsilon(x) - H(x)$. Fix a constant $R_0 > 0$, independent of $\varepsilon$, such that $D_1^* \cup D_2^* \subset B_{R_0/2}(0)$, and let

$$\tilde{w}_\varepsilon(y) := \frac{1}{|y|^{n-2}} w_\varepsilon\left(\frac{y}{|y|^2}\right), \quad 0 < |y| < \frac{1}{R_0}.$$ 

Then $\tilde{w}_\varepsilon$ is harmonic in $B_{1/R_0}\{0\}$. By the last line of (2.45), there exists a positive constant $C(\varepsilon)$ such that

$$|\tilde{w}_\varepsilon(y)| \leq C(\varepsilon)|y|, \quad 0 < |y| < \frac{1}{R_0}.$$ 

Therefore, $\Delta \tilde{w}_\varepsilon = 0$ in $B_{1/R_0}$ and $\tilde{w}_\varepsilon(0) = 0$. By (ii), we have $|\tilde{w}_\varepsilon| \leq C$, on $\partial B_{1/R_0}$, for some positive constant $C$ independent of $\varepsilon$. Hence, $|\tilde{w}_\varepsilon| \leq C$, $|\nabla \tilde{w}_\varepsilon| \leq C$ in $B_{1/(2R_0)}$, then

$$|\tilde{w}_\varepsilon(y)| \leq C|y|, \quad |y| < \frac{1}{2R_0}.$$ 

Therefore, also using (ii), (2.47) holds.

Step 2. For $R > R_0$, let $\Omega = B_R(0)$. Let $\varphi_\varepsilon := u_\varepsilon|_{\partial \Omega}$, then by Corollary 2.1 and Theorem 0.2 it is enough to show, for some $R$, that $Q^*[\varphi^*] \neq 0$, where $\varphi^*$ is defined at the beginning of this section. By symmetry, we have

$$Q^*[\varphi^*] = \int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu} \left( \int_{\partial D_1^*} \frac{\partial v_3^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_3^*}{\partial \nu} \right).$$

Without loss of generality, we may assume $H_{odd}(x) > 0$ on $\mathbb{R}^n_+$. Recall that $v_3^*$ is the solution of (2.36) with boundary data $\varphi^*$. In the following we use notation $(v_3^*)_h$ to denote the the solution of (2.36) with boundary data $h$. Since $Q^*[\varphi^*]$ is linear on $\varphi^*$ and by symmetry $Q^*[H_{even}] = H[\varphi_{even}] = 0$, where $H_{even}(x) := H(x) - H_{odd}(x) = \frac{1}{2}[H(x_1, x') + H(-x_1, x')]$ and similar for $\varphi_{even}$, we may assume $H(x) = H_{odd}(x)$.
Now consider \( w(x) = H(x) - (v^*_3)_H(x) \). Then \( w(x) \) is harmonic in \( \tilde{\Omega}^* \) which is defined at the beginning of this section. By symmetry, \( w(-x_1, x') = -w(x_1, x') \), \( w(x) = H(x) \) on \( \partial D_1^* \cup \partial D_2^* \) and \( w(x) = 0 \) on \( \partial \Omega \). Therefore,

\[
-2 \int_{\partial D_2^*} H \frac{\partial w}{\partial \nu} = \int_{\tilde{\Omega}^*} w(x) \Delta w(x) + \int_{\tilde{\Omega}^*} |\nabla w|^2 = \int_{\tilde{\Omega}^*} |\nabla w|^2 \geq 0.
\]

On the other hand, \( (v^*_3)_H = 0 \) on \( \partial D_2^* \), \( (v^*_3)_H > 0 \) on \( (\partial \Omega)^+ \) and, by the oddness of \( (v^*_3)_H \), \( (v^*_3)_H = 0 \) on \( \{(x_1, x') \mid x_1 = 0\} \). Thus, by the maximum principle and the strong maximum principle, \( (v^*_3)_H > 0 \) in \( \tilde{\Omega}^* \) and in turn, using the Hopf lemma, \( \frac{\partial (v^*_3)_H}{\partial \nu} > 0 \) on \( \partial D_2^* \). Hence, using the harmonicity of \( H \),

\[
\max_{\partial D_2^*} H \int_{\partial D_2^*} \frac{\partial (v^*_3)_H}{\partial \nu} \geq \int_{\partial D_2^*} H \frac{\partial (v^*_3)_H}{\partial \nu} \geq \int_{\partial D_2^*} H \frac{\partial H}{\partial \nu} - \int_{\partial D_2^*} H \frac{\partial w}{\partial \nu} \geq \int_{D_2^*} |\nabla H|^2 \geq \frac{1}{C},
\]

Therefore,

\[
\int_{\partial D_2^*} \frac{\partial (v^*_3)_H}{\partial \nu} \geq \frac{1}{C}
\]

for positive constant \( C \) independent of \( R \).

For \( s_\varepsilon := \varphi_\varepsilon - H \) on \( \partial \Omega \), by step 1, there exists a constant \( C > 0 \) which is independent of \( \varepsilon \) and \( R \), such that \( \|s_\varepsilon\|_{L^\infty(\partial \Omega)} \leq CR^{1-n} \). By Remark 1.1 we have \( \|\nabla (v^*_3)s^*\|_{L^\infty(\partial D_1^* \cup \partial D_2^*)} \leq C\|s^*\|_{L^\infty(\partial \Omega)} \), thus,

\[
\left| \int_{\partial D_1^*} \frac{\partial (v^*_3)s^*}{\partial \nu} \right| \leq C \int_{\partial D_1^*} \|s^*\|_{L^\infty(\partial \Omega)} \leq CR^{1-n},
\]

for some positive constant \( C \) independent of \( \varepsilon \) and \( R \).

Therefore, for large enough \( R \),

\[
\int_{\partial D_2^*} \frac{\partial (v^*_3)_\varphi^*}{\partial \nu} = \int_{\partial D_2^*} \frac{\partial (v^*_3)_H}{\partial \nu} + \int_{\partial D_2^*} \frac{\partial (v^*_3)s^*}{\partial \nu} \geq \frac{1}{C} \neq 0.
\]

It is also clear that \( \int_{\partial \Omega} \frac{\partial w^*_\varepsilon}{\partial \nu} < 0 \). Thus,

\[
Q^*[\varphi^*] = -2 \int_{\partial \Omega} \frac{\partial v^*_1}{\partial \nu} \int_{\partial D_2^*} \frac{\partial (v^*_3)_\varphi^*}{\partial \nu} \neq 0.
\]

This proof is completed. \( \square \)

**Remark 2.2** In \( \mathbb{R}^2 \), when \( D_{1\varepsilon} \) and \( D_{2\varepsilon} \) are identical balls of radius 1, the estimate \( (2.46) \) was established in [2] under a weaker assumption \( \partial_{x_1} H(0) \neq 0 \).
3 Proof of Theorem 0.3 and 0.4

In the introduction, similar to the harmonic case, we still decompose \( u = C_1 V_1 + C_2 V_2 + V_3 \) as in (0.13).

Proposition 1.1 holds since Lemma 1.1-1.3 hold for \( V_1, V_2, V_3 \) defined by (0.14)-(0.16) and \( \rho \in C^2(\tilde{\Omega}) \) which is the solution to:

\[
\begin{cases}
\partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} \rho \right) = 0 & \text{in } \tilde{\Omega}, \\
\rho = 0 & \text{on } \partial D_1 \cup \partial D_2, \\
\rho = 1 & \text{on } \partial \Omega.
\end{cases}
\]

The proofs are essentially the same.

Now we start to estimate \( |C_1 - C_2| \). By the decomposition formula (0.13), instead of (1.20), we denote

\[
a_{lm} = \int_{\partial D_l} a^{ij}_2(x) \partial_{x_i} V_m \nu_j \quad (l, m = 1, 2),
\]

\[
b_l = \int_{\partial D_l} a^{ij}_2(x) \partial_{x_i} V_3 \nu_j \quad (l = 1, 2).
\]

Then Lemma 1.4 and (1.21)-(1.23) still hold for \( a_{lm} \) and \( b_l \) defined above.

In fact, to prove Lemma 1.4 with general coefficients, we only need to change \( \partial_x^* \) to \( a^{ij}_2(x) \partial_{x_i}^* \nu_j \), change \( \Delta^* \) in \( \partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i}^* \right) \) and change \( v_1, v_2, v_3 \) in \( V_1, V_2, V_3 \), respectively, in the original proof of Lemma 1.4. For instance, (1.24) is changed to

\[
0 = \int_{\tilde{\Omega}} \partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} V_1 \right) \cdot V_2 - \int_{\tilde{\Omega}} \partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} V_2 \right) \cdot V_1
\]

\[
= - \int_{\partial D_2} a^{ij}_2(x) \partial_{x_i} V_1 \nu_j \cdot 1 + \int_{\partial D_1} a^{ij}_2(x) \partial_{x_i} V_2 \nu_j \cdot 1
\]

\[
= - a_{11}^2 + a_{12}.
\]

Therefore, to estimate \( |C_1 - C_2| \), it is equivalent to estimating \( |a_{11} - \alpha a_{12}| \) and \( |b_1 - \alpha b_2| \).

For \( |a_{11} - \alpha a_{12}| \), Lemma 1.5-1.7 still hold for \( a_{ij}(l = 1, 2) \) defined by (3.52). The proof is quite similar and the only thing which needs to be shown is the following:

\[
0 = \int_{\tilde{\Omega}} \partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} V_1 \right) \cdot V_1
\]

\[
= - \int_{\tilde{\Omega}} a^{ij}_2(x) \partial_{x_i} V_1 \partial_{x_j} V_1 - \int_{\partial D_1} a^{ij}_2(x) \partial_{x_i} V_1 \nu_j \cdot 1
\]

\[
= - \int_{\tilde{\Omega}} a^{ij}_2(x) \partial_{x_i} V_1 \partial_{x_j} V_1 - a_{11}.
\]
i.e.

\[ a_{11} = - \int_{\Omega} a^{ij}_2(x) \partial_{x_i} V_1 \partial_{x_j} V_1. \]

Then by the uniform ellipticity of \( a^{ij}_2(x) \) and the harmonicity of \( v_1 \),

\[ |a_{11}| \geq \lambda \int_{\Omega} |\nabla V_1|^2 \geq \lambda \int_{\Omega} |\nabla v_1|^2, \]

and

\[ |a_{11}| \leq \int_{\Omega} a^{ij}_2(x) \partial_{x_i} w \partial_{x_j} w \leq \Lambda \int_{\Omega} |\nabla w|^2 \leq \Lambda \int_{\Omega_{b=O_{r/2}}} |\nabla w|^2 + C, \]

where \( w \) is defined in the proof of Lemma 1.5 with the same boundary data of \( V_1 \) and \( \overline{w} \) is defined by (1.26) and (1.31).

Thus, Lemma 1.5–1.7 follow by the same computations. Then Lemma 1.8 and Proposition 1.2 hold with the same proofs.

For \( |b_1 - \alpha b_2| \), Proposition 1.3 also holds for \( b_l(l = 1, 2) \) defined by (3.52) and \( Q_\varepsilon[\varphi] \) defined by (1.17). The proof is the same after changing \( \frac{\partial}{\partial v} \) to \( a^{ij}_2(x) \partial_{x_i}^* \nu_j \).

Combining the above propositions, we obtain our theorems.

4 Appendix

Some elementary results for the conductivity problem

Assume that in \( \mathbb{R}^n \), \( \Omega \) and \( \omega \) are bounded open sets with \( C^{2,\alpha} \) boundaries, \( 0 < \alpha < 1 \), satisfying

\[ \overline{\omega} = \bigcup_{s=1}^{m} \overline{\omega}_s \subset \Omega, \]

where \( \{\omega_s\} \) are connected components of \( \omega \). Clearly, \( m < \infty \) and \( \omega_s \) is open for all \( 1 \leq s \leq \omega \). Given \( \varphi \in C^2(\partial\Omega) \), the conductivity problem we consider is the following transmission problem with Dirichlet boundary condition:

\[
\begin{align*}
\partial_{x_j} \left\{ \left[ (ka^{ij}_1(x) - a^{ij}_2(x)) \chi_\omega + a^{ij}_2(x) \right] \partial_{x_i} u_k \right\} &= 0 \quad \text{in } \Omega, \\
u_k &= \varphi \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( k = 1, 2, 3, \ldots \), and \( \chi_\omega \) is the characteristic function of \( \omega \).

The \( n \times n \) matrixes \( A_1(x) := (a^{ij}_1(x)) \) in \( \omega \), \( A_2(x) := (a^{ij}_2(x)) \) in \( \Omega \setminus \overline{\omega} \) are symmetric and \( \exists \) a constant \( \Lambda \geq \lambda > 0 \) such that

\[ \lambda |\xi|^2 \leq a^{ij}_1(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad (\forall x \in \omega), \quad \lambda |\xi|^2 \leq a^{ij}_2(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad (\forall x \in \Omega \setminus \omega) \]
for all $\xi \in \mathbb{R}^n$ and $a_{ij}^1(x) \in C^2(\overline{\omega})$, $a_{ij}^2(x) \in C^2(\overline{\Omega \setminus \omega})$.

Equation (4.54) can be rewritten in the following form to emphasize the transmission condition on $\partial \omega$:

$$
\begin{cases}
\partial_{x_j} \left( a_{ij}^1(x) \partial_{x_i} u_k \right) = 0 & \text{in } \omega, \\
\partial_{x_j} \left( a_{ij}^2(x) \partial_{x_i} u_k \right) = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
u_k \big|_+ = u_k \big|_- & \text{on } \partial \omega, \\
a_{ij}^2(x) \partial_{x_i} u_k \partial_{x_j} \nu_j \bigg|_+ = ka_{ij}^1(x) \partial_{x_i} u_k \partial_{x_j} \nu_j \bigg|_- & \text{on } \partial \omega,
\end{cases}
\tag{4.55}
$$

Here and throughout this paper $\nu$ is the outward unit normal and the subscript $\pm$ indicates the limit from outside and inside the domain, respectively.

We list the following results which are well known and omit the proofs.

**Theorem 4.1** If $u_k \in H^1(\Omega)$ is a solution of equation (4.54), then $u_k \in C^1(\Omega \setminus \omega) \cap C^1(\overline{\omega})$ and satisfies equation (4.55).

If $u_k \in C^1(\Omega \setminus \omega) \cap C^1(\overline{\omega})$ is a solution of equation (4.55), then $u_k \in H^1(\Omega)$ and satisfies equation (4.54).

**Theorem 4.2** There exists at most one solution $u_k \in H^1(\Omega)$ to equation (4.54).

The existence of the solution can be obtained by using the variational method. For every $k$, we define the energy functional

$$I_k[v] := \frac{k}{2} \int_\omega a_{ij}^1(x) \partial_{x_i} v \partial_{x_j} v + \frac{1}{2} \int_{\Omega \setminus \overline{\omega}} a_{ij}^2(x) \partial_{x_i} v \partial_{x_j} v,
\tag{4.56}$$

where $v$ belongs to the set

$$H^1_\varphi(\Omega) := \{ v \in H^1(\Omega) \mid v = \varphi \text{ on } \partial \Omega \}.$$

**Theorem 4.3** For every $k$, there exists a minimizer $u_k \in H^1(\Omega)$ satisfying

$$I_k[u_k] = \min_{v \in H^1_\varphi(\Omega)} I_k[v].$$

Moreover, $u_k \in H^1(\Omega)$ is a solution of equation (4.54).

Comparing equation (4.55), when $k = +\infty$, the perfectly conducting problem turns out to be:

$$
\begin{cases}
\partial_{x_j} \left( a_{ij}^2(x) \partial_{x_i} u \right) = 0 & \text{in } \Omega \setminus \overline{\omega}, \\
u_+ = u_- & \text{on } \partial \omega, \\
\nabla u = 0 & \text{in } \omega, \\
\int_{\partial \omega_s} a_{ij}^2(x) \partial_{x_i} u \nu_j \bigg|_+ = 0 & (s = 1, 2, \ldots, m), \\\nu = \varphi & \text{on } \partial \Omega.
\end{cases}
\tag{4.57}
$$
We also have similar results:

**Theorem 4.4** If \( u \in H^1(\Omega) \) satisfies equation (4.57) except for the fourth line, then \( u \in C^1(\Omega \setminus \omega) \cap C^1(\overline{\omega}). \)

**Proof:** By the third line of equation (4.57), we have \( u \equiv \text{const} \) on each component of \( \omega \), so \( u \equiv \text{const} \) on each component of \( \partial \omega \). Thus \( u \equiv \text{const} \) on each component of \( \partial (\Omega \setminus \omega) \).

Since \( u \in H^1(\Omega) \) satisfies \( \partial_x \left( a^{ij}_2(x) \partial_x u_k \right) = 0 \) in \( \Omega \setminus \omega \), \( u_{|\partial \Omega} = \varphi \in C^2(\partial \Omega) \) and \( u \equiv \text{const} \) on each component of \( \partial (\Omega \setminus \omega) \), by the elliptic regularity theory, we have \( u \in C^1(\Omega \setminus \omega) \cap C^1(\overline{\omega}). \) \( \square \)

**Theorem 4.5** There exists at most one solution \( u \in H^1(\Omega) \cap C^1(\Omega \setminus \omega) \cap C^1(\overline{\omega}) \) of equation (4.57).

**Proof:** It is equivalent to showing that if \( \varphi = 0 \), equation (4.57) only has the solution \( u \equiv 0 \). Integrating by parts in the first line of equation (4.57), we have

\[
0 = -\int_{\Omega \setminus \omega} \partial_x \left( a^{ij}_2(x) \partial_x u_k \right) \cdot u
= \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_x u \partial_x u - \int_{\partial \Omega} u \cdot a^{ij}_2(x) \partial_x u \nu^j + \int_{\partial \omega} u \cdot a^{ij}_2(x) \partial_x u \nu^j
\geq \lambda \int_{\Omega \setminus \omega} |\nabla u|^2 - \int_{\partial \Omega} \varphi \cdot a^{ij}_2(x) \partial_x u \nu^j + C_s \int_{\partial \omega} a^{ij}_2(x) \partial_x u \nu^j
= \lambda \int_{\Omega \setminus \omega} |\nabla u|^2.
\]

Thus \( \nabla u = 0 \) in \( \Omega \setminus \omega \). And since \( u = \varphi = 0 \) on \( \partial \Omega \), we have \( u \equiv 0 \) in \( \Omega \setminus \omega \). Since \( u_{|+} = u_{|-} \) on \( \partial \omega \) and \( u \equiv C \) on \( \overline{\omega} \), we get \( u = 0 \) on \( \overline{\omega} \). Hence \( u \equiv 0 \) in \( \Omega \), i.e. \( u \equiv 0 \) is the only solution of (4.57) when \( \varphi = 0 \). \( \square \)

Define the energy functional

\[
I_\infty[v] := \frac{1}{2} \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_x v \partial_x v, \tag{4.58}
\]

where \( v \) belongs to the set

\[
A := \{ v \in H^1_\varphi(\Omega) \mid \nabla v \equiv 0 \text{ in } \omega \}.
\]

**Theorem 4.6** There exists a minimizer \( u \in A \) satisfying

\[
I_\infty[u] = \min_{v \in A} I_\infty[v].
\]

Moreover, \( u \in H^1(\Omega) \cap C^1(\Omega \setminus \omega) \cap C^1(\overline{\omega}) \) is a solution of equation (4.57).
\textbf{Proof:} By the lower-semi continuity of $I_\infty$ and the weakly closed property of $\mathcal{A}$, it is easy to see that the minimizer $u \in \mathcal{A}$ exists and satisfies $\partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} u \right) = 0$ in $\Omega \setminus \Omega_s$. The only thing which needs to be shown is the fourth line in equation (4.57), i.e.
\[
\int_{\partial \Omega_s} a^{ij}_2(x) \partial_{x_i} u \nu_j \big|_+ = 0, \quad s = 1, 2, \ldots, m.
\]
In fact, since $u$ is a minimizer, for any $\phi \in C_c^\infty(\Omega)$ satisfying $\phi \equiv 1$ on $\Omega_s$ and $\phi \equiv 0$ on $\Omega_t (t \neq s)$, let
\[
i(t) := I_\infty[u + t\phi] \quad (t \in \mathbb{R}),
\]
we have
\[
i'(0) := \frac{di}{dt} \bigg|_{t=0} = \int_{\Omega \setminus \Omega_s} a^{ij}_2(x) \partial_{x_i} u \phi \nu_j = 0.
\]
Therefore
\[
0 = -\int_{\Omega \setminus \Omega_s} \partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} u \right) \phi = \int_{\Omega \setminus \Omega_s} a^{ij}_2(x) \partial_{x_i} u \phi \nu_j + \int_{\partial \Omega_s} \phi \cdot a^{ij}_2(x) \partial_{x_i} u \nu_j \big|_+
\]
for all $s = 1, 2, \ldots, m$. \hfill \square

Finally, we give the relationship between $u_k$ and $u$.

\textbf{Theorem 4.7} Let $u_k$ and $u$ in $H^1(\Omega)$ be the solutions of equations (4.55) and (4.57), respectively. Then
\[
u_k \rightharpoonup u \quad \text{in} \quad H^1(\Omega), \quad \text{as} \quad k \to +\infty,
\]
and
\[
\lim_{k \to +\infty} I_k[u_k] = I_\infty[u],
\]
where $I_k$ and $I_\infty$ are defined as (4.56) and (4.58).

\textbf{Proof:} Step 1. By the uniqueness of the solution to equation (4.57), we only need to show that there exists a weak limit $u$ of a subsequence of $\{u_k\}$ in $H^1(\Omega)$ and $u$ is the solution of equation (4.57).

(1) To show that after passing to a subsequence, $u_k$ weakly converges in $H^1(\Omega)$ to some $u$.

Let $\eta \in H^1_0(\Omega)$ be fixed and satisfy $\eta \equiv 0$ on $\Omega_s$, then since $u_k$ is the minimizer of $I_k$ in $H^1_0(\Omega)$, we have
\[
\frac{\lambda}{2} \| \nabla u_k \|_{L^2(\Omega)}^2 \leq I_k[u_k] \leq I_k[\eta] = \frac{1}{2} \int_{\Omega \setminus \Omega_s} a^{ij}_2(x) \eta_{x_i} \eta_{x_j} \leq \frac{\lambda}{2} \| \eta \|_{H^1(\Omega)}^2,
\]
i.e.
\[
\| \nabla u_k \|_{L^2(\Omega)} \leq \| \eta \|_{H^1(\Omega)} = M,
\]

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where $\overline{M}$ is independent of $k$.

Since $u_k = \varphi$ on $\partial \Omega$ and $\sup_k \|u_k\|_{H^1(\Omega)} < \infty$, we have $u_k \to u$ in $H^1(\Omega)$.

(2) To show that $u$ is a solution of equation (4.57).

In fact, we only need to prove the following three conditions:

$$
\partial_{x_j} \left( a^{ij}_2(x) \partial_{x_i} u \right) = 0 \quad \text{in } \Omega \backslash \overline{\omega}, 
$$

$$
\nabla u = 0 \quad \text{in } \omega, 
$$

$$
\left. \int_{\partial \omega} a^{ij}_2(x) \partial_{x_i} u_k \nu_j \right|_+ = 0, \quad s = 1, 2, \ldots, m. 
$$

(i) For every $k$, since $u_k \in H^1(\Omega)$ is the solution of equation (4.54), then

$$
\forall \phi \in C^\infty_c(\Omega), \quad \int_{\Omega \backslash \overline{\omega}} a^{ij}_1(x) \partial_{x_i} u_k \phi_{x_j} + \int_{\Omega \backslash \overline{\omega}} a^{ij}_2(x) \partial_{x_i} u_k \phi_{x_j} = 0.
$$

Thus, $\forall \phi \in C^\infty_c(\Omega \backslash \overline{\omega}) \subset C^\infty_c(\omega)$,

$$
0 = \int_{\Omega \backslash \overline{\omega}} a^{ij}_2(x) \partial_{x_i} u_k \phi_{x_j} \longrightarrow \int_{\Omega \backslash \overline{\omega}} a^{ij}_2(x) \partial_{x_i} u \phi_{x_j},
$$

since $u_k \to u$ in $H^1(\overline{\omega}) \subset H^1(\Omega)$.

Therefore,

$$
\int_{\Omega \backslash \overline{\omega}} a^{ij}_2(x) \partial_{x_i} u \phi_{x_j} = 0, \quad \forall \phi \in C^\infty_c(\Omega \backslash \overline{\omega}),
$$

i.e. (4.59).

(ii) Let $\eta \in H^1_\varphi(\Omega)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$, then since $u_k$ is the minimizer of $I_k$ in $H^1_\varphi(\Omega)$, we have

$$
\frac{k\lambda}{2} \|\nabla u_k\|^2_{L^2(\omega)} \leq I_k[u_k] \leq I_k[\eta] = \frac{1}{2} \int_{\Omega \backslash \overline{\omega}} a^{ij}_2(x) \partial_{x_i} \eta \partial_{x_j} \eta \leq \frac{\Lambda}{2} \|\eta\|^2_{H^1(\Omega)},
$$

which implies

$$
\|\nabla u_k\|^2_{L^2(\omega)} \to 0, \quad \text{as } k \to \infty.
$$

By (1), since $u_k \to u$ in $H^1(\Omega)$, then $u_k \to u$ in $H^1(\omega)$. Therefore, by the lower-semi continuity, we get

$$
0 \leq \lambda \int_{\omega} |\nabla u|^2 \leq \int_{\omega} a^{ij}_1(x) \partial_{x_i} u \partial_{x_j} u \leq \int_{\omega} a^{ij}_1(x) \partial_{x_i} u_k \partial_{x_j} u_k \\
\leq \Lambda \|\nabla u_k\|^2_{L^2(\omega)} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.
$$

Hence, $\int_{\omega} |\nabla u|^2 = 0 \implies \nabla u \equiv 0$ in $\omega$, which is just (4.60).

(iii) By (i) and (ii), $u$ satisfies (4.59) and is either constant or $\varphi$ on each component
of $\partial(\Omega \setminus \omega)$. Thus, $u \in C^2(\Omega \setminus \omega)$. For each $s = 1, 2, \ldots, m$, we construct a function $\varrho \in C^2(\Omega \setminus \omega)$, such that $\varrho = 1$ on $\partial \omega_s$, $\varrho = 0$ on $\partial \omega_t (t \neq s)$, and $\varrho = 0$ on $\partial \Omega$.

By Green’s Identity, we have the following:

$$0 = -\int_{\Omega \setminus \omega} \partial_j \left( a^{ij}_2(x) \partial_i u_k \right) \varrho
= \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_i u_k \partial_j \varrho - \int_{\partial \Omega} \varrho \cdot a^{ij}_2(x) \partial_i u_k \nu_j \bigg|_+ + \int_{\partial \omega} \varrho \cdot a^{ij}_2(x) \partial_i u_k \nu_j \bigg|_+
= \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_i u_k \partial_j \varrho + k \int_{\partial \omega_s} a^{ij}_1(x) \partial_i u_k \nu_j \bigg|_-
= \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_i u_k \partial_j \varrho.$$

Similarly,

$$0 = -\int_{\Omega \setminus \omega} \partial_j \left( a^{ij}_2(x) \partial_i u \right) \varrho = \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_i u \partial_j \varrho + \int_{\partial \omega} a^{ij}_2(x) \partial_i u \nu_j \bigg|_+.$$

Since $u_k \rightharpoonup u$ in $H^1(\Omega)$, it follows

$$0 = \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_i u_k \partial_j \varrho \rightharpoonup \int_{\Omega \setminus \omega} a^{ij}_2(x) \partial_i u \partial_j \varrho.$$

Thus,

$$\int_{\partial \omega_s} a^{ij}_2(x) \partial_i u \nu_j \bigg|_+ = 0,$$

for any $s = 1, 2, \ldots, m$. Therefore, we finish the proof of the first part.

Step 2. Since $u_k$ is a minimizer of $I_k$ and $\nabla u = 0$ in $\omega$, for any $k \in \mathbb{N},$

$$I_k[u_k] \leq I_k[u] = I_\infty[u].$$

Then $\limsup_{k \to +\infty} I_k[u_k] \leq I_\infty[u]$.

On the other hand, by Theorem 4.7, since $u$ is the weak limit of $\{u_k\}$ in $H^1(\Omega)$, we obtain

$$I_\infty[u] = \int_{\Omega} a^{ij}_2(x) \partial_i u \partial_j u \leq \liminf_{k \to +\infty} \int_{\Omega} a^{ij}_2(x) \partial_i u_k \partial_j u_k \leq \liminf_{k \to +\infty} I_k[u_k].$$

Therefore,

$$\lim_{k \to +\infty} I_k[u_k] = I_\infty[u].$$

□
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