Nonmonotonous magnetic field dependence and scaling of the thermal conductivity for superconductors with nodes of the order parameter

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(January 20, 2022)

Abstract

We show that there is a new mechanism for nonmonotonous behavior of magnetic field dependence of the electronic thermal conductivity $\kappa$ of clean superconductors with nodes of the order parameter on the Fermi surface. In particular, for unitary scatterers the nonmonotony of relaxation time takes place. Contribution from the intervortex space turns out to be essential for this effect even at low temperatures. Our results are in a qualitative agreement with recent experimental data for superconducting $UPt_3$. For $E_{2u}$-type of pairing we find approximately the scaling of the thermal conductivity in clean limit with a single parameter $x = \frac{T}{T_c} \sqrt{\frac{E}{B}}$ at low fields and low temperatures, as well as weak low-temperature dependence of the anisotropy ratio $\kappa_{zz}/\kappa_{yy}$ in zero field. For $E_{1g}$-type of pairing deviations from the scaling are more noticeable and the anisotropy ratio is essentially temperature dependent.

PACS numbers: 74.25.Fy, 74.70.Tx
I. INTRODUCTION

Growing amount of experimental data indicates that many of high-temperature and heavy-fermion superconductors have highly anisotropic order parameter on the Fermi surface. The possibility for the order parameter to have opposite signs in different regions on the Fermi surface, lines of nodes between those regions (and, possibly, point nodes too) has attracted much attention. Since the presence of nodes of the order parameter leads to the existence of low-energy excitations in spectrum, this strongly modifies low-temperature behavior of thermodynamic and transport characteristics both in the absence and under the applied magnetic field (see reviews \[3,4\] and for more recent literature, for example, \[5–8\]). For these reasons measurements of temperature and magnetic field dependences of those characteristics are important experimental tools for probe the anisotropic structure of the order parameter \[1,2,9–12,42\]. As compared to thermodynamic characteristics like specific heat, an important advantage of studying transport properties, in particular, the thermal conductivity is that it is a directional probe, sensitive to relative orientations of the thermal flow, the magnetic field and directions to nodes of the order parameter.

For isotropic $s$-wave superconductors low temperatures are obviously defined as satisfying the condition $T \ll \Delta_{\text{max}}$, when the number of quasiparticles thermally activated above the gap is exponentially small. In the presence of nodes of the order parameter on the Fermi surface an additional small energy scale $\gamma$ appears, describing the bandwidth of impurity-induced quasiparticle bound states. Then the low-temperature region should be divided into two parts. For temperatures $T \lesssim \gamma$ transport properties are dominated by bound states. Non-zero density of these states on the Fermi surface results in linear temperature dependence of the thermal conductivity in this case. At the same time under the condition $\gamma \lesssim T \ll \Delta_{\text{max}}$, which can be satisfied for sufficiently clean superconductors, one can disregard the influence of bound states. Then the presence of nodes of the order parameter leads to characteristic higher order power-law behaviors of the thermal conductivity with temperature \[14,21\].
The thermal conductivity of $s$-wave isotropic superconductors in the presence of the applied magnetic field exhibits nonmonotonic dependence upon the field $[22, 23]$. In high fields the thermal conductivity $\kappa(B)$ rises rapidly due to suppression of the order parameter as the magnetic field approaches $B_{c2}$. At low fields $\kappa(B)$ decreases with increasing magnetic field. This phenomenon is usually attributed to the scattering of electrons (or/and phonons) by the fluxoids, which was theoretically considered in $[26]$. However, in isotropic $s$-wave superconductors at low temperatures the number of quasiparticles thermally activated into scattering states are exponentially small, so that the contribution to the thermal conduction along the magnetic field from bound states within the vortex cores become essential. In contrast to isotropic superconductors, for anisotropically paired superconductors with nodes of the order parameter on the Fermi surface, the intervortex space can dominate thermodynamic and transport characteristics at low temperatures due to low energy excitations with momentum directions near the nodes $[6, 7, 27–29]$. Under these conditions the characteristics usually become quite sensitive to the applied magnetic field even if it is of relatively small value. For example, quasiparticle density of states on the Fermi surface for clean superconductors with nodes, takes nonzero value just due to the applied magnetic field (as well as impurities). Since low-temperature behavior of thermodynamic quantities like the specific heat is directly associated with the behavior of the density of states on the Fermi surface $N_s(0)$, this is the reason for their sensitivity to the magnetic field $[5, 6]$. For transport characteristics additional quantities are of importance in this respect. These are scattering relaxation times for various channels. For instance, it is known that the relaxation time $\tau_s(\omega)$ at sufficiently low energies for elastic impurity scattering in clean superconductors with nodes, may be quite small for unitary scatterers and extremely large in the case of Born scatterers $[30, 31, 19, 20]$. While $\tau_s(\omega)$ for Born scatterers is directly associated with $N_s(\omega)$, in the unitary limit $\tau_s(\omega)$, as a function of energy, is not reduced entirely to the density of states and should be considered as an independent quantity.

Below we show that for anisotropically paired clean superconductors with nodes of the order parameter on the Fermi surface, at low temperatures there is important additional
mechanism for nonmonotonous dependence of the electronic thermal conductivity upon the magnetic field. This mechanism is associated with the electronic contribution to \( \kappa(B) \) mostly from the intervortex space due to the presence there of extended quasiparticle states of low energies with momentum directions near nodes of the order parameter. Contrary to isotropic s-wave superconductors, the main effect comes in this case from the influence of condensate flow field (even of relatively small value) on the quasiparticle impurity scattering in the unitary limit, rather than from scattering of quasiparticles directly on vortex cores. For these two types of scattering inverse relaxation times could be added under certain conditions \[26\]. However, the contribution from scattering of quasiparticles by vortex cores is supposed to be negligibly small under the conditions considered below, as compared to the one from scattering by impurities in the intervortex space.

In the unitary limit relaxation time for scattering by nonmagnetic impurities of low-energy quasiparticles, is found below to be a nonmonotonous function upon the condensate flow field. This takes place even for uniform superfluid flow, that is in the absence of any scattering by vortices. Furthermore, for type II superconductors with large Ginzburg-Landau parameter, the superflow induced by magnetic field in the intervortex space can be considered on sufficiently large distances from vortex cores (much greater than the coherence length) as quasihomogeneous flow. This allows one to consider approximately the thermal conductivity as a function of the local value of condensate flow field, as this would be for the uniform flow. Magnetic field dependence of the thermal conductivity obtained in the first approximation as spatial averaging of the result over the intervortex space, is found below to be nonmonotonous under certain conditions.

This effect can be important, in particular, for the analysis of recently observed nonmonotonous behavior of the thermal conductivity in \( UPt_3 \), since the electronic contribution to \( \kappa \) is known to dominate there below 1K \[1,2\]. We consider \((1,i)\) phases both for the \( E_{2u} \) representation and for the \( E_{1g} \) one as candidates for the type of superconducting pairing in \( UPt_3 \) at low temperatures and under the weak applied magnetic field \( B_{c1} < B \ll B_{c2} \) (see, for example, \[3,4,32\]). Our theoretical results, basing on this consideration, are in a
qualitative agreement with those experimentally observed in [1,2]. Mostly the both models give rise to such an agreement. The difference between corresponding predictions is not too great, although it is of importance permitting to distinguish between them. Recent experimental data [1,2], in particular, allow for determining the low-temperature interval, where the power law temperature dependence of the thermal conductivity takes place. We find, that the behavior of the zero-field anisotropy ratio for the thermal conductivity for these temperatures seems to indicate in favor of $E_{2u}$ type of pairing. At higher temperatures the behavior of the thermal conductivity becomes essentially depending upon the particular form of the order parameter all over the Fermi surface, not only near its nodes. Under this condition there are various possibilities to fit experimental data within the framework of both models, so that the problem to distinguish between them becomes ambiguous one [18].

By contrast, at sufficiently low temperatures $T \ll T_c$ the behavior of the thermal conductivity is governed mainly by the behavior of the order parameter near nodes, as well as by the strength of scatterers. This leads, in principle, to the possibility to identify the behavior near the nodes and, hence, the type of superconducting pairing, but not a particular form of the order parameter all over the Fermi surface. This circumstance was already emphasized earlier in [17], where the accent was made on the ultra low temperatures $T \ll \gamma$. Quite a small value $\gamma \approx 0.017K$ taking place for clean samples of Ref. [1,2] give the possibility to determine and concentrate on the low-temperature region $\gamma \lesssim T \ll T_c$, while the ultra low temperatures in this case seem not to be sufficiently studied experimentally yet.

The article is organized as follows. Basic equations are listed in the next section. We consider clean superconductors at low temperatures under the applied magnetic field ($H_{c1} \lesssim B \ll H_{c2}$) and disregard the contribution from impurity bound states up to Sec.6. In Sec.3 we determine the nonmonotony of the relaxation time in the unitary limit. It is shown, that for the energy dominating regime $p_f v_s \ll \omega$ the relaxation time $\tau_s(\omega, v_s)$ decreases with increasing superflow in the presence of the line of nodes of the order parameter, in contrast to the case of superflow dominating condition $\omega \ll p_f v_s$, when the relaxation time rises. Analytical calculations are carried out for sufficiently small quantities.
\( p_f v_s, \ \omega \ll \Delta_{\text{max}} \). In Sec.4 the thermal conduction as a function of the superflow as well as along the applied magnetic field is analyzed. We examine three important examples of anisotropic gap functions: the polar state and \((1,i)\) superconducting states for \(E_{1g}\) and \(E_{2u}\) types of superconducting pairing in hexagonal crystals (like \(UPt_3\)). We find that in the temperature dominating region \(T_c \sqrt{\frac{B}{B_{c2}}} \ll T \ll \Delta_{\text{max}}\) the thermal conductivity diminishes with the magnetic field for all three types of pairing considered, if the magnetic field is applied within the basal plane. This turns out to be valid as well for \(\kappa_{zz}\) and the magnetic field along \(z\)-axis (of high symmetry) for \(E_{1g}\) and \(E_{2u}\) types of pairing, not for the polar state. The contribution to the magnetic field dependence of \(\kappa\) from the intervortex space \(r \gg \xi\) dominates in this case, which justifies the semiclassical approximation we use. For higher fields or lower temperatures the thermal conductivity always rises with the increasing magnetic field, although in this case intervortex space dominates only for the magnetic field oriented within the basal plane. This ensures the nonmonotonous magnetic field dependence of \(\kappa\) (at least for \(E_{2u}\) and \(E_{1g}\) models) under the condition \(\beta T_c \sqrt{\frac{B_{c1}}{B_{c2}}} \leq T \ll T_c\), where numerical factor \(\beta\) may be of the order of unity. In Sec.5 the low-temperature behavior of the thermal conductivity in zero field and the magnetic field dependence of \(\kappa_{ii}(T, B)\) at low fields and low temperatures are studied in clean limit (the \(i\)-th axis is aligned along the magnetic field). We show for the temperature region \(\gamma \lesssim T \ll \Delta_{\text{max}}\), that scaling of the thermal conductivity with a single parameter \(x = \frac{T}{T_c} \sqrt{\frac{B_{c2}}{B}}\) at low fields and low temperatures (both for \(i = y\) and for \(i = z\)) as well as weak low-temperature dependence of the anisotropy ratio \(\kappa_{zz}/\kappa_{yy}\) in zero field, are approximately valid in clean limit for \((1,i)\)-phase of \(E_{2u}\)-type of pairing. Qualitatively it is quite close to what it is observed experimentally for \(UPt_3\) in \([1,2]\). Under the same conditions \(E_{1g}\) model results in more noticeable deviations from the scaling, and in essential temperature dependence of the ratio \(\kappa_{zz}(T, B = 0)/\kappa_{yy}(T, B = 0)\).

New test is suggested for discrimination between candidates for the type of pairing in \(UPt_3\), based on the dependence of \(\kappa_{zz}\) upon the value of transport supercurrent flowing in thin films or whiskers along the hexagonal axis in the absence of the magnetic field. In Sec.6 we describe the behavior of the thermal conductivity under the condition \(p_f v_s, T \lesssim \gamma\), when
contribution from impurity bound states dominates the thermal conductivity. We find in this limit, that the thermal conductivity monotonously decreases with increasing condensate flow field (at least for $\gamma \ll \Delta_{\text{max}}$) and does not satisfy the above-mentioned scaling behavior. This is also in agreement with the experimental results for $UPt_3$.

II. BASIC EQUATIONS

Taking into account influence of the magnetic field (satisfying the condition $B_{c1} \lesssim B \ll B_{c2}$) on the thermal conductivity, we are interested in the contribution from large distances from vortex cores ($\xi(T) \lesssim r$), where the problem can be considered approximately as locally quasihomogeneous one on the basis of semiclassical approximation $[6]$. Being justified for those cases, when the contribution from the intervortex space $\xi \lesssim r$ turns out to be dominating, such an approach simplifies greatly all analytical considerations permitting to obtain correct results up to numerical coefficients of the order of unity. Besides, we assume the particle-hole symmetry and consider superconducting states to be unitary: $\hat{\Delta}\hat{\Delta}^\dagger = |\Delta|^2 \hat{\sigma}_0$, where $\hat{\sigma}_0$ is the $2 \times 2$ unit matrix. Then the expression for the thermal conductivity under the applied magnetic field can be written in the form

$$\kappa_{ij} = \frac{N_f}{2} \int_0^\infty d\omega \left( \frac{\omega}{T} \right)^2 \frac{1}{\cosh^2\left( \frac{\omega}{2T} \right)} \frac{1}{\text{Im} \tilde{\omega}} \left\langle \frac{v_{f,i}v_{f,j}}{2(\text{Re} \tilde{\omega} - v_f v_s)} \times \right.$$  

$$\left. \times \text{Re} \left\{ 2\sqrt{(\tilde{\omega} - v_f v_s)^2 - |\tilde{\Delta}(p_f)|^2} \right. \left. + \frac{|\tilde{\omega} - v_f v_s|^2 - (\tilde{\omega} - v_f v_s)^2}{\sqrt{(\tilde{\omega} - v_f v_s)^2 - |\tilde{\Delta}(p_f)|^2}} \right\} \right\rangle_{S_f}. \quad (1)$$

Here $v_s = (1/2)(\nabla \chi + (2e/c)A)$ is the gauge-invariant condensate flow field induced by the magnetic field (see, for example, $[33]$; $\chi$ is the phase of the order parameter), $\tilde{\omega}$ and $\tilde{\Delta}_p$ are the quasiparticle energy and the order parameter renormalized by impurities; notation $\langle \ldots \rangle_{S_f}$ means the averaging over the Fermi surface. Further we ignore contributions to the quasiparticle group velocity due to the momentum direction dependence of the order parameter, since for small enough parameter $\Delta_{\text{max}}/\varepsilon_f$ this would lead to a quite small corrections to the thermal conductivity $[34]$. Also we assume below for simplicity a spherical Fermi
surface and introduce the superfluid velocity $v_s = v_s/m$, so that under this simplification $v_f v_s = m v_f v_s = p_f v_s$. The branch of square root function in (I) should be chosen to have nonnegative imaginary part.

Considering sufficiently clean superconductors in the temperature region $\gamma \lesssim T$, one can disregard the contribution from impurity bound states. Then Eq. (I) reduces to

$$\kappa_{ij} = \frac{9}{2\pi^2 T_c} \int_0^\infty d\omega \left( \frac{\omega}{T} \right)^2 \frac{1}{\cosh^2 \left( \frac{\omega}{2T} \right)} \frac{\tau_s(\omega, v_s)}{\tau_N} I_{ij}(\omega, v_s).$$

Here $\kappa_N(T_c)$ is the thermal conductivity in the normal state at $T = T_c$; $\tau_s(\omega)$ and $\tau_N$ are relaxation times for quasiparticle scattering on impurities in superconducting and normal-metal states respectively. The quantity $I_{ij}(\omega, v_s)$ in the case of spherical Fermi surface is defined as follows

$$I_{ij}(\omega, v_s) = \int_{|\Delta(p_f)| \leq |\omega - p_f v_s|} \frac{d\Omega}{4\pi} \frac{p_i}{p_j} \sqrt{1 - \frac{|\Delta(p_f)|^2}{(\omega - p_f v_s)^2}}.$$  

Assuming particle-hole symmetry, one can represent the relaxation time in the Born approximation in the form:

$$\frac{\tau_s(\omega)}{\tau_N} = \frac{1}{\text{Im} G_0(\omega)},$$

while in the unitarity limit the corresponding expression is

$$\frac{\tau_s(\omega)}{\tau_N} = \frac{1}{\text{Im} \frac{G_0(\omega)}{G_1^2(\omega) - G_0^2(\omega)}}.$$  

The quantities $G_0(\omega), G_1(\omega)$ are

$$G_0(\omega) = i \int \frac{d\Omega}{4\pi} \frac{\omega - p_f v_s}{\sqrt{(\omega - p_f v_s)^2 - |\Delta(p_f)|^2}},$$

$$G_1(\omega) = i \int \frac{d\Omega}{4\pi} \frac{\Delta(p_f)}{\sqrt{(\omega - p_f v_s)^2 - |\Delta(p_f)|^2}}.$$  

In the absence of the superfluid velocity above expressions are reduced to well-known relations \[14\]. For the chosen branch of square root function the real part of the integrand in (I) (that is the density of states) is a nonnegative quantity.
III. RELAXATION TIME IN THE UNITARITY LIMIT

A. Magnetic field, aligned along \( z \)-axis

Let us consider superconducting phases with order parameters having line of nodes in the equatorial plane on a (spherical) Fermi surface, and the applied magnetic field satisfying the condition \( B_{c1} < B \ll B_{c2} \) and aligned along the crystalline axis of high symmetry (\( z \)-axis). As particular examples, there may be the polar phase and \((1, i)-superconducting\) phases of \( E_{1g} \) and \( E_{2u} \)-types of pairing in hexagonal superconductors. Last two phases are widely discussed as candidates for the type of superconducting pairing taking place in the heavy fermion superconductor \( UPt_3 \) \cite{3,4,32}. All these particular order parameters change their signs under the reflection across the equatorial plane on the Fermi surface: \( \Delta(\pi - \theta) = -\Delta(\theta) \). Due to this property function \( G_1(\omega) \) is equal to zero for the magnetic field parallel to \( z \)-axis.

The applied magnetic field influences the relaxation time mainly due to its dependence upon the superflow, induced by the field and aligned in the particular case parallel to the basal plane. In calculating the dependence of the relaxation time \( \tau_s(\omega) \) upon the superflow under the condition \( p_f v_s, \omega \ll \Delta_{max} \), it is essential that only narrow regions near nodes of the order parameter on the Fermi surface govern the dependence of \( G_0(\omega, v_s) - \text{Re} G_0(\omega, v_s = 0) \) upon the superfluid velocity. At the same time the quantity \( \text{Re} G_0(\omega, v_s = 0) \), generally speaking, is formed by the whole Fermi surface. We assume that the order parameter near the line of nodes (\( |\frac{\pi}{2} - \theta| \ll 1 \)) may be represented as \( |\Delta| = \Delta_1|\frac{\pi}{2} - \theta| \), while near the point node on one of the poles (\( |\theta| \ll 1 \)) it has the form \( |\Delta| = \Delta_2|\theta|^n, \ n = 1, 2 \). Calculations show, that if one keeps terms up to \( p_f v_s/\Delta_{1,2} \) in the expansion in powers of the superfluid velocity, taking into account as well the first power of small parameter \( \omega/\Delta_{1,2} \), then contributions from the line of nodes and quadratic points are essential. At the same time contributions from linear points may be neglected as compared to ones from the line of nodes. As a result
we obtain the following expressions for the relaxation time in the unitarity limit

\[
\frac{\tau_s(\omega)}{\tau_N} = \frac{\pi \omega}{2 \Delta_1} \left( 1 + \frac{\Delta_1}{2 \Delta_2} \right) + \frac{2 \omega}{\pi \Delta_1} \left( 1 + \frac{\Delta_1}{2 \Delta_2} \right) \left[ \frac{\Delta_1}{\omega} \text{Re} G_0(\omega, v_s = 0) + \right.
\]

\[
+ \ln \left( \frac{2 \omega}{e \left( \omega + \sqrt{\omega^2 - (pfv_s)^2} \right)} \right) + \sqrt{1 - \left( \frac{pfv_s}{\omega} \right)^2} \right] ^2, \quad pfv_s \leq \omega \ll \Delta_{\text{max}} \text{, } (8)
\]

\[
\frac{\tau_s(\omega)}{\tau_N} = \frac{pfv_s}{\Delta_1} \left( \sqrt{1 - \left( \frac{\omega}{pfv_s} \right)^2} + \frac{\omega}{pfv_s} \sin^{-1} \left( \frac{\omega}{pfv_s} \right) + \frac{\pi \omega}{4 \Delta_2} + \right.
\]

\[
\omega^2 \left( \frac{\Delta_1}{\omega} \text{Re} G_0(\omega, v_s = 0) + \ln \left( \frac{2 \omega}{epfv_s} \right) \right) \left( \sqrt{1 - \left( \frac{\omega}{pfv_s} \right)^2} + \frac{\omega}{pfv_s} \sin^{-1} \left( \frac{\omega}{pfv_s} \right) + \frac{\pi \omega \Delta_1}{4 pfv_s \Delta_2} \right) \right], \quad \omega \leq pfv_s \ll \Delta_{\text{max}} \text{. } (9)
\]

The most important feature of the dependence upon the superflow, describing by Eqs.(8), (9), is its nonmonotonous behavior. Indeed, one can easily see under the conditions \( pfv_s \ll \omega \ll \Delta_{\text{max}} \), that the relaxation time decreases with the increasing superfluid velocity according to the relation

\[
\frac{\tau_s(\omega, v_s)}{\tau_N} = \frac{\pi \omega}{2 \Delta_1} \left( 1 + \frac{\Delta_1}{2 \Delta_2} \right) + \frac{2 \omega}{\pi \left( 1 + \frac{\Delta_1}{2 \Delta_2} \right) \left( \frac{\Delta_1}{\omega} \text{Re} G_0(\omega, v_s = 0) \right) ^2 - \frac{(pfv_s)^2}{2 \omega^2} \text{Re} G_0(\omega, v_s = 0) \right) \text{. } (10)
\]

At the same time under the opposite condition \( \omega \ll pfv_s \ll \Delta_{\text{max}} \) we see from (9), that \( \tau_s/\tau_N = pfv_s/\Delta_1 \) increases linearly with the parameter \( pfv_s \).

Expressions (8)-(10) describe directly the relaxation time for \((1, i)\) superconducting state of \(E_{2\alpha}\)-type of pairing, for which the order parameter has the line of nodes on the equator and quadratic point nodes on the poles. Results for \(E_{1g}\)-type of pairing ( line of nodes on
the equator and linear point nodes on the poles) and for the polar phase (nodes are only on the line on the equator) follow from (8)-(10) in the limit \( \Delta_2 \to \infty \).

In calculating \( \text{Re}G_0(\omega, v_s = 0) \) one should describe the particular form of the order parameter all over the Fermi surface. In the particular case of polar phase \( \Delta = \Delta_1 \cos \theta \) explicit integration in (6) results in the following expression

\[
\text{Re}G_0(\omega, v_s = 0) = \frac{\omega}{\Delta_1} \ln \left( \frac{\Delta_1}{\omega} + \sqrt{\left( \frac{\Delta_1}{\omega} \right)^2 - 1} \right). \tag{11}
\]

Substituting (11) into (8) and (9), one gets that at the point of minimum of the relaxation time the superfluid velocity satisfies the condition

\[
\left( \frac{p_f v_s}{\omega} \right) \sim \omega \ln \left( \frac{\Delta_1}{\omega} \right). \tag{12}
\]

One can see, that nonmonotonous behavior of the relaxation time with superfluid velocity takes place in the unitary limit due to different behaviors of \( \text{Re}G_0 \) and \( \text{Im}G_0 \). At sufficiently low energy the density of states (and, hence, \( \text{Im}G_0 \)) increases with increasing magnetic field, while \( \text{Re}G_0 \) turns out to decrease. In the case in question the relaxation time is described by the relation

\[
\frac{\tau_s(\omega)}{\tau_N} = \frac{p_f v_s}{\Delta_1} \left[ 1 + \left( \frac{\omega}{p_f v_s} \right)^2 \ln^2 \left( \frac{4\Delta_1}{e p_f v_s} \right) \right]. \tag{13}
\]

If \( \omega \ln \left( \frac{\Delta_1}{\omega} \right) \lesssim p_f v_s \), expression (13) increases with increasing \( p_f v_s \).

One can see, that nonmonotonous behavior of the relaxation time with superfluid velocity takes place in the unitary limit due to different behaviors of \( \text{Re}G_0 \) and \( \text{Im}G_0 \). At sufficiently low energy the density of states (and, hence, \( \text{Im}G_0 \)) increases with increasing magnetic field, while \( \text{Re}G_0 \) turns out to decrease. In the case in question the relaxation time is described by the relation

\[
\frac{\tau_s(\omega)}{\tau_N} = \text{Im}G_0(\omega) + \frac{(\text{Re}G_0(\omega))^2}{\text{Im}G_0(\omega)} \tag{14}
\]

and may manifest nonmonotonous magnetic field dependence (see Fig. 1).

**B. Magnetic field parallel to the basal plane**

Nonmonotony of the relaxation time as a function of the applied magnetic field may take place for various orientations of the field, due to the same reasons as for the field parallel to
the z-axis. In this subsection we present shortly some results on magnetic field dependence of relaxation time for the field lying within the basal plane for polar phase (\(\Delta(p_f) = \Delta_p(\theta)\)), (1, i)-phase of \(E_{1g}\)-representation of \(D_{6h}\) point group (\(\Delta(p_f) = \Delta_{1g}(\theta) \exp(i\varphi)\)), and (1, i)-phase of \(E_{2u}\)-representation (\(\Delta(p_f) = \Delta_{2u}(\theta) \exp(2i\varphi)\)). Spherical angles \(\theta\) and \(\varphi\) characterize here momentum directions on a (spherical) Fermi surface, and functions \(\Delta_p(\theta), \Delta_{1g}(\theta), \Delta_{2u}(\theta)\) are supposed to be real.

Let y-axis be parallel to the magnetic field. We consider some fixed spatial point far enough from the vortex core, where induced superfluid velocity, lying in xz-plane, constitutes angle \(\theta_0\) with z-axis. Although for the field orientation within the basal plane \(G_1(\omega)\) differs from zero, the analysis shows that this quantity may be neglected as compared to \(G_0(\omega)\), at least for the types of pairing considered and under the condition \(\omega, pfv_s \ll \Delta_{\text{max}}\). This allows one to use Eq.(14) in calculating \(\tau_s(\omega)\). Assuming the same relations for the order parameter near the line of nodes (\(|\Delta| = \Delta_1|\frac{\pi}{2} - \theta|\)) and point node (\(|\Delta| = \Delta_2|\theta|^n, n = 1, 2\)) as earlier, we get again (within the same accuracy) that the relaxation time diminishes with increasing superfluid velocity under the conditions \(pfv_s \ll \omega \ll \Delta_{\text{max}}\):

\[
\frac{\tau_s(\omega, v_s)}{\tau_N} = \frac{\pi \omega}{2\Delta_1} \left(1 + \frac{\Delta_1}{2\Delta_2}\right) + \frac{2}{\pi\left(1 + \frac{\Delta_1}{2\Delta_2}\right)} \left[\frac{\Delta_1}{\omega} \left(\text{Re}G_0(\omega, v_s = 0)\right)^2 - \left(1 + \frac{\Delta_1}{\Delta_2}\right) \frac{(pfv_s)^2 \cos^2 \theta_0}{2\omega^2} \text{Re}G_0(\omega, v_s = 0)\right].
\]

(15)

In the opposite case \(\omega \ll pfv_s \ll \Delta_{\text{max}}\) relaxation time increases with \(v_s\):

\[
\frac{\tau_s(v_s)}{\tau_N} = pfv_s \left(\frac{\sin \theta_0}{\Delta_1} + \frac{\pi \cos \theta_0}{4\Delta_2}\right).
\]

(16)

Eqs.(15), (16) describe the relaxation time for (1, i) superconducting state of \(E_{2u}\)-type of pairing. As it was in the case of Eqs.(8)-(10), results for \(E_{1g}\)-type of pairing and for the polar phase follow from (15), (16) in the limit \(\Delta_2 \to \infty\). For the polar phase we obtain under the condition \(\omega \ll \Delta_1\) the explicit expression \(\text{Re}G_0(\omega, v_s = 0) = (\omega/\Delta_1) \ln(2\Delta_1/\omega)\). Then Eq.(17) reduces to

\[
\frac{\tau_s(\omega, v_s)}{\tau_N} = \frac{2\omega}{\pi \Delta_1} \left[\frac{\pi^2}{4} + \ln^2 \left(\frac{2\Delta_1}{\omega}\right) - \frac{1}{2} \left(\frac{pfv_s \cos \theta_0}{\omega}\right)^2 \ln \left(\frac{2\Delta_1}{\omega}\right)\right].
\]

(17)
So, magnetic field dependence of the relaxation time for the magnetic field oriented within the basal plane, turns out to be qualitatively quite close to the one for magnetic field orientation along the $z$-axis.

### IV. THERMAL CONDUCTIVITY

In this section we consider the electronic thermal conduction along the direction of the applied magnetic field. Let the magnetic field be aligned firstly along the $z$-axis, so that the supercurrent flows parallel to the basal plane.

For low temperatures and small superfluid velocities satisfying the conditions $T, p_f v_s \ll \Delta_{\text{max}}$, there are two characteristic regions $p_f v_s \ll T \ll \Delta_{\text{max}}$ and $T \ll p_f v_s \ll \Delta_{\text{max}}$, where dependences of the thermal conductivity upon $v_s$ may differ essentially. While within the region $T \ll p_f v_s \ll \Delta_{\text{max}}$ the thermal conductivity always rises with increasing $v_s$, it may be possible both increase or decrease under the condition $p_f v_s \ll T \ll \Delta_{\text{max}}$. Thus, nonmonotonic behavior of the relaxation time may result in respective nonmonotonic dependence of the thermal conductivity upon the superflow (and eventually upon the magnetic field). However, nonmonotony is not a inevitable consequence for the thermal conductivity, which would be valid for any types of pairing in the presence of such a behavior of the relaxation time. The point is that the quantity $I_{ij}(\omega, v_s)$, which forms the behavior of $\kappa$ along with the relaxation time (see (4)), usually turns out to be monotonously increasing function of $v_s$. As a result, the product $\tau_s(\omega, v_s)I_{ij}(\omega, v_s)$, containing nonmonotonic function $\tau_s(\omega, v_s)$, may manifest both monotonous or nonmonotous behavior, depending on the particular behaviors of $\tau_s(\omega, v_s)$ and $I_{ij}(\omega, v_s)$.

For sufficiently low temperatures $T \ll p_f v_s \ll \Delta_{\text{max}}$ (or, in other words, for large enough superfluid flow), it follows from (2) in the first approximation

$$\frac{\kappa_{ij}}{\kappa_N(T_c)} = \frac{3T}{T_c} \frac{\tau_s(\omega = 0, v_s)}{\tau_N} I_{ij}(\omega = 0, v_s).$$

(18)

The possibility to disregard the contribution from impurity induced bound states in this case is justified under the condition $p_f v_s \gg \gamma$. 

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One can show, that the contribution to $I_{zz}(\omega = 0, v_s)$ from narrow region around the line of nodes is $(pfv_s)^3/12\Delta_1^3$. Linear point nodes do not contribute to $I_{zz}(\omega = 0, v_s)$, so that Eq. (18) is not the appropriate approximation for the case of the order parameter having only linear point nodes. It is important, however, that the contribution from linear point nodes in this limit is evidently less than from the line of nodes. At the same time the contribution from the quadratic point nodes dominates being equal to $(pfv_s)^2/6\Delta_2^2$. Furthermore, since Re$G_0(\omega = 0, v_s = 0) = 0$, we find from (19): $\tau_s(\omega = 0, v_s)/\tau_N = pfv_s/\Delta_1$. The contribution to $\tau_s(\omega = 0, v_s)/\tau_N$ from quadratic point node is $(pfv_s)^2/2\Delta_2^2$, which can be disregarded as compared to the above value of $\tau_s(\omega = 0, v_s)/\tau_N$ formed by the line of nodes. Subsequently, under the condition $T \ll pfv_s \ll \Delta_{max}$ we find in the case of polar phase and for (1,i)-phase of $E_{1g}$-type of paring:

$$\frac{\kappa_{zz}}{\kappa_N(T_c)} = \frac{T}{4T_c} \left( \frac{pfv_s}{\Delta_1} \right)^4. \tag{19}$$

For (1,i)-phase of $E_{2u}$-type of paring it follows, that

$$\frac{\kappa_{zz}}{\kappa_N(T_c)} = \frac{T}{2T_c} \left( \frac{pfv_s}{\Delta_1} \right)^2. \tag{20}$$

These higher power law dependences upon the local superfluid velocity result after averaging over the intervortex space in linear dependence upon the magnetic field, formed by the lower border $r \sim \xi$ of the intervortex space considered. Indeed, superfluid velocity $v_s(r)$ depends upon the distance from the vortex core and in the simplest case of circular supercurrents one has $v_s(r) = 1/2m_e r$ for $\xi \ll r \ll \lambda$. Then we obtain after spatial averaging:

$$\kappa_{zz}(T, H) \simeq \frac{1}{4} \left( \frac{pfv_s(\xi)}{\Delta_1} \right)^4 \frac{BT}{B_{c2}T_c} \kappa_N(T_c) \tag{21}$$

for polar phase and for (1,i)-phase of $E_{1g}$-type of paring, and

$$\kappa_{zz}(T, H) \simeq \left( \frac{pfv_s(\xi)}{\Delta_1\Delta_2} \right)^3 \frac{BT}{B_{c2}T_c} \kappa_N(T_c) \tag{22}$$

for (1,i)-phase of $E_{2u}$-type of paring.
Since regions close to vortex cores \((r \sim \xi)\) dominate in spatial averaging of the thermal conductivity under the particular conditions \(T \ll p_f v_s, B || Oz\), the approximation of locally quasiumiform superflow, generally speaking, is not directly applied to this case. In addition, the relation \(p_f v_s(\xi) \sim T_c\) guarantees validity of inequality \(T \ll p_f v_s\) for all \(T \ll T_c\) and, from the other side, violates the condition \(p_f v_s \ll \Delta_{max}\), which leads to the conclusion about rough qualitative character of the particular results derived above. Strictly speaking, in this particular case it is necessary to check the contribution to \(\kappa\) not only from the delocalized quasiparticles, but from quasiparticles bound to vortex cores as well. That problem is, however, beyond the present article. It is important, that in any case the thermal conductivity increases with the increasing magnetic field in the limit considered. This is, fortunately, the only particular example in the article, where the intervortex space doesn’t dominate. Furthermore, it is known for s-wave superconductors, that contribution from bound states (localized within vortex cores) to the low-temperature behavior of the thermal conductivity along the magnetic field, is negligibly small as compared to the normal state thermal conductivity \(\kappa_N\) [24]. In the presence of nodes of the order parameter extended (scattering) low-energy quasiparticle states contribute much greater. Indeed, as \(p_f v_s(\xi)/\Delta_{max} \sim 1\), we get from Eqs. (21), (22) \(\kappa_{zz}(T) \sim (B/B_c^2)\kappa_N(T)\). This estimate is quite close to that observed experimentally for \(UPt_3\) for low temperatures [1,2] and about on two orders greater, than the one realized for conventional superconductors [24]. Also we note, that for the field within the basal plane large distances from the vortex cores dominate for any relation between \(T\) and \(p_f v_s\) (see below) ensuring the applicability of the semiclassical approximation in that case both in temperature dominating regime \(T \gg p_f v_s\) and under the superflow dominating condition \(T \ll p_f v_s\) for three types of pairing discussed. At last, for Born scatterers the situation changes and in spatial averaging in the superflow dominating limit large distances from vortex cores dominate both for \(\kappa_{zz}\) and the field along the \(z\)-axis and for \(\kappa_{yy}\) and the field along the \(y\)-axis.

Let us discuss now the other case \(p_f v_s \ll T \ll \Delta_{max}\). It is worth noting that this condition can’t be valid close to vortex cores (for \(r \sim \xi\)). However, as we show below, under
the condition \( p_f v_s \ll T \ll \Delta_{\text{max}} \) fairly large distances from vortex cores (of the order of the intervortex distance \( R \sim \xi \sqrt{\frac{B^2}{B^2}} \gg \xi \)) may dominate in the spatially averaged thermal conductivity.

Considering firstly the thermal conductivity for the polar phase and substituting \( \Delta = \Delta_1 \cos \theta \) into (3), we obtain under the condition \( \omega, p_f v_s \ll \Delta_1 \):

\[
I_{zz}(\omega, v_s) = \frac{1}{16\Delta_1^3} \left\{ \omega \left[ 2\omega^2 + 3(p_f v_s)^2 \right] \sin^{-1} \left( \frac{\omega}{p_f v_s} \right) + [3\omega^2 + 2(p_f v_s)^2] \sqrt{(p_f v_s)^2 - \omega^2} - \frac{2}{3} \left[ (p_f v_s)^2 - \omega^2 \right]^{3/2}, \omega \leq p_f v_s, \right.
\]

\[
\left. \pi \omega^3 + \frac{3\pi}{2} \omega (p_f v_s)^2, \omega \geq p_f v_s. \right\}
\]

For \( \omega \lesssim p_f v_s \) it follows from (23), (9), that \( I(\omega, v_s) \lesssim \left( \frac{p_f v_s}{\Delta_1} \right)^3, \tau_s \sim \left( \frac{p_f v_s}{\Delta_1} \right) \). Taking account of these estimates, we find, that under the condition \( p_f v_s \ll T \ll \Delta_1 \) the contribution to the thermal conductivity (2) from the integration over the region \( \omega \lesssim p_f v_s \) turns out to be \( \lesssim \left( \frac{p_f v_s}{\Delta_1 T_c} \right)^7 \). Since we are interested in contributions to \( \kappa \) up to \( (p_f v_s)^2 \), one should consider the integration over \( \omega \geq p_f v_s \) and substitute into (2) equations (12), (23):

\[
\frac{\kappa_{zz}}{\kappa_N(T_c)} = \frac{161.74 T^5}{T_c \Delta_1^4} \left[ \ln^2 \left( \frac{2\Delta_1}{T} \right) - 1.63 \ln \left( \frac{2\Delta_1}{T} \right) + 3.16 \right] +
\]

\[
+ \frac{7.77 T^3 (p_f v_s)^2}{T_c \Delta_1^4} \left[ \ln^2 \left( \frac{2\Delta_1}{T} \right) - 1.67 \ln \left( \frac{2\Delta_1}{T} \right) + 3.18 \right].
\]

(24)

We see from (21) and (24), that \( \kappa_{zz} \) for polar state is a monotonously increasing function of the superfluid velocity. According to (24) the thermal conductivity for polar state in the presence of magnetic field roughly \( \propto T^3 \), while in the absence of magnetic field \( \propto T^5 \).

By contrast, the thermal conductivity of \((1, i)\)-phases for \( E_{1g} \) and \( E_{2u} \) types of pairing turns out to be nonmonotonous one. The point is that the contribution to \( I_{zz} \) from linear point nodes is approximately \( \omega^2 / 3\Delta_2^2 \), while from quadratic point nodes is \( \pi \omega / 8\Delta_2 \). These terms dominate as compared to the contribution from line of nodes. This means that magnetic field dependence of the thermal conductivity is formed in the cases in question entirely by the relaxation time:
\[
\frac{\kappa_{zz}}{\kappa_N(T_c)} = \frac{\kappa_{zz}(T_c)}{\kappa_N(T_c)} \bigg|_{v_s=0} - \frac{9(p_f v_s)^2}{\pi^3 T_c (2\Delta_2 + \Delta_1)} \int_0^\infty \frac{\omega d\omega}{\cosh^2 \left( \frac{\omega}{2} \right)} \Re G_0(T\omega, 0) \left\{ \begin{array}{ll}
T\omega & , \text{ for } E_{1g} \\
\frac{\pi}{8} & , \text{ for } E_{2u} \end{array} \right.
\]

(25)

This nonmonotony is seen in Fig. 2 where we plot the thermal conductivity for three particular types of pairing and various values of temperature as a function of the superfluid velocity \[39\].

After averaging over intervortex space, where expressions (24), (25) are valid, we obtain

\[
\frac{\kappa_{zz}}{\kappa_N(T_c)} = \frac{\kappa_{zz}(T_c)}{\kappa_N(T_c)} \bigg|_{B=0} \pm A_{zz}(T) \frac{B}{B_{c2}} \ln \left( \frac{B_{c2}}{B} \right).
\]

(26)

Sign plus corresponds here to the case of polar phase, while sign minus - to \(E_{1g}\) and \(E_{2u}\) types of pairing. Since \(\Re G_0(T\omega, 0)\) is roughly proportional to \(T\) (neglecting the logarithmic factors; see below Eq.(33)), positive dimensionless function \(A_{zz}(T)\) turns out to have a linear temperature dependence for \(E_{2u}\) model, a quadratic temperature dependence for \(E_{1g}\), while for the polar phase \(A_{zz}(T)\) is roughly a cubic function of temperature (see Eq.(24) as well).

Large logarithmic factor in magnetic field dependence \(26\) is formed by contributions from sufficiently large distances from vortex cores (of the order of intervortex distance \(R\)). This justifies the approach we use, which is based on the consideration of quasihomogeneous superconducting intervortex space and valid for large distances from vortex cores \(r \gg \xi\).

So, in spatial averaging of the thermal conductivity the intervortex space, generally speaking, is divided into two parts. The first one is defined by the condition \(T \lesssim p_f v_s(r)\) and contributes to the monotonously increasing thermal conductivity with the magnetic field. The other part of the intervortex space \(p_f v_s(r) \lesssim T\) includes fairly large distances from vortex cores. For \(E_{1g}\) and \(E_{2u}\) representations it contributes to decreasing thermal conductivity with increasing magnetic field and turns out to be essential (that is may compete with the contribution from the first part) only under the condition \(p_f v_s(R) \ll T\). This condition reduces to the inequality \(T \gg T_c \sqrt{\frac{B}{B_{c2}}}\) which has to be satisfied ensuring the possibility for the nonmonotinous magnetic field dependence of the thermal conductivity. Thus,
in the presence of nonmonotony the minimum of \( \kappa(B) \) has to be shifted gradually to lower fields with decreasing temperature. Below some characteristic temperature \( \sim \beta T_c \sqrt{\frac{|B_1|}{B_{c2}}} \) this nonmonotony disappears at all. Rough estimates show that numerical factor \( \beta \) may be of the order of unity. This is in agreement with that observed experimentally in \( UPt_3 \) [12]. Naturally, the relation \( \beta T_c \sqrt{\frac{|B_1|}{B_{c2}}} \ll T \ll T_c \) can be satisfied only for large enough value of the Ginzburg-Landau parameter as this takes place, in particular, in \( UPt_3 \).

Let now the applied magnetic field lie in the basal plane along \( y \)-axis. While the relaxation time behavior is quite similar for both orientations of the magnetic field, it is not always the case for \( I_{ij} \). As a consequence, a qualitative behavior of the thermal conduction along the magnetic field for \( B \parallel Oz \) and \( B \parallel Oy \) may differ as well. In particular, we show below for the field orientation \( B \parallel Oy \), that nonmonotonous magnetic field dependence of the thermal conductivity \( \kappa_{yy} \) takes place for all three types of pairing in question, including the polar phase (in contrast to \( \kappa_{zz} \) for the orientation \( B \parallel Oz \)).

Under conditions \( \omega, p_f v_s \ll \Delta_{max} \) we get, that the main contribution to \( I_{yy} \) comes only from the narrow region of the line of nodes for all types of pairing considered (including \( E_{2u} \) – representation with quadratic point nodes on the poles of the Fermi surface):

\[
I_{yy} = \begin{cases} 
\frac{\pi \omega}{8 \Delta_1} & , \quad p_f v_s |\sin \theta_0| \leq \omega \\
\frac{2}{3} \left( 1 - \frac{\omega^2}{p_f v_s^2 \sin^2 \theta_0} \right)^{3/2} + \frac{\omega}{p_f v_s \sin \theta_0} \sin^{-1} \frac{\omega}{p_f v_s |\sin \theta_0|} + \frac{\omega^2}{p_f v_s^2 \sin^2 \theta_0} \left[ 1 - \frac{\omega^2}{p_f v_s^2 \sin^2 \theta_0} \right] & , \quad p_f v_s |\sin \theta_0| \geq \omega
\end{cases}
\]  

(27)

As it follows from (27), \( I_{yy}(\omega = 0, v_s) = p_f v_s |\sin \theta_0| / 6 \Delta_1 \). Substituting this value together with (16) into (18), we obtain the thermal conductivity at low temperatures \( T \ll p_f v_s \ll \Delta_{max} \) for \( (1, i) \)-phase of \( E_{2u} \)-type of pairing:

\[
\frac{\kappa_{yy}}{\kappa_N(T_c)} = (p_f v_s)^2 \frac{T}{2 T_c} \left( \frac{\sin^2 \theta_0}{\Delta_1^2} + \frac{\pi |\sin 2 \theta_0|}{8 \Delta_1 \Delta_2} \right) .
\]

(28)

As earlier, the answer for polar phase and for \( (1, i) \)-phase of \( E_{1g} \)-type of pairing is obtained from (28) in the limit \( \Delta_2 \to \infty \).
For fairly low superfluid flow, when the condition \( p_f v_s \ll T \ll \Delta_{\text{max}} \) is satisfied, the expression for the thermal conductivity can be obtained, substituting \( I_{yy} = \frac{\pi \omega}{8 \Delta_1} \) together with Eq.(15) into Eq.(12):

\[
\frac{\kappa_{yy}}{\kappa_N(T_c)} = \left. \frac{\kappa_{yy}}{\kappa_N(T_c)} \right|_{v_s=0} - \frac{9(p_f v_s \cos \theta_0)^2 (\Delta_2 + \Delta_1)}{8 \pi^2 T_c \Delta_1 (2 \Delta_2 + \Delta_1)} \int_0^\infty \frac{\omega d\omega}{\cosh^2 \left( \frac{\omega}{2} \right)} \text{Re} G_0(T \omega, 0). \tag{29}
\]

In the particular case of polar phase we obtain from (29), (11) the following dependence of the thermal conductivity upon the low enough superfluid flow:

\[
\frac{\kappa_{yy}}{\kappa_N(T_c)} = \frac{10.36 T^3}{T_c \Delta_1^2} \left[ \ln^2 \left( \frac{2 \Delta_1}{T} \right) - 1.34 \ln \left( \frac{2 \Delta_1}{T} \right) + 2.95 \right] - \frac{0.38(p_f v_s \cos \theta_0)^2 T}{T_c \Delta_1^2} \ln \left( \frac{2 \Delta_1}{T} \right) - 0.45. \tag{30}
\]

Spatial averaging of \( \kappa_{yy} \) over that part of the intervortex space, where Eq.(24) is valid (that is over its exterior part \( p_f v_s(r) \lesssim T \)), leads to the thermal conductivity, which diminishes with increasing weak magnetic field for all three types of pairing according to Eq.(26) (where one should change indices \( z \rightarrow y \) and retain only sign minus in front of \( A_{yy} \)). Disregarding the logarithmic terms, we find that the function \( A_{yy}(T) \) is roughly a linear function of temperature for each type of pairing discussed. Furthermore, the interior part of the intervortex space (\( T \lesssim p_f v_s(r) \)), where one can use Eq.(28), results in \( \kappa_{yy} \), which rises with magnetic field \( \propto B \frac{B_{c2}}{B_{c1}} \ln \left( \frac{B_{c2}}{B} \right) \). These two regions compete with each other as it was above for \( \kappa_{zz} \). But even in the case \( T \lesssim p_f v_s(r) \) large distances from vortex cores \( \xi \ll r \lesssim \xi \sqrt{B_{c2}/B} \) dominate here leading to additional logarithmic factor, as compared to Eqs.(21), (22). Possibly, this logarithmic factor is responsible for exceeding in several times of \( \kappa_{yy} \) as compared to \( \kappa_{zz} \), which is experimentally observed in \( UPt_3 \). For quantitative consideration of this problem, one should evidently go beyond the approximation of locally quasuniform superflow in describing the mixed state.
V. SCALING OF THE THERMAL CONDUCTIVITY AT LOW FIELDS AND LOW TEMPERATURES

Thermodynamic and transport characteristics of superconductors with nodes may exhibit scaling behavior at low fields and low temperatures. This possibility for high-temperature superconductors was studied theoretically in [7,27–29]. In particular, it was shown, that the electronic thermal conductivity of a two-dimensional $d$-wave superconductor with four lines of nodes on a cylindrical Fermi surface may be represented as $\kappa_{ij} \sim T F_{ij}(\alpha T/B^{1/2})$ [28]. One of the important assumptions underlying this result is that contributions to the thermal conductivity from quasiparticles in the intervortex space far enough from vortex cores dominate. In contrast to the thermodynamic quantities, for transport phenomena scattering processes are of importance even in clean limit. We note, that properties of disorder impurity potentials considered in Ref. [28] may be suitable for the Born scatterers, not in the unitary limit.

Superconducting $UPt_3$ essentially differs from the superconductors just mentioned above. It is three-dimensional hexagonal superconductor with the order parameter, which is believed to have both line of nodes on the equator of the Fermi surface and point nodes on its poles. While the strength of impurity scattering for high-temperature superconductors is not yet definitely determined, for heavy-fermion superconductors (in particular, for $UPt_3$) there are various experimental results and the physical background [31,40] indicating to the impurity scattering very close to the unitarity limit. Thus, the problem of scaling behavior of the thermal conductivity in superconductors like $UPt_3$ still has not been theoretically studied properly.

At the same time the behavior quite close to the scaling one has been experimentally established recently for the thermal conductivity at low fields and low temperatures for superconducting $UPt_3$ both for the component $\kappa_{zz}$ under the magnetic field parallel to $z$-axis and for $\kappa_{yy}$ in the case of the field applied along $y$-axis [2]. These are just the cases which we consider in the article. Two forms of scaling are of interest for discussion of the
experimental data of Ref. [2]:

\[ \kappa_{ii} = T^3 F_{ii}(x) , \quad x = \frac{T}{T_c} \sqrt{\frac{B_{c2}}{B}} ; \]  

(31)

\[ \frac{\kappa_{ii}(T, B)}{\kappa_{ii}(T, B = 0)} = g_{ii}(x) . \]  

(32)

In order to check whether our results are in agreement with any of these scaling behaviors, let us specify firstly power law dependences of the thermal conductivity upon temperature (under the condition \( \gamma \lesssim T \ll \Delta_{\text{max}} \)) for three particular types of pairing in question in the absence of the magnetic field.

According to Eq.(14), in order to estimate frequency dependence of the relaxation time in the unitarity limit, one should find the corresponding behavior of \( \text{Im} G_0, \text{Re} G_0 \). Contributions to \( \text{Im} G_0, \text{Re} G_0 \) from the line of nodes in the equatorial plane (subscript \( l \)), linear and quadratic point nodes at the poles of the Fermi surface (subscripts \( p1 \) and \( p2 \) respectively) are as follows

\[
\begin{align*}
\text{Im} G_{0,l}(\omega) &= \frac{\pi \omega}{2 \Delta_1} , \\
\text{Im} G_{0,p1}(\omega) &= \frac{\omega^2}{\Delta_2} , \\
\text{Im} G_{0,p2}(\omega) &= \frac{\pi \omega}{4 \Delta_2} , \\
\text{Re} G_{0,l}(\omega) &\approx \frac{\omega}{\Delta_1} \ln \left( \frac{A_l \Delta_1}{\omega} \right) , \\
\text{Re} G_{0,p1}(\omega) &\approx \frac{A_{p1} \omega}{\Delta_2} , \\
\text{Re} G_{0,p2}(\omega) &\approx \frac{\omega}{\Delta_2} \ln \left( \frac{A_{p2} \Delta_2}{\omega} \right) .
\end{align*}
\]  

(33)

One should emphasize that for all three cases we consider, the behavior of \( \text{Re} G_0 \) (in contrast to \( \text{Im} G_0 \)) is governed, even at low frequencies, by the behavior of the order parameter not only near the nodes, but over the whole Fermi surface. So, the given expressions for \( \text{Re} G_0 \) are approximate ones and constants \( A_l, A_{p1}, A_{p2} \) can’t be determined unambiguously unless the behavior of the order parameter all over the Fermi surface is known. For the polar state \( \Delta = \Delta_1 \cos \theta \) and a spherical Fermi surface we get \( A_l = 2 \) from the comparison of (11) and (33) at \( \omega \ll \Delta_1 \). Let the order parameter in more general case have the form \( |\Delta| = \Delta_0 f(x) \), where \( x = \cos \theta \) and \( f(x) \approx f'(0)x \) for \( |x| \ll 1, |f(x)| \approx |f'(1)|(1 \mp x) \) for \( (1 \mp x) \ll 1 \). Then one can show for sufficiently small frequencies

\[
\begin{align*}
\text{Re} G_0(\omega) &= \zeta \frac{\omega}{\Delta_0} \ln \left( \frac{A \Delta_0}{\omega} \right) , \\
\text{Im} G_0(\omega) &= \zeta \frac{\pi \omega}{2 \Delta_0} ,
\end{align*}
\]  

(34)
where
\[
\zeta = \frac{1}{|f'(0)|} + \frac{1}{|f'(1)|}, \quad \zeta \ln A = \lim_{\alpha \to 0} \left( \int_{\alpha}^{1} \frac{dx}{f(x)} + \frac{\ln(2\alpha|f'(0)|)}{|f'(0)|} + \frac{\ln(2\alpha|f'(1)|)}{|f'(1)|} \right). \tag{35}
\]

So, in calculating \( A \) one has to integrate \( 1/f(x) \) over the whole Fermi surface.

On the other side, if one does not fix from the very beginning the behavior of the order parameter all over the Fermi surface but only near the nodes, constants of this origin may be considered as fitting parameters in comparison of the theoretical results with experimental data under the corresponding conditions. These fitting parameters, generally speaking, may manifest weak temperature dependence in the low temperature region, associated with the respective low-temperature dependence of the order parameter. Any particular choice of the parameters corresponds yet with a large number of basis functions of the representation, rather than only with a unique particular one.

One can easily see from Eqs.\((33), (14)\), that if only one kind of nodes is present, then at low frequencies
\[
\frac{\tau_{s,l}(\omega)}{\tau_N} \approx \frac{2\omega}{\pi \Delta_1} \left[ \frac{\pi^2}{4} + \ln^2 \left( \frac{\omega}{A_1 \Delta_1} \right) \right], \quad \frac{\tau_{s,p1}(\omega)}{\tau_N} \approx A_{p1}^2 + \frac{\omega^2}{\Delta_2^2}, \quad \frac{\tau_{s,p2}(\omega)}{\tau_N} \approx \frac{4\omega}{\pi \Delta_2} \left[ \frac{\pi^2}{16} + \ln^2 \left( \frac{\omega}{A_{p2} \Delta_2} \right) \right]. \tag{36}
\]

Low temperature dependence of the relaxation time of the form \( T \ln^2(\Delta/T) \), found for the polar state in \([31]\), is in agreement with the expression for \( \tau_{s,l} \).

For \((1, i)\)-phases of \( E_{1g} \)- and \( E_{2u} \)-types of pairing with hybrid gap functions, having both the line of nodes and point nodes, the low-frequency dependences of the relaxation times are similar to the case of the polar state. Although the particular form of the order parameter all over the Fermi surface, for instance, point nodes result in the change of constants in the approximate expression for \( \tau_{s,l} \), which then differ for each type of pairing. So, we write
\[
\frac{\tau_{s,1g}(\omega)}{\tau_N} \approx \frac{2\omega}{\pi \Delta_1} \left[ \ln^2 \left( \frac{A_{1g} \omega}{\Delta_1} \right) + \frac{\pi^2}{4} \right], \quad \frac{\tau_{s,2u}(\omega)}{\tau_N} \approx \frac{2\omega}{\pi \Delta_{e_f}} \left[ \ln^2 \left( \frac{A_{2u} \omega}{\Delta_{e_f}} \right) + \frac{\pi^2}{4} \right], \tag{37}
\]
where \( \Delta_{e_f} = 2\Delta_1 \Delta_2/(\Delta_1 + 2\Delta_2) \).

Contributions from the line and point nodes to low-frequency behaviors of \( I_{ij}(\omega) \) are
Substituting Eqs. (36) – (38) into (2), we obtain the following leading low-temperature terms for $\kappa_{zz}$ under the condition $\gamma \lesssim T \ll T_c$:

\[
\begin{align*}
\frac{\kappa_{zz,p1}(\omega)}{\kappa_N(T_c)} & \approx \frac{13.87 T^3 A_{p1}^2}{T_c \Delta_2^4} \\
\frac{\kappa_{zz,p2}(\omega)}{\kappa_N(T_c)} & \approx \frac{20.77 T^3}{T_c \Delta_2^4} \left[ \ln^2 \left( \frac{A_{p2} \Delta_2}{4.67 T} \right) + 0.82 \right] \\
\frac{\kappa_{zz,l}(\omega)}{\kappa_N(T_c)} & \approx \frac{16.177 T^5}{T_c \Delta_1^5} \left[ \ln^2 \left( \frac{A_{l} \Delta_1}{6.99 T} \right) + 2.61 \right] \\
\frac{\kappa_{zz,1g}(\omega)}{\kappa_N(T_c)} & \approx \frac{45.14 T^4}{T_c \Delta_2^4 \Delta_1} \left[ \ln^2 \left( \frac{\Delta}{5.61 A_{1g} T} \right) + 2.64 \right] \\
\frac{\kappa_{zz,2u}(\omega)}{\kappa_N(T_c)} & \approx \frac{10.36 T^3}{T_c \Delta_{ef} \Delta_2} \left[ \ln^2 \left( \frac{\Delta_{ef}}{4.67 A_{2u} T} \right) + 2.67 \right]
\end{align*}
\]

and for $\kappa_{yy}$:

\[
\begin{align*}
\frac{\kappa_{yy,p1}(\omega)}{\kappa_N(T_c)} & \approx \frac{43.1 T^5 A_{p1}^2}{T_c \Delta_2^4} \\
\frac{\kappa_{yy,p2}(\omega)}{\kappa_N(T_c)} & \approx \frac{11.3 T^4}{T_c \Delta_2^4} \left[ \ln^2 \left( \frac{A_{y2} \Delta_2}{5.61 T} \right) + 0.78 \right] \\
\frac{\kappa_{yy,l}(\omega)}{\kappa_N(T_c)} & \approx \frac{10.36 T^3}{T_c \Delta_1^4} \left[ \ln^2 \left( \frac{A_{l} \Delta}{4.67 T} \right) + 2.67 \right] \\
\frac{\kappa_{yy,1g}(\omega)}{\kappa_N(T_c)} & \approx \frac{10.36 T^3}{T_c \Delta_2^4 \Delta_1} \left[ \ln^2 \left( \frac{\Delta}{4.67 A_{1g} T} \right) + 2.67 \right] \\
\frac{\kappa_{yy,2u}(\omega)}{\kappa_N(T_c)} & \approx \frac{10.36 T^3}{T_c \Delta_{ef} \Delta_1} \left[ \ln^2 \left( \frac{\Delta_{ef}}{4.67 A_{2u} T} \right) + 2.67 \right].
\end{align*}
\]

The first three results in Eq. (39) (as well as in Eq. (40)) are in agreement with those obtained in [21], where numerical coefficients were not specified. They concern the cases when only one kind of nodes is presented (both the line of nodes or one kind of point nodes, but not a hybrid gap function). The results in Eqs. (39), (40) concerning hybrid gap functions, are new. The particular example of $\kappa_{zz,1g}$-component of the thermal conductivity demonstrates, that in the unitary limit in the presence of several kinds of nodes (that is for a hybrid gap function) the index of power law behavior of the low-temperature thermal conductivity may be greater than the least index among those taking place for superconductors with one
separate kind of the nodes (in particular, in discussing of the $E_{1g}$-case – for superconductors with the linear point nodes).

Since the same relaxation time enters the expressions for both $\kappa_{zz}$ and $\kappa_{yy}$, then for each particular type of pairing there are relations between the coefficients in those expressions, which govern the behavior of the anisotropy ratio $\kappa_{zz}/\kappa_{yy}$. For instance, as for $E_{2u}$ type of pairing the both quantities $I_{zz,2u}$ and $I_{yy,2u}$ are proportional to $\omega$, the anisotropy ratio of the thermal conductivity in leading approximation is determined simply by the ratio $I_{zz,2u}/I_{yy,2u} = \Delta_1(T)/\Delta_2(T)$, that is only weakly depends upon temperature.

For $E_{1g}$ case we have $I_{zz,1g} \propto \omega^2$, while $I_{yy,1g} \propto \omega$. As a consequence, the anisotropy ratio of the thermal conductivity for $E_{1g}$ type of pairing essentially depends upon temperature:

$$\frac{\kappa_{zz,1g}}{\kappa_{yy,1g}} = \frac{4.36T\Delta_1}{\Delta_2^2} \left( 1 - \frac{0.37\ln\left(\frac{\Delta_1}{4.67A_{1g}T}\right)}{\ln^2\left(\frac{\Delta_1}{4.67A_{1g}T}\right) + 2.67} \right).$$

Essential temperature dependence of the ratio for $E_{1g}$ model was noticed earlier on the basis of numerical results in [18,41].

The analysis of experimental data of Ref. [1] on the anisotropy ratio of the thermal conductivity in $UPt_3$ at low temperatures $\gamma \lesssim T \ll T_c$ seems to permit one to distinguish between $E_{2u}$ and $E_{1g}$ representations in favor of $(1,i)$-phase of $E_{2u}$-type of pairing. As it follows from [1] for their particular clean samples, the temperature interval for which the low-temperature power law behavior of the thermal conductivity takes place is approximately $0.07 < T/T_{c-} < 0.15$. According to Eq.(41), the anisotropy ratio should increase in more than 1.78 times, when the temperature changes from $T = 0.07T_{c-}$ to $T = 0.14T_{c-}$. This seems to be in a contradiction with the experiment (see, Fig. 2 in [1]), which shows the increase of the anisotropy ratio only on $10 - 12$ per cent with the temperature change discussed. In other words, the fitting parameters available for the $E_{1g}$ case in Eqs. (39), (40) allow to describe properly experimental results for the given temperature interval both for $\kappa_{zz}$ or for $\kappa_{yy}$, and not for both components simultaneously.

While $E_{1g}$ type of pairing leads to noticeable overestimation of the low-temperature
dependence of the anisotropy ratio $\kappa_{zz}/\kappa_{yy}$ in $UPt_3$, within the framework of $E_{2u}$ type of pairing we get, for the first sight, the underestimation of that dependence. Although for superconductors with nodes low-temperature deviation of the order parameter from its zero-temperature value is not exponentially small (as for $s$-wave isotropic case), but manifests power-law dependences, the temperature dependence of the quantity $\Delta_1(T)/\Delta_2(T)$ is yet too weak in order to explain the observed change of the ratio on $10-12$ per cent. However, the temperature dependence discussed can be described if one keeps in the expression for the anisotropy ratio, obtained within $E_{2u}$ type of pairing, the first low-temperature correction to the leading term. For this purpose one should specify next terms in the expansion of the momentum dependence of the order parameter near nodes. We let $|\Delta| \approx \Delta_1((\pi/2 - \theta) + L(\pi/2 - \theta)^3|$ for $|\theta - (\pi/2)| \ll 1$, and $|\Delta| \approx \Delta_2(\theta^2 - D\theta^4)$ for $|\theta| \ll 1$. Then the calculations of the anisotropy ratio under the condition $\gamma \lesssim T \ll T_c$ result in

$$\frac{\kappa_{zz,2u}}{\kappa_{yy,2u}} = \frac{\Delta_1}{\Delta_2} \left( 1 + 4.36 \left( D - 0.58 - 0.12\frac{\Delta_1}{\Delta_2} \right) \frac{T}{\Delta_2 S(T)} \right), \quad (42)$$

where the quantity

$$S(T) = \frac{\ln^2(\Delta_{ef}/(5.61A_{2u}T)) + 2.64}{\ln^2(\Delta_{ef}/(4.67A_{2u}T)) + 2.67} \quad (43)$$

is close to unity.

We see, that this relation is sufficiently sensitive to the value of the coefficient $D$, while the effect of $L$ is beyond the first correction to the leading term. Taking $D$ as a fitting parameter, one can easily describe the observed change of the anisotropy ratio for $0.07 < T/T_{c-} < 0.15$. For example, let $\Delta_2 \approx 3\Delta_1 \approx 12T_{c-}$, then we find $D \approx 5.2$. Particular values for coefficients are obtained here for a spherical Fermi surface and they, of course, depend upon the form of the Fermi surface. It is of importance to emphasize, however, that the anisotropy of the Fermi surface doesn’t change our qualitative conclusion itself. It is based on the fact, that $\kappa_{zz,1g}/\kappa_{yy,1g}$ is roughly proportional to the temperature, while $\kappa_{zz,2u}/\kappa_{yy,2u}$ weakly depends on $T$, being at the same time sufficiently sensitive to the coefficient $D$ in the next term of the expansion of momentum dependence of the order parameter near point nodes. Note
relatively large values of $D$, to which our analysis results in. The necessity for such a value of $D$ should be taken into account when the particular basis function is chosen for the numerical study. There is, of course, the problem of describing the temperature dependence of the anisotropy ratio at ultra low temperatures $T \lesssim \gamma$, which should take account of steeper slope of the curve, which the experiments show for $\kappa_{zz}/\kappa_{yy}$ in the crossover between two regimes. This problem is not considered here.

For higher temperatures the behavior of the anisotropy ratio becomes much more sensitive to the particular form of the order parameter all over the Fermi surface, not to its behavior mostly near nodes. As a consequence, the problem of discrimination between types of pairing in considering the anisotropy ratio of the thermal conductivity is essentially ambiguous in that case [18].

Furthermore, it follows from Eqs. (21), (22), (26), (39), (40) (in particular, from temperature dependences of $A_{zz}(T)$, $A_{yy}(T)$), that for temperatures in question scaling behavior (31) (as well as (32)) is valid (in disregarding logarithmic terms) for both components $\kappa_{ii}(T, B)$ ($i = x, y$) and for both limiting cases $\gamma \lesssim T \lesssim T_c \sqrt{B/B_{c2}}$, $T_c \sqrt{B/B_{c2}} \ll T \ll T_c$ just for $E_{2u}$-representation. For $E_{1g}$ type of pairing deviations from scaling are more noticeable for $\kappa_{zz}$-component both in the temperature dominating region $T_c \sqrt{B/B_{c2}} \ll T \ll T_c$ (if one uses the form (31)), or in the field dominating region $\gamma \lesssim T \lesssim T_c \sqrt{B/B_{c2}}$ (for the form (32)). In order to distinguish between two models, considering the magnetic field dependence of $\kappa$, one could introduce, along with the zero-field anisotropy ratio discussed just now, the other characteristic as well. It is the anisotropy ratio of the form $(\kappa_{zz}(T, B) - \kappa_{zz}(T, B = 0)) / (\kappa_{yy}(T, B) - \kappa_{yy}(T, B = 0)) = A_{zz}(T)/A_{yy}(T)$ under the temperature dominating condition. Analogously to the anisotropy ratio in zero field, quantity $A_{zz}(T)/A_{yy}(T)$, in accordance with Eqs.(25), (29), (33), is only weakly temperature dependent function within the framework of $E_{2u}$ model, while it is roughly proportional to $T$ for the $E_{1g}$ type of pairing. The anisotropy of the Fermi surface doesn’t change the conclusion. Unfortunately, for $UPt_3$ under the field $B > B_{c1}$ low temperatures discussed do not belong
to the temperature dominating region $T_c \sqrt{B/B_{c2}} \ll T \ll T_c$.

The other test for the discrimination between possible types of pairing in superconducting $UPt_3$, which is based on the presence of superflow, is as follows. Since the only important qualitative difference between order parameters of $E_{2u}$ and $E_{1g}$ types of pairing is the multiplicity of point nodes on the poles of the Fermi surface, it looks reasonable, for maximizing the difference between the effects, to consider $\kappa_{zz}$ component of thermal conductivity under the presence of the supercurrent along the $z$-axis. In order not to mix various directions of the superflow in this case, let it be the uniform transport supercurrent in the absence of the magnetic field. Such a problem is associated with possible experiments on thin films or whiskers [12]. One can use for both types of pairing Eq. (15) for the relaxation time, inserting there $\theta_0 = 0$. Further, under the given conditions

$$I_{zz,2u}(\omega, v_s) = \begin{cases} \frac{\pi \omega}{8\Delta_2}, & p_f v_s \ll \omega , \\ \frac{\pi p_f v_s}{8\Delta_2}, & p_f v_s \gg \omega ; \end{cases} \quad I_{zz,1g}(\omega, v_s) = \begin{cases} \frac{\omega^2 + (p_f v_s)^2}{3\Delta_2^2}, & p_f v_s \ll \omega , \\ \frac{(p_f v_s)^2}{3\Delta_2^2}, & p_f v_s \gg \omega . \end{cases}$$

Inserting these results into (2), we obtain under the temperature dominating conditions $p_f v_s \ll T \ll \Delta_{max}$:

$$\frac{\kappa_{zz,2u}}{\kappa_N(T_c)} = \frac{\kappa_{zz,1g}}{\kappa_N(T_c)} \bigg|_{v_s=0} + \frac{3T^2}{\pi^2 \Delta_1 T_c^2} \frac{(p_f v_s)^2}{\Delta_2^2} \int_0^\infty \frac{\omega d\omega}{\cosh^2 \left( \frac{\omega}{2} \right)} \left[ \left( \text{Re} G_0(T\omega, 0) \frac{\Delta_1}{T} - \frac{\omega}{4} \right)^2 + \frac{\omega^2}{4} \left( \pi^2 - \frac{1}{4} \right) \right],$$

and for the superflow dominating regime $T \ll p_f v_s \ll \Delta_{max}$:

$$\frac{\kappa_{zz,2u}}{\kappa_N(T_c)} = \frac{3\pi^2}{32} \frac{T p_f v_s^2}{T_c \Delta_2^2}, \quad \frac{\kappa_{zz,1g}}{\kappa_N(T_c)} = A_{p1}^2 \frac{T p_f v_s^2}{T_c \Delta_2^2}. \quad (47)$$

We see for the given direction of the supercurrent, that $\kappa_{zz}$ is the nonmonotonous function upon the value of the supercurrent only in the case of $E_{2u}$ type of pairing, while for the
model the thermal conductivity at low temperatures monotonously changes with the superflow velocity (see Fig. 3). This kind of experiments seems to be quite useful for the discrimination between candidates for the type of pairing in superconducting $UPt_3$.

One of the characteristic features of clean superconductors with nodes is the strong low energy dependence of the relaxation time, both in the Born approximation and in the unitary limit \[30,31,19,20\] (see as well Eqs. (8), (12), (13), (17) in the above). This is essential for finding the low temperature behavior of the thermal conductivity in clean limit. It is worth noting that in the unitary limit the relaxation time may manifest weak dependence upon energy for $\omega/\Delta_{max} > 0.1$, so that for not very clean samples with $0.1\Delta_{max} \lesssim \gamma$ a model with energy independent relaxation time may present a reasonable approximation \[40,19\]. For these values of $\gamma$ the conditions $\gamma \lesssim T \ll T_c$, we are interested in in the article, are not satisfied. However, the low temperature region $\gamma \lesssim T \ll T_c$ does exist for samples of Ref. \[1,2\], for which their authors give the estimate $\gamma \approx 0.017\text{K}$ (one gets for this value $\gamma \approx 0.038T_{c-} \sim 0.01\Delta_{max}$).

We note, that in the presence of scattering by impurities, which is close to the unitarity limit, both the relaxation time and the electronic thermal conductivity of two-dimensional sufficiently clean d-wave superconductors would manifest analogous nonmonotonous dependences upon the magnetic field, which are found above for the three-dimensional superconductors with nodes.

VI. THERMAL CONDUCTIVITY DOMINATED BY IMPURITY-INDUCED BOUND STATES

Impurity-induced bound states dominate the thermal conductivity under the condition $T, v_f v_s \lesssim \gamma$. It is essential, that in this limit results turn out to be independent of the relation between $T$ and $v_f v_s$. The spatial averaging over the intervortex space would contain both the region where $v_f v_s \lesssim \gamma$ and a part of the intervortex space where the opposite condition is valid (since $v_f v_s(\xi) \sim T_c$). Hence impurity-induced bound states could influ-
ence essentially the spatially averaged thermal conductivity only for large enough values of $\gamma$. So, for clean superconductors with sufficiently small $\gamma$ one can disregard the influence of impurity bound states on the magnetic field dependence of the thermal conductivity. Nevertheless, it is of interest to consider how the uniform superfluid flow influences the thermal conductivity dominated by impurity induced bound states. Below we show, that in the unitarity limit the thermal conductivity dominated by impurity induced bound states diminishes with increasing superflow field under various conditions. Hence, there is no non-monotonicities in the superflow field dependence of the thermal conductivity at least unless the relation $T, v_f v_s \lesssim \gamma$ is broken.

Excitation energy renormalized by impurities satisfies the relation $\tilde{\omega}(-\omega) = -\tilde{\omega}^*(\omega)$ \cite{16,43}. Thus, retaining for sufficiently small $\omega$ a pair of terms in the expansion of $\tilde{\omega}(\omega)$ in powers of $\omega$, we have:

$$\tilde{\omega}(\omega) \simeq i\gamma + a\omega,$$ \hspace{1cm} (48)

where $\gamma$ and $a$ are real positive parameters. Parameter $a$ is a function of $\gamma$ and, in particular, in the unitary limit in the absence of the condensate flow field it takes the form

$$a = \frac{\left\langle \left( \gamma^2 + |\tilde{\Delta}|^2 \right)^{-1/2} \right\rangle_{S_f}}{\left\langle \left( \gamma^2 + 2|\tilde{\Delta}|^2 \right) \left( \gamma^2 + |\tilde{\Delta}|^2 \right)^{-3/2} \right\rangle_{S_f}}.$$ \hspace{1cm} (49)

For sufficiently small $\gamma$ one gets $a = 1/2$. In the presence of the superflow field parameter $\gamma$ is a function of $v_s$, and up to the first correction to its zero field value we get

$$\gamma \simeq \gamma_0 + b v_s^2.$$ \hspace{1cm} (50)

To estimate $b$ we make use of the results of Ref. \cite{8}, and obtain that in the unitary limit this coefficient is negative and $|b| \sim v_f^2/\gamma_0$, at least within the logarithmic accuracy.

We use the expansion of the integrand in Eq.(1) over a parameter $(a\omega - v_f v_s)\gamma/(\gamma^2 + |\tilde{\Delta}|^2)$, which is supposed to be small. Renormalized order parameter can be taken in this limit for $\omega = 0$, since we are interested in most important terms which are linear in temperature.
Retaining the corrections to the zero field term up to second order in the superflow field, we obtain

\[
\kappa_{ii} = \frac{\pi^2 N_f T}{3} \left\langle v_{f,i}^2 \left[ \frac{\gamma_0^2}{(\gamma_0^2 + |\Delta|^2)^{3/2}} + (\mathbf{v}_f \mathbf{v}_s)^2 \left( \frac{\gamma_0^2}{(\gamma_0^2 + |\Delta|^2)^{5/2}} - \frac{5\gamma_0^4}{2(\gamma_0^2 + |\Delta|^2)^{7/2}} \right) \right] \right\rangle_{S_f}. \tag{51}
\]

In the presence of nodes of the order parameter only narrow regions on the Fermi surface near those nodes are of importance in Eq.\,(51) under the condition \(\gamma \ll \Delta_{\text{max}}\). This permits to consider contributions from each kind of nodes separately. Due to the same reason corrections containing \(\mathbf{v}_s\) and coming from the corresponding dependence of \(|\tilde{\Delta}|\) in the first term in Eq.\,(51), may be disregarded in this case, since they would have an additional small factor \(\sim (\gamma_0/\Delta_{\text{max}})^2\) associated with small characteristic angular regions near the nodes.

Let the order parameter have the line of nodes on the equator of a spherical Fermi surface: \(|\Delta| = \Delta_1 \sqrt{2} - \theta|\). Then we get from Eq.\,(51) the contribution from this line to the thermal conductivity \(\kappa_{yy}\) in the presence of the superflow, corresponding to the magnetic field along \(y\)-axis:

\[
\kappa_{yy} = \frac{\pi^2 N_f v_f^2 T}{6\Delta_1} \left( 1 - \frac{v_{s,x}^2 v_f^2}{6\gamma_0^2} \right). \tag{52}
\]

Contributions from point nodes on the poles (both linear and quadratic) are negligibly small for \(\kappa_{yy}\). It is not the case for \(\kappa_{zz}\). In the presence of magnetic field along \(z\)-axis we find from Eq.\,(51) (within the logarithmic approximation) the following contribution from the line of nodes for this component of the thermal conductivity:

\[
\kappa_{zz} \simeq \frac{\pi^2 \gamma_0^2 N_f v_f^2 T}{3 \Delta_1^3} \left[ \ln \left( \frac{2A\Delta_1}{e\gamma_0} \right) + \frac{2b v_s^2}{\gamma_0} \ln \left( \frac{2A\Delta_1}{e^{3/2}\gamma_0} \right) \right]. \tag{53}
\]

At the same time the contribution from the linear point node is

\[
\kappa_{zz} \simeq \frac{\pi^2 N_f v_f^2 T}{3} \left[ \frac{\gamma_0}{\Delta_2^2} + \frac{b v_s^2}{\Delta_2^2} \right]. \tag{54}
\]

Making use of the estimation made above for the coefficient \(b\), we have disregarded here the term, which is in \((\gamma_0/\Delta_2)^2\) times less than the last term in (54).
Furthermore, for quadratic point node one gets

\[ \kappa_{zz} \simeq \frac{\pi^2 N_f v_f^2 T}{3} \left[ \frac{1}{2\Delta_2} - \frac{v_s^2 v_T^2}{24\gamma_0 \Delta_2^2} \right]. \]  

(55)

In the absence of the superfluid flow field results (52) - (55) coincide with the ones obtained in [16,17]. According to Eqs.(52) and (55), thermal conductivity diminishes with increasing superflow under the corresponding conditions. Negative value of \( b \) ensures the same qualitative conclusion for Eqs.(53), (54). Scaling relation for the thermal conductivity discussed in previous section is obviously broken in the limit \( T, v_f v_s \lesssim \gamma \) under the conditions we consider. We note nonmonotonous superflow dependence of the thermal conductivity upon condensate flow field under the condition \( T \lesssim \gamma \), since for \( v_f v_s \lesssim \gamma \) the thermal conductivity diminishes while for \( v_f v_s \gg \gamma \) it increases with increasing superflow field.

VII. CONCLUSION

In conclusion we have examined the possibility for nonmonotonous magnetic field dependence of the electronic thermal conduction along the magnetic field at low temperatures for sufficiently clean superconductors with nodes of the order parameter on the Fermi surface. We found that the contribution from low energy quasiparticles in the intervortex space is quite important in this respect and specific for superconductors with nodes. The effect comes from the influence of condensate flow field in the intervortex space on the scattering by nonmagnetic impurities of low-energy extended quasiparticles with momentum directions near nodes of the order parameter. The scattering is considered to be sufficiently close to the unitarity limit. We showed that the relaxation time at low energy is a nonmonotonous function upon the condensate flow field.

Nonmonotonous magnetic field dependence of the thermal conductivity due to this contribution may take place for type II superconductors with large Ginzburg-Landau parameter within the temperature interval \( \beta T_c \sqrt{\frac{B_1}{B_2}} \leq T \ll T_c \), where numerical factor \( \beta \) may be of the order of unity. These results are in a good qualitative agreement with recent experiments [2] on the heavy-fermion superconductor \( UPt_3 \). We explain the anomalously strong
influence of the magnetic field on the quasiparticle scattering, observing in this compound [12]. We obtained as well that scaling behavior of the thermal conductivity with a single parameter $x = \frac{T}{T_c} \sqrt{\frac{B}{B_c}}$ as well as weak low-temperature dependence of the anisotropy ratio $\kappa_{zz}/\kappa_{yy}$ in zero field, are valid with logarithmic accuracy within the temperature interval $\gamma \lesssim T \ll T_c$ for $(1, i)$-phase of $E_{2u}$-type of pairing. Qualitatively it is quite close to that observed for $UPt_3$ in Ref. [1,2]. Under the same conditions $E_{1g}$ model results in more noticeable deviations from the scaling, and in essential temperature dependence of the ratio $\kappa_{zz}(T, B = 0)/\kappa_{yy}(T, B = 0)$. New test is proposed for discrimination between candidates for the type of pairing in $UPt_3$, based on the dependence of $\kappa_{zz}$ upon the value of transport supercurrent flowing in thin films or whiskers along the hexagonal axis.

ACKNOWLEDGMENTS

We would like to thank V.P. Mineev, who was the initiator of our work on the subject of this article, for helpful and stimulating discussions. We are grateful to the authors of Refs. [1,2] for sending us the preprints of their works before publications. One of us (Yu.S.B.) also thanks I. Fomin, A. Huxley and H. Suderow for useful discussions. Yu.S.B. is grateful to Département de Physique de l’ENS for financial support of his stay in France. This research is supported in part by the Russian Foundation for Basic Research under grants No. 96-02-16249 and No. 97-02-17545.
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The effect of the supercurrent flow on the relaxation time and on the thermal conductivity for superconductors with nodes was discussed about ten years ago in [36]. However, only the limit $\omega \ll p_f v_s$ was considered there and the nonmonotony was not noticed. Furthermore, those authors considered $T$-matrix to be diagonal, which in our notation means, in particular, $G_1 = 0$. If so, then for consistency one should assume that the both coherence factors, introduced in [36], are reduced to $1/2$ for all cases in question. Then the expression (25) in [36] for the relaxation time reduces to Eq. (14) of our work. Besides, the quantity $\lambda$, introduced in [36], is equal to zero under the conditions we discuss, due to the equality $|g(E)| = |g(-E)|$. Then the basic formula of [36] for the thermal conductivity in the presence of the supercurrent reduces to our relation (2). Note, that when the relaxation time by impurities is associated with one-particle propagator (see, for example [19,20]), the processes of quasiparticle scattering and of the pair creation and annihilation by impurities are automatically considered together, and there is no need to separate them as it is done in [36]. Complications of this kind can appear, however, in the case of more number of quasiparticles participating in the scattering event [37,38]. The authors are grateful to H. Suderow for pointing out the Ref. [36] at the end stage of the work at this article. We do not discuss here the other problems considered in [36].

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$\Delta_0 \lesssim p_f v_s$ it may essentially depend upon $v_s$. We are concentrating, however, mostly
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FIGURES

FIG. 1. The relaxation time $\tau_s$ as a function of superfluid velocity $v_s$, directed parallel to the basal plane, in the unitary limit for various energies $\omega/\Delta_0(T) = 0.01$ (1), 0.05 (2), 0.2 (3), 1 (4), 5 (5) and three types of pairing: (a) – polar phase, $\Delta = \Delta_0(T)\cos \theta$, (b) – $(1, i)$-state of $E_{1g}$-pairing, $|\Delta| = \Delta_0(T)|\cos \theta|\sin \theta$, (c) – $(1, i)$-state of $E_{2u}$-pairing, $|\Delta| = \Delta_0(T)|\cos \theta|\sin^2 \theta$.

FIG. 2. The thermal conductivity $\kappa_{zz}$ as a function of superfluid velocity $v_s$, directed parallel to the basal plane, for various temperatures and three types of pairing: (a) – polar phase, (b) – $E_{1g}$-pairing, (c) – $E_{2u}$-pairing. The same basis functions are used as for Fig. 1.

FIG. 3. Normalized thermal conductivity $\kappa_{zz}(T, v_s)/\kappa_{zz}(T, 0)$ as a function of superfluid velocity $v_s$, directed along z-axis, for unitary scatterers, various temperatures $T/\Delta_0(T) = 0.01$ (1), 0.025 (2), 0.05 (3), 0.1 (4), 0.2 (5) and two types of pairing: (a) – $E_{1g}$-pairing – $|\Delta| = \Delta_0(T)|\cos \theta|\sin \theta$, (b) – $E_{2u}$-pairing – $|\Delta| = \Delta_0(T)|\cos \theta|\sin^2 \theta$. 
\[ K_{zz}(T) T_c / K_{zz}(T_c) T \]

![Graph showing the relationship between \( K_{zz}(T) T_c / K_{zz}(T_c) T \) and \( p_f v_s / \Delta_0(T) \). The graph has multiple curves labeled 1, 2, and 3.](b)
The diagram shows the relationship between $k_{zz}(T, \nu_s)/k_{zz}(T, 0)$ and $p_f \nu_s / \Delta_0(T)$ for different curves labeled 1, 2, 3, 4, and 5.
