MEAN CURVATURE FLOW WITH GENERIC LOW-ENTROPY INITIAL DATA

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Abstract. We prove that sufficiently low-entropy closed hypersurfaces can be perturbed so that their mean curvature flow encounters only spherical and cylindrical singularities. Our theorem applies to all closed surfaces in \( \mathbb{R}^3 \) with entropy \( \leq 2 \) and to all closed hypersurfaces in \( \mathbb{R}^4 \) with entropy \( \leq \lambda(S^1 \times \mathbb{R}^2) \). When combined with recent work of Daniels-Holgate, this strengthens Bernstein–Wang’s low-entropy Schoenflies-type theorem by relaxing the entropy bound to \( \lambda(S^1 \times \mathbb{R}^2) \).

Our techniques, based on a novel density drop argument, also lead to a new proof of generic regularity result for area-minimizing hypersurfaces in eight dimensions (due to Hardt–Simon and Smale).

1. Introduction

Mean curvature flow is the natural heat equation for submanifolds. A family of hypersurfaces \( M(t) \subset \mathbb{R}^{n+1} \) flows by mean curvature flow if

\[
\left( \frac{\partial}{\partial t} x \right)^\perp = H_{M(t)}(x),
\]

where \( H_{M(t)}(x) \) denotes the mean curvature vector of \( M(t) \) at \( x \). When \( M(0) \) is compact, mean curvature flow is guaranteed to become singular in finite time. Understanding the potential singularities is thus a fundamental problem. One approach to this issue is to study the flow in the generic case: a well-known conjecture of Huisken suggests that the singularities of a generic mean curvature flow should be as simple as possible, namely, spherical and cylindrical [Huisken 2003, #8].

The main results of this note completely resolve Huisken’s conjecture in three and four dimensions for low-entropy initial data (see (1.2) for the definition of entropy). Informally stated (see Corollaries 1.8 and 1.9 for precise statements) we prove the following results.

**Theorem 1.1** (Low-entropy generic flow in \( \mathbb{R}^3 \), informal). If \( M^2 \subset \mathbb{R}^3 \) is a closed embedded surface with entropy \( \lambda(M) \leq 2 \) then there exist arbitrarily small \( \mathcal{C}^\infty \) graphs \( M' \) over \( M \) so that the mean curvature flow starting from \( M' \) has only multiplicity-one spherical and cylindrical singularities.

**Theorem 1.2** (Low-entropy generic flow in \( \mathbb{R}^4 \), informal). If \( M^3 \subset \mathbb{R}^4 \) is a closed embedded hypersurface with entropy \( \lambda(M) \leq \lambda(S^1 \times \mathbb{R}^2) \) then there exist arbitrarily small \( \mathcal{C}^\infty \) graphs \( M' \) over \( M \) so that the mean curvature
flow starting from $M'$ has only multiplicity-one spherical and cylindrical singularities.

In an earlier version of this paper, we conjectured that Theorem 1.2 could be combined with a surgery construction to yield a strengthened version of Bernstein–Wang’s low-entropy Schoenflies theorem [BW22a] (cf. Theorem 1.4 below). This surgery construction has been recently carried out by Daniels-Holgate [DH22] who showed that if a mean curvature flow has only spherical and neckpinch singularities, then one can construct a mean curvature flow with surgery. As such, combining these results leads to the following:

**Corollary 1.3** (Strengthened low-entropy Schoenflies-type theorem). If $M^3 \subset \mathbb{R}^4$ is an embedded 3-sphere with entropy $\lambda(M) \leq \lambda(S^1 \times \mathbb{R}^2)$ then $M$ is smoothly isotopic to the round $S^3$.

See Sections 1.2 and 1.4 for an expanded discussion of this result.

1.1. **Previous work on generic mean curvature flow.** Trailblazing work of Colding–Minicozzi demonstrated that spheres and cylinders are the only linearly stable singularity models for mean curvature flow [CM12]. In particular, the remaining singularity models are unstable so should not generically occur (as conjectured by Huisken). In a previous paper [CCMS20], the authors introduced new methods to the study of generic mean curvature flow, proving that a large class of singularity models (specifically, singularities with tangent flows modeled on multiplicity one compact or asymptotically conical self-shrinkers) can be indeed avoided by a slight perturbation of the initial conditions.

In particular, our previous work shows that for a generic initial surface in $\mathbb{R}^3$, either the mean curvature flow has only spherical and cylindrical singularities or at the first singular time it has a tangent flow with a cylindrical end or higher multiplicity (both possibilities are conjectured not to happen).

We refer the reader to the introduction to our previous article [CCMS20] for further discussion of generic mean curvature flows and related work.

1.1.1. **Relationship between this paper and our previous work.** In [CCMS20], we proved a classification of ancient one-sided flows (analogous to the minimal surface results of Hardt–Simon [HSS5]; see Appendix D for further discussion) which led to a complete understanding of flows on either side of a neighborhood of a non-generic (compact or asymptotically conical) singularity. In particular, we showed that nearby flows to either side do not have such singularities nearby.

In $\mathbb{R}^3$, to understand generic mean curvature flow without a low-entropy condition (in contrast with this note), one must work at the first non-generic time rather than globally in space-time. However, two serious issues arise
when working this way. First, there is no partial regularity known for tangent flows past the first singular time without a low-entropy bound. Second, the possibility that a small perturbation of the initial data increases the first singular time slightly without improving the flow in an effective way. To that end, in \cite{CCMS20} we had to additionally prove that the nearby flows strictly decrease genus as they avoid the non-generic singularity. This genus-loss property is crucial for tackling Huisken’s conjecture in $\mathbb{R}^3$ without a low-entropy condition and is a consequence of the classification of ancient one sided flows, as obtained in \cite{CCMS20}.

On the other hand, by including a low-entropy condition, here we are able to work \textit{globally} in space-time. This allows for significantly simplified arguments. In fact, the key observation of this paper is that in this setting one can completely avoid the classification of one-sided ancient flows and instead rely on a soft argument based on compactness and a new geometric property of non-generic shrinkers (see Proposition 2.2). We emphasize that a drawback of the methods used in this note as compared to our previous work is that the arguments used here give no indication as to the local dynamics near a non-generic singularity (such information was obtained near asymptotically conical and compact shrinkers; see also \cite{CM19, CM22}).

Remark. After the first version of this paper (as well as our previous paper \cite{CCMS20}) were posted, another approach to the generic perturbation of the initial data was pursued by Sun–Xue \cite{SX21b, SX21a}. This approach is in the spirit of local ODE dynamics, as suggested by the Colding–Minicozzi program, cf. \cite{CM19}. The analytic framework in \cite{SX21b, SX21a} has the interesting feature that non-one-sided perturbations are analyzed, but the applications are currently limited to locally perturbing away singularities that arise at the first singular time. Conversely, our geometric approach (first developed in \cite{CCMS20}) is motivated by global results such as the ones stated in Theorems 1.1 and 1.2. Of course, our approach also admits localizations; see Appendix C.

1.2. Entropy. To state our main results, we first recall Colding–Minicozzi’s definition \cite{CM12} of entropy of $M^n \subset \mathbb{R}^{n+1}$:

\[
\lambda(M) := \sup_{x_0 \in \mathbb{R}^{n+1}} \sup_{t_0 > 0} \int_M (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{1}{4t_0} \|x-x_0\|^2}.
\]

By Huisken’s monotonicity of Gaussian area, we see that $t \mapsto \lambda(M(t))$ is non-increasing when $M(t)$ is flowing by mean curvature flow. A computation of Stone \cite{Sto94} shows that the entropies of the self-shrinking cylinders

\footnote{At the first singular time, work of Ilmanen \cite{Ilm95} and Wang \cite{Wan16} show that the support of any tangent flow is a smooth self-shrinker with only conical/cylindrical ends.}
Several fundamental results have been obtained about hypersurfaces with sufficiently small entropy, starting with work of Colding–Ilmanen–Minicozzi–White [CIMW13] who proved that the round sphere $S^n(\sqrt{2n})$ has minimal entropy among all closed self-shrinkers. This was extended by Bernstein–Wang [BW16] who showed that the round sphere minimizes entropy among all closed hypersurfaces (see also [Zhu20, HW19]). Moreover, Bernstein–Wang have also proven [BW17] that the cylinder $S^1(\sqrt{2}) \times \mathbb{R} \subset \mathbb{R}^3$ has second least entropy among all self-shrinkers in $\mathbb{R}^3$ (their result crucially relies on Brendle’s classification of genus zero self-shrinkers [Bre16]).

Subsequent work of Bernstein–Wang provides a robust picture of hypersurfaces with sufficiently small entropy [BW18b, BW18a, BW22b] (see also [BW21]). In particular, they obtained the following low-entropy Schoenflies result:

**Theorem 1.4** (Bernstein–Wang [BW22a]). If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(S^2 \times \mathbb{R})$ then $M$ is smoothly isotopic to the round $S^3$.

In [BW22a], this is proven by flowing $M$ by mean curvature flow and then smoothing out any potential non-generic singularities to construct the desired isotopy. Our previous work [CCMS20] on generic mean curvature flow gave an alternative approach to this result by showing that if one perturbs $M$ slightly, the mean curvature flow directly provides the isotopy:

**Theorem 1.5** ([CCMS20]). If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(S^2 \times \mathbb{R})$ then after a small $C^\infty$-perturbation to a nearby hypersurface $M'$, the mean curvature flow $M'(t)$ is completely smooth until it disappears in a round point.

One of the consequences of this paper is a simplified proof of Theorem 1.5 (see also the stronger version stated in Corollary 1.3).

### 1.3. Main results

We now describe our main results in full generality. We construct generic mean curvature flows of sufficiently low-entropy hypersurfaces in all dimension. To quantify the low-entropy condition we make several definitions.\(^2\) Let $S_n$ denote the set of smooth self-shrinkers in $\mathbb{R}^{n+1}$ with $\lambda(\Sigma) < \infty$, i.e., properly embedded hypersurfaces $\Sigma$ satisfying $H + \frac{\Sigma}{2} = 0$ with finite Gaussian area. Let $S_n^*$ denote the non-flat elements of $S_n$. For $\Lambda > 0$, let

$$S_n(\Lambda) := \{ \Sigma \in S_n : \lambda(\Sigma) < \Lambda \}, \quad S_n^*(\Lambda) := S_n(\Lambda) \cap S_n^*.$$  

\(^2\)Note that $\lambda(S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}) = \lambda(S^k)$.

\(^3\)The definitions here are closely related to the hypotheses $(\ast_n, \Lambda), (\ast\ast_n, \Lambda)$ introduced by Bernstein–Wang (cf. [BW18b, BW22a]), but our second hypothesis is less restrictive.
We also define
\[ S_n^{\text{gen}} := \left\{ O(S^j(\sqrt{2})) \times \mathbb{R}^{n-j} \in S_n : j = 1, \ldots, k, \ O \in O(n+1) \right\} \]
to be the set of (round) self-shrinking spheres and cylinders in \( \mathbb{R}^{n+1} \).

Similarly, we let \( \mathcal{RMC}_n \) denote the space of regular minimal cones in \( \mathbb{R}^{n+1} \), i.e., the set of \( C \subset \mathbb{R}^{n+1} \) with \( C \setminus \{0\} \) a smooth properly embedded hypersurface invariant under dilations and having vanishing mean curvature. Let \( \mathcal{RMC}_n^* \) denote the non-flat elements of \( \mathcal{RMC}_n \). Define
\[ \mathcal{RMC}_n(\Lambda) := \{ C \in \mathcal{RMC}_n : \lambda(C) < \Lambda \}, \quad \mathcal{RMC}_n^*(\Lambda) := \mathcal{RMC}_n(\Lambda) \cap \mathcal{RMC}_n^* \]
For a dimension \( n \geq 2 \) and entropy bound \( \Lambda \in (\lambda(S^n), 2] \), our first hypothesis is
\( (\dagger,n,\Lambda) \)
For \( 3 \leq k \leq n \), \( \mathcal{RMC}_k^*(\Lambda) = \emptyset \)
while our second hypothesis is
\( (\dagger\dagger,n,\Lambda) \)
\[ S_{n-1}^*(\Lambda) \subset S_n^{\text{gen}}. \]

Finally, we define certain notation that will be used throughout.

**Definition 1.6.** For a closed embedded hypersurface \( M^n \subset \mathbb{R}^{n+1} \) we denote by \( \mathcal{F}(M) \) the set of cyclic\(^4\) unit-regular integral Brakke flows \( \mathcal{M} \) with \( \mathcal{M}(0) = \mathcal{H}^n[M] \), and for each \( \mathcal{M} \in \mathcal{F}(M) \), we define \( \text{sing}_{\text{gen}} \mathcal{M} \subset \text{sing} \mathcal{M} \) to be the set of singular points \((x,t)\) so that some\(^5\) tangent flow to \( \mathcal{M} \) at \((x,t)\) is a multiplicity-one flow associated to elements of \( S_n^{\text{gen}} \).

Having given these definitions, we can now state our main technical result. By convention we take \( \lambda(S^0) = 2 \). Everywhere below, \( M \) is taken to be closed and embedded.

**Theorem 1.7.** Assume that \( n \geq 2 \) and \( \Lambda \in (\lambda(S^n), \lambda(S^{n-2})] \) satisfy hypothesis \((\dagger,n,\Lambda)\) and \((\dagger\dagger,n,\Lambda)\). If \( M^n \subset \mathbb{R}^{n+1} \) has \( \lambda(M) \leq \Lambda \) then there exist arbitrarily small \( C^\infty \) graphs \( M' \) over \( M \) so that \( \lambda(M') < \Lambda \) and all \( \mathcal{M}' \in \mathcal{F}(M') \) have \( \text{sing} \mathcal{M}' = \text{sing}_{\text{gen}} \mathcal{M}' \). In particular, the level set flow of \( M' \) does not fatten.

See [CCMS20, Section 1.2] for a discussion of results related to the regularity of flows satisfying \( \text{sing} \mathcal{M}' = \text{sing}_{\text{gen}} \mathcal{M}' \).

In low dimensions, the hypothesis \((\dagger,n,\Lambda)\) and \((\dagger\dagger,n,\Lambda)\) can be understood more concretely. This leads to the following results.

**Corollary 1.8.** If \( M^2 \subset \mathbb{R}^3 \) has \( \lambda(M) \leq 2 \) then there exist arbitrarily small \( C^\infty \) graphs \( M' \) over \( M \) so that the level-set flow of \( M' \) is non-fattening and the associated Brakke flow \( M' \in \mathcal{F}(M') \) has \( \text{sing} \mathcal{M}' = \text{sing}_{\text{gen}} \mathcal{M}' \).

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\(^4\)Recall that a integral varifold \( V \) is cyclic if the unique mod 2 flat chain \([V]\) has \( \partial [V] = 0 \). Work of White [Whi99] shows that this property is preserved under varifold (and Brakke flow) convergence.

\(^5\)Note that if some tangent flow is a multiplicity one element of \( S_n^{\text{gen}} \) then all are by [CM15, CM15], cf. [BW15].
Proof. Condition (†2.2) is vacuous while (††2.2) holds by the classification of self-shrinking curves [AL86].

Corollary 1.9. If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ then there exist arbitrarily small $C^\infty$ graphs $M'$ over $M$ so that the level-set flow of $M'$ is non-fattening and the associated Brakke flow $M' \in \mathfrak{T}(M')$ has $\text{sing} M' = \text{sing}_{\text{gen}} M'$.

Proof. By the resolution of the Willmore conjecture [MN14], $RMC^* (\Lambda_C) = \emptyset$ for $\Lambda = \frac{2\pi^2}{4\pi} \approx 1.57 > \lambda(\mathbb{S}^1) \approx 1.52$.

Thus (†3.Λ) holds for all $\Lambda \leq \Lambda_C$. Furthermore, by the classification of low-entropy shrinkers in $\mathbb{R}^3$ from [BW17], it holds that $S^*_2(\lambda(\mathbb{S}^1)) = S^\text{gen}_2$. Thus (††3,Λ) holds.

1.4. Generic mean curvature flow with surgery. As already observed in [CCMS20], we can apply Corollary 1.9 to give a direct proof of Theorems 1.4 and 1.5. Moreover, Daniels-Holgate has recently proven that if an initial hypersurface admits a (cyclic, unit-regular, integral) Brakke flow with only spherical and neckpinch type singularities, then it is possible to construct a smooth mean curvature flow with surgery starting from this initial condition (see [DH22] for the precise definition of mean curvature flow with surgery).

As such, Corollaries 1.8 and 1.9 combined with [DH22, Theorem 1.2] yields the following generic surgery construction.

Corollary 1.10 (Generic mean curvature flow with surgery). Assume that $n \geq 2$ and $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2}))$ satisfy (†n,Λ) and (††n,Λ). If $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \Lambda$, then there is an arbitrarily small $C^\infty$ graph $M'$ over $M$ and a smooth mean curvature flow with surgery starting from $M'$.

In particular, when $M^3 \subset \mathbb{R}^4$ is an embedded 3-sphere with $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, the mean curvature flow with surgery can be used (see [DH22, Theorem 6.4]) to construct an isotopy to the round 3-sphere. This yields the strengthened version of the low-entropy Schoenflies theorem stated in Corollary 1.3.

Remark. In the setting of 2-convex mean curvature flow with surgery (see [HS99, HS99, Bre15, BHH16, HK17a, HK17b, ADS19, ADS20, BC19, BC21]) the surgery to isotopy construction has been studied in several works [HS09, BHH16, BHH19, Mra18, MW21]. (We also mention related work using Ricci flow with surgery [Mar12, CL19] and singular Ricci flow [BK22, BK23, BK19].)

6 The spherical and neckpinch singularities are the tangent flows for which a canonical neighborhood theorem is proven, thanks to [CHI22, CHHW22].

7 Note that if $M'$ is such a Brakke flow in $\mathbb{R}^{n+1}$ and $\text{sing} M' = \text{sing}_{\text{gen}} M'$, then the condition “$M'$ has only spherical and neckpinch singularities” is a consequence of $\lambda(M') < \lambda(\mathbb{S}^{n-2})$. 


1.5. Generic regularity of area-minimizing hypersurfaces in eight dimensions. We remark that the study of generic mean curvature flow in our previous work [CCMS20] can be viewed as the parabolic analogue of the work of Hardt–Simon [HS85] and Smale [Sma93] concerning the generic regularity of area-minimizing hypersurfaces in eight dimensions. In particular, the existence and uniqueness of the ancient one-sided mean curvature flow [CCMS20] is a direct analogue of the existence and uniqueness of the foliation on either side of a regular area minimizing cone, as proven in [HS85] (see also [Wan22]).

In this paper, we develop a new technique based on density drop, that avoids the classification of the ancient one-sided flow. As one might expect, this also yields a new proof of the generic regularity results of Hardt–Simon [HS85] and Smale [Sma93] that avoids the need to classify the foliation. This is discussed further in Appendix D.

1.6. Organization. See [CCMS20, Section 2] for the conventions used in this paper. In Section 2 we prove entropy drop near non-generic singularities and we use this to prove Theorem 1.7 in Section 3. Appendices A and B recall some standard stability results. Appendix C contains a localized perturbative result. In Appendix D we discuss how the arguments here relate to generic regularity of area-minimizing hypersurfaces in eight dimensions.

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2. Entropy drop near non-generic singularities

**Lemma 2.1.** Assume that \((\mathfrak{n}, \Lambda)\) holds for some \(\Lambda \leq 2\). Suppose that \(V\) is a \(F\)-stationary cyclic integral \(n\)-varifold in \(\mathbb{R}^{n+1}\) satisfying \(F(V) < \Lambda\). Then, there is \(\Sigma \in S_n(\Lambda)\) so that \(V = \mathcal{H}^n|\Sigma\).

**Proof.** This follows from the proof of [BW18b, Lemma 3.1 and Proposition 3.2] except the cyclic property of \(V\) is used to rule out three half-spaces as a potential iterated tangent cone (cf. [Whi09, Corollary 4.5]).

Recall that Huisken has classified the cylinders \(S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}\) as the unique smooth embedded self-shrinkers with non-negative mean curvature \(H \geq 0\) [Hui90, Hui93] (the technical assumption of bounded curvature was later removed by Colding–Minicozzi [CM12]). The following result can be viewed as a geometric consequence of Huisken’s result. It will serve as our key mechanism for perturbing away “non-generic” singularities.
Proposition 2.2. For $\Sigma \in S^*_n$, fix an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\Sigma = \partial \Omega$. Assume that there is a space-time point $(x_0, t_0) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (0, 0)$ so that
\begin{equation}
\sqrt{t_0 - t} \Sigma + x_0 \subset \sqrt{-t} \bar{\Omega}
\end{equation}
for all $t < \min \{0, t_0\}$. Then, one of the following holds:

1. $\Sigma = S^n(\sqrt{2n})$, or
2. $\Sigma = O(\tilde{\Sigma} \times \mathbb{R})$ for $\tilde{\Sigma} \in S^*_{n-1}$ and $O \in O(n+1)$.

Note that if we replaced condition (2.1) with
\begin{equation}
\sqrt{t_0 - t} \Sigma + x_0 \subset \sqrt{-t} \Omega
\end{equation}
(i.e., if we replaced the closure of $\Omega$ with the interior of $\Omega$), we could use an inductive argument to conclude that $\Sigma \in S^*_{n+1}$.

Let us give the geometric intuition underlying our proof strategy. Let $M_0$ denote the spacetime track of $t \mapsto \sqrt{-t} \Sigma$ and $M$ denote the spacetime track of $t \mapsto \sqrt{t_0 - t} \Sigma + x_0$. For $\lambda \in (0, 1]$, let $M_\lambda$ be the parabolic rescaling of $M$ by a factor of $\lambda$; thus, $M_1 = M$ and, as $\lambda \to 0$, $M_\lambda \to M_0$ smoothly locally away from $(0, 0)$. Note that $M_0$ is invariant under parabolic dilations, so $M_\lambda$ always lies weakly to one side of $M_0$.

If $M_\lambda$ touches $M_0$ for some $\lambda > 0$ (equivalently, for all $\lambda > 0$ due to $M_0$'s parabolic dilation invariance), it is then a simple consequence of the strong maximum principle and monotonicity that $\Sigma$ splits a line.

Otherwise, $M_\lambda$ was disjoint from $M_0$ for all $\lambda \in (0, 1]$. It is then standard to use the height of $M_\lambda$ over $M_0$ at time $t = -1$, for $\lambda > 0$ small, to produce a kernel element of the linearized operator that is everywhere nonnegative ($M_\lambda$ always lies weakly to one side of $M_0$). By studying the geometry of parabolic dilations, the kernel element produced is $x_0 \cdot \nu_\Sigma$ if $x_0 \neq 0$ or $x \cdot \nu_\Sigma$ if $x_0 = 0$ ($\implies t_0 \neq 0$). It turns out that the former case implies splitting once again, while the latter implies the mean-convexity of $\Sigma$.

The proof we give below is a more succinct version of the argument above: it handles both cases in a unified way.

Proof of Proposition 2.2. Observe that the set $\cup_{t < 0} \sqrt{-t} \bar{\Omega} \times \{t\}$ is invariant under parabolic dilation around the space-time origin. We thus conclude that for all $\lambda \in [0, \infty)$ and $t < \min \{0, \lambda^2 t_0\}$,
\begin{equation}
\sqrt{\lambda^2 t_0 - t} \Sigma + \lambda x_0 \subset \sqrt{-t} \bar{\Omega}
\end{equation}
In particular, taking $t = -1$ and $\lambda \geq 0$ small, we have that
$$\lambda \mapsto \Sigma_\lambda := \sqrt{1 + \lambda^2 t_0} \Sigma + \lambda x_0 \subset \bar{\Omega}$$
is a 1-parameter family of hypersurfaces with $\Sigma_0 = \Sigma = \partial \Omega$. The normal speed at $\lambda = 0$ is $x_0 \cdot \nu_\Sigma \geq 0$ (where $\nu_\Sigma$ is the unit normal pointing into $\Omega$). Because
$$\Delta_\Sigma(x_0 \cdot \nu_\Sigma) - \frac{1}{2} x \cdot \nabla_\Sigma(x_0 \cdot \nu_\Sigma) + |A_\Sigma|^2(x_0 \cdot \nu_\Sigma) = 0$$
(cf. [CM12 Theorem 5.2]), the maximum principle implies that either \(x_0 \cdot \nu_\Sigma > 0\) along \(\Sigma\) or \(x_0 \cdot \nu_\Sigma = 0\) along \(\Sigma\). (Note that \(\Sigma\) is connected thanks to the Frankel property of shrinkers, cf. [CCMS20 Corollary C.4].)

In the first case (i.e., \(x_0 \cdot \nu_\Sigma > 0\)), each component of \(\Sigma\) is a graph over the \(x^j\)-hyperplane. By [Wan11 (cf. EH89)], each component of \(\Sigma\) must be a hyperplane, so there is only one component and \(\Sigma\) is a flat hyperplane. This contradicts the assumption that \(\Sigma \in S^*_n\) (the set of non-flat shrinkers).

In the second case (i.e., \(x_0 \cdot \nu_\Sigma = 0\)), we see that \(x_0 \neq 0\), then \(\Sigma\) splits a line in the \(x_0\)-direction. It thus remains to consider the situation in which \(x_0 = 0\). If this is the case, then it must hold that \(t_0 \neq 0\) and we have

\[
\tilde{\Sigma}_\mu := (1 + \mu t_0)\Sigma \subset \bar{\Omega}.
\]

for \(\mu \geq 0\) sufficiently small. The normal speed at \(\mu = 0\) is \(t_0 x \cdot \nu_\Sigma \geq 0\). Using the shrinker equation, we thus find that \(t_0 H_\Sigma \geq 0\). Since \(t_0 \neq 0\), we can assume that \(H_\Sigma \geq 0\). Thus, up to a rotation, \(\Sigma = \mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}\) for \(k = 1, \ldots, n\) by [CM12 Theorem 10.1]. This completes the proof. \(\square\)

Recall the the definition of smoothly crossing Brakke flows in Definition B.1.

**Proposition 2.3.** Fix \(n \geq 2\), \(\varepsilon > 0\) and \(\Lambda \in (\mathfrak{S}^n, 2]\) so that \(\langle \Lambda, \Lambda \rangle\) and \(\langle \Lambda, n, \Lambda \rangle\) hold. There is \(\delta = \delta(n, \varepsilon, \Lambda) > 0\) with the following property.

Consider \(\Sigma \in S^*_n(\Lambda - \varepsilon) \setminus S^*_n\) and \(\mathcal{M}\) an ancient cyclic unit-regular integral \(n\)-dimensional Brakke flow in \(\mathbb{R}^{n+1}\) with \(\lambda(\mathcal{M}) \leq F(\Sigma)\) so that \(\mathcal{M}\) does not smoothly cross the flow \((-\infty, 0) \ni t \mapsto \mathcal{H}^n[\sqrt{-t} \Sigma]\). Then, \(\Theta_{\mathcal{M}}(x, t) \leq F(\Sigma) - \delta\) for all \((x, t) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (0, 0)\).

**Proof.** We argue by contradiction. Consider a sequence of \(\Sigma_i \in S^*_n(\Lambda - \varepsilon) \setminus S^*_n\) and \(\mathcal{M}_i\) ancient cyclic unit-regular integral Brakke flows in \(\mathbb{R}^{n+1}\) with \(\lambda(\mathcal{M}_i) \leq F(\Sigma)\) so that \(\mathcal{M}_i\) does not smoothly cross the flow \((-\infty, 0) \ni t \mapsto \mathcal{H}^n[\sqrt{-t} \Sigma_i]\) and so that there are points \((x_i, t_i) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (0, 0)\) with

\[
(2.3) \quad \Theta_{\mathcal{M}_i}(x_i, t_i) \geq F(\Sigma_i) - o(1)
\]
as \(i \to \infty\). We can assume that \(|(x_i, t_i)| = 1\).

By Lemma 2.2, and Allard’s theorem [All72, Sim83], we can pass to a subsequence so that \(\Sigma_i\) converges in \(C^\infty_{\text{loc}}\) to \(\Sigma \in S_n(\Lambda)\). By Brakke’s theorem [Bra75, Whi05], \(\Sigma\) is non-flat. Because cylinders are isolated in \(C^\infty_{\text{loc}}\) by [CIM15], we thus see that \(\Sigma \in S^*_n(\Lambda) \setminus S^*_n\). Note that \(F(\Sigma_i) \to F(\Sigma)\).

We now pass to a further subsequence so that \((x_i, t_i) \to (x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}\) with \(|(x_0, t_0)| = 1\) and the Brakke flows \(\mathcal{M}_i\) converge to an ancient cyclic unit-regular integral Brakke flow \(\mathcal{M}\) with \(\lambda(\mathcal{M}) \leq F(\Sigma)\). By upper semi-continuity of Gaussian density, (2.3) implies that \(\Theta_{\mathcal{M}}(x_0, t_0) \geq F(\Sigma)\). Because \(\lambda(\mathcal{M}) \leq F(\Sigma)\), \(\mathcal{M}\) is a self-similar flow around \((x_0, t_0)\). By stability of smoothly crossing flows, Lemma 2.2 \(\mathcal{M}\) does not smoothly cross \((-\infty, 0) \ni t \mapsto \mathcal{H}^n[\sqrt{-t} \Sigma].\)
Consider any tangent flow to $\tilde{M}$ at $t = -\infty$. By Huisken’s monotonicity formula and Lemma 2.1 there is a smooth shrinker $\tilde{\Sigma}$ so that this tangent flow at $t = -\infty$ corresponds to some $\tilde{\Sigma} \in S_\Lambda$ with multiplicity-one. By the Frankel property for self-shrinkers (cf. [CCMS20, Corollary C.4]) and the strong maximum principle, if $\tilde{\Sigma} \neq \Sigma$ then the flows $t \mapsto \mathcal{H}^n[\sqrt{-t}\Sigma]$ and $t \mapsto \mathcal{H}^n[\sqrt{-t}\tilde{\Sigma}]$ smoothly cross each other at some point. This contradicts the stability of smooth crossings.

We conclude that any tangent flow to $\tilde{M}$ at $t = -\infty$ is the flow associated to $\Sigma$. Since $\tilde{M}$ is self-similar around $(x_0, t_0)$ we find $\tilde{M}(t) = \mathcal{H}^n[\sqrt{t_0 - t}\Sigma + x_0]$ for $t < t_0$. Since $\tilde{M}$ does not smoothly cross $t \mapsto \mathcal{H}^n[\sqrt{-t}\Sigma]$, we see that there is an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\partial \Omega = \Sigma$ so that $\sqrt{t_0 - t}\Sigma + x_0 \subset \sqrt{-t}\tilde{\Omega}$ for $t < \min\{0, t_0\}$. We can thus apply Proposition 2.2 to conclude that (up to a rotation) $\Sigma = \tilde{\Sigma} \times \mathbb{R}$ for $\tilde{\Sigma} \in S_{n-1}^\text{gen}(\Lambda)$. By hypothesis $$(\dagger \dagger n, \Lambda), \tilde{\Sigma} \in S_{n-1}^\text{gen}(\Lambda).$$ This is a contradiction. □

3. Proof of Theorem 1.7

For $M' \subset \mathbb{R}^{n+1}$ a smooth closed hypersurface, recall that $\mathfrak{F}(M')$ is the set of cyclic unit-regular integral Brakke flows $M'$ with $M'(0) = \mathcal{H}^n[M']$. Note that [Ilm93, Whi09] implies that $\mathfrak{F}(M') \neq \emptyset$ (see also [HW20, Appendix B]).

We define $D(M') := \sup\{\Theta_{M'}(x, t) : M' \in \mathfrak{F}(M'), (x, t) \in \text{sing} M' \setminus \text{sing}_{\text{gen}} M'\}$. Recall that by convention $\sup\emptyset = -\infty$.

Assume that hypotheses $(\dagger n, \Lambda)$ and $(\dagger \dagger n, \Lambda)$ hold for $\Lambda \in (\lambda(S^n), \lambda(S^{n-2})$ fixed. Consider a smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$ with $\lambda(M) \leq \Lambda$. Flowing $M$ by mean curvature flow for a short time strictly decreases the entropy unless $M$ is homothetic to a self-shrinker. If $M$ is homothetic to a self-shrinker other than $S^n(\sqrt{2n})$ then by [CM12], a small $C^\infty$-perturbation of $M$ has strictly smaller entropy.

As such, either $M = S^n(r)$ in which case the Theorem 1.7 trivially holds or we can perform an initial perturbation and assume that $\lambda(M) \leq \Lambda - 2\varepsilon$ for some $\varepsilon > 0$. Choose a foliation $\{M_s\}_{s \in (-1, 1)}$ of a tubular neighborhood of $M$ so that $M_0 = M$ and so that $\lambda(M_s) \leq \Lambda - \varepsilon$. Fix $\delta = \delta(n, \varepsilon, \Lambda) > 0$ from Proposition 2.3.

Lemma 3.1. We have

$$\limsup_{s \to s_0} D(M_s) \leq D(M_{s_0}) - \delta.$$ for all $s_0 \in (-1, 1)$. 

Lemma \[\text{3.1}\] implies Theorem \[\text{1.7}\] by a straightforward iteration argument since by Brakke’s regularity theorem \[\text{[Bra75, Whi05]}\], if \(\mathcal{D}(M') \leq 1\) then \(\mathcal{D}(M') = -\infty\) implying that \(\text{sing}(\mathcal{M}') = \text{sing}_{\text{gen}}(\mathcal{M}')\) for all \(\mathcal{M}' \in \mathfrak{S}(\mathcal{M}')\). Since \(\lambda(M') < \Lambda \leq \lambda(\mathbb{S}^{n-2} \times \mathbb{R}^2)\), any \(\mathcal{M}' \in \mathfrak{S}(\mathcal{M}')\) has only (multiplicity one) \(\mathbb{S}^n\) and \(\mathbb{S}^{n-1} \times \mathbb{R}\)-type singularities. Thus, the resolution of the mean convex neighborhood conjecture for \(\mathbb{S}^{n-1} \times \mathbb{R}\) singularities \[\text{[CHH22, CHHW22]}\] (cf. \[\text{[HW20]}\]) implies non-fattening of the flow of \(M'\).

**Proof of Lemma \[\text{3.1}\]** Assume there is \(s_i \to s_0 \in (-1, 1)\) with \(s_i \neq s_0\) but

\[
\lim_{i \to \infty} \mathcal{D}(M_{s_i}) > \mathcal{D}(M_{s_0}) - \delta.
\]

Fix \(\mathcal{M}_i \in \mathfrak{S}(M_{s_i})\) and \((x_i, t_i) \in \text{sing}(\mathcal{M}_i) \setminus \text{sing}_{\text{gen}}(\mathcal{M}_i)\) with

\[
\lim_{i \to \infty} \Theta_{\mathcal{M}_i}(x_i, t_i) > \mathcal{D}(M_{s_0}) - \delta.
\]

Pass to a subsequence \(\mathcal{M}_i\) converging to \(\mathcal{M} \in \mathfrak{S}(M_{s_0})\) and \((x_i, t_i) \to (x_0, t_0) \in \text{sing}(\mathcal{M})\). Since \(s_i \neq s_0\) for all \(i\), we have that \(M_{s_i}\) is disjoint from \(M_{s_0}\) for all \(i\). In particular, \(\text{supp} \mathcal{M}_i \cap \text{supp} \mathcal{M} = \emptyset\) (by the avoidance principle for Brakke flows \[\text{[lim94]}\] 10.6)). Thus, \((x_i, t_i) \neq (x_0, t_0)\).

Observe that that if \((x_0, t_0) \in \text{sing}_{\text{gen}}(\mathcal{M})\) then since \(\lambda(M) < \Lambda \leq \lambda(\mathbb{S}^{n-2})\), we see that \((x_0, t_0)\) must be a \(\mathbb{S}^n\) or \(\mathbb{S}^{n-1} \times \mathbb{R}\)-type singularity. Proposition \[\text{A.1}\] then implies that \((x_i, t_i) \in \text{sing}_{\text{gen}}(\mathcal{M}_i)\), a contradiction. Thus, it must hold that \((x_0, t_0) \in \text{sing}(\mathcal{M}) \setminus \text{sing}_{\text{gen}}(\mathcal{M})\).

Translate \((x_0, t_0)\) to the space-time origin and parabolically dilate to yield \(\tilde{\mathcal{M}}_i\) and \((\tilde{x}_i, \tilde{t}_i)\) with \(|(\tilde{x}_i, \tilde{t}_i)| = 1\) and

\[
\lim_{i \to \infty} \Theta_{\tilde{\mathcal{M}}_i}(\tilde{x}_i, \tilde{t}_i) > \mathcal{D}(M_{s_0}) - \delta.
\]

Pass to a subsequence so that \(\tilde{\mathcal{M}}_i \to \tilde{\mathcal{M}}\) and \((\tilde{x}_i, \tilde{t}_i) \to (\tilde{x}, \tilde{t}) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (0, 0)\). By upper semicontinuity of density

\[
\Theta_{\tilde{\mathcal{M}}}(\tilde{x}, \tilde{t}) > \mathcal{D}(M_{s_0}) - \delta.
\]

On the other hand, we can perform the same translation and parabolic dilation to \(\mathcal{M}\) and by extracting a further subsequence, the resulting flows converge to a tangent flow to \(\mathcal{M}\) at \((x_0, t_0)\). By Lemma \[\text{2.1}\], the tangent flow is the multiplicity-one flow associated to a smooth shrinker \(\Sigma\). Note that

\[
F(\Sigma) \leq \lambda(\mathcal{M}) \leq \limsup_{s \to s_0} \lambda(M_s) \leq \Lambda - \varepsilon.
\]

Since \((x_0, t_0) \in \text{sing}(\mathcal{M}) \setminus \text{sing}_{\text{gen}}(\mathcal{M})\) it must hold that \(\Sigma \in S^*_n(\Lambda - \varepsilon) \setminus S^*_n\). Huiskens’ monotonicity formula implies that \(\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma) = \Theta_{\mathcal{M}}(x_0, t_0)\) (cf. the proof of Proposition 10.6 in \[\text{[CCMS20]}\]). Finally, since the supports of \(\mathcal{M}\) and \(\tilde{\mathcal{M}}\) are disjoint, \(\mathcal{M}\) does not smoothly cross \(\tilde{\mathcal{M}}\). As such (using Lemma \[\text{3.2}\]), \(\tilde{\mathcal{M}}\) does not smoothly cross \(t \mapsto \mathcal{H}^n|\sqrt{-t}\Sigma\). We can now apply Proposition \[\text{2.3}\] to conclude that

\[
\Theta_{\tilde{\mathcal{M}}}(\tilde{x}, \tilde{t}) \leq F(\Sigma) - \delta = \Theta_{\mathcal{M}}(x_0, t_0) - \delta \leq \mathcal{D}(M_{s_0}) - \delta.
\]
This contradicts (3.1), completing the proof. □

Appendix A. Stability of generic singularities

Based on [CHHW22], the following stability of generic singularities was proven in [SS20, Proposition 2.3] (see [CCMS20, Lemma 10.4] for the simple argument when the singularity is modeled on $S^n$). When $n = 2$ this also follows via density considerations using [BW17].

**Proposition A.1.** Suppose that $M_i \rightarrow M$ are unit-regular integral Brakke flows in $\mathbb{R}^{n+1}$ and that $(x_i, t_i) \in \text{sing} M_i$ converge to $(0, 0) \in \text{sing}_{\text{gen}} M$. If the singularity at $(0, 0)$ is modeled on $S^n$ or $S^{n-1} \times \mathbb{R}$, then for $i$ sufficiently large $(x_i, t_i) \in \text{sing}_{\text{gen}} M_i$.

Appendix B. Stability of crossing points

**Definition B.1.** Given two integral unit Brakke flows $M^{(1)}$ and $M^{(2)}$, we say that $M^{(1)}$ and $M^{(2)}$ smoothly cross at $(x, t)$ if there is $r > 0$ with $M^{(j)}|B_r(x) = \mathcal{H}^n|\Gamma^{(j)}(s)$ for $s \in (t - r^2, t + r^2)$ where $\Gamma^{(j)}(s)$ are smooth connected mean curvature flows so that in any small neighborhood of $x$ there are points of $\Gamma^{(1)}(0)$ on both sides of $\Gamma^{(2)}(0)$.

The following is a straightforward consequence of Brakke’s regularity theorem [Bra75, Whi05].

**Lemma B.2.** For $j = 1, 2$, suppose that $M^{(j)}_i \rightarrow M^{(j)}$ are integral unit-regular $n$-dimensional Brakke flows in $\mathbb{R}^{n+1}$. Assume that $M^{(1)}$ smoothly crosses $M^{(2)}$ at $(x, t)$. Then, for $i$ sufficiently large, there is $(x_i, t_i) \rightarrow (x, t)$ so that $M^{(1)}_i$ smoothly crosses $M^{(2)}_i$ at $(x_i, t_i)$.

Appendix C. Local results

In this appendix we prove the following local perturbative result.

**Proposition C.1.** Suppose that $M^n \subset \mathbb{R}^{n+1}$ is a closed embedded hypersurface, $M \in F(M)$ is a cyclic unit-regular integral Brakke flow starting at $M$. Assume that for $(x_0, t_0) \in \text{sing} M$, the following holds:

- $\text{reg} M \cap \{t < t_0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is connected
- any tangent flow $N$ to $M$ at $(x_0, t_0)$ has $N(-1) = \mathcal{H}^n|\Sigma$, for $\Sigma \in S^n_\ast \backslash S^{\text{gen}}_n$ that does not split a line.

Then, there is $r = r(M, x_0, t_0) > 0$ so that for $M_j = \text{graph}_M(u_j)$, $u_j > 0$ with $u_j \rightarrow 0$ in $C^\infty$ it holds that any $(x, t) \in B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$ has $\Theta_{M_j}(x, t) \leq \Theta_M(x_0, t_0) - r$ for $j$ sufficiently large.
In particular, no tangent flow to $\mathcal{M}$ at $(x_0, t_0)$ can arise as the tangent flow to $\mathcal{M}_j$ at some point in $B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$, for $j$ large.

**Proof.** If this failed, there is $(x_j, t_j) \rightarrow (x_0, t_0)$ with

$$\Theta_{\mathcal{M}_j}(x, t) \geq \Theta_{\mathcal{M}}(x_0, t_0) - o(1).$$

The assumption on the connectedness of the regular set implies that $\mathcal{M}_j|\{t < t_0\} \rightarrow \mathcal{M}|\{t < t_0\}$. Thus, by rescaling around $(x_0, t_0)$ so that $(x_j, t_j)$ is scaled to a unit distance from $(0, 0)$, we obtain $\Sigma \in S_n^+ \setminus S_n^{\text{gen}}$ that does not split a line and an ancient Brakke flow $\tilde{M}$ that does not smoothly cross $t \mapsto \mathcal{H}^n|\sqrt{-t} \Sigma$, so that $\lambda(\tilde{M}) \leq F(\Sigma)$, but for some $(\tilde{x}, \tilde{t}) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus \{(0, 0)\}$ it holds that $\Theta_{\tilde{M}}(\tilde{x}, t) \geq F(\Sigma)$.

The argument in the second half of the proof of Proposition 2.3 carries over without change to show that there is an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\partial \Omega = \Sigma$ and

$$\sqrt{\tilde{t} - t} \Sigma + \tilde{x} \subset \sqrt{-t} \Omega$$

for $t < \min\{0, t_0\}$. By Proposition 2.2, we have that either $\Sigma = S^n(\sqrt{-2n}) \in S_n^{\text{gen}}$ or $\Sigma$ splits a line. Either case contradicts the assumption that $\mathcal{M}$ has no such tangent flow at $(x_0, t_0)$. This completes the proof. \[\square\]

Note that Proposition C.1 does not give any indication as to how the perturbation avoids the singularity (the trade-off is that the proof is very short). On the other hand, the results in [CCMS20] give a rather complete description of how the perturbed flow avoids a compact/asymptotically conical singularity. The works [SX21b, SX21a] also obtain some information along these lines, but only as long as the perturbed flow remains graphical over the original flow.

**Appendix D. The setting of area-minimizing hypersurfaces**

We recall the following fundamental result:

**Theorem D.1** (Hardt–Simon [HS85, Theorem 2.1]). If $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a regular area minimizing cone then there exists smooth area-minimizing hypersurfaces $S_\pm$ in each component of $\mathbb{R}^{n+1} \setminus \mathcal{C} = U_+ \cup U_-$ so that if $S'$ is area minimizing and contained in $U_\pm$ then $S' = \lambda S_\pm$.

The uniqueness statement in Theorem D.1 implies smoothness of solution to the Plateau problem for seven-dimensional currents in $\mathbb{R}^8$ with generic boundary data (see [HS85, Theorem 5.6]). Later, Smale used Theorem D.1 to prove that for $(M^8, g)$ a closed Riemannian manifold and $\alpha \in H_7(M; \mathbb{Z})$, there is a $C^k$-close metric $g'$ so that the least area representative of $\alpha$ is smooth [Sma93].

**Remark.** Besides their role in generic regularity of area-minimizing hypersurfaces in eight-dimensional manifolds, the surfaces $S_\pm$ are important objects in their own right, cf. [IW15, CLS22, Wan20, LW20, Sim21, Sim23]. In our previous paper [CCMS20], we proved the parabolic analogue of Theorem
for compact/asymptotically-conical self-shrinkers) by constructing and classifying ancient one-sided flows analogous to the surfaces $S_{\pm}$.

We explain here how the main idea of this note can be used to prove the generic regularity results from \cite{HS85, Sm93} using the following result in lieu of Theorem D.1 (compare with Proposition 2.3):

**Proposition D.2.** There is $\delta > 0$ with the following property. Suppose that $C \subset R^8$ is a non-flat area-minimizing cone. If $S'$ is area-minimizing with support contained in $\bar{U}_{\pm}$, where $R^8 \setminus C = U_+ \cup U_-$, then

$$\Theta_{S'}(x) \leq \Theta_C(0) - \delta.$$ 

**Proof.** Using smooth compactness of the links of area minimizing cones in $R^8$ it suffices to rule out the case where $S' \subset \bar{U}_{\pm}$ is area-minimizing and there is $|x_0| = 1$ so that

$$\Theta_{S'}(x_0) = \Theta_C(0).$$

Because $S'$ is contained in $\bar{U}_{\pm}$, its tangent cone at $\infty$ must be $C$ (e.g., using the Frankel property of minimal hypersurfaces in $S^n$). Thus, $S' = C + x_0$.

This implies that $C + \lambda x_0 \subset \bar{U}_{\pm}$ as $\lambda \to 0$, so $x_0 \cdot \nu_C \geq 0$. It cannot hold that $x_0 \cdot \nu_C = 0$ since $C$ does not split a line, so $x_0 \cdot \nu_C > 0$. This would imply that $C$ is a graph, which is impossible since $C$ is non-flat. \qed

Using this, we obtain the following density drop result (compare with Lemma 3.1):

**Corollary D.3.** There is $\delta > 0$ with the following property. Suppose that $\Sigma = \partial \Omega \subset B_2 \subset R^8$ is an area minimizing boundary with $\text{sing} \Sigma = \{0\}$. Suppose that $\Omega_1, \Omega_2, \cdots \supset \Omega$ is a sequence of sets of finite perimeter in $B_2$ with $\Sigma_i := \partial \Omega_i$ area minimizing, $\Sigma_i \cap \Sigma = \emptyset$, and $\Omega_i \to \Omega$. Then, for $x_i \in \Sigma_i \cap B_1$, we have

$$\limsup_{i \to \infty} \Theta_{\Sigma_i}(x_i) \leq \Theta_{\Sigma}(0) - \delta.$$

Note that this result can be iterated exactly in the proof of Theorem 1.7 to obtain generic regularity of area-minimizing hypersurfaces in eight dimensions:

**Corollary D.4** (cf. \cite{HS85} Theorem 5.6)). For $\Gamma^6 \subset R^8$ a smooth compact oriented submanifold without boundary, there is an arbitrarily small $C^\infty$-perturbation of $\Gamma$ to $\Gamma'$ so that any area-minimizing integral current bounded by $\Gamma'$ is completely smooth.

**Corollary D.5** (cf. \cite{Sm93}). For $(M^8, g)$ a closed oriented Riemannian manifold and $\alpha \in H_7(M; Z)$ a codimension-one integral homology class, there is an arbitrarily small $C^k$-perturbation of $g$ to $g'$ so that there is a unique $g'$-area-minimizing representative $\Sigma$ of $\alpha$ and $\Sigma$ is completely smooth.
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