Quantum entanglement and contextuality with complexifications of $E_8$ root system

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Abstract

The Witting configuration with 40 complex rays was suggested as a possible reformulation of Penrose model with two spin-3/2 systems based on geometry of dodecahedron and used for analysis of nonlocality and contextuality in quantum mechanics. Yet another configuration with 120 quantum states is considered in presented work. Despite of different number of states both configurations can be derived from complexification of 240 minimal vectors of 8D real lattice corresponding to root system of Lie algebra $E_8$. An analysis of properties of suggested configuration of quantum states is provided using many analogies with properties of Witting configuration.

1 Introduction

The Witting configuration with 40 quantum states in 4D Hilbert space is useful for analysis of contextuality and quantum entanglement [1, 2]. The configuration was discussed earlier [1] as a convenient reformulation of a model based on geometry of dodecahedron with two entangled spin-3/2 system introduced by Penrose [3, 4]. There are 40 bases (tetrads with orthonormal states) important for analysis of quantum contextuality in such configuration [1, 2, 3, 4].

The configuration can be derived from complex Witting polytope introduced by Coxeter [5]. The Witting polytope has 240 vertexes corresponding to only 40 different quantum states or rays in complex space due to six common phase multipliers $e^{i\pi k/3}$, $k = 0,\ldots,5$.

Symmetry group of Witting configuration is a discrete subgroup of unitary group $SU(4)$ with 51840 transformations [2, 5]. It may be useful to consider
other discrete subgroups of \( SU(4) \) as a possible tool for construction of similar configurations. Classification of such subgroups may be found in \([6]\). The symmetries of Witting configuration correspond to the subgroup VI from the list and the subgroup investigated in presented work has index IV.

Such a group with 5040 transformations is a 4D unitary representation of a double cover of alternating group \( A_7 \) \([6]\), \( i.e. \), group of 2520 even permutations of seven elements. The unitary equivalent representation of the group with different choice of generators is denoted \( \tilde{A}_7 \) in Section 2.

For such representation \( \tilde{A}_7 \) is the symmetry group of a configuration \( \tilde{\Lambda}_{A_7} \) with 240 complex vectors introduced in Section 3. The configuration has some analogy with Witting polytope, because in both cases natural maps into configuration of 240 8D real vectors correspond to the root system of Lie algebra \( E_8 \) after some orthogonal transformation. Despite of the equivalence in 8D real space the complex configurations provide quite different models of system with quantum states in 4D Hilbert space. The Witting configuration includes 40 rays or quantum states with 40 different bases (orthogonal tetrads) and \( \tilde{\Lambda}_{A_7} \) corresponds to 120 states and 210 bases.

Construction and table with 210 bases of \( \tilde{\Lambda}_{A_7} \) are discussed in Section 4 together with problem of contextuality. Entanglement and measurement of two configurations have some similarity with properties of Witting configuration \([2]\) and analysed in Section 5.

### 2 Group \( \tilde{A}_7 \)

Let us consider two matrices

\[
P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

The matrices \( P_1 \) and \( P_2 \) generate group of 24 transformations isomorphic with double cover \( \tilde{A}_4 \) of group \( A_4 \) of even permutation of \textit{four elements} in quite natural way as permutations of four basic states.

An isomorphism of the 4D unitary representation of group \( \tilde{A}_7 \) of 5040 transformations with double cover of group \( A_7 \) of even permutation of \textit{seven elements} is less trivial, but it can be generated by \( P_1 \) and \( P_2 \) together with
matrix of seventh order

\[ S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2i & i' & i \\
0 & i' & i'' & -i'' \\
0 & i & -i'' & -i'
\end{pmatrix}, \]

(2)

where

\[ i = \frac{i}{\sqrt{7}}, \quad i' = \frac{1+i}{2}, \quad i'' = \frac{1+3i}{2}. \]

(3)

Let us note \(|i|^2 = \frac{1}{7}, \quad |i'|^2 = \frac{2}{7} \text{ and } |i''|^2 = \frac{4}{7}|.\]

Formally the symmetries of quantum states as rays in Hilbert space could be described more naturally by group of 2520 projective transformations \( \tilde{A}_7 = \tilde{A}_7/\left\{+1,-1\right\} \). The group \( \tilde{A}_7 \) is isomorphic with group \( A_7 \) of even permutations of set \( \{1, \ldots , 7\} \) already mentioned earlier. With system of computer algebra GAP \([7]\) the 2-1 homomorphism \( \tilde{\mathcal{H}}_7 : \tilde{A}_7 \mapsto A_7 \) can be defined on generators as

\[
\begin{align*}
\tilde{\mathcal{H}}_7 : \pm P_1 & \rightarrow (1, 2, 4)(3, 6, 5), \\
\tilde{\mathcal{H}}_7 : \pm P_2 & \rightarrow (1, 6)(3, 4), \\
\tilde{\mathcal{H}}_7 : \pm S & \rightarrow (1, 2, 3, 4, 5, 6, 7).
\end{align*}
\]

(4)

Due to homomorphism \( \tilde{\mathcal{H}}_7 \) any projective transformation from \( \tilde{A}_7 \) can be naturally encoded into even permutation of \( \{1, \ldots , 7\} \).

3 States

The group \( \tilde{A}_7 \) represents symmetries of 4D complex configuration \( \tilde{\Lambda}_{A_7} \) with 240 vectors. The configuration can be produced from a basic vector \((1, 0, 0, 0)\) by application different transformations from \( \tilde{A}_7 \). Each vector is included together with an opposite one and it corresponds to configuration \( \Lambda_{A_7} \) with 120 different quantum states represented on Table 1 with notation for coordinates from Eq. (3) and indexing explained below. Complex Witting polytope also has 240 vertices, but only 40 different quantum states can be obtained from such vectors due to six common phase multipliers \( \pm \omega^k, \quad k = 0, 1, 2 \), where

\[ \omega = e^{\frac{2\pi}{3}} = \frac{-1 + \sqrt{3}i}{2}. \]

(5)
Table 1: Coefficients and indexes of 120 states.

| $n$ | $\ket{\psi_n}$ | $\ket{\psi_{n+10}}$ | $\ket{\psi_{n+20}}$ | $\ket{\psi_{n+30}}$ | $t$ |
|-----|-----------------|---------------------|--------------------|--------------------|-----|
| 1   | $(1, 0, 0, 0)$  | $(0, 1, 0, 0)$      | $(0, 0, 1, 0)$     | $(0, 0, 0, 1)$     | 1   |
| 2   | $(2t, t, 0, t)$ | $(t, -2t, -t, t)$  | $(t, t, -2t, -t)$ | $(t, -t, t, -2t)$ | 2   |
| 3   | $(t, 0, t', t''')$ | $(t', t, 0, t''')$ | $(t', t', 0, t'')$ | $(t', t', t', 0)$ | 4   |
| 4   | $(t, 0, -t', -t''')$ | $(t', -t, 0, t'')$ | $(t', t', -t, 0)$ | $(t', -t', -t', 0)$ | 4   |
| 5   | $(0, t', -t, t'')$ | $(t', 0, -t', 0)$  | $(t', -t', 0, t')$ | $(t', -t', 0, t')$ | 4   |
| 6   | $(t, -t', -t, t')$ | $(t', -t', -t, t')$ | $(t', -t', -t, t')$ | $(t', -t', -t, t')$ | 3   |
| 7   | $(t, -t', -t', 0)$ | $(0, t, t', 0)$  | $(t', 0, -t', t')$ | $(t', -t', 0, t')$ | 4   |
| 8   | $(0, t', 0, t'')$ | $(t', 0, -t', 0)$  | $(t', -t', 0, t')$ | $(t', -t', 0, t')$ | 4   |
| 9   | $(t, t', -t', 0)$ | $(0, t, -t', -t')$ | $(t', 0, t, 0)$  | $(t', 0, t, 0)$  | 4   |
| 10  | $(t, t', -t', -t')$ | $(-t, -t', -t', t')$ | $(-t, -t', -t', t')$ | $(-t, -t', -t', t')$ | 3   |

| 41  | $(t', 2t, -t, 0)$ | $(2t, -t', 0, -t)$ | $(t, 0, t', 2t)$  | $(0, t, 2t, -t')$ | 4   |
| 42  | $(t, t', -t', t')$ | $(-t, -t', -t', t')$ | $(t', -t', t', t')$ | $(t', -t', t', t')$ | 3   |
| 43  | $(t, t', t', 0)$ | $(0, t, -t', t')$  | $(t', 0, -t', -t')$ | $(t', -t', 0, t')$ | 4   |
| 44  | $(0, t', t, -t'')$ | $(t', 0, t', 0)$  | $(-t', 0, t', 0)$ | $(-t', 0, t', 0)$ | 4   |
| 45  | $(t, t, t, -t')$ | $(t, -t', t, t')$  | $(-t', -t', t, t')$ | $(t, -t', -t', t)$ | 2   |
| 46  | $(0, t', 1, t')$ | $(t', 0, -t', 0)$  | $(t', -t', 0, t')$ | $(0, t', 0, t')$  | 4   |
| 47  | $(2t, 0, -t', t')$ | $(t', 0, 2t, -t)$ | $(t', 0, 2t, 0)$  | $(2t, -t', 0)$  | 3   |
| 48  | $(-t, t', t', t')$ | $(t, t', 0, 0)$  | $(t', t', 0, -t')$ | $(t', t', 0, -t')$ | 3   |
| 49  | $(t, -t', t', t')$ | $(t', -t', -t', t')$ | $(t', -t', -t', t')$ | $(t', -t', -t', t')$ | 3   |
| 50  | $(0, t', t, 2t)$ | $(t, 2t, 0, -t')$ | $(t, 2t, 0, -t')$ | $(t, 2t, 0, -t')$ | 4   |
| 51  | $(t', -2t, t, 0)$ | $(2t, t', 0, -t)$ | $(t, 0, -t', 2t)$ | $(0, t, 2t, t')$ | 4   |
| 52  | $(t, -t', -t', t')$ | $(t', t, -t', -t')$ | $(t', -t', t', t')$ | $(t', -t', t', t')$ | 3   |
| 53  | $(0, t', -t, t'')$ | $(t', 0, t', 0)$  | $(t, t', -t', 0)$ | $(t, t', -t', 0)$ | 4   |
| 54  | $(-t, t', t', t')$ | $(t, t', -t', -t')$ | $(t, t', -t', -t')$ | $(t, t', -t', -t')$ | 4   |
| 55  | $(t, t', t', 0)$ | $(0, t', t', 0)$  | $(t', 0, t', 0)$  | $(t', 0, t', 0)$  | 4   |
| 56  | $(t, t', t', 0)$ | $(0, t', t', 0)$  | $(t', 0, t', 0)$  | $(t', 0, t', 0)$  | 4   |
| 57  | $(0, -t', t, 2t)$ | $(0, 2t, -t', t)$ | $(t, 2t, 0, t')$  | $(t, 2t, 0, t')$  | 4   |
| 58  | $(t, -t', -t', t')$ | $(t', t', -t', -t')$ | $(t', t', -t', -t')$ | $(t', t', -t', -t')$ | 3   |
| 59  | $(0, -t', t, 0)$ | $(2t, 0, -t', t)$ | $(t, 0, -t', 2t)$ | $(0, 2t, -t', 0)$ | 4   |
| 60  | $(t, -t', t', t')$ | $(t', t', t', t')$ | $(t', t', t', t')$ | $(t', t', t', t')$ | 3   |
| 61  | $(2t, 0, -t, 0)$ | $(t', 0, -2t, t)$ | $(t, -t', 2t, 0)$ | $(2t, -t', 0, 2t)$ | 4   |
| 62  | $(t, -t', t', t')$ | $(t', t', t', t')$ | $(t', t', t', t')$ | $(t', t', t', t')$ | 3   |
| 63  | $(2t, 0, -t, 0)$ | $(t', 0, -2t, t)$ | $(t, -t', 2t, 0)$ | $(2t, -t', 0, 2t)$ | 4   |
| 64  | $(t, 0, t', -t'')$ | $(-t, -t', -t', t')$ | $(-t, -t', -t', t')$ | $(-t, -t', -t', t')$ | 4   |
Despite of such difference both configurations considered as 240 8D real vectors up to some orthogonal transformation are equivalent with $E_8$ root system or minimal vectors of $E_8$ lattice denoted further simply as $E_8$. For Witting polytope relation with the 8D real uniform polytope $4_{21}$ (equivalent with $E_8$) was mentioned by Coxeter \[5, 8\]. The construction of a map between realifications of both complex configurations is provided below.

For certainty let us fix 40 vectors producing whole Witting configuration by multiplication on six ‘phases’ $\pm \omega^k$ Eq. (5), $k = 0, 1, 2$

\begin{align*}
&(1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1), \quad (6a) \\
&\frac{i}{\sqrt{3}}(0, 1, -\omega^\mu, \omega^\nu), \quad \frac{i}{\sqrt{3}}(1, 0, -\omega^\mu, -\omega^\nu), \\
&\frac{i}{\sqrt{3}}(1, -\omega^\mu, 0, \omega^\nu), \quad \frac{i}{\sqrt{3}}(1, \omega^\mu, \omega^\nu, 0) \quad (6b)
\end{align*}

with $\mu, \nu = 0, 1, 2$.

Let us note, that four vectors Eq. (6b) with $\mu = \nu = 0$ have pure imaginary coordinates

\begin{align*}
\frac{i}{\sqrt{3}}(0, 1, -1, 1), \quad \frac{i}{\sqrt{3}}(1, 0, -1, -1), \\
\frac{i}{\sqrt{3}}(1, -1, 0, 1), \quad \frac{i}{\sqrt{3}}(1, 1, 1, 0) \quad (7)
\end{align*}

and together with Eq. (6a) can be used as convenient basis for realification of Witting polytope. $\tilde{\Lambda}_4$, also includes four states Eq. (6a) and four vectors with pure imaginary components

\begin{align*}
\frac{i}{\sqrt{7}}(2, 1, 1, 1), \quad \frac{i}{\sqrt{7}}(1, -2, -1, 1), \\
\frac{i}{\sqrt{7}}(1, 1, -2, -1), \quad \frac{i}{\sqrt{7}}(1, -1, 1, -2) \quad (8)
\end{align*}

After some technical manipulations with GAP \[7\] a simple transformation of Witting polytope into $\tilde{\Lambda}_4$, can be found. It saves all real parts of coordinates and multiply four-vectors with imaginary parts on the orthogonal matrix

\[
T^\Lambda_{\tilde{\omega}} = \frac{1}{\sqrt{21}} \begin{pmatrix}
1 & 4 & -2 & 0 \\
-4 & 1 & 0 & 2 \\
2 & 0 & 1 & 4 \\
0 & -2 & -4 & 1
\end{pmatrix}.
\]
The transformation also converts Eq. (7) into Eq. (8).

Let us use indexation of 40 states Eq. (6) in Witting configuration introduced (up to normalization) in Ref. [2] to produce 120 vectors by appending two extra blocks multiplied on $\omega$ Eq. (5) and $\omega^2$ respectively. All 120 states $\Lambda_7$ can be obtained now by transformation of imaginary parts of all vectors using $T_{W}^{A_7}$.

The result of such operation is presented in Table 1. The choice between two possible directions of a vector is often due to some typographical reasons. All states are normalized on unit and two copies of Table 1 with opposite signs produces complete set $\tilde{\Lambda}_7$ with 240 complex vectors.

Index $t$ in last column of Table 1 denotes type of four states in a row similar with used for description of lattice for group $\tilde{A}_7$ denoted as $2A_7$ in Ref. [9, p. 10] and up to 24 ‘signed’ permutations from $\tilde{A}_4$ it corresponds to

1. 4 basic states (8 vectors) from $(1, 0, 0, 0)$
2. 12 states (24 vectors) from $(ab, 1, 1, 1) \cdot i/\sqrt{7}$
3. 32 states (64 vectors) from $(1, -a, -b, -c) \cdot i/\sqrt{7}$
4. 72 states (144 vectors) from $(0, 1, -ab, c) \cdot i/\sqrt{7}$

Here $a, b, c = -1 \pm i\sqrt{7}/2$, i.e., the expressions include all possible combinations with two complex roots of quadratic equation $z^2 + z + 2 = 0$ [10]. Let us note, the $\omega$ and $\bar{\omega}$ used for construction of Witting polytope are roots of quadratic equation $z^2 + z + 1 = 0$.

4 Bases and contextuality

Number of bases (orthonormal tetrads) for the 120 states is 210. Consequent numbering of such bases in lexicographic order with respect to indexes of states is shown in Table 2. Each state belongs to seven different bases collecting all 21 states orthogonal to it. A diagram with links corresponding to orthogonal states is known as Kochen-Specker graph [11 3]. All 210 bases can be found using search for maximal 4-cliques in the graph.

Yet another method used earlier in Ref. [2] for Witting configuration is due to relation between bases and $4 \times 4$ unitary matrices representing elements of symmetry group $\tilde{A}_7$. Each bases corresponds to $5040/210 = 24$
Table 2: 210 bases with states from Table 1

| n   | B_n          | n   | B_n          | n   | B_n          |
|-----|--------------|-----|--------------|-----|--------------|
| 1   | (1, 5, 33, 111) | 2   | (1, 8, 34, 71) | 3   | (1, 11, 21, 31) |
| 4   | (1, 17, 83, 96) | 5   | (1, 19, 44, 60) | 6   | (1, 50, 53, 120) |
| 7   | (1, 76, 86, 94) | 8   | (2, 7, 26, 83) | 9   | (2, 9, 30, 44) |
| 10  | (2, 12, 22, 32) | 11  | (2, 25, 52, 53) | 12  | (2, 28, 92, 94) |
| 13  | (2, 33, 40, 90) | 14  | (2, 34, 36, 46) | 15  | (3, 10, 77, 118) |
| 16  | (3, 11, 27, 57) | 17  | (3, 12, 38, 82) | 18  | (3, 24, 78, 95) |
| 19  | (3, 39, 53, 66) | 20  | (3, 58, 81, 112) | 21  | (3, 73, 75, 88) |
| 22  | (4, 6, 75, 119) | 23  | (4, 11, 29, 99) | 24  | (4, 12, 35, 42) |
| 25  | (4, 23, 58, 115) | 26  | (4, 37, 94, 110) | 27  | (4, 41, 72, 95) |
| 28  | (4, 45, 114, 118) | 29  | (5, 7, 91, 97) | 30  | (5, 18, 106, 119) |
| 31  | (5, 22, 72, 73) | 32  | (5, 28, 54, 113) | 33  | (5, 36, 38, 49) |
| 34  | (5, 37, 52, 67) | 35  | (6, 15, 56, 81) | 36  | (6, 17, 30, 39) |
| 37  | (6, 22, 27, 103) | 38  | (6, 26, 59, 79) | 39  | (6, 50, 74, 78) |
| 40  | (6, 65, 82, 87) | 41  | (7, 14, 104, 120) | 42  | (7, 20, 39, 86) |
| 43  | (7, 23, 31, 77) | 44  | (7, 29, 60, 116) | 45  | (7, 62, 100, 112) |
| 46  | (8, 9, 51, 59) | 47  | (8, 15, 70, 77) | 48  | (8, 22, 112, 114) |
| 49  | (8, 25, 74, 93) | 50  | (8, 35, 40, 87) | 51  | (8, 39, 92, 109) |
| 52  | (9, 13, 63, 76) | 53  | (9, 16, 37, 50) | 54  | (9, 24, 31, 119) |
| 55  | (9, 27, 80, 96) | 56  | (9, 56, 72, 102) | 57  | (10, 18, 41, 100) |
| 58  | (10, 19, 26, 37) | 59  | (10, 22, 29, 64) | 60  | (10, 30, 97, 117) |
| 61  | (10, 42, 49, 108) | 62  | (10, 86, 113, 115) | 63  | (11, 15, 101, 104) |
| 64  | (11, 18, 61, 63) | 65  | (11, 46, 47, 54) | 66  | (11, 89, 90, 93) |
| 67  | (12, 17, 63, 107) | 68  | (12, 19, 69, 104) | 69  | (12, 54, 56, 117) |
| 70  | (12, 79, 93, 100) | 71  | (13, 20, 22, 110) | 72  | (13, 21, 25, 91) |
| 73  | (13, 34, 65, 88) | 74  | (13, 40, 67, 105) | 75  | (13, 43, 68, 115) |
| 76  | (13, 45, 81, 92) | 77  | (14, 16, 22, 66) | 78  | (14, 21, 28, 51) |
| 79  | (14, 33, 45, 108) | 80  | (14, 36, 68, 109) | 81  | (14, 41, 52, 88) |
| 82  | (14, 78, 84, 105) | 83  | (15, 20, 28, 107) | 84  | (15, 24, 32, 62) |
| 85  | (15, 38, 64, 83) | 86  | (15, 42, 66, 116) | 87  | (16, 18, 25, 69) |
| 88  | (16, 34, 48, 89) | 89  | (16, 40, 49, 107) | 90  | (16, 55, 60, 113) |
| 91  | (16, 75, 92, 97) | 92  | (17, 24, 90, 114) | 93  | (17, 25, 47, 72) |
| 94  | (17, 29, 36, 70) | 95  | (17, 51, 101, 110) | 96  | (18, 23, 32, 102) |
| 97  | (18, 35, 44, 103) | 98  | (18, 80, 82, 110) | 99  | (19, 23, 46, 73) |
| 100 | (19, 27, 40, 106) | 101 | (19, 28, 89, 112) | 102 | (19, 61, 66, 91) |
| 103 | (20, 33, 47, 85) | 104 | (20, 36, 69, 87) | 105 | (20, 52, 59, 118) |
Table 2 (continued): 210 bases with states from Table 1

| $n$  | $B_n$          | $n$  | $B_n$          | $n$  | $B_n$          |
|------|----------------|------|----------------|------|----------------|
| 106  | (20, 74, 96, 98)| 107  | (21, 37, 103, 116)| 108  | (21, 39, 64, 80)|
| 109  | (21, 56, 106, 114)| 110  | (21, 70, 73, 100)| 111  | (23, 30, 57, 98)|
| 112  | (23, 53, 55, 108)| 113  | (23, 78, 92, 101)| 114  | (24, 26, 55, 99)|
| 115  | (24, 52, 61, 115)| 116  | (24, 65, 94, 98)| 117  | (25, 27, 111, 117)|
| 118  | (25, 38, 86, 99)| 119  | (26, 35, 76, 101)| 120  | (26, 45, 102, 107)|
| 121  | (26, 54, 58, 70)| 122  | (27, 34, 84, 100)| 123  | (27, 42, 92, 120)|
| 124  | (28, 29, 71, 79)| 125  | (28, 35, 50, 57)| 126  | (29, 33, 43, 56)|
| 127  | (29, 52, 76, 82)| 128  | (30, 38, 61, 120)| 129  | (30, 62, 69, 88)|
| 130  | (30, 93, 95, 106)| 131  | (31, 35, 81, 84)| 132  | (31, 38, 41, 43)|
| 133  | (31, 66, 67, 74)| 134  | (31, 109, 110, 113)| 135  | (32, 37, 43, 87)|
| 136  | (32, 39, 49, 84)| 137  | (32, 59, 113, 120)| 138  | (32, 74, 76, 97)|
| 139  | (33, 48, 63, 95)| 140  | (33, 65, 101, 112)| 141  | (34, 58, 85, 104)|
| 142  | (34, 61, 72, 108)| 143  | (35, 46, 62, 96)| 144  | (36, 55, 112, 117)|
| 145  | (36, 75, 80, 93)| 146  | (37, 71, 81, 90)| 147  | (38, 60, 90, 102)|
| 148  | (39, 41, 46, 111)| 149  | (40, 54, 116, 118)| 150  | (40, 72, 79, 98)|
| 151  | (41, 57, 85, 116)| 152  | (41, 71, 91, 101)| 153  | (42, 43, 51, 98)|
| 154  | (42, 52, 102, 112)| 155  | (42, 71, 73, 105)| 156  | (43, 55, 104, 118)|
| 157  | (43, 78, 80, 107)| 158  | (44, 48, 74, 115)| 159  | (44, 54, 75, 108)|
| 160  | (44, 65, 107, 109)| 161  | (44, 77, 79, 85)| 162  | (45, 50, 73, 117)|
| 163  | (45, 64, 87, 89)| 164  | (45, 98, 103, 113)| 165  | (46, 79, 97, 110)|
| 166  | (46, 107, 119, 120)| 167  | (47, 49, 88, 103)| 168  | (47, 60, 67, 80)|
| 169  | (47, 71, 106, 115)| 170  | (47, 76, 100, 109)| 171  | (48, 53, 105, 114)|
| 172  | (48, 80, 81, 99)| 173  | (48, 83, 117, 119)| 174  | (48, 91, 94, 102)|
| 175  | (49, 56, 57, 110)| 176  | (49, 95, 96, 107)| 177  | (50, 58, 91, 109)|
| 178  | (50, 77, 99, 106)| 179  | (51, 53, 62, 85)| 180  | (51, 61, 81, 111)|
| 181  | (51, 67, 86, 95)| 182  | (53, 65, 70, 97)| 183  | (54, 64, 68, 95)|
| 184  | (55, 64, 74, 88)| 185  | (55, 82, 84, 91)| 186  | (56, 67, 89, 120)|
| 187  | (57, 59, 64, 105)| 188  | (57, 70, 86, 119)| 189  | (58, 60, 63, 87)|
| 190  | (58, 93, 103, 105)| 191  | (59, 66, 90, 117)| 192  | (59, 94, 106, 108)|
| 193  | (60, 68, 101, 119)| 194  | (61, 77, 96, 105)| 195  | (62, 63, 71, 118)|
| 196  | (62, 72, 82, 92)| 197  | (63, 75, 84, 98)| 198  | (65, 83, 93, 118)|
| 199  | (66, 87, 99, 100)| 200  | (67, 69, 83, 108)| 201  | (68, 73, 85, 94)|
| 202  | (68, 82, 111, 114)| 203  | (68, 97, 99, 103)| 204  | (69, 76, 77, 90)|
| 205  | (69, 84, 115, 116)| 206  | (70, 78, 89, 111)| 207  | (75, 102, 104, 111)|
| 208  | (78, 83, 85, 113)| 209  | (79, 86, 88, 114)| 210  | (89, 96, 109, 116)|
different elements of $\tilde{A}_7$ and such ambiguity corresponds to 24 permutations and changes directions of vectors in the bases described by group $\tilde{A}_4$. Unlike Witting configuration there is no subgroup of symmetry group useful for creation of full list with 210 bases in such a way.

Let us consider contextuality problem for given configuration of quantum states $\Lambda_{A_7}$ with a method used earlier for Witting configuration [2]. A noncontextual classical model reproducing principles of quantum mechanics would require a map of 120 vectors into $\{0, 1\}$ with property: *one and only one vector for any basis is mapped into 1*.

Let us consider some partition of 120 states on 30 bases, *e.g.*, 30 bases underlined in Table[2] All 30 states mapped into 1 in such partitions should be non-orthogonal, because any two orthogonal states in the $\Lambda_{A_7}$ could be extended into orthogonal basis and it would contain more than one state mapped into one.

Thus, the noncontextual model would require existence of 30 mutually nonorthogonal states. A brute-force search used in Ref. [2] for Witting configuration supposes to find all maximal non-orthogonal cliques. Such a method is not effective, but still can be used for $\Lambda_{A_7}$. The list with sizes of all such cliques obtained with GAP software [7] is presented below.

| Size | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|------|-----|-----|-----|-----|-----|-----|-----|
| Num. | 3528| 15120| 277130| 1117200| 3802620| 7018440| 12077100|

The nonorthogonal cliques with size 30 do not exist and maximal size is 24.

## 5 Entanglement of two configurations

Let us consider an entangled state

\[
|\Omega\rangle = \frac{1}{2} (|0\rangle|3\rangle - |1\rangle|2\rangle + |2\rangle|1\rangle - |3\rangle|0\rangle).
\]

In more general case an entangled state can be described by some matrix $J$ as

\[
|\Omega_J\rangle = \frac{1}{\sqrt{\text{Tr}(J J^*)}} \sum_{jk} J_{jk} |j\rangle |k\rangle.
\]
For state $|\Omega\rangle$ in Eq. (11)

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

For consideration of different scenarios of measurement with such kind of configurations it is useful to consider transformations of both systems respecting state Eq. (12). A natural example [2] is a measurement bases for both systems obtained by the same unitary transformation $A$ with property

$$(A \otimes A)|\Omega_J\rangle = \pm|\Omega_J\rangle \implies \bar{A}J = \pm JA, \tag{14}$$

where $\bar{A}$ is complex conjugation of all coefficients in $A$. It is enough to check property Eq. (14) for generators of some subgroup, because products also would respect the same relation. For matrix Eq. (13) there are subgroups of $\tilde{A}_7$ with transformations satisfying Eq. (14) and respecting entangled state Eq. (11). The first group is $\tilde{A}_5$ with 120 transformation generated by matrix $J$ together with

$$H = \begin{pmatrix} 0 & \iota' & \iota & -\iota'' \\ \iota' & -\iota' & \iota' & -\iota \\ \iota & -\iota' & -\iota' & \iota' \\ \iota'' & -\iota & -\iota' & 0 \end{pmatrix}, \tag{15}$$

Let us note analogues of maps Eq. (4) for generators of the subgroup

$$\tilde{S}_7 : \pm J \to (1, 6)(2, 5), \quad \tilde{S}_7 : \pm H \to (1, 6, 5, 2, 7). \tag{16}$$

The second subgroup is $\tilde{S}_5$ with 240 transformations generated by $J$, $H$ and $P_2$. The subgroups $\tilde{A}_5$ and $\tilde{S}_5$ are double covers of $A_5$ and $S_5$ representing even and all permutations of five elements respectively.

Symmetries of Witting configuration also include similar bigger subgroups with 720 and 1440 transformations useful for analysis of entanglement [2]. However for Witting configuration both such subgroups act transitively on all 40 vectors, but for $\Lambda_{A_7}$ only orbit of $\tilde{S}_5$ includes all 120 vectors and $\tilde{A}_5$ has two orbits with 60 vectors. The first orbit includes basic vectors $|0\rangle$ and $|3\rangle$ and the second one $|1\rangle$ and $|2\rangle$. Despite only group $\tilde{S}_5$ includes orbit with whole system of 120 states, both groups can be used for construction from the four basic states $|0\rangle, \ldots, |3\rangle$ all 30 bases underlined in Table 2.
Let us consider two scheme of measurements of entangled systems similar with discussed in Ref. [2]. In a simpler one measurement bases for both systems are chosen by application of some transformation from $\tilde{A}_5$ or $\tilde{S}_5$. However in such a case only 30 bases between 210 can be used. It is similar with analogue property of Witting configuration with only 10 bases between 40 are available for equivalent measurement of both systems.

Let us also recollect construction of ‘$J$-opposite state’ defined by anti-unitary transformation up to unsignificant phase [2]

$$|\psi_J\rangle \simeq J|\bar{\psi}\rangle.$$  \hspace{1cm} (17)

The result of measurement for entangled state Eq. (11) is always pair of such ‘$J$-opposite states’. The indexes for such pairs for configuration $\Lambda_{47}$ are collected in Table 3.

Table 3: Pairs of indexes for $J$-opposite states.

| 1 ↔ 31 | 2 ↔ 32 | 3 ↔ 73 | 4 ↔ 114 | 5 ↔ 38 | 6 ↔ 79 |
| 7 ↔ 120 | 8 ↔ 35 | 9 ↔ 76 | 10 ↔ 117 | 11 ↔ 21 | 12 ↔ 22 |
| 13 ↔ 63 | 14 ↔ 104 | 15 ↔ 28 | 16 ↔ 69 | 17 ↔ 110 | 18 ↔ 25 |
| 19 ↔ 66 | 20 ↔ 107 | 21 ↔ 11 | 22 ↔ 12 | 23 ↔ 53 | 24 ↔ 94 |
| 25 ↔ 18 | 26 ↔ 59 | 27 ↔ 100 | 28 ↔ 15 | 29 ↔ 56 | 30 ↔ 97 |
| 31 ↔ 1 | 32 ↔ 2 | 33 ↔ 43 | 34 ↔ 84 | 35 ↔ 8 | 36 ↔ 49 |
| 37 ↔ 90 | 38 ↔ 5 | 39 ↔ 46 | 40 ↔ 87 | 41 ↔ 111 | 42 ↔ 112 |
| 43 ↔ 33 | 44 ↔ 74 | 45 ↔ 118 | 46 ↔ 39 | 47 ↔ 80 | 48 ↔ 115 |
| 49 ↔ 36 | 50 ↔ 77 | 51 ↔ 101 | 52 ↔ 102 | 53 ↔ 23 | 54 ↔ 64 |
| 55 ↔ 108 | 56 ↔ 29 | 57 ↔ 70 | 58 ↔ 105 | 59 ↔ 26 | 60 ↔ 67 |
| 61 ↔ 91 | 62 ↔ 92 | 63 ↔ 13 | 64 ↔ 54 | 65 ↔ 98 | 66 ↔ 19 |
| 67 ↔ 60 | 68 ↔ 95 | 69 ↔ 16 | 70 ↔ 57 | 71 ↔ 81 | 72 ↔ 82 |
| 73 ↔ 3 | 74 ↔ 44 | 75 ↔ 88 | 76 ↔ 9 | 77 ↔ 50 | 78 ↔ 85 |
| 79 ↔ 6 | 80 ↔ 47 | 81 ↔ 71 | 82 ↔ 72 | 83 ↔ 113 | 84 ↔ 34 |
| 85 ↔ 78 | 86 ↔ 119 | 87 ↔ 40 | 88 ↔ 75 | 89 ↔ 116 | 90 ↔ 37 |
| 91 ↔ 61 | 92 ↔ 62 | 93 ↔ 103 | 94 ↔ 24 | 95 ↔ 68 | 96 ↔ 109 |
| 97 ↔ 30 | 98 ↔ 65 | 99 ↔ 106 | 100 ↔ 27 | 101 ↔ 51 | 102 ↔ 52 |
| 103 ↔ 93 | 104 ↔ 14 | 105 ↔ 58 | 106 ↔ 99 | 107 ↔ 20 | 108 ↔ 55 |
| 109 ↔ 96 | 110 ↔ 17 | 111 ↔ 41 | 112 ↔ 42 | 113 ↔ 83 | 114 ↔ 4 |
| 115 ↔ 48 | 116 ↔ 89 | 117 ↔ 10 | 118 ↔ 45 | 119 ↔ 86 | 120 ↔ 7 |
The bases underlined in Table 2 correspond to examples with two pairs of \( J \)-opposite states included in the same basis. The other method of finding such bases is to consider application of transformations from group \( \tilde{S}_5 \) to initial basis \(|k\rangle\), \( k = 0, \ldots, 3 \). In the Table 1 the basic states have indexes \( 10k + 1 \), \textit{i.e.}, \((1, 11, 21, 31)\) and initial basis has index 3 in the Table 2.

The measurement with only 30 bases is not appropriate for analysis of contextuality \[2, 3\] and the second measurement scheme can be used instead. In such a case different transformations are used for preparation of measurement bases. For any \( A \in \tilde{A}_7 \) the transformation of second system \( B \) can be obtained \[2\] due equations

\[
(A \otimes B)|\Omega_j\rangle = |\Omega_j\rangle \implies BJ = JA, \quad B = JAJ^{-1}.
\]

Let us note, that \( B \in \tilde{A}_7 \) because \( J \in \tilde{A}_7 \) and generators Eq. (11) and Eq. (2) as well as any other transformation \( A \in \tilde{A}_7 \) corresponding to a product of the generators have property \( \bar{B} \in \tilde{A}_7 \).

Permutations and directions of vectors in measurement bases described by group \( \tilde{A}_4 \) are not essential and thus \( 5040 \) transformations from \( \tilde{A}_7 \) correspond to \( 210 \) different measurement bases. All pairs of such bases obtained by transformations \( A \) and \( B \) from Eq. (18) are represented in Table 4. For any such pair second basis is corresponding to collection of \( J \)-opposite states and \textit{vice versa}. For 30 underlined bases in Table 2 both indexes in a pair coincide.

Similarly with Witting configuration in Ref. \[2\] alternative choice of entangled states and matrices \( J \) can be obtained by consideration of decomposition of \( \tilde{A}_7 \) into cosets of subgroup \( \tilde{S}_5 \). There are \( 5040/240 = 21 \) different cosets \( C\tilde{S}_5 \) and alternative entangled states Eq. (12) could be constructed with matrices

\[
J^C = CJC^{-1} = CJC^T.
\]

Except \( J \) Eq. (13) only two matrices between 20 have simplest form with coefficients 0 and ±1

\[
J_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} \quad (20)
\]

and

\[
J_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}. \quad (21)
\]
Table 4: Pairs of indexes for bases with $J$-opposite states.

| 1 ↔ 132 | 2 ↔ 131 | 3 = 3 | 4 ↔ 134 | 5 ↔ 133 | 6 ↔ 43 |
| 7 ↔ 54 | 8 ↔ 137 | 9 ↔ 138 | 10 = 10 | 11 ↔ 96 | 12 ↔ 84 |
| 13 ↔ 135 | 14 ↔ 136 | 15 ↔ 162 | 16 ↔ 110 | 17 ↔ 31 | 18 ↔ 201 |
| 19 ↔ 99 | 20 ↔ 155 | 21 = 21 | 22 ↔ 209 | 23 ↔ 109 | 24 ↔ 48 |
| 25 ↔ 171 | 26 ↔ 92 | 27 ↔ 202 | 28 = 28 | 29 ↔ 128 | 30 ↔ 118 |
| 31 ↔ 17 | 32 ↔ 85 | 33 = 33 | 34 ↔ 147 | 35 ↔ 124 | 36 ↔ 165 |
| 37 ↔ 70 | 38 = 38 | 39 ↔ 161 | 40 ↔ 150 | 41 = 41 | 42 ↔ 166 |
| 43 ↔ 6 | 44 ↔ 186 | 45 ↔ 123 | 46 ↔ 119 | 47 ↔ 125 | 48 ↔ 24 |
| 49 ↔ 97 | 50 = 50 | 51 ↔ 143 | 52 = 52 | 53 ↔ 204 | 54 ↔ 7 |
| 55 ↔ 170 | 56 ↔ 127 | 57 ↔ 117 | 58 ↔ 191 | 59 ↔ 69 | 60 = 60 |
| 61 ↔ 144 | 62 ↔ 173 | 63 ↔ 78 | 64 ↔ 72 | 65 ↔ 108 | 66 ↔ 107 |
| 67 ↔ 71 | 68 ↔ 77 | 69 ↔ 59 | 70 ↔ 37 | 71 ↔ 67 | 72 ↔ 64 |
| 73 ↔ 197 | 74 ↔ 189 | 75 ↔ 139 | 76 ↔ 195 | 77 ↔ 68 | 78 ↔ 63 |
| 79 ↔ 156 | 80 ↔ 176 | 81 ↔ 207 | 82 ↔ 141 | 83 = 83 | 84 ↔ 12 |
| 85 ↔ 32 | 86 ↔ 101 | 87 = 87 | 88 ↔ 205 | 89 ↔ 104 | 90 ↔ 200 |
| 91 ↔ 129 | 92 ↔ 26 | 93 ↔ 98 | 94 ↔ 175 | 95 = 95 | 96 ↔ 11 |
| 97 ↔ 49 | 98 ↔ 93 | 99 ↔ 19 | 100 ↔ 199 | 101 ↔ 86 | 102 = 102 |
| 103 ↔ 157 | 104 ↔ 89 | 105 ↔ 120 | 106 ↔ 160 | 107 ↔ 66 | 108 ↔ 65 |
| 109 ↔ 23 | 110 ↔ 16 | 111 ↔ 182 | 112 = 112 | 113 ↔ 179 | 114 ↔ 192 |
| 115 ↔ 174 | 116 = 116 | 117 ↔ 57 | 118 ↔ 30 | 119 ↔ 46 | 120 ↔ 105 |
| 121 ↔ 187 | 122 = 122 | 123 ↔ 45 | 124 ↔ 35 | 125 ↔ 47 | 126 = 126 |
| 127 ↔ 56 | 128 ↔ 29 | 129 ↔ 91 | 130 ↔ 203 | 131 ↔ 2 | 132 ↔ 1 |
| 133 ↔ 5 | 134 ↔ 4 | 135 ↔ 13 | 136 ↔ 14 | 137 ↔ 8 | 138 ↔ 9 |
| 139 ↔ 75 | 140 ↔ 153 | 141 ↔ 82 | 142 ↔ 185 | 143 ↔ 51 | 144 ↔ 61 |
| 145 ↔ 167 | 146 = 146 | 147 ↔ 34 | 148 = 148 | 149 ↔ 163 | 150 ↔ 40 |
| 151 ↔ 206 | 152 ↔ 180 | 153 ↔ 140 | 154 = 154 | 155 ↔ 20 | 156 ↔ 79 |
| 157 ↔ 103 | 158 = 158 | 159 ↔ 184 | 160 ↔ 106 | 161 ↔ 39 | 162 ↔ 15 |
| 163 ↔ 149 | 164 ↔ 198 | 165 ↔ 36 | 166 ↔ 42 | 167 ↔ 145 | 168 = 168 |
| 169 ↔ 172 | 170 ↔ 55 | 171 ↔ 25 | 172 ↔ 169 | 173 ↔ 62 | 174 ↔ 115 |
| 175 ↔ 94 | 176 ↔ 80 | 177 ↔ 194 | 178 = 178 | 179 ↔ 113 | 180 ↔ 152 |
| 181 ↔ 193 | 182 ↔ 111 | 183 = 183 | 184 ↔ 159 | 185 ↔ 142 | 186 ↔ 44 |
| 187 ↔ 121 | 188 = 188 | 189 ↔ 74 | 190 = 190 | 191 ↔ 58 | 192 ↔ 114 |
| 193 ↔ 181 | 194 ↔ 177 | 195 ↔ 76 | 196 = 196 | 197 ↔ 73 | 198 ↔ 164 |
| 199 ↔ 100 | 200 ↔ 90 | 201 ↔ 18 | 202 ↔ 27 | 203 ↔ 130 | 204 ↔ 53 |
| 205 ↔ 88 | 206 ↔ 151 | 207 ↔ 81 | 208 = 208 | 209 ↔ 22 | 210 = 210 |
The matrices and entangled states coincide with derived earlier for Witting configuration in Ref. [2]

\[ |\Omega_1\rangle = \frac{1}{2}(|0\rangle|2\rangle - |2\rangle|0\rangle + |1\rangle|3\rangle - |3\rangle|1\rangle) \]  
(22)

and

\[ |\Omega_2\rangle = \frac{1}{2}(|0\rangle|1\rangle - |1\rangle|0\rangle + |3\rangle|2\rangle - |2\rangle|3\rangle). \]  
(23)

The indexes for \( J_1 \)- and \( J_2 \)-opposite states are listed in Table 5 and Table 6 respectively.

| Table 5: Pairs of indexes for \( J_1 \)-opposite states. |
|---------------------------------|
| 1 ↔ 11 | 2 ↔ 12 | 3 ↔ 53 | 4 ↔ 94 | 5 ↔ 18 | 6 ↔ 59 |
| 7 ↔ 100 | 8 ↔ 15 | 9 ↔ 56 | 10 ↔ 97 | 11 ↔ 1 | 12 ↔ 2 |
| 13 ↔ 43 | 14 ↔ 84 | 15 ↔ 8 | 16 ↔ 49 | 17 ↔ 90 | 18 ↔ 5 |
| 19 ↔ 46 | 20 ↔ 87 | 21 ↔ 31 | 22 ↔ 32 | 23 ↔ 73 | 24 ↔ 114 |
| 25 ↔ 38 | 26 ↔ 79 | 27 ↔ 120 | 28 ↔ 35 | 29 ↔ 76 | 30 ↔ 117 |
| 31 ↔ 21 | 32 ↔ 22 | 33 ↔ 63 | 34 ↔ 104 | 35 ↔ 28 | 36 ↔ 69 |
| 37 ↔ 110 | 38 ↔ 25 | 39 ↔ 66 | 40 ↔ 107 | 41 ↔ 91 | 42 ↔ 92 |
| 43 ↔ 13 | 44 ↔ 54 | 45 ↔ 98 | 46 ↔ 19 | 47 ↔ 60 | 48 ↔ 95 |
| 49 ↔ 16 | 50 ↔ 57 | 51 ↔ 81 | 52 ↔ 82 | 53 ↔ 3 | 54 ↔ 44 |
| 55 ↔ 88 | 56 ↔ 9 | 57 ↔ 50 | 58 ↔ 85 | 59 ↔ 6 | 60 ↔ 47 |
| 61 ↔ 111 | 62 ↔ 112 | 63 ↔ 33 | 64 ↔ 74 | 65 ↔ 118 | 66 ↔ 39 |
| 67 ↔ 80 | 68 ↔ 115 | 69 ↔ 36 | 70 ↔ 77 | 71 ↔ 101 | 72 ↔ 102 |
| 73 ↔ 23 | 74 ↔ 64 | 75 ↔ 108 | 76 ↔ 29 | 77 ↔ 70 | 78 ↔ 105 |
| 79 ↔ 26 | 80 ↔ 67 | 81 ↔ 51 | 82 ↔ 52 | 83 ↔ 93 | 84 ↔ 14 |
| 85 ↔ 58 | 86 ↔ 99 | 87 ↔ 20 | 88 ↔ 55 | 89 ↔ 96 | 90 ↔ 17 |
| 91 ↔ 41 | 92 ↔ 42 | 93 ↔ 83 | 94 ↔ 4 | 95 ↔ 48 | 96 ↔ 89 |
| 97 ↔ 10 | 98 ↔ 45 | 99 ↔ 86 | 100 ↔ 7 | 101 ↔ 71 | 102 ↔ 72 |
| 103 ↔ 113 | 104 ↔ 34 | 105 ↔ 78 | 106 ↔ 119 | 107 ↔ 40 | 108 ↔ 75 |
| 109 ↔ 116 | 110 ↔ 37 | 111 ↔ 61 | 112 ↔ 62 | 113 ↔ 103 | 114 ↔ 24 |
| 115 ↔ 68 | 116 ↔ 109 | 117 ↔ 30 | 118 ↔ 65 | 119 ↔ 106 | 120 ↔ 27 |
Table 6: Pairs of indexes for $J_2$-opposite states.

|    |     |     |     |     |     |     |
|----|-----|-----|-----|-----|-----|-----|
| 1  | 21  | 2   | 22  | 3   | 24  | 4   |
| 7  | 29  | 8   | 25  | 9   | 27  | 10  |
| 13 | 34  | 14  | 33  | 15  | 38  | 16  |
| 19 | 37  | 20  | 36  | 21  | 1   | 22  |
| 25 | 8   | 26  | 10  | 27  | 9   | 28  |
| 31 | 11  | 32  | 12  | 33  | 14  | 34  |
| 37 | 19  | 38  | 15  | 39  | 17  | 40  |
| 43 | 104 | 44  | 103 | 45  | 108 | 46  |
| 49 | 107 | 50  | 106 | 51  | 111 | 52  |
| 55 | 118 | 56  | 120 | 57  | 119 | 58  |
| 61 | 81  | 62  | 82  | 63  | 84  | 64  |
| 67 | 89  | 68  | 85  | 69  | 87  | 70  |
| 73 | 94  | 74  | 93  | 75  | 98  | 76  |
| 79 | 97  | 80  | 96  | 81  | 61  | 82  |
| 85 | 68  | 86  | 70  | 87  | 69  | 88  |
| 91 | 71  | 92  | 72  | 93  | 74  | 94  |
| 97 | 79  | 98  | 75  | 99  | 77  | 100 |
| 103| 44  | 104 | 43  | 105 | 48  | 106 |
| 109| 47  | 110 | 46  | 111 | 51  | 112 |
| 115| 58  | 116 | 60  | 117 | 59  | 118 |
|    |     |     |     |     |     |     |
| 5  | 28  | 6   | 30  | 7   | 29  | 8   |
| 11 | 8    | 16  | 3    | 12  | 17  | 20  |
| 26 | 18  | 23  | 19  | 24  | 27  | 30  |
| 32 | 13  | 33  | 14  | 31  | 11  | 12  |
| 38 | 15  | 39  | 17  | 37  | 19  | 21  |
| 43 | 18  | 44  | 19  | 42  | 20  | 22  |
| 49 | 23  | 50  | 24  | 48  | 26  | 27  |
| 55 | 28  | 56  | 29  | 57  | 30  | 31  |
|    |     |     |     |     |     |     |
| 61 | 32  | 62  | 33  | 64  | 34  | 65  |
| 67 | 37  | 68  | 38  | 69  | 39  | 70  |
| 73 | 40  | 74  | 41  | 75  | 42  | 76  |
| 79 | 43  | 80  | 44  | 81  | 45  | 82  |
| 85 | 46  | 86  | 47  | 87  | 48  | 88  |
| 91 | 49  | 92  | 50  | 93  | 51  | 94  |
| 97 | 52  | 98  | 53  | 99  | 54  | 100 |
| 103| 55  | 104 | 56  | 105 | 57  | 106 |
| 109| 58  | 110 | 59  | 111 | 60  | 112 |
| 115| 61  | 116 | 62  | 117 | 63  | 118 |
|    |     |     |     |     |     |     |
6 Conclusion

New configuration with 120 quantum states useful for analysis of principles of quantum mechanics such as nonlocality and contextuality was introduced in presented work. Many properties of the configuration has analogies with models based on Witting configuration [1, 2] and geometry of dodecahedron [1, 2, 3, 4] already known earlier.

Let us compare some properties of both complex configurations:

|                        | Witting configuration | $A_7$ |
|------------------------|----------------------|-------|
| Dimension              | 4                    | 4     |
| States                 | 40                   | 120   |
| orthogonal to a state  | 12                   | 21    |
| Bases                  | 40                   | 210   |
| with a state           | 4                    | 7     |
| N-cliques              | 2970                 | 31475754 |
| minimal size           | 4                    | 10    |
| maximal size           | 7                    | 24    |
| required               | 10                   | 30    |
| Symmetry group         | $U_4(F_2)$            | $A_7$ |
| Size of the group      | 25920                | 2520  |
| Equal bases            | 10                   | 30    |
| Size of a subgroup     | 1440                 | 240   |
| $|\langle \psi_j | \psi_k \rangle|^2$, $j \neq k$ | 0, $\frac{1}{3}$     | 0, $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$ |
| Polynomial             | $z^2 + z + 1 = 0$    | $z^2 + z + 2 = 0$ |

Here two roots of polynomial may be used for construction of coordinates in the configurations, N-cliques are maximal cliques with nonorthogonal states used in brute-force search for (impossible) classical noncontextual models and equal bases respected by an appropriate subgroup suggest possibility to apply for entangled configurations the same transformation to both systems in a simpler measurement scheme.

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