HOMOTOPICALLY EQUIVALENT SIMPLE LOOPS ON 2-BRIDGE SPHERES IN 2-BRIDGE LINK COMPLEMENTS (II)

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Abstract. This is the second of series of papers which give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in a 2-bridge link complement to be homotopic in the link complement. The first paper [5] treated the case of the 2-bridge torus links. In this paper, we treat the case of 2-bridge links of slope $n/(2n + 1)$ and $(n + 1)/(3n + 2)$, where $n \geq 2$ is an arbitrary integer.

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1. Introduction

Let $K$ be a 2-bridge link in $S^3$ and let $S$ be a 4-punctured sphere in $S^3 - K$ obtained from a 2-bridge sphere of $K$. In [4], we gave a complete characterization of those essential simple loops in $S$ which are null-homotopic in $S^3 - K$. The purpose of the series of papers starting from [5] and ending with [6], including the present paper as the second one, is to give a necessary and sufficient condition for two essential simple loops on $S$ to be homotopic in $S^3 - K$. These results will be used in [7] to show the existence of a variation of McShane’s identity for 2-bridge links, proving a conjecture proposed in [12]. For an overview of this series of works, we refer the reader to the research announcement [8]. In the first paper [5] of the series, we treated the case when the 2-bridge link is a $(2, p)$-torus link. In this paper, we treat the case of 2-bridge links of slope $n/(2n + 1)$ and $(n + 1)/(3n + 2)$, where $n \geq 2$ is an arbitrary integer. These two families play special roles in our project in the sense that the treatment of these links form a base step of an inductive proof of a theorem for general 2-bridge links giving an answer to the problem treated in this series of papers. We note that the figure-eight knot is both a 2-bridge link of slope $n/(2n + 1)$ with $n = 2$ and a 2-bridge link of slope $(n + 1)/(3n + 2)$ with $n = 1$. Surprisingly, the treatment of the figure-eight knot, the simplest hyperbolic 2-bridge knot, is the most complicated. In fact, the figure-eight knot group admits various unexpected reduced annular diagrams (see Section 8). This reminds us of the phenomenon in the theory of exceptional Dehn filling that the figure-eight knot attains the maximal number of exceptional Dehn fillings.

It has been proved by Weinbaum [15] and Appel and Schupp [1] that the word and conjugacy problems for prime alternating link groups are solvable, by using small cancellation theory (see also [2] and references in it). Moreover, it was also shown by Sela [14] and Préaux [11] that the word and conjugacy problems for any link group are solvable. A characteristic feature of this series of papers including [4] is that we give a complete answer to special (but also natural) word and conjugacy problems for the groups of 2-bridge links, which form a special (but also important) family of prime alternating links. The key tool used in the proofs is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups.

This paper is organized as follows. In Section 2 we recall basic facts concerning 2-bridge links, and describe the main results of this paper (Main Theorems 2.4 and 2.5). In Section 3 we recall the upper presentation of a 2-bridge link group, and recall key facts established in [4] and [5] concerning the upper
presentation. In Section 4, we set up Hypotheses A, B and C, under which we establish technical lemmas used for the proofs in Sections 5–7. The special case of Main Theorem 2.4 (namely, the case of a 2-bridge link of slope 2/5) is treated in Section 5, and the remaining case of Main Theorem 2.4 (namely, the case of a 2-bridge link of slope $n/(2n + 1)$ with $n \geq 3$) in Section 6. The proof of Main Theorem 2.5 is contained in Section 7. In the final section, Section 8, we prove Theorems 2.7 and 2.8.

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2. Main results

For a rational number $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope $r$, which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope $\infty$ and $r$. The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the Conway sphere $(S^2, P) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where $H$ is the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the $\pi$-rotations around the points in the lattice $\mathbb{Z}^2$ (see [10, Section 3] or [4, Section 2] for details). Let $S$ be the 4-punctured sphere $S^2 - P$ in the link complement $S^3 - K(r)$. Any essential simple loop in $S$, up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbb{Q}}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto $S$. The (unoriented) essential simple loop in $S$ so obtained is denoted by $\alpha_s$. We also denote by $\alpha_s$ the conjugacy class of an element of $\pi_1(S)$ represented by (a suitably oriented) $\alpha_s$. Then the link group $G(K(r)) := \pi_1(S^3 - K(r))$ is identified with $\pi_1(S)/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$.

Let $D$ be the Farey tessellation, whose ideal vertex set is identified with $\hat{\mathbb{Q}}$. For each $r \in \hat{\mathbb{Q}}$, let $\Gamma_r$ be the group of automorphisms of $D$ generated by reflections in the edges of $D$ with an endpoint $r$, and let $\hat{\Gamma}_r$ be the group generated by $\Gamma_r$ and $\Gamma_\infty$. Assume that $r \neq \infty$. Then the region, $R$, bounded by a pair of Farey edges with an endpoint $\infty$ and a pair of Farey edges with an endpoint $r$ forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $\mathbb{H}^2$ (see Figure 1). Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of $R$. Suppose that $r$ is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write
\[ r = \frac{1}{m_1 + \frac{1}{m_2 + \cdots + \frac{1}{m_k}}} =: [m_1, m_2, \ldots, m_k], \]

where \( k \geq 1 \), \((m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k\), and \( m_k \geq 2 \). Then the above intervals are given by \( I_1(r) = [0, r_1] \) and \( I_2(r) = [r_2, 1] \), where

\[
\begin{align*}
    r_1 &= \begin{cases} 
      [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is odd}, \\
      [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even},
    \end{cases} \\
    r_2 &= \begin{cases} 
      [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd}, \\
      [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is even}.
    \end{cases}
\end{align*}
\]

![Figure 1. A fundamental domain of \( \hat{\Gamma}_r \) in the Farey tessellation (the shaded domain) for \( r = 5/17 = [3, 2, 2] \).](image)

The following theorem was established by [10] and [4], which describes the role of \( \hat{\Gamma}_r \) in the study of 2-bridge link groups.

**Theorem 2.1.** (1) [10] Proposition 4.6] If two elements \( s \) and \( s' \) of \( \hat{\mathbb{Q}} \) belong to the same orbit \( \hat{\Gamma}_r \)-orbit, then the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(r) \).

(2) [4] Lemma 7.1] For any \( s \in \hat{\mathbb{Q}} \), there is a unique rational number \( s_0 \in I_1(r) \cup I_2(r) \cup \{\infty, r\} \) such that \( s \) is contained in the \( \hat{\Gamma}_r \)-orbit of \( s_0 \). In this
case, \( \alpha_s \) is homotopic to \( \alpha_{s_0} \) in \( S^3 - K(r) \). Thus if \( s_0 \in \{\infty, r\} \), then \( \alpha_s \) is null-homotopic in \( S^3 - K(r) \).

(3) [4 Main Theorem 2.3] The loop \( \alpha_s \) is null-homotopic in \( S^3 - K(r) \) if and only if \( s \) belongs to the \( \Gamma_{\infty} \)-orbit of \( \infty \) or \( r \). In particular, if \( s \in I_1(r) \cup I_2(r) \), then \( \alpha_s \) is not null-homotopic in \( S^3 - K(r) \).

Thus the following question naturally arises.

**Question 2.2.** Consider a 2-bridge link \( K(r) \) with \( r \neq \infty \). For two distinct rational numbers \( s, s' \in I_1(r) \cup I_2(r) \), when are the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) homotopic in \( S^3 - K(r) \)?

If \( r = \infty \), then \( G(K(\infty)) \) is a rank 2 free group, and the work of Komori and Series [3 Theorem 1.2] implies that \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(\infty) \) if and only if \( s \) and \( s' \) belong to the same orbit of \( \hat{\Gamma}_{\infty} = \Gamma_{\infty} \).

The purpose of this series of papers is to solve the above question. In the first paper [5], we treated the case when \( r = 1/p \) for some \( p \in \mathbb{Z} \), and obtained the following answer.

**Theorem 2.3** ([5 Main Theorem 2.7]). Suppose \( r = 1/p \), where \( p \geq 2 \) is an integer. Then, for two distinct rational numbers \( s, s' \in I_1(r) \cup I_2(r) \), the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(r) \) if and only if \( s = q_1/p_1 \) and \( s' = q_2/p_2 \) satisfy \( q_1 = q_2 \) and \( q_1/(p_1 + p_2) = 1/p \), where \((p_i, q_i)\) is a pair of relatively prime positive integers.

In the present paper, we solve the above question for the 2-bridge links \( K(n/(2n + 1)) \) and \( K((n + 1)/(3n + 2)) \), where \( n \geq 2 \) is an arbitrary integer.

**Main Theorem 2.4.** Suppose \( r = n/(2n + 1) = [2, n] \), where \( n \geq 2 \) is an integer. Then, for any two distinct rational numbers \( s, s' \in I_1(r) \cup I_2(r) \), the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are never homotopic in \( S^3 - K(r) \).

**Main Theorem 2.5.** Suppose \( r = (n + 1)/(3n + 2) = [2, 1, n] \), where \( n \geq 2 \) is an integer. Then, for two distinct rational numbers \( s, s' \in I_1(r) \cup I_2(r) \), the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(r) \) if and only if both \( r = 3/8 \) (i.e., \( n = 2 \)) and the set \( \{s, s'\} \) equals either \( \{1/6, 3/10\} \) or \( \{3/4, 5/12\} \).

**Remark 2.6.** The exceptional pairs \( \{1/6, 3/10\} \) and \( \{3/4, 5/12\} \) have the following geometric properties in the Farey tessellation. Let \( \tau \) be the reflection of the hyperbolic plane in the geodesic with endpoints 1/2 and 1/4, which bisects the Farey edge \((0/1, 1/3)\). Then \( \tau \) preserves the Farey tessellation and interchanges \( \infty \) and \( r = 3/8 \). The members of each of the exceptional pairs are interchanged by the involution \( \tau \).
We prove these main theorems by interpreting the situation in terms of combinatorial group theory. In other words, we prove that two words representing the free homotopy classes of \( \alpha_s \) and \( \alpha_{s'} \) are conjugate in the 2-bridge link group \( G(K(r)) \) if and only if \( s \) and \( s' \) satisfy the conditions given in the statements of the theorems. The key tool used in the proofs is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups. The proofs of the main theorems also imply the following theorems.

**Theorem 2.7.** Suppose \( r = n/(2n+1) = [2, n] \), where \( n \geq 2 \) is an integer. Then the following hold for a rational number \( s \in I_1(r) \cup I_2(r) \).

1. The loop \( \alpha_s \) is peripheral if and only if one of the following holds.
   - (a) \( n = 2 \), i.e., \( r = 2/5 \), and \( s = 1/5 \) or \( s = 3/5 \).
   - (b) \( s = (n + 1)/(2n + 1) \).

2. The free homotopy class \( \alpha_s \) is primitive with the following exceptions.
   - (a) \( r = 2/5 \), and \( s = 2/7 \) or \( s = 3/4 \). In this case, \( \alpha_s \) is the third power of some primitive element in \( G(K(r)) \).
   - (b) \( r = 3/7 \) and \( s = 2/7 \). In this case, \( \alpha_s \) is the second power of some primitive element in \( G(K(r)) \).

**Theorem 2.8.** Suppose \( r = (n + 1)/(3n + 2) = [2, 1, n] \), where \( n \geq 2 \) is an integer. Then the loop \( \alpha_s \) is non-peripheral and primitive for any rational number \( s \in I_1(r) \cup I_2(r) \).

Here, a closed loop \( \alpha_s \) in \( S^3 - K(r) \) is said to be *peripheral* if it is homotopic to a loop on a peripheral torus. A loop \( \alpha_s \) is said to be *primitive* if there is no element in the 2-bridge link group \( G(K(r)) \) whose proper power is conjugate to \( \alpha_s \).

### 3. Review of basic results from [4] and [5]

Throughout this paper, the set \( \{a, b\} \) denotes the standard meridian-generator of the rank 2 free group \( \pi_1(B^3 - t(\infty)) \), which is specified as in [4] Section 3 or [5] Section 3. For a positive rational number \( q/p \), let \( u_{q/p} \) be the word in \( \{a, b\} \) representing the (suitably oriented) loop \( \alpha_{q/p} \) defined by the following rule (see [4] Lemma 4.7):

\[
u_{q/p} = a^{\varepsilon_1}b^{\varepsilon_2} \cdots a^{\varepsilon_{2p-1}}b^{\varepsilon_{2p}}.
\]

Here \( \varepsilon_i = (-1)^{i(i-1)/2} \cdot ([t]^* - 1) \), with \( [t]^* \) the smallest integer greater than \( t \). In fact, \( u_{q/p} \) is obtained from a line of slope \( q/p \) in \( \mathbb{R}^2 - \mathbb{Z}^2 \) by reading its intersection with vertical lattice lines (cf. [4] Remark 1)].
The link group $G(K(r))$ with $r > 0$ has the following presentation, called the upper presentation:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(B^3 - t(\infty))/\langle \langle a_r \rangle \rangle$$

$$\cong F(a,b)/\langle \langle u_r \rangle \rangle \cong \langle a,b \mid u_r \rangle.$$ 

### 3.1. S- and T-sequences of slope $r$.

We recall the definition of the sequences $S(r)$ and $T(r)$ and the cyclic sequences $CS(r)$ and $CT(r)$ of slope $r$ defined in [4], all of which are read from the single relator $u_r$ of the upper presentation of $G(K(r))$, and review several important properties of these sequences from [4] so that we can adopt small cancellation theory. To this end we fix some definitions and notation. Let $X$ be a set. By a word in $X$, we mean a finite sequence $x_1 x_2 \cdots x_n$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i$ the $i$-th letter of the word. For two words $u, v$ in $X$, by $u \equiv v$ we denote the visual equality of $u$ and $v$, meaning that if $u = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \cdots y_m^{\delta_m}$, then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \ldots, n$. The length of a word $v$ is denoted by $|v|$. A word $v$ in $X$ is said to be reduced if $v$ does not contain $xx^{-1}$ or $x^{-1}x$ for any $x \in X$. A word is said to be cyclically reduced if all its cyclic permutations are reduced. A cyclic word is defined to be the set of all cyclic permutations of a cyclically reduced word. By $(v)$ we denote the cyclic word associated with a cyclically reduced word $v$. Also by $(u) \equiv (v)$ we mean the visual equality of two cyclic words $(u)$ and $(v)$. In fact, $(u) \equiv (v)$ if and only if $v$ is visually a cyclic shift of $u$.

**Definition 3.1.** (1) Let $v$ be a nonempty reduced word in $\{a, b\}$. Decompose $v$ into

$$v \equiv v_1 v_2 \cdots v_t,$$

where, for each $i = 1, \ldots, t - 1$, all letters in $v_i$ have positive (resp. negative) exponents, and all letters in $v_{i+1}$ have negative (resp. positive) exponents. Then the sequence of positive integers

$$S(v) := (|v_1|, |v_2|, \ldots, |v_t|)$$

is called the $S$-sequence of $v$.

(2) Let $(v)$ be a nonempty cyclic word in $\{a, b\}$. Decompose $(v)$ into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in $v_i$ have positive (resp. negative) exponents, and all letters in $v_{i+1}$ have negative (resp. positive) exponents (taking subindices modulo $t$). Then the cyclic sequence of positive integers

$$CS(v) := (|v_1|, |v_2|, \ldots, |v_t|)$$

is called the cyclic $S$-sequence of $(v)$. Here, the double parentheses denote that the sequence is considered modulo cyclic permutations.
A nonempty reduced word \( v \) in \( \{a, b\} \) is said to be alternating if \( a^\pm 1 \) and \( b^\pm 1 \) appear in \( v \) alternately, i.e., neither \( a^\pm 2 \) nor \( b^\pm 2 \) appears in \( v \). A cyclic word \( (v) \) is said to be alternating if all cyclic permutations of \( v \) are alternating. In the latter case, we also say that \( v \) is cyclically alternating.

**Definition 3.2.** For a rational number \( r \) with \( 0 < r \leq 1 \), let \( G(K(r)) = \langle a, b \mid u_r \rangle \) be the upper presentation. Then the symbol \( S(r) \) (resp. \( CS(r) \)) denotes the \( S \)-sequence \( S(u_r) \) of \( u_r \) (resp. cyclic \( S \)-sequence \( CS(u_r) \) of \( u_r \)), which is called the \( S \)-sequence of slope \( r \) (resp. the cyclic \( S \)-sequence of slope \( r \)).

In the remainder of this paper unless specified otherwise, we suppose that \( r \) is a rational number with \( 0 < r \leq 1 \), and write \( r \) as a continued fraction:

\[
r = [m_1, m_2, \ldots, m_k],
\]

where \( k \geq 1 \), \( (m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k \) and \( m_k \geq 2 \) unless \( k = 1 \). For brevity, we write \( m \) for \( m_1 \).

**Lemma 3.3 ([4, Proposition 4.3]).** The following hold.

1. Suppose \( k = 1 \), i.e., \( r = 1/m \). Then \( S(r) = (m, m) \).
2. Suppose \( k \geq 2 \). Then each term of \( S(r) \) is either \( m \) or \( m + 1 \), and \( S(r) \) begins with \( m + 1 \) and ends with \( m \). Moreover, the following hold.
   a. If \( m_2 = 1 \), then no two consecutive terms of \( S(r) \) can be \( (m, m) \), so there is a sequence of positive integers \( (t_1, t_2, \ldots, t_s) \) such that
      \[
      S(r) = (t_1(m + 1), m, t_2(m + 1), m, \ldots, t_s(m + 1), m).
      \]
      Here, the symbol “\( t_i(m + 1) \)” represents \( t_i \) successive \( m + 1 \)'s.
   b. If \( m_2 \geq 2 \), then no two consecutive terms of \( S(r) \) can be \( (m + 1, m + 1) \), so there is a sequence of positive integers \( (t_1, t_2, \ldots, t_s) \) such that
      \[
      S(r) = (m + 1, t_1(m), m + 1, t_2(m), \ldots, m + 1, t_s(m)).
      \]
      Here, the symbol “\( t_i(m) \)” represents \( t_i \) successive \( m \)'s.

**Definition 3.4.** If \( k \geq 2 \), the symbol \( T(r) \) denotes the sequence \( (t_1, t_2, \ldots, t_s) \) in Lemma 3.3, which is called the \( T \)-sequence of slope \( r \). The symbol \( CT(r) \) denotes the cyclic sequence represented by \( T(r) \), which is called the cyclic \( T \)-sequence of slope \( r \).

**Lemma 3.5 ([4, Proposition 4.4]).** Let \( \tilde{r} \) be the rational number defined as

\[
\tilde{r} = \begin{cases}
  [m_3, \ldots, m_k] & \text{if } m_2 = 1; \\
  [m_2 - 1, m_3, \ldots, m_k] & \text{if } m_2 \geq 2.
\end{cases}
\]
Then we have

\[ T(r) = \begin{cases} S(\bar{r}) & \text{if } m_2 = 1; \\ \overline{S(\bar{r})} & \text{if } m_2 \geq 2, \end{cases} \]

where \( \overline{S(\bar{r})} \) denotes the sequence obtained from \( S(\bar{r}) \) reversing its order.

**Proposition 3.6** ([4, Proposition 4.5]). The sequence \( S(r) \) has a decomposition \((S_1, S_2, S_1, S_2)\) which satisfies the following.

1. Each \( S_i \) is symmetric, i.e., the sequence obtained from \( S_i \) by reversing the order is equal to \( S_i \). (Here, \( S_1 \) is empty if \( k = 1 \).)
2. Each \( S_i \) occurs only twice in the cyclic sequence \( CS(r) \).
3. \( S_1 \) begins and ends with \( m + 1 \).
4. \( S_2 \) begins and ends with \( m \).

**Corollary 3.7** ([4, Corollary 4.6]). The cyclic \( S \)-sequence \( CS(r) \) is symmetric, i.e., the cyclic sequence obtained from \( CS(r) \) by reversing its cyclic order is equivalent to \( CS(r) \) (as a cyclic sequence). In particular, in Lemma 3.6, we actually have

\[ CT(\bar{r}) = CS(\bar{r}). \]

**Remark 3.8.** By using the fact that \( u_r \) is obtained from of the line of slope \( r \) in \( \mathbb{R}^2 - \mathbb{Z}^2 \) by reading its intersection with the vertical lattice lines, we see that the slope \( s = q/p \) is recovered from \( CS(s) = ([S_1, S_2, S_1, S_2]) \) by the rule that \( p \) is the sum of the components of \( S_1 \) and \( S_2 \) whereas \( q \) is the sum of the lengths of \( S_1 \) and \( S_2 \).

**Lemma 3.9** ([4, Proof of Proposition 4.5]). Let \( \bar{r} \) be the rational number defined as in Lemma 3.6. Also let \( S(\bar{r}) = (T_1, T_2, T_1, T_2) \) and \( S(r) = (S_1, S_2, S_1, S_2) \) be decompositions described as in Proposition 3.6. Then the following hold.

1. If \( m_2 = 1 \) and \( k = 3 \), then \( T_1 = \emptyset, T_2 = (m_3), \) and \( S_1 = (m_3(m + 1)), S_2 = (m) \).
2. If \( m_2 = 1 \) and \( k \geq 4 \), then \( T_1 = (t_1, \ldots , t_{s_1}), T_2 = (t_{s_1 + 1}, \ldots , t_{s_2}), \) and
   \( S_1 = (t_1(m + 1), m, t_2(m + 1), \ldots , t_{s_1 - 1}(m + 1), m, t_{s_1}(m + 1)), \)
   \( S_2 = (m, t_{s_1 + 1}(m + 1), m, \ldots , m, t_{s_2}(m + 1), m). \)
3. If \( k = 2 \), then \( T_1 = \emptyset, T_2 = (m_2 - 1), \) and \( S_1 = (m + 1), S_2 = ((m_2 - 1)(m)). \)
4. If \( m_2 \geq 2 \) and \( k \geq 3 \), then \( T_1 = (t_1, \ldots , t_{s_1}), T_2 = (t_{s_1 + 1}, \ldots , t_{s_2}), \) and
   \( S_1 = (m + 1, t_{s_1 + 1}(m), m + 1, \ldots , m + 1, t_{s_2}(m), m + 1), \)
   \( S_2 = (t_1(m), m + 1, t_2(m), \ldots , t_{s_1 - 1}(m), m + 1, t_{s_1}(m)). \)
Example 3.10. (1) If \( r = [2, 1, n] \) with \( n \geq 2 \), then \( \tilde{r} = [n] \) by Lemma 3.5. So by Lemma 3.3(i), \( S(\tilde{r}) = (\emptyset, n, \emptyset, n) \). Thus by Lemma 3.9(i), \( S(r) = (n(3), 2, n(3), 2) \), where \( S_1 = (n(3)) \) and \( S_2 = (2) \).

(2) If \( r = [2, n] \) with \( n \geq 2 \), then \( \tilde{r} = [n - 1] \) by Lemma 3.5. So by Lemma 3.3(i), \( S(\tilde{r}) = (\emptyset, n - 1, \emptyset, n - 1) \). Thus by Lemma 3.9(iii), \( S(r) = (3, (n - 1)(2), 3, (n - 1)(2)) \), where \( S_1 = (3) \) and \( S_2 = ((n - 1)(2)) \).

By Lemmas 3.3 and 3.9 we can easily observe the following lemma.

Lemma 3.11. Let \( S(r) = (S_1, S_2, S_1, S_2) \) be as in Proposition 3.10. Then the following hold.

1. If \( m_2 = 1 \), then \((m + 1, m + 1)\) appears in \( S_1 \).
2. If \( m_2 \geq 2 \) and if \( m \neq m_2 = 2/(2m + 1) \), then \((m, m)\) appears in \( S_2 \).

The following is a refinement of [4, Lemma 7.3 and Remark 5].

Proposition 3.12. Let \( S(r) = (S_1, S_2, S_1, S_2) \) be as in Proposition 3.10. For a rational number \( s \) with \( 0 < s \leq 1 \), suppose that the cyclic \( S \)-sequence \( CS(s) \) contains both \( S_1 \) and \( S_2 \) as subsequences. Then \( s \notin I_1(r) \cup I_2(r) \).

In the above proposition (and throughout this paper), we mean by a subsequence a subsequence without leap. Namely a sequence \((a_1, a_2, \ldots, a_p)\) is called a subsequence of a cyclic sequence, if there is a sequence \((b_1, b_2, \ldots, b_n)\) representing the cyclic sequence such that \( p \leq n \) and \( a_i = b_i \) for \( 1 \leq i \leq p \).

Proof. Recall that \( r = [m_1, m_2, \ldots, m_k] \). Write \( s = [n_1, n_2, \ldots, n_t] \), where \( t \geq 1 \), \( (n_1, \ldots, n_t) \in (\mathbb{Z}_+)^t \) and \( n_t \geq 2 \) unless \( t = 1 \). We show that the following three conditions hold by induction on \( k \geq 1 \), refining the proof of [4, Lemma 7.3]. (As noted in [4, Remark 5], this is equivalent to the desired conclusion that \( s \notin I_1(r) \cup I_2(r) \).

(i) \( t \geq k \).

(ii) \( n_i = m_i \) for each \( i = 1, \ldots, k - 1 \).

(iii) Either \( n_k \geq m_k \) or both \( n_k = m_k - 1 \) and \( t > k \).

First let \( k = 1 \). Then \( r = [m] \) and \( S(r) = (m, m) = (S_2, S_2) \), where \( S_1 \) is empty. By hypothesis, \( CS(s) \) contains a term \( m \). So if \( t = 1 \), then \( n_1 = m \), while if \( t \geq 2 \), then \( n_1 \) is either \( m \) or \( m - 1 \). Thus the three conditions hold, proving the base step.

Now let \( k \geq 2 \). Then \( CS(r) = (S_1, S_2, S_1, S_2) \) consists of \( m \) and \( m + 1 \) by Lemma 3.3. This yields that \( CS(s) \) consists of \( m \) and \( m + 1 \). This happens only when \( t \geq 2 \) and \( n_1 = m_1 \). For the rational numbers \( r \) and \( s \), define the rational numbers \( \tilde{r} \) and \( \tilde{s} \) as in Lemma 3.5.
We consider three cases separately.

**Case 1.** \( m_2 = 1 \).

In this case, \( k \geq 3 \) and, by Lemma 3.11 \((m + 1, m + 1)\) appears in \( S_1 \) as a subsequence, so in \( CS(s) \) as a subsequence. Thus by Lemma 3.3 \( n_2 = 1 \) and so \( t \geq 3 \). So, we have

\[
\tilde{r} = [m_3, \ldots, m_k] \quad \text{and} \quad \tilde{s} = [n_3, \ldots, n_t].
\]

Let \( S(\tilde{r}) = (T_1, T_2, T_1, T_2) \) be the decomposition of \( S(\tilde{r}) \) given by Proposition 3.6.

**Case 1.a.** \( k = 3 \).

By Lemma 3.9(1), \( S_1 = (m_3\langle m + 1 \rangle) \) and \( S_2 = (m) \). Since \( S_1 \) is contained in \( CS(s) \) by assumption, \( CS(\tilde{s}) = CT(s) \) contains a component \( m_3 + d \) for some \( d \in \mathbb{Z}_+ \cup \{0\} \). If \( t = 3 \) then \( CS(\tilde{s}) = (n_3, n_3) \) and hence we have \( n_3 \geq m_3 \). If \( t \geq 4 \) then \( CS(\tilde{s}) \) consists of \( n_3 \) and \( n_3 + 1 \) and hence \( n_3 \geq m_3 - 1 \). Thus in either case, the three conditions hold, as desired.

**Case 1.b.** \( k \geq 4 \).

Since \( S_2 \) is contained in \( CS(s) \) by assumption and since \( S_2 \) begins and ends with \( m \), we see by Lemma 3.9(2) that \( CS(\tilde{s}) = CT(s) \) contains \( T_2 \). Similarly, by using the assumption that \( S_1 \) is contained in \( CS(s) \), we see that \( CS(\tilde{s}) = CT(s) \) contains a subsequence of the form

\[
(t_1 + d', t_2, \ldots, t_{s_1-1}, t_{s_1} + d''),
\]

where \((t_1, t_2, \ldots, t_{s_1-1}, t_{s_1}) = T_1 \) and \( d', d'' \in \mathbb{Z}_+ \cup \{0\} \). Since \( t_1 = t_{s_1} = m_3 + 1 \) by Proposition 3.6, this actually implies that \( CS(\tilde{s}) \) contains \( T_1 \) as a subsequence. Thus \( CS(\tilde{s}) \) contains both \( T_1 \) and \( T_2 \) as subsequences. Since \( S(\tilde{r}) = (T_1, T_2, T_1, T_2) \) is the decomposition described as in Proposition 3.6, the inductive hypothesis implies that the following three conditions hold.

(i) \( t \geq k \).

(ii) \( n_i = m_i \) for each \( i = 3, \ldots, k - 1 \).

(iii) Either \( n_k \geq m_k \) or both \( n_k = m_k - 1 \) and \( t > k \).

Since \( n_1 = m_1 \) and \( n_2 = m_2 \), this implies that the original three conditions hold, as desired.

**Case 2.** Both \( m_2 = 2 \) and \( k = 2 \).

In this case, the three conditions always hold, because if \( n_2 = 1 \) then we must have \( t \geq 3 \), otherwise \( n_2 \geq 2 = m_2 \). (Recall that we already proved that \( n_1 = m_1 \).)
Case 3. Either $m_2 \geq 3$ or both $m_2 = 2$ and $k \geq 3$.

In this case, by Lemma 3.11 $(m, m)$ appears in $S_2$ as a subsequence, so in $CS(s)$ as a subsequence. Thus $n_2 \geq 2$ by Lemma 3.3 and so we have

$$\tilde{r} = [m_2 - 1, m_3, \ldots, m_k] \quad \text{and} \quad \tilde{s} = [n_2 - 1, n_3, \ldots, n_t].$$

Let $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ be the decomposition of $S(\tilde{r})$ given by Proposition 3.6. Since $S_1$ is contained in $CS(s)$ by assumption and since $S_1$ begins and ends with $m + 1$, we see by Lemma 3.9(4) that $CS(\tilde{s}) = CT(s)$ contains $T_2$. Similarly, by using the assumption that $S_2$ is contained in $CS(s)$, we see that $CS(\tilde{s}) = CT(s)$ contains a subsequence of the form

$$(t_1 + d', t_2, \ldots, t_{s_1 - 1}, t_{s_1} + d''),$$

where $(t_1, t_2, \ldots, t_{s_1 - 1}, t_{s_1}) = T_1$ and $d', d'' \in \mathbb{Z}_+ \cup \{0\}$. Since $t_1 = t_{s_1} = (m_2 - 1) + 1 = m_2$ by Proposition 3.6, this actually implies that $CS(\tilde{s})$ contains $T_1$ as a subsequence. Thus $CS(\tilde{s})$ contains both $T_1$ and $T_2$ as subsequences. Since $S(\tilde{r}) = (T_1, T_2, T_1, T_2)$ is the decomposition described as in Proposition 3.6 the inductive hypothesis implies that the following three conditions hold.

(i) $t \geq k$.
(ii) $n_i = m_i$ for each $i = 2, \ldots, k - 1$.
(iii) Either $n_k \geq m_k$ or both $n_k = m_k - 1$ and $t > k$.

Since $n_1 = m_1$, this implies that the original three conditions hold, as desired. This completes the proof of Proposition 3.12. □

3.2. Small cancellation theory. We now recall the small cancellation conditions for the 2-bridge link groups established in [4]. Let $F(X)$ be the free group with basis $X$. A subset $R$ of $F(X)$ is said to be symmetrized, if all elements of $R$ are cyclically reduced and, for each $w \in R$, all cyclic permutations of $w$ and $w^{-1}$ also belong to $R$.

**Definition 3.13.** Suppose that $R$ is a symmetrized subset of $F(X)$. A nonempty word $b$ is called a piece if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where $p$ and $q$ are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [9]).

1. **Condition $C(p)$**: If $w \in R$ is a product of $n$ pieces, then $n \geq p$.
2. **Condition $T(q)$**: For $w_1, \ldots, w_n \in R$ with no successive elements $w_i, w_{i+1}$ an inverse pair $(i \mod n)$, if $n < q$, then at least one of the products $w_1w_2, \ldots, w_{n-1}w_n, w_nw_1$ is freely reduced without cancellation.
The following proposition enables us to apply the small cancellation theory to our problem.

**Proposition 3.14 ([4 Theorem 5.1]).** Suppose that \( r \) is a rational number with \( 0 < r < 1 \). Let \( R \) be the symmetrized subset of \( F(a, b) \) generated by the single relator \( u_r \) of the upper presentation of \( G(K(r)) \). Then \( R \) satisfies \( C(4) \) and \( T(4) \).

We recall a key fact concerning the cyclic word \((u_r)\), which is used in the proofs of the main theorems

**Definition 3.15.** For a positive integer \( n \), a non-empty subword \( w \) of the cyclic word \((u_r)\) is called a maximal \( n \)-piece if \( w \) is a product of \( n \) pieces and if any subword \( w' \) of \( u_r \) which properly contains \( w \) as an initial subword is not a product of \( n \)-pieces.

**Lemma 3.16 ([4 Corollary 5.4(2)]).** Suppose that \( r \) is a rational number such that \( 0 < r < 1 \) and \( r \neq 1/p \) for any integer \( p \geq 2 \). Let \( u_r \) be the single relator of the upper presentation of \( G(K(r)) \), and let \( S(r) = (S_1, S_2, S_1, S_2) \) be as in Proposition 3.6. Decompose

\[
\begin{align*}
 u_r &\equiv v_1v_2v_3v_4, \\
 \text{where } S(v_1) &= S(v_3) = S_1 \quad \text{and} \quad S(v_2) = S(v_4) = S_2. \\
 \text{Let } v_{ib}^* \text{ be the maximal proper initial subword of } v_i, \text{ i.e., the initial subword of } v_i \text{ such that } |v_{ib}^*| = |v_i| - 1 \quad (i = 1, 2, 3, 4). \text{ Then the following hold, where } v_{ib} \text{ and } v_{ie} \text{ are nonempty initial and terminal subwords of } v_i \text{ with } |v_{ib}|, |v_{ie}| \leq |v_i| - 1, \text{ respectively.}
\end{align*}
\]

1. The following is the list of all maximal 1-pieces of \((u_r)\), arranged in the order of the position of the initial letter:

\[
 v_{1b}^*, v_{1e}v_2, v_{2b}v_{3b}^*, v_{2e}v_{3e}^*, v_{3b}^*, v_{3e}v_4, v_{4b}v_{1b}^*, v_{4e}v_{1e}^*.
\]

2. The following is the list of all maximal 2-pieces of \((u_r)\), arranged in the order of the position of the initial letter:

\[
 v_1v_2, v_{1e}v_2v_{3b}^*, v_2v_3v_4, v_{2e}v_3v_4, v_3v_4, v_{3e}v_4v_{1b}^*, v_4v_1v_2, v_{4e}v_1v_2.
\]

**Corollary 3.17.**

1. A subword \( w \) of the cyclic word \((u_r^{\pm 1})\) is a piece if and only if \( S(w) \) does not contain \( S_1 \) as a subsequence and does not contain \( S_2 \) in its interior; i.e., \( S(w) \) does not contain a subsequence \((\ell_1, S_2, \ell_2)\) for some \( \ell_1, \ell_2 \in \mathbb{Z}_+ \).

2. For a subword \( w \) of the cyclic word \((u_r^{\pm 1})\), if \( S(w) \) either contains \((S_1, S_2)\) as a proper initial subsequence or contains \((S_2, S_1)\) as a proper terminal subsequence, then \( w \) is not a product of two pieces.
Proof. (1) We prove the assertion for a subword of the cyclic word $(u_r)$. (The assertion for a subword of the cyclic word $(u_r^{-1})$ follows from this and the facts that $S_1$ and $S_2$ are symmetric and that $w$ is a piece if and only if $w^{-1}$ is a piece.) To show the only if part, let $w$ be a piece which is a subword of $(u_r)$. Suppose on the contrary that $S(w)$ contains $r$ or $(r, S_2, \ell) \ (\ell, \ell_2 \in \mathbb{Z}_+)$ as a subsequence. Let $w'$ be a subword of $w$ corresponding to the subsequence. Then, since each of $S_1$ and $S_2$ appears only twice in the cyclic sequence $CS(r)$ by Proposition 3.6(2), we have the following.

(i) If $S(w') = S_1$, then $w' = v_1$ or $v_3$.
(ii) If $S(w') = (\ell_1, S_2, \ell_2)$, then $w' = v_1v_2v_3b$ or $v_3v_4v_1b$.

In either case, $w'$ cannot be a subword of any of the maximal 1-pieces of $(u_r)$ listed in Lemma 3.16(1). Hence, $w$ is not a piece, a contradiction. To see the if part, let $w$ be a subword of $(u_r)$ whose $S$-sequence does not contain $S_1$ nor $(\ell_1, S_2, \ell_2) \ (\ell, \ell_2 \in \mathbb{Z}_+)$ as a subsequence. Then we see by using Proposition 3.6(2) that $w$ does not contain $v_1$, $v_3$, $v_1v_2v_3b$, nor $v_3v_4v_1e$ as a subword. Since $w$ is a subword of $(u_r)$, this implies that $w$ is a subword of one of the maximal 1-pieces of $(u_r)$ listed in Lemma 3.16(1). Hence $w$ is a piece.

(2) As in (1), we prove the assertion for a subword of the cyclic word $(u_r)$. Suppose that $w$ is a subword of $(u_r)$ such that $S(w)$ contains either $(S_1, S_2, \ell)$ or $(\ell, S_2, S_1)$ with $\ell \in \mathbb{Z}_+$. Let $w'$ be a subword of $w$ corresponding to the subsequence. Then, since $S_1$ and $S_2$ appears only twice in the cyclic sequence $CS(r)$ by Proposition 3.6(2), we have the following.

(i) If $S(w') = (S_1, S_2, \ell)$, then $w' = v_1v_2v_3b$ or $v_3v_4v_1b$.
(ii) If $S(w') = (\ell, S_2, S_1)$, then $w' = v_1v_2v_3b$ or $v_3v_4v_1e$.

In either case, $w'$ cannot be a subword of any of the maximal 2-pieces of $(u_r)$ listed in Lemma 3.16(2). Hence, $w$ is not a product of two pieces, as desired.

We recall the following well-known classical result in combinatorial group theory.

Lemma 3.18 (Lemmas V.5.1 and V.5.2). Suppose $G = \langle X \mid R \rangle$ with $R$ being symmetrized. Let $u, v$ be two cyclically reduced words in $X$ which are not trivial in $G$ and which are not conjugate in $F(X)$. Then $u$ and $v$ represent conjugate elements in $G$ if and only if there exists a reduced nontrivial annular $R$-diagram $M$ such that $u$ is an outer boundary label and $v^{-1}$ is an inner boundary label of $M$.

Let us quickly recall the terminologies in the above lemma. An annular map $M$ is a connected finite 2-dimensional cell complex embedded in $\mathbb{R}^2$, such that
\( \mathbb{R}^2 - M \) has exactly two connected components. It is said to be nontrivial if it contains 2-cells. An annular R-diagram is an annular map \( M \) and a function \( \phi \) assigning to each oriented edge \( e \) of \( M \), as a label, a reduced word \( \phi(e) \) in \( X \) such that the following hold.

(i) If \( e \) is an oriented edge of \( M \) and \( e^{-1} \) is the oppositely oriented edge, then \( \phi(e^{-1}) = \phi(e)^{-1} \).

(ii) For any boundary cycle \( \delta \) of any face of \( M \), \( \phi(\delta) \) is a cyclically reduced word representing an element of \( R \). (Here a boundary cycle \( \delta \) is a closed edge path \( e_1, \ldots, e_n \) in \( M \) of minimal length which includes all the edges of the boundary, \( \partial D \), of \( D \). We define \( \phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_n) \).

An outer boundary label of an annular \( R \)-diagram \( M \) is defined to be the word \( \phi(\alpha) \), where \( \alpha \) is an outer boundary cycle of \( M \), namely a cycle of minimal length going around once along the boundary of the unbounded component of \( \mathbb{R}^2 - M \). An inner boundary label and an inner boundary cycle are similarly defined from the bounded component of \( \mathbb{R}^2 - M \). Unlike the convention in [9, p.253], we assume that the outer boundary cycles are clockwise and inner boundary cycles are counterclockwise (cf. [5, Convention 4.6]). For the definition of an (annular) \( R \)-diagram to be reduced, see [9, p.241] (cf. [4, Section 6]).

It is easy to observe that we may assume the following convention (see [4, Convention 1]).

**Convention 3.19.** Let \( R \) be the symmetrized subset of \( F(a, b) \) generated by the single relator \( u_r \) of the upper presentation of \( G(K(r)) \). For any \( R \)-diagram \( M \), we assume that \( M \) satisfies the following.

1. \( d_M(v) \geq 3 \) for every vertex \( v \in M - \partial M \).
2. For every edge \( e \) of \( \partial M \), the label \( \phi(e) \) is a piece.
3. For a path \( e_1, \ldots, e_n \) in \( \partial M \) of length \( n \geq 2 \) such that the vertex \( e_i \cap e_{i+1} \) has degree 2 for \( i = 1, 2, \ldots, n - 1 \), \( \phi(e_1)\phi(e_2)\cdots\phi(e_n) \) cannot be expressed as a product of less than \( n \) pieces.

The following theorem giving a geometric description of the annular diagrams forms a foundation of the whole papers in this series.

**Theorem 3.20** ([5, Theorem 4.9 and Corollary 4.11]). Suppose that \( r \) is a rational number with \( 0 < r < 1 \). Let \( R \) be the symmetrized subset of \( F(a, b) \) generated by the single relator \( u_r \) of the upper presentation of \( G(K(r)) \), and let \( S(r) = (S_1, S_2, S_1, S_2) \) be as in Proposition 3.6. Suppose that \( M \) is a nontrivial reduced annular \( R \)-diagram such that

(i) the words \( \phi(\alpha) \) and \( \phi(\delta) \) are cyclically reduced;
(ii) the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically alternating;
(iii) the cyclic $S$-sequences of the cyclic words $(\phi(\alpha))$ and $(\phi(\delta))$ do not contain $(S_1, S_2)$ nor $(S_2, S_1)$ as a subsequence,

where $\alpha$ and $\delta$ are, respectively, arbitrary outer and inner boundary cycles of $M$. Let the outer and inner boundaries of $M$ be denoted by $\sigma$ and $\tau$, respectively. Then the following hold.

(1) The outer and inner boundaries $\sigma$ and $\tau$ are simple, i.e., they are homeomorphic to the circle, and there is no edge contained in $\sigma \cap \tau$.
(2) $d_M(v) = 2$ or $4$ for every vertex $v \in \partial M$. Moreover, on both $\sigma$ and $\tau$, vertices of degree $2$ appear alternately with vertices of degree $4$.
(3) $d_M(v) = 4$ for every vertex $v \in M - \partial M$.
(4) $d_M(D) = 4$ for every face $D \in M$.

In particular, Figure 2(a) illustrates the only possible type of the outer boundary layer of $M$, while Figure 2(b) illustrates the only possible type of whole $M$. (The number of faces per layer and the number of layers are variable.)

![Figure 2](image)

4. TECHNICAL LEMMAS

In this section, we set up Hypotheses A, B and C, under which we establish technical lemmas used for the proofs in Sections 5–7.

4.1. Hypothesis A.
Hypothesis A. Let $r$ be a rational number such that $0 < r < 1$ and $r \neq 1/p$ for any integer $p \geq 2$. For two distinct elements $s, s' \in I_1(r) \cup I_2(r)$, suppose that the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$. Then $u_s$ and $u_{s'}^{\pm 1}$ are conjugate in $G(K(r))$. Let $R$ be the symmetrized subset of $F(a, b)$ generated by the single relator $u_s$ of the upper presentation of $G(K(r))$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be the decomposition as in Proposition 3.12. Due to Lemma 3.18 there is a reduced annular $R$-diagram $M$ such that $u_s$ and $u_{s'}^{\pm 1}$ are, respectively, outer and inner boundary labels of $M$. Then we see from Proposition 3.12 that $M$ satisfies the three hypotheses (i), (ii) and (iii) of Theorem 3.20.

Let $J$ be the outer boundary layer of $M$ (see Figure 2(a)). Also let $\alpha$ and $\delta$ be, respectively, the outer and inner boundary cycles of $J$ starting from $v_0$, where $v_0$ is a vertex lying in both the outer and inner boundaries of $J$. Here, recall from [5, Convention 4.6] that $\alpha$ is read clockwise and $\delta$ is read counterclockwise. Let $\alpha = e_1, e_2, \ldots, e_{2t}$ and $\delta^{-1} = e'_1, e'_2, \ldots, e'_2$ be the decompositions into oriented edges in $\partial J$. Then clearly for each $i = 1, \ldots, t$, there is a face $D_i$ of $J$ such that $e_{2i-1}, e_{2i}, e'_{2i}, e'_{2i-1}$ are consecutive edges in a boundary cycle of $D_i$. We denote the path $e_{2i-1}, e_{2i}$ by $\partial D_i^+$ and the path $e'_{2i-1}, e'_{2i}$ by $\partial D_i^-$. In particular, if $J \subseteq M$ (see Figure 2(b)), then, for each $i = 1, \ldots, t$, there is a face $D'_i$ in $M - J$ such that $e'_{2i}$ and $e'_{2i+1}$ are two consecutive edges in $\partial D'_i \cap \delta^{-1}$. Here the indices for the 2-cells are considered modulo $t$, and the indices for the edges are considered modulo $2t$.

Lemma 4.1. Under Hypothesis A, both of the following hold for every $i$.

1. None of $S(\phi(e_{2i-1}))$, $S(\phi(e_{2i}))$, $S(\phi(e'_{2i}))$ and $S(\phi(e'_{2i-1}))$ contains $S_1$ as a subsequence.

2. None of $S(\phi(e_{2i-1}))$, $S(\phi(e_{2i}))$, $S(\phi(e'_{2i}))$ and $S(\phi(e'_{2i-1}))$ contains a subsequence of the form $(\ell, S_2, \ell')$, where $\ell, \ell' \in \mathbb{Z}^+$. 

Proof. By Convention 3.19 each of the words $\phi(e_{2i-1})$, $\phi(e_{2i})$, $\phi(e'_{2i})$ and $\phi(e'_{2i-1})$ is a piece of the cyclic word $(u_s^{\pm 1})$. So, the assertion follows from (the only if part) of Corollary 3.17(1).

Lemma 4.2. Under Hypothesis A, only one of the following holds for each face $D_i$ of $J$.

1. Both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ contain $S_1$ as their subsequence.

2. Both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ contain subsequences of the form $(\ell, S_2, \ell')$, where $\ell, \ell' \in \mathbb{Z}^+$. 

Proof. By Convention 3.19, each of the words $\phi(\partial D_i^+)$ and $\phi(\partial D_i^-)$ is not a piece. So, by (the if part of) Corollary 3.17(1), each $S(\phi(\partial D_i^+))$ contains
S_1 or (ℓ, S_2, ℓ') (ℓ, ℓ' ∈ Z_+) as a subsequence. On the other hand, since
CS(ϕ(∂D_i^+)ϕ(∂D_i^-)^{-1}) = CS(ϕ(∂D_i)) is equal to CS(r) = (S_1, S_2, S_1, S_2),
if S(ϕ(∂D_i^+)) contains S_1 (resp. (ℓ, S_2, ℓ')) then S(ϕ(∂D_i^-)) cannot contain
(ℓ, S_2, ℓ') (resp. S_1). Hence we obtain the desired result. □

Lemma 4.3. Under Hypothesis A, only one of the following holds.

(1) For every face D_i of J, S(ϕ(∂D_i^+)) contains S_1 as its subsequence.
(2) For every face D_i of J, S(ϕ(∂D_i^+)) contains a subsequence of the form
(ℓ, S_2, ℓ'), where ℓ, ℓ' ∈ Z_+.

Proof. Suppose on the contrary that (1) holds for j and (2) holds for j' ≠ j.
Then CS(ϕ(α)) = CS(u_s) = CS(s) contains both S_1 and S_2 as subsequences.
By Proposition 3.12, s ≈ I_1(r) ∪ I_2(r), contradicting the hypothesis of the
lemma. □

4.2. Hypothesis B. Assuming the following Hypothesis B, we will estab-
lish several technical lemmas concerning important properties of CS(ϕ(α)) =
CS(u_s) = CS(s).

Hypothesis B. Suppose under Hypothesis A that Lemma 4.3(1) holds, namely,
for every face D_i of J, suppose that S(ϕ(∂D_i^+)) contains S_1 as its subsequence.
Then we can decompose the word ϕ(α) (clearly (u_s) ≡ (ϕ(α))) into
ϕ(α) ≡ y_1w_1z_1y_2w_2z_2⋯y_tw_tz_t,
where ϕ(∂D_i^+) ≡ ϕ(c_{i_1}⋯c_{i_2}) ≡ y_1w_1z_1, y_i and z_i may be empty, S(w_i) = S_1,
and where S(y_iw_i z_i) = (S(y_i), S_1, S(z_i)) (here S(y_i) and S(z_i) are possibly
empty), for every i. By Lemma 3.12, we also have the decomposition of the
word ϕ(δ^{-1}) as follows (clearly (u_s^{±1}) (ϕ(δ^{-1})) if J = M):
ϕ(δ^{-1}) ≡ y'_1w'_1z'_1y'_2w'_2z'_2⋯y'_tw'_tz'_t,
where ϕ(∂D_i^-) ≡ ϕ(c_{i_1}⋯c_{i_2}) ≡ y'_1w'_1z'_1, y'_i and z'_i may be empty, S(w'_i) = S_1,
and where S(y'_iw'_ iz'_i) = (S(y'_i), S_1, S(z'_i)) (here S(y'_i) and S(z'_i) are possibly
empty), for every i. Then S(y'_i^{i_1}y_i) = S(z_i z'_i^{i_1}) = S_2 for every i.

Notation 4.4. Let v be a reduced word in {a, b}. If v is not an empty word,
then, by v_b and v_e, respectively, we denote a beginning subword and an ending
subword of v such that |v_b| is the first term of the sequence S(v) and |v_e| is
the last term of S(v). On the other hand, if v is empty, then v_b and v_e are
also empty words. (Though similar symbols, v_b and v_e (1 ≤ i ≤ 4), are
used in different meanings in Lemma 3.16, we believe this does not cause any
confusion, because these symbols are not used in the remainder of this paper.)
Remark 4.5. (1) If \( r = [2, 2] = 2/5 \), then, by Example 3.10(2), \( CS(r) = ((3, 2, 3, 2)) \), where \( S_1 = (3) \) and \( S_2 = (2) \). So, in Hypothesis B, both \( S(\phi(\partial D_i^+)) \) and \( S(\phi(\partial D_i^-)) \) are exactly of the form \((\ell, 3, \ell')\), where \( 0 \leq \ell, \ell' \leq 2 \) are integers.

(2) If \( r = [2, n] \) with \( n \geq 3 \), then, again by Example 3.10(2), \( CS(r) = ((3, (n-1)2, 3, (n-1)2)) \), where \( S_1 = (3) \) and \( S_2 = ((n-1)2) \). So, in Hypothesis B, both \( S(\phi(\partial D_i^+)) \) and \( S(\phi(\partial D_i^-)) \) are exactly of the form \((\ell_1, n_1(2), 3, n_2(2), \ell_2)\), where \( 0 \leq \ell_1, \ell_2 \leq 1 \) and \( 0 \leq n_1, n_2 \leq n-1 \) are integers such that if \( n_j = n-1 \) then \( \ell_j \) is necessarily 0 for \( j = 1, 2 \). In particular, \( S(y_{i,b}) = (1) \) or \( (2) \) unless \( y_i \) is an empty word. The same is true for \( S(z_{i,e}) \), \( S(y'_{i,b}) \) and \( S(z'_{i,e}) \).

(3) If \( r = [2, 1, n] \) with \( n \geq 2 \), then, by Example 3.10(1), \( CS(r) = ((n(3), 2, n(3), 2)) \), where \( S_1 = (n(3)) \) and \( S_2 = (2) \). So, in Hypothesis B, both \( S(\phi(\partial D_i^+)) \) and \( S(\phi(\partial D_i^-)) \) are exactly of the form \((\ell, n(3), \ell')\), where \( 0 \leq \ell, \ell' \leq 2 \) are integers. In particular, \( S(w_{i,b}) = S(w_{i,e}) = (3) \) and \( S(w'_{i,b}) = S(w'_{i,e}) = (3) \).

Lemma 4.6. Let \( r = n/(2n+1) = [2, n] \), where \( n \geq 2 \) is an integer. Under Hypothesis B, suppose that \( v \) is a subword of the cyclic word represented by \( \phi(\alpha) \equiv y_1w_1z_1y_2w_2z_2\cdots y_kw_kz_k \) such that \( v \) corresponds to a component of \( CS(\phi(\alpha)) = CS(s) \). Then, after a cyclic shift of indices, \( v \) is equal to one of the following subwords:

\[
z_{0,e}w_1w_2\cdots w_qy_{q+1,b}, \quad z_{0,e}w_1w_2\cdots w_q, \quad w_1w_2\cdots w_qy_{q+1,b}, \quad w_1w_2\cdots w_q,
\]

where \( q \in \mathbb{Z}_+ \cup \{0\} \) in the first three cases and \( q \in \mathbb{Z}_+ \) in the last case. In each of the above, the “intermediate subwords” are empty; to be precise, when we say that \( z_{0,e}w_1w_2\cdots w_qy_{q+1,b} \), for example, is a subword of \((u_s)\), we assume that \( y_1, z_iy_{i+1} (1 \leq i \leq q-1) \) and \( z_q \) are empty words.

Proof. Recall from Hypothesis B that \( S(y_iw_{i,z_i}) = (S(y_i), S_1, S(z_i)) \), where \( S(y_i) \) and \( S(z_i) \) are possibly empty and \( S_1 = (3) \). In other words, if \( y_i \) (resp. \( z_i \)) is not an empty word, then there is a sign change between \( y_i \) and \( w_i \) (resp. between \( w_i \) and \( z_i \)). The desired result follows from this observation. \( \square \)

Throughout the remainder of this paper, we will assume the following convention.

Convention 4.7. In Figures 3–25, the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words, and a dot represents a vertex whose position is clearly identified. Also an Arabic number represents the length of the
corresponding positive (or negative) word. In Figures 3–14, the upper complementary region is regarded as the unbounded region of \( \mathbb{R}^2 - M \). Thus the outer boundary cycles runs the upper boundary from left to right.

The following Lemmas 4.8 and 4.11 show that there are strong restrictions for the shape of the word \( v \) in Lemma 4.6.

**Lemma 4.8.** Let \( r = 2/5 = [2, 2] \). Under Hypothesis B, the following hold for every \( i \).

1. \( S(z_1 y_{i+1}) \neq (3) \).
2. \( S(w_i z_i y_{i+1}) \neq (4) \) and \( S(z_i y_{i+1} w_{i+1}) \neq (4) \).
3. If \( J \subseteq M \), then \( S(w_i z_i y_{i+1} w_{i+1}) \neq (6) \).
4. \( S(z_i-1 y_{i+1} w_{i+1}) \neq (7) \).
5. If \( J = M \), then \( S(z_i-1 y w_i z_i y_{i+1} w_{i+1}) \neq (8) \) and \( S(w_i-1 z_i-1 y w_i z_i y_{i+1}) \neq (8) \).
6. \( S(w_i z_i y_{i+1} w_{i+1}) \neq (3, 3) \).

**Proof.** (1) Suppose on the contrary that \( S(z_1 y_{i+1}) = (3) \) for some \( i \). Without loss of generality, we may assume \( S(z_1 y_2) = (3) \). Then, since \( 0 \leq |z_1|, |y_2| \leq 2 \), we have either \((|z_1|, |y_2|) = (1,2)\) or \((|z_1|, |y_2|) = (2,1)\). We now assume \((|z_1|, |y_2|) = (2,1)\). (The other case can be treated similarly.) Then by using the fact that \( CS(\phi(\partial D_i)) = CS(2/5) = ([3, 2, 3, 2]) \), we see that \( J \) is locally as illustrated in Figure 3(a) which follows Convention 3.4.7. The Arabic numbers 3, 2, 1 and 3 near the upper boundary represent the lengths of the words \( w_1, z_1, y_2 \) and \( w_2 \), respectively, whereas the Arabic numbers 3, 1 and 3 near the lower boundary represent the lengths of the words \( w_1', y_2' \) and \( w_2' \) respectively (in particular \(|z_1'| = 0\)), and the change of directions of consecutive arrowheads represents the change from positive (negative, resp.) words to negative (positive, resp.) words.

Suppose first that \( J = M \). Then we see from Figure 3(a) that \( CS(\phi(\delta^{-1})) = CS(u_{n+1}^\pm) = CS(s') \) involves both a term 1 and a term of the form \( 3 + c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \), contradicting Lemma 3.3 which says that either \( CS(s') \) is equal to \(([m, m]) \) or \( CS(s') \) consists of \( m \) and \( m+1 \) for some \( m \in \mathbb{Z}_+ \). Suppose next that \( J \subseteq M \). Then by Lemma 4.4(1), none of \( S(\phi(e_1')) \) and \( S(\phi(e_2')) \) contains \( S_1 = (3) \) as a subsequence. This means that the initial vertex of \( e_2' \) lies in the interior of the segment of \( \partial D_1^- \) with weight 3. (See Figure 3(b), where the initial vertex of \( e_2' \) is the left-most vertex.) Similarly, the terminal vertex of \( e_3' \) lies in the interior of the segment of \( \partial D_3^- \) with weight 3; in particular, it does not lie in the segment of \( \partial D_2^- \) with weight 1. Hence, we see from Figure 3(b) that \( S(\phi(e_2'e_3')) \) is of the form \((\ell_1, 1, \ell_2)\) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \). This yields that a
term 1 occurs in \( CS(\phi(\partial D_1')) = CS(2/5) = ((3, 2, 3, 2)) \), which is obviously a contradiction.

\[
\begin{array}{c}
\text{(a)} \\
\begin{array}{c}
D_1 \\
3 \\
\hline
3 \\
\end{array} & \begin{array}{c}
D_2 \\
1 \\
\hline
3 \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
D_1' \\
1 \\
\hline
3 \\
\end{array} & \begin{array}{c}
D_2 \\
3 \\
\hline
3 \\
\end{array} \\
\end{array}
\]

**Figure 3.** Lemma 4.8(1) where \( S(z_1y_2) = (2 + 1) \).

(2) Suppose on the contrary that \( S(w_1z_1y_2) = (4) \). (The other case is treated similarly.) Then, since \( w_1 \) and \( z_1 \) have different signs when \( z_1 \) is non-empty and since \( |w_1| = 3 \) and \( 0 \leq |z_1|, |y_2| \leq 2 \), the only possibility is that \( |z_1| = 0 \) and \( S(w_1y_2) = (4) \). If \( J = M \), then we see from Figure 4(a) that \( CS(s') \) involves both a term 1 and a term of the form \( 3 + c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \), contradicting Lemma 3.3. On the other hand, if \( J \subsetneq M \), then we see, by using Lemma 4.1(1) as in the proof of Lemma 4.8(1), that \( S(\phi(e_2'e_3')) \) is of the form \( (\ell_1, 2, 1, \ell_2) \) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \) (see Figure 4(b)). This implies that \( CS(\phi(\partial D_1')) = CS(2/5) \) has a term 1, a contradiction.

\[
\begin{array}{c}
\text{(a)} \\
\begin{array}{c}
D_1 \\
3 \\
\hline
3 \\
\end{array} & \begin{array}{c}
D_2 \\
1 \\
\hline
3 \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
D_1' \\
1 \\
\hline
3 \\
\end{array} & \begin{array}{c}
D_2 \\
3 \\
\hline
3 \\
\end{array} \\
\end{array}
\]

**Figure 4.** Lemma 4.8(2) where \( S(w_1z_1y_2) = (3 + 0 + 1) \).

(3) Suppose \( J \subsetneq M \) and suppose on the contrary that \( S(w_0z_0y_1w_1) = (6) \). Then \( |z_0| = |y_1| = 0 \) and we see, by using Lemma 4.1(1) as in the proof of Lemma 4.8(1), that \( S(\phi(e_3'e_3')) \) is of the form \( (\ell_1, 4, \ell_2) \) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \), as illustrated in Figure 5. This implies that \( CS(\phi(\partial D_1')) = CS(2/5) \) contains a term 4, a contradiction.
(4) Suppose on the contrary that \( S(z_0 y_1 w_1 z_1 y_2) = (7) \). Then we have \( |z_0| = |y_2| = 2 \) and \( |y_1| = |z_1| = 0 \). If \( J = M \), then we see from Figure 5(a) that \( CS(s') \) includes both a term 3 and a term of the form \( 5 + c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \), contradicting Lemma 3.3. So, we must have \( J \not\subseteq M \). Note that \( S(\phi(\partial D_1^\ominus)) = (2, 3, 2) \) and that the terminal vertex of \( e_1' \) lies in the interior of the segment of \( \partial D_1^\ominus \) with weight 3, by Lemma 4.1(1). We may assume the vertex divides the segment into segments with weights 1 and 2 as in Figure 6(b). (The other case where the weights of 1 and 2 are interchanged is treated similarly.) Then, as illustrated in Figure 6(b), \( CS(\phi(\delta_1^{-1})) \) includes a subsequence \( (3, 1, 3) \), where \( \delta_1 \) is an inner boundary cycle of the outer boundary layer, say \( J_1 \), of \( M - J \). If \( M = J \cup J_1 \), then \( CS(\phi(\delta_1^{-1})) = CS(s') \) contains both a term 1 and a term 3, contradicting Lemma 3.3. On the other hand, if \( M \supseteq J \cup J_1 \), then, arguing as in Figure 6(b), we obtain a contradiction.
(5) Suppose on the contrary that \( J = M \) and \( S(z_0y_1w_1z_1y_2w_2) = (8) \). (The other case is treated similarly.) Then \(|z_0| = 2\), \(|y_1| = |z_1| = |y_2| = 0\), and we see from Figure 7 that \( CS(s') \) includes both a term 3 and a term of the form \( 5 + c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \), contradicting Lemma 3.3.

![Figure 7](image)

**Figure 7.** Lemma 4.8(5) where \( S(z_0y_1w_1z_1y_2w_2) = (2 + 0 + 3 + 0 + 0 + 3) \).

(6) Suppose on the contrary that \( S(w_1z_1y_2w_2) = (3, 3) \). Then \(|z_1| = |y_2| = 0\).

If \( J = M \) (see Figure 8(a)), then \( CS(s') \) contains both a term 2 and a term of the form \( 3 + c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \). Here, if \( c = 0 \), then \( s' \notin I_1(2/5) \cup I_2(2/5) \) by Proposition 3.12 contradicting the hypothesis of the theorem, while if \( c > 0 \), then we have a contradiction to Lemma 3.3. On the other hand, if \( J \nsubseteq M \) (see Figure 8(b)), then we see, by using Lemma 4.1(1) as in the proof of Lemma 4.8(1), that \( S(\phi(e'_1e'_3)) \) is of the form \((\ell_1, 2, 2, \ell_2)\) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \). This implies that a subsequence \((2, 2)\) occurs in \( CS(\phi(\partial D'_1)) = CS(2/5) \), which is a contradiction.

![Figure 8](image)

**Figure 8.** Lemma 4.8(6) where \( S(w_1z_1y_2w_2) = (3, 3) \).

**Lemma 4.9.** Let \( r = 2/5 = [2, 2] \). Under Hypothesis B, the following hold, where \( d \in \mathbb{Z}_+ \cup \{0\} \).

1. No two consecutive terms of \( CS(s) \) can be \((3, 3)\).
(2) No two consecutive terms of $CS(s)$ can be $(4, 4)$.

(3a) No two consecutive terms of $CS(s)$ can be of the form $(6 + 3d, 6 + 3d)$.
(3b) If $J \subset M$, then no term of $CS(s)$ can be of the form $6 + d$.
(4) No term of $CS(s)$ can be of the form $7 + 3d$.
(5) No term of $CS(s)$ can be of the form $8 + 3d$.

Proof. (1) Suppose on the contrary that $CS(\phi(\alpha)) = CS(s)$ contains $(3, 3)$ as a subsequence. Let $v = v'v''$ be a subword of the cyclic word $(u_s)$ corresponding to a subsequence $(3, 3)$, where $S(v') = S(v'') = (3)$. By using Lemma 4.6 and the facts that $0 \leq |z_i|, |y_i| \leq 2$ and $|w_i| = 3$, we see that one of the following holds after a shift of indices.

(i) $(v', v'') = (z_1y_2, w_2)$, where $S(z_1y_2) = (3)$.
(ii) $(v', v'') = (w_1, z_1y_2)$, where $S(z_1y_2) = (3)$.
(iii) $(v', v'') = (w_1, w_2)$.

However, (i) and (ii) are impossible by Lemma 4.8(1), and (iii) is impossible by Lemma 4.8(6).

(2) Suppose on the contrary that $CS(s)$ contains $(4, 4)$ as a subsequence. Let $v = v'v''$ be a subword of the cyclic word $(u_s)$ corresponding to a subsequence $(4, 4)$, where $S(v') = S(v'') = (4)$. If $v'$ contains $w_i$ for some $i$, then we see, by using Lemma 4.6 and the identity $|w_i| = 3$, that either $v' = w_iz_1y_{i+1}$ with $|(z_i, y_{i+1})| = (0, 1)$ or $v' = z_{i-1}y_iw_i$ with $|(z_{i-1}, y_i)| = (1, 0)$. However both cases are impossible by Lemma 4.8(2). Thus $v'$ cannot contain $w_i$. Since $S(v') = (4)$ is a component of $CS(s)$, this implies that $v'$ is disjoint from $w_i$ for every $i$. The same conclusion also holds for $v''$, and hence for $v = v'v''$. Thus $v$ is a subword of $z_{i}y_{i+1}$ for some $i$. This is a contradiction, because $|v| = 8$ whereas $|z_{i}y_{i+1}| \leq 4$.

(3a) Suppose on the contrary that $CS(s)$ contains $(6 + 3d, 6 + 3d)$ as a subsequence. Let $v = v'v''$ be a subword of the cyclic word $(u_s)$ corresponding to a subsequence $(6 + 3d, 6 + 3d)$, where $S(v') = S(v'') = (6 + 3d)$. By using Lemma 4.8 and the facts that $0 \leq |y_i|, |z_i| \leq 2$ and $|w_i| = 3$, we see that one of the following holds after a cyclic shift of indices.

(i) $v' = w_{1}w_{2} \cdots w_{q}$ with $q = d + 2$.
(ii) $v' = z_{0}w_{1}w_{2} \cdots w_{q}y_{q+1}$ with $q = d + 1$, where $(|z_0|, |y_{q+1}|) = (1, 2)$ or $(2, 1)$.

If (ii) holds, then either $S(z_{0}y_{1}w_{1}) = (4)$ or $S(w_{q}z_{q}y_{q+1}) = (4)$, a contradiction to Lemma 4.8(2). So (i) holds. By applying the same argument to $v''$ and by using the fact that $v'v''$ is a subword of $(u_s)$, we see that $v'' = w_{q+1}w_{q+2} \cdots w_{2q}$,
where $z_q y_{q+1}$ is empty. Hence we see $S(w_q z_q y_{q+1} w_{q+1}) = S(w_q w_{q+1}) = (3, 3)$. This contradicts Lemma 4.8(6).

(3b) Suppose $J \subsetneq M$ and suppose on the contrary that $CS(s)$ contains a term $6 + d$. First suppose that $CS(s)$ contains a term $6$. Let $v$ be a subword of the cyclic word $(u_s)$ corresponding to a term $6$. By arguing as in the proof of Lemma 4.9(3a), we see that one of the following holds.

(i) $v = w_1 w_2$.
(ii) $v = z_0 w_1 y_2$, where $(|z_0|, |y_{q+1}|) = (1, 2)$ or $(2, 1)$.

However, (i) is impossible by Lemma 4.8(3), and (ii) is impossible by Lemma 4.8(2), as in the proof of Lemma 4.9(3a).

Next suppose that $CS(s)$ involves a term $7$. Let $v$ be a subword of the cyclic word $(u_s)$ corresponding to a term $7$. Arguing as in the proof of Lemma 4.9(3a), we may assume that one of the following holds.

(i) $v = z_0 w_1 w_2$ with $|z_0| = 1$.
(ii) $v = w_1 w_2 y_3$ with $|y_3| = 1$.
(iii) $v = z_0 w_1 y_2$, where $|z_0| = |y_2| = 2$.

However, (i) and (ii) are impossible by Lemma 4.8(2), and (iii) is impossible by Lemma 4.8(4).

Finally suppose that $CS(s)$ contains a term of the form $8 + d$. Let $v$ be a subword of the cyclic word $(u_s)$ corresponding to a term $8 + d$. By using Lemma 4.6 and the facts that $0 \leq |y_i|, |z_i| \leq 2$ and $|w_i| = 3$, we see that $v$ must contain a subword $w_i w_{i+1}$ for some $i$. This contradicts Lemma 4.8(3).

(4) By Lemma 4.9(3b), it remains to prove the assertion for $J = M$. Suppose $J = M$ and suppose on the contrary that $CS(s)$ contains a term of the form $7 + 3d$. Let $v$ be a subword of the cyclic word $(u_s)$ corresponding to a term $7 + 3d$. Arguing as in the proof of Lemma 4.9(3a), we may assume that one of the following holds.

(i) $v = z_0 w_1 w_2 \cdots w_q$ where $|z_0| = 1$ and $q = 2 + d$.
(ii) $v = w_1 w_2 \cdots w_q y_{q+1}$ where $|y_{q+1}| = 1$ and $q = 2 + d$.
(iii) $v = z_0 w_1 w_2 \cdots w_q y_{q+1}$, where $|z_0| = |y_{q+1}| = 2$ and $q = 1 + d$.

However, (i) and (ii) are impossible by Lemma 4.8(2), and (iii) is impossible by Lemma 4.8(4) and (5).

(5) By Lemma 4.9(3b), it remains to prove the assertion for $J = M$. Suppose $J = M$ and suppose on the contrary that $CS(s)$ contains a term of the form $8 + 3d$. Let $v$ be a subword of the cyclic word $(u_s)$ corresponding to a term $8 + 3d$. Arguing as in the proof of Lemma 4.9(3a), we may assume that one of the following holds.
(i) $v = z_0 w_1 w_2 \cdots w_q$ where $|z_0| = 2$ and $q = 2 + d$.
(ii) $v = w_1 w_2 \cdots w_q y_{q+1}$ where $|y_{q+1}| = 2$ and $q = 2 + d$.
(iii) $v = z_0 w_1 w_2 \cdots w_q y_{q+1}$, where $|z_0| = |y_{q+1}| = 1$ and $q = 2 + d$.

However, (i) and (ii) are impossible by Lemma 4.8(5), and (iii) is impossible by Lemma 4.8(2).

**Corollary 4.10.** Let $r = 2/5 = [2, 2]$. Under Hypothesis B, $CS(s)$ satisfies one of the following conditions.

1. $CS(s) = ([5, 5])$.
2. $CS(s)$ has the form consisting of $m$ and $m+1$, where $m$ is one of $2, 3, 4$ and $5$.

**Proof.** By Lemma 3.3, either $CS(s) = ([m, m])$ or $CS(s)$ consists of $m$ and $m + 1$ with $m \in \mathbb{Z}_+$. By Hypothesis B together with Remark 4.5(1), $\phi(\alpha)$ involves a subword $w_i$ whose $S$-sequence is (3), so $CS(\phi(\alpha)) = CS(u_s) = CS(s)$ must contain a term of the form $3 + c$, where $c \in \mathbb{Z}_+ \cup \{0\}$. If $CS(s) = ([m, m])$, then $m \geq 3$ by this observation. Moreover, by Lemma 4.9, $m$ is not equal to $3, 4, 6 + 3d, 7 + 3d$ nor $8 + 3d$ for any $d \in \mathbb{Z}_+ \cup \{0\}$. Hence $m = 5$. On the other hand, if $m$ consists of $m$ and $m + 1$, then by Lemma 4.9, none of $m$ and $m + 1$ is equal to $7 + 3d$ nor $8 + 3d$ for any $d \in \mathbb{Z}_+ \cup \{0\}$. Thus $m$ is less than 5. Since $CS(s)$ involves a term $3 + c$, we have $m + 1 \geq 3$. Hence we see $2 \leq m \leq 5$. □

Next, we study the case where $r = n/(2n + 1) = [2, n]$ with $n \geq 3$. Recall from Remark 4.5(2) that $CS(r) = ([3, (n-1)(2), 3, (n-1)(2)])$, where $S_1 = (3)$ and $S_2 = ((n-1)(2))$. Recall also that $S(y_{i,b}) = (1)$ or (2) unless $y_i$ is an empty word, and that $S(z_{i,c}) = (1)$ or (2) unless $z_i$ is an empty word, for every $i$. The following lemma is a counterpart of Lemma 4.8.

**Lemma 4.11.** Let $r = n/(2n + 1) = [2, n]$, where $n \geq 3$ is an integer. Under Hypothesis B, the following hold for every $i$.

1. $S(z_{i,c} y_{i+1,b}) \neq (3)$.
2. $S(w_i z_{i+1,b}) \neq (4)$ and $S(z_{i,c} y_{i+1,w_i+1}) \neq (4)$.
3. $S(w_i z_{i+1,b}) \neq (5)$ and $S(z_{i,c} y_{i+1,w_i+1}) \neq (5)$.
4. $S(w_i z_{i+1,w_i+1}) \neq (6)$.
5. $S(w_i z_{i+1,w_i+1}) \neq (3, 3)$.

**Proof.** The proofs of (1), (2) and (5), respectively, are parallel to those of (1), (2) and (6) in Lemma 4.8. We only have to replace the subsequence (2) with the sequence ($(n - 1)(2)$).

(3) Suppose on the contrary that $S(z_{i,c} y_{2} w_2) = (5)$. Then $|z_{i,c}| = 2$ and $|y_2| = 0$, and $J$ is as depicted in Figure 9(a). If $J = M$, then we see from
Figure 9(a) that \(CS(\phi(\delta^{-1})) = CS(s')\) includes both a term 2 and a term 4, contradicting Lemma 3.3. If \(J \subsetneq M\), then by using Lemma 4.1(1) as in the proof of Lemma 4.8(1), we can see that \(S(\phi(c'_2e'_3))\) contains \((\ell_1, 4, \ell_2)\) as a subsequence, where \(\ell_1, \ell_2 \in \mathbb{Z}_+\). This implies that a term 4 occurs in \(CS(r)\), a contradiction.

(4) Suppose on the contrary that \(S(w_1z_1y_2w_2) = (6)\). Then \(|z_1| = |y_2| = 0\) and \(J\) is as depicted in Figure 9(b). We obtain a contradiction as in the proof of Lemma 4.11(3). □

![Figure 9](image-url)

**Figure 9.** (a) Lemma 4.11(3) where \(S(z_1y_2w_2) = (2 + 0 + 3)\), and (b) Lemma 4.11(4) where \(S(w_1z_1y_2w_2) = (3 + 0 + 0 + 3)\).

**Lemma 4.12.** Let \(r = n/(2n + 1) = [2, n]\), where \(n \geq 3\) is an integer. Under Hypothesis B, the following hold.

1. No two consecutive terms of \(CS(s)\) can be \((3, 3)\).
2. No two consecutive terms of \(CS(s)\) can be \((4, 4)\).
3. No term of \(CS(s)\) can be of the form \(5 + d\), where \(d \in \mathbb{Z}_+ \cup \{0\}\).

**Proof.** The proofs of (1) and (2) are parallel to those of (1) and (2) in Lemma 4.9 where we have only to use Lemma 4.11(1), (2) and (5) instead of Lemma 4.8(1), (2) and (6).

(3) Suppose on the contrary that \(CS(s)\) has a term of the form \(5 + d\). Let \(v\) be a subword of the cyclic word \((u_s)\) corresponding to a term \(5 + d\). By using Lemma 4.6 and Remark 4.5(2), we see that one of the following holds after a cyclic shift of indices.

- (i) \(v\) contains \(z_0, e w_1\) with \(S(z_0, e w_1) = (4)\) or (5).
- (ii) \(v\) contains \(w_1y_2, b\) with \(S(w_1y_2, b) = (4)\) or (5).
- (iii) \(v\) contains \(w_1w_2\) with \(S(w_1w_2) = (6)\).

However, (i) and (ii) are impossible by Lemma 4.11(2) and (3), and (iii) is impossible by Lemma 4.11(4). □
Finally, we study the case where \( r = (n + 1)/(3n + 2) = [2, 1, n] \) with \( n \geq 2 \).
Recall from Remark 4.5(3) that \( CS(r) = ((n(3), 2, n(3), 2)) \), where \( S_1 = (n(3)) \) and \( S_2 = (2) \). Recall also \( S(w_{i,\varepsilon}) = S(w_{i,\varepsilon}) = (3) \) for every \( i \). The following lemma is a counterpart of Lemmas 4.8 and 4.11.

**Lemma 4.13.** Let \( r = (n + 1)/(3n + 2) = [2, 1, n] \), where \( n \geq 2 \) is an integer. Under Hypothesis B, the following hold for every \( i \).

1. \( S(z_{iy_i+1}) \neq (3) \).
2. \( S(w_{i,\varepsilon}z_{iy_i+1}) \neq (4) \) and \( S(z_{iy_i+1}w_{i+1,\varepsilon}) \neq (4) \).
3. \( S(w_{i,\varepsilon}z_{i}y_{i+1}) \neq (5) \) and \( S(z_{i}y_{i+1}w_{i+1,\varepsilon}) \neq (5) \).
4. \( S(w_{i,\varepsilon}z_{iy_i+1}w_{i+1,\varepsilon}) \neq (3, 3) \).

**Proof.** The proofs of (1), (2) and (4) are parallel to those of Lemma 4.8(1), (2) and (6), respectively. We only have to replace the component 3 with the sequence \((n(3))\).

(3) Suppose on the contrary \( S(w_{1,\varepsilon}z_{1}y_{2}) = (5) \). (The other case is treated similarly.) Then \( |z_1| = 0 \) and \( |y_2| = 2 \). Thus \( S(y_2w_2) = (2, n(3)) \), and so \( CS(s) \) has a term 3. (Here we use the assumption \( n \geq 2 \).) This contradicts Lemma 3.3 because \( CS(s) \) contains a term \( 5 + d \) with \( d \geq 0 \) by the assumption. \( \square \)

**Lemma 4.14.** Let \( r = (n + 1)/(3n + 2) = [2, 1, n] \), where \( n \geq 2 \) is an integer. Under Hypothesis B, the following hold.

1. No two consecutive terms of \( CS(s) \) can be \((4, 4)\).
2. No term of \( CS(s) \) can be \( 5 \).
3. No term of \( CS(s) \) can be of the form \( 7 + d \), where \( d \in \mathbb{Z}_+ \cup \{0\} \).

**Proof.** (1) The proof is parallel to that of Lemma 4.9(2), where we have only to use Lemma 4.13(2) instead of Lemma 4.8(2).

(2) Suppose on the contrary that 5 occurs in \( CS(s) \). Let \( v \) be a subword of the cyclic word \((u_s)\) corresponding to a term 5. Then, by using Lemma 4.6 and Remark 4.5(3), we see that one of the following holds after a shift of indices.

(i) \( v = z_0w_{1,\varepsilon} \) with \( |z_0| = 2 \).
(ii) \( v = w_{1,\varepsilon}y_2 \) with \( |y_2| = 2 \).

However, this contradicts Lemma 4.13(3).

(3) Note that every term of \( CS(s) \) is at most 6, and that this happens only when \( S(w_{i}w_{i+1}) = ((n - 1)(3), 6, (n - 1)(3)) \), where \( |z_i| = |y_{i+1}| = 0 \), for some \( i \). Hence we obtain the desired result. \( \square \)

### 4.3 Hypothesis C
Assuming the following Hypothesis C, we will establish three technical lemmas (Lemmas 4.10, 4.13, 4.18) concerning the sequence \( S(z_{iy_i+1}) \) accordingly as \( r = [2, 2] \), \( r = [2, n] \) with \( n \geq 3 \), and \( r = [2, 1, n] \) with \( n \geq 2 \).
Hypothesis C. Suppose under Hypothesis A that Lemma 4.3(2) holds, namely, for every face $D_i$ of $J$, suppose that $S(\phi(\partial D_i^+))$ contains a subsequence of the form $(\ell, S, \ell')$, where $\ell, \ell' \in \mathbb{Z}_+$. Then we can decompose the word $\phi(\alpha)$ (clearly $(u_{s}) \equiv (\phi(\alpha))$) into

$$\phi(\alpha) \equiv y_1w_1z_1y_2w_2z_2 \cdots y_tw_tz_t,$$

where $\phi(\partial D_i^+ \equiv \phi(e_{2i-1}e_{2i}) \equiv y_iw_i\bar{z}_i$, $y_i$ and $z_i$ are nonempty words, $S(w_i) = S_2$, and where $S(y_iw_i\bar{z}_i) = (S(y_i), S_2, S(z_i))$, for every $i$. By Lemma 4.16 we also have the decomposition of the word $\phi(\delta^{-1})$ as follows (clearly $(u_{s}^{-1}) \equiv (\phi(\delta^{-1}))$ if $J = M$):

$$\phi(\delta^{-1}) \equiv y'_1w'_1z'_1y'_2w'_2z'_2 \cdots y'_tw'_tz'_t,$$

where $\phi(\partial D_i^- \equiv \phi(e_{2i-1}e_{2i}) \equiv y'_iw'_i\bar{z}'_i$, $y'_i$ and $z'_i$ are nonempty words, $S(w'_i) = S_2$, and where $S(y'_iw'_i\bar{z}'_i) = (S(y'_i), S_2, S(z'_i))$, for every $i$. Then $S(y_i^{-1}y_i) = S(z_i^{-1}z_i) = S_1$ for every $i$.

**Remark 4.15.** (1) If $r = [2, 2] = 2/5$, then $CS(r) = (1, 2, 3, 2)$, where $S_1 = (3)$ and $S_2 = (2)$. So, in Hypothesis C, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are exactly of the form $(\ell, 2, \ell')$, where $1 \leq \ell, \ell' \leq 2$ are integers.

(2) If $r = [2, n]$ with $n \geq 3$, then $CS(r) = ((n-1)(2), 3, (n-1)(2))$, where $S_1 = (3)$ and $S_2 = ((n-1)(2))$. So, in Hypothesis C, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are exactly of the form $(\ell, (n-1)(2), \ell')$, where $1 \leq \ell, \ell' \leq 2$ are integers.

(3) If $r = [2, 1, n]$ with $n > 2$, then $CS(r) = ((n-3), 2, n(3), 2)$, where $S_1 = (n(3))$ and $S_2 = (2)$. So, in Hypothesis C, both $S(\phi(\partial D_i^+))$ and $S(\phi(\partial D_i^-))$ are integers.

**Lemma 4.16.** Let $r = 2/5 = [2, 2]$. Under Hypothesis C, the following hold for every $i$.

1. $S(z_iy_{i+1}) = (2, 2)$ is not possible.
2. $S(z_iy_{i+1}) = (2)$ with $|z_i| = |y_{i+1}| = 1$ is not possible.

**Proof.** (1) Suppose on the contrary that $S(z_1y_2) = (2, 2)$. Since $1 \leq |z_1|, |y_2| \leq 2$, we have $|z_1| = |y_2| = 2$. Suppose first that $J = M$. Then it follows from Figure 10(a) that $CS(\phi(\delta^{-1})) = CS(s')$ involves two consecutive 1’s. It then follows that $|z_2| = 2$, for otherwise, $|z_2| = 1$ and hence we see $S(y_2w'_2z'_2) = (1, 2, 2)$, which in turn implies that $CS(s')$ contains $(2, 2 + c)$ with $c \in \mathbb{Z}_+ \cup \{0\}$ as a subsequence, contradicting Lemma 3.3 because $CS(s')$ also contains $(1, 1)$ as a subsequence. We next observe that $z_2$ and $y_3$ have different signs,
as depicted in Figure 10(b). If otherwise, we have $S(y_1^jw_1^jz_2^jy_3^j) = (1, 2, 1 + (3 − d)) = (1, 2, 4 − d)$ where $d = |y_3| \in \{1, 2\}$, and hence $CS(s')$ contains $(2, 2 + c)$ with $c \in \mathbb{Z}_+ \cup \{0\}$ as a subsequence, again contradicting Lemma 3.3. Hence $S(z_2^jy_3^j) = (2, d)$ with $d = |y_3| \in \{1, 2\}$ (see Figure 10(c)). If $d = 1$, then we have $S(z_2^jy_3^j) = (1, 3 − d, 2) = (1, 2, 2)$, which again yields a contradiction to Lemma 3.3. Hence we see $S(z_2^jy_3^j) = (2, 2)$. By repeating this argument, we see $S(z_i^jy_{i+1}^j) = (2, 2)$ with $|z_i| = |y_{i+1}| = 2$ for every $i$. But then $CS(\phi(\alpha)) = CS(s)$ becomes $((\ell(2)))$ with $\ell \geq 3$, yielding a contradiction to Lemma 3.3. Suppose next that $J \subset M$ (see Figure 11). Note that the assumption $S(z_1^jy_2^j) = (2, 2)$ implies that $S(w_1^jz_1^jy_2^j) = (2, 1, 1, 2)$. By using this fact, we can see that $|\phi(e_2^j)| = |\phi(e_3^j)| = 1$, for otherwise a subsequence of the form $(\ell_1, 1, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ would occur in $S(\phi(e_2^j))$, which in turn implies that $CS(\phi(\partial D_1^j)) = CS(2/5)$ would contain a term 1, a contradiction to $CS(2/5) = ((3, 2, 3, 2))$. Assuming that $e_2^j, e_3^j, e_2^j, e_3^j$ is a boundary cycle of $D_1$, we have $S(\phi(e_2^j)) = (1, 3, 2, 2)$ as depicted in Figure 11. But this is impossible, because Corollary 3.17 shows that any subword $w$ of the cyclic word $(\phi(\partial D_1^j))^{-1} = (u_{2/5})$ with $S(w) = (1, 3, 2, 2) = (1, S_1, S_2, 2)$ cannot be a product of two pieces.

![Figure 10](image-url)

**Figure 10.** Lemma 4.10(1) where $S(z_1^jy_2^j) = (2, 2)$ and $J = M$.

(2) Suppose on the contrary that $S(z_1^jy_2^j) = (2)$ with $|z_1| = |y_2| = 1$. Here, if $J = M$ (see Figure 12(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 2 and a term 4, contradicting Lemma 3.3. On the other hand, if $J \subset M$, then, by Lemma 4.11(2), none of $S(\phi(e_2^j))$ contains $S_2$ in its interior. This implies that the initial vertex of $e_2^j$ lies in the (central) segment of $\partial D_1^j$ corresponding to $S_2 = (2)$ and that the terminal vertex of $e_2^j$ lies in the (central) segment of $\partial D_2^j$. 

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corresponding to $S_2 = (2)$. Then we see that $CS(\phi(\partial D_1')) = CS(2/5)$ contains a term of the form $4 + c$ with $c \in \mathbb{Z}_+ \cup \{0\}$, as illustrated in Figure 12(b), a contradiction. \[\square\]
Proof. (1) Suppose on the contrary that \( S(z_1y_2) = (1, 2) \). (The other case is proved similarly.) Then \(|z_1| = 1 \) and \(|y_2| = 2 \). Here, if \( J = M \), then \( CS(\phi(\delta^{-1})) = CS(s') \) contains both a term 1 and a term 3 by Figure 13(a), contradicting Lemma 3.3. On the other hand, if \( J \subset M \), then by Lemma 4.1(2) the initial vertex of \( e'_2 \) lies in the segment of \( \partial D \) corresponding to \( S_2 = (2) \) and the terminal vertex of \( e'_3 \) lies in the segment of \( \partial D \) corresponding to \( S_2 = (2) \). This implies that a subsequence of the form \((\ell_1, 1, \ell_2, 2)\) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \) occurs in \( S(\phi(e'_2e'_3)) \) (see Figure 13(b)), so in \( CS(\phi(\partial D_1')) = CS(r) \), a contradiction. □

(2) Suppose on the contrary that \( S(z_1y_2) = (n_1\langle 3 \rangle, 2, n_2\langle 3 \rangle) \). Then one of the following holds.

(i) Either both \( S(z_1) = (n_1\langle 3 \rangle) \) and \( S(y_2) = (2, n_2\langle 3 \rangle) \), or both \( S(z_1) = (n_1\langle 3 \rangle, 2) \) and \( S(y_2) = (n_2\langle 3 \rangle) \) (see Figure 14(a)).

(ii) \( S(z_1) = (n_1\langle 3 \rangle, 1) \) and \( S(y_2) = (1, n_2\langle 3 \rangle) \) (see Figure 14(b)).

Suppose \( J = M \). Then \( CS(\phi(\delta^{-1})) = CS(s') \) contains a term 4 by Figure 14. On the other hand, by the assumption that Hypothesis C holds, \( CS(s') \) also contains a term 2. This contradicts Lemma 3.3. Suppose \( J \subset M \). Then, by using Lemma 4.1(2) as in the proof of Lemma 4.18(1), we see that a term 4 occurs in \( S(\phi(e'_2e'_3)) \). This implies that \( CS(\phi(\partial D_1')) = CS(r) \) contains a term of the form \( 4 + c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \), a contradiction. □

5. Proof of Main Theorem 2.4 for \( K(2/5) \)

Suppose \( r = 2/5 = [2, 2] \). Recall from Example 3.10(2) that \( CS(2/5) = ((3, 2, 3, 2)) \), where \( S_1 = (3) \) and \( S_2 = (2) \). For two distinct elements \( s, s' \in I_1(2/5) \cup I_2(2/5) \), suppose on the contrary that the unoriented loops \( \alpha_s \) and \( \alpha_{s'} \) are homotopic in \( S^3 - K(2/5) \), namely we suppose Hypothesis A. We will
derive a contradiction in each case to consider. By Lemma 4.3, there are two big cases to consider.

**Case 1.** Hypothesis $B$ holds.

By Corollary 4.10, Case 1 is reduced to the following five cases.

**Case 1.a.** $CS(s) = ((5, 5))$.

Let $v'$ and $v''$ be subwords of the cyclic word $(\phi(\alpha)) = (u_s)$ such that $(v'v'') = (u_s)$ and $S(v') = S(v'') = (5)$. Then by using Lemma 4.6 and the facts that $0 \leq |z_i|, |y_i| \leq 2$ and $|w_i| = 3$, we may assume, after a cyclic shift of indices, that $v' = w_1z_1y_2$ where $|z_1| = 0$ and $|y_2| = 2$. This implies $v'' = w_2z_2y_1$ where $|z_2| = 0$ and $|y_1| = 2$. Thus $J$ is as illustrated in Figure 15(a). If $J = M$, then $CS(\phi(\delta^{-1})) = CS(s')$ also becomes $((5, 5))$, which gives $s' = s$, contradicting the hypothesis of the theorem. Suppose $J \subset M$. By using Lemma 4.11(1) as in the last step of the proof of Lemma 4.3(1), we see that the interior of each of the two segments of $\partial D^+$ with weight 3 contains a (unique) vertex of $M$. Moreover by using the fact that $CS(2/5)$ consists of 3 and 2, we see that the position of two vertices is as illustrated in Figure 15(b). This implies that $J \cup J_1$ is as illustrated in Figure 15(b), where $J_1$ is the outer boundary layer of $M - J$. Thus the inner boundary label of $J_1$ is again $((5, 5))$. By repeating this argument, we see that the cyclic $S$-sequence of an inner boundary label of $M$, namely $CS(s')$, also becomes $((5, 5))$ regardless of the number of layers of $M$, and so $s' = s$, contradicting the hypothesis of the theorem.

**Case 1.b.** $CS(s)$ consists of 2 and 3.

By Proposition 3.12, $s \notin I_1(2/5) \cup I_2(2/5)$, contradicting the hypothesis of the theorem.

**Case 1.c.** $CS(s)$ consists of 3 and 4.

By Lemmas 4.6 and 4.9(1) and (2), $CS(s)$ must be $((4, 3, 4, 3))$. Then by Lemma 4.8(2), there is only one possibility: $J$ consists of two 2-cells and $CS(\phi(\alpha)) = CS(\phi(\partial D^+_1 \partial D^+_2)) = (5, 5)$. 

![Figure 14. Lemma 4.18(2) where (a) (i) occurs, and (b) (ii) occurs.](image)
Figure 15. Case 1.a

\((S(z_2y_1), S(w_1), S(z_1y_2), S(w_2)) = ((4, 3, 4, 3))\), where \(|y_j| = |z_j| = 2\) for \(j = 1, 2\). Here, if \(J = M\) (see Figure 16(a)), then \(CS(\phi(\delta^{-1})) = CS(s')\) becomes \((6)\), contradicting Lemma 3.3. On the other hand, if \(J \subset M\), then, by an argument similar to that in Case 1.a, we see that \(J_1 \cup J_2\) is as illustrated in Figure 16(b), where \(J_1\) is the outer boundary layer of \(M - J\). Thus if the number of layers of \(M\) is two, then \(CS(s')\) also becomes \((4, 3, 4, 3)\), which gives \(s' = s\), contradicting the hypothesis of the theorem. If the number of layers of \(M\) is bigger than 2, then, by an argument using Lemma 4.1 as in the last step of the proof of Lemma 4.8(1), a subsequence of the form \((\ell_1, 4, \ell_2)\) with \(\ell_1, \ell_2 \in \mathbb{Z}_+\) occurs in the inner boundary of \(J_1\), so in \(CS(2/5) = ((3, 2, 3, 2))\), a contradiction.

Figure 16. Case 1.c

Case 1.d. \(CS(s)\) consists of 4 and 5.
By Lemmas 4.6 and 4.8(2), every subword of the cyclic word \((u_\alpha)\) corresponding to a term 4 in \(CS(s)\) must be \(z_jy_{j+1}\) with \(|z_j| = |y_{j+1}| = 2\) for some \(j\). Without loss of generality, we may assume that \(S(z_1y_2) = (4)\), where \(|z_1| = |y_2| = 2\). Since \(CS(s)\) consists of 4 and 5, we have either \(S(w_2z_2y_3) = (4)\) or \(S(w_2z_2y_3) = (5)\). But by Lemma 4.8(2), \(S(w_2z_2y_3) = (5)\), where \(|z_2| = 0\) and \(|y_3| = 2\). By the repetition of this argument, we have \(S(w_iz_iy_{i+1}) = (5)\), where \(|z_i| = 0\) and \(|y_{i+1}| = 2\) for every \(i\). This is obviously a contradiction, since we assumed that \(|z_1| = 2\).

**Case 1.e.** \(CS(s)\) consists of 5 and 6.

In this case, \(J = M\) by Lemma 4.9(3b). By Lemmas 4.6 and 4.8(2), every subword of the cyclic word \((u_\alpha) = (\phi(\alpha))\) corresponding to a term 6 in \(CS(s)\) must be \(w_1w_{j+1}\), where \(|z_j| = |y_{j+1}| = 0\), for some \(j\) (cf. proof of Lemma 4.9(3a)). Without loss of generality, we may assume that \(S(w_1w_2) = (6)\), where \(|z_1| = |y_2| = 0\). Since \(CS(s)\) consists of 5 and 6, we have the following three possibilities by Lemmas 4.6 and 4.8(2):

(i) \(S(z_2w_3) = (5)\), where \(|z_2| = 2\) and \(|y_3| = 0\);
(ii) \(S(w_3y_4) = (5)\), where \(|z_2| = |y_3| = |z_3| = 0\) and \(|y_4| = 2\);
(iii) \(S(w_3w_4) = (6)\), where \(|z_2| = |y_3| = |z_3| = |y_4| = 0\).

If (ii) or (iii) occurs, then \(S(w_2z_2y_3w_3) = (3, 3)\), contradicting Lemma 4.8(6). Hence only (i) can occur. By the repetition of this argument, we have \(S(z_2w_{i+1}) = (5)\), where \(|z_i| = 2\) and \(|y_{i+1}| = 0\), for every \(i\). This is obviously a contradiction, since we assumed \(|z_1| = 0\).

**Case 2. Hypothesis C holds.**

By Remark 4.15(1), the cyclic \(S\)-sequence \(CS(\phi(\alpha)) = CS(s)\) contains a term 2. Hence, by Lemma 3.3, Case 2 is reduced to the following three subcases: if \(CS(s)\) has the form \(((m, m))\), then \(m = 2\), while if \(CS(s)\) has the form consisting of \(m\) and \(m + 1\), then \(m\) is either 1 or 2.

**Case 2.a.** \(CS(s) = ((2, 2))\).

In this case, there is only one possibility: \(J\) consists of one 2-cell, namely \(CS(\phi(\alpha)) = CS(\phi(\partial D^+)) = (S(z_0y_1), S(w_1)) = ((2, 2))\) with \(|y_1| = |z_0| = 1\). But this contradicts Lemma 4.16(2).

**Case 2.b.** \(CS(s)\) consists of 1 and 2.

Recall from Remark 4.15(1) that \(S(\phi(\partial D^+_i)) = (S(y_i), S(w_i), S(z_i)) = (\ell_{i, 1}, 2, \ell_{i, 2})\), where \(1 \leq \ell_{i,j} \leq 2\) are integers. By using Lemma 4.16(2) and the assumption that \(CS(s)\) consists of 1 and 2, we see \(S(z_iy_{i+1}) = (\ell_{i, 2}, \ell_{i+1, 1})\) and therefore
\[ CS(s) = (\ell_{1,1}, 2, \ell_{1,2}, \ldots, \ell_{t,1}, 2, \ell_{t,2}) \]. By Lemma 4.16(1), \((\ell_{i,2}, \ell_{i+1,1})\) is one of \((1,1), (1,2)\) and \((2,1)\) for every \(i\). Thus if the number \(t\) of the 2-cells of \(J\) is one, then \(CS(\phi(\alpha)) = CS(s)\) is either \((1,2,2)\) or \((1,2,1)\), both yielding a contradiction to Proposition 3.6. Hence \(t \geq 2\).

First, assume that \(S(z_1y_2) = (\ell_{1,2}, \ell_{2,1}) = (1,1)\). Then \((\ell_{i,2}, \ell_{i+1,1})\) is \((1,1)\) for every \(i\), for otherwise \(CS(s)\) would contain consecutive 1’s and consecutive 2’s, contradicting Lemma 3.3. Then we have \(CT(s) = (t(2))\) and therefore we have \(t = 2\) by Lemma 3.3 and Corollary 3.7. Thus \(J\) is as illustrated in Figure 17(a) and we have \(CS(\phi(\alpha)) = CS(\phi(\partial D_i^+ \partial D_j^-)) = ((1,2,1,1,2,1)) = ((2,1,2,1,2,1,1))\). If \(J = M\), then \(CS(\phi(\partial D_i^-)) = CS(s') = ((2,2,2,2,2,2,2))\), contradicting Lemma 3.3. On the other hand, if \(J \subset M\), then, by using Lemma 4.1(2) as in the last step of the proof of Lemma 4.16(2), we see that the unique vertex of \(M\) in the interior of \(\partial D_i^+\) must lie in the central segment among the three segments of \(\partial D_i^-\) with weight 2 for each \(i = 1, 2\). By using this fact and the identity \(CS(2/5) = ((3,2,3,2))\), we see that \(J \cup J_1\), where \(J_1\) is the outer layer of \(M - J\), is as illustrated in Figure 17(b). If the number of layers of \(M\) is two, then \(CS(s')\) also becomes \(((1,2,1,1,2,1,2))\), which gives \(s' = s\), contradicting the hypothesis of the theorem. If the number of layers of \(M\) is bigger than 2, then by an argument using Lemma 4.1(2) as in the last step of the proof of Lemma 4.16(2), \(CS(2/5)\) would contain a term 1, a contradiction.

**Figure 17.** Case 2.b where \(S(z_1y_2) = (1,1)\).

Next, assume that \(S(z_1y_2) = (\ell_{1,2}, \ell_{2,1}) = (1,2)\). (The case for \(S(z_1y_2) = (2,1)\) is similar.) Then \((\ell_{i,2}, \ell_{i+1,1})\) is \((1,2)\) for every \(i\), for otherwise there would exist some \(j\) and \(j'\) such that \((\ell_{j-1,2}, \ell_{j,1}, 2, \ell_{j,2}, \ell_{j+1,1}) = (1,2,2,1,1)\).
and such that \((\ell_{j'-1,2}, \ell_{j',1,2}, \ell_{j',2}, \ell_{j'+1,1}) = (2, 1, 2, 1, 2)\), so \(CT(s)\) would contain both a term 1 and a term 3, which together with Corollary 3.7 would yield a contradiction to Lemma 3.3. Thus \(CS(s) = ((2, 2, 1, \ldots, 2, 2, 1))\) and therefore \(CT(s) = ((t(2)))\). Hence, by Lemma 3.3 and Corollary 3.7 we have \(t = 2\). Thus \(J\) is as illustrated in Figure 18(a), and \(CS(\phi(\alpha)) = CS(\phi(\partial D_1^+ \partial D_2^+)) = ((2, 2, 1, 2, 2, 1))\). Here, if \(J = M\), then \(CS(\delta^{-1}) = CS(s') = ((1, 2, 2, 1, 2, 2))\), which gives \(s' = s\), contradicting the hypothesis of the theorem. On the other hand, if \(J \subset M\), then, by using Lemma 4.1(2) (cf. the last step of the proof of Lemma 4.16(2)) and the identity \(CS(2/5) = ((3, 2, 3, 2))\), we see that \(J \cup J_1\), where \(J_1\) is the outer layer of \(M - J\), is as illustrated in Figure 18(b). By repeating this argument, we see that the cyclic \(S\)-sequence of an inner boundary label of \(M\), namely \(CS(s')\), is \(((1, 2, 2, 1, 2, 2))\) regardless of the number of layers of \(M\), which also gives \(s' = s\), contradicting the hypothesis of the theorem.

![Figure 18. Case 2.b where \(S(z_1 y_2) = (1, 2)\).](image)

**Case 2.c.** \(CS(s)\) consists of 2 and 3.

In this case, by Proposition 3.12 \(s \notin I_1(2/5) \cup I_2(2/5)\), contradicting the hypothesis of the theorem. \(\square\)

6. **Proof of Main Theorem 2.4 for \(K(n/(2n + 1))\) with \(n \geq 3\)**

Suppose \(r = n/(2n + 1) = [2, n]\), where \(n \geq 3\) is an integer. Recall from Example 3.10(2) that \(CS(r) = ((3, (n - 1)(2), 3, (n - 1)(2)))\), where \(S_1 = (3)\) and \(S_2 = ((n-1)(2))\). For two distinct elements \(s, s' \in I_1(r) \cup I_2(r)\), suppose on the contrary that the unoriented loops \(\alpha_s\) and \(\alpha_{s'}\) are homotopic in \(S^3 - K(r)\),
namely we suppose Hypothesis A. We will derive a contradiction in each case to consider. By Lemma 4.3 there are two big cases to consider.

**Case 1. Hypothesis B holds.**

By Hypothesis B together with Remark 4.5(2), \( \phi(\alpha) \) involves a subword \( w_i \) whose \( S \)-sequence is (3), so the cyclic \( S \)-sequence \( CS(\phi(\alpha)) = CS(u_s) = CS(s) \) must contain a term of the form \( 3+c \), where \( c \in \mathbb{Z}_+ \cup \{0\} \). This together with Lemmas 3.3 and 4.12 implies that \( CS(s) \) cannot have the form \( (m, m) \) and that Case 1 is reduced to the following two subcases: either \( CS(s) \) consists of 2 and 3 or \( CS(s) \) consists of 3 and 4.

**Case 1.a. \( CS(s) \) consists of 2 and 3.**

Without loss of generality, we may assume that 2 occurs in \( S(z_1y_2) \). There are three possibilities:

(i) \( S(z_1y_2) \) consists of only 2, where \( S(z_1) = (n_1(2)) \), \( S(y_2) = (n_2(2)) \), and \( S(z_1y_2) = ((n_1 + n_2)(2)) \) with \( n_1, n_2 \in \mathbb{Z}_+ \cup \{0\} \);

(ii) \( S(z_1y_2) \) consists of only 2, where \( S(z_1) = (n_1(2), 1) \), \( S(y_2) = (1, n_2(2)) \), and \( S(z_1y_2) = ((n_1 + n_2 + 1)(2)) \) with \( n_1, n_2 \in \mathbb{Z}_+ \cup \{0\} \);

(iii) \( S(z_1y_2) \) consists of 2 and 3.

First assume that (i) occurs. Then \( S(z_1' y_2') = ((n_1' + n_2')(2)) \) where \( n_1' = (n - 1) - n_1 \) and \( n_2' = (n - 1) - n_2 \). So \( n_1' + n_2' = 2(n - 1) - (n_1 + n_2) \) and hence either \( S(z_1y_2) \) or \( S(z_1' y_2') \) contains \( n - 1 \) consecutive \( 2 \)'s. If \( J = M \), then this implies that either \( s \notin I_1(r) \cup I_2(r) \) or \( s' \notin I_1(r) \cup I_2(r) \) by Proposition 3.12, contradicting the hypothesis of the theorem. On the other hand, if \( J \subsetneq M \), then the above observation implies that either \( S(z_1y_2) \) contains \( n - 1 \) consecutive \( 2 \)'s and so \( s \notin I_1(r) \cup I_2(r) \), or otherwise \( S(z_1' y_2') \) contains \( n \) consecutive \( 2 \)'s. The former case is impossible by the assumption. In the latter case, we see, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), that a subsequence of the form \( (\ell_1, n(2), \ell_2) \) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \) occurs in \( S(\phi(\epsilon_2' \epsilon_3')) \), so in \( CS(\phi(\partial D'_1)) = CS(\partial D'_1) \), a contradiction.

Next assume that (ii) occurs. Then \( S(z_1' y_2') = ((n_1' + n_2' + 1)(2)) \), where \( n_1' = (n - 2) - n_1 \) and \( n_2' = (n - 2) - n_2 \). By using the identity \( n_1 + n_2 + 1 = 2(n - 1) - (n_1 + n_2 + 1) \), this case is treated as in the case when (i) occurs.

Finally assume that (iii) occurs. If \( J = M \) (see Figure 19(a)), then \( CS(\phi(\delta^{-1})) = CS(s') \) includes both a term 1 and a term of the form \( 3+c \) with \( c \in \mathbb{Z}_+ \cup \{0\} \), contradicting Lemma 3.3. On the other hand, if \( J \subsetneq M \) (see Figure 19(b)), then, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), a subsequence of the form \( (\ell_1, 1, \ell_2) \) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \) occurs in \( S(\phi(\epsilon_2' \epsilon_3')) \), so in \( CS(\phi(\partial D'_1)) = CS(\partial D'_1) \), a contradiction.
Figure 19. Case 1.a where (iii) occurs.

Case 1.b. $CS(s)$ consists of 3 and 4.

By Lemma 4.12(1) and (2), $CS(s)$ must be $((4, 3, 4, 3))$. Then by Lemmas 4.6 and 4.11(2), there is only one possibility: $J$ consists of two 2-cells, namely $CS(\phi(\alpha)) = CS(\phi(\partial D_1^+ \partial D_2^+)) = ((S(z_2y_1), S(w_1), S(z_1y_2), S(w_2)) = ((4, 3, 4, 3))$ with $|y_j| = |z_j| = 2$ for each $j = 1, 2$.

First suppose $n = 3$. If $J = M$ (see Figure 20(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ also becomes $((4, 3, 4, 3))$ implying that $s' = s$, a contradiction to the hypothesis of the theorem. On the other hand, if $J \subsetneq M$, then, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), a subsequence of the form $(\ell_1, 4, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs in $S(\phi(e_2' e_3'))$, so in $CS(\phi(\partial D_1^+)) = CS(r)$, a contradiction.

Figure 20. Case 1.b where (a) $r = [2, 3]$ and (b) $r = [2, n]$ with $n = 4$.

Next suppose $n \geq 4$. If $J = M$ (see Figure 20(b)), then $CS(\phi(\delta^{-1})) = CS(s')$ includes both a term 2 and a term 4, contradicting Lemma 3.3. On the other hand, if $J \subseteq M$, then we obtain a contradiction by the same reason as for $n = 3$. 

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Case 2. Hypothesis C holds.

By Remark 4.15(2), the cyclic S-sequence $CS(\phi(\alpha)) = CS(s)$ properly contains a subsequence $(2, 2)$. Hence, by Lemma 3.3, Case 2 is reduced to the following two subcases: either $CS(s)$ consists of 1 and 2 or $CS(s)$ consists of 2 and 3.

Case 2.a. $CS(s)$ consists of 1 and 2.

Repeat the argument of Case 2.b of Section 5 replacing the reference to Lemma 4.16 with the reference to Lemma 4.17 to obtain that $J$ consists of at least two 2-cells and that $S(z_iy_{i+1})$ is one of $(1, 1), (1, 2)$ and $(2, 1)$ for every $i$, so that $CS(s) = ((\ell_{1,1}, (n-1)(2), \ell_{1,2}, \ldots, \ell_{t,1}, (n-1)(2), \ell_{t,2})), \ldots$ where $(\ell_{i,2}, \ell_{i+1,1})$ is one of $(1, 1), (1, 2)$ and $(2, 1)$ for every $i$.

First, assume that $S(z_1y_2) = (\ell_{1,2}, \ell_{2,1}) = (1, 1)$. Then $CS(s)$ contains consecutive 1’s and consecutive 2’s, contradicting Lemma 3.3.

Next, assume that $S(z_1y_2) = (\ell_{1,2}, \ell_{2,1}) = (1, 2)$. (The case for $S(z_1y_2) = (2, 1)$ is similar.) The same argument of Case 2.b of Section 5 implies that both $CS(\phi(\alpha)) = CS(s)$ and $CS(\phi(\delta^{-1})) = CS(s')$ become $((n(2), 1, n(2), 1))$ (see Figure 21(a) and (b)), so that $s' = s$, contradicting the hypothesis of the theorem.

![Figure 21](image)

**Figure 21.** Case 2.a where $S(z_1y_2) = (1, 2)$ and $r = [2, n]$ with $n = 3$.

Case 2.b. $CS(s)$ consists of 2 and 3.

In this case, $CS(s)$ contains $S_1 = (3)$ as a subsequence. Moreover, $CS(s)$ also contains $S_2 = ((n-1)(2))$ by Hypothesis C. Hence, $s \notin I_1(r) \cup I_2(r)$ by Proposition 3.12 contradicting the hypothesis of the theorem. □
7. Proof of Main Theorem 2.5

Let $r = (n + 1)/(3n + 2) = [2, 1, n]$, where $n \geq 2$ is an integer. Recall from Example 3.10(1) that $CS(r) = ((n\langle 3 \rangle, 2, n\langle 3 \rangle, 2))$, where $S_1 = (n\langle 3 \rangle)$ and $S_2 = (2)$. For two distinct elements $s, s' \in I_1(r) \cup I_2(r)$, suppose that the unoriented loops $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$, namely we suppose Hypothesis A. We will prove the assertion by showing that $\alpha_s$ and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ for both $r = 3/8$ and the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$ and that we obtain a contradiction in the other cases. By Lemma 4.3, there are two big cases to consider.

**Case 1.** Hypothesis B holds.

By Hypothesis B along with Remark 4.3, $\phi(\alpha)$ involves a subword $w_i$ whose $S$-sequence is $(n\langle 3 \rangle)$, so the cyclic $S$-sequence $CS(\phi(\alpha)) = CS(s)$ must contain a term of the form $3 + c$, where $c \in \mathbb{Z}_+ \cup \{0\}$. This together with Lemmas 3.3 and 4.14 implies that Case 1 is reduced to the following three subcases: if $CS(s)$ has the form $((m, m))$, then $m$ is either 3 or 6, while if $CS(s)$ consists of $m$ and $m + 1$, then $m$ is either 2 or 3.

**Case 1.a.** $CS(s) = ((3, 3))$.

There is only one possibility: $n = 2$ and $J$ consists of one 2-cell, namely $CS(\phi(\alpha)) = CS(\phi(\partial D_1^+)) = CS(w_1) = ((3, 3))$, where $|y_1| = |z_1| = 0$. Here, if $J = M$ (see Figure 22(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ becomes $((2, 3, 3, 2))$, contradicting Lemma 3.3. On the other hand, if $J \subset M$ (see Figure 22(b)), then, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), a subsequence of the form $(\ell_1, 2, 2, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs in $S(\phi(e_2' e_1'))$, so in $CS(\phi(\partial D_1^+)) = CS(r)$, a contradiction.

![Figure 22. Case 1.a where $r = [2, 1, 2]$](image)
Case 1.b. $CS(s) = \langle (6, 6) \rangle$.

There is only one possibility: $n = 2$ and $J$ consists of two 2-cells, namely $CS(\phi(\alpha)) = CS(\phi(\partial D_1^+ \partial D_2^+)) = CS(w_1w_2) = \langle (6, 6) \rangle$, where $|y_j| = |z_j| = 0$ for $j = 1, 2$. So $r = [2, 1, 2] = 3/8$ and $s = 1/6$. Here, if $J = M$ (see Figure 23(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ becomes $\langle (4, 3, 3, 4, 3, 3) \rangle$, and so $s' = 3/10$ by Remark 3.8. This shows that the loops $\alpha_{1/6}$ and $\alpha_{3/10}$ are homotopic in $S^3 - K(3/8)$. On the other hand, if $J \subset M$ (see Figure 23(b)), then, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), a subsequence of the form $(\ell_1, 4, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs in $S(\phi(\partial D_1'))$, so in $CS(\phi(\partial D_1')) = CS(r)$, a contradiction.

![Figure 23](image.png)

**Figure 23.** Case 1.b where $r = [2, 1, 2]$.

Case 1.c. $CS(s)$ consists of 2 and 3.

In this case, $CS(s)$ contains $S_2 = (2)$ as a subsequence. Moreover, $CS(s)$ also contains $S_1 = \langle n(3) \rangle$ by Hypothesis B. Hence, $s \notin I_1(r) \cup I_2(r)$ by Proposition 3.12, contradicting the hypothesis of the theorem.

Case 1.d. $CS(s)$ consists of 3 and 4.

We first observe that either $|z_i| \neq 0$ or $|y_{i+1}| \neq 0$ for every $i$. Suppose on the contrary that $|z_i| = |y_{i+1}| = 0$ for some $i$. Then, since $CS(s)$ consists of 3 and 4 by assumption, we have $S(w_{i,e}z_i y_{i+1}w_{i+1,b}) = S(w_{i,e}w_{i+1,b}) = (3, 3)$, contradicting Lemma 4.13(4).

Next, we observe $|z_i| \neq 0$ and $|y_{i+1}| \neq 0$ for every $i$. Suppose on the contrary that $|z_i| = 0$ for some $i$. (The case $|y_{i+1}| = 0$ is treated similarly.) Then, by the preceding observation and Remark 4.5(3), we see $|y_{i+1}| = 1$ or 2. Since
$S(w, e, z, y_{i+1})$ is not equal to (4) nor (5) by Lemma 4.13(2) and (3), this implies that $S(w, e, z, y_{i+1})$ is equal to (3, 1) or (3, 2) and hence $S(w, e, z, y_{i+1}, w_{i+1, b})$ is equal to (3, 1, 3) or (3, 2, 3). This contradicts the assumption that $CS(s)$ consists of 3 and 4.

Thus we have shown that $|z_i| \neq 0$ and $|y_{i+1}| \neq 0$ for every $i$. So the assumption that $CS(s)$ consists of 3 and 4 implies that $S(w, e, z, y_{i+1}) = (3, 1)$ or $(3, 2)$ and hence $S(w, e, z, y_{i+1}, w_{i+1, b})$ is equal to $(3, 1, 3)$ or $(3, 2, 3)$. This contradicts the assumption that $CS(s)$ consists of 3 and 4.

Suppose first that $n = 2$. Then $r = [2, 1, 2] = 3/8$ and $J$ is obtained from the map in Figure 23(a) by reversing the outer and inner boundaries. Thus $CS(s) = \langle 4, 3, 3, 4, 3, 3 \rangle$ and so $s = 3/10$, and the inner boundary label of $J$ is $\langle 6, 6 \rangle$. If $J = M$, then $CS(s') = CS(\phi(\delta^{-1}) = \langle 6, 6 \rangle$, and hence $s' = 1/6$. This again shows that the loops $\alpha_{3/10}$ and $\alpha_{1/6}$ are homotopic in $S^3 - K(3/8)$.

If $J \subseteq M$, then, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), $CS(r)$ must contain a component $4 + c$ with $c \in \mathbb{Z}_+ \cup \{0\}$, a contradiction.

Suppose next that $n \geq 3$. If $J = M$ (see Figure 24(a)), then $CS(\phi(\delta^{-1})) = CS(s')$ contains both a term 3 and a term 6, contradicting Lemma 3.3. On the other hand, if $J \subseteq M$ (see Figure 24(b)), then, by an argument using Lemma 4.1(1) as in the last step of the proof of Lemma 4.8(1), a subsequence of the form $(\ell_1, 4 + c, \ell_2)$, where $\ell_1, \ell_2 \in \mathbb{Z}_+$ and $c \in \mathbb{Z}_+ \cup \{0\}$, occurs in $S(\phi(e_2' e_3'))$, so $CS(\phi(\delta D'_1)) = CS(r)$, a contradiction.

![Figure 24. Case 1.d where $r = [2, 1, n]$ with $n = 3$](image)
Case 2. Hypothesis C holds.

By Remark 4.13(3), the cyclic $S$-sequence $CS(\phi(\alpha)) = CS(s)$ includes a term 2. Hence, by Lemma 3.3, Case 2 is reduced to the following three subcases: if $CS(s)$ has the form $((m, m))$, then $m$ is 2, while if $CS(s)$ consists of $m$ and $m + 1$, then $m$ is either 1 or 2.

Case 2.a. $CS(s) = (2, 2)$.

There is only one possibility: $J$ consists of one 2-cell, namely $CS(\phi(\alpha)) = CS(\phi(\partial D_1^+)) = ((z_i y_i), S(w_1)) = (2, 2)$. Then $S(z_i y_i) = (2)$, contradicting Lemma 4.18(2).

Case 2.b. $CS(s)$ consists of 1 and 2.

By Remark 4.15(3) and the assumption that $CS(s)$ consists of 1 and 2, we see $S(\phi(\partial D_1^+)) = (S(y_i), S(w_i), S(z_i)) = (\ell_{i,1}, 2, \ell_{i,2})$, where $1 \leq \ell_{i,j} \leq 2$ are integers. By using Lemma 4.18(2) and the assumption that $CS(s)$ consists of 1 and 2, we see $S(z_i y_i+1) = (\ell_{i,2}, \ell_{i+1,1})$ and therefore $CS(s) = ((\ell_{i,1}, 2, \ell_{i,2}, \ldots, \ell_{i,j}, 2, \ell_{i,2}))$. By Lemma 4.18(1), $(\ell_{i,2}, \ell_{i+1,1})$ is either $(1, 1)$ or $(2, 2)$ for every $i$. Thus if the number $t$ of the 2-cells of $J$ is one, then $CS(\phi(\alpha)) = CS(s)$ is either $(1, 2, 1)$ or $(2, 2, 2)$, both yielding a contradiction to Proposition 3.6. Hence $t \geq 2$. By Lemma 3.3, we see either $S(z_i y_i+1) = (1, 1)$ for all $i$, or $S(z_i y_i+1) = (2, 2)$ for all $i$. However, if the latter holds, then $CS(s) = ((3t(2)))$, a contradiction to Lemma 3.3. So, $S(z_i y_i+1) = (1, 1)$ for all $i$, and therefore $CS(s) = ((1, 2, 1, \ldots, 1, 2, 1))$ and $CT(s) = (t(2))$. Hence we have $t = 2$ by Lemma 3.3 and Corollary 3.7. Thus $CS(\phi(\alpha)) = CS(\phi(\partial D_1^+ \partial D_2^+)) = (1, 2, 1, 1, 2, 1))$. So $s = 3/4$.

First suppose $n = 2$, namely $r = 3/8$. If $J = M$ (see Figure 25(a)), then $CS(\delta^{-1}) = CS(s')$ becomes $((3, 2, 3, 2, 2, 3, 2, 3, 2, 2))$, and so $s' = 5/12$ by Remark 3.8. This shows that the loops $\alpha_{3/4}$ and $\alpha_{5/12}$ are homotopic in $S^3 - K(3/8)$. If $J \not\subseteq M$, then by Lemma 4.11(2), the initial vertex of $e_2'$ (resp. the terminal vertex of $e_3'$) must lie in the central segment of $\partial D_1^-$ (resp. $\partial D_2^+$) with weight 2 (see Figure 25(b)). Thus a subsequence of the form $(\ell_1, 2, 2, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs in $S(\phi(e_2' e_3'))$, so in $CS(\phi(\partial D_1^+)) = CS(r)$, a contradiction.

Now let $n \geq 3$. If $J = M$, then $CS(s')$ contains consecutive 2's and consecutive 3's, contradicting Lemma 3.3. On the other hand, if $J \not\subseteq M$, then we see, by using Lemma 4.11(2) as in the case $n = 2$ (cf. Figure 25(b)), that a subsequence of the form $(\ell_1, 2, 2, \ell_2)$ with $\ell_1, \ell_2 \in \mathbb{Z}_+$ occurs in $S(\phi(e_2' e_3'))$, so in $CS(\phi(\partial D_1^+)) = CS(r)$, a contradiction.
Figure 25. Case 2.b where \( r = [2, 1, 2] \)

**Case 2.c.** \( CS(s) \) consists of 2 and 3.

Without loss of generality, we may assume that 3 occurs in \( S(z_1y_2) \). There are three possibilities:

(i) \( S(z_1y_2) \) consists of only 3, where \( S(z_1) = (n_1\langle 3 \rangle) \), \( S(y_2) = (n_2\langle 3 \rangle) \), and \( S(z_1y_2) = ((n_1 + n_2)\langle 3 \rangle) \) with \( n_1, n_2 \in \mathbb{Z}_+ \cup \{0\} \);

(ii) \( S(z_1y_2) \) consists of only 3, where \( S(z_1) = (n_1\langle 3 \rangle, \ell_1) \), \( S(y_2) = (\ell_2, n_2\langle 3 \rangle) \), and \( S(z_1y_2) = ((n_1 + n_2 + 1)\langle 3 \rangle) \) with \( \ell_1, \ell_2 \in \{1, 2\} \), \( \ell_1 + \ell_2 = 3 \) and \( n_1, n_2 \in \mathbb{Z}_+ \cup \{0\} \);

(iii) \( S(z_1y_2) \) consists of 2 and 3.

By an argument as in Case 1.a(i) and (ii) in Section 6, we can see that neither (i) nor (ii) can happen. (In the above, we need to appeal to Lemma 4.1(2) instead of Lemma 4.1(1).)

So, we may assume (iii) holds. By Lemma 4.18(2) and Remark 4.15(3), we see that \( S(z_1y_2) = (n_1\langle 3 \rangle, 2, 2, n_2\langle 3 \rangle) \), where \( 0 \leq n_i \leq n - 1 \) is an integer for \( i = 1, 2 \). We show that \( n_1 = n_2 = n - 1 \). Suppose that this does not hold. If \( J = M \), then \( CS(\phi(\delta^{-1})) = CS(s') \) includes both a term 1 and a term 3, contradicting Lemma 3.3. On the other hand, if \( J \subset M \), then we see, by an argument using Lemma 4.1(2) as in Case 2.b, that a subsequence of the form \((\ell_1, 1, \ell_2)\) with \( \ell_1, \ell_2 \in \mathbb{Z}_+ \) occurs in \( S(\phi(\ell_2')) \), so in \( CS(\phi(\partial D_1')) = CS(r) \), a contradiction. Thus we have \( S(z_1y_2) = ((n - 1)\langle 3 \rangle, 2, 2, (n - 1)\langle 3 \rangle) \). To avoid a contradiction to Lemma 3.3, we must have \( n = 2 \), namely \( r = 3/8 \). Again by Lemma 3.3 and the above argument, we see \( S(z_iy_{i+1}) = (3, 2, 2, 3) \) for every \( i \). Hence \( CS(s) = (2, 3, 2, 2, 3, \ldots, 2, 3, 2, 2, 3) \), and therefore \( CT(s) = (1, 2, \ldots, 1, 2) \). Thus by using Lemma 3.3 and Corollary 3.7, we see \( CT(s) = \)}
and so $CS(s) = \langle 2, 3, 2, 3, 2, 3, 2, 3 \rangle$. Then $s = 5/12$, and hence $J$ is obtained from the map in Figure 25(a) by reversing the outer and inner boundaries. Thus the inner boundary label of $J$ is $\langle 2, 1, 1, 2, 1, 1 \rangle$. If $J = M$, then $CS(s') = CS(\phi(\delta^{-1})) = \langle 2, 1, 1, 2, 1, 1 \rangle$, and so $s' = 3/4$. This again shows that the loops $\alpha_{5/12}$ and $\alpha_{3/4}$ are homotopic in $S^3 - K(3/8)$. If $J \subsetneq M$, then, we see by using Lemma 4.1(2) as in Case 2.b or as in the proof of Lemma 4.18(1), that a term 1 appears in $CS(r) = CS(3/8) = \langle 3, 3, 2, 3, 3, 2 \rangle$, a contradiction. \hfill \square

8. Proof of Theorems 2.7 and 2.8

In order to prove Theorems 2.7 and 2.8 we need the following refinement of Lemma 3.18 (cf. [9, Lemma V.5.2]).

Lemma 8.1. Suppose $G = \langle X \mid R \rangle$ with $R$ being symmetrized. Let $u, v$ be two cyclically reduced words in $X$ which are not trivial in $G$, and let $w$ be a non-trivial reduced word in $X$ such that $u = wvw^{-1}$ in $G$ but $u \neq wvw^{-1}$ in $F(X)$. Then the relation $u = wvw^{-1}$ in $G$ is realized by a nontrivial reduced annular $R$-diagram. To be precise, there is a reduced annular $R$-diagram $(M, \phi)$ and an edge path, $\gamma$, in $M$, joining a vertex $O_+$ of the outer boundary and a vertex $O_-$ of the inner boundary which satisfy the following conditions.

(i) There is an outer boundary cycle $\alpha$ with base point $O_+$ such that $\phi(\alpha)$ is visibly equal to the word $u$.

(ii) There is an inner boundary cycle $\delta$ with base point $O_-$ such that $\phi(\delta)$ is visibly equal to the word $v^{-1}$.

(iii) The word $\phi(\gamma)$ is equal to $w$ in $G$.

Proof. We prove the lemma by imitating [9, Proof of Lemma V.5.2]. By the assumption that $u = wvw^{-1}$ in $G$, $u$ is equal to some product $pp_1 \cdots p_n$ in $F(X)$, where $p = wvw^{-1}$ and $p_i = c_i r_i c_i^{-1}$ with $r_i \in R$. We may assume the number $n$ is minimal among all such products. Let $M'_0$ be a simply connected diagram over $F(X)$ consisting of $n + 1$ disks $D, D_1, \cdots, D_n$ with stems $\gamma, \gamma_1, \cdots, \gamma_n$ joined to a distinguished vertex $O_+$, satisfying the following conditions (see [9, Figure V.1.1]).

(i) Let $O_-$ be the endpoint of $\gamma$ in $\partial D$. Then label of the boundary cycle of $D$ with base point $O_-$ is visibly equal to $v$.

(ii) The label of the boundary cycle of $D_i$, whose base point is the endpoint of $\gamma_i$ in $\partial D_i$, is visibly equal to $r_i$ for each $i \in \{1, \cdots, n\}$.

(iii) Let $\alpha$ be the boundary cycle of $M'_0$ with base point $O_+$ and with initial segment $\gamma$. Then the label of $\alpha$ is visibly equal to the product $pp_1 \cdots p_n$. 

[46]
By applying the operations in [9, Proof of Theorem V.1.1] to $M'$, obtain a simply connected diagram $M'$ whose boundary label is visibly equal to the reduced word $u$. Since $u \neq 1$ in $G$, the 2-cell $D$ with label $v$ was not deleted in the construction of $M'$. Form an annular diagram $M$ from $M'$ by deleting the (interior of) the 2-cell $D$. Since $n$ is minimal, $M$ is reduced by the argument in [9, Proof of Lemma V.2.1]. Continue to denote by $O_{\pm}$ the vertices of $M$ determined by the vertices $O_{\pm}$ of $M'_0$. (It should be noted that both vertices are not removed during the construction.) Continue to denote by $\gamma$ the edge path in $M'$ obtained from the edge $\gamma$ of $M'_0$. (During the construction subsegments of $\gamma$ may be replaced with homotopic segments.) Then $\gamma$ joins $O_+$ and $O_-$, and we have $\phi(\gamma) = w$ in $G$. Continue to denote by $\alpha$ the outer boundary cycle of $M$ obtained from the outer boundary cycle $\alpha$ of $M'_0$ with base point $O_+$, and let $\delta$ be inner boundary cycle of $M$ obtained from the inverse of the boundary cycle of $D$ in $M'_0$ with base point $O_-$. Then we see $\phi(\alpha) \equiv u$, $\phi(\delta) \equiv v^{-1}$. Finally $M$ is nontrivial because the product $wwv^{-1}$ is not equal to $u$ in $F(X)$. This completes the proof of Lemma 8.1.

We also need the following fact.

**Lemma 8.2.** Suppose $r = q/p$, where $p$ and $q$ are relatively prime integers such that $q \not\equiv \pm 1 \pmod{p}$. Then, for a nontrivial element $u \in G(K(r))$, the centralizer $Z(u) = \{w \in G(K(r)) \mid wuw^{-1} = u\}$ of $u$ in $G(K(r))$ is described as follows.

1. If $u$ is non-peripheral, then $Z(u)$ is an infinite cyclic group generated by some primitive element $u_0$ such that $u = u_0^k$ for some integer $k \geq 1$.
2. If $u$ is peripheral, then $Z(u)$ is conjugate to the peripheral subgroup, $\langle m, l \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by a meridian and longitude pair $\{m, l\}$. In particular, $u$ is peripheral if and only if it commutes with a (conjugate of a) meridian.

**Proof.** By the assumption, $K(r)$ is hyperbolic and hence $G(K(r))$ is identified with a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. Thus the desired fact follows from [13, Lemma 4.5] □

**Proof of Theorems 2.7 and 2.8.** Consider a 2-bridge link $K(r)$ with $r = n/(2n+1)$ and $(n + 1)/(3n + 2)$, where $n \geq 2$, and assume that the loop $\alpha_s$ with $s \in I_1(r) \cup I_2(r)$ is either imprimitive or peripheral. Then, by Lemma 8.2 there is a nontrivial element $w \in G(K(r))$ such that $w \not\in \langle u_s \rangle$ and $wu_w^{-1} = u_s$. This identity cannot hold in $F(a, b)$, since $u_s$ is not a nontrivial cyclic permutation of itself. So by Lemma 8.1, the identity $wu_w^{-1} = u_s$ in $G(K(r))$ is realized by a nontrivial reduced annular $R$-diagram, $M$, with outer and inner
labels $u_s$ and $u_s^{-1}$, respectively. Then $M$ satisfies the assumption of Theorem 3.20 and hence its conclusion. Thus the arguments in Sections 5, 6 and 7 reveal all possible shapes of the annular diagram $M$.

**Case 1.** $r = 2/5$.

In this case, the arguments in Section 5 imply that one of the following holds.

(i) $CS(s) = ((5, 5))$ and hence $s = 1/5$ (see Remark 3.8). In this case, $M$ is equivalent to one of the reduced annular diagrams in Figure 15 and their natural generalizations.

(ii) $CS(s) = ((4, 3, 4, 3))$ and hence $s = 2/7$. In this case, $M$ is equivalent to one of the reduced annular diagram in Figure 16(b).

(iii) $CS(s) = ((2, 1, 1, 2, 1, 1))$ and hence $s = 3/4$. In this case, $M$ is equivalent to one of the reduced annular diagram in Figure 17(b).

(iv) $CS(s) = ((2, 2, 1, 2, 1))$ and hence $s = 3/5$. In this case, $M$ is equivalent to one of the reduced annular diagrams in Figure 18 and their natural generalizations.

Suppose that condition (i) holds and that $M$ is as in Figure 15(a). Starting from the uppermost vertex, read the outer boundary label in the clockwise direction with the initial label $a$. Then we obtain the word

$$aba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}bab = wu_{1/5}^{-1}w^{-1},$$

where $w = b^{-1}a^{-1}b^{-1}$ and $u_{1/5} = ababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$. Then the inner boundary label with the same base vertex and with the clockwise direction is equal to

$$baba^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ba = bwu_{1/5}^{-1}w^{-1}b^{-1}$$

Since these two words determine the same element of $G(K(2/5))$, we see that $b$ and $wu_{1/5}^{-1}w^{-1}$ commute with each other. So, $u_{1/5}$ commutes with $w^{-1}bw$, which is conjugate to a meridian $b$. Hence $u_{1/5}$ is peripheral by Lemma 8.2.

The annular diagram in Figure 15(b) shows only that $u_{1/5}$ commutes with $w^{-1}b^2w$, and similarly any natural generalization of the annular diagram in Figure 15 imply only that $u_{1/5}$ commutes with a power of $w^{-1}bw$. Hence the centralizer $Z(u_{1/5})$ is the rank 2 free abelian group generated by $u_{1/5}$ and $w^{-1}bw$. Hence $u_{1/5}$ is primitive.

Suppose that condition (ii) holds. By reading the annular diagram in Figure 16(a) as above, we see that $(ab)^{-1}u_{2/7}(ab)$ is equal to $(ba)^3$. Hence
\( u_{2/7} = ((ab)(ba)(ab)^{-1})^3 \) is imprimitive. On the other hand, the annular diagram in Figure 16(b) shows

\[
(ab)^{-1}u_{2/7}(ab) = (b^2ab^{-1}a^{-1}b^{-1})u_{2/7}(b^2ab^{-1}a^{-1}b^{-1})^{-1}.
\]

Letting \( w := b^2ab^{-1}a^{-1}b^{-1} \), we see that

\[
w = b(bab^{-1}a^{-1}b^{-1}) = (babab^{-1}a^{-1}b^{-1})^{-1}u_{2/7}(b^2ab^{-1}a^{-1}b^{-1}),
\]

(note that the second equality of the above identity comes from \( 1 = u_{2/7}abab^{-1}a^{-1}b^{-1} = (babab^{-1}a^{-1}b^{-1})^{-1}u_{2/7}(b^2ab^{-1}a^{-1}b^{-1}) \)), so that

\[
(ab)^{-1}u_{2/7}(ab) = (ba)^2(ab)^{-1}u_{2/7}(ab)(ba)^{-2},
\]

namely, \((ba)^2\) commutes with \((ab)^{-1}u_{2/7}(ab)\). Since this diagram is the unique annular reduced diagram realizing self conjugacies for \( u_{2/7} \), we see that the centralizer \( Z(u_{2/7}) \) is the infinite cyclic group generated by \((ab)(ba)(ab)^{-1}\). So, we can conclude that \( u_{2/7} \) is not peripheral by Lemma 8.2.

Suppose that condition (iii) holds. By reading the annular diagram in Figure 17(a) as above, we see that \( w^{-1}u_{3/4}^{-1}w \) with \( w = aba^{-1} \) is equal to \((b^{-1}a^{-1}ba)^3\). Hence \( u_{3/4} = (w(a^{-1}b^{-1}ab)w^{-1})^3 \) is imprimitive. On the other hand, the annular diagram in Figure 17(b) shows that \( a^{-1}b^{-1}ab \) commutes with \( w^{-1}u_{3/4}^{-1}w \). Moreover, by an argument as in (ii), we see that \( Z(u_{3/4}) \) is the infinite cyclic group generated by \( w(a^{-1}b^{-1}ab)w^{-1} \). Hence \( u_{3/4} \) is not peripheral by Lemma 8.2.

Suppose that condition (iv) holds. By reading the annular diagram in Figure 18(a) as above, we see that \( u_{3/5} = bu_{3/5}b^{-1} \). Hence, \( u_{3/5} \) is peripheral by Lemma 8.2. We can also see that \( u_{3/5} \) is primitive by an argument as in (i).

Since we have checked all possible cases for the 2-bridge knot \( K(2/5) \), the proof of Theorem 2.7 for \( K(2/5) \) is complete.

**Case 2.** \( r = n/(n + 1) \) with \( n \geq 3 \).

In this case, the arguments in Section 6 imply that one of the following holds.

(i) \( n = 3 \) and \( CS(s) = \{(4, 3, 4, 3)\} \), i.e., \( r = 3/7 \) and \( s = 2/7 \). In this case, \( M \) is equivalent to the diagram in Figure 20(a).

(ii) \( CS(s) = \{(n(2), 1, n(2), 1)\} \), i.e., \( s = (n + 1)/(2n + 1) \). In this case, \( M \) is equivalent to one of the diagrams in Figure 21 and their natural generalizations.
Suppose that condition (i) holds. By reading the annular diagram in Figure 20(a) as in Case 1(i), we see that

\[(ab)^{-1}u_{2/7}(ab) = (bab^{-1}a^{-1}b^{-1})u_{2/7}(bab^{-1}a^{-1}b^{-1})^{-1}.\]

So \(w := ab^2ab^{-1}a^{-1}b^{-1}\) belongs to the centralizer \(Z(u_{2/7})\). Since this diagram is the unique annular reduced diagram realizing self conjugacies for \(u_{2/7}\), we see that \(Z(u_{2/7})\) is the infinite cyclic group generated by \(w\). This implies that \(u_{2/7}\) is not peripheral. On the other hand, we see

\[w^2 = ab(bab^{-1}a^{-1}b^{-1}ab)bab^{-1}a^{-1}b^{-1} = ab(aba^{-1}b^{-1}a^{-1}ba)bab^{-1}a^{-1}b^{-1} = u_{2/7}.\]

In the above identity, the second equality follows from the relation

\[1 = u_{3/7} = abab^{-1}a^{-1}(bab^{-1}a^{-1}b^{-1}ab)a^{-1}b^{-1}.\]

Hence \(u_{2/7} = w^2\) is imprimitive.

Suppose that condition (ii) holds. By reading the annular diagram in Figure 21(a) as in Case 1(i), we see \(u_s = b^{-1}u_s b\), where \(s = (n + 1)/(2n + 1)\). Hence \(u_s\) is peripheral. We can also see as in Case 1(i) that \(u_s\) is primitive.

This completes the proof of Theorem 2.7 for \(K(n/(n + 1))\) with \(n \geq 3\).

**Case 3.** \(r = (n + 1)/(3n + 2)\) with \(n \geq 2\).

In this case, we see from the arguments in Section 7 that there is no such annular diagram. Hence every \(\alpha_s\) with \(s \in I_1(r) \cup I_2(r)\) is primitive and is not peripheral. This completes the proof of Theorem 2.8. \(\square\)

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