ON SUFFICIENCY ISSUES, FIRST INTEGRALS AND EXACT SOLUTIONS OF UZAWA-LUCAS MODEL WITH UNSKILLED LABOR

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Abstract. In this article, the sufficiency issues, first integrals and exact solutions for the Uzawa-Lucas model with unskilled labor are investigated. The sufficient conditions are established by utilizing Arrow’s Sufficiency theorem. The non-negativeness conditions for the balanced growth path (BGP) are provided and growth rate is explicitly given in terms of parameters of the model. The first integrals are established by the partial Hamiltonian approach. Then first integrals are utilized to construct the exact solutions for all the variables. The growth rates of all variables and graphical representation of exact solutions are provided for the special case when the inverse of the intertemporal elasticity of substitution is the same as the share of physical capital.

1. Introduction. The optimal control theory is widely used method to deal with dynamic optimization in economic literature. The optimal control problem involves three types of variables control, state and costate variables. The Pontryagin’s maximum principle [29] provides a set of necessary conditions (first-order conditions) for the optimal control and in general these are not sufficient. The Mangasarian’s sufficiency theorem [18, 6] and Arrow’s Sufficiency theorem [6, 1, 11] are widely used to determine the sufficiency.

Cysne [7, 8] pointed out that several works in economic growth literature (see e.g., Uzawa [31]; Lucas [14]; Caballe and Santos [2]; Mulligan and Sala-I-Martin [19]; Chari et al [3]; Lucas [13]) utilize only first-order conditions in optimal control problems to characterize necessity for the optimum path but not the sufficiency. In several models in economic literature the Mangasarian’s sufficiency theorem fails due to the non-concavity of the constraints. Cysne [7] illustrated how one can utilize Arrow’s theorem for those models to establish sufficiency. Robertson [30] studied the Uzawa-Lucas model for the unskilled workers. Later on, Ferrara and Guerrini [9] re-visited this model and proved some results of Robertson [30] analytically. The sufficiency conditions were not analyzed in [30, 9] and will be established in this article.

The partial Hamiltonian approach [20, 21] and a partial Lagrangian approach [12, 22] are elegant approaches to derive the exact solutions for the models arising in mathematical biology, physics, mechanics, economic growth theory and some other fields as well. The exact solutions for the basic Lucas-Uzawa model and its extension

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for the externalities case are studied in [23, 24, 4, 25] by the partial Hamiltonian approach. The partial Hamiltonian approach is successfully applied in mechanics, physics, epidemiology and other fields of applied mathematics in some recent papers (see e.g. [17, 26, 10]). Several methods exist for computation of conservation laws and first integrals of differential equations which are summarized in [27, 28, 5, 32]. The conservation laws can be obtained by using pairs of symmetries and adjoint symmetries [15]. Hamiltonian formulation can be obtained by using variational identity [16]. In this work, I will utilize partial Hamiltonian approach to compute first integrals and then exact solutions of the Uzawa-Lucas model for the unskilled workers.

The paper is organized in the following manner. In section 2, the necessary and sufficient conditions are established for the Uzawa-Lucas model for the unskilled workers. The non-negativeness conditions for the BGP of model are provided and growth rate is explicitly given in terms of parameters of the model. The first integrals and then exact solution by utilizing these first integrals are derived in Section 3. In Section 4, the exact solution for the special case when inverse of intertemporal elasticity of substitution (IIES) is the same as the share of physical capital is analyzed. A detailed graphical analysis is provided as well. The conclusions are given in Section 5.

2. Uzawa-Lucas model with unskilled labor. Robertson [30] considered an economy with production function of form \( y = k^\alpha (uh)^\beta \) where \( k(t) \) is physical capital, \( h(t) \) is human capital, \( u(t) \) is the share of human capital essential for production of final good, \( \alpha \) is the share of physical capital, \( \beta \) is the share of human capital and \( \alpha, \beta \in (0, 1) \) with \( \alpha + \beta < 1 \). Ferrara and Guerrini [9] re-visited this model and proved some results of Robertson [30] analytically. The representative consumer wants to maximize the lifetime utility and the dynamic optimization problem is

\[
\max_{c, u} \int_0^\infty \frac{c^{1-\theta} - 1}{1 - \theta} e^{-\rho t} dt,
\]

subject to

\[
\dot{k} = k^\alpha (uh)^\beta - c - nk, \quad k(0) = k(0)
\]
\[
\dot{h} = \nu (1 - u)h - (n - m)h, \quad h(0) = h(0)
\]

where \( c(t) \) is consumption per worker, \( \rho > 0 \) represents the discount factor, \( \theta \) is the IIES, \( n \) is the exogenous growth rate of unskilled labor, \( m \leq n \) is the natural growth rate of human capital, \( \nu > 0 \) measures the efficiency of the human capital creation process and \( \nu > n - m \). Robertson [30], Ferrara and Guerrini [9] used \( \theta = \frac{1}{\sigma} \) as the IIES.

2.1. Necessary and sufficient conditions. The current value Hamiltonian function \( H(t, c, u, k, h, \lambda_1, \lambda_2) \) for this problem is defined as (see Appendix A)

\[
H = \frac{c^\theta - 1}{1 - \theta} + \lambda_1 [k^\alpha (uh)^\beta - c - nk] + \lambda_2 h [\nu (1 - u) - (n - m)],
\]

where \( c(t), u(t) \) are two control variables, \( k(t), h(t) \) are two state variables and \( \lambda_1(t), \lambda_2(t) \) are two costate variables for \( k \) and \( h \) respectively. The Pontryagin’s maximum principle [29] (see Appendix A) provides following set of necessary conditions (first-order conditions) for the optimal control:

\[
\lambda_1 = c^{-\theta},
\]
\[ \beta \lambda_1 u^{\beta - 1} k^\alpha h^\beta = \lambda_2 \nu h, \quad (5) \]

\[ \dot{k} = k^\alpha (uh)^\beta - c - nk, \quad (6) \]

\[ \dot{h} = \nu(1 - u) - (n - m)h, \quad (7) \]

\[ \dot{\lambda}_1 = \rho \lambda_1 - \lambda_1 \lambda_2 \nu (1 - u)^{\beta - 1} - \lambda_2 [\nu(1 - u) - (n - m)]. \quad (8) \]

Equation (9), with the aid of equation (5) simplifies to the following equation:

\[ \dot{\lambda}_2 = \rho \lambda_2 - \lambda_1 \beta \nu \lambda_1^{\alpha - 1} k^\alpha u^\beta - \lambda_2 \nu(1 - u) - (n - m). \quad (9) \]

Equation (5) yields following expression for the control variable \( u \)

\[ u = \left( \frac{\lambda_2 \nu}{\beta \lambda_1} \right) \frac{1}{\nu h} k^{\frac{\alpha}{1 - \beta} - 1} h^{-1}. \quad (11) \]

The time derivatives of equations (4) and (11) yield following growth rates of variables \( c \) and \( u \)

\[ \dot{c} = \alpha \frac{\nu}{\beta} (uh)^{\beta - 1} - \frac{1}{\beta} (\rho + n), \quad (12) \]

\[ \dot{u} = \frac{1}{1 - \beta} [(1 - \alpha - \beta)n + \beta(m + \nu)] + \nu u - \frac{\alpha}{1 - \beta} \frac{c}{k}. \quad (13) \]

The transversality conditions are optimality conditions which are used to characterize the optimal paths of economic growth models. The following form of transversality conditions is considered (see e.g. [6]):

\[ \lim_{t \to \infty} e^{-\rho t} \lambda_1 k = 0, \quad \lim_{t \to \infty} e^{-\rho t} \lambda_2 h = 0. \quad (14) \]

The constraint (6) for the variable \( k \) is non-concave in variables \((c, k, u, h)\) due to term \( k^\alpha (uh)^\beta \). The constraint (7) for the variable \( h \) is non-concave in the variables \((c, k, u, h)\) due to term \((1 - u)h\). I conclude that the Mangasarian’s sufficiency theorem [18] fails for this model. I check sufficient conditions using Arrow’s Sufficiency theorem [6, 1, 11] and first I construct the maximized Hamiltonian.

Substitution of \( c \) and \( u \) from (4) and (11), in the current value Hamiltonian (3), yields (see e.g. [6, 1, 11])

\[ H^0(t, k, h, \lambda_1, \lambda_2) = \frac{(\lambda - 1)^{1 - \theta} - 1}{1 - \theta} \]

\[ + \lambda_1 \left( \frac{\lambda_2 \nu}{\beta \lambda_1} \right) \frac{1}{\nu h} k^{\frac{\alpha}{1 - \beta}} - \lambda_1 \frac{1}{\beta} - nk \]

\[ + \lambda_2 [\nu h - \nu \left( \frac{\lambda_2 \nu}{\beta \lambda_1} \right) \frac{1}{\nu h} k^{\frac{\alpha}{1 - \beta}} - (n - m)h]. \quad (15) \]

It is straightforward to check that the maximized Hamiltonian \( H^0 \) given in (15) is concave in state variables \((k, h)\) as \( \alpha + \beta < 1 \). Thus, the first-order conditions given in (4)-(10) are sufficient as well.
2.2. Balanced growth path. In order to determine the BGP for the model, Robertson [30], Ferrara and Guerrini [9] introduced two new variables $x(t) = \frac{c(t)}{k(t)}$ and $z(t) = u(t)\beta h(t)\beta k(t)\alpha^{-1}$. The variable $x(t)$ is the consumption to physical capital ratio and variable $z(t)$ is the average product of physical capital. The dynamics of the economy can be then represented by the following three differential equations in terms of $x$, $z$ and $u$:

$$\frac{\dot{x}}{x} = x + \frac{(\theta - 1)n - \rho + \alpha - \theta}{\theta} z,$$

$$\frac{\dot{z}}{z} = \left(\frac{1 - \alpha - \beta}{1 - \beta}\right) x - (1 - \alpha) z + \frac{(1 - \alpha - \beta)n + \beta(m + \nu)}{1 - \beta},$$

$$\frac{\dot{u}}{u} = \frac{1}{1 - \beta} [(1 - \alpha - \beta)n + \beta(m + \nu)] + \nu u - \frac{n}{1 - \beta}.$$

The BGP of model is determined by setting equations (16)-(18) to zero and then solving the correspond linear system for $x$, $z$ and $u$. Robertson [30], Ferrara and Guerrini [9] provided following expressions for the equilibrium values of $x^*$, $z^*$ and $u^*$:

$$x^* = \frac{(1 - \alpha)(1 - \beta)(\rho + n) + \beta(m + \nu)(\theta - \alpha)}{\alpha(1 - \alpha - \beta + \beta\theta)} - n,$$

$$z^* = \frac{(1 - \alpha - \beta)(\rho + n) + (m + \nu)\theta\beta}{\alpha(1 - \alpha - \beta + \beta\theta)},$$

$$u^* = \frac{(1 - \alpha)(\rho + n) + \beta(\theta - 1)(m + \nu)}{\nu(1 - \alpha - \beta + \beta\theta)} - \frac{n}{\nu}.$$

**Proposition 1.** Let $\gamma_c, \gamma_k, \gamma_h$ and $\gamma_u$ be the growth rates of $(c, k, h, u)$. Assume that $\frac{\beta}{1 - \beta}(\nu + m - n) < \rho < \nu + m - n$ and $\theta \geq \alpha$, then along the BGP equilibrium

$$\gamma_c = \gamma_k = \gamma = \frac{\beta(\nu + m - n - \rho)}{(1 - \alpha)(1 - \beta) + \beta(\theta - \alpha)} > 0,$$

$$\gamma_h = \frac{1 - \alpha}{\beta} \left[ \frac{\beta(\nu + m - n - \rho)}{(1 - \alpha)(1 - \beta) + \beta(\theta - \alpha)} \right], \quad \gamma_u = 0,$$

while $x^* > 0$, $z^* > 0$ and $u^* \in [0, 1]$, are respectively given by (19), (20) and (21).

**Proof.** Take $\frac{\dot{c}}{c} = \gamma_c$, then growth rate of consumption given in (12) yields

$$\alpha k^{\alpha-1}(uh)^\beta = \theta \gamma_c + \rho + n,$$

and thus along BGP the marginal product of physical capital remains constant which is a well-known result in economic growth theory. Take $\frac{\dot{k}}{k} = \gamma_k$, then equation (6) yields

$$\gamma_k = k^{\alpha-1}(uh)^\beta - \frac{c}{k} - n.$$

From equations (24) and (25), it follows that $\frac{c}{k} = \frac{\theta \gamma_c + \rho + n - \gamma_k - n}$ which is constant. Thus $\frac{\dot{c}}{c}$ the ratio of consumption to physical capital is constant along BGP. I arrive at $\frac{\dot{c}}{c} = \frac{\dot{k}}{k}$ and thus $\gamma_c = \gamma_k = \gamma$. Using $\frac{\dot{k}}{k} = \gamma_h$ in equation (7) yields

$$u = 1 - \frac{\gamma_h + n - m}{\nu},$$
which yields $\gamma_u = \frac{u}{u} = 0$. Differentiate equation (24) with respect to time $t$, one gets $\frac{h}{h} = \frac{1}{\sigma} \frac{\dot{\gamma}}{\dot{\gamma}} i.e.$

$$\gamma_h = \frac{1 - \alpha}{\beta} \gamma. \quad (27)$$

Ferrara and Guerrini [9] established conditions (24)-(27) but the value of $\gamma$ was not determined. To complete the proof, one has to find value of $\gamma$. The differentiation of (4) and (5) with respect to time yields $\frac{\dot{\lambda}}{\lambda} = -\sigma \gamma$ and

$$\frac{\dot{\lambda}}{\lambda} = \left(\alpha - \sigma - (1 - \alpha)(1 - \beta)\right) \gamma. \quad (28)$$

Eliminate $\frac{\dot{\lambda}}{\lambda}$ between (10) and (28), and solve for $\gamma$, to obtain value of $\gamma$ given in (22). Equation (27) provides value of $\gamma_h$ same as given in equation (23). The growth rate is positive provided $\rho < \nu + m - n$ and $\theta \geq \alpha$. This completes the first half of proof of proposition 1.

Next, I provide parameter restrictions such that $x^* > 0$, $z^* > 0$ and $u^* \in [0, 1]$. Along BGP, the consumption to physical capital $x^* > 0$ provided

$$\rho + n > \frac{\alpha m (1 - \alpha - \beta + \theta \beta) + \beta (m + n)(\alpha - \theta)}{(1 - \alpha)(1 - \beta)}, \quad (29)$$

and the average product of physical capital $z^*$, is clear positive as $\alpha + \beta < 1$. The variable $u^* \in [0, 1]$ provided

$$\frac{\beta(1 - \theta)(\nu + m - n)}{1 - \alpha} < \rho < \nu + \frac{(1 - \theta)(m - n)\beta}{1 - \alpha}, \quad (30)$$

provided $\theta > 1$ for $\alpha \neq \theta$ and $0 < \theta < 1$ for $\alpha = \theta$. This completes the proof. \Box

3. Exact solution under no parameter restrictions. In this section, first I construct the first integrals of the model by partial Hamiltonian approach. Then I use these first integrals to establish the exact solution of model.

3.1. First integrals for the Uzawa-Lucas model with unskilled labor. The partial Hamiltonian operators determining equation (B-6) for the current value (partial) Hamiltonian (3) can be written as

$$\lambda_1(\eta_1^1 + \dot{k}n_h^1 + \dot{h}n_h^1) + \lambda_2(\eta_2^2 + \dot{k}n_h^2 + \dot{h}n_h^2) - \eta^1 \left[\alpha \lambda_1 k^{\alpha - 1} u^{\beta} h^\beta - n\lambda_1\right]$$

$$-\eta^2 \left[\lambda_1 k^{\alpha} u^{\beta} h^\beta - 1 + \lambda_2(1 - u) - \lambda_2(n - m)\right]$$

$$-H(\xi_t + \dot{k}\xi_k + \dot{h}\xi_h) = B_t + \dot{k}B_k + \dot{h}B_h$$

$$+ \left[\eta^1 - \xi \left(k^{\alpha}(uh)^\beta - c - nk\right)\right](\lambda_1 \rho)$$

$$+ \left[\eta^2 - \xi (\nu(1 - u)h - (n - m)h)\right](\lambda_2 \rho). \quad (31)$$

I assume that $\xi, \eta^1, \eta^2, B$ are functions of $t, k, h$ and these do not depend on $\lambda_1, \lambda_2$. With the aid of equations (4) and (5) one can eliminate variables $\lambda_1$ and $\lambda_2$. Then the partial Hamiltonian determining equation (31) can be expressed only in terms of state variables $k, h$ and control variables $c, u$. I expand the partial Hamiltonian operators determining equation (31) and then separate it with respect to $(c, u)$.
Finally, I arrive at the following system of determining equations for \( \xi, \eta^1, \eta^2, B \):

\[
\begin{align*}
\xi_k &= 0, \quad \text{(32)} \\
\xi_h &= 0, \quad \text{(33)} \\
\eta^1_h &= 0, \quad \text{(34)} \\
\eta^2_h &= 0, \quad \text{(36)} \\
\eta^1_k &= \frac{\beta}{\theta} \eta^2_k - (\beta - 1)\xi + \rho\xi = 0, \quad \text{(35)} \\
\eta^2_k &= (\beta - 1)\xi_t + \rho\xi(\beta - 1) = 0, \quad \text{(37)} \\
\eta^1_h - nk\eta^1_k + (\rho + n)\eta^1 + nk\xi_t + \rho nk\xi &= 0, \quad \text{(38)} \\
-(\nu - n + m)h\xi_t + \rho n^2 + \rho\xi h(\nu - n + m) &= 0, \quad \text{(39)} \\
\xi_1 &= 0, \quad \eta^1_1 = 0, \quad \eta^2_1 = e^{(\nu - \rho - n + m)t}, \quad B_1 = 0, \\
\xi_2 &= e^{-\rho t}, \quad \eta^1_2 = \frac{1}{1 - \theta} \rho e^{-\rho t}k, \quad \eta^2_2 = \frac{(1 - \alpha)\rho}{(1 - \theta)\beta} e^{-\rho t}h, \quad B_2 = \frac{1}{1 - \theta} e^{-\rho t}. \quad \text{(43)}
\end{align*}
\]

The solution of equations (32)-(42) yield following expressions for the partial Hamiltonian operators \( \xi, \eta^1, \eta^2 \) and gauge term \( B \):

\[
\begin{align*}
\xi_1 &= 0, \quad \eta^1_1 = 0, \quad \eta^2_1 = e^{(\nu - \rho - n + m)t}, \quad B_1 = 0, \\
\xi_2 &= e^{-\rho t}, \quad \eta^1_2 = \frac{1}{1 - \theta} \rho e^{-\rho t}k, \quad \eta^2_2 = \frac{(1 - \alpha)\rho}{(1 - \theta)\beta} e^{-\rho t}h, \quad B_2 = \frac{1}{1 - \theta} e^{-\rho t}. \quad \text{(43)}
\end{align*}
\]

The first integrals associated with the partial Hamiltonian operators \( \xi, \eta^1, \eta^2 \) and gauge term \( B \) given in (43) can be computed by utilizing formula (B-7). The first integrals are given as follows:

\[
I_1 = \lambda_2 e^{(\nu - \rho - n + m)t}, \quad \text{(44)}
\]

\[
I_2 = \frac{e^{-\rho t}e^{-\rho t}}{1 - \theta} \left[ (\rho + n - n\theta)k - \theta c - (1 - \beta)(1 - \theta)k^\alpha (hu)^\beta \\
+ \frac{\beta}{\rho} k^\alpha h^\beta u^{\beta - 1} \left( \frac{(1 - \alpha)\rho}{(1 - \theta)\beta} - (1 - \theta)(\nu - n + m) \right) \right]. \quad \text{(45)}
\]

The partial Hamiltonian approach provided two first integrals for the model. The costate variable \( \lambda_1 \) can be view as the shadow price of physical capital and the costate variable \( \lambda_2 \) can be view as the shadow price of human capital capital. The first integral is a constant of motion and thus the first integral \( I_1 \) yields

**Shadow price of human capital \times e^{(\nu - \rho - n + m)t} = constant,**

where \( \nu - \rho - n + m \) can be interpreted as the efficiency of the human capital creation process minus the discount factor minus the natural growth rate of human capital. It is difficult to provide some economic meaning of the second constant of motion \( I_2 \). In next Section, I will utilize both first integrals \( I_1 \) and \( I_2 \) to establish the exact solution of the model.

3.2. **Exact solution by utilizing first integrals when \( \alpha \neq \theta \).** Now, I provide the exact solution for the model with the aid of first integrals \( I_1 \) and \( I_2 \). Setting \( I_1 = c_1 \), I have

\[
\lambda_2 = c_1 e^{-(\nu - \rho - n + m)t}. \quad \text{(46)}
\]
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Setting \( I_2 = c_2 \), I have

\[
\dot{c} + \frac{\beta z}{\nu} \left( \frac{1 - \alpha}{\beta} \rho - (1 - \theta)(\nu - n + m) \right) + \frac{c_2 e^{\theta t} e^{\rho t} (1 - \theta)}{k} \frac{1}{u} = 0
\]

(47)

where I have used variables \( x(t) = \frac{c(t)}{k(t)} \) and \( z(t) = u(t)^\beta h(t)^\beta k(t)^{\alpha - 1} \).

\[
\frac{1}{u} = \frac{\nu}{\beta z} \left( \frac{1 - \alpha}{\beta} \rho - (1 - \theta)(\nu - n + m) \right)
\]

\[
x(t) = x + \left( \frac{\theta - 1}{\theta} \right) n - \rho + \frac{\alpha - \theta}{\theta} z,
\]

(49)

and

\[
\frac{z}{\dot{z}} = \left( \frac{1 - \alpha - \beta}{1 - \beta} \right) x - (1 - \alpha) z + \frac{(1 - \alpha - \beta)n + \beta(m + \nu)}{1 - \beta}.
\]

(50)

The differential equation (12) for the consumption \( c \) can be re-written as

\[
\dot{c} + \frac{\rho + n - \alpha z}{\theta} c = 0,
\]

(51)

which yields

\[
c(t) = c_3 e^{-\left(\frac{\rho + n - \alpha z}{\theta}\right)t} e^{\int z dt},
\]

(52)

where \( c_3 \) is arbitrary constant. The differential equation (6) for the physical capital \( k \) can be re-written as

\[
\dot{k} + (x - z + n)k = 0,
\]

(53)

which yields

\[
k(t) = c_4 e^{-nt} e^{-\int (x - z) dt},
\]

(54)

where \( c_3 \) is arbitrary constant. Finally, \( h(t) \) can be determined from \( z(t) = u(t)^\beta h(t)^\beta k(t)^{\alpha - 1} \) and is given by

\[
h(t) = \frac{z(t)^{\frac{1}{\beta}}}{u(t) k(t)^{\frac{\alpha - 1}{\beta}}},
\]

(55)

The exact solutions for all variables of the model can be summarized as follows:

\[
c(t) = c_3 e^{-\left(\frac{\rho + n - \alpha z}{\theta}\right)t} e^{\int z dt},
\]

(56)

\[
k(t) = c_4 e^{-nt} e^{-\int (x - z) dt},
\]

(57)

\[
u \left( \frac{c_2 e^{\theta t} e^{\rho t} (1 - \theta)}{k(t)} \frac{1}{u(t)} \right) + \theta x(t) - (\rho + n - n\theta) + (1 - \beta)(1 - \theta)z(t)
\]

\[
h(t) = \frac{z(t)^{\frac{1}{\beta}} k(t)^{\frac{\alpha - 1}{\beta}}}{u(t)}
\]

(58)

\[
\lambda(t) = c(t)^{-\theta},
\]

(60)
\[ \lambda_2(t) = c_1 e^{-(\nu - \rho - n + m) t}, \]  

where variables \( x(t) \) and \( z(t) \) satisfy equations (16) and (17). The arbitrary constants \( c_1, c_2, c_3 \) and \( c_4 \) can be determined from initial and transversally conditions. The exact solution provided (56)-(61) can be further analyzed by solving equations (16) and (17) numerically for \( x(t) \) and \( z(t) \). In present form, it cannot be analyzed in economics sense and is interesting for only mathematical point of view. In next section, I provide exact solution of model with some economics interpretation for the special case when \( \alpha = \theta \). This assumption is often used in economic growth literature to study the behavior of the economy.

4. **An exact solution when** \( \alpha = \theta \). The assumption \( \alpha = \theta \) simplifies the equations (16) and (17) for the variables \( x(t) \) and \( z(t) \) as follows:

\[
\frac{\dot{x}}{x} = x - \frac{(n + \rho - n\alpha)}{\alpha}, \tag{62}
\]

and

\[
\frac{\dot{z}}{z} = \left(1 - \alpha - \beta \right) x - (1 - \alpha) z + \frac{(1 - \alpha - \beta)n + \beta(m + \nu)}{1 - \beta}. \tag{63}
\]

The equation (62) is solved subject to the initial condition \( x(0) = x_0 = \frac{c_0}{k_0} \) which finally results in

\[
x(t) = \frac{x_0 x^*}{x_0 + (x^* - x_0)e^{(x^*)t}}, \tag{64}
\]

where \( x^* = \frac{\rho + n(1 - \alpha)}{\alpha} > 0 \). The expression for \( x(t) \) approaches to \( x^* \) provided \( x^* = x_0 \) and thus equation (64) yields

\[
x(t) = x^* = \frac{\rho + n(1 - \alpha)}{\alpha}. \tag{65}
\]

The solution of equation (63) with initial condition \( z(0) = z_0 \) is

\[
z(t) = \frac{z^* z_0}{z_0 + (z^* - z_0)e^{-(1 - \alpha) z^* t}} \tag{66}
\]

where \( z^* \) is

\[
z^* = \frac{(1 - \alpha - \beta)(\rho + n) + \beta\alpha(m + \nu)}{\alpha(1 - \alpha)(1 - \beta)}. \tag{67}
\]

Note that

\[
e^{\int z(t)} = e^{z^* t} \left[ (z_0 - z^*) e^{z^* (\alpha - 1) t} - z_0 \right]^{\frac{1}{1 - \alpha}} = e^{z^* t} \left[ (1 - z_0) e^{z^* t} - 1 \right]^{\frac{1}{1 - \alpha}} = \left( \frac{z_0 z^*}{z(t)} \right)^{\frac{1}{1 - \alpha}} \tag{68}
\]

The initial condition \( c(0) = c_0 \) yields \( c_3 = \frac{c_0}{(-1)^{\frac{1}{1 - \alpha}} z^*^{\frac{1}{1 - \alpha}}} \) and equation (56) for \( c(t) \) takes following form

\[
c(t) = c_0 z_0^{\frac{1}{1 - \alpha}} e^{-\frac{\beta(\rho - \nu - m + n)}{(1 - \alpha)(1 - \beta)} t} \left( z(t) - \frac{1}{1 - \alpha} \right). \tag{69}
\]

The initial condition \( k(0) = k_0 \) yields \( c_4 = \frac{k_0}{(-1)^{\frac{1}{1 - \alpha}} z^{\frac{1}{1 - \alpha}}} \) and equation (57) for \( k(t) \) takes following form

\[
k(t) = k_0 z_0^{\frac{1}{1 - \alpha}} e^{-\frac{\beta(\rho - \nu - m + n)}{(1 - \alpha)(1 - \beta)} t} \left( z(t) - \frac{1}{1 - \alpha} \right). \tag{70}
\]
With the aid of equations (65) and (66), Equation (58) for \( u(t) \) simplifies to
\[
u(t) = \frac{\rho - (\nu - n + m)\beta}{\nu(1-\beta) + c_2 \nu(1-\alpha) e^{\beta t}}.
\]
(71)

which approaches to equilibrium value \( u^* = \frac{\rho - (\nu - n + m)\beta}{\nu(1-\beta)} \) provided \( c_2 = 0 \). Equation (59) for \( h(t) \) with initial condition \( h(0) = h_0 \) yields
\[
h(t) = h_0 e^{-\frac{(\nu - n + m) e^{\beta t}}{(1-\beta)}}.
\]
(72)

where \( h_0 = \frac{k_0}{e^{\beta t}} \). The initial conditions yield \( c_1 = \lambda_2(0) = \frac{\beta}{\nu} c_0 \theta u_0^{\beta - 1} k_0 h_0^{\beta - 1} \) and thus equation (61) for \( \lambda_2 \) gives
\[
\lambda_2 = \frac{\beta}{\nu} c_0 \theta u_0^{\beta - 1} k_0 h_0^{\beta - 1} e^{-(\nu - n + m)t}.
\]
(73)

The exact solution for all variables of the model for \( \alpha = \theta \) can be summarized as follows:
\[
c(t) = c_0 z_0^{\frac{1}{\alpha}} e^{\frac{\beta (\nu - n + m) t}{(1-\alpha)(1-\beta)}} z(t)^{\frac{\beta}{1-\alpha}},
\]
(74)
\[
k(t) = k_0 z_0^{\frac{1}{\alpha}} e^{\frac{\beta (\nu - n + m) t}{(1-\alpha)(1-\beta)}} z(t)^{\frac{\beta}{1-\alpha}},
\]
(75)
\[
u(t) = \frac{\rho - (\nu - n + m)\beta}{\nu(1-\beta)} = u^* = u_0,
\]
(76)
\[
h(t) = h_0 e^{-\frac{(\nu - n + m) e^{\beta t}}{(1-\beta)}}.
\]
(77)
\[
\lambda_1 = c(t)^{-\alpha} = c_0^{-\alpha} z_0^{\frac{1}{\alpha}} e^{\frac{\beta (\nu - n + m) t}{(1-\alpha)(1-\beta)}} z(t)^{\frac{\beta}{1-\alpha}},
\]
(78)
\[
\lambda_2 = \frac{\beta}{\nu} c_0 \theta u_0^{\beta - 1} k_0 h_0^{\beta - 1} e^{-(\nu - n + m)t},
\]
(79)

where
\[
\beta(\nu + m - n) < \rho < \nu + m - n,
\]
(80)
\[
\frac{c_0}{k_0} = \frac{(\rho + n - n\alpha)}{\alpha} > 0,
\]
(81)
\[
h_0 = \frac{k_0^{\frac{1}{1-\alpha}}}{z_0^{\frac{1}{1-\alpha}}},
\]
(82)
\[
z(t) = \frac{z_0}{z_0 + (z^* - z_0) e^{-(1-\alpha)z^* t}},
\]
(83)
\[
z^* = \frac{(1 - \alpha - \beta)(\rho + n) + \alpha \beta (m + \nu)}{\alpha (1 - \alpha)(1 - \beta)}.
\]
(84)
The exact solution of all variables of the model is provided in equations (74)-(79) satisfying conditions (80)-(84). The exact solutions (74)-(79) are graphically represented in Fig 1 for the Benchmark values of parameters given in table 1. The variables \( c(t) \) and \( k(t) \) decrease in short run but increase in long-run and attain the equilibrium value. The variable \( u \) remains constant and variable \( h(t) \) is an increasing function of time.
Table 1. Parameters values

| \( \theta \) = | \( \alpha \) | \( \beta \) | \( \rho \) | \( \nu \) | \( n \) | \( m \) | \( k_0 \) | \( h_0 \) |
|----------------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.33           | 0.25   | 0.05   | 0.1    | 0.04   | 0.01   | 40     | 10     |

Figure 1. Evolution over time of \( c(t) \), \( u(t) \), \( k(t) \) and \( h(t) \).

4.1. Effect of change of share of human capital \( \beta \). Next, I check the effect of change of share of human capital \( \beta \) on the variables of model and it is graphically provided in Figure 2. I use values of all other parameters from Table 1 except for \( \beta \) and for \( \beta \) I have considered three values 0.25, 0.4 and 0.67. Note that 0.67 represents the case \( \beta = 1 - \alpha \), the basic Lucas-Uzawa model. Figure 2 shows that as \( \beta \) increases the consumption and physical capital decreases in short run but increases in long run. The variable \( u \) decreases with increase in \( \beta \). The human capital increases with an increase in \( \beta \).

4.2. Properties of exact solution and convergence to BGP. Now, I prove that the exact solution of all variables of the model provided in equations (74)-(79) satisfying conditions (80)-(84) converges to the BGP of model provided in proposition 1 for \( \alpha = \theta \) case. The growth rates of control \((c, u)\), state \((k, h)\) and costate variables \((\lambda_1, \lambda_2)\) are

\[
\frac{\dot{c}}{c} = \frac{\beta (\nu - \rho + m - n)}{(1 - \alpha)(1 - \beta) - 1 - \alpha \frac{\dot{z}}{\dot{z}}}.
\]
Figure 2. Effect of change of $\beta$ on evolution over time of $c(t)$, $u(t)$, $k(t)$ and $h(t)$.

\[
\dot{k} = \frac{\beta(\nu - \rho + m - n)}{(1-\alpha)(1-\beta)} - \frac{1}{1-\alpha}\frac{\dot{z}}{z},
\]
\[
\dot{h} = \frac{(\nu - \rho + m - n)}{(1-\beta)},
\]
\[
\dot{u} = 0,
\]
\[
\dot{\lambda}_1 = -\frac{\alpha \beta(\nu - \rho + m - n)}{(1-\alpha)(1-\beta)} + \frac{\alpha}{1-\alpha}\frac{\dot{z}}{z},
\]
\[
\dot{\lambda}_2 = -(\nu - \rho + m - n),
\]
where
\[
\frac{\dot{z}}{z} = \frac{z^*(z^* - z_0)(1-\alpha)e^{-(1-\alpha)z^* t}}{z_0 + (z^* - z_0)e^{-(1-\alpha)z^* t}}.
\]

Note that as $t \to \infty$ the growth rate of $z$ i.e. $\frac{\dot{z}}{z}$ given in (91) approaches to zero. The growth rates of consumption given in (85) and the physical capital given in (86)
decrease over time and in long run attain the same equilibrium value $\frac{\beta(\nu - \rho + m - n)}{(1 - \alpha)(1 - \beta)}$. The growth rate of $h$ given in (87) is $\frac{(\nu - \rho + m - n)}{(1 - \beta)}$ and growth rate of $u$ given in (88) is zero. The growth rate of $\lambda_1$ given in (89) attains value $-\frac{\alpha \beta (\nu - \rho + m - n)}{(1 - \alpha)(1 - \beta)}$ as $t \to \infty$. As $t \to \infty$ the growth rate of $\lambda_2$ given in (90) approaches to $-(\nu - \rho + m - n)$. Thus exact solutions of variables given in equations (74)-(79) satisfy all properties of BGP provided in proposition 1 for $\alpha = \theta$ case.

5. Conclusions. In this article, the sufficiency issues, first integrals and exact solutions for the Uzawa-Lucas model with unskilled labor were investigated. The sufficient conditions were established by utilizing Arrow’s Sufficiency theorem. The BGP was completely determined and non-negativeness conditions for the BGP were provided. The first integrals were established by the partial Hamiltonian approach. The first constant of motion was interpreted as “shadow price of human capital $\times e^{(\nu - \rho + m - n)t}$ = constant,” where $\nu - \rho - n + m$ denotes the efficiency of the human capital creation process minus the discount factor minus the natural growth rate of human capital. It is difficult to provide some economic meaning of the second constant of motion. Then the exact solutions for all the variables were constructed with the aid of first integrals. The growth rate of all variables and graphical representation of exact solutions were provided for the special case when IIES is the same as the share of physical capital. The graphical representation of the exact solutions demonstrated that the variables $c(t)$ and $k(t)$ decreased in short run but increased in long run and attained the equilibrium value. The variable $u$ remained constant and variable $h(t)$ was an increasing function of time. The growth rates of consumption and the physical capital decreased over time and in long run attained the same equilibrium value $\frac{\beta(\nu - \rho + m - n)}{(1 - \alpha)(1 - \beta)}$. The growth rate of $h$ was $\frac{(\nu - \rho + m - n)}{(1 - \beta)}$ and growth rate of $u$ was zero. The growth rate of $\lambda_1$ attained value $-\frac{\alpha \beta (\nu - \rho + m - n)}{(1 - \alpha)(1 - \beta)}$ as $t \to \infty$. As $t \to \infty$ the growth rate of $\lambda_2$ approached to $-(\nu - \rho + m - n)$. Moreover, the effect of change of share of human capital $\beta$ on the variables of model was studied as well. As the share of human capital $\beta$ increased the consumption and physical capital decreased in short run but increased in long run. The variable $u$ decreased with increase in $\beta$. The human capital increased with an increase in $\beta$.

Appendix A.

Necessary and sufficient conditions for optimal control theory. Let $t$ be the independent variable which is usually taken to be time and $(q, p) = (q^1, ..., q^n, p_1, ..., p_n)$ the phase space coordinates. In the applications to equations of economics, $q^1, ..., q^n$ are the state variables and $p_1, ..., p_n$ the costate variables. The following results are adopted from [29, 18, 6, 1, 11].

Necessary conditions for optimal control theory: Pontryagin’s maximum principle.

Definition 1. In economic growth theory, the optimal control problem is of the form

$$\text{Maximize } \int_0^\infty F(t, q^i, u_i) \, dt$$

subject to $q^i = f^i(t, q^i, u_i), \quad i = 1, \ldots, n,$ (A-1)

where $u$ is the control vector $u = (u^1, \ldots, u^m)$, $m \leq n$ or $m > n$ as well as appropriate transversality and initial conditions are imposed.
The present value Hamiltonian is defined as
\[ H = F(t, q^i, u_i) + p_i f^i(t, q^i, u_i). \] (A-2)

The Pontryagin’s maximum principle [29] gives necessary conditions for optimal control for \( H \)
\[ \frac{\partial H}{\partial u_i} = 0, \]
\[ q^i = \frac{\partial H}{\partial p_i}, \]
\[ \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \] (A-3)

**Definition 2.** The optimal control problem involving a discount factor \( e^{-\rho t} \) is of the form
\[
\text{Maximize } \int_0^\infty F(t, q^i, u_i) e^{-\rho t} dt \\
\text{subject to } \dot{q}^i = f^i(t, q^i, u_i), \quad i = 1, \ldots, n, \] (A-4)

where \( u \) is the control vector \( u = (u^1, \ldots, u^m) \), \( m \leq n \) or \( m > n \) as well as appropriate transversality and initial conditions are imposed. The integrand contains the discount factor \( e^{-\rho t} \). The current value Hamiltonian \( H^c(t, q, p, u) \) (see, e.g. [6]) is
\[ H^c = F(t, q^i, u_i) + p_i f^i(t, q^i, u_i), \] (A-5)

The Pontryagin’s maximum principle [29] gives necessary conditions for optimal control for \( H^c \)
\[ \frac{\partial H^c}{\partial u_i} = 0, \]
\[ q^i = \frac{\partial H^c}{\partial p_i}, \]
\[ \dot{p}_i = -\frac{\partial H^c}{\partial q^i} + \Gamma_i, \quad i = 1, \ldots, n, \] (A-6)

where \( \Gamma_i = \rho p_i \) and condition \( \frac{\partial H^c}{\partial u_i} = 0 \) yields \( u_i = h_i(p_i, q^i) \). Note that each \( \Gamma_i \) is mostly taken as a linear function of \( p_i \) in economic applications in the earlier literature of optimal control theory in which the functional maximized contains the discount factor \( e^{-\rho t} \) in the integrand [6] and this is known as constant time preferences in economic growth theory. For endogenous time preferences, \( \Gamma_i \) is taken more generally as a nonzero function of \( t, p_i, q^i \). In the correspondence to mechanics they deal with non-conservative forces if the \( \Gamma_i \) are nonzero.

We can have analogue minimization problems as well. It is worthy to mention here that one can also define the present value Hamiltonian for optimal control problem \((A-4)\) involving a discount factor \( e^{-\rho t} \) viz.
\[ H = F(t, q^i, u_i)e^{-\rho t} + p_i f^i(t, q^i, u_i), \] (A-7)
and necessary conditions are given by \((A-3)\).

The Hamiltonian evaluated along the optimal paths i.e. evaluated at \( q^*i, p^*_i \) and \( u^*_i \) is termed as the optimal Hamiltonian and is denoted by \( H^*(t) \).
Sufficient conditions for optimal control theory. The Pontryagin’s maximum principle provides a set of necessary conditions for the optimal control and in general these are not sufficient. These become sufficient if one of following two theorem hold (i) The Mangasarian sufficiency theorem (ii) The Arrow Sufficiency theorem.

The Mangasarian sufficiency Theorem. The Mangasarian sufficiency theorem [18, 6] states that for the optimal control problem (A-1) the necessary conditions of maximum principle are also sufficient if

(i) functions $F$ and $f^i$ are differentiable and concave in the variables $(q^i, u_i)$ jointly, and

(ii) in the optimal solution it is true that $p_i(t) \geq 0$ for all $t \in [0, \infty)$ if $f^i$ are nonlinear functions.

The Arrow sufficiency Theorem. The Arrow’s sufficiency theorem [6, 1, 11] provides a weaker condition than Mangasarian’s theorem. The condition $\frac{\partial H}{\partial u_i} = 0$ yields

$$u^*_i = h_i(t, p_i, q^i).$$

The present value Hamiltonian (A-2) with the aid of (A-8) becomes

$$H^0(t, q^i, p_i) = F(t, q^i, u^*_i) + p_i f^i(t, q^i, u^*_i),$$

and this is refereed to as a maximized Hamiltonian. Note that maximized Hamiltonian $H^0(t, q^i, p_i)$ is different from optimal Hamiltonian $H^*(t)$.

The Arrow’s theorem states that for the optimal control problem (A-1) the necessary conditions of maximum principle are also sufficient if the maximized Hamiltonian function $H^0$ defined in (A-9) is concave in the variables $q^i$ for all $t \in [0, \infty)$, for given $p_i$.

Note that $H^0$ can be concave in $q^i$ even if $F$ and $f^i$ are not concave in $(q^i, u_i)$ and this makes Arrow’s condition a weaker requirement than Mangasarian’s condition.

The Mangasarian sufficiency theorem and Arrow sufficiency theorem can be rephrased in terms of current value Hamiltonian $H^c$ as well.

Appendix B. Let time $t$ be the independent variable and $(q, p) = (q^1, ..., q^n, p_1, ..., p_n)$ the phase space coordinates. We provide an overview of partial Hamiltonian approach [20, 21]. The following definitions and results are adapted from [20, 21, 22, 6].

The total derivative operator with respect to the time $t$ is given by

$$D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \cdots.$$  \hfill (B-1)

The Euler operator $\delta/\delta q^i$ is defined as

$$\delta \delta q^i = \frac{\partial}{\partial q^i} - D \frac{\partial}{\partial q^i}, \quad i = 1, 2, \cdots, n,$$  \hfill (B-2)

and the variational operator $\delta/\delta p_i$ is given by

$$\delta \delta p_i = \frac{\partial}{\partial p_i} - D \frac{\partial}{\partial p_i}, \quad i = 1, 2, \cdots, n.$$  \hfill (B-3)

The summation convention applies for repeated indices.

The variables $t, q, p$ are independent and satisfy

$$\dot{p}_i = D(p_i), \quad \dot{q}^i = D(q^i), \quad i = 1, 2, \cdots, n.$$  \hfill (B-4)
The following result is adapted from [20] and is essential for construction of first integrals via partial Hamiltonian approach:

**Theorem 1.** (see [20]) An operator $X$ of the form

$$X = \xi(t, q^i, p_i) \frac{\partial}{\partial t} + \eta^i(t, q^i, p_i) \frac{\partial}{\partial q^i} + \zeta_i(t, q^i, p_i) \frac{\partial}{\partial p_i}. \quad (B-5)$$

is said to be a partial Hamiltonian operator corresponding to a current value Hamiltonian $H(t, q^i, p_i, c_i)$ which satisfies (A-6), if there exists a function $B(t, q^i, p_i)$ such that

$$\zeta_i \frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - HD(\xi) = D(B) + (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma_i) \quad (B-6)$$

holds. The first integral corresponding to the system (A-6) associated with a partial Hamiltonian operator $X$ of the current value Hamiltonian $H(t, q^i, p_i, c_i)$ is determined from

$$I = p_i \eta^i - \xi H - B, \quad (B-7)$$

where $B(t, p_i, q^i)$ is a gauge-like function.

**REFERENCES**

[1] K. J. Arrow, Applications of control theory to economic growth, *Mathematics of the Decision Sciences, Part 2*, American Mathematical Society, Providence, R.I., (1968), 85–119.
[2] J. Caballé and M. S. Santos, On endogenous growth with physical and human capital, *Journal of Political Economy*, (1993), 1042–1067.
[3] V. V. Chari, L. E. Jones and R. E. Manuelli, The growth effects of monetary policy, *Federal Reserve Bank of Minneapolis, Quarterly Review-Federal Reserve Bank of Minneapolis*, 19 (1995), 18.
[4] A. Chaudhry and R. Naz, Closed-form solutions for the Lucas-Uzawa Growth model with logarithmic utility preferences via the partial Hamiltonian approach, *Discrete Contin. Dyn. Syst. Ser. S*, 11 (2018), 643–654.
[5] A. F. Cheviakov and R. Naz, A recursion formula for the construction of local conservation laws of differential equations, *Journal of Mathematical Analysis and Applications*, 448 (2017), 198–212.
[6] A. C. Chiang, *Elements of Dynamic Optimization*, Illinois: Waveland Press Inc., 2000.
[7] R. P. Cysne, A note on the non-convexity problem in some shopping-time and human-capital models, *Journal of Banking & Finance*, 30 (2006), 2737–2745.
[8] R. P. Cysne, A note on Inflation and Welfare, *Journal of Banking & Finance*, 32 (2008), 1984–1987.
[9] M. Ferrara and L. Guerrini, A note on the Uzawa-Lucas model with unskilled labor, *Applied Sciences*, 12 (2010), 90–95.
[10] B. U. Haq and I. Naeem, First integrals and analytical solutions of some dynamical systems, *Nonlinear Dynamics*, (2018), 1–19.
[11] M. I. Kamien and N. L. Schwartz, Sufficient conditions in optimal control theory, *Journal of Economic Theory*, 3 (1971), 207–214.
[12] A. H. Kara, F. M. Mahomed, I. Naeem and C. Wafo Soh, Partial Noether operators and first integrals via partial Lagrangians, *Mathematical Methods in the Applied Sciences*, 30 (2007), 2079–2089.
[13] R. E. Lucas Jr., Inflation and welfare, *Monetary Theory as a Basis for Monetary Policy*, Palgrave Macmillan UK, (2001), 96–142.
[14] R. E. Lucas Jr., On the mechanics of economic development, *Journal of Monetary Economics*, 22 (1988), 3–42.
[15] W.-X. Ma, Conservation laws by symmetries and adjoint symmetries, *Discr. Cont. Dyn. Sys. Ser. S*, 11 (2018), 707–721.
[16] W.-X. Ma and M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras, *Journal of Physics A: Mathematical and General*, 39 (2006), 10787–10801.
[17] K. S. Mahomed and R. J. Moitsheki, First integrals of generalized Ermakov systems via the Hamiltonian formulation, *International Journal of Modern Physics-B*, 30 (2016), 1640019, 12 pp.

[18] O. L. Mangasarian, Sufficient conditions for the optimal control of nonlinear systems, *SIAM Journal on Control*, 4 (1966), 139–152.

[19] C. B. Mulligan and X. Sala-i-Martin, *Transitional Dynamics in Two-Sector Models of Endogenous Growth* (No. w3986), National Bureau of Economic Research, 1992.

[20] R. Naz, F. M. Mahomed and A. Chaudhry, A partial Hamiltonian approach for current value Hamiltonian systems, *Communications in Nonlinear Science and Numerical Simulation*, 19 (2014), 3600–3610.

[21] R. Naz, The applications of the partial Hamiltonian approach to mechanics and other areas, *International Journal of Non-Linear Mechanics*, 86 (2016), 1–6.

[22] R. Naz, F. M. Mahomed and A. Chaudhry, A partial Lagrangian method for dynamical systems, *Nonlinear Dynamics*, 84 (2016), 1783–1794.

[23] R. Naz, A. Chaudhry and F. M. Mahomed, Closed-form solutions for the Lucas-Uzawa model of economic growth via the partial Hamiltonian approach, *Communications in Nonlinear Science and Numerical Simulation*, 30 (2016), 299–306.

[24] R. Naz and A. Chaudhry, Comparison of closed-form solutions for the Lucas-Uzawa model via the partial Hamiltonian approach and the classical approach, *Mathematical Modelling and Analysis*, 22 (2017), 464–483.

[25] R. Naz and A. Chaudhry, Closed-form solutions of Lucas-Uzawa model with externalities via partial Hamiltonian approach, *Computational and Applied Mathematics*, 37 (2018), 5146–5161.

[26] R. Naz and I. Naeem, The artificial hamiltonian, first integrals, and closed-form solutions of dynamical systems for epidemics, *Zeitschrift fr Naturforschung A*, 73 (2018), 323–330.

[27] R. Naz, F. M. Mahomed and D. P. Mason, Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics, *Applied Mathematics and Computation*, 205 (2008), 212–230.

[28] R. Naz, I. L. Freire and I. Naeem, Comparison of different approaches to construct first integrals for ordinary differential equations, *Abstr. Appl. Anal.*, (2014), Art. ID 978636, 15 pp.

[29] L. S. Pontryagin, *Mathematical Theory of Optimal Processes*, CRC Press, 1987.

[30] P. E. Robertson, Demographic shocks and human capital accumulation in the Uzawa-Lucas model, *Economics Letter*, 74 (2002), 151–156.

[31] H. Uzawa, Optimum technical change in an aggregative model of economic growth, *International Economic Review*, 6 (1965), 18–31.

[32] T. Wolf, A comparison of four approaches to the calculation of conservation laws, *European Journal of Applied Mathematics*, 13 (2002), 129–152.

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