LAPLACE INVARIANTS OF DIFFERENTIAL OPERATORS

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Abstract. We identify conditions giving large natural classes of partial differential operators for which it is possible to construct a complete set of Laplace invariants. In order to do that we investigate general properties of differential invariants of partial differential operators under gauge transformations and introduce a sufficient condition for a set of invariants to be complete. We also give a slightly stricter condition that guarantees existence of such a set. The proof is constructive. The method gives many examples of invariants previously known in the literature as well as many new examples including multidimensional.

1. Introduction

Gauge transformations \((\varphi \mapsto e^\theta \varphi)\) of differential operators are important transformations that preserve algebraic structure of an operator, such as its “factorizability” into factors of some fixed form or existence of Darboux transformations. Invariant properties like these are best described by gauge invariants (algebraic expressions in the coefficients of the operator and their derivatives).

The first examples of gauge invariants for differential operators are the Laplace invariants of the hyperbolic second order operator \(L = \partial_{xy} + a\partial_x + b\partial_y + c\) [4]. These are the gauge invariants \(h\) and \(k\) that can be thought of as derived from incomplete factorizations of this operator as \(L = (\partial_x + b)(\partial_y + a) + h\) and \(L = (\partial_y + a)(\partial_x + b) + k\). These invariants uniquely define the gauge class of the operator, and so are a complete set of invariants.

This is the starting point of a method for solution of \(Lu = 0\) for the above operator in the closed form. Two Laplace transformations \(L \mapsto L_1\) and \(L \mapsto L_{-1}\) are defined by intertwining relations \(N_1L = L_1(\partial_x + b)\) and \(N_{-1}L = L_{-1}(\partial_y + a)\). Each of the transformations swaps the values of \(h\) and \(k\) and then changes one of them. In addition, the two Laplace transformations are (up to the gauge equivalence class) inverses of each other. So as the result of consecutive application of Laplace transformations to some operator \(L\) we have a chain of the corresponding pairs of invariants (not a lattice as may be expected). This “Laplace chain” is finite if one of the invariants is zero at some point in each direction of the chain. This corresponds to factorizability of the transformed operator. The original equation \(Lu = 0\) then can be solved in closed form invoking the invertibility of Laplace transformations.

Laplace transformations are members of a larger group of transformations — Darboux transformations — which can be defined algebraically by the means of an intertwining relation \(NL = L_1M\). For operators \(L = \partial_{xy} + a\partial_x + b\partial_y + c\) it was proved [20, 21] (and then a discrete and a semi-discrete analogues of this result was proved by S. Smirnov [27]) that Laplace transformations are the only invertible Darboux transformations and all the others, even corresponding to a higher order operator \(M\), are not. These non-invertible Darboux transformations induce a map of kernels \(\text{ker } L \rightarrow \text{ker } L_1\) which is not monomorphic, so some solutions are lost. A new construction of invertible Darboux transformations for a large class of operators was discovered in [19] and for even larger class in [10]. Another method using pseudodifferential operators was proposed in [8]. Multidimensional Darboux transformations are proposed by G. Hovhannisyan et al. [11, 12]. Complete
classification of Darboux transformations on the superline (operators of arbitrary order and Darboux transformations of arbitrary order) was obtained in \[9, 14\]. Note that with every manifold one can naturally associate a commutative algebra consisting of formal sums of densities of arbitrary real weights. It is useful for geometric analysis of differential operators. In \[23\], we studied factorization of differential operators on such algebra in the case of the line, with an eye at extending Darboux transformations theory to them.

Gauge invariants can be found using regularized moving frames method of M. Fels and P. Olver \[6, 7\], see also E. Mansfield’s book \[15\], which was developed later also for pseudo-groups by P. Olver and J. Pohjanpelto \[17\]. They also proved that the algebra of invariants can be generated by a finite number of invariants and a finite number of invariant derivatives (which are particular invariant differential operators on the algebra of invariants). The specifics of the use of the regularized moving frames method for gauge invariants of differential operators is described by the second author with E. Mansfield in \[22\].

Concerning Laplace invariants for differential operators the following results are known. Dzhokhadze’s 2004 \[5\] and Mironov’s 2009 \[16\] for 4th order operators and Ch. Athorne and H. Yilmaz’s 2016 \[2\] for arbitrary order operators of the form $\sum_{|\nu|=0}^d \left( \sum_{\forall i,j,v \neq v} a_{\nu} \partial^\nu \right)$, where $d$ is the order of the operator. Thus the order of the operator cannot be larger than the number of the independent variables available. For example, in bivariate case the highest possible order is two and such operators have form $\partial_x \partial_y + a_1 \partial_x + a_2 \partial_y + a_3$; for dimension three the highest possible order is three and such operators have form $\partial_x \partial_y \partial_z + a_1 \partial_x \partial_y + a_2 \partial_x \partial_z + a_3 \partial_y \partial_z + a_4 \partial_x + a_5 \partial_y + a_6 \partial_z + a_7$. Ch. Athorne and H. Yilmaz’s 2016 \[2\] found some Laplace invariants for such operators of arbitrary order and of arbitrary dimension. Afterwards they constructed and investigated the corresponding Darboux (Laplace) transformations \[1, 3\].

In 2007, the second author with F. Winkler \[25\] proposed an algebraic structure, a ring of obstacles, where the remainders of incomplete factorizations for operators of arbitrary order and arbitrary number of variables become invariants. The method gave in particular Laplace invariants for bivariate operators with principal symbols $(p\partial_x + q\partial_y)\partial_x \partial_y$, $\partial_x^2 \partial_y$, and $\partial_y^3$. The Laplace invariants set given by this method is not complete; however, we managed to find an extra (“non-Laplace”) invariant for each case making the resulting sets complete \[24, 26, 18\].

M. van Hoeij with students and collaborators, see e.g., \[13\], works on the solution methods for linear homogeneous ordinary differential equations with rational function or polynomial coefficients. Such is for example, the problem of hypergeometric solutions. In \[13\] and other works, the authors use gauge transformations (they are called there exponential transformations) and construct Darboux transformations and the corresponding Laplace invariants and use them to simplify the equations. Note that there is difference between finding a ring of invariants as specified by some arbitrary choice of a generating set and finding a “distinguished ” generating set whose elements can carry extra information. (The reader can have in mind classical examples of distinguished invariants in differential geometry such as curvature or torsion, or e.g. particular characteristic classes such as Chern classes, etc.) In the literature, Laplace invariants typically mean gauge invariants distinguished in this way. Unlike their classical prototype, they cannot be obtained by a direct generalization of the Laplace method as “remainders” of incomplete factorizations: it is known \[25\] that such “remainders” are not invariants for a general operator. (If this remainder is an operator, then even its principal symbol is not invariant in the general situation.) Nevertheless, Laplace type
invariants are distinguished in the sense that they control representability of an operator in some “generalized” factorized form as we show here.

In the present paper, the main results are contained in Theorem 1 and Theorem 7 which together show that under certain rather general and natural conditions an operator has a complete set of Laplace invariants.

The proofs are constructive and provide a general method of constructing complete sets of Laplace invariants for a very large class of operators which include previously considered classes. We show that examples of Laplace invariants existing in the literature can be obtained by our method, and we also have examples with new types of operators.

The paper is organized as follows. After preliminaries in Sec. 2, in Sec. 3 we define maximally generated and approximately flat classes of operators, which impose some natural restrictions on operators (can be multidimensional and of arbitrary high order). From the perspective of our method, known Laplace invariants can be classified into four types (classification may be incomplete but we do not need that here), we call them maximal, extra, compatibility, and upward invariants. We illustrate them with examples. For these classes of operators we prove Theorem 1 that if one has enough number of invariants of each type, then the set of invariants is complete. In Sec. 4 we introduce the method, first by demonstrating it on known and new examples. Informally, a complete set of gauge invariants obtained by any classical method consists of “nice/short formula” low degree invariants and of some “huge formula” ones of high degrees. The proposed method replaces “huge” ones with other “huge” that are now associated with some generalized “incomplete factorization” of an operator, so now they have structure and meaning. These are what are known as Laplace invariants in the literature. In Sec. 5 we give a theoretical justification of the method and prove the main result, Theorem 7, that under some natural conditions (maximally generated, framed, approximately flat) an operator has a complete set of (Laplace) invariants. The proof is constructive and is essentially the method that is illustrated by examples in Sec. 3.

2. Preliminaries

Let $K$ be arbitrary commutative differential field of characteristic zero with commuting derivations $\partial_1, \ldots, \partial_n$. We consider $K$ to be differentially closed. We denote by $\mathcal{D}(K)$ the corresponding algebra of differential operators over $K$. For any integral vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}_0^n$ we write $\partial^\mathbf{v}$ for the differential monomial $\partial_1^{v_1} \cdots \partial_n^{v_n}$. Many concepts below are conveniently expressed if we treat the usual multi-indices used to denote derivatives as vectors. In particular we will be using the standard basis vectors $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$.

There is a natural partial order on these vectors, where we write $\mathbf{u} \preceq \mathbf{v}$ iff every entry of $\mathbf{u}$ is less than or equal to the corresponding entry of $\mathbf{v}$. If $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$, we say that $\mathbf{u}$ is below $\mathbf{v}$, or that $\mathbf{v}$ is above $\mathbf{u}$. We extend this terminology to terms of an operator $L \in \mathcal{D}(K)$, saying that $a_\mathbf{u} \partial^\mathbf{u}$ is below $a_\mathbf{v} \partial^\mathbf{v}$ iff $\mathbf{u}$ is below $\mathbf{v}$. We define the order of $\mathbf{v}$ as the number $\sum_{i=1}^n v_i$, and likewise extend this terminology to “differential monomials”. Thus a constant term is of order 0, a term with a single derivative is of order 1, and so on.

The principal symbol of an operator is the sum of the highest-order terms. We also define the leading part of operator $L = \sum a_\mathbf{v} \partial^\mathbf{v}$ as

$$Lead_{\preceq}(L) = \sum_{\partial^\mathbf{v} \in \max_{\preceq}(L)} a_\mathbf{v} \partial^\mathbf{v},$$

where $\max_{\preceq}(L)$ is the set of natural numbers with zero
where $\text{max}_{\preceq}(L)$ denotes the set of all differential monomials $\partial^v$ of $L$ that are below no other differential monomial in $L$. Note that the notions of principal symbol and the leading part are not the same. We will call elements of $\text{max}_{\preceq}(L)$ maximal. We call a vector $v$ maximal iff the differential monomial $\partial^v$ is maximal.

**Example 1.** Let $L = \partial_{xx} + a\partial_x + b\partial_y + c$. Then $\partial_{xx}$ is its principal symbol, and $\partial_{xx} + b\partial_y$ is its leading part.

**Definition 1.** The down set of a set of terms $T$, written $\downarrow T$, is the set consisting of $T$ together with all terms that are below any term in $T$ relative to $\preceq$. A set of terms is downward closed iff it is equal to its own down set. Given terms $s$ and $t$ with $s \prec t$ but where there is no term $t'$ with $s \prec t' \prec t$, we say that $t$ covers $s$. A term that is covered by a maximal term, and covered by nothing but maximal terms, will be called submaximal. Definitions in the literature of when an element of a partial order is submaximal vary, but the above definition is best for this paper.

**Example 2.** The down set of $\partial_x \partial_y$ is $\{\partial_x \partial_y, \partial_x, \partial_y, 1\}$, and the set of terms $\{\partial_x \partial_y, \partial_y, 1\}$ is not downward closed.

We will be looking at invariants for operators, or more precisely, for invariants of classes of operators with a given set of maximal terms. Since the gauge transformation of a term with vector $v$ usually contains terms with every vector below $v$, we restrict our investigation to classes of operators with sets of terms that are downward closed. So given a set of terms $T$, which may possibly have arbitrary coefficients, we let $L$ be the set of all operators with terms in the downward closure of $T$.

**Definition 2.** Let a set of terms $T$ be given, where none is below any of the others in $\preceq$. Let $L(T)$ be the set of all operators $L$ which have $T$ as their sets of maximal terms. A set of operators of this form will be said to be generated by its maximal terms, or maximally generated.

Let $L$ be a set of operators that is generated by its maximal terms. Observe that the set of terms that may appear in operators in $L$ is downward closed, and that $L$ is closed under gauge transformations. Our problem will be to obtain differential invariants for $L$.

Given a set $L$ that is closed under gauge transformations, an expression $I$ in terms of the coefficients of terms of members of $L$ and their derivatives is a (differential) invariant of $L$ iff all elements of $L$ that are related by a gauge transformation have the same value of $I$.

### 3. Complete sets of invariants

In this section our goal is to give some sufficient condition for a finite set of invariants to be complete. (In the next section we will turn this into a constructive method.)

A complete set of invariants is such that whenever two operators agree on them, there is a gauge transformation that relates them: if $L$ is a set of operators that is closed under gauge transformations, then the set of invariants $\{I_1, I_2, \ldots I_k\}$ is complete iff every invariant in the set is equal for two operators $L$ and $L'$, there is a function $g \in K$ so that $L' = e^{-g}L e^g$.

We need to distinguish between various kinds of invariants for a maximally generated class of operators $L$.

**Definition 3.** Let the class of operators $L$ be maximally generated, with set of maximal terms $T$. The coefficients of maximal terms are (trivial) invariants for $L$; we call these invariants maximal.
Recall that we are treating multi-indices as vectors.

**Definition 4.** For any term $a_{\nu} \partial^{\nu}$, we call the multi-index $\nu$, the vector of $a_{\nu} \partial^{\nu}$, and write $\nu = v(a_{\nu} \partial^{\nu})$.

Let $L$ be a maximally generated class of operators, and let $T$ be its set of maximal terms. Temporarily make the simplifying assumption that every term in $T$ is of the same order $k$. Consider for the moment a particular operator $L$ in $L$. When $L'$ is obtained from $L$ by a gauge transformation, we have $L' = e^{-g} L e^g$ for some $g \in K$.

Now look at a particular term $a \partial_{\nu}$ in $L$ of degree $k - 1$. It is covered in the partial order $\preceq$ by some of the maximal terms in $T$. We have that the vectors of maximal terms covering $a \partial_{\nu}$ are of the form $v + e_i$ for $i$ in some subset $S$ of $\{1, 2, \ldots, n\}$. Now let $a' \partial_{\nu}$ be the term corresponding to $a \partial_{\nu}$ in $L'$. We have that $a' - a$ is given by

$$a' - a = \sum_{i \in S} (v(i) + 1) b_i g_{x_i}$$

where $b_i$ is the coefficient of the maximal term in $L$ with vector $v + e_i$.

**Example 3.** Let $n = 3$, write $x$ for $x_1$, $y$ for $x_2$, and $z$ for $x_3$. Let

$$T = \{ p \partial_{xyy z}, q \partial_{xyy z}, \partial_{yzzz} \}$$

where $p, q \in K$. Then the term $a_{121} \partial_{xyyz}$ is covered by $p \partial_{xyy z}$ and $q \partial_{xyy z}$ in $\preceq$, but is not below $\partial_{yzzz}$. In this case, $S = \{1, 2\}$, and we have $a'_{121} = a_{121} + 2p_g + 3q_g$.

Other terms that are covered by those in $T$ are those with derivative symbols $\partial_{xyyz}$, $\partial_{xyy y}, \partial_{yy y}, \partial_{xyy y}, \partial_{yzzz}$, $\partial_{zzzz}$ and $\partial_{yzzz}$. The corresponding terms in the operator $L'$ will have $g_x$ in them when their derivative symbols are $\partial_{xyyz}$, $\partial_{yy y}$ and $\partial_{yzzz}$. Similarly, three terms in $L'$ will have $g_y$ in them and three will have $g_z$ in them. We can rewrite $a'_{121} = a_{121} + 2p_g + 3q_g$ as $2p_g + 3q_g = a'_{121} - a_{121}$, and view it as a linear equation in the unknowns $g_x$ and $g_y$. We have similar equations for each of the other 7 terms that are covered by terms in $T$, giving these 8 equations.

$$pg_x = a'_{220} - a_{220}$$
$$2pg_y = a'_{211} - a_{211}$$
$$2pg_x + 3qg_y = a'_{121} - a_{121}$$
$$qg_x = a'_{130} - a_{130}$$
$$qg_y = a'_{103} - a_{103}$$
$$3g_x = a'_{112} - a_{112}$$
$$g_y = a'_{103} - a_{103}$$
$$g_x = a'_{103} - a_{103}$$

We only need three equations to solve for $g_x, g_y$ and $g_z$, so have five “extra” equations. For example, equations (2), (5) and (7) give us $(a'_{220} - a_{220})/p = g_z = (a'_{112} - a_{112})/3$ and $(a'_{130} - a_{130})/q = g_z = (a'_{112} - a_{112})/3$. Rearranging these gives $a'_{220}/p - a_{112}/3 = a_{220}/p - a_{112}/3$ and $a'_{130}/q - a_{112}/3 = a_{130}/q - a_{112}/3$, respectively. This shows that $a_{220}/p - a_{112}/3$ and $a_{130}/q - a_{112}/3$ are invariants. Similar calculations with expressions for $g_x$ and $g_y$ would yield three more invariants. We will call invariants like these extra invariants.

Next we take three equations where we have solved for $g_x, g_y$ and $g_z$. In the presence of the five extra invariants, it does not matter what they are; we will obtain an equivalent set of invariants. So we will use equations (7), (8) and (9). Concentrating on (8) and (9), the compatibility condition stating partial derivatives are equal gives us $a'_{103x} - a_{103x} = \ldots$
by the nonmaximal term $S$ of Definition 6. Let $S$ be maximally generated, with set of maximal terms $M$, and we denote them by $I_m$. (Letting $s$ be the number of submaximal terms, there are $s$ different equations where linear combinations of the $g_{x_s}$ are equal to some difference of the form $a'_v - a_v$, so there will be $s - n$ many extra invariants.) If we have expressions $E_i$ and $E_j$ with $g_{x_s} = E_i$ and $g_{x_s} = E_j$ and obtain an invariant by rearranging the equation $E_{ix_s} = E_{jx_s}$, this invariant is a compatibility invariant, which we will denote $I_c$.

Figure 1. For Definition 6

\[ g_{xy} = a'_{013y} - a_{013y}. \] Now we rearrange this, putting primed quantities on one side, and get $a'_{103x} - a'_{013y} = a_{103x} - a_{013y}$, showing that $a_{103x} - a_{013y}$ is an invariant. Similar calculations with the other pairs of equations give us two more invariants. We call invariants of this kind compatibility invariants. (In general we have $n$ variables, and get $n(n - 1)/2$ compatibility invariants.)

While all the maximal terms of $L$ were at the same degree in the above procedure, it is enough to require the following.

Definition 5. Consider the class of operators $L$, maximally generated by $T$. Let $M$ be the set of maximal terms of $L$, and let $S$ be the set of submaximal terms. Then $L$ is approximately flat iff there are $n$ distinct elements of $S$, $s_1, s_2, \ldots, s_n$ so that for every $s_i$ there is a maximal term $m_i \in M$ where the vector of $m_i$ is the sum of $e_i$ and the vector of $s_i$.

For example, when $T = \{\partial_{xx}, \partial_y\}$, then $L = \{\partial_{xx} + a_{10} \partial_x + \partial_y + a_{00} : a_{10}, a_{00} \in K\}$ is not approximately flat. We have that $S$ is only $\{a_{10} \partial_x\}$ because the constant term is covered by the nonmaximal term $a_{10} \partial_x$. Then there can not be $n = 2$ distinct elements of $S$.

Whenever all the elements of $T$ have the same degree, $L$ is approximately flat. But when $T = \{\partial_{xy}, \partial_y\}$ for example, $L$ is still approximately flat, since we may take $s_1 = a_{11} \partial_{xy}$, $m_1 = \partial_{xy}$, $s_2 = a_{20} \partial_{xx}$, and $m_2 = \partial_{xy}$. See Figure 1.

Another example of a class that is not approximately flat is obtained by taking $T$ to be $\{\partial_{xx}, \partial_{yy}\}$, so $L = \{\partial_{xx} + a_{100} \partial_x + a_{010} \partial_y + a_{001} \partial_z + a_{000} : a_{100}, a_{010}, a_{001}, a_{000} \in K\}$. Then $M = \{\partial_{xx}, \partial_{yy}\}$, while $S = \{a_{100} \partial_x, a_{010} \partial_y, a_{001} \partial_z\}$ is a set with $n = 3$ elements. But $a_{001} \partial_z$ is our only possible choice for both $s_1$ and $s_2$, and we fail to have distinct elements of $S$ for each $i$.

Definition 6. Let $L$ be maximally generated, with set of maximal terms $T$. Assume that $L$ is approximately flat. The coefficients of terms in $T$ are maximal invariants of $L$, and we denote them by $I_m$. Invariants obtained by equating expressions for some $g_{x_s}$ are extra invariants, which we denote by $I_e$. (Letting $s$ be the number of submaximal terms, there are $s$ different equations where linear combinations of the $g_{x_s}$ are equal to some difference of the form $a'_v - a_v$, so there will be $s - n$ many extra invariants.) If we have expressions $E_i$ and $E_j$ with $g_{x_s} = E_i$ and $g_{x_s} = E_j$ and obtain an invariant by rearranging the equation $E_{ix_s} = E_{jx_s}$, this invariant is a compatibility invariant, which we will denote $I_c$.

Finally, suppose that $a_v \partial_e$ is a term that is neither maximal or submaximal, and that $E$ is an expression only involving the coefficients of terms that are above $a_v \partial_e$, or that
are maximal or submaximal (and their derivatives). Then an invariant of the form $a_{\nu} - E$ is an upward invariant, we call it an upward invariant for $a_{\nu} \partial_{\nu}$, and denote it $I_{\nu}$.

Examples of all of these types of invariants appear in the literature.

**Theorem 1.** Let $\mathcal{L}$ be maximally generated, and let $T$ be its set of maximal terms. Assume that $\mathcal{L}$ is approximately flat. Suppose that $I$ is a set of invariants for $\mathcal{L}$ so that the following hold:

1. $I$ contains all the maximal invariants of $\mathcal{L}$.
2. $I$ contains $s - n$ extra invariants, where $s$ is the number of submaximal terms.
3. $I$ contains $n(n - 1)/2$ compatibility invariants, one for each possible second-order compatibility partial of $g$.
4. $I$ contains an upward invariant for every term that is not maximal or submaximal.

Then the set of invariants $I$ is complete.

**Proof.** Let $I$ be a set of invariants as above. Let $L \in \mathcal{L}$ be given, and let $L'$ be an element of $\mathcal{L}$ where $L$ and $L'$ have the same values for all invariants in $I$. We must show there exists a function $g \in K$ so that $L' = e^{-g}L e^g$.

Consider solving for the derivatives of $g$ in $L'$. Since $\mathcal{L}$ is approximately flat, we only obtain linear equations in the first partial derivatives of $g$. The values of these $g_{x_i}$ are the same no matter which equations we use, because $I$ contains enough extra invariants.

Since $I$ contains enough compatibility invariants, we have a (compatible) system of first order partial differential equations for $g$. This gives a value for $g$ in $K$ that is unique up to an additive constant, and the additive constant does not change what our candidate gauge transformation is.

So we have $g \in K$ where $L'' = e^{-g}L e^g$ agrees with $L'$ on the coefficients of all maximal and submaximal terms. It remains to show that $L''$ and $L'$ agree on their remaining terms. We do this by downward induction on the degree of terms. For our basis, let $m$ be the highest degree of a term in $\mathcal{L}$ that is not maximal or submaximal, and consider any term $a_{\nu} \partial^{\nu}$ of degree $m$. If $a_{\nu} \partial^{\nu}$ is maximal or submaximal, we already have that it has the same value in $L'$ and $L''$. So suppose $a_{\nu} \partial^{\nu}$ is not maximal or submaximal. By our choice of $m$, every term of $\mathcal{L}$ that is above $a_{\nu} \partial^{\nu}$ is submaximal or maximal, and all of these terms have the same value in $L'$ and $L''$. Now $a_{\nu} \partial^{\nu}$ has an upward invariant of the form $a - E$, where $E$ is an expression only involving terms above $a_{\nu} \partial^{\nu}$. Since $L'$ and $L$ have the same values for all invariants in $I$, this upward invariant is equal in $L'$ and $L$. And since $L''$ is obtained from $L$ by a gauge transformation, the invariant has the same value in $L$ and in $L''$, implying that it has the same value in $L'$ and $L''$. Since $E$ and the invariant both have the same value in $L'$ and $L''$, we have that $a$ has the same value in $L'$ and $L''$.

The inductive argument now continues. In the next stage, terms of degree $m - 1$ are only below terms that have the same value in $L'$ and $L''$, and are either maximal, submaximal, or have upward invariants. In any event, all terms of degree $m - 1$ have the same value in $L'$ and $L''$. The process continues, eventually showing that the term of degree 0 has the same value in $L'$ and $L''$. \hfill $\Box$

4. **Constructing complete sets of invariants**

**Theorem 2.** Let $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{N}$ be classes of operators that are closed under gauge transformations. Assume that for every $L \in \mathcal{L}$, there is a unique $C \in \mathcal{C}$ so that $N = L - C$ is in $\mathcal{N}$. Then all of the invariants of $\mathcal{N}$ are invariants for $\mathcal{L}$.

**Proof.** Let $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{N}$ be closed under gauge transformations, and assume that for each $L \in \mathcal{L}$ there is a unique $C \in \mathcal{C}$ with $L - C \in \mathcal{N}$. That is, every $L \in \mathcal{L}$ determines a
unique $N \in \mathcal{N}$. Now let a particular $L \in \mathcal{L}$ be given, where $N = L - C$ is in $\mathcal{N}$. Gauging by any nonzero $g \in K$, we have $N' = g^{-1}Ng \in \mathcal{N}$, $L' = g^{-1}Lg \in \mathcal{L}$ and $C' = g^{-1}Cg \in \mathcal{C}$ with $N' = L' - C'$. By uniqueness, $N'$ must be the element of $\mathcal{N}$ determined by $L'$. The invariants of $N' = g^{-1}Ng$ are the same as the corresponding invariants in $N$, making them invariants of $\mathcal{L}$. \hfill \square

In an application, $\mathcal{L}$ would be the class of all operators of a particular form, and the classes $\mathcal{C}$ and $\mathcal{N}$ would be tailored to produce a family of invariants for $\mathcal{L}$. The invariants of $\mathcal{N}$ that we will use are usually the coefficients of its maximal terms.

**Example 4.** Suppose $n = 2$ and let $\mathcal{L}$ be the class of operators maximally generated by $T$, where

$$T = \{ \partial_{xy} \}.$$

Consider the operator

$$L = \partial_{xy} + a_{20}\partial_{xx} + a_{11}\partial_{xy} + a_{10}\partial_{x} + a_{01}\partial_{y} + a_{00} \in \mathcal{L}.$$  

To obtain a complete set of invariants, we first include all the needed maximal, extra, and compatibility invariants. The coefficient 1 of $\partial_{xy}$ is a maximal invariant. There are two submaximal terms, $a_{20}\partial_{xx}$ and $a_{11}\partial_{xy}$. Since the dimension is $n = 2$, there are no extra invariants. There is one compatibility invariant, which we get by observing that $e^{-q}Le^q = L' = \partial_{xy} + a'_{20}\partial_{xx} + a'_{11}\partial_{xy} + \ldots$ has $a'_{20} = a_{20} + g_y$ and $a'_{11} = a_{11} + 2g_x$. Then solving and differentiating, $2a'_{20} - 2a_{20} = 2g_{xy} = (a'_{11} - a_{11})y$. This shows that

$$I_c = 2a_{20}y - a_{11}x$$

is the desired compatibility invariant.

We now let

$$\mathcal{C} = \{(\partial_x + q)^2(\partial_y + r): q, r \in K\},$$

and take $\mathcal{N}$ to be the set of elements of $\mathcal{L}$ with coefficients of $\partial_{xy}$, $\partial_y$ and $\partial_x$ all zero. This will give the first batch of upward invariants. (See Figure 2)

We have that $L - C$ is $(a_{20} - r)\partial_{xx} + (a_{11} - 2q)\partial_{xy} + (a_{10} - (2qr + 2r_x))\partial_x + (a_{01} - (q^2 + q) + a_{00} - ((q^2 + q)r + 2qr_x + r_{xx})).$

For this to be in $\mathcal{N}$, we must have $a_{20} - r = a_{11} - 2q = 0$, giving $r = a_{20}$ and $q = a_{11}/2$, which uniquely determines $C$. Substituting these in, we get that $N = L - C = (a_{10} - (a_{11}a_{20} + 2a_{20}r))\partial_x + (a_{01} - (a_{11}^2/4 + a_{11}r/2))\partial_y + (a_{00} - ((a_{11}^2/4 + a_{11}r/2)a_{20} + a_{11}a_{20}r + a_{20}xx)).$
The coefficients of $\partial_x$ and $\partial_y$ are two invariants,

\[ I_{10} = a_{10} - (a_{11}a_{20} + 2a_{20x}) \quad \text{and} \quad I_{01} = a_{01} - (a_{11}^2/4 + a_{11x}/2). \]

In particular, we get the associated representation

\[ L = (\partial_x + q)^2(\partial_y + r) + I_{10}\partial_x + I_{01}\partial_y + s \]

for some $q, r, s \in K$. This gives meaning to the invariants. We have that $I_{10}$ is zero iff there is a representation of $L$ without a $\partial_x$ term, and so on.

Next we add more terms to the form of $C$, setting

\[ C' = \{(\partial_x + q)^2(\partial_y + r) + (\partial_x + s)(\partial_y + t) : q, r, s, t \in K\} \]

and letting $\mathcal{N}'$ be the set of elements of $\mathcal{N}$ with coefficients of $\partial_x$ and $\partial_y$ both zero. Letting $C' \in C'$, we have that $L - C'$ is

\[ (a_{20} - r)\partial_{xx} + (a_{11} - 1 - 2q)\partial_{xy} + (a_{10} - (2qr + 2r_x) - t)\partial_x + (a_{01} - (q^2 + q_z) - s)\partial_y + (a_{00} - ((q^2 + q)r + 2qr_x + r_{xx} + (st + t_x)). \]

There is one way to make this be an element of $\mathcal{N}'$. As before, we let $r = a_{20}$.

With a slight change, we let $q = (a_{11} - 1)/2$. Now that $q$ and $r$ are determined, we let $s = a_{01} - (q^2 + q_z)$ and $t = a_{10} - (2qr + 2r_x)$.

Then the constant term of $\mathcal{N}' = L - C'$ is $a_{00} - ((q^2 + q)r + 2qr_x + r_{xx} + (st + t_x))$. It is an invariant of $L$. Expanding, we get

\[ I_{00} = a_{00} - ((a_{11} - 1)^2/4 + (a_{11} - 1)/2)a_{20} - (a_{11} - 1)a_{20x} - a_{20xx} - (a_{01} - ((a_{11} - 1)/2 + (a_{11} - 1)/2)x)(a_{10} - ((a_{11} - 1))a_{20} + 2a_{20x}) - (a_{10} - (a_{11} - 1)a_{20} - 2a_{20x}). \]

In particular, we get the associated representation,

\[ L = (\partial_x + q)^2(\partial_y + r) + (\partial_x + s)(\partial_y + t) + I_{00} \]

for some $q, r, s, t \in K$, where $q$ and $r$ are possibly different from $q$ and $r$ in incomplete factorization. \[\square\]
We will now work through a more complicated example in less detail, commenting on the process as we go.

**Example 5.** Suppose \( n = 2 \) and let \( \mathcal{L} \) be the class of operators maximally generated by \( T \), where

\[
T = \{ \partial_{xxxxy} \}.
\]

So elements of \( \mathcal{L} \) have the form \( \partial_{xxxxy} + a_{31}\partial_{xxx} + a_{30}\partial_{xx} + a_{21}\partial_{x} + a_{12}\partial_{xy} + a_{20}\partial_{xy} + a_{11}\partial_{y} + a_{10}\partial_{x} + a_{01}\partial_{x} + a_{00} \). The coefficient of \( \partial_{xxxxy} \) is the maximal invariant and there are no extra invariants. Letting primes denote the gauge action on the coefficients, we have \( 3a'_{32} = 3g_x \) and \( 3a'_{31} = 3a_{31} = 2g_y \). Then \( 2a'_{22} - 2a_{22} = 6g_x \) and \( 3a'_{31} - 3a_{31} = 6g_y \). Thus \( 2a'_{22y} - 2a_{22y} = 6g_{xy} \) and \( 3a'_{31x} - 3a_{31x} = 6g_{xy} \). So \( 2a'_{22y} - 2a_{22y} = 3a'_{31x} - 3a_{31x} \), and

\[
I_c = 2a_{22y} - 3a_{31x}
\]

is the desired compatibility invariant.

We now let

\[
C = \{ (\partial_x + q)^3(\partial_y + r)^2 : q, r \in K \},
\]

and take \( \mathcal{N} \) to be the set of elements of \( \mathcal{L} \) with coefficients of \( \partial_{xxxxy} \), \( \partial_{xxx} \), and \( \partial_{xy} \) all zero. We see that \((\partial_x + q)^3(\partial_y + r)^2\) is \( \partial_{xxxxy} + 2r\partial_{xxx} + 3q\partial_{xy} + (r^2 + q)\partial_{x} + 6(qr + r x)\partial_{xy} + 3(q^2 + q(x))\partial_{y} \), so for \( L = \partial_{xxxxy} + a_{31}\partial_{xxx} + a_{30}\partial_{xx} + a_{21}\partial_{x} + a_{22}\partial_{xy} + a_{11}\partial_{y} + a_{10}\partial_{x} + a_{01}\partial_{x} + a_{00} \), there is a unique choice of \( q \) and \( r \) to get \( L - (\partial_x + q)^3(\partial_y + r)^2 \in \mathcal{N} \), we let \( r = a_{31}/2 \) and \( q = a_{22}/3 \). This gives the first batch of three upward invariants, for \( a_{30}, a_{21} \) and \( a_{12} \).

\[
I_{30} = a_{30} - (r^2 + r_q).
\]

Now we add more terms to the form of \( \mathcal{C} \), setting

\[
C' = \{ (\partial_x + q)^3(\partial_y + r)^2 + (\partial_x + s)^2(\partial_x + t)^2(\partial_y + u) : q, r, s, t, u \in K \}
\]

and letting \( \mathcal{N}' \) be the set of elements of \( \mathcal{N} \) with coefficients of \( \partial_{xxx}, \partial_{xy} \), and \( \partial_{xy} \) all zero. One has to be careful in the choice of \( \mathcal{C}' \). Intuitively, our choice of the additional term \((\partial_x + s)^2(\partial_x + t)^2(\partial_y + u)\) was good because it had three free parameters, precisely the number of terms we were trying to “zero out” when going to \( \mathcal{N}' \). While it is fine to use a factor such as \((\partial_x + \partial_y + s)\), this is not necessary.

The added term of \((\partial_x + s)^2(\partial_x + t)^2(\partial_y + u)\) has principal symbol \( \partial_{xxxx} + \partial_{xy} \). If we were to use the same values of \( q \) and \( r \) as in the previous step, any element of \( L - C' \) would have coefficients of \( \partial_{xxxx} \) and \( \partial_{xy} \) that were \(-1\). So we modify \( q \) and \( r \) to make these coefficients zero. We already have that for any \( L \in \mathcal{L} \), \( q \) and \( r \) can be chosen to make the coefficients of \( \partial_{xxxx} \) and \( \partial_{xy} \) zero in \( L - C \), so we merely use \( L - (\partial_{xxxx} + \partial_{xy}) \in \mathcal{L} \) to determine our new \( q \) and \( r \).

Since \((\partial_x + \partial_y + s)(\partial_x + t)^2(\partial_y + u)\) is \( \partial_{xxxx} + \partial_{xy} + u\partial_{xxxx} + (u + 2t + s)\partial_{xy} + 2t\partial_{xy} \), there will be a unique choice of \( s \), \( t \), and \( u \) that gives an element of \( \mathcal{N}' \). We first take \( u \) so that the coefficient of \( \partial_{xxx} \) is zero, then choose \( s \) so the coefficient of \( \partial_{xy} \) is zero, and finally choose \( s \) so that the coefficient of \( \partial_{xy} \) is zero. Thus the hypotheses of Theorem 2 are still met with \( \mathcal{C}' \) and \( \mathcal{N}' \), and the coefficients of the principal symbol of the unique \( \mathcal{N}' \in \mathcal{N}' \) give three more upward invariants, for \( a_{20}, a_{11} \) and \( a_{02} \).

To get upward invariants for \( a_{10} \) and \( a_{01} \), we need to add another term to \( \mathcal{C}' \). This term should have three free parameters, which we can choose to make the coefficients of \( \partial_{xxx}, \partial_{xy} \) and \( \partial_{xy} \) all zero. While we could use the term \((\partial_x + \partial_y + f)(\partial_x + g)(\partial_y + h)\), we will instead show another possibility. Suppose we first focus on a term that would make the coefficients of \( \partial_{xxx} \) and \( \partial_{xy} \) zero. Since \((2,0) \) and \((1,1) \) are both covered by the vector \((2,1) \), we can use a term with principal symbol \( \partial_{xx} \) and two free parameters, such as \((\partial_x + f)^2(\partial_y + g)\). But what will we use for a term that makes the coefficient of \( \partial_{yy} \) zero,
particularly since we should only use one more free parameter? One solution is to reuse parameters from higher levels, for instance by adding the term \((\partial_x + h)(\partial_y + r)^2\).

So this leads us to taking

\[
C'' = \{ (\partial_x + q)^2(\partial_y + r)^2 + (\partial_x + \partial_y + s)(\partial_x + t)^2(\partial_y + u) + (\partial_x + f)^2(\partial_y + g) + (\partial_x + h)(\partial_y + r)^2 : q, r, s, t, u \in K \}
\]

and letting \(N''\) be the set of elements of \(N\) with coefficients of zero for all derivatives of order above 1. The reader may verify that \(C''\) is closed under gauge transformations. (This would not have been the case if we had instead added the term \((\partial_x + h)(\partial_y + q)^2\).)

Letting some \(L \in \mathcal{L}\) be given, we have that as before, there is a unique way to choose \(q, r, s, t\) and \(u\) in \(C''\) so that \(L - C'' \in N'\). Ignoring the term \((\partial_x + f)^2(\partial_y + g) + (\partial_x + h)(\partial_y + r)^2\) for the moment, let \(N'\) be the unique element of \(N'\) determined by using \(q, r, s, t, u\) as above, and letting \(f, g, h\) be zero. Then the coefficient of \(\partial_x, \partial_y\) in \(N'\) consists of \(a_{20}\) minus an expression in \(q, r, s, t, u\). For simplicity, call this coefficient \(a_{11}'\) and define \(a_{02}'\) similarly.

We now need to change \(f, g, h\) to non-zero values so that the coefficients of the degree 2 terms are zero. Since \((\partial_x + f)^2(\partial_y + g) + (\partial_x + h)(\partial_y + r)^2\) is \(\partial_x xy + \partial_x yy + g\partial_{xx} + (2f + 2r)\partial_{xy} + h\partial_{yy}\), taking \(f, g, h\) non-zero gives us that the coefficients of \(\partial_{xx}, \partial_{xy}\) and \(\partial_{yy}\) are \(a_{20}' - g, a_{11}' - (2f + 2r)\) and \(a_{02}' - h\), respectively. There is a unique way to make these coefficients zero, we take \(g = a_{20}', f = (a_{11}' - 2r)/2\) and \(h = a_{02}'\). This uniquely determines the coefficients of \(\partial_x, \partial_y\) in our element of \(N''\), giving upward invariants for \(a_{10}\) and \(a_{01}\).

This process would continue, until \(N''' \in N''\) is just a function, giving an upward invariant for the constant term.

**Example 6.** Consider the class that is the downward closure of \(\{\partial_{xy}, \partial_{yy}\}\). This class of operators was considered in [24], where obstacles to factorizations [25] were used to compute four Laplace invariants. A fifth invariant was then found to complete the set but it was not a Laplace invariant. This fifth invariant had a long complicated formula and no meaning or structure. Our new method allows us to construct a complete set of Laplace invariants.

So \(n = 2\) and let \(T\) be the class of operators maximally generated by \(T\), where

\[
T = \{ \partial_{xy}, \partial_{yy} \}
\]

Let \(L = \partial_{xy} + \partial_{yy} + a_{20}\partial_{xx} + a_{11}\partial_{xy} + a_{02}\partial_{yy} + a_{10}\partial_x + a_{01}\partial_y + a_{00} \in \mathcal{L}\). We have two maximal invariants, but they are both 1. There are three submaximal terms, \(a_{20}\partial_{xx}, a_{11}\partial_{xy}\) and \(a_{02}\partial_{yy}\). So there will be \(3 - n = 3 - 2\) extra invariants, and one compatibility invariant. We have that when \(L = \partial_{xy} + \partial_{yy} + a_{20}\partial_{xx} + a_{11}\partial_{xy} + a_{02}\partial_{yy} + \ldots\), that \(e^{-g}Le^g = L' = \partial_{xy} + \partial_{yy} + a_{20}'\partial_{xx} + a_{11}'\partial_{xy} + a_{02}'\partial_{yy} + \ldots\). Here, \(a_{20}' = a_{20} + g_y, a_{11}' = a_{11} + 2g_y + 2g_x\) and \(a_{02}' = a_{02} + g_x\). These are three equations in the two unknowns \(g_x\) and \(g_y\). We get \(a_{20}' - a_{20} = g_y\) and \(a_{02}' - a_{02} = g_x\), which we substitute into \(a_{11}' - a_{11} = 2g_y + 2g_x\) to get \(a_{11}' - a_{11} = 2(a_{20}' - a_{20}) + 2(a_{02}' - a_{02})\). Rearranging this, \(a_{11}' - 2a_{20}' - 2a_{02}' = a_{11} - 2a_{20} - 2a_{02}\). This gives the extra invariant,

\[
I_c = a_{11} - 2a_{20} - 2a_{02}.
\]

Returning to \(a_{20}' - a_{20} = g_y\) and \(a_{02}' - a_{02} = g_x\), we differentiate both and get \(a_{20}'' - a_{20} = g_{xy} = a_{02}' - a_{02}\). Thus \(a_{20}' - a_{02}' = a_{20} - a_{02}\), showing that

\[
I_c = a_{20} - a_{02}
\]

is a compatibility invariant.
To get upward invariants, we first take $C$ to be the class of operators of the form $(\partial_x + p)(\partial_y + q)(\partial_x + \partial_y + r)$. Expanding, these operators have the form

$$
\partial_{xxy} + \partial_{xyy} + q\partial_{xx} + (p + q + r)\partial_{xy} + p\partial_{yy} + (q_x + r_y + q(p + r))\partial_x + (q_x + r_x + p(q + r))\partial_y + ((pq + q_x)\partial_x + qr_x + pr_y + r_{xy})
$$

(11)

Taking $N = \{b_{10}\partial_x + b_{01}\partial_y + b_{00} : b_{10}, b_{01}, b_{00} \in K\}$, we get $q = a_{20}$, $p = a_{02}$ and $p + q + r = a_{11}$. The last equation becomes $r = a_{11} - p - q = a_{11} - a_{20} - a_{02}$. The coefficients $b_{10}$ and $b_{01}$ are then invariants. We get

$$
I_{10} = a_{10} - (q(p + r) + q_x + r_y)
$$

and

$$
I_{01} = a_{01} - (p(q + r) + q_x + r_x)
$$

Substituting in our values of $p$, $q$ and $r$, these invariants agree with those obtained given in Theorem 4 of [24].

In particular, we have the associated representation

$$
L = (\partial_x + p)(\partial_y + q)(\partial_x + \partial_y + r) + I_{10}\partial_x + I_{01}\partial_y + b_{00}
$$

(12)

for some $p, q, r, b_{00} \in K$.

To obtain an upward invariant involving $a_{00}$, we let $C'$ be the class of operators of the form $(\partial_x + p)(\partial_y + q)(\partial_x + \partial_y + r) + (\partial_x + s)(\partial_y + t)$ and take $N'$ to be $K$. The first group of equations is almost the same as before; we have $q = a_{20}$, $p = a_{02}$ and $p + q + r + 1 = a_{11}$. Thus $p$ and $q$ have the same values as before while the value of $r$ is $r' = a_{11} - a_{20} - a_{02} - 1$, one less than the previous value. We take our expansion (11), substitute $r'$ for $r$ in it and add $(\partial_x + s)(\partial_y + t) = \partial_{xy} + \partial_x s + \partial_y t + \partial_x t$, giving that the following must be in $N'$.

$$
(a_{10} - (q_x + r_y' + r + q(p + r'))\partial_x + (a_{01} - (q_x + r_x' + p(q + r') + s))\partial_y + a_{00} - (pq + q_x)r' + qr_x' + pr_y' + r_{xy}' + st + t_x)
$$

(13)

Thus $t = a_{10} - (q_x + r_y' + q(p + r')) = a_{10} - (q_x + r_y + q(p + (r - 1))) = I_{10} + q$. Similarly, $s = a_{01} - (q_x + r_x' + p(q + r')) = I_{01} + p$. The terms without derivation operators make our desired invariant, it is

$$
I_{00} = a_{00} - ((pq + q_x)r' + qr_x' + pr_y' + r_{xy}' + st + t_x)
$$

In particular, for this invariant we have the associated representation

$$
L = (\partial_x + p)(\partial_y + q)(\partial_x + \partial_y + r) + (\partial_x + s)(\partial_y + t) + I_{00}
$$

for some $p, q, r, s, t \in K$, where again $p, q, r$ here can be different from $p, q, r$ in (12).

Here is another interesting example from the literature; its maximal terms do not all have the same degree. A complete set of invariants for it was obtained in [18].

**Example 7.** So $n = 2$ and let $C$ be the class of operators maximally generated by $T$, where

$$
T = \{\partial_{xxx}, a_{11}\partial_{xy}, a_{02}\partial_{yy}\}
$$

So elements of $C$ have the form

$$
\partial_{xxx} + a_{20}\partial_{xx} + a_{11}\partial_{xy} + a_{02}\partial_{yy} + a_{10}\partial_x + a_{01}\partial_y + a_{00}.
$$

(14)

We have that 1, $a_{11}$, $a_{02}$ are maximal invariants. Applying a gauge transformation to (14), we get $L' = e^{-a}Le^a = \partial_{xxx} + a_{20}'\partial_{xx} + a_{11}'\partial_{xy} + a_{02}'\partial_{yy} + a_{10}'\partial_x + a_{01}'\partial_y + a_{00}'$, where

$$
a_{20}' = a_{20} + 3g_x, a_{10}' = a_{10} + 3(g_x^2 + g_{xx}) + 2a_{20}g_x + a_{11}g_y \text{ and } a_{01}' = a_{01} + a_{11}g_x + 2a_{02}g_y.
$$

Solving for $g_x$ and $g_y$, we get $(a_{20}' - a_{20})/3 = g_x$ and $(a_{01}' - a_{01} - a_{11}(a_{20}' - a_{20})/3)/(2a_{02}) = g_y$, if we choose $a_{11}' = a_{11}$ and $a_{02}' = a_{02}$.
Compatibility condition \((g_{xy} = g_{yx})\) then gives us
\[2a'_{20y} - 3(a'_{01}/a_{02})_x + (a_{11}a'_{20}/a_{02})_x = 2a_{20y} - 3(a_{01}/a_{02})_x + (a_{11}a_{20}/a_{02})_x.\]
This implies that
\[I_{10} = 2a_{20y} - 3(a_{01}/a_{02})_x + (a_{11}a_{20}/a_{02})_x\]
is a compatibility invariant.

To get an upward invariant for \(a_{10}\), we need at least \(n = 2\) free parameters in \(C\), which will zero out two submaximal terms in going from \(L\) to \(N\). Accordingly, we take the classes of operators
\[C = \{(\partial_x + p)^3 + a_{11}(\partial_x + p)(\partial_y + q) + a_{02}(\partial_y + q)^2\}, \quad N = \{b_{10}\partial_x + b_{00}\}.
Subtracting an element of \(C\) from \((14)\), we get
\[\begin{align*}
(a_{20} - 3p)\partial_{xx} + (a_{10} - 3p^2 + p^2 - a_{11}q)\partial_x + (a_{01} - a_{11}p - 2a_{02}q)\partial_y + \\
a_{00} - (p^3 + 3ppx + pxx + a_{11}(pq + qx) + a_{02}(q^2 + qx)).
\end{align*}\]
(15)

For this to be in \(N\), we need \(a_{20} = 3p\) and \(a_{01} - a_{11}p = 2a_{02}q\). These give \(\frac{p}{a_{20}} = \frac{3}{3}\) and \(q = (a_{01} - a_{11}a_{20}/3)/(2a_{02})\). The coefficient of \(\partial_x\) in \((15)\) then becomes the invariant
\[I_{10} = a_{10} - 3p^2 - a_{11}q = a_{10} - 3((a_{20}/3)_x + (a_{20}/3)^2) - a_{11}(a_{01} - a_{11}a_{20}/3)/(2a_{02})\]
Similarly, we get the invariant \(I_{01}\) by taking \(N\) to be operators of the form \(b_{01}\partial_y + b_{00}\), and making \((15)\) an element of this \(N\). This gives us \(\frac{p}{a_{01}} = \frac{1}{a_{10}}, \quad q = \frac{1}{2a_{11}a_{20}/3}\) as before, and \(\frac{q}{a_{11}} = \frac{3}{2a_{11}a_{02}}\). The coefficient of \(\partial_y\) then becomes the invariant
\[I_{01} = a_{01} - a_{11}p - 2a_{02}q = a_{01} - a_{11}a_{20}/3 - 2a_{02}(a_{10} - 3((a_{20}/3)_x + (a_{20}/3)^2))/a_{11}.
In particular, we have associated representation
\[L = (\partial_x + p)^3 + a_{11}(\partial_x + p)(\partial_y + q) + a_{02}(\partial_y + q)^2 + I_{10}\partial_x + I_{01}\partial_y + b_{00}
(16)\]
for some \(p, q, r, b_{00} \in K\).

These are not strictly speaking upward invariants, since \(I_{10}\) also involves \(a_{01}\) and \(I_{01}\) also involves \(a_{10}\). However, they can be manipulated to yield two upward invariants. We have
\[I_{10} = (a_{10} - a_{11}a_{01}) - (3((a_{20}/3)_x + (a_{20}/3)^2) - a_{11}a_{11}a_{20}/((6a_{02})) \quad \text{and} \quad I_{01} = (a_{01} - 2a_{02}a_{10}) - (a_{11}a_{20}/3 - 6a_{02}((a_{20}/3)_x + (a_{20}/3)^2))/a_{11}\]
Thus \((I_{10} + a_{11}I_{01})/(1 - 2a_{11}a_{02})\) is an upward invariant for \(a_{10}\), and \((I_{01} + 2a_{02}I_{10})/(1 - 2a_{11}a_{02})\) is an upward invariant for \(a_{01}\). (We leave the case where \((1 - 2a_{11}a_{02}) = 0 \) to the reader.)

To get an upward invariant for \(a_{00}\), we slightly modify \(C\) by changing the \(a_{11}(\partial_x + p)(\partial_y + q)\) term to \(a_{11}(\partial_x + r)(\partial_y + q)\). This gives that \(C'\) is the class of operators of the form \((\partial_x + p)^3 + a_{11}(\partial_x + r)(\partial_y + q) + a_{02}(\partial_y + q)^2\). This changes the difference in \((15)\) to become
\[\begin{align*}
(a_{20} - 3p)\partial_{xx} + (a_{10} - 3p^2 - a_{11}q)\partial_x + (a_{01} - a_{11}r - 2a_{02}q)\partial_y + \\
a_{00} - (p^3 + 3ppx + pxx + a_{11}(rq + qx) + a_{02}(q^2 + qx)).
\end{align*}\]
(17)

Now we take \(N\) to be the set of terms of the form \(b_{00}\). Then \(p = a_{20}/3\) as before, and \(q = (a_{10} - 3p^2)/a_{11}\) as in the derivation of \(I_{01}\). Setting the coefficient of \(\partial_y\) equal to zero, we get \(r = (a_{01} - 2a_{02}q)/a_{11}\). Then
\[I_{00} = a_{00} - (p^3 + 3ppx + pxx + a_{11}(rq + qx) + a_{02}(q^2 + qx)).\]
In particular, we have associated representation
\[ L = (\partial_x + p)^3 + a_{11}(\partial_x + r)(\partial_y + q) + a_{02}(\partial_y + q)^2 + I_{00} \]
for some \( p, q, r, s, t \in K \), where again \( p, q, r \) here can be different from \( p, q, r \) in (12).

The next example is a simple version of one treated by Mironov in [16] and by Athorne and Yilmaz in [4]. Consider the operator that is the downward closure of \( \{\partial_{x_1 x_2 \ldots x_n}\} \) in dimension \( n \). Mironov gets invariants for the case \( n = 4 \), while Athorne and Yilmaz produce invariants for all cases through \( n = 6 \).

**Example 8.** Let \( n = 3 \), and call independent variables \( x, y \) and \( z \). Suppose \( \mathcal{L} \) is the class of operators maximally generated by \( T \), where
\[ T = \{\partial_{xyz}\} \]
so elements of \( \mathcal{L} \) have the form \( \partial_{xyz} + a_{110}\partial_{xy} + a_{101}\partial_{xz} + a_{011}\partial_{yz} + a_{100}\partial_x + a_{010}\partial_y + a_{001}\partial_z + a_{000} \).

Since \( n = 3 \) and there are 3 submaximal terms, we have 1 as a maximal invariant, no extra invariants, and three compatible invariants.
\[
\begin{align*}
I_{cxy} &= a_{011}y - a_{101}x \\
I_{czz} &= a_{011}z - a_{110}z \\
I_{cycl} &= a_{101}z - a_{110}y \\
\end{align*}
\]

We take \( \mathcal{C} \) to be the set of operators of the form \((\partial_x + p)(\partial_y + q)(\partial_z + r)\), and get that \( L - C \) is
\[
\begin{align*}
(a_{110} - r)\partial_{xy} + (a_{101} - q)\partial_{xz} + (a_{011} - p)\partial_{yz} + (a_{100} - qr - r_y)\partial_x + \\
(a_{010} - pr - r_x)\partial_y + (a_{001} - pq - q_x)\partial_z + \\
a_{000} - (pqr + rq_x + pr_y + qr_x + r_y) \\
\end{align*}
\]

Taking \( \mathcal{N} \) to be the set of operators of the form \( b_{100}\partial_x + b_{010}\partial_y + b_{001}\partial_z + b_{000} \), we get
\[ p = a_{011}, \quad q = a_{101}, \quad r = a_{110}. \]
The coefficients of \( \partial_x, \partial_y \) and \( \partial_z \) now give us the three upward invariants
\[
\begin{align*}
I_{100} &= a_{100} - a_{101}a_{110} - a_{110}y \\
I_{010} &= a_{010} - a_{011}a_{110} - a_{110}x \\
I_{001} &= a_{001} - a_{001}a_{101} - a_{101}x \\
\end{align*}
\]
These are essentially the same as those in the literature, and they correspond to the representation
\[ L = I_{100}\partial_x + I_{010}\partial_y + I_{001}\partial_z + b_{000} \]
for some \( b_{000} \in K \).

To get \( I_{000} \), there are several possibilities for a class of expressions to add to those in \( \mathcal{C} \).
\[
(\partial_x + s)(\partial_y + t) + (\partial_x + s)(\partial_z + u) + (\partial_y + t)(\partial_z + u) =
\]
\[ \partial_{xy} + \partial_{xz} + \partial_{yz} + (t + u)\partial_x + (s + u)\partial_y + (s + t)\partial_z + \\
(st + tu + su + u_x + tu + u_y) \]
would work, but substituting \( s, t \) and \( u \) into \( st + tu + su + tu + u_y \) could produce complicated expressions. So we use the following, which avoids products such as \( st \).
\[
(\partial_x + s)(\partial_y + q) + (\partial_y + t)(\partial_z + r) + (\partial_z + u)(\partial_x + p) =
\]
\[ \partial_{xy} + \partial_{xz} + \partial_{yz} + (q + u)\partial_x + (r + s)\partial_y + (p + t)\partial_z + \\
(sq + qz + tr + ry + pu + pz) \]
Note that the order of the factors in the three terms is chosen to produce $q_x$, $r_y$ and $p_z$. The other order would produce $s_x$, $t_y$ and $u_z$, which would yield more complicated expressions.

So we let $C'$ be the set of operators of the form

$$(\partial_x + p)(\partial_y + q)(\partial_z + r) + (\partial_x + s)(\partial_y + q) + (\partial_y + t)(\partial_z + r) + (\partial_z + u)(\partial_x + p)$$

This gives us that $L - C$ is

$$(a_{110} - r - 1)\partial_{xy} + (a_{101} - q - 1)\partial_{xz} + (a_{011} - p - 1)\partial_{yz} +$$

$$(a_{100} - qr - r_y - q - u)\partial_x + (a_{101} - pr - r_x - r - s)\partial_y +$$

$$(a_{001} - pq - q_x - p - t)\partial_z +$$

$$(pq + r_qz + pr_y + qr_x + r_{xy} + sq + q_x + tr + r_y + pu + p_z)$$

Letting $N'$ be the set of operators of the form $b_{000}$, we have

$$p = a_{011} - 1 \quad \quad q = a_{101} - 1 \quad \quad r = a_{110} - 1.$$  

Next we get

$$s = a_{010} - pr - r_x - r$$

$$t = a_{001} - pq - q_x - p$$

$$u = a_{100} - qr - r_y - q$$

Substituting these all into the constant term of (20), we get

$$I_{000} = a_{000} - (pq + r_qz + pr_y + qr_x + r_{xy} + sq + q_x + tr + r_y + pu + p_z),$$

which is associated to the representation

$$L = (\partial_x + p)(\partial_y + q)(\partial_z + r) + (\partial_x + s)(\partial_y + q) + (\partial_y + t)(\partial_z + r) + (\partial_z + u)(\partial_x + p) + I_{000}$$

for some $p, q, r, s, t, u \in K$.

Unlike Athorne and Yilmaz’s technique, our method does not naturally produce invariants that are symmetric in all variables. Adding $I_{czz} - (1/3)(I_{xzz})y - (1/3)(I_{yzz})x - I_{100} - I_{010} - I_{001} - 1$ and simplifying, we get

$$a_{000} - (a_{100}a_{011} + a_{010}a_{101} + a_{001}a_{110} - 2a_{011}a_{101}a_{110} + (a_{110}a_{xy} + a_{101}a_{xz} + a_{011}a_{yz})/3)$$

This is essentially the same as the corresponding invariant in [1].

**Example 9.** As an interesting application, we can also obtain an inductive definition of upward invariants for the bottom terms of any totally hyperbolic operator as in [8]. We let $L_n$ be the downward closure of $\partial_{x_1x_2...x_n}$ for some $n \geq 2$. As noted in [1], the form of an upward invariant for a given term $t$ only depends on how far it is below the maximal element of $L_n$. This is because an upward invariant for $t$ only depends on the coefficient of $t$ and terms above it, so an upward invariant for any term the same distance below $\partial_{x_1x_2...x_n}$ as $t$ is can be obtained by substituting the corresponding coefficients in an upward invariant for $t$. For example, when $n = 2$, an upward invariant for $a_{00}$ is

$$I_{00} = a_{00} - (a_{10}a_{01} + a_{10x_1}),$$

and for any $n \geq 2$, upward invariants for terms that are two levels below $\partial_{x_1x_2...x_n}$ can be obtained by substitution in it. For example, when $n = 3$ we have the invariant

$$I_{001} = a_{001} - (a_{101}a_{011} + a_{101x_1}).$$
This means that once we have upward invariants for the bottom terms for every \( n \geq 2 \), we can easily construct a complete set of invariants for any \( n \). So suppose we have some \( n \geq 2 \) and an upward invariant for the bottom term of \( \mathcal{L}_n \),

\[
I_{00\ldots 0} = a_{00\ldots 0} - E,
\]

where \( E \) is an expression in the other coefficients of \( \mathcal{L}_n \). For clarity, let us denote the coefficients in \( \mathcal{L}_n \) using \( b \)'s, while still using \( a \)'s for the coefficients in \( \mathcal{L}_{n+1} \).

Now we seek an upward invariant for the bottom term in \( \mathcal{L}_{n+1} \), and accordingly let \( C \) be the class of operators of the form

\[
(\partial_z + p)\mathcal{L}_n = (\partial_z + p)(\partial_{x_1x_2\ldots x_n} + b_{011\ldots 1}\partial_{x_2x_3\ldots x_n} + \ldots + b_{00\ldots 0}),
\]

where we use \( z \) to denote \( x_{n+1} \). We will take \( \mathcal{N} \) to be the class of all operators in \( \mathcal{L}_{n+1} \) that have \( a_{11\ldots 110} = 1 \) and all terms involving \( \partial_z \) equal to zero. Note that \( \mathcal{N} \) is the same as \( \mathcal{L}_n \), except that its coefficients have different names. In particular, we have the invariant

\[
I_{00\ldots 0} = b_{00\ldots 0} - E
\]

where \( E \) is an expression in the \( b_\alpha \) for \( \alpha \) an \( n \)-long string of 0’s and 1’s that is not all 0’s.

Expanding \( \mathcal{C} \), we have that it is

\[
\partial_{x_1x_2\ldots x_nz} + b_{011\ldots 1}\partial_{x_2x_3\ldots x_nz} + \ldots b_{00\ldots 0}\partial_z + \\
\alpha \partial_{x_1x_2\ldots x_n} + (pb_{011\ldots 1} + b_{011\ldots 1}z)\partial_{x_2x_3\ldots x_n} + \ldots (pb_{00\ldots 0} + b_{00\ldots 0}z)\partial_z
\]

Thus to make \( L - C \) be in \( \mathcal{N} \), we take \( b_\alpha = a_\alpha \) for each \( n \)-long string of 0’s and 1’s \( \alpha \) other than \( 11\ldots 11 \), and we also take \( p = a_{11\ldots 110} - 1 \).

This gives us that \( N = L - C \) is

\[
\partial_{x_1x_2\ldots x_n} + (a_{011\ldots 10} - pb_{011\ldots 1} - b_{011\ldots 1})\partial_{x_2x_3\ldots x_n} + \ldots (a_{00\ldots 0} - pb_{00\ldots 0} - b_{00\ldots 0})\partial_z
\]

Substituting the coefficients of (23) into (21) gives the desired upward invariant for the bottom term of \( \mathcal{L}_{n+1} \).

For example, when \( n = 2 \) we have

\[
I_{00} = a_{00} - (a_{10}a_{01} + a_{10}x_1).
\]

To get an invariant for \( n = 3 \) from this, we have \( p = a_{110} - 1 \), \( b_{ij} = a_{ij} \), and \( a_{ij} = a_{ij0} - pb_{ij} - b_{ijz} = a_{ij0} - (a_{110} - 1)a_{ij1} - a_{ij3} \). Substituting these into (24), we get

\[
I_{000} = (a_{000} - (a_{110} - 1)a_{001} + a_{001z}) - ((a_{100} - (a_{110} - 1)a_{010} - a_{101z})(a_{010} - (a_{110} - 1)a_{011} - a_{011z}) + (a_{100} - (a_{110} - 1)a_{101} - a_{101z})x_1),
\]

where we are using \( z \) for \( x_3 \).

While this is not a symmetrical expression in the coefficients, this recursive definition could be of interest.

5. Complete Sets of Laplace Invariants: A Constructive Proof

We will be working toward a proof that when \( \mathcal{L} \) is maximally generated and approximately flat, that there is a complete set of invariants for \( \mathcal{L} \). Our construction of invariants will start with the highest degree terms in \( \mathcal{L} \), and work down. We will of course include the maximal invariants in our complete set of invariants \( I \). Next, we have the following.

Definition 7. Let \( \mathcal{L} \) be maximally generated, let \( L \) and \( L' \) be two arbitrary elements of \( \mathcal{L} \). Assume \( L' \) is a gauge transform of \( L \), so \( L' = e^{-g}Le^g \) for some \( g \in K \). Let \( E \) be some expression in coefficients of \( L \) (which may involve algebraic operations and
differentiation), and let \( E' \) be the same expression in the corresponding coefficients in \( L' \). Then the difference of \( E, \Delta E \), is given by \( \Delta E = E' - E \).

**Theorem 3.** Let \( E \) and \( F \) be expressions as above, and let \( a_v \partial_v \) be a term of \( \mathcal{L} \). Then the following hold.

1. \( \Delta(E + F) = \Delta E + \Delta F \).
2. \( \Delta(E_{x_i}) = (\Delta E)_{x_i} \) for any variable \( x_i \).
3. \( E \) is invariant iff \( \Delta E = 0 \) for all \( L \) and \( L' \).
4. Suppose that the term of \( v \) is submaximal, so every vector that covers it is maximal. Write the set of maximal vectors covering \( v \) as \( \{v + e_i : i \in S\} \), where \( S \subseteq \{1, 2, \ldots, n\} \) and \( e_i \) is the vector with all entries 0 except that its \( i \)-th component is 1. Then \( \Delta a_v = \sum_{i \in S} (v(i) + 1)a_{v+e_i}, g_{x_i} \).

The proof is straightforward. With assumptions as in Definition \( \mathbb{I} \) make the additional assumption that \( \mathcal{L} \) is approximately flat. Let \( s \) be the number of submaximal terms in \( \mathcal{L} \). Then we have \( s \) equations of the form \( \Delta a_v = \sum_{i \in S} (v(i) + 1)a_{v+e_i}, g_{x_i} \) as in (4) of Theorem \( \mathbb{I} \) although \( v \) and \( S \) will vary between equations. Note that all of the vectors \( v + e_i \) are maximal, so the corresponding coefficients \( a_{v+e_i} \) are all invariants. Since \( \mathcal{L} \) is approximately flat, \( s \geq n \) and each of the \( n \) partial derivatives \( g_{x_i} \) of \( g \) appears in at least one of the \( s \) equations.

These \( s \) equations are linear in the \( n \) partial derivatives \( g_{x_i} \). Solving any \( n \) of them usually gives each \( g_{x_i} \) equal to a linear expression in various \( \Delta a_v \) with invariant coefficients. There are cases where the coefficients of the maximal terms are such that the linear system does not completely determine the values of the \( g_{x_i} \). We need to investigate these further. (Everything from here on has been rewritten a bit.)

We use \( \nabla g = (g_{x_1}, g_{x_2}, \ldots, g_{x_n}) \) for the vector of partial derivatives of \( g \), and proceed to write our equations in vector form. Using (4) of Theorem \( \mathbb{I} \) we have that \( \Delta a_v = \sum_{i \in S} (v(i) + 1)a_{v+e_i}, g_{x_i} = (\sum_{i \in S} (v(i) + 1)a_{v+e_i}, e_i) \cdot \nabla g \). Letting \( \phi(v) \) denote the vector \( (\sum_{i \in S} (v(i) + 1)a_{v+e_i}, e_i) \) for every submaximal vector \( v \), we need that the set of the \( \phi(v) \) spans \( \mathbb{R}_n \).

**Definition 8.** For a maximally generated class \( \mathcal{L} \), let \( S \) be the set of its submaximal vectors. For each \( v \in S \), let \( \phi(v) \) be the vector \( (\sum_{i \in S} (v(i) + 1)a_{v+e_i}, e_i) \) as above. We say that \( \mathcal{L} \) is framed iff \( \{\phi(v) : v \in S\} \) spans \( \mathbb{R}_n \).

Assuming \( \mathcal{L} \) is framed, we have a set of \( s \) equations of the form \( \Delta a_v = \sum_{i \in S} (v(i) + 1)a_{v+e_i}, g_{x_i} \) that determine all of the derivatives \( g_{x_i} \), in terms of \( n \) of the \( \Delta a_v \). In addition to these \( n \) equations, we have \( s - n \) “extra” equations which give other of the \( \Delta a_v \) as linear expressions in the \( g_{x_i} \) with invariant coefficients.

All of the above equations yield invariants. To simplify notation, we illustrate this by letting \( n = 3 \), calling the three variables \( x, y \) and \( z \), letting \( a, b \) and \( c \) be coefficients of submaximal terms, and letting \( \alpha, \beta, \gamma \) and \( \delta \) be invariant coefficients. Then from the \( n \) equations that look like \( g_x = \alpha \Delta a + \beta \Delta b, g_y = \gamma \Delta b + \delta \Delta c \), and so on, we construct invariants as follows. Differentiating, and setting compatibility partials equal, we get equations like \( (\alpha \Delta a + \beta \Delta b)_y = g_{xy} = (\gamma \Delta b + \delta \Delta c)_x \). This becomes \( \alpha_y \Delta a + \alpha \Delta a_y + \beta_y \Delta b + \beta \Delta b_y = \gamma_x \Delta b + \gamma \Delta b_x + \delta_x \Delta c + \delta \Delta c_x \). Which is \( \alpha_y (a' - a) + \alpha (a'_y - a_y) + \beta_y (b' - b) + \beta (b'_y - b_y) = \gamma_x (b' - b) + \gamma (b'_x - b_x) + \delta_x (c' - c) + \delta (c'_x - c_x) \). Rearranging, we get \( \alpha_y a_y + \alpha a'_y + \beta_y b_y + \beta b'_y + \gamma_x b_x + \gamma b'_x + \delta_x c + \delta c'_x \). Noting that \( \alpha_y a_y + \alpha a'_y + \beta_y b_y + \beta b'_y + \gamma_x b_x + \gamma b'_x + \delta_x c + \delta c'_x \) is an invariant. In general there are \( n(n - 1)/2 \) compatibility invariants that look like this, one for each pair of variables.

Hello
For the $s-n$ “extra” equations which look like $\alpha \Delta a + \beta \Delta b = \gamma \Delta b + \delta \Delta c$, we proceed as follows. We expand the $\Delta$s, and get $\alpha(a' - a) + \beta(b' - b) = \gamma(b' - b) + \delta(c' - c)$. Rearranging, $\alpha a' + \beta b' - \gamma b' - \delta c' = \alpha a + \beta b - \gamma b - \delta c$, which makes $\alpha a + \beta b - \gamma b - \delta c$ invariant.

The above discussion gives us the following.

**Theorem 4.** If $\mathcal{L}$ is maximally generated and approximately flat, it has all of the maximal, extra and compatibility invariants needed to produce a set of invariants that is complete by Theorem 7.

Our next step is to produce a set of upward invariants for all the terms that are not maximal or submaximal. Our method will be to repeatedly apply Theorem 2. The first step is to construct a class $\mathcal{C}$ so that the unique $N$ produced as in Theorem 2 is in the class $\mathcal{N}$ of operators in $\mathcal{L}$ that have all their coefficients of maximal and submaximal terms equal to zero. Our construction will have a distinguished set of submaximal terms, which need a certain property so that we can use them to build a “framework”.

**Definition 9.** Let $\mathcal{L}$ be a maximally generated class of operators. Let $M$ be the set of vectors of maximal terms of $\mathcal{L}$, and let $S$ be the set of vectors of submaximal terms of $\mathcal{L}$. For each $s \in S$, let $T(s)$ be $\{i \leq n: s + e_i \in M\}$, and let $\phi(s)$ be the vector $\sum_{i \in T(s)}(s(i) + 1)e_{s+e_i}$. Then $\mathcal{L}$ is framed if there is an $n$-element subset $\{s_1, s_2, \ldots s_n\}$ of $S$ so that the set $\{\phi(s_1), \phi(s_2), \ldots, \phi(s_n)\}$ is linearly independent. In this case, we call $\{s_1, s_2, \ldots s_n\}$ a framing set for $\mathcal{L}$.

The vast majority of operators in the literature give maximally generated classes that are framed.

**Theorem 5.** Let $\mathcal{L}$ be the class of operators that is generated by a single nonzero term $a_v \partial^v$, where $v(i) > 0$ for all $i \leq n$. Then $\mathcal{L}$ is framed.

**Proof.** A framing set consists of the $n$ vectors for submaximal terms $\{v - e_i: i \leq n\}$, since we have that each $\phi(v - e_i)$ is a nonzero multiple of $e_i$. □

**Example 10.** Here is an example of a class $\mathcal{L}$ that is not framed. We take $n = 2$, write $x$ for $x_1$ and $y$ for $x_2$. Then we let $\mathcal{L}$ be maximally generated by $\{\partial_{xx}, 2\partial_{xy}, \partial_{yy}\}$, so operators in $\mathcal{L}$ have the form $\partial_{xx} + 2\partial_{xy} + \partial_{yy} + a_{10}\partial_x + a_{01}\partial_y + a_{00}$. There are two submaximal vectors, $(1,0)$ and $(0,1)$, and $\phi((1,0)) = \phi((0,1)) = (2,2)$.

**Theorem 6.** Let $\mathcal{L}$ be maximally generated, framed, and approximately flat. Let $N$ be the class of $L \in \mathcal{L}$ where all the coefficients of maximal and submaximal terms of $L$ are zero. Then there is a class $\mathcal{C}$ of operators so that for every $L \in \mathcal{L}$ there is a unique $C \in \mathcal{C}$ so that $N = L - C$ is in $\mathcal{N}$.

**Proof.** The notation is simplified if we work with vectors, where the vector $v$ corresponds to the term $a_v \partial^v$. We let $e_i$ denote the vector that is all zeroes, except that its $i$-th component is 1. Let $M$ be the set of maximal vectors for $\mathcal{L}$, and let $S$ be the set of submaximal vectors. Then every vector in $S$ is covered by at least one vector in $M$, and every vector in $M$ covers at least one vector in $S$. (If a vector such as $ke_i = (k,0,0,\ldots,0)$ is maximal, it only covers the one submaximal vector $(k - 1)e_1$.)

We will first produce a correspondence between elements of $M$ and subsets of $S$ that has the properties needed to construct expressions in $\mathcal{C}$. We may assume that the set of vectors for $\mathcal{L}$ contains nonzero multiples of all the $e_i$, since we may simply ignore variables whose derivative symbols do not appear in $\mathcal{L}$. Since $\mathcal{L}$ is framed, there is an $n$-element subset $\{s_1, s_2, \ldots, s_n\}$ of $S$ so that the set $\{\phi(s_1), \phi(s_2), \ldots, \phi(s_n)\}$ is linearly independent. We will use this to define a set of $n$ distinguished parameters, $\{c_1, c_2, \ldots, c_n\}$.
Let $S' = S - \{s_1, s_2, \ldots, s_n\}$. Now fix some function $f : S' \to M$ which takes every submaximal vector to a maximal vector that covers it. To each $v \in M$, we associate the set $f^{-1}(v)$, the preimage of $v$. Some sets $f^{-1}(v)$ may be empty, but the ones that are not partition $S$.

We will construct the class $C$ as a set of sums of operators, where there will be one operator for each vector in $M$. For each $m \in M$, the corresponding operator will have principal symbol $a_m \partial^m$. This guarantees that for $L \in \mathcal{L}$ and $C \in \mathcal{C}$, all operators of the form $N = L - C$ will have coefficients of zero in all terms corresponding to maximal vectors.

For each of the submaximal vectors $s_i$, we will make sure that the operator for each maximal vector $m = s_i + e_j$ that covers it has a factor of $(\partial_{x_j} + c_j)^{m(i)}$. Looking at some particular $s_i$, only the operators for maximal vectors $m$ that cover $s_i$ will contribute terms in $\mathcal{C}$ corresponding to the vector $s_i$. When $m = s_i + e_j$, the term contributed by the operator for $\mathcal{C}$ will be $m(j) a_m c_j \partial^{n_j} = (s(i) + 1) a_{s_i + e_j} c_j$. To make the term corresponding to $s_i$ zero in $L - C$, we must have $\sum_{j \in T(s_i)} (s(j) + 1) a_{s_i + e_j} c_j = a_{s_i}$, where $T(s_i)$ is the set of $j$ so that $s_i + e_j$ covers $s_i$. Letting $c = (c_1, c_2, \ldots, c_n)$, this is the equation $\phi(s_i) \cdot c = a_{s_i}$.

Then to make all the coefficients of all the $s_i$ terms zero in $L - C$, we have $\phi(s_i) \cdot c = a_{s_i}$ for all $i \leq n$. Since $\{s_1, s_2, \ldots, s_n\}$ is a framing set, the $n$ vectors $\phi(s_i)$ are linearly independent, and this system has a unique solution for the $c_i$’s. The values of the $c_i$’s are fixed by this, so we may henceforth treat them as constants.

Next consider any maximal vector $v \in M$ that does not cover any of the $s_i$. Let $S(v) = \{i : v - e_i \in f^{-1}(v)\}$. Let $u_v = v - \sum_{i \in S(v)} e_i$, and let $U(v) = \{i : u_v > 0\}$. Finally, let the operator $F_v$ be $a_v \prod_{i \in U(v)} (\partial_{x_i} + c_i)^{u_v(i)} \prod_{i \in S(v)} (\partial_{x_i} + p_i)$.

It has principal symbol $a_v \partial^v$, so adding it to $\mathcal{C}$ will make the coefficient of $\partial^v$ equal to zero in $N$. And there are unique values of the unknown coefficients $p_i$ that make the coefficients of the terms corresponding to $f^{-1}(v)$ zero in $N$, since each of these coefficients has a free term of the form $a_v p_i$ in it.

If a maximal vector $v$ does cover some of the $s_i$, the same construction works. Suppose that $v$ covers exactly the $s_i$, where $i \in P$. We must confirm that for each $i \in P$ the operator for $v$ has a factor of $(\partial_{x_j} + c_j)^{v(i)}$, where $j$ is such that $v(j) = s_i(j) + 1$. Now $S(v) = \{i : v - e_i \in f^{-1}(v)\}$ contains no elements of $P$, since the $s_i$ are not in the domain of $f$. As before, we let $u_v = v - \sum_{i \in S(v)} e_i$, let $U(v) = \{i : u_v > 0\}$, and let the operator $F_v$ be $a_v \prod_{i \in U(v)} (\partial_{x_i} + c_i)^{u_v(i)} \prod_{i \in S(v)} (\partial_{x_i} + p_i)$. Suppose that $i$ and $j$ are such that $v(j) = s_i(j) + 1$. Then $j \notin S(v)$, so $u_v(j) = v(j)$, and $F_v$ has the desired factor of $(\partial_{x_j} + c_j)^{v(j)}$.

We now let $\mathcal{C}$ be the class of all expressions of the form $\sum_{v \in M} F_v$, where the $c_i$ and the $p_i$ for all $v \in M$ are free parameters. We will show that for each $L \in \mathcal{L}$, there is a unique $C \in \mathcal{C}$ with $N = L - C \in \mathcal{N}$.

The effects of adding one of the operators $F_v$ to $\mathcal{C}$ are almost perfect. The one problem is that they may contain submaximal terms that do not correspond to vectors in $f^{-1}(v)$. Fortunately, the coefficients of these terms only depend on the $c_i$, and not on the $p_i$. For each submaximal vector $s$, we let $T(s)$ be $\{i \leq n : s + e_i \in M\}$. If $s$ is not one of the $s_i$, we let $j$s be the value of $j \leq n$ where $f(s) = s + e_j$. If $s$ is one of the $s_i$, $f(s)$ and $j$s both do not exist. The coefficient of the term in $\mathcal{C}$ corresponding to $s$ is $p_{j}s + \sum_{i \in T(s) - \{j}s} (s(i) + 1) a_{s + e_i} c_i$, where the summand of $p_{j}s$ is present if $s$ is not one of the $s_i$. In this case, we see that there is a unique value of $p_{j}s$ that makes the term zero.

The above theorem gives us that all of the coefficients of maximal terms in the unique $N \in \mathcal{N}$ are invariants. By the construction in the proof, each of these coefficients depends
only on coefficients of maximal and submaximal terms of $\mathcal{L}$, and is thus an upward invariant.

**Definition 10.** In a maximally generated class of operators, we define the *level* of a vector recursively as follows. Maximal vectors have level 0, and the level of all other vectors is 1 less than the minimum of the levels of the vectors that cover them.

In other words, the level of a vector is the length of the longest upward path from it to a maximal vector. Submaximal vectors have level less than or equal to −1. If the class of operators is approximately flat, the levels of all submaximal vectors are actually −1.

In the above theorem, all of the vectors for nonmaximal terms in $\mathcal{N}$ have level −3 or less, since they are covered by maximals of $\mathcal{N}$, which are covered by submaximals, which are covered by maxinals.

**Theorem 7.** Let $\mathcal{L}$ be maximally generated, framed and approximately flat. Then $\mathcal{L}$ has a complete set of invariants.

**Proof.** By Theorem 4, $\mathcal{L}$ has enough maximal, extra and compatibility invariants. Let these form the set of invariants $I_0$, choosing particular extra and compatibility invariants for definiteness. It remains to produce upward invariants for all terms of $\mathcal{L}$ that are not maximal or submaximal. We will do this by downward induction on the level of terms.

Terms of level greater than or equal to −2 form our basis. Terms of level 0 are maximal. Since $\mathcal{L}$ is approximately flat, terms are of level −1 iff they are submaximal. With $\mathcal{C}$ and $\mathcal{N}$ as in Theorem 6, the maximal terms in $\mathcal{N}$ are precisely those of level −2 in $\mathcal{L}$, and by Theorem 2, all the coefficients of these terms in the unique $N \in \mathcal{N}$ are invariant. This yields upward invariants for every term in $\mathcal{L}$ of level −2.

Now let $m \leq −2$, and suppose that the class $\mathcal{N}$ consists of all operators in $\mathcal{L}$ that have all terms of level greater than $m$ equal to zero. Assume that there is a class $\mathcal{C}_m$, containing operators which uniquely determine the parameters $\{c_1, c_2, \ldots, c_n\}$, where each $c_i$ only appears in factors of $(\partial_i + c_i)$. Also assume that for every $L \in \mathcal{L}$ there is a unique $C \in \mathcal{C}_m$ with $N = L - C \in \mathcal{N}$. We will show that the class $\mathcal{N}'$ consisting of all operators in $\mathcal{N}$ with coefficients of level $m$ equal to zero has a class $\mathcal{C}_m - 1 \supseteq \mathcal{C}_m$, so that for all $L \in \mathcal{L}$ there is a unique $C \in \mathcal{C}_{m-1}$ with $L - C \in \mathcal{N}'$.

Our argument will contain many of the same elements found in the proof of Theorem 3. We let $L_m$ be the set of vectors of level $m$, and let $L_{m+1}$ be the set of vectors of level $m + 1$. We let $f: L_m \to L_{m+1}$ be a function that takes each vector to one that covers it. Then for each $v \in L_{m+1}$, there is a subset $f^{-1}(v)$ of $L_m$, and the nonempty $f^{-1}(v)$ partition $L_m$.

For each $v \in L_{m+1}$, let $S(v) = \{i: v - e_i \in f^{-1}(v)\}$. Let $u_v = v - \sum_{i \in S(v)} e_i$, and let $U(v) = \{i: u_v > 0\}$. Finally, let the operator $F_v$ be $\prod_{i \in U(v)} (\partial_i + c_i)^{w(i)} \prod_{i \in S(v)} (\partial_i + p_i)$.

It has principal symbol $\partial^x$, so we will need to compensate for adding it to $\mathcal{C}_m$. There will be unique values of the unknown coefficients $p_i$ that make the coefficients of the terms corresponding to $f^{-1}(v)$ zero in $\mathcal{N}$, since each of these coefficients has a free term of the form $p_i$, in it.

Now we let $\mathcal{C}_{m-1}$ be the class consisting of all possible sums of elements of $\mathcal{C}_m$ and operators of the form $\sum_{v \in L_{m+1}} F_v$. To show that $\mathcal{C}_{m-1}$ has the desired property, let $L \in \mathcal{L}$ be given. Let $L'$ be the operator constructed from $L$ by subtracting 1 from the coefficients of all terms corresponding to vectors in $L_{m+1}$. Let $C$ be the unique element of $\mathcal{C}_m$ with $N = L' - C \in \mathcal{N}$. Now $\sum_{v \in L_{m+1}} F_v$ has coefficients at all levels above $L_{m+1}$ equal to zero, and coefficients at level $m + 1$ equal to 1. So $L' + \sum_{v \in L_{m+1}} F_v$ has all its coefficients at levels $m + 1$ and above equal to those of $L$. Then $C = C + \sum_{v \in L_{m+1}} F_v$.
is an element of $C_{m-1}$ where for all coefficients at levels $m + 1$ and above, $L - C' = (L' + \sum_{v \in L_{m+1}} F_v) - C' = L - C$. Thus $L - C' \in \mathcal{N}$.

Every coefficient of level $m$ in $L - C'$ has one free parameter $p_i$, so there is a unique way to choose these parameters so that $L - C'$ in $\mathcal{N}$.

This completes the induction step, and shows that all terms at or below level $-2$ have upward invariants. Adding all these upward invariants to the set $I_0$ gives a set of invariants that is complete by Theorem 1.

\begin{flushright} \Box \end{flushright}

\textbf{ACKNOWLEDGEMENT}

This material is based upon work supported by the National Science Foundation under grant No.1708033.

\section*{References}

[1] Ch. Athorne. Laplace maps and constraints for a class of third-order partial differential operators. \textit{Journal of Physics A: Mathematical and Theoretical}, 51(8), 2018.

[2] Ch. Athorne and H. Yilmaz. Invariants of hyperbolic partial differential operators. \textit{Journal of Physics A: Mathematical and Theoretical}, 49(13):135201, 2016.

[3] Ch. Athorne and H. Yilmaz. Twisted Laplace maps. \textit{Journal of Physics A: Mathematical and Theoretical}, 52(22):225201, 2019.

[4] G. Darboux. \textit{Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal}, volume 2. Gauthier-Villars, 1889.

[5] O. Dzhokhadze. Laplace invariants for some classes of linear partial differential equations. \textit{Differential Equations}, 40(1):63–74, 2004.

[6] M. Fels and P. J. Olver. Moving coframes. I. A practical algorithm. \textit{Acta Appl. Math.}, 51(2):161–213, 1998.

[7] M. Fels and P. J. Olver. Moving coframes. II. Regularization and theoretical foundations. \textit{Acta Appl. Math}, 55:127–208, 1999.

[8] Elena I. Ganzha. Intertwining laplace transformations of linear partial differential equations. In Moulay Barkatou, Thomas Cluzeau, Georg Regensburger, and Markus Rosenkranz, editors, \textit{Algebraic and Algorithmic Aspects of Differential and Integral Operators}, pages 96–115. Springer Berlin Heidelberg, 2014.

[9] S. Hill, E. Shemyakova, and Th. Voronov. Darboux transformations for differential operators on the superline. \textit{Russian Mathematical Surveys}, 70(6):207–208, 2015. \texttt{arXiv:1505.05194 [math.MP]}.

[10] D. Hobby and E. Shemyakova. Classification of multidimensional Darboux transformations: first order and continued type. \textit{SIGMA (Symmetry, Integrability and Geometry: Methods and Applications)}, 13(10):20 pages, 2017. \texttt{arXiv:1605.04362 [math.DG]}.

[11] G. Hovhannisyan and O. Ruff. Darboux transformations on a space scale. \textit{Journal of Mathematical Analysis and Applications}, (2):1690–1718, 2016.

[12] G. Hovhannisyan, O. Ruff, and Z. Zhang. Higher dimensional Darboux transformations. \textit{Journal of Mathematical Analysis and Applications}, (1):776–805, 2018.

[13] E. Imamoglu and M. van Hoeij. Computing hypergeometric solutions of second order linear differential equations using quotients of formal solutions and integral bases. \textit{Journal of Symbolic Computation}, 83:254–271, 2017.

[14] S. Li, E. Shemyakova, and Th. Voronov. Differential operators on the superline, Berezinians, and Darboux transformations. \textit{Lett. Math. Phys.}, 107(9):1689–1714, 2017. \texttt{arXiv:1605.07286 [math.DG]}.

[15] E. L. Mansfield. \textit{A Practical Guide to the Invariant Calculus}. Cambridge University Press, 2010.

[16] A Mironov. On the laplace invariants of a fourth-order equation. \textit{Differential Equations}, 45(8), 2009.

[17] P. J. Olver and J. Pohjanpelto. Differential invariant algebras of Lie pseudo-groups. \textit{Adv. Math.}, 222(5):1746–1792, 2009.

[18] E. Shemyakova. Invariant properties of third-order non-hyperbolic linear partial differential operators. In \textit{Lecture Notes in Computer Science}, volume 5625, pages 154–169. Springer Berlin Heidelberg, 2009.

[19] E. Shemyakova. Invertible Darboux transformations. \textit{SIGMA (Symmetry, Integrability and Geometry: Methods and Applications)}, 9:Paper 002, 10, 2013.
[20] E. Shemyakova. Proof of the completeness of Darboux Wronskian formulae for order two. Canad. J. Math., 65(3):655–674, 2013.
[21] E. Shemyakova. Classification of Darboux transformations for operators of the form $\partial_x\partial_y + a\partial_x + b\partial_y + c$. Illinois J. Math., 64(1):71–92, 04 2020. arXiv:1304.7063 [math.MP].
[22] E. Shemyakova and E. L. Mansfield. Moving frames for Laplace invariants. In Proceedings of the twenty-first international symposium on symbolic and algebraic computation, pages 295–302, New York, 2008. ACM.
[23] E. Shemyakova and Th. Voronov. Differential operators on the algebra of densities and factorization of the generalized Sturm–Liouville operator. Lett. Math. Phys., 109(2):403–421, 2019. arXiv:1710.09542 [math.DG].
[24] E. Shemyakova and F. Winkler. A Full System of Invariants for Third-Order Linear Partial Differential Operators in General Form. Lecture Notes in Comput. Sci., 4770:360–369, 2007.
[25] E. Shemyakova and F. Winkler. Obstacles to the factorization of linear partial differential operators into several factors. Programming and Computer Software, 33(2):67–73, 2007. http://arxiv.org/abs/1010.2652 [math.AP].
[26] E. Shemyakova and F. Winkler. On the Invariant Properties of Hyperbolic Bivariate Third-Order Linear Partial Differential Operators. In Deepak Kapur, editor, ASCM, volume 5081 of Lecture Notes in Computer Science, pages 199–212. Springer, 2007.
[27] S. V. Smirnov. Factorization of Darboux—Laplace transformations for discrete hyperbolic operators. Theor. Math. Phys., 199(2):621–636, 2019.

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