Knot Weight Systems from Graded Symplectic Geometry

Jian Qiu\textsuperscript{a} and Maxim Zabzine\textsuperscript{b}

\textsuperscript{a}I.N.F.N. and Dipartimento di Fisica  
Via G. Sansone 1, 50019 Sesto Fiorentino - Firenze, Italy

\textsuperscript{b}Department of Physics and Astronomy, Uppsala university,  
Box 516, SE-751 20 Uppsala, Sweden

Abstract

We show that from an even degree symplectic $NQ$-manifold, whose homological vector field $Q$ preserves the symplectic form, one can construct a weight system for tri-valent graphs with values in the $Q$-cohomology ring, satisfying the IHX relation. Likewise, given a representation of the homological vector field, one can construct a weight system for the chord diagrams, satisfying the IHX and STU relations. Moreover we show that the use of the 'Grinthendieck connection' in the construction is essential in making the weight system dependent only on the choice of the $NQ$-manifold and its representation.
1 Introduction

The Vassiliev knot invariants [1, 2] compute the cohomology of the space of embeddings of one or several circles into $S^3$, and can be conveniently kept track of by means of the so called chord diagrams. The precise definition of the chord diagrams will appear in sec.5, and the weight system for knot invariants is basically the scheme of assigning weights to the chord diagrams. The key criterium that a weight system has to meet is the IHX relation and STU relation; the IHX relation can be thought of as some kind of Jacobi identity and the STU relation is analogous to the matrix representation of these Jacobi identities. Thus Lie algebras and their matrix representations furnish the most important class of weight systems. While some preliminary calculation by Bar-Natan (see ref.[3]) at low dimension showed that the Lie algebra weight systems exhaust the Vassiliev knot invariants, it has been established later [4] that at sufficient high dimension, the Lie algebras fail to provide all the weight systems.

In the work of Rozansky and Witten [5], and later in greater detail by Sawon [6], it was pointed out that a hyperKähler manifold plus a holomorphic vector bundle on it also gives a valid weight system. Kapranov [7] showed that one may associate an $L_\infty$-algebra structure to a hyperKähler manifold and one is led to the realization that an $L_\infty$-algebra structure in conjunction with a proper notion of representation provides a wider class of weight systems, which may take value in certain rings. As an example, the Rozansky-Witten (RW) weight-system takes value in the Dolbeault cohomology ring.

In an earlier publication [8] of the authors, we have built up a class of topological field theories (TFT’s) that are associated with the $L_\infty$-algebra structures in the same way the Chern-Simons theory is associated with a Lie algebra. One can use some path integral manipulations and Ward identities to prove that these $L_\infty$ structures do give valid weight systems at the level of ‘physics rigour’. Thus we feel that we owe the reader a purely mathematical and more careful account of our formalism. Besides, the treatment of an $L_\infty$-structure arising from a curved $NQ$-manifold is more subtle than we were able to appreciate at the writing of ref.[8], and we shall rectify this remiss in this paper.

To summarize, the main results of the paper are theorems 5.1 and 5.2. We take an even degree symplectic N-manifold $(\mathcal{M}, \omega)$ and use an exponential map to identify a neighbourhood round any point with a graded vector space $\mathcal{C}A$ of the same dimension. Theorem 5.2 is a recipe for constructing cocycles of Lie algebra of vector fields on $\mathcal{M}$ from a cocycle on $\mathcal{C}A$. And the use of the Grothendieck connection in the formalism is to render the cohomology class to which the cocycle belongs independent of the connection chosen to perform the exponential map. When $\mathcal{M}$ has an $\Omega$-preserving homological vector field $Q$, and a graded bundle $\mathcal{E}$ over $\mathcal{M}$ equipped with a lift $\tilde{Q}$ of $Q$, one can construct two objects $\hat{Q}$ and $\hat{T}$ that give rise to an $L_\infty$-structure (up to $Q$-exact terms) and its representation. $\hat{Q}$ is a vector field on $\mathcal{C}A$ and can be lifted to a Hamiltonian function $\Theta$, since $\mathcal{C}A$ inherits a symplectic structure from $\mathcal{M}$. Theorem 5.1 states that, one can use the third (resp. first) Taylor coefficient of $\Theta$ (resp. $\hat{T}$) as vertices and form a weight-system for the chord diagrams, valued in the $Q$-cohomology ring of $\mathcal{M}$. The result is again independent of any non-canonical choices: connections, trivializations and the like. Our treatment was inspired by the appearance of the Grothendieck connection in quantum field theory, see earlier work [9] and [10] for the recent discussion.

As examples, we generalize the RW weight system based on hyperKähler manifolds to include also the holomorphic symplectic manifolds, and in particular, the equivariant version of our construction illustrates the absolute indispensability of the Grothendieck connection.

The paper is organized as follows, we first present in sec.2 the formula for the Gronthendeck connection and explore its geometrical meaning, in particular the Bott-Haefliger construction. Then in sec.3 we review
the Chevalley-Eilenberg (CE) complex of Lie algebra of vector fields and give the recipe for converting a CE cochain on a flat space to a cochain on a curved space with value in the ring of functions. In sec 4, we give the formula for the exponential map and Grothendieck connection in this special case and verify all properties claimed in sec 2 and 3. Section 5 contains the recipe of the mapping between CE complex and graph complex, which is a useful book keeping tool for the computation. Also our main theorems are stated in that section. Finally, in sec 6, we give plenty of examples to endow substance to our construction.

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2 The Grothendieck Connection

As a general preamble, a graded manifold $\mathcal{M}$ consists of a smooth manifold $M$ with a sheaf of freely generated graded commutative algebra on $M$. The sections of this sheaf, denoted $C^\infty(\mathcal{M})$, are often called the functions on the graded manifold. The underlying manifold $M$ is called the reduced manifold or body of $\mathcal{M}$, denoted as $M = |\mathcal{M}|$. We will employ the ‘functor of points’ perspective in handling the graded manifolds, which allows one to treat the generators of the local graded algebra as coordinates and talk about ‘points’ on $\mathcal{M}$, for more expositions regarding this, see ref. [11].
Consider a non-negatively graded manifold (N-manifold) $\mathcal{M}$, which is locally of the type

$$\mathbb{R} = \mathbb{R}^n \times \mathbb{R}^{n_1}[1] \times \mathbb{R}^{n_2}[2] \times \cdots$$

Choose the coordinates of $\mathcal{M}$ to be $x^A$, and the Greek letters $\xi^A$ for the corresponding coordinates of $\mathbb{R}$. We will reserve $A, B, \cdots$ for the general coordinates, while $\mu, \nu, \cdots$ for the zero degree coordinates, namely $x^\mu$ is the coordinate of the body $|\mathcal{M}|$ and $\xi^\mu$ is the coordinate of $\mathbb{R}^n$. We will also borrow a general relativity jargon by calling $x^A$ curved coordinates and $\xi^A$ flat coordinates.

In a neighborhood $U \subset |\mathcal{M}|$, there is an isomorphism of algebras

$$\varphi : C^\infty(\mathcal{M}|_U) \rightarrow C^\infty(\mathbb{R}). \tag{1}$$

If we pick locally a family of such isomorphisms $\phi_x$, depending smoothly on $x$, such that the origin of $\mathbb{R}$ is mapped to $x$ by $\phi_x$ (the 'functor of points' view is being enforced here!), then we can consider the following differential operator

$$D = dx^A \frac{\partial}{\partial x^A} - dx^A \left[ \frac{\partial \phi_x}{\partial x} \right]_A \left[ \left( \frac{\partial \phi_x}{\partial \xi} \right)^{-1} \right]_B \frac{\partial}{\partial \xi^C} = dx^A \left( \frac{\partial}{\partial x^A} - \mathfrak{g}^C_A \frac{\partial}{\partial \xi^C} \right). \tag{2}$$

The second term in $D$ is called the Grothendieck connection, it is a 1-form defined on the total space of a bundle $\mathcal{B}$ to be introduced shortly, along with its other properties. For now we just take its expression at its face value and we have

**Proposition 2.1** The Grothendieck connection is flat

$$[u \cdot D, v \cdot D] = u \cdot D v \cdot D - (-1)^{|u||v|} v \cdot D u \cdot D = [u, v] \cdot D, \quad u, v \in vect(\mathcal{M}), \quad u \cdot D = u^A D_A. \tag{3}$$

**Proof 1:** A direct calculation does the job.

**Proof 2:** We present a second proof which exhibits the Grothendieck connection as the pull back of a left invariant form $g^{-1} dg$ on a pseudogroup, which would immediately imply the flatness. The following is a special case of a construction due to Bott and Haefliger [12], who used it to construct characteristic classes for foliations. The ensuing discussion is valid for graded manifolds, but for first reading, one may consider only a smooth manifold.

### 2.1 Bott-Haefliger Construction

For an $N$-manifold $\mathcal{M}$, let $M$ be its body. For each $x \in \mathcal{M}$, we can attach a bundle (the Haefliger) structure. Over a point $x$, the fibre consists of local isomorphisms $\phi_{x,U}$, as in Eq.1 defined on some open neighborhood $U \in M$ containing the body of $x$: $U \ni x|_M$, such that the point $x$ is mapped to the origin of $\mathbb{R}$. The pseudogroup $\Gamma$ consists of diffeomorphisms between open sets in $\mathbb{R}$, that is, between $C^\infty(\mathbb{R}|_U)$ and $C^\infty(\mathbb{R}|_V)$, $U, V \subset \mathbb{R}^n$. And let $\Gamma_0$ consist of those diffeomorphisms fixing $0 \in \mathbb{R}$. Note $\Gamma_0$ acts transitively on the fibre, so we have a principle bundle structure.

$$\mathcal{B} \leftarrow \Gamma_0 \downarrow \mathcal{M}$$

---

1Unless otherwise declared, all derivatives are left derivatives; the right derivatives differ from the left ones by a sign $x^A \frac{\partial}{\partial x^A} = (-1)^{|x^A|} x^A \frac{\partial}{\partial x^A}$.

2To avoid difficulty with infinite dimension, one can consider first the bundle $\mathcal{B}^k$ of $k$-jets of local diffeomorphisms with structure group the $k$-jets $\Gamma_0^k$, then take the inverse limit.
For two points $x$ and $y$ nearby in $M$ and two such isomorphisms $\varphi_{x',U}'$, $\varphi_{y',V}'$ (each of which is a point of the fibre over $x$ and $y$), there is an element $g \in \Gamma$ defined on $\varphi_{x',U}' \cap \varphi_{y',V}'(V)$ relating $\varphi_{x',U}'$ and $\varphi_{y',V}'$:

$$\varphi_{y',V}' = g^{-1} \circ \varphi_{x',U}'$$

where $\circ$ denotes composition and we have used $g^{-1}$ for later convenience. Eq.4 allows us to identify $B_W$ with the pseudogroup $\Gamma$ for small enough $W$. To do so, pick a point $x$ and a fiducial isomorphism $\varphi_{x,U}$, let $W \subset U$ be a small open neighbourhood of $x|_M$, the identification $\psi : B_W \rightarrow \Gamma$ goes as

$$\forall y, y|_M \in W \subset V \text{ and } \varphi_{y,V}, \text{ let } \psi(\varphi_{y,V}) = g, \text{ where } g \text{ satisfies } \varphi_{y,V}^{-1} = g^{-1} \circ \varphi_{x,U}^{-1}.$$ 

If we had started from a different fiducial $\varphi_{x',U}'$, with $\varphi_{x',U}' = h^{-1} \circ \varphi_{x',U}'$, then clearly

$$\psi' = h \circ \psi.$$ 

We can consider the pull back through $\psi$ of a left invariant 1-form $\Theta = \psi^*(g^{-1}dg)$, $g \in \Gamma$. One can see that this is a well defined 1-form on $B$, i.e. independent of the choice $\varphi_{x,U}$. Indeed, the change of $\varphi_{x,U}$ multiplies (rather, composes) an element $h \in \Gamma$ to $\psi$ from the left as in Eq.4. Since $d$ here does not act on $h$, $\psi^*(g^{-1}dg)$ is unchanged and $\Theta$ is a well-defined 1-form of $B$. We shall write $\psi^*(g)$ simply as $g$.

We show next that this 1-form leads to the Grothendieck connection Eq.2 thereby establishing the flatness. To pull back the 1-form $\Theta$ to $M$, one chooses a local section $\phi^{-1} : M \rightarrow B$. We shall dispense with the subscript $U$, and also remark that $\varphi^{-1}$ is used to denote points of $B$, but the symbol $\phi^{-1}$ is reserved for sections of $B$. We also write the transition function $g^{-1}$ in Eq.4 as

$$\phi_y = \Gamma_{yx} \circ \phi_x^{-1}.$$ 

Thus we need to compute the quantity

$$\Theta = dy^A \Gamma_{yx} \left( \frac{\partial}{\partial y^A} \Gamma_{yx}^{-1} \right),$$

which by construction is independent of $x$. Taking advantage of this we have

$$\Gamma_{yx}(\partial_y \Lambda \Gamma_{yx}^{-1}) = \Gamma_{yx}(\partial_y \Lambda \Gamma_{yx}^{-1}) \big|_{x=y} = -\partial_y \Lambda \Gamma_{yx} \big|_{x=y}.$$ 

To compute this efficiently, we do the following

$$0 = \partial_y \Lambda \phi_x = \partial_y \Lambda (\phi_y \circ \Gamma_{yx}) \Rightarrow \left( \partial_y \Lambda \phi_y^B \right)(\eta) + \frac{\partial \Gamma_{yx}^C}{\partial y^A} \left( \frac{\partial \phi_y^B}{\partial \xi^C} \right) \bigg|_{\eta = \Gamma_{yx}(\xi)} = 0.$$ 

Evaluating the last equation at $x = y$,

$$\left( \partial_y \Lambda \phi_y^B \right)(\xi) + \frac{\partial \Gamma_{yx}^C}{\partial y^A} \left( \frac{\partial \phi_y^B}{\partial \xi^C} \right) \bigg|_{x=y} = 0,$$

leads to

$$\Theta^C = dy^A \left( \frac{\partial \phi_y^B}{\partial y^A} \right)^D \left[ \left( \frac{\partial \phi_y^B}{\partial \xi^C} \right)^{-1} \right]_C.$$ 

This is the formula given in Eq.2 up to a minus sign.

The sign difference comes about in the following way: to get at Eq.2, one represents $\text{Lie}_\Gamma$ on $C^\infty(\mathbb{R})$, and the action is defined

$$(hf)(\xi) = h^{-*} f(\xi) = f(h^{-1}(\xi)), \text{ } h \in \Gamma, \text{ } f \in C^\infty(\mathbb{R}).$$

in order for it to compose correctly. This is the origin of the minus sign.
Remark While a change of trivialization for $B$ effects a left multiplication $g \to f \circ g$, one can move about the fibre over $y$ by a right multiplication $g \to g \circ h^{-1}_y$, $h^{-1}_y \in \Gamma_0$. Indeed, under a change of local isomorphism, $	ilde{\varphi}_{y,V}^{-1} = h_y \circ \varphi_{y,V}^{-1}$, the group element $g \in \Gamma$ relating $\varphi_{y,V}$ and the fiducial $\varphi_{x,U}$ transforms as
\[
\tilde{\varphi}_{y,V}^{-1} = h_y \circ \varphi_{y,V}^{-1} \quad \Rightarrow \quad g \to g \circ h^{-1}_y.
\]
Similarly if one computes the Grothendieck connection with a different local section $\tilde{\varphi}_{x,U}^{-1} = h_x \circ \varphi_{x,U}^{-1}$, then
\[
\mathcal{G} \to h_x \circ dh^{-1}_x + h_x \circ \mathcal{G} \circ h^{-1}_x,
\]
which gives how $\mathcal{G}$ depends on the fibre of $B$.

Remark The bundle $B$ will not in general have a global section, since, as a principle bundle, the possession of a global section would trivialize the bundle. To forestall a possible confusion, the 1-form $\mathcal{G}$, albeit called a connection, is not a connection of $B$, because it is valued in $\text{Lie } \Gamma_0$, instead of $\text{Lie } \Gamma_0$.

However, one can take a group $K \subset \Gamma_0$ maximally compact, and consider the quotient $B/K$ with fibre $\Gamma_0/K$, which is contractible. Thus $B/K$ possesses a global section (this is a classic result, see theorem 12.2 [13]). In sec 4 we will construct a section of $B/\text{SO}$ using an exponential map of a flow equation, and in special cases we also have sections of $B/\text{GL}$, and $B/\text{SP}$, even though the latter two groups are not maximally compact.

Finally, for application of weight systems for knots we need the notion of graded vector bundles over graded manifolds, defined as sheaves of freely generated $C^\infty(M)$-modules over the body of $M$, the coordinates of the fibre are the generators of this module, see ref. [11].

\[
\begin{array}{ccc}
Q^\uparrow & \mathcal{E} & \downarrow \pi \\
\downarrow \pi_\ast & \mathcal{V} & \\
 Q & \mathcal{M} \end{array}
\]

In particular, we are interested in the case when $M$ is an $NQ$-manifold with homological vector field $Q$, and there is a lifting of $Q$ to $Q^\uparrow$ acting on $\mathcal{E}$. The lift $Q^\uparrow$ can be thought of as the representation of $Q$ [14].

We can choose a trivialization locally on a patch $\mathcal{E}|_U = \mathcal{M}|_U \times \mathcal{V}$, and denote the coordinate of the fibre $\mathcal{V}$ as $z_{\alpha}$. Locally, we can write $Q^\uparrow$ as
\[
Q^\uparrow = Q^\uparrow(x) \frac{\partial}{\partial x^\alpha} + (-1)^{\alpha^\beta + \beta^\gamma} T^\beta_\gamma(x) z_{\beta} \frac{\partial}{\partial z_{\alpha}},
\]
where notations like $(-1)^{\alpha}$ denote $(-1)^{|z_{\alpha}|}$. The nilpotency of $Q^\uparrow$ entails
\[
(-1)^{\beta^\gamma} Q T^\beta_\gamma + (-1)^{\gamma^\beta} T^\beta_\gamma = 0.
\]

The splitting of $Q^\uparrow$ is not independent of local trivializations of $\mathcal{E}$: under $z_{\alpha} \to z_{\alpha} + z_{\beta} \epsilon_\beta^\alpha$, the $T$ matrix transforms as
\[
\delta T^\beta_\alpha = Q^\beta_\alpha + T^\beta_\gamma \epsilon_\gamma^\alpha - (-1)^{\beta + \gamma} \epsilon_\beta^\gamma T^\gamma_\alpha,
\]
thus $T$ is not a section of the bundle $\text{End } \mathcal{E}$. The trivialization independence problem will be handled properly in sec 3.2.
The local model for $\mathcal{E}$ is $\mathbb{R} \times \mathcal{V}$ for some fixed graded vector space $\mathcal{V}$. To build a local isomorphism of $\mathcal{E}|_U$ with $\mathbb{R} \times \mathcal{V}$ is almost verbatim to Eq. except that we demand $\varphi$ be independent of the fibre of $\mathcal{E}$. And we use $z_\alpha$ as the fibre coordinate of $\mathcal{E}$ under a trivialization, and $\zeta_\alpha$ the corresponding coordinate in local model.

## 3 Covariant Cocycles

### 3.1 Chevalley-Eilenberg Complex for Graded Lie Algebra

Consider a graded Lie algebra $\mathfrak{g}$ equipped with the bracket of degree $n$,

$$
\deg[u,v] = \deg u + \deg v - n,
$$

$$
[u,v] = \sum_{i<j} (-1)^{|v_i|+|v_j|} (v_i ,v_j ,v_1 ,\cdots ,v_{i-1} ,v_{i+1} ,v_{i+2} ,\cdots ,v_{j-1} ,v_{j+1} ,v_{j+2} ,\cdots ,v_k) .
$$

The Chevalley-Eilenberg (CE) chain complex is defined as $c_* = S^\bullet(\mathfrak{g}[n+1])$, i.e. the symmetric algebra of $\mathfrak{g}$ with degree shifted by $n+1$ (sometimes, this shifting of degree is called suspension). Note that if $\mathfrak{g}$ is of degree 0, then $S^\bullet(\mathfrak{g}[1]) = \wedge^\bullet\mathfrak{g}$. An element of $c_k$ is written as

$$
(c_1 ,\cdots ,c_{k}) \in c_k , \quad v_i \in \mathfrak{g},
$$

with the symmetry property

$$
(v_1 ,\cdots ,v_{i} ,v_{i+1} ,\cdots ,v_{v}) = \sum_{i<j} (-1)^{|v_i|+|v_j|} (v_i ,v_j ,v_1 ,\cdots ,v_{i-1} ,v_{i+1} ,v_{i+2} ,\cdots ,v_{j-1} ,v_{j+1} ,v_{j+2} ,\cdots ,v_k) .
$$

The coboundary operator $\partial : c_* \to c_* -1$ is defined as follows

$$
\partial t_i (v_1 ,\cdots ,v_{k}) = \sum_{i<j} (-1)^{s_{ij}} (v_i ,v_j ,v_1 ,\cdots ,\hat{v}_i ,\hat{v}_j ,\cdots ,v_{k+1} ) ,
$$

$$
t_i = (|v_i |+n+1)(|v_i |+\cdots |v_{i-1}| + (i-1)(n+1)) ,
$$

$$
s_{ij} = t_i + t_j - (|v_i |+n+1)(|v_j |+n+1) .
$$

The sign factor $s_{ij}$ is called the Kozul sign; it is incurred by moving $v_i ,v_j$ to the very front; $\hat{i}$ means skipping $v_i$.

Define the CE cochain complex $c^*$ as a multi-linear map from $c_*$ to certain $\mathfrak{g}$-module $\mathcal{M}$

$$
c^k(v_1 ,\cdots ,v_k) \in \mathcal{M},
$$

The coboundary operator $\delta : c^* \to c^{*+1}$ is defined as follows

$$
(\delta c^k)(v_1 ,\cdots ,v_{k}) = c^k(\partial t (v_1 ,\cdots ,v_{k+1} ) )
$$

$$
- \sum_i (-1)^{t_i + |v_i |+|v_i |+n} v_i \circ c^k(v_1 ,\cdots ,\hat{v}_i ,\cdots ,v_{k+1} ) ,
$$

where $\circ$ denotes $\mathfrak{g}$-module action. The only condition on $\deg c^k$ from demanding $\delta^2 = 0$ is $\deg \delta c^k = \deg c^k + 1$.

In sec.\ref{sec:5} when the cochains are represented as graphs, $\deg c^k$ will be given in terms of the data of a graph

$$
\deg c^k = nE - (n+1)V ,
$$

\footnote{We will only be interested in the Lie algebra of vector fields, whose bracket has degree 0, and Lie algebra of Hamiltonian functions, whose bracket has degree $-2$, but the formula given below is valid for general degrees.}
where $V$ and $E$ are the number of the vertices and edges of a graph. For now it is convenient to think of $\deg e^k$ naively as the degree carried by the symbol $e^k$.

We will also need of the cyclic bar complex \[15\]. Let $A$ be a graded algebra, then at degree $k$, $B_k = \otimes^k A[1]$ mod a relation of cyclic permutation. A typical element is commonly written as $[g_1|g_2|\cdots|g_k] \in B_k$, with the relation

\[ [g_1|g_2|\cdots|g_k] \sim (-1)^{(\sum|g_i|+1)+\sum|g_i|+k-1}|g_1|\cdots|g_k|g_1] \tag{15} \]

The differential $\partial_H$ acts according to

\[ \partial_H [g_1|g_2|\cdots|g_k] = - \sum_{1 \leq j \leq k} (-1)^{u_j+\sum|g_i|+1}[g_jg_{j+1}|g_{j+2}|\cdots|g_k|g_1|\cdots|g_{j-1}], \quad k+1 \equiv 1 \]

\[ u_j = \sum_i (|g_i|+1) \sum_{l=j}^k (|g_l|+1) \tag{16} \]

If $g$ acts on $A$, then we define an anti-$g$-module action on $B_k$ as

\[ u \circ [g_1|g_2|\cdots|g_k] = - \sum_{1 \leq j \leq k} (-1)^{u_j+\sum|g_i|+1+n}[g_1|\cdots|g_{j-1}|u \circ g_j|g_{j+1}|\cdots|g_k], \tag{17} \]

\[ w_j = (|u|+n) \sum_{i=1}^{j-1} (|g_i|+1), \]

satisfying $u \circ v \circ -(1)^{|u|+n}|v|+n+1 \circ u \circ v = -|u|, v|\circ$.

The extended CE complex is the tensor product of $c_\ast$ and $B_\ast$.

\[ c_{p,q} = c_p \otimes B_q. \]

The total differential for the cochain complex $c^{p,q}$ is induced from $\partial_I$, $\partial_H$ and the $g$-action Eq\[17\]. its full expression is relegated to the appendix.

### 3.2 Lie Algebra of Vector Fields

Consider several variants of the Lie algebra of tangent vector fields. Let $\mathcal{M}$ be an $N$-manifold, locally of type $\mathbb{R}$, and $\mathcal{E}$ is a graded vector bundle over $\mathcal{M}$ with fibre $\mathbb{V}$.

1. Let $\mathfrak{g}$ be the Lie algebra of formal $\Gamma_0$ vector-fields (see sec.2 of ref.\[12\]), which can simply be thought of as vector fields on $\mathbb{R}$ with formal power series as coefficients. Let $K$ be a subgroup of $GL(\mathbb{R})$, in particular $K = SP(\mathbb{R})$. We consider the relative CE complex $c^\ast(g, K)$ taking value in the real or complex number (trivial $g$-module).

1'. $\mathfrak{g} = \text{vect}(\mathcal{M})$, and the CE complex $c^\ast$ taking value in functions of $\mathcal{M}$ with the obvious $g$-module action.

2. Lie algebra $\mathfrak{g}$ as in 1, and $A = \text{End}(\mathcal{V})$, we consider the extended CE complex $c^{\ast\ast}(\mathfrak{g}, K; A)$ valued in $\mathbb{R}$ or $\mathbb{C}$, basic w.r.t $K$ and invariant under diagonal action of $\text{End}(\mathcal{V})$ on $A$ (by conjugation).

2'. $\mathfrak{g} = \text{vect}(\mathcal{M})$ and $A = \text{End}(\mathcal{E})$, we consider the extended CE complex $c^{\ast\ast}$ taking value in $C^\infty(\mathcal{M})$.

The central result of this paper is the following prescription that turns the cocycles of type 1, 2 into cocycles of type 1', 2'. To keep the discussion uniform, we do not distinguish between Hamiltonian vector fields or Hamiltonian functions, one can think of the latter as a Lie algebra with Lie bracket of degree 0, or a Lie algebra with Poisson bracket of degree $-n$, with $n$ being the degree of the symplectic form.
Pick a base point \( x \), let \( u \in \text{vect}(\mathcal{M}) \), and \( \phi^{-1}_{x,u} \in \text{vect}(\mathbb{R}) \) be the push forward. Define

\[
\hat{u} = \phi^{-1}_{x,u} u - u_x \Theta.
\] (18)

Since \( \hat{u} \) always vanishes at the origin of \( \mathbb{R} \), it takes value in the Lie algebra of \( \Gamma_0 \). Furthermore, as \( \Theta \) is a 1-form on \( \mathcal{B} \), \( \hat{u} \) is a function on \( \mathcal{B} \) valued in \( \text{Lie}(\Gamma_0) \). We have

**Proposition 3.1**

\[
\left[ \hat{u}, \hat{v} \right] = [u, v] - u \circ \hat{v} + (-1)^{(|u|+n)(|v|+n)} v \circ \hat{u},
\] (19)

where \( u \circ \) denotes the differentiation w.r.t the basis point

\[
uo = u^A(x) \frac{\partial}{\partial x^A}.
\] (20)

**Proof:** Clearly we have

\[
D_A \phi^B_x = \left[ \frac{\partial}{\partial x^A} - \Theta^A B \frac{\partial}{\partial \xi^A} \right] \phi^B_x = 0
\]

from the chain rule. Thus we have

\[
\partial_A (\phi^{-1}_{x,u}) - [\Theta_A, \phi^{-1}_{x,u}] = 0.
\]

This, in conjunction with the flatness of \( \Theta \) given in Eq.3, gives us the desired result. \( \blacksquare \)

Choose any global section \( \phi \) of \( \mathcal{B}/K \), the exponential map to be introduced later is one possible choice, then prop[3.1] leads to.

**Proposition 3.2** Pick any cochain \( c^k \) of the type 1, the following cochain of type 1’ is well-defined

\[
c^k (u_1, \ldots, u_k) = c^k (\hat{u}_1, \ldots, \hat{u}_k), \quad u_i \in \text{vect}(\mathcal{M}).
\] (21)

i.e. the mapping \( c^* \) to \( c^* \) is a morphism of complexes and \( c \) is valued in \( C^\infty(\mathcal{M}) \).

**Proof** In general the lhs of Eq[21] only gives a function on \( \mathcal{B} \). But if \( c^* \) is basic w.r.t the group \( K \), the lhs of Eq[21] gives a function on the quotient \( \mathcal{B}/K \). Then through a global section of \( \mathcal{B}/K \), one can pull back this function to \( \mathcal{M} \). For similar discussions, see also sec.4 [16].

To show the mapping is a morphism

\[
\delta c^k (\hat{v}_1, \ldots, \hat{v}_{k+1}) = \sum_{i<j} (-1)^{s_{ij}} c^k ((-1)^{|v_i|}[\hat{v}_i, \hat{v}_j], \hat{v}_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, \hat{v}_{k+1}),
\]

\[
= \sum_{i<j} (-1)^{s_{ij}} c^k ((-1)^{|v_i|}[v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}) + \sum_{i<j} (-1)^{s_{ij}} c^k ((-1)^{|v_i|}(-v_i \circ \hat{v}_j + (-1)^{|v_i|+n} v_j \circ \hat{v}_i), \hat{v}_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}),
\]

\[
= \sum_{i<j} (-1)^{s_{ij}} c^k ((-1)^{|v_i|}[v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k+1}) - \sum_{i \neq j} (-1)^{|v_i|+n+|v_j|+n}(-v_i \circ \hat{v}_j, \ldots, v^i \circ \hat{v}_j, \ldots, \hat{v}_{k+1}).
\] (22)

\(^4\)Hopefully, the hat here will not be confused with the hat in Eq[13]
where the symbols $t_i, s_{ij}$ were defined in Eq.12. Note that the last term of Eq.22 can be written so regardless of whether $i < j$ or not, because $(|v_i| + n)(|v_i| + n + 1) = 0 \mod 2$.

Thus if we set the $g$-module action to be

$$(u \circ c^k)(v_1, \cdots, v_k) = \begin{pmatrix} -1 \end{pmatrix}^{(n+|v|)} \deg c^k \times \sum_i (-1)^{(n+|v|)} c^k(\hat{v}_1, \cdots, (\hat{-1}(n+1))u \circ \hat{v}_i, \cdots, \hat{v}_k), \quad (23)$$

The last term of Eq.22 is written as

$$- \sum_i (-1)^{i,j} n |v_i| + (|v| + n) \deg c^k v_i \circ c^k(v_1, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k),$$

in agreement with Eq.13. Thus the rhs of Eq.22 is the differential of a cochain $c^k$ valued in $C^\infty(M)$ with action defined in Eq.23. Note that in this equation, the sign $(-1)^{(n+1)(|v|+n)}$ can be understood as resulting from commuting the action of $u$ (of degree $|v|+n$) across the suspension (of degree $n+1$).

**Remark** In the end of section 4.1, the importance of $c^*$ being basic w.r.t $K$ will be understood at a much more concrete level.

Next we investigate how does $\hat{u}$ respond to a change of the global section $\hat{x}_x^{-1} = h_x \circ \phi_x^{-1}$, where $h_x \in \Gamma_0$, i.e. a diffeomorphism of $\mathbb{R}$: $\xi \rightarrow h_x(\xi)$ preserving the origin. From Eq.11 we read off (the sign flip as well as the change $h \rightarrow h^{-1}$ has been explained there too)

$$\hat{u} \rightarrow h_x \phi_x^{-1} u + \iota_u (h_x^{-1} dh_x^* - h_x^{-1} \mathcal{G} h_x^*).$$

Letting $h$ be infinitesimal and generated by $\Psi \in \text{vect}(\mathbb{R})$, the change in $\hat{u}$ is

$$\delta_\Psi \hat{u} |_{h_x(\xi)} = (h_x \phi_x^{-1} u - \phi_x^{-1} u |_{h_x(\xi)}) + \iota_u (d\Psi - [\mathcal{G}, \Psi]).$$

In this limit, the first round brace constitutes the definition of a Lie derivative and hence gives $-L_{\phi_x^{-1} u} = -[\Psi, \phi_x^{-1} u]$, leading to

**Proposition 3.3** Under an infinitesimal change of local isomorphism as above, $\hat{u}$ changes according to

$$\delta_\Psi \hat{u} = [\hat{u}, \Psi] + u \circ \Psi,$$

where $u \circ$ is as in Eq.20.

If we pick $c^k$ to be a cocycle, then from prop.3.2 $c^k$ is also a cocycle. Now we can show that the cohomology class of $c^k$ is independent of the choice of the section $\phi$.

**Proposition 3.4** Under an infinitesimal change of the section $\phi_x^{-1} \rightarrow h_x \circ \phi_x^{-1}$, where $h$ is generated by $\Psi \in \text{vect}(\mathbb{R})$ fixing the origin. And pick $c^k$-closed, then the class defined by

$$c^k(u_1, \cdots, u_k) = c^k(\hat{u}_1, \cdots, \hat{u}_k), \quad u_i \in \text{vect}(M)$$

is invariant.

For clarity, we prove the proposition for a smooth manifold and $k = 2$, which is more than sufficient to make clear the idea.
Define $c_{\Psi}^{1}(\hat{u}) = c^{2}(\hat{u}, \Psi)$, and using that $c^{2}$ is closed, we can compute

$$0 = \delta c^{2}(\hat{u}, \hat{v}) = c^{2}([\hat{u}, \hat{v}], \Psi) + c^{2}(\hat{u}, [\hat{u}, \Psi]) - c^{2}([\hat{u}, \hat{v}], \Psi). \quad (25)$$

Use Eq.19 to convert the first term of Eq.25

$$c^{2}([\hat{u}, \hat{v}], \Psi) = c^{2}([u, v], \Psi) - c^{2}(u \circ \hat{v}, \Psi) + c^{2}(v \circ \hat{u}, \Psi).$$

Add and subtract on the rhs the following terms $-c^{2}(\hat{v}, u \circ \Psi) + c^{2}(\hat{u}, v \circ \Psi)$, we get

$$c^{2}([\hat{u}, \hat{v}], \Psi) = \delta c_{\Psi}^{1}(u, v) - c^{2}(\delta_{\Psi} \hat{u}, v) - c^{2}([\delta_{\Psi} \hat{u}, v], \Psi).$$

Thus we can conclude

$$\delta_{\Psi} c^{2}(u, v) = \delta c_{\Psi}^{1}(u, v),$$

thus the cohomology class is unaltered.

Finally, we include the representation of a $Q$-structure introduced in Eq.7. Since $Q^{\uparrow}$ is a globally defined vector field on $\mathcal{E}$, one can define similarly the quantity

$$\hat{T} = Q^{\uparrow} - \hat{Q}, \quad (26)$$

and call it the representation matrix of $Q$. Note that on the rhs $\hat{Q} \in \text{vect}(\mathbb{R})$ is regarded as a vector field of $\mathbb{R} \times V$ trivially through the direct product structure.

Since both $\hat{Q}$ and $Q^{\uparrow}$ satisfy a similar relation derived from Eq.11

$$(\hat{Q})^{2} = -Q^{\uparrow} \circ \hat{Q}, \quad \hat{Q}^{2} = -Q \circ \hat{Q}, \quad (27)$$

we can derive

$$\hat{T}^{2} = (Q^{\uparrow} - \hat{Q})^{2} = -[\hat{Q}, Q^{\uparrow} - \hat{Q}] + (\hat{Q})^{2} - Q^{\uparrow} \circ \hat{Q} = -[\hat{Q}, \hat{T}] - Q \circ \hat{Q} + Q^{\uparrow} \circ \hat{Q}. \quad (28)$$

And we note that $Q^{\uparrow} = Q$ when restricted to $M$, as $Q^{\uparrow}$ is a lift of $Q$ and the fibre dependence of $Q^{\uparrow}$ is linear. One can evaluate Eq.28 at $z_{\alpha} = 0$ (not $z_{\alpha} = 0$!, see the end of sec.2.1 for notations)

$$\hat{T}^{2}\big|_{M} = -([\hat{Q}, \hat{T}] + Q \circ \hat{T})\big|_{M}. \quad (29)$$

Besides, using Eq.24 we conclude that $\hat{T}$ transforms

$$\delta_{\gamma} \hat{T}^{\beta} = \hat{T}^{\beta}_{\gamma} e^{\gamma}_{\alpha} - (-1)^{\beta+\gamma} e^{\beta}_{\gamma} \hat{T}^{\alpha}_{\alpha}, \quad (30)$$

in contrast to Eq.10. Thus

$$\hat{T} \in \Gamma(\text{End}\mathcal{E}).$$
4 Exponential Maps

It is high time that we gave some concrete constructions.

By the folk-theorem for graded and super manifolds (for supermanifold case proved independently by Batchelor [17], by Berezin [18], by Gawedzki [19] and with straightforward generalization to the graded case), one can always split $\mathcal{M}$ non-canonically into a direct sum of graded vector bundles

$$
\mathcal{M} \sim E_1[1] \oplus E_2[2] \oplus \cdots,
$$

(31)

where each $E_i$ is over $M = |\mathcal{M}|$. For each vector bundle we choose a connection $A_\mu$, and also choose a connection $\Gamma^\mu_{\nu\rho}$ for $T\mathcal{M}$. For convenience, we assume $\Gamma$ torsionless $\Gamma^\mu_{\nu\rho} = 0$. We build a local isomorphism Eq.1 by first building a local isomorphism $\mathcal{M} \supset U \rightarrow \mathbb{R}^n$ through certain exponential map and then attach the graded vector bundle structure using parallel-transport.

Consider the body $\mathcal{M} = |\mathcal{M}|$ first. The isomorphism $U \rightarrow \mathbb{R}^n$ can be done in many ways. One way is to use a geodesic exponential map.

$$
\phi^\mu_x = x^\mu + \xi^\mu - \frac{1}{2} \Gamma^\mu_{\alpha\beta} \xi^\alpha \xi^\beta + \left( -\frac{1}{6} \partial_\gamma \Gamma^\mu_{\alpha\beta} + \frac{1}{3} \Gamma^\mu_{\kappa\gamma} \Gamma^\kappa_{\alpha\beta} \right) \xi^\gamma \xi^\alpha \xi^\beta + \mathcal{O}(\xi^4),
$$

(32)

where all the $\Gamma$ are evaluated at $x$.

Alternatively one can choose a metric $g_{\mu\nu}$ for $|\mathcal{M}|$, and locally fixes a vielbein—a set of $n$ orthonormal basis for $\Gamma(T\mathcal{M})$, $e^\mu_a$ with $g_{\mu\nu} e^\mu_a e^\nu_b = \delta_{ab}$. The vielbeins are patched together with structure group $SO(n)$ instead of $GL(n)$

$$
e^\mu_a(y) \frac{\partial x^\nu}{\partial y^\mu} = e^\nu_b(x) U^b_a, \quad U^a_b \in SO(n).
$$

Now one can form a different exponential map by using $e^\mu_a$ as the velocity-field for the flow equation, leading to

$$
\phi^\mu_{so} = x^\mu + \xi^\mu + \frac{1}{2} \left( e^\sigma_c \partial_\sigma (e^\mu_a e^\nu_a) \right) \xi^\alpha \xi^\beta + \mathcal{O}(\xi^4).
$$

(33)

where as before the $e^\mu_a$ are evaluated at $x$.

To investigate the property of $\phi$ under a diffeomorphism, let $\tilde{x}^\mu = x^\mu - v^\mu$ be infinitesimal diffeomorphism, where $v^\mu$ is a vector field vanishing at $x$, we check up to order $\xi^3$ that Eq.32 satisfies

$$
\phi^\mu_x (\tilde{\xi}) = \phi^\mu_x (\xi) + \mathcal{O}(v^2),
$$

(34)

where $\tilde{\xi} = \xi^\mu - \xi^\mu \nabla_\rho v^\rho$ is an exact formula. That the result Eq.34 persists to all orders in $\xi$ is not hard to understand. Once the connection is fixed, the flow that defines the exponential map is uniquely fixed by the initial velocity $\xi^\mu$, thus the flow computed in $x$-coordinate system equals the flow computed in $\tilde{x}$-coordinate with initial velocity $(\partial \tilde{x}^\mu / \partial x^\nu) \xi^\nu$ (of course, one has to transform the connection to the $\tilde{x}$ coordinates too). Thus we have the

**Proposition 4.1** The exponential map Eq.32 gives a global section of the bundle $\mathcal{B}/GL$ over $\mathcal{M}$, similarly Eq.33 gives a global section of $\mathcal{B}/SO$

**Proof** Once the connection is chosen, Eq.34 shows that $\phi_x$ is (up to a $GL$ rotation) independent of the coordinate system, thus it only depends on the point $x$. 

For later use, consider $M$ with a symplectic structure $\Omega$, one can always pick a torsionless connection that preserves $\Omega$. Then one can demand that the exponential map preserves $\Omega$, the required map is

$$\phi_{sp}^\mu = x^\mu + \xi^\mu - \frac{1}{2} \Gamma^\mu_{\alpha\beta} \xi^\alpha \xi^\beta + \left\{ - \frac{1}{6} \partial_{\gamma} \Gamma^\mu_{\alpha\beta} + \frac{1}{3} \Gamma^\mu_{\kappa\gamma} \Gamma^\kappa_{\alpha\beta} - \frac{1}{24} R^\mu_{\gamma\alpha\beta} \right\} \xi^\gamma \xi^\alpha \xi^\beta + O(\xi^4),$$

(35)

$R^\mu_{\gamma\alpha\beta} = (\Omega^{-1})^\nu_{\mu} R^\nu_{\gamma\beta \lambda \alpha}$,

where the quantities $R$, $\Omega$ etc are evaluated at the point $x$. It may be checked that the induced symplectic form in $\mathbb{R}^{2n}$ is $\Omega(x)$. The map $\phi_{sp}$ has the same equivariance property as Eq.34 and by picking a vielbein similarly as above, we easily shows

**Proposition 4.2** $\phi_{sp}$ is a section of the bundle $\mathfrak{B}/SP(n)$.

Next we bring in the rest of the graded structure, as a matter of notation, let the underlined indices $\underline{A}$ denote all coordinates other than that of the body $x^\mu$, and we use the typewriter font to denote the combination

$$x^\underline{A} = x^\underline{A} + \xi^\underline{A}. \quad (36)$$

The exponential map for the body is as Eq.32, while for the graded sector, it is

$$\phi^\underline{A} = x^\underline{A} - \xi^\alpha A^\underline{A}_\alpha x^B + \frac{1}{2} \xi^\alpha \xi^\beta (A^\underline{A}_\alpha A_\beta + \Gamma^\rho_{\alpha\beta} A_\rho - \partial_\alpha A_\beta) \underline{A}^B + \frac{1}{6} \xi^\alpha \xi^\beta \xi^\gamma \left( \partial_\gamma (A^\underline{A}_\alpha A_\beta + A_\rho \Gamma^\rho_{\alpha\beta} - \partial_\alpha A_\beta) - \Gamma^\rho_{\alpha\beta} (A_\rho A_\gamma + 2 A_\mu \Gamma^\mu_{\alpha\beta} - \partial_\mu A_\gamma) - (A^\underline{A}_\alpha A_\beta + A_\rho \Gamma^\rho_{\alpha\beta} - \partial_\alpha A_\beta) \right) \underline{A}^B + O((\xi^4)^4),$$

(37)

Notice that since the connection $A$ acts within each $E_i$, we always have $\deg A^\underline{A}_{\mu B} = 0$.

The Gronthendieck connection defined in Eq.2 is given by the formula

$$\mathcal{G} = [\mathcal{G}^\mu, \mathcal{G}^\underline{A}] = [dx^\nu, dx^B] \left[ \begin{array}{ccc} \partial x^\nu / \partial \phi^\mu & \partial x^\nu / \partial \phi^\alpha & \partial x^\nu / \partial \phi^\beta \\ \partial x^\nu / \partial \phi^\alpha & \partial x^\nu / \partial \phi^\beta & \partial x^\nu / \partial \phi^\gamma \\ \partial x^\nu / \partial \phi^\beta & \partial x^\nu / \partial \phi^\gamma & \partial x^\nu / \partial \phi^\delta \end{array} \right]^{-1},$$

which is calculated to be

$$\mathcal{G} = dx^\gamma \left( \delta^\mu_{\gamma} + \Gamma^\mu_{\beta\gamma} \xi^\beta - \frac{1}{3} R^\nu_{\gamma \alpha\beta} \xi^\alpha \xi^\beta \right) \frac{\partial}{\partial \xi^\mu} + dx^A \frac{\partial}{\partial \xi^A} + dx^\gamma x^C \left( A^\gamma_{\beta} + \frac{1}{2} \xi^\beta F^\gamma_{\beta} - \frac{1}{6} \xi^\alpha \xi^\beta \nabla_\alpha F_{\beta\gamma} \right) \frac{\partial}{\partial \xi^A} + O(\xi^3),$$

(38)

where $\nabla$ is the covariant derivative with connection $\Gamma + A$ and $F_{\alpha\beta}$ is the curvature for $A$. If one uses the map Eq.35 then the first line of $\mathcal{G}$ becomes

$$\mathcal{G}_{sp} = dx^\gamma \left( \delta^\mu_{\gamma} + \Gamma^\mu_{\beta\gamma} \xi^\beta - \frac{1}{8} R^\mu_{\gamma \alpha\beta} \xi^\alpha \xi^\beta - \frac{1}{4} R^\mu_{\gamma \alpha\beta} \xi^\alpha \xi^\beta \right) \frac{\partial}{\partial \xi^\mu} + \cdots ,$$

(39)

in particular, $\mathcal{G}$ has a Hamiltonian lift, since it is valued in $\mathfrak{sp}$.

As an example, we take $M = T[1]M$ with body coordinates $x^\mu$ of deg 0 and deg 1 coordinates $v^\mu$, as well as their flat space counterparts $\xi^\mu$, $\nu^\mu$, and $\nu = v + \nu$ as above. For the homological vector field $Q = v^\mu \partial_\mu$, we can compute $\hat{Q}$

$$\hat{Q} = \frac{1}{2} v^\mu \xi^\alpha \xi^\beta R^\mu_{\rho \alpha \beta} + \nu^\mu \nu^\nu \left( \xi^\alpha R^\nu_{\mu \alpha \kappa} + \frac{1}{2} \xi^\gamma \xi^\delta \nabla_\alpha R^\nu_{\mu \beta \gamma} \xi^\delta \right) \frac{\partial}{\partial \nu^\lambda} + O((\xi^4)^3).$$

We will attach some bundle structures to get a representation of $Q$ in sec.6.4.

---

5 As a note on computation, it is much easier to compute $\hat{Q}$ directly than to compute $\phi^{-1}\phi$ and $\iota_Q\mathcal{G}$ separately.
4.1 Connection Dependence

In obtaining the expression Eq.38, we have made some non-canonical choices, such as the splitting in Eq.31 as well as the choices of $\Gamma^\mu_{\nu\rho}$, $A_\mu$. The resulting change of the Grothendieck connection is predicted by the general formula Eq.6, since all of the choices above merely serve to define the local isomorphism Eq.1. However, we would like to check Eq.6 explicitly, which also serves to strengthen the credibility of Eq.38.

Under a change of the connection for $TM$, we write $\delta \Gamma^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta}$, and we continue to assume $\gamma^\mu_{[\alpha\beta]} = 0$.

Let

$$\Psi = \left( \frac{1}{2} \gamma^\mu_{\alpha\beta} \xi^\alpha \xi^\beta + \frac{1}{6} (\nabla_\alpha \gamma^\mu_{\beta\gamma}) \xi^\alpha \xi^\beta \xi^\gamma + \frac{1}{12} \gamma^\mu_{\alpha\beta} F^\rho_{\mu\alpha \beta} \xi^\alpha \xi^\beta \xi^\gamma \right) \frac{\partial}{\partial \xi^\mu},$$

and we can check that

$$[-d + \Theta, \Psi] = \delta \Gamma \Theta,$$

which is the infinitesimal version of Eq.6. The sign discrepancy has been explained earlier.

Likewise we can vary $A$, $\delta A_\mu = a_\mu$, and let

$$\Psi = a_\alpha \xi^\alpha + \frac{1}{2} (\nabla_\alpha a_\beta) \xi^\alpha \xi^\beta + \frac{1}{6} (\nabla_\alpha \nabla_\beta a_\gamma) \xi^\alpha \xi^\beta \xi^\gamma,$$

it may be checked that

$$[-d + \Theta, \Psi] = \delta A \Theta.$$

To check Eq.24, we use the geodesic exponential map Eq.32 as an example, and compute the push-forward of a vector field

$$(\phi^{-1}_x)_* u = \left( u^\mu + \xi^\alpha \nabla_\alpha u^\mu + \xi^\alpha \xi^\beta \left( \frac{1}{6} u^\rho R^\mu_{\rho \alpha \beta} + \frac{1}{2} \nabla_\alpha \nabla_\beta u^\mu \right) \right) \frac{\partial}{\partial \xi^\mu} + O(\xi^3).$$

Thus

$$\hat{u}^\mu = \xi^\alpha \partial_\alpha u^\mu + \frac{1}{2} \xi^\alpha \xi^\beta \left( u^\rho R^\mu_{\rho \alpha \beta} + \nabla_\alpha \nabla_\beta u^\mu \right),$$

(40)

and if one changes the connection $\delta \Gamma^\mu_{\nu\rho} = \gamma^\mu_{\nu\rho}$

$$\delta \Gamma \hat{u}^\mu = \frac{1}{2} \xi^\alpha \xi^\beta \left( 2 \gamma^\mu_{\nu\alpha\gamma} \nabla_\beta u^\gamma - \gamma^\mu_{\nu\alpha} \nabla_\gamma u^\nu + \gamma^\mu_{\nu\beta} \nabla_\alpha u^\beta \right) = u \circ \Psi^\mu + [\hat{u}, \Psi]^\mu.$$

This is the verification of the assertion Eq.24 in the case when the change of local isomorphism is caused by a change of the connection.

A change of the splitting Eq.31 corresponds to a shift

$$x^A = x^A + \xi^A,$$

the verification of Eq.34 is almost trivial given that the exponential map depends linearly on the combination $x^A = x^A + \xi^A$.

Remark We notice that in the expression Eq.19 for $\hat{u}$, only the first term is non-covariant, and it takes value in the lie algebra $gl$ (or $so$ and $sp$ if one uses the exponential map Eq.33 and Eq.35). Thus if the CE cochain into which we plug $\hat{u}$ is basic w.r.t $GL$ (resp. $SO$, $SP$), the result is manifestly covariant. This is a more elementary way of understanding the necessity of basic-ness promised on page 11.

So far, we have checked explicitly all claims made in sec.2.1 and 3.2.
5 Graph Complex

A cochain in the CE complex of Lie algebra of formal vector fields or Hamiltonian functions can be presented as a graph, and the cohomology of CE complex is subsequently computed as the cohomology of graph complex. Unfortunately, the CE cohomology of vector fields are quite banal, see ref. [20], while the same cohomology for Hamiltonian functions is much more interesting, so is its associated graph complex.

As such we will be mostly studying the Lie algebra of Hamiltonian functions, which has a bracket of degree \( n = 0, -2, \cdots \), and the associated graph complex is what we call the plain graph complex. The key is the correspondence and its extension

\[
\begin{align*}
\text{Chevalley-Eilenberg complex} & \xrightarrow{\beta} \text{Graph complex}, \\
\text{CE complex valued in cyclic bar complex} & \xrightarrow{\beta} \text{Extended graph complex}.
\end{align*}
\]

Thus one can construct CE cocycles from graph cocycles and conversely construct graph cycles from CE cycles. For more details, see the lecture notes [21].

A graph is a finite 1-dimensional CW complex, in simpler words, it is a collection of vertices (0-cells) connected by edges (1-cells). The vertices are labelled \( 1, 2, \ldots \) and each edge is oriented. The graph complex is the formal linear combination of graphs mod the relation: flipping the orientation of an edge or exchanging the labelling of two vertices flips the sign of the graph.

For weight systems of knots, we will also need the extended graph complex, which includes an extra oriented circle in addition to the above ingredients. Vertices can be placed on the circle (the peripheral ones) or away from the circle (internal ones); these two types of vertices are labelled separately. The edges can now also run between the two types of vertices. One can still flip the orientation of an edge at the cost of a minus sign. The peripheral vertices can also be cyclically permuted with the appropriate sign.

**Definition** A particular type of the extended graph complex is the so called chord diagrams, consisting of graphs that have tri-valent internal vertices and uni-valent peripheral vertices.

These diagrams naturally appear in the finite type Vassiliev knot invariants [1], and their weights is what we aim to construct eventually.

![Graphs](image)

Figure 1: Examples of graphs, the \(|\text{Aut } \Gamma|\) factor for them are: 24, 4, 2 and 2, 4, 3, 2, 1, 1.
The graph differential acts through 1. shrinking an edge running from internal vertices \(i\) to \(j\) (assuming \(i < j\)), naming the new vertex \(i\) and decreasing the labels of the vertices after \(j\) by one; 2. collapsing two adjacent peripheral vertices together and decreasing the labels of the ensuing peripheral vertices by one; 3. shrinking an edge connecting one internal vertex and one peripheral vertex, and decreasing the labels of the ensuing internal vertices by one; see fig 2. All three operations carry a sign factor given in fig 3.

Thus for the graphs of fig 1

\[
\partial \Gamma_1 = 6\Gamma_3, \quad \partial \Gamma_2 = 2\Gamma_3; \\
\partial \Gamma_4 = -2\Gamma_8, \quad \partial \Gamma_5 = 4\Gamma_8, \quad \partial \Gamma_6 = -3\Gamma_8 + 3\Gamma_9, \quad \partial \Gamma_7 = 2\Gamma_9.
\]  

Figure 2: Differential of a graph

Figure 3: Sign factor for the differential, here \(I\) is the number of internal vertices. In the third picture, if \(j\) is the \(l\)th-the last-vertex on the circle, then we rename the new vertex 1 with sign factor \((-1)^{I+l+1}\). The overall minus sign from that of ref. [8] is due to the difference in labelling schemes.

5.1 Recipe for the CE-Graph Correspondence

We shall only give the recipe, the proof may be found in refs. [22, 23, 8], in particular, see ref. [21], where the graph complex was presented as a polynomial and the proof was written in a few strokes.

Let \(\mathbb{R}\) be a non-negatively graded vector space with constant symplectic form \(\Omega\) of degree \(n\), and \(V\) is a fixed graded vector space. In the following \(f\) are formal polynomials on \(\mathbb{R}\) vanishing at the origin, while \(g\) are \(\text{Mat}(V)\)-valued formal polynomials on \(\mathbb{R}\). But for now, one can consider the \(g\)'s entry by entry and treat them on the same footing as \(f\). Given an element in the extended CE chain \(c_{p,q}\), written as

\[
(f_1, f_2, \cdots, f_p) \otimes [g_1|g_2|\cdots|g_q],
\]

form the formal product

\[
F = (t_1 f_1(\xi_1)) \cdots (t_p f_p(\xi_p))(s_1 g_1(\eta_1)) \cdots (s_q g_q(\eta_q)),
\]

where \(t_i, i = 1 \sim p\) is of degree \(n + 1\) and \(s_i, i = 1 \sim q\) is of degree 1 (which explains the suspension operation, see first paragraph of sec 3.1). Furthermore, both \(\xi, \eta\) are coordinates of \(\mathbb{R}\), but given different names to distinguish the vertices.

\[\text{The entire discussion about CE-graph correspondence is valid only for } n\text{-even, but we have retained } n \text{ here for the coherence of notation.}\]
Let $[\Gamma_{p,q}]$ be a representative of an equivalence class of graphs with $p$ internal and $q$ peripheral vertices. Let $E_{ij}$ be an edge from internal vertices $i$ to $j$, $E_{ij}$ be an edge from internal vertex $i$ to peripheral vertex $j$ and finally $E_{ij}$ an edge from peripheral vertex $i$ to $j$. With each type of vertices we associate a differential

$$E_{ij} \rightarrow \beta_{ij} = (\Omega^{-1})^{AB} \frac{\partial}{\partial \xi_i^A} \frac{\partial}{\partial \xi_j^B}, \quad E_{ij} \rightarrow \beta_{ij} = (\Omega^{-1})^{AB} \frac{\partial}{\partial \xi_i^A} \frac{\partial}{\partial \xi_j^B}, \quad E_{ij} \rightarrow \beta_{ij} = (\Omega^{-1})^{AB} \frac{\partial}{\partial \eta_i^A} \frac{\partial}{\partial \eta_j^B}.$$  

The fact that $\beta_{ij}$ is anti-symmetric under exchange of $i, j$ (this requires $n$ be even) reflects the equivalence relation concerning the flipping of an edge in the previous section.

Form the product of operators

$$\tilde{\beta}_T = \frac{1}{|\text{Aut}\Gamma|} \partial s_1 \cdots \partial s_t \partial t_1 \cdots \partial t_s \prod_{E_{ij}} \beta_{ij} \prod_{E_{ij}} \beta_{ij} \prod_{E_{ij}} \beta_{ij};$$

where $|\text{Aut}\Gamma|$ is the order of automorphism group of the vertices of $\Gamma$.

The graph chain associated with the CE chain Eq.14 is the formal sum over equivalence classes of graphs with $p$ internal and $q$ peripheral vertices

$$(f_1, f_2, \cdots, f_p) \otimes [g_1 | g_2 | \cdots | g_q] \Rightarrow \sum_{[\Gamma_{p,q}]} \left( \tilde{\beta}_{\Gamma_{p,q}} F \right)[\Gamma_{p,q}] \bigg|_{t=s=\xi=\eta=0} + \text{perm}, \quad (45)$$

where perm denotes the permutations among $f$’s and cyclic permutations among $g$’s with the sign factor as in Eq.11 and Eq.16.

We shall call all of the operations above together as $\beta$, forming one way of the correspondence Eq.42.

Trivially, by restricting $q = 0$, all the above results works for the non-extended graph complex.

Remark We also observe that the definition of $\text{deg}c\cdot\cdot$ in Eq.14 is nothing but the $(\text{mod}\ 2)$ degree of the operator $\beta$.

Here is the first main theorem of the paper concerning weight-systems for knot invariants. Let $(\mathcal{M}, \Omega)$ be an even degree symplectic $NQ$-manifold, with an $\Omega$ preserving homological vector field $Q$, and $\mathcal{E}$ a graded vector bundle over $\mathcal{M}$ with a lift $Q^\dagger$. Pick an exponential map which is a global section of $\mathfrak{B}/SP(\mathbb{R})$ (e.g. Eq.5), construct the quantities $\hat{\Omega}$ and $\hat{T}$ as in Eqs.18, 26. Since $Q$ preserves $\Omega$ and the local model $\mathbb{R}$ inherits a symplectic structure from $\mathcal{M}$, thus $\hat{Q}$ will have a Hamiltonian lift $\Theta$. We form the extended chain

$$c = \sum_{p+q=m} c_{p,q}, \quad c_{p,q} = \frac{1}{pq} \left( \Theta_3, \cdots, \Theta_3 \right) \otimes \text{Tr} \left[ \hat{T}_1 | \cdots | \hat{T}_1 \right],$$

where the subscripts 3 and 1 denote the cubic and linear term in the expansion of $\Theta$ and $\hat{T}$ in terms of $\xi$. We have

Theorem 5.1 The graph chain $\beta c$ by applying the above recipe to $c$ is closed with values in the ring $H_Q(\mathcal{M})$.

And the graph cohomology class is independent of the choice of the connection in defining the exponential map, as well as the trivialization of $\mathcal{E}$. Thus we have a well-defined weight-system for chord diagrams valued in $H_Q(\mathcal{M})$.\footnote{Usually, when one computes Feynman diagrams one also includes a factor of $1/p!$ for $p$ edges running between the same pair of vertices. In fact, this factor is implicitly included in the current formalism when one applies the operators $\beta_{ij}$.}
By reading Eq.27 at order 2, we see that $Q \circ \Theta_3 = 0$, and by reading Eq.29 at order 1, we have $Q \circ \hat{T}_1 = 0$. Thus the chain $c$ is trivially $Q$-closed. That the graph chain $\beta c$ is closed is because $\beta$ is a map of complexes: $\partial \beta c = \beta \partial c$, and $\partial c$ is $Q$-exact by using again Eqs.27, 29. Thus $\partial \beta c = \beta Q \circ \hat{c}$ for certain $\hat{c}$, and finally $\partial \beta c = Q \circ \beta \hat{c}$. The proof of the connection independence is similar to the proof of prop.3.4; the trivialization independence is proved by using Eq.30 and the property of trace.

### 5.2 The Dual Story

As the graph chains are formal linear combinations of graphs, the dual graph cochains are written as a formal linear combination

$$b = \sum_{[\Gamma]} b_{\Gamma} [\Gamma]^*, \quad (46)$$

where the ‘cographs’ are defined through the obvious paring $\langle [\Gamma]^*, [\Gamma'] \rangle = \pm 1$ if $[\Gamma] = \pm [\Gamma']$ and zero otherwise.

From a graph cochain $b$, we can construct a CE cochain, denoted $\beta^! b$ using the recipe

$$\langle \beta^! b \rangle((f_1, f_2, \cdots, f_p) \otimes [g_1|g_2|\cdots|g_q]) = \sum_{[\Gamma]} b_{\Gamma} (\beta_{\Gamma} F) \bigg|_{t=s=\xi=\eta=0}, \quad (47)$$

where the lhs denotes the evaluation of $\beta^! b$ on $(f_1, f_2, \cdots, f_p) \otimes [g_1|g_2|\cdots|g_q]$.

We are now in possession of all the technical tools to give the second main theorem of the paper. Taking $q = 0$,

**Theorem 5.2** Let $M$ be an even degree symplectic $N$-manifold, then from any graph cochain made of tri-valent graphs, one can construct a covariant cocycle of Lie algebra of Hamiltonian function on $M$, with values in $C^\infty(M)$. Besides, the cohomology class of the resulting CE cocycle is independent of the choice of connections in defining the exponential map.

**Proof** The procedure is to use recipe 47 to convert the graph cocycle first to a CE cocycle of Hamiltonian functions on a flat space $\mathbb{R}$ of the same dimension as $M$. The tri-valent condition, guarantees amongst other things that the CE cochain is basic w.r.t $SP(\mathbb{R})$. Then one uses the recipe 3.2 to convert the type 1 cocycle to a cocycle of type $1'$. The connection independence is demonstrated in prop.3.4.

### 6 Examples

**Example Type 1 CE Cocycles**

It can be shown that if in a graph cochain Eq.46 $b_{\Gamma} \neq 0$ only for 3-valent graphs, then the graph cochain

![Figure 4: The simplest cocycle](image)

is a cocycle. By applying the recipe Eq.17 to such a cocycle, one obtains a type 1 CE cocycle.
Take the graph cochain defined by fig[4] that is, $b_1 = 0$ except for $b_2 = 1$. The prescription gives the following cochain $c^2 = \beta^I[\otimes]^*$ (setting as in sec.5.1)

$$c^2(f_1, f_2) = \frac{1}{6} (-1)^{|f_1|} (f_1 \partial_{ABC}) (\Omega^{-1})^{AD} (\Omega^{-1})^{BE} (\Omega^{-1})^{CF} (\partial_{FED} f_2),$$  \hspace{1cm} (48)

where the entire rhs is evaluated at the origin. The sign factor $(-1)^{|f_1|}$ follows from the recipe and is crucial for $c^2$ to satisfy Eq.[11] A skeptical reader may wish to check that $c^2$ is closed for himself.

**Example Type 1′ CE Cocycles**

For clarity, we focus on the smooth case $\mathcal{M} = M^{2n}$, with symplectic form $\Omega$. We use the exponential map $\text{Eq.}[35]$ for a symplectic manifold, thus $\mathfrak{g}$ is valued in $\mathfrak{sp}$. We define $u_f$ to be the vector field generated by $f \in C^\infty(M)$. Since $\mathbb{R}^{2n}$ has a symplectic form $\Omega(x)$, which is pulled back from $M^{2n}$ by the exponential map. Then let $\hat{f}$ the Hamiltonian lift of $u_f$ w.r.t $\Omega(x)$. One has the freedom to set the constant term of $\hat{f}$ to be zero; this done, $\hat{f}$ vanishes at the origin as $\xi^2$.

Using the prescription of Eq.[24] we construct a covariant cochain of type $1'$

$$c^2(f, h) = c^2(\hat{f}, \hat{h}) = \frac{1}{6} (f^{\mu\nu\rho} + f^{\lambda} R^{\mu\nu\rho}_{\lambda}) (h_{\mu\nu\rho} + h^\kappa R_{\kappa\mu\nu\rho}) \in C^\infty(M),$$  \hspace{1cm} (49)

where $f_{i_1 \cdots i_n} = \nabla_i \cdots \nabla_{i_n} - \partial_{i_n} f|_x$ and all indices are raised (resp. lowered) with $\Omega^{-1}(x)$ (resp. $\Omega(x)$).

To check explicitly $\text{Eq.}[49]$ is closed requires some effort, the key step is to show

$$\frac{1}{6} \xi^\alpha \xi^\beta \xi^\gamma (f^{\rho} \nabla_{\rho} h_{\alpha\beta\gamma}) = \frac{1}{6} \xi^\alpha \xi^\beta \xi^\gamma \{ - f^{\rho} \partial_{\rho} h_{\gamma} - 2 f^{\rho} R_{\rho\sigma\gamma} h_{\alpha\beta} + \nabla_\sigma f^{\rho} h_{\rho\gamma} - f^{\rho} h^{\rho}_{\alpha\beta\gamma} \\
- f^{\rho} h_{\alpha\rho\gamma} - f^{\rho} R_{\rho\beta\gamma} h_{\alpha\sigma} - f^{\rho} (\nabla_\alpha R_{\rho\beta\gamma}) h_{\sigma} - f^{\rho} (R_{\rho\beta\gamma}) h_{\sigma} - f^{\rho} R_{\rho\beta\gamma} h_{\alpha\sigma} \}$$

by commuting $f^{\rho} \nabla_{\rho}$ over the derivatives $\nabla_\alpha \nabla_\beta \partial_\gamma$; this will then lead to

$$f^{\rho} \partial_{\rho} \hat{h} - h^{\rho} \partial_{\rho} \hat{f} + \{ \hat{f}, \hat{h} \} = 0,$$

where the first curly brace $\{ - , - \}$ is the Poisson bracket on $C^\infty(\mathbb{R}^{2n})$, the second on $C^\infty(M^{2n})$. This equation is the Hamiltonian version of $\text{Eq.}[19]$

**Example Type 2 cycles of extended graph complex from extended CE cycles**

Let $\mathfrak{g}$ be a Lie algebra $\mathfrak{su}$, $\mathfrak{sp}$ or $\mathfrak{so}$. Let $\mathcal{M} = \mathfrak{g}[1]$ with coordinates $\ell^\alpha$. Let the matrices $T_\alpha$ be a representation of $\mathfrak{g}$, and the killing metric $\eta_{\alpha\beta} = \text{Tr}[T_\alpha T_\beta]$ plays the role of the symplectic form on $\mathfrak{g}[1]$. Finally let \( \Theta = 1/2 \eta_{\alpha\beta} f^\alpha \ell^\beta \ell^\beta \), then one can check that

$$c = \frac{1}{4} \text{Tr}[T_\alpha T_\beta T_\gamma T_\delta] \ ( ) \otimes [\ell^\alpha | \ell^\beta | \ell^\gamma | \ell^\delta] + \frac{1}{3} \text{Tr}[T_\alpha T_\beta T_\gamma] \ ( \Theta ) \otimes [\ell^\alpha | \ell^\beta | \ell^\gamma] + \frac{1}{2 \cdot 2!} \text{Tr}[T_\alpha T_\beta] \ ( \Theta , \Theta ) \otimes [\ell^\alpha | \ell^\beta]$$

is a cycle in the extended CE complex.

Apply the recipe Eq.[35] to $c$

$$\beta c = \frac{d_c C_2(G)}{2} \frac{1}{4} [\Gamma_5] + \frac{1}{3} [\Gamma_6] - \frac{1}{2} [\Gamma_7] - \frac{d_c C_2(r)}{4} ( [\Gamma_5] + 2[\Gamma_4] ) ,$$

where $d_c(d_c)$ is the dimension of the (adjoint) representation. One may use Eq.[13] to check that each combination in the two braces is a graph cycle. This is the *Lie algebra weight-system.*
6.1 Weights for Graphs Valued in $Q$-cohomology

This is the central (and non-trivial) application of our result and therefore deserves a separate section.

Let $M^{4n}$ be a holomorphic symplectic manifold with complex coordinates $x^i, \bar{x}^i$. The holomorphic symplectic form is $\Omega_{ij}dx^i dx^j$, and one can always pick a torsionless connection simultaneously preserving the complex structure and the symplectic structure, thus in the complex basis, only the combination $\Gamma_{jk}^i, \Gamma_{jk}^\bar{i}$ is non-zero.

Consider the $NQ$-manifold

$$ M = T_{(0,1)}[1]M, $$

(50)

where $T_{(0,1)}$ denotes the anti-holomorphic tangent bundle. Locally, the coordinates are

$$ \text{deg 0 : } x^i, \bar{x}^i; \quad \text{deg 1 : } v^\bar{i}, $$

and the 2-form $\Omega_{ij}$ is assigned degree 2. The $Q$-vector field

$$ Q = v^\bar{i} \frac{\partial}{\partial x^i} $$

(51)

corresponds to the Dolbeault differential.

The local model $\mathcal{R}$ for $M$ is $\mathbb{C}^{2n} \times \mathbb{C}^{2n}[1]$, and we denote the flat coordinates as $\xi^i, \bar{\xi}^i$ and $\nu^\bar{i}$. One may proceed to apply the exponential map, but since eventually we shall only need a CE cochain of Hamiltonian function in the variable $\xi^i$, with symplectic form $\Omega_{ij}$ (see the next remark), we may set $\xi^i = 0$ and $\nu^\bar{i} = 0$ (not $v^\bar{i}$).

The exponential map is the holomorphic half of Eq.[35]

$$ \phi^i_{\text{hol}} = x^i + \xi^i - \frac{1}{2} \Gamma^i_{mn} \xi^m \xi^n + \left( \frac{1}{6} \partial_j \Gamma^i_{mn} + \frac{1}{3} \Gamma^i_{j} \Gamma^j_{mn} - \frac{1}{24} R^i_{jmn} \right) \xi^j \xi^m \xi^n + \mathcal{O}(\xi^4), $$

$$ R^i_{jmn} = (\Omega^{-1})^{it} R_{tjk} \Omega^{km}, $$

and $\phi^\bar{i}_{\text{hol}} = x^\bar{i}, \phi^i_{\text{hol}} = v^\bar{i}$.

One may compute the quantity (where the Poisson bracket is taken with $\Omega^{-1}(x)$),

$$ \tilde{Q} = \{ v^\bar{i} \Theta_i, - \}, \quad \Theta_i = \frac{1}{6} \xi^i \xi^j \xi^k R_{ijk} + \mathcal{O}(\xi^4). $$

And $\Theta$ satisfies the Maurer-Cartan equation

$$ \tilde{\partial}_i [\Theta_j] = - \{ \Theta_i, \Theta_j \}, \quad \text{i.p.} \quad \tilde{\partial}_i [\Theta_j] \big|_{\xi^3} = 0. $$

(52)

In fact, if $M$ is hyperKähler, $\Theta$ can be computed to all orders in $\xi$ (see ref.[24])

$$ \Theta_i(\xi) = \sum_{n=3}^{\infty} \frac{1}{n!} \nabla_{\xi^1} \cdots \nabla_{\xi^n} R_{\xi^1 \xi^2 \xi^3} \xi^1 \cdots \xi^n. $$

(53)

**Remark** It may seem that we are not following our own recipe in theorem 5.1 as we only have a holomorphic symplectic form $\Omega_{ij}$. To go by the rules, one has to take $M = T^{(0,1)}[2]T_{(0,1)}[1]M$ to give $x^\bar{i}$ and $v^\bar{i}$ their conjugate momenta. But it is fairly clear that, this will lead to a Hamiltonian lift of $\tilde{Q}$ that includes $\Theta$ above, plus terms at most linear in the momenta conjugate to $\xi^i, \nu^\bar{i}$. These terms will vanish once we plug the Hamiltonian function into a tri-valent graph. Thus setting $\xi^i, \nu^\bar{i}$ to zero from the beginning is a slight short-cut we take.
where $M$ as accidentally discovering that shifting $-\nu$ in fact, this was how we first came to realize the necessity of the Grothendieck connection in ref. [25], after

entry if one twists the $Q$-vector field $\Theta + \xi$ slightly. Suppose there is a group $G$ acting on $M$ through a holomorphic moment map $\mu$, $\alpha = 1 \sim \text{rk}_G$, $\partial_i \mu = 0$. Let now $M$ be

$$T_{(0,1)}[1] \times \mathfrak{g}[1],$$

and we let the coordinates of $\mathfrak{g}[1]$ be $\ell^\alpha$ of degree 1. The twisted $Q$-vector field is now

$$Q = v^i \frac{\partial}{\partial x^i} + \ell^\alpha (\partial_j \mu_\alpha)(\Omega^{-1})^{ji} \frac{\partial}{\partial x^j} - \frac{1}{2} f_{\alpha\beta}^\gamma \ell^\alpha \ell^\beta \frac{\partial}{\partial \ell^\gamma}. \quad (54)$$

One can again compute the quantity

$$\tilde{Q} = \{ v^i \Theta_i, -\} + \{ \ell^\alpha M_\alpha, -\},$$

where $M_\alpha = \frac{1}{2} (\partial_i \partial_j \mu_\alpha) \xi^i \xi^j + \frac{1}{6} (\nabla_i \nabla_j \partial_k \mu_\alpha - (\partial_i \partial_j \mu_\alpha) R^l_{ijk} \xi^i \xi^j \xi^k$, \quad (55)

as usual all the terms involving $\mu_\alpha$, $R$ etc are evaluated at $x$. In this expression, the effect of the Grothendieck connection is to correct the coefficient of the curvature term from $-1/24$ to $-1/6$, which is crucial for the

**Proposition 6.1** Under a change of the connection $\Gamma^i_{ij} \rightarrow \Gamma^i_{ij} + \gamma^i_{ij}$ (but retaining the torsionless and $J, \Omega$-preserving property), we have

$$\delta_\Gamma (v^i \Theta_i + \ell^\alpha M_\alpha) = Q \circ \Psi + \{ \ell^\alpha M_\alpha, \Psi \}, \quad \Psi = \frac{1}{6} \gamma^l_{ijk} \Omega_j \xi^i \xi^j \xi^k,$$

where $M_\alpha$ is the quadratic term in the expansion of $M_\alpha$ in Eq. (55).

This result is a special case of Eq. (44) truncated at order $\xi^3$. One can of course do a direct computation, in fact, this was how we first came to realize the necessity of the Grothendieck connection in ref. [25], after accidentally discovering that shifting $-1/24$ to $-1/6$ made everything work.

If $c^k$ is a cocycle constructed from tri-valent graphs, then by evaluating the corresponding $c^k$ on the functions $\Theta + M$, one obtains an element

$$c^k (\Theta + M, \cdots , \Theta + M) \in \bigoplus_{p+q=k} \Omega_M^{(0,p)} \otimes \wedge^q \mathfrak{g}^*,$$

which is annihilated by the differential Eq. (51). These are the equivariant Rozansky-Witten classes. It is equivariant in the sense that, in the real setting, the complex $\Omega^p_M \otimes \wedge^q \mathfrak{g}^*$ is a model for the de Rham complex of $M \times_G EG$ with $EG$ being the universal $G$-bundle. By applying the result of ref. [26], one can turn this complex into the more familiar Cartan model of equivariant cohomology.

To see what is special about the holomorphic setting chosen above, let us try to take $M^{2n}$ to be a symplectic manifold with symplectic form $\Omega_{\mu}$.. Pick as before a torsionless connection $\Gamma_{\nu\rho}$ preserving $\Omega$, and $Q = v^\nu \partial_\mu$ now corresponds to the de Rham differential. Using the exponential map Eqs. (35) and (37) to compute $\phi_*^{-1} Q$ gives

$$\phi_*^{-1} Q = v^\nu \frac{\partial}{\partial x^\mu} + \frac{1}{8} v^\rho (R^{\mu}_{\rho \alpha \beta} - 2 R^{\mu}_{\alpha \rho \beta}) \xi^\alpha \xi^\beta \frac{\partial}{\partial \xi^\mu}, \quad R^{\mu}_{\alpha \beta \gamma} = R^{\mu}_{\alpha \beta \gamma} \Omega_{\alpha \beta}.$$

The second term fails to be Hamiltonian. This is simply because the vector field $v^\nu \partial_\mu$ does not preserve $\Omega_{\alpha \beta}$ as $v^i \partial_\xi$ does $\Omega_{ij}$.
It was pointed out in ref. [6] that from any holomorphic vector bundle $E$ over $M$ with connection $A$ and curvature $K$ (of type (1,1), naturally), one can construct a representation for $Q$ Eq.51 (we will not consider the twisted case). First denote the deg 0 coordinate of the fibre of $E$ as $z_\alpha$ and its flat counterpart as $\zeta_\alpha$.

The lift of $Q$ to $E$ is

$$Q^\uparrow = v^i \frac{\partial}{\partial x^i} + v^j (A_i)^\alpha_\beta z_\alpha \frac{\partial}{\partial z_\beta},$$

corresponding to the (0,1)-covariant derivative $dx^i \nabla^i$.

In applying the exponential map Eq.37, we will expand around $z_\alpha = 0$ (not $\zeta_\alpha$). We can compute

$$\hat{T} = \hat{Q}^\uparrow - \hat{Q} = -(v^i K^\alpha_{\bar{i}\beta}) \zeta_\alpha \frac{\partial}{\partial \zeta_\beta}, \quad K^\alpha_{\bar{i}\beta} = \xi^j (K_{\bar{i}j})^\alpha_\beta + \frac{1}{2} \xi^j \xi^k (\nabla_j K_{\bar{i}k})^\alpha_\beta + O(\xi^3).$$

Again, if the manifold is hyperKähler, $K$ can be computed to all orders

$$K^\alpha_{\bar{i}\beta} = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} (\nabla_{\bar{i}1} \cdots \nabla_{\bar{i}p} K_{\bar{i}p+1})^\alpha_\beta \xi^{\bar{\ell}_1} \cdots \xi^{\bar{\ell}_{p+1}},$$

$K$ satisfies a neat relation

$$\nabla_{[\bar{i}j]} K_{\bar{i}j] \Big|_{\xi^1} = 0, \quad \text{(56)}$$

To complete the discussion, we use $\Theta$ and $K$ to form the cycle

$$c = \sum_{p+q=m} c_{p,q} \epsilon^{p,q} = \frac{(-1)^q}{p!q!} (v_1^i \Theta^1_{\bar{i}} \cdots v_p^i \Theta^p_{\bar{i}}) \otimes \text{Tr}[v^j K_{\bar{i}j}] \cdots [v^j K_{\bar{i}j}] \in \Omega^{(0,m)}_{(M)},$$

and by applying the prescription of sec 5.1 we will obtain a graph chain with coefficients in $\Omega^{(0,m)}(M)$, and the boundary of this graph chain is $\bar{\partial}$-exact. This is the Rozansky-Witten weight-system.

If one plugs this extended graph chain into a cochain made of chord diagrams, one obtains a cohomology class in $H^{\partial\partial}_{(M)}$. See refs. [6] and [8] for some more detailed calculation. The cohomology class is independent of the choice of the connections $\Gamma$ or $A$, these are the so called Rozansky-Witten classes.

Finally, to obtain even more examples, one may try to use the Courant algebroids and construct certain representations on them, though non-trivial examples are at the time being unknown. One may also take a symplectic manifold with some commuting circle action, and construct weight-systems similar to the twisted RW case (though dropping the Dolbeault part), and these are expected to carry information of the foliation of the manifold by these circle actions.

### A The Chevalley-Eilenberg Differential

We give the complete expression of the differential of an extended CE cochain, valid for general $n$. The setting is as follows: $\mathbb{R}$ is a non-negatively graded vector space, $f_i$ are the formal vector fields thereon. Fix a graded vector space $V$, $g_i$ are formal polynomials on $\mathbb{R}$ valued in Mat($V$). Just as in sec 5.1 we consider the $g$’s entry-wise, and treat them as mere polynomials on $\mathbb{R}$.
The full differential is presented as a sum $\delta = \delta_f + \delta_V + \delta_H$, each of which is dual to the operators in fig 2. The sign factors below are obtained from combining Eqs.12, 13, 16 and 17. Let $c = \sum_{p,q} c^{p,q}$ be a cochain,

$$(\delta_f c)(f_1, \ldots, f_p; g_1, \ldots, g_q) = \sum_{1 \leq i < j \leq p} (-1)^{i+j} c((-1)^{|f_i|}|f_i, f_j|, f_1 \cdots \hat{i} \cdots \hat{j} \cdots f_p; g_1, \ldots, g_q),$$

$$(\delta_V c)(f_1, \ldots, f_p; g_1, \ldots, g_q) = - \sum_{1 \leq i \leq p, 1 \leq j \leq q} (-1)^{i+j} (f_1, \ldots, \hat{i} \cdots f_p; g_1, \ldots, g_j-1, f_i \circ g_j, g_{j+1} \cdots g_q),$$

$$(\delta_H c^{p,q})(f_1, \ldots, f_p; g_1, \ldots, g_q) = - \sum_{1 \leq j \leq q} (-1)^{u_j} (f_1, \ldots, f_p; g_j g_{j+1}, g_{j+2} \cdots g_q, g_1, \ldots, g_{j-1}),$$

(57)

where $|\bar{f}| = |f| + n + 1$ and $|\bar{g}| = |g| + 1$. The formidable signs disappear for most of our applications, since $f$, $g$ will mostly be of odd degree and $n$ even in the text.

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