Distant 2-Colored Components on Embeddings: Part III

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Abstract

This is the third (and most technical!) in a sequence of seven papers in which we prove the following: Suppose we have a graph $G$ embedded on a surface of genus $g$. Then $G$ can be $L$-colored, where $L$ is a list-assignment for $G$ in which every vertex has a 5-list except for a collection of pairwise far-apart components, each precolored with an ordinary 2-coloring, as long as the face-width of $G$ is at least $2^{\Omega(g)}$ and the precolored components are of distance at least $2^{\Omega(g)}$ apart. This provides an affirmative answer to a generalized version of a conjecture of Thomassen and also generalizes a result from 2017 of Dvořák, Lidický, Mohar, and Postle about distant precolored vertices. As an application of this result, we prove (in a follow-up to this sequence of papers) a generalization of a result of Dvořák, Lidický, and Mohar which states that if a graph drawn in the plane so that all crossings in $G$ are pairwise of distance at least 15 apart, then $G$ is 5-choosable. In our generalization, we prove an analogous result in which planar drawings with pairwise far-apart crossings have been replaced by drawings on arbitrary surfaces with pairwise far-apart matchings with many crossings, where the graph obtained by deleting these matching edges is a high-face-width embedding.

1 Background

Given a graph $G$, a list-assignment for $G$ is a family of sets $\{L(v) : v \in V(G)\}$ indexed by the vertices of $G$, such that $L(v)$ is a finite subset of $\mathbb{N}$ for each $v \in V(G)$. The elements of $L(v)$ are called colors. A function $\phi : V(G) \to \bigcup_{v \in V(G)} L(v)$ is called an $L$-coloring of $G$ if $\phi(v) \in L(v)$ for each $v \in V(G)$, and, for each pair of vertices $x, y \in V(G)$ such that $xy \in E(G)$, we have $\phi(x) \neq \phi(y)$. Given a set $S \subseteq V(G)$ and a function $\phi : S \to \bigcup_{v \in S} L(v)$, we call $\phi$ an $L$-coloring of $S$ if $\phi(v) \in L(v)$ for each $v \in S$ and $\phi$ is an $L$-coloring of the induced graph $G[S]$. A partial $L$-coloring of $G$ is a function of the form $\phi : S \to \bigcup_{v \in S} L(v)$, where $S$ is a subset of $V(G)$ and $\phi$ is an $L$-coloring of $S$. Likewise, given a set $S \subseteq V(G)$, a partial $L$-coloring of $S$ is a function $\phi : S' \to \bigcup_{v \in S'} L(v)$, where $S' \subseteq S$ and $\phi$ is an $L$-coloring of $S'$. Given an integer $k \geq 1$, a graph $G$ is called $k$-choosable if, for every list-assignment $L$ for $G$ such that $|L(v)| \geq k$ for all $v \in V(G)$, $G$ is $L$-colorable. In 1994, Thomassen demonstrated in [13] that all planar graphs are 5-choosable. Actually, Thomassen proved something
Theorem 1.1. Let $G$ be a planar graph with facial cycle $C$. Let $xy$ be an edge of $G$ with $x, y \in V(C)$. Let $L$ be a list assignment for $V(G)$ such that each vertex of $V(G \setminus C)$ has a list of size at least five and each vertex of $V(C) \setminus \{x, y\}$ has a list of size at least three, where $xy$ is $L$-colorable. Then $G$ is $L$-colorable.

Theorem 1.1 has the following useful corollary, which we use frequently.

Corollary 1.2. Let $G$ be a planar graph with outer cycle $C$ and let $L$ be a list-assignment for $V(G)$ where each vertex of $G \setminus C$ has a list of size at least five. If $|V(C)| \leq 4$ then any $L$-coloring of $V(C)$ extends to an $L$-coloring of $G$.

We also rely on the following very useful result from [11] which is an analogue of Theorem 1.1 where the precolored edge has been replaced by two lists of size two.

Theorem 1.3. Let $G$ be a planar graph, let $F$ be a facial subgraph of $G$, and let $v, w \in V(F)$. Let $L$ be a list-assignment for $V(G)$ where $|L(v)| \geq 2$, $|L(w)| \geq 2$, and furthermore, for each $x \in V(F) \setminus \{v, w\}$, $|L(x)| \geq 3$, and, for each $x \in V(G \setminus F)$, $|L(x)| \geq 5$. Then $G$ is $L$-colorable.

Given an embedding $G$ on surface $\Sigma$, the deletion of $G$ partitions $\Sigma$ into a collection of disjoint, open path-connected components called the faces of $G$. Our main objects of study are the subgraphs of $G$ bounding the faces of $G$. Given a subgraph $H$ of $G$, we call $H$ a facial subgraph of $G$ if there exists a path-connected component $U$ of $\Sigma \setminus G$ such that $H = \partial(U)$. We call $H$ is called a cyclic facial subgraph (or, more simply, a facial cycle) if $H$ is both a facial subgraph of $G$ and a cycle. Given a cycle $C \subseteq G$, we say that $C$ is contractible if it can be contracted on $\Sigma$ to a point, otherwise we say it is noncontractible. More generally, given a subgraph $H$ of $G$, we say that $H$ is contractible if it does not contain any noncontractible cycles, and otherwise $H$ is noncontractible. We now introduce two standard parameters that measure the extent to which an embedding deviates from planarity.

Definition 1.4. Let $\Sigma$ be a surface and let $G$ be an embedding on $\Sigma$.

1) The edge-width of $G$, denoted by $\text{ew}(G)$, is the length of the shortest noncontractible cycle in $G$.

2) The face-width of $G$, denoted by $\text{fw}(G)$, is the smallest integer $k$ such that there exists a noncontractible closed curve of $\Sigma$ which intersects with $G$ on $k$ points.

If $G$ has no noncontractible cycles, then we define $\text{ew}(G) = \infty$, and if $g(\Sigma) = 0$, then we define $\text{fw}(G) = \infty$. The face-width of $G$ is also sometimes called the representativity of $G$. Some authors consider the face-width to be undefined if $G$ has no noncontractible cycles (and, in particular, if $\Sigma = S^2$), but, for our purposes, adopting the convention that $\text{fw}(G) = \infty$ in this case is much more convenient. The notion of face-width was introduced by Robertson and Seymour in their work on graph minors and has been studied extensively.
**Definition 1.5.** Let \( \Sigma \) be a surface and let \( G \) be an embedding on \( \Sigma \). We say that \( G \) is a 2-cell embedding if each component of \( \Sigma \setminus G \) is homeomorphic to an open disc.

If \( G \) is a 2-cell embedding, then \( \text{fw}(G) \) can be alternatively regarded as the smallest integer \( k \) such that there exists a collection of \( k \) facial subgraphs of \( G \) whose union contains a noncontractible cycle of \( G \). In practice, we are usually working with a 2-cell embedding, and in that case, we mostly use the above definition of \( \text{fw}(G) \) rather than that of Definition 1.4, as it is usually easier to work with for our purposes. We have the following simple standard facts.

**Observation 1.6.** Let \( \Sigma \) be a surface and let \( G \) be an embedding on \( \Sigma \). Then \( \text{ew}(G) \geq \text{fw}(G) \) and, for any subgraph \( H \) of \( G \), we have \( \text{ew}(H) \geq \text{ew}(G) \).

Our main result is Theorem 1.7 below, which is slightly stronger than the formulation in which the pairwise-far apart faces with lists of size less than five are replaced by pairwise-far apart components with ordinary 2-colorings. The proof of Theorem 1.7 makes up this series of papers and generalizes Theorem 1.1 to arbitrarily many faces in an embedding \( G \) on an arbitrary surface \( \Sigma \) such that representativity of \( G \) is at least \( 2^{\Omega(g(\Sigma))} \) and the special faces of \( G \) are of distance at least \( 2^{\Omega(g(\Sigma))} \) apart. Note that the pairwise-distance lower bound does not depend on the number of special faces or their sizes.

**Theorem 1.7.** There exist constants \( c, c' \geq 0 \) such that the following hold. Let \( \Sigma \) be a surface of genus \( g := g(\Sigma) \), let \( G \) be an embedding on \( \Sigma \) of face-width at least \( c \cdot 3^g \), and let \( F_1, \ldots, F_m \) be a collection of facial subgraphs of \( G \) such that \( d(F_i, F_j) \geq c' \cdot 3^g \) for each \( 1 \leq i < j \leq m \). Let \( x_1y_1, \ldots, x_my_m \) be a collection of edges in \( G \), where \( x_iy_i \in E(F_i) \) for each \( i = 1, \ldots, m \). Let \( L \) be a list-assignment for \( G \) such that the following hold.

1) For each \( v \in V(G) \setminus (\bigcup_{i=1}^m V(C_i)) \), \( |L(v)| \geq 5 \); AND

2) For each \( i = 1, \ldots, m \), \( x_iy_i \) is \( L \)-colorable, and, for each \( v \in V(F_i) \setminus \{x_i, y_i\} \), \( |L(v)| \geq 3 \).

Then \( G \) is \( L \)-colorable.

## 2 Conventions of this Paper

Unless otherwise specified, all graphs are regarded as embeddings on a previously specified surface, and all surface are compact, connected, and have zero boundary. If we want to talk about a graph \( G \) as an abstract collection of vertices and edges, without reference to sets of points and arcs on a surface then we call \( G \) an abstract graph.

**Definition 2.1.** Let \( \Sigma \) be a surface, let \( G \) be an embedding on \( \Sigma \), and let \( C \) be a contractible cycle in \( G \). Let \( U_0, U_1 \) be the two open connected components of \( \Sigma \setminus C \). The unique natural \( C \)-partition of \( G \) is the pair \( \{G_0, G_1\} \) of subgraphs of \( G \) where, for each \( i \in \{0, 1\}, G_i = G \cap \text{Cl}(U_i) \).
**Definition 2.2.** Given a graph $G$, a subgraph $H$ of $G$, a subgraph $P$ of $G$, and an integer $k \geq 1$, we call $P$ a $k$-chord of $H$ if $|E(P)| = k$ and $P$ is of the following form.

1) $P := v_1 \cdots v_kv_1$ is a cycle with $v_1 \in V(H)$ and $v_2, \ldots, v_k \notin V(H)$; OR

2) $P := v_1 \cdots v_{k+1}$, and $P$ is a path with distinct endpoints, where $v_1, v_{k+1} \in V(H)$ and $v_2, \ldots, v_k \notin V(H)$.

Given a $k \geq 1$ and a $k$-chord $P$ of $H$, $P$ is called a proper $k$-chord of $H$ if $P$ is not a cycle, i.e. $P$ intersects $H$ on two distinct vertices. Otherwise it is called a cyclic or an improper $k$-chord. Note that, for any $1 \leq k \leq 2$, any $k$-chord of $H$ is a proper $k$-chord of $H$, since $G$ has no loops or duplicated edges. A 1-chord of $H$ is simply referred to as a chord of $H$. In some cases, we are interested in analyzing $k$- chords of $H$ in $G$ where the precise value of $k$ is not important.

We thus introduce the following definition. We call $P$ a generalized chord of $H$ if there exists an integer $k \geq 1$ such that $P$ is a $k$-chord of $H$. We call $P$ a proper generalized chord of $H$ if there exists an integer $k \geq 1$ such that $P$ is a proper $k$-chord of $H$. We define improper (or cyclic) generalized chords of $H$ analogously. For any two vertex sets $A, B \subseteq V(G)$, an $(A,B)$-path is a path $P = x_0 \cdots x_k$ such that $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$.

Given a surface $\Sigma$, an embedding $G$ on $\Sigma$, a cyclic facial subgraph $C$ of $G$, and a proper generalized $Q$ of $C$, there is, under certain circumstances, a natural way to talk about one or the other “side” of $Q$ in $G$. That is, analogous to Definition 2.1, there is a natural topological way to partition $G$ into two sides of $Q$, which is made precise below.

**Definition 2.3.** Let $\Sigma$ be a surface, let $G$ be an embedding on $\Sigma$, let $C$ be a cyclic facial subgraph of $G$ and let $Q$ be a generalized chord of $C$, where each cycle in $C \cup Q$ is contractible. The unique natural $(C,Q)$-partition of $G$ is the pair $\{G_0, G_1\}$ of subgraphs of $G$ such that the following hold.

1) $G = G_0 \cup G_1$ and $G_0 \cap G_1 = Q$; AND

2) For each $i \in \{0,1\}$, there is a unique open path-connected region $U$ of $\Sigma \setminus (C \cup Q)$ such that $G_i$ consists of all the edges and vertices of $G$ in the closed region $\text{Cl}(U)$.

If the facial cycle $C$ is clear from the context then we usually just refer to $\{G_0, G_1\}$ as the natural $Q$-partition of $G$.

Note that this is consistent with Definition 2.1 in the sense that, if $Q$ is not a proper generalized chord of $C$ (i.e $Q$ is a cycle) then the natural $Q$-partition of $G$ is the same as the natural $(C,Q)$-partition of $G$. If $\Sigma$ is the sphere (or plane) then the natural $(C,Q)$-partition of $G$ is always defined for any $C, Q$. We also adopt the following standard notation.

**Definition 2.4.** For any graph $G$, vertex set $X \subseteq V(G)$, integer $j \geq 0$, and real number $r \geq 0$, we have the following.

1) We set $D_j(X,G) := \{v \in V(G) : d(v, X) = j\}$.

2) We set $B_r(X,G) := \{v \in V(G) : d(v, X) \leq r\}$.

3) For any subgraph $H$ of $G$, we usually just write $D_j(H,G)$ to mean $D_j(V(H),G)$, and likewise, we usually
write $B_r(H, G)$ to mean $B_r(V(H), G)$.

If the underlying graph $G$ is clear from the context, then we drop the second coordinate from the above notation in order to avoid clutter. We now introduce some additional notation related to list-assignments for graphs. We very frequently analyze the situation where we begin with a partial $L$-coloring $\phi$ of a subgraph of a graph $G$, and then delete some or all of the vertices of $\text{dom}(\phi)$ and remove the colors of the deleted vertices from the lists of their neighbors in $G \setminus \text{dom}(\phi)$. We thus make the following definition.

**Definition 2.5.** Let $G$ be a graph, let $\phi$ be a partial $L$-coloring of $G$, and let $S \subseteq V(G)$. We define a list-assignment $L^S_\phi$ for $G \setminus (\text{dom}(\phi) \setminus S)$ as follows.

\[
L^S_\phi(v) := \begin{cases}
\{\phi(v)\} & \text{if } v \in \text{dom}(\phi) \cap S \\
L(v) \setminus \{\phi(w) : w \in N(v) \cap (\text{dom}(\phi) \setminus S)\} & \text{if } v \in V(G) \setminus \text{dom}(\phi)
\end{cases}
\]

If $S = \emptyset$, then $L^S_\phi$ is a list-assignment for $G \setminus \text{dom}(\phi)$ in which the colors of the vertices in $\text{dom}(\phi)$ have been deleted from the lists of their neighbors in $G \setminus \text{dom}(\phi)$. The situation where $S = \emptyset$ arises so frequently that, in this case, we simply drop the superscript and let $L_\phi$ denote the list-assignment $L^S_\phi$ for $G \setminus \text{dom}(\phi)$. In some cases, we specify a subgraph $H$ of $G$ rather than a vertex-set $S$. In this case, to avoid clutter, we write $L^H_\phi$ to mean $L^V(H)_\phi$. Finally, given two partial $L$-colorings $\phi$ and $\psi$ of $V(G)$, where $\phi(x) = \psi(x)$ for all $x \in \text{dom}(\phi) \cap \text{dom}(\psi)$, and $\phi(x) \neq \psi(y)$ for all edges $xy$ with $x \in \text{dom}(\phi)$ and $y \in \text{dom}(\psi)$, there is a natural well-defined $L$-coloring $\phi \cup \psi$ of $\text{dom}(\phi) \cup \text{dom}(\psi)$, where

\[
(\phi \cup \psi)(x) = \begin{cases}
\phi(x) & \text{if } x \in \text{dom}(\phi) \\
\psi(x) & \text{if } x \in \text{dom}(\psi)
\end{cases}
\]

### 3 Definitions from Paper I

We begin by recalling the following notation introduced in Paper I of this sequence ([5]).

**Definition 3.1.** Let $k, \alpha \geq 1$ be integers. A tuple $T = (\Sigma, G, C, L, C^*)$ is called an $(\alpha, k)$-chart if $\Sigma$ is a surface, $G$ is an embedding on $\Sigma$ with list-assignment $L$, and $C$ is a family of facial subgraphs of $G$, where $C^*$ is an element of $C$ and the following conditions are satisfied:

1) For any distinct $H_1, H_2 \in C$, we have $d(H_1, H_2) \geq \alpha$; AND

2) $|L(v)| \geq 5$ for all $v \in V(G \setminus (\bigcup_{H \in C} V(H)))$; AND

3) For each $H \in C$, there exists a connected subgraph $P_{T,H}$ of $H$ satisfying the following.
The definition above entails that \( C \neq \emptyset \), but, given an \( H \in C \), we possibly have \( P_{T,H} = \emptyset \). We also recall the following terminology.

**Definition 3.2.** Given a surface \( \Sigma \), we define the following.

1) A tuple \( T \) is called a **chart** if there exist integers \( k, \alpha \geq 1 \) such that \( T \) is an \((\alpha, k)\)-chart.

2) A chart \( T = (\Sigma, G, C, L, C^\ast) \) is called **colorable** if \( G \) is \( L \)-colorable. We call \( G \) the **underlying graph** of the chart and we call \( \Sigma \) the **underlying surface** of the chart, and

   a) for each \( H \in C \), the uniquely specified subgraph \( P_{T,H} \) of \( H \) satisfying 3) of Definition 3.1 is called the **precolored subgraph** of \( H \).

   b) The elements of \( C \) are called the **rings** of the chart. In particular, the elements of \( C \setminus \{C_x\} \) are called the **inner** rings of the chart and \( C_x \) is called the **outer** ring of the chart.

If the underlying chart is clear from the context, we usually drop the \( T \) from the notation \( P_{T,H} \) to avoid clutter, i.e we just write \( P_H \).

**Definition 3.3.** Let \( T = (\Sigma, G, C, L, C_x) \), be a chart and let \( H \in C \). We say that \( H \) is a **closed** \( T \)-ring if \( P_H = H \).

Otherwise, we say that \( C \) is an **open** \( T \)-ring. If the chart \( T \) is clear from the context then we just call \( C \) a closed ring or open ring respectively.

We now restate Theorem 1.7 in the language of charts.

**Theorem 3.4.** There exist constants \( c, c' \) such that the following hold. Let \( \Sigma \) be a surface of genus \( g := g(\Sigma) \), let \( G \) be an embedding on \( \Sigma \) with \( \text{fw}(G) \geq c \cdot 3^g \), and let \( T \) be a \((c' \cdot 3^g, 1)\)-chart with underlying surface \( \Sigma \) and underlying graph \( G \). Then \( T \) is colorable.

Of particular importance to us over the course of the proof of Theorem 1.7 are embeddings which do not have separating cycles of length 3 or 4, so we give them a special name.

**Definition 3.5.** Let \( \Sigma \) be a surface and let \( G \) be an embedding on \( \Sigma \).

1) A **separating cycle** in \( G \) is a contractible cycle \( C \) in \( G \) such that each of the two connected components of \( \Sigma \setminus C \) has nonempty intersection with \( V(G) \).
2) We call $G$ short-separation-free if $G$ does not contain any separating cycle of length 3 or 4. Likewise, given a chart $T$, we call $T$ a short-separation-free chart if the underlying graph of $T$ is a short-separation-free.

One of the key ingredients in the proof of Theorem 1.7 is the reduction to a particular class of embeddings on surfaces which is easier to study. We recall the following definitions from Paper I.

**Definition 3.6.** Let $T = (\Sigma, G, C, L, C_\alpha)$ be a chart.

1) We call $T$ a tessellation if it is short-separation-free and every facial subgraph of $G$ not lying in $C$ is a triangle.

2) Given integers $k, \alpha \geq 1$, we call $T$ an $(\alpha, k)$-tessellation if it is both a tessellation and an $(\alpha, k)$-chart.

In Papers I-VI, we show that Theorem 3.4 holds for tessellations. Actually, we prove something stronger by defining a structure called a mosaic and showing that all mosaics are colorable, where a mosaic is a special kind of a tessellation satisfying some additional properties. We now recall the setup and definition of mosaic from Paper I. We begin by fixing constants $N_{mo}, \beta$, where $N_{mo} \geq 100$ and $\beta := 100N_{mo}^2$, and furthermore, $N_{mo}$ is a multiple of 3 (this is just for convenience so that $N_{mo}/3$ is an integer). We now recall the following definition.

**Definition 3.7.** Let $\Sigma$ be a surface, let $G$ be an embedding on $\Sigma$, and let $L$ be a list-assignment for $V(G)$. Given a facial subgraph $H$ of $G$, we say that $H$ is $L$-predictable if, letting $S := \{v \in V(H) : |L(v)| = 1\}$, the following hold.

1) For every subgraph $K \subseteq G[V(S)]$ and every $v \in D_1(K) \setminus V(H)$, where $K$ is induced in $G$ and $K$ is either a path or a cycle, the graph $G[N(v) \cap V(K)]$ is either a proper subpath of $K$ or all of $K$; AND

2) There is a vertex $v \in D_1(S) \setminus V(H)$ such that, for any proper $L$-coloring $\phi$ of $V(S)$, every vertex of $D_1(S) \setminus (V(H) \cup \{v\})$ has an $L_\phi$-list of size at least three, and $v$ has an $L_\phi$-list of size at least two.

We say that $H$ is highly $L$-predictable if it satisfies 1) above and Condition 2) has been replaced by the following slightly stronger condition.

3) For any proper $L$-coloring $\phi$ of $V(S)$, every vertex of $D_1(S) \setminus V(H)$ has an $L_\phi$-list of size at least three.

In order to state the distance conditions we impose on our tessellations, we introduce the following notation.

**Definition 3.8.** Let $T = (G, C, L, C_\alpha)$ be a chart. We define a rank function $\operatorname{Rk}(T|\cdot) : C \rightarrow \mathbb{R}$, and for each $H \in C$, a subset $\omega_T(H)$ of $V(C)$ as follows.

$$\omega_T(H) := \begin{cases} V(H) \text{ if } H \text{ is a closed } T\text{-ring} \\ V(H \setminus P_H) \text{ if } H \text{ is an open } T\text{-ring} \end{cases}$$

$$\operatorname{Rk}(T|H) := \begin{cases} |V(H)| \text{ if } H \text{ is a closed } T\text{-ring} \\ 2N_{mo} \text{ if } H \text{ is an open } T\text{-ring} \end{cases}$$

If the underlying chart $T$ is clear from the context then we drop the symbol $T$ from the notation above. We also recall the following notation from Paper I, which is a generalization of standard notation used to denote the subpath of a path.
consisting of its internal vertices.

**Definition 3.9.** Given a graph $H$, we let $\hat{H}$ be the graph obtained from $H$ by deleting the vertices of degree at most one. In particular, if $H$ is a path of length at least two, then $\hat{H}$ is the path obtained by deleting the endpoints of $H$, and if $H$ is a cycle, then $\hat{H} = H$.

We now state our induction hypothesis.

**Definition 3.10.** A chart $\mathcal{T} := (\Sigma, G, C, L, C_*)$ is called a **mosaic** if $\mathcal{T}$ is a tessellation such that, letting $g$ be the genus of $\Sigma$, the following additional conditions hold.

$M_0)$ For each $H \in C$, if $H$ is open, then $|E(\mathcal{P}_H)| \leq \frac{2N_{mo}}{3}$, and if $H$ is closed, then $|E(\mathcal{P}_H)| \leq N_{mo}$; **AND**

$M_1)$ For each open ring $H$, $\mathcal{P}_H$ is a path and there is no chord of $H$ with an endpoint in $\hat{\mathcal{P}}_H$, and furthermore, $H$ is highly $L$-predictable; **AND**

$M_2)$ Each closed ring is $L$-predictable; **AND**

$M_3)$ For each $C \in C \setminus \{C_*\}$, we have $d(w(C_*), w(C)) \geq 2.9\beta \cdot 6^{g-1} + \text{Rk}(C) + \text{Rk}(C_*)$; **AND**

$M_4)$ For any distinct $C_1, C_2 \in C \setminus \{C_*\}$, we have $d(w(C_1), w(C_2)) \geq \beta \cdot 6^g + \text{Rk}(C_1) + \text{Rk}(C_2)$; **AND**

$M_5)$ $\text{fw}(G) \geq 2.1\beta \cdot 6^{g-1}$ and $\text{ew}(G) \geq 2.1\beta \cdot 6^g$.

Papers I-VI consist of the proof of the following result.

**Theorem 3.11.** **All mosaics are colorable.**

We recall the following definition from Paper I.

**Definition 3.12.** Let $\mathcal{T} := (\Sigma, G, C, L, C_*)$ be a mosaic. We say that $\mathcal{T}$ is **critical** if the following hold.

1) $G$ is not $L$-colorable; **AND**

2) Any mosaic $(\Sigma', G', C', L', D)$ with $|V(G')| < |V(G)|$ is colorable; **AND**

3) Any mosaic $(\Sigma', G', C', L', D)$ with $|V(G')| = |V(G)|$ and $\sum_{v \in V(G)} |L(v)| < \sum_{v \in V(G)} |L(v)|$ is colorable.

Theorem 3.11 is the main step in the proof of main step in the proof of Theorem 3.4. In Paper VI we use the work of Papers I-V to produce a smaller counterexample from a critical mosaic by carefully coloring and deleting a contractible subgraph of a minimal counterexample. More precisely, given a minimal counterexample $(\Sigma, G, C, L, C_*)$, the subgraph of $G$ which we color and delete to produce a smaller counterexample is “almost” a path between $C_*$ and a $C \in C \setminus \{C_*\}$, in the sense that it differs from such a path only in some regions which are on the small sides of short generalized chords of either $C_*$ or $C$. We have to be careful when we perform the deletion described above because the resulting smaller embedding still needs to have high face-width in order to be the underlying graph of a smaller
counterexample. While edge-width is monotone with respect to subgraphs, this is not true of face-width. When we delete a subgraph of an embedding on some surface, the face-width of the resulting smaller embedding can go down. By choosing our induction hypothesis correctly, we ensure that the subgraph obtained from a minimal counterexample in the manner described above has high enough face-width to be a smaller counterexample. The main purpose of Paper III is show that the underlying graph of a critical mosaic has high enough face-width that this holds.

4 An Overview of Paper III

The main result of this paper is Theorem 4.1 below.

**Theorem 4.1.** Let \( T = (\Sigma, G, C, L, C_*) \) be a critical mosaic. Then \( \text{fw}(G) > 4.21\beta \cdot 6^{g(\Sigma)} - 1 \).

To provide an overview of the proof of Theorem 4.1, we first recall another definition which we introduced in Paper II ([6]). We frequently deal with situations where we have a set of vertices that we want to delete, and it is desirable to color as few vertices in the deletion set as possible, so that we can safely delete the remaining vertices in our specified vertex set without coloring them. We thus introduce the following definition.

**Definition 4.2.** Let \( G \) be a graph with a list-assignment \( L \). Given a subset \( Z \subseteq V(G) \) and a partial \( L \)-coloring \( \phi \) of \( V(G) \), we say that \( Z \) is \((L, \phi)\)-inert in \( G \) if every extension of \( \phi \) to an \( L \)-coloring of \( G \setminus (Z \setminus \text{dom}(\phi)) \) extends to an \( L \)-coloring of all of \( G \). If the ambient graph \( G \) is clear from the context, then we just say that \( Z \) is \((L, \phi)\)-inert.

If \( \phi \) is the empty coloring, then we just say that \( Z \) is \( L \)-inert in \( G \). Note that, for a partial \( L \)-coloring \( \phi \) of \( G \), if \( Z \subseteq V(G) \setminus \text{dom}(\phi) \), then \( Z \) is \((L, \phi)\)-inert in \( G \) if and only if \( Z \) is \( L_{\phi} \)-inert in \( G \setminus \text{dom}(\phi) \).

The following observation is immediate, and we use it repeatedly.

**Observation 4.3.** Let \( G \) be a graph with a list-assignment \( L \) and let \( \phi, \psi \) be partial \( L \)-colorings of \( G \), where \( \phi \cup \psi \) is a well-defined proper \( L \)-coloring of \( \text{dom}(\phi) \cup \text{dom}(\psi) \). Let \( Z, Z' \subseteq V(G) \), where \( Z \) is \((L, \phi)\)-inert in \( G \) and \( Z' \) is \((L, \psi)\)-inert in \( G \). Suppose further that

1) \( Z \setminus \text{dom}(\phi) \) and \( Z' \setminus \text{dom}(\psi) \) are distance at least two apart; AND

2) \( (Z \setminus \text{dom}(\phi)) \cap \text{dom}(\psi) = \emptyset \) and \( (Z' \setminus \text{dom}(\psi)) \cap \text{dom}(\phi) = \emptyset \).

Then \( Z \cup Z' \) is \((L, \phi \cup \psi)\)-inert in \( G \).

We now give a brief overview of the proof of Theorem 4.1. Let \( T = (\Sigma, G, C, L, C_*) \) be a critical mosaic. It is straightforward to check that, if \( T \) does not satisfy Theorem 4.1, then there is an open ring \( C \in C \) and a proper generalized chord \( P \) of \( C \) such that \(|E(P)| < 4.21\beta \cdot 6^{g(\Sigma)} - 1 \), where each cycle of \( C \cup P \), other than \( C \), is noncontractible. Now, given such a \( P \) and \( C \), we partially color and delete a subgraph \( A \) of \( G \), where \( A \) consists mostly of
vertices of $C \cup P$ (we make this precise later). More precisely, we find a partial $L$-coloring $\phi$ of $V(A)$ such that $A$ is $(L, \phi)$-inert in $G$, and almost all of the leftover vertices of $D_1(A)$ have $L_\phi$-lists of size three. We then show that $G \setminus A$ is $L_\phi$-colorable (which implies that $G$ is $L$-colorable, producing a contradiction) by showing that it is the underlying graph of a mosaic with list-assignment $L_\phi$, where $G \setminus A$ is embedded on a (not necessarily) surface obtained from $\Sigma$ by deleting a noncontractible closed curve. To perform the procedure described above, we need a few intermediate results that we prove in a more general context than that of mosaics. We use some results from [7], [8], and [9] as black boxes, which we state in Section 5. In particular, we rely on the main result of [9], which is restated in Section 5 as Theorem 5.8. In the setting above, we want to apply Theorem 5.8 to a 4-chord of $C$ whose midpoint is a vertex of $P$ of distance two from $C$, but there is an additional complication we have to consider, which is that, given a vertex $v \in V(P)$ of distance two from $C$, the structure of $B_2(C)$ near $v$ might be “degenerate” in the sense that $v$ is not the midpoint of any proper 4-chord of $C$, so we need a modified version of Theorem 5.8 to deal with this special case, we prove this modified version in Section 6. We stress that the proof of the result in Section 6, which relies on Theorem 5.8 as a black box, is not very long (unlike Theorem 5.8 itself, which spans three papers!).

During the proof of Theorem 4.1, we need to be able to partially $L$-color a given subpath $Q$ of $C$ in such a way that we do not take too many colors away from the vertices of $D_1(C)$. We make this precise in Section 7. The results of Section 7 are not stated in terms of mosaics. Rather, they are stated in terms of a more general framework of an embedding $G$ on a surface $\Sigma$, where $G$ has sufficiently highly face-width and $G$ has a list-assignment $L$ and a cyclic facial subgraph $G$ with the special property that, for an appropriately chosen integer $k$, any $k$-chord of $C$ has one side which is contractible and contains only 5-lists. This is made precise at the start of Section 7. In Section 8, we develop some more results in this general framework. We use the results of Sections 7-8 to prove the main result, Theorem 4.1, and also in later papers. This general framework is useful because this is precisely the special property of critical mosaics that we proved in Paper I (in particular, in Theorem 12.6 of Paper I). At the end of Section 8, we are almost ready to prove Theorem 4.1, but we also need some results which we proved in Papers I-II of this sequence. In Section 9, we restate these results, which are used as black boxes to finish the proof of Theorem 4.1.

5 Black Boxes from [7], [8], and [9] About Paths of Length $\leq 4$

We first recall several definitions used in [7], [8], and [9].

**Definition 5.1.** A *rainbow* is a tuple $(G, C, P, L)$, where $G$ is a planar graph with outer cycle $C$, $P$ is a path on $C$ of length at least two, and $L$ is a list-assignment for $V(G)$ such that $|L(v)| \geq 3$ for each $v \in V(C \setminus P)$ and $|L(v)| \geq 5$ for each $v \in V(G \setminus C)$. We say that a rainbow is *end-linked* if, letting $p, p^*$ be the endpoints of $P$, each of $L(p)$ and $L(p^*)$ is nonempty and $|L(p)| + |L(p^*)| \geq 4$. 
**Definition 5.2.** Given a graph $G$ with list-assignment $L$, a subgraph $H$ of $G$, and a partial $L$-coloring $\phi$ of $G$, we say that $\phi$ is ($H, G$)-sufficient if any extension of $\phi$ to an $L$-coloring of $\text{dom}(\phi) \cup V(H)$ extends to $L$-color all of $G$.

**Definition 5.3.** Let $G$ be a graph and let $L$ be a list-assignment for $G$. Let $P$ be a path in $G$ with $|V(P)| \geq 3$, let $H$ be a subgraph of $G$ and let $\{p, p'\}$ be the endpoints of $P$. We let $\text{End}_{L}(H, P, G)$ be the set of $L$-colorings $\phi$ of $\{p, p'\} \cup V(H)$ such that $\phi$ is ($P, G$)-sufficient.

In the setting above, if $H = \emptyset$, then we just write $\text{End}_{L}(P, G)$. That is, $\text{End}_{L}(P, G)$ is the set of $L$-colorings $\phi$ of the endpoints of $P$ such that any extension of $\phi$ to an $L$-coloring of $V(P)$ extends to $L$-color all of $G$. In Section 5 of [7], we proved the following result.

**Theorem 5.4.** Let $(G, C, P, L)$ be an end-linked rainbow, where $P$ is a 2-path. Then $\text{End}_{L}(P, G) \neq \emptyset$.

In Section 5 of [8], we proved the following result.

**Theorem 5.5.** Let $(G, C, P, L)$ be an end-linked rainbow, where $p_1p_2p_3p_4$ be a subpath of $C$ of length three. Suppose further that $|L(p_3)| \geq 5$. Then

1) there is an $L$-coloring $\psi$ of $\{p_1, p_4\}$ which extends to $|L_{\psi}(p_3)| - 2$ elements of $\text{End}_{L}(p_3, P, G)$; AND

2) If $p_3$ is incident to no chord of $C$, then there is an $L$-coloring $\psi$ of $\{p_1, p_4\}$ which extends to $|L_{\psi}(p_3)| - 1$ elements of $\text{End}_{L}(p_3, P, G)$.

We now introduce one more definition.

**Definition 5.6.** Let $G$ be a planar graph with outer cycle $C$, let $L$ be a list-assignment for $G$, and let $P$ be a path in $C$ with $|V(P)| \geq 3$. Let $pq$ and $p'q'$ be the terminal edges of $P$, where $p, p'$ are the endpoints of $P$. We let $\text{Crown}_L(P, G)$ be the set of partial $L$-colorings $\phi$ of $V(C) \setminus \{q, q'\}$ such that

1) $V(P) \setminus \{q, q'\} \subseteq \text{dom}(\phi)$ and, for each $x \in \{q, q'\}$, $|L_{\phi}(x)| \geq |L(x)| - 2$; AND

2) $\phi$ is ($P, G$)-sufficient.

In Section 6 of [7] we proved the following result.

**Theorem 5.7.** Let $(G, C, P, L)$ be an end-linked rainbow, where $P := p_1p_2p_3p_4$ is a 3-path. Then

1) $\text{Crown}_L(P, G) \neq \emptyset$. Actually, something stronger holds. There is a subgraph $H$ of $G$ with $V(H) \subseteq V(C \setminus P) \cap N(p_1) \cup N(p_4)$, where $|V(H) \cap N(p)| \leq 1$ for each endpoint $p$ of $P$ and $\text{End}_{L}(H, P, G) \neq \emptyset$; AND

2) If there is no chord of $C$ incident to a vertex of $\bar{P}$, then $\text{End}_{L}(P, G) \neq \emptyset$.

With Definition 5.6 in hand, we recall the main result of [9].
Claim 6.3. The second part of the claim follows from the fact that \( G \) is short-separation-free. Lastly, we introduce the following notation. In particular, we recall Definition 5.10, which is standard.

Definition 5.9. Given a planar graph \( G \) with outer face \( C \) and an \( H \subseteq G \), we let \( C^H \) denote the outer face of \( H \).

Definition 5.10. Given a path \( Q \) in a graph \( G \), we let \( \hat{Q} \) denote the subpath of \( Q \) consisting of the internal vertices of \( Q \). In particular, if \( |E(Q)| \leq 2 \), then \( \hat{Q} = \emptyset \). Furthermore, for any \( x, y \in V(Q) \), we let \( xQy \) denote the unique subpath of \( Q \) with endpoints \( x \) and \( y \).

6 A “Degenerate” Version of Theorem 5.8

As indicated in Section 4, we need a slightly modified version of Theorem 5.8 to deal with some of the cases of the main proof of Theorem 4.1. We prove this result below.

Theorem 6.1. (Degenerate Holepunch Theorem) Let \((G, C, P, L)\) be an end-linked rainbow, where \( P = p_0q_0v_0v_1q_1p_1 \) is a path of length five, and suppose further that each vertex of \( \hat{P} \) is incident to a chord of \( C \) whose other endpoint lies in \( C \setminus \hat{P} \). Then, for each \( i \in \{0, 1\} \), there is a partial \( L \)-coloring \( \phi \) of \( V(C) \), where \( \{p_0, p_1, v_i\} \subseteq \text{dom}(\phi) \cap V(P) \subseteq \{p_0, p_1, v_i, v_{1-i}\} \), such that \( \phi \) is \((P, G)\)-sufficient and each vertex of \( P \setminus \text{dom}(\phi) \) has an \( L_\phi \)-list of size at least three.

Proof. Suppose the theorem does not hold, and let \((G, C, P, L)\) be a counterexample to the Theorem, where \( G \) is vertex-minimal with respect to this property. For each \( i \in \{0, 1\} \), we say that a partial \( L \)-coloring \( \phi \) of \( V(C) \) is \( v_i \)-centered if \( \{p_0, p_1, v_i\} \subseteq \text{dom}(\phi) \cap V(P) \subseteq \{p_0, p_1, v_i, v_{1-i}\} \) and \( \phi \) is \((P, G)\)-sufficient and each vertex of \( P \setminus \text{dom}(\phi) \) has an \( L_\phi \)-list of size at least three. Thus, for some \( i \in \{0, 1\} \), there is no \( v_i \)-centered partial \( L \)-coloring of \( V(C) \), say \( i = 0 \) for the sake of definiteness. By removing colors from the lists of some of the vertices if necessary, we may suppose that each vertex of \( C \setminus \hat{P} \) has an \( L \)-list of size precisely three, and we may also suppose that \( L(p_0) \) and \( L(p_1) \) are nonempty sets with \( |L(p_0)| + |L(p_1)| = 4 \).

Claim 6.2. \( G \) is short-separation-free. Furthermore, any chord of \( P \), if it exists, has at least one endpoint in \( \{p_0, p_1\} \).

Proof: The vertex-minimality of \( G \), together with Corollary 1.2, immediately imply that \( G \) is short-separation-free. The second part of the claim follows from the fact that \( G \) is planar and each of \( v_0, v_1 \) has a neighbor in \( C \setminus \hat{P} \). ■

Claim 6.3. \( p_0q_1 \notin E(G) \) and \( p_1q_0 \notin E(G) \).
Proof: There are two edges to rule out. Consider the following cases.

Case 1: $p_0q_1 \in E(G)$

In this case, since each of $v_0, v_1$ has a neighbor in $C \setminus \hat{P}$, it follows that $E(G)$ contains $v_0p_0$ and $v_1p_0$. Let $G' = G - \{q_0, v_0, v_1\}$. As $G$ is short-separation-free, $G'$ has outer face $p_0(C \setminus \hat{P})p_1q_1$ which contains the 2-path $p_0q_1p_1$.

By Theorem 5.4, there is an $L$-coloring $\phi$ of $\{p_0, p_1\}$ with $\phi \in \text{End}(p_0q_1p_1, G')$. Each of $q_0, v_1$ has an $L_\phi$-list of size at least four, and any extension of $\phi$ to an $L$-coloring of $\{p_0, v_0, q_0\}$ is $v_0$-centered, contradicting our assumption.

Case 2: $p_1q_0 \in E(G)$

In this case, analogous to the above, we let $G'' = G - \{q_1, v_0, v_1\}$ and, applying Theorem 5.4 again, we choose a $\psi \in \text{End}(p_0q_0p_1, G'')$. Note that $G$ contains the edges $v_0p_1, v_1p_1$, as each of $v_0, v_1$ has a neighbor in $C \setminus \hat{P}$. Now, $|L_\psi(q_0)| \geq 3$ and $|L_\psi(v_0)| \geq 4$, so $\psi$ extends to an $L$-coloring $\psi^*$ of $\{p_0, v_0, p_1\}$ with $|L_{\psi^*}(q_0)| \geq 3$, and each of $v_1, q_1$ also has an $L_{\psi^*}$-list of size at least three as well. Thus, $\psi^*$ is $v_0$-centered, contradicting our assumption. ■

Claim 6.4. $p_0v_1 \notin E(G)$ and $p_1v_1 \notin E(G)$.

Proof: There are two possible edges to rule out:

Subclaim 6.4.1. $p_0v_1 \notin E(G)$

Proof: Suppose that $p_0v_1 \in E(G)$ and let $G' = G - \{q_0, v_0\}$. As $G$ is short-separation-free, $G'$ has outer cycle $p_0(C \setminus \hat{P})p_1q_1v_1$ and this cycle contains the 3-path $P' = p_0v_1q_1p_1$. Since $v_0$ has a neighbor in $C \setminus \hat{P}$, we have $v_0p_0 \in E(G)$. By 1) of Theorem 5.7, there is a $\psi \in \text{Crown}(P', G')$. By definition, $p_0, p_1 \in \text{dom}(\psi)$ and $\text{dom}(\psi) \subseteq V(C^{G'} \setminus \hat{P}') \subseteq V(C \setminus P)$, and each of $v_1, q_1$ has an $L_\psi$-list of size at least four. Since $N(q_0) = \{p_0, v_0\}$, we extend $\psi$ to a $v_0$-centered partial $L$-coloring of $V(C)$ by coloring $v_0$ with $c$, contradicting our assumption. ■

To finish the proof of Claim 6.4, we just need to check that $p_1v_1 \notin E(G)$. Suppose that $p_1v_1 \in E(G)$.

Subclaim 6.4.2. $q_0, v_0, v_1$ have a common neighbor in $C \setminus P$.

Proof: Suppose not, and let $G^* = G - q_1$. Note that $G^*$ has outer face $p_0(C \setminus \hat{P})p_1v_1v_0q_0$. In particular, the outer face of $G^*$ contains the 4-path $R := p_0q_0v_0v_1p_1$. Now, $q_0, v_0, v_1$ have no common neighbor in $C^{G^*} \setminus R$, so, by Theorem 5.8, there is a $\phi \in \text{Crown}(R, G^*)$. We have $\text{dom}(\phi) \subseteq V(C^{G^*}) \subseteq V(C)$ and $\text{dom}(\phi) \cap V(P) = \{p_0, v_0, p_1\}$. Each of $q_0, v_1$ has an $L_\phi$-list of size at least three, and $N(q_1) = \{p_1, v_1\}$, so $|L_\phi(q_1)| \geq 3$ as well. Thus, $\phi$ is $v_0$-centered, contradicting our assumption. ■
It follows from Subclaim 6.4.2 that there is a unique \( u \in V(C \setminus \hat{P}) \) adjacent to each of \( q_0, v_0, v_1 \). Let \( H_0 \) be the subgraph of \( G \) bounded by outer cycle \( p_0(C \setminus \hat{P})w_0 \) and let \( H_1 \) be the subgraph of \( G \) bounded by outer cycle \( u(C \setminus \hat{P})p_1v_1 \). Note that \( V(G) = V(H_0 \cup H_1) \cup \{v_0, q_1\} \). Since \( |L(u)| = 3 \), it follows from two applications of Theorem 5.4 that there is an \( L \)-coloring \( \psi \) of \( \{p_0, u, p_1\} \), where the restriction of \( \psi \) to \( \{p_0, u\} \) lies in \( \text{End}(p_0q_0u, H_0) \) and the restriction of \( \psi \) to \( \{u, p_1\} \) lies in \( \text{End}(uv_1p_1, H_1) \). Now, each of \( q_0, v_1 \) has an \( L|_{\psi} \)-list of size at least three, and \( |L_{\psi}(v_0)| \geq 4 \). Since \( v_1q_0 \notin E(G) \), it follows that \( \psi \) extends to an \( L \)-coloring \( \psi^* \) of \( \{p_0, u, p_1\} \cup \{v_0, v_1\} \), where \( |L_{\psi^*}(v_0)| \geq 3 \). Since \( q_1 \) only has two neighbors, we have \( |L_{\psi^*}(q_1)| \geq 3 \) as well, and \( \psi^* \) is \( v_0 \)-centered, contradicting our assumption. 

\[ \begin{array}{c}
 q_0 & \quad & v_0 & \quad & v_1 & \quad & q_1 \\
 p_0 & \longrightarrow & u & \longrightarrow & p_1 \\
 K_0 & \quad & K_1 \\
 \end{array} \]

Figure 6.1

Since \( v_1 \) has a neighbor in \( C \setminus \hat{P} \), let \( u \) be the unique neighbor of \( v_1 \) which is closest to \( p_1 \) on the path \( p_0(C \setminus \hat{P})p_1 \). By Claim 6.4, \( u \) is an internal vertex of this path. Let \( K_0 \) be the subgraph of \( G \) bounded by outer cycle \( p_0(C \setminus \hat{P})uv_1v_0q_0 \) and let \( K_1 \) be the subgraph of \( G \) bounded by outer cycle \( u(C \setminus \hat{P})p_1q_1v_1 \). This is illustrated in Figure 6.1.

Claim 6.5. \( q_0, v_0, v_1 \) have a common neighbor which is an internal vertex of the path \( p_0(C \setminus \hat{P})u \).

Proof: Suppose not. The outer cycle of \( K_0 \) contains the 4-path \( R = p_0q_0v_0v_1u \), and, by assumption, the three internal vertices of \( R \) have no common neighbor in \( C^{K_0} \setminus R \). Since \( |L(u)| = 3 \), it follows from Theorem 5.8 that there is a family of \( |L(p_0)| \) different elements of \( \text{Crown}(R, K_0) \), each using a different color on \( u \), and likewise, by 1) of Theorem 5.7, there is a family of \( |L(p_1)| \) different elements of \( \text{Crown}(uv_1q_1p_1, K_1) \), each using a different color on \( u \). Thus, there is a \( \psi \in \text{Crown}(R, K_0) \) and a \( \phi \in \text{Crown}(uv_1q_1, K_1) \) with \( \psi(u) = \phi(u) \). By our choice of \( u \), the outer cycle of \( K_1 \) has no chords incident to \( v_1 \), so \( N(v_1) \cap \text{dom}(\psi) \subseteq \text{dom}(\phi) \), and thus \( |L_{\psi \cup \phi}(v_1)| \geq 3 \). Likewise, by definition of \( \phi \) and \( \psi \), each of \( q_0, q_1 \) has an \( L_{\psi \cup \phi} \)-list of size at least three. Since \( \text{dom}(\psi \cup \phi) \cap V(P) = \{p_0, v_0, p_1\} \), it follows that \( \psi \cup \phi \) is \( v_0 \)-centered, contradicting our assumption. 

By Claim 6.5, there is a unique internal vertex \( u' \) of the path \( p_0(C \setminus \hat{P})u \), where \( u' \) is adjacent to all of \( q_0, v_0, v_1 \). Let \( H_0 \) be the subgraph of \( G \) bounded by outer cycle \( p_0(C \setminus \hat{P})u'q_0 \), and let \( H_1 \) be the subgraph of \( G \) bounded by outer cycle \( u'(C \setminus \hat{P})uv_1 \). Note that \( K_0 - v_0 = H_0 \cup H_1 \), and the outer cycle of \( H_1 \cup K_1 \) contains the 3-path \( R' = v_1q_1p_1 \). Let \( Y' \) be the set of \( L \)-colorings \( \psi \) of \( \{u', p_1\} \) such that \( \psi \) extends to at least \( |L_{\psi}(v_1)| - 2 \) elements of \( \text{End}(v_1, R', H_1 \cup K_1) \). Since \( |L(u')| = 3 \), it follows from 1) of Theorem 5.5 that there is a family of \( |L(p_1)| \) different
elements of \(Y\), each of which uses a different color on \(u'\). By Theorem 5.4, since \(|L(p_0)| + |L(p_1)| = 4\), there is a \(\phi \in \text{End}(p_0q_0u', H_0)\) and a \(\psi \in Y'\) with \(\phi(u') = \psi(u')\). Since \(|L_{\phi\cup\psi}(v_0)| \geq 4\) and \(|L_{\phi\cup\psi}(q_0)| \geq 3\), there is a \(c \in L_{\phi\cup\psi}(v_0)\) with \(|L_{\phi\cup\psi}(q_0) \setminus \{c\}| \geq 3\). Since \(v_1p_1 \not\in E(G)\), we have \(|L_{\psi}(v_1)| - 2 \geq 2\), so \(\psi\) extends to an element of \(\text{End}(v_1, R', H_1 \cup K_1)\) using a color other than \(c\) on \(v_1\). As \(p_0v_1 \not\in E(G)\), \(\phi \cup \psi\) extends to an \(L\)-coloring \(\tau\) of \(\{p_0, u', p_1\} \cup \{v_0, v_1\}\) with \(\tau(v_0) = c\), where the restriction of \(\tau\) to \(\{u', v_1, p_1\}\) lies in \(\text{End}(v_1, R', H_1 \cup K_1)\). Since \(u' \neq u\), we have \(u'q_1 \not\in E(G)\), so \(|L_{\tau}(q_1)| \geq 3\). By our choice of \(c\), \(|L_{\tau}(q_0)| \geq | \geq 3\) as well. Thus, \(\tau\) is \(v_0\)-centered, contradicting our assumption. \(\Box\)

7 Link Colorings

As indicated in the overview, in Sections 7-8, we prove some results in a slightly different framework than that of charts. In particular, we often analyze the region near a single ring of a chart.

Definition 7.1. A tuple \(K = [\Sigma, G, C, L]\) is called a collar if \(\Sigma\) is a surface, \(G\) is an embedding on \(\Sigma\), \(C\) is a facial cycle of \(G\), and \(L\) is a list-assignment for \(V(G)\).

Definition 7.2. Let \(k \geq 1\) be an integer and let \(K = [\Sigma, G, C, L]\) be a collar, where \(\text{fw}(G) \geq k + 2\). For any subpath \(P\) of \(C\), we associate to \(P\) a vertex set \(\text{Sh}^k(P, K)\), where \(v \in \text{Sh}^k(P, K)\) if there is a proper generalized chord \(Q\) of \(C\) of length at most \(k\), and with both endpoints in \(P\), such that, letting \(G = G_0 \cup G_1\) be the natural \((C, Q)\)-partition of \(G\), there exists an \(i \in \{0, 1\}\) such that the following hold.

1) \(v \in V(G_i \setminus C)\) and every vertex of \(G_i \setminus C\) has an \(L\)-list of size at least five; AND

2) \(G_i \cap P\) has one connected component and \(G_{1-i} \cap P\) has two connected components; AND

3) \(G_i\) is contractible.

Note that, in the setting above, the natural \((C, Q)\)-partition of \(G\) is well-defined since \(\text{fw}(G) \geq k + 2\), and, in particular, at least one of \(G_0, G_1\) is contractible (possibly both). Condition 2) above uniquely specified the index \(i \in \{0, 1\}\). If the collar \(K\) is clear from the context then we just write \(\text{Sh}^k(P)\) in place of \(\text{Sh}^k(P, K)\). We have one more definition, and then we can state and prove the main result of Section 7.

Definition 7.3. Let \(k \geq 1\) be an integer and let \(K = [\Sigma, G, C, L]\) be a collar, where \(\text{fw}(G) \geq k + 2\). For any subpath \(P\) of \(C\), we define the following.

1) We say \(P\) is \(k\)-short in \(K\) if, for any proper generalized chord \(Q\) of \(C\) with both endpoints in \(P\) and length at most \(k\), letting \(G = G_0 \cup G_1\) be the natural \((C, Q)\)-partition of \(G\), there is an \(i \in \{0, 1\}\) such that

   a) Every vertex of \(G_i \setminus C\) has an \(L\)-list of size at least five; AND

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b) If at least one endpoint of $Q$ lies in $\bar{P}$, then $G_i \cap P$ has one connected component and $G_{1-i} \cap P$ has two connected components.

2) We define $\text{Link}(P, \mathcal{K})$ to be the set of proper $L$-colorings $\phi$ of $V(P) \setminus \text{Sh}^2(P)$ such that $\text{Sh}^2(P)$ is $L_\phi$-inert in $G \setminus \text{dom}(\phi)$.

In most uses of the terminology above, the collar $\mathcal{K}$ is clear from the context and we simply say that $P$ is $k$-short to mean that $P$ is $k$-short in $\mathcal{K}$, and likewise, we just write $\text{Link}(P)$ to mean $\text{Link}(P, \mathcal{K})$.

Figure 7.1: This domain of an element of $\text{Link}(C - y)$ consists of $\{x, z\}$ and the black vertices

Figure 7.1 shows an example of the situation described above. Let $\mathcal{K} = [\Sigma, G, C, L]$ be a collar, where $fw(G) \geq 4$ and the path $C - y$ is 2-short in $\mathcal{K}$. For any $1 \leq k \leq 2$ and any $k$-chord $Q$ with both endpoints in $C - y$, the unique side of $Q$ which has only of 5-lists outside of $C$ and intersects $C - y$ on a connected subgraph of $C - y$ is also the side of $Q$ which is contractible, as indicated by the dotted lines. The elements of $\text{Sh}^2(C - y)$ are indicated by the red vertices. The following observation is immediate.

**Observation 7.4.** Let $[\Sigma, G, C, L]$ be a collar, where $fw(G) \geq 4$, and let $P$ be a subpath of $C$, where $P$ is 2-short. Then, for any $\phi \in \text{Link}(P)$, each vertex of $D_1(C) \setminus \text{Sh}^2(P)$ has at most two neighbors in $\text{dom}(\phi)$.

The main result of Section 7 provides some conditions under which the colorings defined above exist.

**Theorem 7.5.** Let $[\Sigma, G, C, L]$ be a collar, where $fw(G) \geq 4$, and let $P$ be a subpath of $C$ with $|V(P)| \geq 2$, where each internal vertex of $P$ has an $L$-list of size at least three, and $P$ is 2-short. Let $p, p'$ be the endpoints of $P$ and let $A \subseteq L(p)$ and $A' \subseteq L(p')$, where $A, A'$ are nonempty and $|A| + |A'| \geq 4$. If either $pp' \notin E(G) \setminus E(C)$ or
A \cap A' = \emptyset, then there is a \phi \in \text{Link}(P) with \phi(p) \in A and \phi(p') \in A.

**Proof.** Let P be a subpath of C which is a minimal counterexample to the theorem. Since P is 2-short and each internal vertex of P has an L-list of size at least three, it follows that, for any subpath P' of P, P' is also 2-short and all internal vertices of P' have L-lists of size at least three. Let P := p_1 \cdots p_r for some r \geq 2. As P is a counterexample, there exist nonempty sets A \subseteq L(p_1) and A' \subseteq L(p_r) such that the following hold.

1) |A| + |A'| \geq 4, and either A \cap A' = \emptyset or |V(P)| < |V(C)|; AND
2) There is no \phi \in \text{Link}(P) with \phi(p_1) \in A and \phi(p_r) \in A'.

By removing colors from A, A' if necessary, we suppose further that |A| + |A'| = 4 and |A'| \leq |A|.

**Claim 7.6.** |V(P)| > 2.

**Proof:** Suppose not. Thus, P is just an edge and r = 2. Since each of A, A' is nonempty and |A| \geq 2, there is an L-coloring \phi of P with \phi(p_1) \in A and \phi(p_2) \in A'. It follows from Corollary 1.2 that \phi \in \text{Link}(P), contradicting our assumption that no such element of Link(P) exists. ■

Note that any generalized chord of C of length at most two is a proper generalized chord of C. For the remainder of the proof of Theorem 7.5, we introduce the following notation.

**Definition 7.7.**

1) For any 1 \leq k \leq 2 and any k-chord Q of C with both endpoints in P, we define the following.

(a) We let \( C_Q^{\text{long}} \) and \( C_Q^{\text{short}} \) be the two contractible cycles intersecting precisely on \( p_iwp_j \) such that \( C_Q^{\text{long}} \cup C_Q^{\text{short}} = C \cup Q \), where \( C_Q^{\text{short}} \cap P \) has one connected component and \( C \cap P \) has two connected components.

(b) We let \( \{G_Q^{\text{long}}, G_Q^{\text{short}}\} \) denote the natural \((C, Q)\)-partition of G, where \( G_Q^{\text{long}} \cap (C \cup Q) = C_Q^{\text{long}} \) and \( G_Q^{\text{short}} \cap (C \cup Q) = C_Q^{\text{short}} \).

2) We let \( \mathcal{P} \) be the set of generalized chords of C of length at most two with one endpoint in \( P - p_1 \) and \( p_1 \) as the other endpoint.

As each element of \( \mathcal{P} \) shares a common endpoint, and this endpoint is also an endpoint of \( P \), we immediately have the following.

**Claim 7.8.** For any \( Q, Q' \in \mathcal{P} \), we have either \( G_{Q'}^{\text{short}} \subseteq G_Q^{\text{short}} \) or \( G_{Q'}^{\text{short}} \subseteq G_Q^{\text{short}} \).

We also have the following simple observation.
Claim 7.9. Let $q,q' \in V(P)$ with $q \neq q'$, where $qq'$ is a chord of $C$ and at least one of $q,q'$ lies in $V(\hat{P})$. Let $P^*$ be the subpath of $P$ with endpoints $qq'$. Then any $L$-coloring of $\{q,q'\}$ lies in $\text{Link}(P^*)$.

Proof: As $P$ is 2-short and at least one of $q,q'$ lies in $V(\hat{P})$, we get that each vertex of $G_{qq'}^{\text{short}} \setminus P^*$ has an $L$-list of size at least five. By Theorem 1.1, any $L$-coloring $\phi$ of $\{q,q'\}$ extends to an $L$-coloring of $G_{qq'}^{\text{short}}$, so $\phi \in \text{Link}(P^*)$. ■

We now define a subset $\mathcal{P}_{\text{end}}$ of $\mathcal{P}$, where $Q \in \mathcal{P}_{\text{end}}$ if $p_1,p_r$ are the endpoints of $Q$ and there is a vertex with an $L$-list of size less than five in $G_Q^{\text{short}} \setminus C_Q^{\text{short}}$.

Claim 7.10. $\mathcal{P} \setminus \mathcal{P}_{\text{end}} \neq \emptyset$

Proof: Let $P^* := P - p_1$ and suppose that $\mathcal{P} \setminus \mathcal{P}_{\text{end}} = \emptyset$. Thus, $\text{Sh}^2(P) = \text{Sh}^2(P^*)$. Consider the following cases.

Case 1: $G$ does not contain $p_2p_r$ as a chord of $C$

In this case, since $|V(P)| > 2$, we have $p_2p_r \not\in E(G) \setminus E(P^*)$. Since $|A| \geq 2$ and $|L(p_r)| \geq 3$, there is a $c \in L(p_r)$ with $|A \setminus \{c\}| \geq 2$. Since $|L(p_2)| \geq 3$, it follows from the minimality of $P$ that there is a $\phi \in \text{Link}(P - p_1)$ with $\phi(p_r) = c$. Since $\mathcal{P} \setminus \mathcal{P}_{\text{end}} = \emptyset$, there is no chord of $C$ with $p_1$ as an endpoint and the other endpoint in $V(\hat{P})$. Possibly $p_1p_r$ is an edge of $G$, either as a chord of $C$ or as the lone edge of $E(C) \setminus E(P)$ if $V(P) = V(C)$, but, in any case, we have $N(p_1) \cap \text{dom}(\phi) \subseteq \{p_1,p_{r-1}\}$. By our choice of $c$, we have $A \cap L_\phi(p_1) \neq \emptyset$. Thus, $\phi$ extends to an $L$-coloring $\phi^*$ of $\text{dom}(\phi) \cup \{p_1\}$ with $\phi^*(p_1) \in A$. Since $\text{Sh}^2(P) = \text{Sh}^2(P - p_1)$, we have $\phi^* \in \text{Link}(P)$, contradicting our assumption that no such element of $\text{Link}(P)$ exists.

Case 2: $G$ contains $p_2p_r$ as a chord of $C$.

Possibly $p_1p_r \in E(G) \setminus E(P)$, but, in any case, since $|A| \geq 2$, there is an $L$-coloring $\psi$ of $\{p_1,p_r\}$ with $\psi(p_1) \in A$ and $\psi(p_r) \in A'$. Since $|L(p_2)| \geq 3$, $\psi$ extends to an $L$-coloring $\psi^\dagger$ of $\{p_1,p_2,p_r\}$. By Claim 7.9, the restriction of $\psi^\dagger$ to $\{p_2,p_r\}$ is an element of $\text{Link}(P^*)$. Since $\text{Sh}^2(P) = \text{Sh}^2(P^*)$, we get that $\psi^\dagger \in \text{Link}(P)$, contradicting our assumption that no such element of $\text{Link}(P)$ exists. ■

Since $\mathcal{P} \setminus \mathcal{P}_{\text{end}} \neq \emptyset$, we choose $Q \in \mathcal{P} \setminus \mathcal{P}_{\text{end}}$ maximize the quantity $|V(G_Q^{\text{short}})|$ among all the elements of $\mathcal{P} \setminus \mathcal{P}_{\text{end}}$. Let $t \in \{2, \cdots, r\}$, where $p_1,p_t$ are the endpoints of $Q$, and let $C^\dagger := C_Q^{\text{short}}$ and $G^\dagger := G_Q^{\text{short}}$.

Claim 7.11. $p_t \in V(\hat{P})$.

Proof: Suppose not. Since $t \neq 1$, we have $t = r$. Since $Q \in \mathcal{P} \setminus \mathcal{P}_{\text{end}}$, each vertex of $G^\dagger \setminus C^\dagger$ has an $L$-list of size at least five. In particular, $\text{Sh}^2(P) = V(G^\dagger \setminus Q)$. Consider the following cases.

Case 1: $Q$ is a chord of $C$
In this case, \( Q = p_1 p_r \). Since \(|A| \geq 2\), there is an \( L \)-coloring \( \phi \) of \( \{p_1, p_r\} \) such that \( \phi(p_1) \in A \) and \( \phi(p_r) \in A' \), and, by Theorem 1.1, \( \phi \) extends to an \( L \)-coloring of \( G^1 \). Thus, we have \( \phi \in \text{Link}(P) \), contradicting our assumption that no such element of \( \text{Link}(P) \) exists.

**Case 2:** \( Q \) is a 2-chord of \( C \)

Since \(|A| + |A'| = 4\) and each of \( A, A' \) is nonempty, there is an \( L \)-coloring \( \phi \) of \( \{p_1, p_r\} \) such that \( \phi(p_1) \in A \) and \( \phi(p_r) \in A' \), where any extension of \( \phi \) to an \( L \)-coloring of \( \{p_1, x, p_r\} \) also extends to an \( L \)-coloring of all of \( G^1 \). Thus, we have \( \phi \in \text{Link}(P) \), contradicting our assumption that no such element of \( \text{Link}(P) \) exists. ■

**Claim 7.12.** There is a set \( \mathcal{F} \) of \(|A'|\) distinct elements of \( \text{Link}(p_1 P p_r) \), each of which uses a different color on \( p_t \), where, for each \( \sigma \in \mathcal{F} \), we have \( \sigma(p_r) \in A' \).

**Proof:** Recall that \(|A'| \in \{1, 2\} \), as \(|A| + |A'| = 4\) and \(|A'| \leq |A| \).

By Claim 7.11, we have \( p_t \in V(\bar{P}) \). If \( p_r p_t \) is a chord of \( C \), then by Claim 7.9, any \( L \)-coloring of \( \{p_t, p_r\} \) is an element of \( \text{Link}(p_t P p_r) \). In particular, we choose a \( c \in A' \), and since \(|L(p_t) \setminus \{c\}| \geq 2\), there is a set of \(|A'|\) \( L \)-colorings of \( \{p_t, p_r\} \) which use \( c \) on \( p_r \) and color \( p_t \) with different colors, so we are done in that case. Now suppose that \( p_r p_t \) is not a chord of \( C \). Since \( 1 < t < r \), we have \( p_r p_t \not\in E(G) \setminus E(p_t P p_r) \). If \(|A'| = 1\), then \(|L(p_3)| + |A'| \geq 4\), and it follows from the minimality of \( P \) that there is a \( \sigma \in \text{Link}(p_t P p_r) \) with \( \sigma(p_r) \in A' \), so we are done in that case. If \(|A'| = 2\), then \(|L(p_3)| + |A'| \geq 5\), so it follows from the minimality of \( P \) that there is a pair of elements \( \sigma_1, \sigma_2 \in \text{Link}(p_t P p_r) \), each of which uses a color of \( A' \) on \( p_r \), where \( \sigma_1(p_t) \neq \sigma_2(p_t) \). Again, we are done. ■

Let \( \mathcal{F} \) be as in Claim 7.12 and let \( B := \{\sigma(p_t) : \sigma \in \mathcal{F}\} \). Since \(|B| = |A'|\), we have \(|A| + |B| = 4\). By the maximality of \( Q \), we have \( \text{Sh}^2(P) = \text{Sh}^2(p_t P p_t) \cup V(G^1 \setminus \{A\}) \) as a disjoint union. In particular, we have the following.

**Claim 7.13.** For any \( \phi \in \text{Link}(p_1 P p_t) \) with \( \phi(p_1) \in A \) and any \( \sigma \in \mathcal{F} \) with \( \phi(p_1) = \sigma(p_t) \), the union \( \phi \cup \sigma \) lies in \( \text{Link}(P) \).

**Proof:** Let \( \phi, \sigma \) be as above. It just suffices to check that \( \phi \cup \sigma \) is a proper \( L \)-coloring of its domain. Suppose not. Thus, we have \( p_1 p_r \in E(G) \) and \( \phi(p_1) = \sigma(p_r) \). Since \(|V(P)| > 2\), we have \( p_1 p_r \in E(G) \setminus E(P) \), and thus \( A \cap A' = \emptyset \) by assumption. Since \( \phi(p_1) \in A \) and \( \sigma(p_r) \in A' \), we have a contradiction. ■

**Claim 7.14.** There is a \( \phi \in \text{Link}(p_1 P p_t) \) with \( \phi(p_1) \in A \) and \( \phi(p_t) \in B \).

**Proof:** We break this into two cases. Suppose first that \( Q \) is a chord of \( C \). Since \(|A| \geq 2\) and \( B \) is nonempty, there is an \( L \)-coloring \( \phi \) of \( \{p_1, p_t\} \) with \( \phi(p_1) \in A \) and \( \phi(p_t) \in B \). By Claim 7.9, \( \phi \in \text{Link}(p_1 P p_t) \), so we are done in that case.

Now suppose that \( Q \) is a 2-chord of \( C \). In this case, we have \( Q = p_1 x p_t \) for some \( x \in V(G \setminus C) \). Since \(|A| + |B| \geq 4\)
and each of $A, B$ is nonempty, it follows from Theorem 5.4 that there is an $L$-coloring $\phi$ of $\{p_1, p_t\}$ such that any extension of $\phi$ to an $L$-coloring of $\{p_1, x, p_t\}$ also extends to an $L$-coloring of all of $G$. Since $V(G^+ \setminus Q) = \text{Sh}^2(Q)$, we have $\phi \in \text{Link}(p_1Pp_t)$. □

Let $\phi$ be as in Claim 7.14. Since $\phi(p_t) \in B$ there is a $\sigma \in \mathcal{F}$ with $\sigma(p_t) = \phi(p_t)$. By Claim 7.13, $\sigma \cup \phi \in \text{Link}(P)$, contradicting our assumption that no element of $\text{Link}(P)$ uses a color of $A$ on $p_1$ and a color of $A'$ of $p_r$. This completes the proof of Theorem 7.5. □

8 Uniquely $k$-Determined Collars

We begin by introducing a “global” analogue of the definition of $k$-shortness from the start of Section 7.

Definition 8.1. Let $k \geq 1$ be an integer and let $\mathcal{K} = [\Sigma, G, C, L]$ be a collar. We say that $\mathcal{K}$ is uniquely $k$-determined if $\text{fw}(G) \geq k + 2$ and the following hold.

1) For each $v \in V(C)$, every facial subgraph of $G$ containing $v$, except possibly $C$, is a triangle; AND

2) For any generalized chord $Q$ of $C$ of length at most $C$, letting $G = G_0 \cup G_1$ be the natural $(C, Q)$-partition of $G$, there is an $i \in \{0, 1\}$ such that

a) $G_i$ is contractible and every vertex of $G_i \setminus C$ has an $L$-list of size at least five; AND

b) Either $G_{1-i}$ contains a non-contractible cycle or $G_{1-i} \setminus C$ contains a vertex $v$ with $|L(v)| < 5$.

Note that, in the setting above, the index $i$ is uniquely specified by Definition 8.1, and, in particular, each vertex of $Q \setminus C$ has an $L$-list of size at least five. In view of Definition 8.1, it is natural to introduce the following notation.

Definition 8.2. Let $k \geq 1$ be an integer and let $\mathcal{K} = [\Sigma, G, C, L]$ be a collar, where $\mathcal{K}$ is uniquely $k$-determined.

1) For any (not necessarily proper) generalized chord $Q$ of $C$ with $|E(Q)| \leq k$, we define subgraphs $G^\text{small}_Q$ and $G^\text{large}_Q$ of $G$, where $G^\text{small}_Q \cup G^\text{large}_Q$ is the natural $(C, Q)$-partition of $G$, and furthermore

a) $G^\text{small}_Q$ is contractible and every vertex of $G \setminus C$ has an $L$-list of size at least five; AND

b) Either $G^\text{large}_Q$ contains a noncontractible cycle or there exists a $v \in V(G^\text{large}_Q)$ with $|L(v)| < 5$.

2) We define $\text{Sh}^k(C)$ to be the union of all sets of the form $V(G^\text{small}_Q \setminus Q)$, where $Q$ runs over all generalized chords $Q$ of $C$ with $|E(Q)| \leq k$.

We also sometimes refer to $G^\text{small}_Q$ as the small side of $Q$ in $G$ and $G^\text{large}_Q$ as the large side of $Q$ in $G$. When we construct the deletion set we need in order to prove Theorem 4.1, there is a special case where, given a critical mosaic $T$ and a $C \in C$, we consider a proper 4-chord of $C$ of the form $Q = p_0q_0zq_1p_1$ such that $q_0q_1 \in E(G)$, but we cannot leave
the edge \( q_0q_1 \) behind, and, since \( q_0q_1 \in E(G) \), the graph \( G_Q^{\text{small}} \setminus \{q_0, q_1\} \) is not connected. To deal with this, we introduce the following terminology.

**Definition 8.3.** Let \( \mathcal{K} = [\Sigma, G, C, L] \) be a uniquely 4-determined collar. Given a \( w \in D_2(C) \), we say that \( w \) is degenerate if \( G[N(w) \cap D_1(C)] \) is a path of length at most one. Otherwise we say that \( w \) is non-degenerate. We define a \( w \)-enclosure in \( \mathcal{K} \) to be a subgraph \( Q \) of \( G \) of the following form.

1) If \( w \) is non-degenerate in \( \mathcal{K} \), then \( Q \) is a proper 4-chord of \( C \) with midpoint \( w \), where the endpoints of \( Q \setminus C \) are not adjacent.

2) If \( w \) is degenerate in \( \mathcal{K} \), then
   
   a) \( Q \) is a (not necessarily proper) 5-chord of \( C \), where each vertex of \( Q \setminus C \) lies in \( D_1(C) \) and \( N(w) \cap D_1(C) \subseteq V(Q) \); AND
   
   b) For each internal vertex \( v \) of the 3-path \( Q \setminus C \), every edge from \( v \) to \( C \) lies in \( E(G_Q^{\text{small}}) \).

In the setting above, if the collar \( \mathcal{K} \) is clear from the context, then we just call \( Q \) a \( w \)-enclosure. Definition 8.3 is illustrated in Figure 8.1, where, for each \( k = 1, 2, 3 \), we let \( Q_k \) denote the path in red of distance at most one from \( w_k \). Then, for each \( k = 1, 2, 3 \), \( Q_k \) is a \( w_k \)-enclosure and \( H_k = G_{Q_k}^{\text{small}} \).

![Figure 8.1](image-url)

We note that, in the setting of Definition 8.1, in the special case where \( Q \) is an improper generalized chord (i.e. a cycle), the definition does not require that \( G_Q^{\text{small}} \) only intersects with \( Q \) on a lone vertex, i.e. it is possible that \( G_Q^{\text{small}} \cap C \) is all
of $C$, but, under some conditions, we can show that this is not possible.

**Observation 8.4.** Let $\mathcal{K} = [\Sigma, G, C, L]$ be a uniquely 4-determined collar, where $G$ is short-separation-free. For each $w \in D_2(C)$ and any $w$-enclosure $Q$, the graph $G^\text{small}_Q \cap C$ is a path, i.e. $G^\text{small}_Q \cap C \neq C$.

**Proof.** Recall that, by Definition 7.1, $\text{ew}(G) \geq \text{fw}(G) \geq 7$. Suppose $H \cap C$ is not a path. Thus, $Q$ is an improper generalized chord of $C$ of length either 4 or 5, and $G^\text{small}_Q \cap C = C$. Since $\mathcal{K}$ is uniquely 4-determined, $G^\text{small}_Q$ is contractible and each vertex of $G^\text{small}_Q$, including the vertices of $Q$, has an $L$-list of size at least five. Since $Q$ is an improper generalized chord of $C$, it follows from Definition 8.3 that there is a triangle of $G$ which separates from $C$ either a noncontractible cycle of length at least 6 or a vertex with an $L$-list of size less than five, contradicting short-separation-freeness. □

**Definition 8.5.** Let $\mathcal{K} = [\Sigma, G, C, L]$ be a uniquely 4-determined collar. Let $uw \in E(G)$, where $u \in D_3(C)$ and $w \in D_2(C)$. Let $Q$ be a $w$-enclosure and set $H = G^\text{small}_Q$. A $(Q, uw)$-target in $\mathcal{K}$ is a partial $L$-coloring $\psi$ of $V(Q + uw) \cup V(H \cap C)$ such that

1) both $u$ and the endpoints of $H \cap C$ lie in $\text{dom}(\psi)$, and furthermore, $V(H + uw)$ is $(L, \psi)$-inert in $G$; AND

2) Every vertex of $Q \setminus \text{dom}(\psi)$ and every vertex of $D_1(Q + uw) \setminus V(C)$ has an $L_\psi$-list of size at least three; AND

3) If $w$ is non-degenerate, then $\text{dom}(\psi) \cap V(Q \setminus C) = \{w\}$. Finally, if $w$ is non-degenerate, then every vertex of $\text{dom}(\psi) \cap V(Q \setminus C)$ is either a neighbor of $w$ or an endpoint of the middle edge of $Q$.

Note that, in the setting of Definition 8.5, $H \cap C$ is indeed a path by Observation 8.4, so everything above is well-defined. In the setting above, if the collar $\mathcal{K}$ is clear from the context, then we just refer to $\psi$ as a $(Q, uw)$-target. We now state and prove the main result of Section 8.

**Theorem 8.6.** Let $[\Sigma, G, C, L]$ be a uniquely 4-determined collar, where $G$ is short-separation-free and each vertex of $B_4(C) \setminus V(C)$ has an $L$-list of size at least five. Let $uw \in E(G)$, where $u \in D_3(C)$ and $w \in D_2(C)$. Suppose there is a $w$-enclosure $Q$ such that

1) $u, w \notin \text{Sh}^4(C)$ and each internal vertex of the path $G^\text{small}_Q \cap C$ has an $L$-list of size at least three; AND

2) Letting $p, p'$ be the endpoints of the path $G^\text{small}_Q \cap C$, each of $L(p)$ and $L(p')$ is nonempty, and furthermore, either $p = p'$ or $|L(p)| + |L'(p)| \geq 4$.

Then there is an $(Q, uw)$-target.

**Proof.** Given a partial $L$-coloring $\phi$ of $G^\text{small}_Q$, we define $U_0$ to be the set of vertices of $G \setminus (G^\text{small}_Q \cup C)$ which have at least three neighbors among $\text{dom}(\phi) \cup \{w_2, w_3\}$. We note that, since $u \notin \text{Sh}^4(C)$, we have $u \notin V(G^\text{small}_Q)$. We first deal with the case where $Q$ is a proper 4-chord of $C$. 22
Subclaim 8.6.1. If $Q$ is a proper 4-chord of $C$, then there is a ($Q$, $uw$)-target.

**Proof:** Suppose that $Q$ is a proper 4-chord of $C$. By definition, $w$ is the midpoint of $Q$ in this case, so $Q = pqwq'w'$ for some vertices $q, q' \in D_1(C)$. Let $D$ be the facial cycle $(G^\text{small}_Q \cap C) + Q$ of $G^\text{small}_Q$. Since $w \in D_2(C)$, the three vertices $q, w, q'$ have no common neighbor in $D \setminus Q$. Thus, by Theorem 5.8, there is a partial $L$-coloring $\phi$ of $V(D) \setminus \{q, q'\}$ such that each of $q, q'$ has an $L_\phi$-list of size at least three, where $p, p', w \in \text{dom}(\phi)$ and any extension of $\phi$ to an $L$-coloring of $V(D)$ extends to $L$-color all of $G^\text{small}_Q$. Furthermore, since $u, w \not\in$, any common neighbor of $p, p', w$, if it exists, lies in $G^\text{small}_Q$. Since $q, q'$ are not colored and $u \in D_3(C)$, it follows that $U_\phi = \emptyset$ and $q, q' \not\in N(u)$. In particular, $|L_\phi(u)| \geq 4$ and any extension of $\phi$ to an $L$-coloring of $\text{dom}(\phi) \cup \{u\}$ is a ($Q$, $uw$)-target, so we are done. ■

Thus, for the remainder of the proof of Theorem 8.6, we suppose that $Q$ is not a proper 4-chord of $C$. By definition, $Q$ is a (not necessarily proper) 5-chord of $C$, and $w$ is a degenerate vertex of $D_2(C)$, so $G[N(w) \cap D_1(C)]$ is a subpath of $Q - p$ of length at most one. We let $Q = pq\overline{z}z'q'p'$ for some vertices $q, z, z', q' \in D_1(C)$.

Subclaim 8.6.2. If $Q$ is an improper 5-chord of $C$, then there is a ($Q$, $uw$)-target.

**Proof:** Suppose that $Q$ is an improper 5-chord of $C$. Thus, $p$ is the lone vertex of $G^\text{small}_Q \cap C$. Since each vertex of $Q$ lies in $B_1(C)$, it follows from 2) of Definition 8.3 that $E(\{z, z'\}, V(C))$ consists precisely of $zp, z'p$. Since $G$ is short-separation-free, it follows from our triangulation conditions that $G^\text{small}_Q$ consists precisely of the cycle $Q$ and the chords $zp, z'p$ of $Q$. Thus, there is an edge of the form $px$ for some $x \in V(Q - p)$, where $x \in N(w)$, and there is an $L$-coloring $\phi$ of $px$. Each vertex of $Q \setminus \{p, x\}$ has an $L_\phi$-list of size at least three.

Possibly $G[N(w) \cap D_1(C)]$ is an edge of $Q$, but, in any case, $w$ has at most one neighbor in $Q - x$, and, since $|L_\phi(w)| \geq 4$, $\phi$ extends to an $L$-coloring $\psi$ of $\{p, x, w\}$ such that each vertex of $Q \setminus \{p, x\}$ has an $L_\psi$-list of size at least three. Since $G$ is short-separation-free and, in particular, $K_{2,3}$-free, we have $|U_\psi| \leq 1$, and any vertex of $U_\psi$, if it exists, is adjacent to $x, w, u$. Since $|L_\psi(u)| \geq 4$, $\psi$ extends to an $L$-coloring of $\text{dom}(\psi) \cup \{u\}$ which is a ($Q$, $uw$)-target. ■

Thus, for the remainder of the proof of Theorem 8.6, we suppose that $Q$ is a proper 5-chord of $C$, i.e $p \neq p'$, so we can apply Theorem 6.1. Let $D$ be the facial cycle $(G^\text{small}_Q \cap C) + Q$ of $G^\text{small}_Q$.

Subclaim 8.6.3. If $N(w) \cap D_1(C) \subseteq \{z, z'\}$, then there is ($Q$, $uw$)-target.

**Proof:** Suppose that $N(w) \cap D_1(C) \subseteq \{z, z'\}$. By Theorem 6.1, there is a partial $L$-coloring $\phi$ of $V(D) \setminus \{q, q'\}$, where $p, p' \in \text{dom}(\phi)$ and $N(w) \cap D_1(C) \subseteq \text{dom}(\phi)$, such that

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1) Each vertex of $Q \setminus$ has an $L_{\phi}$-list of size at least three; AND

2) Any extension of $\phi$ to an $L$-coloring of $\text{dom}(\phi) \cup V(Q)$ extends to $L$-color all of $G_Q^{\text{small}}$.

Now consider the following cases.

Case 1: $N(w) \cap D_1(C) = \{z', z'\}$

In this case, if $\{z, z'\} \not\subseteq \text{dom}(\phi)$, then $|U_\phi| \leq 1$, and it is immediate that $\phi$ extends to an $L$-coloring of $\text{dom}(\phi) \cup \{w, u\}$ which is a $(Q, uw)$-target, so now suppose that $\{z, z'\} \subseteq \text{dom}(\phi)$. Now, we have $|U_\phi| \leq 2$, and, for each $x \in U_\phi$, the set $N(x) \cap (\text{dom}(\phi) \cup \{u, w\})$ is either $\{z, w, u\}$ or $\{z', w, u\}$. If $|U_\phi| \leq 1$, then, since $|L_\phi(u)| \geq 5$, it is immediate that there exists an extension of $\phi$ to an $L$-coloring of $\text{dom}(\phi) \cup \{u, w\}$ which is a $(Q, uw)$-target, so suppose now that $|U_\phi| = 2$. Since $G$ is $K_{2,3}$-free, there exist vertices $x, y$ with $U_\phi = \{x, y\}$, where $y \in N(z')$ and $x \in N(z)$. Since $G$ is short-separation-free, the graph $G[N(w)]$ is the 5-cycle $zz'yux$. Since $|L_\phi(u)| \geq 5$ and $|L_\phi(w)| \geq 3$, $\phi$ extends to an $L$-coloring $\psi$ of $\text{dom}(\phi) \cup \{u\}$ such that $|L_\psi(w)| \geq 4$, so $\{w\}$ is $(L, \psi)$ inert in $G$. Furthermore, $U_\psi = U_\phi$, and, since we have not colored $w$, every vertex of $U_\psi$ has an $L_\psi$-list of size at least three, and $\psi$ is a $(Q, uw)$-target.

Case 2: $N(w) \cap D_1(C) \neq \{z, z'\}$

In this case, we suppose without loss of generality that $z' \notin N(w)$. Possibly both of $z, z'$ are colored by $\phi$. Since $N(w) \cap D_1(C) = \{z\}$, the path $P = uzz'$ is an induced path in $G$. Since $G$ is short-separation-free, it follows from our triangulation conditions that, for each $x \in U_\phi$, the graph $G[N(x) \cap (\text{dom}(\phi) \cup \{u, w\})]$ is a subpath of $P$. In particular, $|U_\phi| \leq 2$ and $|U_\phi \cap N(u)| \leq 1$. Furthermore, we have $|L_\phi(w)| \geq 4$ and $|L_\phi(u)| \geq 5$, so it is immediate that $\phi$ extends to an $L$-coloring of of $\{u, w\}$ which is a $(Q, uw)$-target.

Applying Subclaim 8.6.3, we suppose now for the remainder of the proof of Theorem 8.6 that $N(w) \cap V(Q) \not\subseteq \{z, z'\}$. Since $G[N(w) \cap D_1(C)]$ is an edge of $qzz'q'$, we suppose without loss of generality that $q \in N(w)$ and $G[N(w) \cap D_1(C)]$ is a subpath of $qz$. By definition of $\phi$, we have $z \in \text{dom}(\phi)$. Now consider the following cases.

Case 1: $N(w) \cap V(Q) = \{q, z\}$

In this case, we leave $q$ uncolored. Since $|L_\phi(q)| \geq 3$ and $|L_\phi(w)| \geq 4$, $\phi$ extends to an $L$-coloring $\psi$ of $\text{dom}(\phi) \cup \{w\}$ such that $|L_\psi(q)| \geq 3$, so each vertex of $Q \setminus \text{dom}(\psi)$ has an $L_\psi$-list of size at least three. Now, $|U_\psi| \leq 1$, and, for any $x \in U_\phi$, if it exists, we have $N(x) \cap (\text{dom}(\psi) \cup \{u\}) = \{z, w, u\}$. Since $|L_\psi(u)| \geq 4$, $\psi$ extends to an $L$-coloring of $\text{dom}(\psi) \cup \{u\}$ which is a $(Q, uw)$-target.

Case 2: $N(w) \cap V(Q) \neq \{q, z\}$

In this case, since $N(w) \cap V(Q) \not\subseteq \{z, z'\}$, we have $N(w) \cap V(Q) = \{q\}$, and $\phi$ extends to an $L$-coloring $\psi$ of
dom(φ) ∪ {q}. Since z ∈ D_1(C) and every edge from z to C lies in G, it follows that q is adjacent to neither z' nor q', so each vertex of Q \ dom(ψ) has an L_ψ-list of size at least three. Furthermore, every extension of ψ to an L-coloring of dom(ψ) ∪ V(Q) extends to L-color all of G^{smal}. Now, |U_ψ| ≤ 2 and |U_ψ ∩ N(u)| ≤ 1. Since |L_ψ(w)| ≥ 4 and |L_ψ(u)| ≥ 5, ψ extends to an L-coloring of dom(ψ) ∪ {w, u} which is a (Q, uw)-target. □

When we perform the main step in the proof of Theorem 4.1 on a critical mosaic, we consider a long path P = w_0w_1 · · · w_ℓ which is a proper generalized chord of some open ring C, and we partially color and delete a w_2-enclosure and a w_{ℓ−2}-enclosure which are each “maximal” in a certain sense which we make precise below.

**Definition 8.7.** Let K = [Σ, G, C, L] be a uniquely 4-determined collar and let w ∈ D_2(C). Given a w-enclosure Q, we define the following.

1) We say that Q is central if either w is non-degenerate, or, if w is degenerate, then N(w) ∩ D_1(C) is contained in the middle edge of the 3-path Q \ C.

2) We say that Q is maximal if there is no w-enclosure Q' such that G_q is a proper subgraph of G_q'.

Note that, in the setting above, if w is degenerate and G[N(w) ∩ D_1(C)] is a lone vertex, then there is possibly more than one w-enclosure which is both central and maximal, otherwise there is a unique w-enclosure which is both central and maximal.

**9 Black Boxes from [5] and [6]***

We now state some results which we proved in Papers I and II of this sequence and which we need for the proof of Theorem 4.1. We first have the following result which we proved in Section 7 of [5] and makes precise the intuition that, if an embedding has high edge-width and consists mostly of triangles, then it also has high face-width.

**Proposition 9.1.** Let Σ be a surface and let G be a 2-cell embedding on Σ. Then the following hold.

1) Let F be a noncontractible cycle and let F be a family of facial subgraphs of G, where F is contained in ∪ F.
   a) If every element of F is a triangle, then ew(G) ≤ |F| + 2; AND
   b) If there is a D ∈ F such that each D' ∈ F \ {D} is a triangle, then |E(F) \ E(D ∩ F)| ≤ |F| + 2; AND

2) Let α ≥ 2 be a constant and let C be a family of facial subgraphs of G, where each facial subgraph of G, other than those of C, is a triangle. Let D ∈ C be of distance at least α from each element of C \ {D}. Let F' ⊆ G be a cycle with V(F' ∩ D) ≠ ∅. Then either F' contains no vertices of any element of C \ {D} or F' cannot be contained in the union of fewer than 2(α − 1) facial walks of G.
We now restate some facts about critical mosaics that we proved in [5] and which we need for the proof of the main result of this paper. In Section 6 and 11 respectively of [5], we proved the following facts.

**Proposition 9.2.** Given a critical mosaic \((\Sigma, G, C, L, C_*)\), the underlying graph \(G\) is 2-connected and a 2-cell embedding, and:

1) For each \(C \in \mathcal{C}\) and \(v \in V(C \setminus P_C)\), \(|L(v)| = 3\); AND

2) For each \(v \in V(G) \setminus (\bigcup_{C \in \mathcal{C}} V(C))\), \(|L(v)| = 5\) and \(\deg(v) \geq 5\).

Furthermore, each \(C \in \mathcal{C}\) is a chordless, and \(|\mathcal{C}| + g(\Sigma) > 1\).

In Section 12 of [5], we proved that, given a critical mosaic \(\mathcal{T}\) and a ring \(C \in \mathcal{C}\), any generalized chord of \(C\) has “small sides” as long as the this generalized chord is not too long. That is, we have the following result from [5].

**Theorem 9.3.** Let \(\mathcal{T} = (\Sigma, G, C, L, C_*)\), let \(C \in \mathcal{C}\), and let \(Q\) be a generalized chord of \(Q\) with \(|E(Q)| \leq \frac{N_m}{3}\). Let \(G = G_0 \cup G_1\) be the natural \(Q\)-partition of \(G\). Then there exists a unique \(j \in \{0, 1\}\) such that

A) \(G_{1-j}\) is contractible and each \(C' \in \mathcal{C} \setminus \{C\}\) lies in \(G_j\); AND

B) If \(C\) is an open ring, then the following hold.

i) \(|E(Q)| + |E(P_C \cap G_j)| > \frac{2N_m}{3}\). In particular, if \(Q\) is disjoint to \(P_C\), then \(P_C \subseteq G_j\); AND

ii) If \(Q\) is a proper generalized chord of \(C\) and both endpoints of \(Q\) lie in \(P_C\), then \(P_C \cap G_{1-j}\) has one connected component and \(P_C \cap G_j\) has two connected components.

In the language of Section 8, given a critical mosaic \(\mathcal{T} = (\Sigma, G, C, L, C_*)\) and a \(C \in \mathcal{C}\), the tuple \([\Sigma, G, C, L]\) is a collar, and furthermore, Proposition 9.2 and Theorem 9.3 together imply that \([\Sigma, G, C, L]\) is uniquely \(N_m/3\)-determined. We also proved the following in Section 12 of [5], which provides some information about the structure of a minimal counterexample near the precolored paths.

**Theorem 9.4.** Let \(\mathcal{T} = (\Sigma, G, C, L, C_*)\) be a critical mosaic and \(C \in \mathcal{C}\) be an open ring. Then \(|E(P_C)| = \frac{2N_m}{3}\) and furthermore, for any \(x, y \in V(P_C)\), there is no \((x, y)\)-path in \(G\) of length at most \(\min\left\{|E(xP_Cy)|, \frac{N_m}{3}\right\}\), except possibly the path \(xP_Cy\). In particular, for each \(v \in D_1(C)\) with a neighbor in \(P_C\), the graph \(G[N(v) \cap V(P_C)]\) is a path of length at most one.

As part of the proof of Theorem 1.7, given a graph \(G\), a list-assignment \(L\) for \(G\), and a path \(P\) in \(G\), we sometimes want to find an \(L\)-coloring \(\psi\) of \(P\) such that \(G\) does not have too many vertices of distance one from \(P\) with \(L_\psi\)-lists of size less than three. We proved such a result in Paper II ([6]). We recall the following notation from [6].

**Definition 9.5.** Let \(G\) be a graph, let \(L\) be a list-assignment for \(V(G)\), let \(H\) be a subgraph of \(G\), and let \(U \subseteq V(H)\).
We let\( \text{Avoid}^{0}_{G,L}(H \mid U) \) denote the set of \( L \)-colorings \( \phi \) of \( V(H \setminus U) \) such that \( U \) is \((L, \phi)\)-inert in \( G \), and, for every \( v \in D_1(H) \), either \(|L(v)| < 5\) or \(|L_\phi(v)| \geq 3\).

We usually drop the subscripts \( G,L \) from the notation above if these are clear from the context. In the special case where \( X = \emptyset \), we drop the \( X \) from the notation above and write \( \text{Avoid}^{0}(H) \). We now define the following.

**Definition 9.6.** Given a graph \( G \) embedded on a surface \( \Sigma \) and a list-assignment \( L \) for \( G \), a **filament** is a tuple \((P,T,T',f)\), where

1) \( P \subseteq G \) is a shortest path between its endpoints and \( T, T' \) are disjoint, nonempty terminal subpaths of \( P \); AND

2) \( P \setminus (T \cup T') \) is a path of length at least 30 and each vertex of distance at most one from \( P \setminus (T \cup T') \) has an \( L \)-list of size at least five; AND

3) Every facial subgraph of \( G \) which contains a vertex of distance at most one from \( P \setminus (T \cup T') \) is a triangle, and \( f \in \text{Avoid}^{0}_{G,L}(T \cup T') \).

The key result we need about coloring and deleting paths is Theorem 9.7 below, which we proved in [6].

**Theorem 9.7.** Let \( \Sigma \) be a surface and let \( G \) be a short-separation-free embedding on \( \Sigma \), where \( \text{ew}(G) > 4 \) and \(|V(G)| > 5\). Let \( L \) be a list-assignment for \( G \). Then, for any filament \((P,T,T',f)\), there exists a subgraph \( K \) of \( G \) and a \( U \subseteq V(K) \) such that

1) \( P \subseteq K \) and \( V(K \setminus P) \subseteq B_1(P \setminus (T \cup T')) \); AND

2) \( d(K \setminus P, T \cup T') \geq 3 \) and \( f \) extends to an element of \( \text{Avoid}^{0}(K \mid U) \).

10 **Completing the Proof of Theorem 4.1**

With all the machinery above in hand, we complete the proof of Theorem 4.1, which we restate below.

**Theorem 4.1.** Let \( \mathcal{T} = (\Sigma,G,C,L,C_\star) \) be a critical mosaic and let \( g = g(\Sigma) \). Then \( \text{fw}(G) > 4.21 \beta \cdot 6^{g-1} \).

**Proof.** We break the proof into several subsections. Suppose toward a contradiction that \( \text{fw}(G) \leq 4.21 \beta \cdot 6^{g-1} \).

10.1 Knots of open rings

**Claim 10.1.1.** There exists a \( \tilde{C} \in C \) and a generalized chord \( P \) of \( C \) such that \(|E(P)| \leq 4.21 \beta \cdot 6^{g-1} \), where each cycles of \( \tilde{C} \cup P \), except for \( \tilde{C} \), is contractible. Furthermore, any such \( P \) is a proper generalized chord of \( \tilde{C} \).

**Proof:** We first prove the following intermediate result.
10.1 Knots of open rings

Subclaim 10.1.1.1. If there is a noncontractible cycle $D$ of $G$ contained in at most $4.21 \beta \cdot 6^{g-1}$ facial subgraphs of $G$, then there is a $\tilde{C} \in \mathcal{C}$ such that $|E(D \cap \tilde{C})| \geq 2$ and $D$ contains no vertices of any element of $\mathcal{C} \setminus \{\tilde{C}\}$.

Proof: If $V(D)$ has no intersection with $\bigcup_{C \in \mathcal{C}} V(C)$, then, since $G$ is a 2-cell embedding, it follows from 1) of Proposition 9.1, together with the triangulation conditions on $\mathcal{T}$, that $\text{ew}(G) \leq 4.21 \beta \cdot 6^{g-1} + 2$, contradicting our edge-width conditions on $\mathcal{C}$. Thus, there is a $\tilde{C} \in \mathcal{C}$, where $V(\tilde{C} \cap D) \neq \emptyset$. Suppose there is a $C \in \mathcal{C} \setminus \{\tilde{C}\}$ with $V(C \cap D) \neq \emptyset$. Possibly $C_{\circ} \in \{C, \tilde{C}\}$, but, in any case, by our distance conditions on $\mathcal{T}$, we have $d(C, \tilde{C}) \geq 2.9 \beta \cdot 6^{g-1}$, and thus, by 2) of Proposition 9.1, we get $4.21 \beta \cdot 6^{g-1} \geq 2(2.9 \beta \cdot 6^{g-1} - 1)$, which is false. Thus, $V(D)$ intersects with no elements of $\mathcal{C} \setminus \{\tilde{C}\}$. If $|E(D \cap \tilde{C})| \leq 1$, then there is a family of at most $4.21 \beta \cdot 6^{g-1}$ facial walks containing $D$, each of which is a triangle, and then it again follows from 1) of Proposition 9.1 that $\text{ew}(G) \leq (4.21 \beta \cdot 6^{g-1}) + 2$, which is false. ■

Since $\text{fw}(G) \leq 4.21 \beta \cdot 6^{g-1}$ and $G$ is a 2-cell embedding, there is a noncontractible cycle $D \subseteq G$ such that $D$ is contained in at most $4.21 \beta \cdot 6^{g-1}$ facial walks of $G$. Let $\tilde{C}$ be as in Subclaim 10.1.1.1. Now, $|E(D \cap \tilde{C})| \geq 2$ and every connected component of $D \setminus E(\tilde{C})$ which is not an isolated vertex is a proper generalized chord of $\tilde{C}$. By 1) of Proposition 9.1, each such generalized chord has length at most $4.21 \beta \cdot 6^{g-1}$. As $\tilde{C}$ is a contractible cycle, there is a component $P$ of $D \setminus E(\tilde{C})$ which is a generalized chord of $\tilde{C}$ satisfying Claim 10.1.1. It also follows from 2) of Subclaim 10.1.1.1 that any generalized chord $P$ of $\tilde{C}$ satisfying Claim 10.1.1 is a proper generalized chord of $\tilde{C}$. ■

Let $\tilde{C} \in \mathcal{C}$ be as in Claim 10.1.1. We now introduce the following terminology.

Definition 10.1.2. A knot is a generalized chord $P$ of $\tilde{C}$ such that each of the cycles of $\tilde{C} \cup P$ which is distinct from $\tilde{C}$ is noncontractible. We say that $P$ is a short knot if it is of minimal length among all knots.

Claim 10.1.3. Any short knot is a proper generalized chord of $\tilde{C}$, and, in particular, a shortest path in $G$ between its endpoints.

Proof: Let $P$ be a knot. By Claim 10.1.1, $P$ is a proper generalized chord of $\tilde{C}$. Suppose toward a contradiction that $P$ is not a shortest path between its endpoints. Thus, there is another path $P'$ in $G$ with the same endpoints as $P$, where $|E(P')| < |E(P)|$. Note that $P'$ possibly intersects with $\tilde{C}$ on many vertices, so $P'$ is not necessarily a generalized chord of $\tilde{C}$, but, in any case, it follows from the minimality of $P$ that no subpath of $P'$ is a knot of $\tilde{C}$, so there is a noncontractible cycle contained in $P \cup P'$. Since $|E(P \cup P')| \leq 2(4.21 \beta) \cdot 6^{g-1} < 2.1 \beta \cdot 6^g$, this contradicts the edge-width conditions satisfied by $\mathcal{T}$. ■

Applying Claim 10.1.3, together with our face-width conditions on $\mathcal{T}$ and the minimality of $P$, we have the following.

Claim 10.1.4. $\tilde{C}$ is an open ring and, for any short knot $P = w_0 \cdots w_t$, the following hold.
10.1 Knots of open rings

1) \(2.1\beta \cdot 6^{g-1} \leq |E(P)| + 1; AND\)

2) For each \(x \in V(P) \setminus \{w_0, w_1, w_2, w_{\ell-2}, w_{\ell-1}, w_{\ell}\}\), we have \(d(x, Sh^4(\tilde{C}) \cup B_1(\tilde{C})) > 1\).

The deletion we perform to prove Theorem 4.1 is essentially shown in Figure 10.1. Let \(B\) be the \((w, w')\)-path in blue edges shown above in Figure 10.1, where each of \(w, w'\) lies in \(D_2(\tilde{C})\), and \(Q\) is a \(w\)-enclosure which is both central and maximal, and, likewise, \(Q'\) is a \(w\)-enclosure which is both central and maximal. The paths \(Q\) and \(Q'\) are indicated in red, and our deletion set consists of some or all of \(\tilde{C}\), together with the union of \(B \cup G^{\text{small}}_Q \cup G^{\text{small}}_{Q'}\), minus some vertices on \(Q \cup Q'\). In an instance like the one shown in Figure 10.1, the crucial part of our construction is that we do not delete the vertices of the red paths lying in \(D_1(\tilde{C})\), i.e the vertices of the red paths which are adjacent to either \(w\) or \(w'\).

In Claim 10.2.2 we show that, when we perform a deletion like the one above, the resulting embedding on the new surface satisfies the distance conditions necessary for each connected component to be the underlying graph of a mosaic. To embed the graph obtained by this deletion on a surface of lower genus than \(\Sigma\), we introduce the following.

**Definition 10.1.5.** Given a noncontractible closed curve \(N \subseteq \Sigma\), we associate to \(N\) a (not necessarily connected) surface \(\Sigma_N\), where \(\Sigma_N\) is obtained from \(\Sigma \setminus N\) in the following way. For each connected component \(\Sigma'\) of \(\Sigma \setminus N\), we glue open discs to \(\Sigma'\) along \(\partial(\Sigma \setminus N)\) to obtain a surface with empty boundary. Note that \(\Sigma_N\) has either one or two connected components. The following observation is immediate.
Claim 10.1.6. Let $N \subseteq \Sigma$ be a noncontractible closed curve and let $X$ be the subgraph of $G$ consisting of all the edges having nonempty intersection with $N$. Let $G'$ be an embedding of a subgraph $H$ of $G \setminus X$ on $\Sigma_N$ in the natural way. Then every cycle of $H$ which is noncontractible in $G'$ is also noncontractible in $G$.

For an appropriately chosen subgraph $A \subseteq G$ and a noncontractible closed curve $N \subseteq \Sigma$, the embedding of $G \setminus A$ on $\Sigma_N$ satisfies the desired distance conditions to be the underlying graph of a mosaic, since the genus has strictly decreased. We make this precise in Subsection 10.2.

10.2 Facial subgraphs in the new embedding

We first note the following, which is immediate from the definition of a knot.

Claim 10.2.1. Let $P$ be a short knot and let $A$ be a subgraph of $G$ with $V(A) \subseteq B_2(\tilde{C} \cup P) \cup \text{Sh}^6(\tilde{C})$, where $\tilde{C} \cup A$ is connected. Then there is a noncontractible closed curve $N$ of $\Sigma$, where $G \cap N \subseteq A$.

We now have the following, which is our main result for Subsection 10.2.

Claim 10.2.2. Let $P$ be a short knot and let $A$ be a subgraph of $G$ with $V(A) \subseteq B_2(\tilde{C} \cup P) \cup \text{Sh}^6(\tilde{C})$, where $\tilde{C} \cup A$ is connected. Let $N$ be a noncontractible closed curve with $G \cap N \subseteq A$ (where $G, A$ are regarded as subsets of $\Sigma$) and let $G'$ be the embedding of $G \setminus A$ on $\Sigma_N$ in the natural way. Then there is a pair $F_0, F_1$ of facial subgraphs of $G'$ such that all of 1)-5) below hold, where all distances below are in $G \setminus A$.

1) $V(F_0 \cup F_1) = V(\tilde{C} \setminus A) \cup D_1(A)$; AND

2) For each $i \in \{0, 1\}$ and $C \in \mathcal{C} \setminus \{\tilde{C}, C_*\}$, we have $d(F_i, C) \geq \beta \cdot 6^g - 1 + 4N_{mo}$ as well; AND

3) Either $\tilde{C} = C_*$ or, for each $i \in \{0, 1\}$, we have $d(F_i, C_*) \geq 2.9 \beta \cdot 6^g - 2 + 4N_{mo}$; AND

4) $d(F_0, F_1) \geq \beta \cdot 6^g - 1 + 4N_{mo}$; AND

5) $\text{fw}(G') \geq 2.1 \beta \cdot 6^g - 2$ and $\text{ew}(G') \geq 2.1 \beta \cdot 6^g - 1$.

Proof: We first note the following.

Subclaim 10.2.2.1.

i) For each $C \in \mathcal{C} \setminus \{\tilde{C}, C_*\}$, we have $d(C, \tilde{C} \cup A) > \beta \cdot 6^g - 1 + 4N_{mo}$; AND

ii) Either $\tilde{C} = C_*$ or $d(C_*, \tilde{C} \cup A) > 2.9 \beta \cdot 6^g - 2 + 4N_{mo}$.
10.2 Facial subgraphs in the new embedding

**Proof:** Let $C \in \mathcal{C} \setminus \{\hat{C}, C_*\}$. By our distance conditions on $T$, we have $d(C, \hat{C}) \geq \beta \cdot 6^g$. By assumption, $|E(P)| \leq 4.21 \cdot 6^{g-1}$, so each vertex of $B_2(P)$ is of distance at most $2.11\beta \cdot 6^{g-1}$ from $C$. Thus, $d(C, \hat{C} \cup A) \geq d(C, \hat{C}) - 2.11\beta \cdot 6^{g-1}$, so we get $d(C, \hat{C} \cup A) \geq (6 - 2.11)\beta \cdot 6^{g-1} > \beta \cdot 6^{g-1} + 4N_{\max}$. This proves i). Now we prove ii). Suppose that $\hat{C} \neq C_*$. As above, we have $d(C_*, \hat{C} \cup A) \geq d(C_*, C) - 2.11\beta \cdot 6^{g-1}$, and, by our distance conditions on $T$, we have $d(C_*, \hat{C} \cup A) \geq 2.9\beta \cdot 6^{g-1}$, so we get $d(C_*, \hat{C} \cup A) \geq 0.88\beta \cdot 6^{g-1}$. Since $6 \cdot (0.88) > 3$, we have $d(C_*, \hat{C} \cup A) \geq 2.9\beta \cdot 6^{g-2} + 4N_{\max}$, so we are done. 

It follows from Subclaim 10.2.2.1, together with our triangulation conditions, that every facial subgraph of $G$ containing a vertex in $B_1(A)$ is either $\hat{C}$ or a triangle. In particular, letting $U$ be the unique open component of $\Sigma \setminus \hat{C}$ with $\partial(U) = \hat{C}$, it follows from the fact that $\hat{C} \cup A$ is connected that there is a unique open component $U'$ of $\Sigma$ with $U' \subseteq U$ and a unique facial subgraph $F$ of $\Sigma$ such that $\partial(U') = F$, where $V(F) = V(\hat{C} \cup A) / D_1(A)$. Since $V(A) \subseteq B_2(\hat{C} \cup P) \cup Sh^6(\hat{C})$ and $G \cap N \subseteq A$, there is a pair $F_0, F_1$ of distinct facial subgraphs of $G'$, where $F_0 \cup F_1 = F$. In particular, $V(F_0 \cup F_1) = V(\hat{C} \cup A) / D_1(A)$, so $F_0, F_1$ satisfy 1). Subclaim 10.2.2.1 also shows that $F_0, F_1$ satisfy 2)-3) of Claim 10.2.2. We now check 4) and 5). Suppose toward a contradiction that $d(F_0, F_1) < \beta \cdot 6^{g-2} + 4N_{\max}$ in $G \setminus A$. Let $Q$ be a shortest path in $G \setminus A$ from $F_0$ to $F_1$.

**Subclaim 10.2.2.2.** At most one endpoint of $Q$ lies outside of $B_3(\hat{P})$.

**Proof:** Suppose each endpoint of $Q$ lies outside of $B_3(\hat{P})$. Thus, each endpoint of $Q$ has a neighbor in $(A \setminus B_2(\hat{P})) \cup \hat{C}$. Since $V(A) \subseteq B_2(\hat{C} \cup P) \cup Sh^6(\hat{C})$, it follows that there is a noncontractible cycle $D \subseteq G$ which is contained in at most $|E(Q)| + 6$ facial walks of $G$. Since $G$ is a 2-cell embedding and $|E(Q)| \leq \beta \cdot 6^{g-2} + 4N_{\max}$, we contradict our face-width conditions on $G$. ■

Applying Subclaim 10.2.2.2, we let $x, x'$ be the endpoints of $Q$, where $x \in B_3(\hat{P})$.

**Subclaim 10.2.2.3.** $d(x', P) > 3$.

**Proof:** Suppose not. Since $P$ is a shortest path between its endpoints and $x \in B_3(\hat{P})$, it follows that $G$ contains a noncontractible cycle of length at most $2|E(P)| + 6$. Since $|E(P)| \leq 4.21 \cdot 6^{g-1}$, we have $\text{ew}(G) \leq 2(4.21\beta \cdot 6^{g-1}) + 6$, contradicting our edge-width conditions on $T$. ■

Since $x' \notin B_3(\hat{P})$, $x'$ either lies $V(\hat{C} \setminus A)$ or has a neighbor in $B_2(\hat{C}) \cup Sh^6(\hat{C})$. Since $d(x, \hat{P}) \leq 3$, it follows that there is a $(P, \hat{C})$-path $Q^*$ which is disjoint to $P$ except for its lone $P$-endpoint, where $|E(Q^* \setminus Q)| \leq 6$. Let $y$ be the unique $P$-endpoint of $Q^*$ and let $R, R'$ be the two subpaths of $P$ such that $R \cup R' = P$, where $R \cap R' = y$. Each of $R + Q^*$ and $R' + Q^*$ is a knot, and, since $P$ is a short knot, we have $|E(P)| + 2|E(Q^*)| \geq 2|E(P)|$. Thus, we get $|E(Q^*)| \geq |E(P)|/2$. On the other hand, we have $|E(Q^*)| \leq |E(Q)| + 6 < \beta \cdot 6^{g-2} + 4N_{\max} + 6$. By Claim 10.1.4, we have $|E(P)|/2 \geq \beta \cdot 6^{g-1}$. Putting these together, we have $\beta \cdot 6^{g-1} < \beta \cdot 6^{g-2} + 4N_{\max} + 6$, which is false, so our
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Now we prove 5). It is immediate from Claim 10.1.6 that $\text{ew}(G') \geq \text{ew}(G) \geq 21.6^g - 1$. Now suppose toward a contradiction that $\text{ew}(G') < 21.6^g - 2$. Since $G$ is a 2-cell embedding, it follows from our construction of $\Sigma_N$ that each biconnected component of $G'$ is a 2-cell embedding on the connected component of $\Sigma_N$ containing $K$, so there is a biconnected component $K$ of $G'$ and a noncontractible cycle $D \subseteq G'$ which is contained in fewer than $21.6^g - 2$ facial walks of $K$. If $V(D)$ has no intersection with $V(F_0 \cup F_1)$, then $D$ is also contained in at most $21.6^g - 2$ facial walks of $G$, contradicting our face-width conditions on $G$, so there is an $i \in \{0, 1\}$ with $V(D \cap F_i) \neq \emptyset$, say $i = 0$ for the sake of definiteness. Now, each element of $C$ is either disjoint to $K$ or contained in $K$. Let $\mathcal{F}_K$ be the set of facial subgraphs $F$ of $K$ such that $F$ either lies in $C$ or is a subgraph of an element of $\{F_0, F_1\}$. Each facial subgraph of $K$, other than those of $\mathcal{F}_K$, is a triangle. It follows from 2)-4) that any two distinct elements of $\mathcal{F}_K$ are of distance at least $2.916^g - 1 + 4N_{\text{stop}}$ apart. Since $D$ is contained in fewer than $21.6^g - 2$ facial walks of $K$, it follows from 2) of Proposition 9.1 that $V(D)$ has no intersection with any element of $\mathcal{F}_K \setminus \{F_0\}$.

If $|E(D \cap F_0)| \leq 1$, then there is a family of fewer than $21.6^g - 2$ facial triangles of $K$ whose union contains $D$, and then it follows from 1) of Proposition 9.1 that $\text{ew}(G') \leq \text{ew}(K) < 21.6^g - 2 + 2$, which is false, as $\text{ew}(G') \geq \text{ew}(G)$. Thus, $|E(D \cap F_0)| > 1$. By our construction of $\Sigma_N$, we get that $F_i$ is a contractible cycle of $K$, so $D \neq F_0$, and furthermore, each component of $D \setminus E(F_0)$ is either an isolated vertex or a proper generalized chord of $F_0$. Furthermore, there is at least one such chord $\tilde{Q}$ of $F_0$ such that each cycle of $F \cup \tilde{Q}$, except for $F_0$, is noncontractible in $G'$. By 1) of Proposition 9.1, since $|E(D \cap F_0)| > 1$, we have $|E(\tilde{Q})| \leq 21.6^g - 2$. Now, each endpoint of $\tilde{Q}$ lies in $V(\tilde{C} \setminus A) \cup D_1(A)$. We note the following.

**Subclaim 10.2.2.4.** At least one endpoint of $\tilde{Q}$ lies in $B_3(\tilde{P})$.

**Proof:** Suppose not. Thus, for each endpoint $p$ of $\tilde{Q}$, either $p$ lies in $\tilde{C} \setminus A$ or $p$ has a neighbor in $B_2(\tilde{C}) \cup \text{Sh}^6(\tilde{C})$. In any case, there is a knot $Q$ of $\tilde{C}$ such that $|E(Q \setminus \tilde{Q})| \leq 6$. Since $|E(\tilde{Q})| \leq 21.6^g - 2$, it follows from Claim 10.1.4 that we contradict the minimality of $P$. ■

We now let $x, x'$ be the endpoints of $\tilde{Q}$, where, applying Subclaim 10.2.2.4, $x \in B_3(\tilde{P})$. Analogous to Subclaim 10.2.2.3, it is straightforward to check that $d(x', P) > 3$. Since $x' \notin B_3(\tilde{P})$ and $x' \in V(F_0)$, either $x' \in V(\tilde{C} \setminus A)$ or $x'$ has a neighbor in $\text{Sh}^6(\tilde{C}) \cup B_2(\tilde{C})$. In any case, $G$ contains a $(P, \tilde{C})$-path $Q^*$, where $|E(Q^* \setminus \tilde{Q})| \leq 6$, and $G$ contains a knot of length at most $|E(P)| + |E(Q^*)|$. By the minimality of the length of $P$, $|E(Q^*)| \geq \frac{|E(P)|}{2}$, so, by Claim 10.1.4, we have $|E(\tilde{Q})| + 6 \geq \beta \cdot 6^{g-1}$, which is false, as $|E(\tilde{Q})| \leq 21.6^g - 2$. This completes the proof of Claim 10.2.2. ■

In the remaining subsections which make up the proof of Theorem 4.1, when we construct a deletion set $A$ as in Claim 10.2.2 and we partially $L$-color $A$, we need to make sure that the lists of the neighbors of $A$ have at least three colors.
left. Our partial \( L \)-coloring of \( A \) is the union of some individual “parts”, where each these individual parts leaves behind lists of size at least three among the neighbors of \( A \). We just need to have some control over the “overlap” between these parts, i.e the vertices which possibly have a neighbors in more than one part. We make use of the following observation, which is just a consequence of our triangulation conditions.

**Claim 10.2.3.** Let \( w \in D_2(\tilde{C}) \) and let \( Q \) be a \( w \)-enclosure which is both central and maximal. Let \( X \) be a subset of \( V(Q \setminus \tilde{C}) \), where \( X \) contains \( w \) if \( w \) is non-degenerate and otherwise \( X \) contains at least one neighbor of \( w \). Then \( G^\text{small}_Q \setminus (Q \setminus X) \) is connected, and furthermore, \( d(X, \tilde{C} \setminus G^\text{small}_Q) > 2 \).

We now introduce some notation which we use for the remainder of the proof of Theorem 4.1.

**Definition 10.2.4.** We let \( P = P \tilde{C} \) and we denote the path \( \tilde{C} \setminus P \) as \( v_0 \cdots v_n \). We denote that path \( P \) by \( p_1 \cdots p_m \), where \( v_0 p_1, p_m v_n \) are the terminal edges of \( P \).

### 10.3 Enclosures which are non-overlapping with the precolored edges

Subsection 10.3 consists of the following result.

**Claim 10.3.1.** Let \( P = w_0 \cdots w_\ell \) be a short knot. Then there exists a \( k \in \{2, \ell - 2\} \) such that, for any \( w_k \)-enclosure \( Q \) which is both central and maximal, we have \( G^\text{small}_Q \cap \hat{P} \neq \emptyset \).

**Proof:** Suppose not. Thus, there exists a \( w_2 \)-enclosure \( Q \) and a \( w_\ell \)-enclosure \( Q' \), each of which is central and maximal, such that, letting \( H = G^\text{small}_Q \) and \( H' = G^\text{small}_{Q'} \), the graph \( H \cup H' \) has empty intersection with \( \hat{P} \). Now, by Theorem 9.3, each of \( H \cap \tilde{C} \) and \( H' \cap \tilde{C} \) is a subpath of \( C \setminus \hat{P} \). Note that each endpoint of \( H \cap C \) is of distance at most four from \( w_2 \). Likewise, each endpoint of \( H' \cap C \) is of distance at most four from \( w_\ell - 2 \). Furthermore, by 2) of Claim 10.1.4, \( P \setminus \{w_0, w_1, w_2, w_\ell - 2, w_\ell - 1, w_\ell \} \) is disjoint to \( H \cup H' \). Since \( P \) is a shortest path between its endpoints, and \( |E(P)| \leq 4.21 \beta \cdot 6^{g - 1} \), we immediately have the following.

**Subclaim 10.3.1.1.** \( d(H, H') \geq (4.21 \beta \cdot 6^{g - 1}) - 8 \). In particular, the paths \( H \cap \tilde{C} \) and \( H' \cap \tilde{C} \) are disjoint.

For the sake of definiteness, we suppose that there exist indices \( 0 \leq i_0 \leq i_1 \leq j_0 \leq j_1 \leq n \), where \( H' \cap \tilde{C} \) is the path \( v_{i_0} \cdots v_{i_1} \) and \( H' \cap \tilde{C} \) is the path \( v_{j_0} \cdots v_{j_1} \). Now, it follows from Theorem 9.3 that every subpath of \( C \setminus \hat{P} \) is 2-short, so we can apply Theorem 7.5.

**Subclaim 10.3.1.2.** There is a \( \phi \in \text{Link}(v_0 \cdots v_{i_n}) \) and there is a \( \phi' \in \text{Link}(v_{j_1} \cdots v_n) \).
These are symmetric so we just need to prove that there is a \( \phi \in \text{Link}(v_0 \cdots v_{i_0}) \). This is immediate if \( v_0 \cdots v_{i_0} \) is a lone vertex, because then \( |L(v_0)| = |L(v_{i_0})| = 1 \) and \( \text{Sh}^2(v_0 \cdots v_{i_0}) = \emptyset \), so the lone \( L \)-coloring of \( \{v_0\} \) lies in \( \text{Link}(v_0 \cdots v_{i_0}) \). Now suppose that \( i_0 > 0 \). As \( H' \cap \tilde{C} \) is also a subpath of \( C \setminus \tilde{P} \), it follows from Subclaim 10.3.1.1 that each vertex of \( v_0 \cdots v_{i_0} \), except for \( v_0 \), lies outside of \( P \). Since \( |L(v_0)| = 1 \) and each vertex of \( v_1 \cdots v_i \) has an \( L \)-list of size three, it follows from Theorem 7.5 that there is a \( \phi \in \text{Link}(v_0 \cdots v_{i_0}) \).

Let \( \phi, \phi' \) be as in Subclaim 10.3.1.2. Applying Theorem 8.6, we immediately get that there is a \( (Q, w_3w_2) \)-target \( \psi \) such that \( \psi(v_{i_0}) = \phi(v_{i_0}) \) and there is a \( (Q, w_{2\ell-3}w_{2\ell-2}) \)-target \( \psi' \) such that \( \psi'(v_{j_0}) = \phi'(v_{j_0}) \). It follows from Claim 10.2.3 that there is no edge with one endpoint in \( v_0 \cdots v_{i_0-1} \) and the other endpoint in \( H \), except for \( v_{i_0-1}v_{i_0} \). Likewise, there is no edge with one endpoint in \( H' \) and the other endpoint in \( v_{j_1+1} \cdots v_n \), except for \( v_{j_1}v_{j_1+1} \).

Combining this with Subclaim 10.3.1.1, we get that the union \( \phi \cup \psi \cup \psi' \cup \phi' \) is a proper \( L \)-coloring of its domain. By Subclaim 10.3.1.1, the endpoints of the path \( v_{i_1} \cdots v_{j_0} \) are of distance at least \( 4.21 \beta \cdot 6^{\sigma-1} - 8 \). In particular, it follows that there is an edge \( v_av_{a+1} \) of \( v_{i_1+2} \cdots v_{j_0-2} \) with the property that no 2-chord of \( \tilde{C} \) has one endpoint in \( v_{i_1} \cdots v_{m-1} \) and the other endpoint in \( v_{m+2} \cdots v_{j_0} \). It follows from Theorem 7.5 that there is a \( \sigma \in \text{Link}(v_{i_1} \cdots v_{m-1}) \), where \( \sigma(v_{i_1}) = \psi(v_{i_1}) \) and, likewise, there is a \( \sigma' \in \text{Link}(v_{m+2} \cdots v_{j_0}) \), where \( \sigma'(v_{j_0}) = \psi'(v_{j_0}) \). Note that, by our choice of edge \( v_av_{a+1} \), there is no 2-chord of \( \tilde{C} \) with one endpoint in \( \text{dom}(\sigma) \) and the other endpoint in \( \text{dom}(\sigma') \).

It follows from Claim 10.2.3 that there is no edge with one endpoint in \( H \cup H' \) and the other in \( v_{i_0} \cdots v_{j_0} \), except for \( v_{i_0}v_{i_0+1} \) and \( v_{j_0-1}v_{j_0} \), and, since \( \tilde{C} \) is an induced cycle, the union \( \phi \cup \psi \cup \sigma \cup \sigma' \cup \psi' \cup \phi' \) is a proper \( L \)-coloring of its domain. After coloring and deleting the vertices of \( \text{dom}(\phi \cup \psi \cup \sigma \cup \sigma' \cup \psi' \cup \phi') \), each of \( v_a, v_{a+1} \) has at least two colors left over, so \( \phi \cup \psi \cup \sigma \cup \sigma' \cup \psi' \cup \phi' \) extends to an \( L \)-coloring \( \tau \) of \( \text{dom}(\phi \cup \psi \cup \sigma \cup \sigma' \cup \psi' \cup \phi') \cup \{v_a, v_{a+1}\} \).

The idea here is that, after we complete the construction of our deletion set and delete this set, \( \tilde{C} \) splits into two open rings in the mosaic on the new surface, where one of these rings has precolored path \( v_av_{a+1} \). We have now colored all the vertices we need to in \( B_2(\tilde{C}) \cup \text{Sh}^1(\tilde{C}) \), so we need to combine \( \tau \) with a partial \( L \)-coloring of \( P \). By Definition 8.5 applied to \( \psi \) and \( \psi' \), \( \text{dom}(\tau) \) contains \( w_3, w_{2\ell-3} \), so we let \( \tilde{P} \) be the unique subpath of \( w_2Pw_{2\ell-2} \) which contains all the vertices of \( \text{dom}(\tau) \cap V(w_2Pw_{2\ell-2}) \) and whose endpoints lie in \( \text{dom}(\tau) \). Note that \( \tau \) precolors nonzero terminal subpaths of \( \tilde{P} \) of length at most one. By 1) of Claim 10.1.4, \( \tilde{P} \) has length at least \( 2.1 \beta \cdot 6^{\sigma-1} - 7 \). Since \( \tilde{P} \) is a shortest path between its endpoints, it follows from Theorem 9.7 that there is a subgraph \( K \) of \( G \) and a \( U \subseteq V(K) \) such that

1) \( \tilde{P} \subseteq K \) and \( V(K \setminus \tilde{P}) \subseteq B_1(\tilde{P} \setminus \text{dom}(\tau)); \) AND
2) \( d(K \setminus \tilde{P}, V(\tilde{P}) \cap \text{dom}(\tau)) \geq 3 \) and \( \tau|_{\tilde{P}} \) extends to a \( \tau^* \) of \( \text{Avoid}^0(K \mid U) \).

It follows from 2) of Claim 10.1.4 that \( \tau \cup \tau^* \) is a proper \( L \)-coloring of its domain. Now we have enough to construct our deletion set. Our deletion set consists of two parts, the vertices colored by \( \tau \cup \tau^* \) and the “inert” vertices which we can safely remove without coloring them. We let \( S \) be the following union:
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\[ \text{We note that}\]

\[ \text{Let}\]

\[ 10.3 \text{ Enclosures which are non-overlapping with the precolored edges}\]

\[ V \]

\[ \text{we now set}\]

\[ H\]

\[ \tau\]

\[ \text{P}\]

\[ \text{dom}(\tau) \cup V(H \cup H' \cup K) \cup \{w_2, w_{l-2}\}\]

We delete all the vertices of \( S \setminus (V(Q \cup Q') \setminus \text{dom}(\psi \cup \psi')) \), except for the endpoints of \( v_av_{a+1} \), which we keep precolored by \( \tau \). Note that \( \text{dom}(\tau \cup \tau^*) \subseteq S \setminus (V(Q \cup Q') \setminus \text{dom}(\psi \cup \psi')) \). Letting \( S^I = S \setminus (V(Q \cup Q') \setminus \text{dom}(\psi \cup \psi')) \), we now set \( A \) to be the subgraph of \( G \) induced by \( S^I \setminus \{v_a, v_{a+1}\} \). It follows from repeated application of Observation 4.3 that \( V(A) \) is \((L, \tau \cup \tau^*)\)-inert in \( G \). It follows from Claim 10.2.3, together with the definition of \( \psi, \psi' \), that each of \( H \cap A \) and \( H' \cap A \) is connected, so, by our triangulation conditions, \( \tilde{C} \cup A \) is connected. Now, note that \( V(A) \subseteq B_2(\tilde{C} \cup P) \cup \text{Sh}^6(\tilde{C}) \). It now follows from Claim 10.2.1 that there is a noncontractible closed curve \( N \) of \( \Sigma \), where \( N \cap G \subseteq A \). Let \( G' \) be the embedding of \( G \setminus A \) on \( \Sigma_N \) in the natural way and let \( F_0, F_1 \) be as in Claim 10.2.2, i.e \( F_0, F_1 \) are facial subgraphs of \( G' \) with \( V(F_0 \cup F_1) = V(\tilde{C} \setminus A) \cup D_1(A) \). One of \( F_0, F_1 \) contains the truncated precolored path \( P \setminus \{v_0, v_n\} = \tilde{P} \), and the other one contains the edge \( v_av_{a+1} \). Say for the sake of definiteness that \( \tilde{P} \subseteq F_0 \) and \( v_av_{a+1} \in E(F_1) \). Recalling the notation of Definition 2.5, we set \( L^+ = L_{\tau \cup \tau^*} \).

**Subclaim 10.3.1.3.** Each vertex of \( F_0 \cup F_1 \) has an \( L^+\)-list of size at least three, except the vertices of \( V(\tilde{P}) \cup \{v_a, v_{a+1}\} \).

**Proof:** We note that \( A \) contains all of \( \tilde{C} \) except for the paths \( \tilde{P} \) and \( v_av_{a+1} \), so we just need to check the vertices of \( (F_0 \cup F_1) \setminus \tilde{C} \), i.e the vertices of \( F_0 \cup F_1 \) with \( L \)-lists of size five. Let \( v \in V(F_0 \cup F_1) \setminus V(\tilde{C}) \).

**Case 1:** \( x \) has a neighbor in \( \text{dom}(\tau^*) \setminus \text{dom}(\tau) \)

In this case, it follows from 2) of Claim 10.1.4 that \( N(x) \cap \text{dom}(\tau) \subseteq V(\tilde{P}) \cap \text{dom}(\tau) \), so \( N(x) \cap \text{dom}(\tau \cup \tau^*) \subseteq \text{dom}(\tau^*) \), and thus, recalling Definition 9.5, we have \( |L^+(x)| \geq 3 \)

**Case 2:** \( x \) has no neighbor in \( \text{dom}(\tau^*) \setminus \text{dom}(\tau) \)

In this case, we have \( N(w) \cap \text{dom}(\tau \cup \tau^*) \subseteq \text{dom}(\tau) \). We call a partial \( L \)-coloring of \( V(G) \) a \textit{part} if it is one of if it is one of \( \phi, \phi', \psi, \psi', \sigma, \sigma' \). Recall that \( v_av_{a+1} \) was chosen so that there is no 2-chord of \( \tilde{C} \) with one endpoint in \( \text{dom}(\sigma) \) and the other endpoint in \( \text{dom}(\sigma') \). It is also straightforward to check (in particular, applying Claim 10.2.3), that, for any two parts \( p \) and \( p' \), the sets \( \text{dom}(p) \setminus \text{dom}(p') \) and \( \text{dom}(p') \setminus \text{dom}(p) \) are of distance at least three apart in \( G \setminus A \), so we just need to check that, for each part \( p \), we have \( |L^+_p(x)| \geq 3 \). This is immediate from Definition 8.5 if \( p \) is either \( \psi \) or \( \psi' \), and it follows from Observation 7.4 if \( p \) is one of \( \phi, \phi', \sigma, \sigma' \).

Applying Subclaim 10.3.1.3, we have the following.

**Subclaim 10.3.1.4.** \( G' \) is \( L^+\)-colorable.

**Proof:** Let \( G'' \) be a component of \( G' \), and let \( \Sigma_N' \) be the unique connected component of \( \Sigma_N \) containing \( G'' \). Let \( C'' \) be the set of \( F \in C \cup \{F_0, F_1\} \) with \( F \subseteq G'' \), where each element of \( C \) is regarded as a cycle of \( G' \). Note that
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\( C'' \neq \emptyset \), since each component of \( G' \) contains at least one of \( F_0, F_1 \). Let \( F_* \in C'' \), where \( F_* = C_* \) if \( C_* \subseteq G'' \), and otherwise \( F_* \) is an arbitrary element of \( C'' \). Now, let \( T^\dagger = (\Sigma_N^\dagger, G'', C'', L^\dagger, F_*) \). It suffices to show that \( T^\dagger \) is a mosaic, since, if this holds for each component of \( G' \), then \( G' \) is \( L^\dagger \)-colorable by the criticality of \( T \).

Note that \( G'' \) is short-separation-free and furthermore, \( \tilde{C} \) is an induced cycle, each of the precolored paths \( p_1 \cdots p_m \) and \( v_nv_{n+1} \) is \( L^\dagger \)-colorable. Thus, since \( T \) is a tessellation, it follows from Subclaim 10.3.1.3 that \( T^\dagger \) is also a tessellation. It is immediate that \( T^\dagger \) satisfies M0), and, since \( N \) is noncontractible, we have \( g(\Sigma_N) < g \), so it follows from Claim 10.2.2 that \( T^\dagger \) satisfies M3), M4), and M5). Each of \( F_0, F_1 \) is an open \( T^\dagger \)-ring, it is also immediate that \( T^\dagger \) satisfies M2). We just need to check that \( F_0, F_1 \) satisfy M1). Since \( F_1 \) has a precolored path of length one, it is immediate that \( F_1 \) satisfies M1). Since \( C \setminus A \subseteq F_0 \cup F_1 \), it follows from Theorem 9.4 that, for each \( x \in D_1(\bar{P}) \setminus V(F_0) \), the graph \( G[N(x) \cap \bar{P}] \) is a path of length at most one, so \( F_0 \) is highly \( L^\dagger \)-predictable. We just need to check that there is no chord of \( F_0 \) with an endpoint which is an internal vertex of \( p_1 \cdots p_m \). Suppose toward a contradiction that there is such a chord \( xy \) of \( F_0 \), where \( x \in \{ p_2, \ldots , p_{m-1} \} \). Since \( T \) satisfies M1) and \( p_2 \cdots p_{m-1} \subseteq \bar{P} \), we have \( y \notin V(C \setminus A) \), so \( y \in D_1(A) \setminus \tilde{C} \). If \( y \) has a neighbor in \( \tilde{C} \), then we have a 2-chord of \( \tilde{C} \) with \( x \) as an endpoint, contradicting B) i) of Theorem 9.3. On the other hand, no vertex of \( A \setminus \tilde{C} \) is reachable from \( p_2 \cdots p_{m-1} \) by a path of length at most two. In particular:

1) For any subpath \( P' \) of \( C \setminus \bar{P} \), there is no path of length at most two from \( Sh^2(P') \) to \( p_2 \cdots p_{m-1} \)

2) Since \( (H \cup H') \cap \tilde{C} \subseteq \tilde{C} \setminus \bar{P} \), it follows from Claim 10.2.3 that there is no path of length at most two from \( (A \setminus \tilde{C}) \cap (H \cup H') \) to \( p_2 \cdots p_{m-1} \) and there is no path of length at most two from \( w_2 \cdots w_{t-2} \) to \( p_2 \cdots p_{m-1} \).

Thus, \( T^\dagger \) satisfies M1) as well.

By Subclaim 10.3.1.4, there is an \( L^\dagger \)-coloring of \( G \setminus A \), and, since \( V(A) \) is \( (L, \tau \cup \pi^*) \)-inert in \( G \), this \( L^\dagger \)-coloring of \( G \setminus A \) extends to \( L \)-color \( G \), contradicting the fact that \( T \) is a counterexample. This proves Claim 10.3.1.

10.4 An enclosure overlapping with the precolored edges: \( P \)-Augmentations

In Subsection 10.3, we showed that any short knot has an endpoint which is close to \( P \). Over the course of Subsections 10.4-10.5, we complete the proof of Theorem 4.1 by ruling out this configuration as well, but this is somewhat more complicated than Claim 10.3.1 because a short knot possibly has some complicated interaction with the precolored path of \( \tilde{C} \). We take care of all of these complicated interactions in this subsection when we prove Claim 10.4.10. In Subsection 10.5, we treat Claim 10.4.10 as a black box which we use to complete the proof of Theorem 4.1. Everything in this subsection before the proof of Claim 10.4.10 is setup for Claim 10.4.10, so the reader can, if they wish, take note of this black box and either skip the details or return to them later. The idea here is to take “one step back” from
the precolored path of \( \tilde{C} \) and analyze the resulting facial cycle of \( G \setminus P \). As \( \tilde{C} \) is an induced cycle of \( G \), the following is immediate from our triangulation conditions.

**Claim 10.4.1.** \( G \setminus P \) is connected, and there is a unique facial cycle \( \tilde{C}^1 \) of \( G \setminus P \) such that \( E(\tilde{C} \setminus P) \subseteq E(\tilde{C}^1) \) and \( V(\tilde{C}^1) = V(\tilde{C} \setminus P) \cup D_1(P) \).

Let \( \tilde{C}^1 \) be as in Claim 10.4.1. It follows from Theorem 9.4 that \( \tilde{C}^1 \) contains a (not necessarily induced) path consisting of all the vertices of \( D_1(P) \setminus \{v_0, v_n\} \). We denote this path by \( P^1 \). We now introduce the following notation.

**Definition 10.4.2.**

1) We let \( \pi \) be the unique \( L \)-coloring of \( V(P) \).

2) Over all \( 1 \leq k \leq 3 \) and all \( k \)-chords of \( \tilde{C} \) with one endpoint in \( \{v_0, p_1, p_2\} \) and the other endpoint in \( C \setminus P \), we let \( R \) be the unique generalized chord of \( \tilde{C} \) of length at most three which maximizes \( |V(G^\text{small}_R)| \). Likewise, over all \( 1 \leq k \leq 3 \) and all \( k \)-chords of \( \tilde{C} \) with one endpoint in \( \{p_{m-1}, p_m, v_n\} \) and the other endpoint in \( C \setminus P \), we let \( R' \) be the unique generalized chord of \( \tilde{C} \) of length at most three which maximizes \( |V(G^\text{small}_{R'})| \).

3) We let \( r, r' \in \{1, \cdots, n\} \), where \( v_r \) is the non-\( P \)-endpoint of \( R \) and \( v_{r'} \) is the non-\( P \)-endpoint of \( R' \). Finally, we let \( p_x \) denote the unique terminal edge of \( R \) with \( p_x \in V(\tilde{P}) \) and we let \( p_{x'} \) denote the unique terminal edge of \( R' \) with \( p_{{x'}} \in V(\tilde{P}) \).

![Figure 10.2](image-url)
Definition 10.4.2 is illustrated in Figure 10.2, where $P$ is indicated in bold and the paths $R$ and $R'$ are indicated in red. In Figure 10.2, $p_x = p_2$ and $p_{x'} = p_m$. Note that $R$ and $R'$ are well-defined. There is at least one 2-chord of $\hat{C}$ with one endpoint in $\{v_0, p_1\}$ and the other endpoint in $\hat{C} \setminus P$, since $\hat{C}$ is an induced cycle and $v_0, v_1$ have a unique common neighbor in $D_1(\hat{C})$ by our triangulation conditions, so $R$ is well-defined, and likewise for $R'$. Since $\hat{C}$ is an induced cycle, each of $R$ and $R'$ is either a 2-chord of a 3-chord of $\hat{C}$. It follows from Theorem 9.3 that, for any $1 \leq k \leq 3$ and any $k$-chord of $\hat{C}$ with one endpoint in $P$ and the other endpoint in $\hat{C} \setminus P$, the $P$-endpoint of this $k$-chord either lies in $\{v_0, p_1, p_2\}$ or $\{p_{m-1}, p_m, v_n\}$, and Theorem 9.3 also immediately implies the following.

**Claim 10.4.3.** For any $1 \leq k \leq 3$ and any $k$-chord $R_1$ of $\hat{C}$ with one endpoint in $P$ and the other endpoint in $C \setminus P$, either $G^\text{small}_{R_1} \subseteq G^\text{small}_R$ or $G^\text{small}_{R_1} \subseteq G^\text{small}_{R'}$.

To deal with the remaining configurations in the proof of Theorem 4.1, we need to distinguish between the collars $[\Sigma, G, C, L]$ and $[\Sigma, G \setminus P^1, \hat{C}^1, L_x]$, so we set $K = [\Sigma, G, C, L]$ and $K^1 = [\Sigma, G \setminus P^1, \hat{C}^1, L_{\pi}]$.

**Claim 10.4.4.** $K^1$ is uniquely 4-determined. Furthermore, the path $P^1$ is 2-short in in $K^1$.

**Proof:** By Theorem 9.3, $K$ is unique $N_{mo}/3$-determined, so if $K^1$ is not uniquely 4-determined, then there is an improper generalized chord of $\hat{C}^1$ of length at most four which is also a separating cycle of $G \setminus P$, which is false. Furthermore, it follows from B ii) of Theorem 9.3 that the path $P$ is 4-short in $K$, so $P^1$ is indeed 2-short in $K^1$. ■

Recalling the terminology of Definition 8.2, we also note the following.

**Claim 10.4.5.** For any generalized chord $Q \subseteq G \setminus P$ of $\hat{C}^1$ of length at most four, if $Q \cap \hat{C}^1 \subseteq xP^1x'$, then, letting $H$ be the small side of $Q$ in $G \setminus P$, the graph $H \cap \hat{C}^1$ is a subpath of $xP^1x'$.

**Proof:** This is immediate if $Q$ is not a proper generalized chord of $\hat{C}^1$, since $G$ is short-separation-free. On the other hand, if $Q$ is a proper generalized chord of $\hat{C}$, then it is a subpath of a generalized chord of $C$ of length at most six, and the claim follows from Theorem 9.3. ■

**Claim 10.4.6.** There exist two extensions of $\pi$ to $L$-colorings of $V(P \cup G^\text{small}_R)$ which use different colors on $x$. Likewise, there exist two extensions of $\pi$ to $L$-colorings of $V(P \cup G^\text{small}_R)$ which use different colors on $x'$.

**Proof:** These are symmetric so it suffices to prove the first claim. Consider the graph $G^* = G^\text{small}_R \setminus P$, regarded as a planar embedding. There is a unique facial subgraph $F$ of $G^*$ containing all the vertices with $L_{\pi}$-lists of size less than five, and, in particular, $|L_{\pi}(x)| \geq 3$. Since $\hat{C}$ is an induced cycle, every vertex of $F$ has an $L_{\pi}$-list of size at least
three, except possibly \(v_1\), which has an \(L_n\)-list of size at least two. Since \(|L_\pi(x)| \geq 3\), it follows from Theorem 1.3 that there exist two extensions of \(\pi\) to \(L\)-colorings of \(V(P \cup G^{\text{small}}_R)\) which use different colors on \(x\). ■

**Claim 10.4.7.** Given a short knot \(P = w_0 w_1 \cdots w_\ell\), there is an \(w \in \{w_2, w_{\ell-2}\}\) with \(d(w, P) \leq 4\), and furthermore:

1) For every \(w\)-enclosure \(Q\) which is both central and maximal, the graph \(G^{\text{small}}_Q\) contains an edge of \(P\) and \(Q\) contains a vertex of \(\tilde{P}\); AND

2) Either \(d(w, v_0 p_1 p_2) > 4\) or \(d(w, p_{m-1} p_m v_n) > 4\)

**Proof:** By Claim 10.1.4, \(|E(P)| \geq 2.1^\ell \cdot 6^q - 1\). As \(P\) is a shortest path between its endpoints and \(|E(P)| \leq 2N_{mn}/3\), there is at most one \(w \in \{w_2, w_{\ell-2}\}\) with \(d(w, P) \leq 4\). If there is no such \(w\), then we contradict Claim 10.3.1. Thus, there is precisely one \(w\) with \(d(w, P) \leq 4\). Letting \(w'\) be the other vertex of \(\{w_2, w_{\ell-2}\}\), we have \(d(w', P) > 4\), so Claim 10.3.1 also implies that \(w\) satisfies 1). It also follows from Theorem 9.4 that \(d(v_0 p_1 p_2, p_{m-1} p_m v_n) > 8\), so the second part of the claim holds as well. ■

In the remainder of this section, we construct an extension of \(\pi\) to an \(L\)-coloring \(\tau\) of a region of \(G\) near the precolored path, where \(\text{dom}(\pi)\) is part of the deletion set we construct to finish the proof of Theorem 4.1. We perform this construction of \(\tau\) in Claim 10.4.10, which is the main result of Subsection 10.4. To avoid repetition when setting up the subgraph near the precolored path which we color and delete, we introduce the following notation.

**Definition 10.4.8.** Let \(P = w_0 \cdots w_\ell\) be a short knot, where \(w_2\) is the unique vertex of \(\{w_2, w_{\ell-2}\}\) of distance at most four from \(P\). Given a subgraph \(H\) of \(G\), we define \(H_P^{\text{aug}}\) to be the subgraph of \(G\) induced by the vertex set \(V(H \cup P \cup P^1 \cup G^{\text{small}}_R \cup G^{\text{small}}_R) \cup \text{Sh}^2(P^1, \mathcal{K}^1) \cup \{w_2, w_3, w_4\}\).

We now define the extensions of \(\pi\) the following.

**Definition 10.4.9.** Given a short knot \(P = w_0 \cdots w_\ell\), where \(d(w_2, P) \leq 4\), a \(P\)-augmentation is a 4-tuple \((Q, \tau, R_\pi, R'_\pi)\) such that

A1) \(Q\) is a generalized chord of \(\tilde{C}^1\) in \(G \setminus P\), where each vertex of \(Q\) has distance at most two from \(C \setminus P\); AND

A2) Each of \(R_\pi\) and \(R'_\pi\) is a nonempty path of length at most one, where each of \(R_\pi\) and \(R'_\pi\) has precisely one endpoint in \(\tilde{C}\), and this endpoint lies in \(v_r v_{r+1} \cdots v_{r'}\). Furthermore, \(d(R_\pi, R) \leq 1\) and \(d(R'_\pi, R') \leq 1\); AND

A3) Let \(H\) be the small side of \(Q\) in \(G \setminus P\), \(\tau\) is an extension of \(\pi\) to a partial \(L\)-coloring of \(V(H_P^{\text{aug}})\). Furthermore, letting \(v_l, v_l'\) be the respective unique \(\tilde{C}\)-endpoints of \(R_\pi\) and \(R'_\pi\), the following hold.

i) \(V(R \cup R') \cup \{w_4\} \subseteq \text{dom}(\tau)\) and \(H_P^{\text{aug}} \setminus (R_\pi \cup R'_\pi \cup (Q \setminus \text{dom}(\tau)))\) is connected; AND
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ii) $V(H^\text{aug}_P)$ is $(L, \tau)$-inert in $G$ and every vertex of $D_1(H^\text{aug}_P)$ has an $L_{\tau}^{R_+ \cup R'_+}$-list of size at least three; AND

iii) For any path of length at most two with one endpoint in $\text{dom}(\tau)$ and the other endpoint in $v_1 v_2 \cdots v_{\ell+1}$, the $\tau$-colored endpoint of this path lies in $R_+ \cup R'_+$. We now that for some short knot $P$, there exists a $P$-augmentation $(Q, \tau, R_+, R'_+)$. We then complete the proof of Theorem 4.1 by reducing from $T$ to a smaller mosaic in which $\hat{C}$ splits into two new open rings, where $R_+$ and $R'_+$ are the respective precolored paths of the two new open rings.

Claim 10.4.10. For some short knot $P$, there exists a $P$-augmentation.

Proof: We first define the following. Let $X$ be the set of $c \in L_\pi(x)$ such that $\pi$ extends to an $L$-coloring of $V(P \cup G^\text{small}_R)$ using $c$ on $x$. Likewise, let $X'$ be the set of $d \in L_\pi(x)$ such that $\pi$ extends to an $L$-coloring of $V(P \cup G^\text{small}_R)$ using $d$ on $x'$. By Claim 10.4.6, $|X| \geq 2$ and $|X'| \geq 2$.

Subclaim 10.4.10.1. Let $P = w_0 \cdots w_k$ be a short knot, where $w_3 \in D_2(\hat{C})$. Suppose there is a $w_3$-enclosure $Q$ in $K^1$ which is both central and maximal, where $Q$ has no intersection with $G^\text{small}_R \cup G^\text{small}_R$, except possibly on at most the vertices $x, x'$. Then there is a $P$-augmentation.

Proof: In this case, we let $H$ be the small side of $Q$ in $G \setminus P$. By our assumption on $Q$, we have $Q \cap \hat{C}^1 \subseteq xP^1x'$, so it follows from Claim 10.4.5, $H \cap \hat{C}^1$ is a subpath of $xP^1x'$. We let $u$ and $u'$ be the endpoints of $H \cap \hat{C}^1$, where $u'$ lies in the path $uP^1x'$. Possibly $u = u'$. This situation is illustrated in Figure 10.3, where the path $w_2 \cdots w_{k-2}$ is indicated in blue. We claim now that there exist colors $c \in L_\pi(x)$ and $d \in L_\pi(x)$ and elements $\phi_c \in \text{Link}(xP^1u, K^1)$ and $\phi_d \in \text{Link}(u'P^1x', K^1)$ such that

i) $\pi$ extends to an $L$-coloring of $V(P \cup G^\text{small}_R)$ using $c$ on $x$ and $\pi$ extends to an $L$-coloring of $V(P \cup G^\text{small}_R)$ using $d$ on $x'$; AND

ii) There is an $(Q, w_3w_3)$-target $\psi$ in $K^1$, where $\phi_c \cup \psi \cup \phi_d$ is a proper $L$-coloring of its domain.

To see that this holds, let $X_+$ be the set of $a \in L_\pi(u)$ such that there is a $\phi \in \text{Link}(xP^1u, K^1)$ with $\phi(u) = a$ and $\phi(x) \in X$. Likewise, let $X'_+$ be the set of $b \in L_\pi(u')$ such that there is a $\phi \in \text{Link}(u'P^1x', K^1)$ with $\phi(u') = b$ and $\phi(x') \in X$. Note that $|X_+| \geq 2$. This is immediate if $x = x_+$, since any $L_\pi$-coloring of $x$ lies in $\text{Link}(xP^1x_+, K^1)$ in that case, and, if $u \neq u'$, then it follows from Theorem 7.5 that $|X_+| \geq 2$. Likewise, $|X'_+| \geq 2$. Note that if $u = u'$, then $X_+ \cap X'_+ \neq \emptyset$, since, in that case, we can remove colors from $L_\pi(u)$ until $L_\pi(u') = 3$ and then apply the argument above again. In any case, it now follows from Theorem 8.6 that there exist $\phi_c, \psi, \phi_d$ satisfying i)-ii) the above. Now, the union $\pi \cup \phi_c \cup \psi \cup \phi_d$ is a proper $L$-coloring of its
domain. Since $d(R, R') > 2$, it follows from our choice of colors used on $x, x'$ that $\pi \cup \psi \cup \phi_d$ extends to an $L$-coloring $\tau$ of $\text{dom}(\pi \cup \phi_c \cup \psi \cup \phi_d) \cup V(G_{R}^{\text{small}} \cup G_{R'}^{\text{small}})$.

Now, we let $R_e = R \setminus \{x, p_x\}$ and $R'_e = R' \setminus \{x', p_{x'}\}$. We claim now that $(Q, \tau, R_e, R'_e)$ is a $P$-augmentation. Let $H$ be the small side of $Q$ in $G \setminus P$. It is immediate that $(Q, \tau, R_e, R'_e)$ satisfies A1)-A2) of Definition 10.4.9. Furthermore, since $\psi$ is a $(Q, w_4w_3)$ target in $K^1$, it follows from our triangulation conditions, together with the conditions of Definition 8.5, that A3 i) is satisfied as well. It is also immediate from our construction of $\tau$ that A3 ii) is satisfied, and A3 iii) follows from the definition of $R$ and $R'$.

Applying Subclaim 10.4.10.1, we have the following.

**Subclaim 10.4.10.2.** If $P$ is a short knot with an endpoint in $P$, then there is a $P$-augmentation.

**Proof:** Let $P = w_0 \cdots w_\ell$, where $w_0 \in V(P)$. By the minimality of $P$, we have $w_3 \in D_2(\tilde{C}^1)$. Now, if there is a central, maximal $w_3$-enclosure $Q$ in $K^1$ satisfying Subclaim 10.4.10.1, then we are done, so suppose that there is no such central, maximal $w_3$-enclosure in $K^1$. It follows from Theorem 9.4 that $d(G_{R}^{\text{small}}, G_{R'}^{\text{small}}) > 8$, so we suppose without loss of generality that $d(w_3, G_{R'}^{\text{small}}) > 4$. Note that $P \cap G_{R'}^{\text{small}} = w_0w_1$. Furthermore, there is a central, maximal $w_3$ enclosure $Q$ such that, letting $H$ be the small side of $Q$ in $\tilde{C}^1$, the following hold.

1) $Q$ is a proper generalized chord of $\tilde{C}^1$ and $G_{R}^{\text{small}} \subseteq H \cup P$; AND
2) $R \setminus \{p_x, x\}$ is a terminal subpath of $Q$ and the non-$v_r$ endpoint of $Q$ is an internal vertex of the path $xP^1x'$. 

Let $Q, H$ be as above and let $u$ be the unique endpoint of $Q$ which is an internal vertex of $xP^1x'$. Let $H^1 = H \setminus (G_R^\text{small} \setminus \tilde{R})$. It follows from our choice of $Q$ that there is a proper generalized chord $Q^1$ of $\tilde{C}^1$ such that $H^1$ is the small side of $Q^1$ in $K^1$, where $Q^1$ is a $w_3$-enclosure in $K^1$ and $Q^1 \setminus G_R^\text{small} = Q \setminus G_R^\text{small}$. In particular, $Q^1$ is a path with endpoints $x, z$. Let $Z$ be the set of $a \in L_\phi(z)$ such that there is a $\phi(z') \in X'$. Since $|X'| \geq 2$, we have $|Z| \geq 2$ by Theorem 7.5. Note that $Q^1$ is not a maximal $w_3$-enclosure in $K^1$, but, in any case, since $|X| \geq 2$, it follows from Theorem 8.6 that there exists a $(Q^1, w_4w_3)$-target $\psi$ in $K^1$ with $\psi(x) \in X$, where $\psi(z) \in Z$, so there is a $\sigma \in \text{Link}(zP^1x', K^1)$ with $\sigma(z) = \psi(z)$ and $\sigma(x') \in X'$. Thus, $\psi \cup \sigma$ is a proper $L$-coloring of its domain. Since $\psi(x) \in X$ and $\sigma(x') \in X'$, and since there is no edge from $G_R^\text{small}$ to $G_R^\text{small}$, it follows that $\psi \cup \sigma$ extends to an $L$-coloring $\tau$ of $\text{dom}(\psi \cup \sigma) \cup V(G_R^\text{small} \cup G_R^\text{small})$. As above, we let $R_* = R \setminus \{p_x, x\}$ and $R'_* = R_* \setminus \{p_x, x\}$, and it is straightforward to check that $(Q^1, \tau, R_*, R'_*)$ is a $P$-augmentation, so we are done in this case. 

Applying Subclaim 10.4.10.2, we suppose for the remainder of the proof of Claim 10.4.10 that there is no short knot with an endpoint in $P$, or else we are done. We now fix a short knot $P = w_0 \cdots w_\ell$. Applying Claim 10.4.7, we suppose without loss of generality that $d(w_2, P) \leq 4$ and $d(w_2, p_{m-1}p_{m+1}) > 4$.

**Subclaim 10.4.10.3.** For each $i = 2, 3, 4$, we have $d(w_i, P) = i + 1$, and furthermore, and $w_2$ is a degenerate vertex of $D_2(\tilde{C})$.

**Proof:** We first note that $d(w_2, P) > 2$, or else it follows from our face-width conditions that there is a short knot of $\tilde{C}$ of the same length as $P$ and one endpoint in $P$, contradicting our assumption that no short knot has an endpoint in $P$. Thus, $d(w_2, P) = 3$. Likewise, $d(w_3, P) = 4$ and $d(w_4, P) = 5$. Suppose that $w_2$ is non-degenerate in $D_2(\tilde{C})$. Thus, there is a unique central, maximal $w_2$-enclosure $Q$ in $K$, and $Q$ is a proper 4-chord of $\tilde{C}$. Since $Q$ is a 4-chord of $\tilde{C}$, and since $G_Q^\text{small}$ contains an edge of $P$ by Claim 10.4.7, we have $d(w_2, P) \leq 2$, which is false, as $d(w_2, P) = 3$. 

**Subclaim 10.4.10.4.** $R$ is a 3-chord, and letting $xx_*$ be the middle edge of $G$, the graph $G[N(w_2) \cap D_1(\tilde{C})]$ is a path of length at most one which has $x_*$ as an endpoint.

**Proof:** Since $w_2$ is a degenerate vertex of $D_2(\tilde{C})$, we get that $G[N(w_2) \cap V(\tilde{C}^1)]$ is a path of length at most one. Since $w_2$ is a degenerate vertex of $D_2(\tilde{C})$, it follows from Claim 10.4.7 that there is a central, maximal $w_2$-enclosure $Q$ in $K$, where $Q$ is a 5-chord of $\tilde{C}$ and $Q$ contains a 2-path with one endpoint in $N(w_2) \cap D_1(\tilde{C})$ and the other endpoint in $\tilde{P}$. By definition, every vertex of $Q$ is of distance at most one from $\tilde{C}$ and $G[N(w_2) \cap D_1(\tilde{C})]$ is contained in the middle edge of the 3-path $Q \setminus \tilde{C}$. If $Q$ is an improper 5-chord of $\tilde{C}$, or if both endpoints of $Q$ lie
in $P$, then it follows from Theorem 9.3 that $G_{Q}^{\text{small}} \cap \hat{C}$ is a subpath of $P$, and since $G_{Q}^{\text{small}}$ also contains all edges of $E(Q, \hat{C})$, it follows that $d(w_2, P) = 2$, which is false. Thus, $Q$ is a proper 5-chord of $\hat{C}$ with one endpoint in $\hat{P}$ and the other endpoint in $\hat{C} \setminus P$. Let $Q = p \cdot x_1, x_2, x_3, v$ for some vertices $x_1, x_2, x_3 \in D_1(\hat{C})$, where $p \in V(\hat{P})$ and $v \in V(\hat{C} \setminus P)$. By our choice of $Q$, we have $x_2 \in N(w_2)$, and $x_2$ has a neighbor in $\hat{C}$. Since $d(w_2, P) = 3$, $x_2$ has a neighbor in $\hat{C}$. By the minimality of $P$, we have $G_{Q}^{\text{small}} \cap P = w_0w_1$, and since $d(p_{m-1}p_mv_n) > 4$, it follows from the definition of $R$ that $R$ is a 3-chord, where the lone neighbor of $v_r$ on $R$ is precisely $x_2$. ■

Applying Subclaim 10.4.10.4, we have $R = p \cdot x \cdot x_2 \cdot v_r$ for some vertex $x_*$. Possibly $G[N(w_2) \cap D_1(\hat{C})]$ is an edge. By Subclaim 10.4.10.3, $w_2 \in D_2(\hat{C}_1)$, and not that there is at least one $w_2$-enclosure $T$ in $K^1$ such that $T \cap \hat{C}_1 \subseteq xP_1x'$. We now define the following.

1) We let $v_t$ be the unique neighbor of $N(w_2) \cap D_1(\hat{C})$ which is farthest from $v_r$ on the path $v_r \cdots v_{r'}$. We also define a vertex $y_*$, where $y_* = x_*$ if $N(w_2) \cap D_1(\hat{C}) = \{x_*\}$, and otherwise $y_*$ is the other vertex of $N(w_2) \cap D_1(\hat{C})$. Note that $y_*v_t \in E(G)$.

2) We let $T$ be the unique $w_2$-enclosure in $K^1$ such that $T \cap \hat{C}_1 \subseteq xP_1x'$, where the number of vertices in the small side of $T$ is maximized among all such $w_2$-enclosures in $K^1$.

3) We let $uz$ be the edge of $T$ which is distinct from $xx_*$ and has an endpoint in $\hat{C}_1$, where $u \in V(xP_1x')$. Finally, we let $J$ be the small side of $T$ in $K^1$.

\[
\begin{array}{c}
\text{Figure 10.4}
\end{array}
\]
10.4 An enclosure overlapping with the precolored edges: P-Augmentations

The definitions above are illustrated in Figure 10.4, where the path $w_2w_3\cdots w_{\ell-2}$ is illustrated in blue. As in Figure 10.2, the paths $R$ and $R'$ are indicated in red. In Figure 10.4, $T = xxz w_2zu$ is a proper 4-chord of $\hat{C}^1$ with midpoint $w_2$, although possibly $T$ is also a 5-chord of $\hat{C}^1$.

By Subclaim 10.4.10.3, $d(w_3,\hat{C}^1) = 3$, so we apply Theorem 8.6 to the edge $w_2w_3$. Since $|X| \geq 2$ and $|X'| \geq 2$, it follows from Theorems 7.5 and 8.6 that there exists a $(T, w_3w_3)$-target $\sigma$ in $K^1$ and a $\psi \in \text{Link}(uP^1x'$, $K^1)$ such that $\sigma(x) \in X$ and $\psi(x') \in X'$, where $\sigma(u) = \psi(u)$. Note that $\sigma \cup \psi$ is a proper $L$-coloring of its domain. Since $d(R, R') > 1$, it follows that $\sigma \cup \psi$ extends to an $L$-coloring $\tau$ of $\text{dom}(\sigma \cup \psi) \cup V(G_R^{\text{small}} \cup G_R^{\text{small}})$. Note that we have now colored $x_*$, but note that it follows from the definition of $T$ that $x_*z \notin E(G)$, so $|L_\tau(z)| \geq 3$. We set $R_*' = R' \setminus \{p_{x'}, x'\}$. We claim now that there is a $P$-augmentation whose last coordinate is $R_*'$. Consider the following cases.

Case 1: $v_t = v_r$

In this case, we take $R_* = v_r$ and we claim that $\tau$ extends to an $L$-coloring $\tau^\dagger$ of $\text{dom}(\tau) \cup \{w_4\}$ such that $(T, \tau^\dagger, R_*, R_*')$ is a $P$-augmentation. Note that $L_\tau(w_4) | = 4$. Possibly $w_2, w_3, w_4$ have a common neighbor $w$, but any such $w$ other neighbors in $\text{dom}(\tau)$, and since there is at most one such vertex, so $\tau$ extends to an $L$-coloring $\tau^\dagger$ of $\text{dom}(\tau) \cup \{w_4\}$ such that any such $w$ has an $L_{\tau^\dagger}$-list of size at least three. Possibly $y_* \neq x_*$, but if that holds, then $|L_{\tau^\dagger}^R \cup R_*'(y_*)| \geq 3$, since we are not deleting $v_r$. Note that we are not deleting $z$, but $|L_{\tau^\dagger}^{R \cup R_*'}(z)| \geq 3$ as well, since $x_*z \notin E(G)$ and $d(w_3, P^1) > 2$. Thus, every vertex of $D_1(J^\text{aug}_P)$ has an $L_{\tau^\dagger}^{R \cup R_*'}$-list of size at least three, and it is straightforward to check that $(T, \tau^\dagger, R_*, R_*')$ satisfies all the other conditions of Definition 10.4.9.

Case 2: $v_t \neq v_r$

In this case, we have $y_* \neq x_*$ and $G[N(w_2) \cap D_1(\hat{C})]$ is the edge $x_*y_*$. In this case, we set $R_* = v_t y_*$, and we claim that there is a $P$-augmentation whose respective last two coordinates are $R_*$ and $R_*'$. We also set $Q^1$ to be the proper 3-chord $v_*x_* y_*v_t$ of $\hat{C}$. If $T$ is a proper 4-chord of $\hat{C}^1$, then we have the situation shown in Figure 10.4. We now define a generalized chord $Q$ of $\hat{C}^1$ in $G \setminus P$ as follows. If $T$ is a proper 4-chord of $\hat{C}^1$, then we take $Q = uz w_2 y_* v_t$. Otherwise, $T$ is a (not necessarily proper) 5-chord of $\hat{C}^1$ and we take $Q = v_t y_* x_* + (T - x)$. Let $H$ be the small side of $Q$ in $K^1$. Note that $J \cup G^{\text{small}}_{Q^1} \subseteq H$, and $G^{\text{small}}_{Q^1} \subseteq H$ as well. Furthermore, $\tau$ precolors the edge $x_*v_r$. Possibly $w_2 \in \text{dom}(\tau)$, but, in any case, since $|L(y_*)| \geq 5$, it follows from Theorem 1.1 that $\tau$ extends to an $L$-coloring $\tau^\dagger$ of $\text{dom}(\tau) \cup V(G^{\text{small}}_{Q^1})$. Since $y_* \neq x_*$, the graph $H^\text{aug}_P \setminus (R_* \cup R_*' \cup (Q \setminus \text{dom}(\tau^\dagger)))$ is connected, and it is straightforward to check that $(Q, \tau^\dagger, R_*, R_*')$ is a $P$-augmentation. This completes the proof of Claim 10.4.10. □
10.5 Completing the proof of Theorem 4.1

Applying Claim 10.4.10, we let $P = w_0 \cdots w_t$ be a short knot which admits a $P$-augmentation. Applying Claim 10.4.7, we suppose without loss of generality that $d(w_2, P) \leq 4$. Let $(Q, \tau, R_s, R'_s)$ be a $P$-augmentation and let $H$ be the small side of $Q$ in $G \setminus P$.

We now let $Q^+$ be a central, maximal $w_{t-2}$-enclosure in $K$, and we fix indices $t, t' \in \{r, \ldots, r'\}$, where $v_t$ is the unique $\tilde{C}$-endpoint of $R_s$ and $t'$ is the unique $\tilde{C}$-endpoint of $R'_s$. By definition, $d(v_t, R_s) \leq 2$ and $d(v_{t'}, R'_s) \leq 2$. Recall that $|E(P)| \geq (2.1\beta \cdot 6^9)^{-1}$, and $P$ is a shortest path between its endpoints. Since each of $v_t, v_{t'}$ has distance at most $\frac{N_t}{3} + 4$ from $w_2$, we have $w_t \in \{v_{t+1}, \ldots, v_{t'-1}\}$, and it follows from Theorem 9.3 that the path $G_{Q^+} \cap \tilde{C}$ is a subpath of $v_{t+1}v_{t+2} \cdots v_{t'-1}$. Thus, there exist indices $i, i'$ with $t < i \leq i' < t'$, where $H^+ \cap \tilde{C}^1 = v_i \cdots v_{t'}$. As in the proof of Claim 10.3.1, we take the union of some partial $L$-colorings of $G$ with desirable properties in order to construct our deletion set.

Claim 10.5.1. There exist a $\sigma_0 \in \text{Link}(v_{t+1} \cdots v_{t'}, K)$ and a $\sigma_1 \in \text{Link}(v_t \cdots v_{t'-1}, K)$ and a $(Q^+, w_{t-3}w_{t-2})$-target $\psi^+$ in $K$ such that the union $\tau \cup \sigma_0 \cup \psi^+ \cup \sigma_1$ is a well-defined proper $L$-coloring of its domain.

Proof: It follows from Theorem 9.3 that any subpath of $v_{t+1}v_{t+2} \cdots v_{t'-1}$ is 2-short in $K^3$, so we apply Theorem 7.5. Let $Y$ be the set of $c \in L(v_t)$ such that there is a $\sigma_0 \in \text{Link}(v_{t+1} \cdots v_i, K)$ with $\sigma_0(v_i) = c$ and $\sigma_0(v_{t+1}) \in L(v_{t+1}) \setminus \{\tau(v_i)\}$. Likewise, let $Y'$ be the set of $c \in L(v_{t'})$ such that there is a $\sigma_1 \in \text{Link}(v_t \cdots v_{t'-1}, K)$ with $\sigma_0(v_{t'}) = c'$ and $\sigma_1(v_{t'-1}) \in L(v_{t'-1}) \setminus \{\tau(v_t)\}$. Since $|L(v_i)| = |L(v_{t'})| = 3$ and since $|L(v_{t+1}) \setminus \{\tau(v_i)\}| \geq 2$ and $|L(v_{t'-1}) \setminus \{\tau(v_t)\}| \geq 2$, it follows from Theorem 7.5 that $|Y| \geq 2$ and $|Y'| \geq 2$. Possibly $v_t = v_{t'}$, in which case $Y \cap Y' \neq \emptyset$, as $|L(v_i)| = 3$. In any case, it follows from Theorem 8.6 that there is $(Q^+, w_{t-3}w_{t-2})$-target $\psi^+$ such that $\psi^+(v_i) \in Y$ and $\psi^+(v_{t'}) \in Y'$. Thus, there is $\sigma_0 \in \text{Link}(v_{t+1} \cdots v_i, K)$ and a $\sigma_1 \in \text{Link}(v_t \cdots v_{t'-1}, K)$ such that $\sigma_0(v_i) = \psi^+(v_i)$ and $\sigma_1(v_{t'}) = \psi^+(v_{t'})$.

Since $\tilde{C}$ is an induced cycle, the union $\sigma_0 \cup \psi^+ \cup \sigma_1$ is a proper $L$-coloring of its domain. Furthermore, it follows from Conditions A1) and A3 iii) of Definition 10.4.9 that $V(H^\text{me})$ is disjoint to dom$(\sigma_0 \cup \psi^+ \cup \sigma_1)$, and that the only edge between $V(H^\text{me})$ and dom$(\sigma_0 \cup \psi^+ \cup \sigma_1)$ are $v_tv_{t+1}$ and $v_{t'}v_{t'-1}$. Thus, $\tau \cup \sigma_0 \cup \psi^+ \cup \sigma_1$ is indeed a proper $L$-coloring of its domain. $

Let $\sigma_0, \psi^+, \sigma_1$ be as in Claim 10.5.1. Let $\varphi$ be the union $\tau \cup \sigma_0 \cup \psi^+ \cup \sigma_1$. Note that $\varphi$ is a proper $L$-coloring of its domain. Let $P$ be the unique subpath of $w_2Pw_{t-2}$ which includes all the vertices of dom$(\varphi \cap V(w_2Pw_{t-2}))$ and whose endpoints lie in dom$(\varphi)$. Now, by Definition 8.5 applied to $\psi^+$ and Definition 10.4.9 applied to $\tau$, the subpath of $P$ precolored by $\psi^+$ is either $w_{t-3}$ or $w_{t-3}w_{t-2}$, and the subpath of $P$ precolored by $\tau$ is contained in $w_2w_3w_4$. Since $P$ is a shortest path between its endpoints, these subpaths are of distance at least $2\beta \cdot 6^9$ from each other. Now,
10.5 Completing the proof of Theorem 4.1

$G[V(\tilde{P}) \cap \text{dom}(\varphi^+)]$ is a path of length at most one, and it follows from A3 ii) of Definition 10.4.9 applied to $\tau$ that $(\tilde{P}, G[\text{dom}(\tau) \cap V(\tilde{P})], G[\text{dom}(\varphi^+) \cap V(\tilde{P})], \varphi|_{\tilde{P}})$ is a filament. By Theorem 9.7, there is a subgraph $K$ of $G$ and a $U \subseteq V(K)$ such that

1) $\tilde{P} \subseteq K$ and $V(K \setminus \tilde{P}) \subseteq B_1(\tilde{P} \setminus \text{dom}(\varphi));$ AND

2) $d(K \setminus \tilde{P}, V(\tilde{P}) \cap \text{dom}(\varphi)) \geq 3$ and $\varphi|_{\tilde{P}}$ extends to an element $\tau^*$ of $\text{Avoid}^0(K \mid U).

We now have all the ingredients necessary to construct our deletion set. Note that $\varphi \cup \tau^*$ is a proper $L$-coloring of its domain. We now set $S$ to be the following union

$$S = S \setminus (V(Q \cup Q^+) \setminus \text{dom}(\varphi)),$$

where $P \subseteq A$, since $\tau$ is an extension of $\pi$. It follows from repeated application of Observation 4.3 that $V(A)$ is $(L_1, \varphi \cup \tau^*)$-inert in $G$. Furthermore, it follows from Condition A3 i) of Definition 10.4.9 that $H^\text{aug}_P \cap A$ is connected, and it follows from Claim 10.2.3 that $H^+ \cap A$ is connected. Thus, $\tilde{C} \cup A$ is connected. Note that $V(H^\text{aug}_P) \subseteq \text{Sh}^4(\tilde{C}^1) \subseteq \text{Sh}^6(\tilde{C})$, where the latter containment holds since $G$ is short-separation-free. Thus, $V(A) \subseteq B_2(\tilde{C} \cup P) \cup \text{Sh}^6(\tilde{C})$. It now follows from Claim 10.2.1 that there is a noncontractible closed curve $N$ of $\Sigma$, where $N \cap G \subseteq A$. Let $G'$ be the embedding of $G \setminus A$ on $\Sigma_N$ in the natural way. We let $F_0, F_1$ be as in Claim 10.2.2, i.e $F_0, F_1$ are facial subgraphs of $G'$ with $V(F_0 \cup F_1) = V(\tilde{C} \setminus A) \cup D_1(A)$. Note that it follows from Condition A2) of Definition 10.4.9 that the two paths $R_s$ and $R'_s$ do not lie in the same element of $\{F_0, F_1\}$, i.e one of $F_0, F_1$ contains the path $R_s$ and the other contains the path $R'_s$. Say for the sake of definiteness that $R_s \subseteq F_0$ and $R'_s \subseteq F_1$. We now set $L^1 = L^{R_s \cup R'_s}_{P, L}.$

Claim 10.5.2. Each vertex of $F_0 \cup F_1$ has an $L^1$-list of size at least three, except the vertices of $R_s \cup R'_s$.

Proof: We note that $A$ contains all of $\tilde{C}$ except for vertices $v_i, v_{i'}$, so we just need to check the vertices of $(F_0 \cup F_1) \setminus \tilde{C}$, i.e the vertices of $F_0 \cup F_1$ with $L$-lists of size five. Let $z \in V(F_0 \cup F_1) \setminus V(\tilde{C})$. We show that $|L^1(z)| \geq 3$. Consider the following cases.

Case 1: $z$ has a neighbor in $\text{dom}(\tau^*) \setminus \text{dom}(\varphi)$

In this case, we note that every vertex of $\text{dom}(\varphi) \setminus \text{dom}(\tau^*)$ either lies in $\text{Sh}^4(\tilde{C}^1)$ or is of distance at most one from $\tilde{C}^1$. Furthermore, each vertex of $\text{dom}(\tau^*) \setminus \text{dom}(\varphi)$ is of distance at least four from $C^1$. Since $z$ has a neighbor in $\text{dom}(\tau^*) \setminus \text{dom}(\varphi)$, we conclude that $N(z) \cap \text{dom}(\varphi \cup \tau^*) \subseteq \text{dom}(\tau^*)$, and thus, recalling Definition 9.5, we get that $|L^1(z)| \geq 3$.

Case 2: $z$ has no neighbor in $\text{dom}(\tau^*) \setminus \text{dom}(\varphi)$

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In this case, we have $N(z) \cap \text{dom}(\tau^{+} \cup \varphi) \subseteq \text{dom}(\varphi)$. Recall that $\varphi = \tau \cup \sigma_0 \cup \psi^{+} \cup \sigma_1$. We call a partial $L$-coloring of $G$ a part if it is one of $\tau, \sigma_0, \psi^{+}, \sigma_1$. It is straightforward to check, applying Claim 10.2.3 and A 3 iii) of Definition 10.4.9, that, for any two distinct parts $p$ and $p'$, there is no path of length at most two between $\text{dom}(p) \setminus \text{dom}(p')$ and $\text{dom}(p') \setminus \text{dom}(p)$, unless that path contains a vertex of $R_{\ast} \cup R'_{\ast}$. Thus, it suffices to prove that, for each part $p$, we have $|L_{\ast} \cup L'_{\ast}(z)| \geq 3$. This is immediate from Observation 7.4 if $p \in \{\sigma_0, \sigma_1\}$. If $p = \tau$ then it follows from A3 ii) of Definition 10.4.9 and, if $p = \psi^{+}$, then it follows from Definition 8.5, so we are done. ■

Applying Claim 10.5.2, we have the following.

Claim 10.5.3. $G'$ is $L^{\dagger}$-colorable.

Proof: Let $G''$ be a connected component of $G'$, and let $\Sigma_{N}^{\dagger}$ be the unique connected component of $\Sigma_{N}$ containing $G''$. Let $C''$ be the set of $F \in C \cup \{F_0, F_1\}$ with $F \subseteq G''$, where each element of $C$ is regarded as a cycle of $G'$. Note that $C'' \neq \emptyset$, since each component of $G'$ contains at least one of $F_0, F_1$. Let $F_{s} \in C''$, where $F_{s} = C_{s}$ if $C_{s} \subseteq G''$, and otherwise $F_{s}$ is an arbitrary element of $C''$. Now, let $T^{\dagger} = (\Sigma_{N}^{\dagger}, G'', C'', L^{\dagger}, F_{s})$. It suffices to show that $T^{\dagger}$ is a mosaic, since, if this holds for each component of $G'$, then $G'$ is $L^{\dagger}$-colorable by the criticality of $T$. Note that $G''$ is short-separation-free and each of the precolored paths $R_{\ast}$ and $R'_{\ast}$ is $L^{\dagger}$-colorable. Thus, since $T$ is a tessellation, it follows from Subclaim 10.3.1.3 that $T^{\dagger}$ is also a tessellation. It is immediate that $T^{\dagger}$ satisfies M0), and, since $N$ is noncontractible, we have $g(\Sigma_{N}^{\dagger}) < g$. It follows from Claim 10.2.2 that $T^{\dagger}$ satisfies M3), M4), and M5). Each of $F_0, F_1$ is an open $T^{\dagger}$-ring, it is also immediate that $T^{\dagger}$ satisfies M2). We just need to check that $F_0, F_1$ satisfy M1). Since each of $F_0, F_1$ has a precolored path of length at most one, it is immediate that they satisfy M1), so $T^{\dagger}$ is indeed a mosaic. ■

It follows from Claim 10.5.3 that there is an $L^{\dagger}$-coloring of $G \setminus A$, and since $V(A)$ is $(L, \varphi \cup \tau^{+})$-inert in $G$, this $L^{\dagger}$-coloring of $G \setminus A$ extends to an $L$-coloring of $G$, contradicting the fact that $T$ is a counterexample. This completes the proof of Theorem 4.1. ■

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