DYNAMICS AND SPATIOTEMPORAL PATTERN FORMATIONS OF A HOMOGENEOUS REACTION-DIFFUSION THOMAS MODEL

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Abstract. In this paper, we are mainly considered with a kind of homogeneous diffusive Thomas model arising from biochemical reaction. Firstly, we use the invariant rectangle technique to prove the global existence and uniqueness of the positive solutions of the parabolic system, and then use the maximum principle to show the existence of attraction region which attracts all the solutions of the system regardless of the initial values. Secondly, we consider the long time behaviors of the solutions of the system; Thirdly, we derive precise parameter ranges where the system does not have non-constant steady states by using some useful inequalities and a priori estimates; Finally, we prove the existence of Turing patterns by using the steady state bifurcation theory.

1. Introduction. In this paper, we consider a reaction-diffusion biochemical Thomas model which was first proposed by Thomas in [9] to explain the observed oscillatory behavior in substrate-inhibition chemical reaction involving the substrates oxygen and uric acid which react in the presence of the enzyme uricase.

The dimensionless form of the empirical rage equations for the uric acid (denoted by $u$) and the oxygen (denoted by $v$) can be written in the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= a - u - \frac{\rho uv}{1 + u + ku^2}, \\ 
\frac{\partial v}{\partial t} - d_2 \Delta v &= \alpha b - \alpha v - \frac{\rho uv}{1 + u + ku^2}, \\ 
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0, \\ 
u(x, 0) &= u_0(x) > 0, v(x, 0) = v_0(x) > 0, \\
&\quad x \in \Omega, t \geq 0,
\end{align*}
\]

(1)

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where $\Omega$ is an open bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial \Omega$; $u = u(x,t)$ and $v = v(x,t)$ stand for the concentrations of the uric acid and the oxygen at the position $x$ and the time $t$ respectively; $d_1$ and $d_2$ are diffusion coefficients of $u$ and $v$ respectively. $u_0, v_0 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and the Neumann boundary conditions indicate that there are no flux of the chemical substances of $u$ and $v$ on the boundary. Here the parameters $a$, $\alpha$, $b$, $\rho$ and $k$ are all positive. Basically, the acid $u$ and the oxygen $v$ are assumed to be supplied at a constant rates $a$ and $\alpha b$, degrade linearly proportional to their concentrations and both are used up in the reaction at a rate $\rho uv/(1 + u + ku^2)$, which exhibits substrate inhibition. The parameter $k$ is a measure of the severity of the inhibition.

The Thomas model (1) has been studied extensively by several authors, but most of the research focuses either on the corresponding ODE system. Thomas [9] considered the boundedness of the solutions of ODE system of system (1) by proving the existence of invariant rectangles. To the best of our knowledge, in the existing literatures, there are few works investigating the dynamics of the reaction diffusion equations (1). It is then our purpose to consider the dynamics and pattern formations of this particular biochemical model.

The Thomas model (1) is similar to but slight different from the Seelig model ([7,10]). In a recent paper due to Yi et al [10], the authors considered the dynamics of the Seelig model. They not only proved the global existence and boundless of the in-time solutions, but also show the existence of attraction regions which attract all the solutions of the system regardless of the initial values. Furthermore, the considered the existence and non-existence of Turing patterns for the Seelig model. Motivated by the observations of [10], we are concerned with whether the Thomas model has the same dynamics as that of the Seelig model.

This paper is organized as follows. In Section 2, we study the boundedness and uniqueness of global-in-time solutions of the parabolic system (1). In particular, we show that an invariant rectangle exists which attracts all the solutions of the parabolic system (1) regardless of the initial values. Then, we consider the long time behaviors of the solutions of system (1), and derive precise conditions so that the solutions of R-D (1) converge exponentially either to its unique constant steady state solution, or to its stable spatially homogeneous orbitally periodic solutions. In Section 3, we derive conditions so that system (1) does not have non-constant positive steady states, including Turing patterns. In Section 4, we use global bifurcation theory to prove the existence of Turing patterns.

2. Attraction region and large time behaviors of the solutions. For convenience of our discussions, we copy (1) here:

$$
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= a - u - \frac{\rho uv}{1 + u + ku^2}, & x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= \alpha b - \alpha v - \frac{\rho uv}{1 + u + ku^2}, & x \in \Omega, t > 0, \\
\partial_n u &= \partial_n v = 0, & x \in \partial \Omega, t \geq 0, \\
u(x,0) &= u_0(x) > 0, v(x,0) = v_0(x) > 0, & x \in \Omega. 
\end{align*}
$$

To begin with, we derive precise conditions so that system (2) has a unique positive constant equilibrium solution $(u_*, v_*)$.

**Lemma 2.1.** Suppose that $a$, $\alpha$, $b$, $\rho$, $k$ are all positive, $\frac{a}{b} < \alpha$ holds. If additionally, either
we have
Proof. By
\[(1 + \frac{\rho}{\alpha} - ak)^2 - 3k(1 + b\rho - a - \frac{\alpha}{\alpha}u) \leq 0\] (3)
or
\[(1 + \frac{\rho}{\alpha} - ak)^2 - 3k(1 + b\rho - a - \frac{\alpha}{\alpha}u) > 0, 1 + \frac{\rho}{\alpha} - ak > 0, 1 + b\rho - a - \frac{\alpha}{\alpha} \geq 0\] (4)
holds, then system (2) has a unique positive constant equilibrium solution \((u_*, v_*)\), with \(u_* \in (0, a)\).

Proposition 1. Suppose that \(a, \alpha, b, \rho, k\) are all positive, \(\frac{a}{b} < \alpha\) holds. If additionally, either (3) or (4) holds so that \((u_*, v_*)\) is the unique positive constant equilibrium solution. Then, for any \(a_1, a_2 > 0\), the parabolic system (2) has a unique solution \((u(x, t), v(x, t))\), defined for all \(x \in \Omega\) and \(t > 0\). Moreover, there exist two positive constants \(C_1\) and \(C_2\), depending on \(a, \alpha, b, \rho, K, u_0(x)\) and \(v_0(x)\), such that
\[C_1 < u(x, t), v(x, t) < C_2, x \in \Omega, t > 0.\] (11)
Proof. By [2], we can prove the existence and uniqueness of local-in-time solutions to the parabolic system (2). We now use the techniques in [2, 3, 10] to prove the global existence and boundedness of the solutions. We consider two cases:

Case 1. Suppose that \( 0 < \max_{x \in \Omega} v_0(x) \leq \frac{a}{\rho} \) holds. We construct the invariant rectangle \( R := [U_1, U_2] \times [V_1, V_2] \) in the following way:

\[
\begin{align*}
U_1 & := \min_{x \in \Omega} \left\{ a - \rho V_2, \min_{x \in \Omega} \{ u_0(x) \} \right\}, \\
U_2 & := \max_{x \in \Omega} \left\{ a, \max_{x \in \Omega} \{ u_0(x) \} \right\}, \\
V_1 & := \min_{x \in \Omega} \left\{ \frac{\alpha b}{\alpha + \rho}, \min_{x \in \Omega} \{ v_0(x) \} \right\}, \\
V_2 & := \max_{x \in \Omega} \left\{ b, \max_{x \in \Omega} \{ v_0(x) \} \right\}.
\end{align*}
\]

(12)

Clearly, \( u_0(x) \) and \( v_0(x) \) are closed by the rectangle \( R \). We now prove that the vector field points inward on the boundary of \( R \). In fact, on the bottom side of \( R \), \( v = V_1, U_1 \leq u \leq U_2 \), by the definition of \( V_1 \), we have

\[
\alpha(b - V_1) - \rho \frac{uV_1}{1 + u + ku^2} > \alpha(b - V_1) - \rho V_1 = ab - (\alpha + \rho)V_1 \geq 0.
\]

(13)

On the top side of \( R \), \( v = V_2, U_1 \leq u \leq U_2 \), by the definition of \( V_2 \), we have

\[
\alpha(b - V_2) - \rho \frac{uV_2}{1 + u + ku^2} < \alpha(b - V_2) \leq 0.
\]

(14)

Thus, \( R := [U_1, U_2] \times [V_1, V_2] \) is the invariant rectangle for the vector field. Choosing \( C_1 = \min\{U_1, V_1\} \) and \( C_2 = \max\{U_2, V_2\} \), we obtain the desired results.

Case 2. Suppose that \( \max_{x \in \Omega} v_0(x) > a/\rho \) holds. In this case, the aforementioned \( R \) is not the invariant rectangle anymore, since the last inequality in (16) fails. Since the inequalities in (12), (13) and (14) still hold, we can conduct the same analysis as in [10] to show that one can construct a new invariant rectangle as we did in Case 1. This leads to another suitable positive constants \( C_1 \) and \( C_2 \). So far, we have proved the global existence and boundedness of the solutions.

We then show that system (1) has an attraction region defined by

\[
A := \left( a - \rho b, a \right) \times \left( \frac{\alpha b}{\alpha + \rho}, b \right)
\]

in the phase plane which actually attracts all solutions of this system, regardless of the initial values \( u_0 \) and \( v_0 \).

Theorem 2.2. Assume that \( a, \alpha, b, \rho, k \) are all positive, \( \rho < \frac{a}{b} < \alpha \) holds. If additionally, either (3) or (4) holds, let \( (u(x, t), v(x, t)) \) be the unique solution of system (1). Then, for any \( x \in \Omega \), we have

\[
a - \rho b < \lim \inf_{t \to \infty} u \leq \lim \sup_{t \to \infty} u < a, \quad \text{and} \quad \frac{\alpha b}{\alpha + \rho} < \lim \inf_{t \to \infty} v \leq \lim \sup_{t \to \infty} v < b.
\]

(18)
Proof. 1). Firstly, we prove that \( \liminf_{t \to \infty} v > \frac{ab}{\alpha + \rho} \). By Proposition 1, there exists a sufficiently small \( \tau > 0 \) such that for all \( x \in \Omega \) and \( t > 0 \), \( \frac{\rho uv}{1 + u + ku^2} + \alpha v + \tau < (\rho + \alpha)v \) holds.

Let \( v_r = \min_{x \in \Omega} v \) be the unique solution of the following ODE:

\[
\frac{dv_r(t)}{dt} = \alpha b + \tau - (\rho + \alpha)v_r(t), \quad v_r(0) = (1 - \tau) \min_{x \in \Omega} v_0(x).
\]  \( (19) \)

Defining \( p_1(x, t) = v(x, t) - v_r(t) \), and by \( (19) \) and \( (19) \), we have

\[-\frac{\partial p_1(x, t)}{\partial t} + d_2 \Delta p_1(x, t) - (\rho + \alpha)p_1(x, t) = \frac{\rho uv}{1 + u + ku^2} - \tau - \rho v < 0, \]

\( p_1(x, 0) > 0. \)

By the maximum principle for parabolic equations, it follows \( p_1(x, t) > 0 \), equival., \( v(x, t) > v_r(t) \), for all \( x \in \Omega \) and \( t \geq 0 \). Since \( \lim_{t \to \infty} v_r(t) = (\alpha b + \tau)/(\rho + \alpha) \), we have \( \liminf_{t \to \infty} v > \frac{ab}{\alpha + \rho} \).

2). We now prove that \( \limsup_{t \to \infty} v < b \). By Proposition 1, there exists a sufficiently small \( 0 < \delta < b \) such that for all \( x \in \Omega \) and \( t > 0 \), \( \delta < \rho uv/(1 + u + ku^2) \) holds.

Let \( v_\delta = v_\delta(t) \) be the unique solution of the following ODE:

\[
\frac{dv_\delta(t)}{dt} = \alpha b - \delta - \alpha v_\delta(t), \quad v_\delta(0) = (1 + \delta) \max_{x \in \Omega} v_0(x).
\]  \( (21) \)

Letting \( p_2(x, t) = v(x, t) - v_\delta(t) \), and by \( (2) \) and \( (21) \), we have

\[-\frac{\partial p_2(x, t)}{\partial t} + d_2 \Delta p_2(x, t) - \alpha p_2(x, t) = \frac{\rho uv}{1 + u + ku^2} - \delta > 0, \quad p_2(x, 0) < 0. \]

By the maximum principle for parabolic equations, we have \( p_2(x, t) < 0 \), equiv., \( v(x, t) < v_\delta(t) \), for all \( x \in \Omega \) and \( t \geq 0 \). Since \( \lim_{t \to \infty} v_\delta(t) = b - \frac{\delta}{\alpha} \), we have \( \limsup_{t \to \infty} v < b \).

3). We then prove that \( \limsup_{t \to \infty} u < a \). By Proposition 1, there exists a sufficiently small \( 0 < \zeta < a \) such that for all \( x \in \Omega \) and \( t > 0 \), \( \zeta < \frac{\rho uv}{1 + u + ku^2} \) holds.

Let \( u_\zeta = u_\zeta(t) \) be the unique solution of the following ODE:

\[
\frac{du_\zeta(t)}{dt} = a - \zeta - u_\zeta(t), \quad u_\zeta(0) = (1 + \zeta) \max_{x \in \Omega} u_0(x).
\]  \( (23) \)

Defining \( w_1(x, t) = u(x, t) - u_\zeta(t) \), and by \( (2) \) and \( (23) \), we have

\[-\frac{\partial w_1(x, t)}{\partial t} + d_1 \Delta w_1(x, t) - w_1(x, t) = \frac{\rho uv}{1 + u + ku^2} - \zeta > 0, \quad w_1(x, 0) < 0. \]

By the maximum principle of parabolic equations, we have \( w_1(x, t) < 0 \), equiv., \( u(x, t) < u_\zeta(t) \), for all \( x \in \Omega \) and \( t \geq 0 \). Since \( \lim_{t \to \infty} u_\zeta(t) = a - \zeta \), we have \( \limsup_{t \to \infty} u < a \).

4). Finally, we prove that \( \liminf_{t \to \infty} u > a - \rho b \). By

\[
\limsup_{t \to \infty} v < b,
\]  \( (25) \)
we can find a finite number $t_0$, depending on $u_0$ and $v_0$, such that for any $t \geq t_0$ and all $x \in \Omega$,

$$v(x, t) < b,$$

(26)

By Proposition 1 and (26), there exists a sufficiently small $\chi > 0$, such that for all $x \in \Omega$ and $t \geq t_0$, one has

$$\frac{\rho uv}{1 + u + ku^2} < \rho b - \chi.$$ 

(27)

This can be done by choosing $\chi > 0$ sufficiently small, since when $\chi = 0$, (27) holds automatically.

Let $u_\chi$ be the unique solution of the following ODE:

$$\frac{du_\chi(t)}{dt} = a - u_\chi - \rho b + \chi, \quad t > t_0,$$

$$u_\chi(t_0) = (1 - \chi) \min_{x \in \Omega} u(x, t_0).$$

(28)

Letting $w_2(x, t) = u(x, t) - u_\chi(t)$, and by (2) and (28), we have

$$-\frac{\partial w_2(x, t)}{\partial t} + d_1 \Delta w_2(x, t) - w_2(x, t) = \frac{\rho uv}{1 + u + ku^2} - \rho b + \chi < 0, \quad t > t_0,$$

$$w_2(x, t_0) > 0.$$ 

(29)

Then by the maximum principle for parabolic equations, we have $w_2(x, t) > 0$, which implies that $u(x, t) > u_\chi(t)$ for all $x \in \Omega$. From (29), it follows that $\lim_{t \to \infty} u_\chi(t) = a - \rho b + \chi$. Thus, we have $\lim_{t \to \infty} \inf u > a - \rho b$. }

In what follows, we consider the long time behaviors of the solutions of the system (2). Following [1], we define $\sigma := d\lambda_1 - Q$, where $\lambda_1$ is the principal eigenvalue of $-\Delta$ on $\Omega$ subject to homogeneous Neumann boundary conditions, $d := \min\{d_1, d_2\}$, and

$$Q := \sup_{(u,v) \in A} \{|J(u, v)|\},$$

(30)

where

$$J(u, v) = \begin{pmatrix} -1 - \rho \rho_1(u, v) & -\rho \rho_2(u, v) \\ -\rho \rho_1(u, v) & -\alpha - \rho \rho_2(u, v) \end{pmatrix},$$

(31)

where

$$\rho_1(u, v) := \frac{v(1 - ku^2)}{(1 + u + ku^2)^2}, \quad \rho_2(u, v) := \frac{u}{1 + u + ku^2}.$$ 

(32)

Obviously, for $u, v > 0$, the following inequalities hold

$$|\rho_1(u, v)| < \frac{u + v}{1 + u + ku^2}, \quad |\rho_2(u, v)| < \frac{u + v}{1 + u + ku^2}.$$ 

(33)

For any $(u, v) \in A$, defined precisely in (17), we have

$$\frac{u + v}{1 + u + ku^2} < \rho := \frac{a + b}{1 + (a - \rho b) + k(a - \rho b)^2}.$$ 

(34)

Thus,

$$Q = \max \left\{ \sup_{(u,v) \in A} \{|\rho \rho_1(u, v)| + \rho \rho_2(u, v)|\}, \sup_{(u,v) \in A} \{|\rho \rho_1(u, v)| + \alpha + \rho \rho_2(u, v)|\} \right\}$$

$$< D := \max\{1 + 2\rho \rho, \alpha + 2\rho \rho\}.$$ 

(35)
Theorem 2.3. Suppose that $a$, $\alpha$, $b$, $\rho$, $k$ are all positive, $\frac{a}{b} < \alpha$ holds. If additionally, either (3) or (4) holds and that $(d_1, d_2) \in [D/\lambda_1, \infty) \times [D/\lambda_1, \infty)$, where $D$ is defined in (35). If (46) holds, then every solution $(u(x,t), v(x,t))$ of system (42) converges exponentially to $(u_*, v_*)$; while if (45) holds, then every solution $(u(x,t), v(x,t))$ of system (2) converges exponentially to the spatially homogeneous periodic solutions.

Proof. By Theorem 2.2 it follows that, there exists $T > 0$, such that for any $t > T$, the solution $(u(x,t), v(x,t)) \in \mathcal{A}$ for all $x \in \overline{\Omega}$. Without loss of generality, we can assume that $T = 0$.

Clearly, from (35), if $(d_1, d_2) \in [D/\lambda_1, \infty) \times [D/\lambda_1, \infty)$, then $\sigma > 0$. Define

$$f(u, v) := a - u - \frac{\rho uv}{1 + u + ku^2}, \quad g(u, v) := ab - \alpha v - \frac{\rho uv}{1 + u + ku^2}. \quad (36)$$

Then by (1),(3), there exist constants $N_i > 0$, $i = 1, 2, 3$, such that, for any solution $(u(t,x), v(t,x))$ of system (2)

$$||\nabla_x (u(\cdot,t), v(\cdot,t))||_{L^2(\Omega)} \leq N_1 e^{-\alpha t}, \quad ||(u(\cdot,t), v(\cdot,t)) - (\bar{u}(t), \bar{v}(t))||_{L^2(\Omega)} \leq N_2 e^{-\sigma t}, \quad (37)$$

where $\bar{u}$, $\bar{v}$ are the average of $u$ and $v$ over $\Omega$ respectively satisfying

$$\begin{cases}
\frac{d\overline{u}}{dt} = f(\overline{u}, \overline{v}) + o_1(t), \quad \overline{u}(0) = \frac{1}{|\Omega|} \int_{\Omega} u_0(x)dx, |o_1(t)| \leq N_3 e^{-\rho t}, \\
\frac{d\overline{v}}{dt} = g(\overline{u}, \overline{v}) + o_2(t), \quad \overline{v}(0) = \frac{1}{|\Omega|} \int_{\Omega} v_0(x)dx, |o_2(t)| \leq N_3 e^{-\rho t}.
\end{cases} \quad (38)$$

Moreover, the $\omega-$limit set of (38) is the subset of the $\omega-$limit set of the following ODEs

$$\begin{cases}
\frac{du}{dt} = f(u, v), \quad u(0) = \frac{1}{|\Omega|} \int_{\Omega} u_0(x)dx, \\
\frac{dv}{dt} = g(u, v), \quad v(0) = \frac{1}{|\Omega|} \int_{\Omega} v_0(x)dx.
\end{cases} \quad (39)$$

We now consider the dynamics of the ODE system of the original system (2), which was given by

$$\frac{du}{dt} = a - u - \frac{\rho uv}{1 + u + ku^2} =: f(u, v), \quad \frac{dv}{dt} = ab - \alpha v - \frac{\rho uv}{1 + u + ku^2} =: g(u, v). \quad (40)$$

The linearized operator of system (40) evaluated at $(u_*, v_*)$ is given by

$$J(u_*, v_*) = \begin{pmatrix} -1 - \rho h_1 & -\rho h_2 \\ -\alpha - \rho h_2 & -\rho h_1 \end{pmatrix}, \quad (41)$$

where

$$h_1 := \frac{v_*(1 - ku_*^2)}{(1 + u_* + ku_*^2)^2}, \quad h_2 := \frac{u_*}{1 + u_* + ku_*^2}. \quad (42)$$

Then, the characteristic equation of (41) is given by

$$\mu^2 + (1 + \rho h_1 + \rho h_2 + \alpha)\mu + \alpha + \rho h_2 + \rho \alpha h_1 = 0. \quad (43)$$
Suppose that $a, \alpha, b, \rho, k$ are all positive, $\frac{a}{b} < \alpha$ holds. If additionally, either (3) or (4) is satisfied so that $(u_*, v_*)$ is the unique positive constant equilibrium of (2). If

$$h_1 > -\frac{1 + \rho h_2 + \alpha}{\rho} \quad (44)$$

holds, then $(u_*, v_*)$ is locally asymptotically stable in system (40). However, if

$$h_1 < -\frac{1 + \rho h_2 + \alpha}{\rho} \quad (45)$$

holds, then $(u_*, v_*)$ is unstable in system (40), and the system (40) has a locally orbitally stable periodic orbit, denoted by $(p(t), q(t))$.

We now argue that if $\frac{a}{b} < \alpha$, additionally, either (3) or (4) holds, $0 < a - \alpha \leq 1$, and

$$k \in \left(0, \frac{2 + \alpha + \rho}{2a}\right] \cup \left[\frac{\epsilon - \sqrt{\epsilon^2 - a^2 k^2(2 + \alpha + \rho)^2}}{2a^2}, \frac{\epsilon + \sqrt{\epsilon^2 - a^2 k^2(2 + \alpha + \rho)^2}}{2a^2}\right],$$

hold, where

$$\epsilon = 4\alpha + \alpha^2 + 3 + \rho \alpha - a,$$

then, $(u_*, v_*)$ is globally asymptotically stable in system (40).

Define $B(u, v) = 1 + u + ku^2$, then, we have

$$\frac{\partial (fB)}{\partial u} + \frac{\partial (gB)}{\partial v} = \mathcal{X}(u) - \rho v,$$

where $\mathcal{X}(u) := -(3 + \alpha)ku^2 - (2 - 2ak + \alpha + \rho)u + a - \alpha - 1$.

Let $u_\mathcal{X}$ be the symmetry axis of the function $\mathcal{X}(u)$. Then, $u_\mathcal{X} = \frac{2ak - (2 + \alpha + \rho)}{2k(3 + \alpha)}$ holds. If $k \in (0, \frac{2 + \alpha + \rho}{2a}]$ holds, we have $u_\mathcal{X} \leq 0$. Thus, $\mathcal{X}(u) \leq 0$, which indicates that under $\partial (fB)/\partial u + \partial (gB)/\partial v < 0$ in the first quadrant.

On the other hand, let $\Delta_\mathcal{X}$ be the discriminant of the function $\mathcal{X}(u)$. Then,

$$\Delta_\mathcal{X} = (2 - 2ak + \alpha + \rho)^2 + 4k(3 + \alpha)(a - 1 - \alpha). \quad (49)$$

Suppose that $k \in \left[\frac{\epsilon - \sqrt{\epsilon^2 - a^2 k^2(2 + \alpha + \rho)^2}}{2a^2}, \frac{\epsilon + \sqrt{\epsilon^2 - a^2 k^2(2 + \alpha + \rho)^2}}{2a^2}\right]$ holds. Then $\Delta_\mathcal{X} \leq 0$. Again, we can conclude that $\mathcal{X}(u) \leq 0$, which indicates that under $\partial (fB)/\partial u + \partial (gB)/\partial v < 0$ in the first quadrant.

So far, under (46) and $0 < a - \alpha \leq 1$, by Dulac criteria, system (40) does not have closed orbits in the first quadrant. By Theorem 2.2, it follows that the solution is bounded. Thus, by Poincare-Bendixson Theorem, we know that $(u_*, v_*)$ is globally asymptotically stable in ODEs. \qed
3. Non-existence of Turing patterns: Some estimates. This part, we discuss the non-existence of the non-constant positive steady state solutions of the system:

\[
\begin{aligned}
-\frac{d_1 \Delta u}{1 + u + ku^2} &= a - u - \frac{\rho uv}{1 + u + ku^2}, & x & \in \Omega, \\
-\frac{d_2 \Delta v}{1 + u + ku^2} &= \alpha b - \alpha v - \frac{\rho uv}{1 + u + ku^2}, & x & \in \Omega, \\
\partial_n u &= \partial_n v = 0, & x & \in \partial \Omega.
\end{aligned}
\] (50)

Lemma 3.1. (A priori estimates) Suppose that \(a, \alpha, b, \rho, k\) are all positive, \(\frac{a}{b} < \alpha\) holds. If additionally, either (3) or (4) is satisfied, Let \((u(x), v(x))\) be any given positive steady state solution of system (1). Then, for any \(x \in \Omega\), the following conclusions hold:

\[
a - \rho b < u(x) < a, \quad \frac{ab}{\alpha + \rho} < v(x) < b.
\] (51)

Remark 1. Lemma 3.1 is the direct consequence of Theorem 2.2.

For a steady state solution pair \((u(x), v(x))\) of system (50), we define

\[
\phi(x) := u(x) - \bar{u}, \quad \psi(x) := v(x) - \bar{v},\text{ where, } \bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u(x)dx, \quad \bar{v} = \frac{1}{|\Omega|} \int_{\Omega} v(x)dx.
\] (52)

Multiplying the first equation of (50) by \(-1\), and adding the second equation of (50), we can obtain that

\[
\Delta(d_1 u - d_2 v) - u + a - \alpha b + \alpha v = 0.
\] (53)

Integrating (53) over \(\Omega\), we obtain

\[
a - \alpha b = \bar{u} - \alpha \bar{v}.
\] (54)

Define

\[
M_1 := \begin{cases} 
\frac{\lambda_1 d_1^2}{d_2(\alpha + \lambda_1 d_2)}, & \text{if } d_2 - \alpha d_1 \geq 0, \\
\frac{4\lambda_1 d_1^2}{(\alpha d_1 + d_2)^2 + 4\lambda_1 d_1 d_2^2}, & \text{if } d_2 - \alpha d_1 < 0.
\end{cases}
\] (55)

and

\[
M_2 := \begin{cases} 
\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 \alpha d_1 d_2^2}{4\lambda_1 \alpha d_1^2}, & \text{if } d_2 - \alpha d_1 \geq 0, \\
\frac{d_1(\lambda_1 d_1 + 1)}{\lambda_1 d_2^2}, & \text{if } d_2 - \alpha d_1 < 0.
\end{cases}
\] (56)

We are now stating the following useful estimates on the steady state solutions:

Lemma 3.2. Suppose that \((u(x), v(x))\) is the solution pair of (50), and let \(\phi(x), \psi(x)\) be defined in (52). Then, we have

\[
M_1 \int_{\Omega} |\nabla \phi|^2 dx \leq \int_{\Omega} |\nabla \psi|^2 dx \leq M_2 \int_{\Omega} |\nabla \phi|^2 dx
\] (57)

where \(\lambda_1\) is the principle eigenvalue of \(-\Delta\) on \(\Omega\) subject to the homogeneous Neumann boundary conditions.

Proof. Rewrite (53) as

\[
\Delta(d_1 u - d_2 v) - \phi(x) + \alpha \psi(x) = 0.
\] (58)
Multiplying (68) by \(d_1u - d_2v\), integrating over \(\Omega\) by parts, and noticing that \[
\int_\Omega \phi dx = \int_\Omega \psi dx = 0,
\] we can yield
\[
-\int_\Omega |\nabla (d_1u - d_2v)|^2 dx = d_1 \int_\Omega \phi^2 dx - (\alpha d_1 + d_2) \int_\Omega \phi \psi dx + \alpha d_2 \int_\Omega \psi^2 dx. \tag{59}
\]
Thus, we have
\[
(\alpha d_1 + d_2) \int_\Omega \phi \psi dx = d_1 \int_\Omega \phi^2 dx + \alpha d_2 \int_\Omega \psi^2 dx + \int_\Omega |\nabla (d_1u - d_2v)|^2 dx \geq 0. \tag{60}
\]
Multiplying (58) by \(\phi\) and integrate over \(\Omega\) by parts, we have
\[
\int_\Omega \phi^2 dx = -d_1 \int_\Omega |\nabla \phi|^2 dx + d_2 \int_\Omega \nabla \phi \nabla \psi dx + \alpha \int_\Omega \phi \psi dx, \tag{61}
\]
which implies that
\[
\alpha \int_\Omega \phi \psi dx = \int_\Omega \phi^2 dx + d_1 \int_\Omega |\nabla \phi|^2 dx - d_2 \int_\Omega \nabla \phi \nabla \psi dx. \tag{62}
\]
On the other hand, the left side of (59) also equals
\[
-\int_\Omega |\nabla (d_1u - d_2v)|^2 dx = -d_1^2 \int_\Omega |\nabla \phi|^2 dx + 2d_1d_2 \int_\Omega \nabla \phi \nabla \psi dx - d_2^2 \int_\Omega |\nabla \psi|^2 dx. \tag{63}
\]
Then, from (59), (62) and (63), we have
\[
\alpha d_2 \int_\Omega \psi^2 dx + d_2^2 \int_\Omega |\nabla \psi|^2 dx = d_1^2 \int_\Omega |\nabla \phi|^2 dx + d_2 \int_\Omega \phi^2 dx + (d_2 - \alpha d_1) \int_\Omega \phi \psi dx, \tag{64}
\]
Next, we discuss in there cases:
1) If \(d_2 - \alpha d_1 > 0\), then
\[
\alpha d_2 \int_\Omega \psi^2 dx + d_2^2 \int_\Omega |\nabla \psi|^2 dx \geq d_1^2 \int_\Omega |\nabla \phi|^2 dx. \tag{65}
\]
By Poincaré inequality, it follows that
\[
\int_\Omega \psi^2 dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla \psi|^2 dx. \tag{66}
\]
Then, we have
\[
\left(\frac{\alpha d_2}{\lambda_1} + d_2^2\right) \int_\Omega |\nabla \psi|^2 dx \geq d_1^2 \int_\Omega |\nabla \phi|^2 dx. \tag{67}
\]
\[
\int_\Omega |\nabla \psi|^2 dx \geq \frac{\lambda_1 d_2^2}{\alpha d_2 + \lambda_1 d_2} \int_\Omega |\nabla \phi|^2 dx. \tag{68}
\]
On the other hand, by Cauchy inequality, we have
\[
\int_\Omega \phi \psi dx \leq \frac{\alpha d_2}{d_2 - \alpha d_1} \int_\Omega \psi^2 dx + \frac{d_2 - \alpha d_1}{4\alpha d_2} \int_\Omega \phi^2 dx, \tag{69}
\]
By Poincaré inequality, it follows that
\[
\int_\Omega \phi^2 dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 dx. \tag{70}
\]
Thus, by (64), (69) and (70) we have
\[
\int_\Omega |\nabla \psi|^2 dx \leq \left[\frac{d_1^2}{d_2} + \frac{d_1}{\lambda_1 d_2} + \frac{(d_2 - \alpha d_1)^2}{4\lambda_1 \alpha d_2}\right] \int_\Omega |\nabla \phi|^2 dx, \tag{71}
\]
which implies that
\[
\int_{\Omega} |\nabla \psi|^2 \, dx \leq \frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 \alpha d_2^2}{4\lambda_1 \alpha d_2^2} \int_{\Omega} |\nabla \phi|^2 \, dx.
\] (72)

2) If \(d_2 - \alpha d_1 < 0\), then
\[
\alpha d_2 \int_{\Omega} \psi^2 \, dx + d_2^2 \int_{\Omega} |\nabla \psi|^2 \, dx + (\alpha d_1 - d_2) \int_{\Omega} \phi \psi \, dx = d_1^2 \int_{\Omega} |\nabla \phi|^2 \, dx + d_1 \int_{\Omega} \phi^2 \, dx.
\] (73)

By Cauchy inequality, and integrating (64) over \(\Omega\), we have
\[
\int_{\Omega} \phi \psi \, dx \leq \frac{d_1}{\alpha d_1 - d_2} \int_{\Omega} \phi^2 \, dx + \frac{\alpha d_1 - d_2}{4d_1} \int_{\Omega} \psi^2 \, dx.
\] (74)

Thus, by (73), (66) and (74)
\[
\int_{\Omega} |\nabla \psi|^2 \, dx \geq \frac{4\lambda_1 d_1^3}{(\alpha d_1 + d_2)^2 + 4\lambda_1 \alpha d_2^2} \int_{\Omega} |\nabla \phi|^2 \, dx.
\] (75)

On the other hand, by (73), (70) and (60)
\[
d_2^2 \int_{\Omega} |\nabla \psi|^2 \, dx \leq d_1 \int_{\Omega} \phi^2 \, dx + d_1^2 \int_{\Omega} |\nabla \phi|^2 \, dx \leq \frac{d_1}{\lambda_1} + d_1^2 \int_{\Omega} |\nabla \phi|^2 \, dx.
\] (76)

Thus, we have
\[
\int_{\Omega} |\nabla \psi|^2 \, dx \leq \frac{d_1(\lambda_1 d_1 + 1)}{\lambda_1 d_2^2} \int_{\Omega} |\nabla \phi|^2 \, dx.
\] (77)

3) If \(d_2 - \alpha d_1 = 0\), then
\[
\alpha d_2 \int_{\Omega} \psi^2 \, dx + d_2^2 \int_{\Omega} |\nabla \psi|^2 \, dx = d_1^2 \int_{\Omega} |\nabla \phi|^2 \, dx + d_1 \int_{\Omega} \phi^2 \, dx.
\] (78)

We have,
\[
d_2^2 \int_{\Omega} |\nabla \psi|^2 \, dx \leq \left(\frac{d_1}{\lambda_1} + d_1^2\right) \int_{\Omega} |\nabla \phi|^2 \, dx.
\] (79)

**Remark 2.** when \(d_2 - \alpha d_1 = 0\), we have
\[
\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 \alpha d_2^2}{4\lambda_1 \alpha d_2^2} = \frac{d_1(1 + \lambda_1 d_1)}{\lambda_1 d_2^2}.
\] (80)

On the other hand, we have
\[
d_1^2 \int_{\Omega} |\nabla \phi|^2 \, dx \leq \left(\frac{\alpha d_2}{\lambda_1} + d_2^2\right) \int_{\Omega} |\nabla \psi|^2 \, dx.
\] (81)

We thus complete the proof. \(\square\)

For the later discussions, we define
\[
\begin{align*}
\ell_1(u) & := \frac{(k\pi u - 1)(\pi + \alpha b - a)}{\alpha(1 + \pi + k\pi^2)(1 + u + ku^2)}, \\
\ell_2(u) & := \frac{\alpha(1 + \pi + k\pi^2)(\pi + \alpha b - a)}{\pi(1 + ku + k\pi)(\pi + \alpha b - a)}, \\
\ell_3(u) & := \frac{\pi}{1 + u + ku^2}
\end{align*}
\] (82)

where \(\pi\) are defined precisely in (52).
From Lemma 3.1, it follows that, any positive solutions \((u, v)\) of system (50) satisfies \( (u, v) \in A \), where \( A \) is defined in (17). Define
\[
L_i := \sup_{u \in (a - \rho b, a)} |l_i(u)|, \quad i = 1, 2, 3,
\]
(83)

We are now in the position to state the following theorem regarding the non-existence of non-constant positive solutions of the system (50):

**Theorem 3.3.** Let \( L_i(u) \) and \( L_i \), \( i = 1, 2, 3 \) be defined in (82) and (83) respectively. Then, for any \((d_1, d_2) \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3\), system (50) does not have non-constant positive solutions, where

\[
\Sigma_1 := \left\{ (d_1, d_2) \in \mathbb{R}^2 : d_2 > 0, \frac{d_2}{\alpha} \geq d_1 > \frac{\rho L_1}{\lambda_1} \sqrt{1 + \frac{\alpha}{\lambda_1^2 d_2}} \right\},
\]

\[
\Sigma_2 := \left\{ (d_1, d_2) \in \mathbb{R}^2 : d_2 > \frac{\rho L_1}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 d_1 d_2^2}{4\lambda_1 d_1^2}} \right\},
\]

\[
d_2 - \alpha d_1 < 0, \quad d_1 > 0, \quad d_2 > 0 \right\},
\]

\[
\Sigma_3 := \left\{ (d_1, d_2) \in \mathbb{R}^2 : d_1 > \frac{\rho L_2}{\lambda_1} + \frac{\rho L_3}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 d_1 d_2^2}{4\lambda_1 d_1^2}} \right\},
\]

\[
d_2 - \alpha d_1 \geq 0, \quad d_1 > 0, \quad d_2 > 0 \right\},
\]

\[
\Sigma_4 := \left\{ (d_1, d_2) \in \mathbb{R}^2 : d_1 > \frac{\rho L_2}{\lambda_1} + \frac{\rho L_3}{\lambda_1} \sqrt{\frac{d_1 (\lambda_1 d_1 + 1)}{\lambda_1 d_1^2}} \right\},
\]

\[
d_2 - \alpha d_1 < 0, \quad d_1 > 0, \quad d_2 > 0 \right\}.
\]

**Proof.** We first prove that if \((d_1, d_2) \in \Sigma_1 \cup \Sigma_2\), then system (50) does not have non-constant positive solutions.

Multiplying the second equation of (50) by \( \psi \) and integrating over \( \Omega \), we have
\[
d_2 \int_\Omega |\nabla \psi|^2 dx = \int_\Omega \alpha (b - v) \psi dx - \rho \int_\Omega \frac{uv}{1 + u + ku^2} \psi dx.
\]
(85)

A direct calculation shows that
\[
d_2 \int_\Omega |\nabla \psi|^2 dx
\]
\[
= -\alpha \int_\Omega \psi^2 dx - \rho \int_\Omega \frac{u(v - \bar{v})}{1 + u + ku^2} + \frac{u\bar{v}}{1 + u + ku^2} - \frac{\bar{v}}{1 + \bar{u} + k\bar{u}^2} |\psi| dx
\]
\[
= -\alpha \int_\Omega \psi^2 dx - \rho \int_\Omega \frac{u}{1 + u + ku^2} \psi^2 dx + \rho \int_\Omega \frac{\bar{v}}{1 + u + ku^2} - \frac{u\bar{v}}{1 + u + ku^2} \psi dx
\]
\[
= -\alpha \int_\Omega \psi^2 dx - \rho \int_\Omega \frac{u}{1 + u + ku^2} \psi^2 dx + \rho \int_\Omega \frac{\bar{v}}{1 + u + ku^2} - \frac{u\bar{v}}{1 + u + ku^2} \phi \psi dx
\]
\[
= -\alpha \int_\Omega \psi^2 dx - \rho \int_\Omega \frac{u}{1 + u + ku^2} \psi^2 dx + \rho \int_\Omega l_1(u) \phi \psi dx
\]
\[
\leq \rho L_1 \int_\Omega |\phi \psi| dx.
\]
(86)
On the other hand, we have
\[
\int_{\Omega} |\phi\psi| dx \leq \left( \int_{\Omega} |\phi|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\psi|^2 dx \right)^{\frac{1}{2}} \leq \frac{1}{\lambda_1} \left( \int_{\Omega} |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \psi|^2 dx \right)^{\frac{1}{2}}. \tag{87}
\]
By (87) and (86), we have
\[
d_2 \int_{\Omega} |\nabla \psi|^2 dx \leq \frac{\rho L_1}{\lambda_1} \left( \int_{\Omega} |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \psi|^2 dx \right)^{\frac{1}{2}}. \tag{88}
\]
Then we discuss in two cases:

**Case 1.** \(d_2 - \alpha d_1 \geq 0\).

By (88) and Lemma 3.2, we have
\[
d_2 \int_{\Omega} |\nabla \psi|^2 dx \leq \frac{\rho L_1}{\lambda_1} \sqrt{\frac{d_2(\alpha + \lambda_1 d_2)}{\lambda_1 d_1^2}} \int_{\Omega} |\nabla \psi|^2 dx. \tag{89}
\]
For any \((d_1, d_2) \in \Sigma_1\), we have
\[
d_2 > \frac{\rho L_1}{\lambda_1} \sqrt{\frac{d_2(\alpha + \lambda_1 d_2)}{\lambda_1 d_1^2}}. \tag{90}
\]
Thus, we have \(\nabla \psi \equiv 0\). By Lemma 3.3, we have \(\nabla \phi \equiv 0\). Then system (50) does not have non-constant positive solutions.

**Case 2.** \(d_2 - \alpha d_1 < 0\).

By (88) and Lemma 3.2, we have
\[
d_2 \int_{\Omega} |\nabla \psi|^2 dx \leq \frac{\rho L_1}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 d_1 d_2}{4\lambda_1 d_1^2}} \int_{\Omega} |\nabla \psi|^2 dx. \tag{91}
\]
For any \((d_1, d_2) \in \Sigma_2\), we have
\[
d_2 > \frac{\rho L_1}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 d_1 d_2}{4\lambda_1 d_1^3}}. \tag{92}
\]

**Remark 3.** In particular, when
\[
(d_1, d_2) \in \left\{ (d_1, d_2) \in \mathbb{R}^2 : d_2 > \frac{\rho L_1 \alpha}{\lambda_1}, d_1 > \max\left\{ \frac{\rho^2 L_1^2 \alpha^2}{\lambda_1 d_2^2 - \lambda_1 \rho^2 L_1^2 \alpha^2}, \frac{d_2}{\alpha} \right\} \right\}, \tag{93}
\]
we have
\[
d_2 > \frac{\rho L_1}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4\lambda_1 d_1 d_2}{4\lambda_1 d_1^3}}. \tag{94}
\]
Thus, we have \(\nabla \psi \equiv 0\). By Lemma 3.3, we have \(\nabla \phi \equiv 0\). Then system (50) does not have non-constant positive solutions.

Multiplying the second equation of (50) by \(\phi\) and integrating over \(\Omega\), we have
\[
d_1 \int_{\Omega} |\nabla \phi|^2 dx = \int_{\Omega} (a - u)\phi dx - \rho \int_{\Omega} \frac{uv}{1 + u + ku^2} \phi dx. \tag{95}
\]
A direct calculation shows that

$$d_1 \int_{\Omega} |\nabla \phi|^2 dx$$

$$= - \int_{\Omega} \phi^2 dx - \rho \int_{\Omega} \left[ \frac{(u - \pi) v}{1 + u + k u^2} + \frac{\pi v}{1 + u + k u^2} - \frac{v}{1 + u + k u^2} \right] \phi dx$$

$$= - \int_{\Omega} \phi^2 dx - \rho \int_{\Omega} \frac{v}{1 + u + k u^2} \phi^2 dx + \rho \int_{\Omega} \left[ \frac{v}{1 + u + k u^2} - \frac{v}{1 + u + k u^2} \right] \phi dx$$

$$= - \int_{\Omega} \phi^2 dx - \rho \int_{\Omega} \frac{v}{1 + u + k u^2} \phi^2 dx$$

$$+ \frac{\rho \pi v}{1 + \pi + k \pi^2} \int_{\Omega} \frac{1}{1 + u + k u^2} \phi^2 dx - \rho \int_{\Omega} \frac{1}{1 + u + k u^2} \phi \psi dx$$

$$= - \int_{\Omega} \phi^2 dx - \rho \int_{\Omega} \frac{v}{1 + u + k u^2} \phi^2 dx + \rho \int_{\Omega} l_2(u) \phi^2 dx - \rho \int_{\Omega} l_3(u) \phi \psi dx$$

$$\leq \rho L_2 \int_{\Omega} \phi^2 dx + \rho L_3 \int_{\Omega} |\phi| \psi dx$$

$$\leq \rho L_2 \lambda_1 \int_{\Omega} |\nabla \phi|^2 dx + \rho L_3 \lambda_1 \left( \int_{\Omega} |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\psi|^2 dx \right)^{\frac{1}{2}}.$$  \hspace{1cm} (96)

Then we discuss in two cases:

**Case 1.** $d_2 - \alpha d_1 \geq 0$.

By (96) and Lemma 3.2, we have

$$d_1 \int_{\Omega} |\nabla \phi|^2 dx \leq \frac{\rho L_2}{\lambda_1} + \frac{\rho L_3}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4 \lambda_1 \alpha d_1^2 d_2}{4 \lambda_1 \alpha d_1^2}} \int_{\Omega} |\nabla \phi|^2 dx.$$  \hspace{1cm} (97)

For any $(d_1, d_2) \in \Sigma_3$, we have

$$d_1 > \frac{\rho L_2}{\lambda_1} + \frac{\rho L_3}{\lambda_1} \sqrt{\frac{(\alpha d_1 + d_2)^2 + 4 \lambda_1 \alpha d_1^2 d_2}{4 \lambda_1 \alpha d_1^2}}.$$  \hspace{1cm} (98)

Thus, we have $\nabla \phi \equiv 0$. By Lemma 3.3, we have $\nabla \psi \equiv 0$. Then system (50) does not have non-constant positive solutions.

**Case 2.** $d_2 - \alpha d_1 < 0$. By (96) and Lemma 3.2, we have

$$d_1 \int_{\Omega} |\nabla \phi|^2 dx \leq \frac{\rho L_2}{\lambda_1} + \frac{\rho L_3}{\lambda_1} \sqrt{\frac{d_3(\lambda_1 d_1 + 1)}{\lambda_1 d_2^2}} \int_{\Omega} |\nabla \phi|^2 dx.$$  \hspace{1cm} (99)

For any $(d_1, d_2) \in \Sigma_4$, we have

$$d_1 > \frac{\rho L_2}{\lambda_1} + \frac{\rho L_3}{\lambda_1} \sqrt{\frac{d_3(\lambda_1 d_1 + 1)}{\lambda_1 d_2^2}}.$$  \hspace{1cm} (100)

For any $(d_1, d_2) \in \Sigma_4$, we have $\nabla \phi \equiv 0$. By Lemma 3.3, we have $\nabla \psi \equiv 0$. Thus, system (50) does not have non-constant positive solutions. \hfill \Box

4. **Existence of Turing patterns: Global steady state bifurcations.** In this section, we use the global steady state bifurcation theory to consider the existence of positive non-constant of steady state system (50), in particular the existence of Turing patterned solutions.
Let \( h_1 \) and \( h_2 \) be defined precisely in \( \{12\} \). Then, if \( h_1 < -\frac{1}{\rho} \) holds, system \( \{2\} \) is a substrate-inhibition system. That is, the Jacobian matrix of the corresponding ODEs evaluated at \((u_*, v_*)\) takes in the form of

\[
\begin{pmatrix}
  +, & - \\
  +, & -
\end{pmatrix}.
\]

And if \( a, \alpha, b, \rho, k \) are all positive, \( \frac{a}{b} < \alpha \) holds. If additionally, either \( \{3\} \) or \( \{4\} \) is satisfied, and

\[-\frac{1 + \alpha + \rho h_2}{\rho} < h_1, \]

is satisfied, then \((u_*, v_*)\) is positive and stable in the corresponding ODEs system.

Thus, in the rest of the paper, we always assume that \( \frac{a}{b} < \alpha \) holds. If additionally, either \( \{3\} \) or \( \{4\} \) is satisfied, and

\[-\frac{1 + \alpha + \rho h_2}{\rho} < h_1 < -\frac{1}{\rho} \]

are satisfied.

The linearized operator of system \( \{50\} \) evaluated at \((u_*, v_*)\) is given by (choosing \( d_1 \) as the bifurcation parameter)

\[
L(d_1) = \begin{pmatrix}
  d_1 \Delta - 1 - \rho h_1, & -\rho h_2 \\
  -\rho h_1 & d_2 \Delta - \alpha - \rho h_2
\end{pmatrix}.
\]

Let \( \lambda_i \) and \( \xi_i(x) \), \( i \in \mathbb{N}_0 \), be the eigenvalues and the corresponding eigenfunctions of \(-\Delta \) in \( \Omega \) subject to Neumann boundary conditions. Then, by \{5\} \{11\}, the eigenvalues of \( L(d_1) \) are given by those of the following operator \( L_i(d_1) \):

\[
L_i(d_1) = \begin{pmatrix}
  -d_1 \lambda_i - 1 - \rho h_1, & -\rho h_2 \\
  -\rho h_1 & -d_2 \lambda_i - \alpha - \rho h_2
\end{pmatrix},
\]

whose characteristic equation is

\[
\mu^2 - \mu T_i(d_1) + D_i(d_1) = 0, \quad i \in \mathbb{N}_0,
\]

where

\[
\begin{cases}
  T_i(d_1) := -(d_1 + d_2) \lambda_i - (1 + \rho h_1 + \alpha + \rho h_2), \\
  D_i(d_1) := d_1 d_2 \lambda_i^2 + (d_1 + \rho h_1 d_2 + \alpha d_1 + \rho h_2 d_1) \lambda_i + \alpha + \rho h_1 \alpha + \rho h_2.
\end{cases}
\]

According to \{8\} \{11\}, if there exist \( i \in \mathbb{N}_0 \) and \( d_1^* > 0 \), such that

\[
D_i(d_1^*) = 0, \quad T_i(d_1^*) \neq 0, \quad T_j(d_1^*) \neq 0, \quad D_j(d_1^*) \neq 0 \quad \text{for all } j \neq i,
\]

and the derivative \( \frac{d}{dd_1} D_i(d_1^*) \neq 0 \), then a global steady state bifurcation occurs at the critical point \( d_1^* \).

By \( \{103\} \), we have \( T_0(d_1) < 0 \). Thus, for all \( i \in \mathbb{N}_0 \), we have \( T_i(d_1) < 0 \). Solving \( D_i(d_1) = 0 \), we have the set of critical values of \((d_1, d_2)\), given by the hyperbolic curves \( C_i \), with \( i \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\} \):

\[
(C_i) : d_2^i = \frac{(\alpha + \rho h_1 \alpha + \rho h_2)/\lambda_i^2}{d_1 + (1 + \rho h_1)/\lambda_i} - \frac{\alpha d_1 + \rho h_2 d_1}{d_1 \lambda_i + 1 + \rho h_1}, \quad i \in \mathbb{N}.
\]

Suppose that \( \lambda_i, i \in \mathbb{N} \), is the simple eigenvalue of \(-\Delta \). Following \{3\}, we call \( B := \bigcup_{i=1}^\infty C_i \) the bifurcation set with respect to \((u_*, v_*)\), and denote by \( \hat{B} \) the countable set of intersection points of two curves of \( \{C_i\}_{i=1}^\infty \), and \( \hat{B} = B \setminus B_0 \).
Clearly, for any fixed $d_2 > 0$, there exists a unique $d_i^0$ such that $(d_1^0, d_2) \in \hat{B} \cap C_i$, and at $d = d_1^0$, both $d \left(\frac{d}{dd_1} D_i(d_i^0)\right) \neq 0$ are satisfied.

Then, from [6, 11], we have the following results regarding the existence of Turing patterns:

**Theorem 4.1.** Suppose that (105) holds and that $C_i$ is defined in (108), where $\lambda_i$, $i \in \mathbb{N}$, is the simple eigenvalue of $-\Delta$. Then for any $(d_1^0, d_2) \in \hat{B} \cap C_i$ with $d_2$ fixed, there is a smooth curve $\Gamma_i$ of positive solutions of (50) bifurcating from $(d_1^0, u, v) = (d_1^0, u_+, v_+)$, with $\Gamma_i$ contained in a global branch $\hat{C}_i$ of the positive solutions of (50). Moreover

1. Near $(d_1^0, u, v) = (d_1^0, u_+, v_+)$, $\Gamma_i = \{(d_1(s), u(s), v(s)) : s \in (-\epsilon, \epsilon)\}$, where $u(s) = u_+ + s \alpha_i \xi_i(x) + \text{so}_i(s)$, $v(s) = v_+ + s \beta_i \xi_i(x) + \text{so}_2(s)$ for $s \in (-\epsilon, \epsilon)$ for some $C^\infty$ smooth functions $d_1(s), a_i(s), o_2(s)$ such that $d_1(0) = d_1^0$ and $a_i(0) = o_2(0) = 0$. Here $a_i$ and $b_i$ satisfy $L_i(d_1)(a_i, b_i)^T = (0, 0)^T$, and $\xi_i(\cdot)$ is the corresponding eigenfunction of the eigenvalue $\lambda_i$ of $-\Delta$.

2. Moreover, the projection of $\Gamma_i$ onto $d_1$-axis contains the interval $(0, d_1^0)$.

**Proof.** By the definition of $d_1^0$, it follows from Theorem 3.2 in [11] that at $d_1 = d_1^0$, global steady state bifurcations exist. Then, by Theorem 2.3 of [6], we can rule out the possibility that $\hat{C}_i$ contains another $(d_1^0, u_+, v_+)$ with $i \neq j$. We thus complete the proof of this theorem.

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