Exact inversion of Funk-Radon transforms with non-algebraic geometries

Victor Palamodov

November 29, 2017

Abstract. Any even function defined on 2-sphere is reconstructed from its integrals over big circles by means of the classical Funk formula. For the non-geodesic Funk transform on the sphere of arbitrary dimension, there is the explicit inversion formula similar to that for the geodesic transform. A function defined on the sphere of radius one is integrated over traces of hyperplanes tangent to a sphere contained in the unit ball. This reconstruction is generalized in the paper for Riemannian hypersurfaces in an affine space.

MSC (2010) Primary 53C65; Secondary 44A12

1 Introduction

A Riemannian manifold $X$ is embedded as a hypersurface in an affine space. A function defined on $X$ is integrated over intersections of $X$ with hyperplanes tangent to an ellipsoid $\Sigma$ (called cam). We prove the reconstruction formula that looks like the inversion formula for the non-geodesic Funk transform on the sphere stated in [2]. The only condition on $X$ and $\Sigma$ is that no three points $x, y, \sigma$ are collinear.

2 Auxiliary

Let $X$ and $\Sigma$ be manifolds of dimension $n > 1$ with volume forms $dX$ and $d\Sigma$ and $\Phi$ be a real smooth function defined on $X \times \Sigma$ such that $d\Phi (x, \sigma) \neq 0$ as $\Phi (x, \sigma) = 0$. The Funk-Radon transform $M_{\Phi}$ generated by this function is defined by

$$
M_{\Phi} f (\sigma) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{|\Phi| \leq \varepsilon} f dX = \int_{Z(x)} f (x) \frac{dX}{d\Phi} \sigma \in \Sigma.
$$

Suppose that (I): the map $D : Z \times \mathbb{R}_{+} \to T^{*} (X) \setminus 0$ is a diffeomorphism, where $Z = \{ \Phi (x, \sigma) = 0 \}$ and $D (x, \sigma, t) = (x, td_{\sigma} \Phi (x, \sigma))$. This implies that for any $x \in X$, set $Z (x) = \{ \sigma ; \Phi (x, \sigma) = 0 \}$ is diffeomorphic to the sphere $S^{n-1}$. Points $x, y \in X$ are called conjugate for a generating function $\Phi$, if $x \neq y$, $\Phi (x, \sigma) =$
Φ(y, σ) = 0 and d_σΦ(x, σ) \parallel d_σΦ(y, σ) for some σ ∈ Σ. Under condition (I) and condition (II): there are no conjugate points, the integral

\[ Q_n(x, y) = \int_{Z(y)} (Φ(x, σ) - i0)^{-n} \frac{dΣ}{d_σΦ(y, σ)} \]

is well defined for any x, y ∈ X, y ≠ x.

**Theorem 1** Let dX be the volume form of a Riemannian metric g on X and Φ be a generating function satisfying (I), (II) and condition (III):

\[ \text{Re } i^nQ_n(x, y) = 0 \text{ for all } x, y ∈ X \text{ such that } x \neq y. \quad (1) \]

For any odd n, an arbitrary function \( f \in C^{n-1}(X) \) with compact support can be reconstructed from data of the Funk-Radon transform by

\[ f(x) = \frac{1}{2j^{n-1}D_n(x)} \int_{Σ} \delta^{(n-1)}(Φ(x, σ)) M_Φ f(σ) dΣ. \quad (2) \]

For even n, any function \( f \in C^{n-1+ε}(X) \), is recovered by

\[ f(x) = \frac{(n-1)!}{j^nD_n(x)} \int_{Σ} M_Φ f(σ) \frac{dΣ}{Φ(x, σ)^n}, \quad (3) \]

where for any n

\[ D_n(x) = \frac{1}{|S^{n-1}|} \int_{Z(x)} \frac{1}{|∇_xΦ(x, σ)|_g^n d_σΦ(x, σ)}. \]

The integrals (3) and (2) converge to f uniformly on any compact set K ⊂ X.

See [2] for a proof. The singular integrals like (3) and (2) are defined as follows

\[ \int \frac{ω}{Φ^n} = \frac{1}{2} \left( \int \frac{ω}{(Φ - i0)^n} + \int \frac{ω}{(Φ + i0)^n} \right), \]
\[ \int_{Σ} δ^{(n-1)}(Φ) ω = (-1)^{n-1} \frac{(n-1)!}{2πi} \left( \int \frac{ω}{(Φ - i0)^n} - \int \frac{ω}{(Φ + i0)^n} \right) \]

for any smooth n-form ω.

### 3 Reconstructions

**Theorem 2** Let \( E^{n+1} \) be an affine space with an invariant volume form dV and X be a smooth hypersurface in \( E^{n+1} \) (occasionally not closed) with a Riemannian metric g. Let Σ be an ellipsoid in \( E^{n+1} \) such that (E): any line that meets X at least twice or is tangent to X does not touch Σ. Then for any odd n, any function \( f \in C_0^{n-1}(X) \) can be recovered from integrals

\[ M_Φ f(σ) = \int_{Σ} δ(⟨x - σ, ∇q(σ)⟩) f(x) d_σX = \int_{Z(σ)} f(x) \frac{d_σX}{⟨dx, ∇q(σ)⟩}, \sigma ∈ Σ \]
by
\[ f(x) = \frac{1}{2^{j-1}D_n(x)} \int_{\Sigma} \delta^{(n-1)}(\langle x - \sigma, \nabla q(\sigma) \rangle) M_\Phi f(\sigma) d\Sigma. \]

For any even \( n \), an arbitrary function \( f \in C^{n-1+\varepsilon}_0(X) \) can be reconstructed by
\[ f(x) = \frac{(n-1)!}{\int^{n} \Delta_n(x) \int_{\Sigma} \frac{M_\Phi f(\sigma)}{\langle x - \sigma, \nabla q(\sigma) \rangle} d\Sigma. \]

For any \( n \), \( Z(\sigma) = \{ x; \langle x - \sigma, \nabla q(\sigma) \rangle = 0 \} \), \( d\Sigma = dV/dq \) and
\[ \Delta_n(x) \equiv \frac{1}{|S^{n-1}|} \int_{Z(x)} \frac{1}{|\nabla_x \Phi|_g} d\sigma \langle x - \sigma, \nabla q(\sigma) \rangle, \ x \in X. \]

The integrals converge uniformly on any compact set in \( X \).

**Remark 1.** Here \( d_g X \) is the volume form of the Riemannian metric \( g \) on \( X \) and \( |\cdot|_g \) is the Riemannian norm of a covector.

**Remark 2.** Generating function \( \Phi \) can be replaced by
\[ \Phi'(x, \sigma) = \langle x - e, \nabla q(\sigma) \rangle - r, \ r = 2 - 2q(e) \quad (4) \]
where \( e \) is the center of \( \Sigma \). It coincides with \( \Phi \) on \( X \times \Sigma \) and is linear in \( \sigma \). The ellipsoid can be replaced by an arbitrary hyperboloid \( H \) in \( E^{n+1} \) and the volume form \( T^*(dV) \) where \( T \) is a projective transform of the ambient projective space \( P^{n+1} \) such that \( T(H) \) is an ellipsoid.

**Remark 3.** If \( Y \) is a closed and convex manifold in \( E^{n+1} \), and the cam is inside of \( Y \), then for any \( \tau \in \Sigma \), the manifold \( X = \{ x \in Y: \Phi(x, \tau) > 0 \} \) fulfills (\( E \)). If \( Y \) is a sphere with the center at a one-point cam then \( X \) is a hemisphere and Theorem 2 provides inversion of the Funk theorem \([1]\). If \( X \) is a hyperplane and the cam is a point in \( E^{n+1} \) or at infinity Theorem 2 is equivalent to the Radon inversion theorem. In the latter case to fulfill (\( I \)) one need to take each hyperplane through the cam point two times with the opposite conormal vectors.

Theorem 2 was obtained in \([2]\) for the case \( X \) is a subset of the sphere \( S^n \) and the cam is a sphere in the inside \( S^n \). This result with one-point cam was considered also in \([3]\). Note that for arbitrary \( X \) and one-point cam \( \{ e \} \), Theorem 2 is reduced to Funk’s result by the central projection of \( X \) to the unit sphere \( S^n \) with the center \( e \).

### 4 Proof

The reconstruction formulas as above are invariant with respect to any affine transformation. Therefore we can assume that \( \Sigma \) is a sphere (which makes some geometric arguments more obvious). The function \( \Phi \) generates the family of hyperplane sections of \( X \) since any hyperplane \( H \) tangent to \( \Sigma \) can be written in the form \( H_\sigma = \{ x; \langle x - \sigma, \nabla q(\sigma) \rangle = 0 \} \) for some \( \sigma \in \Sigma \). Now we check that \( \Phi(x, \sigma) = \langle x - \sigma, \nabla q(\sigma) \rangle \) satisfies conditions (\( I \)), (\( II \)) and (\( III \)) as in Sect.2.
Lemma 3 $\Phi$ satisfies (I).

Proof. We have $d_x \Phi \neq 0$ on $Z$ since of (E). For any point $x \in X$ and any covector $v \in T^*_x (X)$, $v \neq 0$, there exists one and only one hyperplane $H_\sigma$ such that $x \in H_\sigma$ and $v = td_x \Phi (x, \sigma)$ on $T^*_x (X)$ for some $t > 0$. It follows that the map $D_X$ is bijective. We prove that $D_X$ is a local diffeomorphism. This condition can be written in the form

$$\det J_{\xi, \tau} (x, \sigma) \neq 0, \quad (x, \sigma) \in Z,$$

where

$$J_{\xi, \tau} = \begin{pmatrix} t \nabla_\xi \Phi & \nabla_\xi \nabla_\tau \Phi \\ 0 & \nabla_\tau \Phi \end{pmatrix}$$

and $\xi, \tau$ are arbitrary local systems of coordinates on $X$ and $\Sigma$ respectively. Let $T = (t, t_0)$ be a $n + 1$-vector such that $TJ = 0$ where

$$J_{\xi, \tau} = \begin{pmatrix} \langle \partial x / \partial \xi, \nabla q \rangle & \langle \partial x / \partial \xi \times \partial \sigma / \partial \tau, \nabla^2 q (\sigma) \rangle \\ \langle (x - \sigma) \times \partial \sigma / \partial \tau, \nabla^2 q (\sigma) \rangle & \langle (x - \sigma) \rangle \end{pmatrix}$$

and $\langle \partial \sigma / \partial \tau, \nabla q (\sigma) \rangle = 0$ since $q$ is constant on $\Sigma$. Equation $TJ = 0$ is equivalent to

$$\langle (t, \partial x / \partial \xi), \nabla q \rangle = 0,$$

$$\langle (t, \partial x / \partial \xi) + t_0 (x - \sigma) \times \partial \sigma / \partial \tau_j, \nabla^2 q (\sigma) \rangle = 0, \quad j = 1, ..., n. \quad (7)$$

Vector $(t, \partial x / \partial \xi)$ is tangent to $X$ and (7) means that it is also tangent to $\Sigma$ at $\sigma$. Vector $x - \sigma$ is also tangent to $\Sigma$ since of $\Phi (x, \sigma) = 0$. Therefore there exist constants $c_1, ..., c_n$ such that

$$\theta \triangleq c_1 \partial \sigma / \partial \tau_1 + ... + c_n \partial \sigma / \partial \tau_n = \langle t, \partial x / \partial \xi \rangle + t_0 (x - \sigma).$$

Taking the corresponding linear combination of equations (7) we get

$$\langle \theta \times \theta, \nabla^2 q (\sigma) \rangle = 0$$

which implies $\theta = 0$ since the form $\nabla^2 q$ is strictly positive. It follows that $\langle t, \partial x / \partial \xi \rangle + t_0 (x - \sigma) = 0$ which implies that both terms vanish since the first one is tangent to $X$ and the second one is transversal to $X$. Finally $t = 0$, $t_0 = 0$ and $T = 0$ which completes the proof of (II) and of the Lemma. \hfill \Box

Condition (II). Check that generating function $\Phi$ coincides with $[I]$. This follows from

$$\Phi' (x, \sigma) - \Phi (x, \sigma) = \langle x - e, \nabla q (\sigma) \rangle - r = 2 (q (\sigma) - q (e)) - r = 0 \quad (8)$$

since $q (\sigma) - q (e)$ is a quadratic form of $\sigma - e$, $\sigma \in \Sigma$. Suppose that this condition violates for some points $x, y \in X$. We have then $a \langle x - e, \nabla^2 q (\sigma) \rangle = b \langle y - e, \nabla^2 q (\sigma) \rangle$ for some vector $(a, b) \neq (0, 0)$ and a point $\sigma \in \Sigma$. This implies that $a (x - e) = b (y - e)$ since the matrix $\nabla^2 q$ is nonsingular. This yields that $x, y,$ and $e$ belong to one line. This line crosses the cam which is impossible since of (E).
Lemma 4  Function $\Phi$ fulfils (III).

Proof. We are going to show that integral

$$Q_n(x,y) = \text{Re} i^n \int_{Z(y)} (\Phi(x,\sigma) - i0)^{-n} \frac{d\Sigma}{d\sigma \langle y - \sigma, \nabla q(\sigma) \rangle}$$

(9)

vanishes for all $x, y \in X, y \neq x$. We have

$$\Phi(x,\sigma) = \Phi(x,\sigma) - \Phi(y,\sigma) = \langle x - y, \nabla q(\sigma) \rangle$$
on $Z(y)$. The right hand side does not change its sign if and only if the point $x$ is contained in the convex closed cone bounded by the lines through points $y$ that are tangent to $Z(y)$. It is not the case since of (E). Therefore $\Phi(x,\sigma)$ does change its sign on $Z(y)$. By (8)

$$d\sigma \langle y - \sigma, \nabla q(\sigma) \rangle - \langle y - e, d\sigma \nabla q(\sigma) \rangle = -d(\sigma - e, \nabla q(\sigma)) = -d(q(\sigma) - q(e)) = 0$$
on $\Sigma$ since $q(\sigma) = 1$. The volume form in (8) equals

$$\frac{d\Sigma}{d\sigma \langle y - \sigma, \nabla q(\sigma) \rangle} = \frac{dV}{dq \wedge \langle y - e, d\sigma \nabla q(\sigma) \rangle}.$$ 

Choose affine coordinates $\sigma = A\xi + e$ on $E^{n+1}$ where $A$ is the diagonal matrix such that $2q(A\xi + e) = |\xi|^2$. Then $dV = \det A d\xi_1 \wedge ... \wedge d\xi_{n+1}$, $dq = \sum \xi_i d\xi_i$ and $\langle y - e, d\nabla q(\sigma) \rangle = \langle s, d\xi \rangle$ for some vector $s \in E^{n+1}$. This yields

$$\frac{dV}{\xi d\xi \wedge \langle s, d\xi \rangle} = \frac{\Omega_n}{\langle s, d\xi \rangle} = |s|^{-1} \Omega_{n-1}$$

up to the factor $\det A$. Here $\Omega_k$ denotes the volume form of the euclidean $k$-sphere $S^k$. Finally, we apply [2] Theorem A.20 to $\Phi$ and to the sphere $Z(y) \cong S^{n-1}$ which implies vanishing of $Q_n(x,y)$. $\blacktriangleleft$

Application of Theorem [1] completes the proof of Theorem 2 for any nondegenerated ellipsoid. In the case of one-point cam $\{e\}$ one can take the generating function $\Phi(x,\sigma) = \langle x - e, \sigma \rangle, \sigma \in S^n$ and follow the above arguments. $\blacktriangleleft$

References

[1] Funk P 1913 Über Flächen mit lauter geschlossenen geodätischen Linien Math. Annal. 74(2), 278–300

[2] Palamodov V 2016 Reconstruction from data of integrals CRC

[3] Salman Y 2016 An inversion formula for the spherical transform in $S^2$ for a special family of circles of integration Anal. Math. Phys. 6 43–58