SONIC-SUBSONIC LIMIT OF APPROXIMATE SOLUTIONS TO MULTIDIMENSIONAL STEADY EULER EQUATIONS

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Abstract. A compactness framework is established for approximate solutions to sonic-subsonic flows governed by the steady full Euler equations for compressible fluids in arbitrary dimension. The existing compactness frameworks for the two-dimensional irrotational case do not directly apply for the steady full Euler equations in higher dimensions. The new compactness framework we develop in this paper applies for both non-isentropic and rotational flows. One of our main observations in this paper is that the compactness can be achieved by using only natural weak estimates for the mass balance and the vorticity, along with the Bernoulli law and the entropy relation, through a more delicate analysis on the phase space. As direct applications, we establish two existence theorems for multidimensional sonic-subsonic full Euler flows through infinitely long nozzles.

1. Introduction

The full Euler equations for steady compressible flows in $\mathbb{R}^n$ read
\begin{equation}
\begin{aligned}
\text{div} (\rho u) &= 0, \\
\text{div} (\rho u \otimes u + p I) &= 0, \\
\text{div} (\rho u E + up) &= 0,
\end{aligned}
\end{equation}
where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, n \geq 2$, $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ is the fluid velocity, and
\[ q := |u| = \left( \sum_{i=1}^{n} u_i^2 \right)^{1/2} \]
is the speed, while $\rho$, $p$, and $E$ represent the density, pressure, and total energy, respectively, and $I$ is the $n \times n$ unit matrix. The nonnegative quantities $\rho$, $q$, $p$, and $E$ are not independent.

For ideal polytropic gas,
\[ E = \frac{q^2}{2} + \frac{p}{\gamma - 1} \rho, \]
with adiabatic exponent $\gamma > 1$. In this case, the Bernoulli law is written as
\begin{equation}
\frac{q^2}{2} + h(\rho, p) = B,
\end{equation}
where $h(\rho, p) = \frac{\gamma p}{(\gamma - 1)\rho}$ is the enthalpy, and $B$ is a Bernoulli function determined by appropriate additional conditions (such as boundary conditions and/or asymptotic conditions at infinity). The sound speed of the flow is
\begin{equation}
c = \sqrt{\frac{\gamma p}{\rho}},
\end{equation}
and the Mach number is defined as
\begin{equation}
M = \frac{q}{c}.
\end{equation}
Then, for a fixed Bernoulli function $B$, there is a critical speed $q_{cr} = \sqrt{\frac{2\gamma - 1}{\gamma + 1}} B$ such that, when $q \leq q_{cr}$, the flow is sonic-subsonic (i.e. $M \leq 1$); otherwise, it is supersonic (i.e. $M > 1$).

When the flow is isentropic, the pressure is a function of the density, and (1.1) then reduces to

$$\begin{cases} 
\text{div}(\rho u) = 0, \\
\text{div}(\rho u \otimes u + pI) = 0.
\end{cases}$$

As usual, we require

$$p'(\rho) > 0, \quad 2p'(\rho) + \rho p''(\rho) > 0 \quad \text{for } \rho > 0,$$

which include the $\gamma$-law flows with $p = \kappa \rho^\gamma$, $\gamma > 1$ and $\kappa > 0$, and the isothermal flows with $p = \kappa \rho$; see [12]. In this case, the Bernoulli law has the same formula as (1.2), while the enthalpy $h(p(\rho), \rho) = h(\rho)$ with $h'(\rho) = \frac{p'(\rho)}{\rho}$. The sound speed of the flow is $c = \sqrt{p'(\rho)}$, and the Mach number is defined as $M = \frac{q}{c}$. Then, for a fixed Bernoulli function $B$, the condition $2p'(\rho) + \rho p''(\rho) > 0$ for $\rho > 0$ implies the existence of a unique critical speed $q_{cr} = q_{cr}(B)$ so that the flow can be classified by sonic-subsonic or supersonic by $q \leq q_{cr}$ or $q > q_{cr}$.

It is well known that the steady Euler equations for compressible fluids are of composite-mixed type, which is determined by the Mach number $M$. That is, the system can be reduced to a system such that two of the equations are elliptic-hyperbolic mixed: elliptic when $M < 1$ and hyperbolic when $M > 1$, while the other $n$ equations are hyperbolic. For the isentropic case, there are two mixed characteristics and $n - 1$ hyperbolic characteristics.

During the 1950s, the effort on system (1.5) was focused mainly on the irrotational case, namely when $u$ is constrained to satisfy the additional equation $\text{curl} \ u = 0$. Since the equations of uniform subsonic flow possess ellipticity, solutions have better regularity than those corresponding to transonic flow or supersonic flow. The airfoil problem for irrotational, two-dimensional subsonic flow was solved; cf. Frankl-Keldysh [22], Shiffman [28], Bers [8], and Finn-Gilbarg [20]. The first result for three-dimensional subsonic flow past an obstacle was given by Finn and Gilbarg [21] with some restrictions on the Mach number. Dong [15] and Dong-Ou [16] extended the results to the maximum Mach number $M < 1$ for arbitrary dimension. Also see Du-Xin-Yan [18] for the construction of a smooth uniform subsonic flow in an infinitely long nozzle in $\mathbb{R}^n$, $n \geq 2$.

For the non-potential case, the global existence of isentropic subsonic flow through two-dimensional, infinitely long nozzles was proved in Xie-Xin [32]. The result was also extended to the two-dimensional periodic nozzles in Chen-Xie [5] and to axisymmetric nozzles in Du-Duan [17]. For the full Euler flow, the first result was given by Chen-Deng-Xiang [7] for two-dimensional infinitely long nozzles. Bae [11] showed the stability of contact discontinuities for subsonic full Euler flow in the two-dimensional, infinitely long nozzles. Duan and Luo [19] recently considered the axisymmetric nozzle problem for the smooth subsonic flow.

On the other hand, few results are currently known for the cases of sonic-subsonic flow and transonic flow, since the uniform ellipticity is lost and shocks may be present. That is, smooth solutions may not exist. Instead one must consider weak solutions. Morawetz [24, 25] introduced an approach via compensated compactness to analyze irrotational steady flow of the Euler equations. Indeed, Morawetz established a compactness framework under the assumption that solutions are free of stagnation points and cavitation points. Morawetz’s result has been improved by Chen-Slemrod-Wang [10] in which the approximate solutions away from cavitation are constructed by a viscous perturbation.

The compactness framework for sonic-subsonic irrotational flow allowing for stagnation in two dimensions was due to Chen, Dafermos, Slemrod, and Wang [9] by combining the mass
conservation, momentum, and irrotational equations. The key observation in [9] is that the two-dimensional steady flow can be regarded as a one-dimensional system of conservation laws, that is, \( x \) is regarded as the time variable, so that the div-curl lemma can be applied to the two momentum equations. In fact, the momentum equations are first employed in [9] to reduce the support of the corresponding Young measure to two points, and then the irrotational equation and the mass equation are used to reduce the Young measure to a Dirac measure. Using a similar idea, Xie and Xin [30] investigated the subsonic-sonic limit of the two-dimensional irrotational, infinitely long nozzle problem. Later, in [31], they extended the result to the three-dimensional axi-symmetric flow through an axi-symmetric nozzle. The compactness framework in the multidimensional irrotational case was established in Huang-Wang-Wang [23].

The compactness framework established for irrotational flow no longer applies directly for the steady full Euler equations in \( \mathbb{R}^n \) with \( n \geq 2 \). When \( n \geq 3 \), the equations cannot be reduced to a one-dimensional system of conservation laws. More importantly, the div-curl lemma is no longer valid for the momentum equations, due to the presence of linear characteristics. One of our main observations in this paper is that it is still possible to achieve the same compactness result, i.e., to reduce the Young measure to a Dirac measure, by using only natural weak estimates for the mass balance and the vorticity, along with the Bernoulli law and entropy relation, through a more delicate analysis on the phase space. In particular, the Bernoulli function and entropy function play a key role in our proof. We then establish a compactness framework for approximate solutions for steady full Euler flows in arbitrary dimension.

The rest of this paper is organized as follows. In Section 2, we establish the compactness framework for sonic-subsonic approximate solutions to sonic-subsonic flows governed by the steady full Euler equations for compressible fluids in \( \mathbb{R}^n \) with \( n \geq 2 \), as well as by the steady isentropic Euler equations with weaker conditions. In Sections 3–4, we give two direct applications of the compactness framework to establish the existence of sonic-subsonic full Euler flow through infinitely long nozzles in \( \mathbb{R}^n \) with \( n \geq 2 \).

2. Compactness Framework for Approximate Steady Full Euler Flows

In this section, we establish the compensated compactness framework for approximate solutions of the steady full Euler equations in \( \mathbb{R}^n \) with \( n \geq 2 \) with the form:

\[
\begin{align*}
\text{div}(\rho^\varepsilon u^\varepsilon) &= e_1(\varepsilon), \\
\text{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + p^\varepsilon I) &= e_2(\varepsilon), \\
\text{div}(\rho^\varepsilon u^\varepsilon E^\varepsilon + u^\varepsilon p^\varepsilon) &= e_3(\varepsilon),
\end{align*}
\]

(2.1)

where \( e_1(\varepsilon), e_2(\varepsilon) = (e_{21}(\varepsilon), \cdots, e_{2n}(\varepsilon))^\top \), and \( e_3(\varepsilon) \) are sequences of functions depending on the parameter \( \varepsilon \).

Let a sequence of functions \( \rho^\varepsilon(x), u^\varepsilon(x) = (u^\varepsilon_1, \cdots, u^\varepsilon_n)(x) \), and \( p^\varepsilon(x) \) be defined on an open subset \( \Omega \subset \mathbb{R}^n \) such that the following qualities:

\[
q^\varepsilon := |u^\varepsilon| = \sqrt{\sum_{i=1}^{n} (u^\varepsilon_i)^2}, \quad c^\varepsilon := \sqrt{\frac{\gamma p^\varepsilon}{\rho^\varepsilon}}, \quad M^\varepsilon := \frac{q^\varepsilon}{c^\varepsilon},
\]

(2.2)

\[
B^\varepsilon := \frac{(q^\varepsilon)^2}{2} + \frac{\gamma p^\varepsilon}{(\gamma - 1)\rho^\varepsilon}, \quad S^\varepsilon := \frac{\gamma p^\varepsilon}{(\gamma - 1)(\rho^\varepsilon)^\gamma}
\]

(2.3)

can be well defined and satisfy the following conditions:

(A.1) \( M^\varepsilon \leq 1 \) a.e. in \( \Omega \);
(A.2). $S^\varepsilon$ and $B^\varepsilon$ are uniformly bounded, and for any compact set $K$, there exists a uniform constant $c(K)$ such that $\inf_{x \in K} S^\varepsilon(x) \geq c(K) > 0$. Moreover, $S^\varepsilon$ and $B^\varepsilon$ converge to bounded $\overline{S}$ and $\overline{B}$ a.e. in $\Omega$.

(A.3). curl $u^\varepsilon$ and $e_1(\varepsilon)$ are in a compact set in $W^{-1,p}_{\loc}$ for some $1 < p \leq 2$.

Then we have

**Theorem 2.1** (Compensated compactness framework for the full Euler case). *Let a sequence of functions $\rho^\varepsilon(x)$, $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$, and $p^\varepsilon(x)$ satisfy conditions (A.1)-(A.3). Then there exists a subsequence (still labeled) $(\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x)$ such that

$\rho^\varepsilon(x) \to \rho(x), \quad u^\varepsilon(x) \to (u_1, \cdots, u_n)(x), \quad p^\varepsilon(x) \to p(x)$ \text{ a.e. in } x \in \Omega \text{ as } \varepsilon \to 0,

and

$$M(x) := \frac{q(x)}{c(x)} \leq 1 \quad \text{a.e. } x \in \Omega.$$*

**Proof.** We divide the proof into three steps.

**Step 1.** The strong convergence of $(\rho^\varepsilon, p^\varepsilon)$ follows from the strong convergence of $q^\varepsilon = |u^\varepsilon|$. We employ (2.3) to obtain

$$\frac{(q^\varepsilon)^2}{2} + S^\varepsilon(\rho^\varepsilon)^{\gamma-1} = B^\varepsilon. \quad (2.4)$$

From these, the three variables $\rho^\varepsilon$, $p^\varepsilon$, and $q^\varepsilon$ are determined by one of them. In other words, the pressure $p^\varepsilon$ and density $\rho^\varepsilon$ can be regarded as functions of $q^\varepsilon$ through $(B^\varepsilon, S^\varepsilon)$ with

$$\rho^\varepsilon = \rho(q^\varepsilon; B^\varepsilon, S^\varepsilon) = \left( \frac{2B^\varepsilon - (q^\varepsilon)^2}{2S^\varepsilon} \right)^{\frac{1}{\gamma-1}}, \quad (2.5)$$

$$p^\varepsilon = p(q^\varepsilon; B^\varepsilon, S^\varepsilon) = \frac{\gamma - 1}{2\gamma} \left( \frac{2B^\varepsilon - (q^\varepsilon)^2}{2S^\varepsilon} \right)^{\frac{1}{\gamma-1}}. \quad (2.6)$$

Since $(B^\varepsilon, S^\varepsilon)$ strongly converge to $(\overline{B}, \overline{S})$ a.e., the strong convergence of the density $\rho^\varepsilon$ and pressure $p^\varepsilon$ becomes a nature consequence of the strong convergence of the speed $q^\varepsilon$, especially the velocity $u^\varepsilon$.

**Step 2.** The $H^{-1}_{\loc}$-compactness. From the uniform boundedness of the Bernoulli function $B^\varepsilon$ and the sonic-subsonic condition $M^\varepsilon \leq 1$, it is easy to see

$$q^\varepsilon \leq \sup_{\varepsilon} \sqrt{\frac{2\gamma - 1}{\gamma + 1}B^\varepsilon},$$

which implies the uniform boundedness of the velocity $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$. This shows that

$$(\text{curl } u^\varepsilon)_{ij} = \partial_{x_i}u_j^\varepsilon - \partial_{x_j}u_i^\varepsilon \quad \text{is bounded in } W^{-1,\infty}.$$ 

On the other hand, curl $u^\varepsilon$ is compact in $W^{-1,p}$ for some $1 < p \leq 2$. By the interpolation theorem,

$$\text{curl } u^\varepsilon \quad \text{is compact in } H^{-1}_{\loc}. \quad (2.7)$$

For a fixed compact set $K$, the density $\rho^\varepsilon$ can be controlled by $\sup_{\varepsilon} \left( \sup_{x \in K} B^\varepsilon / \inf_{x \in K} S^\varepsilon \right)^{\frac{1}{\gamma-1}}$. Similarly, we have

$$\text{div}(\rho^\varepsilon u^\varepsilon) \quad \text{is compact in } H^{-1}_{\loc}. \quad (2.8)$$
Step 3. The strong convergence of \( u^\varepsilon(x) \), which also leads to the strong convergence of \( (\rho^\varepsilon, p^\varepsilon)(x) \) from Step 1. By the div-curl lemma of Murat [26] and Tartar [24], the Young measure representation theorem for a uniformly bounded sequence of functions (cf. Tartar [20]; also Ball [2]), and (2.7)–(2.8), we have the following commutation identity:

\[
\sum_{i=1}^{n} \langle \nu(u), \rho(q) u_i \rangle \langle \nu(u), u_i \rangle = \langle \nu(u), \sum_{i=1}^{n} \rho(q) u_i u_i \rangle. \tag{2.9}
\]

Here and hereafter, for simplicity of notation, we have used that \( \nu(u) := \nu_x(\rho(q; B(x), \mathcal{S}(x)), u) \) denotes the associated Young measure (a probability measure) for the sequence \( u^\varepsilon(x) \), and \( \rho(q) := \rho(q; B(x), \mathcal{S}(x)) \) for the limits \( B(x) \) and \( \mathcal{S}(x) \) of \( B^\varepsilon(x) \) and \( S^\varepsilon(x) \) respectively.

Then the main point in this step for the compensated compactness framework is to prove that \( \nu \) is in fact a Dirac measure, which in turn implies the compactness of the sequence \( u^\varepsilon(x) \).

Combining both sides of (2.9) together, we have

\[
\langle \nu(u^{(1)}) \otimes \nu(u^{(2)}), \sum_{i=1}^{n} \rho(q^{(1)}) u_i^{(1)} (u_i^{(1)} - u_i^{(2)}) \rangle = 0. \tag{2.10}
\]

Exchanging indices (1) and (2), we obtain the following symmetric commutation identity:

\[
\langle \nu(u^{(1)}) \otimes \nu(u^{(2)}), I(u^{(1)}, u^{(2)}) \rangle = 0, \tag{2.11}
\]

where

\[
I(u^{(1)}, u^{(2)}) = \sum_{i=1}^{n} (\rho(q^{(1)}) u_i^{(1)} - \rho(q^{(2)}) u_i^{(2)}) (u_i^{(1)} - u_i^{(2)}). \tag{2.12}
\]

Then it remains to prove the strong convergence of the velocity \( u^\varepsilon \) from the above identity.

Notice that

\[
I(u^{(1)}, u^{(2)}) = \sum_{i=1}^{n} (\rho(q^{(1)}) u_i^{(1)} - \rho(q^{(2)}) u_i^{(2)}) (u_i^{(1)} - u_i^{(2)})
\]

\[
= \sum_{i=1}^{n} (\rho(q^{(1)}) |u_i^{(1)}|^2 - \rho(q^{(1)}) u_i^{(1)} u_i^{(2)} - \rho(q^{(2)}) u_i^{(2)} u_i^{(1)} - \rho(q^{(2)}) |u_i^{(2)}|^2)
\]

\[
= \rho(q^{(1)}) ((q^{(1)})^2 - \sum_{i=1}^{n} u_i^{(1)} u_i^{(2)}) + \rho(q^{(2)}) ((q^{(2)})^2 - \sum_{i=1}^{n} u_i^{(1)} u_i^{(2)}). \tag{2.13}
\]

The Cauchy inequality implies

\[
I(u^{(1)}, u^{(2)}) \geq \rho(q^{(1)}) ((q^{(1)})^2 - q^{(1)} q^{(2)}) + \rho(q^{(2)}) ((q^{(2)})^2 - q^{(1)} q^{(2)})
\]

\[
= (q^{(1)} - q^{(2)}) (\rho(q^{(1)}) q^{(1)} - \rho(q^{(2)}) q^{(2)})
\]

\[
= (q^{(1)} - q^{(2)})^2 \frac{d(\rho q)}{dq} (\tilde{q}), \tag{2.14}
\]

where \( \tilde{q} \) lies between \( q^{(1)} \) and \( q^{(2)} \) by the mean-value theorem.

Taking derivative with respect to \( q \) on (2.14), we obtain

\[
\frac{d\rho}{dq} = -\frac{q}{(\gamma - 1) \rho^{\gamma-2} S} = -\frac{\rho q}{c^2}.
\]

Then

\[
\frac{d(\rho q)}{dq} = \rho(1 - M^2).
\]

For sonic-subsonic flows, i.e., \( q^{(1)}, q^{(2)} \leq q_{cr}(\tilde{B}) \), we have

\[1 - M^2(\tilde{q}) \geq 0.\]
On the other hand,
\[ \rho(\tilde{q}) \geq \left( \frac{2B}{(\gamma + 1)S} \right)^{\frac{1}{\gamma-1}} \geq 0. \]

Notice that \( M^2(\tilde{q}) = 1 \) if and only if \( q^{(1)} = q^{(2)} = q_{cr}(B) \), while \( \rho(\tilde{q}) = 0 \) if and only if \( B = 0 \) and \( q^{(1)} = q^{(2)} = q_{cr}(B) \). Then
\[ I(u^{(1)}, u^{(2)}) \geq (q^{(1)} - q^{(2)})^2 \rho(\tilde{q})(1 - M^2(\tilde{q})) \geq 0. \] (2.15)

With (2.11), it implies that
\[ q^{(1)} = q^{(2)}, \]
which also deduces
\[ \rho^{(1)} = \rho^{(2)}, \quad p^{(1)} = p^{(2)}. \]

Again using (2.11) and (2.12), we further obtain
\[ 0 = \langle \nu(u^{(1)}) \otimes \nu(u^{(2)}), I(u^{(1)}, u^{(2)}) \rangle \]
\[ = \langle \nu(u^{(1)}) \otimes \nu(u^{(2)}), \rho(q^{(1)}) \sum_{i=1}^{n} \left( u_i^{(1)} - u_i^{(2)} \right)^2 \rangle, \] (2.16)
which immediately implies
\[ u^{(1)} = u^{(2)}, \]
i.e., \( \nu(u) \) concentrates on a single point. If this would not be the case, we could suppose to have two different points \( \tilde{u} \) and \( \tilde{v} \) in the support of \( \nu \). Then \( (\tilde{u}, \tilde{u}), (\tilde{v}, \tilde{u}), (\tilde{u}, \tilde{v}), \) and \( (\tilde{v}, \tilde{v}) \) would be in the support of \( \nu \otimes \nu \), which contradicts with \( u^{(1)} = u^{(2)} \). Therefore, the Young measure \( \nu \) is a Dirac measure, which implies the strong convergence of \( (\rho^\varepsilon, u^\varepsilon, p^\varepsilon) \). This completes the proof.

For the isentropic case, the entropy function \( S \) is constant. Then the pressure \( p \) may be regarded as a \( C^1(\mathbb{R}^+ \cup \{0\}) \cap C^2(\mathbb{R}^+) \) function of the density \( \rho \), which satisfies \( p'(\rho) > 0 \) and \( 2p'(\rho) + pp''(\rho) > 0 \) for \( \rho > 0 \).

Without loss of generality, we set the enthalpy \( h(\rho) \) as
\[ h(\rho) := \int_{1}^{\rho} \frac{p'(s)}{s} ds, \]
so that \( h(1) = 0 \). It is noticeable that, in this case, we do not have the property that the Bernoulli function is greater or equal to zero. To replace the nonnegative property, we introduce a lower bound
\[ B_{min} = \lim_{\rho \to 0^+} f(\rho), \]
while
\[ f(\rho) := \frac{p'(\rho)}{2} + h(\rho) \]
belongs to \( C^1(\mathbb{R}^+ \cup \{0\}) \). Since \( 2p'(\rho) + pp''(\rho) > 0 \) for \( \rho > 0 \), \( f(\rho) \) is an increase function in \( \rho > 0 \). Then \( f(\rho) > B_{min} \) for \( \rho > 0 \) and \( B_{min} \) is lower bound so that \( B_{min} < f(1) = \frac{p'(1)}{2} \).

In the isentropic case, conditions (A.1)–(A.3) can be reformulated as:

(B.1). \( M^\varepsilon \leq 1 \) a.e. in \( \Omega; \)

(B.2). \( B^\varepsilon = \frac{(q^\varepsilon)^2}{2} + \int_{1}^{\rho^\varepsilon} \frac{p'(s)}{s} ds \) are uniformly bounded and \( B^\varepsilon \geq B_{min}; \)

(B.3). \( \text{curl} \, u^\varepsilon, \frac{e_1(\varepsilon)}{\rho^\varepsilon}, \) and \( \frac{e_2(\varepsilon)}{\rho} \) are uniformly bounded measures.

Similar to Theorem 2.1, we have
**Theorem 2.2.** (Compensated compactness framework for the isentropic case) Let a sequence of functions $\rho^\varepsilon(x)$ and $u^\varepsilon(x) = (u_1^\varepsilon, \cdots, u_n^\varepsilon)(x)$ satisfy conditions (B.1)–(B.3). Then there is a subsequence (still labeled) $(\rho^\varepsilon, u^\varepsilon)(x)$ such that

$$\rho^\varepsilon(x) \rightarrow \rho(x), \quad u^\varepsilon(x) \rightarrow u(x) = (u_1, \cdots, u_n)(x) \quad a.e. \ x \in \Omega \ as \ \varepsilon \rightarrow 0$$

and

$$M(x) \leq 1 \quad a.e. \ x \in \Omega.$$

**Proof.** First, since $p'(\rho)$ for $\rho > 0$, we can employ the implicit function theorem to conclude $\rho(q; B) = h^{-1}(B - \frac{q^2}{2})$. Then we can regard $\rho^\varepsilon$ as a function of $q^\varepsilon$ and $B^\varepsilon$, that is, $\rho^\varepsilon = \rho(q^\varepsilon; B^\varepsilon)$. As a consequence, the sequence $\rho^\varepsilon$ is nonnegative and uniformly bounded. Conditions (B.1) and (B.2) indicate directly that the speed sequence $q^\varepsilon$ is uniformly bounded.

Differentiating the Bernoulli functions $B^\varepsilon$ with respect to $x_i$ yields

$$\partial_{x_i}B^\varepsilon = \sum_{j=1}^{n} u_j^\varepsilon \partial_{x_i}u_j^\varepsilon + \frac{p'(\rho^\varepsilon)}{\rho^\varepsilon} \partial_{x_i}\rho^\varepsilon$$

$$= \sum_{j=1}^{n} u_j^\varepsilon \partial_{x_i}u_j^\varepsilon + \left( e_{2i}(\varepsilon) - \sum_{j=1}^{n} \frac{\rho^\varepsilon u_j^\varepsilon \partial_{x_j}u_i^\varepsilon - u_i^\varepsilon e_1(\varepsilon)}{\rho^\varepsilon} \right)$$

$$= \sum_{j=1}^{n} u_j^\varepsilon \partial_{x_i}u_j^\varepsilon + \frac{e_{2i}(\varepsilon) - e_i^\varepsilon e_1(\varepsilon)}{\rho^\varepsilon}$$

(2.17)

$$= \sum_{j=1}^{n} u_j^\varepsilon \omega_{ij}^\varepsilon + \frac{e_{2i}(\varepsilon) - e_i^\varepsilon e_1(\varepsilon)}{\rho^\varepsilon}.$$  

From the boundedness of $q^\varepsilon$ and condition (B.3), we conclude that

$$\nabla B^\varepsilon$$

is uniformly bounded measures.

Since $B^\varepsilon$ is uniformly bounded, the total-variation norm of $B^\varepsilon$ is uniformly bounded, which implies that $B^\varepsilon(x)$ converges to $B(x)$ in $L^1_{loc}$, as $\varepsilon$ tends to 0.

From the definition, $(\text{curl } u^\varepsilon)_{ij} = \omega_{ij}^\varepsilon = \partial_{x_i}u_j^\varepsilon - \partial_{x_j}u_i^\varepsilon$ is bounded in $W^{-1,\infty}$. On the other hand, $\text{curl } u^\varepsilon$ is a uniformly bounded measure sequence, which implies that $\text{curl } u^\varepsilon$ is compact in $W^{-1,q}$ for each $1 \leq p < \frac{n}{n-1}$. From the interpolation theorem,

$$\text{curl } u^\varepsilon$$

is compact in $H^{-1}_{loc}$.

Similarly, we have

$$\text{div}(\rho^\varepsilon u^\varepsilon)$$

is compact in $H^{-1}_{loc}$.

Next, we will discuss the strong convergence in two cases.

**Case 1:** $B > B_{\min}$. From the monotonicity of $f(\rho)$, we get

$$\rho_{cr} > 0,$$

which implies $f(\rho_{cr}) = B$ and on the support of $\nu$,

$$\rho(q) \in [\rho_{cr}, h^{-1}(B)].$$

Following similar argument for Theorem 2.1, we obtain the following commutation identity:

$$\langle \nu(u^{(1)}), \nu(u^{(2)}) \rangle = 0,$$

(2.18)

and

$$I(u^{(1)}, u^{(2)}) = \sum_{i=1}^{n} (\rho(q^{(1)})u_i^{(1)} - \rho(q^{(2)})u_i^{(2)})(u_i^{(1)} - u_i^{(2)}).$$

(2.19)
By the same calculation as in the previous argument, we have

$$I(u^{(1)}, u^{(2)}) \geq (q^{(1)} - q^{(2)})^2 \frac{d\rho(q)}{dq}(\bar{q}),$$

(2.20)

where $\bar{q}$ lies between $q^{(1)}$ and $q^{(2)}$.

From the definition of $\rho(q)$, we obtain $\frac{d\rho}{dq} = -\frac{\rho}{q^2}$ which implies $\frac{d\rho(q)}{dq} = \rho(1 - M^2)$. From sonic-subsonic flows, i.e., $q^{(1)}, q^{(2)} \leq q_{cr}$, we have

$$\rho(\bar{q})(1 - M^2(\bar{q})) \geq 0.$$

Notice that $M^2(\bar{q}) = 1$ if and only if $q^{(1)} = q^{(2)} = q_{cr}$. Then

$$I(u^{(1)}, u^{(2)}) \geq (q^{(1)} - q^{(2)})^2 \rho(\bar{q})(1 - M^2(\bar{q})) \geq 0.$$

(2.21)

Thus, from (2.18), we obtain

$$q^{(1)} = q^{(2)}.$$

Again using (2.18) and (2.19), we have

$$0 = (\nu(u^{(1)}) \otimes \nu(u^{(2)}), I(u^{(1)}, u^{(2)}))$$

$$= (\nu(u^{(1)}) \otimes \nu(u^{(2)}), \rho q^{(1)} \sum_{i=1}^{n} (u^{(1)}_i - u^{(2)}_i)^2),$$

(2.22)

which immediately implies $u^{(1)} = u^{(2)}$, i.e., the Young measure $\nu$ is a Dirac measure. Thus, we conclude the strong convergence of $(\rho^\varepsilon, u^\varepsilon)(x)$ to $(\rho, u)(x)$ with $M(x) \leq 1$ a.e $x \in \Omega$.

Case 2. $\overline{B} = B_{min}$. Considering the boundedness of $\overline{B}$, we can see $\lim_{\rho \to 0^+} \int_1^\rho \frac{p'(s)}{s} ds$ is finite, which implies $p'(0) = 0$. On the other hand, the monotonicity of $f(\rho)$ shows that $\rho(q) = 0$ for any $q$ belonging to the support of the Young measure $\nu$. For this speed, $0 \leq q^2 \leq p'(0) = 0$, from the sonic-subsonic condition. Then the strong convergence of $(\rho^\varepsilon, u^\varepsilon)$ is shown in this case.

Remark 2.1. The main difference between Theorems 2.1 and 2.2 is that the isentropic case has the Bernoulli-vortex relation (2.17), while the full Euler case has

$$\partial_x B^\varepsilon = \sum_{j=1}^{n} u^\varepsilon_j \omega^\varepsilon_j + \frac{(\rho^\varepsilon)^{\gamma-1}}{\gamma} \partial_x \xi^\varepsilon + \frac{e_{2i}(\varepsilon) - u^\varepsilon e_1(\varepsilon)}{\rho^\varepsilon},$$

(2.23)

so that the gradient of $B^\varepsilon$ cannot be achieved only by the vorticity $\omega^\varepsilon$.

Remark 2.2. The main theorem in Huang-Wang-Wang [23] is included in Theorem 2.2. Condition (B.1) is the same as the one in [23], while $B^\varepsilon$ is assumed to be constant in [23] which clearly satisfies condition (B.2). As a consequence, condition (B.3) could be regarded as the $H^1_{loc}$ compactness of $\text{curl} u^\varepsilon$. The irrotational condition in [23] is removed. Thus, Theorem 2.2 includes more physical consideration. From (2.17), the irrotational condition implies that the Bernoulli function is a constant in the flow field.

Remark 2.3. Consider any function $Q(\rho, u, p) = (Q_1, \cdots, Q_n)(\rho, u, p)$ satisfying

$$\text{div}(Q(\rho^\varepsilon, u^\varepsilon, p^\varepsilon)) = o_Q(\varepsilon),$$

(2.24)

where $o_Q(\varepsilon) \to 0$ in the distributional sense as $\varepsilon \to 0$. We can see from the strong convergence of $(\rho^\varepsilon, u^\varepsilon, p^\varepsilon)$ ensured by Theorem 2.1 that $\text{div}(Q(\rho, u, p)) = 0$ holds in the distributional sense. Thus, if

$$\text{div} (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + p^\varepsilon I) = e_2(\varepsilon) \to 0 \quad \text{in the sense of distributions},$$

(2.25)
the weak solution also satisfies the momentum equations in (1.12) and the energy equation (1.13) in the distributional sense. The statement is also valid for Theorem 2.2.

Then, as corollaries, we conclude the following theorems.

**Theorem 2.3** (Convergence of approximate solutions for the full Euler flow). Let \( \rho(x), u(x) = (u_1(x), \ldots, u_n(x)) \), and \( p(x) \) be a sequence of approximate solutions satisfying (A.1)–(A.3) and \( \varepsilon_j(x) \to 0 \), \( j = 1, 2, 3 \), in the distributional sense as \( \varepsilon \to 0 \). Then there exists a subsequence (still labeled) \( (\rho^\varepsilon, u^\varepsilon, p^\varepsilon) \) that converges a.e. as \( \varepsilon \to 0 \) to a weak solution \( (\rho, u, p) \) to the Euler equations of (1.1), which satisfies \( M(x) \leq 1 \), a.e. \( x \in \Omega \).

**Theorem 2.4** (Convergence of approximate solutions for the isentropic Euler flow). Let \( \rho(x) \) and \( u(x) = (u_1(x), \ldots, u_n(x)) \) be a sequence of approximate solutions satisfying (B.1)–(B.3) and \( e_j(x) \to 0 \), \( j = 1, 2 \), in the distributional sense as \( \varepsilon \to 0 \). Then there exists a subsequence (still labeled) \( (\rho^\varepsilon, u^\varepsilon) \) that converges a.e. as \( \varepsilon \to 0 \) to a weak solution \( (\rho, u) \) to the Euler equations of (1.15), which satisfies \( M(x) \leq 1 \) a.e. \( x \in \Omega \).

There are various ways to construct approximate solutions by either numerical methods or analytical methods such as vanishing viscosity methods. As direct applications, we show two examples in Sections 3 and 4 to apply the compactness framework built above in establishing existence theorems for multidimensional sonic-subsonic full Euler flows through infinitely long nozzles.

### 3. Subsonic-Sonic Limit for Two-Dimensional Steady Full Euler Flows in an Infinitely Long Nozzle

In this section, as a direct application of the compactness framework established in Theorem 2.1, we obtain the subsonic-sonic limit of steady subsonic full Euler flows in a two-dimensional, infinitely long nozzle.

The infinitely long nozzle is defined as

\[
\Omega = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1), -\infty < x_1 < \infty\},
\]

with the nozzle walls \( \partial \Omega := W_1 \cup W_2 \), where

\[
W_i = \{(x_1, x_2) : x_2 = f_i(x_1) \in C^{2,\alpha}, -\infty < x_1 < \infty\}, \quad i = 1, 2.
\]

Suppose that \( W_1 \) and \( W_2 \) satisfy

\[
f_2(x_1) > f_1(x_1) \quad \text{for } x_1 \in (-\infty, \infty),
\]

\[
f_1(x_1) \to 0, \quad f_2(x_1) \to 1 \quad \text{as } x_1 \to -\infty,
\]

\[
f_1(x_1) \to a, \quad f_2(x_1) \to b > a \quad \text{as } x_1 \to \infty,
\]

and there exists \( \alpha > 0 \) such that

\[
\|f_i\|_{C^{2,\alpha}(\mathbb{R})} \leq C, \quad i = 1, 2,
\]

for some positive constant \( C \). It follows that \( \Omega \) satisfies the uniform exterior sphere condition with some uniform radius \( r > 0 \). See Fig. 3.1.

Suppose that the nozzle has impermeable solid walls so that the flow satisfies the slip boundary condition:

\[
u \cdot \nu = 0 \quad \text{on } \partial \Omega,
\]

where \( u = (u_1, u_2) \in \mathbb{R}^2 \) is the velocity and \( \nu = (\nu_1, \nu_2) \) is the unit outward normal to the nozzle wall. In the flow without vacuum, it can be written as

\[(\rho u) \cdot \nu = 0 \quad \text{on } \partial \Omega.
\]
It follows from (3.1) and (3.4) that
\[
\int_s (\rho u) \cdot l \, ds = m
\] (3.5)
holds for some constant \( m \), which is the mass flux, where \( s \) is any curve transversal to the \( x_1 \)-direction, and \( l \) is the normal of \( s \) in the positive \( x_1 \)-axis direction.

We assume that the upstream entropy function is given, i.e.,
\[
\frac{\gamma p}{\gamma - 1} \rho \rightarrow S_-(x_2) \quad \text{as } x_1 \rightarrow -\infty,
\] (3.6)
and the upstream Bernoulli function is given, i.e.,
\[
\frac{\gamma^2}{2} + \frac{\gamma p}{\gamma - 1} \rho \rightarrow B_-(x_2) \quad \text{as } x_1 \rightarrow -\infty,
\] (3.7)
where \( B(x_2) \) is a function defined on \([0,1] \).

**Problem 1(m):** Solve the full Euler system (1.1) with the boundary condition (3.3), the mass flux condition (3.5), and the asymptotic conditions (3.6)–(3.7).

Set
\[
S = \inf_{x_2 \in [0,1]} S_-(x_2), \quad B = \inf_{x_2 \in [0,1]} B_-(x_2).
\]

For this problem, the following theorem has been established in Chen-Deng-Xiang [7].

**Theorem 3.1.** Let the nozzle walls \( \partial \Omega \) satisfy (3.1)–(3.2), and let \( S > 0 \) and \( B > 0 \). Then there exists \( \delta_0 > 0 \) such that, if \( \| (S_- - S, B_- - B) \|_{C^{1,1}([0,1])} \leq \delta \) for \( 0 < \delta \leq \delta_0 \), \( (S_-B_-^{-\gamma})'(0) \geq 0 \), and \( (S_-B_-^{-\gamma})'(1) \leq 0 \), there exists \( \bar{m} \geq 2\delta_0^{\frac{1}{\gamma - 1}} \) such that, for any \( m \in (\delta_0^{\frac{1}{\gamma - 1}}, \bar{m}) \), there exists a global solution (i.e., a full Euler flow) \( (\rho, u, p) \in C^{1,\alpha}(\bar{\Omega}) \) of **Problem 1(m)** such that the following hold:

(i) Subsonicity and positivity of the horizontal velocity: The flow is uniformly subsonic with positive horizontal velocity in the whole nozzle, i.e.,
\[
\sup_{\bar{\Omega}} (q^2 - c^2) < 0, \quad u_1 > 0 \quad \text{in } \bar{\Omega};
\] (3.8)

(ii) The flow satisfies the following asymptotic behavior in the far fields: As \( x_1 \rightarrow -\infty \),
\[
p \rightarrow p_- > 0, \quad u_1 \rightarrow u_-(x_2) > 0, \quad (u_2, \rho) \rightarrow (0, \rho_-(x_2;p_-)),
\] (3.9)
\[
\nabla p \rightarrow 0, \quad \nabla u_1 \rightarrow (0, u'_-(x_2)), \quad \nabla u_2 \rightarrow 0, \quad \nabla \rho \rightarrow (0, \rho'_-(x_2;p_-))
\] (3.10)
uniformly for \( x_2 \in K_1 \subset (0,1) \), where \( \rho_-(x_2;p_-) = \left( \frac{\gamma p_-}{\gamma - 1} S_-(x_2) \right)^{1/\gamma} \), the constant \( p_- \) and function \( u_-(x_2) \) can be determined by \( m, S_-(x_2), \) and \( B_-(x_2) \) uniquely;

(iii) Uniqueness: The full Euler flow of **Problem 1(m)** satisfying (3.8) and the asymptotic behavior (3.9)–(3.10) is unique.
(iv) Critical mass flux: \( \hat{m} \) is the upper critical mass flux for the existence of subsonic flow in the following sense: either \( \sup_{\Omega}(q_1^2 - c_1^2) \to 0 \) as \( m \to \hat{m} \), or there is no \( \sigma > 0 \) such that, for all \( m \in (\hat{m}, \hat{m} + \sigma) \), there are full Euler flows of Problem 1(m) satisfying (3.8), the asymptotic behavior (3.9)–(3.11), and \( \sup_{m \in (\hat{m}, \hat{m} + \sigma)} \sup_{\Omega}(c_1^2 - q_1^2) > 0 \).

We note that Theorem 3.1 does not apply to the critical flows, that is, those flows for which \( m = \hat{m} \) must be sonic at some point. Now we can employ Theorem 2.1 to establish a more general result.

**Theorem 3.2** (Subsonic-sonic limit of two-dimensional full Euler flows). Let \( \delta^{\frac{4}{3}} < m^\varepsilon < \hat{m} \) be a sequence of mass fluxes, and let \((\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x)\) be the corresponding sequence of solutions to Problem 1(\( m^\varepsilon \)). Then, as \( m^\varepsilon \to \hat{m} \), the solution sequence possesses a subsequence (still denoted by) \((\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x)\) that converges strongly a.e. in \( \Omega \) to a vector function \((\rho, u, p)(x)\) which is a weak solution of Problem 1(\( \hat{m} \)). Furthermore, the limit solution \((\rho, u, p)(x)\) also satisfies (1.1) in the distributional sense and the boundary conditions (3.4) as the normal trace of the divergence-measure field \( \rho u \) on the boundary in the sense of Chen-Frid [8].

**Proof.** We divide the proof into three steps.

1. We first need to show that \((\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x)\) satisfy condition (A.1)–(A.3). For \( B^\varepsilon \) and \( S^\varepsilon \), we have

\[
\begin{align*}
\partial_{x_1}(\rho^\varepsilon u^\varepsilon_1) + \partial_{x_2}(\rho^\varepsilon u^\varepsilon_2) &= 0, \\
\partial_{x_1}(\rho^\varepsilon u^\varepsilon_1 B^\varepsilon) + \partial_{x_2}(\rho^\varepsilon u^\varepsilon_2 B^\varepsilon) &= 0, \\
\partial_{x_1}(\rho^\varepsilon u^\varepsilon_1 S^\varepsilon) + \partial_{x_2}(\rho^\varepsilon u^\varepsilon_2 S^\varepsilon) &= 0.
\end{align*}
\]

From (3.11), we introduce the following stream function \( \psi^\varepsilon \):

\[
\begin{align*}
\partial_{x_1}\psi^\varepsilon &= -\rho^\varepsilon u^\varepsilon_2, \\
\partial_{x_2}\psi^\varepsilon &= \rho^\varepsilon u^\varepsilon_1,
\end{align*}
\]

which means that \( \psi^\varepsilon \) is constant along the streamlines.

From the far-field behavior of the Euler flows, we define \( \psi^\varepsilon(x_2) \) as \( \lim_{x_1 \to -\infty} \psi^\varepsilon(x_1, x_2) \). Since both the upstream Bernoulli and entropy functions are given, \( B^\varepsilon \) and \( S^\varepsilon \) have the following expression:

\[
B^\varepsilon(x) = B_-(((\psi^\varepsilon)_-)^{-1}(\psi^\varepsilon(x))), \quad S^\varepsilon(x) = S_-(((\psi^\varepsilon)_-)^{-1}(\psi^\varepsilon(x))),
\]

where \((\psi^\varepsilon)_-^{-1}\psi^\varepsilon(x)\) is a function from \( \Omega \) to \([0, 1]\). For fixed \( x_1 \), it can be regarded as a backward characteristic map with

\[
\frac{\partial((\psi^\varepsilon)_-^{-1}\psi^\varepsilon)}{\partial x_2} = \frac{\rho^\varepsilon u^\varepsilon_1}{\rho^\varepsilon u^\varepsilon_2} > 0.
\]

The boundedness and positivity of \( \rho^\varepsilon u^\varepsilon_2 \) and \( \rho^\varepsilon u^\varepsilon_1 \) show that the map is not degenerate. Thus, we have

\[
\begin{align*}
\partial_{x_1}B^\varepsilon(x) &= -B'_-(((\psi^\varepsilon)_-)^{-1}(\psi^\varepsilon(x))) \frac{\rho^\varepsilon u^\varepsilon_1}{\rho^\varepsilon u^\varepsilon_2}, \\
\partial_{x_2}B^\varepsilon(x) &= B'_-(((\psi^\varepsilon)_-)^{-1}(\psi^\varepsilon(x))) \frac{\rho^\varepsilon u^\varepsilon_2}{\rho^\varepsilon u^\varepsilon_1}.
\end{align*}
\]

Then \( B^\varepsilon \) is uniformly bounded in \( BV \), which implies its strong convergence. The similar argument can lead to the strong convergence of \( S^\varepsilon \).
2. For the corresponding vorticity sequence \( \omega^\varepsilon \), (2.23) can be written as

\[
\begin{align*}
\partial x_1 B^\varepsilon &= u^\varepsilon_2 \omega^\varepsilon + \frac{1}{\gamma}(\rho^\varepsilon)^{\gamma-1} \partial x_1 S^\varepsilon, \\
\partial x_2 B^\varepsilon &= -u^\varepsilon_1 \omega^\varepsilon + \frac{1}{\gamma}(\rho^\varepsilon)^{\gamma-1} \partial x_2 S^\varepsilon.
\end{align*}
\]

(3.14)

By direct calculation, we have

\[
\omega^\varepsilon = \frac{1}{\rho^\varepsilon} \left( u^\varepsilon_2 (\partial x_1 B^\varepsilon - \frac{1}{\gamma}(\rho^\varepsilon)^{\gamma-1} \partial x_1 S^\varepsilon) - u^\varepsilon_1 (\partial x_2 B^\varepsilon - \frac{1}{\gamma}(\rho^\varepsilon)^{\gamma-1} \partial x_2 S^\varepsilon) \right)
\]

(3.15)

which implies that \( \omega^\varepsilon \) as a measure sequence is uniformly bounded, which is compact in \( H^{-1}_{\text{loc}} \).

Then Theorem 2.1 immediately implies that the solution sequence has a subsequence (still denoted by) \( (\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x) \) that converges a.e. in \( \Omega \) to a vector function \( (\rho, u, p) \).

Since (1.11) holds for the sequence of subsonic solutions \( (\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x) \), it is straightforward to see that \( (\rho, u, p) \) also satisfies (1.11) in the distributional sense.

3. The boundary condition is satisfied in the sense of Chen-Frid [8], which implies

\[
\int_{\partial \Omega} \phi(w)(\rho u)(w) \cdot \nu(w) d\mathcal{H}^1(w) = \int_{\Omega} (\rho u)(x) \cdot \nabla \phi(x) \, dx + \langle \text{div}(\rho u)(\Omega), \phi \rangle
\]

(3.16)

for \( \psi \in C^1_0 \). From above, we can see that \( \langle \text{div}(\rho u)(\Omega), \phi \rangle = 0 \). Also,

\[
\int_{\Omega} (\rho u)(x) \cdot \nabla \phi(x) \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} (\rho^\varepsilon u^\varepsilon)(x) \cdot \nabla \phi(x) \, dx = 0.
\]

(3.17)

Then we have

\[
\int_{\partial \Omega} \phi(w)(\rho u)(w) \cdot \nu(w) d\mathcal{H}^1(w) = 0,
\]

(3.18)

that is, \( (\rho u) \cdot \nu = 0 \) on \( \partial \Omega \) in \( \mathcal{D}' \).

This completes the proof.

4. Subsonic-Sonic Limit for the Full Euler Flows in an Infinitely Long Axisymmetric Nozzle

We consider flows through an infinitely long axisymmetric nozzle given by

\[
\Omega = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq \sqrt{x_2^2 + x_3^2} < f(x_1), \ -\infty < x_1 < \infty \},
\]

where \( f(x_1) \) satisfies

\[
\begin{align*}
f(x_1) &\to 1 \quad \text{as} \ x_1 \to -\infty, \\
f(x_1) &\to r_0 \quad \text{as} \ x_1 \to \infty, \\
\|f\|_{C^{2,\alpha}(\mathbb{R})} &\leq C \quad \text{for some} \ \alpha > 0 \ \text{and} \ C > 0, \\
\inf_{x_1 \in \mathbb{R}} f(x_1) &= b > 0.
\end{align*}
\]

(4.1)

(4.2)

See Fig. 4.2

The boundary condition is set as follows: Since the nozzle wall is solid, the flow satisfies the slip boundary condition:

\[
u = 0 \quad \text{on} \ \partial \Omega,
\]

(4.3)
where \( u = (u_1, u_2, u_3) \), and \( \nu = (\nu_1, \nu_2, \nu_3) \) is the unit outward normal to the nozzle wall. In the flow without vacuum, it can be written as

\[
(pu) \cdot \nu = 0 \quad \text{on } \partial \Omega. \tag{4.4}
\]

The continuity equation in (1.1) and the boundary condition (4.4) imply that the mass flux

\[
\int_{\Sigma} (pu) \cdot l \, ds \equiv m_0 \tag{4.5}
\]

remains for some positive constant \( m_0 \), where \( \Sigma \) is any surface transversal to the \( x_1 \)-axis direction, and \( l \) is the normal of \( \Sigma \) in the positive \( x_1 \)-axis direction.

In Duan-Luo [19], the axisymmetric flows without swirl are considered for the fluid density \( \rho = \rho(x_1, r) \), the velocity

\[
u = (u_1, u_2, u_3) = (U(x_1, r) \frac{x_2}{r}, V(x_1, r) \frac{x_3}{r}),
\]

and the pressure \( p = p(x_1, r) \) in the cylindrical coordinates, where \( u_1, u_2, u_3 \) are the axial velocity, radial velocity, and swirl velocity, respectively, and \( r = \sqrt{x_2^2 + x_3^2} \). Then, instead of (1.1), we have

\[
\begin{align*}
\partial_{x_1}(r \rho U) + \partial_r (r \rho V) &= 0, \\
\partial_{x_1}(r \rho U^2) + \partial_r (r \rho UV)_r + r \partial_{x_1} p &= 0, \\
\partial_{x_1}(r \rho UV) + \partial_r (r \rho V^2)_r + r \partial_r p &= 0, \\
\partial_{x_1}(r \rho U(E + \frac{p}{\rho})) + \partial_r (r \rho V(E + \frac{p}{\rho})) &= 0.
\end{align*}
\]

Rewrite the axisymmetric nozzle as

\[
\Omega = \{(x_1, r) : 0 \leq r < f(x_1), -\infty < x_1 < \infty\}
\]

with the boundary of the nozzle:

\[
\partial \Omega = \{(x_1, r) : r = f(x), -\infty < x_1 < \infty\}.
\]

The boundary condition (4.3) becomes

\[
(U, V, 0) \cdot \tilde{\nu} = 0 \quad \text{on } \partial \Omega, \tag{4.7}
\]

where \( \tilde{\nu} \) is the unit outer normal of the nozzle walls in the cylindrical coordinates. The mass flux condition (4.5) can be rewritten in the cylindrical coordinates as

\[
\int_{\Sigma} (r \rho U, r \rho V, 0) \cdot l \, dS \equiv m := \frac{m_0}{2\pi}, \tag{4.8}
\]

where \( \Sigma \) is any curve transversal to the \( x_1 \)-axis direction, and \( l \) is the unit normal of \( \Sigma \).
The quantities \( B = h(\rho, p) + \frac{U^2 + V^2}{2} \) and \( S = \frac{\gamma p}{(\gamma - 1)\rho^\gamma} \) are both constants along each streamline. For the full Euler flows in the axisymmetric nozzle, we assume that the upstream Bernoulli and entropy functions are given, that is,

\[
h(\rho, p) + \frac{U^2 + V^2}{2} \to B_-(r) \quad \text{as} \quad x_1 \to -\infty,
\]

\[
\frac{\gamma p}{(\gamma - 1)\rho^\gamma} \to S_-(r) \quad \text{as} \quad x_1 \to -\infty,
\]

where \( B_-(r) \) and \( S_-(r) \) are smooth functions defined on \([0, 1]\).

Set

\[
B = \inf_{r \in [0, 1]} B_-(r), \quad S_1 = \|B_-'\|_{C^{0,1}([0,1])},
\]

\[
S = \inf_{r \in [0, 1]} S_-(r), \quad S_2 = \|S_-'\|_{C^{0,1}([0,1])}.
\]

We denote the above problem as Problem 2. It is shown in [19] that

**Theorem 4.1.** Suppose that the nozzle satisfies [14]. Let the upstream Bernoulli function \( B(r) \) and entropy function \( S(r) \) satisfy \( B > 0, B'(r) \in C^{1,1}([0,1]), B'(0) = 0, B'(r) \geq 0 \) on \( r \in [0,1] \); and \( S > 0, S'(r) \in C^{1,1}([0,1]), S'(0) = 0, S'(r) \leq 0 \) on \( r \in [0,1] \). Then we have

(i) There exists \( \delta_0 > 0 \) such that, if \( \delta := \max\{\sigma_1, \sigma_2\} \leq \delta_0 \), then there is \( \hat{m} \leq 2\delta_0^{\frac{1}{3}} \) so that, for any \( m \in (\delta^{\frac{1}{3}}, \hat{m}) \), there exists a global \( C^1 \)-solution (i.e. a full Euler flow) \((\rho, U, V, p) \in C^1(\overline{\Omega})\) through the nozzle with mass flux condition [18] and the upstream asymptotic condition [14]. Moreover, the flow is uniformly subsonic, and the axial velocity is always positive, i.e.,

\[
\sup_{\Omega}(U^2 + V^2 - c^2) < 0 \quad \text{and} \quad U > 0 \quad \text{in} \quad \Omega.
\]

(ii) The subsonic flow satisfies the following properties: As \( x_1 \to -\infty \),

\[
\rho \to \rho_- > 0, \quad \nabla \rho \to 0, \quad p \to \frac{\gamma - 1}{\gamma} S_-(r)\rho_-^\gamma, \quad \nabla p \to (0, \frac{\gamma - 1}{\gamma} S'_-(r)\rho_-^\gamma),
\]

\[
(U, V) \to (U_-(r), 0), \quad \nabla U \to (0, U'_-(r)), \quad \nabla V \to 0
\]

uniformly for \( r \in K_1 \subset (0,1) \), where \( \rho_- \) is a positive constant, and \( \rho_- \) and \( U_-(r) \) can be determined by \( m \) and \( B \) uniquely.

(iii) There exists at most one smooth axisymmetric subsonic flow through the nozzle which satisfies [14] and the properties in (ii).

(iv) There exists a critical mass flux \( \hat{m} \) such that, for any \( m \in (\delta^{\frac{1}{3}}, \hat{m}) \), there exists a unique axisymmetric subsonic flow through the nozzle with the mass flux condition [18] and the asymptotic behavior [14]. Moreover, \( \hat{m} \) is the upper critical mass flux for the existence of subsonic flow in the following sense: Either \( \sup_{\Omega}(U^2 + V^2 - c^2) \to 0 \) as \( m \to \hat{m} \), or there is no \( \sigma > 0 \) such that, for all \( m \in (\hat{m}, \hat{m} + \sigma) \), there is an Euler flow with the mass flux \( m \) through the nozzle which satisfies the upstream asymptotic condition [14], the downstream asymptotic behavior [14], and

\[
\sup_{m \in (\hat{m}, \hat{m} + \sigma)} \Omega \sup_{\Omega}(c^2 - (U^2 + V^2)) > 0.
\]

As above, we have the subsonic-sonic limit theorem for this case.

**Theorem 4.2** (Subsonic-sonic limit of three-dimensional Euler flows through an axisymmetric nozzle). Let \( \delta^{\frac{1}{3}} < m^e < \hat{m} \) be a sequence of mass fluxes, and let \( \rho^e, u^e = (u_1^e, u_2^e, u_3^e) \), and \( p^e \) be the corresponding solutions to Problem 2 \((m^e)\). Then, as \( m^e \to \hat{m} \), the solution sequence possesses a subsequence (still denoted by) \((\rho^e, u^e, p^e)\) that converges strongly a.e. in \( \Omega \) to a vector function \((\rho, u, p)\) with \( u = (u_1, u_2, u_3) \) which is a weak solution of Problem 2 \((\hat{m})\).
Furthermore, the limit solution \((\rho, u, p)\) also satisfies (1.1) in the distributional sense and the boundary conditions (3.2) as the normal trace of the divergence-measure field \((\rho u_1, \rho u_2, \rho u_3)\) on the boundary in the sense of Chen-Frid [8].

**Proof.** First, we need to show that \((\rho^\varepsilon, u^\varepsilon, p^\varepsilon)\) satisfy condition (A.1)–(A.3) in \(\Omega\).

For the approximate solutions, \(B^\varepsilon\) and \(S^\varepsilon\) satisfy
\[
\begin{align*}
\partial_x (r \rho^\varepsilon U^\varepsilon B^\varepsilon) + \partial_t (r \rho^\varepsilon V^\varepsilon B^\varepsilon) &= 0, \\
\partial_x (r \rho^\varepsilon U^\varepsilon S^\varepsilon) + \partial_t (r \rho^\varepsilon V^\varepsilon S^\varepsilon) &= 0.
\end{align*}
\] (4.15)
(4.16)

From \(\partial_x (r \rho^\varepsilon U^\varepsilon) + \partial_t (r \rho^\varepsilon V^\varepsilon) = 0\), we introduce \(\psi^\varepsilon\) as
\[
\begin{align*}
\partial_x \psi^\varepsilon &= -r \rho^\varepsilon V^\varepsilon, \\
\partial_t \psi^\varepsilon &= r \rho^\varepsilon U^\varepsilon.
\end{align*}
\] (4.17)

From the far-field behavior of the Euler flows, we define \(\psi^\varepsilon_-(r) := \lim_{x_1 \to -\infty} \psi^\varepsilon(x_1, r)\). Similar to the argument in Theorem 3.2, \((\psi^\varepsilon_-)^{-1}(\psi^\varepsilon)\) are nondegenerate maps. A direct calculation yields
\[
B^\varepsilon(x_1, x_2, x_3) = B_-((\psi^\varepsilon_-)^{-1}(\psi^\varepsilon(x_1, \sqrt{x_2^2 + x_3^2}))),
\]
\[
S^\varepsilon(x_1, x_2, x_3) = S_-((\psi^\varepsilon_-)^{-1}(\psi^\varepsilon(x_1, \sqrt{x_2^2 + x_3^2}))).
\]

Therefore, we have
\[
\begin{align*}
\partial_x B^\varepsilon &= -r \rho^\varepsilon V^\varepsilon \frac{B'}{(\psi^\varepsilon')^2}, \\
\partial_x B^\varepsilon &= x_2 \rho^\varepsilon U^\varepsilon \frac{B'}{(\psi^\varepsilon')^2}, \\
\partial_x B^\varepsilon &= x_3 \rho^\varepsilon U^\varepsilon \frac{B'}{(\psi^\varepsilon')^2}.
\end{align*}
\] (4.18)

Notice that
\[
\frac{B'_-(\psi^-_1)^{-1}(\psi^\varepsilon))}{(\psi^-_1)^{-1}(\psi^\varepsilon))} = \frac{B'_-(\psi^-_1)^{-1}(\psi^\varepsilon))}{(\psi^-_1)^{-1}(\psi^\varepsilon))}.
\] (4.19)

Since \(B'_-(0) = 0\) and \(B_- \in C^{1,1}\), we conclude that \(\frac{B'(s)}{s}\) is bounded. Then the sequence \(B^\varepsilon\) is uniformly bounded in \(BV\), which implies its strong convergence. The similar argument can lead to the strong convergence of \(S^\varepsilon\).

On the other hand, the vorticity \(\omega^\varepsilon\) has the following expression:
\[
\begin{align*}
\omega^\varepsilon_{1,2} &= \partial_{x_1} u^\varepsilon_2 - \partial_{x_2} u^\varepsilon_1 = \frac{24}{7} (\partial_{x_1} V^\varepsilon - \partial_{x_1} U^\varepsilon), \\
\omega^\varepsilon_{2,3} &= \partial_{x_2} u^\varepsilon_3 - \partial_{x_3} u^\varepsilon_2 = 0, \\
\omega^\varepsilon_{3,1} &= \partial_{x_3} u^\varepsilon_1 - \partial_{x_1} u^\varepsilon_3 = \frac{24}{7} (\partial_{x_1} V^\varepsilon - \partial_{x_1} U^\varepsilon).
\end{align*}
\] (4.20)

A direct computation gives
\[
\partial_{x_1} V^\varepsilon - \partial_{x_1} U^\varepsilon = \frac{r}{(\psi^-_1)}(\rho^\varepsilon B'_- - (\rho^\varepsilon)^\gamma S'_-),
\] (4.21)

which implies that \(\omega^\varepsilon\) is uniformly bounded in the bounded measure space.

Since (1.1) holds for the sequence of subsonic solutions \((\rho^\varepsilon, u^\varepsilon, p^\varepsilon)(x)\), it is straightforward to see from Theorem 2.4 that there exists a subsequence (still denoted by) \((\rho^\varepsilon, u^\varepsilon, p^\varepsilon)\) which converges to a vector function \((\rho, u, p)\) a.e. in \(\Omega\) satisfying (1.4) in the distributional sense.
The boundary condition is satisfied in the sense of Chen-Frid [8], which implies
\[ \int_{\partial \Omega} \phi(w)(\rho u)(w) \cdot \nu(w) \, dH^1(w) = \int_{\Omega} (\rho u)(x) \cdot \nabla \phi(x) \, dx + \langle \text{div}(\rho u) \rangle_{\Omega}, \phi \] (4.22)
for \( \psi \in C^1_0 \). From above, we can see \( \langle \text{div}(\rho u) \rangle_{\Omega, \phi} = 0 \). Furthermore, we have
\[ \int_{\Omega} (\rho u)(x) \cdot \nabla \phi(x) \, dx = \lim_{\varepsilon \to 0} \int_{\Omega} (\rho u^\varepsilon)(x) \cdot \nabla \phi(x) \, dx = 0, \] (4.23)
which yields
\[ \int_{\partial \Omega} \phi(w)(\rho u)(w) \cdot \nu(w) \, dH^1(w) = 0, \] (4.24)
that is, \( (\rho u) \cdot \nu = 0 \) on \( \partial \Omega \) in \( D' \).

This completes the proof.

**Remark 4.1.** In the isentropic case, the subsonic results of [5, 17, 32] can be also extended to the sonic-subsonic limit by using Theorem 2.2.

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