Using a New Modification of Trust Region Spectral (TRS) Approach to Solve Optimization Problems

Hussain Ali Mueen ¹ and Mushtak A.K. Shiker ²*
¹, ²Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Hilla-Iraq.
E-mail: ¹ hussain.moeen.pure234@student.uobabylon.edu.iq , ² mmttmhh@yahoo.com , pure.mushtaq@uobabylon.edu.iq

Abstract. A trust-region spectral (TRS) technique is an important strategy to solve optimization problems. In this work, a new modification of (TRS) is introduced by using a new trust-region radius (TRR). We proved the global convergence of the new algorithm. The numerical experiment was made by comparing the proposed algorithm with famous algorithms depending on the number of iterations, time required to find the solution and functions evaluation. Based on these results, we can conclude that the proposed algorithm is better than the three algorithms that were compared with.

Keywords: Optimization problem. Line search method. Global convergence. Trust-region method. Trust-region spectral.

1. Introduction

Consider the following optimization problem:

\[ \min_{x \in \mathbb{R}^n} f(x) \quad (1) \]

Where \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuously differentiable.

The TRSM has been greatly utilized to solve many problems, such as discrete boundary value problems, network flow problems, etc. [1].

There are several approaches to solve Eq. 1 depended on Newton's method as a base, it need to find the second derivative in all iteration. La Cruz presented the spectroscopic method for solving Eq. 1, he used the nonmonotone technique to develop TRM to avoid the disadvantage mentioned previously, for each iteration, this technique takes \( x_k - \lambda_k f(x_k) \) to be the trial point and the residual \( \pm f(x_k) \) to be the search direction, where \( \lambda_k \) is the spectral coefficient that satisfies the condition of Grippo- Lampariello [2], he used the following line search method

\[ f(x_k - \lambda_k d_k) \leq \max_{0 \leq j \leq m-1} f(x_{j-1} + \alpha \lambda_k \nabla f(x_k)^T d_k) \quad (2) \]
where \( f(x) = \frac{1}{2} \| F(x) \|^2 \), \( \alpha \) are nonnegative integer and \( d_k = -g_k \).

This work aims to introduce TRSM to solve the optimization problem. In the classic TRM at each iterative point \( x_k \), we get the direction \( d_k \) by solving the following equation [3]:

\[
\min q_k(d) = f_k + g_k^T d + \frac{1}{2} d^T B_k d, \quad s.t \quad \| d \| \leq \Delta_k,
\]

where \( \Delta_k \) is a TRR.

\( d_k \) must holds

\[
d_k = -g_k \quad (4)
\]

That is agreement \( d_k \) is a descent direction of \( f(x) \) at \( x_k \) [4].

**Armijo rule:** Let \( \lambda > 0 \) be a constant, \( \rho \in (0, 1) \) and \( \mu \in (0, 1) \). Take \( \alpha_k \) to be the largest \( \alpha \) in \( \{ \lambda, \lambda \rho, \lambda \rho^2, \ldots \} \) such that

\[
f_k - f(x_k + \alpha d_k) \geq -\alpha \mu g_k^T d_k \| g_k \|.
\]

The most important step in Armijo rule is how to determine \( \lambda \).

* At each iteration, \( f(x) \) is increasing if \( \lambda \) is greater than the function computations number.

* The consequence is increasing if \( \lambda \) is less than the iterations number and the effectiveness is decreasing.

For (1), we use \( y_k \) as an approximately matrix of \( f_k \).

After multiplying both sides by \( y_k^T \), the classic quasi-Newton equation will be:

\[
B_{k+1} d_k y_k^T = y_k y_k^T \quad (6)
\]

By using \( B_{k+1} = y_{k+1} \), (6) will be:

\[
y_{k+1} = \frac{y_k y_k^T}{y_k^T d_k} \quad (7)
\]

where \( d_k = x_{k+1} - x_k \) and \( y_k = F_{k+1} - F_k \).

The rest sections are arranged as: In section 2 the suggested algorithm is presented. Section 3 contains the convergence of the new suggested algorithm. In Section 4, the numerical tests reported.

## 2. New Modification

A new TRS Algorithm will be introduced for solving (1) in this section. Let \( d_k \) be a solution of Eq. 3. We define \( r_k \) as:

\[
r_k = \frac{f(x_k) - f(x_{k+1})}{q_k(0) - q_k(d_k)},
\]

where \( f(x_k) - f(x_{k+1}) \) is the actual reduction and \( q_k(0) - q_k(d_k) \) is the predicate reduction. \( r_k \) is used to indicate acceptance or rejection of step length. The length step is accepted and trust-region extended when \( r_k \) closed enough to 1. The trial step accepted when \( r_k \) is a positive, but not closed to 1, leading to
$x_{k+1} = x_k + d_k$ and the TRR kept the same without altering. Otherwise, the search area should be shrunk and (3) returned to find an acceptable step length indicating the continuation of the procedure [4, 5]. For the development of TRM, many techniques were used, most notably is line search methods (LSM), which greatly contributed to reduce the re-solving of sub-problems in case of refusing the suggested step length [6, 7]. The authors introduced many papers in various field of science such as optimization (see [8- 14]), reliability (see[15- 20]), line search methods (see [21- 24]), and transportation problems (see [25- 28]).

Now, we introduce the new suggested algorithm to solve (1).

**Algorithm 1. (New suggested algorithm)**

**Step0:** Given constants $0 < \varphi_1 < \varphi_2 < 1, 0 < \beta_1 < \beta_2 < 1,$ and $0 < \Delta_0 < \hat{\Delta}$.

Let $k := 0$;

**Step1:** Compute $F_k$, if $F_k \leq \varepsilon$, then terminate.

**Step 2:** Let $\Delta = \Delta_k$ and solve (3) to get $\varphi_1$;

**Step 3:** Calculate $\varphi_2$ and $\Delta_k$ and solve (3) to get $\varphi_3$;

**Step 4:** Let $x_{k+1} = x_k + d_k$,

Compute $\Delta_{k+1} = \begin{cases} \min\{\beta_2, \Delta_k, \hat{\Delta}\}, & \text{if } r_k < \varphi_1 \\ \Delta_k, & \text{otherwise} \end{cases}$

Compute $\gamma_{k+1}$ by (7). Set $k := k + 1$, go to Step 1.

3. **Convergence analysis:**

To prove the convergence of algorithm 1. Some axioms and assumptions are needed.

**Assumption 1.** [29]

(1) The level set $\Omega = \{x \in R^n | f(x) \leq f(x_0)\}$ is bounded.

(2) $\|J_k - \gamma_k I\| = O (\|d_k\|^2)$ holds.

**Lemma 3.1:** [30]

If $d_k$ is a solution of (3), then

$$\text{pred}_k (d_k) \geq \frac{1}{2} \|\gamma_k F_k\| \min\left\{\Delta_k, \frac{\|F_k\|}{\|\gamma_k\|}\right\}$$

**Proof**

Since $d_k$ is a solution of (3), for any $\alpha \in [0, 1]$, it follows that
\[ \text{pred}_k (d_k) = \frac{1}{2} (\| F_k \|^2 - \| F_k + \gamma_k d_k \|^2) \geq \frac{1}{2} (\| F_k \|^2 - \| F_k - \gamma_k \frac{\alpha \Delta_k}{\| F_k \|} y_k \|^2) = \alpha \Delta_k \| y_k \| F_k \| - \frac{1}{2} \alpha^2 \Delta_k^2 y_k^2 \] (10)

So,
\[ \text{pred}_k (d_k) \geq \frac{1}{2} \max_{0 \leq \alpha \leq 1} \left[ \alpha \Delta_k \| y_k \| F_k \| - \left[ \frac{1}{2} \alpha^2 \Delta_k^2 y_k^2 \right] \right] \geq \frac{1}{2} \| y_k \| F_k \| \min \Delta_k \left( \frac{\| F_k \|}{\| y_k \|} \right) \] (11)

**Lemma 3.2:** [31]

Let assumption 1 holds, and \( \{ x_k \} \) be generated by Algorithm 1, then \( \{ x_k \} \subset \Omega \}. Furthermore, \( \{ f(x_k) \} \) converges.

**Proof**

Algorithm 1 gives:
\[ y_k \geq \varphi_1 > 0 \] (12)

This implies,
\[ f(x_{k+1}) \leq f(x_k) \leq \cdots \leq f(x_0) \].

Therefore, \( \{ x_k \} \subset \Omega \). Since \( (x_k) \geq 0 \), this implies that \( \{ f(x_k) \} \) converges. \( \square \)

**Theorem 3.3** (Global convergence of Algorithm 1)

Let assumption 1 hold, Then Algorithm 1 either stops finitely or generates \( \{ x_k \} \) such that
\[ \lim_{k \to \infty} \| F_k \| = 0 \] (13)

**Proof**

Suppose algorithm 1, not terminate in \( n \) steps, assume it is not stop after finite steps, and (13) not hold, so, \( \exists \varepsilon > 0 \) and a subsequence \( \{ k_j \} \) Verifying,
\[ \| F_{k_j} \| \geq \varepsilon. \] (14)

Let \( K = \{ k | \| F_k \| \geq \varepsilon \} \).
Let \( S_0 = \{ k | \gamma_k \geq \varphi_2 \} \).

From Algorithm 1 and Lemma 3.1, we get
\[ \sum_{k \in S_0} [f(x_k) - f(x_{k+1})] \geq \sum_{k \in S_0} \varphi_2 \cdot \text{pred}_k (d_k) \geq \sum_{k \in K} \varphi_2 \cdot \frac{\varepsilon}{\| y_k \|} \min \Delta_k \left( \frac{\varepsilon}{\| y_k \|} \right) \]

from Lemma 3.2, we have \( \{ f(x_k) \} \) is convergent, so
Then, we get
\[
\sum_{k \in S_0} \varphi_2 \frac{\varepsilon |y_k|}{2} \min \left\{ \Delta_k, \frac{\varepsilon}{|y_k|} \right\} < \infty
\]

By using 3-4 from algorithm 1, we get

\[
\Delta_{k+1} \leq \Delta_k
\]

for all \( k \notin s_0, S_0 \), (15) will be

\[
\sum_{k \in K} \Delta_k < \infty
\]

So \( \exists x^* \) s.t

\[
\lim_{k \to \infty} x_k = x^*
\]

from (17), we get \( \Delta_k \to 0 \). that means

\[
\text{pred}_k (d_k) \geq \frac{\varepsilon |y_k|}{2} \Delta_k
\]

for all sufficiently large \( k \), then

\[
|\text{Ared}_k (d_k) - \text{pred}_k (d_k)| = O(\|d_k\|)^2
\]

That is

\[
\lim_{k \to \infty} r_k = 1.
\]

That show us, the sufficiently \( k \) and \( k \in K \).

\[
\Delta_{k+1} \geq \Delta_k
\]

Which contradicts (16). That's mean (13) is held.

4- Numerical Tests

We will use the following algorithms TTR2 [32], TTR3 [33], TTR4 [34] taken for comparison with our proposed algorithm TTR1, the test results were performed on PC has 8.00 GB Ram and CPU 2.30 with 4 GH, and by using Matlab program R2014a.

Let \( \mu_1 = 0.7; \mu_2 = 0.8; \varepsilon = 0.6; p = 0.4 \); epsilon \( = 10^{-5} \) and 20000 is the total number of iteration exceeds. The algorithm will solve the following optimization problem [32]:

\[
p: f = (x_2 - x_1^2)^3 + (1 - x_1)^2
\]

Table 4.1, 4.2, and 4.3 contain the results, as follows:

4.1: Table of iterations.
The tables above indicate complete preference for TTR1 over the rest of the algorithms TTR2, TTR3, TTR4 according to the time, the functions evaluation number and iterations number, because in all of them the numbers obtained by the algorithm TTR1 is less than that obtained by other three methods.

5- Conclusion:

TRSM is a globalization strategy for solving optimization problems. In this work, a new modification of TRSM is suggested. Then the convergence of the new algorithm is proved. We compared the proposed algorithm with three well-known algorithms, and the computational results show that the new technique outperforms the three methods in terms of efficiency and robustness, and is particularly useful for solving optimization problems.

6. Reference

[1] Shiker M A K and Sahib Z 2018 A modified trust-region method for solving unconstrained optimization, *Journal of Engineering and Applied Sciences*. 13: 22, p 9667–9671.

[2] Cruz W and Marcos R 2003 Nonmonotone spectral methods for large-scale nonlinear systems, *Optimization Methods and Software*. 18: 5, p 583-599.

[3] Dwail H H and Shiker M A K 2020 Reducing the time that TRM requires to solve systems of nonlinear equations, *IOP Conf. Ser.: Mater. Sci. Eng*. 928 042043.

[4] Dwail H H and Shiker M A K 2020 Using a trust region method with nonmonotone technique to solve unrestricted optimization problem, *J. Phys.: Conf. Ser*. 1664 012128.

[5] Mahdi M M and Shiker M A K 2020 Three-term of new conjugate gradient projection approach under Wolfe condition to solve unconstrained optimization problems, *Journal of Advanced Research in Dynamical and Control Systems*. 12: 7, p 788-795.
[6] Dwail H H et al. 2021 A new modified TR algorithm with adaptive radius to solve a nonlinear systems of equations, *J. Phys.: Conf. Ser.* 1804 012108.

[7] Dwail H H and Shiker M A K 2021 Using trust region method with BFGS technique for solving nonlinear systems of equations, *J. Phys.: Conf. Ser.* 1818 012022.

[8] Mahdi M M and Shiker M A K 2020 A new projection technique for developing a Liu-Storey method to solve nonlinear systems of monotone equations, *J. Phys.: Conf. Ser.* 1591 012030.

[9] Mahdi M M et al. 2021 Solving systems of nonlinear monotone equations by using a new projection approach, *J. Phys.: Conf. Ser.* 1804 012107.

[10] Wasi H A and Shiker M A K 2021 Proposed CG method to solve unconstrained optimization problems, *J. Phys.: Conf. Ser.* 1804 012024.

[11] Wasi H A and Shiker M A K 2020 A new hybrid CGM for unconstrained optimization problems, *J. Phys.: Conf. Ser.* 1664 012077.

[12] Wasi H A and Shiker M A K 2021 Nonlinear conjugate gradient method with modified Armijo condition to solve unconstrained optimization, *J. Phys.: Conf. Ser.* 1818 012021.

[13] Wasi H A and Shiker M A K 2021 A modified of FR method to solve unconstrained optimization, *J. Phys.: Conf. Ser.* 1804 012023.

[14] Shiker M A K and Amini K 2018 A new projection-based algorithm for solving a large scale nonlinear system of monotone equations, *Croatian operational research review*, 9: 1, p 63-73.

[15] Hassan Z A H and Mutar E K 2017 Geometry of reliability models of electrical system used inside spacecraft, *Second Al-Sadiq International Conference on Multidisciplinary in IT and Communication Science and Applications (AIC-MITCSA)*, pp. 301-306.

[16] Hassan Z A H and Balan V 2017 Fuzzy T-map estimates of complex circuit reliability, *International Conference on Current Research in Computer Science and Information Technology (ICCIT-2017)*, IEEE, Special issue, pp. 136-139.

[17] Hassan Z A H and Balan V 2015 Reliability extrema of a complex circuit on bi-variate slice classes, *Karbala International Journal of Modern Science*, 1: 1, pp. 1-8.

[18] Hassan Z A H and Shiker M A K 2018 Using of generalized baye’s theorem to evaluate the reliability of aircraft systems, *Journal of Engineering and Applied Sciences*, (Special Issue13), 10797–10801.

[19] Abdullah G and Hassan Z A H 2020 Using of particle swarm optimization (PSO) to addressed reliability allocation of complex network, *J. Phys.: Conf. Ser.* 1664: 012125.

[20] Abdullah G and Hassan Z A H 2020 Using of Genetic Algorithm to Evaluate Reliability Allocation and Optimization of Complex Network, *IOP Conf. Ser.: Mater. Sci. Eng.* 928 042033.

[21] Hashim K H and Shiker M A K 2021 Using a new line search method with gradient direction to solve nonlinear systems of equations, *J. Phys.: Conf. Ser.* 1804 012106.
[22] Hashim K H, et al. 2021 Solving the Nonlinear Monotone Equations by Using a New Line Search Technique, *J. Phys.: Conf. Ser.* **1818** 012099.

[23] Hashim L H, et al. 2021 An application comparison of two negative binomial models on rainfall count data, *J. Phys.: Conf. Ser.* **1818** 012100.

[24] Hashim L H, et al. 2021 An application comparison of two Poisson models on zero count data, *J. Phys.: Conf. Ser.* **1818** 012165.

[25] Hussein H A, Shiker M A K and Zabiba M S M 2020 A new revised efficient of VAM to find the initial solution for the transportation problem, *J. Phys.: Conf. Ser.* **1591** 012032.

[26] Zabiba M S M, Al-Dallal H A H, Hashim, Hashim K H, Mahdi M M and Shiker M A K 2021 A new technique to solve the maximization of the transportation problems. “in press”, ICCEPS - April.

[27] Hussein H A and Shiker M A K 2020 Two New Effective Methods to Find the Optimal Solution for the Assignment Problems, *Journal of Advanced Research in Dynamical and Control Systems*, **12**: 7, p 49-54.

[28] Hussein H A and Shiker M A K 2020 A modification to Vogel’s approximation method to Solve transportation problems, *J. Phys.: Conf. Ser.* **1818** 012029.

[29] Dreeb N K, et al. 2021, Using a new projection approach to find the optimal solution for nonlinear systems of monotone equation, *J. Phys.: Conf. Ser.* **1818** 012101.

[30] Mahdi M M and Shiker M A K 2020 Three terms of derivative free projection technique for solving nonlinear monotone equations, *J. Phys.: Conf. Ser.* **1591** 012031.

[31] Mahdi M M and Shiker M A K 2020 A New Class of Three-Term Double Projection Approach for Solving Nonlinear Monotone Equations, *J. Phys.: Conf. Ser.* **1664** 012147.

[32] La Cruz W, José M, and Marcos R 2006 Spectral residual method without gradient information for solving large-scale nonlinear systems of equations, *Mathematics of Computation*. **75**: 255, p 1429-1448.

[33] Wu Y 2017 A modified three-term PRP conjugate gradient algorithm for optimization models, *Journal of inequalities and applications*. **97**:1, p 219-225.

[34] Apkarian, P, Dominikus N, and Olivier P 2008 A trust region spectral bundle method for nonconvex eigenvalue optimization, *SIAM Journal on Optimization*. **19**:1, p 281-306.