Projective transformations of rotation sets

François Béguin¹, Sylvain Crovisier² ¹ and Frédéric Le Roux³

¹ LAGA, CNRS UMR 7539, Université Paris 13, 93430 Villetaneuse, France
² LMO, CNRS UMR 8628, Université Paris-Sud 11, 91405 Orsay, France
³ IMJ-PRG, CNRS UMR 7586, Université Marie et Pierre Curie, 75005 Paris, France

E-mail: béguin@math.univ-paris13.fr, sylvain.crovisier@math.u-psud.fr and frederic.le-roux@imj-prg.fr

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Abstract

We give a new proof and extend a result of Kwapisz: whenever a set C is realized as the rotation set of some torus homeomorphism, the image of C under certain projective transformations is also realized as a rotation set.

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The concept of rotation set, introduced by Misiurewicz and Ziemian in [5], is one of the most important tools to study the global dynamics of homeomorphisms of the torus $\mathbb{T}^2$. If $f$ is a homeomorphism of $\mathbb{T}^2$ isotopic to the identity, and $F$ is a lift of $f$ to $\mathbb{R}^2$, the rotation set of $F$ is a compact convex subset of the plane which describes ‘at what speeds and in what directions the orbits of $f$ rotate around the torus’. One of the main problems in the theory is to determine which compact convex subsets of $\mathbb{R}^2$ can be realized as the rotation sets of some torus homeomorphisms. For compact convex subsets with empty interiors (i.e. singletons and segments), a conjectural answer to the problem has been formulated by Franks and Misiurewicz (see [1]). Fifteen years ago, Kwapisz has introduced a technical tool which allows to simplify the problem. Namely, he observed that, if a compact convex set $C \subset \mathbb{R}^2$ is realized as the rotation of a certain torus diffeomorphism, and if a projective transformation $L$ maps $C$ to a bounded set of the plane, then $L(C)$ can be realized as the rotation set of another torus diffeomorphism (see [2, section 2]).

Kwapisz’s proof requires to consider the suspension of the initial torus homeomorphism, and to apply a theorem of Fried to find a new surface of section for this flow, in the appropriate cohomology class. Fried’s theorem works only for $C^1$ flows; this forces Kwapisz to consider only rotation sets of $C^1$ diffeomorphisms, whereas the natural setting for his result would

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be rotation sets of homeomorphisms. The purpose of the present note is to provide a more elementary proof of Kwapisz’s result. Our proof remains at the level of surfaces homeomorphisms, i.e. does not require to consider a flow on a three-dimensional manifold. It does not make use of Fried’s theorem (in some sense, we replace it by the more classical fact that the only surface with fundamental group isomorphic to $\mathbb{Z}^2$ is the torus $\mathbb{T}^2$). As a consequence, it works for surface homeomorphisms without any differentiability assumption. This might be of interest in relation with some recent works related to the Franks–Misiurewicz conjecture (see [3, 4], and the example of Avila quoted in these papers).

**Theorem.** Let $C$ be a compact subset of the plane which is realized as the rotation set of some torus homeomorphism. Let $L \in \text{SL}(3, \mathbb{Z})$ be a projective transformation such that the image $C'$ of $C$ under $L$ is a bounded subset of the plane. Then $C'$ is also realized as the rotation set of some torus homeomorphism.

In this statement, we use the usual affine chart to embed the plane in the projective plane. The requirement that the image of $C$ under $L$ is a bounded subset of the plane means that we demand that $L(C)$ does not meet the line at infinity. A more precise version of the above theorem will be given below.

We now recall the classical definition of the rotation set by Misiurewicz and Ziemian. We consider a self-homeomorphism $f$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$. We assume that $f$ is isotopic to the identity which amounts to saying that $F$ commutes with the deck transformations $S : (x, y) \mapsto (x + 1, y)$ and $T : (x, y) \mapsto (x, y + 1)$. The rotation set of $F$ is defined as the set of $w \in \mathbb{R}^2$ such that there exists a sequence $(z_k)_{k \geq 0}$ of points of the plane, and a sequence $(n_k)_{k \geq 0}$ of integers tending to $\infty$ such that

$$\frac{F^n(z_k) - z_k}{n_k}$$

converges to $w$ as $k$ goes to infinity.

Note that this definition depends on the choice of coordinates on the torus (in order to identify the universal cover of the torus with $\mathbb{R}^2$). In particular, it depends on the choice of a basis $(S, T)$ of the fundamental group of the torus. To make things clear we need a definition of the rotation set that makes this dependence explicit.

**Definition.** Consider an action of $\mathbb{Z}^3$ on $\mathbb{R}^2$ generated by three commuting homeomorphisms $G, U, V$. We define the rotation set of $G$ with respect to $U$ and $V$ as the set $\rho_{U,V}(G) \subset \mathbb{R}^2$ of all vectors $w$ such that there exists a compact subset $K$ of the plane and a sequence $(m_k, n_k, p_k)_{k \geq 0}$ of elements of $\mathbb{Z}^3$ so that:

1. For every $k$, $U^{-m_k}V^{-n_k}G^{p_k}(K) \cap K \neq \emptyset$.
2. The sequence $(m_k, n_k, p_k)$ tends to infinity.
3. The sequence $(\frac{m_k}{p_k}, \frac{n_k}{p_k})$ tends to $w$.

**Remark 1.** In the case where $F$ is a lift of a homeomorphism of $\mathbb{T}^2$, and $S, T$ are the elementary translations $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (x, y + 1)$, one easily checks that the rotation set $\rho_{S,T}(F)$ coincides with the classical rotation set of $F$.

In order to prove the above theorem, we will consider a lift $F$ of a torus homeomorphism whose rotation set (in the sense of Misiurewicz and Ziemian) is the given compact convex set $C$. In order to realize the set $C' = L(C)$, we will not only replace $F$ by a new homeomorphism $G$; we will also replace the elementary translations $S : (x, y) \mapsto (x + 1, y)$ and $T : (x, y) \mapsto (x, y + 1)$ by some ‘non-linear translations’ $U, V$. 

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Remark 2. The above definition immediately extends to the case of a $\mathbb{Z}^p$ action on a noncompact topological space $X$. In this more general setting, to get a more symmetric definition, it is tempting to replace $U^{-m_1}V^{-m_2}G^{m_3}$ in the first item by $U^{-m_1}V^{-m_2}G^{-m_3}$, and to define the ‘rotation set’ as a subset of $\mathbb{R}^p$, instead of looking in a specific affine chart. The definition depends on a choice of basis of $\mathbb{Z}^p$, but two different choices give two ‘rotation sets’ that differ under a projective transformation, thus we get a conjugacy invariant which is a subset of $\mathbb{R}^p$ up to projective isomorphisms (see the argument at the end of the paper). Going back to the case of an action of $\mathbb{Z}^3$ on $\mathbb{R}^2$, one could wonder which results of the classical rotation set theory for torus homeomorphisms (in the sense of Misiurewicz and Ziemian) can be generalized to rotation sets of $\mathbb{Z}^3$ actions on the plane.

Now we are in a position to give a more precise statement of the theorem above. We denote by $\Delta_\infty = \{[x : y : 0]\}$ the ‘line at infinity’ in $\mathbb{R}^2$, and by $\Phi : \mathbb{R}^2 \setminus \Delta_\infty \to \mathbb{R}^2$ the affine chart mapping $[x : y : z]$ to $(x/z, y/z)$. If $L \in \text{SL}(3, \mathbb{R})$ is a projective transformation, we denote by $\hat{L}$ the ‘restriction of this map to the affine plane’: more formally,

$$\hat{L} = \Phi L \Phi^{-1} : \mathbb{R}^2 \setminus (\Phi L^{-1}(\Delta_\infty)) \to \mathbb{R}^2.$$

Theorem. Let $S : (x, y) \mapsto (x + 1, y)$ and $T : (x, y) \mapsto (x, y + 1)$. Let $F$ be a lift of a homeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ isotopic to the identity. Let $L \in \text{SL}(3, \mathbb{Z})$ be a projective transformation such that $\rho_{S,T}(F)$ is disjoint from the line $\Phi L^{-1}(\Delta_\infty)$. Let

$$L^{-1} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \quad \text{and} \quad U = S^{n_1} T^{n_2} F^{-c_1} \quad \text{and} \quad V = S^{n_2} T^{n_3} F^{-c_2} \quad \text{and} \quad G = S^{-a_1} T^{-a_2} F^{c_3}.$$

1. The quotient space $\mathbb{R}^2 / \langle U, V \rangle$ is homeomorphic to the torus $\mathbb{T}^2$.
2. The rotation set $\rho_{U,V}(G)$ is equal to $\hat{L}(\rho_{S,T}(F))$.

Remark 3. Note that, since $G$ obviously commutes with $U$ and $V$, it can be seen as a lift of a homeomorphism $g$ of $\mathbb{T}^2$ which is isotopic to the identity. Thus this second theorem implies the first one. From the definition, one easily deduces that $g$ is as smooth as $f$; if $f$ is $C^r$ for some $r \in \mathbb{N} \cup \{\infty\}$ or analytical, then so is $g$. Moreover, every invariant finite measure for $f$ induces an invariant finite measure for $g$. For example, if $f$ preserves a measure in the Lebesgue class, then so does $g$.

Remark 4. Consider a $\mathbb{Z}^2$ action on $\mathbb{R}^2$ generated by some homeomorphisms $U$ and $V$. Assume this action is properly discontinuous. Then the quotient space $\mathbb{R}^2 / \langle U, V \rangle$ is a topological surface (i.e. a separated topological manifold of dimension 2) whose fundamental group is isomorphic to $\mathbb{Z}^2$. According to the classification of surfaces (see e.g. [6]), it follows that this quotient space must be homeomorphic to $\mathbb{T}^2$. This is a key ingredient of the following proof that will play the part of Fried’s theorem in Kwapisz’s original proof.

Proof of Item 1 of the theorem. In view of remark 4, it is enough to prove that the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ generated by the homeomorphisms $U$ and $V$ is properly discontinuous: we consider a ball $B(0, R)$ in $\mathbb{R}^2$, and we aim to prove that $U^m V^n (B(0, R))$ is disjoint from $B(0, R)$ whenever $\| (m,n) \|$ is large enough.

We denote by $D(H)$ the displacement set of the homeomorphism $H$ of the plane, that is, the set of all vectors of the type $H(z) - z$ where $z$ ranges over $\mathbb{R}^2$. Obviously $D(S) = \{ (1,0) \}$ and
D(T) = \{(0, 1)\}. By assumption, the rotation set \(\rho_{S,T}(F)\) is disjoint from the line \(\Phi L^{-1}(\Delta_\infty)\). Therefore, we may consider a compact neighbourhood \(\mathcal{O}\) of \(\rho_{S,T}(F)\) so that
\[
\text{dist}(\mathcal{O}, \mathbb{R}^2 \cap \Phi L^{-1}(\Delta_\infty)) > \epsilon > 0.
\]
From the definition of the rotation set, one immediately sees that there exists an integer \(k_0\) so that \(D(F^k) \subset k\mathcal{O}\) for \(|k| \geq k_0\). And since \(D(F^k)\) is bounded for every \(|k| < k_0\), one gets that there exists \(R'\) so that, for every \(k \in \mathbb{Z}\)
\[
D(F^k) \subset B(0, R') + k\mathcal{O}.
\]
Now recall that \(U = S^{\mu_1} T^{1} F^{-c_1}\) and \(V = S^{\mu_2} T^{1} F^{-c_2}\). Since \(S, T\) and \(F\) commute, one immediately gets, for every \((m, n) \in \mathbb{Z}^2\),
\[
D(U^m V^n) = (ma_1 + na_2, mb_1 + nb_2) - D(F^{mc_1 + nc_2}).
\]
Using the inclusion above, we obtain that, for every \((m, n) \in \mathbb{Z}^2\),
\[
D(U^m V^n) \subset B(0, R') + (ma_1 + na_2, mb_1 + nb_2) - (mc_1 + nc_2)\mathcal{O},
\]
and therefore
\[
U^m V^n(B(0, R)) \subset B(0, R + R') + (ma_1 + na_2, mb_1 + nb_2) - (mc_1 + nc_2)\mathcal{O}
\]
\[
= B(0, R + R') + (mc_1 + nc_2)\Phi L^{-1}([m : n : 0]) - (mc_1 + nc_2)\mathcal{O}
\]
\[
\subset B(0, R + R') + (mc_1 + nc_2) (\Phi L^{-1}(\Delta_\infty) - \mathcal{O})
\]
\[
\subset B(0, R + R') + (mc_1 + nc_2) (\mathbb{R}^2 \setminus B(0, \epsilon)).
\]
(The last inclusion comes from the definition of the neighbourhood \(\mathcal{O}\).)

On the first hand, if \(|mc_1 + nc_2|\) is larger than \(\frac{2R + K}{\epsilon}\), the last inclusion above implies that \(U^m V^n(B(0, R))\) is disjoint from \(B(0, R)\), as desired. On the other hand, since \(\mathcal{O}\) is compact, we can find \(R''\) so that \((mc_1 + nc_2)\mathcal{O} \subset B(0, R'')\) whenever \(|mc_1 + nc_2| \leq \frac{2R + K}{\epsilon}\). As a consequence, if \(|mc_1 + nc_2|\) is smaller than \(\frac{2R + K}{\epsilon}\), but \(||(ma_1 + na_2, mb_1 + nb_2)||\) is larger than \(2R + R' + R''\), then the first inclusion above implies that \(U^m V^n(B(0, R))\) is disjoint from \(B(0, R)\) as desired.

To conclude, it remains to notice that since the vectors \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) are non-colarinear (recall that \(L^{-1}\) has rank three), the map \((m, n) \mapsto (ma_1 + na_2, mb_1 + nb_2, mc_1 + nc_2)\) is a proper embedding of \(\mathbb{Z}^2\) into \(\mathbb{R}^3\). Thus the two quantities \(|mc_1 + nc_2|\) and \(||(ma_1 + na_2, mb_1 + nb_2)||\) cannot remain bounded at the same time when \(||(m, n)||\) is large. This shows that \(U^m V^n(B(0, R))\) is disjoint from \(B(0, R)\) provided that \(||(m, n)||\) is bigger than some constant. In other words, the action of \(\mathbb{Z}^2\) on \(\mathbb{R}^2\) generated by the homeomorphisms \(U\) and \(V\) is properly discontinuous. According to remark 4, this implies that \(\mathbb{R}^2 / (U, V)\) is homeomorphic to \(\mathbb{T}^2\).

\textbf{Proof of Item 2 of the theorem.} Consider a compact subset \(K\) of \(\mathbb{R}^2\) and two sequences \((m_k, n_k, p_k)_{k \geq 0}\) and \((\mu_k, \nu_k, \pi_k)_{k \geq 0}\) of elements of \(\mathbb{Z}^3\) which are related by
\[
(\mu_k, \nu_k, \pi_k) = L(m_k, n_k, p_k).
\]
Obviously, \((m_k, n_k, p_k)\) tends to infinity if and only if \((\mu_k, \nu_k, \pi_k)\) tends to infinity. Now observe that
\[
S^{-m_k} T^{-n_k} F^{p_k} = U^{-\mu_k} V^{-\nu_k} G^{\pi_k}.
\]
In particular, \((S^{−m_k} T^{−n_k} F^{\ell_k} (K)) \cap K \neq \emptyset\) if and only if \((U^{−\mu_k} V^{−\nu_k} G^{\pi_k} (K)) \cap K \neq \emptyset\). Finally, \((m_k/p_k, n_k/p_k)\) converges to \(w \in \mathbb{R}^2\) if and only if \((\mu_k/\pi_k, \nu_k/\pi_k) = \hat{L}(m_k/p_k, n_k/p_k)\) converges to the vector \(\hat{L}(w)\). This shows that \(\rho_{U,V}(G) = \hat{L}(\rho_{S,T}(F))\).

\[\square\]

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