Orthonormal systems in spaces of number theoretical functions

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Abstract. In this paper, we consider some examples of set algebras $\mathcal{A}$ on $\mathbb{N}$. If $\mathcal{E}(\mathcal{A})$ is the set of simple functions on $\mathcal{A}$, then $L^{\ast\alpha}(\mathcal{A})$ denotes the $\|\cdot\|_{\alpha}$-closure of $\mathcal{E}(\mathcal{A})$. Our aim is to determine a complete orthonormal system for the Hilbert space $L^{\ast 2}(\mathcal{A})$ in each regarded case. Here $L^{\ast 2}(\mathcal{A})$ denotes the quotient space $L^{\ast 2}(\mathcal{A})$ modulo null-functions.

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1 Introduction

For a function $f : \mathbb{N} \to \mathbb{C}$, we define $\|\cdot\|_{\alpha}$ by

$$\|f\|_{\alpha} := \left\{ \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^{\alpha} \right\}^{1/\alpha}, \quad 1 \leq \alpha < \infty.$$ 

Let $L^{\alpha} := \{ f : \mathbb{N} \to \mathbb{C} : \|f\|_{\alpha} < \infty \}$ be the linear space of functions on $\mathbb{N}$ with bounded seminorm $\|f\|_{\alpha}$. By $L^{\alpha}$ we denote the quotient space $L^{\alpha}$ modulo null-functions (i.e., functions $f$ with $\|f\|_{\alpha} = 0$). For $\alpha \geq 1$, the norm space $L^{\alpha}$ is complete [7].

Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{N}$. Then

$$\mathcal{E}(\mathcal{A}) := \left\{ s \in \mathcal{E} : s = \sum_{j=1}^{m} \alpha_j \mathbf{1}_{A_j}, \quad \alpha_j \in \mathbb{C}, \quad A_j \in \mathcal{A}, \quad j = 1, \ldots, m, \quad m \in \mathbb{N} \right\}$$

denotes the space of simple functions on $\mathcal{A}$.

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DEFINITION 1. For a given algebra \( A \) and \( 1 \leq \alpha < \infty \), the space \( L^{\star \alpha}(A) \) is defined as the \( \cdot \|_{\alpha} \)-closure of \( E(A) \). A function \( f \in L^{\star \alpha}(A) \) is called uniformly \( (A) - \alpha \) summable. By \( L^\alpha(A) \) we denote the quotient space \( L^{\star \alpha}(A) \) modulo null functions.

Remark 1. If \( A = \mathcal{P}(\mathbb{N}) \) is the algebra of all subsets of \( \mathbb{N} \), then \( L^{\star 1}(A) \) is the \( \cdot \|_{1} \)-closure of \( l^\infty \) is the space \( L^\star \) of uniformly summable functions introduced by Indlekofer [2].

Here we consider algebras \( A \) where every \( A \in \mathcal{A} \) possesses an asymptotic density \( \delta(A) \) defined by

\[
\delta(A) := \lim_{n \to \infty} \frac{1}{n} \sum_{m \in A} 1
\]

if the limit exists. Then \( \delta \) is finitely additive on \( \mathcal{A} \), that is, \( \delta \) is a content on \( \mathcal{A} \).

We say that an arithmetical function \( f \) possesses an (arithmetical) mean value \( M(f) \) if

\[
M(f) := \lim_{n \to \infty} \frac{1}{n} \sum_{m \leq n} f(m)
\]

exists. If every \( A \in \mathcal{A} \) possesses an asymptotic density, then every \( f \in L^{\star 1}(A) \) possesses a mean value. Further, we define an inner product on \( L^{\star 2}(A) \) by

\[
\langle f, g \rangle := M(f \bar{g}), \quad f, g \in L^{\star 2}(A).
\]

This product is well-defined. For this, let \( f, g \in L^{\star 2}(A) \). If \( \varepsilon > 0 \), then there exist \( s_1, s_2 \in E(A) \) such that \( \|f - s_1\|_2 < \varepsilon \) and \( \|g - s_2\|_2 < \varepsilon \). Put \( \varepsilon := \varepsilon^*/(\|f\|_2 + 2\|g\|_2) \). Then

\[
\|f \bar{g} - s_1 \bar{s}_2\|_1 \leq \|f(\bar{g} - s_2)\|_1 + \|(f - s_1)\bar{s}_2\|_1
\]

\[
\leq \|f\|_2 \|\bar{g} - s_2\|_2 + \|f - s_1\|_2 \|\bar{s}_2\|_2
\]

\[
\leq \varepsilon(\|f\|_2 + 2\|g\|_2) \leq \varepsilon^*,
\]

and \( f \bar{g} \in L^{\star 1}(A) \). Since \( L^{\star 2}(A) \) is complete, the space \( L^{\star 2}(A) \) is a Banach space. Therefore the space \( L^{\star 2}(A) \) is a Hilbert space with the inner product defined above.

In this paper, we investigate examples of Hilbert spaces \( L^{\star 2}(A) \) together with associated (complete) orthonormal systems.

Remark 2. The described construction of \( L^{\star \alpha}(A) \) was the starting point of an integration theory by Indlekofer (see [4, 5]).

Embedding \( \mathbb{N} \), endowed with the discrete topology, in the compact space \( \beta \mathbb{N} \), the Stone–Čech compactification of \( \mathbb{N} \), we get:

\[
\tilde{\mathcal{A}} := \{ \tilde{A} \colon A \in \mathcal{A} \}, \quad \text{where } \tilde{A} := \text{cl}_{\beta \mathbb{N}} A,
\]

is an algebra in \( \beta \mathbb{N} \) (for details, see [4, 5]).

Let \( \delta \) be a content on \( \mathcal{A} \), that is, \( \delta : \mathcal{A} \to \mathbb{R}_{\geq 0} \) is finitely additive, and define \( \tilde{\delta} \) on \( \tilde{\mathcal{A}} \) by

\[
\tilde{\delta}(\tilde{A}) = \delta(A), \quad \tilde{A} \in \tilde{\mathcal{A}}.
\]

Then \( \tilde{\delta} \) is a pseudo-measure on \( \tilde{\mathcal{A}} \) and can be extended to a measure on \( \sigma(\tilde{\mathcal{A}}) \), which we also denote by \( \tilde{\delta} \). This leads to the measure space \((\beta \mathbb{N}, \sigma(\tilde{\mathcal{A}}), \tilde{\delta})\).
2 Some Hilbert spaces and corresponding orthonormal systems

2.1 A simple case

Let $\mathcal{A}_0$ be the algebra generated by the sets $A_p := \{n \in \mathbb{N} : p \mid n\}$, $p$ prime, and put

$$\delta(A_p) := M(1_{A_p}) = \lim_{n \to \infty} \frac{1}{n} \sum_{m \leq n, p \mid m} 1 = \frac{1}{p}.$$ 

Note that the following relations of the characteristic functions

$$1_{A \cap B} = 1_A \cdot 1_B, \quad 1_{A \setminus B} = 1_A - 1_A \cdot 1_B, \quad 1_{A \cup B} = 1_A + 1_B - 1_A \cdot 1_B$$

imply that the characteristic function of a set $A \in \mathcal{A}$ is a finite linear combination of products $1_{A_{p_1}} \cdots 1_{A_{p_r}}$.

Thus the asymptotic density $\delta(A)$ exists for all $A \in \mathcal{A}_0$.

For every prime $p$, put

$$h_p := p1_{A_p} - 1$$

and define $h_n : \mathbb{N} \to \mathbb{Z}$ by $h_n = 1$ for $n = 1$ and

$$h_n := \prod_{p \mid n} h_p \quad \text{for every square-free } n \in \mathbb{N}.$$ 

Obviously, for every prime $p$,

$$M(h_p) = 0, \quad M(h_p^2) = p - 1.$$ 

Now, if $f : \mathbb{N} \to \mathbb{C}$ is such that $M(f)$ exists and $f(pm) = f(m)$ for all $m \in \mathbb{N}$, then we conclude that

$$\sum_{m \leq x} h_p(m)f(m) = p \sum_{pm \leq x} f(pm) - \sum_{m \leq x} f(m) = p \sum_{m \leq x/p} f(m) - \sum_{m \leq x} f(m)$$

and $M(h_p f) = 0$, that is,

$$M(h_n) = 0 \quad \text{if } \mu^2(n) = 1, \quad n > 1,$$

and

$$M(h_n h_{n'}) = 0 \quad \text{if } \mu^2(n) = \mu^2(n') = 1 \text{ and } n \neq n'.$$

In the same way, we obtain

$$M(h_n^2 f) = (p - 1)M(f).$$

By induction this leads to

$$M(h_n^2) = \varphi(n) \quad \text{if } \mu^2(n) = 1.$$ 

Putting $h_n^* := (\varphi(n))^{-1/2}h_n$ ($\mu^2(n) = 1$), we have shown the following:

**Theorem 1.** The set $\{h_n^* : n \text{ square-free}\}$ is a complete orthonormal system for $L^2(\mathcal{A}_0)$.

**Remark 3.** We easily see that the function $h_n : \mathbb{N} \to \mathbb{Z}$ satisfies $h_{n_1 n_2} = h_{n_1} \cdot h_{n_2}$ if $(n_1, n_2) = 1$ and $\mu(n_1) = \mu(n_2) = 1$, that is, $n = 1$, or $n$ is a product of an even number of different primes.

**Remark 4.** Every $f \in \mathcal{E}(\mathcal{A}_0)$ can be written as a linear combination of multiplicative $g_j$ such that $g_j(p^l) = 1$ for all $p \geq k_j$ and $l \in \mathbb{N}$, since $g = 1 - 1_{A_p}$ is multiplicative.
2.2 Almost even functions

For primes $p$ and $k = 0, 1, 2, \ldots$, let $A_{p^k} := \{n \in \mathbb{N} \mid p^k \mid n\}$ be the set of natural numbers divisible by $p^k$. Let $A_1$ be the algebra generated by the sets $\{A_{p^k}\}$. Then, for all $A_{p^k}$, the asymptotic density $\delta(A_{p^k})$ exists and equals $1/p^k$; and, as before, the asymptotic density $\delta(A)$ exists for all $A \in A_1$.

Schwarz and Spilker [8, Chap. VI] considered the space $\mathcal{B}$ of even functions and characterized the sets of $\alpha$-almost even functions (see also [3]).

It is well known that $\mathcal{E}(A_1)$ equals $\mathcal{B}$ and $L^\alpha(A_1)$ is exactly the space of $\alpha$-almost even functions (see [5]).

Remark 5. Every $f \in \mathcal{E}(A_1)$ can be written as a linear combination of multiplicative functions $g_j$ such that $g_j(p^l) = 1$ for all $p \geq k_j$ and $l \in \mathbb{N}$.

Remark 6. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $|f| \leq 1$. Then the following statements hold.

(i) If $M(f) \neq 0$, then $f \in L^{\alpha^*}(A_1)$ for every $\alpha \geq 1$.
(ii) $M(|f|) = 0$ if and only if $\sum_{p \text{ prime}} (1 - |f(p)|)/p = \infty$; especially, if $\sum_{p \text{ prime}, f(p) = 0} 1/p = \infty$, then $M(|f|) = 0$.
(iii) $M(|f|) = 0$ if and only if $M(|f|^2) = 0$.

Put

$$h_p := p1_{A_p} - 1 \quad \text{for prime } p$$

and

$$h_{p^k} := p^k 1_{A_{p^k}} - p^{k-1} 1_{A_{p^k-1}} \quad \text{for } k > 2.$$ 

Define $h_n = 1$ for $n = 1$ and

$$h_n := \prod_{p^k \mid n} h_{p^k} \quad \text{for } n > 1.$$ 

Putting

$$h_n^* = \frac{1}{\varphi(n)^{1/2}} h_n,$$ 

(2.1)

where $\varphi$ is Euler’s function, it is easy to show (see above) that $\{h_n^*\}$ is an orthonormal system. We conclude by the following:

Theorem 2. The set $\{h_n^* : n \in \mathbb{N}\}$ is a complete orthonormal system for $L^{*2}(A_1)$.

Remark 7. The functions $h_n$ appear in a very natural way. It is not difficult to show (see [8, pp. 16–17]) that $h_n$ is just the Ramanujan sum $c_n$ for every $n$.

2.3 Limit periodic functions

Let $A_2$ be the algebra generated by all residue classes $A_{a,r} := \{n \in \mathbb{N} : n \equiv a \mod r\}$, $1 \leq a \leq r$, $r \in \mathbb{N}$.

Here again the asymptotic density $\delta$ is a finite additive function on $A_2$. Then we have the following lemma.

Lemma 1. $\mathcal{E}(A_2)$ is the space of all periodic functions on $\mathbb{N}$.

The space $L^{*\alpha}(A_2)$ is the space of $\alpha$-limit-periodic functions.

Defining $e_{a/r} : \mathbb{N} \to \mathbb{C}$ by

$$e_{a/r}(n) := \exp\left(2\pi i \frac{a}{r} n\right),$$

we have the following result (see [8, p. 207]).

Theorem 3. The set $\{e_{a/r} : 1 \leq a \leq r, \gcd(a,r) = 1, r = 1,2,\ldots\}$ is a complete orthonormal system in $L^{*2}(A_2)$.
The asymptotic density \( \delta \)

Let \( L \in \mathbb{R} \), the function \( e_{\beta} : \mathbb{N} \rightarrow \mathbb{C} \) defined by

\[
e_{\beta}(n) = \exp(2\pi i \beta n), \quad n \in \mathbb{N},
\]

possesses a mean value \( M(e_{\beta}) \).

Let \( C \) be the family of all half-open subsets of \([0, 1]\) and denote by \( A_3 \) the algebra generated by the sets

\[
A(\beta, E) := \{ n \in \mathbb{N} : \{ \beta n \} \in E \}, \text{ where } \beta \in [0, 1), \ E \in C, \text{ and } \beta n = [\beta n] + \{ \beta n \} \ (0 \leq \beta n < 1).
\]

Then (see [8, p. 207]) we have the following:

**Theorem 4.** The set \( \{ e_{\beta} : \beta \in [0, 1] \} \) is a complete orthonormal system in \( L^2(A_3) \).

### 2.4 Almost periodic functions

For \( \beta \in \mathbb{R} \), the function \( e_{\beta} : \mathbb{N} \rightarrow \mathbb{C} \) defined by

\[
e_{\beta}(n) = \exp(2\pi i \beta n), \quad n \in \mathbb{N},
\]

we have the following:

**Theorem 5.** Let \( f \) with characteristic functions \( A(f) \) implies that \( A \) is multiplicative. Since \( \| \cdot \| \) is a complete orthonormal system in \( \mathbb{R} \), we have \( \{ e_{\beta} \} \) exists. Let \( E(A_4) \) be the vector space of simple functions on \( A_4 \). Let \( L^{*\alpha}(A_4) \) be the \( \| \cdot \|_{\alpha} \)-closure of \( E(A_4) \).

**Definition 2.** A function \( f \in L^{*\alpha}(A_4) \) is called an \( \alpha \)-almost multiplicative function.

First, we show \( A_1 \subset A_4 \). For the proof, consider

\[
f^{*}(n) := (1 - 1_{A_{p}})(n) = \begin{cases} 0, & p^k \mid n, \\ 1 & \text{otherwise.} \end{cases}
\]

Then \( f^{*} \) is multiplicative. Since \( 1_{\mathbb{N} \setminus A_{p}} = 1_{\mathbb{N}} - 1_{A_{p}} = 1 - 1_{A_{p}} \in E(A_4) \), we have \( \mathbb{N} \setminus A_{p} \in A_4 \). This implies that \( A_{p} \in A_4 \).

Since \( h_n \in E(A_1) \), we have \( h_n \in E(A_4) \). Every \( h_n \) can be written as a finite linear combination of \( 1_{A_{p}} \), \( \cdots 1_{A_{m}} \), where \( m \in \mathbb{N} \).

**Theorem 5.** Let \( f : \mathbb{N} \rightarrow \mathbb{R} \) be multiplicative with \( |f| \leq 1 \). Then \( f \in L^{*\alpha}(A_4) \) for all \( \alpha \geq 1 \).

**Proof.** Put \( f = |f| \text{sign}_f \), where \( \text{sign}_f \) is multiplicative with

\[
(\text{sign}_f)(p^k) = \begin{cases} 1 & \text{if } f(p^k) > 0, \\ 0 & \text{if } f(p^k) = 0, \\ -1 & \text{if } f(p^k) < 0. \end{cases}
\]
Since \(|f| \in \mathcal{L}^\alpha(\mathcal{A}_4)\) and sign \(f \in \mathcal{E}(\mathcal{A}_4)\), we find \(s_1, s_8 \in \mathcal{E}(\mathcal{A}_4)\) such that \(||f| - s_1||_\alpha^\alpha \leq \varepsilon^\alpha\) and \(||\text{sign } f - s_2||_\alpha^\alpha \leq \varepsilon^\alpha\). Note that there exist \(c_1(\alpha), c_2(\alpha) > 0\) such that

\[
||s_2||_\alpha^\alpha \leq \|1 + (s_2 - |f|)||_\alpha^\alpha \leq c_1(\alpha)
\]

and

\[
(|a| + |b|)^\alpha \leq c_2(\alpha)(|a|^\alpha + |b|^\alpha).
\]

Put \(\varepsilon^\alpha := \varepsilon^\alpha/(1 + c_1(\alpha)c_2(\alpha))\). Then

\[
||f - s_1s_2||_\alpha^\alpha = c_2(\alpha)\left\{||f||_{(\text{sign } f - s_2)||_\alpha^\alpha + ||(f - s_1)s_2||_\alpha^\alpha}\right\}
\]

\[
\leq c_2(\alpha)\left\{||\text{sign } f - s_2||_\alpha^\alpha + ||(f - s_1)||_\alpha^\alpha\right\}||s_2||_\alpha^\alpha \leq c_2(\alpha)\varepsilon^\alpha + \varepsilon^\alpha c_2(\alpha)c_1(\alpha) < \varepsilon^\alpha.
\]

This proves Theorem 5. \(\square\)

Next, we construct an orthonormal system for the space \(L^{*2}(\mathcal{A}_4)\).

Let \(\mathcal{R}_0\) be the set of all multiplicative functions with \(f(\mathbb{N}) \subset \{-1, 0, 1\}\) and \(M(|f|) \neq 0\). Define the relation \(\sim\) on \(\mathcal{R}_0\) by

\[
f \sim g \text{ if and only if } \sum_{p \mid f(p) \neq g(p)} \frac{1}{p} < \infty.
\]

Observe, that in this case, by (ii) of Remark 6, \(\sum_{p \mid f(p) = 1/p < \infty}. \) Obviously, \(\sim\) is an equivalence relation on \(\mathcal{R}_0\).

Now choose a representative from each residue class that takes only the values \(\pm 1\) and denote this set by \(\mathcal{F}_1\).

Then \(\mathcal{F}_1\) forms an orthonormal system. For this, let \(f, g \in \mathcal{F}_1\) and observe that \(\sum_{p \mid f(p) \neq g(p)} 1/p = \infty\). Then, by (ii) of Remark 6, \(M(fg) = \langle f, g \rangle = 0\). Furthermore, for \(f \in \mathcal{F}_1\), we have \(f^2 = 1\) and \(\langle f, f \rangle = M(f^2) = 1\).

This shows that \(\mathcal{F}_1\) is an orthonormal system in \(L^{*2}(\mathcal{A}_4)\). Consider, for \(f \in \mathcal{F}_1\), the system

\[
\mathcal{F}_2 := \{fh^*_n: f \in \mathcal{F}_1, n \in \mathbb{N}\},
\]

where \(h^*_n\) is the normalized function (2.1).

**Theorem 6.** \(\mathcal{F}_2\) is a complete orthonormal system for \(L^{*2}(\mathcal{A}_4)\).

**Proof.** First, we show that \(\mathcal{F}_2\) is an orthonormal system. For this, let \(h^*_nf \neq h^*_ng\). This holds if and only if \(f \neq g\) and \(n, \bar{n}\) are arbitrary or \(f = g\) and \(n \neq \bar{n}\). Assume that \(f \neq g\) and \(n, \bar{n}\) are arbitrary. Then

\[
\langle h^*_nf, h^*_ng \rangle = M(h^*_nfh^*_n) = M(fgh^*),
\]

where \(h^*\) is (see Remark 6) a finite linear combination of multiplicative functions \(g_j\) with \(|g_j| = 1\) and \(g_j(p) = 1\) for \(p > k_j\). Therefore

\[
\sum_{p \mid f(p) \neq g(p)} \frac{1 - f(p)g(p)g_j(p)}{p} = \infty \quad \text{and} \quad M(fg_{k_j}) = 0.
\]

So we obtain \(\langle h^*_nf, h^*_ng \rangle = 0\) if \(f \neq g\). In the case \(f = g\) and \(n \neq \bar{n}\), obviously, \(\langle h^*_nf, h^*_nf \rangle = M(h^*_nh^*_n) = 0\).

Since \(\langle h^*_nf, h^*_nf \rangle = M(|h^*_nf|^2) = 1\), \(\mathcal{F}_2\) is an orthonormal system.
For the proof of the completeness of $\mathcal{F}_2$, let $g \in \mathcal{L}^*(A_4)$. Then $g$ can be approximated in the $\|\cdot\|_2$ norm by some $g^* \in \mathcal{E}(A_4)$,

$$g^* = \sum_{j=1}^{m} \alpha_j 1_{A_j}, \quad \alpha_j \in \mathbb{C}, \ A_j \in \mathcal{A}_4.$$ 

Note that $1_A$ for $A \in \mathcal{A}_4$ is a finite linear combination of products of multiplicative functions $f$ taking only the values $\{-1, 0, 1\}$.

Therefore it suffices to prove that each real-valued multiplicative function $f$ with values $f(\mathbb{N}) \subset \{-1, 0, 1\}$ can be approximated by a linear combination of functions from $\mathcal{F}_2$. Choose $g \in \mathcal{F}_1$ that is equivalent to $f$. Then $f = hg$ where $h = fg$, since $g^2 = 1$. Then

$$\sum_p \frac{1 - h(p)}{p} = \sum_p \frac{2}{p} < \infty$$

and $h \in \mathcal{L}^2(\mathcal{A}_1)$. Thus $h$ can be approximated by a linear combination of functions $h_1, \ldots, h_m$, that is, for $\varepsilon > 0$, there exist $\alpha_j \in \mathbb{C}$ such that

$$\left\| h - \sum_{j=1}^{m} \alpha_j h_j \right\|_2 < \varepsilon, \quad \text{which implies} \quad \left\| f - \sum_{j=1}^{m} \alpha_j h_j \right\|_2 < \varepsilon.$$

This ends the proof of the completeness of $\mathcal{F}_2$. \qed

2.6 $q$-ary almost even functions

First, we introduce $q$-multiplicative functions. Let $q \geq 2$ be an integer, and let $\mathbb{A} = \{0, 1, \ldots, q-1\}$. The $q$-ary expansion of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \ldots$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n)q^j, \quad \varepsilon_j(n) \in \mathbb{A}.$$ 

The numbers $\varepsilon_0(n), \varepsilon_1(n), \ldots$ are called the digits in the $q$-ary expansion of $n$. In fact, $\varepsilon_r(n) = 0$ if $r > \log n / \log q$. A function $f : \mathbb{N}_0 \to \mathbb{C}$ is called $q$-multiplicative if $f(0) = 1$ and for every $n \in \mathbb{N}_0$,

$$f(n) = \prod_{j=0}^{\infty} f(\varepsilon_j(n)q^j).$$

Let the algebra $\mathcal{A}_5$ be generated by the sets $A_{j,a} := \{n \in \mathbb{N} : \varepsilon_j(n) = a\}, j \in \mathbb{N}_0, a \in \mathbb{A}$. Every $A \in \mathcal{A}_5$ possesses an asymptotic density $\delta(A)$.

Let $\mathcal{L}^*(\mathcal{A}_5)$ be the $\|\cdot\|_1$-closure of $\mathcal{E}(\mathcal{A}_5)$. Here $\mathcal{E}(\mathcal{A}_5)$ is called the space of $q$-ary even functions. Then $\mathcal{L}^*(\mathcal{A}_5)$ is called the space of $q$-ary almost even functions.

Remark 8. Let $f$ be a real-valued $q$-multiplicative function of modulus $\leq 1$. Then the mean values $M(\|f\|)$ and $M(f)$ always exist (see [6]). Especially, we have:

(i) If $\|f\|_1 = M(\|f\|) > 0$, then

$$\sum_{j=0}^{\infty} \sum_{a \in \mathbb{A}} (1 - |f(aq^j)|) < \infty.$$
(ii) If
\[ \sum_{a \in A} f(aq^j) \neq 0 \quad \text{for all } j \in \mathbb{N}_0 \quad \text{and} \quad \sum_{j=0}^{\infty} \sum_{a \in A} (1 - f(aq^j)) < \infty, \]
then \( M(f) \neq 0. \)

As an immediate consequence, we have the following:

**Corollary 1.** Let \( f \) be a real-valued \( q \)-multiplicative function of modulus \( \leq 1. \) If
\[ \sum_{j=0}^{\infty} \sum_{a \in A} (1 - f(aq^j)) < \infty, \]
then \( f \in L^*_{1}(A_5). \)

This ends Remark 8.

Let \( L^*_2(A_5) \) be the \( \| \cdot \|_2 \)-closure of \( E(A_5). \) Then we define a complete orthonormal system for the space \( L^*_{2}(A_5). \)

**Theorem 7.** The set \( \{h_{a_0, \ldots, a_r}\} \) of \( q \)-multiplicative functions with
\[ h_{a_0, \ldots, a_r}(n) := \prod_{j=0}^{r} \exp \left( \frac{2\pi i a_j}{q} \varepsilon_j(n) \right), \]
a\( j \in \mathbb{A}, j = 0, \ldots, r, r \in \mathbb{N}_0, \) is a complete orthonormal system for \( L^*_2(A_5). \)

The proof is easy and is left to the reader.

### 2.7 Almost \( q \)-multiplicative functions

Let \( f \) be a \( q \)-multiplicative function taking only the values \( \{-1, 0, 1\} \) and define the sets
\[ A_f^+ := \{n: f(n) = 1\}, \quad A_f^0 := \{n: f(n) = 0\}, \quad \text{and} \quad A_f^- := \{n: f(n) = -1\} \]
with characteristic functions \( f^+ \), \( f^0 \), and \( f^- \), respectively. We denote by \( A_6 \) the algebra generated by the sets \( A_f^+, A_f^0, A_f^- \) for all \( q \)-multiplicative \( f \) with \( f(\mathbb{N}) \subset \{-1, 0, 1\} \).

An arbitrary element \( A \) of \( A_6 \) has a characteristic function that is a linear combination of \( q \)-multiplicative functions. From this and by the theorem of Delange [1] the asymptotic density \( \delta(A) \) exists. Let \( E(A_6) \) be the space of simple functions on \( A_6. \) Let \( L^*(A_6) \) be the \( \| \cdot \|_1 \)-closure of \( E(A_6). \)

**Definition 3.** Functions \( f \in L^*(A_6) \) are called **almost \( q \)-ary multiplicative** functions.

Next, we define a complete orthonormal system for \( L^*_{2}(A_6). \)

Let \( \mathcal{G} := \{f : \mathbb{N} \to \mathbb{R}: f \text{ \( q \)- multiplicative, } f(n) \in \{-1, 0, 1\} \text{ for all } n \in \mathbb{N} \text{ with } \|f\|_2 \neq 0 \}. \) Define the relation \( \sim \) on \( \mathcal{G} \) by
\[ f \sim g \quad \text{if and only if} \quad \sum_{l=0}^{q-1} \sum_{a=0}^{g-1} (1 - f(aq^l)g(aq^l)) < \infty. \]

Obviously, \( \sim \) is an equivalence relation on \( \mathcal{G}. \)
Now from each equivalence class we choose a representative that is \( \neq 0 \) for all \( n \in \mathbb{N} \). We denote this set of representatives by \( F_3 \). We consider \( F_4 := \{ h_{a_0,\ldots,a_r} f : f \in F_3, a_j \in A, j = 0,\ldots,r, r \in \mathbb{N} \} \) and show the following:

**Theorem 8.** \( F_4 \) is a complete orthonormal system for \( L^2(\mathcal{A}_6) \).

**Proof.** First, we show that \( F_4 \) is an orthonormal system. In the case \( g_1 = g_2 = g \), since \( g^2 = 1 \), we have

\[
M(h_{a_0,\ldots,a_r} g \bar{h}_{b_0,\ldots,b_r} g) = M(h_{a_0,\ldots,a_r} \bar{h}_{b_0,\ldots,b_r}) = 0
\]

if \( h_{a_0,\ldots,a_r} \neq \bar{h}_{b_0,\ldots,b_r} \) and 1 otherwise. If \( g_1 \neq g_2 \), then

\[
(h_{a_0,\ldots,a_r} g_1 \bar{h}_{b_1,\ldots,b_r} g_2)(aq^j) = (g_1 g_2)(aq^j)
\]

if \( j \) is large enough. Obviously, \( M(g_1 g_2) = 0 \), and \( F_4 \) is an orthonormal system.

To prove the completeness of \( F_4 \), it suffices to show that every \( q \)-multiplicative \( f \) with \( M(|f|) \neq 0 \) and \( f(n) \in \{-1,0,1\} \) for all \( n \in \mathbb{N}_0 \) can be approximated by linear combinations of elements of \( F_4 \).

Let \( f \) be such a function. Then

\[
f = |f| \text{sign}_f \quad \text{and} \quad \text{sign}_f \sim g, \quad g \in F_3,
\]

and

\[
f = |f| \text{sign}_f g^2 = (|f| \text{sign}_f g) g.
\]

Now \( |f| \text{sign}_f g \) is a \( q \)-ary even function and can therefore be approximated by a linear combination of some \( h_{a_0,\ldots,a_r} \). This proves Theorem 8. \( \square \)

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