UPPER BOUNDS FOR DIMENSIONS OF SINGULARITY CATEGORIES

HAILONG DAO AND RYO TAKAHASHI

Abstract. This paper gives upper bounds for the dimension of the singularity category of a Cohen-Macaulay local ring with an isolated singularity. One of them recovers an upper bound given by Ballard, Favero and Katzarkov in the case of a hypersurface.

1. Results

The notion of the dimension of a triangulated category has been introduced by Bondal, Rouquier and Van den Bergh [3, 12], which measures the number of extensions necessary to build the category out of a single object. The singularity category \( \mathcal{D}_{\text{sg}}(R) \) of a noetherian ring/scheme \( R \) is one of the most crucial triangulated categories. This has been introduced by Buchweitz [4] by the name of stable derived category. There are many studies on singularity categories by Orlov [8, 9, 10, 11] in connection with the Homological Mirror Symmetry Conjecture.

It is a natural and fundamental problem to find upper bounds for the dimension of the singularity category of a noetherian ring. In general, the dimension of the singularity category is known to be finite for large classes of excellent rings containing fields [1, 12], but only a few explicit upper bounds have been found so far. The Loewy length is an upper bound for an artinian ring [12], and so is the global dimension for a ring of finite global dimension [6, 7]. Recently, an upper bound for an isolated hypersurface singularity has been given [2].

The main purpose of this paper is to give upper bounds for a Cohen-Macaulay local ring with an isolated singularity. The main result of this paper is the following theorem.

**Theorem 1.1.** Let \((R, \mathfrak{m}, k)\) be a complete equicharacteristic Cohen-Macaulay local ring with \( k \) perfect. Suppose that \( R \) is an isolated singularity. Then the sum \( \mathfrak{N}^R \) of the Noether differentials of \( R \) is \( \mathfrak{m} \)-primary. Let \( I \) be an \( \mathfrak{m} \)-primary ideal of \( R \) contained in \( \mathfrak{N}^R \).

1. One has \( \mathcal{D}_{\text{sg}}(R) = \langle k \rangle_{(\nu(I) - \dim R + 1) \ell(R/I)} \). Hence there is an inequality \( \dim \mathcal{D}_{\text{sg}}(R) \leq (\nu(I) - \dim R + 1) \ell(R/I) - 1 \).

2. Assume that \( k \) is infinite. Then \( \mathcal{D}_{\text{sg}}(R) = \langle k \rangle_{e(I)} \), and hence one has \( \dim \mathcal{D}_{\text{sg}}(R) \leq e(I) - 1 \).
Here we explain the notation used in the above theorem. Let \((R, \mathfrak{m}, k)\) be a commutative noetherian complete equicharacteristic local ring. Let \(A\) be a Noether normalization of \(R\), that is, a formal power series subring \(k[[x_1, \ldots, x_d]]\), where \(x_1, \ldots, x_d\) is a system of parameters of \(R\). Let \(R^e = R \otimes_A R\) be the enveloping algebra of \(R\) over \(A\). Define a map \(\mu : R^e \to R\) by \(\mu(a \otimes b) = ab\) for \(a, b \in R\). Then the ideal \(\mathfrak{N}_A^d = \mu(\text{Ann}_{R^e} \text{Ker} \mu)\) of \(R\) is called the Noether different of \(R\) over \(A\). We denote by \(\mathfrak{N}_R^d\) the sum of \(\mathfrak{N}_A^d\), where \(A\) runs through the Noether normalizations of \(R\). For an \(\mathfrak{m}\)-primary ideal \(I\) of \(R\), let \(\nu(I) = \dim_k(I \otimes_R k)\) be the minimal number of generators of \(I\) and \(e(I) = \lim_n \frac{d}{n^2} \ell(R/I^{n+1})\) the multiplicity of \(I\). The Loewy length of an artinian ring \(\Lambda\) is denoted by \(\ell(\Lambda)\), that is, the minimum positive integer \(n\) with \((\text{rad} \Lambda)^n = 0\).

Our Theorem 1.1 yields the following result.

**Corollary 1.2.** Let \(k\) be a perfect field, and let \(R = k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m)\) be a Cohen-Macaulay ring having an isolated singularity. Let \(J\) be the Jacobian ideal of \(R\), namely, the ideal generated by the \(h\)-minors of the Jacobian matrix \((\frac{\partial f_j}{\partial x_i})\), where \(h = \text{ht}(f_1, \ldots, f_m)\).

1. One has \(D_{\text{sg}}(R) = \langle k \rangle^\nu(J) - \dim R + 1 \ell(R/J)\). Hence there is an inequality \(\dim D_{\text{sg}}(R) \leq (\nu(J) - \dim R + 1) \ell(R/J) - 1\).
2. If \(k\) is infinite, then \(D_{\text{sg}}(R) = \langle k \rangle^e(J)\), and it holds that \(\dim D_{\text{sg}}(R) \leq e(J) - 1\).

Corollary 1.2 immediately recovers the following result, which is stated in [2].

**Corollary 1.3** (Ballard-Favero-Katzarkov). Let \(k\) be an algebraically closed field of characteristic zero. Let \(R = k[[x_1, \ldots, x_n]]/(f)\) be an isolated hypersurface singularity. Then \(D_{\text{sg}}(R) = \langle k \rangle^2 \ell(R/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}))\), and hence \(\dim D_{\text{sg}}(R) \leq 2 \ell(R/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})) - 1\).

As another application of Theorem 1.1, we obtain upper bounds for the dimension of the stable category \(\mathcal{CM}(R)\) of maximal Cohen-Macaulay modules over an excellent Gorenstein ring \(R\):

**Corollary 1.4.** Let \(R\) be an excellent Gorenstein equicharacteristic local ring with perfect residue field \(k\), and assume that \(R\) is an isolated singularity. Then \(\mathfrak{N}_R^d\) is a \(\hat{\mathfrak{m}}\)-primary ideal of the completion \(\hat{R}\) of \(R\). Let \(I\) be an \(\hat{\mathfrak{m}}\)-primary ideal contained in \(\mathfrak{N}_R^d\). Put \(d = \dim R\), \(n = \nu(I)\), \(l = \ell(\hat{R}/I)\) and \(e = e(I)\).

1. One has \(\mathcal{CM}(R) = \langle \Omega^d k \rangle_{(n-d+1)l}\), and \(\dim \mathcal{CM}(R) \leq (n - d + 1)l - 1\).
2. If \(k\) is infinite, then \(\mathcal{CM}(R) = \langle \Omega^d k \rangle_e\), and one has \(\dim \mathcal{CM}(R) \leq e - 1\).

## 2. Proofs

This section is devoted to proving our results stated in the previous section. For the definition of the dimension of a triangulated category and related notation, we refer the reader to [12, Definition 3.2]. We denote by \(D(\mathcal{A})\) the derived category of an abelian category \(\mathcal{A}\). Let \(H^iX\) (respectively, \(Z^iX, B^iX\)) denote the \(i\)-th homology (respectively, cycle, boundary) of a complex \(X\) of objects of \(\mathcal{A}\), and set \(H^*X = \bigoplus_{i \in \mathbb{Z}} H^iX\).

**Lemma 2.1.** Let \(\mathcal{A}\) be an abelian category and \(X\) a complex of objects of \(\mathcal{A}\).
(1) Let $n$ be an integer. If $H^i X = 0$ for all $i > n$, then there exists an exact triangle

$$Y \to X \to H^n X[-n] \to$$

in $D(A)$ such that $H^i Y \cong \begin{cases} 0 & (i \geq n) \\ H^i X & (i < n). \end{cases}$

(2) Let $n \geq m$ be integers. If $H^i X = 0$ for all $i > n$ and $i < m$, then $X \in \langle HX \rangle_{n-m+1}^{D(A)}$.

**Proof.** (1) Truncating $X = (\cdots \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots)$, we get complexes

$$X' = (\cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} Z^n X \to 0),$$

$$Y = (\cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} B^n X \to 0).$$

There are natural morphisms $Y \xrightarrow{f} X' \xrightarrow{g} X$, where $f$ is a monomorphism and $g$ is a quasi-isomorphism. We have a short exact sequence $0 \to Y \xrightarrow{f} X' \to H^n X[-n] \to 0$ of complexes, which induces an exact triangle as in the assertion.

(2) Applying (1) repeatedly, for each $0 \leq j \leq n - m$ we obtain an exact triangle

$$X_{j+1} \to X_j \to H^{n-j} X[-(n-j)] \to$$

in $D(A)$ with $X_0 = X$ such that $H^i X_j \cong 0$ for $i > n - j$ and $H^i X \cong H^i X$ for $i \leq n - j$. Hence $X_{n-m+1} \cong 0$ in $D(A)$, which implies that $X_{n-m}$ is in $\langle H^m X \rangle$. Inductively, we observe that $X = X_0$ belongs to $\langle H^m X \oplus H^{m+1} X \oplus \cdots \oplus H^n X \rangle_{n-m+1} = \langle HX \rangle_{n-m+1}$. ■

For a commutative noetherian ring $R$, we denote by $\text{mod } R$ the category of finitely generated $R$-modules, and by $D^b(\text{mod } R)$ the bounded derived category of $\text{mod } R$. For a sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of $R$ and an $R$-module $M$, let $K(\mathbf{x}, M)$ denote the Koszul complex of $\mathbf{x}$ on $M$.

**Proposition 2.2.** Let $(R, \mathfrak{m})$ be a commutative noetherian local ring and $I$ an $\mathfrak{m}$-primary ideal of $R$. Let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of $R$ that generates $I$. Then for any finitely generated $R$-module $M$ one has $K(\mathbf{x}, M) \in \langle k \rangle_{(n-t+1)l}$ in $D^b(\text{mod } R)$, where $t = \text{depth } M$ and $l = \ell \ell(R/I)$.

**Proof.** Set $K(\mathbf{x}, M) = K = (0 \to K^{-n} \to \cdots \to K^0 \to 0)$. By [5, Proposition 1.6.5(b)], each homology $H^i = H^i K$ is annihilated by $I$, and $H^i$ is regarded as a module over $R/I$. There is a filtration $0 = \mathfrak{m}^t(R/I) \subsetneq \cdots \subsetneq \mathfrak{m}(R/I) \subsetneq R/I$ of ideals of $R/I$. For each integer $i$ we have a filtration

$$0 = \mathfrak{m}^t H^i \subsetneq \cdots \subsetneq \mathfrak{m} H^i \subsetneq H^i$$

of submodules of $H^i$, which shows $H^i \in \langle k \rangle_l$ in $D^b(\text{mod } R)$. We see from [7, Theorem 1.6.17(b)] that $H^i = 0$ for all $i < t$ and $i > 0$. It follows from Lemma [2.11.2] that $K$ is in $\langle \bigoplus_{i=t-n} H^i \rangle_{n-t+1}$ in $D^b(\text{mod } R)$, which is contained in $\langle k \rangle_{(n-t+1)l}$. ■

Recall that the singularity category $D_{sg}(R)$ of a (commutative) noetherian ring $R$ is defined as the Verdier quotient of $D^b(\text{mod } R)$ by the full subcategory of perfect complexes. (A perfect complex is by definition a bounded complex of finitely generated modules.)
Proposition 2.3. Let $R$ be a commutative noetherian ring, and let $M$ be a finitely generated $R$-module. Let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of $R$ such that the multiplication map $M \xrightarrow{x_i} M$ is a zero morphism in $\mathcal{D}_{sg}(R)$ for every $1 \leq i \leq n$. Then $M$ is isomorphic to a direct summand of $K(\mathbf{x}, M)$ in $\mathcal{D}_{sg}(R)$.

Proof. By definition the Koszul complex $K(\mathbf{x}, M) = (0 \to M \xrightarrow{x_i} M \to 0)$ is the mapping cone of the multiplication map $M \xrightarrow{x_i} M$, and there is an exact triangle $M \xrightarrow{x_i} M \to K(\mathbf{x}, M) \to \cdots$ in $\mathcal{D}_{sg}(R)$. By assumption, we have an isomorphism $M \oplus M[1] \cong K(\mathbf{x}, M) = K(x_i, R) \otimes_R M$ in $\mathcal{D}_{sg}(R)$. We observe that

$$K(\mathbf{x}, M) = K(x_1, R) \otimes_R \cdots \otimes_R K(x_{n-1}, R) \otimes_R (K(x_n, R) \otimes_R M)$$

$$\implies K(x_1, R) \otimes_R \cdots \otimes_R K(x_{n-2}, R) \otimes_R (K(x_{n-1}, R) \otimes_R M)$$

$$\cdots$$

$$\implies K(x_1, R) \otimes_R M \simeq M,$$

where $A \simeq B$ means that $A$ has a direct summand isomorphic to $B$ in $\mathcal{D}_{sg}(R)$. ■

Lemma 2.4. (1) Let $\mathcal{A}$ be an abelian category. Let $P = (\cdots \xrightarrow{d_b-1} P_b \xrightarrow{d_b} \cdots \xrightarrow{d_1} P^a \to 0)$ be a complex of projective objects of $\mathcal{A}$ with $\mathcal{H}^i P = 0$ for all $i < b$. Then one has an exact triangle

$$F \to P \to C[-b] \to \cdots$$

in $\mathcal{D}(\mathcal{A})$, where $F = (0 \to P^{b+1} \xrightarrow{d^{b+1}} \cdots \xrightarrow{d^1} P^a \to 0)$ and $C = \text{Coker } d^{b-1}$.

(2) Let $R$ be a commutative noetherian ring.

(a) For any $X \in \mathcal{D}_{sg}(R)$ there exist $M \in \text{mod } R$ and $n \in \mathbb{Z}$ such that $X \cong M[n]$ in $\mathcal{D}_{sg}(R)$.

(b) Let $M$ be a finitely generated $R$-module. Then for an integer $n \geq 0$ there exists an exact triangle

$$F \to M \to \Omega^n M[n] \to \cdots$$

in $\mathcal{D}^b(\text{mod } R)$, where $F = (0 \to F^{-(n-1)} \to \cdots \to F^0 \to 0)$ is a perfect complex.

Proof. (1) There is a short exact sequence $0 \to F \to P \to Q \to 0$ of complexes, where $Q = (\cdots \xrightarrow{d^2} P^{b-1} \xrightarrow{d^{b-1}} P^b \to 0)$. Then $Q \cong C[-b]$ in $\mathcal{D}(\mathcal{A})$.

(2) The assertion (a) is immediate from (1). Setting $a = 0 \geq -n = b$ and letting $P$ be a projective resolution of $M$ in (1) implies (b). ■

Recall that a commutative noetherian ring $R$ is called an isolated singularity if the local ring $R_p$ is regular for every nonmaximal prime ideal $p$ of $R$.

Proposition 2.5. Let $R$ be a complete equicharacteristic Cohen-Macaulay local ring. Then for an element $x \in \mathfrak{m}^R$ and a maximal Cohen-Macaulay $R$-module $M$, the multiplication map $M \xrightarrow{x} M$ is a zero morphism in $\mathcal{D}_{sg}(R)$.

Proof. Lemma 2.4(2) implies that there is an exact triangle

$$F \xrightarrow{\cdot x} M \xrightarrow{\cdot y} \Omega M[1] \to \cdots$$

in $\mathcal{D}^b(\text{mod } R)$, where $F$ is a finitely generated free $R$-module. By virtue of [14, Corollary 5.13], the ideal $\mathfrak{m}^R$ annihilates $\text{Ext}_R^1(M, \Omega M) = \text{Hom}_{\mathcal{D}^b(\text{mod } R)}(M, \Omega M[1])$. Hence $xg = 0$
in $D^b(\text{mod } R)$, and there exists a morphism $h : M \to F$ such that $fh = (M \xrightarrow{\varphi} M)$ in $D^b(\text{mod } R)$. Send this equality by the localization functor $D^b(\text{mod } R) \to D_{sg}(R)$, and note that $F \cong 0$ in $D_{sg}(R)$. Thus the multiplication map $M \xrightarrow{\varphi} M$ is zero in $D_{sg}(R)$.  

Now we can prove the results given in the previous section.

**Proof of Theorem 1.1**  As $k$ is a perfect field and $R$ is an isolated singularity, $\mathfrak{M}^R$ is $m$-primary by [13, Lemma (6.12)].

(1) Put $d = \dim R$, $n = \nu(I)$, $l = \ell(R/I)$ and $e = e(I)$. We have $I = (\mathfrak{x})$ for some sequence $\mathfrak{x} = x_1, \ldots, x_n$ of elements in $I$. Let $X \in D_{sg}(R)$. Then, using Lemma [2.4(2)], we see that $X \cong \Omega^d N[n]$ for some $N \in \text{mod } R$ and $n \in \mathbb{Z}$. Note that $M := \Omega^d N$ is a maximal Cohen-Macaulay $R$-module. Proposition [2.2] implies that $K(\mathfrak{x}, M)$ belongs to $\langle k \rangle_{(n-d+1)l}$ in $D^b(\text{mod } R)$. Applying the localization functor $D^b(\text{mod } R) \to D_{sg}(R)$, we have $K(\mathfrak{x}, M) \in \langle k \rangle_{(n-d+1)l}$ in $D_{sg}(R)$. Since $M$ is isomorphic to a direct summand of $K(\mathfrak{x}, M)$ in $D_{sg}(R)$ by Propositions [2.3] and [2.5] we get $M \in \langle k \rangle_{(n-d+1)l}$ in $D_{sg}(R)$. Therefore $D_{sg}(R) = \langle k \rangle_{(n-d+1)l}$ follows.

(2) Since $k$ is infinite, there exists a parameter ideal $Q$ of $R$ that is a reduction of $I$ (cf. [5, Corollary 4.6.10]). We have $(\nu(Q) - \dim R + 1) \ell(R/Q) = \ell(R/Q) \leq \ell(R/Q) = e(Q) = e(I).

The assertion is a consequence of (1). ■

**Proof of Corollary 1.2**  We see from [14, Lemmas 4.3, 5.8 and Propositions 4.4, 4.5] that $J$ is contained in $\mathfrak{M}^R$ and defines the singular locus of $R$. Hence the assertion follows from Theorem 1.1. ■

**Proof of Corollary 1.4**  We notice that $\widehat{R}$ is an isolated singularity. Suppose that $D_{sg}(\widehat{R}) = \langle k \rangle_r$ holds for some $r \geq 0$. Then it follows from [4, Theorem 4.4.1] that $\mathcal{CM}(\widehat{R}) = (\Omega^d_{\widehat{R}} k)_r = (\Omega^d_{\widehat{R}} k)_r$. The proof of [11, Theorem 5.8] shows that $\mathcal{CM}(\widehat{R}) = (\Omega^d_{\widehat{R}} k)_r$. Thus, Theorem 1.1 completes the proof. ■

**Acknowledgments**

The authors thank Luchezar Avramov and Srikanth Iyengar for their valuable comments.

**References**

[1] T. Aihara; R. Takahashi, Generators and dimensions of derived categories, Preprint (2011), arXiv:1106.0205v3.
[2] M. Ballard; D. Favero; L. Katzarkov, Orlov spectra: bounds and gaps, Invent. Math. (in press), http://arxiv.org/abs/1012.0864.
[3] A. Bondal; M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. 3 (2003), no. 1, 1–36, 258.
[4] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, Preprint (1986), http://hdl.handle.net/1807/16682.
[5] W. Bruns; J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1998.
[6] J. D. Christensen, Ideals in triangulated categories: phantoms, ghosts and skeleta, Adv. Math. 136 (1998), no. 2, 284–339.
[7] H. Krause; D. Kussin, Rouquier’s theorem on representation dimension, Trends in representation theory of algebras and related topics, 95–103, Contemp. Math., 406, Amer. Math. Soc., Providence, RI, 2006.

[8] D. O. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, Proc. Steklov Inst. Math. 246 (2004), no. 3, 227–248.

[9] D. O. Orlov, Triangulated categories of singularities, and equivalences between Landau-Ginzburg models, Sb. Math. 197 (2006), no. 11-12, 1827–1840.

[10] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 503–531, Progr. Math., 270, Birkhäuser Boston, Inc., Boston, MA, 2009.

[11] D. Orlov, Formal completions and idempotent completions of triangulated categories of singularities, Adv. Math. 226 (2011), no. 1, 206–217.

[12] R. Rouquier, Dimensions of triangulated categories, J. K-Theory 1 (2008), 193–256.

[13] R. Takahashi, Classifying thick subcategories of the stable category of Cohen-Macaulay modules, Adv. Math. 225 (2010), no. 4, 2076–2116.

[14] H.-J. Wang, On the Fitting ideals in free resolutions, Michigan Math. J. 41 (1994), no. 3, 587–608.

[15] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, 1990.

Department of Mathematics, University of Kansas, Lawrence, KS 66045-7523, USA
E-mail address: hdao@math.ku.edu
URL: http://www.math.ku.edu/~hdao/

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan/Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, USA
E-mail address: takahashi@math.nagoya-u.ac.jp
URL: http://www.math.nagoya-u.ac.jp/~takahashi/