A game–theoretic approach for Generative Adversarial Networks

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Abstract—Generative adversarial networks (GANs) are a class of generative models, known for producing accurate samples. The key feature of GANs is that there are two antagonistic neural networks: the generator and the discriminator. The main bottleneck for their implementation is that the neural networks are very hard to train. One way to improve their performance is to design reliable algorithms for the adversarial process. Since the training can be cast as a stochastic Nash equilibrium problem, we rewrite it as a variational inequality and introduce an algorithm to compute an approximate solution. Specifically, we propose a stochastic relaxed forward–backward algorithm for GANs. We prove that when the pseudogradient mapping of the game is monotone, we have convergence to an exact solution or in a neighbourhood of it.

I. INTRODUCTION

Generative adversarial networks (GANs) are an example of generative models. Specifically, the model takes a training set, consisting of samples drawn from a probability distribution, and learns how to represent an estimate of that distribution. GANs focus primarily on sample generation, but it is also possible to design GANs that can estimate the probability distribution explicitly [1].

The subject has been recently studied, especially because it has many practical applications on various topics. For instance, they can be used for medical purposes, i.e., to improve the diagnostic performance for the low-dose computed tomography method [2], for polishing images taken in unfavourable weather conditions (as rain or snow) [3]. Other applications range from speech and language recognition, to playing chess and vision computing [4].

The idea behind GANs is to train the generative model via an adversarial process, in which also the opponent is simultaneously trained. Therefore, there are two neural network classes: a generative model that captures the data distribution, and a discriminative model that estimates the probability that a sample came from the training data rather than from the generator. The generative model can be thought of as a team of counterfeiters, trying to produce fake currency, while the discriminative model, i.e., the police, tries to detect the counterfeit money. The competition drives both teams to improve their methods until the counterfeit currency is indistinguishable from the original. To succeed in this game, the counterfeiter must learn to make money that are indistinguishable from original currency, and the generator network must learn to create samples that are drawn from the same distribution as the training data [5].

Since each agent payoff depends on the variables of the other agent, this problem can be described as a game. Therefore, these networks are called adversarial. However, GANs can be also thought as a game with cooperative players since they share information with each other [1]. Since there are only the generator and the discriminator, the problem is an instance of a two-player game. Moreover, depending on the cost functions, it can also be considered as a zero-sum game.

From a mathematical perspective, the class of games that suits the GAN problem is that of stochastic Nash equilibrium problems (SNEPs) where each agent aims at minimizing its expected value cost function which is approximated via a number of samples of the random variable.

Given their connection with robust optimization and game theory, GANs have received theoretical attention as well, both for modelling as Nash equilibrium problems [6], [7] and for designing algorithms that improve the training process [8], [7].

From a game theoretic point perspective, an elegant approach to compute a SNE is to cast the problem as a stochastic variational inequality (SVI) [9] and to use an iterative algorithm to find a solution. The two most used methods for SVIs studied in the literature for GANs [8] are the gradient method [10], known in monotone operator theory as forward–backward (FB) algorithm [11], and the extragradient (EG) method [12], [13]. The iterates of the FB algorithm involve an evaluation of the pseudogradient and a projection step. These iterates are known to converge if the pseudogradient mapping is cocoercive or strongly monotone [14], [15]. However, such technical assumptions are quite strong if we consider that in GANs the mapping is rarely monotone. In contrast, the EG algorithm converges for merely monotone operators but taking two projections into the local constraint set per iteration, thus making the algorithm slow and computationally expensive. Other algorithms for VI that can be applied to GANs can be found in [8].

In this paper we propose a stochastic relaxed FB (SRFB) algorithm, inspired by [16], for GANs. A first analysis of the algorithm for stochastic (generalized) NEPs is currently under review [17]. The SRFB requires a single projection and single evaluation of the pseudogradient algorithm per iteration. The advantage of our proposed algorithm is that it is less computationally demanding that the EG algorithm even if it converges under the same assumptions. Indeed, we prove its convergence under mere monotonicity of the pseudogradient mapping when a huge number of samples is available. Alternatively, if only a finite number of samples is accessible, we prove that averaging can be used to converge to a neighbourhood of the solution.
Notation. Let $\mathbb{R}$ indicate the set of real numbers and let $\mathbb{R} = \mathbb{R} \cup \{+\infty\}$. $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the standard inner product and $||\cdot||$ represents the associated euclidean norm. Given $N$ vectors $x_1, \ldots, x_N \in \mathbb{R}^n$, $x := \text{col}(x_1, \ldots, x_N) = [x_1^T, \ldots, x_N^T]^T$. For a closed set $C \subseteq \mathbb{R}^n$, the mapping $\text{proj}_C : \mathbb{R}^n \rightarrow C$ denotes the projection onto $C$, i.e., $\text{proj}_C(x) = \text{argmin}_{y \in C} ||y - x||$.

II. Generative Adversarial Networks

The basic idea of generative adversarial networks (GANs) is to set up a game between two players: the generator and the discriminator. The generator creates samples that are intended to come from the same distribution as the training data. The discriminator examines the samples to determine whether they are real or fake. The generator is therefore trained to fool the discriminator. Typically, a deep neural network is used to represent the generator and the discriminator. Accordingly, the two players are denoted by two functions, each of which is differentiable both with respect to its inputs and with respect to its parameters.

The generator is represented by a differentiable function $g$, that is, a neural network class with parameter vector $x_g \in \Omega_g \subseteq \mathbb{R}^{n_g}$. The (fake) output of the generator is denoted with $g(z, x_g) \in \mathbb{R}^q$ where the input $z$ is a random noise drawn from the model prior distribution, $z \sim p_z$, that the generator uses to create the fake output $g(z, x_g)$ [6]. The actual strategies of the generator are the parameters $x_g$ that allows $g$ to produce the fake output.

The discriminator is a neural network class as well, with parameter vector $x_d \in \Omega_d \subseteq \mathbb{R}^{n_d}$ and a single output $d(v, x_d) \in [0, 1]$ that indicates the accuracy of the input $v$. We interpret the output as the probability that the discriminator $d$ assigns to an element $v$ to be real. Similarly to the generator $g$, the strategies of the discriminator are the parameters $x_d$.

The problem can be cast as a two player game, or, depending on the cost functions, as a zero sum game. Specifically, in the latter case the mappings $J_g$ and $J_d$ should satisfy the following relation

$$J_g(x_g, x_d) = -J_d(x_g, x_d). \quad (1)$$

In most cases [8], [5], the payoff of the discriminator is given by

$$J_d(x_g, x_d) = \mathbb{E}[\phi(d(\cdot, x_d))] - \mathbb{E}[\phi(d(g(\cdot, x_g), x_d))] \quad (2)$$

where $\phi : [0, 1] \rightarrow \mathbb{R}$ is a measuring function (typically a logarithm [5]). The mapping in (2) can be interpreted as the distance between the real value and the fake one.

In the context of zero sum games, the problem can be rewritten as a minmax problem

$$\min_{x_g} \max_{x_d} J_d(x_g, x_d). \quad (3)$$

In words, (3) means that the generator aims at minimizing the distance from the real value while the discriminator wants to maximize it, i.e. to recognize the fake data.

When the problem is not a zero sum game, the generator has its own cost function, usually given by [8]

$$J_g(x_g, x_d) = \mathbb{E}[\phi(d(g(\cdot, x_g), x_d))]. \quad (4)$$

It can be proven that the two-player game with cost functions (2) and (4) and the zero-sum game with cost function (2) and relation (1) have the same equilibria [6, Theorem 10].

III. Stochastic Nash Equilibrium Problems

In this section we formalize the two player game in a more general form that will support our analysis. Specifically, we consider the problem as a general stochastic Nash equilibrium problem since our analysis is independent on the choice of the cost functions.

We consider a set of two agents $\mathcal{I} = \{g, d\}$, that represents the two neural network classes. The local cost function of agent $i \in \mathcal{I}$ is defined as

$$J_i(x_i, x_j) = \mathbb{E}_\xi[J_i(x_i, x_j, \xi(\omega))], \quad (5)$$

for some measurable function $J_i : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$ where $n = n_g + n_d$. The cost function $J_i$ of agent $i \in \mathcal{I}$ depends on the local variable $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$, the decisions of the other player $x_j, j \neq i$, and the random variable $\xi : \Xi \rightarrow \mathbb{R}^q$ that express the uncertainty. Such uncertainty arises in practice when it is not possible to have access to the exact mapping, i.e., when only a finite number of estimates are available. $\mathbb{E}_\xi$ represent the mathematical expectation with respect to the distribution of the random variable $\xi(\omega)$ in the probability space $(\Xi, \mathcal{F}, \mathbb{P})$. We assume that $\mathbb{E}[J_i(x, \xi)]$ is well defined for all the feasible $x = \text{col}(x_g, x_d) \in \Omega = \Omega_g \times \Omega_d$ [18]. For our theoretical analysis, we postulate the following assumptions on the cost function and on the feasible set which are standard in game theory [19], [18].

Assumption 1: For each $i, j \in \mathcal{I}$, $i \neq j$, the function $J_i(\cdot, x_j)$ is convex and continuously differentiable. ■

Assumption 2: For each $i \in \mathcal{I}$, the set $\Omega_i$ is nonempty, compact and convex. ■

Given the decision variables of the other agent, each player $i$ aims at choosing a strategy $x_i$, that solves its local optimization problem, i.e.,

$$\forall i \in \mathcal{I} : \min_{x_i \in \Omega_i} J_i(x_i, x_j). \quad (6)$$

Given the coupled optimization problems in (6), the solution concept that we are seeking is that of Stochastic Nash equilibrium (SNE) [18].

Definition 1: A stochastic Nash equilibrium is a collective strategy $x^* = \text{col}(x_g^*, x_d^*) \in \Omega$ such that for all $i \in \mathcal{I}$

$$J_i(x_i^*, x_j^*) \leq \inf_{x_i \in \Omega_i} J_i(x_i, x_j) \quad | y \in \Omega_j \}.$$

Thus, a SNE is a set of strategies where no agent can decrease its cost function by unilaterally deviating from its decision.

To guarantee that a SNE exists, we make further assumptions on the cost functions [18, Ass. 1].

Assumption 3: For each $i \in \mathcal{I}$ and for each $\xi \in \Xi$, the function $J_i(\cdot, x_j, \xi)$ is convex, Lipschitz continuous,

\(^1\)From now on, we use $\xi$ instead of $\xi(\omega)$ and $\mathbb{E}$ instead of $\mathbb{E}_\xi$. 


and continuously differentiable. The function $J_i(x_i, x_j, \cdot)$ is measurable and for each $x$ and its Lipschitz constant $\ell_i(x_j, \xi)$ is integrable in $\xi$.

Existence of a SNE of the game in (6) is guaranteed, under Assumptions [13] by [18, Section 3.1] while uniqueness does not hold in general [18, Section 3.2].

For seeking a Nash equilibrium, we rewrite the problems as a stochastic variational inequality. To this aim, let us denote the pseudogradient as

$$
F(x) = \left[ \mathbb{E}[\nabla x_i f_i(x_i, x_d)] \mathbb{E}[\nabla x_j g_j(x_j, x_d)] \right],
$$

where the possibility to exchange the expected value and the pseudogradient in (7) is assured by Assumption 3. Then, the associated stochastic variational inequality (SVI) reads as

$$
\langle F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in \Omega.
$$

We call variational equilibria (v-SNE) the SNE that are also solution of the associated SVI, namely, the solution of the SVI $(\Omega, F)$ in (8) where $F$ is as in (7).

Remark 1: If Assumptions [13] hold, then a tuple $x^* \in \Omega$ is a Nash equilibrium of the game in (6) if and only if $x^*$ is a solution of the SVI in (8) [9, Prop. 14.2], [18, Lem. 3.3].

Moreover, under Assumptions [13], the solution set of SVI $(\Omega, F)$ is non empty and compact, i.e. $\text{SOL}(\Omega, F) \neq \emptyset$ [9, Corollary 2.2.5] and a v-SNE exists.

IV. STOCHASTIC RELAXED FORWARD–BACKWARD WITH AVERAGING

The first algorithm that we propose is inspired by [16], [17] and it is a stochastic relaxed forward backward algorithm with averaging (aSRFB). The iterations reads as in Algorithm [1]

**Algorithm 1** Stochastic Relaxed Forward–Backward with averaging (aSRFB)

Initialization: $x^0_i \in \Omega_i$

Iteration $k \in \{1, \ldots, K\}$: Agent $i \in \{g, d\}$ receives $x^k_j$, $j \neq i$, then updates:

$$
\hat{x}^k_i = (1 - \delta)x^k_i + \delta x^{k-1}_i
$$

(9a)

$$
x^k_{i+1} = \text{proj}_{\Omega_i}([\hat{x}^k_i - \lambda_i F^{\text{SA}}(x^k_i, x^k_j, \xi^k_i)])
$$

(9b)

Iteration $K$: $X^K = \sum_{k=1}^K \frac{\lambda_k x^k_i}{S_K}$

(10)

We note that the averaging step

$$
X^K = \sum_{k=1}^K \lambda_k x^k_i, \quad S_K = \sum_{k=1}^K \lambda_k,
$$

where $X^K = \text{col}(X^K_g, X^K_d)$, was first proposed for VIs in [10], and it can be implemented in an online fashion as

$$
X^K = (1 - \lambda_K)X^{K-1} + \lambda_K x^K
$$

(11)

where $0 \leq \lambda_K \leq 1$. Even if they look similar, (11) is different from (9a). Indeed, in Algorithm [1] (9a) is a convex combination of the two previous iterates $x^k$ and $\hat{x}^{k-1}$, with a fixed parameter $\delta$, while the averaging in (11) is a weighted cumulative sum over all the decision variables $x^k$ for all $k \in \{1, \ldots, K\}$ with time varying weights $\lambda_k$. The parameter $\lambda_K$ can be tuned to obtain uniform, geometric or exponential averaging [8]. The relaxation parameter $\delta$ instead should satisfy the following assumption.

**Assumption 4:** In Algorithm [1] $\delta \in [0, 1]$.

To continue our analysis, we postulate the following monotonicity assumption on the pseudogradient mapping which is standard for VI problems [13], [17], also when applied to GANs [8].

**Assumption 5:** $F$ as in (7) is monotone, i.e. $\langle F(x) - F(y), x - y \rangle \geq 0$ for all $x, y \in \Omega$.

Next, let us define the stochastic approximation of the pseudogradient [11] as

$$
F^{\text{SA}}(x, \xi) = \left[ \nabla x_i f_i(x_i, x_d, \xi), \nabla x_j g_j(x_j, x_d, \xi_d) \right].
$$

(12)

$F^{\text{SA}}$ uses one or a finite number, called mini-batch, of realizations of the random variable. Given the approximation, we postulate the following assumption which is quite strong yet reasonable in our game theoretic framework [8]. Let us first define the filtration $\mathcal{F} = \{\mathcal{F}_k\}$, that is, a family of $\sigma$-algebras such that $\mathcal{F}_0 = \sigma(x_0)$ and $\mathcal{F}_k = \sigma(x_0, \xi_1, \xi_2, \ldots, \xi_k)$ for all $k \geq 1$, such that $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all $k \geq 0$.

**Assumption 6:** $F^{\text{SA}}$ in (12) is bounded, i.e., there exists $B > 0$ such that for $x \in \Omega$, $\mathbb{E}[\|F^{\text{SA}}(x, \xi)\|^2 | \mathcal{F}_k] \leq B$.

For the sake of our analysis, we make an explicit bound on the feasible set.

**Assumption 7:** The local constraint set $\Omega$ is such that $\max \|x - y\|^2 \leq R^2$, for some $R \geq 0$.

For all $k \geq 0$, we define the stochastic error as

$$
\epsilon_k = F^{\text{SA}}(x^k, \xi_k) - F(x^k),
$$

(13)

that is, the distance between the approximation and the exact expected value. Then, we postulate that the stochastic error satisfies the following assumption.

**Assumption 8:** The stochastic error in (13) is such that, for all $k \geq 0$, $\mathbb{E}[\|\epsilon^k\| | \mathcal{F}_k] = 0$ and $\mathbb{E}[\|\epsilon^k\|^2 | \mathcal{F}_k] \leq \sigma^2$ a.s.. Essentially, Assumption 8 states that the error has zero mean and bounded variance, as usual in SVI [8], [13], [17].

As a measure of the quality of the solution, we define the following error

$$
\text{err}(x) = \max_{x^*_i \in \Omega} \langle F(x^*_i), x - x^* \rangle,
$$

(14)

which is known as gap function and it is equal 0 if and only if $x^*$ is a solution of the (S)VI in (9) [9, Eq. 1.5.2]. Another measure function specific for the zero-sum game and other possible measures can be found in [8].

We are now ready to state our first result.

**Theorem 1:** Let Assumptions [13] hold. Let $X^K = \sum_{k=1}^K \frac{\lambda_k x^k_i}{\sum_{k=1}^K \lambda_k}$ as in Assumption 6 $c$ as in Assumption 7 and $\sigma^2$ as in Assumption 8. Then Algorithm [1] with a constant step size and $F^{\text{SA}}$ as in (12) gives

$$
\mathbb{E}[\text{err}(X^K)] = \frac{c R}{\lambda_K} + (2B^2 + \sigma^2)\lambda.
$$

Thus, $\lim_{K \to \infty} \mathbb{E}[\text{err}(X^K)] = (2B^2 + \sigma^2)\lambda$.

**Proof:** See Appendix [11]
Remark 2: The average defined in Theorem 1 is not in conflict with the definition in (10) because if we consider a fixed step size, it holds that
\[ X^K = \frac{\sum_{k=1}^{K} \lambda_k x^k}{\sum_{k=1}^{K} \lambda_k} = \frac{\lambda \sum_{k=1}^{K} x^k}{K \lambda} = \frac{1}{K} \sum_{k=1}^{K} x^k. \]

V. SAMPLE AVERAGE APPROXIMATION

If a huge number of samples is available or it is possible to compute the exact expected value, one can consider using a different approximation scheme or a deterministic algorithm. We discuss these two situations in this section.

In the SVI framework, using a finite, fixed number of samples is called stochastic approximation (SA). It is widely used in the literature but it often requires conditions on the step sizes to control the stochastic error. Usually, the step size sequence should be diminishing with the results that the iterations slow down considerably. The approach that is instead used to keep a fixed step size is the sample average approximation (SAA) scheme. In this case, an increasing number of samples is taken at each iteration and this helps having a diminishing error.

With the SAA scheme, it is possible to prove convergence to the exact solution without using the averaging step. We show this result in Theorem 2 but first we provide more details on the approximation scheme and state some assumptions. The algorithm that we are proposing is presented in Algorithm 2. The differences with Algorithm 1 are the absence of the averaging step and the approximation \( F^{\text{SAA}} \).

Algorithm 2 Stochastic Relaxed Forward–Backward (SRFB)

Initialization: \( x_i^0 \in \Omega_i \)

Iteration \( k \): Agent \( i \) receives \( x_j^k \) for \( j \neq i \), then updates:
\[
\begin{align*}
\bar{x}_i^k &= (1 - \delta) x_i^k + \delta x_i^{k-1} \\
x_i^{k+1} &= \text{proj}_{\Omega_i} [\bar{x}_i^k - \lambda_i F_i^{\text{SAA}}(x_i^k, x_j^k, \xi_i^k)] \tag{15a}
\end{align*}
\]

Formally, the approximation that we use is given by
\[
F^{\text{SAA}}(x, \xi^k) = \left[ \frac{1}{N_k} \sum_{s=1}^{N_k} \nabla x_s f_i(x_i^k, x_d^k, \xi_s^k) \right] \\
N_k = \sum_{s=1}^{N_k} \nabla x_s f_i(x_i^k, x_d^k, \xi_s^k) \tag{16}
\]

where \( N_k \) is the batch size that should be increasing [13].

Assumption 9: The batch size sequence \( (N_k)_{k \geq 1} \) is such that \( N_k \geq b(k + k_0)^{a+1} \), for some \( b, k_0, a > 0 \).

With a little abuse of notation, let us denote the stochastic error also in this case as
\[
e^k = F^{\text{SAA}}(x^k, \xi^k) - F(x^k).
\]

Remark 3: Using the SAA scheme, it is possible to prove that, for some \( C > 0 \), \( \mathbb{E} \left[ e^k \right] \leq C \varepsilon^k \), i.e., the error diminishes as the size of the batch increases. Details on how to obtain this result can be found in [13].

To obtain convergence, we have to make further assumptions on the pseudogradient mapping [16], [13].
B. Classic GAN zero-sum game

A classic cost function for the zero-sum game [1] proposed for GANs reads as

\[
\min_{x_g} \max_{x_d} -\log(1 + e^{x_d^T \omega}) - \log(1 + e^{x_g^T x_d}).
\]

This cost function is hard to optimize because it is concave-concave [8]. Here we take \( \omega = -2 \), thus the equilibrium is \( (x_g, x_d) = (-2, 0) \). In Figure 2a, 2b and 2c we show the distance from the solution, the distance of the average from the solution, and the computational cost respectively. Interestingly, all the considered algorithms converge even if there are no theoretical guarantees.

VII. CONCLUSION

The stochastic relaxed forward–backward algorithm can be applied to Generative Adversarial Networks. Given a
fixed mini-batch, under monotonicity of the pseudogradient, averaging can be considered to reach a neighbourhood of the solution. On the other hand, if a huge number of samples is available, under the same assumptions, convergence to the exact solution holds.

**APPENDIX I**

**PRELIMINARY RESULTS**

We here recall some facts about norms, some properties of the projection operator and a preliminary result.

We start with the norms. We use the cosine rule

\[
\langle x, y \rangle = \frac{1}{2} \left( \|x\|^2 + \|y\|^2 - \|x - y\|^2 \right)
\]

(17)

and the following two property of the norm [21, Corollary 2.15], \(\forall a, b \in \mathcal{E}, \forall \alpha \in \mathbb{R}\)

\[
\|a + (1 - \alpha)b\|^2 = \alpha \|a\|^2 + (1 - \alpha)\|b\|^2 - \alpha(1 - \alpha)\|a - b\|^2,
\]

(18)

\[
\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2.
\]

(19)

Concerning the projection operator, by [21, Proposition 12.26], it satisfies the following inequality: let \(C\) be a nonempty closed convex set, then, for all \(x, y \in C\)

\[
\bar{x} = \text{proj}_C(x) \iff \langle \bar{x} - x, y - \bar{x} \rangle \geq 0.
\]

(20)

The projection is also firmly non expansive [21, Prop. 4.16], and consequently, quasi firmly non expansive [21, Def. 4.1].

The Robbins-Siegmund Lemma is widely used in literature to prove a.s. convergence of sequences of random variables.

**Lemma 1 (Robbins-Siegmund Lemma, [22]):** Let \(\mathcal{F} = (\mathcal{F}_k)_{k \in \mathbb{N}}\) be a filtration. Let \(\{\alpha_k\}_{k \in \mathbb{N}}, \{\theta_k\}_{k \in \mathbb{N}}\) and \(\{\eta_k\}_{k \in \mathbb{N}}\) be non negative sequences such that \(\sum_k \eta_k < \infty, \sum_k \alpha_k < \infty\) and let \(\sum_k \theta_k < \infty\) and \(\{\alpha_k\}_{k \in \mathbb{N}}\) converges a.s. to a non negative random variable.

The next lemma collects some properties that follow from the definition of the SRFB algorithm.

**Lemma 2:** Given Algorithm 1 the following hold.

1) \(x^k - \bar{x}^k = \frac{1}{\theta_k}(x^k - \bar{x}^k)\)

2) \(x^k - \bar{x}^k = \frac{\theta_k}{1 - \lambda}(x^{k+1} - \bar{x}^k) - \lambda(\bar{x}^k - x^k)\)

3) \(\frac{\delta}{(1 - \delta)^2} \|x^k - x^k\|^2 = \|x^k - x^k\|^2\)

**Proof:** Straightforward from Algorithm 1.

**APPENDIX II**

**PROOF OF THEOREM 1**

**Proof:** [Proof of Theorem 1] We start by using the fact that the projection is firmly quasinonexpansive.

\[
\|x^{k+1} - x^*\|^2 \leq \|x^* - x^k + \lambda F^\mathbb{S}(x^k, \bar{x}^k) - x^k\|^2 - \|x^k - \lambda F^\mathbb{S}(x^k, \bar{x}^k) - x^k\|^2
\]

\[
\leq \|x^* - x^k\|^2 - \|x^k - x^{k+1}\|^2 + 2\lambda_k F^\mathbb{S}(x^k, \bar{x}^k, x^* - \bar{x}^k) + 2\lambda_k F^\mathbb{S}(x^k, \bar{x}^k, x^k - \bar{x}^k)
\]

Now we apply Lemma 2 and Lemma 3 to \(\|x^{k+1} - x^*\|^2\):

\[
\|x^{k+1} - x^*\|^2 \leq \frac{1}{1 - \delta} \|x^k - x^*\|^2 - \frac{\delta}{1 - \delta} \|x^k - x^*\|^2 + \frac{\delta}{1 - \delta} \|x^k - x^*\|^2 + \delta \|x^k - x^*\|^2.
\]

(21)

Then, we can rewrite the inequality as

\[
\frac{1}{1 - \delta} \|x^{k+1} - x^*\|^2 \leq \frac{1}{1 - \delta} \|x^k - x^*\|^2 + 2\lambda_k (F^\mathbb{S}(x^k, \bar{x}^k, x^* - \bar{x}^k) + 2\lambda_k (F^\mathbb{S}(x^k, \bar{x}^k, x^k - \bar{x}^k) - (\delta + 1) \|x^{k+1} - x^k\|^2
\]

(22)

Applying the Young’s inequality we obtain

\[
2\lambda_k (F^\mathbb{S}(x^k, \bar{x}^k, x^k - \bar{x}^k) \leq \leq \lambda_k^2 \|F^\mathbb{S}(x^k, \bar{x}^k, \bar{x}^k - x^{k+1})\|^2 + \|x^k - x^k\|^2
\]

\[
2\lambda_k (F^\mathbb{S}(x^k, \bar{x}^k, x^k - \bar{x}^k) \leq \leq \lambda_k^2 \|F^\mathbb{S}(x^k, \bar{x}^k, \bar{x}^k - x^{k+1})\|^2 + \|x^k - x^k\|^2
\]

(23)

Then (22) becomes

\[
\frac{1}{1 - \delta} \|x^{k+1} - x^*\|^2 \leq \frac{1}{1 - \delta} \|x^k - x^*\|^2 + 2\lambda_k (F^\mathbb{S}(x^k, \bar{x}^k, x^k - \bar{x}^k) + 2\lambda_k^2 \|F^\mathbb{S}(x^k, \bar{x}^k, \bar{x}^k - x^{k+1})\|^2 + \|x^k - x^k\|^2 + \|x^k - x^{k+1}\|^2
\]

Reordering, adding and subtracting \(2\lambda_k (F(x^k), x^k - x^k)\) and using Lemma 2 we obtain

\[
\frac{1}{1 - \delta} \|x^{k+1} - x^*\|^2 + \delta \|x^{k+1} - x^k\|^2 \leq \frac{1}{1 - \delta} \|x^k - x^*\|^2 + 2\lambda_k (F(x^k, \bar{x}^k, x^k - x^k) - 2\lambda_k (F(x^k), x^k - x^k) + 2\lambda_k^2 \|F(x^k, \bar{x}^k, \bar{x}^k - x^{k+1})\|^2 + \|x^k - x^k\|^2 + \|x^k - x^{k+1}\|^2
\]

(24)

Then, by the definition of \(\epsilon^k\), reordering leads to

\[
2\lambda_k (F(x^k), x^k - x^k) \leq \leq \frac{1}{1 - \delta} \|x^k - x^*\|^2 + \|x^{k+1} - x^*\|^2 + \|x^k - x^{k+1}\|^2
\]

(25)

Summing over all the iterations, (25) becomes

\[
\sum_{k=1}^{K} \lambda_k (F(x^k), x^k - x^k) \leq \sum_{k=1}^{K} \lambda_k (\epsilon^k, x^k - x^*) \leq \frac{1}{1 - \delta} \|x^k - x^k\|^2 + \|x^{k+1} - x^k\|^2 + \|x^{k+1} - x^k\|^2 + 2\lambda_k^2 \|F(x^k, \bar{x}^k, \bar{x}^k - x^{k+1})\|^2
\]

(26)
Therefore, Noticing that if By definition Using Assumption 5 and resolving the sums, we obtain

\[
2 \sum_{k=1}^{K} \lambda_k \langle \mathcal{F}(x^*), x^k - x^* \rangle \leq 2 \sum_{k=1}^{K} \lambda_k \langle \epsilon_k, x^k - x^* \rangle \\
\leq \frac{1}{1 - \delta} \|x^0 - x^*\|^2 + \delta \|x^0 - \tilde{x}^0\|^2 + 2 \lambda_k^2 \|F^{SA}(x^k, \xi^k)\|^2
\]

Now we notice that \( \langle \epsilon_k, x^k - x^* \rangle = \langle \epsilon_k, x^k - u^k \rangle + \langle \epsilon_k, u^k - x^* \rangle \). We define \( u^0 = x^0 \) and \( u^{k+1} = \text{proj}(u^k - \lambda_k \epsilon_k) \), thus

\[
\|u^{k+1} - x^*\|^2 = \|\text{proj}(u^k - \lambda_k \epsilon_k) - x^*\|^2 \\
\leq \|u^k - \lambda_k \epsilon_k - x^*\|^2 \\
\leq \|u^k - x^*\|^2 + \lambda_k \|\epsilon_k\|^2 - 2 \lambda_k \langle \epsilon_k, u^k - x^* \rangle
\]

Therefore, \( 2 \lambda_k \langle \epsilon_k, x^k - x^* \rangle = 2 \lambda_k \langle \epsilon_k, x^k - u^k \rangle + \|u^k - x^*\|^2 + \lambda_k \|\epsilon_k\|^2 - \|u^{k+1} - x^*\|^2 \). Including this in (27) and doing the sum, we obtain

\[
2 \sum_{k=1}^{K} \lambda_k \langle \mathcal{F}(x^*), x^k - x^* \rangle \leq \\
\leq \frac{1}{1 - \delta} \|x^0 - x^*\|^2 + \delta \|x^0 - \tilde{x}^0\|^2 + 2 \sum_{k=1}^{K} \lambda_k^2 \|F^{SA}(x^k, \xi^k)\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|\epsilon_k\|^2 + 2 \lambda_k \langle \epsilon_k, x^k - u^k \rangle + \|u^0 - x^*\|^2 + 2 \sum_{k=1}^{K} \lambda_k \langle \epsilon_k, x^k - u^k \rangle
\]

By definition \( \|u^0 - x^*\|^2 = \|x^0 - x^*\|^2 \), then, taking the expected value in (29) and using Assumption 8

\[
2 \sum_{k=1}^{K} \lambda_k \langle \mathcal{F}(x^*), x^k - x^* \rangle \leq \\
\leq \frac{1}{1 - \delta} \|x^0 - x^*\|^2 + \delta \|x^0 - \tilde{x}^0\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|F^{SA}(x^k, \xi^k)\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|\epsilon_k\|^2 + \frac{1}{1 - \delta} \|x^0 - x^*\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|F^{SA}(x^k, \xi^k)\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|\epsilon_k\|^2 + \frac{1}{1 - \delta} \|x^0 - x^*\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|F^{SA}(x^k, \xi^k)\|^2 + \sum_{k=1}^{K} \lambda_k^2 \|\epsilon_k\|^2
\]

Let us define \( S = \sum_{k=1}^{K} \lambda_k \), \( X^k = \sum_{k=1}^{K} \lambda_k x^k \), \( \lambda = \frac{1}{S} \sum_{k=1}^{K} \lambda_k \).

Then,

\[
2S \langle \mathcal{F}(x^*), X^k - x^* \rangle \leq \\
\leq \frac{2 - \delta}{1 - \delta} \|x^0 - x^*\|^2 + \delta \|x^0 - \tilde{x}^0\|^2 + \frac{2}{1 - \delta} \|x^0 - x^*\|^2 + \delta \|x^0 - \tilde{x}^0\|^2 + \frac{1}{1 - \delta} \|x^0 - x^*\|^2 + \delta \|x^0 - \tilde{x}^0\|^2
\]

Noticing that if \( \lambda_k \) is constant \( S = K \lambda \) and \( \sum_{k=1}^{K} \lambda_k^2 = K \lambda^2 \)

\[
\langle \mathcal{F}(x^*), X^k - x^* \rangle \leq \frac{cR}{K \lambda} + (2B^2 + \sigma^2) \lambda.
\]
Similarly we can bound the term involving the stochastic errors
\[ 2\lambda (\epsilon^k - \epsilon^{k-1}, x^k - x^{k+1}) \]
\[ \leq 2\lambda \| \epsilon^k - \epsilon^{k-1} \| \| x^k - x^{k+1} \| \]
\[ \leq \lambda \| \epsilon^k - \epsilon^{k-1} \|^2 + \lambda \| x^k - x^{k+1} \|^2 . \]

Substituting in (37), it yields
\[ \frac{1}{1-\delta} \| x^{k+1} - x^* \|^2 + \frac{1}{\delta} \| x^k - x^{k-1} \|^2 \leq \]
\[ \leq \frac{1}{1-\delta} \| x^* - x^k \|^2 + \frac{1}{\delta} \| x^k - x^{k-1} \|^2 + \]
\[ + \lambda \left( \| x^k - x^{k-1} \|^2 + \| x^k - x^k \|^2 \right) + \]
\[ + \lambda \| x^k - x^{k-1} \|^2 + \lambda \| x^k - x^{k+1} \|^2 + 2\lambda \| x^k, x^k - x^k \|^2 \]
\[ \leq \frac{1}{1-\delta} \| x^k - x^{k+1} \|^2 + \frac{1}{\delta} \| x^k - x^{k-1} \|^2 \leq \]
\[ \leq 2\| x^k - x^{k+1} \|^2 + 2 \| x^k - x^k \|^2 + \lambda \| x^k - x^k \|^2 \]
\[ \leq 2 \| x^k - x^{k+1} \|^2 + 4 \| x^k - x^{k+1} \|^2 + \lambda^2 \| x^k \|^2 \]

where we added and subtracted \( x^{k+1} = \text{proj} (x^k - \lambda FSA(kx)) \) in the first inequality and used the firmly non-expansive property of the projection and (19). It follows that
\[ \| x^k - x^k \|^2 \geq 1 \| x^k - x^{k+1} \|^2 - \frac{1}{2} \| x^k - x^{k-1} \|^2 - 4\lambda^2 \| x^k \|^2 \]

Substituting in (38)
\[ \frac{1}{1-\delta} \| x^{k+1} - x^* \|^2 + \frac{1}{\delta} \| x^k - x^{k-1} \|^2 \leq \]
\[ \leq \frac{1}{1-\delta} \| x^* - x^k \|^2 + \frac{1}{\delta} \| x^k - x^{k-1} \|^2 + \lambda \| x^k - x^k \|^2 \]
\[ + \lambda \| x^k - x^{k-1} \|^2 + \lambda \| x^k - x^{k+1} \|^2 + 2\lambda (x^k, x^k - x^k) \]

Taking the expected value, grouping and using Remark 3 and Assumptions 8 and 12, we have
\[ E \left[ \frac{1}{1-\delta} \| x^{k+1} - x^* \|^2 + \frac{1}{\delta} \| x^k - x^{k-1} \|^2 \right] \leq \]
\[ \leq \frac{1}{1-\delta} \| x^* - x^k \|^2 + \lambda \| x^k - x^k \|^2 + \lambda \| x^k - x^{k-1} \|^2 + \]
\[ + \frac{2\lambda C_0}{N_k} + \frac{2\lambda C_0}{N_{k-1}} + \frac{\lambda C_0}{\delta N_k} - \frac{\lambda}{\delta} \| x^k - x^k \|^2 - \frac{1}{\delta} \| x^k - x^{k+1} \|^2 \]

To use Lemma 1, let \( \alpha_k = \frac{1}{2} \| x^k - x^k \|^2 + \lambda \| x^k - x^k \|^2 \), \( \theta_k = \frac{1}{\delta} \| x^k - x^k \|^2 + \frac{1}{\delta} \| x^k - x^{k+1} \|^2 \), \( \eta_k = \frac{2\lambda C_0}{N_k} + \frac{2\lambda C_0}{N_{k-1}} + \frac{\lambda C_0}{\delta N_k} \). Applying the Robbins Siegmund Lemma we conclude that \( \alpha_k \) converges and that \( \sum \theta_k \) is summable. This implies that the sequence \( \{ x^k \} \) is bounded and that \( \| x^k - x^k \| \to 0 \) (otherwise \( \sum \frac{1}{\delta} \| x^k - x^k \|^2 = \infty \)). Therefore \( \{ x^k \} \) has at least one cluster point \( \bar{x} \). Moreover, since \( \sum \theta_k < \infty \), \( \text{res}(x^k) \to 0 \) and \( \text{res}(\bar{x}) = 0 \)

REFERENCES

[1] I. Goodfellow, “Nips 2016 tutorial: Generative adversarial networks,” arXiv preprint arXiv:1701.00160, 2016.
[2] Q. Yang, P. Yan, Y. Zhang, H. Yu, Y. Shi, X. Mou, M. K. Kalra, Y. Zhang, L. Sun, and G. Wang, “Low-dose ct image denoising using a generative adversarial network with wasserstein distance and perceptual loss,” IEEE transactions on medical imaging, vol. 37, no. 6, pp. 1348–1357, 2018.
[3] H. Zhang, V. Sindagi, and V. M. Patel, “Image de-ruining using a conditional generative adversarial network,” IEEE/CAA Journal of Automatica Sinica, vol. 4, no. 4, pp. 588–598, 2017.
[4] K. Wang, C. Gou, Y. Duan, Y. Lin, X. Zheng, and F.-Y. Wang, “Generative adversarial networks: introduction and outlook,” IEEE/CAA journal of automatica sinica, vol. 6, no. 3, pp. 422–439, 2019.
[5] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio, “Generative adversarial nets,” in Advances in neural information processing systems, 2014, pp. 2672–2680.
[6] F. A. Oliehoek, R. Savani, J. Gallego-Posada, E. Van der Pol, E. D. De Jong, and R. Größ, “Gang: Generative adversarial network games,” arXiv preprint arXiv:1712.00679, 2017.
[7] E. Mazumdar, J. L. Ratliff, and S. S. Sastry, “On gradient-based learning in continuous games,” SIAM Journal on Mathematics of Data Science, vol. 2, no. 1, pp. 103–131, 2020.
[8] G. Gidel, H. Berard, G. Vignoud, P. Vincent, and S. Lacoste-L突击on, “A variational inequality perspective on generative adversarial networks,” arXiv preprint arXiv:1802.10551, 2018.
[9] F. Facchinei and J.-S. Pang, Finite-dimensional variational inequalities and complementarity problems. Springer Science & Business Media, 2007.
[10] R. E. Bruck Jr, “On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in hilbert space,” Journal of Mathematical Analysis and Applications, vol. 61, no. 1, pp. 159–164, 1977.
[11] H. Robbins and S. Monro, “A stochastic approximation method,” The annals of mathematical statistics, pp. 400–407, 1951.
[12] G. Kropelie, “The extragradient method for finding saddle points and other problems,” Matecon, vol. 12, pp. 747–756, 1976.
[13] A. Iusem, A. Jofré, R. I. Oliveira, and P. Thompson, “Extragradient method with variance reduction for stochastic variational inequalities,” SIAM Journal on Optimization, vol. 27, no. 2, pp. 686–724, 2017.
[14] L. Rosasco, S. Villa, and B. C. Vù, “Stochastic forward–backward splitting for monotone inclusions,” Journal of Optimization Theory and Applications, vol. 169, no. 2, pp. 388–406, 2016.
[15] B. Franci and S. Grammatico, “Distributed forward-backward algorithms for stochastic generalized Nash equilibrium seeking,” arXiv preprint arXiv:1912.04165, 2019.
[16] Y. Malitsky, “Golden ratio algorithms for variational inequalities,” Mathematical Programming. Jul 2019. [Online]. Available: https://doi.org/10.1007/s10107-019-01416-w
[17] B. Franci and S. Grammatico, “Stochastic generalized Nash equilibrium seeking in merely monotone games,” arXiv preprint arXiv:2002.08318, 2020.
[18] U. Ravat and U. V. Shanbhag, “On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games,” SIAM Journal on Optimization, vol. 21, no. 3, pp. 1168–1199, 2011.
[19] F. Facchinei, A. Fischer, and V. Piccialli, “On generalized Nash games and variational inequalities,” Operations Research Letters, vol. 35, no. 2, pp. 159–164, 2007.
[20] D. P. Kingma and J. Ba, “Adam: A method for stochastic optimization,” arXiv preprint arXiv:1412.6980, 2014.
[21] H. H. Bauschke, P. L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces. Springer, 2011, vol. 408.
[22] H. Robbins and D. Siegmund, “A convergence theorem for non negative almost supermartingales and some applications,” in Optimizing methods in statistics. Elsevier, 1971, pp. 233–257.