A TORRES CONDITION FOR TWISTED ALEXANDER POLYNOMIALS

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Abstract. As a generalization of a fundamental result about the Alexander polynomial of links, we give a description of a Torres condition for the twisted Alexander polynomial of links associated to a unimodular representation.

1. Introduction

The theory of twisted Alexander polynomial was introduced by Lin [11] and Wada [15] independently. Lin defined it for knots in the 3-sphere using regular Seifert surfaces. On the other hand, Wada defined the twisted Alexander polynomial for finitely presentable groups, which include the link groups. In particular, as an application, Wada told the Kinoshita-Terasaka knot from the Conway knot by means of his invariant. Shortly afterward, several significant results on the original Alexander polynomial were generalized to the twisted case. For example, equivalence of the twisted Alexander polynomial and the Reidemeister torsion, and its symmetry [3], [6], sliceness obstruction for knots and a relation to the Casson-Gordon invariant [6], [7], monicness of the twisted Alexander polynomial for fibered knots [1], [2] and so on. Recently the twisted Alexander polynomials are extensively investigated. See for instance [3], [4], [5], [9], [10], [12] and [13].

However, almost all results mentioned above are basically about knots in the 3-sphere and it seems that there are few generalized results on links. The purpose of the present paper is to give a generalization of the following well-known formula for the Alexander polynomial of links.

**Theorem 1.1** (Torres [14]). For the Alexander polynomial \( \Delta_L(t_1, \ldots, t_\mu) \) of a \( \mu \)-component link \( L = L_1 \cup \cdots \cup L_\mu \), it holds that

\[
\Delta_L(t_1, \ldots, t_{\mu-1}, 1) = \begin{cases} 
\frac{t_1^l - 1}{t_1 - 1} \Delta_{L'}(t_1) & \text{if } \mu = 2 \\
(t_1^l \cdots t_{\mu-1}^l - 1) \Delta_{L'}(t_1, \ldots, t_{\mu-1}) & \text{if } \mu > 2,
\end{cases}
\]

where \( L' = L_1 \cup \cdots \cup L_{\mu-1} \) is the link obtained from \( L \) by removing \( L_\mu \) and \( l_i \) denotes the linking number of the components \( L_i \) and \( L_\mu \).

More precisely, we give a description of a Torres condition for the twisted Alexander polynomial of links associated to a unimodular representation. In the next section, we briefly recall the definition of the twisted Alexander polynomial for the link group. The precise statement and the proof of the main theorem of this paper are given in Section 3.

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2. Twisted Alexander polynomial for links

Let $L = L_1 \cup \cdots \cup L_\mu$ be a $\mu$-component link in the 3-sphere. We denote the fundamental group of its exterior $E$ by $G(L)$. Namely, we put $G(L) = \pi_1(E)$ and call it the link group. We choose and fix a Wirtinger presentation of $G(L)$:

$$G(L) = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_{u-1} \rangle.$$ 

Then the abelianization homomorphism

$$\alpha : G(L) \to H_1(E; \mathbb{Z}) \cong \mathbb{Z}^\oplus \mu = \langle t_1 \rangle \oplus \cdots \oplus \langle t_\mu \rangle$$

is given by assigning to each generator $x_i$ the meridian element $t_k \in H_1(E; \mathbb{Z})$ of the corresponding component $L_k$ of $L$. In this paper, we consider a linear representation $\rho : G(L) \to SL(n; F)$, where $F$ denotes a field.

These maps naturally induce two ring homomorphisms $\tilde{\rho} : \mathbb{Z}[G(L)] \to M(n; F)$ and $\tilde{\alpha} : \mathbb{Z}[G(L)] \to \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$, where $\mathbb{Z}[G(L)]$ is the group ring of $G(L)$ over $\mathbb{Z}$ and $M(n; F)$ is the matrix algebra of degree $n$ over $F$. Taking the tensor of $\tilde{\rho}$ and $\tilde{\alpha}$, we obtain a ring homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}[G(L)] \to M(n; F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]).$$

Let $F_u$ denote the free group on generators $x_1, \ldots, x_u$ and

$$\Phi : \mathbb{Z}[F_u] \to M(n; F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}])$$

the composite of the surjection $\mathbb{Z}[F_u] \to \mathbb{Z}[G(L)]$ induced by the presentation and the map $\tilde{\rho} \otimes \tilde{\alpha}$.

Let us consider the $(u-1) \times u$ matrix $M = M(t_1, \ldots, t_\mu)$ whose $(i, j)$th component is the $n \times n$ matrix

$$\Phi \left( \frac{\partial r_i}{\partial x_j} \right) \in M(n; F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]),$$

where $\partial/\partial x$ denotes the free differential calculus. This matrix $M$ is called the Alexander matrix of $G(L)$ associated to the representation $\rho$.

For $1 \leq j \leq u$, let us denote by $M_j = M_j(t_1, \ldots, t_\mu)$ the $(u-1) \times (u-1)$ matrix obtained from $M$ by removing the column corresponding to a generator $x_j$. We also regard $M_j$ as an $n(u-1) \times n(u-1)$ matrix with coefficients in $F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$.

Then Wada’s twisted Alexander polynomial of a link $L$ for a representation $\rho : G(L) \to SL(n; F)$ is defined to be a rational function

$$\Delta_{L, \rho}(t_1, \ldots, t_\mu) = \frac{|M_j|}{\Phi(x_j - 1)},$$

where $|M_j|$ denotes the determinant of the matrix $M_j$, and it is well-defined up to a factor $\pm t_1^{n_k} \cdots t_\mu^{n_k}$ ($k_i \in \mathbb{Z}$) if $n$ is odd and up to only $t_1^{n_k} \cdots t_\mu^{n_k}$ if $n$ is even (see [15] Section 5 for details).

Remark 2.1. In general, the twisted Alexander polynomial for finitely presentable groups is a rational function, but it is actually a polynomial for the link groups (see [15] Proposition 9 and [9] Theorem 3.1).
3. A Torres condition

In this section, we state and prove a generalized Torres condition for the twisted Alexander polynomial of links. An advantage of our description here is that we need not separate the case for \( \mu = 2 \) from \( \mu > 2 \). We first prove the theorem in the case of an \( SL(2;F) \)-representation. After reading the proof for it, one can easily show the similar result for general cases.

**Theorem 3.1.** Let \( L = L_1 \cup \cdots \cup L_\mu \) be a \( \mu \)-component link and \( L' = L_1 \cup \cdots \cup L_{\mu-1} \). For a given representation \( \rho' : G(L') \to SL(2;F) \), it holds that

\[
\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1) = \{ (t_1^{\mu} \cdots t_{\mu-1}^{\mu}) - \varepsilon_{\rho} t_1^{\mu} \cdots t_{\mu-1}^{\mu} + 1 \} \Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1}),
\]

where \( \rho : G(L) \to SL(2;F) \) is the composite of the natural surjection \( G(L) \to G(L') \) and \( \rho' \), \( l_i \) denotes the linking number of \( L_i \) and \( L_{\mu} \), and \( \varepsilon_{\rho'} \) is an element of \( F \).

**Proof.** For the link group \( G(L) \), we choose a Wirtinger presentation:

\[
G(L) = \langle x_{ij} \mid r_{kl} \rangle,
\]

where \( x_{i1}, x_{i2}, \ldots, x_{ii}, \ (1 \leq i \leq \mu) \) are generators corresponding to the component \( L_i \) and the relation

\[
r_{kl} = x_{k,l}^{\pm 1} x_{kl} x_{k',l} x_{k,l+1}^{-1}
\]

corresponds to a crossing of \( L_{k'} \) over \( L_k \). We should note that the link group \( G(L) \) has the deficiency one.

Let us consider the Alexander matrix of \( G(L) \) associated to the representation \( \rho : G(L) \to SL(2;F) \):

\[
M(t_1, \ldots, t_\mu) = \left( \Phi \frac{\partial r_{kl}}{\partial x_{ij}} \right).
\]

Then we know that if we remove the column corresponding to a generator \( x_{ij} \),

\[
|M_{ij}(t_1, \ldots, t_\mu)| = |\Phi(x_{ij} - 1)| |\Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1})|
\]

holds. Thus if we set \( t_\mu = 1 \) in \( M(t_1, \ldots, t_\mu) \), it follows that

\[
|M_{ij}(t_1, \ldots, t_{\mu-1}, 1)| = |\Phi(x_{ij} - 1)| |\Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1}, 1)|
\]

if \( i \neq \mu \).

Now the generators \( \{ x_{ij} \} \) appear in the following two kinds of relations:

(i) \( r_{ij} = x_{r,s}^{\pm 1} x_{j} x_{j+1}^{-1} \) and (ii) \( r_{pq} = x_{\mu}^{\pm 1} x_{pq} x_{pq+1}^{\pm 1} \),

where the relation (i) corresponds to crossings of \( L_r \) over \( L_\mu \) and (ii) corresponds to that of \( L_\mu \) over \( L_p \). Let us see which are the contributions of these relations to the matrix \( M(t_1, \ldots, t_{\mu-1}, 1) \). First the contributions of \( r_{ij} \) are as follows:

\[
\Phi \left( \frac{\partial r_{ij}}{\partial x_{rs}} \right) _{t_r = 1} = O,
\]

\[
\Phi \left( \frac{\partial r_{ij}}{\partial x_{ij}} \right) _{t_s = 1} = \begin{cases} t_r^{\pm 1} \rho(x_{rs})^{\pm 1} & \text{if } \mu \neq r \\ I & \text{if } \mu = r, \end{cases}
\]

\[
\Phi \left( \frac{\partial r_{ij}}{\partial x_{ij+1}} \right) _{t_s = 1} = -I,
\]

where \( O \) and \( I \) denote the zero and the identity matrix respectively. We have used here the fact that \( \rho(x_{ij}) = I \) for \( 1 \leq j \leq j_\mu \) (because the generators \( \{ x_{ij} \} \) are in
the kernel of the natural surjective homomorphism from $G(L)$ to $G(L')$). Next the contributions of $r_{pq}$ are as follows:

$$
\Phi \left( \frac{\partial r_{pq}}{\partial x_{\mu l}} \right)_{t_{\nu} = 1} = \pm (I - t_{p} \rho(x_{pq})),
$$

$$
\Phi \left( \frac{\partial r_{pq}}{\partial x_{pq}} \right)_{t_{\nu} = 1} = I,
$$

$$
\Phi \left( \frac{\partial r_{pq}}{\partial x_{p, q+1}} \right)_{t_{\nu} = 1} = -\rho(x_{pq}) \rho(x_{p, q+1})^{-1} \text{ if } p \neq \mu,
$$

and the case $p = \mu$ has already been considered. Therefore we see that the matrix $M(t_{1}, \ldots, t_{\mu-1}, 1)$ has the following form:

$$
M(t_{1}, \ldots, t_{\mu-1}, 1) = \begin{pmatrix} A & B \\ O & C \end{pmatrix},
$$

where

$$
A = \left( \Phi \left( \frac{\partial r_{kl}}{\partial x_{ij}} \right) \right)_{t_{\nu} = 1} (k, i \neq \mu), \quad B = \left( \Phi \left( \frac{\partial r_{kl}}{\partial x_{ij}} \right) \right)_{t_{\nu} = 1} (k \neq \mu, 1 \leq j \leq j_{\mu}),
$$

and

$$
C = \left( \Phi \left( \frac{\partial r_{kl}}{\partial x_{ij}} \right) \right)_{t_{\nu} = 1} (1 \leq j, l \leq j_{\mu})
$$

$$
= \begin{pmatrix}
\delta_{r_{i1}^*} & -I \\
\delta_{r_{i2}^*} \rho(x_{r_{1} s_1}) & -I \\
\delta_{r_{i3}^*} \rho(x_{r_{2} s_2}) & -I \\
\vdots & \ddots & -I \\
\delta_{r_{i\mu}^*} \rho(x_{r_{\mu} s_\mu}) & -I \\
\end{pmatrix}.
$$

In the submatrix $C$, there is an appearance of $t_{i}^\delta$ for each crossing of $L_i$ over $L_{\mu} (1 \leq i \leq \mu)$ and $\delta_i = 1$ or $-1$ according as $L_i$ crosses over $L_{\mu}$ from left to right or from right to left. Thereby, we obtain

$$
|C| = \prod_{i=1}^{j_{\mu}} |\rho(x_{r_{i}s_{i}})\delta_{r_{i}}|^{2} |\delta_{r_{i}}^{2} - \sum_{\sigma \in S} c_{\sigma}^1 \cdots c_{\sigma}^{j_{\mu}} \delta_{r_{1}}^{\delta_{r_{1}}^*} \cdots \delta_{r_{\mu}}^{\delta_{r_{\mu}}^*} + 1
$$

$$
= (t_{1}^1 \cdots t_{\mu-1}^1)^2 - \varepsilon_{\rho} t_{1}^1 \cdots t_{\mu-1}^1 + 1,
$$

where $c_{\sigma}^i \in F$ denotes a component of the matrix $\rho(x_{r_{i}s_{i}})\delta_{r_{i}}$ determined by a permutation $\sigma$ and $S$ is a subset of the symmetric group $S_{2\mu}$ consisting of permutations which choose just one component from each submatrix $\rho(x_{r_{i}s_{i}})\delta_{r_{i}}$. Furthermore the submatrix $A$ is equivalent to the Alexander matrix $M'(t_{1}, \ldots, t_{\mu-1})$ of $G(L')$ associated to the representation $\rho' : G(L') \to SL(2; F)$. Hence if we remove a column corresponding to a generator $x_{ij} (i \neq \mu)$, then we have

$$
|M_{ij}(t_{1}, \ldots, t_{\mu-1}, 1)| = |A_{ij}| |C|
$$

$$
= ((t_{1}^1 \cdots t_{\mu-1}^1)^2 - \varepsilon_{\rho} t_{1}^1 \cdots t_{\mu-1}^1 + 1) |M'_{ij}(t_{1}, \ldots, t_{\mu-1})|,
$$

where
where $A_{ij}$ is the matrix obtained from $A$ by removing the column corresponding to $x_{ij}$. Therefore, by definition of the twisted Alexander polynomial, we see that

$$\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1) = \{(t_1^1 \cdots t_{\mu-1}^1)^2 - \varepsilon_{\rho} t_1^1 \cdots t_{\mu-1}^1 + 1\} \Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1}).$$

This completes the proof of Theorem 3.1.

\[ \square \]

**Remark 3.2.** The fact that $\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1)$ is divisible by $\Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1})$ also follows from a recent result of Kitano, Suzuki and Wada in [10]. However, we can have no detailed information on the quotient from their result.

A linear representation $\rho : G(L) \to GL(n; F)$ is called reducible if it has a nontrivial invariant subspace in $F^n$. In this case, we can obtain a piece of information about the coefficient $\varepsilon_{\rho'}$.

**Corollary 3.3.** Under the setting as in Theorem 3.1, if $\rho' : G(L') \to SL(2; F)$ is a reducible representation, then we have

$$\varepsilon_{\rho'} = \lambda^l + \lambda^{-l} \quad (l = l_1 + \cdots + l_{\mu-1}),$$

where $\lambda$ is an eigenvalue of the image of a generator of $G(L')$.

**Proof.** First we can assume that the images of generators in a Wirtinger presentation of $G(L)$ have the following forms:

$$\rho(x_{ij}) = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & a_{ij}^{-1} \end{pmatrix} \quad (i \neq \mu) \quad \text{and} \quad \rho(x_{\mu j}) = I,$$

where $a_{ij} \in F^\times$ and $b_{ij} \in F$. Because the representation $\rho'$ has a 1-dimensional invariant subspace in $F^2$.

Since $x_{ij}x_{kl}^{-1} \ (i, k \neq \mu)$ is an element of the commutator subgroup $[G(L), G(L)]$, we see that $a_{ij} = a_{kl}$ holds for these generators. We then put $\lambda = a_{ij}$ for simplicity. Each lower left component of $\rho(x_{ij})$ is zero, so that the nontrivial terms appeared in the coefficient of $t_{l_1}^1 \cdots t_{l_{\mu-1}}^1$ are just

$$-\lambda^{\delta_{r_1} + \cdots + \delta_{r_{\mu}}} - \lambda^{-(\delta_{r_1} + \cdots + \delta_{r_{\mu})}} = -(\lambda^l + \lambda^{-l}),$$

where $l = \sum t_l$. This completes the proof. \[ \square \]

**Example 3.4.** Let $\rho' : G(L') \to SL(2; F)$ be a reducible representation of the knot $L' = L_1$. Then the twisted Alexander polynomial of $L'$ associated to $\rho'$ is given by

$$\Delta_{L',\rho'}(t_1) = \frac{\Delta_{L'}(\lambda t_1)\Delta_{L'}(\lambda^{-1} t_1)}{(t_1 - \lambda)(t_1 - \lambda^{-1})},$$

where $\Delta_{L'}(t_1)$ is the original Alexander polynomial of $L'$ (see the proof of [9] Theorem 3.1 for instance). Hence we have

$$\Delta_{L,\rho}(1, 1) = \frac{(1 - \lambda^t)(1 - \lambda^{-1}t)}{(1 - \lambda)(1 - \lambda^{-1})} \Delta_{L'}(\lambda)\Delta_{L'}(\lambda^{-1})$$

$$= (1 + \lambda + \cdots + \lambda^{t-1})(1 + \lambda^{-1} + \cdots + \lambda^{-(t-1)}) \Delta_{L'}(\lambda)\Delta_{L'}(\lambda^{-1}).$$

In particular, if $\rho'$ is trivial (namely, $\lambda = 1$), then we obtain $\Delta_{L,\rho}(1, 1) = l_1^2$ (because $\Delta_{L'}(1) = \pm 1$).
Example 3.5. Let \( \rho' : G(L') \to SL(2; F) \) be the trivial representation. In this case \( \varepsilon_{\rho'} = 2 \) holds, so that we have
\[
\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1) = (t_1^{1} \cdots t_{\mu-1}^{n-1} - 1)^2 \Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1}).
\]
This formula corresponds to the square of Torres' original formula in Theorem 1.1. In particular, \( \Delta_{L,\rho}(1, \ldots, 1) = 0 \) holds for \( \mu > 2 \).

If we slightly modify the proof of Theorem 3.1, we obtain the following general formula for a unimodular representation \( \rho' : G(L') \to SL(n; F) \). We omit here the repetitious proof.

**Theorem 3.6.** Let \( L = L_1 \cup \cdots \cup L_\mu \) be a \( \mu \)-component link and \( L' = L_1 \cup \cdots \cup L_{\mu-1} \). For a given representation \( \rho' : G(L') \to SL(n; F) \), it holds that
\[
\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1) = \{(t_1^1 \cdots t_{\mu-1}^{n-1})^n + \sum_{k=1}^{n-1} \varepsilon_{k,\rho'}(t_1^1 \cdots t_{\mu-1}^{n-1})^n - k(-1)^n \} \times \Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1}),
\]
where \( \rho : G(L) \to SL(n; F) \) is the composite of the natural surjection \( G(L) \to G(L') \) and \( \rho' \), \( l_i \) denotes the linking number of \( L_i \) and \( L_{\mu} \), and \( \varepsilon_{k,\rho'} \) (\( 1 \leq k \leq n - 1 \)) are elements of \( F \).

Finally, we extend Corollary 3.3 when all the images of the representation \( \rho' : G(L') \to SL(n; F) \) are upper triangle matrices.

**Corollary 3.7.** Under the setting as in Theorem 3.6, if \( \text{Im}(\rho') \) are upper triangle matrices, then the coefficient \( \varepsilon_{k,\rho'} \) is given by
\[
\varepsilon_{k,\rho'} = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} (\lambda_{i_1} \cdots \hat{\lambda}_{i_k} \cdots \lambda_{i_n})^k,
\]
where \( \lambda_k \) (\( 1 \leq k \leq n \)) are the eigenvalues of the image of a generator of \( G(L') \) and \( \hat{\lambda}_k \) implies that \( \lambda_k \) is removed from the product.

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