On the capitulation problem of some pure metacyclic fields
of degree 20 II

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Abstract

Let \( n \) be a 5\(^{th} \) power-free natural number and \( k_0 = \mathbb{Q}(\zeta_5) \) be the cyclotomic field generated by a primitive 5\(^{th} \) root of unity \( \zeta_5 \). Then \( k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5) \) is a pure metacyclic field of absolute degree 20. In the case that \( k \) possesses a 5-class group \( C_{k,5} \) of type \((5,5)\) and all the classes are ambiguous under the action of \( \text{Gal}(k/k_0) \), the capitulation of 5-ideal classes of \( k \) in its unramified cyclic quintic extensions is determined.

Key words: pure metacyclic fields, 5-class groups, Hilbert 5-class field, Capitulation.

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1 Introduction

Let \( k \) be a number field, and \( L \) be an unramified abelian extension of \( k \). We say that an ideal \( I \) of \( k \) or its class capitulates in \( L \) if \( I \) or its prime factors capitate in \( L \).

Let \( \Gamma = \mathbb{Q}(\sqrt[5]{n}) \) be a pure quintic field, where \( n \) is a 5\(^{th} \) power free natural number and \( k_0 = \mathbb{Q}(\zeta_5) \) be the cyclotomic field generated by a primitive 5\(^{th} \) root of unity \( \zeta_5 \). Then \( k = \Gamma(\zeta_5) \) is the normal closure of \( \Gamma \). Let \( k_5^{(1)} \) be the Hilbert 5-class field of \( k \), \( C_{k,5} \) be the 5-ideal classes group of \( k \), and \( C_{k,5}^{(\sigma)} \)
be the group of ambiguous ideal classes under the action of $Gal(k/k_0) = \langle \sigma \rangle$. In the case that $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, the capitulation of the 5-ideal classes of $k$ in the six intermediate extensions of $k_5^{(1)}/k$ is determined in [2].

In this paper, we investigate the capitulation of 5-ideal classes of $k$ in the unramified cyclic quintic extensions of $k_5^{(1)}/k$, whenever $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 2$, which mean that all classes are ambiguous. Let $p$ and $q$ primes such that $p \equiv 1 \pmod{5}$, $q \equiv \pm 2 \pmod{5}$. According to [1, theorem 1.1], if $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 2$, we have three forms of the radicand $n$ as follows:

- $n = 5^ep \not\equiv \pm 1 \pm 7 \pmod{25}$ with $e \in \{1, 2, 3, 4\}$ and $p \not\equiv 1 \pmod{25}$.

- $n = p^e q \equiv \pm 1 \pm 7 \pmod{25}$ with $e \in \{1, 2, 3, 4\}$, $p \not\equiv 1 \pmod{25}$ and $q \not\equiv \pm 7 \pmod{25}$.

- $n = p^e \equiv \pm 1 \pm 7 \pmod{25}$ with $e \in \{1, 2, 3, 4\}$ and $p \equiv 1 \pmod{25}$.

We will study the capitulation of $C_{k,5}$ in the six intermediate extensions of $k_5^{(1)}/k$ in these cases. The theoretical results are underpinned by numerical examples obtained with the computational number theory system PARI/GP [16].

**Notations.**

Throughout this paper, we use the following notations:

- The lower case letter $p$ and $q$ denote a prime numbers such that, $p \equiv 1 \pmod{5}$ and $q \equiv \pm 2 \pmod{5}$.

- $\Gamma = \mathbb{Q}(\sqrt[5]{n})$: a pure quintic field, where $n \not\equiv 1$ is a 5th power-free natural number.

- $k_0 = \mathbb{Q}(\zeta_5)$: the cyclotomic field, where $\zeta_5 = e^{2\pi i/5}$ is a primitive 5th root of unity.

- $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$: the normal closure of $\Gamma$, a quintic Kummer extension of $k_0$.

- $\langle \tau \rangle = Gal(k/\Gamma)$ such that $\tau$ is identity on $\Gamma$, and sends $\zeta_5$ to its square. Hence $\tau$ has order 4.

- $\langle \sigma \rangle = Gal(k/k_0)$ such that $\sigma$ is identity on $k_0$, and sends $\sqrt[5]{n}$ to $\zeta_5\sqrt[5]{n}$. Hence $\sigma$ has order 5.

- For a number field $L$, denote by:
  - $\mathcal{O}_L$: the ring of integers of $L$;
  - $C_L, h_L, C_{L,5}$: the class group, class number, and 5-class group of $L$.
  - $L_5^{(1)}, L^*$: the Hilbert 5-class field of $L$, and the absolute genus field of $L$.
  - $[\mathcal{I}]$: the class of a fractional ideal $\mathcal{I}$ in the class group of $L$.

- $(\frac{a}{b})_5 = 1 \iff X^5 \equiv a \pmod{b}$ resolved on $\mathcal{O}_{k_0}$, where $a, b$ are primes in $\mathcal{O}_{k_0}$.
2 Preliminaries

2.1 Decomposition laws in Kummer extension

Since the extensions of $k$ and $k_0$ are all Kummer’s extensions, we recall the decomposition laws of ideals in these extensions.

**Proposition 2.1.** Let $L$ a number field contains the $l^{th}$ root of unity, where $l$ is prime, and $\theta$ element of $L$, such that $\theta \neq \mu^l$, for all $\mu \in L$, therefore $L(\sqrt[l]{\theta})$ is cyclic extension of degree $l$ over $L$. We note by $\zeta$ a $l^{th}$ primitive root of unity.

1. We assume that a prime $P$ of $L$, divides exactly $\theta$ to the power $P^a$.
   - If $a = 0$ and $P$ don’t divides $l$, then $P$ split completely in $L(\sqrt[l]{\theta})$ when the congruence $\theta \equiv X^l \pmod{P}$ has solution in $L$.
   - If $a = 0$ and $P$ don’t divides $l$, then $P$ is inert in $L(\sqrt[l]{\theta})$ when the congruence $\theta \equiv X^l \pmod{P}$ has no solution in $L$.
   - If $l \nmid a$, then $P$ is totally ramified in $L(\sqrt[l]{\theta})$.

2. Let $B$ a prime factor of $1 - \zeta$ that divides $1 - \zeta$ exactly to the $a^{th}$ power. Suppose that $B \nmid \theta$, then $B$ split completely in $L(\sqrt[l]{\theta})$ if the congruence

$$\theta \equiv X^l \pmod{B^{a+1}} \quad (*)$$

has solution in $L$. the ideal $B$ is inert in $L(\sqrt[l]{\theta})$ if the congruence

$$\theta \equiv X^l \pmod{B^{a}} \quad (**)$$

Figure 1: The unramified quintic sub-extensions of $k^{(1)}_5/k$
has solution in $L$, without (*) has. The ideal $\mathcal{B}$ is totaly ramified in $L$ if the congruence (**) has no solution.

Proof. see [3].

2.2 Relative genus field $(k/k_0)^*$ of $k$ over $k_0$

Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, $k_0 = \mathbb{Q}(\zeta_5)$ the $5^{th}$-cyclotomic field and $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma$. The relative genus field $(k/k_0)^*$ of $k$ over $k_0$ is the maximal abelian extension of $k_0$ which is contained in the Hilbert 5-class field $k_5^{(1)}$ of $k$. Let $q^* \in \{0, 1, 2\}$ such that

$$q^* = \begin{cases} 
2 & \text{if } \zeta_5, \zeta_5 + 1 \text{ are norm of element in } k - \{0\}, \\
1 & \text{if } \zeta_5^i(\zeta_5 + 1)^j \text{ is the norm of an element in } k - \{0\} \text{ for some exponents i and j.} \\
0 & \text{if for no exponents } i, j \text{ the element } \zeta_5^i(\zeta_5 + 1)^j \text{ is a norm from } k - \{0\}. 
\end{cases}$$

Proposition 2.2. Let $k = k_0(\sqrt[5]{n})$ such that $n = \mu \lambda^{e_1} \pi_1^{e_1} \ldots \pi_f^{e_f} \pi_{f+1}^{e_{f+1}} \ldots \pi_g^{e_g}$ in $k_0$, where $\mu$ is unity of $O_{k_0}$, $\lambda = 1 - \zeta_5$ the unique prime above 5 in $k_0$ and each prime $\pi_i \equiv \pm 1, \pm 7 \pmod{\lambda^5}$ for $1 \leq i \leq f$ and $\pi_j \not\equiv \pm 1, \pm 7 \pmod{\lambda^5}$ for $f + 1 \leq j \leq g$. Then we have:

(i) there exists $h_i \in \{1, \ldots, 4\}$ such that $\pi_{f+1} \pi_i^{h_i} \equiv \pm 1, \pm 7 \pmod{\lambda^5}$, for $f + 2 \leq i \leq g$.

(ii) if $n \not\equiv \pm 1 \pm 7 \pmod{\lambda^5}$ and $q^* = 1$, then the genus field $(k/k_0)^*$ is given as:

$$(k/k_0)^* = k(\sqrt[5]{\pi_1}, \ldots, \sqrt[5]{\pi_f}, \sqrt[5]{\pi_{f+1} \pi_{f+2}^{h_{f+2}}}, \ldots, \sqrt[5]{\pi_{f+1} \pi_g^{h_g}})$$

where $h_i$ is chosen as in (i).

(iii) in the other cases of $q^*$ and the congruence of $n$, the genus field $(k/k_0)^*$ is given by deleting an appropriate number of $5^{th}$ root from the right side of (ii).

Proof. see [3] proposition 5.8.

3 Study of capitulation

This being the case, let $\Gamma$, $k_0$ and $k$ as above. If $C_{k,5}$ is of type $(5, 5)$ and the group of ambiguous classes $C_{k,5}^{(s)}$ under the action of $\text{Gal}(k/k_0) = \langle \sigma \rangle$ has rank 2, we have $C_{k,5} = C_{k,5}^{(s)}$. By class field theory $C_{k,5}^{-1-\sigma}$ correspond to $(k/k_0)^*$, and since $C_{k,5} = C_{k,5}^{(s)}$ we get that $C_{k,5}^{-1-\sigma} = \{1\}$, hence $(k/k_0)^* = k_5^{(1)}$ is the Hilbert 5-class field of $k$.

When $C_{k,5}$ is of type $(5, 5)$, it has 6 subgroups of order 5, denoted $H_i$, $1 \leq i \leq 6$. Let $K_i$ be the intermediate extension of $k_5^{(1)}/k$, corresponding by class field theory to $H_i$. Its easy to see that $C_{k,5} \cong C_{k,5}^+ \times C_{k,5}^-$ such that $C_{k,5}^+ = \{A \in C_{k,5} \mid A^2 = A\}$ and $C_{k,5}^- = \{X \in C_{k,5} \mid X^{2} = X^{-1}\}$ with $\text{Gal}(k/\Gamma) = \langle \tau \rangle$. As each $K_i$ is cyclic of order 5 over $k$, there is at least one subgroup of order 5 of
When we have corresponding relations for the subgroups $\tau$ - Since $K$ and since
Throught the paper we order the subgroups $H_i$ of $C_{k,5}$ as follows:

$H_1 = C_{k,5}^+ = \langle A \rangle$, $H_6 = C_{k,5}^- = \langle \chi \rangle$, $H_2 = \langle AX \rangle$, $H_3 = \langle AX^2 \rangle$, $H_4 = \langle AX^3 \rangle$ and $H_5 = \langle AX^4 \rangle$.

By class field theory we have $H_6$ correspond to $K_6 = k\Gamma_5^{(1)}$, with $\Gamma_5^{(1)}$ is the Hilbert 5-class field of $\Gamma$.

By the action of $Gal(k/Q)$ on $C_{k,5}$, we can give the following:

**Proposition 3.1.** For all continuations of the automorphisms $\sigma$ and $\tau$ we have:

1. $K_i^\sigma = K_i$ ($i = 1, 2, 3, 4, 5, 6$) i.e $\sigma$ sets all $K_i$
2. $K_i^{\tau^2} = K_1$, $K_6^{\tau^2} = K_6$, $K_2^{\tau^2} = K_5$ and $K_3^{\tau^2} = K_4$. i.e $\tau^2$ sets $K_1$, $K_6$ and permutes $K_2$ with $K_5$ and $K_3$ with $K_4$.

**Proof.** We will agree that for all $1 \leq i \leq 6$, and for all $w \in Gal(k/Q)$ we have $H_i^w = \{C^w | C \in H_i\}$.

1. Since all classes are ambiguous because $C_{k,5} = C_{k,5}^{(\sigma)}$, then $\sigma$ sets all $H_i$.
2. We have $H_1 = C_{k,5}^+ = \langle A \rangle$ and $H_6 = C_{k,5}^- = \langle \chi \rangle$, then $H_1^{\tau^2} = H_1$ and $H_6^{\tau^2} = H_6$.
   - Since $(AX)^{\tau^2} = AX^{\tau^2} = AX^{-1} = AX^3 \in H_5$ then $H_2^{\tau^2} = H_5$.
   - Since $(AX^2)^{\tau^2} = AX^{\tau^2}(X^2)^{\tau^2} = AX^{\tau^2} = AX^3 \in H_4$ then $H_3^{\tau^2} = H_4$.
   - Since $\tau^4 = 1$ we get that $H_5^{\tau^2} = H_2$ and $H_4^{\tau^2} = H_3$.

The relations between the fields $K_i$ in (1) and (2) are nothing else than the translations of the corresponding relations for the subgroups $H_i$ via class field theory. \( \square \)

To study the capitulation problem of $k$ whenever $C_{k,5}$ is of type $(5,5)$ and $C_{k,5} = C_{k,5}^{(\sigma)}$, we will investigate the three forms of the radicand $n$ proved in [1] theorem 1.1] and mentioned above.

### 3.1 The case $n = p^e \equiv \pm1 \pm 7 \pmod{25}$, where $p \equiv 1 \pmod{25}$

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = p^e$ such that $p \equiv 1 \pmod{25}$ and $e \in \{1, 2, 3, 4\}$. Since $p \equiv 1 \pmod{5}$ we have that $p$ splits completely in $k_0 = \mathbb{Q}(\zeta_5)$ as $p = \pi_1\pi_2\pi_3\pi_4$, with $\pi_i$ are primes in $k_0$ such that $\pi_i \equiv 1 \pmod{5\mathcal{O}_{k_0}}$, then the primes of $k_0$ ramified in $k$ are $\pi_i$.

If $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ are respectively the prime ideals of $k$ above $\pi_1, \pi_2, \pi_3$ and $\pi_4$, then $\mathcal{P}_5 = \pi_i\mathcal{O}_{k}(i = 1, 2, 3, 4)$ and since $\tau$ acte transitively on $\pi_i$, we have that $\tau^2$ permutes $\pi_1$ with $\pi_3$, hence $\tau^2$ permutes $\mathcal{P}_1$ with $\mathcal{P}_3$. Since $\pi_i^\tau = \pi_i$, we have $\mathcal{P}_i^\tau = \mathcal{P}_i$. In fact $[\mathcal{P}_i](i = 1, 2, 3, 4)$ generate the group of strong ambiguous ideal classes denoted $C_{k,s}^{(\sigma)}$. The next theorem allow us to determine explicitly the intermediate extensions of $k_5^{(1)}/k$.
Theorem 3.1. Let $k$ and $n$ as above. Let $\pi_1, \pi_2, \pi_3$ and $\pi_4$ a primes of $k_0$ congrus to 1 modulo $\lambda^5$ such that $p = \pi_1\pi_2\pi_3\pi_4$, then:

1. $k_5^{(1)} = k(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3})$.

2. The six intermediate extensions of $k_5^{(1)}/k$ are: $k(\sqrt[5]{\pi_1}, k(\sqrt[5]{\pi_3}), k(\sqrt[5]{\pi_1\pi_3}), k(\sqrt[5]{\pi_1\pi_3^2})$, and $k(\sqrt[5]{\pi_1\pi_3^3})$. Furthermore $\tau^2$ permutes $k(\sqrt[5]{\pi_1})$ with $k(\sqrt[5]{\pi_3})$, and $k(\sqrt[5]{\pi_1\pi_3})$ with $k(\sqrt[5]{\pi_1\pi_3^2})$, and sets $k(\sqrt[5]{\pi_1\pi_3^3})$, $k(\sqrt[5]{\pi_1\pi_3^4})$.

Proof.

(1) We have that $k_5^{(1)} = (k/k_0)^*$. Since $k = k_0(\sqrt[5]{n})$ with $n = p = \pi_1\pi_2\pi_3\pi_4$ in $k_0$ and $\pi_i \equiv 1 \pmod{\lambda^5}$ $(i = 1, 2, 3, 4)$, then by proposition 2.2 we have $(k/k_0)^* = k(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3})$.

(2) If $k_5^{(1)} = k(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3})$, then the six intermediate extensions are: $k(\sqrt[5]{\pi_1}), k(\sqrt[5]{\pi_3}), k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3^2})$, $k(\sqrt[5]{\pi_1\pi_3^3})$, and $k(\sqrt[5]{\pi_1\pi_3^4})$. We have $\tau^2(\pi_1) = \pi_3$ then its easy to see that $\tau^2$ sets the fields $k(\sqrt[5]{\pi_1\pi_3}), k(\sqrt[5]{\pi_1\pi_3^2}), k(\sqrt[5]{\pi_1\pi_3^3})$. Since $\tau^2(\pi_1) = \tau^2(\sqrt[5]{\pi_1}) = (\tau^2(\sqrt[5]{\pi_1}))^5 = \pi_3$, then $\tau^2(\sqrt[5]{\pi_1})$ is 5th root of $\pi_3$. Hence $k(\sqrt[5]{\pi_3}) = k(\tau^2(\sqrt[5]{\pi_1}))$ i.e $k(\sqrt[5]{\pi_3}) = k(\sqrt[5]{\pi_1})^{\tau^2}$. By the same reasoning we prove that $k(\sqrt[5]{\pi_1}) = k(\sqrt[5]{\pi_1})^{\tau^2}$, Hence $\tau^2$ permutes $k(\sqrt[5]{\pi_1})$ with $k(\sqrt[5]{\pi_1})$.

We have $\tau^2(\pi_1\pi_3^3) = \pi_1^2\pi_3$, then $\tau^2(\pi_1\pi_3^3) = \tau^2(\sqrt[5]{\pi_1\pi_3^3}) = \tau^2(\sqrt[5]{\pi_1\pi_3^3})^5 = \pi_3^2$, hence $\tau^2(\sqrt[5]{\pi_1\pi_3^3})$ is 5th root of $\pi_3$. Then $k(\sqrt[5]{\pi_1\pi_3}) = k(\tau^2(\sqrt[5]{\pi_1\pi_3}))$ i.e $k(\sqrt[5]{\pi_1\pi_3}) = k(\sqrt[5]{\pi_1\pi_3^3}) = k(\sqrt[5]{\pi_1\pi_3^4})$. By the same reasoning we prove that $k(\sqrt[5]{\pi_1\pi_3^3}) = k(\sqrt[5]{\pi_1\pi_3^4})^{\tau^2}$. Hence $\tau^2$ permutes $k(\sqrt[5]{\pi_1\pi_3^3})$ with $k(\sqrt[5]{\pi_1\pi_3^4})$.

The generators of $C_{k,5}$ when its of type $(5,5)$ and the radicand n is as above are determined as follows:

**Theorem 3.2.** Let $k$ and $n$ as above. Let $\pi_1, \pi_2, \pi_3$ and $\pi_4$ a primes of $k_0$ congrus to 1 (mod $\lambda^5$) such that $n = p = \pi_1\pi_2\pi_3\pi_4$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ prime ideals of k such that $\mathcal{P}_1^5 = \pi_i\mathcal{O}_{k_0}$ $(i = 1, 2, 3, 4)$.

Then:

$$C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3], [\mathcal{P}_1\mathcal{P}_4^4] \rangle$$

Proof. According to [1] theorem 1.1, for that case of the radicand n, we have that $\zeta_5^{(1)} = \sqrt[5]{\zeta_5}$ is norm of element in $k - \{0\}$. By [2] section 5.3, if $\zeta_5$ is not norm of unit of k we have $C_{k,5} = C_{k,5}^{(1)} \neq C_{k,5}^{(s)}$, so $C_{k,5}$ contained in $C_{k,5}^{(s)}$. Hence we discuss two cases:

- 1st case: $C_{k,5} = C_{k,5}^{(s)} \neq C_{k,5}^{(1)}$: In this case, $C_{k,5}^{(s)}$ is contained in $C_{k,5} = C_{k,5}^{(1)}$, and by [2] section 5.3 we have $C_{k,5}^{(1)}/C_{k,5}^{(s)} = C_{k,5}/C_{k,5}^{(s)}$ is cyclic group of order 5. Since $C_{k,5}$ has order 25 then $C_{k,5}^{(s)}$ is cyclic of order 5. We have that $C_{k,5}^{(s)} = \langle [\mathcal{P}_1], [\mathcal{P}_2], [\mathcal{P}_3], [\mathcal{P}_4] \rangle$, $\mathcal{P}_1^5 = \mathcal{P}_3^5 = \mathcal{P}_4^5 = \mathcal{P}_3^5$, and $\mathcal{P}_2^5 = \mathcal{P}_4^5$, so $\mathcal{P}_1$ and $\mathcal{P}_2$ can not be both principals in k, otherwise $\mathcal{P}_3 = \mathcal{P}_1^5$ and $\mathcal{P}_4 = \mathcal{P}_2^5$ will be principals too, hence $C_{k,5}^{(s)} = \{1\}$, which is impossible. by the same reasoning we have that $\mathcal{P}_3$ and $\mathcal{P}_4$ can not be both principals in k. Since $C_{k,5}^{(s)}$ is cyclic of order 5 and without loosing generality we get that $C_{k,5}^{(s)} = \langle [\mathcal{P}_1] \rangle$, so $\mathcal{P}_1$ and $\mathcal{P}_3 = \mathcal{P}_1^5$ are not principals. Since $C_{k,5} \cong C_{k,5}^+ \times C_{k,5}^-$ its sufficient to
find generators of $C_{k,5}^+$ and $C_{k,5}^-$. As $[P_1P_3]^r = [(P_1P_3)^r] = [P_1P_3] \text{ then } C_{k,5}^+ = \langle [P_1P_3] \rangle$, and $[P_1P_3]^r = [(P_1P_3)^r] = [P_1P_3] = [P_1P_3]^{−1}$ then $C_{k,5}^- = \langle [P_1P_3] \rangle$. Hence $C_{k,5} = \langle [P_1P_3], [P_1P_3]^r \rangle$.

- 2nd case: $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,\pi}^{(\sigma)}$: We admit the same reasoning of 1st case because none of $P_i$ ($i = 1, 2, 3, 4$) is principal, otherwise $C_{k,5} = C_{k,\pi}^{(\sigma)} = \{1\}$, which is impossible. Hence $C_{k,5} = \langle [P_1P_3], [P_1P_3]^r \rangle$.

Now we are able to state the main theorem of capitulation in this case.

**Theorem 3.3.** We keep the same assumptions as theorem 3.2 Then:

1. If $(\frac{K}{\pi})_5 = 1$ then $K_1 = k(\sqrt[5]{\pi_1\pi_3})$ and $K_2 = k(\sqrt[5]{\pi_1\pi_3})$, $K_3 = k(\sqrt[5]{\pi_1\pi_3})$ and $K_4 = k(\sqrt[5]{\pi_1\pi_3})$. $K_5 = k(\sqrt{\pi_1\pi_3})$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3})$ or $k(\sqrt[5]{\pi_1\pi_3})$. Otherwise we just permute $K_2$ and $K_5$.

2. $[P_1P_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$, $[P_1P_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$ $(i = 1, 3)$, $[P_1P_3]^r$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$, $[P_1P_3]^r$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$.

3. (i) If $(\frac{K}{\pi})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3})$ then the possible types of capitulation are: $(0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0), (0, 2, 0, 5, 0), (1, 2, 0, 5, 0), ((0, 0, 3, 4, 0, 0) or (0, 0, 3, 0, 0)), ((0, 1, 0, 4, 0, 0) or (1, 0, 4, 3, 0, 0)), ((0, 0, 2, 4, 5, 0) or (0, 2, 4, 3, 5, 0)), ((1, 2, 4, 5, 0) or (1, 2, 3, 5, 0))$.

(ii) If $(\frac{K}{\pi})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3})$ then the same possible types of capitulation occur as in (i) with $i_6 = 0$ or $1$ and $i_1 = 0$ or $6$.

(iii) If $(\frac{K}{\pi})_5 \neq 1$ then the same possible types of capitulation occur as (i) and (ii) by permuting 2 and 5.

**Proof.**

1. According to theorem 3.1 we have that $\tau^2$ permutes $k(\sqrt{\pi_1\pi_3})$ with $k(\sqrt[5]{\pi_1\pi_3})$ and $k(\sqrt[5]{\pi_1\pi_3})$ with $k(\sqrt[5]{\pi_1\pi_3})$, and sets $k(\sqrt[5]{\pi_1\pi_3})$, $k(\sqrt[5]{\pi_1\pi_3})$. By class field theory $K_1$ correspond to $H_1$ ($i = 1, 2, 3, 4, 5, 6$), for that we determine explicitly the six subgroups $H_i$ of $C_{k,5}$ as follows.

We have that $C_{k,5} = \langle A, X \rangle$, where $H_1 = C_{k,5}^+ = \langle A \rangle$ and $H_6 = C_{k,5}^- = \langle X \rangle$. By theorem 3.2 we have $A = [P_1P_3]$ and $X = [P_1P_3]^r$, then $AX = [P_1P_3]^2$, $AX^2 = [P_1P_3]^3$, $AX^3 = [P_1P_3]^4$ and $AX^4 = [P_3]^4$. Hence $H_2 = \langle [P_1P_3] \rangle$, $H_3 = \langle [P_1P_3]^2 \rangle$, $H_4 = \langle [P_1P_3]^3 \rangle$, $H_5 = \langle [P_1P_3]^4 \rangle$. Since $\tau^2$ sets $k(\sqrt[5]{\pi_1\pi_3})$ and $k(\sqrt[5]{\pi_1\pi_3})$, if $K_1 = k(\sqrt[5]{\pi_1\pi_3})$ then $K_6 = k(\sqrt[5]{\pi_1\pi_3})$ and vise versa. If $(\frac{K}{\pi})_5 = 1$ then $X^5 \equiv \pi_1 \pmod{\pi_3}$ resolved on $\mathcal{O}_{k_0}$ and by proposition 2.1 we have that $\pi_1$ splits completely in $k_0(\sqrt{\pi_3})$, which equivalent to say that $P_1$ splits completely in $k(\sqrt{\pi_3})$, hence $K_2 = k(\sqrt{\pi_3})$ and we get that $K_5 = k(\sqrt{\pi_1\pi_3})$ and if $K_3 = k(\sqrt[5]{\pi_1\pi_3})$ then $K_4 = k(\sqrt[5]{\pi_1\pi_3})$ and vise versa. Since $\pi_1$ and $\pi_3$ divide $\pi_1\pi_3, \pi_1\pi_3^2, \pi_1\pi_3^3$ and $\pi_1\pi_3^4$, if $(\frac{\pi_1}{\pi_3})_5 \neq 1$ then $K_2 = k(\sqrt{\pi_1\pi_3})$ and $K_5 = k(\sqrt{\pi_3})$.

2. Since $P_i^5 = \pi_1\pi_3^{(i)}O_k \ i = 1, 3$ we have $(P_1P_3)^5 = \pi_1\pi_3O_k$, then $(P_1P_3)^5$ $= \pi_1\pi_3O_k(\sqrt{\pi_1\pi_3})$ in $k(\sqrt{\pi_1\pi_3})$ and $\pi_1\pi_3O_k(\sqrt{\pi_1\pi_3}) = (\sqrt{\pi_1\pi_3}O_k(\sqrt{\pi_1\pi_3}))^5$, hence $P_1P_3O_k(\sqrt{\pi_1\pi_3}) = \sqrt{\pi_1\pi_3}O_k(\sqrt{\pi_1\pi_3})$.  

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Thus $P_1P_3$ seen in $O_{k(\sqrt[5]{\pi_1\pi_3})}$ becomes principal, i.e. $[P_1P_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$.

- Since $(P_1P_3)^5 = \pi_1\pi_3^2O_k$, we have $(P_1P_3)^5 = \pi_1\pi_3^2O_{k(\sqrt[5]{\pi_1\pi_3})}$ in $k(\sqrt[5]{\pi_1\pi_3})$ and $\pi_1\pi_3^2O_{k(\sqrt[5]{\pi_1\pi_3})} = (\sqrt[5]{\pi_1\pi_3})^5$, hence $P_1P_3^2O_{k(\sqrt[5]{\pi_1\pi_3})} = \sqrt[5]{\pi_1\pi_3}O_{k(\sqrt[5]{\pi_1\pi_3})}$. Thus $P_1P_3$ seen in $O_{k(\sqrt[5]{\pi_1\pi_3})}$ becomes principal, i.e. $[P_1P_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$. By the same reasoning we have $[P_1P_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$ and $[P_1P_3]$ capitulates in $k(\sqrt[5]{\pi_1\pi_3})$.

- We have $P_1 = \pi_1O_k$, then $P_1O_{k(\sqrt[5]{\pi_1\pi_3})} = \sqrt[5]{\pi_1}O_{k(\sqrt[5]{\pi_1\pi_3})}$. Hence $[P_1]$ capitulates in $k(\sqrt[5]{\pi_1})$. By the same reasoning we have $[P_3]$ capitulates in $k(\sqrt[5]{\pi_3})$.

(iii) If $(\frac{4}{3})_5 = 1$ and $K_6 = k(\sqrt[5]{\pi_1\pi_3})$ we have $[P_1P_3]$ capitulates in $K_6$ then if $i_6 \neq 0$ we have $i_6 = 1$. $[P_1P_3]$ capitulates in $K_1$ then if $i_1 \neq 0$ we have $i_1 = 6$, so the same possible types of capitulation occur as in (i) with $i_6 = 0$ or 1 and $i_1 = 0$ or 6.

(iii) If $(\frac{4}{3})_5 \neq 1$, by (1) we have $K_2 = k(\sqrt[5]{\pi_3})$ and $K_5 = k(\sqrt[5]{\pi_1})$ then the same possible types of capitulation occur as (i) and (ii) by permuting 2 and 5.

3.2 The case $n = p^e q \equiv \pm 1 \pm 7 \pmod{25}$, where $p \neq 1 \pmod{25}$, $q \neq \pm 7 \pmod{25}$

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = Q(\sqrt[5]{\pi})$, where $n = p^e q \equiv \pm 1 \pm 7 \pmod{25}$ such that $p \neq 1 \pmod{25}, q \neq \pm 7 \pmod{25}$ and $e \in \{1, 2, 3, 4\}$.

Since $q \equiv \pm 2 \pmod{25}$ we have that $q$ is inert in $k_0 = Q(\zeta_5)$, so we can take in the following $q = \pi_5$ a prime in $k_0$. As before, by $P_1, P_2, P_3, P_4$ and $P_5$ we denote respectively the prime ideals of $k$ above $\pi_1, \pi_2, \pi_3, \pi_4$ and $\pi_5$ in $k_0$, such that $P_i^5 = \pi_iO_k$ $(i = 1, 2, 3, 4, 5)$. We have that $\tau^2$ permutes $\pi_1$ with $\pi_3$, then $\tau^2$ permutes $P_1$ with $P_3$, but $\tau^2$ sets $q = \pi_5$ and also $P_5$.

The six intermediate extensions of $k_5^{(1)}/k$ are determined as follows:

**Theorem 3.4.** Let $k, n, \pi_1, \pi_2, \pi_3, \pi_4$ and $\pi_5$ as above. Put $x_1 = \pi_1\pi_5^{h_1}$ and $x_2 = \pi_1\pi_3^{h_2}$ are chosen such that $x_1 \equiv x_2 \equiv 1 \pmod{\lambda^5}$, where $h_1 \in \{1, 2, 3, 4\}$. Then:
(1) \( k_5^{(1)} = k(√x_1, √x_2) \).

(2) The six intermediate extensions of \( k_5^{(1)}/k \) are: \( k(√x_1), k(√x_2), k(√\pi_1\pi_3\pi_5^{2h_1}), k(√\pi_1^2\pi_3\pi_5^{h_1}), k(√\pi_1^4\pi_3^{2h_1}) \) and \( k(√\pi_3\pi_5^{h_1}) \). Furthermore \( τ^2 \) permutes \( k(√\pi_1\pi_3\pi_5^{2h_1}) \) with \( k(√\pi_1^4\pi_3^{2h_1}) \) and \( k(√x_1) \) with \( k(√\pi_3\pi_5^{h_1}) \), and sets \( k(√x_2), k(√\pi_1\pi_3\pi_5^{2h_1}) \).

Proof. Since \( k = k_0(√n) \) we can write \( n \) in \( k_0 \) as \( n = \pi_1^ep_2^e\pi_3^e\pi_4^e\pi_5 \) with \( \pi_i \not\equiv 1 (mod \lambda^5) \) because \( p \not\equiv 1 (mod 25) \) and \( q \not\equiv 1 (mod 25) \). By proposition 2.2 there exists \( h_1, h_2 \in \{1, 2, 3, 4\} \) such that \( \pi_1\pi_5^{h_1} \equiv ±1, ±7 (mod \lambda^5) \) and \( \pi_1\pi_3^{h_2} \equiv ±1, ±7 (mod \lambda^5) \). To investigate the correspondence between the six intermediate extension of \( k_5^{(1)}/k \) and the six subgroups of \( C_{k,5} \), we assume that \( h_2 = 4 \). Put \( x_1 = \pi_1\pi_5^{h_1} \) and \( x_2 = \pi_1\pi_3 \).

(1) The fact that \( k_5^{(1)} = k(√x_1, √x_2) \) follows from proposition 2.2.

(2) The six intermediate extensions are: \( k(√x_1), k(√x_2), k(√x_1x_2), k(√x_1^2x_2) \) and \( k(√x_2x_2) \).

Since \( x_1 = \pi_1\pi_5^{h_1} \) and \( x_2 = \pi_1\pi_3 \), we have \( k(√x_1x_2) = k(√\pi_1^2\pi_3\pi_5^{h_1}), k(√x_1^2x_2) = k(√\pi_1\pi_3\pi_5^{2h_1}), k(√x_2^2) = k(√\pi_1^4\pi_3^{2h_1}) \) and \( k(√x_1^2) = k(√\pi_3\pi_5^{h_1}) \). Since \( \pi_1^2 = \pi_3, \pi_3^2 = \pi_1 \) and \( \pi_3^2 = \pi_5 \), and by the same reasoning as theorem 3.1 we prove that \( τ^2 \) permutes \( k(√\pi_1\pi_3\pi_5^{2h_1}) \) with \( k(√\pi_1^4\pi_3^{2h_1}) \) and \( k(√x_1) \) with \( k(√\pi_3\pi_5^{h_1}) \), and sets \( k(√x_2), k(√\pi_1\pi_3\pi_5^{2h_1}) \).

The generators of \( C_{k,5} \) in this case are determined as follows:

**Theorem 3.5.** Let \( k, n, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5 \) and \( h_1 \) as above. Let \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4 \) and \( \mathcal{P}_5 \) prime ideals of \( k \) such that \( \mathcal{P}_i^5 = \pi_iO_{k_0} (i = 1, 2, 3, 4, 5) \). Then:

\[
C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}], [\mathcal{P}_1\mathcal{P}_3^4] \rangle
\]

Proof. According to 11 theorem 1.1, for this case of the radicand \( n \), we have that \( ζ_5^i(1 + ζ_5)^j \) is not norm of element in \( k - \{0\} \) for any exponents \( i \) and \( j \), then by 9 section 5.3, we have \( C_{k,5} = C_{k,s}^{'(s)} = C_{k,s}^{'(s)} = \langle [\mathcal{P}_1], [\mathcal{P}_2], [\mathcal{P}_3], [\mathcal{P}_4], [\mathcal{P}_5] \rangle \). Since \( \mathcal{P}_1^2 = \mathcal{P}_3, \mathcal{P}_2^2 = \mathcal{P}_4 \) and \( \mathcal{P}_5^2 = \mathcal{P}_5 \), as the proof of theorem 3.2 we have that \( \mathcal{P}_1, \mathcal{P}_3 \) and \( \mathcal{P}_5 \) are non principals. As \( [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]^2 = [(\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1})^2] = [\mathcal{P}_3\mathcal{P}_1\mathcal{P}_5^{2h_1}] = [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}] > \mathcal{P}_1^2 \mathcal{P}_3 \mathcal{P}_5^{2h_1} \) then \( C_{k,5}^+ = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}] \rangle \), and we have that \( C_{k,5}^- = \langle [\mathcal{P}_1\mathcal{P}_3^4] \rangle \). Hence \( C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}], [\mathcal{P}_1\mathcal{P}_3^4] \rangle \).

The main theorem of capitulation in this case is as follows:

**Theorem 3.6.** We keep the same assumptions as theorem 3.5 Then:

(1) \( K_1 = k(√\pi_1^3\pi_3^3) \) or \( k(√\pi_1\pi_3\pi_5^{2h_1}) \), \( K_2 = k(√\pi_1\pi_5^{h_1}) \) or \( k(√\pi_3\pi_5^{h_1}) \), \( K_3 = k(√\pi_1\pi_3^2\pi_5^{h_1}) \) or \( k(√\pi_1^2\pi_3\pi_5^{2h_1}) \), \( K_4 = k(√\pi_1^4\pi_3^2\pi_5^{h_1}) \) or \( k(√\pi_1^2\pi_3\pi_5^{4h_1}) \), \( K_5 = k(√\pi_3\pi_5^{h_1}) \) or \( k(√\pi_1\pi_5^{h_1}) \) and \( K_6 = k(√\pi_1\pi_3\pi_5^{2h_1}) \) or \( k(√\pi_1\pi_3^2) \).
(2) $[P_1 P_3 P_5^{2h_1}]$ capitulates in $k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$, $[P_1 P_5^{h_1}]$ capitulates in $k(\sqrt[3]{\pi_1 \pi_5^{h_1}})$, $[P_1^2 P_3^2 P_5^{h_1}]$ capitulates in $k(\sqrt[3]{\pi_2 \pi_4 \pi_5^{h_1}})$, $[P_1 P_3 P_5^{h_1}]$ capitulates in $k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$, $[P_1^4 P_2^3 P_5^{h_1}]$ capitulates in $k(\sqrt[3]{\pi_4^2 \pi_3 \pi_5^{h_1}})$, $[P_3 P_5^{h_1}]$ capitulates in $k(\sqrt[3]{\pi_3 \pi_5^{h_1}})$ and $[P_1 P_3^4]$ capitulates in $k(\sqrt[3]{\pi_1 \pi_3})$.

(3) - If $K_1 = k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$, then the possible types of capitulation are: $(0,0,0,0,0,0,0)$, $(1,0,0,0,0,0,0)$, $(0,5,0,0,2,0)$ or $(0,2,0,5,0)$, $(1,5,0,0,2,0)$ or $(1,2,0,5,0)$, $(0,5,4,3,2,0)$ or $(0,2,4,3,5,0)$, $(1,5,4,3,2,0)$ or $(1,2,4,3,5,0)$, $(0,5,3,4,2,0)$ or $(0,2,3,4,5,0)$, $(1,5,3,4,2,0)$ or $(1,2,3,4,5,0)$, $(0,0,3,4,0,0)$ or $(0,0,4,3,0,0)$, $(1,0,3,4,0,0)$ or $(1,0,4,3,0,0)$.

- If $K_1 = k(\sqrt[3]{\pi_1 \pi_3})$, then the same possible types occur, with $i_6$ takes value 0 or 1.

Proof.

(1) According to theorem [3.4], we have that $\tau^2$ permutes $k(\sqrt[3]{\pi_2 \pi_3 \pi_5^{h_1}})$ with $k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$ and $k(\sqrt[3]{\pi_1 \pi_5^{h_1}})$ with $k(\sqrt[3]{\pi_3 \pi_5^{2h_1}})$, and sets $k(\sqrt[3]{\pi_2 \pi_3 \pi_5})$. We determine first the six subgroups $H_i$ of $C_{k,5}$.

We have that $C_{k,5} = \langle A, X \rangle$, where $H_1 = C_{k,5} = \langle A \rangle$ and $H_6 = C_{k,5} = \langle X \rangle$. By theorem [3.5] we have $A = [P_1 P_3 P_5^{2h_1}]$ and $X = [P_1 P_3]$, then $AX = [P_1 P_5^{h_1}]$, $AX^2 = [P_2^2 P_3^4 P_5^{h_1}]$, $AX^3 = [P_1 P_3^4 P_5^{h_1}]$ and $AX^4 = [P_3 P_5^{h_1}]$. Hence $H_2 = \langle [P_1 P_5^{h_1}] \rangle$, $H_3 = \langle [P_2^2 P_3^4 P_5^{h_1}] \rangle$, $H_4 = \langle [P_1 P_3^4 P_6^{h_1}] \rangle$ and $H_5 = \langle [P_3 P_5^{h_1}] \rangle$. Since $\tau^2$ sets $k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$ and $k(\sqrt[3]{\pi_1 \pi_3})$, so if $K_1 = k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$ then $K_6 = k(\sqrt[3]{\pi_1 \pi_3})$ and inversely. By class field theory, the fact that $H_i (i = 2, 5)$ correspond to $K_i (i = 2, 5)$ mean that $P_1 P_5^{h_1}$ splits completely in $K_2$ and $P_3 P_5^{h_1}$ splits completely in $K_5$. As $\pi_1 \pi_5^{h_1}$ divides $\pi_2 \pi_3 \pi_5^{h_1}$ and $\pi_4 \pi_3 \pi_5^{h_1}$, by proposition [2.3] $\pi_1 \pi_5^{h_1}$ can not split in $k_0 \sqrt[3]{\pi_2 \pi_3 \pi_5^{h_1}}$ and $k_0 \sqrt[3]{\pi_4 \pi_3 \pi_5^{h_1}}$, this equivalent to say that $P_1 P_3^{h_1}$ can not split completely in $k_0 \sqrt[3]{\pi_2 \pi_3 \pi_5^{h_1}}$ and $k_0 \sqrt[3]{\pi_4 \pi_3 \pi_5^{h_1}}$. By the same reasoning we have that $P_3 P_5^{h_1}$ can not split completely in $k_0 \sqrt[3]{\pi_2 \pi_3 \pi_5^{h_1}}$ and $k_0 \sqrt[3]{\pi_4 \pi_3 \pi_5^{h_1}}$. Hence if $K_2 = k(\sqrt[3]{\pi_2 \pi_3 \pi_5^{h_1}})$ then $K_5 = k(\sqrt[3]{\pi_4 \pi_3 \pi_5^{h_1}})$ and inversely, which allow us to deduce that if $K_3 = k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$ then $K_5 = k(\sqrt[3]{\pi_2 \pi_3 \pi_5^{h_1}})$ and inversely.

(2) We keep the same proof as (2) theorem [3.3].

(3) -If $K_1 = k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$, then $K_6 = k[\Gamma_5^{(1)}] = k(\sqrt[3]{\pi_1 \pi_3})$ and we have that $[P_1 P_3]$ capitulates in $K_6$, moreover since $C_{k,5} = \langle [P_1 P_3 P_5^{2h_1}] \rangle \cong C_{\Gamma,5}$ then $P_1 P_3 P_5^{2h_1} = j_k/\tau(J)$ such that $C_{\Gamma,5} = \langle J \rangle$, then $[P_1 P_3 P_5^{2h_1}]$ capitulates in $K_6$. As $C_{k,5} = \langle [P_1 P_3 P_5^{2h_1}], [P_3 P_5] \rangle$, then all classes capitulate in $K_6 = k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$. We determine the possible types of capitulation $(i_1, i_2, i_3, i_4, i_5, i_6)$. We have that $i_6 = 0$, $K_2 = K_5^{x_2}$, $K_3 = K_4^{x_2}$ and $C_{k,5} = C_{k,5}^{x_2}$. If $i_1 \neq 0$ we have $i_1 = 1$. $i_2$ and $i_5$ are both nulls or non nulls, so if $i_2$ and $i_5 \neq 0$, then $(i_2, i_5) = (2,5)$ or $(5,2)$ depending on $P_1 P_5^{h_1}$ splits completely in $k(\sqrt[3]{\pi_1 \pi_5^{h_1}})$ or in $k(\sqrt[3]{\pi_3 \pi_5^{h_1}})$. Similarly if $i_3$ and $i_4 \neq 0$, then $(i_3, i_4) = (3,4)$ or $(4,3)$. Hence the possible types given are proved.

-If $K_1 = k(\sqrt[3]{\pi_1 \pi_3})$ then $K_6 = k[\Gamma_5^{(1)}] = k(\sqrt[3]{\pi_1 \pi_3 \pi_5^{2h_1}})$ and we have $C_{k,5} = \langle [P_1 P_3 P_5^{2h_1}] \rangle$ capitulates in $K_6$, the possible values of $i_2, i_3, i_4, i_5$ are as above, $(i_2, i_5) = (2,5)$ or $(5,2)$ if they are non nulls, $(i_3, i_4) = (3,4)$ or $(4,3)$ if they are non nulls. If $i_1 \neq 0$ then $i_1 = 6$ because $H_6 = \langle [P_3 P_5] \rangle$, and if
$i_6 \neq 0$ then $i_1 = 1$ because $H_1 = \langle [P_1 P_3 P_5^{2h1}] \rangle$. Hence the possible types given are proved.

3.3 The case $n = 5^ep \not\equiv \pm 1 \pm 7 \pmod{25}$, where $p \not\equiv 1 \pmod{25}$

Let $k = \mathbb{Q}(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt{\frac{n}{e}})$, where $n = 5^ep$ such that $p \not\equiv 1 \pmod{25}$ and $e \in \{1, 2, 3, 4\}$. Since $n = 5^ep \not\equiv \pm 1 \pm 7 \pmod{25}$ then $\lambda = 1 - \zeta_5$ is ramified in $k/k_0$. Let $\pi_1, \pi_2, \pi_3$ and $\pi_4$ primes of $k_0$ such that $p = \pi_1 \pi_2 \pi_3 \pi_4$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and $\mathcal{I}$ prime ideals of $k$ above $\pi_1, \pi_2, \pi_3, \pi_4$ and $\lambda$, we have $\mathcal{P}_5^\lambda = \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3 \mathcal{P}_4$ and $\mathcal{I}^\lambda = \lambda \mathcal{O}_k$. According to [1, theorem 1.1], for this case of the radicand $n$, we have that $\zeta_5^i(1 + \zeta_5^j)$ is not norm of element in $k - \{0\}$ for any exponents $i$ and $j$, then we have $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,5}^{(\sigma)}$. Hence the results about the six intermediate extensions of $k_5^{(1)}/k$, the generators of $C_{k,5}$ and the capitulation problem in this case are the same as case 2 by substituting $q$ by 5, $\pi_5$ by $\lambda = 1 - \zeta_5$ and $\mathcal{P}_5$ by $\mathcal{I}$.

4 Numerical examples

The task to determine the capitulation in a cyclic quintic extension of a base field of degree 20, that is, in a field of absolute degree 100, is definitely far beyond the reach of computational algebra systems like MAGMA and Pari/GP. For this reason we give exemples of a pure metacyclic fields $k = \mathbb{Q}(\sqrt[n]{n}, \zeta_5)$ such that $C_{k,5}$ is of type (5, 5) and $C_{k,5} = C_{k,5}^{(\sigma)}$.

| $n$  | $h_{k,5}$ | $C_{k,5}$  | Rank ($C_{k,5}^{(\sigma)}$) | $n$  | $h_{k,5}$ | $C_{k,5}$  | Rank ($C_{k,5}^{(\sigma)}$) |
|------|-----------|-------------|-----------------------------|------|-----------|-------------|-----------------------------|
| 55   | 25        | (5, 5)      | 2                           | 1457 | 25        | (5, 5)      | 2                           |
| 655  | 25        | (5, 5)      | 2                           | 6943 | 25        | (5, 5)      | 2                           |
| 1775 | 25        | (5, 5)      | 2                           | 8507 | 25        | (5, 5)      | 2                           |
| 1555 | 25        | (5, 5)      | 2                           | 12707| 25        | (5, 5)      | 2                           |
| 2155 | 25        | (5, 5)      | 2                           | 151  | 25        | (5, 5)      | 2                           |
| 5125 | 25        | (5, 5)      | 2                           | 1301 | 25        | (5, 5)      | 2                           |
| 8275 | 25        | (5, 5)      | 2                           | 2111 | 25        | (5, 5)      | 2                           |
| 30125| 25        | (5, 5)      | 2                           | 251$^2$| 25        | (5, 5)      | 2                           |
| 38125| 25        | (5, 5)      | 2                           | 601$^3$| 25        | (5, 5)      | 2                           |
| 113125| 25      | (5, 5)      | 2                           | 2131$^2$| 25        | (5, 5)      | 2                           |
| 93   | 25        | (5, 5)      | 2                           | 1901$^4$| 25        | (5, 5)      | 2                           |
| 382  | 25        | (5, 5)      | 2                           | 1051$^4$| 25        | (5, 5)      | 2                           |
| 943  | 25        | (5, 5)      | 2                           | 1801$^3$| 25        | (5, 5)      | 2                           |
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