ROBUST SPARSE RECOVERY WITH SPARSE BERNOUlli MATRICES VIA EXPANDERS

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ABSTRACT. Sparse binary matrices are of great interest in the field of sparse recovery, nonnegative compressed sensing, statistics in networks, and theoretical computer science. This class of matrices makes it possible to perform signal recovery with lower storage costs and faster decoding algorithms. In particular, Bernoulli(\(p\)) matrices formed by independent identically distributed (i.i.d.) Bernoulli(\(p\)) random variables are of practical relevance in the context of noise-blind recovery in nonnegative compressed sensing.

In this work, we investigate the robust nullspace property of Bernoulli(\(p\)) matrices. Previous results in the literature establish that such matrices can accurately recover \(n\)-dimensional \(s\)-sparse vectors with \(m = O\left(c(p) \log \frac{en}{s}\right)\) measurements, where \(c(p) \leq p\) is a constant dependent only on the parameter \(p\). These results suggest that in the sparse regime, as \(p\) approaches zero, the (sparse) Bernoulli(\(p\)) matrix requires significantly more measurements than the minimal necessary, as achieved by standard isotropic subgaussian designs. However, we show that this is not the case.

Our main result characterizes, for a wide range of sparsity levels \(s\), the smallest \(p\) for which sparse recovery can be achieved with the minimal number of measurements. We also provide matching lower bounds to establish the optimality of our results and explore connections with the theory of invertibility of discrete random matrices and integer compressed sensing.

1. INTRODUCTION

The theory of compressed sensing, introduced in the seminal works [7,8,11], predicts that a sparse vector can be accurately recovered from a small number of linear corrupted measurements. In a mathematical framework, the goal is to recover an \(s\)-sparse vector \(x \in \mathbb{R}^n\) from linear corrupted measurements \(y = Ax + e \in \mathbb{R}^m\), where \(A \in \mathbb{R}^{m \times n}\) is a known matrix often referred to as the “measurement” matrix, and \(e \in \mathbb{R}^m\) is a noise vector satisfying \(\|e\|_2 \leq \eta\), for some constant \(\eta > 0\). The parameter \(m\) denotes the number of measurements and is typically assumed to be much smaller than the ambient dimension, i.e., \(m \ll n\).

It is well established in the literature that \(m = \Theta(s \log(en/s))\) measurements are sufficient and necessary to recover an \(s\)-sparse vector \(x \in \mathbb{R}^n\) [15]. Precisely, when the measurement matrix \(A\) fulfills appropriate conditions, for example a robust version of the nullspace property (NSP), the target sparse vector can be recovered via the so called quadratically constrained basis pursuit (QCBP), namely

\[ \min_{x} \|x\|_1 \quad \text{subject to} \quad Ax = y. \]
\( \hat{x} = \arg \min_{z \in \mathbb{R}^n} \|z\|_1 \text{ subject to } \|Az - y\|_2 \leq \eta. \)

To date, every measurement matrix requiring the minimal number of measurements for sparse recovery, \( m = \Theta(s \log en/s) \), has been generated at random. It is well known that matrices with i.i.d. isotropic subgaussian rows satisfy the so called \( \ell_2 \) restricted isometry property (RIP) with high probability - a sufficient, though not necessary, condition to establish recovery guarantees through the optimization program (1). The assumption that the rows are subgaussian was significantly relaxed to mild moment assumptions in a line of work based on the Mendelson’s small ball method [15].

Another line of work focused on additional desired properties of randomly generated measurement matrices, such as sparsity. In this context, sparse binary matrices have been considered, and random matrices with i.i.d. entries following a Bernoulli(\( p \)) distribution provide a natural model for generating such matrices. The first result in this direction is due to Chandar [9], who demonstrated that no binary matrix can satisfy the \( \ell_2 \) RIP property with the number of rows proportional to \( s \log(en/s) \). However, Berinde, Gilbert, Indyk, Karloff, and Strauss [3] introduced a class of sparse measurement matrices that achieve sparse recovery with \( m = \Theta(s \log(en/s)) \) measurements through the so-called lossless expanders. This class of graphs is a fundamental tool in theoretical computer science, with plenty of applications in linear sketching and dimension reduction [18].

Mathematically, lossless expanders are defined as left regular bipartite graphs with an adjacency matrix that satisfies an \( \ell_1 \) version of the restricted isometry property. Moreover, it can be shown that a random left \( d \)-regular bipartite graph, for an appropriate value of \( d \), is a lossless expander with high probability (see [15], Chapter 13 and the references therein).

Nachin [27] showed that, under the assumption that \( m = \Theta(s \log(en/s)) \), such matrices achieve the minimum possible column sparsity (number of ones per column) up to an absolute constant. In a similar vein, Prasad and Rudelson [21] investigated the restricted isometry property of sparse subgaussian matrices—matrices generated by the Hadamard product (entrywise product) between a random matrix with mean zero isotropic subgaussian entries and a Bernoulli (\( p \)) matrix. They proved that \( m = \Theta(s \log(en/s)/p) \) measurements are sufficient to establish the RIP property and, consequently, the NSP property.

1.1. Nonnegative Compressed Sensing. The line of work of nonnegative compressed sensing aims to study recovery guarantees under the assumption that the target vector is nonnegative or the measurement matrix is nonnegative, i.e, the entries of the target vector or the measurement matrix are nonnegative. Donoho and Tanner [12] studied the central question: Which assumptions on the measurement matrix \( A \) are necessary to ensure the uniqueness of sparse solution under the nonnegative assumption?

The authors introduced the notion of outwardly \( s \)-neighborly polytopes, and proved that such condition is necessary and sufficient to establish perfect recovery in the noiseless case. Another approach to the same question is due to Bruckstein, Elad and Zibulevsky [3]. They introduced the fundamental notion of positive orthant condition (see Section 5 for more details), and analysed the performance of (1) under the additional constraint \( x \geq 0 \) (all entries of \( x \) are nonnegative) using a coherence
type analysis. Wang, Xu and Tang [35], analysed the signal recovery problem with binary matrices formed by i.i.d. Bernoulli(1/2) entries concatenated with a row of all ones. By doing that, the authors were able to derive sharp guarantees in the noiseless case by relying on the RIP property of matrices with i.i.d. random signs. The downside of this approach is that it only provides guarantees for the noiseless setting, and the measurement matrix is not really sparse.

Kueng and Jung in [23], motivated by concrete applications in wireless network detection, analysed the measurement matrices formed by i.i.d. entries Bernoulli(\(p\)) using Tropp’s version of the Mendelson’s small ball method [32]. They obtained that, with high probability, for

\[
m = \Theta \left( \frac{s}{p^2 \log \frac{p}{1-p}} \log \left( \frac{en}{s} \right) \right),
\]

an \(m \times n\) measurement Bernoulli(\(p\)) matrix satisfy the \(\ell_2\) robust null space property of order \(s\). They also showed that such measurement matrix satisfies a suitable positive orthant condition with high probability.

Combining these two facts together, Kueng and Jung [23] were able to show that, if we assume that the vector to be recovered is nonnegative and \(s\)-sparse, then it is possible to prove recovery guarantees for a simple nonnegative least squares:

\[
\hat{x} = \arg \min _{z \in \mathbb{R}^n, z \geq 0} \|Az - y\|_2.
\]

The assumption that the target vector is nonnegative holds for certain problems in network statistics [23]. The first important feature of this result is that the algorithm does not require any knowledge of the noise level \(\eta\), such results are called in the literature noise blind results. The second important feature is that it gives rise to significant computational savings because the least squares algorithm is considerably cheaper than the QCBP (1) in terms of computational costs. We highlight that Bernoulli(\(p\)) matrices have the important property that the entries are i.i.d. in contrast to the lossless expanders designs in which the rows are not independent, we hope that such advantage will be useful to prove noise-blinds results for general sparse recovery problems. Also, matrices with i.i.d. entries are easier to be sampled.

In this work, the main question raised by Kueng and Jung is the following: What is the correct behaviour of the number of measurements in the practically relevant regime when \(p \to 0^+\)? [23, Remark 10].

We first observe that an important downside of the small ball method is the need that the rows of the measurement matrix requires a certain small ball condition that gives sub-optimal estimates when the distribution is sparse. Indeed, the small ball condition has been analysed in a recent line of work dedicated to remove such condition in many different problems in mathematical statistics and mathematical signal processing [22,26,29,33]. Arguably, random matrices formed by Bernoulli(\(p\)) entries with \(p \to 0^+\) are the most important example not fully captured by the small ball method. It was pointed out by Kueng and Jung [23, Section B] that the small ball method cannot give sharper dependency on \(p\) (up to an absolute constant).
Recently, Jeong, Li, Plan and Yilmaz [20] improved the dependency on $p$ in the result of Kueng and Jung as an application of their refined concentration inequalities. More accurately, they proved that, with high probability,

$$m = \Theta \left( \frac{s}{p(1-p)} \log \left( \frac{en}{s} \right) \right)$$

measurements are enough to establish the $\ell_2$ robust null space property of order $s$. We remark that the authors in [20] also proved that $mp > 1/2$ is necessary and they explicitly wrote that the dependence on $p$ should be sharp up to absolute constants. It turns out this only true in the regime when $s$ is a constant (see the discussion after Theorem 1). In this paper, we address the following question:

(Q): In the regime $m \ll n$, let $A \in \{0, 1\}^{m \times n}$ be a random matrix formed by i.i.d. entries Bernoulli$(p)$. What is the smallest order of $p$ such that the matrix $A$ satisfies a robust null space property of order $s$ (see Definition 1) with $m = Cs \log(en/s)$ rows, where $C > 0$ is an absolute constant?

1.2. Main Results. Our main result provides a sharp answer for the question raised (Q) for a wide range of sparsity levels $s$. The main theorem is the following:

**Theorem 1.** (Main Theorem) Let $A \in \{0, 1\}^{m \times n}$ be a random matrix formed by i.i.d. entries following a Bernoulli$(p)$ distribution with $n \geq 2$. For every pair of positive absolute constants $(\theta, \delta)$ satisfying that $6\theta + 19\delta < 1$, the following holds. If

$$m \geq \left\lceil \frac{50s(2-\theta)}{\theta^3\delta^2} \log n \right\rceil \quad \text{and} \quad p \leq \frac{\theta}{s(2-\theta)},$$

then with probability tending to one as $n$ goes to infinity, the matrix $A$ satisfies the $\ell_1$ robust null space property of order $s$ with respect to the $\ell_1$ norm, with parameters

$$\rho = \frac{2\theta + 6\delta}{1 - 4\theta - 13\delta} \quad \text{and} \quad \tau = \frac{1}{mp(1 - 4\theta - 13\delta)}.$$
Our main theorem shows that the state of the art result \( m = \Theta(s \log(en/s)/p) \) is not optimal when \( p \) vanishes. To see this, consider for simplicity the noiseless case. If such result were optimal then for \( p = O(1/s) \) and \( s = O(n^{1/2}) \) (say) we would need \( m = \Omega(s^2 \log(en/s)) \) measurements to perform exact recovery, however Theorem 1 guarantees that \( m = O(s \log(en/s)) \) measurements are enough to perform exact recovery in the same regime of \( p \). Our result reveals a new phase transition not captured by the previous results. Finally, we also present a conjecture in Section 4 about the precise answer for the question (Q) in full generality.

As an application of Theorem 1 we derive noise blind guarantees for nonnegative compressed sensing with sparse Bernoulli matrices. In a few words, a simple \( \ell_1 \) minimization algorithm suffices to perform signal recovery. No knowledge of the noise level is required. After submitting this manuscript, the author was notified that a similar result was obtained with a different measurement matrix, see [28] for more details. We refer the reader to Section 5 for the formal statement and discussion.

1.3. Organization of the paper. The rest of this paper is organised as follows. In Section 2, we provide some background results. In Section 3, we provide the main ideas of this work and the proof of Theorem 1. In Section 4, we establish lower bounds and connections with discrete random matrix theory. Section 5 is dedicated to applications of our main results to noise blind compressed sensing. The Appendix is dedicated to technical proofs.

2. Preliminaries & Background

We start by introducing some notation. The set \( \{1, \ldots, n\} \) is denoted by \( [n] \). For \( 1 \leq q \leq \infty \), we write \( \| \cdot \|_q \) for the standard \( \ell_q \) norm for vectors and \( B_q^n, S_q^{n-1} \) for the \( n \)-dimensional unit ball and \((n - 1)\)-dimensional unit sphere with respect to the \( \ell_q \) norm, respectively. For the standard Euclidean sphere \( S_2^{n-1} \), we just write \( S^{n-1} \).

For functions \( f(s, m, n) \) and \( g(s, m, n) \) we write \( f(s, m, n) \lesssim g(s, m, n) \) if there exists an absolute constant \( C > 0 \) such that \( f(s, m, n) \leq Cg(s, m, n) \). Sometimes we shall use the standard Landau notation to hide the absolute constant, i.e, to replace \( f(s, m, n) \leq Cg(s, m, n) \) by \( f = O(g) \). The notation \( f(s, m, n) \gtrsim g(s, m, n) \) or \( f = \Omega(g) \) is defined analogously, and we write \( f(s, m, n) \sim g(s, m, n) \) or \( f = \Theta(g) \) if \( f(s, m, n) \lesssim g(s, m, n) \) and \( f(s, m, n) \gtrsim g(s, m, n) \). The symbol \( f \gg g \) means that \( f/g \) diverges when \( n \) goes to infinity while \( f \ll g \) means that \( f/g \) converges to zero as \( n \) goes to infinity.

For a set \( S \subset [n] \) we denote its complement over \([n]\) by \( S^c \) and its cardinality by \( |S| \). For a vector \( x \in \mathbb{R}^n \), \( x_S \) is the vector such that \((x_S)_j = x_j \) for all \( j \in S \) and \((x_S)_j = 0 \) otherwise. The notation \( x \geq 0 \) means that the coordinates is nonnegative and \( 1_m \) denotes all ones (row) vector in \( \mathbb{R}^m \). The best \( s \)-term approximation of the vector \( x \) with respect to the \( \ell_1 \) norm is defined as \( \sigma_s(x)_1 := \inf\{\|z - x\|_1 : z \text{ is } s\text{-sparse}\} \).

For matrices \( W \in \mathbb{R}^{n \times n} \), \( \|W\| \) denotes the standard (\( \ell_2 \to \ell_2 \)) operator norm and \( \text{diag}(w) \) denotes the diagonal matrix formed by the vector \( w \in \mathbb{R}^n \). A Bernoulli(\( p \)) random variable takes the value one with probability \( p \) and the value zero, otherwise. A Binomial(\( m, p \)) random variable is distributed as the sum of \( m \) independent Bernoulli(\( p \)) random variables. A Bernoulli(\( p \)) matrix \( A \) is a random matrix formed by i.i.d. Bernoulli(\( p \)) random variables.
Remark 1. We are interested in the regime when $p$ vanishes. Therefore, for the rest of this paper, we explicitly assume that $p \leq 1/2$.

Next, we recall the definition of robust nullspace property that plays a key role in this work.

**Definition 1.** [13, Definition 4.21] Given $q \geq 1$, the matrix $A \in \mathbb{R}^{m \times n}$ satisfies the $\ell_q$ robust nullspace property of order $s$ with respect to a norm $\| \cdot \|$, with constants $0 < \rho < 1$ and $\tau > 0$ if, for any set $S \subset [n]$ with cardinality $|S| \leq s$ and vector $v \in S^{n-1}$,

$$
\|v_S\|_q \leq \frac{\rho}{s^{1-1/q}}\|v_{S^c}\|_1 + \tau\|Av\|.
$$

The next theorem is a classical result that relates nullspace property with recovery guarantees.

**Theorem 2.** [15, Theorem 4.25] Suppose that the matrix $A \in \mathbb{R}^{m \times n}$ satisfies the $\ell_q$ robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$ with respect to a norm $\| \cdot \|$. Then for any vectors $x, z \in \mathbb{R}^n$ and $1 \leq p \leq q$,

$$
\|z - x\|_p \leq \frac{C}{s^{1-1/p}}(\|z\|_1 - \|x\|_1 + 2\sigma_s(x)_1) + D\tau s^{1/p-1/q}\|A(z - x)\|.
$$

Here $C := (1 + \rho)^2/(1 - \rho)$ and $D := (3 + \rho)/(1 - \rho)$. In particular, when $\| \cdot \|$ is the Euclidean norm, the the solution of the optimization program (1) $\hat{x}$ satisfies,

$$
\|\hat{x} - x\|_p \leq \frac{C}{s^{1-1/p}}2\sigma_s(x)_1 + 2D\tau s^{1/p-1/q}\eta.
$$

**Remark 2.** We slightly modified the original statement by making the dependency on $\tau$ a bit more explicitly, thus making $D$ a constant depending only on $\rho$ instead of depending on $\tau$.

In a nutshell, our approach to tackle the question (Q) is based on spectral graph theory. The key idea is to view the Bernoulli($p$) matrix, for a certain regime of $p$, as the adjacency matrix of a bipartite graph that behaves similarly to a lossless expander. In order to provide a precise description of this idea, we present some background results about lossless expander graphs.

To this end, consider a bipartite graph $G = (L, R, E)$, where the set of left nodes $L$ should be identified with $[n]$, the set of right nodes $R$ should be identified with $[m]$ and each edge $e \in \overline{ji} \in E$ connects a vertex $j \in L$ to other vertex $i \in R$. A bipartite graph $G = G(L, R, E)$ is said to be $d$-left regular if all nodes in $L$ have degree $d$.

**Definition 2.** (Lossless expanders) A $d$-left regular bipartite graph is an $(s, d, \theta)$-lossless expander if, for all subsets $J \subset L$ with $|J| \leq s$, the set

$$
R(J) := \{i \in R : \text{there is } j \in J \text{ with } \overline{ji} \in E\},
$$

satisfies the following expansion property

$$
|R(J)| \geq (1 - \theta)d|J|.
$$

The smallest $\theta > 0$ for which (3) holds is denoted by $\theta_s$.

A natural concept associated with a graph is its adjacency matrix, the next definition and theorem establish the connection between the lossless expanders and compressed sensing. To state them accurately, recall that a bipartite graph is graph
whose vertices can be split into two disjoint sets, \( L \) and \( R \), where every edge connects a vertex on \( L \) to a vertex on \( R \) (see [36] for more information about bipartite graphs).

**Definition 3.** (Adjacency matrix)

The adjacency matrix of a bipartite graph \( G = G(L, R, E) \) with \( |L| = n \) and \( |R| = m \) is the \( m \times n \) matrix \( M \) defined entrywise by

\[
M_{i,j} := \begin{cases} 
1, & \text{if } ji \in E \\
0, & \text{otherwise.}
\end{cases}
\]

An important observation is that the adjacency matrix of a lossless expander is a binary matrix with \( d \) ones per column in which every \( m \times k \) submatrix with \( k \leq s \) has at least \((1 - \theta)dk\) nonzero rows. The next result establishes that lossless expanders provide remarkable guarantees for sparse recovery via \( \ell_1 \) minimization.

**Theorem 3.** ([15, Theorems 13.11 and 13.6.]) The adjacency matrix \( M \in \{0,1\}^{m \times n} \) of a \((2s, d, \theta)\)-lossless expander satisfies the \( \ell_1 \) robust nullspace property of order \( s \) with respect to the \( \ell_1 \) norm provided that \( \theta^2 s < 1/6 \). Precisely, for all \( v \in \mathbb{R}^n \) and subset \( S \subset [n] \) for which \( |S| \leq s \), it holds that

\[
\|v_S\|_1 \leq \frac{2\theta_2 s}{1 - 4\theta_2 s} \|v_S\|_1 + \frac{1}{(1 - 4\theta_2 s)d} \|Av\|_1.
\]

Moreover, if \( G \) is a random graph sampled uniformly at random among all left \( d \)-regular bipartite graphs with \( n \) left nodes and \( m \) right vertices, then with probability \( 1 - \varepsilon \), \( G \) is a \((s, d, \theta)\)-lossless expander provided that

\[
d = \left\lceil \frac{1}{\theta} \log \left( \frac{en}{\varepsilon s} \right) \right\rceil \quad \text{and} \quad m \geq c_\theta s \log \left( \frac{en}{\varepsilon s} \right).
\]

Here \( c_\theta > 0 \) is constant that depends only on \( \theta \).

### 3. The quasi-regular lossless expanders

A closer look at the idea of viewing a matrix \( A \in \{0,1\}^{m \times n} \) formed by i.i.d. entries Bernoulli(\( p \)) as an adjacency matrix of a lossless expander, as defined in (4), reveals that this interpretation is not accurate. Indeed, on average, the matrix \( A \) has \( mp \) ones per column, but it is crucial to consider the fluctuations into account. On the other hand, the left degree is a binomial random variable that concentrates well around the mean.

The key fact is that, due to the combinatorial nature of the definition of lossless expanders, a straightforward extension to non-regular expanders is possible. Such an extension is not clear for other classes of expanders; for a detailed discussion related to Ramanujan graphs, see [4]. Below, we introduce our proposed extension of lossless expanders to the non-regular case. To the best of our knowledge, this definition is new.

**Definition 4.** (Quasi-regular lossless expanders) A bipartite graph \( G = G(L, R, E) \) is said to be a \((\delta, d)\)-left quasi-regular if the left degree of all vertices in \( L \) lies on the interval \([1 - \delta)d, (1 + \delta)d] \). Moreover, a \((\delta, d)\)-left quasi-regular bipartite graph is an \((s, d, \delta, \theta)\)-quasi-regular lossless expander if it satisfies the expansion property \( (3) \) for all sets \( J \) of left vertices, with cardinality \( |J| \leq s \). The smallest \( \theta \) for which the expansion property of such quasi-regular graph holds is denoted by \( \theta_{s, \delta} \).
Remark 3. It should be clear that for \( \delta = 0 \), the quasi-regular lossless expander becomes the lossless expander. In particular, \( \theta_{s,0} = \theta_s \).

In this work, it is more appropriate to think that the quasi-regular lossless expanders are represented by matrices with \( (1 \pm \delta)d \) ones per column and each \( m \times k \) submatrix with \( k \leq s \) has at least \( (1 - \theta)dk \) nonzero rows.

The next natural step is to obtain a version of Theorem 3 for Bernoulli(p) matrices. We split our analysis in two parts: The first one is purely deterministic, it establishes that the adjacency matrix of the quasi-regular lossless expander satisfies the \( \ell_1 \) robust nullspace property. The second part is about the random construction, it shows that the Bernoulli(p) matrix \( A \) is the adjacency matrix of a quasi-regular lossless expander with high probability. Let us state the formal result related to the first part.

**Theorem 4.** The adjacency matrix \( A \in \{0,1\}^{m \times n} \) of a \((2s,d,\delta,\theta)\) quasi-regular lossless expander satisfies the \( \ell_1 \) null space property of order \( s \) with respect to the \( \ell_1 \) norm, with parameters

\[
\rho = \frac{2\theta_{2s,\delta} + 6\delta}{1 - 4\theta_{2s,\delta} - 13\delta} \quad \text{and} \quad \tau = \frac{1}{d(1 - 4\theta_{2s,\delta} - 13\delta)},
\]

provided that the constants \( \theta_{2s,\delta} \) and \( \delta \) satisfy \( 6\theta_{2s,\delta} + 19\delta < 1 \).

In particular, for \( \delta = 0 \), we recover the first part of the statement in Theorem 3. It is worth noting that the condition \( 6\theta_{2s,\delta} + 19\delta < 1 \) ensures that \( \rho < 1 \). We postpone the proof to the Appendix.

We note that, while the original proof for the deterministic part of Theorem 3 relies on the intuitive concept of \( \ell_1 \) restricted isometry property, we choose the proof strategy similar to [15]. This approach directly addresses the nullspace property without intermediate steps involving restricted isometry property and also takes the robustness against noise into account.

We now present the main result of this paper, confirming the intuition that Bernoulli(p) matrices are suitable candidates for extending the notion of lossless expanders to the non-regular case. In what follows, \( \delta \) and \( \theta \) will always be (small) absolute constants.

**Theorem 5.** Let \( A \in \{0,1\}^{m \times n} \) be a Bernoulli(p) matrix with \( n \geq 2 \). For every triple of absolute constants \( 0 < \theta < 2/3, 0 < \delta < 1 \) and \( C_m \geq 25 \), if

\[
m \geq \left\lceil C_m \log \frac{n}{p^2\delta^2} \right\rceil \quad \text{and} \quad ps \leq \frac{2\theta}{2 - \theta},
\]

then the matrix \( A \) is the adjacency matrix of a \((s,mn,p,\delta,\theta)\) quasi-regular lossless expander, with probability at least \( 1 - 2n^{1-C_m/3} - 2n^{1-C_m/24} \) that, in particular, converges to one as \( n \) goes to infinity.

**Remark 4.** Notice that our result only contains explicit constants, while the state of the art result [20] relies on deep results in generic chaining in which explicit constants are hard to be obtained. On the other hand, our analysis did not optimized the value of such constants.

**Proof.** It is sufficient to establish that, with high probability, the matrix \( A \) has \( (1 \pm \delta)d \) ones per column (condition 1) and every \( m \times k \) submatrix, with \( k \leq s \), has at least \( (1 - \theta)dk \) nonzero rows (condition 2).
Condition 1: Each column has \( m \rho \) ones on average. By Chernoff deviation inequality [19, Corollary 2.3], the probability that the number of ones deviates from the average more than \( \delta m \rho \) is at most \( 2e^{-\delta^2 m \rho / 3} \). By union bound, the probability that exists a column with number of ones that is either larger than \((1 + \delta) m \rho \) or less than \((1 - \delta) m \rho \) is at most
\[
2ne^{-\delta^2 m \rho / 3} = 2n^{1-C_m / 3},
\]
which clearly vanishes provided that \( C_m > 3 \).

Condition 2: Now we proceed to the second requirement. For each \( k \in [s] \), the number of nonzero rows in a \( m \times k \) submatrix follows a Binomial distribution\((m, q)\) with \( q = 1 - (1 - \rho)^k \). We fix a \( m \times k \) submatrix and then the number of nonzero rows has average \( mq \). Let \( \varepsilon > 0 \) be a constant to be chosen later. We write,
\[
\begin{align*}
mq(1 - \varepsilon) &= m(1 - (1 - \rho)^k)(1 - \varepsilon) > m(1 - e^{-pk})(1 - \varepsilon) \\
&> m(pk - \frac{p^2 k^2}{2!})(1 - \varepsilon) \quad \text{(by Taylor expansion)} \\
&\geq m(1 - \theta)pk,
\end{align*}
\]
where the last estimate holds by setting \( \varepsilon := \theta / 2 \), and recalling that \( \rho k < 2\theta / (2 - \theta) \). Next, by Chernoff deviation inequality, we get the probability that the number of nonzero rows is less than \((1 - \theta)mq\) is \( e^{-\varepsilon^2 mq / 3} \). By union bound, the probability that exists a submatrix \( m \times k \) with less than \((1 - \theta)dk\) nonzero rows is at most
\[
\sum_{k=1}^s \binom{n}{k} e^{-\varepsilon^2 mq / 3} \leq \sum_{k=1}^s e^{k \log(en/k) - \varepsilon^2 mq / 3},
\]
where the latter step follows from the standard estimate for binomial coefficients [34, Exercise 0.05]. By assumption, \( \theta < 2 / 3 \) which implies that \( pk \leq ps < 1 \). Therefore, by Taylor’s expansion we have that \( mq > mpk / 2 \). Also, recalling that \( m \geq C_m \log n / \theta^2 \) and that \( \varepsilon = \theta / 2 \), the estimate on \((5)\) becomes
\[
\sum_{k=1}^s e^{k \log(en/k) - \varepsilon^2 mq / 3} \leq \sum_{k=1}^s e^{-k \log n(C_m / 24 - 1)} \leq 2n^{(1-C / 24)},
\]
where the last inequality follows from the standard summation formula for geometric progression \((n \geq 2)\). \( \square \)

Combining Theorems 4 and 5 we immediately obtain Theorem 1.

4. Lower Bound and Phase Transition

In this section, we investigate the optimality of our results and their connections with the theory of invertibility of discrete random matrices. The presented lower bound suggests a phase transition for the number of measurements, \( m \), based on the range of \( p \). The proof of this bound is straightforward, but the intuition behind it is more delicate.

To start, notice that a necessary condition for establishing exact recovery guarantees by minimizing the \( \ell_0 \) norm is that every collection of \( 2s \) columns must be linearly independent. Similarly, for \( \ell_1 \) minimization to work, it is necessary that every collection of \( s \) columns is linearly independent. We draw attention to two lines of work related to investigating whether there exists a collection of \( s \) linearly
dependent columns with high probability. The focus here is to highlight a common phenomenon.

The first line of work that we mention here is dedicated to estimate the probability that a square random matrix is invertible \[^{25, 30, 31}\]. The results suggest that the phenomenon is local, i.e., the linear dependence is caused by one or two columns, in particular the presence of a zero column. For example, Basak and Rudelson \[^{2, \text{Corollary 1.3}}\] proved the following precise phase transition: For a square \(n \times n\) Bernoulli(\(p\)) matrix, if \(p < \log n/n\) then the matrix has a zero column with positive probability. On the other hand, if \(p \geq (1 + c) \log n/n\), for some positive constant \(c > 0\), then the matrix is invertible with high probability. We should mention that we stated here a slightly simpler version of the original result.

The other line of work considers the following problem \[^{13, 16, 17}\] inspired by integer compressed sensing: Given an integer valued random matrix \(A\) with positive probability. On the other hand, if \(p \geq (1 + c) \log n/n\), for some positive constant \(c > 0\), then the matrix is invertible with high probability. The focus here is to highlight a common phenomenon.

The state of the art is due to Ferber, Sah, Sawhney and Zhu \[^{13}\]. The authors proved that if the entries of \(A\) is a zero column, then it fails to recover the vector \(e_j\), the \(j\)-th vector of the canonical basis. More accurately,

\[
Ae_j = 0
\]

where \(\varepsilon \gg \log \log n / \log n\).

Based on the phase transition due to Rudelson and Basak, it is natural to estimate the probability to have a zero column in a rectangular Bernoulli matrix and use it as a lower bound. Clearly, if the matrix has a zero column, it cannot be \(s\)-robust for any \(s \geq 1\), and consequently, the \(\ell_1\) minimization cannot work.

**Proposition 1.** (Lower bound) Let \(A \in \{0, 1\}^{m \times n}\) be a Bernoulli(\(p\)) matrix with \(p \leq 1/2\). If the matrix \(A\) satisfies a nullspace property of order \(s\) with high probability, then

\[
m \geq \frac{1}{2} \left( \frac{1}{\log 9} s \log \left( \frac{n}{4s} \right) + \frac{1}{\log \sqrt{2}} \frac{\log n}{p} \right).
\]

The (standard) lower bound \(m \geq s \log(en/4s)/\log 9\) is from \[^{15}\] Theorem 10.11.

**Proof.** We evaluate the probability to have a zero column. If the \(j\)-th column of \(A\) is a zero column, then it fails to recover the vector \(e_j\), the \(j\)-th vector of the canonical basis. More accurately,

\[
\mathbb{P}(\exists j \in [n] : Ae_j = 0) = 1 - \mathbb{P}(\forall j \in [n] : Ae_j \neq 0) = 1 - \mathbb{P}(Ae_1 \neq 0)^n \quad \text{(by independence)}
\]

\[
= 1 - (1 - (1 - p)^m)^n.
\]

If we obtain that, for a certain \(p\), \((1 - (1 - p)^m)^n \to C < 1\), then we get a lower bound on \(p\). Notice that \((1 - (1 - p)^m)^n \leq e^{-(1-p)^m n}\). We fix for a moment \(m = c \log n/p\), where \(c > 0\) is a constant to be chosen later. Since \(p \leq 1/2\), we have \((1 - p)^{1/p} \geq 2^{-1/2}\) (the function \(1 - x^{1/x}\) is non-increasing from 0 to 1), therefore
\((1 - p)^{c \log n / p} \geq 2^{-c \log n / 2} = e^{-c(\log \sqrt{2}) \log n} = e^{-\log n} = n^{-1}, \) for \(c = 1 / \log \sqrt{2} \).

Finally, we obtain that \( (e^{-(1-p)m})^n \leq (e^{-\frac{1}{2}})^n = e^{-1}. \) We obtained that

\[ m \geq \frac{1}{\log \sqrt{2}} \left( \frac{\log n}{p} \right), \]

is necessary. We average the two lower bounds to finish the proof. \(\square\)

We point out that the lower bound and Theorem\(\square\) show that, in the very sparse regime \(p \lesssim 1 / s\), the main reason that causes the failure of \(\ell_1\) relaxation is to have a zero column. In the context of invertibility of square random matrices, this shows that the existence of an absolute constant \(C > 1\) such that, for \(p = C \log n / n\), every collection of \(n / (C \log n)\) columns of an \(n \times n\) Bernoulli\(\langle p \rangle\) matrix is linearly independent with high probability. The factor \(C \log n\) may be removed if we extend the range of \(p\) in Theorem\(\square\) (see the conjecture below).

Another immediate consequence of our result is the phase transition for \(A\) being \(s\)-robust when \(A\) has i.i.d. Bernoulli\(\langle p \rangle\) entries with \(p \lesssim 1 / s\), namely

\[
\begin{cases}
  n \leq e^{mp - \varepsilon}, & A \text{ is } s\text{-robust}, \\
  n \geq e^{mp}, & A \text{ is not } s\text{-robust},
\end{cases}
\]

where \(\varepsilon = \Theta(\log \log n / \log n)\). Observe that our result is even stronger because if \(A\) satisfies the \(\ell_1\) robust null space property then it satisfies the \(s\)-robustness property. An interesting open problem is to fully characterize the phase transition above for all values of \(p \in (0, 1)\). Another question is to study if the robust null space property allows the matrix \(A\) to be \(s\)-robust even if add adversarial noise, i.e, some adversary modifies a small fraction of the entries.

Back to the theory of sparse recovery, we present the following conjecture. Since the lower bound has a matching upper bound up to an absolute constant for \(p\) constant and also for \(p \lesssim 1 / s\), we conjecture the following result:

**Conjecture 1.** Let \(A \in \{0, 1\}^{m \times n}\) be a Bernoulli\(\langle p \rangle\) matrix. There exists constants \(C_1, C_2 > 0\) such that if

\[ m \geq \max \left\{ C_1 s \log \frac{en}{s}, C_2 \frac{\log n}{p} \right\}, \]

then with high probability on \(n\), the \(A\) satisfies the \(\ell_1\) robust nullspace property of order \(s\) for some \(\rho < 1\) and \(\tau > 0\). Moreover, we can take any constant \(C_2 > 1\).

The "moreover" part of the conjecture would capture the phase transition in \(\square\) Corollary 1.3. Notice that, if the conjecture is true, then a natural phase transition for sparse recovery occurs:

\[
m(p) \sim \begin{cases}
  s \log \frac{en}{s}, & \text{for } \frac{\log(n)}{s \log \frac{n}{p}} \lesssim p \leq \frac{1}{2} \\
  \log \frac{n}{p}, & \text{for } p \ll \frac{\log(n)}{s \log \frac{n}{p}}.
\end{cases}
\]

It seems that the best choice of \(p\) is

\[ p^* \sim \frac{\log(n)}{s \log \left( \frac{n}{p^*} \right)}, \]

to balance the optimality in terms of the number of measurements and the smallest possible column sparsity of the measurement matrix.
We end this section with some remarks about the conjecture: The small ball method and the standard RIP techniques fail in the sparse regime ($p \to 0^+$). Moreover, the expander technique used in this manuscript fails in the regime $p \gg 1/s$ because the expansion property requires more edges than the maximum number of edges possible. The graph becomes too dense. A natural approach would be to interpolate between $\ell_2$ RIP used in the dense regime and a modified version of $\ell_1$ RIP. Unfortunately, $\ell_p$ RIP fails to give sharp results for $p > 1$ and $p \neq 2$ [1]. Perhaps the conjecture can be tackled by adapting the tools used to establish the invertibility results. We leave this as an opportunity for future work.

5. Applications in nonnegative compressed sensing

In this section, we present a practical application of our main Theorem 1. In many compressed sensing applications, information about the noise level is often unavailable. Noise-blind compressed sensing addresses this challenge by studying algorithms that perform sparse recovery without utilizing any information about the noise as an input. In particular, a line of work [6,14,37] investigated guarantees for the $\ell_1$-minimization as in (1) under the equality constraint instead. Mathematically, this correspond to setting $\eta = 0$ in (1) even though the measurements are contaminated with noise.

One important outcome of this line of work is that, in addition to the nullspace property, the so-called quotient property suffices to achieve guarantees similar to the ones that we would obtain if we used the quadratically constrained basis pursuit. However, the design matrices used so far crucially relies on the fact that the law of random variables are symmetric (in particular it needs to be centred random variables) which clearly is not suitable to our case. For more information about the quotient property and related results, we refer the reader to [15, Chapter 11].

As mentioned in the introduction, in some practical relevant problems, we can assume that the target vector is nonnegative, i.e., it has nonnegative entries. In this case, we establish noise-blind guarantees for Bernoulli$(p)$ matrices studied in the previous sections. Our approach is based on [23].

The starting point of our analysis is the following notion, known as positive orthant condition: A matrix $A \in \{0,1\}^{m \times n}$ satisfies the positive orthant condition if it belongs to the following set,

$$\mathcal{M}^+ := \{A : \exists t \in \mathbb{R}^m \text{ such that } A^T t > 0\}.$$

The following result establishes that Bernoulli$(p)$ random matrices satisfies the positive orthant condition (with high probability).

**Proposition 2.** [23, Theorem 12] Suppose that $A \in \{0,1\}^{m \times n}$ is a Bernoulli$(p)$ matrix. Set

$$w = A^T t \in \mathbb{R}^n \quad \text{with} \quad t := \frac{1}{pm}1_m^T \in \mathbb{R}^m.$$

Then, with probability at least $1 - ne^{-\frac{1}{2}p(1-p)m}$,

$$\max_{i \leq n} |\langle e_i, w \rangle| \leq \frac{3}{2} \quad \text{and} \quad \min_{i \leq n} |\langle e_i, w \rangle| \geq \frac{1}{2}.$$

In particular, on this event, $A \in \mathcal{M}^+$. 
We proceed with two easy lemmas that are crucial to our result. For notation simplicity, we write
\[ \kappa(A) := \{ \|W\|\|W^{-1}\| : \exists t \in \mathbb{R}^m, W = \text{diag}(A^T t) > 0 \} \]
Recall that \( w \geq 0 \) means that the vector \( w \) only has nonnegative entries, we may refer to such a vectors a “nonnegative”.

**Lemma 1.** Suppose a matrix \( A \in \mathbb{R}^{m \times n} \) satisfies the \( \ell_1 \) robust nullspace property of order \( s \), with parameters \( \rho \) and \( \tau \), with respect to the \( \ell_1 \) norm. Let \( W = \text{diag}(w) \) be a matrix, where \( w \geq 0 \). Then \( AW^{-1} \) also satisfies the \( \ell_1 \) robust nullspace property of order \( s \) with respect to the \( \ell_1 \) norm, with parameters \( \tilde{\rho} := \kappa(W) \rho \) and \( \tilde{\tau} := \|W\| \tau \).

**Proof.** We write,
\[ \|w_S\|_1 = \|WW^{-1}w_s\|_1 \leq \sup_{z: \|z\|_1 = 1} \|Wz\|_1 \|W^{-1}v_S\|_1 = \|W\|\|(W^{-1}v)_S\|_1, \]
where the last equality follows from the fact that \( W \) is a diagonal matrix. Now, we use the \( \ell_1 \) robust nullspace property to obtain that
\[ \|v_S\|_1 \leq \|W\|\|\rho\|(W^{-1}v)_S\|_1 + \tau \|AW^{-1}v\|_1 \]
\[ \leq (\|W\|\|W^{-1}\|\|v_S\|_1) + (\tau\|W\|)\|AW^{-1}v\|_1, \]
where in the last inequality, we used the fact that \( W^{-1} \) is also a diagonal matrix. \( \Box \)

**Lemma 2.** Let \( A \in \mathbb{R}^{m \times n} \) be a matrix and let \( W = \text{diag}(w) \). Suppose the existence of a vector \( t \in \mathbb{R}^m \) for which \( w = A^T t \) has only nonnegative entries. Then for all nonnegative vectors \( x, z \in \mathbb{R}^n \),
\[ \|Wz\|_1 - \|Wx\|_1 \leq \|t\|_\infty \|A(z - x)\|_1. \]

**Proof.** Observe by the construction of \( W \), we have
\[ \|Wz\|_1 = \langle Wz, 1_n \rangle = \langle z, W1_n \rangle = \langle z, \text{diag}(A^T t) 1_n \rangle = \langle A^T t, z \rangle = \langle t, Az \rangle. \]
Similarly, the same holds for \( \|Wx\|_1 \). Therefore, we obtain that
\[ \|Wz\|_1 - \|Wx\|_1 = \langle t, A(z - x) \rangle \leq \|t\|_\infty \|A(z - x)\|_1, \]
where the last step follows from Hölder’s inequality. \( \Box \)

We are now in position to state the main result of this section. A similar result using the standard lossless design was recently obtained in [28] with a non-binary matrix. We hope that in a future work that the independence among rows can be used as an advantage to establish noise-blind results with a sharp number of measurements and with no restriction on the target vector.

**Theorem 6.** Let \( A \in \{0,1\}^{m \times n} \) be a Bernoulli\((p)\) matrix, where \( m \) and \( p \) satisfy the assumptions from Theorems 4 and 5, for some positive constants \( \theta, C_m, \) and \( \delta \). Suppose that we receive a vector \( y = Ax + e \), where \( x \in \mathbb{R}^n \) is an unknown vector with nonnegative entries and \( e \in \mathbb{R}^m \) is an unknown noise vector. Then, with probability at least
\[ 1 - ne^{-3p(1-p)m/8 - 6m^{1-C_m/24}}, \]
any solution of
\[ \hat{x} = \arg \min_{z \geq 0} \|Az - y\|_1, \]
(6)
satisfies
\[ \|\hat{x} - x\|_1 \leq 3 \frac{(1 + \rho)^2}{1 - \rho} \sigma_s(x)_1 + \frac{2}{(1 - \rho)pm} \|e\|_1 \left(2(1 + \rho)^2 + \frac{3(3 + \rho)}{1 - 4\rho - 13\delta}\right). \]

Here \( \rho = (2\theta + 6\delta)/(1 - 4\theta - 13\delta) < 1 \) (as in Theorem 4). Moreover, under the same assumptions, on that event, it also holds that
\[ (7) \quad \hat{x} = \arg\min_{z \in \mathbb{R}^n; z \geq 0} \|Az - y\|_2 \]
satisfies
\[ \|\hat{x} - x\|_1 \leq 3 \frac{(1 + \rho)^2}{1 - \rho} \sigma_s(x)_1 + \frac{2}{(1 - \rho)pm} \|e\|_2 \left(2(1 + \rho)^2 + \frac{3(3 + \rho)}{1 - 4\theta - 13\delta}\right). \]

Our theorem should be compared with [23, Theorem 3]. For simplicity, we assume \( p \sim 1/s \). All results are about noise blind nonnegative signal recovery, i.e, neither [6] nor [7] require any knowledge of the noise as an input. Besides, [23, Theorem 3] provides guarantees for the optimization program (7) with the error bound scaling with \( 1/m \) instead of \( 1/\sqrt{m} \), however the error is measured in the \( \ell_2 \) norm instead of the \( \ell_1 \) norm. Theorem 3 requires slightly more measurements than the optimal, the difference between \( \log(en/s) \) and \( \log n \). Again, they are of the same order in the wide range \( s = O(n^{1/\beta}) \) for some constant \( 0 < \beta < 1 \). On the other hand, the measurement matrix \( A \) in Theorem 3 is much sparser as \( p \sim 1/s \) instead of just being a constant.

As a final remark, we hope that some sub-linear time algorithms designed for the lossless expanders can be easily adapted to the Bernoulli(\( p \)) matrix because the latter is a quasi-regular expander. We do not pursue this direction here. We now proceed to the proof of Theorem 6.

Proof. Thanks to Proposition 2, we know that setting \( t = (pm)^{-1} \mathbf{1}_m^T \), we guarantee that, with probability at least \( 1 - ne^{-3p(1-p)m/8} \), \( w = A^T t > 0 \). By Chernoff deviation inequality and union bound, with probability at least \( 1 - 2ne^{-mp/12} \), the matrix \( W := \text{diag}(w) \) satisfies that \( 1/2 \leq \|W\| \leq 3/2 \). In particular, on this event, \( W \) is invertible with condition number \( 1/3 \leq \|W\|\|W^{-1}\| \leq 3 \). Next, since the assumptions from Theorem 4 are satisfied, we have that the matrix \( A \) satisfies (with the probability given in that statement) the \( \ell_1 \) robust nullspace property of order \( s \) with parameters \( \rho \) and \( \tau \) (with respect to the \( \ell_1 \) norm), where \( \rho \) and \( \tau \) as given in Theorem 4.

For the rest of the proof, we assume that all the events above hold. Clearly, the announced probability follows by a simple union bound. Now, we apply Lemma 1 to obtain that \( AW^{-1} \) also satisfies robust NSP of order \( s \) with parameters \( \rho \leq 3\rho \) and \( \tau \leq 3\tau/2 \). Now, by Theorem 2 applied to the vectors \( Wz \) and \( Wx \) (\( C, D \) are the constants from there), followed by Lemma 2 we have
\[ \|W(z - x)\|_1 \leq C(\|Wz\|_1 - \|Wx\|_1 + \sigma_s(Wx)_1) + \|W\|\|D\tau\|\|A(x - z)\|_1 \]
\[ \leq C(\|t\|_\infty\|A(z - x)\|_1 + C\sigma_s(Wx)_1) + \|W\|\|D\tau\|\|A(x - z)\|_1 \]
\[ \leq \|W\|C\sigma_s(x)_1 + \|A(z - x)\|_1 \left( \frac{C}{pm} + \|W\|\|D\tau\| \right), \]
where the last inequality follows from the fact that \( \sigma_s(Wx)_1 \leq \|W\|\sigma_s(x)_1 \) and that \( \|t\|_\infty = 1/pm \). Next, recalling that \( \|W\|\|W^{-1}\| \leq 3 \) and that \( \|W^{-1}\| \leq 2 \), it
follows immediately from the estimate above that
\[
\|x - z\|_1 \leq \|W^{-1}\| \|W(z - x)\|_1 \\
\leq \|W^{-1}\| \left( \|W\|C\sigma_s(x)_1 + \|A(z - x)\|_1 \left( \frac{C}{pm} + \|W\|D\tau \right) \right) \\
\leq 3C\sigma_s(x)_1 + \|A(z - x)\|_1 \left( \frac{2C}{pm} + 3D\tau \right).
\]
To get the first result, choose \(z = z^*\) to be the minimizer of the optimization program (6), therefore \(\|Az^* - y\|_1 \leq \|Ax^* - y\|_1 \leq 2\|e\|_1\). The second result follows from the same arguments with an extra application of the Cauchy-Schwarz inequality.

5.1. Numerical Experiments. We now turn to numerical experiments to support our theory. To this end, we considered the following setup: A vector \(y = Ax + e\) is received, where the ground truth \(x\) is a nonnegative \(s\)-sparse vector (a priori generated at random) in \(\mathbb{R}^n\) with \(n = 2000\), \(A\) is a Bernoulli(\(p\)) matrix with \(m = \lceil \log(n)/p \rceil\), where \(p = 1/2s\), and the noise vector \(e\) is generated at random by taking entrywise absolute value of a normalized isotropic multivariate gaussian. The value of \(s\) varies between 50 to 200.

To recover \(x\) based on the measurement \(y\) and the measurement matrix \(A\), we computed \(\hat{x}\) via the nonnegative least squares (7) in Matlab.

For each experiment, we run ten times and averaged the output. For the first experiment, we settled \(e \equiv 0\). One could observe exact recovery was achieved up to an error of \(10^{-10}\) for \(s = 50, 100, 200\). For the second experiment, we considered noisy measurements. More accurately, we considered \(e\) as defined above, and varied the value of \(s = 50, 100, 200\) again. The following figures show that \(\hat{x}\) is quite close to the ground truth \(x\) which support that is possible to recover the signal with a sparse matrix (the number of ones decreasing with \(s\)
In the third experiment, we fixed $s = 200$ and increased the noise, five times greater and ten times greater. One can see that even five times greater, the non-negative least squares could accurately recover $x$, however for ten times greater, its performance started to become inaccurate, meaning that one would need to increase the number of measurements $m$. 
6. Acknowledgement

The author would like to thank Afonso Bandeira, Hendrik Peterson, Victor Souza and Nikita Zhivotovsky for helpful discussions. The author is grateful to Geoffrey Chinot and Sara van de Geer for a careful revision of the manuscript.

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7. Appendix

The appendix is dedicated to the proof of Theorem 4. In this section, we assume that $G = (L, R, E)$ is an $(s, d, \delta, \theta)$ quasi-regular lossless expander and all the statements are (implicitly) about $G$.

To start, we present two lemmas that reflect the intuition that a lossless expander possesses a small number of collisions. To this end, recall that $| \cdot |$ denote the cardinality of a set, and consider the following sets: For a set $J \subseteq [n]$, the set of all edges emanating from $J$

$$E(J) := \{ji \in E \text{ with } j \in J\},$$

the set of all right vertices connect to $J$

$$R(J) := \{i \in R : \text{ there is a } j \in J \text{ for which } ji \in E\},$$

and the set of edges emanating from $K$ and whose right vertices are also connected to the left vertices in $J$

$$E(J; K) := \{ji \in E(K) \text{ with } i \in R(J)\}.$$

**Lemma 3.** Let $J$ and $K$ be two disjoint sets of left vertices satisfying the estimate $|J| + |K| \leq s$. Then,

$$|E(J; K)| \leq (\theta_{s, \delta} + 3\delta)ds.$$

**Proof.** Notice that $|J \cup K| = |J| + |K|$ because $J$ and $K$ are disjoint sets, we shall use it several times in the proof. Next, let $E_0$ be the set of edges emanating from $J \cup K$. We split $E_0$ into three distinct sets (See Figure 1):

1. The set $E_1$ of edges emanating from $J$.
2. The set $E_2$ of edges emanating from $K$ and whose right vertices are not connected to any left vertex in $J$.
3. The set (of interest) $E(J; K)$.

We know that $|E_0| \leq (1 + \delta)d(|J| + |K|)$ and that $|E_1| \geq (1 - \delta)d|J|$ because of the quasi-regularity of the graph. From the fact that $J$ and $K$ are disjoint sets, we have

$$(8) \quad |E(J; K)| = |E_0| - |E_1| - |E_2| \leq 2\delta d|J| + (1 + \delta)d|K| - |E_2|.$$  

We proceed to bound the cardinality of $E_2$. Observe that each right vertex $i \in R(K) \setminus R(J)$ gives rise to at least one edge emanating from $K$ whose right vertex is not connected to any left vertex in $J$. Therefore,

$$(9) \quad |E_2| \geq |R(K) \setminus R(J)| = |R(J \cup K)| - |R(J)|.$$  

Since $|R(J)| \leq (1 + \delta)d|J|$ (quasi-regularity) and by the expansion property (3), we have

$$|R(J \cup K)| \geq (1 - \theta_{s, \delta})d(|J \cup K|) = (1 - \theta_{s, \delta})d(|J| + |K|),$$

which combined with (9) implies that

$$|E_2| \geq (1 - \theta_{s, \delta})d(|J| + |K|) - (1 + \delta)d|J| = |J|d(-\theta_{s, \delta} - \delta) + (1 - \theta_{s, \delta})d|K|.$$  

Finally, we plug this into (8) to obtain that

$$|E(J; K)| \leq 2\delta d|J| + |J|d(\theta_{s, \delta} + \delta) + (1 + \delta)d|K| - (1 - \theta_{s, \delta})d|K| = |J|d(\theta_{s, \delta} + 3\delta) + |K|d(\delta + \theta_{s, \delta}).$$

Recalling that $|J| + |K| \leq s$ (by assumption), we obtain that $|E(J; K)| \leq sd(\theta_{s, \delta} + 3\delta)$, as claimed. \qed
Figure 1. The decomposition from Lemma 3. The blue vertices belong to $J$ and the green ones to $K$. $E_0$ is the set of all colored edges: The ones in blue belong to $E_1$, the ones in red belong to $E_2$, and the ones in green belong to $E(J; K)$.

Lemma 4. Let $S$ be a set of $s$ left vertices. For each index $i \in R(S)$, set $l(i)$ to be a fixed left vertex connected to $i$. Then the set $E^* := \{ji \in E(S) : j \neq l(i)\}$, has cardinality at most $(\theta_{s,\delta} + \delta)ds$.

Proof. The set $E(S)$ of the edges emanating from $S$ can be partitioned into two sets $E^*(S)$ and $E^c := \{l(i)i, i \in R(S)\}$. Since $|E(S)| \leq (1 + \delta)ds$ (quasi-regularity) and $E^c \geq (1 - \theta_{s,\delta})ds$ (expansion property), we reach the claim.

The next two lemmas are about technical properties of lossless expanders. To state it accurately, recall that $A \in \{0, 1\}^{m \times n}$ is the adjacency matrix of (quasi-regular expander) $G$.

Lemma 5. Let $S$ and $T$ be two disjoint subsets of $[n]$ with cardinalities $s$ and $t$, respectively. Then for any $x \in \mathbb{R}^n$, the following holds

$$\|(Ax_S)_{R(T)}\|_1 \leq (\theta_{s+t,\delta} + 3\delta)d(s + t)\|x_S\|_\infty.$$  

Proof. We estimate the term $\|(Ax)_{R(T)}\|_1$ as follows

$$\|(Ax)_{R(T)}\|_1 = \sum_{i \in R(T)} |(Ax_S)_i| = \sum_{i=1}^m \sum_{j \in S} |A_{ij}x_j| \leq \sum_{i=1}^m \sum_{j \in S} 1(i \in R(T))1(j \in E) |x_j| = \sum_{j \in E(S; T)} |x_j| \leq |E(S; T)|\|x_S\|_\infty.$$  

We now apply Lemma 3 to conclude the proof.
Lemma 6. Given an $s$-sparse vector $w \in \mathbb{R}^n$, let $w^* \in \mathbb{R}^m$ be defined by $w^*_i := w_{l(i)}$, where $l(i) := \arg \max_j \{ w_j, \overline{ji} \in E \}$. Then,

$$\|Aw - w^*\|_1 \leq (\theta_{s, \delta} + \delta)d\|w\|_1.$$ 

Proof. Without loss of generality, assume that the entries of $w$ are ordered in a non-increasing order of magnitude, namely $|w_1| \geq \ldots \geq |w_s| \ (w_{s+1} = \ldots = w_n = 0)$. By doing that, we interpret the edge $\overline{(i)i}$ as the first edge arriving at the right vertex $i$. Since the support of the vector $w$ is $[s]$ and from the fact that $l(i) \in [s]$ for every $i \in R([s])$, we obtain that

$$(Aw - w^*)_i = \sum_{j=1}^n A_{ij} w_j - w_{l(i)} = \sum_{j \in S} \mathbb{1}_{(\overline{ji} \in E \text{ and } j \neq l(i))} w_j.$$ 

Therefore, it follows that

$$\|Aw - w^*\|_1 = \sum_{i=1}^m \left| \sum_{j \in S} \mathbb{1}_{\overline{ji} \in E \text{ and } j \neq l(i)} w_j \right| \leq \sum_{j \in S} \sum_{i=1}^m \mathbb{1}_{\overline{ji} \in E \text{ and } j \neq l(i)} \|w_j\| =: \sum_{j=1}^m c_j |w_j|,$$

where $c_j := \sum_{i=1}^m \mathbb{1}_{\overline{ji} \in E \text{ and } j \neq l(i)}$. For all $k \in [s]$, we have

$$(10) \quad C_k := \sum_{j=1}^k c_j = |\{ \overline{ji} \in E([k]), j \neq l(i) \}| \leq (\theta_{s, \delta} + \delta)dk,$$

where the last inequality follows from Lemma 4. We now perform summation by parts (assuming $C_0 := 0$),

$$\sum_{j=1}^s c_j |w_j| \leq \sum_{j=1}^s C_j |w_j| - C_{j-1} |w_j| = \sum_{j=1}^{s-1} C_j (|w_j| - |w_{j+1}|) + C_s |w_s|$$

Since $|w_j| - |w_{j+1}| \geq 0$ due to the non-increasing order, we apply (10) in each summand to obtain that

$$\sum_{j=1}^s c_j w_j \leq \sum_{j=1}^{s-1} (\theta_{s, \delta} + \delta) d_j (|w_j| - |w_{j+1}|) + (\theta_{s, \delta} + \delta) ds |w_s| = \sum_{j=1}^s (\theta_{s, \delta} + \delta) d |w_j|,$$

and the claim follows as the right-hand side is exactly $(\theta_{s, \delta} + \delta)d\|w\|_1$.}

We are now in position to prove Theorem 4.

Proof. (of Theorem 4) Let $v \in \mathbb{R}^n$ be a fixed vector. We define $S_0$ to be the set of $s$ largest entries of $v$ in absolute value, $S_1$ to be the set of $s$ largest entries not in $S_0$, and the sets $S_2, S_3, \ldots, S_{[\frac{n}{2}]}$ are defined analogously. Now, we write

$$(11) \quad (Av)_i = \sum_{j \in [n]} A_{ij} v_j = \sum_{k \geq 0} \sum_{j \in S_k \overline{ji} \in E} v_{l(i)} + \sum_{k \geq 1} \sum_{j \in S_k \overline{ji} \in E} v_j.$$ 

Moreover, for $i \in R(S_0)$,

$$(Av)_i = \sum_{j \in [n]} A_{ij} v_j = \sum_{k \geq 0} \sum_{j \in S_k \overline{ji} \in E} v_{l(i)} + \sum_{k \geq 1} \sum_{j \in S_k \overline{ji} \in E} v_j.$$
It follows that
\[
|v_{l(i)}| \leq \sum_{\{j \in S_0 / l(i), j \in E\}} |v_j| + \sum_{k \geq 1} \sum_{\{j \in S_k, j \in E\}} |v_j| + |(Av)_i|
\]
Summing over all \(i \in R(S_0)\) and using equation (11),
\[
(1 - \delta)d\|v_{S_0}\|_1 \leq \sum_{i \in R(S_0)} \left(2 \sum_{\{j \in S_0 / l(i), j \in E\}} v_j + \sum_{k \geq 1} \sum_{\{j \in S_k, j \in E\}} v_j\right) + \|Av\|_1.
\]
In order to handle the first term in the right hand side, we apply Lemma 6 with \(w = \|v_{S_0}\|_1\) (taken entrywise) to obtain that
\[
2 \sum_{i \in R(S_0)} \sum_{\{j \in S_0 / l(i), j \in E\}} |v_j| = 2\|Av - w^*\|_1 \leq 2(\theta_s, \delta + \delta)d\|v_{S_0}\|_1.
\]
For the second term, Lemma 5 implies that
\[
\sum_{k \geq 1} \sum_{i \in R(S_k), j \in S_k, j \in E} |v_j| = \sum_{k \geq 1} \|(|A|v_{S_k})_{R(S_k)}\|_1 \leq \sum_{k \geq 1} 2(\theta_{2s, \delta} + 3\delta)d\|v_{S_k}\|_\infty \\
\leq 2(\theta_{2s} + 3\delta)d \sum_{k \geq 1} \|v_{S_{k-1}}\|_1 = 2(\theta_{2s, \delta} + 3\delta)d\|v\|_1,
\]
where in the last inequality we used that \(s\|v_{S_k}\|_\infty \leq \|v_{S_{k-1}}\|_1\). Putting the estimates together,
\[
(1 - \delta)d\|v_{S_0}\|_1 \leq 2(\theta_s + \delta)d\|v_{S_0}\|_1 + 2(\theta_{2s, \delta} + 3\delta)d\|v\|_1 + \|Av\|_1 \\
\leq 4(\theta_{2s, \delta} + 3\delta)d\|v_{S_0}\|_1 + 2(\theta_{2s, \delta} + 3\delta)d\|v_{S_{c-1}}\|_1 + \|Av\|_1.
\]
Rearranging the inequality we conclude the proof. \(\square\)