ON SPLITTING POLYNOMIALS WITH NONCOMMUTATIVE COEFFICIENTS

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Abstract. It is shown that for every splitting of a polynomial with noncommutative coefficients into linear factors \((X - a_k)\) with \(a_k\)'s commuting with coefficients, any cyclic permutation of linear factors gives the same result and all \(a_k\) are roots of that polynomial. It implies that although the set of \(a_k\)'s appearing in a splitting of a polynomial with commutative coefficients in some noncommutative extension does not determine the splitting (in general), the cyclic order consisting of roots appearing in the splitting does. Examples of this phenomenon are given.

1. Introduction. Let \(f(X) = f_nX^n + f_{n-1}X^{n-1} + \cdots + f_0 \in A[X]\) be a polynomial with coefficients in a commutative unital ring \(A\). Suppose there is given a splitting of \(f(X)\) in \(A[X]\)

\[
f(X) = f_n(X - a_1) \cdots (X - a_n).
\]

Then by the substitution homomorphism argument one sees that all \(a_k\)'s are roots of \(f(X)\) and by commutativity of \(A[X]\) any permutation of them defines the same splitting. Therefore the problem of splitting of a given polynomial reduces to the problem of finding the set of its roots. This fact is fundamental for Galois theory and algebraic geometry.

In the case of noncommutative coefficients of a given polynomial the situation is much worse. First of all, a given splitting does not reduces to the set of elements \(a_k\), since we cannot permute linear factors because of noncommutativity of \(A[X]\). Moreover, if \(a \in A\) is not central in \(A\) then the substitution homomorphism of rings

\[
Z[X] \to A, \quad X \mapsto a
\]

does not extend to a homomorphism of \(A\)-algebras

\[
A[X] \to A, \quad X \mapsto a,
\]

because \(X\) is central in \(A[X]\). This means that one can not use the substitution \(A\)-algebra homomorphism argument to prove that elements \(a_1, \ldots, a_n\) appearing in the decomposition

\[
f(X) = f_n(X - a_1) \cdots (X - a_n)
\]

are roots of \(f(X)\). The problem of such splittings in terms of relationships between coefficients of a given polynomial with a generic set of its (left or right) roots and elements \(a_k\) (so called pseudoroots) was related to quadratic algebras with structure encoded by graphs in \([1][3][2][5]\). However, these relationships are

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much more complicated than in the commutative case and make sense only if some elements of the algebra are invertible.

The interest for splitting polynomials in noncommutative algebras started in 1921 when Wedderburn proved \[6\] that any minimal polynomial \(f(X) \in K[X]\) of an element of a central division algebra \(A\) algebraic over the center \(K\) of \(A\) splits in \(A[X]\) into linear factors which can be permuted cyclically and every pseudoroot appearing in this splitting is a root of \(f(X)\). This fact was very helpful in determining the structure of division algebras of small order \[6\] and found many other applications (see e.g. \[4\] for references).

The aim of this paper is to show that under the assumption that coefficients (which do not have to commute one with each other) commute with pseudoroots (which do not have to commute one with each other) the situation is much closer to the commutative case. We show that then pseudoroots are roots and any cyclic permutation of them gives the same splitting. This means that instead of finite sets of commutative roots (ordered \(n\)-tuples up to all permutations) we obtain finite cyclically ordered sets (ordered \(n\)-tuples up to all cyclic permutations) of noncommutative roots.

We give examples of such splittings, where in spite of cyclic symmetry, transposition of any two consecutive linear factors is impossible. These examples are minimal in the sense that the algebra in which we split our polynomial is generated by the roots appearing in the splitting.

It does not seem that this basic (elementary, trivially provable etc but not trivial at all) fact could be easily derived from the known theory of splitting polynomials in noncommutative algebras (Gelfand-Retakh-Wilson \[3\]). The reason is that the condition of commutativity between coefficients and pseudoroots is a closed condition while the general theory works for generic elements. We will see in examples that our result is true even if differences of pseudoroots are not invertible or even nilpotent.

2. Results.

**Lemma 1.** Let \(g, h\) be elements of a monoid \(G\), where \(h\) is right cancelable and the product \(gh\) commutes with \(h\). Then \(g\) and \(h\) commute.

*Proof.* Since \(gh\) commutes with \(h\), we have

\[(5) \quad ghh = hgh.\]

But \(h\) is right cancelable, hence consequently

\[(6) \quad gh = hg. \quad \Box\]

**Lemma 2.** Let \(A\) be a ring and \(A^a\) be its subring of elements commuting with a fixed \(a \in A\). If \(f(X) \in A^a[X]\) decomposes in \(A[X]\) as follows

\[(7) \quad f(X) = g(X)(X - a)\]

then \(g(X) \in A^a[X]\) and \(f(a) = 0\).

*Proof.* To prove that \(g(X) \in A^a[X] = A[X]^{X-a}\) take \(G = A[X]\) with multiplication of polynomials, \(g = g(X),\ h = h(X) = X - a\) and apply Lemma 1.
To prove the root property apply to (7) the substitution homomorphism

\[ A^a[X] \to A^a, \quad X \mapsto a. \]

**Theorem 1.** Let \( A \) be a unital ring and \( A^{a_1, \ldots, a_n} \) be its subring of elements commuting with \( a_1, \ldots, a_n \in A \). If \( f(X) \in A^{a_1, \ldots, a_n}[X] \) splits in \( A[X] \) as follows

\[ f(X) = f_n(X - a_1)(X - a_2) \cdots (X - a_n) \]

then

\[ f(X) = f_n(X - a_n)(X - a_1) \cdots (X - a_{n-1}) \]

and

\[ f(a_1) = \cdots = f(a_n) = 0. \]

**Proof.** To prove the cyclic property of the splitting take \( a = a_n, \ g(X) = f_n(X - a_1)(X - a_2) \cdots (X - a_{n-1}) \) and apply Lemma 2. Then Lemma 2 also implies the root property for \( a_n \). By the cyclic property the same holds for other \( a_k \)'s. □

**Corollary 1.** For any splitting as in Theorem 1 substitution homomorphisms

\[ A^{a_1, \ldots, a_n}[X] \to A^{a_1, \ldots, a_n}[a_k] \subset A^{a_k} \subset A \]

\[ X \mapsto a_k \]

define a ring homomorphism

\[ A^{a_1, \ldots, a_n}[X]/(f(X)) \to A^{a_1, \ldots, a_n}[a_1] \times \cdots \times A^{a_1, \ldots, a_n}[a_n] \]

where cyclic permutations of roots in the splitting correspond to cyclic permutations of factors in the cartesian product on the right hand side.

### 3. Examples.

**Example 1.** Let \( A \) be the ring of upper triangular \( 2 \times 2 \) matrices over a nonzero commutative ring \( K \) and take \( f(X) = X^2(X - 1) \in K[X] \subset A[X] \). Although it has a double root in \( K \) it can be split in \( A \) as follows

\[ f(X) = (X - a_1)(X - a_2)(X - a_3), \]

where

\[ a_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \]

are pairwise distinct roots in \( A \). This can be viewed as a kind of resolution of singularity by passing to a noncommutative extension.

The linear factors can be cyclically permuted, but none two of them can be transposed, because

\[ [a_1, a_2] = [a_2, a_3] = [a_3, a_1] = -a_2 \neq 0. \]
Since the Vandermonde matrix

$$\begin{pmatrix}
a_1^2 & a_2^2 & a_3^2 \\
a_1 & a_2 & a_3 \\
1 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}$$

is not invertible, we are far from the theory presented in [2]. However, it is a remarkable fact, that this example shares many properties with splitting of a polynomial in its Galois extension.

First of all, it is obvious that $A^{a_1,a_2,a_3} = K$. Moreover, $A$ is freely generated by $a_1, a_2, a_3$ as a module over $K$ with a multiplication table

$$
\begin{array}{c|ccc}
& a_1 & a_2 & a_3 \\
\hline
a_1 & 0 & 0 & 0 \\
a_2 & 0 & 0 & 0 \\
a_3 & 1 & 1 & 0 \\
\end{array}
$$

If $K$ doesn’t contain nontrivial idempotents all endomorphisms of the extension $K \subset A$ come in families $\varepsilon, \varepsilon', \varepsilon^\sigma_s, \varepsilon_s$ parameterized by elements $\sigma$ and $s$, where $\sigma$’s are elements of the multiplicative monoid of $K$ acting (from the right) on elements $s$ of the (right) $K$-module $K$ by right multiplication. The logic of this notation will be clear later, when the rules of matrix multiplication

$$
\begin{pmatrix}
\tau & 0 \\
t & 1
\end{pmatrix}
\begin{pmatrix}
\sigma & 0 \\
t & 1
\end{pmatrix} =
\begin{pmatrix}
\tau \sigma & 0 \\
t \sigma + s & 1
\end{pmatrix},
\begin{pmatrix}
t & 1 \\
\sigma & 0 \\
1 & 1
\end{pmatrix} =
\begin{pmatrix}
t \sigma + s & 1
\end{pmatrix}
$$

will appear in the structure of the endomorphism monoid and its various actions.

The endomorphism monoid is determined by its values on basic elements $a_1, a_2, a_3$ as follows

$$
\begin{array}{c|ccc}
& a_1 & a_2 & a_3 \\
\hline
\varepsilon & 1 & 0 & 0 \\
\varepsilon' & 0 & 0 & 1 \\
\varepsilon^\sigma_s & a_1 + sa_2 & \sigma a_2 & (1 - \sigma - s)a_2 + a_3 \\
\varepsilon_s & (1 - s)a_2 + a_3 & 0 & a_1 + sa_2 \\
\end{array}
$$

hence the monoid structure is

$$
\begin{array}{cccccc}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon' & \varepsilon' & \varepsilon' & \varepsilon' & \varepsilon' \\
\varepsilon^\sigma_s & \varepsilon^\sigma_s & \varepsilon^\sigma_s & \varepsilon^\sigma_s & \varepsilon^\sigma_s \\
\varepsilon_s & \varepsilon_s & \varepsilon_s & \varepsilon_s & \varepsilon_s \\
\end{array}
$$

with the neutral element $\varepsilon_0^t$. Note that the only invertible elements are $\varepsilon_t^\tau$ with $\tau$ invertible in $K$.

If $K$ is a domain all roots and all cycles of $f(X)$ also come in families parameterized by elements $\tau$ and $t$. 
The families of roots are
\[ r^\tau = \tau a_2, \]
\[ r^\tau_t = (1 - \tau - t)a_2 + a_3, \]
\[ r_t = a_1 + ta_2, \]
\[ r = a_1 + a_2 + a_3. \]

The families of cycles are
\[ c^\tau = (r^\tau, r^{-\tau}, r), \]
\[ c_t^\tau = (r^\tau, r^\tau_t, r_t), \]
\[ c_t = (r^0, r_t, r^0_t). \]

We see that
(a) The set of all roots is the union of supports of all cycles.

One can check that
(b) The only cycles whose roots form a basis of \( A \) as a free \( K \)-module are \( c^\tau_t \) with \( \tau \) invertible in \( K \).

The action of the endomorphism monoid on roots is

| \( \varepsilon \) | \( r^\tau \) | \( r^\tau_t \) | \( r_t \) | \( r \) |
|---|---|---|---|---|
| \( \varepsilon \) | \( r^\tau \) | \( r^\tau_t \) | \( r_t \) | \( r \) |
| \( \varepsilon' \) | \( r^\tau \) | \( r \) | \( r^0 \) | \( r \) |
| \( \varepsilon_s^\sigma \) | \( r^\tau \sigma + s \) | \( r^\tau \sigma + s \) | \( r_t \sigma + s \) | \( r \) |
| \( \varepsilon_s \) | \( r \) | \( r \) | \( r_s \) | \( r \) |

which implies that
(c) Every root is a translate of \( r^1 \) or \( r^1_0 \).
(d) Every root can be translated to \( r^0 \) or \( r \).

The action of the endomorphism monoid on cycles is

| \( \varepsilon, \varepsilon' \) | \( c^\tau \) | \( c_t^\tau \) | \( c_t \) |
|---|---|---|---|
| \( \varepsilon, \varepsilon' \) | \( c^\tau \) | \( c_t^\tau \) | \( c_t \) |
| \( \varepsilon_s^\sigma \) | \( c^\tau \sigma + s \) | \( c_t^\tau \sigma + s \) | \( c_t \) |
| \( \varepsilon_s \) | \( c^0 \) | \( c_s \) | \( c_s^0 \) |

which implies that
(e) Every cycle is a translate of \( c^1 \) or \( c^1_0 \).
(f) Every cycle can be translated to \( c^0 \).

Points (e) and (f) replace the fact that the Galois group of a splitting field of a separable polynomial permutes the roots. Points (c) and (d) replace the fact that this action is transitive.

Note however that in our case there are many cycles and endomorphisms can move roots from one cycle to another. It turns out that roots are no more equivalent. Instead of strict equivalence of roots induced by the transitive Galois action we have the action of endomorphisms on some lattice of roots. This can be described as follows.
The lattice of ideals in $K[X]$ defines a partial order on roots by taking the minimal polynomial $r_0 \succ r_\tau$, $\tau \neq 0$ which looks as follows

\[
\begin{array}{c|c|c|c|c}
\hline
r_0 & r_\tau, \tau \neq 0 & r_t & r \\
\hline
X & X^2 & X(X-1) & X(X-1) & X-1 \\
\hline
\end{array}
\]

which looks as follows

\[
\begin{array}{c|c|c|c|c}
\hline
r_0 & r_\tau, \tau \neq 0 & r_t & r \\
\hline
X & X^2 & X(X-1) & X(X-1) & X-1 \\
\hline
\end{array}
\]

One can check that all endomorphisms act along the above poset of roots moving them at most upwards (it is obvious that no endomorphism can diminish a root with respect to the partial order induced by the minimal polynomial).

Among all endomorphisms only $\varepsilon$ and $\varepsilon'$ do not preserve this partial order (there are essentially four exceptions: $r_0 \succ r_t^\tau$, $r_0 \succ r_t$, $r \succ r_t^\tau$, $r \succ r_t$ but $\varepsilon'(r_0) \not\succ \varepsilon'(r_t^\tau)$, $\varepsilon(r_0) \not\succ \varepsilon(r_t)$, $\varepsilon(r) \not\succ \varepsilon'(r_t^\tau)$, $\varepsilon'(r) \not\succ \varepsilon'(r_t)$), although they do preserve the weaker partial order opposite to that which is induced by the degree (levels in the poset of roots). In particular, all automorphisms do preserve the above partial order on roots. This replaces transitivity of the Galois action on roots.

Example 2. Let $A$ be the ring of $3 \times 3$ matrices over a nonzero commutative ring $K$ and take $f(X) = X^3 - 4 \in K[X] \subset A[X]$. It can be split as follows

(19) \[ f(X) = (X - a_1)(X - a_2)(X - a_3), \]

with three pairwise distinct roots in $A$

(20) \[ a_1 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ -2 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -4 \\ 1 & 0 & 0 \end{pmatrix}. \]

Note that after reduction modulo 3 $f(X) \equiv (X - 1)^3$ and $a_1 \equiv a_2 \equiv a_3$. Therefore this can be viewed as a kind of resolution of singularity in positive characteristic by passing to a noncommutative extension of a lifting to rational integers. Again, the linear factors can be cyclically permuted. If $3 \neq 0$ in $K$ none two of them can be transposed, because

(21) \[ [a_1, a_2] = [a_2, a_3] = [a_3, a_1] = \begin{pmatrix} 0 & 0 & 6 \\ -6 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \neq 0. \]
Moreover, if 2 and 3 are not zero divisors in $K$, again $A^{a_1,a_2,a_3} = K$. Indeed, $A^{a_1,a_2,a_3} \subset A$ consists of elements
\begin{equation}
(a) = \begin{pmatrix}
\alpha & \beta & 2\gamma \\
-2\gamma & \alpha & \beta \\
-\beta & \gamma & \alpha
\end{pmatrix},
\end{equation}
where $3\beta = 0$, $6\gamma = 0$.

If $2, 3, 5$ and $7$ are invertible in $K$, $A$ is generated by $a_1, a_2, a_3$ as an algebra over $K = A^{a_1,a_2,a_3}$. Indeed, then we have
\begin{align*}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} &= -\frac{1}{9}a_1^2a_2 - \frac{1}{18}a_2^2a_3, \\
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} &= -\frac{2}{3}a_2 + \frac{1}{15}a_1^2a_2^2 - \frac{1}{10}a_2^2a_3^2,
\end{align*}
\begin{align*}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} &= -\frac{1}{3}a_1^2 + \frac{5}{6}a_2^2 + a_3^2, \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} &= \frac{1}{2}a_1^2 - a_2^2 - a_3^2,
\end{align*}
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} &= \frac{1}{3}a_1a_2^2 + \frac{5}{14}a_1^2a_2 + \frac{2}{21}a_2^2a_3, \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} &= \frac{2}{9}a_1 + \frac{2}{9}a_2 - \frac{1}{36}a_1^2a_2^2,
\end{align*}
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} &= -\frac{1}{2}a_2 - \frac{1}{20}a_1^2a_2^2 - \frac{1}{20}a_2^2a_3^2, \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} &= \frac{2}{3}a_1^2 - \frac{2}{3}a_2^2 - a_3^2,
\end{align*}
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} &= \frac{1}{6}a_1a_2^2 - \frac{1}{9}a_1^2a_2 - \frac{2}{9}a_2^2a_3.
\end{align*}

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