EMBEDDING THEOREMS IN THE FRACTIONAL ORLICZ-SOBOLEV SPACE AND APPLICATIONS TO NON-LOCAL PROBLEMS

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ABSTRACT. In the present paper, we deal with a new continuous and compact embedding theorems for the fractional Orlicz-Sobolev spaces, also, we study the existence of infinitely many nontrivial solutions for a class of non-local fractional Orlicz-Sobolev Schrödinger equations whose simplest prototype is

$$(-\triangle)^s_m u + V(x)m(u) = f(x,u), \quad x \in \mathbb{R}^d,$$

where $0 < s < 1$, $d \geq 2$ and $(-\triangle)^s_m$ is the fractional $M$-Laplace operator. The proof is based on the variant Fountain theorem established by Zou.

1. Introduction. In this paper, we are concerned with the study of the fractional $M$-Laplacian equation:

$$(-\triangle)^s_m u + V(x)m(u) = f(x,u), \quad x \in \mathbb{R}^d,$$  \hfill (1)

where $(-\triangle)^s_m$ is the fractional $M$-Laplace operator, $0 < s < 1$, $d \geq 2$, $m : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, $V : \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are given functions.

In the last years, problem (1) has received a special attention for the case $s = 1$ and $m(t) = t$, that is, when it is of the form

$$-\triangle u + V(x)u = f(x,u), \quad x \in \mathbb{R}^d.$$  \hfill (2)

We do not intend to review the huge bibliography related to the equations like (2), we just emphasize that the potential $V : \mathbb{R}^d \to \mathbb{R}$ has a crucial role concerning the existence and behaviour of solutions. For example, when $V$ is radially symmetric, it is natural to look for radially symmetric solutions, see [38, 40]. On the other hand, after the paper of Rabinowitz [33] where the potential $V$ is assumed to be coercive, several different assumptions are adopted in order to obtain existence and multiplicity results (see [6, 10, 23, 41, 42]).

For the case $s = 1$, problem (1) becomes

$$-\triangle_m u + V(x)m(u) = f(x,u), \quad x \in \mathbb{R}^d,$$

where the operator $\triangle_m u = \text{div}(m(|\nabla u|) \nabla u)$ named $M$-Laplacian. This class of problems arises in a lot of applications, such as, Nonlinear Elasticity, Plasticity, ...
Generalized Newtonian Fluid, Non-Newtonian Fluid, Plasma Physics. The reader can find more details involving this subject in [2, 12, 28, 29] and the references therein.

Notice that when $0 < s < 1$ and $m(t) = |t|^{p-2}t$, $p \geq 2$, problem (1) gives back the fractional Schrödinger equation

$$(-\Delta)^s_p u + V(x)|u|^{p-2}u = f(x, u), \quad x \in \mathbb{R}^d,$$

(3)

where $(-\Delta)^s_p$ is the non-local fractional $p$-Laplacian operator. The literature on non-local operators and on their applications is quite large. We can quote [7, 19, 20, 36, 37] and the references therein. We also refer to the recent monographs [19, 31] for a thorough variational approach of non-local problems. In the last decade, many several existence and multiplicity results have been obtained concerning the equation (3), (see [4, 16, 39]). In [11], the authors studied the existence of multiple solutions where the nonlinear term $f$ is assumed to have a superlinear behaviour at the origin and a sublinear decay at infinity. In [3], Vincenzo studied the existence of infinitely many solutions for the problem (3), when $f$ is superlinear and $V$ can change sign.

Contrary to the classical fractional Laplacian Schrödinger equation that is widely investigated, the situation seems to be in a developing state when the new fractional $M$-Laplacian is present. In this context, the natural setting for studying problem (1) are fractional Orlicz-Sobolev spaces. Currently, as far as we know, the only results for fractional Orlicz-Sobolev spaces and fractional $M$-Laplacian operator are obtained in [5, 8, 9, 13, 14, 15, 32, 35]. In particular, in [13], Bonder and Salort define the fractional Orlicz-Sobolev space associated to an $N$-function $M$ and a fractional parameter $0 < s < 1$ as

$$W^{s,M}(\Omega) = \left\{ u \in L^M(\Omega) : \int_{\Omega} \int_{\Omega} M\left(\frac{u(x) - u(y)}{|x - y|^s}\right) \frac{dxdy}{|x - y|^d} < \infty \right\},$$

where $\Omega$ is an open subset of $\mathbb{R}^d$ and $L^M(\Omega)$ is the Orlicz space. They also define the fractional $M$-Laplacian operator as,

$$(-\Delta)^s_{M} u(x) = P.V. \int_{\mathbb{R}^d} m\left(\frac{u(x) - u(y)}{|x - y|^s}\right) \frac{dy}{|x - y|^{d+s}},$$

(4)

this operator is a generalization of the fractional $p$-Laplacian. They established compact embedding result referred to under the blanket title “the Fréchet-Kolmogorov compactness theorem” which gives the compact embedding of $W^{s,M}(\Omega) \hookrightarrow L^M(\Omega)$ when $\Omega$ is a bounded in $\mathbb{R}^d$. They also deduce some consequences such as $\Gamma$-convergence of the modulants and convergence of solutions for some fractional versions of the $(\Delta)_m$ operator as the fractional parameter $s \uparrow 1$.

Motivated by these above results, our first aim is to prove the compact embedding $W^{s,M}(\Omega) \hookrightarrow L^{M^*}(\Omega)$ where $M^*$ is the Sobolev conjugate of $M$ and $\Omega$ is bounded. Furthermore, we state the continuous embedding $W^{s,M}(\mathbb{R}^d) \hookrightarrow L^{M^*}(\mathbb{R}^d)$. Hence the compact embedding $W^{s,M}(\Omega) \hookrightarrow L^{\Phi}(\Omega)$ and the continuous embedding $W^{s,M}(\mathbb{R}^d) \hookrightarrow L^{\Phi}(\mathbb{R}^d)$ remain true for any $N$-function $\Phi$ such that $M^*$ is essentially stronger than $\Phi$. (see Definition 4).

Our next aim is to study the existence and the multiplicity of nontrivial weak solutions of problem (1), where the new fractional $M$-Laplacian is present. Under suitable conditions on the potentials $V$ and $f$ (will be fixed below), we deal with a new compact embedding theorem on the whole space $\mathbb{R}^d$. Also we establish some
useful inequalities which yields to apply a variant of Fountain theorem due to Zou [43]. As far as we know, all these results are new.

Related to functions \( m, M, V \) and \( f \), our hypotheses are the following:

**Conditions on \( m \) and \( M \):**

1. \( 1 < m_0 = \inf_{t>0} \frac{tm(t)}{M(t)} \leq \frac{tm(t)}{M(t)} \leq m^0 = \sup_{t>0} \frac{tm(t)}{M(t)} < m^*_0 < \infty \), for all \( t \neq 0 \) where

   \[ M(t) = \int_0^{|t|} m(s)ds \quad \text{and} \quad m^*_0 = \frac{dm_0}{d-m_0}. \]

2. \( (M_1) \) There exists \( 1 < \mu < m_0 \), such that

   \[ \lim_{|t| \to +\infty} \frac{|t|^\mu}{M(t)} = 0. \]

3. \( (M_2) \) The function \( t \mapsto M(\sqrt{t}), \ t \in [0, \infty[ \) is convex.

4. \( (M_3) \) \( \int_0^1 \frac{M^{-1}(\tau)}{\tau^{\frac{s}{d}}} d\tau < \infty \) and \( \int_1^\infty \frac{M^{-1}(\tau)}{\tau^{\frac{s}{d}}} d\tau = \infty \), where \( 0 < s < 1 \).

**Conditions on \( V \):**

1. \( (V_1) \) \( V \in C(\mathbb{R}^d, \mathbb{R}) \) and \( \inf_{x \in \mathbb{R}^d} V(x) \geq V_0 > 0. \)

2. \( (V_2) \) \( \text{meas}\{x \in \mathbb{R}^d: V(x) \leq L\} < \infty \), for all \( L > 0 \), where \( \text{meas}(\cdot) \) denotes the Lebesgue measure in \( \mathbb{R}^d \).

**Conditions on \( f \):**

1. \( (f_1) \) \( f(x, u) = p\xi(x)|u|^{p-2}u, 1 < p < \mu \) and \( \xi : \mathbb{R}^d \to \mathbb{R} \) is a positive continuous function such that \( \xi \in L^{\frac{d}{\mu}}(\mathbb{R}^d) \).

**Remark 1.** We mention some examples of functions \( m \) which are increasing homeomorphisms and satisfy conditions \( (m_1), (M_1) \) - \( (M_2) \):

1. \( m(t) = qt^{q-2}t, \) for all \( t \in \mathbb{R} \), with \( 2 < q < d \) (also satisfies condition \( (M_3) \)).
2. \( m(t) = pt^{p-2}t + qt^{q-2}t, \) for all \( t \in \mathbb{R} \), with \( 2 < p < q < d \).
3. \( m(t) = qt^{q-2}t \log(1 + |t|) + \frac{|t|^{q-1}}{1 + |t|}, \) for all \( t \in \mathbb{R} \), with \( 2 < q < d \).

Under the above hypotheses, we state our main results.

**Theorem 1.** Let \( M \) be an \( N \)-function and \( s \in (0, 1) \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^d \) with \( C^{0,1} \)-regularity and bounded boundary.

1. If \( (m_1) \) and \( (M_3) \) hold, then the embedding

   \[ W^{s, M}(\Omega) \hookrightarrow L^M(\Omega), \]  

   is continuous.

2. Moreover, for any \( N \)-function \( B \) such that \( M_* \) is essentially stronger than \( B \), denoted \( B \ll M_* \) (see Definition 4), the embedding

   \[ W^{s, M}(\Omega) \hookrightarrow L^B(\Omega), \]  

   is compact.
The boundedness of $\Omega$ in Theorem 1 is a natural requirement for the compactness theorem, but, as we shall show in the next theorem, not necessary for the continuous embedding.

**Theorem 2.** Let $M$ be an $N$-function and $s \in (0, 1)$.

1. If $(m_1)$ and $(M_3)$ hold, then the embedding
   \[ W^{s,M}(\mathbb{R}^d) \hookrightarrow L^M(\mathbb{R}^d), \]
   is continuous.

2. Moreover, for any $N$-function $B$ such that $B \prec \prec M^*$, the embedding
   \[ W^{s,M}(\mathbb{R}^d) \hookrightarrow L^B(\mathbb{R}^d), \]
   is continuous.

In studying the existence of solution of problem (1), it is common to relax the notion of solution by considering weak solutions. By these we understand functions in $W^{s,M}(\Omega)$ that satisfy (1) in sense of distribution.

**Theorem 3.** Suppose that $(m_1)$, $(M_1) - (M_2)$, $(V_1) - (V_2)$ and $(f_1)$ hold. Then, problem (1) possesses infinitely many nontrivial weak solutions.

This paper is organized as follows. In Section 2, we give some definitions and fundamental properties of the spaces $L^M(\Omega)$ and $W^{s,M}(\Omega)$. In Section 3, we prove Theorems 1 and 2. In Section 4, we introduce our abstract framework related to problem (1). Finally, in Section 5, using a variant Fountain theorem [43], we prove Theorem 3.

2. Preliminaries. We start by recalling some basic facts about Orlicz spaces $L^M(\Omega)$. For more details we refer to the books by Adams [1], Kufner et al. [26], Rao and Ren [34] and the papers by Clément et al. [17, 18], Fukagai et al. [21], García-Huidobro et al. [22] and Gossez [24].

2.1. Orlicz spaces. Notation:

- $B_R^c(0) = \mathbb{R}^d \setminus B_R(0)$.
- $\|u\|_{L^\mu(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |u(x)|^\mu \, dx \right)^{1/\mu}$.

Let $M : \mathbb{R} \to \mathbb{R}_+$ be an $N$-function, i.e,

1. $M$ is even, continuous, convex,
2. $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$.

Equivalently, $M$ admits the representation:

\[ M(t) = \int_0^{|t|} m(s) \, ds, \]

where $m : \mathbb{R} \to \mathbb{R}$ is non-decreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ for all $t > 0$ and $m(t) \to \infty$ as $t \to \infty$ (see [25], page 9). We call the conjugate function of $M$, the function denoted $\overline{M}$ and defined by

\[ \overline{M}(t) = \int_0^{|t|} \overline{m}(s) \, ds, \]

where $\overline{m} : \mathbb{R} \to \mathbb{R}$, $\overline{m}(t) = \sup\{s : m(s) \leq t\}$. We observe that $\overline{M}$ is also an $N$-function and the following Young’s inequality holds true.
Proposition 2 \(\) satisfies the convex space and thus, a reflexive Banach space (see \([30]\), Proposition 2.

Let \(\Omega\) be an open subset of \(\mathbb{R}^d\) and for each \(u\) if there is no confusion we shall write \(\|\cdot\|_\infty\) such that \(\int_\Omega |u(x)|^s dx\) for some constant \(s\). Let \(M\) be a function \(\Lambda\) holds in (7) if and only if either \(t = m(s)\) or \(s = \bar{m}(t)\).

If \((M_2)\) is satisfied, another important function related to function \(M\), it is the Sobolev conjugate \(N\)-function \(M^*_N\) of \(M\) defined by,

\[
M^*_N(t) = \int_0^t \frac{M^{-1}\left(\frac{\tau}{\tau + s}\right)}{\tau + s} d\tau.
\]

In what follows, we say that an \(N\)-function \(M\) satisfies the \(\Delta_2\)-condition, if

\[
M(2t) \leq K^2 M(t) \text{ for all } t \geq 0,
\]

for each \(s > 0\), there exists \(K_s > 0\) such that

\[
M(st) \leq K_s M(t), \text{ for all } t \geq 0, \quad \text{(see \([25]\), page 23).}
\]

Definition 4. Let \(A\) and \(B\) be two \(N\)-functions, we say that \(A\) is essentially stronger than \(B\), \(B \ll A\) in symbols, if for each \(a > 0\) there exists \(x_a \geq 0\) such that

\[
B(x) \leq A(ax), \quad x \geq x_a.
\]

The previous definition 4 is equivalent to,

\[
\lim_{t \to +\infty} \frac{B(kt)}{A(t)} = 0, \text{ for all positive constant } k \text{ (see \([34]\, Theorem 2\)).}
\]

Let \(\Omega\) be an open subset of \(\mathbb{R}^d\) and \(\rho(u, M) = \int_\Omega M(u(x)) dx\). The Orlicz space \(L^M(\Omega)\) is the set of equivalence classes of real-valued measurable functions \(u\) on \(\Omega\) such that \(\rho(\lambda u, M) < \infty\) for some \(\lambda > 0\).

\(L^M(\Omega)\) is a Banach space under the Luxemburg norm

\[
\|u\|_{L^M(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega M\left(\frac{u}{\lambda}\right) dx \leq 1 \right\},
\]

if there is no confusion we shall write \(\|\cdot\|_M\) instead of \(\|\cdot\|_{L^M(\Omega)}\), whose norm is equivalent to the Orlicz norm

\[
\|u\|_{L^M(\Omega)} = \sup_{\rho(v,M) \leq 1} \int_\Omega |u(x)||v(x)| dx,
\]

and for each \(u \in L^M(\Omega), \quad \|u\|_M \leq \|u\|_{L^M(\Omega)} \leq 2\|u\|_M\) \(\text{ (see \([26]\, Theorem 4.85\)).}\)

The \(\Delta_2\)-condition with \((M_2)\) ensures that the Orlicz space \(L^M(\Omega)\) is a uniformly convex space and thus, a reflexive Banach space (see \([30]\, Proposition 2.2\)).

The Orlicz spaces Hölder's inequality reads as follows: (see \([26]\, Theorem 4.75\))

\[
\int_\Omega |uv| dx \leq \|u\|_{L^M(\Omega)} \|v\|_{L^M(\Omega)} \quad \text{for all } u \in L^M(\Omega) \text{ and } v \in L^M(\Omega).
\]

In the following, we recall a few results which will be useful in the sequel.

Proposition 1 \(([1]\)). Let \((u_n)_{n \in \mathbb{N}}\) be a sequence in \(L^M(\Omega)\) and \(u \in L^M(\Omega)\). If \(M\) satisfies the \(\Delta_2\)-condition and \(\rho(u_n - u, M) \to 0\), then \(u_n \to u\) in \(L^M(\Omega)\).

Proposition 2 \(([26]\)). Let \(M\) be an \(N\)-function. Then

\[
\|u\|_{L^M(\Omega)} \leq \rho(u, M) + 1, \quad \text{for all } u \in L^M(\Omega).
\]
Lemma 5 ([18]). Let $G$ be an $N$-function satisfying
\[
1 < g_0 := \inf_{t > 0} \frac{tg(t)}{G(t)} \leq \frac{tg(t)}{G(t)} \leq g^0 := \sup_{t > 0} \frac{tg(t)}{G(t)} < \infty
\]
where $g = G'$ and let $\xi_0(t) = \min\{t^{g_0}, t^{\alpha_0}\}$, $\xi_1(t) = \max\{t^{g_0}, t^{\alpha_0}\}$, for all $t \geq 0$. Then
\[
\xi_0(\beta)G(t) \leq G(\beta t) \leq \xi_1(\beta)G(t) \quad \text{for } \beta, t \geq 0,
\]
and
\[
\xi_0(\|u\|_{(G, \Omega)}) \leq \int_{\Omega} G(|u|) dx \leq \xi_1(\|u\|_{(G, \Omega)}) \quad \text{for } u \in L^G(\Omega).
\]

Lemma 6 ([21]). Let $M$ be an $N$-function satisfying $(m_1)$ and $(M_3)$, then the function $M_*$ satisfies the following inequality
\[
m_0^* = \frac{dm_0}{d - m_0} \leq \frac{tm_*(t)}{M_*(t)} \leq (m_0^*)_* = \frac{dm_0}{d - m_0},
\]
where $m_*$ is such that $M_*(t) = \int_0^t m_*(s) ds$.

2.2. Fractional Orlicz-Sobolev spaces. In this subsection we give a brief overview on the fractional Orlicz-Sobolev spaces studied in [13], and the associated fractional $M$-laplacian operator.

Definition 7. Let $M$ be an $N$-function. For an open subset $\Omega$ in $\mathbb{R}^d$ and $0 < s < 1$, we define the fractional Orlicz-Sobolev space $W^{s,M}(\Omega)$ as follows,
\[
W^{s,M}(\Omega) = \left\{ u \in L^M(\Omega) : \int_{\Omega} \int_{\Omega} M \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dxdy}{|x - y|^d} < \infty \right\}.
\]
This space is equipped with the norm,
\[
\|u\|_{(s,M,\Omega)} = \|u\|_{(M,\Omega)} + [u]_{(s, M, \Omega)},
\]
where $[\cdot]_{(s, M, \Omega)}$ is the Gagliardo semi-norm, defined by
\[
[u]_{(s, M, \Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} M \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dxdy}{|x - y|^d} \leq 1 \right\},
\]
if there is no confusion we shall write $[\cdot]_{(s, M)}$ and $\|\cdot\|_{(s, M)}$ instead of $[\cdot]_{(s, M, \Omega)}$ and $\|\cdot\|_{(s, M, \Omega)}$ respectively.

Proposition 3 ([13]). Let $M$ be an $N$-function such that $M$ and $\overline{M}$ satisfy the $\triangle_2$-condition, and consider $s \in (0, 1)$. Then $W^{s,M}(\mathbb{R}^d)$ is a reflexive and separable Banach space. Moreover, $C^{\infty}_0(\mathbb{R}^d)$ is dense in $W^{s,M}(\mathbb{R}^d)$.

A variant of the well-known Fréchet-Kolmogorov compactness theorem gives the compactness of the embedding of $W^{s,M}$ into $L^M$.

Theorem 8 ([13]). Let $M$ be an $N$-function, $s \in (0, 1)$ and $\Omega$ a bounded open set in $\mathbb{R}^d$. Then the embedding
\[
W^{s,M}(\Omega) \hookrightarrow L^M(\Omega)
\]
is compact.
Let \( K > \Omega \) be a bounded subset of \( \mathbb{R}^d \).

**Lemma 10.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) and \( s \in (0,1) \). If \( u \in W^{s,M}(\Omega) \) and \( f \) is Lipschitz continuous on \( \mathbb{R} \) such that \( f(0) = 0 \), then \( f \circ u \) belongs to \( W^{s,M}(\Omega) \).

**Proof.** Let \( u \in W^{s,M}(\Omega) \), then there exists \( \theta > 0 \) such that
\[
\int_{\Omega} M(\theta u(x)) \, dx < \infty \quad \text{and} \quad \int_{\Omega} \int_{\Omega} M\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^d} < \infty.
\]
Let \( K > 0 \) denotes the Lipschitz constant of \( f \), since \( f(0) = 0 \), then
\[
\int_{\Omega} M\left(\frac{\theta}{K} (f \circ u(x))\right) \, dx = \int_{\Omega} M\left(\frac{\theta}{K} (f \circ u(x) - f(0))\right) \, dx \leq \int_{\Omega} M(\theta u(x)) \, dx < \infty.
\]
Therefore \( f \circ u \in L^{M}(\Omega) \).

Now, let \( \lambda = \frac{1}{K} \), we have
\[
\int_{\Omega} \int_{\Omega} M\left(\frac{\lambda |f(u(x)) - f(u(y))|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^d} \leq \int_{\Omega} \int_{\Omega} M\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^d} < \infty.
\]
this implies that \( \lambda (f \circ u) \in W^{s,M}(\Omega) \), thus \( \lambda^{-1} \lambda (f \circ u) = f \circ u \in W^{s,M}(\Omega) \). This ends the proof.

**Lemma 11.** Let \( \Omega \) be a bounded subset of \( \mathbb{R}^d \) with \( C^{0,1} \)-regularity and bounded boundary. Let \( M \) be an \( N \)-function satisfying condition \((m_1)\). Then, given \( 0 < s' < s < 1 \), it holds that the embedding
\[
W^{s,M}(\Omega) \hookrightarrow W^{s',1}(\Omega),
\]
For arbitrary \( t \leq M(t) \), for every \( t \geq 1 \).

Proof. We closely follow the method employed in [14, Proposition 2.9]. The normalization condition \( M(1) = 1 \) is by no means restrictive. From Lemma 5 it is inferred that,

\[
J_1 \leq \int_{\Omega} \int_{\Omega} \frac{dx dy}{|x - y|^{d - (s-s')}} \leq \text{meas} |\Omega| \omega_d \frac{\delta(s-s')}{s - s'},
\]

where \( \delta \) is the diameter of \( \Omega \). To estimate the second term, we invoke (20) and we obtain

\[
J_2 \leq \delta^{(s-s')} \int_{\Omega} \int_{\Omega} M(h_u^s(x,y)) \frac{dx dy}{|x - y|^{d}} < \infty.
\]

Since \([u]_{(s,M)} = 1\), then \([u]_{|s|,1} \leq (\text{meas} |\Omega| \omega_d) \frac{s}{s - s'} + 1 \delta^{(s-s')}.\] (21)

For arbitrary \( u \in W^{s,M}(\Omega) \setminus \{0\} \), let \( v = \frac{u}{[u]_{(s,M)}} \), using inequality (21) we get

\[
[v]_{s',1} \leq \left( \frac{\text{meas} |\Omega| \omega_d}{s - s'} + 1 \right) \delta^{(s-s')}.
\]

By homogeneity of the seminorm \([\cdot]_{s',1}\), we obtain

\[
[u]_{s',1} \leq \left( \frac{\text{meas} |\Omega| \omega_d}{s - s'} + 1 \right) \delta^{(s-s')} [u]_{(s,M)}.
\] (22)

On the other hand, since \( \Omega \) is bounded, there exist \( C \geq 0 \) such that

\[
\|u\|_{L^1(\Omega)} \leq C\|u\|_{(M)} \quad \text{(see [34, Corollary 3])}.
\] (23)

Combining (22) and (23) we get the desired result.

Proof of theorem 1. Let \( u \in W^{s,M}(\Omega) \setminus \{0\} \) and suppose for the moment that \( u \) is bounded on \( \Omega \). Then \( \lambda \to \int_{\Omega} M_s(\|u(x)\|/\lambda) dx \) decreases continuously from infinity to zero as \( \lambda \) increases from zero to infinity. So that

\[
\int_{\Omega} M_s \left( \frac{\|u(x)\|}{k} \right) dx = 1 \quad \text{for some } k > 0.
\] (24)

By the definition of the norm (11), we see that \( k = \|u\|_{(M_s)} \).
Let \( \omega(t) = [M_s(t)]^\frac{d-s'}{d-x'} \) and set \( f(x) = \omega\left(\frac{|u(x)|}{k}\right) \). According to Lemma 9, \( \omega \) is Lipschitz continuous, by lemma 10 and Lemma 11, \( f \in W^{s,M}(\Omega) \cap W^{s',1}(\Omega) \). The well known embedding theorem of the classical fractional Sobolev space \( W^{s,1}(\Omega) \) (see \([20], \text{Theorem } 6.7\) ), gives

\[
W^{s',1}(\Omega) \hookrightarrow L^{\frac{d}{d-s'}}(\Omega).
\]

So there is a constant \( C_1 > 0 \) such that

\[
1 = \left( \int_\Omega M_s\left(\frac{|u(x)|}{k}\right) \right)^{\frac{d-x'}{d-s'}} = \|f\|^\frac{d}{d-s'}_{L^{\frac{d}{d-s'}}(\Omega)} \leq C_1(\|f\|_{L^1(\Omega)} + [f]_{s',1})
\]

\[
= C_1\left(\int_\Omega \omega\left(\frac{|u(x)|}{k}\right) \right) dx + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{d+s'}} \, dxdy
\]

\[
:= C_1 I_1 + C_1 I_2.
\]

On one hand, by (18) and the Hölder inequality, for \( \epsilon = \frac{1}{2C_1} \), we have

\[
C_1 I_1 \leq \frac{1}{2} \int_\Omega M_s\left(\frac{|u(x)|}{k}\right) \right) dx + \frac{C_1 K}{k} \int_\Omega |u(x)| \, dx
\]

\[
\leq \frac{1}{2} + \frac{C_2}{k} \|u\|_{(M)},
\]

where \( C_2 = 2C_1 K\|\chi\|_{(M)} \).

On the other hand, since \( \omega \) is Lipschitz continuous, there exists \( K > 0 \) such that,

\[
C_1 I_2 \leq \frac{C_1 K}{k} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|^{N+s'}} \, dxdy = \frac{C_1 K}{k} [u]_{s',1}.
\]

By (22), since \( s' < s \), there exists \( C_3 > 0 \) such that

\[
[u]_{s',1} \leq C_3[u]_{(s,M)},
\]

so

\[
C_1 I_2 \leq \frac{C_4}{k} [u]_{(s,M)},
\]

where \( C_4 = KC_1C_3 \). Combining (26) and (27), we obtain

\[
1 \leq \frac{1}{2} + \frac{C_2}{k} \|u\|_{(M)} + \frac{C_4}{k} [u]_{(s,M)},
\]

this implies that,

\[
\frac{k}{2} \leq C_2 \|u\|_{(M)} + C_4 [u]_{(s,M)},
\]

from which it follows that

\[
\|u\|_{(M,s)} \leq C_5 \|u\|_{(s,M)},
\]

where \( C_5 = \max\{2C_2, 2C_4\} \).

To extend (28) to arbitrary \( u \in W^{s,M}(\Omega) \), let

\[
f_n(y) = \begin{cases} y & \text{if } |y| \leq n \\ n \operatorname{sgn}(y) & \text{if } |y| > n \end{cases}
\]

and \( u_n(x) = f_n \circ u(x) \).

Clearly \( f_n \) is 1-Lipschitz continuous function. By Lemma 10, \( (u_n) \) belongs to \( W^{s,M}(\Omega) \). So in view of (28)

\[
\|u_n\|_{(M,s)} \leq C_5 \|u_n\|_{(s,M)}.
\]
On the other hand, we have
\[ \|u_n\|_{(s,M)} \leq \|u\|_{(s,M)}, \quad (30) \]
indeed, since \(|u_n(x)| \leq |u(x)|\), for all \(x \in \Omega\), then
\[ \int_\Omega \int_\Omega M\left(\frac{|u_n(x) - u_n(y)|}{|u|_{(s,M)} |x-y|^s}\right) \, dx \, dy \leq \int_\Omega \int_\Omega M\left(\frac{|u(x) - u(y)|}{|u|_{(s,M)} |x-y|^s}\right) \, dx \, dy \leq 1, \]
and
\[ \int_\Omega M\left(\frac{|u_n(x)|}{\|u\|_{(M)}}\right) \, dx \leq \int_\Omega M\left(\frac{|u(x)|}{\|u\|_{(M)}}\right) \, dx \leq 1, \]
then
\[ [u_n]_{(s,M)} \leq [u]_{(s,M)} \quad \text{and} \quad \|u_n\|_{(M)} \leq \|u\|_{(M)}; \]
thus (30) is deduced. Combining (29) and (30), we get
\[ \|u_n\|_{(M)} \leq C_5\|u_n\|_{(s,M)} \leq C_5\|u\|_{(s,M)}. \quad (31) \]
Let \(k_n = \|u_n\|_{(M)}\), the sequence \((k_n)\) is non-decreasing and converges in view of (31). Put \(k' = \lim_{n \to +\infty} k_n\), by Fatou’s Lemma we get
\[ \int_\Omega M\left(\frac{u(x)}{k'}\right) \, dx \leq \lim_{n \to +\infty} \int_\Omega M\left(\frac{u_n(x)}{k_n}\right) \, dx \leq 1, \]
whence \(u \in M^{k'}(\Omega)\) and
\[ \|u\|_{(M)} \leq k' = \lim_{n \to +\infty} \|u_n\|_{(M)} \leq C_4\|u\|_{(s,M)}. \]
Thus the first assertion of the theorem is proved. Now, let’s turn to the compactness embedding.

Let \(S\) be a bounded subset of \(W^{s,M}(\Omega)\). According to the embedding (5), \(S\) is also a bounded subset of \(L^{M^*}(\Omega)\). On the other hand, by a classical compact embedding theorem of \(W^{s,1}(\Omega)\) (see [20], Corollary 7.2) and Lemma 10, we have
\[ W^{s,M}(\Omega) \hookrightarrow W^{s,1}(\Omega) \hookrightarrow L^1(\Omega). \]
Then, \(S\) is precompact in \(L^1(\Omega)\). Hence, by [1, Theorem 8.25], \(S\) is precompact in \(L^{B}(\Omega)\) whenever \(B \prec \prec M^*\). The theorem is proved completely. \(\square\)

3.2. Equivalent norm in \(W^{s,M}(\Omega)\). Let \(\Omega\) be an open subset of \(\mathbb{R}^d\) and \(u \in W^{s,M}(\Omega)\). Let
\[ \rho(u) = \int_\Omega M(u(x)) \, dx, \quad \bar{\rho}(u) = \int_\Omega \int_\Omega M\left(\frac{u(x) - u(y)}{|x-y|^s}\right) \, dx \, dy, \quad \tilde{\rho}(u) = \rho(u) + \bar{\rho}(u) \]
and
\[ |u|_{(s,M,\Omega)} = \inf \left\{ \lambda > 0 : \tilde{\rho}\left(\frac{u}{\lambda}\right) \leq 1 \right\}. \]

Remark 2. We can notice using the Fatou’s lemma that
\[ \tilde{\rho}\left(\frac{u}{|u|_{(s,M,\Omega)}}\right) \leq 1, \quad \text{for all} \ u \in W^{s,M}(\Omega). \]

Lemma 12. \[ |u|_{(s,M,\Omega)} \] is an equivalent norm to \(\|u\|_{(s,M,\Omega)}\) with the relation
\[ \frac{1}{2}\|u\|_{(s,M,\Omega)} \leq |u|_{(s,M,\Omega)} \leq 2\|u\|_{(s,M,\Omega)}, \quad \text{for all} \ u \in W^{s,M}(\Omega). \quad (32) \]
Lemma 13. The following properties hold true:

(i) If $|u|_{(s,M,\Omega)} > 1$, then $|u|^{\alpha u}_{(s,M,\Omega)} \leq \hat{\rho}(u) \leq |u|_{(s,M,\Omega)}^{\alpha u}$.

(ii) If $|u|_{(s,M,\Omega)} < 1$, then $|u|_{(s,M,\Omega)}^{u} \leq \hat{\rho}(u) \leq |u|_{(s,M,\Omega)}^{u}$. 

Proof. We begin by proving (32). Evidently, we have

$$\rho\left(\frac{u}{|u|_{(s,M,\Omega)}}\right) \leq \hat{\rho}\left(\frac{u(x)}{|u|_{(s,M,\Omega)}}\right) \leq 1 \quad \text{and} \quad \rho\left(\frac{u}{|u|_{(s,M,\Omega)}}\right) \leq \hat{\rho}\left(\frac{u}{|u|_{(s,M,\Omega)}}\right) \leq 1,$$

then

$$|u|_{(s,M,\Omega)} \leq |u|_{(s,M,\Omega)} \quad \text{and} \quad |u|_{(s,M,\Omega)} \leq |u|_{(s,M,\Omega)},$$

therefore

$$|u|_{(s,M,\Omega)} \leq 2|u|_{(s,M,\Omega)}.$$

For the second inequality of (32), we have

$$\hat{\rho}\left(\frac{u}{2|u|_{(s,M,\Omega)}}\right) \leq \frac{1}{2} \hat{\rho}\left(\frac{u}{|u|_{(s,M,\Omega)}}\right) + \frac{1}{2} \hat{\rho}\left(\frac{u}{|u|_{(s,M,\Omega)}}\right) \leq 1,$$

then

$$|u|_{(s,M,\Omega)} \leq 2|u|_{(s,M,\Omega)}.$$

This ends the proof of (32).

Let now prove that $\|u\|_{(s,M,\Omega)}$ is a norm in $W^{s,M}(\Omega)$.

(i) It is clear that, if $|u|_{(s,M,\Omega)} = 0$ then $u = 0$, a.e.

(ii) For $\alpha \in \mathbb{R}$, we have

$$|\alpha u|_{(s,M,\Omega)} = \inf \left\{ \lambda > 0 : \hat{\rho}\left(\frac{\alpha u}{\lambda}\right) \leq 1 \right\} = |\alpha| \inf \left\{ \lambda > 0 : \hat{\rho}\left(\frac{u}{\lambda}\right) \leq 1 \right\} = |\alpha| |u|_{(s,M,\Omega)}.$$

(iii) Finally for the triangle inequality, let $u, v \in W^{s,M}(\Omega)$, we compute

$$\hat{\rho}\left(\frac{u + v}{|u|_{(s,M,\Omega)} + |v|_{(s,M,\Omega)}}\right) \leq \hat{\rho}\left(\frac{|u|_{(s,M,\Omega)}}{|u|_{(s,M,\Omega)} + |v|_{(s,M,\Omega)}}\right) + \hat{\rho}\left(\frac{|v|_{(s,M,\Omega)}}{|u|_{(s,M,\Omega)} + |v|_{(s,M,\Omega)}}\right) \leq 1.$$

Thus

$$|u + v|_{(s,M,\Omega)} \leq |u|_{(s,M,\Omega)} + |v|_{(s,M,\Omega)}.$$

The proof of Lemma 12 is completed. \qed
Proof. (1) Assume that $|u|_{(s,M,Ω)} > 1$, then by Lemma 5 and Remark 2,

$$
\hat{\rho}(u) = \int_Ω M\left(\frac{|u|_{(s,M,Ω)}}{|u|_{(s,M,Ω)}}\right) dx + \int_Ω \int_Ω M\left(\frac{|u|_{(s,M,Ω)}}{|u|_{(s,M,Ω)}}\right) \frac{h_u(x,y)}{|x-y|^N} dxdy
\leq |u|_{(s,M,Ω)}^{\alpha} \left[ \int_Ω M\left(\frac{u}{|u|_{(s,M,Ω)}}\right) dx + \int_Ω \int_Ω M\left(\frac{h_u(x,y)}{|u|_{(s,M,Ω)}}\right) \frac{dxdy}{|x-y|^N} \right]
\leq |u|_{(s,M,Ω)}^{\alpha} \hat{\rho}\left(\frac{u}{|u|_{(s,M,Ω)}}\right) \leq |u|_{(s,M,Ω)}^{\alpha},
$$

where $h_u(x,y) = \frac{u(x) - u(y)}{|x-y|^s}$.

Let $1 < \sigma < |u|_{(s,M,Ω)}$, by definition of the norm $|.|_{(s,M,Ω)}$,

$$
\hat{\rho}\left(\frac{u}{\sigma}\right) > 1,
$$

then, according to Lemma 5, we infer

$$
\hat{\rho}(u) \geq \sigma^{\alpha} \left[ \int_Ω M\left(\frac{u}{\sigma}\right) dx + \int_Ω \int_Ω M\left(\frac{h_u(x,y)}{\sigma}\right) \frac{dxdy}{|x-y|^N} \right] = \sigma^{\alpha} \hat{\rho}\left(\frac{u}{\sigma}\right) \geq \sigma^{\alpha},
$$

letting $\sigma \nearrow |u|_{(s,M,Ω)}$ in the above inequality, we obtain (i).

(2) Assume that $|u|_{(s,M,Ω)} < 1$. Using Lemma 5, we get

$$
\hat{\rho}(u) = \hat{\rho}\left(\frac{|u|_{(s,M,Ω)}}{|u|_{(s,M,Ω)}}\right) \leq |u|_{(s,M,Ω)}^{\alpha} \hat{\rho}\left(\frac{|u|_{(s,M,Ω)}}{|u|_{(s,M,Ω)}}\right) \leq |u|_{(s,M,Ω)}^{\alpha},
$$

On the other hand, as above in (1), let $0 < \sigma < |u|_{(s,M,Ω)}$, by Lemma 5,

$$
\hat{\rho}(u) \geq \sigma^{\alpha} \hat{\rho}\left(\frac{u}{\sigma}\right) \geq \sigma^{\alpha}.
$$

Letting $\sigma \nearrow |u|_{(s,M,Ω)}$ in the last inequality, we obtain (ii).

Proof of Theorem 2. Let $u \in W^{s,M}(\mathbb{R}^d)$ such that $|u|_{(s,M,\mathbb{R}^d)} = 1$. By Lemma (13), $\hat{\rho}(u) = 1$. Let $B_i = \{x \in \mathbb{R}^d / i \leq |x| < i + 1\}$, $i \in \mathbb{N}$, such that $\mathbb{R}^d = \bigcup_{i \in \mathbb{N}} B_i$ and $B_i \cap B_j \neq \emptyset$ if $i \neq j$. Then, for all $i \in \mathbb{N}$, we have

$$
\int_{B_i} M(u(x)) dx + \int_{B_i} \int_{B_j} M\left(\frac{u(x) - u(y)}{|x-y|^s M^{-1}|x-y|^N}\right) dxdy \leq \hat{\rho}(u) = 1. \quad (33)
$$

In what follows we show that

$$
C = \sup \left\{ \int_{\mathbb{R}^d} M_s(w(x)) dx : w \in W^{s,M}(\mathbb{R}^d), |w|_{(s,M,\mathbb{R}^d)} = 1 \right\} < \infty.
$$

Indeed, in view of (5) and (32), there exists $C_0 >$ such that

$$
\|u\|_{(M,B_i)} \leq C_0 \|u\|_{(s,M,B_i)} \leq 2C_0 |u|_{(s,M,B_i)} \leq 2C_0 |u|_{(s,M,\mathbb{R}^d)} = 2C_0, \text{ for all } i \in \mathbb{N}.
$$

Let $i \in \mathbb{N}$, we distingue two cases:
Cas 1: If \(1 \leq \|u\|_{(s,M,B_i)} \leq 2C_0\), then, by Lemma 5, (5), (32), Lemma 13 and (33), we have

\[
\int_{B_i} M_s(u(x))dx \leq \|u\|^{(m^0)_*}_{(s,M,B_i)} \leq C_0^{m^0} \|u\|_{(s,M,B_i)}^{(m^0)_*} \quad ((m^0)_* \text{ fixed by Lemma 6})
\]

\[
\leq (2C_0)^{(m^0)_*} |u|^{(m^0)_*}_{(s,M,B_i)}
\]

\[
\leq (2C_0)^{(m^0)_*} \left( \int_{B_i} M(u(x))dx + \int_{B_i} \int_{B_i} M(h_u(x,y))dxdy \right)^{\frac{(m^0)_*}{m^0}}
\]

\[
\leq (2C_0)^{(m^0)_*} \left( \int_{B_i} M(u(x))dx + \int_{B_i} \int_{B_i} M(h_u(x,y))dxdy \right). \tag{34}
\]

Cas 2: If \(\|u\|_{(s,M,B_i)} < 1\), then, also by Lemma 5, (5), (32), Lemma 13 and (33), we have

\[
\int_{B_i} M_s(u(x))dx \leq \|u\|^{m^*_0}_{(s,M,B_i)} \leq C_0^{m^*_0} \|u\|_{(s,M,B_i)}^{m^*_0} \quad (m^*_0 \text{ fixed by Lemma 6})
\]

\[
\leq (2C_0)^{m^*_0} |u|^{m^*_0}_{(s,M,B_i)}
\]

\[
\leq (2C_0)^{m^*_0} \left( \int_{B_i} M(u(x))dx + \int_{B_i} \int_{B_i} M(h_u(x,y))dxdy \right)^{\frac{m^*_0}{m^0}}
\]

\[
\leq (2C_0)^{m^*_0} \left( \int_{B_i} M(u(x))dx + \int_{B_i} \int_{B_i} M(h_u(x,y))dxdy \right). \tag{35}
\]

Combining (34) and (35), we obtain

\[
\int_{\mathbb{R}^d} M_s(u(x))dx = \sum_{i \in \mathbb{N}} \int_{B_i} M_s(u(x))dx
\]

\[
\leq [(2C_0)^{(m^0)_*} + (2C_0)^{m^*_0}] \sum_{i \in \mathbb{N}} \left( \int_{B_i} M(u(x))dx + \int_{B_i} \int_{B_i} M(h_u(x,y))dxdy \right)
\]

\[
= [(2C_0)^{(m^0)_*} + (2C_0)^{m^*_0}] \tilde{p}(u) = (2C_0)^{(m^0)_*} + (2C_0)^{m^*_0}.
\]

Hence \(C < (2C_0)^{(m^0)_*} + (2C_0)^{m^*_0}\).

Now, let \(u \in W^{s,M}(\mathbb{R}^d) \setminus \{0\}\) and \(v = \frac{u}{\|u\|_{(s,M,\mathbb{R}^d)}}\).

Then, by using Proposition 2, we infer

\[
\|v\|_{(s,M)} \leq \int_{\mathbb{R}^d} M_s(v(x))dx + 1 \leq C + 1,
\]

from where it follows that

\[
\|u\|_{(s,M)} \leq (C + 1)\|u\|_{(s,M,\mathbb{R}^d)},
\]

and for all \(N\)-function \(B \prec \prec M_s\), we have

\[
W^{s,M}(\mathbb{R}^d) \hookrightarrow L^M(\mathbb{R}^d) \hookrightarrow L^B(\mathbb{R}^d).
\]

The proof of Theorem 2 is completed.
4. Variational setting of problem (1) and some useful tools. In this section, we will first introduce the variational setting for problem (1). In view of the presence of potential $V$, our working space is

$$E = \{ u \in W^{s,M}(\mathbb{R}^d); \int_{\mathbb{R}^d} V(x)M(u)dx < \infty \},$$

equipped with the following norm

$$\|u\| = [u]_{s,M} + \|u\|_{V,M},$$

where

$$\|u\|_{V,M} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^d} V(x)M\left(\frac{u(x)}{\lambda}\right)dx \leq 1 \right\}.$$ 

We define the functional $G : E \to \mathbb{R}$ by

$$G(u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M(h_u(x,y))d\mu,$$

where $h_u(x,y) = \frac{u(x) - u(y)}{|x - y|^s}$ and $d\mu = \frac{dxdy}{|x - y|^N}$.

For any $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, denote by

$$F(x,t) = \int_0^t f(x,s)ds = \xi(x)|t|^p.$$ 

We consider the following family of functionals on $E$

$$I_\lambda(u) = A(u) - \lambda B(u), \lambda \in [1,2],$$

where

$$A(u) = G(u) + \Psi(u),$$

$$\Psi(u) = \int_{\mathbb{R}^d} V(x)M(u)dx, \quad B(u) = \int_{\mathbb{R}^d} F(x,u)dx.$$

**Lemma 14.** The functional $I_\lambda$ is well defined on $E$, moreover $I_\lambda \in C^1(E,\mathbb{R})$ and for all $v \in E$,

$$\langle I'_\lambda(u),v \rangle = \langle A'(u),v \rangle - \lambda \langle B'(u),v \rangle
= \langle (-\triangle)^s u, v \rangle + \int_{\mathbb{R}^d} V(x)m(u)vdx - \lambda \int_{\mathbb{R}^d} f(x,u)vdx. \quad (38)$$

**Proof.** The proof follows from [41], Proposition 2.2 and [35], Proposition 3.3. \qed

Now we give the definition of weak solution for the problem (1).

**Definition 15.** We say that $u \in E$ is a weak solution to (1) if $u$ is critical point of $I_1$, which means that $u$ satisfies

$$\langle I'_1(u),v \rangle = \langle A'(u),v \rangle - \langle B'(u),v \rangle = 0$$

for all $v \in E$.

**Lemma 16.** Assume that $(m_1)$ and $(V_1)$ are satisfied. Then, the following properties hold true:

(i) $\xi_0([u]_{s,M}) \leq G(u) \leq \xi_1([u]_{s,M})$ for all $u \in E,$
(ii) \( \xi_0(\|u\|_{(V,M)}) \leq \int_{\mathbb{R}^d} V(x)M(u)dx \leq \xi_1(\|u\|_{(V,M)}) \) for all \( u \in E \).

**Proof.** The proof of the first assertion is given by [9, Lemma 3.4]. For the second assertion, let \( u \in E \), on one hand, choosing \( \beta = \|u\|_{(V,M)} \) in Lemma 5, we obtain

\[
M(u) \leq \xi_1(\|u\|_{(V,M)}) M\left(\frac{u}{\|u\|_{(V,M)}}\right),
\]

then

\[
V(x)M(u) \leq \xi_1(\|u\|_{(V,M)})V(x)M\left(\frac{u}{\|u\|_{(V,M)}}\right), \quad \text{for } x \in \mathbb{R}^d.
\]

From the definition of the norm (11), we deduce that

\[
\int_{\mathbb{R}^d} V(x)M(u)dx \leq \xi_1(\|u\|_{(V,M)}) \int_{\mathbb{R}^d} V(x)M\left(\frac{u}{\|u\|_{(V,M)}}\right)dx \leq \xi_1(\|u\|_{(V,M)}).
\]

On the other hand, let \( \epsilon > 0, \beta = \|u\|_{(V,M)} - \epsilon \) in Lemma 5, we get

\[
\xi_0(\|u\|_{(V,M)} - \epsilon)V(x)M\left(\frac{u}{\|u\|_{(V,M)} - \epsilon}\right) \leq V(x)M(u),
\]

then,

\[
\int_{\mathbb{R}^d} V(x)M(u)dx \geq \xi_0(\|u\|_{(V,M)} - \epsilon) \int_{\mathbb{R}^d} V(x)M\left(\frac{u}{\|u\|_{(V,M)} - \epsilon}\right)dx
\]

\[
\geq \xi_0(\|u\|_{(V,M)} - \epsilon).
\]

Letting \( \epsilon \to 0 \) in the above inequality, we obtain

\[
\xi_0(\|u\|_{(V,M)}) \leq \int_{\mathbb{R}^d} V(x)M(u)dx.
\]

Thus the assertion (ii) and the proof of Lemma 16 is complete. \( \square \)

**Lemma 17.** Let \( \varphi, \psi : \mathbb{R} \to \mathbb{R} \) be increasing homeomorphisms such that their associated \( N \)-functions \( \Phi, \Psi \) satisfy

\[ 1 < \varphi_0 := \inf_{t > 0} \frac{t \varphi(t)}{\Phi(t)} \leq \frac{t \varphi(t)}{\Phi(t)} \leq \varphi^0 := \sup_{t > 0} \frac{t \varphi(t)}{\Phi(t)} < m_0, \quad (m_0 \text{ fixed by } (m_1)), \quad (39) \]

and

\[ 1 < \frac{t \psi(t)}{\Psi(t)} \leq \psi^0 := \sup_{t > 0} \frac{t \psi(t)}{\Psi(t)} < \frac{m_0}{\varphi^0}. \quad (40) \]

Then \( \Psi \) satisfies the \( \Delta_2 \)-condition and

\[ \Psi \circ \Phi \prec \prec M. \]

**Proof.** Using Lemma 5, we have, for all \( \lambda > 0 \) and \( t > 1 \),

\[ \Phi(\lambda t) \leq \Phi(\lambda) t^{\varphi_0} \text{ and } M(1) t^{m_0} \leq M(t), \]

then

\[ \frac{\Phi(\lambda t)}{M(t)} \leq \frac{\Phi(\lambda) t^{\varphi_0}}{M(1) t^{m_0}}. \]

Again with Lemma 5, we get

\[ \Psi(\Phi(\lambda t)) \leq \Psi(\Phi(\lambda) t^{\varphi_0}) \leq \Psi(\Phi(\lambda)) t^{\varphi_0} \psi^0. \]
Combining (44), (45) and Lemma 16, we get
\begin{equation*}
\Phi(t) \leq \Phi(t) e^{t \psi(t)} \rightarrow 0, \text{ as } t \rightarrow +\infty.
\end{equation*}
Thus \( \Phi \) is \( \Phi \)-compact
\[ \diamond \]

Now we state our embedding compactness result.

**Lemma 18.** Let \( \Phi \) be an \( \mathcal{N} \)-function satisfying (39) such that
\[ \Phi \prec \prec M. \]  
Under the assumption \( (m_1), (V_1) \) and \( (V_2) \), the embedding from \( E \) into \( L^{\Phi}(\mathbb{R}^d) \) is compact.

**Proof.** Let \( \Phi \) be an \( \mathcal{N} \)-function satisfying (39) such that \( \Phi \prec \prec M \) and \( (v_n) \) be a bounded sequence in \( E \), since \( E \) is reflexive, up to subsequence, \( v_n \rightarrow v \) in \( E \). Let \( u_n = v_n - v, u_n \rightarrow 0 \) in \( E \). We have to show that \( u_n \rightarrow 0 \) in \( L^{\Phi}(\mathbb{R}^d) \), by Proposition 1 this means that
\[ \int_{\mathbb{R}^d} \Phi(u_n) dx \rightarrow 0, \text{ as } n \rightarrow +\infty. \]  
(42)

According to Vitali’s theorem it suffices to show that, the sequence \( (\Phi(u_n)) \) is equi-integrable, which means:
\[ (a) \quad \begin{cases} \forall \epsilon > 0, \exists \omega \subset \mathbb{R}^d \text{ measurable with } \text{meas}(\omega) < \infty \\ \text{such that } \int_{\mathbb{R}^d \setminus \omega} \Phi(u_n(x)) dx < \epsilon, \forall n \in \mathbb{N}, \end{cases} \]
\[ (b) \quad \begin{cases} \forall \epsilon > 0, \exists \delta > 0 \text{ such that } \int_E \Phi(u_n(x)) dx < \epsilon, \\ \forall n, \forall E \subset \mathbb{R}^d, E \text{ measurable and } \text{meas}(E) < \delta. \end{cases} \]

We do the proof in two steps. We start by checking (a). Let \( L > 0 \) and \( A_L = \{ x \in \mathbb{R}^d : V(x) \leq L \} \).

By (41) and [26], Proposition 4.17.6., there exists \( C > 0 \) such that
\[ \| u_n \|_{(\Phi,A_L^L)} \leq C \| u_n \|_{(M,A_L^L)}, \text{ for all } n \in \mathbb{N}. \]  
(43)

Since \( \Phi \) satisfying (39), then by Lemma 5 and (43), we infer
\[ \int_{A_L^L} \Phi(u_n) dx \leq \xi_1(\| u_n \|_{(\Phi,A_L^L)}) \leq \| u_n \|^{\psi_0}_{(\Phi,A_L^L)} + \| u_n \|^{\psi_0}_{(\Phi,A_L^L)} \]
\[ \leq C^{\psi_0} + \xi_1(\| u_n \|^{\psi_0}_{(M,A_L^L)} + \| u_n \|^{\psi_0}_{(M,A_L^L)}). \]  
(44)

Applying again Lemma 5, we obtain
\[ \| u_n \|^{\psi_0}_{(M,A_L^L)} \leq 2^{\psi_0-1} \left[ \left( \int_{A_L^L} M(u_n) dx \right)^{\frac{\psi_0}{m}} + \left( \int_{A_L^L} M(u_n) dx \right)^{\frac{\psi_0}{m}} \right]. \]  
(45)

Combining (44), (45) and Lemma 16, we get
\[ \int_{A_L^L} \Phi(u_n) dx \leq (2^{\psi_0-1} + 2^{\psi_0-1}) (C^{\psi_0} + \xi_1) \left[ \left( \int_{A_L^L} M(u_n) dx \right)^{\frac{\psi_0}{m}} \right]
\[ + \left( \int_{A_L^L} M(u_n) dx \right)^{\frac{\psi_0}{m}} + \left( \int_{A_L^L} M(u_n) dx \right)^{\frac{\psi_0}{m}} \]
this can be made arbitrarily small by choosing $L$ large enough. Thus (a) is verified.

Now we check (b). Let $B \subset \mathbb{R}^d$ measurable subset of $\mathbb{R}^d$. Let $\Psi$ be the $N$-function satisfying (40) and $\Psi$ be the conjugate of $\Psi$. By Lemma 5, $\Psi \circ \Phi < M$. In the light of [[26], Theorem 4.17.6] there exists $K > 0$ such that

$$\|u\|_{(\Psi \circ \Phi)} \leq K \cdot \|u\|_{(M)}$$

for all $u \in L^M(\mathbb{R}^d)$. (46)

**Claim 1.** We claim that $\xi_1(\|u_n\|_{(M)}) \leq \xi_1\left(\frac{1}{V_0}\xi_1(\|u_n\|) + 1\right)$.

Indeed, using (12), Proposition 2 and Lemma 16, we deduce

$$\xi_1(\|u_n\|_{(M)}) \leq \xi_1(\|u_n\|_{L^M(\Omega)}) \leq \xi_1\left(\int_{\mathbb{R}^d} M(u_n)dx + 1\right)$$

$$\leq \xi_1\left(\frac{1}{V_0} \int_{\mathbb{R}^d} V(x)M(u_n)dx + 1\right)$$

$$\leq \xi_1\left(\frac{1}{V_0} \xi_1(\|u_n\|_{(V,M)}) + 1\right)$$

$$\leq \xi_1\left(\frac{1}{V_0} \xi_1(\|u_n\|) + 1\right),$$

thus the claim.

Combining (46), claim 1, Lemma 5 and applying the Hölder inequality, we infer that

$$\int_B \Phi(u_n)dx \leq \|\Phi(u_n)\|_{L^\Psi(\mathbb{R}^d)} \|\chi_B\|_{L^\Psi(\mathbb{R}^d)}$$

$$\leq \left(\int_{\mathbb{R}^d} \Psi(\Phi(u_n))dx + 1\right) \|\chi_B\|_{L^\Psi(\mathbb{R}^d)}$$

$$\leq (\xi_1(\|u_n\|_{(\Psi \circ \Phi)}) + 1) \|\chi_B\|_{L^\Psi(\mathbb{R}^d)}$$

$$\leq (K'' \xi_1(\|u_n\|_{(M)}) + 1) \|\chi_B\|_{L^\Psi(\mathbb{R}^d)}$$

$$\leq (K'' \xi_1\left(\frac{1}{V_0} \xi_1(\|u_n\|) + 1\right) + 1) \|\chi_B\|_{L^\Psi(\mathbb{R}^d)}$$

$$\leq C\|\chi_B\|_{L^\Psi(\mathbb{R}^d)}$$ (47)
where $C = K'' \xi_1 \left[ \frac{1}{m_0} \xi_1(T) + 1 \right] + 1$, $K'' > 0$ and $T = \sup_n \|u_n\| < \infty$.

On the other hand, the following limit holds

$$\lim_{t \to +\infty} \frac{\Psi^{-1}(t)}{t} = 0,$$

then, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that if $|t| \leq \delta_\epsilon$, we have

$$t \Psi^{-1} \left( \frac{1}{\epsilon} \right) \leq \frac{\epsilon}{C}.$$  \hfill (48)

By [26], Proposition 4.6.9, we know that

$$\|\chi_B\|_{L^p(\mathbb{R}^d)} = \text{meas}(B) \Psi^{-1} \left( \frac{1}{\text{meas}(B)} \right).$$  \hfill (49)

If $\text{meas}(B) \leq \delta_\epsilon$, combining (48) and (49), we get

$$\text{meas}(B) \Psi^{-1} \left( \frac{1}{\text{meas}(B)} \right) \leq \frac{\epsilon}{C}.$$  \hfill (48)

Therefore, for all measurable subset of $\mathbb{R}^d$ such that $\text{meas}(B) \leq \delta_\epsilon$,

$$\int_B \Phi(u_n) dx \leq \epsilon.$$

We conclude that $(\Phi(u_n))$ is uniformly integrable and tight over $\mathbb{R}^d$. Thus the Lemma 18 is proved. \hfill \Box

**Corollary 1.** Under $(m_1)$ and $(M_1)$, the embedding from $E$ into $L^p(\mathbb{R}^d)$ is compact.

**Proof.** Let $\Phi(t) = |t|^\mu$. By condition $(M_1)$, $\Phi \prec \prec M$. Applying Lemma 18, we deduce that $E$ is compactly embedded into $L^p(\mathbb{R}^d)$. \hfill \Box

**Lemma 19.** Assume that $(m_1)$ and $(M_1)$ are satisfied. Then the functional $A$ is weakly lower semi-continuous on $E$.

**Proof.** By [9], Lemma 3.3, $G$ is weakly lower semi-continuous, so it is enough to show that $\Psi$ is too. Let $(u_n) \subset E$ be a sequence which converges weakly to $u$ in $E$. Since $E$ is compactly embedded in $L^p(\mathbb{R}^d)$, it follows that $(u_n)$ converges strongly to $u$ in $L^p(\mathbb{R}^d)$. Up to a subsequence,

$$u_n(x) \to u(x), \text{ a.e in } \mathbb{R}^d.$$  

Using Fatou’s lemma, we get

$$\Psi(u) \leq \liminf_{n \to \infty} \Psi(u_n).$$

Therefore, $A$ is weakly lower semi-continuous. Thus the proof. \hfill \Box

**Lemma 20.** Suppose that $(m_1)$, $(M_1) - (M_2)$, $(V_1) - (V_2)$ and $(f_1)$ hold. Then $I'_\lambda$ maps bounded sets to bounded sets uniformly in $\lambda \in [1, 2]$.

**Proof.** Let $u, v \in E$. Using H"older and Young inequalities, we compute

$$|\langle I'_\lambda(u), v \rangle| = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(h_u) h_v dx \mu + \int_{\mathbb{R}^d} V(x)m(u)vdx - \lambda \int_{\mathbb{R}^d} \xi(x) |u|^{p-2}uvdx \right|$$

$$\leq \|m(h_u)\|_{L^{\rho'_M}(\mathbb{R}^d)} \|h_v\|_{(M)} + \int_{\mathbb{R}^d} V(x)M(m(u))dx + \int_{\mathbb{R}^d} V(x)M(v)dx$$

$$+ 2\rho\|\xi\|_{L^{\rho'_M}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)}^{-1} \|v\|_{L^p(\mathbb{R}^d)}.$$
then according to Lemma 5, we obtain

\[ |⟨I_λ(u), v⟩| \leq \left[ \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(h) d\mu \right)^{\frac{1}{p'}} + \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(h) d\mu \right)^{\frac{1}{p}} \right] [v]_{(s,M)} \]

\[ + \int_{\mathbb{R}^d} V(x) M(h) dx + \int_{\mathbb{R}^d} V(x) M(h) dx + 2p\|\xi\|_{L^{\frac{p}{1-p}}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \]

\[ + \int_{\mathbb{R}^d} V(x) M(h) dx \]

combining [Lemma 2.9, [13]], Lemma 16 and Corollary 1, we get

\[ |⟨I_λ(u), v⟩| \leq \left[ \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(h) d\mu \right)^{\frac{1}{p'}} + \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(h) d\mu \right)^{\frac{1}{p}} \right] [v]_{(s,M)} \]

\[ + m^0 \int_{\mathbb{R}^d} V(x) M(h) dx + \int_{\mathbb{R}^d} V(x) M(h) dx + 2p\|\xi\|_{L^{\frac{p}{1-p}}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \]

\[ \leq \left[ \left( m^0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M(h) d\mu \right)^{\frac{1}{p'}} + \left( m^0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M(h) d\mu \right)^{\frac{1}{p}} \right] \|v\|_{L^{\frac{p}{1-p}}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \]

\[ + m^0 \xi^1(\|v\|_{(V,M)}) + \xi_1(\|v\|_{(V,M)}) + 2p\|\xi\|_{L^{\frac{p}{1-p}}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \]

thus

\[ \|I_λ(u)\| \leq \left[ \left( m^0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M(h) d\mu \right)^{\frac{1}{p'}} + \left( m^0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M(h) d\mu \right)^{\frac{1}{p}} \right] \|v\|_{L^{\frac{p}{1-p}}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \]

\[ + m^0 \xi^1(\|v\|_{(V,M)}) + \xi_1(\|v\|_{(V,M)}) + 2 + 2p\|\xi\|_{L^{\frac{p}{1-p}}(\mathbb{R}^d)} \|u\|_{L^p(\mathbb{R}^d)} \]

From the last inequality, we conclude that \(I_λ\) maps bounded sets to bounded sets for \(λ \in [1, 2]\). \(\square\)

**Lemma 21.** If \(u_n \rightharpoonup u\) in \(E\) and

\[ ⟨A'(u_n), u_n - u⟩ \to 0, \text{ as } n \to +\infty, \]  

(50)

then \(u_n \rightharpoonup u\) in \(E\).

**Proof.** Since \((u_n)\) converges weakly to \(u\) in \(E\), then \([u_n]_{(s,M)}\) and \([u_n]_{(V,M)}\) are bounded sequences of real numbers. That fact and relations (i) and (ii) from lemma 16, imply that the sequences \((G(u_n))\) and \((\Psi(u_n))\) are bounded. This means that the sequence \((A(u_n))\) is bounded. Then, up to a subsequence, \(A(u_n) \to c\). Furthermore, Lemma 19 implies

\[ A(u) \leq \liminf_{n \to \infty} A(u_n) = c. \]  

(51)

Since \(A\) is convex, we have

\[ A(u) \geq A(u_n) + ⟨A'(u_n), u - u_n⟩. \]  

(52)

Therefore, combining (50), (51) and (52), we conclude that \(A(u) = c\).

Taking into account that \(\frac{u_n + u}{2}\) converges weakly to \(u\) in \(E\) and using again the weak lower semi-continuity of \(A\), we find

\[ c = A(u) \leq \liminf_{n \to \infty} A\left(\frac{u_n + u}{2}\right). \]  

(53)

We argue by contradiction, and suppose that \((u_n)\) does not converge to \(u\) in \(E\). Then, there exists \(β > 0\) and a subsequence \((u_{n_m})\) of \((u_n)\) such that
by (i) and (ii) in lemma 16, we infer that
\[
A\left(\frac{\|u_{n_m} - u\|_{(V,M)}}{2}\right) \geq \xi_0\left(\frac{\|u_{n_m} - u\|_{(V,M)}}{2}\right) + \xi_0\left(\frac{\|u_{n_m} - u\|_{(s,M)}}{2}\right)
\]
\[
\geq \xi_0\left(\frac{\|u_{n_m} - u\|_{(V,M)}}{2}\right)
\]
\[
\geq \xi_0(\beta).
\]

On the other hand, the \(\Delta_2\) condition and relation \((M_2)\) enable us to apply [27, Theorem 2.1], in order to obtain
\[
\frac{1}{2}A(u) + \frac{1}{2}A(u_{n_m}) - A\left(\frac{u_{n_m} + u}{2}\right) \geq \frac{\|u_{n_m} - u\|}{2} \geq \xi_0(\beta), \text{ for all } m \in \mathbb{N}. \quad (54)
\]

Letting \(m \to \infty\) in the above inequality, we get
\[
c - \xi_0(\beta) \geq \limsup_{m \to \infty} A\left(\frac{u_{n_m} + u}{2}\right) \geq c.
\]

That is a contradiction. It follows that \((u_n)\) converges strongly to \(u\) in \(E\). Thus lemma 21 is proved.

5. **Proof of Theorem 3.** Let \((E, \|\cdot\|)\) be a Banach space and \(E = \bigoplus_{j \in \mathbb{N}} X_j\) with \(\text{dim } X_j < \infty\) for any \(j \in \mathbb{N}\). Set \(Y_k = \bigoplus_{j=1}^k X_j\), \(Z_k = \bigoplus_{j=k+1}^\infty X_j\) and \(B_k = \{u \in Y_k : \|u\| \leq \rho_k\}\), \(\text{for } \rho_k > 0\).

Consider a \(C^1\)-functional \(I_\lambda : E \to \mathbb{R}\) defined as
\[
I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2].
\]

Let, for \(k \geq 2\),
\[
\Gamma_k := \{\gamma \in \mathcal{C}(B_k, E) : \gamma \text{ is odd, } \gamma|_{\partial B_k} = \text{id}\},
\]
\[
c_k := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)).
\]

In order to prove Theorem 3, we apply the following variant of fountain Theorem due to Zou [43].

**Theorem 22.** Assume that \(I_\lambda\) satisfies the following assumptions:

(i) \(I_\lambda\) maps bounded sets to bounded sets for \(\lambda \in [1, 2]\) and \(I_\lambda(-u) = I_\lambda(u)\) for all \((\lambda, u) \in [1, 2] \times E\).

(ii) \(B(u) \geq 0, B(u) \to +\infty\) as \(\|u\| \to +\infty\) on any finite dimensional subspace of \(E\).

(iii) There exist \(r_k < \rho_k\) such that
\[
ak(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0 > bk(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u), \quad \forall \lambda \in [1, 2],
\]

and
\[
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1, 2].
\]

Then there exist \(\lambda_n \to 1, u_{\lambda_n} \in Y_n\) such that
\[
I_{\lambda_n}'[Y_n](u_{\lambda_n}) = 0, \quad I_{\lambda_n}(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)] \text{ as } n \to \infty.
\]

(56)
Particularly, if \((u_{\lambda_n})\) has a convergent subsequence for every \(k\), then \(I_1\) has infinitely many nontrivial critical points \(\{u_k\} \in E \setminus \{0\}\) satisfying \(I_1(u_k) \to 0^-\) as \(k \to \infty\).

Since \(E\) is reflexive and separable, we choose a basis \(\{e_j : j \in \mathbb{N}\}\) of \(E\) and \(\{e_j^* : j \in \mathbb{N}\}\) of \(E^*\) such that \(\langle e_i^*, e_j \rangle = \delta_{i,j}, \forall i, j \in \mathbb{N}\). Let \(X_j = \langle e_j \rangle\) for all \(j \in \mathbb{N}\) and

\[
Y_k = \bigoplus_{j=1}^k X_j = \bigoplus_{j=1}^k \langle e_j \rangle, \quad Z_k = \bigoplus_{j=k+1}^\infty X_j = \bigoplus_{j=k}^\infty \langle e_j \rangle \text{ for all } k \in \mathbb{N}.
\]

In order to apply Theorem 22, we need the following lemmas.

**Lemma 23.** Let \((u_{\lambda_n})_{n \in \mathbb{N}}\) be a bounded sequence of \(E\) satisfying (56), \(u_{\lambda_n} \to u_0\) as \(n \to +\infty\) for some \(u_0 \in E\). Then

\[
\lim_{n \to +\infty} \langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} - u_0 \rangle = 0.
\]

**Proof.** Using Lemma 20, we observe that \((I'_{\lambda_n}(u_{\lambda_n}))_{n \in \mathbb{N}}\) is bounded in \(E^*\). As \(E = \bigcup_n Y_n\), we can choose \(w_n \in Y_n\) such that \(w_n \to u_0\) as \(n \to +\infty\).

\[
\lim_{n \to +\infty} |\langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} - u_0 \rangle| \leq \lim_{n \to +\infty} |\langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} - w_n \rangle| + \|I'_{\lambda_n}(u_{\lambda_n})\| \|w_n - u_0\| = 0.
\]

Hence

\[
\lim_{n \to +\infty} \langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} - u_0 \rangle = 0.
\]

Thus the proof. \(\square\)

**Lemma 24.** Let \((m_1), (V_1), (V_2), (M_1)\) and \((f_1)\) be satisfied. Then \(B(u) \geq 0\) for all \(u \in E\). Furthermore, \(B(u) \to \infty\) as \(\|u\| \to \infty\) on any finite dimensional subspace of \(E\).

**Proof.** Evidently \(B(u) \geq 0\) for all \(u \in E\) follows by \((f_1)\). We claim that for any finite dimensional subspace \(H \subset E\), there exists a constant \(c_H > 0\) such that

\[
\text{meas}(\Lambda_u) \geq c_H, \text{ for all } u \in H \setminus \{0\},
\]

where

\[
\Lambda_u = \{x \in \mathbb{R}^d : \xi(x) |u(x)|^p \geq c_H \|u\|^p\}.
\]

We argue by contradiction, suppose that for any \(n \in \mathbb{N}\) there exists \(u_n \in H \setminus \{0\}\) such that

\[
\text{meas}(\{x \in \mathbb{R}^d : \xi(x) |u_n(x)|^p \geq \frac{1}{n} \|u_n\|^p\}) < \frac{1}{n}.
\]

For each \(n \in \mathbb{N}\), let \(v_n = \frac{u_n}{\|u_n\|} \in H\), \(\|v_n\| = 1\), then

\[
\text{meas}(\{x \in \mathbb{R}^d : \xi(x) |v_n(x)|^p \geq \frac{1}{n} \}) < \frac{1}{n}.
\]

(58)

Up to a subsequence, we may assume that \(v_n \to v\) for some \(v \in H\) and \(\|v\| = 1\).

Furthermore, there exists a constant \(\delta_0 > 0\) such that

\[
\text{meas}(\{x \in \mathbb{R}^d : \xi(x) |v(x)|^p \geq \delta_0 \}) \geq \delta_0.
\]

In fact, if not,

\[
\text{meas}(\mathcal{A}_n) < \frac{1}{n}, \text{ for all } n \in \mathbb{N},
\]

where

\[
\mathcal{A}_n = \{x \in \mathbb{R}^d : \xi(x) |v(x)|^p \geq \frac{1}{n} \}.
\]
Let \( m > n \), then \( A_n \subset A_m \) and \( \text{meas}(A_n) \leq \text{meas}(A_m) < \frac{1}{m} \to 0 \) as \( m \to +\infty \), it yields

\[
\text{meas}\{x \in \mathbb{R}^d : \xi(x)|v(x)|^p \geq \frac{1}{n}\} = 0, \text{ for all } n \in \mathbb{N}.
\]

Therefore

\[
0 \leq \int_{\mathbb{R}^d} \xi(x)|v|^p \mu dx \leq \frac{\|v\|_{L^p(\mathbb{R}^d)}}{n} \to 0 \text{ as } n \to +\infty.
\]

This together with (f1) yields \( v = 0 \), a.e. which is in contradiction to \( \|v\| = 1 \).

Thus (59) is proved.

By Hölder inequality and Corollary 1, it holds that

\[
\int_{\mathbb{R}^d} \xi(x)|v_n - v|^p \mu dx \leq \|\xi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |v_n - v|^p \mu dx \right)^{\frac{p}{p-1}} \to 0 \text{ as } n \to \infty. \tag{60}
\]

Set

\[
\Lambda_0 := \{x \in \mathbb{R}^d : \xi(x)|v(x)|^p \geq \delta_0\}
\]

and for all \( n \in \mathbb{N} \),

\[
\Lambda_n := \{x \in \mathbb{R}^d : \xi(x)|v_n(x)|^p < \frac{1}{n}\}, \quad \Lambda_n^c := \mathbb{R}^d \setminus \Lambda_n.
\]

Taking into account (58) and (59), for \( n \) large enough, we get

\[
\text{meas}(\Lambda_n \cap \Lambda_0) \geq \text{meas}(\Lambda_0) - \text{meas}(\Lambda_n^c) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.
\]

Therefore, for \( n \) large enough, we obtain

\[
\int_{\mathbb{R}^d} \xi(x)|v_n - v|^p \mu dx \geq \int_{\Lambda_n \cap \Lambda_0} \xi(x)|v_n - v|^p \mu dx
\]

\[
\geq \frac{1}{2^p} \int_{\Lambda_n \cap \Lambda_0} \xi(x)|v|^p dx - \int_{\Lambda_n \cap \Lambda_0} \xi(x)|v_n|^p dx
\]

\[
\geq \left( \frac{\delta_0}{2^p} - \frac{1}{n} \right) \text{meas}(\Lambda_n \cap \Lambda_0)
\]

\[
\geq \frac{\delta_0}{2^{p+2}} > 0
\]

which contradicts (60). Thus the claim.

By (57), we have

\[
B(u) = \int_{\mathbb{R}^d} \xi(x)|u|^p \mu dx \geq \int_{\Lambda_n} c_H\|u\|^p dx \geq c_H\|u\|^p \text{meas}(\Lambda_n) = c_H^2\|u\|^p.
\]

This implies that \( B(u) \to \infty \) as \( \|u\| \to \infty \) in \( H \). The proof of Lemma 24 is complete.

**Lemma 25.** Let \((l_k)_{k \in \mathbb{N}}\) the sequence defined by

\[
l_k := \sup_{u \in Z_k, \|u\| = 1} \|u\|_{L^p(\mathbb{R}^d)}.
\]  

Then

\[
l_k \to 0 \text{ as } k \to +\infty. \tag{62}
\]
Proof. It is clear that \((l_k)\) is non-increasing positive sequence. So there exists \(z \geq 0\) such that \(l_k \to z\) as \(k \to +\infty\). For any \(k \in \mathbb{N}\), there exists \(u_k \in Z_k\) such that \(\|u_k\| = 1\) and \(\|u_k\|_{L^p(\mathbb{R}^d)} \geq \frac{1}{k}\). We observe that \(u_k \to u\) in \(E\) and \(\langle e_n^*, u_k \rangle = 0\) for \(k > n\). So \((e_n^*, u) = \lim_{k \to +\infty} \langle e_n^*, u_k \rangle = 0\), for all \(n \in \mathbb{N}\), which gives \(u = 0\). Corollary 1 implies that \(u_k \to 0\) in \(L^p(\mathbb{R}^d)\). Thus \(z = 0\).

Lemma 26. Assume that \((m_1), (M_1), (V_1) - (V_2)\) and \((f_1)\) are satisfied. Then there exists a sequence \(\rho_k \to 0^+\) as \(k \to \infty\) such that

\[
a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) > 0, \quad \text{for all } k \in \mathbb{N},
\]

and

\[
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \quad \text{as } k \to \infty \quad \text{uniformly for } \lambda \in [1, 2].
\]

Proof. Using Lemma 16 and Hölder inequality, for any \(u \in Z_k\) and \(\lambda \in [1, 2]\),

\[
I_\lambda(u) \geq \xi_0(|u|_{(s, M)}) + \xi_0(\|u\|_{(V, M)}) - 2 \int_{\mathbb{R}^d} \xi(x)|u|^p dx
\]

\[
\geq \xi_0(|u|_{(s, M)}) + \xi_0(\|u\|_{(V, M)}) - 2\|\|\|_{L^{p^*}} u\|_{L^p(\mathbb{R}^d)}^p.
\]

Combining (61) and (63), we get

\[
I_\lambda(u) \geq \xi_0(|u|_{(s, M)}) + \xi_0(\|u\|_{(V, M)}) - 2\|\|\|_{L^{p^*}} u\|_{L^p(\mathbb{R}^d)}^p,
\]

for all \(k \in \mathbb{N}\) and \((\lambda, u) \in [1, 2] \times Z_k\). Choose \(\theta > 0\) (\(\theta\) will be fixed later) and

\[
\rho_k = (4\theta \xi_0(\|\|_{L^{p^*}} u\|_{L^p(\mathbb{R}^d)}^p))^{1/p}.
\]

By (62), we have

\[
\rho_k \to 0 \quad \text{as } k \to +\infty,
\]

and so, for \(k\) large enough, \(\rho_k \leq 1\). Then,

\[
\xi_0(|u|_{(s, M)}) + \xi_0(\|u\|_{(V, M)}) \geq \frac{1}{2^{m+1}} \|\|_{m^o} \text{ for } u \in Z_k \text{ and } \|\| = \rho_k.
\]

By (64), (65), (67) and choosing \(\theta > 2^{m^o-2}\), direct computation shows

\[
a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq \left(2^{m^o-2} - \frac{1}{2\theta}\right) \rho_k^{m^o} > 0, \quad \text{for all } k \in \mathbb{N}.
\]

Besides, by (64), for each \(k \in \mathbb{N}\), we have

\[
I_\lambda(u) \geq -2\|\|_{L^{p^*}} u\|_{L^p(\mathbb{R}^d)}^p \rho_k^p, \quad \text{for all } \lambda \in [1, 2] \text{ and } u \in Z_k \text{ with } \|\| \leq \rho_k.
\]

Therefore,

\[
-2\|\|_{L^{p^*}} u\|_{L^p(\mathbb{R}^d)}^p \rho_k^p \leq \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \leq 0, \quad \text{for all } \lambda \in [1, 2] \text{ and } k \in \mathbb{N}.
\]

Combining (62) and (68), we have

\[
d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \quad \text{as } k \to \infty \quad \text{uniformly for } \lambda \in [1, 2].
\]

The proof of Lemma 26 is complete.

Lemma 27. Assume that \((m_1), (M_1), (V_1) - (V_2)\) and \((f_1)\) hold. Then for any \(k \in \mathbb{N}\) there are, \(r_k < \rho_k\) (fixed by Lemma 26) such that

\[
b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) < 0.
\]
Proof. Since $Y_k$ is with finite dimensional and by the claim (57), there exists $\epsilon_k > 0$ such that
\begin{equation}
\text{meas}(\Lambda^k_n) \geq \epsilon_k, \quad \text{for all } u \in Y_k \setminus 0,
\end{equation}
where $\Lambda^k_n := \{ x \in \mathbb{R}^d : \xi(x)|u(x)|^p \geq \epsilon_k |u|^p \}$. By Lemma 16, for any $k \in \mathbb{N}$,
\begin{equation}
I_\lambda(u) \leq \xi_1([u]_{(s,M)}) + \xi_1([u]_{(V,M)}) - \int_{\mathbb{R}^d} \xi(x)|u|^p \, dx
\end{equation}
\begin{equation}
\leq \xi_1([u]_{(s,M)}) + \xi_1([u]_{(V,M)}) - \int_{\Lambda^k_n} \epsilon_k|u|^p \, dx
\end{equation}
\begin{equation}
\leq \xi_1([u]_{(s,M)}) + \xi_1([u]_{(V,M)}) - \epsilon_k|u|^p, \text{meas}(\Lambda^k_n)
\end{equation}
\begin{equation}
\leq \xi_1([u]_{(s,M)}) + \xi_1([u]_{(V,M)}) - \epsilon_k^2|u|^p
\end{equation}
\begin{equation}
\leq 2||u||^{m_0} - \epsilon_k^2|u|^p \leq -2||u||^{m_0}
\end{equation}
for all $u \in Y_k$ with $||u|| \leq \min\{\rho_k, 4^{-\frac{1}{m_0-p}} \epsilon_k \frac{2}{m_0-p}, 1\}$. We choose
\begin{equation}
0 < r_k < \min\{\rho_k, 4^{-\frac{1}{m_0-p}} \epsilon_k \frac{2}{m_0-p}, 1\}, \quad \text{for all } k \in \mathbb{N}.
\end{equation}
Using (70), we deduce that
\begin{equation}
b_k(\lambda) = \max_{u \in Y_k, ||u|| = r_k} I_\lambda(u) < -2r_k^{m_0} < 0, \quad \text{for all } k \in \mathbb{N}.
\end{equation}
Thus the proof.

Proof of Theorem 3: Since $I_\lambda(u) \leq I_1(u)$ for all $u \in E$ and $I_1$ maps bounded sets to bounded sets, we see that $I_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover, $I_1$ is even. Then the condition (i) in Theorem 22 is satisfied. Besides, Lemma 24 shows that the condition (ii) in Theorem 22 holds. While Lemma 26 together with Lemma 27 implies that the condition (iii) holds.

Therefore, by Theorem 22, for each $k \in \mathbb{N}$, there exist $\lambda_n \to 1$, $u_{\lambda_n} \in Y_n$ such that
\begin{equation}
I_\lambda_n|_{Y_n}(u_{\lambda_n}) = 0, \quad I_\lambda_n(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty.
\end{equation}
For the sake of notational simplicity, in what follows we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$.

Claim 3. We claim that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $E$.

In fact, if $||u_n|| \leq 1$, for all $n \in \mathbb{N}$, nothing to prove. If not, we define the following sets:
\begin{align*}
N_1 &= \{ n \in \mathbb{N}, \quad ||u_n||_{(V,M)} \leq 1 \quad \text{and} \quad [u_n]_{(s,M)} \leq 1 \}, \\
N_2 &= \{ n \in \mathbb{N}, \quad ||u_n||_{(V,M)} \leq 1 \quad \text{and} \quad [u_n]_{(s,M)} \geq 1 \}, \\
N_3 &= \{ n \in \mathbb{N}, \quad ||u_n||_{(V,M)} \geq 1 \quad \text{and} \quad [u_n]_{(s,M)} \leq 1 \}, \\
N_4 &= \{ n \in \mathbb{N}, \quad ||u_n||_{(V,M)} \geq 1 \quad \text{and} \quad [u_n]_{(s,M)} \geq 1 \}.
\end{align*}

It is clear that
\begin{equation}
||u_n|| \leq 2 \quad \text{for all } \quad n \in N_1.
\end{equation}
Let $n \in N_2$, combining (71), Lemmas 16, Lemma 18 and the Hölder inequality, we obtain
\begin{equation}
||u_n||_{(V,M)}^{m_0} + [u_n]_{(s,M)}^{m_0} \leq I_{\lambda_n}(u_n) + \lambda_n \int_{\mathbb{R}^d} \xi(x)|u_n(x)|^p \, dx
\leq C_1 + 2||\xi||_{L^{\frac{m_0}{m_0-p}}(\mathbb{R}^d)} ||u_n||_p^p.
\end{equation}
for some constants $C_1, C_2 > 0$. Since $p < m_0$, there exists $D_1 > 0$ such that $[u_n]_{(s,M)} \leq D_1$ for all $n \in N_2$. It follows that

$$
\|u_n\| \leq 1 + D_1 \quad \text{for all} \quad n \in N_2.
$$

(73)

By the same argument as above, we can see that for some $D_2 > 0$,

$$
\|u_n\|_{(V,M)} \leq D_2 \quad \text{for all} \quad n \in N_3.
$$

(74)

Let $n \in N_4$, combining (71), Lemmas 16, Lemma 18 and the Hölder inequality, we obtain

$$
\frac{1}{2m_0-1}\|u_n\|^{m_0} \leq I_{\lambda_n}(u_n) + \lambda_n \int_{\mathbb{R}^d} \xi(x)|u_n(x)|^p dx \leq C_1 + \frac{2}{p}\|u_n\|_{L^\frac{p}{n}\mathbb{R}^d}^p \|u_n\|^p.
$$

Then there exists $D_3 > 0$ such that

$$
\|u_n\| \leq D_3 \quad \text{for all} \quad n \in N_4.
$$

(75)

By accumulating all the preceding cases (72), (73), (74) and (75), we deduce that the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $E$.

**Claim 4.** The sequence $(u_n)$ admits a strongly convergent subsequence in $E$.

In fact, in view of Claim 3 and up to subsequence, $u_n \rightarrow u_0$ as $n \rightarrow +\infty$, for some $u_0 \in E$. On one hand, according to Lemma 23, we have

$$
\lim_{n \rightarrow +\infty} \langle I'_{\lambda_n}(u_n) - I'_{\lambda_n}(u_0), u_n - u_0 \rangle = \lim_{n \rightarrow +\infty} \langle I'_{\lambda_n}(u_n), u_n - u_0 \rangle - \langle I'_{\lambda_n}(u_0), u_n - u_0 \rangle = \lim_{n \rightarrow +\infty} -\langle I'_{\lambda_n}(u_0), u_n - u_0 \rangle.
$$

Since

$$
I'_{\lambda_n}(u_0) \to I'_{\lambda}(u_0) \quad \text{in} \quad E^* \quad \text{as} \quad n \to +\infty \quad \text{and} \quad u_n - u_0 \to 0 \quad \text{in} \quad E \quad \text{as} \quad n \to +\infty,
$$

so

$$
\lim_{n \rightarrow +\infty} \langle I'_{\lambda_n}(u_0), u_n - u_0 \rangle = 0,
$$

then

$$
\lim_{n \rightarrow +\infty} \langle I'_{\lambda_n}(u_n) - I'_{\lambda_n}(u_0), u_n - u_0 \rangle = 0.
$$

On the other hand, by Hölder inequality and Lemma 18, we get

$$
\int_{\mathbb{R}^d} |f(x, u_n) - f(x, u_0)||u_n - u_0| dx
$$

$$
= p \int_{\mathbb{R}^d} \xi(x)||u_n|^{p-2}u_n - |u|^{p-2}u||u_n - u_0| dx
$$

$$
\leq p \int_{\mathbb{R}^d} \xi(x)(||u_n|^{p-1} + |u|^{p-1})|u_n - u_0| dx
$$

$$
\leq ||\xi||_{L^\frac{p}{n}\mathbb{R}^d} \left( \int_{\mathbb{R}^d} (||u_n|^{p-1} + |u|^{p-1})^\frac{n}{p} dx \right)^{\frac{p-1}{p}} \|u_n - u_0\|_{L^\infty}.
$$
\[
\leq 2\|\xi\|_{L^{\frac{1}{1-p}}} \left( \|u_n\|_{L^p}^p + \|u_0\|_{L^p}^p \right)^{\frac{p-1}{p}} \|u_n - u_0\|_{L^p} \to 0, \ n \to +\infty,
\]

since
\[
\langle I_{\lambda_n}'(u_n) - I_{\lambda_n}'(u_0), u_n - u_0 \rangle
= \langle G'(u_n) - G'(u_0), u_n - u_0 \rangle
+ \int_{\mathbb{R}^d} V(x)[m(u_n) - m(u_0)](u_n - u_0)dx
- \lambda_n \int_{\mathbb{R}^d} [f(x,u_n) - f(x,u_0)](u_n - u_0)dx \to 0, \ n \to +\infty.
\]

Therefore
\[
\langle G'(u_n) - G'(u_0), u_n - u_0 \rangle + \int_{\mathbb{R}^d} V(x)[m(u_n) - m(u_0)](u_n - u_0)dx \to 0, \ as \ n \to \infty.
\]

According to Lemma 21, \((u_n)\) converges strongly to \(u_0\) in \(E\). Thus the claim.

Now by the last assertion of Theorem 22, we conclude that \(I_1\) has infinitely many nontrivial critical points. Therefore, (1) possesses infinitely many nontrivial solutions. The proof of Theorem 3 is complete.

\section*{REFERENCES}

[1] R. A. Adams, \textit{Sobolev Spaces}, Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.
[2] C. O. Alves, G. M. Figueiredo and J. A. Santos, Strauss and Lions type results for a class of Orlicz-Sobolev spaces and applications, \textit{Topol. Methods Nonlinear Anal.}, 44 (2014), 435–456.
[3] V. Ambrosio, Multiple solutions for a fractional \(p\)-Laplacian equation with sign-changing potential, \textit{Electron. J. Differential Equations}, 2016 (2016), 12 pp.
[4] G. Autuori and P. Pucci, Elliptic problems involving the fractional Laplacian in \(\mathbb{R}^d\), \textit{J. Differ. Equ.}, 255 (2013), 2340–2362.
[5] E. Azroul, A. Benkirane and M. Srati, Introduction to fractional Orlicz-Sobolev spaces, \texttt{arXiv:1807.11753}.
[6] A. Bahrouni, H. Ounaies and V. D. Rădulescu, Infinitely many solutions for a class of sublinear Schrödinger equations with indefinite potentials, \textit{Proc. Roy. Soc. Edinburgh Sect. A}, 145 (2015), 445–465.
[7] A. Bahrouni, Trudinger-Moser type inequality and existence of solution for perturbed nonlocal elliptic operators with exponential nonlinearity, \textit{Commun. Pure Appl. Anal.}, 16 (2017), 243–252.
[8] A. Bahrouni, S. Bahrouni and M. Q. Xiang, On a class of nonvariational problems in fractional Orlicz-Sobolev spaces, \textit{Nonlinear Analysis}, 190 (2020), 111595, 13 pp.
[9] S. Bahrouni, H. Ounaies and L. S. Tavares, Basic results of fractional Orlicz-Sobolev space and applications to non-local problems, \textit{Topol. Methods Nonlinear Anal.}, accepted for publication.
[10] T. Bartsch and Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problems in \(\mathbb{R}^d\), \textit{Comm. Partial Differ. Equ.}, 20 (1995), 1725–1741.
[11] G. M. Bisci and V. D. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations, \textit{Calc. Var. Partial Differential Equations}, 54 (2015), 2985–3008.
[12] G. Bonanno, G. Molica Bisci and V. Rădulescu, Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz-Sobolev spaces, \textit{C. R. Math. Acad. Sci. Paris}, 349 (2011), 263–268.
[13] J. Fernández Bonder and A. M. Salort, Fractional order Orlicz-Sobolev spaces, \textit{Journal of Functional Analysis}, 277 (2019), 333–367.
[14] J. F. Bonder, M. P. LLanos and A. M. Salort, A Hölder infinity Laplacian obtained as limit of Orlicz fractional Laplacians, \texttt{arXiv:1807.01669}.
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[15] J. F. Bonder and A. M. Salort, Magnetic Fractional order Orlicz-Sobolev spaces, J. Funct. Anal., 277 (2019), 333–367, arXiv:1812.05998.

[16] X. J. Chang, Ground state solutions of asymptotically linear fractional Schrödinger equations, J. Math. Phys., 54 (2013), 061504, 10 pp.

[17] Ph. Clément, M. García-Huidobro, R. Manásevich and K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, Calc. Var. Partial Differential Equations, 11 (2000), 33–62.

[18] Ph. Clément, B. de Pagter, G. Sweers and F. de Thelin, Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces, Mediterr. J. Math., 1 (2004), 241–267.

[19] S. Dipierro, M. Medina and E. Valdinoci, Fractional Elliptic Problems with Critical Growth in the Whole of $\mathbb{R}^n$, Lecture Notes, Scuola Normale Superiore di Pisa, 15. Edizioni della Normale, Pisa, 2017.

[20] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), 521–573.

[21] M. García-Huidobro, V. K. Le, R. Manásevich and K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: An Orlicz-Sobolev space setting, Nonlinear Differ. Equat. Appl., 6 (1999), 207–225.

[22] M. Mihăilescu and V. Rădulescu, Nonhomogeneous Neumann problems in Orlicz-Sobolev spaces, C. R. Acad. Sci. Paris., 346 (2008), 401–406.

[23] G. Molica Bisci, V. D. Rădulescu and R. Servadei, Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications, 162. Cambridge University Press, Cambridge, 2016.

[24] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys., 43 (1992), 270–291.

[25] M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.

[26] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys., 55 (1977), 149–162.

[27] C. Torres, On superlinear fractional $p$-Laplacian in $\mathbb{R}^d$, (2014), arXiv:1412.3392.

[28] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

[29] Q. Y. Zhang and Q. Wang, Multiple solutions for a class of sublinear Schrödinger equations, J. Math. Anal. Appl., 389 (2012), 887–898.

[30] Q. Y. Zhang and B. Xu, Multiplicity of solutions for a class of semilinear Schrödinger equations with sign-changing potential, J. Math. Anal. Appl., 377 (2011), 834–840.
[43] W. M. Zou, Variant fountain theorems and their applications, Manuscripta Math., 104 (2001), 343–358.

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