L\textsuperscript{p} DECAY FOR GENERAL HYPERBOLIC-PARABOLIC SYSTEMS OF BALANCE LAWS

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Abstract. We study time asymptotic decay of solutions for a general system of hyperbolic-parabolic balance laws in multi space dimensions. The system has physical viscosity matrices and a lower order term for relaxation, damping or chemical reaction. The viscosity matrices and the Jacobian matrix of the lower order term are rank deficient. For Cauchy problem around a constant equilibrium state, existence of solution global in time has been established recently under a set of reasonable assumptions. In this paper we obtain optimal L\textsuperscript{p} decay rates for p \geq 2. Our result is general and applies to physical models such as gas flows with translational and vibrational non-equilibrium. Our result also recovers or improves the existing results in literature on the special cases of hyperbolic-parabolic conservation laws and hyperbolic balance laws, respectively.

1. Introduction. We are interested in a general class of partial differential equations arising from continuum mechanics. They are hyperbolic-parabolic balance laws in the following form:

\[ w_t + \sum_{j=1}^{m} f_j(w)x_j = \sum_{j,k=1}^{m} [B_{jk}(w)w_{x_k}]x_j + r(w), \quad m \geq 1, \]  

where \( w, f_j, r \in \mathbb{R}^n \) and \( B_{jk} \in \mathbb{R}^{n \times n} \). The unknown function \( w = w(x,t) \) depends on the space variable \( x = (x_1, \ldots, x_m)^t \in \mathbb{R}^m \) and the time variable \( t \in \mathbb{R}^+ \), and stands for physical densities such as mass density, momentum density, energy density, etc. As given functions of \( w \), \( f_j \) are flux functions, \( 1 \leq j \leq m \), and \( r \) represents external forces, relaxation, chemical reactions and so forth. The matrices \( B_{jk}, 1 \leq j, k \leq m \), are also known functions of \( w \). They are viscosity matrices, describing viscosity, heat conduction, species diffusion, etc. Equation (1.1) describes the balance of physical quantities, such as mass, momentum and energy, of a flow. The flux functions usually satisfy an entropy condition so that the corresponding inviscid system is completely hyperbolic [2]. We are interested in physical models (not artificial models) thus the viscosity matrices are rank deficient. This implies (1.1)
is hyperbolic-parabolic, not uniformly parabolic. Similarly, the Jacobian matrix of the lower order term, \( r'(w) \), is also rank deficient.

We now give several important examples from physics:

**Example 1.** Navier-Stokes equations for compressible flows:

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho uu^t) + \nabla p = \text{div}(2\mu \mathcal{P}) + \nabla (\mu' \text{div} u), \\
(\rho E)_t + \text{div}(\rho Eu + pu) = \text{div}[2\mu \mathcal{P} \cdot u + \mu' \text{div} uuI + \kappa \nabla T].
\end{cases}
\] (1.2)

Here \( \rho \) is the gas density, \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \) the velocity, \( p \) the pressure, \( E = e + \frac{1}{2} |u|^2 \) the total energy, with the internal energy \( e \), \( I \in \mathbb{R}^{m \times m} \) the identity matrix, and \( T \) the temperature. The strain rate tensor \( \mathcal{P} \in \mathbb{R}^{m \times m} \) has entries

\[
\mathcal{P}_{jk} = \frac{1}{2} (u_{jx_k} + u_{kx_j}), \quad 1 \leq j, k \leq m.
\]

The dissipation parameters are the shear viscosity coefficient \( \mu \), the second viscosity coefficient \( \mu' \), and the thermal conductivity \( \kappa \). The thermodynamic equation is

\[
T \text{ds} = de + pdv, \quad v = 1/\rho,
\] (1.3)

where \( v \) is the specific volume, and \( s \) is the entropy. Therefore, only two of the thermodynamic variables are independent. The other variables and the dissipation parameters are regarded as known functions of these two.

The Navier-Stokes equations can be derived from the Boltzmann equation by Chapman-Enskog expansion assuming the gas molecules are under translational non-equilibrium while they have no internal structure or other forms of non-equilibrium [10]. The equations are an example of (1.1) with \( r(w) = 0 \). That is, they are an example of the special case of hyperbolic-parabolic conservation laws,

\[
w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} [B_{jk}(w)w_{x_k}]_{x_j}, \quad m \geq 1.
\] (1.4)

**Example 2.** The following equations describe the motion of a polyatomic gas in vibrational non-equilibrium:

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho uu^t) + \nabla p = 0, \\
(\rho E)_t + \text{div}(\rho Eu + pu) = 0, \\
(\rho e_2)_t + \text{div}(\rho e_2 u) = \frac{\rho e_2^* - e_2}{\tau},
\end{cases}
\] (1.5)

where the notations are as defined in Example 1 except

\[
E = e + \frac{1}{2} |u|^2, \quad e = e_1 + e_2,
\]

with \( e_1 \) and \( e_2 \) being the total of equilibrium internal energy and vibrational energy, respectively. The first three equations in (1.5) are exactly the Euler equations for inviscid, compressible flows, describing the conservation of mass, momentum and energy, respectively. The last equation in (1.5) describes the relaxation of the vibrational energy, the non-equilibrium internal energy, towards its local equilibrium value \( e_2^* \) in the time scale \( \tau \). Here \( \tau \) is called the relaxation time.

Because we are considering a polyatomic gas whose molecules have an internal structure, and the structure is not in dynamical equilibrium, we have to use two sets
of thermodynamic variables to describe the flow: one for the total of equilibrium modes, and one for the non-equilibrium vibrational mode. We use subscript “1” for the first set and “2” for the second one, respectively. Thus
\[ s = s_1 + s_2, \] (1.6)
with \( s_1 \) and \( s_2 \) being the equilibrium entropy and vibrational entropy, respectively.

The two sets of variables obey different thermodynamic equations:
\[ T_1 ds_1 = de_1 + pdv, \quad T_2 ds_2 = de_2. \] (1.7)

We note that the second equation is volume independent. It is clear that two variables in mode 1 and one in mode 2 are independent. With \( m \) components of the velocity, (1.5) is a system of \( m + 3 \) equations for \( m + 3 \) unknowns.

As in the case of Navier-Stokes equations, the equations for inviscid, vibrational non-equilibrium flow (1.5) can be derived from the Boltzmann equation by Chapman-Enskog expansion [10]. In this case, the only non-equilibrium mode is the vibrational mode. In particular, the flow is under translational equilibrium.

Equation (1.5) is in the form (1.1) with \( B_{jk} = 0 \). That is, it is an example of the special case of hyperbolic balance laws:
\[ w_t + \sum_{j=1}^{m} f_j(w)x_j = r(w), \quad m \geq 1. \] (1.8)

The following example is also of this type.

**Example 3.** Euler equations with damping for inviscid, compressible, isentropic or isothermal flows:
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho uu) + \nabla p &= -\rho u.
\end{align*}
\] (1.9)

Here since the flow is isentropic or isothermal, we regard the pressure \( p \) as a known function of the density \( \rho \). Equation (1.9) describes the motion of a gas through a porous medium, which induces a friction force proportional to the momentum.

Other examples of (1.8) with \( r'(w) \) rank deficient are the Kerr-Debye model for the propagation of electromagnetic waves in a nonlinear Kerr medium, and the equations for the motion of an unbounded, homogeneous, viscoelastic bar with fading memory when the kernel of the memory is a finite sum of exponential decay functions [14].

The next example of (1.1) is not of the special cases of (1.4) or (1.8). It has both nontrivial viscosity matrices and the lower order term.

**Example 4.** The following equations describe the motion of a polyatomic gas in both translational and vibrational non-equilibrium:
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho uu') + \nabla p &= \text{div}(2\mu \mathcal{P} \cdot \mathcal{P}) + \nabla (\mu' \text{div} u), \\
(\rho E)_t + \text{div}(\rho Eu + pu) &= \text{div}[2\mu \mathcal{P} \cdot u + \mu' \text{div} u I \cdot u + \kappa \nabla T_1 + \nu \rho \nabla e_2], \\
(\rho e_2)_t + \text{div}(\rho e_2 u) &= \text{div}(\nu \rho \nabla e_2) + \frac{e_2^2 - e_2}{\tau}.
\end{align*}
\] (1.10)

Here the notations are the same as in Examples 1 and 2, with a new dissipation parameter \( \nu \), which is the self-diffusion coefficient. Equation (1.10) is a system of \( m+3 \) equations for \( m+3 \) unknowns, supplemented by the thermodynamic equations.
In this case, mode 1 represents the combination of the translational mode and rotational mode of the gas molecules, while mode 2 is for the vibrational mode.

Similar to Examples 1 and 2, (1.10) is derived from Boltzmann equation by Chapman-Enskog expansion under the assumption of translational and vibrational non-equilibrium [1]. The translational non-equilibrium results in viscosity, heat conduction and self-diffusion, while the vibrational non-equilibrium gives rise to the relaxation. The other internal modes, such as rotation, are assumed to be in equilibrium hence share the same temperature $T_1$ of the translational mode. Otherwise, new relaxation equations can be added for those modes. This extends the system (1.10) but does not change its structure.

We are interested in the Cauchy problem of (1.1) with prescribed initial data:

$$w(x,0) = w_0(x).$$

(1.11)

Here $w_0$ is assumed to be a small perturbation of a constant equilibrium state $ar{w}$, $r(ar{w}) = 0$.

In a recent paper the author has proposed a set of structural conditions for (1.1), which leads to the existence of global solution of the Cauchy problem near an equilibrium state [13]. The general theorem there applies to Example 4 under physical assumptions. It also recovers the known results in the literature of the hyperbolic-parabolic conservation laws (1.4) and of the hyperbolic balance laws (1.8) as special cases.

In this paper we show that under the same set of structural conditions, we may obtain $L^p$ ($p \geq 2$) convergence rates of small solutions of (1.1) to an equilibrium state. The rates are optimal, and consistent with those obtained in [3] for the hyperbolic-parabolic conservation laws (1.4), as well as those in [4] for the hyperbolic balance laws (1.8). Our result is true for all space dimensions $m \geq 1$. However, the strategies used to prove the cases $m \geq 2$ and $m = 1$ are different. (Neither applies to the other.) Therefore, we focus on multi space dimensions ($m \geq 2$) here, and leave the case of one space dimension to a future paper.

We now state our basic structural conditions for (1.1). Consider a neighborhood $\mathcal{O}$ of the constant equilibrium state $\bar{w}$, and define the equilibrium manifold $E$ in $\mathcal{O}$ as

$$E = \{ w \in \mathcal{O} \mid r(w) = 0 \}. \tag{1.12}$$

The functions $f_j(w)$, $B_{jk}(w)$ and $r(w)$ are assumed to be smooth in $\mathcal{O}$. In the following we use $f_j'$ to denote the Jacobian matrix of $f_j$ with respect to $w$, etc.

**Assumption 1.1.**

1. There exists a strictly convex entropy function $\eta$, which is a scalar function of $w$ in $\mathcal{O}$, satisfying the following properties.
   (i) $\eta'' f_j'$, $1 \leq j \leq m$, are symmetric in $\mathcal{O}$, where $\eta''$ is the Hessian of $\eta$ with respect to $w$.
   (ii) In $\mathcal{O}$, $(\eta'' B_{jk})' = \eta'' B_{kj}$, $1 \leq j, k \leq m$, and $\eta'' \sum_{j,k=1}^m B_{jk} \xi_j \xi_k$ is symmetric, semi-positive definite for all unit vectors $\xi = (\xi_1, \ldots, \xi_m)^t \in S^{m-1}$.
   (iii) On $E$, $\eta'' r'$ is symmetric, semi-negative definite.

2. Equation (1.1) has $n_1$ conservation laws. That is, there is a partition $n = n_1 + n_2$, $n_1, n_2 \geq 0$, such that

$$r(w) = \begin{pmatrix} 0_{n_1 \times 1} \\ r_2(w) \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1.13}$$

with $w_1 \in \mathbb{R}^{n_1}$, $r_2, w_2 \in \mathbb{R}^{n_2}$, and $(r_2)_{w_2} \in \mathbb{R}^{n_2 \times n_2}$ is nonsingular. Here $(r_2)_{w_2}$ denotes the Jacobian matrix of $r_2$ with respect to $w_2$, etc.
3. There is a diffeomorphism \( \varphi \to w \) from an open set \( \hat{\Omega} \subset \mathbb{R}^n \) to \( \Omega \) and a constant orthogonal matrix \( P \in \mathbb{R}^{n \times n} \) such that

\[
P^tB_{jk}(w(\varphi))w'_\varphi(\varphi)P = \begin{pmatrix} 0_{n_3 \times n_3} & 0_{n_3 \times n_4} \\ 0_{n_4 \times n_3} & B_{jk}^* \end{pmatrix}, \quad 1 \leq j, k \leq m. \tag{1.14}
\]

Here \( n_3, n_4 \geq 0 \) are two constant integers such that \( n_3 + n_4 = n \), and \( \sum_{j,k=1}^m B_{jk}^* \xi_k \xi_j \in \mathbb{R}^{n_4 \times n_4} \) is nonsingular (if \( n_4 > 0 \)) for all \( \varphi \in \hat{\Omega} \) and all \( \xi = (\xi_1, \ldots, \xi_m)^t \in \mathbb{S}^{m-1} \).

4. \([9]\) For \( \xi = (\xi_1, \ldots, \xi_m)^t \in \mathbb{S}^{m-1} \) let

\[
A(\xi) = \sum_{j=1}^m f_j(\bar{w})\xi_j, \quad B(\xi) = \sum_{j,k=1}^m b_{jk}(\bar{w})\xi_k \xi_j. \tag{1.15}
\]

Let \( \mathbb{N}_1 \) be the null space of \( B(\xi) \) and \( \mathbb{N}_2 \) be the null space of \( r'(\bar{w}) \). Then for each \( \xi, \mathbb{N}_1 \cap \mathbb{N}_2 \) contains no eigenvectors of \( A(\xi) \).

We comment that the partitions \( n = n_1 + n_2 \) and \( n = n_3 + n_4 \) in conditions 2 and 3 of Assumption 1.1 are independent. The locations of the conservation laws and rate equations are also independent of the locations of the hyperbolic equations and parabolic equations. (Matrix \( P \) is usually a permutation and is for such a purpose.) In the appendix we use Example 1.4 to illustrate the independence by different choices of dissipation parameters.

We introduce the following notations to abbreviate the norms of Sobolev spaces with respect to \( x \):

\[
\| \cdot \|_s = \| \cdot \|_{H^s(\mathbb{R}^m)}, \quad \| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^m)}. \tag{1.16}
\]

With \( \varphi \) and \( P \) given in condition 3 of Assumption 1.1, we define

\[
\bar{w} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = P^t \varphi(w), \tag{1.17}
\]

where \( \bar{w}_1 \in \mathbb{R}^{n_3} \) and \( \bar{w}_2 \in \mathbb{R}^{n_4} \).

**Theorem 1.2.** \([13]\) Let \( \bar{w} \) be a constant equilibrium state, Assumption 1.1 be satisfied, \( s > \frac{n}{2} + 1 \) (\( m \geq 1 \)) be an integer, and \( w_0 - \bar{w} \in H^s(\mathbb{R}^m) \). Then there exists a constant \( \varepsilon > 0 \) such that if \( \|w_0 - \bar{w}\|_s \leq \varepsilon \), the Cauchy problem (1.1), (1.11) has a unique global solution \( w \). The solution satisfies \( w - \bar{w} \in C([0, \infty); H^s(\mathbb{R}^m)), \quad D_x w \in L^2([0, \infty); H^{s-1}(\mathbb{R}^m)), \quad D_x \bar{w}_2(w) \in L^2([0, \infty); H^s(\mathbb{R}^m)), \quad r(w) \in L^2([0, \infty); H^s(\mathbb{R}^m)) \), and

\[
\sup_{t \geq 0} \|w - \bar{w}\|_s(t) + \int_0^\infty \left[ \|D_x w\|_{s-1}^2(t) + \|D_x \bar{w}_2(w)\|_s^2(t) + \|r_2(w)\|_s^2(t) \right] dt \leq C \|w_0 - \bar{w}\|_s^2, \tag{1.18}
\]

where \( C > 0 \) is a constant. Here \( D_x w \) denotes first partial derivatives of \( w \) with respect to \( x \), etc.

Let \( D_x^l \) be partial derivatives \((\partial/\partial x)^\alpha\) with a multi index \( \alpha \) such that \( |\alpha| = l \). Our main result is the following \( L^2 \) decay estimates for \( w \) in Theorem 1.2 when \( m \geq 2 \).

**Theorem 1.3.** Let \( \bar{w} \) be a constant equilibrium state of (1.1), and Assumption 1.1 be true. Let \( m \geq 2, \ s > \frac{n}{2} + 1 \) be an integer, and \( w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m) \).
Then there exists a constant $\varepsilon > 0$ such that if $\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, the solution of (1.1), (1.11) given in Theorem 1.2 has the following estimates for $t \geq 0$:
\[
\|D_2^l (w - \bar{w})\|_1(t) \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1})(t + 1)^{-\frac{m}{p} - \frac{1}{4}} \tag{1.19}
\]
for $0 \leq l \leq s - 2$, and
\[
\|D_2^l r_2(w)\|_1(t) \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1})(t + 1)^{-\frac{m}{p} - \frac{1}{4} + \frac{1}{4}} \tag{1.20}
\]
for $0 \leq l \leq s - 4$. Here $C > 0$ in (1.19) and (1.20) is a constant.

To obtain $L^p$ decay rates with $p \geq 2$ we recall Gagliardo-Nirenberg inequality [8]: There is a constant $C > 0$ such that for $g \in H^k(\mathbb{R}^m)$,
\[
\|D_2^l g\|_{L^p} \leq C\|D_2^k g\|_1^\theta \|g\|^{1 - \theta} \tag{1.21}
\]
where $0 \leq l \leq k, p \in [2, \infty]$, and $\theta = \frac{k + m(1/2 - 1/p)}{k} \leq 1$ ($\theta < 1$ if $p = \infty$). Applying (1.21) to $g = w - \bar{w}$ with $k = s - 2$, and to $g = r_2(w)$ with $k = s - 4$, we have the following corollary of Theorem 1.3:

**Corollary 1.4.** Under the assumptions of Theorem 1.3, the solution of (1.1), (1.11) has the following $L^p$ estimates with $p \geq 2$: For $t \geq 0$,
\[
\|D_2^l (w - \bar{w})\|_{L^p}(t) \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1})(t + 1)^{-\frac{m}{p}(1 - \frac{1}{p}) - \frac{1}{4}} \tag{1.22}
\]
for $0 \leq l \leq s - 2 - m(1/2 - 1/p)$, and
\[
\|D_2^l r_2(w)\|_{L^p}(t) \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1})(t + 1)^{-\frac{m}{p}(1 - \frac{1}{p}) - \frac{1}{4} + \frac{1}{4}} \tag{1.23}
\]
for $0 \leq l \leq s - 4 - m(1/2 - 1/p)$. If $p = \infty$, we further require $l \neq s - 2 - m/2$ for (1.22), and $l \neq s - 4 - m/2$ for (1.23). Here $C > 0$ is a constant.

As applications, in the appendix we give the reduced versions of Assumption 1.1, Theorem 1.3 and Corollary 1.4 for the special cases (1.4) and (1.8). See Assumptions A.1 and A.3, and Theorems A.2 and Theorems A.4. There we also apply our main results to Example 4.

For hyperbolic-parabolic conservation laws (1.4), $L^2$-decay rates similar to those in Theorem A.2 and convergence rates to the solution of the corresponding linear system have been obtained in [3] under similar assumptions. For hyperbolic balance laws (1.8), a parallel result of Theorem A.4 has been obtained in [4]. In fact, Theorem A.4 simplifies and slightly weakens the assumptions in [4] for the same decay rates, see [11] for a discussion of the two sets of hypotheses. Here we extend the study of $L^p$ decay rates to the general hyperbolic-parabolic system of balance laws (1.1), which leads to our main results, Theorem 1.3 and Corollary 1.4.

The plan of the paper is as follows. Section 2 is for preliminaries needed in our analysis. Section 3 is devoted to estimates of the linearized system. In Section 4 we give a weighted energy estimate. In Section 5 we carry out the proof of Theorem 1.3. Finally, in the appendix we discuss applications of the main results to Example 4 for the thermal non-equilibrium flow, and to the two special cases of hyperbolic-parabolic conservation laws (1.4) and hyperbolic balance laws (1.8).

Throughout this paper we use $C$ to denote a universal positive constant. Also, we use the bar accent for the value of a variable taken at the constant equilibrium state $\bar{w}$, e.g., $\bar{\varphi} = \varphi(\bar{w})$, etc.
2. Preliminaries. In this section we assume that condition 2 of Assumption 1.1 holds. In (1.1) the lower order term \( r(w) \) represents the part of solution with faster decay rate, as evidenced by (1.18). We separate this part from the leading term by introducing a new variable \( \psi \) using the notations in (1.13):

\[
\psi = \psi(w) = \begin{pmatrix} w_1 \\ r_2(w) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (w),
\]

(2.1)

where \( \psi_1 = w_1 \in \mathbb{R}^{n_1} \) and \( \psi_2 = r_2 \in \mathbb{R}^{n_2} \). Under condition 2 of Assumption 1.1, \( \psi \) is a diffeomorphism, with the Jacobian matrices

\[
\psi_w = \begin{pmatrix} I_{n_1 \times n_1} \\ (r_2)_w \end{pmatrix}, \quad w_\psi = \psi_w^{-1} = \begin{pmatrix} I_{n_1 \times n_1} \\ -(r_2)_w^{-1} \end{pmatrix}.
\]

(2.2)

Next we linearize (1.1) around the constant equilibrium state \( \bar{w} \) using the new variable \( \psi \). Let

\[
\tilde{\psi} = \psi - \bar{w} = \begin{pmatrix} w_1 - \bar{w}_1 \\ r_2(w) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix},
\]

(2.3)

be the perturbation. Multiplying from the left by \( \psi_w \), (1.1) can be written as

\[
\tilde{\psi}_t + \sum_{j=1}^{m} \psi_{w_j} f_j \psi_{x_j} = \sum_{j,k=1}^{m} \psi_{w_j} [B_{jk} \psi_{w_k}]_{x_j} + \psi_{w_r}.
\]

(2.4)

Multiplying the equation by a constant matrix \( \tilde{A}_0 \) to be defined in (2.6), (1.1) can be further written as

\[
\tilde{A}_0 \tilde{\psi} + \sum_{j=1}^{m} \tilde{A}_j \tilde{\psi}_{x_j} = \sum_{j,k=1}^{m} \tilde{B}_{jk} \tilde{\psi}_{x_k x_j} + \tilde{L} \tilde{\psi} + \tilde{R},
\]

(2.5)

where

\[
\tilde{A}_0 = (w_\psi \eta'' \psi_{w_j})(\bar{w}), \quad \tilde{A}_j = (w_\psi \eta'' f_j \psi_{w_j})(\bar{w}),
\]

\[
\tilde{B}_{jk} = (w_\psi \eta'' B_{jk} \psi_{w_j})(\bar{w}), \quad \tilde{L} = (w_\psi \eta'' r' \psi_{w_j})(\bar{w}),
\]

(2.6)

with \( 1 \leq j, k \leq m \), and

\[
\tilde{R} = R_1 + R_2 + R_3,
\]

\[
R_1 = \tilde{A}_0 \sum_{j=1}^{m} [(\psi_{w_j} f_j \psi_{w_j})(\bar{w}) - (\psi_{w_j} \psi_{w_j})(\bar{w})] \psi_{x_j},
\]

\[
R_2 = \tilde{A}_0 \sum_{j,k=1}^{m} \{ \psi_{w_j} (B_{jk} \psi_{w_k})_{x_j} - (\psi_{w_j} B_{jk} \psi_{w_k})(\bar{w}) \} \psi_{x_k x_j},
\]

\[
R_3 = \tilde{A}_0 [(\psi_{w_r})(\bar{w}) - (\psi_{w_r} \psi_{w_j})(\bar{w})] \psi_{x_j}.
\]

(2.7)

Without the nonlinear source \( \tilde{R} \), (2.5) would be a linear system of \( \tilde{\psi} \) with constant coefficients. The coefficients possess important properties, which we cite from Lemma 2.8 of [13]:

**Lemma 2.1.** [13] Under conditions 1, 2 and 4 of Assumption 1.1, we have the following.

(i) \( \tilde{A}_0 \) and \( \tilde{L} \) are real, symmetric. \( \tilde{A}_0 \) is positive definite while \( \tilde{L} \) is semi-negative definite.
(ii) For $\xi = (\xi_1, \ldots, \xi_m)^t \in S^{m-1}$, let
\[
\tilde{A}(\xi) = \sum_{j=1}^{m} \tilde{A}_j \xi_j, \quad \tilde{B}(\xi) = \sum_{j,k=1}^{m} \tilde{B}_{jk} \xi_k \xi_j.
\] (2.8)

Then $\tilde{A}(\xi)$ is real, symmetric, and $\tilde{B}(\xi)$ is real, symmetric and semi-positive definite. They satisfy $\tilde{A}(-\xi) = -\tilde{A}(\xi)$ and $\tilde{B}(-\xi) = \tilde{B}(\xi)$.

(iii) If $\zeta \in \mathbb{R}^n \backslash \{0\}$ and $\tilde{B}(\xi) \zeta = \tilde{L} \zeta = 0$ for some $\xi \in S^{m-1}$, then $\lambda \tilde{A}_0 \zeta + \tilde{A}(\xi) \zeta \neq 0$ for any $\lambda \in \mathbb{R}$.

When (i) and (ii) of Lemma 2.1 hold, there are several equivalent forms of (iii), see Theorem 1.1 and Remark 1.2 in [9]. Among them there is the existence of a so-called compensating function. As it is needed in our analysis, we state the following lemma as a consequence of Lemma 2.1 above, together with Theorem 1.1 and Remark 1.2 in [9]:

**Lemma 2.2.** Let conditions 1, 2, and 4 of Assumption 1.1 be true. Then there exists a smooth function $K(\xi) \in \mathbb{R}^{n \times n}$, defined on $S^{m-1}$ and called a compensating function, with the following properties: For each $\xi \in S^{m-1}$,
\[
S = S(\xi) = \frac{1}{2} [K(\xi) \tilde{A}(\xi) + \tilde{A}^t(\xi) K^t(\xi)] + \tilde{B}(\xi) - \tilde{L}
\] (2.9)
is real, symmetric, positive definite.

We may use Fourier transform to study a linear system with constant coefficients. We use the hat accent to denote the Fourier transform of a function in $x$:
\[
\hat{\psi}(\xi, t) = \int_{\mathbb{R}^m} \tilde{\psi}(x, t) e^{-ix \cdot \xi} \, dx,
\]
\[
\tilde{\psi}(x, t) = \frac{1}{(2\pi)^{m}} \int_{\mathbb{R}^m} \hat{\psi}(\xi, t) e^{ix \cdot \xi} \, d\xi.
\] (2.10)

Taking Fourier transform of (2.5) we have
\[
\tilde{A}_0 \hat{\psi}_t + \sum_{j=1}^{m} \tilde{A}_j i \xi_j \hat{\psi} + \sum_{j,k=1}^{m} \tilde{B}_{jk} \xi_k \xi_j - \tilde{L} \hat{\psi} = \hat{R},
\]
which can be further written as
\[
\tilde{A}_0 \hat{\psi}_t + [i|\xi| \tilde{A}(\frac{\xi}{|\xi|}) + |\xi|^2 \tilde{B}(\frac{\xi}{|\xi|}) - \tilde{L}] \hat{\psi} = \hat{R},
\] (2.11)
using (2.8).

Next we simplify $\hat{R}$ defined in (2.7). We write
\[
R_1 = \tilde{A}_0 \begin{pmatrix} R_{11} \\ R_{12} \end{pmatrix}, \quad R_2 = \tilde{A}_0 \begin{pmatrix} R_{21} \\ R_{22} \end{pmatrix}, \quad R_3 = \tilde{A}_0 \begin{pmatrix} R_{31} \\ R_{32} \end{pmatrix},
\] (2.12)
with $R_{11}, R_{21}, R_{31} \in \mathbb{R}^{n_1}$, and $R_{12}, R_{22}, R_{32} \in \mathbb{R}^{n_2}$. Since
\[
\begin{pmatrix} R_{11} \\ R_{12} \end{pmatrix} = \sum_{j=1}^{m} \left[ (\psi_w f_j^w w)\bar{w} - (\psi_w f_j^w w)(w) \right] \psi_j,
\] (2.13)
by (2.7), using (2.2) we have

\[ R_{11} = (I_{n_1 \times n_1} 0_{n_1 \times n_2}) \sum_{j=1}^{m} [(f_j w_\psi)(\tilde{w}) - (f_j w_\psi)(w)] \psi_{x_j} \]

\[ = (I_{n_1 \times n_1} 0_{n_1 \times n_2}) \sum_{j=1}^{m} [(f_j w_\psi)(\tilde{w})\psi - f_j(w)]_{x_j}. \]

If we write

\[ f_j = \begin{pmatrix} f_{j1} \\ f_{j2} \end{pmatrix}, \quad f_{j1} \in \mathbb{R}^{n_1}, \quad f_{j2} \in \mathbb{R}^{n_2} \quad (2.14) \]

for \(1 \leq j \leq m\), then

\[ R_{11} = \sum_{j=1}^{m} (\tilde{f}_{j1})_{x_j}, \quad (2.15) \]

\[ \tilde{f}_{j1} = -[f_{j1}(w) - f_{j1}(\tilde{w}) - (f_{j1} w_\psi)(\tilde{w})\tilde{\psi}] = O(1)|\tilde{\psi}|^2. \quad (2.16) \]

Similarly, from (2.7) and (2.2),

\[ \begin{pmatrix} R_{21} \\ R_{22} \end{pmatrix} = \sum_{j,k=1}^{m} \{\psi_{x_j}(w)[B_{jk}(w)w_{x_k}]_{x_j} - (\psi_{x_k} B_{jk} w_\psi)(\tilde{w})\psi_{x_k} \}, \quad (2.17) \]

with

\[ R_{21} = \sum_{j=1}^{m} (b_{j1})_{x_j}, \quad (2.18) \]

\[ b_{j1} = (I_{n_1 \times n_1} 0_{n_1 \times n_2}) \sum_{k=1}^{m} [B_{jk}(w)w_{x_k} - (B_{jk} w_\psi)(\tilde{w})\psi_{x_k}]. \quad (2.19) \]

On the other hand, from (2.7), (2.2), (2.3) and (1.13), and by direct calculation, we have

\[ R_{31} = 0, \quad R_{32} = [(r_2)_{w_2}(w) - (r_2)_{w_2}(\tilde{w})]r_2(w) = O(1)|w - \tilde{w}|^2. \quad (2.20) \]

Therefore,

\[ \tilde{R} = \sum_{j=1}^{3} R_j = \tilde{A}_0 \sum_{j=1}^{m} \begin{pmatrix} \tilde{f}_{j1} + b_{j1} \\ 0_{n_2 \times 1} \end{pmatrix}_{x_j} + \tilde{A}_0 \begin{pmatrix} 0_{n_1 \times 1} \\ R_{12} + R_{22} + R_{32} \end{pmatrix}. \quad (2.21) \]

To simplify \(\tilde{A}_0\) we cite Lemma 3.4 in [11]:

**Lemma 2.3.** [11] If \((\eta'' \tau')(\tilde{w})\) is symmetric then

\[ (\eta'' w_\psi)(\tilde{w}) = \begin{pmatrix} \eta_{w_1 w_1} - \eta_{w_2 w_1} (r_2)^{-1} (r_2)_{w_1} & \eta_{w_2 w_1} (r_2)^{-1} (r_2)_{w_1} \\ 0_{n_2 \times n_1} & \eta_{w_2 w_2} (r_2)^{-1} (r_2)_{w_2} \end{pmatrix}(\tilde{w}). \quad (2.22) \]

**Proof.** We calculate \((\eta'' \tau')(\tilde{w})\) and use the symmetry to relate the corresponding blocks. Substituting the result into \((\eta'' w_\psi)(\tilde{w})\) gives us (2.22), see [11]. \(\square\)

We take \((n_1, n_2)\) partition of \(\tilde{A}_0\) in rows and columns. From (2.2), (2.6) and (2.22), its \((2, 1)\) block is zero. By definition, \(\tilde{A}_0\) is symmetric thus block diagonal. That is, under the assumption of Lemma 2.3 we have

\[ \tilde{A}_0 = \text{diag}(A_{01}, A_{02}), \]

\[ A_{01} = [\eta_{w_1 w_1} - \eta_{w_2 w_1} (r_2)^{-1} (r_2)_{w_1}](\tilde{w}) \in \mathbb{R}^{n_1 \times n_1}, \]
Lemma 2.6 follows those in [11]. With energy estimates in multi space dimensions, e.g. [3]. Here our formulation Lemma 2.6 is a special case of (1.21). They can be found in many papers dealing Nirenberg inequality or its applications (Moser-type calculus inequalities). (Here as Lemmas 2.6-2.8 as follows. The lemmas are either special cases of Gagliardo-ξ vectors.)

\[
A_{02} = \{(r_2)^{-1}\}_{1}^{t} (\eta_{w_2}(r_2)^{-1}) (\bar{w}) \in \mathbb{R}^{n_2 \times n_2}. \tag{2.23}
\]

Together with (2.21), \( \tilde{R} \) is simplified as
\[
\tilde{R} = \begin{pmatrix} A_{01} \sum_{j=1}^{m} (f_{j1} + b_{j1}) x_j \end{pmatrix} + \begin{pmatrix} 0_{n_1 \times 1} \end{pmatrix} \begin{pmatrix} A_{02}(R_{12} + R_{22} + R_{32}) \end{pmatrix}, \tag{2.24}
\]
where \( A_{01}, A_{02}, f_{j1} \) and \( b_{j1} \) with \( 1 \leq j \leq m \), and \( R_{nk} \) with \( 1 \leq k \leq 3 \) are given by (2.23), (2.16), (2.19), (2.13), (2.17) and (2.20).

Taking transpose of (2.22) and applying (1.13), similar to (2.23) we simplify \( \tilde{L} \) in (2.6) as
\[
\tilde{L} = \text{diag}(0_{n_1 \times n_1}, (\eta_{w_2}(r_2)^{-1})(\bar{w})) \tag{2.25}
\]
under the assumption of Lemma 2.3.

In our analysis of (1.1) in later sections, the treatment of the lower order term \( r \) relies on the following crucial estimate obtained in [11]:

Lemma 2.4. [11] Under conditions 1(iii) and 2 of Assumption 1.1, in a small neighborhood of \( \bar{w} \) we have
\[
(\eta_{w_1, w_2} - \eta_{w_2, w_1})(r_2)^{-1}(r_2)w_1 = O(1) |r_2(w)|. \tag{2.26}
\]

To handle the viscosity term in (1.1) we need the diffeomorphism \( \varphi \) defined in condition 3 of Assumption 1.1. The direct use of \( \varphi \) is the main difference in our approach to handle the general system (1.1). This is to compare to the traditional approach of converting the system under study into a “normal form”, commonly used in the study of the special cases (1.4) and (1.8), see [3, 4] and references therein. For the application in later sections we cite the following lemma describing properties related to \( \varphi \) from [13]:

Lemma 2.5. [13] Conditions 1(iii) and 3 in Assumption 1.1 imply that \( P^t w_{\varphi}^t \eta^t P \) is block-upper triangular, and \( P^t w_{\varphi}^t \eta^t B_{jk} w_{\varphi} P \) is block diagonal in the partition \( n = n_3 + n_4 \):
\[
P^t w_{\varphi}^t \eta^t P = \begin{pmatrix} \tilde{\eta_1} & 0_{n_3 \times n_4} \\ \tilde{\eta_3} & \eta_4 \end{pmatrix}, \tag{2.27}
\]
\[
P^t w_{\varphi}^t \eta^t B_{jk} w_{\varphi} P = \text{diag}(0_{n_3 \times n_3}, \tilde{\eta_4} B_{jk}). \tag{2.28}
\]
Besides, \( \sum_{j=1}^{m} \tilde{\eta_4} B_{jk} \xi_j \xi_j \) is symmetric, positive definite for all \( w \in \mathbb{D} \) and unit vectors \( \xi = (\xi_1, \ldots, \xi_m)^t \in \mathbb{S}^{m-1} \).

Our study in later sections needs some tools from analysis. We summarize them as Lemmas 2.6-2.8 as follows. The lemmas are either special cases of Gagliardo-Nirenberg inequality or its applications (Moser-type calculus inequalities). (Here Lemma 2.6 is a special case of (1.21).) They can be found in many papers dealing with energy estimates in multi space dimensions, e.g. [3]. Here our formulation follows those in [11].

Lemma 2.6. [8] If \( w \in H^s(\mathbb{R}^m) \) with \( s > m/2 \) then
\[
\|w\|_{L_\infty} \leq C \|D_x w\|^\alpha \|w\|^{1-\alpha} \leq C \|w\|_s, \tag{2.29}
\]
where \( \alpha = m/(2s) < 1 \) and \( C > 0 \) is a constant depending only on \( m \) and \( s \).

Lemma 2.7. Let \( g \) be a given smooth function of \( w \) in a neighborhood of \( \bar{w} \). If \( w - \bar{w} \in H^s(\mathbb{R}^m) \) with \( \|w - \bar{w}\|_s \leq \varepsilon \) and \( s > m/2 \), then
\[
\|D_x^l g\| \leq C \|D_x^l w\|, \quad 1 \leq l \leq s, \tag{2.30}
\]
or
\[ \|D_x g\|_{s-1} \leq C \|D_x w\|_{s-1}, \] (2.31)
where \( C > 0 \) is a constant depending only on \( m, s \) and \( \varepsilon \).

**Lemma 2.8.** If \( D_x g, \dot{g} \in H^{-1}(\mathbb{R}^m) \cap L^{\infty}(\mathbb{R}^m) \) then
\[ \|D_x^l (g \dot{g}) - g D_x^l \dot{g}\| \leq C \|D_x g\|_{L^\infty} \|D_x^{l-1} \dot{g}\| + \|D_x^l \dot{g}\| \|\dot{g}\|_{L^{\infty}}, \] (2.32)
where \( C > 0 \) is a constant depending only on \( m \) and \( l \).

### 3. Estimates for linear system.
Motivated by the linear system (2.11) (in the Fourier space) and the expression for \( \bar{R} \) in (2.24), we consider the following linear system with sources:
\[ \hat{A}_0 \hat{\psi} + [i|\xi| \hat{A} \xi] + |\xi|^2 \hat{B} \xi - \hat{L} \dot{\psi} = i \hat{H} \xi + \hat{h}, \] (3.1)
where \( \hat{A}_0, \hat{A}, \hat{B} \) and \( \hat{L} \) are defined by (2.6) and (2.8), \( \hat{\xi} = \xi/|\xi| \in \mathbb{S}^{m-1}, H = H(x, t) \in \mathbb{R}^{n \times m}, h = h(x, t) \in \mathbb{R}^n \), and
\[ h = \begin{pmatrix} 0_{n_1 \times 1} \\ h_2 \end{pmatrix}. \] (3.2)

Similar systems have been studied before. In fact, the case without \( i \hat{H} \xi \) in the source was considered in [3], see Lemma 3.A.1 therein. As evidenced by (2.24), the viscosity \( \hat{B} \) was considered in [4] for hyperbolic balance laws. Here we extend the previous works to our general system of hyperbolic-parabolic balance laws. The first result of this section is the following lemma.

**Lemma 3.1.** Under conditions 1, 2 and 4 of Assumption 1.1, the solution of (3.1) with \( h \) satisfying (3.2) has the decay estimate: For \( \xi \in \mathbb{R}^m, t \geq 0, \)
\[ |\hat{\psi}(\xi, t)|^2 \leq C e^{-2c_1 \varrho(\xi)|t|} |\hat{\psi}(\xi, 0)|^2 \]
\[ + C \int_0^t e^{-2c_1 \varrho(\xi)(t-\tau)} [(1 + |\xi|^2)|\hat{H}(\xi, \tau)|^2 + |\hat{h}_2(\xi, \tau)|^2] d\tau, \] (3.3)
where \( C \) and \( c_1 \) are positive constants, \( \varrho(r) = \frac{r^2}{1 + r^2} \).

and the matrix/vector norms are the Euclidean norms.

**Proof.** Multiply (3.1) by \( \hat{\psi}^* \), the conjugate transpose of \( \hat{\psi} \), and note that \( \hat{A}_0, \hat{A} \hat{\xi}, \hat{B} \hat{\xi} \) and \( \hat{L} \) are real, symmetric by Lemma 2.1. Taking the real part of the result we have
\[ \frac{1}{2} (\hat{\psi}^* \hat{A}_0 \hat{\psi})_t + |\xi|^2 \hat{\psi}^* \hat{B} \hat{\xi} \hat{\psi} - \hat{\psi}^* \hat{L} \hat{\psi} = \Re(i \hat{\psi}^* \hat{H} \xi + \hat{\psi}^* \hat{h}), \] (3.5)
Let \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \), with \( \psi_1 \in \mathbb{R}^{n_1} \) and \( \psi_2 \in \mathbb{R}^{n_2} \). Noting (3.2), the right-hand side of (3.5) can be written as
\[ \Re(i \hat{\psi}^* \hat{H} \xi + \hat{\psi}^* \hat{h}_2). \] (3.6)
Applying (2.25), we have
\[ -\hat{\psi}^* \hat{L} \hat{\psi} = -\hat{\psi}^*_2 (\eta \psi_2 (r \psi_2)^{-1} (\bar{w}) \hat{\psi}_2 \geq c_1 |\hat{\psi}_2|^2 \]
for some constant $c_1 > 0$ since by Lemma 2.1, $-\hat{L}$ is real, symmetric, semi-positive definite, and $\hat{w}_{w_2w_2}(r_2)^{-1}$ is non-singular. Together with (3.5) and (3.6), we have

$$\frac{1}{2}(\tilde{\psi}^* \tilde{A}_0 \tilde{\psi})_t + \frac{1}{2} \tilde{\psi}^* \tilde{L} \tilde{\psi} \leq \Re(\tilde{\psi}^* \tilde{H} \tilde{\psi}) + C|\hat{h}_2|^2$$

for a constant $C > 0$. Using the fact that $\tilde{B}$ is real, symmetric, semi-positive definite (from Lemma 2.1) and by Cauchy-Schwarz inequality, we further have

$$(\tilde{\psi}^* \tilde{A}_0 \tilde{\psi})_t + \frac{1}{2} |\tilde{\psi}|^2 \tilde{B}(\tilde{\xi}) \tilde{\psi} - \tilde{\psi}^* \tilde{L} \tilde{\psi} \leq \frac{\delta |\xi|^2}{1 + |\xi|^2} |\tilde{\psi}|^2 + \frac{1 + |\xi|^2}{\delta} |\hat{H}|^2 + C_1|\hat{h}_2|^2,$$

where $\delta > 0$ is a constant to be chosen, and $C_1 > 0$ is some constant.

Let $\tilde{K}$ be the compensating function defined in Lemma 2.2. We multiply (3.1) by $-i|\tilde{\psi}|^2 K(\tilde{\xi})$ and take the real part. This gives us

$$[-\frac{i}{2}|\xi|^2 K(\tilde{\xi}) \tilde{A}_0 \tilde{\psi}]_t + \frac{1}{2} |\xi|^2 |K(\tilde{\xi}) \tilde{A}(\tilde{\xi}^t \tilde{K}^t(\tilde{\xi}))| \tilde{\psi}$$

$$= \Re[|\xi|^2 \tilde{\psi}|K(\tilde{\xi}) \tilde{B}(\tilde{\xi}) \tilde{\psi} - i|\xi|^2 K(\tilde{\xi}) \tilde{L} \tilde{\psi} + |\xi|^2 K(\tilde{\xi}) \tilde{H} \xi - i|\xi|^2 K(\tilde{\xi}) \tilde{h}].$$

From Lemma 2.1, $\tilde{B}$ and $-\tilde{L}$ are real, symmetric, and semi-positive definite. We may take $\tilde{B}^\frac{1}{2}$ and $(-\tilde{L})^\frac{1}{2}$ that are real, symmetric, and semi-positive definite. Thus the right-hand side of (3.8) is bounded above by

$$|\xi|^2 |K(\tilde{\xi}) \tilde{B}^\frac{1}{2}(\tilde{\xi})||\tilde{B}^\frac{1}{2}(\tilde{\xi}) \tilde{\psi}| + |\xi||\tilde{\psi}||K(\tilde{\xi})(-\tilde{L})^\frac{1}{2}||(-\tilde{L})^\frac{1}{2} \tilde{\psi}|$$

$$+ C|\xi|^2 |\tilde{\psi}||\tilde{H}| + C|\xi||\tilde{\psi}||\hat{h}_2|. (3.9)$$

By Lemma 2.2,

$$S = \frac{1}{2} |K(\tilde{\xi}) \tilde{A}(\tilde{\xi}) + \tilde{A}(\tilde{\xi}) K^t(\tilde{\xi}) + B(\tilde{\xi}) - \tilde{L}$$

is real, symmetric, positive definite. Let $\tilde{c}_2 > 0$ be the minimum value of its smallest eigenvalue on $\mathbb{S}^{n-1}$. We further bound (3.9) above by

$$\frac{\tilde{c}_2}{2} |\xi|^2 |\tilde{\psi}|^2 + C|\xi|^4 |\tilde{B}^\frac{1}{2}(\tilde{\xi}) \tilde{\psi}|^2 + C|(-\tilde{L})^\frac{1}{2} \tilde{\psi}|^2 + C|\xi|^2 |\tilde{H}|^2 + C|\hat{h}_2|^2$$

$$= \frac{\tilde{c}_2}{2} |\xi|^2 |\tilde{\psi}|^2 + C|\xi|^4 |\tilde{B}(\tilde{\xi}) \tilde{\psi} + C\tilde{\psi}^*(-\tilde{L}) \tilde{\psi} + C|\xi|^2 |\tilde{H}|^2 + C|\hat{h}_2|^2.$$

Equation (3.8) thus gives us

$$[-\frac{i}{2} |\xi|^2 K(\tilde{\xi}) \tilde{A}_0 \tilde{\psi}]_t + \frac{1}{2} |\xi|^2 |\tilde{S}(\tilde{\xi})| \tilde{\psi}$$

$$\leq |\xi|^2 \tilde{\psi}^* \tilde{B}(\tilde{\xi}) - \tilde{L} \tilde{\psi} + C|\xi|^4 \tilde{B}(\tilde{\xi}) \tilde{\psi} + C\tilde{\psi}^*(-\tilde{L}) \tilde{\psi} + C|\xi|^2 |\tilde{H}|^2 + C|\hat{h}_2|^2$$

$$\leq C_2(1 + |\xi|^2) |\xi|^2 \tilde{\psi}^* \tilde{B}(\tilde{\xi}) \tilde{\psi} + C_2(1 + |\xi|^2) \tilde{\psi}^*(-\tilde{L}) \tilde{\psi} + C_2|\xi|^2 |\tilde{H}|^2 + C_2|\hat{h}_2|^2,$$

with some constant $C_2 > 0$.

We multiply (3.7) by $(1 + |\xi|^2)$, and (3.10) by $\alpha > 0$ to be determined. Adding the results gives us

$$(1 + |\xi|^2) |\tilde{\psi}|^2 \tilde{A}_0 \tilde{\psi} - \frac{\alpha}{2} \frac{|\xi|}{1 + |\xi|^2} |\tilde{\psi}|^2 K(\tilde{\xi}) \tilde{A}_0 \tilde{\psi}$$

$$+ \frac{\alpha}{2} |\xi|^2 |\tilde{S}(\tilde{\xi})| \tilde{\psi} + (1 + |\xi|^2)|\xi|^2 |\tilde{B}(\tilde{\xi}) \tilde{\psi} - \tilde{\psi}^* \tilde{L} \tilde{\psi}|$$

$$\leq \delta |\xi|^2 |\tilde{\psi}|^2 + \frac{(1 + |\xi|^2)}{\delta} |\tilde{H}|^2 + C_1(1 + |\xi|^2)|\hat{h}_2|^2$$

where $\delta > 0$ is a constant to be chosen, and $C_1 > 0$ is some constant.
We take $\alpha = 4\hat{c}_2/\delta$, i.e., $\alpha = 2\hat{c}_2/\delta$, and let

$$E_\delta = \hat{\psi}^* \hat{A}_0 \hat{\psi} - \alpha i \frac{|\xi|}{1 + |\xi|^2} \hat{\psi}^* K(\hat{\xi}) \hat{A}_0 \hat{\psi}. \tag{3.12}$$

Using the definition of $\tilde{c}_2$ we obtain the following from (3.11):

$$\begin{align*}
(1 + |\xi|^2) (E_\delta t) + \hat{\delta} |\xi|^2 |\hat{\psi}|^2 + (1 + |\xi|^2) \hat{\psi}^* |\xi|^2 \hat{B}(\hat{\xi}) - \hat{L} |\hat{\psi}|
\leq & \left[ \frac{(1 + |\xi|^2)^2}{\delta} + \alpha C_2 |\xi|^2 |\hat{H}|^2 + [C_1 (1 + |\xi|^2) + \alpha C_2 |\hat{h}_2|^2]ight. \\
& \left. + \alpha C_2 (1 + |\xi|^2) \hat{\psi}^* |\xi|^2 \hat{B}(\hat{\xi}) - \hat{L} |\hat{\psi}| \right].
\end{align*} \tag{3.13}$$

Taking $\delta$ small such that $\delta \leq \tilde{c}_2/(4C_2)$ hence $\alpha C_2 \leq 1$, and dividing both sides by $(1 + |\xi|^2)$, (3.13) becomes

$$(E_\delta t) + \frac{|\xi|^2}{1 + |\xi|^2} |\hat{\psi}|^2 \leq \left( \frac{(1 + |\xi|^2)}{\delta} + \frac{|\xi|^2}{1 + |\xi|^2} \right) |\hat{H}|^2 + (C_1 + \frac{1}{1 + |\xi|^2}) |\hat{h}_2|^2. \tag{3.14}$$

Note that

$$\left| i \frac{|\xi|}{1 + |\xi|^2} \hat{\psi}^* K(\hat{\xi}) \hat{A}_0 \hat{\psi} \right| \leq \frac{1}{2} |\hat{\psi}|^2 |K(\hat{\xi})| |\hat{A}_0| \leq C_3 |\hat{\psi}|^2$$

for some constant $C_3 > 0$. From (3.12) we have

$$\lambda_m - \frac{2\delta}{\hat{c}_2} C_3 |\hat{\psi}|^2 \leq E_\delta \leq (\lambda_m + \frac{2\delta}{\hat{c}_2} C_3) |\hat{\psi}|^2, \tag{3.15}$$

where $\lambda_m > 0$ and $\lambda_m > 0$ are the smallest and the largest eigenvalues of $\hat{A}_0$, respectively. We choose $\delta$ small such that $\delta \leq \lambda_m \tilde{c}_2/(4C_3)$, thus $2\delta C_3/\hat{c}_2 \leq \lambda_m/2$. This and (3.15) imply that $E_\delta$ is equivalent to $|\hat{\psi}|^2$. Now we fixed a small $\delta$ so chosen. Equation (3.14) becomes

$$(E_\delta t) + 2c_1 \varrho(|\xi|) E_\delta \leq C_4 (1 + |\xi|^2) |\hat{H}|^2 + C_4 |\hat{h}_2|^2, \tag{3.16}$$

where $c_1 > 0$ and $C_4 > 0$ are constants, and $\varrho$ is defined in (3.4). Solving (3.16) we have

$$E_\delta(\xi, t) \leq e^{-2c_1 \varrho(|\xi|) t} E_\delta(\xi, 0) + C \int_0^t e^{-2c_1 \varrho(|\xi|)(t - \tau)} [(1 + |\xi|^2) |\hat{H}(\xi, \tau)|^2 + |\hat{h}_2(\xi, \tau)|^2] d\tau.$$

Since $E_\delta$ is equivalent to $|\hat{\psi}|^2$, we obtain (3.3). \hfill \Box

Our next result is to apply Plancherel’s theorem to obtain estimates in the physical space.

**Lemma 3.2.** Let $\psi(\cdot, \cdot) \in L^1([0, T]; L^\infty, H \in C([0, T]; L^1(\mathbb{R}^m)), \hat{H} \in C([0, T]; L^1(\mathbb{R}^m))$, $H \in C([0, T]; L^1(\mathbb{R}^m))$, $\hat{H} \in C([0, T]; L^1(\mathbb{R}^m))$, and $\hat{h}_2 \in C([0, T]; L^1(\mathbb{R}^m))$ under conditions 1, 2 and 4 of Assumption 1.1, the solution of (3.1) with $h$ satisfying (3.2) has the following estimate in the physical space: For $0 \leq t \leq T$,

$$\begin{align*}
\|D_x^t \psi(t)\|^2 & \leq C[(t + 1)^{-\frac{m}{2} - 1} \|\psi\|^2_{L^2} (0) + e^{-c_1 t} \|D_x^t \psi(0)\|^2] \\
+ C \int_0^t (t - \tau + 1)^{-\frac{m}{2} - 1} (\|H\|^2_{L^2} + \|\hat{h}_2\|^2_{L^2}) d\tau.
\end{align*}$$
where $C$ and $c_1$ are positive constants.

**Proof.** For a multi index $\alpha$ with $|\alpha| = l$, by Plancherel’s theorem and (3.3) we have

$$\|D_x^\alpha \psi\|^2(t) = |(i\xi)^\alpha \hat{\psi}(\xi)|^2.$$

$$\leq C \int_{\mathbb{R}^m} |\xi|^{2l} e^{-2c_1 \rho(|\xi|) t} |\hat{\psi}(\xi, 0)|^2 \, d\xi$$

$$+ C \int_{0}^{t} \int_{|\xi| \leq 1} e^{-2c_1 \rho(|\xi|)(t-\tau)} |\xi|^{2l} ((1 + |\xi|^2) \hat{H}(\xi, \tau))^2 + |\hat{h}_2(\xi, \tau)|^2) \, d\xi d\tau,$$

where $\rho$ is defined in (3.4).

Noting $\rho(r) \geq r^2/2$ for $0 \leq r \leq 1$ and $\rho(r) \geq 1/2$ for $r \geq 1$, the first term on the right-hand side of (3.18) is bounded by

$$C \int_{|\xi| \leq 1} |\xi|^{2l} e^{-c_1 \rho(|\xi|)^t} |\hat{\psi}(\xi, 0)|^2 \, d\xi + C \int_{|\xi| \geq 1} |\xi|^{2l} e^{-c_1 \rho(|\xi|)^t} |\hat{\psi}(\xi, 0)|^2 \, d\xi$$

$$\leq C \|\hat{\psi}\|_{L^\infty}(0) \int_{|\xi| \leq 1} |\xi|^{2l} e^{-c_1 \rho(|\xi|)^t} \, d\xi + Ce^{-c_1 t} \|\psi\|_{L^\infty}(0)^2$$

$$\leq C(t + 1)^{-\frac{m}{2} - l} \|\psi\|_{L^\infty}(0)^2 + Ce^{-c_1 t} \|D_x^l \psi\|^2(0).$$

This gives the first term on the right-hand side of (3.17).

Similarly, we bound the second term on the right-hand side of (3.18) by

$$C \int_{0}^{t} \int_{|\xi| \leq 1} |\xi|^{2l} e^{-c_1 \rho(|\xi|)^t} ((\hat{H}(\xi, \tau))^2 + |\hat{h}_2(\xi, \tau)|^2) \, d\xi d\tau$$

$$+ C \int_{0}^{t} \int_{|\xi| \geq 1} e^{-c_1 (t-\tau)} |\xi|^{2l} ((1 + |\xi|^2) \hat{H}(\xi, \tau)^2 + |\hat{h}_2(\xi, \tau)|^2) \, d\xi d\tau$$

$$\leq C \int_{0}^{t} \left( \left( \|\hat{H}\|_{L^\infty}^2 + \|\hat{h}_2\|_{L^\infty}^2 \right)(\tau) \int_{|\xi| \leq 1} |\xi|^{2l} e^{-c_1 \rho(|\xi|)^t} \, d\xi d\tau \right.$$

$$+ C \int_{0}^{t} \int_{|\xi| \geq 1} (\|\hat{H}\|_{L^\infty}^2 + \|\hat{h}_2\|_{L^\infty}^2)(\tau) \int_{|\xi| \leq 1} e^{-c_1 \rho(|\xi|)^t} \, d\xi d\tau$$

$$+ C \int_{0}^{t} e^{-c_1 (t-\tau)} (\|D_x^l H\|^2 + \|D_x^{l+1} H\|^2 + \|D_x^l h_2\|^2)(\tau) \, d\tau$$

$$\leq C \int_{0}^{t} (t - \tau + 1)^{-\frac{m}{2} - l} \left( \|H\|_{L^1}^2 + \|h_2\|_{L^1}^2 \right)(\tau) \, d\tau$$

$$+ C \int_{0}^{t} (t - \tau + 1)^{-\frac{m}{2} - l} \left( \|D_x^l H\|_{L^1}^2 + \|D_x^l h_2\|_{L^1}^2 \right)(\tau) \, d\tau$$

$$+ C \int_{0}^{t} e^{-c_1 (t-\tau)} (\|D_x^l H\|^2 + \|D_x^l h_2\|^2)(\tau) \, d\tau.$$

This gives the last three terms on the right-hand side of (3.17).
4. Weighted energy estimates. In this section we use weighted energy estimates to derive decay rates for the nonlinear system. Although the rates are not optimal, they help to obtain the optimal ones in next section. This is possible because these rates are needed in the nonlinear source when performing a priori estimate, and can be enhanced by the other decay factors in the nonlinear terms. Similar techniques have been used for obtaining both pointwise estimates and $L^2$-decay rates, e.g., see [5, 6, 4, 7].

**Theorem 4.1.** Let $\bar{w}$ be a constant equilibrium state of (1.1), and Assumption 1.1 be true. Let $m \geq 2$, $s > \frac{m}{2} + 1$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\|w_0 - \bar{w}\|_s \leq \varepsilon$, the solution of (1.1), (1.11) given in Theorem 1.2 has the following estimates:

\[
\|D_x^l(w - \bar{w})\|_s(t) \leq C\|w_0 - \bar{w}\|_s(t + 1)^{-\frac{1}{2}}, \quad t \geq 0, \quad 0 \leq l \leq s,
\]

\[
\int_0^\infty \sum_{l=0}^{s-1} (t + 1)^l \|D_x^{l+1}w\|_{s-l}^2(t) dt + \int_0^\infty \sum_{l=0}^{s-1} (t + 1)^l \|D_x^{l+1}w\|_{s-l}^2 \int_0^t d\tau \leq C\|w_0 - \bar{w}\|_s^2.
\]

**Proof.** For $t \geq 0$ we define

\[
M^2(t) = \sum_{l=0}^{s} \sup_{0 \leq \tau \leq t} [(\tau + 1)^l \|D_x^l(w - \bar{w})\|_{s-l}^2(\tau)] + \int_0^t \sum_{l=0}^{s-1} (\tau + 1)^l \|D_x^{l+1}w\|_{s-l-1}(\tau) d\tau + \int_0^t \sum_{l=0}^{s-1} (\tau + 1)^l \|D_x^{l+1}\bar{w}\|_{s-l}^2 + \|D_x^l r_2(w)\|_{s-l}^2(\tau) d\tau.
\]

(4.3)

Our goal is to prove

\[
M^2(t) \leq C\|w_0 - \bar{w}\|_s^2
\]

for some constant $C > 0$. Equations (4.1) and (4.2) are then direct consequences of (4.3) and (4.4). Below we assume that $M(t)$ is small.

We apply $D_x^l$ to (1.1) and multiply the result by $(D_x^l w)^{t} \eta''(w)$. This gives us

\[
(D_x^l w)^{t} \eta''(w) D_x^l w_t + (D_x^l w)^{t} \eta''(w) \sum_{j=1}^{m} D_x^{l}[f_j'(w)w_{x_j}] = (D_x^l w)^{t} \eta''(w) \sum_{i,j=1}^{m} D_x^l[B_{ij}(w)w_{x_j}]_x + (D_x^l w)^{t} \eta''(w) D_x^l \tau(w).
\]

We replace the time variable $t$ by $\tau$, multiply the equation by $(\tau + 1)^k$, and integrate the result over $\mathbb{R}^m \times [0, t]$. Then we have, for $1 \leq k \leq l \leq s$,

\[
\frac{1}{2} \int_{\mathbb{R}^m} (\tau + 1)^k [(D_x^l w)^{t} \eta''(w) D_x^l w](x, t) dx = \frac{1}{2} \int_{\mathbb{R}^m} [(D_x^l w)^{t} \eta''(w) D_x^l w](x, 0) dx + \sum_{j=1}^{5} I_j,
\]

(4.5)

where

\[
I_1 = \frac{1}{2} \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k [(D_x^l w)^{t} \eta''(w) D_x^l w](x, \tau) dx d\tau
\]
\[ + \frac{k}{2} \int_0^t \int_{R^n} (\tau + 1)^{k-1} [(D_x w)^t \eta''(w) D_x w] \, dx \, d\tau, \]

\[ I_2 = \sum_{j=1}^m \int_0^t \int_{R^n} (\tau + 1)^k \left\{ \frac{1}{2} (D_x w)^t [\eta''(w)f'_j(w)] \right\} D_x w \, dx \, d\tau, \]

\[ I_3 = - \sum_{j=1}^m \int_0^t \int_{R^n} (\tau + 1)^k \left\{ (D_x w)^t \eta''(w) [D_x (f'_j(w) w_x)] \right\} D_x w \, dx \, d\tau, \]

\[ I_4 = - \sum_{i,j=1}^m \int_0^t \int_{R^n} (\tau + 1)^k \left\{ [(D_x w_{x_i})^t \eta''(w) + (D_x w)^t \eta''(w)_{x_i}] \times D_x [B_{ij}(w) w_x] \right\} D_x w \, dx \, d\tau, \]

\[ I_5 = \int_0^t \int_{R^n} (\tau + 1)^k [(D_x w)^t \eta''(w) D_x r(w)] \, dx \, d\tau = \int_0^t \int_{R^n} (\tau + 1)^k [D_x^t w_{i,j}^t \eta_{w_{2,w_i}} D_x^t r_2(w) + D_x^t w_{i,j}^t \eta_{w_{2,w_i}} D_x^t r_2(w)] \, dx \, d\tau. \] \hspace{1cm} (4.6)

Note that we have used the symmetry of \((\eta'' f'_j)\) in Assumption 1.1 to obtain \(I_3\), and (1.13) to derive the second expression of \(I_5\).

To estimate \(I_1\) we need to treat \(w_1\). Applying condition 3 of Assumption 1.1 and (1.17), we have

\[ D_x^t [B_{ij}(w) w_x] = PD_x^t [P^t B_{ij}(w) w_x P\bar{w}_x], \]

\[ = PD_x^t \left( \begin{array}{c} 0_{n\times 1} \\ B_{ij}^t(w) \bar{w}_{2,x} \end{array} \right). \] \hspace{1cm} (4.7)

From (1.1), (4.7) with \(l = 1\), (2.29), (2.31) and (1.18), we have

\[ \| w_1 \|_{L^\infty} \leq C [\| D_x w \|_{L^\infty} + \sum_{i,j=1}^m \| [B_{ij}(w) w_x] \|_{L^\infty} + \| r(w) \|_{L^\infty}] \]

\[ \leq C [\| D_x w \|_{L^\infty} + \| D_x w \|_{L^\infty} \| D_x \bar{w}_2 \|_{L^\infty} + \sum_{i,j=1}^m \| B_{ij}^t(w) \bar{w}_{2,x} \|_{L^\infty} + \| r_2(w) \|_{L^\infty}] \]

\[ \leq C [\| D_x w \|_{L^\infty} + \| D_x^2 \bar{w}_2 \|_{L^\infty} + \| r_2(w) \|_{L^\infty}]. \] \hspace{1cm} (4.8)

Applying (2.29) with \(w\) replaced by \(D_x w\) and \(s\) by \(s - 1\), we have

\[ \| D_x w \|_{L^\infty} \leq C \| D_x^s w \|^{\alpha} \| D_x w \|^{1-\alpha} \leq C(t + 1)^{-\frac{1}{2} \left( \frac{s}{\alpha} + 1 \right)} M(t), \] \hspace{1cm} (4.9)

where \(\alpha = m/(2s - 2) < 1\), and we have used the definition of \(M(t)\) given in (4.3). Similarly,

\[ \| D_x^2 \bar{w}_2 \|_{L^\infty} \leq C \| D_x^s \bar{w}_2 \|^{\alpha} \| D_x^2 \bar{w}_2 \|^{1-\alpha} \]

\[ = C(t + 1)^{-\frac{1}{2} \left( \frac{s}{\alpha} + 1 \right)} [(t + 1)^s \| D_x^s \bar{w}_2 \|^{\alpha}]^{\frac{1}{\alpha}} \| D_x^2 \bar{w}_2 \|^{\frac{1}{1-\alpha}} \] \hspace{1cm} (4.10)

Also, applying (2.29) with \(w\) replaced by \(w - \bar{w}\) and noting \(\alpha = m/(2s) < 1\), we have

\[ \| w - \bar{w} \|_{L^\infty} \leq C \| D_x^s (w - \bar{w}) \|^{\alpha} \| w - \bar{w} \|^{1-\alpha} \leq C(t + 1)^{-\frac{m}{2}} M(t). \] \hspace{1cm} (4.11)
To estimate $\|r_2(w)\|_{L^\infty}$ we consider the equation satisfied by $r_2(w)$, which is $\dot{\psi}_2(w)$ by (2.3). Taking the second part of (2.4), we have
\[
\dot{\psi}_2(t) - (r_2)_{w_2}(\tilde{w})\psi_2 = R,
\]
where by applying (2.2),
\[
R = \left((r_2)_{w_1} - (r_2)_{w_2}\right)\left(-\sum_{j=1}^m f_j(w)x_j + \sum_{j,k=1}^m [B_{jk}(w)w_{x_k}]x_j\right) + [(r_2)_{w_2}(w) - (r_2)_{w_2}(\tilde{w})]r_2(w).
\]
Writing $\bar{(r_2)_{w_2}}$ for $(r_2)_{w_2}(\tilde{w})$ and solving the linear system (4.12), we have
\[
\dot{\psi}_2(x, t) \equiv e^{(r_2)_{w_2} \psi_2(x, 0)} + \int_0^t e^{(t-\tau)(r_2)_{w_2}} R(x, \tau) \, d\tau.
\]

From Lemma 2.1 and (2.25), $(\eta_{w_2}^{-1}(r_2)_{w_2}^{-1})(\tilde{w})$ is real, symmetric and semi-negative definite. In fact, it is negative definite by Assumption 1.1. This implies $[\eta_{w_2}^{-1}(r_2)_{w_2}^{-1}\eta_{w_2}^{-1}](\tilde{w})$ is real, symmetric and negative definite, hence the eigenvalues of $\bar{(r_2)_{w_2}}$ are all negative. Let $-c_2$ with $c_2 > 0$ be the largest eigenvalue of $\bar{(r_2)_{w_2}}$. By triangle inequality,
\[
\|r_2(w)\|_{L^\infty} = \|\dot{\psi}_2(\cdot, t)\|_{L^\infty} 
\leq Ce^{-c_2t} \|r_2(w_0)\|_{L^\infty} + C \int_0^t e^{-c_2(t-\tau)} \|R(\cdot, \tau)\|_{L^\infty} \, d\tau.
\]

To estimate $\|R\|_{L^\infty}$ we apply triangle inequality to the right-hand side of (4.13), and compare the result with the right-hand side of (4.8). This gives us
\[
\|R\|_{L^\infty} \leq C(\|D_x w\|_{L^\infty} + \|D_x^2 \tilde{w}\|_{L^\infty} + \|w - \tilde{w}\|_{L^\infty} \|r_2(w)\|_{L^\infty}).
\]

Substituting (4.9)-(4.11) into (4.16) we have
\[
\|r_2(w)\|_{L^\infty} \leq \int_0^t e^{-c_2(t-\tau)} \|R(\cdot, \tau)\|_{L^\infty} \, d\tau
\leq CM(t)(t + 1)^{-\frac{1}{2}(\frac{3}{2} + 1)} + C \int_0^t e^{-c_2(t-\tau)}(\tau + 1)^{-\frac{1}{2}(\frac{3}{2} + 1)} \times \left[|\tau + 1|^\alpha \|D_x^{\alpha + 1} \tilde{w}\|_{L^2(\tau)}^2 \|D_x^2 \tilde{w}\|_{L^2(\tau)}^2 \right] \, d\tau
\leq CM(t)(t + 1)^{-\frac{3}{2}} \|r_2(w)\|_{L^\infty} \|r_2(w)\|_{L^\infty} (\tau) \, d\tau.
\]

With Cauchy-Schwarz inequality, the second term on the right-hand side is further bounded by
\[
C \left[ \int_0^t e^{-c_2(t-\tau)}(\tau + 1)^{-\frac{1}{2}(\frac{3}{2} + 1)} \, d\tau \right]^\frac{1}{2}
\times \left\{ \int_0^t [(\tau + 1)^\alpha \|D_x^{\alpha + 1} \tilde{w}\|_{L^2(\tau)}^2]^{\beta}(\tau + 1)|D_x^2 \tilde{w}|_{L^2(\tau)}^2 \right\} \frac{1}{2},
\]
which is absorbed into the first term on the right-hand side of (4.17), noting (4.3) and applying Hölder’s inequality. Substituting the result into (4.15) we have
\[
\|r_2(w)\|_{L^\infty} \leq Ce^{-c_2t} \|r_2(w_0)\|_{L^\infty} + CM(t)(t + 1)^{-\frac{1}{2}(\frac{3}{2} + 1)}
\]
This implies

$$\left[ 1 - C M(t) \right] \sup_{0 \leq \tau \leq t} \left[ (\tau + 1)^{\frac{3}{2}(\frac{m}{4} + 1)} \| r_2(\omega) \|_{L^\infty}(\tau) \right] \leq C \| r_2(\omega_0) \|_{L^\infty} + M(t).$$

We have assumed $M(t)$ being small. Thus

$$\sup_{0 \leq \tau \leq t} \left[ (\tau + 1)^{\frac{3}{2}(\frac{m}{4} + 1)} \| r_2(\omega) \|_{L^\infty}(\tau) \right] \leq C \| r_2(\omega_0) \|_{L^\infty} + M(t).$$

Applying (2.29) we have

$$\| r_2(\omega) \|_{L^\infty}(t) \leq C \| w_0 - \bar{w} \|_{s-1} + M(t)(t + 1)^{-\frac{3}{2}(\frac{m}{4} + 1)}. \quad (4.18)$$

Now we substitute (4.9), (4.10) and (4.18) into (4.8) to obtain

$$\| w_i \|_{L^\infty}(t) \leq C \| w_0 - \bar{w} \|_{s-1} + M(t)(t + 1)^{-\frac{3}{2}(\frac{m}{4} + 1)} + C(t + 1)^{-\frac{3}{2}(\frac{m}{4} + 1)}$$

$$\times \left[ (t + 1)^{\alpha} \| D_x^{s+1} \hat{w}_2 \|^2(t) \right]^{\frac{3}{2}} \left[ (t + 1) \| D_x^2 \hat{w}_2 \|^2(t) \right]^{\frac{1}{2}}, \quad (4.19)$$

where $\alpha = m/(2s-2) < 1$. Applying (4.19), (4.3) and Cauchy-Schwarz and Hölder’s inequalities, and noting $m \geq 2$ and $1 \leq k \leq l \leq s$, (4.6) gives rise to

$$|I_1| \leq C \int_0^t \| w_i \|_{L^\infty}(\tau)(\tau + 1)^k \| D_x^2 w \|^2(\tau) d\tau + C \int_0^t (\tau + 1)^{k-1} \| D_x^2 w \|^2(\tau) d\tau$$

$$\leq C \| w_0 - \bar{w} \|_{s-1} + M(t)M^2(t) + C \int_0^t [(\tau + 1)^{-\frac{3}{2}(\frac{m}{4} + 1)} + 2^k \| D_x^2 w \|^4(\tau) d\tau]^{\frac{3}{2}}$$

$$\times \left\{ \int_0^t [(\tau + 1)^k \| D_x^{s+1} \hat{w}_2 \|^2(\tau)] \alpha [(\tau + 1) \| D_x^2 \hat{w}_2 \|^2(\tau)]^{1-\alpha} d\tau \right\}^{\frac{3}{2}}$$

$$+ C \int_0^t (\tau + 1)^{k-1} \| D_x^2 w \|^2(\tau) d\tau$$

$$\leq C \| w_0 - \bar{w} \|_{s-1} + M(t)M^2(t) + C \int_0^t (\tau + 1)^{k-1} \| D_x^2 w \|^2(\tau) d\tau,$$

(4.20)

where $\alpha$ is the same as in (4.19).

Similarly, from (4.3), (4.6) and (4.9),

$$|I_2| \leq C \int_0^t (\tau + 1)^k \| D_x w \|_{L^\infty}(\tau) \| D_x^2 w \|^2(\tau) d\tau \leq CM^3(t). \quad (4.21)$$

From (4.3), (4.6), (2.32), (2.30) and (4.9),

$$|I_3| \leq C \sum_{j=1}^m \int_0^t (\tau + 1)^k \| D_x^j w \|(\tau) \| D_x^l f_j(w)w_x \| - f_j(w)D_x^l w_x \| d\tau$$

$$\leq C \sum_{j=1}^m \int_0^t (\tau + 1)^k \| D_x^j w \|(\tau) \| D_x f_j(w)w_x \| \| D_x^l w \|$$

$$+ \| D_x^l f_j(w) \| \| D_x w \|_{L^\infty}(\tau) d\tau$$

$$\leq C \int_0^t (\tau + 1)^k \| D_x^2 w \|^2 \| D_x w \|_{L^\infty} \| D_x w \| d\tau \leq CM^3(t).$$

(4.22)
To estimate $I_4$ we consider

$$
(D_x^l w_{x_i})^t \eta''(w) = [D_x^l (w\varphi \varphi_{x_i}) - w_D D_x^l \varphi_{x_i}]^t \eta''(w) + D_x^l \varphi_{x_i}^t w^t \eta''(w),
$$

where $\varphi$ is defined in condition 3 of Assumption 1.1. Using $\tilde{w}$ defined in (1.17), the right-hand side of (4.23) can be written as

$$
(D_x^l \tilde{w}_{x_i})^t [D^t w^t \eta''(w)] + [D_x^l (w\varphi \varphi_{x_i}) - w_D D_x^l \varphi_{x_i}]^t \eta''(w).
$$

(4.24)

Applying (1.14) and (2.27), we have from (4.23) and (4.24) the following:

$$
(D_x^l w_{x_i})^t \eta''(w) (D_x^l [B_{ij}(w) w_{x_j}] = (D_x^l \tilde{w}_{2x_i})^t \tilde{\eta}_4 D_x^l [B_{ij}^*(w) \tilde{w}_{2x_j}]
$$

$$
+ [D_x^l (w\varphi \varphi_{x_i}) - w_D D_x^l \varphi_{x_i}]^t \eta''(w) D_x^l [B_{ij}(w) w_{x_j}].
$$

(4.25)

Substituting (4.25) into (4.6), we have

$$
I_4 = -\int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k \sum_{i,j=1}^m [(D_x^l \tilde{w}_{2x_i})^t (\tilde{\eta}_4 B_{ij}^*)(\tilde{w}) D_x^l \tilde{w}_{2x_j}] (x, \tau) \, dx \, d\tau
$$

$$
+ O(1) \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^{k+1} \sum_{i,j=1}^m \left| |D_x^l \tilde{w}_{2x_i}|^2 \right| (x, \tau) \, dx \, d\tau
$$

$$
+ O(1) \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k \sum_{i,j=1}^m \left| |D_x^l (w\varphi \varphi_{x_i}) - w_D D_x^l \varphi_{x_i}|^2 \right| (x, \tau) \, dx \, d\tau.
$$

(4.26)

Note that Fourier transform preserves inner products and that $\sum_{i,j=1}^m (\tilde{\eta}_4 B_{ij}^*)(\tilde{w})$ $\tilde{\xi}_j \tilde{\xi}_i$ is symmetric, positive definite by Lemma 2.5, where $\tilde{\xi}_i = \xi_i / |\xi|$, and $\xi = (\xi_1, \ldots, \xi_m)^t$ stands for Fourier variables. We conclude that there is a constant $c_3 > 0$ such that

$$
- \int_{\mathbb{R}^m} \sum_{i,j=1}^m [(D_x^l \tilde{w}_{2x_i})^t (\tilde{\eta}_4 B_{ij}^*)(\tilde{w}) D_x^l \tilde{w}_{2x_j}] (x, \tau) \, dx \leq -c_3 \|D_x^{l+1} \tilde{w}_2\|^2(\tau).
$$

(4.27)

Substituting (4.27) into (4.26) gives us

$$
I_4 \leq -c_3 \int_0^t (\tau + 1)^k \|D_x^{l+1} \tilde{w}_2\|^2(\tau) \, d\tau
$$

$$
+ C \int_0^t (\tau + 1)^k \|w - \tilde{w}\|_{L^\infty} \|D_x^{l+1} \tilde{w}_2\|^2(\tau) \, d\tau
$$

$$
+ C \int_0^t (\tau + 1)^k \|D_x^{l+1} \tilde{w}_2\| \sum_{i,j=1}^m \left| |D_x^l [B_{ij}^*(w) \tilde{w}_{2x_j}] - B_{ij}^*(w) D_x^l \tilde{w}_{2x_j}|^2 \right| (\tau) \, d\tau
$$

$$
+ C \int_0^t (\tau + 1)^k \sum_{i,j=1}^m \left| |D_x^l (w\varphi \varphi_{x_i}) - w_D D_x^l \varphi_{x_i}|^2 \right| + \|D_x^l w\| \|D_x w\|_{L^\infty} \right| (\tau) \|D_x^l [B_{ij}(w) w_{x_j}]\| (\tau) \, d\tau.
$$

(4.28)
From (4.7) we have
\[
\|D^l_x[B_{ij}(w)w_{x_j}]\| = \|D^l_x[B^*_x(w)\hat{w}_{x_j}]\|
\leq \|D^l_x[B^*_x(w)\hat{w}_{x_j}] - B^*_x(w)D^l_x\hat{w}_{x_j}\| + \|B^*_x(w)D^l_x\hat{w}_{x_j}\|.
\] (4.29)
Substituting (4.29) into the last integral in (4.28), applying (2.29), (4.3), (2.32), (2.30) and (4.9), and noting $1 \leq k \leq l \leq s$, we arrive at
\[
I_4 \leq -c_3 \int_0^t (\tau + 1)^k \|D^{l+1}_x\hat{w}_2\|^2(\tau) \, d\tau + CM^3(t),
\] (4.30)
where $c_3 > 0$ is a constant.

To estimate $I_5$ in (4.6) we need the key estimate (2.26), which implies
\[
\eta_{w_1w_2} D^l_x[w_1] + \eta_{w_2w_2} D^l_x[w_2] = \eta_{w_2w_2} [(r_2)_w^{-1}(r_2)_w] D^l_x[w_1] + (1)|r_2(w)| \|D^l_x[w_1].
\] (4.31)
Noting $l \geq 1$, we write $D^l_x = D^{l-1}_x D_{x_j}$ for some $1 \leq j \leq m$. Thus
\[
D^l_x[w_2] = D^{l-1}_x[w_{2x_j}] = D^{l-1}_x[(w_2)_x^j] = \eta_{w_2w_2} [(r_2)_w^{-1}(r_2)_w] w_{1x_j} + (r_2)_w^{-1}(w_2)_x^j,
\] (4.32)
where we have applied (2.1) and (2.2). Substituting (4.32) into the right-hand side of (4.31), we simplify it as
\[
\eta_{w_2w_2} [(r_2)_w^{-1}(r_2)_w] D^{l-1}_x[w_{1x_j}] - D^{l-1}_x[(r_2)_w^{-1}(r_2)_w] D^{l-1}_x[w_{1x_j}] + O(1)|r_2(w)| \|D^l_x[w_1].
\] (4.33)
We replace the right-hand side of (4.31) by (4.33) and substitute the result into $I_5$ in (4.6). Noting the leading term in (4.33) is $\eta_{w_2w_2} D^{l-1}_x[(r_2)_w^{-1}r_2(w)_x^j]$, which can be further linearized, we have
\[
I_5 = \int_0^t (\tau + 1)^k \int_\mathbb{R} D^l_x r_2(w)[\eta_{w_2w_2} D^l_x[w_1] + \eta_{w_2w_2} D^l_x[w_2](x, \tau) \, dx \, d\tau
\leq \int_0^t (\tau + 1)^k \int_\mathbb{R} \{D^l_x r_2(w)[\eta_{w_2w_2} (r_2)_w^{-1}](\hat{w}) D^l_x r_2(w)\}(x, \tau) \, dx \, d\tau
+ C \int_0^t (\tau + 1)^k \|D^l_x r_2(w)\| \|w - \hat{w}\|_{L^\infty} \|D^l_x r_2(w)\|
+ \|D^{l-1}_x[(r_2)_w^{-1}r_2(w)_x^j] - (r_2)_w^{-1}D^l_x r_2(w)\|
+ \|[(r_2)_w^{-1}(r_2)_w] D^{l-1}_x[w_{1x_j}] - D^{l-1}_x[(r_2)_w^{-1}(r_2)_w] w_{1x_j}\|
+ \|w_2(w)\|_{L^\infty} \|D^l_x[w_1]\|(\tau) \, d\tau.
\] (4.34)
Here we note that for $l = 1$ the terms involving $D^l_x$ on the right-hand side of (4.34) disappear.

As discussed above in deriving (4.15), we have concluded that $[\eta_{w_2w_2} (r_2)_w^{-1}](\hat{w})$ is real, symmetric, and negative definite. Let its largest eigenvalue be $-c_4$ with $c_4 > 0$. We apply this fact to the right-hand side of (4.34). Together with (2.32), (2.30), (4.11), (4.3), (4.9) and (4.18), noting $m \geq 2$, we have for $1 \leq k \leq l \leq s$,
\[
I_5 \leq -c_4 \int_0^t (\tau + 1)^k \|D^l_x r_2(w)\|^2(\tau) \, d\tau + C \int_0^t (\tau + 1)^k \|D^l_x r_2(w)\|
\times \{|w - \hat{w}|_{L^\infty} \|D^l_x r_2(w)\| + \|D_x w\|_{L^\infty} \|D^{l-1}_x[w]\| + \|r_2(w)\|_{L^\infty} \|D^l_x[w]\|\}(\tau) \, d\tau
\]
\[ L^p \text{ decay for hyperbolic-parabolic balance laws} \]

\[ \leq - c_4 \int_0^t (\tau + 1)^k \| D_x^2 r_2(w) \|^2(\tau) d\tau + CM^3(t) + \| w_0 - \bar{w} \|_{s-1} M^2(t). \quad (4.35) \]

Combining (4.5), (4.20)-(4.22), (4.30) and (4.35) and noting \( \eta'' \) is symmetric, positive definite, we arrive at

\[ (t + 1)^k \| D_x^2 w \|^2(t) + \int_0^t (\tau + 1)^k \| D_x^{l+1} \bar{w}_2 \|^2 + \| D_x^l r_2(w) \|^2(\tau) \, d\tau \]

\[ \leq C \| D_x^2 w_0 \|^2 + CM^3(t) + \| w_0 - \bar{w} \|_{s-1} M^2(t) + C \int_0^t (\tau + 1)^{k-1} \| D_x^l w \|^2(\tau) \, d\tau \]

(4.36)

for \( 1 \leq k \leq l \leq s \), where \( C > 0 \) is a constant.

Equation (2.5) is for the unknown \( \hat{\psi} \) defined in (2.3), and is equivalent to (1.1). Taking the Fourier transform with respect to the space variables we have (2.11). Comparing (2.11) with the linear system (3.1) we see the obvious similarity. Therefore, following the same argument in Section 3 we have the counterpart of (3.8) for (2.11):

\[ \left[ -\frac{i}{2} |\xi|^2 \hat{\psi}^* K(\xi) \hat{A}_0 \hat{\psi} \right]_t + \frac{1}{2} |\xi|^2 \hat{\psi}^* [K(\xi) \hat{A}(\xi) + \hat{A}'(\xi) K^t(\xi)] \hat{\psi} \]

\[ = \Re \{ i |\xi|^3 \hat{\psi}^* K(\xi) \hat{B}(\xi) \hat{\psi} - i |\xi|^3 \hat{\psi}^* K(\xi) \hat{L} \hat{\psi} - i |\xi|^3 \hat{\psi}^* K(\xi) \hat{R} \}, \quad (4.37) \]

where \( \hat{\xi} = \xi / |\xi| \). Here we note that all the terms in (4.37) are real since \( K(\xi) \hat{A}_0 \) is real, skew symmetric by Lemma 2.2. With (2.7), the right-hand side of (4.37) is simplified to

\[ \Re \{ -i |\xi|^3 \hat{\psi}^* K(\xi)(\hat{R}_1 + \hat{R}_4 + \hat{R}_5) \}, \]

where

\[ R_1 = \hat{A}_0 \sum_{j=1}^m (\psi_{w, f_j w_\psi})(\hat{w}) - (\psi_{w, f_j w_\psi})(\hat{w}), \]

\[ R_4 = \hat{A}_0 \sum_{j,k=1}^m \psi_{w, w_{x_k}} (\hat{w}), \quad R_5 = \hat{A}_0 (\psi_{w, r}) \]

(4.38)

Using the notation \( S \) defined in (2.9) we write (4.37) as

\[ |\xi|^2 \hat{\psi}^* S(\hat{\psi}) \]

\[ = \left| \frac{i}{2} |\xi|^2 \hat{\psi}^* K(\xi) \hat{A}_0 \hat{\psi} \right|_t + |\xi|^2 \hat{\psi}^* [\hat{B}(\xi) - \hat{L}] \hat{\psi} \]

\[ + \Re \{ -i |\xi|^3 \hat{\psi}^* K(\xi)(\hat{R}_1 + \hat{R}_4 + \hat{R}_5) \}, \quad (4.39) \]

with \( R_1, R_4 \) and \( R_5 \) defined in (4.38).

We change \( t \) to \( \tau \) in (4.39), multiply the equation by \( (\tau + 1)^k |\xi|^{2l} \) for \( 1 \leq k \leq l \leq s - 1 \), and integrate the result over \( \mathbb{R}^m \times [0, t] \). From Lemma 2.2, \( S \) is real, symmetric, and positive definite. Thus we have, for some constant \( c_5 > 0 \),

\[ c_5 \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k |\xi|^{2l+2} |\hat{\psi}|^2(\xi, \tau) \, d\xi d\tau \leq \sum_{j=6}^{9} I_j, \quad (4.40) \]

with

\[ I_6 = \frac{i}{2} \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k |\xi|^{2l+1} |\hat{\psi}^* K(\xi) \hat{A}_0 \hat{\psi}|_l(\xi, \tau) \, d\xi d\tau, \]
Applying (2.28) and (1.17) to the first term on the right-hand side and substituting

\[ I_7 = \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k [\xi]^{2l+2} |\hat{\psi} \hat{\psi} \hat{\psi} \hat{B}(\xi) \hat{\psi}] (\xi, \tau) \, d\xi \, d\tau, \]
\[ I_8 = -\int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k [\xi]^{2l+2} (\hat{\psi} \hat{L} \hat{\psi})(\xi, \tau) \, d\xi \, d\tau, \]
\[ I_9 = -\int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k \Re \{i[\xi]^{2l+1} \hat{\psi} * K(\xi)(\hat{R}_1 + \hat{R}_4 + \hat{R}_5)\} (\xi, \tau) \, d\xi \, d\tau. \]

(4.41)

Noting \( I_6 \) is real and by direct calculation, we have

\[ I_6 = \frac{i}{2} \int_{\mathbb{R}^m} (t + 1)^k [\xi]^{2l+1} |\hat{\psi} * K(\xi) \hat{A}_0 \hat{\psi}| (\xi, t) \, d\xi - \frac{i}{2} \int_{\mathbb{R}^m} [\xi]^{2l+1} |\hat{\psi} * K(\xi) \hat{A}_0 \hat{\psi}| (\xi, 0) \, d\xi \]
\[ - \frac{ki}{2} \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^{k-1} [\xi]^{2l+1} |\hat{\psi} \hat{K}(\xi) \hat{A}_0 \hat{\psi}| (\xi, \tau) \, d\xi \, d\tau \]
\[ \leq C(t + 1)^k (||D^{l+1}_x \hat{\psi}||^2 + ||D^{l+1}_x \hat{\psi}||^2)(t) + C(||D^{l+1}_x \hat{\psi}||^2 + ||D^{l+1}_x \hat{\psi}||^2)(0) \]
\[ + \frac{c_5}{4} \int_0^t \int_{\mathbb{R}^m} (\tau + 1)^k [\xi]^{2l+2} |\hat{\psi}|^2(\xi, \tau) \, d\xi \, d\tau + C \int_0^t (\tau + 1)^{k-2} ||D^l_x \hat{\psi}||^2(\tau) \, d\tau, \]

(4.42)

where in the last step we have applied Plancherel theorem.

To estimate \( I_7 \) we note that \( \phi \) is the diffeomorphism for viscosities, and we need to convert \( \psi \) via the mappings \( \psi \to w \to \phi \) as follows. From (2.8), (2.6) and Parseval’s identity, noting \( \xi = \xi / ||\xi|| \) we have

\[ I_7 = \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} [\xi]^{2l} \sum_{i,j=1}^m \xi_i \xi_j \hat{\psi}^* (\xi, \tau)(w^{l''}_i \eta'' B_{ij} w^l \hat{\psi})(\bar{\bar{w}}) \hat{\psi}(\xi, \tau) \, d\xi \, d\tau \]
\[ = \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} \sum_{i,j=1}^m (D^{l}_x \hat{\psi}_x)^i (x, \tau)(w^{l''}_i \eta'' B_{ij} w^l \hat{\psi})(\bar{\bar{w}})(D^{l}_x \hat{\psi}_x)(x, \tau) \, dx \, dt. \]

(4.43)

Consider a typical term in the summation on the right-hand side. With (2.3) and by direct calculation, we have

\[ (D^l_x \hat{\psi}_x)^i (w^{l''}_i \eta'' B_{ij} w^l \hat{\psi})(\bar{\bar{w}})(D^l_x \hat{\psi}_x), \]
\[ = (D^l_x \hat{\psi}_x)^i (w^{l''}_i \eta'' B_{ij} w^l \hat{\psi})(\bar{\bar{w}})(D^l_x \hat{\psi}_x) + O(1)||D^l_x \hat{\psi}_x|| ||w \bar{\bar{w}}|| ||D^l_x \hat{\psi}_x|| \]
\[ = D^l_x (\psi \phi \phi x)^i w^{l''}_i \eta'' B_{ij} w^l \hat{\psi} - \psi \phi \phi D^l_x (\psi \phi \phi x) + O(1)||D^l_x \hat{\psi}_x|| ||w \bar{\bar{w}}|| ||D^l_x \hat{\psi}_x|| \]
\[ + O(1)||D^l_x \hat{\psi}_x|| ||D^l_x (\psi \phi \phi x)|| ||w \bar{\bar{w}}|| ||D^l_x \hat{\psi}_x||. \]

Applying (2.28) and (1.17) to the first term on the right-hand side and substituting the result into (4.43), we have

\[ I_7 \leq \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} \sum_{i,j=1}^m (D^l_x \hat{\psi}_x)^i (x, \tau) \, dx \, dt \]
\[ + C \int_0^t (\tau + 1)^k \sum_{i,j=1}^m ||D^l_x (\psi \phi \phi x)|| ||\psi \phi \phi D^l_x (\psi \phi \phi x)|| ||(\tau)|| ||D^l_x \hat{\psi}_x|| ||(\tau)|| \, d\tau \]
\[ + C \int_0^t (\tau + 1)^k \sum_{i,j=1}^m ||D^l_x \hat{\psi}_x|| ||(\tau)|| ||D^l_x (\psi \phi \phi x)|| ||\psi \phi \phi D^l_x (\psi \phi \phi x)|| ||(\tau)|| \, d\tau \]
Using (2.32), (2.30) and (4.41), and noting \(1 \leq k \leq l \leq s - 1\), similar to (4.22) we have

\[
I_7 \leq C \int_0^t (\tau + 1)^k \| D_x^{l+1} \tilde{w}_2 \|^2(\tau) \| w - \tilde{w} \|_{L^\infty}(\tau) \, d\tau.
\]

For \(I_8\) we apply (2.3) and (2.25) to have

\[
I_8 = -\int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} \left( D_x^{l+1} \tilde{\psi}_j LD_x^{l+1} \tilde{\psi}\right)(x, \tau) \, dx \, d\tau
\]

\[
= -\int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} \{ D_x^{l+1} \tilde{\psi}_2 [\eta_{w_2 w_2}(r_2)_{w_2}^{-1}](\tilde{w}) D_x^{l+1} \tilde{\psi}_2\}(x, \tau) \, dx \, d\tau
\]

\[
\leq C \int_0^t (\tau + 1)^k \| D_x^{l+1} r_2(w) \|^2(\tau) \, d\tau.
\]

To estimate \(I_9\), from (4.38), (2.32) and (2.30) we note that

\[
\| D_x^l R_1 \| \leq C \sum_{j=1}^m \| D_x^l (\psi_{w_j} f_j w_{\psi} \psi_{x_j}) - \psi_{w_j} f_j w_{\psi} D_x^l \psi_{x_j} \| + \| w - \tilde{w} \|_{L^\infty} \| D_x^l \psi_{x_j} \|
\]

\[
\leq C(\| D_x w \|_{L^\infty} \| D_x^l w \| + \| w - \tilde{w} \|_{L^\infty} \| D_x^{l+1} w \|).
\]

Similarly, with (4.7) and (4.29) we also have

\[
\| D_x^l R_4 \| \leq C \sum_{i,j=1}^m \{ \| D_x^l [\psi_{w_i} B_{ij} w_{x_j}] \| - \psi_{w_i} D_x^l (B_{ij} w_{x_j}) \| + \| D_x^l (B_{ij} w_{x_j}) \| \}
\]

\[
\leq C(\| D_x w \|_{L^\infty} (\| D_x w \|_{L^\infty} \| D_x^l w \| + \| D_x^{l+1} \tilde{w}_2 \|) + \| D_x^l w \| \| D_x^l \tilde{w}_2 \|_{L^\infty}
\]

\[
\leq C(\| D_x w \|_{L^\infty} (\| D_x w \|_{L^\infty} \| D_x^l w \| + \| D_x^{l+1} w \|)
\]

\[
\| D_x^l R_5 \| \leq C(\| D_x w \|_{L^\infty} \| D_x^{l-1} r_2(w) \| + \| D_x^l w \| \| r_2(w) \|_{L^\infty} + \| D_x^l r_2(w) \|).
\]

For \(I_9\) in (4.41) we have

\[
I_9 \leq \frac{C_4}{4} \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} \xi \delta^{2l+2} |\tilde{\psi}|^2(\xi, \tau) \, d\xi \, d\tau
\]

\[
+ C \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} |\xi|^{2l} (|\tilde{\psi}_1|^2 + |\tilde{\psi}_4|^2 + |\tilde{\psi}_5|^2)(\xi, \tau) \, d\xi \, d\tau.
\]

The second integral on the right-hand side can be written as

\[
C \int_0^t (\tau + 1)^k (\| D_x^l R_1 \|^2 + \| D_x^l R_4 \|^2 + \| D_x^l R_5 \|^2) \, d\tau \leq C \int_0^t (\tau + 1)^k \| | w - \tilde{w} \|_{L^\infty} \| D_x^{l+1} w \|^2 + \| D_x w \|_{L^\infty} \| D_x^l w \|^2 + \| D_x^{l+1} w \|^2
\]

\[
+ \| D_x^{l-1} r_2(w) \|^2 + \| D_x^l \tilde{w}_2 \|_{L^\infty} \| D_x^l w \|^2 + \| D_x^{l+2} \tilde{w}_2 \|^2
\]

\[
+ \| r_2(w) \|_{L^\infty} \| D_x^l w \|^2 + \| D_x^l r_2(w) \|^2) \, d\tau,
\]
applying (4.46)-(4.48). By (4.3), (4.9)-(4.11) and (4.18), noting

\[ I_0 \leq \frac{c_5}{4} \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} |\xi|^{2l+2} |\tilde{\psi}|^2(\xi, \tau) \, d\xi \, d\tau \\
+ C \int_0^t (\tau + 1)^k (\|D_{x}^{l+2} \tilde{w}_2\|^2 + \|D_{x}^l r_2(w)\|^2)(\tau) \, d\tau \\
+ C M^4(t) + C \|w_0 - \tilde{w}\|^2_{s-1} M^2(t). \]  

(4.49)

Combining (4.40), (4.42), (4.44), (4.45) and (4.49), and with (2.30) and (2.1)-(2.3), we arrive at

\[ \int_0^t (\tau + 1)^k \|D_{x}^{l+1} w\|^2(\tau) \, d\tau \leq C \int_0^t (\tau + 1)^k \|D_{x}^{l+1} \tilde{\psi}\|^2(\tau) \, d\tau \]

\[ \leq C \int_0^t (\tau + 1)^k \int_{\mathbb{R}^m} |\xi|^{2l+2} |\tilde{\psi}|^2(\xi, \tau) \, d\xi \, d\tau \]

\[ \leq C \|w_0 - \tilde{w}\|^2 + M^3(t) + (t + 1)^k \|D_{x}^l w\|^2(t) \]

\[ + \int_0^t (\tau + 1)^k (\|D_{x}^{l+1} \tilde{w}_2\|^2 + \|D_{x}^l r_2(w)\|^2(\tau) \, d\tau \\
+ \int_0^t (\tau + 1)^{k-2} \|D_{x}^l w\|^2(\tau) \, d\tau \]

for \( 1 \leq k \leq l \leq s - 1 \).

Recalling from Theorem 1.2 we have

\[ \sup_{0 \leq \tau \leq t} \|w - \tilde{w}\|^2(\tau) + \int_0^t \left[ \|D_{x} w\|^2_{s-1} + \|D_{x} \tilde{w}_2\|^2 + \|r_2(w)\|^2_{s} \right] (\tau) \, d\tau \leq C \|w_0 - \tilde{w}\|^2. \]  

(4.51)

Summing up (4.36) for \( k \leq l \leq s \), we have for \( 1 \leq k \leq s \),

\[ \sup_{0 \leq \tau \leq t} \left[ (\tau + 1)^k \|D_{x}^k w\|^2_{s-k}(\tau) \right] + \int_0^t (\tau + 1)^k (\|D_{x}^{k+1} \tilde{w}_2\|^2_{s-k} + \|D_{x}^k r_2(w)\|^2_{s-k})(\tau) \, d\tau \]

\[ \leq C \|w_0 - \tilde{w}\|^2 + M^3(t) + \int_0^t (\tau + 1)^{k-1} \|D_{x}^k w\|^2_{s-k}(\tau) \, d\tau. \]  

(4.52)

Summing up (4.50) for \( k \leq l \leq s - 1 \), we have for \( 1 \leq k \leq s - 1 \,

\[ \int_0^t (\tau + 1)^k \|D_{x}^{k+1} w\|^2_{s-k-1}(\tau) \, d\tau \]

\[ \leq C \|w_0 - \tilde{w}\|^2 + M^3(t) + (t + 1)^k \|D_{x}^k w\|^2_{s-k}(t) \]

\[ + \int_0^t (\tau + 1)^k (\|D_{x}^{k+1} \tilde{w}_2\|^2_{s-k} + \|D_{x}^k r_2(w)\|^2_{s-k})(\tau) \, d\tau \\
+ \int_0^t (\tau + 1)^{k-2} \|D_{x}^k w\|^2_{s-k-1}(\tau) \, d\tau. \]  

(4.53)

Inductively, (4.51)-(4.53) imply

\[ \sup_{0 \leq \tau \leq t} \left[ (\tau + 1)^k \|D_{x}^k (w - \tilde{w})\|^2_{s-k}(\tau) \right] + \int_0^t (\tau + 1)^k (\|D_{x}^{k+1} \tilde{w}_2\|^2_{s-k} \\
+ \|D_{x}^k r_2(w)\|^2_{s-k})(\tau) \, d\tau \leq C \|w_0 - \tilde{w}\|^2 + M^3(t), \quad 0 \leq k \leq s, \]  

(4.54)
In fact, (4.51) implies (4.54) and (4.55) for $k = 0$. Taking $k = 1$ in (4.52) and applying (4.55) with $k = 0$ gives us (4.54) for $k = 1$. Taking $k = 1$ in (4.53) and applying (4.54) with $k = 1$ and (4.55) with $k = 0$ gives us (4.55) for $k = 1$. By induction, we can show that (4.54) is true for $0 \leq k \leq s$, and (4.55) is true for $0 \leq k \leq s - 1$. Finally, we sum up (4.54) and (4.55) to have

$$M^2(t) \leq C[\|w_0 - \bar{w}\|^2_s + M^3(t)],$$

which implies

$$[1 - CM(t)]M^2(t) \leq C\|w_0 - \bar{w}\|^2_s.$$  \hspace{1cm} (4.56)

If $M(t)$ is sufficiently small, (4.56) implies

$$M^2(t) \leq C\|w_0 - \bar{w}\|^2_s$$  \hspace{1cm} (4.57)

for some constant $C > 0$. Therefore, by a standard continuity argument, (4.57) is true if $\|w_0 - \bar{w}\|_s$ is sufficiently small.

5. **A priori estimates.** In this section we prove our main result, Theorem 1.3. Let

$$N^2_k(t) = \sup_{0 \leq \tau \leq t} [((\tau + 1)\frac{m}{2}) \sum_{l=0}^{k}((\tau + 1)^{l}\|D^l_\tau(w - \bar{w})\|^2(\tau))]$$ \hspace{1cm} (5.1)

for $t \geq 0$ and $0 \leq k \leq s$, where $w$ is the solution given by Theorem 1.2 and $\bar{w}$ is the constant equilibrium state. By a standard continuity argument, to prove (1.19) in Theorem 1.3 under smallness assumption on the initial data, we only need to prove the following proposition.

**Proposition 5.1.** Under the hypotheses of Theorem 1.3, if $N_{s-2}(T)$ is bounded by a small positive constant, which is independent of $T > 0$, then

$$N_{s-2}(T) \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1}),$$ \hspace{1cm} (5.2)

where $C > 0$ is a constant independent of $T$.

**Proof.** We recall from (2.1) and (2.3),

$$\psi = \begin{pmatrix} w_1 \\ r_2(w) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{\psi} = \psi - \bar{\psi} = \begin{pmatrix} w_1 - \bar{w}_1 \\ r_2(w) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix},$$ \hspace{1cm} (5.3)

where $w_1$ and $r_2$ are from condition 2 of Assumption 1.1, and the partition of the vectors are $n = n_1 + n_2$. The unknown $\tilde{\psi}$ satisfies (2.5), with coefficient matrices defined in (2.6), and the nonlinear source $\bar{R}$ given by (2.21). Taking Fourier transform with respect to $x$, we have found that $\tilde{\psi}$ satisfies (2.11).

With the notation changed from $\tilde{\psi}$ to $\psi$, (2.11) with $\bar{R}$ given by (2.21) is in the form of (3.1), where

$$H = \tilde{A}_0 \begin{pmatrix} \tilde{f}_{11} + b_{11} & \cdots & \tilde{f}_{m1} + b_{m1} \\ 0_{n_2 \times m} \end{pmatrix}, \quad h = \tilde{A}_0 \begin{pmatrix} 0_{m \times 1} \\ R_{12} + R_{22} + R_{32} \end{pmatrix},$$ \hspace{1cm} (5.4)
while $\tilde{A}_0, \tilde{f}_{j1}$ and $b_{j1}$ for $1 \leq j \leq m$, $R_{12}, R_{22}$ and $R_{32}$ are given in (2.6), (2.16), (2.19), (2.13), (2.17) and (2.20). Applying Lemma 3.2, we thus have

$$
\|D_x^2\tilde{\psi}\|^2(t) \leq C[(t + 1)^{-\frac{2}{m} - \frac{1}{2}}\|\tilde{\psi}\|_{L^1}^2(0) + e^{-ctl}\|D_x^2\tilde{\psi}\|^2(0)] \\
+ C \int_0^t (t - \tau + 1)^{-\frac{2}{m} - \frac{1}{2}}(\|H\|^2_{L^2} + \|h_2\|^2_{L^2})(\tau) \, d\tau \\
+ C \int_0^t (t - \tau + 1)^{-\frac{2}{m} - \frac{1}{2}}(\|D_x^2H\|^2_{L^2} + \|D_x^2h_2\|^2_{L^2})(\tau) \, d\tau \\
+ C \int_0^t e^{-c(t - \tau)}(\|D_x^2H\|^2_{L^2} + \|D_x^2h_2\|^2_{L^2})(\tau) \, d\tau
$$

(5.5)

for $t \geq 0$ and $l \geq 0$, where $C$ and $c_1$ are positive constants. Here from (2.23) and (5.4),

$$
h = \left(\begin{array}{c}
0_{n_2 \times 1} \\
A_{02}(R_{12} + R_{22} + R_{32})
\end{array}\right).
$$

By (3.2), we thus have

$$
h_2 = A_{02}(R_{12} + R_{22} + R_{32}),
$$

(5.6)

where from (2.23),

$$
A_{02} = \{(r_2)^{-1}t_{1}\eta_{2}w_2(r_2)^{-1}\} \in \mathbb{R}^{n_2 \times n_2}.
$$

From (5.3) and (2.30), we have

$$
\|\tilde{\psi}\|_{L^1} = O(1)\|w - \tilde{w}\|_{L^1}; \quad \|D_x^l\tilde{\psi}\| = O(1)\|D_x^l(w - \tilde{w})\|, \quad 0 \leq l \leq s.
$$

(5.7)

From (5.4), (2.16), (2.19), (5.6), (2.13), (2.17), (2.20) and (5.7), we also have

$$
\|H\|_{L^1} \leq C \sum_{j=1}^{m} (\|\dot{f}_{j1}\|_{L^1} + \|b_{j1}\|_{L^1}) \leq C(\|\tilde{\psi}\|^2 + \|w - \tilde{w}\|\|D_x\tilde{\psi}\|) \\
\leq C(\|w - \tilde{w}\|^2 + \|w - \tilde{w}\|\|D_xw\|) \\
\|h_2\|_{L^1} \leq C(\|R_{12}\|_{L^1} + \|R_{22}\|_{L^1} + \|R_{32}\|_{L^1}) \\
\leq C(\|w - \tilde{w}\|\|D_x\tilde{\psi}\| + \|D_xw\|\|D_x\tilde{\psi}\| + \|w - \tilde{w}\|\|D_x^2\tilde{\psi}\| + \|w - \tilde{w}\|^2) \\
\leq C(\|w - \tilde{w}\|\|D_xw\| + \|D_xw\|^2 + \|w - \tilde{w}\|\|D_x^2w\| + \|w - \tilde{w}\|^2).
$$

(5.8)

Note that (5.1) implies

$$
\|D_x^2(w - \tilde{w})\|_{L^1}(t) \leq (t + 1)^{-\frac{m}{2} - \frac{1}{2}}N_k(t), \quad 0 \leq l \leq k.
$$

(5.9)

Thus (5.8), (5.9) and (4.1) imply that for $l \geq 0$,

$$
\int_0^t (t - \tau + 1)^{-\frac{2}{m} - \frac{1}{2}}(\|H\|^2_{L^2} + \|h_2\|^2_{L^2})(\tau) \, d\tau \\
\leq C \int_0^t (t - \tau + 1)^{-\frac{2}{m} - \frac{1}{2}}(\|w - \tilde{w}\|^4 + \|D_xw\|^4 + \|D_x^2w\|^4)(\tau) \, d\tau \\
\leq C(1)^{-\frac{2}{m} - \frac{1}{2}}[N_0^2(t) \int_0^t (\tau + 1)^{-m} \, d\tau + \|w - \tilde{w}\|^4 \int_0^t (\tau + 1)^{-2} \, d\tau] \\
\leq C(1)^{-\frac{2}{m} - \frac{1}{2}}[N_0^2(t) + \|w - \tilde{w}\|^4],
$$

noting $m \geq 0$. 

Next we consider higher derivatives. From (2.16), (2.30) and (5.9),
\[
\| D_x^l f_{j1} \|_{L^1} = \| - D_x^l f_{j1}(w) + (f_{j1})_0(\bar{w})D_x^l \bar{\psi} \|_{L^1} \\
\leq C \| w - \bar{w} \| \| D_x^l \bar{\psi} \| + \| D_x(f_{j1})_0 \| \| D_x^{l-1} \bar{\psi} \| + \cdots + \| D_x^{l-1}(f_{j1})_0 \| \| D_x \bar{\psi} \| \\
\leq C \sum_{k=0}^l \| D_x^k (w - \bar{w}) \| \| D_x^{l-k} (w - \bar{w}) \| \leq C (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t) \tag{5.11}
\]
Here we have derived (5.11) for $1 \leq l \leq s$. However, it is easy to verify that (5.11) is true for $0 \leq l \leq s$. Similarly, from (2.19) we have
\[
\| D_x^l b_{j1} \|_{L^1} \leq C \sum_{k=0}^l \| D_x^k (w - \bar{w}) \| \| D_x^{l-k} w \|
\]
for $0 \leq l \leq s - 1$. Now we apply (5.9) and (4.1) to the first term on the right-hand side and (5.9) to the other terms. This gives us, for $0 \leq l \leq s - 1$,
\[
\| D_x^l b_{j1} \|_{L^1} \leq C [(t+1)^{-\frac{d}{2} - \frac{1}{2}} N_0(t) \| w_0 - \bar{w} \|_{s} + (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t)] \tag{5.12}
\]
From (2.13), (2.17) and (2.20), we have the following results parallel to (5.11) and (5.12):
\[
\| D_x^l R_{12} \|_{L^1} \leq C [(t+1)^{-\frac{d}{2} - \frac{1}{2}} N_0(t) \| w_0 - \bar{w} \|_{s} + (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t)], \quad 0 \leq l \leq s - 1,
\]
\[
\| D_x^l R_{22} \|_{L^1} \leq C \sum_{k=0}^{l-1} \| D_x^k w \| \| D_x^{l-k} w \| + \| D_x w \|_{s-1} \sum_{k=1}^l \| D_x^k w \| \| D_x^{l-k} w \| \\
\leq C [(t+1)^{-\frac{d}{2} - \frac{1}{2}} N_1(t) \| w_0 - \bar{w} \|_{s} + (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t)], \quad 0 \leq l \leq s - 2,
\]
\[
\| D_x^l R_{32} \|_{L^1} \leq C (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t), \quad 0 \leq l \leq s \tag{5.13}
\]
Here in deriving the estimate for $\| D_x^l R_{22} \|_{L^1}$, we have applied (2.32), (2.29) and (2.31) as well.
Combining (5.4), (5.6) and (5.11)-(5.13), we have
\[
\| D_x^l H \|_{L^1} \leq C \sum_{j=1}^m (\| D_x^l \tilde{f}_{j1} \|_{L^1} + \| D_x^l b_{j1} \|_{L^1}) \\
\leq C [(t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t) + (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_0(t) \| w_0 - \bar{w} \|_{s}], \quad 0 \leq l \leq s - 1, \tag{5.14}
\]
\[
\| D_x^l h_2 \|_{L^1} \leq C (\| D_x^l R_{12} \|_{L^1} + \| D_x^l R_{22} \|_{L^1} + \| D_x^l R_{32} \|_{L^1}) \\
\leq C [(t+1)^{-\frac{d}{2} - \frac{1}{2}} N_1^2(t) + (t+1)^{-\frac{d}{2} - \frac{1}{2}} N_2^2(t) \| w_0 - \bar{w} \|_{s}], \quad 0 \leq l \leq s - 2.
\]
These further imply that for $0 \leq l \leq s - 2$,
\[
\int_0^t (t-\tau+1)^{-\frac{d}{2}} (\| D_x^l H \|_{L^1}^2 + \| D_x^l h_2 \|_{L^1}^2)(\tau) \, d\tau \\
\leq C \int_0^t (t-\tau+1)^{-\frac{d}{2}} [(t+1)^{-m-l} N_1^2(\tau) + (t+1)^{-m-l} N_2^2(\tau) \| w_0 - \bar{w} \|_{s}^2] \, d\tau
\]
\[
\leq C(t + 1)^{-\frac{m}{2} + l + 1} [N_s^2(t) + N_s^2(t)] \|w_0 - \bar{w}\|^2 \|1 + \ln(t + 1)\]. \tag{5.15}
\]

Now we estimate the last term in (5.5). Applying (2.16), (2.32), (2.30) and (2.29), we have
\[
\|D^l_t f_{\tilde{j}_1}\| \leq \|D^{l+1}_t (f_{\tilde{j}_1} w_\psi \bar{\psi}) - f_{\tilde{j}_1} w_\psi D^l_t \bar{\psi}\| + C\|w - \bar{w}\|_{L^\infty} \|D^l_t \bar{\psi}\|
\leq C(\|D_x^l w\|_{L^\infty} \|D^l_x w\| + \|w - \bar{w}\|_{L^\infty} \|D^l_x (w - \bar{w})\|)
\leq C(\|D_x^l w\|_{s-1} \|D^{l-1}_x w\| + \|w - \bar{w}\|_{s-1} \|D^l_x (w - \bar{w})\|),
\tag{5.16}
\]
where \(0 \leq l \leq s\), and the first term on the right-hand side disappears when \(l = 0, 1\). Therefore,
\[
\|D^l_t f_{\tilde{j}_1}\|^2 = \|D^l_x f_{\tilde{j}_1}\|^2 + \|D^{l+1}_x f_{\tilde{j}_1}\|^2
\leq C(\|w - \bar{w}\|^2_{s-1} \|D^l_x w\|^2 + \|D_x w\|^2_{s-1} \|D^l_x w\|^2 + \|D_x w\|^2_{s-1} \|D^{l-1}_x w\|^2)
\tag{5.17}
\]
for \(0 \leq l \leq s\), where we take only the first term on the right-hand side if \(l = 0\), and the first two terms if \(l = 1\).

To estimate \(\|w - \bar{w}\|^2_{s-1}\), we apply (4.1) to its last term, and (5.9) to the others. Noting \(s > m/2 + 1\), or \(s - 1 > m/2\), we have
\[
\|w - \bar{w}\|^2_{s-1} = \sum_{k=0}^{s-2} \|D^l_x (w - \bar{w})\|^2 + \|D^{s-1}_x (w - \bar{w})\|^2
\leq C[(t + 1)^{-\frac{m}{2}} N^2_{s-2}(t) + (t + 1)^{-s+1} \|w_0 - \bar{w}\|^2]\tag{5.18}
\leq C(t + 1)^{-\frac{m}{2}} [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2].
\]
Since \(m\) and \(s\) are integers, we have \(s - 1 \geq m/2 + 1/2\). (In fact, \(s - 1 \geq m/2 + 1/2\) if \(m\) is odd, and \(s - 1 \geq m/2 + 1\) if \(m\) is even.) Thus similar to (5.18) we also have
\[
\|D_x w\|^2_{s-1} \leq C(t + 1)^{-\frac{m}{2} - \frac{1}{2}} [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2]. \tag{5.19}
\]
Substituting (5.18) and (5.19) into (5.17) and applying (4.1) and (5.9) give us
\[
\|D^l_t f_{\tilde{j}_1}\|^2 \leq C(t + 1)^{-\frac{m}{2} - l - \frac{1}{2}} [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2] [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2] \tag{5.20}
\]
for \(0 \leq l \leq s - 1\).

Similarly, we have the following from (2.19), (2.13) and (2.20):
\[
\|D^l_t b_{\tilde{j}_1}\|^2 \leq C(\|w - \bar{w}\|^2_{s-1} \|D^{l+1}_x w\|^2 + \|D_x w\|^2_{s-1} \|D^{l+1}_x w\|^2 + \|D_x w\|^2_{s-1} \|D^l_x w\|^2)
\leq C(t + 1)^{-\frac{m}{2} - l - 1} [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2] [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2],
\tag{5.21}
\]
\(0 \leq l \leq s - 2\),
\[
\|D^l_x R_{12}\|^2 \leq C(\|w - \bar{w}\|^2_{s-1} \|D^{l+1}_x w\|^2 + \|D_x w\|^2_{s-1} \|D^l_x w\|^2)
\leq C(t + 1)^{-\frac{m}{2} - l - 1} [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2] [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2],
\tag{5.22}
\]
\(0 \leq l \leq s - 1\),
\[
\|D^l_x R_{32}\|^2 \leq C(\|w - \bar{w}\|^2_{s-1} \|D^l_x (w - \bar{w})\|^2 + \|D_x w\|^2_{s-1} \|D^{l-1}_x w\|^2)
\leq C(t + 1)^{-m - l - \frac{1}{2}} [N^2_{s-2}(t) + \|w_0 - \bar{w}\|^2] [N^2_{s-2}(t), \quad 0 \leq l \leq s. \tag{5.23}
\]
Here in (5.21) and (5.22), the terms involving \(\|D^l_x w\|^2\) disappear if \(l = 0\), while in (5.23) the term containing \(\|D^{l-1}_x w\|^2\) is omitted if \(l = 0, 1\).
To estimate $R_{22}$ we recall (4.7). Thus from (2.17) and similar to (5.21)-(5.23) we have
\[
\|D^l_x R_{22}\| \\
\leq \sum_{j,k=1}^m \left\{ \|D^l_x \psi_x \|_\infty \|w_j \psi_{xj} \| + \|D^l_x (\psi_x \psi_{xj} \| + \| D^l_x (\psi_j \psi_{xj}) \| \right\} \\
\leq C(\|a_x \|_\infty \|D^{l+1}_x w\| + \|w - \bar{w}\|_\infty \|D^{l+2}_x w\| + \|D_x w\|_2^2 \|D^l_x w\| \\
+ \|D^2_x \bar{w}_2\|_\infty \|D^{l}_x w\|) \\
\leq C(\|w - \bar{w}\|_{s-1} \|D^{l+2}_x w\| + \|D_x w\|_{s-1} \|D^{l+1}_x w\| + \|D_x w\|_{s-1} \|D^{l}_x w\|) \\
+ \|D^2_x \bar{w}_2\|_{s-1} \|D^{l}_x w\|, \quad 0 \leq l \leq s - 2,
\] (5.24)
where the terms containing $\|D^l_x w\|$ do not exist when $l = 0$. Here we have used (4.7) to obtain $\|D^2_x \bar{w}_2\|_{s-1}$. This is necessary since only $\bar{w}_2$ has better regularity. Applying (4.1), (5.9), (5.18) and (5.19), we have
\[
\|D^l_x R_{22}\|^2 \leq C(t + 1)^{-\frac{n}{2} - l - \frac{1}{2}} \left\{ N_{s-2}^2(t) + \|w_0 - \bar{w}\|^2_s \|D^l_x w\|^2 \right\} \\
+ C(t + 1)^{-\frac{n}{2} - l} \|D^l_x \bar{w}_2\|^2_{s-1}(t), \quad 0 \leq l \leq s - 2.
\] (5.25)
Combining (5.20)-(5.23) and (5.25), we have
\[
\int_0^t e^{-c_1(t-\tau)} \left( \|D^l_x H\|^2 + \|D^l_x \bar{w}_2\|^2 \right) d\tau \\
\leq C \int_0^t e^{-c_1(t-\tau)} \sum_{j=1}^m \left\{ \|D^l_j \psi_j \|^2 + \|D^l_x b_j\|^2 \right\} + \sum_{j=1}^m \|D^l_x R_{j2}\|^2 \right\} d\tau \\
\leq C(\|N_{s-2}^2(t) + \|w_0 - \bar{w}\|^2_s \|D^l_x w\|^2 \right\} + \int_0^t e^{-c_1(t-\tau)} \left( C(t + 1)^{-\frac{n}{2} - l} \|D^l_x \bar{w}_2\|^2_{s-1} \right) d\tau, \quad 0 \leq l \leq s - 2.
\] (5.26)
Integrating over $[0, t/2]$ then $[t/2, t]$, the first integral on the right-hand side is $O(1)(t + 1)^{-\frac{n}{2} - l - \frac{1}{2}}$. For the second integral, we apply (4.2) as well, which results in $O(1)(t + 1)^{-\frac{n}{2} - l - \frac{1}{2}} \|w_0 - \bar{w}\|^2_s$. Therefore, for $0 \leq l \leq s - 2$,
\[
\int_0^t e^{-c_1(t-\tau)} \left( \|D^l_x H\|^2 + \|D^l_x \bar{w}_2\|^2 \right) d\tau \\
\leq C(t + 1)^{-\frac{n}{2} - l - \frac{1}{2}} \left\{ N_{s-2}^2(t) + \|w_0 - \bar{w}\|^2_s \|D^l_x w\|^2 + N_{s-2}^4(t) \right\}.
\] (5.27)
Substituting (5.7), (5.10), (5.15) and (5.27) into (5.5), noting $\psi$ is a diffeomorphism, and applying (2.30), we arrive at
\[
\|D^l_x (w - \bar{w})\|^2(t) \leq C \|D^l_x \tilde{\psi}\|^2(t) \\
\leq C(t + 1)^{-\frac{n}{2} - l} \left\{ \|w_0 - \bar{w}\|^2_s + \|w_0 - \bar{w}\|^2_s + N_{s-2}^4(t) \right\}, \quad 0 \leq l \leq s - 2.
\] (5.28)
We multiply both sides by $(t + 1)^{\frac{n}{2} + l}$ and sum up the result over $l$ for $0 \leq l \leq s - 2$. Taking supremum for $0 \leq t \leq T$, by (5.1) we have
\[
N_{s-2}^4(T) \leq C[\|w_0 - \bar{w}\|^2_{L^1} + \|w_0 - \bar{w}\|^2_s + N_{s-2}^4(T)].
\]
or
\[1 - CN_{s-2}(T)N_{s-2}(T) \leq C(\|w_0 - \bar{w}\|_s^2 + \|w_0 - \bar{w}\|_{L_1}^2).\]
This implies \(N_{s-2}(T) \leq 2C(\|w_0 - \bar{w}\|_s^2 + \|w_0 - \bar{w}\|_{L_1}^2),\) or (5.2), if \(N_{s-2}(T)\) is bounded by a small positive constant \(1/\sqrt{2C}\), which is independent of \(T\).

**Remark 5.2.** In the special case of hyperbolic balance laws (1.8), we may replace \(N_{s-2}(T)\) by \(N_{s-1}(T)\) in Proposition 5.1. This justifies (A.2) for \(0 \leq l \leq s - 1 - m(1/2 - 1/p)\) in the appendix. In fact, in such a special case we have \(B_{jk} = 0\) for \(1 \leq j, k \leq m\). As a consequence, \(b_{j1} = 0\) for \(1 \leq j \leq m\) and \(R_{22} = 0\), see (2.17) and (2.19). Thus (5.14) and (5.26), hence (5.15) and (5.27), are true for \(0 \leq l \leq s - 1\). This implies that (5.28) is true for \(0 \leq l \leq s - 1\) as well. After multiplying (5.28) by \((t + 1)^{\frac{s}{2} + l}\), we sum up the result for \(0 \leq l \leq s - 1\) instead. This results in \(N_{s-1}(T) \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L_1}).\)

The rest of this section is to obtain (1.20) using (1.19), hence complete the proof of Theorem 1.3. Note that \(r_2 = \bar{\psi}_2\) from (5.3). Recall (4.14), which gives us
\[
\|D_t^l r_2(w)(t)\| \leq C[e^{-c_2 t} \|D_t^l r_2(w_0)\| + \int_0^t e^{-c_2 (t-\tau)} \|D_t^l r_2(\tau)\| d\tau],
\] (5.29)
where \(c_2 > 0\) is a constant and \(R\) is defined in (4.13).

To estimate \(\|D_t^l r_2\|\) we apply (2.30) and note that \(r_2(\bar{w}) = 0\). This gives us
\[
\|D_t^l r_2(w_0)\| \leq C \|D_t^l (w_0 - \bar{w})\|, \quad 0 \leq l \leq s.
\] (5.30)

To estimate \(\|D_t^l R\|\) we recall (4.13) and apply (2.32). Similar to (5.16), (5.24) and (5.23), we have
\[
\|D_t^l R\| \leq \sum_{j=1}^m \|D_t^l [(r_2)_w f_j(w)_{x_j}]\| + \sum_{j,k=1}^m \|D_t^l [(r_2)_w (B_{jk} w_{x_k})_{x_j}]\|
\]
\[
\leq C \|D_x w\|_{L^\infty} \|D_t^l w\| + \|D_t^{l+1} w\|
\]
\[
+ \|D_x w\|_{L^\infty} \|D_t^l w\|_{L^\infty} \|D_t^l w\| + \|D_t^{l+1} \bar{w}_2\|
\]
\[
+ \|D_x^2 w\|_{L^\infty} \|D_t^l \bar{w}_2\|_{L^\infty} + \|D_x w\|_{L^\infty} \|D_t^l w\| + \|D_t^{l+2} \bar{w}_2\|
\]
\[
\leq C \|D_t^{l+1} w\| + \|w - \bar{w}\|_s \|D_t^l (w - \bar{w})\| + \|D_t^{l+2} \bar{w}_2\|
\]
\[
+ \|D_x w\|_{L^\infty} \|D_t^{l-1} (w - \bar{w})\| + \|D_t^{l+2} \bar{w}_2\|
\]
\[
\leq C \|D_t^{l+1} w\| + \|w - \bar{w}\|_s \|D_t^l (w - \bar{w})\| + \|D_t^2 \bar{w}_2\|_{s-1} \|D_t^l (w - \bar{w})\|
\]
\[
+ \|D_x w\|_{L^\infty} \|D_t^{l-1} (w - \bar{w})\| + \|D_t^{l+2} \bar{w}_2\|, \quad 0 \leq l \leq s - 1,
\] (5.31)
where the terms with \(\|D_t^l w\|, \|D_t^{l-1} r_2\|\) or \(\|D_t^{l-1} (w - \bar{w})\|\) disappear if \(l = 0\). Applying (1.19), (5.18), (5.19) and (5.2) to the right-hand side of (5.31), we further have
\[
\|D_t^l R(t)\| \leq C(\|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L_1})
\]
\[
\times [(t + 1)^{-\frac{m}{2}} + 2^{-\frac{m}{2}} + (t + 1)^{-\frac{m}{2}} \|D_x^2 \bar{w}_2\|_{s-1}(t)]
\] (5.32)
for $0 \leq l \leq s - 4$. Here in deriving (5.32) we have used the fact $m \geq 2$. Also, the decay rate for $\|D_x w\|_{s-1}$ is not worse than $(t+1)^{-\frac{s}{2}}$, see the comment on $m$ being even number above (5.19).

Now we substitute (5.30) and (5.32) into (5.29): For $0 \leq l \leq s - 4$,

$$\|D^l_{xx} r_2(w)\|_2 \leq C e^{-c_2 t} \|D^l_{xx} (w_0 - \tilde{w})\| + C (\|w_0 - \tilde{w}\|_s + \|w_0 - \tilde{w}\|_{L^1}) \times \int_0^t e^{-c_2 (t-\tau)} [(\tau + 1)^{-\frac{m}{2} - \frac{s}{2}} + (\tau + 1)^{-\frac{m}{2} - \frac{l}{2}}] \|D^2_2 \tilde{w}_2\|_{s-1}(\tau) d\tau$$

$$\leq C (\|w_0 - \tilde{w}\|_s + \|w_0 - \tilde{w}\|_{L^1}) (t + 1)^{-\frac{m}{2} - \frac{s}{2}}$$

$$\times [1 + \int_0^t e^{-c_2 (t-\tau)} [(\tau + 1)^{-\frac{m}{2} - \frac{l}{2}}] \|D^2_2 \tilde{w}_2\|_{s-1}(\tau) d\tau].$$

Applying Cauchy-Schwarz inequality and (4.2) to the integral on the right-hand side, we obtain (1.20).

As a final comment of this section, we note that in the special case of hyperbolic balance laws (1.8), the terms containing $\tilde{w}_2$ on the right-hand side of (5.31) disappear. In particular, we no longer have $\|D^l_{xx} \tilde{w}_2\|$, and (5.32) becomes

$$\|D^l_{xx} R\|_2 (t) \leq C (\|w_0 - \tilde{w}\|_s + \|w_0 - \tilde{w}\|_{L^1}) (t + 1)^{-\frac{m}{2} - \frac{l}{2}}$$

for $0 \leq l \leq s - 2$. This justifies (A.3) in the appendix.

Appendix A. Applications. In this appendix we discuss applications, first to the special cases of (1.4) and (1.8), then to Example 4. For the hyperbolic-parabolic conservation laws (1.4), Assumption 1.1 is simplified to Assumption A.1, and Theorem 1.3 and Corollary 1.4 are reduced to Theorem A.2 below.

Assumption A.1. 1. There exists a strictly convex entropy function $\eta$ of $w$ such that in $\Omega$, all $\eta'' f_j$ are symmetric for $1 \leq j \leq m$, $(\eta'' B_{jk})^t = \eta'' B_{jk}$ for $1 \leq j, k \leq m$, and $\eta'' \sum_{j,k=1}^m B_{jk} \xi_k \xi_j$ is symmetric, semi-positive definite for all $\xi = (\xi_1, \ldots, \xi_m)^t \in S^{m-1}$.

2. There is a diffeomorphism $\varphi \rightarrow w$ from an open set $\hat{\Omega} \subset \mathbb{R}^n$ to $\Omega$ such that

$$B_{jk}(w(\varphi)) w_{\varphi}(\varphi) = \text{diag}(0_{n_3 \times n_3}, B^*_j), \quad 1 \leq j, k \leq m,$$

where $B^*_{jk} \in \mathbb{R}^{n_4 \times n_4}$, $n_3$ and $n_4 = n - n_3 > 0$ are constant integers, and $\sum_{j,k=1}^m B^*_{jk} \xi_k \xi_j$ is nonsingular for all $\varphi \in \hat{\Omega}$ and $\xi = (\xi_1, \ldots, \xi_m)^t \in S^{m-1}$.

3. For each $\xi = (\xi_1, \ldots, \xi_m)^t \in S^{m-1}$, let $A(\xi) = \sum_{j=1}^m f^t_j(\tilde{w}) \xi_j$ and $B(\xi) = \sum_{j,k=1}^m B_{jk}(\tilde{w}) \xi_k \xi_j$. Then the null space of $B(\xi)$ contains no eigenvectors of $A(\xi)$.

Theorem A.2. Let $\tilde{w}$ be a constant state and Assumption A.1 be true. Let $m \geq 2$, $s > \frac{m}{2} + 1$ be an integer, and $w_0 - \tilde{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\|w_0 - \tilde{w}\|_s + \|w_0 - \tilde{w}\|_{L^1} \leq \varepsilon$, the Cauchy problem (1.4), (1.11) has a unique solution for $t \geq 0$, satisfying

$$\|D^l_{xx} (w - \tilde{w})\|_{L^p}(t) \leq C (\|w_0 - \tilde{w}\|_s + \|w_0 - \tilde{w}\|_{L^1})(t + 1)^{-\frac{m}{2} - \frac{l}{2} + \frac{1}{p}}$$

(A.1)

for $0 \leq l \leq s - 2 - m(1/2 - 1/p)$ with $p \geq 2$ ($l \neq s - 2 - m/2$ if $p = \infty$). In particular, the decay rate in $L^2$ is $(t + 1)^{-\frac{m}{2} - \frac{l}{2}}$ in (A.1).
We note that the constant orthogonal matrix \( P \) in Assumption 1.1 is not needed in this special case since there is no intertwining of dissipation mechanism from second order and zeroth order derivatives, see the comment after Assumption 1.1. We also set \( n_4 > 0 \) since otherwise (1.4) would be a system of hyperbolic conservation laws and condition 3 in Assumption A.1 would not be satisfied.

Similarly, for the special case of hyperbolic balance laws (1.8), our general assumptions and results are reduced to Assumption A.3 and Theorem A.4 as follows.

**Assumption A.3.** 1. There exists a strictly convex entropy function \( \eta \) of \( w \) in \( \mathbb{R}^n \) such that \( \eta'' f_j' \), \( 1 \leq j \leq m \), are symmetric in \( \mathbb{R} \), and \( \eta'' f' \) is symmetric, semi-negative definite on \( \mathbb{R} \).

2. Equation (1.8) has \( n_1 \) conservation laws, i.e., there is a partition \( n = n_1 + n_2 \), \( n_1, n_2 > 0 \), such that

\[
\begin{align*}
  r(w) &= \begin{pmatrix} n_1 \times 1 \\ r_2(w) \end{pmatrix}, \\
  w &= \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},
\end{align*}
\]

with \( w_1 \in \mathbb{R}^{n_1} \), \( r_2, w_2 \in \mathbb{R}^{n_2} \), and \( r_2(w_2) \) is nonsingular.

3. The null space of \( r'(\bar{w}) \) contains no eigenvectors of \( A(\xi) = \sum_{j=1}^m f_j'(\bar{w}) \xi_j \) for all \( \xi = (\xi_1, \ldots, \xi_m)^t \in \mathbb{S}^{m-1} \).

**Theorem A.4.** Let \( \bar{w} \) be a constant equilibrium state and Assumption A.3 be true. Let \( m \geq 2, s > m/2 + 1 \) be an integer, and \( w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^2(\mathbb{R}^m) \). Then there exists a constant \( \varepsilon > 0 \) such that if \( \| w_0 - \bar{w} \|_s + \| w_0 - \bar{w} \|_{L^1} \leq \varepsilon \), the Cauchy problem (1.8), (1.11) has a unique solution for \( t \geq 0 \), satisfying

\[
\| D_s^l (w - \bar{w}) \|_{L^p} (t) \leq C(\| w_0 - \bar{w} \|_s + \| w_0 - \bar{w} \|_{L^1}) (t + 1)^{-\frac{p}{2} \left(1 - \frac{1}{s} - \frac{m}{2} \right) - \frac{1}{2}} \tag{A.2}
\]

for \( 0 \leq l \leq s - 2 - m(1/2 - 1/p) \), and

\[
\| D_s^l r_2(w) \|_{L^p} (t) \leq C(\| w_0 - \bar{w} \|_s + \| w_0 - \bar{w} \|_{L^1}) (t + 1)^{-\frac{p}{2} \left(1 - \frac{1}{s} - \frac{m}{2} \right) - \frac{1}{2}} \tag{A.3}
\]

for \( 0 \leq l \leq s - 2 - m(1/2 - 1/p) \). Here \( p \geq 2 \), \( C > 0 \) is a constant, and if \( p = \infty \), \( l \neq s - 1 - m/2 \) for (A.2), and \( l \neq s - 2 - m/2 \) for (A.3). In particular, the \( L^2 \) rates in (A.2) and (A.3) are \( (t + 1)^{-\frac{1}{2} - \frac{1}{2} - \frac{m}{2}} \) and \( (t + 1)^{-\frac{1}{2} - \frac{1}{2} - \frac{m}{2}} \), respectively.

In Assumption A.3 we set \( n_1, n_2 > 0 \). The case \( n_1 = 0 \) leads to better decay rates than those in (A.2) and (A.3) while physical models dictate \( n_1 > 0 \) usually. The case \( n_2 = 0 \) is precluded as otherwise the system is one of hyperbolic conservation laws.

In (A.2) we allow \( 0 \leq l \leq s - 1 - m(1/2 - 1/p) \) rather than \( 0 \leq l \leq s - 2 - m(1/2 - 1/p) \). Similarly, in (A.3) we let \( 0 \leq l \leq s - 2 - m(1/2 - 1/p) \) instead of \( 0 \leq l \leq s - 4 - m(1/2 - 1/p) \). This is due to the lack of second derivatives in (1.8), see Remark 5.2 and the final comment of Section 5.

Assumption A.1 is satisfied by Example 1 under physical assumptions, thus Theorem A.2 applies. Similarly, Assumption A.3 is satisfied by Example 3 and Theorem A.4 applies. The resulting \( L^p \) decay rates are available in the literature. Example 2, on the other hand, is a system of hyperbolic balance laws with extremely weak dissipation, in the sense that it violates condition 3 of Assumption A.3. Indeed, there are up to \( m \) linearly independent eigenvectors of \( A(\xi) \) that are contained in the null space of \( r'(\bar{w}) \) [12]. Finally, having both viscosity and a lower order term, Example 4 is an important physical model that fits the general framework (1.1). It has been shown that under physical assumptions, it satisfies Assumption 1.1 [13], thus the main result in this paper applies. This is to be detailed as follows.
To give precise statements on Example 4, we introduce notations based on the relation among thermodynamic variables discussed in Section 1:

\[ p = p(v, e_1) = \tilde{p}(v, T_1), \quad T_1 = T_1(v, e_1), \quad e_2 = \omega(T_2). \]  

(A.4)

Note that a state is an equilibrium state if and only if \( T_2 = T_1 \). Thus the equilibrium manifold is characterized as

\[ \mathcal{E} = \{ T_2 = T_1 \} \cap \mathcal{O}, \]  

(A.5)

and \( e_2 \) satisfies

\[ e_2^* = \omega(T_1), \quad \bar{e}_2 = \omega(\bar{T}_1). \]  

(A.6)

Here \( \bar{e}_2 \) is \( e_2 \) at the constant equilibrium state \( \bar{w} \), etc. Without loss of generality, we take \( \bar{u} = 0 \in \mathbb{R}^m \) hence

\[ \bar{w} = (\bar{\rho}, 0_{1 \times m}, \bar{\rho} e, \bar{\rho} \bar{e}_2)^t. \]  

(A.7)

The physical assumptions to be imposed on Example 4 are

\[ \frac{\partial}{\partial \rho} \bar{p}(v, T_1) < 0, \quad T_{1e_1} = \frac{\partial}{\partial e_1} T_1(v, e_1) > 0, \quad \rho_{e_1} = \frac{\partial}{\partial e_1} \rho(v, e_1) \neq 0, \quad \omega'(T) > 0. \]  

(A.8)

The following proposition is from Propositions 4.1 and 4.2 in [13].

**Proposition A.5.** Let (A.8) be true, and the dissipation parameters in (1.10) at \( \bar{w} \) satisfy

\[ \kappa > 0, \quad \bar{\nu} > 0, \quad \bar{\mu} > 0, \quad 2\bar{\mu} + \bar{\nu}' > 0. \]  

(A.9)

Then (1.10) satisfies Assumption 1.1 in a small neighborhood \( \mathcal{O} \) of \( \bar{w} \).

Equation (1.10) is supplemented by the following initial condition:

\[ w(x, 0) = (\rho, \rho u^t, \rho E, \rho e_2)^t(x, 0) = (\rho_0, \rho_0 u_0^t, \rho_0 (e_0 + \frac{1}{2} |u_0|^2), \rho_0 e_20)^t(x) \equiv w_0(x), \]  

(A.10)

with prescribed functions \( \rho_0(x), e_0(x), e_20(x) \in \mathbb{R} \) and \( u_0(x) \in \mathbb{R}^m \). For (1.10) and (A.10) we have the following theorem as an application of Theorem 1.3 and Corollary 1.4.

**Theorem A.6.** Let \( \bar{\rho}, \bar{e}_1 > 0 \) be constants, \( \bar{T}_1 = T_1(\bar{\rho}, \bar{e}_1) \), \( \bar{e}_2 = \omega(\bar{T}_1) \) and \( \bar{e} = \bar{e}_1 + \bar{e}_2 \). Let (A.8) and (A.9) be true, \( s > \frac{m}{2} + 1 \) be an integer (with \( m \geq 2 \)), and \( w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m) \) for \( w_0 \) and \( \bar{w} \) in (A.10) and (A.7), respectively. Then there exists a constant \( \varepsilon > 0 \) such that if \( \| w_0 - \bar{w} \|_s + \| w_0 - \bar{w} \|_{L^1} \leq \varepsilon \), the Cauchy problem (1.10), (A.10) has a unique solution, with \( (\rho - \bar{\rho}, \rho u^t, \rho E - \bar{\rho} \bar{e}, \rho e_2 - \bar{\rho} \bar{e}_2) \in C([0, \infty); H^s(\mathbb{R}^m)) \). The solution satisfies the following \( L^p \) decay properties for \( p \geq 2 \):

\[ \| D^l_x (\rho - \bar{\rho}, \rho u^t, \rho E - \bar{\rho} \bar{e}, \rho e_2 - \bar{\rho} \bar{e}_2) \|_{L^p} (t) \leq C(\| w_0 - \bar{w} \|_s + \| w_0 - \bar{w} \|_{L^1}) (t+1)^{-\frac{m}{2} (1 - \frac{1}{p}) - \frac{1}{4}} \]  

(A.11)

for \( 0 \leq l \leq s - 2 - m(1/2 - 1/p) \), and

\[ \| D^l_x (\rho e_2^* - e_2) \|_{L^p} (t) \leq C(\| w_0 - \bar{w} \|_s + \| w_0 - \bar{w} \|_{L^1}) (t+1)^{-\frac{m}{2} (1 - \frac{1}{p}) - \frac{1}{4}} \]  

(A.12)

for \( 0 \leq l \leq s - 4 - m(1/2 - 1/p) \), where \( C \) in (A.11) and (A.12) is a constant. In particular, the \( L^2 \) decay rates in (A.11) and (A.12) are \( (t+1)^{-\frac{m}{4} - \frac{1}{4}} \) and \( (t+1)^{-\frac{m}{4} - \frac{1}{4}} \), respectively.
As a final remark, we comment on the partitions \( n = n_1 + n_2 = n_3 + n_4 \). It is clear from (1.10) that we have \( n = m + 3, n_1 = m + 2 \) and \( n_2 = 1 \). If \( \nu > 0 \) in (A.9) then \( n_3 = 1 \) and \( n_4 = m + 2 \). In this case the orthogonal matrix \( P \) in (1.14) is the identity matrix. On the other hand, if \( \nu = 0 \) then \( \nu \equiv 0 \) for \( n_3 \) and \( n_4 \) to be constant integers. In this case, \( n_3 = 2 \) and \( n_4 = m + 1 \). The matrix \( P \) is a permutation that can move the last equation of (1.10) up above the momentum equation.

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