LOGARITHMIC POTENTIALS ON $\mathbb{P}^n$

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Abstract. We study the projective logarithmic potential $G_{\mu}$ of a Probability measure $\mu$ on the complex projective space $\mathbb{P}^n$. We prove that the Range of the operator $\mu \rightarrow G_{\mu}$ is contained in the (local) domain of definition of the complex Monge-Ampère operator acting on the class of quasi-plurisubharmonic functions on $\mathbb{P}^n$ with respect to the Fubini-Study metric. Moreover, when the measure $\mu$ has no atom, we show that the complex Monge-Ampère measure of its Logarithmic potential is an absolutely continuous measure with respect to the Fubini-Study volume form on $\mathbb{P}^n$.

1. Introduction and statement of the results

Logarithmic potentials of Borel measures in the complex plane play a fundamental role in Logarithmic Potential Theory. This due to the fact that this theory is associated to the Laplace operator which is a linear elliptic partial differential operator of second order. It is well known that in higher dimension plurisubharmonic functions are rather connected to the complex Monge-Ampère operator which is a fully non-linear second order partial differential operator. Therefore Pluripotential theory cannot be described by logarithmic potential. However the class of logarithmic potentials gives a nice class of plurisubharmonic functions which turns out to be in the local domain of definition of the complex Monge-Ampère operator. This study was carried out by Carlehed [5] in the case of a compactly supported measures on $\mathbb{C}^n$ or a bounded hyperconvex domain in $\mathbb{C}^n$.

Our main goal is to extend this study to the complex projective space motivated by the fact that the complex Monge-Ampère operator plays an important role in Kähler geometry (see [13]). A large class of singular potentials on which the complex Monge-Ampère is well defined was introduced (see [12], [8], [4]). However the global domain of definition of the complex Monge-Ampère operator on compact Kähler manifolds is not yet well understood. Using the characterization of the local domain of definition given by Cegrell and Blocki (see [2], [3], [7]), we show that it is contained in the local domain of definition of the complex Monge-Ampère operator on the complex projective space $(\mathbb{P}^n, \omega)$ equipped with the Fubini-Study metric.

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Let \( \mu \) be a probability measure on \( \mathbb{P}^n \). Then its projective logarithmic potential is defined on \( \mathbb{P}^n \) as follows:

\[
G_{\mu}(\zeta) := \int_{\mathbb{P}^n} G(\zeta, \eta) d\mu(\eta) \quad \text{where} \quad G(\zeta, \eta) := \log \frac{\mid \zeta \land \eta \mid}{\mid \zeta \mid \mid \eta \mid}
\]

**Theorem 1.1.** Let \( \mu \) be a probability measure on \( \mathbb{P}^n \). Then the following properties hold.

1. The potential \( G_{\mu} \) is a negative \( \omega \)-plurisubharmonic function on \( \mathbb{P}^n \) normalized by the following condition:

\[
\int_{\mathbb{P}^n} G_{\mu} \omega_{FS}^n = -\alpha_n,
\]

where \( \alpha_n \) is a numerical constant.

2. \( G_{\mu} \in W^{1,p}(\mathbb{P}^n) \) for any \( 0 < p < 2n \).

3. \( G_{\mu} \in DMA_{loc}(\mathbb{P}^n, \omega) \).

We also show a regularizing property of the operator \( \mu \to G_{\mu} \) acting on probability measures on \( \mathbb{P}^n \).

**Theorem 1.2.** Let \( \mu \) be a probability measure on \( \mathbb{P}^n \) with no atoms. Then the Monge-Ampère measure \( (\omega + dd^c G_{\mu})^n \) is absolutely continuous with respect to the Fubini-Study volume form on \( \mathbb{P}^n \).

2. THE LOGARITHMIC POTENTIAL, PROOF OF THEOREM 1.1

The complex projective space can be covered by a finite number of charts given by \( \mathcal{U}_k := \{[\zeta_0, \zeta_1, \cdots, \zeta_n] \in \mathbb{P}^n : \zeta_k \neq 0 \} \) (\( 0 \leq k \leq n \)) and the corresponding coordinate chart is defined on \( \mathcal{U}_k \) by the formula

\[
z^k(\zeta) = z^k := (z_j^k)_{0 \leq j \leq n, j \neq k} \quad \text{where} \quad z_j^k := \frac{\zeta_j}{\zeta_k} \quad \text{for} \quad j \neq k
\]

The Fubini-Study metric \( \omega = \omega_{FS} \) is given on \( \mathcal{U}_k \) by \( \omega|_{\mathcal{U}_k} = \frac{1}{2}dd^c \log(1 + |z^k|^2) \). The projective logarithmic kernel on \( \mathbb{P}^n \times \mathbb{P}^n \) is naturally defined by the following formula

\[
G(\zeta, \eta) := \log \frac{\mid \zeta \land \eta \mid}{\mid \zeta \mid \mid \eta \mid} = \log \sin \frac{d(\zeta, \eta)}{\sqrt{2}} \quad \text{where} \quad |\zeta \land \eta|^2 = \sum_{0 \leq i < j \leq n} |\zeta_0 \eta_j - \zeta_j \eta_i|^2
\]

and \( d \) is the geodesic distance associated to the Fubini-Study metric (see [15],[6]).

We recall some definitions and give a useful characterization of the local domain of definition of the complex Monge-Ampère operator given by Z. Blocki (see [2],[3]).
Definition 2.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. By definition the set $\text{DMA}_{\text{loc}}(\Omega)$ denotes the set of plurisubharmonic functions $\phi$ on $\Omega$ for which there a positive Borel measure $\sigma$ on $\Omega$ such that for all open $U \subset X$ and $\forall (\phi_j) \in PSH(U) \cap C^\infty(U) \setminus \phi$ in $U$, the sequences of measures $(dd^c \phi_j)^n$ converges weakly to $\sigma$ in $U$. In this case, we put $(dd^c \phi)^n = \sigma$.

The following result of Blocki gives a useful characterization of the local domain of definition of the complex Monge-Ampère operator.

Theorem 2.2. (Z. Blocki [2], [3]). 1. If $\Omega \subset \mathbb{C}^2$ is an open set then $\text{DMA}_{\text{loc}}(\Omega) = PSH(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)$.

2. If $n \geq 3$, a plurisubharmonic function $\phi$ on a open set $U \subset \mathbb{C}^n$ belong to $\text{DMA}_{\text{loc}}(\Omega)$ if and only if for any $z \in \Omega$ there exists a neighborhood $U_z$ of $z$ in $\Omega$ and a sequence $(\phi_j) \subset PSH(U_z) \cap C^\infty(U_z) \setminus \phi$ in $U_z$ such that the sequences

$$|\phi_j|^{n-p-2}d\phi_j \wedge d^c \phi_j \wedge (dd^c \phi_j)^p \wedge (dd^c |z|^2)^{n-p-1}, \quad p = 0, 1, \ldots, n - 2$$

are locally weakly bounded in $U_z$.

Observe that by Bedford and Taylor [1], the class of locally bounded plurisubharmonic functions in $\Omega$ is contained in $\text{DMA}_{\text{loc}}(\Omega)$. By the work of J.-P. Demailly [9], any plurisubharmonic function in $\Omega$ bounded near the boundary $\partial \Omega$ is contained in $\text{DMA}_{\text{loc}}(\Omega)$. Let $(X, \omega)$ be a Kähler manifold of dimension $n$. We denote by $PSH(X, \omega)$ the set of $\omega$-plurisubharmonic functions in $X$. Then it is possible to define in the same way the local domain of definition $\text{DMA}_{\text{loc}}(X, \omega)$ of the complex Monge-Ampère operator on $(X, \omega)$. A function $\varphi \in PSH(X, \omega)$ belongs to $\text{DMA}_{\text{loc}}(X, \omega)$ iff for any local chart $(U, z)$ the function $\hat{\varphi} := \varphi + \rho \in \text{DMA}_{\text{loc}}(U)$ where $\rho$ is a Kähler potential of $\omega$. Then the previous theorem extends trivially to this general case.

Let $(\chi_j)_{0 \leq j \leq n}$ be a fixed partition of unity subordinated to the covering $(U_j)_{0 \leq j \leq n}$. We define $m_j = \int \chi_j \, d\mu$ and $J = \{j \in \{0, 1, \ldots, n\} : m_j \neq 0\}$. The $J \neq \emptyset$ and for $j \in J$, the measure $\mu_j := \frac{1}{m_j} \chi_j \mu$ is a probability measure on $\mathbb{P}^n$ supported in $U_j$ and we have the following convex decomposition of $\mu$

$$\mu = \sum_{j \in J} m_j \mu_j$$

Therefore the potential $\mathcal{G}_\mu$ can be written as a convex combination

$$\mathcal{G}_\mu = \sum_{j \in J} m_j \mathcal{G}_{\mu_j}.$$ 

To show that $\mathcal{G}_\mu \in \text{DMA}_{\text{loc}}(\mathbb{P}^n, \omega)$, it suffices to consider the case of a compact measure supported in an affine chart. Without loss of generality, we may always assume that $\mu$ is compactly supported in $U_0$ and we are reduced to the study of the potential $\mathcal{G}_\mu$ on the open set $U_0$. The restriction
of \( G(\zeta, \eta) \) to \( U_0 \times U_0 \) can expressed in the affine coordinates as

\[
G(\zeta, \eta) = N(z, w) - \frac{1}{2} \log(1 + |z|^2)
\]

where

\[
N(z, w) := \frac{1}{2} \log \frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2}
\]

will be called the projective logarithmic kernel on \( \mathbb{C}^n \).

**Lemma 2.3.**
1. The kernel \( N \) is upper semicontinuous in \( \mathbb{C}^n \times \mathbb{C}^n \) and smooth off the diagonal of \( \mathbb{C}^n \times \mathbb{C}^n \).
2. For any fixed \( w \in \mathbb{C}^n \), the function \( N(., w) : z \mapsto N(z, w) \) is plurisubharmonic in \( \mathbb{C}^n \) and satisfies the following inequality

\[
\frac{1}{2} \log \frac{|z - w|^2}{1 + |w|^2} \leq N(z, w) \leq \frac{1}{2} \log(1 + |z|^2), \quad \forall (z, w) \in \mathbb{C}^n \times \mathbb{C}^n
\]

From lemma 2.3, we have the following properties of the projective logarithmic kernel \( G \) on \( \mathbb{P}^n \times \mathbb{P}^n \).

**Corollary 2.4.**
1. The kernel \( G \) is a non positive upper semi continuous function on \( \mathbb{P}^n \times \mathbb{P}^n \) and smooth off the diagonal of \( \mathbb{P}^n \times \mathbb{P}^n \).
2. For any fixed \( \eta \in \mathbb{P}^n \), the function \( G(., \eta) : \zeta \mapsto G(\zeta, \eta) \) is a non positive \( \omega \)-plurisubharmonic function in \( \mathbb{P}^n \) and smooth in \( \mathbb{P}^n \setminus \{\eta\} \), hence \( G(., \eta) \in DMA_{loc}(\mathbb{P}^n, \omega) \). Moreover \( (\omega + dd^c G(., \eta))^{\mathbb{P}^n} = \delta_\eta \).

For a probability measure \( \nu \) on \( \mathbb{C}^n \), we define the projective logarithmic potential of \( \nu \) as follows

\[
\mathbb{V}_\nu(z) := \frac{1}{2} \int_{\mathbb{C}^n} \log \frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2} d\nu(w)
\]

**Proposition 2.5.** Let \( \nu \) be a probability measure \( \nu \) on \( \mathbb{C}^n \). Then the function \( \mathbb{V}_\nu(z) \) is plurisubharmonic in \( \mathbb{C}^n \) and for all \( z \in \mathbb{C}^n \)

\[
\mathbb{V}_\nu(z) \leq \frac{1}{2} \log(1 + |z|^2).
\]

Also \( \mathbb{V}_\nu \in DMA_{loc}(\mathbb{C}^n) \) and

\[
(dd^c\mathbb{V}_\nu)^{\mathbb{P}^n} = \int_{\mathbb{C}^n \times \cdots \times \mathbb{C}^n} dd^c z_1 N(., w_1) \wedge \cdots \wedge dd^c z_n N(., w_n) d\nu(w_1) \cdots d\nu(w_n)
\]

**Proof of theorem 1.1** As we have seen we have

\[
\mathcal{G}_\mu = \sum_{j \in J} m_j \mathcal{G}_{\mu_j},
\]

where \( \mu_j \) is compactly supported in the affned chart \( U_j \).

Observe that for a fixed \( k \) one can write on \( U_k \)

\[
\mathcal{G}_{\mu_k}(\zeta) + 1/2 \log(1 + |z|^2) = \mathbb{V}_{\mu_k}(z), \quad \text{where} \quad z := z^k(\zeta) \in \mathbb{C}^n,
\]

which is plurisubharmonic in \( \mathbb{C}^n \). Hence \( \mathcal{G}_\mu \) is \( \omega \)-plurisubharmonic in \( \mathbb{P}^n \).
2. By the co-area formula (see [10])

\[\int_{\mathbb{P}^n} G_\mu(\zeta)dV(\zeta) = \int_0^{\pi/\sqrt{2}} \log \sin \frac{r}{\sqrt{2}} A(r)dr = -c_n/\sqrt{2n^2}\]

where \(A(r) := c_n \sin^{2n-2}(r/\sqrt{2}) \sin(\sqrt{2}r)\) is the area of the sphere about \(\eta\) and radius \(r\) on \(\mathbb{P}^n\) and \(c_n\) is a numerical constant (see [14] page 168 or [11] lemma 5.6).

Let \(p \geq 1\). Since \(|\nabla d\omega| = 1\), also by the co-area formula

\[\int_{\mathbb{P}^n} |\nabla G_\mu(\zeta)|^p dV(\zeta) \leq \int_{\mathbb{P}^n} \cot^p \left(\frac{d(\zeta, \eta)}{\sqrt{2}}\right) d\mu(\eta) dV(\zeta) \leq 2\sqrt{2}c_n \int_0^{\pi/2} \sin^{2n-1-p} tdt\]

which is finite if and only if \(p < 2n\). Hence for all \(p \in ]0, 2n[ : G_\mu \in W^{1, p}(\mathbb{P}^n)\) (by concavity of \(x^p\)).

3. When \(n = 2\), we can apply the previous result to conclude that \(G_\mu \in DMA_{loc}(\mathbb{P}^2)\). When \(n \geq 3\), we apply Blocki's characterization stated above to show that \(G_{\mu_k} \in DMA_{loc}(U_k)\). We consider the following approximating sequence

\[\mathbb{V}_\mu(z) = \frac{1}{2} \int_{\mathbb{C}^n} \log \left(\frac{|z-w|^2 + |z \wedge w|^2}{1 + |w|^2} + \epsilon^2\right) d\mu_k(w) \setminus \mathbb{V}_\mu(z),\]

and use the next classical lemma on Riesz potentials to show a uniform estimates on their weighted gradients as required in Blocki’s theorem.

**Lemma 2.6.** Let \(\mu\) be a probability measure on \(\mathbb{C}^n\). For \(0 < \alpha < 2n\), define the Riesz potential of \(\mu\) by

\[J_{\mu, \alpha}(z) := \int_{\mathbb{C}^n} \frac{d\mu(w)}{|z-w|^{\alpha}}\]

If \(0 < p < 2n/\alpha\) then \(J_{\mu, \alpha} \in L^p_{loc}(\mathbb{C}^n)\).

3. Regularizing property and proof of theorem 1.2

We prove a regularizing property of the operator \(\mu \rightarrow G_\mu\). By the localization process explained before, the proof of theorem 1.2 follows from the following theorem which generalizes and improves a result of Carlehed (see [5]).

**Theorem 3.1.** Let \(\mu\) be a probability measure on \(\mathbb{C}^n\) with no atoms and let \(\psi \in L(\mathbb{C}^n)\). Assume that \(\psi\) is smooth in some open subset \(U \subset \mathbb{C}^n\). Then for any \(0 \leq m \leq n\), the Monge-Ampère measure \((dd^c \mathbb{V}_\mu)_m \wedge (dd^c \psi)^{n-m}\) is absolutely continuous with respect to the Lebesgue measure on \(U\).

The proof is based on the following elementary lemma.
Lemma 3.2. Let \((w_1, \ldots, w_n) \in (\mathbb{C}^n)^n\) fixed such that \(w_1 \neq w_2\). Let \(\psi \in \mathcal{L}(\mathbb{C}^n)\). Assume that \(\psi\) is smooth in some open subset \(U \subset \mathbb{C}^n\). Then for any integer \(0 \leq m \leq n\), the measure

\[
\bigwedge_{1 \leq j \leq m} d\bar{c} \log(|z - w_j|^2 + |\cdot \wedge w_j|^2) \wedge (d\bar{c} \psi)^{n-m}
\]

is absolutely continuous with respect to the Lebesgue measure on \(U\).

Proof of theorem 3.1: We first assume that \(m = n\). Let \(K \subset \mathbb{C}^n\) be a compact set such that \(\langle d\bar{c}|z|^2 \rangle^n(K) = 0\). Set \(\Delta = \{(w, w, \ldots, w) : w \in \mathbb{C}^n\}\). Since \(\mu\) puts no mass at any point, it follows by Fubini’s theorem that \(\mu^{\otimes n}(\Delta) = 0\). By proposition 2.5

\[
\int_K (d\bar{c} \nu_{\mu})^n = \int_{(\mathbb{C}^n)^n \setminus \Delta} f(w_1, \ldots, w_n) d\mu^{\otimes n}(w_1, \ldots, w_n)
\]

where

\[
f(w_1, \ldots, w_n) = \int_K d\bar{c} \log(|z - w_1|^2 + |z \wedge w_1|^2) \wedge \cdots \wedge d\bar{c} \log(|z - w_n|^2 + |z \wedge w_n|^2)
\]

By Lemma 3.2, for any \((w_1, \ldots, w_n) \notin \Delta, f(w_1, \ldots, w_n) = 0\), hence \(\langle d\bar{c} \nu_{\mu} \rangle^n(K) = 0\). The case \(1 \leq m \leq n\) follows from Lemma 3.2 in the same way. The proof is complete.

Proof of theorem 1.2: As we have seen in the proof of Theorem 1.1, one can write on each coordinate chart \(U_k\),

\[
\mathcal{G}_{\mu}(\zeta) = m_k \mathcal{G}_{\mu_k} + \psi_k(z),
\]

where \(\psi_k \in \mathcal{L}(\mathbb{C}^n)\) is a smooth function in \(\mathbb{C}^n\). Using Theorem 3.1 again we conclude that \(\mathcal{G}_{\mu} \in DMA_{\text{loc}}(U_k)\). Therefore \(\mathcal{G}_{\mu} \in DMA_{\text{loc}}(\mathbb{P}^n)\).

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