Non-Universal Aging in the Long-Range Ising Model

Henrik Christiansen,1 Suman Majumder,1 Malte Henkel,2,3 and Wolfhard Janke1

1 Institut für Theoretische Physik, Universität Leipzig, IPF 231101, 04081 Leipzig, Germany
2 Laboratoire de Physique et Chimie Théoriques (CNRS UMR 7019), Université de Lorraine Nancy, 54506 Vandœuvre-lès-Nancy Cedex, France
3 Centro de Física Teórica e Computacional, Universidade de Lisboa, 1749-016 Lisboa, Portugal

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Aging in phase-ordering kinetics of the long-ranged $d = 2$ Ising model is studied via Monte Carlo simulations. The dynamical scaling and aging behavior is analyzed. The dynamical scaling of the spin-spin two-time autocorrelation function is best described by sub-aging in the regime of long-range interactions and by simple aging for effective short-range interactions. The sub-aging exponent $\mu$ and the nonequilibrium autocorrelation exponent $\lambda$ depend explicitly on the range parameter $\sigma$. 

Our data support the non-universal relation $\lambda = \sigma$ for $\sigma \leq 1$ and $\lambda = 1.25$ for $\sigma > 1$.

Change in properties of a system with its growing age, i.e., aging, is an omnipresent phenomenon, and, hence, is a subject of interest in diverse fields of research ranging from biology to physics [1–4]. Physical aging has been studied extensively in nonequilibrium systems from ferromagnets and spin glasses [5–7] to interface growth [8–11] as well as structural glasses [12, 13] and even collapsing polymers [14–16]. Here, we study aging of the phase-ordering kinetics of the long-ranged Ising model with long-range power-law decaying potential. This model can be seen as the most archetypical system for investigating the influence of interaction range, serving as a template, e.g., for Coulombic or dipolar interactions.

Aging can be probed via the two-time autocorrelation function

$$ C(t, t_w) = \langle \psi(\vec{r}, t) \psi(\vec{r}, t_w) \rangle $$

where $\psi$ is the space ($\vec{r}$) and time ($t$)-dependent order parameter, and $t_w (\leq t)$ is the waiting time. For coarsening dynamics, i.e., the nonequilibrium dynamics of a system subjected to a quench from a high-temperature phase into a low-temperature ordered phase or to the critical phase at the critical temperature $T_c$, simple aging is characterized by the absence of time-translation invariance and power-law scaling of $C(t, t_w)$ as

$$ C(t, t_w) = t_w^{-b} f_C(y), $$

where $y \equiv t/t_w$ and asymptotically for $t \gg t_w$ one has

$$ f_C(y) \simeq f_C(\infty) y^{-\lambda/z}. $$

Here $b$ is an a priori non-trivial aging exponent, $\lambda$ is the corresponding nonequilibrium autocorrelation exponent, and $z$ is the dynamical critical exponent. For quenches to temperature $T < T_c$, generally the exponent $b = 0$ which simplifies the scaling form (2) to $C(t, t_w) = f_C(t/t_w)$ [2].

The understanding of phase-ordering kinetics and aging in systems with strong long-range interactions has rarely been studied [17], largely due to the computational difficulties one encounters in their simulation. The “simplest” model one can think of incorporating long-range interactions is the power-law interacting long-range Ising model (LRIM) described by the Hamiltonian

$$ \mathcal{H} = - \sum_{i \neq j} J(r_{ij}) s_i s_j, \quad \text{with } J(r_{ij}) = \frac{1}{r_{ij}^{d+\sigma}}, $$

where the interaction strength $J(r_{ij})$ depends on the distance $r_{ij}$ between the spins $i$ and $j$, and can be tuned by the exponent $\sigma$. Here $s_i = \pm 1$ are the spins and $d$ is the spatial dimension. Although the equilibrium properties of the LRIM have been of increasing interest [13, 19], various aspects related to the nonequilibrium dynamics still remain elusive. For the time dependence of the growing characteristic length scale of the LRIM, Bray and Rutenberg predicted that

$$ \ell(t) \propto t^\alpha = t^{1/2} \begin{cases} t & \sigma < 1 \\ (t \ln t)^{1/2} & \sigma = 1 \\ t^{1/4} & \sigma > 1 \end{cases} $$

This prediction states that in the true long-range case of $\sigma < 1$, the growth exponent $\alpha = 1/z$ is $\sigma$-dependent, for $\sigma > 1$ the growth is identical to that in the nearest-neighbor Ising model (NNIM) with $z = 2$ [23, 24], and for the crossover value of $\sigma = 1$ the growth law is the same as in the NNIM, but with a multiplicative logarithmic correction. Note that (5) does not depend on $d$. This long-standing prediction has only recently been confirmed by us in $d = 2$ [25] employing a fast approach of doing Monte Carlo (MC) simulations of the LRIM and in Ref. [26] for $d = 1$. In the phase-ordering long-range spherical model, simple aging with $\lambda = d/2$, independently of $\sigma$, holds true [17]. For the $d = 1$ LRIM there also exists a recent study of aging in phase-ordering [27]. Based on data for $\sigma = 0.8$ and $\sigma = 3$, simple aging of the two-time spin-spin autocorrelator was reported. The
autocorrelation exponent was found to read $\lambda = 1/2$ for $\sigma \leq 1$ and $\lambda = 1$ for $\sigma > 1$. On the other hand, for the aging phenomena of the $d = 2$ LRIM there exists no understanding neither theoretically nor via simulations. Motivated by the above facts here we aim to explore the aging and dynamical scaling in the $d = 2$ LRIM numerically for the first time.

For the MC simulation of the LRIM given by the Hamiltonian in Eq. (1) we introduce the kinetics via single spin flips. A randomly chosen spin is flipped according to the standard Metropolis update with probability $\min\{1, \exp(-\Delta E/k_B T)\}$. Here $k_B = 1$ is the Boltzmann factor, $T$ is the temperature and $\Delta E$ is the change in energy before and after the flip. $N = L^d$ (where $L$ is the linear size of a square lattice) such attempts constitute one MC sweep, setting the time scale. Obviously, for LRIM the calculation of the energy change is the rate limiting step, as it involves all the spins in the considered lattice. However, following our recent approach of storing the effective field for each spin and updating it only when a spin flip is accepted makes such simulation significantly faster [23]. As an initial configuration at high temperature, we chose a square lattice with randomly distributed equal proportion of up and down spins. Since systems with long-range interaction suffer severely from finite-size effects we use Ewald summation [13, 14, 28, 29] to calculate an effective $J_{ij} = J(r_{ij})$ once at the beginning of the simulation. We chose $T_q = 0.1T_c$ as the quench temperature, where we extract $T_c$ from the data presented in Ref. [13]. All presented results are averaged over 50 independent runs for $L \leq 2048$ and 30 for $L = 4096$ (using different random number seeds).

Before exploring two-time quantities, we first illustrate in Fig. 1 the ordering kinetics for the LRIM with $\sigma = 0.6$ and linear system size $L = 4096$. The images within this figure are typical evolution snapshots showing the growth of domains of like spins. This growth is a scaling phenomenon, characterized by the scaling of various morphology-characterizing functions. For example the equal-time spin-spin correlation function scales as $C(r, t) \equiv C[r/\ell(t)]$, where $\ell(t)$ is the length scale at time $t$. Using this scaling property one can estimate $\ell(t)$ from the decay of $C(r, t)$ as an intersection with a constant value where we choose $C[r = \ell(t), t] = 0.5$. The main plot in Fig. 2 shows the time dependence of $\ell(t)$ in the LRIM for $\sigma = 0.6$. There the solid line represents the corresponding prediction made in Eq. (5) illustrating consistency of our data with the power law [13] for $\lambda = 0.6$. Statistical errors are of the order of the line width. The insets in (a) and (c) illustrate the quality of data collapse by plotting $\Delta C = C(t, t_w = 20) - C(t, t_w)$ for $t_w = 50, 100$, and 200 on a linear scale.

![FIG. 1. Plot of the characteristic length scale $\ell(t)$ as a function of time $t$, for the LRIM with $\sigma = 0.6$ and $L = 4096$ quenched to $T = 0.1T_c$. The solid line depicts the corresponding theoretical prediction in Eq. (5). The images within the frame of the plot are typical snapshots at the mentioned times obtained from a single run with identical parameters. In the snapshots, spins pointing up are marked in blue.](image1)

![FIG. 2. Double-log plot of the order-parameter autocorrelation function $C(t, t_w)$ against the scaling variables (a) $t/t_w$, (b) $\ell(t)/\ell(t_w)$, and (c) $h(t)/h(t_w)$ with $\mu = 0.976$ for the long-range Ising model with $\sigma = 0.6$ and $L = 4096$ when quenched to $T = 0.1T_c$. Here $h(t)$ is defined as in Eq. (4). The solid line in (c) shows the consistency of our data with the power law [13] for $\lambda = 0.6$. Statistical errors are of the order of the line width. The insets in (a) and (c) illustrate the quality of data collapse by plotting $\Delta C = C(t, t_w = 20) - C(t, t_w)$ for $t_w = 50, 100$, and 200 on a linear scale.](image2)
we suggest that sub-aging with corrections to scaling and simple aging was recovered when diluted ferromagnets was seen to be caused by coupling aging. Simple aging is the limit case and hence cannot be considered as a viable explanation of impossible since it violates basic consistency conditions on the data. However, according to Kurchan, super-aging is where and with respect to .

This prompted us to look for more general scaling forms. We checked the autocorrelator scaling with respect to

\[ C(t, t_w) = \tilde{f}_C \left( \frac{h(t)}{h(t_w)} \right) \simeq \tilde{f}_C \left( \frac{h(t)}{h(t_w)} \right)^{-\lambda/z} \]

where

\[ h(t) = h_0 \exp \left( \frac{t^{1-\mu} - 1}{1-\mu} \right) \]

and the exponent characterises the deviation from simple aging. Simple aging is the limit case , when . Un-recognised corrections to scaling might suggest that sub-aging with or even “super-aging” with might describe simulational or experimental data. However, according to Kurchan, super-aging is impossible since it violates basic consistency conditions and hence cannot be considered as a viable explanation of numerical data. Indeed, the apparent value of for diluted ferromagnets was seen to be caused by corrections to scaling and simple aging was recovered when was used as the scaling variable. Similar recovery, but from sub-aging to simple aging, was observed for the collapse of polymers. Sub-aging with for , however, has indeed been encountered many times in analytical, numerical, and experimental investigations, see for a list of examples. In long-range systems treated by the Hamiltonian mean-field (HMF) model, sub-aging was observed and traced to the contribution of the momenta. We show the observed scaling with respect to above , where provides the best data collapse. The overall scaling is greatly improved which is evidence for “true” sub-aging in the sense that no scaling is found when plotting against , but only when assuming sub-aging. To further consolidate this, we show \( \Delta C = C(t, t_w = 20) - C(t, t_w = 50, 100, 200) \) in the insets of Fig. (a) and (c). The data collapse in the early-time regime is clearly superior for sub-aging. Due to finite-size effects, we do not use the long-time data for this analysis. The finite-size effect can be perceived from the downward tendency of the data for large in Fig. (c). Figure shows the scaling of for (a) , (b) , (c) , (d) and (e). Here we find that the sub-aging exponent grows from for to for and for simple aging like, e.g., in the NNIM. Although the values appear very close to , the data collapse is highly sensitive to the choice of , as is also indirectly indicated by the three digit precision given for . It is, however, difficult to estimate reliable error bars, as the best data collapse cannot be (easily) quantified algorithmically. Since plotting vs. does not provide a clear functional dependence, we abstain from such an analysis.

Finally we turn our attention to the nonequilibrium autocorrelation exponent in Eq. ([31]). For the NNIM a lower bound exists and most simulations are compatible with . It
is unknown if a similar bound exists for systems with long-range interactions, however, one clearly expects to recover these results for large $\sigma$. To get a rough idea of the value of $\lambda$ for different $\sigma$ for the LRIM, we analyze the effective instantaneous exponent

$$
\lambda_{\text{eff}} = -z \begin{cases} 
\frac{d \ln[C(t, t_w)]}{d \ln(t)} & \sigma \leq 1 \\
\frac{d \ln[C(t, t_w)]}{d \ln(t)} & \sigma > 1 
\end{cases}
$$

(8)

When plotted against $1/t$, $\lambda_{\text{eff}}$ is expected to go to $\lambda_{\text{eff}} = \lambda$ for $1/t \to 0$. In Fig. 4 such a plot is shown for different values of $\sigma$ with fixed $t_w = 50$. Finite-size effected data are excluded from an analysis for different system sizes. Clearly, in $d = 2$ the lower bound $\lambda \geq 1$ mentioned above for the NNIM [3, 37] does not hold for the LRIM. By extrapolating the data to $1/t \to 0$, we conjecture that

$$
\lambda = \begin{cases} 
\sigma & \sigma \leq 1 \\
1.25 & \sigma > 1 
\end{cases}
$$

(9)

To further reinforce this result, we next check the asymptotic consistency of our data with the power law \[9\]. In Fig. 4(c) the solid black line is a power law decay with $\lambda = 0.6 = \sigma$ fixed and $z = 1.6$ taken from the prediction \[5\]. The data for $L = 4096$ thus indeed appear consistent with our conjecture. To get a more complete idea, we repeat this analysis also for the other values of $\sigma$. The solid lines in Fig. 4 are power laws of form \[6\] with $\lambda = \sigma$ and the dotted lines are of the same form with $\lambda = 1.25$. For $\sigma \neq 1$, $z$ was taken from the prediction \[5\], whereas for $\sigma = 1$ we take $z = 2$ without the logarithmic correction as it is well known that logarithmic factors which arise in single-time quantities such as the length scale $\ell(t)$ do cancel out in two-time quantities \[12, 43\]. In Fig. 4(c) for $\sigma = 1$ we plot both the power law for $\lambda = 1.25$ and $\lambda = \sigma$ as it is \textit{a priori} not clear whether the jump in $\lambda$ occurs at $\sigma = 1$, as could be inferred from prediction \[5\]. From this figure, it is apparent that $\lambda = 1.0$ is consistent with our data, while $\lambda = 1.25$ is not.

To summarize, we have performed the first investigation of aging in the two-dimensional long-ranged Ising model. While \textit{simple aging} does not lead to a satisfactory data collapse for strongly long-ranged interactions, a good data collapse is obtained assuming a sub-aging scaling form, where the sub-aging exponent $\mu$ approaches the short-range limit for increasing $\sigma$. For the nonequilibrium autocorrelation exponent, we empirically find the non-universal relation of $\lambda = \sigma$ for $\sigma \leq 1$ and $\lambda = 1.25$ for $\sigma > 1$.

It would of course be very intriguing to find a theoretical explanation for the relationship between $\lambda$ and $\sigma$ or the apparent discontinuity of $\lambda$ at $\sigma = 1$, also in sight of the recent results for the one-dimensional long-range Ising model [27]. To complete the dimensional analysis an extension to the investigation of aging in the long-range Ising model with dimension $d = 3$ [44] is worthwhile. Also of interest is the extension to other models with long-range interactions, e.g., the long-range XY model or long-range interacting polymers [44].

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