QUANTUM SHARED BROADCASTING

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Abstract

A generalization of quantum broadcasting protocol is presented. Here the goal is to copy an unknown input state into two subsystems which partially overlap. We show that the possibility of implementing these protocols strongly depends upon the overlap among the subsystems. Conditions for approximated shared broadcasting are analyzed.

1 Introduction

The no-cloning theorem [1] is an important property of quantum mechanics which follows from the linear structure of the theory. In its weaker version the no-cloning theorem formalizes the physical impossibility of creating a machine that produces exact copies of an unknown, given quantum state. More generally it prevents us to construct a machine such that, randomly choosing one of two non-orthogonal states of a system, it will produce perfect copies of such state.

Together with entanglement (of which it is a direct consequence) the no-cloning theorem contributed in modifying our approach to information theory and in refining our intuition about quantum-based information processing [2] with profound consequences both in quantum computation and in quantum communication. In particular it played an important role in the development of quantum error correction techniques [3] by preventing one from having codes that create redundant copies of every state of a quantum system.

In recent years several generalizations of the no-cloning theorem have been proposed to include the possibility of imperfect copies. In particular this yielded a proof of the impossibility of cloning with arbitrary high fidelity [4, 5, 6, 7] an unknown quantum state. Furthermore, Barnum et al. [8] introduced the idea of quantum broadcasting to deal with mixed input state. Differently from
the original setup [1], in a broadcasting scenario it is not required to produce factorizable copies of the original input states. Instead one is allowed to create a joint (possible entangled) many-body output state composed by subsystems which locally reproduce the original input state. In this settings it has been shown that quantum broadcasting is still prohibited in the case of single input copy and two output copies [8] but can be done when starting from a sufficient number of copies [9].

No-cloning and quantum broadcasting proved to be an important investigation tool for characterizing the quantum capacity [10] of quantum communication channels [11]. Using these results in conjunction with the degradability properties [12] of a quantum communication line one can show that channels which are anti-degradable [13] must have zero quantum capacity (see for instance Ref. [14]).

In the present paper we would like to discuss a weaker version of broadcasting that we name Quantum Shared Broadcasting (QSB). In its simplest form the scheme is described in Fig. 1. As in the standard broadcasting scenario we have a source system $S$ which provide us with a set of unknown input states $|\psi_S\rangle$ that we would like to duplicate into two output systems I and II. In the present case however we do not require the output systems to be independent. Instead we assume a partial overlap between them. As shown in the picture, we parameterize such overlap by introducing a subsystem $A$ which belongs to both the output systems and representing I and II as the composed systems $AB$ and $AC$ respectively (here $B$ and $C$ are two independent spaces). QSB succeeds when, for a given unknown input state $|\psi\rangle$ of $S$, the output density matrix of $ABC$ is such that both its restrictions on $I=AB$ and $II=AC$ contain a copy of $|\psi\rangle$. As we will see the possibility of realizing such a transformation strongly depends upon the dimensionality of the shared subsystem $A$. In particular, perfect shared broadcasting (i.e. a QSB which produces perfect copies of the states $|\psi\rangle$) cannot be achieved if $A$ is smaller than the source system $S$, while it is trivially allowed if $S$ fits in $A$. Similar results apply in the case of imperfect QSB where one is interested in getting only approximated copies of the $|\psi\rangle$'s: in this case we provide a threshold which connects the achievable fidelity and the dimension of $A$.

As in the case of no-cloning, the possibility of performing QSB transformations has profound consequences in quantum communication. In particular this appears to be a fundamental step for analyzing the communication performances of joint channels. For instance QSB can be used to determine whether or not it is possible to boost the quantum communication efficiency of a given quantum channel by adding an anti-degradable (zero-quantum capacity) quantum channel to it. Even though the quantum channel capacity seems to be a super-additive quantity [15], anti-degradable channels play probably the role of neutral elements in this context.

The paper is organized as follows. We start in Section 2 analyzing the case of perfect broadcasting showing that this is possible only if the source system $S$ can fit within the overlap subsystem $A$. In Section 3 instead we focus on the imperfect broadcasting case providing a threshold which connects the dimension
Figure 1: Shared broadcasting scheme: states of the source system $S$ are simultaneously copied into the output systems I and II which share a common subsystem $A$. In this setting $B$ and $C$ represent the parts of I and II which do not overlap (in other words I is the joint system $AB$ while II is $AC$). Standard broadcasting is a particular instance of shared broadcasting in which $A$ is kept into a fix reference state. Perfect shared broadcasting is possible if and only if $A$ is bigger or equal to the source space $S$.

of $A$ with the fidelity of the two copies with the input state. The paper then ends with a perspectives and conclusions section.

2 Perfect Quantum Shared Broadcasting

Let $S$, $A$, $B$ and $C$ be quantum systems described by the Hilbert spaces $\mathcal{H}_S$, $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ having dimensions $d_S$, $d_A$, $d_B$ and $d_C$ respectively. We call $S$ the source system and $A$, $B$, and $C$ the output systems. Without loss of generality we will assume that $d_S \leq d_A d_B$ and $d_S \leq d_A d_C$. Under these hypothesis it is always possible to find isometries $V_{ABS}$ and $V_{ACS}$ connecting $\mathcal{H}_S$ to $\mathcal{H}_{AB}$ and $\mathcal{H}_{AC}$ respectively, that allow us to “represent” states of $S$ as states of $AB$ and $AC$.

We are interested in finding a completely positive, trace preserving (CPT) map $\mathcal{N}$ from $S$ to $A, B, C$ which would allow us to “copy” any vector of $\mathcal{H}_S$ into $AB$ and $AC$ (see Fig. 1). Specifically, given $|\psi_S\rangle \in \mathcal{H}_S$ let us define the density
matrix

\[ \rho_{ABC} \equiv \mathcal{N}(|\psi_S\rangle\langle\psi_S|) \]  

with \( \rho_{AB} \equiv \text{Tr}_C[\rho_{ABC}] \) and \( \rho_{AC} \equiv \text{Tr}_B[\rho_{ABC}] \) its reduced density matrices associated with \( AB \) and \( AC \). Our goal is to find \( \mathcal{N} \) and isometries \( V_{ABS} : \mathcal{H}_S \to \mathcal{H}_{AB} \) and \( V_{ACS} : \mathcal{H}_S \to \mathcal{H}_{AC} \) such that the fidelities \[ \frac{|\langle S' | \rho_{AB}|S\rangle|^2}{\langle S | \rho_{AB}|S\rangle} \] \( \text{among} \) the input state \(|\psi_S\rangle\) and \( \rho_{AB}, \rho_{AC} \) are equal to one for all input states \(|\psi_S\rangle \in \mathcal{H}_S\), i.e.

\[
\begin{align*}
F(\rho_{AB}:|\psi_{AB}\rangle) &\equiv \langle \psi_{AB}|\rho_{AB}|\psi_{AB}\rangle = 1 \\
F(\rho_{AC}:|\psi_{AC}\rangle) &\equiv \langle \psi_{AC}|\rho_{AC}|\psi_{AC}\rangle = 1 ,
\end{align*}
\]

where \( |\psi_{AB}\rangle \equiv V_{ABS}|\psi_S\rangle \) and \( |\psi_{AC}\rangle \equiv V_{ACS}|\psi_S\rangle \) are respectively the “representations” of \(|\psi_S\rangle\) defined in terms of the isometries \( V_{ABS} \) and \( V_{ACS} \). A channel \( \mathcal{N} \) which satisfies Eq. (2) for all \(|\psi_S\rangle \) of \( \mathcal{H}_S \) is said to be a perfect Quantum Shared Broadcasting map. Standard broadcasting \[8\] can be obtained from this by constraining \( \mathcal{N} \) to act trivially on \( A \) requiring for instance that \( \rho_{ABC} \) to be of the form \( \rho_{BC} \otimes \rho_A \) with \( \rho_A \) fixed.

We shall see that the possibility of achieving shared broadcasting strongly depends upon the ratio between the dimensions of the source space \( S \) and \( A \): if \( S \) is small enough to entirely fit inside \( A \) then a perfect QSB map exists, if instead \( S \) is bigger than \( A \) such a map cannot be defined.

**Theorem 1:** Maps \( \mathcal{N} \) which perform perfect QSB exist if and only if \( A \) is sufficiently big to contain \( S \), i.e. if and only if \( d_S \leq d_A \). When this happens \( \mathcal{N} \) can be chosen to be an isometry.

**Proof:** Showing that \( \mathcal{N} \) exists when \( d_S \leq d_A \) is trivial. Indeed when this happens there exists always an isometry \( V_{AS} \) which connects \( \mathcal{H}_S \) with \( \mathcal{H}_A \), i.e. \( V_{AS}|\psi_S\rangle = |\psi_A\rangle \), for all \(|\psi_S\rangle\). Expand then \( V_{AS} \) to construct the isometry \( V_{ABS} \) from \( \mathcal{H}_S \) and \( \mathcal{H}_A \otimes \mathcal{H}_B \) introduced in Eq. (2). This can be done for instance by imposing the conditions \( V_{ABS}|\psi_S\rangle = |\psi_A \otimes 0_B\rangle \), with \(|0_B\rangle\) being some reference vector of \( B \). Do the same for \( AC \) by introducing a reference vector \(|0_C\rangle\) on \( C \), i.e. \( V_{ACS}|\psi_S\rangle = |\psi_A \otimes 0_C\rangle \). Consider then the transformation which for \(|\psi_S\rangle \in \mathcal{H}_S \) gives \(|\psi_S\rangle \to |\psi_A \otimes 0_B \otimes 0_C\rangle \). This is clearly CPT since it is an isometry. Moreover it satisfies the conditions (2).

Let us now consider the case in which \( d_S \geq d_A + 1 \). We will prove the thesis by contradiction showing that if such \( \mathcal{N} \) does exist then one can violate the standard no-broadcasting theorem for vectors belonging to a two-dimensional subspace \( \mathcal{H}_0 \) of \( \mathcal{H}_S \) — i.e. it would be possible to construct a broadcasting machine that creates two perfect copies (one in \( B \) and the other in \( C \)) of any vectors of \( \mathcal{H}_0 \).

We start observing that the condition (2) implies that the density matrices \( \rho_{AB} \) and \( \rho_{AC} \) are pure. Specifically, for \( X = B,C \) we must have \( \rho_{AX} = \)
This implies immediately that the global density matrix $\rho_{ABC}$ can be written as
\[
\rho_{ABC} = |\psi_{AB}\rangle\langle \psi_{AB}| \otimes \rho_C = |\psi_{AC}\rangle\langle \psi_{AC}| \otimes \rho_B ,
\]
with $\rho_C$ and $\rho_B$ density matrices which (in principle) may still depend upon $|\psi_S\rangle$. Take then the partial trace with respect to $B$ (or $C$) of both the second and the third term of Eq. (3). By doing so one arrives to the conclusion that for all $|\psi_S\rangle$, \(i\) $\rho_{ACB}$ must be pure, \(ii\) $\rho_{ACB}$ must be separable with respect to $A$, $B$ and $C$. That is
\[
\rho_{ABC} = |\phi_A\rangle\langle \phi_A| \otimes |\phi_B\rangle\langle \phi_B| \otimes |\phi_C\rangle\langle \phi_C| ,
\]
where $|\phi_A\rangle$, $|\phi_B\rangle$ and $|\phi_C\rangle$ are (not necessarily identical) normalized vectors of $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ whose dependence upon the input $|\psi_S\rangle$ will be determined in the following. For $X = B, C$ this gives
\[
\rho_{AX} = |\phi_A\rangle\langle \phi_A| \otimes |\phi_X\rangle\langle \phi_X| ,
\]
which, due to Eq. (4), fixes the action of the isometries $V_{ABS}$ and $V_{ACS}$ by imposing the conditions
\[
V_{AXS}|\psi_S\rangle = |\phi_A \otimes \phi_X\rangle .
\]
Apply this to an orthonormal basis $\{|k_S\rangle; k = 1, \cdots , d_S\}$ of $\mathcal{H}_S$, i.e.
\[
V_{AXS}|k_S\rangle = |\phi^{(k)}_A \otimes \phi^{(k)}_X\rangle ,
\]
with $|\phi^{(k)}_A\rangle$ and $|\phi^{(k)}_X\rangle$ defined as in Eq. (5). Since isometries preserve inner product we must have
\[
\langle \phi^{(k)}_A | \phi^{(k')}_A \rangle \langle \phi^{(k)}_X | \phi^{(k')}_X \rangle = \delta_{kk'} ,
\]
for all $k, k' \in 1, \cdots , d_S$. Now we remind that the maximum number of mutually orthonormal vectors in $A$ is $d_A$. Therefore since $d_S > d_A$, there must exist at least a couple of $k, k'$ (say $k = 1$ and $k' = 2$) such that $\langle \phi^{(1)}_A | \phi^{(2)}_A \rangle \neq 0$. Consequently Eq. (5) implies that
\[
\langle \phi^{(1)}_B | \phi^{(2)}_B \rangle = \langle \phi^{(1)}_C | \phi^{(2)}_C \rangle = 0 .
\]
Consider then the two-dimensional subspace $\mathcal{H}_0 \subseteq \mathcal{H}_S$ generated by $|1_S\rangle$ and $|2_S\rangle$, i.e. the set of normalized vectors $|\psi_S\rangle = \alpha |1_S\rangle + \beta |2_S\rangle$. From Eq. (7) and the linearity of $V_{AXS}$ follows that their “representations” on $AX$ are
\[
|\psi_{AX}\rangle = V_{AXS}|\psi_S\rangle = \alpha |\phi^{(1)}_A \otimes \phi^{(1)}_X\rangle + \beta |\phi^{(2)}_A \otimes \phi^{(2)}_X\rangle .
\]
According to Eq. (6) this should be separable with respect to the bipartition $A - X$. From the orthogonality conditions (9) this is only possible if $|\phi_A^{(1)}\rangle = e^{i\varphi} |\phi_A^{(2)}\rangle$ for some (irrelevant) phase $\varphi$. Hence for all vectors of $\mathcal{H}_0$ we can write

$$V_{AXS}|\psi_S\rangle = |\phi_A^{(1)}\rangle \otimes \left(\alpha |\phi_X^{(1)}\rangle + e^{i\varphi} \beta |\phi_X^{(2)}\rangle\right)$$

$$= |\phi_A^{(1)}\rangle \otimes W_{XS}|\psi_S\rangle,$$  \hspace{1cm} (11)

with $W_{BS}$ and $W_{CS}$ being isometries which map $\mathcal{H}_0$ into $B$ and $C$ according to the rules $W_{XS}|1_s\rangle = |\phi_X^{(1)}\rangle$, $W_{XS}|2_s\rangle = e^{i\varphi} |\phi_X^{(2)}\rangle$. Replacing this into Eq. (9) we have finally

$$\rho_{AB} = |\phi_A^{(1)}\rangle\langle\phi_A^{(1)}| \otimes W_{BS}|\psi_S\rangle\langle\psi_S|W_{BS}^{\dagger}$$

$$\rho_{AC} = |\phi_A^{(1)}\rangle\langle\phi_A^{(1)}| \otimes W_{CS}|\psi_S\rangle\langle\psi_S|W_{CS}^{\dagger},$$ \hspace{1cm} (12)

which shows that the vectors $|\psi_S\rangle \in \mathcal{H}_0$ have been copied into $B$ and $C$. This is impossible due to the no-broadcasting theorem. Therefore $\mathcal{N}$ cannot exist. □

## 3 Approximated Quantum Shared Broadcasting

A simple generalization of the perfect QSB protocol introduced in the previous section is obtained by requiring the fidelities among the output copies and the input states to be higher than a certain fixed threshold:

**Definition 1:** Given $\varepsilon \in (0,1]$ we say that $\varepsilon$-QSB from S to ABC is possible if there exists a channel $\mathcal{N}: S \rightarrow ABC$ and isometries $V_{ABS}: S \rightarrow AB, V_{ACS}: S \rightarrow AC$ such that for an arbitrary input state $|\psi_S\rangle \in \mathcal{H}_S$ we have

$$\begin{cases}
F(\rho_{AB};|\psi_{AB}\rangle) > 1 - \varepsilon \\
F(\rho_{AC};|\psi_{AC}\rangle) > 1 - \varepsilon,
\end{cases}$$

where $\rho_{AB}, \rho_{AC}, |\psi_{AB}\rangle$, and $|\psi_{AC}\rangle$ are defined as in Eq. (8). We say that Approximated Quantum Shared Broadcasting (AQB) from S is possible if for any $\varepsilon \in (0,1]$ one can find subsystems $A_\varepsilon \subseteq A, B_\varepsilon \subseteq B, C_\varepsilon \subseteq C$, channel $\mathcal{N}^{(\varepsilon)}: S \rightarrow A_\varepsilon B_\varepsilon C_\varepsilon$ and isometries $V_{ABS}^{(\varepsilon)}: S \rightarrow A_\varepsilon B_\varepsilon, V_{ACS}^{(\varepsilon)}: S \rightarrow A_\varepsilon C_\varepsilon$ which realize an $\varepsilon$-QSB of S.

In the following sections we will give conditions which relate $\varepsilon$ to $d_A$ and $d_S$ that are necessary for implementing $\varepsilon$-QSB protocols and hence AQB. Before doing so we notice however that if $d_A < d_S$ and the dimensionality of $BC$ is bounded then impossibility of AQB follows from the impossibility of perfect QSB. This can be shown by contradiction by noticing that if AQB would be possible than the sets of channels $\mathcal{N}^{(\varepsilon)}$ and isometries $V_{ABS}^{(\varepsilon)}, V_{ACS}^{(\varepsilon)}$ would be compact. Therefore letting $\varepsilon \rightarrow 0$ one will find a limiting channel $\mathcal{N}^{(0)}$ and limiting isometries $V_{ABS}^{(0)}, V_{ACS}^{(0)}$, which fulfill the perfect QSB impossibility of which was established in Sec. 2.
3.1 Notation and preliminary results

To deal with the approximations of Eq. (13) we find it useful to review some basic properties of the fidelity that will be extensively used in the remaining part of the manuscript:

i) **Transitivity:** Let $\rho$, $\omega$ and $\sigma$ be density matrices, then the following triangle inequality holds

$$\sqrt{F(\rho; \omega)} \geq 1 - \sqrt{1 - F(\rho; \sigma)} - \sqrt{1 - F(\sigma; \omega)},$$

(14)

which shows that if $\rho$ and $\omega$ are “close” to $\sigma$ then they must be “close” to each other too. Equation (14) can be established by relating $F(\rho; \sigma)$ with the trace distance $D(\rho; \sigma) \equiv (1/2)\text{Tr}|\rho - \sigma|$ through the inequality $1 - \sqrt{F(\rho; \sigma)} \leq D(\rho; \sigma) \leq \sqrt{1 - F(\rho; \sigma)}$. Furthermore Eq. (13) can be strengthened if at least one of the density matrices $\rho$ and $\omega$ represents a pure states. Indeed in this case one gets

$$F(\rho; |\psi\rangle) \geq 1 - \sqrt{1 - F(\rho; \sigma)} - \sqrt{1 - F(\sigma; |\psi\rangle)}.$$

(15)

ii) **Monotonicity under partial trace and purification:** By Bures-Uhlmann theorem [16] one can easily verify that the fidelity of the density matrices $\rho_{AB}$ and $\sigma_{AB}$ of a joint system $AB$ is always smaller than or equal to the fidelity of the corresponding reduced density matrices $\rho_A \equiv \text{Tr}_B[\rho_{AB}]$ and $\sigma_A \equiv \text{Tr}_B[\sigma_{AB}]$, i.e.

$$F(\rho_{AB}; \sigma_{AB}) \leq F(\rho_A; \sigma_A).$$

(16)

In converse direction, by the same theorem one can also verify that for any purification $|\varphi_{AB}\rangle$ of $\rho_A$ there exists a purification $|\chi_{AB}\rangle$ of $\sigma_A$ such that

$$F(\rho_A; \sigma_A) \leq F(|\varphi_{AB}\rangle; |\chi_{AB}\rangle).$$

(17)

iii) **Convexity:** Given a density matrix $\rho$ and a vector $|\psi\rangle$,

$$F(\rho; |\psi\rangle) \leq F(|\phi\rangle; |\psi\rangle),$$

(18)

$$F(\rho; |\psi\rangle) \leq \lambda_{\text{max}},$$

(19)

where $\lambda_{\text{max}}$ is the maximal eigenvalue of $\rho$ and $|\phi\rangle$ the corresponding eigenvector.

Using the above properties it is relatively easy to generalize the identity (1). Specifically one can show that if Eq. (13) holds for all input states $|\psi_S\rangle$, then there should exist (not necessarily identical) pure states $|\phi_A\rangle$, $|\phi_B\rangle$ and $|\phi_C\rangle$ of
A, B, and C such that the output states $\rho_{ABC}$ of the channel are uniformly “close” to the tensor product states $|\phi_A \otimes \phi_B \otimes \phi_C\rangle$, while the isometric representations $|\psi_{AB}\rangle$ and $|\psi_{AC}\rangle$ of $|\psi_S\rangle$ are uniformly “close” to $|\phi_A \otimes \phi_B\rangle$ and $|\phi_A \otimes \phi_C\rangle$, respectively.

Lemma 1: If $\epsilon$-QSB from $S$ to $ABC$ is possible for some given value $\epsilon \in (0, 1]$, then for all input states $|\psi_S\rangle$ one can find $|\phi_A\rangle$, $|\phi_B\rangle$, and $|\phi_C\rangle$ such that

$$F(\rho_{ABC}; |\phi_A \otimes \phi_B \otimes \phi_C\rangle) > 1 - 3 \epsilon^{1/8},$$
$$F(|\psi_{AX}\rangle; |\phi_A \otimes \phi_X\rangle) > 1 - \epsilon',$$

where $\rho_{ABC}$ and $|\psi_{AX}\rangle$ are defined as in Eqs. (7) and (3), while $\epsilon' = 2 \epsilon^{1/2}$ for $X = B$, and $\epsilon' = 3.4 \epsilon^{1/8}$ for $X = C$. (The asymmetry among the B and C is a consequence of the relative freedom one has in defining the vectors $|\phi_A\rangle$, $|\phi_B\rangle$, and $|\phi_C\rangle$).

Proof: Purify the output state $\rho_{ABC}$ to the vector $|\varphi_{ABCE}\rangle$ with $E$ being an auxiliary system. Since $|\varphi_{ABCE}\rangle$ is purification of $\rho_{AB}$ and $\rho_{AC}$, then by the condition (13) and by the property ii) of the fidelity we have

$$F(|\varphi_{ABCE}\rangle; |\psi_{AB} \otimes \psi'_{CE}\rangle) > F(\rho_{AB}; |\psi_{AB}\rangle) > 1 - \epsilon,$$
$$F(|\varphi_{ABCE}\rangle; |\psi_{AC} \otimes \psi'_{BE}\rangle) > F(\rho_{AC}; |\psi_{AC}\rangle) > 1 - \epsilon,$$

(22)

for some $|\psi'_{CE}\rangle$, $|\psi'_{BE}\rangle$ (this simply follows from the fact that any purification of a pure state is factorizable). By triangle inequality (15) and by monotonicity under partial trace (16) we then obtain

$$F(|\psi_{AB} \otimes \psi'_{CE}\rangle; |\psi_{AC} \otimes \psi'_{BE}\rangle) > 1 - 2\sqrt{\epsilon},$$

(23)

$$F(|\psi_{AB}\rangle; \sigma_A \otimes \sigma'_B) > 1 - 2\sqrt{\epsilon},$$
$$F(|\psi'_{CE}\rangle; \sigma_C \otimes \sigma'_E) > 1 - 2\sqrt{\epsilon},$$

(24) (25)

with $\sigma_A = \text{Tr}_C(|\psi_{AC}\rangle\langle\psi_{AC}|)$, $\sigma_C = \text{Tr}_A(|\psi_{AC}\rangle\langle\psi_{AC}|)$, $\sigma'_B = \text{Tr}_E(|\psi'_{BE}\rangle\langle\psi'_{BE}|)$, and $\sigma'_E = \text{Tr}_B(|\psi'_{BE}\rangle\langle\psi'_{BE}|)$. Let us now define $|\phi_A\rangle$, $|\phi_B\rangle$, $|\phi_C\rangle$, and $|\phi_E\rangle$ as the eigenvectors associated with the maximal eigenvalues of the density matrices $\sigma_A$, $\sigma_B$, $\sigma_C$ and $\sigma_E$, respectively. According to the property (13) we then have

$$F(|\psi_{AB}\rangle; |\phi_A \otimes \phi_B\rangle) > F(|\psi_{AB}\rangle; \sigma_A \otimes \sigma_B)$$
$$> 1 - 2\sqrt{\epsilon},$$
$$F(|\psi_{CE}\rangle; |\phi_C \otimes \phi_E\rangle) > F(|\psi_{CE}\rangle; \sigma_C \otimes \sigma_E)$$
$$> 1 - 2\sqrt{\epsilon}.$$

(26) (27)

The first one already proves Eq. (21) for $X = B$. To proceed apply the triangle inequality (15) to Eqs. (22) and (26). This yields

$$F(|\varphi_{ABCE}\rangle; |\phi_A \otimes \phi_B \otimes \psi'_{CE}\rangle) > 1 - (\epsilon^{1/4} + \sqrt{2})\epsilon^{1/4}$$
$$> 1 - 2.5 \epsilon^{1/4}.$$
Again by triangle inequality between (27) and (28) we get

\[
F(|\varphi_{ABCE}; |\phi_A \otimes \phi_B \otimes \phi_C \otimes \phi_E)\rangle > 1 - \left(\sqrt{2} \varepsilon^{1/8} + \sqrt{\varepsilon^{1/4} + \sqrt{2}}\right)^{1/8}
\]

\[
> 1 - 3.0 \varepsilon^{1/8},
\]

which, by partial trace with respect to \(E\), gives Eq. (20).

To derive the case \(X = C\) of Eq. (21) we first apply to Eqs. (23) and (26) the triangle inequality (15) and then the monotonicity under partial trace (16), obtaining

\[
F(|\psi_{AC} \otimes \psi_{BE}; |\phi_A \otimes \phi_B \otimes \psi'_C \otimes \phi_E)\rangle > 1 - 2\sqrt{2} \varepsilon^{1/4},
\]

(30)

\[
F(|\psi_{AC}; |\phi_A \otimes \psi'_C\rangle \langle \phi_A | \otimes \sigma_C) > 1 - 2\sqrt{2} \varepsilon^{1/4},
\]

(31)

with \(\sigma_C \equiv \text{Tr}_E[|\psi'_C\rangle \langle \psi'_C|]\). Taking \(|\psi'_C\rangle\) the eigenvector of \(\sigma_C\), associated with its maximal eigenvalue and invoking the property (18) we then have,

\[
F(|\psi_{AC}; |\phi_A \otimes \sigma_C\rangle \langle \phi_A | \otimes \omega_C) > 1 - 2\sqrt{2} \varepsilon^{1/4}.
\]

(32)

By tracing with respect to \(A\) we then get

\[
F(|\phi_C; |\phi'_C\rangle) > F(\sigma_C; |\phi'_C\rangle) > 1 - 2\sqrt{2} \varepsilon^{1/4},
\]

(33)

with \(\sigma_C\) and \(|\phi_C\rangle\) as in Eq. (27). This together with Eq. (32) finally gives

\[
F(|\psi_{AC}; |\phi_A \otimes \phi_C\rangle > 1 - 2\sqrt{2} \varepsilon^{1/4}.
\]

(34)

which proves Eq. (21) for \(X = C\).

\[\Box\]

### 3.2 Conditions for approximated \(\varepsilon\)-QSB

Starting from the results of the previous section we now show that for \(d_S > d_A\), \(\varepsilon\)-QSB transformations cannot be realized if the parameter \(\varepsilon\) is below a certain finite threshold \(\varepsilon_0\) that we have estimated as

\[
\varepsilon_0 = \min \left\{0.6 \times 10^{-175}; 2.4 \times 10^{-14} d_A^{-8}\right\}.
\]

(35)

This bound is not optimal and can probably be improved. However, it does not depend upon the dimension of the output states \(B\) and \(C\) and implies that AQSB transformations are not physical for \(d_S > d_A\).

**Theorem 2:** Let \(d_S > d_A\), then it is not possible to have \(\varepsilon\)-QSB transformations from \(S\) to \(ABC\) with \(B\) and \(C\) generic quantum systems, for values of
ε which are smaller than or equal to ε₀ of Eq. (35).

For clarity we split the proof into three parts. In part one we show that the output states are "close" to a set of factorizable and "almost" orthogonal states of the output states (1) associated with an orthonormal basis of the source system elements of the selected basis of S. As in the perfect QSB case, the finite size of A will allows us to identify two elements of the selected basis of S whose output images on B and C separately are described by almost orthogonal vectors. In part two we will use this result to identify a two-dimensional subspace of S whose image on A is "almost" constant. Finally, in part three we will show that the channel N yields "good" copies of such subspace on B and C. The threshold on ε will follows by direct comparison of the resulting fidelities with the optimal cloning values [6, 7]. In Fig. 2 we summarize the main passages of the derivation.

Proof of Theorem 2: Part one

Assuming that ε-QSB from S to ABC is possible for some given ε ∈ (0, 1], consider an arbitrary orthonormal basis \{k_S: k = 1, \cdots, d_S\} of \(H_S\) and denote with \(k_{AB}\) and \(k_{AC}\) their isometric representations on \(AB\) and \(AC\), and with \(\phi_A^{(k)}, \phi_B^{(k)}\) and \(\phi_C^{(k)}\) the corresponding pure vectors of A, B and C which satisfy the conditions of Lemma 1, e.g.

\[
F(k_{AX}; |\phi_A^{(k)} \otimes \phi_X^{(k)}|) > 1 - \varepsilon',
\]

for \(X = B, C\) and for all \(k\). Exploiting the orthogonality of \(k_S\) and using the triangle inequality one can show that \{\(\phi_A^{(1)} \otimes \phi_B^{(1)}\), \cdots, \(\phi_A^{(d_S)} \otimes \phi_B^{(d_S)}\)\} and \{\(\phi_A^{(1)} \otimes \phi_C^{(1)}\), \cdots, \(\phi_A^{(d_S)} \otimes \phi_C^{(d_S)}\)\} are two sets of "almost" orthogonal vectors, i.e.

\[
|\langle \phi_A^{(k')} \otimes \phi_X^{(k')} | \phi_A^{(k)} \otimes \phi_X^{(k)} \rangle| = |\langle \phi_A^{(k')} | \phi_A^{(k)} \rangle \langle \phi_X^{(k')} | \phi_X^{(k)} \rangle| < \varepsilon'',
\]

for all \(k \neq k'\) and with \(\varepsilon''\) being a small quantity depending on ε. In particular, a rough estimation from Eq. (21) gives \(\varepsilon'' = 2\sqrt{\varepsilon' + \varepsilon'}\), i.e. \(\varepsilon'' \approx 4.9 \varepsilon^{1/4}\) for \(X = B\) and \(\varepsilon'' \approx 7.1 \varepsilon^{1/16}\) for \(X = C\). Now we invoke the following result.

Lemma 2: If \{\(\phi^{(i)}\), \cdots, \(\phi^{(m)}\)\} is a collection of \(m > d\) unit vectors in a Hilbert space \(H\) of dimension \(d\) satisfying the condition

\[
|\langle \phi^{(i)} | \phi^{(j)} \rangle| < \xi,
\]

for all \(i \neq j\), then \(\xi > \sqrt{m-d \over d(m-1)}\).

Proof: Define the state \(\rho = \sum_{j=1}^m |\phi^{(j)}\rangle \langle \phi^{(j)}| / m\) and use the equation \(\text{Tr}[\rho^2] \geq 1/d\).
Lemma 2 requires that for any collection of \( m \geq d + 1 \) vectors of a \( d \)-dimensional space there must be at least two whose scalar product is greater than \( 1/d \). Since in our case \( d_S \geq d_A + 1 \) this implies that among the vectors \(|\phi_A^{(1)}\rangle, \ldots, |\phi_A^{(d_S)}\rangle\) of \( \mathcal{H}_A \) there must exist at least two elements (say \( k = 1 \) and \( k = 2 \)) which satisfy the inequality

\[
F(|\phi_A^{(1)}\rangle; |\phi_A^{(2)}\rangle) = |\langle \phi_A^{(2)} | \phi_A^{(1)} \rangle|^2 \geq 1/d_A^2 .
\]

(39)

Taking then \( \varepsilon'' \leq 1/d_A^2 \) and replacing it into Eq. (37) we get

\[
|\langle \phi_X^{(2)} | \phi_X^{(1)} \rangle| < \sqrt{\varepsilon''} ,
\]

(40)

which shows that \(|\phi_B^{(1)}\rangle\) and \(|\phi_C^{(1)}\rangle\) are almost orthogonal to \(|\phi_B^{(2)}\rangle\) and \(|\phi_C^{(2)}\rangle\), respectively.

**Proof of Theorem 2: Part two**

We now use Eq. (40) to improve the inequality (39) showing that for small \( \varepsilon \) the vectors \(|\phi_A^{(1)}\rangle\) and \(|\phi_A^{(2)}\rangle\) are indeed close. To do so let us focus on the two-dimensional subspace \( \mathcal{H}_0 \) of \( \mathcal{H}_S \) formed by the superpositions

\[
|\psi_S\rangle = \alpha|1_S\rangle + \beta|2_S\rangle ,
\]

(41)

with \( \alpha \) and \( \beta \) complex amplitudes. Their representations \(|\psi_{AX}\rangle = V_{AXS}|\psi_S\rangle\) can be expressed as

\[
|\psi_{AX}\rangle = \alpha|1_{AX}\rangle + \beta|2_{AX}\rangle ,
\]

(42)

where we used the linearity of \( V_{AXS} \). Let us now define the state

\[
|\tilde{\phi}_X^{(2)}\rangle \equiv e^{i\theta} \left| \phi_X^{(2)} \right\rangle - \frac{\langle \phi_X^{(1)} | \phi_X^{(2)} \rangle |\phi_X^{(1)}\rangle}{\sqrt{1 - |\langle \phi_X^{(1)} | \phi_X^{(2)} \rangle|^2}} ,
\]

(43)

with \( \theta \) being a phase factor that will be defined later on. The vector \(|\tilde{\phi}_X^{(2)}\rangle\) is orthogonal to \(|\phi_X^{(1)}\rangle\) and, thanks to Eq. (40), is close to \(|\phi_X^{(2)}\rangle\), i.e. \(|\tilde{\phi}_X^{(2)}\rangle \approx |\phi_X^{(2)}\rangle\). We will now show that, for all \(|\psi_S\rangle \in \mathcal{H}_0\), the representations (12) can be faithfully expressed in terms of superpositions of the orthonormal states \(|\phi_A^{(1)} \otimes \phi_X^{(1)}\rangle\) and \(|\phi_A^{(2)} \otimes \tilde{\phi}_X^{(2)}\rangle\). Indeed, taking into account the inequalities Eq. (21) and the conditions (37) and (40), one can verify that there is a suitable choice of phase \( \theta \), such that for all \( \alpha \) and \( \beta \) we have

\[
F(|\psi_{AX}\rangle; \alpha|\phi_A^{(1)} \otimes \phi_X^{(1)}\rangle + \beta|\phi_A^{(2)} \otimes \tilde{\phi}_X^{(2)}\rangle) > 1 - \varepsilon'''
\]

(44)

with \( \varepsilon''' = 3\sqrt{\varepsilon''} + \sqrt{\varepsilon''} + \varepsilon'' \), i.e. \( \varepsilon''' \approx 11.4 \varepsilon^{1/8} \) for \( X = B \) and \( \varepsilon'' \approx 15.2 \varepsilon^{1/32} \) for \( X = C \). According to Eq. (21) the vector \(|\psi_{AX}\rangle\) should also be close to a
The idea is now to use Eq. (48) together with the transitivity and monotonicity of the fidelity we can hence derive the following inequalities

\[
F(\ket{\phi_A} \otimes \ket{\phi_X}; \alpha \ket{\phi_A^{(1)}} \otimes \ket{\phi_X^{(1)}} + \beta \ket{\phi_A^{(2)}} \otimes \ket{\phi_X^{(2)}}) \\
> 1 - \left( \sqrt{\varepsilon'} + \sqrt{\varepsilon''} \right),
\]

(45)

\[
F(\ket{\phi_A}; |\alpha|^2 \ket{\phi_A^{(1)}}, |\beta|^2 \ket{\phi_A^{(2)}}) \\
> 1 - \left( \sqrt{\varepsilon'} + \sqrt{\varepsilon''} \right).
\]

(46)

In the limit of small \(\varepsilon\), the latter expression shows that, independently of the coefficient \(\alpha\) and \(\beta\) the density matrix on the left hand side is close to a pure state. This can only happen if the vectors \(\ket{\phi_A^{(1)}}\) and \(\ket{\phi_A^{(2)}}\) are indeed almost parallel. The easiest way to verify this is by computing the maximal eigenvalue \(\lambda_{\text{max}}\) of the density matrix \(|\alpha|^2 \ket{\phi_A^{(1)}}, |\beta|^2 \ket{\phi_A^{(2)}}\) and by requiring that, for all choices of \(\alpha\) and \(\beta\) it should be greater than or equal to the fidelity associated with Eq. (46) — see the convexity property (12) of the fidelity. This gives

\[
\lambda_{\text{max}} = \frac{1 + \sqrt{1 - 4 |\alpha\beta|^2 \left[ 1 - F(\ket{\phi_A^{(1)}}; |\phi_A^{(2)}\rangle) \right]}}{2},
\]

(47)

and hence

\[
F(\ket{\phi_A^{(1)}}; |\phi_A^{(2)}\rangle) > 1 - 4(\sqrt{\varepsilon'} + \sqrt{\varepsilon''}) \\
> 1 - 19.4 \varepsilon'^{1/16},
\]

(48)

where the values of \(\varepsilon'\) and \(\varepsilon''\) of the case \(X = B\) has been employed to get the best scaling.

**Proof of Theorem 3: Part three**

The idea is now to use Eq. (48) together with the transitivity and monotonicity conditions of the fidelity to show that for all \(|\psi_S\rangle\) of Eq. (11) the reduced density matrices \(\rho_{AX}\) are close to \(\ket{\phi_A^{(1)}} \otimes \left( \alpha \ket{\phi_X^{(1)}} + \beta e^{i\theta'} \ket{\phi_X^{(2)}} \right)\) with the constant \(\theta'\) accounting for the relative phase between \(|\phi_A^{(1)}\rangle\) and \(|\phi_A^{(2)}\rangle\). Indeed we first notice that

\[
F\left( \alpha \ket{\phi_A^{(1)}} \otimes \ket{\phi_X^{(1)}} + \beta \ket{\phi_A^{(2)}} \otimes \ket{\phi_X^{(2)}}; \right. \\
\left. \alpha \ket{\phi_A^{(1)}} \otimes \ket{\phi_X^{(1)}} + \beta e^{i\theta'} \ket{\phi_A^{(1)}} \otimes \ket{\phi_X^{(2)}} \right) \\
\geq F(\ket{\phi_A^{(1)}}; |\phi_A^{(2)}\rangle) > 1 - 19.4 \varepsilon'^{1/16}.
\]

(49)

Exploiting the triangle inequality (15) twice we can thus use Eqs. (13) and (44) to show that

\[
F\left( \rho_{AX}; \alpha \ket{\phi_A^{(1)}} \otimes \ket{\phi_X^{(1)}} + \beta e^{i\theta'} \ket{\phi_A^{(1)}} \otimes \ket{\phi_X^{(2)}} \right) \\
> 1 - \varepsilon_{iv},
\]

(50)
where for $X = B$ one has $\varepsilon_{iv} = 3.8 \varepsilon^{1/64}$ while for $X = C$ one has $\varepsilon_{iv} = 3.9 \varepsilon^{1/128}$. The monotonicity property of the fidelity can then be invoked to verify that the reduced density matrices $\rho_X$ are close to the vector $\alpha |\phi_X^{(1)} + \beta e^{i\theta'}|\tilde{\phi}_X^{(2)}\rangle$, where similarly to Eq. (12), $W_{XS}$ is an isomorphism from $H_0 \in H_S$ to $H_X$ which maps $W_{XS}|1_S\rangle = |\phi_X^{(1)}\rangle$, $W_{XS}|2_S\rangle = e^{i\theta'}|\tilde{\phi}_X^{(2)}\rangle$, i.e.

$$F(\rho_X; W_{XS}|\psi_S\rangle) > 1 - \varepsilon_{iv}. \quad (51)$$

The above expression shows that the QSB channel $N$ produces output states (1) whose reduced density matrices on $B$ and $C$ are approximated copies of the states $|\psi_S\rangle$ of the two-dimensional subspace $H_0 \in H_S$. The resulting transformation

$$|\psi_S\rangle \rightarrow \begin{cases} \rho_B = \text{Tr}_C[N(|\psi_S\rangle\langle\psi_S|)] \\ \rho_C = \text{Tr}_B[N(|\psi_S\rangle\langle\psi_S|)] \end{cases} \quad (52)$$

is indeed an (approximated) $1 \rightarrow 2$ quantum cloner. Accordingly if $\varepsilon$-QSB could be realized for arbitrarily small $\varepsilon$, then the fidelity of such copies with the input states would become arbitrarily close to one. This however is prevented by the fact that the fidelities of any $1 \rightarrow 2$ cloning devices are bounded from above by the value $5/6$ [6, 7]. By comparing this with Eq. (51), and by taking into account the condition $\varepsilon'' \leq 1/d_A^2$ introduced in Eq. (10), we get the threshold of Eq. (35): for values of $\varepsilon$ smaller than such $\varepsilon_0$ we can hence conclude that $\varepsilon$-QSB maps cannot be implemented.

This complete the proof of Theorem 2. ■
4 Conclusions

Quantum Broadcasting protocols have been generalized to include the possibility of producing output copies on partially overlapping systems. In this context we have shown that perfect Quantum Shared Broadcasting is possible if and only if the overlap among the output systems is sufficiently large to include all possible input states. We have also analyzed the case of imperfect copies proving the existence of a finite upper bound $\varepsilon_0$ on the achievable fidelities below which no approximated QSB can be performed when $d_S > d_A$. Since the derivation of such threshold has been obtained by imposing only some of (but not all) the necessary conditions on the output states of the channels the reported value for $\varepsilon_0$ is probably not optimal. The characterization of the ultimate value for $\varepsilon_0$ and its application in the context of quantum capacity characterization is currently under investigation.

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