Supplementary Materials for

Network structural origin of instabilities in large complex systems

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Supplementary Text

In Sec. S1 below, we establish a general relation between the Jacobian and adjacency matrices for a broad class of nonlinear network systems with multi-dimensional node dynamics. In Sec. S2, we provide mathematical proofs of almost-sure nonnormality and reactivity for random weighted directed networks generated by the GCLV model. In Sec. S3, we detail computational procedures used to obtain the results in fig. S4 and table S1.

S1. Nonnormality and reactivity of Jacobian vs. adjacency matrices

Here, we consider a class of network systems that are more general than Eq. (1):

\[ \dot{x}_i = f(x_i) + \sum_{j=1}^{n} A_{ij} h(x_j), \quad i = 1, \ldots, n, \quad (S1) \]

where vector \( x_i \in \mathbb{R}^m \) represents the dynamical state of node \( i \), function \( f(\cdot) \in \mathbb{R}^m \) describes the node dynamics, function \( h(\cdot) \in \mathbb{R}^m \) captures how the node is coupled to the rest of the network, and \( A \) is the possibly weighted \( n \times n \) adjacency matrix of the network (but note that the arguments below would remain unchanged if \( A \) is replaced with the Laplacian matrix \( L \)). We assume that the system has an equilibrium \( x^* = (x_1^*T, x_2^*T, \ldots, x_n^*T)^T \in \mathbb{R}^N \), \( N := mn \), and seek to analyze its stability. We linearize the system at \( x^* \) and assume that \( D_x f(x_1^*) = D_x f(x_2^*) = \cdots = D_x f(x_n^*) =: F \). This assumption is valid under either of the following conditions: 1) \( f(\cdot) \) is a linear function (which is the case, e.g., for power grids and many mechanical networks); 2) \( x^* \) is a synchronous state, i.e., \( x_1^* = x_2^* = \cdots x_n^* \), or can be transformed into a synchronous state by a suitable change of coordinates (as often done, e.g., for biological clocks). We also assume that \( D_x h(x) = \zeta(x)H \) for some scalar-valued function \( \zeta(\cdot) \) and constant matrix \( H \). Under these assumptions, the \( N \times N \) Jacobian matrix \( M \) of the network system takes the form

\[ M = I_n \otimes F + A^* \otimes H, \quad (S2) \]

where \( \otimes \) denotes the Kronecker product, \( A^* \) is defined by \( A_{ij}^* = \zeta(x_j^*)A_{ij} \), and we recall that \( I_n \) is the identity matrix of size \( n \). Equation (1) corresponds to a special case of this formulation with \( m = 1 \), \( F = -\alpha \), and \( \zeta(x) = H = 1 \) (and hence \( M = -\alpha I_n + A \)).

Letting \( A^* = VAV^{-1} \) be the eigen-decomposition of \( A^* \), we have

\[ M = (V \otimes I_m)[I_n \otimes F + \Lambda \otimes H](V^{-1} \otimes I_m). \quad (S3) \]
It follows from this that, for each eigenvalue \( \lambda \) of \( A^* \), the eigenvalues of the matrix
\[
C(\lambda) := F + \lambda H
\]  
are also eigenvalues of the Jacobian matrix \( M \). Writing the Jordan decomposition of \( C(\lambda) \) as
\[
C(\lambda) = Y(\lambda) \Sigma(\lambda) Y(\lambda)^{-1},
\]
we can express the Jordan decomposition of the Jacobian matrix \( M \) in terms of \( Y(\lambda) \) and \( \Sigma(\lambda) \) as
\[
M = R \Phi R^{-1},
\]
where
\[
R = [v_1 \otimes Y(\lambda_1), v_2 \otimes Y(\lambda_2), \cdots, v_n \otimes Y(\lambda_n)],
\]
\[
\Phi = \begin{bmatrix}
\Sigma(\lambda_1) & O & \cdots & O \\
O & \Sigma(\lambda_2) & \ddots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & \Sigma(\lambda_n)
\end{bmatrix},
\]
and \( v_i \) denotes the right eigenvector associated with the \( i \)th eigenvalue \( \lambda_i \) of \( A^* \). We note that the Jacobian matrix \( M \) is normal if and only if all of its eigenvectors are orthogonal to each other (i.e., \( R^\dagger R = I_N \), where \( R^\dagger \) denotes the conjugate transpose of \( R \)). Since the \( (i, j) \) block of \( R^\dagger R \) can be written as
\[
(R^\dagger R)_{ij} = (v_i^\dagger v_j) \cdot (Y(\lambda_i)^\dagger Y(\lambda_j)),
\]
a necessary and sufficient condition for \( M \) to be nonnormal can be written as
\[
\| R^\dagger R - I_N \|_F^2 = \sum_i \| Y(\lambda_i)^\dagger Y(\lambda_i) - I_m \|_F^2 + 2 \sum_{i<j} (v_i^\dagger v_j)^2 \cdot \| Y(\lambda_i)^\dagger Y(\lambda_j) \|_F^2 > 0,
\]
where the first sum represents the contribution coming from the nonnormality of \( C(\lambda) \) while the second represents the contribution from the nonnormality of \( A^* \). Note that \( |v_i^\dagger v_j| \) measures the non-orthogonality of the eigenvectors \( v_i \) and \( v_j \) and that we generically have \( \| Y(\lambda_i)^\dagger Y(\lambda_j) \|_F^2 > 0 \) if \( \lambda_i \neq \lambda_j \). Consequently, if \( A^* \) for a given network is nonnormal, i.e., \( v_i^\dagger v_j \neq 0 \) for some \( i \) and \( j \), then generically the corresponding Jacobian matrix \( M \) must also be nonnormal. While the converse does not generally hold (i.e., the nonnormality of \( M \) does not imply that of \( A^* \)), the decomposition in Eq. (S8) clearly shows that the nonnormality of \( M \) must come from the node dynamics if it does not come from the network structure (i.e., if \( A^* \) is normal).

An analogous decomposition holds true also for reactivity. To see this, we first note that having distinct left and right eigenvectors associated with the rightmost eigenvalue is a sufficient
condition for reactivity (which is proved in materials and methods). Let \( y_k(\lambda) \) denote the \( k \)th column of the matrix \( Y(\lambda) \), i.e., the \( k \)th eigenvector of \( C(\lambda) \). Let \( \sigma_1(\lambda_1) \) denote the rightmost eigenvalue of \( M \), which is also an eigenvalue of \( C(\lambda_1) \). Then, \( v_1 \otimes y_1(\lambda_1) \) is the eigenvector of \( M \) associated with \( \sigma_1(\lambda_1) \), where \( y_1(\lambda_1) \) is the eigenvector of \( C(\lambda_1) \) associated with \( \sigma_1(\lambda_1) \). Since having distinct left and right eigenvectors associated with \( \sigma_1(\lambda_1) \) is equivalent to the non-orthogonality between \( v_1 \otimes y_1(\lambda_1) \) and some other right eigenvector of \( M \), a sufficient condition for \( M \) to be reactive can be expressed as

\[
\sum_{k=1}^{m} (y_1(\lambda_1)\dagger y_k(\lambda_1))^2 + \sum_{j=2}^{n} (v_1\dagger v_j)^2 \cdot \|y_1(\lambda_1)\dagger Y(\lambda_j)\|_F^2 > 0.
\]

(S9)

Similarly to the case of nonnormality, this condition shows that \( A^* \) being reactive (i.e., \( |v_1\dagger v_j| > 0 \) for some \( j \)) generically implies \( M \) being reactive. (In the derivation of Eqs. (S8) and (S9), we implicitly assumed that \( A^* \) is diagonalizable for simplicity, but this assumption can be lifted using the Jordan transformation to derive similar decomposition.)

Thus, if \( A^* \) is nonnormal and reactive with high probability in a class of network systems, then the Jacobian matrix \( M \) is nonnormal and reactive with high probability as well. For the weighted GCLV model, the almost-sure nonnormality and reactivity of \( A \) proved in Sec. S2.4 below suggest that \( \tilde{A} \) (with the modified link weight distribution) and thus the Jacobian matrix \( M \) in Eq. (S2) would also be nonnormal and reactive almost surely in the limit of large network sizes.

S2. Proof of nonnormality and reactivity for almost all large networks

Here, we prove that, for the GCLV model with a given joint distribution of in- and out-degrees and a distribution of link weights, the probability that the network’s adjacency matrix \( A \) is nonnormal and the probability that \( A \) is reactive both converge to one in the limit of large network size, \( n \to \infty \). The proofs are valid regardless of whether we allow for self-links or not, as they play no role in the arguments. We first prove key convergence properties of the connection probabilities \( \rho_{ij} \) in the model (Sec. S2.1). We then present proofs for the almost-sure nonnormality (Sec. S2.2) and the almost-sure reactivity (Sec. S2.3) for the case of unweighted GCLV model (i.e., assuming \( A_{ij} = 0 \) or \( 1 \)). Finally, we show how the proofs can be extended to weighted networks and also to Laplacian-coupled networks (Sec. S2.4).
S2.1. Convergence properties of connection probabilities in the GCLV model

Here, we first show that both \( \max_{1 \leq i \leq n} \rho_{ij} \) for any fixed \( j \) and \( \max_{1 \leq j \leq n} \rho_{ij} \) for any fixed \( i \) converge to zero in probability as \( n \to \infty \). This property is essential for the proofs in the sections below. To prove this property for \( \max_{1 \leq i \leq n} \rho_{ij} \), we first note that \( \rho_{ij} \geq 0 \), which implies that the convergence in probability is equivalent to

\[
\lim_{n \to \infty} \mathbb{P} \left( \max_{1 \leq i \leq n} \rho_{ij} > \epsilon \right) = 0 \quad \text{for all } \epsilon > 0. \tag{S10}
\]

Since \( \max_{1 \leq i \leq n} \rho_{ij} = \max_{1 \leq i \leq n} \tilde{d}_{i}^{\text{in}} / \sum_{k=1}^{n} \tilde{d}_{k}^{\text{in}} \) by the definition of \( \rho_{ij} \) in Eq. (8), and thus \( \max_{1 \leq i \leq n} \tilde{d}_{i}^{\text{in}} / \sum_{k=1}^{n} \tilde{d}_{k}^{\text{in}} = \max_{1 \leq i \leq n} \sum_{k=1}^{n} \tilde{d}_{k}^{\text{in}} / d_{j}^{\text{out}} \), it follows from Eq. (9) that

\[
\lim_{n \to \infty} \mathbb{P} \left( \max_{1 \leq i \leq n} \rho_{ij} > \epsilon \right) = \lim_{n \to \infty} \mathbb{P} \left( \frac{\tilde{d}_{j}^{\text{out}}}{d_{j}^{\text{out}}} > \frac{\tilde{d}_{j}^{\text{out}} - b_{n}}{a_{n}} \right) \]
\[
= \lim_{n \to \infty} \left[ 1 - G_{\gamma} \left( \frac{\tilde{d}_{j}^{\text{out}} - b_{n}}{a_{n}} \right) \right] \]
\[
= \lim_{n \to \infty} \left[ 1 - G_{\gamma} \left( \frac{\tilde{d}_{j}^{\text{out}} - b_{n}}{a_{n}} \right) \right] = 0 \tag{S11}
\]

for any \( \epsilon > 0 \), where the last equality is due to the assumption \( \lim_{n \to \infty} a_{n} / n = \lim_{n \to \infty} b_{n} / n = 0 \), and further due to strong law of large numbers, \( \frac{\epsilon}{n} \sum_{k=1}^{n} \tilde{d}_{k}^{\text{in}} / d_{j}^{\text{out}} \to \epsilon d_{i}^{\text{in}} / d_{j}^{\text{out}} > 0 \) as \( n \to \infty \) (recalling the definition \( \tilde{d} = \mathbb{E}(\tilde{d}_{k}^{\text{in}}) \)). This proves that \( \max_{1 \leq i \leq n} \rho_{ij} \) converges to 0 in probability as \( n \to \infty \).

The convergence of \( \max_{1 \leq j \leq n} \rho_{ij} \) in probability can be proved following the same argument, with index \( i \) replaced by \( j \) and the “in” superscript replaced by “out” in appropriate places.

In addition, \( \rho_{ij} \) satisfies a similar but slightly different property: \( \max_{1 \leq i \leq n} \rho_{in} \) converges to zero in probability as \( n \to \infty \). To see this, we first note that, for any \( a > 0 \), we have

\[
\left\{ \max_{1 \leq i \leq n} \rho_{in} > \epsilon \right\} = \left\{ \max_{1 \leq i \leq n} \rho_{in} > \epsilon \text{ and } a_{n} \leq a \right\} \cup \left\{ \max_{1 \leq i \leq n} \rho_{in} > \epsilon \text{ and } a_{n} > a \right\}, \tag{S12}
\]
and thus
\[ P\left( \max_{1 \leq i \leq n} \rho_{in} > \varepsilon \right) = P\left( \max_{1 \leq i \leq n} \rho_{in} > \varepsilon \text{ and } \tilde{d}_{in}^n \leq a \right) + P\left( \max_{1 \leq i \leq n} \rho_{in} > \varepsilon \text{ and } \tilde{d}_{out}^n > a \right) \]
\[ \leq P\left( \max_{1 \leq i \leq n} \rho_{in} > \varepsilon \text{ and } \tilde{d}_{in}^n \leq a \right) + P\left( \tilde{d}_{out}^n > a \right) \leq P\left( \tilde{d}_{max}^n > \frac{\varepsilon \sum_{k=1}^{n} \tilde{d}_{in}^k}{a} \right) + P\left( \tilde{d}_{out}^n > a \right) \]
\[ \leq P\left( \tilde{d}_{max}^n > \frac{\varepsilon \sum_{k=1}^{n} \tilde{d}_{in}^k}{a} \right) + \tilde{d} \]  
(S13)
where the last inequality is due to Markov’s inequality. Following the same procedure as in Eq. (S11), we have
\[ \lim_{n \to \infty} P\left( \max_{1 \leq i \leq n} \rho_{in} > \varepsilon \right) \leq \lim_{n \to \infty} \left[ 1 - G_{\gamma}\left( \frac{\varepsilon \sum_{k=1}^{n} \tilde{d}_{in}^k / (a - b_n / n)}{a_n / n} \right) \right] + \frac{\tilde{d}}{a} = \frac{\tilde{d}}{a}. \]  
(S14)
Since \( a > 0 \) can be chosen arbitrarily, we conclude that
\[ \lim_{n \to \infty} P\left( \max_{1 \leq i \leq n} \rho_{in} > \varepsilon \right) = 0, \]  
(S15)
i.e., \( \max_{1 \leq i \leq n} \rho_{in} \) converges to 0 as \( n \to \infty \). Again, the convergence of \( \max_{1 \leq i \leq n} \rho_{nj} \) in probability can be proved by following the same argument with index \( i \) replaced by \( j \) and the “in” superscript replaced by “out” in appropriate places. Combining the results above, we see that \( \varepsilon_i(n) := \max_{1 \leq j \leq n} (\rho_{ij} + \rho_{ji}) \to 0 \) for any fixed \( i \) and that \( \varepsilon_n(n) \to 0 \) as \( n \to \infty \). If the support of the distribution of expected degrees is bounded (i.e., constrained to the finite interval \([0, d_{max}]\)), the convergence \( \rho_{ij} \to 0 \) occurs with probability one and is uniform over all \( 1 \leq i, j \leq n \), since we have \( \sum_{k=1}^{n} \tilde{d}_{in}^k / n \to \tilde{d} > 0 \) almost surely and thus have \( \max_{1 \leq i, j \leq n} \rho_{ij} \leq \frac{1}{n} d_{max}^2 / (\sum_{k=1}^{n} \tilde{d}_{in}^k / n) \to 0 \) (independently of \( i \) and \( j \)).

### S2.2. Proof of nonnormality

We now show that \( A \) is nonnormal with probability approaching one as \( n \to \infty \) under the assumption that \( A_{ij} \in \{0, 1\} \) (the proof will be extended to general weighted \( A_{ij} \) in Sec. S2.4). Since a sufficient condition for \( A \) to be nonnormal is that there exists \( 1 \leq i \leq n \) such that \( d_{in}^i \neq d_{out}^i \) (i.e., at least one term is strictly positive in the first summation in Eq. (3)), we have
\[ P(A \text{ is nonnormal}) \geq P(d_{in}^i \neq d_{out}^i \text{ for some node } i) = E(Q(n)), \]  
(S16)
where we defined

\[
Q(n) := \mathbb{P}(d^\text{in}_i \neq d^\text{out}_i \text{ for some node } i \mid \tilde{d}^\text{in}_1, \ldots, \tilde{d}^\text{in}_n, \tilde{d}^\text{out}_1, \ldots, \tilde{d}^\text{out}_n),
\]

i.e., the conditional probability that the sufficient condition is satisfied given a realization of the (random) expected degrees \(d^\text{in}_1, \ldots, \tilde{d}^\text{in}_n, \tilde{d}^\text{out}_1, \ldots, \tilde{d}^\text{out}_n\). We note that \(Q(n)\) itself is a random variable because the expected degrees are random in the GCLV model. Below, we will show that \(Q(n) \to 1\) in probability as \(n \to \infty\), implying that its expected value \(\mathbb{E}(Q(n)) \to 1\), and thus that \(\mathbb{P}(A \text{ is nonnormal}) \to 1\), in view of Eq. (S16).

To estimate \(Q(n)\), we first seek to estimate an analogous conditional probability for a given node \(i\):

\[
Q_i(n) := \mathbb{P}(d^\text{in}_i \neq d^\text{out}_i \mid \tilde{d}^\text{in}_1, \ldots, \tilde{d}^\text{in}_n, \tilde{d}^\text{out}_1, \ldots, \tilde{d}^\text{out}_n).
\]

We note that, for a given realization of the expected degrees, the difference between the (actual) in- and out-degrees of node \(i\) is a sum of independent random variables: \(d^\text{in}_i - d^\text{out}_i = \sum_j Y_{ij}\), where \(Y_{ij} := A_{ij} - A_{ji}\) has mean \(\mu_{ij} := \rho_{ij} - \rho_{ji}\), variance \(\sigma^2_{ij} := \rho_{ij} + \rho_{ji} - \rho^2_{ij} - \rho^2_{ji}\), and finite third moment \(h_{ij} := \mathbb{E}(|Y_{ij} - \mu_{ij}|^3)\) for \(i \neq j\). Since \(Y_{ii} = 0\), its mean \(\mu_{ii}\), variance \(\sigma^2_{ii}\), and third moment \(h_{ii}\) are all equal to zero. Since \(Y_{ij} \in \{-1, 0, 1\}\), we have

\[
\frac{h_{ij}}{\sigma^2_{ij}} = \frac{\mathbb{E}(|Y_{ij} - \mu_{ij}|^3)}{\mathbb{E}(|Y_{ij} - \mu_{ij}|^2)} \leq \max_{x \in \{-1,0,1\}} \frac{|x - \mu_{ij}|^3}{|x - \mu_{ij}|^2} = \max_{x \in \{-1,0,1\}} |x - \rho_{ij} + \rho_{ji}| \leq 1 + \rho_{ij} + \rho_{ji}
\]

for \(i \neq j\), where the first inequality follows from the general inequality \(\sum_i a_i / \sum_i b_i \leq \max_i (a_i / b_i)\). We note that the standard deviation \(s_i\) of \(\sum_j Y_{ij}\) is given by \(s^2_i := \sum_j \sigma^2_{ij}\). According to the Berry–Esseen Theorem (59), the distribution of the standardized sum \(\sum_j (Y_{ij} - \mu_{ij}) / s_i\) converges to the standard normal distribution with the approximation error bounded as

\[
\sup_{x \in \mathbb{R}} |F_i(x) - \Phi(x)| \leq C_0 \sum_{j=1}^n \mathbb{E}\left(\left|\frac{Y_{ij} - \mu_{ij}}{s_i}\right|^3\right) = \frac{C_0}{s^3_i} \sum_{j=1}^n h_{ij} \leq \frac{C_0}{s^3_i} \sum_{j=1}^n (\rho_{ij} + \rho_{ji})^2 + 1,
\]

where we denote the CDF of the random variable \(\sum_j (Y_{ij} - \mu_{ij}) / s_i\) by \(F_i(x)\) and the CDF of standard normal distribution by \(\Phi(x)\), and \(C_0\) is a constant, which is known (60) to satisfy \(0.40 \leq C_0 \leq 0.56\). The last inequality in Eq. (S20) follows from Eq. (S19) and the definition.
of $\sigma_{ij}$. Observing that
\[
Q_i(n) \geq 1 - \mathbb{P}(-\xi < d_i^{\text{in}} - d_i^{\text{out}} \leq \xi) \\
= 1 - \mathbb{P}(\sum_j Y_{ij} \leq \xi) + \mathbb{P}(\sum_j Y_{ij} \leq -\xi) \\
\geq 1 - \Phi\left(\frac{\xi - \sum_j \mu_{ij}}{s_i}\right) + \Phi\left(-\frac{\xi - \sum_j \mu_{ij}}{s_i}\right) - \frac{2C_0}{s_i^3} \sum_{j=1}^n (\rho_{ij} + \rho_{ji})^2 ,
\]
(S21)
for any $\xi > 0$ and taking the limit $\xi \to 0$, we obtain a lower bound for $Q_i(n)$:
\[
Q_i(n) \geq P_i(n) := 1 - \frac{2C_0}{s_i} - \frac{2C_0}{s_i^3} \sum_{j=1}^n (\rho_{ij} + \rho_{ji})^2 .
\]
(S22)

We now estimate $Q(n)$ using a recursive argument involving $Q_i(n)$ and $P_i(n)$. For a given $1 \leq i \leq n$, let $f_i(n)$ denote the conditional probability that the subnetwork induced by nodes $1, \ldots, i$ satisfies the sufficient condition for nonnormality, i.e., there exists $1 \leq j \leq i$ for which the in- and out-degrees of node $j$ defined within the subnetwork are distinct (given a realization of all the expected degrees). For $i = n$, we have $f_n(n) = Q(n)$. For $i = 1$, we have $f_1(n) = 0$, since the adjacency matrix of a single isolated node is always normal. For $i = 2$, a direct calculation yields $f_2(n) = \rho_{12}(1 - \rho_{21}) + \rho_{21}(1 - \rho_{12})$, noting that the adjacency matrix elements $A_{12}$ and $A_{21}$ are the only random variables involved. For the general case, by considering the addition of node $i$ to the subnetwork induced by nodes $1, \ldots, i-1$ (and the links between node $i$ and the subnetwork in both directions), we have the following inequality for any $i \geq 2$:
\[
f_i(n) \geq f_{i-1}(n) \cdot \min_{1 \leq j < i} ((1 - \rho_{ji})(1 - \rho_{ij}) + \rho_{ji}\rho_{ij}) + (1 - f_{i-1}(n))Q_i(n),
\]
(S23)
which implies
\[
f_i(n) \geq f_{i-1}(n) \cdot (1 - \bar{\epsilon}_i(n) - Q_i(n)) + Q_i(n),
\]
(S24)
where we define $\bar{\epsilon}_i(n) := \max_{1 \leq j < i}(\rho_{ji} + \rho_{ij})$. Noting that the r.h.s. of Eq. (S24) is monotonically increasing in $Q_i(n)$ and using $Q_i(n) \geq P_i(n)$ from Eq. (S22), this leads to
\[
f_i(n) \geq f_{i-1}(n) \cdot (1 - \bar{\epsilon}_i(n) - P_i(n)) + P_i(n) \\
\geq f_{i-1}(n) \cdot (1 - P_i(n)) + P_i(n) - \bar{\epsilon}_i(n),
\]
(S25)
which is equivalent to
\[
1 - f_i(n) \leq (1 - f_{i-1}(n)) \cdot (1 - P_i(n)) + \bar{\epsilon}_i(n).
\]
(S26)
Recursively applying this inequality, we obtain

\[
1 - Q(n) = 1 - f_n(n) \leq \tilde{\varepsilon}_n(n) + \prod_{i=2}^{n} (1 - P_i(n)) + \sum_{j=3}^{n} \tilde{\varepsilon}_{j-1}(n) \prod_{i=j}^{n} (1 - P_i(n)). \quad \text{(S27)}
\]

Since the derivation of the bound in Eq. (S27) does not depend on a particular ordering of the nodes. By reversing the order of the nodes, i.e., interchanging node \(i\) with node \(n - i + 1\), we can rewrite this relation as

\[
1 - Q(n) \leq \tilde{\varepsilon}_n(n) + \prod_{i=1}^{n-1} (1 - P_i(n)) + \sum_{j=1}^{n-2} \tilde{\varepsilon}_{j+1}(n) \prod_{i=1}^{j} (1 - P_i(n)). \quad \text{(S28)}
\]

We would thus prove our claim if we show that the r.h.s. converges to zero in probability (recalling that all these terms are random since \(\rho_{ij}\) and \(s_i\) are).

To help estimate the r.h.s. of Eq. (S28), we consider the double sequence \(g_{jn}\) defined by

\[
g_{jn} = \begin{cases} 
\frac{1}{j} \sum_{i=1}^{j} P_i(n), & j \leq n, \\
\frac{1}{n} \sum_{i=1}^{n} P_i(n), & j > n. 
\end{cases} \quad \text{(S29)}
\]

According to the Moore-Osgood Theorem (61), if \(\lim_{j \to \infty} g_{jn}\) exists for every \(n\) and \(\lim_{n \to \infty} g_{jn}\) exists for every \(j\), with the convergence in the latter limit uniform in \(j\), then \(\lim_{j,n \to \infty} g_{jn}\) exists (regardless of how \(j\) and \(n\) are taken to \(\infty\)) and can be calculated as iterated limits, i.e.,

\[
\lim_{j,n \to \infty} g_{jn} = \lim_{j \to \infty} \lim_{n \to \infty} g_{jn} = \lim_{n \to \infty} \lim_{j \to \infty} g_{jn}. \quad \text{(S30)}
\]

We now show that the required conditions are satisfied, so that the theorem can be applied. First, we see that \(\lim_{j \to \infty} g_{jn} = \frac{1}{n} \sum_{i=1}^{n} P_i(n)\) for every fixed \(n\), since \(g_{jn}\) itself equals \(\frac{1}{n} \sum_{i=1}^{n} P_i(n)\) and does not vary with \(j\) for any \(j > n\) by the definition of \(g_{jn}\). Next, we compute \(\lim_{n \to \infty} g_{jn}\) for a given \(j\). From the definition of \(g_{jn}\) for \(j \leq n\) in Eq. (S29) and the definition of \(P_i(n)\) in Eq. (S22), we have

\[
\lim_{n \to \infty} g_{jn} = \frac{1}{j} \sum_{i=1}^{j} \lim_{n \to \infty} P_i(n)
\]

\[
= 1 - \frac{2C_0}{j} \sum_{i=1}^{j} \lim_{n \to \infty} \frac{1}{s_i} - \frac{2C_0}{j} \sum_{i=1}^{j} \lim_{n \to \infty} \frac{1}{s_i} \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki})^2. \quad \text{(S31)}
\]
To compute the limit of $1/s_i$, we first consider $s_i^2$ and observed that

$$s_i^2 = \sum_{k=1}^{n} \sigma_{ik}^2 = \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki} - \rho_{ik}^2 - \rho_{ki}^2)$$

$$= \frac{1}{n} \sum_k d_{ik}^{\text{out}} \tilde{d}_{ik} = \tilde{d}_{i} - \frac{1}{n} \sum_k (d_{ik}^{\text{out}})^2 - \frac{1}{n} \sum_k (d_{ik}^{\text{in}})^2 - \frac{1}{n} \sum_k (d_{ik}^{\text{out}})(d_{ik}^{\text{in}})^2.$$  \hspace{1cm} (S32)

We note that $\frac{1}{n} \sum_k d_{ik}^{\text{in}} \to \bar{d}$ and $\frac{1}{n} \sum_k d_{ik}^{\text{out}} \to \bar{d}$ as $n \to \infty$ by the strong law of large numbers, implying that the first term in Eq. (S32) converges to $\tilde{d}_{i}$. For the last two terms, the strong law of large numbers can be used again to see that $\frac{1}{n} \sum_k (d_{ik}^{\text{out}})^2 \to \mathbb{E}((d_{ik}^{\text{out}})^2)$ and $\frac{1}{n} \sum_k (d_{ik}^{\text{out}})(d_{ik}^{\text{in}})^2 \to \mathbb{E}((d_{ik}^{\text{out}})(d_{ik}^{\text{in}})^2)$, where $\tilde{d}_{ik}$ and $\tilde{d}_{ik}^{\text{out}}$ denote random variables drawn from the joint distribution of $\tilde{d}_{ik}$ and $d_{ik}^{\text{out}}$, which is independent of $i$. With a factor of $\frac{1}{n}$ in the denominators, we see that both of these two terms converge to zero. We thus have $s_i \to (\tilde{d}_{i} + d_{i}^{\text{out}})^{1/2}$ almost surely. For the last term in Eq. (S31), we note that

$$\sum_{k=1}^{n} (\rho_{ik} + \rho_{ki})^2 = \sum_{k=1}^{n} (\rho_{ik}^2 + \rho_{ki}^2 + 2 \rho_{ik} \rho_{ki})$$

$$= \frac{(\tilde{d}_{i}^{\text{out}})^2 \cdot \frac{1}{n} \sum_k (d_{ik}^{\text{out}})^2 + (\tilde{d}_{i}^{\text{in}})^2 \cdot \frac{1}{n} \sum_k (d_{ik}^{\text{in}})^2}{n \cdot \frac{1}{n} \sum_k (d_{ik}^{\text{in}})^2}$$ \hspace{1cm} (S33)

converges to zero as $n \to \infty$ almost surely, since $\frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{in}} \tilde{d}_{ik}^{\text{out}} \to \mathbb{E}(\tilde{d}_{ik}^{\text{in}} \tilde{d}_{ik}^{\text{out}})$ by the strong law of large numbers. Thus, Eq. (S31) becomes

$$\lim_{n \to \infty} g_{jn} = 1 - \frac{2C_0}{\sqrt{\sum_{i=1}^{j} \frac{1}{\tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}}}}}.$$ \hspace{1cm} (S34)

To see that this convergence is uniform in $j$, we note that the random variables

$$\frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{out}}, \quad \frac{1}{n} \sum_k (d_{ik}^{\text{in}})^2, \quad \frac{1}{n} \sum_k (d_{ik}^{\text{out}})^2, \quad \frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{in}} \tilde{d}_{ik}^{\text{out}}, \quad \frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{in}} d_{ik}^{\text{out}}$$ \hspace{1cm} (S35)

appearing in Eqs. (S32) and (S33) are all independent of $j$, and so is the convergence to their respective limits (1 for the first one and 0 for the other three). Thus, there exists a function $\varepsilon(n)$ satisfying $\lim_{n \to \infty} \varepsilon(n) = 0$ and

$$\left| \frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{out}} - \frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{in}} \right| \leq \varepsilon(n), \quad \left| \frac{1}{n} \sum_k (d_{ik}^{\text{out}})^2 \right| \leq \varepsilon(n), \quad \left| \frac{1}{n} \sum_k (d_{ik}^{\text{in}})^2 \right| \leq \varepsilon(n),$$

$$\left| \frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{in}} \tilde{d}_{ik}^{\text{out}} \right| \leq \varepsilon(n), \quad \left| \frac{1}{n} \sum_k \tilde{d}_{ik}^{\text{in}} d_{ik}^{\text{out}} \right| \leq \varepsilon(n).$$ \hspace{1cm} (S36)
This, combined with Eqs. (S32) and (S33), leads to the estimates

\[ |s_i^2 - (\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out})| \leq \left( d_{i}^{in} + (d_{i}^{out})^2 \right) \cdot \varepsilon(n) \]  

(S37)

and

\[ \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki})^2 \leq \left( (d_{i}^{in})^2 + (d_{i}^{out})^2 + 2d_{i}^{in}d_{i}^{out} \right) \cdot \varepsilon(n). \]  

(S38)

We also have a constant lower bound for \( s_i^2 \):

\[ s_i^2 = \sum_{k=1}^{n} \sigma_{ik}^2 = \sum_{k=1}^{n} \left( (1 - \rho_{ik})\rho_{ik} + (1 - \rho_{ki})\rho_{ki} \right) \geq (1 - c) \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki}) \geq (1 - c) \sum_{k=1}^{n} \rho_{ki} = (1 - c)d_{i}^{out} \geq 1 - c, \]  

(S39)

where we recall from the definition of the GCLV model that \( c < 1 \) is a constant and that we have \( 0 \leq \rho_{ij} \leq c \) and \( \tilde{d}_{i}^{out} \geq 1 \). For the convergence of \( 1/s_i \), we have the estimate

\[ \left| \frac{1}{s_i} - \frac{1}{\sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} \right| = \frac{|s_i^2 - (\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out})|}{s_i \sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} \leq \frac{(d_{i}^{in} + (d_{i}^{out})^2 \cdot \varepsilon(n)}{2\sqrt{1 - c}}, \]  

(S40)

since \( s_i \geq \sqrt{1 - c} > 0 \) from Eq. (S39) and \( \tilde{d}_{i}^{in}, \tilde{d}_{i}^{out} \geq 1 \) from the model definition. Combining Eqs. (S22), (S38), (S39), and (S40) for \( j \leq n \), we have

\[
\left| g_{jn} - 1 - \frac{2C_0}{j} \sum_{i=1}^{j} \frac{1}{\sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} \right| \\
= \left| \frac{1}{j} \sum_{i=1}^{j} P_i(n) - 1 + \frac{2C_0}{j} \sum_{i=1}^{j} \frac{1}{\sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} \right| \\
= \left| \frac{1}{j} \sum_{i=1}^{j} \left( 1 - 2C_0 \frac{1}{s_i} + 2C_0 \frac{1}{s_i^2} \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki})^2 \right) - 1 + \frac{2C_0}{j} \sum_{i=1}^{j} \frac{1}{\sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} \right| \\
= \left| \frac{2C_0}{j} \sum_{i=1}^{j} \left( -\frac{1}{s_i} + \frac{1}{\sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} - \frac{1}{s_i^3} \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki})^2 \right) \right| \\
\leq \left| \frac{2C_0}{j} \sum_{i=1}^{j} \left( \frac{1}{s_i} - \frac{1}{\sqrt{\tilde{d}_{i}^{in} + \tilde{d}_{i}^{out}}} + \frac{1}{s_i^3} \sum_{k=1}^{n} (\rho_{ik} + \rho_{ki})^2 \right) \right| 
\]
\[ \leq 2C_0 \varepsilon(n) \cdot \frac{1}{j} \sum_{i=1}^{j} \left( \frac{\tilde{d}_{i}^{\text{in}} + (\tilde{d}_{i}^{\text{in}})^2 + (\tilde{d}_{i}^{\text{out}})^2}{2\sqrt{1 - c}} + \frac{(\tilde{d}_{i}^{\text{in}})^2 + (\tilde{d}_{i}^{\text{out}})^2 + 2\tilde{d}_{i}^{\text{in}} \tilde{d}_{i}^{\text{out}}}{(1 - c)^{3/2}} \right) \]

\[
\leq 2C_0 \varepsilon(n) M,
\]

where the average over \( j \) in the second to the last line above is bounded by a finite constant \( M \) because the averages of the individual terms \( \tilde{d}_{i}^{\text{in}}, (\tilde{d}_{i}^{\text{in}})^2, (\tilde{d}_{i}^{\text{out}})^2 \), and \( \tilde{d}_{i}^{\text{in}} \tilde{d}_{i}^{\text{out}} \) all converge to finite values as \( j \to \infty \) due to the strong law of large numbers. Since \( 2C_0 \varepsilon(n) M \) converges to zero as \( n \to \infty \) with a rate that does not depend on \( j \) (since \( \varepsilon(n) \to 0 \)), the convergence in Eq. (S34) is indeed uniform in \( j \). We can now apply the Moore-Osgood Theorem to conclude that the limit of the double sequence \( g_{jn} \) exists and can be computed as an iterated limit:

\[
\lim_{j,n \to \infty} g_{jn} = \lim_{j \to \infty} \lim_{n \to \infty} g_{jn} = 1 - 2C_0 \cdot \frac{1}{j} \sum_{i=1}^{j} \frac{1}{\sqrt{\tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}}}}
\]

\[
= 1 - 2C_0 \cdot \mathbb{E} \frac{1}{\sqrt{\tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}}}} \geq 1 - \frac{2 \cdot 0.56}{\sqrt{1 + 1}} \geq 0.2,
\]

where we also used \( C_0 \leq 0.56 \) and \( \tilde{d}_{i}^{\text{in}}, \tilde{d}_{i}^{\text{out}} \geq 1 \). Thus, the double sequence \( g_{jn} \) converges to the same value regardless of how \( j \) and \( n \) are taken to \( \infty \), and the limit is bounded away from zero.

We now return to Eq. (S28) and show the convergence of \( 1 - Q(n) \) to zero by estimating the terms on the r.h.s. one by one. For the first term, we have \( \bar{\varepsilon}_n(n) = \max_{1 \leq j \leq n-1} (\rho_{jn} + \rho_{nj}) \leq \max_{1 \leq j \leq n} (\rho_{jn} + \rho_{nj}) = \varepsilon_n(n), \) and hence the term converges to zero in probability as \( n \to \infty \), since we proved \( \varepsilon_n(n) \to 0 \) in the previous section.

For the second term, we have

\[
\prod_{i=1}^{n-1} (1 - P_i(n)) \leq \exp \left( - \sum_{i=1}^{n-1} P_i(n) \right) = \exp \left( -(n - 1) \cdot g_{n-1,n} \right),
\]

where we used the fact that \( 1 - x \leq e^{-x} \) for any \( 0 \leq x \leq 1 \). According to Eq. (S42), the factor \( g_{n-1,n} \) above converges to a strictly positive value while the factor \( n - 1 \) diverges, implying that \( \exp \left( -(n - 1) \cdot g_{n-1,n} \right) \), and hence the second term on the r.h.s. of Eq. (S28), converges to zero as \( n \to \infty \).
For the third term, we choose a positive integer $N < n - 2$ and split the sum to obtain

$$\sum_{j=1}^{n-2} \tilde{e}_{j+1}(n) \prod_{i=1}^{j} (1 - P_i(n)) \leq \sum_{j=1}^{n-2} \tilde{e}_{j+1}(n) \exp \left( - \sum_{i=1}^{j} P_i(n) \right)$$

$$< \sum_{j=1}^{N} \tilde{e}_{j+1}(n) \exp \left( - \sum_{i=1}^{j} P_i(n) \right) + 2 \sum_{j=N+1}^{n-2} \exp(-j \cdot g_{jn}),$$

(S44)

where we used the fact that $\rho_{ij} < 1$ and thus $\tilde{e}_i(n) = \max_{1 \leq j < i} (\rho_{ji} + \rho_{ij}) < 2$ for any $i$. Taking the limit $n \to \infty$ on the r.h.s., we see that the first term converges to zero in probability, since each $\tilde{e}_{j+1}(n)$ converges to zero, the argument of the exponential function is bounded (as it converges; see Eq. (S34)), and the number of terms in the sum is finite. For the second term, we see that

$$\sum_{j=N+1}^{n-2} \exp(-j \cdot g_{jn}) \leq \sum_{j=N+1}^{n-2} e^{-0.1j} \leq \sum_{j=N+1}^{\infty} e^{-0.1j} = \frac{e^{-0.1(N+1)}}{1 - e^{-0.1}},$$

(S45)

where we used Eq. (S41) and $\tilde{d}_{in}, \tilde{d}_{out} \geq 1$ to estimate $g_{jn}$ as

$$g_{jn} \geq 1 - \frac{2C_0}{\tilde{d}_{i} \tilde{d}_{out}} - 2C_0 \varepsilon(n) \geq 0.2 - 2C_0 \varepsilon(n) \geq 0.1$$

(S46)

for sufficiently large $n$ (and sufficiently small $\varepsilon(n)$). Taking the limit $n \to \infty$ in Eq. (S45) and noting that the r.h.s. can be made arbitrarily small by choosing sufficiently large $N$, we see that the last sum in Eq. (S44) converge to zero. We thus conclude that the third term of the r.h.s. of Eq. (S28) also converges to zero in probability.

Putting everything together, we have proved that $Q(n)$, and thus its expected value $\mathbb{E}(Q(n))$, converges to one in probability, i.e., the probability that $A$ is nonnormal converges to one as the network size $n$ approaches infinity.

S2.3. Proof of reactivity

Now we show that $A$ is reactive with probability converging to one as $n \to \infty$, assuming that $A_{ij} \in \{0, 1\}$ (which will be relaxed to allow for weighted $A_{ij}$ in Sec. S2.4). We first present a sufficient condition for reactivity and then show that the probability of satisfying this condition approaches one with increasing $n$. 

S2.3.1 Sufficient condition for reactivity

Consider a square matrix $X = (X_{ij})$ with $X_{ij} \geq 0$ for $i \neq j$ and denote by $G$ the network in which link $j \rightarrow i$ exists if and only if $X_{ij} \neq 0$. We will prove that the following is a sufficient condition for $X$ to be reactive:

(C1) The network $G$ has a strongly connected component $\tilde{G}$ containing a node with a nonzero eigenvector in-centrality, and the adjacency matrix $\tilde{A}$ of $\tilde{G}$ has at least one column that strictly dominates the corresponding row.

Here, given column/row vectors $a$ and $b$, we say that $a$ dominates $b$ if their $i$th components $a_i$ and $b_i$ satisfy $a_i \geq b_i$ for all $i$. If at least one of these inequalities is strict, then we say $a$ strictly dominates $b$.

Thus, for a network satisfying this condition, we have $v_{1i} \neq 0$ for some $i \in \tilde{G}$ (recalling that the eigenvector in-centrality $v_{1i}$ is the $i$th component of the right eigenvector associated with $\lambda_1(A)$) and $\tilde{A}_{kj} \geq \tilde{A}_{jk}, \forall k$ for some $j$ (with at least one of these inequalities being strict). For an unweighted directed network $G$, the latter part of the condition regarding $\tilde{A}$ can be interpreted in terms of network topology: there is at least one node whose in-neighbors form a proper subset of the out-neighbors within the strongly connected component containing that node.

To establish the sufficiency of condition (C1) for the reactivity of $A$, suppose that (C1) is satisfied. Then, $A$ can be transformed by an appropriate node permutation into the block form (62)

$$A = \begin{bmatrix} C & B & O \\ O & \tilde{A} & D \\ O & O & E \end{bmatrix},$$

where $O$ denotes the matrix of all zeros (of an appropriate size), and the middle block $\tilde{A}$ is the adjacency matrix of the strongly connected component $\tilde{G}$ in (C1). Such a permutation can be constructed by re-indexing the nodes in the following order: those nodes $i$ outside $\tilde{G}$ to which there is a directed path from a node in $\tilde{G}$, the nodes in $\tilde{G}$, and all the remaining nodes (where the ordering within each group can be arbitrary).

To prove that $A$ must be reactive, we now assume that $A$ is not reactive and show that this assumption leads to a contradiction. We first partition the right eigenvector associated with $\lambda_1(A)$ as $v_1 = [v^T_c, v^T_1, v^T_e]^T$ according to the block structure in Eq. (S47), where (C1) guarantees that $\tilde{v}_1^T$ is nonzero. By explicitly writing the eigenvalue relation $Av_1 = \lambda_1(A)v_1$ for the second and third rows of the block form in Eq. (S47), we see that $\lambda_1(A)$ is also an eigenvalue of the subma-
atrix $A' := \begin{bmatrix} \tilde{A} & D \\ O & E \end{bmatrix}$ with eigenvector $[\tilde{v}_i^T, v_e^T]^T$. Since the largest (Perron-Frobenius) eigenvalue $\lambda_1(A')$ of the submatrix $A'$ cannot exceed that of the entire matrix $A$, we must have $\lambda_1(A) = \lambda_1(A')$. If $A'$ were reactive, then we would have $\lambda_1(A) = \lambda_1(A') < \lambda_1\left(\frac{A' + (A')^T}{2}\right)$, implying that $A$ is also reactive, contradicting the assumption we made above. Hence, $A'$ must be non-reactive, and thus the right eigenvector $[\tilde{v}_i^T, v_e^T]^T$ is also a left eigenvector of $A'$ corresponding to $\lambda_1(A')$. This further implies $\tilde{v}_i^T \tilde{A} = \lambda_1(A')\tilde{v}_i^T$, and thus $\lambda_1(A')$ is an eigenvalue of $\tilde{A}$ with left eigenvector $\tilde{v}_1$. With the arguments we used above for $A'$ now applied to $\tilde{A}$, we see that $\lambda_1(A) = \lambda_1(\tilde{A})$ (the largest eigenvalue of $\tilde{A}$) and that $\tilde{A}$ must be non-reactive. This implies that $\tilde{v}_1$ is not only the left eigenvector but also the right eigenvector corresponding to $\lambda_1(\tilde{A})$, and thus

$$\tilde{A}^T \tilde{v}_1 = \lambda_1(\tilde{A}) \tilde{v}_1 = \tilde{A}\tilde{v}_1.$$  \hspace{1cm} (S48)

Because $\tilde{G}$ is strongly connected, $\tilde{A}$ is irreducible, and the components of $\tilde{v}_1$, which we denote by $\tilde{v}_{1i}$, are all strictly positive by the Perron-Frobenius Theorem. Since (C1) is satisfied, there exists an index $j$ for which the $j$th column of $\tilde{A}$ (and hence the $j$th row of $\tilde{A}^T$) strictly dominates the $j$th row of $\tilde{A}$. By the positivity of $\tilde{v}_{1i}$, this implies $(\tilde{A}^T \tilde{v}_1)_j = \sum_i (\tilde{A}^T)_{ji} \tilde{v}_{1i} > \sum_i \tilde{A}_{ji} \tilde{v}_{1i} = (\tilde{A}\tilde{v}_1)_j$, contradicting Eq. (S48). Therefore, $A$ must be reactive.

### S2.3.2 Proof that condition (C1) is satisfied for almost all large networks

The GCLV model is a special case of the more general model discussed in Ref. 63 for which the two functions $\kappa$ and $\varphi_n$ defining the model are given by

$$\kappa(x_i, x_j) = \frac{\tilde{n}_i d_{\text{out}}}{\tilde{d}}$$ \hspace{1cm} (S49)

and

$$\varphi_n(x_i, x_j) = \frac{n \tilde{d}_{\text{out}}}{\sum_{k=1}^{n \tilde{d}} \tilde{d}_{\text{out}}(\tilde{d}_k^n \tilde{d}_{i_{\text{out}}})} - 1,$$ \hspace{1cm} (S50)

where we used the notations $x \wedge y := \min\{x, y\}$ and $x_i := (\tilde{n}_i, \tilde{d}_{i_{\text{out}}})$ for each node $i$ and recall that $\tilde{d} = \mathbb{E}(\tilde{d}_i^n) = \mathbb{E}(\tilde{d}_i^{\text{out}})$ for any $i$. By Proposition 3.13 in Ref. 63 and our model assumption $\tilde{d} > 1$, there exists a constant $\tau > 0$ such that the largest strongly connected component $C_n$ has approximately $\tau n$ nodes in the limit of large network size $n$. In other words, if we denote the size of that component by $\tilde{n} = |C_n|$, we have $\tilde{n}/n \to \tau$ as $n \to \infty$ (a defining property of a giant strongly connected component of the network). In addition, according to Ref. 49, the largest eigenvalue $\lambda_1(A)$ is also the largest eigenvalue of the adjacency matrix $\tilde{A}$ of $C_n$ in...
the limit \( n \to \infty \). From this, it follows that the probability of satisfying the in-centrality part of condition (C1) (that there is a node \( i \) in \( \tilde{G} = C_n \) for which \( v_{1i} \neq 0 \) ) approaches one as \( n \to \infty \). Therefore, the probability that \( A \) satisfies (C1) is asymptotically bounded from below by the probability that there is a column of \( A \) that strictly dominates the corresponding row of \( A \) within \( C_n \).

We can now estimate the probability that \( A \) is reactive as

\[ \mathbb{P}(A \text{ is reactive}) \geq \mathbb{E}(p(n)), \tag{S51} \]

where \( p(n) \) denotes the conditional probability that there exists a node \( i \) in the network for which the \( i \)th column of \( A \) strictly dominates the \( i \)th row of \( A \) within \( C_n \) given realizations of \( \{\tilde{d}_i^\text{in}\}_{i=1}^n \) and \( \{\tilde{d}_i^\text{out}\}_{i=1}^n \). We thus seek to establish a lower bound for \( p(n) \) and use it to show that \( p(n) \) approaches one in the large network limit. For that purpose, we first consider an analogous probability for a given node \( i \). Specifically, we define \( p_i(n) \) to be the conditional probability that the \( i \)th column strictly dominates the \( i \)th row in \( \tilde{A}(n) \) (again, given a realization of \( \{\tilde{d}_i^\text{in}\}_{i=1}^n \) and \( \{\tilde{d}_i^\text{out}\}_{i=1}^n \)). This probability can be computed as

\[
p_i(n) = \mathbb{P}(A_{ji} \geq A_{ij}, \forall j \in C_n) - \mathbb{P}(A_{ji} = A_{ij}, \forall j \in C_n) \nonumber
= \prod_{j \in C_n, j \neq i} (1 - r_{ij}(n)) - \prod_{j \in C_n, j \neq i} (1 - s_{ij}(n)), \tag{S52}
\]

where \( r_{ij}(n) \) is the probability that \( A_{ji} < A_{ij} \) for given \( i, j \in C_n \), which can be expressed using the definition of the model as

\[
r_{ij}(n) = (1 - \rho_{ji})\rho_{ij} = \frac{\tilde{d}_i^\text{in} \tilde{d}_j^\text{out}}{\sum_{k=1}^n \tilde{d}_k^\text{in}} - \frac{\tilde{d}_j^\text{in} \tilde{d}_i^\text{out}}{\sum_{k=1}^n \tilde{d}_k^\text{in}}, \tag{S53}
\]

and \( s_{ij}(n) \) is the probability that \( A_{ji} \neq A_{ij} \), which can be written as

\[
s_{ij}(n) = (1 - \rho_{ij})\rho_{ij} + (1 - \rho_{ij})\rho_{ji} = \frac{\tilde{d}_i^\text{in} \tilde{d}_j^\text{out}}{\sum_{k=1}^n \tilde{d}_k^\text{in}} + \frac{\tilde{d}_j^\text{in} \tilde{d}_i^\text{out}}{\sum_{k=1}^n \tilde{d}_k^\text{in}} - 2 \cdot \frac{\tilde{d}_j^\text{in} \tilde{d}_i^\text{out}}{\sum_{k=1}^n \tilde{d}_k^\text{in}}, \tag{S54}
\]

Note that \( p_i(n) \) in Eq. (S52) is strictly positive for any given \( i \) and \( n \), since \( 0 < \rho_{ij} < 1 \) and thus \( s_{ij}(n) > r_{ij}(n) \) for all \( i \) and \( j \).

To derive a lower bound for \( p_i(n) \) in Eq. (S52), we will use several elementary inequalities. We first note that the derivative of the function \(-x - \ln(1 - x)\) is \( \frac{x}{1-x} \), which is monotonically increasing on the interval \([0, 1]\). Applying the Mean Value Theorem to this function, we obtain
the inequality \(-x - \ln(1 - x) \leq \frac{x^2}{1-x}\). We further note that the inequalities \(x(1-y) + y(1-x) \leq c\) and \(x(1-y) \leq c\) can be shown to hold true for any \(0 \leq x, y \leq c\) under the model assumption \(c \geq 1/2\). From this, together with \(\rho_{ij} \leq c\) as well as Eqs. (S53) and (S54), we see that \(r_{ij}(n) \leq c\) and \(s_{ij}(n) \leq c\). Further noting that \(|e^{-x} - e^{-y}| \leq |x - y|\) for any \(x, y > 0\), we estimate the second term in Eq. (S52) using an exponential function:

\[
\exp \left( - \sum_{j \in C_n, j \neq i} s_{ij}(n) \right) - \prod_{j \in C_n, j \neq i} (1 - s_{ij}(n)) \leq - \sum_{j \in C_n, j \neq i} s_{ij}(n) - \ln \left( \prod_{j \in C_n, j \neq i} (1 - s_{ij}(n)) \right) \leq \sum_{j \in C_n, j \neq i} \frac{s_{ij}^2(n)}{1 - s_{ij}(n)} \leq \frac{1}{c} \sum_{j \in C_n, j \neq i} (\rho_{ij} + \rho_{ji})^2 \leq \frac{1}{c} \sum_{1 \leq j \leq n, j \neq i} (\rho_{ij} + \rho_{ji})^2 \leq \frac{\left( (\bar{d}_i^n)^2 + (\bar{d}_i^\text{out})^2 + 2 \bar{d}_i^n \bar{d}_i^\text{out} \right)}{1 - c} \cdot \varepsilon(n),
\]

where the third and last line follow from the inequality \(-x - \ln(1 - x) \leq \frac{x^2}{1-x}\) and Eq. (S38), respectively. We also have a similar estimate for \(r_{ij}(n)\):

\[
\exp \left( - \sum_{j \in C_n, j \neq i} r_{ij}(n) \right) - \prod_{j \in C_n, j \neq i} (1 - r_{ij}(n)) \leq - \sum_{j \in C_n, j \neq i} r_{ij}(n) - \ln \left( \prod_{j \in C_n, j \neq i} (1 - r_{ij}(n)) \right) \leq \sum_{j \in C_n, j \neq i} \frac{r_{ij}^2(n)}{1 - r_{ij}(n)} \leq \frac{1}{c} \sum_{j \in C_n, j \neq i} ((1 - \rho_{ji})\rho_{ij})^2 \leq \frac{1}{c} \sum_{1 \leq j \leq n, j \neq i} \rho_{ij}^2 \leq \frac{(\bar{d}_i^n)^2}{1 - c},
\]
where the last inequality follows from the lower left inequality in Eq. (S36). Combining Eqs. (S52), (S55), and (S56), we obtain a lower bound for \( p_i(n) \):

\[
p_i(n) \geq \exp\left(- \sum_{j \in C_n, j \neq i} r_{ij}(n)\right) - \exp\left(- \sum_{j \in C_n, j \neq i} s_{ij}(n)\right) - \frac{2(d_i^n)^2 + (d_i^\text{out})^2 + d_i^n d_i^\text{out}}{1 - c} \cdot \varepsilon(n). \tag{S57}
\]

To further estimate the r.h.s., we note that

\[
\sum_{j \in C_n, j \neq i} r_{ij}(n) = \sum_{j \in C_n, j \neq i} \frac{d_i^n d_j^n}{\sum_{k=1}^n d_k^n} - \frac{d_i^n d_j^\text{out}}{\sum_{k=1}^n d_k^n} + \frac{d_i^\text{out} d_j^\text{out}}{\sum_{k=1}^n d_k^n}.
\]

and

\[
\sum_{j \in C_n, j \neq i} s_{ij}(n) = \sum_{j \in C_n, j \neq i} \frac{d_i^n d_j^\text{out}}{\sum_{k=1}^n d_k^n} + \frac{d_i^n d_j^\text{out}}{\sum_{k=1}^n d_k^n} - 2 \frac{d_i^n d_j^\text{out}}{\sum_{k=1}^n d_k^n} - 2 \frac{d_i^n d_j^\text{out}}{\sum_{k=1}^n d_k^n}.
\]

from which we see that these sums satisfy

\[
\sum_{j \in C_n, j \neq i} r_{ij}(n) < \sum_{j \in C_n, j \neq i} s_{ij}(n) \leq d_i^n (1 + \varepsilon(n)) + d_i^\text{out}, \tag{S60}
\]

where \( \varepsilon(n) \) is defined in Eq. (S36). Since \( \varepsilon(n) \) converges to zero as \( n \to \infty \), it is also bounded, implying that there is a constant \( \eta > 1 \) for which \( 1 + \varepsilon(n) < \eta \) for all \( n \). Hence, we have

\[
\sum_{j \in C_n, j \neq i} r_{ij}(n) < \sum_{j \in C_n, j \neq i} s_{ij}(n) < \eta d_i^n + d_i^\text{out}. \tag{S61}
\]

Now, noting that \( e^{-x} \) is a monotonically decreasing function, and its derivative, \( -e^{-x} \), is mono-
tonically increasing, we obtain the following estimate for the first two terms of Eq. (S57):

\[
\exp\left(-\sum_{j \in \mathcal{C}_n, j \neq i} r_{ij}(n)\right) - \exp\left(-\sum_{j \in \mathcal{C}_n, j \neq i} s_{ij}(n)\right) \\
\geq \exp\left(- (\eta \tilde{d}_i^{\text{in}} + \tilde{d}_i^{\text{out}})\right) \cdot \left(\sum_{j \in \mathcal{C}_n, j \neq i} s_{ij}(n) - \sum_{j \in \mathcal{C}_n, j \neq i} r_{ij}(n)\right) \\
= \exp\left(- (\eta \tilde{d}_i^{\text{in}} + \tilde{d}_i^{\text{out}})\right) \cdot \left(\tilde{d}_i^{\text{out}} \cdot \sum_{j \in \mathcal{C}_n, j \neq i} \tilde{d}_j^{\text{in}} - \tilde{d}_i^{\text{out}} \sum_{j \in \mathcal{C}_n, j \neq i} \tilde{d}_j^{\text{in}} \sum_{k=1}^{n} \tilde{d}_k^{\text{in}} \cdot \frac{1}{\sum_{k=1}^{n} \tilde{d}_k^{\text{in}}}\right) \\
\geq \tilde{d}_i^{\text{out}} \exp\left(- (\eta \tilde{d}_i^{\text{in}} + \tilde{d}_i^{\text{out}})\right) \cdot \left(\sum_{j \in \mathcal{C}_n, j \neq i} \tilde{d}_j^{\text{in}} - \tilde{d}_i^{\text{out}} \cdot \tilde{d}_i^{\text{in}}\right) \\
\geq \tilde{d}_i^{\text{out}} \exp\left(- (\eta \tilde{d}_i^{\text{in}} + \tilde{d}_i^{\text{out}})\right) \cdot \left(\sum_{1 \leq j \leq \hat{n}, j \neq i} \min\{\tilde{d}_j^{\text{in}}, \tilde{d}_i^{\text{in}}\} - \tilde{d}_i^{\text{out}} \cdot \tilde{d}_i^{\text{in}}\right) \\
= \tilde{d}_i^{\text{out}} \exp\left(- (\eta \tilde{d}_i^{\text{in}} + \tilde{d}_i^{\text{out}})\right) \cdot \left(\frac{\hat{n} - 1}{\hat{n}} \cdot \tilde{d} - \frac{1}{\hat{n}} \cdot \sum_{1 \leq j \leq \hat{n}, j \neq i} \tilde{d}_j^{\text{in}} \cdot \tilde{d}_i^{\text{in}} \cdot \min\{\tilde{d}_j^{\text{in}}, \tilde{d}_i^{\text{in}}\} - \tilde{d}_i^{\text{out}} \cdot \tilde{d}_i^{\text{in}}\right), (S62)
\]

where \(\{\tilde{d}_j^{\text{in}}\}_{j=1}^{\hat{n}}\) denotes the order statistics of \(\{\tilde{d}_j^{\text{in}}\}_{j=1}^{\hat{n}}\), i.e., the re-indexed version of \(\{\tilde{d}_j^{\text{in}}\}_{j=1}^{n}\) in which \(d_1^{\text{in}} \geq d_2^{\text{in}} \geq \cdots \geq d_n^{\text{in}}\), and we define \(\tilde{h}_i^{\text{in}} := \tilde{d} - \min\{\tilde{d}_j^{\text{in}}, \tilde{d}_i^{\text{in}}\}\).

To further estimate the term \(\frac{1}{\hat{n}} \sum_{1 \leq j \leq \hat{n}, j \neq i} \tilde{h}_j^{\text{in}}\) in Eq. (S62), we consider the so-called conditional value at risk (64), given by

\[
c_r := \mathbb{E}\left(\tilde{h}_i^{\text{in}} \mid \tilde{h}_i^{\text{in}} \geq v_r\right) (S63)
\]

in the case of the random variable \(\tilde{h}_i^{\text{in}} := \tilde{d} - \min\{\tilde{d}_j^{\text{in}}, \tilde{d}_i^{\text{in}}\}\), where \(v_r := \sup\{\xi : \mathbb{P}(\tilde{h}_i^{\text{in}} > \xi) \geq r\}\) is called the value at risk and \(\tilde{d}_i^{\text{in}}\) is a random variable following the same distribution as \(\tilde{d}_i^{\text{in}}\) (for any \(i\)). We note that both \(c_r\) and \(v_r\) are constants determined by the parameter \(\tau\) and the distribution of \(\tilde{d}_i^{\text{in}}\). A finite-sample estimator for \(c_r\) is given by

\[
\hat{c}_r := \frac{1}{\tau n} \sum_{1 \leq j \leq [\tau n]} \tilde{h}_{ij}^{\text{in}} (S64)
\]
and can be used to approximate \( \frac{1}{n} \sum_{1 \leq j \leq \tilde{n}, \ j \neq i} \tilde{h}^i_{[j]} \) as

\[
\left| \frac{1}{n} \sum_{1 \leq j \leq \tilde{n}, \ j \neq i} \tilde{h}^i_{[j]} - \hat{c}_r \right| 
\leq \left| \frac{1}{n} \sum_{1 \leq j \leq \tilde{n}, \ j \neq i} \tilde{h}^i_{[j]} - \frac{1}{n} \sum_{1 \leq j \leq [\tau n]} \tilde{h}^i_{[j]} \right| 
+ \left| \frac{1}{n} \sum_{1 \leq j \leq [\tau n]} \tilde{h}^i_{[j]} - \frac{1}{\tau n} \sum_{1 \leq j \leq [\tau n]} \tilde{h}^i_{[j]} \right| 
\tag{S65}
\]

where we used \( \tilde{h}^i_{[j]} \leq \tilde{d} \) and we have \( \lim_{n \to \infty} \delta(n) = 0 \) (since \( \lim_{n \to \infty} \tilde{n}/n = \tau \)). Using this in Eq. (S62), we obtain

\[
\exp \left( - \sum_{j \in \mathcal{C}_n, \ j \neq i} r_{ij}(n) \right) - \exp \left( - \sum_{j \in \mathcal{C}_n, \ j \neq i} s_{ij}(n) \right) 
\geq \tilde{d}^\text{out}_i \exp \left( -(\eta \tilde{d}^\text{in}_i + \tilde{d}^\text{out}_i) \right) \left( \frac{\tilde{n} - 1}{n} \tilde{d} - \hat{c}_r - \delta(n) \right) - \tilde{d}^\text{in}_i \varepsilon(n). \tag{S66}
\]

Using the concentration bounds proved in Ref. 64 and noting the fact that \( x \geq \epsilon \) if and only if \( \max\{x, 0\} \geq \epsilon \) for any given \( \epsilon > 0 \), we have

\[
\mathbb{P} \left( (\hat{c}_r - c_r)^+ > \epsilon \right) \leq 3 \exp \left( - \frac{n \tau \epsilon^2}{11 \tilde{d}^2} \right) \tag{S67}
\]

for any \( \epsilon > 0 \), where we use the notation \( (x)^+ = \max\{x, 0\} \). Applying Borel-Cantelli lemma (Proposition 2.6 in Ref. 65) and noting that \( \sum_{n=1}^{\infty} \exp \left( - \frac{n \tau \epsilon^2}{11 \tilde{d}^2} \right) < \infty \), Eq. (S67) implies

\[
\chi(n) := (\hat{c}_r - c_r)^+ \longrightarrow 0 \tag{S68}
\]

as \( n \to \infty \) almost surely. Combining

\[
\hat{c}_r \leq \max\{\hat{c}_r, c_r\} = c_r + (\hat{c}_r - c_r)^+ = c_r + \chi(n). \tag{S69}
\]

with Eqs. (S57), (S66), and (S68), we see that

\[
p_i(n) \geq \tilde{p}_i(n) := \tilde{d}^\text{out}_i \exp \left( -(\eta \tilde{d}^\text{in}_i + \tilde{d}^\text{out}_i) \right) \left( \frac{\tilde{n} - 1}{n} \tilde{d} - c_r - \chi(n) - \delta(n) \right) - \tilde{d}^\text{in}_i \varepsilon(n) 
- \frac{2 (\tilde{d}^\text{in}_i)^2 + (\tilde{d}^\text{out}_i)^2 + \tilde{d}^\text{in}_i \tilde{d}^\text{out}_i \cdot \varepsilon(n)}{1 - c} 
\longrightarrow \tilde{d}^\text{out}_i \exp \left( -(\eta \tilde{d}^\text{in}_i + \tilde{d}^\text{out}_i) \right) \cdot \frac{\tau (\tilde{d} - c_r)}{\tilde{d}} \tag{S70}
\]
as $n \to \infty$ almost surely. Without loss of generality, we may modify the function $\varepsilon(n)$ in Eq. (S36) to additionally satisfy

$$\frac{n-1}{n} \frac{\hat{d} - c_r - \chi(n) - \delta(n)}{\frac{1}{n} \sum_{k=1}^{n} \tilde{a}_{ik}^k} - \frac{\tau(\hat{d} - c_r)}{\hat{d}} \leq \varepsilon(n),$$

(S71)

and thus the deviation from the limit in Eq. (S70) can be estimated as

$$\left| \hat{p}_i(n) - \tilde{d}_i^\text{out} \exp\left(-\eta \tilde{a}_i^\text{in} + \tilde{d}_i^\text{out}\right) \cdot \frac{\tau(\hat{d} - c_r)}{\hat{d}} \right| \leq \left(\tilde{d}_i^\text{out} + \tilde{a}_i^\text{in} \tilde{d}_i^\text{out}\right) \exp\left(-\eta \tilde{a}_i^\text{in} + \tilde{d}_i^\text{out}\right) \cdot \varepsilon(n) + \frac{2(\tilde{d}_i^\text{in})^2 + (\tilde{d}_i^\text{out})^2 + \tilde{a}_i^\text{in} \tilde{d}_i^\text{out}}{1 - c} \cdot \varepsilon(n).$$

(S72)

We now seek to estimate $p(n)$ by a recursive argument involving $p_i(n)$ and $\hat{p}_i(n)$. Let $\tilde{A}_k(n)$ denote the principal submatrix of $A$ obtained by keeping only those rows $i$ and columns $i$ for which $i \leq k$ and $i \in \mathcal{C}_n$. We further let $q_k(n)$ denote the probability that there is at least one column strictly dominating the corresponding row in $\tilde{A}_k(n)$. We thus have $q_n(n) = p(n)$. Then, we have the following recursive inequality:

$$q_k(n) \geq q_{k-1}(n) \cdot \min_{1 \leq j \leq k-1} \left(1 - r_{kj}(n)\right) + (1 - q_{k-1}(n))p_k(n).$$

(S73)

Let $\varepsilon_k(n) := \max_{1 \leq j \leq k} \rho_{kj} \geq \max_{1 \leq j \leq k} r_{kj}(n)$. Using this notation, Eq. (S73) can be rewritten as

$$1 - q_k(n) \leq (1 - q_{k-1}(n)) \cdot (1 - p_k(n)) + \varepsilon_k(n).$$

(S74)

Since the r.h.s. of Eq. (S74) is monotonically decreasing in $p_k(n)$, the inequality remains true if $p_k(n)$ is replaced with its lower bound $\hat{p}_k(n)$. We thus have

$$1 - q_k(n) \leq (1 - q_{k-1}(n)) \cdot (1 - \hat{p}_k(n)) + \varepsilon_k(n).$$

(S75)

Recursive application of this inequality yields

$$1 - q_n(n) \leq \varepsilon_n(n) + \prod_{j=2}^{n}(1 - \hat{p}_j(n)) + \sum_{j=3}^{n} \varepsilon_{j-1}(n) \prod_{i=j}^{n} (1 - \hat{p}_i(n)).$$

(S76)

Applying the same reordering of nodes we used for Eq. (S27) to obtain Eq. (S28), we obtain

$$1 - q_n(n) \leq \varepsilon_n(n) + \prod_{j=1}^{n-1}(1 - \hat{p}_j(n)) + \sum_{j=1}^{n-2} \varepsilon_{j+1}(n) \prod_{i=1}^{n} (1 - \hat{p}_i(n)).$$

(S77)
To help estimate the r.h.s. of Eq. (S77), we consider the double sequence defined by
\[
m_{jn} = \begin{cases} 
\frac{1}{j} \sum_{i=1}^{j} \hat{p}_i(n), & j \leq n, \\
\frac{1}{n} \sum_{i=1}^{n} \hat{p}_i(n), & j > n.
\end{cases}
\] (S78)

As was done in Sec. S2.2 for a similar double sequence (see Eq. (S29)), we will use the Moore-Osgood theorem ([61]) to calculate the limit of this double sequence. We first note that
\[
\lim_{j \to \infty} m_{jn} = m_{nn} 
\]
for each fixed \( n \) by definition and that
\[
\lim_{n \to \infty} m_{jn} = \frac{1}{j} \sum_{i=1}^{j} \hat{p}_i(n) = \frac{1}{j} \sum_{i=1}^{j} \tilde{d}_{i}^{\text{out}} \exp\left( -\left( \eta \tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}} \right) \right) \cdot \frac{\tau (\bar{d} - c_{\tau})}{d} 
\] (S79)
for each fixed \( j \). To see that this convergence is uniform over all \( j \), we note that Eq. (S72) implies
\[
\left| \frac{1}{j} \sum_{i=1}^{j} \hat{p}_i(n) - \frac{1}{j} \sum_{i=1}^{j} \tilde{d}_{i}^{\text{out}} \exp\left( -\left( \eta \tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}} \right) \right) \cdot \frac{\tau (\bar{d} - c_{\tau})}{d} \right|
\leq \frac{1}{j} \sum_{i=1}^{j} \left| \left( \tilde{d}_{i}^{\text{out}} + \tilde{d}_{i}^{\text{in}} \tilde{d}_{i}^{\text{out}} \right) \exp\left( -\left( \eta \tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}} \right) \right) + \frac{2 (\tilde{d}_{i}^{\text{in}})^2 + (\tilde{d}_{i}^{\text{out}})^2 + \tilde{d}_{i}^{\text{in}} \tilde{d}_{i}^{\text{out}}}{1 - c} \right| \cdot \varepsilon(n) 
\] (S80)
where the average over \( j \) on the second line is bounded by a finite constant \( M' \) because the averages of the individual terms all converge to finite values as \( j \to \infty \) due to the strong law of large numbers. Since \( M' \varepsilon(n) \) converges to zero as \( n \to \infty \) with a rate independent of \( j \), the convergence in Eq. (S79) is indeed uniform in \( j \). Applying the Moore-Osgood theorem and using the strong law of large numbers again, we conclude that the limit \( m_{\infty} \) of the double sequence \( m_{jn} \) exists and can be computed using the following iterated limit:
\[
\lim_{j,n \to \infty} m_{jn} = \lim_{j \to \infty} \lim_{n \to \infty} m_{jn}
= m_{\infty} := \mathbb{E}\left( \tilde{d}_{i}^{\text{out}} \exp\left( -\left( \eta \tilde{d}_{i}^{\text{in}} + \tilde{d}_{i}^{\text{out}} \right) \right) \right) \cdot \frac{\tau (\bar{d} - c_{\tau})}{d} > 0. 
\] (S81)

We note that the limit \( m_{\infty} \) is strictly positive because \( \tilde{d}_{i}^{\text{out}} \geq 1, \tau > 0, \) and \( \bar{d} - c_{\tau} \geq 1 \) (which follows from \( \tilde{d}_{i}^{\text{in}} \geq 1 \) and the definition of \( c_{\tau} \)).

With the limit \( m_{\infty} \) in hand, we now return to Eq. (S77) and show the convergence of \( 1 - q_n(n) \) to zero as \( n \to \infty \) by estimating the terms on the r.h.s. one by one. We first note that,
from Eqs. (S11) and (S15), both \( \epsilon_k(n) \in [0, 1] \) and \( \epsilon_n(n) \in [0, 1] \) converge to zero in probability as \( n \to \infty \). Thus, the first term in Eq. (S77) converges to zero. For the second term, we have

\[
\prod_{j=1}^{n-1} (1 - \hat{p}_j(n)) \leq \exp\left(-\sum_{j=1}^{n-1} \hat{p}_j(n)\right) = \exp\left(-(n-1) \cdot m_{n-1,n}\right). \tag{S82}
\]

According to Eq. (S81), the factor \( m_{n-1,n} \) above converges to a strictly positive value while the factor \( n - 1 \) diverges, implying that \( \exp\left(-(n-1) \cdot m_{n-1,n}\right) \), and hence the second term on the r.h.s. of Eq. (S77), converges to zero as \( n \to \infty \).

For the third term on the r.h.s. of Eq. (S77), we first note that, for any \( 0 < N < n - 2 \), we have

\[
\sum_{j=1}^{n-2} \epsilon_{j+1}(n) \prod_{i=1}^{j} (1 - \hat{p}_i(n)) \leq \sum_{j=1}^{n-2} \epsilon_{j+1}(n) \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right) = \sum_{j=1}^{n-2} \epsilon_{j+1}(n) \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right)
= \sum_{j=1}^{N} \epsilon_{j+1}(n) \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right) + \sum_{j=N+1}^{n-2} \epsilon_{j+1}(n) \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right)
\leq \sum_{j=1}^{N} \epsilon_{j+1}(n) \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right) + \sum_{j=N+1}^{n-2} \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right)
\leq \sum_{j=1}^{N} \epsilon_{j+1}(n) \exp\left(-\sum_{i=1}^{j} \hat{p}_i(n)\right) + \sum_{j=N+1}^{n-2} \exp\left(-j \cdot m_{jn}\right). \tag{S83}
\]

The first sum on the last line converges to zero in probability as \( n \to \infty \), since each \( \epsilon_{j+1}(n) \) converges to zero in probability, and the sum has only a finite number of terms. For the second sum, we observe that

\[
\sum_{j=N+1}^{n-2} \exp\left(-j \cdot m_{jn}\right) \leq \sum_{j=N+1}^{\infty} \exp\left(-j \cdot m_{\infty}/2\right) = \frac{\exp\left(-(N + 1) \cdot m_{\infty}/2\right)}{1 - \exp\left(-m_{\infty}/2\right)}, \tag{S84}
\]

where we used Eq. (S80) to estimate \( m_{jn} \) for \( j, n > N \) with sufficiently large \( N \) as

\[
m_{jn} \geq \frac{1}{j} \sum_{i=1}^{j} \tilde{d}_i^{\text{out}} \exp\left(-\eta \tilde{a}_i^{\text{in}} + \tilde{a}_i^{\text{out}}\right) \cdot \frac{\tau(d - c_r)}{d} - M' \epsilon(n) \geq \frac{m_{\infty}}{2}. \tag{S85}
\]

Noting that the r.h.s. of Eq. (S84) can be made arbitrarily close to zero by sufficiently increasing \( N \), we conclude that the third term on the r.h.s. of Eq. (S77) also converges to zero in probability.

Therefore, all three terms on the r.h.s. of Eq. (S77) converge to zero in probability as \( n \to \infty \), implying \( \lim_{n \to \infty} p(n) = \lim_{n \to \infty} q_n(n) = 1 \). In view of Eq. (S51), this proves that the probability that \( A \) is reactive converges to one in the limit of large networks.
S2.4. Extension to weighted networks with self-links and Laplacian-coupled networks

We now show that a weighted adjacency matrix $A$ is nonnormal and reactive with probability tending to one as $n \to \infty$ for the weighted GCLV model, in which the link weights are independently and identically distributed. We also allow arbitrary nonzero diagonal elements in the matrix $A$, which does not affect the column-row strict dominance condition and thus condition (C1). For these weighted networks, the probability $r_{ij}(n)$ that $A_{ji} < A_{ij}$ and the probability $s_{ij}(n)$ that $A_{ji} \neq A_{ij}$ have upper and lower bounds, respectively:

\[
    r_{ij}(n) \leq (1 - \rho_{ji})\rho_{ij} + \frac{1}{2}\rho_{ij}\rho_{ji} = \rho_{ij} - \frac{1}{2}\rho_{ij}\rho_{ji},
\]

\[
    s_{ij}(n) \geq (1 - \rho_{ji})\rho_{ij} + (1 - \rho_{ij})\rho_{ji} = \rho_{ij} + \rho_{ji} - 2\rho_{ij}\rho_{ji}.
\]

Using Eq. (S86) in Eq. (S52) yields a lower bound for the probability $p_i(n)$ that the $i$th column strictly dominates the $i$th row for $i \in C_n$. The rest of the proof in Sec. S2.3.2 would then remain valid if we simply redefine $r_{ij}(n) := \rho_{ij} - \frac{1}{2}\rho_{ij}\rho_{ji}$ and $s_{ij}(n) := \rho_{ij} + \rho_{ji} - 2\rho_{ij}\rho_{ji}$, since the difference it creates in the coefficient of the high-order term $\rho_{ij}\rho_{ji}$ in Eq. (S56) does not affect this and the subsequent inequalities. This shows that almost all random weighted directed networks are reactive and thus also nonnormal.

In the case of Laplacian-coupled networks, the adjacency matrix $A$ in Eq. (1) is replaced by $-L = -K + A$, the negative of the Laplacian matrix. Since condition (C1) is satisfied for $X = A$ with probability approaching one as $n \to \infty$ and the addition of diagonal elements from the term $-K$ does not affect condition (C1), we conclude that the probability that $-L$ satisfies condition (C1), and thus the probability that $-L$ is reactive approaches one in the limit of large networks. Since reactivity implies nonnormality, we also conclude that $-L$ is also nonnormal with probability approaching one as $n \to \infty$.

S3. Computational details for fig. S4 and table S1

For a given random network model and for each network size $n$, the probabilities were estimated from $10^6$ network realizations. The unweighted networks were generated using four different network topology models. For the first two, we used the GCLV model with two different in- and out-degree distributions: the gamma distribution $p(x) \sim x^{a-1}e^{-bx}$ with parameters $a = 2$ and $b = 1/2$ and the Dirac delta distribution concentrated at $d = 2$ (which is equivalent to the Erdős–Rényi (ER) model with connection probability $p = d/n$ and fixed $d$). In both cases, the in- and out-degrees were uncorrelated. The remaining two models are the ER model
with a fixed connection probability $p$ and the random $d$-regular networks. For the ER model, we used $p = 0.8$. For the $d$-regular networks, we used $d = 3$ and generated realizations using the configuration model (66). When using these four models in fig. S4 and table S1, we prohibit self-links (i.e., we set $A_{ii} = 0$ for all $i$), which play a limited role in the condition for nonnormality based on Eq. (3) and the condition $\theta_1 > 0$ for reactivity and thus are not expected to significantly affect the probability estimates. In addition, Eq. (31) indicates that neglecting self-links (when they are present) would not overestimate the probability that $A$ is nonnormal (it would, in fact, underestimate it if the $A_{ii}$ are not all identical). The estimated probabilities plotted in fig. S4, A to C are also shown in table S1 under “unweighted networks.” For weighted networks in table S1, each realization was generated by first creating the network topology using one of the models described above and then assigning to each link a random weight drawn from the (discrete) Poisson distribution with mean 9. The numerical results are presented using a threshold of $10^{-8}$ for both nonnormality $\|D\|_F$ and reactivity $\lambda_\Delta(A)$. In the absence of any threshold, it follows from the expected impact of link weights on the imbalances underlying nonnormality and reactivity that networks with continuously distributed random weights are nonnormal and reactive with probability one.
Fig. S1: **Version of Fig. 1 indicating network types.** Both the marker symbols and colors encode the five types of networks as labeled in the data set. We observe that the distributions of nonnormality and reactivity are comparable for (and with large overlaps between) different types of networks.
**Fig. S2:** Topological and spectral imbalances in representative networks. Here, we show the examples of biological, social, and technological networks indicated in Fig. 1. (A) Network of feeding relations between species or groups of species in the cypress wetlands of South Florida during the dry season. (B) Network of friendship relations between boys in an Illinois high school in 1957–1958. (C) Network of connections between Gnutella host computers in 2002. In the first column, the color of node $i$ indicates the generalized degree imbalance $\delta_i^\text{in} - \delta_i^\text{out}$ (positive and negative imbalances in shades of blue and red, respectively), showing that the imbalances are heterogeneously distributed across the network. The second column shows the matrix whose $(i, j)$ component is $d_{ij}^\text{in} - d_{ij}^\text{out}$ if $i \neq j$ and $\delta_i^\text{in} - \delta_i^\text{out}$ if $i = j$. Each matrix component is color coded in (A) and (B), while only the sign of the component is shown in (C) (to highlight the small but numerous imbalances). The node indices are ordered so that $\delta_i^\text{in} - \delta_i^\text{out}$ is non-increasing with respect to $i$. In the third column, the color of node $i$ represents the imbalance $v_1i - u_1i$ between the node’s in- and out-centralities (the corresponding components of left and right eigenvectors normalized so that $\|v_1\| = \|u_1\| = 1$), which is distributed heterogeneously across the network.
Fig. S3: **Nonnormality of adjacency vs. Laplacian matrices.** For a given network, nonnormality of the adjacency matrix $A$ does not necessarily imply nonnormality of the Laplacian matrix $L = K - A$, and vice versa, where we recall that $K$ denotes the diagonal matrix with $\sum_{k\neq i} A_{ik}$ on the diagonal. Indeed, the Laplacian matrix $L$ is nonnormal if and only if $LL^T - L^T L = (AA^T - A^TA) + [K(A - A^T) - (A - A^T)K]$ is nonzero, which is different from the nonnormality condition for $A$, i.e., $AA^T - A^TA \neq 0$. (A) Example (unweighted) network for which $A$ is normal but $L$ is nonnormal. In this case, $AA^T - A^TA$ is zero, but $K(A - A^T) - (A - A^T)K$ is nonzero, rendering $L$ nonnormal. (B) Example (unweighted) network for which $A$ is nonnormal but $L$ is normal. In this case, $AA^T - A^TA$ is nonzero but is canceled exactly by $K(A - A^T) - (A - A^T)K$, rendering $L$ normal.

\[ L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \]
Fig. S4: **Nonnormality and reactivity of typical random networks.** (A to C) Probability that $A$ is nonnormal (A), probability that $A$ is reactive (B), and conditional probability that $A$ is reactive given that it is nonnormal (C), plotted as functions of the network size $n$ for four different models of random unweighted networks (see table S1 for the corresponding cases of weighted networks). The error bars indicate the estimated standard deviation (too small to be visible in some cases). The estimates were computed for each $4 \leq n \leq 12$ but not shown if they equal one, since the vertical axis is on logarithmic scale. The estimates for $n > 12$ are one within numerical precision. The results suggest that, in each case, all three probabilities approach one at least exponentially as $n$ increases. (D to F) Typical ER network with 30 nodes and a connection probability of $p = 0.2$, for which $A$ is both nonnormal and reactive. Node-level imbalances are visualized as in fig. S2. See supplementary text, Sec. S3 for computational details.
Fig. S5: Validating the approximations underlying Eqs. (5) and (6). (A) Leading eigenvalues of $H$ and $H_1$ are plotted against each other for the real networks used in Fig. 3B, validating the approximation $\lambda_1(H) \approx \lambda_1(H_1)$ underlying Eq. (5) in most cases. (B to D) Comparing the three quantities in Eq. (6) with $A_1$ replaced by $A$ for the same set of networks as in (A). In all panels, the quantities are normalized by $\sqrt{\langle w^2 \rangle}$. We observe good agreement between these quantities for most of the networks.
Table S1: Probability that $A$ is normal, probability that $A$ is non-reactive, and conditional probability that $A$ is non-reactive given that $A$ is nonnormal for both weighted and unweighted random networks. See supplementary text, Sec. S3 for computational details.

|                | $n = 4$     | $n = 6$     | $n = 8$     | $n = 10$    | $n = 12$    |
|----------------|-------------|-------------|-------------|-------------|-------------|
| **Unweighted networks:** |             |             |             |             |             |
| $\mathbb{P}(A \text{ is normal})$ |             |             |             |             |             |
| GCLV model (gamma) | 0.04        | 0.0002      | 0           | 0           | 0           |
| GCLV model (ER w/ fixed $d$) | 0.02        | 0.0003      | $9 \times 10^{-6}$ | 0           | 0           |
| ER w/ fixed $p$ | 0.1         | 0.005       | $5 \times 10^{-5}$ | 0           | 0           |
| $d$-regular networks | 1           | 0.09        | 0.0002      | 0           | 0           |
| $\mathbb{P}(A \text{ is non-reactive})$ |             |             |             |             |             |
| GCLV model (gamma) | 0.04        | 0.0002      | $3 \times 10^{-6}$ | 0           | 0           |
| GCLV model (ER w/ fixed $d$) | 0.03        | 0.002       | $0.0003 \times 6 \times 10^{-5}$ | $2 \times 10^{-5}$ |
| ER w/ fixed $p$ | 0.1         | 0.005       | $5 \times 10^{-5}$ | 0           | 0           |
| $d$-regular networks | 1           | 1           | 1           | 1           | 1           |
| $\mathbb{P}(A \text{ is non-reactive} \mid A \text{ is nonnormal})$ |             |             |             |             |             |
| GCLV model (gamma) | 0.0002      | $3 \times 10^{-5}$ | $3 \times 10^{-6}$ | 0           | 0           |
| GCLV model (ER w/ fixed $d$) | 0.006       | 0.001       | $0.0003 \times 6 \times 10^{-5}$ | $2 \times 10^{-5}$ |
| ER w/ fixed $p$ | 0.0002      | $7 \times 10^{-5}$ | $2 \times 10^{-6}$ | 0           | 0           |
| $d$-regular networks | –           | 1           | 1           | 1           | 1           |
| **Weighted networks:** |             |             |             |             |             |
| $\mathbb{P}(A \text{ is normal})$ |             |             |             |             |             |
| GCLV model (gamma) | 0.0001      | $10^{-6}$   | 0           | 0           | 0           |
| GCLV model (ER w/ fixed $d$) | 0.0004      | $7 \times 10^{-6}$ | 0           | 0           | 0           |
| ER w/ fixed $p$ | $2 \times 10^{-6}$ | 0           | 0           | 0           | 0           |
| $d$-regular networks | 0.003       | 0.0001      | $10^{-6}$   | 0           | 0           |
| $\mathbb{P}(A \text{ is non-reactive})$ |             |             |             |             |             |
| GCLV model (gamma) | 0.0001      | $4 \times 10^{-6}$ | 0           | 0           | 0           |
| GCLV model (ER w/ fixed $d$) | 0.0007      | $8 \times 10^{-5}$ | $2 \times 10^{-5}$ | $8 \times 10^{-6}$ | $4 \times 10^{-6}$ |
| ER w/ fixed $p$ | $2 \times 10^{-6}$ | 0           | 0           | 0           | 0           |
| $d$-regular networks | 0.03        | 0.02        | 0.009       | 0.008       | 0.005       |
| $\mathbb{P}(A \text{ is non-reactive} \mid A \text{ is nonnormal})$ |             |             |             |             |             |
| GCLV model (gamma) | $2 \times 10^{-5}$ | 3 \times 10^{-6} | 0           | 0           | 0           |
| GCLV model (ER w/ fixed $d$) | 0.0003      | $8 \times 10^{-5}$ | $2 \times 10^{-5}$ | $8 \times 10^{-6}$ | $4 \times 10^{-6}$ |
| ER w/ fixed $p$ | 0           | 0           | 0           | 0           | 0           |
| $d$-regular networks | 0.03        | 0.02        | 0.009       | 0.008       | 0.005       |
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