Whittaker modules for the Schrödinger-Virasoro algebra

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Abstract

In this paper, Whittaker modules for the Schrödinger-Virasoro algebra $\mathfrak{sv}$ are defined. The Whittaker vectors and the irreducibility of the Whittaker modules are studied. $\mathfrak{sv}$ has a triangular decomposition according to the Cartan algebra $\mathfrak{h}$:

$$\mathfrak{sv} = \mathfrak{sv}^- \oplus \mathfrak{h} \oplus \mathfrak{sv}^+.$$

For any Lie algebra homomorphism $\psi : \mathfrak{sv}^+ \to \mathbb{C}$, we can define Whittaker modules of type $\psi$. When $\psi$ is nonsingular, the Whittaker vectors, the irreducibility and the classification of Whittaker modules are completely determined. When $\psi$ is singular, by constructing some special Whittaker vectors, we find that the Whittaker modules are all reducible. Moreover, we get some more precise results for special $\psi$.

2000 Mathematics Subject Classification: 17B10, 17B35, 17B65, 17B68

Keywords: Schrödinger-Virasoro algebra, Whittaker vector, Whittaker module, induced module, irreducible module.

1 Introduction

The Schrödinger-Virasoro algebra $\mathfrak{sv}$, playing important roles in mathematics and statistical physics, is a infinite-dimensional Lie algebra first introduced by M. Henkel in [7] by looking at the invariance of the free Schrödinger equation. This infinite-dimensional Lie algebra contains both the Lie algebra of invariance of the free Schrödinger equation and the centerless Virasoro algebra (Witt algebra) as

* Supported by the National Natural Science Foundation of China (No. 10671160).
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subalgebras. As natural deformations of the Schrödinger-Virasoro algebra \( \mathfrak{sv} \), the twisted Schrödinger-Virasoro algebra, \( \varepsilon \)-deformation Schrödinger-Virasoro algebra, the extended Schrödinger-Virasoro algebra and the generalized Schrödinger-Virasoro algebras are introduced in [20]-[22]. The derivations, the 2-cocycles, the central extensions and the automorphisms for these algebras have been well studied by many authors (e.g., [6], [11], [20]-[23]).

With respect to the representation theory for Schrödinger-Virasoro algebra, the weight modules is well studied in [10], there it is proved that an irreducible weight module with finite-dimensional weight spaces over the Schrödinger-Virasoro algebras is a highest/lowest weight module or a uniformly bounded module. This is the analogue of a well known classical result in the Virasoro algebra setting conjectured by V. Kac and proved or partially proved by many authors (see [12], [13] and [19]).

In this paper, we construct and study the so called Whittaker modules for the Schrödinger-Virasoro algebra \( \mathfrak{sv} \) which are not weight modules.

The notion of Whittaker modules is first introduced by D. Arnal and G. Pinczon in [1] in the process of construction of a very vast family of representations for \( \mathfrak{sl}(2) \). The versions of Whittaker modules of the complex semisimple Lie algebras are generalized by Kostant in [9]. The prominent role played by Whittaker modules is illustrated by the main result in [3] about the classification of the irreducible modules for \( \mathfrak{sl}_2(\mathbb{C}) \). The result illustrate that the irreducible \( \mathfrak{sl}_2(\mathbb{C}) \)-modules fall into three families: highest (lowest) weight modules, Whittaker modules, and a third family obtained by localization. Since the construction of Whittaker modules depends on the triangular decomposition of a finite-dimensional complex semisimple Lie algebras, it is natural to consider Whittaker modules for other algebras with a triangular decomposition. Recently, the Whittaker modules for Virasoro algebras, Heisenberg algebras, affine Lie algebras as well as generalized Weyl algebras are studied by M. Ondrus, E. Wiesner, K. Christodouloupolou, G. Benkart, etc. (see [2], [5], [14], [16] and [17]).

The Schrödinger-Virasoro algebra \( \mathfrak{sv} \) has a triangular decomposition: \( \mathfrak{sv} = \mathfrak{sv}^- \oplus \mathfrak{h} \oplus \mathfrak{sv}^+ \). For any Lie algebra homomorphism \( \psi : \mathfrak{sv}^+ \rightarrow \mathbb{C} \), we can define Whittaker modules of type \( \psi \) for \( \mathfrak{sv} \). Moreover, for \( \xi \in \mathbb{C} \), we can construct two special Whittaker modules \( W_\psi \) and \( L_{\psi,\xi} \) for \( \mathfrak{sv} \) (see section 2). In section 3 and section 4 we will study the Whittaker modules of nonsingular type. In section 3, the Whittaker vectors of \( W_\psi \) and \( L_{\psi,\xi} \) are studied. In section 4, The classification of the irreducible Whittaker modules of nonsingular type is studied. In the final section, we study the Whittaker modules of singular type. The Whittaker vectors of \( W_\psi \) and \( L_{\psi,\xi} \) are studied. By constructing some special Whittaker vectors, we see that \( L_{\psi,\xi} \) are all reducible. We also get some more precise results for special \( \psi \).

Throughout this paper the symbols \( \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \) and \( \sum \) represent for the complex field, the set of nonnegative integers, the set of integers, the set of positive integers and the sum with finite summands respectively.
2 Definitions and Notations

The Schrödinger-Virasoro algebra $\mathfrak{sv}$ is defined to be a Lie algebra with $\mathbb{C}$-basis 
\[
\{ L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z} \}
\]
subject to the following Lie brackets:
\[
\begin{align*}
[L_m, L_n] &= (n-m)L_{n+m}, \\
[L_m, Y_{n+\frac{1}{2}}] &= (n+\frac{1-m}{2})Y_{m+n+\frac{1}{2}}, \\
[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] &= (n-m)M_{m+n+1}, \\
[M_m, M_n] &= [M_m, Y_{n+\frac{1}{2}}] = 0.
\end{align*}
\]

It is easy to see the following facts about $\mathfrak{sv}$:

(i) The center of $\mathfrak{sv}$ is $\mathbb{C}M_0$.

(ii) $\mathfrak{sv}$ is a semi-direct product of the Witt algebra $\mathfrak{Vir}_0 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n$ and the two-step nilpotent infinite-dimensional Lie algebra $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}M_n \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C}Y_{\frac{1}{2}+n}$.

(iii) $\mathfrak{sv}$ has a triangular decomposition according to the Cartan algebra $\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}M_0$:
\[
\mathfrak{sv} = \mathfrak{sv}^- \oplus \mathfrak{h} \oplus \mathfrak{sv}^+,
\]
where
\[
\begin{align*}
\mathfrak{sv}^+ &= \text{span}_\mathbb{C}\{ L_n, M_n, Y_{\frac{1}{2}+m} \mid m \in \mathbb{N}, n \in \mathbb{Z}+ \}, \\
\mathfrak{sv}^- &= \text{span}_\mathbb{C}\{ L_{-n}, M_{-n}, Y_{-\frac{1}{2}-m} \mid m \in \mathbb{N}, n \in \mathbb{Z}+ \}.
\end{align*}
\]

(iv) $\mathfrak{sv}^+$ (resp. $\mathfrak{sv}^-$) is generated by $L_1, L_2, M_1$ and $Y_{\frac{1}{2}}$ (resp. $L_{-1}, L_{-2}, M_{-1}$ and $Y_{-\frac{1}{2}}$).

In the following of this section we give some notations which will be frequently used to describe the basis of the universal enveloping algebra $U(\mathfrak{sv})$ and the basis of Whittaker modules for the Schrödinger-Virasoro algebra. Set
\[
\mathfrak{b}^+ = \mathfrak{sv}^+ \oplus \mathfrak{h}, \quad \mathfrak{b}^- = \mathfrak{sv}^- \oplus \mathfrak{h}.
\]

Let $\mathbb{C}[M_0]$ be the polynomial algebra generated by $M_0$. Obviously, $\mathbb{C}[M_0]$ is contained in $Z(\mathfrak{sv})$, the center of $U(\mathfrak{sv})$.

As in [18], for a non-decreasing sequence of positive integers: $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_s$, we call $\mu = (\mu_1, \mu_2, \cdots, \mu_s)$ a partition, and for a non-decreasing sequence of non-negative integers: $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$, we call $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ a pseudopartition. Let $\mathcal{P}$ denote the set of partitions, and let $\tilde{\mathcal{P}}$ represent the set of pseudopartitions. Then $\mathcal{P} \subset \tilde{\mathcal{P}}$. For $\lambda \in \mathcal{P}$, we also write $\tilde{\lambda} = (0^{\lambda(0)}, 1^{\lambda(1)}, 2^{\lambda(2)}, \ldots)$, where $\lambda(k)$ is the number of times of $k$ appears in the pseudopartition and $\lambda(k) = 0$ for $k$ sufficiently large. Then a pseudopartition $\tilde{\lambda}$ is a partition whenever $\lambda(0) = 0$. 

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For \( \mu = (\mu_1, \mu_2, \cdots, \mu_s) \in \mathcal{P}, \) \( \tilde{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_r) \) and \( \tilde{\nu} = (\nu_1, \nu_2, \cdots, \nu_t) \in \tilde{\mathcal{P}}, \) we define

\[
|\tilde{\lambda}| = \lambda_1 + \lambda_2 + \cdots + \lambda_r,
\]

\[
\frac{1}{2} + \tilde{\nu} = (\frac{1}{2} + \nu_1, \frac{1}{2} + \nu_2, \cdots, \frac{1}{2} + \nu_t),
\]

\[
|\frac{1}{2} + \tilde{\nu}| = (\frac{1}{2} + \nu_1) + (\frac{1}{2} + \nu_2) + \cdots + (\frac{1}{2} + \nu_t),
\]

\[
\#(\tilde{\lambda}) = \lambda(0) + \lambda(1) + \cdots,
\]

\[
\#(\mu, \tilde{\nu}, \tilde{\lambda}) = \#(\mu) + \#(\tilde{\nu}) + \#(\tilde{\lambda}),
\]

\[
L_{-\tilde{\lambda}} = L_{-\lambda_1} \cdots L_{-\lambda_2} L_{-\lambda_1} = \cdots L_{-2} L_{-1} L_{-2}^{-1} L_{-1}^{\lambda(0)},
\]

\[
M_{-\mu} = M_{-\mu_1} \cdots M_{-\mu_2} M_{-\mu_1} = \cdots M_{-2} M_{-1}^{\mu(1)},
\]

\[
Y_{-\frac{1}{2} - \tilde{\nu}} = Y_{-\frac{1}{2} - \nu_1} \cdots Y_{-\frac{1}{2} - \nu_2} Y_{-\frac{1}{2} - \nu_1} = \cdots Y_{-1}^{(1)} Y_{-1}^{(0)}.
\]

For the sake of convenience, we define \( \bar{0} = (0^0, 1^0, 2^0, \cdots) \) and set \( L_0 = M_0 = M_{\frac{1}{2} + \bar{0}} = 1 \in U(\mathfrak{sv}). \) In the following, we regard \( \bar{0} \) as an element of \( \mathcal{P} \) and \( \tilde{\mathcal{P}}. \)

For any \( (\mu, \tilde{\nu}, \tilde{\lambda}) \in \mathcal{P} \times \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \) and \( p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) \in \mathbb{C}[M_0], \) it is obvious that

\[
p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) M_{-\mu} Y_{-\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}} \in U(\mathfrak{sv})_{-(|\mu| + |\frac{1}{2} + \tilde{\nu}| + |\tilde{\lambda}|)},
\]

where \( U(\mathfrak{sv})_a = \{ x \in U(\mathfrak{sv}) | [L_0, x] = ax \} \) is the \( a \)-weight space of \( U(\mathfrak{sv}). \)

**Definition 2.1.** Let \( V \) be a \( \mathfrak{sv} \)-module and let \( \psi : \mathfrak{sv}^+ \to \mathbb{C} \) be a Lie algebra homomorphism. A vector \( v \in V \) is called a Whittaker vector if \( xv = \psi(x)v \) for every \( x \in \mathfrak{sv}^+ \). A \( \mathfrak{sv} \)-module \( V \) is called a Whittaker module of type \( \psi \) if there is a Whittaker vector \( w \in V \) which generates \( V \). In this case we call \( w \) the cyclic Whittaker vector.

The Lie algebra homomorphism \( \psi \) is called nonsingular if \( \psi(M_1) \) is nonzero, otherwise \( \psi \) is called singular. The Lie brackets in the definition of \( \mathfrak{sv} \) force \( \psi(L_n) = \psi(M_m) = \psi(Y_{\frac{1}{2} + k}) = 0 \) for \( n \geq 3, m \geq 2, k \geq 1. \)

For a Lie algebra homomorphism \( \psi : \mathfrak{sv}^+ \to \mathbb{C} \), we define \( \mathbb{C}_\psi \) to be the one-dimensional \( \mathfrak{sv}^+ \)-module given by \( x\alpha = \psi(x)\alpha \) for \( x \in \mathfrak{sv}^+ \) and \( \alpha \in \mathbb{C} \). Then we have an induced \( \mathfrak{sv} \)-module

\[
W_\psi = U(\mathfrak{sv}) \otimes U(\mathfrak{sv}^+) \mathbb{C}_\psi.
\]  

(2.1)

For \( \xi \in \mathbb{C}, (M_0 - \xi)W_\psi \) is a submodule of \( W_\psi \) since \( M_0 \) is in the center of \( \mathfrak{sv} \). Set

\[
L_{\psi, \xi} := W_\psi / (M_0 - \xi)W_\psi.
\]  

(2.2)

Then \( L_{\psi, \xi} \) is a quotient module for \( \mathfrak{sv} \). The following facts about \( W_\psi \) are obvious:
(i) $W_\psi$ is a Whittaker module of type $\psi$, with cyclic Whittaker vector $w := 1 \otimes 1$;

(ii) The set
\[
\{ M_0^k M_{-\nu} Y_\frac{1}{2} \bar{\nu} L_{-\bar{\lambda}} w \mid (\mu, \tilde{\nu}, \bar{\lambda}) \in \mathcal{P} \times \tilde{\mathcal{P}} \times \tilde{\mathcal{P}}, k \in \mathbb{N} \}
\] (2.3)
forms a basis of $W_\psi$. This follows from the PBW theorem and the fact that
\[
\{ M_0^k M_{-\nu} Y_\frac{1}{2} \bar{\nu} L_{-\bar{\lambda}} w \mid (\mu, \tilde{\nu}, \bar{\lambda}) \in \mathcal{P} \times \tilde{\mathcal{P}} \times \tilde{\mathcal{P}}, k \in \mathbb{N} \}
\] (2.4)
is a basis of $U(b^-)$;

(iii) $W_\psi$ has the universal property in the sense that for any Whittaker module $V$ of type $\psi$ generated by $w'$, there is a surjective homomorphism $\varphi : W_\psi \to V$ such that $uw \mapsto uw'$, $\forall u \in U(b^-)$. Hence we call $W_\psi$ the universal Whittaker module of type $\psi$.

For any $0 \neq v = \sum p_{\mu, \tilde{\nu}, \bar{\lambda}}(M_0) M_{-\nu} Y_\frac{1}{2} \bar{\nu} L_{-\bar{\lambda}} w \in W_\psi$, we define
\[
\text{maxdeg}(v) := \max\{|\mu| + \frac{1}{2} + \tilde{\nu} + |\bar{\lambda}| |p_{\mu, \tilde{\nu}, \bar{\lambda}}(M_0) \neq 0\},
\]
\[
\text{max}_{L_0}(v) := \max\{|\lambda(0)| |p_{\mu, \tilde{\nu}, \bar{\lambda}}(M_0) \neq 0\}.
\]
We set \text{maxdeg}(w) = 0, \text{maxdeg}(0) = -\infty.

**Remark 2.2.** For any $x \in U(su^+)$, $w' = uw$, $u \in U(b^-)$, we have
\[
(E_n - \psi(E_n))w' = [E_n, u]w.
\]
In particular,
\[
(E_n - \psi(E_n))w' = [E_n, u]w,
\]
where $E_n = L_n$ or $M_n$ or $Y_{\frac{1}{2} + (n-1)}$, $\forall n \in \mathbb{Z}_+$.

For $m \in \mathbb{Z}_+, n, k \in \mathbb{N}$, $\mu \in \mathcal{P}$, $\tilde{\lambda}, \tilde{\nu} \in \tilde{\mathcal{P}}$, we give some identities of $U(su)$, each of them can be checked by induction on $\#(\lambda)$ or $\#(\bar{\nu})$ or $a \in \mathbb{N}$:
\[
M_m L_{-\bar{\lambda}} = \sum a_i M_{-m_i} L_{-\bar{\lambda}_i} + \sum b_i L_{-\bar{\lambda}_i} M_{n_i} + L_{-\bar{\lambda}} M_m,
\] (2.5)
where $a_i, b_i \in \mathbb{C}$, $m_i \geq 0$, $0 < n_i \leq m$, $|\tilde{\lambda}_i| + m_i = |\tilde{\lambda}_i| - n_i = |\tilde{\lambda}| - m$, and $\lambda''(i)(0) < \lambda(0)$ if $n_i = m$.

\[
M_m L_{-k}^a = \sum_{i=0}^{a} (-1)^i \left( \prod_{j=0}^{i-1} (m - jk) \right) \binom{a}{i} L_{-k}^a M_{-ik}.
\] (2.6)
\[
Y_{\frac{1}{2} + n} L_{-\bar{\lambda}} = \sum a_i Y_{\frac{1}{2} - m_i} L_{-\bar{\lambda}_i} + \sum b_i L_{-\bar{\lambda}_i} Y_{\frac{1}{2} + n_i} + L_{-\bar{\lambda}} Y_{\frac{1}{2} + n},
\] (2.7)
where $a_i, b_i \in \mathbb{C}, 0 \leq n_i \leq n$, $|\tilde{\lambda}'(i)| + (\frac{1}{2} + m_i) = |\tilde{\lambda}''(i)| - (\frac{1}{2} + n_i) = |\tilde{\lambda}| - (\frac{1}{2} + n)$, and $\lambda''(0) < \lambda(0)$ if $n_i = n$.

$$Y_{\frac{1}{2} + n}Y_{-\frac{1}{2} - \tilde{\nu}} = \sum b_i Y_{-\frac{1}{2} - \tilde{\nu}'(i)} M_{n_i} + Y_{-\frac{1}{2} - \tilde{\nu}} Y_{\frac{1}{2} + n},$$

(2.8)

where $b_i \in \mathbb{C}, n_i \leq n, |\frac{1}{2} + \tilde{\nu}'(i)| - n_i = |\frac{1}{2} + \tilde{\nu}| - (\frac{1}{2} + n)$.

$$L_n L_{-\tilde{\lambda}} = \sum a_i L_{-\tilde{\lambda}'(i)} L_{n_i} + L_{-\tilde{\lambda}} L_n,$$

(2.9)

where $a_i \in \mathbb{C}, n_i \leq n, \tilde{\lambda}'(i) - n_i = |\tilde{\lambda}| - n$, and $\lambda''(0) < \lambda(0)$ if $n_i = n$.

$$L_n M_{-\mu} = \sum a_i M_{-\mu'(i)} + \sum b_i M_{-\mu''(i)} M_{m_i} + M_{-\mu} L_n,$$

(2.10)

where $a_i, b_i \in \mathbb{C}, m_i < n, |\mu'(i)| = |\mu''(i)| - m_i = |\mu| - n$.

$$L_n Y_{\frac{1}{2} - \tilde{\nu}} = \sum a_i Y_{-\frac{1}{2} - \tilde{\nu}'(i)} + \sum b_i Y_{-\frac{1}{2} - \tilde{\nu}'(i)} Y_{\frac{1}{2} + n_i} + \sum c_i Y_{-\frac{1}{2} - \tilde{\nu}''(i)} M_{m_i} + Y_{-\frac{1}{2} - \tilde{\nu}} L_n,$$

(2.11)

where $a_i, b_i, c_i \in \mathbb{C}, m_i, n_i < n, |\frac{1}{2} + \tilde{\nu}'(i)| = |\frac{1}{2} + \tilde{\nu}''(i)| - (\frac{1}{2} + n_i) = |\frac{1}{2} + \tilde{\nu}| - n$.

### 3 Whittaker vectors for Whittaker modules of non-singular type

In this section we always assume that the Lie homomorphism $\psi$ is nonsingular, that is $\psi(M_1) \neq 0$. Let $W_\psi$ and $L_{\psi, \xi}$ be the Whittaker modules for Schrödinger-Virasoro $\mathfrak{sv}$ defined by (2.1) and (2.2) respectively. The main results of this section are given in Theorem 3.5 and Theorem 3.7 in which we characterize the Whittaker vectors in $W_\psi$ and $L_{\psi, \xi}$. For this purpose, we first give a series lemmas which will be used to prove our main results.

**Lemma 3.1.** Let $E_n$ be defined in Remark 2.2, $w = 1 \otimes 1 \in W_\psi$ be the cyclic Whittaker vector. For $n \in \mathbb{Z}_+$,

$$E_n M_{-\mu} Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}} w = v' + v'' + \psi(E_n) M_{-\mu} Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}} w,$$

where $\text{maxdeg}(v') < |\mu| + |\frac{1}{2} + \tilde{\nu}| + |\tilde{\lambda}|$, $\text{maxdeg}(v'') < \lambda(0)$.

**Proof.** If $E_n = M_n$, the result follows from (2.5). If $E_n = Y_{\frac{1}{2} + (n-1)}$, it follows from (2.8), (2.5) and (2.7). If $E_n = L_n$, it follows from (2.10), (2.11), (2.9), (2.5) and (2.7).

**Lemma 3.2.** (i) For $m \in \mathbb{Z}_+, \tilde{\lambda} \in \tilde{P}$, then $\text{maxdeg}(M_m L_{-\tilde{\lambda}} w) \leq |\tilde{\lambda}| - m + 1$;
(ii) For $a, k \in \mathbb{N}$, then

$$[M_{k+1}, L_{-k}^a]w = v - a(k + 1)\psi(M_1)L_{-k}^{a-1}w,$$

where $\maxdeg(v) < (a - 1)k$ if $k > 0$, and $\maxdeg_{La}(v) < a - 1$ if $k = 0$;

(iii) Suppose $\tilde{\lambda} = (k^{\lambda(k)}, (k + 1)^{\lambda(k+1)}, \ldots)$, $\lambda(k) \neq 0$. Then

$$[M_{k+1}, L_{-\tilde{\lambda}}]w = v - (k + 1)\lambda(k)\psi(M_1)L_{-\tilde{\lambda}}w,$$

where $\tilde{\lambda}'$ satisfies $\lambda'(k) = \lambda(k) - 1, \lambda'(i) = \lambda(i)$ for all $i > k$, $\maxdeg(v) < |\tilde{\lambda}| - k$ if $k > 0$ or $v = v' + v''$ with $\maxdeg(v') < |\tilde{\lambda}| - k$ and $\maxdeg_{La}(v'') < \lambda(k) - 1$ if $k = 0$;

**Proof.** (i) follows from (2.5) and the fact that $\psi(M_1) = 0$ if $i \geq 2$. (ii) follows from (2.6). For (iii), we denote $L_{-\tilde{\lambda}} = L_{-\tilde{\lambda}}^{\lambda(k)}$. Then

$$[M_{k+1}, L_{-\tilde{\lambda}}]w = [M_{k+1}, L_{-\tilde{\lambda}}]\lambda^{\lambda(k)} + L_{-\tilde{\lambda}}[M_{k+1}, L_{-\tilde{\lambda}}^{\lambda(k)}]w. \tag{3.1}$$

By using the assumption of $k$, we see that $[M_{k+1}, L_{-\tilde{\lambda}}] \in U(b^-)$ and

$$\maxdeg([M_{k+1}, L_{-\tilde{\lambda}}^{\lambda(k)}]w) < |\tilde{\lambda}| - k.$$

For the second term on the right hand side of (3.1), by using (ii) we see that

$$L_{-\tilde{\lambda}}[M_{k+1}, L_{-\tilde{\lambda}}^{\lambda(k)}]w = L_{-\tilde{\lambda}}v - a(k + 1)\psi(M_1)L_{-\tilde{\lambda}}^{a-1}w,$$

where

$$\maxdeg(L_{-\tilde{\lambda}}v) < (a - 1)k + |\tilde{\lambda}| = |\tilde{\lambda}| - k$$

if $k > 0$, and

$$\maxdeg_{La}(L_{-\tilde{\lambda}}v) < a - 1 = \lambda(0) - 1$$

if $k = 0$. Thus (iii) holds. \qed

**Lemma 3.3.** For $m, k \in \mathbb{N}$, $\tilde{\nu}, \tilde{\tilde{\lambda}} \in \tilde{P}$, we have

(i) \(\maxdeg([Y_{\frac{1}{2}+m}, Y_{\frac{1}{2}+\tilde{\nu}}L_{-\tilde{\tilde{\lambda}}}])w \leq \left| \frac{1}{2} + \tilde{\nu} \right| + |\tilde{\lambda}| - \left( \frac{1}{2} + m \right) + 1;\)

(ii) If \(\nu(i) = \lambda(i) = 0\) for all \(0 \leq i \leq k\), then

$$\maxdeg([Y_{\frac{1}{2}+k+1}, Y_{\frac{1}{2}+\tilde{\nu}}L_{-\tilde{\lambda}}]w) \leq \left| \frac{1}{2} + \tilde{\nu} \right| + |\tilde{\lambda}| - k - 1;$$

(iii) If \(\lambda(i) = 0\) for all \(0 \leq i \leq k\), \(\nu(j) = 0\) for all \(0 \leq j < k\) and \(\nu(k) \neq 0\), then

$$[Y_{\frac{1}{2}+k+1}, Y_{\frac{1}{2}+\tilde{\nu}}L_{-\tilde{\lambda}}]w = v - 2(1 + k)\psi(M_1)\nu(k)Y_{\frac{1}{2}+\tilde{\nu}}L_{-\tilde{\lambda}},$$
where $\text{maxdeg}(v) < |\tilde{\lambda}| + 1 + \nu - k$, $\nu'$ satisfies that $\nu'(i) = \nu(i)$ for all $i \neq k$ and $\nu'(k) = \nu(k) - 1$.

**Proof.** For (i), note that
\[
[Y_{\frac{1}{2}+m}, Y_{-\frac{1}{2}-\nu}]L_{-\tilde{\lambda}}w = [Y_{\frac{1}{2}+m}, Y_{-\frac{1}{2}-\nu}]L_{-\tilde{\lambda}}w + Y_{-\frac{1}{2}+\nu}[Y_{\frac{1}{2}+m}, L_{-\tilde{\lambda}}]w.
\]
By using (2.8) and Lemma 3.2 (i) to the first term on the right-hand side of (3.2), we see that
\[
\text{maxdeg}([Y_{\frac{1}{2}+m}, Y_{-\frac{1}{2}-\nu}]L_{-\tilde{\lambda}}w) \leq \frac{1}{2} + \nu + |\tilde{\lambda}| - (\frac{1}{2} + m) + 1.
\]
By using (2.7) to the second term on the right-hand side of (3.2), we see that
\[
\text{maxdeg}(Y_{-\frac{1}{2}+\nu}[Y_{\frac{1}{2}+m}, L_{-\tilde{\lambda}}]w) \leq \frac{1}{2} + \nu + |\tilde{\lambda}| - (\frac{1}{2} + m) + \frac{1}{2}.
\]
Thus (i) holds.

For (ii), by using the assumption of $k$, we see that $[Y_{\frac{1}{2}+k+1}, Y_{-\frac{1}{2}-\nu}] \in U(b^-)$. Thus
\[
\text{maxdeg}([Y_{\frac{1}{2}+k+1}, Y_{-\frac{1}{2}-\nu}]L_{-\tilde{\lambda}}w) \leq |\tilde{\lambda}| + \frac{1}{2} + \nu - (\frac{1}{2} + k + 1).
\]
By using (2.7), we see that
\[
\text{maxdeg}(Y_{-\frac{1}{2}+\nu}[Y_{\frac{1}{2}+k+1}, L_{-\tilde{\lambda}}]w) \leq |\tilde{\lambda}| + \frac{1}{2} + \nu - (k + 1).
\]
Thus (ii) follows.

Finally, for (iii), we denote $Y_{-\frac{1}{2}-\nu} = Y_{-\frac{1}{2}-\nu} Y^{(\nu(k))}$. Then
\[
[Y_{\frac{1}{2}+k+1}, Y_{-\frac{1}{2}-\nu}]L_{-\tilde{\lambda}}w = [Y_{\frac{1}{2}+k+1}, Y_{-\frac{1}{2}-\nu}] Y^{(\nu(k))}_{-\frac{1}{2}-k} L_{-\tilde{\lambda}}w + Y_{-\frac{1}{2}-\nu}[Y_{\frac{1}{2}+k+1}, Y^{(\nu(k))}_{-\frac{1}{2}-k}] L_{-\tilde{\lambda}}w
\]
\[+ Y_{-\frac{1}{2}-\nu} Y^{(\nu(k))}_{-\frac{1}{2}-k}[Y_{\frac{1}{2}+k+1}, L_{-\tilde{\lambda}}]w.
\]
For the first term on the right-hand side of (3.3), since $[Y_{\frac{1}{2}+k+1}, Y_{-\frac{1}{2}-\nu}] \in U(b^-)$, we see that
\[
\text{maxdeg}([Y_{\frac{1}{2}+k+1}, Y_{-\frac{1}{2}-\nu}] Y^{(\nu(k))}_{-\frac{1}{2}-k} L_{-\tilde{\lambda}}w) \leq |\tilde{\lambda}| + \frac{1}{2} + \nu - (\frac{1}{2} + k + 1).
\]
For the second term, since $[Y_{\frac{1}{2}+k+1}, Y^{(\nu(k))}_{-\frac{1}{2}-k}] = -2(k + 1) Y^{(\nu(k)-1)}_{-\frac{1}{2}-k} M_1$ and $[M_1, L_{-\tilde{\lambda}}] \in U(b^-)$ according to the assumption of $k$, by using (2.5), we see that
\[
Y_{-\frac{1}{2}-\nu} Y^{(\nu(k))}_{-\frac{1}{2}-k}[Y_{\frac{1}{2}+k+1}, L_{-\tilde{\lambda}}]w = v' - 2(k + 1) \psi(M_1) \nu(k) Y_{-\frac{1}{2}-\nu} L_{-\tilde{\lambda}}w.
\]
where \( \text{maxdeg}(v') \leq |\lambda| + \frac{1}{2} + |\tilde{\nu}| - (\frac{1}{2} + k + 1) \), \( v' \) satisfies that \( v'(i) = \nu(i) \) for all \( i \neq k \) and \( v'(k) = \nu(k) - 1 \).

For the third term, note that \( \psi(Y_{\frac{1}{2} + k + 1}) = 0 \), by using (2.7), we see that

\[
\text{maxdeg}(Y_{\frac{1}{2} - \tilde{\nu}}^{-\nu(k)}[Y_{\frac{1}{2} + k + 1}, L_{-\lambda}]w) \leq \frac{1}{2} + |\tilde{\nu}| + |\lambda| - k - 1.
\]

Thus (iii) follows. \( \square \)

**Lemma 3.4.** For \( m \in \mathbb{N}, \mu \in \mathcal{P}, \bar{\nu} \in \tilde{\mathcal{P}}, \) we have

\[
\text{maxdeg}([L_m, M_{-\mu}Y_{\frac{1}{2} - \bar{\nu}}]w) \leq |\mu| + \frac{1}{2} + |\bar{\nu}| - m + 1.
\]

**Proof.** By (2.10) and (2.11), we can write \( L_m M_{-\mu}Y_{\frac{1}{2} - \bar{\nu}} \) as a linear combination of the PBW basis (2.4) of \( U(\mathfrak{su}) \):

\[
L_m M_{-\mu}Y_{\frac{1}{2} - \bar{\nu}} = \sum_{\mu', \bar{\nu}', \bar{\nu}''} p_{\mu', \bar{\nu}'}(M_0)M_{-\mu'}Y_{\frac{1}{2} - \bar{\nu}'} + \sum_{\mu'', \bar{\nu}'', n, E_n} p_{\mu'', \bar{\nu}'', n}(M_0)M_{-\mu''}Y_{\frac{1}{2} - \bar{\nu}''}E_n,
\]

where \( n \in \mathbb{Z}, \mu', \mu'' \in \mathcal{P}, \bar{\nu}', \bar{\nu}'', \bar{\nu}'' \in \tilde{\mathcal{P}} \) satisfying \( |\mu'| + |\frac{1}{2} + \bar{\nu}'| = |\mu''| + |\frac{1}{2} + \bar{\nu}''| - n = |\mu| + |\frac{1}{2} + \bar{\nu}|- m; E_n = M_n \) or \( Y_{\frac{1}{2} + (n-1)} \). Noting that \( M_i w = Y_{\frac{1}{2} + j} w = 0 \) for \( i > 1, j > 0 \), we see that Lemma 3.4 holds. \( \square \)

**Theorem 3.5.** Suppose \( \psi(M_1) \neq 0 \) and \( W_\psi \) is the universal Whittaker module for \( \mathfrak{su} \) with cyclic Whittaker vector \( w = 1 \otimes 1 \). Then \( v \in W_\psi \) is a Whittaker vector if and only if \( v = uw \) for some \( u \in \mathbb{C}[M_0] \).

**Proof.** It is obvious that \( uw \) is a Whittaker vector if \( u \in \mathbb{C}[M_0] \) as \( M_0 \) is in the center of \( \mathfrak{su} \).

Let \( w' \in W_\psi \) be an arbitrary vector. We can write \( w' \) as a linear combination of the basis (2.3) of \( W_\psi \):

\[
w' = \sum_{\mu, \tilde{\nu}, \lambda} p_{\mu, \tilde{\nu}, \lambda}(M_0)M_{-\mu}Y_{\frac{1}{2} - \tilde{\nu}}L_{-\lambda}w, \tag{3.4}
\]

where \( p_{\mu, \tilde{\nu}, \lambda}(M_0) \in \mathbb{C}[M_0] \). Set

\[
N := \text{max}\{|\mu| + \frac{1}{2} + |\tilde{\nu}| + |\lambda| p_{\mu, \tilde{\nu}, \lambda}(M_0) \neq 0\},
\]

\[
\Lambda_N := \{(\mu, \tilde{\nu}, \lambda)|p_{\mu, \tilde{\nu}, \lambda}(M_0) \neq 0, |\mu| + \frac{1}{2} + |\tilde{\nu}| + |\lambda| = N\}.
\]
We first show that the Whittaker vectors in $W_\psi$ are all of type $\psi$. In fact, let $\psi' : \mathfrak{so}^+ \to \mathbb{C}$ be a Lie algebra homomorphism which is different from $\psi$. Then there exists at least one element in $\{L_1, L_2, M_1, Y_{\frac{\lambda}{2}}\}$, denoted by $E$, such that $\psi(E) \neq \psi'(E)$. Assume $w'$ is a Whittaker vector of type $\psi'$, then by the definition we have

$$Ew' = \psi'(E)w' = \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \notin \Lambda_N} p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) M_{-\mu} Y_{-\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}} \psi'(E)w.$$

On the other hand, if we denote $K := \max \{\lambda(0)|((\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_N, p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) \neq 0\}$, then by Remark 2.2 and Lemma 3.1 we have

$$Ew' = v' + v'' + \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_N} p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) M_{-\mu} Y_{-\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}} \psi(E)w,$$

where $\maxdeg(v') < N$, $\maxdeg(v'') < K$. By comparing (3.5) and (3.6) we obtain $\psi'(E) = \psi(E)$, which is a contradiction to our assumption that $\psi'(E) \neq \psi(E)$.

Next, for $w'$ defined in (3.4), we want to show that if there is $(0, 0, 0) \neq (\mu, \tilde{\nu}, \tilde{\lambda}) \in \mathcal{P} \times \tilde{\mathcal{P}} \times \mathcal{P}$ such that $p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) \neq 0$, then there is $E_n \in \{L_n, M_n, Y_{\frac{1}{2} + (n-1)}|n \in \mathbb{Z}_+\}$ such that $(E_n - \psi(E_n))w' \neq 0$, which will prove the necessity.

Assume that $p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) \neq 0$ for some $(\mu, \tilde{\nu}, \tilde{\lambda}) \neq (0, 0, 0)$. By Remark 2.2,

$$(E_n - \psi(E_n))w' = \sum_{\mu, \tilde{\nu}, \tilde{\lambda}} p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) [E_n, M_{-\mu} Y_{-\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}]w.$$

Set

$$\kappa := \min \{n \in \mathbb{N}|\mu(n) \neq 0 \text{ or } \nu(n) \neq 0 \text{ or } \lambda(n) \neq 0 \text{ for some } (\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_N\}.$$

We divide the argument into three cases.

**Case I.** $\kappa$ satisfies $\lambda(\kappa) \neq 0$ for some $(\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_N$.

We have

$$(M_{\kappa+1} - \psi(M_{\kappa+1}))w' = \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \notin \Lambda_N} p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) [M_{\kappa+1}, M_{-\mu} Y_{-\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}]w$$

$$+ \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_N} p_{\mu, \tilde{\nu}, \tilde{\lambda}}(M_0) [M_{\kappa+1}, M_{-\mu} Y_{-\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}]w.$$
\[ + \sum_{(\mu, \nu, \lambda) \in \Lambda_N \atop \lambda(\bar{k}) \neq 0} p_{\mu, \nu, \lambda}(M_0)[M_{\bar{k}+1}, M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}]w \tag{3.7} \]

For the first term on the right hand side of (3.7), by using Lemma 3.2 (i), we know that the degree of it is strictly smaller than \( N - \bar{k} \). For the second term on the right hand side of (3.7), note that \( \lambda(i) = 0 \) for \( 0 \leq i \leq \bar{k} \), we have

\[ [M_{\bar{k}+1}, M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}] = M_{-\mu}Y_{-\frac{1}{2} - \nu}[M_{\bar{k}+1}, L_{-\lambda}] \in U(b^-). \]

Thus the degree of it is also strictly smaller than \( N - \bar{k} \). Now using Lemma 3.2 (iii) to the third term on the right hand side of (3.7), we know that it is of the form

\[ v - \sum_{(\mu, \nu, \lambda) \in \Lambda_N \atop \lambda(\bar{k}) \neq 0} (k + 1)\lambda(k)\psi(M_1)p_{\mu, \nu, \lambda}(M_0)M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}w, \]

where if \( \bar{k} = 0 \) then \( v = v' + v'' \) such that \( \text{maxdeg}(v') < N - \bar{k} \) and \( \text{maxdeg}(v'') < \lambda(0) - 1 \), if \( \bar{k} > 0 \) then \( \text{maxdeg}(v) < N - \bar{k} \); \( \bar{X} \) satisfies \( \lambda(\bar{k}) = \lambda(k) - 1 \), for all \( i > \bar{k} \). Thus the degree of the third term is equal to \( N - \bar{k} \). This proves \( (M_{\bar{k}+1} - \psi(M_{\bar{k}+1}))w' \neq 0 \).

**Case II.** \( \bar{k} \) satisfies \( \nu(\bar{k}) = 0 \) for some \((\mu, \nu, \lambda) \in \Lambda_N \) and \( \lambda(\bar{k}) = 0 \) for any \((\mu, \nu, \lambda) \in \Lambda_N \).

In this case, we use \( Y_{\frac{1}{2}+\bar{k}+1} - \psi(Y_{\frac{1}{2}+\bar{k}+1}) \) to act on both sides of (3.4), then

\[
(Y_{\frac{1}{2}+\bar{k}+1} - \psi(Y_{\frac{1}{2}+\bar{k}+1}))w' = \sum_{(\mu, \nu, \lambda) \notin \Lambda_N} p_{\mu, \nu, \lambda}(M_0)[Y_{\frac{1}{2}+\bar{k}+1}, M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}]w \\
+ \sum_{(\mu, \nu, \lambda) \in \Lambda_N \atop \nu(\bar{k}) = 0} p_{\mu, \nu, \lambda}(M_0)[Y_{\frac{1}{2}+\bar{k}+1}, M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}]w \\
+ \sum_{(\mu, \nu, \lambda) \in \Lambda_N \atop \nu(\bar{k}) \neq 0} p_{\mu, \nu, \lambda}(M_0)[Y_{\frac{1}{2}+\bar{k}+1}, M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}]w. \tag{3.8} \]

By using Lemma 3.3 (i) to the first term on the right hand side of (3.8), Lemma 3.3 (ii) to the second term and Lemma 3.3 (iii) to the third term, we have

\[
(Y_{\frac{1}{2}+\bar{k}+1} - \psi(Y_{\frac{1}{2}+\bar{k}+1}))w' = v - \sum_{(\mu, \nu, \lambda) \in \Lambda_N \atop \nu(\bar{k}) \neq 0} 2(\bar{k} + 1)\nu(\bar{k})\psi(M_1)p_{\mu, \nu, \lambda}(M_0)M_{-\mu}Y_{-\frac{1}{2} - \nu}L_{-\lambda}w, \]

where \( \text{maxdeg}(v) < N - \frac{1}{2} - \bar{k} \); \( \nu'(i) = \nu(i) \) for all \( i \neq \bar{k} \) and \( \nu'(\bar{k}) = \nu(\bar{k}) - 1 \). Thus \( (Y_{\frac{1}{2}+\bar{k}+1} - \psi(Y_{\frac{1}{2}+\bar{k}+1}))w' \neq 0 \).
Case III. \( k \) satisfies \( \mu(k) \neq 0 \) for some \((\mu, \bar{\nu}, \bar{\lambda}) \in \Lambda_N \) and \( \lambda(k) = \nu(k) = 0 \) for any \((\mu, \bar{\nu}, \bar{\lambda}) \in \Lambda_N \). Note that in this case \( k > 0 \) since \( \mu \in \mathcal{P} \).

Subcase 1. \( \bar{\lambda} = 0 \) for any \((\mu, \bar{\nu}, \bar{\lambda}) \) with \( p_{\mu, \bar{\nu}, \bar{\lambda}}(M_0) \neq 0 \).

In this subcase, \( w' = \sum p_{\mu, \bar{\nu}}(M_0)M_{-\mu}Y_{-\frac{1}{2}-\bar{\nu}}w \). By using \( L_{k+1} - \psi(L_{k+1}) \) to act on \( w' \), we have

\[
(L_{k+1} - \psi(L_{k+1}))w' = \sum_{(\mu, \bar{\nu}) \in \Lambda_N} p_{\mu, \bar{\nu}}(M_0)[L_{k+1}, M_{-\mu}]Y_{-\frac{1}{2}-\bar{\nu}}w + \sum_{(\mu, \bar{\nu}) \in \Lambda_N} p_{\mu, \bar{\nu}}(M_0)M_{-\mu}[L_{k+1}, Y_{-\frac{1}{2}-\bar{\nu}}]w
\]

\[
+ \sum_{(\mu, \bar{\nu}) \in \Lambda_N} p_{\mu, \bar{\nu}}(M_0)[L_{k+1}, M_{-\mu}Y_{-\frac{1}{2}-\bar{\nu}}]w + \sum_{(\mu, \bar{\nu}) \in \Lambda_N} p_{\mu, \bar{\nu}}(M_0)[L_{k+1}, M_{-\mu}Y_{-\frac{1}{2}-\bar{\nu}}]w. \tag{3.9}
\]

We denote the four terms on the right hand side of (3.9) by \( v_1, v_2, v_3 \) and \( v_4 \) respectively. For

\( \mu = (k^{\mu(k)}, (k+1)^{\mu(k+1)}, \cdots), \mu(k) \neq 0, \)

we denote \( M_{-\mu} = M_{-\mu'}M_{-\mu(k)}, \) where \( \mu' = (k+1)^{\mu(k+1)}, (k+2)^{\mu(k+2)}, \cdots. \) Note that \([L_{k+1}, M_{-\mu'}] \in U(b^-) \) and \([L_{k+1}, M_{-\mu(k)}] = -\mu^{\mu(k)} - 1 M_1, \) we have

\[ v_1 = v_1' - \sum_{(\mu, \bar{\nu}) \in \Lambda_N} (k^{\mu(k)}k\psi(M_1)p_{\mu, \bar{\nu}}(M_0)M_{-\mu(k)}^{\mu(k)-1}Y_{-\frac{1}{2}-\bar{\nu}}w, \]

where \( \maxdeg(v_1') < N - k. \) Thus \( \maxdeg(v_1) = N - k. \) For \( v_i (i = 2, 3), \) note that \([L_{k+1}, Y_{-\frac{1}{2}-\bar{\nu}}] \in U(b^-) \) and \([L_{k+1}, M_{-\mu}Y_{-\frac{1}{2}-\bar{\nu}}] \in U(b^-), \) we have \( \maxdeg(v_1) < N - k. \) Finally, for \( v_4, \) by using Lemma 3.4, we have \( \maxdeg(v_4) < N - k. \) Thus \((L_{k+1} - \psi(L_{k+1}))w' \neq 0. \)

Subcase 2. There exists some \( \bar{\lambda} \neq 0 \) for which \( p_{\mu, \bar{\nu}, \bar{\lambda}}(M_0) \neq 0. \)

Denote

\[ N' := \max\{|\mu| + |\frac{1}{2} + \bar{\nu}| + |\bar{\lambda}| \neq 0, p_{\mu, \bar{\nu}, \bar{\lambda}}(M_0) \neq 0\}, \]

and set

\[ \Lambda_{N'} := \{(\mu, \bar{\nu}, \bar{\lambda})|\bar{\lambda} \neq 0, p_{\mu, \bar{\nu}, \bar{\lambda}}(M_0) \neq 0, |\mu| + |\frac{1}{2} + \bar{\nu}| + |\bar{\lambda}| = N'\}, \]

\[ l := \min\{n| \bar{\lambda} = (n^{\lambda(n)}, (n+1)^{\lambda(n+1)}, \cdots) such that |\mu| + |\frac{1}{2} + \bar{\nu}| + |\bar{\lambda}| = N' \text{ and } p_{\mu, \bar{\nu}, \bar{\lambda}}(M_0) \neq 0}\}. \]
Note that $\tilde{\lambda} = 0$ for those $(\mu, \tilde{\nu}, \tilde{\lambda})$ satisfying $N' < |\mu| + \frac{1}{2} + \tilde{\nu} + |\tilde{\lambda}| \leq N$ and $p_{\mu,\tilde{\nu},\tilde{\lambda}}(M_0) \neq 0$. Thus we have

$$w' = \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_{N'} \atop \lambda(l) \neq 0} p_{\mu,\tilde{\nu},\tilde{\lambda}}(M_0)M_{-\mu}Y_{-\frac{1}{2}-\tilde{\nu}}L_{-\tilde{\lambda}}w$$

$$+ \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_{N'} \atop \lambda(l) = 0} p_{\mu,\tilde{\nu},\tilde{\lambda}}(M_0)M_{-\mu}Y_{-\frac{1}{2}-\tilde{\nu}}L_{-\tilde{\lambda}}w$$

$$+ \sum_{|\mu| + \frac{1}{2} + \tilde{\nu} + |\tilde{\lambda}| < N'} p_{\mu,\tilde{\nu},\tilde{\lambda}}(M_0)M_{-\mu}Y_{-\frac{1}{2} - \tilde{\nu}}L_{-\tilde{\lambda}}w$$

$$+ \sum_{N' < |\mu| + \frac{1}{2} + \tilde{\nu} \leq N} p_{\mu,\tilde{\nu}}(M_0)M_{-\mu}Y_{-\frac{1}{2} - \tilde{\nu}}w.$$ (3.10)

We apply $(M_{l+1} - \psi(M_{l+1}))$ to act on both sides of (3.10) and write the resulting four terms on the right hand side by $v_1$, $v_2$, $v_3$ and $v_4$ respectively. It is obvious that $v_4 = 0$. For $v_1$, note that $l > k > 0$, by Lemma 3.2 (iii), we see that

$$v_1 = v - \sum_{(\mu, \tilde{\nu}, \tilde{\lambda}) \in \Lambda_{N'} \atop \lambda(l) \neq 0} p_{\mu,\tilde{\nu},\tilde{\lambda}}(M_0)(l + 1)\lambda(l)\psi(M_1)M_{-\mu}Y_{-\frac{1}{2} - \tilde{\nu}}L_{-\tilde{\lambda}}w,$$

where $\text{maxdeg}(v) < N' - l$ and $\text{maxdeg}(v_1) = N' - l$. For $v_2$, since $[M_{l+1}, L_{-\tilde{\lambda}}] \in U(b^-)$, we have $\text{maxdeg}(v_2) \leq N' - l - 1$. Finally for $v_3$, by using Lemma 3.2 (i), we see that $\text{maxdeg}(v_3) < N' - l$. These imply that $(M_{l+1} - \psi(M_{l+1}))w' \neq 0$. The proof of Theorem 3.5 is completed. \[\square\]

**Corollary 3.6.** The center of $U(\mathfrak{sv})$ is $\mathbb{C}[M_0]$.

**Proof.** For any $z \in Z(U(\mathfrak{sv}))$, the center of $U(\mathfrak{sv})$, $zw$ is a Whittaker vector, so $z \in \mathbb{C}[M_0]$ by Theorem 3.5. This means $Z(U(\mathfrak{sv})) \subseteq \mathbb{C}[M_0]$ and then $Z(U(\mathfrak{sv})) = \mathbb{C}[M_0]$.

**Theorem 3.7.** Suppose $\psi(M_1) \neq 0$ and $\bar{w} = \overline{1 \otimes 1} \in L_{\psi,\xi}$. Then $v \in L_{\psi,\xi}$ is a Whittaker vector if and only if $v = u\bar{w}$ for some $u \in \mathbb{C}1$.

**Proof.** It is easy to see that the set

$$\{M_{-\mu}Y_{-\frac{1}{2} - \tilde{\nu}}L_{-\tilde{\lambda}}\bar{w}|\mu \in \mathcal{P}, \tilde{\nu}, \tilde{\lambda} \in \tilde{\mathcal{P}}, k \in \mathbb{N}\}$$

forms a basis of $L_{\psi,\xi}$. Then we can use the same argument as in Theorem 3.5 to complete the proof of Theorem 3.7. \[\square\]

**Theorem 3.8.** Let $\psi_1, \psi_2$ be Lie algebra homomorphisms from $\mathfrak{su}^+$ to $\mathbb{C}$ and $\xi_1, \xi_2 \in \mathbb{C}$. Then the $\mathfrak{su}$-modules $L_{\psi_1,\xi_1}$ and $L_{\psi_2,\xi_2}$ are isomorphic if and only if $\psi_1 = \psi_2, \xi_1 = \xi_2$. 13
Proof. Suppose that \( \tilde{w}_1 \) is a cyclic Whittaker vector of \( L_{\psi, \xi_i} \) \((i = 1, 2)\), \( f : L_{\psi, \xi_1} \to L_{\psi, \xi_2} \) is an isomorphism of modules. Then
\[
E_n f(\tilde{w}_1) = f(E_n \tilde{w}_1) = \psi_1(E_n) f(\tilde{w}_1), \quad \forall \ n \in \mathbb{Z}_+.
\]
Thus \( f(\tilde{w}_1) \) is a Whittaker vector of type \( \psi_1 \). This implies \( \psi_1 = \psi_2 \) since there are no Whittaker vectors of type other than \( \psi_2 \) in \( L_{\psi, \xi_2} \) by the proof of Theorem 3.5. Moreover, \( \xi_1 f(\tilde{w}_1) = f(M_0 \tilde{w}_1) = M_0 f(\tilde{w}_1) = \xi_2 f(\tilde{w}_1) \), we get \( \xi_1 = \xi_2 \). This completes the proof. \( \square \)

4 Irreducible Whittaker modules of nonsingular type

In this section, the Lie algebra homomorphism \( \psi \) is assumed to be non-singular, that is \( \psi(M_1) \neq 0 \), we prove that the Whittaker module \( L_{\psi, \xi} \), defined by (2.2), is irreducible, and we also prove that every irreducible Whittaker module of type \( \psi \) for the Schrödinger Virasoro algebra \( \mathfrak{sv} \) is isomorphic to \( L_{\psi, \xi} \) for some \( \xi \in \mathbb{C} \).

Fix a Whittaker module \( V \) of type \( \psi \) with cyclic Whittaker vector \( w \). \( V \) is naturally a \( \mathfrak{sv}^+ \)-module. Following [10] and [18] we define a new action, called dot action, of \( \mathfrak{sv}^+ \) on \( V \) by setting
\[
x \cdot v = xv - \psi(x)v, \quad \text{for } x \in \mathfrak{sv}^+ \text{ and } v \in V. \tag{4.1}
\]
Then it is clear that \( V \) is a \( \mathfrak{sv}^+ \)-module under the dot action, and we have \( E_n \cdot v = E_n v - \psi(E_n)v = [E_n, u]w \) for \( n \in \mathbb{Z}_+ \) and \( v = uw \in V \).

**Lemma 4.1.** If \( n \in \mathbb{Z}_+ \), then \( E_n \) acts locally nilpotent on \( V \) under the dot action.

**Proof.** By the Lie products of \( \mathfrak{sv} \), we see that
\[
ad^2 M_n(L_i) = 0 = adM_n(M_i) = adM_n(Y_{\frac{n}{2}+i}), \forall i \in \mathbb{Z}.
\]
\[
ad^3 Y_{\frac{n}{2}+(n-1)}(L_i) = 0 = adY_{\frac{n}{2}+(n-1)}(M_i) = ad^2 Y_{\frac{n}{2}+(n-1)}(Y_{\frac{n}{2}+i}), \forall i \in \mathbb{Z}.
\]
Thus, for any basis element \( u = M_0^k M_{-\mu} Y_{-\frac{1}{2}+\delta} L_{-\lambda} \) of \( U(\mathfrak{b}^-) \), it is clear that \( ad^2 M_n \), \( ad^3 Y_{\frac{n}{2}+(n-1)} \) act on \( u \) as zero. To prove \( ad^m L_n(u) = 0 \) for \( m \) sufficiently large, we note that \( ad^m L_n(M_{-\mu} Y_{-\frac{1}{2}+\delta} L_{-\lambda}) \in U(\mathfrak{sv})_{-((|\mu|+|\frac{1}{2}+\delta|)+|\lambda|)+nm} \), so \( ad^m L_n(M_{-\mu} Y_{-\frac{1}{2}+\delta} L_{-\lambda}) \) is a combination of basis elements of \( U(\mathfrak{sv}) \) of the form
\[
M_0^k M_{-\mu_1} Y_{-\frac{1}{2}-\bar{\nu}_1} L_{-\bar{\lambda}_1} M_{m_1} Y_{\frac{1}{2}+n_1} \cdots M_{m_l} Y_{\frac{1}{2}+n_l} L_{h} \cdots L_{i_1}, \tag{4.2}
\]
where \( -((|\mu_1|+|\frac{1}{2}+\bar{\nu}_1|+|\bar{\lambda}_1|)+\sum_{i=1}^{p} m_i + \sum_{i=1}^{q} (\frac{1}{2}+n_i) + \sum_{i=1}^{h} l_i = -(|\bar{\lambda}|+|\mu|+|\frac{1}{2}+\bar{\nu}|)+nm, \#(\mu_1) + \#(\bar{\nu}_1) + \#(\lambda_1) + p + q + h \leq \#(\mu, \bar{\nu}, \bar{\lambda}). \) Recall that \( E_n w = 0 \) if
Lemma 4.2. Let \((\mu, \tilde{\nu}, \tilde{\lambda}) \in \mathcal{P} \times \tilde{\mathcal{P}} \times \tilde{\mathcal{P}}, k \in \mathbb{N}\).

(i) For all \(n > 0\), \(E_n M^k_0 M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w \in \text{span}_\mathbb{C} \{M^0_0 M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w \mid |\mu'| + |\tilde{\nu}'| + |\tilde{\lambda}'| + \lambda(0) \leq |\mu| + |\frac{1}{2} + \tilde{\nu}| + |\tilde{\lambda}| + \lambda(0); i = k, k + 1\}.\)

(ii) If \(n > |\mu| + |\frac{1}{2} + \tilde{\nu}| + |\tilde{\lambda}| + 2\), then \(E_n \cdot (M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w) = 0\).

Proof. (i) Since

\[
E_n \cdot (M^k_0 M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w) = M^k_0 (E_n \cdot (M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w)),
\]

we only need to prove (i) for \(k = 0\). The result for \#(\mu, \tilde{\nu}, \tilde{\lambda}) = 0\) is obvious. Now we prove the result for \#(\mu, \tilde{\nu}, \tilde{\lambda}) > 0\) by induction.

For the case \(\mu \neq 0\), set \(m = \max\{i | \mu(i) > 0\}\). Then

\[
M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} = M_m M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-},
\]

where \(\mu'(m) = \mu(m) - 1, \mu'(i) = \mu(i)\) for all \(i \neq k\). Therefore

\[
E_n \cdot M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w
\]

\[
= [E_n, M_m] M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-} w + M_m [E_n, M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^{-}] w. \tag{4.3}
\]

For the first term on the right hand side of (4.3), note that \([E_n, M_m] = 0\) for \(E_n = M_n\) or \(Y_{\frac{1}{2} + (n-1)}\); we only need to consider the case for \(E_n = L_n\). If \(n - m \leq 0\), it is obvious that

\[
[L_n, M_m] M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w = -m M_{n-m} M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w
\]

has the desired form. If \(n - m > 0\),

\[
M_{n-m} M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w = M_{n-m} \cdot (M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w) + \psi(M_{n-m}) M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w.
\]

By assumption, \(M_{n-m} \cdot (M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w)\), and therefore \(M_{n-m} M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w\), has the desired form. For the second term on the right hand side of (4.3), we have, by the induction hypothesis, that

\[
[E_n, M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^-] w \in \text{span}_\mathbb{C} \{M^0_0 M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^- w \mid |\mu''| + |\frac{1}{2} + \tilde{\nu}'| + |\tilde{\lambda}'| + \lambda(0) \leq |\mu| + |\frac{1}{2} + \tilde{\nu}| + |\tilde{\lambda}| + \lambda(0); i = 0, 1; j = 0, 1\}.
\]

Thus

\[
M_m [E_n, M_\mu Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}}^-] w
\]

\[\]
has the desired form since $-m < 0$ and $m + |\tilde{\lambda}| + |\mu| + \frac{|1|}{2} + \tilde{\nu} = |\tilde{\lambda}| + |\mu| + \frac{|1|}{2} + \tilde{\nu}$.

For the case $\mu = 0, \tilde{\nu} \neq 0$ or $\mu = \tilde{\nu} = 0, \tilde{\lambda} \neq 0$, one can prove the result by a similar argument as we did in the first case. This is omitted for shortness.

(ii) Note that
\[
\begin{align*}
\left[ E_n, M_{-\mu} Y_{\frac{1}{2} - \tilde{\nu}} L_{-\tilde{\lambda}} \right] &= \sum_{\mu^1, \tilde{\nu}^1, \tilde{\lambda}^1, E_m} p_{\mu^1, \tilde{\nu}^1, \tilde{\lambda}^1, E_m} M_{-\mu^1} Y_{\frac{1}{2} - \tilde{\nu}^1} L_{-\tilde{\lambda}^1} E_m,
\end{align*}
\]
where $p_{\mu^1, \tilde{\nu}^1, \tilde{\lambda}^1, E_m} \in \mathbb{C}$, and $m = n - (|\mu| + \frac{|1|}{2} + \tilde{\nu} + |\tilde{\lambda}|) + (|\mu^1| + \frac{|1|}{2} + \tilde{\nu}^1 + |\tilde{\lambda}^1|) > 2$.

This implies (ii) as $E_m w = 0$ for any $m > 2$.

**Lemma 4.3.** Suppose $V$ is a Whittaker module for $\mathfrak{su}$, and let $v \in V$. Regarding $V$ as an $\mathfrak{su}^+$-module under the dot action, then $U(\mathfrak{su}^+) \cdot v$ is a finite-dimensional $\mathfrak{su}^+$-submodule of $V$.

**Proof.** This is a direct result of Lemma 4.2. \hfill $\square$

**Lemma 4.4.** Let $V$ be a Whittaker module for $\mathfrak{su}$, and let $S \subseteq V$ be a nonzero submodule. Then there is a nonzero Whittaker vector $w' \in S$.

**Proof.** $\forall 0 \neq v \in S$, by Lemma 4.3, $U(\mathfrak{su}^+) \cdot v$ is a finite-dimensional submodule of $S$. Then by Lemma 4.1, we know that every element $E_n$ of $\mathfrak{su}^+$ is nilpotent on $U(\mathfrak{su}^+) \cdot v$ under the dot action. Then the result of the lemma follows from Engel’s Theorem (Theorem 3.3 in [8]). \hfill $\square$

**Proposition 4.5.** For any $\xi \in \mathbb{C}$, the Whittaker module $L_{\psi, \xi}$ for the Schrödinger Virasoro algebra $\mathfrak{sv}$ is irreducible.

**Proof.** It follows from Lemma 4.4 and Theorem 3.7. \hfill $\square$

It is known (see Lemma 2.1.3 in [4]) that the Schur’s Lemma can be generalized to infinite dimensional irreducible modules with countable cardinality.

**Theorem 4.6.** Let $S$ be an irreducible Whittaker module of type $\psi$ for the Schrödinger Virasoro algebra $\mathfrak{su}$. Then $S \cong L_{\psi, \xi}$ for some $\xi \in \mathbb{C}$.

**Proof.** Let $w_s \in S$ be a cyclic Whittaker vector corresponding to $\psi$. Since $M_0$ acts by a scalar by Schur’s Lemma, there exists $\xi \in \mathbb{C}$ such that $M_0 s = \xi s$ for all $s \in S$. Now by the universal property of $W_\psi$, there exists a module homomorphism $\varphi : W_\psi \to S$ with $uw \mapsto uw_s$. This map is surjective since $w_s$ generates $S$. But then
\[
\varphi((M_0 - \xi)W_\psi) = (M_0 - \xi)\varphi(W_\psi) = (M_0 - \xi)S = 0,
\]
so it follows that
\[
(M_0 - \xi)W_\psi \subseteq \ker \varphi \subseteq W_\psi.
\]
Because $L_{\psi,\xi}$ is irreducible by Proposition 4.5 and $ker\varphi \neq W_\psi$, this forces $ker\varphi = (M_0 - \xi) W_\psi$. □

For a given $\psi : \mathfrak{su}^+ \to \mathbb{C}$ and $\xi \in \mathbb{C}$, note that

$$I = U(\mathfrak{su})(M_0 - \xi) + \sum_{i \in \mathbb{Z}_+} U(\mathfrak{su})(L_i - \psi(L_i)) + \sum_{i \in \mathbb{Z}_+} U(\mathfrak{su})(M_i - \psi(M_i)) + \sum_{i \in \mathbb{N}} U(\mathfrak{su})(Y_{\frac{1}{4}+i} - \psi(Y_{\frac{1}{4}+i}))$$

is a left ideal of $U(\mathfrak{su})$. For $u \in U(\mathfrak{su})$, let $\bar{u}$ denote the coset $u + I \in U(\mathfrak{su})/I$. Then we may regard $U(\mathfrak{su})/I$ as a Whittaker module of type $\psi$ with cyclic Whittaker vector $\bar{1}$. We have the following result.

**Lemma 4.7.** The Whittaker module $V = U(\mathfrak{su})/I$ is irreducible, and thus $V \cong L_{\psi,\xi}$.

**Theorem 4.8.** Suppose that $V$ is a Whittaker module of type $\psi$ such that $M_0$ acts by a scalar $\xi \in \mathbb{C}$, then $V$ is irreducible.

**Proof.** Let $K$ denote the kernel of the natural surjective map $U(\mathfrak{su}) \to V$ given by $u \mapsto uw$, where $w$ is a cyclic Whittaker vector of $V$. Then $K$ is a proper left ideal containing $I$. By Lemma 4.7, $I$ is maximal, and thus $K = I$ and $V \cong U(\mathfrak{su})/I$ is irreducible. □

### 5 Whittaker modules of singular type

From now on, we assume that the Lie algebra homomorphism $\psi$ is singular, that is $\psi(M_1) = 0$.

As in Theorem 3.5 and Theorem 3.7, we use the notation $w$ (resp. $\bar{w}$) to denote the cyclic Whittaker vector $1 \otimes 1$ (resp. $1 \otimes 1$) for $W_\psi$ (resp. $L_{\psi,\xi}$). The following facts about $W_\psi$ and $L_{\psi,\xi}$ are obvious:

(i) $W_\psi = U(\mathfrak{su})w$ is free as a $U(\mathfrak{b}^-)$-module and the set

$$\{M_0^k M_{-\mu} Y_{-\frac{1}{2}-\bar{\nu}} L_{-\frac{1}{2}} \bar{w} | (\mu, \bar{\nu}, \bar{\lambda}) \in \mathcal{P} \times \bar{\mathcal{P}} \times \bar{\mathcal{P}}, k \in \mathbb{N}\} (5.1)$$

forms a basis for $W_\psi$ by the PBW Theorem.

(ii) $L_{\psi,\xi} = U(\mathfrak{su})\bar{w}$ is free as a $U(\mathfrak{su}^- \oplus \mathbb{C} L_0)$-module and the set

$$\{M_{-\mu} Y_{-\frac{1}{2}-\bar{\nu}} L_{-\frac{1}{2}} \bar{w} | (\mu, \bar{\nu}, \bar{\lambda}) \in \mathcal{P} \times \bar{\mathcal{P}} \times \bar{\mathcal{P}}\} (5.2)$$

forms a basis for $L_{\psi,\xi}$.

(iii) $W_\psi$ and $L_{\psi,\xi}$ are $\mathfrak{su}^+$-modules under the dot action defined in (4.1).
Denote $\psi(L_1) = \eta_1, \psi(L_2) = \eta_2$ and $\psi(Y^2) = \eta_3$. Set
\[
z = \begin{cases} 
L_0 & \text{if } \eta_1 = \eta_2 = \eta_3 = 0, \\
L_0 M_0^2 - \eta_2 M_{-2} M_0 - \eta_1 M_{-1} M_0 + \eta_2 M_{-1}^2 - \frac{\eta_2}{2} Y_{-\frac{1}{2}} M_0 + \frac{\eta_2}{2}^2 M_{-1} & \text{if } \eta_2 \neq 0 \text{ or } \eta_3 \neq 0, \\
L_0 M_0 - \eta_1 M_{-1} & \text{if others.}
\end{cases}
\]

**Proposition 5.1.** For $u \in \mathbb{C}[z, M_0]$, $v = uw$ is a Whittaker vector of $W_\psi$.

**Proof.** It is easy to check that $L_1 \cdot v = L_2 \cdot v = M_1 \cdot v = Y^2 \cdot v = 0$. Thus for any $y \in su^+$, we have $y \cdot v = 0$. \qed

**Proposition 5.2.** If we set
\[
z = \begin{cases} 
L_0 & \text{if } \eta_1 = \eta_2 = \eta_3 = 0, \\
\xi^2 L_0 - \xi \eta_2 M_{-2} - \xi \eta_1 M_{-1} + \eta_2 M_{-1}^2 - \frac{\eta_2}{2} Y_{-\frac{1}{2}} + \frac{\eta_2}{2}^2 M_{-1} & \text{if } \eta_2 \neq 0 \text{ or } \eta_3 \neq 0, \\
\xi L_0 - \eta_1 M_{-1} & \text{if others,}
\end{cases}
\]

then for $u \in \mathbb{C}[z]$, $u\bar{w}$ is a Whittaker vector of $L_{\psi, \xi}$.

**Proof.** It is easy to check, we omit the details. \qed

**Theorem 5.3.** If $\psi$ is singular, then $L_{\psi, \xi}$ is reducible for any $\xi \in \mathbb{C}$.

**Proof.** By Proposition 5.2, we can easily see that the submodule $V$ generated by $z\bar{w} \in L_{\psi, \xi}$ is a proper Whittaker submodule. \qed

If $\psi$ is identically zero, that is $\eta_1 = \eta_2 = \eta_3 = \psi(M_1) = 0$, then we have the following more precise results.

**Theorem 5.4.** If $\psi$ is identically zero, then the set of Whittaker Vectors of $W_\psi$ is $\mathbb{C}[L_0, M_0]w$.

**Proof.** By Proposition 5.1, we see that each element of $\mathbb{C}[M_0, L_0]w$ is a Whittaker vector of $W_\psi$. For any $v \in W_\psi \setminus \mathbb{C}[M_0, L_0]w$, noting that $W_\psi$ has a basis given by (5.1), we can write it as
\[
v = \sum a_{k,l}^{\mu, \nu, \lambda} M_{-\mu} Y_{-\frac{1}{2} - \nu} L_{-\lambda} L_0^k M_0^l w, \quad (5.3)
\]
where $\mu, \lambda \in \mathbb{P}, \nu \in \overline{\mathbb{P}}$ satisfying $\mu \neq 0$ or $\nu \neq 0$ or $\lambda \neq 0$ for some $a_{k,l}^{\mu, \nu, \lambda} \neq 0$. For $v$ defined in (5.3), we need to prove that there exists $x \in su^+$ such that $x \cdot v \neq 0$.

**Case 1.** There exists $\lambda \neq 0$ such that $a_{k,l}^{\mu, \nu, \lambda} \neq 0$ in (5.3).

We denote
\[
p = \max \{ \#(\lambda) \mid a_{k,l}^{\mu, \nu, \lambda} \neq 0 \}
\]

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and

\[ N = \max \{ \lambda_p \mid a_{k,l}^{\mu,\bar{\nu},\lambda} \neq 0 \}. \]

Then \( N \geq 1 \) and \( v \) is of the form

\[
v = \sum_{\lambda_p = N} a_{k,l}^{\mu,\bar{\nu},\lambda} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_{-\lambda} L_0^k M_0^l w + \sum_{\lambda_p < N} a_{k,l}^{\mu,\bar{\nu},\lambda} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_{-\lambda} L_0^k M_0^l w. \tag{5.4} \]

By using \( M_N \) to act on both sides of (5.4) by dot action, we see that

\[
M_N \cdot v = - \sum_{\#(\lambda') = p-1} \lambda(N) N a_{k,l}^{\mu,\bar{\nu},\lambda} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_{-\lambda'} L_0^k M_0^{l+1} w + \sum_{\#(\lambda'') < p-1} \sum_{\lambda''} b_{k,l}^{\mu,\bar{\nu},\lambda''} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_{-\lambda''} L_0^k M_0^l w \neq 0,
\]

where \( \lambda'(i) = \lambda(i) \) if \( i \neq p \) and \( \lambda'(p) = \lambda(p) - 1, b_{k,l}^{\mu,\bar{\nu},\lambda''} \in \mathbb{C} \).

**Case 2.** \( \lambda = \bar{\nu} \) for any \( a_{k,l}^{\mu,\bar{\nu},\lambda} \neq 0 \) and there exists \( \bar{\nu} \neq \bar{\nu} \) such that \( a_{k,l}^{\mu,\bar{\nu},\lambda} \neq 0 \) in (5.3).

In this case, (5.3) becomes

\[
v = \sum a_{k,l}^{\mu,\bar{\nu}} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_0^k M_0^l w. \tag{5.5} \]

For (5.5), we set

\[
b := \max \{ \nu_s \mid \bar{\nu} = (\nu_1, \ldots, \nu_s), a_{k,l}^{\mu,\bar{\nu}} \neq 0 \}.\]

Then (5.5) can be rewritten as

\[
v = \sum_{\nu(b) = 0} a_{k,l}^{\mu,\bar{\nu}} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_0^k M_0^l w + \sum_{\nu(b) = 0} a_{k,l}^{\mu,\bar{\nu}} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} L_0^k M_0^l w. \tag{5.6} \]

By using \( Y_{\frac{i}{2} + b} \) to act on both sides of (5.6) by dot action, we have

\[
Y_{\frac{i}{2} + b} \cdot v = \sum_{\nu(b) \neq 0} (-1 - 2b)\nu(b) a_{k,l}^{\mu,\bar{\nu}} M_{-\mu} Y_{\frac{i}{2} - \bar{\nu}} Y_{\nu(b)}^{-1} L_0^k M_0^{l+1} w \neq 0,
\]

where \( \bar{\nu}'(i) = \bar{\nu}(i) \) for \( i \neq b \) and \( \bar{\nu}'(b) = 0 \).

**Case 3.** \( \lambda = \bar{\nu} = \bar{\nu} \) for any \( a_{k,l}^{\mu,\bar{\nu},\lambda} \neq 0 \) in (5.3).

In this case, (5.3) becomes

\[
v = \sum a_{k,l}^{\mu} M_{-\mu} L_0^k M_0^l w. \tag{5.7} \]

For (5.7), we set

\[
c := \max \{ \mu_t \mid \mu = (\mu_1, \ldots, \mu_t), a_{k,l}^{\mu} \neq 0 \}.\]
Then (5.7) can be rewritten as

\[ v = \sum_{\mu(c) \neq 0} a_{k,l}^c M_{-\mu} L_0^k M_0^l w + \sum_{\mu(c) = 0} a_{k,l}^c M_{-\mu} L_0^k M_0^l w. \] (5.8)

By using \( L_c \) to act on both sides of (5.8) by dot action, we have

\[ L_c \cdot v = \sum_{\mu(c) \neq 0} (-c)\mu(c) a_{k,l}^c M_{-\mu} M_{\mu(c)}^{-1} L_0^k M_0^l w \neq 0, \]

where \( \mu'(i) = \mu^i \) for \( i \neq a \) and \( \mu'(a) = 0 \). This completes the proof.

Theorem 5.5. If \( \psi \) is identically zero, \( \xi \neq 0 \), then the set of Whittaker vectors of \( L_{\psi,\xi} \) is \( C[L_0] \bar{w} \).

Proof. Noting that \( L_{\psi,\xi} \) has a basis defined by (5.2), we can repeat the proof of Theorem 5.4 word for word except that \( M_0 \) should be replaced by \( \xi \) and \( w \) replaced by \( \bar{w} \).

Recall the definition of Verma module of generalized Schrödinger-Virasoro algebras given in [21]. We observe that if \( \psi \) is identically zero and \( V_\zeta \) is the submodule of \( L_{\psi,\xi} \) generated by \( (L_0 - \zeta) \bar{w} \), where \( \zeta \in C \), then the quotient module

\[ V(\xi, \zeta) := L_{\psi,\xi}/V_\zeta \]

is the Verma module for \( \mathfrak{su} \). Denote by \( \bar{\bar{w}} \) the homomorphic image of \( \bar{w} \), we immediately obtain the following Lemma by Theorem 4.6 of [21]:

Lemma 5.6. The Verma module \( V(\xi, \zeta) \) is irreducible if and only if \( \xi \neq 0 \).

Theorem 5.7. If \( \psi \) is identically zero, \( \xi \neq 0 \), then

(i) For each \( \zeta \in C \), the Whittaker module \( L_{\psi,\xi} \) has the following filtration

\[ L_{\psi,\xi} = V^0 \supseteq V^1 \supseteq \cdots \supseteq V^i \supseteq \cdots \]

where \( V^i \) is a Whittaker submodule of \( L_{\psi,\xi} \) defined by \( V^i = U(\mathfrak{su})(L_0 - \zeta)^i \bar{w} \), and \( V^{i+1} \) is a maximal submodule of \( V^i \). More precisely, \( V^i/V^{i+1} \) is isomorphic to the Verma module \( V(\xi, \zeta) \).

(ii) \( L_{\psi,\xi} \) is isomorphic to \( V^i \) as \( \mathfrak{su} \)-modules for each \( i \in N \).

Proof. For (i), it is obvious that \( V^i/V^{i+1} \cong V(\xi, \zeta) \) according to the definitions of \( V^i, i \in N \). Then \( V^{i+1} \) is a maximal submodule of \( V^i \) by Lemma 5.6. Thus (ii) holds.

For (ii), since \( \psi \) is identically zero, we can easily check that the linear map

\[ f : L_{\psi,\xi} \to V^i, \]

\[ u\bar{w} \mapsto u(L_0 - \zeta)^i \bar{w}, \]
where \( u \in U(\mathfrak{sv}^- \oplus \mathbb{C}L_0) \), is an isomorphism of modules.

\[ \square \]

**Proposition 5.8.** If \( \psi \) is identically zero, \( \xi = 0 \), then the submodule \( V \) of \( L_{\psi,0} \) generated by \( L_{-2}\bar{w} \) is a maximal proper submodule. Moreover, \( L_{\psi,0}/V \) is a one-dimensional trivial module.

**Proof.** Note that \( L_{-i}\bar{w}, M_{-i-1}\bar{w}, Y_{\frac{1}{2}-i}\bar{w} \in V \) for all \( i \in \mathbb{N} \). Thus \( M_{-\mu}Y_{-\frac{1}{2}-\tilde{\nu}}L_{-\tilde{\lambda}}\bar{w} \in V \) for all \( (\mu, \tilde{\nu}, \tilde{\lambda}) \in \mathcal{P} \times \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \) with \( \#(\mu, \tilde{\nu}, \tilde{\lambda}) > 0 \). Since \( M_0\bar{w} = \mathfrak{sv}^+\bar{w} = 0 \), we see that each element of \( V \) is a linear combination of elements with form \( M_{-\mu}Y_{-\frac{1}{2}-\tilde{\nu}}L_{-\tilde{\lambda}}\bar{w} \), \( \#(\mu, \tilde{\nu}, \tilde{\lambda}) > 0 \). Thus \( \bar{w} \notin V \). So \( L_{\psi,0}/V \) is a one-dimensional trivial quotient module and \( V \) is a maximal proper submodule of \( L_{\psi,0} \). \( \square \)

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