ALMOST-KÄHLER ANTI-SELF-DUAL METRICS ON $K3\#3\mathbb{CP}^2$

INYOUNG KIM

Abstract. Donaldson-Friedman constructed anti-self-dual classes on $K3\#3\mathbb{CP}^2$ using twistor space. We show that some of these conformal classes have almost-Kähler representatives.

1. Introduction

On a smooth, oriented riemannian 4-manifold $(M, g)$, 2-forms decomposes as self-dual and anti-self-dual 2-forms $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, according to the eigenvalue of the Hodge star operator $\ast$. By definition, a 2-form $\alpha$ is called self-dual if $\ast\alpha = \alpha$. Then the curvature operator takes the form according to this decomposition of 2-forms $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$,

$$R = \begin{pmatrix} W_+ + \frac{s}{12} & \dot{r} \\ \dot{r} & W_- + \frac{s}{12} \end{pmatrix},$$

where $\dot{r}$ comes from the trace-free Ricci curvature. If $W_+ = 0$, then $g$ is called to be an anti-self-dual metric.

Let $(M, \omega)$ be a 4-dimensional symplectic manifold. The space of almost-complex structures which are compatible with the symplectic form, $\omega(v, w) = \omega(Jv, Jw)$ is nonempty and contractible [23]. If we define $g(v, w) := \omega(x, Jy)$, then $g$ is a metric which is compatible with $J$, $g(v, w) = g(Jv, Jw)$. We call such a metric $g$ an almost-Kähler metric. Note that $\omega$ is a self-dual harmonic 2-form of length $\sqrt{2}$ with respect to $g$. On the other hand, by conformal invariant properties, if we have an anti-self-dual metric $g$ and a nondegenerate self-dual harmonic 2-form, then, there exists a unique almost-Kähler anti-self-dual metric in the conformal class of $g$ [5].

Let $(M, g)$ be an oriented, smooth, compact Riemannian 4-manifold. Then there is Weitzenböck formula for a self-dual 2-form $\omega$,

$$\Delta\omega = \nabla^*\nabla\omega - 2W_+ (\omega, \cdot) + \frac{s}{3} \omega,$$

where $s$ is the scalar curvature.

Let $(M, g, \omega)$ be an almost-Kähler anti-self-dual 4-manifold. Then $\omega$ is a self-dual harmonic 2-form of length $\sqrt{2}$, we get

$$0 = |\nabla\omega|^2 + \frac{2s}{3}.$$

Thus, $s \leq 0$ in case of almost-Kähler anti-self-dual metrics. Moreover, $s \equiv 0$ if and only if $(g, J, \omega)$ is a Kähler manifold. If an almost-Kähler anti-self-dual metric is not Kähler, we call it a strictly almost-Kähler anti-self-dual metric. Let $(M, g, J)$ be a Kähler manifold with $\int_M sd\mu_g \geq 0$. Then either the first Chern class $c_1^R \in H^2(M, \mathbb{R}) = 0$ or its Kodaira dimension is $-\infty$ [29]. Using the formula

$$c_1^2 = (2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2\left| W_+ \right|^2 - \frac{\left| \dot{\rho} \right|^2}{2} \right) d\mu_g,$$

if $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ admit scalar-flat Kähler metrics, then $n \geq 10$. In [14], it was shown that there exist strictly almost-Kähler anti-self-dual metrics by deforming scalar-flat Kähler metrics on certain manifolds. Using the Seiberg-Witten invariant, it was shown that if $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ admits an almost-Kähler anti-self-dual metric, then $n \geq 10$ [14].

In this respect, it might be an interesting question whether there exists a manifold which admits almost-Kähler anti-self-dual metrics but not scalar-flat Kähler metrics. $K3\# n\overline{\mathbb{CP}^2}$ for $n \geq 3$ are candidates since they do not admit scalar-flat Kähler metrics [29] but it was shown that there exists anti-self-dual metrics on them [6]. In this paper, we show that some of anti-self-dual conformal classes constructed by Donaldson-Friedman in [6] are almost-Kähler, using twistor interpretation of self-dual harmonic 2-forms.

**Theorem 1.** There exist strictly almost-Kähler anti-self-dual metrics on $K3\# n\overline{\mathbb{CP}^2}$ for $n \geq 3$.

**Acknowledgement:** The author is most grateful to Prof. Claude LeBrun for suggesting this problem and precious advices. The author is thankful to Jongsu Kim for helpful discussions. The author would like to thank Eui-Sung Park for helpful comment regarding Lemma 5. The author is very thankful to Chanyoung Sung and Korea National University of Education for supports and opportunities, by which this work has been able to be carried out. This article is supported by NRF-2018R1D1A3B07043346.

2. **Twistor spaces**

An oriented, riemannian 4-manifold $(M, g)$ with an anti-self-dual metric (ASD) corresponds to the complex 3-manifold, which is called the twistor space [2, 24]. Consider the unit sphere bundle of self-dual 2-form $p: S(\Lambda^+) \to M$. Using the Levi-Civita connection, we can split the tangent bundle of $Z := S(\Lambda^+)$ by

$$T_z(Z) = V_z \oplus (p^*TM)_z.$$
H. Kalafat showed that if \((M,\text{entended manifold by replacing singularity of})\) \(\text{V}\) are anti-self-dual. K3 surface is known to admit such metrics \([30]\) and we consider the

\[\text{Theorem 2. (Donaldson-Friedman)}\]
\[
\text{There exist anti-self-dual metrics on } \text{V}\text{ that admit an isometric }\).
\[\text{admits an isometric }\text{that }\]. \text{Also, there is a fixed-point free anti-holomorphic involution }\sigma\text{, defined by the quaternionic structure on }\text{V}_+, \sigma^2 = \text{Id}\ [2].\text{ }\sigma\text{ is the antipodal map on each fiber and }\sigma\text{ preserves each }
\text{Conversely, let }\text{Z}\text{ be a complex 3-manifold which has a fixed-point free anti-holomorphic involution }\sigma\text{ such that }\sigma^2 = \text{Id}.\text{ Suppose further }\text{Z}\text{ is fibered by }\sigma\text{-invariant holomorphic curves }\text{C}\text{, which are called the real twistor lines and normal bundle of each real twistor line is isomorphic to }\mathcal{O}(1) \oplus \mathcal{O}(1).\text{ Then there is a corresponding 4-manifold with the anti-self-dual metric }[2, 24].

Let \((M_i,g_i)\) be anti-self-dual 4-manifolds and let \(Z_i\) be twistor spaces corresponding to \(M_i\). Take a twistor line \(l_i \subset Z_i\) and by blowing up this line, we get an exceptional divisor \(Q_i = \text{CP}_1 \times \text{CP}_1\) on \(Z_i\) and we denote blown up manifolds by \(\tilde{Z}_i\). We identify \(\tilde{Z}_1\) and \(\tilde{Z}_2\) along \(Q_i\) by interchanging factors \(Q_1\) and \(Q_2\) and we denote the identified singular manifold with normal crossing divisor \(Q\) by \(Z_0\). A real structure \(\sigma_i\) on \(Z_i\) extends to \(\tilde{Z}_i\) such that \(\tilde{\sigma}_i|_{Z_0} = \sigma_i\) and therefore induces the real structure \(\sigma_0\) on \(Z_0\). It was shown in [6] that if \(H^2(Z_0,\Theta_{Z_0}) = 0\), then there exists a complex deformation of \(Z_0\) and this deformation produces anti-self-dual metrics on \(M_1 \# M_2\). From this, it was shown that \(n\text{CP}_2\) admit anti-self-dual metrics [6].

In this paper, we are in particular interested in the existence of almost-Kähler anti-self-dual metrics on \(K3\#3\text{CP}_2\), more generally \(K3\#n\text{CP}_2\), \(n \geq 3\). Ricci-flat Kähler metrics are anti-self-dual. K3 surface is known to admit such metrics [30] and we consider the corresponding twistor space \(Z\). Note that \(H^2(Z,\Theta_{Z}) \neq 0\) [6]. However, by overcoming the obstruction, it was shown that \(NK3\#n\text{CP}_2\) admit anti-self-dual metrics for \(N > 0\) and \(n \geq 2N + 1\) [6].

**Theorem 2. (Donaldson-Friedman)** There exist anti-self-dual metrics on \(NK3\#n\text{CP}_2\) for \(N > 0\) and \(n \geq 2N + 1\) [6].

This method was developed further by LeBrun and Singer [21] when a 4-manifold \(M\) admits an isometric \(Z_2\)-action with \(k\)-isolated fixed points. Moreover, by considering cohomological interpretation of positive scalar curvature condition [1], it was shown in [19] that \(X\#n\text{CP}_2\) admit an anti-self-dual metric of positive scalar curvature if \(M\) does and \(H^2(Z,\Theta_{Z}) = 0\), where \(Z\) is the twistor space of \(M\). Here \(X = M/\mathbb{Z}_2\) and \(X\) be the oriented manifold by replacing singularity of \(M/\mathbb{Z}_2\) by 2-sphere of self-intersection \(-2\). Also, Kalafat showed that if \((M_i,g_i)\) are anti-self-dual 4-manifolds of positive scalar curvature
such that their twistor spaces $Z_i$ satisfy $H^2(Z_i, \Theta_{Z_i}) = 0$, then $M_1 \# M_2$ admits an anti-self-dual metric of positive scalar curvature [13]. In this paper, we use this method of LeBrun in the construction of almost-Kähler ASD metrics on $K3\#n\mathbb{CP}_2$ for $n \geq 3$.

Let $M$ be an anti-self-dual space and let $Z$ be its twistor space. It was shown there are correspondences between certain cohomology groups on a twistor space $Z$ and solutions of differential equations on a 4-manifold $M$ [7], [11]. One of these correspondences we need in this paper is the following. Consider $Spin(4) = SU(2) \times SU(2)$. Let $V_\pm$ be the basic Spin representations of two factors. We denote $S^m_+$ for $S^mV_+$, where $S^m$ denote the symmetric power and $S^m_+$ for $S^mV_-$. By following [11], we note the following operator,

$$D_m : \Gamma(S^m_+) \to \Gamma(S^{m-1}_+ \otimes S_-)$$

with weight $\frac{1}{2}(m + 2)$.

**Theorem 3** (7,11). Let $M$ be an anti-self-dual space and $Z$ be its twistor space. Then

$$T : H^1(Z, \mathcal{O}(-m - 2)) \to \Gamma(S^m_+)$$

defines an isomorphism onto the space of solutions to $D_m \phi = 0$, for $m \geq 0$. In particular, when $m = 2$, a real element of $H^1(Z, \mathcal{O}(-4))$ corresponds to a real self-dual closed 2-form.

Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be complex spaces. A map between complex spaces $X, Y$ is given by $(f, f^\#)$, where $f : X \to Y$ and $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_X$. Here $f_* \mathcal{O}_X$ is the direct image sheaf. Then a map $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called flat over $y$ if $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{Y,f(x)}$ via the map $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. A sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ on $X$ is said to be flat if $\mathcal{F}_x$ is flat module over $\mathcal{O}_{Y,f(x)}$.

If $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-module, then $f^{-1}\mathcal{G}$ is a $f^{-1}\mathcal{O}_Y$-module. Moreover, via the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, $f^* \mathcal{G}$ is defined to be $f^{-1}\mathcal{G} \otimes f^{-1}\mathcal{O}_Y \mathcal{O}_X$. Then $f^* \mathcal{G}$ is an $\mathcal{O}_X$-module.

Let $X \subset Y$ and $i : X \to Y$ be the inclusion map. For a sheaf of $\mathcal{O}_Y$-module $\mathcal{G}$, we consider the following map

$$r_X : H^k(Y, \mathcal{G}) \to H^k(X, i^* \mathcal{G}).$$

We denote $\alpha|_X := r_X(\alpha)$, where $\alpha \in H^i(Y, \mathcal{G})$.

Let $\alpha \in H^1(Z, K)$. Then, $\alpha|_{\mathbb{CP}_1} \in H^1(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(-4))$. By Serre duality,

$$H^1(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(-4)) = H^0(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(2)).$$

Then $\phi \in H^0(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(2))$ correspond to $S^2_+$. Since $\mathbb{CP}_1 = P((S^*_+)_x \setminus 0)$, a section $\phi$ gives rise to a homogenous polynomial of degree 2 on $(S^*_+)_x \setminus 0$, which gives an element of $(S^2_+)_x$ [11]. When $m = 2$, we have $S^2_+ = \Gamma(A_+)$ and $D_2$ on $\Gamma(A_+)$ is the exterior derivative $d$ [11]. Thus, an element of $H^1(Z, \mathcal{O}_Z(K))$ corresponds to a self-dual closed 2-form. Since $\Delta = dd^* + d^*d$ and $d^* = *d*$ for a 2-form on a 4-manifold, a self-dual closed 2-form is in particular harmonic. A real element of $H^1(Z, \mathcal{O}_Z(K))$ corresponds to a self-dual closed real 2-form on a 4-manifold.
Remark 1. Let \( Z_t \) be the twistor space of \((K3#3\mathbb{C}P_2, g_t)\), where \( g_t \) is the family of anti-self-dual metrics constructed by Donaldson-Friedman. If we have a real cohomology class \( \alpha \in H^1(Z_t, \mathcal{O}_{Z_t}(K_1)) \) such that \( \alpha|_l \neq 0 \) for any twistor line \( l \in Z_t \), we get a nondegenerate real self-dual closed 2-form. In particular, this gives an almost-Kähler anti-self-dual metric on \( K3#3\mathbb{C}P_2 \).

3. Extension of a Cohomology class

Let \((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\) be complex spaces with maps \((f, f^\#)\), where \( f : X \to Y \) and \( f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X \). Let \( \mathcal{F} \) be a sheaf of \( \mathcal{O}_X \)-module on \( X \). Let us define higher direct image sheaf \( R^q f_* (\mathcal{F}) \). This is the sheaf associated to the following presheaf on \( Y \),

\[
V \mapsto H^q(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}).
\]

Theorem 4 (3, 4). Let \( X, Y \) be reduced complex spaces. Suppose \( f : X \to Y \) be a proper morphism and \( \mathcal{F} \) be a coherent sheaf on \( X \), which is flat over \( y \) for all \( y \in Y \). Let \( \mathcal{I}_y \) be the ideal sheaf of \( y \). Then we have

1. For all \( q \geq 0 \), \( h^q(X_y, \mathcal{F}_y) \) is an upper semi-continuous function of \( y \).
2. \( R^q f_* (\mathcal{F}) \) is locally free if \( h^q(X_y, \mathcal{F}_y) \) is constant.
3. If \( h^q(X_y, \mathcal{F}_y) \) is constant, then the map \( (R^q f_* \mathcal{F})_y / \mathcal{I}_y (R^q f_* \mathcal{F})_y \to H^q(X_y, \mathcal{F}_y) \) is bijective.

In Theorem 4, on \( f^{-1}(y) \), we consider the inclusion map \( i_y : f^{-1}(y) \to X \) and we define \( \mathcal{F}_y := i_y^* \mathcal{F} \).

In this paper, we are in particular interested in the case \( K3#3\mathbb{C}P_2 \). Let \( Z \) be the twistor space of \( K3 \) with a Ricci-flat Kähler metric and let \( \mathbb{C}P_2 \) be \( \mathbb{C}P_2 \) with the non-standard orientation and \( g_{FS} \) be the Fubini-Study metric. Let \( F_1 \) be the twistor space of \((\mathbb{C}P_2, g_{FS})\). Then take a twistor line \( l_i \), \( i = 1, 2, 3 \) on \( Z \) and by blowing up \( l_i \), we get \( \tilde{Z} \) with an exceptional divisor \( Q'_i \) for \( i = 1, 2, 3 \), which is \( \mathbb{C}P_1 \times \mathbb{C}P_1 \). The normal bundle of \( Q'_i \) in \( \tilde{Z} \) is \( \mathcal{O}(1, -1) \). In this paper, \( \mathcal{O}(a, b) \) means some power of tautological line bundle on \( \mathbb{C}P_n \) or the sheaf of sections of it according to the context. Let \( \pi_1 \) be the projection of \( \mathbb{C}P_1 \times \mathbb{C}P_1 \) to the first factor and \( \pi_2 \) the second factor. By \( \mathcal{O}(a, b) \), we mean \( \pi_1^* \mathcal{O}(a) \otimes \pi_2 \mathcal{O}(b) \).

Similarly, by blowing up a twistor line, we get \((\tilde{F}_1, Q''_1)\) such that the normal bundle of \( Q''_1 \) in \( \tilde{F}_1 \) is \( \mathcal{O}(1, -1) \). We identify \( \tilde{Z} \) with \( \tilde{F}_1 \) by identifying \( Q'_1 \subset \tilde{Z} \) and \( Q''_1 \subset \tilde{F}_1 \) by switching each factor in the \( Q'_1 \) and \( Q''_1 \) and denote the identification of \( Q'_1 \) and \( Q''_1 \) by \( Q_1 \).

Let \( \tilde{\mathcal{F}} = \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{F}_3 \) and \( \Omega = Q_1 \cup Q_2 \cup Q_3 \). Then we get a singular space \( Z_0 = \tilde{Z} \cup \Omega \tilde{\mathcal{F}} \) with normal crossing divisor \( \Omega \).

In this paper, we consider the following type of deformation, as suggested by LeBrun in [19]. A 1-parameter family of standard deformation of a singular complex space \( Z_0 \) is a flat, proper, holomorphic map \( \varpi : Z \to \mathcal{U} \) with an anti-holomorphic involution \( \sigma : Z \to Z \).
such that $\sigma|_{Z_0} = \sigma_0$ and $U \subset \mathbb{C}$ is an open neighborhood of 0. $Z$ is a complex 4-manifold and when $v \in U$ is non-zero real, $Z_u = \varpi^{-1}(u)$ is a twistor space. For a precise definition, we refer to [19]. By the construction of anti-self-dual metrics in [6], there exists a standard deformation of $Z_0$. We denote this standard deformation by

$$\varpi : Z \to U,$$

with fiber $Z_t$ which is a smoothing of a singular twistor space $Z_0$ and $U \in \mathbb{C}$ is a neighborhood of the origin.

Let $K_Z$ be the canonical bundle of $Z$ and let $\mathcal{I}_Z$ be the ideal sheaf of $\tilde{Z} \subset Z$. Then we consider the invertible sheaf $O_Z(K_Z) \otimes 2\mathcal{I}_Z$. We use $K$ instead of $K_Z$. We apply Leray spectral sequence to this map $\varpi$.

Since $O_Z(K) \otimes 2\mathcal{I}_Z$ is an invertible sheaf, it is coherent and $Z$ can be covered by open sets such that $O_Z(K) \otimes 2\mathcal{I}_Z|_U$ is a free $O_Z|_U$-module. Since $\varpi : Z \to U$ is a flat, proper morphism, $O_Z(K) \otimes 2\mathcal{I}_Z$ is flat over all $t \in U$. Thus, by the theorem 4, we have $h^1(Z_t, (O_Z(K) \otimes 2\mathcal{I}_Z)_t)$ is an upper semi-continuous function of $t \in U$ and if $h^1(Z_t, (O_Z(K) \otimes 2\mathcal{I}_Z)_t)$ is constant, then $R^1\varpi_*(O_Z(K) \otimes 2\mathcal{I}_Z)$ is locally free.

In Leray spectral sequence, we have

$$E_2^{p,q} = H^p(U, R^q\varpi_*(O_Z(K) \otimes 2\mathcal{I}_Z)),$$

and

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}.$$  

By Cartan’s Theorem B, $H^i(U, R^q\varpi_*(O_Z(K) \otimes 2\mathcal{I}_Z)) = 0$ for $i > 0$. Thus, we get $d_2 = 0$ for all $q$. Therefore, Leray spectral sequence degenerates at $E_2$-level.

Then, we have

$$E_2^{p,q} = E_\infty^{p,q} = Gr_p H^{p+q}(Z, O_Z(K) \otimes 2\mathcal{I}_Z),$$

where $Gr_p$ is the $p$-th filtration of $H^{p+q}(Z, O_Z(K) \otimes 2\mathcal{I}_Z)$. In particular, for $p = 0, q = 1$, we get

$$E_2^{0,1} = Gr_0 H^1(Z, O_Z(K) \otimes 2\mathcal{I}_Z) = H^1(Z, O_Z(K) \otimes 2\mathcal{I}_Z).$$

Thus, we get

$$H^0(U, R^1\varpi_*(O_Z(K) \otimes 2\mathcal{I}_Z)) = H^1(Z, O_Z(K) \otimes 2\mathcal{I}_Z).$$

By Theorem 4, if $h^1(Z_t, (O_Z(K) \otimes 2\mathcal{I}_Z)_t)$ is constant, then the following restriction map

$$r_t : H^1(Z, O_Z(K) \otimes 2\mathcal{I}_Z) \to H^1(Z_t, (O_Z(K) \otimes 2\mathcal{I}_Z)_t)$$

is surjective.

Let $\omega$ be a self-dual harmonic 2-form on a smooth, oriented, compact Riemannian manifold. Then

$$0 = \int_M <dd^* + d^*d\omega, \omega> = \int_M <d\omega, d\omega> + \int_M <d^*\omega, d^*\omega>.$$  

Thus, $d\omega = 0$. Therefore, $H^1(Z_t, O_{Z_t}(K_t))$ corresponds to the space of self-dual harmonic 2-forms on $K3\#3\mathbb{CP}_2$, which has dimension 3.
Theorem 5. Let \( \varpi : Z \to U, U \subset \mathbb{C} \), be a 1-parameter family of standard deformation of \( Z_0 \) such that each fiber \( Z_t \) corresponds to the twistor space of \( (K3\#3\mathbb{C}P^2, g_t) \) constructed in [6]. Then for \( t \neq 0 \), \( h^1(Z_t, \mathcal{O}_{Z_t}(K_t)) = 3 \). If \( h^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{Z}})_0) = 3 \), then a given real element \( H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{Z}})_0) \) can be extended nearby fiber so that we get a real element in \( H^1(Z_t, \mathcal{O}_{Z_t}(K_t)) \) for \( t \neq 0 \).

4. First Cohomology of the singular fiber and a nondegenerate element

In this section, we show that \( h^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{Z}})_0) = 3 \) and there is a nondegenerate element in \( H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{Z}})_0) \). Then using Theorem 5, we prove Theorem 1.

Let \( \tilde{F} = \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{F}_3 \) and \( I_{\tilde{Z}} = I_{\tilde{F}_1} + I_{\tilde{F}_2} + I_{\tilde{F}_3} \).

Lemma 1.
1. \( [-\tilde{F}]|_{\tilde{F}_1} = [Q_1] \).
2. \( [-\tilde{F}]|_{\tilde{Z}} = [-\varpi] \).

Proof. First, we assume that the first factor of \( Q_i = \mathbb{C}P_1 \times \mathbb{C}P_1 \) is the twistor line and the second factor is the blown up line so that the normal bundle of \( Q_i \) in \( \tilde{Z} \) is \( \mathcal{O}(1, -1) \) and in \( \tilde{F}_1 \) is \( \mathcal{O}(-1, 1) \). Note that by adjunction formula, \( K_{\tilde{Z}_t} = K_{\tilde{Z}} \otimes [Z_t]|_{\tilde{Z}} \). But the normal bundle of \( Z_t \) for \( t \neq 0 \) in \( Z \) is trivial. Thus, \( K_{\tilde{Z}_t} = K_{\tilde{Z}}|_{\tilde{Z}_t} \) for \( t \neq 0 \). We also note that normal bundle of \( \tilde{F}_1 \) restricted to \( \tilde{F}_1 - Q_i \) is trivial.

We consider the following.

\[
0 \to V_{Q_1, \tilde{F}_1} \to V_{Q_1} \to V_{\tilde{F}_1}|_{Q_1} \to 0
\]

Here \( V_{Q_1, \tilde{F}_1} \) is the normal bundle of \( Q_1 \) in \( \tilde{F}_1 \), which is \( \mathcal{O}(-1, 1) \) and \( V_{Q_1} \) is the normal bundle of \( Q_1 \) in \( Z \), which is \( \mathcal{O}(1, -1) \oplus \mathcal{O}(-1, 1) \) and \( V_{\tilde{F}_1}|_{Q_1} \) is the normal bundle of \( \tilde{F}_1 \) in \( Z \) restricted to \( Q_1 \). From this, we get that \( V_{\tilde{F}_1}|_{Q_1} \) is \( \mathcal{O}(1, -1) \). Note that \( Q_1 \) can be seen as a divisor of \( \tilde{F}_1 \). Then, \( [-Q_1]|_{Q_1} = \mathcal{O}(1, -1) \) and \( [-Q_1]|_{\tilde{F}_1 - Q} \) is trivial, where \( [-] \) denotes the line bundle corresponding to a divisor. Thus, \( [-Q_1] \) is the same with the normal bundle of \( \tilde{F}_1 \subset Z \).

Similarly, we can show \( [-\tilde{F}]|_{\tilde{Z}} = [-\varpi] \). We can check the result does not depend on the choice of the factor of \( Q_i = \mathbb{C}P_1 \times \mathbb{C}P_1 \).

The same proof with Lemma 1 implies that

\[
[\tilde{Z}]|_{\tilde{Z}} = [-\varpi] \text{ and } [\tilde{F}]|_{\tilde{F}_1} = [-Q_1].
\]

Thus, we have

\[
K_{\tilde{Z}}|_{\tilde{Z}} = K_{\tilde{Z}} \otimes [-\tilde{Z}]|_{\tilde{Z}} = K_{\tilde{Z}} \otimes [\varpi].
\]
Lemma 2. 1. \((\mathcal{O}_Z(K_Z) \otimes 2I_f)_t = (\mathcal{O}_Z(K_Z) \otimes 2I_f)|_{\mathcal{Z}_t} \otimes \mathcal{O}_{\mathcal{Z}_t} = \mathcal{O}_{\mathcal{Z}_t}(K_t)\) for \(t \neq 0\).
2. \((\mathcal{O}_Z(K_Z) \otimes 2I_f)_t \otimes \mathcal{O}_{\mathcal{Z}} = \mathcal{O}_{\mathcal{Z}}(K_\mathcal{Z} \otimes [-Q])\)
3. \((\mathcal{O}_Z(K_Z) \otimes 2I_f)|_{\hat{F}_t} \otimes \mathcal{O}_{\hat{F}_t} = \mathcal{O}_{\hat{F}_t}(K_{\hat{F}_t} \otimes 3[Q_i])\)

Lemma 3. Let \(\pi : \hat{S} \to S\) be the blowing up along a submanifold \(W\) with codimension \(k + 1\) and let \(I\) be the ideal sheaf of \(W\). Then we have \(\mathcal{O}_\hat{S}(\pi^*K_S) = \mathcal{O}_\hat{S}(K_{\hat{S}}) \otimes I_W^k\).

Remark 2. From Lemma 3, we get
\[\mathcal{O}_{\hat{S}}(K_{\hat{S}} \otimes [-Q]) = \mathcal{O}_{\hat{S}}(\pi^*K_Z)\]

Lemma 4. \(\mathcal{O}_{\hat{S}}(K_{\hat{S}} \otimes [-Q])\) is non-trivial along the twistor line direction and trivial along the blown up direction of \(Q_i = \mathbb{CP}_1 \times \mathbb{CP}_1\). On the other hand, \(\mathcal{O}_{\hat{F}_t}(K_{\hat{F}_t} \otimes [3Q_i])\) is non-trivial along the blown up direction and trivial along the twistor line direction.

Proof. Suppose we have chosen the factor of \(Q_i\) so that the normal bundle of \(Q_i\) in \(\hat{Z}\) is \(\mathcal{O}(1, -1)\) and in \(\hat{F}_t\) is \(\mathcal{O}(-1, 1)\). Then the first factor of \(Q_i\) is the twistor line direction in \(\hat{Z}\) and the blown up direction in \(\hat{F}_t\).

Using this, we get
\[K_{\hat{Z}}|_{Q_i} = K_{Q_i} \otimes [-Q_i]|_{Q_i} = \mathcal{O}(-2, -2) \otimes \mathcal{O}(-1, 1) = \mathcal{O}(-3, -1),\]
\[K_{\hat{Z}} \otimes [-Q]|_{Q_i} = K_{\hat{Z}}|_{Q_i} \otimes [-Q]|_{Q_i} = \mathcal{O}(-3, -1) \otimes \mathcal{O}(-1, 1) = \mathcal{O}(-4, 0).\]

Similarly, we have
\[K_{\hat{F}_t}|_{Q_i} = K_{Q_i} \otimes [-Q_i]|_{Q_i} = \mathcal{O}(-2, -2) \otimes \mathcal{O}(1, -1) = \mathcal{O}(-1, -3),\]
\[K_{\hat{F}_t} \otimes 3[Q_i]|_{Q_i} = \mathcal{O}(-1, -3) \otimes \mathcal{O}(-3, 3) = \mathcal{O}(-4, 0).\]

\[\square\]

Below, we state Künneth formula in order to calculate the cohomology \(H^i(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(a, b))\).

Theorem 6 (26). Let \(\mathcal{F}, \mathcal{G}\) be coherent sheaves on \(X\) and \(Y\) respectively, which are projective varieties over a field \(k\). Let \(\pi_1\) is the projection map from \(X \times Y\) to \(X\) and similarly, \(\pi_2\) to \(Y\). Then the following holds.
\[H^n(X \times Y, \pi_1^\ast \mathcal{F} \otimes \mathcal{O}_{X \times Y} \pi_2^\ast \mathcal{G}) \cong \bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).\]

Our goal is to show that \(H^1(Z_0, (\mathcal{O}_Z(K_Z) \otimes 2I_f)_0)\) is 3-dimensional and there is a nondegenerate element in \(H^1(Z_0, (\mathcal{O}_Z(K_Z) \otimes 2I_f)_0)\). The reason to choose \(\mathcal{O}_Z(K) \otimes 2I_f\) instead of \(\mathcal{O}_Z(K) \otimes I_f\) is that we would like to get an element \(\alpha \in H^1(Z_0, (\mathcal{O}_Z(K_Z) \otimes 2I_f)_0)\), such that \(\alpha|_Q\) is nonzero. Let \(i : Q_i \to \hat{Z}\) and \(i : Q_i \to \hat{F}_t\) be inclusion maps. Then we note that
\[h^1(Q_i, i^\ast (\mathcal{O}_Z(\pi^*K_Z))) = h^1(Q_i, \mathcal{O}_{Q_i}(-4, 0)) = 3.\]

If we use \(K \otimes I_f\), we get
\[H^1(Q_i, i^\ast (\mathcal{O}_Z(\pi^*K_Z \otimes [Q_i]))) = H^1(Q_i, \mathcal{O}_{Q_i}(-3, -1)) = 0.\]
Lemma 5. Let $Z$ be the twistor space of $K3$-surface with a Ricci-flat Kähler metric and $\tilde{Z}$ be the blown up of $Z$ along three twistor lines. Let $F$ be the twistor space of $\mathbb{CP}_2$ with Fubini-Study metric and $\tilde{F}$ be the blown up of $F$ along a twistor line. Then we have

\[ h^1(\tilde{Z}, \mathcal{O}_Z(\pi_1^*K_Z)) = 3 \]
\[ h^1(\tilde{F}, \mathcal{O}_F(\pi_2^*K_F)) = 0. \]

Proof. Below we show that $h^1(\tilde{Z}, \mathcal{O}_Z(\pi_1^*K_Z)) = h^1(Z, \mathcal{O}_Z(K_Z))$ and $h^1(\tilde{F}, \mathcal{O}_F(\pi_2^*K_F)) = h^1(F, \mathcal{O}_F(K_F))$. $H^1(Z, \mathcal{O}_Z(K_Z))$ corresponds to the space of self-dual harmonic 2-forms on $K3$-surface. Since $b_+ = 3$ on this surface, we get $h^1(Z, \mathcal{O}_Z(K_Z)) = 3$. Similarly, since there is no self-dual harmonic 2-form on $\mathbb{CP}_2$, we have $h^1(F, \mathcal{O}_F(K_F)) = 0$. □

Lemma 6. Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{G}$ be a sheaf of abelian groups on $X$. If $R^if_*\mathcal{G} = 0$ for $i > 0$, then for all $i \geq 0$, there is a following isomorphism.

\[ H^i(X, \mathcal{G}) \cong H^i(Y, f_*\mathcal{G}). \]

Proof. This follows from the Leray spectral sequence argument. We refer to ([13], Proposition 3.0.6) for details of the proof. □

Therefore, in order to prove Lemma 5, we need to prove $R^i\pi_*\mathcal{O}_{\tilde{Z}}(\pi^*K) = 0$ for $i > 0$ and $\pi_*\mathcal{O}_{\tilde{Z}}(\pi^*K) = K$. For this, we use following Propositions ([28] V2. p.124, [13] Proposition 3.0.8)

Proposition 1. Let $X$ and $Y$ be complex manifolds and suppose $f : X \to Y$ be a holomorphic proper and submersive map and $\mathcal{G}$ be a coherent analytic sheaf on $X$. If $H^i(f^{-1}(y), \mathcal{G}|_{f^{-1}(y)}) = 0$ for all $y \in Y$, then $R^i f_*(\mathcal{G}) = 0$.

Proposition 2. Let $\pi : (\tilde{Z}, Q) \to (Z, l)$ be blowing up of a twistor line $l$ and $K$ be the canonical bundle on $Z$. Then, we have $R^i\pi_*\mathcal{O}_{\tilde{Z}}(\pi^*K) = 0$.

Proof. This follows from $\pi^{-1}(y)$ is a point or $\mathbb{P}^1$ and $H^1(\mathbb{P}^1, \mathcal{O}) = 0$. □

Then we get

\[ H^i(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\pi^*K)) = H^i(Z, \pi_*\mathcal{O}_{\tilde{Z}}(\pi^*K)). \]

Using the Projection formula and Zariski’s Main Theorem [10], we get $\pi_*\mathcal{O}_{\tilde{Z}}(\pi^*K) = \mathcal{O}_Z(K)$ ([13], Lemma 3.0.9, 3.0.10).

Lemma 7. (Projection formula) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is $\mathcal{O}_X$-module and $\mathcal{E}$ is locally free $\mathcal{O}_Y$-module of finite rank, then

\[ f_*(\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) = f_*\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{E}. \]

For $\mathcal{G} = \mathcal{O}_X$, we get

\[ f_* f^* \mathcal{E} = f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{E}. \]
Lemma 8. (Zariski’s Main Theorem, weak version) Let $X$ and $Y$ be noetherian integral schemes and let $f : X \to Y$ be a birational projective morphism. If $Y$ is normal, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.

First, we consider $H^1(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\pi^*K_Z))$.

Lemma 9. Let $(M, g)$ be an oriented, smooth, compact 4-dimensional Riemannian manifold and $g$ has the property $s = W_+ = 0$, where $s$ is the scalar curvature of $g$. Then any self-dual harmonic 2-form on $M$ is parallel.

Proof. From the following the Weitzenböck formula for self-dual 2-forms, we have

$$\Delta \omega = \nabla^* \nabla \omega - W_+(\omega, \cdot) + \frac{s}{3} \omega.$$  

Thus, if $s = W_+ = 0$ and $M$ is compact, we get $\nabla \omega = 0$ for a self-dual harmonic 2-form. 

By Yau’s theorem, a K3 surface admits a Ricci-flat Kähler metric [30]. Note that for a Kähler metric, the self-dual Weyl tensor $W_+$ is determined by the scalar curvature $s$. Namely, $W_+$ takes the following form in a Kähler case.

$$W_+ = \begin{pmatrix} -\frac{s}{12} & 0 & 0 \\ 0 & -\frac{s}{12} & 0 \\ 0 & 0 & \frac{s}{6} \end{pmatrix}.$$

Thus, $s = 0$ if and only if $W_+ = 0$. Therefore, K3 surface with Ricci-flat Kähler metric has $W_+ = 0$. In particular, a self-dual harmonic 2-form on K3 surface with Ricci-flat Kähler metric is parallel.

Lemma 10 (5). Let $(Y, g)$ be a smooth, oriented Riemannian $n$-manifold, $n \geq 2$ and let $P$ be a point of $Y$. Let $\phi$ be a differential l-form on $Y - p$ such that $d\phi = 0$ and $d^* \phi = 0$. If there is a neighborhood $U$ of $p$ and a positive constant $C$ such that $|\phi| < C$ on $U - p$, then $\phi$ extends to $Y$ uniquely and smoothly and $d\phi = 0$ and $d^* \phi = 0$ on $Y$.

Remark 3. We note that if we assume $\phi$ is self-dual on $Y - \{p\}$ in the Lemma 10, the extended $\phi$ is also self-dual. Let $*$ be the Hodge-star operator of $(Y, g)$. Then

$$*\phi(p) = \lim_{z \to p} \phi(z) = \lim_{z \to p} \phi(z) = \phi(p)$$

since $*\phi$ and $\phi$ are smooth sections of $\Lambda^2$ and $\phi$ is self-dual on $Y - \{p\}$.

Lemma 11. Let $\alpha \in H^1(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\pi^*K_Z))$ be a real element and suppose that $\alpha$ is not identically zero. Then $\alpha|_l \neq 0$ for any real twistor line $l \in \tilde{Z} - Q$ and $\alpha|_{Q_i} \in H^1(Q_i, \mathcal{O}_{Q_i}(-4,0))$ is not zero.
Proof. Since \([Q]\) is trivial on \(\tilde{Z} - Q\), by restricting cohomology on the subset, we get
\[\alpha|_{\tilde{Z} - Q} \in H^1(\tilde{Z} - Q, \mathcal{O}_{\tilde{Z} - Q}(K_{\tilde{Z} - Q})) = H^1(Z - \mathcal{L}, \mathcal{O}_{Z - \mathcal{L}}(K_{Z - \mathcal{L}})),\]
where \(\mathcal{L} = l_1 \cup l_2 \cup l_3\) and it is real. Thus, by Theorem 4, \(\alpha|_{\tilde{Z} - Q}\) corresponds to a closed real self-dual 2-form \(\phi\) on \(K3 - \{p_1, p_2, p_3\}\). We claim \(\phi\) is bounded near \(p_i\). So it is enough to show that \(\alpha|_l \in H^1(\mathbb{CP}_1, \mathcal{O}(-4))\) is bounded, where \(l\) is a twistor line on \(Z\) near \(l_i\). Since \(\pi^*K_Z\) is non-trivial along the twistor line direction, \(\alpha|_{l_i \times \{z\}}\) is bounded for any \(z \in \mathbb{CP}_1\). Thus, for a twistor line \(l\) near \(l_i\), \(\alpha|_l\) is bounded. Thus, \(\phi\) is bounded near \(p_i\). Then by the Lemma 10 and Remark 3, \(\phi\) is extended smoothly as a self-dual 2-form and \(d\phi = 0\) on \(K3\). In particular, it is harmonic. By the Lemma 9, \(\phi\) is parallel on \(K3\). Then \(||\phi||\) is constant and at a point \(q\), there exists an orthonormal basis such that \(\phi(q) = e_1 \wedge e_2 + e_3 \wedge e_4\). Then, it can be easily checked that \(\phi\) is nondegenerate at \(q\). \(\square\)

As a Corollary of Lemma 11, we get the following.

**Corollary 1.** Let \(i : Q \to \tilde{Z}\) be the inclusion map. Then the restriction map \(r_1 : H^1(\tilde{Z}, \mathcal{O}_Z(\pi^*K_Z)) \to H^1(Q, i^*(\mathcal{O}_Z(\pi^*K_Z)))\) is an isomorphism.

**Proof.** Note that \(h^1(\tilde{Z}, \mathcal{O}_Z(\pi^*K_Z)) = h^1(Q, i^*(\mathcal{O}_Z(\pi^*K_Z))) = 3\) and \(r_1\) is injective by Lemma 11. Thus, \(r_1\) is an isomorphism. \(\square\)

We need to calculate \(h^1(Z_0, (\mathcal{O}_Z(K_Z) \otimes 2\mathcal{I}_Z))_0\). First, we consider one of \((\tilde{F}_i, Q_i)\), which we denote by \((\tilde{F}, Q)\). Let us consider the following exact sequence.

\[0 \to \mathcal{O}_{\tilde{F}}(\pi^*K_F) \to \mathcal{O}_{\tilde{F}}(K_{\tilde{F}}) \to \mathcal{O}_Q(K_{\tilde{F}}|Q) \to 0.\]

Using the fact \(H^0(Q, \mathcal{O}_Q(K_{\tilde{F}}|Q)) = H^1(Q, \mathcal{O}_Q(K_{\tilde{F}}|Q)) = 0\), we get
\[H^1(\tilde{F}, \mathcal{O}_F(\pi^*K_F)) = H^1(F, \mathcal{O}_F(K_F)) = 0.\]

Similarly, we get \(H^1(\tilde{F}, \mathcal{O}_F(K_F \otimes n|Q|)) = 0\) for \(n = 1, 2\). From the exact sequence below,
\[0 \to \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 2|Q|) \to \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 3|Q|) \to \mathcal{O}_Q(K_{\tilde{F}} \otimes 3|Q||Q) \to 0,\]
we get
\[0 \to H^1(\tilde{F}, \mathcal{O}_F(K_{\tilde{F}} \otimes 3|Q|)) \to H^1(Q, \mathcal{O}_Q(K_{\tilde{F}} \otimes 3|Q||Q))\]
\[\to H^2(\tilde{F}, \mathcal{O}_F(K_{\tilde{F}} \otimes 2|Q|)) \to H^2(\tilde{F}, \mathcal{O}_F(K_{\tilde{F}} \otimes 3|Q|)) \to 0.\]

In order to calculate \(H^2(\tilde{F}, \mathcal{O}_F(K_{\tilde{F}} \otimes 2|Q|))\), we describe the twistor space \(F\). Let \(V\) be the vector space which is isomorphic to \(\mathbb{C}^3\) and \(V^*\) is the dual vector space of \(V\). Then \(F\) is given by \([2], [6], [18]\).
\[{([v], [w]) \in P(V) \times P(V^*) \cong \mathbb{CP}_2 \times \mathbb{CP}_2|v \cdot w = 0}\].

Thus, the twistor space \(F\) is a hypersurface of \(\mathbb{CP}_2 \times \mathbb{CP}_2\) given by a linear system \(\mathcal{O}(1,1)\).
Thus, we get $g_i$ gives $g_{i+1}$.

Note that all terms are zero using the cohomology of $\mathbb{C}P^2$ and $\mathbb{C}P^2$ and $\mathbb{C}P^2$ and $\mathbb{C}P^2$.

Proof. Let $P := \mathbb{C}P^2 \times \mathbb{C}P^2$. By adjunction formula, we have

$$K_P = K_P \otimes \mathcal{O}(1,1)|_P,$$

and $K_P = \mathcal{O}_P(-3, -3)$. Thus, in particular, we get $K_P = \mathcal{O}_P(-2, -2)$.

We consider the following exact sequence.

$$0 \to \mathcal{O}_P(K_P) \to \mathcal{O}_P(K_P \otimes \mathcal{O}(1,1)) \to \mathcal{O}_P(K_P \otimes \mathcal{O}(1,1)|_P) \to 0.$$

Then from this, we get the following long exact sequence,

$$0 \to H^0(P, \mathcal{O}(-3, -3)) \to H^0(P, \mathcal{O}(-2, -2)) \to H^0(F, \mathcal{O}_F(K_F)) \to$$

$$\to H^1(P, \mathcal{O}(-3, -3)) \to H^1(P, \mathcal{O}(-2, -2)) \to H^1(F, \mathcal{O}_F(K_F)) \to$$

$$\to H^2(P, \mathcal{O}(-3, -3)) \to H^2(P, \mathcal{O}(-2, -2)) \to H^2(F, \mathcal{O}_F(K_F)) \to$$

$$\to H^3(P, \mathcal{O}(-3, -3)) \to H^3(P, \mathcal{O}(-2, -2)) \to H^3(F, \mathcal{O}_F(K_F)) \to H^4(P, \mathcal{O}(-3, -3)) \to 0.$$

Note that all terms are zero using the cohomology of $\mathbb{C}P^2$ and $\mathbb{C}P^2$ and $\mathbb{C}P^2$ and $\mathbb{C}P^2$.

Thus, $h^4(P, \mathcal{O}_P(-3, -3)) = 1$ and therefore, we get

$$h^3(F, \mathcal{O}_F(K_F)) = 1.$$

□

Again from the following short exact sequence

$$0 \to \mathcal{O}_{\tilde{F}}(\pi^* K_F) \to \mathcal{O}_{\tilde{F}}(K_{\tilde{F}}) \to \mathcal{O}_Q(K_{\tilde{F}}|Q) \to 0,$$

we get

$$0 \to H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(\pi^* K_F)) \to H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}})) \to 0$$

$$\to H^3(\tilde{F}, \mathcal{O}_{\tilde{F}}(\pi^* K_F)) \to H^3(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}})) \to 0.$$

Using $H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(\pi^* K_F)) = H^2(F, \mathcal{O}_F(K_F)) = 0$, we get $H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}})) = 0$. Since $h^3(\tilde{F}, \mathcal{O}_{\tilde{F}}(\pi^* K_F)) = h^3(F, \mathcal{o}_F(K_F)) = 1$, we get $h^3(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}})) = 1$.

The following short exact sequence

$$0 \to \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q]) \to \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 2[Q]) \to \mathcal{O}_Q(K_{\tilde{F}} \otimes 2[Q]|Q) \to 0,$$

gives

$$0 \to H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) \to H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 2[Q])) \to$$

$$\to 0 \to H^3(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) \to H^3(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 2[Q])) \to 0.$$

Thus, we get $H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) \cong H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 2[Q]))$. 

Also from the following short exact sequence

\[ 0 \to \mathcal{O}_F(K_F) \to \mathcal{O}_F(K_F \otimes [Q]) \to \mathcal{O}_Q(K_F \otimes [Q]|_Q) \to 0, \]

we get

\[ 0 \to H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) \to H^2(\tilde{F}, \mathcal{O}_Q(-2, -2)) \to H^3(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) \to 0. \]

Note that \( h^2(Q, \mathcal{O}_Q(-2, -2)) = 1 \) and \( h^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 0 \). Thus, we get \( h^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 0 \). If \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 0 \), then \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 0 \) and therefore in this case, \( r_2 : H^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) \to H^1(Q, \mathcal{O}_Q(K_F \otimes 3[Q]|_Q) \) is an isomorphism. If \( h^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 1 \), then \( h^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 2[Q])) = 1 \).

From the following short exact sequence

\[ 0 \to \mathcal{O}_F(K_F \otimes 2[Q]) \to \mathcal{O}_F(K_F \otimes 3[Q]) \to \mathcal{O}_Q(K_F \otimes 3[Q]|_Q) \to 0, \]

we get

\[ 0 \to H^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) \to H^1(Q, \mathcal{O}_Q(K_F \otimes 3[Q]|_Q) \to H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) \to 0. \]

Thus, if \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) \neq 0 \) and \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) \neq 0 \), then \( h^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 1 \) and in this case, \( r_2 \) is an isomorphism. If \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) \neq 0 \) and \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 0 \), then \( h^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 2 \) and \( r_2 \) is injective.

Thus, we can conclude that either \( h^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = h^1(Q, \mathcal{O}_Q(K_F \otimes 3[Q]|_Q) = h^1(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(-4, 0)) = 3 \) and \( r_2 \) is an isomorphism, or \( h^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 2 \) and \( r_2 \) is injective.

Remark 4. Note that if \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes [Q])) = 0 \), then \( H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 2[Q])) = H^2(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 0 \). In this case, \( h^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 3 \) and \( r_2 \) is an isomorphism.

Lemma 13. Let \( F \) be the twistor space of \( \mathbb{CP}_2 \) with Fubini-Study metric with nonstandard orientation and let \( \pi : (\tilde{F}, Q) \to (F, l) \) be the blowing up along a twistor line \( l \subset F \). Let \( r_2 : H^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) \to H^1(Q, \mathcal{O}_Q(K_F \otimes 3[Q]|_Q) \) be the restriction map. Then either \( h^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 3 \) and \( r_2 \) is an isomorphism or \( h^1(\tilde{F}, \mathcal{O}_F(K_F \otimes 3[Q])) = 2 \) and \( r_2 \) is injective.

Lemma 14. \( h^1(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{L}_3)|_0) \geq 3 \)

Proof. Consider \( \varpi : Z \to \mathcal{U} \) be the standard complex deformation. Each fiber \( Z_t \) for \( t \neq 0 \) is the twistor space of \( (K3#3\mathbb{CP}_2, g_t) \), where \( g_t \) is a family of ASD metrics constructed in [6]. Note that \( (\mathcal{O}_Z(K) \otimes 2\mathcal{L}_3)|_t = K_{Z_t} \) for \( t \neq 0 \). By Lemma 5, we have \( h^1(Z_t, \mathcal{O}_{Z_t}(K_{Z_t})) = 3 \). By upper semicontinuity, we get \( \dim h^1(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{L}_3)|_0) \geq 3 \).

□
Proposition 3. \( h^1(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{I}_\mathcal{F})) = 3 \).

Proof. Let \( a : \tilde{Z} \to Z_0 \) be the normalization map. We consider the following exact sequence.

\[ 0 \to \mathcal{O}_Z(K \otimes 2\mathcal{I}_\mathcal{F})_0 \to a_*(\mathcal{O}_Z(K_\mathcal{F} \otimes [-\mathcal{Q}]) \oplus \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes [3\mathcal{Q}])) \to \mathcal{O}_0(\pi^*K_{Z}|_0) \to 0. \]

Note that each restriction of \( \mathcal{O}_Z(K_\mathcal{F} \otimes [-\mathcal{Q}]) \) and \( \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes [3\mathcal{Q}]) \) to \( \mathcal{Q} \) are the same.

Note that \( H^0(Q, \mathcal{O}_Q(\pi^*K_{Z}|_Q)) = H^0(Q, \mathcal{O}_Q(-4,0)) = 0 \). Therefore, we get the following long exact sequence

\[ 0 \to H^1(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{I}_\mathcal{F})) \to H^1(Z, \mathcal{O}_Z(K_\mathcal{Z} \otimes [-\mathcal{Q}])) \oplus H^1(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes [3\mathcal{Q}])) \to H^2(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{I}_\mathcal{F})) \to \cdots \]

Since \( h^1(Q, \mathcal{O}_Q(\pi^*K_{Z}|_Q)) = h^1(Q, \mathcal{O}_Q(-4,0)) = 3 \), we have \( h^1(Q, \mathcal{O}_Q(\pi^*K_{Z}|_Q)) = 9 \). The map \( r \) is given by

\[ r = (\alpha|_Q - \beta_1|_Q, \alpha|_Q - \beta_2|_Q, \alpha|_Q - \beta_3|_Q), \]

where \( \alpha \in H^1(\tilde{Z}, \mathcal{O}_Z(\pi^*K_{Z})) \) and \( \beta_i \in H^1(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes 3[Q_i])) \). Since \( r_2 : H^1(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes 3[Q_i])) \to H^1(Q, \mathcal{O}_Q(K_\mathcal{F} \otimes 3[Q_i])) \) is injective, we get \( \dim \ker r \geq h^1(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes [3\mathcal{Q}])) \).

Thus, we get \( \dim \ker r \leq \dim H^1(\tilde{Z}, \mathcal{O}_Z(\pi^*K_{Z} \otimes [-\mathcal{Q}])) \), which is 3. Since \( f \) is injective and \( \dim H^1(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{I}_\mathcal{F})) \geq 3 \), we get \( \dim \ker f \geq 3 \). Thus, we get \( \dim \ker f = \dim \ker r \leq 3 \). Thus, \( \dim \ker f = 3 \) and therefore, \( \dim H^1(Z_0, \mathcal{O}_Z(K \otimes 2\mathcal{I}_\mathcal{F})) = 3 \).

\[ \square \]

Corollary 2. The restriction map \( r_2 : H^1(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes [3\mathcal{Q}])) \to H^1(Q, \mathcal{O}_Q(K_\mathcal{F} \otimes 3[Q]_{|Q})) \) is an isomorphism.

Proof. Suppose \( r_2 \) is not an isomorphism. Then \( h^1(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes [-\mathcal{Q}])) < 9 \). In this case, \( \dim \ker f = \dim \ker r < 3 = \dim H^1(\tilde{Z}, \mathcal{O}_Z(K_\mathcal{Z} \otimes [-\mathcal{Q}])) \), which is a contradiction since \( \dim \ker f = 3 \) from Proposition 3. Thus, \( r_2 \) is an isomorphism.

\[ \square \]

From the argument before Remark 4 and Corollary 2, we get the following corollary.

Corollary 3. Either \( H^2(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes n[Q])) = 0 \) or \( h^2(\tilde{\mathcal{F}}, \mathcal{O}_\mathcal{F}(K_\mathcal{F} \otimes n[Q])) = 1 \) for \( n = 1, 2, 3 \).

We show that there is a nondegenerate element in \( H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathcal{I}_\mathcal{F}))_0 \). By lemma 11, if \( \alpha \in H^1(\tilde{Z}, \pi^*K_{Z}) \) is not identically zero, there \( \alpha|_Q \neq 0 \) and \( \alpha|_l \neq 0 \) for every twistor.
line $l \subset \tilde{Z} - Q$. Below, we prove the similar one for a cohomology class in $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 3[Q]))$.

Lemma 15. Let $\beta$ be a real element of $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 3[Q]))$ and suppose $\beta$ is not identically zero. Then $\beta|_l \neq 0$ for any real twistor line $l \in \tilde{F} - Q$ and $\beta|Q \neq 0$.

Proof. Since $[Q]$ is trivial on $\tilde{F} - Q$, by restricting a real cohomology $\beta \in H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 3[Q]))$ to $\tilde{F} - Q$, we get a real element in $H^1(\tilde{F} - Q, K_{\tilde{F} - Q})$. By Theorem 4, this element corresponds to a real self-dual harmonic 2-form on $(\mathbb{CP}_2 - \{y\}, g_{FS})$, where $g_{FS}$ is the restriction of Fubini-Study metric. Moreover, $g_{FS}$ on $\mathbb{CP}_2 - \{y\}$ with the non-standard orientation is conformal to Burns metric. By Corollary 2, the restriction map $r_2 : H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 3[Q])) \to H^1(Q, \mathcal{O}_Q(K_{\tilde{F}} \otimes 3[Q]|_Q))$ is an isomorphism. Note that $H^1(Q, \mathcal{O}_Q(K_{\tilde{F}} \otimes 3[Q]|_Q)) = H^1(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(0, -4))$. Thus, on $Q$, there is a rational curve $\mathbb{CP}_1$ such that the restriction of $K_{\tilde{F}} \otimes 3[Q]|_Q$ on it is $\mathcal{O}(-4)$ and $H^1(\mathbb{CP}_1, \mathcal{O}(-4)) \neq 0$. On $\tilde{F} - Q$, $K_{\tilde{F}} \otimes 3[Q]|_Q$ is $K_{\tilde{F} - Q}$, and for a twistor line $l$ on $\tilde{F} - Q$, which is $\mathbb{CP}_1$, $K_{\tilde{F} - Q}|_l = \mathcal{O}(-4)$ [2]. Namely, the restriction of the sheaf are the same for the rational curve on $Q$ and any twistor line on $\tilde{F} - Q$. From this, we get $r : H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes 3[Q])) \to H^1(\mathbb{CP}_1, \mathcal{O}(-4))$ is an isomorphism for any twistor line on $\tilde{F} - Q$. 

\[\square\]

Proposition 4. There is a real element $\gamma \in H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{j}})_{0})$ such that $\gamma|_l \neq 0$, for any real twistor line in $\tilde{Z} - Q$ and $\tilde{J} - Q$.

Proof. From the description of $H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{j}})_{0})$ in the following long exact sequence,

$$0 \rightarrow H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{j}})_{0}) \rightarrow H^1(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}} \otimes [-Q])) \oplus H^1(\tilde{J}, \mathcal{O}_{\tilde{J}}(K_{\tilde{J}} \otimes [3Q])) \rightarrow$$

\[\rightarrow H^1(Q, \mathcal{O}_Q(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(0, -4))) \rightarrow,\]

an element of $H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{j}})_{0})$ is given by the kernel of the map $r = (\alpha|_{Q_i} - \beta|_{Q_i})$, where $\alpha \in H^1(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(K_{\tilde{Z}} \otimes [-Q]))$ and $\beta \in H^1(\tilde{J}, \mathcal{O}_{\tilde{J}}(K_{\tilde{J}} \otimes [3Q]))$. Since Ker $r = \mathbb{C}^3$, we take $(\alpha, \beta) \in$ Ker $r$, which is real and not identically zero.

First, we assume that $\alpha$ is not identically zero. Then by Lemma 11, $\alpha|_l \neq 0$ for any twistor line $l \in \tilde{Z} - Q$ and $\alpha|_{Q_i} \neq 0$ for any $i$. Then we have $\alpha|_{Q_i} = \beta_i|_{Q_i} \neq 0$. By Lemma 15, $\beta_i|_l \neq 0$ for any twistor line in $\tilde{F}_i - Q_i$. If we assume $\beta_j$ is not identically zero for some $j$, then by Lemma 15, $\beta_j|_l \neq 0$ for any twistor line in $\tilde{F}_j - Q_j$ and $\beta_j|_{Q_j} \neq 0$. Then $\alpha|_{Q_j} \neq 0$. By the same argument using Lemma 11 and 15, we get $\alpha|_l \neq 0$ for any twistor line $l \in \tilde{Z} - Q$ and $\beta|_l \neq 0$ for any twistor line $l \in \tilde{F}_i - Q_i$ for any $i$. 

\[\square\]

We have shown that $h^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{j}})_{0}) = 3$ for all $t$ including the singular fiber. Thus, by Theorem 5, a given element in $H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2I_{\tilde{j}})_{0})$ can be extended to nearby fiber. If we take an element given in Proposition 4, then we get a nondegenerate real element of $H^1(Z_t, \mathcal{O}_Z(K_t))$ for $t$ near 0 since nondegeneracy is an open condition.
Thus, we get a nondegenerate self-dual harmonic 2-form on \((K3#3CP_2, g_t)\) and therefore an almost-Kähler anti-self-dual metric in the conformal class of \(g_t\). Let \(\alpha \in H^1(\tilde{Z}, \pi^*K_Z)\) and \(Q = \bigcup_{1 \leq i \leq n} Q_i\) for any \(n \geq 4\). A self-dual harmonic 2-form corresponding to \(\alpha|_{\tilde{Z} - Q}\) can be extended to \(K3\) by the argument of Lemma 11. The same argument is easily extended to cover cases \(K3#nCP_2\) for \(n \geq 4\). Since \(K3#3CP_2\) does not admit a scalar-flat Kähler metric, we get a strictly almost-Kähler anti-self-dual metric on \(K3#3CP_2\) for \(n \geq 3\). This finishes the proof of Theorem 1.

5. Scalar curvatures of almost-Kähler anti-self-dual metrics

Recall that the anti-self-duality is a conformal invariant. By the solution of Yamabe problem [22], each conformal class on a compact manifold of \(\dim \geq 3\) has a representative whose scalar curvature is constant. There are three types according to the sign of the scalar curvature. It is interesting to note the type of almost-Kähler anti-self-dual metrics on \(K3#nCP_2\) for \(n \geq 3\).

From the Weitzenböck formula for a self-dual 2-form,

\[ \Delta \omega = \nabla^* \nabla \omega - 2W_+ (\omega, \cdot) + \frac{s}{3} \omega, \]

if an anti-self-dual compact manifold \((M, g)\) is positive type, then \(b_+(M) = 0\) (Corollary 1, [27]). Otherwise, there is a self-dual harmonic 2-form with respect to an anti-self-dual metric on \(M\) with constant positive scalar curvature. Then we have

\[ 0 = \langle \nabla^* \nabla \omega, \omega \rangle + \frac{s}{3} < \omega, \omega >. \]

Then we get \(\int_M \frac{s}{3} < \omega, \omega > \leq 0\), which is a contradiction. Since \(K3#nCP_2\) for \(n \geq 3\) has self-dual harmonic 2-forms, an anti-self-dual conformal class on \(K3#nCP_2\) for \(n \geq 3\) cannot be positive type.

Moreover, from the following result in [19, Proposition 3.5], \(K3#nCP_2\) for \(n \geq 3\) cannot admit an anti-self-dual metric with zero scalar curvature.

Proposition 5 (19). Suppose \(M\) be a smooth, oriented, compact four-dimensional manifold. If \(M\) admits a scalar-flat anti-self-dual metric, then \(M\) is homeomorphic to \(kCP_2\) for \(k \geq 5\) or \(M\) is diffeomorphic to \(CP_2#nCP_2\) for \(n \geq 10\), or diffeomorphic to \(K\) surface.

Thus, we can conclude that an anti-self-dual conformal class on \(K3#3CP_2\) for \(n \geq 3\) is negative type. The following result was proven in case \(b_1(M) = 0\) in [27] and in general in [9].

Theorem 7 (9, 27). Suppose \((M, c)\) be a compact, oriented anti-self-dual conformal manifold and its conformal class contains a metric of constant negative scalar curvature. Then the corresponding twistor space does not have a nontrivial divisor.
From this, we get the twistor space \( Z \) of \((K3\#n\mathbb{CP}^2, g)\) for \( n \geq 3 \) does not admit a nontrivial divisor, where \( g \) is an anti-self-dual metric with negative type. This is Corollary 2 in [27].

Note that the property of \( K3 \)-surface we need in this paper in the construction of almost-Kähler anti-self-dual metrics is that \( b_+ \neq 0 \) and the metric is anti-self-dual and has vanishing scalar curvature. Then among the list given in Proposition 5, \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n \geq 10 \) with scalar-flat Kähler metrics have these properties. Thus, instead of \( K3 \) surface, we may use \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n \geq 10 \).

We note that it was shown that \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n \geq 14 \) admit scalar-flat Kähler metrics by twistor method [15, 16]. The optimal case \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n = 10 \) was successful using gluing method in [25].

**Theorem 8** (20). Suppose \( M \) be a compact scalar-flat Kähler surface such that \( c_1 \neq 0 \). Let \( Z \) be its twistor space and \( D \) be the corresponding divisor. Suppose \( M \) be not a minimal ruled surface of genus \( \gamma \geq 2 \) such that \( H^0(M, \Theta_M) \neq 0 \) and \( M \) be not of the form \( \mathbb{P}(L \oplus \mathcal{O}) \to S_\gamma \), where \( S_\gamma \) is a riemann surface of genus \( \gamma \geq 2 \). Then \( H^2(Z, \Theta_Z \otimes I_D) = 0 \).

**Proposition 6** (20, 14). Suppose \( M \) be a compact scalar-flat Kähler surface such that \( c_1 \neq 0 \). Let \( Z \) be its twistor space with the corresponding divisor \( D \). If \( H^2(Z, \Theta_Z \otimes I_D) = 0 \), then \( H^2(Z, \Theta_Z) = 0 \).

Let \( g \) be a scalar-flat Kähler metric on \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n \geq 10 \). Then by Theorem 8 and Proposition 6, the twistor space of \( (\mathbb{CP}^2\#n\mathbb{CP}^2, g) \) has \( H^2(Z, \Theta_Z) = 0 \). Thus, Donaldson-Friedman construction can be applied to the pair \( (\mathbb{CP}^2\#n\mathbb{CP}^2, g) \) for \( n \geq 10 \) and \( (\mathbb{CP}^2, g_{FS}) \). Moreover, our construction of nondegenerate self-dual harmonic 2-form also applies in these cases. The existence of strictly almost-Kähler anti-self-dual metrics on \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n \geq 11 \) is already shown by deforming scalar-flat Kähler metrics [14]. The method of showing existence of almost-Kähler anti-self-dual metrics on such manifolds in this paper is different from this case. On the other hand, since \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) admit scalar-flat Kähler metrics unlike \( K3\#n\mathbb{CP}^2 \), we can only state the theorem in the following way.

**Theorem 9.** There is an almost-Kähler anti-self-dual metric on \( \mathbb{CP}^2\#n\mathbb{CP}^2 \) for \( n \geq 11 \).

6. Appendix: Calculation of the second cohomology of the singular fiber

In this section, we consider \( H^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathcal{I}_\mathcal{Z})_0) \), where \( Z_0 = \tilde{Z} \cup_0 \tilde{\mathcal{F}} \) is obtained from the twistor space of K3 surface with Ricci-flat Kähler metric and 3 copies of twistor space of \( (\mathbb{CP}^2, g_{FS}) \).

By Serre Duality

\[
H^2(Z, K_Z) = H^1(Z, \mathcal{O})^*.
\]
Lemma 16.\[ H_\alpha := \] Proof. By the above argument, it suffices to show that if \( H \) is an 4-manifold with an anti-self-dual metric. Since \( H \) was shown that \([8], [11], [17]\) that.

Theorem 10 (8, 11, 17). For the twistor space \( Z \) of a compact, smooth, oriented riemannian 4-manifold with an anti-self-dual metric \((M, g), H^1(Z, \mathcal{O}) = H^1(M, \mathbb{C})\).

Proof. By the above argument, it suffices to show that if \( H_\alpha = 0 \). Define \( \alpha := d\omega \). Then \( \ast \alpha = -\alpha \) by definition of \( d_+ \) and \( d_+ \omega = 0 \). Then we have

\[
||\alpha||^2 = \int_M \alpha \wedge \ast \alpha = -\int_M \alpha \wedge \alpha = -\int_M d\omega \wedge \alpha = -\int_M \omega \wedge d\alpha = 0.
\]

□

Lemma 16. For a twistor space \( Z \) of K3 surface with Ricci-flat Kähler metric, we have \( H^2(Z, \mathcal{O}_Z(K_Z)) = 0 \). For \( Z_t \), which is a twistor space of \((K3\#n\mathbb{CP}^2, g_t)\), where \( g_t \) is a family of anti-self-dual metrics constructed in [6], we have \( H^2(Z_t, \mathcal{O}_{Z_t}(K_{Z_t})) = 0 \).

Proof. Since K3 surface and \( K3\#n\mathbb{CP}^2 \) are simply connected, we get immediately the conclusion from Theorem 10 and Serre Duality. □

In Corollary 3, it is shown that either \( H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes n[Q])) = 0 \) or \( h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes n[Q])) = 1 \) for \( n = 1, 2, 3 \). We claim if \( H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes n[Q])) = 0 \), then \( H^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) = 0 \) and if \( h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes n[Q])) = 1 \), then \( h^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) = 3 \).

Proposition 7. If \( H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) = 0 \), then \( h^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) = 0 \).

Proof. We consider again the long exact sequence

\[
0 \longrightarrow H^1(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) \stackrel{\partial}{\longrightarrow} H^1(\tilde{Z}, \mathcal{O}_Z(K_Z \otimes [-Q])) \longrightarrow H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) \longrightarrow H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [3Q])) \longrightarrow H^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) \longrightarrow H^2(\tilde{Z}, \mathcal{O}_Z(K_Z \otimes [-Q])) \otimes H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [3Q])) \longrightarrow 0.
\]

From Remark 4, if \( H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) = 0 \), then \( H^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [3Q])) = 0 \). Moreover, from Lemma 16, we get \( H^2(Z_0, (\mathcal{O}_Z(K_Z \otimes [-Q])) = H^2(\tilde{Z}, \mathcal{O}_Z(K_Z \otimes 3Q)) \). From Corollary 2, we get \( r \) is surjective. Therefore, we get \( H^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) = 0 \).

Proposition 8. If \( h^2(\tilde{F}, \mathcal{O}_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) = 1 \), then \( h^2(Z_0, (\mathcal{O}_Z(K) \otimes 2\mathbb{I}_3)) = 3 \).
Proof. Again we consider the long exact sequence given in the proof of Proposition 7. Note that from Corollary 2, we get \( r \) is surjective and from Corollary 3, we get \( h^2(\tilde{F}, O_{\tilde{F}}(K_{\tilde{F}} \otimes 3[Q])) = 1 \). From this, we get \( h^2(\tilde{F}, O_{\tilde{F}}(K_{\tilde{F}} \otimes [3Q])) = 3 \). Thus, we get \( h^2(Z_0, (O_Z(K) \otimes 2I_{\tilde{F}})_0) = h^2(\tilde{F}, O_{\tilde{F}}(K_{\tilde{F}} \otimes [3Q])) = 3 \).

\[ \square \]

Remark 5. From Lemma 16, we have \( H^2(Z_t, O_{Z_t}(K_t)) = H^1(Z_t, O)^* = 0 \). Then depending on \( h^2(\tilde{F}, O_{\tilde{F}}(K_{\tilde{F}} \otimes [Q])) \), \( h^2(Z_0, (O_Z(K) \otimes 2I_{\tilde{F}})_0) = 0 \) or 3. Thus, we cannot conclude about \( h^2(Z_0, (O_Z(K) \otimes 2I_{\tilde{F}})_0) \).
Bibliography

[1] M. F. Atiyah, *Green’s functions for self-dual four-manifolds*, in: Mathematical Analysis and Applications, Part A, vol. 7 of Adv. in Math. Suppl. Stud., Academic Press, New York, 1981, pp. 129-158.

[2] M. Atiyah, N. Hitchin and I. Singer, *Self-duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A 362 (1978), 425 -461.

[3] C. Bănică and O. Stănășilă, *Algebraic methods in the global theory of complex spaces*, John Wiley & Sons, New York (1976).

[4] W. Barth, K. Hulek, C. Peters, and A. V. de Ven, *Compact Complex Surfaces*, 2nd ed. 2004, Springer.

[5] C. Bishop and C. LeBrun, *Anti-Self-Dual 4-Manifolds, Quasi-Fuchsian Groups and Almost-Kähler Geometry*, arXiv: 1708. 03824 [math DG]; to appear in Comm. An. Geom.

[6] S. Donaldson and R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, 1989 Nonlinearity, 2 (1989), pp. 197-239.

[7] M. G. Eastwood, R. Penrose, and R. O. Wells, Jr., *Cohomology and Massless Fields*, Commun. Math. Phys. 78. 305-351 (1981).

[8] M. G. Eastwood and M. A. Singer, *The Fröhlicher spectral sequence on a twistor space*, J. Differ. Geom. 38 (3), 653-669 (1993).

[9] P. Gauduchon, *Weyl structures on a self-dual conformal manifolds*, Proc. Symp. Pure Math. 54 (1993) Part 2, pp. 259-270.

[10] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52.

[11] N. J. Hitchin, *Linear field equations on self-dual spaces*, Proc. R. Soc. Lond. A 370, 173-191 (1980).

[12] N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) 43 (1981), 133-150.

[13] M. Kaledat, *Self-dual Metrics on 4-Manifolds*, Thesis, Stony Brook University.

[14] I. Kim, *Almost-Kähler anti-self-dual metrics*, Ann. Glob. Anal. Geom. 49 (2016), 369 -391.

[15] J. -S. Kim and M. Pontecorvo, *A new method of constructing scalar-flat Kähler surfaces*, J. Differ. Geom. 41(2), 449-477 (1995).

[16] J. -S. Kim, C. LeBrun and M. Pontecorvo, *Scalar-flat Kähler surfaces of all genera*, J. Reine Angew. Math. 486, 69-95 (1997).

[17] C. LeBrun, *Twistors, Kähler Manifolds, and Bimeromorphic Geometry I*, J. Am. Math. Soc. 5 (1992) 289-316.

[18] C. LeBrun, *Twistors for Tourists: A Pocket Guide for Algebraic Geometers*, Proc. Symp. Pure Math. 62.2 (1997) 361-385.

[19] C. LeBrun, *Curvature functionals, optimal metrics, and the differential topology of 4-manifolds*, in Differential Faces of Geometry, Kluwer Academic/Plenum, 2004.

[20] C. LeBrun and M. Singer, *Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces*, Invent. math. 112, 273-313 (1993).

[21] C. LeBrun and M. Singer, *A Kummer-type construction of self-dual 4-manifolds*, Math. Ann. 300. 165-180(1994).

[22] J. Lee and T. Parker, *The Yamabe problem*, Bull. Am. Math. Soc. 17 (1987), 37-91.

[23] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, Oxford (1995).

[24] R. Penrose, *Nonlinear gravitons and curved twistor theory*, General Relativity and Gravitation 7 (1976), 31-52.

[25] Y. Rollin and M. A. Singer, *Non-minimal scalar-flat Kähler surfaces and parabolic stability*, Invent. Math. 162(2), 235-270 (2005).

[26] J. H. Sampson and G. Washnitzer, *A Künnett formula for coherent algebraic sheaves*, Illinois J. Math., Volume 3, Issue 3 (1959), 389-402.

[27] M. Ville, *Twistor examples of algebraic dimension zero threefolds*, Invent. math. 103 (1991) 537-545.

[28] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I, 2*, Cambridge University Press 2003.

[29] S. -T. Yau, *On the curvature of compact Hermitian manifolds*, Invent. Math. 25, 213 -239 (1974)
[30] S. -T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. 31 (1978), 339-411.

Department of Mathematics Education, Korea National University of Education
Email address: kijysd@snu.ac.kr