Certain properties of contra-$T^*_{12}$-continuous functions

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Received July 28, 2019. Accepted for publication Sep. 2, 2019

Abstract—The concept of contra function was introduced by Dontchev [2], in this work, we use the notion of $T^*_{12}$-open to study a new class of function called a contra-$T^*_{12}$-continuous function as a generalization of contra-continuous.

Keywords: $T^*_{12}$-open sets; contra-$T^*_{12}$-continuous function; operator topological space; contra-$T^*_{12}$-closed graph.

1. INTRODUCTION

In 1996, Dontchev [2] introduced contra-continuous functions. In [10], the authors introduced the concept of almost contra-$T^*$-continuous function. In this paper, we introduce a new class of function called contra-$T^*_{12}$-continuous function where $T_1, T_2$ are operators associated with the topology $\tau$ on $X$. Throughout the paper, the space $X$ and $Y$ or $(X, Y)$ and $(Y, \delta)$ stand for topological space, let $A$ be a subset of $X$, the closure of $A$ and the interior of $A$ will be denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively.

2.1 Definition: A subset $A$ of a space $X$ is said to be:

i) Semi-open [6] if $A \subseteq \text{Cl}(\text{Int}(A))$,

ii) Pre-open [7] if $A \subseteq \text{Int}(\text{Cl}(A))$,

iii) b-open [1] if $A \subseteq \text{Cl}(\text{Int}(A) \cup \text{Int}(\text{Cl}(A)))$.

The complement of semi-open (pre-open, b-open) is said to be semi-closed (pre-closed, b-closed). The family of all semi-open (pre-open, b-open, semi-closed, pre-closed, b-closed) subset of a space $X$ is denoted by $SO(X)|PO(X)|BO(X)|SC(X)|PC(X)|BC(X)$, respectively.

II. PRELIMINARIES

In this section, we recall the basic facts and definitions needed in this work.
\textbf{2.2 Definition [4]}: A function \( f : X \rightarrow Y \) is called semi-continuous (pre-continuous, b-continuous) if for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \text{SO}(X) \) \((U \in \text{PO}(X), U \in \text{BO}(X)) \) such that \( f(U) \subseteq V \).

\textbf{2.3 Definition}: A function \( f : X \rightarrow Y \) is called contra-continuous [2] (contra-semi continuous [4], contra-pre-continuous [3], contra-b-continuous [5]) if \( f^{-1}(V) \) is closed (semi-closed, pre-closed, b-closed, resp.) in \( X \) for each open set \( V \) of \( Y \).

\section{III. OPERATOR TOPOLOGICAL SPACES}

\textbf{3.1 Definition [8]}: Let \( (X, \tau) \) be a topological space and let \( T : p(X) \rightarrow p(X) \) be a function (where \( p(X) \) is the power set of \( X \)) we say that \( T \) is an operator associated with the topology \( \tau \) on \( X \) if \( W \subseteq T(W) \ (W \in \tau) \) and the triple \((X, \tau, T)\) is called an operator topological space.

\textbf{3.2 Definition [9]}: Let \( (X, \tau, T) \) be an operator topological space, let \( A \subseteq X \)

\begin{enumerate}
  \item A is called T-open if given \( x \in A \), then there exists \( V \in \tau \) there exists \( x \in V \subseteq T(V) \subseteq A \).
  \item A is called T*-open if \( A \subseteq T(A) \) (\( A \) is not necessarily open).
\end{enumerate}

\textbf{3.3 Remarks}:

\begin{enumerate}
  \item Every T-open set is open.
  \item Every open set is T*-open, so we have the following implications:
  \[ \text{T-open} \rightarrow \text{open} \rightarrow \text{T*-open} \]
  \item Let \( (X, \tau) \) be a topological space define \( T : p(X) \rightarrow p(X) \) as follows: \( T(A) = \text{Int Cl}(A) \) then \( T \) is an operator associated with the topology \( \tau \) on \( X \) and the triple \((X, \tau, T)\) is an operator topological space.
\end{enumerate}

As an example, we can suppose \( X = \mathbb{R} \), \( \tau = \tau_{\text{u}} \) the usual topology on \( \mathbb{R} \), if \( T(A) = \text{Int Cl}(A) \), then the triple \((\mathbb{R}, \tau_{\text{u}}, T)\) is an operator topological space,

notice that \( Q \subseteq \mathbb{R} \) satisfies \( Q \subseteq \text{Int Cl}(Q) \), so \( Q \) is a T*-open (pre-open) which is not open.

\textbf{3.3 Definition}: Let \( (X, \tau) \) be a topological space and let \( T_1, T_2 \) be two operators associated with the topology \( \tau \) on \( X \) then \( (X, \tau, T_1, T_2) \) is called a bi operator topological space.

\textbf{3.4 Definition}: Let \( (X, \tau, T_1, T_2) \) be an operator topological space and let \( A \subseteq X \), we say that \( A \) is a T*_{12}-open if \( A \subseteq T_1(A) \cup T_2(A) \), the complement of T*_{12}-open is called T*_{12}-closed for example if:

\[ T_1(A) = \text{Cl}(\text{Int}(A)), \]
\[ T_2(A) = \text{Int}(\text{Cl}(A)), \]

then:
\[ A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(A), \]

this is the definition of b-open set.

Notice that every T*_{1,1}-open (T*_{2,2}-open) is T*_{12}-open because if \( A \) is a T*_{1,1}-open then \( A \subseteq T_1(A) \subseteq T_1(A) \cup T_2(A) \), so \( A \) will be T*_{12}-open.

\section{IV. CONTRA-T*_{12}-CONTINUOUS FUNCTIONS}

In this section, we obtain some properties of contra-T*_{12}-continuous functions.

\textbf{4.1 Lemma [1]}: Let \( (X, \tau) \) be a topological space then:

\begin{enumerate}
  \item The intersection of an open set and a b-open set is a b-open set.
  \item The union of any family of b-open sets is a b-open set.
\end{enumerate}

Now, we generalize Lemma 4.1 as follows:

\textbf{4.2 Lemma}: Let \( (X, \tau, T_1, T_2) \) be a bi operator topological space assume that

\[ T_1(W \cap B) = T_1(W) \cap T_1(B), \forall \tau, B \subseteq X, \]
\[ T_2(W \cap B) = T_2(W) \cap T_2(B), \] where \( \tau, B \subseteq X. \] Therefore:

1) The intersection of an open set and a \( T_{*12}\)-open set is \( T_{*12}\)-open.

2) The union of any family \( T_{*12}\)-open sets is a \( T_{*12}\)-open set.

**Proof:**

1) Let \( W \subseteq X \) be an open set and let \( V \) be a \( T_{*12}\)-open set we have to prove that \( W \cap V \) is also a \( T_{*12}\)-open set. Since \( W \) is open then:

\[
 W \subseteq T_1(W) \quad \ldots (1) \\
 W \subseteq T_2(W) \quad \ldots (2) 
\]

Since \( V \) is a \( T_{*12}\)-open then

\[
 V \subseteq T_1(V) \cap T_2(V) \quad \ldots (3) 
\]

\[
 W \cap V \subseteq W \cap (T_1(V) \cap T_2(V)) = (W \cap T_1(V)) \cup (W \cap T_2(V)) \subseteq (T_1(W) \cap T_1(V)) \cup (T_2(W) \cap T_2(V)) = (T_1(W \cap V)) \cup (T_2(W \cap V)) 
\]

Then \( W \cap V \) is \( T_{*12}\)-open set.

2) Let \( \mathcal{L} = \{ w_\alpha \mid \alpha \in I \} \) be any family of \( T_{*12}\)-open sets we must prove that \( \bigcup_\alpha w_\alpha \) is also a \( T_{*12}\)-open

\[
 w_\alpha \subseteq T_1(w_\alpha) \cup T_2(w_\alpha) \text{ for each } \alpha \in I \\
 \bigcup_\alpha w_\alpha \subseteq \bigcup_\alpha (T_1(w_\alpha) \cup T_2(w_\alpha)) = \bigcup_\alpha T_1(w_\alpha) \cup \bigcup_\alpha T_2(w_\alpha) 
\]

Now \( \bigcup_\alpha T_1(w_\alpha) = T_1(\bigcup_\alpha w_\alpha) \)

Also \( \bigcup_\alpha T_2(w_\alpha) = T_2(\bigcup_\alpha w_\alpha) \)

Then \( \bigcup_\alpha w_\alpha \subseteq T_1(\bigcup_\alpha w_\alpha) \cup T_2(\bigcup_\alpha w_\alpha) \) and \( \bigcup_\alpha w_\alpha \) is a \( T_{*12}\)-open.

**4.3 Remarks:**

i) The intersection of two \( T_{*12}\)-open is not necessarily \( T_{*12}\)-open, so the collection of all \( T_{*12}\)-open sets is not necessarily a topology on \( X. \)

Let \( \tau_{*12} \) be the topology generated by the collection of all \( T_{*12}\)-open sets.

ii) The intersection of any collection of \( T_{*12}\)-closed sets is \( T_{*12}\)-closed. Let \( T_{*12}\)-Cl(\( B \))-intersection of all \( T_{*12}\)-closed sets containing \( B \).

Recall that for a function \( f: X \rightarrow Y \), the subset \{ \( (x, f(x)) \mid x \in X \} \subseteq X \times Y \) is called the graph of \( f \) and denoted by \( G(f) \).

**4.4 Definition:** Let \( f:(X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) be a function the graph \( G(f) \) of \( f \) is said to be contra- \( T_{*12}\)-closed graph if for each \( (x, y) \in (X \times Y) \) \( G(f) \) there exists \( U \) which is \( T_{*12}\)-open containing \( x \) and a closed set \( V \) of \( Y \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \). The implies that \( f(U) \cap V = \emptyset \).

**4.5 Definition:** A space \( X \) is said to be contra-compact if every closed cover of \( X \) has a finite sub cover.

**4.6 Theorem:** Let \( (X, \tau, T_1, T_2) \) be a bi operator topological space and suppose \( f:(X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) has a contra- \( T_{*12}\)-closed graph, then the inverse image of a contra-compact set \( A \) of \( Y \) is \( T_{*12}\)-closed in \( X \).

**Proof:** Assume that \( A \) is contra-compact set of \( A \) and \( x \notin f^{-1}(A) \) for each \( a \in A \), \( (x, a) \notin G(f) \).

Then there exists \( U_a \) which is \( T_{*12}\)-closed containing \( x \) and \( V_a \) which is closed in \( Y \) containing a such that

\[
 f(U_a) \cap V_a = \emptyset. 
\]

Consider \( \mathcal{L} = \{ \bigcap_{\alpha} V_a \mid a \in A \} \) and \( \mathcal{L} \) is a closed cover of the subspace \( A \), but \( A \) is contra-compact then there exists \( a_1, a_2, a_3 \ldots a_n \) such that

\[
 A \subseteq \bigcup_{i=1}^n V_{a_i} 
\]

Let \( U = \bigcap_{i=1}^n U_{a_i} \), then \( U \) is \( T_{*12}\)-closed containing \( x \) and \( f(U) \cap A = \emptyset \), therefore
Since \( f \) is \( \text{Cl}(V) \cap \), such that \( f(x) \) is Urysohn then there exists open sets \( V \) and \( W \) and let \( x \in E \) such that \( E = \{ x \in X \mid f(x) = g(x) \} \) is \( \text{Cl}(W) \) is open in \( X \). Then \( f \) is \( T^*_{12} \)-closed.

**Proof:** First we show that an open set \( U \) of \( Y \) is contra-\( T^*_{12} \)-compact by (theorem 4.6) \( f^1(U) \) is a \( T^*_{12} \)-closed in \( X \) then for \( f \) is contra-\( T^*_{12} \)-continuous.

4.7 **Theorem:** Let \( Y \) be contra-\( T^*_{12} \)-compact space and let \( (X, \tau_{(12)}, T_1, T_2) \) be operator topological space, suppose \( f : (X, \tau_{(12)}, T_1, T_2) \rightarrow (Y, \delta) \) has a contra-\( T^*_{12} \)-closed graph then \( f \) is contra-\( T^*_{12} \)-continuous.

**Proof:** Since \( f \) is contra-\( T^*_{12} \)-continous then \( f^1(U) = g^1(X \times U) \) is a \( T^*_{12} \)-closed in \( X \). Then \( f \) is contra-\( T^*_{12} \)-continuous.

4.8 **Theorem:** Let \( f : (X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) be a function and \( g : X \rightarrow X \times Y \) the graph function of \( f \) defined by \( g(x) = (x, f(x)) \) for every \( x \in X \), if \( g \) is contra-\( T^*_{12} \)-continuous then \( f \) is contra-\( T^*_{12} \)-continuous.

**Proof:** Since \( g \) is contra-\( T^*_{12} \)-continous then \( f^1(U) = g^1(X \times U) \) is a \( T^*_{12} \)-closed in \( X \). Then \( f \) is contra-\( T^*_{12} \)-continuous.

4.9 **Theorem:** If \( f : (X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) is contra-\( T^*_{12} \)-continuous and \( g : (X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) is contra-continuous and \( Y \) is Urysohn space then \( E = \{ x \in X \mid f(x) = g(x) \} \) is \( T^*_{12} \)-closed in \( X \).

**Proof:** Let \( x \in E^c \), then \( f(x) \neq g(x) \), since \( Y \) is a Urysohn space then there exists open sets \( V \) and \( W \) such that \( f(x) \in V, g(x) \in W \), and \( \text{Cl}(V) \cap \text{Cl}(W) = \emptyset \).

Since \( f \) is contra-\( T^*_{12} \)-continous then \( f^1(\text{Cl}(V)) \) is \( T^*_{12} \)-open in \( X \) and \( g \) is contra-continuous and \( f(x) \) is open in \( X \), let \( U = f^1(\text{Cl}(V)), G = g^1(\text{Cl}(W)) \).

Then \( x \in U \cap G = A \), where \( A \) is \( T^*_{12} \)-open in \( X \) and \( f(A) \cap g(A) \subseteq f(U) \cap g(G) \subseteq \text{Cl}(V) \cap \text{Cl}(W) = \emptyset \), hence \( f(A) \cap g(A) = \emptyset \) and \( A \cap E = \emptyset, A \subseteq E^c \).

where \( A \) is \( T^*_{12} \)-open there for \( x \notin T^*_{12} \)-Cl(\( E \)), then \( E \) is \( T^*_{12} \)-closed in \( X \).

4.10 **Definition:** A subset \( A \) of operator topological space \( (X, \tau, T_1, T_2) \) is said to be \( T^*_{12} \)-dense in \( X \) if \( T^*_{12} \)-Cl(\( A \)) = \( X \).

4.11 **Remarks:** Let \( (X, \tau) \) be a topological space define:

\[ T_1 : p(X) \rightarrow p(X) \]

\[ T_2 : p(X) \rightarrow p(X) \text{ as follows} \]

\[ T_1 (A) = \text{Int} (\text{Cl} (A)) \]

\[ T_2 (A) = \text{Cl} (\text{Int}(A)) \text{, then } T^*_{12} \text{-dense subset will be b-dense and } T^*_{12} \text{-Cl}(A) \text{ will be b-Cl}(A) \text{ so b-dense in } X \text{ mean that b-Cl}(A) = X. \]

4.12 **Corollary:** Let \( f : (X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) is contra-\( T^*_{12} \)-continuous and \( g : (X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) is contra-continuous if \( Y \) is Urysohn and \( f = g \) on \( T^*_{12} \)-dense set \( A \subseteq X \) then \( f = g \) on \( X \).

**Proof:** since \( f \) is contra-\( T^*_{12} \)-continuous and is contra continuous and \( Y \) is Urysohn by previous Theorem \( E = \{ x \in X \mid f(x) = g(x) \} \) is a \( T^*_{12} \)-closed in \( X \). We have \( f = g \) on \( T^*_{12} \)-dense set \( A \subseteq E \), then \( X = T^*_{12} \text{-Cl} (A) \subseteq T^*_{12} \text{-Cl} (E) = E \). Hence \( f = g \) on \( X \).

4.13 **Definition:** A bi operator topological space \( (X, \tau, T_1, T_2) \) is called \( T^*_{12} \)-connected if \( X \) is not the Union of two non-empty \( T^*_{12} \)-open sets.

4.14 **Theorem:** If \( f : (X, \tau, T_1, T_2) \rightarrow (Y, \delta) \) is contra-\( T^*_{12} \)-continuous from a \( T^*_{12} \)-connected space onto \( Y \), then \( Y \) is not a discrete space.
Proof: Suppose that $Y$ is discrete. Let $\emptyset \neq A \subset Y$ then $A$ is proper nonempty open and closed subset of $Y$. Then $f^{-1}(A)$ is a proper nonempty $T_{12}^*$-clopen ($T_{12}^*$-open and $T_{12}^*$-closed) subset of $X$ such that $X = f^{-1}(A) \cup (f^{-1}(A))^c$ which means that $X$ is $T_{12}^*$-disconnected which is a contradiction. Hence $Y$ is not discrete.

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