Algebraic $K$-theory of stable $\infty$-categories via binary complexes

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Abstract

We adapt Grayson’s model of higher algebraic $K$-theory using binary acyclic complexes to the setting of stable $\infty$-categories. As an application, we prove that the $K$-theory of stable $\infty$-categories preserves infinite products.

1. Introduction

Using binary acyclic complexes, Grayson [5] gave a description of higher algebraic $K$-theory for exact categories in terms of generators and relations. The present article shows that Grayson’s picture of higher algebraic $K$-theory admits a concise description in the context of stable $\infty$-categories, using the language introduced by Blumberg–Gepner–Tabuada in [2]; see Section 2 for a quick recollection of the notions of additive and localizing invariants as well as the associated categories of motives. Let $U_{add}$ and $U_{loc}$ denote the universal additive and localizing invariants from [2].

In Section 3, we define analogs $F^qC$ and $B^qC$ of acyclic complexes and Grayson’s binary acyclic complexes in the setting of stable $\infty$-categories. Following Grayson’s argument, we obtain the following analog of [5, Corollary 6.5].

**Theorem 1.1.** The cofiber $\Gamma C$ of the ‘diagonal’ $U_{loc}(\Delta): U_{loc}(F^qC) \to U_{loc}(B^qC)$ is naturally equivalent to $\Omega U_{loc}(C)$.

Each binary acyclic complex has a length. One may ask to which extent the abelian group $\pi_0K(\Gamma \mathcal{C})$ defined in terms of binary acyclic complexes of a fixed length $r$ differs from the abelian group $\pi_0K(\Gamma \mathcal{C})$. Following the construction from [8], we show in Section 4 that binary acyclic complexes of length 5 generate the whole group $\pi_0K(\Gamma \mathcal{C})$, and we construct a split to binary acyclic complexes of length 7. More explicitly, we show the following.

**Theorem 1.2.** Let $\mathcal{C}$ be an idempotent complete stable $\infty$-category. The canonical map $\pi_0K(\Gamma_5\mathcal{C}) \to \pi_0K(\Gamma \mathcal{C})$ is a surjection and the canonical map $\pi_0K(\Gamma_7\mathcal{C}) \to \pi_0K(\Gamma \mathcal{C}) \cong K_1(\mathcal{C})$ admits a natural section.

The $K$-theory functor preserves filtered colimits and finite products for essentially trivial reasons. By work of Carlsson [3], it is known that the algebraic $K$-theory functor commutes with infinite products of exact categories and of Waldhausen categories with a cylinder functor. This result plays an essential role in proofs of the integral Novikov conjecture, for example, for virtually polycyclic groups [4, 14] and linear groups with a finite-dimensional model for the classifying space for proper actions [6, 7, 12]. As an application of Theorem 1.2, we will imitate the proof of Carlsson’s theorem given in [8] and show the following result in Section 5.
Theorem 1.3. For every family \( \{\mathcal{C}_i\}_{i \in I} \) of small stable \( \infty \)-categories, the natural map

\[
\text{K}\left( \prod_{i \in I} \mathcal{C}_i \right) \to \prod_{i \in I} \text{K}(\mathcal{C}_i)
\]
is an equivalence. The same result holds for connective algebraic \( K \)-theory \( K \) instead of non-connective \( K \)-theory \( K \).

The case of connective algebraic \( K \)-theory admits the following generalization to the universal additive invariant.

Theorem 1.4. The universal additive invariant \( U_{\text{add}} : \text{Cat}_{\infty}^{\text{ex}} \to \text{M}_{\text{add}}^{\text{comm}} \) commutes with arbitrary products. In particular, any additive invariant that becomes corepresentable on the category of non-commutative motives \( \text{M}_{\text{add}}^{\text{comm}} \) commutes with arbitrary products.

2. Algebraic \( K \)-theory of stable \( \infty \)-categories

We understand algebraic \( K \)-theory as an invariant of stable \( \infty \)-categories in the sense of Lurie [10]. In this context, the work of Blumberg–Gepner–Tabuada [2] provides a conceptual approach to algebraic \( K \)-theory which, in addition to some elementary facts about the class group \( K_0 \), is sufficient for the purposes of this article. The current section serves to recall the necessary statements and terminology.

2.1. Algebraic \( K \)-theory via universal invariants

We work in the setting of quasi-categories, the term \( \infty \)-category being used synonymously throughout. A quasi-category is stable if it admits all finite limits and colimits, and pushout squares and pullback squares coincide [10, Proposition 1.1.3.4]. A functor between stable \( \infty \)-categories is exact if it preserves finite limits and colimits [10, Proposition 1.1.4.1]. We denote by \( \text{Cat}_{\infty}^{\text{ex}} \) the \( \infty \)-category of small stable \( \infty \)-categories and exact functors. Inside \( \text{Cat}_{\infty}^{\text{ex}} \) we have the full subcategory of idempotent complete stable \( \infty \)-categories \( \text{Cat}_{\infty}^{\text{perf}} \).

The idempotent completion of any small stable \( \infty \)-category is stable [10, Corollary 1.1.3.7], and the idempotent completion functor \( \text{Idem} : \text{Cat}_{\infty}^{\text{ex}} \to \text{Cat}_{\infty}^{\text{perf}} \) provides a left adjoint to the inclusion functor.

We are mainly interested in localizing and additive invariants in the sense of Blumberg–Gepner–Tabuada [2]. In order to give the definitions of these notions, we recall some additional terminology.

Definition 2.1. Let \( \mathcal{A} \) and \( \mathcal{B} \) be stable \( \infty \)-categories, and let \( \mathcal{A} \subseteq \mathcal{B} \) be a full subcategory. We define the Verdier quotient \( \mathcal{B}/\mathcal{A} \) as the localization of \( \mathcal{B} \) at the collection of arrows in \( \mathcal{B} \) whose cofiber is equivalent to an object in \( \mathcal{A} \).

Definition 2.2 (cf. [2, Definition 5.12 and Proposition 5.13]). An exact sequence in \( \text{Cat}_{\infty}^{\text{ex}} \) is a sequence of exact functors

\[
\mathcal{A} \to \mathcal{B} \to \mathcal{C}
\]
satisfying the following properties.

1. The functor \( \mathcal{A} \to \mathcal{B} \) is fully faithful.
2. The composed functor \( \mathcal{A} \to \mathcal{C} \) is trivial.
3. The induced exact functor \( \text{Idem}(\mathcal{B}/\mathcal{A}) \to \text{Idem}(\mathcal{C}) \) is an equivalence.
Our definition of an exact sequence of stable ∞-categories does not match exactly the definition and characterization in \cite{2}. However, \cite[Proposition I.3.5]{11} shows that Definition 2.2 is equivalent to the one in \cite{2}. We prefer the present formulation because it applies more directly in the arguments of Section 3.

**Definition 2.3** \cite[Definition 8.1]{2}. Let $\mathcal{D}$ be a presentable stable ∞-category. A functor $F : \text{Cat}^\text{ex}_{\infty} \to \mathcal{D}$ is a localizing invariant if it satisfies the following properties.

1. $F$ commutes with filtered colimits.
2. $F$ sends exact sequences to cofiber sequences.

**Remark 2.4.** Note that \cite[Definition 8.1]{2} also requires that $F$ sends the canonical functor $\mathcal{C} \to \text{Idem}(\mathcal{C})$ to an equivalence for every $\mathcal{C} \in \text{Cat}^\text{ex}_{\infty}$. But this is a consequence of (2) by considering $0 \to \mathcal{C} \to \text{Idem}(\mathcal{C})$.

**Theorem 2.5** \cite[Theorem 8.7]{2}. There exists a presentable stable ∞-category $\mathcal{M}_{\text{loc}}$ and a localizing invariant $U_{\text{loc}} : \text{Cat}^\text{ex}_{\infty} \to \mathcal{M}_{\text{loc}}$ such that

\[
(U_{\text{loc}})^* : \text{Fun}^{\text{Lex}}(\mathcal{M}_{\text{loc}}, \mathcal{D}) \to \text{Fun}^{\text{loc}}(\text{Cat}^\text{ex}_{\infty}, \mathcal{D})
\]

is an equivalence for every presentable stable ∞-category $\mathcal{D}$, where $\text{Fun}^{\text{Lex}}$ denotes the ∞-category of colimit-preserving functors, and $\text{Fun}^{\text{loc}}$ denotes the ∞-category of localizing invariants.

For the moment, we contend ourselves with the characterization of non-connective algebraic $K$-theory $K : \text{Cat}^\text{ex}_{\infty} \to \text{Sp}$ as the localizing invariant corepresented by $U_{\text{loc}}(\text{Sp}^\omega)$, where $\text{Sp}^\omega$ denotes the stable ∞-category of compact spectra \cite[Theorem 9.8]{2}. That is,

\[
K(A) \simeq \text{Map}(U_{\text{loc}}(\text{Sp}^\omega), U_{\text{loc}}(A)).
\] (2.6)

The notion of an additive invariant is closely related to that of a localizing invariant. The difference lies in the fact that we require fewer exact sequences to be turned into cofiber sequences.

**Definition 2.7** \cite[Definition 5.18]{2}. An exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of stable ∞-categories is split-exact if $f$ and $g$ admit right adjoints $i : B \to A$ and $j : C \to B$, respectively, such that the unit $\text{id} \to if$ and counit $gj \to \text{id}$ are equivalences.

**Remark 2.8.** Note that in the above definition the unit $\text{id} \to if$ is automatically an equivalence since $f$ is fully faithful. One can also prove that the counit $gj \to \text{id}$ is an equivalence by showing that $j$ is fully faithful.

**Example 2.9.** The following is the canonical non-trivial example of a split-exact sequence of stable ∞-categories. Let $\mathcal{C}$ be a stable ∞-category, and consider the stable ∞-category $\mathcal{E}(\mathcal{C})$ of cofiber sequences in $\mathcal{C}$. Then, $\mathcal{E}(\mathcal{C})$ fits into a split-exact sequence

\[
\mathcal{C} \xrightarrow{k} \mathcal{E}(\mathcal{C}) \xrightarrow{q} \mathcal{C},
\]

in which the functor $q$ projects to the cofiber, while $k$ sends an object $X$ to the canonical cofiber sequence $X \xrightarrow{\text{id}} X \to 0$.

**Definition 2.10** \cite[Definition 6.1]{2}. Let $\mathcal{D}$ be a presentable stable ∞-category. A functor $F : \text{Cat}^\text{ex}_{\infty} \to \mathcal{D}$ is an additive invariant if it satisfies the following properties.
(1) $F$ commutes with filtered colimits.
(2) $F$ sends the canonical functor $\mathcal{C} \to \text{Idem}(\mathcal{C})$ to an equivalence for every $\mathcal{C} \in \text{Cat}_{\infty}^{\text{ex}}$.
(3) For every split-exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\text{Cat}_{\infty}^{\text{ex}}$, the morphism $F(f) + F(j) : F(A) \oplus F(\mathcal{C}) \to F(B)$ is an equivalence.

**Theorem 2.11** [2, Theorem 6.10]. There exists a presentable stable $\infty$-category $\mathcal{M}_{\text{add}}$ and an additive invariant $U_{\text{add}} : \text{Cat}_{\infty}^{\text{ex}} \to \mathcal{M}_{\text{add}}$ such that

$$(U_{\text{add}})^* : \text{Fun}^{\text{Lex}}_{\infty}(\mathcal{M}_{\text{add}}, \mathcal{D}) \to \text{Fun}^{\text{add}}_{\infty}(\text{Cat}_{\infty}^{\text{ex}}, \mathcal{D})$$

is an equivalence for every presentable stable $\infty$-category $\mathcal{D}$, where $\text{Fun}^{\text{Lex}}_{\infty}$ denotes the $\infty$-category of colimit-preserving functors, and $\text{Fun}^{\text{add}}_{\infty}$ denotes the $\infty$-category of additive invariants.

Assuming the non-connective algebraic $K$-theory functor $K$ to be known, connective algebraic $K$-theory $K : \text{Cat}_{\infty}^{\text{ex}} \to \text{Sp}$ can be characterized as the functor $\tau_{\geq 0} \circ K$, where $\tau_{\geq 0} : \text{Sp} \to \text{Sp}$ denotes the functor taking connective covers. By [2, Theorem 7.13], connective algebraic $K$-theory, considered as a colimit-preserving functor on $\mathcal{M}_{\text{add}}$, is corepresented by $U_{\text{add}}(\text{Sp}^\omega)$. Note that with the above definition, the connective algebraic $K$-theory of $\mathcal{C}$ only depends on $\text{Idem}(\mathcal{C})$. It is possible to define connective algebraic $K$-theory such that the values at $\mathcal{C}$ and $\text{Idem}(\mathcal{C})$ may differ, but here we are following [2].

2.2. Lower algebraic $K$-theory

Our arguments do require some additional information about the lower algebraic $K$-groups of a small stable $\infty$-category. For $\mathcal{C}$ a small stable $\infty$-category, we denote by $K_0(\mathcal{C})$ the abelian group generated by equivalence classes of objects in $\mathcal{C}$, subject to the relation $[C_0] + [C_2] = [C_1]$ for every cofiber sequence $C_0 \to C_1 \to C_2$ in $\mathcal{C}$. In general, we have an isomorphism

$$K_0(\text{Idem}(\mathcal{C})) \cong \pi_0 K(\mathcal{C}) \cong \pi_0 K(\mathcal{C}).$$

The next lemma gives a convenient criterion for equality of elements in $K_0(\mathcal{C})$.

**Lemma 2.12** (Heller’s criterion). Let $X$ and $Y$ be objects in a stable $\infty$-category $\mathcal{C}$. Then, $[X] = [Y] \in K_0(\mathcal{C})$ if and only if there are cofiber sequences

$$A \to X \oplus S \to B \quad \text{and} \quad A \to Y \oplus S \to B$$

for some objects $A$, $B$, $S \in \mathcal{C}$.

**Proof.** The proof of [8, Lemma 3.2] applies almost verbatim. \qed

Fix an uncountable regular cardinal $\kappa$. The suspension of a stable $\infty$-category $\mathcal{C}$ is defined to be

$$\Sigma \mathcal{C} := \text{Ind}_{\kappa}(\mathcal{C})^\kappa / \mathcal{C},$$

where $\text{Ind}_{\kappa}(\mathcal{C})^\kappa$ denotes the full subcategory of $\kappa$-compact objects in the Ind-completion of $\mathcal{C}$, see also [2, Section 2.4, Section 9.1]. The stable $\infty$-category $\text{Ind}_{\kappa}(\mathcal{C})^\kappa$ admits a canonical Eilenberg swindle since it admits countable coproducts, hence has contractible $K$-theory. Non-connective $K$-theory can be constructed from connective $K$-theory by the formula

$$K(\mathcal{C}) \simeq \colim_{n \in \mathbb{N}} \Omega^n K(\Sigma^n \mathcal{C}),$$
see [2, Definition 9.6]. It follows that $K(C) \simeq \Omega^n K(\Sigma^n C)$ for every $n \in \mathbb{N}$, and hence the negative $K$-groups of any stable $\infty$-category can be described by the formula

$$\pi_{-n} K(C) \cong K_0(\text{Idem}(\Sigma^n C)).$$

3. Grayson’s model for stable $\infty$-categories

Let $C$ be a stable $\infty$–category.

**Definition 3.1** (cf. [10, Definition 1.2.2.2]). For a linearly ordered set $I$, denote by $I^{[1]}$ the subposet of $I \times I$ containing all pairs $(i,j)$ satisfying $i \leq j$. An $I$-complex in $C$ is a functor $X: N(I^{[1]}) \to C$ such that $X_{i,i}$ is a zero object for all $i$, and such that for $i \leq j \leq k$ the square

$$
\begin{array}{ccc}
X_{i,j} & \longrightarrow & X_{i,k} \\
\downarrow & & \downarrow \\
X_{j,j} & \longrightarrow & X_{j,k}
\end{array}
$$

is a pushout. Denote by $S_I C$ the full subcategory of $\text{Fun}(I^{[1]}, C)$ spanned by the $I$-complexes. For $I = [n]$, we also write $S_n C$ instead of $S_{[n]} C$.

An $\mathbb{N}$-complex $X$ is called bounded if there exists a natural number $r \in \mathbb{N}$ such that $X_{0,i} \to X_{0,i+1}$ is an equivalence for $i \geq r$. In this case, we say that $X$ is supported on $[0, r]$. Let $FC$ denote the full subcategory of $S_{\mathbb{N}} C$ spanned by the bounded $\mathbb{N}$-complexes. We call $FC$ the stable $\infty$-category of bounded complexes.

Let $F^\delta C \subseteq FC$ be the full stable subcategory spanned by those bounded complexes $X$ such that $X_{0,i} \simeq 0$ for $i \gg 0$.

**Definition 3.2.** Let $\mathbb{N}^\delta$ denote the set of natural numbers considered as a discrete poset. The map $\mathbb{N}^\delta \to \mathbb{N}^{[1]}$, $i \mapsto (i, i+1)$ induces the functor

$$\text{gr}: FC \to \bigoplus_N C$$

which associates to a bounded complex its (underlying) graded object.

**Definition 3.3.** The category of binary complexes $BC$ is defined by the following pullback square of $\infty$-categories:

$$
\begin{array}{ccc}
BC & \xrightarrow{\top} & FC \\
\downarrow & & \downarrow \text{gr} \\
FC & \xrightarrow{\text{gr}} & \bigoplus_N C
\end{array}
$$

The functor $\top$ is called the top projection, while $\bot$ is the bottom projection. By substituting $F^\delta C$ at appropriate places, we obtain the following variations of $BC$:

$$
\begin{array}{ccc}
B^\top C & \xrightarrow{\top} & F^\top C \\
\downarrow & & \downarrow \text{gr} \\
FC & \xrightarrow{\text{gr}} & \bigoplus_N C
\end{array} \quad \begin{array}{ccc}
B^\bot C & \xrightarrow{\top} & FC \\
\downarrow & & \downarrow \text{gr} \\
F^\delta C & \xrightarrow{\text{gr}} & \bigoplus_N C
\end{array} \quad \begin{array}{ccc}
B^\delta C & \xrightarrow{\top} & F^\delta C \\
\downarrow & & \downarrow \text{gr} \\
F^\delta C & \xrightarrow{\text{gr}} & \bigoplus_N C
\end{array}
$$

The objects of $B^\top C$, $B^\bot C$ and $B^\delta C$ are called binary $\top$-acyclic complexes, binary $\bot$-acyclic complexes and binary acyclic complexes, respectively.
Remark 3.4. Occasionally, we will require an explicit description of maps $K \to BC$, where $K$ is some simplicial set. Since limits of stable $\infty$-categories may be computed in the category of all $\infty$-categories \cite[Theorem 1.1.4.4]{10}, it follows from \cite[Example 17.7.3 and Remark 17.7.4]{13} that maps $p: K \to BC$ may be described by a pair $(p^+, p^-)$ of maps $K \to FC$ together with a natural equivalence $gr \circ p^+ \sim gr \circ p^-$ in $\bigoplus\mathbb{N} C$. To specify a map $K \to B^i C$, $K \to B^b C$ or $K \to B^q C$, we additionally have to require that $p^+$, $p^-$ or both of them map to the full subcategory $F^0 C$.

Note that $B^q C$, $B^i C$, $B^b C$ and $BC$ are all stable. Moreover, these categories fit into a commutative square

$$
\begin{array}{ccc}
B^q C & \longrightarrow & B^i C \\
\downarrow & & \downarrow \\
B^b C & \longrightarrow & BC
\end{array}
$$

in which all functors are fully faithful and exact. The identity functor on $FC$ induces the diagonal functor $\Delta: FC \to BC$. Similarly, the identity functor on $F^q C$ induces a functor $\Delta: F^q C \to B^q C$, and these fit into a commutative square of exact functors

$$
\begin{array}{ccc}
F^q C & \overset{\Delta}{\longrightarrow} & B^q C \\
\downarrow & & \downarrow \\
FC & \overset{\Delta}{\longrightarrow} & BC
\end{array}
$$

By definition, the diagonal functor $\Delta$ is split by both $\top$ and $\bot$.

Definition 3.5. Let $C$ be a small stable $\infty$-category. Define the Grayson construction $\Gamma C$ on $C$ to be the motive

$$
\Gamma C := \cofib(U_{loc}(F^q C) \overset{U_{loc}(\Delta)}{\longrightarrow} U_{loc}(B^q C)).
$$

Our goal is to show that $\Gamma C$ represents $\Omega U_{loc}(C)$. The proof follows closely Grayson’s arguments in \cite{5}. Note that the functor $\Delta: F^q C \to B^q C$ is not fully faithful and hence we cannot directly form the Verdier quotient in $\text{Cat}_{\infty}^{\text{ex}}$.

Objects in the $\infty$-category $FC$ are filtered objects in $C$ with a choice of all possible filtration quotients. To make some of the upcoming proofs easier to read, we also define versions of these categories which do not include choices of filtration quotients, which will make it easier to map into these categories.

Let $f C \subseteq \text{Fun}(N(N), C)$ be the full subcategory of bounded filtrations, that is, the full subcategory spanned by those functors $X$ which are essentially constant in all but finitely many degrees and satisfy $X_0 \simeq 0$. By \cite[Lemma 1.2.2.4]{10}, the forgetful functor $u: FC \to f C$ induced by the map $\mathbb{N} \to N[1]$, $i \mapsto (0, i)$ is an equivalence.

Denote by $f^i C \subseteq f C$ the full stable subcategory spanned by those $X$ satisfying $\colim X \simeq 0$. By choosing an inverse of the equivalence $u: FC \overset{\sim}{\longrightarrow} f C$, we also obtain a functor $gr: f C \to \bigoplus\mathbb{N} C$. This allows us to define stable $\infty$-categories $b C$, $b^i C$, $b^q C$ and $b^i C$ in analogy to Definition 3.3.

The next lemma, which is reminiscent of the Gillet–Waldhausen theorem, but much easier to prove, tells us that we may concentrate on describing the $K$-theory of $FC/F^q C$. Let $\iota: C \to f C$ denote the functor induced by the projection $N(N) \to \Delta^0$, and let $\pi: f C \to C$ denote the colimit functor (which exists since filtrations in $f C$ become essentially constant). The structure maps of the colimit provide a natural transformation $\tau: \text{id} \to \iota \circ \pi$. 
Lemma 3.6. The functors \( \iota: \mathcal{C} \to fC/f^qC \) and \( \pi: fC/f^qC \to \mathcal{C} \) induced by \( \iota \) and \( \pi \) are equivalences.

Proof. There is an evident equivalence \( \pi \circ \iota \simeq \text{id}_C \). Since \( \pi \) vanishes on \( f^qC \), there is an induced exact functor \( \pi: fC/f^qC \to \mathcal{C} \) as well as a natural transformation \( \pi: \text{id} \to \iota \circ \pi \). We still have \( \pi \circ \iota \simeq \text{id} \). Moreover, \( \pi \) is a natural equivalence by Definition 2.1 of the Verdier quotient since \( \text{cofib}(\tau_X) \) lies in \( f^qC \) for every \( X \in f^qC \). The claim follows. \( \square \)

In what follows, we will omit the overline decoration on \( \iota \) and \( \pi \); it should always be clear from context whether we consider these as functors to/from the quotient category or the original category.

Lemma 3.7. There is a natural equivalence
\[
\Gamma C \simeq \text{fib}(U_{\text{loc}}(B^qC) \xrightarrow{U_{\text{loc}}(\top)} U_{\text{loc}}(F^qC)).
\]

Proof. Since \( \top \circ \Delta \simeq \text{id} \), we have the following commutative diagram, natural in \( C \):
\[
\begin{array}{ccc}
0 & \to & U_{\text{loc}}(F^qC) \\
\downarrow & & \downarrow \text{id} \\
\text{fib}(U_{\text{loc}}(B^qC) \xrightarrow{U_{\text{loc}}(\top)} U_{\text{loc}}(F^qC)) & \to & U_{\text{loc}}(B^qC) \xrightarrow{U_{\text{loc}}(\top)} U_{\text{loc}}(F^qC) \\
& \downarrow \text{id} & \downarrow \\
\text{fib}(U_{\text{loc}}(B^qC) \xrightarrow{U_{\text{loc}}(\top)} U_{\text{loc}}(F^qC)) & \simeq & \Gamma C & \to & 0
\end{array}
\]
\( \square \)

Definition 3.8. Let \( f_rC \subseteq fC \) denote the full subcategory of bounded filtrations supported on \([0, r]\), and define \( b_rC \) as the pullback
\[
\begin{array}{ccc}
b_rC & \xrightarrow{\top} & f_rC \\
\downarrow & & \downarrow \text{gr} \\
f_rC & \xrightarrow{\text{gr}} & \bigoplus C
\end{array}
\]

Definition 3.9. Let \( p_r: N(\mathbb{N}) \to \Delta^1 \) denote the map characterized by sending \( i \) to 0 for \( i \leq r \) and \( i \) to 1 for \( i \geq r + 1 \). Furthermore, choose a zero object \( 0 \) in \( \mathcal{C} \) and a section \( s \) of the trivial fibration \( \mathcal{C}_0/ \to \mathcal{C} \). Define the functor \( \iota_{r+1}: \mathcal{C} \to f_{r+1}C \) as the composition
\[
\mathcal{C} \xrightarrow{s} \mathcal{C}_0/ \subseteq \text{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{p_r^*} f_{r+1}C.
\]

Lemma 3.10. The map \( U_{\text{loc}}(\top): U_{\text{loc}}(BC) \to U_{\text{loc}}(FC) \) induced by the top projection is an equivalence.

Proof. Since we have a commutative square
\[
\begin{array}{ccc}
BC & \xrightarrow{\top} & FC \\
\downarrow \simeq & & \downarrow \simeq \\
bC & \xrightarrow{\top} & fC
\end{array}
\]
it suffices to prove the claim for the lower horizontal arrow.
Since \(fC \simeq \colim r f_rC\) and finite limits commute with directed colimits (combine \(10\), Theorem 1.1.4.4 and Proposition 1.1.4.6 with \(9\), Lemma 5.4.5.6), we also have \(bC \simeq \colim r b_rC\). As \(\mathcal{U}_{loc}\) commutes with filtered colimits, it is enough to prove that \(\mathcal{U}_{loc}(\top) : \mathcal{U}_{loc}(b_rC) \to \mathcal{U}_{loc}(f_rC)\) is an equivalence for all \(r\). We do this by induction on \(r\), the case \(r = 0\) being trivial.

Let \(gr_{r+1} : f_{r+1}C \to C\) be the composition of \(gr : f_{r+1}C \to \bigoplus C\) with the projection onto the \((r + 1)\)th component. Then, \(gr_{r+1} \circ tr_{r+1} \simeq \text{id}_C\), and there exists a natural transformation \(\text{id} \to gr_{r+1}\). Since \(gr_{r+1}\) vanishes on \(f_rC\), there is an induced functor \(\overline{gr}_{r+1} : f_{r+1}C/f_rC \to C\). On the quotient, the natural transformation \(\text{id} \to \text{id}_C\) becomes an equivalence since its fiber is contained in \(f_rC\). Hence, \(\overline{gr}_{r+1} : f_{r+1}C/f_rC \to C\) is an equivalence.

Consider the functor \(gr_{r+1} \circ \top : b_{r+1}C \to C\) next. Then, \(\Delta \circ \top \circ gr_{r+1}\) defines a functor in the opposite direction which satisfies \(\overline{gr}_{r+1} \circ \top \circ gr_{r+1} \simeq \text{id}\). There also exists a natural transformation \(\text{id} \to \Delta \circ \top \circ gr_{r+1} \circ \top\) since \(gr_{r+1} \circ \top \simeq gr_{r+1} \circ \bot\), which induces a natural equivalence on \(b_{r+1}C/b_rC\) by similar reasoning to the above.

We thus have a commutative diagram

\[
\begin{array}{ccc}
b_{r+1}C/b_rC & \to & f_{r+1}C/f_rC \\
\overline{gr}_{r+1} \circ \top \downarrow & & \downarrow \overline{gr}_{r+1} \\
C & & C
\end{array}
\]

Since \(\overline{gr}_{r+1} \circ \top\) and \(\overline{gr}_{r+1}\) are equivalences, \(\top\) is also an equivalence.

In particular, we obtain a map of split cofiber sequences

\[
\begin{array}{ccc}
\mathcal{U}_{loc}(b_rC) & \to & \mathcal{U}_{loc}(b_{r+1}C) \\
\downarrow & & \downarrow \\
\mathcal{U}_{loc}(f_rC) & \to & \mathcal{U}_{loc}(f_{r+1}C)
\end{array}
\]

By induction, we conclude that \(\mathcal{U}_{loc}(b_rC) \to \mathcal{U}_{loc}(f_rC)\) is an equivalence for all \(r\).

**Corollary 3.11.** There exists a natural equivalence

\[\Gamma C \simeq \Omega \text{fib}(\mathcal{U}_{loc}(BC/B^qC) \xrightarrow{\mathcal{U}_{loc}(\top)} \mathcal{U}_{loc}(FC/F^qC)).\]

**Proof.** Consider the map of cofiber sequences

\[
\begin{array}{ccc}
\mathcal{U}_{loc}(B^qC) & \to & \mathcal{U}_{loc}(BC) \\
\downarrow & & \downarrow \\
\mathcal{U}_{loc}(F^qC) & \to & \mathcal{U}_{loc}(FC)
\end{array}
\]

Since \(\mathcal{U}_{loc}(BC) \xrightarrow{\mathcal{U}_{loc}(\top)} \mathcal{U}_{loc}(FC)\) is an equivalence by Lemma 3.10, we obtain a natural equivalence

\[\Omega \text{fib}(\mathcal{U}_{loc}(BC/B^qC) \xrightarrow{\mathcal{U}_{loc}(\top)} \mathcal{U}_{loc}(FC/F^qC)) \simeq \text{fib}(\mathcal{U}_{loc}(B^qC) \xrightarrow{\mathcal{U}_{loc}(\top)} \mathcal{U}_{loc}(F^qC)).\]

The claim follows by combining this with Lemma 3.7.

**Definition 3.12.** Define the shift \(\langle - \rangle[1] : bC \to bC\) as the functor induced by the functor \(fC \to fC\) arising from the map of posets \(\mathbb{N} \to \mathbb{N}, i \mapsto \max\{0, i - 1\}\).

The next lemma is straightforward.
Lemma 3.13. The cofiber of the canonical natural transformation \((-)[1] \to \text{id}_{bC}\) takes values in the essential image of the diagonal functor \(\Delta : fqC \to bC\).

Recall that \(u : FC \to fC\) denotes the forgetful functor.

Proposition 3.14. The functor \(\pi \circ u \circ \top : BC/BqC \to C\) is an equivalence.

Proof. It suffices to prove that \(\pi \circ \top : bC/bC \to C\) is an equivalence. The functor \(\Delta \circ \iota : C \to bC/bC\) provides a right-inverse to \(\pi \circ \top\).

Note that \(\text{gr} \Delta \iota \pi \top(X)\) is zero in all but the lowest degree. Therefore, the transformation \(\tau \top : \top \to \iota \circ \pi \circ \top\) and the zero transformation \(0 : \bot \to \iota \circ \pi \circ \top\) induce a natural transformation \((-)[1] \to \Delta \circ \iota \circ \pi \circ \top\). In the resulting zig-zag \(\text{id} \leftarrow (-)[1] \to \Delta \circ \iota \circ \pi \circ \top\), the left-hand transformation becomes an equivalence in \(bC/bC\) by Lemma 3.13. Since the cofiber of the right-hand transformation lies in \(bC\), this transformation also becomes an equivalence in \(bC/bC\).

Since \(\Delta \circ \iota\) is a right-inverse to \(\pi \circ \top\), the claim follows. \(\square\)

Remark 3.15. By interchanging the roles of \(\top\) and \(\bot\), Proposition 3.14 also shows that the functor \(\pi \circ u \circ \bot : BC/BqC \to C\) is an equivalence.

Corollary 3.16. The functor \(\top : BC/BqC \to FC/FqC\) is an equivalence.

Proof. Since \(\pi\) is an equivalence by Lemma 3.6 and \(\pi \circ \top\) is an equivalence by Proposition 3.14, the claim follows. \(\square\)

Corollary 3.17. There exists a natural equivalence

\[ U_{\text{loc}}(B\top C/BqC) \simeq \text{fib}(U_{\text{loc}}(BC/BqC) \xrightarrow{U_{\text{loc}}(\top)} U_{\text{loc}}(FC/FqC)). \]

Proof. Consider the map of cofiber sequences

\[
\begin{array}{ccc}
U_{\text{loc}}(B\top C/BqC) & \xrightarrow{U_{\text{loc}}(\top)} & U_{\text{loc}}(BC/BqC) \\
\downarrow & & \downarrow \text{id} \\
\text{fib} \left(U_{\text{loc}}(BC/BqC) \xrightarrow{U_{\text{loc}}(\top)} U_{\text{loc}}(FC/FqC)\right) & \xrightarrow{} & U_{\text{loc}}(BC/BqC) \to U_{\text{loc}}(FC/FqC)
\end{array}
\]

Since \(U_{\text{loc}}(\top) : U_{\text{loc}}(BC/BqC) \to U_{\text{loc}}(FC/FqC)\) is an equivalence by Corollary 3.16, the claim follows. \(\square\)

Combining Corollary 3.11 and Corollary 3.17 we obtain the following result.

Corollary 3.18. There is a natural equivalence

\(\Gamma C \simeq \Omega U_{\text{loc}}(B\top C/BqC)\).

Lemma 3.19. Let \(X\) be a binary complex, that is, an object of \(BC\). Then,

\([X^\top_{0,k}] = [X^\bot_{0,k}] \in K_0(C)\)

for all \(k \in \mathbb{N}\).

Proof. This follows by an easy induction since \(\text{gr}(X^\top) \simeq \text{gr}(X^\bot)\). \(\square\)
**Definition 3.20.** Denote by $C^x$ the full subcategory of $C$ spanned by those objects $X$ satisfying $[X] = 0 \in K_0(C)$.

By Lemma 3.19, the functor $\pi \circ \perp$ restricts to a functor $\perp^x : B^iC \to C^x$ which vanishes on $B^0C$.

**Proposition 3.21.** The functor $\perp^x : B^iC/B^0C \to C^x$ induced by $\pi \circ \perp$ is an equivalence.

The proof of Proposition 3.21 relies on the following construction.

**Construction 3.22.** Let $X \in bC$ be a binary complex supported on $[0,r]$ satisfying $[\text{colim } X^\top] = 0 \in K_0(C)$, and hence also $[\text{colim } X^\perp] = 0 \in K_0(C)$ by Lemma 3.19. Let $k \geq r$. Choose objects $A, B, S \in C$ which fit into cofiber sequences

$$A \xrightarrow{a} X^\top_k \oplus S \xrightarrow{b} B$$

and

$$A \xrightarrow{a'} S \xrightarrow{b'} B.$$

These exist by virtue of Lemma 2.12.

Define $C \in fC$ as the filtered object

$$0 \to \cdots \to 0 \to A \xrightarrow{a} X^\top_k \oplus S \to 0 \to 0 \to \cdots,$$

where $A$ sits in degree $k - 1$ and $X^\top_k \oplus S$ occupies degree $k$.

Define a second filtered object $Y^\top$ as

$$X_0^\top \to \cdots \to X_{k-2}^\top \to X_{k-1}^\top \oplus A \xrightarrow{X^\top(k-1\leq k)\oplus a'} X_k^\top \oplus S \to 0 \to \cdots$$

We observe that $\text{gr}(Y^\top)$ and $\text{gr}(X^\perp \oplus C)$ are canonically equivalent. Hence, the given data combine to a new binary complex which we denote by $\mu(X)$. Note that $\mu(X)$ depends on the choice of $k$ and the cofiber sequences $A \to X_k \oplus S \to B$ and $A \to S \to B$.

Moreover, there is a canonical morphism $m : X \to \mu(X)$ such that $m^\perp$ is given by the inclusion $X^\perp \to X^\perp \oplus C$, and such that $m^\top$ is given by the inclusion of a direct summand up to degree $k$.

**Proof of Proposition 3.21.** It suffices to show that $\perp^x : b^iC/b^0C \to C^x$ is an equivalence. By taking vertical Verdier quotients in the commutative square of exact functors

$$\begin{array}{ccc} b^0C & \rightarrow & b^iC \\
\downarrow & & \downarrow \\
b^iC & \rightarrow & bC \end{array}$$

we obtain an exact functor $i : b^iC/b^0C \to bC/b^iC$. This functor fits into the commutative square of exact functors

$$\begin{array}{ccc} b^iC/b^0C & \rightarrow & bC/b^iC \\
\downarrow & & \downarrow \\
C^x & \rightarrow & C \end{array}$$

We make the following claims.

1. $\perp^x$ is essentially surjective.
2. $i$ is fully faithful.

Since these claims in conjunction with Remark 3.15 imply that $\perp^x$ is an equivalence, it suffices to prove the claims.
Let us first show claim (1). For any $X \in C^\chi$, apply Construction 3.22 to $\Delta(\iota(X))$ to obtain a preimage of $X$ under $\bot^\chi$.

For claim (2), we rely on [11, Theorem I.3.3] to reduce the claim to showing that the canonical map (induced by the inclusion of indexing categories)

$$\text{colim}_{Z \to Y \in b^bC/Y} \text{Map}_{b^bC}(X, \text{cofib}(Z \to Y)) \to \text{colim}_{Z \to Y \in b^bC/Y} \text{Map}_{b^bC}(X, \text{cofib}(Z \to Y))$$

is an equivalence. Since $b^bC \to bC$ is fully faithful, it suffices to show that the inclusion $b^bC/Y \subseteq b^bC/Y$ is cofinal. Using [9, Theorem 4.1.3.1], this in turn can be reduced to showing that

$$F := b^bC/Y \times_{b^bC/Y} (b^bC/Y)(\leq_k)$$

is weakly contractible for all $Z \to Y \in b^bC/Y$. In fact, we claim that $F$ is filtered.

Let $K$ be a finite simplicial set, and suppose we are given a map $f : K \to F$. Then, $f$ corresponds to a diagram $\Delta^0 \star K \to b^bC/Y$ with the following properties.

- The restriction to $\Delta^0 \star \emptyset$ classifies the object $Z \to Y$.
- The restriction to $K$ factors via $b^bC/Y$.

Since $b^bC$ admits finite colimits, there is a universal cone $\hat{f} : (\Delta^0 \star K)^\triangleright \to b^bC/Y$.

Let $W \in b^bC$ denote the object classified by the cone point of $\hat{f}$. Choose $k \in \mathbb{N}$ such that both $Y$ and $W$ are supported on $[0, k]$. Now apply Construction 3.22 to $W$ to obtain $\mu(W)$. We claim that the dashed arrow in the following diagram can be filled in such that the resulting triangle commutes:

$$W \xrightarrow{m} \mu(W) \xrightarrow{\cdot} Y$$

To construct the required 2-simplex in $bC$, we rely on Remark 3.4.

For any filtered object $V$, let $V^{\leq_k}$ denote the restriction of $V$ along the inclusion $[0, k] \to \mathbb{N}$. By construction, we find commutative diagrams

$$\text{Id}(W)^{\leq_k} \xrightarrow{m} \text{Id}(\mu(W))^{\leq_k} \xrightarrow{\cdot} \text{Id}(Y)^{\leq_k}$$

for $\text{Id} \in \{\top, \bot\}$ since $\text{Id}(W)^{\leq_k}$ is a direct summand in $\text{Id}(\mu(W))^{\leq_k}$. These diagrams extend essentially uniquely to commutative diagrams in $fC$ since $\text{Id}(Y)(i) \simeq 0$ for $i > k$. By definition, the chosen equivalences of graded objects patch together to yield a 2-simplex in $bC$.

It follows that we can replace the cone point $W$ by $\mu(W)$ and still obtain a diagram $\hat{f} : (\Delta^0 \star K)^\triangleright \to b^bC/Y$. Since $\mu(W) \in b^bC$, this corresponds to a diagram $K^\triangleright \to F$, so $F$ is filtered.

This proves that the inclusion $b^bC/Y \subseteq b^bC/Y$ is cofinal; thus, we have shown claim (2). □

Proof of Theorem 1.1. Combining Corollary 3.18 and Proposition 3.21, we obtain the following sequence of natural equivalences:

$$\Gamma C \simeq \Omega \mathcal{L}_{\text{loc}}(B^bC/B^qC) \simeq \Omega \mathcal{L}_{\text{loc}}(C^\chi).$$

Since every object in $C$ is a retract of an object in $C^\chi$, it follows that $\mathcal{L}_{\text{loc}}(C^\chi) \simeq \mathcal{L}_{\text{loc}}(C)$. This proves the theorem. □
Since the diagonal \( F^q \to B^q \) defines a natural transformation of endofunctors on \( \text{Cat}^\infty \), the construction \( \Gamma \mathcal{C} \) can be iterated. Any word \( W \) of length \( n \) over the alphabet \( \{ F^q, B^q \} \) defines a stable \( \infty \)-category \( W \mathcal{C} \). Letting the word \( W \) vary, the \( 2^n \) possible choices of \( W \) assemble into a commutative cube of dimension \( n \). By taking the \( n \)-fold iterated cofiber of this cube, we obtain a motive \( \Gamma^n \mathcal{C} \). To make sense of the naturality of the assignment \( \mathcal{C} \mapsto \Gamma^n \mathcal{C} \), we choose, once and for all, one preferred order of taking cofibers, say reading the word \( W \) from left to right.

**Corollary 3.23.** For all \( n \geq 1 \) there is a natural equivalence

\[
\Gamma^n \mathcal{C} \simeq \Omega^n \mathcal{U}_{\text{loc}}(\mathcal{C}).
\]

**Proof.** The proof is by induction, with the case \( n = 1 \) being Theorem 1.1. By induction hypothesis, and using Theorem 1.1 again, we have

\[
\Gamma^{n+1} \mathcal{C} \simeq \text{cofib}(\Gamma^n \mathcal{F}^q(\mathcal{C} \xrightarrow{\Gamma^q \Delta} \Gamma^n B^q \mathcal{C}))
\]

\[
\simeq \text{cofib}(\Omega^n \mathcal{U}_{\text{loc}}(\mathcal{F}^q \mathcal{C}) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \Omega^n \mathcal{U}_{\text{loc}}(B^q \mathcal{C}))
\]

\[
\simeq \Omega^n \Gamma \mathcal{C}
\]

\[
\simeq \Omega^{n+1} \mathcal{U}_{\text{loc}}(\mathcal{C}). \quad \Box
\]

**Corollary 3.24.** For all \( n \geq 1 \) there are natural equivalences

\[
\mathbf{K}(\Gamma^n \mathcal{C}) \simeq \Omega^n \mathbf{K}(\mathcal{C}) \quad \text{and} \quad K(\Gamma^n \mathcal{C}) \simeq \Omega^n \tau_{\geq n} K(\mathcal{C}).
\]

**Proof.** Using Corollary 3.23, the statement about non-connective \( K \)-theory is immediate because \( \mathbf{K} \) is a localizing invariant. The claim about connective \( K \)-theory follows by taking connective covers. \( \Box \)

From Corollary 3.24, we can now deduce a more algebraic description of higher \( K \)-groups resembling [5, Corollary 7.4].

**Proposition 3.25.** Let \( \mathcal{C} \) be an idempotent complete stable \( \infty \)-category and let \( n \geq 1 \). Then, \( \pi_n \mathbf{K}(\mathcal{C}) \cong \pi_0 \mathbf{K}(\Gamma^n \mathcal{C}) \) admits the following presentation.

It is the abelian group generated by equivalence classes \([X]\) of objects in \((B^q)^n \mathcal{C}\), subject to the relations

1. \( [X] = [X'] + [X''] \) whenever there exists a cofiber sequence \( X' \to X \to X'' \) in \((B^q)^n \mathcal{C}\);
2. \( [X] = 0 \) if \( X \) lies in the essential image of some diagonal functor

\[
(\mathcal{F}^q)^k \mathcal{F}^q(B^q)^{n-k-1} \mathcal{C} \xrightarrow{\Delta} (\mathcal{B}^q)^n \mathcal{C}, \quad 0 \leq k \leq n-1.
\]

By abuse of notation, we will refer to the above group also by \( K_0(\Gamma^n \mathcal{C})\).

**Proof.** Consider first the case \( n = 1 \). Since idempotent completeness can be characterized by the existence of certain colimits in \( \mathcal{C} \) (cf. [9, Section 4.4.5]), it follows from [9, Corollary 5.1.2.3] that \( \mathcal{F}^q \mathcal{C} \) is idempotent complete. As \( gr: \mathcal{F}^q \mathcal{C} \to \bigoplus_{\tau} \mathcal{C} \) preserves colimits, \( \mathcal{B}^q \mathcal{C} \) is idempotent complete by [9, Lemma 5.4.5.5]. This allows us to identify \( K_0(\mathcal{F}^q \mathcal{C}) \cong \pi_0 \mathbf{K}(\mathcal{F}^q \mathcal{C}) \) and \( K_0(\mathcal{B}^q \mathcal{C}) \cong \pi_0 \mathbf{K}(\mathcal{B}^q \mathcal{C}) \).

Since the diagonal functor \( \Delta: \mathcal{F}^q \mathcal{C} \to \mathcal{B}^q \mathcal{C} \) admits a retraction (by \( \top \) or \( \bot \)), we obtain a split exact sequence of abelian groups

\[
0 \to K_0(\mathcal{F}^q \mathcal{C}) \to K_0(\mathcal{B}^q \mathcal{C}) \to \pi_0 \mathbf{K}(\Gamma \mathcal{C}) \to 0.
\]
By Corollary 3.23, we have \( \pi_1 K(C) \cong \pi_0 K(\Gamma C) \), and the claim follows immediately.

The general case follows by considering cubes of higher dimensions.

4. Shortening binary complexes

Our next goal is to show that the explicit description of \( K_n \) in Proposition 3.25 includes a large number of superfluous generators and relations. To make this statement precise, we introduce the following variations of the categories \( F^q C, B^q C \) and the motive \( \Gamma C \). In this section, we always assume that \( C \) is idempotent complete.

**Definition 4.1.** Let \( r \geq 0 \). Denote by \( F^q r \subset F^q C \) the full subcategory of bounded filtrations supported on \([0, r]\). Define \( B^q r C \) as the pullback of the diagram

\[
\begin{array}{ccc}
F^q r C & \xrightarrow{\mathcal{U}_{\text{loc}}(F^q r C)} & F^q r C \\
\mathcal{U}_{\text{loc}}(\Delta) & \downarrow & \downarrow \\
B^q r C & \xleftarrow{\mathcal{U}_{\text{loc}}(B^q r C)} & B^q r C
\end{array}
\]

and set

\[
\Gamma_r C := \text{cofib} \left( \mathcal{U}_{\text{loc}}(F^q r C) \longrightarrow \mathcal{U}_{\text{loc}}(B^q r C) \right) \in M_{\text{loc}}.
\]

The evident inclusion functors induce a map \( \Gamma_r C \to \Gamma C \). The key result of this section, Proposition 4.3, states that the induced homomorphism \( K_0(\Gamma_r C) \to K_0(\Gamma C) \) admits a natural section for \( r = 7 \).

For convenience, we regard \( \Gamma_r C \) as the cofiber of the map \( \mathcal{U}_{\text{loc}}(f^q r C) \to \mathcal{U}_{\text{loc}}(b^q r C) \).

Note that the complexes in \( f^r \subset C \) start with a zero object in degree 0. In this section, we will suppress this zero and write complexes starting with degree 1.

Let \( X \in b^q C \). By Lemmas 2.12 and 3.19, we can choose objects \( A_k, B_k, S_k \) fitting into cofiber sequences \( A_k \to X_k^\top \oplus S_k \to B_k \) and \( A_k \to X_k^\bot \oplus S_k \to B_k \) for each \( k \geq 3 \). If \( X_k^\top \) and \( X_k^\bot \) are both trivial, also choose \( A_k, B_k \) and \( S_k \) to be trivial. Let \( f_k^\top : B_k \to \Sigma A_k \) and \( f_k^\bot : B_k \to \Sigma A_k \) be the induced maps. Define \( Y_2^\top \) to be the complex with

\[
Y_2^\top := \begin{array}{cccccccc}
X_1^\top & \longrightarrow & X_2^\top & \longrightarrow & X_3^\top & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\oplus & & & & & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & B_3 & \stackrel{f_3^\top}{\longrightarrow} & \Sigma A_3 & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
\]

and

\[
Y_2^\bot := \begin{array}{cccccccc}
X_1^\bot & \longrightarrow & X_2^\bot & \longrightarrow & X_3^\bot & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\oplus & & & & & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & B_3 & \stackrel{f_3^\bot}{\longrightarrow} & \Sigma A_3 & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
\]

By construction, there is a canonical equivalence \( \text{gr}(Y_2^\top) \cong \text{gr}(Y_2^\bot) \).

Similarly, define \( Y_k \) for \( k \geq 3 \) to be the complex with

\[
Y_k^\top := \begin{array}{cccccccc}
\Sigma^{-1} B_k & \xrightarrow{\Sigma^{-1} f_k^\top} & A_k & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\oplus & & & & & & & & & & \\
0 & \longrightarrow & X_k^\top & \longrightarrow & X_{k+1}^\top & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
\]

and

\[
Y_k^\bot := \begin{array}{cccccccc}
\Sigma^{-1} B_k & \xrightarrow{\Sigma^{-1} f_k^\bot} & A_k & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\oplus & & & & & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & B_{k+1} & \xrightarrow{f_{k+1}^\bot} & \Sigma A_{k+1} & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
\]
and

\[ \Sigma^{-1} B_k \xrightarrow{\Sigma^{-1} f_k^\top} A_k \xrightarrow{\oplus} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots \]

\[ Y_k^\perp := 0 \xrightarrow{x_k^\perp} X_k^\perp \xrightarrow{X_{k+1}^\perp} 0 \xrightarrow{0} 0 \xrightarrow{\cdots} \]

\[ 0 \xrightarrow{0} 0 \xrightarrow{B_{k-1}} \xrightarrow{f_{k+1}} \xrightarrow{\Sigma A_{k+1}} 0 \xrightarrow{\cdots} \]

It follows again from the choice of \( f_k^\top \) and \( f_k^\perp \) that we have canonical equivalences \( \text{gr}(\top(Y_k)) \simeq \text{gr}(\perp(Y_k)) \). Note that \( Y_k \) is an object of \( b_0^rC \).

**Proposition 4.2.** In \( K_0(\Gamma C) \), we have

\[ [X] = \sum_{k \geq 2} [Y_k]. \]

In particular, \( K_0(\Gamma_5 C) \rightarrow K_0(\Gamma C) \) is surjective.

**Proof.** We denote the morphisms \( X_k^\top \rightarrow X_{k+1}^\top \) by \( x_k^\top \). Let \( X_{k+1}^\top / X_k^\top := \text{cofib}(x_k^\top) \) and \( X_{k+1}^\perp / X_k^\perp := \text{cofib}(x_k^\perp) \). We have a morphism \( X[1] \rightarrow Y_2 \) with top component given as follows and bottom component given analogously. The induced map on the underlying graded is trivial.

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{x_1^\top} & X_1^\top & \xrightarrow{x_2^\top} & X_2^\top & \xrightarrow{x_3^\top} & X_3^\top & \xrightarrow{x_4^\top} & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
& x_1^\top & X_1^\top & \xrightarrow{(x_2^\top, 0)} & X_2^\top \oplus B_3 \xrightarrow{0 + f_3^\perp} \xrightarrow{\Sigma A_3} 0 & \xrightarrow{0} & \cdots \\
\end{array}
\]

By inspection, we see that the fiber has top component

\[ \Sigma^{-1} X_1^\top \xrightarrow{0} \Sigma^{-1} X_2^\top / X_1^\top \xrightarrow{0} \Sigma^{-1} X_3^\top / X_2^\top \oplus \Sigma^{-1} B_3 \xrightarrow{0 \oplus \Sigma^{-1} f_3^\perp} X_3^\top \oplus A_3 \xrightarrow{x_3^\top + 0} X_4^\top \rightarrow \cdots \]

and bottom component

\[ \Sigma^{-1} X_1^\perp \xrightarrow{0} \Sigma^{-1} X_2^\perp / X_1^\perp \xrightarrow{0} \Sigma^{-1} X_3^\perp / X_2^\perp \oplus \Sigma^{-1} B_3 \xrightarrow{0 \oplus \Sigma^{-1} f_3^\top} X_3^\perp \oplus A_3 \xrightarrow{x_3^\perp + 0} X_4^\perp \rightarrow \cdots \]

Note that the fiber splits as the direct sum of the diagonal of the complex

\[ \Sigma^{-1} X_1^\top \xrightarrow{0} \Sigma^{-1} X_2^\top / X_1^\top \xrightarrow{0} \Sigma^{-1} X_3^\top / X_2^\top \rightarrow 0 \rightarrow \cdots \]

and the complex \( F_1 \) with top component

\[ 0 \rightarrow 0 \rightarrow \Sigma^{-1} B_3 \xrightarrow{(0, \Sigma^{-1} f_3^\perp)} X_3^\top \oplus A_3 \xrightarrow{x_3^\top + 0} X_4^\top \rightarrow X_5^\top \rightarrow \cdots \]

and bottom component

\[ 0 \rightarrow 0 \rightarrow \Sigma^{-1} B_3 \xrightarrow{(0, \Sigma^{-1} f_3^\top)} X_3^\perp \oplus A_3 \xrightarrow{x_3^\perp + 0} X_4^\perp \rightarrow X_5^\perp \rightarrow \cdots \]

We conclude that \( [X[1]] = [F_1] + [Y_2] \in K_0(\Gamma C) \). By Lemma 3.13, we have \( [X] = [X[1]] \) and we can shift \( F_1 \) down. The claim now follows by induction on the length of the support of \( X \). \( \square \)
Proposition 4.3. Sending $[X]$ to $\sum_{k \geq 2} [Y_k]$ yields a natural splitting of the natural map

$$K_0(\Gamma_7 \mathcal{C}) \to K_0(\Gamma \mathcal{C}).$$

Proof. We first show that $\sum_{k \geq 2} [Y_k]$ is independent of the choices of $A_k, B_k$ and $S_k$. Let $k \geq 3$ be given.

Consider the following cofiber sequence of bounded filtrations supported on $[0,7]$ which is obtained by considering the right-hand map and taking its fiber:

$$\begin{align*}
\Sigma^{-2}B_k & \to 0 \to \Sigma^{-1}B_k \\
\Sigma^{-1}X^T_k \oplus \Sigma^{-1}X^\perp_k \oplus \Sigma^{-1}S & \to \Sigma^{-1}B_k \oplus (\Sigma^{-1}f^T_{k+2}) \oplus (\Sigma^{-1}f^\perp_{k+2}) X^T_k \oplus A_k \\
0 & \to (\Sigma^{-1}f^T_{k+2}) \oplus (\Sigma^{-1}f^\perp_{k+2}) X^T_k \oplus A_k \\
\Sigma^{-1}X^T_{k+1} \oplus A_k \oplus \Sigma^{-1}B_k \oplus A_k & \to X^T_k \oplus A_k \oplus (\Sigma^{-1}f^T_{k+2}) X^T_{k+1} \oplus B_{k+2} \\
0 & \to (\Sigma^{-1}f^T_{k+2}) X^T_k \oplus A_k \oplus (\Sigma^{-1}f^T_{k+2}) X^T_{k+1} \oplus B_{k+2} \\
X^T_{k+2} \oplus B_{k+2} & \to 0 \\
0 & \to (\Sigma^{-1}f^T_{k+2}) X^T_k \oplus A_k \oplus (\Sigma^{-1}f^T_{k+2}) X^T_{k+1} \oplus B_{k+2} \\
\Sigma A_{k+2} & \to 0 \\
0 & \to (\Sigma^{-1}f^T_{k+2}) X^T_k \oplus A_k \oplus (\Sigma^{-1}f^T_{k+2}) X^T_{k+1} \oplus B_{k+2}
\end{align*}$$

Note that the 2-cells on the right are all trivial, but this is not true for the 2-cells on the left. Interchanging $\top$ and $\perp$ in (4.4) defines a second cofiber sequence of bounded filtrations supported on $[0,7]$. Note that the two right-hand maps define a morphism in $b^b \mathcal{C}$ where the map on the underlying graded objects is trivial. Taking the fiber of this map yields an object $F_k$ in $b^b \mathcal{C}$ whose top component is given by the left-hand column in (4.4) and whose bottom component is given analogously.

Note that the right-hand column in the two versions of (4.4) is $Y_k$, while the fiber $F_k$ defines the sum of a shifted copy of $Y_{k+1}$ and a binary complex contained in the essential image of the diagonal functor. Using Lemma 3.13, it follows that $[Y_k] + [Y_{k+1}] \in K_0(\Gamma_7 \mathcal{C})$ is equal to the class of the binary acyclic complex defined by the middle column. Now it is enough to observe that the middle column is independent of the choice of $A_{k+1}, B_{k+1}$ and $S_{k+1}$.

The independence of $A_3, B_3$ and $S_3$ follows in the same way.

This shows that the assignment $X \mapsto \sum_{k \geq 2} [Y_k]$ gives a well-defined map

$$\text{ob } b^b \mathcal{C} \to K_0(\Gamma_7 \mathcal{C}),$$

which evidently sends equivalent objects to the same class.

Let $X \to X' \to X''$ be a cofiber sequence in $b^b \mathcal{C}$. Consider the cofiber sequences $X^T_k \to (X')^T_k \to (X'')^T_k$ and $X^\perp_k \to (X')^\perp_k \to (X'')^\perp_k, k \in \mathbb{N}$, as elements in $K_0(\mathcal{E} \mathcal{C})$. Since $[X^T_k] = [X^T_k]$ and $[(X'')^T_k] = [(X'')^T_k]$ in $K_0(\mathcal{C})$ by Lemma 3.19, we conclude from Example 2.9 that

$$[X^T_k \to (X')^T_k \to (X'')^T_k] = [X^T_k \to (X')^T_k \to (X'')^T_k] \in K_0(\mathcal{E} \mathcal{C}).$$
From Lemma 2.12, it follows that there are cofiber sequences \( A_k \to A'_k \to A''_k, B_k \to B'_k \to B''_k \) and \( S_k \to S'_k \to S''_k \) fitting into cofiber sequences of cofiber sequences as follows:

\[
\begin{array}{ccc}
A_k & \to & X'_k \oplus S_k \to B_k \\
& \downarrow & \downarrow \\
A'_k & \to & (X'_k)'' \oplus S'_k \to B'_k \\
& \downarrow & \downarrow \\
A''_k & \to & (X''_k)'' \oplus S''_k \to B''_k \\
& \downarrow & \downarrow \\
& & B_k
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
A_k & \to & X''_k \oplus S_k \to B_k \\
& \downarrow & \downarrow \\
A'_k & \to & (X''_k)'' \oplus S'_k \to B'_k \\
& \downarrow & \downarrow \\
A''_k & \to & (X''_k)'' \oplus S''_k \to B''_k \\
& \downarrow & \downarrow \\
& & B_k
\end{array}
\]

Using these for the construction of the objects \( Y_k, Y'_k, Y''_k \) of \( b^k \) as above, we obtain cofiber sequences \( Y_k \to Y'_k \to Y''_k \). Hence, (4.5) induces a well-defined homomorphism \( K_0(b^k) \to K_0(\Gamma_7 C) \).

If \( X^\perp \simeq X^\perp \), we can make our choices such that \( f_k^\perp \simeq f_k^\perp \) for all \( k \geq 3 \) and hence \( Y_k^\perp \simeq Y_k^\perp \) for all \( k \geq 2 \). Therefore, the map induces a well-defined homomorphism \( K_0(\Gamma C) \to K_0(\Gamma_7 C) \).

It is a split of the natural map \( K_0(\Gamma_7 C) \to K_0(\Gamma C) \) by Proposition 4.2. \( \square \)

The preceding Propositions 4.2 and 4.3 prove Theorem 1.2. We obtain the following generalization to higher algebraic K-theory.

**Theorem 4.6.** The canonical map \( K_0(\Gamma^n C) \to K_0(\Gamma^n C) \) is a surjection and the canonical map \( K_0(\Gamma^n C) \to K_0(\Gamma^n C) \cong K_n(C) \) admits a natural section.

**Proof.** We argue by induction. Proposition 4.2 and Proposition 4.3 prove the case \( n = 1 \), which is the start of the induction. The induction step is analogous to [5, Remark 8.1], [8, Proof of Theorem 1.4] and Corollary 3.23 above.

We will only describe the induction for the existence of the natural section, the induction for the surjectivity statement is completely analogous.

In order to extend Proposition 4.3 from \( K_1 \) to higher \( K \)-groups, we require the additional observation that \( \Gamma \) and \( \Gamma_7 \) commute. This can be morally seen by permuting the two factors in \( \mathbb{N} \times \mathbb{N} \), but the formal argument is rather lengthy. Hence, we will first complete the proof using this claim before giving the formal argument.

The map \( K_0(\Gamma^n C) \to K_0(\Gamma^n - 1 C) \) admits a natural section by Proposition 4.3 because it is a natural retract of the homomorphism \( K_0(\Gamma_7 B_7^n - 1 C) \to K_0(\Gamma B_7^n - 1 C) \). Since \( \Gamma \) and \( \Gamma_7 \) commute, it suffices to show that \( K_0(\Gamma_7^n - 1 C) \to K_0(\Gamma^n C) \) admits a natural section. Since this map is in turn a natural retract of the map \( K_0(\Gamma_7^n - 1 B_7 C) \to K_0(\Gamma_7^n - 1 B_7 C) \), this follows from the induction assumption.

What is left to do is to provide an argument why \( \Gamma \) and \( \Gamma_7 \) may be permuted. Fix \( r \in \mathbb{N} \).

Let \( W \) be a word of length \( n \) over the alphabet \( \{B^q, B^r\} \), and let \( \sigma \in \mathbb{S}_n \) be a permutation. We claim that there is a canonical equivalence \( WC \simeq W_r C \), where \( W_\sigma \) denotes the word \( W \) permuted according to \( \sigma \).

Recall that the natural transformation \( gr : F^q \to \bigoplus \mathbb{N} C \) is obtained via pullback with a map of posets \( \gamma : \mathbb{N}^2 \to \mathbb{N}^{[1]} \). Letting \( \mathbb{N}(0) := \mathbb{N}^{[1]} \) and \( \mathbb{N}(1) := \mathbb{N}^\delta \), define for \( x = (x_1, \ldots, x_n) \in \{0, 1\}^n \)

\[
\mathbb{N}(x) := \prod_{i=1}^n \mathbb{N}(x_i).
\]

Consider the functor

\[
\mathbb{N} : [1]^n \to \{\text{posets}\}, x \mapsto \mathbb{N}(x)
\]
induced by $\gamma$. Then, $\sigma$ induces a natural isomorphism $N \xrightarrow{\cong} \sigma^* N$ to the diagram of posets obtained by permuting the coordinates according to $\sigma$.

Let $\text{Fun}(N, C): (\Delta^1)^n \to \text{Cat}^\infty_\infty$ denote the induced $n$-cube of stable $\infty$-categories. Then, we obtain an induced equivalence of functors

$$\text{Fun}(\sigma^* N, C) \xrightarrow{\sim} \text{Fun}(N, C),$$

which contains as a full subfunctor those cubes in which we restrict to $F^q$ and $F^r$ at the appropriate places (according to the original choice of word $W$).

Let $\tilde{\text{Fun}}(N, C): (\text{sd}\Delta^1)^n \to \text{Cat}^\infty_\infty$ denote the induced $n$-cube of stable $\infty$-categories. Then, $W^C$ is the limit of $\tilde{\text{Fun}}(N, C)$, while $W_\sigma C$ can be obtained as the limit of $\tilde{\text{Fun}}(\sigma^* N, C)$. Since we have seen that the diagrams $\tilde{\text{Fun}}(N, C)$ and $\tilde{\text{Fun}}(\sigma^* N, C)$ are equivalent, we obtain

$$W^C \simeq W_\sigma C.$$

Let $W'$ denote the word over the alphabet $\{F^q, F^r\}$ obtained from $W$ by replacing $B$ with $F$. Then, there is an evident diagonal transformation $W'C \xrightarrow{\Delta} WC$ which fits into a commutative square

$$\begin{array}{ccc}
W'C & \xrightarrow{\Delta} & WC \\
\downarrow \cong & & \downarrow \cong \\
W_\sigma'C & \xrightarrow{\Delta} & W_\sigma C
\end{array}$$

Applying $U_{loc}$ and taking horizontal cofibers proves that

$$VC \simeq V_\sigma C$$

for any word $V$ over the alphabet $\{\Gamma, \Gamma_r\}$. In particular, $\Gamma$ and $\Gamma_r$ commute. □

5. Infinite products

This section is devoted to the proof of Theorem 1.3. The proof of Theorem 1.3 for connective $K$-theory is almost verbatim the same as the proof of [8, Theorem 4.1].

**Lemma 5.1.** The functor $K_0: \text{Cat}^\infty_\infty \to \text{Ab}$ commutes with infinite products.

**Proof.** Let $\{C_i\}_{i \in I}$ be a family of stable $\infty$-categories. The natural comparison map $K_0(\prod_{i \in I} C_i) \to \prod_{i \in I} K_0(C_i)$ is obviously surjective. Injectivity follows from Lemma 2.12. □

The next lemma, whose proof is similar to the argument in the proof of [8, Theorem 1.2], shows that Verdier quotients are compatible with the formation of products.

**Lemma 5.2.** Let $\{D_i \to C_i \to C_i/D_i\}_{i \in I}$ be a family of Verdier sequences in $\text{Cat}^\infty_\infty$.

Then, there is a natural equivalence

$$\prod_{i \in I} C_i /\prod_{i \in I} D_i \simeq \prod_{i \in I} C_i / D_i.$$
Proof. Consider the commutative diagram of stable ∞-categories and exact functors

\[
\begin{array}{ccc}
\prod_{i \in I} C_i & \xrightarrow{t} & \prod_{i \in I} C_i/\prod_{i \in I} D_i \\
\rho & \nearrow & \searrow \rho'
\end{array}
\]

Since the localization functor \( \ell \) is essentially surjective, so is \( f \). Therefore, it suffices to show that \( f \) is fully faithful. Using [11, Theorem I.3.3] to compute mapping spaces in the localization and referring to [1, Lemma 3.10] for the fact that filtered colimits distribute over products in spaces, we conclude that

\[
\text{Map}_{\prod_{i \in I} C_i/\prod_{i \in I} D_i}((X_i)_i, (Y_i)_i) \\
\cong \text{colim}_{((Z_i)_i \to (Y_i)_i) \in \prod_{i \in I} D_i/(Y_i)_i} \text{Map}_{\prod_{i \in I} C_i}((X_i)_i, \text{cofib}(Z_i_1 \to (Y_i)_i)) \\
\cong \text{colim}_{((Z_i)_i \to (Y_i)_i) \in \prod_{i \in I} D_i/(Y_i)_i} \prod_{i \in I} \text{Map}_{\prod_{i \in I} C_i}(X_i, \text{cofib}(Z_i \to Y_i)) \\
\cong \prod_{i \in I} \text{Map}_{\prod_{i \in I} C_i}(X_i, \text{cofib}(Z_i \to Y_i)) \\
\cong \prod_{i \in I} \text{Map}_{\prod_{i \in I} C_i/\prod_{i \in I} D_i}(X_i, Y_i) \\
\cong \text{Map}_{\prod_{i \in I} C_i/\prod_{i \in I} D_i}((X_i)_i, (Y_i)_i),
\]

so \( f \) is also fully faithful. \( \square \)

**Lemma 5.3.** Let \( \{C_i\}_{i \in I} \) be a family of stable ∞-categories. The canonical functor

\[
\text{Idem} \left( \prod_{i \in I} C_i \right) \to \prod_{i \in I} \text{Idem}(C_i)
\]

is an equivalence.

**Proof.** The canonical functor \( \prod_{i \in I} C_i \to \prod_{i \in I} \text{Idem}(C_i) \) exhibits \( \prod_{i \in I} \text{Idem}(C_i) \) as an idempotent completion of \( \prod_{i \in I} C_i \) in the sense of [9, Definition 5.1.4.1]: Since idempotent completeness amounts to the existence of certain colimits ([9, Section 4.4.5]) and colimits in a product category can be computed componentwise, \( \prod_{i \in I} \text{Idem}(C_i) \) is idempotent complete; moreover, every object in \( \prod_{i \in I} \text{Idem}(C_i) \) is a retract of an object in \( \prod_{i \in I} C_i \) because this is true for each individual component. \( \square \)

**Proposition 5.4.** Let \( \{C_i\}_{i \in I} \) be a family of stable ∞-categories. The comparison map

\[
\pi_n K\left( \prod_{i \in I} C_i \right) \to \prod_{i \in I} \pi_n K(\mathcal{C}_i)
\]

is an isomorphism for all \( n \in \mathbb{Z} \).

**Proof.** By Lemma 5.3 and the fact that \( K \) is a localizing invariant, we may assume without loss of generality that all \( C_i \) are idempotent complete.
The case \( n = 0 \) is provided by Lemma 5.1. For \( n \geq 1 \), we consider the commutative diagram
\[
\begin{array}{c}
K_0(\Gamma^n \prod_{i \in I} C_i) \rightarrow \prod_{i \in I} K_0(\Gamma^n C_i) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
\[
\approx \prod_{i \in I} K(\text{Fun}^{ex}(\mathcal{B}, \text{Idem}(\mathcal{C}_i)))
\]

\[
\approx \prod_{i \in I} \text{Map}(\mathcal{U}_{\text{add}}(\mathcal{B}), \mathcal{U}_{\text{add}}(\mathcal{C}_i))
\]

where (**) follows from Lemma 5.3 and (***) follows from Theorem 1.3. Since \( \mathcal{M}_{\text{add}} \) is a localization of \( \text{PreSp}((\text{Cat}_{\infty}^{\text{perf}})) \) by \([2, \text{Remark 6.8}]\), it is generated by the images of compact idempotent complete stable \( \infty \)-categories under \( \mathcal{U}_{\text{add}} \). This verifies the universal property of the product. Hence,

\[
\mathcal{U}_{\text{add}} \left( \prod_{i \in I} \mathcal{C}_i \right) \approx \prod_{i \in I} \mathcal{U}_{\text{add}}(\mathcal{C}_i).
\]

□

Remark 5.5. The proof of Theorem 1.4 breaks down for \( \mathcal{U}_{\text{loc}} \) since the identification of mapping spectra in \( \mathcal{M}_{\text{loc}} \) holds only under stricter assumptions on \( \mathcal{B} \), cf. \([2, \text{Theorem 9.36}]\).

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