SUPER-REPLICABLE FUNCTIONS $N(j_{1,N})$ AND PERIODICALLY VANISHING PROPERTY

CHANG HEON KIM AND JA KYUNG KOO

Abstract. We find the super-replication formulae which would be a generalization of replication formulae. And we apply the formulae to derive periodically vanishing property in the Fourier coefficients of the Hauptmodul $N(j_{1,12})$ as a super-replicable function.

1. Introduction

Let $\mathcal{H}$ be the complex upper half plane and let $\Gamma_1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to \((\frac{1}{0} \ \frac{1}{1})\) mod $N$ ($N = 1, 2, \cdots$). Since the group $\Gamma_1(N)$ acts on $\mathcal{H}$ by linear fractional transformations, we get the modular curve $X_1(N) = \Gamma_1(N) \setminus \mathcal{H}^*$, as a projective closure of the smooth affine curve $\Gamma_1(N) \setminus \mathcal{H}$, with genus $g_{1,N}$. Here, $\mathcal{H}^*$ denotes the union of $\mathcal{H}$ and $\mathbb{P}^1(\mathbb{Q})$.

Ishida and Ishii showed in [11] that for $N \geq 7$, the function field $K(X_1(N))$ is generated over $\mathbb{C}$ by the modular functions $X_2(z,N)^{\epsilon_N} \cdot N$ and $X_3(z,N)^N$, where $X_r(z,N) = e^{2\pi i (r-1)(N-1)/4N} \prod_{s=0}^{N-1} \frac{K_{r,s}(z)}{K_{1,s}(z)}$ and $\epsilon_N$ is 1 or 2 according as $N$ is odd or even. Here, $K_{r,s}(z)$ is a Klein form of level $N$ for integers $r$ and $s$ not both congruent to 0 mod $N$. On the other hand, since the genus $g_{1,N} = 0$ only for the eleven cases $1 \leq N \leq 10$ and $N = 12$ ([23], [13]), the function field $K(X_1(N))$ in this case is a rational function field $\mathbb{C}(j_{1,N})$ for some modular function $j_{1,N}$ (Table 3, Appendix).

The element \((\frac{1}{0} \ \frac{1}{1})\) of $\Gamma_1(N)$ takes $z$ to $z + 1$, and in particular a modular function $f$ in $K(X_1(N))$ is periodic. Thus it can be written as a Laurent series in $q = e^{2\pi i z}$ ($z \in \mathcal{H}$), which is called a $q$-series (or $q$-expansion) of $f$. We call $f$ normalized if its $q$-series starts

\begin{itemize}
  \item Supported by KOSEF Research Grant 98-0701-01-01-3
  \item AMS Classification : 11F03, 11F22
\end{itemize}
with $q^{-1} + a_1 q + a_2 q^2 + \cdots$. By a Hauptmodul $t$ we mean the normalized generator of a genus zero function field $K(X_1(N))$ and we write $t = q^{-1} + \sum_{k \geq 1} H_k q^k$ for its $q$-series.

For a Fuchsian group $\Gamma$, let $\overline{\Gamma}$ denote the inhomogeneous group of $\Gamma (= \Gamma / \pm I)$. Let $\Gamma_0(N)$ be the Hecke subgroup given by $\{(a, b, c, d) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$. Also, let $t = \mathcal{N}(j_{1,N})$ be the Hauptmodul of $\Gamma_1(N)$ and $X_n(t)$ be a unique polynomial in $t$ of degree $n$ such that $X_n(t) - \frac{1}{n} q^{-n}$ belongs to the maximal ideal of the local ring $\mathbb{C}[[q]]$. Polynomials with this property are known as the Faber polynomials ([5], Chapter 4).

Write $X_n(t) = \frac{1}{n} q^{-n} + \sum_{m \geq 1} H_{m,n} q^m$.

When $\overline{\Gamma}_1(N) = \overline{\Gamma}_0(N)$, $\mathcal{N}(j_{1,N})$ becomes a replicable function, that is, it satisfies the following replication formulae

\[(*) \quad H_{a,b} = H_{c,d} \quad \text{whenever} \quad ab = cd \quad \text{and} \quad (a,b) = (c,d)\]

([1], [3], [20]). Given a replicable function $f$ the $n$-plicate of $f$ is defined iteratively by

$$f^{(n)}(nz) = - \sum'_{a,d = n \atop 0 \leq b < d} f^{(a)} \left( \frac{az + b}{d} \right) + nX_n(f)$$

where the primed sum means that the term with $a = n$ is omitted ([3]). We call $f$ completely replicable if $f$ is a replicable function with rational integer coefficients and has only a finite number of distinct replicates, which are themselves replicable functions. According to [1] there are, excluding the trivial cases $q^{-1} + aq$, 326 completely replicable functions of which 171 are monstrous functions, i.e., modular functions whose $q$-series coincide with the Thompson series $T_g(q) = \sum_{n \in \mathbb{Z}} \text{Tr}(g|V_n)q^n$ for some element $g$ of the monster simple group $M$ whose order is approximately $8 \cdot 10^{53}$. Here we observe that $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is the infinite dimensional graded representation of $M$ constructed by Frenkel et al. ([8], [9]). Furthermore, in [3] Cummins and Norton showed that if $f$ is replicable, it can be determined only by the 12 coefficients of its first 23 ones.
If $\Gamma_1(N) \neq \Gamma_0(N)$, unlike those replicable functions mentioned above, we show in §3 that the Fourier coefficients of $X_n(t)$ with $t = \mathcal{N}(j_1,N)$ ($N \neq 7, 9$) satisfy a twisted formula (10) by a character $\psi$ (see Corollary 11). Here we note that when we work with the Thompson series, it is reduced to replication formulae in (*) by viewing $\psi$ as the trivial character. Thus in this sense it gives a more general class of modular functions, which we propose to call $\mathcal{N}(j_1,N)$ a super-replicable function.

There would be certain similarity between some of replicable functions and super-replicable ones as follows. We derived in [17] the following self-recursion formulas for the Fourier coefficients of $\mathcal{N}(j_1,N)$ without the aid of its 2-plicate when $N = 2, 6, 8, 10, 12$: for $k \geq 1$,

$$H_{4k-1} = \frac{H_{2k-1}}{2} + 2 \sum_{1 \leq j < k-1} H_{2j}H_{4k-2j-2} + \alpha \cdot H_{4k-2} - \frac{H_{2k-1}^2}{2} - \sum_{1 \leq j < 2k-2} H_jH_{4k-j-2}$$

$$H_{4k} = -\beta \cdot H_{4k-2} - \sum_{1 \leq j < 2k-1} H_jH_{2(2k-j-1)}$$

$$H_{4k+1} = \frac{H_{2k}}{2} + 2 \sum_{1 \leq j < k} H_{2j}H_{4k-2j} + \alpha \cdot H_{4k} + \frac{H_{2k}^2}{2} - \sum_{1 \leq j < 2k} H_jH_{4k-j}$$

$$H_{4k+2} = -\beta \cdot H_{4k} - \sum_{1 \leq j < 2k} H_jH_{2(2k-j)}$$

where $\alpha = -\mathcal{N}(j_1,N)(\frac{1+N^2}{N})$ and $\beta = -\mathcal{N}(j_1,N)(\frac{1}{N^2})$. Furthermore, we verified in [18] that the above recursion can be also applied to 14 monstrous functions of even levels (including $\mathcal{N}(j_{1,2})$ and $\mathcal{N}(j_{1,6})$) which are Thompson series of type $2B, 6C, 6E, 6F, 10B, 10E, 14B, 18C, 18D, 22B, 30C, 30G, 42C, 46AB$ (these are all replicable functions) and one monster-like function of type $18e$ (for the definition of monster-like function, we refer to [6]). Therefore the Hauptmoduln mentioned above which have self-recursion formulas can be determined just by the first four coefficients $H_1, H_2, H_3$ and $H_4$ without the aid of 2-plicate.

What is more interesting would be the fact that there seems to be a connection between super-replicable functions and infinite dimensional Lie superalgebras. That is, considering
the arguments from Borcherds [2], Kang [12] and Koike [20] we believe that the super-
replication formulae in (10) might suggest the existence of certain infinite dimensional Lie superalgebra whose denominator identity implies such formulae.

Lastly, as an application of super-replication formulae we consider the following periodically vanishing property. Many of monstrous functions, for example, Thompson series of type 4B, 4C, 4D, 6F, 8B, 8C, 8D, 8E, 8F, 9B, etc have periodically vanishing properties among the Fourier coefficients (see the Table 1 in [22]). This result must be known to experts, but we could not find a reference. Hereby we describe it in Theorem 13. Meanwhile, as for the case of super-replicable functions, we see from the Appendix, Table 4 that only the Haupmodul $N(j_{1,12})$ seems to have such property. To this end, we shall first derive in §4 an identity (22) which is analogous to the “$2^k$-plication formula” ([7], [20]) satisfied by replicable functions. And, combining this with the super-replication formulae we are able to verify that the Fourier coefficients $H_m$ of $N(j_{1,12})$ vanish whenever $m \equiv 4 \mod 6$ (Corollary 19).

Through the article we adopt the following notations:

- $S_{\Gamma_1(N)}$ the set of $\Gamma_1(N)$-inequivalent cusps
- $q_h = e^{2\pi i z/h}$, $z \in \mathbb{H}$
- $f|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\left(\frac{az+b}{cz+d}\right)$
- $f(z) = g(z) + O(1)$ means that $f(z) - g(z)$ is bounded as $z$ goes to $i\infty$.

2. Hauptmodul of $\Gamma_1(12)$

In this section we investigate the generalities of the modular function $j_{1,12}$ which is under primary consideration and construct the Hauptmodul $N(j_{1,12})$. We also examine some number theoretic property of $N(j_{1,12})$. As for more arithmetic properties, we refer to [10].
Lemma 1. Let $\frac{a}{c}$ and $\frac{a'}{c'}$ be fractions in lowest terms. Then $\frac{a}{c}$ is $\Gamma_1(N)$-equivalent to $\frac{a'}{c'}$ if and only if $\pm\left(a'\right) \equiv \left(a+nc\right) \mod N$ for some $n \in \mathbb{Z}$.

Proof. Straightforward. \hfill \Box

Using the above lemma we can check that the cusps $0, 1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9, \infty, 5/12$ are $\Gamma_1(12)$-inequivalent. But from \[13\] we know that the cardinality of $S_{\Gamma_1(12)}$ is 10, whence

$$S_{\Gamma_1(12)} = \{0, 1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9, \infty, 5/12\}.$$ 

For later use we are in need of calculating the widths of the cusps of $\Gamma_1(12)$.

Lemma 2. Let $a/c \in \mathbb{P}^1(\mathbb{Q})$ be a cusp where $(a, c) = 1$. Then the width of $a/c$ in $X_1(N)$ is given by $N/(c, N)$ if $N \neq 4$.

Proof. If $N \mid 4$, the statement is obvious. Hence, we assume that $N$ does not divide 4, i.e., $N \neq 1, 2, 4$. First, choose $b$ and $d$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Let $h$ be the width of the cusp $a/c$. Then $h$ is the smallest positive integer such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \pm\Gamma_1(N).$$

Thus we have

$$\begin{pmatrix} 1-calh & * \\ -c^2h & 1+calh \end{pmatrix} \in \pm\Gamma_1(N).$$

If $\begin{pmatrix} 1-calh & * \\ -c^2h & 1+calh \end{pmatrix}$ is an element of $-\Gamma_1(N)$, by taking trace $2 \equiv -2 \mod N$; hence $N \mid 4$. Thus when $N \neq 1, 2, 4$, $\begin{pmatrix} 1-calh & * \\ -c^2h & 1+calh \end{pmatrix} \in \Gamma_1(N)$. This condition is equivalent to saying that

$$h \in \frac{N}{(c^2, N)}\mathbb{Z} \cap \frac{N}{(ca, N)}\mathbb{Z} = \frac{N}{(c, N)}\mathbb{Z}. $$
We then have the following table of inequivalent cusps of $\Gamma_1(12)$:

| cusp  | width |
|-------|-------|
| $\infty$ | 1 |
| 0     | 12   |
| 1/2   | 6    |
| 1/3   | 4    |
| 1/4   | 3    |
| 1/5   | 12   |
| 1/6   | 2    |
| 1/8   | 4    |
| 1/9   | 3    |
| 5/12  | 1    |

Recall the Jacobi theta functions $\theta_2, \theta_3,$ and $\theta_4$ defined by

\[
\theta_2(z) = \sum_{n \in \mathbb{Z}} q^{(n+\frac{1}{2})^2} \\
\theta_3(z) = \sum_{n \in \mathbb{Z}} q^{n^2} \\
\theta_4(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}
\]

for $z \in \mathfrak{H}$. We have the following transformation formulas ([26] pp.218-219).

1. $\theta_2(z + 1) = e^{\frac{1}{4\pi i}} \theta_2(z)$
2. $\theta_3(z + 1) = \theta_4(z)$
3. $\theta_4(z + 1) = \theta_3(z)$
4. $\theta_2 \left( -\frac{1}{z} \right) = (-iz)^{\frac{1}{2}} \theta_4(z)$
5. $\theta_3 \left( -\frac{1}{z} \right) = (-iz)^{\frac{1}{2}} \theta_3(z)$
6. $\theta_4 \left( -\frac{1}{z} \right) = (-iz)^{\frac{1}{2}} \theta_2(z)$.

**Lemma 3.** Let $k$ be an odd positive integer and $N$ be a multiple of 4. Then for $F(z) \in M_{1/2}(\tilde{\Gamma}_0(N))$ and $m \geq 1$, $F(mz) \in M_{1/2}(\tilde{\Gamma}_0(mN), \chi_m)$ with $\chi_m(d) = (\frac{m}{d})$ and $(d,m) = 1$.

**Proof.** [29], Proposition 1.3.

Put $j_{1,12}(z) = \theta_3(2z)/\theta_3(6z)$.

**Theorem 4.** (a) $\theta_3(2z) \in M_{1/2}(\tilde{\Gamma}_0(4))$ and $\theta_3(6z) \in M_{1/2}(\tilde{\Gamma}_0(12), \chi_3)$.
(b) $K(X_1(12))$ is equal to $\mathbb{C}(j_{1,12}(z))$. $j_{1,12}$ takes the following value at each cusp: $j_{1,12}(\infty) = 1$, $j_{1,12}(0) = \sqrt{3}$, $j_{1,12}(\frac{1}{2}) = 0$ (a simple zero), $j_{1,12}(\frac{1}{3}) = i$, $j_{1,12}(\frac{1}{4}) = \sqrt{3}i$, $j_{1,12}(\frac{1}{5}) = -\sqrt{3}$, $j_{1,12}(\frac{1}{7}) = \infty$ (a simple pole), $j_{1,12}(\frac{1}{8}) = -\sqrt{3}i$, $j_{1,12}(\frac{1}{9}) = -i$, $j_{1,12}(\frac{1}{11}) = -1$.

**Proof.** For the first part, we recall that ([19], p.184)

$$\theta_3(2z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(4)).$$

Then by Lemma 3 we immediately get that

$$\theta_3(6z) \in M_{\frac{1}{2}}(\widetilde{\Gamma}_0(12), \chi_3).$$

By the assertion (a), it is clear that $j_{1,12}(z) \in K(X_1(12))$. Thus for (b), it is enough to show that $j_{1,12}(z)$ has only one simple zero and one simple pole on the curve $X_1(12)$. As is well-known, $\theta_3(z)$ never vanishes on $\mathfrak{H}$. Hence we are forced to investigate the zeroes and poles of $j_{1,12}$ at each cusp of $\Gamma_1(12)$. Let $S = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix})$ and $T = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$.

(i) $s = \infty$:

$$j_{1,12}(\infty) = \lim_{z \to \infty} \frac{\theta_3(2z)}{\theta_3(6z)} = \lim_{q \to 0} \frac{1 + 2q + 2q^4 + \cdots}{1 + 2q^3 + 2q^6 + \cdots} = 1.$$ 

(ii) $s = 0$:

$$j_{1,12}(0) = \lim_{z \to i\infty} \frac{\theta_3(2z)}{\theta_3(6z)} \bigg|_S = \lim_{z \to i\infty} \frac{\sqrt{-i\frac{z}{2}} \theta_3(\frac{z}{2})}{\sqrt{-i\frac{z}{6}} \theta_3(\frac{z}{6})} \text{ by (3)} = \sqrt{3}.$$ 

(iii) $s = \frac{1}{2}$: We observe that $(ST^{-2}S)\infty = (\begin{smallmatrix} -1 & 0 \\ -2 & -1 \end{smallmatrix}) \cdot \infty = \frac{1}{2}$. 


Considering the identities

\[ \theta_3(2z)^2|_{S_1} = z^{-1} \theta_3 \left( \frac{-2}{z} \right)^2 = z^{-1} \left( -\frac{iz}{2} \right)^{\frac{1}{2}} \theta_3 \left( \frac{z}{2} \right)^2 \text{ by (3)} \]

\[ = -\frac{i}{2} \theta_3 \left( \frac{z}{2} \right)^2 \]

\[ \theta_3(2z)^2|_{ST^{-2}z_1} = -\frac{i}{2} \theta_3 \left( \frac{z}{2} \right)^2 |_{T^{-2}z_1} = -\frac{i}{2} \theta_4 \left( \frac{z}{2} \right)^2 \text{ by (4)} \]

\[ \theta_3(2z)^2|_{ST^{-2}S_1} = -\frac{i}{2} \theta_4 \left( \frac{z}{2} \right)^2 |_{S_1} = -\frac{i}{2} z^{-1} \left\{ (-2iz)^{\frac{1}{2}} \theta_2(2z) \right\}^2 \text{ by (5)} \]

we get that

\[ \theta_3(2z)^2|_{z=\frac{i}{2}} = \lim_{z \to i\infty} \theta_3(2z)^2|_{ST^{-2}S_1} = \lim_{z \to i\infty} -\theta_2(2z)^2 \]

\[ = \lim_{z \to i\infty} -2^2 q_2(1 + q^2 + q^6 + q^{12} + \cdots)^2 \text{ since } \theta_2(z) = 2q_8(1 + q + q^3 + \cdots) \]

\[ = 0 \text{ (a triple zero)}. \]

On the other hand

\[ \theta_3(2z)^2|_{-1 0 \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}} = \frac{1}{\sqrt{3}} \theta_3(2z)^2|_{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}_{-1 0 \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}} = \frac{1}{\sqrt{3}} \theta_3(2z)^2|_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{-1 0 \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}} \]

\[ = \frac{1}{\sqrt{3}} \theta_3(2z)^2|_{\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}}_{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}} = -\frac{1}{\sqrt{3}} \theta_2(2z)^2|_{\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}} \]

\[ = -3^{-1} \theta_2 \left( 2 \cdot \frac{z-1}{3} \right)^2 = -3^{-1} e^{-\frac{\pi i}{3}} q_6(1 + \cdots)^2, \]

so that \( \theta_3(6z)^2 \) has a simple zero at \( \frac{1}{2} \). Thus \( j_{1,12}^2 \equiv \frac{\theta_4(2x)^2}{\theta_3(6x)^2} \) has a double zero at \( \frac{1}{2} \), whence \( j_{1,12} \) has a simple zero at \( \frac{1}{2} \).

(iv) \( s = \frac{1}{3} \): \( (ST^{-3}S)\infty = \frac{1}{3} \). First we recall that \( \theta_3(2z) \theta_3(6z) \left| S \right. = \sqrt{3} \theta_3(\frac{z}{3}) \theta_3(\frac{z}{4}) \) by (4). Observe that

\[ \theta_2(2z) = \frac{1}{2} (\theta_3(\frac{z}{2}) - \theta_4(\frac{z}{2})) \text{ and } \theta_3(2z) = \frac{1}{2} (\theta_3(\frac{z}{3}) + \theta_4(\frac{z}{3})). \]

From these identities we can write
\( \theta_3(\frac{x}{2}) = \theta_2(2z) + \theta_3(2z) \) and \( \theta_3(\frac{x}{6}) = \theta_2(\frac{2}{3}z) + \theta_3(\frac{2}{3}z) \). Then we have that

\[
\left. \frac{\theta_3(2z)}{\theta_3(6z)} \right|_{ST^{-3}} = \sqrt{3} \cdot \left. \frac{\theta_2(2z) + \theta_3(2z)}{\theta_2(\frac{2}{3}z) + \theta_3(\frac{2}{3}z)} \right|_{T^{-3}}
\]

\[
= \sqrt{3} \cdot \frac{(e^{-\frac{2\pi i}{3}})^6 \theta_2(2z) + \theta_3(2z)}{(e^{-\frac{2\pi i}{3}})^2 \theta_2(\frac{2}{3}z) + \theta_3(\frac{2}{3}z)} \quad \text{by (1), (2) and (3),}
\]

and

\[
\left. \frac{\theta_3(2z)}{\theta_3(6z)} \right|_{ST^{-3}S} = \sqrt{3} \cdot \left. \frac{i \theta_2(2z) + \theta_3(2z)}{-i \theta_2(\frac{2}{3}z) + \theta_3(\frac{2}{3}z)} \right|_{S}
\]

which goes to \( \frac{i+1}{i+1} = i \) as \( z \to i\infty \), so that

\[
\hat{j}_{1,12} \left( \frac{1}{3} \right) = i.
\]

(v) \( s = \frac{1}{4} \): \( (\frac{1}{4} \ 0) \infty = \frac{1}{4} \). In this case we use the following well-known fact from [19] p.148: For \( \gamma \in \Gamma_0(4) \) and \( z \in \mathcal{F} \),

\[
\Theta(\gamma z) = \left( \frac{c}{d} \right) \sqrt{-1} \sqrt{cz + d}
\]

where \( \Theta(z) = \theta_3(2z) \). Then

\[
\left. \frac{\theta_3(2z)}{\theta_3(6z)} \right|_{(\frac{1}{4} \ 0)} = \frac{\Theta(\frac{1}{4} \ 1) z}{\Theta(\frac{3}{4} \ 0) (\frac{1}{4} \ 0) z} = \frac{\Theta(\frac{1}{4} \ 1) z}{\Theta(\frac{3}{4} \ 0) (\frac{1}{0} \ -2) z}
\]

\[
= \frac{\sqrt{4z + 1}}{\Theta(z)} \Theta(\frac{z}{3})
\]

which tends to \( \sqrt{3}i \) when \( z \) goes to \( i\infty \). Therefore

\[
\hat{j}_{1,12} \left( \frac{1}{4} \right) = \sqrt{3}i.
\]
(vi) $s = \frac{1}{5}$: Because $(\frac{5}{24}, \frac{1}{5}) \cdot (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$ sends $\infty$ to $\frac{1}{5}$,

$$j_{1,12} \left( \frac{1}{5} \right) = \lim_{z \to \infty} \frac{\theta_3(2z)}{\theta_3(6z)} \left( \frac{5}{24}, \frac{1}{5} \right) \cdot (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$$

$$= \lim_{z \to \infty} -\frac{\theta_3(2z)}{\theta_3(6z)} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$$

by (a)

$$= -j_{1,12}(0) = -\sqrt{3}.$$

(vii) $s = \frac{1}{6}$: Observe that $(ST^{-6}S)\infty = \frac{1}{6}$.

Considering the identities

$$\frac{\theta_3(2z)}{\theta_3(6z)} \left( \begin{pmatrix} 24 \frac{1}{5} \\ 0 \end{pmatrix} \right) = \lim_{z \to \infty} \theta_3(6z) \right)$$

by (3)

$$\frac{\theta_3(2z)}{\theta_3(6z)} \left( \begin{pmatrix} 24 \frac{1}{5} \\ 0 \end{pmatrix} \right) = \lim_{z \to \infty} -\theta_3(6z) \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$$

we have that

$$j_{1,12} \left( \frac{1}{6} \right) = \lim_{z \to \infty} \frac{\theta_3(2z)}{\theta_3(6z)}$$

$$= \lim_{z \to \infty} \frac{2q^2(1 + q^2 + q^6 + \cdots)}{2q^2(1 + q^6 + q^{18} + \cdots)}$$

since $\theta_2(z) = 2q^2(1 + q + q^3 + \cdots)$.

Thus by Table 1, $j_{1,12}$ has a simple pole at $\frac{1}{6}$.

(viii) $s = \frac{1}{8}$: Because $(\frac{-5}{-36}, \frac{1}{7}) \cdot (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ sends $\infty$ to $\frac{1}{8}$,

$$j_{1,12} \left( \frac{1}{8} \right) = \lim_{z \to \infty} \frac{\theta_3(2z)}{\theta_3(6z)} \left( \frac{-5}{-36}, \frac{1}{7} \right) \cdot (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

$$= \lim_{z \to \infty} -\frac{\theta_3(2z)}{\theta_3(6z)} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

by (a)

$$= -j_{1,12}(0) = -\sqrt{3}i.$$
(ix) \( s = \frac{1}{9} \): Because \(( -5 \frac{2}{48} ) ( \frac{1}{3} \frac{1}{19} )\) sends \( \infty \) to \( \frac{1}{9} \),

\[
j_{1,12} \left( \frac{1}{9} \right) = \lim_{z \to \infty} \frac{\theta_3(2z)}{\theta_3(6z)} \left( -5 \frac{2}{48} \right) \left( \frac{1}{3} \frac{1}{19} \right) = \lim_{z \to \infty} -\frac{\theta_3(2z)}{\theta_3(6z)} \left( \frac{1}{3} \frac{1}{19} \right) \quad \text{by (a)}
\]

\[
= -j_{1,12} \left( \frac{1}{3} \right) = -i.
\]

(x) \( s = \frac{5}{12} \): \(( \frac{5}{12} \frac{2}{5} ) \cdot \infty = \frac{5}{12} \).

\[
j_{1,12} \left( \frac{5}{12} \right) = \lim_{z \to \infty} \frac{\theta_3(2z)}{\theta_3(6z)} \left( \frac{5}{12} \frac{2}{5} \right) = \lim_{z \to \infty} -\frac{\theta_3(2z)}{\theta_3(6z)} \quad \text{by (a)}
\]

\[
= -1.
\]

\[\square\]

We will now construct the Hauptmodul \( \mathcal{N}(j_{1,12}) \) from the modular function \( j_{1,12} \) mentioned in Theorem 4.

\[
\frac{2}{j_{1,12}(z) - 1} = \frac{2 \theta_3(6z)}{\theta_3(2z) - \theta_3(6z)} = \frac{2(1 + 2q^3 + 2q^{12} + 2q^{27} + \cdots)}{2q - 2q^3 + 2q^4 + 2q^9 - 2q^{12} + \cdots}
\]

\[= \frac{1}{q} + q + q^2 + q^3 - q^6 - q^7 - q^8 - q^9 + q^{11} + 2q^{12} + \cdots,
\]

which is in \( q^{-1}\mathbb{Z}[[q]] \). From the uniqueness of the normalized generator it follows that

\[
\mathcal{N}(j_{1,12}) = \frac{2}{j_{1,12} - 1}.
\]

By Theorem 4(b) we have the following table:

| \( s \) | \infty | 0 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/8 | 1/9 | 5/12 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( j_{1,12}(s) \) | 1 | \( \sqrt{3} \) | 0 | \( i \) | \( \sqrt{3}i \) | \(-\sqrt{3} \) | \( \infty \) | \(-\sqrt{3}i \) | \(-i \) | \(-1 \) |
| \( \mathcal{N}(j_{1,12})(s) \) | \infty | \( \sqrt{3} + 1 \) | \(-2 \) | \(-1 - i \) | \(-1 - \sqrt{3} \) | 1 | \( -\sqrt{3} \) | 0 | \( -1 + \sqrt{3} \) | \( -1 + i \) | \(-1 \) |

**Theorem 5.** Let \( d \) be a square free positive integer and \( t = \mathcal{N}(j_{1,N}) \) be the normalized generator of \( K(X_1(N)) \). Let \( s \) be a cusp of \( \Gamma_1(N) \) whose width is \( h_s \). If \( t \in q^{-1}\mathbb{Z}[[q]] \) and
\[
\prod_{s \in S_{\Gamma_1(N) \setminus \{\infty\}}} (t(z) - t(s))^{h_s} \text{ is a polynomial in } \mathbb{Z}[t], \text{ then } t(\tau) \text{ is an algebraic integer for } \tau \in \mathbb{Q}(\sqrt{-d}) \cap \mathcal{H}.
\]

**Proof.** Let \( j(z) = \frac{1}{q} + 744 + 196884q + \cdots \), the elliptic modular function. It is well-known that \( j(\tau) \) is an algebraic integer for \( \tau \in \mathbb{Q}(\sqrt{-d}) \cap \mathcal{H} \) ([21], [28]). For algebraic proofs, see [4], [24], [27] and [30]. Now, we view \( j \) as a function on the modular curve \( X_{1}(N) \). Then \( j \) has a pole of order \( h_s \) at the cusp \( s \). On the other hand, \( t(z) - t(s) \) has a simple zero at \( s \). Thus

\[
j \times \prod_{s \in S_{\Gamma_1(N) \setminus \{\infty\}}} (t(z) - t(s))^{h_s}
\]

has a pole only at \( \infty \) whose degree is \( \mu_N = [\Gamma(1) : \Gamma_1(N)] \), and so by the Riemann-Roch Theorem it is a monic polynomial in \( t \) of degree \( \mu_N \) which we denote by \( f(t) \). Since \( \prod_{s \in S_{\Gamma_1(N) \setminus \{\infty\}}} (t(z) - t(s))^{h_s} \) is a polynomial in \( \mathbb{Z}[t] \) and \( j, t \) have integer coefficients in the \( q \)-expansions, \( f(t) \) is a monic polynomial in \( \mathbb{Z}[t] \) of degree \( \mu_N \). This shows that \( t(\tau) \) is integral over \( \mathbb{Z}[j(\tau)] \). Therefore \( t(\tau) \) is integral over \( \mathbb{Z} \) for \( \tau \in \mathbb{Q}(\sqrt{-d}) \cap \mathcal{H} \).

**Corollary 6.** For \( \tau \in \mathbb{Q}(\sqrt{-d}) \cap \mathcal{H} \), \( \mathcal{N}(j_{1,12})(\tau) \) is an algebraic integer.

**Proof.** \( \mathcal{N}(j_{1,12}) \) has integral Fourier coefficients. And by Table 1 and 2,

\[
\prod_{s \in S_{\Gamma_1(12) \setminus \{\infty\}}} (t(z) - t(s))^{h_s} = (t^2 - 2t - 2)^{12} (t+2)^6 (t^2 + 2t + 2)^4 (t^2 + t + 1)^3 t^2 (t + 1) \in \mathbb{Z}[t].
\]

Now the assertion is immediate from Theorem 5. 

\( \square \)
3. Super-replication formulae

Let $\Delta^n$ be the set of $2 \times 2$ integral matrices $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ where $a \in 1 + N\mathbb{Z}$, $c \in N\mathbb{Z}$, and $ad - bc = n$. Then $\Delta^n$ has the following right coset decomposition: (See [19], [23], [28])

$$\Delta^n = \bigcup_{\nu|n} \bigcup_{(a,N)=1}^{\frac{n}{\nu}-1} \Gamma_1(N)\sigma_a \left( \begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix} \right)$$

where $\sigma_a \in SL_2(\mathbb{Z})$ such that $\sigma_a \equiv \left( \begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix} \right) \mod N$. Let $f(z)$ be a modular function with respect to $\Gamma_1(N)$. For brevity, let us call it $f(z)$ is on $\Gamma_1(N)$. For $f \in K(X_1(N))$ we define an operator $U_n$ and $T_n$ by

$$f|_{U_n} = n^{-1}\sum_{i=0}^{n-1} f\left( \begin{smallmatrix} \frac{1}{n} & i \\ 0 & n \end{smallmatrix} \right)$$

and

$$f|_{T_n} = n^{-1}\sum_{a|n} \sum_{(a,N)=1}^{\frac{n}{a}-1} f|_{\sigma_a}\left( \begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix} \right).$$

**Lemma 7.** For $f \in K(X_1(N))$ and $\gamma_0 \in \Gamma_0(N)$, $(f|_{T_n})|_{\gamma_0} = (f|_{\gamma_0})|_{T_n}$ for any positive integer $n$. In particular, $f|_{T_n}$ is again on $\Gamma_1(N)$.

**Proof.** First we claim that

$$\Delta^n\gamma_0 = \gamma_0\Delta^n \quad \text{for} \quad \gamma_0 = \left( \begin{smallmatrix} x & y \\ z & w \end{smallmatrix} \right) \in \Gamma_0(N).$$

Let $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Delta^n$. Then

$$\gamma_0^{-1}(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \gamma_0 = \left( \begin{smallmatrix} x & y \\ z & w \end{smallmatrix} \right) \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} x & y \\ z & w \end{smallmatrix} \right) \equiv \left( \begin{smallmatrix} 1 & * \\ 0 & d \end{smallmatrix} \right) \mod N.$$

Hence $\gamma_0^{-1}\Delta^n\gamma_0 \subset \Delta^n$ so that $\Delta^n\gamma_0 \subset \gamma_0\Delta^n$. By the same argument we can show the reverse inclusion. We note that

$$\Delta^n\gamma_0 = \bigcup_{a|n} \bigcup_{(a,N)=1}^{\frac{n}{a}-1} \Gamma_1(N)\sigma_a \left( \begin{smallmatrix} a & i \\ 0 & \frac{n}{a} \end{smallmatrix} \right) \gamma_0$$

13
\[ \gamma_0 \Delta^n = \bigcup_{a | n} \bigcup_{i=0}^{\frac{n}{a}-1} \gamma_0 \Gamma_1(N) \sigma_a \begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \]

\[ = \bigcup_{a | n} \bigcup_{i=0}^{\frac{n}{a}-1} \Gamma_1(N) \gamma_0 \sigma_a \begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \] because \( \Gamma_1(N) \triangleleft \Gamma_0(N) \).

Here we note that \( \sigma_a \begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \)’s are the matrices appearing in the definition of \( (f|T_n)|\gamma_0 \) and \( \gamma_0 \sigma_a \begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \)’s are those appearing in the definition of \( (f|\gamma_0)|T_n \). Now the assertion follows. \( \square \)

For a positive integer \( N \) with \( g_{1,N} = 0 \), we let \( t \) (resp. \( t_0 \)) be the Hauptmodul of \( \Gamma_1(N) \) (resp. \( \Gamma_0(N) \)). And, we write \( X_n(t) = \frac{1}{n} q^{-n} + \sum_{m \geq 1} H_{m,n} q^m \) and \( X_n(t_0) = \frac{1}{n} q^{-n} + \sum_{m \geq 1} h_{m,n} q^m \).

**Lemma 8.** For positive integers \( m \) and \( n \), \( H_{m,n} = H_{n,m} \) and \( h_{m,n} = h_{n,m} \).

**Proof.** Let \( p = e^{2\pi i y} \) and \( q = e^{2\pi i z} \) with \( y, z \in \mathbb{H} \). Note that \( X_n(t) \) can be viewed as the coefficient of \( p^n \)-term in \( -\log p - \log(t(y) - t(z)) \) \([2]\). Thus \( H_{m,n} \) becomes the coefficient of \( p^n q^m \)-term of

\[- \log p - \log(t(y) - t(z))
\]

\[= - \log(1 - p/q) + \log(p^{-1} - q^{-1}) - \log(t(y) - t(z))
\]

\[= \sum_{i \geq 1} \frac{1}{i} (p/q)^i - F(p, q)
\]

where \( F(p, q) = \log \left( \frac{p^{-1} - q^{-1} + \sum_{i \geq 1} H_i (p^i - q^i)}{p^{-1} - q^{-1}} \right) \). We then come up with \( F(p, q) = F(q, p) \), which implies that \( H_{m,n} = H_{n,m} \). Similarly if we work with \( t_0 \) instead of \( t \), the identity \( h_{m,n} = h_{n,m} \) follows. \( \square \)
Theorem 9. For positive integers $n$ and $l$ such that $(n, N) = (l, n) = 1$,

$$X_l(t)|_{T_n} = X_{tn}(t)|_{\sigma_n} + c$$

where $c$ is a constant. In particular,

$$t|_{T_n} = X_n(t)|_{\sigma_n} + c.$$  

Proof. Since $X_l(t)$ has poles only at $\Gamma_1(N)\infty$, the poles of $X_l(t)|_{T_n}$ can occur only at

$$\left(\begin{smallmatrix} a & i \\ 0 & n \end{smallmatrix}\right)^{-1} \sigma_a^{-1} \Gamma_1(N)\infty$$

where $a$ and $i$ are the indices appearing in the definition of $T_n$. On the other hand, we have

$$\left(\begin{smallmatrix} n^{-1} & -i \\ 0 & n \end{smallmatrix}\right) \Gamma_1(N)\infty = n^{-1} \left(\begin{smallmatrix} n^{-1} & -i \\ 0 & a \end{smallmatrix}\right) \sigma_a^{-1} \Gamma_1(N)\infty$$

$$= \left(\begin{smallmatrix} n^{-1} & -i \\ 0 & a \end{smallmatrix}\right) \sigma_a^{-1} \Gamma_1(N)\infty.$$  

Let $\gamma$ be an element in $\Gamma_1(N)$. Then

$$\left(\begin{smallmatrix} n^{-1} & -i \\ 0 & a \end{smallmatrix}\right) \sigma_a^{-1} \gamma \infty \equiv \left(\begin{smallmatrix} n^{-1} & -i \\ 0 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \infty \mod N$$

$$\equiv \left(\begin{smallmatrix} n \gamma \mod N \\ \gamma \mod N \end{smallmatrix}\right) \equiv \frac{n + Nm}{Nk}$$

for some $k, m \in \mathbb{Z}$. If there exists an integer $x > 1$ with $x \mid (n + Nm, Nk)$, then $x$ must divide $\det \left[ \left(\begin{smallmatrix} a & i \\ 0 & n \end{smallmatrix}\right) \sigma_a^{-1} \gamma \right] = n$. In this case $x \nmid N$ because $(n, N) = 1$. Therefore $\left(\begin{smallmatrix} a & i \\ 0 & n \end{smallmatrix}\right)^{-1} \sigma_a^{-1} \gamma \infty$ is of the form $\gamma_0 \infty$ for some $\gamma_0 \in \Gamma_0(N)$. Now we conclude that $X_l(t)|_{T_n}$ can have poles only at $\gamma_0 \infty$ for some $\gamma_0 \in \Gamma_0(N)$. By Lemma 7,

$$n(X_l(t)|_{T_n})|_{\gamma_0} = n(X_l(t)|_{\gamma_0})|_{T_n}$$

$$= \sum_{a|n} \sum_{i=0}^{\frac{n}{a} - 1} (X_l(t)|_{\gamma_0})|_{\sigma_a\left(\begin{smallmatrix} a & i \\ 0 & n \end{smallmatrix}\right)} = \sum_{a|n} \frac{n}{a} (X_l(t)|_{\gamma_0})|_{\sigma_a} u_{a^2}(az)$$

$$= \sum_{a|n} \frac{n}{a} (X_l(t)|_{\gamma_0})|_{\sigma_a} u_{a^2}(az) + X_l(t)|_{\gamma_0 \sigma_a(nz)}.$$
Here we note that
\[ \sum_{a|n \atop a \neq n} \frac{n}{a} (X_i(t)|_{\gamma_0})|_{U_{\frac{a}{n}}} (az) = O(1). \]

In fact, if \( \gamma_0 \sigma_a \notin \pm \Gamma_1(N) \), it is clear that \( (X_i(t)|_{\gamma_0})|_{U_{\frac{a}{n}}} \) has a holomorphic \( q \)-expansion. Otherwise
\[
X_i(t)|_{\gamma_0 \sigma_a U_{\frac{a}{n}}} = X_i(t)|_{U_{\frac{a}{n}}} = (l^{-1} q^{-l} + \text{terms of positive degree})|_{U_{\frac{a}{n}}} = O(1)
\]
because \((l, n) = 1\). Now we have
\[
(8) \quad n(X_i(t)|_{T_n})|_{\gamma_0} = X_i(t)|_{\gamma_0 \sigma_n (nz)} + O(1).
\]

This implies that \( X_i(t)|_{T_n} \) has a pole at \( \gamma_0 \infty \) if and only if \( \gamma_0 \sigma_n \in \pm \Gamma_1(N) \), that is, \( \gamma_0 \in \pm \Gamma_1(N) \sigma_n^{-1} \). Hence \( X_i(t)|_{T_n} \) has poles only at cusps \( \Gamma_1(N) \sigma_n^{-1} \infty \). In this case we derive from \( \Box \)
\[
n(X_i(t)|_{T_n})|_{\sigma_n^{-1}} = X_i(t)|_{\sigma_n^{-1} \sigma_n (nz)} + O(1) = l^{-1} q^{-ln} + O(1).
\]

Thus \( (X_i(t)|_{T_n})|_{\sigma_n^{-1}} = (nl)^{-1} q^{-ln} + O(1) \) and \( (X_i(t)|_{T_n})|_{\sigma_n^{-1}} \) has poles only at cusps \( \sigma_n \Gamma_1(N) \sigma_n^{-1} \infty = \Gamma_1(N) \infty \). On the other hand, \( X_{ln}(t) \) has poles only at \( \Gamma_1(N) \infty \) too and \( X_{ln}(t) = (ln)^{-1} q^{-ln} + O(1) \). Therefore \( (X_i(t)|_{T_n})|_{\sigma_n^{-1}} = X_{ln}(t) + c \) for some constant \( c \).

Then we have \( X_i(t)|_{T_n} = X_{ln}(t)|_{\sigma_n} + c \), as desired. \( \Box \)

**Corollary 10.** Let \( N \) be a positive integer such that the genus \( g_{1,N} \) is zero and \([\Gamma_0(N) : \Gamma_1(N)] \leq 2\). For positive integers \( n, l \) and \( m \) such that \( (n, N) = (l, n) = 1 \), we have
\[
\sum_{e \in (\mathbb{Z}/N\mathbb{Z})^\times} e^{-1} \left\{ \psi(e) \left( 2H_{\frac{mn}{e^2},l} - h_{\frac{mn}{e^2},l} \right) + h_{\frac{mn}{e^2},l} \right\} = \psi(n)(2H_{m,ln} - h_{m,ln}) + h_{m,ln}
\]
where \( \psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\} \) is a character defined by
\[
\psi(e) = \begin{cases} 
1, & \text{if } e \equiv \pm 1 \mod N \\
-1, & \text{otherwise.}
\end{cases}
\]
Proof. It follows from Theorem 9 that
\[
X_l(t) | T_n = \sum_{\substack{e \mid n \cr e > 0}} e^{-1} (X_l(t)|_{\sigma_e}) |_{\nu_{\neq}} (ez) = X_{ln}(t)|_{\sigma_n} + \text{constant}.
\]

Note that for each positive integer \(r\),

\[
X_r(t)|_{\sigma_e} + X_r(t) = \begin{cases} 
2X_r(t), & \text{if } e \equiv \pm 1 \mod N \\
X_r(t_0) + \text{constant}, & \text{otherwise.}
\end{cases}
\]

In the above when \(e\) is not congruent to \(\pm 1 \mod N\), \(X_r(t)|_{\sigma_e} + X_r(t)\) is on \(\Gamma_0(N)\) and has poles only at \(\Gamma_0(N) \infty\) with \(r^{-1}q^{-r}\) as its pole part, which guarantees the above equality. We then have

\[
X_r(t)|_{\sigma_e} = \frac{1}{2} \{\psi(e)(2X_r(t) - X_r(t_0)) + X_r(t_0)\} + \text{constant}.
\]

Now (9) reads

\[
\sum_{\substack{e \mid n \cr e > 0}} e^{-1} \cdot \frac{1}{2} \{\psi(e)(2X_l(t) - X_l(t_0)) + X_l(t_0)\} |_{\nu_{\neq}} (ez)
\]

\[
= \frac{1}{2} \{\psi(n)(2X_{ln}(t) - X_{ln}(t_0)) + X_{ln}(t_0)\} + \text{constant}.
\]

Comparing the coefficients of \(q^m\)-terms on both sides, we get the corollary. \(\square\)

**Corollary 11.** Let \(N\) be a positive integer such that the genus \(g_{1,N}\) is zero and \([\Gamma_0(N) : \Gamma_1(N)] \leq 2\). For positive integers \(a, b, c, d\) with \(ab = cd\), \((a, b) = (c, d)\) and \((b, N) = (d, N) = 1\),

\[
H_{a,b} = \psi(bd)H_{c,d} + \frac{(1 - \psi(bd))}{2} h_{c,d}.
\]
Proof. In Corollary 10 we take $n = b$, $l = 1$ and $m = a$. Then it follows from the conditions and the replicability of $h_{m,n}$ that
\[
\psi(b)(2H_{a,b} - h_{a,b}) = \psi(d)(2H_{c,d} - h_{c,d}).
\]
Now the assertion follows. \qed

Corollary 12. Let $N$ be a positive integer with $g_{1,N} = 0$ and $[\Gamma_0(N) : \Gamma_1(N)] = 2$. If $(mn, N) = 1$ and $mn \not\equiv \pm 1 \mod N$, then $h_{m,n} = 2H_{m,n}$.

Proof. In Corollary 11 we take $a = m, b = n$ and $c = n, d = m$. The condition that $\psi(mn) = -1$ implies
\[
H_{m,n} = -H_{n,m} + h_{n,m}
= -H_{m,n} + h_{m,n} \quad \text{by Lemma 8}
\]
This proves the corollary. \qed

4. Vanishing property in the Fourier coefficients of $N(j_{1,12})$

As mentioned in the introduction, many of the Thompson series have periodically vanishing properties among the Fourier coefficients. Now we will give a more theoretical explanation for these phenomena.

Let $T_g$ be the Thompson series of type $g$ and $\Gamma_g$ be its corresponding genus zero group. To describe $\Gamma_g$ we are in need of some notations. Let $N$ be a positive integer and $Q$ be any Hall divisor of $N$, that is, $Q$ be a positive divisor of $N$ for which $(Q,N/Q) = 1$. We denote by $W_{Q,N}$ a matrix \[
\begin{pmatrix}
Qx & y \\
Nz & Qw
\end{pmatrix}
\]
with $\det W_{Q,N} = Q$ and $x, y, z$ and $w \in \mathbb{Z}$, and call it an Atkin-Lehner involution. Let $S$ be a subset of Hall divisors of $N$ and let $\Gamma = N + S$
be the subgroup of $\text{PSL}_2(\mathbb{R})$ generated by $\Gamma_0(N) = \{(a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$ and all Atkin-Lehner involutions $W_{Q,N}$ for $Q \in S$. For a positive divisor $h$ of 24, let $n$ be a multiple of $h$ and set $N = nh$. When $S$ is a subset of Hall divisors of $n/h$, we denote by $\Gamma_0(n|h) + S$ the group generated by $(h \ 0 \ 0 \ 1)^{-1} \circ \Gamma_0(n/h) \circ (h \ 0 \ 0 \ 1)$ and $(h \ 0 \ 0 \ 1)^{-1} \circ W_{Q,n/h} \circ (h \ 0 \ 0 \ 1)$ for all $Q \in S$. If there exists a homomorphism $\lambda$ of $\Gamma_0(n|h) + S$ into $\mathbb{C}^*$ such that

\begin{align}
(11) \quad &\lambda(\Gamma_0(N)) = 1, \\
(12) \quad &\lambda((1/h \ 0 \ 0 \ 1)) = e^{-2\pi i/h}, \\
(13) \quad &\lambda((1/n \ 0 \ 0 \ 1)) = \begin{cases} e^{2\pi i/h} & \text{if } n/h \in S, \\ e^{-2\pi i/h} & \text{if } n/h \not\in S, \end{cases} \\
(14) \quad &\lambda \text{ is trivial on all Atkin-Lehner involutions of } \Gamma_0(N) \text{ in } \Gamma_0(n|h) + S,
\end{align}

then we let $n|h + S$ be the kernel of $\lambda$ which is a subgroup of $\Gamma_0(n|h) + S$ of index $h$. Ferenbaugh\cite{6} found out a necessary and sufficient condition for the homomorphism $\lambda$ to exist and calculated the genera of groups of type $n|h + S$. All the genus zero groups of type $n|h + S$ are listed in \cite{6}, Table 1.1 and 1.2. Now we have the following theorem.

**Theorem 13.** Write $T_g(q) = q^{-1} + \sum_{m \geq 1} c_g(m) q^m$.

(i) If $\Gamma_g = \Gamma_0(N)$ and $h$ is the largest integer such that $h \mid 24$ and $h^2 \mid N$, then $c_g(m) = 0$ unless $m \equiv -1 \pmod{h}$.

(ii) If $\Gamma_g = N + S$ and there exists a prime $p$ such that $p^2 \mid N$ and $p \nmid Q$ for all $Q \in S$, then $c_g(m) = 0$ whenever $m \equiv 0 \pmod{p}$.

(iii) If $\Gamma_g = n|h + S$, then $c_g(m) = 0$ unless $m \equiv -1 \pmod{h}$.

**Proof.** (i) Note that $(1/h^{-1})$ belongs to the normalizer of $\Gamma_0(N)$. Thus $T_g|_{(1/h^{-1})}$ has poles only at $\infty$, which is a simple pole. This enables us to write $T_g|_{(1/h^{-1})} = c \cdot T_g$ for some constant $c$. By comparing the coefficients of $q^m$-terms on both sides, the assertion follows.
(ii) From Corollary 3.1 [20] it follows that $T_g|_{U_p} = 0$. Thus (ii) is clear.

(iii) Considering the identity in [20], p.27 we have 
\[ T_g(z + 1|h) = e^{-2\pi i/h} \cdot T_g(z) \]. 
Then 
\[ T_g(q) = q^{-1} + \sum_{0 < c \in \mathbb{Z}} c_g(lh - 1)q^{lh - 1} \], which implies (iii). \[ \square \]

Unlike the cases of Thompson series, when we handle the super-replicable function $\mathcal{N}(j_{12})$ we can not directly use the ingredients adopted in Theorem 13 Therefore we start with

**Lemma 14.** For \((a \ b \ c \ d) \in \Gamma_0(12), (a \ b \ c \ d) = (\frac{3}{a}) j_{1,12}.\) Here \((\cdot)\) denotes the generalized quadratic residue symbol.

**Proof.** Immediate from Lemma 3 and Theorem 4 \[ \square \]

We fix $N = 12$ and let $t$ denote the Hauptmodul $\mathcal{N}(j_{1,12})$ in what follows.

**Lemma 15.** For \((a \ b \ c \ d) \in \Gamma_0(24),\)

\[ (t | u_2)_\chi (a \ b \ c \ d) = (-1)^{\frac{a}{4}} \cdot \left( \frac{3}{a + \frac{c}{2}} \right) \cdot (t | u_2)_\chi \]

where $\chi = (\frac{-1}{\cdot})$ is the Jacobi symbol and $(t | u_2)_\chi$ is the twist of $t | u_2$ by $\chi$ ([19], p.127).

**Proof.** From [19], p.128 we observe that

\[ (t | u_2)_\chi = \frac{1}{\sqrt{-4}} \left( t | u_2 \left( z + \frac{1}{4} \right) - t | u_2 \left( z + \frac{3}{4} \right) \right) \]

\[ = \frac{1}{\sqrt{-4}} \sum_{i \in (\mathbb{Z}/42) \times} \left( \frac{-1}{i} \right) \left( t | u_2 + \frac{1}{2} \right) |_{(\frac{4}{i} \frac{0}{4})} . \]

It then follows from [17], Corollary 28 that

\[ \left( t - t \left( \frac{1}{6} \right) \right) \times t | u_2 = H_2. \]

If we compare the coefficients of $q$-term on both sides, we get $H_4 = t \left( \frac{1}{6} \right) \cdot H_2 = 0$. And, substituting $H_2 = 1$ and $H_4 = 0$ we get $t \left( \frac{1}{6} \right) = 0$. Now

\[ t | u_2 = \frac{1}{t} = \frac{j_{1,12} - 1}{20}. \]
Then for \((a \ b \\
c \ d) \in \Gamma_0(12)\)

\[
(t|U_2 + \frac{1}{2})|(a \ b \\
c \ d) = \frac{1}{2} j_{1,12}(a \ b \\
c \ d) = \frac{1}{2} \left(\frac{3}{d}\right) j_{1,12} \quad \text{by Lemma } 14
\]

\[
= \left(\frac{3}{d}\right) \left(t|U_2 + \frac{1}{2}\right) = \left(\frac{3}{d}\right) \left(t|U_2 + \frac{1}{2}\right)
\]

since \(ad \equiv 1 \mod 12\). Let \((a \ b \\
c \ d) \in \Gamma_0(24)\). For \(i = 1, 3\), we consider \((\frac{4}{0} \ i \ 4 \ 0) \ (a \ b \\
c \ d) = (\frac{4}{4} z_i \ 0 \ 0 \ 4)\) for some integer \(z_i\). Then we have

\[
(\frac{4}{0} \ i \ 4 \ 0) \ (a \ b \\
c \ d) = (\frac{4}{4} z_i \ 0 \ 0 \ 4)
\]

for some integer \(z_i\). Comparing (1,2)-component on both sides we get

\[4b + id = (a + ic/4)z_i + 4x_i.\]

Thus \(id \equiv (a + ic/4)z_i \mod 4\). Then

\[z_i \equiv (a + ic/4)id \mod 4 \quad \text{because } n^2 \equiv 1 \mod 4 \quad \text{for every odd integer } n\]

\[\equiv aid + i^2 \cdot \frac{c}{4} \cdot d \mod 4 \equiv i + \frac{c}{4} \cdot d \mod 4 \quad \text{due to } ad \equiv 1 \mod 4\]

\[\equiv i + 6c_1d \mod 4, \quad \text{where we write } c = 24c_1.
\]

Therefore we derive that for \((a \ b \\
c \ d) \in \Gamma_0(24)\),

\[
\sqrt{-4} \cdot (t|U_2)_x \left|(\frac{a \ b}{c \ d}\right) = \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(t|U_2 + \frac{1}{2}\right) \left|\frac{4}{4} \ i \ 0 \ 0 \ 4\right\right. (a \ b \\
c \ d) \quad \text{by } (15)
\]

\[
= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(t|U_2 + \frac{1}{2}\right) \left|\frac{4}{4} \ i \ 0 \ 0 \ 4\right\right. (a + ic/4 x_i \ 0 \ 0 \ 4)
\]

\[
= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left(\frac{-1}{i}\right) \left(\frac{3}{a + ic/4}\right) \left(t|U_2 + \frac{1}{2}\right) \left|\frac{4}{4} \ i \ 0 \ 0 \ 4\right\right. \quad \text{by } (17).
\]
Now if we set \( c = 24 \) as before, then we have

\[
\sqrt{-4} \cdot (t|_{U_2^k})_\chi \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|
\]

\[
= \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left( \frac{1}{i} \right) \left( \frac{3}{a + 6c_1i} \right) \left( t|_{U_2^k} + \frac{1}{2} \right) \left| \begin{bmatrix} 4 & 0 \\ i + 6c_1d & 4 \end{bmatrix} \right|
\]

\[
= \left( \frac{3}{a + 6c_1} \right) \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left( \frac{1}{i} \right) \left( t|_{U_2^k} + \frac{1}{2} \right) \left| \begin{bmatrix} 4 & 0 \\ i + 6c_1d & 4 \end{bmatrix} \right|
\]

since \( a + 6c_1i \equiv a + 6c_1 \mod 12 \)

\[
= \left( \frac{3}{a + 6c_1} \right) \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \left( \frac{1}{i} \right) \left( \frac{1}{i + 6c_1d} \right) \left( \frac{1}{i + 6c_1d} \right) \left( t|_{U_2^k} + \frac{1}{2} \right) \left| \begin{bmatrix} 4 & 0 \\ i + 6c_1d & 4 \end{bmatrix} \right|
\]

\[
= (-1)^{c_1} \cdot \left( \frac{3}{a + 6c_1} \right) \cdot \sqrt{-4} \cdot (t|_{U_2^k})_\chi
\]

because \( \left( \frac{1}{i} \right) \left( \frac{-1}{i+6c_1i} \right) = \left( \frac{-1}{i^2+6c_1id} \right) = \left( \frac{-1}{i+2c_1id} \right) = (-1)^{c_1id} = (-1)^{c_1} \) and \( i + 6c_1d \) runs over \((\mathbb{Z}/4\mathbb{Z})^\times\). This completes the lemma. \( \square \)

**Lemma 16.**  
(i) For each \( k \geq 1 \), we put \( g(z) = (-1)^{k-1} \cdot \left( \frac{1}{2} \right) \cdot (X_{2^k}(t)(z) - X_{2^k}(t)(z + \frac{1}{2})) \). Then \( g \) belongs to \( K(X_1(24)) \).

(ii) For \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(24) \) and \( k \geq 1 \),

\[
(t|_{U_{2^k}})_\chi \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = (-1)^{\chi} \cdot \left( \frac{3}{a + \frac{1}{4}} \right) \cdot (t|_{U_{2^k}})_\chi.
\]

In particular, \( (t|_{U_{2^k}})_\chi \) lies in \( K(X_1(24)) \).

**Proof.** First we note that for \( n \mid N^\infty \), \( T_n = U_n \). Here, by \( n \mid N^\infty \) we mean that \( n \) divides some power of \( N \). To show \( g \in K(X_1(24)) \), we observe that

\[
g = (-1)^{k-1} \left( X_{2^k}(t)(z) - X_{2^k}(t)|_{U_{2^k}(2z)} \right).
\]
Then using Lemma 7 we obtain that $g \in K(X_1(24))$. For $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(12)$,

\begin{equation}
(t|_{U_{2^k}^1} + \frac{1}{2}) \bigg| \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \left( t|_{U_{2^k}^2} + \frac{1}{2} \right) \bigg|_{U_{2^k}^{2k-1}} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \left( t|_{U_{2^k}^2} + \frac{1}{2} \right) \bigg|_{U_{2^k}^{2k-1}} \text{ by Lemma 7}
\end{equation}

\begin{equation}
= \left( \frac{3}{a} \right) \left( t|_{U_{2^k}^2} + \frac{1}{2} \right) \bigg|_{U_{2^k}^{2k-1}} \text{ by (17)}
\end{equation}

\begin{equation}
= \left( \frac{3}{a} \right) \left( t|_{U_{2^k}^2} + \frac{1}{2} \right).
\end{equation}

Now we can proceed in the same manner as in the proof of Lemma 15. □

Lemma 17. For $k \geq 1$,

\begin{enumerate}
  \item \((t|_{U_{2^k}^1}) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix} \right)} = \begin{cases} \frac{1}{2}q_2^{-1} + O(1), & \text{if } k = 1 \\
O(1), & \text{otherwise.} \end{cases}\)
  \item \((t|_{U_{2^k}^1}) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix} \right)} = \begin{cases} O(1), & \text{if } k = 1 \\
\frac{1}{4}q_4^{-1} + O(1), & \text{if } k = 2 \\
-\frac{1}{8}q_2^{-1} + O(1), & \text{if } k = 3 \\
(-1)^{k-1} \cdot \frac{1}{2^k} \cdot q^{-2^{k-4}} + O(1), & \text{if } k \geq 4. \end{cases}\)
\end{enumerate}

Proof.  (i) First, \(t|_{\left( \begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix} \right)}\) is holomorphic at \(\infty\) because \(t\) has poles only at the cusps \(\Gamma_1(12)\infty\). Now for \(k \geq 1\),

\begin{equation}
(t|_{U_{2^k}^1}) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix} \right)} = ((t|_{U_{2^k}^{2k-1}})|_{U_{2^k}^2}) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix} \right)} = \frac{1}{2} \left( t|_{U_{2^k}^{2k-1}} \right) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 6 & 1 \end{smallmatrix} \right)}(\frac{1}{2}) + \frac{1}{2} \left( t|_{U_{2^k}^{2k-1}} \right) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix} \right)}(\frac{1}{6})
\end{equation}

\begin{equation}
= \frac{1}{2} \left( t|_{U_{2^k}^{2k-1}} \right) \big|_{\left( \begin{smallmatrix} 1 & 0 \\ 12 & 7 \end{smallmatrix} \right)}(\frac{1}{2}) + \frac{1}{2} \left( t|_{U_{2^k}^{2k-1}} \right) \bigg|_{U_{2^k}^{2k-1}} \left( \begin{smallmatrix} 1 & 0 \\ 7 & 2 \end{smallmatrix} \right) \bigg|_{U_{2^k}^{2k-1}}(\frac{1}{2}) \text{ by Lemma 7}
\end{equation}

\begin{equation}
= \begin{cases} \frac{1}{2}q_2^{-1} + O(1), & \text{if } k = 1 \\
O(1), & \text{otherwise.} \end{cases}
\end{equation}
(ii) We observe that \( t|_{\frac{1}{3} \frac{0}{1}} \in O(1) \). And for \( k \geq 1 \),

\[
(19) \quad (t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} = ((t|_{U_{2k-1}})|_{U_2})|_{\frac{1}{3} \frac{0}{1}}
\]

\[
= \frac{1}{2} (t|_{U_{2k-1}})|_{\frac{1}{0} \frac{0}{1}} (\frac{1}{3} \frac{0}{1}) + \frac{1}{2} (t|_{U_{2k-1}})|_{\frac{1}{1} \frac{1}{2}} (\frac{1}{3} \frac{0}{1})
\]

\[
= \frac{1}{2} (t|_{U_{2k-1}})|_{\frac{1}{0} \frac{0}{1}} (\frac{1}{3} \frac{0}{1}) + \frac{1}{2} (t|_{U_{2k-1}})|_{\frac{2}{3} \frac{1}{2}} (\frac{2}{0} \frac{0}{1})
\]

has a holomorphic Fourier expansion if \( k = 1 \). Thus we suppose \( k \geq 2 \). We then derive that

\[
(t|_{U_{2k-1}})|_{\frac{2}{3} \frac{1}{2}} = ((t|_{U_{2k-2}})|_{U_2})|_{\frac{2}{3} \frac{1}{2}}
\]

\[
= \frac{1}{2} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{2} (t|_{U_{2k-2}})|_{\frac{1}{1} \frac{1}{2}} (\frac{2}{3} \frac{1}{2})
\]

\[
= \frac{1}{2} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{2} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{1}{2}} (\frac{1}{0} \frac{1}{2})
\]

If we substitute the above into (19), for \( k \geq 2 \),

\[
(20) \quad (t|_{U_2})|_{\frac{1}{3} \frac{0}{1}}
\]

\[
= \frac{1}{2} (t|_{U_{2k-1}})|_{\frac{1}{0} \frac{0}{1}} (\frac{1}{3} \frac{0}{1}) + \frac{1}{4} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{0}{1}} (\frac{1}{3} \frac{0}{1}) (\frac{1}{0} \frac{1}{2}) + \frac{1}{4} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) (\frac{2}{0} \frac{0}{1})
\]

\[
= \frac{1}{2} (t|_{U_{2k-1}})|_{\frac{1}{0} \frac{0}{1}} (\frac{1}{3} \frac{0}{1}) + \frac{1}{4} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{4} (t|_{U_{2k-2}})|_{\frac{1}{0} \frac{1}{2}} (\frac{1}{0} \frac{1}{2})
\]

When \( k = 2 \),

\[
(t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} = \frac{1}{2} (t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{4} (t|_{U_2})|_{\frac{1}{0} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{4} (t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} (4z + 1)
\]

\[
= \frac{1}{4} q_4^{-1} + O(1) \quad \text{by (i)}.
\]

If \( k = 3 \),

\[
(t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} = \frac{1}{2} (t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{4} (t|_{U_2})|_{\frac{1}{0} \frac{0}{1}} (\frac{2}{3} \frac{1}{2}) + \frac{1}{4} (t|_{U_2})|_{\frac{1}{3} \frac{0}{1}} (4z + 1)
\]

\[
= \frac{1}{8} e^{-\pi(z + \frac{1}{2})} + O(1) = -\frac{i}{8} q_2^{-1} + O(1) \quad \text{by (i) and the case } k = 1 \text{ in (ii)}.
\]
For $k \geq 4$, we will show by induction on $k$ that

$$
(21) \quad \left( t | U_{2^k} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) = (-1)^{k-1} \cdot \frac{i}{2^k} \cdot q^{-2k-4} + O(1).
$$

First we note that by (i) and (20)

$$
\left( t | U_{2^k} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) = 1
$$

$$
= \frac{1}{4} \left( t | U_{2^{k-2}} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) (4z + 1) + O(1).
$$

If $k = 4$,

$$
\left( t | U_{2^4} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) = \frac{1}{4} \left( t | U_{2^{4-2}} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) (4z + 1) + O(1)
$$

$$
= \frac{1}{16} e^{-\frac{\pi i}{2}(4z+1)} + O(1) \quad \text{by the case } k = 2
$$

$$
= (-1)^{4-1} \cdot \frac{i}{2^4} \cdot q^{-2^4-4} + O(1).
$$

Thus when $k = 4$, (21) holds. Meanwhile, if $k = 5$ then we get that

$$
\left( t | U_{2^5} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) = \frac{1}{4} \left( t | U_{2^{5-2}} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) (4z + 1) + O(1)
$$

$$
= -\frac{i}{32} e^{-\pi i(4z+1)} + O(1) \quad \text{by the case } k = 3
$$

$$
= (-1)^{5-1} \cdot \frac{i}{2^5} \cdot q^{-2^5-4} + O(1).
$$

Therefore in this case (21) is also valid. Now for $k \geq 6$,

$$
\left( t | U_{2^k} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) = \frac{1}{4} \left( t | U_{2^{k-2}} \right) \left| \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right) (4z + 1) + O(1)
$$

$$
= \frac{1}{4} \cdot (-1)^{k-2-1} \cdot \frac{i}{2^{k-2}} e^{-2\pi i(4z+1)} \cdot 2^{k-2-4} + O(1)
$$

by induction hypothesis for $k - 2$

$$
= (-1)^{k-1} \cdot \frac{i}{2^k} \cdot q^{-2^k-4} + O(1).
$$

This proves the lemma. $\square$

**Theorem 18.** For $k \geq 1$,

$$
(22) \quad \left( t | U_{2^k} \right) \chi (z) = (-1)^{k-1} \cdot \frac{1}{2^k} \cdot \left( X_{2^k}(t)(z) - X_{2^k}(t) \left( z + \frac{1}{2} \right) \right).
$$
This identity twisted by a character $\chi$ is analogous to the "$2^k$-plication formula" \([7, 20]\) satisfied by the Hauptmodul of $\Gamma_0(N)$.

**Proof.** Put $g(z) = (-1)^{k-1} \cdot \frac{1}{2} \left( X_2(t)(z) - X_2(t) \left( z + \frac{1}{2} \right) \right)$ as before. We see by Lemma [16] that both $(t|U_{2k})_\chi$ and $g$ sit in $K(X_1(24))$. For $(t|U_{2k})_\chi = g$, we will show that $(t|U_{2k})_\chi - g$ has no poles in $\mathfrak{H}$. Recall that

$$(t|U_{2k})_\chi = \frac{1}{\sqrt{-4}} \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \chi(i) \left. \left( t|U_{2k} \right) \right| (\begin{smallmatrix} 4i & j \\ 0 & 2k \end{smallmatrix}) = \frac{1}{\sqrt{-4}} \sum_{i \in (\mathbb{Z}/4\mathbb{Z})^\times} \sum_{j=0}^{2^{k-1}} \chi(i) t|\left( \begin{smallmatrix} 1 & j \\ 0 & 2k \end{smallmatrix} \right)(\begin{smallmatrix} 4i & j \\ 0 & 2k \end{smallmatrix}) \cdot$$

Since $t$ has poles only at $\Gamma_1(12)\infty$, $(t|U_{2k})_\chi$ can also have poles only at

$$(\begin{smallmatrix} 4i & j \\ 0 & 2k \end{smallmatrix})^{-1} \Gamma_1(12)\infty = 16^{-1} \cdot 2^{-k} \cdot (\begin{smallmatrix} 4i & j \\ 0 & 2k \end{smallmatrix}) \Gamma_1(12)\infty = (\begin{smallmatrix} 2^{k+2} - 4j - i \\ 0 & 4 \end{smallmatrix}) \Gamma_1(12)\infty.$$

Let $(a \ b \ c \ d) \in \Gamma_1(12)$. Then

$$(\begin{smallmatrix} 2^{k+2} - 4j - i \\ 0 & 4 \end{smallmatrix}) (a \ b \ c \ d) = (\begin{smallmatrix} 2^{k+2}a - 4jc - ic \\ 4c \end{smallmatrix}) * 2^l = \frac{2^{k+2}a - 4jc - ic}{4c}. \quad (2^{k+2}a - 4jc - ic \in \mathbb{C})$$

Observe that $(2^{k+2}a - 4jc - ic) \in \mathbb{Z}$, thus $(a \ b \ c \ d) = (2^{k+2}a - 4jc - ic) \in \mathbb{Z}$. Write $(2^{k+2}a - 4jc - ic, 4c) = 2^l$ for some integer $l \geq 0$. Then

$$s = \frac{(2^{k+2}a - 4jc - ic)/2^l}{4c/2^l}$$

is in lowest terms. Since $12 | c$, $s$ is of the form $s = \frac{n}{3m}$ for some integers $m$ and $n$. We assume $(3m, n) = 1$. Here we consider two cases.

(i) $2^2 \nmid m$:

Choose integers $x$ and $y$ such that $(\frac{n}{3m} \ x \ y) \in SL_2(\mathbb{Z})$ and consider

$$(t|U_{2k})_\chi \left| \left( \frac{n}{3m} \ x \ y \right) \right. = \frac{1}{\sqrt{-4}} \left( t|U_{2k} \left( \frac{4}{0} \right) \left( \frac{n}{3m} \ x \ y \right) - t|U_{2k} \left( \frac{4}{0} \right) \left( \frac{n}{3m} \ x \ y \right) \right)$$

Since $(3m, n) = 1$ and $2^2 \nmid m$, we can write $(\frac{4n + 3m}{12m} \ x \ y) = \gamma_0 U_1$ and $(\frac{4n + 9m}{12m} \ x \ y) = \gamma_0' U_2$ where both $\gamma_0$ and $\gamma_0'$ are in $\Gamma_0(12)$ and $U_1, U_2$ are upper triangular matrices. Then by
\( (t|_{U_{2k}}) \chi \mid \left( \frac{n}{3m} \frac{x}{y} \right) \in O(1). \) Hence, if \( 2^2 \nmid m \) then \((t|_{U_{2k}}) \chi \) is holomorphic at the cusp \( s = \frac{n}{3m}. \)

(ii) \( 2^2 \mid m: \)

If \( 2^2 \mid m \), then \( s \) is of the form \( s = \left( \frac{n}{3m} \frac{x}{y} \right) \in \Gamma_0(24). \) Thus by Lemma 16,

\[
(t|_{U_{2k}}) \chi \mid \left( \frac{n}{3m} \frac{x}{y} \right) = (-1)^{3m} \cdot \left( \frac{3}{n+3m/4} \right) \cdot (t|_{U_{2k}}) \chi \in O(1).
\]

As for the other cases, if we use Lemma 11 it is easy to see that \( s \) is equivalent to \( \frac{1}{12} \) or \( \frac{5}{12} \) under \( \Gamma_1(24). \)

Thus we conclude that \((t|_{U_{2k}}) \chi \) can have poles only at \( \frac{1}{12}, \frac{5}{12} \) under \( \Gamma_1(24)-\text{equivalence}. \)

Next, let us investigate the poles of \( g. \) Recall that \( X_{2k}(t) \) has poles only at \( \Gamma_1(12) \infty. \)

Therefore \( g \) can have poles only at \( \left( \frac{3}{5} \frac{i}{2} \right)^{-1} \Gamma_1(12) \infty \) for \( i = 0, 1. \) And, for \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_1(12), \)

\[
\left( \begin{array}{cc} 2 & i \\ 0 & 2 \end{array} \right)^{-1} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \infty = \left( \begin{array}{cc} 2 & -i \\ 0 & 2 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \infty = \frac{2a - ic}{2c} = \frac{a - ic/2}{c} \text{ in lowest terms.}\]

Hence by Lemma 11, \( g \) can have poles only at \( \frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}, \frac{1}{12}, \frac{5}{12} \) under \( \Gamma_1(24)-\text{equivalence}. \)

At \( \frac{1}{24}, \frac{5}{24}, \frac{7}{24} \) and \( \frac{11}{24}, \) it is easy to check that \( g \) is holomorphic. For example, at \( \frac{5}{24}, \)

\[
g|_{\left( \frac{5}{24} \frac{1}{5} \right)} = (-1)^{k-1} \cdot \frac{1}{2} \cdot \left( X_{2k}(t)|_{\left( \frac{5}{24} \frac{1}{5} \right)} - X_{2k}(t)|_{\left( \frac{2}{0} \frac{1}{2} \right)} \cdot (\frac{5}{24} \frac{1}{5}) \right)
= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left( X_{2k}(t)|_{\left( \frac{5}{24} \frac{1}{5} \right)} - X_{2k}(t)|_{\left( \frac{34}{48} \frac{7}{10} \right)} \right)
= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left( X_{2k}(t)|_{\left( \frac{5}{24} \frac{1}{5} \right)} - X_{2k}(t)|_{\left( \frac{17}{24} \frac{12}{17} \right)} \cdot (\frac{2}{0} \frac{1}{2}) \right)
\in O(1) \text{ since } \frac{5}{24}, \frac{17}{24} \notin \Gamma_1(12) \infty.
Now it remains to show that \((t|_{U^k}) \chi - g\) has no poles at the cusps equivalent to \(\frac{1}{12}, \frac{5}{12}\) under \(\Gamma_1(24)\). At \(\frac{1}{12}\),

\(t|_{U^k}) \chi \big|_{\left(\frac{1}{12} \ 0 \big)} = \frac{\sqrt{-4}}{1} \left( t|_{U^k(\frac{4}{12} \ 0)}(\frac{4}{12} \ 0) - t|_{U^k(\frac{4}{12} \ 0)}(\frac{4}{12} \ 0) \right)
\]

\(= \frac{\sqrt{-4}}{1} \left( t|_{U^k(\frac{1}{3} \ 0)}(\frac{1}{3} \ 0) - t|_{U^k(\frac{1}{3} \ 0)}(\frac{1}{3} \ 0) \right)
\]

\(= \frac{\sqrt{-4}}{1} \left( t|_{U^k(\frac{1}{3} \ 0)}(\frac{1}{3} \ 0) - t|_{U^k(\frac{1}{3} \ 0)}(\frac{1}{3} \ 0) \right)
\]

\(= \frac{\sqrt{-4}}{1} \left( t|_{U^k(\frac{1}{3} \ 0)}(16z + 1) - t|_{U^k(\frac{1}{3} \ 0)}(4z + \frac{1}{2}) \right)
\]

By (23) and Lemma 17, we have the following:

If \(k = 1\),

\[ (t|_{U^k}) \chi \big|_{\left(\frac{1}{12} \ 0 \big)} = \frac{\sqrt{-4}}{1} \left( -\frac{1}{2} e^{-\pi i(4z + \frac{1}{2})} + O(1) \right) = \frac{1}{4} q^{-2} + O(1). \]

If \(k = 2\),

\[ (t|_{U^k}) \chi \big|_{\left(\frac{1}{12} \ 0 \big)} = \frac{\sqrt{-4}}{1} \left( -\frac{1}{8} e^{-\pi i(16z + 1)} + O(1) \right) = \frac{1}{8} q^{-4} + O(1). \]

If \(k = 3\),

\[ (t|_{U^k}) \chi \big|_{\left(\frac{1}{12} \ 0 \big)} = \frac{\sqrt{-4}}{1} \left( -\frac{i}{8} e^{-\pi i(16z + 1)} + O(1) \right) = \frac{1}{16} q^{-8} + O(1). \]

If \(k \geq 4\),

\[ (t|_{U^k}) \chi \big|_{\left(\frac{1}{12} \ 0 \big)} = \frac{\sqrt{-4}}{1} \left( -(-1)^{k-1} \cdot \frac{i}{2k} \cdot e^{-\pi i(2k-4)(16z + 1)} + O(1) \right) \]

\[= (-1)^{k-1} \cdot \frac{1}{2k+1} \cdot q^{-2k} + O(1). \]

Observe that the identities for \(k = 1, 2, 3\) are the same as the last one when \(k \geq 4\).

Hence we conclude that for all \(k \geq 1\),

\[ (t|_{U^k}) \chi \big|_{\left(\frac{1}{12} \ 0 \big)} = (-1)^{k-1} \cdot \frac{1}{2k+1} \cdot q^{-2k} + O(1). \]

On the other hand,

\[ g \big|_{\left(\frac{1}{12} \ 0 \big)} = (-1)^{k-1} \cdot \frac{1}{2} \cdot X_{2k}(t) \big|_{\left(\frac{1}{12} \ 0 \big)} - X_{2k}(t) \big|_{\left(\frac{2}{12} \ 0 \big)}(\frac{1}{12} \ 0) \big) \]

\[= (-1)^{k-1} \cdot \frac{1}{2} \cdot \left( X_{2k}(t) - X_{2k}(t) \big|_{\left(\frac{7}{12} \ 0 \big)}(\frac{2}{12} \ 0) \big) \right) \]

\[= (-1)^{k-1} \cdot \frac{1}{2k+1} \cdot q^{-2k} + O(1). \]
Thus

\[ \left( (t | U_{2k}) \chi - g \right) | (\frac{1}{12} \ 0) \in O(1). \]

At \( \frac{5}{12} \), we see that \( (\frac{5}{12} \ 2) = (\frac{-19}{2} \ 5) (\frac{1}{12} \ 0) \) sends \( \infty \) to \( \frac{5}{12} \). Then

\[ (t | U_{2k}) \chi | (\frac{5}{12} \ 2) = (t | U_{2k}) \chi | (\frac{-19}{2} \ 5) (\frac{1}{12} \ 0) \]

by Lemma 16

\[ = (-1) \cdot (t | U_{2k}) \chi | (\frac{1}{12} \ 0) = (-1)^{k} \cdot \frac{1}{2^{k+1}} \cdot q^{-2k} + O(1). \]

And

\[ g | (\frac{5}{12} \ 2) = (-1)^{k-1} \cdot \frac{1}{2} \cdot \left( X_{2k}(t) | (\frac{5}{12} \ 2) - X_{2k}(t) | (\frac{2}{0} \ 2) (\frac{5}{12} \ 2) \right) \]

\[ = (-1)^{k-1} \cdot \frac{1}{2} \cdot \left( X_{2k}(t) | (\frac{5}{12} \ 2) - X_{2k}(t) | (\frac{11}{12} \ -1) (\frac{2}{0} \ 2) \right) \]

\[ = (-1)^{k} \cdot \frac{1}{2} \cdot X_{2k}(t) | (\frac{2}{0} \ 2) + O(1) = (-1)^{k} \cdot \frac{1}{2^{k+1}} \cdot q^{-2k} + O(1). \]

This implies that

\[ \left( (t | U_{2k}) \chi - g \right) | (\frac{5}{12} \ 2) \in O(1), \]

from which the theorem follows. \( \square \)

Now, we are ready to show periodically vanishing property of \( N(j_{1,12}). \)

**Corollary 19.** As before we let \( t \) be the Hauptmodul of \( \Gamma_{1}(12) \) and write \( X_{n}(t) = \frac{1}{n} q^{-n} + \sum_{m \geq 1} H_{m,n} q^{m}. \) Then we have

(i) \( H_{m,2k} = (-1)^{k-1} \left( \frac{1}{m} \right) H_{2^k m,1} \) for odd \( m. \)

(ii) \( H_{m} = 0 \) whenever \( m \equiv 4 \mod 6, \) and \( m = 5. \)

**Proof.** First if we compare the coefficients of \( q^{m} \)-terms on both sides of the identity in Theorem 18 we get (i). We see from the Appendix, Table 4 that \( H_{5} = 0. \) On the other
hand, by the super-replication formula (Corollary 11) it follows that for \( m \) relatively prime to 12,

\[
H_{m,2^k} = H_{2^k,m} = \begin{cases} 
H_{2^k,m,1}, & \text{if } m \equiv \pm 1 \mod 12 \\
-H_{2^k,m,1}, & \text{if } m \equiv \pm 5 \mod 12
\end{cases}
\]

because \( h_{2^k,m,1} = 0 \) in this case ([20], Corollary 3.1). Then \( H_{2^k,m,1} = 0 \) when \( k \) is odd and \( m \equiv 5 \mod 6 \), or \( k \) is even and \( m \equiv 1 \mod 6 \). It is easy to see that

\[
\{2^k m \mid k, m \geq 1, \text{ } k \text{ odd}, m \equiv 5 \mod 6\} \cup \{2^k m \mid k, m \geq 1, \text{ } k \text{ even}, m \equiv 1 \mod 6\} = \{l \in \mathbb{Z} \mid l \equiv 4 \mod 6\}.
\]

This proves (ii).

\[\square\]

**Appendix. Fourier coefficients of the Hauptmodul \( \mathcal{N}(j_{1,N}) \)**

We shall make use of the following modular forms to construct \( j_{1,N} \). For \( z \in \mathbb{H} \),

- \( \eta(z) \) the Dedekind eta function
- \( G_2(z) \) Eisenstein series of weight 2
- \( G_2(pz) = G_2(z) - pG_2(pz) \) for each prime \( p \)
- \( E_2(z) = G_2(z)/(2\zeta(2)) \) normalized Eisenstein series of weight 2
- \( E_2(pz) = E_2(z) - pE_2(pz) \) for each prime \( p \)
- \( \mathcal{P}_{N,a}(z) = \mathcal{P}_{L_z} \left( \frac{a_1 z + a_2}{N} \right) \) \( N \)-th division value of \( \mathcal{P} \) where \( a = (a_1, a_2), L_z = \mathbb{Z}z + \mathbb{Z} \) and \( \mathcal{P}_{L}(\tau) \) is the Weierstrass \( \mathcal{P} \)-function (relative to a lattice \( L \))

Now we get the following tables due to [14]-[16]:

**Table 3. Hauptmoduls \( \mathcal{N}(j_{1,N}) \)**
When \( N = 1, 2, 3, 4, 6 \), \( N(j_{1,N}) \) becomes a Thompson series \( T_g \) with \( \Gamma_g = \Gamma_0(N) \). Hence, if \( N = 4 \), \( N(j_{1,4}) \) has periodically vanishing property by Theorem \( \text{13-}(\text{i}) \). Otherwise, the Fourier coefficients of \( N(j_{1,N}) \) do not vanish (see [22], Table 1). Therefore, we consider only the following cases \( N \) for which \( \Gamma_1(N) \neq \Gamma_0(N) \).

**Table 4. Fourier coefficients \( H_m \) of \( N(j_{1,N}) \) for \( 1 \leq m \leq 60 \)**

| \( N \) | \( j_{1,N} \) | \( N(j_{1,N}) \) |
|---|---|---|
| 1 | \( j(z) \) | \( j(z) - 744 \) |
| 2 | \( \theta_2(z)^5 / \theta_4(2z)^6 \) | \( 256/j_{1,2} + 24 \) |
| 3 | \( E_4(z) / E_4(3z) \) | \( 240/(j_{1,3} - 1) + 9 \) |
| 4 | \( \theta_2(2z)^3 / \theta_3(2z)^4 \) | \( 16/j_{1,4} - 8 \) |
| 5 | \( 4\eta(z)^2 / \eta(5z) + \eta_j(3z)^2 / \eta(z)^2 \) | \(-8/(j_{1,5} + 44) - 5 \) |
| 6 | \( \frac{G_j(2z)^2(z)G_j(2z)(3z)}{2G_j(2z)(z) - G_j(2z)(3z)} \) | \( 2/(j_{1,6} - 1) - 1 \) |
| 7 | \( \frac{P_j(1,0)(1z) - P_j(2,0)(1z)}{P_j(1,0)(1z) - P_j(2,0)(1z)} \) | \(-1/(j_{1,7} - 1) - 3 \) |
| 8 | \( \frac{\theta_2(2z)}{\theta_3(4z)} \) | \( 2/(j_{1,8} - 1) - 1 \) |
| 9 | \( \frac{P_9(1,0)(9z) - P_9(2,0)(9z)}{P_9(1,0)(9z) - P_9(2,0)(9z)} \) | \(-1/(j_{1,9} - 1) - 2 \) |
| 10 | \( \frac{P_{10}(1,0)(10z) - P_{10}(2,0)(10z)}{P_{10}(1,0)(10z) - P_{10}(2,0)(10z)} \) | \(-1/(j_{1,10} - 1) - 2 \) |
| 12 | \( \frac{\theta_3(2z)}{\theta_3(6z)} \) | \( 2/(j_{1,12} - 1) \) |
| H  | N(j₁,₁₁) | N(j₁,₁₇) | N(j₁,₁₈) | N(j₁,₁₉) | N(j₁,₁₀) | N(j₁,₁₂) |
|----|----------|----------|----------|----------|----------|----------|
| H₁₁| 420      | -29      | -1       | 5        | 4        | 1        |
| H₁₂| 180      | -35      | -12      | 1        | 4        | 2        |
| H₁₃| -615     | -10      | -20      | -5       | 1        | 2        |
| H₁₄| -826     | 37       | -16      | -11      | -2       | 2        |
| H₁₅| 410      | 70       | 1        | -12      | -6       | 1        |
| H₁₆| 1760     | 53       | 22       | -7       | -8       | 0        |
| H₁₇| 705      | -21      | 38       | 3        | -7       | 2        |
| H₁₈| -2415    | -106     | 30       | 15       | -3       | 3        |
| H₁₉| -3100    | -126     | 1        | 22       | 4        | 4        |
| H₂₀| 1530     | -38      | -40      | 19       | 10       | -4       |
| H₂₁| 6270     | 119      | -64      | 5        | 14       | -2       |
| H₂₂| 2460     | 226      | -52      | -15      | 12       | 0        |
| H₂₃| -8090    | 164      | -2       | -32      | 6        | 3        |
| H₂₄| -10174   | -70      | 68       | -36      | -6       | 5        |
| H₂₅| 4840     | -326     | 107      | -22      | -16      | 7        |
| H₂₆| 19570    | -378     | 88       | 8        | -22      | 6        |
| H₂₇| 7500     | -106     | -2       | 40       | -20      | 4        |
| H₂₈| -24360   | 353      | -112     | 58       | -8       | 0        |
| H₂₉| -30024   | 652      | -180     | 50       | 8        | -4       |
| H₃₀| 14130    | 469      | -144     | 12       | 26       | -8       |
| H₃₁| 55970    | -189     | 3        | -41      | 34       | -10      |
| H₃₂| 21155    | -885     | 182      | -84      | 31       | -9       |
| H₃₃| -67380   | -1015    | 292      | -93      | 12       | -6       |
| H₃₄| -81926   | -290     | 228      | -54      | -14      | 0        |
| H₃₅| 37895    | 910      | 4        | 22       | -41      | 6        |
| H₃₆| 148410   | 1664     | -286     | 103      | -54      | 12       |
| H₃₇| 55305    | 1179     | -452     | 148      | -47      | 14       |
| H₃₈| -174500  | -483     | -356     | 124      | -20      | 14       |
| H₃₉| -209577  | -2205    | -4       | 32       | 23       | 8        |
| H₄₀| 96025    | -2492    | 440      | -96      | 61       | 0        |
| H₄₁| 371620   | -692     | 686      | -200     | 84       | -10      |
| H₄₂| 137160   | 2212     | 544      | -219     | 72       | -18      |
| H₄₃| -427665  | 3998     | -5       | -128     | 31       | -22      |
| H₄₄| -508800  | 2809     | -668     | 46       | -32      | -20      |
| H₄₅| 230670   | -1120    | -1044    | 231      | -90      | -12      |
| H₄₆| 885070   | -5119    | -816     | 330      | -122     | 0        |
| H₄₇| 323605   | -5754    | 5        | 275      | -107     | 15       |
| H₄₈| -1001340 | -1598    | 996      | 67       | -44      | 26       |
| H₄₉| -1181123 | 4992     | 1563     | -216     | 45       | 33       |
| H₅₀| 531545   | 8968     | 1210     | -439     | 133      | 29       |
|    | \( N(j_{1.5}) \) | \( N(j_{1.7}) \) | \( N(j_{1.8}) \) | \( N(j_{1.9}) \) | \( N(j_{1.10}) \) | \( N(j_{1.12}) \) |
|----|----------------|----------------|----------------|----------------|----------------|----------------|
| \( H_{51} \) | 20222670 | 6251 | 6 | -477 | 174 | 19 |
| \( H_{52} \) | 734130 | -2506 | -1464 | -275 | 154 | 0 |
| \( H_{53} \) | -2253515 | -11285 | -2276 | 107 | 61 | -20 |
| \( H_{54} \) | -2639348 | -12579 | -1768 | 501 | -68 | -37 |
| \( H_{55} \) | 1178880 | -3455 | -8 | 708 | -192 | -45 |
| \( H_{56} \) | 4456650 | 10812 | 2128 | 590 | -254 | -42 |
| \( H_{57} \) | 1606500 | 19278 | 3284 | 146 | -220 | -26 |
| \( H_{58} \) | -4901250 | 13362 | 2552 | -447 | -90 | 0 |
| \( H_{59} \) | -5703676 | -5278 | -9 | -911 | 100 | 27 |
| \( H_{60} \) | 2532720 | -23765 | -3056 | -987 | 272 | 52 |

**References**

[1] D. Alexander, C. Cummins, J. Mckay and C. Simons, Completely replicable functions. In: Groups, Combinatorics and Geometry, Cambridge Univ. Press, 87-95, 1992.

[2] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math., 109, 405-444, 1992.

[3] C. J. Cummins and S. P. Norton, Rational Hauptmoduls are replicable, Canadian J. Math., 47 (6), 1201-1218, 1995.

[4] M. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, Abh. Math. Sem. Univ. Hamburg, 14, 197-272, 1941.

[5] P. L. Duren, Univalent Functions, Springer-Verlag, 1983.

[6] C. R. Ferenbaugh, The genus-zero problem for \( n|h \)-type groups, Duke Math. J., 72 (1), 31-63, 1993.

[7] I. B. Frenkel, J. Lepowsky, A. Meurman, A natural representation of the Fischer-Griess monster with the modular function \( J \) as character, Proc. Natl. Acad. Sci. USA 81, 3256-3260, 1984.

[8] K. J. Hong and J. K. Koo, Generation of class fields by the modular function \( j_{1.12} \), Acta Arith., 93, 257-291, 2000.

[9] N. Ishida and N. Ishii, The equation of the modular curve \( X_1(N) \) derived from the equation of the modular curve \( X(N) \), Tokyo J. Math., 22, 167-175, 1999.

[10] S. J. Kang, Graded Lie superalgebras and the superdimension formula, J. Algebra, 204, 597-655, 1998.

[11] C. H. Kim and J. K. Koo, On the genus of some modular curve of level \( N \), Bull. Australian Math. Soc., 54, 291-297, 1996.

[12] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag, 1984.

[13] M. Koike, On replication formula and Hecke operators, Nagoya University (preprint).

[14] S. Lang, Elliptic Functions, Springer-Verlag, 1987.

[15] J. Mckay and H. Strauss, The \( q \)-series of monstrous moonshine and the decomposition of the head characters, Comm. Algebra, 18 (1), 253-278, 1990.

[16] T. Miyake, Modular Forms, Springer-Verlag, 1989.
[24] A. Néron, Modeles minimaux des variétés abéliennes sur les corps locaux et globaux, Publ. Math. I.H.E.S., 21, 5-128, 1964.

[25] S. P. Norton, More on moonshine. In: Computational Group Theory, 185-193, Academic Press, 1984.

[26] R. Rankin, Modular Forms and Functions, Cambridge: Cambridge University press 1977.

[27] J. P. Serre and J. Tate, Good reduction of abelian varieties, Ann. Math., 88, 492-517, 1968.

[28] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton Univ. Press, 1971.

[29] ————. On modular forms of half-integral weight, Ann. Math., 97, 440-481, 1973.

[30] J. H. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Springer-Verlag, 1994

CHANG HEON KIM, DEPARTMENT OF MATHEMATICS, SEOUL WOMEN’S UNIVERSITY, 126 KONGNUNG 2-DONG, NOWON-GU, SEOUL, 139-774 KOREA
E-mail address: chkim@swu.ac.kr

KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, DEPARTMENT OF MATHEMATICS, TAEJON, 305-701 KOREA
E-mail address: jkkoo@math.kaist.ac.kr