CONSTRUCTING SEMISIMPLE SUBALGEBRAS OF REAL SEMISIMPLE LIE ALGEBRAS

PAOLO FACCIN AND WILLEM A. DE GRAAF

Abstract. We consider the problem of constructing semisimple subalgebras of real (semi-) simple Lie algebras. We develop computational methods that help to deal with this problem. Our methods boil down to solving a set of polynomial equations. In many cases the equations turn out to be sufficiently “pleasant” to be able to solve them. In particular this is the case for $S$-subalgebras.

1. Introduction

The subject of this paper is the problem of finding semisimple subalgebras of real semisimple Lie algebras. The analogous problem for complex Lie algebras has been widely studied (see for example [7], [8], [15], [12]). In order to describe the main results in this area we need to introduce some terminology. Let $\tilde{g}$ be a semisimple complex Lie algebra, with adjoint group $\tilde{G}$ (this is the group of inner automorphisms). Two subalgebras $g_1^c, g_2^c \subset \tilde{g}^c$ are said to be equivalent if there is an $\eta \in \tilde{G}$ with $\eta(g_1^c) = g_2^c$. They are called linearly equivalent if for all representations $\rho : \tilde{g}^c \to gl(V^c)$ we have that the subalgebras $\rho(g_1^c), \rho(g_2^c)$ are conjugate under $GL(V^c)$. A subalgebra of $\tilde{g}^c$ is called regular if it is normalized by a Cartan subalgebra of $\tilde{g}^c$. An $S$-subalgebra is a subalgebra not contained in a regular subalgebra. We have the following:

- There is an algorithm to determine the regular semisimple subalgebras of $\tilde{g}^c$, up to equivalence ([8]).
- The maximal semisimple $S$-subalgebras of the simple Lie algebras of classical type ([7]), and the semisimple $S$-subalgebras of the simple Lie algebras of exceptional type ([8]) have been classified up to equivalence.
- The simple subalgebras of the Lie algebras of exceptional type have been classified up to equivalence ([18]).
- The semisimple subalgebras of the simple Lie algebras of ranks not exceeding 8 have been classified up to linear equivalence ([12]).

Now let $\tilde{g}$ be a real semisimple Lie algebra with adjoint group $\tilde{G}$. A classification of the semisimple subalgebras of $\tilde{g}$, up to $\tilde{G}$-conjugacy, appears to be completely out of reach. Therefore we consider a weaker problem. Note that if $\tilde{g} \subset \tilde{g}$, then also for the complexifications, $g^c = C \otimes \tilde{g}, \tilde{g}^c = C \otimes \tilde{g}$ we have that $g^c \subset \tilde{g}^c$. So assume that we know an inclusion $g^c \subset \tilde{g}^c$. This leads to the following problem: let $\tilde{g}^c$ be a complex semisimple Lie algebra, and $g^c$ a complex semisimple subalgebra of it. Let $\tilde{g} \subset g^c$ be a real form of $\tilde{g}^c$. List, up to isomorphism, all real forms $\tilde{g} \subset \tilde{g}^c$ such that $\tilde{g} \subset \tilde{g}$.

We recall the following fact ([19], §2, Proposition 1): let $\tilde{g}, \tilde{g}' \subset \tilde{g}^c$ be two real forms of $\tilde{g}^c$. Then $\tilde{g}$ and $\tilde{g}'$ are isomorphic if and only if there is a $\phi \in Aut(\tilde{g}^c)$ such that $\phi(\tilde{g}) = \tilde{g}'$.

Because of this we can reformulate the problem as follows: let $\varepsilon : g^c \hookrightarrow \tilde{g}^c$ be an embedding of complex semisimple Lie algebras. Let $\tilde{g} \subset g^c$ be a real form.
up to isomorphism, all real forms $\tilde{g}$ of $\tilde{g}^c$ such that there is a $\phi \in \text{Aut}(\tilde{g}^c)$ with $\phi(\varepsilon(g)) \subset \tilde{g}$. This is the main problem that we consider in this paper.

Let $\tilde{g}_1, \ldots, \tilde{g}_m$ be the non-compact real forms of $\tilde{g}^c$ (i.e., each non-compact real form of $\tilde{g}^c$ is isomorphic to exactly one $\tilde{g}_i$). In our setting the $\tilde{g}_i$ are given by a basis and a multiplication table (see Section 1.1). In this paper we describe algorithmic methods that help to solve the following problem: given an embedding $\varepsilon : g^c \hookrightarrow \tilde{g}^c$, and a real form $g$ of $g^c$, find all $i$ such that there is an automorphism $\phi$ of $\tilde{g}^c$ such that $\phi(\varepsilon(g)) \subset \tilde{g}_i$, along with a basis of the subalgebra $\phi(\varepsilon(g))$ of $\tilde{g}_i$ in terms of a basis of $\tilde{g}_i$. Our algorithms reduce this problem to finding the solution to a set of polynomial equations. We show some nontrivial examples where it is possible to deal with these polynomial equations. Our approach is particularly well suited for $S$-subalgebras; at the end of the paper we give a list of all $\tilde{g}_i$, when $\tilde{g}^c$ is of exceptional type and the image of $\varepsilon$ is an $S$-subalgebra of $\tilde{g}^c$.

For real semisimple Lie algebras the problem of finding and classifying the semisimple subalgebras has previously been considered in the literature. Cornwell has published a series of papers on this topic, [1], [2], [9], [10], the last two in collaboration with Ekins. Their methods require detailed case-by-case calculations, and it is not entirely clear whether they are applicable to every subalgebra. For example, no $S$-subalgebras are considered in these publications (except for some $S$-subalgebras of type $A_1$ in the Lie algebras of types $G_2$ and $F_4$).

Komrakov ([17]) classified the maximal proper semisimple Lie subalgebras of a real simple Lie algebra. However, his paper does not give an account of the methods used. He also has a list of the real forms which contain a maximal $S$-subalgebra, for $\tilde{g}^c$ of exceptional type. We find the same inclusions as Komrakov, except that in type $E_6$ we find a few more (see Section 5).

Now we give an outline of the paper. The next section contains concepts and constructions from the literature that we use. We also give an algorithm to compute equivalences of representations of semisimple Lie algebras, which may not have been described before, but follows immediately from the representation theory of such algebras. In Section 3 we describe our method. Section 4 has some examples computed using our implementation. Finally, in Section 5 we give the list of real semisimple subalgebras of the real simple Lie algebras of exceptional type, that correspond to $S$-subalgebras of the corresponding complex Lie algebras.

1.1. Computational set up. We have implemented the algorithms in the language of the computer algebra system GAP4 ([11]), using the package CoReLG ([4]). In this system a Lie algebra is given by a basis and a multiplication table. The package CoReLG contains functionality for constructing all real forms of a simple complex Lie algebra (see [5]). So in our implementations we work with Lie algebras given in that way. An element of an algebra is represented by its coefficient vector relative to the given basis of the algebra. Subspaces (in particular, subalgebras) are given by a basis. Linear maps (in particular, automorphisms) are defined with respect to the given basis of the Lie algebra. And so on.

Also we use the GAP4 package SLA ([13]), which contains a database of the semisimple subalgebras of the simple complex Lie algebras of ranks not exceeding 8. We use this database to obtain the starting data for our algorithms: the embeddings $\varepsilon : g^c \hookrightarrow \tilde{g}^c$.

1.2. Notation. Throughout we endow symbols denoting vector spaces or algebras over the complex numbers by a superscript $c$. If this superscript is absent, then the vector space, or algebra, is defined over the reals. In the above discussion we have already used this convention.
We use standard notation and terminology for Lie algebras, as can for instance be found in the books of Humphreys ([13]) and Onishchik ([19]). Lie algebras will be denoted by fraktur symbols (like $\mathfrak{g}$). The adjoint representation of a Lie algebra $\mathfrak{g}$ is defined by $\text{ad}_g x(y) = [x, y]$. We also just use ad if no confusion can arise about which Lie algebra is meant.

We denote the real forms of the simple Lie algebras using the convention of [16], Appendix C.3 and C.4, see also [19], Table 5.

We denote the imaginary unit in $\mathbb{C}$ by $i$.

2. Preliminaries

2.1. Semisimple real Lie algebras. Let $\mathfrak{g}^c$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}^c$ be a fixed Cartan subalgebra of $\mathfrak{g}^c$, and let $\Phi$ denote the corresponding root system. By $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ we denote a basis of simple roots of $\Phi$, corresponding to a choice of positive roots $\Phi^+$. For $\alpha, \beta \in \Phi$ we let $r, q$ be the maximal integers such that $\beta - r\alpha$ and $\beta + q\alpha$ lie in $\Phi$, and we define $\langle \beta, \alpha^\vee \rangle = r - q$. For $\alpha \in \Phi$ we denote by $\mathfrak{g}^c_\alpha$ the corresponding root space in $\mathfrak{g}^c$.

There is a basis of $\mathfrak{g}^c$ formed by elements $h_1, \ldots, h_\ell \in \mathfrak{h}^c$, along with $x_\alpha \in \mathfrak{g}^c_\alpha$ for $\alpha \in \Phi$ such that

\begin{align*}
[h_1, h_2] &= 0 \\
[h_1, x_\alpha] &= (\alpha, \alpha^\vee) x_\alpha \\
[x_\alpha, x_{-\alpha}] &= h_\alpha \\
[x_\alpha, x_\beta] &= N_{\alpha, \beta} x_{\alpha + \beta},
\end{align*}

where $h_\alpha$ is the unique element in $[\mathfrak{h}^c_\alpha, \mathfrak{g}^c_{-\alpha}]$ with $[h_\alpha, x_\alpha] = 2x_\alpha$. This implies that $h_{\alpha_i} = h_i$ for $1 \leq i \leq \ell$. Furthermore, $N_{\alpha, \beta} = \pm (r + 1)$, where $r$ is the maximal integer with $\alpha - r\beta \in \Phi$. Also we define $x_\gamma = 0$ if $\gamma \notin \Phi$.

A basis with these properties is called a Chevalley basis of $\mathfrak{g}^c$ (see [13], §25.2).

Let $\iota \in \mathbb{C}$ denote the imaginary unit, and consider the elements

\begin{equation}
(2.1) \quad \iota h_1, \ldots, \iota h_\ell \text{ and } x_\alpha - x_{-\alpha}, \iota(x_\alpha + x_{-\alpha}) \text{ for } \alpha \in \Phi^+.
\end{equation}

Let $u$ denote the $\mathbb{R}$-span of these elements. Then $u$ is closed under the Lie bracket, and hence is a real Lie algebra. This Lie algebra is compact, and called a compact form of $\mathfrak{g}^c$. We have $\mathfrak{g}^c = u + \mathfrak{u}$ and we define a corresponding map $\tau : \mathfrak{g}^c \to \mathfrak{g}^c$ by $\tau(x + i y) = x - i y$, for $x, y \in u$. This map is called the conjugation of $\mathfrak{g}^c$ with respect to $u$.

Let $\theta : \mathfrak{g}^c \to \mathfrak{g}^c$ be an automorphism of order 2, commuting with $\tau$. Then $\theta$ maps $u$ into itself, and hence $u = u_1 + u_{-1}$, where $u_k$ denotes the $\theta$-eigenspace with eigenvalue $k$. Set $\mathfrak{z} = u_1$ and $\mathfrak{p} = u_{-1}$, and $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{p}$. Then $\mathfrak{g}$ is a real subspace of $\mathfrak{g}^c$, closed under the Lie bracket. So it is a real form of $\mathfrak{g}^c$. Also here we get a conjugation, $\sigma : \mathfrak{g}^c \to \mathfrak{g}^c$, by $\sigma(x + i y) = x - i y$ for $x, y \in \mathfrak{g}$. The maps $\sigma, \tau$ and $\theta$ pairwise commute, all have order 2 and $\tau = \theta \sigma$.

Every real form of $\mathfrak{g}^c$ can be constructed in this way (see [19]). The decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{p}$ is called a Cartan decomposition. The restriction of $\theta$ to $\mathfrak{g}$ is called a Cartan involution of $\mathfrak{g}$.

2.2. Canonical generators. For $1 \leq i \leq \ell$ let $g_i, x_i, y_i$ be elements of $\mathfrak{g}^c$ such that

\begin{align*}
[g_i, g_j] &= 0 \\
[g_i, x_j] &= (\alpha_j, \alpha_i^\vee) x_j \\
[g_i, y_j] &= - (\alpha_j, \alpha_i^\vee) y_j \\
x_i y_j &= \delta_{ij} g_i.
\end{align*}

\begin{equation}
(2.2)
\end{equation}
A set of $3\ell$ elements with these commutation relations is called a canonical generating set of $\mathfrak{g}^c$ (\cite{13}, §IV.3). We have the following:

- A canonical generating set of $\mathfrak{g}^c$ generates $\mathfrak{g}^c$.
- Sending one canonical generating set to another one uniquely extends to an automorphism of $\mathfrak{g}^c$.

An example of a canonical generating set is the following: let $g_i = h_i$, $x_i = x_{\alpha_i}$, $y_i = x_{-\alpha_i}$, (where we use the notation of Section 2.1).

2.3. Computing endomorphism spaces. Here $\mathfrak{g}^c$ is a complex semisimple Lie algebra with canonical generators $h_i$, $x_i$, $y_i$ for $1 \leq i \leq \ell$. Let $\mathfrak{h}^c$ denote the span of the $h_i$ (a Cartan subalgebra of $\mathfrak{g}^c$). First we review some of the basic facts of the representation theory of $\mathfrak{g}^c$ (see \cite{14}, §20).

Let $\rho : \mathfrak{g}^c \to \mathfrak{gl}(V^c)$ be a finite-dimensional representation of $\mathfrak{g}^c$. For $\mu \in (\mathfrak{h}^c)^\ast$ we set $V^c_\mu = \{v \in V^c \mid \rho(h)v = \mu(h)v\}$. If $V^c_\mu \neq 0$ then $\mu$ is called a weight of $\rho$ (or of the $\mathfrak{g}^c$-module $V^c$), and $V^c_\mu$ is the corresponding weight space. Elements of $V^c_\mu$ are called weight vectors of weight $\mu$. We have that $V^c$ is the sum of its weight spaces. Let $v \in V^c_\mu$ and suppose that $\rho(x_i)v = 0$ for $1 \leq i \leq \ell$. Then $v$ is called a highest weight vector, and $\mu$ a highest weight of $\rho$.

Suppose that $\rho$ is irreducible. Then there is a unique highest weight $\lambda$. Moreover, $\dim V^c_\lambda = 1$. Let $v_\lambda \neq 0$ be a highest weight vector of weight $\lambda$. Then there is a set $S_\lambda$ of sequences $(i_1, \ldots, i_k)$, with $k \geq 0$ and $1 \leq i_1 \leq \ell$ such that the elements $\rho(y_{i_1}) \cdots \rho(y_{i_k})v_\lambda$ form a basis of $V^c_\lambda$. We note that $S_\lambda$ is not uniquely determined. But for each $\lambda$ we fix one $S_\lambda$.

Now let $\varphi : \mathfrak{g}^c \to \mathfrak{gl}(W^c)$ be another irreducible representation of $\mathfrak{g}^c$ with the same highest weight $\lambda$. Let $w_\lambda \neq 0$ be a highest weight vector of weight $\lambda$. Define the linear map $A : V^c \to W^c$ that maps $\rho(y_{i_1}) \cdots \rho(y_{i_k})v_\lambda$ to $\varphi(y_{i_1}) \cdots \varphi(y_{i_k})w_\lambda$, for all $(i_1, \ldots, i_k) \in S_\lambda$.

**Lemma 2.1.** We have $A\rho(x) = \varphi(x)A$ for all $x \in \mathfrak{g}^c$.

*Proof.* Since $\rho$, $\varphi$ are irreducible representations of $\mathfrak{g}^c$ with the same highest weight, there exists an isomorphism, that is, a bijective linear map $A' : V^c \to W^c$ with $A'\rho(x)v = \varphi(x)A'v$ for all $x \in \mathfrak{g}^c$ and $v \in V^c$. This implies that $A'v_\lambda = aw_\lambda$, where $a \in \mathbb{C}$, $a \neq 0$. It also follows that $A = \frac{1}{a}A'$, whence the statement. \qed

Now we drop the assumption that $\rho$ is irreducible. Let $\lambda_1, \ldots, \lambda_r$ be the distinct highest weights of $\rho$. For $1 \leq j \leq r$ let $v_{j,1}, \ldots, v_{j,m_j}$ be a linearly independent set of highest weight vectors of highest weight $\lambda_j$. So each $v_{j,1}$ generates an irreducible $\mathfrak{g}^c$-submodule, denoted $V(\lambda_j, t)$, of $V^c$, and $V^c$ is their direct sum. We use the basis of $V^c$ consisting of the elements $\rho(y_{i_1}) \cdots \rho(y_{i_k})v_{j,t}$, for $(i_1, \ldots, i_k) \in S_{\lambda_j}$. For $1 \leq j \leq r$ and $1 \leq s, t \leq m_j$ we let $A_{j}^{s,t}$ be the linear map $V^c \to V^c$ that maps $\rho(y_{i_1}) \cdots \rho(y_{i_k})v_{j,s}$ to $\rho(y_{i_1}) \cdots \rho(y_{i_k})v_{j,t}$ for $(i_1, \ldots, i_k) \in S_{\lambda_j}$, and it maps all other basis elements to 0. Then $A_{j}^{s,t}$ is an isomorphism of $V(\lambda_j, s)$ to $V(\lambda_j, t)$, and it maps all other submodules $V(\lambda_k, u)$ to 0. So by Lemma 2.1 $A_{j}^{s,t} \rho(x) = \rho(x)A_{j}^{s,t}$ for all $x \in \mathfrak{g}^c$, i.e., it is contained in

$$\text{End}_\rho(V^c) = \{A \in \text{End}(V^c) \mid A\rho(x) = \rho(x)A \text{ for all } x \in \mathfrak{g}^c\}.$$ 

**Lemma 2.2.** The $A_{j}^{s,t}$ for $1 \leq j \leq r$ and $1 \leq s, t \leq m_j$ form a basis of $\text{End}_\rho(V^c)$.

*Proof.* Let $A \in \text{End}_\rho(V^c)$. Then $A$ is determined by the images $A_{j}^{s,t}$ for $1 \leq j \leq r$, $1 \leq s \leq m_j$. But $A$ maps (highest) weight vectors to (highest) weight vectors of the same weight. So there are $\alpha_{j}^{s,t} \in \mathbb{C}$ such that

$$A_{j}^{s,t} = \alpha_{j}^{s,1} v_{j,1} + \cdots + \alpha_{j}^{s,m_j} v_{j,m_j}.$$
It follows that $A = \sum_{j,s,t} a_{j,s,t} A_{j,s,t}^{s,t}$. It is obvious that the $A_{j,s,t}^{s,t}$ are linearly independent. □

Now consider a second representation $\varphi : \mathfrak{g}^c \to \mathfrak{gl}(V^c)$ that is equivalent to $\rho$, i.e., there is a bijective linear map $A_0 : V^c \to V^c$ such that $A_0\rho(x) = \varphi(x)A_0$ for all $x \in \mathfrak{g}^c$. In particular, $A_0$ lies in the space

$$\text{End}_{\rho,\varphi}(V^c) = \{ A \in \text{End}(V^c) \mid A\rho(x) = \varphi(x)A \text{ for all } x \in \mathfrak{g}^c \}.$$ We want to find a basis of $\text{End}_{\rho,\varphi}(V^c)$. A first observation is that $\text{End}_{\rho,\varphi}(V^c) = \{ A_0A \mid A \in \text{End}_{\rho}(V^c) \}$. So since above we have seen how to construct a basis of $\text{End}_{\rho}(V^c)$, the problem boils down to constructing $A_0$. Since $\varphi$ is equivalent to $\rho$ there are $w_{j,1}, \ldots, w_{j,m}$ forming a basis of the weight space with weight $\lambda_j$, relative to the representation $\varphi$. Applying Lemma 2.1 to each submodule $V(\lambda_j, l)$ we see that mapping $v_{j,l}$ to $w_{j,l}$ (for all $j,l$) uniquely extends to a bijective linear map $A_0 : V^c \to V^c$, contained in $\text{End}_{\rho,\varphi}(V^c)$.

### 2.4. On solving polynomial equations.

In the end, the solution to our problem will be given by a set of polynomial equations, which we need to solve. For this, to the best of our knowledge, no good algorithm is available. So in each particular case we have to look at the equations and see whether we can solve them. However, there are some algorithms that can help with that, most importantly the algorithm for constructing a Gröbner basis (see [3]). Let $F$ be a field, and $R = F[x_1, \ldots, x_m]$ the polynomial ring in $m$ indeterminates over $F$. Let $P \subset R$ be a finite set of polynomials, and consider the polynomial equations $p = 0$ for $p \in P$. We want to determine the set $V = \{ v \in F^m \mid p(v) = 0 \text{ for all } p \in P \}$. Let $G$ be any other generating set of the ideal $I$ of $R$ generated by $P$. Then solving $p = 0$ for all $p \in P$ is equivalent to solving $g = 0$ for all $g \in G$ (the set of solutions is the same). A convenient choice for $G$ is a Gröbner basis of $I$ with respect to a lexicographical monomial order. Then $G$ has a triangular form, which, in most cases, makes solving the equations easier. We refer to [3] for a more detailed discussion.

### 3. Construction of embeddings

Here we turn to our main problem, stated in Section 1.

Let $\mathfrak{g}^c$, $\tilde{\mathfrak{g}}^c$ be complex semisimple Lie algebras, and suppose that we have an embedding $\varepsilon : \mathfrak{g}^c \to \tilde{\mathfrak{g}}^c$. Let $\mathfrak{h}^c$ be a fixed Cartan subalgebra of $\mathfrak{g}^c$, and let $\Phi$ denote the corresponding root system. Let $h_1, \ldots, h_r$, and $x_\alpha$ for $\alpha \in \Phi$ be a Chevalley basis of $\mathfrak{g}^c$. Let $\mathfrak{u}$ be the compact form spanned by the elements (2.3), with corresponding conjugation $\tau$. Let $\mathfrak{g}$ be a real form of $\mathfrak{g}^c$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and corresponding involution $\theta$, and conjugation $\sigma$. We assume that $\mathfrak{g}$ and $\mathfrak{u}$ are compatible, i.e., $\tau$ and $\sigma$ commute, and $\theta = \sigma \tau$ and $\mathfrak{u} = \mathfrak{t} \oplus \mathfrak{p}$.

#### Proposition 3.1.

Let $\tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}^c$ be a real form of $\tilde{\mathfrak{g}}^c$ such that $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$. Then there are a compact form $\tilde{\mathfrak{u}} \subset \tilde{\mathfrak{g}}^c$ of $\tilde{\mathfrak{g}}^c$, with conjugation $\tilde{\tau} : \tilde{\mathfrak{g}}^c \to \tilde{\mathfrak{g}}$, and an involution $\tilde{\theta}$ of $\tilde{\mathfrak{g}}^c$ such that

1. $\varepsilon(\mathfrak{u}) \subset \tilde{\mathfrak{u}}$,
2. $\varepsilon(\theta) = \tilde{\theta} \varepsilon$,
3. $\tilde{\theta} \tilde{\tau} = \tilde{\tau} \tilde{\theta}$,
4. there is a Cartan decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{p}}$, such that the restriction of $\tilde{\theta}$ to $\tilde{\mathfrak{g}}$ is the corresponding Cartan involution, and $\tilde{\mathfrak{u}} = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{p}}$.

Conversely, if $\tilde{\mathfrak{u}} \subset \tilde{\mathfrak{g}}$ is a compact form, with corresponding conjugation $\tilde{\tau}$, and $\tilde{\theta}$ is an involution of $\tilde{\mathfrak{g}}^c$ such that (1), (2) and (3) hold, then $\theta$ leaves $\tilde{\mathfrak{u}}$ invariant, and setting $\mathfrak{t} = \tilde{\mathfrak{t}}_1$, $\mathfrak{p} = \tilde{\mathfrak{t}}_{r-1}$ (where $\tilde{\mathfrak{t}}_k$ is the $k$-eigenspace of $\tilde{\theta}$), we get that $\tilde{\mathfrak{g}} = \mathfrak{t} \oplus \mathfrak{p}$ is a real form of $\tilde{\mathfrak{g}}^c$ with $\varepsilon(\mathfrak{g}) \subset \tilde{\mathfrak{g}}$. 

Proof. There is a Cartan decomposition $\tilde{g} = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{p}}$ such that $\varepsilon(\mathfrak{t}) \subset \tilde{\mathfrak{t}}$, $\varepsilon(\mathfrak{p}) \subset \tilde{\mathfrak{p}}$ (this is the Karpelevich-Mostow theorem, see [19, §6, Corollary 1]). We let $\tilde{\theta}$ be the involution of $\tilde{g}^c$ such that $\tilde{\theta}(x) = x$ for all $x \in \tilde{\mathfrak{t}}^c$, and $\tilde{\theta}(x) = -x$ for all $x \in \tilde{g}^c$. Finally we set $\tilde{u} = \tilde{\mathfrak{t}} \oplus \tilde{\mathfrak{p}}$. Then the statements (1), (2), (3), and (4) are all obvious. The converse is clear as well. \qed

Throughout this section let $\mathfrak{h}^c$ be a fixed Cartan subalgebra of $\mathfrak{g}^c$. We let $\Psi$ denote the root system of $\tilde{g}^c$ with respect to $\mathfrak{h}^c$. By $g_1, \ldots, g_m$ together with $y_{\beta}$, for $\beta \in \Psi$ we denote a fixed Chevalley basis of $\tilde{g}^c$. We let $\tilde{u}$ be the compact form of $\tilde{g}^c$ spanned by $g_i$, $1 \leq i \leq m$, $y_{\beta}$, $i(y_{\beta} + y_{-\beta})$ for $\beta \in \Psi^+$. From the formulation of the main problem we see that it does not make a difference if we replace $\varepsilon$ by $\phi \varepsilon$, where $\phi \in \text{Aut}(\tilde{g}^c)$. The first step of our procedure is to replace $\varepsilon$ by a $\phi$ to ensure that $\varepsilon(\mathfrak{u}) \subset \tilde{u}$. This is the subject of Section 3.1.

In Section 3.2 we show how to find the involutions $\tilde{\theta}$ with Proposition 3.1(2) and (3). Then Proposition 3.1 shows how to construct the corresponding real forms of $\tilde{g}^c$.

We recall ([8], see also [18], [12]) that two embeddings $\varepsilon, \varepsilon' : \mathfrak{g}^c \hookrightarrow \tilde{g}^c$ are called equivalent if there is an inner automorphism $\phi$ of $\tilde{g}^c$ such that $\varepsilon = \phi \varepsilon'$. They are called linearly equivalent if for all representations $\rho : \tilde{g}^c \rightarrow \mathfrak{gl}(V^\mathfrak{c})$ the induced representations $\rho \circ \varepsilon, \rho \circ \varepsilon'$ are equivalent. Equivalence implies linear equivalence, but the converse is not always true. However, the cases where the same linear equivalence class splits into more than one equivalence class are rather rare (cf. [18, Theorem 7]).

3.1. Embedding the compact form. Suppose that $\varepsilon(\mathfrak{h}^c) \subset \mathfrak{h}^c$. Then for $\alpha \in \Phi$ there is a subset $A_{\alpha} \subset \Psi$ such that

$$\varepsilon(x_{\alpha}) = \sum_{\beta \in A_{\alpha}} a_{\alpha,\beta} y_{\beta}$$
$$\varepsilon(x_{-\alpha}) = \sum_{\beta \in A_{\alpha}} b_{\alpha,\beta} y_{-\beta},$$

(3.1)

where $a_{\alpha,\beta}, b_{\alpha,\beta} \in \mathbb{C}$ (in fact, $A_{\alpha}$ consists of all $\beta$ which restricted to $\varepsilon(\mathfrak{h}^c)$ equal $\alpha$).

We say that the embedding $\varepsilon$ is balanced if $\varepsilon(\mathfrak{h}^c) \subset \mathfrak{h}^c$ and for all $\alpha \in \Phi$, and $\beta \in A_{\alpha}$, we have $b_{\alpha,\beta} = \bar{a}_{\alpha,\beta}$ (complex conjugation). Of course, this notion depends on the choices of Cartan subalgebras and Chevalley bases in $\mathfrak{g}^c$, $\tilde{g}^c$. If we use the term “balanced” without mentioning these, then we use the choices fixed at the outset. Otherwise we explicitly mention a different choice made.

Lemma 3.2. If $\varepsilon$ is balanced then $\varepsilon(\mathfrak{u}) \subset \tilde{u}$. Conversely, if $\varepsilon(\mathfrak{h}^c) \subset \mathfrak{h}^c$ and $\varepsilon(\mathfrak{u}) \subset \tilde{u}$, then $\varepsilon$ is balanced.

Proof. By standard arguments one can show that $\varepsilon(h_i)$ is a $\mathbb{Q}$-linear combination of the $g_j$. (Set $x = \varepsilon(x_{\alpha_1})$, $y = \varepsilon(x_{-\alpha_2})$, $h = \varepsilon(h_i)$. Then $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$. So by $\mathfrak{sl}_2$-representation theory the eigenvalues of $\text{ad}_{\mathfrak{h}_i} h$ are integers. Let $\{\beta_1, \ldots, \beta_m\}$ be a basis of simple roots of $\Psi$, with corresponding Cartan matrix $\tilde{C}$. Then $\beta_j(h) \in \mathbb{Z}$ for all $j$. Furthermore, if we write $h = a_1 g_1 + \cdots + a_m g_m$, then we get that the vector $(a_1, \ldots, a_m)$ is $\tilde{C}^{-1}$ times the vector $(\beta_1(h), \ldots, \beta_m(h))$. So $a_j \in \mathbb{Q}.$) In particular, $\varepsilon(h_i)$ lies in the $\mathbb{R}$-span of $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$. 

Also, for \( \alpha \in \Phi^+ \) we have
\[
\varepsilon(x_\alpha - x_{-\alpha}) = \sum_{\beta \in A_\alpha} a_{\alpha,\beta} y_\beta - b_{\alpha,\beta} y_{-\beta}
\]
\[
= \sum_{\beta \in A_\alpha} \frac{a_{\alpha,\beta} + b_{\alpha,\beta}}{2} (y_\beta - y_{-\beta}) - \frac{a_{\alpha,\beta} - b_{\alpha,\beta}}{2} i(y_\beta + y_{-\beta}).
\]

We see that all coefficients lie in \( \mathbb{R} \), whence \( \varepsilon(x_\alpha - x_{-\alpha}) \in \hat{\mathfrak{u}} \). The argument for \( \varepsilon(i(x_\alpha + x_{-\alpha})) \) is exactly similar.

For the converse, from (3.2) we get that \( a_{\alpha,\beta} + b_{\alpha,\beta} \in \mathbb{R} \) and \( a_{\alpha,\beta} - b_{\alpha,\beta} \in i\mathbb{R} \). That implies \( b_{\alpha,\beta} = \bar{a}_{\alpha,\beta} \).

The next lemma says that the automorphism that we are after exists.

**Lemma 3.3.** There exists an inner automorphism \( \phi \) of \( \hat{\mathfrak{g}}^c \) such that \( \phi \varepsilon \) is balanced.

**Proof.** There is a compact form \( \hat{\mathfrak{u}}' \) of \( \hat{\mathfrak{g}}^c \) such that \( \varepsilon(\mathfrak{u}) \subset \hat{\mathfrak{u}}' \) ([19], §6, Proposition 3). There is an inner automorphism \( \phi' \) of \( \hat{\mathfrak{g}}^c \) such that \( \phi'(\hat{\mathfrak{u}}') = \hat{\mathfrak{u}} \) ([19], §3, Corollary to Proposition 6). Moreover, the span of the elements \( \phi'(\varepsilon(\mathfrak{h})) \) lies in a Cartan subalgebra of \( \hat{\mathfrak{u}} \), which is conjugate to the span of the \( i\mathfrak{g}_j \) by an inner automorphism of \( \hat{\mathfrak{u}} \). This automorphism extends to an inner automorphism of \( \hat{\mathfrak{g}}^c \). So we get an inner automorphism \( \phi \) of \( \hat{\mathfrak{g}}^c \) such that \( \phi(\varepsilon(\mathfrak{h})) \subset \hat{\mathfrak{u}} \), and \( \phi(\varepsilon(\mathfrak{h}^c)) \subset \hat{\mathfrak{h}}^c \). So by Lemma (3.2) we conclude that \( \phi \varepsilon \) is balanced. \( \square \)

Now suppose that \( \varepsilon \) has the property that \( \varepsilon(\mathfrak{h}^c) \subset \hat{\mathfrak{h}}^c \), but \( \varepsilon \) is not balanced. Let \( \Delta = \{\alpha_1, \ldots, \alpha_\ell\} \) be a fixed basis of simple roots of \( \Phi \). Then we set up a system of polynomial equations. The indeterminates are \( s_{\alpha,\beta}, t_{\alpha,\beta}, \) where \( \alpha, \beta \in \Delta, \beta \in A_\alpha \). For \( 1 \leq i \leq \ell \) we set
\[
X_i = \sum_{\beta \in A_\alpha} (s_{\alpha,\beta} + t_{\alpha,\beta}) y_\beta
\]
\[
Y_i = \sum_{\beta \in A_\alpha} (s_{\alpha,\beta} - t_{\alpha,\beta}) y_{-\beta}
\]

Next we require that the \( 3\ell \) elements \( \varepsilon(h_i), X_i, Y_i \) satisfy the relations (2.2) (where in place of \( g_i \) we take \( \varepsilon(h_i) \), in place of \( x_i, y_i \) we take \( X_i, Y_i \)). This leads to a set of polynomial equations in the indeterminates \( s_{\alpha,\beta}, t_{\alpha,\beta} \), which we solve over \( \mathbb{R} \). Let \( \bar{s}_{\alpha,\beta}, \bar{t}_{\alpha,\beta} \in \mathbb{R} \) be the values that we obtain. Let \( \hat{X}_i, \hat{Y}_i \) be the same as \( X_i, Y_i \), but with these values substituted. Then mapping \( h_i \) to \( \varepsilon(h_i) \), \( x_{-\alpha} \) to \( \hat{X}_i \), \( x_{-\alpha} \) to \( \hat{Y}_i \) defines an embedding \( \hat{\varepsilon} : \mathfrak{g}^c \to \hat{\mathfrak{g}}^c \) (see Section 2.2).

**Lemma 3.4.** \( \hat{\varepsilon} \) is balanced.

**Proof.** Consider the elements \( x_{-\alpha} - x_{-\alpha}, i(x_{-\alpha} + x_{-\alpha}) \), for \( \alpha \in \Delta \) and \( ih_i \), for \( 1 \leq i \leq \ell \). The span of these over \( \mathbb{C} \) is the same as the span of the canonical generating set consisting of the \( x_{-\alpha}, x_{-\alpha}, h_i \). So they generate \( \mathfrak{g}^c \) over \( \mathbb{C} \), and since they lie in \( \mathfrak{u} \), they generate \( \mathfrak{u} \) over \( \mathbb{R} \). Moreover, their images under \( \hat{\varepsilon} \) lie in \( \hat{\mathfrak{u}} \), so \( \varepsilon(\mathfrak{u}) \subset \hat{\mathfrak{u}} \). Since also \( \varepsilon(\mathfrak{h}^c) \subset \hat{\mathfrak{h}}^c \) we conclude by Lemma (3.2) \( \square \)

Since \( \hat{\varepsilon} \) agrees with \( \varepsilon \) on \( \mathfrak{h}^c \), we have that \( \varepsilon \) and \( \hat{\varepsilon} \) are linearly equivalent (see [8], Theorem 1.5, see also [12], Theorem 4). If the linear equivalence class of \( \varepsilon \) does not split into more than one equivalence class, then we are done: \( \varepsilon \) and \( \hat{\varepsilon} \) are equivalent. If we are in a rare case where there are more equivalence classes, then we have to find more solutions to the polynomial equations: one for each equivalence class contained in the linear equivalence class of \( \varepsilon \).
Remark 3.5. For the embeddings that have been determined with the methods of \cite{12}, the following trick often works. Let $\Pi = \{\beta_1, \ldots, \beta_m\}$ be a fixed basis of simple roots of $\Psi$. Let $\delta_1, \ldots, \delta_m \in \mathbb{C} \setminus \{0\}$, and let $\phi$ be the automorphism of $\mathfrak{g}^c$ mapping $g_j \mapsto g_j$, $\beta_\rho \mapsto \delta_\beta g_\beta$. Then the images of the $g_j$, and $\beta_\rho$ under $\phi$ also form a Chevalley basis of $\mathfrak{g}^c$. Moreover, $\phi(\beta_\rho) = \delta_\beta^{-1} \delta_\rho^{-1} \beta_\rho$. This then yields a set of polynomial equations for the $\delta_\beta$. It is by no means guaranteed that this set is consistent (i.e., has any solution at all). However, from our experience, we get that in many cases the set is not only consistent, but also a reduced Gröbner basis is of the form $\{\delta_1^2 - r_1, \ldots, \delta_m^2 - r_m\}$, with $r_i \in \mathbb{R}$, $r_i > 0$, which makes solving the equations extremely easy.

A solution of the equations yields an automorphism $\phi$ of $\mathfrak{g}^c$ such that $\phi(u) = \tilde{u}'$, where $\tilde{u}'$ is the compact form spanned by the elements $g_j$, $y'_\beta = y'_\beta - y'_\beta$, $(y'_\beta + y'_\beta)$. Moreover, $\epsilon$ is balanced with respect to the Chevalley basis consisting of the $y'_\beta$, so that $\epsilon(u) \subset \tilde{u}'$. So if we set $\epsilon' = \phi^{-1} \epsilon$, then $\epsilon'$ is equivalent to $\epsilon$ and $\epsilon'(u) \subset \tilde{u}$.

3.2. Finding $\tilde{\theta}$. Here we assume that we have an embedding $\epsilon : \mathfrak{g} \hookrightarrow \mathfrak{g}^c$ such that $\epsilon(b^c) \subset b^c$ and $\epsilon(u) \subset \tilde{u}$. Now we focus on the problem of finding the involutions $\tilde{\theta}$ of $\tilde{\mathfrak{g}}$ such that $\epsilon \theta = \epsilon \tilde{\theta}$.

Let $ad : \tilde{\mathfrak{g}} \to \mathfrak{g}(\mathfrak{g}^c)$ be the adjoint representation, i.e., $ad(x)(y) = [x, y]$. Set

$$A = \{A \in \text{End}(\tilde{\mathfrak{g}}) \mid A(\epsilon \theta(y)) = \epsilon(\theta(y))A \text{ for all } y \in \mathfrak{g}^c\}.$$

Proposition 3.6. Let $\tilde{\theta} \in \text{End}(\tilde{\mathfrak{g}})$. Then $\tilde{\theta}$ is an involution of $\tilde{\mathfrak{g}}$ with $\epsilon \theta = \epsilon \tilde{\theta}$ if and only if $\tilde{\theta} \in A$ and

1. $\tilde{\theta}^2 = I$, where $I \in \text{End}(\tilde{\mathfrak{g}})$ is the identity,
2. $\tilde{\theta}(ad x)\tilde{\theta} = ad \tilde{\theta}(x)$ for all $x \in \tilde{\mathfrak{g}}$.

Proof. Suppose that $\tilde{\theta}$ is an involution of $\tilde{\mathfrak{g}}$. Then \((1)\) is immediate. Also for $y \in \mathfrak{g}^c$ we have $\tilde{\theta}(ad x)\tilde{\theta}(y) = \tilde{\theta}(x, \tilde{\theta}(y)) = \tilde{\theta}(x)\tilde{\theta}(y)$, so \((2)\) follows. Together with $\epsilon \theta = \epsilon \tilde{\theta}$ this also implies that $\tilde{\theta} \in A$.

For the converse we first show that $\tilde{\theta}$ is an involution of $\tilde{\mathfrak{g}}$. From \((1)\) it follows that it is bijective and that it has order 2. Using \((2)\) we get $\tilde{\theta}(x, y) = \tilde{\theta}(x)\tilde{\theta}(y) = \tilde{\theta}(x, \tilde{\theta}(y)) = [\tilde{\theta}(x), \tilde{\theta}(y)]$. Secondly, $\epsilon \tilde{\theta} = \epsilon \theta$ is equivalent to $ad \epsilon \tilde{\theta}(y) = ad \epsilon \theta(y)$ for all $y \in \mathfrak{g}^c$. Using \((1)\) and \((2)\) it is straightforward to see that this is the same as $\tilde{\theta} \in A$. \(\square\)

We let $a_1, \ldots, a_n$ be a fixed basis of $\mathfrak{g}^c$ (for example, the Chevalley basis fixed at the start). The idea now is to translate the conditions of Proposition 3.6 into polynomial equations. For that we proceed as follows:

1. Compute a basis $A_1, \ldots, A_s$ of $A$ (see Section 2.3) note that, if we let $\rho, \varphi : \mathfrak{g} \to \mathfrak{g}(\mathfrak{g}^c)$ be the representations given by $\rho(y) = ad \epsilon \theta(y)$, $\varphi(y) = ad \epsilon \theta(y)$, then $A = \text{End}_{\mathfrak{g}}(\mathfrak{g}^c)$.
2. Let $z_1, \ldots, z_s$ be indeterminates over $\mathbb{C}$, and set $A = z_1 A_1 + \cdots + z_s A_s$. Then $A^2 = I$ is equivalent to a set of polynomial equations in the $z_i$. Let $P_1$ denote the corresponding set of polynomials.
3. We note that Proposition 3.6(2) is equivalent to $Aad\alpha_j A = ad\alpha_j$ for $1 \leq j \leq n$. Also this is equivalent to a set of polynomial equations in the $z_i$. Let $P_2$ denote the corresponding set of polynomials.
Now we consider the compact form $\tilde{u}$, and the corresponding conjugation $\tilde{\tau} : \tilde{\mathfrak{g}}^c \rightarrow \tilde{\mathfrak{g}}^c$. We want to construct involutions $\tilde{\theta}$ of $\tilde{\mathfrak{g}}^c$ that commute with $\tilde{\tau}$ (or, equivalently, that leave $\tilde{u}$ invariant). First we observe that it is straightforward to compute $\tilde{\tau}(x)$ for an $x \in \tilde{\mathfrak{g}}^c$. Indeed, let $u_1, \ldots, u_n$ be a basis of $\tilde{u}$, and write $x = \sum \alpha_i u_i$, with $u_i \in \mathbb{C}$. Then $\tilde{\tau}(x) = \sum \alpha_i u_i$.

Let $R = \mathbb{R}[x_1, \ldots, x_s, y_1, \ldots, y_t]$. We substitute $x_i + iy_i$ for $z_i$ in the polynomials in the sets $P_1, P_2$. A polynomial $f$ in one of these sets then transforms into $g + ih$, with $g, h \in R$. The polynomial equation $f = 0$ is equivalent to two polynomial equations, this time over $\mathbb{R}$, $g = h = 0$. This way we obtain a set of polynomials $Q_1 \subset R$.

Let $A = \sum_{i=1}^n (x_i + iy_i)A_i$, then $\tilde{\tau}A(a_j) = A\tilde{\tau}(a_j)$ is the same as

$$\sum_{i=1}^n (x_i - iy_i)\tilde{\tau}(A_i a_j) = \sum_{i=1}^n (x_i + iy_i)A_i\tilde{\tau}(a_j).$$

Again we split the real and imaginary parts. Doing this for $1 \leq j \leq n$ we obtain a system of (linear) polynomial equations. The corresponding set of polynomials is denoted by $Q_2$.

Finally we solve the system of polynomial equations $q = 0$ for $q \in Q_1 \cup Q_2$. Let $\tilde{\mathfrak{g}}_1, \ldots, \tilde{\mathfrak{g}}_m$ be fixed noncompact real forms of $\tilde{\mathfrak{g}}^c$, such that each noncompact real form of $\tilde{\mathfrak{g}}^c$ is isomorphic to exactly one of the $\tilde{\mathfrak{g}}_i$. Each solution of the polynomial equations yields an involution $\tilde{\theta}$ of $\tilde{\mathfrak{g}}^c$, and we construct the corresponding real form $\tilde{\mathfrak{g}}$ as in Proposition 3.1. The using the methods of [5] we find an isomorphism $\tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_i$, and hence we can map $\mathfrak{g}$ to a subalgebra of an appropriate $\tilde{\mathfrak{g}}_i$.

**Remark 3.7.** This method works best when the polynomial equations have a finite set of solutions: we list them all, and obtain all $\tilde{\mathfrak{g}}_i$ such that $\mathfrak{g}$ maps to a subalgebra by an automorphism of $\tilde{\mathfrak{g}}^c$. However, it can happen that the set of solutions is infinite. Example 4.1 describes a situation where we can deal with that.

### 4. Implementation and Examples

As stated in the introduction, we have implemented the algorithms described here in the computer algebra system GAP4, using the package CoReLG. The main bottleneck of the method is the need to solve a system of polynomial equations. One of the main parameters influencing the complexity of this system is the dimension of the space $\mathcal{A}$, since the number of indeterminates is $2 \dim \mathcal{A}$. (Although, of course, there are also some linear equations, effectively reducing the number of indeterminates.) From Section 2.3 we see that $\dim \mathcal{A} = \sum m_i^2$, where the $m_i$ are the multiplicities of the irreducible $\mathfrak{g}^c$-submodules of $\tilde{\mathfrak{g}}^c$. It can happen that $\dim \mathcal{A}$ is so large that the polynomial equations become unwieldy. For example, if $\mathfrak{g}^c$ is the regular subalgebra of type $A_1 + A_1$ of $F_4$, then $\dim \mathcal{A} = 159$. On the other hand, there are many subalgebras that lead to equations systems that we can deal with. In this section we give some examples. An especially favourable situation arises when $\mathfrak{g}^c$ is an $S$-subalgebra. That will be the subject of the next section.

In the last two examples we also report on the running times. They have been obtained on a 3.16 GHz processor. We remark here that there are two fundamental inefficiencies affecting these running times: firstly, we work over a field containing the square root of all integers. This field has been implemented by ourselves in GAP (see [8]); however, since there is no GAP kernel support for it, computations using this field tend to take markedly longer that, say, over $\mathbb{Q}$. Secondly, we create a lot of polynomials, and also the polynomial arithmetic in GAP is not the most efficient possible (essentially for the same reason as for our field).
Example 4.1. Let \( \tilde{\mathfrak{g}}^c, \mathfrak{g}^c \) be the Lie algebras of type \( A_3 \) and \( A_2 \) respectively. We consider the simplest possible embedding: Let \( \alpha_1, \alpha_2, \alpha_3 \) denote the simple roots of the root system of \( \tilde{\mathfrak{g}}^c \); ordered as usual; then the subalgebra generated by \( x_{\alpha_i}, \) \( x_{-\alpha_i} \) for \( i = 1, 2 \) is isomorphic to \( \mathfrak{g}^c \). We consider the real form of \( \mathfrak{g}^c \) isomorphic to \( \mathfrak{sl}(\mathbb{R}) \) (i.e., the split form).

Since the image of \( \mathfrak{g}^c \) in \( \tilde{\mathfrak{g}}^c \) is regular, i.e., is generated by root vectors of \( \tilde{\mathfrak{g}}^c \), it is automatic that \( \varepsilon(u) \subset \tilde{u} \).

In this case \( \mathcal{A} \) has dimension 4. We get a set of 46 polynomial equations in the unknowns \( x_i, y_i, 1 \leq i \leq 4 \). The reduced Gröbner basis of the ideal generated by these polynomials is

\[
\{ x_1 - 1, x_2 - x_3, x_3^2 + y_2^2 - 1, x_4 + 1, y_1, y_2 + y_3, y_4 \}.
\]

So there is an infinite number of solutions. Now we set \( z_1 = 1, z_2 = x_3 - y_3, z_3 = x_3 + y_3, z_4 = -1 \) (i.e., we work symbolically with \( x_3, y_3 \)) and \( \mathcal{A} = z_1 A_1 + \cdots + z_4 A_4 \). Then the characteristic polynomial of \( \mathfrak{g} \) is

\[
T^{15} + 3T^{14} + (-3x_3^2 - 3y_3^2)T^{13} + \cdots + (3x_3^2 + 9x_3^4 + 3y_3^4)T^3 + 3y_3^4 + 3x_3^2 y_3^4 + y_3^6.
\]

However, using \( x_3^2 + y_3^2 = 1 \), this reduces to

\[
T^{15} + 3T^{14} - 3T^{13} - 17T^{12} - 3T^{11} + 39T^9 - 25T^8 - 45T^7 + 25T^6 + 39T^5 - 37T^4 - 17T^3 - 3T^2 - 3T + 1
\]

which is \((T - 1)^9(T + 1)^9\). From this we conclude that if we take any solution of the equations and construct the corresponding real form \( \tilde{\mathfrak{g}} \), then its Cartan decomposition will be \( \tilde{\mathfrak{g}} = \mathfrak{f} \oplus \mathfrak{p} \) with \( \dim \mathfrak{f} = 6 \) and \( \dim \mathfrak{p} = 9 \). Now there is, up to isomorphism, only one real form of \( \tilde{\mathfrak{g}}^c \) with a Cartan decomposition satisfying this, namely \( \mathfrak{sl}(\mathbb{R}) \). Also, up to equivalence, \( \tilde{\mathfrak{g}}^c \) contains exactly one subalgebra isomorphic to \( \mathfrak{g}^c \). So we conclude that \( \mathfrak{sl}(\mathbb{R}) \) is the only real form of \( \tilde{\mathfrak{g}}^c \) containing a subalgebra isomorphic to \( \mathfrak{sl}(\mathbb{R}) \).

Example 4.2. Let \( \tilde{\mathfrak{g}}^c, \mathfrak{g}^c \) be the Lie algebras of type \( E_8 \) and \( A_1 + G_2 + G_2 \) respectively. As real form \( \mathfrak{g} \) we took the direct sum of the noncompact real forms of \( A_1 \) and \( G_2 \) (twice) respectively. In this case \( \mathcal{A} \) was computed in 2058 seconds, and \( \dim \mathcal{A} = 6 \). The polynomial equations were computed in 36783 seconds. The set \( \mathcal{Q}_1 \cup \mathcal{Q}_2 \) contains 37460 polynomials. However, a reduced Gröbner basis of the ideal generated by them is

\[
\{ x_1 + 1, x_2, x_3 - 1, x_4 + 1, x_5 - 1, x_6 + 1, y_1, y_2, y_3, y_4, y_5, y_6 \}.
\]

So there is only one solution. The corresponding real form of \( E_8 \) turned out to be \( \mathfrak{evii} \).

Example 4.3. Let \( \tilde{\mathfrak{g}}^c \) be of type \( E_6 \). Then, up to equivalence, \( \tilde{\mathfrak{g}}^c \) contains a unique subalgebra of type \( B_4 \). So let \( \mathfrak{g}^c \) be of type \( B_4 \) and let \( \mathfrak{g} = \mathfrak{so}(4,5) \). In this example \( \mathcal{A} \) was computed in 55 seconds, and \( \dim \mathcal{A} = 7 \). The polynomial equations were computed in 510 seconds, the reduced Gröbner basis of the ideal generated by them is

\[
\{ x_5^2 - x_7, x_5 x_6, x_6^2 + y_5^2 + x_7 - 1, x_5 x_7 - x_6, x_5 x_7, x_7^2 - x_7, x_5 y_6, x_7 y_6, x_1 + x_5, x_2 + x_6, x_3 + 1, x_4 + x_7, y_1, y_2 - y_6, y_3, y_4, y_5, y_7 \}.
\]

We see that \( x_7 \) can have the values 0, 1. Adding \( x_7 \) to the generating set, the Gröbner basis becomes

\[
\{ x_5^2 + y_5^2 - 1, x_1, x_2 + x_6, x_3 + 1, x_4, x_5, x_7, y_1, y_2 - y_6, y_3, y_4, y_5, y_7 \}.
\]

Here the value of \( x_6, y_6 \) determines the solution completely. Furthermore, there is an infinite number of possible values for those indeterminates. However, with
the same method as in Example 4.1 we established that all solutions lead to the inclusion \( \mathfrak{so}(4,5) \subset E_1 \).

Adding \( x_7 - 1 \) to the generating set, we get the Gröbner basis
\[
\{ x_5^2 - 1, x_1 + x_5, x_2, x_3 + 1, x_4 + 1, x_6, x_7 - 1, y_1, y_2, y_3, y_4, y_5, y_6, y_7 \}.
\]
Here we get two solutions, which both yield the inclusion \( \mathfrak{so}(4,5) \subset E_1 \).

5. \( S \)-subalgebras of the exceptional Lie algebras

In this section we consider embeddings \( \varepsilon : \mathfrak{g}^c \hookrightarrow \tilde{\mathfrak{g}}^c \), such that \( \varepsilon(\mathfrak{g}^c) \) is a maximal \( S \)-subalgebra of \( \tilde{\mathfrak{g}}^c \), and the latter is of exceptional type.

Let \( \mathfrak{g} \) be a real form of \( \mathfrak{g}^c \). By [19], §6, Theorem 2, if \( \varepsilon(\mathfrak{g}^c) \) is an \( S \)-subalgebra of \( \tilde{\mathfrak{g}}^c \), then there are at most two real forms of \( \tilde{\mathfrak{g}}^c \) that contain \( \varepsilon(\mathfrak{g}) \). And if \( \tilde{\mathfrak{g}}^c \) has no outer automorphisms there is at most one such real form. This explains why our method works particularly well in this case: the polynomial equations have at most two solutions. Example 4.2 illustrates this phenomenon (there the subalgebra is a non maximal \( S \)-subalgebra).

Table 1 contains the results that we obtained using our programs (for the situation described above, i.e., \( \varepsilon(\mathfrak{g}^c) \) is a maximal \( S \)-subalgebra of \( \tilde{\mathfrak{g}}^c \)). We describe the subalgebras of the complex Lie algebras by giving the type of their root systems, with an upper index denoting the Dynkin index (see [20]).

Komrakov ([17]) has also published a list of the \( S \)-subalgebras of the real simple Lie algebras of exceptional type. In type \( E_6 \) we find a few differences: the inclusions marked by a \((\ast)\) are not contained in Komrakov’s list. About all other inclusions Komrakov’s list and ours agree.

| complex inclusion | real inclusion |
|-------------------|---------------|
| \( A_2^4 \subset E_6 \) | \( \{ \mathfrak{su}(1,2) \subset E_{11} \} \) |
| \( G_2^3 \subset E_6 \) | \( \{ \mathfrak{sl}(3,\mathbb{R}) \subset E_{11} \} \) |
| \( A_2^3 \oplus G_2^1 \subset E_6 \) | \( \{ \mathfrak{su}(1,2) \oplus G^{cmp} \subset E_{11} \} \) |
| \( C_4^1 \subset E_6 \) | \( \{ \mathfrak{sp}(2,2) \subset E_{11} \} \) |
| \( F_4^1 \subset E_6 \) | \( \{ \mathfrak{sp}(1,3) \subset E_{11} \} \) |
| \( A_4^{11} \subset E_7 \) | \( \{ \mathfrak{sl}(2,\mathbb{R}) \subset EV \} \) |
| \( A_1^{399} \subset E_7 \) | \( \{ \mathfrak{sl}(3,\mathbb{R}) \subset EV \} \) |
| \( A_2^{31} \subset E_7 \) | \( \{ \mathfrak{su}(1,2) \subset EV \} \) |
| Subalgebras | Description |
|-------------|-------------|
| $A_1^2 + A_1^4 \subset E_7$ | $su(2) \oplus sl(2, \mathbb{R}) \subset E V$
| & $sl(2, \mathbb{R}) \oplus su(2) \subset EV I$
| & $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \subset EV I$
| & $su(2) \oplus G \subset EV I$
| $A_1^7 + G_2^2 \subset E_7$ | $sl(2, \mathbb{R}) \oplus G^{cmp} \subset EV$
| & $sl(2, \mathbb{R}) \oplus G \subset EV$
| & $sp(3) \oplus G \subset EV I$
| & $sp(1, 2) \oplus G^{cmp} \subset EV I$
| & $sp(3, \mathbb{R}) \oplus G^{cmp} \subset EV I I$
| & $sp(3, \mathbb{R}) \oplus G \subset EV$
| & $su(2) \oplus FI \subset EV I$
| & $su(2) \oplus FII \subset EV I I$
| $C_3^1 \oplus G_2^2 \subset E_7$ | $sl(2, \mathbb{R}) \oplus F_4^{cmp} \subset EV II$
| & $sl(2, \mathbb{R}) \oplus FI \subset EV$
| & $sl(2, \mathbb{R}) \oplus FII \subset EV II$
| $A_1^2 \oplus G_2^{8} \subset E_8$ | $sl(2, \mathbb{R}) \subset EV I I$
| $A_1^{40} \subset E_8$ | $sl(2, \mathbb{R}) \subset EV I I$
| $A_1^{40} \subset E_8$ | $sl(2, \mathbb{R}) \subset EV I I$
| $B_2^{40} \subset E_8$ | $so(2, 3) \subset EV I I I$
| & $so(4, 1) \subset EV I I I$
| & $su(2) \oplus su(1, 2) \subset EV I I I$
| & $su(2) \oplus sl(3, \mathbb{R}) \subset E I X$
| $A_1^{16} \oplus A_2^{2} \subset E_8$ | $sl(2, \mathbb{R}) \oplus su(3) \subset EV I I I$
| & $sl(2, \mathbb{R}) \oplus su(1, 2) \subset EV I I I$
| & $sl(2, \mathbb{R}) \oplus sl(3, \mathbb{R}) \subset EV I I I$
| & $F_4^{cmp} \oplus G \subset E I X$
| $F_4^{1} \oplus G_2^{1} \subset E_8$ | $F I \oplus G^{cmp} \subset E I X$
| & $F I \oplus G \subset EV I I I$
| & $F II \oplus G^{cmp} \subset EV I I I$
| & $F II \oplus G \subset E I X$
| $A_1^{28} \subset F_4$ | $sl(2, \mathbb{R}) \subset F I$
| & $su(2) \oplus G \subset F I$
| $A_1^{3} \oplus G_2^{1} \subset F_4$ | $sl(2, \mathbb{R}) \oplus G^{cmp} \subset F I I$
| & $sl(2, \mathbb{R}) \oplus G \subset F I$
| $A_1^{28} \subset G_2$ | $sl(2, \mathbb{R}) \subset G$

References

[1] J. F. Cornwell. Semi-simple real subalgebras of non-compact semi-simple real Lie algebras. I. II. Rep. Mathematical Phys., 2(4):239–261; ibid. 2 (1971), no. 4, 289–309, 1971.
[2] J. F. Cornwell. Semi-simple real subalgebras of non-compact semi-simple real Lie algebras. III. Rep. Mathematical Phys., 3(2):91–107, 1972.
[3] D. Cox, J. Little, and D. O’Shea. Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra. Springer Verlag, New York, Heidelberg, Berlin, 1992.
[4] Heiko Dietrich, Paolo Faccin, and Graaf. CoReLG. Computation with Real Lie Groups. A GAP4 package, 2013. in preparation, (http://science.unitn.it/~corelg/index.html)
[5] Heiko Dietrich, Paolo Faccin, and Willem A. de Graaf. Computing with real Lie algebras: real forms, Cartan decompositions, and Cartan subalgebras. J. Symbolic Comput., 56:27–45, 2013.

[6] Heiko Dietrich and Willem A. de Graaf. A computational approach to the kostant-sekiguchi correspondence. Pacific Journal of Mathematics, 265(2):349–379, 2013.

[7] E. B. Dynkin. Maximal subgroups of the classical groups. Trudy Moskov. Mat. Obšč., 1:39–166, 1952. English translation in: Amer. Math. Soc. Transl. (6), (1957), 245–378.

[8] E. B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. Mat. Sbornik N.S., 30(72):549–646 (3 plates), 1952. English translation in: Amer. Math. Soc. Transl. (6), (1957), 111–244.

[9] J. M. Ekins and J. F. Cornwell. Semi-simple real subalgebras of non-compact semi-simple real Lie algebras. IV. Rep. Mathematical Phys., 5(1):17–49, 1974.

[10] J. M. Ekins and J. F. Cornwell. Semi-simple real subalgebras of non-compact semi-simple real Lie algebras. V. Rep. Mathematical Phys., 7(2):167–203, 1975.

[11] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.5, 2012. (\protect\url{http://www.gap-system.org}).

[12] Willem A. de Graaf. Constructing semisimple subalgebras of semisimple Lie algebras. J. Algebra, 325(1):416–430, 2011.

[13] Willem A. de Graaf. SLA - computing with Simple Lie Algebras. a GAP package, 2013. (\protect\url{http://science.unitn.it/~degraaf/sla.html}).

[14] J. E. Humphreys. Introduction to Lie Algebras and Representation Theory. Springer Verlag, New York, Heidelberg, Berlin, 1972.

[15] N. Jacobson. Lie Algebras. Dover, New York, 1979.

[16] A. W. Knapp. Lie groups beyond an introduction, volume 140 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[17] B. P. Komrakov. Maximal subalgebras of real Lie algebras and a problem of Sophus Lie. Dokl. Akad. Nauk SSSR, 311(3):528–532, 1990.

[18] A. N. Minchenko. Semisimple subalgebras of exceptional Lie algebras. Tr. Mosk. Mat. Obs., 67:256–293, 2006. English translation in: Trans. Moscow Math. Soc. 2006, 225–259.

[19] Arkady L. Onishchik. Lectures on Real Semisimple Lie Algebras and Their Representations. European Mathematical Society, Zürich, 2004.