A STRUCTURE THEOREM ON DOUBLING MEASURES WITH DIFFERENT BASES: A NUMBER THEORETIC APPROACH

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ABSTRACT. In seemingly unrelated statements, yet interconnected proofs, we provide two distinct structural classifications related to normal numbers and $n$-adic doubling measures. The motivation for this is originally due to refined questions about unions of certain doubling measures important in analysis; in particular we are able to classify $n$-adic doubling measures in a way that extends results of Wu from a single measure to an infinite union. However, the proof strategy that we employ is completely different from Wu’s and heavily utilizes many number theoretic elements, which leads to the second, unexpected, classification related to normal numbers.

1. INTRODUCTION

The goal of this paper is to establish a structure theorem for $n$-adic doubling measures (including dyadic doubling measures $n = 2$), a key area of importance in harmonic analysis and related fields, as discussed below. As a byproduct of our work, we prove a statement involving a classification related to normal numbers, a popular topic in number theory, that has nothing to do with measures.

We begin with recalling several definitions. A doubling measure $\mu$ is a measure for which there exists a positive constant $C$ such that for every interval $I \subset \mathbb{R}$, $\mu(2I) \leq C\mu(I)$, where $2I$ is the interval which shares the same midpoint of $I$ and twice the length of $I$.

We will focus on $n$-adic intervals, $n \in \mathbb{N}$:

$$I = \left[\frac{k-1}{n^\ell}, \frac{k}{n^\ell}\right), \quad \ell, k \in \mathbb{Z}.$$ 

The $n$-adic children of the interval defined above are

$$I_j = \left[\frac{k-1}{n^\ell} + \frac{j-1}{n^{\ell+1}}, \frac{k-1}{n^\ell} + \frac{j}{n^{\ell+1}}\right), \quad 1 \leq j \leq n.$$ 

A measure $\mu$ is a $n$-adic doubling measure on $\mathbb{R}$ if there exists a positive constant $C$, independent of all parameters, such that for any $n$-adic $I$,

$$\frac{1}{C} \leq \frac{\mu(I_{j_1})}{\mu(I_{j_2})} \leq C, \quad 1 \leq j_1, j_2 \leq n,$$

where both $I_{j_1}$ and $I_{j_2}$ are some $n$-adic children of $I$ defined in (1.1). We denote the collection of all $n$-adic doubling measures by $\mathcal{D}_n$.

Doubling measures are a classical topic in analysis and they have many deep connections to other fields, such as PDE. The well-known Muckenhoupt $A_p$ weights
and reverse Hölder weights are all automatically doubling, and the doubling property is the key point of the definition of spaces of homogeneous type. For more background and applications of doubling measures (particularly from a more modern perspective), see, for example [4], [5], [7], [9], [10], [11], [12], [13]. In particular, dyadic doubling measures have been central to a rich area of study, see, for example [6]. The study of the union as well as the intersection of \( n \)-adic doubling measures is a recent topic. It dates back to Wu’s work [15, 16] in 90s on using the null set to characterize the \( n \)-adic doubling measures, in particular, she proved the following result.

**Theorem 1.1** ([15]). For any two integers \( A \) and \( B \) greater than 2, \( \frac{\log A}{\log B} \) is irrational if and only if \( D_A \nsubseteq D_B \) and \( D_B \nsubseteq D_A \).

A natural question to ask is whether we can extend the above result to intersection or unions of \( n \)-adic measures. Here is some recent work along this line of research.

1. In unpublished work of Jones, he proved that any finite intersection of the prime BMO function classes is never equal to the full BMO function class (for more details about this see [8]). Since then, a folkloric question has been: “does an analogue of Jones’s result hold for \( p \)-adic doubling measures?” This proved a difficult extension that was answered recently by [3] and [2], described in items 2. and 3. below.

2. The first work which extends Wu’s type of results to the union or the intersections of \( D_n \)’s was due to Boylan, Mills and Ward [3], which also was the first step to answering the folkloric analogue of Jones’s question above;

3. In our recent work [2], we answer the analogue of Jones’s question for measures by proving that for any finite family of primes \( p_i \), there exists a measure that is \( p_i \)-adic doubling yet not doubling. Additionally we extend this result to the setting of Muckenhoupt \( A_p \) and reverse Hölder weights.

The results in [2] left several open question (both implicit and explicitly stated). In this paper, we completely resolve one of these by proving the following structure theorem for unions of \( n \)-adic measures. Here is the main result.

**Theorem 1.2.** Let \( \{n_i\} \) and \( \{m_j\} \) be any sequences of integers greater than 2. Then the following statements are equivalent:

1. There exists some \( i \geq 1 \) such that

\[
\frac{\log n_i}{\log m_j} \in \mathbb{R} \setminus \mathbb{Q}, \quad \forall j \geq 1;
\]

2. \[
\bigcup_{i \geq 1} D_{n_i} \nsubseteq \bigcup_{j \geq 1} D_{m_j}.
\]

**Remark 1.3.** Here are some remarks for Theorem 1.2.

1. Clearly, Theorem 1.2 generalizes Theorem 1.1.
2. The proof of Theorem 1.1 by Wu is based on Kronecker’s theorem on irrational numbers, namely, for \( r \) being irrational, the set \( \{ kr \mod 1 : k \in \mathbb{Z} \} \) is dense on \([0, 1)\). To our best knowledge, Wu’s approach seems difficult to extend to the situation when we try to understand the behavior of the union or the intersection of \( D_n \). More precisely, for \( x \in \mathbb{R} \) with \( \| x \| > \frac{1}{10} \), where \( \| x \| \) denotes the distance from \( x \) to the nearest integer, let

\[
\mathcal{J} := \left\{ j : j \in \mathbb{N}, \| jx \| > \frac{1}{20} \right\}
\]

The proof in [15] relies heavily on defining corresponding \( \mathcal{J} \) for a specially chosen sequence of \( x_i \in \mathbb{R} \) and noting that for all these \( x_i \) (where \( \| x_i \| > \frac{1}{10} \)), \( \mathcal{J}^c \) contains no consecutive integers. To establish a result as in Theorem 1.2, we likely would need to do the same for the sets

\[
\mathcal{J}' := \left\{ j : j \in \mathbb{N}, \| jx_{1,i} \|, \| jx_{2,i} \| > \frac{1}{20} \right\}
\]

where \( \| x_{1,i} \|, \| x_{2,i} \| > \frac{1}{m} \), which would in turn involve showing that unions of sets like \( \mathcal{J}' \) do not contain consecutive integers. This appears quite challenging. The rest of Wu’s approach relies on an intricate construction that uses several highly technical lemmas that she had developed. This allowed for a precise, explicit construction, but made direct generalizations of her techniques even less amenable. Hence we decided to take a completely different approach;

3. Our approach is different from Wu and based a systematic study of far numbers on \( \mathbb{R} \) (see [1]). The advantage of our method is two-fold. First, our approach is simpler and less technically reliant than the argument in [15], avoiding the construction of certain auxiliary \( n \)-adic doubling measures and iteration arguments. Second, we are able to extend this type of structure theorem to the union of \( n \)-adic measures (namely, Theorem 1.2);

4. Note that \( D_m = D_{ma} \) for any integer \( a \geq 1 \). This fact follows easily from the definition, but will contribute to an important reduction step in our proof (see Section 3.2). Note that this fact can be thought of as a version of Hensel’s lemma in analysis: that is, that \( m \)-adic doubling measures (and in particular \( p \)-adic doubling measures for any prime \( p \)) can be “lifted” to \( m^a \)-adic doubling measures for \( a \geq 1 \).

We prove this via an explicit construction that heavily weaves in number theory.

Remark 1.4. An important observation for the construction (see the next section for details) is that all the intervals involved only depend on \( \{ n_i \} \) and are independent of \( \{ m_j \} \). This is very different from

1. The null set construction in [15];

2. The exotic measures in [2, 3].

This is why we are allowed to prove a structure theorem for infinite unions of \( n \)-adic measures instead of simply finite ones.

In fact, another novelty of our approach is that we can prove a number theoretic classification related to normal numbers (see [14], [15]). Recall
Theorem 1.5 ([1]), \(\frac{\log m}{\log n}\) is rational if and only if every number normal in base \(m\) is also normal in base \(n\).

Indeed, as a byproduct of our proof, we will have proved the following result which classifies a pair of numbers \((m, n)\) with \(\frac{\log m}{\log n}\) being irrational, that is, pairs \((n, m)\) not satisfying Theorem 1.5.

Theorem 1.6. Let \(m \geq n\) be two integers with \(\frac{\log m}{\log n}\) being irrational. Then there are only two possible cases:

1. \(\frac{1}{n}\) is \(m\)-far (see, Definition 1.7);
2. There exists some positive integers \(a, b \geq 1\), such that \((m^a, n^b)\) is a good pair (see, Definition 3.2).

This result is of independent interest. First of all, it has nothing to do with the construction of doubling measures. Secondly, we can see that this result has potential further applications in studying normal numbers (see [14]) and other objects in number theory. Thirdly, it suggests a possible connection between the adjacency of general dyadic grids and the collection of \(n\)-adic doubling measures.

We begin with the definition of \(n\)-far numbers.

Definition 1.7. A real number \(\delta\) is \(n\)-far if the distance from \(\delta\) to each given rational \(k/n\) is at least some fixed multiple of \(1/n\), where \(m \geq 0\) and \(k \in \mathbb{Z}\). That is, if there exists \(C > 0\) such that

\[
|\delta - \frac{k}{n^m}| \geq \frac{C}{n^m}, \quad \forall m \geq 0, k \in \mathbb{Z},
\]

where \(C\) may depend on \(\delta\) but independent of \(m\) and \(k\). Note also that \(0 < C < 1\).

Remark 1.8. Note that in the first condition in Theorem 1.6, we cannot interchange the role of \(m\) and \(n\). Indeed, for example, \(\frac{1}{6}\) is 2-far while \(\frac{1}{2}\) is not 6-far.

We need the following result from [1].

Proposition 1.9. All rationals except those of the form \(k/n^m\), \(m \geq 0, k \in \mathbb{Z}\) are \(n\)-far numbers.

The rest of the paper is devoted to prove Theorem 1.2.

2. Proof of the main result: Part I.

We begin with observing that it is suffices to check that (2) implies (1). Indeed, assuming (1) fails, we see that for each \(i \geq 1\), there exists some \(j_i \geq 2\), such that \(\frac{\log n_i}{\log m_{j_i}} \in \mathbb{Q}\), which means there exists some \(n_i \in \mathbb{Z}\) such that both \(n_i\) and \(m_{j_i}\), are some powers of \(n_i\). This further implies for each \(i \geq 1\), \(D_{n_i} = D_{n_i} = D_{m_{j_i}}\) (this fact is easy to see using the definition), which contradicts (2).

In the rest of this paper, we prove (1) implies (2). The desired result clearly follows from the following result.

Theorem 2.1. Let \(n := n_i\) and \(\{m_{j_i}\} \subseteq \mathbb{Z}\) satisfy the assumption in (1). Then there exists a measure \(\mu\) on \(\mathbb{R}\), such that \(\mu\) is \(n\)-adic doubling but not \(m_{j_i}\)-adic doubling for any \(j \geq 1\).
2.1. Construction of $\mu$. To begin with, we simply let the restriction of $\mu$ on $(-\infty, 0)$ be the Lebesgue measure, and we will re-distribute the weights on $[0, \infty)$. Take any $a, b > 0$ with

$$a + b = 2 \quad \text{and} \quad 0 < a < 1 < b.$$  

Now on each $\ell \in \mathbb{N}$, we re-distribute the weight on the interval $[\ell, \ell + 1)$ as follows.

Step I: Note that $[\ell, \ell + 1)$ is an $n$-adic interval. We assign the weight $a$ to its leftmost $n$-adic child and $b$ to its rightmost $n$-adic child, that is

$$\mu|_{[\ell, \ell + \frac{1}{n^j}]} = adx \quad \text{and} \quad \mu|_{[\ell + \frac{1}{n^j}, \ell + 1]} = bdx.$$

While for all other $n$-adic children, the weights there remain unchanged;

Step II: Repeat the procedure $\ell + 1$ times in Step I to all the $n$-adic children whose weights have been redistributed from the previous step. For example, for the $n$-adic child $[\ell, \ell + \frac{1}{n^2}]$ which has been selected in Step I, we let

$$d\mu|_{[\ell, \ell + \frac{1}{n^2}]} = (a^2)dx \quad \text{and} \quad d\mu|_{[\ell + \frac{1}{n^2}, \ell + \frac{1}{n^1}]} = (ab)dx,$$

and

$$d\mu|_{[\ell + \frac{1}{n^2}, \ell + \frac{j}{n^2}]} = adx, \quad j \in \{1, \ldots, n - 2\}.$$

We plot the measure $\mu$ when $n = 3$ (see, Figure 1).

| $\ell$ | $a$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $b$ | $a^2$ | $\frac{4}{9}$ | $\frac{11}{9}$ | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{16}{9}$ | $\frac{17}{9}$ | $b^2$ | $\frac{1}{4}$ | $b^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | |

Figure 1. $\mu$ with $n = 3$, where the blue parts refer to the weights associated to each interval.

We have the following observation.

**Lemma 2.2.** $\mu$ is $n$-adic doubling but not doubling.

**Proof.** It is clear that $\mu$ is $n$-adic doubling, as the ratio $\frac{1}{n^2}$ for any $n$-adic interval is either $\frac{a}{b}$, 1 or $\frac{b}{a}$.

Next we argue that $\mu$ is not doubling. Indeed, this follows by comparing the weights near the integer points. More precisely, for each $\ell \geq 1$, note that on its left hand side, there is an $n$-adic interval with weight $b^\ell$ with sidelength $\frac{1}{n^\ell}$; while on its right hand side, there is an $n$-adic interval with weight $a^{\ell+1}$ with sidelength $\frac{1}{n^{\ell+1}}$. Then, for example, we might consider the interval

$$\left[\ell - \frac{1}{n^\ell}, \ell + \frac{1}{n^{\ell+1}}\right],$$

whose left half has weight $b^\ell$, while right half has weight $a^{\ell+1}$. The desired claim is clear. \qed
In the rest of the paper, we show that the measure $\mu$ constructed above is not $m$-adic doubling with $m$ satisfying

\begin{equation}
\frac{\log n}{\log m} \in \mathbb{R} \setminus \mathbb{Q}.
\end{equation}

Without loss of generality, we may assume $n \geq 3$, otherwise, that is if $n = 2$, we simply replace it by $n = 4$, as $D_2 = D_4$.

We include some motivation before we proceed. The key observation is that condition (2.2) is indeed not equivalent to the far number characterization. For example, $\frac{\log 12}{\log 18}$ is irrational, however, neither $\frac{1}{12}$ is 18-far nor $\frac{1}{18}$ is 12-far, since $\frac{1}{12} = \frac{27}{18^2}$ and $\frac{1}{18} = \frac{8}{12^2}$, respectively. This suggests us to consider two different cases:

I. $\frac{1}{n}$ is $m$-far;

II. $\frac{1}{n}$ is not $m$-far.

For the first case, since $\frac{1}{n}$ is $m$-far, this means $\frac{1}{n}$ is “far away” from all rationals of the form $\frac{k}{m^\ell}$, and hence it suffices to consider sufficient small $m$-adic intervals near the points $\ell + \frac{1}{n}$ for $\ell$ large. While for the second case, we shall later see that this reduces to the solvability of a certain equation (see (3.1)), which allows us to modify the argument from the first case so that the ratio (1.1) will “blow up” as desired.

In the rest of this section, we will focus on the case when $\frac{1}{n}$ is $m$-far, while we postpone the more complex second case to the next section.

2.2. Proof of Theorem 2.1: the far number case. For $\ell$ sufficiently large, we consider the point $\ell + \frac{1}{n}$ and the interval

$$I_\ell := \left[ \ell + \frac{1}{n} - \frac{1}{n^{\ell+1}}, \ell + \frac{1}{n} + \frac{1}{n^{\ell+1}} \right].$$

By the construction of $\mu$, the weights associated to $I_\ell$ are $ab^\ell$ on its left half and 1 on its right half.

Take any integer $\ell' > \frac{(\ell+1)\log n}{\log m}$ and let $J_{\ell'}$ be the unique $m$-adic interval which contains $\ell + \frac{1}{n}$ with sidelength $m^{-\ell'}$. Note that by the choice of $\ell'$, $J_{\ell'} \subset I_\ell$ and moreover, since $\frac{1}{n}$ is $m$-far, $\ell + \frac{1}{n}$ does not coincide with any endpoints of $J_{\ell'}$. We consider three cases. To this end, we denote

$$J_{\ell',L} := \text{the leftmost } m\text{-adic child of } J_{\ell'}$$

and

$$J_{\ell',R} := \text{the rightmost } m\text{-adic child of } J_{\ell'}.$$

Case I: $\ell + \frac{1}{n} \notin J_{\ell',L} \cup J_{\ell',R}$. In this case, it suffices to note that

\begin{equation}
\frac{\mu(J_{\ell',L})}{\mu(J_{\ell',R})} = ab^\ell.
\end{equation}

(see, Figure 2).
Case II: $\ell + \frac{1}{n} \in J_{\ell'}^L$. Let $l(J_{\ell'})$ be the left endpoint for $J_{\ell'}$. Note that $l(J_{\ell'})$ is of the form $\frac{k}{m}$ for some $k \in \mathbb{Z}$, this implies that

$$|l(J_{\ell'}) - \left(\ell + \frac{1}{n}\right)| \geq \frac{C}{m^{e+1}},$$

for some $C > 0$ only depends on $m$ and $n$, where we have used the fact that $\frac{1}{n}$ is $m$-far. Moreover, we can also conclude that $C < \frac{1}{m}$. Indeed, since $\ell + \frac{1}{n} \in J_{\ell'}^L$ and $l(J_{\ell'}) + \frac{1}{m^{e+1}} = \frac{k+1}{m^{e+1}} \neq \ell + \frac{1}{n}$ (since $\frac{1}{n}$ is $m$-far), it follows that

$$|l(J_{\ell'}) - \left(\ell + \frac{1}{n}\right)| < \frac{1}{m^{e+1}},$$

which together with (2.3) gives the desired assertion.

Therefore, we have

$$\frac{\mu(J_{\ell'}^L)}{\mu(J_{\ell'}^R)} \geq \frac{ab^e \cdot \frac{m}{m^{e+1}} + \left(\frac{1}{m} - C\right) \cdot \frac{1}{m^{e+1}}}{m^{e+1}} \geq Cm \cdot ab^e.$$  
(see, Figure 3).

Case III: $\ell + \frac{1}{n} \in J_{\ell'}^R$. The third case can be treated as an application of the previous two cases. Indeed, by a similar argument as in Case II, we can see the ratio (1.1) with respect to any two of the $m$-adic children of $J_{\ell'}$ is of size 1. Therefore, instead of considering $J_{\ell'}$, we consider $J_{\ell'}^R$ and check whether the point $\ell + \frac{1}{n}$ locates in the rightmost $m$-adic child of $J_{\ell'}^R$ or not: if not, then we can apply the argument in Case I and Case II at this smaller scale to get estimates as in (2.3) and (2.4); otherwise, we simply go down to the next smallest scale and repeat this procedure. Note that this procedure will stop in finite steps as the distance between $\ell + \frac{1}{n}$ and the right endpoint of $J_{\ell'}$ is fixed (see, Figure 4).
In conclusion, if \( \frac{1}{n} \) is \( m \)-far, we have proved that, for each interval \([\ell, \ell + 1)\), there exists a sufficiently small \( m \)-adic interval containing the point \( \ell + \frac{1}{n} \), such that the maximum of the ratio (1.1) is bounded below by \( \min\{Cm, 1\} \cdot ab^\ell \), which clearly blows up when \( \ell \) converges to infinity.

### 3. Proof of the main result: Part II.

In this section, we prove Theorem 2.1 under the assumption that \( \frac{1}{n} \) is not \( m \)-far. To begin with, we first make a remark that our approach in the previous section might not work: indeed, since \( \frac{1}{n} \) is not \( m \)-far, by Proposition 1.9, for each \( \ell \in \mathbb{N} \), we have

\[
\ell + \frac{1}{n} = \frac{k_\ell}{m s_\ell},
\]

for some \( k_\ell, s_\ell \in \mathbb{N} \).

This implies \( \ell + \frac{1}{n} \) can indeed be one of the endpoints when we restrict our attention to those small \( m \)-adic interval containing \( \ell + \frac{1}{n} \) and hence we are not able to benefit any more from the fact that the weights on both sides of \( \ell + \frac{1}{n} \) differ dramatically (see, e.g., Figure 2). Therefore, we have to refine the choice of \( m \)-adic interval for Case II above, which will entail a more concerted effort.

We shall see later that the desired refinement reduces to study the solubility of the equation

\[
(3.1) \quad \frac{k}{m^{\ell-1}} = \frac{1}{n^\ell},
\]

where \( m, n \geq 2 \). More precisely, if (3.1) is unsolvable, then we can refine our argument in the first case by considering sufficiently small \( m \)-adic intervals near points \( \left\lfloor \frac{\log m}{\log n} \right\rfloor - 1 + \frac{1}{n^\ell} \) for \( \ell \) sufficiently large.

For this purpose, we have the following definition.

**Definition 3.1.** Let \( m, n \) be two integers greater than 2. We say a pair \((m, n)\) is **solvable** if there exists integers \( k, \ell \geq 1 \) such that (3.1) holds. Otherwise, we say \((m, n)\) is **unsolvable**.

The plan of this section is as follows. In the first part, we study the solubility of the equation (3.1), more precisely, we shall show that one can “transform” every solvable pair \((m, n)\) into an unsolvable pair while this “transformation” is invariant under Theorem 2.1. In the second part, we use the insolubility of the equation (3.1) to complete the proof of Theorem 2.1.
3.1. **Good pairs and semi-good pairs.** We first understand the solubility of the equation (3.1), with the assumption that $\frac{1}{n}$ is not $m$-far.

Recall that by Proposition 1.9, there exists some $k_0, s_0 \in \mathbb{N}$, such that
\[
\frac{1}{n} = \frac{k_0}{m^{s_0}},
\]
that is, $k_0n = m^{s_0}$. This implies that if $p$ is a prime factor of $n$, then so is $m$, and hence we can write the prime decomposition of $n$ and $m$ as follows:

\[
(3.2) \quad n = p_1^{a_1} \cdots p_N^{a_N} \quad \text{and} \quad m = p_1^{b_1} \cdots p_N^{b_N},
\]
where $p_i, 1 \leq i \leq N$ are all primes and $a_i, b_i \geq 0, 1 \leq i \leq N$. Moreover, if for some $i \in \{1, \ldots, N\}$, $a_i > 0$, then $b_i > 0$.

We now introduce the concept of *good pair* and *semi-good pair*.

**Definition 3.2.** Let $m$ and $n$ be defined as in (3.2). We say $(m, n)$ is a *semi-good pair* if

(a). $m > n$;  
(b). $b_i > a_i, \ 1 \leq i \leq N$.

Moreover, we say $(m, n)$ is a *good pair* if the second condition above is replaced by the following:

(c). $b_i \geq a_i$ for all $1 \leq i \leq N$, and there exists some $i \in \{1, \ldots, N\}$, such that $a_i = b_i > 0$.

**Example 3.3.** $(m, n) = (108, 6)$ is a semi-good pair and $(m, n) = (108, 36)$ is a good pair.

We have the following easy but important observation.

**Lemma 3.4.** A good pair is unsolvable.

**Proof.** Without loss of generality, we assume $a_1 = b_1 > 0$ in condition (c) above. Then for each $\ell \geq 1$, (3.2) is equivalent to
\[
\frac{k_p^2 \cdot p_2^{2\ell} \cdots p_N^{2\ell}}{p_2^{b_1(\ell - 1)} \cdots p_N^{b_N(\ell - 1)}} = \frac{1}{p_1^{a_1}},
\]
which clearly has no solutions. \qed

**Remark 3.5.** Lemma 3.4 has a certain geometric interpretation: indeed, let $(m, n)$ be a good pair, then Lemma 3.4 asserts that the point $\frac{1}{n^\ell}$ is an endpoint of a $m$-adic interval with sidelength $m^{-\ell}$, since condition (c) implies $\frac{1}{n} = \frac{1}{m^{k'}}$ for some $k' \in \mathbb{N}$, while it is not an endpoint of a $m$-adic interval with sidelength $m^{-\ell + 1}$ since $\frac{1}{n^\ell} \neq \frac{1}{m^{k'}}$ for all $k \in \mathbb{N}$. This exactly suggests us how to find a pair of $m$-adic siblings with the ratio (1.2) between them “blowing up”.

However, Lemma 3.4 in general is not true for a semi-good pair. For example, if $(m, n) = (108, 6)$, then for $\ell > 2$, there always holds
\[
\frac{2^{\ell - 2} \cdot 3^{2\ell - 3}}{108^{\ell - 1}} = \frac{1}{6^\ell}.
\]

Our next result suggests that we can always modify a semi-good pair into a good pair.
Proposition 3.6. Let \( N \) be some positive integers and
\[
\begin{align}
b_1 > a_1 & \geq 0; \\
b_2 & > b_1; \\
\vdots & \\
b_N & > a_N \geq 0.
\end{align}
\] (3.3)

Then one of the following statements hold:
\[
\begin{align}
b_1a_1 & = a_1b_1 \\
b_2a_1 & \geq a_2b_1 \\
\vdots & \\
b_Na_1 & \geq a_Nb_1; \\
\end{align}
\]
\( \vdots \) (i)
\[
\begin{align}
b_1a_i & = a_ib_i \\
b_2a_i & \geq a_2b_i \\
\vdots & \\
b_Na_i & \geq a_Nb_i; \\
\end{align}
\]
\( \vdots \) (N)
\[
\begin{align}
b_1a_N & \geq a_1b_N \\
b_Na_1 & = a_Nb_N.
\end{align}
\]

Proof. To begin with, we may assume all \( a_i \)'s are strictly bigger than 0. Otherwise, for example, if \( a_N = 0 \), it is then easy to see that the statement \( (N - 1) \) implies the statement \( (N) \) and hence it reduces the case when there are only \( N - 1 \) pairs of \( (a_i, b_i) \) which satisfy (3.3). Therefore, the desired claim follows easily from induction (note that the case when \( N = 1 \) is trivial).

Let us write all these \( N \) statements in a more compact way.

| (1) | (2) | \cdots | (i) | \cdots | (N) |
|-----|-----|-----------|-----|-------|------|
| \( b_1a_1 = a_1b_1 \) | \( b_1a_2 \geq a_1b_2 \) | \cdots | \( b_1a_i \geq a_1b_i \) | \cdots | \( b_1a_N \geq a_1b_N \) |
| \( \vdots \) | \( \vdots \) | \cdots | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( b_1a_1 \geq a_1b_1 \) | \( b_1a_2 \geq a_1b_2 \) | \cdots | \( b_1a_i = a_i b_i \) | \cdots | \( b_1a_N \geq a_1 b_N \) |
| \( \vdots \) | \( \vdots \) | \cdots | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( b_Na_1 \geq a_Nb_1 \) | \( b_Na_2 \geq a_Nb_2 \) | \cdots | \( b_Na_i \geq a_Nb_i \) | \cdots | \( b_Na_N = a_Nb_N \) |

We prove the desired claim by contradiction and assume none of statements (1) \(-\) (N) hold.

Step 1: Since statement (1) fails, then without loss of generality, we may assume \( b_2a_1 \geq a_2b_1 \) fails, namely,
\[
(3.4) \quad b_2a_1 < a_2b_1.
\]
(otherwise, we just re-label these pairs). This implies the first estimate in statement (2) holds, which is \( b_1a_2 \geq a_1b_2 \). Let us view this by using the table above, which is

\[
\begin{array}{ll}
b_1a_1 = a_1b_1 & b_1a_2 \geq a_1b_2 \\
b_2a_1 < a_2b_1 & b_2a_2 = a_2b_2\
\end{array}
\]

Step 2: Since statement (2) fails, then without loss of generality, we may assume \( b_3a_2 \geq a_3b_2 \) fails, namely,
\[
(3.5) \quad b_3a_2 < a_3b_2.
\]
This means the following:

(2.a). The second estimate in statement (3) holds, which is \( b_2a_3 \geq a_2b_3 \);
(2, b). The first estimate in statement (3) also holds, which is $b_1a_3 \geq a_1b_3$. Indeed, since 

$$a_1a_2a_3b_1b_2b_3 = a_1a_2a_3b_1b_2b_3,$$

this together with (3.4) and (3.5) yields the desired claim.

In summary, for Step 2, we have the following table, where blue indicates true statements that follow from our argument and red indicates corrected statements induced by those we assumed to fail:

| $b_1a_1 = a_1b_1$ | $b_1a_2 \geq a_1b_2$ | $b_1a_3 \geq a_1b_3$ | $b_1a_4 \geq a_1b_4$ |
|-------------------|-------------------|-------------------|-------------------|
| $b_2a_1 < a_2b_1$ | $b_2a_2 = a_2b_2$ | $b_2a_3 \geq a_2b_3$ | $b_2a_4 \geq a_2b_4$ |
| $b_3a_2 < a_3b_2$ | $b_3a_3 = a_3b_3$ | $b_3a_4 \geq a_3b_4$ |
| $b_4a_3 < a_4b_3$ | $b_4a_4 = a_4b_4$ |

**Step 3:** Since statement (3) fails, then without loss of generality, we assume $b_4a_3 \geq a_4b_3$ fails, that is

(3.6) $b_4a_3 < a_4b_3$.

This means the following

(3, s). The first estimate in statement (4) holds, which is $b_3a_4 \geq a_3b_4$;

(3, b). The second estimate in statement (4) holds, which is $b_2a_4 \geq a_2b_4$. Indeed, this is due to (3.5), (3.6) and

$$a_2a_3a_4b_2b_3b_1 = a_2a_3a_4b_2b_3b_4;$$

(3, c). The first estimate in statement (4) holds, which is $b_1a_4 \geq a_1b_4$. Indeed, this is due to (3.4), (3.5), (3.6) and

$$a_1a_2a_3a_4b_1b_2b_3b_1 = a_1a_2a_3a_4b_1b_2b_3b_4.$$ 

In summary, for Step 3, we have the following table:

| $b_1a_1 = a_1b_1$ | $b_1a_2 \geq a_1b_2$ | $b_1a_3 \geq a_1b_3$ | $b_1a_4 \geq a_1b_4$ |
|-------------------|-------------------|-------------------|-------------------|
| $b_2a_1 < a_2b_1$ | $b_2a_2 = a_2b_2$ | $b_2a_3 \geq a_2b_3$ | $b_2a_4 \geq a_2b_4$ |
| $b_3a_2 < a_3b_2$ | $b_3a_3 = a_3b_3$ | $b_3a_4 \geq a_3b_4$ |
| $b_4a_3 < a_4b_3$ | $b_4a_4 = a_4b_4$ |

In general, we may assume at Step $i - 1$, there holds

| $b_1a_1 = a_1b_1$ | $b_1a_2 \geq a_1b_2$ | $\cdots$ | $b_1a_{i-1} \geq a_1b_{i-1}$ | $b_1a_i \geq a_1b_i$ |
|-------------------|-------------------|-----|-------------------|-------------------|
| $b_2a_1 < a_2b_1$ | $b_2a_2 = a_2b_2$ | $\cdots$ | $b_2a_{i-1} \geq a_2b_{i-1}$ | $b_2a_i \geq a_2b_i$ |
| $b_3a_2 < a_3b_2$ | $\cdots$ | $b_{i-1}a_{i-1} = a_{i-1}b_{i-1}$ | $b_{i-1}a_i \geq a_{i-1}b_i$ | $b_{i-1}a_i = a_{i-1}b_i$ |
| $\cdots$ | $b_{i-1}a_{i-1} < a_{i-1}b_{i-1}$ | $b_{i-1}a_i < a_{i-1}b_i$ |

That is, we have

(3.7) $b_{j+1}a_j < a_{j+1}b_j,$ \quad $j = 1, 2, \ldots, i - 1,$

which corresponds to the red part in the table above, and

$$b_{s'}a_s \geq a_{s'}b_s \quad \text{and} \quad b_s a_s = a_s b_s, \quad 1 \leq s \leq i, 1 \leq s' < s.$$

which corresponds to the blue part in the table above.

We now proceed to the next step.
Step $i$. Since the statement (i) fails, without loss of generality, we assume $b_{i+1}a_i \geq a_{i+1}b_i$ fails, that is
\[(3.8) \quad b_{i+1}a_i < a_{i+1}b_i.\]
This means the following:

(i.a) The $i$-th estimate in statement (i + 1) holds, that is, $b_i a_{i+1} \geq a_i b_{i+1}$;

(i.b) For any $j \in \{1, 2, \ldots, i-1\}$, the $j$-th statement (i + 1) holds, that is, $b_j a_{i+1} \geq a_j b_{i+1}$
holds. Indeed, this is due to the equation
\[a_j a_{j+1} \ldots a_{i+1} b_j b_{j+1} \ldots b_i = a_j a_{j+1} \ldots a_{i+1} b_j b_{j+1} \ldots b_i+1\]
holds the estimates
\[b_{\ell+1}a_{\ell} < a_{\ell+1}b_{\ell}, \quad \ell = j, \ldots, i.\]

In summary, this gives us the table for Step $i$.

| $b_i a_1 = a_1 b_1$ | $b_i a_2 = a_2 b_2$ | $\ldots$ | $b_i a_{i-1} = a_{i-1} b_{i-1}$ | $b_i a_i = a_i b_i$ | $b_i a_{i+1} = a_{i+1} b_{i+1}$ |
|----------------------|----------------------|------------|----------------------|----------------------|----------------------|
| $b_{i+1} a_1 = a_1 b_{i+1}$ | $b_{i+1} a_2 = a_2 b_{i+1}$ | $\ldots$ | $b_{i+1} a_{i-1} = a_{i-1} b_{i+1}$ | $b_{i+1} a_i = a_i b_{i+1}$ | $b_{i+1} a_{i+1} = a_{i+1} b_{i+1}$ |
| $b_{i+1} a_{i+1} = a_{i+1} b_{i+1}$ | $b_{i+1} a_{i+2} = a_{i+2} b_{i+1}$ | $\ldots$ | $b_{i+1} a_{i-1} = a_{i-1} b_{i+1}$ | $b_{i+1} a_i = a_i b_{i+1}$ | $b_{i+1} a_{i+1} = a_{i+1} b_{i+1}$ |
| $b_{i+1} a_i = a_i b_{i+1}$ | $b_{i+1} a_{i+1} = a_{i+1} b_{i+1}$ | $\ldots$ | $b_{i+1} a_{i-1} = a_{i-1} b_{i+1}$ | $b_{i+1} a_i = a_i b_{i+1}$ | $b_{i+1} a_{i+1} = a_{i+1} b_{i+1}$ |

Continuing the above procedure until $i = N - 1$, it is then clear that the table of Step $N - 1$ implies the statement (N) holds, which contradicts our assumption. \[\square\]

**Corollary 3.7.** Let $(m, n)$ be a semi-good pair and $\frac{\log m}{\log n} \in \mathbb{R} \setminus \mathbb{Q}$, then there exists some $a, b \in \mathbb{N}$, such that $(m^a, n^b)$ is a good pair.

**Proof.** First of all, note that there does not exist pairs of positive integers $(a, b)$, such that $m^a = n^b$, since $\frac{\log m}{\log n}$ is irrational. Consider the prime decomposition of $m$ and $n$ as in (3.2). Then Proposition 3.6 asserts that there exists a pair $(a_i, b_i)$ for some $i \in \{1, \ldots, N\}$, such that $(m^{a_i}, n^{b_i})$ is a good pair. \[\square\]

3.2. **Proof of the main result: the non-far number case.** In the second half of this section, we complete the proof of Theorem 2.1 under the assumption $\frac{1}{a}$ is not $m$-far. Recall from (3.2) that this means if $p$ is a prime factor of $n$, then $p$ also divides $m$. We make several reductions.

**Step I:** Make $(m, n)$ into a semi-good pair. This is simple. Indeed, we can just take some $a$ sufficiently large, such that

1. $m^a > n$;
2. $b_i a > a_i$ for all $1 \leq i \leq N$.

This is possible due to (3.2). To this end, it suffices to replace $m$ by $m^a$ and $n$ unchanged;

**Step II:** Make $(m, n)$ into a good pair. This is guaranteed by Corollary 3.7.

We make a remark that Theorem 2.1 is indeed invariant under the operation $(m, n) \rightarrow (m^a, n^b)$ for any integers $a, b \geq 1$. This is indeed due to the basic fact
that $D_m = D_m^a$ for any $a \geq 1$ (similarly, $D_n = D_n^b$ for any $b \geq 1$). Therefore, it suffices to prove Theorem 2.1 under the assumption where $(m, n)$ is a good pair.

For $\ell$ sufficiently large, we consider the point

$$P_\ell := \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1 + \frac{1}{n^{\ell}},$$

together with two $m$-adic intervals associated to it:

$$K_\ell := \left[ P_\ell - \frac{1}{m^{\ell}}, P_\ell \right] = \left[ \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1 + \frac{1}{n^{\ell}} - \frac{1}{m^{\ell}}, \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1 + \frac{1}{n^{\ell}} \right],$$

and

$$L_\ell := \left[ P_\ell, P_\ell + \frac{1}{m^{\ell}} \right] = \left[ \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1 + \frac{1}{n^{\ell}}, \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1 + \frac{1}{n^{\ell}} + \frac{1}{m^{\ell}} \right].$$

We collect several basic facts about $P_\ell, K_\ell$ and $L_\ell$:

1. $P_\ell = r(K_\ell) = l(L_\ell)$, where recall that for any interval $I$, $r(I)$ is the right endpoint and $l(I)$ is the left endpoint of $I$, respectively;

2. Both $K_\ell$ and $L_\ell$ are $m$-adic intervals with sidelength $m^{-\ell}$. Indeed, by the definition of good pair $n | m$, which suggests $\frac{1}{n^{\ell}} = \left(\frac{m}{n}\right)^{\ell}$, which gives the desired assertion;

3. $K_\ell$ and $L_\ell$ are $m$-adic siblings, that is, there exists some $m$-adic interval with sidelength $m^{-\ell+1}$, such that it contains both $K_\ell$ and $L_\ell$. Indeed, by Lemma 3.4 for each $\ell \geq 1$, there does not exists a $k \in \mathbb{N}$, such that

$$\left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1 + \frac{1}{n^{\ell}} = \frac{k}{m^{\ell-1}}.$$

Note that the solubility of the above equation is equivalent to the solubility of the equation (3.1) since $\left\lfloor \frac{\ell \log m}{\log n} \right\rfloor$ is a positive integer. This implies that $P_\ell$ can not be an endpoint for any $m$-adic intervals with sidelength $m^{-\ell+1}$, which implies $K_\ell$ and $L_\ell$ are $m$-adic siblings with a common $m$-adic parent $R$ with sidelength $m^{-\ell+1}$.

Finally, we show that the ratio

$$\frac{\mu(K_\ell)}{\mu(L_\ell)}$$

diverges when $\ell$ tends to infinity, which will then imply Theorem 2.1 with $\frac{1}{n}$ being assumed not $m$-far. Note that on the interval

$$\left[ \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - 1, \left\lfloor \frac{\ell \log m}{\log n} \right\rfloor \right],$$

the construction procedure presented in Section 2.1 repeats $\left\lfloor \frac{\ell \log m}{\log n} \right\rfloor$ times. In particular, this means that near $P_\ell$, we can find two $n$-adic intervals, which are

$$\tilde{K}_\ell := \left[ P_\ell - \frac{1}{n^{\ell}}, P_\ell \right]$$

and

$$\tilde{L}_\ell := \left[ P_\ell, P_\ell + \frac{1}{n^{\ell}} \right].$$
with the associated weights \( a^\ell b^{\frac{\ell \log m}{\log n}} \) and \( a^\ell \), respectively. Moreover, it is also easy to see that
\[
K_\ell \subseteq \bar{K}_\ell \quad \text{and} \quad L_\ell \subseteq \bar{L}_\ell.
\]
This is clear from the fact that
\[
\ell \frac{\log m}{\log n} \leq m^\ell.
\]
Therefore,
\[
\frac{\mu(K_\ell)}{\mu(L_\ell)} = a^\ell b^{\frac{\ell \log m}{\log n} - \ell} \cdot \frac{1}{m^\ell} = b^{\frac{\ell \log m}{\log n} - \ell}.
\]
It is clear that the last term diverges when \( \ell \) converges to infinity since \( m > n \) and
\[
\left\lfloor \frac{\ell \log m}{\log n} \right\rfloor - \ell \geq \left( \frac{\log m}{\log n} - 1 \right) \ell - 1.
\]
(see, Figure 5). The proof is complete.

Figure 5. The non-far number case: \( K_\ell, L_\ell, \bar{K}_\ell \) and \( \bar{L}_\ell \).

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