A new formulation of the linear sampling method: spatial resolution and post-processing

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Abstract. A new formulation of the linear sampling method is described, which requires the regularized solution of a single functional equation set in a direct sum of $L^2$ spaces. This new approach presents the following notable advantages: it is computationally more effective than the traditional implementation, since time consuming samplings of the Tikhonov minimum problem and of the generalized discrepancy equation are avoided; it allows a quantitative estimate of the spatial resolution achievable by the method; it facilitates a post-processing procedure for the optimal selection of the scatterer profile by means of edge detection techniques. The formulation is described in a two-dimensional framework and in the case of obstacle scattering, although generalizations to three dimensions and penetrable inhomogeneities are straightforward.

1. Introduction

The linear sampling method [12] provides an approach for the visualization of scatterers from measurements of the far-field pattern under fixed-frequency scattering conditions. The mathematical basis of this method [5] is given by the far-field equation: the latter is an ill-posed integral equation of the first kind, defined for each point in $\mathbb{R}^2$ or $\mathbb{R}^3$, whose integral kernel is the far-field pattern of the scattered field and whose datum is a known analytical function. A basic theorem, from now on named the general theorem, states that there exists an approximate solution to the far-field equation such that its $L^2$-norm blows up to infinity for all points approaching the boundary of the scatterer from inside and stays arbitrarily large outside. This behaviour inspires a visualization algorithm [14] based on the two following main steps: for each point of a computational grid covering the object to be detected

(i) determine an approximate solution of a discretized (and noisy) version of the far-field equation;

(ii) plot the value of the indicator function, which is an appropriate monotonically increasing or decreasing function $I : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ of the Euclidean norm of the approximate solution computed at the previous step (i); then, depending on the decreasing or increasing monotonicity of $I$, the profile of the scatterer will be highlighted by all the grid points in which the indicator function is respectively small or large in comparison with the surrounding points.
In step (i) of the method it is crucial to account for the ill-conditioning of the linear inverse problem of solving the discretized (and noisy) far-field equation (which is a consequence of the ill-posedness of its continuous and exact version). Hence we need to apply a regularization procedure: the most common one in this framework is Tikhonov method [19], in which the ill-conditioned or ill-posed equation is replaced by a convex minimum problem and the optimal trade-off between the stability of the solution and its reliability in reproducing the data function is realized by a judicious choice of the regularization parameter, according to some optimality criterion (typically, the generalized discrepancy principle by Morozov). In most traditional sampling implementations [6, 10, 11, 14, 18] the regularization algorithm is applied for each point in a computational grid containing the scatterer, so that a different optimal regularization parameter and a different regularized solution are determined for each one of these sampling points. More recently [7] heuristic recipes have been utilized for reducing this multiple regularization approach to a single procedure for the choice of the regularization parameter. In a new rigorous formulation of the method [2] the set of far-field equations parameterized over the grid has been replaced by a single functional equation formulated in a functional space that is the direct sum of many $L^2$ spaces. This “no-sampling” approach does not require to sample over the region containing the object, so that a single regularization procedure (Tikhonov method) is applied to the functional equation and a single regularization parameter is determined by means of some optimality criterion (generalized discrepancy principle). This entails the following three essential advantages. First, the computation of the method is now much faster, since pointwise regularization is avoided together with a big number of time consuming zero-finding processes for the generalized discrepancy function; this may have important implications in the case of three-dimensional anisotropic scattering, when the inversion requires a notable computational effort. Second, the indicator function now has an explicit analytical form and its Fourier transform, at least for reasonable choices of the function $I$ introduced above, can be rather easily computed; from this Fourier analysis, a quantitative estimate of the spatial resolution achievable by the method is now possible, according to the well-known Shannon theorem [16]. Third, knowing the analytical form of the indicator function facilitates, as explained soon below, the implementation of edge detection based on active contours.

Step (ii) of the linear sampling method contains a still open issue, which holds for both the “sampling” and the “no-sampling” implementation of the algorithm: indeed, a threshold criterion must be introduced in order to automatically decide which level curve of the indicator function best approaches the profile of the scatterer or, equivalently, what “small” (or “large”) means in this context. In order to address this problem, in the present paper we shall follow the approach introduced in [1] where an edge detection algorithm, based on active contours, is applied to the visualization maps provided by our linear “no-sampling” method. Active contours [8, 9, 17] essentially consist in the iterative minimization of a functional representing the energy of a deformable model which is fit to the profile. They have proved their effectiveness in addressing several issues in learning, filtering and problems in visual clutter, and found applications from computer graphics, user-interface design, medical imaging and robotics. As most edge detection techniques, also deformable models require at some stage the computation of the gradient of the visualization map. In the new “no-sampling” formulation of the linear sampling method, the pixel content distribution, given by the point values of the indicator function, is analytically known: it follows that also the gradient of the image can be analytically determined and thus the accuracy of the contour computation is increased.

For sake of simplicity, throughout the paper we shall only deal with two-dimensional inverse scattering by impenetrable objects satisfying Dirichlet boundary conditions; however, all the results concerning the “no-sampling” implementation of the linear sampling method hold for more general two-dimensional and three-dimensional (possibly inhomogeneous) scattering frameworks. Moreover, active surfaces [20] can be applied to three-dimensional volumes provided
by the linear sampling method: even in this case the use of the new “no-sampling” approach is in principle advantageous, by virtue of its higher computational effectiveness.

The plan of this paper is the following. In the next section we briefly recall the traditional implementation of the linear sampling method: to this end, we introduce some preliminary notations and state the general theorem from which the method itself gets its inspiration. In Section 3 we illustrate our “no-sampling” implementation of the linear sampling method, while in Section 4 we describe deformable models for edge detection and how they can be applied to the visualization maps provided by the “no-sampling” version of the method.

2. The traditional implementation of the linear sampling method

For each \( z \in \mathbb{R}^2 \), let us consider the following Fredholm integral equation of the first kind, called far-field equation:

\[
\int_{\Omega} u_\infty(\hat{x}; d) g_z(d) ds(d) = \Phi_\infty(\hat{x}, z),
\]

where: \( \Omega = \{ x \in \mathbb{R}^2 \mid |x| = 1 \} \); \( \hat{x} \) and \( d \) are unit vectors denoting the observation and incidence directions respectively; \( u_\infty(\hat{x}; d) \) is the far-field pattern associated with the scattered field; the unknown \( g_z(\cdot) \) is a function in \( L^2(\Omega) \) for each \( z \); finally, \( \Phi_\infty(\hat{x}, z) \) is the far-field pattern of the fundamental solution \( \Phi(x, z) \) of the Helmholtz equation in \( \mathbb{R}^2 \) and is given by

\[
\Phi_\infty(\hat{x}, z) = \frac{e^{\pi i/4}}{\sqrt{8\pi k}} e^{-ik\hat{x} \cdot z},
\]

where \( k \) is the wavenumber. We also introduce the far-field operator \( F : L^2(\Omega) \to L^2(\Omega) \) defined as

\[
(Fg(\cdot))(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d) \quad \forall g(\cdot) \in L^2(\Omega).
\]

Then, if we consider the case in which the scatterer is a perfect conductor satisfying the Dirichlet boundary condition and taking up an unknown spatial region which is the closure of a \( C^2 \) domain \( D \subset \mathbb{R}^2 \) such that \( \mathbb{R}^2 \setminus \bar{D} \) is connected, we can state the general theorem [5] at the basis of the linear sampling method in the following way.

**Theorem 2.1 [general theorem]** Let us assume that \( k^2 \) is not an eigenvalue for the negative Laplacian in \( D \subset \mathbb{R}^2 \). Then, if \( F \) is the far-field operator (3), we have that

(i) if \( z \in D \), then for every \( \epsilon > 0 \) there exists a solution \( g_z(\cdot) \in L^2(\Omega) \) of the inequality

\[
\|(Fg_z(\cdot))(\cdot) - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon
\]

such that

\[
\lim_{z \to \partial D} \|g_z(\cdot)\|_{L^2(\Omega)} = \infty
\]

and

\[
\lim_{z \to \partial D} \|v_{g_z(\cdot)}(\cdot)\|_{H^1(D)} = \infty,
\]

where \( v_{g_z(\cdot)}(\cdot) \) is the Herglotz wave function with kernel \( g_z(\cdot) \);

(ii) if \( z \notin D \), then for every \( \epsilon > 0 \), every solution \( g_z(\cdot) \in L^2(\Omega) \) of the inequality

\[
\|(Fg_z(\cdot))(\cdot) - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon
\]

is such that

\[
\lim_{\epsilon \to 0} \|g_z(\cdot)\|_{L^2(\Omega)} = \infty
\]

and

\[
\lim_{\epsilon \to 0} \|v_{g_z(\cdot)}(\cdot)\|_{H^1(D)} = \infty.
\]
In a real scattering experiment, we can assume that the far-field pattern \( u_\infty \) of the scattered field is measured for \( N \) observation angles \( \{ \varphi_i \}_{i=0}^{N-1} \) and \( N \) incidence angles \( \{ \theta_j \}_{j=0}^{N-1} \), i.e. for observation and incidence directions respectively given by \( \{ \hat{x}_i = (\cos \varphi_i, \sin \varphi_i) \}_{i=0}^{N-1} \) and \( \{ d_j = (\cos \theta_j, \sin \theta_j) \}_{j=0}^{N-1} \), where

\[
\varphi_i = \frac{2\pi i}{N}, \quad \theta_j = \frac{2\pi j}{N}, \quad i, j = 0, \ldots, N - 1.
\]

These values are placed into the far-field matrix \( \mathbf{F} \), whose elements are defined as

\[
\mathbf{F}_{ij} := u_\infty(\hat{x}_i, d_j).
\]

In practical applications the far-field matrix is affected by the measurement noise, and therefore only a noisy version \( \mathbf{F}_h \) of the far-field matrix is at disposal, such that

\[
\mathbf{F}_h = \mathbf{F} + \mathbf{H},
\]

where \( \mathbf{H} \) is the noise matrix. If, as usual, we denote with \( \| \cdot \| \) the operatorial norm of a linear continuous operator and regard the matrix \( \mathbf{H} \) as a linear continuous operator in \( \mathbb{C}^N \), we assume to know that \( \| \mathbf{H} \| \leq h \).

We can now summarize the traditional implementation of the linear sampling method in the following recipe:

- take a grid \( \mathcal{Z} := \{ z_l \}_{l=0}^{L-1} \) of \( L \) sampling points covering a region which is known to contain the scatterer;

- for each grid point \( z_l = r_l(\cos \psi_l, \sin \psi_l) \in \mathcal{Z} \), perform a discretization of \( \Phi(\hat{x}, z) \) by defining the column vector

\[
\Phi_\infty(z_l) := \frac{\alpha^{\frac{1}{2}}}{\sqrt{8\pi k}} \left[ e^{-ikr_0 \cos(\varphi_0 - \psi_l)}, \ldots, e^{-ikr_{N-1} \cos(\varphi_{N-1} - \psi_l)} \right]^T;
\]

- for each grid point \( z_l \in \mathcal{Z} \), consider the following square and ill-conditioned linear system (obtained as a discretization of the far-field equation (1) over the incidence and observation angles) in the unknown \( \mathbf{g}(z_l) \):

\[
\mathbf{F}_h \mathbf{g}(z_l) = \frac{N}{2\pi} \Phi_\infty(z_l);
\]

- find the Tikhonov regularized solution \( \mathbf{g}_{\alpha(z_l)}(z_l) \) [19] of system (14), i.e. determine:

\[
\mathbf{g}_{\alpha(z_l)}(z_l) = \arg\min \left\{ \left\| \mathbf{F}_h \mathbf{g}(z_l) - \frac{N}{2\pi} \Phi_\infty(z_l) \right\|_{\mathbb{C}^N}^2 + \alpha(z_l) \| \mathbf{g}(z_l) \|_{\mathbb{C}^N}^2 \right\},
\]

where we have denoted with \( \| \cdot \|_{\mathbb{C}^N} \) the euclidean norm on \( \mathbb{C}^N \);

- for each grid point \( z_l \in \mathcal{Z} \), fix the optimal value \( \alpha^*(z_l) \) of the regularization parameter \( \alpha(z_l) \) by means of the generalized discrepancy principle [19];

- chosen a suitable monotonic continuous function \( I : \mathbb{R}^+ \cup \{0\} \to \mathbb{R} \), for each grid point \( z_l \in \mathcal{Z} \) plot, as indicator function, \( I(\| \mathbf{g}_{\alpha^*(z_l)}(z_l) \|_{\mathbb{C}^N}) \).

Then the linear sampling method, getting its inspiration from limits (5) and (8), consists in visualizing the scatterer profile as the set of all the grid points in which, depending on the decreasing or increasing monotonicity of \( I \), the indicator function is respectively small or large in comparison with the surrounding points. For sake of simplicity, here we take \( I(t) = t^2 \), i.e.
we choose the map \( z_l \mapsto \|g_{a^*(z_l)}(z_l)\|_{\mathbb{C}^N}^2 \) as our indicator function: it is defined over the grid \( \mathcal{Z} \) and it will be denoted with \( \Psi(z_l) \). An explicit form for this function can be determined by using the Singular Value Decomposition (SVD) \( \{\sigma_p^h, u_p^h, v_p^h\}_{p=0}^{r_h-1} \) of the far-field matrix \( F_h \), where \( r_h \) is the rank of \( F_h \). In fact, if \( (\cdot, \cdot)_{\mathbb{C}^N} \) is the scalar product in \( \mathbb{C}^N \), then the Tikhonov regularized solution is given by

\[
g_{a^*(z_l)}(z_l) = \frac{N}{2\pi} \sum_{p=0}^{r_h-1} \frac{\sigma_p^h}{(\sigma_p^h)^2 + \alpha^*(z_l)} (\Phi_\infty(z_l), v_p^h)_{\mathbb{C}^N} u_p^h \tag{16}
\]

and therefore the indicator function can be written as

\[
\Psi(z_l) := \|g_{a^*(z_l)}(z_l)\|_{\mathbb{C}^N}^2 = \frac{N^2}{4\pi^2} \sum_{p=0}^{r_h-1} \frac{(\sigma_p^h)^2}{((\sigma_p^h)^2 + \alpha^*(z_l))^2} \left| (\Phi_\infty(z_l), v_p^h)_{\mathbb{C}^N} \right|^2. \tag{17}
\]

### 3. A new implementation of the linear sampling method

The indicator function (17) is known only on the grid \( \mathcal{Z} \). On the other hand, the knowledge of its analytical form on \( \mathbb{R}^2 \) or, better, over a rectangle \( T_A^B := (-A, A) \times (-B, B) \) containing the scatterer, can open new perspectives on both the computational effectiveness of the method and the quantitative assessment of its performances in terms of spatial resolution, as we shall see in the following. Then we are interested in regarding expression (17) as a sampled version of a function \( \Psi(z) \) defined over \( T_A^B \); nevertheless, this is not at all a straightforward task, since the dependence of expression (17) on \( z_l \), and therefore on any \( z \), is explicit for \( \Phi_\infty(z_l) \), but only implicit, and in general not known explicitly, for \( \alpha^*(z_l) \).

In order to overcome this drawback, in the present section we propose a new implementation of the algorithm, again based on the general theorem 2.1, whereby the optimal value of the regularization parameter does not depend on \( z \in T_A^B \). The starting point is to replace the finite set of equations (14) by an infinite set of equations

\[
F_h g(z) = \frac{N}{2\pi} \Phi_\infty(z) \quad \forall z \in T_A^B. \tag{18}
\]

In this framework \( T_A^B \) can be regarded as a continuous grid, in which the generic sampling point \( z_l \) has become a continuous variable \( z \in T_A^B \). Then we want to modify the approach to the method from the pointwise algebraic setting represented in equation (18) to a unifying functional context, whereby regularization consists of a single procedure, which gives rise to a single value of the regularization parameter.

This result can be accomplished within the following mathematical framework. Let us consider the direct sum of Hilbert spaces:

\[
\left[ L^2(T_A^B) \right]^N := L^2(T_A^B) \oplus \ldots \oplus L^2(T_A^B),
\]

where \( L^2(T_A^B) \) denotes the usual set of Lebesgue square-integrable functions defined for almost all (f.a.a.) \( z \in T_A^B \) and with values in \( \mathbb{C} \). If \( f(\cdot) = \{f_i(\cdot)\}_{i=0}^{N-1} \) and \( g(\cdot) = \{g_i(\cdot)\}_{i=0}^{N-1} \) are two generic elements of \( \left[ L^2(T_A^B) \right]^N \), then this is a Hilbert space equipped with the scalar product

\[
(f(\cdot), g(\cdot))_{2,N} := \sum_{i=0}^{N-1} (f_i(\cdot), g_i(\cdot)) \quad \forall f(\cdot), g(\cdot) \in \left[ L^2(T_A^B) \right]^N, \tag{20}
\]
and the induced norm
\[ \|f(\cdot)\|_{2,N} = \sqrt{\int_{T^R_A} \|f(z)\|^2_{C^N} dz}. \]  

(21)

**Definition 3.1** The linear operator \( F_h : \left[ L^2(T^R_A)^N \right] \rightarrow \left[ L^2(T^R_A)^N \right] \) is given by
\[ [F_h g(\cdot)](\cdot) := \left\{ \sum_{j=0}^{N-1} (F_h)_{ij} g_j(\cdot) \right\}^{N-1}_{i=0}, \quad \forall g(\cdot) \in \left[ L^2(T^R_A)^N \right], \]

(22)

where the \((F_h)_{ij}\) are the elements of the noisy far-field matrix.

It can be proved [2] that the operator \( F_h \) is continuous and \( \|F_h\| = \sigma_h^0 \): this implies that \( \|F_h\| = \|F_h\| \) and therefore the bound \( h \) on the noise affecting the matrix \( F_h \) is the same as the one on the noise affecting the operator \( F_h \).

From a practical viewpoint, the introduction of the operator \( F_h \) allows one to express the infinitely many algebraic systems (18) as the single functional equation in \( [L^2(T^R_A)^N] \)
\[ [F_h g(\cdot)](\cdot) = \frac{N}{2\pi} \Phi_\infty(\cdot), \]

(23)

where \( \Phi_\infty(\cdot) \) is the element in \( [L^2(T^R_A)^N] \) trivially obtained from \( \Phi_\infty(z) \) simply regarding \( z \) as a variable on \( T^R_A \) instead of a fixed point in \( \mathbb{R}^2 \). The regularization of this equation occurs in a way which is independent from \( z \) and therefore provides a single value of the regularization parameter (we point out, however, that such a value will depend on the choice of the rectangle \( T^R_A \)). At this stage there is a final computational open issue to address, which is concerned with how to determine the Tikhonov regularized solution of equation (23) in practice. This problem has been solved in [2], by proving that the regularized solution in the new functional context is obtained from the regularized solution of (18) when \( z \) is thought as an independent variable, i.e.
\[ g_\alpha(\cdot) = \frac{N}{2\pi} \sum_{p=0}^{r_h-1} \frac{\sigma_p^h}{(\sigma_p^h)^2 + \alpha} \left( \Phi_\infty(\cdot), v_p^h \right)_{C^N} u_p^h, \]

(24)

where we have denoted with \((\Phi_\infty(\cdot), v_p^h)_{C^N}\) the element in \( L^2(T^R_A)^N \) defined f.a.a. \( z \in T^R_A \) as \((\Phi_\infty(z), v_p^h)_{C^N}\).

In expression (24) the parameter \( \alpha \) is a generic real positive number which can be fixed by applying (only once) the generalized discrepancy principle in this new functional context. If \( \alpha^* \) is the optimal value provided by this criterion, then
\[ g_{\alpha^*}(\cdot) = \frac{N}{2\pi} \sum_{p=0}^{r_h-1} \frac{\sigma_p^h}{(\sigma_p^h)^2 + \alpha^*} \left( \Phi_\infty(\cdot), v_p^h \right)_{C^N} u_p^h \]

(25)

is the optimal regularized solution of the functional problem (23) and the new indicator function is
\[ \Psi(z) := \|g_{\alpha^*}(z)\|^2_{C^N} = \frac{N^2}{4\pi^2} \sum_{p=0}^{r_h-1} \frac{(\sigma_p^h)^2}{((\sigma_p^h)^2 + \alpha^*)^2} \left| \left( \Phi_\infty(z), v_p^h \right)_{C^N} \right|^2 \]

(26)

We point out that in the analytical expression (26) no sampling is performed, since \( \alpha^* \) does not depend on \( z \) and all the other terms are known once the observed far-field matrix is at disposal;
then, in particular, the problem of deciding the parameters of the optimal grid containing the scatterer (i.e. number of points and/or sampling distances) is removed.

At this point we can conceive a new implementation of the linear sampling method, whereby the contour of the scatterer is detected, for example, by all the points in which the indicator function (26) becomes mostly large. Numerical simulations show [2] that the visualizations of the scatterer profiles obtained by means of the traditional implementation of the linear sampling method are essentially indistinguishable from the ones obtained with our new “no-sampling” implementation, which consequently does not seem to improve the performances of the method itself. However, the advantages of the new approach are three-fold and have been already explained in the introduction; in particular, as far as an estimate of the achievable resolution is concerned, we observe that the knowledge of the analytical expression (26) of $\Psi(z)$ in $T^R_A$ allows one to realize that it can be analytically continued onto all $\mathbb{R}^2$ to a function, still denoted with $\Psi(z)$, whose Fourier transform is quite easily computable. To this end, we introduce the polar coordinates of the $N$ components $v^h_{p,j}$ of each vector $v^h_p$, i.e.

$$v^h_{p,0} = (\rho^h_{p,0}, \epsilon^h_{p,0}) = \rho^h_{p,0} e^{i \epsilon^h_{p,0}},$$

$$\ldots$$

$$v^h_{p,N-1} = (\rho^h_{p,N-1}, \epsilon^h_{p,N-1}) = \rho^h_{p,N-1} e^{i \epsilon^h_{p,N-1}};$$

then, by means of some algebraic computations and putting $z = (z_1, z_2) \in \mathbb{R}^2$, one gets the following expression for $\Psi(z)$:

$$\Psi(z_1, z_2) = \sum_{p=0}^{r_h-1} \sum_{i,j=0}^{N-1} R^h_{p,i,j} \cos \left( \hat{\omega}_{1,i,j} z_1 + \hat{\omega}_{2,i,j} z_2 + \Delta^h_{p,i,j} \right),$$

where

$$R^h_{p,i,j} := \frac{N^2}{32\pi^2 \rho^2_p} \cdot \frac{(\sigma^h_p)^2}{(\sigma^h_p)^2 + \alpha^2} \cdot \rho^h_{p,i} \rho^h_{p,j}$$

and

$$\hat{\omega}_{1,i,j} := k(\cos \varphi_i - \cos \varphi_j), \quad \hat{\omega}_{2,i,j} := k(\sin \varphi_i - \sin \varphi_j), \quad \Delta^h_{p,i,j} := \epsilon^h_{p,i} - \epsilon^h_{p,j}. \tag{30}$$

If we adopt the following definition

$$F(f)(\omega_1, \omega_2) := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1, x_2) e^{-i(\omega_1 x_1 + \omega_2 x_2)} dx_1 dx_2$$

for the Fourier transform, it follows that the Fourier transform of function (28) is

$$F(\Psi)(\omega_1, \omega_2) = \sum_{p=0}^{r_h-1} \sum_{i,j=0}^{N-1} R^h_{p,i,j} \cdot \left\{ 2\pi^2 \cos \Delta^h_{p,i,j} \left[ \delta(\omega_1 - \hat{\omega}_{1,i,j}) \delta(\omega_2 - \hat{\omega}_{2,i,j}) + \delta(\omega_1 + \hat{\omega}_{1,i,j}) \delta(\omega_2 + \hat{\omega}_{2,i,j}) \right] + \right.$$  

$$
\left. + 2i\pi^2 \sin \Delta^h_{p,i,j} \left[ \delta(\omega_1 - \hat{\omega}_{1,i,j}) \delta(\omega_2 - \hat{\omega}_{2,i,j}) - \delta(\omega_1 + \hat{\omega}_{1,i,j}) \delta(\omega_2 + \hat{\omega}_{2,i,j}) \right] \right\}. \tag{32}$$

The previous expression (32) shows that the Fourier transform of the indicator function analytically extended to all $\mathbb{R}^2$ is a distribution, which can be named “Dirac brush”, whose support (which is, by definition, the band of $\Psi(z)$) is the compact set

$$S = \{(\hat{\omega}_{1,i,j}, \hat{\omega}_{2,i,j}) \}_{i,j=0}^{N-1}. \tag{33}$$
We point out that the support (33) of distribution (32) is independent of the scatterer and only depends on the wavenumber \( k \) and on the number \( N \) of the observation/incidence angles (of course, the Fourier transform of the indicator function does depend on the scatterer characteristics, in particular through the singular system of the far-field matrix). Moreover, if we define \( \Omega_1 := \max_{i,j} [\hat{\omega}_{i,j}] \) and \( \Omega_2 := \max_{i,j} [\hat{\omega}_{i,j}] \), we can say that the indicator function (26) is \((\Omega_1, \Omega_2)\)-bandlimited. Hence, if we put

\[
sinc(t) := \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0, \end{cases}
\]  

the following Shannon-Nyquist representation of \( \Psi(z) \) [3, 4]

\[
\Psi(z_1, z_2) = \sum_{n_1, n_2 = -\infty}^{+\infty} \Psi(n_1 d_1, n_2 d_2) \text{sinc} \left( \frac{z_1 - n_1 d_1}{d_1} \right) \text{sinc} \left( \frac{z_2 - n_2 d_2}{d_2} \right)
\]  

(35)

holds, provided that the sampling distances \( d_1 \) and \( d_2 \) along the \( z_1 \)-axis and \( z_2 \)-axis respectively satisfy the following conditions:

\[
d_1 < \frac{\pi}{\Omega_1}, \quad d_2 < \frac{\pi}{\Omega_2},
\]  

(36)

where \( \frac{\pi}{\Omega_1} \) and \( \frac{\pi}{\Omega_2} \) are called Nyquist distances. For example, if \( N \) is a multiple of 4, then \( \Omega_1 = \Omega_2 = 2k \) and therefore the Nyquist distances are \( \frac{\pi}{\Omega_1} = \frac{\pi}{\Omega_2} = \frac{\lambda}{4} \), where \( \lambda = \frac{2\pi}{k} \) is the wavelength. If \( N \) is not a multiple of 4, then \( \frac{\pi}{4} \) is a strict lower bound for the Nyquist distance.

The previous results provide some hints about how to estimate the ability of the linear sampling method to recover close objects from the superposition of their noisy discretized far-field patterns. Indeed, the Shannon-Nyquist representation (35) implies that \( \Psi(z) \) cannot vary significantly on length scales smaller than the Nyquist distance \( \lambda/4 \), since such representation consists of a superposition of sinc-like functions that are peaked at a distance smaller than \( \lambda/4 \) from one another and are very smooth between adjacent sampling points. On the other hand, it should be pointed out that, in whatever implementation of the linear sampling method, the visualization of the scatterer profile is obtained by choosing a cut-off section for the 3D plot of \( \Psi(z) \) and the relation between the Nyquist distance for \( \Psi(z) \) and the spatial resolution achievable for the scatterer profile on this section is still an open issue. Therefore the value \( \lambda/4 \) is only an approximate estimate of the true resolution power achievable by the method.

4. Deformable models

Deformable models provide an effective tool for detecting edges in an image. From a geometrical viewpoint, a deformable contour is a curve (although in the applications we have in mind the curve is closed, the general approach we are sketching does not require this hypothesis) \( \gamma^0 : [0, 1] \rightarrow \mathbb{R}^2 \), while a deformable model is a couple formed by a space \( A_d \) of admissible deformations of \( \gamma^0 \) and a functional \( \mathcal{E} : A_d \rightarrow \mathbb{R} \) to be minimized. This functional represents the energy of the model and has the following form:

\[
\mathcal{E}(\gamma) := \int_0^1 \left[ \frac{1}{2} \left( w_1(s) \| \gamma'(s) \|_{\mathbb{R}^2}^2 + w_2(s) \| \gamma''(s) \|_{\mathbb{R}^2}^2 \right) + E_{ext}(\gamma(s)) \right] ds,
\]  

(37)

where
(i) the maps \( w_1 : [0, 1] \to \mathbb{R} \) and \( w_2 : [0, 1] \to \mathbb{R} \) are weight functions that respectively control the importance of the first-order and second-order terms imposing the regularity of the curve; their choice determines the mechanical properties or, more precisely, the internal forces, i.e. elasticity and rigidity respectively, of the model;

(ii) \( E_{ext} \) denotes the potential energy associated to the external forces deriving from the image map and pushing the curve to the significant lines which correspond to the desired attributes (i.e., in our case, edges); in our LSM-oriented application, we choose \([8, 9, 17]\)

\[
E_{ext} := -\|\nabla J(\Psi)\|_{\mathbb{R}^2}^2,
\]

where \( J(\Psi) \) is an equivalent and shorter way for denoting the generic indicator function \( I(\|g_{w^2}(z)\|_{\mathbb{R}^2}) \), provided that \( J = I \circ \sqrt{\cdot} \).

The functions \( w_1, w_2, \gamma \) and \( E_{ext} \) are assumed to be smooth enough for computational purposes. Moreover, we shall restrict the space \( A_d \) of admissible deformations by assigning the boundary conditions \( \gamma(0), \gamma'(0), \gamma(1) \) and \( \gamma'(1) \); we can also use periodic curves or, in general, other kinds of boundary conditions.

A necessary condition for \( \gamma \) to be a local minimum for the functional \( E(\gamma) \) is that [15] it satisfies the Euler (vectorial) equation:

\[
(w_1(s) \gamma'(s))'' - (w_2(s) \gamma''(s))'' - \nabla E_{ext}(\gamma(s)) = 0
\]

with given boundary conditions \( \gamma(0), \gamma'(0), \gamma(1) \) and \( \gamma'(1) \). In order to determine the solution to (39), we replace it with the evolution problem

\[
\begin{align*}
\frac{\partial}{\partial t} \gamma(s, t) &= (w_1(s) \gamma'(s, t))' - (w_2(s) \gamma''(s, t))'' - G(\gamma(s, t)), \\
\gamma(s, 0) &= \gamma^0(s), \\
\gamma(0, t) &= \gamma^0(0), \quad \gamma(1, t) = \gamma^0(1), \\
\gamma'(0, t) &= (\gamma^0)'(0), \quad \gamma'(1, t) = (\gamma^0)'(0),
\end{align*}
\]

where the boundary or initial conditions (40)(b)-(d) impose that \( \gamma \) is “close” enough to the initial guess \( \gamma^0 \). The term \( G \) in (40)(a) is a modification of \( \nabla E_{ext} \), which is introduced to reduce the instabilities arising with the discretization. In our application we assume

\[
G(\gamma(s)) := -\kappa \frac{\nabla E_{ext}(\gamma(s))}{\sup_{\gamma} \|\nabla E_{ext}(\gamma(s))\|_{\mathbb{R}^2}},
\]

for an appropriate \( \kappa \in \mathbb{R}^+ \), but other choices are possible [9]. A numerical solution to (39) can be found by discretizing, in both \( s \) and \( t \), equation (40)(a) and solving iteratively (in \( t \)) the discretized system until a stable solution is found [17]. We discretize the time domain with the time points \( t_j = j \Delta t \), for \( j = 0, 1, \ldots \), and the space domain with \( M \) equidistant knots \( s_i = i \Delta s \), for \( i = 0, \ldots, M - 1 \) and a distance \( \Delta s = 1/M \). Then, if we put \( \gamma^j_i := \gamma(s_i, t_j) \), \( w_{1i} := w_1(s_i) \), \( w_{2i} := w_2(s_i) \), we can write the finite difference approximation of the equation (40)(a) as:

\[
\frac{\gamma^j_i - \gamma^{j-1}_i}{\Delta t} = \frac{1}{\Delta s^2} [w_{1i+1}(\gamma^j_{i+1} - \gamma^j_i) - w_{1i}(\gamma^j_i - \gamma^{j-1}_i)] + \\
- \frac{1}{\Delta s^2} [w_{2i-1}(\gamma^j_{i-2} - 2\gamma^j_{i-1} + \gamma^j_i) - 2w_{2i}(\gamma^j_{i-1} - 2\gamma^j_i + \gamma^j_{i+1}) + \\
w_{2i+1}(\gamma^j_i - 2\gamma^j_{i+1} + \gamma^j_{i+2})] + G(\gamma^j_i).
\]
With the approximation $G(\gamma^j_1) \simeq G(\gamma^j_{1-1})$, equation (42) has solution

$$\gamma^j = (I - A)^{-1}(\gamma^{j-1} + \Delta t \, G(\gamma^{j-1})),$$  \hfill (43)

where $\gamma^j := (\gamma^j_1, \ldots, \gamma^j_{M-1})$ and $I - A$ is a diagonally dominant, banded and sparse matrix, whose entries depend on the weights $w_1$ and $w_2$, as well as on the spatial discretization step $\Delta s$.

Numerical analysis considerations show that reliable results are obtained when $w_1$ and $w_2$ are constants and of the order of $(\Delta s)^2$ and $(\Delta s)^4$ respectively, while $\kappa \Delta t$ is smaller than $\Delta s$ [9]. A computational recipe for choosing $w_1, w_2$ and $\kappa$ in an automatic way for all numerical tests is the following: $w_1 = b_1 (\Delta s)^2$, $w_2 = b_2 (\Delta s)^4$, $\kappa = c \Delta s^2$, where $b_1, b_2$ and $c$ are arbitrarily chosen in these intervals: $b_1, b_2 \in (1, 10)$, $c \in (0, 1)$.

Finally, we point out that in the “no-sampling” implementation of the linear sampling method the function $G : T \to \mathbb{R}$ can be analytically determined. In fact

$$\nabla E_{ext} = \nabla \left[ - (J'(\Psi))^2 (\Psi^2_{z_1} + \Psi^2_{z_2}) \right] = (f_1, f_2),$$  \hfill (44)

with, for any $z = (z_1, z_2) \in \mathbb{R}^2$ and for any $m, n \in \{1, 2\}$,

$$f_m := -2 J'(\Psi) [J''(\Psi) \Psi_{zm}(\Psi^2_{z_1} + \Psi^2_{z_2}) + J'(\Psi) (\Psi_{z_1} \Psi_{z_1 z_m} + \Psi_{z_2} \Psi_{z_2 z_m})],$$  \hfill (45)

where an usual notation for partial derivatives is adopted, i.e. $\Psi_{zm} := \frac{\partial \Psi}{\partial z_m}$, $\Psi_{z_m z_n} := \frac{\partial^2 \Psi}{\partial z_m \partial z_n}$. By using the explicit form (28) of $\Psi(z)$, we can analytically compute the partial derivatives in (45):

$$\Psi_{zm}(z_1, z_2) = - \sum_{p,i,j=0}^{N-1} R^h_{p,ij} \hat{\omega}_{m,ij} \sin \left( \hat{\omega}_{1,ij} z_1 + \hat{\omega}_{2,ij} z_2 + \Delta \hat{e}_{p,ij} \right),$$

$$\Psi_{z_m z_n}(z_1, z_2) = - \sum_{p,i,j=0}^{N-1} R^h_{p,ij} \hat{\omega}_{m,ij} \hat{\omega}_{n,ij} \cos \left( \hat{\omega}_{1,ij} z_1 + \hat{\omega}_{2,ij} z_2 + \Delta \hat{e}_{p,ij} \right).$$

Of course, relation (45) can be made much simpler if $J$ is chosen as the identity in $\mathbb{R}^+ \cup \{0\}$: this clearly corresponds to taking $\Psi(z)$ as indicator function and, in such a case, definition (45) becomes $f_m := -2 (\Psi_{z_1} \Psi_{z_1 z_m} + \Psi_{z_2} \Psi_{z_2 z_m})$.

We now show an example of edge detection applied to a visualization map provided by our “no-sampling” method in a numerical simulation. The scatterer is a conducting kite, whose boundary is described by the following parametric equations:

$$x_1(t) = 1.5 \sin(t), \quad x_2(t) = \cos(t) + 0.65 \cos(2t) - 1, \quad t \in [0, 2\pi].$$  \hfill (46)

We employ incident plane waves having wavenumber $k = 1$ and coming from $N = 15$ incidence directions uniformly distributed over $[0, 2\pi]$. The far-field pattern is computed, by means of the Nyström method [13], at the same 15 angles and 10% Gaussian noise is added to each entry of the far-field matrix. The function $J$ is chosen as the opposite of the logarithm normalized between 0 and 1. Figure 1 shows the axial view of the indicator function with, superimposed, the initial guess (a circumference of radius 2) and some iterations of the deformable model algorithm; Figure 2 presents a comparison between the original profile (dotted) and the one provided by the last iteration (solid) shown in the previous figure.

To give an idea about the computational speed of the “no-sampling” approach, we point out that the axial view of the indicator function in Figure 1 is obtained after a CPU time of around 1s, while, under the same conditions, a traditionally implemented linear sampling code would employ around 60s of CPU time in the case of a grid formed by $60 \times 60$ equidistant points (both experiments have been performed on a PC with 1.73 GHz clock).
Figure 1. Axial view of the indicator function for a conducting kite and, superimposed, some iterations of the active contours algorithm. This figure has been taken from [1]

Figure 2. True (dotted) and reconstructed (solid) profile of the conducting kite. This figure has been taken from [1]

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