A Multi-set Identity for Partitions

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Guo-Niu Han kindly pointed out to us (something that we should have noticed ourselves if we would have been in the habit of reading care\(^\text{fully}\) all the papers that we cite), that our main result is contained in [B.H.].

Introduction

Given an integer-partition \(\lambda \vdash n\) and a box (a cell) \(v = [i,j] \in \lambda\) it determines the arm length \(a_v\) \((= \lambda_i - j)\), the leg length \(l_v\) \((= \lambda'_j - i)\), and the left length \(f_v\) \((= j - 1)\). Thus, for example, the hook length \(h_v\) is given by \(h_v = a_v + l_v + 1\). Denote \(p_v = a_v + f_v + 1\). C. Bessenrodt [B], and R. Bacher and L. Manivel [B.M] (see also [B.H]) proved the following identity:

\[
\sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{h_v} = \sum_{\lambda \vdash n} \sum_{v \in \lambda} x^{p_v},
\]

which is equivalent to the multi-set identity:

\[
\bigcup_{\lambda \vdash n} \{h_v \mid v \in \lambda\} = \bigcup_{\lambda \vdash n} \{p_v \mid v \in \lambda\}.
\]

In this note we prove the following refinement of (2).

Fill \(v\) with a pair of numbers in two different ways:

**First Filling:** Fill \(v\) with \((a_v, l_v)\).

**Second Filling:** Fill \(v\) with \((a_v, f_v)\).

This yields the following two multi-sets of pairs:

\[
A_1(n) = \bigcup_{\lambda \vdash n} \{(a_v, l_v) \mid v \in \lambda\},
\]

\[
A_2(n) = \bigcup_{\lambda \vdash n} \{(a_v, f_v) \mid v \in \lambda\}.
\]

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Theorem 1: For all non-negative integers \( n \) we have the multi-set identity, \[ A_1(n) = A_2(n). \]

The proof here is by applying the technique of generating functions. Theorem 1 indicates that for each \( n \) there is a map \( \varphi \) on the cells of the partitions of \( n \), \( \varphi : v \rightarrow \varphi(v) \), such that \((a_v, f_v) = (a_{\varphi(v)}, l_{\varphi(v)})\). The construction of an explicit such \( \varphi \) – for all \( n \) – would yield a bijective proof of Theorem 1.

The proof.

As usual, \((z)_a := (1 - z)(1 - qz) \cdots (1 - q^{a-1}z)\).

The proof would follow from the following two lemmas.

Lemma 1: Let \( M_1(c, d)(n) \) be the number of times the pair \((c, d)\) shows up in \( A_1(n) \), then
\[
\sum_{n=0}^{\infty} M_1(c, d)(n)q^n = \frac{q^{c+d+1}}{1 - q^{c+d+1}} \cdot \frac{1}{(q)_\infty}. \tag{1}
\]

Lemma 2: Let \( M_2(c, d)(n) \) be the number of times the pair \((c, d)\) shows up in \( A_2(n) \), then
\[
\sum_{n=0}^{\infty} M_2(c, d)(n)q^n = \frac{q^{c+d+1}}{1 - q^{c+d+1}} \cdot \frac{1}{(q)_\infty}. \tag{2}
\]

Proof of Lemma 2: \( M_2(c, d)(n) \) counts the number of Ferrers diagrams of \( n \) where one of the cells that has (right) arm \( c \) and left-arm \( d \) is marked. Obviously it belongs to a row of length \( c + d + 1 \), and each such row has exactly one such cell. Hence this is the same as counting the number of Ferrers diagrams of \( n \) where one of the rows of length \( c + d + 1 \) is marked. We can construct such a Ferrers diagram (with any number of cells) by first drawing that row of length \( c + d + 1 \) (weight \( q^{c+d+1} \)) then putting below it an arbitrary Ferrers diagram with largest part \( \leq c + d + 1 \), whose generating function is \( 1/((1-q)(1-q^2) \cdots (1-q^{c+d+1})) \), and then placing above the above-mentioned fixed row any Ferrers diagram whose smallest part is \( \geq c + d + 1 \), whose generating function is \( 1/((1-q^{c+d+1})(1-q^{c+d+2}) \cdots) \). Combining, we get that the generating function of such marked creatures, which is the left side of (2), is the right side of (2) \( \square \).

Before proving Lemma 1 we have to recall certain basic facts from \( q \)-land.

Fact 1 (The \( q \)-Binomial Theorem [essentially Theorem 2.1 of [A]3]).
\[
\frac{1}{(z)_{a+1}} = \sum_{j=0}^{\infty} \frac{(q)_{a+j}}{(q)_a (q)_j} z^j.
\]

But the “conditions” \(|q| < 1, |t| < 1\), stated by Andrews, are, in our world-view, a category mistake.
(This is easily proved by induction on $a$).

When $a = \infty$ this simplifies to

**Fact 2**

\[
\frac{1}{(z)_{\infty}} = \sum_{j=0}^{\infty} \frac{z^j}{(q)_j}.
\]

**Fact 3**: The generating function for Ferrers diagrams bounded in an $m$ by $n$ rectangle is $\frac{(q)_{m+n}}{(q)_m (q)_n}$.

This is Proposition 1.3.19 in [St] and Theorem 3.1 of [A]. Here is a proof by induction of this elementary fact. Let the generating function be $F(m, n; q)$. Consider the last cell of the top row. If it is occupied, the generating function of these diagrams is $q^n F(m-1, n)$ (remove the fully-occupied top row), if it is not, it is $F(m, n-1)$ (delete the empty rightmost column), getting the recurrence $F(m, n; q) = q^n F(m-1, n; q) + F(m, n-1; q)$. Then verify that the same recurrence is satisfied by $\frac{(q)_{m+n}}{(q)_m (q)_n}$, and check the trivial initial conditions $m = 0$ and $n = 0$.

By sending $n$ to infinity we obtain

**Fact 4**: The generating function for Ferrers diagrams with parts bounded by $m$ is $\frac{1}{(q)_m}$. By conjugation, this is also the generating function for Ferrers diagrams with at most $m$ parts.

**Proof of Lemma 1**: The left-side of (1) is the generating function for Ferrers diagrams where one hook with arm-length $c$ and leg-length $d$ is marked. Let’s figure out the generating function (weight-enumerator) for all such $(c, d)$-hook-marked Ferrers diagrams.

Suppose the corner of that hook is at cell $(i+1, j+1)$ (i.e. the $(i+1)$-row and the $(j+1)$-column). Here $0 \leq i < \infty$ and $0 \leq j < \infty$. Let’s look at its anatomy. It consists of seven parts. (See diagram in [http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/TeunnaFerrers.html](http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/TeunnaFerrers.html)).

1. **Strictly left of and above** cell $(i+1, j+1)$. This is a fully occupied $i$ by $j$ rectangle with weight $q^{ij}$.

2. **Above the arm** (of length $c+1$). This is a fully occupied $i$ by $c+1$ rectangle with weight $q^{(c+1)i}$.

3. **To the left of the leg** (of length $d+1$). This is a fully occupied $d+1$ by $j$ rectangle with weight $q^{(d+1)j}$.

4. The Ferrers diagram with $\leq i$ rows lying **above and to the right** of the arm. By Fact 4, the generating function of this is $1/(q)_i$.

5. The Ferrers diagram with $\leq j$ columns lying **below and to the left** of the leg. By Fact 4, the generating function of this is $1/(q)_j$. 

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6. The hook itself. This gives generating function $q^{c+d+1}$.

7. The Ferrers diagram formed inside the hook, i.e. lying below the arm and to the right of the leg. By Fact 3 its generating function is $rac{(q)_{c+d}}{(q)_{c}(q)_{d}}$.

Combining, we see that the generating function for these $(c,d)$-hook-marked Ferrers diagrams is

$$
\frac{(q)_{c+d}}{(q)_{c}(q)_{d}} \cdot q^{c+d+1} \cdot q^{i+j+i(c+1)+j(d+1)} \cdot \frac{1}{(q)_{i}} \cdot \frac{1}{(q)_{j}}.
$$

Summing over all $0 \leq i, j < \infty$, we get that the generating function on the left of (1) equals

$$
\frac{(q)_{c+d}}{(q)_{c}(q)_{d}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{i+j+i(c+1)+j(d+1)} \cdot \frac{1}{(q)_{i}} \cdot \frac{1}{(q)_{j}}.
$$

by Fact 2 with $z = q^{d+i+1}$. This, in turn, equals

$$
\frac{(q)_{c+d}}{(q)_{c}(q)_{d}} \cdot q^{c+d+1} \cdot \sum_{i=0}^{\infty} \frac{1}{(q)_{i}} \cdot \sum_{j=0}^{\infty} q^{i(c+1)}(q)_{j}^{(d+1)} \cdot \frac{1}{(q)_{j}}
$$

by Fact 1 with $z = q^{c+1}$. Finally, this equals

$$
= \frac{(q)_{c+d}}{(q)_{c}(q)_{d}} \cdot q^{c+d+1} \cdot \frac{1}{(q)_{c}} \cdot \frac{1}{(q)_{c}^{c+1+d+1}};
$$

by Fact 1 with $z = q^{c+1}$. Finally, this equals

$$
= \frac{(q)_{c+d}}{(q)_{c}(q)_{d}} \cdot \frac{1}{(q)_{c}} \cdot \frac{1}{(q)_{c}^{c+1+d+1}};
$$

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