Introduction

Let $L$ be an ample line bundle of type $(d_1; d_2; \ldots; d_g)$ on an abelian variety $A$ of dimension $g$. Consider the associated rational map $\nu : A \dashrightarrow \mathbf{P} H^0(L)$. Suppose $L = M^n$, for some ample line bundle $M$ on $A$. Then Koizumi and Ohbuchi have shown that $L$ gives a projectively normal embedding if $n \geq 3$ and when $n = 2$, no point of $K(L)$ is a base point for $M$, (see [BL], 7.3.1). Consider the case when $g = 2$, and $L$ is an ample line bundle of type $(1; d)$ on $A$, i.e. $L \in \mathcal{M}^n$ for any ample line bundle $M$ on $A$, $n > 1$. Then it has been shown by Lazarsfeld (see [L]) that whenever $L$ is birational onto its image and $d$ odd and $d$ even, then $L$ gives a projectively normal embedding. We showed that if the Neron-Severi group of $A$, $\text{NS}(A)$, is $\mathbb{Z}$, generated by $L$, and $d$ even, then $L$ gives a projectively normal embedding, (see [I]).

In this article we show

Theorem 1.1 Suppose $L$ is an ample line bundle on a $g$-dimensional simple abelian variety $A$. If $h^0(L) > 2^g g!$, then $L$ gives a projectively normal embedding, for all $g \geq 1$.

Since projective normality is an open condition, our theorem is, therefore true for a generic pair $(A; L)$, as above.
We outline the proof of 1.1.

We first show that for abelian varieties, it is enough to show the surjectivity of the homomorphism

$$\text{Sym}^2 H^0(L) \to H^0(L^2)$$

to give a projectively normal embedding, (see 2.3).

To show 2-normality and hence projective normality of $L$, we first consider a finite isogeny $A \to B = A/H$, where $H$ is a maximal isotropic subgroup of the mixed group of $L,K(L)$. Then $L$ descends down to a principal polarization $M$ on $B$. We then show that the surjectivity of the map $2$ is equivalent to showing the dual subgroup $H^0$ of $H$, in $B$ (P is $B$) generates the linear system of $M^2$ (and hence its translates also) i.e. the images of points of $H^0$, under the morphism $B \to K(B) \to M^2 j: b \mapsto t_b + t_{b^+}$, have their linear span as $jM^2 j$ for all $2 H^0$. (Here is the unique divisor in $jM j$ and $K(B)$ (respectively $K(B)$) denotes the Kummer variety of $B$ in $M^2 j$ (respectively in $jM^2 j$).

We show (see 3.2)

Proposition 1.2 Let $L$ be an ample line bundle on a simple abelian variety $Z$ and consider the associated rational map $Z \to \text{Pic}^0(L)$. Then any finite subgroup $G$ of $Z$, of order strictly greater than $h^0(L)g^+$, generates the linear system $\text{Pic}^0(L)$. More precisely, the points $L(g)$ where $g$ runs over all elements of $G$ not in the base locus of $L$ span $\text{Pic}^0(L)$.

We then apply above proposition to $L = tM^2$, to obtain bounds as asserted for a polarized abelian variety $(A;L)$, in 1.1.

Acknowledgements: We thank A. Hirschowitz and M. Waldschmidt for a useful conversation. We are grateful to Marc Hindry for a discussion which helped us in 3.1 and the referee for suggestions. We also thank Institut de Mathématiques, Univ. Paris-6, for their hospitality where this work was done and the French Ministry of National Education, Research and Technology and DFG ‘Arithmetik und Geometrie’ Essen, for their support.

Notations: Let $L$ be an ample line bundle on an abelian variety $Z$, of dimension $g$.

The mixed group of $L$ is $K(L) = f(a; Z): L \to t_a Lg; t_a : A \to A; x \mapsto a + x$.

The theta group of $L$ is $G(L) = f(a; L): L \to t_a Lg$. 

2
The Weil form $e^z: K(L) \times K(L) \to \mathbb{C}$, is the commutator map $(x; y) \mapsto x^y x^0 y^0 x^1 y^1$, for any lifts $x^0; y^0 \in \mathbb{G}(L)$, of $x; y \in K(L)$.

$h^0(L) = \dim H^0(Z; L)$

If $G$ is a finite subgroup of $Z$, then $\text{Card}(G) = \text{order}(G)$.

2 $r$-normality' of $L$

Consider an abelian variety $A$ of dimension $g$ and an ample line bundle $L$ on $A$.

Consider the multiplication maps

$$H^0(L)^r \to H^0(L^r); \text{ for } r \geq 2.$$

Definition 2.1 $L$ is said to be $r$-normal, if $r$ is surjective.

Definition 2.2 $L$ is normally generated, if $L$ is $r$-normal for all $r \geq 2$.

The main result of this section is the following.

Proposition 2.3 Suppose $L$ is an ample line bundle on an abelian variety $A$. If $L$ is $2$-normal, then $L$ is $r$-normal, for all $r \geq 2$. In particular, $L$ is normally generated.

Firstly, we will see

Proposition 2.4 Suppose $L$ and $M$ are ample line bundles on an abelian variety $A$.

1) The multiplication map

\[
\begin{array}{c}
X \\
\rightarrow \\
\downarrow \\
H^0(L) \quad H^0(M) \quad ! \quad H^0(L \cdot M)
\end{array}
\]

is surjective, for any open subset $U$ of $\text{Pic}^0(A)$.

2) If the multiplication map $H^0(L) \quad H^0(M) \quad ! \quad H^0(L \cdot M)$ is surjective, then the maps

\[
\begin{array}{c}
(a) \quad H^0(L) \quad H^0(M) \quad ! \quad H^0(L \cdot M)
\end{array}
\]

and

\[
\begin{array}{c}
(b) \quad H^0(L^1) \quad H^0(M) \quad ! \quad H^0(L \cdot M)
\end{array}
\]

are also surjective, for in some open subset $U$ of $\text{Pic}^0(A)$. 

3
Proof: 1) See [BL], 7.3.3.

2) Denote $\hat{A} = \text{Pic}^0(A)$. Consider the projections $p_h : A \to \hat{A}$ and $p_\chi : A \to \hat{A}$, and the sheaves $E_0 = p_\chi^{-1}(\pi_h L)$, $E_1 = p_\chi^{-1}(\pi_h L \cdot \pi^{-1})$, $E_2 = p_\chi^{-1}(\pi_h M \cdot \pi^{-1})$, where $\pi$ is the Poincare bundle on $A$. Since the sheaves $E_0(\cdot) = H^0(L)$, $E_1(\cdot) = H^0(L \cdot \pi^{-1}$) and $E_2(\cdot) = H^0(M \cdot \pi^{-1})$ have constant dimension, for all $2 \hat{A}$, $E_0$, $E_1$, and $E_2$ are vector bundles on $\hat{A}$.

Consider the natural maps

$$E_0 \to E_2 \stackrel{\pi_\chi}{\to} p_\chi^{-1}(\pi_h (L \cdot M \cdot \pi^{-1}))$$

and

$$E_1 \to E_2 \stackrel{\pi_\chi}{\to} H^0(L \cdot M \cdot \pi^{-1})$$

Since the map

$$\pi_2(0) = \pi_1(0) : H^0(L) \to H^0(M)$$

by assumption, is surjective, by semicontinuity, $\pi_2(\cdot)$ and $\pi_1(\cdot)$ are surjective, for in some open subset $U$ of $\hat{A}$. 2

Proof of 2.3: We prove by induction on $r$. Suppose the map $\pi : H^0(L) \to H^0(L \cdot \pi^{-1})$ is surjective.

Consider the map

$$H^0(L) \to H^0(L \cdot \pi^{-1}) \to H^0(L \cdot \pi^{-1} : H^0(L \cdot \pi^{-1}'))$$

To see the surjectivity of the map $\pi_{r+1} = \pi \cdot (\text{Id} \cdot \pi)$, we need to show that the map $\pi_{r+1}$ is surjective.

By 2.41),

$$H^0(L) \cdot H^0(L \cdot \pi^{-1}) = \bigotimes_{2U} H^0(L) \cdot H^0(L \cdot \pi^{-1} : H^0(L \cdot \pi^{-1}))$$

Since $L$ is 2-normal, by 2.42) (a),

$$H^0(L) \cdot H^0(L \cdot \pi^{-1}) = H^0(L \cdot \pi^{-1})$$

which implies (using 2.41)) that

$$H^0(L) \cdot H^0(L \cdot \pi^{-1}) = \bigotimes_{2U} H^0(L \cdot \pi^{-1}) \cdot H^0(L \cdot \pi^{-1})$$

which equals

$$H^0(L \cdot \pi^{-1}) = H^0(L \cdot \pi^{-1})$$

2
3 '2-normality' of L

Consider the multiplication map

\[ H^0(L) \otimes H^0(L) \to H^0(L^2) \]

This map factors via

\[ H^0(PH^0(L);O(2)) = \text{Sym}^2 H^0(L) \to H^0(L^2) \]

and \( \text{Ker } 2 = I_2 \) is the vector space of quadrics containing \( L(A) \) in \( PH^0(L) \).

We will use the following.

Proposition 3.1 Let \( D \) be an ample divisor on a \( g \)-dimensional simple abelian variety \( Z \). Suppose \( G \) is a finite subgroup of \( Z \), of order \( k \) \( (= \text{Card}(G)) \), in the sequel, and contained in \( D \). Then \( k \mid D^g \). (Here \( D^g \) denotes the self intersection number of \( D \), which by Riemann-Roch is \( h^0(O(D)) = k \).

Proof: Let \( G \mid Z \) be a subgroup of order \( k \) with \( G \mid D \). This implies that \( G \mid Y = 2gD + s \), where \( D_+ = fx + : x \to 2Dg \), and now \( Y \) is invariant under the group \( G \). If \( s = \dim Y = 0 \), then our proof ends here. Otherwise, let \( Y = Y_1 \mid \cdots \mid Y_r \), where \( Y_j \) are the irreducible components, with \( s_j = \dim Y_j \). Let \( Y_1 \) be an irreducible component of \( Y \), with \( \dim Y_1 = \dim Y \). Choose \( g \mid s \) translates \( D + h_j \), \( h_j \in 2Z \), of \( D \) which intersect properly with \( Y \), i.e. \( Y \setminus \bigcup_{j=1}^r Y_j + h_j \) is a finite set of points. Then \( Y_1 \setminus \bigcup_{j=1}^r Y_j + h_j \so \setminus \bigcup_{j=1}^r Y_j + h_j \) and hence \( \text{deg}Y_1 \mid \text{deg}Y \mid D^g \). (Here \( \text{deg}Y \) is the intersection number of \( Y \) with the class of \( D^g \), in \( H^2(Z) \)).

Since \( Y \) is \( G \)-invariant, \( 2gY_1 + Y \). Consider the subgroup \( G_{Y_1} = \text{fg} 2G : Y_1 + g = Y_1g \) of \( G \). Since \( F \bigcap_{g \in G_{Y_1}} \text{deg}(Y_1 + ) \) \( \text{deg}Y \) and \( \text{deg}Y_1 = \text{deg}Y_1 \) + we get the inequalities \( \text{Card} \big( G_{Y_1} \big) \text{deg}Y_1 \text{deg}Y \text{D}^g \), i.e. \( \text{Card}(G) \text{D}^g \big( \text{Card}(G_{Y_1}) \text{deg}Y_1 \big) \). To complete our proof, it suffices to show that \( \text{Card}(G_{Y_1}) \text{deg}Y_1 = 2 \). Now \( G_{Y_1} \mid \text{Stab}(Y_1) = \text{fa} 2Z : Y_1 + a = Y_1g \). Observe that \( \text{Stab}(Y_1) = \bigcup_{y \in Y_1} Y_1 \). Now for \( a \not\in 2Z \), \( \text{Stab}(Y_1) = (Y_1 \ y_0) \setminus \bigcup_{y \in Y_1} Y_1 \). Hence it is now clear that \( \text{degStab}(Y_1) \text{deg}Y_1 \). Since \( Z \) is simple, \( \text{Stab}(Y_1) \) is zero-dim ensional and we get \( \text{Card}(G_{Y_1}) \text{degStab}(Y_1) \text{deg}Y_1 \). We will require the geometric interpretation of 3.1, namely,
Proposition 3.2 Let \( L \) be an ample line bundle on a simple abelian variety \( Z \) and consider the associated rational map \( Z \to \mathbb{P} H^0(L) \). Then any finite subgroup \( G \) of \( Z \), of order strictly greater than \( h^0(L) \), generates the linear system \( \mathbb{P} H^0(L) \). More precisely, the points \( L(g) \) where \( g \) runs over all elements of \( G \) not in the base locus of \( L \) span \( \mathbb{P} H^0(L) \).

Proof of 1:1:

Consider a polarized simple abelian variety \((A;L)\), of type \((d_1; \ldots; d_g)\). Let \( H = K(L) \) be a maximal isotropic subgroup, for the Weil form \( \varepsilon \). Consider the isogeny \( A \to B = \frac{A}{H} \). Then \( L \) descends down to a principal polarization \( M \) on \( B \). We may assume \( M \) is symmetric, i.e., \( M \in M, i(b) = b \cdot b \cdot 2 \cdot B \). By Projection formula and using the fact that \( O_A = 2H \cdot L \), where \( L \) denotes the degree 0 line bundle on \( B \), corresponding to the character \( \varepsilon \) on \( H \), we get

\[
H^0(L) = 2H \cdot H^0(M \cdot L) = 2H \cdot H^0(t_{M}
\]

via the isomorphism \( B \to \text{Pic}^0(B), b \mapsto t_{M} + 1 \). Let \( M^1, H^0 = M^1(\varepsilon) \).

Similarly,

\[
H^0(L^2) = 2H \cdot H^0(t_{M^2})
\]

Consider the multiplication map

\[
\text{Sym}^2 H^0(L) \to H^0(L^2)
\]

Then \( \text{Sym}^2 H^0(L) = P : 2H \cdot H^0(t_{M}) \cdot H^0(t_{M}) \) and we can write the map \( 2 \) as

\[
\begin{array}{c}
\text{Sym}^2 H^0(L) \\
\to
\end{array}
\]

where

\[
X \cdot H^0(t_{M}) \cdot H^0(t_{M})
\]

To show surjectivity of \( 2 \), it is enough to show surjectivity of \( 1 \), for each \( 2H^0 \).

Now it is well known that the morphism associated to the line bundle \( M^2 \), embeds the Kummer variety of \( B, K(B) \), in the linear system \( \mathcal{M} \), and is given as

\[
\begin{array}{c}
B \to K(B) \\
\to \mathcal{M} \cdot t_{b} + t_{b}
\end{array}
\]

where \( t_{b} \) is the unique symmetric divisor in \( \mathcal{M} \).

Under translation by \( 2B \), we have the corresponding morphism

\[
\begin{array}{c}
B \to K(B) \\
\to \mathcal{M} \cdot t_{b} + t_{b}
\end{array}
\]

6
(Here \( K(B) = \text{Image}(B) \) in \( \mathbb{M}^2 \)). So showing surjectivity of \( \iota \) in (I) is equivalent to showing that the image of \( H^0 \) in \( K(B) \), generates the linear system \( \mathbb{M}^2 \) and hence the translates \( \mathbb{M}^2 \). Since the pair \((A;L)\) is a simple polarized abelian variety, with \( h^0(L) = \text{Card}(H^0) > 2^g \cdot \sharp! = h^0(\mathbb{M}^2) \cdot \sharp! \) by 3.2 and (I), each \( \iota \) is surjective. Hence, by 2.3, our proof is now complete.

Remark 3.3 Notice that if \( g = 1 \), any line bundle of degree strictly greater than 2 on an elliptic curve, gives projectively normal embedding. Hence the bound is sharp.

If \( g = 2 \) and \( L(1;d) \) is an ample line bundle on an abelian surface \( A \), with \( h^0(L) = 7;8 \), then by [I] and [L], we know \( L \) gives projectively normal embedding (generically). By the method of proof of 1.1, we cannot expect to obtain a sharp bound since if \( L \) were to be of type \((2;4)\), then it does not give a projectively normal embedding.

Consider the situation when \( g \geq 3 \). It is clear, by Kunneth, that if \( A = A_1 \ldots A_r \) and \( L = p_1L_1 \ldots p_rL_r \), where \((A_j;L_j)\) are polarized abelian varieties and \( L_j \) give projectively normal embedding then \( L \), of type \( \langle 1 \rangle \), also gives a projectively normal embedding. Hence for a generic pair \((A;L)\) of type \( \langle \rangle \), \( L \) gives a projectively normal embedding. This will show that there exists line bundles \( L \) with \( h^0(L) = 2^g \cdot \sharp! \) on \( A \), such that \( A \) is projectively normal in \( \text{PH}^0(L) \).

But we can hope to improve the bound in 3.1, for \( g > 1 \), by taking into consideration the structure of the finite subgroup \( H^0 \), which essentially distinguishes the type of \( L \), for instance, types \((1;8)\) and \((2;4)\), when \( g = 2 \). Here \( H^0 \) is in the former case and in the latter case either \( \frac{2}{22} \) or \( \frac{2}{42} \) satisfying certain condition with respect to the Weil form \( e^{i \cdot 2} \).

Also, in 3.1, if the ample divisor \( D \) is moreover a symmetric divisor, i.e. \( j(D) = D \), then the subset \( Y = \{ \mathcal{g}u \}^2 D + \mathbb{g} \) is in fact invariant for the action of \( G \). So one may get better bounds in some cases and hence for the pair \((A;L)\), in 1.1, (since all the divisors in the linear system of \( \mathbb{M}^2 \), are symmetric for the action of \( i \)).

References

[BL] Birkenhake, Ch., Lange, H.: Complex abelian varieties, Springer-Verlag, Berlin, (1992).
[I] Iyer, J.: Projective normality of abelian surfaces given by primitive line bundles, Manuscr Math. 98, 139-153, (1999).

[L] Lazarsfeld, R.: Projectivité normale des surface abéliennes, Redige par O Debarre. Prepublication No.14, Europroject-CIMA, Nice, (1990).