Cosmological Evolution in 1/R-Gravity Theory

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Recently, corrections of the \( L(R) \) type to Einstein-Hilbert action that become important at small curvature are proposed. Those type of models intend to explain the observed cosmic acceleration without dark energy. We derive the full Modified Friedmann equation in the Palatini formulation of those modified gravity model of the \( L(R) \) type. Then, we discuss various cosmological predictions of the Modified Friedmann equation.

1. Introduction

It now seems well-established that the expansion of our universe is currently in an accelerating phase. The most direct evidence for this is from the measurements of type Ia supernova \cite{1}. Other indirect evidences such as the observations of CMB by the WMAP satellite \cite{2}, large-scale galaxy surveys by 2dF and SDSS \cite{3} also seem supporting this.

But now the mechanisms responsible for this acceleration are not very clear. Many authors introduce a mysterious cosmic fluid called dark energy to explain this (see Ref.\cite{4} for a review). On the other hand, some authors suggest that maybe there does not exist such mysterious dark energy, but the observed cosmic acceleration is a signal of our first real lack of understanding of gravitational physics \cite{5}. An example is the braneworld theory of Dvali et al. \cite{6}. Recently, there are active discussions in this direction by modifying the action for gravity \cite{7-17}. Specifically, a \( 1/R \) term is suggested to be added to the action \cite{7}: the so called \( 1/R \) gravity. It is interesting that such term may be predicted by string/M-theory \cite{8}. In Ref.\cite{12}, Vollick used Palatini variational principle to derive the field equations for \( 1/R \) gravity. In Ref.\cite{11}, Dolgov et al. argued that the fourth order field equations following from the metric variation suffer serious instability problem. If this is indeed the case, the Palatini formulation appears quite appealing, because the second order field equations following from Palatini variation are free of this sort of instability \cite{13}. In our previous paper \cite{13}, we have derived the first and second order Modified Friedmann (MF) equations by directly computing the various order Ricci tensors. We have shown that the first and second order MF equation can fit the SNe Ia data at an acceptable level. However, for cosmological predictions involving the derivative of the Hubble parameter such as the deceleration parameter, there were obvious differences in the first order and second order MF equations. This indicated that to make predictions about those quantities, the approximated Friedmann equation is not suitable. In this paper, we will derive the full Modified Friedmann equation for Palatini formulation of the \( 1/R \) gravity. We will show that the full Modified Friedmann equation can fit the SNe Ia data at an acceptable level, which is not surprising from our previous study. But now we can compute exactly the quantities involving the derivatives of the Hubble parameter such as the deceleration parameter and effective equation of state. In this paper, when computing various cosmological predictions, we will always use \( H_0 = 70 \text{ km/(sec-Mpc)} \) as the current value of the Hubble parameter.

2. The Modified Friedmann Equation

The field equations follow from the variation in Palatini approach of the generalized Einstein-Hilbert action (See Ref.\cite{12} for details)

\[
S = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} L(R) + \int d^4x \sqrt{-g} L_M
\]

where \( \kappa = 8\pi G \), \( L \) is a function of the scalar curvature \( R \) and \( L_M \) is the Lagrangian density for matter. In the Palatini formulation, the connection is not associated with \( g_{\mu\nu} \), but with \( h_{\mu\nu} \equiv L'(R)g_{\mu\nu} \), which is known from varying the action with respect to \( \Gamma^\lambda_{\mu\nu} \).

The field equations in Palatini formulation are

\[
L'(R)R_{\mu\nu} - \frac{1}{2} L(R)g_{\mu\nu} = -\kappa T_{\mu\nu}
\]

\[
R_{\mu\nu} = R_{\mu\nu}(g) = \frac{3}{2} (L')^{-2} \nabla_\rho L' \nabla_\nu L' + (L')^{-1} \nabla_\rho \nabla_\nu L' + \frac{1}{2} (L')^{-1} g_{\mu\nu} \nabla_\sigma \nabla^\sigma L'
\]

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\[ R = R(g) + 3(L')^{-1} \nabla_{\mu} \nabla_{\nu} L' - \frac{3}{2} (L')^{-2} \nabla_{\mu} L' \nabla_{\nu} L' \]  \hspace{1cm} (4)

where a prime denotes differentiation with respect to \( R \), \( R_{\mu\nu}(g) \) is the Ricci tensor with respect to \( g_{\mu\nu} \) and \( R = g^{\mu\nu} R_{\mu\nu} \), \( T_{\mu\nu} \) is the energy-momentum tensor given by

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \]  \hspace{1cm} (5)

We assume that the universe contains dust and radiation, thus \( T_{\mu\nu} = \{-\rho_m - \rho_r, p_r, p_r, p_r\} \) where \( \rho_m \) and \( \rho_r \) are the energy densities for dust and radiation respectively, \( p_r \) is the pressure of the radiation. Note that \( T = g^{\mu\nu} T_{\mu\nu} = -\rho_m \) because of the relation \( p_r = \rho_r / 3 \).

Note by contracting Eq. (2), we get:

\[ L'(R)R - 2L(R) = -\kappa T \]  \hspace{1cm} (6)

Assume we can solve \( R \) as a function of \( T \) from Eq. (6). Thus Eqs. (3), (4) do define the Ricci tensor with respect to \( h_{\mu\nu} \).

Now consider the Robertson-Walker metric describing the cosmological evolution,

\[ ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (7)

We only consider a flat metric, which is favored by present observations \[2\].

From Eqs. (7), (8) we can get the non-vanishing components of the Ricci tensor:

\[ R_{00} = 3 \frac{\ddot{a}}{a} - \frac{3}{2} (L')^{-2} (\partial_0 L')^2 + \frac{3}{2} (L')^{-1} \nabla_0 \nabla_0 L' \]  \hspace{1cm} (8)

\[ R_{ij} = -[a \ddot{a} + 2 \dot{a}^2 + (L')^{-1} (\partial_0 L')^2 \partial_0 L' + \frac{\dot{a}^2}{2} (L')^{-1} \nabla_0 \nabla_0 L'] \delta_{ij} \]  \hspace{1cm} (9)

Substituting equations (8) and (9) into the field equations (2), we can get

\[ 6H^2 + 3H (L')^{-1} \partial_0 L' + \frac{3}{2} (L')^{-2} (\partial_0 L')^2 = \frac{\kappa (\rho + 3p) + \frac{L}{L'}}{L'} \]  \hspace{1cm} (10)

where \( H = \dot{a}/a \) is the Hubble parameter, \( \rho \) and \( p \) are the total energy density and total pressure respectively. This is the general form of the MF equation for Palatini formulation of the modified gravity of the \( L(R) \) type. Assume that we can solve \( R \) in term of \( T \) from Eq. (9) for a specifical model, substitute it into the expressions for \( L' \) and \( \partial_0 L' \), we can get the MF equation.

In this paper we will study the special \( L(R) \) suggested in Ref. \[5\], the so called \( 1/R \) gravity:

\[ L(R) = R - \frac{\alpha^2}{3R} \]  \hspace{1cm} (11)

where \( \alpha \) is a positive constant with the same dimension as \( R \) and following Ref. \[12\], the factor of \( 1/3 \) is introduced to simplify the field equations.

The field equations follow from Eq. (2)

\[ (1 + \frac{\alpha^2}{3R^2}) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - \frac{\alpha^2}{3R}) = -\kappa T_{\mu\nu} \]  \hspace{1cm} (12)

Contracting the indices gives

\[ R = \frac{1}{2} \alpha [\frac{\kappa T}{\alpha} - 2 \sqrt{1 + \frac{1}{4} (\frac{\kappa T}{\alpha})^2}] \]  \hspace{1cm} (13)

From Eqs. (12), (13) we can see that the field equations reduce to the Einstein equations if \( \kappa T \gg \alpha \). On the other hand, when \( \alpha \gg \kappa T \), deviations from the Einstein’s theory will be large. This is exactly the case we are interested in. We hope it can explain today’s cosmic acceleration.
From the conservation equation $\dot{\rho} + 3H\rho = 0$, we can get

$$\partial_0 L' = \frac{(\frac{\alpha}{\pi})^2 \kappa \rho}{\sqrt{1 + \frac{1}{4}(\frac{\alpha}{\pi})^2}} H$$

(14)

Substituting this into Eq. (10), we can get the Modified Friedmann (MF) equation in this modified gravity theory:

$$H^2 = \frac{\kappa \rho - \alpha (G(\frac{\alpha}{\pi}) - \frac{1}{3G(\frac{\alpha}{\pi})})}{(1 + \frac{1}{3G(\frac{\alpha}{\pi}))}(6 + 3F(\frac{\alpha}{\pi}))\sqrt{1 + \frac{1}{4}F(\frac{\alpha}{\pi})})$$

(15)

where the two functions $G$ and $F$ are defined as

$$G(x) = -\left(\frac{1}{2}x + \sqrt{1 + \frac{1}{4}x^2}\right)$$

(16)

$$F(x) = \frac{x}{(G(x)^2 + \frac{1}{4})\sqrt{1 + \frac{1}{4}x^2}}$$

(17)

From Eqs. (16), (17) we can see that when $x \gg 1, G(x) \sim -x, F(x) \sim 0$. Thus when $\kappa \rho \gg \alpha$, Eq. (15) reduces to the standard Friedmann equation

$$H^2 = \frac{\kappa}{3\rho}$$

(18)

This confirms the assertion we made below equation (13). Below we will use the full Modified Friedmann equation (15) to compute numerically some cosmological parameters, namely, age of the universe in Sec.3, redshift-luminosity relationship in Sec.4, deceleration parameter in Sec.5, effective equation of state in Sec.6.

3. Age of the universe

The age of the universe is given by

$$t_0 = \int_0^1 \frac{da}{aH(a)}$$

(19)

Thus from the MF equation (15) we can numerically draw the dependence of the age of the universe on $\Omega_m$, see Fig.1. If we take $t_0 > 10$ Gyr, this implies $H_0t_0 > 0.71$; if we take $t_0 > 11$ Gyr, this implies $H_0t_0 > 0.79$. From Fig.1 this correspond to $\Omega_m > 0.07$ and $\Omega_m > 0.1$ respectively.
FIG. 2: The dependence of luminosity distance on redshift computed from the MF equation (15). The solid, dotted, dashed lines correspond to $\Omega_m = 0.1, 0.2, 0.3$, respectively. The little crosses are the observed data.

FIG. 3: The dependence of the $\chi^2$ on the parameter $\Omega_m$. It can be seen that $\chi^2$ gets a little smaller for smaller value of $\Omega_m$.

4. Data Fitting with SNe Ia Observations

It is the observations of the SNe Ia that first reveal our universe is in an accelerating phase. It is still the most important evidence for acceleration and the best discriminator between different models to explain the acceleration. Thus, any model attempting to explain the acceleration should fit the SNe Ia data as the basic requirement.

We have shown in Ref. [13] that the first order MF equation fits the SNe Ia data at an acceptable level and the second order correction to the luminosity distance is small, thus we have concluded that this strongly implies that the full MF equation also fits the SNe Ia at an acceptable level. Now with the full MF equation in hand, we can explicitly show this is true and it can be checked numerically that the deviations between the full MF equation and the approximated MF equation is very small.

We will use data listed in Ref. [19], which contains 25 SNe Ia observations. Since our purpose is to show the MF equations fit the data at an acceptable level, also because the differences in various $\Omega_m$ get larger when redshift is larger and trustable observations around redshift 1 is very few, we think 25 low redshift samples is enough. Also we did not perform a detailed $\chi^2$ analysis, this is suitable when we get more high redshift samples.

Fig.2 shows the prediction of the full MF equation for $\Omega_m = 0.1, 0.2, 0.3$ respectively. Fig.3 shows the dependence of the $\chi^2$ (See Ref. [13] for the definition of this quantity) on the $\Omega_m$. We can see that
FIG. 4: The dependence of deceleration parameter on redshift for $\Omega_m = 0.1, 0.2, 0.3$ respectively. The solid, dotted, dash-dotted, dashed lines correspond to the full, first order, second order and third order MF equations respectively.

$\chi^2$ gets smaller for smaller value of $\Omega_m$. This property has already been observed when data fitting with the first order MF equation [13]. We think this is an interesting feature of this modified gravity theory: It has already provided the possibility of eliminating the necessity of dark energy for explanation of cosmic acceleration, now it seems that a universe without dark matter is favored when fit this model to SNe Ia data. Thus we boldly suggest that maybe this modified gravity theory can provide the possibility of eliminating dark matter. This is surely an interesting thing and worth some investigations.

However, we should also note that we only use SNe Ia data smaller than redshift 1 to draw the above conclusions. It can easily been seen in Fig.2 that the differences between predictions drawn from the different values of $\Omega_m$ become larger when redshift is around 1. Thus, future high redshift supernova observations may give a more conclusive discrimination between the parameters.

5. The Deceleration Parameter

Given the observation that our universe is currently expanding in an accelerating phase, the deceleration parameter should become negative in recent cosmological times. The deceleration parameter is defined by

$$q = \frac{-\ddot{a}}{aH^2}.$$

The deceleration parameter for the MF equation (15) is too complicated and thus we will not present its analytic expression here. We just draw it in Fig.4. In Fig.4 it can be also noted that the redshift $z_q=0$ which gives $q(z_q=0) = 0$ gets smaller for larger value of $\Omega_m$. Current observation of SNe Ia gives a constraint: $0.6 < z_q=0 < 1.7$ [1]. Thus $\Omega_m$ can not be too small.

5. The Effective Equation of State

Although in the modified gravity theory there is no need of introducing dark energy. Following the general framework of Linder and Jenkins [20], we can define the effective Equation of State (EOS) of dark energy. Written in this form, it also has the advantage that many parametrization work that has been done for EOS of dark energy can be compared with the modified gravity.

Now following Linder and Jenkins, the additional term in Modified Friedmann equation really just describes our ignorance concerning the physical mechanism leading to the observed effect of acceleration. Let us take a empirical approach, we just write the Friedmann equation formerly as

$$\frac{H^2}{H_0^2} = \Omega_m (1 + z)^3 + \delta H^2 / H_0^2$$

where we now encapsulate any modification to the standard Friedmann equation in the last term, regardless of its nature.

Define the effective EOS $\omega_{eff}(z)$ as

$$\omega_{eff}(z) = -1 + \frac{1}{3} \frac{d \ln \delta H^2}{d \ln (1+z)}$$

Then in terms of $\omega_{eff}$, equation (20) can be written as

$$\frac{H^2}{H_0^2} = \Omega_m (1 + z)^3 + (1 - \Omega_m) \int_0^z d \ln (1+z') [1 + \omega_{eff}(z')]$$

(22)
Fig. 5 shows the effective EOS computed with the MF equation \[15\]. We can see that it is consistent for all three values of $\Omega_m$ with current bound: $-1.45 < \omega_{DE} < -0.74$ (95\% C.L.), \[21\]. A special property of the full EOSs are that it would undergo a divergence at roughly $z = 1.9, 1.25, 0.9$ for $\Omega_m = 0.1, 0.2, 0.3$ respectively. After the divergence point, the EOSs reduce to zero (We have verified this numerically, but have not drawn them in Fig.5). What do these divergences imply deserve further studies: maybe it is a catastrophe of this modified gravity theory; maybe it merely indicate that the definition of effective EOS is not suitable here: we just can not consider the effects of modified MF equation as an equivalent effects of dark energy.

6. Discussions and Conclusions.

In this paper we derived the full Modified Friedmann equation of $1/R$ gravity in Palatini formulation. Although we only considered the $1/R$ gravity, our derivation is actually quite general and can be easily applied to other models of modified gravity of the $L(R)$ type. For example, the modifications by adding $1/R + R^2$ \[16\] or $\ln R$ terms \[16\] are also suggested recently. Those two models may explain both the current cosmic acceleration and early inflation without dark energy and inflaton. Also quite interestingly, in those two models, the instability of the metric formulation of the $1/R$ gravity may be resolved. The Palatini formulation of those two models have been studied in Ref. \[17\] using the framework in this paper.

We have also shown that the Palatini formulation of $1/R$ gravity may accommodate the observational data indicating that our universe expansion is accelerating without introducing dark energy. On the other hand, the CMBR anisotropy recently measured by WMAP gives another important data set. Any reasonable cosmological model should fit those data. Whether the MF equation derived in this paper can pass this test is our work ongoing. Also, it is important to find whether the $1/R$ gravity can give distinctive predictions from other models intending to explain the cosmic acceleration such as cosmological constant, quintessence, Chaplygin gas, etc. This is possible following the general framework of Lue et al. \[3\]. There, the authors showed that mechanisms of acceleration due to dark energy or Modified Friedmann equation will give different predictions on the large scale structure power spectra and ISW effects. Applying their framework to the MF equation \[16\] is straightforward. This is also an interesting future working direction.

Whether the current observed cosmic acceleration is an indication that gravity should be modified at large scale or there exist some form of dark energy is surely one of the most important problem in modern physics. Both of those two possibilities deserve be investigated further.

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