Supplementary for “Imaging surface plasmons: the fingerprint of leaky waves on the far field”

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I. INTRODUCTION

We start with a scalar potentials expansion of the electromagnetic field in the three media $j = 1, 2, 3$ corresponding respectively to air metal and substrate (i.e. glass or fused silica). There is no source in the substrate and the dipole (point source) is located in medium $j = 1$. We write for the field in each medium:

$$
\begin{align*}
D_j &= \nabla \times \nabla \times [\hat{\mathbf{z}} \Psi_{TM,j}] + ik_0 \varepsilon_j \nabla \times [\hat{\mathbf{h}} \Psi_{TE,j}], \\
B_j &= \nabla \times \nabla \times [\hat{\mathbf{h}} \Psi_{TE,j}] - ik_0 \varepsilon \nabla \times [\hat{\mathbf{z}} \Psi_{TM,j}]
\end{align*}
$$

with

$$
[\nabla^2 + k_0^2 \varepsilon_j] \Psi_{TM,TE,j} = 0.
$$

Using Boundary conditions we show that the only non vanishing scalar potentials for the dipole direction perpendicular to the film is in medium $j = 3$

$$
\Psi_{TM,\perp}(x, z) = \frac{i \mu \lambda}{4 \pi} \int_{-\infty}^{+\infty} \frac{dk_1}{k_1} \tilde{T}_{13,23}(k) e^{ik_1 h} e^{ik_3 z} J_0(k_0 y)
$$

where we defined $k_i = \sqrt{k_0^2 \varepsilon_i - k^2}$ with $\text{Im} g(k_j) \geq 0$, $k_0 = 2\pi/\lambda = \omega/c$, and $z \geq d$. To obtain Eq. 3 we also used the formula $H_0^{(1)}(u) - H_0^{(1)}(-u) = 2J_0(u)$ which is valid in the complex plane $u = u' + iu''$ (if $|\text{arg}(z)| < \pi$). The Fresnel coefficient characterizing the transmission of the metal film is for TM waves defined by

$$
\tilde{T}_{13}^{TM}(k) = \frac{T_{23}^{TM} T_{12}^{TM}}{1 + R_{23}^{TM} R_{12}^{TM} e^{2ik_3 d} e^{i(k_2 - k_3) d}}
$$

where

$$
R_{ij}^{TM} = \frac{k_i / \varepsilon_i - k_j / \varepsilon_j}{k_i / \varepsilon_i + k_j / \varepsilon_j},
$$

$$
T_{ij}^{TM} = \frac{2k_i / \varepsilon_i}{k_i / \varepsilon_i + k_j / \varepsilon_j}.
$$

We then introduce the variables $k = k_0 n \sin \xi$, $k_3 = k_0 n \cos \xi$ with $\xi = \xi' + i\xi''$ and write

$$
\Psi_{TM,\perp}(x, z) = \int d\xi F_{\perp}^{TM}(\xi) e^{ik_0 n ((z-d) \cos \xi + p \sin \xi)}
$$

with

$$
F_{\perp}^{TM}(\xi) = \frac{i \mu \lambda}{8 \pi} \frac{k_0 n \sin \xi \cos \xi}{\sqrt{(\xi''^2 - \sin^2 \xi)}},
$$

$$
e^{i\sqrt{\xi''^2 - \sin^2 \xi}} h e^{ik_0 n \cos \xi} H_{0}^{(1)}(k_0 n \sin \xi) e^{-ik_0 n \varepsilon \sin \xi}
$$

(8)
(we point out that the $\varrho$ and $\varphi$ dependencies are here and in the following implicit in our notation: $F_+^\| (\xi) := F_+ (\xi, \varrho, \varphi)$). Similar expressions can be obtained for the components $\mu_\parallel$ of the dipole parallel to the interface. More precisely for the TM modes we have

$$\Psi^{\text{TM}, \|} (x, z) = \int_{\Gamma} d\xi F_+^{\text{TM}, \|} (\xi) e^{ik_0 n (z - d) \cos \xi + \varrho \sin \xi} \quad (9)$$

with

$$F_+^{\text{TM}, \|} (\xi) = \frac{\mu_\parallel . \hat{\varrho}}{8\pi} k_0 n \cos \xi \tilde{T}_{13}^{\text{TM}} (k_0 n \sin \xi)$$

$$+ e^{\sqrt{(\varepsilon_{\xi} - \sin^2 \xi) n} \cdot \xi} e^{ik_0 n \varrho \cos \xi} H_{11}^{(1)} (k_0 n \varrho \sin \xi) e^{-ik_0 n \varrho \sin \xi} \quad (10)$$

Similarly for the TE waves we obtain:

$$\Psi^{\text{TE}, \|} (x, z) = \int_{\Gamma} d\xi F_+^{\text{TE}, \|} (\xi) e^{ik_0 n (z - d) \cos \xi + \varrho \sin \xi} \quad (11)$$

with

$$F_+^{\text{TE}, \|} (\xi) = -\frac{\mu_\parallel . \hat{\varrho}}{8\pi} k_0 n \cos \xi \tilde{T}_{13}^{\text{TE}} (k_0 n \sin \xi)$$

$$+ e^{\sqrt{(\varepsilon_{\xi} - \sin^2 \xi) n} \cdot \xi} e^{ik_0 n \varrho \cos \xi} H_{11}^{(1)} (k_0 n \varrho \sin \xi) e^{-ik_0 n \varrho \sin \xi} \quad (12)$$

We used the formula $H_{11}^{(+)} (u) + H_{11}^{(-)} (-u) = 2J_1 (u)$. Here the Fresnel coefficients are defined by

$$\tilde{T}_{13}^{\text{TE}} (k) = \frac{T_{13}^{\text{TE}} T_{12}^{\text{TE}}}{1 + R_{13}^{\text{TE}} R_{12}^{\text{TE}} e^{2ik_2 d}} e^{i(k_2 - k_3) d} \quad (13)$$

with

$$R_{ij}^{\text{TE}} = \frac{k_i - k_j}{k_i + k_j} \quad (14)$$

$$T_{ij}^{\text{TE}} = \frac{2k_i}{k_i + k_j} \quad (15)$$

The presence of the singular Hankel functions $H_{11}^{(+)}$ and $H_{01}^{(+)}$ in all these expressions imply the existence of a branch cut starting at the origin and associated with the function $1/\sqrt{\sin \xi}$. This branch cut is chosen in order to have no influence during subsequent contour deformations and is running just below the actual path $\Gamma$ slightly off the real axis $\xi'$ and to the left of the vertical line $\xi'' = -\pi/2$ (the original branch cut is composed of the line $\xi'' = -\pi/2$ and of the half-axis $[\xi'' = 0, \xi' \leq 0]$). We also introduce the polar coordinates $\varrho = r \sin \vartheta, z = d + r \cos \vartheta$ leading to $(z - d) \cos \xi + \varrho \sin \xi = r \cos (\xi - \vartheta)$ and therefore:

$$\Psi (x, z) = \int_{\Gamma} d\xi F_+ (\xi) e^{ik_0 n r \cos (\xi - \vartheta)} \quad (16)$$

The definition of the square root $k_1 = k_0 n \sqrt{\varepsilon_{\xi} - \sin^2 \xi}$, with $\varepsilon_3 = n^2$ real and $\varepsilon_1 = \varepsilon_2 + i\varepsilon_2'' \sim 1 + i\delta$ with $\delta \to 0^+$, implies the presence of a branch cut which must be chosen carefully in order i) to be consistent with the choice for $k_1$ made in Eq. 3 during integration along the contour $\Gamma$, ii) to allow further contour deformations leading to convergent calculations. The branch cut adapted to our problem is shown in Figs. 1,2 and correspond to the choice $\text{Imag}[k_1] \geq 0$ in the whole complex $\xi$-plane. The branch cut starts at the branch point $M$ of coordinate $\xi$, defined by the condition $k_1 = 0$. We point out that due to invariance of the Fresnel coefficient $\tilde{T}_{13}^{\text{TM,TE}} (k_0 n \sin \xi)$ over the permutation $k_2 \leftrightarrow -k_2$ we don’t actually need an additional branch cut for $k_2$ (this important property will survive for a larger number of layers).
II. THE DIFFERENT CONTRIBUTIONS ALONG THE CLOSED CONTOUR

After introducing the function \( f(\xi) = i \cos(\xi - \vartheta) = i \cos(\xi' - \vartheta) \cosh \xi'' + \sin(\xi' - \vartheta) \sinh \xi'' \) we define the steepest descent path \( SDP \) by the condition

\[
\text{Imag}[f(\xi)] = \cos(\xi' - \vartheta) \cosh \xi'' = 1.
\]  

SDP goes through the saddle point \( \xi_0 \) defined by the condition \( \frac{df(\xi)}{d\xi} = 0 \) which has a solution at \( \xi_0 = \vartheta \). Importantly, there are actually two trajectories solutions of Eq. 17 and going through \( \xi_0 \). We choose the one such that the real part of \( f(\xi) \) decay uniformly along SDP when going away arbitrarily to the left or to the right from the saddle point (see Fig. 1).

Cauchy theorem allows us to deform the original \( \Gamma \) contour to include SDP as a part of the integration path. For this we label \( \Gamma \) by the letter \( ABCD \) (see Fig. 1). The integral in Eq. 10 is thus written \( \int_\Gamma = \int_{ABCD} \). We will consider two cases depending whether \( \vartheta \) is or not larger than the real part of the branch point \( \xi_\vartheta \simeq \arcsin(1/n) = \vartheta_c \).

A. Closing the contour in the case \( \vartheta > \xi_c \)

If \( \vartheta > \xi_c \) the closed integration contour contain eight contributions (see Fig. 1) and we have:

\[
0 = \int_F + \int_{DE} + \int_{EF} + \int_{FG} + \int_{GH} + \int_{HI} + \int_{IA} - I_{SDP}.
\]

The contribution

\[
\int_{DE} := \int_{\pi/2+\vartheta-i\infty}^{\pi/2+\vartheta+i\infty} d\xi F_+(\xi) e^{ik_0nr \cos(\xi - \vartheta)}
\]

\[
\int_{DE} := \int_{\pi/2+\vartheta-i\infty}^{\pi/2+\vartheta+i\infty} d\xi F_+(\xi) e^{ik_0nr \cos(\xi - \vartheta)}
\]

approaches zero asymptotically and can therefore be neglected.

Similarly, we can neglect \( \int_{IA} := \int_{\pi/2+i\infty}^{\pi/2+i\infty} d\xi F_+(\xi) e^{ik_0nr \cos(\xi - \vartheta)} \) which approaches also zero asymptotically.

The contribution \( \int_{EF} \) and \( \int_{FG} \) are calculated along the SDP. However, due to the presence of the branch cut crossing SDP at \( F \) the integration along \( FG \) actually corresponds to a change of Riemann sheet associated with the second determination for the square root \( k_1 \) (we point out that since the branch cut is very close to the imaginary axis at \( F \) we have at the limit \( \xi_\vartheta \simeq i\arccosh(1/\cos \vartheta) \). More precisely if we call “+” the Riemann sheet in which \( \text{Imag}[g_1] \geq 0 \) the second Riemann surface “−” associated with the condition \( \text{Imag}[g_1] \leq 0 \) is connected to “+” through the branch cut represented in Fig. 1. Therefore, crossing the branch cut at \( F \) corresponds actually to a change in sign of the square root \( k_1 \rightarrow -k_1 \). We have consequently the contributions

\[
\int_{EF} := \int_{\pi/2+\vartheta-i\infty}^{\pi/2+\vartheta+i\infty} d\xi F_+(\xi) e^{ik_0nr \cos(\xi - \vartheta)}
\]

\[
\int_{FG} := \int_{\pi/2+\vartheta+i\infty}^{\pi/2+\vartheta-i\infty} d\xi F_+(\xi) e^{ik_0nr \cos(\xi - \vartheta)}
\]

where \( F_-(\xi) \) is the same function of \( k \) as \( F_+(\xi) \) but \( \sqrt{(\xi_3 - \sin^2 \xi)} \) (defined with \( \text{Imag}[\sqrt{(\xi_3 - \sin^2 \xi)}] \geq 0 \)) is now replaced by \(-\sqrt{(\xi_3 - \sin^2 \xi)} \). More precisely the square root \( z_+ = \sqrt{g} \) of the complex variable \( g' + ig'' \) is defined on the “+” Riemann sheet by \( z_+ = \text{sign}(g'') \sqrt{(g' + |g'|)/2 + i \sqrt{(-g' + |g'|)/2} \) where \( \text{sign}(x) = 1 \) if \( x > 0 \), \( \text{sign}(x) = -1 \) if \( x < 0 \) and \( \text{sign}(x) = 0 \) if \( x = 0 \). On the “−” sheet we have therefore \( z_- = -z_+ \). An important remark concerns here the integration convergence along the SDP when approaching the vertical asymptotes \( \pm \vartheta + i \infty \). Indeed, due to the presence of the coefficient \( e^{\pm ik_0nh} \sqrt{(\xi_3 - \sin^2 \xi)} \) in the definition of \( F_\pm(\xi) \) it is not obvious that the integrand will take a finite value at infinity. Actually a careful study of the limit behaviour of \( F_\pm(\xi) \) including the exponentials terms as well as the Hankel function contribution shows that there is no convergence problem for \( F_+(\xi) \) at infinity (this also explains why \( \int_{DE} \) and \( \int_{IA} \) goes to zero asymptotically). However when going on the “−” Riemann sheet the convergence is not always ensured. We found that however no problem occurs on this second sheet as soon as
the condition \( z + q \tan \vartheta > h \) is verified. In particular, no problem appears if we impose \( z > h \). Since here we are interested in the asymptotic behavior valid for \( z \gg h \) this condition will be automatically satisfied.

This point is particularly relevant when we consider the contribution

\[
\tilde{\int}_{GH} := \int_{-\pi/2+\vartheta}^{\infty} d\xi F_-(\xi) e^{ik_0nr \cos (\xi - \vartheta)}
\]

which approaches zero if the previous condition \( z > h \) is fulfilled. From \( H \) we thus cross the branch cut and go back to the “+” sheet. We thus obtain a contour \( \int_{HI} = \int_{HI} d\xi F_+(\xi) e^{ik_0nr \cos (\xi - \vartheta)} \) along the branch cut in the original “+” space and contouring the branch point \( k_1 = 0 \) (corresponding nearly to \( \xi_c \simeq \arcsin (1/n) = \vartheta_c \)). We will see in the subsection D that this contribution corresponds to a lateral wave associated with a Goos-Hänchen effect in transmission.

Finally, due to the presence of isolated singularities in the complex plane (i.e. poles) for the TM waves we must subtract a residue contribution \( I_{SP} \) which value will precisely depends on the position \( \vartheta \) along the real axis (i.e. whether or not the poles are encircled by the closed contour in the complex \( \xi \)-plane). A complete analysis of these singularities show that we can in principle extract from the transmission coefficient \( \tilde{T}^{TM}_{13}(k) \) four poles corresponding to the four SP modes guided along the metal slab. However, the branch cut choice made here allows only the existence of three solutions called respectively symmetric leaky (sl), symmetric bound (sb) and asymmetric bound (ab) modes. The two bound modes are always well outside the region of integration and are never encircled by the contour. Only the leaky mode sl can eventually contribute as a residue depending whether or not the angle \( \vartheta \) is larger than the leakage radiation angle \( \vartheta_{LR} \) defined by the condition \( \cos (\xi_p - \vartheta_{LR}) \cosh \xi''_p = 1 \) (with \( \xi_p \) the complex coordinate of the SP pole sl). This implies:

\[
\vartheta_{LR} = \xi'_p + \arccos (1/\cosh \xi''_p) \simeq \xi'_p,
\]

and therefore the residue contribution is written:

\[
I_{SP} = 2\pi i \text{Res}[F_+(\xi_p)] e^{ik_0nr \cos (\xi_p - \vartheta)} \Theta(\vartheta - \vartheta_{LR}).
\]

In the following we write \( k_p = k_0 n \sin \xi_p, k_{3,p} = k_0 n \cos \xi_p \) and \( k_{1,p} = k_0 n \sqrt{\left(\frac{\mu_1}{\mu_3} - \sin^2 \xi_p\right)} \) the pole wavevectors associated with this sl mode. The calculation of the different residues is straightforward and leads for the vertical dipole case to:

\[
\text{Res}[F^{TM \perp}_{+}(\xi_p)] = \frac{i \mu_1}{8\pi} \frac{k_0 n \sin \xi_p \cos \xi_p}{\sqrt{\left(\frac{\mu_1}{\mu_3} - \sin^2 \xi_p\right)}} \text{Res}[\tilde{T}^{TM}_{13}(k_0 n \sin \xi_p)] e^{i \sqrt{\left(\frac{\mu_1}{\mu_3} - \sin^2 \xi_p\right)} h} e^{ik_0nz \cos \xi_p H_{0}^{(+)}(k_0 n Q \sin \xi_p)}
\]

(23)
We now write $\hat{T}_{13}^\text{TM}(k)$ as a rational fraction $\frac{N_{13}(k_p)}{D_{13}(k_p)}$ (with polynomial functions $N_{13}(k)$, $D_{13}(k)$ of the variable $k$) and therefore for the single pole $\xi_p$ we get

$$\text{Res}[\hat{T}_{13}^\text{TM}(k_0 \sin \xi_p)] = \frac{N_{13}(k_p)}{D_{13}(k_0 \sin \xi_p)} = \frac{1}{k_{3,p}} \frac{N_{13}(k_p)}{\partial D_{13}(k_p) / \partial k_p}.$$ 

We thus have finally

$$\text{Res}[F^\text{TM,+}(\xi_p)] = \frac{i \mu_\perp k_p e^{ik_{1,p}h} e^{ik_{3,p}z} N_{13}(k_p)}{8\pi k_{3,p}} \frac{\partial D_{13}(k_p)}{\partial k_p} H_0^+(k_p \vartheta).$$

A similar expression is obtained for the horizontal dipole:

$$\text{Res}[F^\text{TM,\perp}(\xi_p)] = \frac{\mu_\perp \vartheta e^{ik_{1,p}h} e^{ik_{3,p}z} N_{13}(k_p)}{8\pi k_{3,p}} \frac{\partial D_{13}(k_p)}{\partial k_p} H_1^+(k_p \vartheta).$$

There is no residue for the TE modes.

Going back to the SDP contribution we define the variable $\tau = e^{i\pi/4} \sqrt{2} \sin ((\xi - \vartheta)/2)$ which leads to $f(\xi) = i - \tau^2$. Along SDP $\tau$ is real such that $\tau^2 = -\sin (\xi' - \vartheta) \sin \xi'' \geq 0$. We thus obtain $\tau = 2 \sin ((\xi' - \vartheta)/2) \cosh (\xi''/2)$. The saddle point corresponds to $\tau = 0$ ($\vartheta > \xi$) and $\tau = 0$ ($\vartheta < \xi$) along SDP). With this new variable the point $F$ has therefore the coordinate $\tau_F \simeq -2 \sin (\vartheta/2) \cosh (\vartheta/2) \arccosh((1/\cos(\vartheta))) = -2 \sin (\vartheta/2) \sqrt{1 + \frac{1}{\cos \vartheta}} < 0$. Defining the term $I_{SDP} = -\int_{EF} -\int_{FG}$ we therefore obtain

$$I_{SDP} = e^{ik_{0nr}} \int_{-\infty}^{\tau_F} d\tau G_-(\tau) e^{-k_{0nr} \tau^2}
+ \int_{\tau_F}^{\infty} d\tau G_+(\tau) e^{-k_{0nr} \tau^2}
+ \int_{-\infty}^{+\infty} d\tau \{G_+(\tau)[1 - \Theta(\tau_F - \tau)]
+ G_-(\tau)[1 - \Theta(\tau - \tau_F)]\} e^{-k_{0nr} \tau^2}$$

where we defined $G_\pm(\tau) = F_\pm(\xi) \frac{d\xi}{d\tau}$ and used $\frac{d\xi}{d\tau} = \sqrt{2e^{i\pi/4} \cosh ((\xi - \vartheta)/2)}$. We introduced the Heaviside step function $\Theta(x)$ defined as: $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ otherwise. Importantly $\lim_{\tau \to \tau_F^+} G_+(\tau) = \lim_{\tau \to \tau_F^-} G_-(\tau)$ and therefore the function $G(\tau) = G_+(\tau)(1 - \Theta(\tau_F - \tau)) + G_-(\tau)(1 - \Theta(\tau - \tau_F))$ which is not defined at $\tau_F$ can be prolonged without difficulties at $F$.

B. Closing the contour in the case $\vartheta < \xi_c$

If $\vartheta < \xi_c'$ the closed integration contour contain 6 contributions (see Fig. 2) and we have:

$$0 = \int_{\Gamma} + \int_{DE} + \int_{EF} + \int_{FG} + \int_{GH} + \int_{HA}.$$ 

All these contribution but $\int_{NM}$ are defined on the “+” Riemann sheet. $\int_{HA}$ and $\int_{DE}$ tends asymptotically to zero for reasons already discussed in the previous paragraph. Importantly there is no contribution along the branch cut since the integration path along the SDP starts and finishes on the proper Riemann sheet “+”. Regrouping the terms we thus have for $\vartheta > \xi_c'$: $\int_{\Gamma} = I_{SDP} = -\int_{EF} -\int_{FG} -\int_{GH}$ with

$$\int_{EF} := \int_{EF} d\xi F_+(\xi) e^{ik_{0nr} \cos (\xi - \vartheta)}
\int_{FG} := \int_{FG} d\xi F_+(\xi) e^{ik_{0nr} \cos (\xi - \vartheta)}
\int_{GH} := \int_{GH} d\xi F_+(\xi) e^{ik_{0nr} \cos (\xi - \vartheta)}.$$
FIG. 2: Integration contour in the complex $\xi$-plane for $\vartheta < \xi'_c$.

We then use the same variable $\tau$ and function $G_{\pm}(\tau)$ and thus obtain

$$I_{SDP} = e^{ik_0nr} \left\{ \int_{-\infty}^{\tau_p} d\tau G_+(\tau)e^{-k_0nr\tau^2} + \int_{\tau_p}^{\tau_G} d\tau G_+(\tau)e^{-k_0nr\tau^2} + \int_{\tau_G}^{\infty} d\tau G_+(\tau)e^{-k_0nr\tau^2} \right\}$$

(29)

which is rewritten as

$$I_{SDP} = e^{ik_0nr} \int_{-\infty}^{\tau_p} d\tau G_+(\tau)e^{-k_0nr\tau^2}$$

(30)

with

$$G(s) = G_+(\tau)[1 - \Theta(\tau - \tau_F)] + G_+(\tau)[1 - \Theta(\tau_G - \tau)] + G_-(\tau)[1 - \Theta(\tau_F - \tau)][1 - \Theta(\tau - \tau_G)].$$

(31)

C. The Steepest descent path contribution

The previous integral $I_{SDP}$ for both $\vartheta > \xi'_c$ and $\vartheta < \xi'_c$ is of the gaussian form and can be evaluated by doing a Taylor expansion of $G(\tau)$ around $\tau = 0$. Using well known integrals we thus obtain

$$I_{SDP} = e^{ik_0nr} \sum_{m \in \text{even}} \frac{\Gamma \left( \frac{m+1}{2} \right)}{m! (k_0nr)^{\frac{m+1}{2}}} \frac{d^m}{d\tau^m} G(0).$$

(32)

It is important to observe that $G(\tau)$ is highly singular in the vicinity of the SP pole $s_l$. Writing $\tau_p$ the coordinate of the pole in the $\tau$-space we thus define

$$G(\tau) := G_0(\tau) + \frac{\text{Res}[G(\tau_p)]}{\tau - \tau_p}$$

(33)
which (together with Eq. 32) immediately implies

$$I_{SDP} = e^{i k_0 n r} \sum_{m \in \text{even}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{(k_0 n r)^{m+1}} \left\{ \frac{d^m}{dx^m} G_0(0) - \text{Res}[G(\tau_p)] \frac{1}{\tau_{p}^{m+1}} \right\}.$$  (34)

Remarkably, the singular integral

$$I_{SDP}^{\text{pole}} := e^{i k_0 n r} \int_{-\infty}^{+\infty} d\tau \frac{\text{Res}[G(\tau_p)]}{\tau - \tau_p} e^{-k_0 n r \tau^2}$$

can be directly calculated and we thus obtain

$$I_{SDP} = e^{i k_0 n r} \sum_{m \in \text{even}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{m!(k_0 n r)^{m+1}} \frac{d^m}{dx^m} G_0(0) + I_{SDP}^{\text{pole}}$$  (35)

with

$$I_{SDP}^{\text{pole}} = -2i \pi \text{Res}[G(\tau_p)] e^{i k_0 n r \cos(\xi_p - \vartheta)} \left\{ \Theta(-\tau_p') - \frac{1}{2} \text{erfc}(-i \tau_p \sqrt{(k_0 n r)}) \right\}$$

$$= -e^{i k_0 n r} \sum_{m \in \text{even}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{(k_0 n r)^{m+1}} \text{Res}[G(\tau_p)] \frac{1}{\tau_{p}^{m+1}}$$  (36)

where erfc(z) = \((2/\pi)^{1/2} \int_{z}^{+\infty} e^{-t^2} dt\) is the Gauss complementary error function. Notably, we have (see appendix) \(\text{Res}[G(\tau_p)] = \text{Res}[F_+(\xi_p)]\) and \(\Theta(-\tau_p') = \Theta(\vartheta - \vartheta_{LR})\) therefore \(I_{SDP}^{\text{pole}}\) contains up to the sign difference the same contribution which already appeared in \(I_{SP}\). Consequently, the sum \(I_{SDP}^{\text{pole}} + I_{SP}\) of the two contributions proportional to the residue represents a simple explicit mathematical expression:

$$I_{SDP}^{\text{pole}} + I_{SP} = i \pi \text{Res}[G(\tau_p)] e^{i k_0 n r \cos(\xi_p - \vartheta)} \text{erfc}(-i \tau_p \sqrt{(k_0 n r)})$$  (37)

This sum is sometimes by definition associated with the surface plasmon mode. We point out however that the error function is highly singular and therefore we should preferably use the equivalent expression:

$$I_{SDP}^{\text{pole}} + I_{SP} = 2i \pi \text{Res}[G(\tau_p)] e^{i k_0 n r \cos(\xi_p - \vartheta)} \Theta(\vartheta - \vartheta_{LR}) - e^{i k_0 n r} \sum_{m \in \text{even}} \frac{\Gamma\left(\frac{m+1}{2}\right)}{(k_0 n r)^{m+1}} \frac{\text{Res}[G(\tau_p)]}{\tau_{p}^{m+1}}.$$  (38)

We also note that most of the discussions and confusions made during the XXth on the role of SPs in the Sommerfeld integral resulted from the above mentioned intricate relationship existing between the two singular terms \(I_{SP}\) and \(I_{SDP}^{\text{pole}}\). For a historical discussion see Collin[3].

D. The lateral wave contribution: Goos-Hänchen effect in transmission

In the case \(\vartheta > \xi'\), the integral \(\int_H\) along the branch cut can be transformed using the method described in Ref. 2. For this we separate the integral \(\int_H d\xi F_+(\xi) e^{i k_0 n r \cos(\xi - \vartheta)}\) into a contribution \(\int_{HM} = \int_H d\xi F_+(\xi) e^{i k_0 n r \cos(\xi - \vartheta)}\) starting at infinity at \(\xi = i \infty + 0^+\) and stopping at the branch-point \(M (\xi_c \approx \vartheta_c)\) and into a contribution \(\int_M = \int_M d\xi F_+(\xi) e^{i k_0 n r \cos(\xi - \vartheta)}\) starting at \(M\) and finishing at infinity \(\xi = i \infty + 0^-\) on the other side of the branch cut.
FIG. 3: Integration contour in the complex $\xi$-plane along the branch cut $HI$ around the branch point $M$ for $\vartheta > \xi'_c$. (A) shows the closed contour used to deform analytically the contour $HM$. (B) shows the closed contour used to deform the contour $MI$.

As shown in Fig. 3(A) in order to calculate $I_{HM}$ the integration contour is closed by longing the modified steepest descent path $MG$ defined by the equation

$$\cos (\xi' - \vartheta) \cosh \xi'' = \cos (\xi'_c - \vartheta) \cosh \xi''_c \simeq \cos (\vartheta_c - \vartheta) = K < 1. \quad (39)$$

The curve $MG$ with a vertical asymptote at $\xi = -\pi/2 + \vartheta$ is thus defined by $\xi'' = \text{arcosh}(K/\cos (\xi' - \vartheta))$. We thus have $0 = \int_{HM} + \int_{MN} + \int_{NG} + \int_{GH}$ where $N$ is the intersection point between the modified steepest descent path $MG$ and the branch cut ($\xi_N \simeq i\text{arcosh}(K/\cos (\vartheta))$). $\int_{NG} := \int_{NG} d\xi F_-(\xi) e^{i k_0 n r \cos (\xi - \vartheta)}$ and $\int_{GH} := \int_{GH} d\xi F_+(\xi) e^{i k_0 n r \cos (\xi - \vartheta)}$ are evaluated on the “−” Riemann sheet while $\int_{HM} := \int_{HM} d\xi F_+(\xi) e^{i k_0 n r \cos (\xi - \vartheta)}$ and $\int_{MN} := \int_{MN} d\xi F_+(\xi) e^{i k_0 n r \cos (\xi - \vartheta)}$ are evaluated on the “+” Riemann sheet. From $H$ we cross a second time the branch cut in order to close the contour on the “+” Riemann sheet.

A similar analysis is done for the integration contour $\int_{MI}$. We have $0 = \int_{MI} + \int_{IG} + \int_{GN} + \int_{NM}$ where $\int_{MI}, \int_{IG}$
and \( \int_{GN} \) are defined as previously on the “+” Riemann sheet while \( \int_{HM} := \int_{M} d\xi F_-(\xi)e^{ik_0nr\cos(\xi-\vartheta)} \) is evaluated on the “-” Riemann sheet. In order to close the contour in “+” we must finally cross the branch cut in the region surrounding \( M \). The infinitesimal loop surrounding \( M \) gives however a vanishing contribution which can be neglected. Regrouping all these expressions we define \( I_{LW} = -\int_{HM} - \int_{M} \) and we obtain

\[
I_{LW} = \int_{\xi_c}^{\xi} d\xi [F_-(\xi) - F_+(\xi)]e^{ik_0nr\cos(\xi-\vartheta)} + \int_{\xi_c}^{-\pi/2+\vartheta+i\infty} d\xi [F_-(\xi) - F_+(\xi)]e^{ik_0nr\cos(\xi-\vartheta)} + \int_{IG} + \int_{GH}. \quad (40)
\]

The contributions \( \int_{IG}, \int_{GH} \) vanish asymptotically as discussed before and therefore can be neglected. Importantly due to the definition of the square root the function \( F_-(\xi) - F_+(\xi) \) tends to vanish at the intersection point \( N \). We then define the function \( \Phi(\xi) = [F_+(\xi) - F_+(\xi)]\text{sign}(\xi_N' - \xi''n) \) vanishing at \( \xi_c \) and write

\[
I_{LW} = \Theta(\vartheta - \xi') \int_{\xi_c}^{-\pi/2+\vartheta+i\infty} d\xi \Phi(\xi)e^{ik_0nr\cos(\xi-\vartheta)}.
\quad (41)
\]

The Heaviside function was introduced in order to remember that \( I_{LW} \) is only defined if \( \xi' < \vartheta \). In the present work we will only evaluate \( I_{LW} \) approximately using the method discussed in Ref. 2. First, we observe that \( e^{ik_0nr\cos(\xi-\vartheta)} = e^{ik_0nrK}e^{ik_0nr\sin(\xi'-\vartheta)\sin\xi''} \). Second, considering that only \( \xi \) values in the vicinity of \( \xi_c \approx \vartheta \) contribute significantly to \( I_{LW} \) we write \( d\xi \approx id\xi'' \), and \( \sin(\xi' - \vartheta)\sin\xi'' \approx -\sin(\vartheta - \vartheta_c)\xi'' < 0 \). We therefore obtain

\[
I_{LW} \approx ie^{ik_0nrK}\Theta(\vartheta - \vartheta_c) \int_{0}^{+\infty} d\xi'' \Phi(\vartheta + i\xi'')e^{-k_0nr\sin(\vartheta - \vartheta_c)\xi''} = 2ie^{ik_0nrK}\Theta(\vartheta - \vartheta_c) \int_{0}^{+\infty} du \Phi(\vartheta + iu^2)e^{-k_0nr\sin(\vartheta - \vartheta_c)u^2} \quad (42)
\]

where we used the variable \( \xi'' = u^2 \). This integral is of the Gaussian kind and can be computed exactly using a Taylor expansion of \( \Phi \) near \( \vartheta_c \). We consequently deduce

\[
I_{LW} \approx ie^{ik_0nrK}\Theta(\vartheta - \vartheta_c) \sum_{m=1}^{+\infty} \frac{\Gamma(1 + m/2)}{(k_0nr \sin(\vartheta - \vartheta_c))^{1+m/2}} \frac{H^{(m)}(0)}{m!} \quad (43)
\]

where we used the series expansion \( \Phi(\vartheta + iu^2) = H(u) = \sum_{m=1}^{+\infty} \frac{u^m}{m!} \frac{d^m}{du^m} H(u)|_{u=0} = \sum_{m=1}^{+\infty} \frac{u^m H^{(m)}(0)}{m!} \) (the term \( m = 0 \) vanishes since \( \Phi(\vartheta_c) = 0 \)).

The phase \( \delta \varphi = k_0nr \cos(\vartheta - \vartheta_c) = k_0nr[\cos \vartheta \cos \vartheta_c + \sin \vartheta \sin \vartheta_c] \) takes a simple interpretation if you define the length \( L_1 \) of \( L_2 \) by:

\[
\begin{align*}
r \sin \vartheta &= \vartheta_1 + L_2 \sin \vartheta_c \\
r \cos \vartheta &= z - d &= L_2 \cos \vartheta_c.
\end{align*} \quad (44)
\]

FIG. 4: Geometric construction of the Goos-Hänchen phase in transmission.
\[
\delta \varphi = k_0 n (L_2 + L_1 \sin \vartheta_c) = k_0 n L_2 + k_0 L_1
\] (45)

where we used \( n \sin \vartheta_c = 1 \). As it is clear from Fig. 4, \( L_1 \) is the path length of a ‘creeping’ wave propagating along the interface before to be re-emitted at the critical angle \( \vartheta_c \). The re-emitted waves propagates along a distance \( L_2 \) in the medium of optical index \( n \) and then reaches the point defined by the coordinates \( (r, \vartheta) \). The phase \( \delta \varphi \) is thus generated by a virtual propagation of length \( L_1 \) along the interface air-dielectric \( z = d \) (supposing no metal is present and that the volume corresponding to the film between \( z = 0 \) and \( z = d \) is filled with the medium of permittivity \( \varepsilon_2 \approx 1 \)) and followed by a re-emission at the critical angle in the glass substrate \( \varepsilon_3 = n^2 \). The previous analysis justifies therefore the name “lateral” we gave to the contribution \( I_{LW} \). This effect can be seen as a kind of Goos-Hänchen deflection in transmission and is somehow equivalent to the already known Goos-Hänchen effect associated with lateral waves in the reflection mode.

E. The Far-field Fraunhofer regime

We are interested into evaluating the different integrals when \( r \to + \infty \). As a first approximation, concerning \( I_{SDP} \) we calculate only the term \( m = 0 \) in the sum which reads:

\[
I_{SDP,m=0} = \frac{2 \pi k_0 n \cos \vartheta}{ir} e^{ik_0nr} G(0) = \frac{2 \pi e^{ik_0nr} e^{-i\pi/4}}{\sqrt{2} k_0nr} F_+(\vartheta). \tag{46}
\]

In the far-field where \( r \gg \lambda \) the Hankel function can be approximated using the asymptotic formulas

\[
H_0^{(+)}(x) = \sqrt{\frac{2}{\pi x}} e^{-i\pi/4}(1 - \frac{i}{8x}) e^{ix} + O(x^{-5/2})
\]

\[
H_1^{(+)}(x) = \sqrt{\frac{2}{\pi x}} e^{-3i\pi/4}(1 + \frac{3i}{8x}) e^{ix} + O(x^{-5/2}) \tag{47}
\]

which are valid for \( x \gg 1 \). Therefore for the vertical dipole we get

\[
I_{SDP,m=0}^\perp = \frac{2 \pi k_0 n \cos \vartheta}{ir} e^{ik_0nr} \tilde{\Psi}_{TM,\perp}[k_0 n \sin \vartheta \hat{\mathbf{n}}, z = d(1 - \frac{i}{8k_0nr \sin \vartheta^2} + ...)] \tag{48}
\]

where

\[
\tilde{\Psi}_{TM,\perp}[k, z = d] = \frac{i \mu_{\perp}}{8 \pi^2 k_1} \tilde{T}_{13}^{TM}(k) e^{ik_3d} e^{ik_1h}
\]

\[
= \frac{i \mu_{\perp}}{8 \pi^2 k_0 \sqrt{1 - n^2 \sin \vartheta^2}} \tilde{T}_{13}^{TM}(k) e^{ik_3d} e^{ik_1h} \tag{49}
\]

is the 2D Fourier transform of \( \Psi_{TM,\perp}(\vartheta, z = d) e^{-i k \cdot x} \) for the wavevector \( k = k_0 n \sin \vartheta \hat{\mathbf{n}} \). Similarly for the horizontal dipole we obtain for TM components:

\[
I_{SDP,m=0}^\parallel = \frac{2 \pi k_0 n \cos \vartheta}{ir} e^{ik_0nr} \tilde{\Psi}_{TM,\parallel}[k_0 n \sin \vartheta \hat{\mathbf{n}}, z = d(1 + \frac{3i}{8k_0nr \sin \vartheta^2} + ...)] \tag{50}
\]

where

\[
\tilde{\Psi}_{TM,\parallel}[k, z = d] = \frac{-i \mu_{\parallel} \cdot k}{8 \pi^2 k^2} \tilde{T}_{13}^{TM}(k) e^{ik_3d} e^{ik_1h}
\]

\[
= \frac{-i \mu_{\parallel} \cdot \hat{\mathbf{n}}}{8 \pi^2 k_0nr \sin \vartheta} \tilde{T}_{13}^{TM}(k) e^{ik_3d} e^{ik_1h}. \tag{51}
\]

For TE components we have also:
\[ I_{SDP,m=0}^{||} = \frac{2\pi k_0 n \cos \vartheta}{ir} e^{ik_0 nr} \tilde{\Psi}_{TE,||}[k_0 n \sin \vartheta \hat{\mathbf{r}}, z = d](1 + \frac{3i}{8k_0 nr \sin \vartheta^2} + \ldots) \] (52)

with now

\[ \tilde{\Psi}_{TE,||}[k, z = d] = \frac{i k_0 \mu || \cdot (\hat{\mathbf{z}} \times \mathbf{k})}{8\pi^2 k_1 k^2} \tilde{I}_{13}^{TE}(k)e^{ik_3 d}e^{ik_1 h} \]
\[ = \frac{i \mu || \tilde{\Phi}}{8\pi^2 k_0 \sqrt{1 - n^2 \sin \vartheta^2}} \tilde{T}_{13}^{TE}(k)e^{ik_3 d}e^{ik_1 h}. \] (53)

In the far field only the term in \(1/r\) survives and (in agreement with the Stratton-Chu formalism and Richards and Wolf [4]) we can always write:

\[ \Psi \approx I_{SDP,m=0}^{||} \approx \frac{2\pi k_0 n \cos \vartheta}{ir} e^{ik_0 nr} \tilde{\Psi}_{TM} \quad \text{or} \quad |k_0 n \sin \vartheta \hat{\mathbf{r}}, z = d|. \] (54)

**F. The intermediate regime: Generalization of the Norton wave**

The next term in the power expansion of \(\Psi\) contributes proportionally to \(1/r^2\). To evaluate this term we must take into account not only \(I_{SDP,m=0}\) but also \(I_{SDP,m=2}\) and \(I_{LW,m=1}\). We use the notation

\[ F_\pm(\xi) = \sqrt{\frac{2\pi k_0 n}{r}} e^{-i\frac{\pi}{4}} k_0 n \cos \xi Q_\pm^\alpha(k_0 n \sin \xi) \]
\[ \cdot \left[1 - i \frac{1 - 4\alpha^2}{k_0 nr \sin (\xi)^2} + \ldots\right] \] (55)

(with \(\alpha = 0\) or \(1\) depending whether the dipole is vertical or horizontal) and we obtain for the SDP contributions proportional to \(1/r^2\):

\[ I_{SDP,m=0} = \frac{2\pi(1 - 4\alpha^2)}{r^2} e^{ik_0 nr} \cos \vartheta Q_\pm^\alpha(k_0 n \sin \xi) \] (56)

and

\[ I_{SDP,m=2} = \frac{e^{ik_0 nr} \sqrt{\pi} \frac{d^2 G(\tau)}{d\tau^2}}{4(k_0 nr)^{3/2}} \bigg|_{\tau = 0} \]
\[ = -\frac{\pi}{r^2} e^{ik_0 nr} \frac{d^2}{d\xi^2} \left(\frac{\cos \xi}{\cos (\xi - \vartheta/2)}\right) Q_\pm^\alpha(k_0 n \sin \xi) \bigg|_{\xi = \vartheta}. \] (57)

We also have to include the lateral wave (i.e. Goos-Hänchen) contribution:

\[ I_{LW,m=1} \approx e^{ik_0 nr K} \frac{i \sqrt{\pi} \Theta(\vartheta - \vartheta_c)}{2(k_0 nr \sin (\vartheta - \vartheta_c))^{3/2}} \frac{dH(u)}{du} \bigg|_{u = 0} \] (58)

which reads

\[ I_{LW,m=1} = \frac{\pi e^{ik_0 (nL_2 + L_1)} \Theta(\vartheta - \vartheta_c)e^{i\vartheta}}{r^2(\sin (\vartheta - \vartheta_c))^{3/2}} \frac{d\{\cos(\vartheta_c + iu^2)[Q_+^\alpha(k_0 n \sin (\vartheta_c + iu^2)) - Q_-^\alpha(k_0 n \sin (\vartheta_c + iu^2))]\}}{du} \bigg|_{u = 0}. \] (59)

The sum \(I_{SDP,m=0} + I_{SDP,m=2} + I_{LW,m=1}\) describes an asymptotic field varying as \(1/r^2\) and which constitutes a generalization of the result obtained by Norton for the radio wave antenna on a conducting earth problem.
III. HOW TO DEFINE THE SURFACE PLASMON MODE?

A. From the near-field to the far-field

As seen in Section 2.E the dominant contribution in the far-field has the form

\[
\Psi(x, z) = \frac{2\pi k_0 n \cos \vartheta}{ir} e^{ik_0 n r} \tilde{\Psi}[k_0 n \sin \vartheta \hat{\vartheta}, z = d].
\]

(60)

From Eq. 53 we also have the relation

\[
\tilde{\Psi}[k_0 n \sin \vartheta \hat{\vartheta}, z = d] := Q(s)
\]

\[
= \sqrt{\left( \frac{r}{2\pi k_0 n} \right)} e^{i\vartheta} \frac{F_+(\xi)}{k_0 n \cos \xi}
\]

\[
= \sqrt{\left( \frac{r}{2\pi k_0 n} \right)} e^{i\vartheta} F_+(\xi) \frac{d\xi}{ds}
\]

(61)

where \(s = k_0 n \sin \xi\) and where \(\xi\) is here identical to \(\vartheta\) (as usual the \(\varrho\) and \(\varphi\) dependencies are here implicit in \(Q(s) := Q(s, \varphi, \varrho)\) and \(F_+(\xi) := F_+(\xi, \varphi, \varrho)\)). In the complex plane \(\xi = \xi' + i\xi''\) and \(s = s' + is''\) we have the singular/regular decomposition: \(Q(s) = Q_0(s) + \text{Res}[Q(s_p)]/(s - s_p)\). Furthermore, from Eq. 59 this implies

\[
\frac{1}{2\pi i} \oint_{C_p} ds Q(s) = \text{Res}[Q(s_p)]
\]

\[
= \sqrt{\left( \frac{r}{2\pi k_0 n} \right)} e^{i\vartheta} \frac{1}{2\pi i} \oint_{C_p} d\xi F_+(\xi)
\]

\[
= \sqrt{\left( \frac{r}{2\pi k_0 n} \right)} e^{i\vartheta} \text{Res}[F_+(\xi_p)]
\]

(62)

where \(C_p\) and \(C_p\) are small closed contours surrounding the plasmon pole in respectively the complex \(s\)-plane and \(\xi\)-plane. Therefore, we can equivalently write

\[
Q(s) = Q_0(s) + \sqrt{\left( \frac{r}{2\pi k_0 n} \right)} e^{i\vartheta} \frac{\text{Res}[F_+(\xi_p)]}{s - s_p}.
\]

(63)

The calculations being done in the far-field limit, where \(r, \varrho \to +\infty\), we have for the vertical dipole case the residue:

\[
\text{Res}[F^\text{TM,1}\!\!\parallel(\xi_p)] = \frac{i\mu_{\perp}}{8\pi} e^{ik_1 \cdot h} e^{ik_3 \cdot d} \frac{N_{13}(k_p)}{\partial D_{13}(k_p) / \partial k_p} H_0^{(+)}(k_p \varrho) e^{-ik_p \varrho} \simeq \frac{i\mu_{\perp}}{8\pi} e^{ik_1 \cdot h} e^{ik_3 \cdot d} \frac{N_{13}(k_p)}{\partial D_{13}(k_p) / \partial k_p} \sqrt{\left( \frac{2}{\pi k_p \varrho} \right)} e^{-i\frac{\pi}{4}}.
\]

(64)

and similarly for the horizontal dipole residue:

\[
\text{Res}[F^\text{TM,1\!\!\perp}(\xi_p)] = \frac{\mu_{\parallel}}{8\pi} e^{ik_1 \cdot h} e^{ik_3 \cdot d} \frac{N_{13}(k_p)}{\partial D_{13}(k_p) / \partial k_p} H_1^{(+)}(k_p \varrho) e^{-ik_p \varrho} \simeq \frac{\mu_{\parallel}}{8\pi} e^{ik_1 \cdot h} e^{ik_3 \cdot d} \frac{N_{13}(k_p)}{\partial D_{13}(k_p) / \partial k_p} \sqrt{\left( \frac{2}{\pi k_p \varrho} \right)} e^{-i\frac{3\pi}{4}}.
\]

(65)

Regrouping all the terms and using the fact that \(Q(s) = \tilde{\Psi}[k, z = d]\) with \(k = k_0 n \sin \vartheta \hat{\vartheta}\) and \(\varrho = r \sin \vartheta\) this allow us to obtain a decomposition of the Fourier field \(\tilde{\Psi}[k, z = d]\) into a singular (i.e. SP) and regular contribution:

\[
\tilde{\Psi}^\|\!\|_S[k, z = d] = \tilde{\Psi}^\|\!\|_S[k, z = d] + \tilde{\Psi}^\perp\!\perp_S[k, z = d]
\]

(66)

with

\[
\tilde{\Psi}^\|\!\|_S[k, z = d] = \frac{i\mu_{\perp}}{8\pi} e^{ik_1 \cdot h} e^{ik_3 \cdot d} \frac{N_{13}(k_p)}{\partial D_{13}(k_p) / \partial k_p} \sqrt{\left( \frac{2}{\pi k_p \varrho} \right)} e^{-i\frac{\pi}{4}}.
\]

(67)
These formulas are rigorously only valid in the propagative sector where $|k| \leq k_0n$ (i.e. from the far-field definition). However, due to the simplicity of the mathematical expressions obtained one is free to extend the validity of Eqs. 65 to the full spectrum of $k \in \mathbb{R}^2$ values including both the propagative sector for which $k_3 = \sqrt{(k_0^2n^2 - |k|^2)}$ and the evanescent sector for which $k_3 = i\sqrt{(|k|^2 - k_0^2n^2)}$ (i.e. if $|k| \geq k_0n$).

It should now be observed that we can slightly modify our current analysis by observing that Eq. 61 is not exactly a $\xi_p$ singularity corresponding to the $s_1$ mode contributed to the integration contours used. Still, for the symmetry of the mathematical expressions it is clearly possible, and actually very useful (as we will see below), to extract a second SP contribution $\xi_{-p} = -\xi_p$ corresponding to $-k_p$. This is clearly the $s_1$ pole associated with propagation in the opposite radial direction. Taking into account this second pole and the symmetries of $k_{1p}$, $k_{3p}$, $N_{13}(k_p)$ and antisymmetry of $\frac{\partial D_{13}(k_p)}{\partial k_p}$ in the substitution $k_p \rightarrow -k_p$ one obtain after straightforward calculations:

$$\tilde{\Psi}^{\perp}_{SP} \left[ k, z = d \right] = \frac{i\mu_{\perp} k_p}{8\pi k_{1p}} e^{ik_1p} e^{ik_3p d} \frac{N_{13}(k_p)}{\sqrt{k_p}} \frac{1}{\sqrt{k - k_p}} \frac{1}{i(k + k_p)} + \frac{1}{1 + i(k + k_p)}.$$  

$$\tilde{\Psi}^{||}_{SP} \left[ k, z = d \right] = \frac{\mu_{\parallel} \cdot \hat{k} e^{ik_1p} e^{ik_3p d} N_{13}(k_p)}{8\pi k_p} \frac{\partial D_{13}(k_p)}{\partial k_p} \frac{1}{\sqrt{k - k_p}} \frac{1}{i(k - k_p)} + \frac{1}{1 + k + k_p}.$$  

From this definition we can calculate the SP field in the complete space. In particular for $z \geq d$ we have $\tilde{\Psi}^{\perp}_{SP} \left( x, z \right) = \int d^2k \tilde{\Psi}^{\perp}_{SP} \left[ k, z = d \right] e^{ik_3Z}$ with $Z = z - d$. More precisely using the symmetry of the system we obtain

$$\Psi^{\perp}_{SP} \left( x, z \right) = \frac{i\mu_{\perp} k_p}{8\pi k_{1p}} e^{ik_1p} e^{ik_3p d} N_{13}(k_p) \frac{\partial D_{13}(k_p)}{\partial k_p} \frac{1}{\sqrt{k - k_p}} \frac{1}{i(k + k_p)} + \frac{1}{1 + k + k_p}.$$  

(69)

and

$$\Psi^{||}_{SP} \left( x, z \right) = \frac{\mu_{\parallel} \cdot \hat{\theta} e^{ik_1p} e^{ik_3p d} N_{13}(k_p)}{8\pi k_p} \frac{\partial D_{13}(k_p)}{\partial k_p} \frac{1}{\sqrt{k - k_p}} \frac{1}{i(k - k_p)}.$$

(70)

with $x = \theta \hat{\theta}$. To obtain these last equations we also used the well known Bessel function properties:

$$\int d\varphi_k e^{ik \cos(\varphi - \varphi_k)} \left\{ \cos(m \varphi_k) \sin(m \varphi_k) \right\} = 2\pi \left\{ \cos(m \varphi) \sin(m \varphi) \right\} J_m(k \varphi).$$

(71)

(m=0,1,...) to integrate over the $\varphi_k$-coordinate of the 2D vector $k$.

We point out that the convergence of integrals 69, 70 is ensured since the Cosine integral $\int_a^{+\infty} dk \cos(k \varphi)/k = -\text{Ci}(a \varphi) \approx \frac{\cos(a \varphi)}{a \varphi} - \frac{\sin(a \varphi)}{a^2 \varphi}$ for $a \varphi \gg 1$ is bounded.

### B. Asymptotic expansion

Remarkably, using the relations $H_0^{(+)}(u) - H_0^{(+)}(-u) = 2J_0(u)$ and $H_1^{(+)}(u) + H_1^{(+)}(-u) = 2J_1(u)$ (valid for $|\arg(z)| < \pi$) as well as the parity properties of the functions $\frac{\cos(a \varphi)}{\sqrt{k - k_p} \pm \frac{\sin(a \varphi)}{|k - k_p|}}$ (i.e. under the transformation
Second, the term \( \text{Res}_{m} \) exactly identical to the pole contribution appearing in Eq. 22. This results from the equality \( \text{Res}_{m} \) and using the complex variable \( \xi \) such as \( k = k_{0}\sin \xi \) and the integration contour \( \Gamma \) used in the previous Sections we obtain

\[
\Psi_{SP}^\perp (x, z) = \int_{\Gamma} d\xi F_{SP}^\perp (\xi) e^{ik_{0}nr \cos (\xi - \vartheta)}
\]

\[
F_{SP}^\perp (\xi) = \frac{i\mu_{1} k_{p}}{8\pi} \sqrt{(k_{0}n)e^{ik_{1}\rho}e^{ik_{3}\rho}} N_{13}(k_{p}) \sin \xi H_{0}^{(+)}(k_{0}n \sin \xi)e^{-ik_{0}n \sin \xi} \frac{\cos \xi}{\sin \xi - \sin \xi_{p}} + \frac{\cos \xi}{i(\sin \xi + \sin \xi_{p})}
\]

\[
F_{SP}^\parallel (\xi) = \frac{\mu_{1}}{8\pi} \frac{\partial}{\partial \xi} \Psi_{SP}^\perp (k_{0}n)e^{ik_{1}\rho}e^{ik_{3}\rho} N_{13}(k_{p}) \sin \xi H_{1}^{(+)}(k_{0}n \sin \xi)e^{-ik_{0}n \sin \xi} \frac{\cos \xi}{\sin \xi - \sin \xi_{p}} - \frac{\cos \xi}{i(\sin \xi + \sin \xi_{p})}
\]

The integral along \( \Gamma \) can be evaluated by using the same contour deformation as in Section 2. However, due to the absence of the square root \( k_{1} \) in Eq. 75 there is no branch cut contribution to the integration contour. The integral can thus be split into one contribution from the residue and one contribution from the SDP. We get therefore:

\[
\Psi_{SP}^\perp (x, z) = 2\pi i \text{Res}[F_{SP}^\perp (\xi_{p})] e^{ik_{0}nr \cos (\xi_{p} - \vartheta)} \Theta (\vartheta - \vartheta_{LR}) + e^{ik_{0}nr} \sum_{m \in \text{even}} \frac{\Gamma(m+1)}{m!(k_{0}nr)^{m+1}} \frac{d^{m}}{d\tau^{m}} G_{SP}^\perp (0)
\]

with \( G_{SP}^\perp (\tau) = F_{SP}^\perp (\xi_{p}) \frac{d\xi}{d\tau} \).

Few remarks are important:

(i) First, the singular term

\[
2\pi i \text{Res}[F_{SP}^\perp (\xi_{p})] e^{ik_{0}nr \cos (\xi_{p} - \vartheta)} \Theta (\vartheta - \vartheta_{LR})
\]

is exactly identical to the pole contribution appearing in Eq. 22. This results from the equality \( \text{Res}[F_{SP}^\perp (\xi_{p})] = \text{Res}[F_{SP}^{TM,\perp} (\xi_{p})] \) (compare with Eqs. 24-25).

(ii) Second, the term \( m = 0 \) in the SDP contribution is dominant in the far-field regime and leads to \( \Psi_{SP}^\perp (x, z) = \Psi_{SP}^\perp (k_{0}n \sin \vartheta, z = d) \) as expected.

(iii) Third, the decomposition \( Q(s) = Q_{0}(s) + \text{Res}[Q(s)]/(s - s_{p}) + \text{Res}[Q(-s)]/(s + s_{p}) \) leads to

\[
G_{SP}^\perp (\tau) = G_{SP,0}^\perp (\tau) + \frac{\text{Res}[G_{SP}^\perp (\tau_{p})]}{\tau - \tau_{p}} + \frac{\text{Res}[G_{SP}^\perp (\tau_{-p})]}{\tau - \tau_{-p}}
\]
where \( \tau_p = -e^{i\pi/4}\sqrt{2}\sin((\xi_p + \vartheta)/2) \). Therefore, if we compare with Eqs. 32-38 we see that \( \Psi_{SP}^{\perp}(x, z) \) is not exactly equal to \( I_{SP} + I_{SP}^{\text{pole}} \) explicitly defined in Eqs. 37 and 36. More precisely we obtain:

\[
\Psi_{SP}^{\perp}(x, z) = 2i\pi \text{Res}[G_{SP}^{\perp}(\tau_p)]e^{ik_0nr\cos(\xi_p - \vartheta)}e^{i\vartheta/2}(\xi_p - \vartheta_{LR})
\]

which differs from Eqs. 37, 38 by the two last lines. We can also rewrite these expressions as

\[
\Psi_{SP}^{\perp}(x, z) = e^{ik_0nr} \sum_{m \in \text{even}} \frac{\Gamma(m+1)}{\Gamma(k_0nr)^{m+1}} \frac{d^m}{d\tau_{m+1}} G_{SP}^{\perp}(0)
\]

where we used Eq. 36. applied to \( \tau_p \) and \( \tau_{SP} \).

**IV. MORE ON INTENSITY AND FIELD IN THE BACK FOCAL PLANE AND IMAGE PLANE OF THE MICROSCOPE**

A general analysis of the imaging process occurring through a microscope objective with high numerical aperture \( NA \) and an ocular tube lens is given in for example Ref. [5]. Here, we give without proofs the calculated field and intensity in the focal plane of the objective and the image plane of the microscope expressed in term of the TE and TM scalar potentials defined in Eqs. 1.2.

For this purpose we use the Fourier transform of the electromagnetic TM and TE field at the \( z = d \) interface defined by:

\[
\Phi_{TM}[k, z] = -\{kk_3(k) - k^2\hat{z}\} \Psi_{TM}[k, z]
\]

\[
\Phi_{TE}[k, z] = -k_0n^2k \times \hat{z} \Psi_{TE}[k, z].
\]

This implies [6] that the electric field recorded in the back focal plane of the objective is (i.e. taking into account the vectorial nature of the field and the transformation of the spherical wave front to a planar wave front):

\[
E_{\text{back focal plane}} = \frac{2\pi e^{ik_0nf}}{if} T_1 \sqrt{k_0k_3(k)} \left\{ \begin{array}{c} -k_0n k \Psi_{TM}[k, d] + k_0n^2k \hat{z} \Psi_{TE}[k, d] \end{array} \right\}
\]

with by definition \( \varphi_1 = -k \times \hat{z}/k \). The geometric coefficient \( k_3(k) \) is reminiscent from the 'sin' condition [3] which lead to strong geometrical aberrations at very large angle \( \vartheta \). As a direct consequence we deduce the intensity in the back focal plane:

\[
|E_{\text{back focal plane}}|^2 = \frac{4\pi^2 T_1}{f^2n^2k_0k_3(k)} |\Phi_{TM}[k, d]|^2 + |\Phi_{TE}[k, d]|^2
\]

which is therefore proportional to the total Fourier field intensity for TM and TE waves taken separately.

Finally, in the image plane we obtain the electric field :

\[
E(x') = N' \int_{|k| \leq k_0NA} d^2k \sqrt{k_3(k)} e^{-ik \cdot \hat{z}'} \cdot \{\Phi_{TM}[k, d] + \Phi_{TE}[k, d]\}
\]
\[ E(x') = N' \int_{|k| \leq k_0N} d^2 k \sqrt{k_3(k)} e^{-ik \cdot x'} \cdot \{-k_0n k \tilde{\Psi}_{TM}[k, d] + k_0 n^2 k \tilde{\varphi}_1 \tilde{\Psi}_{TE}[k, d]\} \]  

where \( N' \) is a constant characterizing the microscope. In the letter we used these formulas for computing fields and intensity in the Fourier and image plane (see Figs. 3, 4 of the letter).

### Appendix A

We have by definition
\[ \tau''_p = \sin \left( \frac{\xi'_p - \vartheta}{2} \right) \cosh \left( \frac{\xi''_p}{2} \right) + \cos \left( \frac{\xi'_p - \vartheta}{2} \right) \sinh \left( \frac{\xi''_p}{2} \right). \]  

(A1)

The condition \( \tau''_p < 0 \) is equivalent to \( \tan \left( \frac{\xi'_p - \vartheta}{2} \right) < -\tanh \left( \frac{\xi''_p}{2} \right) \), i.e. to
\[ \frac{\xi'_p - \vartheta}{2} < -\arctan \left( \tanh \left( \frac{\xi''_p}{2} \right) \right) = -\arccos \left( \frac{1}{\cosh \left( \frac{\xi''_p}{2} \right)} \right). \]  

(A2)

We therefore obtain \( \vartheta_{LR} < \vartheta \) where holds the relation
\[ \cos \left( \frac{\xi'_p - \vartheta_{LR}}{2} \right) \cosh \xi''_p = 1. \]  

(A3)

This is clearly the definition of the leakage radiation angle introduced in the discussion of the singular term \( I_{SP} \). This therefore implies the equality
\[ \Theta(\vartheta - \vartheta_{LR}) = \Theta( -\tau''_p). \]  

(A4)

### Appendix B

We have the relation \( G(\tau) = F(\xi) \frac{dF}{d\tau} \) and we define
\[ F(\xi) = F_0(\xi) + \text{Res}[F(\xi)] \]  

\[ G(\tau) = G_0(\tau) + \text{Res}[G(\tau)] \]  

(B1)

Therefore we obtain for the residues the relation:
\[ \text{Res}[G(\tau_p)] = \frac{1}{2 \pi i} \oint_{C_p} d\tau G(\tau) \]
\[ = \frac{1}{2 \pi i} \oint_{C_p} d\xi F(\xi) = \text{Res}[F(\xi_p)]. \]  

(B2)

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Imaging surface plasmons: from leaky waves to far-field radiation

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We show that, contrary to the common wisdom, surface plasmon poles are not involved in the imaging process in leakage radiation microscopy. Identifying the leakage radiation modes directly from a transverse magnetic potential leads us to reconsider the surface plasmon field and unfold the non-plasmonic contribution to the image formation. While both contributions interfere in the imaging process, our analysis reveals that the reassessed plasmonic field embodies a pole mathematically similar to the usual surface plasmon pole. This removes a long-standing ambiguity associated with plasmonic signals in leakage radiation microscopy.

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Surface plasmon optics has become a mature and extended field of research, ranging from the development of new optical nanodevices and nanoantennas to the renewal of integrated quantum optics [1, 2]. In this context, surface plasmon imaging techniques are of critical importance to the researcher, among which leakage radiation microscopy (LRM) is now emerging as a powerful tool [3, 4]. As a far-field optical method, LRM is used for analyzing SP modes both in direct and Fourier (momentum) spaces and it has been successfully implemented in various plasmonic systems, both at the classical and quantum level [5–10]. Yet, there is still no satisfying theoretical definition of the SP field in an imaging context. Instead, recent work on leaky waves have focused on semi-infinite air-metal interfaces, a configuration not relevant to LRM [14–17]. This has fuelled recent debates concerning the precise relation between experimentally recorded images and SP modes [11–13].

In this Letter, we instead propose a novel approach to the problem of leaky waves that provides a full analytical theory of the coherent SP imaging process in the case of a point-like radiating electric dipole located in air above the thin metal film. This leads us to a new definition of the SP field as a Fano-type interfering component of the imaged radiation. We derive analytical expressions for the far-field radiation that meets all the necessary conditions prescribed to the leakage field symmetries [18]. Importantly, we show that our approach naturally makes the SP field free from the long standing ambiguities of the historical Zenneck and Sommerfeld solutions [19, 20] and removes the field discontinuity at the LR angle, until now problematic. Doing so, we also identify the contribution of a lateral wave thus far unnoticed that we associate with a new type of Goos-Hänchen effect in transmission.

In the geometry considered in Fig. 1, a harmonically radiating point-like dipole $\mu e^{-i\omega t}$ drives an SP wave that leaks through the film in medium 3 and then propagates in the matching oil of the high numerical aperture (NA) immersion objective required for LRM. Due to the specific dispersion relation of SP waves, leakage radiation (LR) is emitted at an angle $\theta_{LR} > \theta_c = \arcsin \sqrt{\varepsilon_1/\varepsilon_3}$ defining a radiation cone in the forbidden-light sector shown on Fig. 1(a) which intersects the reference sphere $\Sigma$ of the LRM objective.

Due to the planar symmetry of the problem, the radiated field in medium 3 can be represented in terms of
transverse magnetic (TM) and electric (TE) scalar potentials $\Psi$ with $[\nabla^2 + k_0^2 \varepsilon_3]\Psi_{TM,TE} = 0$ where $k_0 = \omega/c$. For simplicity, the case of a dipole normal to the film $\mathbf{\mu} = \mu_{\perp} \hat{z}$ (i.e. $\Psi_{TE} = 0$) is only discussed here. A general and detailed calculation is given in [23]. Using boundary conditions at the different interfaces, we expand the potential at $[x = (x, y), z]$ as

$$
\Psi_{TM}(x, z) = \int_{-\infty}^{+\infty} k dk A(k, z) H_0^{(1)}(k|x|) + i \sum_{m=1}^{3} F_{13}(m) e^{ik_{1m}x} e^{ik_{3m}z} + H_0^{(+)}(k|x|) \text{ is the zeroth order radiating-like Hankel function (evolving asymptotically as } e^{+ik|x|}/\sqrt{k|x|} \text{ for large } |x|). \text{ The Fresnel coefficient } F_{13}(m) \text{ gives the transmission through the film of TM radiation with a wave vector } |k| = k. \text{ Stability imposes complex square roots } k_j(k) = \sqrt{k_0^2 - k^2} \text{ (} j = 1 \text{ or } 3) \text{ with } \text{Im}[k_j] \geq 0.
$$

The precise computation of such a Sommerfeld-like integral is extraordinarily involved due to the presence of two branch-cuts associated with $k_{13}(k)$ and several SP poles in the complex $k$-plane [24]. To simplify at most the problem, we chose an alternative parametrization of the integral through the complex variable $\xi$ defining $k = k_0 \sin \xi$. This leaves only one branch-cut $k = k_0 \sqrt{\varepsilon_3 - \varepsilon_3 \sin^2 \xi}$ with a branch-point $\xi = \vartheta_c$. We impose $\text{Im}[k_3] \geq 0$ in the whole complex $\xi$-plane as the choice of the Riemann sheet $R_+$. The integral becomes

$$
\Psi_{TM}(x, z) = \int_{\Gamma} d\xi F(\xi) e^{ik_{0m}nr \cos(\xi - \vartheta)} \text{ with } \xi = r \sin \vartheta, z = d + r \cos \vartheta \text{ leading to } (z - d) \cos \xi + |x| \sin \xi = r \cos(\xi - \vartheta). \text{ The initial contour } \Gamma \text{ corresponding to the condition } \sin \xi \text{ real is represented on Fig. 1(b).}
$$

To evaluate Eq. (1), we deform the contour $\Gamma$ in order to include the steepest descent path (SDP) determined by $\text{Im}[f_{\hat{k}}(\xi)] = 1$ with $f_{\hat{k}}(\xi) = i \cos(\xi - \vartheta)$. The SDP crosses $\Gamma$ at the saddle point $\xi_0 = \vartheta$ defined by $df(\xi)/d\xi = 0$. The SDP contribution $\Psi_{SDP}$ to the field is calculated using the steepest-descent method discussed below. Two additional contributions $\Psi_p$ and $\Psi_{BC}$ associated respectively with the SP poles and the BC have to be accounted for when deforming the contour. The most relevant here is a single SP pole $k_p = k_0 \sin \xi_p$ resulting from the divergency of $T_{13}(k) = N_{13}(k)/D_{13}(k)$ when $D_{13}(k_p) = 0$. Such a transcendental equation is known to possess four kinds of SP modes corresponding to leaky waves and bound modes in medium 1 or 3 [23]. Importantly on the $R_+$ sheet, only the leaky mode in medium 3 (labeled symmetric leaky in [18]) is possibly encircled during the contour integration (see Fig. 1(b)) depending on whether $\xi_0 > \vartheta_{LR}$ or not. This implies that the residue contribution to $\Psi_{TM}$ associated with the SP pole reads

$$
\Psi_p = 2\pi i \text{Res}[F(\xi_p)] e^{ik_{0m}nr \cos(\xi_p - \vartheta)} \Theta(\vartheta - \vartheta_{LR}), \quad \text{where } \Theta(x) \text{ is the Heaviside unit-step function and where the LR angle is precisely defined by } \text{Im}[f_{\hat{k}_{LR}}(\xi_p)] = 1, \text{ i.e. } \vartheta_{LR} = \text{Re}[\xi_p] + \arccos(1/\cosh \text{Im}[\xi_p]) \approx \text{Re}[\xi_p]. \text{ An intensity plot of this contribution is displayed on Fig. 2(a) which clearly shows the conical wave front structure emitted at the angle arctan(Re[k_p]/Re[k_3]) = Re[\xi_p] \approx \vartheta_{LR}.}$

FIG. 2: Intensity contour-plots corresponding to the contributions (a) $\Psi_p$ and (b) $\Psi_{SDP}$, with their coherent superposition displayed in (c). To remove the asymptotic divergence at infinity we calculated $\text{Im}\left[F(\xi_p)\right]^2$ in (a), while we calculated instead $\tau^2(\text{Re}[\xi_p])^2$ in (b, c).

Beyond $\vartheta_{LR}$, $\Psi_p \propto e^{ik_{0p}z} H_0^{(+)}(k_p|x|)$ (where $k_{3p} = k_3(k_p)$) in direct relation to the original derivation by Zenneck and Sommerfeld of surface waves [19, 20]. It corresponds to what modern literature coined leaky SP mode [4, 18] belonging to the general family of leaky waves discussed for years in the radio antenna community [23]. It is important to realize however that this contribution is actually non-physical, due to the field discontinuity at $\vartheta_{LR}$ introduced by the $\Theta(\vartheta - \vartheta_{LR})$ pre-factor. It thus appears necessary to find a genuinely physical definition of a leaky SP wave. We now show that this is only possible by including in our discussion both $\Psi_{SDP}$ and $\Psi_{BC}$ contributions.

The central result of the Letter is that the LRM imaging process is essentially determined by $\Psi_{SDP}$ which is evaluated as

$$
\Psi_{SDP} = e^{ik_{0m}nr} \sum_{m \in \text{even}} \frac{\Gamma(m+1)}{m!(k_0nr)^m} \frac{d^m}{d\tau^m} G(0) \quad (3)
$$

with the variable $\tau = e^{ir/4} \sqrt{2} \sin ((\xi - \vartheta)/2)$ and the function $G(\tau) = F(\xi) \frac{dF}{d\xi}$ in the vicinity of the saddle point $\tau = 0$ (i.e. $\xi_0 = \vartheta$). A second contribution must be accounted for when $\vartheta > \vartheta_c$ because in this case, the close integration path have to surround the branch-cut in the
\( R \) sheet (see Fig. 1(b)). Following 26, this contribution can be given as a series expansion

\[
\Psi_{\text{BC}} \approx e^{i \Delta \varphi \Theta(\vartheta - \vartheta_c)} \sum_{m=1}^{+\infty} \frac{\alpha_m(r, \vartheta_c)}{(k_0nr \sin(\vartheta - \vartheta_c))^{1+m/2}} \tag{4}
\]

where the coefficients \( \alpha_m(r, \vartheta_c) \) can be explicitly computed and \( \Delta \varphi = k_0nr \cos(\vartheta - \vartheta_c) \) is interpreted as the phase accumulated by a wave creeping along the metal film (at velocity \( c \)) and re-emitted at the critical angle \( \vartheta_c \) (at velocity \( c/n \)) 23. Such a wave is associated in our case to a Goos–Hänchen-like effect in transmission 26.

This contribution is not specific to the LRM geometry but it has never been discussed whereas the \( m = 1 \) far-field dominant term in Eq. 4 evolves as \( \sim 1/r^2 \), i.e. as a Norton wave defined on the same \( 1/r^2 \) order from \( \Psi_{\text{SDP}} \) 27.

These terms however can be neglected in the far field \( (r \gg 2\pi/k_0) \) where only survives the dominant \( 1/r \) term in the power expansion of \( \Psi_{\text{SDP}, m=0} \) in Eq. 3. The radiated far field is thus

\[
\Psi_{\text{SDP}, m=0}(x, z) \approx \frac{2\pi k_0nr \cos \vartheta}{ir} e^{i k_0nr \Psi_{\text{TM}}(k, d)} \tag{5}
\]

where \( \Psi_{\text{TM}}[k, d] = A(k, d)/\pi \) is the bidimensional Fourier transform of \( \Psi_{\text{TM}}(x, z) \) calculated at \( z = d \) for the in-plane wave vector \( k = k_0n \sin \vartheta/|x| \). This expression shows in Fig. 2(b) a conical structure peaked on \( \vartheta_{\text{LR}} \) which should be compared with the one obtained from \( \Psi_p \) alone in Fig. 2(a). The comparison with Fig. 2(c) combining both contributions coherently clearly expresses a central result for LRM: \( \Psi_{\text{SDP}, m=0} \) strongly dominates not only over \( \Psi_{\text{BC}} \) as discussed above, but also over \( \Psi_p \), consistently with the finite value of the SP propagation length \( l_p = (2\pi|k_p|)^{-1} \) that overdamps the exponential tail of \( \Psi_p \) for angles \( \vartheta \geq \vartheta_{\text{LR}} \).

From an imaging perspective, the radiating field given by Eq. 5 is detected in the objective back-focal plane \( \Pi \) sketched in Fig. 1 (see 21). We plot \( I_{\text{TM}, \Pi}(k) \) in Fig. 3 for a dipole either perpendicular or parallel to the interface 22. We point out that these intensity maps are in quantitative agreement with experiments 4, 10 and clearly reveal a bidimensional ring with radius \( k_r \approx \text{Re}[k_p] \) and width \( \delta k \approx 2\pi|k_p| \).

As a fundamental paradox, this ring is nowadays associated with the detection of the SP mode 4, 22, 30 despite the fact that it is \( \Psi_{\text{SDP}} \) and certainly not \( \Psi_p \) which is involved in the measurement process. This paradox stems from \( G(\tau) \) being singular (thus \( \Psi_{\text{SDP}} \) too) at the SP pole \( \tau_p \) (i.e. \( \xi_p \)), with a polar contribution \( \Psi_{\text{SDP}}^{\text{pole}} \) sharing a closed mathematical relation with \( \Psi_p \). This generated an historical confusion regarding the actual role of SP modes in the Sommerfeld integral, an issue debated since the work of Zenneck and Sommerfeld 14, 19, 20.

In order to remove the ambiguity, we reconsider the very definition of what the SP field is, by going back to Eq. 4 and observing that \( \Psi_{\text{TM}}(k, d) \) is an explicit function of \( k = k_0n \sin \vartheta \) with \( \vartheta \in [0, \pi/2] \). This function can be easily continued over the complex \( k \)-plane analytically. This function presents some isolated poles such as \( k_p \) and \( -k_p \) which allow us to decompose \( \Psi_{\text{TM}}(k, d) \) into a regular and polar part. It is this polar part

\[
\Psi_{\text{TM}}^{\text{pole}}(k, d) = \eta_p \frac{1}{\sqrt{k - k_p}} + \frac{1}{i(k + k_p)} \tag{6}
\]

with \( \eta_p = \frac{i\mu_e}{8\pi} k_p e^{ik_p x} e^{ik_p d} \frac{N_{13}(k_p)}{\sqrt{k_p}} \frac{1}{\Im(k_p)} \) that we will define as the SP field \( \Psi_{\text{SP}}(k, d) := \Psi_{\text{TM}}^{\text{pole}}(k, d) \) (see 22 for the general case). To justify this definition, we point out that from Eq. 4 we deduce a SP field \( \Psi_{\text{SP}}(x, z) = \int d^2k \Psi_{\text{TM}}^{\text{pole}}(k, d) e^{i k x} e^{i k_p (z - d)} \) which, in analogy with Eq. 3, is alternatively defined by a contour integral over \( \xi \) along \( \Gamma \)

\[
\Psi_{\text{SP}}(x, z) = \int_{\Gamma} d\xi F_{\text{SP}}(\xi) e^{i k_0nr \cos(\xi - \vartheta)} \tag{7}
\]

where \( F_{\text{SP}}(\xi) = \Psi_{\text{TM}}^{\text{pole}}(k, d) k |H_{13}^{(2)}(k|x)| e^{-ik|x|} \). The remarkable fact about Eq. 7 is that it can be evaluated by the procedures used for Eq. 3, without any branch cut, and split into two contributions from the residue
another from the SDP:
\[
\Psi_{SP}(x, z) = 2\pi \text{Res}[F_{SP}(\xi_p)] e^{ik_{0}nr \cos(\xi_p - \beta)} \Theta(\beta - \theta_{LR}) + \sum_{m \in \mathbb{N}} \frac{\Gamma(m + \frac{1}{2})}{m! (k_{0}nr)^{m+\frac{1}{2}}} d^m G_{SP}(0)
\]
with \( G_{SP}(\tau) = F_{SP}(\xi_{G}) \). Importantly, the residue term in Eq. (8) is identical to \( \Psi_p \) given that \( \text{Res}[F_{SP}(\xi_p)] = \text{Res}[F(\xi_{G})] \).

In the far field, the \( m = 0 \) term in the sum dominates and we have \( \Psi_{SP}(x, z) \approx 2\pi k_{0}cos(\beta) n_{r} e^{ik_{0}nr \cos(\xi_p - \beta)} \Psi_{SP}[k, d] \). In Fig. 2 we compare this expression for \( \Psi_{SP} \) to Eq. (8) by computing the intensity in the back-focal plane. In the case of a vertical dipole -Fig. 3(a)- the SP term \( I_{SP, \text{II}}[k] \) is quasi-identical to \( I_{TM, \text{II}}[k] \). We define the non-plasmonic signal by \( I_{SS}[k] \propto |\Psi_{TM}[k, d]|^2 \). In the case of a horizontal dipole -Fig. 3(b)- there is also an additional TE contribution \( I_{TE, \text{II}}[k] \propto \left| \Psi_{TE}[k, d] \right|^2 \) to \( I_{SS} \). The intensity dip observed for such a horizontal dipole is attributed to a Fano-type interference effect in the k-space between the peaked SP contribution and the broad non-plasmonic signal made explicit by our analysis [31, 32].

In a last step, we calculate direct space images through a microscope ocular (see Fig. 1(a)) associated with SP propagation on the metal film by an inverse Fourier transform of the field signal in the II plane, taking into account the finite angular aperture of the objective [23]. We compare in Fig. 4(a,b) the images calculated from Eqs. (8) and (9) respectively for a vertical and a horizontal dipole. Signal differences are more important for a horizontal dipole where TE and TM fields interfere, and decrease for distances larger than \( 2\pi/k_{0} \).

We point out that the pure SP field at a point \( x' \) of the image plane is given by a simple expression
\[
E_{SP}(x') \propto \int_{|k| \leq k_{0}NA} d^2 k \sqrt{k_3} \Psi_{SP}[k, d] e^{ik_{0}x'}
\]
\[
= - \int d^2 x D_{SP, \parallel}(x, d) \chi(x + \frac{x'}{M})/\sqrt{k_{SP}}
\]
where \( D_{SP, \parallel}(x, d) = \frac{\partial^2}{\partial u^2} \Psi_{SP}(x, d) \) is the in-plane component of the SP displacement field along the interface \( z = d \) and \( \chi(u) \approx \frac{k_{0}NA}{2\pi|u|} J_{1}(k_{0}NA|u|) \) is the (scalar) point-spread function of the microscope objective. Taking a large microscope magnification \( M = nf'/f < 1 \) (with \( f', f \) being respectively the objective and ocular focal lengths) enables us to analyze the recorded images simply using the paraxial-like Eq. (9), despite that leaky waves are emitted in a non-paraxial regime at \( \theta_{LR} \).

Additionally, since \( \Psi_{SP}[k, d] \) defines a sharp ring-like distribution, we can approximately write \( E_{SP}(x') \propto D_{SP, \parallel}(-x'/M, d) \) for large \(|x'|/M\). Therefore, as shown in Fig. 4 the difference between the real image and the SP field vanishes asymptotically when \(|x'| \) increases. Finally, this analysis shows that only in-plane components of the SP field participate to the image, therefore resolving definitively the current controversy [11, 12].

To conclude, we have removed the long-standing ambiguity associated with the very definition of a SP mode as probed in LRM through a revision of the SP field. We have shown how this field interferes, in a Fano-type way, with a broad non-plasmonic radiative signal in the LR imaging process. We expect our findings to have important impact in the ever-growing field of plasmonics in its different variants: classical to quantum through molecular to non-linear plasmonics.

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