Chain Polytopes and Algebras with Straightening Laws

Takayuki Hibi · Nan Li

Dedicated to Professor Ngô Viêt Trung on the occasion of his sixtieth birthday

Received: 25 June 2014 / Revised: 10 September 2014 / Accepted: 7 October 2014 / Published online: 19 February 2015
© Institute of Mathematics, Vietnam Academy of Science and Technology (VAST) and Springer Science+Business Media Singapore 2015

Abstract It will be shown that the toric ring of the chain polytope of a finite partially ordered set is an algebra with straightening laws on a finite distributive lattice. Thus, in particular, every chain polytope possesses a regular unimodular triangulation arising from a flag complex.

Keywords Algebra with straightening laws · Chain polytope · Partially ordered set

Mathematics Subject Classification (2010) Primary 52B20 · Secondary 13P10 · 03G10

1 Introduction

In [6], the order polytope $O(P)$ and the chain polytope $C(P)$ of a finite poset (partially ordered set) $P$ are studied in detail from a viewpoint of combinatorics. Toric rings of order polytopes are studied in [1]. In particular, it is shown that the toric ring $K[O(P)]$ of the order polytope $O(P)$ is an algebra with straightening laws ([2, p. 124]) on a finite distributive lattice. In the present paper, it will be proved that the toric ring $K[C(P)]$ of the chain polytope $C(P)$ is also an algebra with straightening laws on a finite distributive lattice. It then follows immediately that $C(P)$ possesses a regular unimodular triangulation arising from a flag complex.

T. Hibi (✉)
Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan
e-mail: hibi@math.sci.osaka-u.ac.jp

N. Li
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
e-mail: nan@math.mit.edu
2 Toric Rings of Order Polytopes and Chain Polytopes

Let \( P = \{x_1, \ldots, x_d\} \) be a finite poset. For each subset \( W \subseteq P \), we associate
\[
\rho(W) = \sum_{i \in W} e_i \in \mathbb{R}^d,
\]
where \( e_1, \ldots, e_d \) are the unit coordinate vectors of \( \mathbb{R}^d \). In particular, \( \rho(\emptyset) \) is the origin of \( \mathbb{R}^d \). A poset ideal of \( P \) is a subset \( I \) of \( P \) such that, for all \( x_i \) and \( x_j \) with \( x_i \in I \) and \( x_j \leq x_i \) one has \( x_j \in I \). An antichain of \( P \) is a subset \( A \) of \( P \) such that \( x_i \) and \( x_j \) belonging to \( A \) with \( i \neq j \) are incomparable.

Recall that the order polytope is the convex polytope \( \mathcal{O}(P) \subseteq \mathbb{R}^d \) which consists of those \((a_1, \ldots, a_d) \in \mathbb{R}^d\) such that \( 0 \leq a_i \leq 1 \) for every \( 1 \leq i \leq d \) together with \( a_i \geq a_j \) if \( x_i \leq x_j \) in \( P \). The vertices of \( \mathcal{O}(P) \) are those \( \rho(I) \) such that \( I \) is a poset ideal of \( P \) ([6, Corollary 1.3]). The chain polytope is the convex polytope \( \mathcal{C}(P) \subseteq \mathbb{R}^d \) which consists of those \((a_1, \ldots, a_d) \in \mathbb{R}^d\) such that \( a_i \geq 0 \) for every \( 1 \leq i \leq d \) together with
\[
a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leq 1
\]
for every maximal chain \( x_{i_1} < x_{i_2} < \cdots < x_{i_k} \) of \( P \). The vertices of \( \mathcal{C}(P) \) are those \( \rho(A) \) such that \( A \) is an antichain of \( P \) ([6, Theorem 2.2]).

Let \( S = [x_1, \ldots, x_d, t] \) denote the polynomial ring over a field \( K \) whose variables are the elements of \( P \) together with the new variable \( t \). For each subset \( W \subseteq P \), we associate the squarefree monomial \( x(W) = \prod_{i \in W} x_i \in S \). In particular, \( x(\emptyset) = 1 \). The toric ring \( K[\mathcal{O}(P)] \) of \( \mathcal{O}(P) \) is the subring of \( R \) generated by those monomials \( t \cdot x(I) \) such that \( I \) is a poset ideal of \( P \). The toric ring \( K[\mathcal{C}(P)] \) of \( \mathcal{C}(P) \) is the subring of \( R \) generated by those monomials \( t \cdot x(A) \) such that \( A \) is an antichain of \( P \).

3 Algebras with Straightening Laws

Let \( R = \bigoplus_{n \geq 0} R_n \) be a graded algebra over a field \( K_0 = K \). Suppose that \( P \) is a poset with an injection \( \varphi : P \to R \) such that \( \varphi(\alpha) \) is a homogeneous element of \( R \) with \( \deg \varphi(\alpha) \geq 1 \) for every \( \alpha \in P \). A standard monomial of \( R \) is a finite product of the form \( \varphi(\alpha_1)\varphi(\alpha_2) \cdots \) with \( \alpha_1 \leq \alpha_2 \leq \cdots \). Then, we say that \( R = \bigoplus_{n \geq 0} R_n \) is an algebra with straightening laws on \( P \) over \( K \) if the following conditions are satisfied:

- The set of standard monomials is a basis of \( R \) as a vector space over \( K \).
- If \( \alpha \) and \( \beta \) in \( P \) are incomparable and if
\[
\varphi(\alpha)\varphi(\beta) = \sum_i r_i \varphi(\gamma_{i_1})\varphi(\gamma_{i_2}) \cdots ,
\]
where \( 0 \neq r_i \in K \) and \( \gamma_{i_1} \leq \gamma_{i_2} \leq \cdots \) is the unique expression for \( \varphi(\alpha)\varphi(\beta) \in R \) as a linear combination of distinct standard monomials, then \( \gamma_{i_1} \leq \alpha \) and \( \gamma_{i_1} \leq \beta \) for every \( i \).

We refer the reader to [2, Chapter XIII] for fundamental material on algebras with straightening laws. The relations (1) are called the straightening relations of \( R \).

Let \( P \) be an arbitrary finite poset and \( \mathcal{J}(P) \) the finite distributive lattice ([7, p. 252]), consisting of all poset ideals of \( P \) ordered by inclusion. The toric ring \( K[\mathcal{O}(P)] \) of the order polytope \( \mathcal{O}(P) \) is a graded ring with \( \deg(t \cdot x(I)) = 1 \) for every \( I \in \mathcal{J}(P) \). We then define the injection \( \varphi : \mathcal{J}(P) \to K[\mathcal{O}(P)] \) by setting \( \varphi(I) = t \cdot x(I) \) for every \( I \in \mathcal{J}(P) \). One
of the fundamental results obtained in [1] is that \(K[\mathcal{O}(P)]\) is an algebra with straightening laws on \(\mathcal{J}(P)\). Its straightening relations are

\[
\varphi(I)\varphi(J) = \varphi(I \cap J)\varphi(I \cup J),
\]

where \(I\) and \(J\) are poset ideals of \(P\) which are incomparable in \(\mathcal{J}(P)\).

**Theorem 3.1** The toric ring of the chain polytope of a finite poset is an algebra with straightening laws on a finite distributive lattice.

**Proof** Let \(P\) be an arbitrary finite poset and \(\mathcal{C}(P)\) its chain polytope. The toric ring \(K[\mathcal{C}(P)]\) is a graded ring with \(\deg(t \cdot x(A)) = 1\) for every antichain \(A\) of \(P\).

For a subset \(Z \subseteq P\), we write \(\max(Z)\) for the set of maximal elements of \(Z\). In particular, \(\max(Z)\) is an antichain of \(P\). The poset ideal of \(P\) generated by a subset \(Y \subseteq P\) is the smallest poset ideal of \(P\) which contains \(Y\).

Now, we define the injection \(\psi : \mathcal{J}(P) \to K[\mathcal{C}(P)]\) by setting \(\psi(I) = t \cdot x(\max(I))\) for all poset ideals \(I\) of \(P\). If \(I\) and \(J\) are poset ideals of \(P\), then

\[
\psi(I)\psi(J) = \psi(I \cup J)\psi(I * J),
\]

where \(I * J\) is the poset ideal of \(P\) generated by \(\max(I \cap J) \cap (\max(I) \cup \max(J))\). Since \(I * J \subseteq I\) and \(I * J \subseteq J\), the relations (3) satisfy the condition of the straightening relations.

It remains to prove that the set of standard monomials of \(K[\mathcal{C}(P)]\) is a \(K\)-basis of \(K[\mathcal{C}(P)]\). It follows from [6, Theorem 4.1] that the Hilbert function ([2, p. 33]) of the Ehrhart ring ([2, p. 97]) of \(\mathcal{O}(P)\) coincides with that of \(\mathcal{C}(P)\). Since \(\mathcal{O}(P)\) and \(\mathcal{C}(P)\) possess the integer decomposition property ([4, Lemma 2.1]), the Ehrhart ring of \(\mathcal{O}(P)\) coincides with \(K[\mathcal{O}(P)]\) and the Ehrhart ring of \(\mathcal{C}(P)\) coincides with \(K[\mathcal{C}(P)]\). Hence, the Hilbert function of \(K[\mathcal{O}(P)]\) is equal to that of \(K[\mathcal{C}(P)]\). Thus, the set of standard monomials of \(K[\mathcal{C}(P)]\) is the \(K\)-basis of \(K[\mathcal{C}(P)]\) as desired.

**4 Flag and Unimodular Triangulations**

The fact that \(K[\mathcal{C}(P)]\) is an algebra with straightening laws guarantees that the toric ideal of \(\mathcal{C}(P)\) possesses an initial ideal generated by squarefree quadratic monomials. We refer the reader to [3] and [5, Appendix] for the background of the existence of squarefree quadratic initial ideals of toric ideals. By virtue of [8, Theorem 8.3], it follows that

**Corollary 4.1** Every chain polytope possesses a regular unimodular triangulation arising from a flag complex.

**5 Further Questions**

Let, as before, \(P\) be a finite poset and \(\mathcal{J}(P)\) the finite distributive lattice consisting of all poset ideals of \(P\) ordered by inclusion. Let \(S = K[x_1, \ldots, x_n, t]\) denote the polynomial ring and \(\Omega = \{w_I\}_{I \in \mathcal{J}(P)}\) a set of monomials in \(x_1, \ldots, x_n\) indexed by \(\mathcal{J}(P)\). We write \(K[\Omega]\) for the subring of \(S\) generated by those monomials \(w_I \cdot t\) with \(I \in \mathcal{J}(P)\) and define
the injection $\varphi : \mathcal{J}(P) \to K[\Omega]$ by setting $\varphi(I) = w_I \cdot t$ for every $I \in \mathcal{J}(P)$.

Suppose that $K[\Omega]$ is an algebra with straightening laws on $\mathcal{J}(P)$ over $K$. We say that $K[\Omega]$ is compatible if each of its straightening relations is of the form $\varphi(I)\varphi(I') = \varphi(J)\varphi(J')$ such that $J \leq I \wedge I'$ and $J' \geq I \vee I'$, where $I$ and $I'$ are poset ideals of $P$ which are incomparable in $\mathcal{J}(P)$.

Let $K[\Omega]$ and $K[\Omega']$ be compatible algebras with straightening laws on $\mathcal{J}(P)$ over $K$. Then, we identify $K[\Omega]$ with $K[\Omega']$ if the straightening relations of $K[\Omega]$ coincide with those of $K[\Omega']$.

Let $P^*$ be the dual poset ([7, p. 247]) of a poset $P$. The toric ring $K[C(P^*)]$ of $C(P^*)$ can be regarded as an algebra with straightening laws on $\mathcal{J}(P)$ over $K$ in the obvious way. Clearly, each of the toric rings $K[O(P)]$, $K[C(P)]$, and $K[C(P^*)]$ is a compatible algebra with straightening laws on $\mathcal{J}(P)$ over $K$.

**Question 5.1** (a) Given a finite poset $P$, find all possible compatible algebras with straightening laws on $\mathcal{J}(P)$ over $K$.

(b) In particular, for which posets $P$, does there exist a unique compatible algebra with straightening laws on $\mathcal{J}(P)$ over $K$?

**Example 5.2** (a) Let $P$ be the poset of Fig. 1. Then, $K[O(P)] = K[C(P)]$, and there exists a unique compatible algebra with straightening laws on $\mathcal{J}(P)$ over $K$. In fact, $\mathcal{J}(P)$ for $P$ is the poset in Fig. 2. In the corresponding algebra, we must have $be = af$. Then, we have either $bc = ad$ or $bc = af$. However, since $bc \neq be$, it follows that $bc = ad$. Similarly, we have $de = cf$. Hence, the ASL relations are unique.

(b) Let $P$ be the poset of Fig. 3. Then, there exist three compatible algebras with straightening laws on $\mathcal{J}(P)$ over $K$. They are $K[O(P)]$, $K[C(P)]$, and $K[C(P^*)]$.

(c) Let $P$ be the poset of Fig. 4. Then, there exist nine compatible algebras with straightening laws on $\mathcal{J}(P)$ over $K$.

**Fig. 1** Example 5.2 (a)
Conjecture 5.3 If $P$ is a disjoint union of chains, then the compatible algebras with straightening laws on $\mathcal{J}(P)$ over $K$ are $K[\mathcal{O}(P)]$, $K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$. 
References

1. Hibi, T.: Distributive lattices, affine semigroup rings and algebras with straightening laws. In: Nagata, M., Matsumura, H. (eds.) Commutative algebra and combinatorics on advanced studies in pure mathematical, vol. 11, pp. 93–109, Amsterdam (1987)
2. Hibi, T.: Algebraic combinatorics on convex polytopes. Carslaw Publications, Glebe (1992)
3. Ohsugi, H., Hibi, T.: Toric ideals generated by quadratic binomials. J. Algebra 218, 509–527 (1999)
4. Ohsugi, H., Hibi, T.: Convex polytopes all of whose reverse lexicographic initial ideals are squarefree. Proc. Am. Math. Soc. 129, 2541–2546 (2001)
5. Ohsugi, H., Hibi, T.: Quadratic initial ideals of root systems. Proc. Am. Math. Soc. 130, 1913–1922 (2002)
6. Stanley, R.: Two poset polytopes. Discrete Comput. Geom. 1, 9–23 (1986)
7. Stanley, R. Enumerative combinatorics, 2nd edn., vol. I. Cambridge University Press, Cambridge (2012)
8. Sturmfels, B.: Gröbner bases and convex polytopes. America Mathematical Society, Providence (1995)