Filtering crosstalk from bath non-Markovianity via spacetime classical shadows

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From an open system perspective non-Markovian effects due to a nearby bath or neighbouring qubits are dynamically equivalent. However, there is a conceptual distinction to account for: neighbouring qubits may be controlled. We combine recent advances in non-Markovian quantum process tomography with the framework of classical shadows to characterise spatiotemporal quantum correlations. Observables here constitute operations applied to the system, where the free operation is the maximally depolarising channel. Using this as a causal break, we systematically erase causal pathways to narrow down the progenitors of temporal correlations. We show that one application of this is to filter out the effects of crosstalk and probe only non-Markovianity from an inaccessible bath. It also provides a lens on spatiotemporally spreading correlated noise throughout a lattice from common environments. We demonstrate both examples on synthetic data. Owing to the scaling of classical shadows, we can erase arbitrarily many neighbouring qubits at no extra cost. Our procedure is thus efficient and amenable to systems even with all-to-all interactions.

INTRODUCTION

In the race to fault tolerant quantum computing, magnified sensitivity to complex dynamics in open quantum systems requires increasingly tailored characterisation and spectroscopic techniques [1–8]. Correlated dynamics are one particularly pernicious class of noise, and can be generated from a variety of sources, including inhomogeneous magnetic fields, coherent bath defects, and nearby qubits, see Figure 1a [9, 10]. Concerningly, these effects are often omitted from quantum error correction noise models despite being ubiquitous in noisy intermediate-scale quantum (NISQ) hardware [5, 6, 11–14].

Temporal – or non-Markovian – correlations are elements of error that are correlated between different points in time, as mediated by interactions with an external system [6, 15]. A process is said to be non-Markovian if the total dynamics do not factorise into a product of dynamical maps [16], a stronger condition than the well-known completely-positive divisibility of dynamics [17]. The specific mediator of these effects is both conceptually and experimentally relevant device information. Is it controllable, or is it part of the inaccessible bath? Relatedly, if the dynamics of two nearby qubits do not spatially factorise, this is known as crosstalk. If one qubit is traced out, then entangling crosstalk – such as the always-on ZZ interactions in transmon qubits [18] – can generate temporal correlations for the second qubit. Whether the dynamics look non-Markovian depends on whether it is feasible or not to dilate the characterisation to multiple qubits and account for the variables responsible for these correlations. Typically, it is not. Since crosstalk and bath non-Markovianity can easily be conflated, it is crucial to find robust methods that can not only account for their behaviour, but distinguish them.

In this Letter, we establish a systematic, concrete, and efficient approach to the two pragmatic questions: (1)
if non-Markovian dynamics are detected across different timescales for a qubit, do they come from neighbouring qubits or a nearby bath? And (2) how can we determine when two qubits are coupled to a shared bath generating common cause non-Markovian effects. The solutions here have highly practical implications. Namely, whether curbing the correlated effects is achievable through control or fabrication methods [19, 20]. Process tensor tomography (PTT) is a recently developed generalisation to quantum process tomography, and can guarantee an answer to these questions and more, but the number of experiments required grows as $O(d^{2kn+N})$ to find correlations across $k$ steps over $N$ qubits [6].

The basic premise of our work is to apply the method of classical shadows [21] to PTT, resolving these problems. The classical shadow philosophy implements randomised single-shot measurements to learn properties of a state, granting access to an exponentially larger pool of observables at fixed locality. Employing this, instead of reconstructing the whole multi-time process for an entire quantum register, we can estimate and analyse each of the fixed-weight process marginals. Marginalising over a measurement is equivalent to measuring and throwing the outcome away. To marginalise over a process input is equivalent to inputting a maximally mixed state. Hence, these are maximal depolarising channels at no extra cost, which act as causal breaks on controllable systems.

When suitably placed, these operations eliminate temporal correlations as mediated on the chosen Hilbert spaces, thus allowing non-Markovian sources to be causally tested. We illustrate this idea in Figure 1b. The end result is the simultaneous determination of the bath-mediated non-Markovianity on all qubits. Our approach hence only depends on the individual system size (in this work, qubits), and is a physics-independent way for us to test the relevant hypotheses. We are also able to simultaneously compute all spacetime marginals, extending the randomised measurement toolkit to the spatiotemporal domain [22].

**SPATIOTEMPORAL CLASSICAL SHADOWS**

By virtue of the state-process equivalence for multi-time processes [15, 23–25], quantum operations on different parts of a system at different times constitute observables on a many-body quantum state. This allows state-of-the-art characterisation techniques to be applied to quantum stochastic processes. Classical shadow tomography [21, 22] is one such technique, and already has many generalisations and applications [26–29]. Measuring classical shadows allows for exponentially greater observables to be determined about a state, provided sufficiently low weight. But this restriction means the technique has limitations for the study of temporal correlations (which are high weight) in contrast to spatial ones, as discussed in Ref. [12]. Our work expands on this to the multi-qubit-multi-time case, and identifies other desirable applications of classical shadows to multi-time processes.

**Definitions and Notation.** Consider a quantum device with a register of qudits $Q := \{q_1, q_2, \cdots, q_N\}$ across a series of times $T_k := \{t_0, t_1, \cdots, t_k\}$. We take the whole quantum device to define the system: $\mathcal{H}_S := \bigotimes_{j=1}^N \mathcal{H}_{q_j}$. The device interacts with an external, inaccessible environment whose space we denote $\mathcal{H}_E$. The $k$-step open process is driven by a sequence $\textbf{A}_{k−1:0}$ of control operations on the whole register, each represented mathematically by completely positive (CP) maps: $\textbf{A}_{k−1:0} := \{\mathcal{A}_0, \mathcal{A}_1, \cdots, \mathcal{A}_{k−1}\}$, after which one obtains a final state $\rho_k^S(\textbf{A}_{k−1:0})$ conditioned on this choice of interventions. Note that where we label an object with time information only, that object is assumed to concern the entire register. These controlled dynamics have the form:

$$\rho_k^S(\textbf{A}_{k−1:0}) = \text{Tr}_E[U_{k:k−1} \mathcal{A}_{k−1:1} \cdots U_{1:0} \mathcal{A}_0(\rho_0^{SE})],$$  

where $U_{k:k−1}(\cdot) = u_{k:k−1}(\cdot) u_{k:k−1}^\dagger$. Now let the Choi representations of each $\mathcal{A}_j$ be denoted by a caret, i.e. $\hat{\mathcal{A}}_j = \mathcal{A}_j \otimes \mathcal{I}[|\Phi^+\rangle\langle\Phi^+|] = \sum_{n,m} \mathcal{A}_{ij}[|n\rangle\langle m|] \otimes |n\rangle\langle m|$. Then, the driven process in Equation (1) for arbitrary $k_1$ uniquely defines a multi-linear mapping across the register $Q$—called a process tensor, $\Upsilon_{k_0}$—via a generalised Born rule [15, 24]:

$$\rho_k^S(\textbf{A}_{k−1:0}) = \text{Tr} \left[ \Upsilon_{k:0} \left( U_{k:k−1} \otimes \hat{\mathcal{A}}_k \otimes \cdots \hat{\mathcal{A}}_0 \right)^T \right],$$  

At each time $t_j$, the process has an output index $\sigma_j$ (which is measured), and input index $i_{j+1}$ (which feeds back into the process). The details of process tensors can be found in the appendix, but are not crucial to understanding this work. The two important properties that we stress are: (i) a sequence of operations constitutes an observable on the process tensor via Equation (2), generating the connection to classical shadows, and (ii) a process tensor forms a collection of possibly correlated completely positive, trace-preserving (CPTP) maps, and hence may be marginalised in both time and space to yield the $j$th CPTP map describing the dynamics of the $j$th qubit $\mathcal{E}_{k:j−1}$. A process is said to be Markovian if and only if its process tensor is a product state across time. The measure of non-Markovianity we use throughout this work is that described in Ref. [16]. Specifically, it is the relative entropy $S[\rho||\sigma] = \text{Tr} [\rho \log \rho - \log \sigma]$ between a process tensor $\Upsilon_{k:0}$ and its closest Markov description, the product of its marginals:

$$\Upsilon_{k:0}^{(\text{Markov})} = \hat{\mathcal{E}}_{k:k−1} \otimes \cdots \otimes \hat{\mathcal{E}}_{1:0} \otimes \rho_0.$$

We denote this generalised quantum mutual information (QMI) for a given process by $\mathcal{N}(\Upsilon_{k:0})$. Classical shadow
The case where \( t \) read out at time \( t \), and each \( t_{j+1} \) signifies the preparation of a new state, also at time \( t_j \).

tomography provides access to a small number of low weight observables, with \( (I) \) on the remainder of the subsystems. The case where \( \langle \hat{I}_{j+1}, \hat{I}_j \rangle \) is evaluated is equivalent to selecting an \( \hat{A}_j = \hat{I}_{j+1} \otimes \hat{I}_j = \hat{I}_{j+1} \hat{I}_j \). This is the Choi state of the maximal depolarising channel, up to normalisation. When marginalising across all but a handful of times or qubits, we will denote the remaining steps or registers by commas, i.e.

\[
\hat{E}_{j_0:j_0-1,j_1:j_1-1} = \text{Tr}_l[\{I_{j_0}, I_{j_0-1}, I_{j_1}, I_{j_1-1}\}[Y_{k:0}]],
\]

where the overlines denote complement, i.e. every qubit except \( q_l : Q \setminus \{q_l\} \), or every time except \( t_j : T_k \setminus \{t_j\} \).

When non-Markovian correlations persist as mediated by the inaccessible bath, we designate this as bath non-Markovianity (BNM). When the correlations are mediated from neighbouring qubits, we designate this as register non-Markovianity (RNM). Naturally, since the bath cannot be controlled by definition, BNM can be probed without RNM, but RNM effects cannot be isolated by themselves. Instead, one might consider the spatial process marginals alone to measure direct crosstalk [26, 30, 31].

Procedure. – To map these correlations on each qubit, the classical shadows procedure naturally extends as follows: at each \( t \in T_k \), on each \( q \in Q \), apply a unitary operation randomly selected from the single qubit Clifford group, followed by a projective measurement in the \( Z \)-basis. This defines a POVM on all qubits across all times \( \{U_l^p | x \rangle \langle x | U_l^p\}_{q \in Q} \). The measurements considered have four defining features: the qubit \( q \) on which they act, the time \( t \) at which they are implemented, and the basis change \( U \) applied prior to a measurement outcome \( x \). To avoid notational overload, we omit these final two labels when writing instruments where the context is clear.

Record both the outcome of the measurement and the random unitary. Reset the qubit to state \( |0\rangle \) and apply a random Clifford gate, recording this operation as well. The intended effect of this is to apply a randomised quantum instrument – i.e. a random measurement with an independently random post-measurement state. See Figure 2 for the circuit diagram. The application of an instrument in each chosen location in space and time constitutes a single-shot piece of information about the process tensor. The single shot is a projection of the process tensor onto the sequence of interleaving measurements \( P \) and preparations \( P_l^q \):
to have the $\sigma$ and $i$ legs alternating, and from which properties can be efficiently determined using the usual median-of-means estimation described in Ref. [21].

**ERASING NON-MARKOVIAN PATHWAYS**

The above procedure suffices to estimate marginals of a process tensor with only logarithmic overhead, which we show for completeness in the appendix. In short, we estimate the required observables to uniquely fix the process marginal, and then employ a maximum likelihood algorithm to determine a physically consistent process tensor. We consider two possible applications of spatiotemporal classical shadows, supplemented by numerical demonstrations.

**Distinguishing between passive crosstalk and bath non-Markovianity.**—First, we consider certifying when non-Markovian correlations originate via an inaccessible bath, or from neighbouring qubits in the register. Certifying bath non-Markovianity means estimating $\gamma_{k:0}$ — the marginal process tensor for a single qubit. This can be simultaneously performed for all $q_i \in $ Q. The Choi state of the operations on the remainder of the qubits at each time will be $I/d$, i.e. a maximally depolarising channel. Because this enacts a causal break any information travelling from the system into the register cannot persist forwards in time. Hence, computing $\mathcal{N}(\gamma_{k:0})$ will be a measure of correlations from an inaccessible bath alone. We formally show this in the appendix.

We demonstrate this numerically in Figure 3. Here, we have 15 qubits and one defect quantum system acting as the bath in a two-step process, and then compute $\mathcal{N}(\gamma_{k:0})$. The qubits each experience a random nearest-neighbour $ZZ$-coupling crosstalk, and the ones geometrically closest to the defect are Heisenberg-coupled to that system: $H = \sum_i \sum_j J_{ij}^{(i)} \sigma_i^{(i)} \sigma_j^{(j)}$ for random $J_{ij}^{(i)}$. Figure 3a shows the standard fare: estimating the process tensor of each qubit and determining its non-Markovianity while the other qubits remain idle. However, the results are not so informative, because they do not distinguish between RNM and BNM effects, and so every qubit experiences temporal correlations. Figure 3b shows the results of a shadow marginal estimation, and we readily identify only the qubits coupled to bath defects have a non-zero $\mathcal{N}(\gamma_{k:0})$.

**Identifying shared baths.**—A second important scenario we consider is where two qubits are correlated via common cause from a shared bath. For example, this might be experiencing the same stray magnetic field inhomogeneities or through a coupling to a common defect. This is sometimes referred to as crosstalk, because the joint

FIG. 3. A numerical simulation demonstrating the proposed technique to isolate environmental effects. A grid of 15 qubits is simulated with crosstalk effects and an inaccessible non-Markovian defect. **a** Determining the non-Markovianity on each qubit individually (with the remainder idle) is not very informative, since each qubit looks non-Markovian due to passive crosstalk. **b** After learning all of the shadow marginals $\gamma_{k:0}$, the crosstalk is filtered out to reveal which qubits possess temporal correlations from the environment.

FIG. 4. A numerical simulation demonstrating the proposed technique to determine qubits with shared baths. A grid of 14 qubits is simulated with crosstalk effects and two inaccessible non-Markovian defects. **a** Shadow filtering may be used to find qubits coupled to an inaccessible bath as before **b** By looking at the correlations between map $\mathcal{E}^{(1)}$ with $\mathcal{E}^{(2)}$, we can infer which qubits share common baths and the extent to which the defects redistribute quantum information. Note, the relationship lines between qubits are not direct crosstalk interactions, but bath-mediated spatiotemporal correlation.
map $\hat{E}^{(q_1,q_2)}_{j,j-1}$ does not factor to $\hat{E}^{(q_1)}_{j,j-1} \otimes \hat{E}^{(q_2)}_{j,j-1}$ [30]. However, we consider this a coarse description because neither system acts as a direct cause for each other’s dynamics. Instead, they are subject to spatiotemporal correlations as mediated by the same non-Markovian bath. The key, therefore, is to measure the relationship between the maps $\hat{E}^{(q)}_{j,j-1}$ and $\hat{E}^{(q)}_{j,j+1}$.

We demonstrate this numerically in Figure 4. We have a similar setup to before, except this time with two bath defects. Performing a shadow filtering (Figure 4a) again reveals which qubits are coupled to the defects. However, in Figure 4b, we look at the spacetime marginals estimated from the shadow data. This fine-grained data indicates which qubits are commonly coupled to bath defects. How-ever, we consider this a coarse description because neither system acts as a direct cause for each other’s dynamics. Instead, they are subject to spatiotemporal correlations as mediated by the same non-Markovian bath. The key, therefore, is to measure the relationship between the maps $\hat{E}^{(q)}_{j,j-1}$ and $\hat{E}^{(q)}_{j,j+1}$.

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Process tensors and process tensor marginals from classical shadow tomography

We review here the background of process tensors, and detail the procedure to obtain process tensor marginals from the classic shadow data. For a more extensive discussion, see Refs. [6, 15, 32].

Process tensor framework. A quantum stochastic process describes the non-deterministic dynamics of an open quantum system $\rho^S \in \mathcal{B}(\mathcal{H}_S)$ across a series of times $\mathbf{T}_k = \{t_0, t_1, \cdots, t_k\}$. Specifically, this represents knowledge of the conditional state $\rho_k^S(\mathbf{A}_{k−1:0})$ subject to an arbitrary sequence $\mathbf{A}_{k−1:0} := \{\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_{k−1}\}$ of control operations at each $t_j$. Note that these controls in full generality constitute non-deterministic maps, such as quantum instruments. These controlled dynamics have the form:

$$\rho_k^S(\mathbf{A}_{k−1:0}) = \text{tr}_E[U_{k,k−1} \mathbf{A}_{k−1} \cdots \mathbf{U}_{1:0} \mathbf{A}_0(\rho_{0E})],$$  \hfill (11)

where $U_{k,k−1}(\cdot) = u_{k,k−1}(\cdot)u_{k,k−1}^\dagger$. Eq. (11) can be used to define a mapping from past controls $\mathbf{A}_{k−1:0}$ to future states $\rho_k(\mathbf{A}_{k−1:0})$, which is the process tensor $\mathbf{T}_{k:0}$:

$$\mathbf{T}_{k:0}[\mathbf{A}_{k−1:0}] = \rho_k^S(\mathbf{A}_{k−1:0}).$$  \hfill (12)

Each intervention $\mathbf{A}_j$ has a Choi representation $\mathbf{A}_j$ via the Choi-Jamiolkowski isomorphism (CJI), given by its action on one half of a maximally entangled state $|\Phi^+\rangle = \sum_{i=1}^d |ii\rangle$:

$$\hat{\mathbf{A}}_j = (\mathbf{A} \otimes I) [|\Phi^+\rangle\langle\Phi^+|] = \sum_{i,j=1}^d \mathbf{A}[|ii\rangle\langle jj|] \otimes |ii\rangle\langle jj|. \hfill (13)$$

The two subsystems of this state can be seen as ‘input’ and ‘output’ spaces: $\mathbf{A} \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_I)$, since the output state under the action of the map is given on the left space after projecting some state onto the right subsystem. That is, $\mathbf{A}[\sigma] = \text{Tr}_I [I_0 \otimes \sigma^T, \mathbf{A}]$.

Through a generalisation of the CJI, process tensors may be represented as a $2k+1$-partite quantum state, $\Psi_{k:0}$. Equation (13) is generalised by swapping in one half of a fresh maximally entangled state at each time. This state then predicts multi-time probabilities in accordance with Equation (11). That is, when the final state $\rho_k^S(\mathbf{A}_{k−1:0})$ is measured with a POVM $\{\Pi_k\}$, the observed probabilities for each effect is given by

$$p_k^S(\mathbf{A}_{k−1:0}) = \text{Tr} \left[ \Psi_{k:0} \left( \Pi_k \otimes \hat{\mathbf{A}}_{k−1} \otimes \cdots \otimes \hat{\mathbf{A}}_0 \right)^T \right].$$  \hfill (14)

Thus, sequences of CP maps constitute observables on the process tensor. If a process output is measured when conditioned on an informationally complete set of instruments, then Equation (14) can be uniquely inverted to reconstruct $\mathbf{T}_{k:0}$ - although in practice when the probabilities are actually noisy frequency estimates, an estimation procedure such as maximum likelihood must be employed to ensure physical conditions are imposed, like positivity of the map and causal direction of the statistics.

The Choi form of a process tensor is an $2k+1$-partite quantum state $\Psi_{k:0}$. Any correlations between these subsystems constitute non-Markovian correlations due to the external environment. In keeping with our channel Choi state convention, we use the notation $\sigma_j$ to denote an output leg of the process at time $t_j$, and $i_j$ for the input leg of the process at time $t_j−1$. The collection of indices is therefore $\{\sigma_k, i_k, \cdots, \sigma_2, i_2, \sigma_1, i_1, \sigma_0\}$.

Assuming now that we have a process tensor defined on a register $\mathbf{Q} = \{q_1, \cdots, q_N\}$, then we can fine-grain not just in time but also in space. We first note that without loss of informational completeness, each instrument $\mathbf{A}_j$ can be factored into local operations $\otimes_{i=1}^N \hat{\mathbf{A}}_j^{(q_i)}$. The dynamical maps of the process can also be further marginalised in the same way to obtain $\hat{\mathbf{E}}_{j;j−1}^{(q_i)}$: the average CPTP map taking the $ith$ qubit from time $t_{j−1}$ to $t_j$. Although process tensors contain information about all multi-time correlations, for simplicity’s (and efficiency’s) sake, in this work we consider only two map correlations. Therefore, spacetime correlations can be probed across the following pairs of marginals:

- Purely spatial: $\hat{\mathbf{E}}_{j;j−1}^{(q_n)}$ and $\hat{\mathbf{E}}_{j;j−1}^{(q_m)}$.
- Purely temporal: $\hat{\mathbf{E}}_{j;j−1}^{(q_n)}$ and $\hat{\mathbf{E}}_{t;t−1}^{(q_n)}$.
- Spatiotemporal: $\hat{\mathbf{E}}_{j;j−1}^{(q_n)}$ and $\hat{\mathbf{E}}_{t;t−1}^{(q_n)}$. 


阴影过滤

FIG. 5. Graphical depiction of causal testing. At the top, we represent the process tensor in terms of its dilated link product representation. Below, we show how the transformation implements depolarising channels, leaving the process as tensor product structure across $\mathcal{B}(\mathcal{H}_C)$.

Pure spatial correlations constitute either direct crosstalk, or presence of an environmental common cause factor. As we show in the next section, purely temporal marginals constitute genuine bath non-Markovian correlations. And finally, spatiotemporal correlations indicate two qubits coupled to the same non-Markovian bath.

Recovering process marginals.–

Although collecting shadows suffices to estimate properties of the process marginals, they will not in general be able to reconstruct a physical estimate of the process marginals. To this effect we estimate enough observables to constitute informationally complete information about the process tensor, and then employ maximum-likelihood PTT to process the data and obtain a physical estimate. The advantages of a physical (positive, causal) estimate is that we make use of information theoretic tools that rely on the positivity of the state.

The total number of shadows required to estimate $M$ observables $O_i$ to a precision of $\epsilon$ was shown by Huang et al. to be $21$

\[
N_{\text{shots}} = \mathcal{O}\left( \frac{\log M}{\epsilon^2} \max_{1 \leq i \leq M} \|O_i\|_{\text{shadow}} \right). \tag{15}
\]

For an $l$-local Pauli observable, this shadow norm is shown to be $3^l$. The numerical simulations in the main text are two step processes, whose Choi forms are equivalent to a five qubit state. The remainder of the observables on the process are trivial, i.e. $I$. In the Choi picture, this is equivalent to a maximally depolarising channel. Explicitly,

\[
\mathcal{R}_\lambda[\rho] := (1 - \lambda)\rho + \lambda I / 2, \tag{16}
\]

then $\mathcal{R}_1 = I / 2$, with normalisation chosen to be $d$.

Causality conditions fix the expectations of $\sum_{j=1}^k (d^2 - 1)d^{4j-2}$ Pauli operators. For a two step process then, this leaves $M = 820$ expectation values to estimate. From the bounds given in Ref. [21], we therefore have $N_{\text{shots}} = \mathcal{O}(\log(N_{\text{qubits}} \cdot M)/\epsilon^2) \approx 2 \times 10^7$ measurements in the worst case, which is what we use in our numerical experiments to achieve a precision of $\epsilon = 0.01$. The median-of-means algorithm from Ref. [21] is used to estimate an informationally complete set of observables for each process marginal, and the maximum likelihood estimation obtained using the methods developed in Ref. [6].

Pathways proof

Distinguishing RNM and BNM.– Here, we give a short graphical proof for the fact that applying depolarising channels to other qubits in the register at each time eliminates them as a potential non-Markovian source. In the dilated open quantum evolution picture, a process tensor may be written as the link product $\star$ of a series of $SE$
unitaries.

\[ \Upsilon_{k,0} = \text{Tr}_E \left[ \bigotimes_{i=1}^k \hat{U}_{i,i-1}^{(SE)} \ast \rho_0^{(SE)} \right], \quad (17) \]

where \( \hat{U}_{i,i-1} \) is the Choi representation of the \( i \)th step unitary \( U_{i,i-1} \). This lives on \( \mathcal{B}(\mathcal{H})_S^i \otimes \mathcal{B}(\mathcal{H})_E^i \otimes \mathcal{B}(\mathcal{H})_S^0 \otimes \mathcal{B}(\mathcal{H})_E^0 \). Suppose the environment factorises into two spaces: \( \mathcal{H}_E \cong \mathcal{H}_C \otimes \mathcal{H}_B \) – the controllable qubits and the uncontrollable bath – then each \( U_{i,i-1} \) has twelve indices: ket and bra for the input and output spaces for each of \( S, C, B \). Equation (17) can be written more intuitively using the graphical calculus formalism [39], as shown at the top of Figure 5.

Because depolarising channels are a product state over \( \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{B}(\mathcal{H}_i) \), they hence implement a causal break. Note that any choice of instrument for which the Choi state is a product state (for example, a projective measurement and fresh preparation) constitutes a valid causal break, but a depolarising channel is the only one for which the locality of the observable does not grow, and hence can be efficient under the classical shadows procedure.

**Identifying systems with a shared non-Markovian bath.** The same logic applies for the observation of correlations between some \( \hat{E}^{(qn)}_{j,j-1} \) and \( \hat{E}^{(qm)}_{l,l-1} \). Obtaining these two correlated marginals means that a causal break is applied at the \( \sigma \) leg of \( \hat{E}^{(qm)}_{j,j-1} \) and at the \( i \) leg of \( \hat{E}^{(qn)}_{l,l-1} \). Hence, this erases any mixing terms between \( \mathcal{H}_{qn} \) and \( \mathcal{H}_{qm} \) in the unitaries. Consequently, non-zero values for the mutual information between these maps must be due to interactions involving \( \mathcal{H}_B \) & \( \mathcal{H}_{qn} \) at the first time step, and then \( \mathcal{H}_B \) & \( \mathcal{H}_{qm} \) at the second second. I.e. the two qubits shared the same bath.