On the Optimization Dynamics of Wide Hypernetworks

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Abstract

Recent results in the theoretical study of deep learning have shown that the optimization dynamics of wide neural networks exhibit a surprisingly simple behaviour. In this work, we study the optimization dynamics of hypernetworks, which are architectures in which a learned meta-network produces the weights of a task-specific primary network. Hypernetworks have been demonstrated repeatedly to obtain state of the art results. However, their theoretical understanding is still lacking.

As can be expected, the optimization process of multiplicative models is much more complicated than optimizing standard ReLU networks. It is shown that for an infinitely wide neural network with a gating layer the cost function cannot be accurately approximated by its first order Taylor approximation. Specifically, for a fixed sized primary network of depth $H$, the first $H$ terms of the Taylor approximation of the cost function are non-zero, even when the meta-network is infinitely wide. However, for an infinitely wide meta and primary networks, the learning dynamics is determined by a linear model obtained from the first-order Taylor expansion of the network around its initial parameters and the kernel of this process is given by the Hadamard product of the kernels induced by the meta and primary networks.

As part of our study, we partially solve an open problem suggested by Dyer & Gur-Ari (2020) and show that the convergence rate of the $r$’th order term of the Taylor expansion of the cost function along the optimization trajectories of SGD is $O\left(1/n^{r-1}\right)$, where $n$ is the width of the learned neural network, improving upon the $O(1/n)$ bound suggested by the conjecture of Dyer & Gur-Ari, while matching their empirical observations.

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1 Introduction

A longstanding question in deep learning is how different architectures affect the ability to train deep networks. A popular approach for studying this question, is the “infinite width” setting (Arora et al., 2019; Sirignano & Spiliopoulos, 2019; Yang & Schoenholz, 2017; Lee et al., 2019), which provides a convenient framework for analyzing deep neural networks. For instance, Lee et al. (2018) suggested that for various architectures, when the weights are assumed to be i.i.d samples, the pre-activations converge to Gaussian Processes (GP).

A recent paper on the Neural Tangent Kernel (NTK) (Jacot et al., 2018) showed that when the width of the network approaches infinity the gradient-descent training dynamics of a fully connected network \( f \) can be characterized by a kernel. Differently said, the evolution through time of the function computed by the network follows the dynamics of kernel gradient descent. To prove this phenomenon, various papers (Lee et al., 2019; Bai et al., 2020; Bai & Lee, 2020) introduce a Taylor expansion of the cost function around the initialization and consider its values. It is shown that the first order term is deterministic during the SGD optimization and the higher order terms converge to zero as the width \( n \) tends to infinity.

A central open problem in the literature is to quantify the convergence rate of the higher order terms as a function of width. Lee et al. (2019) showed a convergence rate of the linear term of order \( O(1/\sqrt{n}) \). The upper bound was later improved by Dyer & Gur-Ari (2020); Hanin & Nica (2020) to \( O(1/n) \). Specifically, Dyer & Gur-Ari (2020) introduced the notion of correlation functions, obtained by taking the ensemble averages of \( f \), its products, and its derivatives with respect to the parameters, evaluated on arbitrary inputs. They provide a conjecture regarding the orders of magnitude of the correlation functions. An implication of the proposed conjecture states that the \( r \)th term of the Taylor expansion of the cost function at the SGD trajectories, expanded around the initialization, is upper bounded, in order \( O(1/n) \). In this paper, we prove (without relying on the conjecture) that this hypothesized upper bound is increasingly loose as \( r \) increases, and prove an asymptotic behaviour in the order of \( 1/n^{r-1} \). This result implies that for a large enough neural network, the higher order terms in the Taylor expansion exponentially diminish and therefore, are irrelevant. This provides a novel insight into the inductive bias of neural networks coupled with gradient descent algorithms, beyond the common linearized regimes.

The existing literature is focused on ReLU networks. In this work we extend the scope and study the dynamics of wide multiplicative models. Examples of such models include networks that employ gating, and networks that output the weights of other networks, often refereed to as hypernetworks.

A typical hypernetwork \( h \) involves two networks, \( f \) and \( g \). The function \( h \) takes two inputs \( x \) and \( z \). The meta-network \( f \) takes the input \( x \) and returns the weights of the primary network, \( g \). The primary network, takes the input \( z \) and returns the output of \( h \).

In Thms. 2 and 3 we prove that for infinitely wide neural networks with gatings and hypernetworks with an infinitely wide meta-network and a fixed sized primary network, the Taylor expansion of the cost function at initialization is non-linear. In particular, for the case of hypernetworks, the degree of the Taylor expansion is determined by the depth \( H \) of the primary network.

In Sec. 5 we show that for a hypernetwork with an infinitely wide meta and primary networks, the NTK decomposes into the Hadamard product of kernels belonging to the meta-network and the primary network, and that its time derivative vanishes.

However, a much more complicated dynamics arise when considering a fixed sized primary network. In Thm. 6 we show that for hypernetworks with an infinitely wide meta-network and a fixed sized primary network, the NTK is a non-constant random variable. Furthermore, the first \( H \) terms of the Taylor expansion around the initialization point are random, and do not vanish.

Overall, our results uncover a tradeoff between the depth and width of the primary network. Adding layers to the primary network can be detrimental to the optimization process, since it introduces high order terms. However, this problem can be strongly mitigated by increasing the width of the primary network.
1.1 Related Work

The connection between infinitely wide neural networks, Gaussian processes and kernel methods, has been the focus of many recent papers. Empirical support has demonstrated the power of CNTK (convolutional neural tangent kernel) on practical datasets, demonstrating new state of the art results for kernel methods. Dyer & Gur-Ari (2020) extended the NTK framework for ResNets (He et al., 2016). In their paper, they showed that despite vanilla ReLU networks, convergence to the NTK may occur when depth and width simultaneously tend to infinity, while giving the exact behaviour as a function of width.

As part of the referred analysis, we provide a partial solution to a conjecture proposed by Dyer & Gur-Ari (2020). In their paper, they conjecture the asymptotic behaviour of general correlation functions involving high order derivative tensors, which arise when analysing the dynamics of gradient descent. Roughly speaking, given inputs \( \{x_i\}_{i=1}^r \) the outputs of a neural network \( f(x_1; w), \ldots, f(x_r; w) \in \mathbb{R} \) with normally distributed parameters \( w \in \mathbb{R}^N \), correlation functions takes the form:

\[
\sum_{\eta_0, \ldots, \eta_r \in [N]} \mathbb{E}_w \left[ \prod_{j=1}^{r} \Gamma_{\eta_{j+1}, \ldots, \eta_{j+1}}(x_j) \right] \tag{1}
\]

where

\[
\Gamma_{\eta_1, \ldots, \eta_k}(x_j) := \frac{\partial^k f(x_j; w)}{\partial w_{\eta_1} \cdots \partial w_{\eta_k}} \tag{2}
\]

Computing these correlation functions involve keeping track of various moments of normally distributed weights along paths, as done in recent finite width correction works. Dyer & Gur-Ari (2020) employ the Feynman diagram to efficiently accomplish this often tedious task, albeit at the cost of only being provably accurate for deep linear, or shallow ReLU networks. Understanding the asymptotic behaviour of these terms can be crucial for understanding training dynamics, as the derivative of the NTK is composed of these terms. In this work, we analyze the asymptotic behaviour correlation functions of the form:

\[
\mathcal{T}'(x_0, \ldots, x_r) := \sum_{\eta_0, \ldots, \eta_r \in [N]} \Gamma_{\eta_1, \ldots, \eta_r}(x_0) \prod_{j=1}^{r} \Gamma_{\eta_{j+1}}(x_j) \tag{3}
\]

where \( \nabla_w^{(r)} f(x_0) \) is a rank \( r \) tensor, representing the \( r \)'th derivative of the output, and \( \bigotimes_{j=1}^{r} \nabla_w f(x_j) \) denotes outer products of the gradients for different examples. For clarity, the following are two examples of correlation functions,

\[
\mathbb{E}_w \left[ f(x_1; w) \cdot \frac{\partial f(x_2; w)}{\partial w_{\mu_1}} \right] \quad \text{and} \quad \mathbb{E}_w \left[ \frac{\partial^2 f(x_1; w)}{\partial w_{\mu_1} \partial w_{\mu_2}} \cdot \frac{\partial f(x_2; w)}{\partial w_{\mu_1}} \right] \tag{4}
\]

As we show in Sec. 4, terms of the form in Eq. 3 represent high order terms in the multivariate Taylor expansion of outputs, and are, therefore, relevant for the full understanding of training dynamics, beyond linearization. In Theorem. 1 we prove that \( \mathcal{T}'(x_0, \ldots, x_r) \sim \frac{1}{r} \) for vanilla neural nets, where \( n \) is the width of the network. The above result is a partial solution to an open problem suggested by Dyer & Gur-Ari (2020). In their paper, they conjecture the asymptotic behaviour of general correlation functions, and predict an upper bound on the asymptotic behaviour of terms of the form in Eq. 3 in the order of \( \frac{1}{r} \). Our results therefore prove a stronger version of the conjecture, while giving the exact behaviour as a function of width.
We parameterize the primary network $g$, where $f$ was used to generate the weights of a second RNN used to perform the actual task, called the primary network. However, the idea of having one network predict the weights of another was proposed earlier. For instance, [Klein et al. 2015; Riegler et al. 2015] proposed a framework in which the convolution weights are specified as the output of a separate neural network based on the input in order to adapt the lower layers to the motion of the image input. This was later extended for multiple layers by [Jia et al. 2016], for video frame and stereo view prediction.

Hypermnetworks can be naturally applied for meta-learning. In [Bertinetto et al. 2016] they employed hypernetworks for few-shot learning tasks. Hypermnetworks were recently used for continuous learning by [von Oswald et al. 2020]. Hypermnetworks can be efficiently used for neural architecture search, as was demonstrated by [Brock et al. 2018; Zhang et al. 2019]. In [Lorraine & Duvenaud 2018] they applied hypermnetworks for hyperparameters selection.

A recent paper [Jayakumar et al. 2020] studies the role of multiplicative interaction within a unifying framework to describe a range of classical and modern neural network architectural motifs, such as gating, attention layers, hypernetworks, and dynamic convolutions amongst others. It is shown that standard neural networks are a strict subset of neural networks with multiplicative interactions. In [Galanti & Wolf 2020] they theoretically compare the expressive power of hypermnetworks and standard embedding methods in an attempt to understand the benefits of multiplicative models.

Despite their success and increasing prominence, little theoretical work was done in order to better understand hypermnetworks and their behavior. [Chang et al. 2020] showed that applying standard initializations on a hypernetwork produces sub-optimal initialization of the primary network. A principled technique for weight initialization in hypermnetworks is then developed.

2 Setup

In this section we introduce the setting of the analysis considered in this paper. We begin by defining fully connected neural networks and hypermnetworks in the context of the NTK framework.

**Neural Networks** In the NTK framework, a fully connected neural network, $f(x; w) = g^L(x)$, is defined in the following manner:

$$
\begin{align*}
    g^l(x) &= \sqrt{\frac{1}{n_l}}W^l d^{l-1}(x) \\
    q^0(x) &= x,
\end{align*}
$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is the activation function of $f$, taken specifically to be the ReLU function $\text{ReLU}(x) := \max(0, x)$, and the weight matrices $W^l \in \mathbb{R}^{n_l \times n_{l-1}}$ are trainable variables, initialized independently according to a standard normal distribution, $W_{i,j}^l \sim \mathcal{N}(0, 1)$. The width of $f$ is denoted by $n := \min(n_1, \ldots, n_{L-1})$. The parameters $w$ are aggregated as a long vector $w = (\text{vec}(W^1), \ldots, \text{vec}(W^L))$. The coefficients $\sqrt{\frac{1}{n_l}}$ serve for normalizing the activations of each layer. This parametrization is nonstandard, and we will refer to it as the NTK parameterization. It has already been employed in several recent works (Karras et al. 2018; van Laarhoven 2017; Park et al. 2019).

**Hypermnetworks** Given the input tuple $u = (x, z) \in \mathbb{R}^{n_0 \times m_0}$, a hypermnetwork is a function of the form:

$$
h(u; w) := g(z; f(x; w))
$$

where $f(x; w)$ and $g(z; v)$ are two neural network architectures with depth $L$ and $H$ respectively.

The function $f(x; w)$ is called meta-network, takes the input $x$ and computes the weights $v = f(x; w)$ of a second neural network $g(z; v)$, referred as the primary network, which is assumed to output a scalar. As before, the variable $w \in \mathbb{R}^{n_0}$ stands for a vector of trainable parameters ($v$ is not trained directly and is given by $f$).

We parameterize the primary network $g(z; v) = g^H(z; v)$ as follows:

$$
\begin{align*}
    g^l(z; v) &= \sqrt{\frac{1}{m_{l-1}}}v^l \cdot a^{l-1}(z; v) \\
    a^0(z) &= z
\end{align*}
$$

and $a^0(z) = z$.  


Here, the weights of the primary network $V^l(x) \in \mathbb{R}^{m_l \times m_{l-1}}$ are given in a concatenated vector form by the output of the meta-network $f(x; w) = v = (\text{vec}(V^1), \ldots, \text{vec}(V^H))$. The output dimension of the meta-network $f$ is therefore $n_L = \sum_{i=1}^{H} m_i \cdot m_{i-1}$. We denote by $f^d(x; w) := V^d(x; w) := V^d$ the $d$'th output matrix of $f(x; w)$. The width of $g$ is denoted by $m := \min(m_1, \ldots, m_{H-1})$.

**Optimization** Let $S = \{(u_i, y_i)\}_{i=1}^{N}$, where $u_i = (x_i, z_i)$ be some dataset and let $\ell(a, b) := \|a - b\|$ be the $\ell_1$-loss function. For a given model $h(u; w)$ (e.g., neural network or hypernetwork), in the supervised learning settings, given a model $h(u; w)$, we are interested in selecting the parameters $w$ that minimize the empirical risk:

$$c(w) := \sum_{i=1}^{N} \ell_i := \sum_{i=1}^{N} \ell(h(u_i; w), y_i) \quad (8)$$

In order to minimize the empirical error $c(w)$, we consider the SGD method with learning rate $\mu > 0$ and step of the form $w_{t+1} \leftarrow w_t - \mu \nabla_w c(w_t)$. A convenient simplification of the SGD method is the Gradient Flow. In the Gradient Flow, also known as continuous-time gradient-descent, the evolution of the parameters over time can be expressed as:

$$\dot{w}_t := \frac{\partial w_t}{\partial t} := -\mu \nabla_w c(w_t) \quad (9)$$

In recent works (Karras et al., 2018; Lee et al., 2019; Arora et al., 2019; van Laarhoven, 2017), the optimization dynamics of the gradient method for standard fully-connected neural networks was analyzed, as the network width tends to infinity. In our work, since hypernetworks consist of two interacting neural networks, there are multiple ways in which the size can tend to infinity. We consider two cases: (i) the width of both $f$ and $g$ tend to infinity and (ii) the width of $f$ tends to infinity and that of $g$ is fixed.

**Terminology and Notations** Throughout the paper we denote by $A \otimes B$ the outer product of the tensors $A$ and $B$. When considering the outer products of a sequence of tensors $\{A_i\}_{i=1}^{k}$, we denote, $\bigotimes_{i=1}^{k} A_i = A_1 \otimes \cdots \otimes A_k$. The notation $X_n = \mathcal{O}_p(a_n)$ states that $X_n/a_n$ is stochastically bounded, i.e., for any $\epsilon > 0$, there exists $M > 0$ and $N \in \mathbb{N}$, such that, for all $n \in \mathbb{N}$, we have: $\mathbb{P}[\|X_n/a_n\| > M] < \epsilon$. A convenient property of the probability big O notation is that it satisfies: $\mathcal{O}_p(a_n) \cdot \mathcal{O}_p(b_n) = \mathcal{O}_p(a_n \cdot b_n)$ and $\mathcal{O}_p(a_n) + \mathcal{O}_p(b_n) = \mathcal{O}_p(a_n + b_n)$.

### 3 Taylor Expansion at Initialization

The Taylor expansion of the cost function for three different models is analyzed: (i) standard neural networks, (ii) neural networks with a gating operation and (iii) hypernetworks. It is shown that at initialization the SGD algorithm behaves differently for standard neural networks than for the two multiplicative models.

Let $F(u; w)$ (e.g., a neural network or a hypernetwork) be a function with parameters initialized to be $w$. After a single gradient step with learning rate $\mu$, the cost is given by:

$$c(w - \mu \nabla_w c(w)) := \sum_{i=1}^{N} \hat{\ell}_i := \sum_{i=1}^{N} \ell(F(u_i; w - \mu \nabla_w c(w)), y_i) \quad (10)$$

Using Taylor approximation at point $w$ with respect to the function $c(w)$, it holds that:

$$c(w - \mu \nabla_w c(w)) - c(w) = \sum_{r=1}^{\infty} \frac{(-\mu)^r \cdot \text{sgn}(\hat{\ell}_i)}{r!} \cdot \left< \nabla_w^{(r)} F(u_i; w), \nabla_w c(w)^r \right>$$

where $\nabla_w^{(r)} F(u_i; w)$ is the $r$'th tensor that holds the $r$'th derivative of the output unit.

For infinitely wide fully-connected networks, it holds that the first order term converges to the neural tangent kernel. And so, at large widths and small learning rates, the loss surface appears deterministic and linear at initialization, and remains so during training. In particular, all of the higher order terms vanish at initialization. As we show for infinitely wide hypernetworks, the behaviour is different. For wide $f$ and $g$, the higher order terms vanish. On the other hand, for wide $f$ and fixed $g$, the first $H$ terms do not vanish while higher order terms do vanish.
3.1 Neural Networks

In the following theorem we show that the high order terms of Eq. \( \frac{1}{n} \) vanish with a rate of \( O \left( \frac{1}{n^2} \right) \). This extends the previous \( O \left( \frac{1}{n} \right) \) upper bound of Dyer & Gur-Ari (2020) and Hanin & Nica (2020) for the second order term.

**Theorem 1.** Let \( f(x; w) \) be a ReLU neural network with \( n_L = 1 \). Let \( S = \{(x_i, y_i)\}_{i=1}^{N} \) be a dataset of labeled samples. Consider a cost function of the form \( c(w) := \sum_{i=1}^{N} \ell(f(x_i; w), y_i) \). Then, for any \( r > 1 \) and \( i \in [N] \), we have:

\[
\langle \nabla_w^{(r)} f(x_i; w), \nabla_w c(w)^r \rangle = O_p \left( \frac{1}{n^{r-1}} \right) \quad \text{as} \quad n \to \infty
\]  

and for \( r \leq 1 \), \( \langle \nabla_w^{(r)} f(x_i; w), \nabla_w c(w)^r \rangle \) converges to a non-constant random variable.

The proofs in this paper are presented in the appendix.

The above results suggests that SGD steps traverses through smooth paths towards the optimum. Higher order terms in the tailor expansion are exponentially diminishing, and are therefore increasingly irrelevant for the optimization. This is an inductive bias that is the result of both the model and SGD, going beyond linearization. We note that this result verifies the empirical observations of Dyer & Gur-Ari (2020) regarding the asymptotic behaviour of these terms. For example, in their fifth row of Tab. 2, they show empirically that when \( r = 3 \), then, \( E_w[\langle \nabla_w^{(3)} f(x_i; w), \nabla_w c(w)^3 \rangle] = O(1/n^3) \).

3.2 Multiplicative Models

In this section, we consider the Taylor expansion of the cost function for multiplicative models. We show that in this case, the Taylor expansion is non-linear.

First, we consider a model \( h \) that is given as the product of the two outputs of a neural network \( f \). This is a very simple case of a neural network with gatings. In the following theorem, we show that in contrast to Thm. 1 in this case, the Taylor approximation of the cost function is non-linear.

**Theorem 2.** Let \( h(x; w) = f^1(x; w) \cdot f^2(x; w) \), where, \( f(x; w) \) is a ReLU neural network with two output neurons \( f^1(x; w) \) and \( f^2(x; w) \). Let \( S = \{(x_i, y_i)\}_{i=1}^{N} \) be a dataset of labeled samples. Consider a cost function of the form \( c(w) := \sum_{i=1}^{N} \ell(h(x_i; w), y_i) \). For any \( r > 2 \) and \( i \in [N] \), we have:

\[
\langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w)^r \rangle = O_p \left( \frac{1}{n^{r-2}} \right) \quad \text{as} \quad n \to \infty
\]  

and for \( r \leq 2 \), \( \langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w)^r \rangle \) converges to a non-constant random variable.

This result demonstrates that even for an infinitely wide neural network with gating, the optimization dynamics is no longer a simple, convex problem. In Thm. 8 in the appendix we extend this theorem for a multiplication of \( H \) functions \( \{f^d(x; w)\}_{d=1}^{H} \).

Next, we consider the case of hypernetworks. We show that when the primary network is finite and of depth \( H \), the Taylor expansion of the loss function behaves as a polynomial of order \( H \).

**Theorem 3.** Let \( h(u; w) = g(z; f(x; w)) \) be a ReLU hypernetwork. Let \( S = \{(u_i, y_i)\}_{i=1}^{N} \) be a dataset of labeled samples, such that, \( u_i = (x_i, z_i) \). Consider a cost function of the form \( c(w) := \sum_{i=1}^{N} \ell(h(u_i; w), y_i) \). For any \( r > H \) and \( i \in [N] \), we have:

\[
\langle \nabla_w^{(r)} h(u_i; w), \nabla_w c(w)^r \rangle = O_p \left( \frac{1}{n^{r-H}} \right) \quad \text{as} \quad n \to \infty
\]  

and for \( r \leq H \), \( \langle \nabla_w^{(r)} h(u_i; w), \nabla_w c(w)^r \rangle \) converges to a non-constant random variable.

Theorem 3 demonstrates the difficulty of training hypernetwork, even when the meta-network \( f \) is infinitely wide, since the loss surface is highly non-linear, and random at initialization. As evident from this theorem, the complexity of the optimization is determined by the depth of the primary network. We note that Thm. 2 is a special case of Thm. 3 where \( f \) is a neural network that returns two outputs \( f^1(x; w) \) and \( f^2(x; w) \) and \( g \) is a two layered neural network with a fixed input 1 and identity activations instead of ReLU.
4 Gradient Flow

When considering Gradient Flow the evolution of the parameters over time can be expressed as Eq. 9
\[ \dot{w}_t := \frac{\partial w_t}{\partial t} := -\mu \nabla_w c(w_t) \]

The evolution of the cost with respect to time is therefore given by:
\[ \dot{c}(w_t) = \frac{\partial c(w_t)}{\partial t} = \nabla_w c(w_t) \frac{\partial^T w}{\partial t} = -\mu \| \nabla_w c(w_t) \|^2 \tag{15} \]

Denoting by \( f(w_t) \) the concatenation of the outputs of \( f \) for each data point into a vector, we have:
\[ \dot{c}(w_t) = -\mu \frac{\partial c(w_t)}{\partial f(w_t)} \frac{\partial^T f(w_t)}{\partial w} \frac{\partial^T c(w_t)}{\partial f(w_t)} = -\mu \frac{\partial c(w_t)}{\partial f(w_t)} K_t^f \frac{\partial^T c(w_t)}{\partial f(w_t)} \tag{16} \]

Provided that the neural tangent kernel \( K_t^f := \frac{\partial f(w_t)}{\partial w} \frac{\partial^T f(w_t)}{\partial w} \) is constant during the evolution of the optimization process, we are left with a simplified kernel regression dynamics. It has been shown that infinitely wide vanilla feed forward neural network give rise to such dynamics, with a deterministic kernel. In addition, it has been shown that wide but finite networks give rise to a kernel \( K_t^f \), such that its time derivative at initialization is inversely proportional to the width \( K_t^f \sim \mathcal{O}(\frac{1}{n}) \).

As such, training dynamics of wide neural networks are well captured by linearized models.

In the following theorem, we show that, hypernetworks with finitely wide meta-networks, exhibit a more complicated training dynamics. Specifically, we show that the kernel \( K_t^f(u, u') \) corresponding to the model \( h(u; w) \) is non-deterministic, and evolves in a highly non linear way during the course of the optimization.

\( \textbf{Theorem 4.} \) Let \( h(u; w) = g(z; f(x; w)) \) be a ReLU hypernetwork, \( S = \{ (u_i, y_i) \}_{i=1}^N \) be a dataset of labeled samples, and \( K_t^h(u, u') := \frac{\partial h(u; w)}{\partial w} \frac{\partial^T h(u'; w)}{\partial w} \). Consider a cost function of the form \( c(w) := \sum_{i=1}^N \ell(h(u_i; w), y_i) \). Then, it holds that for any \( r \leq H \) and any \( u, u' \):
\[ \frac{\partial^r K_t^h(u, u')}{\partial t^r} \bigg|_{t=0} \sim \mathcal{O}_p(1) \quad \text{as} \ n \to \infty \tag{17} \]

5 Wide Hypernetworks

In Sec. 3 we showed that the Taylor expansion of the cost function is non-linear when the size of the primary network is finite. In this section, we consider the case of infinitely wide primary networks.

In this case, it is shown that the optimization trajectories of the SGD are determined by a linear model obtained from the first-order Taylor expansion of the cost function around its initialization.

5.1 Hypernetworks As Gaussian Processes

Previous work have shown the equivalence between popular architectures, and Gaussian processes, when the width of the architecture tends to infinity. This equivalence has sparked renewed interest in kernel methods, through the corresponding GP kernel, and the Neural Tangent Kernel (NTK) induced by the architecture, which fully characterise the training dynamics of infinitely wide networks. This equivalence has recently been unified to encompass most architectures which use a pre-defined set of generic computational blocks (Yang, 2019). Multiplicative interactions, however, of the kind found in hypernetworks, have not been previously addressed. As a first step towards understanding the dynamics of hypernetworks, we consider the limit where both the base and meta-networks are infinitely wide. As we next show, the equivalence to GP argument extends naturally to hypernetworks.

\( \textbf{Theorem 5 (Hypernetworks as GPs).} \) Let \( h(u; w) = g(z; f(x; w)) \) be a hypernetwork. Then,
\[ \lim_{\min(n, m) \to \infty} h(u; w) \overset{D}{=} \mathcal{G} \tag{18} \]

where \( \mathcal{G} \) is some centered Gaussian distribution.
Next, we would like to prove the existence, and identify the corresponding Neural Tangent Kernel of a hypernetwork. Recall the definition of the Neural Tangent Kernel as the infinite width limit of the Jacobian inner product given by:

\[
K_t^{h}(u, u') = \frac{\partial h(u; w_t)}{\partial w} \cdot \frac{\partial^\top h(u'; w_t)}{\partial w}
\]

\[
= \frac{\partial g(z; f(x; w_t))}{\partial f(x; w_t)} \cdot K_f^{f}(x, x') \cdot \frac{\partial^\top g(z'; f(x'; w_t))}{\partial f(x'; w_t)}
\]

where \(K_f^{f}(x, x') = \frac{\partial f(x; w_t)}{\partial w} \cdot \frac{\partial^\top f(x'; w_t)}{\partial w}\). In the following theorem we show that \(K_t^{h}(u, u')\) converges at initialization \(t = 0\) to a limiting kernel in the infinite width limit, denoted by \(\Theta_h(u, u')\), and that \(K_t^{h}(u, u') \approx \Theta_h(u, u')\) for \(t > 0\). Furthermore, we show that the hyperkernel is decomposed as the Hadamard product between the kernels corresponding to \(f\) and \(g\).

**Theorem 6** (Hyperkernel convergence at initialization and composition). Let \(h(u; w) = g(z; f(x; w))\) be a ReLU hypernetwork. Then,

\[
\lim_{\min(n, m) \to \infty} K_0^{h}(u, u') = \Theta_h(u, u')
\]

(20)

where:

\[
\Theta_h(u, u') := \Theta_f(x, x') \cdot \Theta_g(u, u')
\]

(21)

such that:

\[
\lim_{n \to \infty} K_0^{f}(x, x') = \Theta_f(x, x') I, \quad \lim_{\min(n, m) \to \infty} K_0^{g}(u, u') = \Theta_g(u, u')
\]

(22)

Under gradient flow, it holds that:

\[
\lim_{n \to \infty} \frac{\partial K_n^{h}(u, u')}{\partial t} \bigg|_{t=0} = O_p(1/m)
\]

(23)

6 Discussion

We claim that the optimization dynamics of multiplicative models are more complicated than the standard ReLU networks. Specifically, for a fixed sized primary network of depth \(H\), the first \(H\) terms of the Taylor approximation of the cost function are non-zero, even when the meta-network is infinitely wide. This is in contrast to standard ReLU networks, where the terms of degree \(\geq 1\) of the Taylor approximation tend to zero as the width tends to infinity.

However, for an infinitely wide meta and primary networks, the learning dynamics is determined by a linear model obtained from the first-order Taylor expansion of the network around its initial parameters and the kernel of this process is given by the Hadamard product of the kernels induced by the meta and primary networks.

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A Proofs of the Main Results

A.1 Useful Lemmas

Lemma 1. Let \( X_n \) be a sequence of random variables converging to a random variable \( X \) in distribution. Then, \( X_n = O_p(1) \).

Proof. For all \( M > 0 \), we have:
\[
\lim_{n \to \infty} P[|X_n| > M] = 1 + \lim_{n \to \infty} (F_{X_n}(-M) - F_{X_n}(M))
= 1 + F_X(-M) - F_X(M)
= P[|X| > M]
\]

For any \( \epsilon > 0 \), we let \( M_\epsilon > 0 \) be a large enough real number, such that, \( P[|X| > M_\epsilon] < \epsilon/2 \). In addition, let \( N_\epsilon > 0 \) be a large enough positive integer, such that, for all \( n > N_\epsilon \), we have:
\[
|P[|X_n| > M_\epsilon] - P[|X| > M_\epsilon]| < \epsilon/2.
\]
Therefore, for all \( n > N_\epsilon \), we have:
\[
P[|X_n| > M_\epsilon] < \epsilon
\]
Hence, \( X_n = O_p(1) \) by definition.

Let \( f(x; w) \) be a neural network with \( H \) outputs \( \{f^d(x; w)\}_{d=1}^H \). To prove Thms. 1, 2 and 3, we would like to estimate the order of magnitude of the following term:
\[
\mathcal{T}_{n,i,d}^{l_1,i,d} := \left\langle \frac{\partial^k f^d(x_i; w)}{\partial W_{i_1} \ldots \partial W_{i_k}}, \frac{\partial f^d(x_i; w)}{\partial W_{i_j}} \right\rangle
\]

For this purpose, we provide an explicit expression for \( \frac{\partial^k f^d(x_i; w)}{\partial W_{i_1} \ldots \partial W_{i_k}} \). For any set \( I := \{l_1, \ldots, l_k\} \), such that, \( l_1 < \cdots < l_k \), it holds that:
\[
\frac{\partial^k f^d(x_i; w)}{\partial W_{i_1} \ldots \partial W_{i_k}} = \frac{1}{\sqrt{n_{l_1-1}}} q^{l_1-1}_i \otimes A_{l_1}^{l_2} \otimes \cdots \otimes A_{l_k}^{l_{k+1}}
\]

We notice that if \( l_i = l_{i+1} \), then, \( \frac{\partial^k f^d(x_i; w)}{\partial W_{i_1} \ldots \partial W_{i_k}} = 0 \), since \( f^d(x_i; w) \) is a ReLU neural network.

where \( A_{l_1}^{l_2} \) is a \( 2k-1 \) tensor, defined as follows:
\[
A_{l_1}^{l_2} = \begin{cases} 
\frac{1}{\sqrt{n_{l_1-1}}} C_{l_1}^{l_1} \otimes A_{l_1}^{l_2} & 1 < j < k-1 \\
\frac{1}{\sqrt{n_{l_1-1}}} C_{l_1}^{l_1} \otimes A_{l_1}^{l_2} & j = k-1 
\end{cases}
\]

where:
\[
C_{l_1}^{l_2} = \begin{cases} 
\sqrt{\frac{n_{l_2}}{n_{l_1}}} Z_{l_1}^{l_2} P_{l_1}^{l_2} & l_{j+1} \neq L \\
1 & \text{else}
\end{cases}
\]

and:
\[
P_{l_1}^{l_2} = \prod_{l=i}^{v-1} \left( \sqrt{\frac{n_{l}}{n_{l-1}}} W^{l+1} Z_{l_1}^{l} \right) \text{ and } Z_{l_1}^{l} = \text{diag}(\sigma(y^l(x_l)))
\]

The individual gradients can be expressed using:
\[
\frac{\partial f^d_j(x_i)}{\partial W_{j}} = \frac{q^{l_j-1}_i \otimes C_{l_1}^{l_2} \otimes \cdots \otimes C_{l_{j-1}}^{l_{j-1}}}{\sqrt{n_{j-1}}}
\]

Note that the following holds for any \( u < v < h \leq L \):
\[
C_{l_1}^{u+h} = C_{l_1}^{u} + \frac{W^{v}}{\sqrt{n_{v-1}}} C_{l_1}^{v} \text{ and } C_{l_1}^{v+h} = C_{l_1}^{v-1} L P_{l_1}^{v+h-1}
\]
In the following, given the sets \( l = \{l_1, \ldots, l_k\}, i = \{i_1, \ldots, i_k\}\) and \( d = \{d_1, \ldots, d_k\}\), we derive the limit of \( T_{n,i,d}^{l,i,d} \) using elementary tensor algebra. By Eqs. [31] and [27] we see that:

\[
T_{n,i,d}^{l,i,d} = \left( \prod_{i=1}^{k} \frac{\partial f_{l,i,d}(x_{i},w)}{\partial W_{i}\iota} \right)_{i=1}^{l_{i}-1} \otimes \frac{C_{i,d}^{l_{i}-1} \otimes \cdots \otimes C_{i,d}^{l_{k}-1} \otimes C_{i,d}^{L}}{n_{l_{i}}^{-1} \sqrt{\Lambda_{l_{i}}^{-1}}} 
\]

\[
= \frac{1}{n_{l_{i}}^{-1}} \left( \prod_{i=1}^{l_{i}-1} q_{i,d} \right) \left( \prod_{j=1}^{k-1} \frac{C_{i,d}^{p_{j}+1} \otimes \cdots \otimes C_{i,d}^{p_{j}+1}}{n_{l_{j}+1}^{-1}} \right) \text{Lemma 2.}
\]

We recall the analysis of Lee et al. [2018] showing that in the infinite width limit, every pre-activation \( y'(x) \) of \( f(x; w) \) at hidden layer \( l \in [L] \) has all its coordinates tending to i.i.d. centered Gaussian processes of covariance \( \Sigma_l(x, x') : \mathbb{R}^{n_0} \times \mathbb{R}^{n_0} \to \mathbb{R} \) defined recursively as follows:

\[
\Sigma_l^0(x, x') = x^\top x',
\]

\[
\Lambda_l^1(x, x') = \begin{bmatrix} \Sigma_l^{-1}(x, x) & \Sigma_l^{-1}(x, x') \\ \Sigma_l^{-1}(x', x) & \Sigma_l^{-1}(x', x') \end{bmatrix} \in \mathbb{R}^{2 \times 2},
\]

\[
\Sigma_l^j(x, x') = E_{(u,v) \sim N(0, \Lambda_l^{-1}(x, x'))} [\sigma(u)\sigma(v)]
\]

In addition, we define the derivative covariance as follows:

\[
\dot{\Sigma}_l^j(x, x') = E_{(u,v) \sim N(0, \Lambda_l^{-1}(x, x'))} [\dot{\sigma}(u)\dot{\sigma}(v)]
\]

when considering \( x = x_i \) and \( x' = x_j \) from the training set, we simply write \( \Sigma_{i,j}^l := \Sigma_l^j(x_i, x_j) \) and \( \dot{\Sigma}_{i,j}^l := \dot{\Sigma}_l^j(x_i, x_j) \).

**Lemma 2.** The following holds:

1. \( \lim_{n_1 \ldots n_k \to \infty} P_{i,d}^{u,v} = \prod_{l=1}^{i-1} \Sigma_l^i I . \)

2. \( \lim_{n_1 \ldots n_k \to \infty} P_{i,d}^{u} = \prod_{l=1}^{i-1} \Sigma_l^i \delta_{l_1=d_1} . \)

3. \( \lim_{n_1 \ldots n_k \to \infty} \left( \frac{q_{i,d}^u q_{i,d}^v}{n_0} \right) = \Sigma_{i,j}^0 . \)

Here, \( \delta_T \) is an indicator that returns 1 if \( T \) is true and 0 otherwise.

**Proof.** See Arora et al. [2019]. \( \square \)

**Theorem 7.** It holds that for any \( k > 1 \) and any sets \( l = \{l_1, \ldots, l_k\}, i = \{i_1, \ldots, i_k\}\) and \( d = \{d_1, \ldots, d_k\}\):

\[
\lim_{n \to \infty} n^{k-1} T_{n,j,d}^{l,i,d} = \delta_d \cdot \prod_{j=1}^{k-1} G_j
\]

where \( G_1, \ldots, G_{k-1} \) are centered Gaussian variables with finite, non-zero variances, and \( \delta_d := \delta(d_1 = \ldots = d_k = d) . \)

**Proof.** First, by Eq. [33] it holds that:

\[
n^{k-1} T_{n,i,d}^{l,i,d} = \left( \prod_{i=1}^{l_{i}-1} q_{i,d} \right) \left( \prod_{j=1}^{k-1} \left( \frac{C_{i,d}^{l_{j}+1} \otimes \cdots \otimes C_{i,d}^{l_{j}+1}}{n_0} \right) \right) \text{Lemma 2.}
\]

\[
= \left( \frac{q_{i,d}^{l_{i}-1}}{n_0} \right) \left( \prod_{j=1}^{k-1} \left( \frac{C_{i,d}^{l_{j}+1} \otimes \cdots \otimes C_{i,d}^{l_{j}+1}}{n_0} \right) \right) \text{Lemma 2.}
\]

\[
= \left( \frac{q_{i,d}^{l_{i}-1}}{n_0} \right) \left( \prod_{j=1}^{k-1} \left( \frac{C_{i,d}^{l_{j}+1} \otimes \cdots \otimes C_{i,d}^{l_{j}+1}}{n_0} \right) \right) \text{Lemma 2.}
\]

\[
= \frac{q_{i,d}^{l_{i}-1}}{n_0} \left( \prod_{j=1}^{k-1} \left( \frac{C_{i,d}^{l_{j}+1} \otimes \cdots \otimes C_{i,d}^{l_{j}+1}}{n_0} \right) \right) \text{Lemma 2.}
\]
Note that intermediate activations do not depend on the index $d_j$, and so we remove the dependency on $d_j$ in the relevant terms. Next, by applying Lemma 2:

$$\lim_{n \to \infty} \frac{q_i^{l-1} \cdot q_i^{l-1}}{n} \leq \delta_d \tag{38}$$

Expanding the second term using Eq. 32:

$$\left\langle C_{i,j} \to L, q_{i,j+1}^{l-1}, C_{i,d} \right\rangle = C_{i,j}^{l-1} \cdot C_{i,d}^{l-1}$$

Denoting $\xi_j$:

$$\xi_j = C_{i,j,d_j}^{l+1-1} \cdot Z_i^{l+1-1} \cdot q_{i,j+1}^{l+1-1} \sum_{l=1}^{l+1} \frac{d_l}{q_{i,j+1}^{l+1-1}}$$

and their limits:

$$G_j = \lim_{n \to \infty} \xi_j \tag{42}$$

and denoting $\xi = [\xi_1, \ldots, \xi_{k-1}]$, and $G = [G_1, \ldots, G_{k-1}]$, it holds using the multivariate Central Limit theorem:

$$\lim_{n \to \infty} \xi = G \tag{43}$$

Using the Mann-Wald theorem [Mann & Wald, 1943] (where we take the mapping as the product pooling of $\xi$), we have that:

$$\lim_{n \to \infty} \prod_{j=1}^{k-1} \xi_j \xrightarrow{n} \prod_{j=1}^{k-1} G_j \tag{44}$$

Finally, by Slutsky’s theorem,

$$\lim_{n \to \infty} n^{k-1} \prod_{j=1}^{k-1} \xi_j \xrightarrow{n} \prod_{j=1}^{k-1} G_j$$

and

$$\lim_{n \to \infty} n^{k-1} \prod_{j=1}^{k-1} \xi_j \xrightarrow{n} \prod_{j=1}^{k-1} G_j$$

$\square$
Corollary 1. It holds that for any $k > 1$, $d \in [H]$, $i \in [N]$ and any sets $l = \{l_1, \ldots, l_k\}$, $i = \{i_1, \ldots, i_k\}$ and $d = \{d_1, \ldots, d_k\}$:

$\mathcal{T}^{l,i,d}_{n,i,d} = \mathcal{O}_p \left( \frac{1}{n^{k-1}} \right)$ (46)

Proof. By Thm. 1, $X_n := n^{k-1} \cdot \mathcal{T}^{l,i,d}_{n,i,d}$ converges to a random variable $X$. Therefore, by Lem. 1, $X_n = \mathcal{O}_p(1)$. Since, $\frac{1}{n^{k-1}} = \mathcal{O}_p \left( \frac{1}{n^{k-1}} \right)$, then, by the properties of the probability big O notation, $\mathcal{T}^{l,i,d}_{n,i,d} = \mathcal{O}_p \left( \frac{1}{n^{k-1}} \right)$. \hfill \Box

A.2 Neural Networks

Theorem 1. Let $f(x; w)$ be a ReLU neural network with $n_L = 1$. Let $S = \{(x_i, y_i)\}_{i=1}^N$ be a dataset of labeled samples. Consider a cost function of the form $c(w) := \sum_{i=1}^N \ell(f(x_i; w), y_i)$. Then, for any $r > 1$ and $i \in [N]$, we have:

$\langle \nabla_w^{(r)} f(x_i; w), \nabla_w c(w) \rangle = \mathcal{O}_p \left( \frac{1}{n^{r-1}} \right)$ as $n \to \infty$ (12)

and for $r \leq 1$, $\langle \nabla_w^{(r)} f(x_i; w), \nabla_w c(w) \rangle$ converges to a non-constant random variable.

Proof. We denote $f^1 := f$ and let $d = (1, \ldots, 1)$. We have:

$\langle \nabla_w^{(r)} f(x_i; w), \nabla_w c(w) \rangle$

$= \sum_{i \in [N]} \sum_{t \in [L]} \left[ \prod_{j=1}^{r} \text{sgn}(\ell_{i,j}) \right] \cdot \left\langle \frac{\partial^r f(x_i; w)}{\partial W_{t_1} \ldots \partial W_{t_r}}, \prod_{t=1}^{r} \frac{\partial f(x_i; w)}{\partial W_{t_t}} \right\rangle$

$= \sum_{i \in [N]} \sum_{t \in [L]} \left[ \prod_{j=1}^{r} \text{sgn}(\ell_{i,j}) \right] \cdot \mathcal{T}^{l,i,d}_{1,i,d} = \mathcal{O}_p \left( \frac{(NL)^r}{n^{r-1}} \right)$ \hfill \Box

A.3 Products

Lemma 3. Let $h(x; w) = \prod_{d=1}^{H} f^d(x; w)$, where, $f^d(x; w)$ is a ReLU neural network with $H$ output neurons $\{f^d(x; w)\}_{d=1}^{H}$. We have:

$\langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w) \rangle$

$= \sum_{\alpha_1 + \cdots + \alpha_H = r} r! \prod_{d=1}^{H} \left\langle \nabla_w^{(\alpha_d)} f^d(x_i; w), \nabla_w c(w)^{\alpha_d} \right\rangle$ (48)

Proof. By the higher order product rule:

$\nabla_w^{(r)} h(x; w) = \sum_{\alpha_1 + \cdots + \alpha_H = r} \frac{r!}{\alpha_1! \cdots \alpha_H!} \bigotimes_{d=1}^{H} \nabla_w^{(\alpha_d)} f^d(x; w)$ (49)

In addition, by elementary tensor algebra, we have:

$\langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w) \rangle$

$= \sum_{\alpha_1 + \cdots + \alpha_H = r} \frac{r!}{\alpha_1! \cdots \alpha_H!} \bigotimes_{d=1}^{H} \left\langle \nabla_w^{(\alpha_d)} f^d(x_i; w), \nabla_w c(w)^{\alpha_d} \right\rangle$ (50)
Lemma 4. Let \( h(x; w) = \prod_{d=1}^{H} f^d(x; w) \), where, \( f(x; w) \) is a ReLU neural network with \( H \) output neurons \( \{f^d(x; w)\}_{d=1}^{H} \). We have:

\[
\langle \nabla^{(\alpha_d)} f^d(x_i; w), \nabla_w c(w)^{\alpha_d} \rangle = \sum_{i \in [N]^{\alpha_d}} \sum_{t \in [L]^{\alpha_d}} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{t_j}) \right] \cdot \sum_{d \in [H]^{\alpha_d}} \left( \prod_{j=1}^{\alpha_d} \prod_{t \in [H]\setminus\{d_j\}} f^t(x_i; w) \right) \mathcal{T}^{l_{i,d}}_{n,i,d}
\]

Proof. We have:

\[
\langle \nabla^{(\alpha_d)} f^d(x_i; w), \nabla_w c(w)^{\alpha_d} \rangle = \sum_{i \in [N]^{\alpha_d}} \sum_{t \in [L]^{\alpha_d}} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{t_j}) \right] \cdot \left( \frac{\partial f^d(x_i; w)}{\partial W_{t}} \right) = \sum_{i \in [N]^{\alpha_d}} \sum_{t \in [L]^{\alpha_d}} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{t_j}) \right] \cdot \left( \frac{\partial f^d(x_i; w)}{\partial W_{t}} \right)
\]

By the product rule:

\[
\frac{\partial h(u_{i,j}; w)}{\partial W_{t}} = \sum_{d=1}^{H} \left( \prod_{t \in [H]\setminus\{d\}} f^t(x_i; w) \right) \cdot \frac{\partial f^d(x_i; w)}{\partial W_{t}}
\]

Hence,

\[
\frac{\alpha_d}{\prod_{t=1}^{\alpha_d}} \frac{\partial h(u_{i,j}; w)}{\partial W_{t}} = \sum_{d \in [H]^{\alpha_d}} \left( \prod_{j=1}^{\alpha_d} \prod_{t \in [H]\setminus\{d_j\}} f^t(x_i; w) \right) \cdot \frac{\partial f^d(x_i; w)}{\partial W_{t}}
\]

In particular,

\[
\langle \nabla^{(\alpha_d)} f^d(x_i; w), \nabla_w c(w)^{\alpha_d} \rangle = \sum_{i \in [N]^{\alpha_d}} \sum_{t \in [L]^{\alpha_d}} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{t_j}) \right] \cdot \sum_{d \in [H]^{\alpha_d}} \left( \prod_{j=1}^{\alpha_d} \prod_{t \in [H]\setminus\{d_j\}} f^t(x_i; w) \right) \mathcal{T}^{l_{i,d}}_{n,i,d}
\]

Theorem 8. Let \( h(x; w) = \prod_{l=1}^{H} f^d(x; w) \), where, \( f(x; w) \) is a ReLU neural network with \( H \) output neurons \( \{f^d(x; w)\}_{d=1}^{H} \). Let \( S = \{(x_i, y_i)\}_{i=1}^{N} \) be a dataset of labeled samples. Consider a cost function of the form \( c(w) := \sum_{i=1}^{N} \ell(h(x_i; w), y_i) \). For any \( r > H \) and \( i \in [N] \), we have:

\[
\langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w)^{r} \rangle = \mathcal{O}_p \left( \frac{1}{n^{r-2}} \right) \text{ as } n \to \infty
\]

and for \( r \leq H \), \( \langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w)^{r} \rangle \) converges to a non-constant random variable.

Proof. By Lemmas 3 and 4, we have:

\[
\langle \nabla_w^{(r)} h(x_i; w), \nabla_w c(w)^{r} \rangle = \sum_{i \in [N]^{\alpha_d}} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{t_j}) \right] \cdot \left( \sum_{d \in [H]^{\alpha_d}} \mathcal{Q}^{l_{i,d}}_{n,d} \cdot \mathcal{T}^{l_{i,d}}_{n,i,d} \right)
\]

where

\[
\mathcal{Q}^{l_{i,d}}_{n,d} := \left( \prod_{j=1}^{\alpha_d} \prod_{t \in [H]\setminus\{d_j\}} f^t(x_i; w) \right)
\]
We notice that by the central limit theorem, \( \{f^{t}(x_{i}; w)\}_{j \in [\alpha d], t \in [H]} \) jointly converges in distribution to a set of Gaussian random variables \( \{G^{t}_{j}\}_{j \in [\alpha d], t \in [H]} \) as \( n \to \infty \). Hence, the product \( Q^{i,d}_{n, d} := (\Pi_{j=1}^{\alpha d} \Pi_{t \in [H] \setminus \{d_{j}\}}^{\alpha d} f^{t}(x_{i}; w)) \) converges to \( \Pi_{j=1}^{\alpha d} \Pi_{t \in [H] \setminus \{d_{j}\}}^{\alpha d} G^{t}_{j} \) in distribution by the Mann-Wald theorem (Mann & Wald, 1943). In particular, by Lem. I we have:

\[
Q^{i,d}_{n, d} = O_{p}(1) \quad (59)
\]

By Cor. I we have:

\[
Q^{i,d}_{n, d} \cdot T^{i,d}_{n, i, d} = O_{p}\left( \frac{1}{n^{d_{a}-1}} \right) \quad (60)
\]

In particular, for any \( r \geq H \), we have:

\[
\langle \nabla^{(r)}_{w} h(x_{i}; w), \nabla_{w} c(w) \rangle^{r} = O_{p}\left( \sum_{\alpha_{1} + \cdots + \alpha_{H} = r} \frac{r!}{\alpha_{1}! \cdots \alpha_{H}!} \cdot \prod_{d=1}^{H} \left( \frac{m \cdot L \cdot H}{n} \right)^{\alpha_{d}-1} \right)
\]

\[
= O_{p}\left( \sum_{\alpha_{1} + \cdots + \alpha_{H} = r} \frac{r!}{\alpha_{1}! \cdots \alpha_{H}!} \cdot \left( \frac{m \cdot L \cdot H}{n} \right)^{r-H} \right)
\]

\[
= O_{p}\left( H^{r} \cdot \left( \frac{m \cdot L \cdot H}{n} \right)^{r-H} \right)
\]

Next, we consider the case \( r \leq H \). We recall that by Eq. (57) we have:

\[
\langle \nabla^{(r)}_{w} h(x_{i}; w), \nabla_{w} c(w) \rangle^{r}
\]

\[
= \sum_{i \in [N]^{n_{d}}} \sum_{l \in [L]^{n_{d}}} \left[ \sum_{j=1}^{\alpha_{d}} \prod_{j=1}^{\alpha_{d}} \text{sgn}(\ell_{i,j}) \right] \cdot \sum_{d \in [H]^{n_{d}}} Q^{i,d}_{n, d} \cdot T^{i,d}_{n, i, d} = 0 \quad (63)
\]

On the other hand, by Lem. II for any \( \alpha_{d} \leq 1 \), we have: \( T^{i,d}_{n, i, d} = O_{p}(1) \). Hence, for any \( \alpha_{1}, \ldots, \alpha_{r} \geq 0 \), such that, at least one \( \alpha_{j} \geq 2 \), we have:

\[
\lim_{n \to \infty} \prod_{d=1}^{H} \left( \sum_{i \in [N]^{n_{d}}} \sum_{l \in [L]^{n_{d}}} \left[ \sum_{j=1}^{\alpha_{d}} \prod_{j=1}^{\alpha_{d}} \text{sgn}(\ell_{i,j}) \right] \cdot \sum_{d \in [H]^{n_{d}}} Q^{i,d}_{n, d} \cdot T^{i,d}_{n, i, d} \right) = 0 \quad (64)
\]

In particular, if the limit exists, we have:

\[
\lim_{n \to \infty} \langle \nabla^{(r)}_{w} h(x_{i}; w), \nabla_{w} c(w) \rangle^{r}
\]

\[
= \lim_{n \to \infty} \sum_{\alpha_{1} + \cdots + \alpha_{H} = r} \frac{r!}{\alpha_{1}! \cdots \alpha_{H}!} \prod_{d=1}^{H} \left( \sum_{i \in [N]^{n_{d}}} \sum_{l \in [L]^{n_{d}}} \left[ \sum_{j=1}^{\alpha_{d}} \prod_{j=1}^{\alpha_{d}} \text{sgn}(\ell_{i,j}) \right] \cdot \sum_{d \in [H]^{n_{d}}} Q^{i,d}_{n, d} \cdot T^{i,d}_{n, i, d} \right)
\]

\[
= \lim_{n \to \infty} \sum_{0 \leq \alpha_{1} + \cdots + \alpha_{H} \leq 1} \frac{r!}{\alpha_{1}! \cdots \alpha_{H}!} \prod_{d=1}^{H} \left( \sum_{i \in [N]^{n_{d}}} \sum_{l \in [L]^{n_{d}}} \left[ \sum_{j=1}^{\alpha_{d}} \prod_{j=1}^{\alpha_{d}} \text{sgn}(\ell_{i,j}) \right] \cdot \sum_{d \in [H]^{n_{d}}} Q^{i,d}_{n, d} \cdot T^{i,d}_{n, i, d} \right) = 0
\]
On the other hand, for any $0 \leq \alpha_1, \ldots, \alpha_H \leq 1$, the terms $\{T_{i,d}^{l,i,d} H_{i,d} \}_{i,d=1}^H$ and $\{Q_{i,d}^{l,i,d} H_{i,d} \}_{i,d=1}^H$ converge jointly in distribution to Gaussian random variables $\{T_{i,d}^{l,i,d} H_{i,d} \}_{i,d=1}^H$ and $\{Q_{i,d}^{l,i,d} H_{i,d} \}_{i,d=1}^H$ as $n \to \infty$. In addition, by the continuous mapping theorem $\ell_{i,j}$ converge to $\ell(G_{i,j}, y_{i,j})$ in distribution, for some normally distributed random variable $G_{i,j}$. In particular, $\lim_{n \to \infty} \mathbb{P}[\ell_{i,j} \geq 0] = \mathbb{P}[\ell(G_{i,j}, y_{i,j}) \geq 0]$. Therefore, $\text{sgn}(\ell_{i,j})$ converges to $\text{sgn}(\ell(G_{i,j}, y_{i,j}))$ in distribution. Hence, by the Mann-Wald theorem [Mann & Wald, 1943],

$$\triangledown \triangledown \alpha \cdot \rho \cdot \sum_{\alpha_1^0 \leq \alpha_1, \ldots, \alpha_H \leq 1} r! \cdot \prod_{d=1}^H \left( \sum_{i \in [N]} \sum_{l \in [k]^d} \left[ \prod_{j=1}^H \text{sgn}(\ell(G_{i,j}, y_{i,j})) \right] \cdot \sum_{d \in [H]} Q_{i,d}^{l,i,d} \cdot T_{i,d}^{l,i,d} \right) \tag{66}$$

\[ \square \]

**Theorem 2.** Let $h(x; w) = f^1(x; w) \cdot f^2(x; w)$, where, $f(x; w)$ is a ReLU neural network with two output neurons $f^1(x; w)$ and $f^2(x; w)$. Let $S = \{(x_i, y_i)\}_{i=1}^N$ be a dataset of labeled samples. Consider a cost function of the form $c(w) := \sum_{i=1}^N f(h(x_i; w), y_i)$. For any $r > 2$ and $i \in [N]$, we have:

$$\langle \nabla_w (r) h(x_i; w), \nabla_w c(w) \rangle = O_p \left( \frac{1}{n^{r-2}} \right) \text{ as } n \to \infty \tag{13}$$

and for $r \leq 2$, $\langle \nabla_w (r) h(x_i; w), \nabla_w c(w) \rangle$ converges to a non-constant random variable.

**Proof.** Follows from Thm. 8 for $H = 2$.

### A.4 Hypernetworks

**Lemma 5.** Let $h(w; u) = g(z; f(x; w))$ be a ReLU hypernetwork. We have:

$$\langle \nabla_w (r) h(u_i; w), \nabla_w c(w) \rangle \tag{67}$$

$$= \sum_{\alpha_1 \cdots \alpha_H \geq 0} \frac{r!}{\alpha_1! \cdots \alpha_H!} \cdot z_i \cdot \prod_{j=1}^{H-1} \sigma(g^j(z_i)) \cdot \prod_{d=1}^H \left\langle \nabla_w (\alpha_d) f^d(x_i; w), \nabla_w c(w) \alpha_d \right\rangle$$

**Proof.** By the higher order product rule and the fact that the second derivative of ReLU is 0 everywhere:

$$\nabla_w (r) h(u_i; w) = \sum_{\alpha_1 \cdots \alpha_H \geq 0} \frac{r!}{\alpha_1! \cdots \alpha_H!} \cdot z_i \cdot \nabla_w (\alpha_H) f^H(x_i; w) \bigotimes_{j=1}^{H-1} D_{H-j} \tag{68}$$

where

$$D_j := \sigma(g^j(z_i)) \cdot \nabla_w (\alpha_j) f^j(x_i; w) \tag{69}$$

In addition, by elementary tensor algebra, we have:

$$\langle \nabla_w (r) h(u_i; w), \nabla_w c(w) \rangle \tag{70}$$

$$= \sum_{\alpha_1 \cdots \alpha_H \geq 0} \frac{r!}{\alpha_1! \cdots \alpha_H!} \cdot z_i \cdot \left\langle \nabla_w (\alpha_H) f^H(x_i; w), \nabla_w c(w) \alpha_H \right\rangle \cdot \prod_{j=1}^{H-1} \left\langle \sigma(g^{H-j}(z_i)) \cdot \nabla_w (\alpha_{H-j}) f^{H-j}(x_i; w), \nabla_w c(w) \alpha_{H-j} \right\rangle$$

$$= \sum_{\alpha_1 \cdots \alpha_H \geq 0} \frac{r!}{\alpha_1! \cdots \alpha_H!} \cdot z_i \cdot \left[ \prod_{j=1}^{H-1} \sigma(g^j(z_i)) \right] \cdot \prod_{d=1}^H \left\langle \nabla_w (\alpha_d) f^d(x_i; w), \nabla_w c(w) \alpha_d \right\rangle$$

\[ \square \]
Lemma 6. Let \( h(u; w) = g(z; f(x; w)) \) be a ReLU hypernetwork. In addition, let,
\[
\forall d \in [H] : \quad h^d(z, x, w) := a^{d-1}(z) \prod_{t=1}^{H-d} f^{H-t+1}(x; w) \cdot \hat{\sigma}_{H-t}(g^{H-t}(z))
\]
(71)
We have:
\[
\left< \nabla^{(\alpha_d)} h^d(x_i; w), \nabla_w c(w)^{\alpha_d} \right> = \sum_{i \in [N]} \sum_{t \in [H]} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{i,j}) \right] \sum_{d \in [H]} \left( \prod_{j=1}^{\alpha_d} h^{d_j}(x_{i,j}, z_{i,j}; w) \right) T_{n,i,d}^{l,i,d}
\]
(72)
Proof. We have:
\[
\left< \nabla^{(\alpha_d)} h^d(x_i; w), \nabla c(w)^{\alpha_d} \right> = \sum_{i \in [N]} \sum_{t \in [L]} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{i,j}) \right] \left< \frac{\partial h^{d_i}(x_{i,j}; w)}{\partial W_{i,j}} \right> \left< \frac{\partial f^d(x_{i,j}; w)}{\partial W_{i,t}} \right>
\]
(73)
By the product rule:
\[
\frac{\partial h(u_{i,j}; w)}{\partial W_{i,j}} = \sum_{d=1}^{H-d} \left[ \prod_{t=1}^{H-d} f^{H-t+1}(x_{i,j}; w) \cdot \hat{\sigma}_{H-t}(g^{H-t}(z_{i,j})) \right] \frac{\partial f^d(x_{i,j}; w)}{\partial W_{i,j}} \cdot a^{d-1}(z_{i,j})
\]
(74)
Hence,
\[
\frac{\partial h(u_{i,j}; w)}{\partial W_{i,j}} = \sum_{d=1}^{H-d} \left[ \prod_{j=1}^{\alpha_d} h^{d_j}(x_{i,j}, z_{i,j}; w) \right] \frac{\partial f^d(x_{i,j}; w)}{\partial W_{i,t}}
\]
(75)
In particular,
\[
\left< \nabla^{(\alpha_d)} h^d(x_i; w), \nabla c(w)^{\alpha_d} \right> = \sum_{i \in [N]} \sum_{t \in [L]} \left[ \prod_{j=1}^{\alpha_d} \text{sgn}(\ell_{i,j}) \right] \sum_{d \in [H]} \left( \prod_{j=1}^{\alpha_d} h^{d_j}(u_{i,j}; w) \right) T_{n,i,d}^{l,i,d}
\]
(76)

Theorem 3. Let \( h(u; w) = g(z; f(x; w)) \) be a ReLU hypernetwork. Let \( S = \{(u_i, y_i)\}_{i=1}^N \) be a dataset of labeled samples, such that, \( u_i = (x_i, z_i) \). Consider a cost function of the form \( c(w) := \sum_{i=1}^N \ell(h(u_i; w), y_i) \). For any \( r > H \) and \( i \in [N] \), we have:
\[
\left< \nabla^{(r)}_w h(u_i; w), \nabla c(w) \right> = O_p \left( \frac{1}{n^{r-H}} \right) \quad \text{as} \quad n \to \infty
\]
(14)
and for \( r \leq H \), \( \left< \nabla^{(r)}_w h(u_i; w), \nabla c(w) \right> \) converges to a non-constant random variable.
Proof. By Lems. 6 and 7, we have:
\[
\left< \nabla^{(r)}_w h(u_i; w), \nabla c(w) \right> = \sum_{\alpha_1 + \cdots + \alpha_H = r} \frac{r!}{\alpha_1! \cdots \alpha_H!} \cdot z_i \cdot \prod_{j=1}^{H-1} \hat{\sigma}(g^j(z_i)) \cdot \prod_{d=1}^{H} \left[ \prod_{t \in [H]} f^t(x_{i,j}; w) \right] T_{n,i,d}^{l,i,d}
\]
(77)
Therefore, by the same arguments as in the proof of Thm. 8, we have the desired. □
A.5 Proofs of the Results in Sec.[4]

Theorem 4. Let \( h(u; w) = g(z; f(x; w)) \) be a ReLU hypernetwork, \( S = \{(u_i, y_i)\}_{i=1}^N \) be a dataset of labeled samples, and \( K_{\alpha}^h(u, u') := \frac{\partial h(u; w)}{\partial w} \frac{\partial^\alpha h(u'; w)}{\partial w} \). Consider a cost function of the form \( c(w) := \sum_{i=1}^N \ell(h(u_i; w), y_i) \). Then, it holds that for any \( r \leq H \) and any \( u, u' \):

\[
\frac{\partial^r K_{\alpha}^h(u, u')}{\partial u^r} \Big|_{u=0} \sim \mathcal{O}_p(1) \text{ as } n \to \infty \tag{17}
\]

Proof. Note that the \( r \)'th order derivative can be expressed as follows:

\[
\frac{\partial^r K_{\alpha}^h(u, u')}{\partial u^r} = \left( \frac{\partial}{\partial t} \right)^r \left[ \frac{\partial h(u; w)}{\partial w} \frac{\partial^\alpha h(u'; w)}{\partial w} \right]
\]

\[
= \sum_{\alpha=0}^r \frac{r!}{\alpha! \cdot (r-\alpha)!} \left[ \left( \frac{\partial}{\partial t} \right)^{r-\alpha} \frac{\partial h(u; w)}{\partial w} \right] \left( \frac{\partial^\alpha h(u'; w)}{\partial w} \right) \left( \frac{\partial^\alpha h(u'; w)}{\partial w} \right) + \mathcal{R}
\]

\[
\left( \frac{\partial}{\partial t} \right)^r \left[ \frac{\partial h(u; w)}{\partial w} \frac{\partial^\alpha h(u'; w)}{\partial w} \right] = (-\mu)^r \left[ \nabla_w c(w) \frac{\partial^\alpha h(u'; w)}{\partial w} \right] \left( \frac{\partial^\alpha h(u'; w)}{\partial w} \right) \tag{79}
\]

\[
= (-\mu)^r \left[ \sum_{i} \text{sgn}(\ell_i) \frac{\partial h(u_i; w)}{\partial w} \frac{\partial^\alpha h(u'; w)}{\partial w} \right] \left( \frac{\partial^\alpha h(u'; w)}{\partial w} \right) \tag{80}
\]

We denote the derivative of the output \( h(u; w) \) with respect to the weights \( w^{\eta_1} \ldots w^{\eta_k} \) by \( \Gamma_{\eta_1 \ldots \eta_k}(\mu) = \frac{\partial^k h(u; w)}{\partial w^{\eta_1} \cdots \partial w^{\eta_k}} \). It then follows:

\[
\left( \sum_{i=1}^N \text{sgn}(\ell_i) \frac{\partial h(u_i; w)}{\partial w} \frac{\partial^\alpha h(u'; w)}{\partial w} \right) \left( \frac{\partial^\alpha h(u'; w)}{\partial w} \right) \tag{81}
\]

\[
= \sum_{\alpha_1 \ldots \alpha_r=1} \prod_j \text{sgn}(\ell_{\alpha_j}) \sum_{\eta_1 \ldots \eta_{r+1}=1} \Delta_{\eta} \prod_{j=0}^{r+1} \Gamma_{\eta_{k_j+1} \ldots \eta_{k_{j+1}}} (u_{\alpha_j}) \tag{82}
\]

where we have assigned \( u' := u_{\alpha_0}, u := u_{\alpha_r+1}, 0 = k_0 \leq k_1 \leq \ldots \leq k_{r+2} = 2r+2 \), and:

\[
\Delta_{\eta} = \begin{cases} 
1 & \{\eta_{k_j+1} \ldots \eta_{k_{j+1}}\} \in \{\eta_1 \ldots \eta_{k_j}, \eta_{k_j+1} \ldots \eta_{k_{j+1}}\} \\
0 & \text{else}
\end{cases} \tag{83}
\]

Denoting by \( \delta = \delta(k_1 = 1) \delta(k_2 = 2) \ldots \delta(k_{r+1} = r + 1) \) it holds that:

\[
\sum_{\eta_1 \ldots \eta_{r+2}=1} |w| \Delta_{\eta} \prod_{j=0}^{r+1} \Gamma_{\eta_{k_j+1} \ldots \eta_{k_{j+1}}} (u_{\alpha_j}) \tag{84}
\]

\[
= \sum_{\eta_1 \ldots \eta_{r+2}=1} (\delta + 1 - \delta) \Delta_{\eta} \prod_{j=0}^{r+1} \Gamma_{\eta_{k_j+1} \ldots \eta_{k_{j+1}}} (u_{\alpha_j}) \tag{85}
\]
Note that:

\[
\sum_{\eta_1 \ldots \eta_{k+2} = 1}^{[w]} \delta \Delta_\eta \prod_{j=0}^{r+1} \Gamma_{\eta_{j+1} \ldots \eta_{j+1}} (u_{\alpha_j})
\]

(86)

\[
= \sum_{\eta_1 \ldots \eta_{k+2} = 1}^{[w]} \Delta_\eta \left( \prod_{j=0}^{r} \Gamma_{\eta_{j+1}} (\mu_{\alpha_j}) \right) \Gamma_{\eta_{k+2} \ldots \eta_{k+2}} (u_{\alpha_{r+1}})
\]

(87)

\[
= (\nabla_w^{(r+1)} h(u_{\alpha_{r+1}}; w), \bigotimes_{j=0}^{r} \nabla_w h(u_{\alpha_j}; w)^r) = O_p(1) \text{ as } n \to \infty
\]

(88)

where we used the results of theorem 3.

\[\Box\]

A.6 Proofs of the Results in Sec.[5]

Theorem 5 (Hypernetworks as GPs). Let \( h(u; w) = g(z; f(x; w)) \) be a hypernetwork. Then,

\[
\lim_{\min(n, m) \to \infty} h(u; w) \overset{D}{=} \mathcal{G}
\]

(18)

where \( \mathcal{G} \) is some centered Gaussian distribution.

Proof. By [Lee et al., 2018], taking the width \( n = \min(n_1, \ldots, n_{L-1}) \) to infinity, the outputs \( V^d(x; w) := f^d(x; w) \) are governed by a centered Gaussian process, such that, the entries \( V^d_{i,j}(x; w) \), given some input \( x \), are independent and identically distributed. For the function \( h(u; w) = g(z; f(x; w)) \), it holds for the first layer:

\[
g^1(z; f(x; w)) = \sqrt{\frac{1}{m_0}} V^1(x; w) z
\]

(89)

In the infinite width limit \( (n \to \infty) \), \( V^1(x; w)_{i,j} \) are i.i.d Gaussian processes, such that,

\[
\mathbb{E} \left[ \text{vec}(V^1(x; w))_i, \text{vec}(V^1(x'; w))_j \right] = \sigma(x, x') \cdot \delta_{i=j}
\]

(90)

And so \( g^1(z; f(x; w)) \) is also a Gaussian process, such that:

\[
[g^1(z; f(x; w)), g^1(z'; f(x'; w))] \sim \mathcal{N}(0, \Lambda^1_g)
\]

(91)

where:

\[
\Lambda^1_g = \begin{pmatrix}
\sigma(x, x) z^\top z & \sigma(x, x') z'^\top z \\
\sigma(x, x') z^\top z' & \sigma(x', x') z'^\top z'
\end{pmatrix}
\]

(92)

In a similar fashion to the standard feed forward case, the pre-activations \( g^l(z; f(x; w)) \) converge to Gaussian processes as we let \( m = \min(m_1, \ldots, m_{H-1}) \) tend to infinity, with a covariance defined recursively:

\[
\Sigma^0_g(z, z') = z^\top z'
\]

(93)

\[
\Lambda^1_g = \begin{pmatrix}
\sigma(x, x) \cdot \Sigma_{g}^{l-1}(u, u) & \sigma(x, x') \cdot \Sigma_{g}^{l-1}(u', u) \\
\sigma(x, x') \cdot \Sigma_{g}^{l-1}(u, u') & \sigma(x', x') \cdot \Sigma_{g}^{l-1}(u', u')
\end{pmatrix}
\]

(94)

\[
\Sigma^H(u, u') = \sqrt{2} \mathbb{E}_{u, v \sim \mathcal{N}(0, \Lambda^1)} \left[ \sigma(u) \sigma(v) \right]
\]

(95)

\[\Box\]

We make use of the following lemma in the proof of Thm.[6]
Lemma 7. Recall the parametrization of the primary network:
\[
\begin{align*}
g_i^l := g_i^l(z_i; v) &= \sqrt{\frac{1}{m_i}} f_i^l(x_i; w) \cdot a_i^{l-1} \\
a_i^l := a_i^l(z_i; v) &= \sqrt{2} \cdot \sigma(g_i^{l-1})
\end{align*}
\] and \(a_i^0 := z_i\)  
(96)

For any pair \(u_i = \{u_i\}\), we denote:
\[
P_i^{l_2-l_1} = \prod_{l=l_1}^{l_2-1} \left( \sqrt{\frac{2}{m_l}} V^{l+1}(x_i; w) \cdot Z^l(z_i) \right) \text{ and } Z^l(z) = \text{diag}(\sigma(g^l(z)))
\] 
(97)

It holds that:
1. \(\lim_{\min(n,m) \to \infty} P_i^{l_2-l_1}(P_j^{l_2-l_1})^\top = \prod_{l=1}^{l_2-1} \Sigma^H(u_i, u_j)I\).
2. \(\lim_{\min(n,m) \to \infty} \frac{\partial h(u, u')}{\partial v} = \sum_{l=0}^{H-1} \left( \Sigma_{l}^H(u_i, u_j) \prod_{h=1}^{H-l-1} \Sigma_{h}^l(u_i, u_j) \right).\)

Proof. We have:
\[
P_i^{l_2-l_1}(P_j^{l_2-l_1})^\top = P_i^{l_2-l_1-1} \frac{2}{m_{l_2-1}} V^{l_2}(x_i; w) \cdot Z^{l_2-1}(z_i) Z^{l_2-1}(z_j) V^{l_2}(x_j; w)^\top (P_j^{l_2-l_1-1})^\top
\] 
(98)

Note that it holds that:
\[
\lim_{n,m \to \infty} \frac{2}{m_{l_2-1}} V^{l_2}(x_i; w) \cdot Z^{l_2-1}(z_i) Z^{l_2-1}(z_j) V^{l_2}(x_j; w)^\top = \sqrt{2} \mathbb{E}_{u,v \sim \mathcal{N}(0, \Lambda^{l_2})} \left[ \sigma(u) \sigma(v) \right]I = \Sigma_{l_2}^H(u_i, u_j)I
\] 
(99)

Applying the above recursively proves the first claim. Using the first claim, along with the derivation of the neural tangent kernel (see [Arora et al. (2019)]) proves the second claim.

\[\square\]

Theorem 6 (Hyperkernel convergence at initialization and composition). Let \(h(u; w) = g(z; f(x; w))\) be a ReLU hypernetwork. Then,
\[
\lim_{\min(n,m) \to \infty} K_0^f(u, u') = \Theta_h(u, u')
\] 
(20)

where:
\[
\Theta_h(u, u') := \Theta_f(x, x') \cdot \Theta_g(u, u')
\] 
(21)

such that:
\[
\lim_{n \to \infty} K_0^f(x, x') = \Theta_f(x, x')I, \quad \lim_{\min(n,m) \to \infty} K_0^g(u, u') = \Theta_g(u, u')
\] 
(22)

Under gradient flow, it holds that:
\[
\lim_{n \to \infty} \frac{\partial K_0^f(u, u')}{\partial t} \bigg|_{t=0} = \mathcal{O}_p(1/m)
\] 
(23)

Proof. Recalling that \(v = \text{vec}(g(x, z)) = [\text{vec}(V^1)\ldots\text{vec}(V^H)]\), concatenated into a single vector of length \(\sum_{l=0}^{H-1} m_l \cdot m_{l+1}\). The components of the inner matrix \(K_i^f(x, x')\) are given by:
\[
K_i^f(x, x')(i, j) = \sum_{l=1}^{L} \left\langle \frac{\partial v_i(x)}{\partial w^l}, \frac{\partial v_j(x')}{\partial w^l} \right\rangle
\] 
(101)

and it holds that in the infinite width limit, \(K_f(x, x')\) is a diagonal matrix:
\[
\lim_{n \to \infty} K_i^f(x, x') = \Theta_f(x, x')I
\] 
(102)
We notice that the two terms are the same up to changing between the inputs with no loss of generality, we can simply prove the convergence of the second term. We have:

\[
\lim_{n,m \to \infty} K_h(x, z, u') = \lim_{n,m \to \infty} \frac{\partial g(z; f(x; w))}{\partial f(x; w)} \cdot \Theta_f(x, x') I \cdot \frac{\partial g(z'; f(x'; w))}{\partial f(x'; w)}^\top
\]

\[
= \Theta_f(x, x') \lim_{n,m \to \infty} \frac{\partial h(u; w)}{\partial v} \cdot \frac{\partial h(u'; w)}{\partial v}^\top
\]

\[
= \Theta_f(x, x') \cdot \Theta_g(u, u')
\]

where we used the results of Lemma. [7]

Next, we would like to prove that:

\[
\lim_{n \to \infty} \frac{\partial K^h_t}{\partial t} (u, u') \bigg|_{t=0} = O_p(1/m)
\]

For this purpose, we would like to show that the derivative of \(K^h_t(u, u')\) with respect to \(t\), tends to zero,

\[
\frac{\partial K^h_t(u, u')}{\partial t} = \frac{\partial h(u; w)}{\partial w} \cdot \frac{\partial}{\partial t} \left( \frac{\partial T h(u'; w)}{\partial w} \right) + \frac{\partial h(u; w)}{\partial w} \cdot \frac{\partial}{\partial t} \left( \frac{\partial f(x; w)}{\partial w} \right) \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

\[
= \frac{\partial h(u; w)}{\partial f(x; w)} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

\[
= \frac{\partial h(u; w)}{\partial f(x; w)} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

\[
= \frac{\partial h(u; w)}{\partial f(x; w)} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

Next, we show that the first term is of order \(O_p(1/m)\). The proof that the second term is of order \(O_p(1/m)\) is similar. Substituting \(\frac{\partial}{\partial t} = -\mu \nabla_w c(w) \frac{\partial f(x; w)}{\partial w} = -\mu \nabla_w c(w) \frac{\partial f(x; w)}{\partial w}^\top\), it follows:

\[
\frac{\partial h(u; w)}{\partial f(x; w)} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

\[
= -\mu \nabla_w c(w) \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

\[
= -\mu \nabla_w c(w) \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

It then follows:

\[
\lim_{n \to \infty} \frac{\partial h(u; w)}{\partial f(x; w)} \frac{\partial f(x; w)}{\partial w} \frac{\partial}{\partial w} \frac{\partial f(x; w)}{\partial w}
\]

\[
= -\mu \sum_i \ell_i \Theta_f(x, x_i) \Theta_f(x, x') \lim_{n \to \infty} \frac{\partial T h(u_i; w)}{\partial f(x_i; w)} \frac{\partial^2 h(u; w)}{\partial f(x; w) \partial f(x; w) \partial f(x; w)} \frac{\partial h(u'; w)}{\partial f(x'; w)}
\]
We notice that:

\[
\lim_{n \to \infty} \frac{\partial^T h(u; w) \partial f(x; w) \partial h(u'; w)}{\partial f(x; w) \partial f(x; w)} = \sum_{l_1, l_2} \lim_{n \to \infty} \left\langle \frac{\partial^2 h(u; w)}{\partial f^{l_1}(x; w) \partial f^{l_2}(x; w)} \frac{\partial h(u; w)}{\partial f^{l_1}(x; w)} \otimes \frac{\partial h(u'; w)}{\partial f^{l_2}(x'; w)} \right\rangle
\]

We recall that \( f^l(x; w) \) converges to a GP (as a function of \( x \)) as \( n \to \infty \) (Lee et al., 2018). Therefore, \( T_{m,l_1,l_2}^{u,u_i,u'}(u, u_i, u') \) are special cases of the terms \( T_{n,i,d}^{l_1,i,d} \) (see Eq. 26) with weights that are distributed according to a GP instead of a normal distribution. In this case, we have: \( k = 2, d = d_1 = \cdots = d_k = 1 \), the neural network \( f^l \) is replaced with \( h \), the weights \( W \) are translated into \( f^l(x; w) \). We notice that the proof of Cor. 1 showing that \( T_{n,i,d}^{l_1,i,d} = O_p(1/n^k) \) is simply based on Lem. 2. Since Lem. 7 extends Lem. 2 to our case, the proof of Cor. 1 can be applied to show that \( T_{m,l_1,l_2}^{u,u_i,u'}(u, u_i, u') = O_p(1/m) \).