Cyclic forms on DG-Lie algebroids and semiregularity

EMMA LEPRI (*)

ABSTRACT – Given a transitive DG-Lie algebroid \((\mathcal{A}, \rho)\) over a smooth separated scheme \(X\) of finite type over a field \(K\) of characteristic 0 we define a notion of connection \(\nabla: R\Gamma(X, \text{Ker}\rho) \rightarrow R\Gamma(X, \Omega^1_X[-1] \otimes \text{Ker}\rho)\) and construct an \(L_\infty\) morphism between DG-Lie algebras \(f: R\Gamma(X, \text{Ker}\rho) \rightarrow R\Gamma(X, \Omega^1_X[2])\) associated to a connection and to a cyclic form on the DG-Lie algebroid.

In this way, we obtain a lifting of the first component of the modified Buchweitz–Flenner semiregularity map in the algebraic context, which has an application to the deformation theory of coherent sheaves on \(X\) admitting a finite locally free resolution. Another application is to the deformations of (Zariski) principal bundles on \(X\).

MATHEMATICS SUBJECT CLASSIFICATION (2020) – 14D15, 17B70

KEYWORDS – semiregularity, connections, DG-Lie algebroids, Atiyah class, L-infinity maps

Introduction

Let \(\mathcal{F}\) be a coherent sheaf admitting a finite locally free resolution on a smooth variety \(X\) over a field \(K\) of characteristic zero. The Buchweitz–Flenner semiregularity map, introduced in [7], and generalising the semiregularity map of Bloch [5], is defined by the formula:

\[
(0.1) \quad \sigma: \text{Ext}_X^2(\mathcal{F}, \mathcal{F}) \rightarrow \prod_{q \geq 0} H^{q+2}(X, \Omega^q_X), \quad \sigma(c) = \text{Tr}(\exp(-\text{At}(\mathcal{F})) \circ c),
\]

where \(\text{Tr}\) denotes the trace maps \(\text{Tr}: \text{Ext}_X^i(\mathcal{F}, \mathcal{F} \otimes \Omega_X^j) \rightarrow H^i(X, \Omega_X^j)\) for \(i, j \geq 0\), and the exponential of the opposite of the Atiyah class \(\text{At}(\mathcal{F}) \in \text{Ext}_X^1(\mathcal{F}, \mathcal{F} \otimes \Omega_X^1)\) is

(*) Indirizzo dell’A.: Università degli studi di Roma La Sapienza, Dipartimento di Matematica Guido Castelnuovo, P.le Aldo Moro 5, I-00185 Rome, Italy;
emma.lepri@uniroma1.it
defined via the Yoneda pairing
\[ \text{Ext}^i_X(\mathcal{F}, \mathcal{F} \otimes \Omega^j_X) \times \text{Ext}^j_X(\mathcal{F}, \mathcal{F} \otimes \Omega^i_X) \to \text{Ext}^{i+j}_X(\mathcal{F}, \mathcal{F} \otimes \Omega^{i+j}_X), \quad (a, b) \mapsto a \circ b, \]
\[ \exp(-\text{At}(\mathcal{F})) \in \prod_{q \geq 0} \text{Ext}^q_X(\mathcal{F}, \mathcal{F} \otimes \Omega^q_X). \]

We refer to [20] for a discussion of the role of the Buchweitz–Flenner semiregularity map in deformation theory and of the reason it is more convenient in this setting to consider the modified Buchweitz–Flenner semiregularity map, obtained as follows. Denote by \( \sigma_q : \text{Ext}^2_X(\mathcal{F}, \mathcal{F}) \to H^{q+2}(X, \Omega^q_X) \) the components of the semiregularity map \( \sigma = \sum \sigma_q \), for every \( q \geq 0 \) denote by \( \Omega^q_X = (\bigoplus_{i=0}^q \Omega^i_X [-i], d_{DR}) \) the truncated de Rham complex, and consider the composition
\[ \tau_q : \text{Ext}^2_X(\mathcal{F}, \mathcal{F}) \xrightarrow{\sigma_q} H^{q+2}(X, \Omega^q_X) = H^2(X, \Omega^q_X [q]) \xrightarrow{i_q} \mathbb{H}^2(X, \Omega^{\geq q}_X [2q]), \]
where the map \( i_q \) is induced by the inclusion of complexes \( \Omega^q_X [q] \subset \Omega^{\geq q}_X [2q] \). The map \( \tau_q \) is the \( q \)-component of the modified Buchweitz–Flenner semiregularity map.

A lifting to an \( L_\infty \) morphism of the first component \( \tau_1 \) of the modified Buchweitz–Flenner semiregularity map was constructed in [20] in the context of complex manifolds: for every connection of type \( (1, 0) \) (also called connections compatible with the holomorphic structure, see e.g. [16]) on a finite complex of locally free sheaves \( \mathcal{E} \) on a complex manifold \( X \) the existence of an \( L_\infty \) morphism between DG-Lie algebras
\[ g : A_{X}^{0,*}(\mathcal{H}om_{\mathcal{O}_X}^*(\mathcal{E}, \mathcal{E})) \twoheadrightarrow A_{X}^{*,*}[2] \]
whose linear component induces \( \tau_1 \) in cohomology was proved. As a consequence, recalling that \( \text{Ext}^2_X(\mathcal{F}, \mathcal{F}) \) is the obstruction space for the functor of deformations of a coherent sheaf \( \mathcal{F} \), the map \( \tau_1 \) annihilates all obstructions to deformations of a coherent sheaf admitting a finite locally free resolution. We refer again to [20] for a survey on the existing literature in this regard; we only note here that this fact was proved by Mukai and Artamkin [1] for the 0th component of the Buchweitz–Flenner semiregularity map, in [17] with some mild assumptions for the Bloch semiregularity map, in [7] for curvilinear obstructions for the map \( \sigma \) when the Hodge to de Rham spectral sequence of \( X \) degenerates at \( E_1 \), and finally in the general case for all obstructions and for the maps \( \tau_q \) in [28].

The fact that the construction of the \( L_\infty \) morphism can also be realised in the algebraic case was outlined in [20, Section 5] and is expanded on here: given a simplicial connection on the finite complex of locally free sheaves \( \mathcal{E} \), the map \( \tau_1 \) can be lifted to an \( L_\infty \) morphism
\[ g : \text{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}^*(\mathcal{E}, \mathcal{E})) \twoheadrightarrow \text{Tot}(\mathcal{U}, \Omega^{\geq 1}_X [2]), \]
where \( \text{Tot} \) denotes the Thom-Whitney totalisation and \( \mathcal{U} \) is an affine open cover of \( X \).

Since \( \text{Hom}^*_X(\mathcal{E}, \mathcal{E}) \) is the kernel of the anchor map of the transitive DG-Lie algebroid of derivations of pairs \( D^*(X, \mathcal{E}) \) of [18], it is natural to generalise this construction to the framework of DG-Lie algebroids. In fact, the main result of this paper, Theorem 2.12, is the construction of an \( L_\infty \) morphism between DG-Lie algebras

\[
f : \text{Tot}(\mathcal{U}, \mathcal{L}) \rightsquigarrow \text{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]),
\]

where \( \mathcal{L} \) denotes the kernel of the anchor map of a transitive DG-Lie algebroid, which is equipped with a connection \( \nabla : \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{L}) \) and a \( d_{\text{Tot}} \)-closed cyclic form.

For this reason, in Section 1 we introduce connections on transitive DG-Lie algebroids, which are \( \mathbb{K} \)-linear operators that play the role of connections of type \( (1, 0) \) in the construction of the \( L_\infty \) morphism. Connections on transitive DG-Lie algebroids are associated to simplicial liftings of the identity; also associated to a simplicial lifting of the identity is the extension cocycle, which generalises the notion of Atiyah cocycle. Section 2 describes cyclic forms on DG-Lie algebroids, cyclic forms induced by DG-Lie algebroid representations and the construction of the \( L_\infty \) morphism. Also contained in Section 2 is the following application to the deformation theory of coherent sheaves, analogous to the one obtained for complex manifolds in [20]:

**Theorem 0.1** (=Corollary 2.16). Let \( \mathcal{F} \) be a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme \( X \) of finite type over a field \( \mathbb{K} \) of characteristic zero. Then every obstruction to the deformations of \( \mathcal{F} \) belongs to the kernel of the map

\[
\tau_1 : \text{Ext}^2_X(\mathcal{F}, \mathcal{F}) \to \mathbb{H}^2(X, \Omega_X^{\leq 1}[2]).
\]

If the Hodge to de Rham spectral sequence of \( X \) degenerates at \( E_1 \), then every obstruction to the deformations of \( \mathcal{F} \) belongs to the kernel of the map

\[
\sigma_1 : \text{Ext}^2_X(\mathcal{F}, \mathcal{F}) \to H^3(X, \Omega_X^1), \quad \sigma_1(a) = - \text{Tr}(\text{At}(\mathcal{F}) \circ a).
\]

Lastly, since to every principal bundle one can naturally associate the Atiyah Lie algebroid of [2], Section 3 contains the following application to the deformation theory of (Zariski) principal bundles.

**Theorem 0.2** (=Corollary 3.12). Let \( P \) be a principal bundle on a smooth separated scheme \( X \) of finite type over an algebraically closed field \( \mathbb{K} \) of characteristic zero and let

\[
\text{Tot}(\mathcal{U}, \Omega_X^1[-i] \otimes a \cdot d(P)) \times \text{Tot}(\mathcal{U}, \Omega_X^1[-j] \otimes a \cdot d(P)) \xrightarrow{(\cdot, -)} \text{Tot}(\mathcal{U}, \Omega_X^{i+j}[-i-j]),
\]
for $i, j \geq 0$, be a $d_{\text{Tot}}$-closed cyclic form. Then every obstruction to the deformations of $P$ belongs to the kernel of the map

$$f_1 : H^2(X, \varpi \omega(P)) \to \mathbb{H}^2(X, \Omega^1_X[2]), \quad f_1(x) = \langle \text{At}(P), x \rangle,$$

where $\text{At}(P)$ denotes the Atiyah class of the principal bundle $P$.

**Notation**

By $\mathbb{K}$ we always denote a characteristic zero field. Given two complexes of $O_X$-modules $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$ denotes $\mathcal{F} \otimes_{O_X} \mathcal{G}$. If $V = \oplus V^i$ is either a graded vector space or a graded sheaf, $\overline{v}$ denotes the degree of a homogeneous element $v \in V$. For every integer $p$ the symbol $[p]$ denotes the shift functor, defined by $V[p]^i = V^{p+i}$. For complexes of $O_X$-modules $\mathcal{E}, \mathcal{F}$ we denote by $\mathcal{H}om^*_O(\mathcal{E}, \mathcal{F})$ the graded sheaf of $O_X$-linear morphisms

$$\mathcal{H}om^*_O(\mathcal{E}, \mathcal{F}) = \bigoplus_i \mathcal{H}om^i_O(\mathcal{E}, \mathcal{F}), \quad \mathcal{H}om^i_O(\mathcal{E}, \mathcal{F}) = \prod_j \mathcal{H}om_O(\mathcal{E}^j, \mathcal{F}^{i+j}).$$

If $L$ is a DG-Lie algebra, $H^*(L)$ always denotes the cohomology of the underlying cochain complex, which inherits a graded bracket from the one on $L$.

**1. DG-Lie algebroids, connections and extension cocycles**

The goal of this section is to define $\mathbb{K}$-linear operators

$$\nabla : \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$$

called connections on the kernel $\mathcal{L}$ of the anchor map of a transitive DG-Lie algebroid. A short review of the Thom-Whitney totalisation functor $\text{Tot}$ is given. In order to construct a connection, we introduce the notion of simplicial lifting of the identity. The section ends with the definition of the extension cocycle associated to a simplicial lifting of the identity, which generalises the notion of Atiyah cocycle. Different notions of Atiyah classes for DG-Lie algebroids have been considered elsewhere in the literature, see e.g. [3, 6, 26].

Let $X$ be a smooth separated scheme of finite type over a field $\mathbb{K}$ of characteristic zero, and let $\Theta_X, \Omega^1_X = \Omega^1_X/\mathbb{K}$ denote its tangent and cotangent sheaves respectively. Often it will be useful to consider the cotangent sheaf as a trivial complex of sheaves concentrated in degree one, so as to have an inclusion $\Omega^1_X[-1] \to \Omega^*_X$, where $\Omega^*_X = \otimes_p \Omega^p_X[-p]$ denotes the de Rham complex.
A DG-Lie algebroid over $X$ is a complex of sheaves of $O_X$-modules $\mathcal{A}$ equipped with a $\mathbb{K}$-bilinear bracket $[-,-]: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which defines a DG-Lie algebra structure on the spaces of sections, and with a morphism of complexes of $O_X$-modules $\rho: \mathcal{A} \to \Theta_X$, called the anchor map, such that the induced map on the spaces of sections is a homomorphism of DG-Lie algebras. Moreover for any sections $a_1, a_2$ of $\mathcal{A}$ and $f$ of $O_X$, the following Leibniz identity holds:

$$[a_1, fa_2] = f[a_1, a_2] + \rho(a_1)(f)a_2.$$ 

Example 1.2. The sheaf $\Theta_X$ is a trivial example of a DG-Lie algebroid concentrated in degree zero, with anchor map given by the identity. A DG-Lie algebroid over $\text{Spec} \, \mathbb{K}$ is exactly a DG-Lie algebra over the field $\mathbb{K}$. Every sheaf of DG-Lie algebras over $O_X$ can be considered as a DG-Lie algebroid over $X$ with trivial anchor map.

Definition 1.3. Let $(\mathcal{A}, \rho)$ and $(\mathcal{B}, \sigma)$ be DG-Lie algebroids over $X$. A morphism of DG-Lie algebroids $\varphi: \mathcal{A} \to \mathcal{B}$ is a morphism of complexes of sheaves which preserves brackets and commutes with the anchor maps:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\
\rho \downarrow & & \sigma \\
\Theta_X. & & \\
\end{array}$$

Let $(\mathcal{A}, \rho)$ be a DG-Lie algebroid over $X$ and assume that $\mathcal{L} = \text{Ker} \, \rho$ is a finite complex of locally free sheaves. Notice that on $\mathcal{L}$ there is a naturally induced graded Lie bracket: for sections $x, y$ of $\mathcal{L}$

$$[x, y] := [i(x), i(y)],$$

where $i: \mathcal{L} \to \mathcal{A}$ denotes the inclusion. This bracket is $O_X$-linear, in fact for any sections $x, y$ of $\mathcal{L}$ and $f$ of $O_X$ one has

$$[x, fy] := [i(x), i(fy)] = [i(x), fi(y)] = f[i(x), i(y)] + \rho(i(x))(f)y = f[i(x), i(y)] = f[x, y],$$

so that $\mathcal{L}$ is a sheaf of DG-Lie algebras over $O_X$.

Definition 1.4. [22, Chapter 3] A DG-Lie algebroid $(\mathcal{A}, \rho)$ over $X$ is transitive if the anchor map $\rho: \mathcal{A} \to \Theta_X$ is surjective.

Let now $(\mathcal{A}, \rho)$ be a transitive DG-Lie algebroid over $X$, consider the short exact sequence of complexes of sheaves

$$0 \longrightarrow \mathcal{L} \xrightarrow{i} \mathcal{A} \xrightarrow{\rho} \Theta_X \longrightarrow 0$$
and tensor it with the shifted cotangent sheaf $\Omega^1_X [-1]$ to obtain the short exact sequence
\begin{equation}
0 \longrightarrow \Omega^1_X [-1] \otimes \mathcal{L} \longrightarrow \Omega^1_X [-1] \otimes \mathcal{A} \longrightarrow \Omega^1_X [-1] \otimes \Theta_X \longrightarrow 0.
\end{equation}

Because of the isomorphism
\[
\Omega^1_X [-1] \otimes \Theta_X \cong \mathcal{H}om^*_O (\Omega^1_X, \Omega^1_X [-1]) \cong \mathcal{H}om^*_O (\Omega^1_X, \Omega^1_X [-1]),
\]
one can consider $\text{Id}_{\Omega^1} \in \Gamma(\mathcal{X}, \Omega^1_X [-1] \otimes \Theta_X)$ as an element of degree one.

**Definition 1.5.** A **lifting of the identity** is a global section $D$ in $\Gamma(\mathcal{X}, \Omega^1_X [-1] \otimes \mathcal{A})$ such that $(\text{Id} \otimes \rho)(D) = \text{Id}_{\Omega^1} \in \Gamma(\mathcal{X}, \Omega^1_X [-1] \otimes \Theta_X)$.

Since the map $\text{Id} \otimes \rho$ is not in general surjective on global sections, a lifting of the identity does not always exist. However a germ of a lifting of the identity, i.e., a preimage of $\text{Id}_{\Omega^1}$ in $\Omega^1_X [-1] \otimes \mathcal{A}$, always exists.

**Example 1.6.** For particular DG-Lie algebroids, the notion of lifting of the identity can be related to the more familiar notion of algebraic connection. Let $(\mathcal{E}, \delta_{\mathcal{E}})$ be a finite complex of locally free sheaves. Following [18, Section 5], define the complex of derivations of pairs
\[
\mathcal{D}^*(\mathcal{X}, \mathcal{E}) = \left\{ (h, u) \in \Theta_X \times \mathcal{H}om^*_\mathcal{E} (\mathcal{E}, \mathcal{E}) \mid u(f e) = f u(e) + h(f) e, \quad \forall f \in \mathcal{O}_{\mathcal{X}}, e \in \mathcal{E}\right\}.
\]
The complex $\mathcal{D}^*(\mathcal{X}, \mathcal{E})$ is a finite complex of coherent sheaves and the natural map
\[
\alpha : \mathcal{D}^*(\mathcal{X}, \mathcal{E}) \to \Theta_X, \quad (h, u) \mapsto h,
\]
which is called the anchor map, is surjective, see [18]. The graded Lie bracket is defined as
\[
[(h, u), (h', u')] = ([h, h'], [u, u']),
\]
where the (graded) Lie brackets on $\Theta_X$ and $\mathcal{H}om^*_\mathcal{E} (\mathcal{E}, \mathcal{E})$ are the (graded) commutators of the composition products. For $f \in \mathcal{O}_{\mathcal{X}}$ we then have that
\[
[(h, u), f(h', u')] = [(h, u), (f h', f u')] = ([h, f h'], [u, f u'])
\]
\[
= (h(f) h' + f h' h, f u' + h(f) u' - (-1)^{\text{deg} f} u' f u)
\]
\[
= (h(f) h' + f [h, h'], h(f) u' + f [u, u'])
\]
\[
= f (\{h, h'\}, [u, u']) + h(f)(h', u')
\]
\[
= f (\{h, u\}, (h', u')) + \alpha((h, u))(f)(h', u'),
\]
hence $(\mathcal{D}^*(X, \mathcal{E}), \alpha)$ is a transitive DG-Lie algebroid over $X$. Recall that an algebraic connection on the complex of locally free sheaves $\mathcal{E}$ is the data for every $i$ of an algebraic connection on $\mathcal{E}^i$, i.e., a $\mathbb{K}$-linear map $D: \mathcal{E}^i \to \Omega^1_X \otimes \mathcal{E}^i$ such that for $e \in \mathcal{E}^i$, $f \in \mathcal{O}_X$

$$D(fe) = d_{dR}f \otimes e + fD(e),$$

where $d_{dR}$ denotes the universal derivation $d_{dR}: \mathcal{O}_X \to \Omega^1_X$. A global algebraic connection on $\mathcal{E}$ need not exist. The kernel of the anchor map $\alpha$ is the sheaf of DG-Lie algebras $\mathcal{H}om^{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$, the graded sheaf of $\mathcal{O}_X$-linear endomorphisms of $\mathcal{E}$, with bracket equal to the graded commutator

$$[f, g] = fg - (-1)^{\bar{f}\bar{g}}gf,$$

and differential given by

$$g \mapsto [\delta, g] = \delta_E g - (-1)^{\bar{g}}g \delta_E.$$

The short exact sequence in (1.1) in this case is isomorphic to

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X[-1] \otimes \mathcal{E}) \xrightarrow{g \mapsto (0, g)} \mathcal{J}_\Omega^* \xrightarrow{(\beta, g) \mapsto \beta} \mathcal{D}er_{\mathbb{K}}^*(\mathcal{O}_X, \Omega^1_X[-1]) \to 0,$$

where the complex $\mathcal{J}_\Omega^*$ is defined as the subcomplex of

$$\mathcal{D}er_{\mathbb{K}}^*(\mathcal{O}_X, \Omega^1_X[-1]) \times \mathcal{H}om_{\mathbb{K}}^*(\mathcal{E}, \Omega^1_X[-1] \otimes \mathcal{E})$$

of elements $(\beta, v)$, with $\beta \in \mathcal{D}er_{\mathbb{K}}^*(\mathcal{O}_X, \Omega^1_X[-1])$ and $v \in \mathcal{H}om_{\mathbb{K}}^*(\mathcal{E}, \Omega^1_X[-1] \otimes \mathcal{E})$ such that

$$v(fx) =fv(x) + \beta(f) \otimes x,$$

for all $x \in \mathcal{E}$ and $f \in \mathcal{O}_X$; see also [20, Section 5]. In this case a lifting of the identity is exactly a global algebraic connection on the complex of sheaves $\mathcal{E}$: via the isomorphism $\Omega^1_X[-1] \otimes \mathcal{D}^*(X, \mathcal{E}) \cong \mathcal{J}_\Omega^*$ a lifting of the identity $D$ corresponds to $\mathbb{K}$-linear maps $D': \mathcal{E}^i \to \Omega^1_X[-1] \otimes \mathcal{E}^i$ for all $i$ such that $D'(fe) = fD'(e) + d_{dR}(f) \otimes e$ for all $f \in \mathcal{O}_X$ and $e \in \mathcal{E}^i$.

Before defining connections on the kernel of the anchor map of a transitive DG-Lie algebroid, it is useful to give a brief review of the definition and of some of the main properties of the Thom-Whitney totalisation functor $\text{Tot}$; for more details see e.g. [11, 12, 17, 24, 27]. The Thom-Whitney totalisation is a functor from the category of semicosimplicial DG-vector spaces to the category of DG-vector spaces. For every $n \geq 0$ consider

$$A_n = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(1 - \sum_i t_i, \sum_i dt_i)}.$$
the commutative differential graded algebra of polynomial differential forms on the affine standard $n$-simplex, and the maps

$$\delta^*_k : A_n \to A_{n-1}, \quad 0 \leq k \leq n \quad \delta^*_k(t_i) = \begin{cases} t_i & i < k \\ 0 & i = k \\ t_{i-1} & i > k. \end{cases}$$

**Definition 1.7.** The Thom-Whitney totalisation of a semicosimplicial DG-vector space $V$

$$V : V_0 \xrightarrow{\delta_0} V_1 \xrightarrow{\delta_1} V_2 \xrightarrow{\delta_2} \cdots$$

is the DG-vector space

$$\text{Tot}(V) = \left\{ (x_n) \in \prod_{n \geq 0} A_n \otimes V_n \mid (\delta^*_k \otimes \text{Id})x_n = (\text{Id} \otimes \delta_k)x_{n-1} \right\},$$

with differential induced by the one on $\prod_{n \geq 0} A_n \otimes V_n$. To simplify notation, we will denote this differential by $d_{\text{Tot}} = d_A + d_V$, where $d_A$ denotes the differential of polynomial differential forms, and $d_V$ the differential on $V$.

If $f : V \to W$ is a morphism of semicosimplicial DG-vector spaces, then $\text{Tot}(f) : \text{Tot}(V) \to \text{Tot}(W)$ is defined as the restriction of the map

$$\prod_{n \geq 0} \text{Id} \otimes f : \prod_{n \geq 0} A_n \otimes V_n \to \prod_{n \geq 0} A_n \otimes W_n.$$
Example 1.8. Let $E$ be a finite complex of quasi-coherent sheaves on $X$, and $\mathcal{U} = \{U_i\}$ an open cover of $X$. Denote by $U_{i_1 \cdots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$, and consider the semicosimplicial DG-vector space of Čech cochains:

$E(\mathcal{U}) = \prod_i E(U_i) \xrightarrow{\delta_0} \prod_{i,j} E(U_{ij}) \xrightarrow{\delta_1} \prod_{i,j,k} E(U_{ijk}) \to \cdots$

The Whitney integration theorem states that there exists a quasi-isomorphism between the Tot complex $\text{Tot}(\mathcal{U}, E)$ and the complex of Čech cochains $C^*(\mathcal{U}, E)$ of $E$ (see [32] for the $C^\infty$ version, [9, 13, 21, 27] for the algebraic version used here). Hence if the open cover $\mathcal{U}$ is affine the cohomology of $\text{Tot}(\mathcal{U}, E)$ is isomorphic to the hypercohomology of the complex of sheaves $E$. If the open cover is affine the complex $\text{Tot}(\mathcal{U}, E)$ is a model for the module $R\Gamma(X, E)$ of derived global sections of $E$, see e.g. [12, 25].

Moreover there is a canonical inclusion of the global sections of $E$ in the totalisation $\text{Tot}(\mathcal{U}, E)$: in fact, the injection $\iota : \Gamma(X, E) \to \prod_i E(U_i)$ is such that $\delta_0 \iota = \delta_1 \iota$ and therefore for every $a \in \Gamma(X, E)$

$$(1 \otimes \iota(a), 1 \otimes \delta_0 \iota(a), 1 \otimes \delta_1^2 \iota(a), \cdots)$$

belongs to $\text{Tot}(\mathcal{U}, E)$. It is easy to see that if the complex $E$ has trivial differential, every global section gives a cocycle in $\text{Tot}(\mathcal{U}, E)$.

We are now ready to define connections on $\mathcal{L} = \text{Ker} \rho$, the kernel of the anchor map of a transitive DG-Lie algebroid $(\mathcal{A}, \rho)$ over $X$. Assume that $\mathcal{L}$ is a finite complex of locally free sheaves and fix an affine open cover $\mathcal{U} = \{U_i\}$ of $X$. The short exact sequence

$$0 \to \Omega^1_X[-1] \otimes \mathcal{L} \xrightarrow{\text{Id} \otimes i} \Omega^1_X[-1] \otimes \mathcal{A} \xrightarrow{\text{Id} \otimes \rho} \Omega^1_X[-1] \otimes \Theta_X \to 0.$$ 

gives a short exact sequence of the corresponding semicosimplicial complexes of Čech cochains, and since the functor $\text{Tot}$ is exact there is an exact sequence

$$0 \to \text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}) \xrightarrow{\text{Id} \otimes \rho} \text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \xrightarrow{\text{Id} \otimes \rho} \text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X) \to 0.$$ 

Denote by $d$ the differential on $\mathcal{A}$ and $\mathcal{L}$, which can be extended to $\Omega^1_X[-1] \otimes \mathcal{A}$ and to $\Omega^1_X[-1] \otimes \mathcal{L}$ by setting

$$d(\eta \otimes x) = (-1)^{\overline{\eta}} \eta \otimes dx = -\eta \otimes dx.$$
Denote by $d_{\text{Tot}}$ the differentials on all the above Tot complexes: for $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L})$ and $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ the differential $d_{\text{Tot}}$ is equal to $d_A + d$, while for $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$ one has that $d_{\text{Tot}}$ is just $d_A$, where $d_A$ is the differential of polynomial differential forms on the affine simplex, see Definition 1.7.

Because of the natural inclusion of global sections in the totalisation remarked upon in Example 1.8, $\text{Id}_{\Omega^1}$ belongs to $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$, where it has degree one.

**Definition 1.9.** A simplicial lifting of the identity is an element $D$ of $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$ such that $(\text{Id} \otimes \rho)(D) = \text{Id}_{\Omega^1}$ in $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \Theta_X)$.

It is clear that a simplicial lifting of the identity always exists and that $D$ has degree one in $\text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A})$.

**Remark 1.10.** Notice that via the isomorphism

$$\Omega^1_X[-1] \otimes \Theta_X = \Omega^1_X[-1] \otimes \text{Der}_K(O_X, O_X) \cong \text{Der}_K^+(O_X, \Omega^1_X[-1]),$$

we have that $(\text{Id} \otimes \rho)(D) = d_{dR} \in \text{Tot}(\mathcal{U}, \text{Der}_K^+(O_X, \Omega^1_X[-1]))$.

In order to define a connection on $\mathcal{L}$, it is necessary to define a Lie bracket

$$[-,-] : \text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{A}) \times \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X[-1] \otimes \mathcal{L}),$$

induced by the bracket of the following lemma.

**Lemma 1.11.** There exists a well defined $K$-bilinear bracket

$$[-,-] : (\Omega^1_X[-1] \otimes \mathcal{A}) \times \mathcal{L} \to \Omega^1_X[-1] \otimes \mathcal{L}.$$  

**Proof.** Denote by $i : \mathcal{L} \to \mathcal{A}$ the inclusion, take $\eta \otimes a$ with $\eta \in \Omega^1_X[-1]$ and $a \in \mathcal{A}$, and define for $x \in \mathcal{L}$

$$[\eta \otimes a, x] := \eta \otimes [a, i(x)].$$

Notice that the Leibniz identity in Definition 1.1 implies that

$$[fa_1, a_2] = f[a_1, a_2] - (-1)^{\overline{a_1} \overline{a_2}} \rho(a_2)(f)a_1.$$

Hence the bracket $[\eta \otimes a, x]$ is well defined: for any $f \in O_X$

$$[\eta \otimes fa, x] = \eta \otimes (fa, x) = \eta \otimes (f[a, x] - (-1)^{\overline{a} \overline{x}} \rho(x)(f)a) = \eta \otimes f[a, x] = f\eta \otimes [a, x] = [f\eta \otimes a, x].$$
It is clear that \([\eta \otimes a, x]\) belongs to \(\Omega^1_X [-1] \otimes \mathcal{L}\):
\[
(\text{Id} \otimes \rho)([\eta \otimes a, x]) = (\text{Id} \otimes \rho)(\eta \otimes [a, x]) = \eta \otimes [\rho(a), \rho(x)] = 0.
\]

Since the functor \(\text{Tot}\) preserves products, the map
\[
[-, -]: (\Omega^1_X [-1] \otimes \mathcal{A}) \times \mathcal{L} \to \Omega^1_X [-1] \otimes \mathcal{L}
\]
induces a \(\mathbb{K}\)-bilinear map
(1.2) \([-, -]: \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A}) \times \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L}),\]
which is defined component-wise as the restriction of
\[
A_n \otimes \prod (\Omega^1_X [-1] \otimes \mathcal{A})(U_{i_1 \ldots i_n}) \otimes A_n \otimes \prod \mathcal{L}(U_{i_1 \ldots i_n})
\]
\[
\downarrow [-, -]
\]
\[
A_n \otimes \prod (\Omega^1_X [-1] \otimes \mathcal{L})(U_{i_1 \ldots i_n})
\]
\[
[\eta_n \otimes (t_{i_1 \ldots i_n}), \phi_n \otimes (u_{i_1 \ldots i_n})] = \eta_n \phi_n \otimes ([(-1)^{\theta_n(i_1 \ldots i_n)}t_{i_1 \ldots i_n}u_{i_1 \ldots i_n}]),
\]
for \(\eta_n, \phi_n\) in \(A_n, t_{i_1 \ldots i_n}\) in \((\Omega^1_X [-1] \otimes \mathcal{A})(U_{i_1 \ldots i_n})\) and \(u_{i_1 \ldots i_n}\) in \(\mathcal{L}(U_{i_1 \ldots i_n})\).

**Definition 1.12.** A **connection** on \(\mathcal{L}\) is the adjoint operator of a simplicial lifting of the identity \(D \in \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A})\)
\[
\nabla = [D, -]: \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L}),
\]
where \([-,-]: \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A}) \times \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L})\) is the bracket in (1.2). It is a \(\mathbb{K}\)-linear operator.

We will now examine the relationship between connections and particular representatives of extension classes. The short exact sequence
(1.3) \[
0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{A} \xrightarrow{\rho} \Theta_X \longrightarrow 0
\]
gives an extension class \([u_\rho] \in \text{Ext}^1_X (\Theta_X, \mathcal{L})\). It is possible to give a representative of \([u_\rho]\) in the totalisation \(\text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A})\) with respect to an affine open cover \(\mathcal{U}\) of \(X\).

**Definition 1.13.** An **extension cocycle** \(u\) of the transitive DG-Lie algebroid \(\mathcal{A}\) is the differential of a simplicial lifting of the identity \(D\) in \(\text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A})\), \(u = d_{\text{Tot}} D\).
Notice that \( u \) belongs to \( \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L}) \):

\[
(\text{Id} \otimes \rho)u = (\text{Id} \otimes \rho)d_{\text{Tot}}(D) = d_{\text{Tot}}(\text{Id} \otimes \rho)D = d_{\text{Tot}} \text{Id}_{\Omega^1} = 0,
\]

where the last equality is a consequence of the fact that \( \text{Id}_{\Omega^1} \) is a global section and \( \Omega^1_X [-1] \otimes \Theta_X \) has trivial differential (see Example 1.8). Note that \( u \) has degree two in \( \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L}) \) and that \( d_{\text{Tot}}u = d_{\text{Tot}}d_{\text{Tot}}D = 0 \).

Using the isomorphisms

\[
\Omega^1_X [-1] \otimes \mathcal{L} \cong \mathcal{H}om^*_O(\Theta_X [1], \mathcal{L}) \cong \mathcal{H}om^*_O(\Theta_X, \mathcal{L}) [-1],
\]

the cohomology class of \( u \) belongs to

\[
\mathcal{H}^2(\text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L})) \cong \mathcal{H}^2(\text{Tot}(\mathcal{U}, \mathcal{H}om^*_O(\Theta_X, \mathcal{L}) [-1])) \cong \\
\mathcal{H}^2(X, \mathcal{H}om^*_O(\Theta_X, \mathcal{L}) [-1]) \cong \mathcal{H}^1(X, \mathcal{H}om^*_O(\Theta_X, \mathcal{L})) \cong \text{Ext}^1_X(\Theta_X, \mathcal{L}).
\]

This cohomology class does not depend on the chosen simplicial lifting of the identity: if \( D \) and \( D' \) are two simplicial liftings of the identity in \( \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A}) \), we have that \( (\text{Id} \otimes \rho)(D - D') = \text{Id}_{\Omega^1} - \text{Id}_{\Omega^1} = 0 \), so \( D - D' \) belongs to \( \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L}) \) and \( d_{\text{Tot}}D \) and \( d_{\text{Tot}}D' \) differ by the coboundary \( d_{\text{Tot}}(D' - D) \). It is easy to see that the cohomology class of \( u \) is trivial if and only if the short exact sequence in (1.3) splits.

**Lemma 1.14.** Let \( \nabla : \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{L}) \) be a connection on \( \mathcal{L} \), associated to the simplicial lifting of the identity \( D \in \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A}) \). Let \( u = d_{\text{Tot}}D \) be the corresponding extension cocycle, then for every \( x \) in \( \text{Tot}(\mathcal{U}, \mathcal{L}) \) we have that

\[
\nabla(d_{\text{Tot}}x) = [u, x] - d_{\text{Tot}}\nabla(x).
\]

**Proof.** Recall that \( d \) denotes the differential of \( \mathcal{A} \) and \( \mathcal{L} \), which can be extended to \( \Omega^1_X [-1] \otimes \mathcal{A} \) and to \( \Omega^1_X [-1] \otimes \mathcal{L} \) by setting \( d(\eta \otimes x) = (-1)\overline{\eta} \otimes dx = -\eta \otimes dx \). It is easy to see that for the \( \mathbb{K} \)-bilinear map \([-, -] : (\Omega^1_X [-1] \otimes \mathcal{A}) \times \mathcal{L} \to \Omega^1_X [-1] \otimes \mathcal{L} \) of Lemma 1.11,

\[
d[\eta \otimes a, x] = [d(\eta \otimes a), x] + (-1)\overline{\eta + a}[\eta \otimes a, dx].
\]

A straightforward calculation then shows that for \( z \in \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A}) \) and \( w \in \text{Tot}(\mathcal{U}, \mathcal{L}) \),

\[
d_{\text{Tot}}[z, w] = [d_{\text{Tot}}z, w] + (-1)\overline{z}[z, d_{\text{Tot}}w],
\]

and the conclusion follows from the fact \( u = d_{\text{Tot}}D \).
2. Cyclic forms and $L_\infty$ morphisms

This section describes cyclic forms on DG-Lie algebroids and illustrates how DG-Lie algebroid representations give rise to cyclic forms. We then discuss induced cyclic forms on the Thom-Whitney totalisation and the property of $d_{\text{Tot}}$-closure. The central result is the construction of a $L_\infty$ morphism associated to a connection and to a $d_{\text{Tot}}$-closed cyclic form for a transitive DG-Lie algebroid. This allows us to state the results of [20] for a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme of finite type over a field $\mathbb{K}$ of characteristic zero.

Let $\mathcal{A}$ be a DG-Lie algebroid over a smooth separated scheme $X$ of finite type over a field $\mathbb{K}$ of characteristic zero, with anchor map $\rho: \mathcal{A} \to \Theta_X$. Assume that the kernel of the anchor map $L$ is a finite complex of locally free sheaves. Notice that for any $a \in \mathcal{A}$ and $x \in L$, the bracket $[a, x]$ belongs to $L$:

$$\rho([a, x]) = [\rho(a), \rho(x)] = 0.$$  

**Definition 2.1.** A cyclic bilinear form on a DG-Lie algebroid $(\mathcal{A}, \rho)$ is a graded symmetric $\mathcal{O}_X$-bilinear product of degree zero on $L = \text{Ker} \rho$

$$\langle - , - \rangle: L \times L \to \mathcal{O}_X$$

such that for all sections $x, y$ of $L$ and $a$ of $\mathcal{A}$

$$\langle [a, x], y \rangle + (-1)^{\bar{a} \bar{y}} \langle x, [a, y] \rangle = \rho(a)(\langle x, y \rangle).$$  

(2.1)

Notice that the definition implies that for all $x, y, z \in L$

$$\langle x, [y, z] \rangle = \langle [x, y], z \rangle.$$  

(2.2)

These two properties will be discussed after giving some examples.

In the following two examples, the cyclicity of the forms will follow from Lemma 2.5.

**Example 2.2.** An example of cyclic form on $(\mathcal{A}, \rho)$ is induced by the Killing form. Consider the adjoint representation as a morphism of sheaves of DG-Lie algebras

$$\text{ad}: L \to \text{Hom}_{\mathcal{O}_X}^*(L, L), \quad a \mapsto [a, -]$$

and consider the trace map $\text{Tr}: \text{Hom}_{\mathcal{O}_X}^*(L, L) \to \mathcal{O}_X$, which is morphism of sheaves of DG-Lie algebras (when considering $\mathcal{O}_X$ as a trivial sheaf of DG-Lie algebras). Then one can define the form

$$\langle - , - \rangle: L \times L \to \mathcal{O}_X, \quad (x, y) \mapsto \text{Tr}(\text{ad} x \text{ ad} y).$$
Example 2.3. For the DG-Lie algebroid $D^*(X, E)$ of Example 1.6,

$$0 \longrightarrow \mathcal{H}om^*_O(E, E) \longrightarrow D^*(X, E) \longrightarrow \Theta_X \longrightarrow 0,$$

a natural bilinear form on $\mathcal{H}om^*_O(E, E)$ is induced by the trace map $\text{Tr}: \mathcal{H}om^*_O(E, E) \to O_X$ as follows:

$$\mathcal{H}om^*_O(E, E) \times \mathcal{H}om^*_O(E, E) \to O_X, \quad \langle f, g \rangle := -\text{Tr}(fg).$$

Example 2.3 explains the definition of cyclic form: (2.2) reflects the cyclicity property of the trace map $\text{Tr}: \mathcal{H}om^*_O(E, E) \to O_X$, $\text{Tr}(ab) = (-1)^{\overline{a\overline{b}}} \text{Tr}(ba)$, while (2.1) is related to the properties of the extension of the trace map to $D^*(X, E)$, for which we refer to [18].

The Leibniz identity of Definition 1.1

$$[a, fx] = f[a,x] + \rho(a)(fx) \quad \forall a \in \mathcal{A}, \ x \in \mathcal{L}, \ f \in O_X$$

can be restated by noticing that for all $a$ in $\mathcal{A}$ the operator $(\rho(a), [a, -])$ belongs to $D^*(X, \mathcal{L})$ of Example 1.6. Hence there is a morphism of DG-Lie algebroids

$$\text{ad}: \mathcal{A} \to D^*(X, \mathcal{L}).$$

The morphism $\text{ad}: \mathcal{A} \to D^*(X, \mathcal{L})$ restricts to the morphism $\text{ad}: L \to \mathcal{H}om^*_O(L, \mathcal{L})$ of Example 2.2, so that the following diagram commutes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \overset{i}{\longrightarrow} & \mathcal{A} & \overset{\rho}{\longrightarrow} & \Theta_X & \longrightarrow & 0 \\
& & \downarrow{\text{ad}} & & \downarrow{\text{ad}} & & \| & & \\
0 & \longrightarrow & \mathcal{H}om^*_O(L, \mathcal{L}) & \longrightarrow & D^*(X, \mathcal{L}) & \overset{\alpha}{\longrightarrow} & \Theta_X & \longrightarrow & 0.
\end{array}
$$

This motivates the following definition:

**Definition 2.4.** A representation of a DG-Lie algebroid $(\mathcal{A}, \rho)$ over $X$ is a morphism of DG-Lie algebroids $\theta: \mathcal{A} \to D^*(X, E)$, where $E$ is a finite complex of locally free sheaves over $X$:

$$
\begin{array}{ccc}
\mathcal{A} & \overset{\theta}{\longrightarrow} & D^*(X, E) \\
\rho \downarrow & & \alpha \downarrow \\
\Theta_X & & 
\end{array}
$$

Every representation $\theta: \mathcal{A} \to D^*(X, E)$ induces a form $\langle -, - \rangle_\theta: \mathcal{L} \times \mathcal{L} \to O_X$: for any $x \in \mathcal{L}$ we have that $\alpha \circ \theta(x) = \rho(x) = 0$, so that

$$\theta|_\mathcal{L}: \mathcal{L} \to \mathcal{H}om^*_O(E, E),$$
and using the trace map $\text{Tr}: \text{Hom}^*_{O_X}(\mathcal{E}, \mathcal{E}) \to O_X$ we can define for $x, y$ sections of $\mathcal{L}$

$$\langle x, y \rangle_\theta := \text{Tr}(\theta(x)\theta(y)).$$

Forms obtained in this way are cyclic:

**Lemma 2.5.** For any DG-Lie algebroid representation $\theta: \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E})$ the induced form $\langle -, - \rangle_\theta: \mathcal{L} \times \mathcal{L} \to O_X$ is cyclic.

**Proof.** For $a \in \mathcal{A}$ and $x, y \in \mathcal{L}$

$$\langle [a, x], y \rangle_\theta + (-1)^{\widetilde{a} \cdot \widetilde{x}} \langle x, [a, y] \rangle_\theta = \text{Tr}(\theta([a, x])\theta(y) + (-1)^{\widetilde{a} \cdot \widetilde{x}}\theta(x)\theta([a, y]))$$

$$= \text{Tr}([\theta(a), \theta(x)]\theta(y) + (-1)^{\widetilde{a} \cdot \widetilde{x}}\theta(x)[\theta(a), \theta(y)])$$

$$= \text{Tr}(\theta(a)\theta(x)\theta(y) - (-1)^{\widetilde{a} \cdot (\widetilde{x} + \widetilde{y})}\theta(x)\theta(y)\theta(a))$$

$$= \text{Tr}([\theta(a), \theta(x)\theta(y)]).$$

Notice that if $\widetilde{a} \neq 0$ then $a$ belongs to $\mathcal{L}$, so that $\theta(a)$ belongs to $\text{Hom}^*_{O_X}(\mathcal{E}, \mathcal{E})$ and it is clear that $\text{Tr}([\theta(a), \theta(x)\theta(y)]) = 0$, by the properties of the trace map.

The only remaining non-trivial case is when $\widetilde{a} = \widetilde{x} + \widetilde{y} = 0$. Let $\{e^k_i\}$ with $i = 1, \cdots, n_k$ be a local basis of $\mathcal{L}^k$ and let

$$\theta(a)(e^k_i) = \sum_j A^k_{ij} e^k_j, \quad \theta(x)\theta(y)(e^k_i) = \sum_j B^k_{ij} e^k_j, \quad A^k_{ij}, B^k_{ij} \in O_X,$$

then

$$[\theta(a), \theta(x)\theta(y)](e^k_i) = \theta(a)\theta(x)\theta(y)(e^k_i) - \theta(x)\theta(y)\theta(a)(e^k_i)$$

$$= \theta(a)(\sum_j B^k_{ij} e^k_j) - \theta(x)\theta(y)(\sum_j A^k_{ij} e^k_j)$$

$$= \sum_j B^k_{ij} \theta(a)(e^k_j) + \sum_j (\alpha \circ \theta)(a)(B^k_{ij}) e^k_j - \sum_{j,s} A^k_{ij} B^k_{js} e^k_s$$

$$= \sum_{j,s} B^k_{ij} A^k_{js} e^k_s + \sum_{j} \rho(a)(B^k_{ij}) e^k_j - \sum_{j,s} A^k_{ij} B^k_{js} e^k_s.$$

The trace of $[\theta(a), \theta(x)\theta(y)]$ is hence equal to

$$\sum_k (-1)^k \left( \sum_{j,i} B^k_{ij} A^k_{ji} + \sum_i \rho(a)(B^k_{ii}) - \sum_{j,i} A^k_{ij} B^k_{ji} \right) = \sum_k (-1)^k \rho(a)(B^k_{ii}) = \rho(a) \left( \sum_{k,i} (-1)^k B^k_{ii} \right) = \rho(a) \text{Tr}(\theta(x)\theta(y)) = \rho(a)(\langle x, y \rangle_\theta).$$
For every \( i \geq 0 \), let \( \Omega^i_X [-i] \) denote the sheaf \( \Omega^i_X \) seen as a trivial complex concentrated in degree \( i \). Any cyclic form \( \langle -, - \rangle : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X \) can be extended to a collection of \( \mathcal{O}_X \)-bilinear forms

\[
\langle -, - \rangle : (\Omega^i_X [-i] \otimes \mathcal{L}) \times (\Omega^j_X [-j] \otimes \mathcal{L}) \to \Omega^{i+j}_X [-i-j], \quad i, j \geq 0,
\]

according to the Koszul sign rule, by setting for \( x, y \in \mathcal{L}, \omega \in \Omega^i_X [-i], \) and \( \eta \in \Omega^j_X [-j] \)

\[
\langle \omega \otimes x, \eta \otimes y \rangle = (-1)^{i+j} \omega \wedge \eta \langle x, y \rangle.
\]

It is immediate to see that this form is cyclic in the sense that

\[
\langle b \cdot x - x \cdot b, \omega \rangle = \langle x \cdot b, \omega \rangle - \langle b \cdot x, \omega \rangle = 0,
\]

where the bracket is the one of Lemma 1.11, and the anchor map has been extended to \( \Omega^1_X [-1] \otimes \mathcal{A} \) by setting \( \langle \text{Id} \otimes \rho \rangle (\omega \otimes a) := \omega \otimes \rho(a) \).

**Definition 2.6.** A cyclic form \( \langle -, - \rangle : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X \) is **d-closed** if for all \( z, w \in \mathcal{L} \)

\[
\langle dz, w \rangle + (-1)^z \langle z, dw \rangle = 0.
\]

**Lemma 2.7.** For any DG-Lie algebroid representation \( \theta : \mathcal{A} \to \mathcal{D}^*(X, \mathcal{E}) \) the induced cyclic form \( \langle -, -, \rangle_\theta : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X \) is d-closed.

**Proof.** Since \( \theta|\mathcal{L} : \mathcal{L} \to \mathcal{H}om^*_\mathcal{O}_X(\mathcal{E}, \mathcal{E}) \) is a morphism of DG-Lie algebroids it commutes with differentials: for \( x \in \mathcal{L}, \)

\[
\theta(dx) = d\mathcal{H}om^*_\mathcal{O}_X(\mathcal{E}, \mathcal{E})(\theta(x)) = [d\mathcal{E}, \theta(x)].
\]

For \( x, y \) sections of \( \mathcal{L} \)

\[
\langle dx, y \rangle_\theta + (-1)^y \langle x, dy \rangle_\theta =
\]

\[
\text{Tr}(\theta(dx)\theta(y) + (-1)^y \theta(x)\theta(dy)) = \text{Tr}([d\mathcal{E}, \theta(x)]\theta(y) + (-1)^y \theta(x)[d\mathcal{E}, \theta(y)])
\]

\[
= \text{Tr}(d\mathcal{E}\theta(x)\theta(y) - (-1)^y \theta(x)\theta(y)d\mathcal{E}) = \text{Tr}([d\mathcal{E}, \theta(x)\theta(y)]) = 0.
\]

It follows from the properties of the Thom-Whitney totalisation functor \( \text{Tot} \) that every collection of cyclic forms \( \langle -, - \rangle : (\Omega^i_X [-i] \otimes \mathcal{L}) \times (\Omega^j_X [-j] \otimes \mathcal{L}) \to \Omega^{i+j}_X [-i-j] \), with \( i, j \geq 0 \), induces a collection of \( \mathbb{K} \)-bilinear forms

\[
\langle -, - \rangle : \text{Tot}(\mathcal{U}, \Omega^i_X [-i] \otimes \mathcal{L}) \times \text{Tot}(\mathcal{U}, \Omega^j_X [-j] \otimes \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^{i+j}_X [-i-j]).
\]

Recalling Definition 1.7, the required forms are induced component-wise by the restriction of
\[ A_n \otimes \prod_{i_1 \ldots i_n} (\Omega^j_X [-i] \otimes \mathcal{L})(U_{i_1 \ldots i_n}) \times A_n \otimes \prod_{i_1 \ldots i_n} (\Omega^j_X [-j] \otimes \mathcal{L})(U_{i_1 \ldots i_n}) \]

\[ \downarrow \]

\[ A_n \otimes \prod_{i_1 \ldots i_n} \Omega^i_X [-i - j](U_{i_1 \ldots i_n}), \]

\[ \langle \eta_n \otimes (x_{i_1 \ldots i_n}), \omega_n \otimes (y_{i_1 \ldots i_n}) \rangle := \eta_n \omega_n (\langle (-1)^{\nu_n} (x_{i_1 \ldots i_n}, y_{i_1 \ldots i_n}) \rangle), \]

with \( \eta_n, \omega_n \) in \( A_n, x_{i_1 \ldots i_n} \) in \( (\Omega^j_X [-i] \otimes \mathcal{L})(U_{i_1 \ldots i_n}) \) and \( y_{i_1 \ldots i_n} \) in \( (\Omega^j_X [-j] \otimes \mathcal{L})(U_{i_1 \ldots i_n}) \).

Let \( (\Omega^*_X = \oplus_p \Omega^p_X [-p], d_{dR}) \) denote the de Rham complex. In the following, when working with \( \text{Tot}(\mathcal{U}, \Omega^*_X) \) the differential is denoted by \( d_{\text{Tot}} \) if \( \Omega^*_X = \oplus_p \Omega^p_X [-p] \) is considered as complex with trivial differential, and by \( d_{\text{Tot}} + d_{dR} \) if it is considered as a complex with the de Rham differential.

**Lemma 2.8.** The form induced on the totalisation by a cyclic form \( \langle -, - \rangle : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X \) is cyclic: for all \( b \in \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \mathcal{A}) \) and \( z, w \in \text{Tot}(\mathcal{U}, \mathcal{L}) \) one has that

\[ \langle [b, z], w \rangle + (-1)^{\bar{b} \bar{w}} \langle z, [b, w] \rangle = (\text{Id} \otimes \rho)(b)(\langle z, w \rangle). \]

Moreover if the form \( \langle -, - \rangle : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X \) is \( d \)-closed (Definition 2.6), then for the induced form, for \( z \in \text{Tot}(\mathcal{U}, \Omega^1_X [-i] \otimes \mathcal{L}) \) and \( w \in \text{Tot}(\mathcal{U}, \Omega^1_X [-j] \otimes \mathcal{L}) \) we have that

\[ \langle d_{\text{Tot}} z, w \rangle + (-1)^{\bar{z} \bar{w}} \langle z, d_{\text{Tot}} w \rangle = d_{\text{Tot}} \langle z, w \rangle. \]

The above condition will be called \( d_{\text{Tot}} \)-closure.

**Proof.** For the first item, since everything is defined component-wise, it suffices to prove that for every \( a \in A_n \otimes \prod (\Omega^1_X [-1] \otimes \mathcal{A})(U_{i_1 \ldots i_n}) \) and every \( x, y \in A_n \otimes \prod \mathcal{L}(U_{i_1 \ldots i_n}) \)

\[ \langle [a, x], y \rangle + (-1)^{\bar{a} \bar{x}} \langle x, [a, y] \rangle = (\text{Id} \otimes \rho)(a)(\langle x, y \rangle) \]

for every \( n \geq 0 \). By linearity let \( a = \omega_n \otimes z_n \), with \( \omega_n \in A_n \) and \( z_n \in \prod (\Omega^1_X [-1] \otimes \mathcal{A})(U_{i_1 \ldots i_n}); \) let \( x = \eta_n \otimes x_n \) and \( y = \phi_n \otimes y_n \), with \( \eta_n, \phi_n \) in \( A_n \) and \( x_n, y_n \) in \( \prod \mathcal{L}(U_{i_1 \ldots i_n}) \).
Then

\[ \langle [a, x], y \rangle + (-1)^a \langle x, [a, y] \rangle = \]

\[ \langle [\omega \otimes z_n, \eta_n \otimes x_n], \phi_n \otimes y_n \rangle + (-1)^a \langle \eta_n \otimes x_n, [\omega \otimes z_n, \phi_n \otimes y_n] \rangle = \]

\[ (-1)^a \bar{\eta}_n \bar{z}_n \langle \omega_n \eta_n \otimes [z_n, x_n], \phi_n \otimes y_n \rangle + \]

\[ + (-1)^a \bar{\phi}_n \bar{z}_n \omega_n \phi_n \otimes [z_n, y_n] \rangle = \]

\[ (-1)^a \bar{\phi}_n (\bar{z}_n + \bar{w}_n) + \bar{\eta}_n \bar{z}_n \langle \omega_n \eta_n \phi_n \langle [z_n, x_n], y_n \rangle + \]

\[ + (-1)^{\omega_n} \bar{\eta}_n \bar{w}_n \omega_n \phi_n (x_n, [z_n, y_n]) \rangle = \]

\[ (-1)^{\omega_n} \bar{\phi}_n (\bar{z}_n + \bar{w}_n) + \bar{\eta}_n \bar{w}_n \omega_n \phi_n \langle (x_n, [z_n, y_n]) \rangle = \]

\[ (-1)^{\omega_n} \bar{\phi}_n (\bar{z}_n + \bar{w}_n) + \bar{\eta}_n \bar{w}_n \omega_n \phi_n (Id \otimes \rho)(z_n) \langle \langle x_n, y_n \rangle \rangle = (Id \otimes \rho)(a)((x, y)). \]

For the second item, recall that \( d_{\text{Tot}} \) is the differential on \( \text{Tot}(\mathcal{U}, \Omega_X^1) \) when considering \( \Omega_X^1 \) as a complex with trivial differential. Again, since everything is defined component-wise, it is sufficient to prove that

\[ \langle d_{\text{Tot}} \langle \eta_n \otimes x_n, \omega_n \otimes y_n \rangle, \rangle \]

\[ = d_{\text{Tot}} \langle \eta_n \otimes x_n, \omega_n \otimes y_n \rangle, \]

for \( \eta_n, \omega_n \in A_n \) and \( x_n \in \prod (\Omega_X^{1} [-i] \otimes \mathcal{L}) (U_{i_1 \ldots i_n}) \) and \( y_n \in \prod (\Omega_X^{1} [-j] \otimes \mathcal{L}) (U_{i_1 \ldots i_n}) \).

Then

\[ \langle d_{\text{Tot}} \langle \eta_n \otimes x_n, \omega_n \otimes y_n \rangle, \rangle \]

\[ + (-1)^{\bar{w}_n} \bar{\eta}_n \langle \eta_n \otimes x_n, d_{\text{Tot}} \langle \omega_n \otimes y_n \rangle \rangle = \]

\[ \langle d_{A_n} \eta_n \otimes x_n + (-1)^{\bar{w}_n} \bar{\eta}_n \otimes dx_n, \omega_n \otimes y_n \rangle + \]

\[ + (-1)^{\bar{w}_n} \bar{\eta}_n \langle \eta_n \otimes x_n, d_{A_n} \omega_n \otimes y_n + (-1)^{\omega_n} \omega_n \otimes dy_n \rangle = \]

\[ = (-1)^{\omega_n} \bar{\eta}_n \langle d_{A_n} \eta_n \omega_n \langle x_n, y_n \rangle + (-1)^{\bar{w}_n} \bar{\eta}_n \langle dx_n, y_n \rangle + \]

\[ + (-1)^{\bar{w}_n} \bar{\eta}_n \langle \omega_n \langle x_n, y_n \rangle + \]

\[ + (-1)^{\bar{w}_n} \bar{\eta}_n \langle d_{A_n} \eta_n \omega_n \langle x_n, dy_n \rangle + \]

\[ (-1)^{\omega_n} \bar{\eta}_n d_{A_n} \langle \eta_n \omega_n \rangle \langle x_n, y_n \rangle + + (-1)^{\bar{w}_n} \bar{\eta}_n \langle dx_n, y_n \rangle + \]

\[ (-1)^{\bar{w}_n} d_{A_n} \langle \eta_n \omega_n \rangle \langle x_n, y_n \rangle + \]

\[ d_{A_n} \langle \eta_n \otimes x_n, \omega_n \otimes y_n \rangle = d_{\text{Tot}} \langle \eta_n \otimes x_n, \omega_n \otimes y_n \rangle, \]

where \( d_{A_n} \) denotes the differential on \( A_n \), the differential graded algebra of polynomial differential forms on the affine \( n \)-simplex.

\[ \square \]
Corollary 2.9. Let $D$ in $\text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{A})$ be a simplicial lifting of the identity and let
\[
\nabla = [D, -, -] : \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{L})
\]
be its associated connection, as in Definition 1.12. Then for any cyclic form
\[
\langle -, - \rangle : \text{Tot}(\mathcal{U}, \Omega_X^i[-i] \otimes \mathcal{L}) \times \text{Tot}(\mathcal{U}, \Omega_X^j[-j] \otimes \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega_X^{i+j}[-i-j]),
\]
with $i, j \geq 0$, we have
\[
\langle \nabla(x), y \rangle + (-1)^{\tau}\langle x, \nabla(y) \rangle = d_{dR}\langle x, y \rangle
\]
for $x, y \in \text{Tot}(\mathcal{U}, \mathcal{L})$.

Proof. It follows from the cyclicity of the form and by Remark 1.10.

The next part is dedicated to defining an $L_\infty$ morphism associated to a connection and to a $d_{\text{Tot}}$-closed cyclic form on a transitive DG-Lie algebroid. We assume that the reader is familiar with the notions and basic properties of DG-Lie algebras and $L_\infty$ morphisms between them; details can be found in [13, 19, 23, 24] and in the references therein. The definition of an $L_\infty$ morphism between a DG-Lie algebra and an abelian DG-Lie algebra, i.e., a DG-Lie algebra with trivial bracket is recalled here, because it will be needed for explicit calculations.

Let $V$ be a graded vector space over a field of characteristic 0. Let $\sigma$ be a permutation of $\{1, \ldots, n\}$, and let $v_1, \ldots, v_n$ be homogeneous vectors of $V$: denote by $\chi(\sigma; v_1, \ldots, v_n) = \pm 1$ the antisymmetric Koszul sign, defined by the relation
\[
v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma; v_1, \ldots, v_n) v_1 \wedge \cdots \wedge v_n
\]
in the $n$th exterior power $V^\wedge n$. If the vectors $v_1, \ldots, v_n$ are clear from the context we will write $\chi(\sigma)$ instead of $\chi(\sigma; v_1, \ldots, v_n)$. Given two non-negative integers $p$ and $q$, $S(p, q)$ denotes the set of $(p, q)$-shuffles, the permutations $\sigma$ of the set $\{1, \ldots, p+q\}$ such that
\[
\sigma(1) < \sigma(2) < \cdots < \sigma(p); \quad \sigma(p+1) < \cdots < \sigma(p+q).
\]
Recall that the cardinality of $S(p, q)$ is $\binom{p+q}{p}$. Because of the universal property of wedge powers, every linear map $V^\wedge p \to \hat{W}$ will be interpreted as a graded skew-symmetric $p$-linear map $V \times \cdots \times V \to W$. 


DEFINITION 2.10. Let \((V, \delta, [\cdot, \cdot])\) be a DG-Lie algebra and \((M, d)\) an abelian DG-Lie algebra. An \(L_\infty\) morphism \(g: V \to M\) is a sequence of maps \(g_n: V^\wedge n \to M, n \geq 1\), with \(g_n\) of degree \(1 - n\) such that, for every \(n\) and every \(v_1, \ldots, v_n \in V\) homogeneous, the following conditions \(C_i\) are satisfied for all \(i \in \mathbb{N}\):

\[
(C_1) : \quad dg_1(v_1) = g_1(\delta v_1),
\]

\[
(C_n) : \quad dg_n(v_1, \ldots, v_n) = (-1)^{1-n} \sum_{\sigma \in S(1,n-1)} \chi(\sigma)g_n(\delta v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})
+ (-1)^{2-n} \sum_{\sigma \in S(2,n-2)} \chi(\sigma)g_{n-1}(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \ldots, v_{\sigma(n)}).
\]

REMARK 2.11. Notice that condition \(C_1\) entails that the linear component \(g_1\) induces a map in cohomology \(g_1: H^*(V) \to H^*(M)\). It is clear that the cohomology \(H^*(M)\) of an abelian DG-Lie algebra \(M\) is an abelian graded Lie algebra. Condition \(C_2\) can be written as

\[
g_1([v_1, v_2]) = dg_2(v_1, v_2) + g_2(\delta v_1, v_2) + (-1)^{|v_1|}g_2(v_1, \delta v_2),
\]

which implies that the map induced by \(g_1\) in cohomology is a morphism of graded Lie algebras.

Recall that since the functor \(\text{Tot}\) sends semicosimplicial DG-Lie algebras to DG-Lie algebras, the complex \(\text{Tot}(U, \mathcal{L})\) is a DG-Lie algebra. The complex of \(O_X\)-modules \(\Omega^{\leq 1}_X = O_X \xrightarrow{dR} \Omega^1_X\) can be considered as a sheaf of abelian DG-Lie algebras, and hence it gives rise to a semicosimplicial abelian DG-Lie algebra; therefore the complex \(\text{Tot}(U, \Omega^{\leq 1}_X[2])\) is an abelian DG-Lie algebra.

THEOREM 2.12. Let \((\mathcal{A}, \rho)\) be a transitive DG-Lie algebroid over a smooth separated scheme \(X\) of finite type over a field \(\mathbb{K}\) of characteristic zero. Let \(\mathcal{L} = \text{Ker}\, \rho\) be a finite complex of locally free sheaves and let \((\cdot, \cdot): \text{Tot}(U, \Omega^i_X[-i] \otimes \mathcal{L}) \times \text{Tot}(U, \Omega^j_X[-j] \otimes \mathcal{L}) \to \text{Tot}(U, \Omega^{i+j}_{X}[-i-j]),\ i, j \geq 0,\) be a cyclic form which is \(d_{\text{Tot}}\)-closed. For every simplicial lifting of the identity \(D \in \text{Tot}(U, \Omega^1_X[-1] \otimes \mathcal{A})\) there exists a \(L_\infty\) morphism between DG-Lie algebras on the field \(\mathbb{K}\):

\[
f : \text{Tot}(U, \mathcal{L}) \rightsquigarrow \text{Tot}(U, \Omega^{\leq 1}_X[2])
\]
with components

\begin{align*}
  f_1(x) &= \langle u, x \rangle, \\
  f_2(x, y) &= \frac{1}{2} \langle \nabla(x), y \rangle - (-1)^y \langle \nabla(y), x \rangle, \\
  f_3(x, y, z) &= -\frac{1}{2} \langle x, [y, z] \rangle, \\
  f_n &= 0 \quad \forall n \geq 4,
\end{align*}

where \( \nabla = [D, -] : \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \Omega^1_X \otimes \mathcal{L}) \) denotes the connection associated to the simplicial lifting of the identity \( D \), and \( u = d_{\text{Tot}} D \) its extension cocycle.

**Proof.** The strategy of the proof is to check that the conditions \( C_i \) of Definition 2.10 hold for \( n = 1, 2, 3, 4 \). In fact, since \( f_n = 0 \) for \( n \geq 4 \), the conditions are automatically satisfied for \( n \geq 5 \).

Denote by \( d_{\text{Tot}} \) the differential on \( \text{Tot}(\mathcal{U}, \mathcal{L}) \), and by \( d_{\text{Tot}} + d_{dR} \) the differential on \( \text{Tot}(\mathcal{U}, \Omega^\otimes_X [2]) \). Condition \( C_1 \) requires that

\[ f_1(d_{\text{Tot}} x) = (d_{\text{Tot}} + d_{dR}) f_1(x); \]

notice however that since \( u \) belongs to \( \text{Tot}(\mathcal{U}, \Omega^\otimes_X [1] \otimes \mathcal{L}) \), \( (d_{\text{Tot}} + d_{dR}) f_1 = d_{\text{Tot}} f_1 \) in \( \text{Tot}(\mathcal{U}, \Omega^\otimes_X [2]) \). Then by the \( d_{\text{Tot}} \)-closure of the cyclic form and by the fact that \( u \) is closed:

\[ f_1(d_{\text{Tot}} x) = \langle u, d_{\text{Tot}} x \rangle = (-1)^x d_{\text{Tot}} \langle u, x \rangle = \langle u, d_{\text{Tot}} x \rangle = d_{\text{Tot}} f_1(x). \]

For \( n = 2 \) the condition is

\[ (C_2) : \quad f_2(d_{\text{Tot}} x, y) + (-1)^y f_2(x, d_{\text{Tot}} y) = f_1([x, y]) - (d_{\text{Tot}} + d_{dR}) f_2(x, y). \]
By definition of $f_2$ we have again that $(d_{\text{Tot}} + d_{dR}) f_2 = d_{\text{Tot}} f_2$, and then, using Lemma 1.14,

$$f_2(d_{\text{Tot}} x, y) + (-1)^x f_2(x, d_{\text{Tot}} y) =$$

$$\frac{1}{2} \left( \langle \nabla(d_{\text{Tot}} x), y \rangle - (-1)^{(x+1)} \langle \nabla(y), d_{\text{Tot}} x \rangle + (-1)^x \langle \nabla(x), d_{\text{Tot}} y \rangle +\right.$$

$$\left. - (-1)^x \langle \nabla(d_{\text{Tot}} y), x \rangle \right) =$$

$$\frac{1}{2} \left( - \langle d_{\text{Tot}} \nabla(x), y \rangle + [u, x], y \rangle - (-1)^{(x+1)} \langle \nabla(y), d_{\text{Tot}} x \rangle +\right.$$

$$\left. + (-1)^x \langle \nabla(x), d_{\text{Tot}} y \rangle + (-1)^x \langle d_{\text{Tot}} \nabla(y), x \rangle - (-1)^x \langle [u, y], x \rangle \right) =$$

$$\frac{1}{2} \left( \langle u, [x, y] \rangle - (-1)^x \langle u, [y, x] \rangle - d_{\text{Tot}} \langle \nabla(x), y \rangle + (-1)^x d_{\text{Tot}} \langle \nabla(y), x \rangle \right) =$$

$$\langle u, [x, y] \rangle - \frac{1}{2} d_{\text{Tot}} \langle \langle \nabla(x), y \rangle - (-1)^x \langle \nabla(y), x \rangle \rangle = f_1([x, y]) - d_{\text{Tot}} f_2(x, y).$$

Condition $C_3$ is the following:

$$(d_{\text{Tot}} + d_{dR}) f_3(x, y, z) =$$

$$f_3(d_{\text{Tot}} x, y, z) - (-1)^x f_3(d_{\text{Tot}} y, x, z) + (-1)^{x+y} f_3(d_{\text{Tot}} z, x, y) +$$

$$- f_2([x, y], z) + (-1)^y f_2([z, x], y) - (-1)^y f_2([x, z], y),$$

and we begin by noting that by the $d_{\text{Tot}}$-closure

$$f_3(d_{\text{Tot}} x, y, z) - (-1)^x f_3(d_{\text{Tot}} y, x, z) + (-1)^{x+y} f_3(d_{\text{Tot}} z, x, y) =$$

$$- \frac{1}{2} \left( \langle d_{\text{Tot}} x, [y, z] \rangle - (-1)^x \langle d_{\text{Tot}} y, [x, z] \rangle + (-1)^{x+y} \langle d_{\text{Tot}} z, [x, y] \rangle \right) =$$

$$- \frac{1}{2} \left( \langle d_{\text{Tot}} x, [y, z] \rangle - (-1)^x \langle [d_{\text{Tot}} y, x], z \rangle + (-1)^{x+y} \langle [x, y], d_{\text{Tot}} z \rangle \right) =$$

$$- \frac{1}{2} \left( \langle d_{\text{Tot}} x, [y, z] \rangle + (-1)^x \langle [x, d_{\text{Tot}} y], z \rangle + (-1)^{x+y} \langle [y, d_{\text{Tot}} z] \rangle \right) =$$

$$- \frac{1}{2} \left( \langle d_{\text{Tot}} x, [y, z] \rangle + (-1)^x \langle d_{\text{Tot}} [y, z] \rangle \right) = - \frac{1}{2} d_{\text{Tot}} \langle x, [y, z] \rangle = d_{\text{Tot}} f_3(x, y, z).$$
On the other hand, by Corollary 2.9

\[ - f_2([x, y], z) + (-1)^{\overline{y}} f_2([x, z], y) - (-1)^{\overline{y}(\overline{y} + \overline{z})} f_2([y, z], x) = \]
\[ - \frac{1}{2} \langle [\nabla(x, y), z] + (-1)^{\overline{y}} \langle [x, \nabla(y)], z \rangle - (-1)^{\overline{y}(\overline{y} + \overline{z})} \langle \nabla([y, z]), x \rangle - \langle \nabla(x), [y, z] \rangle \rangle = \]
\[ - \frac{1}{2} \langle \langle \nabla(x), [y, z] \rangle + (-1)^{\overline{y}} \langle [y, \nabla(z)], [x, y] \rangle - (-1)^{\overline{y}(\overline{y} + \overline{z})} \langle \nabla([y, z]), [x, y] \rangle \rangle = \]
\[ + (-1)^{\overline{y}} \langle [\nabla(y), [x, z]], y \rangle - (-1)^{\overline{y}} \langle [\nabla(z), y], y \rangle + (-1)^{\overline{y}} \langle [\nabla(y), [x, z]], y \rangle + \]
\[ + (-1)^{\overline{y}(\overline{y} + \overline{z})} \langle [\nabla(y), [x, z]], x \rangle + (-1)^{\overline{y}(\overline{y} + \overline{z})} \langle [y, \nabla(z)], x \rangle - \langle \nabla(x), [y, z] \rangle \rangle = \]
\[ - \frac{1}{2} \langle \langle \nabla(x), [y, z] \rangle + (-1)^{\overline{y}} \langle \nabla([y, z]), [x, y] \rangle \rangle = - \frac{1}{2} d_{dR} \langle x, [y, z] \rangle = d_{dR} f_3(x, y, z). \]

Lastly, condition C4 is

\[ f_3([a_1, a_2], a_3, a_4) + (-1)^{\overline{a_1}(\overline{a_1} + \overline{a_2})} f_3([a_3, a_4], a_1, a_2) + \]
\[ + (-1)^{\overline{a_1}(\overline{a_1} + \overline{a_2})} f_3([a_2, a_3], a_1, a_4) + (-1)^{\overline{a_1} \overline{a_2} \overline{a_1} \overline{a_2} \overline{a_1} \overline{a_2} \overline{a_1} \overline{a_2}} f_3([a_2, a_4], a_1, a_3) \]
\[ - (-1)^{\overline{a_1} \overline{a_2}} f_3([a_1, a_3], a_2, a_4) + (-1)^{\overline{a_1}(\overline{a_2} + \overline{a_3})} f_3([a_1, a_4], a_2, a_3) = 0. \]

By the graded Jacobi identity we have that

\[ - \frac{1}{2} \langle [a_1, a_2], [a_3, a_4] \rangle + (-1)^{\overline{a_2} \overline{a_1}} \langle [a_1, a_3], [a_2, a_4] \rangle + \]
\[ + (-1)^{\overline{a_2}(\overline{a_2} + \overline{a_1})} \langle [a_1, a_4], [a_2, a_3] \rangle + (-1)^{\overline{a_1}(\overline{a_2} + \overline{a_1})} \langle [a_3, a_4], [a_1, a_2] \rangle + \]
\[ - (-1)^{\overline{a_1} \overline{a_2} + \overline{a_1} + \overline{a_2}} \langle [a_2, a_4], [a_1, a_3] \rangle + (-1)^{\overline{a_1}(\overline{a_1} + \overline{a_2})} \langle [a_2, a_3], [a_1, a_4] \rangle \rangle = \]
\[ = - \langle [a_1, a_2], [a_3, a_4] \rangle - (-1)^{\overline{a_2} \overline{a_1}} \langle [a_1, a_3], [a_2, a_4] \rangle + \]
\[ + (-1)^{\overline{a_2}(\overline{a_2} + \overline{a_1})} \langle [a_1, a_4], [a_2, a_3] \rangle = - \langle [a_1, [a_2, [a_3, a_4]] \rangle + \]
\[ - (-1)^{\overline{a_2} \overline{a_1}} \langle [a_1, [a_3, [a_2, a_4]] \rangle + (-1)^{\overline{a_1}(\overline{a_1} + \overline{a_2})} \langle [a_1, [a_4, [a_2, a_3]]] \rangle = \]
\[ - \langle a_1, [a_2, [a_3, a_4]] \rangle - (-1)^{\overline{a_2} \overline{a_1}} [a_3, [a_2, a_4]] - ([a_2, a_3], a_4) = 0. \]

We can now state the results of [20] for a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme $X$ of finite type on a field $K$ of characteristic zero.

Remark 2.13. It is not very restrictive to require that a coherent sheaf on $X$ has a finite locally free resolution: in fact, by [14, III, Exercises 6.8, 6.9] every coherent sheaf on a smooth, Noetherian, integral, separated scheme admits a finite locally free resolution.
Let \((E, d_E)\) be a finite complex of locally free sheaves. Consider the DG-Lie algebroid of derivations of pairs \(D^*(X, E)\) of Example 1.6, [18], and the short exact sequence

\[0 \rightarrow \mathcal{H}om^*_O X(E, E) \rightarrow D^*(X, E) \xrightarrow{\alpha} \Theta_X \rightarrow 0;\]

it was noted in Example 1.6 that by tensoring with \(\Omega^1_X[-1]\) one obtains

\[0 \rightarrow \mathcal{H}om^*_O X(E, \Omega^1_X[-1] \otimes E) \xrightarrow{g \mapsto (0, g)} \mathcal{J}^*_\Omega^1 \xrightarrow{(\beta, g) \mapsto \beta} \mathcal{D}er_K(O_X, \Omega^1_X[-1]) \rightarrow 0,\]

Fixing an affine open cover \(U\) of \(X\) and applying the Tot functor, we get the short exact sequence

\[0 \xrightarrow{\text{Tot}} \mathcal{H}om^*_O X(E, \Omega^1_X[-1] \otimes E) \xrightarrow{\text{Tot}} \mathcal{D}er_K(O_X, \Omega^1_X[-1]) \rightarrow 0.\]

We have already remarked that a lifting of the identity in \(\mathcal{J}^*_\Omega^1\) is equivalent to a global algebraic connection on every component \(E^i\); hence a lifting to \(\text{Tot}(U, \mathcal{J}^*_\Omega^1)\) of the universal derivation \(d_{dR} : O_X \rightarrow \Omega^1_X[-1]\) in \(\text{Tot}(U, \mathcal{D}er_K(O_X, \Omega^1_X[-1]))\) can be termed a simplicial connection on the complex of locally free sheaves \(E\). As seen in Example 2.3, a natural cyclic form to consider is the one induced by

\[\mathcal{H}om^*_O X(E, E) \times \mathcal{H}om^*_O X(E, E) \rightarrow O_X, \quad (a, b) \mapsto -\text{Tr}(ab),\]

where \(\text{Tr} : \mathcal{H}om^*_O X(E, E) \rightarrow O_X\) is the usual trace map. Then the \(L_\infty\) morphism of Theorem 2.12 yields:

**Corollary 2.14.** Let \(E\) be a finite complex of locally free sheaves on a smooth separated scheme \(X\) of finite type over a field \(\mathbb{K}\) of characteristic zero. For every simplicial connection \(D \in \text{Tot}(U, \mathcal{J}^*_\Omega^1)\) there exists an \(L_\infty\) morphism between DG-Lie algebras on the field \(\mathbb{K}\)

\[g : \text{Tot}(U, \mathcal{H}om^*_O X(E, E)) \rightsquigarrow \text{Tot}(U, \Omega^1_X[-1][2])\]

with components

\[g_1(x) = -\text{Tr}(ux),\]

\[g_2(x, y) = \frac{1}{2} \text{Tr}(\nabla(x)y - (-1)^x \nabla(y)x),\]

\[g_3(x, y, z) = \frac{1}{2} \text{Tr}(x, [y, z]),\]

\[g_n = 0 \forall n \geq 4.\]
Hence the applications to deformation theory of [20], stated in the context of complex manifolds, are also valid in the algebraic context as announced.

Let $F$ be a coherent sheaf on $X$ admitting a finite locally free resolution, and denote by

$$
\sigma = \sum_{q \geq 0} \sigma_q: \operatorname{Ext}^2_X(F, F) \to \prod_{q \geq 0} H^{q+2}(X, \Omega^q_X), \quad \sigma(c) = \operatorname{Tr}(\exp(-\operatorname{At}(F)) \circ c),
$$

the Buchweitz–Flenner semiregularity map of [7]. Above, $\operatorname{At}(F) \in \operatorname{Ext}^1_X(F, F \otimes \Omega^1_X)$ denotes the Atiyah class of $F$, the exponential of its opposite

$$
\exp(-\operatorname{At}(F)) \in \prod_{q \geq 0} \operatorname{Ext}^q_X(F, F \otimes \Omega^q_X)
$$

is obtained via the Yoneda pairing

$$
\operatorname{Ext}^i_X(F, F \otimes \Omega^j_X) \times \operatorname{Ext}^j_X(F, F \otimes \Omega^i_X) \to \operatorname{Ext}^{i+j}_X(F, F \otimes \Omega^{i+j}_X), \quad (a, b) \mapsto a \circ b,
$$

and $\operatorname{Tr}$ denotes the trace maps

$$
\operatorname{Tr}: \operatorname{Ext}^i_X(F, F \otimes \Omega^j_X) \to H^i(X, \Omega^j_X), \quad i, j \geq 0.
$$

For every $q \geq 0$ one can consider the composition

$$
\tau_q: \operatorname{Ext}^2_X(F, F) \xrightarrow{\sigma_q} H^{q+2}(X, \Omega^q_X) = H^2(X, \Omega^q_X[q]) \xrightarrow{i_q} \mathbb{H}^2(X, \Omega^{\leq q}_X[2q]),
$$

where $\Omega^{\leq q}_X = (\oplus_{i=0}^q \Omega^i_X[-i], d_{dR})$ is the truncated de Rham complex and $i_q$ is induced by the inclusion of complexes $\Omega^q_X[q] \subset \Omega^{\leq q}_X[2q]$. The map $\tau_q$ is the $q$-component of the modified Buchweitz–Flenner semiregularity map. It is convenient to also consider the maps

$$
\sigma_q: \operatorname{Ext}^2_X(F, F) \to H^*(X, \Omega^q_X[q])
$$

$$
\tau_q: \operatorname{Ext}^2_X(F, F) \xrightarrow{\sigma_q} H^*(X, \Omega^q_X[q]) \xrightarrow{i_q} \mathbb{H}^*(X, \Omega^{\leq q}_X[2q])
$$

defined by the same formulas.

**Corollary 2.15.** Let $F$ be a coherent sheaf admitting a finite locally free resolution $E$ on a smooth separated scheme $X$ of finite type over a field $\mathbb{K}$ of characteristic zero. Then every simplicial connection on the resolution $E$ gives a lifting of the map

$$
\tau_1: \operatorname{Ext}^2_X(F, F) \to \mathbb{H}^*(X, \Omega^{\leq 1}_X[2])
$$

to an $L_\infty$ morphism

$$
g: \text{Tot}(U, \text{Hom}^*_\mathcal{O}_X(E, E)) \rightsquigarrow \text{Tot}(U, \Omega^{\leq 1}_X[2]).$$
Recall that to a DG-Lie algebra $M$ over a field $\mathbb{K}$ of characteristic zero we can associate a functor $\text{Def}_M : \text{Art}_K \to \text{Set}$, the functor of Maurer-Cartan solutions modulo gauge action (for more details see e.g. [11, 17, 18, 23, 24]). It is well known that the second cohomology $H^2(M)$ of the cochain complex underlying $M$ is an obstruction space for the deformation functor $\text{Def}_M$. Recall also that an $L_1$ morphism between DG-Lie algebras $g : V \hookrightarrow M$ gives a morphism of deformation functors $g : \text{Def}_V \to \text{Def}_M$ such that the map induced in cohomology commutes with obstruction maps. If the DG-Lie algebra $M$ has trivial bracket, every obstruction in $\text{Def}_V$ belongs to the kernel of the map $g : H^2(V) \to H^2(M)$.

**Corollary 2.16.** Let $F$ be a coherent sheaf admitting a finite locally free resolution on a smooth separated scheme $X$ of finite type over a field $\mathbb{K}$ of characteristic zero. Then every obstruction to the deformations of $F$ belongs to the kernel of the map $\tau_1 : \text{Ext}^2_X(F, F) \to H^2(X, \Omega^1_X \leq 1) \subset [2]$.

If the Hodge to de Rham spectral sequence of $X$ degenerates at $E_1$, then every obstruction to the deformations of $F$ belongs to the kernel of the map $\sigma_1 : \text{Ext}^2_X(F, F) \to H^3(X, \Omega^1_X)$, $\sigma_1(a) = -\text{Tr} (\text{At}(F) \circ a)$.

**Proof.** If $E$ is a finite locally free resolution of $F$, the DG-Lie algebra $\text{Tot}(\mathcal{U}, \mathcal{H}om^*_O_X(E, E))$ controls the deformations of $F$, see e.g. [11]. According to Corollary 2.15, the map $\tau_1 : \text{Ext}^2_X(F, F) \to H^2(X, \Omega^1_X \leq 1) \subset [2])$

lifts to an $L_\infty$ morphism $g : \text{Tot}(\mathcal{U}, \mathcal{H}om^*_O_X(E, E)) \hookrightarrow \text{Tot}(\mathcal{U}, \Omega^1_X \leq 1) \subset [2])$, whose linear component $g_1$ commutes with obstruction maps of the associated deformation functors. By construction the DG-Lie algebra $\text{Tot}(\mathcal{U}, \Omega^1_X \leq 1) \subset [2])$ is abelian and therefore every obstruction of the associated deformation functor is trivial.

If the Hodge to de Rham spectral sequence of $X$ degenerates at $E_1$ then the inclusion of complexes $\text{Tot}(\mathcal{U}, \Omega^1_X \leq 1) \subset [1]) \to \text{Tot}(\mathcal{U}, \Omega^1_X \leq 1) \subset [2])$ is injective in cohomology, so that $H^3(X, \Omega^1_X) \hookrightarrow H^2(X, \Omega^1_X \leq 1) \subset [2])$ and the maps $\sigma$ and $\tau$ have the same kernel. ■

**Remark 2.17.** In the setting of Theorem 2.12, if the cyclic form is induced by a DG-Lie algebroid representation $\theta : \mathcal{A} \to \mathcal{D}^*(X, E)$, the $L_\infty$ morphism can be obtained up to a sign from the the $L_\infty$ morphism of Corollary 2.14 as follows. Let
Cyclic forms on DG-Lie algebroids and semiregularity

\[ D \in \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{A}) \] denote a simplicial lifting of the identity, and denote by \( \text{Id} \otimes \theta : \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{A}) \to \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{D}^*(X, \mathcal{E})) \cong \text{Tot}(\mathcal{U}, \mathcal{J}_\Omega) \) the induced map on the totalisation. Denoting as usual by \( \alpha \) the anchor map of the transitive DG-Lie algebroid \( \mathcal{D}^*(X, \mathcal{E}) \), it is clear that \( (\text{Id} \otimes \theta)(D) \) is a simplicial lifting of the identity in \( \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{D}^*(X, \mathcal{E})) \):

\[
(\text{Id} \otimes \alpha)(\text{Id} \otimes \theta)(D) = \text{Id} \otimes (\alpha \circ \theta)(D) = (\text{Id} \otimes \rho)(D) = \text{Id}_{\Omega^1} \in \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \Theta_X).
\]

Let \( u = d_{\text{Tot}}D \in \text{Tot}(\mathcal{U}, \Omega_X^1[-1] \otimes \mathcal{L}) \) denote the extension cocycle associated to \( D \), then

\[
(\text{Id} \otimes \theta)(u) = (\text{Id} \otimes \theta)(d_{\text{Tot}}D) = d_{\text{Tot}}(\text{Id} \otimes \theta)(D).
\]

Therefore the \( L_\infty \) morphism \( f : \text{Tot}(\mathcal{U}, \mathcal{L}) \rightsquigarrow \text{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2]) \) associated to \( D \) and to \( (\cdot, \cdot)_\theta \) is the composition of the DG-Lie algebra morphism

\[
\theta : \text{Tot}(\mathcal{U}, \mathcal{L}) \to \text{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}^*(\mathcal{E}, \mathcal{E}))
\]

and of the \( L_\infty \) morphism

\[
-g : \text{Tot}(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}^*(\mathcal{E}, \mathcal{E})) \rightsquigarrow \text{Tot}(\mathcal{U}, \Omega_X^{\leq 1}[2])
\]

associated to the simplicial lifting of the identity \( (\text{Id} \otimes \theta)(D) \) and to the cyclic form \( (a, b) \mapsto \text{Tr}(ab) \).

3. The \( L_\infty \) morphism for the Atiyah Lie algebroid of a principal bundle

Since Lie algebroids arise naturally in connection with principal bundles, we give an application of the \( L_\infty \) morphism constructed in Theorem 2.12 to the deformation theory of principal bundles.

Let \( X \) be a smooth separated scheme of finite type over an algebraically closed field \( \mathbb{K} \) of characteristic zero, let \( G \) be an affine algebraic group with Lie algebra \( \mathfrak{g} \), and let \( P \to X \) be a principal \( G \)-bundle on \( X \). By \( G \)-principal bundle we mean a \( G \)-fibration which is locally trivial for the Zariski topology, see e.g. [30]. We begin by finding a DG-Lie algebra that controls the deformations of \( P \), using an argument similar to those in [4, 24, 31]. Let \( \text{Art}_\mathbb{K} \) be the category of Artin local \( \mathbb{K} \)-algebras with residue field \( \mathbb{K} \). For any \( A \) in \( \text{Art}_\mathbb{K} \) denote by \( \mathfrak{m}_A \) its maximal ideal and by \( 0 \) the closed point in \( \text{Spec} \ A \).

To every semicosimplicial Lie algebra \( \mathfrak{h} \) over \( \mathbb{K} \)
there are associated two functors \( Z^1_b, H^1_b : \text{Art}_\mathbb{K} \to \text{Set} \), which here are described in brief; for more details see [12,24]. The functor of non-abelian cocycles \( Z^1_b \) is defined as

\[
Z^1_b(A) = \{ e^x \in \exp(b_1 \otimes m_A) \mid e^{\delta_1(x)} = e^{\delta_2(x)} e^{\delta_0(x)} \}.
\]

For every \( A \in \text{Art}_\mathbb{K} \) there is a left action of \( \exp(b_0 \otimes m_A) \) on \( Z^1_b(A) \)

\[
\exp(b_0 \otimes m_A) \times Z^1_b(A) \to Z^1_b(A), \quad (e^a, e^x) \mapsto e^{\delta_1(a)} e^x e^{-\delta_0(a)}.
\]

The functor \( H^1_b : \text{Art}_\mathbb{K} \to \text{Set} \) is then defined as

\[
H^1_b(A) = \frac{Z^1_b(A)}{\exp(b_0 \otimes m_A)}.
\]

Consider the Thom-Whitney totalisation functor \( \text{Tot} \) from semicosimplicial DG-vector spaces to DG-vector spaces (Definition 1.7), and recall it takes semicosimplicial Lie algebras to DG-Lie algebras. We then have the following result, see [12,15,24]:

**Proposition 3.1.** For every semicosimplicial Lie algebra \( b \) there exists a natural isomorphism of functors \( H^1_b \cong \text{Def}_{\text{Tot}(b)} \).

**Definition 3.2.** [4,8] An infinitesimal deformation of \( P \) over \( A \in \text{Art}_\mathbb{K} \) is the data of a principal \( G \)-bundle \( P_A \to X \times \text{Spec} A \) and an isomorphism \( \theta : i^*(P_A) \cong P \).

\[
\begin{array}{ccc}
P & \longrightarrow & P_A \\
\downarrow P & & \downarrow P_A \\
X & \longrightarrow & X \times \text{Spec} A.
\end{array}
\]

Two deformations \( (P_A, \theta) \) and \( (P_A', \theta') \) are isomorphic if there exists an isomorphism of principal \( G \)-bundles \( \lambda : P_A \to P_A' \) such that \( \theta = \theta' \circ i^*(\lambda) \).

This defines a functor \( \text{Def}_P : \text{Art}_\mathbb{K} \to \text{Set} \) such that \( \text{Def}_P(A) \) is the set of isomorphism classes of deformations of \( P \) over \( A \in \text{Art}_\mathbb{K} \). For every \( A \in \text{Art}_\mathbb{K} \), the set \( \text{Def}_P(A) \) contains the trivial deformation \( P \times \text{Spec} A \to X \times \text{Spec} A \).

If \( M \) is a DG-Lie algebra such that \( \text{Def}_P \cong \text{Def}_M \), where \( \text{Def}_M \) is the functor of Maurer-Cartan solutions modulo gauge action, one says that \( M \) controls the deformations of \( P \).
Fix an open cover \( \mathcal{U} = \{ U_i \} \) of \( X \) such that \( P \) is trivial on every \( U_i \), and let \( \{ g_{ij} : U_{ij} \to G \} \) denote the transition functions for \( P \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \).

- Let \( \text{ad} P = P \times^G \mathfrak{g} \) denote the adjoint bundle of \( P \), with transition functions \( \{ \text{Ad}_{g_{ij}} \} \), and let \( \mathcal{A}(P) \) denote the sheaf of sections of the vector bundle \( \text{ad} P \).
- The group \( G \) acts on itself by conjugation; denote by \( \text{Ad} P = P \times^G G \) the associated bundle corresponding to this action. Recall that \( \Gamma(X, \text{Ad}(P)) \cong \text{Gauge}(P) \), where \( \text{Gauge}(P) \) is the group of bundle automorphisms of \( P \).

There is a one to one correspondence between first order deformations of \( P \), i.e., deformations over \( K \to \mathbb{K} \), and \( H^1(X, \mathcal{A}(P)) \), see e.g. [8, 31]. This implies that on every affine open set the deformations of \( P \) are trivial.

**Lemma 3.3.** Let \( P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A) \) be the associated bundle induced by the action \( \text{Ad} \otimes \text{Id} : G \times \mathfrak{g} \otimes \mathfrak{m}_A \to \mathfrak{g} \otimes \mathfrak{m}_A \). Then there is an isomorphism

\[
\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)) \cong \Gamma(\mathcal{A}(P) \otimes \mathfrak{m}_A).
\]

**Proof.** A section of \( P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A) \) is the data of

\[
\{ \omega_i : U_i \to \mathfrak{g} \otimes \mathfrak{m}_A \mid \omega_i(p) = (\text{Ad}_{g_{ij}(p)} \otimes \text{Id})\omega_j(p) \ \forall p \in U_{ij} \}.
\]

Let \( t_1, \cdots, t_n \) be a basis of the finite dimensional vector space \( \mathfrak{m}_A \), then for every \( p \in U_i \) one can write \( \omega_i(p) = \sum_{k} h_{i,k}(p) \otimes t_k \). Since the action of \( G \) on \( \mathfrak{g} \otimes \mathfrak{m}_A \) is defined as

\[
g \cdot (x \otimes t) = \text{Ad}_g(x) \otimes t,
\]

the maps \( h_{i,k} \) are such that \( h_{i,k}(p) = \text{Ad}_{g_{ij}(p)} h_{j,k}(p) \) for every \( p \in U_{ij} \).

An element of \( \Gamma(\mathcal{A}(P) \otimes \mathfrak{m}_A) \) is a finite sum \( \sum_{k} \eta_k \otimes t_k \), with \( \eta_k \) sections of \( \text{ad} P \), so that each \( \eta_k \) is the data of

\[
\{ \eta_{k,i} : U_i \to \mathfrak{g} \mid \eta_{k,i}(p) = \text{Ad}_{g_{ij}(p)} \eta_{k,j}(p) \ \forall p \in U_{ij} \}.
\]

Then, setting \( (\eta_{k,i} \otimes t_k)(p) = \eta_{k,i}(p) \otimes t_k \) for every \( p \in U_i \), the data \( \{ \eta_{k,i} \otimes t_k : U_i \to \mathfrak{g} \otimes \mathfrak{m}_A \} \) is exactly a section of \( P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A) \).

**Lemma 3.4.** For every \( A \in \text{Art}_K \) there is an isomorphism of groups

\[
\exp(\Gamma(\mathcal{A}(P) \otimes \mathfrak{m}_A)) \cong \left\{ \begin{array}{l}
\text{automorphisms of the trivial} \\
\text{deformation } P \times \text{Spec } A
\end{array} \right\}.
\]
PROOF. Denote by $G^0(A)$ the group of morphisms $f : \text{Spec } A \to G$ such that $f(0) = \text{Id}_G$, and recall that there is an isomorphism of groups $\exp(\mathfrak{g} \otimes \mathfrak{m}_A) \cong G^0(A)$ (see e.g. [29, Section 10]). The group structure on $G^0(A)$ is induced by the group structure on $G$, while $\exp(\mathfrak{g} \otimes \mathfrak{m}_A)$ is a group with the Baker-Campbell-Hausdorff product. By Lemma 3.3

$$\Gamma(\mathfrak{a} \cdot \mathfrak{d}(P)) \otimes \mathfrak{m}_A \cong \Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)),$$

so that we can work with $\exp(\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)))$. Consider the associated bundle $P \times^G G^0(A)$, induced by the adjoint action of $G$ on $G^0(A)$; the isomorphism $\exp(\mathfrak{g} \otimes \mathfrak{m}_A) \cong G^0(A)$ induces an isomorphism $\exp(\Gamma(P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A))) \cong \Gamma(P \times^G G^0(A))$. In fact, a section of $P \times^G (\mathfrak{g} \otimes \mathfrak{m}_A)$ is the data of

$$\{\eta_i : U_i \to \mathfrak{g} \otimes \mathfrak{m}_A \mid \eta_i(p) = (\text{Ad}_{g_{ij}(p)} \otimes \text{Id})\eta_j(p) \ \forall p \in U_{ij}\},$$

and composing with the exponential $\exp : \mathfrak{g} \otimes \mathfrak{m}_A \to G^0(A)$ we obtain

$$\{\exp \circ \eta_i : U_i \to G^0(A) \mid \exp \circ \eta_i(p) = g_{ij}(p) \exp \circ \eta_j(p)g_{ij}(p)^{-1} \ \forall p \in U_{ij}\}.$$ 

Notice that this data is equivalent to

$$\left\{\lambda_i : U_i \times \text{Spec } A \to G \mid \begin{array}{l}
\lambda_i(p,0) = \text{Id}_G \ \forall p \in U_i, \\
\lambda_i(p) = g_{ij}(p)\lambda_j(p)g_{ij}(p)^{-1} \ \forall p \in U_{ij}
\end{array}\right\},$$

which is a section of the associated bundle $\text{Ad}(P \times \text{Spec } A) = (P \times \text{Spec } A) \times^G G$, where $G$ acts on itself by conjugation.

For any $G$-principal bundle $Q$ the global sections of the associated bundle $\text{Ad}(Q) = Q \times^G G$ correspond to bundle automorphisms of $Q$. Therefore the $\{\lambda_i\}$ give an element $F \in \text{Gauge}(P \times \text{Spec } A)$, and the condition $\lambda_i(p,0) = \text{Id}_G$ for all $p \in U_i$ is equivalent to the fact that the automorphism $F$ induces the identity when restricted to $P$, so that $F$ is an automorphism of the trivial deformation. 

\begin{proposition}
Let $\mathcal{U} = \{U_i\}$ be an affine open cover of $X$ and let $\mathfrak{a} \cdot \mathfrak{d}(P)(\mathcal{U})$ be the semicosimplicial Lie algebra of Čech cochains:

$$\prod_i \mathfrak{a} \cdot \mathfrak{d}(P)(U_i) \Rightarrow \prod_{i,j} \mathfrak{a} \cdot \mathfrak{d}(P)(U_{ij}) \Rightarrow \prod_{i,j,k} \mathfrak{a} \cdot \mathfrak{d}(P)(U_{ijk}) \Rightarrow \cdots.$$ 

There is a natural isomorphism of functors $H^1_{\mathfrak{a} \cdot \mathfrak{d}(P)(\mathcal{U})} \to \text{Def}_P$.
\end{proposition}

PROOF. Recall that all deformations of $P$ on an affine open set are trivial, as mentioned above. Fix $A \in \text{Art}_K$, by Lemma 3.4 an element $f$ of $Z^1_{\mathfrak{a} \cdot \mathfrak{d}(P)(\mathcal{U})}(A)$ is the data
for every $U_{ij}$ of isomorphisms $f_{ij} : P|_{U_{ij}} \times \text{Spec } A \to P|_{U_{ij}} \times \text{Spec } A$, which restrict to the identity $P|_{U_{ij}} \to P|_{U_{ij}}$ and such that $f_{ik} = f_{ij}f_{jk}$ for all $i, j, k$.

The last condition means that the $\{f_{ij}\}$ glue to obtain a principal $G$-bundle $P_A \to X \times \text{Spec } A$ and isomorphisms $f_i : P_A|_{U_i \times \text{Spec } A} \to P|_{U_i} \times \text{Spec } A$ such that $f_{ij} = f_{ij}^{-1}$. Such isomorphisms coincide when restricted to $\overline{f_i} : i^*(P_A|_{U_i \times \text{Spec } A}) \to P|_{U_i}$ and hence glue to an isomorphism of principal bundles $i^*(P_A) \to P$. This means that an element of $Z^1_{\alpha, d}(P)(U_i)(A)$ gives a locally trivial deformation of $P$ over $A \in \text{Art}_k$.

An element of $\exp(\prod_i \alpha, d(P)(U_i) \otimes m_A)$ is again by Lemma 3.4 the data, for every $U_i$, of automorphisms $\lambda_i : P|_{U_i} \times \text{Spec } A \to P|_{U_i} \times \text{Spec } A$ which restrict to the identity $P|_{U_i} \to P|_{U_i}$. Two elements $f = \{f_{ij}\}, h = \{h_{ij}\}$ of $Z^1_{\alpha, d}(P)(U)(A)$ are equivalent under the action of $\lambda \in \exp(\prod_i \alpha, d(P)(U_i) \otimes m_A)$ if and only if $h_{ij} = \lambda_i f_{ij} \lambda_j^{-1}$ for all $i, j$.

\[
P_A|_{U_i \times \text{Spec } A} \xrightarrow{\lambda} P|_{U_i} \times \text{Spec } A
\]

This can be expressed as $h_i^{-1}\lambda_i f_i = h_j^{-1}\lambda_j f_j$, which means that the $\{\lambda_i\}$ glue to a bundle isomorphism $\lambda : P_A \to P'_A$, where $P_A$ is the deformation corresponding to $\{f_{ij}\}$, and $P'_A$ to $\{h_{ij}\}$. Since each $\lambda_i$ restricts to the identity on $P|_{U_i}$, $\lambda$ is an isomorphism of deformations.

**Corollary 3.6.** If $U = \{U_i\}$ is an affine open cover of $X$, there is an isomorphism $\text{Def}_P \cong \text{Def}_{\text{Tot}(U, \alpha, d(P))}$, i.e., the DG-Lie algebra $\text{Tot}(U, \alpha, d(P))$ controls the deformations of $P$.

**Proof.** Consequence of Propositions 3.5 and 3.1.

We now specialise the $L_\infty$ morphism of Section 2 to the Atiyah Lie algebroid of the principal $G$-bundle $P$.

A Lie algebroid is DG-Lie algebroid (Definition 1.1) concentrated in degree zero. Consider the **Atiyah Lie algebroid** of the principal bundle $P$ introduced in [2], which is a Lie algebroid structure on the sheaf $Q$ of sections of the vector bundle $Q = T_P/G$, the quotient of the tangent bundle of the total space $T_P$ by the canonical induced $G$-action. There is a canonical short exact sequence of locally free sheaves over $X$

\[
0 \longrightarrow \alpha, d(P) \longrightarrow Q \xrightarrow{\rho} \Theta_X \longrightarrow 0,
\]
where \( \alpha \mathcal{d} (P) \) denotes the sheaf of sections of the adjoint bundle \( \text{ad} P = P \times^G \mathfrak{g} \) and \( \rho : Q \to \Theta_X \) is the anchor map. The vector bundle \( Q \) is the bundle of invariant tangent vector fields on \( P \), and the Lie bracket on \( Q \) is induced by the Lie bracket of vector fields.

**Definition 3.7.** [2] A connection on the principal bundle \( P \to X \) is a splitting of the exact sequence in (3.1). The **Atiyah class** of \( P \) is the extension class \( \text{At}_X (P) \in \text{Ext}^1_X (\Theta_X, \alpha \mathcal{d} (P)) \cong H^1 (X, \Omega^1_X \otimes \alpha \mathcal{d} (P)) \) of the short exact sequence (3.1).

Therefore the Atiyah class \( \text{At}_X (P) \) is trivial if and only if there exists a connection on \( P \).

Let \( \Omega^1_X \) denote the cotangent sheaf, and \( \Omega^1_X [-1] \) the cotangent sheaf considered as a trivial complex of sheaves concentrated in degree one. As in Section 1, one can tensor the short exact sequence (3.1) with \( \Omega^1_X \to 1 \) to obtain a short exact sequence of complexes of sheaves

\[
0 \to \Omega^1_X [-1] \otimes \alpha \mathcal{d} (P) \to \Omega^1_X [-1] \otimes Q \xrightarrow{\text{Id} \otimes \rho} \Omega^1_X [-1] \otimes \Theta_X \to 0.
\]

Fix an affine open cover \( \mathcal{U} = \{ U_i \} \) of \( X \); as in Section 1 the short exact sequence above induces a short exact sequence of DG-vector spaces

\[
0 \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \alpha \mathcal{d} (P)) \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes Q) \xrightarrow{\text{Id} \otimes \rho} \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \Theta_X) \to 0,
\]
and we denote by \( d_{\text{Tot}} \) the differentials of the above complexes.

It is easily seen that a lifting of the identity \( \text{Id}_{\Omega^1} \in \Gamma(X, \Omega^1_X [-1] \otimes \Theta_X) \) to \( D \in \Gamma(X, \Omega^1_X [-1] \otimes Q) \) is equivalent to a splitting of the exact sequence in (3.1). Hence in the case of a principal bundle \( P \), a lifting of the identity can be identified with a connection on \( P \). Therefore we call a preimage of \( \text{Id}_{\Omega^1} \) in \( \Omega^1_X [-1] \otimes Q \) a germ of a connection on \( P \), and we use the following terminology:

**Definition 3.8.** A **simplicial connection** on the principal bundle \( P \) is a lifting \( D \) in \( \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes Q) \) of the identity \( \text{Id}_{\Omega^1} \) in \( \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \Theta_X) \).

**Definition 3.9.** The **Atiyah cocycle** of \( P \) is

\[
u = d_{\text{Tot}} D \in \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \alpha \mathcal{d} (P)).
\]

It is natural to use the name Atiyah cocycle instead of extension cocycle of Definition 1.13, because its cohomology class is equal to the Atiyah class of Definition 3.7.

As in Definition 1.12, given a simplicial connection \( D \in \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes Q) \) it is possible to define an adjoint operator

\[
\nabla = [D, -]: \text{Tot}(\mathcal{U}, \alpha \mathcal{d} (P)) \to \text{Tot}(\mathcal{U}, \Omega^1_X [-1] \otimes \alpha \mathcal{d} (P)).
\]
A cyclic form on the Atiyah Lie algebroid $Q$ is a symmetric bilinear form $\langle - , - \rangle : \mathfrak{a} \mathfrak{d} (P) \times \mathfrak{a} \mathfrak{d} (P) \to \mathcal{O}_X$ such that for all $x, y \in \mathfrak{a} \mathfrak{d} (P)$ and $q \in Q$,

$$\langle [q, x], y \rangle + \langle x, \{ q, y \} \rangle = \rho (q) (\langle x, y \rangle),$$

where $\rho : Q \to \Theta_X$ is the anchor map of the Atiyah Lie algebroid $Q$.

**Example 3.10.** The cyclic form induced by the adjoint representation of a DG-Lie algebroid of Example 2.2 in this case can be constructed in an equivalent way, starting from the Killing form of the Lie algebra $\mathfrak{g}$ of the group $G$:

$$K : \mathfrak{g} \otimes \mathbb{K} \mathfrak{g} \to \mathbb{K}, \quad K (g, h) = \text{Tr} (\text{ad} g \ \text{ad} h).$$

Take $x, y$ in $\mathfrak{a} \mathfrak{d} (P) (U)$ and let $U = \bigcup_i U_i$ with $U_i$ open sets trivialising the principal bundle $P$, then

$$x = \{ x_i : U_i \to \mathfrak{g} \mid x_i (p) = \text{Ad}_{g_{ij} (p)} x_j (p) \forall p \in U_{ij} \},$$

and analogously for $y$. Define $\langle x, y \rangle$ as $\{ \langle x_i, y_i \rangle : U_i \to \mathbb{K} \}$, where for $p \in U_i$,

$$\langle x_i, y_i \rangle (p) = K (x_i (p), y_i (p)).$$

This is well defined because the Killing form is invariant under automorphisms of the Lie algebra $\mathfrak{g}$, so that for $p \in U_{ij}$

$$K (x_i (p), y_i (p)) = K (\text{Ad}_{g_{ij} (p)} x_j (p), \text{Ad}_{g_{ij} (p)} y_j (p)) = K (x_j (p), y_j (p)).$$

Recall that Tot preserves multiplicative structures, hence $\text{Tot} (\mathcal{U} , \mathfrak{a} \mathfrak{d} (P))$ is a DG-Lie algebra. In the sequel, $\text{Tot} (\mathcal{U} , \Omega^\leq_X [2]) = \text{Tot} (\mathcal{U} , \Omega^1_X [2] \xrightarrow{d_{dR}} \Omega^1_X [1])$ is considered as a DG-Lie algebra with trivial bracket; its differential is denoted $d_{\text{Tot}} + d_{dR}$. Theorem 2.12 then yields the following.

**Corollary 3.11.** For every simplicial connection $D$ on a principal bundle $P$ on a smooth separated scheme $X$ of finite type over an algebraically closed field $\mathbb{K}$ of characteristic zero, endowed with a $d_{\text{Tot}}$-closed cyclic form $\langle - , - \rangle : \text{Tot} (\mathcal{U} , \Omega^i_X [-i] \otimes \mathfrak{a} \mathfrak{d} (P)) \otimes \text{Tot} (\mathcal{U} , \Omega^j_X [-j] \otimes \mathfrak{a} \mathfrak{d} (P)) \to \text{Tot} (\mathcal{U} , \Omega_X^{i+j} [-i-j]), i, j \geq 0$, there exists an $L_\infty$ morphism of DG-Lie algebras on the field $\mathbb{K}$

$$f : \text{Tot} (\mathcal{U} , \mathfrak{a} \mathfrak{d} (P)) \cong \text{Tot} (\mathcal{U} , \Omega^\leq_X [2]),$$
with components

\[ f_1(x) = \langle u, x \rangle, \]
\[ f_2(x, y) = \frac{1}{2} (\langle \nabla(x), y \rangle - (-1)^{\overline{x} \overline{y}} \langle \nabla(y), x \rangle), \]
\[ f_3(x, y, z) = -\frac{1}{2} (x, [y, z]), \]
\[ f_n = 0 \quad \forall n \geq 4. \]

As seen in Remark 2.11, the linear component \( f_1 \) of the \( L_\infty \) morphism induces a map of graded Lie algebras

\[ f_1 : H^*(\text{Tot}(U, \mathcal{d}(P)) \to H^*(\text{Tot}(\mathcal{U}, \Omega^\leq 1_X [2])), \]

which, since the open cover \( \mathcal{U} \) is affine, becomes

\[ f_1 : H^*(X, \mathcal{d}(P)) \to H^*(X, \Omega^\leq 1_X [2]). \]

**Corollary 3.12.** Let \( P \) be a principal bundle on a smooth separated scheme \( X \) of finite type over an algebraically closed field \( \mathbb{K} \) of characteristic zero and let

\[ \text{Tot}(\mathcal{U}, \Omega^i_X [-i] \otimes \mathcal{d}(P)) \times \text{Tot}(\mathcal{U}, \Omega^j_X [-j] \otimes \mathcal{d}(P)) \xrightarrow{(-, -)} \text{Tot}(\mathcal{U}, \Omega^{i+j}_X [-i - j]), \]

for \( i, j \geq 0 \), be a \( d_{\text{Tot}} \)-closed cyclic form. Then every obstruction to the deformations of \( P \) belongs to the kernel of the map

\[ f_1 : H^2(X, \mathcal{d}(P)) \to H^2(X, \Omega^\leq 1_X [2]), \quad f_1(x) = \langle \text{At}(P), x \rangle, \]

where \( \text{At}(P) \) denotes the Atiyah class of the principal bundle \( P \).

**Proof.** The proof is analogous to the one of Corollary 2.16: the linear component of the \( L_\infty \) morphism of DG-Lie algebras of Corollary 3.11 induces a morphism in cohomology which commutes with obstruction maps of the associated deformation functors, and the deformation functor associated to an abelian DG-Lie algebra has trivial obstructions. By Corollary 3.6, if \( \mathcal{U} = \{ U_i \} \) is an affine open cover of \( X \), the DG-Lie algebra \( \text{Tot}(\mathcal{U}, \mathcal{d}(P)) \) controls the deformations of \( P \) and an obstruction space is \( H^2(\text{Tot}(\mathcal{U}, \mathcal{d}(P))) \cong H^2(X, \mathcal{d}(P)) \). Since the DG-Lie algebra \( \text{Tot}(\mathcal{U}, \Omega^\leq 1_X [2]) \) is abelian, we obtain that \( f_1 \) annihilates all obstructions.

\[ \blacksquare \]

**Acknowledgements** – I thank my supervisor Marco Manetti for the help during the preparation of this paper.
Cyclic forms on DG-Lie algebroids and semiregularity

REFERENCES

[1] I. V. ARTAMKIN, On deformations of sheaves. Izv. Math. 32 663 (1989).
[2] M. F. ATIYAH, Complex analytic connections in fibre bundles. Trans. Amer. Math. Soc. 85 (1957), 181–207.
[3] P. BATAKIDIS – Y. VOGLAIRE, Atiyah classes and dg-Lie algebroids for matched pairs, J. Geom. Phys., (2018) 123, 156-172.
[4] I. BISWAS – S. RAMANAN, An Infinitesimal Study of the Moduli of Hitchin Pairs. J. Lond. Math. Soc. (2), 49, Issue 2, (1994) 219–231.
[5] S. BLOCH, Semi-regularity and de Rham cohomology. Invent. Math., 17, 51-66 (1972).
[6] F. BOTTACINI, Atiyah classes of Lie algebroids. In Current Trends in Analysis and its Applications, Proceedings of the 9th ISAAC Congress, Krakow, 2013. Trends Math, Birkhauser (2015), 375-393.
[7] R.-O. BUCHWEITZ – H. FLENNER, A Semiregularity Map for Modules and Applications to Deformations. Compos. Math., 137, 135-210, (2003).
[8] I. F. DONIN, Construction of a versal family of deformations for holomorphic bundles over a compact complex space. Sb. Math. 23 405 (1974).
[9] J. L. DUPONT, Curvature and characteristic classes. Lecture Notes in Mathematics 640, Springer-Verlag (1978).
[10] Y. FÉLIX – S. HALPERIN – J. THOMAS, Rational homotopy theory. Graduate Texts in Mathematics 205, Springer-Verlag (2001).
[11] D. FIORENZA – D. IACONO – E. MARTINENGO, Differential graded Lie algebras controlling infinitesimal deformations of coherent sheaves, J. Eur. Math. Soc. (JEMS) 14 (2012), no. 2, 521-540.
[12] D. FIORENZA – M. MANETTI – E. MARTINENGO, Semicosimplicial DGLAs in deformation theory. Comm. Algebra, 40, Iss. 6 (2012).
[13] E. GETZLER, Lie theory for nilpotent L∞ algebras. Ann. of Math. 170 (1), 271-301 (2009).
[14] R. HARTSHORNE, Algebraic geometry, Graduate Texts in Mathematics, 52, Springer-Verlag, (1977).
[15] V. HINICH, Descent of Deligne groupoids, Int. Math. Res. Not. IMRN, (1997), n. 5, 223-239.
[16] D. HUYBRECHTS, Complex Geometry: An Introduction. Springer (2005).
[17] D. IACONO – M. MANETTI, Semiregularity and obstructions of complete intersections. Adv. Math. 235 (2013) 92-125.
[18] D. IACONO – M. MANETTI, On deformations of pairs (manifold, coherent sheaf). Canad. J. Math. 71, 1209-1241 (2019).
[19] M. KONTSEVICH, Deformation quantization of Poisson manifolds, I. Lett. Math. Phys. 66, 157-216 (2003).
[20] E. Lepri – M. Manetti, Connections and $L_\infty$ liftings of semiregularity maps. *J. Geom. Phys.*, **168**, 104303 (2021).

[21] L. Lunardon, Some remarks on Dupont contraction. *Rend. Mat. Appl. (7)* **39**, 79-96 (2018).

[22] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, **213**, Cambridge University Press, (2005).

[23] M. Manetti, Lectures on deformations of complex manifolds. *Rend. Mat. Appl. (7)* **24**, 1-183 (2004).

[24] M. Manetti, *Lie methods in deformation theory*. Springer Monographs in Mathematics (2022).

[25] F. Meazzini, A DG-enhancement of $D(\text{QCoh}(X))$ with applications in deformation theory. 2018, [arXiv:1808.05119](https://arxiv.org/abs/1808.05119).

[26] R. A. Mehta – M. Stiénon – P. Xu, The Atiyah class of a dg-vector bundle. *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 4, 357-362.

[27] V. Navarro Aznar, Sur la théorie de Hodge-Deligne. *Invent. Math.* **90**, 11-76 (1987).

[28] J. P. Pridham, Semiregularity as a consequence of Goodwillie’s theorem. 2012, [arXiv:1208.3111](https://arxiv.org/abs/1208.3111).

[29] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*, Publ. Math. Inst. Hautes Etudes Sci., **80** (1994), p. 5-79.

[30] C. Sorger, Lectures on moduli of principal $G$-bundles over algebraic curves, 1999, [https://inis.iaea.org/collection/NCLCollectionStore/_Public/38/005/38005695.pdf](https://inis.iaea.org/collection/NCLCollectionStore/_Public/38/005/38005695.pdf)

[31] K. Ueno, Infinitesimal deformation of principal bundles, determinant bundles and affine Lie algebras. *RIMS Kôkyûroku Bessatsu*, **778**, p.36-41 (1992).

[32] H. Whitney, *Geometric integration theory*. Princeton University Press, Princeton, N. J., (1957).