Knizhnik-Zamolodchikov-Bernard equations on Riemann surfaces

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October 12, 1994.

Knizhnik-Zamolodchikov-Bernard equations for twisted conformal blocks on compact Riemann surfaces with marked points are written explicitly in a general projective structure in terms of correlation functions in the theory of twisted b-c systems. It is checked that on the moduli space the equations provide a flat connection with the spectral parameter.

1. Introduction.

In addition to the conformal symmetry, the Wess-Zumino-Novikov-Witten (WZNW) model possesses the symmetry of an affine Lie algebra \( \hat{G} \) [1]. The Virasoro algebra is embedded in \( U(\hat{G}) \) by the Sugawara construction. This additional symmetry leads to certain equations for the conformal blocks in the WZNW theory. When the theory is defined on the sphere, these are the well-known Knizhnik-Zamolodchikov (KZ) equations [2]:

\[
\left( \frac{\partial}{\partial z_\alpha} + \frac{1}{k + h^*} \sum_{\beta \neq \alpha} \frac{t^a_\alpha t^a_\beta}{z_\alpha - z_\beta} \right) \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = 0. \tag{1.1}
\]

These equations relate the dependence of the conformal block \( \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle \) on the positions of the marked points \( z_\alpha \) to the action of the currents \( j^a(z) \) on the fields \( \Phi_\alpha(z_\alpha) \):

\[
t^a \Phi(w) = \oint_w j^a(z) \Phi(w) \frac{dz}{2\pi i}. \tag{1.2}
\]

We may also speak of the KZ equations as of the connection

\[
\nabla_\alpha = \frac{\partial}{\partial z_\alpha} + \frac{1}{k + h^*} \sum_{\beta \neq \alpha} \frac{t^a_\alpha t^a_\beta}{z_\alpha - z_\beta} \tag{1.3}
\]

on the bundle of conformal blocks over the moduli space of the sphere with marked points.

The subject of this paper is the generalization of the KZ equations (1.1) to surfaces of higher genera and the discussion of their properties. First these equations for the surfaces of nonzero genera were obtained by Bernard [3,4]. To define the action of the zero modes of the current he included twists on the handles of the surface. Conformal blocks become functions of the twists, and their derivatives along the twists give the action of the zero modes of the current. The moduli space contains not only the positions of the marked points, but also the moduli of the complex structure of the surface itself. The connection form becomes a differential operator with respect to the twists. We shall call such a system of the equations the Knizhnik-Zamolodchikov-Bernard (KZB) equations, and the corresponding connection — the KZB connection.

In the paper we write out the KZB equations in a simple form. All terms in the equations can be expressed explicitly as Poincaré series in the Schottky parametrization, and at the same time admit an invariant description in an arbitrary projective structure. We also briefly discuss the transformation properties of the equations as the projective structure changes. A remarkable feature of the KZB equations is their relationship to the twisted b-c system of spin 1. After multiplying the conformal block by the square root of the b-c holomorphic partition function, the connection depends on the level \( k \) only through a spectral parameter \( (k + h^*)^{-1} \); besides the connection form becomes symmetric. It was pointed out by Losev that this relationship to the b-c systems can be explained in terms of the BRST construction in the \( G/G \) coset.
model [5]. The presence of the spectral parameter in the connection ensures that from its flatness at integer
k (for "physical" reasons) the flatness at an arbitrary k will follow [5]. This imposes certain strong conditions
on the connection form. We explicitly check these conditions and prove the flatness.

At higher genera the KZB equations contain an interesting "potential" term. This term vanishes at
genus one (on a torus), and Bernard originally claimed that it was zero at any genus [4]. However there exist
indications that the potential should not vanish at higher genera [5]. So far we cannot say much about the
potential term, except its closeness as a 1-form on the moduli space, which is one of the conditions for the
compatibility of the KZB equations (cf. [8]).

The paper is organized as follows. In section 2 we introduce the twisted WZNW model on a compact
Riemann surface. In section 3 we recall how the stress-energy tensor defines the connection on the moduli
space and how this connection depends on the choice of the projective structure. Section 4 deals with twisted
1-forms on the surface; we also define the twisted b-c system of spin 1 in this section. These sections provide
a necessary kit for working with the KZB equations. Here we tried to follow the ideas and the notation of
Bernard [3,4] and Losev [5]. Finally, section 5 contains the main results of the paper. In this section we write
out the equations and discuss their properties. Section 6 summarizes the discussion and presents several
questions for further investigation. Some auxiliary information and calculations are gathered in Appendices.

2. Twisted WZNW theory.

We consider the WZNW model at level k on a compact Riemann surface of genus N. The model is
defined for a simple compact Lie group G with the Lie algebra G [2]. We shall deal only with the holomorphic
part of the theory, thus all fields are holomorphic except at locations of other fields. The holomorphic theory
contains the currents j(z) taking values in the Lie algebra G, the stress-energy tensor T(z), and primary
fields Φ(z). The primary fields take values in irreducible finite dimensional representations of G and are
multivalued on the surface. The fields obey the operator product expansion (OPE):

\[ j^a(z)j^b(w) = -k \frac{\delta^{ab}}{(z-w)^2} - f^{abc}(w)j^c(w) + O(1), \]

\[ j^a(z)\Phi(w) = \frac{1}{z-w} f^a \Phi(w) + O(1), \]

where \( \delta^{ab} \) is the invariant bilinear form on G, \( f^a \) is the action of \( a \in G \) in the representation of the primary
field \( \Phi \), \( f^{abc} \) are the structure constants of G. We shall work in an orthonormal basis of G normalized so that

\[ f^{abc} f^{abd} = 2h^* \delta^{cd}, \]

where \( h^* \) is the dual Coxeter number of G. The stress-energy tensor \( T(z) \) is expressed in terms of the currents
by the Sugawara construction:

\[ T(z) = -\frac{1}{2(k + h^*)} :j^a(z)j^a(z): = -\frac{1}{2(k + h^*)} \lim_{w \to z} \left( j^a(z)j^a(w) + \frac{k \dim G}{(z-w)^2} \right). \]

To be specific about notation, we shall label normalized correlation functions (divided by the partition
function) with the subscript \( N \), like \( \langle X \rangle_N \), non-normalized correlators being without any subscript: \( \langle X \rangle = \langle X \rangle_N Z \), where \( Z = (1) \) is the partition function.

Our ultimate goal is to study the connection on the bundle of correlation functions over the moduli
space. To write this connection is the same as to compute the correlation functions with the insertion of
\( T(z) \) (see section 3). Since \( T(z) \) is constructed of the currents, to write the KZB connection we first need to
compute correlation functions with the currents inserted. On the sphere any correlation function \( \langle j^a(z)X \rangle \)
is a 1-form in $z$ which can be reconstructed from its singularities (singular terms in Laurent expansion at the poles). On Riemann surfaces of higher genera there exist global holomorphic 1-forms, and to restore a meromorphic 1-form, we need extra information, besides singularities. For example, any meromorphic 1-form is uniquely defined by its singularities and integrals over A-cycles. Here we encounter difficulties, since zero modes

$$\oint_{A_i} (j^\alpha(z))X \, dz$$

(2.5)

are not determined by the OPE. For this reason Bernard suggested to include twists on A-cycles [3,4]. Given a set of $N$ elements of $G$ ($N$ is the genus of the surface)

$$g_i = \exp\left(\sum_a \xi^a_i \, t^a\right), \quad i = 1, \ldots, N,$$

(2.6)

we insert in all correlation functions the twists

$$\hat{g}_i = \exp \oint_{A_i} \xi^a_i j^\alpha(z) \, dz.$$  

(2.7)

From now on we shall suppress the twists in notation, thus the correlators $\langle X \prod_{i=1}^N \hat{g}_i \rangle$ will be written simply as $\langle X \rangle$.

Following Bernard, we write equations not for a single partition function or a correlation function, but for them as functions of the twists. This allows us to express zero modes of the currents as the derivatives along the twists:

$$\delta_{\hat{g}_i} \langle X \rangle = \mathcal{L}^{\alpha i} \langle X \rangle,$$

(2.8)

where $\mathcal{L}^{\alpha i}$ is the right-invariant derivative along the $i$-th twist:

$$\mathcal{L}^{\alpha i} f(g_1, \ldots, g_N) = \frac{\partial}{\partial \xi} \bigg|_{\xi=0} f(g_1, \ldots, e^{\xi a_i} g_i, \ldots, g_N)$$

(2.9)

for any function $f$ of $N$ group elements $g_i$.

3. 1-forms on the moduli space and projective structures on the surface.

The variation of a correlation function under an infinitesimal shift of the moduli is described by inserting the stress-energy tensor $T(z)$. Indeed, let us consider an infinitesimal change of complex structure induced by a coordinate transformation

$$z \mapsto z + \varepsilon(z, \bar{z}),$$

(3.1)

where $\varepsilon(z, \bar{z})$ is defined only locally (otherwise it would give a reparametrization). The Beltrami differential

$$\mu(z, \bar{z}) = \partial z(z, \bar{z})$$

(3.2)

is defined globally and has the transformation properties of a (-1,1)-form on the surface. At the same time it may be thought of as a tangent vector to the moduli space. To compute the variation of a (non-normalized) correlation function under the change of moduli described by $\mu$, we need to insert the stress-energy tensor coupled to $\mu$ inside the correlator:

$$\delta_{\mu} \langle X \rangle = \int_{\Sigma} \langle T(z)X \rangle \, \mu(z, \bar{z}) \, dz d\bar{z}.$$  

(3.3)

This expression would be well-defined if $T(z)$ were a 2-differential on the surface $\Sigma$. In fact, the holomorphic stress-energy tensor $T(z)$ is not a 2-differential, but transforms with the Schwarzian term:

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12} \{z; w\},$$

(3.4)
where
\[
\{z; w\} = \left( \frac{d^3 z}{dw^3} / \frac{dz}{dw} \right) - \frac{3}{2} \left( \frac{d^2 z}{dw^2} / \frac{dz}{dw} \right)^2,
\]
(3.5)
c is the Virasoro central charge. The Schwarzian derivative \(\{z; w\}\) vanishes for projective transformations, therefore after fixing the projective structure \(T(z)\) becomes a 2-differential. Thus the values of the correlation function depend on the choice of the family of projective structures on the surfaces [7]. Physically, this is due to the anomaly; prescriptionally, this dependence appears in the regularization of the stress-energy tensors (2.4), (4.18). One easily checks that the normal ordering in (2.4) and (4.18) depends on the choice of the local coordinate, and that this dependence reproduces exactly the transformation law (3.4).

The projective structures should be chosen in such a way that the system of equations (3.3) is compatible. Then the KZB equations would be compatible, i.e. they would define a flat connection (not just projectively flat). Such families of projective structures exist. One of them is the projective structure defined by the Schottky parametrization (see Appendix A). The Schottky parametrization is also convenient for writing out explicit formulas, and we shall present the expressions for all terms of the equations in the Schottky representation. However, we must stress that the whole treatment is parametrization independent, and can be performed in any “compatible” projective structure (i.e. such that the equations (3.3) are integrable). Any change of the projective structure is defined by adding a 1-form on the moduli space to \(T(z)\), therefore one can easily construct a “non-compatible” projective structure by adding a non-closed 1-form. From now on we fix a “compatible” projective structure and speak of \(T(z)\) as of a 2-differential, and of correlation functions as of functions on the moduli space.

Remark now that \(T(z)\) being a 2-differential on the surface serves as a 1-form on the moduli space [7]. Indeed, by (3.3) it defines the response to any change of the moduli. In particular, if we change the moduli by moving a marked point \(\xi\), then
\[
\frac{\partial}{\partial \xi} \langle X \rangle = \oint_{\xi} \frac{dz}{2\pi i} \langle T(z)X \rangle.
\]
(3.6)
It allows us to view any meromorphic 2-differential with poles at marked points as a 1-form on the moduli space. For a surface of genus \(N \geq 2\) with \(n\) marked points there exist \((3N - 3 + n)\) linearly independent meromorphic 2-differentials with simple poles at marked points. They form a basis in the cotangent space to the moduli space of the surface with marked points. Remark that the second-order residues of the stress-energy tensor are fixed by the field dimensions, higher orders vanish if the fields are primary with respect to Virasoro algebra.

Due to the equivalence between 2-differentials on the surface and 1-forms on the moduli space, the differential \(d_m\) on the moduli space mapping functions to 1-forms can be treated as an operator mapping functions on the moduli space to 2-differentials on the surface. We denote this operator by \(d_m(z)\) so that
\[
\delta_{\mu} F = \int_{\Sigma} \mu(z, \bar{z}) d_m(z) F dz d\bar{z}
\]
(3.7)
for any function \(F\) of the moduli. Notice that for \(F\) being a correlation function,
\[
d_m(z) \langle X \rangle = \langle T(z)X \rangle.
\]
(3.8)

4. Twisted 1-forms on the surface and the twisted b-c system.

**Twisted 1-forms.**

Any correlation function of the form \(\langle j^a(z)X \rangle\) is a meromorphic twisted 1-form in \(z\) with values in the Lie algebra \(G\) (we shall also call it a twisted 1-form in \(z, a\)). Twisting means that passing across A-cycles (where the twists \(\hat{g}_i\) are placed) this 1-form is conjugated by the corresponding twists. More formally twisted 1-forms can be described with the help of the “Schottky-type” covering. Namely, let us consider the covering \(\Sigma^*\) of the surface \(\Sigma\) such that the set of B-generators of \(\pi_1(\Sigma)\) lifts to an infinite tree on \(\Sigma^*\), and A-generators
lift again to closed loops. Recall that the fundamental group \( \pi_1(\Sigma) \) is generated by the \( 2N \) elements \( A_i \) and \( B_i, i = 1, \ldots, N \), with the only defining relation
\[
A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots A_N B_N A_N^{-1} B_N^{-1} = 1. \tag{4.1}
\]
\( \pi_1(\Sigma^*) \) is the subgroup of \( \Sigma \) generated by all \( A_i \) and group elements conjugate to \( A_i \). Evidently, it is a normal subgroup, and the group
\[
\Gamma = \pi_1(\Sigma)/\pi_1(\Sigma^*) \tag{4.2}
\]
is the group of covering transformations of \( \Sigma^* \), interchanging points with the same projections to \( \Sigma \). For our choice of the covering, \( \Gamma \) is the free group with \( N \) generators. Choose the generators to be the classes of \( B_i \) and denote the corresponding transformations of \( \Sigma^* \) by \( \gamma_i \).

This covering arises in the Schottky construction (see Appendix A). In that case \( \Gamma \) is the Schottky group, \( \Sigma^* \) is the Riemann sphere without fixed points of \( \Gamma \), \( \gamma_i \) are the projective transformations generating \( \Gamma \). We should emphasize though, that the construction of this covering does not restrict our choice of parametrizations of the surface and the moduli nor the choice of projective structures.

After introducing the covering \( \Sigma^* \) we can define the twisted 1-form as a 1-form on the covering obeying proper transformation rules. Assume that the local coordinates on different sheets of the covering are related by the action of \( \Gamma \). Then we call \( f^a(z) \) a twisted 1-form in \( z, a \) if for all \( i = 1, \ldots, N \)
\[
\gamma_i f^a(\gamma_i(z)) = (g_i^*)_b^a f^b(z), \tag{4.3}
\]
where \((g_i)_b^a\) are the matrix elements of the twists in the adjoint representation.

**Main examples of meromorphic twisted 1-forms.**

To reconstruct twisted 1-forms from their singularities and integrals over A-cycles, we shall decompose them in a sum of suitable finite-dimensional spaces. For our construction we need to fix an auxiliary point \( w_0 \) on the surface.

(i) **Twisted 1-forms** \( \omega^a_{ib}(z; w_0) \). Consider first the space of all meromorphic twisted 1-forms with the only simple pole at \( w_0 \). This space is \((N \dim G)\)-dimensional, and any 1-form from this space is uniquely determined by its integrals over A-cycles. Define a basis \( \omega^a_{ib}(z; w_0) \) in this space by
\[
\oint_{A_i} \omega^a_{ib}(z; w_0) dz = \delta^a_b \delta_{ij}. \tag{4.4}
\]
The residues at \( z = w_0 \) are
\[
\text{Res}_{w_0} \omega^a_{ib}(z; w_0) = (g_i^{-1})_b^a. \tag{4.5}
\]
In the Schottky parametrization such twisted 1-forms are given by the Poincaré series
\[
\omega^a_{ib}(z; w_0) = \sum_{\gamma \in \Gamma} (g_{\gamma^{-1}})_b^a \left[ \frac{\gamma'(z)}{\gamma(z) - \gamma_i(w_0)} - \frac{\gamma'(z)}{\gamma(z) - w_0} \right]. \tag{4.6}
\]

(ii) **Twisted 1-forms** \( \theta^a_b(z; w, w_0) \). Another useful example of twisted 1-forms are 1-forms \( \theta^a_b(z; w, w_0) \) with zero integrals over A-cycles and a given simple pole at a given point \( w \) (which is compensated by a simple pole at the auxiliary point \( w_0 \)):
\[
\oint_{A_i} \theta^a_b(z; w, w_0) dz = 0, \tag{4.7}
\]
\[
\text{Res}_w \theta^a_b(z; w, w_0) = -\text{Res}_{w_0} \theta^a_b(z; w, w_0) = \delta^a_b. \tag{4.8}
\]
In the Schottky parametrization the Poincaré series for \( \theta^a_b(z; w, w_0) \) is
\[
\theta^a_b(z; w, w_0) = \sum_{\gamma \in \Gamma} (g_{\gamma^{-1}})_b^a \left[ \frac{\gamma'(z)}{\gamma(z) - w} - \frac{\gamma'(z)}{\gamma(z) - w_0} \right]. \tag{4.9}
\]
Here we assumed that $w$ and $w_0$ belong to the fundamental domain of $\Gamma$. Holomorphically extending these expressions to other sheets of the covering $\Sigma^*$, we can write

$$\omega_{ib}^a(z; w_0) = \theta_{b}^a(z; \gamma_i(w), w_0).$$  \hspace{1cm} (4.10)

The transformation properties of $\theta_{b}^a(z; w, w_0)$ as a function of the parameters $w$ or $w_0$ are described by

$$\theta_{b}^a(z; \gamma_j(w), w_0) = \theta_{b}^a(z; w, w_0)(g_j^{-1})^b_c + \omega_{ib}^a(z; w_0).$$  \hspace{1cm} (4.11)

The proof of this equality follows from the observation that l.h.s. and r.h.s. are both twisted 1-forms in $z, a$ and have equal singularities and equal integrals over $A$-cycles.

(iii) Twisted 1-forms $\Omega_{ab}^c(z, w)$. Differentiating $\theta_{b}^a(z; w, w_0)$ with respect to the parameters we obtain another twisted 1-form $\Omega_{ab}^c(z, w)$ with a pole of second order:

$$\Omega_{ab}^c(z, w) = \frac{\partial}{\partial w} \theta_{b}^a(z; w, w_0) = -\frac{\partial}{\partial w} \theta_{b}^a(z; w_0, w),$$  \hspace{1cm} (4.12)

The first order residue of $\Omega_{ab}^c(z, w)$ at $z = w$ vanishes as well as integrals over $A$-cycles. A remarkable property of $\Omega_{ab}^c(z, w)$ is its symmetry under interchanging $(z, a) \leftrightarrow (w, b)$:

$$\Omega_{ab}^c(z, w) = \Omega_{ba}^c(w, z),$$  \hspace{1cm} (4.13)

in particular, $\Omega_{ab}^c(z, w)$ is a twisted 1-form both in $z, a$ and in $w, b$.

The Poincaré series for $\Omega_{ab}^c(z, w)$ in Schottky parametrization is

$$\Omega_{ab}^c(z, w) = \sum_{\gamma \in \Gamma} (g_\gamma^{-1})_{b}^{c} \frac{\gamma'(z)}{(\gamma(z) - w)^2}.$$  \hspace{1cm} (4.14)

The twisted 1-forms $\omega_{ib}^a(z; w_0)$ and $\theta_{b}^a(z; w, w_0)$ can be expressed in terms of $\Omega_{ab}^c(z, w)$ as

$$\omega_{ib}^a(z; w_0) = \int_{w_0}^{\gamma_i(w_0)} d\xi \Omega_{ab}^c(z, \xi),$$  \hspace{1cm} (4.15)

$$\theta_{b}^a(z; w, w_0) = \int_{w_0}^{w} d\xi \Omega_{ab}^c(z, \xi).$$  \hspace{1cm} (4.16)

In our discussion we shall need to look at these forms at their singular points. For this reason we wish to treat the twisted 1-forms we defined above as correlation functions in the twisted b-c system of spin 1. Then the regularization of singularities will correspond to the normal ordering of the fields.

Twisted b-c system.

The b-c system of spin 1 contains anticommuting fields $b$ of spin 1 and $c$ of spin 0 (i.e. $b$ field transforms as a 1-form and $c$ field — as a function on the surface). For constructing the twisted version we take $\dim G$ copies of b-c systems. The fields $b^a$ and $c^a$ obey the OPE

$$b^a(z)c^b(w) = \frac{\delta^{ab}}{z - w} + o(1),$$  \hspace{1cm} (4.17)

and the stress-energy tensor is given by

$$T(z) = - :b^a(z)\partial c^a(z): = - \lim_{w \to z} \left( b^a(z)\partial c^a(w) - \frac{\dim G}{(z - w)^2} \right).$$  \hspace{1cm} (4.18)

We define b-c currents by

$$j^a(z) = f^{abc}b^b(z)c^c(z),$$  \hspace{1cm} (4.19)
Then these currents form the affine algebra $\hat{G}$ at the level $2h^*$:

$$j^a(z)j^b(w) = -2h^* \frac{\delta^{ab}}{(z-w)^2} - \frac{f^{abc}}{z-w}j^c(w) + O(1).$$

(4.20)

On the handles of the surface we insert the twists

$$\hat{g}_i = \exp \left( \oint_{A_i} \xi_i^a j^a(z) \, dz \right)$$

by the same group elements as in the WZNW model.

To eliminate zero modes of $b$ and $c$ fields we include in correlation functions, along with the twists, the product of all $c$ fields in the auxiliary point $w_0$ and the integrals of all $b$ fields over all $A$-cycles:

$$\prod_{a=1}^{\text{dim} \mathcal{G}} c^a(w_0) \prod_{i=1}^N \oint_{A_i} b^a(z) \, dz.$$  

(4.22)

Similarly to the twisted WZNW model, we shall suppress in notation both the twists (4.21) and the insertions (4.22), thus writing $\langle X \rangle^{b-c}$ instead of $\langle X \rangle^{\dim \hat{G}} c^a(0) \prod_{i=1}^N \oint_{A_i} b^a(0) \, dz$.

**Correlation functions.**

Consider the normalized correlation function $(b^a(z)c^b(w))^{b-c}_N$. The twists (4.21) make it a twisted 1-form in $z, a$. Its integrals over $A$-cycles vanish due to the insertion of $\prod_{i=1}^N \oint_{A_i} b^a(z) \, dz$. It has two poles: at $z = w$

$$\text{Res}_{w}(b^a(z)c^b(w))^{b-c}_N = \delta^{ab}$$

(4.23)

and at $z = w_0$ (due to the insertion of $\prod_a c^a(w_0)$). Therefore,

$$\langle b^a(z)c^b(w) \rangle^{b-c}_N = \theta^b_a(z; w, w_0).$$

(4.24)

Also

$$\Omega^{ab}(z, w) = \langle b^a(z) \partial c^b(w) \rangle^{b-c}_N = \langle b^b(w) \partial c^a(z) \rangle^{b-c}_N.$$  

(4.25)

Many-point correlation functions obey the Wick rule. For 4-point correlation functions it reads:

$$\langle b^a(z_1)b^b(z_2)c^c(z_3)c^d(z_4) \rangle^{b-c}_N = \langle b^a(z_1)c^d(z_4) \rangle^{b-c}_N \langle b^b(z_2)c^c(z_3) \rangle^{b-c}_N - \langle b^a(z_1)c^c(z_3) \rangle^{b-c}_N \langle b^b(z_2)c^d(z_4) \rangle^{b-c}_N.$$  

(4.26)

This identity can be checked easily by comparing the transformation properties and the singularities of the l.h.s. and the r.h.s. We shall need this formula in further computations.

**Regularization.**

We can regularize singularities by normally ordering the fields inside the correlation functions. Mathematically, this amounts to discarding the negative power terms in Laurent series, i.e.

$$\Omega^{ab}(z, w)_{\text{reg}} = \langle b^a(z) \partial c^b(z) \rangle^{b-c}_N = \lim_{w \to z} \left( \Omega^{ab}(z, w) - \frac{\delta^{ab}}{(z-w)^2} \right),$$  

(4.27)

$$\theta^b_a(z; w, w_0)_{\text{reg}} = \langle b^a(z) c^b(z) \rangle^{b-c}_N = \lim_{w \to z} \left( \theta^b_a(z; w, w_0) - \frac{\delta^{ab}}{z-w} \right).$$

(4.28)

This way of regularization depends on the choice of the local coordinate. Although we shall admit such a dependence in intermediate calculations, we can check afterwards that the final result transforms properly under changes of coordinates. We shall indicate the regularization by the subscript “reg”.

**Stress-energy tensor and partition function.**

Using this notation,

$$\langle T(z) \rangle^{b-c}_N = -\Omega^{aa}(z, z)_{\text{reg}}.$$  

(4.29)
If we choose a proper family of projective structures on the surface (e.g. the projective structure of the Schottky representation), the stress-energy tensor can be integrated (on a covering of the moduli space) to a holomorphic partition function $Z^{b,c}$ [6,5]:

$$d_m(z) \log Z^{b,c} = \langle T(z) \rangle^b_N$$  \hspace{1cm} (4.30)

(see also section 3). The square root $\Pi$ of this partition function will play an important role in our discussion:

$$Z^{b,c} = \Pi^2.$$  \hspace{1cm} (4.31)

Notice that $\Pi$ depends both on the moduli of the surface and on the twists $g_i$.

The following property of $\Pi$ follows immediately from its definition:

$$d_m(z)\Pi = -\frac{1}{2} \Pi \Omega^{a\alpha}(z,z)_{\text{reg}}$$  \hspace{1cm} (4.32)

In the Schottky representation $\Pi$ can be computed explicitly [6]:

$$\Pi_{\text{Schottky}} = \prod_{k=1}^{\infty} \prod_{\gamma \in \text{prim adj}} \det(1 - g_{\gamma} K_{\gamma}^k).$$  \hspace{1cm} (4.33)

In (4.33) the product is taken over the primitive conjugate classes of the Schottky group ($\gamma$ and $\gamma^{-1}$ are elements of the same primitive class), the determinants are computed in the adjoint representation, $K_{\gamma}$ is the multiplier of the transformation $\gamma$ (see Appendix A). However, we shall not further need any explicit expression for $\Pi$, and reproduce the formula (4.33) only to simplify comparing with other works.

5. Knizhnik-Zamolodchikov-Bernard equations.

**KZB equation.**

Arising from the Sugawara construction (2.4), the KZB equation must have the form

$$(d_m + A) F = 0,$$  \hspace{1cm} (5.1)

where $d_m$ is the differential on the moduli space, the connection form $A$ is an operator acting on the twisted conformal block $F$. It seems useful to write equations for the product ($F \Pi$) instead of $F$, since in this case the dependence on the level $k$ reduces to the factor $(k + h^*)^{-1}$ in front of the connection form.

Let $F = \langle \Phi_1(z_1) \ldots \Phi_n(z_n) \rangle$ be a non-normalized correlation function (conformal block), with the twists implicitly included, $\Pi$ be the square root of the partition function of the twisted $b,c$ system (see section 4). Let $\theta^a_i(z;w_0)$, $\omega^b_i(z;w_0)$ and $\Omega^{ab}(z,w)$ be the twisted 1-forms defined in section 4, $d_m(z)$ be the differential on the moduli space which maps functions on the moduli to 2-differentials on the surface (see section 3). Then the KZB equation looks like follows:

$$\left(d_m(z) + \frac{1}{2(k + h^*)} \left[ \left( \sum_{a=1}^{n} t_{a}^{k} \theta^a_i(z;\xi_a,w_0) + \sum_{i=1}^{N} \mathcal{L}^{ia}(z;w_0) \right) \left( \sum_{\beta=1}^{n} \theta^b_{\beta}(z;\xi_\beta,w_0) t_{\beta}^{c} + \sum_{j=1}^{N} \omega^b_j(z;w_0) \mathcal{L}^{jc} \right) \right] + U(z) \right) (F \Pi) = 0,$$  \hspace{1cm} (5.2)

where all right-invariant derivatives with respect to the twists $\mathcal{L}^{ia}$ must be thought of as acting also outside the brackets (in particular, on $F \Pi$), and the “potential” term

$$U(z) = h^* \Omega^{a\alpha}(z,z)_{\text{reg}} - \frac{1}{\Pi} \mathcal{L}^{ia} \omega^b_ia(z;w_0) \omega^b_j(z;w_0) \mathcal{L}^{jc} \Pi$$

$$= - \frac{2h^*}{\Pi} \left( d_m(z) + \frac{1}{2h^*} \mathcal{L}^{ia} \omega^b_ia(z;w_0) \mathcal{L}^{jc} \right) \Pi$$  \hspace{1cm} (5.3)
is just a scalar factor. The derivation of this equation is presented in Appendix B. To make the structure of
the equation more transparent we need to introduce more notation. Namely, we generalize the construction
of pairing 1-forms with 0-boundaries to the twisted setup at higher genera.

**Pairing 1-forms with 0-boundaries.**

First recall how one can pair 1-forms with 0-boundaries. Let $\omega$ be a holomorphic 1-form on a simply
connected domain. Let $C$ be a 0-boundary on this domain with values in a linear space $L$, i.e. a set of points
$z_1, \ldots, z_n$ labelled with elements $l_1, \ldots, l_n$ of $L$ such that

$$l_1 + \ldots + l_n = 0.$$  

(5.4)

Define the pairing between $C$ and $\omega$

$$\int_C \omega = \sum_{\alpha=1}^{n} l_{\alpha} \int_{w_0}^{z_{\alpha}} \omega,$$  

(5.5)

taking values in $L$. Here $w_0$ is an arbitrary auxiliary point inside the domain. Condition (5.4) along with
the holomorphicity of $\omega$ makes this definition independent of the choice of $w_0$ and paths of integration.
In other words, we integrate $\omega$ over a 1-chain with the boundary $C$, and the result does not depend on the
particular choice of the 1-chain.

This definition obviously extends to the case of multiply connected domains and meromorphic forms $\omega$, if all integrals of $\omega$ along noncontractible contours and first order residues at all poles vanish. We shall
further extend this construction to twisted 1-forms on Riemann surfaces, but first let us look more closely
at the KZ equation to motivate our discussion.

Usually we write the KZ equation in the form (1.1) although this expression is not quite invariant under
the action of $SL(2, \mathbb{C})$ projective transformations. Without using invariant properties of the conformal block
$F = (\Phi_1(z_1) \ldots \Phi_n(z_n))$ equation (1.1) gets transformed to the most general form

$$\left( \frac{\partial}{\partial z_{\alpha}} + \frac{1}{k + h^2} t_\alpha^a \sum_{\beta \neq \alpha} \left( \frac{1}{z_{\alpha} - z_{\beta}} - \frac{1}{z_{\alpha} - w_0} \right) t_{\beta}^a \right) F = 0.$$  

(5.6)

Thus the equation (1.1) uses the choice of the auxiliary point $w_0 = \infty$. Of course, equations (5.6) and (1.1)
are equivalent on the class of $G$-invariant functions

$$\left( \sum_{\alpha=1}^{n} t_{\alpha}^a \right) F = 0,$$  

(5.7)

therefore we may choose any convenient value for $w_0$. The proof of (5.7) is obvious: we surround each point
$z_{\alpha}$ by a contour so that the sum of these contours is homologically equivalent to zero. Then, integrating the
current $j^a(z)$ along these contours we obtain

$$\left( \sum_{\alpha=1}^{n} t_{\alpha}^a \right) F = \left( \sum_{\alpha=1}^{n} \oint_{C_{\alpha}} \frac{dz}{2\pi i} j_{\alpha}^a(z) \Phi_1(z_1) \ldots \Phi_n(z_n) \right) = 0.$$  

(5.8)

To make the independence of $w_0$ explicit in notation we rewrite the equation in terms of pairing with 0-
boundaries. Namely, the points $z_{\alpha}$ labelled with the operators $t_{\alpha}^a$ form a 0-boundary with values in the space
of operators on $G$-invariant elements $f \in V_1 \otimes \ldots \otimes V_n$, where $V_\alpha$ is the representation space of the field $\Phi_\alpha$.

We denote such a 0-boundary by $C^a$. Then the KZ equations take the form

$$\left[ \frac{\partial}{\partial z_{\alpha}} + \frac{1}{k + h^2} t_{\alpha}^a \left( \oint_{C_{\alpha}} \frac{dz}{(z_{\alpha} - z)^2} \right)_{\text{reg}} \right] F = 0.$$  

(5.9)

Here $dz/(z_{\alpha} - z)^2$ is a meromorphic 1-form on the sphere with the only pole of second order at $z = z_{\alpha}$ and
with zero first order residue. The subscript “reg” means discarding singular powers when we integrate to
the singular point $z_{\alpha}$:

$$\left( \oint_{C_{\alpha}} \frac{dz}{(z_{\alpha} - z)^2} \right)_{\text{reg}} = \lim_{z_{\alpha} \to z_{\alpha}' \to \infty} \left( \oint_{C_{\alpha}} \frac{dz}{(z_{\alpha}' - z)^2} - \frac{t_{\alpha}^a}{z_{\alpha}' - z} \right).$$  

(5.10)
The regularization depends on the choice of the coordinate \( z \) around \( z_\alpha \), which is consistent with the transformation properties of \( \Phi_\alpha \) as a conformal field with the conformal dimension

\[
\Delta_\alpha = \frac{t^\alpha t_{\overline{\alpha}}}{2(k + h^*)}.
\] (5.11)

**Generalization to higher genera.**

This notation admits a generalization to higher genera. Now the consideration analogous to (5.7) leads to the \( \mathcal{G} \)-invariance condition

\[
\left( \sum_{\alpha=1}^{n} t^\alpha - \sum_{i=1}^{N}(\mathcal{L}^{ia} - \mathcal{R}^{ia}) \right) F(g) = 0.
\] (5.12)

Here \( F \) is the twisted conformal block depending on the twists \( g_i \), \( N \) is the genus of the surface. \( \mathcal{L}^{ia} \) and \( \mathcal{R}^{ia} = (g_i^{-1})_b^a \mathcal{L}^b \) are the right- and the left-invariant derivatives along the twists \( (g_i^{-1})_b^a \) are the matrix elements in the adjoint representation). The analogue of the 0-boundary of the previous construction will be a new object (“generalized 0-boundary”) consisting of the points \( z_a \) labelled with \( t^a_a \) and the operators \( \mathcal{L}^{ia} \) labelling handles. Again, denote this object by \( C^a \). If \( \omega^a \) is a meromorphic twisted 1-form with vanishing first order residues and zero integrals over \( A \)-cycles, then the operator

\[
\int_{C^a} \omega^a = \sum_{\alpha=1}^{n} \int_{w_0}^{z_\alpha} \omega^a t^\alpha + \sum_{i=1}^{N} \int_{w_0}^{\gamma_i(w_0)} \omega^a \mathcal{L}^{ia},
\] (5.13)

when restricted to \( \mathcal{G} \)-invariant functions (5.12) is independent of the choice of \( w_0 \) and paths of integration (summation over \( a \) is assumed). The integrals in (5.13) are defined on the covering \( \Sigma^* \), therefore there is no ambiguity in the projections of the homology classes of integration paths onto \( B \)-cycles. The arrow above the integral sign indicates that we place differentiation operators \( \mathcal{L}^{ia} \) to the right, thus they do not act on \( \omega \) itself. Similarly

\[
\int_{C^\alpha} \conjugate{\omega}^a = \sum_{\alpha=1}^{n} \int_{w_0}^{z_\alpha} \conjugate{\omega}^a t^\alpha + \sum_{i=1}^{N} \int_{w_0}^{\gamma_i(w_0)} \omega^a \mathcal{L}^{ia},
\] (5.14)

is the conjugate (up to sign, since \( t^a_a \) and \( \mathcal{L}^{ia} \) are antisymmetric) operator.

More formally, in our construction we integrate 1-forms over an element of the operator-valued \( \Gamma \)-invariant relative homology group \( H^1(\Gamma)(\Sigma^*, M, L) \), where \( \Sigma^* \) is the “Schottky-type” covering (see section 4), \( M \) is the set of preimages of the marked points on \( \Sigma^* \), \( L \) is the space of operators acting on twisted conformal blocks. \( \Gamma \) acts on 1-chains by mapping in \( \Sigma^* \) and simultaneously conjugating by the corresponding twist \( g_\gamma \).

The element of \( H^1(\Gamma)(\Sigma^*, M, L) \) as a 1-chain on \( \Sigma^* \) has the boundary consisting of the marked points with the corresponding operators. This condition determines the cycle in \( H^1(\Gamma)(\Sigma^*, M, L) \) unambiguously after we fix the basis of \( A \)- and \( B \)-cycles on the surface by placing twists on the handles and specify the \( B \)-projections of the cycle to be the differentiation operators along the twists*.

**Final form of the equation.**

Using this notation we rewrite the KZB equations as

\[
\left( d_m(z) + \frac{1}{2(k + h^*)} \left[ \int_{C^a} \Omega^{ab}(\eta, z) \, dn + \int_{C^\alpha} \Omega^{bc}(z, \xi) \, d\xi + U(z) \right] \right) (F \Pi) = 0.
\] (5.15)

Now recall that \( \Omega^{ab}(z, w) = \langle b^a(z) \partial_c b^c(w) \rangle_N \) in the twisted b-c system. Therefore, we naturally define

\[
d_m(z) \Omega^{ab}(u, w) = \langle T(z) b^a(u) \partial_c b^c(w) \rangle_N - \langle T(z) \rangle_N \langle b^a(u) \partial_c b^c(w) \rangle_N.
\] (5.16)

* The author thanks A. Losev for elucidating this point.
Since \( T = - b \partial c \); the Wick rule (4.26) gives

\[
d_m(z) \Omega^{ab}(u, w) = \Omega^{ac}(u, z) \Omega^{cb}(z, w). \tag{5.17}
\]

This enables us to rewrite (5.15) as

\[
\left( d_m(z) + \frac{1}{2(k + h^*)} \left( \int_{C^a} \int_{C^b} d\xi \Omega^{ab}(\eta, \xi) \bigg|_{reg} + U(z) \right) \right) (F\Pi) = 0. \tag{5.18}
\]

Here the regularization is introduced to eliminate logarithmic divergences when we perform both integrations to the same marked point \( z_\alpha \):

\[
\left( \int_{C^a} \int_{C^b} d\xi \Omega^{ab}(\eta, \xi) \bigg|_{reg} = \lim_{z'_{\alpha} \to z_\alpha} \left[ \int_{C^a} \int_{C^b} d\xi \Omega^{ab}(\eta, \xi) - \sum_{\alpha=1}^n t_{\alpha} a_{\alpha} \log(z'_\alpha - z_\alpha) \right]. \tag{5.19}
\]

Recall that a connection \( \nabla = d + \lambda A \) is flat (the equations \((\partial_{\mu} + \lambda A_{\mu})F = 0\) are compatible) for any \( \lambda \) if and only if \( dA = 0 \) and \( [A_\mu, A_\nu] = 0 \) for any directions \( \mu \) and \( \nu \) on the moduli space. The first of these two conditions is almost checked (the first summand is locally integrated), except the potential term \( U(z) \). \( U(z) \) is a holomorphic 2-differential on the surface, and therefore represents a 1-form on the moduli space (since it can be coupled to Beltrami differentials, see section 3). Appendix C contains the proof that this 1-form on the moduli space is closed, i.e. \( d_m(w)U(z) - d_m(z)U(w) = 0 \). Therefore, there locally exists a function \( W \) of the moduli and the twists such that

\[
U(z) = d_m(z)W. \tag{5.20}
\]

We may define the “universal” operator

\[
A_{KZB} = \left( \int_{C^a} \int_{C^b} d\xi \Omega^{ab}(\eta, \xi) \bigg|_{reg} + W \right). \tag{5.21}
\]

Using this operator, we rewrite the KZB equations in the most invariant form

\[
\left[ d_m + \frac{1}{2(k + h^*)} (d_m A_{KZB}) \right] (F\Pi) = 0 \tag{5.22}
\]

for any variation of the moduli.

\textit{Dependence on the coordinates.}

Let us discuss how the terms of this equation depend on the choice of the coordinates on the surface. Notice that \( d_m A_{KZB} \) is not quite a 1-form on the moduli space (or, equivalently, \( A_{KZB} \) is not a function). Indeed, there are two coordinate dependent regularizations in this term. One of them is performed in (5.19) and depends on the local coordinates at the marked points. This dependence gives the correct conformal dimensions (5.11) of the fields \( \Phi_\alpha \). The other regularization appears in \( U(z) \) and depends on the projective structure on the surface. This regularization provides for \( d_m A_{KZB} \) the transformation properties (3.4) of the stress-energy tensor with the Virasoro central charge

\[
c = - \frac{h^* \dim \mathcal{G}}{k + h^*} = \frac{k \dim \mathcal{G}}{k + h^*} - \dim \mathcal{G} = c\text{wznw} + \frac{1}{2} c_{b-c}, \tag{5.23}
\]

which is exactly the central charge of \((F\Pi)\).

Remark that since \( U(z) \) depends on the choice of the projective structure, the question of whether it is zero does not make sense. However, if \( U(z) \) does not depend on the twists, then there exists a projective structure (induced by \( U(z) \)) in which \( U(z) = 0 \). This happens in the case of torus, but we believe that at higher genera \( U(z) \) depends on the twists nontrivially.
Examples.
We can illustrate our construction by three simple examples. In the case of zero genus (sphere) we have
\[ \Omega_{\text{sphere}}^{ab}(z, w) = \frac{\delta_{ab}}{(z - w)^2}, \]  
(5.24)
and
\[ A_{KZB}^{\text{sphere}} = \sum_{\alpha \neq \beta} t^a_{\alpha} t^b_{\beta} \log(z_\alpha - z_\beta). \]  
(5.25)

In the case of torus (genus one) the potential term vanishes in the Schottky parametrization[3]:
\[ U_{\text{Schottky}}^{\text{torus}} = 0. \]  
(5.26)
If there are no marked points, a torus is characterized by a single complex moduli parameter \( q \). The torus with a given \( q \) is defined as the quotient of \( \mathbb{C}^* \) by the equivalence \( z \sim qz \). Elementary computations show that on a torus without marked points
\[ A_{KZB}^{\text{torus}} = \log q \Delta, \]  
(5.27)
where \( \Delta = L^a L^a \) is the Laplacian on the group \( G \) of the twist [3]. In the abelian case \( (G = U(1)) \) \( \Pi \) does not depend on the twists, and the potential \( U(z) \) vanishes. In this case
\[ A_{KZB}^{\text{abel}} = \tau_{ij} L^i L^j, \]  
(5.28)
where \( \tau_{ij} \) is the period matrix of the surface:
\[ \tau_{ij} = \int_{w_0}^{\gamma_j(w_0)} \omega_i(z) \, dz \]  
(5.29)
in the abelian case \( \omega_{ij}^a(z; w_0) \) becomes proportional to \( \delta_{ij}^a \) and independent of \( w_0 \).

Symmetry.
Our computations are consistent with Losev’s observation that the KZB connection form must be symmetric [5]. The symmetricity of \( A_{KZB} \) and, therefore, of the connection form \( d_m A_{KZB} \) indicates that the connection preserves a certain pairing between solutions of KZB equations with central charges \( k \) and \( (-k - 2h^*) \).
Let solutions \( F_k \) and \( F_{-k-2h^*} \) of KZB equations for central charges \( k \) and \( (-k - 2h^*) \) respectively take values in tensor products \( V_1 \otimes \ldots \otimes V_n \) and \( V_1^* \otimes \ldots \otimes V_n^* \) of representations of \( G \) (here \( V_n^* \) is the representation dual to \( V_n \)). Then we can define a pairing between \( F_k \) and \( F_{-k-2h^*} \) by
\[ (F_k, F_{-k-2h^*})^{b-c} = \int dg \, \Pi^2(g) \langle F_k, F_{-k-2h^*} \rangle \]
\[ = \prod_{i=1}^N dg_i \, \Pi^2(g) F_k^{i_1 \ldots i_N} (g) F_{-k-2h^*}^{i_1 \ldots i_N} (g) \eta_{i_1 j_1} \cdots \eta_{i_N j_N}, \]  
(5.30)
where \( \eta_{ij}^{(a)} \) are the matrices of the pairings \( V_a \otimes V_a^* \rightarrow \mathbb{C} \), the integration is performed over all twists, \( dg_i \) is the invariant measure on \( G \). With respect to this bilinear form operators \( t^a_i, L^a \) and \( R^a \) are antisymmetric, therefore \( A_{KZB} \) and \( d_m A_{KZB} \) are symmetric. Hence, for the solutions \( F_k \) and \( F_{-k-2h^*} \) of the KZB equations we have
\[ d_m(F_k, F_{-k-2h^*})^{b-c} = (d_m F_k, F_{-k-2h^*})^{b-c} + (F_k, d_m F_{-k-2h^*})^{b-c} \]
\[ = \frac{1}{k + h^*} (d_m A_{KZB} F_k, F_{-k-2h^*})^{b-c} + \frac{1}{(-k - 2h^*) + h^*} (F_k, d_m A_{KZB} F_{-k-2h^*})^{b-c} = 0 \]  
(5.31)
due to the symmetricity of \( d_m A_{KZB} \). Losev [5] interpreted the symmetricity of the connection as a consequence of an operator formalism in \( G/G \) WZNW coset theory. Referring to the operator formalism we could
treat $F_{-k-2h^*}$ as a conformal block with the central charge $(-k-2h^*)$, and the twisted b-c system — as ghosts in BRST construction [5].

Integrability.

Now let us return to the integrability conditions. To explicitly prove the flatness of the connection it remains to show that

$$[\partial_\mu A_{KZB}, \partial_\nu A_{KZB}] = 0$$

(5.32)

for any directions $\mu$ and $\nu$ in the moduli space. In Appendix D we prove this for $\partial_\mu$ and $\partial_\nu$ being derivatives with respect to the coordinates of inserted fields. We also derive the corresponding generalization of the classical Yang-Baxter equation, which though does not seem to be very instructive (see Appendix D). Instead of commuting the most general operators with marked points, in Appendix E we prove the compatibility of the equations for the partition function (with no marked points). We claim that since marked points can be obtained in a degeneration of the surface (double points), we indeed proved the general form of the statement. The compatibility of the equations for the setup with marked points (and the result of Appendix D in particular) can be obtained as a limiting case of the statement of Appendix E (we present an independent derivation in Appendix D just to study the generalization of the classical Yang-Baxter equation). Thus we proved that the KZB connection is flat. Remark that our proof did not refer to any particular projective structure; the only property we used was that the holomorphic b-c partition function exists (i.e. the projective structure is “compatible” — see section 3).

6. Conclusion.

To summarize, we have written the KZB equations and proved their integrability. The concise form (5.21), (5.22) of the equations contrasts sharply with our way of proving the compatibility (Appendices C,D,E). We still hope that the structure of the equations and their relation to the b-c systems can suggest a more elegant treatment of the problem and help to avoid the tedious computations. Another question remains open about the meaning of the potential term. One may look for its geometric description or for its expression in the WZNW coset construction [5]. We also admit the possibility to explicitly integrate the potential $U(z)$ to $W$ (see equation (5.20)). Another approach to the problem can be inspired by the works on geometric quantization of the moduli space of flat connections, which arrive to similar equations [8].

Acknowledgements.

The author wishes to thank A.Losev for initiating this research and friendly guidance throughout the work, A.Morozov, N.Nekrasov and V.Fock for stimulating discussions. The author acknowledges the support of the Russian Foundation of Fundamental Research under the Grant No.93-02-14365.

Appendix A. Schottky parametrization of Riemann surfaces.

In the Schottky parametrization the surface is constructed as the quotient of the Riemann sphere (more strictly, of the sphere without the fixed points of the group) by the action of a Schottky group. The Schottky group $\Gamma$ is a group freely generated by $N$ projective maps $\gamma_i$ such that one can find $2N$ circles $A_i$ and $A'_i = \gamma_i(A_i)$, $i = 1, \ldots, N$ — all external to each other, and $\gamma_i$ maps the exterior of $A_i$ onto the interior of $A'_i$. The exterior to all the circles $A_i$ and $A'_i$ is a fundamental domain of $\Gamma$. The surface is obtained by gluing each circle $A'_i$ to $A_i$ by the action of $\gamma_i$. Then $A_i$ become A-cycles on the surface.

For future use we should introduce two more definitions. The first one is the parametrization of a projective transformation $\gamma$ by its fixed points $u_\gamma$, $v_\gamma$ (repulsive and attractive) and its multiplier $K_\gamma$ defined by

$$\frac{\gamma(z) - u_\gamma}{\gamma(z) - v_\gamma} = K_\gamma \frac{z - u_\gamma}{z - v_\gamma}, \quad |K_\gamma| < 1.$$  

(A.1)

Finally, we shall need to extend the twists introduced on the handles of the surface to the group homomorphism between $\Gamma$ and $G$:

$$g : \Gamma \to G,$$

$$g_{\gamma_i} = g_i,$$

$$g_{\gamma_\mu} = g_\gamma g_\mu.$$  

(A.2)
The global coordinate $z$ on the Riemann sphere defines naturally a projective structure on the surface. It is remarkable that in this family of projective structures the stress-energy tensor of b-c systems and of
the WZNW model can be integrated to partition functions (or conformal blocks) according to (3.3); i.e. in
Schottky parametrization the KZB connection becomes flat.

Appendix B. Derivation of the KZB equations.

To derive the KZB equations we first express correlation functions with inserted currents in terms of the
themselves $\theta^a_d(z; w, w_0)$, $\omega^a_{ib}(z; w_0)$, $\Omega^{ab}(z, w)$, introduced in section 4. Let $\Phi$
represent the product $\Phi_1(\xi_1) \ldots \Phi_n(\xi_n)$. Then

$$\langle j^a(z) \Phi \rangle = \left( \sum_{\alpha} \theta^a_d(z; \xi_\alpha, w_0) t^\alpha_a + \sum_{b} \omega^a_{ib}(z; w_0) L^{ib} \right) \langle \Phi \rangle , \quad (B.1)$$

since r.h.s. and l.h.s. are both twisted 1-forms in $z, a$, have equal singularities at the poles $z = \xi_\alpha$ and equal
integrals over A-cycles. Although each of the two terms on the r.h.s. depends on the choice of the auxiliary
point $w_0$ and has a pole at $z = w_0$, their sum does not. For further convenience we introduce more notation:

$$L^a(z; w_0) = \omega^a_{ib}(z; w_0) L^{ib}, \quad (B.2)$$

$$\bar{L}^a(z; w_0) = L^{ib} \omega^a_{ib}(z; w_0) = L^a(z; w_0) + \{ L^{ib} \omega^a_{ib}(z; w_0) \} . \quad (B.3)$$

Here we use braces to specify the range of action for the differentiation operators $L^{ia}$. By convention, all
differentiation operators act only inside the braces. Also the summation over all repeating indices (including
those labelling marked points and handles) is assumed if not specified otherwise.

For the correlation function with the insertion of two currents we have

$$\langle j^a(z) j^b(w) \Phi \rangle = - k \Omega^{ab}(z, w) \langle \Phi \rangle - \theta^a_d(z; w, w_0) f^{abc} j^c(w) \Phi $$

$$+ \theta^a_e(z; \xi_\alpha, w_0) \epsilon^e_{\alpha} j^e(w) \Phi + L^a(z; w_0) j^b(w) \Phi \quad (B.4)$$

for the same reason of the equality of singularities and integrals over A-cycles. Using (B.1) this becomes

$$\langle j^a(z) j^b(w) \Phi \rangle = - k \Omega^{ab}(z, w) \langle \Phi \rangle $$

$$- \theta^a_d(z; w, w_0) f^{abc} \theta^c_d(w; \xi_\alpha, w_0) t^\alpha_a \langle \Phi \rangle $$

$$- \theta^a_d(z; w, w_0) f^{abc} \theta^c_d(w; \xi_\alpha, w_0) t^\alpha_a \langle \Phi \rangle $$

$$+ \theta^a_e(z; \xi_\alpha, w_0) \theta^e_d(w; \xi_\beta, w_0) t^\beta_e \langle \Phi \rangle $$

$$+ \theta^a_e(z; \xi_\alpha, w_0) \theta^e_d(w; \xi_\beta, w_0) t^\beta_e \langle \Phi \rangle $$

$$+ L^a(z; w_0) \theta^b_e(w; \xi_\beta, w_0) t^\beta_e \langle \Phi \rangle $$

$$+ L^a(z; w_0) L^b(w; w_0) \langle \Phi \rangle . \quad (B.5)$$

Introduce

$$\Xi^a(z; w_0) = f^{abc} \theta^b_e(z; w, w_0) = \langle j^a(z) \rangle_{N-c} . \quad (B.6)$$

Then, taking the limit $w \to z,$

$$\langle j^a(z) j^a(z) \rangle = - k \Omega^{aa}(z, z) \langle \Phi \rangle $$

$$+ \Xi^a(z; w_0) \theta^a_d(z; \xi_\alpha, w_0) t^\alpha_a \langle \Phi \rangle $$

$$+ \Xi^a(z; w_0) L^a(w; w_0) \langle \Phi \rangle $$

$$+ \theta^a_e(z; \xi_\alpha, w_0) \theta^e_d(z; \xi_\beta, w_0) t^\alpha_e t^\beta_d \langle \Phi \rangle $$

$$+ \theta^a_e(z; \xi_\alpha, w_0) \theta^e_d(z; \xi_\beta, w_0) t^\alpha_e t^\beta_d \langle \Phi \rangle $$

$$+ L^a(z; w_0) \theta^a_d(z; \xi_\beta, w_0) t^\beta_d \langle \Phi \rangle $$

$$+ L^a(z; w_0) L^a(z; w_0) \langle \Phi \rangle . \quad (B.7)$$
By the Sugawara construction (2.4) and the identity (4.32) we have
\[ \Xi^a(z; w_0) = \{ 2L^a(z; w_0) \log \Pi + L^b \omega^b_0(z; w_0) \}. \]  
(B.8)
The proof is based on checking that r.h.s. and l.h.s. are twisted 1-forms with additive B-periods with equal jumps at A-cycles and integrals over A-cycles.
This allows us to rewrite (B.7) as
\[
\langle j^a(z)j^a(z) : \Phi \rangle = \left[ -k\Omega^{aa}(z, z)_{\text{reg}} \right. \\
+ \left( L^a(z; w_0) + t^b_a \theta^a_0(z; \xi_a, w_0) \right) \left( L^a(z; w_0) + t^b_\beta \theta^a_\beta(z; \xi_\beta, w_0) \right) \\
+ 2 \left\{ L^a(z; w_0) \log \Pi \right\} \left( L^a(z; w_0) + t^b_\beta \theta^a_\beta(z; \xi_\beta, w_0) \right) \langle \Phi \rangle. 
\]  
(B.9)
By the Sugawara construction (2.4) and the identity (4.32) we have
\[
d_m(z) \langle (\Phi) \Pi \rangle = \frac{-1}{2(k + h^*)} \left[ h^* \Pi \Omega^{aa}(z, z)_{\text{reg}} \right. \\
+ \Pi \left( L^a(z; w_0) + t^b_a \theta^a_0(z; \xi_a, w_0) \right) \left( L^a(z; w_0) + t^b_\beta \theta^a_\beta(z; \xi_\beta, w_0) \right) \\
+ 2 \left\{ L^a(z; w_0) \Pi \right\} \left( L^a(z; w_0) + t^b_\beta \theta^a_\beta(z; \xi_\beta, w_0) \right) \langle \Phi \rangle \\
- \langle \Phi \rangle L^a(z; w_0) L^a(z; w_0) \Pi \\
\left. \right] \langle \Phi \rangle \Pi \\
= \frac{-1}{2(k + h^*)} \left( L^a(z; w_0) + t^b_a \theta^a_0(z; \xi_a, w_0) \right) \left( L^a(z; w_0) + t^b_\beta \theta^a_\beta(z; \xi_\beta, w_0) \right) + U(z) \right) \langle \Phi \rangle \Pi, \\
\right.
(B.10)
where
\[ U(z) = \{ h^* \Omega^{aa}(z, z)_{\text{reg}} - \frac{1}{2} \Pi \bar{L}^a(z; w_0) L^a(z; w_0) \Pi \}. \]  
(B.11)

Appendix C. "Potential" term is a closed 1-form on the moduli space.

Closeness of the potential term \( U(z) \) is equivalent to
\[ d_m(w) U(z) - d_m(z) U(w) = 0, \]  
(C.1)
where the operators \( d_m(z) \) should be understood as in (5.16), (5.17). Before proving this let us derive several useful identities. We shall use the notation introduced in Appendix B.
First, we wish to prove that
\[
L^a(z; w_0) \Omega^{bc}(w, \xi) - L^b(w; w_0) \Omega^{ac}(z, \xi) \\
+ \left[ \Omega^{ad}(z, \xi) \theta^b_0(w; \xi, w_0) - \Omega^{ad}(w, \xi) \theta^a_0(z; \xi, w_0) \right] f^{cde} \\
+ \Omega^{ec}(w; \xi) \theta^b_\beta(z; w, w_0) f^{ade} - \Omega^{ec}(z, \xi) \theta^b_\beta(w; z, w_0) f^{ade} = 0 \]  
(C.2)
The proof consists of the following steps: (i) Observe that l.h.s. is a twisted 1-form in \( z, a \). To check this we use (4.11) along with the identity
\[ g t^a g^{-1} = (g)_a^b t^b \]  
(C.3)
in the adjoint representation. (ii) L.h.s. is regular at \( z = w \). (iii) L.h.s. is regular at \( z = \xi \). (iv) Integrals over A-cycles vanish. Assuming that \( w, \xi, w_0 \) belong to the fundamental domain,

\[
\oint_{A_i} (l.h.s.)dz = L^i_{ab} \Omega^{bc}(w, \xi) - \oint_{A_i} \Omega^{ce}(z, \xi) \theta^b_d(w; z, w_0) f^{ade} dz.
\]  

(C.4)

We can prove that this is identically zero by considering correlation functions in the twisted b-c system:

\[
L^i_{ab} \Omega^{bc}(w, \xi) = L^i_{ab} (b^b(w) \partial c^c(\xi))_{N}^{b-c} \\
= (\oint_{A_i} j^a(z) dz b^b(w) \partial c^c(\xi))_{N}^{b-c} - (\oint_{A_i} j^a(z) dz)_{N}^{b-c} (b^b(w) \partial c^c(\xi))_{N}^{b-c}.
\]

(C.5)

Recalling that \( j^a(z) = f^{abc} b^b(z) c^c(z) \) and using the Wick rule, we arrive to

\[
L^i_{ab} \Omega^{bc}(w, \xi) = -f^{ade} \oint_{A_i} dz (b^b(w) c^c(z))_{N}^{b-c} (b^d(z) \partial c^c(\xi))_{N}^{b-c} \\
= -f^{ade} \oint_{A_i} dz \Omega^{dc}(z, \xi) \theta^b_d(w; z, w_0),
\]

(C.6)

which finishes the proof.

We can obtain more identities by integrating (C.2) in \( \xi \) along different paths. If \( \xi \) runs from \( w_0 \) to \( u \), we have

\[
L^i(z; w_0) \theta^b_d(w; u, w_0) - L^i_b(w; w_0) \theta^a_d(z; u, w_0) \\
+ \theta^a_d(z; u, w_0) \theta^b_d(w; u, w_0) f^{ade} \\
+ \theta^a_d(w; u, w_0) \theta^b_d(z; w, w_0) f^{bde} - \theta^a_d(z; u, w_0) \theta^b_d(w; z, w_0) f^{ade} = 0
\]

(C.7)

Setting \( u = \gamma(w_0) \), this easily leads to the commutation relation

\[
[L^i(z; w_0), L^b(z; w_0)] = f^{acd} \theta^b_c(w; z, w_0) L^d(z; w_0) - f^{bed} \theta^a_c(z; w, w_0) L^d(w; w_0).
\]

(C.8)

Let us now return back to proving (C.1). Since we are interested in the antisymmetric with respect to interchanging \( z \leftrightarrow w \) part, we shall use the symbol \( \simeq \) to indicate that the antisymmetric parts of the two expressions are equal.

\[
d_m(z) U(w) = h^a d_m(z) \Omega^{aa}(w, w)_{reg} - \frac{1}{2} d_m(z) \left( \frac{1}{\Pi} \tilde{L}^a(w; w_0) L^a(w; w_0) \Pi \right).
\]

(C.9)

The first summand has zero antisymmetric part, since \( \Omega^{aa}(w, w)_{reg} = -2d_m(w) \log \Pi \) is an exact 1-form on the moduli space, and \( d_m^2 = 0 \). Also, since

\[
\tilde{L}^a(w; w_0) L^a(w; w_0) = d_m(w) \left( L^{ib} \int_{w_0}^{\gamma(w_0)} d\xi \int_{w_0}^{\gamma(w_0)} d\eta \Omega^{bc}(\xi, \eta) L^{jc} \right),
\]

(C.10)

we arrive to

\[
d_m(z) U(w) \simeq -\frac{1}{2} \left[ \left( d_m(z) \frac{1}{\Pi} \tilde{L}^b(w; w_0) L^b(w; w_0) \Pi + \frac{1}{\Pi} \tilde{L}^b(w; w_0) L^b(w; w_0) \left( d_m(z) \Pi \right) \right) \right] \\
= \frac{1}{4} \left[ -\Omega^{aa}(z, z)_{reg} \tilde{L}^b(w; w_0) L^b(w; w_0) \Pi + \frac{1}{\Pi} \tilde{L}^b(w; w_0) L^b(w; w_0) \left( \Pi \Omega^{aa}(z, z)_{reg} \right) \right] \\
= \frac{1}{4} \left[ \frac{1}{2} \left\{ \tilde{L}^b(w; w_0) \Omega^{aa}(z, z)_{reg} \right\} \left\{ \tilde{L}^b(w; w_0) \Pi \right\} + \tilde{L}^b(w; w_0) L^b(w; w_0) \Omega^{aa}(z, z)_{reg} \right] \\
= \frac{1}{4} \left[ \tilde{L}^b(w; w_0) L^b(w; w_0) \Omega^{aa}(z, z)_{reg} + \Xi^b(w; w_0) L^b(w; w_0) \Omega^{aa}(z, z)_{reg} \right]
\]

(C.11)
From (C.2) we deduce that

\[ L^b(w; w_0) \Omega^{aa}(z, z)_{\text{reg}} = L^a(z; w_0) \Omega^{ba}(w, z) + \left[ \Omega^{ea}(w, z) \theta^a_d(z; w, w_0) f^{bde} + \Omega^{bd}(w, z) \Xi^d(z; w_0) \right]. \] (C.12)

Using it in rewriting the first summand in (C.11), we obtain

\[ 4d_m(z) U(w) \simeq L^b(w; w_0) L^a(z; w_0) \Omega^{ab}(z, w) \]

\[ + L^b(w; w_0) \left[ \Omega^{ea}(w, z) \theta^a_d(z; w, w_0) f^{bde} + \Omega^{bd}(w, z) \Xi^d(z; w_0) \right] - \Xi^b(z; w_0) L^b(z; w_0) \Omega^{aa}(w, w)_{\text{reg}}. \] (C.13)

Commuting \( L^a(z; w_0) \) and \( L^b(w; w_0) \) in the first term according to (C.8),

\[ 4d_m(z) U(w) \simeq f^{bcd} \theta^a_c(z; w, w_0) L^d(w; w_0) \Omega^{ab}(z, w) - f^{cde} \theta^a_d(z; w, w_0) L^b(w; w_0) \Omega^{ae}(z, w) \]

\[ + \Omega^{ea}(w, z) L^b(w; w_0) \theta^a_d(z; w, w_0) f^{bde} \]

\[ + \Omega^{bd}(w, z) L^b(w; w_0) \Xi^d(z; w_0) \]

\[- \Xi^b(z; w_0) \left[ L^b(z; w_0) \Omega^{aa}(w, w)_{\text{reg}} - L^a(w; w_0) \Omega^{ba}(z, w) \right] = \Omega^{ea}(w, z) L^b(w; w_0) \theta^a_d(z; w, w_0) f^{bde} \]

\[ + \Omega^{bd}(w, z) L^b(w; w_0) \Xi^d(z; w_0) \]

\[- \Xi^b(z; w_0) \Omega^{aa}(z, w) \theta^a_d(z; w, w_0) f^{bde} \]

\[- \Xi^b(z; w_0) \Omega^{bb}(z, w) \Xi^d(z; w_0). \] (C.14)

The last term is symmetric, therefore,

\[ 4d_m(z) U(w) \simeq \Omega^{ea}(w, z) L^b(w; w_0) \theta^a_d(z; w, w_0) f^{bde} \]

\[ - \Omega^{ea}(w, z) L^a(z; w_0) \theta^a_d(w; w, w_0) f^{bde} \]

\[ - \Xi^b(z; w_0) \Omega^{ea}(z, w) \theta^a_d(z; w, w_0) f^{bde} \]

\[ = \Omega^{ea}(w, z) \left[ \theta^a_f(z; w, w_0) \theta^b_f(w; w, w_0)_{\text{reg}} f^{dfe} \right] \]

\[ + \theta^a_f(w; w, w_0)_{\text{reg}} \theta^a_d(z; w, w_0) f^{fde} - \theta^a_d(z; w, w_0) \theta^b_f(w; w, w_0) f^{afe} \]

\[ - \Xi^b(z; w_0) \Omega^{ea}(z, w) \theta^a_d(w; w, w_0) f^{bde} \]

\[ \simeq \Omega^{ea}(w, z) \theta^a_f(z; w, w_0) \theta^b_f(w; w, w_0)_{\text{reg}} f^{dfe} f^{dfe} \]

\[ + \Omega^{ea}(w, z) \theta^a_d(z; w, w_0)_{\text{reg}} \theta^a_f(z; w, w_0) f^{dfe} f^{dfe} \]

\[- f^{ehf} f^{beg} - f^{bde} f^{heg} \]

\[ + \Omega^{ea}(w, z) \Xi^b(w; w_0) \theta^a_d(z; w, w_0) f^{bde} = 0. \] (C.15)

We used (C.7) and the Jacobi identity

\[ f^{abc} f^{ade} + f^{abd} f^{ace} + f^{abe} f^{acd} = 0. \] (C.16)

Appendix D. Compatibility of the KZB equations for moving marked points.

We check the compatibility of the equations for the derivatives of the conformal block with respect to the coordinates of marked points. It is slightly simplifies computations if we deal with the equations for "bare" conformal blocks \( F = \langle \Phi_1(\xi_1) \ldots \Phi_n(\xi_n) \rangle \), not for (IF). The equations for \( F \) look like follows:

\[ \left[ \frac{\partial}{\partial \xi_\alpha} + \frac{1}{k + h^*} t^\alpha_\beta(\xi_\beta; \xi_\alpha, w_0)_{\text{reg}} t^\beta_\alpha(w_0; \xi_\alpha) + L^a(\xi_\alpha; w_0) \right] F = 0. \] (D.1)
The regularization of $\theta_0^a(\xi_\alpha; \xi_\beta, w_0)$ is assumed when $\beta = \alpha$. Our goal is to show that the $n$-point operators

$$A_\alpha^{(n)} = t_\alpha^a \left( \theta_0^a(\xi_\alpha; \xi_\beta, w_0)_{\text{reg}} t_\beta^b + L^a(\xi_\alpha; w_0) \right) \quad (D.2)$$

commute:

$$[A_\alpha^{(n)}, A_\beta^{(n)}] = 0. \quad (D.3)$$

The commutativity property is sufficient to check only for 2- and 3-point operators. The structure of the operators $A_\alpha^{(n)}$ ensures that any $n$-point operators will then also commute. One easily observes this property for the KZ equations (1.1). In this case the commutativity of the 2-point operators is trivial, since

$$A_\alpha^{(2)} = - A_\beta^{(2)} \quad (D.4)$$

for two marked points $\alpha$ and $\beta$. The 3-point operators commute due to the classical Yang-Baxter equation

$$[R_{\alpha\beta}^{\text{sphere}}, R_{\beta\gamma}^{\text{sphere}}] + [R_{\beta\gamma}^{\text{sphere}}, R_{\gamma\alpha}^{\text{sphere}}] + [R_{\gamma\alpha}^{\text{sphere}}, R_{\alpha\beta}^{\text{sphere}}] = 0, \quad (D.5)$$

where the R-matrix is

$$R_{\alpha\beta}^{\text{sphere}} = \frac{t_\alpha^a t_\beta^b}{\xi_\alpha - \xi_\beta}. \quad (D.6)$$

This argument slightly changes in the case of higher genera. Let

$$S_\alpha = t_\alpha^a L^a(\xi_\alpha; w_0) + \theta_0^a(\xi_\alpha; \xi_\beta, w_0)_{\text{reg}} t_\beta^b, \quad (D.7)$$

$$R_{\alpha\beta} = \theta_0^a(\xi_\alpha; \xi_\beta, w_0) t_\alpha^b t_\beta^c. \quad (D.8)$$

(no summation over $\alpha$ and $\beta$). Then

$$A_\alpha^{(n)} = S_\alpha + \sum_{\beta \neq \alpha} R_{\alpha\beta}. \quad (D.9)$$

The commutativity of the 2-point operators implies

$$[S_\alpha + R_{\alpha\beta}, S_\beta + R_{\beta\alpha}] = 0. \quad (D.10)$$

For the 3-point operators the commutativity means

$$[A_\alpha^{(2)} + R_{\alpha\gamma}, A_\beta^{(2)} + R_{\beta\gamma}] = 0, \quad (D.11)$$

where $A_\alpha^{(2)}$ and $A_\beta^{(2)}$ are the 2-point operators for the marked points $\xi_\alpha$ and $\xi_\beta$. Up to (D.10) this is equivalent to

$$[A_\alpha^{(2)}, R_{\beta\gamma}] + [R_{\alpha\gamma}, A_\beta^{(2)}] + [R_{\alpha\gamma}, R_{\beta\gamma}] = 0 \quad (D.12)$$

or being rewritten in another way:

$$[R_{\alpha\beta}, R_{\beta\gamma}] + [R_{\alpha\gamma}, R_{\beta\gamma}] + [R_{\alpha\gamma}, R_{\beta\alpha}] = [R_{\beta\gamma}, S_\alpha] + [S_\beta, S_\alpha]. \quad (D.13)$$

This is a generalization of the classical Yang-Baxter equation (D.5) to higher genera. Notice that here we do not have the symmetry $R_{\alpha\beta} = - R_{\beta\alpha}$ any more.

Our proof of the compatibility condition (D.3) will proceed in two steps. First we check (D.13), then (D.10). The generalized classical Yang-Baxter equation (D.13) follows from (C.7):

$$[R_{\alpha\beta}, R_{\beta\gamma}] + [R_{\alpha\gamma}, R_{\beta\gamma}] + [R_{\alpha\gamma}, R_{\beta\alpha}] - [R_{\beta\gamma}, S_\alpha] - [S_\beta, S_\alpha]$$

$$= t_\alpha^a t_\beta^b t_\gamma^c \left( \theta_0^a(\xi_\alpha; \xi_\gamma, w_0) \theta_0^b(\xi_\beta; \xi_\gamma, w_0) f_{\text{de}}^{\gamma} \right.$$  

$$+ \theta_0^a(\xi_\alpha; \xi_\beta, w_0) \theta_0^c(\xi_\gamma; \xi_\beta, w_0) f_{\text{bd}}^{\gamma} - \theta_0^a(\xi_\beta; \xi_\alpha, w_0) \theta_0^c(\xi_\gamma; \xi_\alpha, w_0) f_{\text{bd}}^{\gamma}$$

$$+ L^a(\xi_\alpha; w_0) \theta_0^b(\xi_\beta; \xi_\gamma, w_0) - L^b(\xi_\beta; w_0) \theta_0^a(\xi_\gamma; \xi_\alpha, w_0) \right)$$

$$= 0. \quad (D.14)$$
Commuting the 2-point operators, obtain
\[
[S_\alpha + R_\alpha \beta, S_\beta + R_\beta \alpha] = K^{ab}_{ic} \partial_{\alpha}^{a} \partial_{\beta}^{b} L^{ic} + X^{abc} \partial_{\alpha}^{a} \partial_{\beta}^{b} \tilde{t}^{ic} + \tilde{X}^{abc} \partial_{\beta}^{b} \tilde{t}_{\alpha}^{c},
\]  
(D.15)
where
\[
K^{ab}_{ic} = f^{ade} \partial_{e}^{b} (\xi; \xi_{\alpha}, w_{0}) \omega^{d}_{ic} (\xi_{\alpha}; w_{0}) - f^{bde} \partial_{e}^{a} (\xi; \xi_{\beta}, w_{0}) \omega^{d}_{ic} (\xi_{\beta}; w_{0})
\]
\[
+ \omega^{a}_{id} (\xi_{\alpha}; w_{0}) \omega^{b}_{ie} (\xi_{\beta}; w_{0}) f^{cde}
\]
\[
+ L^{a} (\xi_{\alpha}; w_{0}) \omega^{b}_{ic} (\xi_{\beta}; w_{0}) - L^{b} (\xi_{\beta}; w_{0}) \omega^{a}_{ic} (\xi_{\alpha}; w_{0})
\]  
= 0.
\]  
(D.16)
\[
X^{abc} (\xi_{\alpha}, \xi_{\beta}) = - \tilde{X}^{abc} (\xi_{\beta}, \xi_{\alpha})
\]
\[
= f^{bde} \partial_{e}^{b} (\xi_{\alpha}; \xi_{\beta}, w_{0}) \omega^{d}_{ic} (\xi_{\beta}; \xi_{\alpha}, w_{0}) + f^{ade} \partial_{e}^{a} (\xi_{\beta}; \xi_{\alpha}, w_{0}) \omega^{d}_{ic} (\xi_{\alpha}; \xi_{\beta}, w_{0})
\]
\[
- L^{a} (\xi_{\beta}; w_{0}) \omega^{b}_{ic} (\xi_{\alpha}; \xi_{\alpha}, w_{0}) + L^{a} (\xi_{\alpha}; \xi_{\alpha}, w_{0}) \omega^{b}_{ic} (\xi_{\beta}; \xi_{\alpha}, w_{0})
\]
\[
+ f^{cde} \partial_{e}^{a} (\xi_{\beta}; \xi_{\alpha}, w_{0}) \omega^{d}_{ic} (\xi_{\alpha}; \xi_{\alpha}, w_{0})
\]  
= 0.
\]  
(D.17)

again due to (C.7). This completes our proof of (D.3).

Appendix E. Compatibility of the equations
for the partition function.

We explicitly show that in the case of no insertions (equations for the partition function) the KZB equations are compatible (the connection is flat). This follows from
\[
[d_{m} (z) A_{KZB}, d_{m} (w) A_{KZB}] = 0.
\]  
(E.1)
We prove it in this Appendix.

Let
\[
A (z) = d_{m} (z) A_{KZB} = \Delta_{B} (z) + U (z),
\]  
(E.2)
where
\[
\Delta_{B} (z) = L^{a} (z; w_{0}) \omega^{a} (z; w_{0})
\]  
(E.3)
is a symmetric second order differential operator, \(U (z)\) is the potential term,
\[
U (z) = h^{*} \Omega^{a} (z, z) \omega^{a} (z, w_{0}) - \frac{1}{\Pi} \Delta_{B} (z) \Pi
\]
\[
= h^{*} \Omega^{a} (z, z) \omega^{a} (z, w_{0}) - \{ \Delta_{B} (z) \log \Pi \} - \{ L^{a} (z) \log \Pi \}^{2}.
\]  
(E.4)
From now on we shall omit the dependence on the auxiliary point \(w_{0}\) in all expressions, so that \(L^{a} (z) \equiv L^{a} (z; w_{0})\), etc. Commuting \(A (z)\) and \(A (w)\), we obtain
\[
[A (z), A (w)] = [\Delta_{B} (z), \Delta_{B} (w)] + [\Delta_{B} (z), U (w)] - [\Delta_{B} (w), U (z)]
\]
\[
= [\Delta_{B} (z), \Delta_{B} (w)] + \left\{ L^{a} (z) U (w) \right\} L^{a} (z) + L^{a} (z) U (w) - \left\{ L^{a} (w) U (z) \right\} L^{a} (w) - L^{a} (w) L^{a} (w) U (z).
\]  
(E.5)
We wish to prove this to be zero.
First compute
\[
\Delta_B(z, \Delta_B(w)) = \bar{L}^a(z)\bar{L}^b(w)[L^a(z), L^b(w)] \\
+ [\bar{L}^a(z), \bar{L}^b(w)]L^a(w)L^a(z) \\
+ \bar{L}^a(z)[L^a(z), \bar{L}^b(w)]L^b(w) \\
+ \bar{L}^b(w)[L^a(z), \bar{L}^b(w)]L^a(z) \\
= \bar{L}^a(z)\bar{L}^b(w)f^{acd}\theta^e_c(w; z)\bar{L}^d(z) \\
- \bar{L}^a(z)\bar{L}^b(w)f^{bdc}\theta^e_c(z; w)L^d(w) \\
+ \bar{L}^d(z)f^{acd}\theta^e_c(w; z)L^b(w)L^a(z) \\
- \bar{L}^d(w)f^{bdc}\theta^e_c(z; w)L^b(w)L^a(z) \\
+ \bar{L}^a(z)f^{acd}\theta^e_c(w; z)L^d(z)L^b(w) \\
- \bar{L}^a(z)\bar{L}^b(w)\theta^e_c(z; w)L^d(w) \\
+ \bar{L}^a(z)\left\{L^a(z)\bar{L}^b(w) + \bar{L}^d(w)f^{bdc}\theta^e_c(z; w)\right\}L^b(w) \\
+ \bar{L}^b(w)\bar{L}^d(z)f^{acd}\theta^e_c(w; z)L^a(z) \\
- \bar{L}^b(w)f^{bdc}\theta^e_c(z; w)L^d(w)L^a(z) \\
- \bar{L}^d(w)\left\{L^a(z)\bar{L}^b(w) + \bar{L}^d(w)f^{bdc}\theta^e_c(z; w)\right\}L^a(z). 
\]

(E.6)

In the above transformations we used the commutation relation (C.8) and the one transposed to it. In the last expression the second term cancels the sixth one, the fourth term cancels the ninth one. We denote the expression in braces by \( P^{ab}(z, w) \). Notice that
\[
P^{ab}(z, w) = \{L^a(z)\bar{L}^b(w) + \bar{L}^d(w)f^{bdc}\theta^e_c(z; w)\} \\
= \{L^b(w)L^a(z) + \bar{L}^d(z)f^{acd}\theta^e_c(w; z)\} = P^{ba}(w, z). 
\]

(E.7)

Before going further let us state a simple consequence of the identity (C.7). Namely, setting \( u \to w \) we obtain
\[
L^a(z)\theta^b_c(w; w)_{\text{reg}} = L^b(w)\theta^a_c(z; w) \\
= -\theta^a_c(z; w)\theta^b_c(w; w)_{\text{reg}}f^{ade} \\
- \theta^a_c(z; w)\theta^b_c(w; w)_{\text{reg}}f^{bde} \\
+ \theta^a_c(w; z)\theta^b_c(z; w)f^{ade} \\
+ f^{bdc}\Omega^{ad}(z, w). 
\]

(E.8)

This lemma allows us to rewrite \( P^{ab}(z, w) \) as
\[
P^{ab}(z, w) = \{-2L^b(w)L^a(z)\log\Pi - L^b(w)f^{acd}\theta^e_c(z; w) \\
+ f^{acd}\theta^b_c(w; z)\bar{L}^d(z) + \bar{L}^d(z)f^{acd}\theta^b_c(w; z)\} \\
= \{-2L^b(w)L^a(z)\log\Pi - 2f^{acd}\theta^b_c(w; z)\bar{L}^d(z)\log\Pi \\
- f^{bdc}\theta^a_c(z; w)\theta^b_c(w; z) + 2h^*\Omega^{ab}(z, w)\}. 
\]

(E.9)

We shall further use this expression, and now return to (E.6), which after cancellations and regrouping the remaining terms becomes
\[
\Delta_B(z, \Delta_B(w)) = [\bar{L}^a(z), \bar{L}^b(w)]f^{acd}\theta^e_c(w; z)\bar{L}^d(z) - \bar{L}^d(z)f^{acd}\theta^b_c(w; z)L^a(z) \\
+ \bar{L}^a(z)P^{ab}(z, w)\bar{L}^b(w) - \bar{L}^b(w)P^{ab}(z, w)L^a(z) \\
= \bar{L}^f(z)f^{acf}\theta^e_c(w; z)f^{acd}\theta^b_c(w; z)\bar{L}^d(z) \\
- \bar{L}^d(z)f^{acd}\theta^b_c(w; z)f^{acd}\theta^e_c(z; w)\bar{L}^f(z) \\
+ \bar{L}^d(z)\left\{P^{df}(z, w) + f^{bef}\theta^a_c(z; w)f^{acd}\theta^b_c(w; z)\right\}\bar{L}^f(w) \\
+ \bar{L}^f(w)\left\{P^{df}(z, w) + f^{bef}\theta^a_c(z; w)f^{acd}\theta^b_c(w; z)\right\}\bar{L}^d(z). 
\]

(E.10)
The first two terms of the last expression cancel each other due to the Jacobi identity (C.16). Introduce now one more quantity \( Q^{ab}(z, w) \) defined by

\[
Q^{ab}(z, w) = P^{ab}(z, w) + f^{acf} f^{bfe} \theta^d_c(z; w) \theta^f_e(w; z).
\]  

(E.11)

Like for \( P^{ab}(z, w) \) we have

\[
Q^{ab}(z, w) = Q^{ba}(w, z).
\]  

(E.12)

The expression (E.9) for \( P^{ab}(z, w) \) gives

\[
Q^{ab}(z, w) = 2\{h^* \Omega'^{ab}(z, w) - L^b(w)L^a(z) \log \Pi - f^{acd} \theta^b_c(w; z)L^d(z) \log \Pi\}.
\]  

(E.13)

With this notation the commutator (E.10) becomes

\[
[\Delta_B(z), \Delta_B(w)] = \tilde{L}^a(z)Q^{ab}(z, w)L^b(w) - \tilde{L}^b(w)Q^{ab}(z, w)L^a(z) = \bigg\{ \tilde{L}^a(z)Q^{ab}(z, w)L^b(w) - \tilde{L}^b(w)Q^{ab}(z, w)L^a(z) \bigg\} = \bigg\{ \tilde{L}^a(z)Q^{ad}(z, w) - Q^{ab}(z, w)f^{bcd} \theta^e_c(w; z) \bigg\} L^d(w) - \{ F^a(w)Q^{ad}(w, z) - Q^{ab}(w, z)f^{bcd} \theta^e_c(w; z) \} L^d(z).
\]  

(E.14)

This is a first order differential operator. It is antisymmetric although this is not seen at once from (E.14). Therefore we can restrict our attention to its symbol only. Returning to the original problem (E.1), we arrive to

\[
[A(z), A(w)] = Y^d(z, w)L^d(z) + \tilde{L}^d(z)Y^d(z, w) - Y^d(w, z)L^d(w) - \tilde{L}^d(w)L^d(z),
\]  

where

\[
Y^d(z, w) = \bigg\{ L^d(z)U(w) - \frac{1}{2} \left( \tilde{L}^a(w)Q^{ad}(w, z) - Q^{ab}(w, z)f^{bcd} \theta^e_c(w; z) \right) \bigg\}
\]  

(E.16)

We shall prove that

\[
Y^d(z, w) = 0,
\]  

(E.17)

then (E.1) will follow.

Writing out \( Y^d(z, w) \) explicitly,

\[
Y^d(z, w) = h^* \left\{ L^d(z)\Omega^{aa}(w, w)_{reg} - \tilde{L}^a(w)\Omega^{da}(z, w) + \Omega^{ab}(w, z)f^{bcd} \theta^e_c(w; z) \right\} - L^d(z) \left( \frac{1}{\Pi} \tilde{L}^b(w)L^b(w) \Pi \right) + \tilde{L}^a(w)L^d(z)L^a(w) \log \Pi + \tilde{L}^a(w)f^{abc} \theta^d_c(z; w)L^b(w) \log \Pi - f^{bcd} \theta^e_c(w; z)L^b(z)L^a(w) \log \Pi - f^{bcd} \theta^e_c(w; z)f^{efg} \theta^d_e(z; w)L^f(w) \log \Pi.
\]  

(E.18)

Transforming the expression in braces, we arrive to

\[
\left\{ L^d(z)\Omega^{aa}(w, w)_{reg} - \tilde{L}^a(w)\Omega^{da}(z, w) + \Omega^{ab}(w, z)f^{bcd} \theta^e_c(w; z) \right\} = \left\{ -\Omega^{da}(z, w)\tilde{L}^a(w) + \Omega^{ba}(z, w)f^{bcd} \theta^e_c(w; z) + \Omega^{ae}(z, w)\theta^d_e(w; z) f^{df} + \Omega^{df}(z, w)\Xi_f(w) \right\} = \left\{ 2\Omega^{df}(z, w)L^f(w) \log \Pi \right\}.
\]  

(E.19)
Two next summands in (E.18) become

\[
\tilde{L}^a(w)L^d(z)\bar{L}^a(w)\log \Pi - L^d(z) \left( \frac{1}{\Pi} \tilde{L}^b(w)L^b(w)\Pi \right)
\]

\[
= [\tilde{L}^a(w), L^d(z)]\bar{L}^a(w)\log \Pi - 2 \left\{ L^a(w)\log \Pi \right\} L^d(z)\bar{L}^a(w)\log \Pi
\]

\[
= \tilde{L}^b(w)f^{acb}\theta^d_c(z; w)L^a(w)\log \Pi
\]

\[- f^{dcb}\theta^a_c(w; z)\bar{L}^b(z)L^a(w)\log \Pi
\]

\[- \left\{ L^d(z)\tilde{L}^a(w) + \tilde{L}^b(w)f^{acb}\theta^d_c(z; w) \right\} L^a(w)\log \Pi
\]

\[- \left\{ 2L^d(z)L^a(w)\log \Pi \right\} \bar{L}^a(w)\log \Pi
\]

\[
= \tilde{L}^b(w)\left( f^{acb}\theta^d_c(z; w)\bar{L}^a(w)\log \Pi
\]

\[- f^{dcb}\theta^a_c(w; z)\bar{L}^b(z)L^a(w)\log \Pi
\]

\[- \left\{ L^d(z)\Xi^a(w) + \tilde{L}^b(w)f^{acb}\theta^d_c(z; w) \right\} L^a(w)\log \Pi.
\]

The first two terms cancel the fourth and the fifth summands in (E.18) thus leading to

\[
Y^d(z, w) = 2h^* \left\{ \Omega^d(z, w)L^f(w)\log \Pi \right\}
\]

\[- \left\{ f^{efa}f^{cbd}\theta^f_c(z; w)\theta^b_c(z; w)
\]

\[+ L^d(z)\Xi^a(w) + \tilde{L}^b(w)f^{acb}\theta^d_c(z; w) \right\} L^a(w)\log \Pi.
\]  

(21)

Notice that

\[
L^d(z)\Xi^a(w) + \tilde{L}^b(w)f^{acb}\theta^d_c(z; w)
\]

\[= f^{acb} \left( L^b(w)\theta^d_c(z; w) - L^d(z)\theta^b_c(w; w)_{reg} \right)
\]

\[+ \theta^d_c(z; w)f^{cba} \left( \Xi^b(w) - 2L^b(w)\log \Pi \right).
\]  

(22)

The very last term \( \left\{ 2\theta^d_c(z; w)f^{cba}L^b(w)\log \Pi \right\} \) vanishes after multiplying by \( L^a(w)\log \Pi \). The first term in (E.22) can be rewritten according to (E.8), which leads to

\[
Y^d(z, w) = 2h^* \Omega^d(z, w)L^f(w)\log \Pi
\]

\[- \left\{ f^{efa}f^{cbd}\theta^f_c(z; w)\theta^b_c(z; w)
\]

\[+ \theta^d_c(z; w)f^{cba}\Xi^b(w)
\]

\[+ f^{acb} \left( \theta^d_c(z; w)f^{bfa}(w; w)_{reg}f^{efa}
\]

\[+ \theta^d_c(z; w)f^{efa}(w; w)_{reg}f^{bfa}
\]

\[- \theta^d_c(z; w)f^{bfa}(w; w)_{reg}f^{def} \right\}
\]

\[L^a(w)\log \Pi
\]

\[= 0.
\]

(23)

Here the first line cancels the last one; the second line — the sixth one; the third to fifth lines cancel due to the Jacobi identity.
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