Derivation of quantum theory with superselection rules

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We reconstruct finite-dimensional quantum theory with superselection rules, which can describe hybrid quantum-classical systems, from four purely operational postulates: symmetric sharpness, complete mixing, filtering, and local equality. It has been shown that each of the classical and fully quantum theories is singled out by an additional postulate.

I. INTRODUCTION

Although quantum theory has enjoyed great success as an approach for explaining the behavior of the microscopic world, we still lack a deep and intuitive understanding of its principles. This is in contrast to special and general relativity, which are based on simple physical principles. In 1932, von Neumann presented an abstract mathematical formulation of quantum theory based on complex Hilbert spaces [1]. However, as he pointed out “I do not believe in Hilbert space anymore” [2], it is thus noteworthy that this formulation is far from a clear understanding of the physical structure of quantum theory. For a proper understanding of the reason why the complex Hilbert space (or $C^*$-algebraic) formalism is relevant in describing the microscopic world, Birkoff and von Neumann [3], Zierler [4], Mackey [5], Jauch and Piron [6], Ludwig [7], and many other researchers have investigated the reconstruction of the mathematical structure of quantum theory (simply referred to as the derivation of quantum theory) from physically meaningful principles.

Around 2000, a research program was launched by Fuchs and Brassard (e.g., [8]), the goal of which was to reconstruct quantum theory from a few simple information-theoretic principles. Since then, an increasing number of studies have adopted an information-theoretic approach [9–31]. Many of these studies are based on the operational probabilistic theory (OPT) or other similar theories such as the generalized probabilistic theory [9–21]. An OPT offers a powerful approach for ensuring a deeper understanding of the operational and information-theoretic aspects of quantum processes. In 2001, Hardy proposed a novel approach for deriving ordinary quantum theory, i.e., quantum theory without superselection rules (which we denote as fully quantum theory), using five informational axioms [9]. However, some of these axioms are expressed in mathematical terms that have no clear physical underpinning. Subsequently, Dakic and Brukner [13] and Masanes and Mueller [14] attempted reconstruction approaches based on more sophisticated axioms. Chiribella et al. [15] and Hardy [16] succeeded in reconstructing fully quantum theory using purely operational postulates. To avoid technical difficulties resulting from infinite dimensions, many studies have focused on finite-dimensional quantum theory, which still exhibits all the essential quantum phenomena.

In quantum information science, effective use of classical systems in combination with fully quantum systems can be crucial. For instance, the control of a fully quantum system using classical information, such as the outcomes of measurements, is crucial in many fields (e.g., local operations and classical communication, quantum teleportation, and one-way quantum computing). Furthermore, fully quantum theory is arguably not self-contained as the outcome of a measuring process is essentially classical in nature. To deal with classical and quantum information in a unified way requires the application of quantum theory to handle hybrid quantum-classical systems. Several studies have been conducted on hybrid quantum-classical systems and we will mention only a few examples of such studies herein: the interaction between quantum and classical systems (e.g., [32, 33]), formulation of hybrid quantum-classical dynamics (e.g., [34–36]), quantumness of correlations in quantum states (e.g., [37, 38]), and quantum coherence (e.g., [39, 40]). However, it is noteworthy that all the results mentioned in the previous paragraph cannot be applied to a quantum theory having hybrid quantum-classical systems. Therefore, in this paper, we will derive quantum theory with superselection rules, which can describe classical systems, fully quantum systems, and hybrid quantum-classical systems, from purely operational postulates only.

Some properties that can be used as postulates (or axioms) to single out fully quantum theory are not satisfied in classical systems: an instance of this is the so-called purification postulate [15, 21, 41], which claims that every state has a purification. Moreover, there exist postulates that can be used to derive both fully quantum theory and classical probability theory (which we refer to as classical theory) but are not satisfied in hybrid quantum-classical systems; instances of such postulates are as follows: (i) there exists a deterministic reversible process between any pair of normalized pure states [9, 13, 14, 18]; (ii) each system is characterized by a natural number that is referred to as dimension or capacity [9, 14]; and (iii) there exists a deterministic reversible process producing any given permutation of any given maximal set of perfectly distinguishable normalized pure states [16, 19]. The derivation of quantum theory with superselection rules from an OPT is presented by Barnum and Wilce [42], Selby et al. [43], Wilce [44], and Jia [45]. However, in Refs. [42, 45], mathematical assumptions that are not stated in operational terms were adopted as postulates. Also, some postulates adopted in Refs. [43, 44] (e.g., symmetric purification or the existence of a conjugate system) seem to be difficult to intuitively comprehend, at least for readers that are unfamiliar with quantum
theory\(^1\). In contrast, the set of our postulates is stated in operational terms and provides an intuitive interpretation; in particular, each of our four postulates is easy to intuitively understand in the context of classical theory. Moreover, we show that each of the fully quantum and classical theories can be singled out using an additional operational postulate. In this paper, we consider only finite-dimensional systems.

The paper is organized as follows. In Sec. II, we present a review of the framework of OPTs. In Sec. III, we present a set of four operational postulates and overview the derivation of quantum theory with superselection rules (which we simply refer to as quantum theory subsequently). In Secs. IV and V, we consider an OPT satisfying the first three postulates and show that each state space is the cone of squares of a Euclidean Jordan algebra (EJA). In Sec. VI, we present a review of the fundamental properties of EJAs that will be used in Sec. VII. In Sec. VII, we discuss an OPT with the four postulates and show that quantum theory is the only theory consistent with these postulates.

II. OPERATIONAL PROBABILISTIC THEORY (OPT)

In this section, we introduce the framework of OPTs. The framework can be explained in several ways, leading to essentially almost the same formalism. Some basic facts about an OPT are reviewed. The proofs of some of the results are not presented in this paper; for the details, we refer the reader to Refs. [9–21, 41]. To provide an intuitive grasp of an operational interpretation, we will use diagrammatic representations, which is motivated by the work of Coecke, Abramsky, and others (see, e.g., [46–48]). Any diagrammatic representation can be faithfully and rigorously described in a mathematical formula.

A. Operational theory

First, we introduce an operational theory. This theory describes the compositional structure of physical processes.

1. Processes, states, and effects

An operational theory consists of a collection of systems and a collection of processes. Systems could represent a physical system such as a photon. Processes could represent a particular behavior of a physical device such as a beam splitter. Systems are labeled by capital letters from the beginning of the alphabet \((A, B, \ldots)\). Syst is defined as the set of all systems. Each process has an input system and an output system; a process having an input system \(A\) and an output system \(B\) is called a process from \(A\) to \(B\). Proc\(_{A\to B}\) is defined as the set of all processes from \(A\) to \(B\).

We introduce an ‘empty’ system, called a trivial system and denoted by \(I\). A process from \(I\) to \(A\) (which is a process with no input) is called a state preparation, or simply a state, of \(A\) and is denoted like \(|\rho\rangle\) in analogy to Dirac’s bra-ket notation (note that \(|\rho\rangle\) could be a mixed state, which will be defined later). \(St_A := \text{Proc}_{I\to A}\) is called the state space of system \(A\). Similarly, a process from \(A\) to \(I\) (i.e., a process with no output) is called an effect of \(A\) and is denoted like \(|e\rangle\). Eff\(_A := \text{Proc}_{A\to I}\) is called the effect space of system \(A\). An effect represents an event associated with a particular outcome of a measurement. A process from \(I\) to \(I\) (i.e., a process with no input and no output) is called a scalar. Let Scalar := \(\text{Proc}_{I\to I}\).

In diagrammatic terms, a process \(f \in \text{Proc}_{A\to B}\), a state \(|\rho\rangle \in St_A\), and an effect \(|e\rangle \in \text{Eff}_A\) and a scalar \(p \in \text{Scalar}\) are depicted as

\[
\begin{array}{c}
\text{\(f\)} \\
\text{\(|\rho\rangle\)} \\
\text{\(|e\rangle\)} \\
\text{\(p\)}
\end{array}
\]

Systems are represented by labeled wires (labels are often omitted). The trivial system \(I\) is represented by ‘no wire’. Processes are represented by boxes that have an input wire at the bottom and an output wire at the top. For a scalar, the box will be omitted. Diagrammatic representations can be considered as something like data flow diagrams with time increasing from the bottom to the top.

Example of quantum theory The state space \(St_A\) and the effect space \(\text{Eff}_A\) are isomorphic to \(\bigoplus_{i=1}^{k_A} S_+(C^n)\), where \(C\) is the set of all complex numbers and \(S_+(C^n)\) is the set of all complex positive semidefinite matrices of order \(n\). The natural numbers \(k, n_1, \ldots, n_k\) are determined by the system \(A\). \(N_A := \sum_{i=1}^{k_A} n_i\) is called the rank of \(A\). System \(A\) is called classical if \(k = N_A\) (i.e., \(St_A \cong Eff_A \cong \bigoplus_{i=1}^{N_A} S_+(C) \cong R_+^{N_A}\)) holds, where \(X \cong Y\) denotes that \(X\) is isomorphic to \(Y\), and \(R_+\) is the set of all nonnegative real numbers. Classical theory is a special case of quantum theory in which every system is classical. System \(A\) is called fully quantum if \(k = 1\) (i.e., \(St_A \cong Eff_A \cong S_+(C^{N_A})\)) holds. Fully quantum theory is a special case of quantum theory in which every system is fully quantum. Note that a system \(A\) with \(N_A = 1\) is classical and fully quantum. The state space of the trivial system, \(St_I = \text{Scalar}\), is isomorphic to \(S_+(C) \cong R_+\). When \(St_A \cong \bigoplus_{i=1}^{k_A} S_+(C^n)\) and \(St_B \cong \bigoplus_{j=1}^{k_B} S_+(C^n)\) hold, Proc\(_{A\to B}\) is isomorphic to the space of all completely positive (CP) maps from \(\bigoplus_{i=1}^{k_A} S(C^n)\) to \(\bigoplus_{j=1}^{k_B} S(C^n)\), where \(S(C^n)\) is the set of all complex Hermitian matrices of order \(n\). In the examples of quantum theories, we will identify a process with its corresponding CP map; in particular, we will identify a state (or effect) with the corresponding positive semidefinite matrix.
Two processes can be composed sequentially if the output system of one is equal to the input system of the other. The sequential composition of \( f \in \text{Proc}_{A \rightarrow B} \) and \( g \in \text{Proc}_{B \rightarrow C} \) is a process from \( A \) to \( C \), which is denoted as \( g \circ f \in \text{Proc}_{A \rightarrow C} \). The composition, \((e| \otimes |p) \in \text{Scalar} \), of a state \([|p| \otimes |r]\) in \( \text{St}_A \) and an effect \((e| \otimes |p) \in \text{Eff}_A \) is also denoted by \((e|p) \otimes (e|p)\). \( g \circ f \) and \((e|p) \otimes (e|p)\) are respectively depicted as

\[
\begin{array}{c|c}
|c & e \\
|g & f \\
\end{array},
\begin{array}{c|c}
|e & \rho \\
|f & p \\
\end{array}.
\]

(2)

Whenever we write \( g \circ f \) for two processes \( f \) and \( g \), we always assume that the output system of \( f \) and the input system of \( g \) are equal. Sequential composition is associative, i.e., \( h \circ (g \circ f) = (h \circ g) \circ f \) holds for any \( f \in \text{Proc}_{A \rightarrow B}, g \in \text{Proc}_{B \rightarrow C}, \) and \( h \in \text{Proc}_{C \rightarrow D} \), where \( f = f' \) denotes that \( f \) and \( f' \) are indistinguishable in an operational theory.

Two systems \( A \) and \( B \) can be composed in parallel to yield a new system, denoted \( A \oplus B \). The composition is associative, i.e., \( A \oplus (B \oplus C) = (A \oplus B) \oplus C \) holds for any \( A, B, C \in \text{Sys} \). \( A \oplus A \) and \( A \oplus I \) are identified with \( A \) itself. The parallel composition of \( f \in \text{Proc}_{A \rightarrow B} \) and \( g \in \text{Proc}_{C \rightarrow D} \), denoted as \( f \otimes g \), is a process from \( A \otimes C \) to \( B \otimes D \). Diagrammatically, \( f \otimes g \) is depicted as a pair of processes arranged side by side:

\[
\begin{array}{c|c|c}
|g & \rho & g \\
|f & p & f \\
\end{array}.
\]

(3)

The parallel composition of processes is also associative, i.e., \( f \otimes (g \otimes h) = (f \otimes g) \otimes h \) holds for processes \( f, g, \) and \( h \).

A collection of connected processes will be called a diagram. Assume that

\[
(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)
\]

(4)

holds for four any processes \( f_1, f_2, g_1, \) and \( g_2 \), which is diagrammatically represented as

\[
\begin{array}{c|c|c}
|g_1 & g_2 \\
|f_1 & f_2 \\
\end{array} =
\begin{array}{c|c|c}
|g_1 & g_2 \\
|f_1 & f_2 \\
\end{array}.
\]

(5)

where the auxiliary lines (dashed lines) are only intended to guide the eye.

**Example of quantum theory** The sequential composition, \( g \circ f \), of two processes \( f \in \text{Proc}_{A \rightarrow B} \) and \( g \in \text{Proc}_{B \rightarrow C} \) is the CP map defined as \( (g \circ f)[|\rho|] := g[f[|\rho|]] \) for any \(|\rho| \in \text{St}_A \). In particular, \((e| \otimes |p) = \text{Tr}[e(\cdot)|\rho|] \) holds, where \( \cdot \) indicates the matrix product. When \( \text{St}_A \equiv \bigoplus_{j=1}^{k_A} S_+(C^{m_A}) \) and \( \text{St}_B \equiv \bigoplus_{k=1}^{k_B} S_+(C^{m_B}) \) hold, \( \text{St}_{AB} \equiv \bigoplus_{j=1}^{k_A} \bigoplus_{k=1}^{k_B} S_+(C^{m_{AB}}) \) holds. The parallel composition, \(|\rho| \otimes |\sigma| \), of two states \(|\rho| \)

and \(|\sigma| \) is the tensor product of the matrices \([|\rho|] \) and \([|\sigma|] \). The parallel composition, \( f \otimes h \), of two processes \( f \in \text{Proc}_{A \rightarrow B} \) and \( h \in \text{Proc}_{C \rightarrow D} \) is the CP map defined as \( (f \otimes h)[|\rho| \otimes |\sigma|] := f[|\rho|] \otimes h[|\sigma|] \) for any \(|\rho| \in \text{St}_A \) and \(|\sigma| \in \text{St}_C \).

We present two examples of systems that are neither classical nor fully quantum. The first one is hybrid quantum-classical systems. Consider a classical system \( A \) and a fully quantum one \( B \) with \( N_A, N_B > 1 \). Since \( \text{St}_A \equiv \bigoplus_{j=1}^{N_A} S_+(C) \) and \( \text{St}_B \equiv S_+(C^{m_B}) \) hold, the state space of the composite system \( A \otimes B \) satisfies \( \text{St}_{AB} \equiv \bigoplus_{j=1}^{N_A} S_+(C^{m_{AB}}) \). The second one is a system \( C \) with superposition rules for the total number of particles. In such a system, coherent superpositions between states with a different number of particles cannot be observed. Assume that \( C \) consists of at most \( K \) particles. The state space \( \text{St}_C \) can be decomposed \( \text{St}_C = \bigoplus_{j=0}^{K} S_+(C^j) \), where \( S_+(C^j) \) is the subspace with the number of particles equal to \( j \). If each subspace \( S_+(C^j) \) can be thought of as a fully quantum system, then \( S_+(C^j) \equiv S_+(C^{m_n}) \) for some natural number \( m_n \). In this case, \( \text{St}_C \) is isomorphic to \( \bigoplus_{j=0}^{K} S_+(C^j) \).

3. Identity processes and swap processes

Assume that, for each system \( A \), there exists a 'do nothing' process \( \text{id}_A \), which is called an identity process and denoted by \( \text{id}_A \in \text{Proc}_{A \rightarrow A} \) or simply \( \text{id} \). Diagrammatically, \( \text{id}_A \) is depicted as

\[
\begin{array}{c|c}
\text{id} \\
\end{array}.
\]

(6)

\( \text{id}_A \) is depicted as the ‘empty space’. Identity processes satisfy the following conditions for any systems \( A \) and \( B \) and any process \( f \in \text{Proc}_{A \rightarrow B} \):

\[
\begin{align*}
(1) \quad f \circ \text{id}_A &= f = \text{id}_B \circ f \\
(2) \quad \text{id}_A \otimes \text{id}_B &= \text{id}_{AB} \\
(3) \quad f \otimes \text{id}_A &= f = \text{id}_B \otimes f \\
\end{align*}
\]

(7)

It follows that there is a unique identity process for each system. The above property (1) is depicted as:

\[
\begin{array}{c|c|c}
|f & f & f \\
\end{array} =
\begin{array}{c|c|c}
|f & f & f \\
\end{array}.
\]

(8)

where the auxiliary boxes indicate the identity processes. It is easily seen that, for any processes \( f \) and \( g \), \((\text{id} \circ f) \otimes \text{id} \otimes (\text{id} \circ g) \) holds, which is diagrammatically represented as

\[
\begin{array}{c|c|c}
|g & f & g \\
|f & g & f \\
\end{array}.
\]

(9)
We also assume that, for any systems $A$ and $B$, there exists a process $X_{A\otimes B} \in \text{Proc}_{A\otimes B \rightarrow B\otimes A}$, called a swap process and diagrammatically depicted as

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (1,0) {$B$};
  \node (C) at (1,1) {$A'$};
  \node (D) at (0,1) {$B'$};
  \draw[->] (A) to node [above] {$f$} (B);
  \draw[->] (B) to node [right] {$g$} (C);
  \draw[->] (C) to node [left] {$h$} (D);
  \draw[->] (D) to node [below] {$i$} (A);
\end{tikzpicture}
\end{array}
\]
(9)

such that

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (1,0) {$B$};
  \node (C) at (1,1) {$A'$};
  \node (D) at (0,1) {$B'$};
  \draw[->] (A) to node [above] {$f$} (B);
  \draw[->] (B) to node [right] {$g$} (C);
  \draw[->] (C) to node [left] {$h$} (D);
  \draw[->] (D) to node [below] {$i$} (A);
\end{tikzpicture}
\end{array}
\end{array}
\]
(10)

(i.e., $X_{A,B} \circ (f \otimes id_B) \circ X_{B,A} = id_B \otimes f$) holds for any systems $A$, $A'$, and $B$ and any process $f \in \text{Proc}_{A \rightarrow A'}$.

Example of quantum theory The identity process is the identity map. The swap process $X_{A,B}$ is the process satisfying $X_{A,B} \circ (|\psi\rangle \otimes |\phi\rangle) = (|\phi\rangle \otimes |\psi\rangle)$ for any $|\psi\rangle \in \text{St}_A$ and $|\phi\rangle \in \text{St}_B$.

Note that although no knowledge of category theory is required to read this paper, this theory provides a suitable framework to describe an operational theory. Specifically, an operational theory can be considered as a category with systems as objects and processes as morphisms. In particular, this category is a strict symmetric monoidal category with the tensor unit $I$ and the tensor product $\otimes$.

4. Properties of scalars

We introduce the following diagram consisting of two processes $u_1 \in \text{Proc}_{C \rightarrow A\otimes E}$ and $u_2 \in \text{Proc}_{B\otimes E \rightarrow D}$

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (1,0) {$B$};
  \node (C) at (1,1) {$C$};
  \node (D) at (0,1) {$D$};
  \draw[->] (A) to node [above] {$f$} (B);
  \draw[->] (B) to node [right] {$g$} (C);
  \draw[->] (C) to node [left] {$h$} (D);
  \draw[->] (D) to node [below] {$i$} (A);
\end{tikzpicture}
\end{array}
\end{array}
\]
(11)

This diagram, denoted as $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D}$, maps $f \in \text{Proc}_{A \rightarrow B}$ to $u(f) := u_2 \circ (f \otimes id_E) \circ u_1 \in \text{Proc}_{D \rightarrow C}$. Any process from $C$ to $D$ that includes $f \in \text{Proc}_{A \rightarrow B}$ is expressed in the form $u(f) \in \text{Proc}_{C \rightarrow D}$ with some diagram $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D}$; for example,

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (1,0) {$B$};
  \node (C) at (1,1) {$C$};
  \node (D) at (0,1) {$D$};
  \draw[->] (A) to node [above] {$f$} (B);
  \draw[->] (B) to node [right] {$g$} (C);
  \draw[->] (C) to node [left] {$h$} (D);
  \draw[->] (D) to node [below] {$i$} (A);
\end{tikzpicture}
\end{array}
\]
(12)

where $u_1$ and $u_2$ are, respectively, the processes enclosed by the lower and upper auxiliary boxes.

For a scalar $a$ and a process $f$, $a \otimes f$ is denoted by $af$ or $a \cdot f$. A process expressed in the form $af$ with a scalar $a$ and a process $f$ will be referred to as scalar multiplication of $f$. Two processes $f, g \in \text{Proc}_{A \rightarrow B}$ will be referred to as proportional, denoted by $f \propto g$, if there exists a scalar $a$ satisfying either $f = ag$ or $af = g$.

It follows that, for any scalar $a$, process $f \in \text{Proc}_{A \rightarrow B}$, and diagram $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D}$, $u(af) = a \cdot u(f)$ holds, which is diagrammatically represented as

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (1,0) {$B$};
  \node (C) at (1,1) {$C$};
  \node (D) at (0,1) {$D$};
  \draw[->] (A) to node [above] {$f$} (B);
  \draw[->] (B) to node [right] {$g$} (C);
  \draw[->] (C) to node [left] {$h$} (D);
  \draw[->] (D) to node [below] {$i$} (A);
\end{tikzpicture}
\end{array}
\end{array}
\]
(13)

This implies that any scalar can freely move around a diagram; for example, $[(af) \otimes g] \otimes h = [f \otimes (ag)] \otimes h = (f \otimes g) \otimes (ah) = a[(f \otimes g) \otimes h]$ holds for any processes $f$, $g$, $h$, and any scalar $a$. Consider the particular case in which $A = B = C = D = E = I$ and $u_2 = id_I$ and let $b := u_1 \in \text{Scalar}$; then, $u : \text{Scalar} \rightarrow \text{Scalar}$ satisfies $u(a) = a \cdot b$ for any $a \in \text{Scalar}$. Also, from $a = a \cdot id_I$ and Eq. (13), $u(a) = a \cdot u(id_I) = ab$ holds. Thus, for any two scalars $a$ and $b$, $a \cdot b = ab$ holds, i.e., the sequential and parallel compositions of two scalars are equal.

Example of quantum theory For any $a, b \in \text{Scalar} = R$, $a \cdot b = a \otimes b = a \cdot b$ is the multiplication of two nonnegative real numbers $a$ and $b$. $af$ with $a \in \text{Scalar}$ and $f \in \text{Proc}_{A \rightarrow B}$ is the CP map that satisfies $(af)[|\psi\rangle] = a \cdot f[|\psi\rangle]$ for any $|\psi\rangle \in \text{St}_A$.

B. OPT

Next, we review an OPT. This theory is an operational theory that assigns probabilities for processes.

1. Tests and feasible processes

A test is a mathematical model that represents the behavior of a physical device. A test having an input system $A$ and an output system $B$ consists of $k$ processes $f_1, \ldots, f_k \in \text{Proc}_{A \rightarrow B}$ that represent mutually exclusive and collectively exhaustive events. $f \in \text{Proc}_{A \rightarrow B}$ is called deterministic if $f$ is a test. Intuitively, a deterministic process represents an event that always happens. Test$_{A \rightarrow B}$ and Proc$_{A \rightarrow B}$ are, respectively, defined as the sets of all tests and deterministic processes from $A$ to $B$. Each element of Meas$_A \subseteq$ Test$_{A \rightarrow I}$ is called a measurement of $A$. Assume that, for each system $A$, there exist a deterministic state of $A$ and a deterministic effect of $A$.

As well as processes, tests can be composed sequentially and in parallel. The sequential composition of $\{ f_j \in$
\( \text{Proc}_{A \rightarrow B}^{j} \) and \( \{g_{l} \in \text{Proc}_{B \rightarrow C} \}_{l=1}^{M} \) are sets of processes \((g_{l} \circ f_{j})^{(l)}_{(j=1, l)} \in \text{Test}_{A \rightarrow C} \) is \( \{g_{l} \circ f_{j}^{(l)} \}_{(j=1, l)}^{M} \in \text{Test}_{A \rightarrow C} \). The parallel composition of \( \{f_{j} \in \text{Proc}_{A \rightarrow B}^{j} \}_{j=1}^{M} \) and \( \{h_{m} \in \text{Proc}_{C \rightarrow D}^{j} \}_{m=1}^{M} \in \text{Test}_{C \rightarrow D} \) is \( \{f_{j} \otimes h_{m}^{(j,m)} \}_{(j,m)=(1,1)}^{M} \in \text{Test}_{A \rightarrow B \otimes C \rightarrow D} \). Since the sequential composition, \( \{g \circ f \} \), of two tests \( \{f \} \) and \( \{g \} \) with \( f \in \text{Proc}_{A \rightarrow B}^{j} \) and \( g \in \text{Proc}_{B \rightarrow C}^{j} \) is a test, the sequential composition of two deterministic processes is deterministic. Also, the parallel composition of two deterministic processes is deterministic.

We will call a process \( f \) feasible if there exists a test including \( f \). Intuitively, a feasible process represents a randomly occurring event. Let \( \text{Proc}_{A \rightarrow B}^{j} \) be the set of all feasible processes from \( A \) to \( B \); then, \( \text{Proc}_{A \rightarrow B}^{j} \subseteq \text{Proc}_{A \rightarrow B}^{j} \). Obviously, it holds. Also, we define \( \text{St}_{A}^{F} := \text{Proc}_{A \rightarrow A}^{0} \), \( \text{Eff}_{A}^{F} := \text{Proc}_{A \rightarrow A}^{1} \), and \( \text{Scalar}^{F} := \text{Proc}_{A \rightarrow A}^{0} \).

**Example of quantum theory** A set of processes (i.e., CP maps), \( \{f_{j}\}_{j=1}^{k} \), is a test if and only if the CP map \( \sum_{j=1}^{k} f_{j} \) is trace-preserving (TP). A set of effects is \( \{\{e_{j}\} \in \text{Eff}_{A}^{k} \}_{j=1}^{k} \) with \( j \) and only if \( \sum_{j=1}^{k} (e_{j}) = \mathbb{1} \) holds, where \( \mathbb{1} \) is the identity matrix of order \( n \). This means that \( \Pi \) is a positive operator-valued measure. \( f \in \text{Proc}_{A \rightarrow B}^{D} \) means that \( f \) is a TP-CP map. Also, \( f \in \text{Proc}_{A \rightarrow B}^{D} \) means that \( f \) is a trace non-increasing CP map. In particular, \( \rho \in \text{St}_{A}^{F} \) is equivalent to \( \rho^{\pi} \leq 1 \), and \( (e) \in \text{Eff}_{A}^{F} \) is equivalent to \( (e) \leq \mathbb{1} \) holds, where \( H \leq H' \) (or \( H' \geq H \) with Hermitian matrices \( H \) and \( H' \) denotes that \( H' - H \) is positive semidefinite.

2. **Assigning Probabilities**

Assume that, for each feasible state \( \rho \), a probability for \( \rho \) to happen is assigned and is dependent only on \( \rho \). Let us denote this probability by \( \text{Pr}[\rho] \), where \( \text{Pr} \) is a function from \( \text{St}_{A}^{F} \) to \( [0, 1] \) (where \( [0, 1] \) denotes the set of all real numbers between \( 0 \) and \( 1 \), inclusive). Also, assume that each \( \rho \in \text{Scalar}^{F} \) is identified with \( \text{Pr}(|\rho|) \) and that \( \text{Scalar}^{F} = [0, 1] \) holds. This means that any feasible scalar can be interpreted as a probability. A feasible state \( |\rho| \) satisfying \( \text{Pr}[|\rho|] = 1 \) is called a normalized state. Let \( \text{St}_{A}^{E} \) be the set of all normalized states of \( A \).

Assume that, for any \( A, B, E \in \text{Syst} \), \( f \in \text{Proc}_{A \rightarrow B}^{F} \), and \( |\sigma| \in \text{St}_{A \otimes E}^{F} \), the probability of the joint occurrence of the state \( |\sigma| \) and the process \( f \) is \( \text{Pr}[f, |\sigma|] := \text{Pr}[f \otimes \text{id}_{E} \circ |\sigma|] \). Also, assume that, for any two feasible scalars \( p, q \) and \( \rho \), the probability of the joint occurrence of \( p \) and \( q \) (i.e., \( \text{Pr}(p, q) = \text{Pr}(pq) = pq \)) is the product of probabilities \( p \) and \( q \) which means that \( pq \) is the product of two real numbers \( p \) and \( q \). This can be interpreted to indicate that an event that happens and one that happens to occur independently. The case \( q = \text{id}_{E} \) yields \( p \cdot \text{id}_{E} = p \), which gives \( \text{id}_{E} = 1 \).

It is natural to assume that \( f_{1}, \ldots, f_{k} \in \text{Proc}_{A \rightarrow B} \) represent mutually exclusive and collectively exhaustive events if, for any feasible state \( |\sigma| \in \text{St}_{A \otimes E}^{F} \), the sum of the probabilities of the joint occurrence of \( |\sigma| \) and \( f_{j} \) is equal to the probability of occurrence of \( |\sigma| \). This assumption implies

\[
\{f_{j}\}_{j=1}^{k} \subseteq \text{Test}_{A \rightarrow B}^{E} \quad \mathrel{\Leftrightarrow} \quad \forall E \in \text{Syst}, |\sigma| \in \text{St}_{A \otimes E}^{F}, \sum_{j=1}^{k} \text{Pr}[f_{j}, |\sigma|] = \text{Pr}[|\sigma|].
\]

(14)

**Example of quantum theory** \( \text{Pr}[|\rho|] = \text{Tr}[|\rho|] \) and \( \text{Pr}[f, |\sigma|] = \text{Tr}[(f \otimes \text{id}_{E}) \circ |\sigma|] \) hold.

3. **Deterministic effects**

We denote one of the deterministic effects of a system \( A \) as \( (\varpi_{A}) \) or simply \( (\varpi) \), which is depicted as

\[
\varpi_{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(15)

From Eq. (14), \( (\varpi_{A})(\varpi_{A}) = \text{Pr}[|\rho|] \) obviously holds for any \( \rho \in \text{St}_{A}^{F} \). This can be interpreted that any deterministic effect always happens. In particular, \( (\varpi_{A}) \) = 1 holds. For any systems \( A \) and \( B \), we choose \( (\varpi_{A \otimes B}) := (\varpi_{A}) \otimes (\varpi_{B}) \), which is depicted as

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(16)

**Example of quantum theory** \( (\varpi_{A}) \) is the identity matrix \( \mathbb{1} \).

4. **Unfeasible processes**

In this paper, for the sake of mathematical convenience, we assume \( \text{Scalar} = \mathbb{R} \). A real number larger than 1 is an unfeasible scalar. Unfeasible scalars are not quite intuitive since they cannot be interpreted as probabilities; however, they allow for a simple mathematical analysis, as will be seen throughout this paper. For any \( a \in \text{Scalar} \) and \( f \in \text{Proc}_{A \rightarrow B}^{F} \subseteq \text{Proc}_{A \rightarrow B}^{E} \), \( a \in \text{Proc}_{A \rightarrow B}^{E} \) obviously holds. Assume that any unfeasible process is expressed as scalar multiplication of a feasible process; i.e.,

\[
\text{Prob}_{A \rightarrow B}^{E} = \{a \in \text{Scalar}, f \in \text{Proc}_{A \rightarrow B}^{F} \}
\]

(17)

holds. Such \( \text{Prob}_{A \rightarrow B}^{E} \) can be interpreted as the smallest conceivable process space that is consistent with an operational theory satisfying \( \text{Scalar} = \mathbb{R} \).

Recall that \( \text{Pr}[|\rho'|] = (\varpi_{A})(\varpi_{A})' \) holds for any \( |\rho'| \in \text{St}_{A}^{F} \). We here extend the domain of the function \( \text{Pr} \) by letting \( \text{Pr}[|\rho|] := (\varpi_{A})(\varpi_{A}) \) for any \( \rho \in \text{St}_{A}^{F} \). Note that \( \text{Pr}[|\rho|] \) can be larger than 1. Also, let \( \text{Pr}[f, |\sigma|] := \text{Pr}[(f \otimes \text{id}_{E}) \circ |\sigma|] = (\varpi_{A})(f \otimes \text{id}_{E}) \circ |\sigma| \) for any \( f \in \text{Proc}_{A \rightarrow B}^{E} \) and \( |\rho| \in \text{St}_{A \otimes E}^{F} \).
5. Properties of tests and feasible processes

Since any $|\sigma\rangle \in \text{St}_{AE}$ is expressed as $|\sigma\rangle = a(|\sigma'\rangle)$ with $a \in \text{Scalar}$ and $|\sigma'\rangle \in \text{St}_{AE}$, $\Pr[|\sigma\rangle] = \Pr[a(|\sigma'\rangle)] = a \cdot \Pr[|\sigma'\rangle]$ and $\Pr[|f, \sigma\rangle] = \Pr[f, a(|\sigma'\rangle)] = a \cdot \Pr[|\sigma'\rangle]$ (by the definition of $\Pr$), obviously hold. Thus, $\Pr[|\sigma\rangle] \leq \Pr[|\sigma'\rangle]$, $\forall E \in \text{Syst}, |\sigma\rangle \in \text{St}_{AE}$.

(18)

holds for any $f \in \text{Proc}_{E}^{F}$. Considering the particular case of $B = E = I$, we have that, for any $|e\rangle \in \text{Eff}_{A}$,

\[ (e|\sigma\rangle \leq (\overline{|\sigma\rangle}) \cdot (\overline{|\sigma\rangle}), \forall |\sigma\rangle \in \text{St}_{A}. \]  

(19)

We have that, for any $|\rho\rangle \in \text{St}_{A}$,

\[ |\rho\rangle \in \text{Proc}_{A}^{D} \iff (\overline{|\rho\rangle}) = 1 \iff |\rho\rangle \in \text{St}_{A}. \]  

(20)

Using Eq. (20), one can easily see that Eq. (14) can be replaced with

\[ (f)_{j=1}^{k} \in \text{Test}_{A \rightarrow B} \iff |\rho\rangle \in \text{St}_{A}, \sum_{j=1}^{k} \Pr[|f,\rho\rangle] = \Pr[|\rho\rangle]. \]  

(21)

We also have

\[ |\rho\rangle \in \text{St}_{A} \iff (\overline{|\rho\rangle}) \leq 1, \]  

(22)

where $\iff$ follows from the fact that $|\rho\rangle, [1 - (\overline{|\rho\rangle})|\sigma\rangle]$ with $|\sigma\rangle \in \text{St}_{A}^{N}$ is a test.

6. Equality and local equality of processes

Recall that any scalar that includes $f \in \text{Proc}_{A \rightarrow B}$ is expressed in the form $u(f)$ with some diagram $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar}$. In an OPT, two processes $f, f' \in \text{Proc}_{A \rightarrow B}$ are said to be equal and denoted by $f = f'$ if

\[ u(f) = u(f'), \forall u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar} \]  

(23)

holds. $f = f'$ means that $f$ and $f'$ are indistinguishable from a probabilistic point of view. It follows that if $f = f'$ holds if and only if

\[ u(f) = u(f'), \forall C, D \in \text{Syst}, u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D} \]  

(24)

holds.  

---

4 Proof: Since $\Pr[|\rho\rangle, |\sigma\rangle] = (\overline{|\rho\rangle}) (\overline{|\sigma\rangle})$ holds for any $|\rho\rangle \in \text{St}_{A}$ and $|\sigma\rangle \in \text{St}_{AE}$, Eq. (14) gives that $|\rho\rangle \in \text{Proc}_{A \rightarrow B}^{D}$ is equivalent to $|\rho\rangle \in \text{St}_{A}$. Also, from the definition of a normalized state, $|\rho\rangle$ is equivalent to $|\rho\rangle \in \text{St}_{A}^{N}$ (note that if $|\rho\rangle$ is a normalized state, then $|\rho\rangle \in \text{Proc}_{A \rightarrow B}^{D}$, and thus $|\rho\rangle$ is feasible).

5 Proof: For any $f \in \text{Proc}_{A \rightarrow B}$ and $|\sigma\rangle \in \text{St}_{AE}$, $\Pr[|f, \sigma\rangle] = [\overline{|f \sigma\rangle}, \overline{|f \sigma\rangle}) \overline{a(|\sigma\rangle)} = \overline{|f \sigma\rangle} + \overline{|f \sigma\rangle} = \overline{|f \sigma\rangle}$. Also, $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. Thus, $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. Substituting these relations into Eq. (14) gives Eq. (21).

6 Proof: The case $C = D = I$ immediately yields the “if” part. To prove the “only if” part, we assume that $u(f) = u(f')$ holds for some diagram $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D}$ and prove $f \neq f'$. From the definition of equality of processes, there exists a diagram $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar}$ such that $u(f) = u(f')$. Let $v : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar}$ be the diagram satisfying $v(|\rho\rangle) = u(|\rho\rangle)$ ($v|\rho\rangle \in \text{St}_{A}$); then, $v(f) \neq v(f')$ holds. Thus, $f \neq f'$ holds.

7 Proof: The “only if” part is obvious. For the “if” part, assume $|\overline{\psi}\rangle = 0$. From Eq. (19), any $|e\rangle \in \text{Eff}_{A}$ satisfies $0 \leq |e\rangle \leq (\overline{|\psi\rangle}) = 0$, i.e., $|e\rangle = 0$. For any $|e\rangle \in \text{Eff}_{A}$, which is in the form $|e\rangle = a|e'\rangle$ with $a \in \text{Scalar}$ and $|e'\rangle \in \text{Eff}_{A}$, we have $|e\rangle = a|e'\rangle$, where $a = 0$. Thus, $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (26) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (25) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (24) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (23) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (22) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (21) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (20) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (19) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (18) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (17) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (16) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (15) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (14) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (13) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (12) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (11) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (10) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (9) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (8) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (7) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (6) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (5) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (4) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (3) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (2) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$. (1) gives $\Pr[|\sigma\rangle] = (\overline{|\sigma\rangle})$.
holds. It is easy to verify that $(\oplus_A)$ is the unique deterministic effect of $A$. Indeed, assume $(e) \in \text{Eff}_A$ is deterministic; then, $(e) = (\oplus_A)(p)$ holds for any $p \in \text{St}_A$, and thus $(e) = (\oplus_A)$ holds. 9

Two processes $f$, $f' \in \text{Proc}_{A \rightarrow B}$ are said to be locally equal and denoted by $f = f'$ if

\[(e \circ f \circ \rho) = (e \circ f' \circ \rho), \quad \forall (e) \in \text{Eff}_B, \rho \in \text{St}_A \]  

(29) holds. We have that, from Eq. (26),

\[f \equiv f' \iff f \circ \rho = f' \circ \rho, \quad \forall \rho \in \text{St}_A. \]  

(30)

$f = f'$ is a sufficient condition for $f$ locally equal $f'$, but not necessary. Note that, in the particular case of $A = I$ or $B = I$, the equality and local equality are equivalent, as already shown in Eqs. (26) and (28). We can easily obtain

\[f = f' \iff f \otimes \text{id}_E \equiv f' \otimes \text{id}_E, \quad \forall E \in \text{Syst.} \]  

(31)

A process $f \in \text{Proc}_{A \rightarrow B}$ is called reversible if there exists $\bar{f} \in \text{Proc}_{B \rightarrow A}$, called an inverse of $f$, such that $f \circ \bar{f} = \text{id}_A$ and $\bar{f} \circ f = \text{id}_B$. $\bar{f}$ is also reversible. Note that, for a reversible process $f$, there may be more than one inverse of $f$.

**Example of quantum theory**

In quantum theory, two processes $f, f' \in \text{Proc}_{A \rightarrow B}$ are equal if and only if they are locally equal. The “only if” part is straightforward, so we prove only “if” part. Assume $f = f'$. From Eq. (30), $f([\rho]) = f'([\rho])$ holds for any $\rho \in \text{St}_A$. It is well known that any $|\sigma\rangle \in \text{St}_{A \otimes E}$ can be expressed in the form $|\sigma\rangle = \sum_{\rho} c_{\rho} |\rho\rangle \otimes |\rho\rangle$, where $c_1, \ldots, c_N \in \mathbb{C}$, $\rho_1, \ldots, \rho_N \in \text{St}_A$, and $|\rho_1\rangle, \ldots, |\rho_N\rangle \in \text{St}_E$ hold. Thus, we have $(f \otimes \text{id}_E)(|\sigma\rangle) = \sum_{\rho} c_{\rho} f(|\rho\rangle) \otimes |\rho\rangle = \sum_{\rho} c_{\rho} f'(|\rho\rangle) \otimes |\rho\rangle = (f' \otimes \text{id}_E)(|\sigma\rangle)$. Therefore, $f \otimes \text{id}_E = f' \otimes \text{id}_E$ holds for any system $E$, which gives $f = f'$ from Eq. (31).

A process $f \in \text{Proc}_{A \rightarrow B}$ is reversible if and only if $N_A = N_B$ holds and $f$ is written in the form $f(\rho) = E \cdot |\rho\rangle \cdot E^{\dagger}$ with an invertible matrix $E$ of order $N_A$, where $^*$ denotes the conjugate transpose. Its inverse, $\bar{f}$, is written as $\bar{f}(\rho) = E^{-1} \cdot |\rho\rangle \cdot (E^{-1})^\dagger$.

### 7. Sum of processes

Assume that, for any two feasible processes $g_1, g_2 \in \text{Proc}_{A \rightarrow B}$ and any $p \in \text{Scalar}$, there exists a feasible process $h \in \text{Proc}_{A \rightarrow B}$ satisfying

\[u(h) = p \cdot u(g_1) + (1 - p) \cdot u(g_2), \quad \forall u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar}. \]  

(32)

Such a process $h$, denoted by $pg_1 + (1 - p)g_2$, can be interpreted as a probabilistic mixture of $g_1$ and $g_2$ with probabilities $p$ and $1 - p$. For any two processes $f_1, f_2 \in \text{Proc}_{A \rightarrow B}$, $f_{\sum} \in \text{Proc}_{A \rightarrow B}$ is called the sum of $f_1$ and $f_2$ and denoted by $f_1 + f_2$ if it satisfies

\[u(f_{\sum}) = u(f_1) + u(f_2), \quad \forall u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar}. \]  

(33)

Since the right-hand side of Eq. (32) equals to $u(pg_1) + u((1 - p)g_2)$, the feasible process $pg_1 + (1 - p)g_2$ equals to the sum of two processes $g_1$ and $(1 - p)g_2$. It is easily seen that, for any $f_1, f_2 \in \text{Proc}_{A \rightarrow B}$, the process $f_1 + f_2$ always exists, i.e., the sum of two processes is always a process. In particular, the sum of two scalars $p_1 + p_2 \in \text{Scalar}$ (i.e., $p_1 + p_2$) is equal to the sum of the real numbers.

It is easily seen that any $f_1, \ldots, f_k \in \text{Proc}_{A \rightarrow B}$ and $a_1, \ldots, a_k \in \text{Scalar}$ satisfy

\[u\left(\sum_{j=1}^{k} a_j f_j\right) = \sum_{j=1}^{k} a_j u(f_j), \quad \forall u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D}, \]  

(34)

which is diagrammatically depicted as

\[\sum_{j=1}^{k} a_j f_j \rightarrow u = \sum_{j=1}^{k} a_j f_j \rightarrow u \]  

(35)

This implies that the diagram $u$ distributes over addition.

We consider a set of process $F = \{ f_j \in \text{Proc}_{A \rightarrow B} \}_{j=1}^{k}$. It follows from Eq. (14) that $f \in \text{Test}_{A \rightarrow B}$ and $\{ \sum_{j=1}^{k} f_j \} \in \text{Test}_{A \rightarrow B}$ are equal, which yields

\[\{ f_j \}_{j=1}^{k} \in \text{Test}_{A \rightarrow B} \iff \sum_{j=1}^{k} f_j \in \text{Proc}_{A \rightarrow B}. \]  

(36)

In the particular case of $B = I$, we have

\[\{ e \}_{j=1}^{k} \in \text{Meas}_A \iff \sum_{j=1}^{k} e_j = (\oplus). \]  

(37)

**Example of quantum theory** The sum of processes is equal to the sum of CP maps. In particular, the sum of states (or effects) is the sum of matrices.

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9 Some papers (e.g., [15, 41]) adopt an OPT in which each system does not necessarily have a unique deterministic effect. Such an OPT has non-fixed causal structure. In this paper, we restrict our attention to fixed causal structure.

10 Proof: For each $i \in \{1, 2\}$, $f_i$ is expressed in the form $f_i = a_i f_{i}'$ with $a_i \in \text{Scalar}$ and $f_{i}' \in \text{Proc}_{A \rightarrow B}$. We assume, without loss of generality, that $a_1, a_2 > 0$. Substituting $g_1 = f_{1}'$, $g_2 = f_{2}'$, and $p = \frac{a_1}{a_1 + a_2}$ into Eq. (32) gives $h = \frac{a_1}{a_1 + a_2} f_{1}' + \frac{a_2}{a_1 + a_2} f_{2}' \in \text{Proc}_{A \rightarrow B}$. Since $u(ah) = a \cdot u(h) = a \cdot (\frac{a_1}{a_1 + a_2} u(f_{1}') + \frac{a_2}{a_1 + a_2} u(f_{2}')) = u(f_1) + u(f_2)$ holds, we have $f_1 + f_2 = ah \in \text{Proc}_{A \rightarrow B}$. 
8. Process spaces are convex cones

Each process space $\text{Proc}_{A \rightarrow B}$ can be regarded as a subset of a real vector space. Let $V_{A \rightarrow B}$ be the set of all formal sums of the form $\sum a_i f_i$ with $a_i \in \mathbb{R}$ and $f_i \in \text{Proc}_{A \rightarrow B}$, where $\sum a_i f_i$ is the element satisfying $u(\sum a_i f_i) = \sum a_i u(f_i)$ for any diagram $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar}$. It follows that $V_{A \rightarrow B}$ is the vector space spanned by the process space $\text{Proc}_{A \rightarrow B}$. In this paper, we only consider the case where $V_{A \rightarrow B}$ is finite-dimensional for any systems $A$ and $B$. Let us define $V_A := V_{I \rightarrow A}$ and $V_A^* := V_{A \rightarrow I}$. We will refer to an element of $V_{A \rightarrow B}$ as an extended process from $A$ to $B$. Similarly, we will refer to elements of $V_A$ and $V_A^*$ as an extended state of $A$ and an extended effect of $A$, respectively. Since each extended effect can be described as a linear functional on an extended state, $V_A^*$ can be regarded as the dual vector space of $V_A$. We use an overline, $\overline{\cdot}$, to denote extended processes such as $\overline{f}$.

Any extended process $\overline{f} \in V_{A \rightarrow B}$, which can be expressed in the form $\overline{f} = \sum a_i f_i$ with $a_i \in \mathbb{R}$ and $f_i \in \text{Proc}_{A \rightarrow B}$, satisfies $\overline{f} = f_0 - f_-$, where $f_0 := \sum_{a_i \geq 0} a_i f_i \in \text{Proc}_{A \rightarrow B}$ and $f_- := \sum_{a_i < 0} a_i f_i \in \text{Proc}_{A \rightarrow B}$. As well as processes, extended processes can be composed sequentially and in parallel. Specifically, for any $\overline{f} := \sum a_i f_i \in V_{A \rightarrow B}$, $\overline{g} := \sum_b g_b j_b \in V_{B \rightarrow C}$, and $\overline{n} := \sum \lambda c_i h_i \in V_{C \rightarrow D}$ with $a_i, b_j, c_k \in \mathbb{R}$, $f_i \in \text{Proc}_{A \rightarrow B}$, $g_j \in \text{Proc}_{B \rightarrow C}$, and $h_k \in \text{Proc}_{C \rightarrow D}$, $\overline{f} \otimes \overline{g} = \sum a_i b_j (g_j \circ f_i)$ and $\overline{f} \otimes \overline{n} = \sum a_i c_k (f_i \otimes h_k)$ hold. Diagrammatically, we have

$$\overline{\sum a_i f_i} \otimes \overline{\sum b_j g_j} = \overline{\sum j a_i b_j} \quad \overline{(f_i \otimes g_j)}$$

and

$$\overline{\sum a_i f_i} \otimes \overline{\sum c_k h_k} = \overline{\sum a_i c_k} \quad \overline{(f_i \otimes h_k)}$$

Since $af + bg \in \text{Proc}_{A \rightarrow B}$ holds for any $a, b \in \text{Scalar} = \mathbb{R}$ and $f, g \in \text{Proc}_{A \rightarrow B}$, $\text{Proc}_{A \rightarrow B}$ is a convex cone. Moreover, $\text{Proc}_{A \rightarrow B}$ is salient, i.e., $\text{Proc}_{A \rightarrow B}$ does not contain both $\overline{f}$ and $-\overline{f}$ for any $\overline{f} \neq 0$.

We here recall some basic properties of convex cones. For a salient convex cone $C$ with $0 \in C$ in a real vector space $V$, the partial ordering on $V$ is defined as follows: for $x, y \in V$,

$$x \leq y \quad \text{or} \quad y \geq x :\Leftrightarrow \quad y - x \in C.$$  

(40)

For $x \in C$, the set defined as

$$\mathcal{F}_x := \{y \in C : \exists \delta \in \mathbb{R}_{++}, \delta y \leq x\}$$

(41)

is called the face of $x$, where $\mathbb{R}_{++}$ is the set of all positive real numbers. $x \in C$ is called an interior point if $\mathcal{F}_x = C$ holds. $x \in C$ is called atomic if $\mathcal{F}_x = \{px : p \in \mathbb{R}\}$ holds.

Since each process space $\text{Proc}_{A \rightarrow B}$ is a convex cone, the partial ordering on $V_{A \rightarrow B}$ and the face of $f \in \text{Proc}_{A \rightarrow B}$ can be, respectively, defined by Eqs. (40) and (41) with $C = \text{Proc}_{A \rightarrow B}$. One can easily verify that $u(\overline{f}) \leq u(\overline{g})$ holds for any $u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Proc}_{C \rightarrow D}$ and $\overline{f}, \overline{g} \in V_{A \rightarrow B}$ satisfying $\overline{f} \leq \overline{g}$. From the definition of faces, we have that, for any $f, g \in \text{Proc}_{A \rightarrow B}$,

$$f \in \mathcal{F}_g \Leftrightarrow \exists \delta \in \mathbb{R}_{++}, \delta f \leq g. \quad (42)$$

We call a state or an effect pure if it is atomic, mixed if it is not atomic, and completely mixed if it is an interior point. $\text{St}_A^N$ is defined as the set of all pure states of system $A$. Let $\text{St}_A^{NP} := \text{St}_A^N \cap \text{St}_A^*$, which is the set of all normalized pure states of $A$.

Recall that an effect $(e)$ is feasible if and only if there exists a measurement including $(e)$. Using Eq. (37), we have

$$(e) \in \text{Eff}_A^f \Leftrightarrow \quad (e) \leq (\overline{\oplus})$$

(43)

From Eq. (37), the following relation also holds:

$$(e) \in \text{Eff}_A^f \Leftrightarrow \quad ((e), (\overline{\oplus}) = (\text{Meas}_A)$$

$$(\overline{\oplus}) \leq (e) \in \text{Eff}_A^f. \quad (44)$$

It follows that $(\overline{\oplus})$ is completely mixed. Indeed, for any $(e) \in \text{Eff}_A$, there exists $d \in \mathbb{R}_{++}$ such that $d(e) \in \text{Eff}_A^f$, which implies from Eq. (43) that $d(e) \leq (\overline{\oplus})$.

Lemma 1 If $|\rho \rangle \in \text{St}_A$ and $(e) \in \text{Eff}_A^f$, satisfy $(e|\rho \rangle = 0$, then $(e|\sigma \rangle = 0$ holds for any $|\sigma \rangle \in \mathcal{F}_{|\rho \rangle}$.

Proof From the definition of $\mathcal{F}_{|\rho \rangle}$, there exists $\delta \in \mathbb{R}_{++}$ such that $\delta(e) \leq |\rho \rangle$. This gives $0 \leq (e|\sigma \rangle \leq \delta^{-1}(e|\rho \rangle) = 0$. ■

Lemma 2 If $|\rho \rangle \in \text{St}_A$ and $(e) \in \text{Eff}_A^f$, satisfy $(e|\rho \rangle = (\overline{\oplus})|\rho \rangle$, then $(e|\sigma \rangle = (\overline{\oplus})|\sigma \rangle$ holds for any $|\sigma \rangle \in \mathcal{F}_{|\rho \rangle}$.

Proof From Eq. (44), $(e') := (\overline{\oplus}) - (e) \in \text{Eff}_A^f$ holds. From $(e'|\rho \rangle = 0$ and Lemma 1, we have $(e'|\sigma \rangle = 0$. Therefore, $(e|\sigma \rangle = (\overline{\oplus})|\sigma \rangle$ holds. ■

It is easily seen that $\text{St}_A^N$ is a convex set. This paper assumes that $\text{St}_A^N$ is closed: in this case, $\text{St}_A$ is a closed convex cone. From Carathéodory’s theorem, any $|\rho' \rangle \in \text{St}_A^N$ can be expressed in the form $|\rho' \rangle = \sum_{i=1}^n p_i |\psi_i \rangle$, where $|\psi_1 \rangle, \ldots, |\psi_n \rangle \in \text{St}_A^{NP}$, $p_1, \ldots, p_n \in \mathbb{R}_{++}$, and $\sum_{i=1}^n p_i = 1$ hold. From Eq. (27), any $|\rho \rangle \in \text{St}_A$ is proportional to some normalized state and thus is expressed in the form $|\rho \rangle = \sum_{i=1}^n c_i |\psi_i \rangle$ with $|\psi_1 \rangle, \ldots, |\psi_n \rangle \in \text{St}_A^{NP}$ and $c_1, \ldots, c_n \in \mathbb{R}_{++}$.

Example of quantum theory Recall that $\text{St}_A \cong \bigoplus_{k_1}^{k_A} \mathcal{S}^m$ and $\text{St}_B \cong \bigoplus_{k_1=1}^{k_B} \mathcal{S}^m$ hold for some natural numbers $k_A, m_1, \ldots, m_k, k_B, n_1, \ldots, n_k$; in this case, $V_{A \rightarrow B}$ is isomorphic to the space of all linear maps from $\bigoplus_{j=1}^{k_A} \mathcal{S}(C^{m_j})$ to $\bigoplus_{j=1}^{k_B} \mathcal{S}(C^{n_j})$ (which are often called Hermitian-preserving maps). $\text{V}_A \cong \text{V}_A^* \cong \bigoplus_{k_1=1}^{k_B} \mathcal{S}^m$ also holds. For extended processes $\overline{f}, \overline{g} \in V_{A \rightarrow B}$, $\overline{f} \leq \overline{g}$ holds if and only if $\overline{g} - \overline{f}$ is a CP map. In particular, for any $|\tau \rangle, |\gamma \rangle \in \text{V}_A$, $|\tau \rangle \leq |\gamma \rangle$ holds.
and only if $|\bar{\gamma}| - |\bar{x}|$ is a positive semidefinite matrix. The same holds for effects. For two states $|\phi\rangle, |\sigma\rangle \in \mathbf{St}_A$, $|\sigma\rangle \in \mathcal{F}(\rho)$ and supp $|\sigma\rangle \subseteq$ supp $|\rho\rangle$ are equivalent, where supp denotes the support of a matrix. A state $|\rho\rangle$ is completely mixed if and only if the matrix $|\rho\rangle$ has full rank. A state $|\rho\rangle$ is pure if and only if rank $|\rho\rangle \leq 1$ holds.

9. Perfectly distinguishable states

A set of states $\Phi := \{|\rho_i\rangle\} \in \mathbf{St}_A^k$ is said to be perfectly distinguishable if there exists a measurement $\Pi := \{|e_i\rangle\}_{i=1}^m \in \mathbf{Meas}_A$ $(m \geq k)$ such that

$$(e_i|\rho_i\rangle = (\mp|\rho_i\rangle), \quad \forall i \in \{1, \ldots, k\},$$

in which case, we say that $\Pi$ perfectly distinguishes between $\Phi$. Equation (45) implies $(e_i|\rho_i\rangle = \delta_{i,j})$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$, where $\delta_{ij}$ is the Kronecker delta. The kernel of $|\sigma\rangle \in \mathbf{St}_A$, denoted by $\mathcal{K}_A$, is defined as the set of all states of $A$ that is perfectly distinguishable from $|\sigma\rangle$. $|\rho\rangle \in \mathcal{K}_A$ and $|\sigma\rangle \in \mathcal{K}_B$ respectively are equivalent. Let $\mathbf{PDS}_A$ and $\mathbf{MPDS}_A$ be, respectively, the sets of all PDSs and MPDSs of $A$. The maximum number of elements of an MPDS for $A$ is called the rank of $A$ and denoted by $N_A$. Since there exist normalized states for any system $A$, $N_A \geq 1$ holds. For an MPDS $\Phi$, a measurement $\Pi \in \mathbf{Meas}_A$ with $|\Pi| = |\Phi|$ that perfectly distinguishes between $\Phi$ is called maximal, where $|\Phi|$ is the number of elements in a set $X$. An effect $(e \in \mathbf{Eff}_A)$ is called maximal if $\Pi$ exists a maximal measurement including $(e)$. Let $\mathbf{Eff}_A$ be the set of all maximal effects of $A$. $\mathbf{Eff}_A \subseteq \mathbf{Eff}_M \subseteq \mathbf{Eff}_A$ obviously holds.

Example of quantum theory A set of states $|\rho_1\rangle \in \mathbf{St}_A^k$ is perfectly distinguishable if and only if $|\rho_i\rangle \cdot |\rho_i\rangle = |\emptyset\rangle$ holds for any distinct $i, j \in \{1, \ldots, k\}$, where $|\emptyset\rangle$ is the zero square matrix of order $N_A$. It follows that $|\rho\rangle \in \mathcal{K}_A$ and $|\rho\rangle \cdot |\rho\rangle = |\emptyset\rangle$ are equivalent. $\Psi := \{|\psi_i\rangle\} \in \mathbf{St}_A^k$ is a PDS if and only if $k \leq N_A$ and $|\psi_i\rangle = |\phi_i\rangle$ $(\forall i \in \{1, \ldots, k\})$ hold for some orthonormal basis (ONB) $|\phi_{1}^{N_A}\rangle_{i=1}^{N_A}$ of $\mathbf{C}^{N_A}$. In particular, $\Psi$ is an MPDS if and only if $k = N_A$ holds. For any $\Phi := \{|\phi_i\rangle\}_{i=1}^{N_A} \in \mathbf{MPDS}_A$, there exists a unique maximal measurement $(|e_i\rangle)_{i=1}^{N_A}$ that perfectly distinguishes between $\Phi$, where $(e)_{i}$ and $|\phi_i\rangle$ are the same matrix. An effect is maximal if and only if it is in the form $|\phi\rangle \langle\phi|$ with some unit vector $|\phi\rangle \in \mathbf{C}^{N_A}$.

III. FOUR POSTULATES

In this section, we present the postulates used in our derivation. For each postulate, examples in classical and quantum theories are also presented. It is noteworthy that all these postulates are satisfied in quantum theory; conversely, quantum theory is uniquely singled out from these postulates.

A. Postulates

1. Symmetric sharpness

Our first postulate concerns the duality between normalized pure states and maximal effects.

Postulate 1 (Symmetric sharpness) To every normalized pure state $|\phi\rangle$, there corresponds one and only one maximal effect, denoted by $(|\phi\rangle\langle\phi|)$, giving unit probability (i.e., $|\phi\rangle\langle\phi| = 1$).

Furthermore, for any two normalized pure states $|\phi\rangle$ and $|\psi\rangle$ and their corresponding maximal effects $(|\phi\rangle\langle\phi|)$ and $(|\psi\rangle\langle\psi|)$, respectively, the probability of the joint occurrence of $|\phi\rangle$ and $(|\psi\rangle\langle\psi|)$ is equal to the probability of the joint occurrence of $|\phi\rangle$ and $|\psi\rangle$ (i.e., $(|\phi\rangle\langle\phi|)(|\psi\rangle\langle\psi|) = (|\phi\rangle\langle\phi|)(|\phi\rangle\langle\phi|))$.

In any OPT, for any $(e \in \mathbf{Eff}_A)$, there exists $|\rho\rangle \in \mathbf{St}_A^{N_A}$ such that $(e|\rho\rangle = 1)$. The symmetric sharpness postulate states that each $(e \in \mathbf{Eff}_A)$ corresponds to one and only one $|\rho\rangle \in \mathbf{St}_A^{N_A}$ such that $(e|\rho\rangle = 1)$. This postulate also states that the probability of the joint occurrence of any normalized pure state and any maximal effect is invariant under the exchange of the normalized pure state and its corresponding maximal effect.

Example of classical theory As previously mentioned, $\mathbf{St}_A \cong \mathbf{R}^{N_A}$ holds. Regarding the examples of classical theory, without loss of generality, we will identify a state of $A$ with the corresponding element of $\mathbf{R}^{N_A}$, which is an $N_A$-dimensional nonnegative column vector. Each system $A$ has exactly $N_A$ normalized pure states $|1\rangle := |1\rangle, \ldots, |N_A\rangle := |N_A\rangle$, where $(|n\rangle)_{n=1}^{N_A}$ is the standard ONB of $\mathbf{R}^{N_A}$. Each state $|\rho\rangle$ of $A$ can be expressed in the form $|\rho\rangle = \sum_{i=1}^{N_A} p_i |i\rangle$ with $p_1, \ldots, p_{N_A} \in \mathbf{R}_+$. We will also identify a process from $A$ to $B$ with its corresponding nonnegative row vector. $(i|) := |i\rangle$ is the unique maximal effect satisfying $(i|j) = |i\rangle \langle j| = 1$ $(i|j) = \delta_{i,j} = (j|i)$ also holds.

Example of quantum theory For any $|\rho\rangle \in \mathbf{St}_A^{N_A}$, which can be expressed in the form $|\rho\rangle = |\psi\rangle \langle\psi|$ with some unit vector $|\rho\rangle \in \mathbf{C}^{N_A}$, $(|\phi\rangle := |\phi\rangle \langle\phi|$ represents the unique maximal effect that satisfies $(\phi|\phi) = (\langle\phi\phi\rangle)^2 = 1$. $(\phi|\phi) = (\langle\psi\psi\rangle)^2 = (\phi|\phi)$ also holds for any $|\rho\rangle = |\phi\rangle \langle\phi| \in \mathbf{St}_A^{N_A}$ and $|\phi\rangle = |\phi\rangle \langle\phi| \in \mathbf{St}_A^{N_A}$.

2. Complete mixing

The second postulate provides a sufficient condition for a state to be completely mixed.

Postulate 2 (Complete mixing) Every state for which there exists no maximal effect giving zero probability is completely mixed.
In any OPT, the converse of this postulate is also true, i.e., for every completely mixed state $|\rho\rangle$, there exists no maximal effect giving zero probability. Indeed, for any $(e) \in \text{Eff}_A^M$, there exists $|\phi\rangle \in \text{St}_A^\text{NP}$ satisfying $(e|\phi\rangle) = 1$. From $|\phi\rangle \in \text{St}_A = \mathcal{F}_\rho$, $\delta(\rho) \leq |\rho\rangle$ holds for some $\delta \in \mathbb{R}^+$, which gives $(e|\rho\rangle \geq \delta(e|\rho\rangle) = \delta > 0$.

**Example of classical theory**  Every $|\rho\rangle \in \text{St}_A$ is in the form $|\rho\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle$ with $p_1, \ldots, p_{N_A} \in \mathbb{R}^+$, and $\text{OR} \{\rho|\phi_i\rangle\}_{i=1}^{N_A}$. If $|j\rangle \rho = p_j > 0$ holds for each $|j\rangle = |j\rangle \in \text{Eff}_A^M$, then $|\rho\rangle$ is completely mixed.

**Example of quantum theory**  The arguments for classical theory can readily be extended to quantum theory. Every $|\rho\rangle \in \text{St}_A$ can be expressed in the form $|\rho\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle$ with $p_1, \ldots, p_{N_A} \in \mathbb{R}^+$ and an ONB $\{\rho|\phi_i\rangle\}_{i=1}^{N_A}$ of $\mathbb{C}^N_A$. If $|\phi_i\rangle = p_j > 0$ holds for each $|\phi_i\rangle = |\phi_j\rangle$ with $j \in \text{Eff}_A^M$, then $|\rho\rangle$ is completely mixed.

### 3. Filtering

The third postulate entails the existence of what we refer to as filters. Herein, we will first define filters before stating the postulate. For any normalized pure state $|\phi\rangle \in \text{St}_A^\text{NP}$ and any $c \in [0,1]$, a filter $F_c^{\phi}$ is defined as a process in $\text{Proc}_A^{\text{E}}$ that satisfies

\[
\begin{align*}
F_c^{\phi} & = \begin{array}{c} \phi \downarrow \rho \end{array} = c \begin{array}{c} \phi \downarrow \rho \end{array};
\end{align*}
\]

(46)

that is, $F_c^{\phi}$ transforms $|\phi\rangle$ to $c|\phi\rangle$ and leaves any state that is perfectly distinguishable from $|\phi\rangle$. Furthermore, suppose that $F_c^{\phi}$ is reversible if $c > 0$ holds. We say that $|\phi\rangle \in \text{St}_A^\text{NP}$ can be arbitrarily filtered if there exists a filter $F_c^{\phi}$ for every $c \in [0,1]$.

**Postulate 3 (Filtering)**  Any normalized pure state can be arbitrarily filtered.

**Example of classical theory**  For any $|j\rangle \in \text{St}_A^\text{NP}$ and $c \in [0,1]$, the filter $F_c^{\phi}$ is in $\text{Proc}_A^{\text{E}}$ can be expressed as the following matrix:

\[
F_c^{\phi} |j\rangle = c |j\rangle + \Upsilon_j,
\]

(47)

where $\Upsilon_j = \mathbb{I}_{N_A} - |j\rangle \langle j|$. For any $|\sigma\rangle \in \text{St}_A$, which is in the form $|\sigma\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle$, we have $F_c^{\phi} |\sigma\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle - (1 - c)p_j |\phi_j\rangle$. Since $\{F_c^{\phi} |\phi_i\rangle, \mathbb{I}_{N_A} - F_c^{\phi} |\phi_i\rangle\}$ is a test, $F_c^{\phi} |\rho\rangle$ is feasible. It is easy to verify that $F_c^{\phi}$ satisfies Eq. (46). Indeed, $F_c^{\phi} |j\rangle = c |j\rangle$ obviously holds. Also, for any $|\rho\rangle \in \mathcal{K}(\phi)$, since $|\rho\rangle$ can be expressed in the form $|\rho\rangle = \sum_{i=1}^{N_A} q_i |\phi_i\rangle$ with $q_j = 0$, we have $F_c^{\phi} |\rho\rangle = |\rho\rangle$. Let $\tilde{F}^{\phi}_c := |j\rangle \langle j| + c \Upsilon_j$; then, $\tilde{F}^{\phi}_c \circ F_c^{\phi} = c \mathbb{I}_N = F_c^{\phi} \circ \tilde{F}^{\phi}_c$ obviously holds (note that $\mathbb{I}_N = \mathbb{I}_{N_A}$ holds). Thus, for any $c > 0$, $F_c^{\phi}$ is reversible and its inverse is $c^{-1} \tilde{F}^{\phi}_c$.

**Example of quantum theory**  For any $|\phi\rangle = |\phi\rangle \langle \phi| \in \text{St}_A^\text{NP}$ and $c \in [0,1]$, the CP map $F_c^{\phi}$ is defined as

\[
F_c^{\phi} |\rho\rangle = E_c^{\phi} |\rho\rangle = c \mathbb{I}_N = F_c^{\phi} \circ \tilde{F}^{\phi}_c
\]

(48)

is a filter, where $\Upsilon_j := \mathbb{I}_{N_A} - |\phi\rangle \langle \phi|$. Since $[E_c^{\phi}] \circ F_c^{\phi} \leq \mathbb{I}_{N_A}$ holds, $F_c^{\phi}$ is trace non-increasing and thus feasible. Again, Eq. (46) can easily be verified. Also, let $\tilde{F}^{\phi}_c$ be the following CP map:

\[
\tilde{F}^{\phi}_c |\rho\rangle = E_c^{\phi} |\rho\rangle = c \mathbb{I}_N = F_c^{\phi} \circ \tilde{F}^{\phi}_c
\]

(49)

then, $F_c^{\phi} \circ \tilde{F}^{\phi}_c = c \mathbb{I}_N = F_c^{\phi} \circ \tilde{F}^{\phi}_c$ holds. Thus, for any $c > 0$, $F_c^{\phi}$ is reversible and its inverse is $c^{-1} \tilde{F}^{\phi}_c$.

### 4. Local equality

The fourth postulate requires that two processes are equal if they are locally equal.

**Postulate 4 (Local equality)**  Any two locally equal processes are equal.

Recall that, in any OPT, the converse is true, i.e., any two equal processes are locally equal. Thus, this postulate implies that, for any $f, f' \in \text{Proc}_{A-B}$, $f \equiv_{\text{local}} f'$ and $f = f'$ are equivalent.

We here consider the following scenario. Suppose that two processes $f, f' \in \text{Proc}_{A-B}$ are equal and we want to prove it, where $A$ and $B$ refer to systems in our laboratory. From the definition of equality of processes, we need to show that two scalars $(e| \circ (f \circ \mathbb{I}_E) \circ |\rho\rangle$ and $(e| \circ (f' \circ \mathbb{I}_E) \circ |\rho\rangle$ are equal for any $E \in \text{Syst}$, $|\rho\rangle \in \text{St}_{A_{idE}}$, and $(e| \in \text{Eff}_{B_{idE}}$. If the local equality postulate does not hold, then since $E$ could be an extremely large system (such as the universe), it may be practically impossible to prove $f = f'$. If the local equality postulate holds, then we only need to show that $(e| \circ f \circ |\rho\rangle = (e| \circ f' \circ |\rho\rangle$ holds for any $|\rho\rangle \in \text{St}_A$ and $(e| \in \text{Eff}_B$, which can be accomplished only in the laboratory. Thus, we can say that this postulate significantly reduces the amount of information required to identify a process.

**Example of classical theory**  Any process $f \in \text{Proc}_{A-B}$ can be expressed in the form $f = \sum_{i=1}^{N_B} \sum_{j=1}^{N_A} f_{i,j} |i_B\rangle \langle j_A|$ with the standard ONBs $|i_B\rangle (i_{B}^\text{N}) \approx |j_A\rangle (j_{A}^\text{N})_A$ of $\mathbb{R}^N$ and $|l_{A}\rangle (l_{A}^\text{N}) \approx |j_{B}\rangle (j_{B}^\text{N})_B$ of $\mathbb{R}^N$, where $f_{i,j} := \langle i_B|f|j_A\rangle = \langle i_B| \circ f \circ |j_A\rangle$. Hence, if two processes $f, f' \in \text{Proc}_{A-B}$ satisfy $(i_{B}^\text{N}) \circ f \circ |j_A\rangle = (i_{B}^\text{N}) \circ f' \circ |j_A\rangle$ for any $i \in [1, \ldots, N_B]$ and $j \in [1, \ldots, N_A]$, then $f = f'$ obviously holds.
Example of quantum theory As we have already shown in Subsubsec. II B 6, any quantum theory can enjoy the local equality postulate. The above four postulates can be thought of as approaches describing the behavior of feasible processes. (Recall that any process is proportional to some feasible process.) Indeed, it can certainly be stated that the complete mixing postulate is equivalent to the statement that every feasible state for which there exists no maximal effect giving zero probability is completely mixed, and that the local equality postulate is equivalent to the statement that any two feasible and locally equal processes are equal. Regarding the filtering postulate, we note that a filter $F_{c} \in \text{Proc}_{A \rightarrow \phi}^{F}$ is reversible if and only if there exist a feasible process $\tilde{F}_{c} \in \text{Proc}_{A \rightarrow \phi}^{F}$ and a nonzero feasible scalar $p$ such that $\tilde{F}_{c} \circ F_{c} = p \cdot \text{id}_{A} \circ F_{c}$. It is noteworthy that, roughly speaking, each of our four postulates is essentially the same as, or similar to, that used in previous studies. Specifically, in studies that focus on reconstructing quantum theory from operational postulates, postulates on symmetric sharpness [43] (or sharpness [16, 17, 44]), complete mixing [15, 18], and filtering [17, 44, 49] have been used. Also, several studies have used the so-called local tomography (or local discriminability) possibility postulate [9, 13–16, 18, 43], which is highly related to the local equality postulate. Indeed, it has been shown that the local equality postulate holds for any OPT having the local tomography postulate [41]. We will show later that, conversely, the local tomography postulate holds for any OPT having the local equality postulate, which means that these two postulates are equivalent. Although each of the four postulates is not new in itself, combining them provides a new way to reconstruct quantum theory.

B. Overview of derivation

In this paper, we show that an OPT having the four proposed postulates satisfies the following properties:

(A) For each system $A$, the state space $\text{St}_{A}$ is isomorphic to a direct sum of spaces of complex positive semidefinite matrices, i.e., $\text{St}_{A} \cong \bigoplus_{i=1}^{k} \text{S}_{i}(\mathbb{C}^{n_{i}})$, where $k, n_{1}, \ldots, n_{k}$ refer to some natural numbers that satisfy $\sum_{i=1}^{k} n_{i} = N_{A}$ (Theorem 41).

(B) For any systems $A$ and $B$ with $\text{St}_{A} \cong \bigoplus_{i=1}^{k} \text{S}_{i}(\mathbb{C}^{n_{i}})$ and $\text{St}_{B} \cong \bigoplus_{j=1}^{k} \text{S}_{j}(\mathbb{C}^{n_{j}})$, $\text{St}_{A \otimes B} \cong \bigoplus_{i=1}^{k_{A}} \bigoplus_{j=1}^{k_{B}} \text{S}_{i}(\text{S}_{j}(\mathbb{C}^{n_{i}n_{j}}))$ holds (Theorem 42).

(C) Let $\text{M}_{A}^{*} := \{ \text{M}_{A}^{(\sigma)} : |\sigma| \in \text{St}_{A} \} \cong \text{St}_{A}$, where $\text{M}_{A}^{(\sigma)}$ refers to a certain full and faithful matrix representation of $|\sigma|$. Also, let $\text{M}_{A}$ be the real vector space spanned by $\text{M}_{A}^{*}$ and $\text{CP}_{A \rightarrow B}$ be the set of all CP maps from $\text{M}_{A}$ to $\text{M}_{B}$. Then, for any systems $A$ and $B$, each process space $\text{Proc}_{A \rightarrow B}$ is isomorphic to $\text{CP}_{A \rightarrow B}$ as convex cones. Also, there exists an isomorphism $\mathcal{L} : \text{Proc}_{A \rightarrow B} \ni f \mapsto L_{f} \in \text{CP}_{A \rightarrow B}$ such that (i) $\text{M}_{f(\phi)}^{(\sigma)} = L_{f}[\text{M}_{\phi}^{(\sigma)}]$ for any $f \in \text{Proc}_{A \rightarrow B}$ and $|\sigma| \in \text{St}_{A}$. It is evident that quantum theory is uniquely singled out from the above three properties. We present a brief comment on these properties. The key step in the derivation of quantum theory is to prove Property (A), which characterizes the state space for each system. Properties (B) and (C) can be easily derived from our postulates and Property (A). Property (B) represents the relation between the state space of a composite system and those of its subsystems. Property (C) implies that each process is fully and faithfully represented by a CP map. This property also characterizes deterministic processes and feasible processes.

We now present an overview of our approach to the derivation of quantum theory. The first half of our approach focuses on analyzing the behavior of an individual system using the first three postulates presented in the previous subsection and shows that the following property holds:

$\Diamond$ For each system $A$, $\text{St}_{A}$ is a symmetric cone and $\text{Eff}_{A}$ is its dual cone.

We will give the definition of a symmetric cone and a dual cone in Subsec. VA. It is a well-known fact that a symmetric cone can be decomposed as a direct sum of five types of irreducible symmetric cones, one of which is the set of complex positive semidefinite matrices, $\text{S}_{1}(\mathbb{C}^{n})$, of a certain order $n$ [50] (see also Theorem 29). The second half of our approach assumes that Property $(\Diamond)$ and the local equality postulate hold. By analyzing the behavior of a composite system, we can conclude that $\text{St}_{A}$ is isomorphic to a direct sum of spaces of complex positive semidefinite matrices, which results in Property (A). Properties (B) and (C) can also be derived. Our approach consists of the following three steps (where the first half is further divided into two steps):

Step 1): Derive some basic properties of PDSs (Sec. IV).
Step 2): Derive Property $(\Diamond)$ (Sec. V).
Step 3): Derive Properties (A)–(C), i.e., single out quantum theory (Sec. VII).

A schematic flow chart of our derivation of quantum theory is shown in Fig. 1. We will briefly explain each step.

Step 1

Firstly, we consider an OPT with the symmetric sharpness and complete mixing postulates and derive some basic properties of PDSs. For instance, we show that, to every MPDS, there corresponds one and only one maximal measurement and that each PDS has certain symmetries.

Step 2

Secondly, we use the filtering postulate in addition to the first two postulates. In this step, we first show that every state
IV. BASIC PROPERTIES OF A PDS

In this section, we present an OPT satisfying the symmetric sharpness and complete mixing postulates and derive some basic properties of PDSs. We list here the main results of our study in this section:

1. To every MPDS, there corresponds one and only one maximal measurement (Lemma 3).
2. Every MPDS of system $A$ has exactly $N_A$ elements (Lemma 7). Also, the sum of all elements is the same for every MPDS of $A$ (Lemma 11).
3. For every PDS $\Phi := (|\phi_i\rangle)_{i=1}^n$, the kernel of $|\chi_{\Phi}\rangle := \Sigma_{i=1}^k |\phi_i\rangle$ is equal to the set of all states $|\psi\rangle$ satisfying $(\chi_{\Phi} | \psi\rangle) = 0$, where $|\chi_{\Phi}\rangle := \Sigma_{i=1}^k |\phi_i\rangle$ (Lemma 9). Also, for every PDS $\Phi$, the face of $|\chi_{\Phi}\rangle$ is equal to the set of all states $|\psi\rangle$ satisfying $(\chi_{\Phi} | \psi\rangle) = (\dagger | \psi\rangle)$ (Lemma 16).

A. Results about symmetric sharpness

From the symmetric sharpness postulate, each maximal effect corresponds to one and only one normalized pure state. Let us define $\dagger$ as the map $\text{St}_A^{\Phi} \ni |\phi\rangle \mapsto |\phi\rangle \dagger := (|\phi\rangle \in \text{Eff}_A^{\Phi})$, where $(|\phi\rangle \dagger$ satisfies $(|\phi\rangle | \phi\rangle = 1$. By a slight abuse of notation, we will use the same symbol $\dagger$ for the inverse map $\text{Eff}_A^{\Phi} \ni (e| \mapsto (|e\rangle \dagger := |e\rangle \in \text{St}_A^{\Phi}$, where $|e\rangle$ satisfies $(e| e\rangle = 1$. We can represent as

$$\left( \begin{array}{c} \dagger \\ \phi \end{array} \right) = \left( \begin{array}{c} \phi \dagger \\ e \end{array} \right), \quad \left( \begin{array}{c} e \dagger \\ \phi \end{array} \right) = \left( \begin{array}{c} \phi \dagger \\ e \end{array} \right). \quad (50)$$

The following lemma ensures that, to every MPDS, there corresponds one and only one maximal measurement.

**Lemma 3** For any MPDS $\Phi := (|\phi_i\rangle)_{i=1}^n$, there exists a unique maximal measurement $\Pi := (|e_i\rangle)_{i=1}^n$ that perfectly distinguishes between $\Phi$. Furthermore, $\Phi$ is the unique MPDS that is perfectly distinguished by $\Pi$.

**Proof** There exists a maximal measurement, denoted by $\Pi' := (|e_i\rangle)_{i=1}^n$, that perfectly distinguishes between $\Phi$. From the symmetric sharpness postulate, $(|\phi_i\rangle \dagger | \phi_i\rangle = 1$. We can represent as

$$|\phi\rangle \mapsto (\phi\rangle \dagger = 0.$$ 

**Lemma 4** We have that, for $|\phi\rangle \in \text{St}_A^{NP}$ and $|\psi\rangle \in \text{St}_A$,

$$|\psi\rangle \in \text{K}_{\Phi}(\psi) \iff (|\phi\rangle | \psi\rangle = 0.$$

**Proof** The case $|\psi\rangle = (\emptyset)$ is obvious; suppose $|\psi\rangle \neq (\emptyset)$. 

$(\Rightarrow)$: Let $(|e_\psi\rangle, (e_\phi\rangle)$ be a measurement that perfectly distinguishes between $(|\phi\rangle, |\psi\rangle)$. Arbitrarily choose $|\varphi\rangle \in \text{St}_A^{NP} \cap \text{K}_{\Phi}(\psi)$. From $(e_\psi | \psi\rangle = (\dagger \psi\rangle$ and Lemma 2, we have $(e_\varphi | \varphi\rangle = (\dagger \varphi\rangle$ =
1. Hence, the set of the pure states \(|\phi, \psi\rangle\) is perfectly distinguished by the measurement \((|e_0\rangle, |e_1\rangle)\) and thus is a PDS. (Note that this implies \(N_A \geq 2\).) Consider an MPDS \((|\phi\rangle, |\psi\rangle, |\phi\rangle, \ldots, |\psi\rangle\) that includes \(|\phi\rangle\) and \(|\psi\rangle\), where \(k\) is a nonnegative integer. From Lemma 3, \(\{(\phi^i), (\psi^i), \ldots, (\psi^i)\}\) is a measurement that perfectly distinguishes between this MPDS, which indicates \((\phi|\psi\rangle = 0\). Since \((\phi|\psi\rangle = 0\) holds for any normalized pure state \(|\psi\rangle \in F_\phi\), we have \((\phi|\phi\rangle = 0\).

\[\varepsilon = \text{Let } \Pi := \{(|\phi^i\rangle, (\phi^i)\} \text{.} \]

Since \((\phi|\psi\rangle = 0\) is feasible, \(\Pi\) is a measurement (see Eq. (44)). From \(\langle\phi^i|\phi\rangle = 1\) and \((\langle\phi^i|\phi\rangle) = (\phi|\phi\rangle\), \(|\phi\rangle\) and \(|\psi\rangle\) are perfectly distinguished by \(\Pi\).

**Lemma 5** Consider \(\{c_i \in R^{+}_{l=1} \}, \{d_j \in R^{+}_{j=1} \}\), \(|\psi_i\rangle \in St^{NP}_{l=1}, \) and \(|\phi_i\rangle \in St^{NP}_{j=1}\). Then, we have

\[
\sum_{i=1}^{l} \frac{c_i}{|\psi_i\rangle} = \sum_{j=1}^{m} \frac{d_j}{|\phi_j\rangle} \Rightarrow \sum_{i=1}^{l} \frac{c_i}{|\psi_i\rangle} = \sum_{j=1}^{m} \frac{d_j}{|\phi_j\rangle}.
\]

**Proof** From the symmetric sharpness postulate, we have that, for any \(|\phi\rangle \in St^{NP}_{l=1}\),

\[
\sum_{i=1}^{l} \frac{c_i}{|\psi_i\rangle} = \sum_{j=1}^{m} \frac{d_j}{|\phi_j\rangle} = \sum_{i=1}^{l} c_i \langle \psi_i | \langle \psi_i | \sum_{j=1}^{m} d_j \langle \phi_j | \phi_j \rangle.
\]

Since any state can be written as a weighted sum of normalized pure states, Eq. (53) gives \(\sum_{i=1}^{l} c_i |\psi^i\rangle = \sum_{j=1}^{m} d_j |\phi^j\rangle\) for any \(|\rho\rangle \in St_A\). Therefore, \(\sum_{i=1}^{l} c_i |\psi^i\rangle = \sum_{j=1}^{m} d_j |\phi^j\rangle\) holds.

For every \(|\rho\rangle \in St_A\), which can be expressed in the form \(|\rho\rangle = \sum_{i=1}^{l} c_i |\psi^i\rangle\) with \(c_1, \ldots, c_l \in R_{++}\) and \(|\psi_1\rangle, \ldots, |\psi_l\rangle \in St^{NP}_{l=1}\), \(|\rho\rangle \in \text{Eff}_{A}^{+}\) is defined as

\[
|\rho\rangle := \left(\sum_{i=1}^{l} c_i |\psi_i\rangle\right) = \sum_{i=1}^{l} c_i |\psi_i\rangle.
\]

(\(0 = 0\) obviously holds. Lemma 5 guarantees that \(|\rho\rangle\) is uniquely determined, regardless of how \(|\rho\rangle\) is decomposed. Lemma 5 can be immediately generalized to extended states. Indeed, it is clear that Eq. (52) holds even if \(c_i\) and \(d_j\) are any real numbers. Thus, for any extended state \(|\bar{\psi}\rangle \in V_A\), which is expressed in the form \(|\bar{\psi}\rangle = \sum_{i=1}^{l} c_i |\psi^i\rangle\) with \(c_1, \ldots, c_l \in R^+\) and \(|\psi_1\rangle, \ldots, |\psi_l\rangle \in St^{NP}_{l=1}\), we can define \(|\bar{\psi}\rangle := \sum_{i=1}^{l} c_i |\psi_i\rangle\). Clearly, \(|\bar{\psi}\rangle\) is uniquely determined, regardless of how \(|\bar{\psi}\rangle\) is decomposed. One can easily verify that the map \(\bar{\psi} : V_A \ni |\bar{\psi}\rangle \mapsto \sum_{i=1}^{l} c_i |\psi_i\rangle \in V_A^*\) is linear. We should note that the converse of Eq. (52) does not necessarily hold.

For every PDS \(\Phi : \{(|\phi_i\rangle)_{i=1}^{k} \in \text{PDS}_A, |\phi_i\rangle\) is defined as

\[
|\Phi\rangle := \sum_{i=1}^{k} c_i |\phi_i\rangle.
\]

Since \((\bar{\psi}|\phi\rangle = k\) holds, \(|\phi\rangle\) is not feasible if \(k\) is larger than 1.

**Lemma 6** For any \(\Phi \in \text{MPDS}_{A}\) and \(|\epsilon\rangle \in \text{Eff}_{A}^{\Phi}\), we have

\[
\langle \epsilon | \Phi \rangle = 1.
\]

**Proof** Let \(\Phi : \{(|\phi_i\rangle)_{i=1}^{k} \in \text{MPDS}_{A}\) with \(|\phi_i\rangle = N_A\). From Lemma 6, for any \(|\phi_i\rangle \in \text{MPDS}_A\), we have

\[
n = \sum_{i=1}^{k} |\phi_i\rangle = \sum_{i=1}^{k} \frac{\phi_i}{|\phi_i\rangle} = N_A.
\]

**Lemma 7** Every MPDS of system \(A\) has exactly \(N_A\) elements.

**Proof** There exists an MPDS \(\Phi : \{|\phi_i\rangle\}_{i=1}^{N_A} \in \text{MPDS}_A\) with \(|\phi_i\rangle = N_A\). From Lemma 6, for any \(|\phi_i\rangle \in \text{MPDS}_A\), we have

\[
n = \sum_{i=1}^{k} |\phi_i\rangle = \sum_{i=1}^{k} \frac{\phi_i}{|\phi_i\rangle} = N_A.
\]

**Lemma 8** For any \(\Phi \in \text{PDS}_A\), both \((\chi^{\downarrow}_{\Phi})\) and \((\bar{\psi} - (\chi^{\uparrow}_{\Phi})\) are feasible effects.

**Proof** Let \(\Phi : \{|\phi_i\rangle\}_{i=1}^{k} \subseteq \text{MPDS}_A\). Consider an MPDS \(\Phi_{ext} : \{|\phi_i\rangle\}_{i=1}^{k} \supseteq \Phi\). Since \(|\phi_i\rangle \in \text{MPDS}_A\) is a measurement from Lemma 3, \((\chi^{\downarrow}_{\Phi})\) is a feasible effect. Also, from Eq. (44), \((\bar{\psi} - (\chi^{\uparrow}_{\Phi})\) is a feasible effect.

The face of a PDS \(\Phi \in \text{PDS}_A\) is defined as \(\mathcal{F}_{\Phi} := \mathcal{F}_{|\phi\rangle}\). Also, the kernel of \(\Phi\) is defined as \(\mathcal{K}_{\Phi} := \mathcal{K}_{|\phi\rangle}\). They can also be expressed by

\[
\mathcal{F}_{\Phi} = \{|\rho\rangle \in St_A : \exists \delta \in R_{++}, \delta |\rho\rangle \leq |\phi\rangle\},
\]

\[
\mathcal{K}_{\Phi} = \{|\rho\rangle : |\rho\rangle \text{ is perfectly distinguishable from } |\phi\rangle\}.
\]

Note that \(\mathcal{F}_{\Phi}\) and \(\mathcal{K}_{\Phi}\) depend only on \(|\phi\rangle\), which implies that, for any \(\Psi \in \text{PDS}_A\) satisfying \(|\phi\rangle = |\phi\rangle\), \(\mathcal{F}_{\Phi} = \mathcal{F}_{\Psi}\) and \(\mathcal{K}_{\Phi} = \mathcal{K}_{\Psi}\) hold.

**Lemma 4** can be generalized as follows:

**Lemma 9** We have that, for \(\Phi \in \text{PDS}_A\) and \(|\phi\rangle \in St_A\),

\[
|\phi\rangle \in \mathcal{K}_{\Phi} \iff (\chi^{\downarrow}_{\Phi}|\phi\rangle = 0\).
\]
Proof Let $\Phi := \{\langle \phi_i \rangle \}_{i=1}^k$. The case $k = 0$ is obvious since $|\psi_0\rangle = |0\rangle$ and $K_0 = S_{A_0}$ holds. Suppose $k \geq 1$.

$\implies$ Let $\Pi := \{(e_{0}, (e_{0}), \ldots, (e_{0}))\}$ be a measurement that perfectly distinguishes between $|\psi_{k}\rangle$, $|\psi_{k}\rangle$. For each $i = \{1, \ldots, k\}$, we have $|\phi_{i}\rangle \in F_{\phi}$, and thus $(e_{0})|\phi_{i}\rangle = (\mp |\phi_{i}\rangle = 1$ holds from Lemma 2. This implies that $\Pi$ perfectly distinguishes between $|\psi_{i}\rangle$, $|\psi_{i}\rangle$, i.e., from Lemma 4, $(\phi_{i})|\rho\rangle = 0$ holds. Therefore, $(\phi_{i})|\rho\rangle = \sum_{i=1}^{k} (\phi_{i})|\rho\rangle = 0$ holds.

$=:\quad$ Since $(\phi_{i})$ is feasible from Lemma 8, $\Pi' := \{(\phi_{i})|\phi_{i}\rangle, (\mp |\phi_{i}\rangle = 0$ holds. We have

\[
(\phi_{i})|\phi_{i}\rangle = \sum_{i=1}^{k} (\phi_{i})|\phi_{i}\rangle = k = (\mp |\phi_{i}\rangle ,
\]

where the second equality in the first line follows from $(\phi_{i})|\phi_{i}\rangle = \delta_{i,j}$. Therefore, $|\phi_{i}\rangle$ and $|\rho\rangle$ are perfectly distinguished by $\Pi'$.

Lemma 4 is the special case of this lemma where $\Phi = \{\langle \phi_i \rangle\}$. From Lemma 9, $K_0$ is rewritten as $K_0 = \{\rho \in S_{A_0} : (\phi_{i})|\rho\rangle = 0\}$. It is easily seen that $(\phi_{i})|\rho\rangle = 0$ holds for any $|\rho\rangle \in K_0$ and $(\phi_{i}) \in F_{\phi}$.

Lemma 10 For any $\Phi_1, \Phi_2 \in PDS_A$ with $\{\phi_{i}'|\phi_{i}\rangle = 0, \Phi_1 \cup \Phi_2 \in PDS_A$ holds.

Proof If $|\phi_{i}\rangle = 0$ or $|\phi_{i}\rangle = 0$, then the lemma is obvious. Suppose $|\phi_{i}\rangle = 0$ or $|\phi_{i}\rangle = 0$. Let $\Phi_1 := \{(\phi_{i})|\phi_{i}\rangle = 0 \}$ and $\Phi_2 := \{(\phi_{i})|\phi_{i}\rangle = 0 \}$ with $0 < k < m$. Also, let $\Psi_{1} := \{P_{i} = (\phi_{i})|\phi_{i}\rangle = 0 \}$ for each $i = \{0, 1, \ldots, m-k\}$. To prove that $\Psi_{m-k} = \Phi_1 \cup \Phi_2$ is a PDS, we proceed by induction on $l$. $\Psi_{l} = \Phi_1 \cup \Phi_2$ holds. Suppose $\Psi_{l} \in PDS_A$ with $0 < l < m-k$. Let $l := k + l + 1$. Since $|\phi_{i}\rangle \in F_{\phi}$, holds from $|\phi_{i}\rangle \in F_{\phi}$, Lemma 2 gives $(\phi_{i})|\phi_{i}\rangle = 0$. Also, from $\Phi_2, (\phi_{i})|\phi_{i}\rangle = 0$ holds for any $i = \{0, 1, \ldots, l\}$. Thus, we have

\[
\begin{align*}
\left(\frac{e}{\phi_{i}}\right)^{\dagger} &= \left(\frac{e}{\phi_{i}}\right)^{\dagger} + \sum_{i=1}^{k} \left(\frac{e}{\phi_{i}}\right)^{\dagger} \\
&= 0.
\end{align*}
\]

Let $(e) := (\mp |\phi_{i}\rangle$; then, from Lemma 8, $(e) \in EF_A^{p}$ holds.

This implies that $\Pi := \{(\phi_{i})|\phi_{i}\rangle, (e)\}$ is a measurement. We also have

\[
\begin{align*}
\left(\frac{e}{\phi_{i}}\right)^{\dagger} &= \left(\frac{e}{\phi_{i}}\right)^{\dagger} - \left(\frac{e}{\phi_{i}}\right)^{\dagger} \\
&= 1.
\end{align*}
\]

Thus, $\Psi_{l+1}$ is perfectly distinguished by $\Pi$ and thus a PDS. Therefore, $\Psi_{m-k}$ is a PDS.

B. Results about symmetric sharpness and complete mixing

We here discuss an OPT satisfying the symmetric sharpness and complete mixing postulates.

Lemma 11 $|\chi_{\Phi}| = |\chi_{\Psi}|$ holds for any $\Phi, \Psi \in MPDS_A$.

Proof Let $p$ be the maximum value of $p' \in R_{+}$ satisfying $p'|\chi_{\Phi}| \leq |\chi_{\Psi}|$. Also, let

\[
\begin{align*}
\left(\frac{e}{\phi_{i}}\right)^{\dagger} &= \left(\frac{e}{\phi_{i}}\right)^{\dagger} - \left(\frac{e}{\phi_{i}}\right)^{\dagger} \\
&= 1 - p.
\end{align*}
\]

Since $|\chi_{\Phi}| \not\in F_{(p', s)}$ holds from the definition of $p$, $(s)$ is not completely mixed. Thus, from the complete mixing postulate, there exists a maximal effect $(e)$ satisfying $(e)s = 0$. Therefore, we have

\[
\left(\frac{e}{s}\right)^{\dagger} = \left(\frac{e}{s}\right)^{\dagger} - \left(\frac{e}{s}\right)^{\dagger} = 1 - p,
\]

i.e., $p = 1$, where the last equality follows from $(e) = (e) = 1$ by Lemma 6. Substituting $p = 1$ into Eq. (64) gives

\[
\begin{align*}
\left(\frac{e}{s}\right)^{\dagger} &= \left(\frac{e}{s}\right)^{\dagger} - \left(\frac{e}{s}\right)^{\dagger} = N_{A} - N_{A} = 0,
\end{align*}
\]

where the second equality follows from Lemma 7. Hence, we have $(s) = |0\rangle$, which means $|\chi_{\Phi}| = |\chi_{\Psi}|$.

By lemma 11, we know that $|\chi_{\Phi}|$ with $\Phi \in MPDS_A$ depends only on $A \in Syst$ and not on $\Phi$. Let us denote such $|\chi_{\Phi}|$ by $|\chi_{A}|$ or simply $|\chi|$ and refer to it as the invariant state of $A$.

Clearly, $(\chi_{A}) = (\mp A)$ holds.

Example of quantum theory $|\chi_A|$ is the identity matrix of order $N_A$.

Lemma 12 The invariant state is completely mixed.

Proof Any $|\rho\rangle \in S_{A_0}$ has the form $|\rho\rangle = \sum_{i=1}^{l} c_{i} |\psi_{i}\rangle$ with $c_{1}, \ldots, c_{l} \in R_{+}$ and $|\psi_{1}\rangle, \ldots, |\psi_{l}\rangle \in S_{A_0}^{p}$. For each $i \in \{1, \ldots, l\}$, consider an MPDS $\Phi$ including $|\psi_{i}\rangle$; then, it follows that $|\psi_{i}| \leq |\chi_{\Phi}| = |\chi_{A}|$, which leads to $|\rho\rangle \in T_{|\chi_{A}|}$. Therefore, $|\chi_{A}|$ is completely mixed.

We will say that a set of $k$ PDSs $\{\Phi_i \in PDS_A\}_{i=1}^{k}$ can compose an MPDS if $\Phi_1, \ldots, \Phi_k$ are disjoint and $\bigcup_{i=1}^{k} \Phi_i$ is an MPDS. In particular, we will call two PDSs $\Phi$ and $\Psi$ complementary if $\Phi, \Psi$ can compose an MPDS. We see at once that, for any $\Phi \in PDS_A$, there exists $\Psi \in PDS_A$ such that $\Phi$ and $\Psi$ are complementary.

Lemma 13 Consider a set of $k$ PDSs $\Phi := \{\Phi_i \in PDS_A\}_{i=1}^{k}$. Then, $\Phi$ can compose an MPDS if and only if $\sum_{i=1}^{k} |\chi_{\Phi_i}| = |\chi_{\Phi}|$ holds.

By this lemma, $\Phi, \Psi \in PDS_A$ are complementary if and only if $|\chi_{\Phi} + \Phi_{\Psi}| = |\chi|$ holds.

Proof The case $k \leq 1$ is obvious; suppose $k > 1$.

"Only if": Since $\Phi_1, \ldots, \Phi_k$ are disjoint and $\Phi := \bigcup_{i=1}^{k} \Phi_i$ is an MPDS, we have $\sum_{i=1}^{k} |\chi_{\Phi_i}| = |\chi_{\Phi}| = |\chi|$. 
Lemma 16

Proof

(54)

where the second equality follows from Lemma 6. Thus, \((\langle x, \psi \rangle \vert \phi) = 0\) holds for any distinct \(i, j \in \{1, \ldots, k\}\).

Lemma 14

If two PDSs \(\Phi\) and \(\Psi\) satisfy \(|\chi_{\Phi}\rangle \in \mathcal{K}_\Phi\), then there exists a PDS that includes \(\Phi\) and is complementary to \(\Psi\).

Proof

Using Lemma 9, we have \(\langle x, \psi \rangle \vert \phi \rangle = 0\). From this and Lemma 10, \(\Phi \cup \Psi \) is a PDS such that \(|\chi_{\Phi \cup \Psi} \rangle = |\chi_{\Phi} \rangle + |\chi_{\Psi} \rangle\). Let \(\Phi'\) be a PDS complementary to \(\Phi \cup \Psi\); then, \(|\chi_{\Phi'} \rangle \vert \phi \rangle = 0\) holds, and thus, from Lemma 10, \(\Phi \cup \Phi'\) is a PDS including \(\Phi\). Since \(|\chi_{\Phi'} \rangle \vert \phi \rangle = |\chi_{\Phi} \rangle + |\chi_{\Psi} \rangle + |\chi_{\Phi'} \rangle + |\chi_{\Psi} \rangle = |\chi \rangle\), holds if we exchange \(\Phi\) and \(\Psi\). Thus, for any two PDSs \(\Phi, \Psi \in \text{PDS}_A\), \(\mathcal{K}_\Phi = \mathcal{F}_\Psi\) and \(\mathcal{F}_\Phi = \mathcal{K}_\Psi\) are obviously equivalent.

V. SYMMETRIC PROPERTIES OF STATE SPACE

In this section, we show Property (6) in an OPT that satisfies the symmetric sharpness, complete mixing, and filtering postulates. The main results in this section are:

1. Every state has a spectral decomposition, which means that any \(|\psi\rangle \in \text{St}_A\) can be expressed in the form \(|\psi\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle\) with \(p_1, \ldots, p_{N_A} \in \mathbb{R}\), and \(|\phi_i\rangle\) \(i=1, \ldots, N_A\) \(\in \text{MPDS}_A\) (Proposition 20).
2. \(\text{St}_A\) is a symmetric cone (Theorem 25).
3. \(\text{Eff}_A\) is the dual cone of \(\text{St}_A\) (Theorem 28).

A. Symmetric cones

We here review the definitions of symmetric cones. A convex cone \(C\) in a real vector space \(V\) is called a symmetric cone if \(C\) is self-dual and homogeneous, whose definitions are given below.

First, we will recall the definition of self-duality. Let \(C\) be a convex cone in a real vector space \(V\).

\[ C^* := \{ f \in V^*: \forall x \in C, f(x) \geq 0 \} \]

is called the dual cone of \(C\), where \(V^*\) is the dual vector space of \(V\). One can easily verify that \(C^*\) is a closed convex cone. For any inner product of \(V\), denoted by \(\langle , \rangle\), there exists an isomorphism \(\# : V^* \cong f \mapsto f^\# \in V\) satisfying \(f(x) = \langle f^\#, x \rangle\) for any \(x \in V\) and \(f \in V^*\).

Let

\[ C^* := \{ f \in V : f(x) \geq 0 \} \]

then, \(C^* = \{ x \in V : \forall y \in C, \langle x, y \rangle \geq 0 \} \) obviously holds. \(C^*\) and \(C^*\) are isomorphic as convex cones. Also, the restriction of \# to \(C^*\) is an isomorphism from \(C^*\) to \(C^*\). If \(C^* = \text{C}\) holds for some inner product \(\langle , \rangle\), then \(C\) is called self-dual with respect to \(\langle , \rangle\). In this case, \(C \cong C^*\) obviously holds. In particular, the state space \(\text{St}_A\) is self-dual if there exists an inner product \(\langle , \rangle\) such that

\[ \text{St}_A = \{ \tau \in \text{V}_A : \forall \rho \in \text{St}_A, \langle \tau, \rho \rangle \geq 0 \} \]

where \((\tau, \rho) \in \text{V}_A\) is a simple notation for \(\langle \tau, \rho \rangle\).

Next, we will give the definition of homogeneity. A convex cone \(C\) is called homogeneous if, for any two interior points \(x, y \in C\), there exists an automorphism \(g_{x,y}\) on \(C\) such that \(g_{x,y}(x) = y\).
B. Projection processes

For any $\Phi \in \text{PDS}_A$, a process $P_\Phi \in \text{Proc}_{A \rightarrow A}$ is called a projection process onto $\mathcal{F}_\Phi$ if

\[
P_\Phi = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall \left( \begin{array}{c} p \\ \sigma \end{array} \right) \in \mathcal{F}_\Phi,
\]

\[
P_\sigma = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall \left( \begin{array}{c} \sigma \\ \sigma \end{array} \right) \in \mathcal{K}_\Phi
\]  \tag{73}

holds.

Example of quantum theory For any PDS $\Phi := \{ |\phi_i \rangle \}_{i=1}^N$, a projection process onto $\mathcal{F}_\Phi$ is expressed by $P_\Phi \circ |\phi \rangle = |\chi_\Phi \rangle \cdot |\phi \rangle \cdot |\chi_\Phi \rangle$. One can easily see that $|\chi_\Phi \rangle$ is a projection matrix.

The following lemma guarantees the existence of projection processes.

Lemma 18 For any $\Phi \in \text{PDS}_A$, there exists a projection process $P_\Phi$ onto $\mathcal{F}_\Phi$.

Proof If $\Phi$ is an MPDS, then $\text{id}_A$ is a projection process onto $\mathcal{F}_\Phi = \mathcal{S}_A$, and thus the lemma is obvious. Suppose that $\Phi$ is not an MPDS. Let $\Phi := \{ |\phi_i \rangle \}_{i=1}^k$ be a projection process onto $\mathcal{F}_\Phi$. It suffices to show that $P_\Phi := F_0^{\phi_1} \circ \cdots \circ F_0^{\phi_k} \circ F_0^{\phi_1}$ is a projection process onto $\mathcal{F}_\Phi$, where $F_0^{\phi_1}, \ldots, F_0^{\phi_k}, F_0^{\phi_1}$ are filters (see Eq. (46)). Arbitrarily choose $|\phi \rangle \in \mathcal{F}_\Phi$ and $i \in \{ k+1, \ldots, N \}$. From $\mathcal{F}_\Phi$ and $\mathcal{F}_\Phi$ holds for $|\sigma \rangle \in \mathcal{K}_\Phi$. Therefore, we have $F_0^{\phi_k} \circ |\phi \rangle = |\phi \rangle$, which gives $P_\Phi \circ |\phi \rangle = |\phi \rangle$. We also arbitrarily choose $|\phi \rangle \in \mathcal{K}_\Phi$. From Lemma 17, we have $|\sigma \rangle \in \mathcal{F}_\Phi$, i.e., $\delta |\sigma \rangle \leq |\chi_\Phi \rangle$ holds for some $\delta \in \mathbb{R}_+$. Thus, $\delta P_\Phi \circ |\sigma \rangle \leq P_\Phi \circ |\chi_\Phi \rangle = \sum_{i=1}^N P_\Phi \circ |\phi_i \rangle = |0 \rangle$ obviously holds, which gives $P_\Phi \circ |\sigma \rangle = |0 \rangle$.

C. Spectral decomposition

We here show that any state and any extended state have spectral decompositions. The following lemma is useful for proving the existence of a spectral decomposition of a state.

Lemma 19 Consider a PDS $\Phi \in \text{PDS}_A \setminus \text{MPDS}_A$ (where \ denotes the set difference operator). Then, for any $|\phi \rangle \in \mathcal{K}_\Phi$, there exist $p \in \mathbb{R}_+$, $|\phi \rangle \in \mathcal{S}_A \cap \mathcal{K}_\Phi$, and $|\phi' \rangle \in \mathcal{K}_\Phi$ satisfying

\[
\begin{bmatrix} \sigma \\ \rho \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \phi \end{bmatrix} \begin{bmatrix} \phi^\dagger \end{bmatrix} = 0.
\]

Furthermore, $\Phi' := \Phi \cup \{ |\phi \rangle \} \in \text{PDS}_A$ and $|\phi' \rangle \in \mathcal{K}_\Phi$ hold.

Proof Let $\Psi$ be a PDS complementary to $\Phi$ and $p$ be the maximum value of $\rho' \in \mathbb{R}_+$ satisfying $|\rho' \rangle \geq p |\chi_\Psi \rangle$. Also, let $|\sigma \rangle := |\rho \rangle - p |\chi_\Psi \rangle$. We have $0 \leq (\chi_\Phi^\dagger |\sigma \rangle \leq (\chi_\Phi^\dagger |\phi \rangle = 0$, i.e., $|\sigma \rangle \in \mathcal{K}_\Phi$.

Firstly, we prove that $|\sigma \rangle + |\chi_\Phi \rangle$ is not completely mixed. Assume, by contradiction, that $|\sigma \rangle + |\chi_\Phi \rangle$ is completely mixed; then, there exists $\delta \in \mathbb{R}_+$ such that

\[
\begin{bmatrix} \sigma \\ \rho \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} \rho \end{bmatrix} = \delta \begin{bmatrix} \chi_\Phi \end{bmatrix},
\]

(75)

Since $\mathcal{F}_\Phi = \mathcal{K}_\Psi$ and $\mathcal{K}_\Phi = \mathcal{F}_\Psi$ hold from Lemma 17, we have $|\sigma \rangle \in \mathcal{K}_\Psi = \mathcal{F}_\Psi$, $|\chi_\Phi \rangle \in \mathcal{F}_\Phi = \mathcal{K}_\Phi$, and $|\chi_\Psi \rangle \in \mathcal{F}_\Psi$. Thus, $P_\Psi \circ |\sigma \rangle = |\sigma \rangle$, $P_\Psi \circ |\chi_\Phi \rangle = |0 \rangle$, and $P_\Psi \circ |\chi_\Psi \rangle = |\chi_\Psi \rangle$ hold, where $P_\Psi$ is a projection process onto $\mathcal{F}_\Psi$. This yields

\[
\begin{bmatrix} \sigma \\ \rho \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} \rho \end{bmatrix} = \delta \begin{bmatrix} \chi_\Psi \end{bmatrix},
\]

(76)

where the inequality follows from Eq. (75). Therefore, we have

\[
\begin{bmatrix} \rho \\ \rho \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} \rho + \delta \end{bmatrix} = \begin{bmatrix} \rho + \delta \end{bmatrix} \geq \begin{bmatrix} 0 \end{bmatrix},
\]

(77)

which contradicts the definition of $p$. Hence, $|\sigma \rangle + |\chi_\Phi \rangle$ is not completely mixed.

Secondly, we show that there exist $|\phi \rangle \in \mathcal{S}_A \cap \mathcal{K}_\Phi$ and $|\phi' \rangle \in \mathcal{K}_\Phi$ satisfying Eq. (74). From the complete mixing postulate, there exists $|\phi \rangle \in \mathcal{S}_A \cap \mathcal{K}_\Phi$ satisfying $(\phi^\dagger |\chi_\Psi \rangle + |\chi_\Psi \rangle) = 0$, i.e., $\phi^\dagger |\sigma \rangle = (\phi^\dagger |\phi \rangle = 0$. From $(\phi^\dagger |\phi \rangle = 0$, we have $|\phi \rangle \in \mathcal{K}_\Phi$. From Lemma 15, $|\phi \rangle \in \mathcal{K}_\Phi$ holds, and thus $|\sigma \rangle + p |\chi_\Psi \rangle \geq p |\chi_\Psi \rangle \geq p |\phi \rangle$ holds. Let $|\rho' \rangle := |\rho \rangle - p |\phi \rangle \in \mathcal{S}_A$; then, $|\rho' \rangle \in \mathcal{K}_\Phi$ holds from $|\rho \rangle \in \mathcal{K}_\Phi$. Moreover, we have

\[
|\rho' \rangle = |\rho \rangle - p |\phi \rangle = |\phi \rangle + p |\phi \rangle - p = 0
\]

(78)

where the last equality follows from the fact that, from Lemma 6, we have $(\phi^\dagger |\chi_\Psi \rangle = 1$ and thus $(\phi^\dagger |\chi_\Psi \rangle = (\phi^\dagger |\rho \rangle - (\phi^\dagger |\phi \rangle = 1$. Therefore, Eq. (74) holds.

Finally, we show $\Phi' \in \text{PDS}_A$ and $|\phi' \rangle \in \mathcal{K}_\Phi$. From $(\phi^\dagger |\rho \rangle = 0$ and Lemma 10, we have $\Phi' \in \text{PDS}_A$. Also, since we have $(\chi_\Phi^\dagger |\rho \rangle = (\chi_\Phi^\dagger |\phi \rangle = 0$, $|\rho' \rangle \in \mathcal{K}_\Phi$ holds from Lemma 9.

Now, we are in a position to show the following proposition.

Proposition 20 (Spectral decompositions of states) For any $|\rho \rangle \in \mathcal{S}_A$, there exist $p_1, \ldots, p_N \in \mathbb{R}_+$ and $\{ |\phi_i \rangle \}_{i=1}^N \in \text{MPDS}_A$ satisfying

\[
|\rho \rangle = \sum_{i=1}^N p_i |\phi_i \rangle.
\]  \tag{79}
The decomposition shown in Eq. (79) is called a spectral decomposition of $|\rho\rangle$.

**Proof** Let $|\rho_1\rangle := |\rho\rangle$. We show that, for each $k \in \{1, \ldots, N_A\}$, $|\rho_k\rangle$ can be expressed in the form

$$|\rho_k\rangle = \frac{1}{\sqrt{\lambda_k}} \sum_{i=1}^{N_k} c_i |\phi_i\rangle,$$

where $c_i$ are complex coefficients,

$$\sum_{i=1}^{N_k} c_i^* c_i = 1,$$

and $|\phi_i\rangle$ are orthonormal states.

(80)

We now proceed by induction on $k$. Let $\Phi_0$ be the empty set. Since $\Phi_0 \in \text{PDS}_A \setminus \text{MPDS}_A$ and $|\rho_1\rangle \in \text{St}_A$, $K_{\Phi_0}$ holds. Lemma 19 gives that there exist $p_1 \in R_+$ and $|\phi_1\rangle \in \text{St}^{\text{NP}}_A$, and $|\rho_2\rangle \in \text{St}_A$ satisfying Eq. (80) with $k = 1$. $|\rho_2\rangle \in K_{\Phi_0}$ also holds, where $\Phi_1 := \{|\phi_1\rangle\}$. Consider the case $k \in \{2, \ldots, N_A\}$; let $\Phi_k := \Phi_{k-1} \cup \{|\phi_k\rangle\} = \{|\phi_i\rangle\}_{i=1}^{k}$. Since $\Phi_{k-1} \in \text{PDS}_A \setminus \text{MPDS}_A$ and $|\rho_k\rangle \in K_{\Phi_{k-1}}$, from Lemma 19, there exist $p_k \in R_+$, $|\phi_k\rangle \in \text{St}^{\text{NP}}_A \cap K_{\Phi_{k-1}}$, and $|\rho_{k+1}\rangle \in K_{\Phi_k}$ satisfying Eq. (80).

From $|\rho_{N_A}\rangle = N_A$, $\Phi_{N_A}$ is obviously an MPDS. Thus, from $|\rho_{N_A+1}\rangle \in K_{\Phi_{N_A}} = \{|\rho\rangle\}$, $|\rho_{N_A+1}\rangle = |\rho\rangle$ holds. By recursively applying Eq. (80), we have

$$|\rho\rangle = \sum_{i=1}^{N_A} c_i |\phi_i\rangle.$$

(81)

Using Proposition 20, we can show that every extended state has also a spectral decomposition.

**Proposition 21** (Spectral decompositions of extended states) For any $|\overline{\rho}\rangle \in \text{V}_A$, there exist $c_1, \ldots, c_{N_A} \in R$ and $|\phi_i\rangle_{i=1}^{N_A} \in \text{MPDS}_A$ satisfying

$$|\overline{\rho}\rangle = \sum_{i=1}^{N_A} c_i |\phi_i\rangle.$$

(82)

The decomposition shown in Eq. (82) is called a spectral decomposition of $|\overline{\rho}\rangle$.

**Proof** $|\overline{\rho}\rangle \in \text{V}_A$ can be expressed in the form $|\overline{\rho}\rangle = |\rho_+\rangle - |\rho_-.\rangle$ for some $|\rho_+\rangle, |\rho_-\rangle \in \text{St}_A$. Arbitrarily choose $q \in R_+$ such that $q|\rho_-\rangle \geq |\rho_-\rangle$. Note that since, from Lemma 12, $|\rho\rangle$ is completely mixed, such $q$ exists. Since $|\rho\rangle := |\overline{\rho}\rangle + q|\rho_-\rangle = |\rho_+\rangle + |\rho_-\rangle \geq |\rho_-\rangle$ holds, $|\rho\rangle \in \text{St}_A$ holds. From Proposition 20, $|\rho\rangle$ has a spectral decomposition as in Eq. (79) with $p_1, \ldots, p_{N_A} \in R_+$ and $|\phi_i\rangle_{i=1}^{N_A} \in \text{MPDS}_A$. Thus, letting $c_i := p_i - q$, we have

$$|\overline{\rho}\rangle = \sum_{i=1}^{N_A} c_i |\phi_i\rangle.$$

(83)

**Example of quantum theory** A spectral decomposition of a state (resp. extended state) is a spectral decomposition of a positive semidefinite matrix (resp. Hermitian matrix).

D. Self-duality

Using Proposition 20, we will derive that each state space is self-dual.

**Lemma 22** The binary operation $\langle , \rangle$ on $\text{V}_A$ defined by

$$\langle \overline{\rho}, \overline{\sigma} \rangle := \langle \overline{\rho} | \overline{\sigma} \rangle, \quad \overline{\rho}, \overline{\sigma} \in \text{V}_A$$

(84)

is an inner product.

**Proof** It suffices to show that $\langle , \rangle$ satisfies (1) symmetry: $\langle \overline{\rho}, \overline{\sigma} \rangle = \langle \overline{\sigma}, \overline{\rho} \rangle$ ($\forall \overline{\rho}, \overline{\sigma} \in \text{V}_A$), (2) linearity in the second argument: $\langle \overline{\rho}, a_1 |\overline{\sigma}\rangle + a_2 |\overline{\sigma}\rangle \rangle = a_1 \langle \overline{\rho}, |\overline{\sigma}\rangle \rangle + a_2 \langle \overline{\rho}, |\overline{\sigma}\rangle \rangle$, $a_1, a_2 \in R$ ($\forall \overline{\rho}, \overline{\sigma}, |\overline{\sigma}\rangle \rangle \in \text{V}_A$), and (3) positivity-definiteness: $\langle \overline{\rho}, \overline{\rho} \rangle > 0$ ($\forall \overline{\rho} \neq |\overline{\sigma}\rangle \rangle \in \text{V}_A$).

(1): $\langle \overline{\rho}, \overline{\sigma} \rangle \in \text{V}_A$ and $|\overline{\sigma}\rangle \rangle$ in $\text{V}_A$ can be expressed in the form $|\overline{\rho}\rangle = \sum_{i=1}^{N_A} c_i |\phi_i\rangle$ and $|\overline{\sigma}\rangle = \sum_{i=1}^{N_A} d_i |\phi_i\rangle$, where $c_1, \ldots, c_{N_A} \in R$ and $|\phi_1\rangle, \ldots, |\phi_{N_A}\rangle \in \text{St}^{\text{NP}}_A$ hold. We have

$$\langle \overline{\rho}, \overline{\sigma} \rangle = \sum_{i=1}^{N_A} c_i d_i = \sum_{i=1}^{N_A} c_i d_i = \sum_{i=1}^{N_A} c_i d_i > 0.$$

(2): For any $|\overline{\rho}\rangle \in \text{V}_A$, since $|\overline{\rho}\rangle \in \text{V}_A^*$ is a linear functional on $\text{V}_A$, the map $\text{V}_A \ni |\overline{\rho}\rangle \langle |\overline{\rho}\rangle \rangle \rightarrow \langle \overline{\rho}, \overline{\sigma} \rangle$ is obviously linear.

(3): For any $|\overline{\rho}\rangle \in \text{V}_A \setminus |\overline{\sigma}\rangle \rangle$, which has a spectral decomposition of the form $|\overline{\rho}\rangle = \sum_{i=1}^{N_A} c_i |\phi_i\rangle$ with $c_1, \ldots, c_{N_A} \in R$ and $|\phi_1\rangle, \ldots, |\phi_{N_A}\rangle \in \text{St}^{\text{NP}}_A$, we have

$$\langle \overline{\rho}, \overline{\rho} \rangle = \sum_{i=1}^{N_A} c_i^2 > 0.$$

**Example of quantum theory** $\langle \overline{\rho}, \overline{\sigma} \rangle := \langle \overline{\rho} | \overline{\sigma} \rangle = \text{Tr} [\overline{\rho} \cdot |\overline{\sigma}\rangle \rangle]$ ($\forall \overline{\rho}, \overline{\sigma} \in \text{V}_A$) is an inner product.

**Proposition 23** For any system $A$, $\text{St}_A$ is self-dual with respect to the inner product $\langle , \rangle$ of Eq. (84).

**Proof** Arbitrarily choose $|\overline{\rho}\rangle \in \text{V}_A$. It suffices to show that $|\overline{\rho}\rangle \in \text{St}_A$ holds if and only if $\langle \overline{\rho}, \overline{\rho} \rangle > 0$ holds for any $|\overline{\mu}\rangle \in \text{St}_A$. The “only if” part is obvious from $|\overline{\rho}\rangle \in \text{EF}_A$. To prove the “if” part, we assume $|\overline{\rho}\rangle \notin \text{St}_A$ and show that there exists $|\overline{\mu}\rangle \in \text{St}_A$ such that $\langle \overline{\rho}, \overline{\mu} \rangle < 0$. $|\overline{\rho}\rangle$ has a spectral decomposition $|\overline{\rho}\rangle = \sum_{i=1}^{N_A} c_i |\phi_i\rangle$ with $c_1, \ldots, c_{N_A} \in R$ and $|\phi_i\rangle_{i=1}^{N_A} \in \text{MPDS}_A$. From $|\overline{\rho}\rangle \notin \text{St}_A$, $c_i < 0$ holds for some $i \in \{1, \ldots, N_A\}$. For such $i$, we have $\langle \overline{\rho}, |\phi_i\rangle \rangle = \langle \overline{\rho} | \phi_i \rangle = c_i < 0$. Thus, the “if” part is proved.
E. Homogeneity and symmetry

We will derive that each state space is homogeneous and thus symmetric. Homogeneity is easily derived from Proposition 20 and the filtering postulate.

**Proposition 24** For any system $A$, $St_A$ is homogeneous.

**Proof** It is sufficient to show that, for any completely mixed state $|\rho\rangle \in St_A$, there exists a reversible process $f_{00} \in Proc_{A\to A}$ satisfying $f_{00} \circ |\gamma\rangle = |\rho\rangle$. Indeed, in this case, let $f_{00}$ be an inverse process of $f_{00}$; then, for any completely mixed states $|\rho_1\rangle, |\rho_2\rangle \in St_A$, $g := f_{00} \circ f_{00}$ is an automorphism on $St_A$ such that $g \circ |\rho_1\rangle = |\rho_2\rangle$. Thus, $St_A$ is homogeneous.

We arbitrarily choose a completely mixed state $|\rho\rangle \in St_A$ and show that there exists a reversible process $f \in Proc_{A\to A}$ satisfying $f \circ |\gamma\rangle = |\rho\rangle$. $|\rho\rangle$ has a spectral decomposition $|\rho\rangle = \sum_{i=1}^{N_{A}} p_i |\phi_i\rangle$ with $p_i, \ldots, p_{N_A} \in \mathbb{R}$ and $\{|\phi_i\rangle\}_{i=1}^{N_A} \in MPDS_A$. Since $|\rho\rangle$ is completely mixed, $p_i > 0$ holds for any $i \in \{1, \ldots, N_A\}$. Here, let $f \coloneqq F_{\{|\phi_i\rangle\}_{i=1}^{N_A}}$. $F_{\{|\phi_i\rangle\}_{i=1}^{N_A}}$ is an inverse process of $F_{\{|\phi_i\rangle\}_{i=1}^{N_A}}$. Then, we can easily see $f \circ f = \text{id}_A = f \circ f$, i.e., $f$ is reversible. Moreover, since $f \circ |\phi_i\rangle = p_i |\phi_i\rangle$ holds for any $i \in \{1, \ldots, N_A\}$, we have

$$f = \sum_{i=1}^{N_A} p_i |\phi_i\rangle \langle \phi_i| = \sum_{i=1}^{N_A} p_i |\phi_i\rangle \langle \phi_i|.$$  \hfill (87)

**Theorem 25** For any system $A$, $St_A$ is a symmetric cone.

**Proof** Combining Propositions 23 and 24, $St_A$ is self-dual and homogeneous.

The following theorem shows that a cone is symmetric if and only if it is the cone of squares of some EJA.

**Theorem 26** (Koecher-Vinberg theorem)[51, 52] For any symmetric cone $C$ and any interior point $e \in C$, there exists an EJA $E$ with the identity element $e$ such that $C$ is the cone of squares of $E$. Furthermore, $E$ is the real vector space spanned by $C$. Conversely, for any EJA, its cone of squares is a symmetric cone.

The definitions of an EJA and its cone of squares are given in Sec. VI. For any system $A$, since $St_A$ is a symmetric cone, the Koecher-Vinberg theorem immediately yields the following theorem:

**Theorem 27** For any system $A$, there exists an EJA $E_A$ such that $St_A$ is the cone of squares of $E_A$. Furthermore, $E_A$ and $V_A$ are the same as real vector spaces.

F. Effect space is dual cone of state space

**Theorem 28** For any system $A$, $Eff_A$ is the dual cone of $St_A$, i.e., $Eff_A = St_A^\ast$ holds.

**Proof** Since $(\epsilon'_{\rho} \geq 0 \, \forall \rho \in St_A)$ holds for any $(\epsilon'_{\rho} \in Eff_A)$, $Eff_A \subseteq St_A^\ast$ is obvious. It remains to prove $Eff_A \supseteq St_A^\ast$.

Arbitrarily choose $(\tilde{\tau}) \in St_A^\ast$. Let $(\cdot, \cdot)$ be the inner product defined by Eq. (84). Since $V_A$ is the dual vector space of $V_A$, there exists $[\tilde{\tau}] \in V_A$ such that $\langle \tilde{\tau}, \cdot \rangle = \langle \tilde{\tau}^\ast, \cdot \rangle \, \forall \tilde{\tau} \in V_A$. Since $\langle \tilde{\tau}, \rho \rangle \geq 0$ holds for any $|\rho\rangle \in St_A$, from Proposition 23, we have $[\tilde{\tau}]^\ast \in V_A$. Consider $[\tilde{\tau}]^\ast \in Eff_A$; then, from Eq. (84), $\langle \tilde{\tau}, \rho \rangle = \langle \tilde{\tau}^\ast, \rho \rangle \, \forall \rho \in St_A$, which yields $[\tilde{\tau}] = [\tilde{\tau}]^\ast \in Eff_A$. Therefore, $Eff_A \supseteq St_A^\ast$ holds.

Theorems 25 and 28 immediately yield Property $(\diamond)$. It follows from $St_A \cong St_A^\ast$ and this theorem that $St_A$ and $Eff_A$ are isomorphic as symmetric cones. The restriction of $\hat{\tau} : V_A \to V_A$ to $St_A$ is an isomorphism from $St_A$ to $Eff_A$, whose inverse is the restriction of # : $V_A \to Eff_A$. We abuse notation by using the same symbol $\hat{\tau}$ for the map #. Let $[\tilde{\tau}^\ast] = [\tilde{\tau}] \in V_A$ for any $[\tilde{\tau}] \in V_A^\ast$. Clearly, $\langle [\tilde{\tau}], [\tilde{\tau}]^\ast \rangle = \langle \tilde{\tau}, \tilde{\tau} \rangle$ holds for any $[\tilde{\tau}] \in V_A$ and $[\tilde{\tau}] \in V_A^\ast$.

VI. EUCLIDEAN JORDAN ALGEBRAS (EJAs)

This section will be devoted to presenting some basic properties of EJAs that are needed in the remainder of this paper.

A. EJAs

We first review the definitions of Jordan algebras, EJAs, and their cones of squares. A vector space $E$ over some field $F$ is called a Jordan algebra if $E$ is equipped with a commutative bilinear map $(x, y) \mapsto x \bullet y$ satisfying

$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x, \quad \forall x, y, z \in E, \quad (88)$$

where $x^2 := x \bullet x$. The operator $\bullet$ is said to be the Jordan product. Note that the Jordan product is not associative in general; Eq. (88) is less restrictive than the associative condition (i.e., $x \bullet (y \bullet z) = (x \bullet y) \bullet z$ for any $x, y, z \in E$). A finite-dimensional Jordan algebra $E$ over $F$ is called an EJA if it is equipped with an inner product $(\cdot, \cdot)$ satisfying

$$\langle x \bullet y, z \rangle = \langle y, x \bullet z \rangle, \quad \forall x, y, z \in E. \quad (89)$$

Clearly, any EJA $E$ is a real Hilbert space with the inner product $(\cdot, \cdot)$. We call

$$E^* := \{x^2 : x \in E\} \quad (90)$$

the cone of squares of $E$.

These are two typical examples of EJAs:
• An $n$-dimensional real vector space, $\mathbb{R}^n$, with the componentwise product as the Jordan product and the usual inner product. The Jordan product is commutative and associative, and thus Eq. (88) holds. The cone of squares of $\mathbb{R}^n$ is nonnegative orthant $\mathbb{R}_+^n$. The state space, $\mathcal{S}_+$, of a classical system $A$ is isomorphic to $\mathbb{R}_+^N$.

• The space, $\mathcal{S}(C^n)$, of all complex Hermitian matrices of order $n$ equipped with the Jordan product $\mathbf{x} \circ \mathbf{y} := (x \cdot y + y \cdot x)/2$ and the inner product $(x, y) := \text{Tr}(x \cdot y)$, where $\cdot$ denotes the matrix product. One can easily verify that the Jordan product is commutative and satisfies Eq. (88), but it is not associative if $n$ is larger than $1$. The cone of squares of $\mathcal{S}(C^n)$ is $\mathcal{S}_+(C^n)$. The state space, $\mathcal{S}_+$, of a fully quantum system $A$ is isomorphic to $\mathcal{S}_+(C^n)$.

### B. Fundamental properties of EJAs

We next present some fundamental properties of EJAs. Many proofs are omitted since they can be found in, e.g., Ref. [53] or can be easily obtained. We will use the notation such as $|\varphi\rangle$ to denote an element of an EJA $E$. If $|\varphi\rangle$ is in $E^+$, then we will often denote it by $|\varphi\rangle$. We will also use the simple notation $|\varphi\rangle, \langle \psi|$ for $|\varphi\rangle, \langle \psi| \in E$ for $(\langle \psi|, |\varphi\rangle)$.

The element, denoted by $|\chi\rangle$, of $E$ that satisfies $|\chi\rangle \cdot |\bar{\varphi}\rangle = |\bar{\varphi}\rangle$ for any $|\bar{\varphi}\rangle \in E$ is called the identity element of $E$. $|\chi\rangle \in E^+$ obviously holds. The Koecher-Vinberg theorem states that the cone of squares $E^+$ of any EJA $E$ is a symmetric cone; conversely, for a given symmetric cone $C$ and its interior point $|\chi\rangle \in E^+$, there exists an EJA $E$ with the identity element $|\chi\rangle$ that satisfies $E^+ = C$. Two elements $|\rho\rangle, |\sigma\rangle \in E^+$ are called orthogonal if $|\rho\rangle \cdot |\sigma\rangle = 0$, which is equivalent to $|\rho\rangle \cdot |\sigma\rangle = |0\rangle$. $|\rho\rangle \in E$ is called an idempotent if $|\rho|^2 = |\rho\rangle$. One can obviously see that any idempotent is in $E^+$ and that the zero element $|0\rangle$ and the identity element $|\chi\rangle$ are idempotents. We say that a nonzero idempotent is primitive if it cannot be expressed as the sum of two nonzero idempotents. Any idempotent can be decomposed into the sum of mutually orthogonal primitive idempotents. A set of mutually orthogonal primitive idempotents, $\Phi := \{|\phi_i\rangle\}_{i=1}^k$, is called a Jordan frame if it satisfies $\sum_{i=1}^k |\phi_i\rangle = |\chi\rangle$. For any set of mutually orthogonal primitive idempotents $\Phi := \{|\phi_i\rangle\}_{i=1}^k$, there exists a Jordan frame $\Phi'$ with $\Phi' \supseteq \Phi$. Each Jordan frame of $E$ has the same number of elements, called the rank of $E$, and denoted by $\text{rank } E$. The dimension of $E$ (as a real vector space) is denoted by $\dim E$. For any $|\varphi\rangle \in E$, there exist a Jordan frame $\{|\phi_i\rangle\}_{i=1}^n (n := \text{rank } E)$ and real numbers $c_1, \ldots, c_n \in \mathbb{R}$, called the eigenvalues, such that

$$
|\varphi\rangle = \sum_{i=1}^n c_i |\phi_i\rangle.
$$

Such a representation is called a spectral decomposition of $|\varphi\rangle$.

The eigenvalues are uniquely determined by $|\varphi\rangle$. $|\varphi\rangle$ is in $E^+$ if and only if the eigenvalues of $|\varphi\rangle$ are all nonnegative. The sum of the eigenvalues of $|\varphi\rangle$, $\sum_{i=1}^n c_i$, is called the trace of $|\varphi\rangle$ and denoted by $\text{tr } |\varphi\rangle$. The number of nonzero eigenvalues of $|\varphi\rangle$ is called the rank of $|\varphi\rangle$ and denoted by $\text{rank } |\varphi\rangle$. Since, from the Koecher-Vinberg theorem, $E^+$ is a convex cone, the properties of convex cones can be applied to $E^+$. For example, the partial ordering on $E$ is defined as in Eq. (40). $|\phi\rangle \in E^+$ is pure if and only if $\text{rank } |\phi\rangle \leq 1$ holds. $|\rho\rangle \in E^+$ is completely mixed if and only if $\text{rank } |\rho\rangle = \text{rank } E$ holds.

A vector subspace of $E$, $E'$, is called a subalgebra of $E$ if $E'$ is closed under the Jordan product $\star$, i.e., $|\bar{\varphi}\rangle \cdot |\bar{\varphi}\rangle \in E'$ holds for any $|\bar{\varphi}\rangle, |\bar{\varphi}\rangle \in E'$. $E'$ is itself an EJA. The identity element, $|\chi'\rangle$, of $E'$ is the idempotent of $E$ satisfying

$$
E' = \{|\rho\rangle \in E : |\chi'\rangle \cdot |\bar{\varphi}\rangle = |\bar{\varphi}\rangle\},
$$

$$
E'^+ = \{|\rho\rangle \in E^+ : |\chi'\rangle \cdot |\bar{\varphi}\rangle = |\bar{\varphi}\rangle\}. \quad (92)
$$

Two subalgebras $\{0\}$ and $E$ itself are called trivial. For any idempotent $|\rho\rangle$ of $E$, $E_{\rho} := \{|\varphi\rangle \in E : |\rho\rangle \cdot |\bar{\varphi}\rangle = |\bar{\varphi}\rangle\}$ is the subalgebra of $E$ with the identity element $|\rho\rangle$. Two subalgebras, $E_1$ and $E_2$, of $E$ are said to be orthogonal if $|\varphi\rangle \cdot |\bar{\varphi}\rangle = |0\rangle$ holds for any $|\varphi\rangle \in E_1$ and $|\bar{\varphi}\rangle \in E_2$. A necessary and sufficient condition for two subalgebras $E_1$ and $E_2$ to be orthogonal is that $|\rho\rangle \cdot |\bar{\varphi}\rangle = |0\rangle$ (i.e., $|\rho\rangle, |\sigma\rangle = |0\rangle$) holds for any $|\rho\rangle \in E_1^+$ and $|\sigma\rangle \in E_2^+$.\footnote{Another equivalent definition of a simple EJA is that it has no non-trivial ideals, where a subalgebra $E'$ of $E$ is called an ideal if $|\varphi\rangle \star |\bar{\varphi}\rangle \in E'$ holds for any $|\varphi\rangle \in E'$ and $|\bar{\varphi}\rangle \in E$.}

### C. Direct sum decomposition of EJAs

We will introduce the direct sum decomposition of EJAs. Let us consider $k (k \geq 1)$ mutually orthogonal non-trivial subalgebras, $E_1, \ldots, E_k$, of $E$. We say that $E$ is a direct sum of $E_1, \ldots, E_k$, denoted by $E = \bigoplus_{i=1}^k E_i$, if any $|\varphi\rangle \in E$ can be expressed in the form

$$
|\varphi\rangle = \sum_{i=1}^k |\varphi_i\rangle, \quad |\varphi_i\rangle \in E_i. \quad (93)
$$

In this case, we have

$$
\text{rank } E = \sum_{i=1}^k \text{rank } E_i, \quad \dim E = \sum_{i=1}^k \dim E_i. \quad (94)
$$

$E = \bigoplus_{i=1}^k E_i$ is called a direct sum decomposition of $E$. $|\varphi_1\rangle, \ldots, |\varphi_k\rangle$ of Eq. (93) are uniquely determined by $|\varphi\rangle$; indeed, by multiplying both sides of Eq. (93) by the identity element of $E_i$, denoted by $|\chi_i\rangle$, we obtain $|\varphi_i\rangle = |\chi_i\rangle \cdot |\bar{\varphi}\rangle$. The identity element $|\chi\rangle$ of $E$ is obviously decomposed as $|\chi\rangle = \sum_{i=1}^k |\chi_i\rangle$. $|\varphi\rangle \in E^+$ holds if and only if $|\varphi_i\rangle \in E_i^+$ holds for each $i$. $E^+ = \left( \bigoplus_{i=1}^k E_i \right)^+$ is often rewritten as $\bigoplus_{i=1}^k E_i^+$, which is referred to as a direct sum decomposition of $E^+$. For each nonzero pure element $|\phi\rangle \in E^+$, there exists one and only one $i \in \{1, \ldots, k\}$ such that $|\phi\rangle \in E_i^+$. The operator $\star$ is commutative and associative.

An EJA $E$ with nonzero rank is called simple if it cannot be expressed by a direct sum of two non-trivial subalgebras of $E^{12}$. In this case, its cone of squares $E^+$ is called irreducible.
For a simple EJA $E$, any inner product $(\vec{v}, \vec{w})$ ($\langle \vec{v}, \vec{w} \rangle \in E$) is proportional to $\text{tr}[\vec{v} \cdot \vec{w}]$. For each EJA $E$, $E = \bigoplus_{i=1}^{k} E_i$ (or $E^* = \bigoplus_{i=1}^{k} E_i^*$) is called a simple decomposition if $E_1, \ldots, E_k$ are all simple. In this case, the decomposition of $\langle \vec{v} \rangle \in E$ in Eq. (93) is also called the simple decomposition. Any EJA with nonzero rank is expressed as a direct sum of simple EJAs in a unique way\(^\text{13}\).

It is well known that there are five types of simple EJAs:

**Theorem 29 (Jordan-von Neumann-Wigner theorem \([50]\))**

Every simple EJA is isomorphic to one of the following\(^\text{14}\):

1. $S(F^n)$ with $n \geq 1$ and $F \in \{R, C, H\}$, where $S(F^n)$ is the set of all Hermitian matrices on the vector space $F^n$ and $H$ is the set of all quaternions.

2. Spin factors $Spin_i$, with $s \geq 5$, where $s$ is the dimension.

3. $S(O^3)$, where $O$ is the set of all octonions.

Table I shows the ranks and dimensions of these simple EJAs. It follows that they are classified by their ranks and dimensions.

| Type     | Rank | Dimension |
|----------|------|-----------|
| $S(F^n)$ | $n$  | $n(n+1)/2$ |
| $S(C^n)$ | $n$  | $n^2$     |
| $S(H^n)$ | $n$  | $n(2n-1)$ |
| Spin     | 2    | $s$       |
| $S(O^3)$ | 3    | 27        |

**D. State spaces as cones of squares of EJAs**

In Secs. IV and V, we showed that an OPT with the symmetric sharpness, complete mixing, and filtering postulates satisfies Property (\(\diamond\)). In this subsection, we will show that the converse is almost true. Specifically, we show that an OPT with Property (\(\diamond\)) satisfies all the properties (i.e., all lemmas, propositions, and theorems) except Lemma 18 that we have derived in Secs. IV and V. We here assume that Property (\(\diamond\)) holds and do not assume that the three postulates hold.

We begin with some preliminaries.

**Lemma 30** If Property (\(\diamond\)) holds, then, for any system $A$, there exists an EJA $E_A$ with $E_A = St_A$ such that

\[ (\pmatrix{\gamma}) = \text{tr}(\rho), \quad \forall\rho \in St_A. \]  

**Proof** $St_A$ is self-dual with respect to some inner product $\langle , \rangle$. Since $Eff_A$ is the dual cone of $St_A$ and $\langle |\rangle$ is completely mixed, there exists a completely mixed state $|\psi\rangle \in St_A$ such that $\langle |\psi\rangle |\psi\rangle = \langle \gamma \rangle$ for any $|\gamma\rangle \in V_A$.

First, we consider the EJA $E_A$ with the identity element that satisfies $E_A^{\perp} = St_A$. From the Koecher-Vinberg theorem, such an EJA exists. Let $\bullet$ and $\cdot$ be, respectively, the Jordan product and the trace in $E_A$. $E_A$ has the simple decomposition of the form $E_A = \bigoplus_{i=1}^{k} E_i$. Arbitrarily choose $|\gamma\rangle \in V_A$, which have the simple decompositions $|\gamma\rangle = \sum_{i=1}^{k} |\gamma_i\rangle$ with $|\gamma_i\rangle \in E_i$ (\(i \in \{1, \ldots, k\}\)). Since $\langle \gamma_i |\gamma_i\rangle = |\gamma_i|^2$ holds for any distinct $i$ and $j$, $\langle \gamma_i |\gamma_i\rangle = \sum_{i=1}^{k} \langle \gamma_i |\gamma_i\rangle$ holds. Since, for each $i$, $\langle \gamma_i |\gamma_i\rangle$ is proportional to $\text{tr}(|\gamma_i\rangle \cdot |\gamma_i\rangle)$,

\[ \langle \gamma_i |\gamma_i\rangle = \sum_{i=1}^{k} c_i \cdot \text{tr}(|\gamma_i\rangle \cdot |\gamma_i\rangle) \]  

holds with some constants $c_1, \ldots, c_k \in R_+.$

Next, we consider the EJA $E_A$, with the Jordan product $\bullet$ defined as

\[ |\gamma\rangle \cdot |\gamma\rangle := \sum_{i=1}^{k} c_i |\gamma_i\rangle \cdot |\gamma_i\rangle, \quad \forall\rho \in St_A, \]  

where $|\gamma\rangle$ and $|\gamma\rangle$ have the simple decompositions $|\gamma\rangle = \sum_{i=1}^{k} |\gamma_i\rangle$ and $|\gamma\rangle = \sum_{i=1}^{k} |\gamma_i\rangle$ with $|\gamma_i\rangle \in E_i$ (\(i \in \{1, \ldots, k\}\)). One can easily verify that $E_A$ has the simple decomposition of the form $E_A = \bigoplus_{i=1}^{k} E_i$ with $E_i^+ = E_i$. Arbitrarily choose a Jordan frame $|\phi_i\rangle_{s=1}^{n}$ of $E_i$, where $n := \text{rank} E_A$, $|\phi_i\rangle$ is obviously a pure state. Let $|\phi_i\rangle := c_i |\phi_i\rangle$, where $i \in \{1, \ldots, k\}$ is the value satisfying $|\phi_i\rangle \in E_i$. Then, since $\sum_{i=1}^{k} |\phi_i\rangle = \sum_{i=1}^{k} c_i |\phi_i\rangle = |\gamma\rangle$ and $|\phi_i\rangle \cdot |\phi_i\rangle = c_i^2 |\phi_i\rangle \cdot |\phi_i\rangle = c_i^2 |\phi_i\rangle$, $|\phi_i\rangle$ hold, $|\phi_i\rangle = |\phi_i\rangle$ is a Jordan frame of $E_A$. Let tr denote the trace in $E_A$; then, any primitive idempotent $|\phi_i\rangle \in E_i$ of $E_A$ satisfies $\text{tr} |\phi_i\rangle = 1 = \text{tr} |\phi_i\rangle = c_i \cdot \text{tr} |\phi_i\rangle$. Arbitrarily choose $|\gamma\rangle \in E_A$, which has the simple decomposition $|\gamma\rangle = \sum_{i=1}^{k} |\gamma_i\rangle$ with $|\gamma_i\rangle \in E_i$. Then, we have

\[ \text{tr} |\gamma\rangle = \sum_{i=1}^{k} \text{tr} |\gamma_i\rangle = \sum_{i=1}^{k} c_i \cdot \text{tr} |\gamma_i\rangle \]  

and thus

\[ \langle \pmatrix{\gamma}\rangle = \langle \gamma \cdot \gamma \rangle = \sum_{i=1}^{k} c_i \cdot \text{tr} |\gamma_i\rangle = \sum_{i=1}^{k} c_i \cdot \text{tr} |\gamma_i\rangle \]  

\[ = \sum_{i=1}^{k} c_i \cdot \text{tr} |\gamma_i\rangle = \text{tr} |\gamma\rangle. \]
Therefore, Eq. (95) holds.

For each system $A$, we will choose an EJA $E_A$ satisfying $E_A^* = St_A$ and Eq. (95). We here present some basic properties of the EJA $E_A$. From Eq. (95), for $(\rho) \in St_A$, $(\rho) \in St_n^\text{NP}$ and $\text{tr} \, (\rho) = 1$ are obviously equivalent. Let $(\cdot, \cdot)$ be the inner product of $E_A$ defined as $\langle \chi, \psi \rangle := \text{tr}(\chi \cdot \psi^\dagger)$; then, we have

$$
(\langle \chi, \psi \rangle = \text{tr} \, (\chi \cdot \psi^\dagger)) \quad \forall (\chi, \psi) \in E_A.
$$

(100)

We will equip $V_A$ with the Jordan product $\cdot$ and the inner product $(\cdot, \cdot)$ to identify $V_A$ with the EJA $E_A$. Any extended state in $V_A$ has a spectral decomposition. Since a state $|\psi\rangle$ is pure if and only if rank $|\psi\rangle \leq 1$ holds, $|\psi\rangle \in St_n^\text{NP}$ holds if and only if $|\psi\rangle$ is a primitive idempotent. Since $Eff_A$ is a symmetric cone, by equipping $V_A$ with an appropriate Jordan product (and an inner product), we can regard $V_A$ as an EJA with the identity element $\langle \chi, \chi \rangle$ that satisfies $V_A^* = Eff_A$. We define the isomorphism $\hat{\cdot} : V_A \ni \langle \chi, \chi \rangle \rightarrow \langle \hat{\chi}, \hat{\chi} \rangle \in V_A^*$ to satisfy

$$
\langle \chi, \hat{\psi} \rangle = \text{tr} \, (\chi \cdot \hat{\psi}^\dagger), \quad \forall (\chi, \hat{\psi}) \in V_A \quad \forall \langle \hat{\chi}, \hat{\psi} \rangle \in E_A.
$$

(101)

Then, it follows from Eq. (100) that $\langle \hat{\chi}, \hat{\chi} \rangle \in St_n$. Also, we use the same notation, $\hat{\cdot} : V_A \rightarrow V_A$, for the inverse of $\hat{\cdot}$.

**Example of quantum theory** The identity element of $V_A$ is $|\chi_A\rangle = 1_{N_A}$. The Jordan product of $|\psi\rangle, |\bar{\psi}\rangle \in V_A$ is $[|\psi\rangle, |\bar{\psi}\rangle] = [\langle \psi, \bar{\psi} \rangle + [\psi, \bar{\psi}] / 2]$, where $[\psi, \bar{\psi}] = \text{tr} \, (\psi \cdot \bar{\psi}^\dagger)$ is equal to the trace of the matrix $[\bar{\psi}], i.e., \text{Tr} \, (\bar{\psi}, \bar{\psi}) = \text{Tr} \, (\psi, \bar{\psi}) = \text{Tr} \, (\psi \cdot \bar{\psi}^\dagger) = \text{Tr} \, (\bar{\psi} \cdot \psi^\dagger)$ holds.

**Lemma 31** Assume that Property $(\cdot)$ holds. Let $E_A$ be an EJA satisfying $E_A^* = St_A$ and Eq. (95). Then, a set of normalized pure states is an MPDS if and only if it is a Jordan frame.

**Proof** Consider a set of normalized pure states, $\Phi := \{|\phi_i\rangle\}_{i=1}^N$. Let $n := \text{rank} \, E_A$. For each $i$, $|\phi_i\rangle := |\chi_i\rangle - |e_i\rangle$ has a spectral decomposition of the form $|\chi_i\rangle = \sum_{j=1}^n a_i |\psi_j\rangle$ where $c_1, \ldots, c_n \in \mathbb{R}_{++}$ and $\{|\psi_j\rangle\}_{j=1}^n$ is a subset of some Jordan frame. From $|\phi_i\rangle = |\chi_i\rangle - |e_i\rangle$, $|\psi_j\rangle, |\bar{\psi}_j\rangle$ are orthogonal to $|\phi_i\rangle$; $|\psi_{j+1}\rangle$ are mutually orthogonal primitive idempotents, there exists a Jordan frame $\{|\psi_i\rangle\}_{i=1}^n$ that includes $|\psi_1\rangle, \ldots, |\psi_{j+1}\rangle$. This yields

$$
|e_i\rangle = |\chi_i\rangle - |a_i\rangle \geq \sum_{j=1}^n |\psi_j\rangle \geq |\psi_{j+1}\rangle = |\phi_i\rangle.
$$

(102)

Since $\langle \phi_i, \phi_j \rangle \leq (e_i |\phi_j\rangle = 0$ holds for any distinct $i, j \in \{1, \ldots, k\}$, $\Phi$ is a set of mutually orthogonal primitive idempotents. Assume, by contradiction, that $\Phi$ is not a Jordan frame; then, there exists a Jordan frame $\Phi_{\text{ex}}$ that truly includes $\Phi$. From the proof of the “if” part, the Jordan frame $\Phi_{\text{ex}}$ is an MPDS, which contradicts that $\Phi$ is an MPDS. Therefore, $\Phi$ is a Jordan frame.

It is easily seen from the proof of this lemma that $|\rho\rangle \in St_A$ and $|\sigma\rangle \in St_A$ are perfectly distinguishable (i.e., $|\rho\rangle \in K_{\text{NP}}$) if and only if $|\rho\rangle \cdot |\sigma\rangle = 0$ holds. $N_A = \text{rank} \, E_A$ holds and any MPDS has $N_A$ elements. It also follows that, for any MPDS $\Phi := \{|\phi_i\rangle\}_{i=1}^N$, $\Pi := \{|\phi_i\rangle\}_{i=1}^N$ is the unique measurement that perfectly distinguishes between $\Phi$. Eq. (82) obviously holds from Eq. (91). The identity element $|\chi\rangle$ is the invariant state. Any PDS $\Phi := \{|\phi_i\rangle\}_{i=1}^N$ satisfies $|\phi_i\rangle \cdot |\phi_j\rangle = \delta_{i,j}$, and thus $|\phi_i\rangle$ is an idempotent. Any maximal effect is pure and expressed in the form $|\psi\rangle \cdot |\phi\rangle$ where $|\phi\rangle \in St_n^\text{NP}$, which satisfies $|\phi\rangle \cdot |\psi\rangle = 1$.

With the above preliminaries, we can easily show that an OPT with Property $(\cdot)$ satisfies all the properties except Lemma 18 that we have derived in Secs. IV and V. To show that all the properties derived in Sec. IV hold, it suffices to show that the symmetric sharpness and complete mixing postulates hold. As for the symmetric sharpness postulate, it is obvious that, to each $|\phi\rangle \in St_n^\text{NP}$, there corresponds one and only one maximal effect, $|\phi\rangle \cdot |\phi\rangle$, giving unit probability and that $(|\phi\rangle \cdot |\phi\rangle) = (|\phi\rangle \cdot |\phi\rangle) = (|\phi\rangle \cdot |\phi\rangle)$ holds for any $|\phi\rangle, |\phi\rangle \in St_n^\text{NP}$. For the complete mixing postulate, since any $|\rho\rangle \in St_A$ that is not completely mixed has a spectral decomposition in the form $|\rho\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle$ with $p_1, \ldots, p_{N_A} \in \mathbb{R}_{++}$, $p_{N_A} = 0$, and $\{|\phi_i\rangle\}_{i=1}^{N_A}$ is an MPDS, the maximal effect $\langle \phi_i | \cdot | \phi_i \rangle$ satisfies $\langle \phi_i | \cdot | \phi_i \rangle = 0$. All the properties except Lemma 18 in Sec. V are immediately obtained by Property $(\cdot)$.

**Lemma 32** For any $\Phi \in PDS_A$, let $E_\Phi$ be the subalgebra of $E_A$ with the identity element $|\chi_\Phi\rangle$, i.e., $E_\Phi := \{|\bar{\psi}\rangle \in E_A : |\chi_\Phi\rangle \cdot |\bar{\psi}\rangle = |\bar{\psi}\rangle\}$. Then, we have $F_\Phi = E_\Phi^\ast$.

**Proof** Let $\Psi$ be a PDS complementary to $\Phi$; then, we have

$$
|\rho\rangle \in F_\Phi \Leftrightarrow \langle \chi_\Phi, \rho \rangle = 0
\Leftrightarrow \langle \chi_\Phi, |\bar{\psi}\rangle = |0\rangle
\Leftrightarrow \langle \chi_\Phi, \cdot |\rho\rangle = |0\rangle
\Leftrightarrow \langle \psi_\Phi, \cdot |\rho\rangle = |0\rangle
\Leftrightarrow \langle \rho \rangle \in E_\Phi^\ast.
$$

(103)

where the first line follows from $F_\Phi = K_\Psi$. The third line follows from $|\chi_\Phi\rangle \cdot |\chi_\Phi\rangle = |\chi_\Phi\rangle$ and $|\chi_\Phi\rangle \cdot |\bar{\psi}\rangle = |\bar{\psi}\rangle$, and the last line follows from $E_\Phi^\ast = \{|\sigma\rangle \in E_A^\ast : |\chi_\Phi\rangle \cdot |\sigma\rangle = |\sigma\rangle\}$, Thus, we have $F_\Phi = E_\Phi^\ast$.

**VII. DERIVATION OF QUANTUM THEORY**

In this section, we consider an OPT having Property $(\cdot)$ and the local equality postulate. We investigated the properties of individual systems in the previous sections; we will investigate the structure of composite systems in this section.

We first derive the following properties:
(1) \( D_{A\otimes B} = D_A D_B \) holds (Lemma 34).
(2) \( N_{A\otimes B} = N_A N_B \) holds (Lemma 37).
(3) The operation \( \dag \) distributes over \( \otimes \) (Lemma 38).
(4) When \( S_A \) and \( S_B \) have simple decompositions \( S_A = \bigoplus_{i=1}^{k_A} E_{A,i} \) and \( S_B = \bigoplus_{j=1}^{k_B} E_{B,j} \), \( S_{A\otimes B} \) has the simple decomposition \( S_{A\otimes B} = \bigoplus_{i,j=1}^{k_A,k_B} E_{A,i} \otimes E_{B,j} \) (Lemmas 39 and 40), where \( E_{A,i} \otimes E_{B,j} \) will be defined in Subsec. VII C.

Subsequently, we derive Properties (A)–(C) listed in Subsec. III B based on the above stated properties.

A. Results about local equality

In this subsection, we show that the following properties are derived only from the local equality postulate (without considering Property (C)):

(i) \( D_{A\otimes B} = D_A D_B \) holds.
(ii) The parallel composition of two pure states is pure, and the parallel composition of two pure effects is pure.

1. \( D_{A\otimes B} = D_A D_B \)

Let us fix a basis of \( V_A \). Since \( V_A^\ast \) is the dual vector space of \( V_A \), there exists a basis, \( \{ |\tilde{v}\rangle \}_{i=1}^{D_A} \) of \( V_A^\ast \) satisfying \( \langle \tilde{v}_i | \tilde{w} \rangle = \delta_{i,j} \). Also, let \( \{|\tilde{w}\rangle\} \in V_{A\otimes B} \) and \( \{|\tilde{e}\rangle\} \in V_{A\otimes B}^\ast \) be defined by

\[
|\tilde{w}\rangle := \sum_{i=1}^{D_A} |\tilde{w}_i\rangle \otimes |\tilde{w}_i\rangle, \quad \langle \tilde{e}_i | \tilde{w} \rangle := \sum_{i=1}^{D_A} \langle \tilde{e}_i | \tilde{w}_i \rangle \otimes \langle \tilde{w}_i | \tilde{w} \rangle.
\]  

(104)

**Example of fully quantum theory** Let \( \{|s\rangle\}_{i=1}^{N_A} \) be an ONB of the complex Hilbert space \( C^{N_A} \). Also, let \( \{ |\tilde{w}_{s,t}\rangle \} \in V_A \) be defined as

\[
|\tilde{w}_{s,t}\rangle := \begin{cases} |s\rangle \otimes |t\rangle, & s = t, \\ \frac{1}{\sqrt{2}}(|s\rangle \otimes |t\rangle + |t\rangle \otimes |s\rangle), & s < t, \\ \frac{i}{\sqrt{2}}(|s\rangle \otimes |t\rangle - |t\rangle \otimes |s\rangle), & s > t \\ \end{cases}
\]  

(105)

and \( \{ |\tilde{w}_{s,t}\rangle \} \in V_A^\ast \) is the same matrix as \( \{ |\tilde{w}_{s,t}\rangle \} \), where \( i := \sqrt{-1} \).

Then, \( B := \{ |\tilde{w}_{s,t}\rangle\}_{(s,t)\in\{0,1\}\times\{0,1\}} \) and \( B^\ast := \{ |\tilde{w}_{s,t}\rangle\}_{(s,t)\in\{0,1\}\times\{0,1\}}^\ast \) are, respectively, ONBs of \( V_A \) and \( V_A^\ast \). Substituting \( B \) and \( B^\ast \), respectively, into \( \{ |\tilde{w}\rangle\}_{i=1} \) and \( \{ |\tilde{e}\rangle\}_{i=1} \) in Eq. (104) gives

\[
|\tilde{w}_{A}\rangle := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (|\tilde{e}_{A}\rangle).
\]  

(106)

Note that since this matrix is not positive semidefinite, \( \langle \tilde{e}_{A} | \tilde{w} \rangle \in \text{St}_{A\otimes B} \) and \( \langle \tilde{e}_{A} | \tilde{w} \rangle \notin \text{Eff}_{A\otimes B} \).

**Lemma 33**

\[
\begin{array}{c}
\begin{array}{c}
\tilde{e}_{A} \quad \tilde{e}_{A} \\
\tilde{w}_{A} \quad \tilde{w}_{A}
\end{array}
\begin{array}{c}
\tilde{e}_{A} \quad \tilde{e}_{A} \\
\tilde{w}_{A} \quad \tilde{w}_{A}
\end{array}
\end{array}
\]  

(107)

**Proof** From the local equality postulate and Eq. (30), it suffices to show, for any \( |\rho\rangle \in S_{A\otimes B} \),

\[
\begin{array}{c}
\begin{array}{c}
\tilde{e}_{A} \quad \tilde{e}_{A} \\
\tilde{w}_{A} \quad \tilde{w}_{A}
\end{array}
\begin{array}{c}
\tilde{e}_{A} \quad \tilde{e}_{A} \\
\tilde{w}_{A} \quad \tilde{w}_{A}
\end{array}
\end{array}
\]  

(108)

Since \( \langle \rho | \tilde{w} \rangle \) can be expressed in the form \( \langle \rho | \tilde{w} \rangle = \sum_{j=1}^{D_B} \rho_j |\tilde{w}_j\rangle \) with \( \rho_j \in \mathbb{R} \), we have

\[
\begin{array}{c}
\begin{array}{c}
\tilde{e}_{A} \quad \tilde{e}_{A} \\
\tilde{w}_{A} \quad \tilde{w}_{A}
\end{array}
\begin{array}{c}
\tilde{e}_{A} \quad \tilde{e}_{A} \\
\tilde{w}_{A} \quad \tilde{w}_{A}
\end{array}
\end{array}
= \sum_{j=1}^{D_B} \rho_j |\tilde{w}_j\rangle = |\rho\rangle.
\]  

(109)

where the first and second equalities follow from Eq. (104) and \( \langle \rho | \tilde{w} \rangle = \delta_{i,j} \), respectively. This proves the first equality of Eq. (108). The second equality of Eq. (108) can be proved in the same way.

**Lemma 34** \( D_{A\otimes B} = D_A D_B \) holds for any systems \( A \) and \( B \).

**Proof** From the local equality postulate, each extended process \( \tilde{f} \in V_{A \rightarrow B} \) is identified by a set of scalars \( \{ |\tilde{e}_j\rangle \langle \tilde{f} | \tilde{w}_i\rangle \} \), where \( \{ |\tilde{e}_j\rangle \} \in S_{A\otimes B} \) and \( \{ |\tilde{w}_i\rangle \} \in \text{Eff}_{A\otimes B} \) are, respectively, sets of some fixed states and effects. Thus, we have dim \( V_{A \rightarrow B} \leq D_A D_B \). One can also see dim \( V_{A \otimes B} \geq D_A D_B \). Indeed, if \( \{ |\tilde{w}_i\rangle\}_{i=1}^{D_A} \) and \( \{ |\tilde{w}_j\rangle\}_{j=1}^{D_B} \) are, respectively, bases of \( V_A \) and \( V_B \), then \( \{ |\tilde{w}_i\rangle \otimes |\tilde{w}_j\rangle \}_{i,j=1}^{D_A,D_B} \) is a set of linearly independent extended states of \( V_{A\otimes B} \). We only need to show that \( \dim V_{A\otimes B} \leq \dim V_{A\rightarrow B} \); indeed, in this case, we have \( D_A D_B \leq \dim V_{A\otimes B} \leq \dim V_{A\rightarrow B} \leq D_A D_B \) and thus \( D_{A\otimes B} = \dim V_{A\otimes B} = D_A D_B \).

We will show \( \dim V_{A\otimes B} \leq \dim V_{A\rightarrow B} \). Consider the linear maps \( F : V_{A \rightarrow B} \ni \tilde{f} \mapsto (\text{id}_B \otimes \tilde{f}) \circ |\tilde{e}_A\rangle \in V_{A \otimes B}, \) i.e.,

\[
F \left( \begin{array}{c}
|\tilde{f}\rangle \\
\end{array} \right) := \begin{array}{c}
|\tilde{f}\rangle \\
\end{array} \cdot \begin{array}{c}
|\tilde{e}_A\rangle \\
\end{array}.
\]  

(110)

and the linear map \( G : V_{A \otimes B} \ni |\tilde{w}\rangle \mapsto (|\tilde{e}_A\rangle \otimes \text{id}_B) \circ (|\tilde{w}\rangle) \in V_{A \rightarrow B}, \) i.e.,

\[
G \left( \begin{array}{c}
|\tilde{w}\rangle \\
\end{array} \right) := \begin{array}{c}
|\tilde{w}\rangle \\
\end{array} \cdot \begin{array}{c}
|\tilde{e}_A\rangle \\
\end{array}.
\]  

(111)
We have that, for any $|\tilde{x}\rangle \in V_{A\otimes B}$,
\begin{equation}
F \left( G \left( \begin{array}{c} \varepsilon_x \\ \eta_A \end{array} \right) \right) = \begin{array}{c} \varepsilon_x \\ \eta_A \end{array} = \begin{array}{c} \varepsilon_x \\ \eta_A \end{array},
\end{equation}
where the second equality follows from $[id_{A\otimes A} \circ [\eta_A]] \otimes [[\tilde{x}] \circ id_{I}] = [[\tilde{x}]] \circ [id_{A\otimes B} \circ [\tilde{x}]]$ (see Eq. (8)) and the last equality follows from Lemma 33. Thus, $F[G(\cdot)]$ is the identity map on $V_{A\otimes B}$, which gives that $G$ must be injective. Therefore, $\dim V_{A\otimes B} \leq \dim V_{A-B}$ holds.

It follows from this lemma that $V_{A-B}$ and $V_{A \otimes B}$ are isomorphic as real vector spaces for any systems $A$ and $B$. One can easily see that $F$ and $G$ defined above are isomorphisms between these vector spaces. Note that if $V_{A \otimes B}$ is an EJA with a Jordan product $\cdot$, then $V_{A-B}$ equipped with the Jordan product $\cdot$, that is defined by $\tilde{\tilde{F}} \cdot \tilde{\tilde{G}} := G[F(\cdot) \cdot F(\tilde{\tilde{G}})] (\tilde{\tilde{F}}, \tilde{\tilde{G}} \in V_{A \otimes B})$ is also an EJA. In this case, $F$ and $G$ are also isomorphisms between these EJAs.

It follows from this lemma that, for two bases $\{|w_i\rangle_i\}_{i=1}^{D_A}$ of $V_A$ and $\{|w_j\rangle_j\}_{j=1}^{D_B}$ of $V_B$, $\{|w_i \otimes w_j\rangle_{i,j=1}^{D_A D_B}\}$ is a basis of $V_{A \otimes B}$. Thus, any $|\tilde{x}\rangle \in V_{A \otimes B}$ is expressed in the form
\begin{equation}
|\tilde{x}\rangle = \sum_{i=1}^{D_A} \sum_{j=1}^{D_B} c_{i,j} |w_i \otimes w_j\rangle, \quad c_{i,j} \in \mathbb{R}.
\end{equation}

Also, one can easily see that, for any $|\rho\rangle, |\rho'\rangle \in \text{St}_{A \otimes B}$, we have
\begin{equation}
\forall \left( \begin{array}{c|c} \varepsilon & \varepsilon' \\ \rho & \rho' \end{array} \right) \quad \left( \begin{array}{c|c} \varepsilon & \varepsilon' \\ \rho & \rho' \end{array} \right) = \left( \begin{array}{c|c} \varepsilon & \varepsilon' \\ \rho & \rho' \end{array} \right)
\end{equation}
which indicates that any bipartite state can be identified from the statistics of local measurements on the individual systems. This property, often referred to as local tomography or local distinguishability, has been discussed since at least the 1980s (see, e.g., [54–56]). It is known that local tomography equivalent to the relation $D_{A \otimes B} = D_A D_B$ and that, in any OPT that satisfies local tomography, the local equality postulate holds [41]. Thus, from Lemma 34, the local equality postulate, local tomography, and the relation $D_{A \otimes B} = D_A D_B$ are all equivalent.

2. Composition of two pure states/effects is pure

Lemma 35 The parallel composition of two pure states is pure. Also, the parallel composition of two pure effects is pure.

Proof Since the case of effects can be treated similarly, we only prove the case of states. Arbitrarily choose $|\psi\rangle \in \text{St}_A^P$ and $|\psi'\rangle \in \text{St}_B^P$. We can express $|\psi\rangle \otimes |\psi'\rangle$ in the following form:
\begin{equation}
|\psi\rangle \otimes |\psi'\rangle = \sum_{i=1}^{\ell} |\varphi_i\rangle, \quad |\varphi_i\rangle \in \text{St}_{A-B}^P.
\end{equation}

To prove $|\psi\rangle \otimes |\psi'\rangle \in \text{St}_{A-B}^P$, we only need to show that $|\varphi_1\rangle \propto |\psi\rangle \otimes |\psi'\rangle$ always holds.

The case $|\psi'\rangle = |\emptyset\rangle$ is obvious; assume $|\psi'\rangle \neq |\emptyset\rangle$. Arbitrarily choose $(e| \in \text{Eff}_A$. Applying $(e| \otimes id_B$ to Eq. (115) and using $|\psi\rangle \in \text{St}_A^P$, one can see that
\begin{equation}
P_{|e\rangle\langle e'|} \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right) = \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right)
\end{equation}
holds for some $p_{|e\rangle\langle e'|} \in \mathbb{R}_+$. In the same way, applying $id_A \otimes (\tilde{\tilde{G}}|\psi'$ to Eq. (115) and using $|\psi\rangle \in \text{St}_A^P$, one can see that
\begin{equation}
P_{|e\rangle\langle e'|} \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right) = \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right)
\end{equation}
holds for some $q \in \mathbb{R}_+$. From Eqs. (116) and (117), we have
\begin{equation}
P_{|e\rangle\langle e'|} \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right) = \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right)
\end{equation}
Thus, Eq. (116) gives
\begin{equation}
P_{|e\rangle\langle e'|} \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right) = \left( \begin{array}{c|c} \rho & \rho' \\ \psi & \psi' \end{array} \right)
\end{equation}
for any $(e'| \in \text{Eff}_B$, where $s := q \cdot (\tilde{\tilde{G}}|\psi')^{-1}$. Since this equation holds for any effects $(e|$ and $(e'|, |\varphi_1\rangle = s|\psi\rangle \otimes |\psi'\rangle$ holds from Eq. (114). Therefore, we have $|\varphi_1\rangle \propto |\psi\rangle \otimes |\psi'\rangle$.

B. Basic properties of composite systems

In what follows, we consider an OPT enjoying Property ($\circ$) and the local equality postulate. For any system $A$, let $E_A$ be an EJA satisfying $E_A^+ = \text{St}_A$ and Eq. (95).

Lemma 36 For any $\{\|\phi_i\|\}_{i=1}^{N_1} \in \text{MPDS}_{A}$ and $\{\|\varphi_i\|\}_{i=1}^{N_2} \in \text{MPDS}_{B}$, we have $\Phi := \{\|\phi_i\| \otimes |\varphi_i\rangle\}_{i=1}^{N_1 N_2} \in \text{MPDS}_{A \otimes B}$. Furthermore, $\{\|\phi_i\| \otimes |\varphi_i\rangle\}_{i=1}^{N_1 N_2}$ is the maximal measurement that perfectly distinguishes between $\Phi$.

Proof We have
\begin{equation}
\sum_i \sum_j \|\phi_i\| \otimes \|\varphi_j\| = \sum_i \sum_j \|\phi_i\| \otimes \|\varphi_j\| = \left\| \begin{array}{c} \phi_i \\ \varphi_j \end{array} \right\| = \left\| \begin{array}{c} \phi_i \rangle \\ \varphi_j \rangle \end{array} \right\|.
\end{equation}
which indicates that \( \Pi := \{ (e_{i,j}^k) = (\phi_i^k) \otimes (\varphi_j^k) (\chi_{A,N}, \chi_{B,N}) \} \) is a measurement. Let \( |\psi_{i,j}\rangle := |\phi_i\rangle \otimes |\varphi_j\rangle \); then, \( (e_{i,j}^k)|\psi_{i,j}\rangle = 1 \) holds, and thus \( \Pi \) perfectly distinguishes between \( \Phi \). Since \( |\theta_{i,j}\rangle \in \text{St}_{\text{NP}}^A \) holds from Lemma 35 and \( \left( \frac{\Phi}{\text{St}_{\text{NP}}^B} \right) |\theta_{i,j}\rangle = (\Phi_A |\theta_{i,j}\rangle |\theta_{i,j}\rangle) = 1, \Phi \) is a PDS. It remains to prove that \( \Phi \) is an MPDS. Assume, by contradiction, that \( \Phi \) is not an MPDS; then, there exists \( |\psi\rangle \in \text{St}_{\text{NP}}^A \) that is perfectly distinguishable from \( |\chi_{A} \rangle \). \( e_{i,j}^k \) is a feasible pure state that satisfies \( (e_{i,j}^k)|\theta_{i,j}\rangle = 1 \), and thus \( (e_{i,j}^k) |\phi_{i,j}\rangle \psi_{i,j} \rangle = 0 \) holds. From \( (e_{i,j}^k) |\phi_{i,j}\rangle \psi_{i,j} \rangle = 0 \), we have \( \langle \phi_{i,j}| \psi_{i,j} \rangle = 0 \), which contradicts \( |\psi_{i,j}\rangle \in \text{St}_{\text{NP}}^B \). This completes our proof.

From this lemma, we can easily see that, for any two maximal measurements \( \{ (e_{i,j}) \} \in \text{Meas}_A \) and \( \{ (e_{i,j}^k) \} \in \text{Meas}_B \), \( \{ (e_{i,j}) \otimes (e_{i,j}^k) \} \) is a measurement and that, for any two maximal effects \( (e_{i,j}) \in \text{Eff}_A^M \) and \( (e_{i,j}^k) \in \text{Eff}_B^M \), the effect \( (e_{i,j}) \otimes (e_{i,j}^k) \) is maximal. This lemma also shows

\[
|\chi_{A,B} = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \langle \phi_i | \otimes | \varphi_j \rangle = |\chi_A \rangle \otimes |\chi_B \rangle.
\]

where \( \{ |\phi_i\rangle \} \in \text{MPDS}_A \) and \( \{ |\varphi_j\rangle \} \in \text{MPDS}_B \) hold.

Lemma 37 \( N_{A,B} = N_A N_B \) holds for any systems A and B.

Proof From Lemma 36, \( \Phi := \{ (|\phi_i\rangle \otimes |\varphi_j\rangle) \} \) is a MPDS for any \( \{ |\phi_i\rangle \} \in \text{MPDS}_A \) and \( \{ |\varphi_j\rangle \} \in \text{MPDS}_B \). Thus, \( N_{A,B} = |\Phi \rangle = N_A N_B \) holds.

Lemma 38 For any \( |\rho\rangle \in \text{St}_A \) and \( |\sigma\rangle \in \text{St}_B \), we have

\[
\begin{pmatrix}
|k\rangle \\
|\rho\rangle \\
|\sigma\rangle
\end{pmatrix}^\dagger
= \begin{pmatrix}
|k\rangle \\
|\rho\rangle \\
|\sigma\rangle
\end{pmatrix} = \begin{pmatrix}
|\rho\rangle \\
|\sigma\rangle \\
|k\rangle
\end{pmatrix}.
\]

Proof \( |\rho\rangle \) and \( |\sigma\rangle \), respectively, have spectral decompositions of the form \( |\rho\rangle = \sum_{i=1}^{N_A} p_i |\phi_i\rangle \) and \( |\sigma\rangle = \sum_{j=1}^{N_B} q_j |\varphi_j\rangle \) with \( p_1, \ldots, p_{N_A}, q_1, \ldots, q_{N_B} \in \mathbb{R}^+ \), \( \{ |\phi_i\rangle \} \in \text{MPDS}_A \), and \( \{ |\varphi_j\rangle \} \in \text{MPDS}_B \). From the proof of Lemma 36, \( |\rho\rangle \otimes |\sigma\rangle \) is an idempotent. Therefore, we can consider the subalgebra of \( \text{St}_{\text{A,B}} \) with the identity element \( |\chi\rangle \).

Lemma 38 implies that \( \dagger \) distributes over \( \otimes \). This result can be easily generalized to extended states and effects, i.e., we have

\[
(\langle \chi \otimes \langle \varphi \rangle \rangle \dagger = (\langle \chi \rangle \otimes \langle \varphi \rangle) \quad \forall \langle \chi \rangle \in \text{St}_A, \langle \varphi \rangle \in \text{St}_B,
\]

\[
(\langle \chi \otimes \langle \varphi \rangle \rangle \dagger = (\langle \chi \rangle \otimes \langle \varphi \rangle) \quad \forall \langle \chi \rangle \in \text{St}_A, \langle \varphi \rangle \in \text{St}_B.
\]

C. Simple decompositions of state spaces of composite systems

Consider two subalgebras \( E_A^\prime \) of \( V_A \) and \( E_B^\prime \) of \( V_B \). Let \( |\chi_A\rangle \) and \( |\chi_B\rangle \) denote the identity elements of \( E_A^\prime \) and \( E_B^\prime \), respectively. One can easily verify that \( |\chi\rangle := |\chi_A\rangle \otimes |\chi_B\rangle \) is an idempotent of \( V_{A,B} \). Indeed, it follows that there exist \( \Phi_A := \{ |\phi_i\rangle \} \in \text{PDS}_A \) satisfying \( |\chi_A\rangle = |\chi_A\rangle \otimes |\chi_B\rangle \) and \( \Phi_B := \{ |\varphi_j\rangle \} \in \text{PDS}_B \) satisfying \( |\chi_B\rangle = |\chi_B\rangle \otimes |\chi_A\rangle \) such that \( \Psi := \{ |\phi_i\rangle \otimes |\varphi_j\rangle \} \) is a PDS of \( A \otimes B \) that satisfies \( |\chi\rangle = |\chi\rangle \). Thus, \( |\chi\rangle \) is an idempotent. Therefore, we can consider the subalgebra of \( \text{St}_{\text{A,B}} \) with the identity element \( |\chi\rangle \).

We will denote this subalgebra by \( E_A^\prime \otimes E_B^\prime \). Clearly,

\[
E_A^\prime \otimes E_B^\prime = \{ |\chi\rangle \in \text{St}_{\text{A,B}} \} \quad \forall \langle \chi \rangle = |\chi\rangle
\]

holds. Let \( E_A^\prime \otimes E_B^\prime = \{ |\rho\rangle \} \), then \( E_A^\prime \otimes E_B^\prime = \mathcal{F}_{|\chi\rangle} \) holds from Lemma 32.

Lemma 39 When \( S_A \) and \( S_B \) have direct sum decompositions \( S_A = \bigoplus_{i=1}^{N_A} E_{A,i} \) and \( S_B = \bigoplus_{j=1}^{N_B} E_{B,j} \), \( S_{A,B} \) has a direct sum decomposition

\[
S_{A,B} = \bigoplus_{i=1}^{N_A} \bigoplus_{j=1}^{N_B} E_{A,i} \otimes E_{B,j}.
\]

Proof Equation (126) holds if and only if

\[
V_{A,B} = \bigoplus_{i=1}^{N_A} \bigoplus_{j=1}^{N_B} E_{i,j}, \quad E_{i,j} := E_{A,i} \otimes E_{B,j}
\]

holds. The case \( k_A = k_B = 1 \) is obvious since \( E_{1,1} \) is the Eja with the identity element \( |\chi_{A,B}\rangle \), i.e., \( |\chi_{A,B}\rangle = V_{A,B} \) holds. Assume \( k_A \geq 2 \) or \( k_B \geq 2 \). To show Eq. (127), it suffices to show the followings: (a) the EJAs \( \{ E_{i,j} \}_{i,j=1}^{k_A,k_B} \) are mutually orthogonal and (b) any \( |\chi\rangle \in V_{A,B} \) can be expressed in the form \( |\chi\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \langle \chi_i \rangle \otimes |\chi_j\rangle \) with \( \langle \chi_i \rangle \in E_{A,i} \).
i.e., $\langle \rho, \rho' \rangle = 0$.

Next, we prove (b). For each $i$ and $j$, let $d_{A,i} := \dim E_{A,i}$ and $d_{B,j} := \dim E_{B,j}$. From Eq. (94), $\sum_{i=1}^{k_A} d_{A,i} = D_A$ and $\sum_{j=1}^{k_B} d_{B,j} = D_B$ hold. Choose a basis, \( \{ |i_j \rangle \}_{i,j=1}^{k_i,k_j} \) of $V_A$ satisfying $|i_j \rangle \in E_{A,i}$ for each $i$ and $s$, and a basis, \( \{ |j \rangle \}_{j=1}^{k_B} \) of $V_B$ satisfying $|j \rangle \in E_{B,j}$ for each $j$ and $t$. $\Psi := \{ |i_j \rangle \} \otimes \{ |j \rangle \}$ is a set of linearly independent extended states. From \( |\Psi \rangle = \sum_{i=1}^{k_A} \sum_{j=1}^{k_B} d_{A,i} d_{B,j} = D_A D_B \) and Lemma 34, \( |\Psi \rangle = D_A D_B \) holds, which implies that $\Psi$ is a basis of $V_{A\otimes B}$. Thus, any $|\phi \rangle \in V_{A\otimes B}$ can be expressed in the form

$$
|\psi \rangle = \sum_{i=1}^{k_A} \sum_{j=1}^{k_B} c_{i,j} |i_j \rangle \otimes |j \rangle,
$$

where $c_{i,j} \in \mathbb{R}$. Since $|i_j \rangle \otimes |j \rangle$ holds for any $i$, $j$, $s$, and $t$, we have $|\psi \rangle \in V_{E_{A\otimes B}}$.

This proof tells us that the dimension of $E_{A,i} \otimes E_{B,j}$, i.e.,

$$\dim(E_{A,i} \otimes E_{B,j}) = (\dim E_{A,i}) \cdot (\dim E_{B,j}).$$

One can also easily obtain

$$\text{rank}(E_{A,i} \otimes E_{B,j}) = (\text{rank } E_{A,i}) \cdot (\text{rank } E_{B,j}).$$

Note that Eq. (126) can also be expressed by

$$
\left( \bigoplus_{i=1}^{k_A} E_{A,i}^{+} \right) \otimes \left( \bigoplus_{j=1}^{k_B} E_{B,j}^{+} \right) = \bigoplus_{i=1}^{k_A} \bigoplus_{j=1}^{k_B} E_{A,i}^{+} \otimes E_{B,j}^{+}.
$$

This indicates that the operation $\otimes$ on $E_{A\otimes B}$ distributes over the operation $\oplus$ on $E$.

We close this subsection with an important result, which will be useful in the sequel.

**Lemma 40** When $S_{A,i}$ and $S_{B,j}$ have the simple decompositions

$$S_{A,i} = \bigoplus_{i=1}^{k_A} E_{A,i}^{+}$$

and

$$S_{B,j} = \bigoplus_{j=1}^{k_B} E_{B,j}^{+},$$

$S_{A\otimes B}$ has the simple decomposition given by Eq. (126).

**Proof** From Lemma 39, $S_{A\otimes B}$ can be expressed in the form of Eq. (126), it is sufficient to show that $E_{A,i} \otimes E_{B,j}$ is simple for each $i \in \{1, \ldots, N_A\}$ and $j \in \{1, \ldots, N_B\}$. Consider arbitrary fixed $i$ and $j$. Assume, by contradiction, that $E_{A,i} \otimes E_{B,j}$ is not simple, i.e., $E_{A,i} \otimes E_{B,j}$ can be decomposed into a direct sum $E_{A,i} \otimes E_{B,j} = E_0 \oplus E_1$, where $E_0$ and $E_1$ are $E_{A,i} \otimes E_{B,j}$ with nonzero ranks. Let $|\psi \rangle \rangle_{i=1}^{n_A}$ (with $|\psi \rangle \rangle_{i=1}^{n_A}$) and $|\phi \rangle \rangle_{j=1}^{n_B}$ (with $|\phi \rangle \rangle_{j=1}^{n_B}$) be, respectively, Jordan frames of $E_{A,i} \otimes E_{B,j}$. Then, one can easily see that $\Phi := \{ |\phi \rangle \rangle_{j=1}^{n_B} \otimes |\psi \rangle \rangle_{i=1}^{n_A} \}$ is a Jordan frame of $E_{A,i} \otimes E_{B,j}$. Since both $E_{A,i}$ and $E_{B,j}$ are simple, there exist two normalized pure states $|\varphi \rangle \in E_{A,i}^{+}$ and $|\varphi' \rangle \in E_{B,j}^{+}$ such that

$$
\langle \varphi' | \varphi \rangle > 0, \quad \forall i \in \{1, \ldots, n\},
$$

$$
\langle \varphi'' | \varphi' \rangle > 0, \quad \forall i' \in \{1, \ldots, n'\},
$$

which can be verified for each simple $E_{A,i}$ classified by the Jordan-von Neumann-Wigner theorem. Let $|\psi \rangle := |\varphi \rangle \otimes |\varphi' \rangle \in E_{A,i}^{+} \otimes E_{B,j}^{+}$; then, from Lemma 35, $|\psi \rangle$ is pure. Since any nonzero pure state $|\psi \rangle \in E_{A,i}^{+}$ satisfies either $|\varphi \rangle \in E_{A,i}^{+}$ or $|\varphi' \rangle \in E_{B,j}^{+}$ (but not both), we can permute $E_0$ and $E_1$, if necessary, so that $|\psi \rangle \in E_{A,i}^{+}$ holds. Then, we have that, for any $l \in \{1, \ldots, n\}$ and $l' \in \{1, \ldots, n'\}$,

$$
\langle \varphi'' | \varphi' \rangle > 0, \quad \forall i \in \{1, \ldots, n\},
$$

which is a contradiction. Therefore, $E_{A,i} \otimes E_{B,j}$ is simple.

**D. State space is isomorphic to direct sum of spaces of complex positive semidefinite matrices**

In this subsection, we derive Properties (A) and (B) given in Subsec. III B.

**Theorem 41** For any system $A$, $S_{A} \cong \bigoplus_{i=1}^{k_S} S_{A}(C^{n_{i}})$ holds for some natural numbers $k, n_{1}, \ldots, n_{k}$ with $\sum_{i=1}^{k} n_{i} = N_{A}^{15}$.

**Proof** We will use the Jordan-von Neumann-Wigner theorem. $S_{A}$ has the simple decomposition of the form $S_{A} = \bigoplus_{i=1}^{k_S} E_{i}^{+}$. We only need to prove $E_{i} \cong S(C)$ for each $i \in \{1, \ldots, k\}$, where $n_{i} := \text{rank } E_{i}^{+}$.

Consider arbitrary fixed $i$. Let $n := n_{i}$. In the case of $n = 1$, $E_{i} \cong S(C)$ obviously holds from $S(R) \cong S(C) \cong S(H)$, so assume $n \geq 2$. $E_{i} \cong S(C) \cong S(H)$. $S(C)$ is simple from Lemma 40. Let $r := \text{rank } E_{i} \in \mathbb{N}^{2}$ and $d := \text{dim } E_{i} \in \mathbb{N}^{2}$. We prove, using Table I, that $S(O^{2})$, $S(R^{2})$, $S(H^{2})$, and $Spin$, with $s \geq 5$ are ruled out. Firstly, considering the case $E_{i} \cong S(O^{2})$, one can see from $n = 3$ that the simple $E_{i} \cong S(O^{2})$ must have $r = 9$ and $d = 27^{2}$, which contradicts Table I. Secondly, we consider the case $E_{i} \cong S(R^{2})$, which leads to $r = 2n^{3}$ and $d = n^{2}(n + 1)^{2}/4$, and easily see that this case is ruled out. In the same way, considering the case $E_{i} \cong S(H^{2})$, which leads to $r = n^{2}$ and $d = n^{2}(2n - 1)^{2}$, one can see that this case is also ruled out. Finally, we consider the case $E_{i} \cong Spin$, with $s \geq 5$, which gives, from $n = 2$, $r = 4$ and $d = s^{2}$. Since $E_{i} \cong Spin$ is not isomorphic to $S(H^{2})$, this case is ruled out. Thus, we conclude that $E_{i} \cong S(C)$ must hold.

Combining Lemma 40 and Theorem 41 gives the following theorem.

**Theorem 42** For any systems $A$ and $B$ with $S_{A} \cong \bigoplus_{i=1}^{k_{A}} S_{A}(C^{n_{i}})$ and $S_{B} \cong \bigoplus_{j=1}^{k_{B}} S_{B}(C^{m_{j}})$, we have $S_{A\otimes B} \cong \bigoplus_{i=1}^{k_{A}} \bigoplus_{j=1}^{k_{B}} S_{A}(C^{n_{i}})$.

---

15 Theorem 41 is somewhat similar to Theorem 4.14 in Ref. [43]. We here point out that there seems to be a gap in the proof of the latter. This proof shows that if $S_{A}$ is simple, then $S_{A} \cong S(C(N_{A}^{2}))$ holds; however, it does not rule out the possibility of, for example, $S_{A} \cong S(H^{2}) \oplus Spin$. (Note that $S_{A}(H^{2}) \oplus Spin$ has the same rank and dimension as $S_{A}(C^{3})$.)
Proof. $\text{St}_A$ and $\text{St}_B$ have the simple decomposition of the form $\text{St}_A = \bigoplus_{i=1}^{k_A} E^*_A$, with $E^*_A \cong S_i(C^n)$ and $\text{St}_B = \bigoplus_{j=1}^{k_B} E^*_B$, with $E^*_B \cong S_i(C^n)$. From Lemma 40, $\text{St}_{AB}$ has the simple decomposition $\text{St}_{AB} = \bigoplus_{i,j=1}^{k_A k_B} E_{A,i} \otimes E_{B,j}$. Since rank$(E_{A,i} \otimes E_{B,j}) = \text{rank}(E_{A,i}) \cdot \text{rank}(E_{B,j}) = m_i n_j$ holds, applying Theorem 41, we have $E_{A,i} \otimes E_{B,j} \cong S_i(C^{m_n})$.

As will be shown here, Theorems 41 and 42 allow us to obtain a simple expression for a state (resp. extended state) in terms of a positive semidefinite matrix (resp. Hermitian matrix). For each system $A$, $\text{St}_A$ has the simple decomposition of the form $\text{St}_A = \bigoplus_{i=1}^{k_A} E_{A,i}$, where $E_{A,i} \cong S_i(C^n)$ holds for some natural number $n_i$. Let us choose a set of $k_A$ mutually orthogonal projection matrices of order $N_A$, denoted by $\mathcal{P}_A := \{P_{A,i}\}_{i=1}^{k_A}$, with $\sum_{i=1}^{k_A} P_{A,i} = I_{N_A}$ obviously holds. $\mathcal{M}_A$ is defined as

$$
\mathcal{M}_A := \left\{ H = \sum_{i=1}^{k_A} H_i : H_i \in \mathcal{M}_{A,i} \right\}.
$$

$\mathcal{M}_A$ is an EJa with the Jordan product $H \cdot H' = (H'H + H'H)/2$ ($H,H' \in \mathcal{M}_A$). Clearly, $I_{N_A}$ is the identity element of $\mathcal{M}_A$. $\text{tr} H = \text{Tr} H$ also holds for any $H \in \mathcal{M}_A$. It follows that the cone of squares, $\mathcal{M}_A^+ \subset \mathcal{M}_A$ is the set of all positive semidefinite matrices in $\mathcal{M}_A$. Each $\mathcal{M}_{A,i}$ is the subalgebra of $\mathcal{M}_A$ with the identity element $P_{A,i}$. One can easily check $\mathcal{M}_{A,i} \cong S_i(C^{n_i}) \cong E_{A,i}$, $\mathcal{M}_A = \bigoplus_{i=1}^{k_A} \mathcal{M}_{A,i} \cong \mathcal{V}_A$, and $\mathcal{M}_A^+ \cong \mathcal{S}_A$. Using an isomorphism $\mathcal{M}_i$ from $\mathcal{V}_A$ to $\mathcal{M}_A$, we can fully and faithfully represent any $|\tilde{\xi}\rangle \in \mathcal{V}_A$ as the corresponding Hermitian matrix $\mathcal{M}_{\tilde{\xi}} \in \mathcal{M}_i$. In particular, $\mathcal{M}_{\tilde{\xi}}^+ \subset \mathcal{M}_{\tilde{\xi}}$ holds if and only if $\psi \in \mathcal{S}_A$ holds. $\mathcal{M}_{\tilde{\xi}}$ is an ONB of the complex Hilbert space $\mathcal{C}_A$. Consider the following extended state of system $A \otimes A$:

$$
|\psi_A\rangle := \sum_{i=1}^{k_A} |\psi_A^{(i)}\rangle,
$$

where $|\psi_A^{(i)}\rangle$ is an ONB of the complex Hilbert space $\mathcal{C}_A$. From Eqs. (138) and (139), we have

$$
|\psi_{AB}\rangle := \sum_{i=1}^{k_A} \sum_{j=1}^{k_B} |\psi_A^{(i)}\rangle \otimes |\psi_B^{(j)}\rangle = |\tilde{\xi}\rangle \otimes |\tilde{\eta}\rangle.
$$

(141)

It follows that $|\psi_{AB}\rangle$ is positive semidefinite, and thus $|\psi_A\rangle \in \mathcal{M}_{AB}$ holds. $|\psi_A\rangle$ can be expressed by

$$
|\psi_A\rangle := \sum_{i=1}^{D_A} y_i |\tilde{\xi}_i\rangle \otimes |\tilde{\eta}_i\rangle,
$$

(142)

where

$$
\mu(l,s,t) := \sum_{j=1}^{l-1} n_j^2 + (s-1)n_l + t \in \{1, \ldots, D_A\},
$$

(143)

and

$$
\gamma_{\mu(l,s,t)} := |\tilde{\xi}_i\rangle \otimes |\tilde{\eta}_i\rangle.
$$

(143)
It follows that \((|\overline{w}\rangle)^{D_1}_{i=1}\) is an ONB of \(V_A\). \(\gamma_i \in \{1, -1\}\) holds from Eq. (140). It is noteworthy that \(|\underline{\eta}_A\rangle\) is somewhat similar to \(|\overline{\eta}_A\rangle\) of Eq. (104); however, \(|\underline{\eta}_A\rangle\) is a state, while \(|\overline{\eta}_A\rangle\) is not in general a state. Let \((\cap A) := |\underline{\eta}_A\rangle^\dagger\).

**Example of quantum theory** Consider a system \(A\) with \(N_A = 2\). If \(A\) is classical, i.e., \(St_A \cong S_2(C) \oplus S_2(C)\), then \(|\underline{\eta}_A\rangle\) is expressed by

\[
M^{\underline{\eta}_A}_{|\underline{\eta}_A\rangle} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

In this case, \(|\overline{\eta}_A\rangle\) is a state; indeed, \(|\overline{\eta}_A\rangle = |\underline{\eta}_A\rangle\) holds. If \(A\) is fully quantum, i.e., \(St_A \cong S_4(C^2)\), then \(|\underline{\eta}_A\rangle\) is expressed by

\[
M^{\underline{\eta}_A}_{|\underline{\eta}_A\rangle} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

**Lemma 43**

\[
\begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array}.
\]

**Proof** Since \((|\overline{w}\rangle)^{D_1}_{i=1}\) of Eq. (143) is an ONB of \(V_A\), any \(|\rho\rangle \in St_A\) can be expressed in the form \(|\rho\rangle = \sum_{i=1}^{D_1} \rho_i |\overline{w}\rangle\) with \(\rho_i \in \mathbb{R}\). Thus, from Eq. (142), we have

\[
\begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array} = \sum_{i=1}^{D_1} \gamma_i |\overline{\rho_i w}\rangle |\overline{w}\rangle = \sum_{i=1}^{D_1} \rho_i |\overline{w}\rangle |\overline{w}\rangle,
\]

where the second equality follows from \((|\overline{w}\rangle |\overline{w}\rangle') = \delta_{w, w'}\) and \(\gamma_i^2 = 1\). This proves, from the local equality postulate, the first equality of Eq. (146). The same is true for the second equality of Eq. (146).

For any systems \(A\) and \(B\), let \(CP_{A-B}\) be the set of all CP maps from \(M_A\) to \(M_B\) and \(CP_{A-B}\) be the set of all linear maps from \(M_A\) to \(M_B\). It is easily seen that \(CP_{A-B}\) is a convex cone in \(M_{A-B}\).

**Theorem 44** For any systems \(A\) and \(B\), \(CP_{A-B}\) and \(CP_{A-B}\) are isomorphic as convex cones. Also, there exists an isomorphism \(L : CP_{A-B} \ni f \rightarrow L_f \in CP_{A-B}\) such that

\[
M^{L_{\rho}}_{L_f} = L_f[M^{\rho}_{|\rho\rangle}], \quad \forall f \in CP_{A-B}, \forall |\rho\rangle \in St_A.
\]

**Proof** Let \(L^{A-B} : V_{A-B} \ni \overline{f} \rightarrow L_f := M^B \circ \overline{f} \circ M^A \in M_{A-B}\), where \(M^A : M_A \rightarrow V_A\) is the inverse of the isomorphism \(M^A : V_A \rightarrow M_A\). We simply denote \(L^{A-B}\) by \(L\). One can easily see that \(L\) is linear and satisfies

\[
L_f [M^A_{|\rho\rangle}] = M^B_{\overline{f}|\rho\rangle}, \quad \forall \overline{f} \in V_{A-B}, \forall |\rho\rangle \in V_A. \quad (149)
\]

It follows that \(V_{A-B}\) and \(M_{A-B}\) are isomorphic as vector spaces and that \(L : V_{A-B} \rightarrow M_{A-B}\) is an isomorphism, whose inverse is \(L^{-1} : M_{A-B} \ni h \rightarrow L^{-1}_f := M^B \circ h \circ M^A \in V_{A-B}\). Indeed, we have that, for any \(\overline{f} \in V_{A-B}\) and \(h \in M_{A-B}\),

\[
(L^{-1} \circ L)(\overline{f}) = M^B \circ (M^B \circ \overline{f} \circ M^A) = \overline{f},
\]

\[
(L \circ L^{-1})(h) = M^B \circ (M^B \circ h \circ M^A) = h. \quad (150)
\]

What is left is to show that the restriction of \(L\) to \(CP_{A-B}\), denoted by the same notation \(L\), is an isomorphism from \(CP_{A-B}\) to \(CP_{A-B}\). It suffices to show \(L_f \in CP_{A-B}\) for any \(f \in CP_{A-B}\) and \(\overline{c}_f \in CP_{A-C-B}\) for any \(c \in CP_{A-B}\).

First, we show \(L_f \in CP_{A-B}\) for any \(f \in CP_{A-B}\). \(L^{A-B} \in M_{A-B}\) is the identity operator on \(M_A\). Also, we have that, for any \(\overline{f} \in V_{A-B}, \overline{g} \in V_{C-D}, \overline{c} \in V_A\), and \(\overline{g} \in V_C\),

\[
L_{\overline{c}} L_{\overline{g}} [M_{\overline{g}, \overline{c}}] = M_{\overline{c}, \overline{g}}[M_{\overline{g}, \overline{c}}],
\]

\[
= M_{\overline{c}, \overline{g}}[M_{\overline{g}, \overline{c}}],
\]

\[
= M_{\overline{c}, \overline{g}} \circ M_{\overline{g}, \overline{c}}.
\]

where the third and last lines follow from Eq. (136). Thus, we have

\[
L_{\overline{c}} L_{\overline{g}} = L_{\overline{c} \overline{g}}, \quad \forall \overline{f} \in V_{A-B}, \forall \overline{g} \in V_{C-D}. \quad (152)
\]

Moreover, we have that, for any \(f \in CP_{A-B}\) and \(E \in St_b\),

\[
(f \otimes \text{id}_E) \circ |\sigma\rangle \in St_{BSE}, \quad \forall |\sigma\rangle \in St_{BSE}, \quad (153)
\]

which implies that \(L_{f \otimes \text{id}_E} = L_f \otimes \text{id}_E\) is positive, and thus \(L_f\) is CP (i.e., \(L_f \in CP_{A-B}\)).

Next, we show \(L^{A-B} \in CP_{A-B}\) for any \(c \in CP_{A-B}\). Arbitrarily choose \(c \in CP_{A-B}\). Let \(|\sigma_c\rangle := (\text{id}_A \otimes L_{c}^{-1}) \circ |\underline{\eta}_A\rangle \in V_{A-\overline{B}}\), i.e.,

\[
\begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array}.
\]

From Eqs. (149) and (152), we have

\[
M_{|\sigma_c\rangle} = M_{(\text{id}_A \otimes L_{c}^{-1})|\underline{\eta}_A\rangle},
\]

\[
= M_{\text{id}_A \otimes L_{c}^{-1}}[M_{|\underline{\eta}_A\rangle}] = (L_{c}^{-1} \circ \text{id}_A)[M_{|\underline{\eta}_A\rangle}]. \quad (155)
\]

From \(M_{|\underline{\eta}_A\rangle} \in M_{A-B}^+\) and \(c \in CP_{A-B}\), we have \(M_{|\sigma_c\rangle} \in M_{A-B}^+\), i.e., \(|\sigma_c\rangle \in St_{A-\overline{B}}\). Let

\[
\begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\cap A \\
\cup A
\end{array}
\end{array}.
\]
Then, since $|\sigma_r|$ and $(\cap\triangle u)$ are, respectively, a state and an effect, $\rho_r$ is a process. Substituting Eq. (154) into Eq. (156) and using Eq. (146) yields $L^{-1}_c = f_r$. Therefore, $L^{-1}_c \in \text{Proc}_{A \rightarrow B}$ holds.

**Theorem 45** Let $L : \text{Proc}_{A \rightarrow B} \ni f \mapsto L_f \in \text{CP}_{A \rightarrow B}$ be an isomorphism satisfying Eq. (148); then, $f \in \text{Proc}_{A \rightarrow B}$ is deterministic if and only if $L_f$ is a TP-CP map satisfying Eq. (148); then, $f \in \text{Proc}_{A \rightarrow B}$ is feasible if and only if $L_f$ is a trace non-increasing CP map.

**Proof** From Eq. (21), $f \in \text{Proc}^D_{A \rightarrow B}$ holds if and only if $(\mathcal{P}_B \circ f \circ |\rho\rangle = (\mathcal{P}_B A |\rho\rangle)$ holds for any $|\rho\rangle \in \mathcal{S}_A$. Since $(\mathcal{P}_B \circ f \circ |\rho\rangle = \text{Tr}_{M_{|\rho\rangle}} |\rho\rangle = \text{Tr}_{M_{|\rho\rangle}} L_f[|\rho\rangle]$ and $(\mathcal{P}_B A |\rho\rangle = \text{Tr}_{M_{|\rho\rangle}} A |\rho\rangle)$ hold, $f \in \text{Proc}^D_{A \rightarrow B}$ holds if and only if $L_f$ is a trace non-increasing CP map.

Assume $f \in \text{Proc}^F_{A \rightarrow B}$; then, from Eq. (18), $\text{Tr}_{M_{|\rho\rangle}} L_f[|\rho\rangle] = (\mathcal{P}_B A |\rho\rangle) \leq (\mathcal{P}_B A |\rho\rangle = \text{Tr}_{M_{|\rho\rangle}} A |\rho\rangle)$ holds for any $|\rho\rangle \in \mathcal{S}_A$, which implies that $L_f$ is a trace non-increasing CP map. Conversely, assume that $L_f$ is a trace non-increasing CP map. $L_f$ can be written in the Kraus representation as

$$L_f(H) = \sum_{i=1}^{k} E_i H E_i^\dagger, \quad \forall H \in \mathcal{M}_A, \quad \sum_{i=1}^{k} E_i^\dagger E_i \leq \mathbb{I}_{N_A}, \quad (157)$$

where $E_1, \ldots, E_k$ are $N_B \times N_A$ matrices. Consider $g \in \text{Proc}_{A \rightarrow B}$ satisfying

$$L_g(H) = \sum_{i=1}^{l} E_i H E_i^\dagger, \quad \forall H \in \mathcal{M}_A, \quad \sum_{i=1}^{l} E_i^\dagger E_i = \mathbb{I}_{N_A}, \quad (158)$$

where $l \geq k$ holds and $E_{k+1}, \ldots, E_l$ are $N_B \times N_A$ matrices; then, one can easily see that $L_{f+g} = L_f + L_g$ is a TP-CP map. Thus, $f + g \in \text{Proc}^D_{A \rightarrow B}$ (i.e., $(f, g) \in \text{Test}_{A \rightarrow B}$) holds. Therefore, $f$ is feasible.

**F. Classical and fully quantum theories**

Up to now, we showed that an OPT with the four postulates introduced in this paper is quantum theory, which includes classical theory and fully quantum theory as special cases. One can single out classical theory and fully quantum theory by introducing appropriate additional postulates. Here, to single out these theories, we introduce postulates about perfectly distinguishable normalized pure states. We say that a system $A$ satisfies perfect distinguishability if any distinct $|\psi\rangle, |\phi\rangle \in \mathcal{S}^\text{NP}_A$ are perfectly distinguishable. Also, we say that a system $A$ satisfies indistinguishability if, for any $|\psi\rangle, |\phi\rangle \in \mathcal{S}^\text{NP}_A$, there exists $|\varphi\rangle \in \mathcal{S}^\text{NP}_A$ such that neither $|\psi\rangle$ nor $|\phi\rangle$ is perfectly distinguishable from $|\varphi\rangle$. It is easily seen that, in quantum theory, a system $A$ is classical if $A$ satisfies perfect distinguishability. Thus, the following postulate singles out classical theory.

**Postulate 5C (Perfect distinguishability)** Any system satisfies perfect distinguishability. Also, in quantum theory, a system $A$ is fully quantum if $A$ satisfies indistinguishability, and thus the following postulate singles out fully quantum theory.

**Postulate 5Q (Indistinguishability)** Any system satisfies indistinguishability.

**VIII. CONCLUSION**

In this study, we imposed four purely operational postulates on the framework of OPTs. We showed that these postulates are sufficient to single out finite-dimensional quantum theory with superselection rules. In an OPT satisfying the first three postulates — the symmetric sharpness, complete mixing, and filtering —, we discovered that each state space is a symmetric cone and the corresponding effect space is its dual cone. We showed that such an OPT is quantum theory if it further satisfies the local equality postulate. Moreover, classical and fully quantum theories can be, respectively, singled out from the perfect distinguishability and indistinguishability postulates.

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