Strong integrability of $\lambda$-deformed models

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Abstract

We study the notion of strong integrability for classically integrable $\lambda$-deformed CFTs and coset CFTs. To achieve this goal we employ the Poisson brackets of the spatial Lax matrix which we prove that it assumes the Maillet $r/s$-matrix algebra. As a consequence the system in question are integrable in the strong sense. Furthermore, we show that the derived Maillet $r/s$-matrix algebras can be realized in terms of twist functions, at the poles of which we recover the underlying symmetry algebras.
1 Introduction and conclusions

A two-dimensional $\sigma$-model is integrable provided that its equations of motion can be packed into a flat Lax connection

$$\partial_+ L_+ - \partial_- L_- = [L_+, L_-], \quad L_\pm = L_\pm (\tau, \sigma; z), \quad z \in \mathbb{C},$$

where $z$ is a spectral parameter and $L_\pm$ are the Lie algebra valued $L_\pm = L_\pm t_A$ Lax matrices. The $t_A$'s are the generators of a semi-simple Lie algebra satisfying the following normalization and commutation relations

$$\text{Tr} (t_A t_B) = \delta_{AB}, \quad [t_A, t_B] = f_{ABC} t_C,$$

where the structure constants $f_{ABC}$ are purely imaginary. Using (1.1), we can design the monodromy matrix

$$M(z) = \text{Pexp} \int_{-\infty}^{\infty} d\sigma L_1, \quad L_1 = L_+ - L_-,$$

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1 The worldsheet coordinates $\sigma^\pm$ and $(\tau, \sigma)$ are related by

$$\sigma^\pm = \tau \pm \sigma, \quad \partial_0 = \partial_+ + \partial_-, \quad \partial_1 = \partial_+ - \partial_-$$

and $*d\sigma^\pm = \pm d\sigma^\pm$ or $*d\sigma = d\tau, *d\tau = d\sigma$ in Lorentzian signature.
which is conserved for all values of \( z \), i.e. \( \partial_0 M(z) = 0 \). Expanding the monodromy matrix introduces infinite conserved charges which a priori are not in involution.

Following \cite{1}, we compute the equal time Poisson brackets of \( L_1 \)

\[
\{L_1^{(1)}(\sigma_1; z), L_1^{(2)}(\sigma_2; w)\}_{\text{PB}} = \{L_1^A(\sigma_1; z), L_1^B(\sigma_2; w)\}_{\text{PB}},
\]

where the superscript in parenthesis specifies the vector spaces on which the matrices act.\footnote{In particular}

It was proven in \cite{2,3}, that the conserved charges are in involution provided that the brackets assume the \( r/s \)-Maillot form

\[
\{L_1^{(1)}(\sigma_1; z), L_1^{(2)}(\sigma_2; w)\}_{\text{PB}} = \left([r_{zw}, L_1^{(1)}(\sigma_1; z)] + [r_{zw}, L_1^{(2)}(\sigma_2; w)]\right)\delta_{12} - 2s_{zw}\delta_{12}',
\]

where \( \delta_{12} = \delta(\sigma_1 - \sigma_2), \delta_{12}' = \partial_0 \delta(\sigma_1 - s_2) \) correspond to ultralocal and non-ultralocal terms, respectively. The \( r_{\pm zw} = r_{zw} \pm s_{zw} \) are matrices on the basis \( t_A \otimes t_B \) which depend on \( (z, w) \) and as a consequence of the anti-symmetry of the Poisson brackets, it follows that \( r_{+zw} + r_{-zw} = 0 \). A sufficient condition for the Poisson structure to satisfy the Jacobi identity is the (non-dynamical) modified Yang–Baxter (mYB) relation

\[
[r_{+z_1z_3}^{(13)} r_{-z_1z_2}^{(12)}] + [r_{+z_2z_3}^{(23)} r_{+z_1z_2}^{(12)}] + [r_{+z_2z_3}^{(23)} r_{+z_1z_3}^{(13)}] = 0.
\]

A consequence of the non-ultralocal terms is that the Poisson brackets for the monodromy matrix are not well defined. Nevertheless, it is possible to regularize the Poisson brackets by using a generalized symmetric limit procedure \cite{2,3} arising at

\[
\{M^{(1)}(z), M^{(2)}(w)\}_{\text{PB}} = [r_{zw}, M(z) \otimes M(w)] + M^{(1)}(z) s_{zw} M^{(2)}(w) - M^{(2)}(w) s_{zw} M^{(1)}(z).
\]

Let also note that the vanishing of the Poisson brackets of \( \text{Tr} \left(M^m(z_1)\right) \) with \( \text{Tr} \left(M^n(z_2)\right) \) \((m, n \text{ are positive integers})\) is independent of the regularization scheme \cite{2,3}.

The plan of the paper is as follows: In Sec. 2, we work out the strong integrability for group spaces. We initiate our study by revisiting a known example, namely the
isotropic $\lambda$-deformed model (§ 2.1). Next we focus on $\lambda$-deformations of two current algebras at unequal levels (§ 2.2). In Sec. 3 we shift gears and we consider the strong integrability for coset spaces. Contrary to the group spaces the $r/s$-Maillet form is satisfied in the weak sense as we need to include first class constraints which are related to the residual gauge invariance of the subgroup. At first we revisit the symmetric coset case in (§ 3.1) and then we move on to the $\lambda$-deformed $G_{k_1} \times G_{k_2} / G_{k_1+k_2}$ (§ 3.2). Finally in Sec. 4 we provide the $r/s$-Maillet form of the most general $\lambda$-deformed models consisting of $n$ commuting copies current algebras, at different levels.

The goal of this paper is to study the notion strong integrability for classically integrable $\lambda$-deformed CFTs and coset CFTs. To achieve this goal we prove that the spatial Lax realizes the $r/s$-Maillet form [2, 3]. In the coset cases, we have two choices to attack the problem. One may either employ the first class constraints, related to the residual gauge symmetry in the subgroup, or use second class constraints by strongly imposing the constraints and projecting to the coset. In this work we employed the first class constraints. It would be very interesting to study the coset cases by imposing the second class constraints. This computation would require an extension of the $r/s$-Maillet form (1.3), in order to include the antisymmetric step function – resulting from a Dirac analysis of the second class constraints.

2 Group spaces

In this section we work out the $r/s$-Maillet form for the isotropic group case at equal and unequal levels. In the equal level case, we review the construction in [4] as a warm up for the unequal level case.

2.1 The isotropic group space

As a warm up we revisit a known example where the above formalism has been already applied in detail. We consider the isotropic $\lambda$-deformed model which is known to be integrable [5], for the $SU(2)$ case this was first shown in [6]. The system of equations of motion takes the form

$$\partial_{\pm} A_{\mp} = \pm \frac{1}{1 + \lambda} [A_{\mp}, A_{\mp}], \quad (2.1)$$
with the corresponding Lax matrices given by

$$L_\pm = \frac{2}{1 + \lambda} \frac{z}{z + 1} A_\pm, \quad z \in \mathbb{C}. \quad (2.2)$$

Using the above we can build the spatial Lax matrix

$$L_1 = -\frac{2z}{(1 + \lambda)(1 - z^2)} ((1 - z)A_- + (1 + z)A_+) . \quad (2.3)$$

To evaluate (1.2), we need the Poisson brackets of $A_\pm$. These can be evaluated by expressing the gauge fields in terms the currents $J_\pm$ and by using two commuting copies of the current algebra [7,8]

$$\{J^A_\pm(\sigma_1), J^B_\pm(\sigma_2)\}_{\text{P.B.}} = \frac{2}{k} \left( f_{ABC} f^C_\pm(\sigma_2) \delta_{12} \pm \delta_{AB} \delta'_{12} \right) ,$$

$$(2.4)$$

where the currents $J_\pm$ are given in terms of $A_\pm$ as [9,10]

$$J_\pm = \lambda^{-1} A_\pm - A_\mp. \quad (2.5)$$

Employing the above into (1.3), we find [4]

$$r_{\pm \pm} \rightarrow r_{\pm \pm} \Pi, \quad \Pi = \sum_A t_A \otimes t_A,$$

$$r_{\pm \pm} = -\frac{2e^2(1 + z^2 + x(1 - z^2))zw}{(z - w)(1 - z^2)} , \quad r_{-\pm \pm} = -r_{+ \pm \pm} , \quad (2.6)$$

where: $x = \frac{1 + \lambda^2}{2\lambda}$, $e = \frac{2\lambda}{\sqrt{k(1 - \lambda)(1 + \lambda)^3}},$

while the mYB equation (1.4) is also satisfied. In addition, we note that the above expressions are invariant under the symmetry of the model

$$\lambda \rightarrow \lambda^{-1}, \quad k \rightarrow -k. \quad (2.7)$$

The above result (2.6), can be written in terms of a twist function $\varphi_{\lambda}(z)$ [11]

$$r_{\pm \pm} = -\frac{2}{z^{-1} - w^{-1}} \varphi_{\lambda^{-1}}^{-1}(z^{-1}), \quad (2.8)$$

3 To get rid of the unphysical pole at $z = 0$, we have transformed $z \rightarrow z^{-1}$. 

4
where
\[ \varphi_\lambda^{-1}(z^{-1}) = -e^2 \frac{1 + x + z^2(1 - x)}{1 - z^2}. \] (2.9)

The twist function has poles of order one at
\[ \varphi_\lambda^{-1}(z^{-1}) = 0 \implies z = \pm \frac{1 + \lambda}{1 - \lambda}, \] (2.10)

provided that \( \lambda \neq 0 \), otherwise the expressions trivialize to zero and the non-ultra local term \( \delta'_{12} \) is removed \([12]\). Evaluating the spatial Lax matrix (2.3) and \( s_{zw} \) at the poles we obtain
\[ L_1(z\pm) = \pm J_\pm, \quad s_{z\pm z\pm} = \mp \frac{1}{k}, \quad s_{z\pm z\mp} = 0. \] (2.11)

Using the above in the Maillet brackets (1.3), we readily obtain the current algebra (2.4).\(^4\)

Next, we focus on two interesting zoom-in limits of the \( \lambda \)-deformed model, namely the non-Abelian T-dual of the PCM and the pseudo-dual chiral models. In the non-Abelian T-dual model, we expand \( \lambda \) around one \([5]\)
\[ \lambda = 1 - \frac{\kappa^2}{k}, \quad k \gg 1, \] (2.12)
yielding a twist function
\[ \varphi_\kappa^{-1}(z^{-1}) = -\frac{1}{\kappa^2} \frac{1}{1 - z^2}, \] (2.13)
which is in agreement with the PCM result Eq.(35) in \([3,5]\) by identifying \( \gamma = 2/\kappa^2 \) and also transforming \( z \to z^{-1} \).

In the pseudodual chiral model, we expand \( \lambda \) around minus one \([13]\)
\[ \lambda = -1 + \frac{b^{2/3}}{k^{1/3}}, \quad k \gg 1, \] (2.14)
yielding a twist function
\[ \varphi_b^{-1}(z^{-1}) = -\frac{4}{b^2} \frac{z^2}{1 - z^2}. \] (2.15)

Note that, this twist function is basically the same as the one in (2.13) for PCM as expected, since the pseudo-chiral and PCM models have their equations of motion
\(^4\)The \( r_{zw} \) terms do not contribute as either they identically vanish or the commutators in (1.3) vanish when they are evaluated at different poles or equal poles, respectively.\(^5\)This is expected since the PCM model is canonically equivalent to its non-Abelian T-dual \([14,16]\).
and Bianchi identities exchanged.

## 2.2 The isotropic deformation at unequal levels

Let us consider \( \lambda \)-deformations of the direct product of two current algebras at levels \( k_1 \) and \( k_2 \). We choose without loss of generality that \( k_2 > k_1 \). The system of equations of motion takes the form \([17]\)

\[
\partial_\pm A_\mp = \pm \frac{1 - \lambda_0^{\pm 1} \lambda_1^\mp}{1 - \lambda_1^\mp} [A_+, A_-], \quad \partial_\pm B_\mp = \pm \frac{1 - \lambda_0^{\pm 1} \lambda_2^\mp}{1 - \lambda_2^\mp} [B_+, B_-],
\]

where \( \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1 \). The corresponding Lax matrices read

\[
L_\pm = \frac{2z}{z \mp 1} \frac{1 - \lambda_0^{\pm 1} \lambda_1^\pm}{1 - \lambda_1^\pm} A_\pm, \quad \hat{L}_\pm = \frac{2z}{z \mp 1} \frac{1 - \lambda_0^{\pm 1} \lambda_2^\pm}{1 - \lambda_2^\pm} B_\pm, \quad z \in \mathbb{C}.
\]

To evaluate \([1.2]\), we need the Poisson brackets of \((A_\pm, B_\pm)\) which can be expressed in terms of two commuting copies of the current algebras

\[
\{J_{i\pm}^A(\sigma_1), J_{j\pm}^B(\sigma_2)\}_\text{P.B.} = \frac{2}{k_i} \delta_{ij} \left( f_{ABC} J_{i\pm}^C(\sigma_2) \delta_{12} \pm \delta_{AB} \delta_{12} \right),
\]

\[
\{J_{i\pm}^A(\sigma_1), J_{j\pm}^B(\sigma_2)\}_\text{P.B.} = 0,
\]

where the currents are given by \([17]\)

\[
J_{1+} = \lambda_0^{-1} \lambda_1^{-1} A_+ - A_-, \quad J_{1-} = \lambda_0^{-1} \lambda_2^{-1} B_+ - B_-,
\]

\[
J_{2+} = \lambda_0 \lambda_2^{-1} B_+ - B_-, \quad J_{2-} = \lambda_0 \lambda_1^{-1} A_- - A._+
\]

Employing the above into \([1.3]\) we find

\[
r_{\pm w} \rightarrow r_{\pm z} \Pi, \quad \hat{r}_{\pm w} \rightarrow \hat{r}_{\pm z} \Pi, \quad \Pi = \sum_A t_A \otimes t_A,
\]

\[
r_{+w} = \frac{4\lambda_1 zw(-2z + z + 1 + (z - 1)\lambda_1^2)(-2z + z + 1 + (z + 1)\lambda_1^2))}{\sqrt{k_1 k_2(z - w)(1 - z^2)(1 - \lambda_1^2)^3}},
\]

\[
\hat{r}_{+w} = \frac{4\lambda_2 zw(-2z + z + 1 + (z - 1)\lambda_2^2)(-2z + z + 1 + (z + 1)\lambda_2^2))}{\sqrt{k_1 k_2(z - w)(1 - z^2)(1 - \lambda_2^2)^3}},
\]

\[
r_{-w} = -r_{+w}, \quad \hat{r}_{-w} = -\hat{r}_{+w}.
\]

(2.20)
We have checked that indeed the mYB equations (1.4) are indeed satisfied. Let us also note that (2.20) is invariant under the symmetry [17]

$$
\lambda_1 \to \lambda_1^{-1}, \quad \lambda_2 \to \lambda_2^{-1}, \quad k_1 \to -k_2, \quad k_2 \to -k_1
$$

and that it matches (2.6) for equal levels. This is so because as was shown in [18] the model with $k_1 = k_2$, constructed in [19], is canonically equivalent to two single $\lambda$-deformed models.

The above results could also be written in terms of two twist functions $(\varphi_\lambda, \hat{\varphi}_\lambda)$ as in (2.8), with

$$
\varphi_\lambda^{-1}(z^{-1}) = \frac{2\lambda_1(-2z\lambda_1 + \lambda_0(z + 1 + (z - 1)\lambda_1^2))(-2z\lambda_1 + \lambda_0^{-1}(z - 1 + (z + 1)\lambda_1^2))}{\sqrt{k_1k_2}(1 - z^2)(1 - \lambda_1^2)^3},
$$
$$
\hat{\varphi}_\lambda^{-1}(z^{-1}) = \frac{2\lambda_2(-2z\lambda_2 + \lambda_0^{-1}(z + 1 + (z - 1)\lambda_2^2))(-2z\lambda_2 + \lambda_0(z - 1 + (z + 1)\lambda_2^2))}{\sqrt{k_1k_2}(1 - z^2)(1 - \lambda_2^2)^3}.
$$

These twist functions have single poles at

$$
z_+ = \frac{1 - \lambda_1^2}{1 - 2\lambda_0\lambda_1 + \lambda_1^2}, \quad z_- = -\frac{1 - \lambda_0^2}{1 - 2\lambda_0^{-1}\lambda_1 + \lambda_1^2},
$$
$$
\hat{z}_+ = \frac{1 - \lambda_2^2}{1 - 2\lambda_0^{-1}\lambda_2 + \lambda_2^2}, \quad \hat{z}_- = -\frac{1 - \lambda_0^2}{1 - 2\lambda_0\lambda_2 + \lambda_2^2},
$$

provided that $\lambda_{1,2} \neq 0$, otherwise the expressions trivialize to zero and the non-ultra local term $\delta_{12}$ is removed [12]. The poles in (2.23) are of order one, provided that $\lambda_{1,2} \neq \lambda_0, \lambda_0^{-1}$ otherwise they become of order two. Assuming they are of order one, one can evaluate the spatial Lax matrices and $(s_{zw}, \hat{s}_{zw})$ at the poles to obtain

$$
L_1(z_+) = J_1+, \quad L_1(z_-) = -J_2-, \quad \hat{L}_1(\hat{z}_+) = -J_1-, \quad \hat{L}_1(\hat{z}_-) = J_2+,
$$
$$
s_{zz_+} = \frac{-1}{k_1}, \quad s_{zz_-} = \frac{1}{k_2}, \quad s_{z+z_+} = 0, \quad \hat{s}_{zz_+} = \frac{1}{k_1}, \quad \hat{s}_{zz_-} = -\frac{1}{k_2}, \quad \hat{s}_{z+z_+} = 0.
$$

Using the above expressions in the Maillet brackets (1.3), we readily obtain the current algebra (2.18).

Let us now analyze the particular case in which $\lambda_1 = \lambda_0 = \lambda_2$. At that point the twist functions have poles of order two at $z = 1$ and $\hat{z} = -1$. Upon evaluating the spatial
Lax matrices and $s_{zw}, \hat{s}_{zw}$ at these poles we obtain that

$$L_1 = \frac{1}{k_1 - k_2} (k_1 J_1^+ + k_2 J_2^-), \quad \hat{L}_1 = -\frac{1}{k_1 - k_2} (k_1 J_1^- + k_2 J_2^+) ,$$

$$s_{zz} = -\frac{1}{k_1 - k_2}, \quad \hat{s}_{z\hat{z}} = \frac{1}{k_1 - k_2} .$$

Using the above in the Maillet brackets (1.3), we obtain two copies of a current algebra at level $k_2 - k_1$

$$\{L^A_1(\sigma_1), L^B_1(\sigma_2)\}_{\text{P.B.}} = \frac{2}{k_1 - k_2} \left( f_{ABC} L^C_1(\sigma_2) \delta_{12} + \delta_{AB} \delta'_{12} \right) ,$$

$$\{\hat{L}^A_1(\sigma_1), \hat{L}^B_1(\sigma_2)\}_{\text{P.B.}} = \frac{2}{k_2 - k_1} \left( f_{ABC} \hat{L}^C_1(\sigma_2) \delta_{12} + \delta_{AB} \delta'_{12} \right) .$$

These current algebras are realised at level $k_2 - k_1$. This is in resonance with the fact that the fixed point in the IR when $\lambda_1 = \lambda_0 = \lambda_2$ is given by the CFT $G_{k_1} \times G_{k_2 - k_1} \times G_{k_2 - k_1} \ [17]$.

3 Coset spaces

In this section we first revisit, as a warm up, the construction of [11] of the $r/s$-Maillet form for the symmetric coset $G_k/H_k$ case. Then we proceed to find the same form for the coset $G_{k_1} \times G_{k_2} / G_{k_1 + k_2}$.

3.1 The isotropic $G_k/H_k$ symmetric coset space

Let us consider a semi-simple group $G$ and its decomposition to a semi-simple group $H$ and a symmetric coset $G/H$. Consider a coupling matrix $\lambda_{AB}$ with elements $\lambda_{ab} = \delta_{ab}$ and $\lambda_{\alpha\beta} = \lambda \delta_{\alpha\beta}$, which is known to be integrable [9]. The subgroup indices are denoted by Latin letters and coset indices by Greek letters. The restriction to symmetric cosets amounts to structure constants with $f_{\alpha\beta\gamma} = 0$. The equations of motion simplify
\[ \partial_+ A_- - \partial_- A_+ = [A_+, A_-] + \lambda^{-1}[B_+, B_-], \]
\[ \partial_\pm B_\mp = -[B_\mp, A_\pm], \]
where: \( A_\pm = A_\pm^a t_a, \quad B_\pm = B_\pm^a t_a, \)
and the corresponding Lax matrices reads
\[ L_\pm = A_\pm + \frac{z^{\pm 1}}{\sqrt{\lambda}} B_\pm, \quad z \in \mathbb{C}. \]

Using the above we construct the spatial Lax matrix
\[ L_1 = A_+ - A_- + \frac{1}{\sqrt{\lambda}} \left( z B_+ - z^{-1} B_- \right). \]

To evaluate (1.2), we split the algebra (2.4) into subgroup and coset components
\[ \{j^a_\pm(\sigma_1), j^b_\pm(\sigma_2)\}_{\text{P.B.}} = \frac{2}{k} \left( f_{abc} j^c_\pm(\sigma_1) \delta_{12} \pm \delta_{ab} \delta_{12}' \right), \]
\[ \{I^a_\pm(\sigma_1), I^b_\pm(\sigma_2)\}_{\text{P.B.}} = \frac{2}{k} \left( f_{abc} I^c_\pm(\sigma_1) \delta_{12} \pm \delta_{ab} \delta_{12}' \right), \]
\[ \{j^a_\pm(\sigma_1), I^b_\pm(\sigma_2)\}_{\text{P.B.}} = \frac{2}{k} f_{a\beta\gamma} J^\gamma_\pm(\sigma_1) \delta_{12}, \]
where the currents
\[ j_\pm = A_\pm - A_\mp, \quad I_\pm = \lambda^{-1} B_\pm - B_\mp. \]

Note that there is a first class constraint
\[ \chi = j_+ + j_- \approx 0. \]

Since \( \{\chi^d(\sigma_1), \chi^b(\sigma_2)\}_{\text{P.B.}} \approx 0, \) this corresponds to the residual gauge symmetry. Before we proceed to the computation of the Poisson brackets of (3.3) we could also include \( \chi \) into the spatial Lax matrix as
\[ \tilde{L}_1 = L_1 + \varphi(\lambda, z) \chi, \]
where \( \varphi \) is an arbitrary, Lagrange multiplier type, function of \( \lambda \) and \( z \) to be later determined. We can now compute the Poisson brackets of \( \tilde{L}_1 \) and after some tedious
where as they transform as

\[ \{ \tilde{L}_1^{(1)}(\sigma_1; z), \tilde{L}_1^{(2)}(\sigma_2; w) \}_{\text{PB}} \approx \left( [r_{-zw}, \tilde{L}_1^{(1)}(\sigma_1; z)] + [r_{+zw}, \tilde{L}_1^{(2)}(\sigma_2; w)] \right) \delta_{12} - 2s_{zw}\delta'_{12}, \]

(3.8)

where

\[ r_{\pm zw} \rightarrow r^\text{sub}_{\pm zw} \Pi_{\text{sub}} + r^\text{coset}_{\pm zw} \Pi_{\text{coset}}, \quad \Pi_{\text{sub}} = \sum_a t_a \otimes t_a, \quad \Pi_{\text{coset}} = \sum_a t_a \otimes t_a, \]

(3.9)

The above expressions are not apparently invariant under the symmetry of the model

\[ \lambda \rightarrow \lambda^{-1}, \quad k \rightarrow -k, \]

(3.10)
as they transform as

\[ r^\text{sub}_{\pm zw} \rightarrow r^\text{sub}_{\pm zw} \pm \frac{1 + \epsilon(\lambda^{-1}, z) + \epsilon(\lambda, z)}{k}, \quad r^\text{coset}_{\mp zw} \rightarrow r^\text{coset}_{\pm zw}. \]

(3.11)

Let us now check if the modified Yang–Baxter equation (1.4) is satisfied for (3.9), yielding that

\[ [r^{(13)}_{+z_1z_3}, r^{(12)}_{-z_1z_2}] + [r^{(23)}_{+z_2z_3}, r^{(12)}_{+z_1z_2}] + [r^{(23)}_{+z_2z_3}, r^{(13)}_{+z_1z_3}] = c_1(z_{1,2}) f_{abc} t_a \otimes t_b \otimes t_c + c_2(z_{1,2}) f_{a\beta\gamma} t_\beta \otimes t_\gamma \otimes t_a, \]

(3.12)

where

\[ c_1(z_{1,2}) = \frac{2}{k} \left( -\epsilon(\lambda, z_2) r^\text{sub}_{+z_1z_2} + \epsilon(\lambda, z_1) \left( r^\text{sub}_{-z_1z_2} - \frac{2}{k} \epsilon(\lambda, z_2) \right) \right), \]

(3.13)

\[ c_2(z_{1,2}) = \frac{2}{k} \left( -\epsilon(\lambda, z_2) r^\text{coset}_{+z_1z_2} + \epsilon(\lambda, z_1) r^\text{coset}_{-z_1z_2} - \frac{4\lambda(z_1 - \lambda^{-1}z_1^{-1})(z_2 - \lambda^{-1}z_2^{-1})}{k^2(\lambda - \lambda^{-1})^2} \right). \]

The non-vanishing right hand side (3.12) can be attribute to the residual gauge symmetry of the subgroup. A similar situation was encountered in [20] where the Maillet algebra was computed in the presence of Wilson lines among the inserted operators. The presence of the Wilson lines leads to non-trivial monodromy properties and the Jacobi identity is weakly satisfied. Another similar situation has appeared before in
studies of parafermions \cite{21,22}.

In order to proceed we demand that the mYB equation is satisfied, i.e. $c_{1,2}(z_{1,2}) = 0$. These two conditions determine $\rho(\lambda, z_1)$ and $\rho(\lambda, z_2)$. It turns that they both originate from the following two solutions

$$
\rho_1(\lambda, z) = \frac{\lambda^{-1} - z^2}{\lambda - \lambda^{-1}} \quad \text{or} \quad \rho_2(\lambda, z) = \frac{z^2 - \lambda}{\lambda - \lambda^{-1}},
$$

which satisfy $\rho_1(\lambda^{-1}, z^{-1}) = \rho_2(\lambda, z)$, hence we analyze the above results for $\rho_1$. Then (3.9) simplifies to

$$
\begin{align*}
    r^{\text{sub}}_{zw} &= \frac{2\lambda (z^2 - \lambda^{-1})(z^2 - \lambda)}{k(z^2 - w^2)(\lambda - \lambda^{-1})}, \\
    r^{\coset}_{zw} &= \frac{2w(z^2 - \lambda^{-1})(z^2 - \lambda)}{k(z^2 - w^2)(\lambda - \lambda^{-1})},
\end{align*}
$$

which is now invariant under the symmetry (3.10).

This result has been also derived using a twist function description \cite{23}

$$
\begin{align*}
    r^{\text{sub}}_{zw} &= -\frac{2z^2}{z^2 - w^2} \varphi^{-1}_\lambda(z), \\
    r^{\coset}_{zw} &= -\frac{2zw}{z^2 - w^2} \varphi^{-1}_\lambda(z),
\end{align*}
$$

with $\varphi_\lambda(z)$ is given by \cite{11}

$$
\varphi_\lambda(z) = -\frac{kz^2(\lambda - \lambda^{-1})}{(z^2 - \lambda)(z^2 - \lambda^{-1})},
$$

with poles of order one at

$$
z_1 = \pm \lambda^{-1/2}, \quad z_2 = \pm \lambda^{1/2},
$$

provided that $\lambda \neq 0$. A comment is in order regarding the twist function description (3.16). Demanding that (3.9) takes the form of (3.16), fixes $\rho$ and $\varphi_\lambda(z)$ to the value $\rho_1$ in (3.14) and (3.17), respectively.

\footnote{We would like to thank Sakura Schäfer-Nameki for discussions on this point.}
Evaluating the spatial Lax matrix \( \tilde{L}_1(z_1^\pm) = j_+ \pm J_+ \), \( \tilde{L}_1(z_2^\pm) = -j_- \mp J_- \),
\[
\begin{align*}
&\tilde{s}_{\text{sub}}_{z_1^\pm z_1^\mp} = -\frac{1}{k}, \quad \tilde{s}_{\text{sub}}_{z_2^\pm z_2^\mp} = \frac{1}{k}, \quad \tilde{s}_{\text{sub}}_{z_1^\pm z_1^\mp} = -\frac{1}{k}, \quad \tilde{s}_{\text{sub}}_{z_2^\pm z_2^\mp} = \frac{1}{k}, \\
&\tilde{s}_{\text{coset}}_{z_1^\pm z_1^\mp} = -\frac{1}{k}, \quad \tilde{s}_{\text{coset}}_{z_2^\pm z_2^\mp} = \frac{1}{k}, \quad \tilde{s}_{\text{coset}}_{z_1^\pm z_1^\mp} = \frac{1}{k}, \quad \tilde{s}_{\text{coset}}_{z_2^\pm z_2^\mp} = \frac{1}{k}, \\
&\tilde{s}_{\text{sub}}_{z_1^\pm z_2^\pm} = 0 = \tilde{s}_{\text{sub}}_{z_1^\pm z_2^\mp}, \quad \tilde{s}_{\text{coset}}_{z_1^\pm z_2^\mp} = 0 = \tilde{s}_{\text{coset}}_{z_1^\pm z_2^\mp}.
\end{align*}
\] (3.19)
Plugging the above in the Maillet brackets (3.8), we recover the current algebra (3.4).

Let us mention that for \( \lambda = 0 \), these expressions drastically simplify to
\[
\begin{align*}
&\tilde{L}_1 = -J_-, \quad r_{\text{sub}}^{z_1^\pm z_1^\mp} = \frac{2z^2}{k(z^2 - w^2)}, \quad r_{\text{coset}}^{z_1^\pm z_1^\mp} = \frac{2zw}{k(z^2 - w^2)}, \quad s_{\text{sub}}_{z_1^\pm z_1^\mp} = \frac{1}{k}, \quad s_{\text{coset}}_{z_1^\pm z_1^\mp} = 0
\end{align*}
\] (3.20)
and the non-ultralocal term is partially removed in the subgroup part of the algebra [24][11].

### 3.2 The isotropic \( G_{k_1} \times G_{k_2} / G_{k_1 + k_2} \) coset space

As a second example we consider the isotropic \( \lambda \)-deformed \( G_{k_1} \times G_{k_2} / G_{k_1 + k_2} \) coset space. This has been proved to be integrable as it admits a flat Lax connection [25]. More concretely, its equations of motion take the form of
\[
\begin{align*}
\partial_+ \tilde{A}_- - \partial_- \tilde{A}_- &= [\tilde{A}_+, \tilde{A}_-] + (\alpha^2 + \beta)[B_+, B_-], \\
\partial_\pm B_\mp &= -[B_\mp, \tilde{A}_\pm],
\end{align*}
\] (3.21)
where
\[
\tilde{A}_\pm = A_\pm + \alpha B_\pm, \quad A_\pm = \frac{1}{2}(A_{1\pm} + A_{2\pm}), \quad B_\pm = \frac{1}{2}(A_{1\pm} - A_{2\pm}),
\] (3.22)
and
\[
\begin{align*}
\alpha &= \frac{(s_1 - s_2)(1 - \lambda)}{1 - \lambda_f^{-1} \lambda}, \quad \beta = \frac{1 + \lambda - 2(1 - 4s_1 s_2)\lambda^2}{\lambda(1 - \lambda_f^{-1} \lambda)}, \\
\lambda_f^{-1} &= 1 - 8s_1 s_2, \quad s_i = \frac{k_i}{k}, \quad k = k_1 + k_2, \quad i = 1, 2.
\end{align*}
\] (3.23)
The above system of equations (3.21) admits a flat Lax connection [25]

\[ L_\pm = \tilde{A}_\pm + z^{\pm 1} \sqrt{\alpha^2 + \beta} B_\pm = A_\pm + \left( \alpha + z^{\pm 1} \sqrt{\alpha^2 + \beta} \right) B_\pm, \quad z \in \mathbb{C}. \]  

(3.24)

Using the above we build the spatial Lax matrix

\[ L_1 = \frac{1}{2} \left( (1 + \alpha) A_{1|1} + (1 - \alpha) A_{2|1} \right) + \sqrt{\alpha^2 + \beta} \left( zB_+ - z^{-1}B_- \right), \]

(3.25)

where \( A_{ij|1} = A_{i+} - A_{i-} \). To evaluate (1.2), we need the Poisson brackets of \( A_{i\pm} \) which in turn can be obtained from two commuting copies of current algebras (2.18), where [25]

\[ J_{1+} = \frac{1}{2s_1} (\lambda^{-1} - 1) B_+ + A_{1|1}, \quad J_{1-} = \frac{1}{2s_1} (\lambda^{-1} - 1) B_+ - A_{1|1}, \]

\[ J_{2+} = -\frac{1}{2s_2} (\lambda^{-1} - 1) B_+ + A_{2|1}, \quad J_{2-} = -\frac{1}{2s_2} (\lambda^{-1} - 1) B_+ - A_{2|1}. \]

(3.26)

To proceed we define the subgroup generators \( \mathcal{H}_\pm \)

\[ \mathcal{H}_\pm = \frac{1}{2} (s_1 J_{1\pm} + s_2 J_{2\pm}), \]

(3.27)

or equivalently through (3.26)

\[ \mathcal{H}_\pm = \pm \frac{1}{2} \left( s_1 A_{1|1} + s_2 A_{2|1} \right). \]

(3.28)

Similarly to (3.6), we find a first class constraint

\[ \chi = \mathcal{H}_+ + \mathcal{H}_- \approx 0. \]

(3.29)

Since \( \{ \chi^A(\sigma_1), \chi^B(\sigma_2) \} \) \( \approx 0 \) we have again a residual gauge symmetry. Before we proceed to the derivation of the Poisson brackets of (3.25), we include \( \chi \) into the spatial Lax matrix as in (3.7)

\[ \tilde{L}_1 = L_1 + \varrho(\lambda, z) \chi. \]

(3.30)

Then we express \( \tilde{L}_1 \) or equivalently \( A_{i|1}, B_\pm \) in terms of \( J_{i\pm} \). We are in position to compute the Poisson brackets of \( \tilde{L}_1 \) and for simplicity we start with the equal level

\[ \mathcal{H}_+ = \frac{1}{2} A_{1|1} - s_2 (B_+ - B_-), \]

7 We first rewrite \( \mathcal{H}_+ \) in (3.28) as
The equal level case

Plugging the spatial Lax matrix (3.30) into the Maillet algebra (3.8) we find

\[ r_{\pm zw} \rightarrow r_{\pm zw} \Pi, \quad \Pi = \sum_A t_A \otimes t_A, \]

\[ r_{+zw} = \frac{1}{k} \left( \frac{2(z - \sqrt{\lambda})(z^2 - \lambda^{-1})(w + \sqrt{\lambda})}{z(z - w)(\lambda - \lambda^{-1})} - q(\lambda, z) \right), \quad r_{-zw} = -r_{+wz}. \]  

(3.31)

The above expression transforms under the symmetry (3.10), as

\[ r_{+zw} \rightarrow r_{+zw} + \frac{2 - \frac{2\sqrt{\lambda}}{1 + \lambda} (z + z^{-1}) + q(\lambda^{-1}, z) + q(\lambda, z)}{k}. \]  

(3.32)

Then we plug (3.31) into the mYB equation (1.4), yielding

\[ [r_{+z_1z_3}, r_{-z_2z_3}] + [r_{+z_2z_3}, r_{-z_1z_2}] + [r_{+z_2z_3}, r_{+z_1z_2}] \] \[ = c(z_{1,2}) f_{ABC} t_A \otimes t_B \otimes t_C, \]

\[ c(z_{1,2}) = \frac{1}{k} \left( q(\lambda, z_1)r_{-z_2z_3} - q(\lambda, z_2)r_{+z_1z_2} \right) - \frac{1}{k^2} q(\lambda, z_1)q(\lambda, z_2). \]  

(3.33)

Let us now demand the right hand side of (3.33) ivanishes for every \( z_{1,2} \), i.e. \( c(z_{1,2}) = 0 \). Demanding that this condition is satisfied in the limit \( z_2 \rightarrow z_1 = z \), yields a first order differential equation for \( q(\lambda, z) \) with respect to \( z \) which has the following solution

\[ q(\lambda, z) = -\frac{2(z - \sqrt{\lambda})(z^2 - \lambda^{-1})}{z(\lambda - \lambda^{-1} - 2\lambda^{-1}(z + \sqrt{\lambda})f(\lambda))}. \]  

(3.34)

where \( f(\lambda) \) is a \( \lambda \)-dependent integration constant. Plugging the above into \( c(z_{1,2}) \) in (3.33), we find that it is satisfied for every \( z_{1,2} \). Then we use (3.34) and we find that (3.31) is invariant under the symmetry (3.10), provided that

\[ f(\lambda) = \lambda^{-1/2}(1 + \lambda^{-1})\frac{\lambda^4 - h(\lambda)}{2\lambda^2}, \quad \text{with} \quad h(\lambda)h(\lambda^{-1}) = 1. \]  

(3.35)

and we also note that (3.28) can be also used to express \( A_{2|1} \) in terms of \( A_{1|1} \) as

\[ A_{2|1} = \frac{1}{s_2} \left( 2s_2 - s_1 A_{1|1} \right). \]

Using the above, (3.27) and \( J_{1\pm} \)'s in (3.26), we can write \( (B_{\pm}, A_{1|1}) \) in terms of \( (J_{1\pm}, J_{2\pm}) \). Therefore, \( \tilde{L}_1 \) is expressed on the \( J_{i\pm} \) basis.
For simplicity we pick up the integration constant to be \( f(\lambda) = 0 \), a choice which is also justified by the results that follow. This choice leads to
\[
\varrho(\lambda, z) = -\frac{2(z - \sqrt{\lambda})(z^2 - \lambda^{-1})}{z(\lambda - \lambda^{-1})}
\]
and
\[
{\tilde{r}}_{+zw} = \frac{2(z^2 - \lambda)(z^2 - \lambda^{-1})}{kz(z - w)(\lambda - \lambda^{-1})}, \quad {\tilde{r}}_{-zw} = -{r}_{+wz}.
\]
The above expression can be written in terms of a twist function \( {\varphi}_\lambda(z) \) [26]
\[
{r}_{+zw} = -\frac{1}{z - w} {\varphi}_\lambda(z)^{-1},
\]
where
\[
{\varphi}_\lambda(z) = -\frac{kz(\lambda - \lambda^{-1})}{2(z^2 - \lambda^{-1})(z^2 - \lambda)}.
\]
The twist function (3.39) has four poles of order one at
\[
z_1^\pm = \pm \lambda^{-1/2}, \quad z_2^\pm = \pm \lambda^{1/2},
\]
provided that \( \lambda \neq 0 \). Evaluating the spatial Lax matrix (3.30) and \( s_{zw} \) at (3.40), using also (3.36), we obtain
\[
\begin{align*}
{\tilde{L}}_1(z_1^+) &= J_1^+, \quad {\tilde{L}}_1(z_1^-) = J_2^+, \quad {\tilde{L}}_1(z_2^+) = -J_1^-, \quad {\tilde{L}}_1(z_2^-) = -J_2^-, \\
{s}_{z_1^\pm z_1^\pm} &= -\frac{2}{k}, \quad s_{z_2^\pm z_2^\pm} = \frac{2}{k}, \quad s_{z_1^\pm z_1^\mp} = 0 = s_{z_2^\pm z_2^\mp}, \quad s_{z_1^\pm z_2^\mp} = 0 = s_{z_2^\pm z_1^\mp}.
\end{align*}
\]
Employing the above in the Maillet brackets (3.8), we obtain the current algebra (2.18). At \( \lambda = 0 \), the expressions drastically simplify respectively to
\[
\begin{align*}
{\tilde{L}}_1 &= -\frac{1}{2} (J_1^- + J_2^-), \quad {r}_{+zw} = \frac{2z}{z - w} s_{zw}, \quad s_{zw} = \frac{1}{k},
\end{align*}
\]
so the non-ultralocal term \( \delta'_{12} \) is partially removed [24].

Finally, two comments are in order regarding the twist function description (3.38). Firstly, demanding that (3.31) takes the form (3.38), uniquely fixes \( \varrho \) and \( {\varphi}_\lambda(z) \) to the values (3.36) and (3.39) respectively. This property will be the guideline in the unequal level case which follows. Secondly, it was argued in [26], using the results of [27], that the one-parameter deformation of the coset \( \sigma \)-model \( G \times G/G_{\text{diag}} \) admits such a
description in terms of the twist function (see Eq.(3.3) of [26])

\[ \phi_{YB}(z) = \frac{16Kz}{(1 - z^2)^2 + \eta^2(1 + z^2)^2}, \]  

(3.43)

which after the analytic continuation

\[ \eta = i\frac{1 - \lambda}{1 + \lambda}, \quad \frac{8iK}{\eta} = k, \]  

(3.44)

(3.43) maps to (3.39). Such a mapping is expected since the \( \lambda \) and the \( \eta \) deformations are related up to a Poisson–Lie T-duality and an analytic continuation, as the one in (3.44) [28–32].

**The unequal level case**

Upon inserting the spatial Lax matrix (3.30) into the Maillet algebra (3.8), we require that \( r_{\pm zw} \) can be expressed in terms of twist function description as in (3.38), which automatically satisfies the mYB (1.4). This is an arduous task whose end result takes the form

\[ \phi_{\lambda}(z) = \frac{1}{\xi^2} \frac{8ks_1s_2\zeta z}{z^4 + 1 + 4\xi^2 \frac{s_1^2 - s_2^2}{\xi^2} z(z^2 + 1) + \left(2 + 4\frac{2s_1s_2 - s_2^2}{\xi^2} z^2 - 1\right) z^2}, \]

\[ \zeta = \frac{\sqrt{\lambda(1 - \lambda_1^{-1}\lambda)(1 - \lambda_2^{-1}\lambda)(1 - \lambda_3^{-1}\lambda)}}{(1 - \lambda_f^{-1}\lambda)^2}, \]

\[ \xi = \frac{1 - \lambda}{1 - \lambda_f^{-1}\lambda}, \]  

(3.45)

\[ \lambda_1 = \frac{1}{s_2 - 3s_1}, \quad \lambda_2 = \frac{1}{s_1 - 3s_2}, \quad \lambda_3 = \frac{1}{(s_1 - s_2)^2}, \quad \lambda_f^{-1} = 1 - 8s_1s_2. \]

For equal levels the above expressions are in agreement with (3.39). The twist function (3.45) is invariant under the remarkable transformation [25]

\[ \lambda \rightarrow \frac{1 - (s_1 - s_2)^2\lambda}{(s_1 - s_2)^2 - \lambda_f^{-1}\lambda}, \quad k_1 \rightarrow -k_1, \quad k_2 \rightarrow -k_2, \]  

(3.46)

as one can readily check using the induced transformations \((\xi, \zeta) \rightarrow (-\xi, \zeta)\).

The twist function (3.45) has four poles of order one at

\[ z_{1\pm} = \pm\zeta^{-1}(1 \mp (s_1 - s_2)\xi)(1 + \xi), \quad z_{2\pm} = \pm\xi^{-1}(1 \pm (s_1 - s_2)\xi)(1 - \xi), \]  

(3.47)
provided that $\zeta \neq 0$ or equivalently that $\lambda$ is different from the CFT values \[25\]

$$\lambda \neq \{0, \lambda_1, \lambda_2, \lambda_3\}. \quad (3.48)$$

Evaluating the spatial Lax matrix \[3.30\] and $s_{zw}$ at the poles \[3.47\], we obtain

$$\tilde{L}_1(z_{1+}) = f_{1+}, \quad \tilde{L}_1(z_{1-}) = f_{2+}, \quad \tilde{L}_1(z_{2+}) = -f_{1-}, \quad \tilde{L}_1(z_{2-}) = -f_{2-},$$

$$s_{z_{1+}z_{1+}} = -\frac{1}{k_1}, \quad s_{z_{1-z_{1-}}} = -\frac{1}{k_2}, \quad s_{z_{2+z_{2+}}} = \frac{1}{k_1}, \quad s_{z_{2-z_{2-}}} = \frac{1}{k_2},$$

$$s_{z_{1\pm}z_{2\mp}} = s_{z_{2\pm}z_{1\mp}}, \quad s_{z_{1\pm}z_{1\mp}} = 0 = s_{z_{2\pm}z_{2\mp}}. \quad (3.49)$$

Using the above in the Maillet brackets \[3.8\], we readily obtain the current algebra \[2.18\]. At the CFT points \[3.48\], the expressions for the spatial Lax and $r_{zw}$ drastically simplify to

$$\tilde{L}_1 = \left\{-s_1f_{1-} - s_2f_{2-}, \frac{s_1f_{1+} + s_2f_{2+}}{s_1 - s_2}, -\frac{s_1f_{1-} + s_2f_{2+}}{s_1 - s_2}, s_1f_{1+} + s_2f_{2+}\right\},$$

$$r_{zw} = \frac{2z}{z - w}s_{zw}, \quad s_{zw} = \left\{\frac{1}{k'}, \frac{1}{k_2 - k_1'}, \frac{1}{k_1 - k_2'}, -\frac{1}{k}\right\}, \quad k = k_1 + k_2, \quad (3.50)$$

in the ordering of CFT points given by \[3.48\]. Therefore, the non-ultralocal term $\delta_{12}'$ is partially removed at the CFT points \[3.48\].

Finally, three comments are in order regarding the twist function description \[3.45\]:

1. The above twist function \[3.45\] matches the twist function for the two parameter deformation of $G \times G/G_{\text{diag}}$, or equivalently the bi-Yang–Baxter model, see Eq.(3.8) of \[26\]

$$\varphi_{\text{bi-YB}}(z) = \frac{16Kz}{\zeta^2 z^4 + 1 + \frac{\eta^2 - \bar{\eta}^2}{\zeta}z (z^2 + 1) + \left(2 + \frac{(\eta^2 - \bar{\eta}^2)^2 - 16}{4\zeta^2}\right)z^2},$$

$$\zeta = \sqrt{\left(1 + \frac{1}{4}(\eta - \bar{\eta})^2\right)\left(1 + \frac{1}{4}(\eta + \bar{\eta})^2\right)}, \quad (3.51)$$

up to the analytic continuation

$$\eta = 2i\bar{s}_2\bar{\zeta}, \quad \bar{\eta} = 2is_1\zeta, \quad K = \frac{1}{2}ks_1s_2\zeta. \quad (3.52)$$

Agreement with the one parameter deformation of the coset $\sigma$-model $G \times G/G_{\text{diag}}$.
of Eqs. (3.43), (3.44) is found for \( \bar{\eta} = \eta \).

2. Using the results of \([27, 26]\) along with (3.52), the monodromy matrix when evaluated at the poles \( z_{i\pm} \) generates a \( q \)-deformed Poisson algebra with

\[
q_1 = e^{-i/k_1}, \quad q_2 = e^{-i/k_2},
\]

being roots of unity \([9]\).

3. Using Eq.(4.9) \([25]\) and (3.52), we can find the RG flows for two-parameter deformed \( G \times G/G_{\text{diag}} \)

\[
\frac{d\eta}{dt} = \frac{c_G \eta}{32K} \left(1 + \frac{1}{4} (\eta + \bar{\eta})^2 \right) \left(1 + \frac{1}{4} (\eta - \bar{\eta})^2 \right),
\]

\[
\frac{d}{dt} \left( \frac{\eta}{\bar{\eta}} \right) = 0, \quad \frac{d}{dt} \left( \frac{\eta}{K} \right) = 0,
\]

where \( t = \ln \mu^2 \) is the logarithmic RG scale. Please note that (3.54) is in agreement with the RG flows of the bi-Yang–Baxter model, Eq.(4.9) in \([30]\).

4 More on group spaces

In this section we study the strong integrability of the most general \( \lambda \)-deformation with \( n \) commuting current algebras at different levels \([33]\). Using the Lax matrices, which were given in Eqs. (3.15)-(3.22) of that work, we find that the corresponding spatial parts take the Maillet \( r/s \)-algebra \([1,3]\) with:

\[
\hat{r}_{zw} = \frac{2zw(z-1/\sqrt{k_1})}{(z-w)(d+\hat{d}_1(z))}, \quad r_{-zw} = -r_{+zw},
\]

\[
\hat{r}_{zw} = \frac{2zw(z-1/\sqrt{k_n})}{(z-w)(\hat{d}+\hat{d}_1(z))}, \quad \hat{r}_{-zw} = -\hat{r}_{+zw},
\]

where \( (d, d_1; \hat{d}, \hat{d}_1) \) were given in Eqs.(3.15), (3.19), (3.20), (3.22) of \([33]\). As a consistency check we have verified that the mYB equations \([1,4]\) are satisfied.

To analyze the underlying symmetries we rewrite (4.1) in terms of twist functions

---

8Under the redefinitions \( (\eta, \bar{\eta}, K) \rightarrow (2\eta, 2\zeta, \frac{1}{2}) \) and for \( c^2 = -1 \) (complex branch).

9The two coupling case studied in \([34]\), can be obtained as a special case of (4.1) for \( n = 2 \).
\((\varphi_\lambda, \hat{\varphi}_\lambda)\) as in (2.8), where
\[
\varphi_\lambda^{-1}(z^{-1}) = \frac{z - 1/\sqrt{k_1}}{d + d_1(z)}, \quad \hat{\varphi}_\lambda^{-1}(z^{-1}) = \frac{z - 1/\sqrt{k_n}}{d + d_1(z)},
\]
with poles of order one respectively at
\[
\varphi_\lambda^{-1}(z^{-1}) = 0 \implies z_1 = \frac{1}{\sqrt{k_1}}, \ldots, z_{n-1} = \frac{1}{\sqrt{k_{n-1}}}, \quad z_n = \frac{1}{\sqrt{k_1}},
\]
\[
\hat{\varphi}_\lambda^{-1}(z^{-1}) = 0 \implies \hat{z}_1 = \frac{1}{\sqrt{k_n}}, \quad \hat{z}_2 = \frac{1}{\sqrt{k_2}}, \ldots, \hat{z}_n = \frac{1}{\sqrt{k_n}}.
\]
Evaluating the spatial Lax matrix and \(s_{zw}\) at the poles we obtain
\[
L_1(z_1) = -J_1, \ldots, L_1(z_{n-1}) = -J_{n-1}, \quad L_1(z_n) = J_1,
\]
\[
\hat{L}_1(\hat{z}_1) = -J_1, \quad \hat{L}_1(\hat{z}_2) = J_2, \ldots, \hat{L}_1(\hat{z}_n) = J_n,
\]
\[
s_{z_1z_1} = \frac{1}{k_1}, \ldots, s_{z_{n-1}z_{n-1}} = \frac{1}{k_{n-1}}, \quad s_{z_nz_n} = -\frac{1}{k_1},
\]
\[
\hat{s}_{\hat{z}_1\hat{z}_1} = \frac{1}{k_n}, \quad \hat{s}_{\hat{z}_2\hat{z}_2} = -\frac{1}{k_2}, \ldots, \hat{s}_{\hat{z}_n\hat{z}_n} = -\frac{1}{k_n},
\]
\[
s_{z_iz_j} = 0 = \hat{s}_{\hat{z}_i\hat{z}_j}, \quad \text{when } i \neq j.
\]
Using the above in the Maillet bracket (1.3), we readily obtain \(n\) commuting copies of the current algebra (2.18).

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