Algebraic-geometric codes from vector bundles and their decoding

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Abstract—Algebraic-geometric codes can be constructed by evaluating a certain set of functions on a set of distinct rational points of an algebraic curve. The set of functions that are evaluated is the linear space of a given divisor or, equivalently, the set of section of a given line bundle. Using arbitrary rank vector bundles on algebraic curves, we propose a natural generalization of the above construction. Our codes can also be seen as interleaved versions of classical algebraic-geometric codes. We show that the algorithm of Brown, Minder and Shokrollahi can be extended to this new class of codes and it corrects any number of errors up to \(t' - g/2\), where \(t'\) is the designed correction capacity of the code and \(g\) is the curve genus.

I. INTRODUCTION

The construction of error correcting codes using methods from algebraic-geometry was first proposed by Goppa [4], [5] in the early ’80s. He constructed codes by evaluating a certain set of differential forms on a set of distinct rational points of an algebraic curve. From a dual point of view, Goppa’s codes can be constructed by evaluating a certain set of functions on a set of distinct rational points of an algebraic curve [1]. The set of functions that are evaluated is the linear space of a given divisor, whose support is disjoint from the set of evaluation points. These codes generalize the Bose-Chaudhuri-Hocquenghem (BCH), Reed-Solomon (RS) and Goppa codes (latest codes being introduced by Goppa in the early ’70s). Unlike the RS codes, algebraic-geometric (AG) codes are not generally MDS codes, but their Singleton defect is upper bounded by the genus of the curve. However, despite their Singleton defect, AG codes are better than RS codes since they allow the construction of longer codes over the same alphabet. Another advantage of AG codes is that for fixed code parameters, their encoding and decoding algorithms run faster as they can be performed in a smaller field. Since the end of the 80s, intensive research has been done on decoding algorithms and most of the methods used to decode BCH, RS or Goppa codes were extended to the class of AG codes [7].

In order to explain our approach to construct codes from vector bundles on projective curves, let us recall the classical construction of AG codes. Let \(C\) be an absolutely irreducible, smooth, projective curve of genus \(g\), defined over a finite base field \(\mathbb{F}_q\). Let \(\mathcal{P} = \{P_1, P_2, \ldots, P_n\}\) be a set of distinct rational points of \(C\) and \(D \in \text{Div}(\mathcal{C})\) be a divisor whose support is disjoint from the set \(\mathcal{P}\). The linear code \(C(\mathcal{P}, D)\) is defined as the image of the evaluation map:

\[
\begin{align*}
\text{ev} : \mathcal{L}(D) & \rightarrow \mathbb{F}_q^n \\
\quad f & \mapsto (f(P_1), f(P_2), \ldots, f(P_n))
\end{align*}
\]

where \(\mathcal{L}(D)\) is the linear space of \(D\). The parameters of the code, or bound on them, can be determined using well-known statements in algebraic-geometry, notably the Hasse-Weil theorem and the Riemann-Roch theorem, and it can be seen that the Singleton defect of the code is upper bounded by the curve genus \(g\).

Interleaved AG codes were defined in [3] as follows. Suppose that \(Q = q^r\) and identify \(\mathbb{F}_q\) and \(\mathbb{F}_q^r\) as \(\mathbb{F}_{q^r}\)-vector spaces by fixing a basis of \(\mathbb{F}_Q\) over \(\mathbb{F}_q\). For any point \(P \in C\) and any \(f = (f_1, f_2, \ldots, f_r) \in \mathcal{L}(D)^{\oplus r}\), the evaluation vector \(f(P) = (f_1(P), f_2(P), \ldots, f_r(P)) \in \mathbb{F}_q^r\) can be identified with an element of \(\mathbb{F}_Q\). The code \(C(\mathcal{P}, D, r)\) is defined as the image of the evaluation map

\[
\begin{align*}
\text{ev} : \mathcal{L}(D)^{\oplus r} & \rightarrow \mathbb{F}_{q^r}^n \\
\quad f & \mapsto (f(P_1), f(P_2), \ldots, f(P_n))
\end{align*}
\]

This code does not generally be \(\mathbb{F}_Q\) linear, however it is a \(\mathbb{F}_{q^r}\)-vector subspace of \(\mathbb{F}_{q^r}^n\).

To explain our approach, we first remark that there is a one-to-one correspondence between the divisor class group \(\text{Cl}(C)\) (the group of equivalence classes of \(\text{Div}(C)\)) and isomorphism classes of line bundles on \(C\). For any divisor \(D\), we denote by \(\text{O}(D)\) the line bundle associated with \(D\). The linear space \(\mathcal{L}(D)\) is isomorphic to the space of global sections of \(\text{O}(D)\), which is generally denoted by \(H^0(C, \text{O}(D))\).

We construct AG codes by evaluating global sections of arbitrary rank vector bundles \(E\) on the points \(P_1, P_2, \ldots, P_n\). Thus, the classical construction of AG codes (1) corresponds to the case \(E = \text{O}(D)\) and the construction of interleaved AG codes (2) corresponds to \(E = \mathcal{L}(D)^{\oplus r}\). In general, if \(E\) is a rank-\(r\) vector bundle, we obtain a code \(C(\mathcal{P}, E)\) over \(\mathbb{F}_Q\), not necessarily linear, where \(Q = q^r\). Further, we investigate the parameters of these codes, or bound on them, such as the code length, dimension, and minimum distance. While the code dimension can still be lower bounded by the Riemann-Roch theorem, it is much harder to compute its exact value or to give a lower bound on the minimum distance. The main reason is that unlike line vector bundles, arbitrary rank vector bundles of negative degree may have non zero sections. To overcome such situations, we need the vector bundle to satisfy some stability condition. When this condition is satisfied, we can compute the code dimension and we can show that the Singleton defect is upper bounded by the curve genus \(g\).

Now we come to the decoding problem. Bleichenbacher, Kiayias and Yung [2] proposed a new decoding algorithm for interleaved Reed-Solomon codes over the \(Q\)-ary symmetric channel, which was later extended by Brown, Minder and
Shokrollahi to the case of interleaved AG codes [3]. One advantage of using interleaved AG codes is that they allow transmissions at rates closer to the channel capacity. If \( Q = q = 2^{hr} \), the \( Q \)-ary symmetric channel model applies to settings where packets of \( hr \) bits are sent and errors are assumed to be bursty. From the coding theory perspective, errors on bits of the same packet are assumed to be correlated. This usually arrives when packets of bits are sent over different transmission channels, some of which may induce errors. When \( Q \) is too large, efficient decoding of codes designed over \( \mathbb{F}_Q \) is impossible, which explains the advantage of interleaved codes, since their decoding algorithms operate over \( \mathbb{F}_q \). In this paper we show that the decoding algorithm of interleaved AG codes can be extended to the class of codes constructed from vector bundles and it corrects any number of errors up to \( g/2 \) from the designed minimum distance of the code.

In the next section we review some of the mathematical background needed to understand the construction of AG codes from vector bundles on algebraic curves. Our aim is not to do an exhaustive or a self-contained presentation, but rather to guide the reader from elementary to more complicated objects and show that objects as vector bundles or stable vector bundles naturally arise in algebraic geometry. In sections III and IV we present respectively the construction of AG codes from vector bundles and their decoding over the \( \mathbb{F}_q \)-ary symmetric channel model applies to \( \mathbb{F}_q \)-ary symmetric channels, since their decoding algorithms operate over \( \mathbb{F}_q \). It is canonically endowed with a \( k \)-vector space structure and its dimension is denoted by \( h^0(\mathcal{C}, E) \).

A \emph{line bundle} \( L \) over \( \mathcal{C} \) is simply a rank-1 vector bundle. To any meromorphic section \( s : \mathcal{C} \to L \) we associate a divisor \( (s) = Z(s) - P(s) \), where \( Z(s) \) and \( P(s) \) denote respectively the set of zeros and the set of poles, counted with multiplicities. Note that if \( s \in H^0(\mathcal{C}, L) \) is a global section, then \( (s) = Z(s) \) is an effective divisor. The \emph{degree} of \( L \) is by definition the degree of \( (s) \).

\[ \text{deg}(L) := \text{deg}(s) \]

The fact that \( \text{deg}(L) \) is well defined follows from the first assertion of the following proposition. The second assertion highlights the connection between global sections of line bundles and linear spaces.

\[ \text{Proposition 1:} \quad (a) \text{ If } s, s' \text{ are meromorphic sections of a line bundle } L, \text{ then } (s) \text{ and } (s') \text{ are linear equivalent divisors.} \]

(b) For any meromorphic section \( s \), the map

\[ \mathcal{L}(s) \to H^0(\mathcal{C}, L) \]

\[ f \mapsto fs \]

defines an isomorphism of vector spaces.

If \( E \) is a rank-\( r \) vector bundle on \( \mathcal{C} \), its \( r \)-th exterior power \( \text{det}(E) := \wedge^r E \) is a line bundle, called the \emph{determinant bundle} of \( E \). The \emph{degree} of \( E \) is by definition the degree of its determinant bundle:

\[ \text{deg}(E) := \text{deg}(\text{det}(E)) \]

The \emph{slope} of \( E \) is defined by \( \mu(E) = \frac{\text{deg}(E)}{\text{rank}(E)} \).

Examples of vector bundles that naturally arise in algebraic-geometry are the tangent bundle and its dual, called the cotangent bundle, or if the curve \( \mathcal{C} \) is embedded in some projective space, the normal bundle. The cotangent bundle of \( \mathcal{C} \) is also called the \emph{canonical bundle} and is denoted by \( \Omega \). Any divisor associated with the canonical bundle is called \emph{canonical divisor}. We can now state the Riemann-Roch theorem [6].
Theorem 2 (Riemann-Roch): Let $E$ be a vector bundle of rank $r$ and degree $e$ on $C$. Then:

$$h^0(C, E) - h^0(C, \Omega \otimes E^*) = e + r(1 - g)$$

where $E^*$ is the dual vector bundle of $E$.

At this point we have introduced the necessary tools for defining AG codes from vector bundles. In order to be able to investigate the parameters of these codes and their decoding algorithm, we need the vector bundles to satisfy some stability condition. The theory of stable vector bundles goes back to the classification problem of vector bundles in the 60s. However, in this paper we need only a weaker version of the stability, which we call weak stability. Before introducing the stability condition, we have to be more specific about morphisms of vector bundles and vector sub-bundles.

Let $E$ and $F$ be two vector bundles on $C$. A morphism of vector bundles is a regular map $\varphi : E \to F$, such that:

- for any $P \in C$ and any $x \in E_P$, $\varphi(x) \in F_P$,
- for any $P \in C$ the induced map $\varphi_P : E_P \to F_P$ is a morphism of vector spaces.

Any morphism of vector bundles $\varphi : E \to F$ induces in an obvious way a morphism between the corresponding vector spaces of global sections, that is $\varphi : H^0(C, E) \to H^0(C, F)$.

We say that $E$ is a sub-bundle of $F$ if there is a morphism of vector bundles $\varphi : E \to F$, such that for any point $P \in C$ the induced morphism $\varphi_P : E_P \to F_P$ is injective. In this case one can define a quotient vector bundle $F/E$, whose fiber in a point $P \in C$ is defined by $(E/F)_P = E_P/F_P$.

**Definition 3:** A vector bundle $E$ is said to be weakly stable if for any line sub-bundle $L \subset E$ the following inequality holds:

$$\deg(L) \leq \mu(E)$$

**Proposition 4:** (a) Any line bundle is weakly stable.
(b) Let $E = \bigoplus_{i=1}^n E_i$ be a direct sum of vector bundles. Then $E$ is weakly stable if and only if $\mu(E_1) = \cdots = \mu(E_n)$ and all $E_i$, $i = 1, \ldots, n$, are weakly stable.
(c) If $E$ is weakly stable then for any line bundle $L$, the tensor product $E \otimes L$ is also weakly stable.

**Proposition 5:** Assume that $E$ is a weakly stable vector bundle on $C$.

(a) Any global section of $E$ vanishes in at most $|\mu(E)|$ points.
(b) If $\deg(E) < 0$ then $h^0(C, E) = 0$.

**III. AG CODES FROM VECTOR BUNDLES**

In this section we define algebraic-geometric codes from vector bundles. Through the rest of this paper, we denote by $C$ an absolutely irreducible, smooth, projective curve of genus $g$, defined over a finite base field $\mathbb{F}_q$. For any vector bundle $E \to C$, let $\bar{E} \to \bar{C}$ be the vector bundle obtained by extending scalars from $\mathbb{F}_q$ to its algebraic closure $\bar{\mathbb{F}}_q$. We define $\deg(E) = \deg(\bar{E})$ and we say that $E$ is weakly stable iff $\bar{E}$ is weakly stable.

Let $P = \{P_1, \ldots, P_n\}$ be a set of distinct rational points of $C$, $E$ be a rank-$r$ vector bundle on $C$, and set $Q = q^r$. We fix once for all basis of $\mathbb{F}_q$ and $E_{P_i}, i = 1, \ldots, n$, as vector spaces over $\mathbb{F}_q$, which allows us to identify $E_{P_i} \cong \mathbb{F}_q \cong \mathbb{F}_q^r$. Henceforth, these identifications will be used without recalling the subjacent basis. The algebraic-geometric code $C(P, E)$ over $\mathbb{F}_q$ is defined as the image of the evaluation map:

$$ev : H^0(C, E) \longrightarrow \bigoplus_{i=1}^n E_{P_i} \cong \mathbb{F}_q^{\deg(E)}$$

$$f \longmapsto \langle f(P_1), f(P_2), \ldots, f(P_n) \rangle$$

(3)

Note that this is not necessarily a $\mathbb{F}_q$ linear code, but it is a $\mathbb{F}_q$ linear subspace of $\mathbb{F}_q^\deg(E)$. The length of the code is $n$ and for the other parameters, the following notations will be used:

- $K$ is the size of the code.
- $k$ is the dimension of the code; since it is not necessarily a linear code its dimension is defined by:

$$k = \log_q K \in \mathbb{R}$$

- $d$ is the minimal distance of the code. For arbitrary non-linear codes, $d$ corresponds to the minimal distance between any two codewords. However, since $C(P, E)$ is $\mathbb{F}_q$ linear, $d$ is also equal to the minimal weight of a non-zero codeword.

Note that if the evaluation map is injective, then $K = q^{\deg(E)}$ and therefore $k = \frac{\deg(E)}{r}$.

**Theorem 6:** Assume that $E$ is a weakly stable vector bundle of degree $e$ and slope $\mu = e/r < n$. Then:

(a) the evaluation map is injective,
(b) $d \geq n - |\mu|$, 
(c) $k \geq \mu + 1 - g$, 
(d) the Singleton defect of the code is upperbounded by the curve genus $g$.

**Proof.** Since $E$ is weakly stable, any section $f \in H^0(C, E)$ vanishes in at most $|\mu|$ points, which proves (a) and (b). Because the evaluation map is injective (a), we also have $k = \deg(E)/r$. By the Riemann-Roch theorem $h^0(C, E) \geq e + r(1 - g)$, therefore $k \geq \mu + 1 - g$. Finally, (d) follows from (b) and (c).

**Proposition 7:** Let $E$ be vector bundle of degree $e$ and slope $\mu = e/r$. Assume that both $E$ and $E^*$ are weakly stable and $\mu > 2g - 2$. Then $k = \mu + 1 - g$.

**Proof.** The tensor product $\Omega \otimes E^*$ is a weakly stable vector bundle of degree:

$$\deg(\Omega \otimes E^*) = r \deg(\Omega) - \deg(E) = r(2g - 2) - e < 0$$

Therefore $h^0(\Omega \otimes E^*) = 0$ and the assertion follows by the Riemann-Roch theorem.

**IV. DECODING ALGORITHM**

Let $C(P, E)$ be a code over $\mathbb{F}_q$ defined by a rank-$r$ vector bundle $E$ and an evaluation set $P = \{P_1, \ldots, P_n\}$. Assume that the codeword $(f(P_1), f(P_2), \ldots, f(P_n))$, defined by some $f \in H^0(C, E)$, is transmitted over the $Q$-ary symmetric channel and let $(y_1, y_2, \ldots, y_n)$ be the received word. Our goal is to decode the codeword and for this we proceed in a way similar to [3]. Let $t$ be a parameter to be
determined latter and let \( L \) be a line bundle of degree \( l := t+g \).
The decoding works in two steps as follows:

(S1) Find a non-zero element

\[ (v, w) \in H^0(\mathcal{C}, E \otimes L) \times H^0(\mathcal{C}, L) \]
such that

\[ v(P_i) = y_i \otimes w(P_i), \quad \forall i = 1, \ldots, n \]

If \((v, w)\) does not exist, output a decoding error.

(S2) If there exists \( f \in H^0(\mathcal{C}, E) \) such that

\[ v = f \otimes w \]
decode \( f \). Otherwise, output a decoding error.

**Proposition 8:** Let \( \epsilon \) denote the number of errors incurred during transmission. If \( \epsilon \leq t \) then there exists a non-zero element \((v, w)\) satisfying (S1).

**Proof.** Assume that errors occur in points \( P_{i_1}, \ldots, P_{i_s} \) and let \( D_{\text{err}} = P_{i_1} + \cdots + P_{i_s} \). Using the Riemann-Roch theorem it can be proved that \( h^0(\mathcal{C}, L(-D_{\text{err}})) > 0 \). Therefore, we can choose a non-zero \( w \in H^0(\mathcal{C}, L(-D_{\text{err}})) \) and define \( v = f \otimes w \). If \( P_i \) is not an error point, meaning that \( P_i \not\in \{P_{i_1}, \ldots, P_{i_s}\} \), then \( y_i = f(P_i) \) and so \( v(P_i) = y_i \otimes w(P_i) \). Otherwise, the equality \( v(P_i) = y_i \otimes w(P_i) \) still holds, because \( v(P_i) = w(P_i) = 0 \) for any \( P_i \in \{P_{i_1}, \ldots, P_{i_s}\} \).

*Theorem 9:* Let \( \epsilon \) denote the number of errors incurred during transmission. Assume that \( E \) is a weakly stable vector bundle of degree \( e \) and slope \( \mu = e/\epsilon \), such that:

\[ \epsilon \leq t \quad \text{and} \quad \epsilon + t \leq n - \mu - g \]

Then the above decoder outputs the transmitted codeword.

**Proof.** From the above proposition, there exists a non-zero element \((v, w) \in H^0(\mathcal{C}, E \otimes L) \times H^0(\mathcal{C}, L)\) satisfying (S1), that is:

\[ v(P_i) = y_i \otimes w(P_i), \quad \forall i = 1, \ldots, n \]

Assume that errors occur in points \( P = \{P_{i_1}, \ldots, P_{i_s}\} \) and let \( D = \sum_{i \not\in P} P_i \). For any \( P_i \not\in P \) we have \( y_i = f(P_i) \) and therefore:

\[ (v - f \otimes w)(P_i) = y_i \otimes w(P_i) - f(P_i) \otimes w(P_i) = 0 \]

It follows that \( v - f \otimes w \in H^0(\mathcal{C}, E \otimes L(-D)) \). On the other hand, knowing that \( \text{deg}(E) = e \), \( \text{deg}(L) = t + g \) and \( \text{deg}(D) = n - \epsilon \), we get:

\[ \text{deg}(E \otimes L(-D)) = e + r(t + g - (n - \epsilon)) = r(\epsilon + t + n - \mu + g) < 0 \]

Consequently, \( E \otimes L(-D) \) is a weakly stable vector bundle of negative degree, so it has no non-zero global sections. Hence \( v = f \otimes w \) and the decoder outputs \( f \).

Note that the designed correction capacity of the code for the \( Q \)-ary symmetric channel is \( t^* = \left\lfloor \frac{n - \mu}{2} \right\rfloor \). From the above theorem, it follows that the decoding algorithm corrects any pattern of \( \epsilon < t^* - \frac{g}{2} \) errors. We note that the above algorithm can easily be extended to correct both errors and erasures.

Throughout the rest of this section we give a possible realization the decoding algorithm. Our goal is just to prove that the decoding algorithm is constructible and executable in polynomial time. We fix once for all:

- a basis of \( \mathbb{F}_q \) over \( \mathbb{F}_q \),
- a basis of \( E_P \) over \( \mathbb{F}_q \) for each \( P \in P \),
- a basis of \( L_P \) over \( \mathbb{F}_q \) for each \( P \in P \),
- \( f_1, \ldots, f_h \) a basis of \( H^0(\mathcal{C}, E) \) over \( \mathbb{F}_q \),
- \( \varphi_1, \ldots, \varphi_a \) a basis of \( H^0(\mathcal{C}, E \otimes L) \) over \( \mathbb{F}_q \).

The first three basis allow us to identify:

\[ E_P \otimes L_P \cong E_P \cong \mathbb{F}_q \]

Let \((y_1, y_2, \ldots, y_n)\) be the received word. For each \( 1 \leq i \leq n \), we consider that

\[ y_i \in \mathbb{F}_q \cong \left\{ \begin{array}{c} y_{i,1} \\ \vdots \\ y_{i,r} \end{array} \right\} \in \mathbb{F}_q^r \]

and we set

\[ Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in M_{nr,n}(\mathbb{F}_q) \]

where each \( y_i \) is identified with the corresponding column vector. Moreover, we define:

\[ FP_i = (f_1(P_i), f_2(P_i), \ldots, f_h(P_i)) \in M_{1,h}(\mathbb{F}_q) \cong M_{r,h}(\mathbb{F}_q) \]

\[ F = \begin{pmatrix} FP_1 \\ FP_2 \\ \vdots \\ FP_n \end{pmatrix} \in M_{n, nh}(\mathbb{F}_q) \]

\[ V = \begin{pmatrix} \varphi_1(P_1) & \varphi_2(P_1) & \cdots & \varphi_a(P_1) \\ \varphi_1(P_2) & \varphi_2(P_2) & \cdots & \varphi_a(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1(P_n) & \varphi_2(P_n) & \cdots & \varphi_a(P_n) \end{pmatrix} \in M_{n, a}(\mathbb{F}_q) \]

where all \( f_j(P_i) \) and \( \varphi_j(P_i) \) are identified with \( r \times 1 \) column vectors in \( \mathbb{F}_q^r \), and

\[ W = \begin{pmatrix} \psi_1(P_1) & \psi_2(P_1) & \cdots & \psi_h(P_1) \\ \psi_1(P_2) & \psi_2(P_2) & \cdots & \psi_h(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(P_n) & \psi_2(P_n) & \cdots & \psi_h(P_n) \end{pmatrix} \in M_{n, h}(\mathbb{F}_q) \]

The decoding algorithm can now be described as follows:

(S1) Find a non-zero solution

\[ (v_1, \ldots, v_a, w_1, \ldots, w_b) \in \mathbb{F}_q^{a+b} \]
of the system

\[
V \begin{pmatrix} v_1 \\ \vdots \\ v_a \end{pmatrix} = Y W \begin{pmatrix} w_1 \\ \vdots \\ w_b \end{pmatrix}
\]

If the system does not have a non-zero solution, output a decoding error.

(S2) Find a solution \( \lambda = (\lambda_1, \ldots, \lambda_h) \in \mathbb{F}_q^h \) of the system

\[
V \begin{pmatrix} v_1 \\ \vdots \\ v_a \end{pmatrix} = F \Lambda W \begin{pmatrix} w_1 \\ \vdots \\ w_b \end{pmatrix}
\]

where

\[
\Lambda = \begin{pmatrix} t_{\lambda} \\ \vdots \\ t_{\lambda} \end{pmatrix} \in M_{nh,n}(\mathbb{F}_q)
\]

and output \( \bar{f} = \lambda_1 f_1 + \cdots + \lambda_h f_h \). If the system does not have any solution, output a decoding error.

Note that the second step of the above realization compute a section \( \bar{f} = \lambda_1 f_1 + \cdots + \lambda_h f_h \) verifying

\[
v(P_i) = \bar{f}(P_i) \otimes w(P_i), \quad \forall i = 1, \ldots, n
\]

where \( v = v_1 \varphi_1 + \cdots + v_a \varphi_a \) and \( w = w_1 \psi_1 + \cdots + w_b \psi_b \). This is a little bit different from the second step of the decoding algorithm, which requires the above equality to hold for any point \( P \) (i.e. \( v = f \otimes w \)). Assuming that the vector bundle \( E \) is weakly stable, any non-zero section of \( E \otimes L \) vanishes in at most \( \mu(E \otimes L) = \mu + t + g \) points. Furthermore, assuming that \( n > \mu + t + g \) and \( v(P_i) = f(P_i) \otimes w(P_i), \quad \forall i = 1, \ldots, n \), it follows that the section \( v - \bar{f} \otimes w \) vanishes in more than \( \mu + t + g \) points, and so \( v = \bar{f} \otimes w \).

V. CONSTRUCTION OF WEAKLY STABLE VECTOR BUNDLES

Most of statements concerning the parameters and the decoding of algebraic codes constructed from vector bundles require weakly stable vector bundles. A trivial example of a weakly stable vector bundle of rank \( r \) and degree \( e \) is given by the direct sum of \( r \) line bundles of degree \( e \). Such a vector bundle is called completely undecomposable. In this section we show that for any curve of genus \( g \geq 2 \) and any integers \( r > 0 \) and \( e \) there exist non-trivial examples of weakly stable vector bundles of rank \( r \) and degree \( e \).

Let \( e = \alpha r + \beta \), with \( \alpha, \beta \in \mathbb{Z} \) such that \( 0 \leq \beta < r \). Let \( F_1, F_2 \) and \( F \) be line bundles on \( C \), such that:

\[
\deg(F_1) = \deg(F_2) = \alpha, \quad \deg(F) = \alpha + 1
\]

Consider a sequence of vector bundles \( E_i \) defined by the following non-trivial extensions:

\[
E_1 = F_1 \\
E_2 = E_1 \oplus F_2 \\
E_3 = E_2 \oplus F_3 \\
\vdots \\
E_i = E_{i-1} \oplus F_i, \quad 0 \leq i \leq r - \beta - 1
\]

\[
E_{r - \beta - 1} = E_{r - \beta} \oplus F_{r - \beta} \\
E_{r - \beta} = E_{r - \beta + 1} \oplus F_{r - \beta + 1} \\
E_r = E_{r - 1} \oplus F_r
\]

The extensions of \( F_2 \) by \( E_i, \quad i \leq r - \beta - 1 \), are classified by \( H^1(C, F_2 \otimes E_i) \approx H^0(C, \Omega \otimes F_2 \otimes E_i^*) \), by Poincaré duality.

Since \( F_2 \otimes E_i \) is a vector bundle of degree \( 0 \) and rank \( i \), by the Riemann-Roch theorem we obtain \( h^0(C, \Omega \otimes F_2 \otimes E_i^*) > 0 \). From this it follows that there exist non-trivial examples of weakly stable vector bundles on algebraic curves. A trivial example of a decoding of algebraic codes constructed from vector bundles is the curve genus.

VI. CONCLUSIONS

A new construction of AG codes from vector bundles on algebraic curves was proposed in this paper, which allows a unified treatment of classical AG codes and more recently interleaved AG codes. In the same time, this construction extends the above class of AG codes to a much larger class of codes. These new codes have very good properties and they can be designed over very large Galois fields with reasonable decoding complexity, since decoding can be performed in a smaller field. We have also provided a decoding algorithm for these codes that corrects any number of errors up to \( t^* - g/2 \), where \( t^* \) is the designed correction capacity of the code and \( g \) is the curve genus.

The aim of this paper is also to relate the construction of AG codes to more sophisticated and powerful concepts in algebraic-geometry. However, this is only a first step and more work has to be done in this area. It is very likely that for suitable choices of vector bundles \( E \) and \( L \), the decoding algorithm will correct errors up to the designed correction capacity of the code. We think that future work could bring out many useful interactions between algebraic geometric codes and vector bundles on algebraic curves.

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