Reference tracking stochastic model predictive control over unreliable channels and bounded control actions

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Abstract

A stochastic model predictive control framework over unreliable Bernoulli communication channels, in the presence of unbounded process noise and under bounded control inputs, is presented for tracking a reference signal. The data losses in the control channel are compensated by a carefully designed transmission protocol, and that of the sensor channel by a dropout compensator. Our proposed class of feedback policies is parametrized in terms of the disturbances appearing in the dynamics of the dropout compensator. A reference governor is employed to generate a trackable reference trajectory and mean-square boundedness of the error between the states of the system and that of the reference governor is achieved with help of stability constraints. The overall approach yields a computationally tractable quadratic program, which is solved iteratively online.

Key words: tracking, stochastic MPC, bounded controls, packet dropouts, networked system.

1 Introduction

Regulation and tracking are two basic problems in control theory and have been studied rigorously in the framework of model predictive control (MPC) [37]. A tracking problem can be considered as the generalization of a regulation problem with a varying set point. However, when the set point varies, the stabilizing and recursively feasible design of a regulating controller may not remain valid [21, Page 5], and consequently a tracking problem generally requires a separate treatment.

The recursive feasibility issue for tracking MPC in linear systems has been tackled in [9] by the construction of a robust control invariant set around a pseudo trajectory. On the other hand [24] considers an artificial target steady state and input with control invariant sets to ensure recursive feasibility for piecewise constant reference signals and [22] extends this further to periodic reference signals. Since networked control systems are gaining prominence due to their flexibility, the reference tracking problem for networked systems without additive process noise is considered in [33]. The above references [9, 22, 24, 33] give a good understanding of the underlying challenges and techniques to solve tracking problems but do not consider process noise in the system dynamics.

Tracking MPC under bounded disturbances is considered in [2, 23, 34]. In many applications the bound on the additive process noise is not known a priori, but the distribution is known. Therefore, a stochastic formulation of MPC is proposed for tracking problems in [10, 38] with probabilistic constraints on the control commands. A stochastic MPC formulation for tracking with hard constraints on control actions is missing in the literature.

It is well known that a class of feedback policies is essential for stochastic MPC in contrast to their deterministic counterpart [20, 26]. Disturbance feedback policies are typically employed for tractability of the underlying optimization program [12]. In order to satisfy hard constraints on control actions [6] employed saturated disturbance feedback policies and ensured stability by using drift conditions for sufficiently large control authority. An interesting class of policies with evolving saturated disturbances is employed in [32] for covariance steering. The approach [6] was extended for
any positive control authority in [29] and the controller was implemented on networks with the help of novel transmission protocols. It is also argued that a stabilizing stochastic MPC formulation over networks has three ingredients – a class of feedback policies, appropriate transmission protocols and stability constraints. In order to take the effect of past dropouts of the control channel in the feedback policy an affine dropout feedback policy was proposed in [31] for a system without process noise, which was extended in [28] to accommodate the presence of process noise. However, [28, 29] assumed that the downlink (a channel from sensors of the plant to the controller) is perfect, although when the downlink is unreliable or perfect measurements are not available, the computation of past disturbances is not possible. Therefore, affine saturated disturbance feedback policies are not applicable in such situations. We fill this particular lacuna in this article.

Moreover, generally, in the case of incomplete and corrupt measurements a Kalman filter is employed. Therefore, the saturated innovation term of the Kalman filter can be used in the feedback policy as in [17, 27]. This approach was further extended under the settings of unreliable channels in [30] by taking feedback of the received innovations. In the aforementioned works [6, 17, 27, 29] the case of unreliable downlink under perfect measurements was omitted. This case cannot be readily treated along the lines of [30] because the choice of a feedback policy class is not straightforward. Moreover, all these approaches above [6, 17, 27, 29, 30] present a constrained stochastic MPC approach for a fixed set-point at the origin and their adaptation under the setting of a time varying reference signal is also missing in the literature. In the current article, we address this gap.

In this article, we adapt and extend the results of [29] by considering an unreliable downlink and time-varying set point. We consider a class of feedback policies in which the feedback term takes the additive uncertainties appearing in the dynamics of the dropout compensator into account. We prove that the proposed class of feedback policies leads to tractability of the underlying constrained optimal control problem. We present minor modifications to the transmission protocols and stability constraints presented in [29] under the settings of the current article in order to get a provably stabilizing and tractable framework. Moreover, the present approach of tracking can also be employed along with the results of [30] with minor modifications for systems with incomplete and corrupt measurements.

The contributions of this article are (a) stochastic MPC formulation of tracking with hard constraints on control actions and (b) a new class of feedback policies in the context of an unreliable downlink that ensures tractability of the underlying constrained stochastic optimal control problem.

This article proceeds as follows: We present the problem formulation and system setup in Section 2. The main ingredients of the tracking problem under unreliable channels are explained in Section 3, Section 4 and Section 5, respectively. We present our main result on tractability and stability in Section 6. We illustrate our results in Section 7 by numerical experiments and conclude in Section 8.

We use the symbol \(0\) to denote a matrix of appropriate dimensions with all elements 0. For any vector sequence \((v_n)_{n \in \mathbb{N}_0}\), let \(v_{nk}\) denote the vector \(\begin{bmatrix} v_n \, v_n^T \, \ldots \, v_{n+k-1}^T \end{bmatrix}^T\), \(k \in \mathbb{Z}_+\). The notations \(E[f]\) and \(E[f|z]\) are interchangeably used for the conditional expectation with given \(z\). Let \(\sigma_1(M)\) denote the largest singular value of \(M\), and \(M^T\) its Moore-Penrose pseudo inverse. A block diagonal matrix \(M\) with diagonal entries \(M_1, \ldots, M_n\) is represented as \(M = \mbox{bdia}(M_1, \ldots, M_n)\). \(I_d\) is the \(d \times d\) identity matrix. We simply use \(I\) for the identity matrix when its dimensions are clear from the context. For a vector \(V \in \mathbb{R}^d\), \(V^{(i)}\) denotes its \(i\)th entry for \(i = 1, \ldots, d\).

2 Problem setup and solution architecture

2.1 System dynamics

We consider a discrete-time dynamical system

\[
x_{t+1} = Ax_t + Bu_t^d + w_t,
\]

where \(x_t \in \mathbb{R}^d\), \(u_t^d \in \mathbb{R}^m\), \(w_t \in \mathbb{R}^d\), are the system state, input and disturbance respectively. The matrix pair \((A, B)\) is assumed to be controllable. The additive disturbance \((w_t)_{t \in \mathbb{N}_0}\) is assumed to be a sequence of i.i.d. zero mean random vectors taking values in \(\mathbb{R}^d\). Each component of \(w_t\) is symmetrically distributed about the origin and \(w_t\) has bounded fourth moments, i.e., \(E[[|w_t|^4]] < \infty\). The bounded fourth moment assumption being much weaker than a Gaussian distribution assumption, allows for more general noise distribution characteristics in this work.

The controller communicates with the dynamical system (plant) through unreliable Bernoulli channels. The parameters of the control policy are transmitted through a control channel (uplink) from the controller to the plant. State information \(x_t\) is transmitted through a sensor channel (downlink) from the plant to the controller. Successful transmissions from an unreliable channel follow a Bernoulli distribution. To model this distribution, we utilize two i.i.d. Bernoulli random variable sequences \((v_t)_{t \in \mathbb{N}_0}\) and \((s_t)_{t \in \mathbb{N}_0}\) for the control channel and the sensor channel, respectively.

At each time instant \(t\), \(v_t = 1\) with probability \(p_v\) and \(v_t = 0\) with probability \(1 - p_v\). Similarly, \(s_t = 1\) with probability \(p_s\) and \(s_t = 0\) with probability \(1 - p_s\). A transmission across the control channel and sensor channel is considered successful when \(v_t = 1\) and \(s_t = 1\), respectively. Each successful transmission from controller to the actuator is accompanied by an acknowledgment of success to the sender as done in TCP like protocols.
The control input $u^o_t$ is required to be bounded and without loss of generality is assumed to be uniformly bounded as
$$\|u^o_t\|_{\infty} \leq u_{\text{max}} \text{ for all } t \in \mathbb{N}_0.$$  \hfill (2)

We use the superscript notation $u^o_t$ in the system dynamics to denote the actual applied control action to the system and highlight the presence of unreliable uplink; See Fig. 1.

2.2 Solution architecture

The aim of a tracking controller in the presence of perfect channels is to design a control sequence $(u^o_t)_{t \in \mathbb{N}_0}$ such that the state process $(x^o_t)_{t \in \mathbb{N}_0}$ converges to the given reference sequence $(r_t)_{t \in \mathbb{N}_0}$. Since $(r_t)_{t \in \mathbb{N}_0}$ may not be trackable due to the dynamical constraints, generally it is approximated by a trackable signal $(x^f_t)_{t \in \mathbb{N}_0}$ typically obtained by a reference governor. Section 3 details the design of the reference governor.

Even in the presence of perfect channels the tracking problem is not trivial when the process noise has unbounded support and control inputs are uniformly bounded. Since the traditional form of asymptotic stability cannot be achieved in the presence of additive stochastic uncertainties, we address the notion of mean-square boundedness. We recall the following definition:

**Definition 1** \[7, \text{III.A}]\ An $\mathbb{R}^d$-valued random process $(x_t)_{t \in \mathbb{N}_0}$ is said to be mean square bounded with respect to the available information $\mathcal{X}_0$ at $t = 0$ at the controller if there exists $\gamma < \infty$ such that
$$\sup_{t \in \mathbb{N}_0} \mathbb{E}_{\mathcal{X}_0}[\|x_t\|^2] \leq \gamma.$$

The above notion of mean-square boundedness implies that the probability of the state-norm being equal to $k$, decays faster than $k^{-2}$ when $k$ grows large, uniformly over time \[7, \text{III.A}]. Moreover, this notion also implies that there does not exist any divergent trajectory with probability one \[19, \text{Lemma 4.1}]. Therefore, for the purpose of tracking in a stochastic environment, we focus on designing a controller such that the underlying error between the state of the system and the reference signal remains mean-square bounded.

Given a reference signal $(r_t)_{t \in \mathbb{N}_0}$, the reference governor in Section 3, generates sequences $(x^f_t)_{t \in \mathbb{N}_0}$ and $(u^f_t)_{t \in \mathbb{N}_0}$ such that the deviation of $x^f_t$ from $r_t$, denoted as the **governor error** $e^G_t = x^f_t - r_t$ is bounded; and the reference control input $u^f_t$ satisfies the bound $\|u^f_t\|_{\infty} \leq \delta u_{\text{max}}$ for some $\delta \in (0, 1)$; see Section 3 for details.

The reference governor signals $x^f_t, u^f_t$ are utilized by the SMPC controller to generate an open-loop control sequence $\eta_t$ and disturbance feedback gain matrix $\Theta_t$; such that the closed-loop reference tracking error, denoted $e_t = x_t - x^f_t$ is guaranteed to be mean square bounded. However, due to the unreliable sensor channel, the state vector $x_t$ is not always available for feedback computation in the controller. Instead, a state estimator is used as a dropout compensator and a state estimate $\tilde{x}_t$ is made available to the controller. The tracking error $e_t$, can thus be split into two errors, (i) the estimator error $e^D_t = x_t - \tilde{x}_t$ and (ii) the controller error $e^C_t = \tilde{x}_t - x^f_t$.

The dropout compensator is designed in Section 4 such that the estimation error $e^D_t = x_t - \tilde{x}_t$ remains mean-square bounded. Further, the SMPC controller is designed in Section 5 such that the controller error $e^C_t = \tilde{x}_t - x^f_t$ remains mean-square bounded. Thus, the **overall error** between the state of the system $x_t$ and the reference signal $r_t$ written as,
$$e^O_t = e_t + e^D_t + e^C_t + e^G_t$$ \hfill (3)

remains mean square bounded, due to each component error being mean-square bounded (Theorem 8). We conclude by showing that the overall approach is computationally tractable and satisfies hard constraints on control actions in Section 6. The error signals are summarized in Table 1.

| Error          | Expression | Governing Equations |
|----------------|------------|---------------------|
| Tracking error | $e_t = x_t - x^f_t$ | (10) |
| Governor error | $e^G_t = x^f_t - r_t$ | (4) and (5) |
| Estimator error| $e^D_t = x_t - \tilde{x}_t$ | (9) |
| Controller error| $e^C_t = \tilde{x}_t - x^f_t$ | (20) |
| Overall error  | $e^O_t = x_t - r_t$ | (3) |

The following sections proceed with further details of the control system architecture components, viz., the reference governor, dropout compensator and SMPC controller.

3 Reference governor

For a given constrained linear system only a class of signals may be trackable \[11, 22\]. This class of trackable signals is generally obtained by a reference governor \[4, 39\]. Although the computation of steady states and terminal sets is not possible in the presence of unbounded disturbances but the construction of a trackable pseudo reference trajectory is feasible with the help of a reference governor. Therefore, in the present article we employ a reference governor to generate a trackable reference trajectory and design a regulating controller that regulates the tracking error. In subsequent analysis, we focus on a class of trackable signals as defined below:

**Definition 2** For a given control authority $u_{\text{max}}$, weight $0 < \delta < 1$ and bound $\gamma^G \geq 0$, a signal $(r_t)_{t \in \mathbb{N}_0}$ is called **trackable**...
if there exist sequences \((x_t^r)_{t \in \mathbb{N}_0}, (u_t^r)_{t \in \mathbb{N}_0}\) such that

\[
x_{t+1}^r = Ax_t^r + Bu_t^r,
\]

\[
\sup_{t \in \mathbb{N}_0} \left\{ \left\| x_t^r - r_t \right\| \leq \gamma^G, \right. \\
\left. \left\| u_t^r \right\|_{\infty} \leq \delta u_{\max} \text{ for each } t \right. \}
\]

The weight \(\delta\) is used to distribute the control authority between the governor input \(u^r_t\) and SMPC controller. The difference between the trackable trajectory \(x_t^r\) and reference signal \(r_t\) is denoted as the governor error

\[
e_t^G = x_t^r - r_t.
\]

The sequences \((x_t^r)_{t \in \mathbb{N}_0}, (u_t^r)_{t \in \mathbb{N}_0}\) can be obtained by solving an optimal control problem of the form

\[
(x_t^r)_{t \in T}, (u_t^r)_{t \in T} = \arg\min_{(x_t^r, u_t^r)} \sum_{t=0}^{T} \left\| x_t - r_t \right\|^2 + \left\| u_t \right\|^2
\]

subject to \(x_{t+1} = Ax_t + Bu_t\)

\[
\left\| u_t \right\|_{\infty} \leq \delta u_{\max}; \quad \forall t \leq T
\]

\[
x_0 = 0.
\]

Given \(r_t\) and \(\delta, \gamma^G\), the problem (6) can then be solved offline for a finite horizon problem or online using a moving horizon, for infinite horizon reference signals as done in [1]. We refer the reader to [1, 3, 18] for other advances in reference governor designs.

4 Dropout compensator

At each time step, the state information \(x_t\) is transmitted from the plant to the controller over an unreliable communication channel as described in Section 2. A Bernoulli random variable \(s_t\) determines the successful transmission of data across the channel. For the event, \(s_t = 1\) (with probability \(p_s\)), \(x_t\) is successfully transmitted and for \(s_t = 0\) (with probability \(1 - p_s\)), the transmission fails.

When a successful transmission is received, \(s_t = 1\), the state estimate for the controller is set to the received value, i.e., \(\hat{x}_t = x_t\). For failed transmission, \(s_t = 0\), the previous state estimate is propagated forward using the system dynamics

\[
\hat{x}_t = A\hat{x}_{t-1} + Bu_{t-1}^D
\]

Fig. 1. A dropout compensator feeds the estimated states to the controller and the controller generates the admissible control sequence by taking reference governor into account. The computed control parameters are transmitted through the control channel by a suitably chosen transmission protocol and the successfully received parameters are stored in a buffer.

initialized with \(\hat{x}_0 = 0\) and \(u_0^D = 0\). For \(t \geq 0\), \(u_t^D\) is given by the disturbance feedback policy (16) discussed in Section 5. The combined state estimate update can then be written as,

\[
\hat{x}_t = s_t x_t + (1 - s_t)(A\hat{x}_{t-1} + Bu_{t-1}^D).
\]

Dropout compensators of the form (8) are widely used in the literature [13] and a justification for (8) can be found in [30, Lemma 3]. Using (1) and (8), the estimator error dynamics can then be written as,

\[
e_t^D := x_t - \hat{x}_t = (1 - s_t)(Ae_{t-1} + u_{t-1}).
\]

5 Controller

Let \(N, N_r \in \mathbb{Z}_+\) be two positive integer constants such that \(N_r \leq N\). The stochastic MPC controller solves a \(N\)-step stochastic optimal control problem after every \(N_r\) steps. Recalling that the controller objective is to minimize the reference tracking error,

\[
e_t := x_t - x_t^r,
\]

the objective function for the SMPC controller is set to

\[
V_t := E_{\tilde{x}_t} \left[ \sum_{i=0}^{N-1} e_{t+i}^T Qe_{t+i} + (u_{t+i}^e)^T Ru_{t+i} + e_{t+N}^T Q_j e_{t+N} \right].
\]
$R$ is taken to be positive definite. The controller transmits the parameters of the policy (defined in Section 5.1) over the control channel with the help of a transmission protocol (defined in Section 5.2) such that a previously transmitted parameters can be used in the case of transmission failures. The backbone of SMPC is to iteratively solve the following constrained stochastic optimal control problem (CSOCP) in the presence of stability constraints (defined in Section 5.3) over the class of policies:

$$\begin{align*}
\text{minimize} & \quad V_t \\
\text{subject to} & \quad \text{dynamics (1)} \\
& \quad \text{control constraint (2)} \\
& \quad \text{stability constraints (22).}
\end{align*}$$

The above CSOCP (12) is re-solved after every $N_r$ time steps with a new initial state estimate and the updated decision variables are transmitted through the control channel to allow the computation of $u_t^o$ at the plant input by using the transmission protocol. We discuss these mechanisms in the following subsections.

### 5.1 Class of feedback policies

The presence of buffer and unreliable control channel is generally ignored at the time of controller design. In this article, we follow the approach of [29] to incorporate both effects at the synthesis stage. For this purpose, we have different notations and equations for the control policy and the applied control to the system, respectively $u_t$ and $u_t^a$. Please notice that the dynamics of the dropout compensator (8) can also be written as

$$\dot{x}_{t+1} = A\tilde{x}_t + Bu_{t+1} + \tilde{w}_t,$$  
(13)

where $\tilde{w}_t := s_{t+1} (Ae^{D} + u_t)$. The class of policies is parametrized in terms of the past disturbances as follows:

$$u_{t+i} = u^r_{t+i} + \eta_{t+i} + \sum_{j=0}^{i} \theta_{t+j} \psi(\tilde{w}_{t+j-1})$$

(14)

where $u^r_{t+i}$ is the reference control input; $(\eta_{t+i})_{i=0,...,N-1}$ and $(\theta_{t+j})_{j<i=0,...,N-1}$ are the nominal input and feedback gain respectively, to be computed by solving CSOCP (12). The estimator disturbance $\tilde{w}_{t+j-1}$ is computed from (13) as

$$\tilde{w}_{t+j-1} = \tilde{x}_{t+j} - (A\tilde{x}_{t+j-1} + Bu^a_{t+j-1}).$$

(15)

The computation is initialized with $\tilde{x}_{-1} = 0$, $u^a_{t+1} = 0$ (as per Section 4). The saturation function $\psi: \mathbb{R} \to \mathbb{R}$ is an anti-symmetric function; such that, $\psi(0) = 0$, $\psi(-x) = -\psi(x)$ and $\sup_{x \in \mathbb{R}} \psi(x) \leq \psi_{\max}$. For vector inputs, like $\tilde{w}_{t+j-1}$, $\psi$ is applied element-wise. The class of policies (14) can be written in compact form as follows:

$$u_{t:N} = u^r_{t:N} + \eta_t + \Theta_t \psi(\tilde{w}_{t-1:N}),$$

(16)

where $\eta_t := \eta_{t:N}$, $\eta_t \in \mathbb{R}^{mN}$ and $\Theta_t$ is a lower block triangular matrix

$$
\Theta_t = \begin{bmatrix}
\theta_{0,t} & 0 & \cdots & 0 & 0 \\
\theta_{1,t} & \theta_{1,t+1} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\theta_{N_{t-1},t} & \theta_{N_{t-1},t+1} & \cdots & \theta_{N_{t-1},t+N_{t-2}} & \theta_{N_{t-1},t+N_{t-1}} \\
\end{bmatrix},
$$

(17)

with each $\theta_{k,\ell} \in \mathbb{R}^{m\times d}$ and $\|\psi(\tilde{w}_{t:N-1})\|_{\infty} \leq \psi_{\max}$.

### 5.2 Transmission protocol

Recall that $u^r_{t:N}$ is obtained from the reference governor, and that $(\eta_t, \Theta_t)$ is the solution of the CSOCP (12) under the class of policies (16), which is solved at time $t$. Therefore, $u^r_{t:N}$, $\eta_t$ and $\Theta_t$ are available at time $t$, which can be used to construct the control policy $u_{t:N}$ with the help of the causally available information of the compensator disturbance. The causal availability of the compensator disturbance motivates the selection of the transmission rule of the policy parameters in a certain way described below. Several transmission protocols designed to mitigate the effects of packet dropouts in the control channel are discussed in [29]. We consider one of them here, which is formally defined below. Our approach remains valid for the other protocols in [29] as well provided minor and straightforward adjustments are made. The idea of transmission protocols is inspired by so-called packetized predictive control techniques [36], where a buffer stores a finite sequence of future control actions. We initialize a buffer to hold $N_r$ number of control inputs and present the transmission protocol below:

(TP1) At each time $t \geq 0$, do:

(a) If $t = kN_r$, for some $k \in \mathbb{N}_0$,

(i) Compute $u^r_{t:N}$, $\eta_t$ and $\Theta_t$ using the optimization based controller from Section 5.

(ii) Compute $u_{t:N}$ using (16).

(iii) Set $\ell = 0$.

(iv) Transmit $\begin{bmatrix} u^r_t \\ (u^r_{t+1:N_r-1} + \eta_t + \eta_{t+1:N_r-1})^T \end{bmatrix}$ to the buffer.

(b) else,

(i) Update $\ell = \ell + 1$.

(ii) If buffer is empty, transmit $\begin{bmatrix} u^r_{t+\ell} \\ (u^r_{t+\ell+1:N_r-\ell-1} + \eta_t + \eta_{t+\ell+1:N_r-\ell-1})^T \end{bmatrix}$.

Otherwise, transmit $u^r_{t+\ell}$.

As mentioned in (TP1), $N_r - \ell$ blocks of control are transmitted at each time $t + \ell$, when buffer is empty. Otherwise only 1 block of the control is transmitted. If the transmitted block is received at the buffer, then it is stored there starting from the beginning. At each time instant the first block of the buffer is applied to the plant and then the buffer is
rotated by using a left shift register. This operation of shift register makes the buffer automatically empty before each optimization instant.

Let $g_t = v_t, g_{t+ℓ} = g_{t+ℓ−1} + (1 − g_{t+ℓ−1})v_{t+ℓ}$. The term $g_{t+ℓ}$ captures the effect of (TP1). The transmission protocol (TP1), thus, effectively applies an input to the actuator given by

$$u_t^{ℓ+1} = g_t(ℓ+1)(u_t^{ℓ+1} + η_t^{ℓ+1}) + v_t^{ℓ+1} \sum_{i=0}^{ℓ} \theta_{ℓ+1,i}ψ_i(\bar{u}_{t+i−1})$$

at time $t + ℓ$. Let us define a block diagonal matrix $S := \text{bdig}(I_m, I_m, \cdots, I_m, I_m)$ and the matrix $G$, which has $N \times N$ blocks in total, each dimension $m \times m$ and for $i, j = 1, \cdots, N$, the matrix $G$ is given in terms of the blocks $G_b^{i,j}$ each of dimension $m \times m$ as

$$G_b^{i,j} = \begin{cases} g_{t+i−1}I_m & \text{if } i = j ≤ N_t, \\ I_m & \text{if } i = j > N_t, \\ 0_m & \text{otherwise.} \end{cases}$$

The stacked control vector in one optimization horizon is written as follows:

$$u_t^{ℓ,N} := ℋu_t^{ℓ,N} + Gη_t + SΘj(\bar{u}_{t−1,N}).$$

where $Θj$ and $ψ(\bar{u}_{t,N+1})$ are as defined in (16) and $ℋ = G$.

Remark 3 In any application where the future reference control signals $u_t^γ$, which are generated by the reference governor, are known a priori, we can transmit the future values of $u_t^γ$ several times until the first successful reception. In such situations we can assume that $u_t^γ$ is available at the actuator at time $t$. Therefore, we can set $ℋ = I$. In the present article, we consider $ℋ = G$.

5.3 Stability constraints

Similar to [16], stability of the proposed approach is independent of the cost function. We employ stability constraints, which are derived from [8] under the following assumptions:

(A1) The zero mean noise sequence $(w_t)_{t \in \mathbb{N}_0}$ is fourth moment bounded, i.e., $E[||w_t||^4] \leq C_4$, for some $C_4 < \infty$.

(A2) The system matrix $A$ has all eigenvalues in the unit disk and those on unit circle are semi-simple\(^2\).

(A3) System matrix pair $(A, B)$ is controllable.

Remark 4 (A2) refers to the largest class of linear time invariant systems known to be stabilizable by bounded control actions under perfect communication channels.

\(^2\) The algebraic and geometric multiplicities are same for the eigenvalues that are on the unit circle.

The above assumptions (A1) - (A3) are also used in [16] for the case of perfect channels. We quickly recall some steps so that the stability constraints along the lines of [30] can be employed under the settings of the present article. Let us first define the controller error $e_t^C := \bar{x}_t − x_t^γ$, then from (13) and (4) we get

$$(e_t^C)^+ := \bar{x}_t − x_t^γ = Aε_t^{C+1} + Bu_{t}^{C+1} + \tilde{w}_t^{C+1},$$

where $u_t^C = u_t^γ − u_t^γ$. Without loss of generality due to the Assumption (A2), we can assume that the error dynamics (20) is of the form

$$\begin{bmatrix} (e_t^{C+1})^o \\ (e_t^{C+1})^r \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ 0 & A_s \end{bmatrix} \begin{bmatrix} (e_t^{C})^o \\ (e_t^{C})^r \end{bmatrix} + \begin{bmatrix} B_o \\ B_s \end{bmatrix} u_t^C + \tilde{w}_t^C,$$

where $A_o ∈ \mathbb{R}^{d_o × d_o}$ is orthogonal and $A_s ∈ \mathbb{R}^{d_s × d_s}$ is Schur stable, with $d = d_o + d_s$. By the controllability assumption (A3), there exists a positive integer $κ$ such that the reachability matrix

$$R_κ(A_o, B_o) := \begin{bmatrix} A_o^{−1}B_o & \cdots & A_o^κB_o & B_o \end{bmatrix}$$

has full row rank; i.e., rank($R_κ(A_o, B_o)$) = $d_o$. We presented our tracking problem in such a way [30, Lemma 7] is applicable in the context of this article and provides the following result:

Lemma 5 Let us consider (21) and let assumptions (A1) - (A2) hold. Suppose for $t = 0, κ, 2κ, \ldots$, the following conditions hold:

$$(A_o^{τ+κ})^T R_κ(A_o, B_o) E_{X_0} [u_t^γ] ≤ −ζ$$

whenever $((A_o^{τ})^T (e_t^C)^o) ≥ c, \quad (22a)$

$$(A_o^{τ+κ})^T R_κ(A_o, B_o) E_{X_0} [u_t^γ] ≥ ζ$$

whenever $((A_o^{τ})^T (e_t^C)^o) < −c, \quad (22b)$

for each $j = 1, \cdots, d_o$, where $ζ, c > 0$. Then under the transmission protocol (TP1) there exists $γ_C > 0$ such that

$$E_{X_0} \left[ ||e_t^C||^2 \right] ≤ γ_C \quad \text{for all } t. \quad (23)$$

Moreover, when $ζ ∈ \left[ 0, \frac{(1−δ)U_{max}}{\sqrt{d_o}σ^2_1(R_κ(A_o, B_o)^r)} \right], \quad \text{there exists a } κ\text{-history dependent class of policies } u_t^γ_{κ,t} \text{ such that } (22) \text{ and } (2) \text{ are satisfied.}$

PROOF. We apply [30, Lemma 7] on (21) to get (22) and the bound (23) when $||u_t^γ|| ≤ (1−δ)U_{max}$ by the choice of $ζ$. The reference governor is designed in (4) such that $||u_t^γ||_{∞} ≤$
\(\delta u_{\text{max}}. \) Therefore, noting \(\|u_t^r\|_\infty \leq \|u_t\|_\infty \leq \|u_t^f\|_\infty \leq u_{\text{max}}, u_t^f \) satisfies (2).

6 Tractability and Stability

In this section we present how the CSOCP (12) can be written as a computationally tractable quadratic program. Recalling \(u_t^f = u_t^a - u_t^f, \) we get

\[
u_{t:N}^f := (H-I)u_{t:N}^f + \mathcal{G}\eta_t + S\Theta_t\psi(\hat{w}_{t-1:N}). \tag{24}\]

The compact form representation of error in (11) over one optimization horizon is as follows:

\[
e_{t:N+1} = \mathcal{A}e_t + \mathcal{B}u_{t:N}^f + \mathcal{D}w_{t:N} \tag{25}\]

where \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \) and \(\mathcal{D} \) are standard matrices of appropriate dimensions. The cost function (11) can also be written in a compact form as follows:

\[
V_t := \mathbb{E}_{\mathcal{X}_{t+1}}\left[\|e_{t:N+1}\|_Q^2 + \|u_{t:N}^f\|_R^2\right], \tag{26}\]

where \(Q \) and \(R \) are standard block diagonal matrices of appropriate dimensions. Let \(\Theta_t(\cdot) := \left[\theta_{t,0}^T, \theta_{t,1}^T, \ldots, \theta_{N-1,t}^T\right]^T\) be the first \(d \) columns of \(\Theta_t\) and \(\theta_t \) be such that \(\Theta_t = \left[\Theta_t(\cdot) \theta_t\right]^T. \) Let \(\Pi_w = \psi(\hat{w}_{t-1})\psi(\hat{w}_{t-1})^T, \) \(\Sigma_w = \mathbb{E}_{\mathcal{X}_t}\left[\psi(\hat{w}_{t-1})(e_t^D)^T\right], \) \(\mu_G = \mathbb{E}[G], \mu_S = \mathbb{E}[S], \Sigma_G = \mathbb{E}[G^T\alpha G], \Sigma_S = \mathbb{E}[S^T\alpha S], \Sigma_{HG} = \mathbb{E}\left[(H-I)^T\alpha G\right] \) and \(\Sigma_{HS} = \mathbb{E}\left[(H-I)^T\alpha S\right]. \) We have the following Lemma:

Lemma 6 The objective function (11) can be written as the following convex quadratic function:

\[
V_t = \eta_t^T\Sigma_g\eta_t + 2\eta_t^T\Sigma_g\Theta_i(\cdot)\Phi(\hat{w}_{t-1}) + \mu_g^T\Pi_w(\Theta_t(\cdot)) + 2\mu_S^T\Sigma_G\Phi(\hat{w}_{t-1})^T\Sigma_G^T\mu_S + 2\mu_G^T\Pi_w^T(\Theta_t(\cdot)) + 2\mu_S^T\Sigma_G\Sigma_G^T(\alpha G) + 2\nu_t^T\Sigma_{\hat{w}t}\Phi(\hat{w}_{t-1})^T\Sigma_{\hat{w}t}\Phi(\hat{w}_{t-1}) + 2\nu_t^T\Sigma_{\hat{w}t}\Phi(\hat{w}_{t-1})^T\Sigma_{\hat{w}t}\Phi(\hat{w}_{t-1}) + 2\nu_t^T\Sigma_{\hat{w}t}\Phi(\hat{w}_{t-1})^T\Sigma_{\hat{w}t}\Phi(\hat{w}_{t-1}) \tag{27}\]

A proof of the Lemma 6 is given in the appendix. Please notice in the proof of Lemma 6 that \(V_t \) is obtained by removing some constant terms of \(V_t \) in (11). Therefore, they are not equal but they serve the same purpose for optimization point of view. The variance and covariance matrices involved in the above cost function are computed offline by using Monte-Carlo simulation to avoid online burden. However, \(\Sigma_g, \Sigma_{\hat{w}}^\text{off}, \) and \(\Sigma_{\hat{w}}^\text{off} \) depend on the number of consecutive dropouts of the sensor channel. If we know that there are at most \(h \) number of consecutive packet dropouts in the sensor channel, we can compute \(h \) number of values of \(\Sigma_g, \Sigma_{\hat{w}}^\text{off}, \) and \(\Sigma_{\hat{w}}^\text{off} \) and use the required one at the time of optimization. Since the sensor channel is assumed to be Bernoulli, the number of consecutive packet dropouts of the sensor channel is unbounded. However, for practical purposes, when data is not received for a long time from a particular channel, we can either give high priority to that channel so that precomputed values of \(\Sigma_g, \Sigma_{\hat{w}}^\text{off} \) can be utilized or we compute new values online in advance. The assumption of an uniform bound on the number of consecutive packet dropouts is standard in literature [35, Assumption 4]. However, we do not need such assumption for any theoretical result in this article. Further, the hard constraint on control actions can be written as affine function of the decision variables. We have the following Lemma:

Lemma 7 For the class of control policies (16), the input constraint (2) is equivalent to:

\[
\left\|u_{r:N}^f(i) + \eta_i(i)\right\|_1 \leq u_{\text{max}}, \tag{28}\]

for \(i = 1, \ldots, N_m.\)

PROOF. Since the saturation function is component-wise symmetric about the origin, the claim follows from [15, Proposition 3] by observing that \(\left\|u_{r:N}^a\right\|_1 \leq u_{\text{max}} \iff \left\|u_{r:N}^f\right\|_\infty \leq u_{\text{max}}. \)

The CSOCP (12) can be written as the following convex quadratic program:

\[
\text{minimize } (27) \quad \text{subject to } (28), (22) \tag{29}\]

The above optimization problem (29) can be solved, for example, using the MATLAB based software package YALMIP [25] and the solver GUROBI [14] or SDPT3 [40].

We have the following result on mean-square boundedness of the overall error:

Theorem 8 Consider a discrete-time dynamical system (1) and overall error (3) and the control sequence is generated by repeatedly solving the program (29). Let assumptions (A1) - (A3) hold, then overall error is mean-square bounded.

A proof of Theorem 8 is given in the appendix.

7 Numerical Experiments

In this section, we present a numerical experiment and record the empirical mean of the states to illustrate our results by taking averages over 50 sample paths for 120 time steps. We
Consider the four-dimensional stochastic LTI system with system matrices

\[
A = \begin{bmatrix}
0.9 & 0 & 0 & 0 \\
0 & 0 & -0.8 & -0.6 \\
0 & 0.8 & -0.36 & 0.48 \\
0 & 0.6 & 0.48 & -0.64
\end{bmatrix}, \quad B = \begin{bmatrix}
0.5 \\
0.5 \\
0 \\
0.5
\end{bmatrix}.
\]

The control is constrained by the hard bound \( u_{\text{max}} = 5 \). The additive process noise is mean zero Gaussian and \( \Sigma_w = 0.5I \). The successful transmission probabilities for uplink and downlink are \( p_c = p_r = 0.9 \), and simulation data is \( Q = I_d, Q_f = I_d, R = 1, N = 5 \). The reachability index of the matrix pair \((A, B)\) is 3. We have chosen the recalculation interval \( N_r \) same as the reachability index \( \kappa \) of the matrix pair \((A_w, B_w)\). The reference signal admits the following recursion:

\[
r_{t+1} = Ar_t + B(2.5 \sin(0.083t)); \quad r_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T. \quad (30)
\]

We choose \( \delta = 0.5, x_r^t = r_t, u_r^t = 2.5 \sin(0.083t) \) and \( x_0 = r_0 \).

In Fig. 2, we plot the trajectory generated by the reference governor and the averaged trajectory of the system considered in this numerical experiment. We observe that the averaged states follow very closely to the states of the reference governor.

We further studied the mean square boundedness of the tracking error (the error between states of the controlled plant and the states of a reference governor \( e_t = x_t - x_r^t \)).

Since in this experiment \( r_t = x_r^t \), we have \( e_t = e_r^t \). We first fix \( p_c = 0.9 \) and record empirical MSB over 200 sample paths when \( p_r \) varies in the set \( \{0.5, 0.6, 0.7, 0.8, 0.9, 1\} \). In the second experiment, we fix \( p_r = 0.9 \) and record the empirical MSB over 200 sample paths when \( p_c \) varies in the set \( \{0.5, 0.6, 0.7, 0.8, 0.9, 1\} \). Our observations are plotted in Fig. 3.
8 Epilogue

The stochastic MPC formulation under unreliable channels is considered in this article and the proposed approach is generalized for tracking problems. A class of trackable signals is restricted with the help of a reference governor and available control authority. This idea can be easily extended in the setting of incomplete and corrupt measurements with the help of a Kalman filter along the lines of [30].

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A Appendix

PROOF. [Proof of Lemma 6] Since $e_t:N+1$ and $u_{t:N}^r$ are affine functions of decision variables and the expectation of a convex function is convex [5], $V_t$ in (26) is convex. We substitute the stacked error vector (25) in the objective function (26).

$$V_t = E_t \left[ \|Ae_t + Bu_{t:N}^r + D_{t,N} w_{t:N} \|^2_Q + \|u_{t:N}^r\|^2_R \right]$$

$$= E_t \left[ \|Ae_t\|^2_Q + \|D_{t,N} w_{t:N}\|^2_Q + \|u_{t:N}^r\|^2_R + 2(e_t^T A^T Q B + w_{t:N}^T D^T Q B)u_{t:N}^r + 2e_t^T A^T Q D w_{t:N} \right].$$

Let $\beta_t := E_t \left[ \|Ae_t\|^2_Q + \|D_{t,N} w_{t:N}\|^2_Q + 2(e_t^T A^T Q B)u_{t:N}^r \right]$. Then $V_t = E_t \left[ \|u_{t:N}^r\|^2_R + 2(e_t^T A^T Q B + w_{t:N}^T D^T Q B)u_{t:N}^r \right] + \beta_t$. We now substitute the stacked control vector (24) in $V_t$ to get $V_t$ as

$$V_t = E_t \left[ \|u_{t:N}^r\|^2_R + 2(e_t^T A^T Q B + (H-I)u_{t:N}^r + \Theta_t \psi(\tilde{w}_{t-1:N}))^2 + (\Theta_t \psi(\tilde{w}_{t-1:N})) \right] + \beta_t.$$  

Let $\beta_t' := \beta_t + 2E_t \left[ e_t^T A^T Q B (H-I)u_{t:N}^r \right]$. Since $E_t \left[ e_t \right] = \tilde{x}_t - x_t = e_t^F$, by removing zero terms we get the following equation:

$$V_t = E_t \left[ \|u_{t:N}^r\|^2_R + 2(e_t^T A^T Q B + w_{t:N}^T D^T Q B)\Theta_t \psi(\tilde{w}_{t-1:N}) + (\Theta_t \psi(\tilde{w}_{t-1:N})) \right] + \beta_t'$$  

(A.1)

Since $\tilde{w}_{t-1:N}$ is known at $t$, we simplify the term

$$E_t \left[ w_{t,N}^T D^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right] = E_t \left[ w_{t,N}^T D^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right] = \text{tr}(D^T Q B \mu \Theta_t \psi(\tilde{w}_{t-1:N})) \right].$$  

(A.2)

We consider the term $E_t \left[ e_t^T A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right]$ in (A.1) as follows:

$$E_t \left[ e_t^T A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right] = E_t \left[ (x_t - \tilde{x}_t)^T A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right] + (\tilde{x}_t - x_t^F)^T E_t \left[ A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right] = E_t \left[ e_t^F \right] + \text{tr}(A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N})) + \beta_t'$$  

(A.3)

We substitute (A.2) and (A.3) in (A.1) to get

$$V_t = E_t \left[ \|u_{t:N}^r\|^2_R + 2\beta_t' \right] + 2\text{tr}(D^T Q B \mu \Theta_t \psi(\tilde{w}_{t-1:N})) + 2\text{tr}(A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}).$$  

(A.4)

Let us define $c := (u_{t:N}^r)^T E_t \left[ (H-I)^T \alpha (H-I) \right] u_{t:N}^r$. We simplify the first term in the right hand side of (A.4) as follows:

$$E_t \left[ \|u_{t:N}^r\|^2_R + 2\beta_t' \right] = E_t \left[ \|u_{t:N}^r\|^2_R + 2\text{tr}(D^T Q B \mu \Theta_t \psi(\tilde{w}_{t-1:N})) \right] + 2\text{tr}(A^T Q B \Theta_t \psi(\tilde{w}_{t-1:N}) \right].$$  

Let us consider the term $E_t \left[ \|u_{t:N}^r\|^2_R \right]$ on the right hand side of (A.5). In order to simplify offline computations, we perform the following manipulation:

$$E_t \left[ \|u_{t:N}^r\|^2_R \right] = E_t \left[ \|u_{t:N}^r\|^2_R \right] = \text{tr}(S \left[ \Theta_t^{-1} \right] \psi(\tilde{w}_{t-1:N}))$$  

(A.5)
Further, the Lemma 9 and the Lemma 5 give us

By (4) we have

\[ (8) \]

Lemma 9

Before the proof of Theorem 8, we need the following result:

Let us consider the term \( E_{\tilde{x}_1} \left[ \eta_i^T \mathcal{G}^T \alpha S \Theta_i \psi(\tilde{w}_{i-1:N}) \right] \) on the right hand side of (A.5). By observing \( E_{\tilde{x}_i} \left[ \psi(\tilde{w}_{i+i-1}) \right] = \mathbf{0} \) for each \( i = 1, \ldots, N-1 \), we get

\[ E_{\tilde{x}_i} \left[ \eta_i^T \mathcal{G}^T \alpha S \Theta_i \psi(\tilde{w}_{i-1:N}) \right] = \eta_i^T \Sigma G S \Theta_i^{(t)} \psi(\tilde{w}_{i-1}). \]  

(A.7) Similar to (A.7), we get

\[ E_{\tilde{x}_i} \left[ (u_{i,N}^T)^T (H - I)^T \alpha S \Theta_i \psi(\tilde{w}_{i-1:N}) \right] \]

\[ = (u_{i,N}^T)^T \Sigma_{H S \Theta_i^{(t)}} \psi(\tilde{w}_{i-1}). \]  

(A.8)

Expression (27) follows by substituting (A.5), (A.6), (A.7) and (A.8) in (A.4), and ignoring the term \( \beta_r^t + c \), which is independent of the decision variables. Therefore, the objective function in (26) is equivalent to (27) for the sake of optimization.

Before the proof of Theorem 8, we need the following result:

**Lemma 9** Suppose that the dropout compensator is driven by the recursion (8) and let assumptions (A1)-(A2) hold, then there exists \( \gamma^D > 0 \) such that

\[ E_{X_0} \left[ \left\| e_t^D \right\|^2 \right] \leq \gamma^D \quad \text{for all } t. \]  

(A.9)

The proof of the above Lemma is along the lines of the proof of [30, Lemma 9]. Therefore, we omit the details for brevity.

**PROOF.** [Proof of Theorem 8] Since \( e_t^D = x_t - r_t = x_t - \tilde{x}_t + \tilde{x}_t - x_t^r + x_t^r - r_t = e_t^D + e_t^C + e_t^G \) and \( \left\| e_t^D \right\|^2 \leq 3 \left( \left\| e_t^D \right\|^2 + \left\| e_t^C \right\|^2 + \left\| e_t^G \right\|^2 \right) \) by using Cauchy-Schwartz inequality. By taking conditional expectation on both sides we get

\[ E_{X_0} \left[ \left\| e_t^D \right\|^2 \right] \leq 3 \left( E_{X_0} \left[ \left\| e_t^D \right\|^2 \right] + E_{X_0} \left[ \left\| e_t^C \right\|^2 \right] + \left\| e_t^G \right\|^2 \right). \]

By (4) we have

\[ E_{X_0} \left[ \left\| e_t^D \right\|^2 \right] \leq 3 \left( E_{X_0} \left[ \left\| e_t^D \right\|^2 \right] + E_{X_0} \left[ \left\| e_t^C \right\|^2 \right] + \gamma^G \right) \]

Further, the Lemma 9 and the Lemma 5 give us

\[ E_{X_0} \left[ \left\| e_t^D \right\|^2 \right] \leq 3 \left( \gamma^D + E_{X_0} \left[ \left\| e_t^C \right\|^2 \right] + \gamma^G \right) \]