A MEAN-REVERTING CURRENCY MODEL WITH FLOATING INTEREST RATES IN UNCERTAIN ENVIRONMENT

Weiwei Wang∗ and Ping Chen
School of Science
Nanjing University of Science and Technology
Nanjing 210094, China

(Communicated by Yuanguo Zhu)

Abstract. Currency option is an important risk management tool in the foreign exchange market, which has attracted the attention of many researchers. Unlike the classical stochastic theory, we investigate the valuation of currency option under the assumption that the risk factors are described by uncertain processes. Considering the long-term fluctuations of the exchange rate and the changing of the interest rates from time to time, we propose a mean-reverting uncertain currency model with floating interest rates to simulate the foreign exchange market. Subsequently, European and American currency option pricing formulas for the new currency model are derived and some mathematical properties of the formulas are studied. Finally, some numerical algorithms are designed to calculate the prices of these options.

1. Introduction. In the twentieth century, the probability theory was first proposed by Kolmogorov to describe the random phenomena. Based on probability theory, Ito founded stochastic calculus theory in 1944 and put forward a concept of stochastic differential equation. In 1973, by using the stochastic differential equation, Black and Scholes [1] proposed the famous Black-Scholes option pricing formulas. From then on, stochastic calculus theory was widely employed to evaluate financial derivatives.

As we know, when the probability distribution is close enough to the real frequency, we can apply the probability theory. But sometimes its difficult to get the real frequency. In this case, we have to invite some domain experts to evaluate the belief degree for each event. According to Kahneman and Tversky [5], people usually overweight unlikely events, so the belief degree cannot be treated as probability distribution. In order to deal with the belief degree mathematically, Liu [6] founded an uncertainty theory in 2007, which has become a branch of axiomatic mathematics for modeling human uncertainty. In 2008, Liu [7] proposed the concept of uncertain process to describe dynamic uncertain systems. As a special type of uncertain process, Liu [8] designed a canonical Liu process in 2009 which could be regarded as a counterpart of Brownian motion. Based on it, Liu [8] developed the uncertain calculus to deal with the integral and differential of an uncertain process.

2010 Mathematics Subject Classification. Primary: 91G20; Secondary: 91G80.
Key words and phrases. Uncertainty theory, uncertain differential equation, currency model, option pricing, finance.

∗ Corresponding author: Weiwei Wang.
As a type of differential equations driven by canonical Liu processes, uncertain differential equations were first proposed by Liu [7] in 2008. Subsequently, the study of uncertain differential equation was followed by many researchers. Nowadays, the uncertain differential equation has achieved fruitful results in both theory and practice. In 2010, Chen and Liu [2] first proved the existence and uniqueness theorem of solution of uncertain differential equation under linear growth condition and Lipschitz continuous condition. Later, the theorem was verified again by Gao [3] under local linear growth condition and local Lipschitz continuous condition. In 2009, the concept of stability of uncertain differential equation was presented by Liu [8], and some stability theorems were proved by Yao et al. [26]. After that, different types of stability of uncertain differential equations were explored, for example, stability in mean (Yao-Ke-Sheng [27]), stability in moment (Sheng-Wang [15]), almost sure stability (Liu-Ke-Fei [13]), and exponential stability (Sheng-Gao [17]).

In order to solve uncertain differential equations, Chen-Liu [2] obtained an analytic solution to linear uncertain differential equations. In addition, Liu [11] and Yao [22] proposed analytic methods for solving two special types of uncertain differential equations, which were generalized by Liu [12] and Wang [18] later. More importantly, Yao and Chen [25] presented the Yao-Chen formula to solve the uncertain differential equations numerically, which builds the relationships between uncertain differential equations and ordinary differential equations. On the basis of Yao-Chen formula, Yao [23] presented some formulas to calculate extreme value, first hitting time, and time integral of solution of uncertain differential equation. Besides, some numerical methods for solving general uncertain differential equations were designed by Yao and Chen [25], Yang and Shen [21], Yang and Ralescu [20], and Gao [4].

Uncertain differential equations have shown great significance in uncertain environments, especially in financial situations. With the increasingly active of the foreign exchange market, currency option, which is regarded as an important risk management tool, has attracted many investors’ attention. And an appropriate pricing formula for currency option is becoming extremely significant. Traditionally, the finance theory assumes that stock price, interest rate, and exchange rate follow stochastic differential equations. But this presumption was challenged by Liu [10] in which a convincing paradox was provided to show why the real stock price is impossible to follow any stochastic differential equation. In this case, some researchers employed uncertain differential equations to model the foreign exchange market and solve the currency option pricing problem. In 2015, Liu et al. [14] first proposed an uncertain currency model and gave the European and American option pricing formulas. Subsequently, Shen and Yao [16] presented a mean-reverting uncertain currency model to describe the foreign exchange rate in the long term. Due to the fluctuations of financial market from time to time, the interest rate is changing instead of a constant. In 2017, Wang and Ning [19] proposed an uncertain currency model with floating interest rates to describe the foreign exchange market. But in Wang-Ning’s currency model, they just focused on the short-term fluctuations of the foreign exchange rate.

Considering the long-term fluctuations of the exchange rate and the changing of the interest rates from time to time, this paper presents a mean-reverting uncertain currency model with floating interest rates to simulate the foreign exchange market. The rest of this paper is organized as follows. In Sect.2, we first review some basic
concepts about uncertainty theory. In Sect.3, we introduce a mean-reverting uncertain currency model with floating interest rates. In Sect.4, European and American currency option pricing formulas are obtained for the new currency model and some properties of the formulas are discussed. In Sect.5, we give some numerical algorithms to calculate the prices of the European and American currency option and some numerical experiments are performed. Finally, a brief conclusion is provided in Sect.6.

2. Preliminary. In this section, we will give a brief introduction of uncertain variable, uncertain differential equation and uncertain currency model.

2.1. Uncertain variable.

Definition 2.1 (Liu [6]). Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( \mathcal{M} : \mathcal{L} \to [0, 1] \) is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (Normality Axiom) \( \mathcal{M}\{\Gamma\} = 1 \) for the universal set \( \Gamma \);

Axiom 2 (Duality Axiom) \( \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1 \) for any event \( \Lambda \);

Axiom 3 (Subadditivity Axiom) For every countable sequence of events \( \{\Lambda_i\} \), we have

\[
\mathcal{M}\left\{ \bigcup_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.
\]

The triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an uncertainty space. Besides, the product uncertain measure on the product \( \sigma \)-algebra \( \mathcal{L} \) was defined by Liu [8] as follows.

Axiom 4 (Product Axiom) Let \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\) be uncertainty spaces for \( k = 1, 2, \cdots \). The product uncertain measure \( \mathcal{M} \) is an uncertain measure satisfying

\[
\mathcal{M}\left\{ \bigotimes_{k=1}^{\infty} \Lambda_k \right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \cdots \), respectively.

Definition 2.2 (Liu [6]). An uncertain variable is a function from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set of real numbers, i.e., for any Borel set \( B \) of real numbers, the set

\[
\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}
\]

is an event.

Definition 2.3 (Liu [6]). The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by

\[
\Phi(x) = \mathcal{M}\{\xi \leq x\}
\]

for any real number \( x \).

The inverse function \( \Phi^{-1} \) of the uncertainty distribution \( \Phi \) of uncertain variable \( \xi \) is called the inverse uncertainty distribution of \( \xi \) if it exists and is unique for each \( \alpha \in (0, 1) \). Inverse uncertainty distribution plays an important role in operations of uncertain differential equation.

Definition 2.4 (Liu [6]). The expected value of an uncertain variable \( \xi \) is defined by

\[
E[\xi] = \int_{0}^{+\infty} \mathcal{M}\{\xi \geq x\} dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\} dx
\]

provided that at least one of the two integrals exists.
Theorem 2.5 (Liu [9]). Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi \). Then
\[
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha.
\]

Theorem 2.6 (Liu [9]). Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent uncertain variables with regular uncertainty distributions \( \Phi_1, \Phi_2, \ldots, \Phi_n \), respectively. If the function \( f(\xi_1, \xi_2, \ldots, \xi_n) \) is strictly increasing with respect to \( \xi_1, \xi_2, \ldots, \xi_m \) and strictly decreasing with respect to \( \xi_{m+1}, \xi_{m+2}, \ldots, \xi_n \), then the uncertain variable
\[
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
\]
has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).
\]

2.2. Uncertain differential equation. An uncertain process is essentially a sequence of uncertain variables indexed by time, which is used to model the evolution of uncertain phenomena.

Definition 2.7 (Liu [8]). An uncertain process \( C_t (t \geq 0) \) is said to be a canonical process if
(i) \( C_0 = 0 \) and almost all sample paths are Lipschitz continuous,
(ii) \( C_t \) has stationary and independent increments,
(iii) every increment \( C_{s+t} - C_s \) is a normal uncertain variable with expected value 0 and variance \( t^2 \), whose uncertainty distribution is
\[
\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3} t}\right)\right)^{-1}, x \in \mathbb{R}.
\]
If \( C_t \) is a canonical process, then the uncertain process \( X_t = \exp(\sigma C_t) \) is called a geometric canonical process.

Definition 2.8 (Liu [7]). Suppose that \( C_t \) is a canonical Liu process, and \( f \) and \( g \) are two real functions. Then
\[
dX_t = f(t, X_t) dt + g(t, X_t) dC_t
\]
is called an uncertain differential equation.

Definition 2.9 (Yao and Chen [25]). Let \( \alpha \) be a number with \( 0 < \alpha < 1 \). An uncertain differential equation
\[
dX_t = f(t, X_t) dt + g(t, X_t) dC_t
\]
is said to have an \( \alpha \)-path \( X_t^\alpha \) if it solves the corresponding ordinary differential equation
\[
dX_t^\alpha = f(t, X_t^\alpha) dt + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) dt,
\]
where \( \Phi^{-1}(\alpha) \) is the inverse uncertainty distribution of a standard normal uncertain variable, i.e.,
\[
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.
\]

Theorem 2.10 (Yao and Chen [25]). Let \( X_t \) and \( X_t^\alpha \) be the solution and \( \alpha \)-path of the uncertain differential equation
\[
dX_t = f(t, X_t) dt + g(t, X_t) dC_t,
\]
respectively. Then
\[
\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha,
\]
A MEAN-REVERTING CURRENCY MODEL

\[ \mathcal{M} \{ X_t > X_t^\alpha, \forall t \} = 1 - \alpha. \]

**Theorem 2.11** (Yao [24]). Let \( X_t \) and \( Y_t \) be the solutions of uncertain differential equations

\[ dX_t = f_1(t, X_t)dt + g_1(t, X_t)dC_{1t} \]

and

\[ dY_t = f_2(t, Y_t)dt + g_2(t, Y_t)dC_{2t}, \]

respectively, where \( C_{1t} \) and \( C_{2t} \) are two independent canonical Liu processes. Then for any positive numbers \( T \) and \( K \), we have

\[ E \left[ \sup_{0 \leq t \leq T} (Y_t - K)^+ \exp \left( - \int_0^t X_s ds \right) \right] = \int_0^1 \sup_{0 \leq t \leq T} (Y_t^\alpha - K)^+ \exp \left( - \int_0^t X_1^{1-\alpha} ds \right) d\alpha, \]

where \( X_t^\alpha \) and \( Y_t^\alpha \) are the \( \alpha \)-paths of the two uncertain differential equations, respectively.

2.3. Uncertain currency model. In modern society, the international business becomes increasingly frequent, more and more people pay their attention to the foreign exchange market. Currency option, which is regarded as an important risk management tool, has attracted the attention of many researchers. An appropriate pricing formula for currency option is becoming extremely significant. Uncertain finance is a powerful tool in solving foreign exchange problems. In 2015, Liu et al. [14] proposed an uncertain currency model

\[
\begin{cases}
    dX_t = uX_t dt \\
    dY_t = vY_t dt \\
    dZ_t = eZ_t dt + \sigma Z_t dC_t,
\end{cases}
\]

where \( X_t \) represents the riskless domestic currency with domestic interest rate \( u \), \( Y_t \) represents the riskless foreign currency with foreign interest rate \( v \), and \( Z_t \) represents the exchange rate that is the domestic currency price of one unit of foreign currency at time \( t \). In other words, one unit of foreign currency equals to \( Z_t \) unit of domestic currency.

Liu-Chen-Ralescu’s currency model discussed the short-term volatility of the foreign exchange rate. But in the real global market, the foreign exchange rate fluctuates around an average level in the long term. Then Shen and Yao [16] gave a mean-reverting uncertain currency model for the long term

\[
\begin{cases}
    dX_t = uX_t dt \\
    dY_t = vY_t dt \\
    dZ_t = (m - a Z_t) dt + \sigma dC_t,
\end{cases}
\]

where \( m, a, \sigma \) are constants and the foreign exchange rate \( Z_t \) stays around the average \( m/a \) in the long run. Besides, they derived the European and American currency option pricing formulas for the currency model (2).

Due to the fluctuations of financial market from time to time, the interest rate is changing instead of a constant. In 2017, Wang and Ning [19] provided an uncertain
currency model with floating interest rates

\[
\begin{align*}
    dr_t &= (m_1 - a_1 r_t)dt + \sigma_1 dC_{1t} \\
    df_t &= (m_2 - a_2 f_t)dt + \sigma_2 dC_{2t} \\
    dZ_t &= (m_3 - a_3 Z_t)dt + \sigma_3 dC_{3t},
\end{align*}
\]

(3)

where the constant \( \sigma_1 \) is the diffusion of the domestic interest rate \( r_t \), the constant \( \sigma_2 \) is the diffusion of the foreign interest rate \( f_t \), \( m_1, m_2, a_1, a_2 \) are constants, \( C_{1t}, C_{2t}, C_{3t} \) are independent canonical Liu processes.

3. Mean-reverting uncertain currency model with floating interest rates.
In Wang-Ning’s currency model, the processes of interest rates were assumed to be the uncertain counterparts of Vasicek model. There is no doubt that Vasicek model may bring a negative value to the interest rate. Besides, in Wang and Ning’s model, the exchange rate was assumed to follow a geometric Liu process. However, a geometric Liu process will increase to infinity as the time extends, it cannot be applied to describe the exchange rate in the long run.

In this section, we will make some improvements to the above currency models. Considering the long-term fluctuations of the exchange rate and the changing of the interest rates from time to time, we introduce a mean-reverting uncertain currency model with floating interest rates

\[
\begin{align*}
    dr_t &= m_1(a_1 - r_t)dt + \sigma_1 \sqrt{r_t} dC_{1t} \\
    df_t &= m_2(a_2 - f_t)dt + \sigma_2 \sqrt{f_t} dC_{2t} \\
    dZ_t &= (m_3 - a_3 Z_t)dt + \sigma_3 dC_{3t},
\end{align*}
\]

(4)

where the domestic and foreign interest rate models are the uncertain counterparts of CIR model, \( \sigma_1 \) is the diffusion of the domestic interest rate \( r_t \), \( \sigma_2 \) is the diffusion of the foreign interest rate \( f_t \), and \( m_1, a_1, m_2, a_2, m_3, a_3, \sigma_1, \sigma_2, \sigma_3 \) are constants, \( C_{1t}, C_{2t}, C_{3t} \) are independent canonical Liu processes. In the model (4), when the exchange rate \( Z_t \) exceeds or is below the average level, it will be pulled to the average \( m_3/a_3 \) at the rate \( a_3 \).

According to Yao-Chen formula [25], it is easy to verify that the \( \alpha \)-path of the domestic interest rate \( r_t \) satisfies the differential equation

\[
    dr_t^\alpha = m_1(a_1 - r_t^\alpha)dt + \sigma_1 \sqrt{r_t^\alpha} \Phi^{-1}(\alpha)dt,
\]  

(5)

the \( \alpha \)-path of the foreign interest rate \( f_t \) satisfies the differential equation

\[
    df_t^\alpha = m_2(a_2 - f_t^\alpha)dt + \sigma_2 \sqrt{f_t^\alpha} \Phi^{-1}(\alpha)dt,
\]  

(6)

and the \( \alpha \)-path of the foreign exchange rate \( Z_t \) is

\[
    Z_t^\alpha = Z_0 \exp(-a_3 t) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3}(1 - \exp(-a_3 t)),
\]  

(7)

where \( \Phi \) is the standard normal uncertainty distribution, i.e.,

\[
    \Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.
\]

4. Currency option pricing formulas. In this section, we will derive the European and American currency option pricing formulas for the uncertain currency model (4).
4.1. European currency option. A European currency option is a contract that gives the holder the right without the obligation to buy one unit of foreign currency with $K$ units of domestic currency at the expiration date $T$.

Assume that the European currency option has a strike price $K$ and an expiration time $T$. Let $f_e$ represent the price of the contract in domestic currency. Then the investor pays $f_e$ for buying the contract at time 0, and has a payoff $(Z_T - K)^+$ in domestic currency at the expiration time $T$. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-\int_0^T r_s ds)(Z_T - K)^+$. Thus the net return of the investor at time 0 is

$$-f_e + \exp(-\int_0^T r_s ds)(Z_T - K)^+. $$

On the other hand, the bank receives $f_e$ for selling the contract at time 0, and pays $(1 - K/Z_T)^+$ in foreign currency at the expiration time $T$. Thus the net return of the bank at the time 0 is

$$f_e - \exp(-\int_0^T f_s ds)Z_0(1 - K/Z_T)^+. $$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$E[-f_e + \exp(-\int_0^T r_s ds)(Z_T - K)^+] = E[f_e - Z_0 \exp(-\int_0^T f_s ds)(1 - K/Z_T)^+]. $$

Thus, the European currency option price is given by the definition below.

**Definition 4.1.** Assume a European currency option has a strike price $K$ and an expiration time $T$. Then the European currency option price is

$$f_e = \frac{1}{2} E[(Z_T - K)^+ \exp(-\int_0^T r_s ds)] + \frac{Z_0}{2} E[(1 - K/Z_T)^+ \exp(-\int_0^T f_s ds)]. $$

**Theorem 4.2.** Assume a European currency option for the uncertain currency model (4) has a strike price $K$ and an expiration time $T$. Then the European currency option price is

$$f_e = \frac{1}{2} \int_0^1 (Z_T^\alpha - K)^+ \exp(-\int_0^T r_s^{1-\alpha} ds) d\alpha + \frac{Z_0}{2} \int_0^1 (1 - K/Z_T)^+ \exp(-\int_0^T f_s^{1-\alpha} ds) d\alpha, $$

where $r_s^{1-\alpha}$ satisfies the following ordinary differential equation

$$dr_s^{1-\alpha} = m_1(a_1 - r_s^{1-\alpha}) ds + \sigma_1 \sqrt{r_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds,$$

$f_s^{1-\alpha}$ satisfies the following ordinary differential equation

$$df_s^{1-\alpha} = m_2(a_2 - f_s^{1-\alpha}) ds + \sigma_2 \sqrt{f_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds,$$

and

$$Z_T^\alpha = Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3}(1 - \exp(-a_3 T)).$$
Proof. According to Definition 4.1, we have
\[ f_e = \frac{1}{2} E \left[ (Z_T - K)^+ \exp(- \int_0^T r_s ds) \right] + \frac{Z_0}{2} E \left[ (1 - \frac{K}{Z_T})^+ \exp(- \int_0^T f_s ds) \right]. \]
Assume
\[ f_1 = \frac{1}{2} E \left[ (Z_T - K)^+ \exp(- \int_0^T r_s ds) \right], \]
\[ f_2 = \frac{Z_0}{2} E \left[ (1 - \frac{K}{Z_T})^+ \exp(- \int_0^T f_s ds) \right], \]
and we have
\[ f_e = f_1 + f_2. \]
Set
\[ \xi = f(Z_T, r_s) = (Z_T - K)^+ \exp(- \int_0^T r_s ds). \]
Obviously, the function \( f(Z_T, r_s) \) is increasing with respect to \( Z_T \) and decreasing with respect to \( r_s \). According to Theorem 2.6, we get the inverse uncertainty distribution of the uncertain variable \( \xi \)
\[ \Psi^{-1}(\alpha) = (Z_T^\alpha - K)^+ \exp(- \int_0^T r_s^{1-\alpha} ds). \]
By Theorem 2.5, we get
\[ f_1 = \frac{1}{2} \int_0^1 (Z_T^\alpha - K)^+ \exp(- \int_0^T r_s^{1-\alpha} ds) d\alpha. \]
Set
\[ \gamma = g(Z_T, f_s) = (1 - \frac{K}{Z_T})^+ \exp(- \int_0^T f_s ds). \]
Obviously, the function \( g(Z_T, f_s) \) is increasing with respect to \( Z_T \) and decreasing with respect to \( f_s \). Similarly, by Theorems 2.5 and 2.6, we get
\[ f_2 = \frac{Z_0}{2} \int_0^1 (1 - \frac{K}{Z_T^\alpha})^+ \exp(- \int_0^T f_s^{1-\alpha} ds) d\alpha. \]
Taking \( f_1 \) and \( f_2 \) into \( f_e \), we have
\[ f_e = \frac{1}{2} \int_0^1 (Z_T^\alpha - K)^+ \exp(- \int_0^T r_s^{1-\alpha} ds) d\alpha \]
\[ + \frac{Z_0}{2} \int_0^1 (1 - \frac{K}{Z_T^\alpha})^+ \exp(- \int_0^T f_s^{1-\alpha} ds) d\alpha. \]
According to the Definition 4.1, Theorems 2.5 and 2.6, the European currency option pricing formula is verified. \( \square \)
4.2. **American currency option.** Unlike the European currency option, an American currency option is a contract that gives the holder the right without the obligation to buy one unit of foreign currency with $K$ units of domestic currency at any time prior to the expiration date $T$.

Assume that the American currency option has a strike price $K$ and an expiration time $T$. Let $f_a$ represent the price of the contract in domestic currency. Then the investor pays $f_a$ for buying the contract at time 0, and has a payoff \( \sup_{0 \leq t \leq T} (Z_t - K)^+ \) in domestic currency at the supreme value. Considering the time value of money resulted from the bond, the present value of the payoff is

\[
\sup_{0 \leq t \leq T} (Z_t - K)^+ \exp(-\int_0^t r_s ds).
\]

Thus the net return of the investor at time 0 is

\[
-f_a + \sup_{0 \leq t \leq T} (Z_t - K)^+ \exp(-\int_0^t r_s ds).
\]

On the other hand, the bank receives $f_a$ for selling the contract at time 0, and pays \( \sup_{0 \leq t \leq T} Z_0(1 - K Z_t)^+ \exp(-\int_0^t f_s ds) \) in foreign currency. Thus the net return of the bank at the time 0 is

\[
f_a - \sup_{0 \leq t \leq T} Z_0(1 - K Z_t)^+ \exp(-\int_0^t f_s ds).
\]

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

\[
E \left[ -f_a + \sup_{0 \leq t \leq T} (Z_t - K)^+ \exp(-\int_0^t r_s ds) \right] = E \left[ f_a - \sup_{0 \leq t \leq T} Z_0(1 - K Z_t)^+ \exp(-\int_0^t f_s ds) \right].
\]

Thus, the American currency option price is given by the definition below.

**Definition 4.3.** Assume an American currency option has a strike price $K$ and an expiration time $T$. Then the American currency option price is

\[
f_a = \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} (Z_t - K)^+ \exp(-\int_0^t r_s ds) \right] + \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} Z_0(1 - K Z_t)^+ \exp(-\int_0^t f_s ds) \right].
\]

**Theorem 4.4.** Assume an American currency option for the uncertain currency model \( (4) \) has a strike price $K$ and an expiration time $T$. Then the American currency option price is

\[
f_a = \frac{1}{2} \int_0^1 \sup_{0 \leq t \leq T} (Z_t^\alpha - K)^+ \exp(-\int_0^t r_s^{1-\alpha} ds) d\alpha + \frac{Z_0}{2} \int_0^1 \sup_{0 \leq t \leq T} (1 - K Z_t^\alpha)^+ \exp(-\int_0^t f_s^{1-\alpha} ds) d\alpha,
\]

where $r_s^{1-\alpha}$ satisfies the following ordinary differential equation

\[
ds_s^{1-\alpha} = m_1 (a_1 - r_s^{1-\alpha}) ds + \sigma_1 \sqrt{r_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds,
\]
\[ f_s^{1-\alpha} \text{ satisfies the following ordinary differential equation} \]
\[ df_s^{1-\alpha} = m_2(a_2 - f_s^{1-\alpha})ds + \sigma_2 \sqrt{f_s^{1-\alpha} \Phi^{-1}(1 - \alpha)} ds, \]
and
\[ Z_t^\alpha = Z_0 \exp(-a_3 t) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 t)). \]

**Proof.** According to Definition 4.3, we have
\[ f_a = \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} (Z_t - K)^+ \exp(-\int_0^t r_s ds) \right] + \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} Z_0 (1 - \frac{K}{Z_t})^+ \exp(-\int_0^t f_s ds) \right]. \]
Assume
\[ f_3 = \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} (Z_t - K)^+ \exp(-\int_0^t r_s ds) \right], \]
\[ f_4 = \frac{1}{2} E \left[ \sup_{0 \leq t \leq T} Z_0 (1 - \frac{K}{Z_t})^+ \exp(-\int_0^t f_s ds) \right], \]
and we have
\[ f_a = f_3 + f_4. \]
It follows from Theorem 2.11 that
\[ f_3 = \frac{1}{2} \int_0^1 \sup_{0 \leq t \leq T} (Z_t^\alpha - K)^+ \exp(-\int_0^t f_s^{1-\alpha} ds) d\alpha, \]
and
\[ f_4 = \frac{Z_0}{2} \int_0^1 \sup_{0 \leq t \leq T} (1 - \frac{K}{Z_t^\alpha})^+ \exp(-\int_0^t f_s^{1-\alpha} ds) d\alpha. \]
Taking \( f_3 \) and \( f_4 \) into \( f_a \), we have
\[ f_a = \frac{1}{2} \int_0^1 \sup_{0 \leq t \leq T} (Z_t^\alpha - K)^+ \exp(-\int_0^t r_s^{1-\alpha} ds) d\alpha \]
\[ + \frac{Z_0}{2} \int_0^1 \sup_{0 \leq t \leq T} (1 - \frac{K}{Z_t^\alpha})^+ \exp(-\int_0^t f_s^{1-\alpha} ds) d\alpha. \]
According to the Definition 4.3 and Theorem 2.11, the American currency option pricing formula is verified.

We have derived the pricing formulas of the European and American currency option, and next we discuss the properties of this pricing formulas.

**Theorem 4.5.** Let \( f_e \) and \( f_a \) be the European and American currency option price of the uncertain currency model (4), respectively. Then,
1. \( f_e \) is an increasing function of \( m_3 \);
2. \( f_e \) is a decreasing function of \( K \);
3. \( f_a \) is an increasing function of \( m_3 \);
4. \( f_a \) is a decreasing function of \( K \).
Proof. By Theorem 4.2, the pricing formula of the European currency option can be expressed as

\[
f_e = \frac{1}{2} \int_0^1 \left( Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 T)) - K \right)^+ \exp(- \int_0^T r_s^{1-\alpha} ds) d\alpha + \frac{Z_0}{2} \int_0^1 \exp(- \int_0^T f_s^{1-\alpha} ds)
\]

\[
\left( 1 - \frac{K}{Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 T))} \right)^+ d\alpha,
\]

where \( r_s^{1-\alpha} \) satisfies the following ordinary differential equation

\[
dr_s^{1-\alpha} = m_1(a_1 - r_s^{1-\alpha}) ds + \sigma_1 \sqrt{r_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds,
\]

\( f_s^{1-\alpha} \) satisfies the following ordinary differential equation

\[
df_s^{1-\alpha} = m_2(a_2 - f_s^{1-\alpha}) ds + \sigma_2 \sqrt{f_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds.
\]

(1) Since \( a_3, 1 - \exp(-a_3 T) > 0 \), the functions

\[
\frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 T)) - K,
\]

and

\[
1 - \frac{K}{Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 T))}
\]

are increasing with respect to \( m_3 \). Thus, the European currency option price \( f_e \) is increasing with respect to the parameter \( m_3 \).

(2) Since the functions

\[
Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 T)) - K,
\]

and

\[
1 - \frac{K}{Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 T))}
\]

are decreasing with respect to \( K \), the European currency option price \( f_e \) is decreasing with respect to the strike price \( K \).

By Theorem 4.4, the pricing formula of the American currency option can be expressed as

\[
f_a = \frac{1}{2} \int_0^1 \sup_{0 \leq t \leq T} \left( Z_0 \exp(-a_3 t) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 t)) - K \right)^+ \exp(- \int_0^t r_s^{1-\alpha} ds) d\alpha + \frac{Z_0}{2} \int_0^1 \exp(- \int_0^t f_s^{1-\alpha} ds)
\]

\[
\sup_{0 \leq t \leq T} \left( 1 - \frac{K}{Z_0 \exp(-a_3 t) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha)}{a_3} (1 - \exp(-a_3 t))} \right)^+ d\alpha,
\]

where \( r_s^{1-\alpha} \) satisfies the following ordinary differential equation

\[
dr_s^{1-\alpha} = m_1(a_1 - r_s^{1-\alpha}) ds + \sigma_1 \sqrt{r_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds,
\]

and

\[
df_s^{1-\alpha} = m_2(a_2 - f_s^{1-\alpha}) ds + \sigma_2 \sqrt{f_s^{1-\alpha}} \Phi^{-1}(1 - \alpha) ds.
\]
Step 3. Set $f_s^{1-\alpha}$ satisfies the following ordinary differential equation
\[
df_s^{1-\alpha} = m_2(a_2 - f_s^{1-\alpha})ds + \sigma_2 \sqrt{f_s^{1-\alpha} \Phi^{-1}(1-\alpha)}ds.
\]

(3) Since $a_3, 1 - \exp(-a_3 t) > 0$, the functions
\[
\sup_{0 \leq t \leq T} \left( m_3 + \sigma_3 \Phi^{-1}(a) \left( 1 - \exp(-a_3 t) \right) - K \right),
\]
and
\[
\sup_{0 \leq t \leq T} \left( 1 - \frac{K}{m_3 + \sigma_3 \Phi^{-1}(a) \left( 1 - \exp(-a_3 t) \right)} \right)
\]
are increasing with respect to $m_3$. Thus, the American currency option price $f_a$ is increasing with respect to the parameter $m_3$.

(4) Since the functions
\[
\sup_{0 \leq t \leq T} \left( Z_0 \exp(-a_3 t) + \frac{m_3 + \sigma_3 \Phi^{-1}(a)}{a_3} \left( 1 - \exp(-a_3 t) \right) - K \right),
\]
and
\[
\sup_{0 \leq t \leq T} \left( 1 - \frac{K}{Z_0 \exp(-a_3 t) + \frac{m_3 + \sigma_3 \Phi^{-1}(a)}{a_3} \left( 1 - \exp(-a_3 t) \right)} \right)
\]
are decreasing with respect to $K$, the American currency option price $f_a$ is decreasing with respect to the strike price $K$.

5. **Numerical algorithms.** In Section 4, we have derived the European and American currency option pricing formulas in the form of inverse uncertainty distribution. In this section, we design some numerical algorithms to calculate the prices of options.

5.1. **European currency option.** According to Theorem 4.2, the algorithm to calculate the price of European currency option for the currency model (4) is designed as follows.

**Step 0.** Choose two large numbers $N$ and $M$ according to the desired precision degree. Set $\alpha_i = \frac{i}{N}$, $t_j = \frac{jT}{M}$, and $r_{t_0}^{\alpha_i} = r_0$, $f_{t_0}^{\alpha_i} = f_0$, $i = 1, 2, \cdots, N - 1$, $j = 1, 2, \cdots, M$.

**Step 1.** Set $i = 0$.

**Step 2.** Set $i \leftarrow i + 1$.

**Step 3.** Set $j = 0$.

**Step 4.** Calculate the inverse uncertainty distribution of the foreign exchange rate $Z_T$
\[
\Psi_T^{-1}(\alpha_i) = Z_0 \exp(-a_3 T) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha_i)}{a_3} \left( 1 - \exp(-a_3 T) \right).
\]

**Step 5.** Calculate
\[
(Z_T^{\alpha_i} - K)^+ = \max \left( 0, \Psi_T^{-1}(\alpha_i) - K \right)
\]
and
\[
\left( 1 - \frac{K}{Z_T^{\alpha_i}} \right)^+ = \max \left( 0, 1 - \frac{K}{\Psi_T^{-1}(\alpha_i)} \right).
\]

**Step 6.** Employ the recursion formulas
\[
r^{1-\alpha_i}_{t_{j+1}} = r^{1-\alpha_i}_{t_j} + m_1(a_1 - r^{1-\alpha_i}_{t_j}) \frac{T}{M} + \sigma_1 \sqrt{r^{1-\alpha_i}_{t_j} \frac{\sqrt{3}}{\pi}} \ln \frac{1 - \alpha_i T}{a_i} \frac{T}{M},
\]
\[ f_{t_j}^{1-\alpha_i} = f_{t_j}^{1-\alpha_i} + m_2(a_2 - f_{t_j}^{1-\alpha_i}) \frac{T}{M} + \sigma_2 \sqrt{f_{t_j}^{1-\alpha_i}} \sqrt{\frac{3}{\pi}} \ln \frac{1-\alpha_i}{\alpha_i} \frac{T}{M}, \]

and calculate \( r_{t_j}^{1-\alpha_i} \), \( f_{t_j}^{1-\alpha_i} \).

**Step 7.** Set \( j \leftarrow j + 1 \).

**Step 8.** If \( j < M \), return to Step 6.

**Step 9.** Calculate the discount rates

\[ \exp \left( - \int_0^T r_s^{1-\alpha_i} ds \right) \left\langle \exp \left( - \frac{T}{M} \sum_{j=1}^{M} r_{t_j}^{1-\alpha_i} \right) \right\rangle \]

and

\[ \exp \left( - \int_0^T f_s^{1-\alpha_i} ds \right) \left\langle \exp \left( - \frac{T}{M} \sum_{j=1}^{M} f_{t_j}^{1-\alpha_i} \right) \right\rangle. \]

**Step 10.** Calculate

\[ \exp \left( - \int_0^T r_s^{1-\alpha_i} ds \right) (Z_T^\alpha_i - K)^+ \]

and

\[ \exp \left( - \int_0^T f_s^{1-\alpha_i} ds \right) (1 - \frac{K}{Z_T^\alpha_i})^+. \]

If \( i < N - 1 \), return to Step 2.

**Step 11.** Calculate the price of European currency option

\[ f_e \leftarrow \frac{1}{2(N-1)} \sum_{i=1}^{N-1} \left[ (\Psi_T^{-1}(\alpha_i) - K)^+ \exp \left( - \frac{T}{M} \sum_{j=1}^{M} r_{t_j}^{1-\alpha_i} \right) \right] + \frac{Z_0}{2(N-1)} \sum_{i=1}^{N-1} \left[ \left( 1 - \frac{K}{\Psi_T^{-1}(\alpha_i)} \right)^+ \exp \left( - \frac{T}{M} \sum_{j=1}^{M} f_{t_j}^{1-\alpha_i} \right) \right]. \]

Example 5.1 Assume that the parameters in the model (4) are \( m_1 = 1, a_1 = 0.05, \sigma_1 = 0.02, m_2 = 1, a_2 = 0.04, \sigma_2 = 0.03, m_3 = 6, a_3 = 1, \sigma_3 = 0.1, r_0 = 0.03, f_0 = 0.02, Z_0 = 7, K = 6, T = 1 \). Then the European currency option price is \( f_e = 0.3696 \).

5.2. **American currency option.** According to Theorem 4.4, the algorithm to calculate the price of American currency option for the currency model (4) is designed as follows.

**Step 0.** Choose two large numbers \( N \) and \( M \) according to the desired precision degree. Set \( \alpha_i = \frac{i}{N}, t_j = \frac{jT}{M}, \) and \( r_{t_0}^{\alpha_i} = r_0, f_{t_0}^{\alpha_i} = f_0, i = 1, 2, \ldots, N - 1, j = 1, 2, \ldots, M. \)

**Step 1.** Set \( i = 0 \).

**Step 2.** Set \( i \leftarrow i + 1 \).

**Step 3.** Set \( j = 1 \).

**Step 4.** Set \( k = 0 \).

**Step 5.** Calculate the inverse uncertainty distribution of the foreign exchange rate \( Z_{t_j} \)

\[ \Psi_j^{-1}(\alpha_i) = Z_0 \exp(-a_3 t_j) + \frac{m_3 + \sigma_3 \Phi^{-1}(\alpha_i)}{a_3} (1 - \exp(-a_3 t_j)). \]
Step 6. Calculate
\[
(Z_{ij}^α - K)^+ = \max (0, \Psi^{-1}_j(α_i) - K)
\]
and
\[
(1 - \frac{K}{Z_{ij}^α})^+ = \max \left(0, 1 - \frac{K}{\Psi^{-1}_j(α_i)}\right).
\]

Step 7. Employ the recursion formulas
\[
r_{lk+1}^{1-α_i} = r_{lk}^{1-α_i} + m_1(a_1 - r_{lk}^{1-α_i}) \frac{T}{M} + \sqrt{\frac{T}{M}} \ln \frac{1 - α_i}{α_i} M,
\]
\[
f_{lk+1}^{1-α_i} = f_{lk}^{1-α_i} + m_2(a_2 - f_{lk}^{1-α_i}) \frac{T}{M} + \sqrt{\frac{T}{M}} \ln \frac{1 - α_i}{α_i} M,
\]
and calculate \(r_{lk+1}^{1-α_i}, f_{lk+1}^{1-α_i}\).

Step 8. Set \(k ← k + 1\).

Step 9. If \(k < j\), return to Step 7.

Step 10. Calculate the discount rates
\[
\exp \left(- \int_0^t r_s^{1-α_i} ds\right) ← \exp \left(- \frac{T}{M} \sum_{k=1}^j r_{lk}^{1-α_i}\right)
\]
and
\[
\exp \left(- \int_0^t f_s^{1-α_i} ds\right) ← \exp \left(- \frac{T}{M} \sum_{k=1}^j f_{lk}^{1-α_i}\right).
\]

Step 11. Calculate
\[
(Z_{ij}^α - K)^+ \exp \left(- \int_0^t r_s^{1-α_i} ds\right)
\]
and
\[
(1 - \frac{K}{Z_{ij}^α})^+ \exp \left(- \int_0^t f_s^{1-α_i} ds\right).
\]

Step 12. Set \(j ← j + 1\).

Step 13. If \(j \leq M\), return to Step 4.

Step 14. Calculate the supremums
\[
\sup_{0 ≤ t ≤ T} (Z_t^α - K)^+ \exp(- \int_0^t r_s^{1-α_i} ds) ← \sup_{1 ≤ j ≤ M} (\Psi^{-1}_j(α_i) - K)^+ \exp(- \frac{T}{M} \sum_{k=1}^j r_{lk}^{1-α_i})
\]
and
\[
\sup_{0 ≤ t ≤ T} (1 - \frac{K}{Z_t^α})^+ \exp(- \int_0^t f_s^{1-α_i} ds) ← \sup_{1 ≤ j ≤ M} (1 - \frac{K}{\Psi^{-1}_j(α_i)})^+ \exp(- \frac{T}{M} \sum_{k=1}^j f_{lk}^{1-α_i}).
\]

If \(i < N - 1\), return to Step 2.

Step 15. Calculate the price of American currency option
\[
f_a ← \frac{1}{2(N-1)} \sum_{i=1}^{N-1} \left[ \sup_{1 ≤ j ≤ M} (\Psi^{-1}_j(α_i) - K)^+ \exp \left(- \frac{T}{M} \sum_{k=1}^j r_{lk}^{1-α_i}\right) \right]
\]
\[
+ \frac{Z_0}{2(N-1)} \sum_{i=1}^{N-1} \left[ \sup_{1 ≤ j ≤ M} \left(1 - \frac{K}{\Psi^{-1}_j(α_i)}\right)^+ \exp \left(- \frac{T}{M} \sum_{k=1}^j f_{lk}^{1-α_i}\right) \right].
\]
Example 5.2 Assume that the parameters in the model (4) are $m_1 = 1, a_1 = 0.05, \sigma_1 = 0.02, m_2 = 1, a_2 = 0.04, \sigma_2 = 0.03, m_3 = 6, a_3 = 1, \sigma_3 = 0.1, r_0 = 0.03, f_0 = 0.02, Z_0 = 7, K = 6, T = 1$. Then the American currency option price is $f_a = 1.0241$.

6. Conclusion. In this paper, we presented a mean-reverting uncertain currency model with floating interest rates, in which the interest rate models were the uncertain counterparts of CIR model. Subsequently, European and American currency option pricing formulas for the proposed model were derived and some properties of the formulas were also studied. Besides, some numerical algorithms were designed to compute the prices of these options and some numerical experiments were performed.

Acknowledgments. This work was supported by the National Natural Science Foundation of China (11271189) and Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX17_0320). Special thanks to two anonymous reviewers, the AE and the editor, Prof. Kok-Lay Teo, for their constructive comments and suggestions to improve the presentation and quality of this paper.

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Received March 2018; revised April 2018.

*E-mail address: 975788285@qq.com*

*E-mail address: prob123@njust.edu.cn*