Quasi-continuous random variables and processes under the $G$-expectation framework

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Abstract

In this paper, we use PDE and probabilistic methods to obtain a kind of quasi-continuous random variables. We also give a characterization of $M^p_G(0,T)$ and get a kind of quasi-continuous processes by applying Krylov's estimates. Furthermore, the Itô-Krylov formula under the $G$-expectation framework is established.

Key words: $G$-expectation, $G$-Brownian motion, quasi-continuous, Krylov’s estimates, Itô-Krylov’s formula.

MSC-classification: 60H10, 60H30

1 Introduction

Let $\Omega = C^d_0(\mathbb{R}^+)$ and $L_{ip}(\Omega)$ be the space of all bounded and Lipschitz cylinder functions on $\Omega$ (see Section 2 for definition). Motivated by model uncertainty in finance, Peng 2004-2005 firstly constructed a kind of dynamically consistent fully nonlinear expectations on $(\Omega, L_{ip}(\Omega))$ by stochastic control and PDE methods (see [9, 10]). An important case is $G$-expectation $\hat{E}[\cdot]$, which is a sublinear expectation. Under $G$-expectation $\hat{E}[\cdot]$, the canonical process $(B_t)_{t\geq 0}$ is called $G$-Brownian motion. The completion of $L_{ip}(\Omega)$ under the norm $||X||_{L^p_G} := (\hat{E}[(|X|^p)]^{1/p}, p \geq 1$, is denoted by $L^p_G(\Omega)$. Under the $G$-expectation framework, the corresponding stochastic calculus of Itô's type and the existence and uniqueness theorem of $G$-SDEs were also established by Peng in [11,12].

Denis et al. [4] obtained a representation theorem of $G$-expectation $\hat{E}[\cdot]$ by using stochastic control method:

\[ \hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{for} \quad X \in L_{ip}(\Omega), \]

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where $\mathcal{P}$ is a family of weakly compact probability measures on $(\Omega, \mathcal{B}(\Omega))$. Furthermore, they gave a characterization of the space $L^p_G(\Omega)$, which shows that every element in $L^p_G(\Omega)$ is quasi-continuous (see Section 2 for definition). The representation theorem of $G$-expectation $\hat{\mathbb{E}}[\cdot]$ were also obtained in [5] by using a simple probabilistic method. Based on representation theorem, the properties of the solutions to $G$-SDEs were discussed in [2]. On the classical probability space $(\Omega, \mathcal{B}(\Omega), P)$, every random variable is quasi-continuous (Lusin’s theorem). But under the $G$-expectation framework, it is difficult to verify that whether a random variable is quasi-continuous or not, because the elements in $\mathcal{P}$ are singular. The question is how big the space $L^p_G(\Omega)$? Whether $f(\xi) \in L^p_G(\Omega)$, where $\xi \in L^p_G(\Omega; \mathbb{R}^n)$ and $f$ is a bounded Borel measurable function on $\mathbb{R}^n$. In particular, whether $I_A(B_t) \in L^p_G(\Omega)$, where $A = [a, b]$ or $A = \{x \in \mathbb{R}^d : |x - x_0| \leq r\}$. The similar questions also exist for processes in $M^p_G(0, T)$ (see Section 2 for definition).

In this paper, we partly solve this kind of problems, but our results imply that the space $L^p_G(\Omega)$ (resp. $M^p_G(0, T)$) is big enough to contain some useful random variables (resp. processes). We first use PDE and probabilistic methods to obtain some polar sets associated to $X_t$, which is the solution to a multi-dimensional $G$-SDE. Then we apply these polar sets to obtain some quasi-continuous random variables. In particular, $I_{\{X_t \in [a, b]\}}$ is quasi-continuous. Next we give the characterization of $M^p_G(0, T)$, which is useful for the study of $G$-stochastic processes. Finally, we use Krylov’s estimates to get a kind of processes in $M^p_G(0, T)$. Moreover, we also obtain dominated convergence theorem and Itô-Krylov’s formula for the $G$-Itô processes.

This paper is organized as follows. In section 2, we recall some necessary notations and results of $G$-expectation theory. In section 3, we study the polar sets and give some useful quasi-continuous random variables. In section 4, we obtain the characterization of $M^p_G(0, T)$, state Krylov’s estimates of $G$-diffusion processes and establish Itô-Krylov’s formula under the $G$-expectation framework.

# 2 Preliminaries

The main purpose of this section is to recall some basic notions and results of $G$-expectation, which are needed in the sequel. The readers may refer to [11], [12], [13] [14] for more details.

Let $\Omega = C^0_0(\mathbb{R}^+)\subseteq \mathbb{R}$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \geq 0}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}\left(\max_{t \in [0, i]}|\omega^1_t - \omega^2_t|\right) \wedge 1.$$ 

For each $t \in [0, \infty)$, we denote

- $B_t(\omega) := \omega_t$ for each $\omega \in \Omega$;
• $\mathcal{B}(\Omega)$: the Borel $\sigma$-algebra of $\Omega$, $\Omega_t := \{\omega \land t : \omega \in \Omega\}$, $\mathcal{F}_t := \mathcal{B}(\Omega_t)$;

• $L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$-measurable real functions;

• $B_b(\Omega)$: all bounded elements in $L^0(\Omega)$; $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$;

• $C_b(\Omega)$: all continuous elements in $B_b(\Omega)$; $C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$;

• $L_{lip}(\Omega) := \{\varphi(B_{t_1}, \ldots, B_{t_k}) : k \in \mathbb{N}, t_1, \ldots, t_k \in [0, \infty), \varphi \in C_b, L_{lip}(\mathbb{R}^{k \times d})\}$, where $C_b, L_{lip}(\mathbb{R}^{k \times d})$ denotes the space of bounded and Lipschitz functions on $\mathbb{R}^{k \times d}$, $L_{lip}(\Omega_t) := L_{lip}(\Omega) \cap L^0(\Omega_t)$.

For each given monotonic and sublinear function $G : S(d) \to \mathbb{R}$, let the canonical process $B_t = (B_t^i)_{i=1}^d$ be the $d$-dimensional $G$-Brownian motion under the $G$-expectation space $(\Omega, L_{lip}(\Omega), \hat{E}[-], (\hat{E}[-])_{t \geq 0})$, where $S(d)$ denotes the space of all $d \times d$ symmetric matrices. For each $p \geq 1$, the completion of $L_{lip}(\Omega)$ under the norm $\|X\|_{L^p_G} := (\hat{E}|X|^p)^{1/p}$ is denoted by $L^p_G(\Omega)$. Similarly, we can define $L^p_{\omega}(\Omega, T)$ for each fixed $T \geq 0$. In this paper, we suppose that $G$ is non-degenerate, i.e., there exist two constants $0 < \underline{\sigma}^2 \leq \overline{\sigma}^2 < \infty$ such that

$$\frac{1}{2}\overline{\sigma}^2 \text{tr}[A - B] \leq G(A) - G(B) \leq \frac{1}{2}\underline{\sigma}^2 \text{tr}[A - B]$$
for $A \succeq B$.

From this we can deduce that $|G(A)| \leq \frac{1}{2}\overline{\sigma}^2 \sqrt{d} \sqrt{\text{tr}[AA^T]}$ for any $A \in S(d)$.

Denis et al. [1] proved that the completions of $C_b(\Omega)$ and $L_{lip}(\Omega)$ under $\| \cdot \|_{L^p_G}$ are the same.

**Theorem 2.1** ([1] [5]) There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega)$, the set of probability measures on $(\Omega, \mathcal{B}(\Omega))$, such that

$$\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi]$$
for all $\xi \in L^1_{\omega}(\Omega)$.

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{E}$. For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).$$

An important property of this capacity is that $c(F_n) \downarrow c(F)$ for any closed sets $F_n \downarrow F$.

A set $A \in \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$ q.s.

**Definition 2.2** A real function $X$ on $\Omega$ is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that $X|_{\Omega^c}$ is continuous.
Definition 2.3 We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ such that $X = Y$, q.s.

Theorem 2.4 (\cite{1, 5}) We have

\[ L^p_G(\Omega) = \{ X \in L^0(\Omega) : \lim_{N \to \infty} \hat{E}[|X|^p \mathbb{I}_{|X| \geq N}] = 0 \text{ and } X \text{ has a quasi-continuous version} \} \]

Theorem 2.5 (\cite{1, 5}) Let $X_k \in L^1_G(\Omega)$, $k \geq 1$, be such that $X_k \downarrow X$ q.s.. Then $\hat{E}[X_k] \downarrow \hat{E}[X]$. In particular, if $X \in L^1_G(\Omega)$, then $\hat{E}[|X_k - X|] \downarrow 0$.

Definition 2.6 Let $M^2_G(0, T)$ be the collection of processes of the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0, T]$,

\[ \eta(t) = \sum_{i=0}^{N-1} \xi_i(t) I_{[t_i, t_{i+1})}(t), \]

where $\xi_i \in L^p_G(\Omega_{t_i})$, $i = 0, 1, 2, \ldots, N - 1$. For each $p \geq 1$, denote by $M^p_G(0, T)$ the completion of $M^2_G(0, T)$ under the norm $||\eta||_{M^p_G} := (\hat{E}[\int_0^T |\eta|^p dt])^{1/p}$.

For each $\eta \in M^2_G(0, T)$, the G-Itô integral $\{ \int_0^t \eta_s dB_s \}_{t \in [0, T]}$ is well defined, see Peng \cite{12} and Li-Peng \cite{8}.

3 Quasi-continuous random variables

In this section, we shall find some sufficient conditions for some Borel measurable functions on $\Omega$ to be quasi-continuous by virtue of a PDE approach. We consider the following type of G-SDEs (in this paper we always use Einstein’s summation convention): for each given $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $q \leq i \leq n$,

\[ X^{i, i}_t = x_i + \int_0^t b_i(s, X^x_s) ds + \int_0^t h^i_j(s, X^x_s) dB^j_s + \int_0^t \sigma_i(s, X^x_s) dB_s, \]

where $b(t, x) = (b_1(t, x), \ldots, b_n(t, x))^T$, $h^i_k(t, x) = (h^i_1(t, x), \ldots, h^i_n(t, x))^T : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $h^i_k(t, x) = h^i_j(t, x)$, $\sigma(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are deterministic functions and $\sigma_i$ is the $i$-th row of $\sigma$. Denote by $X^x = (X^{x, i}_1, \ldots, X^{x, n}_n)^T$.

Then the above G-SDE can be written as

\[ X^x_t = x + \int_0^t b(s, X^x_s) ds + \int_0^t h^i_j(s, X^x_s) dB^j_s + \int_0^t \sigma(s, X^x_s) dB_s. \]

In this paper, we shall use the following assumptions on the coefficients of G-SDE (\textbf{H1}):

(\textbf{H1}) There exist two nonnegative constants $C$ and $C'$ such that for each $(t, x), (t, x') \in [0, \infty) \times \mathbb{R}^n$,

\[ |b(t, x) - b(t, x')| + |h^i_k(t, x) - h^i_k(t, x')| \leq C|x - x'|, \]

\[ |\sigma(t, x) - \sigma(t, x')| \leq C'|x - x'|; \]
(H2) \( b, h^{jk}, \sigma \) are continuous in \( t \);

(H3) There exist two constants \( 0 < \lambda < \Lambda < \infty \) such that for each \((t, x) \in [0, \infty) \times \mathbb{R}^n \),
\[
\lambda I_{n \times n} \leq \sigma(t, x)(\sigma(t, x))^T \leq \Lambda I_{n \times n} \text{ if } n \leq d, \\
\lambda I_{d \times d} \leq (\sigma(t, x))^T \sigma(t, x) \leq \Lambda I_{d \times d} \text{ if } n > d;
\]

(H4) There exists a constant \( L > 0 \) such that for each \((t, x) \in [0, \infty) \times \mathbb{R}^n \),
\[
|b_i(t, x)| \leq L, \\
|h^{jk}(t, x)| \leq L \text{ for } j, k \leq d \text{ and } i \leq n;
\]

(H5) There exist two constants \( 0 < \gamma < \Gamma < \infty \) such that for each \((t, x) \in [0, \infty) \times \mathbb{R}^n \),
\[
\gamma \leq |\sigma_i(t, x)|^2 = \sigma_i(t, x)(\sigma_i(t, x))^T \leq \Gamma \text{ for } i \leq n.
\]

Remark 3.1 If \( n \leq d \), then (H3) is stronger than (H5).

For each fixed \( t \geq 0 \) and for each given \( \xi \in L^2_G(\Omega_t; \mathbb{R}^n) \), consider the following G-SDE:
\[
X^{t, \xi}_s = \xi + \int_t^s b(r, X^{t, \xi}_r)dr + \int_t^s h^{jk}(r, X^{t, \xi}_r)d\langle B^j, B^k \rangle_r + \int_t^s \sigma(r, X^{t, \xi}_r)dB_r. \tag{2}
\]

Theorem 3.2 (\cite{13}) Assume (H1) and (H2) hold. Then G-SDE (2) has a unique solution \((X^{t, \xi}_s)_{s \in [t, T]} \in M^2_G(t, T; \mathbb{R}^n)\) for each \( T > t \).

For each fixed \( T > t \) and \( \Phi \in C_{b, \text{Lip}}(\mathbb{R}^n) \), we define
\[
Y^{t, \xi}_s = \hat{E}_s[\Phi(X^{t, \xi}_T)].
\]

In particular, for each \( x \in \mathbb{R}^n \), we set
\[
u(t, x) = Y^{t, x}_t.
\]

It is important to note that \( \nu(0, x) = \hat{E}[\Phi(X^{0, x}_T)] = \hat{E}[\Phi(X^{0, x}_T)] \).

Theorem 3.3 (\cite{4, 13}) Assume (H1) and (H2) hold. Then we have

(1) \( \nu(t, x) \) is a deterministic continuous function of \((t, x)\);

(2) For each \( \xi \in L^2_G(\Omega_t; \mathbb{R}^n) \), \( Y^{t, \xi}_t = \nu(t, \xi) \);

(3) \( \nu \) is the unique viscosity solution of the following PDE:
\[
\begin{cases}
\partial_t \nu + G(\sigma^T D^2_x \nu + H(D_x \nu, x, t)) + \langle b, D_x \nu \rangle = 0, & (t, x) \in [0, T) \times \mathbb{R}^n, \\
\nu(T, x) = \Phi(x),
\end{cases}
\]

where \( H_{jk} = 2\langle h^{jk}(t, x), D_x u \rangle \).
Lemma 3.4 Assume (H1), (H2) and (H3) hold. Let $T > 0$, $\alpha = (n \land d) \lambda \sigma^2 (8d\bar{\sigma}^2 \Lambda)^{-1}$, $\beta = (2d\bar{\sigma}^2 \Lambda)^{-1}$, $\varepsilon = (8\kappa)^{-1} \land T$, $m \geq 8\kappa$ and $u_m$ be the solution of PDE (3) with the terminal condition $u_m(T,x) = \exp(-\frac{m\beta|x-a|^2}{2})$, where $a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n$, $n \land d = \min\{n, d\}$.

$$\kappa = C(\bar{\sigma}^2 d\sqrt{d} + 1) + \delta_{a,T}(\bar{\sigma}^2 d\sqrt{d} + 1)^2 ((n \land d) \lambda \sigma^2)^{-1},$$

$$\delta_{a,T} = \max_{t \leq T}\{|h^j(t,a)|, |b(t,a)| : j, k = 1, \ldots, d\}.$$ Then for any $(t,x) \in [T - \varepsilon, T) \times \mathbb{R}^n$, we have

$$0 \leq u_m(t,x) \leq (1 + m(T-t))^{-\alpha}. \quad (4)$$

**Proof.** It is easy to check that $\bar{u}_m(t,x) = 0$ is a viscosity subsolution of PDE (3). Thus by comparison theorem we get $u_m(t,x) \geq 0$ for each $(t,x) \in [0,T] \times \mathbb{R}^n$. Set

$$\tilde{u}_m(t,x) = (1 + m(T-t))^{-\alpha} \exp(-\frac{m\beta|x-a|^2}{2(1 + m(T-t))}). \quad (5)$$

It is obvious that $\tilde{u}_m(T,x) = \exp(-\frac{m\beta|x-a|^2}{2})$. In the following, we show that $\tilde{u}_m$ is a viscosity supersolution of PDE (3) on $t \geq T - \varepsilon$. It is easy to verify that

$$\partial_t \tilde{u}_m = \frac{\alpha m}{1 + m(T-t)} \tilde{u}_m - \frac{m^2 \beta |x-a|^2}{2(1 + m(T-t))^2} \tilde{u}_m,$$

$$\partial_{x_i} \tilde{u}_m = \frac{m\beta(x_i - a_i)}{1 + m(T-t)} \tilde{u}_m,$$

$$\partial_{x_i x_i} \tilde{u}_m = -\frac{m\beta}{1 + m(T-t)} \tilde{u}_m + \frac{m^2 \beta |x_i - a_i|^2}{(1 + m(T-t))^2} \tilde{u}_m,$$

$$\partial_{x_i x_j} \tilde{u}_m = \frac{m^2 \beta (x_i - a_i)(x_j - a_j)}{(1 + m(T-t))^2} \tilde{u}_m, \quad i \neq j.$$ By the assumptions (H1)-(H3), we obtain that

$$G(-\sigma^T \sigma) \leq -\frac{\sigma^2}{2} \text{tr}[\sigma^T \sigma] \leq -\frac{1}{2}(n \land d) \lambda \sigma^2,$$

$$G(\sigma^T (x-a)(x-a)^T \sigma) \leq \frac{\sigma^2}{2} |x-a|^2 \text{tr}[\sigma^T \sigma] \leq \frac{1}{2} d\Lambda \sigma^2 |x-a|^2,$$

$$G((-\langle h^j(t,x), x-a \rangle)^d_{j,k=1}) \leq G((-\langle h^j(t,x) - h^j(t,a), x-a \rangle)^d_{j,k=1}) + G((-\langle h^j(t,a), x-a \rangle)^d_{j,k=1})$$

$$\leq \frac{1}{2} \sigma^2 d\sqrt{d}(C|x-a|^2 + \delta_{a,T}|x-a|),$$

$$-\langle b(t,x), x-a \rangle = -\langle b(t,x) - b(t,a) + b(t,a), x-a \rangle \leq C|x-a|^2 + \delta_{a,T}|x-a|. \quad (6)$$
Theorem 3.7  The proof is the same without any difficulty.

where \( f \) is a Lipschitz continuous function satisfying \( f_1(t, D_xu) + f_2(t, D_xu) = 0 \), is a Lipschitz continuous function satisfying \( f_1(t, 0) = 0 \). The proof is the same without any difficulty.

Remark 3.6  We remark that there is a potential to extend our results to a much more general nonlinear expectation setting. In particular, by slightly more involved estimates, our results still hold for the following PDE (see [3, 4]):

\[
\begin{align*}
\partial_t u &+ G(\sigma^T D_x^2 u)\sigma + H(D_x u, x, t) + f_1(t, D_x u) + \langle b, D_x u \rangle + f_2(t, D_x u) = 0, \\
u(T, x) & = \Phi(x),
\end{align*}
\]

where \( f_i, i = 1, 2 \), is a Lipschitz continuous function satisfying \( f_i(t, 0) = 0 \). The proof is the same without any difficulty.

Theorem 3.7  Under the same assumptions as in Lemma 3.4, we have for each \( T > 0 \)

\[
\mathbb{E} [\exp(-\frac{m\beta |X_T^a - a|^2}{2})] \leq (1 + m(T \wedge \varepsilon))^{-\alpha}. \tag{6}
\]

Furthermore, we have

\[
c(|\{X_T^a = a\}) = 0. \tag{7}
\]
Proof. If \( T \leq \varepsilon \), it follows form Lemma 3.4 and \( \hat{E}[\exp(-\frac{m\beta}{2}|X_T^x-a|^2)] = u_m(0, x) \) that \( \hat{E}[\exp(-\frac{m\beta}{2}|X_T^x-a|^2)] \leq (1 + mT)^{-\alpha} \). If \( T > \varepsilon \), by Theorem 3.3 and Lemma 3.4, we get that

\[
\hat{E}[\exp(-\frac{m\beta}{2}|X_T^x-a|^2)] = \hat{E}[\hat{E}_{T-\varepsilon}[\exp(-\frac{m\beta}{2}|X_{T-\varepsilon}^x-a|^2)]]
\]

\[
= \hat{E}[u_m(T - \varepsilon, X_{T-\varepsilon}^x)]
\]

\[
\leq \hat{E}[(1 + m\varepsilon)^{-\alpha}]
\]

\[
= (1 + m\varepsilon)^{-\alpha}.
\]

Thus we obtain equation (6). Note that \( \exp(-\frac{m\beta}{2}|X_T^x-a|^2) \geq I_{\{X_T^x=a\}} \), then

\[
c(\{X_T^x=a\}) \leq \hat{E}[\exp(-\frac{m\beta}{2}|X_T^x-a|^2)] \leq (1 + m(T \wedge \varepsilon))^{-\alpha}.
\]

Thus we can get \( c(\{X_T^x=a\}) = 0 \) by letting \( m \to \infty \). ■

Corollary 3.8 Assume \( b = h^k = 0 \) and (H1), (H2), (H3) hold. If \( \phi \) is a continuous function with a compact support on \( \mathbb{R}^n \), then

\[
\lim_{t \to \infty} \hat{E}[\phi(X_T^x)] = 0.
\]

Proof. Since \( \phi \) is a continuous function with a compact support, we can assume \( |\phi(x)| \leq MI_{\{|x| \leq N\}} \) for two constants \( N, M > 0 \). By the scaling property of \( G \)-Brownian motion, it is easy to check that \( X_T^x \) and \( \sqrt{t}X_1^{t,x} \) are identically distributed, where

\[
\hat{X}_1^{t,x} = \frac{x}{\sqrt{t}} + \int_0^1 \sigma(ts, \sqrt{t}\hat{X}_s^{t,x})dB_s.
\]

Note that \( |\phi(X_T^x)| \leq M I_{\{|X_T^x| \leq N\}} \). Consequently,

\[
\hat{E}[|\phi(X_T^x)|] \leq M \hat{E}[I_{\{|X_T^x| \leq N\}}] = M \hat{E}[I_{\{|X_1^{t,x}| \leq N\}}].
\]

By Remark 3.3 and Theorem 3.7, we obtain for each \( m \geq 0 \),

\[
\hat{E}[\exp(-\frac{m\beta}{2}|X_1^{t,x}|^2)] \leq \frac{1}{(1 + m)^\alpha}.
\]

Thus we get for each \( m \),

\[
\hat{E}[I_{\{|X_1^{t,x}| \leq \frac{N}{\sqrt{t}}\}}] \leq \exp(\frac{m\beta N^2}{2t})\hat{E}[\exp(-\frac{m\beta}{2}|X_1^{t,x}|^2)] \leq \exp(\frac{m\beta N^2}{2t})(1 + m)^\alpha.
\]

By equation (8) and sending \( t \to \infty \), we get

\[
\lim_{t \to \infty} \hat{E}[|\phi(X_T^x)|] \leq M \lim_{t \to \infty} \exp\left(\frac{m\beta N^2}{2t}\right) = \frac{M}{(1 + m)^\alpha}.
\]

Letting \( m \to \infty \), we conclude the desired result. ■
Corollary 3.9 Assume \( b = h^k = 0 \) and (H1), (H2), (H3) hold. Then for each \( t > 0, \ y \in \mathbb{R}^n \) and \( \epsilon > 0 \), we have

\[
c(|X_t^x - y| \leq \epsilon) \leq \exp\left(\frac{\beta \epsilon^2}{2} t^{-\alpha}\right),
\]

where \( \alpha = (n \wedge d)\lambda \gamma^2 (2d\bar{\sigma}^2\Lambda)^{-1}, \ \beta = (d\bar{\sigma}^2\Lambda)^{-1}. \) In particular,

\[
\lim_{\epsilon \downarrow 0} \sup_{y \in \mathbb{R}^d} c(|X_t^x - y| \leq \epsilon) = 0.
\]

Proof. By Remark 3.5 and Theorem 3.7, we obtain for each \( y \in \mathbb{R}^n \) and \( m \geq 0, \)

\[
\mathbb{E}[\exp(-\frac{m\beta|X_t^x - y|^2}{2})] \leq \frac{1}{(1 + mt)^\alpha}.
\]

Thus we get for each \( m \) and \( \epsilon > 0, \)

\[
\mathbb{E}[I_{\{|X_t^x - y| \leq \epsilon\}}] \leq \exp\left(\frac{m\beta\epsilon^2}{2}\right)\mathbb{E}[\exp(-\frac{m\beta|X_t^x - y|^2}{2})] \leq \exp\left(\frac{m\beta\epsilon^2}{2}\right) \frac{1}{(1 + mt)^\alpha}.
\]

In particular, taking \( m = \frac{1}{\epsilon^2}, \) we get for each \( y \in \mathbb{R}^n, \)

\[
c(|X_t^x - y| \leq \epsilon) \leq \exp\left(\frac{\beta \epsilon^2}{2} t^{-\alpha}\right),
\]

which completes the proof. \( \blacksquare \)

Remark 3.10 From the Corollary 3.9, we can obtain that for each \( t > 0, \ y \in \mathbb{R}^d \) and \( \epsilon > 0 \), we have

\[
c(|B_t - y| \leq \epsilon) \leq \exp\left(\frac{\beta \epsilon^2}{2} t^{-\alpha}\right),
\]

where \( \alpha = \frac{\gamma^2}{2d^2}, \ \beta = (d\bar{\sigma}^2)^{-1}. \)

Next we shall show that \( c(\{X_t^x; i = a_i\}) = 0 \) for any \( t > 0, \ i \leq n \) and \( a_i \in \mathbb{R}. \)

Lemma 3.11 Assume (H1), (H2), (H4) and (H5) hold. Let \( T > 0, \ \alpha = \gamma^2 (8\bar{\sigma}^2\Gamma)^{-1}, \ \beta = (2\bar{\sigma}^2\Gamma)^{-1}, \ \epsilon = (8\kappa)^{-1} \wedge T, \ m \geq 8\kappa \) and \( u_m \) be the solution of PDE (3) with terminal condition \( u_m(T, x) = \exp(-\frac{m\beta|x_i - a_i|^2}{2}), \) where \( a_i \in \mathbb{R}, \ i \leq n, \ \kappa = L^2(\bar{\sigma}^2d\sqrt{d+1})^2(\gamma \sigma^2)^{-1}. \) Then for any \( (t, x) \in [T - \epsilon, T) \times \mathbb{R}^n, \) we have

\[
0 \leq u_m(t, x) \leq (1 + m(T - t))^{-\alpha}.
\]

Proof. The proof of \( u_m(t, x) \geq 0 \) is the same as in Lemma 3.4. Set

\[
\tilde{u}_m(t, x) = (1 + m(T - t))^{-\alpha} \exp(-\frac{m\beta|x_i - a_i|^2}{2(1 + m(T - t))}).
\]
Furthermore, we have \( \hat{u}_m(T, x) = \exp\left(-\frac{m\beta|x_i - a_i|^2}{2}\right) \). In the following, we show that \( \hat{u}_m \) is a viscosity supersolution of PDE (3) on \( t \geq T - \varepsilon \). It is easy to verify that

\[
\begin{align*}
\partial_t \hat{u}_m &= \frac{\alpha m}{1 + m(T - t)} \hat{u}_m - \frac{m^2 \beta |x_i - a_i|^2}{2(1 + m(T - t))^2} \hat{u}_m, \\
\partial_{x_i} \hat{u}_m &= -\frac{m \beta (x_i - a_i)}{1 + m(T - t)} \hat{u}_m, \\
\partial_{x_i,x_i}^2 \hat{u}_m &= -\frac{m \beta}{1 + m(T - t)} \hat{u}_m + \frac{m^2 \beta^2 |x_i - a_i|^2}{(1 + m(T - t))^2} \hat{u}_m, \\
\partial_{x_j} \hat{u}_m &= 0, \quad \partial_{x_i,x_j}^2 \hat{u}_m = 0, \quad j \neq i, \\
\sigma^T D^2 \hat{u}_m \sigma &= (\partial_{x_i,x_i}^2 \hat{u}_m) \sigma_i^T \sigma_i, \\
G(-\sigma_i^T \sigma_i) &\leq -\frac{\bar{\sigma}_i^2}{2}; \quad G(\sigma_i^T \sigma_i) \leq \frac{\bar{\sigma}_i^2}{2}, \\
(h^{jk}(t, x), D_x \hat{u}_m)^{d}_{j,k=1} = (\partial_{x_i} \hat{u}_m)(h^{jk}_i(t, x))^{d}_{j,k=1}.
\end{align*}
\]

Then for each \( (t, x) \in [T - \varepsilon, T) \times \mathbb{R}^n \), we have

\[
\begin{align*}
\partial_t \hat{u}_m + G(\sigma^T D^2 \hat{u}_m \sigma + (2(h^{jk}(t, x), D_x \hat{u}_m)^{d}_{j,k=1}) + \langle b, D_x \hat{u}_m \rangle \\
\leq \partial_t \hat{u}_m + \frac{m \beta \hat{u}_m}{1 + m(T - t)} G(-\sigma_i^T \sigma_i) + \frac{m^2 \beta^2 \hat{u}_m |x_i - a_i|^2}{(1 + m(T - t))^2} G(\sigma_i^T \sigma_i) \\
+ \frac{2m \beta \hat{u}_m}{1 + m(T - t)} G(\langle -(x_i - a_i) h^{jk}_i(t, x) \rangle^{d}_{j,k=1}) - \frac{m \beta \hat{u}_m}{1 + m(T - t)} (x_i - a_i) b_i \\
\leq -\frac{m \beta \hat{u}_m}{1 + m(T - t)} |x_i - a_i|^2 \left(\frac{m}{4(1 + m\varepsilon)} - \kappa\right) \\
\leq 0,
\end{align*}
\]

which implies that \( \hat{u}_m \) is a viscosity supersolution of PDE (3) on \( t \geq T - \varepsilon \). Thus by comparison theorem we obtain for \( (t, x) \in [T - \varepsilon, T) \times \mathbb{R}^n \),

\[
u_m(t, x) \leq \hat{u}_m(t, x) \leq (1 + m(T - t))^{-\alpha}.
\]

The proof is complete. \( \blacksquare \)

**Remark 3.12** From the above proof, we know that the above result still holds if assumptions (H4) and (H5) just hold for some \( i \).

**Theorem 3.13** Under the same assumptions as in Lemma 3.11, we obtain that for each \( i \leq n \) and \( T > 0 \)

\[
\mathbb{E}[\exp\left(-\frac{m \beta |X^x_{T,i} - a_i|^2}{2}\right)] \leq (1 + m(T \wedge \varepsilon))^{-\alpha}.
\] (11)

Furthermore, we have

\[
c(\{X^x_T = a_i\}) = 0.
\] (12)
Proof. The proof is the same as in Theorem 3.7 and we omit it. ■

Finally, we shall study the capacity of a curve.

Theorem 3.14 Assume (H1), (H2), (H4) and (H5) hold. Suppose \( f \) satisfies \( \partial_x f, \partial^2_{x,x_j} f \in \mathcal{C}_{\text{b}, \text{Lip}}(\mathbb{R}^n) \) and there exist two constants \( 0 < \delta \leq \Delta < \infty \) such that

\[
\delta \leq \left| \sum_{i=1}^n \partial_x f \sigma^i \right|^2 \leq \Delta.
\]

Then for each \( T > 0 \) we have

\[
c(\{f(X^x_T) = 0\}) = 0.
\]

Proof. Applying the \( G \)-Itô formula yields that

\[
f(X^x_T) = f(x) + \int_0^T \partial_x f b_i(s, X^x_s) ds + \int_0^T \left[ \partial_x f h_{j}^k + \frac{1}{2} \partial^2_{x,x_i} f \sigma^i \sigma^k \right] (s, X^x_s) d(B^j, B^k)_s
\]

\[
+ \int_0^T \partial_x f \sigma^i (s, X^x_s) dB_s.
\]

Thus \( \tilde{X}^x_t = ((X^x_t)^T, f(X^x_t))^T \) can be seen as the solution to the \( G \)-SDE (1) with

\[
\tilde{b}(t, x) = \begin{pmatrix} b(t, x) \\ \partial_x f b_i(t, x) \end{pmatrix}, \tilde{\sigma}(t, x) = \begin{pmatrix} \sigma(t, x) \\ \partial_x f \sigma^i(t, x) \end{pmatrix}
\]

and

\[
\tilde{h}^{jk}(t, x) = \begin{pmatrix} h_{j}^k(t, x) \\ [\partial_x f h_{j}^k(t, x) + \frac{1}{2} \partial^2_{x,x_i} f \sigma^i \sigma^k(t, x)] \end{pmatrix}.
\]

It follows from Theorem 3.13 that \( c(\{f(X^x_T) = 0\}) = 0 \) and this completes the proof. ■

Remark 3.15 It is very difficult to find a supersolution for a curve. Here we deal with this question by a probabilistic method instead of a PDE approach.

Remark 3.16 If we take \( n = d, x = 0, b = 0, h^j = 0 \) and \( \sigma = I_{d \times d} \), then \( X^x_T = B_T \). Thus the above results still hold for \( G \)-Brownian motion. In particular, [17] studies some sample path properties of 1-dimensional \( G \)-Brownian motion.

Example 3.17 If we take \( n = 2, d = 1, x = 0, b = 0, h^j = 0, \sigma = (1, -1)^T \) and \( f(x, y) = x - y \). Then \( f(B_T, B_T) = 0, q.s. \). However \( \partial_x f \sigma^1 + \partial_y f \sigma^2 = 0 \).

3.2 Some applications

In this subsection, we shall prove that some Borel measurable functions on \( \Omega \) are quasi-continuous. In the sequel, we always assume (H1), (H2), (H4) and (H5) hold.
Theorem 3.18 Let $\xi \in L^1_G(\Omega; \mathbb{R}^k)$ and $A \in \mathcal{B}(\mathbb{R}^k)$ with $c(\{\xi \in \partial A\}) = 0$. Then $I_{\{\xi \in A\}} \in L^1_G(\Omega)$.

**Proof.** For each $\epsilon > 0$, since $\xi \in L^1_G(\Omega; \mathbb{R}^k)$, we can find an open set $O \subset \Omega$ with $c(O) \leq \frac{\epsilon}{2}$ such that $\xi|_O$ is continuous. Set $D_\epsilon = \{x \in \mathbb{R}^k : d(x, \partial A) \leq \frac{\epsilon}{2}\}$ and $A_\epsilon = \{x \in \mathbb{R}^k : d(x, \partial A) < \frac{\epsilon}{2}\}$, it is easy to check that $\{\xi \in D_\epsilon\} \cap O^c$ is closed, $\{\xi \in A_\epsilon\} \subset \{\xi \in D_\epsilon\}$ and $\{\xi \in D_\epsilon\} \cap O^c \subset \{\xi \in \partial A\} \cap O^c$. Then we have

$$c(\{\xi \in D_\epsilon\} \cap O^c) = c(\{\xi \in \partial A\} \cap O^c) = 0.$$  

Thus we can find an $i_0$ such that $c(\{\xi \in A_{i_0}\} \cap O^c) \leq \frac{\epsilon}{2}$. Set $O_1 = \{\xi \in A_{i_0}\} \cup O$, it is easy to verify that $c(O_1) \leq \epsilon$, $O^c_1 = \{\xi \in A_{i_0}^c\} \cap O^c$ is closed and $I_{\{\xi \in A\}}$ is continuous on $O^c_1$. Thus $I_{\{\xi \in A\}}$ is quasi-continuous, which implies $I_{\{\xi \in A\}} \in L^1_G(\Omega)$. $\blacksquare$

We first consider the capacity of $X^t_s\xi$ on the boundary of cubes. Then by the above theorem, we can get a kind of quasi-continuous random variables associated to $G$-SDEs.

**Lemma 3.19** Let $A = [a, b]$, where $a, b \in \mathbb{R}^n$ with $a \leq b$. Then $c(\{X^t_s \in \partial A\}) = 0$ for any $t > 0$ and $x \in \mathbb{R}^n$.

**Proof.** It is easy to verify that

$$\{X^t_s \in \partial A\} \subset \bigcup_{i=1}^n (\{X^t_s;i = a_i\} \cup \{X^t_s;i = b_i\}).$$

Thus by Theorem 3.13 we can get $c(\{X^t_s \in \partial A\}) = 0$. $\blacksquare$

**Lemma 3.20** Let $A = [a, b]$, where $a, b \in \mathbb{R}^n$ with $a \leq b$. Then for each given $t \geq 0$, $\xi \in L^2_G(\Omega_t; \mathbb{R}^n)$, $s > t$, we have $c(\{X^t_s\xi \in \partial A\}) = 0$.

**Proof.** We can choose a sequence $\varphi_k \in C_{b, lip}(\mathbb{R}^n)$, $k \geq 1$, such that $\varphi_k \downarrow I_{\partial A}$. By Theorem 3.3 we have

$$\tilde{E}[\varphi_k(X^t_s\xi)] = \tilde{E}[\tilde{E}[\varphi_k(X^t_s\xi)|x = \xi]] = \tilde{E}[\phi_k(\xi)],$$

where $\phi_k(x) = \tilde{E}[\varphi_k(X^t_s\xi)|x = x]$. For each fixed $x \in \mathbb{R}^n$, by Theorem 2.5 and Lemma 3.19 we obtain

$$\tilde{E}[\varphi_k(X^t_s\xi)|x = x] \leq \tilde{E}[\tilde{E}[\varphi_k(X^t_s\xi)|x = x]] = c(\{X^t_s\xi \in \partial A\}) = 0,$$

which implies $\phi_k(\xi) \leq 0$. By Theorem 2.5 we get $\tilde{E}[\phi_k(\xi)] \downarrow 0$. Note that $c(\{X^t_s\xi \in \partial A\}) \leq \tilde{E}[\varphi_k(X^t_s\xi)]$, then we get $c(\{X^t_s\xi \in \partial A\}) = 0$. $\blacksquare$

**Theorem 3.21** Let $A_i = [a^i, b^i]$ with $a^i \leq b^i$ for $i \geq 1$ and $D \in \mathcal{B}(\mathbb{R}^n)$ with $\partial D \subset \bigcup_{i=1}^\infty \partial A_i$. Then for each given $t \geq 0$, $\xi \in L^2_G(\Omega_t; \mathbb{R}^n)$, $s > t$, we have $I_{\{X^t_s\xi \in D\}} \in L^1_G(\Omega_s)$. In particular, $I_{\{X^t_s \in \partial D\}} \in L^1_G(\Omega_s)$. 

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Proof. This is a direct consequence of Lemma 3.20 and Theorem 3.18.

In the following, we only consider the capacity of $B_t$ on the spheres for simplicity. But the method can be extended to $X_t$.

**Lemma 3.22** Let $D$ be a $d$-dimensional sphere. Then we have for each $t > 0$,

$$c(\{B_t \in \partial D\}) = 0.$$  

**Proof.** Without loss of generality, we assume $D$ is the unit sphere. Set $\bar{x} = (x_1, \ldots, x_{d-1})$ and denote functions

$$f(\bar{x}) := \sqrt{1 - |\bar{x}|^2} I_{\{|\bar{x}|^2 \leq 1\}}.$$  

For each $\epsilon > 0$, there exists a nonnegative function $J^*(\bar{x}) \in C_0^\infty(\mathbb{R}^{d-1})$ such that

$$J^*(\bar{x}) = \begin{cases} 1, & \text{if } |\bar{x}| \leq 1 - 2\epsilon; \\ 0, & \text{if } |\bar{x}| \geq 1 - \epsilon. \end{cases}$$  

Then define function $f^*(x) := x_d - J^*(\bar{x}) f(\bar{x})$. It is easy to check that $J^*(\bar{x}) f(\bar{x}) \in C_0^\infty(\mathbb{R}^{d-1})$. Moreover, $|\sum_{i=1}^d \partial_x f^*(x) e_i|^2 = \sum_{i=1}^{d-1} |\partial_x f^*(x)|^2 + 1$. Then applying Theorem 3.14, we obtain for each given $t > 0$,

$$c(\{B_t^d - J^*(\bar{B}_t) f(\bar{B}_t) = 0\}) = 0,$$

where $\bar{B}_t = (B_t^1, \ldots, B_t^{d-1})$. Consequently,

$$c(\{B_t^d - f(\bar{B}_t) = 0\} \cap \{|\bar{B}_t|^2 \leq 1 - 2\epsilon\}) = 0.$$  

As $\{B_t^d - f(\bar{B}_t) = 0\} \cap \{|\bar{B}_t|^2 \leq 1 - 2\epsilon\} \cup \{B_t^d - f(\bar{B}_t) = 0\} \cap \{|\bar{B}_t|^2 < 1\}$, then by taking $\epsilon \downarrow 0$ we get that

$$c(\{B_t^d - f(\bar{B}_t) = 0\} \cap \{|\bar{B}_t|^2 < 1\}) = 0.$$  

From Theorem 3.13 we get $c(\{B_t^d = 0\}) = 0$. Therefore, we deduce that

$$c(\{B_t^d - f(\bar{B}_t) = 0\}) \leq c(\{B_t^d - f(\bar{B}_t) = 0\} \cap \{|\bar{B}_t|^2 < 1\}) + c(\{B_t^d = 0\} = 0).$$  

By a similar analysis, we also get $c(\{B_t^d + f(\bar{B}_t) = 0\}) = 0$. Thus

$$c(\{B_t \in \partial D\}) \leq c(\{B_t^d - f(\bar{B}_t) = 0\}) + c(\{B_t^d + f(\bar{B}_t) = 0\}) = 0,$$

which is the desired result.

**Remark 3.23** For each $0 \leq t_1 \leq t_2 < \infty$, by Lemmas 3.20 and 3.22, we can also obtain that $c(\{|B_{t_1}|^2 + |B_{t_2}|^2 = 1\}) = 0$.

The following result is a direct consequence of Lemmas 3.19 and 3.22.

**Theorem 3.24** Suppose $A_t$ is a $d$-dimensional sphere or $|a^i, b^i|$ with $a^i$, $b^i \in \mathbb{R}^d$, $a^i \leq b^i$ for $i \geq 1$ and $D \in \mathcal{B}(\mathbb{R}^d)$ with $\partial D \subset \cup_{i=1}^\infty \partial A_i$. Then $I_{\{B_t \in D\}} \in L_2(\Omega_t)$ for any $t > 0$.
4 Quasi-continuous processes

In this section, we first consider the characterization of $M_p^G(0, T)$. And then we study the applications of Krylov’s estimates under $G$-expectation. In particular, we show that some Borel measurable functions on $[0, T] \times \Omega$ are in $M_p^G(0, T)$, $p \geq 1$.

4.1 Characterization of $M_p^G(0, T)$

We shall give a characterization of the space $M_p^G(0, T)$ for each $T > 0$ and $p \geq 1$, which generalizes the results in [1].

Set $F_t = B(\Omega_t)$ for $t \in [0, T]$ and the distance

$$\rho((t, \omega), (t', \omega')) = |t - t'| + \max_{s \in [0, T]} |\omega_s - \omega'_s|$$

for $(t, \omega), (t', \omega') \in [0, T] \times \Omega_T$.

Define, for each $p \geq 1$,

$$M_p^G(0, T) = \{ \eta : \text{progressively measurable on } [0, T] \times \Omega_T \text{ and } \hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt] < \infty \}$$

and the corresponding capacity

$$\hat{c}(A) = \frac{1}{T} \hat{\mathbb{E}}[\int_0^T I_A(t, \omega) dt]$$

for each progressively measurable set $A \subset [0, T] \times \Omega_T$.

**Proposition 4.1** Let $A$ be a progressively measurable set in $[0, T] \times \Omega_T$. Then $I_A = 0$ $\hat{c}$-q.s. if and only if $\int_0^T I_A(t, \cdot) dt = 0$ c-q.s.

**Proof.** It is obvious $\int_0^T I_A(t, \cdot) dt \geq 0$. Thus we can easily get $\hat{\mathbb{E}}[\int_0^T I_A(t, \omega) dt] = 0$ if and only if $c(\{ \int_0^T I_A(t, \cdot) dt > 0 \}) = 0$, which completes the proof. ■

In the following, we do not distinguish two progressively measurable processes $\eta$ and $\eta'$ if $\hat{c}(\{ \eta \neq \eta' \}) = 0$.

**Proposition 4.2** For each $p \geq 1$, $M_p^G(0, T)$ is a Banach space under the norm

$$||\eta||_{M_p} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}.$$

**Proof.** The proof is similar to the classical results and we omit it. ■

It is clear that $M_p^G(0, T) \subset M_p^G(0, T)$ for any $p \geq 1$. Thus $M_p^G(0, T)$ is a closed subspace of $M_p^G(0, T)$. Also we set

$$M_c(0, T) = \{ \text{all adapted processes } \eta \text{ in } C_b([0, T] \times \Omega_T) \}.$$
Theorem 4.7
For each \( R \) has a quasi-continuous version if there exists a quasi-continuous process \( \hat{\eta} \) such that

\[
\text{in } [0, T] \text{ the one in [15, 16]}. \]

Remark 4.5
This definition of quasi-continuous of a process is different from

Thus \( \eta \) is quasi-continuous (q.c.) if for each \( \varepsilon > 0 \), since \( \mathcal{P} \) is weakly compact, there exists a compact set \( K \subset \Omega_T \) such that \( \mathbb{E}[I_{K^c}] \leq \varepsilon \).

Therefore

\[
\mathbb{E}\left[\int_0^T |\eta - \eta_k^p| \, dt\right] \leq \mathbb{E}[I_K] \int_0^T |\eta - \eta_k^p|^p \, dt + \mathbb{E}[I_{K^c}] \int_0^T |\eta - \eta_k^p|^p \, dt \\
\leq \sup_{(t, \omega) \in [0, T] \times K} T|\eta_t(\omega) - \eta_k^p(\omega)|^p + (2l)^p T \varepsilon,
\]

where \( l \) is the upper bound of \( \eta \). Noting that \( [0, T] \times K \) is compact and \( \eta \in C_b([0, T] \times \Omega_T) \), thus

\[
\lim_{k \to \infty} \mathbb{E}\left[\int_0^T |\eta - \eta_k^p| \, dt\right] \leq (2l)^p T \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we get \( \|\eta_k - \eta\|_{\mathcal{M}^p} \to 0 \) as \( k \to \infty \). Thus \( \eta \in \mathcal{M}_G^0(0, T) \), which implies the desired result.

Proof. We first prove that the completion of \( \mathcal{M}_c(0, T) \) under the norm \( \|\cdot\|_{\mathcal{M}^p} \) belongs to \( \mathcal{M}_G^0(0, T) \). For each fixed \( \varepsilon > 0 \), since \( \mathcal{P} \) is weakly compact, there exists a compact set \( K \subset \Omega_T \) such that \( \mathbb{E}[I_{K^c}] \leq \varepsilon \).

\[
\mathbb{E}\left[\int_0^T |\eta - \eta_k^p| \, dt\right] \leq \mathbb{E}[I_K] \int_0^T |\eta - \eta_k^p|^p \, dt + \mathbb{E}[I_{K^c}] \int_0^T |\eta - \eta_k^p|^p \, dt \\
\leq \sup_{(t, \omega) \in [0, T] \times K} T|\eta_t(\omega) - \eta_k^p(\omega)|^p + (2l)^p T \varepsilon,
\]

where \( l \) is the upper bound of \( \eta \). Noting that \( [0, T] \times K \) is compact and \( \eta \in C_b([0, T] \times \Omega_T) \), thus

\[
\lim_{k \to \infty} \mathbb{E}\left[\int_0^T |\eta - \eta_k^p| \, dt\right] \leq (2l)^p T \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we get \( \|\eta_k - \eta\|_{\mathcal{M}^p} \to 0 \) as \( k \to \infty \). Thus \( \eta \in \mathcal{M}_G^0(0, T) \), which implies the desired result.

Now we prove the converse part. For each given \( \eta_k = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(t) \in \mathcal{M}_G^0(0, T) \), it is easy to find \( \{\phi_i^k : k \geq 1\} \subset C([0, \infty)) \), \( i < N \), \( k \geq 1 \) such that \( supp(\phi_i^k) \subset (t_i, t_{i+1}) \) and \( \int_0^T |\phi_i^k(t) - I_{[t_i, t_{i+1})}(t)|^p \, dt \to 0 \) as \( k \to \infty \). Set \( \eta^k = \sum_{i=0}^{N-1} \xi_i \phi_i^k(t) \), it is easy to check that \( \eta^k \in \mathcal{M}_c(0, T) \) and \( \|\eta^k - \eta\|_{\mathcal{M}^p} \to 0 \) as \( k \to \infty \). Thus \( \mathcal{M}_G^0(0, T) \) belongs to the completion of \( \mathcal{M}_c(0, T) \) under the norm \( \|\cdot\|_{\mathcal{M}^p} \), which completes the result. \( \blacksquare \)

Definition 4.4 A progressively measurable process \( \eta : [0, T] \times \Omega_T \to \mathbb{R} \) is called quasi-continuous (q.c.) if for each \( \varepsilon > 0 \), there exists a progressive open set \( G \) in \( [0, T] \times \Omega_T \) such that \( \mathbb{P}(G) < \varepsilon \) and \( \eta|_{\partial G} \) is continuous.

Remark 4.5 This definition of quasi-continuous of a process is different from the one in [10].

Definition 4.6 We say that a progressively measurable process \( \eta : [0, T] \times \Omega_T \to \mathbb{R} \) has a quasi-continuous version if there exists a quasi-continuous process \( \eta' \) such that \( \mathbb{P}(\{\eta \neq \eta'\}) = 0 \).

Theorem 4.7 For each \( p \geq 1 \),

\[
\mathcal{M}_G^p(0, T) = \{\eta \in \mathcal{M}^p(0, T) : \lim_{N \to \infty} \mathbb{E}\left[\int_0^T |\eta| \, dt \right] = 0 \text{ and } \eta \text{ has a quasi-continuous version}\}.
\]
\textbf{Proof.} We denote

\[ J_p = \{ \eta \in M^p(0,T) : \lim_{N \to \infty} \hat{E}\left[ \int_0^T |\eta_t|^p I_{\{|\eta_t| \geq N\}} dt \right] = 0 \} \]

\eta \text{ has a quasi-continuous version}. \]

Noting that the completion of \( M_c(0,T) \) under the norm \( \| \cdot \|_{M^p_c} \) is \( M^p_G(0,T) \), then, by the same analysis as in Propositions 18 and 24 in [1], we can get \( M^p_G(0,T) \subset J_p \).

On the other hand, for each \( \eta \in J_p \), we want to prove that \( \eta \in M^p_G(0,T) \).

We may assume that \( \eta \) is quasi-continuous. For each \( N > 0 \), set \( \eta^N = (\eta \wedge N) \vee (-N) \), since \( \hat{E}\left[ \int_0^T |\eta_t - \eta_t^N|^p dt \right] \leq \hat{E}\left[ \int_0^T |\eta_t|^p I_{\{|\eta_t| \geq N\}} dt \right] \to 0 \) as \( N \to \infty \), we only need to show that \( \eta^N \in M^p_G(0,T) \) for each fixed \( N > 0 \). For each \( \varepsilon > 0 \), there exist a compact set \( K_\varepsilon \subset \Omega_T \) such that \( \hat{E}[I_{K_\varepsilon}] \leq \varepsilon \) and a progressive open set \( G_\varepsilon \subset [0,T] \times \Omega_T \) such that \( \hat{E}(G_\varepsilon) < \varepsilon \) and \( \eta^N|_{G_\varepsilon} \) is continuous. By Tietze’s extension theorem, there exists an \( \hat{\eta}^{N,\varepsilon} \in C_b([0,T] \times \Omega_T) \) such that \( |\hat{\eta}^{N,\varepsilon}| \leq N \) and \( \hat{\eta}^{N,\varepsilon}|_{G_\varepsilon} = \eta^N|_{G_\varepsilon} \). For each \( k \geq 1 \), set \( t^k_i = \frac{i}{k}T \) for \( i = 0, \ldots, k \). We also set \( F^{i,k}_\varepsilon = G^\varepsilon \cap ([t^k_i, t^{k+1}_i] \times \Omega_T) \) for \( i \leq k-1 \). Since \( G^\varepsilon \) is progressively measurable, we can get \( F^{i,k}_\varepsilon \in B([0,t^{k+1}_i]) \times B(\Omega_{t^{k+1}_i}) \). It is easy to check that \( F^{i,k}_\varepsilon \) is closed.

By Tietze’s extension theorem, there exists a \( \zeta^{N,i,k} \in C_b([0,t^{k+1}_i] \times \Omega_T) \) such that \( \zeta^{N,i,k} \in B([0,t^{k+1}_i]) \times B(\Omega_{t^{k+1}_i}) \), \( |\zeta^{N,i,k}| \leq N \) and \( \zeta^{N,i,k}|_{F^{i,k}_\varepsilon} = \eta^N|_{F^{i,k}_\varepsilon} \). We denote \( \tilde{\eta}^{N,i,k}_t(\omega) = \sum_{i=0}^{k-1} \zeta^{N,i,k}(t,\omega)I_{[t^k_i, t^{k+1}_i]}(t) \) and

\[ \hat{\eta}^{N,i,k}_t = \hat{\eta}^{N,k}_t(t - \frac{T}{k},\omega)I_{[t^k_i, t^{k+1}_i]}(t), \quad \tilde{\eta}^{N,i,k}_t = \tilde{\eta}^{N,k}_t(t - \frac{T}{k},\omega)I_{[t^k_i, t^{k+1}_i]}(t). \]

Similar to the analysis as in Proposition [4,3], we can get \( \hat{\eta}^{N,i,k}_t \in M^p_G(0,T) \). We have also

\[ \hat{E}\left[ \int_0^T |\eta^N_t - \hat{\eta}^{N,i,k}_t|^p dt \right] \]

\[ \leq 3^{p-1} \hat{E}\left[ \int_0^T |\eta^N_t - \tilde{\eta}^{N,i,k}_t|^p dt \right] + \hat{E}\left[ \int_0^T |\tilde{\eta}^{N,i,k}_t - \hat{\eta}^{N,i,k}_t|^p dt \right] + \hat{E}\left[ \int_0^T |\hat{\eta}^{N,i,k}_t - \tilde{\eta}^{N,i,k}_t|^p dt \right] \]

\[ \leq 3^{p-1} \left( 2(2N)^pT \varepsilon + \hat{E}\left[ \int_0^T |\eta^N_t - \tilde{\eta}^{N,i,k}_t|^p dt \right] \right) \]

\[ \leq 3^{p-1} \left( 2(2N)^pT \varepsilon + (2N)^pT \varepsilon + \hat{E}\left[ \int_0^T |\eta^N_t - \tilde{\eta}^{N,i,k}_t|^p dt \right] \right) \]

\[ \leq 3^{p-1} \left( 2(2N)^pT \varepsilon + (2N)^pT \varepsilon + \sup_{(t,\omega) \in [0,T] \times \Omega} \|T|\hat{\eta}^{N,i,k}_t(t,\omega) - \tilde{\eta}^{N,i,k}_t(t,\omega)|^p \right). \]

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Noting that \([0, T] \times K_\varepsilon\) is compact and \(\tilde{\eta}^N, \varepsilon \in C_b([0, T] \times \Omega_T)\), thus
\[
\limsup_{k \to \infty} \hat{\varepsilon}\int_0^T |\eta_t^N - \tilde{\eta}_t^{N,k}|^p dt \leq (6N)^p T \varepsilon,
\]
which implies \(\eta^N \in M^p_G(0, T)\). The proof is complete. \(\blacksquare\)

**Corollary 4.8** Let \(\eta \in M^1_G(0, T)\) and \(f \in C_b([0, T] \times \mathbb{R})\). Then \((f(t, \eta_t))_{t \leq T} \in M^p_G(0, T)\) for any \(p \geq 1\).

**Theorem 4.9** Let \(\eta^k \in M^1_G(0, T)\), \(k \geq 1\), be such that \(\eta^k \downarrow \eta\) \(\hat{\varepsilon}\)-q.s.. Then \(\hat{\mathbb{E}}[\int_0^T \eta^k_t dt] \downarrow \hat{\mathbb{E}}[\int_0^T \eta_t dt]\). Moreover, if \(\eta \in M^1_G(0, T)\), then \(\hat{\mathbb{E}}[\int_0^T |\eta^k_t - \eta_t| dt] \downarrow 0\).

**Proof.** Since \(\eta^k \in M^1_G(0, T)\), we can choose \(\eta^{k,N} \in M^1_G(0, T)\) such that \(\hat{\mathbb{E}}[\int_0^T |\eta^k_t - \eta^{k,N}_t| dt] \to 0\) as \(N \to \infty\). It is easy to check that \(\int_0^T \eta^{k,N}_t \in L^1_G(\Omega_T)\) and \(\hat{\mathbb{E}}[\int_0^T \eta^{k,N}_t dt] \leq \hat{\mathbb{E}}[\int_0^T |\eta^k_t - \eta^{k,N}_t| dt]\). Thus \(\int_0^T \eta^k_t \in L^1_G(\Omega_T)\) for \(k \geq 1\). By Proposition 4.1 and Theorem 4.7, it is easy to verify that \(\int_0^T \eta^k_t dt \downarrow \int_0^T \eta_t dt\) \(\hat{\varepsilon}\)-q.s.. Thus by Theorem 2.5 we get \(\hat{\mathbb{E}}[\int_0^T |\eta^k_t - \eta_t| dt] \downarrow 0\). If \(\eta \in M^1_G(0, T)\), then \(|\eta^k - \eta| \in M^1_G(0, T)\) and \(|\eta^k - \eta| \downarrow 0\) \(\hat{\varepsilon}\)-q.s.. Thus \(\hat{\mathbb{E}}[\int_0^T |\eta^k_t - \eta_t| dt] \downarrow 0\). \(\blacksquare\)

The following example shows that \(M^p_G(0, T)\) is strictly contained in \(M^p(0, T)\).

**Example 4.10** Suppose \(0 < \sigma^2 < \bar{\sigma}^2 < \infty\), \(T > 0\). We consider 1-dimensional G-Brownian motion \((B_t)_{t \geq 0}\). \((B_t)_{t \geq 0}\) is the quadratic process of \((B_t)_{t \geq 0}\). Let
\[
\eta_t = I_{\{(B_t) = \sigma^2 + \bar{\sigma}^2 \}}\text{ for } t \leq T.
\]

In the following we show that \(\eta \notin M^1_G(0, T)\). We can choose \(f^k(t, x) \in C_b([0, T] \times \mathbb{R})\), \(k \geq 1\), such that
\[
f^k(t, x) = 1 \text{ for } |x - \left(\frac{\sigma^2 + \bar{\sigma}^2}{2}\right)t| \leq \frac{T}{k}, f^k(t, x) = 0 \text{ for } |x - \left(\frac{\sigma^2 + \bar{\sigma}^2}{2}\right)t| \geq \frac{2T}{k}.
\]
Set \(g^k = \bigwedge_{i=1}^k f^i\), it is easy to check that \(g^k \in C_b([0, T] \times \mathbb{R})\), \(g^k(t, x) = 1 \text{ for } |x - \left(\frac{\sigma^2 + \bar{\sigma}^2}{2}\right)t| \leq \frac{T}{k}\) and \(g^k \downarrow I_{\{x = \left(\frac{\sigma^2 + \bar{\sigma}^2}{2}\right)t\}}\). Thus \(g^k(t, (B_t)) \downarrow \eta_t\). By Corollary 4.8, we have \(g^k(t, (B_t)) \in M^1_G(0, T)\). If \(\eta \in M^1_G(0, T)\), then by Theorem 4.9 we get \(\hat{\mathbb{E}}[\int_0^T |g^k(t, (B_t)) - \eta_t| dt] \downarrow 0\). On the other hand, by the representation of \(\hat{\mathbb{E}}[\cdot]\) in (4.1), there exists a probability measure \(P \in \mathcal{P}\) such that \(\langle B_t \rangle = ((\sigma^2 + \bar{\sigma}^2) - \frac{1}{k}) \wedge \sigma^2)t\) \(P\)-a.s.. Thus \(\hat{\mathbb{E}}[\int_0^T |g^k(t, (B_t)) - \eta_t| dt] \geq E_P[\int_0^T g^k(t, (B_t)) - \eta_t| dt] = 1\). This contradiction implies that \(\eta \notin M^1_G(0, T)\).
4.2 Some applications of Krylov’s estimates

Assume \( n \leq d \) and consider the following \( n \)-dimensional G-Itô process: for each \( t \geq 0 \),

\[
X_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \beta^{jk}_s d(B^j, B^k)_s + \int_0^t \sigma_s dB_s,
\]

where \( x_0 \in \mathbb{R}^n \) and the processes \( \alpha, \beta^{jk} \in \mathcal{M}^2_G(0, T; \mathbb{R}^n), \sigma \in \mathcal{M}^2_G(0, T; \mathbb{R}^{n \times d}) \).

In this subsection, we make the following assumptions:

(B1) There exists a constant \( L > 0 \) such that for each \( t \geq 0 \),

\[
|\alpha_t| \leq L, \quad |\beta^{jk}_t| \leq L \text{ for } j, k \leq d;
\]

(B2) There exists a constant \( \bar{\lambda} > 0 \) such that for each \( t > 0 \),

\[
\sigma_t(\sigma_t)^T \geq \bar{\lambda} I_{n \times n};
\]

(B3) There exists a constant \( \bar{\Lambda} > 0 \) such that for each \( t > 0 \),

\[
\sigma_t(\sigma_t)^T \leq \bar{\Lambda} I_{n \times n}.
\]

Definition 4.11 A stopping time \( \tau \) relative to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) is a map on \( \Omega \) with values in \([0, \infty)\), such that for every \( t \),

\[
\{ \tau \leq t \} \in \mathcal{F}_t.
\]

Theorem 4.12 Assume (B1) and (B2) hold. Let \( D \) be a bounded region in \( \mathbb{R}^n \) and \( \tau \) be a stopping time with \( \tau \leq \tau_D \), where \( \tau_D \) is the first exit time of \( X_t \) from \( D \). Then for each \( x_0 \in \mathbb{R}^n \), \( T \geq 0 \) and \( p \geq n \), there exists a constant \( \bar{N} \) depending on \( p, \bar{\lambda}, L, G, T \) and \( D \) such that for each \( t \in [0, T] \) and all Borel function \( f(t, x), g(x) \),

\[
\hat{E}[\int_0^{t \wedge \tau} |f(t, X_t)| dt] \leq \bar{N} \| f \|_{L^{p+1}([0, T] \times D)};
\]

\[
\hat{E}[\int_0^{t \wedge \tau} |g(X_t)| dt] \leq \bar{N} \| g \|_{L^p(D)}.
\]

Proof. Let \( \mathcal{P} \) be the weakly compact set that represents \( \hat{E} \). By Corollary 5.7 in Chapter 3 of [13], we obtain that \( d(B^j, B^k)_t = \gamma_t^{jk} dt \) q.s. and \( \sigma^2 t I_{d \times d} \leq \gamma_t = (\gamma_t^{jk})_{j, k=1}^d \leq \sigma^2 t I_{d \times d} \). Note that \( B \) is a martingale on the probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, P)\) for each \( P \in \mathcal{P} \). Then it is easy to check that

\[
W_t^P := \int_0^t \gamma_s^{-\frac{1}{2}} dB_s, \quad P - a.s.
\]

is a Brownian motion on \((\Omega, (\mathcal{F}_t)_{t \geq 0}, P)\). Thus we have

\[
X_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \beta^{jk}_s \gamma_t^{jk} ds + \int_0^t \sigma(s, X_s) \gamma_t^{\frac{1}{2}} dW_t^P, \quad P - a.s.
\]
Applying Theorem 2.4 in Chapter 2 of Krylov [6] (see also [7]), we can find a constant $\bar{N}$ depending on $p, \bar{\lambda}, L, G, T$ and $D$ such that for all Borel function $f(t, x)$,

$$E_P[\int_0^{t \wedge \tau} |f(t, X_t)|dt] \leq \bar{N}\|f\|_{L^{p+1}([0, T] \times D)}.$$

Therefore, we have

$$\hat{\mathbb{E}}[\int_0^{t \wedge \tau} |f(t, X_t)|dt] = \sup_{P \in \mathbb{P}} E_P[\int_0^{t \wedge \tau} |f(t, X_t)|dt] \leq \bar{N}\|f\|_{L^{p+1}([0, T] \times D)}$$

and the second inequality can be proved in a similar way. 

**Theorem 4.13** Assume (B1), (B2) and (B3) hold. Then for each $\delta > 0$ and $p \geq n$, there exists a constant $N$ depending on $p, \bar{\lambda}, \bar{\Lambda}, L, G$ and $\delta$ such that for all Borel function $f(t, x)$ and $g(x)$,

$$\hat{\mathbb{E}}[\int_0^{\infty} \exp(-\delta t)|f(t, X_t)|dt] \leq N\|f\|_{L^{p+1}([0, \infty] \times \mathbb{R}^n)},$$

$$\hat{\mathbb{E}}[\int_0^{\infty} \exp(-\delta t)|g(X_t)|dt] \leq N\|g\|_{L^p(\mathbb{R}^n)}.$$

**Proof.** The proof is immediate from the proof of Theorem 4.12 and Theorem 3.4 in Chapter 2 of [6].

**Remark 4.14** Theorems 4.12 and 4.13 are called Krylov’s estimates.

The following results are direct consequences of Theorem 4.13.

**Corollary 4.15** Under the assumptions (B1)-(B3), for each $T > 0$ and $p \geq n$, there exists a constant $N_T$ depending on $p, \bar{\lambda}, \bar{\Lambda}, L, G$ and $T$ such that for all Borel function $f(t, x)$ and $g(x)$,

$$\hat{\mathbb{E}}[\int_0^{T} |f(t, X_t)|dt] \leq N_T\|f\|_{L^{p+1}([0, T] \times \mathbb{R}^n)},$$

$$\hat{\mathbb{E}}[\int_0^{T} |g(X_t)|dt] \leq N_T\|g\|_{L^p(\mathbb{R}^n)}.$$

**Corollary 4.16** Assume (H1), (H2), (H4) and (H5) hold. Then for each $p \geq 1, i \leq n$ and the solution $X_t^{x;i}$ to $G$-SDE (1), we can find a constant $N_T$ depending on $p, \gamma, \Gamma, L, G$ and $T$ such that for all Borel function $g(x)$,

$$\hat{\mathbb{E}}[\int_0^{T} |g(X_t^{x;i})|dt] \leq N_T\|g\|_{L^p(\mathbb{R}^1)}.$$

In the following, we first use Krylov’s estimates to get quasi-continuous processes.
Lemma 4.17 Assume (B1)-(B3) hold.

(i) If $\psi$ is in $L^p([0, T] \times \mathbb{R}^n)$ with $p \geq n + 1$, then for each $T > 0$, we have $(\psi(t, X_t))_{t \leq T} \in M^1_{\mathcal{L}}(0, T)$. Moreover, for each $\psi' = \psi$, a.e., we have $\psi' : (\cdot, X) = \psi(\cdot, X)$.

(ii) If $\varphi$ is in $L^p(\mathbb{R}^n)$ with $p \geq n$, then for each $T > 0$, we have $(\varphi(X_t))_{t \leq T} \in M^1_{\mathcal{L}}(0, T)$. Moreover, for each $\varphi' = \varphi$, a.e., we have $\varphi'(X) = \varphi(X)$.

Proof. We only prove (ii) and (i) can be proved similarly. Note that there exists a sequence bounded and continuous functions $\varphi^k$ such that $\varphi^k$ converges in $L^p(\mathbb{R}^n)$ to $\varphi$. Then by Corollary 4.15 there exists a constant $C'$ such that

$$
\lim_{k \to \infty} \hat{E}\int_0^T |\varphi^k - \varphi|(X_t)dt \leq C' \lim_{k \to \infty} \|\varphi^k - \varphi\|_{L^p(\mathbb{R}^n)} = 0.
$$

By Theorem 4.4 we can get $(\varphi^k(X_t))_{t \leq T} \in M^1_{\mathcal{L}}(0, T)$ for each $k \geq 1$. Thus we conclude $(\varphi(X_t))_{t \leq T} \in M^1_{\mathcal{L}}(0, T)$.

Assume $\varphi = \varphi'$, a.e. Then

$$
\hat{E}\int_0^T |\varphi' - \varphi|(X_t)dt \leq C'\|\varphi' - \varphi\|_{L^p(\mathbb{R}^n)} = 0,
$$

which completes the proof. \hfill \blacksquare

Theorem 4.18 Assume (B1)-(B3) hold. Let $(\varphi^k)_{k \geq 1}$ be a sequence of $\mathbb{R}^n$-Borel measurable functions and $|\varphi^k(x)| \leq \bar{C}(1 + |x|^l)$, $k \geq 1$ for some constants $C$ and $l$. If $\varphi^k \to \varphi$, a.e., then for each $T > 0$ and $p \geq 1$,

$$
\lim_{k \to \infty} \hat{E}\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^pdt = 0.
$$

Proof. By Lemma 4.17, we may assume that $|\varphi(x)| \leq \bar{C}(1 + |x|^l)$. For each fixed $N > 0$, we have

$$
\hat{E}\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^pdt \leq \hat{E}\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^pI_{\{|X_t| \leq N\}}dt
$$
$$
+ \hat{E}\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^pI_{\{|X_t| \geq N\}}dt.
$$

By Corollary 4.15 there exists a constant $C'$ independent of $k$ such that

$$
\hat{E}\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^pI_{\{|X_t| \leq N\}}dt \leq C'\int_{\{|x| \leq N\}} |\varphi^k(x) - \varphi(x)|^pdxd\nu^{1/p}.
$$

Then applying dominated convergence theorem yields that

$$
\hat{E}\int_0^T |\varphi^k(X_t) - \varphi(X_t)|^pI_{\{|X_t| \leq N\}}dt \to 0 \text{ as } k \to \infty.
$$
Noting that \( |\varphi^k(X_t) - \varphi(X_t)|^p I_{\{|X_t| \geq N\}} \leq \frac{(2\tilde{C})^p}{N}(1 + |X_t|^p)|X_t| \), then we get
\[
\limsup_{k \to \infty} \mathbb{E}\left[ \int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p dt \right] \leq \frac{(2\tilde{C})^p}{N} \int_0^T \mathbb{E}(1 + |X_t|^p)|X_t| dt.
\]
Since \( N \) can be arbitrarily large, we obtain
\[
\lim_{k \to \infty} \mathbb{E}\left[ \int_0^T |\varphi^k(X_t) - \varphi(X_t)|^p dt \right] = 0,
\]
which is the desired result. ■

Theorem 4.18 can be seen as the dominated convergence theorem of the G-Itô processes.

**Theorem 4.19** Assume (B1)-(B3) hold. If \( \varphi \) is a \( \mathbb{R}^n \)-Borel measurable function of polynomial growth, then for each \( T > 0 \), we have \( (\varphi(X_t))_{t \leq T} \in \mathcal{M}_G^n(0, T) \)

**Proof.** We can find a sequence of continuous functions \( \varphi^k, k \geq 1 \), with compact support, such that \( \varphi^k \) converges to \( \varphi \) a.e. and \( |\varphi^k(x)| \leq \tilde{C}(1 + |x|^l) \), where \( \tilde{C} \), \( l \) are constants independent of \( k \). Then by Theorem 4.18 for each \( T > 0 \), we conclude that
\[
\lim_{k \to \infty} \mathbb{E}\left[ \int_0^T |\varphi^k - \varphi|^2(X_t) dt \right] = 0.
\]
Since \( (\varphi^k(X_t))_{t \leq T} \in \mathcal{M}_G^n(0, T) \) for each \( k \) by Theorem 4.18 we derive that \( (\varphi(X_t))_{t \leq T} \in \mathcal{M}_G^n(0, T) \) and this completes the proof. ■

Now we give Itô-Krylov’s formula for \( G \)-diffusion processes. Let us recall some notations.

- \( \mathcal{W}_p^2(\mathbb{R}^n) \): the space of all functions \( u \) defined on \( \mathbb{R}^n \) such that \( u \in C(\mathbb{R}^n) \) and its generalized derivatives \( \partial_x u, \partial^2_{x,x} u \) belong to \( L^p(\mathbb{R}^n) \),
- \( \mathcal{W}_{p,\text{loc}}^2(\mathbb{R}^n) \): the space of all functions \( u \) defined on \( \mathbb{R}^n \) such that \( u \in C(\mathbb{R}^n) \) and its generalized derivatives \( \partial_x u, \partial^2_{x,x} u \) belong to \( L^p_{\text{loc}}(\mathbb{R}^n) \),
- \( \mathcal{W}^{1,2}_p([0, T] \times \mathbb{R}^n) \): the space of all functions \( u \) defined on \( \mathbb{R}^n \) such that \( u \in C([0, T] \times \mathbb{R}^n) \) and its generalized derivatives \( \partial_t u, \partial_x u, \partial^2_{x,x} u \in L^p([0, T] \times \mathbb{R}^n) \),
- \( \mathcal{W}^{1,2}_{p,\text{loc}}([0, T] \times \mathbb{R}^n) \): the space of all functions \( u \) defined on \( \mathbb{R}^n \) such that \( u \in C([0, T] \times \mathbb{R}^n) \) and its generalized derivatives \( \partial_t u, \partial_x u, \partial^2_{x,x} u \in L^p_{\text{loc}}([0, T] \times \mathbb{R}^n) \).

**Remark 4.20** We remark that \( \mathcal{W}_{p,\text{loc}}^2(\mathbb{R}^n) \) and \( \mathcal{W}^{1,2}_{p,\text{loc}}([0, T] \times \mathbb{R}^n) \) are subsets of the ordinary Sobolev spaces (see also Section 2.1 in Krylov [3]).

**Theorem 4.21** Assume (B1)-(B3) hold. Then, for each function \( u \in \mathcal{W}_{p,\text{loc}}^2(\mathbb{R}^n) \) with \( p > n \), we have
\[
u(X_t) = u(x_0) + \int_0^t \partial_x u(X_s) \sigma^i_s dB^i_s + \int_0^t \partial_{x,x} u(X_s) \beta^{ijk}_s dB^j_s B^k_s \tag{13}
\]
\[
+ \int_0^t \partial_x u(X_s) \sigma^i_s dB^i_s + \frac{1}{2} \int_0^t \partial_{x,x} u(X_s) \sigma^{ij}_s \sigma^{kl}_s dB^j s B^k_s, \text{ q.s.}
\]
Proof. Applying Sobolev's embedding theorem yields that $\partial_x u \in C(\mathbb{R}^n)$, a.e.. Thus recalling Lemma 14.7 it suffices to prove the result under assumption

$$u \in C^1(\mathbb{R}^n) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^n).$$

For each $R > 0$, we define $\tau_R := \inf \{ t > 0 : |X_t^x| \geq R \}$. Then there exists a sequence of functions $u^\rho \in C^2(\mathbb{R}^n)$ such that:

(i) $u^\rho$ converges uniformly to $u$ in the sphere $\{ x : |x| \leq R \}$;

(ii) $\partial_x u^\rho$ converges uniformly to $\partial_x u$ in the sphere $\{ x : |x| \leq R \}$;

(iii) $\partial^2_{x,x} u^\rho$ converges in $L^p(\{ x : |x| \leq R \})$ to $\partial^2_{x,x} u$.

Applying G-Itô's formula given in [3] yields that

$$u^\rho(X_{t\wedge \tau_R}) = u^\rho(0) + \int_0^{t\wedge \tau_R} \partial_x u^\rho(X_s) \sigma_s^i \sigma_s^j \sigma_s^k d\langle B^j, B^k \rangle_s + \int_0^{t\wedge \tau_R} \partial_x u^\rho(X_s) \beta_s^i \sigma_s^j d\langle B^j \rangle_s$$

By the above properties (i) and (ii), we obtain

$$\lim_{\rho \to \infty} u^\rho(X_{t\wedge \tau_R}) \to u(X_{t\wedge \tau_R})$$

and

$$\lim_{\rho \to \infty} \mathbb{E} \left[ \int_0^{t\wedge \tau_R} |\partial_x u^\rho(X_s) - \partial_x u(X_s)|^2 |\sigma_s^i|^2 ds \right] \leq C t \lim_{\rho \to \infty} \sup_{|x| \leq R} |\partial_x u^\rho(x) - \partial_x u(x)|^2 = 0,$$

where $C$ is some constant depending only on $\Lambda$. Thus we get

$$\lim_{\rho \to \infty} \int_0^{t\wedge \tau_R} \partial_x u^\rho(X_s) \sigma_s^i \sigma_s^j dB_s^j = \int_0^{t\wedge \tau_R} \partial_x u(X_s) \sigma_s^i \sigma_s^j dB_s^j, \text{ q.s..}$$

By a similar analysis, we also derive that

$$\int_0^{t\wedge \tau_R} \partial_x u^\rho(X_s) \sigma_s^i \sigma_s^j \sigma_s^k dB_s^j = \int_0^{t\wedge \tau_R} \partial_x u(X_s) \sigma_s^i \sigma_s^j \sigma_s^k dB_s^j, \text{ q.s..}$$

as $\rho \to 0$.

Then from the above property (iii) and Theorem 4.12 we can find some constant $C'$ depending on $p, \lambda, \Lambda, L, G$ and $R$ so that

$$\lim_{\rho \to \infty} \mathbb{E} \left[ \int_0^{t\wedge \tau_R} |\partial^2_{x,x} u^\rho(X_s) \sigma_s^i \sigma_s^j \sigma_s^k d\langle B^j, B^k \rangle_s - \partial^2_{x,x} u(X_s) \sigma_s^i \sigma_s^j \sigma_s^k d\langle B^j, B^k \rangle_s| \right]$$

$$\leq C' \lim_{\rho \to \infty} \int_{\{|x| \leq R\}} |\partial^2_{x,x} u^\rho(x) - \partial^2_{x,x} u(x)|^p dx \leq 0.$$
Thus
\[
\lim_{\rho \to \infty} \int_0^{t \wedge \tau_R} \partial_{x,x_1}^2 u(x,s) \sigma_s^{ij} \sigma_s^{lk} \, dB^i_s \, dB^k_s = \int_0^{t \wedge \tau_R} \partial_{x,x_1}^2 u(x,s) \sigma_s^{ij} \sigma_s^{lk} \, dB^i_s \, dB^k_s, \quad q.s.
\]

Consequently, sending \(\rho \to \infty\), we derive that
\[
u(X_{t \wedge \tau_R}) = u(x_0) + \int_0^{t \wedge \tau_R} \partial_x u(X_s) \alpha_s^i \, ds + \int_0^{t \wedge \tau_R} \partial_x u(X_s) \beta_s^{ijk} \, dB^i_s \, dB^j_s \, dB^k_s
+ \int_0^t \partial_x u(X_s) \sigma_s^{ij} \, dB^j_s + \frac{1}{2} \int_0^t \partial_{x,x_1}^2 u(X_s) \sigma_s^{ij} \sigma_s^{lk} \, dB^i_s \, dB^j_s \, dB^k_s.
\]

Note that \(\tau_R \to \infty\) as \(R \to \infty\). Then we can get equation (13) by sending \(R \to \infty\).

**Remark 4.22** Following Li-Peng [8], the definition of Itô’s integral is defined by
\[
\int_0^t \partial_x u(X_s) \sigma_s^{ij} \, dB^j_s = \lim_{R \to \infty} \int_0^{t \wedge \tau_R} \partial_x u(X_s) \sigma_s^{ij} \, dB^j_s.
\]

**Corollary 4.23** Assume (B1)-(B3) hold. Then, for each function \(u \in \mathcal{W}_p^2(\mathbb{R}^n)\) with \(p \geq 2n\), we have in \(L^2_G(\Omega_t)\),
\[
u(X_t) = u(x_0) + \int_0^t (\partial_x u(X_s) \alpha_s^i) \, ds + \int_0^t \partial_x u(X_s) \beta_s^{ijk} \, dB^i_s \, dB^j_s \, dB^k_s
+ \int_0^t \partial_x u(X_s) \sigma_s^{ij} \, dB^j_s + \frac{1}{2} \int_0^t \partial_{x,x_1}^2 u(X_s) \sigma_s^{ij} \sigma_s^{lk} \, dB^i_s \, dB^j_s \, dB^k_s.
\]

**Proof.** By Lemma 4.17 one can easily check that the right hand side of the above equations is in \(L^2_G(\Omega_t)\). Thus by Theorem 4.21 \(u(X_t) \in L^2_G(\Omega_t)\) and the proof is complete.

By a similar analysis, we can also obtain the following generalized Itô-Krylov’s formula of \(G\)-Itô processes.

**Theorem 4.24** Assume (B1)-(B3) hold. Then, for each function \(u \in \mathcal{W}_{p,loc}^{1,2}([0,T] \times \mathbb{R}^n)\) (resp. \(\mathcal{W}_{p,loc}^{1,2}([0,T] \times \mathbb{R}^n)\)) with \(p > n + 2\) (resp. \(\geq 2(n+1)\)), we have q.s.

(\(L^2_G(\Omega_t)\)),
\[
u(t,X_t) = u(0,x_0) + \int_0^t (\partial_s u(s,X_s) + \partial_x u(s,X_s) \alpha_s^i) \, ds + \int_0^t \partial_x u(s,X_s) \beta_s^{ijk} \, dB^i_s \, dB^j_s \, dB^k_s
+ \int_0^t \partial_x u(s,X_s) \sigma_s^{ij} \, dB^j_s + \frac{1}{2} \int_0^t \partial_{x,x_1}^2 u(s,X_s) \sigma_s^{ij} \sigma_s^{lk} \, dB^i_s \, dB^j_s \, dB^k_s.
\]

In particular, we have the following dominated convergence theorem for \(G\)-random variables.
Theorem 4.25 Assume (B1)-(B3) hold. Let $u, u^\rho \in W^{1,2}_{p,loc}([0, T] \times \mathbb{R}^n)$ with $p > (n+2)$ for $\rho \geq 1$. Moreover, $u^\rho$ converges pointwise to $u$ and $\partial_t u^\rho, \partial_x u^\rho, \partial^2_{x,i} u^\rho$ a.e. converge to $\partial_t u, \partial_x u, \partial^2_{x,i} u$. If there exist some constants $\tilde{C}$ and $l$ such that

$$(|\partial_t u^\rho| + |\partial_x u^\rho| + |\partial^2_{x,i} u^\rho|)(t, x) \leq \tilde{C}(1 + |x|^l),$$

then for each $p \geq 1$,

$$\lim_{\rho \to \infty} \mathbb{E}[|u^\rho(t, X_t) - u(t, X_t)|^p] = 0.$$  

Proof. The proof is immediate from Theorems 4.18 and 4.24. \[\blacksquare\]

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