Kinematics Analysis and Workspace Calculation of a 3-DOF Manipulator

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Abstract. Robot working space is an important kinematic indicator. Exact computation of the boundary shape and volume or area of the manipulator workspace is very important for its optimum design and application. This paper describes the kinematics analysis of a 3-DOF manipulator and the calculation of its workspace. The kinematics model of the three degree of freedom (3-DOF) robot was set up with the D-H parameters method. The effectiveness of kinematics equations was verified by the Robotics Toolbox simulation of the MATLAB. The robotics workspace of the manipulator was analyzed and simulated based on the Monte-Carlo method and MATLAB software programming. The analysis results show that the manipulator’s working space is well-knit and this method is feasible and practical, prove the trajectory planning of workspace, and can well perform task, which lay a foundation to further study of the trajectory planning and motion control.

1. Introduction

The kinematics analysis on the manipulator is an important part of the robot study [1]. The manipulator is composed of a serial of rigid links connected to each other with revolute or prismatic joints. Each joint location is usually defined relative to neighboring joint. Calculating the position and orientation of the end-effector of the manipulator by the given joint angles is known as forward kinematics [2, 3]. Forward kinematics problem is straightforward and there is no complexity in deriving the equations. Inverse kinematics is a mapping form the operational space to the joint space, so it can be considered as the inverse of forward kinematics. An efficient Inverse kinematic algorithm for robot manipulator is very useful [4, 5]. Working space of robotic arm is the reference point at the end of the robot arm to achieve the set point of space. Working space is discussed from the geometric aspects of the work performance of the robot arm. Analytical work is used to determine the spatial configuration of the robot arm. Working space represents the range of activities, which is a parameter of the ability to work the robot arm kinematics, and is also an important indicator [6]. Robot arm and the sizes of work space according to the design requirements, must reach the required working space, which is reasonable to judge about the structure, operability and flexibility of the robot arms. Work space model for robot planning and control information provide important constraints, such as working space was determined according to the mechanical interference between the robot arm and the robot arm motion planning to avoid obstacles and so on [7, 8].

In the literature, several methods have been proposed to determine the workspace of manipulators by using analytic, geometric, or numerical approaches. Early studies on the serial manipulator
workspace were developed by Roth (1975) [9]. A geometric algorithm to compute the parallel manipulator workspace has been used (Boneveand and Ryu, 2001) [10]. A discretization method is presented to determine the workspace of both serial and parallel manipulators by Gianni (2008) [11] and a new software tool designed to compute and allow visualization of the different types of workspaces of parallel manipulators by E. Macho (2011) [12].

In this paper, the kinematics of a 3-DOF manipulator was studied and proposed a numerical method to solve the problem. Monte Carlo method [13] was then used to simulate the working space, which can straightforwardly and visually describe the working space of robot with high speed. It was indicated that the working points in the workspace of the robot were distributed compactly and uniformly, which can meet design requirements with high efficiency.

2. Structure of the 3-DOF Manipulator
The manipulator has three degrees of freedom: base, shoulder, elbow, and end-effector. The length of the Link1 is 1000mm, Link 2 is 550mm, and the distance from center of the end-effector to elbow (Link3) is 550mm. The structure of the manipulator is described through the Robotics Toolbox Link() functions [14], in Fig. 1.

![Figure 1. The three dimension image of manipulator.](image)

2.1. D-H Coordination System
Denavit and Hartenberg had put forward a matrix method to build the attached coordinate system on each link in the joint chains of the robot for describing the relationship of translation or rotation between the contiguous links in 1955. The configuration of the manipulator was built with three degrees of freedom: base, shoulder, elbow, and wrist, as shown in Fig.2. D-H coordinate system of the manipulator is found in Fig. 3. Table 1 is the link parameters.
Figure 2. The structure of the manipulator.

Figure 3. Robot arm coordinate frame of link.

Table 1. Link parameters.

| i | \( a_{i-1}(^\circ) \) | \( a_i(\text{mm}) \) | \( d_i(\text{mm}) \) | \( \theta_i(\text{\textdegree}) \) | Variable range |
|---|------------------|------------------|------------------|----------------|----------------|
| 1 | 0 | 0 | 0 | \( \theta_1 \) | -180\textdegree-180\textdegree |
| 2 | 0 | 90\textdegree | 0 | \( \theta_2 \) | 0°-90° |
| 3 | \( a_2 \) | 0 | 0 | \( \theta_3 \) | -60\textdegree-60\textdegree |
| 4 | 0 | 90\textdegree | \( d_4 \) | 0 | 0 |

The transformation matrix of the manipulator is given according to Danevit-Hartenberg method.
3. Forward Kinematic Analysis

The forward kinematic analysis means that the location and pose of the end of manipulator in a given references coordinates of the links and the variables of the joints for the manipulator. Its kinematic equation could be expressed as the follows.

$$\mathbf{J} = \begin{bmatrix} o_x & a_x & p_x \\ o_y & a_y & p_y \\ o_z & a_z & p_z \end{bmatrix}$$

where \((n_x, n_y, n_z)\) is the normal vector, \((a_x, a_y, a_z)\) is a approach vector, \((o_x, o_y, o_z)\) is a orientation vector, \((p_x, p_y, p_z)\) is the end-effector position vector. Eventually, \(\mathbf{J}\) is

$$\mathbf{J} = \begin{bmatrix} s_{23} & c_{23} & c_{23}d_4 + c_{23}d_2 \\ c_{23} & s_{23} & s_{23}d_4 + s_{23}d_2 \\ 0 & -c_{23} & -c_{23}d_4 + s_{23}d_2 \end{bmatrix}$$

And,

\(p_x = c_1s_2d_4 + c_1c_2d_2\)
\(p_y = s_1s_2d_4 + s_1c_2d_2\)
\(p_z = -c_2d_4 + s_2d_2\)

When the structural parameter of the manipulator is obtained, the workspace of the manipulator depends on generalized coordinate \(q_i\). For the limit of the actual structure and driving device, so, \(q_i^{\min} \leq q_i \leq q_i^{\max}\). The assemble of point P can be defined as the workspace of the manipulator, namely

$$w = \begin{bmatrix} x(q_i) \\ y(q_i) \\ z(q_i) \end{bmatrix}$$

The position of the end-effector of the manipulator relative to the base coordinate system was obtained from (2).

$$p = \begin{bmatrix} x(q) = c_1s_2d_4 + c_1c_2d_2 \\ y(q) = s_1s_2d_4 + s_1c_2d_2 \\ z(q) = -c_2d_4 + s_2d_2 \end{bmatrix}$$
4. Monte Carlo Method for Constrained Workspace

The Monte Carlo Method for the workspace of manipulator does not use inverse kinematics but needs only forward kinematics. Constraints including the joint limits can be easily coped by this method. It is relatively simple and flexible. It is also convenient to use this method to directly obtain graphical depiction of the workspace [15].

4.1. Simulation

In this paper, the Monte Carlo method was used to determine the workspace of the manipulator. Values could be yielded randomly in their rotational range, from (5).

\[ \theta^\text{min}_i \leq \theta_i \leq \theta^\text{max}_i + (\theta^\text{max}_i - \theta^\text{min}_i)? \text{Rand}(i) \]  

\( \theta^\text{min}_i \): Minimum range of motion joint \( i \), \( \theta^\text{max}_i \): Maximum range of motion joint \( i \), \( i \) means joint number.

These random points from the graphical form, that is, the robot arm workspace cloud, namely the robot arm of Monte Carlo work space.

Computer programming can be realized by above algorithm, and the results that come out with the graphics, as shown in Fig 4. Where, \( n \) means iteration number.

![Diagram](image)

**Figure 4.** Working space simulation flow based on Monte Carlo method.
4.2. Application

The following figures are results of the robot arm work space simulation. Fig 5a describes the end of the three degrees of freedom robot arm based on the coordinate reference point formed in the workspace. Fig 5b, Fig 5c and Fig 5d show the three Cartesian coordinates in the base coordinates of the projection plane, such as XOY, XOZ, YOZ plane.

![Figure 5. Workspace.](image)

As the Monte Carlo method itself limits the work space by the robot arm simulation results with the theoretical value gap. In order to make simulation results more close to the actual working space, increasing the iteration numbers. We can get as close to the actual working space as possible.

5. Inverse Kinematic Analyses

Inverse kinematics is used to calculate the corresponding joint angles when the posture of the end effect is known. In this work, the algebraic method is adopted, and we can separate all joint angles. Namely,

\[
\mathbb{T} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
o_x & a_x & p_x \\
o_y & a_y & p_y \\
o_z & a_z & p_z \\
0 & 0 & 1
\end{bmatrix}
= \mathbb{T}(q_1)\mathbb{T}(q_2)\mathbb{T}(q_3)\mathbb{T}
\]

(6)

\[
\theta_i = \mathbb{T}^{-1}(q_i)\mathbb{T}
\]

(7)
\[ s_1 0 0 \]
\[ a_x \]
\[ a_y \]
\[ p_x \]
\[ c_1 0 0 \]
\[ a_x \]
\[ a_y \]
\[ p_y \]
\[ 0 1 0 \]
\[ o_x \]
\[ a_y \]
\[ p_z \]
\[ 0 0 1 \]
\[ 0 0 1 \]

In matrix equation (8), element (2, 4) is corresponding equal, so:

\[- s_y p_x + c_y p_y = 0 \] (9)

According to the Trigonometric substitution formula, as following:

\[ p_x = r_1 \cos f ; \quad p_y = r_1 \sin f \] (10)

Where \( r_1 = \sqrt{p_x^2 + p_y^2} \), \( f = \arctan(p_y / p_x) \),

\[ \begin{align*}
\sin(f - q_1) &= 0 \\
\cos(f - q_1) &= 0 \\
f - q_1 &= 0 \\
q_1 &= \arctan(p_y / p_x) \\
\end{align*} \] (11)

(2) \( \theta_2 \)

Matrix equation (8) element (1, 4) and (3, 4) is corresponding equal,

\[ \begin{align*}
c_y p_x + s_y p_y &= s_2 d_1 + a_x x_1 \\
p_z &= -c_2 d_1 + s_2 \theta_2 \\
\end{align*} \] (12)

As following,

\[- s_y p_x + c_y (c_y p_x + s_y p_y) = h \] (13)

In, \( h = \frac{(c_y p_x + s_y p_y)^2 + p_x^2 + a_x^2 - d_2^2}{2r_2} \),

\[- p_z = r_2 \cos \theta_2 ; \quad c_y p_x + s_y p_y = r_2 \sin \theta_2 \] (14)

Where, \( r_2 = \sqrt{p_x^2 + (c_y p_x + s_y p_y)^2} \); \( \theta_2 = \arctan((c_y p_x + s_y p_y) / (- p_z)) \);

\[ \begin{align*}
\sin(\theta_2 - q_2) &= h / r_2 \\
\cos(\theta_2 - q_2) &= \sqrt{1 - (h / r_2)^2} \\
f_2 - q_2 &= \arctan((h / r_2) / \sqrt{1 - (h / r_2)^2}) \\
q_2 &= \arctan((c_y p_x + s_y p_y) / (- p_z)) - \arctan(h / \sqrt{r_2^2 - h^2}) \\
\end{align*} \] (15)
(3) $\theta_3$

According to equation (12), the $\theta_3$ equation showing,

$$\tan(q_2 + q_3) = (c_xp_x + s_xp_y/c_x) / (-p_z + s_xp_z)$$

(16)

$$q_3 = \arctan((c_xp_x + s_xp_y/c_x) / (-p_z + s_xp_z)) - q_2$$

(17)

$$q_3 = \arctan((c_xp_x + s_xp_y/c_x) / (-p_z + s_xp_z)) - \arctan((c_xp_x + s_xp_y/c_x) / (-p_z + s_xp_z)) + \arctan(h \theta \sqrt{p_z^2 - h^2})$$

(18)

All joint variables are as follows,

$$q_1 = \arctan(p_x / p_z)$$

$$q_2 = \arctan((c_xp_x + s_xp_y) / (-p_z)) - \arctan(h \theta \sqrt{p_z^2 - h^2})$$

$$q_3 = \arctan((c_xp_x + s_xp_y/c_x) / (-p_z + s_xp_z)) - \arctan((c_xp_x + s_xp_y/c_x) / (-p_z + s_xp_z)) + \arctan(h \theta \sqrt{p_z^2 - h^2})$$

(19)

According to the workspace that already given on above analysis, it verified the end-effector location $p_{e0}$ (800, 0, 0) in Robotics Toolbox, as shown in Fig.6.

![Figure 6. The end-effector of 3-DOF manipulator at $P_{e0}$ point.](image)

6. Conclusion

In this paper, the forward and inverse kinematics solutions of the manipulator were analyzed. The range of joint variables was obtained by the simulation of forward kinematics, it lays the foundation for the workspace analysis, the motion control and the trajectory planning of manipulator in future.

Monte Carlo method was used to simulate the working space, which can straightforwardly and visually describe the working space of robot with high speed. It was indicated that the working points in the workspace of the robot were distributed compactly and uniformly, which can meet design requirements with high efficiency. The effectiveness of the working space model was testified by the simulation analysis on MATLAB and experimental validation.
The kinematics of manipulator was simulated in the Robotics toolbox on the workspace, and the inverse kinematics of manipulator was verified to be correct by the use of a point, and this end-effector can work well with the manipulator. So the 3-DOF manipulator can meet the working requirement in factory.

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