Infrared Structure of $e^+e^- \to 2$ jets at NNLO

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Abstract

The production of two jets is the simplest exclusive quantum chromodynamics process in electron-positron annihilation. Using this process, we examine the structure of next-to-next-to-leading order (NNLO) corrections to jet production observables. We derive a subtraction formalism including double real radiation at tree level and single real radiation at one loop. For two-jet production, these subtraction terms coincide with the full matrix elements, thus highlighting the phase space structure of the subtraction procedure. We then analytically compute the infrared singularities arising from each partonic channel. For the purely virtual (two-parton) NNLO corrections, these take the well known form predicted by Catani’s infrared factorization formula. We demonstrate that individual terms in the infrared factorization formula can be identified with infrared singular terms from three- and four-parton final states, leaving only single poles and a contribution from the one-loop soft gluon current, which subsequently cancels between the three- and four-parton final states. Summing over all different final states, we observe an explicit cancellation of all infrared poles and recover the known two-loop correction to the hadronic $R$-ratio.
1 Introduction

Jet production observables can be measured very accurately at present high energy colliders. By confronting these data with theoretical calculations, one can determine the strong coupling constant or parton distribution functions [1]. Analyzing the different sources of error on these determinations, it becomes clear that the largest source of uncertainty are currently the insufficiently precise theoretical calculations of QCD corrections to jet observables, which mostly include terms only to the next-to-leading order (NLO) in perturbation theory.

To improve this situation, the calculation of next-to-next-to-leading order (NNLO) corrections to jet observables is mandatory. For an \( n \)-jet observable, several ingredients are required: the two-loop \( n \)-parton matrix elements, the one-loop \( (n+1) \)-parton matrix elements and the tree level \( (n+2) \)-parton matrix elements. In the recent past, enormous progress has been made in the calculation of two-loop \( 2 \rightarrow 2 \) and \( 1 \rightarrow 3 \) QCD matrix elements, which are now known for all massless parton–parton scattering processes [2–5] relevant to hadron colliders as well as for \( \gamma^* \rightarrow q\bar{q}g \) [6, 7] and its crossings [8]. For the corresponding partonic processes, the one-loop matrix elements with one additional parton [9, 10] and the tree-level matrix elements with two more partons are also known and form part of NLO programs for \( 1 \rightarrow 4 \) [11, 12] and \( 2 \rightarrow 3 \) reactions [13]. Since these matrix elements lead to infrared singularities due to one or two partons becoming theoretically unresolved (soft or collinear), one needs to find one- and two-particle subtraction terms which account for these singularities in the matrix elements, and which are sufficiently simple to be integrated analytically over the unresolved phase space. One-particle subtraction at tree level is well understood from NLO calculations [14–18] and general algorithms are available for one-particle subtraction at one loop [19–22], in a form that has recently been integrated analytically [21, 22].

Tree-level two-particle subtraction terms have been extensively studied in the literature [23–25]. However their integration over the unresolved phase space remains an outstanding issue. The problem of integrating out double real emission contributions has so far only been addressed in specific calculations [26–29], each of which requires a subset of the ingredients needed for generic jet observables at NNLO.

In this paper, we aim to contribute to a formulation of a general subtraction scheme including double real emission by studying the simplest QCD process, two-jet production in \( \gamma^* \rightarrow q\bar{q} \) annihilation at NNLO. Using the iterated sector decomposition [30, 31] for the treatment of all real emission contributions [32–34] and without the need to define subtraction terms, the NNLO corrections to this process have been obtained in a numerical form [35] very recently.

In this work, we extract the infrared divergent parts of all partonic contributions to two-jet final states analytically and identify them with known structures in the virtual two-loop [36] and one-loop single unresolved [20] corrections. These identifications might help to construct a general NNLO subtraction formalism, which would allow the NNLO computation of three-jet production in \( e^+e^- \)-annihilation and, in the longer term, the inclusion of subtraction terms for double initial state emission, as required for the NNLO corrections to jet observables at lepton-hadron and hadron-hadron colliders.

This paper is organized as follows. In Section 2 we describe the different partonic contributions yielding two-jet final states at NNLO. Section 3 describes a formalism to subtract the infrared singularities from single and double real emission present in these contributions. All partonic channels are computed in Section 4 and decomposed into infrared finite and infrared divergent parts, the latter being identified with known structures. The integrated partonic terms represent the analytic integrals of the subtraction terms accounting for all real singularities. The structure of the infrared cancellations is illustrated in Section 5, where we also check the correctness of our results by rederiving the NNLO corrections to the hadronic \( R \)-ratio. An appendix listing all integrals needed in this calculation is enclosed.

2 Perturbative corrections to two-jet final states

Two-jet final states in \( e^+e^- \) annihilation are produced by the primary process \( \gamma^* \rightarrow q\bar{q} \), the decay of a virtual photon into a quark–antiquark pair,

\[
\gamma^*(q) \rightarrow q(p_1) + \bar{q}(p_2) .
\]
At higher orders in perturbation theory, this process receives corrections from the exchange of virtual particles and from real radiation. While the NLO corrections involve up to three final state partons, one finds at NNLO that two-jet final states are produced through partonic final states including up to four partons. The individual partonic channels are:

- **LO** \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) \) tree level
- **NLO** \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) \) one loop
  \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) g(p_3) \) tree level
- **NNLO** \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) \) two loop
  \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) g(p_3) \) one loop
  \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) q(p_3) \bar{q}(p_4) \) tree level
  \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) g(p_3) \bar{q}(p_4) \) tree level
  \( \gamma^* (q) \to q(p_1) \bar{q}(p_2) g(p_3) g(p_4) \) tree level

The contributions of these processes are weighted by so-called jet functions (described in detail in Section III below), which select two-jet final states from the partonic final state momenta. For the analytic extraction of infrared poles present in all partonic channels, we will determine the inclusive \( i \)-parton production cross sections for these processes. They are described by a single Lorentz-invariant

\[
q^2 = \left( \sum_i p_i \right)^2 \quad (2.2)
\]

The \( n \)-particle phase space in dimensional regularization with \( d = 4 - 2\epsilon \) space-time dimensions reads,

\[
d\Phi_n = \frac{d^{d-1}p_1}{2E_1(2\pi)^{d-1}} \cdots \frac{d^{d-1}p_n}{2E_n(2\pi)^{d-1}} (2\pi)^d \delta^d(q - p_1 - \ldots - p_n) \quad (2.3)
\]

Parameterizations of the three- and four-particle massless phase space appropriate to analytic integration are discussed in [33]. In the following, we frequently normalize the three- and four-particle phase space to the volume of the two-particle phase space,

\[
P_2 = \int d\Phi_2 = 2^{-3+2\epsilon} \pi^{-1+\epsilon} \Gamma(1 - \epsilon) \Gamma(2 - 2\epsilon) (q^2)^{\epsilon} \quad (2.4)
\]

### 3 Infrared subtraction terms

To obtain the perturbative corrections to a jet observable at a given order, all partonic multiplicity channels contributing to this order have to be summed. Each partonic channel contains infrared singularities. However, after summation, all singularities cancel among each other [37].

While infrared singularities from purely virtual corrections are obtained immediately after integration over the loop momenta, their extraction is more involved for real emission (or mixed real-virtual) contributions. Here, the infrared singularities become only explicit after integrating the real radiation matrix elements over the phase space appropriate to the jet observable under consideration. In general, this integration involves the (often iterative) definition of the jet observable, such that an analytic integration is not feasible (and also not appropriate). Instead, one would like to have a flexible method that can be easily adapted to different jet observables. Therefore, the infrared singularities of the real radiation contributions should be extracted using infrared subtraction terms. These subtraction terms are constructed such that they approximate the full real radiation matrix elements in all singular limits while still being sufficiently simple to be integrated analytically over a section of phase space that encompasses all regions corresponding to singular configurations.

To specify the notation, we define the tree level \( m \)-parton contribution to the \( J \)-jet cross section in \( d \)-dimensions by,

\[
d\sigma^H = N_{in} \sum_m d\Phi_m(p_1, \ldots, p_m, Q) \frac{1}{S_m} |M_m(p_1, \ldots, p_m)|^2 F_J^{(m)}(p_1, \ldots, p_m) \quad (3.1)
\]
$N_n$ includes all QCD-independent factors, $\sum_m$ denotes the sum over all configurations with $m$ partons, $d\Phi_m$ is the phase space for $m$ partons, $S_m$ is a symmetry factor for identical partons in the final state and finally $|M_m|$ is the tree level $m$-parton matrix element. The jet function $F_j^{(m)}$ defines the procedure for building $J$-jets out of $m$ partons. The main property of $F_j^{(m)}$ is that the jet observable defined above is collinear and infrared safe as explained in [18]. In general $F_j^{(m)}$ contains $\theta$ and $\delta$-functions and $F_2^{(2)} = 1$.

### 3.1 NLO infrared subtraction terms

At NLO, we consider the following $m$-jet cross section,

$$d\sigma_{NLO} = \int d\Phi_{m+1} \left( d\sigma_{NLO}^R - d\sigma_{NLO}^S \right) + \int d\Phi_{m+1} d\sigma_{NLO}^S + \int d\Phi_m d\sigma_{NLO}^V . \quad (3.2)$$

The cross section $d\sigma_{NLO}^R$ has the same expression as the Born cross section $d\sigma_{NLO}^B$ above except that $m \to m + 1$, while $d\sigma_{NLO}^S$ is the one-loop virtual correction to the $m$-parton Born cross section $d\sigma^B$. The cross section $d\sigma_{NLO}^S$ is a local counter-term for $d\sigma_{NLO}^R$. It has the same unintegrated singular behavior as $d\sigma_{NLO}^R$ in all appropriate limits. Their difference is free of divergences and can be integrated over the $(m + 1)$-parton phase space numerically. The subtraction term $d\sigma_{NLO}^S$ has to be integrated analytically over all singular regions of the $m + 1$-parton phase space. The resulting cross section added to the virtual contribution yields an infrared finite result.

The subtraction term $d\sigma_{NLO}^S$ can be constructed in a number of different ways, and even the phase space used with these subtraction terms can vary from method to method. In general, two different methods are applied at next-to-leading order. In the slicing method [15, 17], one exploits the fact that both real radiation matrix element and final state phase space factorize in all soft and collinear limits. Therefore, a subtraction term for a given singular limit can be obtained by simply expanding the full matrix element around the limit under consideration. This subtraction term is then integrated over a small slice of phase space, which is tailored to contain only one singular configuration. In the subtraction method [14, 16, 18], the infrared subtraction terms are integrated over the whole final state phase space (or at least over the full sub-phase space appropriate to one or more final state momenta). In this method, care has to be taken in order to construct subtraction terms which fully account for the limit they are aimed at without introducing spurious infrared singularities in other limits.

A systematic procedure for finding NLO infrared subtraction terms in the second method is the dipole formalism derived by Catani and Seymour [18]. Their subtraction terms are obtained as sum of dipoles $\sum D_{ijk}$ (where each dipole corresponds to a single infrared singular configuration) such that,

$$d\sigma_{NLO}^R - d\sigma_{NLO}^S = N_m \sum_{m+1} d\Phi_{m+1} (p_1, ..., p_{m+1}, Q) \frac{1}{S_{m+1}} \left| M_{m+1} (p_1, ..., p_{m+1}) \right|^2 F_j^{(m+1)} (p_1, ..., p_{m+1})$$

$$- \sum_{\text{pairs } i,j} \sum_{k \neq i,j} D_{ijk} \left| M_m ((p_1, ..., \tilde{p}_{ij}, \tilde{p}_k, ..., p_{m+1}) \right|^2 F_j^{(m)} (p_1, ..., \tilde{p}_{ij}, \tilde{p}_k, ..., p_{m+1}) \quad (3.3)$$

The dipole contribution $D_{ijk}$ involves the $m$-parton amplitude depending only on the redefined on-shell momenta $p_1, ..., \tilde{p}_{ij}, \tilde{p}_k, p_{m+1}$ and the splitting matrix $V_{ij,k}/s_{ij}$ which depends only on $p_i, p_j, \tilde{p}_k$. The momenta $p_i, p_j$ and $\tilde{p}_k$ are respectively the emitter, unresolved parton and the spectator momenta corresponding to a single dipole term. The redefined on-shell momenta $\tilde{p}_{ij}, \tilde{p}_k$ are linear combinations of them.

For the NLO two-jet cross section, the real corrections involve three partons in the final state. The single unresolved configurations occur when the invariants $s_{13}$ or $s_{23}$ vanish. We notice that the subtraction term built as the sum of the dipoles is very close to the full three-parton matrix element. This argument can in fact be turned around: the dipole subtraction terms can be obtained by an appropriate partial fractioning of the full matrix element squared into terms which can be uniquely attributed to individual single unresolved configurations. In the following, we shall use this formulation as definition of a dipole term, without specifying its precise form.
The jet function $\mathcal{F}_j^{(m)}$ in (3.3) depends not on the individual momenta $p_i, p_j$ and $p_k$, but only on $\tilde{p}_{ij}, \tilde{p}_k$. One can therefore carry out the integration over the unresolved dipole phase space appropriate to $p_i, p_j$ and $p_k$ analytically, exploiting the dipole factorization of the phase space [18],

$$d\Phi_{m+1}(p_1,\ldots,p_{m+1},Q) = d\Phi_m(p_1,\tilde{p}_{ij},\tilde{p}_k,\ldots,p_{m+1}) \cdot d\Phi_D(p_i,p_j,p_k), \quad (3.4)$$

where $d\Phi_D$ is symmetric under permutations of $i, j, k$. The dipole phase space is proportional to the three-parton phase space, as can be seen by using $m = 2$ in the above formula and exploiting the fact that the two-particle phase space is a constant,

$$P_2 = \int d\Phi_2, \quad (3.5)$$

such that

$$d\Phi_3 = P_2 \ d\Phi_D. \quad (3.6)$$

It should be noted in this context, that the sum of dipoles is much easier to integrate over the dipole phase space than an individual dipole. In what follows, we shall take into account this fact, and choose to use the full three-parton matrix element as subtraction term [12] to the real corrections to the two-jet rate at NLO (and, in Section 3.2 below, the full four-parton matrix element as candidate subtraction term at NNLO). Taking the full matrix element as the subtraction term provides a clear visualization of the rate at NLO (and, in Section 3.2 below, the full four-parton matrix element as candidate subtraction term to the real corrections to the two-jet phase space than an individual dipole. In what follows, we shall take into account this fact, and choose to use the full three-parton matrix element as subtraction term [12] to the real corrections to the two-jet rate at NLO (and, in Section 3.2 below, the full four-parton matrix element as candidate subtraction term at NNLO). Taking the full matrix element as the subtraction term provides a clear visualization of the phase space regions covered by the different contributions involving the jet-functions $\mathcal{F}_j^{(m)}$. At NLO the difference of the unintegrated real and subtracted contributions to the two-jet cross section reads,

$$d\sigma_{NLO}^R - d\sigma_{NLO}^S = N_{in} \ d\Phi_3(p_1,\ldots,p_3,\mathcal{P}) |\mathcal{M}_3(p_1,\ldots,p_3)|^2 \times \left( \mathcal{F}_2^{(3)}(p_1,p_2) - \frac{1}{2} \mathcal{F}_2^{(2)}(\tilde{p}_{13},\tilde{p}_2) - \frac{1}{2} \mathcal{F}_2^{(2)}(\tilde{p}_{23},\tilde{p}_1) \right). \quad (3.7)$$

The factors $1/2$ in front of the $\mathcal{F}_2^{(2)}$ are obtained by first writing out both dipoles, i.e. both terms obtained by partial fractioning of the three-parton matrix element which are appropriate to the configurations where parton 3 is unresolved with respect to either parton 1 or 2. Exploiting the fact that $\mathcal{F}_2^{(2)} = 1$, these dipoles can be added together to yield the full matrix element. Reintroducing both jet functions consequently yields a factor $1/2$ for each of the momentum redefinitions.

Equation (3.7) can be interpreted as follows: out of the three-parton final state phase space one considers the two-jet configuration (defined by $\mathcal{F}_2^{(2)}$) minus the fully inclusive phase space (defined by $\mathcal{F}_2^{(2)} = 1$), which covers to this order the two- and three-jet configurations.

For the analytic integration, the subtraction term is rewritten as,

$$d\sigma_{NLO}^S = d\Phi_3 |\mathcal{M}_3|^2 = d\Phi_2 |\mathcal{M}_2|^2 \int_{d\Phi_D} |\mathcal{M}_3|^2, \quad (3.8)$$

where the three-parton phase space is now a product of the dipole phase space and a two-parton phase space and where the three-parton matrix element is normalized to the two-parton matrix element such that

$$|\mathcal{M}_j|^2 \equiv \frac{1}{|\mathcal{M}_2|^2} |\mathcal{M}_3|^2. \quad (3.9)$$

The analytic integral of the subtraction term is therefore the three-parton contribution integrated over the fully inclusive phase space. The three-parton contribution to two-jet final states is thus obtained by adding and subtracting the three-parton contribution to the inclusive cross section. The subtracted term is then used in the numerical integration, accounting for all infrared singularities in the two-jet final states, while the added term is integrated analytically to make the infrared singularities explicit.

To summarize schematically the contributions to the two-jet cross section at NLO, one finds the following two- and three-parton phase space contributions

$$d\sigma_{NLO}^{2j} = d\Phi_3 |\mathcal{M}_3(p_1,\ldots,p_3)|^2 \left( \mathcal{F}_2^{(3)} - \mathcal{F}_2^{(2)} \right) + d\Phi_2 |\mathcal{M}_2|^2 \mathcal{F}_2^{(2)} \left( \int_{d\Phi_D} |\mathcal{M}_3|^2 + |\mathcal{M}_2^{V,1}|^2 \right), \quad (3.10)$$

where $|\mathcal{M}_j^{V,1}|^2$ is the normalized one-loop virtual correction to the $j$-parton matrix element, which exactly cancels all infrared poles obtained from the integral of the subtraction term. Both virtual and subtraction terms are proportional to the two-parton matrix element and phase space.
3.2 NNLO infrared subtraction terms

At NNLO, the two-jet production is induced by final states containing up to four partons, including the one-loop virtual corrections to three-parton final states. As at NLO, one has to introduce subtraction terms for the three- and four-parton contributions. Schematically the NNLO two-jet cross section reads,

$$d\sigma^{NNLO}_{\gamma\gamma} = \int_{d\Phi_4} (d\sigma^{R}_{NLO} - d\sigma^{S}_{NLO}) + \int_{d\Phi_4} d\sigma^{S}_{NLO}$$

$$+ \int_{d\Phi_3} (d\sigma^{V,1}_{NLO} - d\sigma^{V,S,1}_{NLO}) + \int_{d\Phi_3} d\sigma^{V,S,1}_{NLO} + \int_{d\Phi_2} d\sigma^{V,2}_{NLO},$$  \hspace{1cm} (3.11)

where \(d\sigma^{S}_{NLO}\) denotes the real radiation subtraction term coinciding with the four-parton tree level cross section \(d\sigma^{R}_{NLO}\) in all singular limits. Likewise, \(d\sigma^{V,S,1}_{NLO}\) is the one-loop virtual subtraction term coinciding with the one-loop three-parton cross section \(d\sigma^{V,1}_{NLO}\) in all singular limits. Finally, the two-loop correction to the two-parton cross section is denoted by \(d\sigma^{V,2}_{NLO}\).

A method to construct subtraction terms for \(n\)-parton cross sections at NNLO has been worked out in detail by Weinzierl for double real radiation [22, 25] and single real radiation off one-loop matrix elements [22]. The method which we will outline in the following differs from that proposed in [22, 25], and we shall highlight the main differences at the end of this section.

Following the line of thought elaborated in the NLO discussion above, the subtraction terms are chosen to be the full matrix elements, i.e. the sum of the full four-parton matrix elements related to the tree level processes \(\gamma^* \rightarrow q\bar{q}gg\), \(\gamma^* \rightarrow q\bar{q}q\bar{q}\) and \(\gamma^* \rightarrow qqq'q'(q \neq q')\) for \(d\sigma^{S}\) and the one-loop corrected matrix element squared of \(\gamma^* \rightarrow q\bar{q}g\) for \(d\sigma^{V,S,1}_{NLO}\).

Clearly, these subtraction terms have the appropriate behavior in all singular limits. Moreover, as explained in [33], they can be integrated analytically by reducing them algebraically to four master integrals which have been derived explicitly in [33]. Likewise the integration of the one-loop correction to \(\gamma^* \rightarrow q\bar{q}g\) over the dipole phase space can be performed analytically by reducing it to three master integrals which are listed in the appendix. These calculations are presented in Section 4 below.

To perform the phase space integration of the subtraction terms, one needs to factorize the phase space into a product of a two-particle phase space multiplying a dipole phase space for the three-particle contribution and multiplying a tripole phase space

$$d\Phi_4 = P_2 d\Phi_T, \hspace{1cm} (3.12)$$

for the four-particle contribution. The latter factorization [24, 27, 33] is obtained by redefining a set of four massless on-shell momenta (emitter, two unresolved partons, spectator) into two on-shell massless momenta.

The construction of the NNLO subtraction term to the four-parton tree level cross section starts from the full four-parton matrix element:

$$d\sigma^{S,0}_{NNLO} = N_m d\Phi_4 (p_1,...,p_4,Q) |M_4(p_1,...p_4)|^2 \left( \frac{1}{2} F^{(2)}_2 (\tilde{p}_{134}, \tilde{p}_2) + \frac{1}{2} F^{(2)}_2 (\tilde{p}_{234}, \tilde{p}_1) \right), \hspace{1cm} (3.13)$$

which is in analogy to the NLO subtraction term. However, in order to have a subtraction term which correctly accounts for all singularities in the two-jet region and can be integrated over the inclusive (i.e. two-jet, three-jet and four-jet) phase space, this is not sufficient. The infrared singularities yielded by the subtraction term in the two-jet region of the four-parton phase space exactly cancel those coming from the double real emission matrix element integrated over the same two-jet region. Furthermore, the subtraction term yields only finite terms when integrated over the four-jet region. However, the subtraction term generates single infrared singularities when integrated over the three-jet region of the four-parton phase space which need to be treated carefully. Indeed, the singular structure in the three-jet region has to be mapped out and subtracted from the subtraction term. This is done by subtracting the sum of all dipole terms appropriate to single-particle singularities from the four-parton matrix element subtraction term. According to (33), these read

$$d\sigma^{S,1}_{NNLO} = N_m d\Phi_4 (p_1,...,p_4,Q) \sum_{\text{pairs } i,j} \sum_{k \neq i,j} D_{ijk} |M_3(p_i, \tilde{p}_{ij}, \tilde{p}_k)|^2 F^{(3)}_3 (p_i, \tilde{p}_{ij}, \tilde{p}_k),$$  \hspace{1cm} (3.14)
where \( l \neq i, j, k \). With this, the full NNLO subtraction term becomes

\[
\delta\sigma_{NLO}^S = \delta\sigma_{NLO}^{S,0} - \delta\sigma_{NLO}^{S,1}.
\]

(3.15)

Together with

\[
\delta\sigma_{NLO}^R = N_{in} \Phi_4(p_1, \ldots, p_4, Q) |\mathcal{M}_4(p_1, \ldots, p_4)|^2 F_2^{(4)}(p_1, p_2, p_3, p_4),
\]

(3.16)

we obtain the difference between double real and subtracted matrix element as,

\[
\delta\sigma_{NLO}^R - \delta\sigma_{NLO}^S = N_{in} \Phi_4(p_1, \ldots, p_4, Q) \times
\]

\[
\left[ |\mathcal{M}_4(p_1, \ldots, p_4)|^2 \left( F_2^{(4)}(p_1, p_2, p_3, p_4) - \frac{1}{2} F_2^{(2)}(\tilde{p}_{134}, \tilde{p}_2) - \frac{1}{2} F_2^{(2)}(\tilde{p}_{234}, \tilde{p}_1) \right)
\]

\[
+ \sum_{\text{pairs } i,j} \sum_{k \neq i,j} D_{ijk} |\mathcal{M}_3(p_i, \tilde{p}_{ij}, \tilde{p}_k)|^2 F_3^{(3)}(p_i, \tilde{p}_{ij}, \tilde{p}_k) \right],
\]

(3.17)

which denotes the full contribution from the four-parton channel to the two-jet production cross section. It is finite and can therefore be integrated numerically.

The subtraction term for the one-loop virtual contribution is constructed in complete analogy to the NLO three-particle subtraction term (3.7) by using the full one-loop three-particle matrix element. We have,

\[
\delta\sigma_{NNLO}^{V,1} - \delta\sigma_{NNLO}^{V,S,1} = N_{in} \Phi_3(p_1, \ldots, p_3, Q) |\mathcal{M}_3^{V,1}(p_1, \ldots, p_3)|^2 \times
\]

\[
\left( F_2^{(3)}(p_1, p_2, p_3) - \frac{1}{2} F_2^{(2)}(\tilde{p}_{13}, \tilde{p}_2) - \frac{1}{2} F_2^{(2)}(\tilde{p}_{23}, \tilde{p}_1) \right).
\]

(3.18)

The difference between the one-loop virtual contribution and its subtraction term, \( \delta\sigma_{NNLO}^{V,1} - \delta\sigma_{NNLO}^{V,S,1} \), is integrated over the two-jet and three-jet regions of the three-parton final state phase space. In the two-jet region, it is finite (in fact, yields a zero contribution) and can be integrated numerically in a straightforward manner. Although \( \delta\sigma_{NNLO}^{V,S,1} \) will not produce infrared poles from real emission in the three-jet region of the final state phase space, it will still generate infrared poles from this region, since \( |\mathcal{M}_3^{V,1}(p_1, \ldots, p_3)|^2 \) contains explicit infrared poles from the loop integration. These poles are proportional to the tree level three-particle matrix element squared, and are canceled by extracting the infrared poles from the real emission subtraction term \( \delta\sigma_{NNLO}^{S,1} \). To see this, we express

\[
\delta\sigma_{NNLO}^{S,1} = N_{in} \Phi_4(p_1, \ldots, p_4, Q) \sum_{\text{pairs } i,j} \sum_{k \neq i,j} D_{ijk} |\mathcal{M}_3(p_i, \tilde{p}_{ij}, \tilde{p}_k)|^2 F_3^{(3)}(p_i, \tilde{p}_{ij}, \tilde{p}_k)
\]

\[
\times \int_{d\Phi_D} D_{ijk}.
\]

(3.19)

Since the sum of integrals of the dipole terms over the dipole phase space exactly cancels the infrared singularities present in the corresponding one-loop matrix element, we find that

\[
- \delta\sigma_{NNLO}^{V,S,1} - \delta\sigma_{NNLO}^{S,1} = N_{in} \Phi_3(p_1, p_2, p_3, Q) \times
\]

\[
\left[ |\mathcal{M}_3^{V,1}(p_1, \ldots, p_3)|^2 \left( -\frac{1}{2} F_2^{(2)}(\tilde{p}_{13}, \tilde{p}_2) - \frac{1}{2} F_2^{(2)}(\tilde{p}_{23}, \tilde{p}_1) \right)
\]

\[
- |\mathcal{M}_3(p_1, p_2, p_3)|^2 F_3^{(3)}(p_1, p_2, p_3) \right) \sum_{\text{pairs } i,j; k \neq i,j} \int_{d\Phi_D} D_{ijk}
\]

(3.20)

is finite if integrated over the three-jet region of the three-parton final state phase space. The full contribution from the three-parton channel to two-jet final states is therefore

\[
\delta\sigma_{NNLO}^{V,1} - \delta\sigma_{NNLO}^{V,S,1} - \delta\sigma_{NNLO}^{S,1} =
\]

6
\[ N_{\text{in}} d\Phi_3(p_1, p_2, p_3, Q) \left[ |M_{3}^{V,1}(p_1, \ldots p_3)|^2 \left( F_2^{(3)}(p_1, p_2, p_3) - \frac{1}{2} F_2^{(2)}(\vec{p}_{13}, \vec{p}_2) - \frac{1}{2} F_2^{(2)}(\vec{p}_{23}, \vec{p}_1) \right) \right. \\
\left. - |M_{3}(p_1, p_2, p_3)|^2 F_3^{(3)}(p_1, p_2, p_3) \sum_{\text{pairs } i,j,k \neq i,j} \int d\Phi_D D_{ijk} \right], \] (3.21)

which is finite over the full three-parton phase space and can be integrated numerically.

Finally, the subtraction terms \( d\sigma_{NNLO}^{S,0} \) and \( d\sigma_{NNLO}^{V,S,1} \) integrated over the triple and dipole phase space cancel the infrared singularities present in the two-loop virtual contribution \( d\sigma_{NNLO}^{V,1} \). Using \( F_2^{(2)} = 1 \), we find

\[ d\sigma_{NNLO}^{S,0} = N_{\text{in}} d\Phi_4 |M_4|^2 = N_{\text{in}} d\Phi_2 |M_2|^2 \int_{d\Phi_T} |M_4|^2, \] (3.22)

\[ d\sigma_{NNLO}^{V,S,1} = N_{\text{in}} d\Phi_4 |M_3^{V,1}|^2 = N_{\text{in}} d\Phi_2 |M_2|^2 \int_{d\Phi_D} |M_3^{V,1}|^2. \] (3.23)

Adding these to the two-loop virtual contributions,

\[ d\sigma_{NNLO}^{V,2} + d\sigma_{NNLO}^{S,0} + d\sigma_{NNLO}^{V,S,1} = N_{\text{in}} d\Phi_2 |M_2|^2 \left[ |M_2^{V,2}|^2 + \int_{d\Phi_T} |M_4|^2 + \int_{d\Phi_D} |M_3^{V,1}|^2 \right], \] (3.24)

we find an infrared finite result for the two-parton channel.

To summarize schematically, the NNLO corrections to the two-jet rate take the following structure

\[ d\sigma_{NNLO} = \left[ d\sigma_{NNLO}^R - d\sigma_{NNLO}^{S,0} + d\sigma_{NNLO}^{V,S,1} \right] + \left[ d\sigma_{NNLO}^{V,1} - d\sigma_{NNLO}^{V,S,1} - d\sigma_{NNLO}^{S,1} \right] + \left[ d\sigma_{NNLO}^{V,2} + d\sigma_{NNLO}^{S,0} + d\sigma_{NNLO}^{V,S,1} \right], \] (3.25)

\[ = d\Phi_4 \left[ |M_4|^2 \left( F_2^{(4)} - F_2^{(2)} \right) + \sum_{ijk} |M_3|^2 D_{ijk} F_3^{(3)} \right] \\
+ d\Phi_4 \left[ |M_3^{V,1}|^2 \left( F_2^{(3)} - F_2^{(2)} \right) - \sum_{ijk} |M_3|^2 \left( \int_{d\Phi_D} D_{ijk} \right) F_3^{(3)} \right] \\
+ d\Phi_2 |M_2|^2 \left[ |M_2^{V,2}|^2 + \int_{d\Phi_T} |M_4|^2 + \int_{d\Phi_D} |M_3^{V,1}|^2 \right] F_2^{(2)}. \] (3.26)

The two-, three- and four-parton final state contributions are made explicit in this formula as those contain respectively the corresponding parton phase space \( d\Phi_i \). These correspond to (3.21), (3.22) and (3.23), which are all separately finite and can be integrated numerically over the two-, three- and four-parton final space respectively.

The interpretation of the individual terms is as follows: the infrared singularities of the four-parton contribution \( d\sigma_{NNLO}^R \) to two-jet final states are subtracted through the full matrix element, which is integrated over the inclusive phase space, \( d\sigma_{NNLO}^{S,0} \). Since this term also contains infrared singularities in the three-jet region, a further subtraction term \( d\sigma_{NNLO}^{V,S,1} \) applicable only in this region, has to be subtracted off \( d\sigma_{NNLO}^{S,1} \). Infrared singularities from the one-loop virtual correction to the three-parton contribution to two-jet final states \( d\sigma_{NNLO}^{V,1} \) are subtracted through the full one-loop matrix element integrated over the inclusive phase space, \( d\sigma_{NNLO}^{V,S,1} \). This term yields explicit infrared poles in the three-jet region, which cancel with the infrared singularities obtained by integrating \( d\sigma_{NNLO}^{S,1} \) over the single unresolved phase space appropriate to three-jet final states. Finally, all infrared poles produced through the two-jet contribution of the inclusive phase space integration of the subtraction terms \( d\sigma_{NNLO}^{S,0} \) and \( d\sigma_{NNLO}^{V,S,1} \) cancel with the poles from the two-loop virtual corrections, \( d\sigma_{NNLO}^{V,2} \).
The construction of the subtraction terms presented here differs in one essential point from the procedure proposed in [25]: we first subtract all double and single unresolved singularities from the real radiation contribution \( d\sigma^{R}_{NNLO} \) using \( d\sigma^{S1}_{NNLO} \) and afterwards add in the oversubtracted single unresolved singularities through \( d\sigma^{S1}_{NNLO} \). In contrast, [25] first subtracts all single unresolved singularities from the real radiation contribution \( d\sigma^{R}_{NNLO} \) and then constructs subtraction terms which account only for the double real emission contributions in the (already singly subtracted) matrix element. The double real emission subtraction terms constructed this way are more involved than the corresponding terms employed here. In particular, the subtraction terms obtained in [25] can not be related to four-particle matrix elements in a straightforward manner, which renders their analytic phase space integration more difficult. As a matter of fact, the subtraction terms constructed here can be integrated using the methods derived in [33], while the subtraction terms of [25] first require the computation of as yet unknown classes of integrals [38].

On the other hand, let us note that taking the subtraction term to be the full four-parton matrix element squared we clearly do not separate singularities arising from individual partonic configurations. Therefore, more work is needed for extending the method derived here to final states with more than two jets, while the subtraction terms of [25] readily generalize to final states with higher jet multiplicity.

In the next section, we analytically compute the infrared poles arising in each parton level contribution discussed here. For simplicity, we set \( F_{J}^{(0)} = 1 \) in the following (corresponding to computing the inclusive cross section) and discard the single-particle subtraction term \( d\sigma^{S1}_{NNLO} \), whose infrared structure is well known from NLO calculations [18] and whose contributions exactly cancel.

### 4 NNLO contributions

As outlined in Section 2 above, two-jet final states at NNLO accuracy are obtained from the production of up to four final state partons. In this section, we present the contributions from two, three and four-parton final states, elaborating the structure of infrared singularities for each channel. The integrated partonic terms derived in this section represent the integrals of the subtraction terms presented in Section 3 and account for all singularities arising from real radiation.

#### 4.1 Two-parton final states

The calculation of the virtual two-loop corrections to \( \gamma^{*} \rightarrow q\bar{q} \) (also called quark form factor) in dimensional regularization with \( d = 4 - 2\epsilon \) space-time dimensions was performed long ago [39]. In the following, we only list the results, and illustrate the application of the infrared factorization formula. Renormalization of ultraviolet divergences is performed in the \( \overline{\text{MS}} \) scheme by replacing the bare coupling \( \alpha_{0} \) with the renormalized coupling \( \alpha_{s} \equiv \alpha_{s}(\mu^{2}) \), evaluated at the renormalization scale \( \mu^{2} \)

\[
\alpha_{0}\mu_{0}^{2}\epsilon = \alpha_{s}\mu^{2} \left[ 1 - \frac{11N - 2N_{f}}{6\epsilon} \left( \frac{\alpha_{s}}{2\pi} \right) + \mathcal{O}(\alpha_{s}^{2}) \right],
\]

where

\[
\epsilon = (4\pi)^{\gamma - 1} \quad \text{with Euler constant} \quad \gamma = 0.5772 \ldots
\]

and \( \mu_{0}^{2} \) is the mass parameter introduced in dimensional regularization to maintain a dimensionless coupling in the bare QCD Lagrangian density.

The renormalized amplitude can be written as

\[
\langle M | M \rangle_{qq} = \sqrt{4\pi \alpha e} \left[ |M^{(0)}\rangle_{qq} + \left( \frac{\alpha_{s}}{2\pi} \right) |M^{(1)}\rangle_{qq} + \left( \frac{\alpha_{s}}{2\pi} \right)^{2} |M^{(2)}\rangle_{qq} + \mathcal{O}(\alpha_{s}^{3}) \right],
\]

where \( \alpha \) denotes the electromagnetic coupling constant, \( e_{q} \) the quark charge, \( \alpha_{s} \) the QCD coupling constant at the renormalization scale \( \mu \), and the \( |M^{(i)}\rangle \) are the \( i \)-loop contributions to the renormalized amplitude. They are scalars in colour space.

The squared amplitude, summed over spins, colours and quark flavours, is denoted by

\[
\langle M | M \rangle_{qq} = \sum |M(\gamma^{*} \rightarrow q\bar{q})|^{2} = T_{qq}(q^{2})
\]

(4.3)
The perturbative expansion of $T_{q\bar{q}}(q^2)$ at renormalization scale $\mu^2 = q^2$ reads:

$$T_{q\bar{q}} = 4\pi\alpha \sum_q e_q^2 \left[ T_{q\bar{q}}^{(2)}(q^2) + \left( \frac{\alpha_s(q^2)}{2\pi} \right) T_{q\bar{q}}^{(4)}(q^2) + \left( \frac{\alpha_s(q^2)}{2\pi} \right)^2 T_{q\bar{q}}^{(6)}(q^2) + \mathcal{O}(\alpha_s^3(q^2)) \right], \quad (4.4)$$

where

$$T_{q\bar{q}}^{(2)}(q^2) = \langle M^{(0)|M^{(0)}}_{q\bar{q}} = 4N(1 - \epsilon)q^2 \rangle, \quad (4.5)$$

$$T_{q\bar{q}}^{(4)}(q^2) = \langle M^{(0)|M^{(1)}}_{q\bar{q}} + \langle M^{(1)|M^{(0)}}_{q\bar{q}} \rangle \right], \quad (4.6)$$

$$T_{q\bar{q}}^{(6)}(q^2) = \langle M^{(1)|M^{(1)}}_{q\bar{q}} + \langle M^{(2)|M^{(0)}}_{q\bar{q}} + \langle M^{(2)|M^{(0)}}_{q\bar{q}} \rangle \right], \quad (4.7)$$

where $N$ is the number of colours.

In the following, we decompose $T_{q\bar{q}}^{(6)}(q^2)$ into the contributions from the interference of two-loop and tree diagrams,

$$T_{q\bar{q}}^{(6,2\times0)}(q^2) = \langle M^{(0)|M^{(2)}}_{q\bar{q}} + \langle M^{(2)|M^{(0)}}_{q\bar{q}} \rangle \right], \quad (4.8)$$

$$+ \left[ \frac{1}{4\pi^4} - \frac{3}{4\pi^2} + \frac{1}{\epsilon^2} \left( \frac{41}{16} + \frac{13\pi^2}{24} \right) \right] - \frac{221}{32} + \frac{3\pi^2}{2} + \frac{8}{3}\zeta_3) + \left( \frac{1151}{64} + \frac{475\pi^2}{96} + \frac{29\pi^4}{288} \right) + N_f \left( \frac{65}{216} + \frac{\pi^2}{24} + \frac{2085}{1296} - \frac{91\pi^2}{216} + \frac{19}{18}\zeta_3 + \mathcal{O}(\epsilon) \right) \right], \quad (4.9)$$

as well as the one-loop self-interference,

$$T_{q\bar{q}}^{(6,1\times1)}(q^2) = \langle M^{(1)|M^{(1)}}_{q\bar{q}} \rangle \right], \quad (4.10)$$

$$+ \left[ \frac{1}{4\pi^4} - \frac{3}{4\pi^2} + \frac{1}{\epsilon^2} \left( \frac{41}{16} + \frac{13\pi^2}{24} \right) \right] - \frac{221}{32} + \frac{3\pi^2}{2} + \frac{8}{3}\zeta_3) + \left( \frac{1151}{64} + \frac{475\pi^2}{96} + \frac{29\pi^4}{288} \right) + N_f \left( \frac{65}{216} + \frac{\pi^2}{24} + \frac{2085}{1296} - \frac{91\pi^2}{216} + \frac{19}{18}\zeta_3 + \mathcal{O}(\epsilon) \right) \right], \quad (4.9)$$

The pole structure of $T_{q\bar{q}}^{(4)}$ and of both contributions to $T_{q\bar{q}}^{(6)}$ can be expressed using an infrared factorization formula, obtained from resummation [36]. To this end, we write

$$T_{q\bar{q}}^{(4)}(q^2) = \text{Poles}_{q\bar{q}}^{(1\times0)}(q^2) + \text{Finite}_{q\bar{q}}^{(1\times0)}(q^2), \quad (4.11)$$

$$T_{q\bar{q}}^{(6,1\times0)}(q^2) = \text{Poles}_{q\bar{q}}^{(1\times0)}(q^2) + \text{Finite}_{q\bar{q}}^{(1\times0)}(q^2). \quad (4.12)$$
\( \mathcal{P}oles_{q\bar{q}} \) contains infrared singularities that will be analytically canceled by those occurring in radiative processes of the same order. \( \mathcal{F}_{\text{finite}} \) is the renormalized remainder, which is finite as \( \epsilon \to 0 \). For simplicity we set the renormalization scale \( \mu^2 = q^2 \).

The infrared factorization formula shows how to predict infrared pole structure of the one-loop and two-loop contributions renormalized in the \( \overline{\text{MS}} \) scheme in terms of the tree and renormalized one-loop amplitudes, \( |\mathcal{M}^{(0)}\rangle \) and \( |\mathcal{M}^{(1)}\rangle \) respectively. At one-loop it yields

\[
\mathcal{P}oles_{q\bar{q}}^{(1 \times 0)} = 2 \Re \langle \mathcal{M}^{(0)} | I^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle,
\]

while at two loops

\[
\mathcal{P}oles_{q\bar{q}}^{(2 \times 0)} = 2 \Re \left[ \frac{1}{2} \langle \mathcal{M}^{(0)} | I^{(1)}(\epsilon) I^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle - \frac{\beta_0}{\epsilon} \langle \mathcal{M}^{(0)} | I^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle \\
+ \langle \mathcal{M}^{(0)} | I^{(1)}(\epsilon) | \mathcal{M}^{(1)} \rangle \\
+ e^{-\epsilon \gamma} \frac{\Gamma(1 - 2\epsilon)}{\Gamma(1 - \epsilon)} \left( \frac{\beta_0}{\epsilon} + K \right) \langle \mathcal{M}^{(0)} | I^{(1)}(2\epsilon) | \mathcal{M}^{(0)} \rangle \\
+ \langle \mathcal{M}^{(0)} | H^{(2)}(\epsilon) | \mathcal{M}^{(0)} \rangle \right],
\]

and

\[
\mathcal{P}oles_{q\bar{q}}^{(1 \times 1)} = \Re \left[ 2 \langle \mathcal{M}^{(1)} | I^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle - \langle \mathcal{M}^{(0)} | I^{(1)}(\epsilon) I^{(1)}(\epsilon) | \mathcal{M}^{(0)} \rangle \right],
\]

where the constant \( K \) is

\[
K = \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{10}{9} T_R N_F.
\]

It should be noted that, in this prescription, part of the finite terms in \( \mathcal{T}^{(4)}_{q\bar{q}} \) and \( \mathcal{T}^{(6,1\times j)}_{q\bar{q}} \) are accounted for by the \( \mathcal{O}(\epsilon^0) \) expansion of \( \mathcal{P}oles_{q\bar{q}}^{(1 \times j)} \).

For the process under consideration, there is only one colour structure present at tree level which is just the contraction of the quark and antiquark colours \( i \) and \( j \), it is simply \( \delta_{ij} \). Adding higher loops does not introduce additional colour structures, and the amplitudes are therefore scalars. Similarly, the infrared singularity operator \( I^{(1)}(\epsilon) \) is a 1 \( \times \) 1 matrix in the colour space. By explicit evaluation of \( \mathcal{P}oles_{q\bar{q}}^{(1 \times j)} \), one can show that \( I^{(1)}(\epsilon) \) is only determined up to a finite constant (which we shall label \( c_1 \) below) that can be chosen arbitrarily. In fact, in [18], where \( I^{(1)}(\epsilon) \) was derived for the first time, a constant term arising from the finite contributions to the dipole matrix elements was found. This term was omitted in [36].

Including \( c_1 \), \( I^{(1)}(\epsilon) \) is given by,

\[
I^{(1)}(\epsilon) = -\frac{e^{\epsilon \gamma}}{2 \Gamma(1 - \epsilon)} \left[ \frac{N^2 - 1}{2N} \left( \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + c_1 \right) S_{12} \right],
\]

(since we have set \( \mu^2 = q^2 \)):

\[
S_{12} = \left( \frac{\mu^2}{q^2} \right)^\epsilon = (-1)^\epsilon.
\]

Note that on expanding \( S_{12} \), imaginary parts are generated, the sign of which is fixed by the small imaginary part \( + i 0 \) of \( q^2 \). Other combinations such as \( \langle \mathcal{M}^{(0)} | I^{(1)}(\epsilon) \rangle \) are obtained by using the hermitian conjugate operator \( I^{(1)}(\epsilon) \), where the only practical change is that the sign of the imaginary part of \( S_{12} \) is reversed.

Finally, the last term of Eq. (4.13) that involves \( H^{(2)}(\epsilon) \) produces only a single pole in \( \epsilon \) and is given by,

\[
\langle \mathcal{M}^{(0)} | H^{(2)}(\epsilon) | \mathcal{M}^{(0)} \rangle = \frac{e^{\epsilon \gamma}}{4 \epsilon \Gamma(1 - \epsilon)} H^{(2)}(\epsilon) \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle,
\]
where in the $\overline{\text{MS}}$ scheme

$$H_q^{(2)} = \left( \frac{7}{4} \zeta_3 + \frac{409}{864} \frac{11 \pi^2}{96} \right) N^2 + \left( -\frac{1}{4} \zeta_3 - \frac{41}{108} - \frac{\pi^2}{96} \right) + \left( -\frac{3}{2} \zeta_3 - \frac{3}{32} + \frac{\pi^2}{8} \right) \frac{1}{N^2}$$

(4.20)

As a result, one finds the finite parts

$$\frac{\text{Finite}_{qg}^{(2 \times 0)}(q^2)}{4Nq^2} = \left( N - \frac{1}{N} \right) \left( -4 + \frac{c_1}{2} \right)$$

(4.21)

$$\frac{\text{Finite}_{qg}^{(1 \times 1)}(q^2)}{4Nq^2} = 4 \left( N - \frac{1}{N} \right)^2 \frac{(8 - c_1)^2}{64}.$$  

(4.22)

As with $\zeta_3$, the process-dependent $H_q^{(2)}$ can be constructed by counting the number of radiating partons present in the event. In our case, there is only a quark–antiquark pair present in the final state, so that

$$H_q^{(2)} = 2H_q^{(2)}$$

where the constant $H_q^{(2)}$ is renormalization-scheme-dependent, but independent of the choice of $c_1$. As with the single pole parts of $I^{(1)}(c)$, the process-dependent $H_q^{(2)}$ can be constructed by counting the number of radiating partons present in the event. In our case, there is only a quark–antiquark pair present in the final state, so that

$$H_q^{(2)} = 2H_q^{(2)}$$

where in the $\overline{\text{MS}}$ scheme

$$H_q^{(2)} = \left( \frac{7}{4} \zeta_3 + \frac{409}{864} \frac{11 \pi^2}{96} \right) N^2 + \left( -\frac{1}{4} \zeta_3 - \frac{41}{108} - \frac{\pi^2}{96} \right) + \left( -\frac{3}{2} \zeta_3 - \frac{3}{32} + \frac{\pi^2}{8} \right) \frac{1}{N^2}$$

(4.20)

As a result, one finds the finite parts

$$\frac{\text{Finite}_{qg}^{(1 \times 0)}(q^2)}{4Nq^2} = \left( N - \frac{1}{N} \right) \left( -4 + \frac{c_1}{2} \right)$$

(4.21)

$$\frac{\text{Finite}_{qg}^{(2 \times 0)}(q^2)}{4Nq^2} = \left( N - \frac{1}{N} \right) \left[ N \left( -\frac{81659}{5184} \frac{5 \pi^2}{36} + \frac{389}{72} \zeta_3 + \frac{49 \pi^4}{1440} + c_1 \left( \frac{31}{36} - \frac{\pi^2}{12} + \frac{c_1}{16} \right) \right) + \frac{1}{N} \left( \frac{255}{64} + \frac{29 \pi^2}{48} + \frac{15}{4} \zeta_3 + \frac{11 \pi^4}{360} + c_1 \left( 1 - \frac{c_1}{16} \right) \right) + N_f \left( \frac{4085}{1296} + \frac{7 \pi^2}{72} - \frac{1}{36} \zeta_3 - \frac{5 c_1}{18} \right) \right].$$

(4.22)

$$\frac{\text{Finite}_{qg}^{(1 \times 1)}(q^2)}{4Nq^2} = 4 \left( N - \frac{1}{N} \right)^2 \frac{(8 - c_1)^2}{64}.$$  

(4.23)

4.2 Three-parton final states

The three-particle final state yielding two jets is $\gamma^* \rightarrow q\bar{q}g$ in the configuration where only two jets are formed from three partons.

The renormalized amplitude for this process can be written as

$$|M|_{q\bar{q}g} = \sqrt{4\pi \alpha_e q} \sqrt{4\pi \alpha_s} \left[ |M^{(0)}|_{q\bar{q}g} + \left( \frac{\alpha_s}{2\pi} \right) |M^{(1)}|_{q\bar{q}g} + \mathcal{O}(\alpha_s^2) \right],$$

(4.25)

where $|M^{(i)}|$ are the $i$-loop contributions to the renormalized amplitude. The squared amplitude, summed over spins, colours and quark flavours, is denoted by

$$\langle M | M \rangle_{q\bar{q}g} = \sum |M(\gamma^* \rightarrow q\bar{q}g)|^2.$$  

(4.26)

The perturbative expansion of the squared amplitude at renormalization scale $\mu^2 = q^2$ reads:

$$\langle M | M \rangle_{q\bar{q}g} = 4\pi \alpha \sum_q e_q^2 8\pi^2 \left[ \left( \frac{\alpha_s(q^2)}{2\pi} \right)^2 \langle M^{(0)} | M^{(0)} \rangle_{q\bar{q}g} \right.$$

$$\left. + \left( \frac{\alpha_s(q^2)}{2\pi} \right)^2 \langle M^{(1)} | M^{(1)} \rangle_{q\bar{q}g} + \mathcal{O}(\alpha_s^3(q^2)) \right].$$

(4.27)

The squared amplitudes are integrated over the three-parton phase space and multiplied by $F_3^{(2)}$, the jet function ensuring that out of three partons a two-jet final state is formed. The two-jet final state phase space for these three-parton configurations contains the regions where the gluon is either collinear or soft, and subtraction terms have to be introduced to extract the resulting infrared singularities from the partonic
cross sections. As outlined in Section 3, these subtraction terms are taken to be the full squared amplitudes associated with $\gamma^* \rightarrow q\bar{q}g$ up to the one-loop order. Those are integrated over the three-parton phase space factor $d\Phi_D$ and the integrated two-parton phase space, $P_2$. The momenta associated with this dipole phase space are the two dipole momenta (emitter and spectator) and one unresolved (collinear or soft gluon). The infrared poles present in the single unresolved contributions related to this virtual matrix element. A subtraction term is required, which is chosen to be the virtual matrix element itself. The analytic integration of this subtraction term over the dipole phase space is obtained by first reducing all terms in the integral to three master integrals, denoted $T_{qg}^{(4)}$, $V_{qg}^{(0)}$, and $V_{qg}^{(1)}$, which is made explicit as follows,

$$T_{qg}^{(4)}(q^2) = 8\pi^2 \int d\Phi_D \langle M^{(0)} | M^{(0)} \rangle_{qg}$$

$$= \left( N - \frac{1}{N} \right) T_{qg}^{(2)}(q^2) \left\{ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7\pi^2}{12} + \left( \frac{109}{8} - \frac{7\pi^2}{8} - \frac{25}{3} \zeta_3 \right) \epsilon \right. $$

$$+ \left. \left( \frac{639}{16} - \frac{133\pi^2}{48} - \frac{25}{2} \zeta_3 - \frac{71\pi^4}{1440} \right) \epsilon^2 + \mathcal{O}(\epsilon^3) \right\}. \quad (4.29)$$

The infrared poles in $T_{qg}^{(4)}$ cancel all infrared poles present in the one-loop two-particle final state contribution $T_{qg}^{(4)}$ in [4,13]:

$$T_{qg}^{(4)} = \text{Poles}_{qg}^{(0,0)} + \text{Finite}_{qg}^{(0,0)} \quad (4.30)$$

with

$$\text{Poles}_{qg}^{(0,0)} = -2R\langle M^{(0)} | I^{(1)}(\epsilon) | M^{(0)} \rangle, \quad (4.31)$$

$$\frac{\text{Finite}_{qg}^{(0,0)}}{4Nq^2} = \left( N - \frac{1}{N} \right) \left\{ \frac{19}{4} - \frac{1}{2} \right\}. \quad (4.32)$$

At $\mathcal{O}(\alpha_s^2)$, the renormalized matrix element is $\langle M^{(0)} | M^{(1)} \rangle_{qg}^{(0,0)} + \langle M^{(1)} | M^{(0)} \rangle_{qg}^{(0,0)}$. To cancel the poles present in the single unresolved contributions related to this virtual matrix element, a subtraction term is required, which is chosen to be the virtual matrix element itself. The analytic integration of this subtraction term over the dipole phase space is obtained by first reducing all terms in the integral to three master integrals, solving iteratively integration-by-parts identities as outlined in [40,41] for two-loop integrals and in [33] for four-partic phase space integrals. For this step, extensive use of the algebraic programming language FORM [42] is made. As a result, one obtains a linear combination of three master integrals, denoted by $V_{a,a}$, $V_{b,b}$, and $V_8$, which are listed in the appendix. Inserting these, one obtains

$$T_{qg}^{(6)}(q^2) = 8\pi^2 \int d\Phi_D \left( \langle M^{(0)} | M^{(1)} \rangle_{qg} + \langle M^{(1)} | M^{(0)} \rangle_{qg} \right)$$

$$= \left( N - \frac{1}{N} \right) T_{qg}^{(2)}(q^2) \left\{ N \left[ -\frac{5}{4\epsilon^3} - \frac{67}{12\epsilon^3} + \frac{1}{\epsilon^2} \left( \frac{141}{8} + \frac{13\pi^2}{8} \right) \right. $$

$$+ \left. \frac{1}{\epsilon} \left( -\frac{1481}{24} + \frac{107\pi^2}{18} + \frac{55}{3} \zeta_3 \right) + \left( -\frac{10885}{48} + \frac{64\pi^2}{3} + \frac{1265}{18} \zeta_3 - \frac{41\pi^4}{96} \right) \right\}$$

$$+ \frac{1}{\epsilon} \left( \frac{79}{2} - \frac{15\pi^2}{4} - \frac{53}{3} \zeta_3 \right) + \left( \frac{1069}{8} - \frac{697\pi^2}{48} - \frac{91}{2} \zeta_3 + \frac{19\pi^4}{72} \right) \right\} + \mathcal{O}(\epsilon) \left\} \right\}. \quad (4.34)$$

The infrared poles in $T_{qg}^{(6)}$ cancel part of the infrared poles from the two-particle final state contribution $T_{qg}^{(6)}$. This cancellation becomes evident if we decompose

$$T_{qg}^{(6)} = \text{Poles}_{qg}^{(1,0)} + \text{Finite}_{qg}^{(1,0)} \quad (4.35)$$
and identify individual terms of $\mathcal{P}oles_{qgq}$ with terms from the infrared factorization formulae (4.14) and (4.15),

$$
\mathcal{P}oles_{qgq}^{(1\times 0)} = 2R \left[ -\langle \mathcal{M}(0)|I^{(1)}(\epsilon)|\mathcal{M}(1)\rangle + \frac{\beta_0}{\epsilon} \langle \mathcal{M}(0)|I^{(1)}(\epsilon)|\mathcal{M}(0)\rangle - \langle \mathcal{M}(1)|I^{(1)}(\epsilon)|\mathcal{M}(0)\rangle \\
-\langle \mathcal{M}(0)|H_V^{(2)}(\epsilon)|\mathcal{M}(0)\rangle + \frac{1}{2} \langle \mathcal{M}(0)|S_V^{(2)}(\epsilon)|\mathcal{M}(0)\rangle \right].
$$

(4.36)

The first two terms in this formula cancel with terms in (4.14), while the third term cancels with a term in (4.15). The last two terms contain the remaining poles, and will be discussed in detail now.

For the process under consideration, the infrared singularity operator $I^{(1)}$ (4.11) contains only a single colour structure $(N^2 - 1)/(2N) = C_F$, as does the one-loop squared matrix element $T_{qg}^{(4)}$. By inserting $I^{(1)}$ (4.17) in (4.36), one finds that all $1/\epsilon$ and $1/\epsilon^3$ poles of $T_{qg}^{(6)}$ which are proportional to the colour structures $[(N^2 - 1)/(2N)]^2 = C_F^2$ and $(N^2 - 1)/(2N)N_f = C_F N_f$ are correctly accounted for. This cancellation is independent of the choice of the constant term $c_1$ in $I^{(1)}$. The $1/\epsilon^2$ terms proportional to these colour structures in (4.36) depend on $c_1$. They match the corresponding terms in $T_{qg}^{(6)}$ if we choose

$$
c_1 = \frac{43}{4} - \frac{\pi^2}{3}.
$$

(4.37)

With this choice, the first three terms of (4.36) correctly account for all $1/\epsilon$ to $1/\epsilon^2$ poles proportional to $[(N^2 - 1)/(2N)]^2 = C_F^2$ and $(N^2 - 1)/(2N)N_f = C_F N_f$ in $T_{qg}^{(6)}$, and all remaining single pole terms in these two colour structures can be attributed to the $H_V^{(2)}(\epsilon)$ term in (4.36).

Pole terms corresponding to the colour structure $N(N^2 - 1)/(2N) = C_F C_A$ can be generated only through the combination of $I^{(1)}$ and $\beta_0/\epsilon$ in (4.36), thus yielding at most $1/\epsilon^3$ poles. In $T_{qg}^{(6)}$, one does however observe $1/\epsilon^4$ terms in this colour structure, as first pointed out in [43]. These terms have no analogue in the two-particle final state contributions, and must thus cancel among three and four-particle final states. We observe that these terms are proportional to the one-loop soft gluon current squared matrix element derived in [20] (which is a scalar in colour space for this particular process), integrated over the dipole phase space. Explicitly, we find,

$$
\langle \mathcal{M}(0)|S_V^{(2)}(\epsilon)|\mathcal{M}(0)\rangle = S_V^{(2)} \langle \mathcal{M}(0)|\mathcal{M}(0)\rangle,
$$

(4.38)

with

$$
S_V^{(2)} = -\frac{\epsilon^{2\gamma}}{1 + \epsilon} \left[ (N^2 - 1) \frac{\Gamma^4(1 - \epsilon)\Gamma^3(1 + \epsilon)}{\Gamma^2(1 - 2\epsilon)\Gamma(1 + 2\epsilon)} \right] \int d\Phi_D \left( \frac{q^2}{s_{13}s_{23}} \right)^{1+\epsilon}
$$

$$
= (N^2 - 1) \left[ -\frac{1}{4\epsilon^4} - \frac{3}{4\epsilon^3} + \frac{1}{\epsilon^2} \left( -\frac{13}{4} + \frac{7\pi^2}{24} \right) + \frac{1}{\epsilon} \left( -\frac{51}{4} + \frac{7\pi^2}{8} + \frac{14}{3} \zeta_3 \right) \right]
$$

$$
+ \left( -\frac{205}{4} + \frac{91\pi^2}{24} + 14\zeta_3 + \frac{7\pi^4}{480} \right) + O(\epsilon).
$$

(4.39)

The integrated soft gluon current accounts for all $1/\epsilon^4$ to $1/\epsilon^2$ poles proportional to $N(N^2 - 1)/(2N) = C_F C_A$ in $T_{qg}^{(6)}$, the remaining single poles are again attributed to $H_V^{(2)}(\epsilon)$, which finally reads,

$$
\langle \mathcal{M}(0)|H_V^{(2)}(\epsilon)|\mathcal{M}(0)\rangle = \frac{\epsilon^{2\gamma}}{4\epsilon \Gamma(1 - \epsilon)} H_V^{(2)} \langle \mathcal{M}(0)|\mathcal{M}(0)\rangle
$$

(4.40)

with

$$
H_V^{(2)} = 2H_{V,q}^{(2)}.
$$

(4.41)
The expression for $H^{(2)}_{\gamma,q}$ in the $\overline{\text{MS}}$ scheme is given by,

$$
H^{(2)}_{\gamma,q} = \left( -11\zeta_3 + \frac{409}{24} + \frac{\pi^2}{18} \right) N^2 + \left( \frac{26\zeta_3}{48} - \frac{1655\pi^2}{48} - \frac{\pi^2}{18} \right) + \left( -15\zeta_3 + \frac{279}{16} \right) \frac{1}{N^2} \\
+ \left( \frac{\pi^2}{18} + \frac{5}{24} \right) \left( N^2 - 1 \right) N_F \left( N - \frac{1}{N} \right) N^2.
$$

(4.43)

To summarize the discussion of the infrared pole terms arising from three-particle final states at this order, we have demonstrated that all double and higher poles can either be identified with terms present in the infrared factorization formulae for the virtual corrections (provided an appropriate finite constant is inserted in the infrared singularity operator $I^{(1)}$) or with the dipole phase space integral of the soft gluon current. The single poles are attributed to a constant $H^{(2)}_{\gamma}$, whose origin (like the origin of the $H^{(2)}_{\gamma}$ appearing in the double virtual corrections) is not fully understood at present. After subtraction of the poles, one is left with,

$$
\frac{F^{\text{final}}_{qgq}}{4Nq^2} = \left( N - \frac{1}{N} \right) \left[ N \left( -\frac{5549}{48} + \frac{41\pi^2}{24} + \frac{143}{3} \zeta_3 + \frac{41\pi^4}{180} \right) + \frac{1}{N} \left( \frac{673}{8} - 7\pi^2 - \frac{75}{2} \zeta_3 - \frac{73\pi^4}{180} \right) + N_f \left( \frac{109}{24} - \frac{8}{3} \zeta_3 \right) \right].
$$

(4.44)

4.3 Four-parton final states

Three different tree level processes can yield four-parton final states relevant to two-jet final states: $\gamma^* \to q\bar{q}q'\bar{q}'$ ($q \neq q'$), $\gamma^* \to q\bar{q}q\bar{q}$ and $\gamma^* \to q\bar{q}g g$. The amplitude for each process can be written as

$$
|M\rangle_{q\bar{q}ij} = \sqrt{4\pi\alpha_s} \left[ \langle M^{(0)}\rangle_{q\bar{q}ij} + \mathcal{O}(\alpha_s) \right],
$$

(4.45)

where $ij = q'\bar{q}'$, $q\bar{q}$, $gg$.

The squared amplitude, summed over spins, colours and quark flavors, is denoted by

$$
\langle M|\langle M\rangle_{q\bar{q}ij} = \sum_i \langle M(\gamma^* \to q\bar{q}ij)\rangle^2.
$$

(4.46)

The perturbative expansion of the squared amplitude at renormalization scale $\mu^2 = q^2$ reads:

$$
\langle M|\langle M\rangle_{q\bar{q}ij} = 4\pi\alpha \sum_q e_q^2 64\pi^4 \left[ \left( \frac{\alpha_s(q^2)}{2\pi} \right)^2 \langle M^{(0)}\rangle_{q\bar{q}ij} + \mathcal{O}(\alpha_s(\alpha_s(q^2))) \right].
$$

(4.47)

These amplitudes need to be integrated over the four-parton phase space and in order to give rise to contributions to the two-jet rate, they need to be multiplied by the corresponding jet-function $F^{(2)}_\gamma$. Indeed, two of the four final state partons are experimentally unresolved. The two-jet final state phase space, however also contain regions where one or two particles are theoretically unresolved; the associated partonic cross sections contain single and double singularities. Those are canceled analytically through the introduction of subtraction terms. The appropriate subtraction terms for the infrared singular structure of these real-real corrections are the corresponding matrix elements themselves as explained in Section 3 above. These are to be integrated over the tripole phase space factor $d\Phi_T$ (5.12), which is a factorized form of the four-parton phase space $d\Phi_4$. These integrals have been discussed in detail in [33], where it was demonstrated that any integral appearing in this context can be expressed as a linear combination of four master integrals $R_4$, $R_6$, $R_{6b}$ and $R_{8b}$, which were derived in [33] and are listed in the appendix for completeness. As a result, one finds the integrated subtraction terms,

$$
\langle M^{(0)}|\langle M^{(0)}\rangle_{q\bar{q}ij} = 64\pi^4 \int d\Phi_T \langle M^{(0)}|\langle M^{(0)}\rangle_{q\bar{q}ij}\rangle_{q'\bar{q}'}
$$

(4.48)
\[ T_{qg}(q^2) = \mathcal{T}_{qg}^{(2)}(q^2) \left[ N - \frac{1}{N} \right] (N_f - 1) \]
\[ + \frac{1}{12} - \frac{7}{8} - \frac{1}{16} \left[ \frac{11\pi^2}{72} \right] + \left[ -\frac{11753}{1296} + \frac{77\pi^2}{108} + \frac{67}{18}\zeta_3 \right] \]
\[ + O(\epsilon) \] 

\[ T_{qqq}(q^2) = 64\pi^4 \int d\Phi_T \langle M^{(0)} | M^{(0)} \rangle_{qgq} \]
\[ + \frac{1}{N_f - 1} T_{qqq}(q^2) + T_{qq}(q^2) \left[ N - \frac{1}{N} \right] \]
\[ + \frac{1}{16} - \frac{1}{8} \left[ \frac{13}{16} \right] + \left[ -\frac{39}{32} - \frac{17\pi^2}{24} - \frac{21}{4}\zeta_3 + \frac{2\pi^4}{45} \right] + O(\epsilon) \]

\[ T_{qgg}(q^2) = 64\pi^4 \int d\Phi_T \langle M^{(0)} | M^{(0)} \rangle_{qgqg} \]
\[ + \frac{1}{N_f - 1} T_{qqq}(q^2) + T_{qq}(q^2) \left[ N - \frac{1}{N} \right] \]
\[ + \frac{1}{N} \left[ \frac{1}{\epsilon} \left( \frac{13}{16} - \frac{8}{32} + \frac{3}{2}\zeta_3 \right) + \left( -\frac{3}{2} + \frac{3\pi^2}{4} \right) \right] \]
\[ + \frac{1}{16} - \frac{1}{8} \left[ \frac{13}{16} \right] + \left[ -\frac{6921}{64} + \frac{473\pi^2}{48} + 40\zeta_3 - \frac{17\pi^4}{144} \right] + O(\epsilon) \] 

\[ T_{qqq}^{(6)} + T_{qqg}^{(6)} + T_{qgg}^{(6)} = \mathcal{P}olcs_{qgqg}^{(0\times 0)} + \mathcal{F}inite_{qgqg}^{(0\times 0)} \] 

The infrared poles from these four-parton final states exactly cancel the infrared poles present in the two- and three-particle final states derived above. To see this cancellation, we consider

\[ \mathcal{T}_{qqq}^{(6)} + \mathcal{T}_{qqg}^{(6)} + \mathcal{T}_{qgg}^{(6)} = \mathcal{P}olcs_{qgqg}^{(0\times 0)} + \mathcal{F}inite_{qgqg}^{(0\times 0)} \] 

and identify

\[ \mathcal{P}olcs_{qgqg}^{(0\times 0)} = \mathcal{R} \left[ \langle M^{(0)} | I^{(1)}(\epsilon) I^{(1)}(\epsilon) | M^{(0)} \rangle - 2e^{-c_0} \frac{\gamma^2}{\Gamma(1 - \epsilon)} \left( \frac{\beta_0}{\epsilon} + K \right) \langle M^{(0)} | I^{(1)}(2\epsilon) | M^{(0)} \rangle \right. 
\[ + \langle M^{(0)} | I^{(1)}(\epsilon) | M^{(0)} \rangle - 2 \langle M^{(0)} | H^{(2)}_R(\epsilon) | M^{(0)} \rangle 
\[ - \langle M^{(0)} | S^{(2)}_V(\epsilon) | M^{(0)} \rangle \right] \] 

As for the three-parton final states above, we have to fix \( c_1 \) according to (4.37) to obtain cancellation of the \( 1/\epsilon^2 \) poles in the \( (N^2 - 1)/(2N)^2 = C_F^2 \) and \( (N^2 - 1)/(2N)N_f = C_F N_f \) colour structures. It can be seen that the \( qgqg \) final state also yields a term canceling the contribution from the one-loop soft gluon current in the three-parton channel. Any remaining \( 1/\epsilon \) poles are collected in

\[ \langle M^{(0)} | H^{(2)}_R(\epsilon) | M^{(0)} \rangle = \frac{c_0^2}{4\epsilon \Gamma(1 - \epsilon)} H^{(2)}_R \langle M^{(0)} | M^{(0)} \rangle, \]

with

\[ H^{(2)}_R = 2H^{(2)}_{R,q} \] 

The expression for \( H^{(2)}_{R,q} \) in the \( \overline{MS} \) scheme reads,

\[ H^{(2)}_{R,q} = \left( \frac{51}{4} \zeta_3 - \frac{14315}{864} - \frac{49\pi^2}{288} \right) N^2 + \left( -\frac{105}{4} \zeta_3 - \frac{14731}{432} + \frac{13\pi^2}{288} \right) + \left( \frac{27}{2} \zeta_3 - \frac{561}{32} + \frac{\pi^2}{8} \right) \frac{1}{N^2} \]
\[ + \left( \frac{11\pi^2}{144} - \frac{35}{108} \right) \frac{(N^2 - 1)N_F}{N}. \]
The single pole constants from single virtual and double real radiation add up to the single pole constant from the double virtual contribution,

\[ H_{V,q}^{(2)} + H_{R,q}^{(2)} = H_q^{(2)}. \]  \hfill (4.59)

Subtraction of the pole terms leaves

\[
\frac{\mathcal{F}_{\text{finite}}^{(0 \times 0)}}{4Nq^2} = \left( N - \frac{1}{N} \right) \left[ N \left( \frac{1264873}{10368} + \frac{19\pi^2}{108} - \frac{4217}{72} \zeta_3 - \frac{437\pi^4}{1440} \right) + \frac{1}{N} \left( -\frac{10637}{128} + \frac{2\pi^2}{3} + \frac{135}{4} \zeta_3 + \frac{7\pi^4}{18} \right) + N_f \left( -\frac{7883}{1296} - \frac{41\pi^2}{216} + \frac{133}{36} \zeta_3 \right) \right].
\]  \hfill (4.60)

5 Structure of infrared cancellations

In any infrared safe physical observable, such as the two-jet cross section or the inclusive hadron production rate, one observes the cancellation of all infrared singularities present in individual partonic channels after adding all channels contributing to the same (infrared safe) final state [37]. At NLO, this cancellation is obvious in

\[ \mathcal{P} \text{oles}_{q\bar{q}}^{(1 \times 0)} + \mathcal{P} \text{oles}_{q\bar{q}\bar{q}}^{(0 \times 0)} = 0. \]  \hfill (5.1)

At NNLO, the cancellation reads

\[ \mathcal{P} \text{oles}_{q\bar{q}}^{(2 \times 0)} + \mathcal{P} \text{oles}_{q\bar{q}}^{(1 \times 1)} + \mathcal{P} \text{oles}_{q\bar{q}\bar{q}}^{(1 \times 0)} + \mathcal{P} \text{oles}_{q\bar{q}(ij)}^{(0 \times 0)} = 0. \]  \hfill (5.2)

In the computation of any jet observable, only the finite terms from each partonic channel will appear.

As a check, we compute the perturbative corrections to the hadronic \( R \)-ratio, which is obtained by simply setting all measurement functions \( \mathcal{F}_j^{(\text{m})} \) to unity,

\[
\hat{R} = \frac{R_{\text{had}}}{R_{\text{tree}}}.
\]  \hfill (5.3)

\[
= 1 + \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{T_{q\bar{q}}^{(4)}(q^2) + T_{q\bar{q}\bar{q}}^{(4)}(q^2)}{T_{q\bar{q}}^{(2)}(q^2)} \right)
+ \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{T_{q\bar{q}}^{(6)}(q^2) + T_{q\bar{q}\bar{q}}^{(6)}(q^2) + T_{q\bar{q}q'}^{(6)}(q^2) + T_{q\bar{q}\bar{q}}^{(6)}(q^2)}{T_{q\bar{q}}^{(2)}(q^2)} \right)
+ \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{\mathcal{F}_{\text{finite}}^{(1 \times 0)} + \mathcal{F}_{\text{finite}}^{(0 \times 0)}}{4Nq^2} \right)
+ \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{\mathcal{F}_{\text{finite}}^{(2 \times 0)} + \mathcal{F}_{\text{finite}}^{(1 \times 1)} + \mathcal{F}_{\text{finite}}^{(1 \times 0)} + \mathcal{F}_{\text{finite}}^{(0 \times 0)}}{4Nq^2} \right)
\]  \hfill (5.4)

\[
= 1 + \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{N^2 - 1}{2N} \right) \frac{3}{2}
+ \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{N^2 - 1}{2N} \right) \left[ N \left( \frac{243}{16} - 11\zeta_3 \right) + \frac{1}{N} \frac{3}{16} + N_f \left( -\frac{11}{4} + 2\zeta_3 \right) \right].
\]  \hfill (5.5)

This result is in agreement with the literature [44], thus providing a further strong check on the correctness of all terms computed here.

6 Summary and Conclusions

In this paper, we analytically examined the infrared singularity structure of two-jet production in \( e^+ e^- \) annihilation.
For this purpose, we developed a subtraction formalism including double real radiation at tree level and single real radiation at one loop. In the case of the two-jet production process considered here, the subtraction terms coincide with the full tree level four-parton and one-loop three-parton matrix elements. The phase space structure of the subtraction extends the NLO dipole subtraction formalism [18] to NNLO. In particular, the subtraction terms are integrated (both numerically and analytically) over the inclusive phase space. For single-particle subtraction at tree level and one loop, this procedure is already sufficient to transfer all infrared singularities from the numerical integration (involving the jet definition) to an analytic expression which is canceled against infrared poles from virtual corrections. This procedure has however to be extended in the case of double-particle subtraction. Indeed, the inclusive integral of the subtraction term will yield infrared singularities from double real emission (which subtract the corresponding terms in the full tree level double real emission cross section) as well as infrared singularities from single real emission. The latter have to be subtracted from the subtraction term in order to avoid spurious infrared poles in the cross section. The infrared singularities from this second level subtraction can in the present calculation be identified to correspond to three-jet final states, which are part of the inclusive four-parton cross section. They are canceled by explicit infrared poles present in the virtual one-loop single emission subtraction term, which is integrated over the inclusive three-parton phase space. In fact, this one-loop single emission subtraction term compensates the infrared singularities of the one-loop three-parton contribution to two-jet final states, but also yields purely virtual infrared poles in the three-jet region.

The infrared pole structure of each hadronic final state contribution to two-jet final states is computed explicitly by analytical integration of the appropriate subtraction terms. To analyze the resulting infrared poles and their cancellation among different contributions, we start from Catani’s infrared factorization formula for two-loop amplitudes [36] and identify individual terms in this formula with corresponding terms in the integrals of the tree-level double real emission and one-loop single real emission subtraction terms. It turns out that an identification up to the double pole terms is possible, provided a particular finite constant (which is irrelevant for the infrared structure of the two-loop virtual corrections) is chosen for the infrared singularity operator. Besides these terms canceling with infrared singularities in the two-loop virtual corrections, we identify one particular contribution which cancels between double real emission and one-loop single real emission. This contribution is found to be proportional to the one-loop correction to the soft gluon current [20]. These identifications may help to find a more detailed insight into the structure of NNLO infrared singularities of processes with more than two jets in the final state. Once this is accomplished, it appears feasible to construct NNLO infrared subtraction terms similar to the ones proposed here for processes with higher partonic multiplicity and/or combined emission in initial and final state.

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A Master Integrals

All loop and phase space integrals in Section 4 were carried out by first reducing them to a small set of master integrals. This reduction is based on an iterative solution of integration-by-parts identities [45] and is explained in detail in [33,40,41]. For reference, we collected the analytic expressions (which can be found in various places in the literature [33, 46, 47]) for all master integrals in this appendix.
A.1 Virtual two-loop corrections

The virtual two-loop vertex master integrals were first derived in \[46\] in the context of the calculation of the two-loop quark form factor \[39\]. Factoring out a common

\[ S_T = \left( \frac{(4\pi)^\epsilon}{16\pi^2 \Gamma(1-\epsilon)} \right)^2, \quad (A.1) \]

they read

\[ A_{2,LO}^2 = \int \frac{dk}{(2\pi)^d} \int \frac{dl}{(2\pi)^d} \frac{1}{k^2(k-p_1-p_2)(l-p_1-p_2)^2} \]

\[ = S_T (-q^2)^{-2\epsilon} \frac{\Gamma^2(1+\epsilon)\Gamma^6(1-\epsilon)-1}{\Gamma^2(2-2\epsilon)} \epsilon^2, \quad (A.2) \]

\[ A_3 = \int \frac{dk}{(2\pi)^d} \int \frac{dl}{(2\pi)^d} \frac{1}{k^2l^2(k-p_1-p_2)^2(l-p_1-p_2)^2} \]

\[ = S_T (-q^2)^{-2\epsilon} \frac{\Gamma(1+2\epsilon)\Gamma^5(1-\epsilon)-1}{\Gamma(3-3\epsilon)} \frac{1}{2(1-2\epsilon)\epsilon^2}, \quad (A.3) \]

\[ A_4 = \int \frac{dk}{(2\pi)^d} \int \frac{dl}{(2\pi)^d} \frac{1}{k^2l^2(k-p_1-p_2)^2(k-l-p_1)^2} \]

\[ = S_T (-q^2)^{-2\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)\Gamma^4(1-\epsilon)\Gamma(1+2\epsilon)-1}{\Gamma(2-3\epsilon)} \frac{1}{2(1-2\epsilon)\epsilon^2}, \quad (A.4) \]

\[ A_6 = \int \frac{dk}{(2\pi)^d} \int \frac{dl}{(2\pi)^d} \frac{1}{k^2l^2(k-p_1-p_2)^2(k-l)^2(l-p_1)^2} \]

\[ = S_T (-q^2)^{-2\epsilon} \left[ \frac{1}{\epsilon^4} + \frac{5\pi^2}{6\epsilon^2} + \frac{27}{4\epsilon}\zeta_3 + \frac{23\pi^4}{36} + \mathcal{O}(\epsilon) \right]. \quad (A.5) \]

A.2 Three-particle phase space integrals of one-loop corrections

The inclusive three- and four-particle phase space integrals appear in this calculation in the form of dipole and tripole phase space integrals. We therefore factor out not only the usual normalization factor of dimensional regularization, but also the volume of the two-particle phase space into

\[ S_{T,2} = P_2 \left( \frac{(4\pi)^\epsilon}{16\pi^2 \Gamma(1-\epsilon)} \right)^2. \quad (A.6) \]

The master integrals appearing as three-particle phase space integrals of one-loop matrix elements are

\[ V_{5,a} = \Re \left[ -i \int d\Phi_3 \int \frac{dk}{(2\pi)^d} \frac{1}{k^2(k-p_1-p_2-p_3)^2} \right] \]

\[ = S_{T,2} (q^2)^{-2\epsilon} \frac{\Gamma^6(1-\epsilon)\Gamma(1+\epsilon)-1}{\Gamma(2-2\epsilon)\Gamma(3-3\epsilon)} \epsilon \Re(-1)^{-\epsilon} \quad (A.7) \]

\[ V_{5,b} = \Re \left[ -i \int d\Phi_3 \int \frac{dk}{(2\pi)^d} \frac{1}{k^2(k-p_1-p_3)^2} \right] \]

\[ = S_{T,2} (q^2)^{-2\epsilon} \frac{\Gamma^5(1-\epsilon)\Gamma(1-2\epsilon)\Gamma(1+\epsilon)-1}{\Gamma(2-2\epsilon)\Gamma(3-4\epsilon)} \epsilon \Re(-1)^{-\epsilon} \quad (A.8) \]

\[ V_8 = \Re \left[ -i \int d\Phi_3 \int \frac{dk}{2p_1 \cdot p_2} \int \frac{dl}{(2\pi)^d} \frac{1}{k^2(k-p_1)^2(k-p_1-p_2)^2(k-p_1-p_2-p_3)^2} \right] \]

\[ = S_{T,2} (q^2)^{-2\epsilon} \left[ \frac{5}{2\epsilon^4} + \frac{9\pi^2}{2\epsilon^2} + \frac{89\zeta_3}{6\epsilon} + \frac{13\pi^4}{180} + \mathcal{O}(\epsilon) \right]. \quad (A.9) \]
The master integral $V_8$ was first derived in [33].

### A.3 Four-particle phase space integrals

Inclusive four-particle phase space integrals of tree level matrix elements can be expressed as linear combination of four master integrals, which were derived in [33]:

$$R_4 = \int \mathrm{d}\Phi 4 = P_4$$
$$= S_{\Gamma,2} (q^2)^{2-2\epsilon} \frac{\Gamma^5(1-\epsilon)\Gamma(2-2\epsilon)}{\Gamma(3-3\epsilon)\Gamma(4-4\epsilon)} ,$$

(A.10)

$$R_6 = \int \mathrm{d}\Phi_4 \frac{1}{s_{134}s_{234}}$$
$$= S_{\Gamma,2} (q^2)^{-2\epsilon} \left[ -1 + \frac{\pi^2}{6} + \epsilon \left( -12 + \frac{5\pi^2}{6} + 9\zeta_3 \right) 
+ \epsilon^2 \left( -91 + \frac{9\pi^2}{2} + 45\zeta_3 + \frac{61\pi^4}{180} \right) + O(\epsilon^3) \right] ,$$

(A.11)

$$R_{8,a} = \int \mathrm{d}\Phi_4 \frac{1}{s_{13}s_{23}s_{14}s_{24}}$$
$$= S_{\Gamma,2} (q^2)^{-2-2\epsilon} \left[ \frac{5}{\epsilon^4} - \frac{20\pi^2}{3\epsilon^2} - \frac{126\zeta_3}{\epsilon} + \frac{7\pi^4}{18} + O(\epsilon) \right] ,$$

(A.12)

$$R_{8,b} = \int \mathrm{d}\Phi_4 \frac{1}{s_{13}s_{134}s_{23}s_{24}}$$
$$= S_{\Gamma,2} (q^2)^{-2-2\epsilon} \left[ \frac{3}{4\epsilon^4} - \frac{17\pi^2}{12\epsilon^2} - \frac{44\zeta_3}{\epsilon} - \frac{61\pi^4}{60} + O(\epsilon) \right] .$$

(A.13)

### References

[1] R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and Collider Physics*, Cambridge University Press (Cambridge, 1996);

G. Dissertori, I.G. Knowles and M. Schmelling, *Quantum Chromodynamics: High Energy Experiments and Theory*, Oxford University Press (Oxford, 2003).

[2] Z. Bern, L.J. Dixon and A. Ghinculov, Phys. Rev. D **63** (2001) 053007 [hep-ph/0010075].

[3] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B **601** (2001) 318 [hep-ph/0010212]; **601** (2001) 347 [hep-ph/0011094]; **605** (2001) 486 [hep-ph/0101304];

E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B **605** (2001) 467 [hep-ph/0102201];

C. Anastasiou, E.W.N. Glover and M.E. Tejeda-Yeomans, Nucl. Phys. B **629** (2002) 255 [hep-ph/0201274];

E.W.N. Glover and M.E. Tejeda-Yeomans, JHEP **0306** (2003) 033 [hep-ph/0304169];

E.W.N. Glover, [hep-ph/0401119].

[4] Z. Bern, A. De Freitas and L.J. Dixon, JHEP **0109** (2001) 037 [hep-ph/0109078]; JHEP **0203** (2002) 018 [hep-ph/0201161]; JHEP **0306** (2003) 028 [hep-ph/0304168].

[5] Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov and H.L. Wong, JHEP **0111** (2001) 031 [hep-ph/0109079];

T. Binoth, E.W.N. Glover, P. Marquard and J.J. van der Bij, JHEP **0205** (2002) 060 [hep-ph/0202266].
[6] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, Nucl. Phys. B 627 (2002) 107 [hep-ph/0112081] and 642 (2002) 227 [hep-ph/0206067].

[7] S. Moch, P. Uwer and S. Weinzierl, Phys. Rev. D 66 (2002) 114001 [hep-ph/0207043].

[8] T. Gehrmann and E. Remiddi, Nucl. Phys. B 640 (2002) 379 [hep-ph/0207020].

[9] Z. Bern, L.J. Dixon and D.A. Kosower, Phys. Rev. Lett. 70 (1993) 2677 [hep-ph/9302280]; Z. Kunszt, A. Signer and Z. Trocsanyi, Phys. Lett. B 336 (1994) 529 [hep-ph/9405386]; Z. Bern, L.J. Dixon and D.A. Kosower, Nucl. Phys. B 437 (1995) 259 [hep-ph/9409393].

[10] E.W.N. Glover and D.J. Miller, Phys. Lett. B 396 (1997) 257 [hep-ph/9609474]; Z. Bern, L.J. Dixon, D.A. Kosower and S. Weinzierl, Nucl. Phys. B 489 (1997) 3 [hep-ph/9610370]; J.M. Campbell, E.W.N. Glover and D.J. Miller, Phys. Lett. B 409 (1997) 503 [hep-ph/9706297]; Z. Bern, L.J. Dixon and D.A. Kosower, Nucl. Phys. B 513 (1998) 3 [hep-ph/9708239].

[11] L.J. Dixon and A. Signer, Phys. Rev. Lett. 78 (1997) 811 [hep-ph/9609460]; Phys. Rev. D 56 (1997) 4031 [hep-ph/9706285]; Z. Nagy and Z. Trocsanyi, Phys. Rev. Lett. 79 (1997) 3604 [hep-ph/9707309]; S. Weinzierl and D.A. Kosower, Phys. Rev. D 60 (1999) 054028 [hep-ph/9901277].

[12] J. Campbell, M.A. Cullen and E.W.N. Glover, Eur. Phys. J. C 9 (1999) 245 [hep-ph/9809429].

[13] W.B. Kilgore and W.T. Giele, Phys. Rev. D 55 (1997) 7183 [hep-ph/9610433]; Z. Nagy and Z. Trocsanyi, Phys. Rev. Lett. 87 (2001) 082001 [hep-ph/0104315]; Z. Nagy, Phys. Rev. Lett. 88 (2002) 122003 [hep-ph/0110313]; Phys. Rev. D 68 (2003) 094002, hep-ph/0307268; J. Campbell and R.K. Ellis, Phys. Rev. D 65 (2002) 113007 [hep-ph/0202176].

[14] R.K. Ellis, D.A. Ross and A.E. Terrano, Nucl. Phys. B 178 (1981) 421.

[15] K. Fabricius, I. Schmitt, G. Kramer and G. Schierholz, Z. Phys. C 11 (1981) 315.

[16] Z. Kunszt and D.E. Soper, Phys. Rev. D 46 (1992) 192; S.D. Ellis, Z. Kunszt and D.E. Soper, Phys. Rev. Lett. 69 (1992) 3615 [hep-ph/9208249].

[17] W.T. Giele and E.W.N. Glover, Phys. Rev. D 46 (1992) 1980.

[18] S. Catani and M.H. Seymour, Nucl. Phys. B 485 (1997) 291; 510 (1997) 503(E) [hep-ph/9605323].

[19] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B 425 (1994) 217 [hep-ph/9403226]; D.A. Kosower, Nucl. Phys. B 552 (1999) 319 [hep-ph/9901201]; D.A. Kosower and P. Uwer, Nucl. Phys. B 563 (1999) 477 [hep-ph/9905315]; Z. Bern, V. Del Duca and C.R. Schmidt, Phys. Lett. B 445 (1998) 168 [hep-ph/9810409]; Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, Phys. Rev. D 60 (1999) 116001 [hep-ph/9903516].

[20] S. Catani and M. Grazzini, Nucl. Phys. B 591 (2000) 435 [hep-ph/0007142].

[21] D.A. Kosower, Phys. Rev. Lett. 91 (2003) 061602 [hep-ph/0301069].

[22] S. Weinzierl, JHEP 0307 (2003) 052 [hep-ph/0306248].

[23] J. Campbell and E.W.N. Glover, Nucl. Phys. B 527 (1998) 264 [hep-ph/9710255]; S. Catani and M. Grazzini, Phys. Lett. B 446 (1999) 143 [hep-ph/9810389]; Nucl. Phys. B 570 (2000) 287 [hep-ph/9908523]; F.A. Berends and W.T. Giele, Nucl. Phys. B 313 (1989) 595; V. Del Duca, A. Frizzo and F. Maltoni, Nucl. Phys. B 568 (2000) 211 [hep-ph/9909464].

[24] D.A. Kosower, Phys. Rev. D 67 (2003) 116003 [hep-ph/0212097].
[25] S. Weinzierl, JHEP 0303 (2003) 062 [hep-ph/0302180].

[26] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Phys. Lett. B 414 (1997) 354 [hep-ph/9705305];
A. Gehrmann-De Ridder and E.W.N. Glover, Nucl. Phys. B 517 (1998) 269 [hep-ph/9707224].

[27] D.A. Kosower and P. Uwer, Nucl. Phys. B 674 (2003) 365 [hep-ph/0307031].

[28] D. de Florian and M. Grazzini, Nucl. Phys. B 616 (2001) 247 [hep-ph/0108273].

[29] A. Daleo, C.A. Garcia Canal and R. Sassot, Nucl. Phys. B 662 (2003) 334 [hep-ph/0303199].

[30] K. Hepp, Commun. Math. Phys. 2 (1966) 301;
M. Roth and A. Denner, Nucl. Phys. B 479 (1996) 495 [hep-ph/9605420].

[31] T. Binoth and G. Heinrich, Nucl. Phys. B 585 (2000) 741 [hep-ph/0004013]; 680 (2004) 375 [hep-ph/0305234].

[32] G. Heinrich, Nucl. Phys. Proc. Suppl. 116 (2003) 368 [hep-ph/0211144];
T. Binoth and G. Heinrich, [hep-ph/0402265]

[33] A. Gehrmann-De Ridder, T. Gehrmann, G. Heinrich, Nucl. Phys. B 682 (2004) 265 [hep-ph/0311276].

[34] C. Anastasiou, K. Melnikov and F. Petriello, [hep-ph/0311311]

[35] C. Anastasiou, K. Melnikov and F. Petriello, [hep-ph/0402280]

[36] S. Catani, Phys. Lett. B 427 (1998) 161 [hep-ph/9802439];
G. Sterman and M. E. Tejeda-Yeomans, Phys. Lett. B 552 (2003) 48 [hep-ph/0210130].

[37] T. Kinoshita, J. Math. Phys. 3 (1962) 650;
T.D. Lee and M. Nauenberg, Phys. Rev. 133 (1964) B1549.

[38] S. Weinzierl, [hep-ph/0402131]

[39] G. Kramer and B. Lampe, Z. Phys. C 34 (1987) 497; 42 (1989) 504(E);
T. Matsuura and W.L. van Neerven, Z. Phys. C 38 (1988) 623;
T. Matsuura, S.C. van der Maarck and W.L. van Neerven, Nucl. Phys. B 319 (1989) 570.

[40] S. Laporta, Int. J. Mod. Phys. A 15 (2000) 5087 [hep-ph/0102033].

[41] T. Gehrmann and E. Remiddi, Nucl. Phys. B 580 (2000) 485 [hep-ph/9912329].

[42] J.A.M. Vermaseren, *Symbolic Manipulation with FORM*, Version 2, CAN, Amsterdam, 1991 and *New features of FORM*, [math-ph/0010025]

[43] W. Giele et al., “The QCD/SM working group: Summary report.”, Proceedings of Les Houches Workshop on “Physics at TeV Colliders”, 2001, p.275 [hep-ph/0204316].

[44] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, Phys. Rept. 277 (1996) 189.

[45] F.V. Tkachov, Phys. Lett. 100B (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159.

[46] R.J. Gonsalves, Phys. Rev. D28 (1983) 1542;
G. Kramer and B. Lampe, J. Math. Phys. 28 (1987) 945.

[47] T. Gehrmann and E. Remiddi, Nucl. Phys. B 601 (2001) 248 [hep-ph/0008287]; 601 (2001) 287 [hep-ph/0101124].