A GENERALIZATION OF DOUADY’S FORMULA

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Abstract. The Douady’s formula was defined for the external argument on
the boundary points of the main hyperbolic component \(W_0\) of the Mandelbrot
set \(M\) and it is given by the map \(T(\theta) = 1/2 + \theta/4\). We extend this formula to
the boundary of all hyperbolic components of \(M\) and we give a characterization
of the parameter in \(M\) with these external arguments.

1. Introduction. In this paper we will restrict our attention to the dynamics of
complex quadratic polynomials \(P_c(z) = z^2 + c\). In this case, the dynamical plane can
be decomposed into two complementary sets: The filled Julia set \(K_c\) which consists
of all points with bounded orbit and the basin of infinity \(A_c(\infty)\). The boundary of
\(K_c\) is called the Julia set \(J_c\).

When we consider the polynomial \(P_c\) in the extended complex plane \(\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\), the point at infinity is a super-attracting fixed point and hence there exists
a neighborhood of infinity \(U_c\) and an analytic isomorphism, known as the Böttcher
map, \(\phi_c : U_c \to \{z \in \mathbb{C} : |z| > R\}\), such that \(\phi_c(\infty) = \infty\), \(\phi'_c(\infty) = 1\) and
\(\phi_c \circ P_c = [\phi_c]^2\). If the critical point 0 is in \(K_c\), then \(K_c\) is connected and \(U_c = \mathbb{C} \setminus K_c\), otherwise \(U_c\) is a neighborhood of infinity, containing the critical value \(c\), see for instance [4].

The Mandelbrot set \(M\) is defined as the set
\[ M = \{c \in \mathbb{C} : K_c \text{ is connected}\}. \]

A. Douady and J.H. Hubbard defined the map \(\Phi_M : \hat{\mathbb{C}} \setminus M \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}\), given by \(\Phi_M(c) = \phi_c(c)\) and they proved that \(\Phi_M\) is an analytic isomorphism satisfying \(\Phi_M(\infty) = \infty\) and \(\Phi'_M(\infty) = 1\), [2, 4].

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For \( \theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \), the external ray of argument \( \theta \) of the Mandelbrot set is the curve
\[
R_M(\theta) = \Phi_M^{-1}\left( \{re^{2\pi i \theta} : r > 1 \} \right).
\]
An external ray is said to land at \( c \) if \( \lim_{r \to 1} \Phi_M^{-1}(re^{2\pi i \theta}) = c \). In this case, we say that \( \theta \) is an external argument of \( c \).

A quadratic polynomial \( P_c \) is called hyperbolic if it has an attracting cycle. A component \( W \) of the interior of \( M \) is hyperbolic if \( P_c \) is hyperbolic for some \( c \in W \). In fact, in this case \( P_c \) is hyperbolic for all \( c \in W \) and it is conjectured that all components of the interior of \( M \) are hyperbolic \([2,4]\).

The main hyperbolic component of \( M \), \( W_0 \), is defined as the set of parameters \( c \in \mathbb{C} \) for which \( P_c \) has an attracting fixed point \( \alpha_c \). The boundary of \( W_0 \) is called the main cardioid of \( M \). It is known that for every hyperbolic component \( W \) of the interior of \( M \) there exists \( k \in \mathbb{N} \) fixed, such that \( P_c \) has an attracting cycle of period \( k \) for every \( c \in W \). The map \( \rho_W : W \to \mathbb{D} \), defined by the derivative of \( P_c^k \) at the attracting periodic point, is an analytic isomorphism which can be extended continuously to the boundary of \( W \). Using \( \rho_W \), we can define the internal argument \( \gamma \) for all \( c \in W \), \([2]\). On the other hand, from Yoccoz’s Theorem, \( M \) is locally connected for all parameters in the boundary of hyperbolic components \([4]\). Hence, a parameter \( c \in M \) in the boundary of a hyperbolic component \( W \) has well defined internal and external arguments, \([3,4]\). The parameter \( c \in \partial W \) with internal argument zero is called the root of \( W \). If \( c = \pm \frac{1}{4} \) is a parabolic parameter, then \( c \) has a rational internal argument and two external arguments \( \theta^- < \theta^+, [4]\).

Douady gives a formula that relates the parameters in the main cardioid with real parameters in \( M \). The map induced by this formula takes a parameter with external argument \( \theta \) and sends it onto a real parameter with external argument \( T(\theta) = 1/2 + \theta/4 \). In [1], the following was proved.

**Theorem 1.1.** If \( c \) is a parabolic point of the boundary of \( W_0 \) with internal argument \( \gamma \) and external arguments \( \theta^- , \theta^+ \), with \( 0 < \theta^- < \theta^+ < \frac{1}{4} \), then \( T(\theta^-) \) is an external argument of a real Misiurewicz parameter and \( T(\theta^+) \) is an external argument of \( c' \in M \cap \mathbb{R} \), the root of a primitive hyperbolic component.

Furthermore, if \( \gamma \) is irrational and satisfies an asymmetrical Diophantine condition then there exists an absolutely continuous invariant measure for \( P_{c'} \), see [1].

In this work, we extend this formula to the boundary of all hyperbolic components and we give a characterization of the parameter with these external arguments.

2. **Generalization of Douady’s formula.** It is known that if \( W \subset M \) is a hyperbolic component of the interior of \( M \) then the root \( c \in W \) has two external arguments \( \theta_W^- \) and \( \theta_W^+ \), with the exception when \( c = \frac{1}{4} \), [4]. Hence we can associate to \( W \) the couple \( (\theta_W^-, \theta_W^+) \).

Let \( W \) be a given hyperbolic component of the interior of \( M \), with period \( k \) and \((\theta^- = a_1 a_2 \ldots a_k, \theta^+ = b_1 b_2 \ldots b_k)\), the external rays in the root of \( W \). If \( \theta \) is an external argument in the boundary of \( W \) we define the map
\[
F(\theta) = \frac{\theta}{4\pi} + \left( \frac{b_1}{2} + \ldots + \frac{b_k}{2^{k+1}} + \frac{a_1}{2^{k+1}} + \ldots + \frac{a_k}{2^{2k}} \right)
= \frac{1}{2} b_1 b_2 \ldots b_k a_1 a_2 \ldots a_k \theta.
\]
As in the main hyperbolic component \( W_0 \), \( k = 1, a_1 = 0 \) and \( b_1 = 1 \), this map generalizes the Douady’s formula and we will show that it has similar properties of the map \( T \). Before, we give some basic concepts and properties of tuning that can be found in [3].

Let \( W \) be a hyperbolic component of \( M \), of period \( k \), and \( c_0 \) the center of \( W \). There is a copy of \( M_W \) inside of \( M \), in which \( W \) corresponds to the main cardioid \( W_0 \). More precisely, there is a continuous bijection \( \psi_W : M \to M_W \), such that \( \psi_W(0) = c_0 \), \( \psi_W(W_0) = W \) and \( \partial M_W \subset \partial M \). For all \( c \in M \) the point \( \psi_W(c) \) is called \( c_0 \) tuned by \( c \) and it is denoted by \( c_0 \perp c \) or \( W \perp c \). The filled Julia set \( K_{c_0 \perp c} \) can be obtained by taking in \( K_{c_0} \) a component \( U \) and replacing \( U \) by a copy of \( K_c \).

In particular, it is known the following result [3].

**Theorem 2.1.** (Douady-85) Let \( W \) be a hyperbolic component of the interior of \( M \) with period \( k \) and \( (\theta^\pm = a_1 a_2 \ldots a_k, \theta^+ = b_1 b_2 \ldots b_k) \) be the external rays in the root \( c_1 \) of \( W \). If \( \theta = s_1 s_2 s_3 \ldots \) is an external argument of \( c \in M \), then the corresponding external argument \( \theta' \) of \( c_0 \perp c \) is given by the following algorithm:

\[
\theta' = (\theta^-, \theta^+) \perp \theta = s'_1 s'_2 s'_3 \ldots ,
\]

where \( s'_i = a_1 a_2 \ldots a_k \) if \( s_i = 0 \) and \( s'_i = b_1 b_2 \ldots b_k \) if \( s_i = 1 \).

From now on, we suppose that \( W \) is a hyperbolic component of the interior of \( M \) with period \( k \) and \( (\theta^\pm = a_1 a_2 \ldots a_k, \theta^+ = b_1 b_2 \ldots b_k) \) are the external rays in the root of \( W \) and \( \theta^- < \theta^+ \).

**Remark 1.** If \( c_t \in \partial W \) has a rational internal argument \( t = \frac{p}{q} \), with \( (p, q) = 1 \), then the external arguments \( (\theta^+_t, \theta^-_t) \) at \( c_t \) can be obtained by tuning. Explicitly,

\[
\theta^+_t = (\theta^-, \theta^+) \perp \theta^+_0 ,
\]

where \( \theta^+_0 \) and \( \theta^-_0 \) are the external arguments in the parameter \( c \in \partial W_0 \) with internal argument \( t \).

In fact, for every external angle \( \theta \) at \( \partial W \) there is an external angle \( \theta_0 \) at the main cardioid. Moreover, the map \( \theta \mapsto \theta_0 \) is monotone, one to one and preserves the type of the landing point.

**Lemma 2.2.** If \( \theta^\pm \) and \( \theta^\pm_0 \), are as above, then

\[
F(\theta^\pm) = (\theta^-, \theta^+) \perp T(\theta^\pm_0) .
\]

**Proof.** Since \( P_{c_t} \) has a parabolic periodic point, with period \( kq \), the external arguments in the parabolic parameters \( c \in \partial W_0 \) and \( c_t \in \partial W \), can be written as \( \theta^\pm_0 = .s_1^i s_2^i \ldots s_q^i \) and \( \theta^\pm_t = .t_1^i t_2^i \ldots t_k^i \), respectively.

By definition we have,

\[
T(\theta^\pm_0) = .10s_1^i s_2^i \ldots s_q^i ,
\]

and

\[
F(\theta^\pm_t) = .b_1 b_2 \ldots b_k a_1 a_2 \ldots a_k t_1^i t_2^i \ldots t_k^i .
\]

From Remark [4] we have

\[
F(\theta^\pm_t) = F((\theta^-, \theta^+) \perp \theta^\pm_0) = (\theta^-, \theta^+) \perp T(\theta^\pm_0) .
\]
The previous lemma can be generalized to all external angles whose rays land at \( \partial W \). Hence, we have that the generalized Douady’s formula is nothing but the tuning of the original formula.

By the Theorem 1.1, Theorem 2.1, Lemma 2.2 and the definition of \( F(\theta) \) we obtain the following result.

**Theorem 2.3.** Let \( W \) be a hyperbolic component of the interior of \( M \) with period \( k \) and \((\theta^- = a_1a_2...a_k, \theta^+ = b_1b_2...b_k)\) be the external rays at the root of \( W \). If \( \theta \) is an external argument in the boundary of \( W \) and \( \theta^- < \theta < \theta^+ \), then \( F(\theta) \) is an external argument of \( c \in M \). In particular, if \( \theta^- a_1...a_k, \theta^+ b_1...b_k \) are the external arguments in a parabolic point \( c \in W \), then

1. \( F(\theta^-) \) is an external argument of the root of a primitive hyperbolic component.  
2. \( F(\theta^+) \) is an external argument of a Misiurewicz parameter.

Let \( W \) be a hyperbolic component. The interval \([-2,0]\) tuned with \( W \) is a curve in \( M \) that we call the main vein of the little Mandelbrot copy starting at \( c_0 \).

**Corollary 1.** The external ray with angles \( F(\theta^-) \) or \( F(\theta^+) \) land in the main vein of the hyperbolic component \( W \).

3. **Summability condition.** Given a real number \( a \), the continued fraction expansion of \( a \) is \([a_1,a_2,a_3,...]\) where

\[
a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
\]

For each \( n \), the truncated continued fraction \([a_0,a_1,...,a_n]\) is a rational number \( p_n/q_n \) known as the convergent of \( a \).

Let \( U \) and \( V \) be two domains with \( U \) compactly contained in \( V \), in notation \( U \subset V \). A quadratic-like map \( g : U \to V \) is a degree 2 branched covering. Given a quadratic-like map, the little filled Julia set is the set of points which can be iterated infinitely many times. The map \( f_c \) is called renormalizable if there is an iterate \( n \), two neighborhoods \( U \) and \( V \) around 0 satisfying \( U \subset V \), such that the restriction of \( f^n_c \) to \( U \) is a quadratic-like map with connected little filled Julia set.

In fact, the operation of tuning can be seen as the inverse operation to renormalization. For a hyperbolic component \( W \), the map \( W \perp c \) is renormalizable with iterate equal to the period of \( W \), and the induced quadratic-like map is quasiconformally conjugated to \( f_c \).

When \( c \) is a real parameter, a central cascade of \( f_c \) is a sequence \( U_m \) of neighborhoods of 0 such that \( U_{m+1} \subset U_m \) and the first return of 0 to \( U_m \) belongs to \( U_{m+1} \). The first time \( n_m \) such that \( f_{c^n}^m(0) \in U_m \) is called the \( m \)-central return of \( f_c \). For \( 0 < \alpha \leq 1 \), a quadratic map \( f_c \) is said to satisfy a summability condition of order \( \alpha \) if the series

\[
\sum \frac{1}{|(f_c^n)'(c)|^\alpha}
\]

converges (see [10]).

Let \( A \) be the set of irrational angles \( \theta \) and such that the external ray with angle \( \theta \) lands at the main cardioid. By Douady-Hubbard and Yoccoz’s theorems, the set landing points of rays with angles in \( A \) consists precisely of the parameters \( c \) with irrational internal angle (see [4] and [6]). In [11], the first author showed that if
θ ∈ A, then the ray \( T(θ) \) lands at the real line at some parameter \( c' \). Consider the set \( \mathcal{R}F \) of all real parameters \( c' \) such that \( c' \) is the landing point of a ray with angle \( T(θ) \) with \( θ ∈ A \). In [1] the first author showed that if \( c ∈ \mathcal{R}F \) then there is a central cascade around the critical point where the \( n \)-central return is equal to \( q_{2n+1} \), the \( n \)-convergent of \( γ \), where \( γ \) is the internal argument of \( c \). (see Theorem 1.3 (iii) and Lemma 5.2 in [1]). Moreover, if the continuous fraction expansion of \( γ \) is of bounded type, then the map \( f_γ \) satisfies a summability condition of order \( \frac{1}{2} \) (see Lemma 5.4 in [1] and the proof of Proposition 3.1 in [8]). We call a parameter \( c' ∈ \mathcal{R}F \) of bounded type whenever the associated parameter \( c \) in the main cardioid has an internal address of bounded type.

By making use of the properties of tuning we show that the generalized formula \( F \) also has these properties.

**Lemma 3.1.** For \( c ∈ \mathcal{R}F \) of bounded type, let \( W \) be a hyperbolic component of the Mandelbrot set, then \( W ⊥ c \) satisfies a summability condition of order \( 1/2 \).

**Proof.** Let \( g = W ⊥ c \). By construction \( g \) is renormalizable of the same period \( m \) of the component \( W \). There exist a neighborhood \( U \) around 0 such that the map \( g^m \) is quasiconformally conjugated to \( f_c \). This implies that the moduli of the central returns of \( g \) are comparable with the moduli \( v_n \) of the central returns of \( f_c \) (for more details see [7]). In Lemma 5.4 of [1] it is shown that the moduli satisfy

\[
\sum_{k=1}^{∞} \left| \frac{v_{k+1}}{v_k} \right|^{1/2} < ∞,
\]

the Martens-Nowicki’s condition, [8]. By quasiconformality, if \( v_k' \) are the moduli of the central returns of \( g \) we have

\[
\frac{1}{K} \sum_{k=1}^{∞} \left| \frac{v_{k+1}}{v_k} \right|^{1/2} \leq \frac{1}{K} \sum_{k=1}^{∞} \left| \frac{v_{k+1}'}{v_k'} \right|^{1/2} \leq K \sum_{k=1}^{∞} \left| \frac{v_{k+1}}{v_k} \right|^{1/2} < ∞.
\]

As in M. Martens and T. Nowicki, the quotient \( \left| \frac{v_{n+1}}{v_n} \right| \) is a lower bound of \( |(g^n)'(c)| \). Hence \( g \) satisfies a summability condition with exponent \( \frac{1}{2} \), (see the proof of Proposition 3.1 in [8]).

The previous lemma has the following consequence:

**Theorem 3.2.** If \( θ \) is an irrational external argument in the boundary of \( W \) between the external arguments \( θ^- \) and \( a_1 b_1 a_2 b_2 ... b_k \) then the external ray with angle \( F(θ) \) lands in a parameter \( c' ∈ M \) which is finitely renormalizable. Furthermore, the corresponding Julia set \( J_{c'} \) is locally connected and the map \( f_{c'} \) admits an absolutely continuous invariant measure with respect to Lebesgue.

**Proof.** The map \( f_{c'} \) renormalizes to a polynomial with parameter \( c \) in \( \mathcal{R}F \) which is non-renormalizable. Then, in fact, \( f_{c'} \) is only 1-renormalizable. By Yoccoz’s Theorem \( M \) is locally connected at \( c' \) and is the landing point of at least one ray. By hypothesis, the angle \( θ' \) of the ray landing at \( c' \) is the tuning of \( W' \) with the ray \( R_θ \) landing at parameter \( c \) in the boundary of the main cardioid. Hence \( T(θ) ∈ A \) and \( F(θ) = W' ⊥ T' (θ) \). By Lemma 3.1 the map \( f_{c'} \) satisfies a summability condition with exponent \( 1/2 \). J. Graczyk and S. Smirnov showed that when a map satisfies a summability condition with exponent \( α < \frac{2}{\sqrt{4 + \mu_{\max}}} \) then admits an absolutely
continuous invariant measure [5]. Here μ_{max} is maximal multiplicity of the critical points, which for this quadratic polynomial is equal to 1.

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