PROMOTION AND CYCLIC SIEVING ON FAMILIES OF SSYT

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Abstract. We examine a few families of semistandard Young tableaux, for which we observe the cyclic sieving phenomenon under promotion.

The first family we consider consists of stretched hook shapes, where we use the cocharge generating polynomial as CSP-polynomial.

The second family we consider consists of skew shapes, consisting of rectangles. Again, the charge generating polynomial together with promotion exhibits the cyclic sieving phenomenon. This generalizes earlier result by B. Rhoades and later B. Fontaine and J. Kamnitzer.

Finally, we consider certain skew ribbons, where promotion behaves in a predictable manner. This result is stated in form of a bicyclic sieving phenomenon.

One of the tools we use is a novel method for computing charge of skew semistandard tableaux, in the case when every number in the tableau occur with the same frequency.

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1. Introduction

The cyclic sieving phenomenon has been studied extensively since its introduction in 2004 [RSW04]. Briefly, this phenomenon relates a cyclic group action on a set of combinatorial objects with the values at roots of unity, on some $q$-analog of the cardinality of the set.

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1.1. Cyclic sieving. The notion of cyclic sieving is defined as follows.

**Definition 1** (Cyclic sieving, Reiner–Stanton–White [RSW04]). Let $X$ be a set of combinatorial objects and $C_n$ be a cyclic group of order $n$ acting on $X$. We say that the triple $(X, C_n, f(q))$ exhibits the cyclic sieving phenomenon, (CSP) if for all $d \in \mathbb{Z}$,

$$|\{x \in X : g^d \cdot x = x\}| = f(\xi^d)$$

where $\xi$ is a primitive $n$th root of unity and $f(q) \in \mathbb{N}[q]$.

A few selected relevant CSP-triples are listed in Table 1, together with the new results in this article.

One particular instance of CSP is the set of semistandard Young tableaux of a fixed rectangular shape, and a $q$-analog given by the $q$-hook-content formula. The cyclic group action is the promotion operator (denoted $\partial$, defined in Section 2.1), see B. Rhoades [Rho10a]. Rhoades’ result was further refined in [FK13] where cyclic sieving on rectangular SSYT with a fixed content vector was considered.

In a recent preprint, [APRU20], it is proved that for a fixed skew shape $\lambda/\mu$,

$$\text{SSYT}(n\lambda/n\mu), C_n, f^{n\lambda/n\mu}(q)$$

is a CSP-triple, where $f^{n\lambda/n\mu}(q)$ is a skew Kostka–Foulkes polynomial. Note that when $\mu = \emptyset$, $f^{n\lambda}(q)$ can be computed by the $q$-hook formula. The notion of multiplying every part in a partition with a factor is commonly referred to as stretching, and has a certain geometric interpretation. It is therefore natural to explore situations similar to (2).

1.2. Main results. There has been less research on non-rectangular shapes—promotion usually have a very high order, which makes it unlikely to find nice instances of CSP.

One case which do behave nicely is the case of hook shapes. Hook semistandard tableaux have been considered in [BMS14, PSV16, OP19]. In this paper, we consider certain semistandard Young tableaux where the shape is a stretched hook, i.e., of the form $\lambda = ((n+1)a, nb)$ for some non-negative $a, b \geq 1$.

The first main theorem is as follows.

**Theorem 2** (Corollary 25 below). Let $\lambda = ((n+1)a, nb)$, and let $\nu = na^{a+b+1}$. Then

$$\text{SSYT}(\lambda, \nu), (\partial), q^{-n}\tilde{K}_{\lambda,\nu}(q)$$

exhibits the cyclic sieving phenomenon, where

$$\tilde{K}_{\lambda,\nu}(q) = \sum_{T \in \text{SSYT}(\lambda, \nu)} q^{cc(T)} = q^{n}\prod_{i=1}^{a}\prod_{j=1}^{b} \frac{[i+j+n-1]_q}{[i+j-1]_q}.$$
order ideals of $2 \times [n]$ posets was considered earlier in [RS12]). For some related open problems on plane partitions and cyclic sieving, we refer to [AKLM05, Hop20].

| Set                              | Group | Polynomial | Ref.  |
|----------------------------------|-------|------------|-------|
| Binary words, $BW(2n,n)$         | rot   | $[2n]_q$  | RSW04 |
| Words, $W(kn,n^k)$               | rot   | $[kn]_{n,n,\ldots,n}_q$ | RSW04 |
| Rectangular, SSYT($a^k$)         | $\partial$ | $q^k s_{\lambda_k}(1,\ldots,q^{k-1})$ | Rho10a |
| Rectangular, SSYT($a^b,\gamma$) | $\partial^d$ | $q^b K_{a^b,\gamma}(q)$ | FK13  |
| Hooks SSYT($(n-m,1^m),\gamma$)  | $\partial^d$ | $[n\gamma(m-1)]_m^{-1}_q$ | BMS14 |
| Plane partitions                 | $\partial^\dagger$ | $\prod_{1 \leq i \leq a \atop 1 \leq j \leq b} [i+j+n-1]_q$ | SW18  |
| Matrices $M(n\nu,n^m)$           | rot   | $\sum_{\lambda \vdash mn} K_{\lambda,n^m}(q) K_{\lambda,\nu}(1)$ | Rho10b |
| Stretched hooks                  | $\partial$ | $\prod_{1 \leq i \leq a \atop 1 \leq j \leq b} [i+j+n-1]_q$ | Corollary 25 |
| Disjoint rows                    | $\partial$ | $K_{n\lambda,n\mu,n^\lambda/\mu}(q)$ | Theorem 30 |
| Disjoint rectangles              | $\partial^d$ | $K_{(a^1\nu_1\oplus\cdots\oplus a^r\nu_r),(\nu+r)}(\gamma,q)$ | Theorem 36 |
| Certain two-row ribbon           | $\partial$ | $[\nu]_\gamma - [\nu]_\gamma + [n-1]_q$ | Corollary 40 |
| Certain three-row ribbon          | $\partial$ | $[n-2]_\gamma + (n-3)[n-1]_q$ | Theorem 43 |

Table 1. The group action is given by $k$-promotion, $\partial$, or an appropriate power of it, depending on the rotational symmetry of the content composition. The action $\partial^\dagger$ is defined via so-called “toggles”, but can be mapped in an equivariant manner to promotion on rectangular SSYT.

We also show cyclic sieving under promotion on skew shapes, consisting of a disjoint union of rows. Let $\nu \vdash m$ and $n \geq 1$. Let $SM(\nu,n)$ be the set of skew semistandard Young tableaux where the shape $\lambda/\mu$ is a disjoint union of $\ell(\nu)$ rows where row $j$ has length $\nu_j$. Moreover, we ask that the content is given by $n^m$, that is, there are $n$ boxes with label $j$, for each $j = 1,\ldots,m$.

**Theorem 3** (Theorem 30 below). We have that

$$(SM(\nu,n), \langle \partial \rangle, K_{\lambda/\mu,n^m}(q))$$

is a CSP-triple, where $K_{\lambda/\mu,n^m}(q)$ is a Kostka–Foulkes polynomial.

In [FK13], a cyclic sieving phenomenon involving $K_{a^b,\gamma}(q)$ is proved, where promotion act on the set SSYT($a^b,\gamma$). This result refines the earlier instance of CSP given in [Rho10a]. In Theorem 36, we give a generalization of the result by Fontaine–Kamnitzer to the case when promotion act on skew shapes consisting of a disjoint union of rectangles. As expected, Kostka–Foulkes polynomial indexed by a skew shape and a composition is the main part of the CSP-polynomial.

Finally, in Section 5, we study promotion on two families of ribbon shapes, where the order of promotion behaves nicely. We prove two instances of cyclic sieving. In the last section, we give a few examples indicating that promotion on general ribbon shapes is most likely hard to analyze.
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2. Preliminaries

2.1. Semistandard tableaux and Jeu-de-taquin promotion. With every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we associate a Young diagram, an array with $\lambda_i$ boxes on row $i$. Given two partitions $\lambda$ and $\mu$ with $\mu \subset \lambda$, the skew-diagram $\lambda/\mu$ is given by the cells in $\lambda$ that are not in $\mu$. A semistandard Young tableau of shape $\lambda$ (or $\lambda/\mu$) is a filling of its boxes with positive integers such that each row is weakly increasing from left to right and each column is strictly increasing from top to bottom. We denote the set of semistandard tableaux of shape $\lambda$ by $\text{SSYT}(\lambda)$. A semistandard tableau that contains each of the numbers from 1 to $n$ exactly once is called standard. The content of a tableau $T$ is the composition $(\nu_1, \nu_2, \ldots, \nu_n)$ where $\nu_i$ is given by the number of occurrences of $i$ in $T$. We denote the set of all semistandard Young tableaux of shape $\lambda$, content $\nu$ by $\text{SSYT}(\lambda, \nu)$.

With any tableau $T$, we associate a reading word denoted $\text{rw}(T)$, a listing of its entries row by row, left to right, bottom to top. Note that if $T$ is standard, its reading word is a permutation of $n$.

Example 4. A semistandard tableau of shape $(5, 2, 2)$ and its reading word.

$$T = \begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 1
\end{array}, \quad \text{rw}(T) = 342311234.$$

We act on semistandard Young tableaux by Jeu-de-taquin promotion (also called $K$-promotion) was introduced in [Sch63, Sch72]. We shall follow the definition given in [BMS14]. Our definition differs from theirs in the sense that we start with removing all 1s—the inverse of the operation in their work. Throughout this paper, we will refer to this simply as promotion, denoted by $\partial$.

Jeu-de-taquin promotion works as follows. First all entries on $T$ labeled 1 are replaced by dots. Then the dots are moved to the outside corners by repeatedly changing places with the smaller of the entries to the right or below. If both are equal, exchange is made with the entry below to maintain tableau rules. Once no more exchanges are possible, all numbers are decremented by one, and dots are replaced by $n$, as seen in the following example.

Example 5. Jeu-de-taquin promotion

$$\begin{array}{c}
1 & 1 & 2 & 3 & 4 \\
2 & 3 & 3 & 4 \\
3 & 4 & 5 & 1
\end{array} \rightarrow \begin{array}{c}
\cdot & 2 & 3 & 4 \\
2 & 3 & 3 & 4 \\
3 & 4 & 5 & 1
\end{array} \rightarrow \begin{array}{c}
\cdot & 2 & 3 & 4 \\
2 & 3 & 3 & 4 \\
3 & 4 & 5 & 1
\end{array} \rightarrow \begin{array}{c}
2 & 2 & 3 & 4 \\
\cdot & 3 & 3 \\
3 & 4 & 5 & 1
\end{array} \rightarrow \begin{array}{c}
2 & 2 & 3 & 4 \\
\cdot & 3 & 3 \\
\cdot & 4 & 5 & 1
\end{array} \rightarrow \begin{array}{c}
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 3 & 4 \\
3 & 3 & 5 & 1
\end{array} \rightarrow \begin{array}{c}
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 3 & 4 \\
3 & 3 & 5 & 1
\end{array}
\end{array}$$
2.2. Charge, cocharge and Kostka–Foulkes polynomials.

Definition 6. For a word $\pi$ define \textit{major index} as $\text{maj}(\pi) := \sum_{d \in \text{Des}(\pi)} d$. If $\pi$ is also a permutation of length $n$, define \textit{charge} as $\text{charge}(\pi) := \text{maj}(\text{rev}(\pi^{-1}))$ and \textit{cocharge} as $\text{cc}(\pi) := \binom{n}{2} - \text{charge}(\pi)$. For a word $w$ with content given by a partition $(v_1, \ldots, v_n)$ we can define $v_1$ \textit{standard subwords} as follows. The first standard subword is obtained by starting with the rightmost 1 and then go left to find a 2 then a 3 etc. If there is no $i + 1$ to the left of $i$ in the word then we wrap and take the rightmost $i + 1$ instead and continue for as long as possible. These letters, in their original order, makes up the first standard subword $w_1$. The standard subword $w_r$ is formed in the same way but not using any letters that already belong to any of the standard subwords $w_1, \ldots, w_{r-1}$. We then extend the definition of charge and cocharge to the word $w$ by $\text{charge}(w) := \sum_i \text{charge}(w_i)$, $\text{cc}(w) := \sum_i \text{cc}(w_i)$. In this work we focus on the case where $w$ is the reading word of a tableaux.

Example 7. The tableau $T \in \text{SSYT}(6, 3, 3)$ below has the standard subwords $w_1 = 35214$, $w_2 = 42135$, $w_3 = 31245$ and $\text{charge}(\text{rw}(T)) = 2 + 4 + 7 = 13$, $\text{cc}(\text{rw}(T)) = 8 + 6 + 3 = 17$.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 2 & 3 & 4 \\
2 & 2 & 3 & 1 & 1 & 1 \\
3 & 1 & 5
\end{array}
\quad \text{rw}(T) = 34522311234455
\]

(3)

We shall also use the following alternative method of computing cocharge. The cocharge of a permutation $\pi \in S_n$ can be computed as follows. We now define \textit{cocharge value} of $j$, denoted $\text{cc}(\pi, j)$.

\[
\text{cc}(\pi, j) := \begin{cases} 
0 & \text{if } j = 1 \\
\text{cc}(\pi, j - 1) & \text{if } j - 1 \text{ appear to the left of } j \text{ in } \pi \\
\text{cc}(\pi, j - 1) + 1 & \text{otherwise.}
\end{cases}
\]

(4)

This uniquely defines the values of $\text{cc}(\pi, 1)$, $\text{cc}(\pi, 2)$, $\ldots$, $\text{cc}(\pi, n)$. Finally, $\text{cc}(\pi)$ is the sum of these cocharge values.

Example 8 (Computing cocharge). Let $\pi = (4, 8, 6, 9, 7, 2, 3, 1, 5)$. The cocharge value $\text{cc}(\pi, j)$ is written as the subscript of $j$:

$4_2, 8_4, 6_3, 9_4, 7_3, 2_1, 3_1, 1_0, 5_2$.

The total sum of the subscripts is 20, so the cocharge of $\pi$ is 20.

Lemma 9. Let $\pi \in S_n$, and let rot act by cyclic shift to the right. Then

\[
\text{cc}(\pi) = \begin{cases} 
\text{cc}(\text{rot}(\pi)) - 1 & \text{if } \pi(n) \neq 1 \\
\text{cc}(\text{rot}(\pi)) + n - 1 & \text{if } \pi(n) = 1.
\end{cases}
\]

In particular $\text{cc}(\pi) \equiv \text{cc}(\text{rot}(\pi)) - 1 \mod n$.

Proof. This follows immediately from the recursion in (4). \qed
The charge and cocharge statistics on tableaux can be used to calculate a special family of polynomials called the Kostka–Foulkes polynomials and modified Kostka–Foulkes polynomials, respectively.

\[ K_{\lambda/\mu,\nu}(q) := \sum_{T \in \text{SSYT}(\lambda/\mu,\nu)} q^{\text{charge}(rw(T))} \]

\[ \tilde{K}_{\lambda/\mu,\nu}(q) := \sum_{T \in \text{SSYT}(\lambda/\mu,\nu)} q^{\text{cc}(rw(T))} \]

The modified and classical Kostka–Foulkes polynomials are related via the relation

\[ \tilde{K}_{\lambda/\mu,\nu}(q) = q^{\kappa(\nu)} K_{\lambda/\mu,\nu}(q^{-1}) \]  

(5)

where

\[ \kappa(\lambda) := \sum_{j} \left( \lambda'_j / 2 \right) \]

for the conjugate partition \( \lambda' \). Since \( \lambda'_j \) is the number of boxes in the \( j \)th column of \( \lambda \), we get \( \kappa(n\lambda) = n \kappa(\lambda) \). Moreover, for \( \lambda = (a + 1, 1^b) \), then \( \kappa(\lambda) = b(b + 1)/2 \).

Finally, permuting the entries of the content partition does not change the Kostka–Foulkes polynomial. That is,

\[ K_{\lambda/\mu,\sigma(\nu)} := K_{\lambda/\mu,\nu} \]

for any permutation \( \sigma \). Warning! Note that we may only compute Kostka–Foulkes polynomials via charge on SSYT if the content is a partition.

**Lemma 10.** Let \( \xi \) be an \( n^{th} \) root of unity, such that \( K_{\lambda/\mu,\nu}(\xi) \) is a real number. Then \( \tilde{K}_{\lambda/\mu,\nu}(\xi) = K_{\lambda/\mu,\nu}(\xi) \).

**Proof.** By using (5), we have that

\[ \tilde{K}_{\lambda/\mu,\nu}(\xi) = \xi^{n \kappa(\nu)} K_{\lambda/\mu,\nu}(\xi^{-1}) = K_{\lambda/\mu,\nu}(\xi). \]  

(6)

In the last equality, we use the fact that \( \xi^{-1} = \overline{\xi} \), and that \( K_{\lambda/\mu,\nu}(\xi) \) is real. \( \square \)

### 3. Promotion on stretched hooks

The main results of this section are a new way of computing the charge of a word with rectangular content and bijection between stretched hooks and plane partitions which allows to prove cyclic sieving for stretched hooks.

Let \( \text{SSYT}(\lambda,\mu) \) denote the set of semistandard Young tableaux of shape \( \lambda \) and content \( \mu \). The Kostka coefficient, \( K_{\lambda\mu} = K_{\lambda\mu}(1) \) is therefore given by \( |\text{SSYT}(\lambda,\mu)| \).

Given \( a, b \geq 0 \) and \( n \geq 1 \), let \( \text{SHST}(a, b, n) \) (stretched hook semistandard tableaux) be the set

\[ \text{SHST}(a, b, n) := \{ T \in \text{SSYT}((na+n)b, n^{a+b+1}) \} \]

We let \( k \)-promotion \( \partial \) act on \( \text{SHST}(a, b, n) \).

**Example 11.** The six tableaux in \( \text{SHST}(1, 2, 2) \) form two 3-cycles under promotion.
Note that in the example above, repeating promotion \( a + b \) times lets us recover our original tableaux. Now we will show that this is always the case.

**Proposition 12** (Order of promotion). For any \( T \in \text{SHST}(a, b, n) \), we have \( \partial^{a+b}(T) = T \).

**Proof.** The \( n \) 1s are all in the first positions of the first row. When they are removed 2s will fill their positions. Thus the promotion operation on stretched hook shapes is independent of the entries of the first row, as the total content is fixed. Promotion acts on the remaining \( b \times n \) shape by doing promotion with alphabet \( 2, 3, \ldots, a + b + 1 \). This is equivalent to doing promotion on rectangular semi standard Young tableaux filled with alphabet \( [a + b] \) which has the property \( \partial^{a+b}(T) = T \). \( \square \)

Deleting the first row and decremented entries by one gives a bijection between \( \text{SHST}(a, b, n) \) and \( \text{SSYT}(n, b, a + b) \) that commutes with promotion, which we used in proving our proposition above. What makes the stretched hook shapes still interesting is the use of cocharge statistic and its relation with the promotion operation.

Let \( |\text{SHST}(a, b, n)|_q := \sum_{T \in \text{SHST}(a, b, n)} q^{cc(T)} \). Note that this is a (modified) Kostka–Foulkes polynomial.

**Example 13.** The six stretched hook semistandard tableaux given in Example 11 above have cocharges 6, 10, 8, 7, 8, and 9, respectively. The corresponding polynomial is

\[
|\text{SHST}(1, 2, 2)|_q = q^6 + q^7 + 2q^8 + q^9 + q^{10} = q^6(1 + q + q^2)(1 + q^2).
\]

Given a word \( w \) on \( [k] \) and an integer \( j \geq 1 \), consider the subword of \( w \) consisting of entries in \( \{j, j+1\} \). By repeatedly removing consecutive pairs \((j+1, j)\), we end up with a word of the form \( j^r \). We let \( \delta_j(w) \) be the value of \( s \), and set

\[
\delta(w) := (\delta_1(w), \delta_2(w), \ldots, \delta_j(w)) \tag{7}
\]

as the depth sequence of \( w \). The depth sequence gives a new way of computing the charge of any word with rectangular content.

**Theorem 14** (Charge for rectangular content). Suppose \( w \) is a word with content \( k^n \). Then

\[
\text{charge}(w) = \sum_j \delta_j(w)(k - j). \tag{8}
\]

**Proof.** To calculate charge, we first divide \( w \) into standard subwords, which in this case all have the same length \( n \), then calculate the charge at each subword. The contribution of a letter \( j \) to charge just depends on whether it comes before or after \( j + 1 \) in its subword. So, limiting our attention to \( js \) we can see the subword selection process as an ordering \( \sigma \) on the \( n \) entries labeled \( j \), each of which then claims the first unclaimed \( j + 1 \) to the left, looping around if necessary. We shall now show that the total contribution of entries \( j \) to the charge is independent of \( \sigma \). It follows that we can use (8), as the pairings with \( j + 1 \) coming before \( j \) do not contribute to the charge.

Consider an ordering \( \sigma \), and exchange the order of two consecutively ordered entries. They either might be paired up with their original \((j + 1)s\), in which case total charge does not change, or the largest entry in each pair are interchanged.
The latter can happen if the relevant entries have one of the following four relative orderings:

\[(j+1)\, (j+1)\, jj, \quad (j+1)\, j\, (j+1), \quad j\, (j+1)\, (j+1), \quad j\, j\, (j+1)\, (j+1).\]

It is straightforward to verify that the order exchange does not change the total contribution to charge. □

Example 15. Note that the first row of tableaux from Example 11 have depth sequences \((2, 0, 0), (0, 0, 2), (0, 2, 0),\) and their charges are 6, 2 and 4 as expected.

For this example promotion rotates the depth sequence. We will next look at an easy way to calculate the depth sequence using only the first row, which will show that promotion always rotates it.

Theorem 16. Let \(T \in \text{SHST}(a, b, n)\). Then the depth sequence of the reading word of \(T\) only depends on the first row of \(T\). In particular, \(\delta_j\) is given by the number of \(j + 1\)s on the first row.

Proof. When we consider the entries \(j\) and \(j + 1\) in the reading word, the \(j + 1\)s on the first row will not be matched to any \(j\)s, as they are at the end of the word, \(\delta_j\) is at least that. We claim that any other \(j + 1\) will be paired to a \(j\) under the algorithm, and therefore will not contribute to charge.

Outside the first row, we have \(b\) rows of length \(n\) each. Consider the \(n\) columns. In each column, there can be a \(j\), and \(j + 1\), both or neither. If we have a \(j + 1\) and a \(j\), \(j\) will appear directly above \(j + 1\), therefore will be after \(j + 1\) in the reading word. Also, if we have \(l\) columns with just \(j + 1\), we can have at most \(n - l\) cells labeled \(j\) in the first \(n\) columns, so there are at least \(l\) corresponding \(j\)s on the first row, meaning all single \(j + 1\)s will be matched as well. □

Corollary 17. For \(T \in \text{SHST}(a, b, n)\) we have that

\[\text{charge}(T) = ((a + b + 2)a + 1)n - \text{sum}_1(T),\]

where \(\text{sum}_1(T)\) is the sum of all entries of the first row.

Proof. By Theorem 14 and 16

\[\text{charge}(T) + \text{sum}_1(T) = \sum_{j \geq 2} \delta_{j-1}(\text{rw}(T))(a + b + 1 - (j-1)) + n + \sum_{j \geq 2} \delta_{j-1}(\text{rw}(T))j = \sum_{j \geq 2} \delta_{j-1}(\text{rw}(T))(a + b + 2) + n = an(a + b + 2) + n,\]

where the last identity uses Theorem 16 once more. □

Corollary 18. For \(T \in \text{SHST}(a, b, n)\) we have that

\[\delta \circ \partial \circ T = \text{rot}^{-1} \circ \delta \circ T.\]

That is, promotion rotates the depth sequence one unit to the left.

Proof. This follows as the promotion on the first row just acts by removing any 2s, subtracting 1 from larger entries, and adding \(a + b + 1\)s to replace the 2s. □

Note that the result in this corollary is reminiscent of the notion of cyclic descent sets.
3.1. Plane partitions, Gelfand–Tsetlin polytopes, and cyclic sieving. A plane partition \( \pi \) is a rectangular array with non-negative integer entries, such that rows and columns are weakly increasing. Let \( \mathcal{PP}(a, b, n) \) denote the set of plane partitions within a \( a \times b \)-rectangle, with maximal entry at most \( n \). A classical result due to MacMahon [Mac96, p. 659] states that

\[
M_q(a, b, n) := \sum_{\pi \in \mathcal{PP}(a, b, n)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[i + j + n - 1]_q}{[i + j - 1]_q}. \tag{9}
\]

Here, \( |\pi| \) denotes the total sum of entries in the plane partition.

A result due to R. Stanley [Sta01] implies that for \( \lambda = b^a \), MacMahon's generating function is essentially a principal specialization of a rectangular Schur polynomial:

\[
M_q(a, b, n) = q^{-\lambda}(1, q, q^2, \ldots, q^{a+b-1}). \tag{10}
\]

**Definition 19.** We now define a bijection \( \rho \) between \( \text{SHST}(a, b, n) \) and \( \mathcal{PP}(a, b, n) \). Given a stretched hook \( T \), we first construct a corresponding GT-pattern, which we denote by \( \text{GT}(T) \). The \( i \)th element (from the left) in row \( j \) (from below) in \( \text{GT}(T) \) is the number of boxes in row \( i \) of \( T \) that are labeled \( j \) or smaller. The GT-pattern will thus have \( a + b + 1 \) rows, the top row starts with \( n(a+1) \), followed by \( b \) entries labeled \( n \) and ending with \( a \) zeros. Below the zeros at the end of the first row there will by definition be a triangle of zeros and below the \( n \)s there will be a triangle of \( n \)s. Furthermore, the leftmost entry on each row can be determined by the rest of the entries since the sum in row \( j \) must be \( jn \). Eliminating these entries leaves a sideways plane partition of size \( a \times b \) with entries less than or equal to \( n \), which we denote by \( \rho(T) \). The inverse of this map is equally easy to define.

**Example 20.** A tableau \( T \in \text{SHST}(2, 3, 4) \), the corresponding GT-pattern \( \text{GT}(T) \) and the corresponding plane partition \( \rho(T) \) (shown in bold).

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 & 3 & 4 & 4 \\
3 & 4 & 4 & 4 & 4 & 4 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{ccccccc}
12 & 4 & 4 & 4 & 4 & 0 & 0 \\
10 & 4 & 4 & 2 & 2 & 0 & 0 \\
9 & 4 & 2 & 2 & 1 & 1 & 1 \\
8 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}
\quad
\begin{array}{ccccccc}
12 & 4 & 4 & 4 & 4 & 0 & 0 \\
12 & 4 & 4 & 4 & 4 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{11}
\]

Now let us compare the charge statistic in stretched hook tableaux with the sum statistic on the corresponding plane partition. Consider the two extremal cases where the pattern consists only of \( 0 \)s or \( n \)s.

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{ccccccc}
12 & 4 & 4 & 4 & 4 & 0 & 0 \\
12 & 4 & 4 & 4 & 4 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{12}
\]

By Corollary [17] this example gives us the maximum charge value 36 on \( \text{SHST}(2, 3, 4) \), which is when the depth sequence is \( (4, 4, 0, 0, 0) \). The corresponding plane partition \( \rho(T) \), is formed of all zeros, and has sum \(|\rho(T)| = 0 \). In general, the reading word with with maximal charge has the minimal depth sequence formed of \( a \)
ns followed by zeros, and minimal sum of the entries in the first row, giving us
\[ \text{charge}(T) = a(a + 2b + 1)n/2. \]

This example gives us the minimum charge value 12 on \( \text{SHST}(2, 3, 4) \) which is when the depth sequence is \((0, 0, 0, 4, 4)\) and the sum of the first row is maximal. The corresponding plane partition \( \rho(T) \), is formed of all ns, and has sum \(|\rho(T)| = 16\).

In general, the reading word has depth sequence formed of \( b \) zeroes followed by \( a \) ns, which by Theorem 14 gives
\[ \text{charge}(T) = \frac{a(a + 1)n}{2}, \]
and the corresponding plane partition has sum
\[ |\rho(T)| = abn, \]
giving us the total \( a(a + 2b + 1)n/2 \).

Note that in both cases the sum of the charge of the tableaux and the sum of the entries on the corresponding rectangle is the same. We will show that this is always the case.

**Theorem 21.** The bijection described above, sends \( T \in \text{SHST}(a,b,n) \) to \( \rho(T) \in \mathcal{PP}(a,b,n) \) such that
\[ \text{charge}(T) + |\rho(T)| = \frac{n(a + 2b + 1)(a)}{2} \]
\[ \text{cc}(T) - |\rho(T)| = \frac{n(b + 1)(b)}{2} \]

**Proof.** We will focus on the first identity, the second follows as \( \text{charge}(T) + \text{cc}(T) = \frac{n(a+b)(a+b+1)}{2} \).

We already showed that the identity holds if \( \rho(T) \) is formed by \( ns \) only. Assume \(|\rho(T)| \) holds when the sum of the entries is larger than or equal to \( M \). Consider a pattern with \(|\rho(T)| = M - 1\). As this is not the maximal case, there is an entry that we can increase by 1 without violating the rules. This operation corresponds to replacing the rightmost \( k \) by \( k + 1 \) for some \( k \) on the first row, and replacing a \( k + 1 \) by \( k \) on a row below, where the number \( k \) and the place of the above entry depend on the choice of the coordinate we increase. This increases the sum of the first row by 1, so by Corollary 17 the sum of charge(T) and \(|\rho(T)| \) stays the same. \( \square \)

An immediate result that follows from this formula is the affect of exchanging \( a \) and \( b \) on the cocharge polynomial.

**Corollary 22** (Conjugation Symmetry). Rotating the rectangle \( \rho(T) \) gives us a bijection between \( \text{SHST}(a,b,n) \) to \( \text{SHST}(b,a,n) \) preserving \(|\rho(T)| \) and thus the cocharge. In particular, we have
\[ |\text{SHST}(a,b,n)|_q = q^{\frac{n(a^2-a-b^2)}{2}} |\text{SHST}(b,a,n)|_q. \]

Also, combining the above result with Equation (9) gives us a way to calculate \( |\text{SHST}(a,b,n)|_q \) directly.
Corollary 23 (Kostka–Foulkes polynomials as plane partitions). We have the identity

\[ |\text{SHST}(a,b,n)_q| = q^{n} \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[i+j+n-1]_q}{[i+j-1]_q} = q^{n} M_q(a,b,n). \]

The main result in [SW18] is the following instance of the cyclic sieving phenomenon on plane partitions, under an operation called toggles.

Theorem 24 (See [SW18]). Let \( \tau \) act on \( \mathcal{PP}(a,b,n) \). Then

\[ (\mathcal{PP}(a,b,n), \langle \tau \rangle, M_q(a,b,n)) \]

is a CSP-triple.

Combining this with Corollary 23 we get our first main result.

Corollary 25 (Cyclic sieving on stretched hook tableaux). We have that \( \partial \) act on \( \text{SHST}(a,b,n) \), with order \( a + b \), and the triple

\[ (\text{SHST}(a,b,n), \langle \partial \rangle, q^{-n} \text{SHST}_q(a,b,n)) \]

is a CSP-triple.

Proof. Note that the promotion action on any \( T \in \text{SHST}(a,b,n) \) is determined by promotion on the rectangular shape obtained by deleting the first row of \( T \). Our bijection with plane partitions matches the one described in [Hop19, Appendix 1] where it is shown that the toggle operation is mapped to \( \partial^{-1} \). The result follows as

\[ q^{-n} \text{SHST}_q(a,b,n) = M_q(a,b,n). \]

□

Note that the \( n = 1 \) correspond to a special case of [BMS14].

3.2. Discussion and background. The earliest reference connecting toggles and promotion is the article by A. Kirillov and A. Berenstein [KB96].

Later in [SW18], using completely different methods, the authors prove that plane partitions in an \( a \times b \)-box with max size \( n \) exhibit CSP under piecewise linear toggles. However, they do not mention the connection with promotion.

The connection between promotion, toggles and cyclic sieving is made explicit in S. Hopkins [Hop19], where he studies plane partitions with additional symmetry. He also discusses the connection with promotion and rowmotion on posets considered in [SW12].

Question 26. There is a promotion-type action on type \( B \) minuscule poset ideals, see [RS12, Hop20]. This is the same as a type of toggle on symmetric plane partitions, and there are \( 2^n \) such plane partitions. Can this action also be realized as an action on SSYT?

4. Promotion and cyclic sieving on skew shapes

Let \( \nu \vdash m \) be an integer partition, and let \( \text{SM}(\nu, n) \) be the set of skew semistandard Young tableaux where the shape is a disjoint union of \( \ell(\nu) \) rows where row \( j \) has length \( n \nu_j \). The content is given by \( n \nu_j \), so that \( \text{SM}(\nu, 1) \) consists of skew standard Young tableaux. Given a tableau \( T \in \text{SM}(\nu, n) \), we associate a matrix \( M = M(T) \) via

\[ M_{ij} = \text{number of entries in row } i \text{ of } T \text{ with value } j. \]
By construction,
\[ \sum_{j \geq 1} M_{ij} = n \nu_i \quad \text{and} \quad \sum_{i \geq 1} M_{ij} = n, \]
and we do in fact have a bijection between \( \text{SM}(\nu, n) \) and \( \mathcal{M}(n \nu, n^m) \), the set of non-negative integer matrices with row sums given by \( n \nu \) and each column summing to \( n \). Given a matrix \( M \in \mathcal{M}(n \nu, n^m) \), we associate a biword \( W \) where \( (i, j) \) appears \( M_{n+1-i,j} \) times and the entries in the biword are sorted lexicographically. We then obtain a bijection between \( \text{SM}(\nu, n) \) and \( \mathcal{M}(n \nu, n^m) \), the set of non-negative integer matrices with row sums given by \( n \nu \) and each column summing to \( n \).

**Example 27.** Here is a \( T \in \text{SM}(211, 3) \), the corresponding matrix \( M(T) \), and the biword \( W(T) \).

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 3 & 4 \\
2 & 1 & 4 & 2 & 1 & 1 & 2
\end{pmatrix}
\quad
\begin{pmatrix}
2 & 1 & 2 & 1 \\
1 & 0 & 1 & 1 \\
0 & 2 & 0 & 1
\end{pmatrix}
\]

**Proposition 28.** Let \( \nu \vdash m \) and \( n \geq 1 \). The Robinson–Schensted–Knuth correspondence (RSK) gives a bijection

\[
\text{SM}(\nu, n) \xrightarrow{\text{RSK}} \bigcup_{\lambda \vdash nm} \text{SSYT}(\lambda, n^m) \times \text{SSYT}(\lambda, n \nu)
\]

where the insertion algorithm is performed on the biword associated with \( T \), and \( \nu \) denotes the reverse of \( \nu \). Furthermore, \( \text{charge}(T) = \text{charge}(P) \).

**Proof.** Let \( w \) be the bottom row of the biword associated with \( T \), i.e., \( w = rw(T) \). By definition, \( \text{charge}(T) = \text{charge}(w) \). Moreover, \( w \) insert to the semistandard tableau \( P \) under RSK, so \( w \) and \( rw(T) \) are Knuth-equivalent. It is a property of \( \text{charge} \) that if \( w_1 \) and \( w_2 \) are Knuth-equivalent, then \( \text{charge}(w_1) = \text{charge}(w_2) \), see e.g. \[But94\]. This last property implies that \( \text{charge}(T) = \text{charge}(P) \).

We let the \( q \)-analogue of \( |\text{SM}(\nu, n)| \) be defined as

\[
|\text{SM}(\nu, n)|_q := \sum_{T \in \text{SM}(\nu, n)} q^{\text{charge}(T)}. \tag{17}
\]

By definition, \( |\text{SM}(\nu, n)|_q \) is the skew Kostka–Foulkes coefficient \( K_{\lambda/\mu, n^m}(q) \), where \( \lambda/\mu \) is the skew shape defined by the disjoint rows of lengths given by the parts of \( n \nu \).

**Corollary 29.** Let \( \nu \vdash m \) and \( n \geq 1 \). Then

\[
|\text{SM}(\nu, n)|_q = \sum_{\lambda \vdash mn} K_{\lambda, n^m}(q) K_{\lambda, n \nu}(1) \tag{18}
\]

**Proof.** The identity follows from Proposition 28 and by using the fact that \( K_{\lambda \nu} = K_{\lambda \nu} \).

Note that \( K_{\lambda \nu}(1) = K_{\lambda \nu} \), the Kostka coefficients defined previously.
**Theorem 30** (Cyclic sieving on stretched skew shapes with disjoint rows). Let \( \nu \vdash m \) and \( n \geq 1 \). Set \( \lambda/\mu \) to be the skew shape defined by the disjoint rows of lengths given by the parts of \( n\nu \). Then

\[
\left( \text{SM}(\nu,n), \langle \partial \rangle, K_{\lambda/\mu,n\nu}(q) \right)
\]

is a CSP-triple.

**Proof.** We first note that \( \partial \) acting on \( T \in \text{SM}(\nu,n) \) simply corresponds to cyclic rotation of the columns one step left in \( M(T) \). Hence, elements in \( \text{SM}(\nu,n) \) fixed under \( \partial \) are in bijection with matrices in \( M(n\nu,n^m) \) which are fixed under cyclic rotation of columns \( j \) steps. A result by B. Rhoades [Rho10b, Thm. 1.3], with a suitable specialization, implies that

\[
\left( M(n\nu,n^m), \langle \text{cyclic rotation of columns} \rangle, \sum_{\lambda \vdash mn} K_{\lambda,n\nu}(q)K_{\lambda,\nu}(1) \right)
\]

is a triple which exhibits the cyclic sieving phenomenon. Together with Corollary 29, the theorem follows. \( \square \)

### 4.1. Disjoint union of rectangles

We shall now extend Theorem 30 to a much larger class of skew shapes and contents, where each disjoint shape is now a rectangle (and not just a row), and arbitrary contents.

In [Rho10a], B. Rhoades proved that promotion acting on rectangular semistandard Young tableaux of shape \( a^b \), and entries bounded by \( m \), gives instance of cyclic sieving together with evaluating \( q^s_{a^b}(1,q,...,q^{m-1}) \). This was later refined in [FK13], where the content of the semistandard tableaux was held fixed, and the CSP-polynomial is essentially given by a Kostka–Foulkes polynomial.

In this subsection, we extend their result slightly, to skew shapes consisting of a disjoint union of rectangles. First some additional notation. We let \( \mu^k \) denote the partition (composition) obtained from \( \mu \), where each part has been repeated \( k \) times, so that if \( \mu = 1^{a_1}2^{a_2}\cdots k^{a_k} \), then \( \mu^k := 1^{ka_1}2^{ka_2}\cdots k^{ka_k} \). For compositions, we concatenate the parts, e.g. \( 125^3 = 125125125 \).

A skew shape with \( k \) boxes is a \( k \)-ribbon if it is connected and does not contain a \( 2 \times 2 \)-arrangement of boxes. The head of a ribbon is the upper right-most box. A collection of \( k \)-ribbons form a horizontal strip if their union is a skew shape and their heads lie in different columns. A semistandard \( k \)-ribbon tableau of shape \( \lambda/\mu \) and content \( \nu \) is a sequence of skew shifted diagrams \( \lambda_1/\mu \subset \lambda_2/\mu \subset \cdots \subset \lambda_n/\mu = \lambda/\mu \) where \( \lambda_1/\mu \) is a horizontal strip containing \( \nu_1 \) \( k \)-ribbons and each \( \lambda_{i+1}/\lambda_i+1 \) is a horizontal \( k \)-ribbon strip containing \( \nu_i \) \( k \)-ribbons.

We let \( K^{(k)}_{\lambda/\mu,\nu} \) be the number of semistandard \( k \)-ribbon tableaux of shape \( \lambda/\mu \) and content \( \nu \). Note that for \( k = 1 \), we recover the usual Kostka coefficient \( K_{\lambda/\mu,\nu} \). Finally, if \( \lambda/\mu \) is a ribbon with \( k \) boxes, we let \( \varepsilon_k(\lambda/\mu) := (-1)^{h-1} \) where \( h \) is the number of rows covered by \( \lambda/\mu \). For arbitrary skew shapes \( \lambda/\mu \), we let

\[
\varepsilon_k(\lambda/\mu) := \prod_j \varepsilon_k(\lambda^{(j)}/\mu^{(j)})
\]

where \( \lambda^{(1)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \ldots \) is any partitioning of \( \lambda/\mu \) with \( k \)-ribbons (the sign can be shown to be independent of the particular choice of partitioning). We let \( \varepsilon_k(\lambda/\mu) \) be 0 if no such partitioning exists.
Example 31. The 3-ribbon tableaux counted by $K^{(3)}_{4422,1111}$ are the following six fillings,

and the 3-ribbon tableaux counted by $K^{(3)}_{4422,2111}$ are given by

It is straightforward to verify that $\varepsilon_3(4422) = -1$, for example, as the last semistandard 3-ribbon tableau gives a partitioning with four 3-ribbons, with the signs $(-1)^{3-1}$, $(-1)^{1-1}$, $(-1)^{1-1}$, and $(-1)^{2-1}$, respectively.

It can be shown (see e.g., [APRU20]) that $\varepsilon_k(\lambda/\mu)$ can be given in terms of a skew character, but we do not need that here.

Theorem 32 (See [DLT94, p.29]). Let $\lambda/\mu$ be a skew shape and $\nu$ a weak composition. Let $\xi$ be a primitive $j$th root of unity. Then

$$K^{(j)}_{\lambda/\mu,\nu} = (-1)^{|\nu|} \varepsilon_j(\lambda/\mu) K_{\lambda/\mu,\nu}(\xi).$$

For example, $K^{(3)}_{4422,21313}(q)$ is given by

$$q^{25} + q^{24} + 4q^{23} + 5q^{22} + 10q^{21} + 13q^{20} + 21q^{19} +$$

$$24q^{18} + 33q^{17} + 34q^{16} + 39q^{15} + 36q^{14} + 36q^{13} +$$

$$27q^{12} + 23q^{11} + 14q^{10} + 9q^9 + 4q^8 + 2q^7$$

and it evaluates to $-3$ at $q = e^{2\pi i/3}$. This is in agreement with the fact that $K^{(3)}_{4422,2111} = 3$, as we saw in Example 31. Note that basic properties of Kostka-Foulkes polynomials allows to reorder the composition giving the content, in this case $K^{(3)}_{4422,21313}(q) = K^{(3)}_{4422,21112121}(q)$.

We now recall the main result in [FK13].

Theorem 33 (Cyclic sieving on rectangular SSYT, fixed content). Let $(\gamma_1, \ldots, \gamma_m)$ be a sequence of non-negative integers with sum ab. Suppose $\text{rot}_m^d(\gamma) = \gamma$ for some $d \mid m$. Then

$$\left(\text{SSYT}(a^b, \gamma), (\partial^d), q^{\frac{1}{2}(a^b-\eta^2+\cdots+\eta^2)}K_{a^b,\gamma}(q)\right)$$

is a CSP-triple. Note that $\langle \partial^d \rangle$ generates a cyclic group of size $m/d$.

Note that $\text{rot}_m^d(\gamma) = \gamma$ implies that $\gamma$ is the concatenation of $m/d$ copies of some composition with $d$ parts. Let $\mu := (\gamma_1, \ldots, \gamma_{jd})$ and with a light abuse of notation we will also write $\mu = \gamma_{jd}/m$. For example, $\gamma = 12121212$, $d = 2$, $j = 2$ gives $\mu = 1212$.

Combining this theorem with Theorem 32 we have that for fixed $a^b$, $d \geq 1$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$ and any $j \mid \frac{m}{d}$, that

$$|\{T \in \text{SSYT}(a^b, \gamma) : \partial^{jd}(T) = T\}| = \begin{cases} 
K^{(m/(jd))}_{a^b,\gamma_{jd/m}}(q) & \text{if } \text{rot}_m^d(\gamma) = \gamma \\
0 & \text{otherwise.}
\end{cases}$$

(19)
Let $\ell = \frac{m}{jd}$ and $\xi$ be a primitive $\ell$th root of unity. From (19) and Theorem 33 (with the assumption that $\rot^{d}(\gamma) = \gamma$) we have due to the CSP that

$$K_{a',\gamma}/\ell = \xi\frac{1}{\ell}(a^{2}\ell-(\gamma_{1}^{2}+\cdots+\gamma_{m}^{2}))K_{a',\gamma}(\xi).$$

(20)

Now, by Theorem 32 $K_{a',\gamma}/\ell = \varepsilon_{\ell}(a^{b})(-1)^{\gamma_{1}/\ell}K_{a',\gamma}(\xi)$, so (unless both sides vanish)

$$(-1)^{\frac{m}{d}(\ell-1)}\varepsilon_{\ell}(a^{b}) = \xi\frac{1}{\ell}(a^{2}\ell-(\gamma_{1}^{2}+\cdots+\gamma_{m}^{2})).$$

(21)

Note that we must have $\ell | ab$ in order for $K_{a',\gamma}/\ell$ to be non-zero, so $(-1)^{\frac{m}{d}(\ell-1)}$ is either $-1$ or $1$. Moreover, $\gamma_{1}^{2} + \cdots + \gamma_{m}^{2} = \ell(\gamma_{1}^{2} + \cdots + \gamma_{d}^{2})$, so there is a factor of $\ell/2$ in the exponent of $\xi$.

Intuitively, we can then think that $\xi\gamma_{1}^{2} + \cdots + \gamma_{m}^{2} = (-1)^{\gamma_{1}^{2} + \cdots + \gamma_{d}^{2}} = (-1)^{\gamma_{1}^{2} + \cdots + \gamma_{d}^{2}} = (-1)^{\frac{m}{d}}$. Hence, the appearance of $\gamma$ in (21) is only used to define a “nice” exponent, where the dependence on $\ell$ is encapsulated in $\xi$.

**Corollary 34.** Given positive integers $a$, $b$ and $M$, where $M | ab$, there is an integer $E = E(a, b, M) > 0$, such that for all $\ell | M$ with $\varepsilon_{\ell}(a^{b}) \neq 0$, we have

$$(-1)^{\frac{m}{d}(\ell-1)}\varepsilon_{\ell}(a^{b}) = \xi^{E},$$

(22)

where $\xi$ is a primitive $\ell$th root of unity.

**Proof.** Use (21), we can choose $m$ and $d$ such that $M = \frac{n}{d}$ and a $\gamma$, satisfying $\rot^{d}(\gamma) = \gamma$, $\sum \gamma_{i} = ab$. \hfill $\square$

We shall now use Corollary 34 to study shapes which are disjoint unions of rectangles. For positive integer vectors $a = (a_{1}, \ldots, a_{r})$, $b = (b_{1}, \ldots, b_{r})$, let

$$a^{b} := (a^{b_{1}}_{1}) \oplus \cdots \oplus (a^{b_{r}}_{r})$$

denote a skew shape consisting of a disjoint union of $r$ rectangles, where rectangle $k$, $1 \leq k \leq r$, has shape $a^{b_{k}}_{k}$.

**Lemma 35.** Let $a^{b}$ be a disjoint union of rectangles, and $\gamma$ be a composition of length $m$ and size $a_{1}b_{1} + \cdots + a_{r}b_{r}$, such that $\rot^{d}(\gamma) = \gamma$ for some $d | m$. Then there is an $E \in \mathbb{N}$ (depending on $a^{b}$, $m$ and $d$), such that for all $j | \frac{m}{d}$ where $K_{a^{b},\gamma}(\xi) \neq 0$,

$$\xi^{E} = \text{sgn} K_{a^{b},\gamma}(\xi),$$

(23)

where $\xi$ is a primitive $j$th root of unity.

**Proof.** First of all, Theorem 32 implies that unless $j$ divides $a_{k}b_{k}$ for all $k \in \{1, \ldots, r\}$, we have $K_{a^{b},\gamma}(\xi) = 0$ where $\xi$ is a primitive $j$th root of unity. Hence, it suffices to consider the case when

$$\frac{m}{d}$$

divides \, $\gcd(a_{1}b_{1}, a_{2}b_{2}, \ldots, a_{r}b_{r})$.

(24)
By Theorem $32$ and some rewriting, we have that
\[
\text{sgn } K_{a^v, \gamma}(\xi) = (-1)^{\frac{1}{2} |\gamma| (j-1)} \varepsilon_j(a^b) \\
= (-1)^{\frac{1}{2} |a_1 b_1 + \cdots + a_r b_r| (j-1)} \prod_{k=1}^r \varepsilon_j(a^b_k) \\
= \prod_{k=1}^r (-1)^{\frac{a^b_k}{j} (j-1)} \varepsilon_j(a^b_k).
\]

We can now use Corollary $34$—since we assume (24)—on each of the factors in the right hand side, and deduce that \(\text{sgn } K_{a^v, \gamma}(\xi) = \xi^{E_1 + \cdots + E_r} = \xi^E\) for some fixed \(E\) which does not depend on \(j\), but only on \(a^v, m\) and \(d\).

We can now prove the main result of this subsection.

**Theorem 36** (Cyclic sieving on disjoint rectangles). Suppose \(\gamma = (\gamma_1, \ldots, \gamma_m)\) is an integer vector with total sum \(a_1 b_1 + \cdots + a_r b_r\), and such that \(\text{rot}^d_m(\gamma) = \gamma\). Then there exists some \(E = E(a^v, \gamma) \in \mathbb{N}\) such that
\[
\left(\text{SSYT}(a^v, \gamma), (\partial^d), q^E K_{a^v, \gamma}(q) \right)
\]
is a CSP-triple.

Before the proof, let us briefly discuss some details. Promotion rotates the content, \(\gamma\), so in order for \(\partial^d\) to fix an element, it is trivially necessary that \(\gamma\) is fixed under \(\text{rot}^d_m\). However, since promotion act independently on each rectangle, we must have that the content on the \(k^{th}\) rectangle, \(\nu^{(k)}\), also has this rotational symmetry in order for a tableau to be a fixed-point, for every \(k = 1, \ldots, r\).

**Proof.** Let \(\xi\) be a primitive \(j^{th}\) root of unity, where \(j \mid \frac{m}{d}\). It is given that \(\gamma\) is the concatenation of \(m/d\) copies of some smaller composition, so we can set \(\mu := \gamma^{jd/m} = (\gamma_1, \ldots, \gamma_{jd})\).

By using Lemma $35$ we can conclude that there is an \(E > 0\), depending only on \(a^v\) and \(\gamma\), such that \(q^E K_{a^v, \gamma}(q)\) is a non-negative integer as \(q = \xi\). Hence, we have that \(\xi^E K_{a^v, \gamma}(\xi)\) is equal to \(K_{a^v, \mu}^{(m/(jd))}\), and it remains to show that
\[
K_{a^v, \mu}^{(m/(jd))} = |\{T \in \text{SSYT}(a^v, \gamma) : \partial^{jd}(T) = T\}|. \tag{25}
\]
Since \(\partial\) act on each rectangle independently, we have that the right hand side of (25) is given by
\[
\sum_{\nu^{(1)} + \cdots + \nu^{(r)} = \gamma} \prod_{k=1}^r |\{T \in \text{SSYT}(a^v_k, \nu^{(k)}) : \partial^{jd}(T) = T\}|,
\]
as we need to distribute the entries in \(\gamma\) among the different rectangles. However, the product is 0 unless each composition \(\nu^{(i)}\) has the rotational symmetry \(\text{rot}^d_m(\nu^{(i)}) = \nu^{(i)}\), so we have
\[
\sum_{\rho^{(1)} + \cdots + \rho^{(r)} = \mu} \prod_{k=1}^r |\{T \in \text{SSYT}(a^v_k, (\rho^{(k)})^{m/(jd)}) : \partial^{jd}(T) = T\}|. \tag{26}
\]
Now, the left hand side of (25) is given by
\[
\sum_{\rho^{(1)} + \cdots + \rho^{(r)}} \prod_{k=1}^{r} K_{a_k^{(m/(jd))}}(\mu_k , \rho^{(k)})
\]
(27)
since any skew semistandard ribbon tableau of shape \(a^b\) and content \(\mu\) is formed from \(r\) semistandard ribbon tableaux of rectangular shape, where the total content is \(\mu\). We can now see that (26) and (27) agree since by Theorem 32, for every \(k \in \{1, \ldots, r\}\),
\[
\{T \in \text{SSYT}(a_k^{h_k}, (\rho^{(k)})^{m/(jd)}) : \partial^{jd}(T) = T\} = K_{a_k^{(m/(jd))}}.
\]
Hence, we have proved the CSP. \(\Box\)

In the above proof, we were able to adjust the sign of \(K_{a_k^h, \gamma(q)}\), by multiplying with an appropriate power of \(q\), so that the result is a CSP-polynomial. The following example illustrates that adjusting the sign of a potential CSP-polynomial is not always possible.

**Example 37.** Let
\[
\begin{align*}
f(q) &= 6 + 2q + 3q^2 + 2q^3 + 3q^4 + 2q^5 \\
g(q) &= 4 + 3q + 4q^2 + 4q^4 + 3q^5.
\end{align*}
\]
If we let \(\xi\) be a primitive 6th root of unity, then
\[
(f(\xi^1), f(\xi^2), f(\xi^3), f(\xi^6)) = (3, 3, 6, 18)
\]
\[
(g(\xi^1), g(\xi^2), g(\xi^3), g(\xi^6)) = (3, -3, 6, 18).
\]
One can show that there is some \(X\) of cardinality 18, such that \(\langle X, \mathbb{Z}/6\mathbb{Z}, f(q)\rangle\) is a CSP-triple. However, there is no \(E \in \mathbb{Z}\) such that \(q^E g(q)\) is a non-negative integer at every 6th root of unity.

### 5. Bi-cyclic sieving on ribbon SYT

A natural generalization of CSP is when the product of two cyclic groups act simultaneously on the set. It is called bicyclic sieving and was first considered in \cite{BRS08} and can be defined as follows. Assume we have two cyclic groups \(C_1, C_2\) with generators \(c_1, c_2\) of order \(k_1, k_2\) respectively, acting on a finite set \(X\). Let \(f(q,t)\) be a bivariate polynomial and \(\xi_1, \xi_2\) be primitive \(k_1\) and \(k_2\)-roots of unity respectively. Then we say that \((X, C_1 \times C_2, f(q,t))\) exhibits the **bicyclic sieving phenomenon**, biCSP for short, if for any \(i, j \in \mathbb{Z}\) we have
\[
f(\xi_1^i, \xi_2^j) = |\{x \in X : c_1^i c_2^j x = x\}|.
\]
That is, we have cyclic sieving for both cyclic groups, not only separately but also jointly. We prove biCSP for two families of ribbon SYT in this subsection.

Let \(\text{SYT}_R(\alpha_1, \ldots, \alpha_\ell)\) denote the set of ribbon standard Young tableaux with \(\alpha_i\) boxes in row \(i\).

**Remark 38.** The cardinality of \(\text{SYT}_R(m-b, b)\) is \(\binom{m}{b} - 1\), as there are \(\binom{m}{b}\) ways to choose the second row, and all choices but 1, 2, \ldots, \(b\) give a valid standard filling.
Theorem 39. The action $\partial$ act on $\text{SYT}_R(m-b,b)$ has a unique orbit of size $m-1$, and all other orbits have sizes dividing $m$. In particular,

$$\left( \text{SYT}_R(m-b,b), \langle \partial^m \rangle, \left( \begin{array}{c} m \\ b \end{array} \right) - m + [m-1]_q \right)$$

and

$$\left( \text{SYT}_R(m-b,b), \langle \partial^{m-1} \rangle, \left[ \begin{array}{c} m \\ b \end{array} \right] - [m]_q + n - 1 \right)$$

are CSP-triples.

Proof. Let us consider the action of promotion on the ribbon. Let $w = w_1 \ldots w_m$ be the reading word of the tableau. Either $w_1 = 1$ or $w_{b+1} = 1$. If $w_1 = 1$ or if $w_{b+1} = 1$ and $w_b < w_{b+2}$, the promotion acts separately on each row.

Only if $w_{b+1} = 1$ and $w_{b+2} < w_b$ an exchange happens between rows, and that is only possible when the first row is $1,b+2,b+3,\ldots,m$. Let us call this particular tableau $T^*$. In the figure below we will use the notation $a := m-b$ for convenience. The orbit of $T^*$ is of size $m-1$:

In any other orbit, there is no exchange between the rows, so promotion affects them independently. As each row is a rectangular tableau and our alphabet has $m$ elements, the order of promotion divides $m$.

The first CSP follows as $\langle \partial^m \rangle$ has order $m-1$, and we have one orbit of size $m-1$, plus $\left( \begin{array}{c} m \\ b \end{array} \right) - m$ elements that are fixed by everything.

For the second CSP, note that $\partial^{m-1}$ fixes the $(m-1)$-cycle described above. For the rest of the tableaux, the action of $\partial^{m-1}$ on the first row matches the rotation action on $b$ element subsets of $m$, except that we are missing the elements belonging to the $(m-1)$-cycle and the subset $\{m-b+1, m-b+2, \ldots, m\}$ which violates the tableau rules.

Note that under the rotation action these element actually form an $m$-cycle whose fixed points can be calculated by plugging in the appropriate root of unity to $[m]_q$. 
Subtracting this from the polynomial $\left\langle \frac{m}{1-m} \right\rangle$, given by the CSP of rotation and adding back the $m-1$ fixed points gives us the desired result. \(\Box\)

These two can be combined to give a CSP for the action of $\partial$, but we will instead give a bicyclic version as the polynomial is nicer.

**Corollary 40.** Promotion $\partial$ acting on $\text{SYT}_R(m-b,b)$ has order $m(m-1)$. Let $\psi_{m,b}(q,t) := [m-1]_q + \binom{m}{b}_t - [m]_t$, let $\xi$ be a primitive $(m-1)^{th}$ root of unity, and $\zeta$ be a primitive $m^{th}$ root of unity. Then for all $r,s \in \mathbb{Z}$,

$$\psi_{m,b}(\xi^r,\xi^s) = |\{T \in \text{SYT}_R(m-b,b) : \partial^{rm+s(m-1)}(T) = T\}|.$$

In other words,

$$\left(\text{SYT}_R(m-b,b), \langle \partial^m \rangle \times \langle \partial^{m-1} \rangle, \left(\frac{m}{b}\right)_t, [m]_t + [m-1]_q \right)$$

exhibits the bi-cyclic sieving phenomenon.

If $b = 1$ or $m - 1$, the total number of tableaux is $m - 1$, so the $(m-1)$-orbit is the only one. If $b = m - b = 2$, we have 5 tableaux in total, which are divided into a 3-orbit and a 2-orbit, so the promotion has order 6. Next, we show that apart from these trivial cases, promotion on two rows has order $m(m-1)$.

**Proposition 41.** If $b,m - b > 1$ and we do not have $b = m - b = 2$, $\partial$ on $\text{SYT}_R(m-b,b)$ has order $m(m-1)$.

**Proof.** We will show that in these cases, there is always an orbit of size $m$. Consider the tableaux with bottom row $1,2,\ldots,b-1,b+1$. This tableau does not come up in the $(m-1)$-cycle, so promotion is applied independently to the two rows. As promotion has order $m$ on the bottom row, it has order $m$ on the tableau. \(\Box\)

Note that the same arguments apply to the two column case by symmetry. Next, we consider the three row ribbon case where the first and last rows consist of one box only.

**Lemma 42.** We have that $|\text{SYT}_R(1,m-2,1)| = (m-1)(m-2) - 1$.

**Proof.** There are $m-1$ ways to pick the entry to go into the first row, as it can not be $m$, and $m-2$ ways to pick the entry in the last row so that it won’t be the smallest entry of the rest. Of these $(m-1)(m-2)$ choices, only one does not give a tableau—choosing $m-1$ for top row and $m$ for the bottom row. \(\Box\)

**Theorem 43.** For $m > 3$, the $\partial$ acting on $\text{SYT}_R(1,m-2,1)$ has order $(m-1)(m-2)$, with one $(m-2)$-cycle, and $m-3$ $(m-1)$-cycles. As a consequence, if we let $\psi_{m,b}(q,t) := [m-2]_t + (m-3)[m-1]_q$, $\xi$ be a primitive $(m-1)^{th}$ root of unity, and $\zeta$ be a primitive $(m-2)^{th}$ root of unity, we have $r,s \in \mathbb{Z}$,

$$\psi_{m,b}(\xi^r,\xi^s) = |\{T \in \text{SYT}_R(1,m-2,1) : \partial^{r(m-2)+s(m-1)}(T) = T\}|.$$

In other words,

$$\left(\text{SYT}_R(1,m-2,1), \langle \partial^{m-1} \rangle \times \langle \partial^{m-2} \rangle, [m-2]_t + (m-3)[m-1]_q \right)$$

exhibits the bi-cyclic sieving phenomenon.
Proof. There are \( m - 2 \) tableaux where the top row is 1. If the bottom row is \( m \), we get the following \( m - 2 \) cycle:

For the other \( m - 3 \) cases where bottom row contains \( k < m \) we get the following \( m - 1 \) cycles (we denote \( m - k \) by \( a \) for visual clarity):

6. Discussion

We have covered promotion on a few new classes of shapes, and in particular, paid some well-deserved attention to skew shapes. In many cases, the Kostka–Foulkes polynomials play a central role, except in the last section where we needed something different.

It is natural to examine other families of ribbon shapes. However, the order of promotion on the set \( \text{SYT}_R(k,k,k) \) for \( k = 1, \ldots, 4 \) is \( 1, 60, 81473960, \) and \( 82008289440 \), respectively, which is discouraging. Similarly, the order of promotion on the set \( \text{SYT}_R(1,1,k,1) \) for \( k \geq 1 \) seem (we have verified this for \( k \leq 15 \)) to be given by the generating function

\[
x \frac{1 + 16x - 19x^2 + 10x^3 - 2x^4}{(1 - x)^4} = x + 20x^2 + 55x^3 + 114x^4 + 203x^5 + \cdots.
\]
The main interesting open case is the staircase. Let $sc_m := (m, m - 1, \ldots, 2, 1)$. It has been shown (see [Hai92]) that $\partial$ acting on $\text{SYT}(sc_m)$ has order $m(m - 1)$.

The staircase together with promotion was first considered in the context of CSP in [PWT1], where the authors give an injection from $\text{SYT}(sc_m)$ to $\text{SYT}(m^{m+1})$, which commutes with promotion. However, there is still no natural polynomial which gives an instance of the cyclic sieving phenomenon.

**Problem 44.** Find some $p_m(q) \in \mathbb{N}[q]$ such that

$$(\text{SYT}(sc_m), \langle \partial \rangle, p_m(q))$$

is a CSP-triple.

Promotion on the *shifted staircase* admits a CSP; this is an example of a minuscule poset which behave nice with respect to promotion and cyclic sieving, see [RS12]. For more background and discussion, see [Hop20].

In [Hop19, Conj. 5.2], a related problem regarding cyclic sieving on triangular *plane partitions* is stated, with an explicit polynomial.

Finally, a different approach would be to keep the Kostka–Foulkes polynomials, and replace promotion with a different group action. In [APRU20], it is shown that the set of standard Young tableaux of shape $m\lambda := (m\lambda_1, m\lambda_2, \ldots, m\lambda_\ell)$ admit a cyclic sieving phenomenon with $K_{m\lambda, 1^{m\lambda}}(q)$ as CSP-polynomial and *some* cyclic group of order $m$.

As a more concrete open problem, the statement of Theorem 36 with $h(a^b, \gamma)$ is a bit unintuitive, as there is a choice of $\delta(j)$s involved.

**Problem 45.** Is there some way to define $h(a^b, \gamma)$ without this seemingly arbitrary choice?

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