The Character of the Principal Series of Representations of the Real Unimodular Group

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Abstract

The character of the principal series of representations of $SL(n, R)$ is evaluated by using Gel’fand and Naimark’s definition of character. This representation is realized in the space of functions defined on the right coset space of $SL(n, R)$ with respect to the subgroup of real triangular matrices. This form of the representations considerably simplifies the problem of determination of the integral kernel of the group ring which is fundamental in the Gel’fand-Naimark theory of character. An important feature of the principal series of representations is that the ‘elliptic’ elements of $SL(n, R)$ do not contribute to its character.
I. INTRODUCTION

One of the principal problems in the theory of representations of the real unimodular group of arbitrary order is the determination of the irreducible unitary representations of the discrete series. In their attempt to determine all the unitary irreducible representations (UIR’s) of this family of groups Gel’fand and Graev \[1\] regarded the group \( G = SL(n, R) \) as the real form of \( G^c = SL(n, C) \). The discrete series of UIR’s which exists only for even \( n \) appears in this construction in an especially interesting way. These are distinguished by the fact that they are defined in the space of purely analytic functions. For \( n = 2 \) the space \( W \) of the complex matrices on which the group acts splits into two transitive subspaces \( W^+ \) and \( W^- \) consisting of functions analytic in the upper and lower half-plane respectively. These are the positive and negative discrete series of the UIR’s of \( SL(2, R) \) (or \( SU(1, 1) \)). Although the Gel’fand - Graev theory is perhaps the only theory capable of explaining the occurrence of the discrete series of representations of \( SL(n, R) \) the complete resolution of this problem still remains open.

For the principal series of representations however, the above involved realization of the representation can be dispensed with, and the character of the representation can be determined by a simple extension \[2\] of Gel’fand and Naimark’s method as outlined in their seminal paper \[3\] on the Lorentz group. In two previous papers \[4, 5\] this problem was completely solved for \( n = 2 \). The character of the discrete series of representations was determined by using the Hilbert space method of Bargmann \[6\] and Segal \[7\] while those of the principal and exceptional series were determined by a simple extension of the method of ref. \[3\]. The main advantage of this method is that the entire analysis can be carried out within the canonical framework of Bargmann \[8\]. It is the object of this paper to extend this method for the evaluation of the character of the principal series of representations of the real unimodular group of arbitrary order.

First of all it is necessary to realize the representations of the principal series of \( SL(n, R) \) in a form suitable for the computation of the character. In what follows the representations
of the principal series will be realized in the form of operators in the space of the functions defined on the right coset space of $SL(n, R)$ with respect to the subgroup of real triangular matrices. This form of the representation is particularly suited for the determination of the integral kernel of the group ring which is fundamental in the Gel’fand - Naimark theory of character. The general outline of the remaining steps of the calculation is similar to ref. [4]. An important feature of the principal series of representations is that the ‘elliptic’ elements of $SL(n, R)$ i.e. the elements of $SL(n, R)$ corresponding to complex eigenvalues do not contribute to its character.

II. SOME SUBGROUPS OF $SL(n, R)$ AND THE IN Variant MEASURE ON THEM

In this analysis the triangular and diagonal subgroups of the real unimodular group will play a very important role. The representations will be constructed in the space of functions defined on the right coset spaces of the group with respect to the subgroup of real triangular matrices.

A. The subgroup $K$ and the invariant measure on it

$K$ is the group of all triangular matrices of the form

\[
K = \begin{pmatrix}
k_{11} & k_{12} & \cdots & k_{1n} \\
0 & k_{22} & \cdots & k_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & k_{nn}
\end{pmatrix}
\]  

subject to the condition

\[
\det K = k_{11}k_{22}\ldots k_{nn} = 1
\]

The above matrix has the following properties: $k_{pq} = 0$ for $p > q$, $k_{pq} \neq 0$ for $p \leq q$. To determine the left and right invariant measures on $K$ we choose as parameters in $K$ the
coordinates $k_{pq}, p < q$ and $k_{pp}$ ($p = 2, 3, \ldots, n$). We note that out of the $n$ diagonal elements $k_{11}, k_{22}, \ldots, k_{nn}$ only $n - 1$ diagonal elements can be regarded as independent because of the unimodularity condition.

We start from the left - invariant differential

$$d\omega = k^{-1}dk$$

where $dk$ is the matrix of the elements $dk_{pq}$. Since the diagonal elements of the matrix $k \in K$ are connected by

$$\frac{dk_{11}}{k_{11}} + \frac{dk_{22}}{k_{22}} + \cdots + \frac{dk_{nn}}{k_{nn}} = 0$$

all the elements $dk_{pq}$ are not independent arbitrary increments. Hence out of the $n$ diagonal elements only $n - 1$ are independent which we choose as, $dk_{22}, dk_{33}, \ldots, dk_{nn}$. The off diagonal elements are all arbitrary.

Proceeding in the usual way and denoting $k^{-1}$ by $Q$ so that $Q_{ij} = K_{ji}$, $K_{ji}$ being the cofactor of $k_{ji}$ we obtain the left invariant measure on $K$,

$$d\mu_l(k) = | \det D | \prod_{p<q} dk_{pq} \prod_{p=2}^{n} dk_{pp}$$

Here $D$ is a triangular matrix whose determinant is given by the product of its diagonal elements

$$\det D = Q_{11}^{n-1} Q_{22}^{n-1} Q_{33}^{n-2} Q_{44}^{n-3} \cdots Q_{n-1,n-1}^2 Q_{nn}$$

$$= k_{33}^2 k_{44}^2 \cdots k_{nn}^{n-2}$$

Hence

$$d\mu_l(k) = |k_{33}| |k_{44}|^2 \cdots |k_{nn}|^{n-2} \prod_{p<q} dk_{pq} \prod_{p=2}^{n} dk_{pp}$$

For the calculation of the right invariant measure it is convenient to make a different choice of basis i.e a different choice of the independent parameters of the subgroup $K$. Of course after carrying out the calculations we shall transform it back to the old set. We now arrange the independent elements of $k \in K$ in the following sequence
The elements of the right invariant differential

\[
d\omega = dk \ k^{-1}
\]  

are arranged in the same sequence

\[
d\omega_{11} ; d\omega_{12} ; d\omega_{22} ; d\omega_{13} , d\omega_{33} ; \ldots ; d\omega_{1n} , d\omega_{2n} , \ldots , d\omega_{n-1} , d\omega_{nn}
\]

Hence

\[
d\mu_r(k) = | \det D | \prod_{p \neq q, p < q} dk_{pq} \prod_{p=1}^{n-1} dk_{pp}
\]

where \( D \) is a triangular matrix so that its determinant is the product of its diagonal elements.

Thus

\[
d\mu_r(k) = | k_{11} |^{-1} | k_{22} |^{-2} | k_{33} |^{-3} \ldots | k_{n-1,n-1} |^{-(n-1)} | k_{nn} |^{-(n-1)} \prod_{p < q} dk_{pq} \prod_{p=1}^{n-1} dk_{pp}
\]

The above measure has been calculated with \( k_{11}, \ldots, k_{n-1,n} \) as the independent elements.

We now transform it back to the old basis in which \( k_{11} \) is not independent but \( k_{nn} \) is. Thus

\[
dk_{11} = \left| \frac{\partial k_{11}}{\partial k_{nn}} \right| dk_{nn} = \left| \frac{k_{11}}{k_{nn}} \right| dk_{nn}
\]

Hence

\[
d\mu_r(k) = | k_{22} |^{-2} | k_{33} |^{-3} \ldots | k_{nn} |^{-n} \prod_{p < q} dk_{pq} \prod_{p=2}^{n} dk_{pp}
\]

Let us now introduce the Radon - Nikodym derivative

\[
\beta(k) = \frac{d\mu(k)}{d\mu_r(k)} = | k_{22} |^2 | k_{33} |^4 \ldots | k_{nn} |^{2n-2}
\]

From the definition of \( \beta(k) \) it follows that

\[
\beta(k_1 k_2) = \beta(k_1) \beta(k_2)
\]
B. The subgroup $H$

We denote by $H$ the set of all matrices $h = \| h_{pq} \|$ satisfying the condition $h_{pq} = 0$ for $q > p$. Thus $h$ is a triangular matrix with the upper triangle zero. Clearly $H$ is a subgroup of $SL(n, R)$. It is also clear that $H$ has properties analogous to those of $K$. These properties can be derived from those of $K$ in the following way. Let us denote by $\tilde{g}$ the transpose of the matrix $g$. It is evident that this operation carries $K$ into $H$ and $H$ into $K$. As a consequence the left shift in $K$ is equivalent to the right shift in $H$. It, therefore, follows that the left invariant measure in $K$ coincides with the right invariant measure in $H$ and the right invariant measure in $K$ coincides with the left invariant measure in $H$:

$$d\mu_r(h) = |h_{33}| |h_{44}|^2 \ldots |h_{nn}|^{n-2} \prod_{p>q} dh_{pq} \prod_{p=2}^n dh_{pp}$$ (15)

$$d\mu_l(h) = |h_{22}|^{-2} |h_{33}|^{-3} \ldots |h_{nn}|^{-n} \prod_{p>q} dh_{pq} \prod_{p=2}^n dh_{pp}$$ (16)

C. The subgroup $X$

Let us denote by $X$ the subgroup of matrices $x = \| x_{pq} \| \in H$ all diagonal elements of which are equal to 1.

$$x_{pq} = 0 \quad \text{for} \quad p < q$$

$$x_{pp} = 1$$ (17)

We choose as parameters determining $x \in X$ the variables $x_{pq}, p > q$. The left and right invariant measure on $X$ are given by

$$d\mu_r(x) = d\mu_l(x) = \prod_{p>q} dx_{pq}$$ (18)
D. The subgroup \( Z \)

We denote by \( Z \) the subgroup of matrices \( \zeta = \| \zeta_{pq} \| \in K \) all diagonal elements of which are equal to 1.

\[
\zeta_{pq} = 0 \quad \text{for} \quad p > q \\
\zeta_{pp} = 1
\]  

(19)

As before the left shift in \( X \) is equivalent to the right shift in \( Z \) and as a consequence

\[
d\mu_r(\zeta) = d\mu_l(\zeta) = \prod_{p<q} d\zeta_{pq}
\]  

(20)

E. The subgroup \( D \)

We denote by \( D \) the subgroup of real diagonal matrices

\[
\delta = \begin{pmatrix}
\delta_1 & 0 & 0 & \cdots & 0 \\
0 & \delta_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \delta_n
\end{pmatrix}
\]  

(21)

satisfying

\[
\delta_1 \delta_2 \cdots \delta_n = 1
\]  

(22)

the left and right invariant measure on \( D \) is given by

\[
d\mu(\delta) = \frac{d\delta_2 \ d\delta_3 \cdots d\delta_n}{| \delta_2 || \delta_3 | \cdots | \delta_n |}
\]  

(23)

F. Some relations among \( G = SL(n, R), H, K, X, Z, D \)

To study the cosets of the real unimodular group with respect to the subgroups mentioned above it is useful to have all the elements of \( SL(n, R) \), with some exceptions, represented in the form of products of these subgroups.
Let us first consider representations of the elements of the group $K$. Since $Z$ and $D$ are subgroups of $K$ each product of the form $\zeta \delta$ and $\delta \zeta$ where $\zeta \in Z$ and $\delta \in D$ is an element of $K$. Conversely each $k \in K$ may be represented in a unique way in the form

$$k = \delta \zeta = \zeta' \delta$$  \hspace{1cm} (24)

where

$$k_{pp} = \delta_p \quad , \quad \zeta_{pq} = \frac{k_{pq}}{k_{pp}} \quad , \quad \zeta'_{pq} = \frac{k_{pq}}{k_{qq}}$$  \hspace{1cm} (25)

We can derive analogous representations for the subgroup $H$:

$$h = \delta x = x' \delta$$  \hspace{1cm} (26)

where $\delta \in D$, $x \in X$ and

$$h_{pp} = \delta_p \quad , \quad x_{pq} = \frac{h_{pq}}{h_{pp}} \quad , \quad x'_{pq} = \frac{h_{pq}}{h_{qq}}$$  \hspace{1cm} (27)

Let us now consider arbitrary elements of $SL(n,R)$. We show that each element $g \in SL(n,R)$, with some exceptions, can be represented in the form

$$g = \zeta h \quad ; \quad \zeta \in Z \quad , \quad h \in H$$  \hspace{1cm} (28)

Only those elements are exceptional for which one (or more) of the minors

$$g_m = \det G_m$$  \hspace{1cm} (29)

where

$$G_m = \begin{pmatrix} g_{mm} & g_{m,m+1} & \cdots & g_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ g_{nm} & g_{n,m+1} & \cdots & g_{nn} \end{pmatrix}$$  \hspace{1cm} (30)

vanishes. To prove this we shall display an element $\zeta'g \in H$. Since $h$ is a triangular matrix whose elements above the main diagonal are zero, $\zeta'g \in H$ if

$$\sum \zeta'_{pq} g_{sq} = 0 \quad \quad q > p$$  \hspace{1cm} (31)
which implies

$$
\tilde{G}_{p+1} \zeta'_{p} = -\eta_{p}
$$

(32)

where

$$
\zeta'_{p} = \begin{pmatrix}
\zeta'_{p,p+1} \\
\zeta'_{p,p+2} \\
\vdots \\
\zeta'_{p,n}
\end{pmatrix}, \quad \eta_{p} = \begin{pmatrix}
g_{p,p+1} \\
g_{p,p+2} \\
\vdots \\
g_{p,n}
\end{pmatrix}
$$

(33)

The existence of nontrivial solutions for \( \zeta'_{mn} \), therefore, requires that the determinant must be nonvanishing:

$$
\det \tilde{G}_{p+1} = \det G_{p+1} = g_{p+1} \neq 0 \quad p = 1, 2, 3, \ldots
$$

(34)

Hence if one (or more) of the minors \( g_{p+1} \) vanishes the matrix \( \tilde{G}_{p+1} \) ceases to be invertible and, therefore, solutions for \( \zeta' \) donot exist and the decomposition \( g = \zeta h \) cannot exist. Hence for \( \zeta' \) thus determined

$$
\zeta' g = h , \quad h \in H \\
g = \zeta^{-1} h = \zeta h
$$

(35)

Analogously it may be shown that it is possible to write

$$
g = k x , \quad k \in K , \quad x \in X
$$

(36)

This decomposition exists if all the minors

$$
g_{m} = \det G_{m} \neq 0 , \quad m = 2, 3, 4, \ldots
$$

(37)

It now follows that multiplication of \( g \) on the left by \( \zeta' \in Z \) keeps \( \det G_{m} = g_{m} \) invariant. This can be written symbolically in the form

$$
g_{m}(\zeta' g) = g_{m}(g)
$$

(38)
Writing \( g = \zeta h \) we have

\[
g_m(g) = g_m(\zeta' g) = g_m(\zeta' \zeta h) = g_m(h)
\]  

(39)

where \( \zeta' \) is completely arbitrary. Hence choosing \( \zeta' = \zeta^{-1} \) we have

\[
g_m(g) = g_m(h) = \det H_m
\]

(40)

where

\[
H_m = 
\begin{pmatrix}
  h_{mm} & 0 & 0 & \cdots & 0 \\
  h_{m+1,m} & h_{m+1,m+1} & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  h_{n,m} & h_{n,m+1} & h_{n,m+2} & \cdots & h_{nn}
\end{pmatrix}
\]

(41)

Hence

\[
g_m = \det H_m = h_{mm} h_{m+1,m+1} \cdots h_{nn}
\]

(42)

which yields

\[
h_{mm} = \frac{g_m}{g_{m+1}}
\]

(43)

Identical analysis can be carried out for the matrix \( gx' \) and it now follows that

\[
k_{mm} = \frac{g_m}{g_{m+1}}
\]

(44)

We have therefore shown for \( g = kx \) and \( g = \zeta h \)

\[
k_{pp} = \frac{g_p}{g_{p+1}}
\]

(45)

Let us now denote by

\[
\begin{pmatrix}
  p_1 , \ p_2 , \cdots , \ p_m \\
  q_1 , \ q_2 , \cdots , \ q_m
\end{pmatrix}
\]

(46)

the submatrix of \( g \) consisting of the elements with row indices \( p_1,p_2,\ldots,p_m \) and column indices \( q_1,q_2,\ldots,q_m \) i.e.
\[
\begin{pmatrix}
  p_1, \ p_2, \ \cdots, \ p_m \\
  q_1, \ q_2, \ \cdots, \ q_m
\end{pmatrix} =
\begin{pmatrix}
  g_{p_1,q_1} & g_{p_1,q_2} & \cdots & g_{p_1,q_m} \\
  g_{p_2,q_1} & g_{p_2,q_2} & \cdots & g_{p_2,q_m} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{p_m,q_1} & g_{p_m,q_2} & \cdots & g_{p_m,q_m}
\end{pmatrix}
\tag{47}
\]

By repeating the previous arguments it can be shown that

\[
\det \begin{pmatrix} p, \ p+1, \ p+2, \ \cdots, \ n \\ q, \ p+1, \ p+2, \ \cdots, \ n \end{pmatrix} = h_{pq} \ h_{p+1,p+1} \ \cdots \ h_{nn} \tag{48}
\]

Thus

\[
h_{pq} = \det \begin{pmatrix} p, \ p+1, \ p+2, \ \cdots, \ n \\ q, \ p+1, \ p+2, \ \cdots, \ n \end{pmatrix} / g_{p+1} , \quad p > q \tag{49}
\]

In a similar manner

\[
k_{pq} = \det \begin{pmatrix} p, \ q+1, \ q+2, \ \cdots, \ n \\ q, \ q+1, \ q+2, \ \cdots, \ n \end{pmatrix} / g_{q+1} , \quad q > p \tag{50}
\]

From the decomposition \( k = \zeta \delta \) it now follows that

\[
\zeta_{pq} = \det \begin{pmatrix} p, \ q+1, \ q+2, \ \cdots, \ n \\ q, \ q+1, \ q+2, \ \cdots, \ n \end{pmatrix} / g_q , \quad q > p \tag{51}
\]

\[
x_{pq} = \det \begin{pmatrix} p, \ p+1, \ p+2, \ \cdots, \ n \\ q, \ p+1, \ p+2, \ \cdots, \ n \end{pmatrix} / g_p , \quad p > q \tag{52}
\]

**III. THE PRINCIPAL SERIES OF REPRESENTATIONS OF** \( SL(n, R) \)

The basic method of Gel’fand and coworkers for deriving irreducible representations is the decomposition of a suitable reducible representation into irreducible parts. We therefore introduce the reducible ‘quasiregular’ representation as follows:

Let us consider the Hilbert space \( G(H) \) of square integrable functions on \( H \) with the scalar product
\[(f_1, f_2) = \int \overline{f_1(h)} f_2(h) \, d\mu_r(h) \quad (53)\]

We first prove the following result. The operator

\[T_g f(h) = f(hg) \quad (54)\]

is a unitary representation of \(SL(n, R)\). The matrix \(h_g \in H\) is a matrix belonging to the coset \(\tilde{h}_g\). By \(\tilde{h}\) we mean the coset \(Z_{g_0}\). Under multiplication on the right by \(g \in SL(n, R)\) all elements of the coset \(\tilde{h}\) go into the elements of a single coset which we denote by \(\tilde{h}_g\). Thus \(h' = h_g\) means that \(h'\) and \(hg\) belong to the same coset i.e. we have

\[\zeta h' = hg \quad (55)\]

It now easily follows that

\[h_{g_1g_2} = (h_{g_1})_{g_2} \quad (56)\]

so that

\[T_{g_1g_2} = T_{g_1}T_{g_2} \quad (57)\]

and \(T_g\) is a representation. We now show that \(T_g\) is unitary i.e.

\[(T_g f_1, T_g f_2) = (f_1, f_2) \quad (58)\]

First we demonstrate that although the mapping \(h \rightarrow h_g\) is not a right translation the right invariant measure remains invariant i.e.

\[d\mu_r(h) = d\mu_r(h') \quad (59)\]

To prove this we introduce the differential invariant

\[d\omega' = dh' \, h'^{-1} \quad (60)\]

\[d\mu_r(h') = \prod d\omega'_{pq} \quad (61)\]
We now arrange $d\omega'_{pq}$ in the following sequence

$$d\omega'_{21}, d\omega'_{31}, \ldots, d\omega'_{n1}; d\omega'_{22}, d\omega'_{32}, \ldots, d\omega'_{n2}; \ldots; d\omega'_{n-1,n-1}, d\omega'_{n,n-1}; d\omega'_{nn}$$ (62)

Now since

$$\zeta h' = h g$$ (63)

$\zeta$ and $h'$ are both functions of $h$. We, therefore, obtain

$$\zeta dh' + d\zeta h' = dh g$$ (64)

This yields

$$dh' h'^{-1} = \zeta^{-1} dh h^{-1} \zeta - \zeta^{-1} d\zeta$$ (65)

so that

$$d\omega' = \zeta^{-1} d\omega \zeta - \zeta^{-1} d\zeta = -\zeta^{-1} d\zeta + du$$ (66)

where

$$du = \zeta^{-1} d\omega \zeta$$ (67)

Now we arrange both $du$ and $d\omega$ in the same sequence as $d\omega'$ so that

$$\prod du_{pq} = |\det D| \prod d\omega_{pq}$$ (68)

where $D$ is the Jacobian matrix

$$D = \frac{\partial }{\partial } \left( \begin{array}{cccccc} u_{21} & u_{31} & \ldots & u_{n1} & u_{22} & u_{32} & \ldots & u_{n2} & \ldots & u_{n-1,n-1} & u_{nn} \\ \omega_{21} & \omega_{31} & \ldots & \omega_{n1} & \omega_{22} & \omega_{32} & \ldots & \omega_{n2} & \ldots & \omega_{n-1,n-1} & \omega_{nn} \end{array} \right)$$ (69)

It can be verified that $D$ is a block triangular matrix in which each diagonal block is itself triangular and has the determinant 1. Thus

$$\det D = 1$$ (70)

Hence
\[ \prod_{p \geq q, p \neq 1} du_{pq} = d\mu_r(h) \quad (71) \]

Now

\[ d\omega'_pq = du_{pq} - \sum_{r = p}^n \zeta'_pr d\zeta_{rq} \quad (72) \]

where we have written \( \zeta^{-1} = \zeta' \). For \( p \geq q \) the second term of eqn. (72) is zero. Hence

\[ d\omega'_pq = du_{pq} , \quad p \geq q \quad (73) \]

We therefore obtain

\[ d\mu_r(h') = \prod_{p \geq q, p \neq 1} d\omega'_pq = \prod_{p \geq q, p \neq 1} du_{pq} = d\mu_r(h) \quad (74) \]

It now immediately follows that the quasiregular representation is unitary

\[ (T_g f_1, T_g f_2) = \int d\mu_r(h) \overline{f_1(h_g)} f_2(h_g) = \int d\mu_r(h') \overline{f_1(h')} f_2(h') = (f_1, f_2) \quad (75) \]

To decompose the quasiregular representation into irreducible representations we start from

\[ \int f(h) \, d\mu_r(h) = \int \beta(h) \, f(h) \, d\mu_l(h) \quad (76) \]

If we introduce the decomposition

\[ h = \delta \, x \quad (77) \]

it then follows that

\[ d\mu_l(h) = d\mu(\delta) \, d\mu(x) \quad (78) \]

Noting further that

\[ \beta(\delta x) = \beta(\delta) \, \beta(x) = \beta(\delta) \quad (79) \]

we have
\[ \int f(h) \, d\mu_r(h) = \int d\mu(x) \int f(\delta x) \beta(\delta) \, d\mu(\delta) \]  

Replacing \( f(h) \) by \( |f(h)|^2 \) we obtain

\[ \int |f(h)|^2 \, d\mu_r(h) = \int d\mu(x) \int |\phi(x, \delta)|^2 \, d\mu(\delta) \]  

where

\[ \phi(x, \delta) = f(\delta x) \beta^{\frac{1}{2}}(\delta) \]  

Let \( \sigma(\delta) \) be the character or equivalently the one dimensional matrix element in a single irreducible unitary representation of the commutative subgrup \( D \). We introduce the ‘Fourier transform’ of the function \( \phi(x, \delta) \) in the following way:

\[ f_\sigma(x) = \int \phi(x, \delta) \overline{\sigma}(\delta) \, d\mu(\delta) \]
\[ = \int \beta^{\frac{1}{2}}(\delta) f(\delta x) \overline{\sigma}(\delta) \, d\mu(\delta) \]  

By Fourier transform we mean the \( \sigma \) - transform. We shall presently see that ‘Fourier’ transform in the context of real unimodular group is essentially the Mellin transform. By Plancherel theorem it now follows that

\[ \int |f_\sigma(x)|^2 \, d\lambda(\sigma) = \int |\phi(x, \delta)|^2 \, d\mu(\delta) \]  

We, therefore, obtain

\[ \| f \|^2 = \int |f(h)|^2 \, d\mu_r(h) \]
\[ = \int d\mu(x) \int |\phi(x, \delta)|^2 \, d\mu(\delta) \]
\[ = \int d\lambda(\sigma) \int |f_\sigma(x)|^2 \, d\mu(x) \]  

Thus

\[ \| f \|^2 = \int d\lambda(\sigma) \| f_\sigma \|^2 \]  

This evidently means that \( G(H) \) is decomposed into a direct continuous sum of unitary spaces \( G_\sigma(X) \). We shall prove that this decomposition is simultaneously a decomposition of the quasiregular representation \( T_g \) into UIR’s in each of the spaces \( G_\sigma(X) \).
We denote the $\sigma$ - component of the transformed function $f'(h) = f(h_g)$ by $f'_\sigma(x)$ so that

$$f'(x) = \int f(h_g) \beta^2(\delta) \overline{\sigma(\delta)} \, d\mu(\delta)$$

(87)

where we have omitted the subscript $\sigma$.

If we now set $h_g = h^1$ it then follows that $hg$ and $h^1$ are in the same coset and we have

$$\zeta^1 h^1 = h g$$

(88)

Introducing $h^1 = \delta^1 x^1$ we have

$$hg = \delta^1 \delta^{1-1} \zeta^1 \delta^1 x^1$$

(89)

so that noting $\delta^{1-1} \zeta^1 \delta^1 = \zeta \in Z$ we obtain

$$\delta x g = \delta^1 \zeta x^1$$

(90)

where we have set $h = \delta x$. Hence

$$x g = \delta^{-1} \delta^1 x^1 = \delta^{(2)} \zeta x^1$$

(91)

Since $\delta^{(2)} \zeta \in K$ setting $k = \delta^{(2)} \zeta$ we have

$$x g = k x^1$$

(92)

This equation implies that $x^1$ and $xg$ lie in the same right coset of $SL(n, R)$ by the subgroup $K$, i.e.

$$x^1 = x_g$$

(93)

Further

$$h_g = h^1 = \delta^1 x^1 = \delta \delta^{(2)} x_g$$

(94)

Hence we have

$$f'(x) = \int f(\delta \delta^{(2)} x_g) \beta^2(\delta) \overline{\sigma(\delta)} \, d\mu(\delta)$$

(95)
Using the right invariance of the above integral and the separability of the Radon-Nikodym derivative

\[ f'(x) = \beta^{-\frac{1}{2}}(\delta^{(2)}) \int f(\delta x_g) \, \beta^{\frac{1}{2}}(\delta) \, \overline{\sigma(\delta\delta^{(2)-1})} \, d\mu(\delta) \quad (96) \]

Let us now consider some basic properties of the character \( \sigma(\delta) \) of the group \( D \). Since the group \( D \) is abelian its UIR’s are one dimensional and \( \sigma(\delta) \) is the one dimensional matrix satisfying

\[ \sigma(\delta\delta^{(2)}) = \sigma(\delta) \, \sigma(\delta^{(2)}) \quad (97) \]

which is the group composition law and

\[ \overline{\sigma(\delta)} = \sigma(\delta^{-1}) \quad (98) \]

which is the condition of unitarity of the representation. These two equations imply that

\[ |\sigma(\delta)|^2 = 1 \quad (99) \]

Using these properties we obtain

\[ \overline{\sigma(\delta\delta^{(2)-1})} = \overline{\sigma(\delta)} \, \overline{\sigma(\delta^{(2)})} \quad (100) \]

Thus

\[ f'(x) = \beta^{-\frac{1}{2}}(\delta^{(2)}) \, \sigma(\delta^{(2)}) \int f(\delta x_g) \, \beta^{\frac{1}{2}}(\delta) \, \overline{\sigma(\delta)} \, d\mu(\delta) \quad (101) \]

the above integral is the \( \sigma \)-transform of the function \( f(\delta x_g) \). Thus

\[ T_g^\sigma \, f(x) = \sigma(\delta^{(2)}) \, \beta^{-\frac{1}{2}}(\delta^{(2)}) \, f(x_g) \quad (102) \]

where \( \delta^{(2)} \) is defined by

\[ x \, g = \delta^{(2)} \zeta x^1, \quad k = \delta^{(2)} \zeta, \quad x^1 = x_g \quad (103) \]

It can be easily verified that the operator \( T_g^\sigma \) is unitary.
\[(f_1, f_2) = (T^\sigma_g f_1, T^\sigma_g f_2)\]  
(104)

To avoid inessential complication in the notation we shall replace \(\delta^{(2)}\) by \(\delta\) so that

\[x g = kx^1 = \delta \zeta x^1, \quad x^1 = x_g\]  
(105)

The finite element of the group is then given by

\[T^\sigma_g f(x) = \sigma(\delta) \beta^{-\frac{1}{2}}(\delta) f(x_g)\]  
(106)

We now determine \(\sigma(\delta)\). We note that the character \(\sigma(\delta)\) of the group \(D\) is the character of the direct sum of \(n - 1\) multiplicative groups of all real numbers \(\delta_2, \delta_3, \ldots, \delta_n\) and can be written as

\[\sigma(\delta) = \sigma_2(\delta_2) \sigma_3(\delta_3) \ldots \sigma_n(\delta_n)\]  
(107)

Supressing the subscript we denote a particular factor \(\sigma_p(\delta_p)\) by \(\sigma(\delta)\) which satisfies

\[\sigma(\delta) = \eta(\delta) \psi(\epsilon)\]  
(108)

where

\[\delta = |\delta| \epsilon\]  
(109)

and \(\epsilon\) is the signature of \(\delta\):

\[\epsilon = \frac{\delta}{|\delta|}\]  
(110)

The function \(\psi(\epsilon)\) must satisfy

\[\psi(\epsilon \epsilon') = \psi(\epsilon) \psi(\epsilon')\]  
(111)

Setting \(\epsilon' = 1\) we have

\[\psi(\epsilon) [\psi(1) - 1] = 0\]  
(112)

Hence \(\psi(1) = 1\). Setting now \(\epsilon = \epsilon'\) and noting that \(\epsilon^2 = 1\) we have \(\psi^2(\epsilon) = 1\). Thus

\[\psi(\epsilon) = \pm 1\]  

This evidently implies that
\[ \psi(\epsilon) = \left( \frac{\delta}{|\delta|} \right)^{\eta}, \quad \eta = 0, 1 \] (113)

To determine \( \eta(\delta) \) we set \(|\delta| = e^t\) so that

\[ \eta(e^t) \eta(e^{t'}) = \eta(e^{t+t'}) \] (114)

which is solved by

\[ \eta(e^t) = e^{it\rho} \quad \text{i.e.} \quad \eta(\delta) = |\delta|^\rho \] (115)

Combining these results we obtain

\[ \sigma(\delta) = \prod_{p=2}^{n} \sigma_p(\delta_p) = \prod_{p=2}^{n} |\delta_p|^\rho_p \left( \frac{\delta_p}{|\delta_p|} \right)^{\eta_p} \] (116)

Thus the finite element of the group is given by

\[ T^\sigma_g f(x) = |\delta_2|^{|\rho_2| - 1} |\delta_3|^{|\rho_3| - 2} \ldots |\delta_n|^{|\rho_n| - n + 1} \left( \frac{\delta_2}{|\delta_2|} \right)^{\eta_2} \left( \frac{\delta_3}{|\delta_3|} \right)^{\eta_3} \ldots \left( \frac{\delta_n}{|\delta_n|} \right)^{\eta_n} f(x^1) \] (117)

The above formula for the representation can be written out in detail if in the groups \( X \) and \( SL(n, R) \) parameters \( x_{pq} (p > q) \) and \( g_{pq} \) are introduced. In order to find \( x^1 = x_g \) we must represent the element

\[ g^1 = x g \] (118)

in the form

\[ g^1 = x g = k x^1 \] (119)

Then \( x^1 = x_g \). Using the eqn. (52) we have

\[ x_{pq}^1 = \det \begin{pmatrix} p, & p + 1, & p + 2, & \ldots, & n \\ q, & p + 1, & p + 2, & \ldots, & n \end{pmatrix}^1 / g_p^1 \] (120)

In other words
\[
x_{pq}^1 = \det \left( \begin{array}{cccc}
g_{pq}^1 & g_{p,p+1}^1 & g_{p,p+2}^1 & \cdots & g_{pn}^1 \\
g_{p+1,q}^1 & g_{p+1,p+1}^1 & g_{p+1,p+2}^1 & \cdots & g_{p+1,n}^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_{n,q}^1 & g_{n,p+1}^1 & g_{n,p+2}^1 & \cdots & g_{n,n}^1 \\
\end{array} \right) / g_p^1 \tag{121}
\]

where
\[
g_p^1 = \det \left( \begin{array}{cccc}
g_{pp}^1 & g_{p,p+1}^1 & \cdots & g_{pn}^1 \\
g_{p+1,p}^1 & g_{p+1,p+1}^1 & \cdots & g_{p+1,n}^1 \\
\vdots & \vdots & \ddots & \vdots \\
g_{n,p}^1 & g_{n,p+1}^1 & \cdots & g_{n,n}^1 \\
\end{array} \right) \tag{122}
\]

In the above equations the elements \( g_{pq}^1 \) are given by,
\[
g_{pq}^1 = g_{pq} + \sum_{r=1}^{p-1} x_{pr} g_{rq} \tag{123}
\]

It now remains to determine \( \delta_p, p = 2, 3, \ldots, n \). From eqns. (26), (27) and (45)
\[
\delta_p = k_{pp} = h_{pp} = \frac{g_p^1}{g_{p+1}^1} \tag{124}
\]

The action of the finite element of the group is, therefore, given by
\[
T_g^\sigma f(x) = \left| \frac{g_2^1}{g_3^1} \right|^{i\rho_2-1} \left| \frac{g_3^1}{g_4^1} \right|^{i\rho_3-2} \cdots \left| \frac{g_{n-1}^1}{g_n^1} \right|^{i\rho_{n-1}-n+2} \left| g_n^1 \right|^{i\rho_n-n+1} \\
\text{sgn}^{\sigma_2} \left( \frac{g_2^1}{g_3^1} \right) \text{sgn}^{\sigma_3} \left( \frac{g_3^1}{g_4^1} \right) \cdots \text{sgn}^{\sigma_n} \left( g_n^1 \right) f(x^1) \tag{125}
\]

IV. THE CHARACTER OF THE PRINCIPAL SERIES OF REPRESENTATIONS OF SL(n, R)

We now proceed to compute the character of the above representations. We introduce the operator of the group ring
\[
T_t^\sigma = \int d\mu(g) \, t(g) \, T_g^\sigma \tag{126}
\]

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where $t(g)$ is an arbitrary test function on the group having a compact support. The action of the group ring is then given by

$$T_t^\sigma f(x) = \int d\mu(g) \ t(g) \ \beta^{-\frac{1}{2}}(\delta) \ \sigma(\delta) \ f(x^1)$$

(127)

Let us now perform the left translation $g \to x^{-1}g$. Under this mapping the equation $xg = kx^1$ is replaced by $g = kx_1$ where

$$
(x_1)_{pq} = \det \begin{pmatrix}
g_{pq} & g_{p,p+1} & \cdots & g_{pn} \\g_{p+1,q} & g_{p+1,p+1} & \cdots & g_{p+1,n} \\
\vdots & \vdots & \ddots & \vdots \\g_{nq} & g_{n,p+1} & \cdots & g_{nn}^1
\end{pmatrix} / g_p
$$

(128)

It can be shown that under the decomposition $g = kx_1$ the invariant measure decomposes as

$$d\mu_1(g) = d\mu_r(g) = d\mu_1(k) \ d\mu(x_1)$$

(129)

Thus

$$T_t^\sigma f(x) = \int d\mu(x_1) \int d\mu_1(k) \ t(x^{-1}kx_1) \ | k_{22} |^{i\rho_2-1} | k_{33} |^{i\rho_3-2} \ \cdots \ | k_{nn} |^{i\rho_n-n+1}$$

$$\text{sgn}^{\eta_2} k_{22} \ \text{sgn}^{\eta_3} k_{33} \ \cdots \ \text{sgn}^{\eta_n} k_{nn} \ f(x_1)$$

(130)

which can be written in the form

$$T_t^\sigma f(x) = \int K(x,x_1) \ f(x_1) \ d\mu(x_1)$$

(131)

The integral kernel of the group ring is, therefore, given by

$$K(x,x_1) = \int_K t(x^{-1}kx_1) \ | k_{22} |^{i\rho_2-1} | k_{33} |^{i\rho_3-2} \ \cdots \ | k_{nn} |^{i\rho_n-n+1}$$

$$\text{sgn}^{\eta_2} k_{22} \ \text{sgn}^{\eta_3} k_{33} \ \cdots \ \text{sgn}^{\eta_n} k_{nn} \ d\mu_1(k)$$

(132)

The above kernel has a trace $\text{Tr} \ (T_t^\sigma)$ given by

$$\text{Tr} \ (T_t^\sigma) = \int K(x,x) \ d\mu(x)$$

(133)
The eqn. (132), therefore yields

\[
\text{Tr } (T^\sigma_t) = \int_K \int_X t(x^{-1}k:x) \big| k_{22} |^{i\rho_2 - 1} \big| k_{33} |^{i\rho_3 - 2} \ldots \big| k_{nn} |^{i\rho_n - n + 1} \\
\text{sgn}^{n_2} k_{22} \text{ sgn}^{n_3} k_{33} \ldots \text{ sgn}^{n_n} k_{nn} \, d\mu_1(k) \, d\mu(x) \quad (134)
\]

We now note that the elements of the group \( SL(n, R) \) with distinct eigenvalues can be divided into two broad classes:

(a) the ‘hyperbolic’ class for which the eigenvalues of the matrix \( g \) are real,

(b) the ‘elliptic’ class for which the eigenvalues of \( g \) are complex.

We shall first show that every hyperbolic element of \( SL(n, R) \) can be represented in the form

\[
g = x^{-1} k \, x \quad (135)
\]

where \( k_{pp} = \lambda_p, (p = 1, 2, 3, \ldots, n) \) are the real eigenvalues of the matrix \( g \) taken in any order.

We recall that every \( g \in SL(n, R) \) belonging to the hyperbolic class can be diagonalized as

\[
g' \, g'^{-1} = \delta \quad (136)
\]

where

\[
\delta = \begin{pmatrix}
\delta_1 & 0 & 0 & \cdots & 0 \\
0 & \delta_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \delta_n
\end{pmatrix}
\quad (137)
\]

belongs to the subgroup of real diagonal matrices of determinant unity and \( g' \in SL(n, R) \).

We now use the decomposition

\[
g' = k' \, x \quad (138)
\]
Thus

\[ g = g' \delta g' = x^{-1} k' \delta k' x \]  

(139)

Now \( k' \delta k' \in K \), so that writing \( k = k' \delta k' \) we have the decomposition \( g = x^{-1} k x \) in which \( k_{pp} = \delta p = \lambda_p, \ p = 1, 2, \ldots, n \). We can, therefore, say that every hyperbolic element \( g \in SL(n, R) \) can be represented in the form

\[ g = x^{-1} k x \]  

(140)

where

\[ k_{pp} = \lambda_{g}^{(p)}, \quad p = 1, 2, 3, \ldots, n. \]  

(141)

are the real eigenvalues of the matrix \( g \) taken in any order. We can therefore assert that for the principal series of representations the trace is concentrated on the hyperbolic elements.

We now proceed to derive the integral relation connected with the representation of \( g \) in the form \( g = x^{-1} k x \) and show that

\[
\text{Tr} \left( T_i^p \right) = \int_X d\mu(x) \int_K d\mu(k) \ t(x^{-1} k x) \ \phi_p(k) \\
= \int t(g) \frac{\sum \phi_p(k_g) \beta_{g}^{\frac{1}{2}}(k_g)}{\prod_{p>q} | \lambda_{g}^{(p)} - \lambda_{g}^{(q)} |} \ d\mu(g)
\]  

(142)

where

\[ \phi_p(k) = | k_{22} |^{i\rho_{2} - 1} | k_{33} |^{i\rho_{3} - 2} \ldots | k_{nn} |^{i\rho_{n} - n + 1} \sgn^{n_2} k_{22} \sgn^{n_3} k_{33} \ldots \sgn^{n_n} k_{nn}, \]  

(143)

\( t(g) \) is a function on \( SL(n, R) \), \( k_g \) are the elements of \( k \) such that \( x^{-1} k_g x = g \), \( \beta(k) = \frac{d\mu(k)}{d\nu(k)} \) and the sum is taken over all \( k_g \) which are derived by all possible permutations on the main diagonal of \( k_g \) of the eigenvalues of the element \( g \); finally \( \lambda_{g}^{(1)}, \lambda_{g}^{(2)}, \ldots, \lambda_{g}^{(n)} \) are the eigenvalues of \( g \).

To prove the above formula we remove from the group \( K \) all those matrices \( k \) for which the moduli of any two eigenvalues coincide. This at the same time cuts \( K \) into \( n! \) connected (but disjoint) regions \( K_s \) such that each of these regions contains no pair of matrices \( k \), the
diagonal elements of which differ only in order. Since in cutting $K$ into $K_s$ only manifolds of lower dimension are removed, the integral over $K$ is decomposed into the sum of $n!$ integrals over $K_s$.

We now set $g = x^{-1}kx$, $x \in X$, $k \in K_s$. If $k$ runs over $K_s$ and $x$ runs over $X$ the element $g$ runs once over all the hyperbolic elements of $SL(n, R)$ except those for which one of the minors $x_m$ equals zero. But these excluded elements make up a manifold of lower dimensions; consequently elements of the form $g = x^{-1}kx$ fill up all the hyperbolic elements of $SL(n, R)$ except of a set of lower dimensions. We now find the relation between the invariant measures in $SL(n, R)$, $K$ and $X$ under the condition $g = x^{-1}kx$.

We first introduce the differential invariant

$$d\omega^g = g^{-1}dg = x^{-1}du$$

where

$$du = d\omega^k + d\omega^x - k^{-1}d\omega^x k$$

$$d\omega^k = k^{-1}dk, \quad d\omega^x = dx x^{-1}$$

we shall first prove

$$d\mu(g) = \prod d\omega_{pq} = \prod du_{pq}$$

where the product is taken over all the independent elements of $d\omega^g$ and $du$.

To prove the above formula we arrange the components of $d\omega^g$ in the following order,

$$d\omega^g_{12}, d\omega^g_{13}, \ldots, d\omega^g_{1n} ; d\omega^g_{22}, d\omega^g_{23}, \ldots, d\omega^g_{2n} ; \ldots ; d\omega^g_{n-1,n-1}, d\omega^g_{n-1,n} ; d\omega^g_{nn}$$

$$d\omega^g_{n1}, d\omega^g_{n-1,1}, \ldots, d\omega^g_{n21} ; d\omega^g_{n2}, d\omega^g_{n-1,2}, \ldots, d\omega^g_{n32} ; \ldots ; d\omega^g_{n,n-1}, d\omega^g_{n-1,n-1} ; d\omega^g_{nn-1}$$

and also the components of $du$ in the same order. Thus

$$d\mu(g) = \prod d\omega^g = |\det D| \prod du$$
Here $D$ is a block triangular matrix in which each diagonal block except the last is itself triangular and has the determinant 1. Thus calling the last block $L$ we have

$$\det D = \det L$$  \hspace{1cm} (149)$$

where $L$ equals to the matrix

$$\begin{pmatrix}
x'_{n1} \frac{\partial u_{11}}{\partial u_{n1}} + 1 & x'_{n-1,1} \frac{\partial u_{11}}{\partial u_{n1}} & \ldots & x'_{21} \frac{\partial u_{11}}{\partial u_{n1}} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
x'_{n1} \frac{\partial u_{11}}{\partial u_{n-1,1}} + x'_{n,n-1} & x'_{n-1,1} \frac{\partial u_{11}}{\partial u_{n-1,1}} + 1 & \ldots & x'_{21} \frac{\partial u_{11}}{\partial u_{n-1,1}} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x'_{n1} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n2} & x'_{n-1,1} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n-2,2} & \ldots & x'_{21} \frac{\partial u_{11}}{\partial u_{21}} + 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
x'_{n1} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n2} & x'_{n-1,1} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n-2,2} & \ldots & x'_{21} \frac{\partial u_{11}}{\partial u_{21}} + 1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x'_{n1} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n3}x_{21} & x'_{n-1,1} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n-3,3}x_{21} & \ldots & x'_{21} \frac{\partial u_{11}}{\partial u_{21}} + x'_{n-3,3}x_{21} & x'_{n3} & x'_{n-3,3} & \ldots & 1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x'_{n1} \frac{\partial u_{11}}{\partial u_{n,n-1}} + x'_{n-1,1} & x'_{n-1,1} \frac{\partial u_{11}}{\partial u_{n,n-1}} & \ldots & x'_{21} \frac{\partial u_{11}}{\partial u_{n,n-1}} & x'_{n-1,2} & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}$$  \hspace{1cm} (150)$$

where we have written

$$x^{-1} = x'$$  \hspace{1cm} (151)$$

We shall show that

$$\frac{\partial u_{11}}{\partial u_{p1}} = 0 \hspace{1cm} p = 2, 3, \ldots, n$$  \hspace{1cm} (152)$$

so that $L$ becomes a triangular matrix with determinant 1. To prove this we start from

$$d\omega_{pq}^{g} = \sum_{r=1}^{n} (g^{-1})_{pr} \, dg_{rq} = \sum_{r=1}^{n} a_{rp} \, dg_{rq}$$  \hspace{1cm} (153)$$

where $a_{rp}$ is the cofactor of $g_{rp}$. Setting $p = q = 1$ we have

$$d\omega_{11}^{g} = a_{11} \, dg_{11} + \sum_{r=2}^{n} a_{r1} \, dg_{r1}$$  \hspace{1cm} (154)$$

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In the above it should be noted that although $dg_{11}$ is not independent all other $dg_{rp}$ are independent differentials. Hence

$$\frac{\partial \omega_{11}^g}{\partial g_{p1}} = a_{11} \frac{\partial g_{11}}{\partial g_{p1}} + a_{p1} \quad (155)$$

To calculate $\frac{\partial g_{11}}{\partial g_{p1}}$ we expand $\det g (= 1)$ with respect to the first column so that

$$g_{11} a_{11} + g_{21} a_{21} + g_{31} a_{31} + \ldots + g_{n1} a_{n1} = 1 \quad (156)$$

We now note that none of the cofactors appearing above contains elements from the first column i.e. all the cofactors are independent of $g_{21}, g_{31}, g_{41}, \ldots, g_{n1}$. Hence differentiating with respect to $g_{p1}$ we have

$$\frac{\partial g_{11}}{\partial g_{p1}} a_{11} + a_{p1} = 0 \quad (157)$$

Thus

$$\frac{\partial g_{11}}{\partial g_{p1}} = -\frac{a_{p1}}{a_{11}} \quad (158)$$

Substituting eqn. (158) in eqn. (155) we have

$$\frac{\partial \omega_{11}^g}{\partial g_{p1}} = a_{11} \frac{\partial g_{11}}{\partial g_{p1}} + a_{p1} = 0 \quad (159)$$

Let us now consider

$$d\omega_{1q}^g = \sum (g^{-1})_{1r} d g_{rq} = a_{11} d g_{1q} + a_{21} d g_{2q} + \ldots + a_{n1} d g_{nq} \quad (160)$$

Thus

$$\frac{\partial \omega_{1q}^g}{\partial g_{p1}} = 0, \quad q = 2, 3, 4, \ldots, n, \quad p = 2, 3, 4, \ldots, n \quad (161)$$

Now we write eqn. (144) in the form $du = x \ d\omega^g \ x^{-1}$ so that

$$du_{11} = d\omega_{11}^g + \sum_{q=2}^n d\omega_{1q}^g \ x'_{q1} \ , \quad x' = x^{-1} \quad (162)$$

Thus
\[
\frac{\partial u_{11}}{\partial g_{p1}} = \frac{\partial \omega_{11}^g}{\partial g_{p1}} + \sum_{q=2}^{n} \frac{\partial \omega_{1q}^g}{\partial g_{q1}} x_{q1}' = 0 , \quad p = 2, 3, 4, \ldots, n \tag{163}
\]

where we have used eqn. (159) and eqn. (161). We now note

\[
\frac{\partial u_{11}}{\partial u_{p1}} = \frac{n}{\sum_{k=2}^{n}} \frac{\partial u_{11}}{\partial g_{k1}} \frac{\partial g_{k1}}{\partial u_{p1}} + \frac{n}{\sum_{k=1}^{n} \sum_{l=2}^{n}} \frac{\partial u_{11}}{\partial g_{kl}} \frac{\partial g_{kl}}{\partial u_{p1}} \tag{164}
\]

The eqn. (163) implies that the first term on the r. h. s. of eqn. (164) is zero so that

\[
\frac{\partial u_{11}}{\partial u_{p1}} = \frac{n}{\sum_{k=1}^{n} \sum_{l=2}^{n}} \frac{\partial u_{11}}{\partial g_{kl}} \frac{\partial g_{kl}}{\partial u_{p1}} \tag{165}
\]

We shall now show that

\[
\frac{\partial g_{kl}}{\partial u_{p1}} = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, n, \quad l = 2, 3, \ldots, n \tag{166}
\]

To prove this we write eqn. (144) in the form

\[
dg = g x^{-1} du x = g' du x \tag{167}
\]

where we have written \(gx^{-1} = g'\). Thus

\[
dg_{kl} = \sum_{r=1}^{n} \sum_{s=l}^{n} g'_{kr} \frac{\partial u_{rs}}{\partial u_{p1}} x_{sl} \tag{168}
\]

so that

\[
\frac{\partial g_{kl}}{\partial u_{p1}} = \sum_{r=1}^{n} \sum_{s=l}^{n} g'_{kr} \frac{\partial u_{rs}}{\partial u_{p1}} x_{sl} \tag{169}
\]

Now, in the sum on the r. h. s. of eqn. (168) \(l\) starts from 2. Hence for \(l = 2, 3, 4, \ldots, n\)

\[
\frac{\partial g_{kl}}{\partial u_{p1}} = \sum_{r=1}^{n} g'_{kr} \sum_{s=l}^{n} \frac{\partial u_{rs}}{\partial u_{p1}} x_{sl} = 0 \tag{170}
\]

because

\[
\frac{\partial u_{rs}}{\partial u_{p1}} = 0 \quad \text{for} \quad s = 2, 3, 4, \ldots, n \tag{171}
\]

Hence finally we have

\[
\frac{\partial u_{11}}{\partial u_{p1}} = 0 \quad \text{for} \quad p = 2, 3, 4, \ldots, n \tag{172}
\]
and the matrix $L$ in eqn. (150) becomes a triangular matrix with 1 along the main diagonal so that $\det L = 1$ and we have

$$d\mu(g) = \prod d\omega^g = \prod du$$

(173)

Here

$$du = d\omega^k + d\omega^x - k^{-1} d\omega^x k = d\omega^k + dv$$

(174)

where

$$dv = d\omega^x - k^{-1} d\omega^x k$$

(175)

and $d\omega^k$ and $d\omega^x$ are given by eqn. (146).

In eqn. (174) $d\omega^k$ is a triangular matrix with $k^{-1}dk_{11}, k^{-1}dk_{22}, \ldots, k^{-1}dk_{nn}$ along the principal diagonal. $dv$ is a square matrix in which the independent elements are

$$dv_{21}, dv_{31}, dv_{32}, dv_{41}, dv_{42}, dv_{43}, \ldots; dv_{n1}, dv_{n2}, \ldots, dv_{n,n-1}.$$  

(176)

The elements like $dv_{12}, dv_{13}, \ldots$ are nonzero but not independent and for all the elements

$$\frac{\partial v_{pq}}{\partial \omega^k_{rs}} = 0$$

(177)

Thus

$$d\mu(g) = \prod du = | \det D | \prod d\omega^k \prod dv$$

(178)

where $D$ is the Jacobian matrix. If we now note that

$$du_{pq} = d\omega^k_{pq} + dv_{pq} \quad p \leq q$$

(179)

$$du_{pq} = dv_{pq} \quad p > q$$

(180)

the matrix $D$ in eqn. (178) becomes a triangular matrix with 1 along the main diagonal so that

$$d\mu(g) = d\mu_1(k) \prod_{p > q} dv_{pq}$$

(181)
where
\[ dv = d\omega^x - k^{-1} d\omega^x k \]  
(182)

Let us now set
\[ k = \zeta^{-1} \delta \zeta \]  
(183)

We then obtain
\[ dv = \zeta^{-1} dp \zeta \]  
(184)

where
\[ dp = dw - \delta^{-1} dw \delta \]  
(185)

\[ dw = \zeta d\omega^x \zeta^{-1} \]  
(186)

Writing \( \zeta^{-1} = \zeta' \) we obtain from eqn. (184)
\[ dv_{pq} = \sum_{r=p}^{n} \sum_{s=1}^{q} \zeta'_{pr} dp_{rs} \zeta_{sq} \quad p > q \]  
(187)

in which \( r_{\text{min}} = p, s_{\text{max}} = q, r_{\text{min}} > s_{\text{max}} \), so that
\[ \frac{\partial v_{pq}}{\partial p_{ml}} = \zeta'_{pm} \zeta_{ql} \]  
(188)

It, therefore, follows that the determinant of the matrix connecting \( dv_{pq} \) \( (p > q) \) and \( dp_{ml} \) \( (m > l) \) is a block triangular determinant in which each diagonal block itself is triangular and has the determinant 1. Thus
\[ \prod_{r>s} dv_{rs} = \prod_{r>s} dp_{rs} \]  
(189)

so that
\[ d\mu(g) = d\mu(k) \prod_{r>s} dp_{rs} \]  
(190)

Now
\[ dp_{rs} = dw_{rs} \left( 1 - \frac{\delta_s}{\delta_r} \right) \]  \hspace{1cm} (191)

and

\[ \prod_{r>s} dp_{rs} = | \det D | \prod_{r>s} dw_{rs} \]  \hspace{1cm} (192)

where \( D \) is a diagonal matrix and \( \det D \) is the product of its diagonal elements. Thus

\[ \det D = \prod_{p>q} (\delta_p - \delta_q) \left[ \delta_2 \delta_3^2 \delta_4^3 \ldots \delta_n^{n-1} \right]^{-1} \]  \hspace{1cm} (193)

Since \( \delta_p = k_{pp} \), the above equation yields

\[ \prod_{r>s} dp_{rs} = \prod_{p>q} \left| \frac{k_{pp} - k_{qq}}{\beta_1^{1/2}(k)} \right| \prod_{r>s} dw_{rs} \]  \hspace{1cm} (194)

Since the denominator is the square root of the Radon - Nikodym derivative we can write

\[ \prod_{r>s} dp_{rs} = \prod_{p>q} \left| \frac{k_{pp} - k_{qq}}{\beta_1^{1/2}(k)} \right| \prod_{r>s} dw_{rs} \]  \hspace{1cm} (195)

It is now easy to check that the transformation from \( d\omega^x \) to \( dw = \zeta d\omega^x \zeta^{-1} \) is a linear mapping of the components of \( d\omega^x \) with determinant 1. Thus

\[ \prod_{r>s} dw_{rs} = \prod_{r>s} d\omega^x_{rs} = d\mu(x) \]  \hspace{1cm} (196)

We, therefore finally obtain

\[ d\mu(g) = \prod_{p>q} \left| \frac{k_{pp} - k_{qq}}{\beta_1^{1/2}(k)} \right| d\mu_1(k) d\mu(x) \]  \hspace{1cm} (197)

so that

\[ d\mu_1(k) d\mu(x) = \frac{\beta_1^{1/2}(k_g)}{\Pi_{p>q} | \lambda_p^{(p)} - \lambda_q^{(q)} |} d\mu(g) \]  \hspace{1cm} (198)

Hence finally

\[ \int_X d\mu(x) \int_{K_s} t(x^{-1}kx) \phi_{\rho}(k) d\mu_1(k) = \int t(g) \frac{\beta_1^{1/2}(k_g)}{\Pi_{p>q} | \lambda_p^{(p)} - \lambda_q^{(q)} |} d\mu(g) \]  \hspace{1cm} (199)

where \( \phi_{\rho}(k_g) \) and \( \beta_1^{1/2}(k_g) \) are given by eqns. (143) and (13) respectively and \( k_g \) is taken from \( K_s \). Summing this equation over all \( K_s \) we obtain the formula in eqn. (142). We can now immediately write down the character \( \pi(g) \) of the representation as
\[ \pi(g) = \frac{\sum_{k_g} \chi_{\rho}(k_g)}{\prod_{p>q} |\lambda_g^{(p)} - \lambda_g^{(q)}|} \]  \hspace{1cm} (200)

where

\[ \chi_{\rho}(k_g) = \prod_{p=2}^{n} |\lambda_g^{(p)}|^{|\rho_p|} \text{sgn}^{|\rho_p|} \lambda_g^p \]  \hspace{1cm} (201)

It can be easily verified that the formula (200) yields the formula for the character of the principal series of representations of \( SL(2, R) \) for \( n = 2 \).

\[ \pi(g) = \frac{|\lambda_g|^{i\rho} + |\lambda_g|^{-i\rho}}{|\lambda_g - \lambda_g^{-1}|} \text{sgn}^n \lambda_g \]  \hspace{1cm} (202)

in agreement with ref. [4].
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