Enhancement of non-contact friction between metal surfaces induced by the electrical double layer

A.I. Volokitin
Samara State Technical University, 443100 Samara, Russia

Casimir and electrostatic non-contact friction between two gold plates, and a gold tip and a gold plate, are calculated taking into account the contribution of the electrical double layer. It is shown that in an extreme-near field (d < 10nm) the contribution from the electrical double layer leads to the enhancement of non-contact friction by many orders of magnitude in comparison to the result of the conventional theory without this contribution. Casimir and electrostatic friction dominate for short and large separations, respectively. The calculated electrostatic friction is in good agreement with experimental data. The results obtained open the way to detect the Casimir friction using Atomic Force Microscope.

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I. INTRODUCTION

One of the important achievements in nanotribology is the discovery of non-contact friction between bodies without direct contact. Beside of the fundamental significance of this phenomenon, it has practical application in biology and quantum computing for ultra-sensitive force registration due to the link between friction and force fluctuations established by the fluctuation-dissipation theorem. Origin of non-contact friction is shown in Fig. 1 (a) Casimir friction. A fluctuating electromagnetic field created by quantum and thermal fluctuations of the current density in one body induces the current density in another body. The interaction between the fluctuating and induced current densities is an origin of the Casimir force. When the bodies are in relative motion the induced current lag behind and pull the fluctuating current back producing the Casimir friction. A theory of the Casimir friction for relative sliding of two plates was developed in Refs. 7,8 using the fluctuation-electrodynamics developed by Rytov et al. 9–11. The Casimir friction exists even at zero temperature due to quantum fluctuations and is denoted as quantum friction. In contrast to the Casimir forces and radiative heat transfer, which were measured in many experiments, the detection of the Casimir friction is still a challenging problem for experimentalists. However, the Casimir frictional drag force between quantum wells and graphene sheets, and the current-voltage dependence of nonsuspended graphene on the surface of the polar dielectric SiO₂, were accurately described using the theory of the Casimir friction. 13–16. In Refs. 17,18 it was shown that the Casimir frictional drag force produced by current density in a graphene sheet can be detected using an

![FIG. 1: Origin of non-contact friction](image-url)
AFM. However, at present there is no direct mechanical detection of the Casimir friction. (b) Interaction of a charged tip with an image charge produces an electrostatic friction during motion of the tip. Theory of the electrostatic friction was developed in Refs. 19, 20. Although the Casimir and electrostatic friction are both of electromagnetic origin, the detailed mechanism is not totally clear. In the conventional theory, where the reflection amplitudes are determined by the Fresnel’s formulas, calculated friction is many orders of magnitudes smaller than experimentally observed values. However the friction is enhanced by many orders of magnitudes by ionic adsorbates.

(c) The van der Waals and electrostatic interactions between fluctuating surface displacements produce a time-dependent stresses acting on the surfaces that excite the acoustic phonons and produce phonon heat transfer and friction for moving bodies.

In this article the Casimir and electrostatic friction are calculated between gold plates, and a gold tip and a gold plate taking into account of surface charge density related with the electrical double layer. Metal consists of positive ions occupying the positions of the crystal lattice and mobile electrons. The electrons can not move away from the crystal because the positive nucleus exerts an attractive force. Thus, the surface of a clean metallic material can be considered as the superposition of two thin layers; one positively charged below the surface of the solid and the other negatively charged adjacent on the surface. This charge separation zone constitutes the electrical double layer (Fig.2). Approximating the double layer by two opposite charged planes, the plane charge density can be estimated from the relation $4\pi \sigma_s \epsilon_0 = \Delta \varphi$ where for gold the separation between planes $d_0 \approx 0.1$nm and the potential step due to the double layer $\Delta \varphi \approx 4.3$eV, thus $\sigma_s \approx 0.38$Cm$^{-2}$. It is shown that in an extreme near-field ($d < 10$nm) the Casimir and electrostatic friction are enhanced by many orders of the magnitudes in comparison to the case when the electrical double layer is not taken into account, and they can be measured using the present state of art equipment. Giant enhancement of heat transfer between gold surfaces in an extreme near-field due to the electrical double layer was studied in Ref. 23.

II. CASIMIR FRICTION

According to the theory of the Casimir friction, for two plates separated by a vacuum gap with thickness $d$ and sliding with relative velocity $v$, for $d < \lambda_T = \hbar c/k_B T$ and non relativistic velocity $v \ll c$, the frictional force between surfaces is dominated by the contribution from evanescent electromagnetic waves and determined by

$$f_x = \hbar \int_0^\infty d\omega \int_{q \gtrsim k} \frac{d^2 q e^{-2kzd}}{(2\pi)^2} \left[ \frac{4\text{Im}R_{1p}(\omega, q)\text{Im}R_{2p}(\omega', q)}{|1 - e^{-2kzd}R_{1p}(\omega, q)R_{2p}(\omega', q)|^2} [n_2(\omega') - n_1(\omega)] + (p \to s) \right], \tag{1}$$

where $n_i(\omega) = [e^{\hbar \omega/k_BT} - 1]^{-1}$, $\omega' = \omega - q_z v$, $R_p$ and $R_s$ are the reflection amplitudes for $p-$ and $s-$ polarized electromagnetic waves, $k = \omega/c$, $q_z = \sqrt{q^2 - k^2}$, $q > \omega/c$ is the component of the wave vector parallel to the surface.
To linear order in the sliding velocity \( f_z = \gamma v \) where the friction coefficient at \( T_1 = T_2 = T \) is determined by \(^{4,6,8}\)

\[
\gamma_{rad} = \frac{h^2}{8\pi^2 k_B T} \int_0^\infty \frac{d\omega}{\sinh^2(h\omega/2k_B T)} \int_0^\infty dk_z k_z (k_z^2 + k_z') e^{-2k_z d} \left[ \frac{\text{Im}R_{1p}(\omega, q)\text{Im}R_{2p}(\omega, q)}{1 - e^{-2k_z d}R_{1p}(\omega, q)R_{2p}(\omega, q)} \right]^2 + (p \to s). \tag{2}
\]

In the presence of the surface charge density, that can be due to a potential difference or the electrical double layer, the reflection amplitude for the \( p \)-polarized electromagnetic waves is determined by \(^24\)

\[
R_p = \frac{i\varepsilon k_z - k_z'}{i\varepsilon k_z + k_z'} + 4\pi i q^2 \alpha_s \varepsilon
\tag{3}
\]

where \( k_z' = \sqrt{\varepsilon k_z^2 - q^2} \), \( \varepsilon \) is the dielectric function of the substrate, and \( \alpha_s \) is the normal component of the surface dipole susceptibility. Without taking into account the contribution of the electric double layer, Eq. (3) reduces to the Fresnel’s formula for the reflection amplitude for \( p \)-polarized electromagnetic waves

\[
R_p = \frac{i\varepsilon k_z - k_z'}{i\varepsilon k_z + k_z'}.	ag{4}
\]

Due to the screening by electrons in surface layer the interaction of an external electromagnetic field with lower of the double layer plane can be neglected. In this case \( \alpha_s = \sigma_z^2 M \) where \( M \) is the surface mechanical susceptibility that determines the surface displacement under the action of external mechanical stress: \( u = M\sigma_{zz} \). In the elastic continuum model \(^{25}\)

\[
M = \frac{i}{\rho\varepsilon t} \left( \frac{\omega}{c_l} \right)^2 \frac{p_t(q, \omega)}{S(q, \omega)}, \tag{5}
\]

where

\[
S(q, \omega) = \left[ \left( \frac{\omega}{c_l} \right)^2 - q^2 \right]^2 + 4q^2 p_t p_t,
\]

\[
p_t = \left[ \left( \frac{\omega}{c_l} \right)^2 - q^2 + i0 \right]^{1/2}, \quad p_t = \left[ \left( \frac{\omega}{c_l} \right)^2 - q^2 + i0 \right]^{1/2},
\]

where \( \rho, c_l, \) and \( c_t \) are the mass density of the medium, the velocity of the longitudinal and transverse acoustic waves. A surface charge density and dipole moment is also induced by the applied potential difference \(^{24,26}\). Due to the screening of the parallel component of the electric field the contribution of the electrical double layer to the reflection amplitude for \( s \)-polarized electromagnetic waves can be neglected and for these waves the reflection amplitude is determined by the the Fresnel’s formula

\[
R_s = \frac{ik_z - k_z'}{ik_z + k_z'}.	ag{6}
\]

**III. PHONONIC FRICTION**

The electrostatic and van der Waals interactions between surfaces 1 and 2 results in stresses that act on surfaces 1 and 2. If \( u_1 \) and \( u_2 \) denote the surface displacements of solid 1 and 2 then equations \(^{24,26}\)

\[
\sigma_1(\omega) = au_1(\omega) - bu_2(\omega'), \tag{7}
\]

\[
\sigma_2(\omega') = au_2(\omega') - bu_1(\omega), \tag{8}
\]

where \( \omega' = \omega - q_z v \),

\[
a = \frac{H}{2\pi d^2} + q\sigma_z^2 \coth qd, \quad b = \frac{H}{4\pi} \frac{q^2 K_2(qd)}{d^2} + \frac{q\sigma_z^2}{\sinh qd},
\]

where \( K_2(qd) \) is the modified Bessel function of the second kind of order 2.
where the first and second terms are due to the van der Waals and electrostatic interactions, respectively. \( H \) is the Hamaker constant, \( K_2(x) \) is the modified Bessel function of the second kind and second order. The surface displacements due to thermal and quantum fluctuations are determined by

\[
    u_1(\omega) = u'_1(\omega) + M_1(\omega)[au_1(\omega) - bu_2(\omega')],
\]

\[
    u_2(\omega') = u'_2(\omega') + M_2(\omega')[au_2(\omega') - bu_1(\omega)],
\]

where according to the fluctuation-dissipation theorem, the spectral density of fluctuations of the surface displacements is determined by

\[
    \langle |u_i'|^2 \rangle = \hbar \text{Im} M_i(\omega, q) \coth \frac{\hbar \omega}{2k_BT_i}
\]

where \( M_i \) is the mechanical susceptibility for surface \( i \). The friction force can be calculated from equation

\[
    f_x v = Q_1 + Q_2
\]

where \( Q_1 \) and \( Q_2 \) are the heat generated in surfaces 1 and 2 which are determined by the rate of the work of the mechanical stress that act on surfaces 1 and 2 in the rest reference frame of surfaces 1 and 2

\[
    Q_1 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2q}{(2\pi)^2} \omega \text{Im} \langle u_1 \sigma_1 \rangle,
\]

\[
    Q_2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2q}{(2\pi)^2} \omega' \text{Im} \langle u_2 \sigma_2 \rangle
\]

From Eqs. (13-14) we get

\[
    f_x = 4 \int_0^\infty \frac{d\omega}{2\pi} \int \frac{d^2q}{(2\pi)^2} qz \frac{b^2 \text{Im} M_1(\omega) \text{Im} M_2(\omega')}{(1 - aM_1(\omega))(1 - aM_2(\omega') - b^2 M_1(\omega)M_2(\omega'))^2} \langle n_2(\omega') - n_1(\omega) \rangle.
\]

At \( T_1 = T_2 = T \) the phonon friction coefficient is given by

\[
    \gamma_{ph} = \frac{\hbar^2}{8\pi^2 k_BT} \int_0^\infty \frac{d\omega}{\sinh^2(\hbar\omega/2k_BT)} \int_0^\infty dq \frac{b^2 \text{Im} M_1(\omega) \text{Im} M_2(\omega)}{(1 - aM_1(\omega))(1 - aM_2(\omega) - b^2 M_1(\omega)M_2(\omega))^2}.
\]

### IV. NUMERICAL RESULTS

Fig 3 shows the dependence of friction coefficient for the Casimir friction between two gold plates on the distance between them for different mechanisms at \( T = 300K \). For gold \( c_l = 3240\text{ms}^{-1} \), \( c_l = 1200\text{ms}^{-1} \), \( \rho = 1.9280 \times 10^4\text{kgm}^{-3} \), \( H = 34.7 \times 10^{-20}\text{J} \) (see Ref.30) and dielectric function

\[
    \varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\nu},
\]

where \( \omega_p = 1.71 \times 10^{16}\text{s}^{-1} \), \( \nu = 4.05 \times 10^{13}\text{s}^{-1} \). In Fig 3 the blue lines are for the Casimir friction associated with the electrical double layer and \( p \)-polarized waves. The green and pink lines are for the phonon friction associated with the van der Waals and the electrostatic interaction between the fluctuating surface displacements at the potential difference \( \phi = 10V \). The red line is the theory result for the Casimir friction without the contribution from the double layer. Full and dashed lines are for the contributions from the bulk and surface (Rayleigh) acoustic modes. The mechanical susceptibility has a pole at \( \omega = \omega_s = c_s q = \xi c_s q \), where \( \omega_s \) and \( c_s \) are the frequency and the propagation velocity of the Rayleigh surface waves, while for gold \( \xi = 0.94 \). Near the pole at \( \omega \approx \omega_s \) the mechanical susceptibility can be written as

\[
    M = -\frac{c}{\rho c_l(\omega - c_s q + i\gamma)},
\]
FIG. 3: The dependence of the Casimir friction coefficient between two gold plates on the distance between them at $T = 300K$. The blue lines is the contributions from fluctuating dipole moment of surface electrical double layer at the double layer potential step $\Delta \phi = 4.3eV$. The brown lines is the contributions from the fluctuating dipole moment induced on surfaces by the potential difference $\phi = 10V$. The pink line is the contribution from van der Waals interaction between fluctuating surface displacements. The Full and dashed lines are the contributions from the bulk and surface (Rayleigh) phonon modes. The red line is the result of the conventional theory of the Casimir friction without the contributions from the fluctuations of the surface displacements.

where for gold $c = 0.36$ and $\gamma$ is the damping constant for surface wave for which can be used estimate $26,32 \gamma = 0.17\omega$. In the extreme near-field ($d < 10nm$) the double layer contribution leads to an enhancement of the Casimir friction by many orders of the magnitude in comparison with the theory which does not include this contribution.

Vibration of a tip charged by bias voltage $\phi$ parallel to metallic substrate produces an electrostatic friction. In the case when the tip is a section of the cylindrical surface with the radius of curvature $R$ and with the width $w$ the friction coefficient is the same as for a wire located at $d_1 = \sqrt{(d + R)^2 - R^2}$ and with the charge per unit length $Q = C\phi$, where $C^{-1} = 2\ln[(d + R + d_1)/R]$, and is given by $6,19,20$

$$\Gamma_c = \lim_{\omega \to 0} \frac{2C^2\phi^2w}{16\omega_p^2d^2} \int_0^\infty dq e^{-2qd_1} \frac{\text{Im}R_p(\omega, q)}{\omega}$$

(19)

where $R_p$ is the reflection amplitude for $p$-polarized waves. Using in Eq. (19) the Fresnel’s formula for the reflection amplitude gives the friction coefficient

$$\Gamma_c^{Fresnel} = \frac{w\nu\phi^2}{16\omega_p^2d^2}$$

(20)

For gold with $w = 7 \cdot 10^{-6} m$, $d = 20$ nm and $\phi = 1V$, Eq. (20) gives $\Gamma_c = 1.69 \cdot 10^{-20}$ kg/s that is eight orders of magnitude smaller than the experimental value $3 \cdot 10^{-12}$ kg/s from Ref. 1. In the case when the surface double layer is taken into account, for complete screening, when the interaction of the external electric field with the subsurface charge distribution is neglected, and $1/\varepsilon \ll 4\piq\sigma^2M \ll 1$ the reflection amplitude (21) can be approximated by

$$R_p^{DL} \approx 1 + 8\piq\sigma^2M$$

(21)

and the friction coefficient is determined by

$$\Gamma_c = \lim_{\omega \to 0} 16\pi\sigma^2C^2\phi^2w \int_0^\infty dq e^{-2qd_1} \frac{\text{Im}M(\omega, q)}{\omega}$$

(22)

The contribution to friction from bulk acoustic waves, which is determined by the integration for $q \leq \omega/c_t$, vanishes in the limit $\omega \to 0$. Thus at low frequency the electrostatic friction coefficient is determined by Rayleigh surface waves. Substitution of (18) in (22) for $R \gg d$ gives

$$\Gamma_c^{DL} = 0.14\frac{\sigma^2\phi^2wR^{1/2}}{d^{3/2}\rho c^2c_t}$$

(23)
The friction coefficient has the same distance dependence as it was observed experimentally\(^1\). At \( R = 1 \mu m \) and with the same parameters as above \( \Gamma_{DL} = 1.9 \cdot 10^{-12} \text{kg/s} \) is in good agreement with the experimental value \( 3 \cdot 10^{-12} \text{kg/s} \) from Ref.\(^1\).

For a spherical tip the friction coefficient\(^6,19,20\)

\[
\Gamma_s = \lim_{\omega \to 0} \frac{C^2 \varphi^2}{2} \int_0^\infty dq q^2 e^{-2qd} \frac{\text{Im} R_p(\omega, q)}{\omega}
\]

where

\[
d_1 = \pm \sqrt{3Rd/2 + \sqrt{(3Rd/2)^2 + Rd^3 + d^4}} \quad C = \frac{d_1^2 - d^2}{2d}
\]

Substituting in (25) the Fresnel’s formula for the reflection amplitude and using Eq.(21) gives

\[
\Gamma_s^{\text{Fresnel}} = \frac{\sqrt{3Rd\varphi^2}}{4\omega^2 d^{3/2}}
\]

and

\[
\Gamma_s^{DL} = 0.14 \frac{R\sigma^2 \varphi^2}{d\rho c t_c^2}
\]

Fig. 4 shows the dependence of the friction coefficient associated with the friction between a gold tip and a gold plate on the separation \( d \). The friction coefficient for the Casimir friction in the tip-plate configuration can be obtained from the plate-plate configuration using the proximity (or Derjaguin) approximation\(^3\). Within this approximation the friction coefficient for a spherical tip with a radius \( R \)

\[
\Gamma_s = 2\pi \int_0^R d\rho \gamma(z(\rho))
\]

and for a cylindrical tip with a radius \( R \) and a width \( w \)

\[
\Gamma_c = 2w \int_0^R d\rho \gamma(z(\rho))
\]

where \( \gamma(z) \) is the friction coefficient between two plates separated by a distance \( z(\rho) = d + R - \sqrt{R^2 - \rho^2} \) denoting the tip-surface distance as a function of the distance \( \rho \) from the tip symmetry. In the extreme-near field the friction...
coefficient $\gamma \sim 10^{-12} - 10^{-13}$ kgs$^{-1}$ which can be measured with the present state-of-the-art equipment. The blue and green lines include the contribution from the electrical double layer, while the brown and pink lines without this contribution, for a cylindrical and spherical tip, respectively. The cylindrical tip has the radius $R = 1 \mu m$ and the width $w = 7 \mu m$, the spherical tip has the radius $R = 30$ nm. For $d < 10$ nm the contribution from the electrical double layer produces the friction coefficient $\gamma \sim 10^{-10} - 10^{-12}$ kgs$^{-1}$ that is by many orders of the magnitude larger than the friction coefficient without this contribution. This friction can be measured with the present state-of-the-art equipment.

V. CONCLUSION

We have shown that the interaction of the electromagnetic field with the surface charge density of the electric double layer produces an enhancement of the Casimir and electrostatic non-contact friction between a gold tip and a gold plate by many orders of magnitude in comparison with the theory without taking into account of this contribution. For electrostatic friction, which are dominated by the excitation of Rayleigh surface waves, the theory is in good agreement with experimental data. The theory also shows that the electrical double layer contribution to the Casimir friction can be measured using AFM.

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*alevolokitin@yandex.ru

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