Incremental Updates of Generalized Hypertree Decompositions

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Structural decomposition methods, such as generalized hypertree decompositions, have been successfully used for solving constraint satisfaction problems (CSPs). As decompositions can be reused to solve CSPs with the same constraint scopes, investing resources in computing good decompositions is beneficial, even though the computation itself is hard. Unfortunately, current methods need to compute a completely new decomposition, even if the scopes change only slightly. In this article, we make the first steps toward solving the problem of updating the decomposition of a CSP $P$ so that it becomes a valid decomposition of a new CSP $P'$ produced by some modification of $P$. Even though the problem is hard in theory, we propose and implement a framework for effectively updating generalized hypertree decompositions. The experimental evaluation of our algorithm strongly suggests practical applicability.

CCS Concepts: • Theory of computation → Design and analysis of algorithms; Constraint and logic programming; Database query processing and optimization (theory);

Additional Key Words and Phrases: Constraint satisfaction, hypergraphs, structural decomposition

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1 INTRODUCTION

Constraint satisfaction problems (CSPs) are fundamental in modeling many problems of artificial intelligence and other areas of computer science. While satisfiability checking of CSPs is generally NP-hard [23], the structure of constraints plays a crucial role in their resolution. This can be represented by a hypergraph $H = (V(H), E(H))$, which consists of a set of vertices $V(H)$ and a set of edges $E(H)$ with $E(H) \subseteq 2^V(H)$. Intuitively, the vertices of $H$ correspond to variables and the edges of $H$ group together variables appearing in the same constraint. It is well

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known that checking if a CSP is satisfiable is tractable for all CSPs that have an underlying acyclic hypergraph [28].

Larger islands of tractability have been discovered by generalizing the concept of hypergraph acyclicity. In this direction, hypergraph decompositions and their associated width proved to be essential concepts [12]. In this work, we focus on generalized hypertree decompositions (GHDs) and generalized hypertree width (ghw) [13]. The ghw of a hypergraph $H$ intuitively measures its degree of cyclicity and a GHD of $H$ can be used to solve the related CSP (definitions in Section 2). Acyclic hypergraphs have $ghw = 1$. Furthermore, CSPs with bounded $ghw$ can be solved in polynomial time [13].

Computing a GHD of width $\leq k$ is NP-hard for any $k \geq 2$ [8, 14], but certain properties of hypergraphs, like bounded intersection width and bounded multi-intersection width, make the problem tractable [8, 11]. This set off a quest for efficient algorithms that led to implementations based on different principles. The aforementioned intersection widths were exploited in References [7, 15] for the implementation of several sequential and parallel algorithms for computing GHDs. Alternative characterizations of width have been used for even more general forms of decompositions in References [6, 20, 24]. Furthermore, the fixed-parameter tractability of the problem has been explored extensively in theory and practice but is outside of the scope of this article [4, 17, 19, 22].

Decompositions have also been employed in commercial systems and research prototypes, both for CSPs and query answering in databases [1–3, 18, 21]. In particular, in Reference [1] GHDs of low width significantly speed up query answering. It is thus worth investing resources in computing a GHD of low width. However, this is a hard task, and we want to avoid the computation of a new GHD whenever possible. Unfortunately, a new decomposition must be computed even if the CSP slightly changes, resulting in a loss on the investment previously made. This is the case in the setting of incremental constraint satisfaction, where constraint solvers handle mutable sets of variables [26] or constraints [9]. The same issue appears in different scenarios as well.

Consider a user modeling a problem in the context of interactive problem solving. In this case, the user interactively models a problem and needs prompt feedback on the effect of her modifications on the resolution process. While investigating alternatives, information about the impact on the decomposition is shown, i.e., an estimation of the computational effort of solving the problem. Similarly, a compositional modeling problem consists in synthesizing the most appropriate model of a physical system for a given analytical query [5]. The construction of the “best” model passes through several phases in which the model is iteratively refined by modifying constraints. Here, support during the modeling process is needed.

In the rest of the article, we will use the crossword puzzles in Figure 1 for our examples. Given a puzzle, we want to fill every contiguous horizontal or vertical line of white cells with words from a certain set. The puzzles are CSPs in which each cell is a variable and there is a constraint over the white cells belonging to the same line. Suppose we want to solve the puzzle $P$, and then the slightly modified puzzle $P'$ with the help of GHDs. Although the two puzzles resemble each other, their resolution requires different decompositions. In particular, even though the hypergraphs of

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Fig. 1. Two similar crossword puzzles $P$ and $P'$. Given a set of words $W$, we want to fill every contiguous horizontal or vertical line of white cells with words from $W$. If two lines intersect, then the words assigned to these lines must intersect in the right positions.
P and P' share a significant part, we have to compute a new GHD to solve P'. Intuitively, it should be possible to obtain a GHD for P' by slightly modifying the already-computed GHD of P. Thus, the question arises naturally: Can we reuse the first GHD and adjust only the parts affected by the modification?

We investigate the problem of incrementally updating GHDs upon constraint modifications. To this aim, we study the behavior of GHDs when the CSP is modified and analyze the problem of computing a GHD of the modified CSP when a GHD of the original CSP is already available. We also propose a set of typical and natural constraint modifications. In this work, we focus on elementary modifications of CSPs, i.e., changes such as binding a variable to a constant, the introduction of a new constraint, and enforcing equality between a set of variables. For each modification, we also include its dual function. Moreover, we briefly discuss how to extend our framework when we are confronted with sequences of updates. In particular, our contributions are as follows:

- We propose a framework for constraint modifications and describe their effect on the hypergraph as well. Moreover, we present the novel SEARCHUPDATEGHD problem.
- We resolve the complexity of SEARCHUPDATEGHD for a set of relevant elementary modifications. For most of these, the problem turns out to not be solvable in polynomial time under standard assumptions and is therefore effectively just as difficult as computing a GHD de novo.
- Despite its complexity, the problem still offers room for practical solutions. To this end, we provide a general framework of mutable subtrees for updating GHDs. This is devised to work under any kind of modification, thus making our approach universally applicable.
- We emphasize the practical applicability of our method by providing an implementation strategy that can be used for any top-down algorithm for computing GHDs. In this way, we can take advantage of the state-of-the-art solutions in the field.
- We extensively compare our method with classical algorithms. Given a modification, we use our algorithm to update an existing decomposition and a reference classical algorithm to compute a GHD for the new hypergraph afresh. Results show that we significantly outperform classical methods for most classes of modifications. In particular, we achieve mean speed-ups between 6 and 50 over the reference algorithm.

The article proceeds as follows. In Section 2, we introduce basic concepts. In Section 3, we formally define the SEARCHUPDATEGHD problem and study its complexity. In Section 4, we describe the framework of mutable subtrees and an actual implementation strategy for any GHD top-down algorithm. In Section 6, we present the results of the experimental evaluation of our methods by comparing it to classical algorithms for computing GHDs. Finally, in Section 7 we draw conclusions and point to relevant questions for future work.

2 PRELIMINARIES

2.1 CSPs and Hypergraphs

A CSP is a triple (V, D, Ct), where V is a set of variables, D is a set of values, and Ct is a set of constraints. A constraint (si, ri) ∈ Ct consists of a tuple of variables si and a constraint relation ri containing valid combinations of values for the variables si. A solution is a mapping from V to D, s.t. for each (si, ri) ∈ Ct the variables si are mapped to a legal combination of values in ri.

A hypergraph H = (V(H), E(H)) is a pair consisting of a set of vertices V(H) and a set of non-empty (hyper)edges E(H) ⊆ 2V(H). We assume w.l.o.g. that there are no isolated vertices, i.e., for each v ∈ V(H), there is at least one edge e ∈ E(H) such that v ∈ e. We will often use H to denote the set of edges E(H). A subhypergraph H' of H is then simply a subset of (the edges of) H. Given
U \subseteq V(H)$ the induced subhypergraph of $H$ w.r.t. $U$ is the hypergraph $H[U]$ s.t. $V(H[U]) = U$ and $E(H[U]) = \{ e \cap U \mid e \in E(H) \}$.

Let $P = (V, D, C_i)$ be a CSP, the hypergraph $H_P$ of $P$ is defined with $V(H_P) = V$ and $E(H_P) = \{ s_i \mid (s_i, r_i) \in C_i \}$.

**Example 2.1.** A crossword puzzle like the ones in Figure 1 can be represented as a CSP. Each cell of the puzzle is a variable in $V$ and, for simplicity, the domain $D$ is the set of letters of the alphabet. Given a relation of words $W$ with characters in $D$, the set $C_i$ of constraints contains a constraint $c_i$ for each contiguous horizontal or vertical line of white cells that can be filled with appropriate words in $W$. For instance, consider the puzzle $P'$ of Figure 1(b). The constraint $c_1$ defined over the variables $w_1 = \langle a, b, c \rangle$ can take values in $r_1 = W$. If $W = \{(d, o, g), (g, o, d), (o, d, d)\}$, then the assignment $\langle a, b, c, d, e, f, g, h, i, j, k, l \rangle = (g, o, d, o, o, d, o, g, d, o, d, d)$ is a solution.

The hypergraphs $H_P, H_{P'}$ underlying the puzzles $P, P'$ are shown in Figure 2. The set of vertices of each hypergraph is the set of variables of the corresponding CSP, while the sets of edges match the related set of constraint scopes.

### 2.2 Generalized Hypertree Decompositions

We use $B(E)$ to denote the set of vertices of $H$ covered by a certain set of edges $E$ of $H$. More precisely, given a hypergraph $H = (V(H), E(H))$ and a set of edges $E \subseteq E(H)$, we define $B(E) = \bigcup_{e \in E} e$ as the set of all vertices of $H$ contained in the set of edges $E$.

A GHD [13] of a hypergraph $H = (V(H), E(H))$ is a tuple $(T, (B_u)_{u \in T}, (\lambda_u)_{u \in T})$, where $T = (N(T), E(T))$ is a tree, every $B_u$ is a subset of $V(H)$, every $\lambda_u$ is a subset of $E(H)$, and the following hold:

1. For every edge $e \in E(H)$, there is a node $u \in T$, such that $e \subseteq B_u$.
2. For every vertex $v \in V(H)$, $\{u \in T \mid v \in B_u\}$ is a connected subtree in $T$.
3. For each $u \in T$, $B_u \subseteq B(\lambda_u)$ holds.

We refer to the vertex sets $B_u$ as the bags of the GHD, while we call the edge sets $\lambda_u$ edge covers. By slight abuse of notation, we write $u \in T$ to express that $u$ is a node in $N(T)$. Condition (2) is also called the connectedness condition.

We use the following notational conventions. To avoid confusion, we will consequently refer to the elements in $V(H)$ as vertices of the hypergraph and to the elements in $N(T)$ as the nodes of the decomposition. For a node $u \in T$, we write $T_u$ to denote the subtree of $T$ rooted at $u$. By slight abuse of notation, we will often write $u' \in T_u$ to denote that $u'$ is a node in the subtree $T_u$ of $T$. Finally, we define $V(T_u) = \bigcup_{u' \in T_u} B_{u'}$.

The width of a GHD is defined as the largest size of any set $\lambda_u$ over all nodes $u \in T$. The generalized hypertree width of $H (ghw(H))$ is the minimum width over all GHDs of $H$.

**Example 2.2.** A GHD for the hypergraph $H_{P'}$ of Figure 2(b) is shown in Figure 3. It is easy to check that conditions (1)–(3) are satisfied. This GHD has width 2, because there is at least one
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Fig. 3. A GHD of width 2 for the hypergraph $H_P'$ of Figure 2(b).

node of the decomposition with $|\lambda_u| = 2$. It is also possible to prove that $ghw(H_P') = 2$. Indeed, the hypergraph $H_P'$ has a cycle and, thus, it cannot have width 1. On the contrary, the hypergraph $H_P$ is acyclic and $ghw(H_P) = 1$ holds.

2.3 Top-Down Construction of GHDs

For a set $U \subseteq V(H)$ of vertices, we define $[U]$-components of a hypergraph $H$ through the following steps:

- We define $[U]$-adjacency as a binary relation on $E(H)$ as follows: Two edges $e_1$ and $e_2$ are $[U]$-adjacent if $(e_1 \cap e_2) \setminus U \neq \emptyset$ holds.
- We define $[U]$-connectedness as the transitive closure of the $[U]$-adjacency relation.
- A $[U]$-component of $H$ is a maximally $[U]$-connected subset $C \subseteq E(H)$.

For a set of edges $S \subseteq E(H)$, we say that $C$ is $[S]$-connected as a short-cut for $C$ is $[U]$-connected with $U = \bigcup_{e \in S} e$. We also call $S$ a separator and $C$ an $[S]$-component. The size of an $[S]$-component $C$ is defined as the number of edges in $C$.

Many top-down GHD algorithms such as in References [7, 15, 16] can be described by the same schema. Here we simply refer to a generic top-down schema with the understanding that each implementation presents its own peculiarities. A pseudo-code description of such a schematic top-down GHD algorithm is given in Algorithm 1.

For a fixed $k \geq 1$, Algorithm 1 takes as input a hypergraph $H$ and builds a GHD of $H$ of width $\leq k$ or rejects if none can be found. Each call of the function $\text{FindDecomp}$ is over a particular set of edges, referred to as $H'$. This recursive function forms the core of this schematic description. A recursive function is chosen for notational simplicity, a real implementation could also just use nested loops or other ways to implement iteration. The function $\text{FindDecomp}$ iterates over a set of valid separators, as seen in line 4. Note that the particulars of how this set is computed can vary greatly between specific implementations. The algorithm picks a vertex set $\text{sep}$ from this set and then finds a subset of edges $\text{cover}$ such that $|\text{cover}| \leq k$ and $\text{sep} \subseteq B(\text{cover})$. These two will form the bag (respectively, edge cover) of the root node $u$ of the GHD found for subgraph $H'$.

Next, in line 6, it proceeds to determine all $[\text{sep}]$-components of $H'$, which we denote as $C_1, \ldots, C_{\ell}$. It was shown in Reference [13] that if $H$ has a GHD of width $\leq k$, then it also has a GHD of width $\leq k$ such that the edges in each $C_i$ are “covered” in separate and distinct subtrees below $u$. We say that such a GHD is in normal form. More precisely, “covered” means that $u$ has $\ell$ child nodes $u_1, \ldots, u_{\ell}$, s.t. for every $i$ and every $e \in C_i$, there exists a node $u_e \in T_{u_i}$ with $e \subseteq B_{u_e}$. Hence, the algorithm recursively searches for a GHD of the subhypergraphs $H_i$ with $E(H_i) = C_i$ and $V(H_i) = \bigcup C_i$. We can see this inside function $\text{FindDecomp}$ in lines 8 to 11. If all recursive calls succeed, then the function terminates by forming a GHD with root $u$ and subtrees covering the components $C_i$, seen in line 12.
Algorithm 1: A schematic top-down GHD algorithm

Type: Node=(Bag: Vertex Set, Edge Cover: Set of Edges, Children: Set of Nodes)
Input: $H$: Hypergraph
Parameter: $k$: width parameter
Output: a GHD of $H$ with width $\leq k$, or Reject if none exists

1 begin
2 return FindDecomp($H$)  \(\triangleright\) initial call
3 function FindDecomp($H'$: Subgraph)
4 for sep $\in$ Separators($H'$, $k$) do \(\triangleright\) iterate over set of possible bags relative to $H'$ with cover number $\leq k$
5 cover := find some cover $\subseteq E(H)$ such that sep $\subseteq B$ (cover) and $|\text{cover}| \leq k$
6 comps := [sep]-components of $H'$
7 children := $\emptyset$  \(\triangleright\) initialize the set of children
8 for $c \in$ comps do
9 if FindDecomp($c$) returns Reject then
10 continue outer loop
11 children = children $\cup$ FindDecomp($c$)  \(\triangleright\) recursive call to compute subtrees
12 return Node(Bag : sep, Edge Cover : cover, Children : children)
13 return Reject  \(\triangleright\) reject if no valid separator could be found

Example 2.3. We decompose the hypergraph $H_P$ of Figure 2(b) so to obtain the GHD shown in Figure 3. For $k = 2$, the generic top-down algorithm $\mathcal{A}$ takes in input $H_P$ and computes a GHD of $H_P$ of width 2, if it exists. First, $\mathcal{A}$ guesses a separator of size $\leq 2$ with edges in $C = E(H_P)$, which will be used as the edge cover $\lambda_u$ for the root node $u$ of the GHD. Suppose that $\lambda_u = \{w_5, w_3\}$ and $B_u = \{c, e, h, i, k\}$ is a suitable choice for the bag. Then the root $u$ of the decomposition will be exactly the same of Figure 3. At this point, $\mathcal{A}$ computes the $[\lambda_u]$-components $C_1, C_2 \subseteq C$ with $C_1 = \{w_1, w_2, w_4\}$, $C_2 = \{w_6\}$ (Figure 4). Each $C_i$ is now recursively decomposed. Since $|C_2| \leq 2$, it can be “covered” by a single node of the decomposition. On the contrary, $|C_1| > 2$, and thus $\mathcal{A}$ starts a recursive call on $C_1$, guesses the separator $S = \{w_1, w_2\}$, and computes the new bag $\{a, b, f, g, h\}$. As the component $C_1$ is split w.r.t. $S$, we obtain a single component $C_3 \subseteq C_1$, with $C_3 = \{w_4\}$. This component can be “covered” by a single node. Finally, all nodes are attached to their respective fathers except for the root. The resulting decomposition is returned.

3 THE GHD UPDATE PROBLEM

We introduce the problem of updating GHDs upon constraint modifications. Here we propose important classes of elementary modifications and describe their effect on both the hypergraph and the CSP. Then, we formally define the update problem and settle its complexity.

3.1 Elementary Constraint Modifications

The modification of a CSP affects its underlying hypergraph and consequently its GHDs. We define a framework that allows us to update the GHD of a CSP in the face of constraint modifications. First, we define modifications.

Definition 3.1. Given a hypergraph $H$, a modification is a function mapping hypergraphs to hypergraphs.

We identify three fundamental hypergraph objects for the computation of a GHD: vertices, edges, and intersections between edges. As modifying a CSP typically implies the modification.
of these objects, we define six classes of modifications that reflect elementary changes on a hypergraph. In this sense, the proposed classes of modifications are natural, even though not necessarily minimal. We show the effect of these modifications on the hypergraph and intuitively explain their correspondence with the related CSP. However, we ignore all those CSP modifications that do not change the hypergraph. For instance, changing constraint relations does not affect the CSP structure, and therefore the same GHD can be reused to solve the modified CSP again. We start with two classes for vertex modifications.

**Definition 3.2.** \( \text{AddVar} \) is the class of modifications \( \delta \) such that for every hypergraph \( H \):
- \( V(\delta(H)) := V(H) \cup \{w\} \)
- \( E(\delta(H)) := E \cup \{e' \cup \{w\} \mid e' \in E'\} \), where \( (E, E') \) is a partition of \( E(H) \) and \( E' \neq \emptyset \).

\( \text{DelVar} \) is the class of modifications \( \delta \) such that for every hypergraph \( H \), \( V(\delta(H)) := V(H) \setminus \{v\} \) and \( E(\delta(H)) := \{e \setminus \{v\} \mid e \in E(H)\} \).

On the CSP level, \( \text{DelVar} \) functions bind a CSP variable to a constant value and could possibly simplify its hypergraph. However, \( \text{AddVar} \) functions remove such binding, i.e., replace a constant with a variable.

Two classes are needed for edge insertion and removal.

**Definition 3.3.** \( \text{AddConstr} \) is the class of modifications \( \delta \) s.t. for every hypergraph \( H \), \( \delta(H) := H \cup f \), where \( f \notin E(H) \) is a new edge. \( \text{DelConstr} \) is the class of modifications \( \delta \) s.t. for every hypergraph \( H \), \( \delta(H) := H \setminus e \), where \( e \in E(H) \).

\( \text{AddConstr} \) and \( \text{DelConstr} \) modifications correspond to alterations of the set of constraints \( C_t \) of a CSP. In particular, \( \delta \in \text{AddConstr} \) introduces a new constraint in \( C_t \), while \( \delta \in \text{DelConstr} \) removes a constraint from \( C_t \).

Finally, we present classes to modify intersections between edges. Let \( H \) be a hypergraph. Given \( U \subseteq V(H) \), we denote with \( E_U = \{e \in E(H) \mid e \cap U \neq \emptyset\} \) the edges incident on \( U \).

**Definition 3.4.** \( \text{AddEq} \) is the class of modifications \( \delta \) such that, for every hypergraph \( H \), some vertices \( U \subseteq V(H) \) are merged into \( w \in V(\delta(H)) \) and the edges in \( E_U \) are incident on \( w \). \( \text{DelEq} \) is the class of modifications \( \delta \) such that, for every hypergraph \( H \), a vertex \( w \in V(H) \) is split into a set \( U \subseteq V(\delta(H)) \) and the edges in \( E_{\{w\}} \) are arbitrarily distributed on \( U \).

Intuitively, an \( \text{AddEq} \) modification introduces an equality constraint between some variables of the CSP. In other words, a new AllEqual constraint is defined over a set of variables of the CSP. However, \( \text{DelEq} \) modifications remove this kind of constraint and thus all equalities between a specific set of variables.
Example 3.5. The hypergraph $H_P'$ of Figure 2(b) is obtained by applying a modification $\delta \in AddVar$ to $H_P$ in Figure 2(a). In particular, $\delta$ adds a new vertex $h$ in the edges $\{f, g\}$ and $\{c, e\}$ as well as in $V(H_P)$. Note that $H_P$ can be obtained from $H_P'$ via a modification $\delta \in DelVar$ removing $h$ from $V(H_P')$.

Figure 5 shows two additional modifications of $H_P'$. The hypergraph $H_{P_2}$ of Figure 5(a) is the result of a modification $\delta \in AddConstr$ introducing a new edge $\{c, i\}$, while $H_{P_3}$ of Figure 5(b) shows the effect on $H_P'$ of a $\delta \in AddEq$ adding an AllEqual constraint between the variables $b, g$ (b is merged into g). Finally, $H_P'$ can be obtained through an appropriate inverse modification $\delta$ to $H_{P_2}$ and $H_{P_3}$ with $\delta \in DelConstr$ and $\delta \in DelEq$, respectively.

Note that the set of all considered elementary modifications is complete, i.e., given two hypergraphs $H, H'$, there exists a sequence of modifications $\delta_1, \ldots, \delta_\ell$ s.t. $H' = \delta_\ell(\cdots(\delta_1(H)))$.

3.2 The Complexity of Updating GHDs

Recall that checking $\text{ghw}(H) \leq k$, and therefore computing a width $k$ GHD of $H$, is NP-hard even when $k > 1$ is constant [8, 11, 14]. In the context of modifications, this naturally presents the question of the complexity of the following task: Given a hypergraph $H$ together with a minimal-width GHD, as well as a modification $\delta$, find a GHD for $\delta(H)$ with the same width if one exists or correctly identify that the width increased. Intuitively, the knowledge of a witness for $H$ could make the problem easier, in particular if $\delta$ is a simple modification. Formally, we extend the standard problem of checking $\text{ghw}(H) \leq k$ for constant $k$ (see, e.g., Reference [8]) by simply adding a modification (from some class of modifications $\Delta$) and a GHD of the original hypergraph to the input.

**SearchUpdateGHD($\Delta$)**

*Instance:* hypergraph $H$, modification $\delta \in \Delta$, a minimal-width GHD of $H$

*Output:* A GHD of $\delta(H)$ with width $\leq \text{ghw}(H)$ if it exists or answer ‘no’ otherwise.

In the future, it might be of interest to further generalise the SearchUpdateGHD to decide whether a modification increases the width of $H$ only up to some constant threshold $c$, i.e., whether $\text{ghw}(\delta(H)) \leq \text{ghw}(H) + c$. However, note that of the elementary modifications introduced above, only DelConstr can actually increase the $\text{ghw}$ of the hypergraph by more than 1. Thus, for all introduced modifications except DelConstr this generalised problem is trivially true for any $c > 0$. For DelConstr, the increase in width can depend fully on the structure that was “hidden” by the deleted constraint. We therefore focus on the analysis of SearchUpdateGHD in this article and leave the generalised version open for future work on settings with more complex modifications.

Importantly, SearchUpdateGHD is a search problem rather than a decision problem. This is motivated from two sides. Our primary motivation stems from practical situation in which small,
iterative updates are consistently made to some CSP and for which we want to maintain a low-width GHD. Since the GHD is necessary to possibly exploit low width for solving the CSP, we are interested in the search problem rather than the decision problem.

The second motivating factor comes from the possibility of certain classes of modifications capturing other classes, i.e., if one can express some modification in a class Δ via a sequence of modifications from another class Δ'. Focusing purely on the decision problem makes it problematic to consider the complexity of sequences of updates, since we have no information on the complexity of obtaining the new input GHDs along the sequence of updates. By studying the search problem instead we can make strong statements for such cases.

The complexity of search problems is a complex topic and the full theoretical framework is not necessary in our context here. Instead, we will be content with showing that even for the simple classes of atomic updates that were discussed previously (except DelVar), SearchUpdateGHD cannot be solved in polynomial time. Note that SearchUpdateGHD trivially reduces to the problem of finding an optimal GHD of δ(H) and all negative results therefore extend also to finding optimal GHDs under modifications.

**Theorem 3.6.** For Δ ∈ {AddEq, DelEq, AddVar, AddConstr, DelConstr}, SearchUpdateGHD (Δ) cannot be solved in polynomial time (assuming P ≠ NP).

**Proof Idea.** The basic strategy for each modification class Δ is simple. We show how to decide an NP-hard decision problem by finding an initial hypergraph H₀, which can be modified by some sequence of δ₁, δ₂, . . . , δℓ ∈ Δ to some target H. The decision problem will be equivalent to the question whether $\text{ghw}(H) \leq \text{ghw}(H₀)$. However, this strategy presents us with two technical challenges. First, the initial H₀ needs to be chosen in such a way that a minimal-width GHD can be constructed in polynomial time. Second, there can be no index i such that after applying the first i modifications to H₀, we get a Hᵢ with $\text{ghw}(Hᵢ) < \text{ghw}(H₀)$ even when $\text{ghw}(Hᵢ) \leq \text{ghw}(H₀)$. That is, the sequence cannot increase the width at intermediate hypergraphs before decreasing again.

To tackle these issues, we do not reduce from $\text{ghw}$ checking where the second challenge is particularly problematic as our operations are not monotonic (wrt. $\text{ghw}$) in general. Instead, we reduce from 3-Sat by building on the proof of NP-hardness of $\text{ghw}$ checking (for constant width) given by Gottlob et al. [11]. There a hypergraph is constructed that has $\text{ghw}$ 3 exactly if some 3-Sat instance is satisfiable and has $\text{ghw}$ 2 otherwise. Using this specific hypergraph, we give concrete H₀ and modification sequences as described above for each Δ. A full proof for each Δ is given in Appendix A.

Updating GHDs is computationally difficult for all of the natural atomic operations that we considered, except for DelVar¹ (where the problem is trivial as DelVar cannot increase width and a new GHD is trivial to construct). As part of the proof of Theorem 3.6 we discuss how to decide 3-Sat via sequences of modifications, as long as those sequences adhere to certain conditions. Using this observation, we can strengthen the statement from Theorem 3.6 to all modification classes that capture any of the hard atomic cases in the following formal sense.

For a sequence of modifications δ₁, δ₂, . . . , δℓ let us write $δ_n'(H)$ as a shorthand for $δ_n(δ_{n-1}(\cdots (δ₁(H)) \cdots ))$. Let Δ, Δ' be two sets of modifications. We say that Δ polynomially captures Δ' if for every hypergraph H and $δ' ∈ Δ'$ there exists a sequence δ₁, δ₂, . . . , δℓ of modifications in Δ such that $\text{ghw}(δ₁(H)) \leq \text{ghw}(δ₁^{i+1}(H))$ for $1 ≤ i < ℓ$, $δ ℓ(H) = δ'(H)$ and ℓ is polynomially bounded in the size of H. In plain terms, every modification in Δ' can equivalently be reached via a polynomial sequence of modifications from Δ.

¹A DelVar modification results in an induced subhypergraph, which is well known to never increase in width (see, e.g., Reference [11]).
COROLLARY 3.7. Let \( \Delta \) be a class of modifications that polynomially captures at least one of AddEq, DelEq, AddVar, AddConstr, or DelConstr. Then SearchUpdateGHD(\( \Delta \)) cannot be solved in polynomial time (assuming P ≠ NP).

4 TOWARD A FRAMEWORK FOR UPDATES

We have seen that SearchUpdateGHD is difficult in general. In the following, we thus focus on making the first steps toward practical solutions for the problem. In this section, we present the theoretical framework of mutable subtrees for the uniform treatment of GHD updates under arbitrary modifications. Moreover, we briefly discuss how our approach extends to sequences of elementary modifications.

4.1 The \( \delta \)-mutable Subtrees of a Decomposition

We lay the theoretical foundations of \( \delta \)-mutable subtrees, a notion that will let us treat updates uniformly. We first introduce some convenient notation. Let \( \langle T, (B_u)_{u \in T}, (\lambda_u)_{u \in T} \rangle \) be a GHD of a CSP \( P \), and let \( T' \) be a subtree of \( T \). We write \( T \setminus T' \) for the forest created by removing the nodes of \( T' \) from \( T \). Since we are interested in the hypergraph structure, we write \( H[T'] \) for the subhypergraph of \( H \) induced by the vertices \( \bigcup_{u \in T'} B_u \).

We are now ready to introduce the central notion of our framework, \( \delta \)-mutable subtrees. Intuitively, these subtrees (of a decomposition) represent a kind of local neighborhood of the modification \( \delta \), i.e., the segment of the decomposition that corresponds to those parts of the hypergraph that are changed by \( \delta \). Note that the definitions and results in this section apply not only to the previously discussed elementary modifications but also to arbitrary modifications in the sense of Definition 3.1.

**Definition 4.1 (\( \delta \)-mutable Subtree).** Let \( G \) be a GHD of hypergraph \( H \) with tree \( T \), and let \( \delta \) be a modification. A subtree \( T^* \) of \( T \) is a \( \delta \)-mutable subtree if the following conditions hold:

- \( H[T \setminus T^*] = \delta(H)[T \setminus T^*] \),
- and no \( v \in V(\delta(H)) \setminus V(H) \) is adjacent (in \( \delta(H) \)) to a vertex in \( B(T \setminus T^*) \).

Thus, we split our existing decomposition in two parts: the mutable subtree \( T^* \), where the corresponding part of the hypergraph has changed, and the outer subtrees that correspond to those subhypergraphs that remain unchanged by the modification. An important reason for considering mutable subtrees is captured by the following Lemma 4.2, namely that all the trees outside of \( T^* \) are still correct GHDs for their respective parts of the new hypergraph. Hence, it is possible to reuse these partial decompositions for \( \delta(H) \) and save the effort of decomposing those parts of the hypergraph again.

**Lemma 4.2.** Let \( \langle T, (B_u)_{u \in T}, (\lambda_u)_{u \in T} \rangle \) be a GHD of hypergraph \( H \) with tree \( T \), let \( \delta \) be a modification, and let \( T^* \) be a \( \delta \)-mutable subtree. For every tree \( T' \) in the forest \( T \setminus T^* \) it holds that \( \langle T', (B_u)_{u \in T'}, (\lambda_u)_{u \in T'} \rangle \) is a GHD of \( \delta(H)[T'] \).

**Proof.** Since we assume that \( T' \) is a tree in the forest \( T \setminus T^* \), we also have that \( B(T') \subseteq B(T \setminus T^*) \). Hence, it must also hold that \( H[T'] = \delta(H)[T'] \) by assumption that \( T^* \) is \( \delta \)-mutable.

We now argue that \( \langle T', (B_u)_{u \in T'}, (\lambda_u)_{u \in T'} \rangle \) is a GHD of \( H[T'] \) and thus, by the previous argument, also of \( \delta(H)[T'] \). First, observe that the connectedness condition is clearly still satisfied in \( T' \), since we never change the bags. For the covers it is clear that if \( e \in E(H) \), then \( e \cap B(T') \) is an edge in \( H[T'] \). Since \( B_u \subseteq B(T') \) we clearly also have that \( B_u \subseteq \bigcup_{e \in \lambda_u} e \cap B(T') \). What is left is to verify

\[ \text{Technically every edge } e \text{ in every edge cover } \lambda_u \text{ is replaced by the edge } e \cap \bigcup_{u \in T'} B_u \text{ of the induced subhypergraph.} \]

ACM Journal of Experimental Algorithmics, Vol. 27, No. 1, Article 1.16. Publication date: February 2023.
that every edge $e$ of $H[T']$ is covered in $T'$. Let $e'$ be one of the edges in $H$ such that $e = e' \cap B(T')$. Since we start from a GHD of $H$, there must be a node $u$ where $e'$ is covered. Hence, all the vertices of $e$ are in $B_u$. Hence, all of the subtrees induced by the vertices in $e$ touch by the connectedness condition. Since all of the vertices of $e$ are in $B(T')$ all those subtrees must have a common node in $T'$. Hence, $\langle T', (B_u)_{u \in T'}, (\lambda_u)_{u \in T'} \rangle$ is a GHD of $H[T']$ and therefore also of $\delta(H)[T']$. □

Example 4.3. In Example 3.5, a $\delta \in \text{AddConstr}$ is used to create the hypergraph $H_{P_1}$ from $H_P$, as in Figure 5(a). We now consider reverting this modification, i.e., the modification $\delta^{-1} \in \text{DelConstr}$ that removes the edge $\{c, i\}$, i.e., we have $\delta^{-1}(H_{P_1}) = H_P$ (recall, the hypergraph of $P'$ is shown in Figure 2(b)). As input for our update example, we use the width 2 GHD $\langle T, (B_u)_{u \in T'}, (\lambda_u)_{u \in T'} \rangle$ of $H_{P_1}$ given in Figure 6(a). The two highlighted nodes in Figure 6(a) represent a $\delta^{-1}$-mutable subtree $T'$ of $T$. Observe that $T \setminus T^*$ consists of two trees that correspond to the induced subhypergraphs in Figure 6(b). By Lemma 4.2, these parts remain correct GHDs for their respective induced subhypergraphs.

We could update the overall decomposition by changing the bag $\{c, e, i, k\}$ to $\{e, i, k\}$ while removing $w_2$ from the $\lambda$ label to update the decomposition to fit $P'$. Mechanically, this can be checked by searching for a GHD of $\delta^{-1}(H_{P_1})[\{(a, b, c, e, i, h, k)\}]$ that is consistent with the surrounding trees in a certain way that will be discussed below.

Since we want to reuse as much of the old decomposition as possible, it naturally becomes interesting to have $T \setminus T^*$ as large as possible. Hence, we are interested in finding minimal $\delta$-mutable subtrees, i.e., those $\delta$-mutable subtrees with the least number of nodes. Fortunately, it is relatively easy to find minimal mutable subtrees. The full tree $T$ is trivially a $\delta$-mutable subtree. We can then start from $T_0 = T$ and greedily eliminate leaves as long as the property from Definition 4.1 remains valid. Once no more leaves can be removed, the procedure will have reached a minimal $\delta$-mutable subtree.\(^3\)

**Lemma 4.4.** For any GHD $G$ of a hypergraph $H$ and any modification $\delta$ there exists a unique minimal $\delta$-mutable subtree. Moreover, there exists an algorithm with input ($G, H, \delta(H)$) that computes the minimal $\delta$-mutable subtree in polynomial time.

**Proof.** We first prove the uniqueness of minimal $\delta$-mutable subtrees. Suppose toward a contradiction that there are two distinct minimal $\delta$-mutable subtrees $T_1$ and $T_2$ of a GHD $\langle T, (B_u)_{u \in T'}, (\lambda_u)_{u \in T'} \rangle$. Recall, in the argument for Lemma 4.2 it was already argued that for every tree $T'$ in $T \setminus T_1$ or $T \setminus T_2$, we have that $H[T'] = \delta(H)[T']$.

\(^3\)A more detailed argument is available in the technical appendix.
ALGORITHM 2: Finding Minimal δ-mutable Subtrees.

Input: H, δ(H), GHD (T, (B_u)_{u∈T}, (λ_u)_{u∈T}) of H
Output: A minimal δ-mutable subtree.

begin
T := T
V_{new} = V(δ(H)) \setminus V(H)
repeat
foreach Leaf u of T do
T’_u := T \ {u}
A := H[T \ T’_u]
B := δ(H)[T \ T’_u]
Adj := all vertices adjacent to B(T \ T’_u) in δ(H)
if A = B and V_{new} \cap Adj = ∅ then
T := T’_u
break
until T’ did not change
Return T
end

Now, from the assumption that both T_1 and T_2 are minimal but distinct there has to exist a tree T’ ∈ T \ T_1 such that T’ ∩ T_2 ≠ ∅. If this were not true, then T_2 would be a subtree of T_1, and we are done. Fix such a T’ and let X be the set of nodes that are in T’ and T_2. Since H[T’] = δ(H)[T’], and X ⊆ T’, also H[X] = δ(H)[X] and it becomes easy to see that

H[T \ (T_2 \ X)] = H[B(X) ∪ B(T \ T_2)] = δ(H)[B(X) ∪ B(T \ T_2)] = δ(H)[T \ (T_2 \ X)].

Thus, T_2 \ X is also a δ-mutable subtree and smaller than T_2, contradicting our initial assumption of minimality. Note that the second condition of Definition 4.1 cannot become unsatisfied by removing nodes from T_2.

The algorithm from the statement is given in Algorithm 2. The algorithm clearly starts with a δ-mutable subtree, and throughout the iterative elimination the working tree T’ remains a δ-mutable subtree. Note that the argument from before can be seen as a method to create a smaller δ-mutable subtree from any disjoint pair of δ-mutable subtrees. Hence, if no more leaves can be removed from T’ in the algorithm, then every smaller tree must not be disjoint. But if the minimal subtree were to be a proper subtree of T’, then removal of some leaf must be possible, since the induced subhypergraphs of T \ T’ grow monotonically. Thus, there can be no smaller δ-mutable subtree that is a subtree of T’ and none that has disjoint vertices from T’. It follows that the returned T’ is minimal and the algorithm is therefore correct.

By Lemma 4.2, we can use the old decomposition to derive correct GHDs for certain induced subgraphs of δ(H). It is not guaranteed that the minimal-width GHD of δ(H) can be constructed in such a way that these pre-solved induced subgraphs correspond to parts of the decomposition. However, the possibility of only having to recompute a decomposition for some small subgraph δ(H)^* is promising in practice. In particular, we are interested in δ(H)^*, which is the part of δ(H) that contains δ(H)[T^*] for the minimal δ-mutable subtree T^*, plus any possible new vertices and edges introduced by δ. Ideally, we want a new GHD for δ(H)^* with which we can replace T^* ⊆ T to arrive at a valid generalized hypertree for δ(H). This way, we can fully reuse the T \ T^* parts of the old GHD. To replace the new decomposition of δ(H)^* in place of T^* in T, we need to enforce some additional constraints on the GHD of H^*.
as a set $C \subseteq V(\delta(H^*))$. A bag constraint $C$ is satisfied by a GHD $\langle T, (B_u)_{u \in T}, (\lambda_u)_{u \in T} \rangle$ if there exists a node $u \in T$, s.t., $C \subseteq B_u$. In particular, given such a GHD and a mutable subtree $T^*$, let $\{u_1, \ldots, u_q\}$ be the set of nodes in $T \setminus T^*$ that have a neighbor in $T^*$. We call the set $\{C_i \mid C_i = B_{u_i} \cap (\bigcup_{u \in T^*} B_u), 1 \leq i \leq q\}$ the $T^*$-induced bag constraints.

**Theorem 4.5.** Let $G$ be a width $k$ GHD of a hypergraph $H$ with tree $T$, let $\delta$ be a modification and let $T^*$ be a $\delta$-mutable subtree of $T$. If $\delta(H^*)$ has a GHD of width $\leq k$ that satisfies all $T^*$-induced bag constraints, then $\text{gwh}(\delta(H)) \leq k$.

**Proof.** We prove the statement by constructing the required new width $k$ GHD of $\delta(H)$ from the GHD of $\delta(H^*)$ and the subtrees $T \setminus T^*$. Hence, not only is the width of $\delta(H)$ at most $k$, but a GHD of $\delta(H)$ can be efficiently constructed by only computing a GHD (with bag constraints) for $\delta(H^*)$.

Suppose $\mathcal{D}^*$ is a width $k$ GHD of $H^*$ that satisfies all $T^*$-induced bag constraints. Let $C_1, \ldots, C_\ell$ be $T^*$-induced bag constraints and recall that every bag constraint is associated one-to-one to a node in $T$ that neighbors a node in $T^*$. Let $u_i$ be the node associated to the constraint $C_i$ in this way for all $i \in [\ell]$.

The final decomposition $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$ is now constructed as follows starting from $\mathcal{D}^*$. For each bag constraint $C_i$, identify the subtree $T_i \in T \setminus T^*$ that contains $u_i$ as well as any node $u_i'$ in $\mathcal{D}^*$ that satisfies $C_i$. Then, attach the tree $T_i$ at node $u_i$ to $\mathcal{D}^*$ at $u_i'$. By attaching subtrees for each bag constraint this way, we obtain our final $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$.

We now argue that $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$ is indeed a width $k$ GHD of $\delta(H)$. Indeed, width $k$ follows immediately from the construction, since $\mathcal{D}^*$ and $\langle T, (B_u)_{u \in T}, (\lambda_u)_{u \in T} \rangle$ both have width $k$ and none of their $\lambda$-labels are modified. For connectedness, recall that by our definition of bag constraints the tree $T_i$ is attached to a node $u_i'$ whose bag contains $B_{u_i} \cap B(T^*)$. Hence, every vertex in $B(T^*)$, and thus also every vertex in bags of $\mathcal{D}^*$, that also occurs in $B(T_i)$ must be in $B_{u_i}$. We see that connectedness cannot be violated by the attaching step of our construction. By Lemma 4.2, all the individual parts that are attached to $\mathcal{D}^*$ already satisfy the connectedness condition, and it therefore holds also for all of $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$.

Finally, we verify that all edges of $\delta(H)$ are covered by some bag of $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$. We partition the set of edges in two sets, edges that are in $\delta(H)^*$ and those that are not. If an edge is in $\delta(H)^*$, then it must be covered by $\mathcal{D}^*$ and thus also in $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$. In the latter case, observe that if an edge $e$ is in $\delta(H)$ but not in $\delta(H)^*$, then $e$ is in $H$ and thus covered by some node of $T \setminus T^*$. Note that there is a $T^*$-induced bag constraint for every tree in $T \setminus T^*$. Hence, by Lemma 4.2 and the above construction reattaching the subtree in which $e$ is covered, $e$ is also covered in $\langle T', (B'_u)_{u \in T'}, (\lambda'_u)_{u \in T'} \rangle$.

Note that we made no explicit use of the second condition in Definition 4.1. The condition effectively enforces that any edges that contain new vertices will be in $H^*$ and in this way implicitly factors into the above argument. \qed

It is possible for no $T^*$-induced bag constraints satisfying GHD of $\delta(H)^*$ with width at most $k$ to exist, even if $\text{gwh}(\delta(H)) \leq k$. Thus, while the discussions of this section—and in particular the ideas of Theorem 4.5—form the foundation of our practical implementation, some adaptations are necessary to efficiently deal with those cases. This will be the topic of the following section.

### 5 Algorithmic Implementation of $\delta$-Mutable Subtree Framework

We now want to focus on how to use the concept of $\delta$-mutable subtrees from Section 4 to define an algorithmic framework that can take an existing GHD and the hypergraph of an updated CSP and provide some data structure that can be used to speed up the computation of a new GHD, even
if parts of the old GHD need to be recomputed. We want this framework to be implementation
agnostic, so that it can be plugged in to essentially any existing decomposition algorithm with few
required adaptations.

To this end, we first define this framework, and afterwards we also give a practical example of
how it has been adapted to an existing state-of-the-art decomposition algorithm. We then report
on its performance in various update tasks in Section 6.

5.1 Introducing the Framework

The goal of our implementation is twofold: We want a strategy built on top of the framework of
δ-mutable subtrees, and we want it to encompass existing algorithms for computing GHDs. In the
following, we make use of the basics of top-down GHD construction, as explained in Section 2.3.
For more detail, we refer the reader to a recent overview on hypergraph decompositions [10].
Before we proceed with explaining our implementation, we will need some way of referring how
the bags and edge covers of the old GHDs are affected by an update δ when we want to use the old
GHD with the modified hypergraph δ(H). We first introduce a function sδ : E(H) → E(δ(H)) ∪ ∅,
which will map edges e ∈ E(H) to their corresponding equivalent e′ ∈ E(δ(H)), if it exists, or ∅, if
δ actually deleted that edge. By slight abuse of notation, given subset X ⊆ E(H), we shall use
δ(X) = {sδ(e) | e ∈ X}. In this same vein, we also introduce for a vertex set Y ⊆ V(H), the notation
δ(Y) = Y ∩ V(δ(H)). Another notational choice we make throughout this section is how to refer to
the inputs of algorithms that deal with updated hypergraphs. Since all algorithms we present only
deal with a single updated hypergraph and its subgraphs and never need the original hypergraph,
we omit the use of the δ function. So instead of δ(H), we just write H. We will still need a δ mutable
subtree, and we assume that it has been computed and is provided to the algorithms as an input.

The idea behind our framework is to try to update the minimal δ-mutable subtree T∗ and reuse
many of the outer subtrees. If this is not possible, due to the way the modification has changed
the hypergraph, then we still want to return a GHD of the updated CSP quickly. Bag constraints
from Theorem 4.5 encode the properties necessary for parts of T∗ to be reused. As was mentioned,
however, it is possible that to successfully find a new GHD of low width, we need to forgo some
of them. For our implementation, we think of them as soft constraints: We make an effort to find
GHDs that reuse T∗ if they exist and, if they do not, use them as a starting point in the search
space.

We realize this behavior via the concept of a scene.

Definition 5.1 (Scene). Let (T, (B_u)_{u ∈ T}, (λ_u)_{u ∈ T}) be a normal-form GHD of a hypergraph H. A
scene mapping σ : 2^{E(H)} → N(T) is a partial mapping from a subhypergraph H′ ⊆ H to a node
u ∈ T. The co-domain element of σ is denoted as a scene. Given a modification δ and a δ-mutable
subtree T∗, we call σ(H) out-scene if σ(H) ∉ T∗ or in-scene if σ(H) ∈ T∗.

Scenes avoid decomposing again parts of the hypergraph for which we already know a GHD.
Lemma 4.2 implies that the use of out-scenes is valid. Using in-scenes is more complex: We try
to utilize them at most once to see if they help in finding a GHD of δ(H). If this leads to a reject
case, then we know that the scene will not be used again. Therefore, in-scenes never harm the
correctness of our approach. We compute a scene mapping via a two-phase traversal of the old
GHD, and we require this old GHD to be in normal form so that we can determine which subtrees
of the GHD “cover” certain components. This procedure can be seen in pseudocode in Algorithm 3
and Algorithm 4, each detailing one of the two phases. We proceed to give an informal explanation
below:

In the first downward phase, the GHD is traversed top-down and the bags of the encountered
nodes are used to “replay a decomposition procedure.” Starting at the root u of the GHD, we create a
new mapping $H \rightarrow u$ for the current hypergraph $H$. This can be seen in line 13. Then, we compute the $[B_u]$-components $C_1, \ldots, C_\ell$, which we assign to the child nodes $u_1, \ldots, u_\ell$ of $u$, where we have that $B_u \cap C_\ell \neq \emptyset$. Finally, we make a recursive call on each pair $(C_i, u_i)$. The recursive call and the matching from components to child nodes happens in Algorithm 3 between the lines 14 to 20. Due to the properties of GHDs, we know that each $[B_u]$-component matches with exactly one child node. If not, then the downward phase stops. This can only happen when considering nodes of $T^*$. The second upward phase generates mappings in a different way. It is called for any subtree below $T^*$ if the downward phase stops at a non-leaf node. In the algorithm, we see this happening at lines 4, 8, and 20. In this phase, instead of simulating a decomposition procedure, we traverse the GHD in a bottom-up fashion, and at every node $u$ we look at the subtree $T_u$ to create the mapping $\{e \in E(H) \mid e \cap (\bigcup_{u \in T} B_u) \neq \emptyset\} \rightarrow u$. In Algorithm 4, this is done by a recursive call on the child nodes of $u$, looking at all edges covered below them, as seen between lines 4 and 7. Between lines 10 to line 12, the algorithm computes the edges covered in the bag of $u$. Combined with the previous set, this gives us all edges that form $H[T_u]$. We are thus mapping $H[T_u]$ to $u$, as seen in line 17. This upward phase ensures we can make full use of Lemma 4.2 to consider all subtrees below $T^*$.

**Example 5.2.** We shall consider here as our initial hypergraph $H_P$, seen in Figure 5. A GHD of $H_P$ is provided in Figure 6(a), and we shall refer to it as $G$ in the sequel. We will use the same modification $\delta^{-1} \in \text{DelConstr}$ as introduced in Example 4.3. Thus, using $\delta^{-1}(H_P)$ and $G$ we will create the scene mapping. We start with the downward phase.
We start with the root node \( u_1 \) of \( G \) and create a scene mapping \( \delta^{-1}(H_{P_1}) \rightarrow u_1 \). Next, we consider the \([B_{u_1}]\)-components of \( \delta^{-1}(H_{P_1}) \), yielding components \( C_2 = \{w_3, w_6\} \) and \( C_3 = \{w_2, w_4\} \). We look for unique matching pairings of child nodes of \( u_1 \) and \([B_{u_1}]\)-components. We see that \( (B_{w_2} \setminus B_{u_1}) \cap V(C_2) = \{i, k\} \) and \( (B_{u_1} \setminus B_{u_1}) \cap V(C_3) = \{d, f, g\} \). Since all components were matched, we proceed on the pairings \((C_2, u_2)\) and \((C_3, u_3)\). Next, we consider the node \( u_2 \) and create the mapping \( C_2 \rightarrow u_2 \). We consider now the \([B_{u_2}]\)-components of \( C_2 \). We get one component, \( C_4 = \{w_5\} \), and we have that \( (B_{u_1} \setminus B_{u_2}) \cap V(C_4) = \{j, l\} \). Thus, we proceed on the pairing \((C_4, u_4)\). We create the mapping \( C_4 \rightarrow u_4 \). We note that there are no \([B_{u_1}]\)-components of \( C_4 \), since \( B_{u_1} \) already covers the entire component \( C_4 \). We continue with \((C_3, u_3)\). We create the mapping \( C_3 \rightarrow u_3 \). As before, we note that there are no \([B_{u_1}]\)-components of \( C_3 \), as \( B_{u_1} \) already fully covers \( C_3 \). Since the downward phase was never stopped at a non-leaf node, we do not proceed to the upward phase. To summarize, we get the following scene mapping: \%((\delta^{-1}(H_{P_1}) \rightarrow u_1), (C_2 \rightarrow u_2), (C_3 \rightarrow u_3), (C_4 \rightarrow u_4))\). We will see in Algorithm 5 how this scene mapping can be used to speed up GHD computation under updates.

Proposition 5.3. The Scene Creation algorithm, detailed in Algorithm 3, has a time complexity of \( O(N^3) \), where \( N \) is the size of the input hypergraph.

Proof. We first analyse the complexity of the SceneCreationUp function from Algorithm 4, as it forms essentially a subroutine of the SceneCreationDown function. We see that SceneCreationUp takes as input the subtree \( T_n \) for a given input node \( n \). Then, for each node \( n' \in T_n \), it computes the induced subgraph \( H[T_{n'}] \). This can be seen by the fact that it first looks at all edges that are directly covered by the bag of \( n' \) and combines this set with the set of all edges covered by any descendants of \( n' \) in the GHD. This operation clearly takes linear space in the size of the subgraph \( H' \), which we can consider to be bounded by the size of the input graph. Next, we compute the connected components over this induced subgraph when separated by the bag of \( n' \).
Computation of connected components on undirected graphs is known to be in linear time\(^\text{[27]}\). Thus, the complexity is \(O(N \ast (N + N))\) for the entire run of SceneCreation\_Up. Thus we get the quadratic runtime \(O(N^2)\).

Next we look at SceneCreation\_Down. This function runs over the entire GHD and takes initially the entire graph as input. For each node \(n'\) it encounters during the recursion, it computes the connected components over the current subgraph. Then it recursively proceeds over the subtree rooted at \(n'\). This continues until we hit a leaf node, or one of the conditions at lines 4 or 8 is met or if the matching of components and nodes fails at line 20, and we stop the recursion and call the function SceneCreation\_Up. Note that this can only happen once for a given subtree, ending the recursion at that point. Thus for a given node \(n'\), we either have a linear operation, or call the quadratic function SceneCreation\_Up. This gives us a simple upper bound of \(O(N^3)\) for SceneCreation\_Down. \(\square\)

Algorithm overview. A pseudo-code representation of our framework can be seen in Algorithm 5, called GH\_Update. As input we expect four items: (1) the hypergraph \(H\) of the updated CSP; (2) a decomposition algorithm \(D\), which we call decomposer; (3) a GHD \(G\) of the CSP before the modification; and, last, (4) the \(\delta\)-mutable subtree \(T^*\). The output is a GHD of \(H\) of width \(\leq k\), or a reject if none can be found. The decomposer \(D\) takes as input a subhypergraph and a scene mapping. It produces a GHD of \(H\) of width \(\leq k\) or rejects if none exists. Our algorithm initially computes a scene mapping \(\sigma\), in line 2, using the aforementioned procedure. Then, the recursive function DecomUpdate is called on \(H\) and \(\sigma\). At line 5, the function checks if a scene \(\sigma(H')\) exists for the current subhypergraph \(H'\). This check is a stateful operation and can change the contents of \(\sigma\) in the following way: For in-scenes, it will remove them from \(\sigma\) after the first time they have been checked and returned. For out-scenes, no such removal takes place. If a scene was reported as being defined, then at line 6, the algorithm immediately fixes the current node of the GHD with \(\sigma(H')\) and avoids the use of the decomposer, which would start an expensive search for a new bag. At line 7, we separate \(H'\) into the same \([B_u]\)-components we encountered while computing the old GHD. Now we make a recursive call on each of these components in lines 8 to 10, adding each GHD produced to the set of children of \(u\). We then return the thus-created GHD with \(u\) at its root. Line 12 is executed only if \(H'\) has never been encountered while building the old GHD. In this case, the decomposer \(D\) is called to find a GHD of \(H'\) of width \(\leq k\).

This design ensures that in either case, whether the \(\delta\)-mutable subtree can be simply updated or an entirely new GHD needs to be computed, we can use the same strategy. Moreover, in both cases we exploit the information provided by the old GHD. The decomposer can be any existing GHD algorithm—it just needs to be adapted to use scene mappings.

5.2 Applying the Framework to an Existing Decomposition Algorithm
To demonstrate that our framework can be applied existing combinatorial algorithms for finding GHDs, we looked at BalancedGo by Gottlob et al.\(^\text{[15]}\). As it was an open source program, we modified it to make use of the \(\delta\)-mutable framework. Our extension of BalancedGo that supports our proposed strategy for update handling is available at https://github.com/cem-okulmus/BalancedGoUpdate.

It is notable that the actual code for the algorithm, specifically the det-\(k\)-decomp algorithm, is taken verbatim from BalancedGo with only a few lines having been added. This is in addition to

\(^4\)Note that this article deals with (strongly) connected components of directed graphs, which is a strict generalization of connected components in the undirected setting. Formally, there is a trivial reduction of connected component computation of undirected graphs to strongly connected component computation of directed graphs.
**Algorithm 5: GHDUpdate**

**Type:** Node=(Bag: Vertex Set, Edge Cover: Set of Edges, Children: Set of Nodes)

**Input:** $H$: Hypergraph, $D$: Decomposer, $G$: GHD, $T^*$: $\delta$-mutable subtree

**Parameter:** $k$: width parameter

**Output:** a GHD of $H$ with width $\leq k$, or **Reject** if none exists

1. begin
2. $\sigma := \text{SceneCreationDown}(\text{root } r \text{ of } G, H, T^*)$
3. return DecompUpdate($H, \sigma$) \Comment{initial call}
4. function DecompUpdate($H'$: Subgraph, $\sigma$: Scene Mapping)
5. if $\sigma(H')$ is defined then \Comment{$\sigma$ will report in-scenes as being defined only once}
6. $u := \sigma(H')$ \Comment{Use node from $T^*$}
7. comps := $[u.\text{Bag}]$-components of $H'$
8. $u.\text{Children} := \emptyset$
9. for $c \in \text{comps}$ do
10. $u.\text{Children} = u.\text{Children} \cup \text{DecompUpdate}(c, \sigma)$
11. return $u$ \Comment{$u$ now forms the root of the output GHD}
12. return $D(H', \sigma)$ \Comment{using Decomposer for this subgraph}

---

...general functionality for extracting the mutable subtree from a given decomposition with respect to an updated graph and for computing the scene mapping. The actual use of the scene mapping inside the algorithm is a trivial process, and we believe almost any existing or future approach that actually computes GHDs via a combinatorial process can make use of it.

**6 EMPIRICAL EVALUATION**

In this section, we explore the potential of updating GHDs with the methods of Section 5. We describe our experiments, show their results and discuss the implications of our findings.

**6.1 Methodology and Synthetic Update Generation**

We compared multiple approaches for updating GHDs upon elementary modifications: **Update** and **Classic**. **Update** consists in our implementation of the general strategy of Section 5 on top of the BalancedGo program from Reference [15]. **Classic** uses the original BalancedGo program to compute a GHD of the modified hypergraph from scratch. In addition to these two, we also compared the htdLE0 program from Reference [25]. Similarly to **Classic**, this program computes a GHD for the modified hypergraph from scratch.

More precisely, given a hypergraph $H$ and a GHD $G$ of $H$ of width $k$, we applied an elementary modification $\delta$ to $H$ and compared the times taken by **Update**, **Classic**, and htdLE0 to output a GHD of $\delta(H)$ of width $\leq k$, if it exists. Recall that **Update** first computes the minimal $\delta$-mutable subtree $T^*$ from $G$ and then tries to build a GHD of $\delta(H)$ of width $\leq k$ by reusing the parts of $G$ that were not affected by $\delta$.

We conducted our experiments on the HyperBench dataset [7]. HyperBench is a large collection of hypergraphs from applications, benchmarks, and random generation that has been successfully used in a variety of hypergraph decomposition experiments. By using the LocalBIP implementation of BalancedGo and the results in Reference [15], we determined the optimal ghw of 1,798 from the 2,035 CSPs of HyperBench with a timeout of 1 hour per instance. Indeed, updating a GHD of optimal width is the hardest case. We thus used these 1,798 hypergraphs, their GHDs, and their ghw as a basis for our experiments.
Table 1. A Breakdown of the Instances Used for the Empirical Evaluation by Their Widths Prior to Modification

| Width | Count | Count Distinct |
|-------|-------|----------------|
| 2     | 2,375 | 95             |
| 3     | 5,800 | 232            |
| 4     | 8,125 | 325            |
| 5     | 8,850 | 354            |
| 6     | 8,875 | 355            |
| 7     | 6,350 | 254            |
| 8     | 2,925 | 117            |
| 9     | 1,225 | 49             |
| 10    | 425   | 17             |
| **Total** | 44,950 | 1,798          |

In addition to the number of instances ("Count"), we also show the number of distinct original hypergraphs on which the synthetic instance is based.

For each hypergraph $H$, we randomly generated five elementary modifications per each class from Section 3.1 as follows. For AddVar, we introduce a new vertex into $\ell$ randomly chosen edges, where $\ell$ is the average (rounded up) degree of the original hypergraph. We generate AddEq modifications by merging two random vertices and DelEq modifications by splitting a vertex $x$ into two vertices $y_1, y_2$. In half of the edges incident to $x$, we replace $x$ by $y_1$ and in the other half we replaced $x$ by $y_2$. Notably, AddConstr adds an edge with average (rounded up) rank such that all vertices in the new edge are already part of some existing edge. That means that we generate challenging cases while avoiding the easy case where most vertices in the new edge have no effect. For DelConstr, a random edge is removed from the hypergraph.

Note that updating a GHD of optimal width in case of $\delta \in$ DelVar is trivial. Indeed, let $v \in V(H)$ be the vertex removed by $\delta$ and consider a GHD $G$ of $H$ of width $k$. A GHD of $\delta(H)$ of width $\leq k$ can be easily obtained by removing $v$ from all bags $B_u$ of $G$. In total, this process produce 44,950 instances, each consisting of the original hypergraph $H$ with a known minimal GHD and a modification $\delta$. We can see a breakdown of the original widths of the instances before the modifications in Table 1.

For each such instance, we compute the hypergraph $\delta(H)$. This $\delta(H)$ is used as input for Classic and htdLEO to check whether $\text{ghw}(\delta(H)) \leq \text{ghw}(H)$. We also compute the minimal $\delta$-mutable subtree from the decomposition of $H$. The subtree and the decomposition, in addition to the hypergraph $\delta(H)$, were provided as input to the Update implementation to solve the corresponding instance of SEARCHUPDATEGHD. Note that the time to compute the $\delta$-mutable subtrees is trivial (under 1 ms) for all of our instances and thus not explicitly reported. The raw data for our experiments are provided here.\(^5\)

All experiments ran on an Intel Xeon CPU E5-2650 at 2.9 GHz with 264 GB RAM. Nevertheless, each instance used only one core of the CPU and 1 GB RAM. We set a timeout of 30 minutes for each run, i.e., we stopped the program if this threshold was crossed.

\(^5\)https://zenodo.org/record/6481125.
## 6.2 Results and Discussion

To reduce the effect of variance, we only report on the 26,013 instances for which it took `Classic` more than 15 ms to compute a decomposition. In the “easier” cases, it is reasonable to just use `Classic` instead of the more sophisticated `Update`. If we move the threshold to any $t > 15$, then the superiority of `Update` becomes even clearer. This suggests that our approach is even more fruitful when applied to “hard” cases.

Since our `Update` approach is built on top of `Classic`, we will use only the latter as a baseline for our experiments. As it will be evident, this is also justified by the fact that `Classic` performs better than `htdLEO` on average.

The results for each class of modification are shown in Table 2. The column `Positive` contains the percentage of cases where the width of the hypergraph did not increase due to the modification. The column `Better` contains the percentage of instances in which `Update` outperformed `Classic`. In the next three columns, we record the geometric means (in milliseconds) for `Classic`, `Update`, and `htdLEO`. We then report on the speedup, which is defined as the ratio between `Classic` and `Update` runtimes via the geometric mean of all speedups. In the last columns, we compare the number of exclusive timeouts for each solver. For instance, the column `Classic` reports on the number of instances that timed out for `Classic` but neither for `Update` nor `htdLEO`. Finally, for each operation, we show the number of instances that timed out for all methods. Since computing $T^*$ takes far less than a ms for all of our instances, the time is not reported explicitly.

To compare the different approaches, we adopt the same methodology that was adopted in Reference [7], i.e., we compare mean running times and number of instances that timed out. Overall, Table 2 clearly demonstrates the significant benefits of using `Update`. For every modification class, the `Update` mean time is significantly lower than the other approaches. The mean speedups are very high throughout all modifications even in the most difficult cases, i.e., `DelEq` and `AddVar` modifications. In theory, `DelConstr` is problematic, since the deleted edge could have covered an arbitrarily complex structure. However, it seems that this occurs rarely in practice, and deleting an edge simplifies the hypergraph instead. This is clearly apparent in the observation that 99.48% of `DelConstr` instances were positive, i.e., the width did not increase by deleting a constraint.

Interestingly, `Update` seems to be particularly well suited for `DelConstr` and `AddEq` modifications. In theory, `DelConstr` is problematic, since the deleted edge could have covered an arbitrarily complex structure. However, it seems that this occurs rarely in practice, and deleting an edge simplifies the hypergraph instead. This is clearly apparent in the observation that 99.48% of `DelConstr` instances were positive, i.e., the width did not increase by deleting a constraint.

The `Better` column shows that `Update` is faster than `Classic` in 78.8% of cases on average. This is despite the fact that many instances were solvable by `Classic` in less than 40 ms and `Update` has an additional overhead because of the scene mapping creation. Another source of slowdowns are negative instances ($ghw(\delta(H)) > k$), where the entire search space needs to be explored. In this case, the scene mapping is of little use, and its creation only causes delays. Moreover, the `Timeout`
Table 3. Statistics for Classic, Update, and htdLEO Shown by the Width of the Update Instances

| Width | Count | Positive (%) | Better (%) | Mean Classic | Mean Update | Mean htdLEO | Mean Speedup |
|-------|-------|--------------|------------|--------------|-------------|-------------|--------------|
| 2     | 732   | 27.87        | 34.29      | 906          | 784         | 59,227      | 1.16         |
| 3     | 1,582 | 66.81        | 70.29      | 773          | 191         | 220,364     | 4.05         |
| 4     | 3,511 | 97.09        | 80.03      | 762          | 24          | 29,063      | 31.17        |
| 5     | 2,241 | 100.00       | 88.26      | 866          | 13          | 9,624       | 69.20        |
| 6     | 4,055 | 100.00       | 82.22      | 1,049        | 34          | 142,452     | 30.71        |
| 7     | 1,056 | 100.00       | 85.61      | 251          | 17          | 201,694     | 14.86        |
| 8     | 28    | 100.00       | 60.71      | 794          | 148         | 773,108     | 5.35         |
| 9     | 3     | 100.00       | 66.67      | 6,336        | 54          | 1,559,313   | 118.41       |

Mean Classic, Mean Update, and Mean htdLEO are in milliseconds. All mean times were rounded to the closest integer, and all other non-integer numbers were rounded to two decimal places.

Table 4. Statistics for Classic, Update, and htdLEO Shown by the Width of the Positive Update Instances

| Width | Count | Better (%) | Mean Classic | Mean Update | Mean htdLEO | Mean Speedup |
|-------|-------|------------|--------------|-------------|-------------|--------------|
| 2     | 204   | 55.39      | 54           | 26          | 103,151     | 2.05         |
| 3     | 1,057 | 83.73      | 122          | 15          | 176,450     | 8.20         |
| 4     | 3,409 | 81.93      | 609          | 18          | 30,376      | 34.62        |
| 5     | 2,241 | 88.26      | 866          | 13          | 9,624       | 69.20        |
| 6     | 4,055 | 82.22      | 1,049        | 34          | 142,452     | 30.71        |
| 7     | 1,056 | 85.61      | 251          | 17          | 201,694     | 14.86        |
| 8     | 28    | 60.71      | 794          | 148         | 773,108     | 5.35         |
| 9     | 3     | 66.67      | 6,336        | 54          | 1,559,313   | 118.41       |

All mean times were rounded to the closest integer.

columns show that Update solves ≈94% of the instances, while Classic and htdLEO solve 89.2% and 60.2% of them, respectively.

The Positive column shows that elementary modifications do not change the width of the hypergraph in 91.26% of cases. This is somewhat surprising, as some of the modifications (e.g., AddConstr and AddEq) can lead to severe structural changes that intuitively increase how connected parts of the hypergraph are. Despite this, we observe that all modifications increase $ghw$ only rarely. We believe that this is because the width of a hypergraph effectively captures only the most structurally complex part. With higher width and larger hypergraphs it becomes more common that large parts of the hypergraph are less complex than the width suggests. Even if a modification makes such a simpler part of the hypergraph more complex, our observations illustrate that this rarely affects the overall width.

Some evidence for these intuitions can be seen in Table 3, where we can observe a clear trend that the width increase is less common (thus higher Positive %) the higher the width (which itself strongly correlates with hypergraph size, see Reference [7]). Note that the problem is not trivial even though the positive rate approaches 100% for the most complex hypergraphs. Even if the width stays the same, a minimal-width GHD for the new hypergraph may be entirely different than the input GHD (cf., the Better (%) column). As we ultimately need a GHD to algorithmically exploit low width, the high rate of positive instances therefore does not simplify the problem. Table 4 and Table 5 give additional insights into positive and negative update instances, respectively.
Table 5. Statistics for Classic, Update, and htdLEO Shown by the Width of the Negative Update Instances

| Width | Count | Better (%) | Mean Classic | Mean Update | Mean htdLEO | Mean Speedup |
|-------|-------|------------|--------------|------------|------------|-------------|
| 2     | 528   | 26.14      | 2,695        | 2,910      | 47,800     | 0.93        |
| 3     | 525   | 43.24      | 31,951       | 32,648     | 344,718    | 0.98        |
| 4     | 102   | 16.67      | 1,337,066    | 1,434,802  | 6,633      | 0.93        |

All mean times were rounded to the closest integer.

Fig. 7. Geometric mean runtimes (log. scale) of Classic and Update w.r.t. ghw and instance size.

We also investigated how our approach behaves with increasing ghw of the input decomposition as well as in relation to hypergraph size (in number of constraints and vertices, separately). The results of both studies are summarized in Figure 7. Note that the runtimes are given on a logarithmic scale. Since htdLEO is more than one order of magnitude slower than the other two methods, we do not report on it. We see that beginning from width 3, Update provides significantly better mean runtimes than Classic, and the speedup generally increases as well.

In Table 6, we show how many instances fall into every bucket used for the plots in Figure 7. Our goal was to find a partitioning into size intervals where each interval has a similar size.

We observe that the superiority of Update becomes more pronounced as the input CSPs (hypergraphs) become larger. Intuitively, this is explained by the fact that the modification usually affects a smaller fraction of the hypergraph as the size increases. Hence, it is possible to replace the

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mutable subtree and reuse much of the old decomposition, as shown in Section 5, then the strengths of Update are emphasized. In practice, this is particularly promising, since recomputation of a GHD is problematic particularly for larger instances.

7 CONCLUSION AND FUTURE WORK

We introduced the problem of updating GHDs under modifications to the underlying CSP. We presented theoretical foundations—in the form of δ-mutable subtrees—as well as concrete algorithmic strategies viable for all modifications and top-down decomposition algorithms. We implemented these strategies on top of an existing competitive implementation from the literature, allowing it to make use of old decompositions to improve computation of GHDs in the update scenario. Though experimental evaluation we verified that our approach, on average, greatly speeds up the computation of GHDs in response to elementary modifications.

This article represents only a first step into this new challenging problem, and much is left to be done. For an immediate next step, we are particularly interested in how the input decomposition affects update performance. Particular decompositions, e.g., balanced separator decompositions [7], may affect the shape and size of mutable subtrees and therefore also affect how easily parts of a tree can be reused. Furthermore, we see potential in identifying specific modifications where SearchUpdateGHD, or some variation of it, can in fact be solved efficiently. Despite our negative results for elementary modifications, it may be of interest to identify relevant special cases of the elementary modifications that allow for easier updates.

APPENDIX

A PROOF OF THEOREM 3.6

The argument will require details of the reduction of 3-Sat to checking whether a hypergraph has $ghw \leq 2$ by Gottlob et al. [11]. The reduction is highly technical and we recall the construction and key facts here for convenience. For full details, we refer to Reference [11]. It will be convenient to use $[n]$ for integer $n$ to refer to the set $\{1, 2, \ldots, n\}$.

A.1 Reducing 3-Sat to Checking $ghw \leq 2$

The hypergraph $H$ to be constructed consists of three main parts: two versions of a gadget introduced below and a subhypergraph encoding the clauses of the 3-Sat instance. We first fix some notation. We write $[n]$ for the set $\{1, 2, \ldots, n\}$. Extending this common notation, we write $[n; m]$ for the set of pairs $[n] \times [m]$. Furthermore, we refer to the element $(1, 1)$ of any set $[n; m]$ as min and $(n, m)$ as max.
For two disjoint sets $M_1, M_2$ and $M = M_1 \cup M_2$, the construction makes use of a gadget with vertices $V = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\} \cup M$ and edges $E_A \cup E_B \cup E_C$ as follows:

\[ E_A = \{\{a_1, b_1\} \cup M_1, \{a_2, b_2\} \cup M_2, \{a_1, b_1\}, \{a_2, b_1\}, \{a_1, a_2\}\} \]

\[ E_B = \{\{b_1, c_1\} \cup M_1, \{b_2, c_2\} \cup M_2, \{b_1, c_1\}, \{b_2, c_1\}, \{b_1, b_2\}, \{c_1, c_2\}\} \]

\[ E_C = \{\{c_1, d_1\} \cup M_1, \{c_2, d_2\} \cup M_2, \{c_1, d_2\}, \{c_2, d_1\}, \{d_1, d_2\}\}. \]

Let $\varphi = \bigwedge_{i=1}^{m} (L_i \lor L_i' \lor L_i^3)$ be an arbitrary instance of 3-Sat with $m$ clauses and variables $x_1, \ldots, x_n$. In addition to the vertices for two of the aforementioned gadgets, the reduction uses the following sets to construct the target hypergraph $H$:

$Y, Y', Y_\ell, Y'_\ell$: The sets $Y = \{y_1, \ldots, y_n\}$ and $Y' = \{y_1', \ldots, y_n'\}$ will encode the truth values of the variables of $\varphi$. $Y_\ell (Y'_\ell)$ are the sets $Y \setminus \{y_\ell\} (Y' \setminus \{y'_\ell\})$.

$A, A', A_p, A'_p$: We have sets $A = \{a_p \mid p \in [2n + 3; m]\}$ and $A' = \{a'_p \mid p \in [2n + 3; m]\}$ with the following important subsets:

\[ A_p = \{a_{\min}, \ldots, a_p\} \]
\[ A'_p = \{a'_{\min}, \ldots, a'_p\} \]
\[ \bar{A}_p = \{a_p, \ldots, a_{\max}\} \]
\[ \bar{A'}_p = \{a'_p, \ldots, a'_{\max}\}. \]

$S$: First define $Q = [2n + 3; m] \cup \{(0, 1), (0, 0), (1, 0)\}$. Then, $S$ is defined as $Q \times \{1, 2, 3\}$. The elements in $S$ are pairs, which we denote as $(q \mid k)$. The values $q \in Q$ are themselves pairs of $(i, j)$.

$S_p$: For $p \in [2n + m; m]$, we write $S_p$ for the set $\{(p, 1), (p, 2), (p, 3)\}$. And $S_p^k$ for the singleton $\{(p \mid k)\}$ for $k \in \{1, 2, 3\}$.

The vertices of $H$ are as follows:

\[ V(H) = S \cup A \cup A' \cup Y \cup Y' \cup \{z_1, z_2\} \cup \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, a'_1, a'_2, b'_1, b'_2, c'_1, c'_2, d'_1, d'_2\}. \]

The edges of $H$ are defined below. First, we take two copies of the gadget $H_0$ described above:

- Let $H_0 = (V_0, E_0)$ be the hypergraph of the lemma described at the beginning of the section with $V_0 = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\} \cup M_1 \cup M_2$ and $E_0 = E_A \cup E_B \cup E_C$, where we set $M_1 = S \setminus S_{(0,1)} \cup \{z_1\}$ and $M_2 = Y \cup S_{(0,1)} \cup \{z_2\}$.

- Let $H'_0 = (V'_0, E'_0)$ be the corresponding hypergraph, with $V'_0 = \{a'_1, a'_2, b'_1, b'_2, c'_1, c'_2, d'_1, d'_2\} \cup M'_1 \cup M'_2$ and $E'_A, E'_B, E'_C$ are the primed versions of the edge sets $M'_1 = S \setminus S_{(1,0)} \cup \{z_1\}$ and $M'_2 = Y' \cup S_{(1,0)} \cup \{z_2\}$.

Beyond the gadget $H$ contains the following edges:

- $e_p = A'_p \cup \bar{A}_p$, for $p \in [2n + 3; m]$, $e_{y_i} = \{y_i, y_i'\}$, for $1 \leq i \leq n$,
- For $p = (i, j) \in [2n + 3; m]$ and $k \in \{1, 2, 3\}$:
  \[ e_p^{k, 0} = \begin{cases} A'_p \cup (S \setminus S_p^k) \cup Y \cup \{z_1\} & \text{if } L_j^k = x_\ell \\ A'_p \cup (S \setminus S_p^k) \cup Y_\ell \cup \{z_1\} & \text{if } L_j^k = \neg x_\ell, \end{cases} \]
  \[ e_p^{k, 1} = \begin{cases} A'_p \cup S_p^k \cup Y_\ell' \cup \{z_2\} & \text{if } L_j^k = x_\ell \\ A'_p \cup S_p^k \cup Y' \cup \{z_2\} & \text{if } L_j^k = \neg x_\ell. \end{cases} \]

- $e_{(0,0)} = \{a_1\} \cup A \cup S \setminus S_{(0,0)} \cup Y \cup \{z_1\}$
- $e_{(0,0)} = S_{(0,0)} \cup Y' \cup \{z_2\}$
The key GHD. The hypergraph construction above is such that only certain (if any) width-2 GHDs are possible for \( H_\varphi \). In particular, in Reference [11] it is shown extensively that any width-2 GHD of \( H_\varphi \) needs to be a line, i.e., it has no branching. Furthermore, the gadget construction is used to specify two blocks of nodes that need to be at the two ends of the line, indirectly fixing the possible nodes between them. Here it is enough to consider the standard GHD that can be constructed when a satisfying assignment \( \sigma \) for \( \varphi \) is known. Recall that \( x_1, \ldots, x_n \) are the variables of \( \varphi \) and let

\[
Z = \{ y_i \in Y \mid \sigma(x_i) = 1 \} \cup \{ y'_i \in Y' \mid \sigma(x_i) = 0 \}.
\]

The following line graph, together with \( B_\mu \) and \( \lambda_\mu \) labels from Table A.1, describe a width-2 GHD \( D_\varphi \) for \( \varphi \),

\[
u_C - u_B - u_A - \mu_{\min} \oplus 1 - u(1,1) - \cdots - u(2n+3,m-1) - u_{\max} - u'_A - u'_B - u'_C.
\]

In the following arguments, we will make use of this basic structure to argue the existence of width-2 GHDs for other cases.

Table A.1. Definition of \( B_\mu \) and \( \lambda_\mu \) for GHD of \( H \)

| \( u \in T \) | \( B_\mu \) | \( \lambda_\mu \) |
|---|---|---|
| \( u_C \) | \( \{ d_1, d_2, c_1, c_2 \} \cup Y \cup S \cup \{ z_1, z_2 \} \) | \( \{ c_1, d_1 \} \cup M_1, \{ c_2, d_2 \} \cup M_2 \) |
| \( u_B \) | \( \{ c_1, c_2, b_1, b_2 \} \cup Y \cup S \cup \{ z_1, z_2 \} \) | \( \{ b_1, c_1 \} \cup M_1, \{ b_2, c_2 \} \cup M_2 \) |
| \( u_A \) | \( \{ b_1, b_2, a_1, a_2 \} \cup Y \cup S \cup \{ z_1, z_2 \} \) | \( \{ a_1, b_1 \} \cup M_1, \{ a_2, b_2 \} \cup M_2 \) |
| \( u_{\min} \oplus 1 \) | \( \{ a_1 \} \cup A \cup Y \cup S \cup Z \cup \{ z_1, z_2 \} \) | \( e^0, e^0 \cup \ldots \cup e^0 \cup e^0 \) |
| \( u_{\max} \in [2n+3,m] \) | \( A' \cup \bar{A}_p \cup Y \cup S \cup Z \cup \{ z_1, z_2 \} \) | \( \{ a_1', b_1' \} \cup M_1', \{ a_2', b_2' \} \cup M_2' \) |
| \( u_A' \) | \( \{ a_1', a_2', b_1', b_2' \} \cup Y' \cup S \cup \{ z_1, z_2 \} \) | \( \{ b_1', b_1' \} \cup M_1', \{ b_2', b_2' \} \cup M_2' \) |
| \( u_B' \) | \( \{ b_1', b_2', c_1', c_2' \} \cup Y' \cup S \cup \{ z_1, z_2 \} \) | \( \{ c_1', c_2' \} \cup M_1', \{ c_2', d_1' \} \cup M_2' \) |
| \( u_C' \) | \( \{ c_1', c_2', d_1', d_2' \} \cup Y' \cup S \cup \{ z_1, z_2 \} \) | \( \{ c_1', d_1' \} \cup M_1', \{ c_2', d_1' \} \cup M_2' \) |

A.2 Adapting the Argument to Updates

We now show how to use the construction for the reduction from 3-Sat to checking \( \text{ghw} \leq 2 \) to the update problem restricted to the stated classes of atomic updates. Recall that for an instance \( \varphi \) of 3-Sat, the constructed hypergraph \( H_\varphi \) has \( \text{ghw}(H_\varphi) = 2 \) if \( \varphi \) is satisfiable and width 3 otherwise. Our plan is to manipulate \( H_\varphi \) in such a way that we can efficiently construct a width 2 GHD of \( H' = \delta^{-1}(H_\varphi) \) (and \( H' \) is not acyclic) for \( \delta \) in the respective classes. If such a modification \( \delta \in \Lambda \) always exists, then the satisfiability of \( \varphi \) many-one reduces to the decision version of SEARCHUPDATEGHDD(\( \Lambda \)) for inputs \( H' \), \( \delta \), and the width 2 GHD of \( H' \). We are able to give such a reduction for the DelConstr and DelEq case. We will describe below how to handle the other operations using a slightly more involved strategy.

**DelConstr.** Here our goal is easy to reach. To obtain \( H' \) it is sufficient to add a large edge \( e^* = V(H) \setminus \{ d_1 \} \) to \( H_\varphi \) is sufficient. Since \( d_1 \) has at least two distinct edges to other vertices (which are in \( e^* \)) we see that the resulting \( H' \) is not acyclic. Clearly, then \( \text{ghw}(H') = 2 \) and it is trivial to construct an appropriate width 2 GHD.

**DelEq.** Let \( H' \) be the hypergraph performing a AddEq modification on vertices \( z_2 \) and \( a_1' \) in \( H_\varphi \), (using \( a_1' \) to represent the vertex after the join). In particular, this will merge edges containing
$M_1', M_2,$ and $M_3'$ in the two gadgets as well as all edges of form $e_p^{k-1}$ (as well as some linking edges in the gadget). The resulting edge $e^*$ is of the form

$$e^* = S \cup Y \cup Y' \cup A' \cup \{a'_1, a'_2, b'_1, b'_2, c'_1, b'_2, c_2, d_2, z_1\}.$$  

The GHD $D_\varphi$ given above can then be adapted in the following manner to yield a GHD of width 2 for $H'$. Replace all edges that contained $z_2$ or $a'_1$ in covers $\lambda_1$ by the new $e^*$. Note that this affects all nodes in the GHD. Then add $e^*$ to the bag of every node. Clearly, all edges of $H_\varphi$ are still covered and our new edge $e^*$ is covered in every node. Finally, $H'$ is not acyclic, as some cycles in the gadgets remain untouched by the merge. For example, the edges $\{c_1, d_2\}, \{c_1, c_2\}$ form an $\alpha$-cycle with $e^*$ (note that $e^*$ does not contain $c_1$).

**A.3 The Complex Cases: AddVar, AddConstr, and AddEq**

For the other modification classes, we will now slightly change our strategy and instead show how to decide satisfiability of 3-Sat via a polynomial number of calls to SearchUpdateGHD. Note that we use the returned new decompositions from the calls and thus cannot directly derive NP-hardness by Turing reduction of the SearchUpdateGHD decision problem. Formally, for a class $C$ of updates, instead of a single $H'$ we construct a sequence $H_0', \ldots, H_\ell'$ with $H_0' = H_\varphi$, $\ell$ polynomial in the size of $\varphi$, and for each $i \in [\ell]$ there is a $\delta_i \in C$ such that $\delta_i(H_{i-1}') = H_i'$. We will show that for all $0 \leq i \leq \ell$, $\text{ghw}(H_i') \leq 2$ if and only if $\varphi$ is satisfiable.

Suppose now that we can construct such $H_0'$, sequence of modifications $\delta_1, \delta_2, \ldots, \delta_\ell$, as well as a width-2 GHD for $H_0'$ efficiently. If SearchUpdateGHD$(C)$ were feasible in polynomial time, then we could verify $\text{ghw}(H_\varphi) \leq 2$ in polynomial time by iteratively constructing a GHD for it from successive calls to SearchUpdateGHD, starting from the known GHD of $H_0'$. As stated previously, $\varphi$ is satisfiable if and only if $\text{ghw}(H_\varphi) \leq 2$ and thus this would yield a polynomial procedure for solving 3-Sat.

**AddVar.** The desired sequence of hypergraphs and modifications is defined via $\delta_i^{-1}$ being the modification that removes vertex $y_i'$ from $H_i'$. Thus, $\ell = n$ and all $\delta_i \in \text{AddVar}$. Observe that AddVar modifications can never decrease $\text{ghw}$, that is, $\text{ghw}(H_{i-1}') \leq \text{ghw}(H_i')$ for all $i \in [\ell]$. This can be easily observed, since their inverse, the modifications of DelVar, produce an induced subhypergraph and thus cannot increase $\text{ghw}$. Thus, we see that if $\text{ghw}(H_0') = 2$ and $\text{ghw}(H_\varphi) = 2$ (which is equivalent to $\varphi$ being satisfiable), then $\text{ghw}(H_i') = 2$ for all $1 \leq i \leq \ell$. We therefore see that this gives us a sequence of modifications as described above.

Recall that in the original GHD $D_\varphi$ given above, the set $Z$ is derived from a satisfying assignment for $\varphi$. For $H_0'$, we can simply set $Z = \emptyset$ (and remove all $y_i'$ from the bags) to obtain a width-2 GHD. Only the edges $e_{y_i}$ relied on $Z$ to be covered in $H_\varphi$, but in $H_0'$ they are all singletons $\{y_i\}$ and thus always covered in the first gadget. Hence, we can construct a width-2 GHD for $H_0'$ (which is cyclic), and, as described above, a linear number of calls to SearchUpdateGHD(AddVar) are sufficient to decide whether $\varphi$ is satisfiable.

**AddConstr.** We again argue via a sequence $H_0', \ldots, H_\ell'$ of hypergraphs and modifications $\delta_1, \ldots, \delta_\ell \in \text{AddConstr}$ with $\delta_i(H_{i-1}') = H_i'$. In contrast to the AddVar case, AddConstr modifications can decrease $\text{ghw}$, and our use of such a sequence thus depends on particular properties of our choice of modification sequence.

We define our sequences via $\delta_i^{-1}$ being the modification (in DelConstr) that deletes edge $e_{y_i}$ from $H_\varphi$. The construction of a width-2 GHD for $H_0'$ is the same as for $D_\varphi$ above but with $Z = \emptyset$. The function of $Z$ is only to connect $u_{\min \oplus 1}$ and $u_{\max}$ in a way such that every $e_{y_i}$ is covered in either one of the respective bags. Since $H_0'$ no longer contains those edges, this is still satisfied with $Z = \emptyset$. It is not difficult to verify that $Z$ was not used to cover any other edges in $H_\varphi$ and therefore the correctness of the resulting GHD. Now, suppose that $\text{ghw}(H_\varphi) = 2$; then there exists some
width-2 GHD $D_\varphi$ of the form shown above. Note that no $e_{y_i}$ edge is used in a $\lambda_u$ set for this GHD, but all of them are covered in some bag. In consequence, $D_\varphi$ is also a GHD for every hypergraph $H'_i$ in our sequence, meaning every hypergraph in the sequence has $\text{ghw}(2) \text{iff } \text{ghw}(H'_i) = 2$. Note that while it is hard to find $D_\varphi$, the key point here is that a width-2 GHD for the special case $H'_0$ can always be found easily.

We can now proceed as in the AddVar case. Start from input $H'_0$, $\delta_1$, and the width-2 GHD of $H'_0$ as described above and call $\text{SEARCHUPDATEGHD}(\text{AddConstr})$ to find a width-2 GHD for $H'_0$. Iterating this process, we either arrive at some $H'_i$ for which $\text{ghw}(H'_i) > 2$ and reject or we show that $\text{ghw}(H'_i) = 2$. In the former case, we have by the argument above that then also $\text{ghw}(H'_i) > 2$. Thus, we can correctly decide whether $\text{ghw}(H'_i) \leq 2$—and therefore also 3-Sat—using linearly many calls of $\text{SEARCHUPDATEGHD}(\text{AddConstr})$.

**AddEq.** We construct the initial hypergraph $H'_0$ from $H_\varphi$ by replacing every edge $e_{y_i} = \{y_i, y'_i\}$ by the edge $e'_i = \{y_i, \star\}$. Consider the sequence $\delta_1, \ldots, \delta_n$ such that $\delta_i \in \text{AddEq}$ merges $\star$ into $y'_i$, i.e., $\star$ is replaced in every edge by $y'_i$. It is easy to see that $H'_n = H_\varphi$, and if $\text{ghw}(H'_n) = 2$ for all $i \in [n]$, then $\text{ghw}(H'_i) = 2$. We will first argue that $H'_0$ has $\text{ghw}(2)$ and that witnessing GHD can be found easily. Then we show that if $\text{ghw}(H'_i) = 2$, then $\text{ghw}(H'_i) = 2$ for all $i \in [n]$. All together, this again means that it is possible to decide 3-Sat using a linear number of calls to $\text{SEARCHUPDATEGHD}(\text{AddEq})$.

The decomposition for $H'_i$ is again based on $D_\varphi$ with $Z = \emptyset$. Observe that $D_\varphi$ does not use any $e_{y_i}$ as a cover, and thus the only concern with adapting it for $H'_i$ is making sure that every $e'_i$ is covered in some bag. To that end, add nodes $u'_i$, for $i \in [n]$ as children of $u_{\text{min} \ominus 1}$ with $B_{u'_i} = e'_i$ and cover $\lambda_{u'_i} = \{e'_i\}$. The connectedness condition is clearly not violated by these new nodes and every $e'_i$ is covered. Let $D_0$ be the GHD described here, and note it clearly has width 2 (and $H'_0$ is not acyclic).

To see that every $H'_i$ for $1 \leq i \leq n$ has $\text{ghw}(H'_i) = 2$, if $\text{ghw}(H'_i) = 2$, then we now proceed in similar fashion to the argument for AddConstr. Since we assume that $\text{ghw}(H'_i) = 2$, there exists a satisfying assignment $\sigma$ for the variables of $\varphi$. Let $Z = \{y_i \mid \sigma(x_i) = 1\} \cup \{y'_i \mid \sigma(x_i) = 0\}$ be the set as in the original definition of $D_\varphi$. Let $D_1$ be the GHD obtained from $D_\varphi$ using this $Z$ and for all $j$ s.t. $i < j \leq n$, add nodes $u'_j$ as children of $u_{\text{min} \ominus 1}$ as in the construction of $D_0$ above. By construction, $H'_i$ contains the edges $e_{y_j}$ for $j \leq i$ and edges $e'_j$ for $j > i$. It is then straightforward to verify that $D_1$ indeed is a width-2 GHD for $H'_i$. Thus, as described above, we can use a linear number of calls to $\text{SEARCHUPDATEGHD}(\text{AddEq})$ to decide 3-Sat. Consequently, if $P \neq \text{NP}$, then $\text{SEARCHUPDATEGHD}(\text{AddEq})$ cannot be solvable in polynomial time.

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