Bounds on density of states and spectral gap in CFT\textsubscript{2}

Shouvik Ganguly,\textsuperscript{q} Sridip Pal\textsuperscript{\check{q}}

\textsuperscript{q}Department of Electrical and Computer Engineering, University of California, San Diego

La Jolla, CA 92093, USA

\textsuperscript{\check{q}}Department of Physics, University of California, San Diego

La Jolla, CA 92093, USA

E-mail: shgangul@eng.ucsd.edu, srpal@ucsd.edu

Abstract: We improve the recently discovered bounds on the $O(1)$ correction to the Cardy formula for the density of states in 2 dimensional conformal field theory at high energy. We prove a conjectured upper bound on the asymptotic gap between two consecutive Virasoro primaries for a central charge greater than 1, demonstrating it to be 1. Furthermore, a systematic method is provided to establish a limit on how tight the bound on the $O(1)$ correction to the Cardy formula can be made using bandlimited functions. The techniques and the functions used here are of generic importance whenever the Tauberian theorems are used to estimate some physical quantities.
1 The premise and the results

Modular invariance is a powerful constraint on the data of 2D conformal field theory (CFT). It relates the low temperature data to the high temperature data. For example, using the fact that the low temperature behavior of the 2D CFT partition function is universal and controlled by a single parameter $c$, the central charge of the CFT, we can deduce the universal behavior of the partition function at high temperature and thereby deduce the asymptotic behavior of the density of states, which controls the high temperature behavior\(^1\) of a 2D CFT [3]. Similar ideas can be extended to one point functions as well, where the low temperature behavior is controlled by the low lying spectra and three point coefficients [4, 5]. Yet another remarkable implication of the modular invariance of the partition function is the existence of infinite Virasoro primaries for CFT with $c > 1$. Significant progress has been made in recent years towards exploiting the modular invariance to deduce results in 2D CFT under the umbrella of modular bootstrap [4–13].

Recently, with the use of complex Tauberian theorem,\(^2\) Mukhametzhanov and Zhiboedov [18] have explored the regime of validity, as well as corrections, to the Cardy formula with great nuance. In particular, they have investigated the entropy

\(^1\)The fact that the modular invariance of CFT can predict the asymptotic density of states is explicitly stated in [1]. One usually takes the inverse Laplace transform of the partition function to deduce such behavior; similar techniques also appeared in [2]. We thank Shouvik Datta for pointing this out.

\(^2\)The usefulness of Tauberian theorems in the context of CFT is pointed out in [14]; subsequently, its importance was emphasized in Appendix C of [5], where the authors used Ingham’s theorem [15]. The fact that going out to the complex plane while using Tauberian theorems would provide extra mileage in controlling the correction terms in various asymptotic quantities of CFT, has been pointed out in [16]. In particular, the use of [17] turned out to be extremely useful in this context.
$S_{\delta}$ associated with a particular energy window of width $\delta$ around a peak value $\Delta$, which is allowed to go to infinity, and found

$$S_{\delta} = \log \left( \int_{\Delta - \delta}^{\Delta + \delta} \rho(\Delta') \, d\Delta' \right) \sim \frac{2\pi}{3} \sqrt{\frac{c\Delta}{3}} + \frac{1}{4} \log \left( \frac{c\delta^4}{3\Delta^3} \right) + s(\delta, \Delta), \quad (1.1)$$

where $\rho(\Delta)$ is the density of states, given by a sum of Dirac delta functions peaked at the positions of the operator dimensions. It is shown in [18] that for $O(1)$ energy width, the $O(1)$ correction $s(\delta, \Delta)$ is bounded from above and below:

$$\delta = O(1) : \ s_-(\delta, \Delta) \leq s(\delta, \Delta) \leq s_+(\delta, \Delta) \quad (1.2)$$

The purpose of the current note is to improve the bound and provide a systematic way to estimate how tight the bounds can be made using bandlimited functions. We also prove the conjectured upper bound on the asymptotic gap between Virasoro primaries, which turns out to be 1. This gap is optimal since for the Monster CFT, the gap is precisely 1.

Our results can be summarized by figure [1], where the green line and dots denote the lower (upper) bound on the upper (lower) bound. The orange lines denote the improved achievable bounds. The brown dots stand for the lower (upper) bound on the upper (lower) bound obtained from implementing the positive definiteness condition on the Fourier transform of $\pm(\phi_\pm - \Theta)$ via Matlab. The bound on bounds represented by the green line is thus weaker than that represented by the brown dots. In short, the brown shaded region is not achievable by any bandlimited function.

In particular, we show that the upper bound on $s(\delta, \Delta)$ is given by

$$\exp[s_+(\delta, \Delta)] = \begin{cases} 
MZ(\delta), & \delta < 0.73 \\
\frac{3}{405\pi} \left(11\delta^2 + \frac{45}{\pi^2}\right), & 0.73 < \delta \leq 0.785 \\
1.7578, & \delta > 0.785
\end{cases} \quad (1.3)$$

where $MZ(\delta)$ is a function introduced in [18] and defined as

$$MZ(\delta) = \begin{cases} 
\frac{\pi}{3} \left(\frac{\delta}{\pi}\right)^3 \left(\sin \left(\frac{\delta}{2}\right)\right)^{-4}, & \delta < \frac{a_*}{2\pi} \sim 0.54 \\
2.02, & \delta > \frac{a_*}{2\pi} \sim 0.54
\end{cases} \quad (1.4)$$

Here, $a_* \sim 3.38$ satisfies $a_* = 3\tan(a_*/4)$. Eq.(1.3) is an improvement of the upper bound for $\delta > 0.73$, as evident from figure [2].
The lower bound $s_-(\delta, \Delta)$ is given by

$$
\exp [s_-(\delta, \Delta)] = \begin{dcases} 
mz(\delta), & \frac{\sqrt{3}}{\pi} \leq \delta < \frac{\sqrt{165}}{19\pi} \sim 0.94, \\
\frac{3(11\delta^2 - 45\pi)}{40\delta^3}, & \frac{\sqrt{165}}{19\pi} < \delta \leq 1, \\
0.5, & \delta > 1.
\end{dcases}
$$

(1.5)

where $mz(\delta)$ is a function, introduced in [18]

$$
mz(\delta) = \begin{dcases} 
2\left(\frac{\delta^2 - 3}{3\delta^3}\right), & \frac{\sqrt{3}}{\pi} \leq \delta < \frac{2}{\pi} \sim 0.95, \\
\frac{4\pi}{27} \sim 0.46, & \delta \geq \frac{3}{2}.
\end{dcases}
$$

(1.6)

The eq. (1.5) is an improvement of the lower bound for $\delta > 0.94$, as evident from figure [2].

The rest of the paper details the derivation of the above. In section 2, we derive the improvement on the bound on the $O(1)$ correction to the Cardy formula. Section 3 describes a systematic way to estimate how tight the bound can be made. We derive the optimal gap on the asymptotic spectra in section 4 and conclude with
2 Derivation of the improvement

The basic ingredients for estimating the asymptotic growth of the density of states are two functions \( \phi_{\pm} \) such that the following holds:

\[
\phi_{-}(\Delta') < \Theta (\Delta' \in [\Delta - \delta, \Delta + \delta]) < \phi_{+}(\Delta'). \tag{2.1}
\]

We refer the readers to section 4 of [18] for details of the procedure leading to a bound when \( \Delta \) goes to infinity. The basic result can be summarized as:

\[
c_{-}\rho_{0}(\Delta) \leq \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') \leq c_{+}\rho_{0}(\Delta), \tag{2.2}
\]

where \( \rho_{0}(\Delta) \) reproduces the contribution from the vacuum at high temperature and is given by

\[
\rho_{0}(\Delta) = \pi \sqrt{\frac{c_{3}}{3}} I_{1} \left( \frac{c_{3}}{\sqrt{\Delta - \frac{c_{12}}{12}}} \right) \Theta \left( \Delta - \frac{c}{12} \right) + \delta \left( \Delta - \frac{c}{12} \right). \tag{2.3}
\]

The above is in fact the leading result for the density of states at high energy. Furthermore, \( c_{\pm} \) is defined as

\[
c_{\pm} = \frac{1}{2} \int_{-\infty}^{\infty} dx \, \phi_{\pm}(\Delta + \delta x). \tag{2.4}
\]

The eq. (2.2) holds if the Fourier transform of \( \phi_{\pm} \) has a support on an interval which lies entirely within \([-2\pi, 2\pi]\). With this constraint in mind, we consider the following functions:

\[
\phi_{+}(\Delta') = \left[ \sin \left( \frac{\Lambda_{+}(\Delta' - \Delta)}{6} \right) \right]^{\delta} \left( 1 + \frac{(\Delta' - \Delta)^{2}}{\delta^{2}} \right), \tag{2.5}
\]

\[
\phi_{-}(\Delta') = \left[ \sin \left( \frac{\Lambda_{-}(\Delta' - \Delta)}{6} \right) \right]^{\delta} \left( 1 - \frac{(\Delta' - \Delta)^{2}}{\delta^{2}} \right). \tag{2.6}
\]

In order to ensure that the indicator function on the interval \([\Delta - \delta, \Delta + \delta]\) is bounded above by \( \phi_{+} \), we need to have

\[
\delta \Lambda_{+} \leq 4.9323. \tag{2.7}
\]
The number in the eq. (2.7) is obtained by requiring that \( \phi_+ (\Delta \pm \delta) > 1 \). The functions \( \phi_\pm \) have Fourier transforms with bounded supports \([-\Lambda_\pm, \Lambda_\pm]\), respectively. Thus, in order for this support to lie within \([-2\pi, 2\pi]\), we also require that \( \Lambda_\pm < 2\pi \). The bound is then obtained by minimizing (or maximizing)

\[
c_\pm = \frac{1}{2\delta} \int dx \ \phi_\pm (\Delta + x) = \frac{3\pi (11\delta^2 \Lambda_\pm^2 \pm 180)}{20\delta^3 \Lambda_\pm^3}
\]

(2.8)

for a given \( \delta \) by varying \( \Lambda_\pm \) subject to the constraint given by the eq. (2.7), as well as \( \Lambda_\pm < 2\pi \). From the eq. (2.2), one can conclude [18] that

\[
c_- \leq \exp [s(\delta, \Delta)] \leq c_+.
\]

(2.9)

Since for a fixed \( \delta \), \( c_+ \) is a monotonically decreasing function of \( \Lambda_+ \), we deduce that \( c_+ \) should be minimized by

\[
\Lambda_+ = \min \left\{ 2\pi, \frac{4.9323}{\delta} \right\} = \begin{cases} 2\pi, & \delta < 0.785, \\ \frac{4.9323}{\delta}, & \delta > 0.785. \end{cases}
\]

(2.10)

This explains the number 0.785 appearing in the bounds in the eq. (1.3). The final bound can be obtained by combining these results with the result of [18]. A similar analysis can be performed on \( c_- \). These procedures yield the eq. (1.3) for the upper bound, while the lower bound is given by

\[
\exp [s_-(\delta, \Delta)] = \begin{cases} m z(\delta), & \sqrt{\frac{3}{\pi}} \leq \delta < \sqrt{\frac{15\pi}{11}} \sim 0.94, \\ \frac{3(11\delta^2 - 45\pi^2)}{40\delta^3}, & \sqrt{\frac{15\pi}{11}} < \delta < \frac{3\sqrt{15\pi}}{\pi} \sim 1.12, \\ \frac{11}{60} \sqrt{\frac{11}{15\pi} \sim 0.49}, & \delta > \frac{3\sqrt{15\pi}}{\pi} \sim 1.12. \end{cases}
\]

(2.11)

The lower bound can be further improved for \( \delta > 1 \) by considering the following function whose Fourier transform has a support over \([-\frac{2\pi}{\delta}, \frac{2\pi}{\delta}]\).

\[
\phi_-^{\text{Sphere}} (\Delta') := \frac{1}{1 - \left( \frac{\Delta'}{\delta} \right)^2} \left( \frac{\sin \left( \frac{\pi (\Delta' - \Delta)}{\delta} \right)}{\pi (\Delta' - \Delta)} \right)^2.
\]

(2.12)

This yields \( c_- = 0.5 \), which is an improvement over the above; see figure 3.

**Serendipity — connection to the sphere packing problem:** The function in the eq. (2.12) also appears in the context of one dimensional sphere packing problem.
\textbf{Figure 2:} \( \text{Exp}[s_{\pm}] \): The orange line denotes the improved lower (upper) bound while the blue line is from [18].

\textbf{Figure 3:} The orange line represents the improvement on the lower bound by using the function \( \phi_{\text{Sphere}} \) appearing in the sphere packing problem.

In fact, there is an uncanny similarity between the functions required in the two problems, especially if we look at the requirements on the function producing the lower bound\(^3\). In the sphere packing problem, one has a Fourier transform pair \( f, \hat{f} \) satisfying

\begin{align}
    f(x) &\leq 0 \text{ for } |x| > 1, \\
    \hat{f}(k) &\geq 0.
\end{align}

In our case, we have \( x \leftrightarrow \Delta' \) and \( k \leftrightarrow t \) and we require that \( \hat{f}(k) \) has bounded support. In both scenarios, the goal is to maximize \( \hat{f}(0) \). In the case of sphere packing, we also normalize \( f(0) \) to one. For more details on the relevance of sphere packing to CFT, we refer the reader to the recent article [20].

It turns out that only in one dimension [19], where the sphere packing problem is trivial, the relevant function as given in the eq. (2.12) has bounded support in the

\(^3\)SP thanks John McGreevy for pointing to [20], where sphere packing plays a pivotal role.
Fourier domain and is positive\(^4\). This seems to suggest that if we want to further improve our bound, we need a bandlimited function whose Fourier transform becomes negative within the band.

Before moving on to the discussion of the bound on bounds, we pause to remark that the following class of functions parameterized by \(\alpha\) can not be used to improve the bound from above:

\[
\phi^{(\alpha)}_+(\Delta') = \left[ \frac{\sin \left( \frac{\Lambda_+ \delta}{\alpha} \right)}{\Lambda_+ \delta / \alpha} \right]^{-\alpha} \left[ \frac{\sin \left( \frac{\Lambda_+ (\Delta' - \Delta)}{\alpha} \right)}{\Lambda_+ (\Delta' - \Delta) / \alpha} \right]^\alpha, \quad \alpha \geq 2. \tag{2.15}
\]

Within this class of functions, \(\alpha = 4\) gives the tightest bound as found in [18].

### 3 Bound on bounds

In this section, we provide a systematic algorithm to estimate how tight the bounds can be made using bandlimited functions \(\phi_\pm\). This provides us with a quantitative estimate of the limitation of the procedure which produces these bounds on the \(O(1)\) correction to the Cardy formula. If one drops the requirement that the function be bandlimited, one might hope to do better. For the rest of this section, we will restrict ourselves to bandlimited functions only.

We recall that the functions \(\phi_\pm\) are chosen in such a way that they satisfy

\[
\phi_-(\Delta') < \Theta (\Delta' \in [\Delta - \delta, \Delta + \delta]) < \phi_+(\Delta'). \tag{3.1}
\]

This inequality gives a trivial bound on \(c_\pm\):

\[
c_- \leq 1 \leq c_+. \tag{3.2}
\]

In what follows, we make this inequality tighter. In this context, the following characterization of the Fourier transform of a positive function in terms of a positive definite function turns out to be extremely useful. Before delving into the proof, let us define the notion of positive definiteness of a function. Unless otherwise specified, here we will be dealing with functions from the real line to the complex plane. A function \(f(t)\) is said to be positive definite if for every positive integer \(n\) and for every set of distinct points \(t_1, \ldots, t_n\) chosen from the real line, the \(n \times n\) matrix \(A\) defined

\(^4\)For higher dimensions too, bandlimited functions are used (see, for example, Proposition 6.1 in [19]); nonetheless, they do not provide the tightest bound for the higher dimensional sphere packing problem. For \(n = 1\), the function appearing in the said proposition is related to the one that we have used. For other values of \(n\), we obtain bounds strictly less than 1/2. We thank Tom Hartman for pointing this out.
by

\[ A_{ij} = f(t_i - t_j) \]  

is positive definite. A function \( g(\Delta) \) is said to be positive if \( g(\Delta) > 0 \) for every \( \Delta \). One can show that the Fourier transform of a positive function is positive definite\(^5\).

Now, let us explore how this characterization can improve the eq. (3.2). Without loss of generality, we set \( \Delta = 0 \) henceforth, and define

\[ g_\pm(\Delta') = \pm [\phi_\pm(\Delta') - \Theta (\Delta' \in [-\delta, \delta])] . \]  

(3.4)

At this point we use the fact that \( \phi_\pm \) is a bandlimited function, i.e., it has a bounded support \([ -\Lambda_\pm, \Lambda_\pm ] \), and that \( \Lambda_\pm < 2\pi \). This requirement stems from the procedure followed in [18]. Thus we arrive at the following:

\[ \tilde{g}_\pm(0) = \pm 2\delta (c_\pm - 1) , \]  

(3.5)

\[ \tilde{g}_\pm(t) = \mp 2\delta \left( \frac{\sin(t\delta)}{t\delta} \right) \text{ for } |t| \geq 2\pi . \]  

(3.6)

The eq. (3.2) states that \( \tilde{g}(0)/2\delta > 0 \). In order to improve this, we construct \( 2 \times 2 \) matrices with \( t_2 > 2\pi \):

\[ G^{(2)}_\pm = \begin{bmatrix} \tilde{g}_\pm(0) & \tilde{g}_\pm(t_2) \\ \tilde{g}_\pm(t_2) & \tilde{g}_\pm(0) \end{bmatrix} . \]  

(3.7)

For a fixed \( \delta \), we consider the first positive peak of \( \tilde{g}_\pm \) outside \( t > 2\pi \). If this occurs at \( t = t(\delta) \), we choose \( t_2 = t(\delta) \). Subsequently, the positive definiteness of the matrix \( G^{(2)}_\pm \) boils down to the inequality

\[ \tilde{g}_\pm(0) > \tilde{g}_\pm(t(\delta)) , \]  

(3.8)

where \( t(\delta) \) is the first positive peak of \( \tilde{g}_\pm \) outside \( t > 2\pi \). For example, we can show that (see the green lines in Fig. 4):

\[ c_+ > \begin{cases} 1.2172 , & \delta < 0.715 , \\ 1.0913 , & 1.735 > \delta > 0.715 , \\ 1.0579 , & 2.74 > \delta > 1.736 , \\ \end{cases} \]  

(3.9)

\[ c_- < \begin{cases} 0.872 , & \delta < 1.229 , \\ 0.9291 , & 2.238 > \delta > 1.229 . \\ \end{cases} \]  

(3.10)

---

\(^5\)The proof is given in a box separately at the end of this subsection for those who are interested.
**Figure 4**: $\text{Exp}[s \pm]$: The green line is the analytical lower and upper bound on upper and lower bound i.e. $c_\pm$ respectively. The green shaded region is not achievable by any bandlimited function.

**Positive function $\Leftrightarrow$ Positive definite Function: Fourier transform**

We will show that the Fourier transform of an even and positive function is a positive definite function. Consider a function $g(\Delta)$ and let us define the Fourier transform as

$$
\tilde{g}(t) = \int_{-\infty}^{\infty} dt \ g(\Delta)e^{-i\Delta t} = 2 \int_{0}^{\infty} dt \ \cos(\Delta t)g(\Delta). \quad (3.11)
$$

Now, we construct the matrix

$$
G_{ij} = g(t_i - t_j) = 2 \int_{0}^{\infty} dt \ \cos [\Delta (t_i - t_j)] g(\Delta). \quad (3.12)
$$

In order to show that $G$ is a positive definite matrix, i.e., $\sum_{ij} v_i v_j G_{ij} > 0$ for $v_i \in \mathbb{R}$ such that $\sum_i v_i^2 \neq 0$, we think of an auxiliary 2 dimensional space with $n$ vectors $\vec{v}_i$, (for clarity, we remark that $i$ labels the vector itself, not its component) such that we have

$$
\vec{v}_i \equiv (|v_i| \cos(\Delta t_i), |v_i| \sin(\Delta t_i)). \quad (3.13)
$$

Thus, we have

$$
\sum_{ij} v_i v_j G_{ij} = 2 \int_{0}^{\infty} dt \ \left( \sum_{ij} v_i v_j \cos [\Delta (t_i - t_j)] \right) g(\Delta) \quad (3.14)
$$

$$
= 2 \int_{0}^{\infty} dt \ \left( \vec{V} \cdot \vec{V} \right) g(\Delta) > 0 \quad (3.15)
$$
if \( t_1, \ldots, t_n \) are distinct. Here, \( \vec{V} \) is given by

\[
\vec{V} = \sum_i \text{sign}(v_i) \vec{v}_i(t_i).
\]

(3.16)

This completes the proof that the Fourier transform of an even positive function is a positive definite function. First of all, it is easy to see that \( c_\pm \), and hence the inequality, is insensitive to the midpoint of the interval, i.e., \( \Delta \), so we set it to 0 and this makes the functions \( \phi_\pm \) and \( \Theta \) even. In particular, we will be applying this theorem to \( \phi_+(\Delta') - \Theta (\Delta' \in [\Delta - \delta, \Delta + \delta]) \) and \( \Theta (\Delta' \in [\Delta - \delta, \Delta + \delta]) - \phi_-(\Delta') \). We make one more remark before exploring the consequences of this. The above result is true for any function, not necessarily even. The converse is also true due to Bochner’s Theorem, but in what follows, we do not require the converse statement.

Matlab implementation

We implement the above argument using more than two points and making sure that \(|t_i - t_j| \geq 2\pi\). For a fixed \( \delta \), we use a random number generator to sample the points \( t_i \) with the mentioned constraint. We do this multiple times and each time, we test the positive definiteness of the matrix \( G \) by providing as an input the value of \( \pm(c_\pm - 1) \). The range of \( \pm(c_\pm - 1) \) is chosen to be from the first peak \( t(\delta) \) till some value larger than the achievable bound given in \( 1.3 \) and \( 2.11 \). This in turn yields a lower bound (or upper bound) for \( c_\pm \) for each trial\(^6\). Subsequently, we pick out the best possible bound among all the trials. For example, we provide a table [1] showing the outputs from a typical run for improving the bound on the upper bound. The tables [1] and [2] improve the lower (upper) bound for \( c_\pm \) and this is shown in the figure [1], where the brown dots are the stronger bounds over the green lines and disallow a larger region.

4 Bound on spectral gap: towards optimality

In this section, we switch gear and explore the asymptotic spectral gap. In [18], it has recently been shown that the asymptotic gap between Virasoro primaries are bounded above by \( 2\sqrt{\frac{3}{\pi^2}} \simeq 1.1 \) and it has been conjectured that the optimal gap should be 1. The example of Monster CFT tells us that the gap can not be below than 1, hence 1 should be the optimal number. In this section, we show that the previous bound \( 2\sqrt{\frac{3}{\pi^2}} \) can be improved and made arbitrarily closer to the optimal value 1. Ideally, to prove this one should find out a function \( f \) (which will eventually

\(^6\)We assume that the mesh size for \( c_\pm - 1 \) is small enough that one can safely find out a lower bound.
| $\delta$ | Number of iterations | # points | Max($c_+$) | Lower Bound |
|---|---|---|---|---|
| 0.4 | 10000 | 300 | 2.2 | 1.7042 |
| 0.5 | 1000 | 300 | 2.02 | 1.6905 |
| 0.5 | 10000 | 200 | 2.02 | 1.7002 |
| 0.5 | 10000 | 300 | 2.02 | 1.7179 |
| 0.6 | 1000 | 200 | 2.02 | 1.6086 |
| 0.6 | 10000 | 200 | 2.02 | 1.5917 |
| 0.7 | 10000 | 200 | 2.02 | 1.4246 |
| 0.7 | 10000 | 250 | 2.02 | 1.4270 |
| 0.8 | 10000 | 200 | 1.757 | 1.3692 |
| 0.8 | 10000 | 200 | 2.757 | 1.3698 |
| 0.9 | 10000 | 200 | 2.757 | 1.3798 |
| 1 | 20000 | 200 | 1.757 | 1.3759 |
| 1.1 | 10000 | 200 | 2.757 | 1.3331 |
| 1.20 | 10000 | 150 | 2.757 | 1.2597 |
| 1.25 | 10000 | 150 | 2.757 | 1.2581 |
| 1.3 | 10000 | 170 | 2.757 | 1.2531 |
| 1.4 | 10000 | 150 | 2.757 | 1.2581 |
| 1.5 | 10000 | 150 | 1.757 | 1.2599 |
| 1.5 | 10000 | 150 | 2.757 | 1.2597 |
| 1.5 | 10000 | 150 | 2.757 | 1.2597 |
| 1.6 | 10000 | 150 | 1.757 | 1.2313 |
| 1.7 | 10000 | 150 | 1.757 | 1.1933 |

Table 1: Typical output from a run yielding lower bounds for the upper bound $c_+$. The Max($c_+$) column contains a number that is greater than or equal to what can already be achieved.

play the role of $\phi_-$ in this game, to be precise $f(\Delta') = \phi_-(\Delta + \Delta')$ such that following holds:

$$f(\Delta') \leq \Theta \left( \Delta' \in \left[ -\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] \right)$$

and

$$\tilde{f}(t) = 0 \text{ for } |t| \geq \frac{2\pi}{\epsilon}, \ \epsilon > 1$$

$$\tilde{f}(0) > 0$$

This would have implied

$$\int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho(\Delta') > 0$$
| $\delta$ | Iteration Number | # points | Min($c_-$) | Upper Bound |
|----------|------------------|----------|------------|-------------|
| 0.6      | 1000             | 200      | 0.173      | 0.5738      |
| 0.6      | 10000            | 200      | 0.173      | 0.5535      |
| 0.7      | 10000            | 200      | 0.362      | 0.5604      |
| 0.7      | 10000            | 250      | 0.362      | 0.5599      |
| 0.8      | 10000            | 200      | 0.44       | 0.5567      |
| 0.9      | 10000            | 200      | 0.46       | 0.5853      |
| 1        | 10000            | 200      | 0.48       | 0.6060      |
| 1.1      | 10000            | 200      | 0.49       | 0.7112      |
| 1.2      | 10000            | 150      | 0.49       | 0.7161      |
| 1.2      | 10000            | 180      | 0.49       | 0.7161      |
| 1.3      | 10000            | 170      | 0.49       | 0.7111      |
| 1.4      | 10000            | 150      | 0.49       | 0.7243      |
| 1.5      | 10000            | 150      | 0.49       | 0.7788      |
| 1.6      | 20000            | 150      | 0.49       | 0.7895      |
| 1.7      | 20000            | 150      | 0.49       | 0.7861      |

**Table 2**: Typical output from a run providing upper bound for the lower bound $c_-$. The Min($c_-$) column contains a number that is smaller than or equal to what can already be achieved.

Now what would happen if $\tilde{f}(0) = 0$? One need to go back to the original derivation and reconsider it carefully. Hence instead of the eq. (2.2), we consider a more basic inequality\[^{18}\]:

$$
\exp \left[ \beta (\Delta - \delta) \right] \int d\Delta' \rho_0(\Delta') e^{-\beta \Delta'} \phi_-(\Delta') - Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_2^2} \right) e^{-\beta \Lambda_2} \int_{-\Lambda_2}^{\Lambda_2} dt |\dot{\phi}(t)| 
\leq \int_{\Delta_0 - \delta}^{\Delta + \delta} d\Delta' \rho(\Delta')
$$

(4.5)

where $\Lambda_2 = \frac{2\pi}{\epsilon}$ and $Z_H(\beta)$ is the contribution from the heavy states and defined as

$$
Z_H(\beta) = \sum_{\Delta > \Delta_H > \frac{\delta}{12}} e^{-\beta \left( \Delta - \frac{\delta}{12} \right)}.
$$

(4.6)

Now we make the following choice for $\phi_-$:

$$
\phi_-(\Delta') = \frac{\cos^2 \left( \frac{\pi (\Delta' - \Delta)}{\epsilon} \right)}{1 - 4 \left( \frac{\Delta' - \Delta}{\epsilon} \right)^2}, \quad f(\Delta') = \frac{\cos^2 \left( \frac{\pi \Delta'}{\epsilon} \right)}{1 - 4 \left( \frac{\Delta'}{\epsilon} \right)^2}
$$

(4.7)
This function $f$ has following properties:

\begin{align}
    f(\Delta') &\leq \Theta \left( \Delta' \in \left[ -\frac{\epsilon}{2}, \frac{\epsilon}{2} \right] \right) \\
    \tilde{f}(t) &= 0 \quad \text{for} \quad |t| \geq \frac{2\pi}{\epsilon} \\
    \tilde{f}(0) &= 0 \quad \Rightarrow \quad c_- = 0
\end{align}

Since $c_- = 0$, one can not readily evaluate the integral appearing in (4.5) by saddle point method and deduce $\exp [\beta(\Delta - \delta)] \int d\Delta' \rho_0(\Delta') e^{-\beta \Delta' \phi_- (\Delta')} = c_- \rho_0(\Delta)$, so we look for subleading corrections to the saddle point approximation. We find that the leading behavior is given by, after setting $\beta = \pi \sqrt{c_3 \Delta}$,

\begin{equation}
    \exp [\beta(\Delta - \delta)] \int d\Delta' \rho_0(\Delta') e^{-\beta \Delta' \phi_- (\Delta')} = C \rho_0(\Delta),
\end{equation}

where $C$ turns out to be

\begin{equation}
    C = \int_0^\infty dx \left( \frac{\cos^2 \left( \frac{\pi x}{\epsilon} \right)}{1 - 4x^2} \right) \exp \left[ -\frac{x^2}{2\pi \sqrt{\frac{c_3}{\epsilon}} \Delta} \right].
\end{equation}

We remark that $C > 0$ for any finite $\Delta$ and it becomes 0 only at infinitely large $\Delta$. The second piece in the eq. (4.5) for large $\Delta$ goes as $\rho_0(\Delta)^{1-\frac{1}{2}(1-\frac{1}{\epsilon})}$. The analysis for this second term is exactly same as done in [18]. For sufficiently large $\Delta$, it can be numerically verified that $\rho_0(\Delta)^{1-\frac{1}{2}(1-\frac{1}{\epsilon})}$ is subleading compared to $C \rho_0(\Delta)$ as long as $\epsilon > 1$ (we also provide an analytical proof later on). Here we have

\begin{equation}
    \rho_0(\Delta) = \Delta \to \infty \left( \frac{c}{48\Delta^3} \right)^{\frac{1}{2}} \exp \left[ 2\pi \sqrt{\frac{c_3}{\frac{3}{\epsilon}}} \right].
\end{equation}

In fact one can analytically show that $\rho_0(\Delta)^{1-\frac{1}{2}(1-\frac{1}{\epsilon})}$ is subleading to $C \rho_0(\Delta)$ for large $\Delta$. One way to show this is to have an estimate for $C$. We start with the observation that the integrand is positive in $\left( 0, \frac{\epsilon}{2} \right)$ and negative in $\left( \frac{\epsilon}{2}, \infty \right)$. Furthermore, we have

\begin{equation}
    \int_0^\infty d\Delta' f(\Delta') = 0
\end{equation}

Using the above facts, one can always choose $0 < \epsilon_1 < \frac{\epsilon}{2}$ and $\frac{\epsilon}{2} < \epsilon_2 < \infty$ such that

\begin{align}
    \int_0^{\epsilon_1} d\Delta' f(\Delta') &= -\int_{\epsilon_2}^{\infty} d\Delta' f(\Delta') \\
    \int_{\epsilon_1}^{\epsilon_2} d\Delta' f(\Delta') &= 0
\end{align}
This is basically guaranteed by the continuity. We choose $\epsilon_1$ such that $0 < \epsilon_1 < \frac{\epsilon}{2}$ and consider the function $F(y) = \int_{\epsilon_1}^{y} dx \ f(x)$. Now $F(y)$ is a continuous function. It is positive when $y = \frac{\epsilon}{2}$ and negative when $y \to \infty$. Thus by continuity, there exists $\frac{\epsilon}{2} < \epsilon_2 < \infty$ such that the eq. (4.15) holds. The shaded region in the figure. 5 is the area under the function $f$ restricted to the interval $[\epsilon_1, \epsilon_2]$ so that the eq. (4.15) is satisfied.

Figure 5: The function \( \left( \frac{\cos^2(\pi \frac{x}{\epsilon})}{1 - 4 \frac{x^2}{\epsilon^2}} \right) \), the shaded region is the area under the function restricted to the interval $[\epsilon_1, \epsilon_2]$. Here $\epsilon_1 = 0.25, \epsilon = 1.01, \epsilon_2 = 0.819$. These are chosen to ensure the shaded area is 0.

Now we note that

$$\int_{\epsilon_1}^{\epsilon_2} dx \ f(x) \exp \left[ \frac{-x^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] \geq 0 \quad \text{(4.17)}$$

and

$$\int_{0}^{\epsilon_1} dx \ f(x) \exp \left[ \frac{-x^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] \geq \exp \left[ \frac{-\epsilon_1^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] \int_{0}^{\epsilon_1} dx \ f(x) \quad \text{(4.18)}$$

$$\int_{\epsilon_2}^{\infty} dx \ f(x) \exp \left[ \frac{-x^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] \geq \exp \left[ \frac{-\epsilon_2^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] \int_{\epsilon_2}^{\infty} dx \ f(x) \quad \text{(4.19)}$$

where in the second inequality, we have used negativity of $f(x)$ for $x > \frac{\epsilon}{2}$. Combining the last four equations i.e (4.15),(4.17),(4.18),(4.19) we can write

$$C \geq \Omega \left( \exp \left[ \frac{-\epsilon_1^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] - \exp \left[ \frac{-\epsilon_2^2}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} \right] \right) \sim_{\Delta \to \infty} \frac{(\epsilon_2^2 - \epsilon_1^2) \Omega}{2\pi \sqrt{\frac{\pi}{3}} \Delta^{\frac{3}{2}}} > 0 \quad \text{(4.20)}$$

where $\Omega = \int_{0}^{\epsilon_1} dx \ f(x) > 0$ is an order one positive number. This clearly proves that
as long as $\epsilon > 1$, we can neglect the second piece i.e. contributions from the heavy states due to its subleading nature. In fact, one can do much better and show that $C$ falls like $\Delta^{-3/4}$ by noting the following:

$$C = \frac{\epsilon \pi}{8} \exp \left[ -\frac{1}{8\pi \sqrt{\frac{2}{3}} \Delta^{3/2}} \right] \operatorname{Erfi} \left( \frac{1}{2\sqrt{2\pi \sqrt{\frac{2}{3}} \Delta^{3/4}}} \right) \right.$$

$$- \frac{\epsilon \pi}{8} e^{-\frac{\sqrt{\pi}}{8 + \sqrt{\pi} \Delta^{3/2}} \text{Im} \left[ \operatorname{Erf} \left( \frac{\sqrt{\pi}}{2} \left( 2\pi + \frac{i \sqrt{3}}{2\pi \sqrt{\pi} \Delta^{3/2}} \right) \right) \right]}$$

$$\approx \frac{\epsilon}{8} \left( \frac{3}{64c} \right)^{1/4} \Delta^{-3/4}.$$  

(4.21)

To summarize, we have proved that for sufficiently large $\Delta$,

$$\int_{\Delta - \frac{\epsilon}{4}}^{\Delta + \frac{\epsilon}{4}} d\Delta' \rho(\Delta') \geq C \rho_0(\Delta) > 0$$  

(4.22)

Therefore we have been able to show that the asymptotic gap between two consecutive operators is bounded above by $\epsilon$, where $\epsilon > 1$. Now one can choose $\epsilon$ to be arbitrarily close to 1, which proves that the optimal bound is exactly 1. The analysis can be carried over to the case for Virasoro primaries, as pointed out in [18]. This implies that the asymptotic gap between two consecutive Virasoro primaries is bounded above by 1, thereby proves the conjecture made in [18].

5 Brief discussion

In this work, we have improved the existing bound on the $O(1)$ correction to the density of states in 2D CFT at high energy and proven the conjectured upper bound on the gap between Virasoro primaries. In particular, we have shown that there always exists a Virasoro primary in the energy window of width greater than 1 at large $\Delta$.

We have provided a systematic way to estimate how tight the bound can be made using bandlimited functions. Since there is still a gap between the achievable bound and the bound on the bound, there is scope for further improvement. Ideally, one would like to close this gap, which might be possible either by sampling more points and leveraging the positive definiteness condition on a bigger matrix, or by choosing some suitable function which would make the achievable bound closer to the bound on the bound. Another possible way to obtain the bound on bound is to use a known 2D CFT partition functions, for example 2D Ising model and explicitly

---

\(^7\)We thank Alexander Zhiboedov for pointing this out in an email exchange.
evaluate $s(\delta, \Delta)$. It would be interesting to see how the bound on bound obtained in this paper compares to the one which can be obtained from the 2D Ising model. For example, one can verify that the bound on bound obtained here is stronger than that could be obtained from 2D Ising model\(^8\) for $\delta = 1$. It would be interesting to further explore this.

The utility of the technique developed here lies beyond the $O(1)$ correction to the Cardy formula. We expect the technique to be useful whenever one wants to leverage the complex Tauberian theorems. As emphasized in [18], the importance of Tauberian theorems lies beyond the discussion of 2D CFT partition functions, especially in investigating Eigenstate Thermalization Hypothesis [21–24] in 2D CFTs[25–35]. We end with a cautious remark that if we relax the condition of using bandlimited functions, the bound on bounds would not be applicable and it might be possible to obtain nicer achievable bounds on the $O(1)$ correction to the Cardy formula.

\section*{Acknowledgements}

The authors thank Denny’s for staying open throughout the night. The authors acknowledge helpful comments and suggestions from Tom Hartman, Baur Mukhametzhanov, and especially, Alexander Zhiboedov. The authors thank Shouvik Datta for encouragement. SG wishes to acknowledge Pinar Sen for some fruitful and illuminating discussions on what makes the Fourier transform of a function positive, which helped him arrive at the answer by thinking about autocorrelation functions and power spectral densities. SP acknowledges a debt of gratitude towards Ken Intriligator and John McGreevy for fruitful discussions and encouragement. SP thanks Shouvik Datta and Diptarka Das for introducing him to the rich literature of CFT in 2017. This work was in part supported by the US Department of Energy (DOE) under cooperative research agreement DE-SC0009919. SP also acknowledges the support from Inamori Fellowship.

\section*{References}

[1] J. L. Cardy, \textit{Operator content and modular properties of higher dimensional conformal field theories}, Nucl. Phys. B366 (1991) 403–419.

[2] W. Nahm, \textit{A semiclassical calculation of the mass spectrum of relativistic strings}, Nuclear Physics B 81 (Oct., 1974) 164–178.

[3] J. L. Cardy, \textit{Operator content of two-dimensional conformally invariant theories}, Nuclear Physics B 270 (1986) 186–204.

\(^8\)We thank Alexander Zhiboedov for raising this question of how our bound compares to $s(1.7, \Delta)$ for the 2D Ising model, as found in [18].
[4] P. Kraus and A. Maloney, *A Cardy formula for three-point coefficients or how the black hole got its spots*, JHEP 05 (2017) 160, arXiv:1608.03284 [hep-th].

[5] D. Das, S. Datta, and S. Pal, *Charged structure constants from modularity*, JHEP 11 (2017) 183, arXiv:1706.04612 [hep-th].

[6] S. Hellerman, *A universal inequality for CFT and quantum gravity*, JHEP 08 (2011) 130, arXiv:0902.2790 [hep-th].

[7] T. Hartman, C. A. Keller, and B. Stoica, *Universal spectrum of 2d conformal field theory in the large c limit*, JHEP 09 (2014) 118, arXiv:1405.5137 [hep-th].

[8] E. Dyer, A. L. Fitzpatrick, and Y. Xin, *Constraints on flavored 2d cft partition functions*, JHEP 02 (2018) 148, arXiv:1709.01533 [hep-th].

[9] D. Das, S. Datta, and S. Pal, *Universal asymptotics of three-point coefficients from elliptic representation of Virasoro blocks*, Phys. Rev. D98 no. 10, (2018) 101901, arXiv:1712.01842 [hep-th].

[10] S. Collier, Y.-H. Lin, and X. Yin, *Modular bootstrap revisited*, JHEP 09 (2018) 061, arXiv:1608.06241 [hep-th].

[11] S. Collier, Y. Gobeil, H. Maxfield, and E. Perlmutter, *Quantum Regge trajectories and the virasoro analytic bootstrap*, arXiv:1811.05710 [hep-th].

[12] M. Cho, S. Collier, and X. Yin, *Genus two modular bootstrap*, JHEP 04 (2019) 022, arXiv:1705.05865 [hep-th].

[13] J.-B. Bae, S. Lee, and J. Song, *Modular constraints on conformal field theories with currents*, JHEP 12 (2017) 045, arXiv:1708.08815 [hep-th].

[14] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, *Operator product expansion convergence in conformal field theory*, Physical Review D 86 no. 10, (2012) 105043.

[15] A. Ingham, *A Tauberian theorem for partitions*, Annals of Mathematics (1941) 1075–1090.

[16] B. Mukhametzhanov and A. Zhiboedov, *Analytic Euclidean bootstrap*, arXiv preprint arXiv:1808.03212 (2018).

[17] M. A. Subhankulov, *Tauberian theorems with remainder*, American Math. Soc. Providence, RI, Transl. Series 2 (1976) 311–338.

[18] B. Mukhametzhanov and A. Zhiboedov, *Modular Invariance, Tauberian Theorems, and Microcanonical Entropy*, arXiv:1904.06359 [hep-th].

[19] H. Cohn and N. Elkies, *New upper bounds on sphere packings I*, Annals of Mathematics (2003) 689–714.

[20] T. Hartman, D. Mazac, and L. Rastelli, *Sphere packing and quantum gravity*, arXiv:1905.01319 [hep-th].

[21] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, Physical Review A 43 no. 4, (1991) 2046.
[22] M. Srednicki, *Chaos and quantum thermalization*, Physical Review E 50 no. 2, (1994) 888.

[23] J. R. Garrison and T. Grover, *Does a single eigenstate encode the full hamiltonian?*, Phys. Rev. X 8 (Apr, 2018) 021026. https://link.aps.org/doi/10.1103/PhysRevX.8.021026.

[24] M. Rigol, V. Dunjko, and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, Nature 452 no. 7189, (2008) 854.

[25] N. Lashkari, A. Dymarsky, and H. Liu, *Eigenstate Thermalization Hypothesis in conformal field theory*, J. Stat. Mech. 1803 no. 3, (2018) 033101, arXiv:1610.00302 [hep-th].

[26] E. M. Brehm and D. Das, *On KdV characters in large c CFTs*, arXiv:1901.10354 [hep-th].

[27] E. M. Brehm, D. Das, and S. Datta, *Probing thermality beyond the diagonal*, Phys. Rev. D98 no. 12, (2018) 126015, arXiv:1804.07924 [hep-th].

[28] P. Basu, D. Das, S. Datta, and S. Pal, *Thermality of eigenstates in conformal field theories*, Phys. Rev. E96 no. 2, (2017) 022149, arXiv:1705.03001 [hep-th].

[29] A. Maloney, G. S. Ng, S. F. Ross, and I. Tsiaras, *Generalized Gibbs ensemble and the statistics of KdV charges in 2d CFT*, JHEP 03 (2019) 075, arXiv:1810.11054 [hep-th].

[30] A. Maloney, G. S. Ng, S. F. Ross, and I. Tsiaras, *Thermal correlation functions of KdV charges in 2d CFT*, JHEP 02 (2019) 044, arXiv:1810.11053 [hep-th].

[31] A. Dymarsky and K. Pavlenko, *Exact generalized partition function of 2D CFTs at large central charge*, JHEP 05 (2019) 077, arXiv:1812.05108 [hep-th].

[32] A. Dymarsky and K. Pavlenko, *Generalized Eigenstate Thermalization in 2d CFTs*, arXiv:1903.03559 [hep-th].

[33] S. Datta, P. Kraus, and B. Michel, *Typicality and thermality in 2d CFT*, arXiv:1904.00668 [hep-th].

[34] Y. Kusuki and M. Miyaji, *Entanglement Entropy, OTOC and bootstrap in 2d CFTs from Regge and light cone limits of multi-point conformal block*, arXiv:1905.02191 [hep-th].

[35] Y. Hikida, Y. Kusuki, and T. Takayanagi, *Eigenstate thermalization hypothesis and modular invariance of two-dimensional conformal field theories*, Phys. Rev. D98 no. 2, (2018) 026003, arXiv:1804.09658 [hep-th].