Scaling Limit and Renormalisation Group in the Critical Point Analysis of General (Quantum) Many Body Systems

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Abstract

We employ the machinery of smooth scaling and coarse-graining of observables, developed recently by us in the context of so-called fluctuation operators (inspired by prior work of Verbeure et al) to make a rigorous renormalisation group analysis of the critical regime. The approach appears to be quite general, encompassing classical, quantum, discrete and continuous systems. One of our central topics is the analysis of the famous ‘scaling hypothesis’, that is, we make a general investigation under what conditions on the l-point correlation functions a scale invariant (non-trivial) limit theory can be actually attained. Furthermore, we study in a rigorous manner questions like the quantum character of the system in the scaling limit, the phenomenon of critical slowing down etc.
1 Introduction

One of the central ideas of the renormalization group analysis of, say, the critical regime, is *scale invariance* of the system in the *scaling limit*. This is the famous *scaling hypothesis* (as to the underlying working philosophy compare any good text book of the subject matter like e.g. [1] and references therein). Central in this approach is the so-called *blockspin transformation*, [2]. That is, observables are averaged and appropriately renormalized over blocks of increasing size. At each intermediate scale a new *effective theory* is constructed and the art consists of choosing (or rather: calculating) the *critical scaling exponents*, so that the sequence of effective theories converge to a (scale invariant) limit theory, provided that the start theory lay on the *critical submanifold* in the (in general infinite dimensional) parameter space of theories or Hamiltonians.

Usually the calculations can only be performed in an approximative way, the main tools being of a perturbative character and being typically model dependent. Frequently, the more general discussion concentrates on spin systems to motivate and explain the calculational steps. While the general working philosophy, based on the concepts of *asymptotic scale invariance*, *correlation length* and the like, is the result of a deep physical analysis of the phenomena, there is, on the other side, no abundance of both rigorous and model independent results.

This applies in particular to the control of the convergence of the scaled $l$-point correlation functions to their respective limits if we start from a microscopic theory, lying on the *critical submanifold*. In this case, correlations are typically long-ranged and the usual heuristic arguments about the interplay between poor clustering, on the one side, and formation of *block variables* of increasing size, on the other side, become rather obscure as one is usually cavalier as to the interchange of various limit procedures. One knows from examples, that this may be a dangerous attitude in such a context.

Furthermore, the clustering of the higher correlation functions in the various channels of phase space may be quite complex and non-uniform in general. A concise and selfcontained discussion of the more general aspects and problems, lurking in the background together with a useful series of notes and references, can be found in [3], section 7.

Usually, the crucial scaling relation (the *scaling hypothesis*)

$$W_l^T(Lx_1, \ldots, Lx_l; \mu^*) = L^{-l n} \cdot L^{l \gamma} \cdot W_l^T(x_1, \ldots, x_l; \mu^*)$$

(1)

which is conjectured to hold at the fixed point (denoted by $\mu^*$ in the parameter space), is the starting point (or physical input) of the analysis. Here, $W_l^T$ denote the truncated $l$-point functions (see below), $L$ is the diameter of the blocks, $Lx_i$ are the respective centers of the blocks, $n$ is the space dimension, $\gamma$ the statistical renormalisation exponent. If it is different from $n/2$, we have an ‘anomalous’ scale dimension.
In the following analysis, one of our aims is a rigorous investigation of such (and similar) scaling relations for the $l$-point functions, starting from the underlying microscopic characteristics of the theory. We will do this in a quite general manner, that is, the underlying model theory can be classical or quantum, discrete or continuous. We try to make only very few and transparent assumptions. Our strategy is it, to deal only with the really characteristic (almost model independent) aspects of the subject matter. Another goal is it, to derive properties of both the intermediate and limit states, observables, dynamics etc., with particular emphasis on the quantum aspects. As a perhaps particularly interesting result we mention a rigorous discussion of the phenomenon called critical slowing down.

What regards the general working philosophy, one should perhaps mention the framework, expounded in e.g. [4] in the context of the analysis of the ultraviolet behavior in algebraic quantum field theory, or, in the classical regime, the approach of e.g. Sinai ([20]). While our framework also comprises the classical regime, it is mainly designed to deal with the quantum case. In so far, it is an extension of the methods, developed by us in [8], which, on their side, have been inspired by prior work of Verbeure et al; see the corresponding references in [8]. Recently we became aware of a nice treatment of the block spin approach in the quantum regime in the book of Sewell ([14]), who employs methods which are different from ours, but are complementing them (quantum (non-) central limits).

The technical analysis of the convergence behavior of the $l$-point correlation functions is mainly contained in sections 4 and 5 of the present paper, which represented the core of a previous preprint version. The aim of this technical analysis is to isolate the critical assumptions, which have to be made, in order that the physical picture comes out correctly. Put differently, we show that the general scaling picture is by no means an automatic consequence of a few general physical assumptions but depends on a number of critical details of the behavior of correlation functions.

We have now added section 3, which contains a discussion of a variety of general concepts and features being of relevance in the renormalisation process. We mention e.g. some subtleties concerning the scale invariance of the limit theory, the (re)construction of the theory and its dynamics from the correlation functions (which is not entirely trivial, as the underlying observable algebras are constantly changing under renormalisation; cf. also [4]), the emerging (non)-quantum character of (parts of) the limit theory, the phenomenon of critical slowing down, which is derived in our setting from the KMS-property of the limit state.

We want to remark that there exists a superficially different approach, which is more related to the well established concepts of renormalisation theory in quantum field theory (see, for example, [4], [6] or [7]), that is, renormalising propagators, Green’s functions and path integrals and in which scale invariance
is present on a more implicit level.

Due to lack of space, we do not intend (and actually feel unable) to relate our approach to the more perturbation theoretically oriented approaches mentioned above. We think, these different aspects are complementing each other.

We concentrate our analysis entirely on the hierarchy of correlation functions which can be used to define the theory. We generate renormalized limit correlation functions from them which happen to be scale invariant (in a sense clarified below), thus defining a new limit theory via a reconstruction process. The nature of this limit depends on the degree of clustering of the original microscopic correlation functions. We do not openly discuss the flow of, say, the renormalized Hamiltonians through parameter space as a sequence of more and more coarse-grained effective Hamiltonians. The characteristics of these renormalised intermediate theories are however implicitly given by their hierarchy of correlation functions as was already explained in e.g. \[8\] or \[4\].

One should therefore emphasize, that this well-known integrating out or decimation of degrees of freedom, which characterizes the ordinary approaches is automatically contained in our approach! The effective time evolution is carried over from the microscopic theory as described in \[8\] or (in a slightly other context) in \[10\], see also \[12\] and is redefined on each intermediate scale, thus implying automatically a rescaling of both the time evolution and the corresponding Hamiltonian; see section 3. In case we work in an scenario, defined by ordinary Gibbs states, our framework would exactly yield these effective Hamiltonians.

2 The Conceptual Framework

2.1 Concepts and Tools

As to the general framework we refer the reader to \[8\]. One of our technical tools is a modified (smoothed) version of averaging (modifications of the ordinary averaging procedure are also briefly mentioned in the notes in \[8\]). Instead of averaging over blocks with a sharp cut off, we employ a smoothed averaging with smooth, positive functions of the type

$$f_R(x) := f(|x|/R) \quad \text{with} \quad f(s) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

(2)

Remark: We will see in the following, that the final result is more or less independent of the particular class of averaging functions!

We note that this class of scaled functions has a much nicer behavior under Fourier transformation, as, for example, functions with a sharp cut off, the main reason being that the tails are now also scaled. We have

$$\hat{f}_R(k) = \text{const} \cdot R^n \cdot \hat{f}(R \cdot k)$$

(3)
Remark: One might perhaps think that this choice of averaging will lead to a different limit theory. This is however not the case. Furthermore, the mathematical differences between the two approaches, that is, using sharp or smooth and scaled cut off functions, are relatively subtle and not so apparent. We are investigating these aspects in [9].

Another point, worth mentioning, are the implications of translation invariance. We have for the correlation functions

$$W(x_1, \ldots, x_n) = W(x_1 - x_2, \ldots, x_{n-1} - x_n)$$ (4)

The truncated correlation functions are defined inductively as follows (see [8])

$$W(x_1, \ldots, x_n) = \sum_{\text{part}} \prod W_T(x_{i_1}, \ldots, x_{i_k})$$ (5)

The (distributional) Fourier transform reads

$$\hat{W}_T(p_1, \ldots, p_l) = \hat{W}_T(p_1, p_1 + p_2, \ldots, p_1 + \cdots p_{l-1}) \cdot \delta(p_1 + \cdots p_l)$$ (6)

The dual sets of variables are

$$y_i := x_i - x_{i+1}, q_i = \sum_{j=1}^{i} p_j \quad i \leq (l - 1)$$ (7)

2.2 The case of Normal Fluctuations

As in [8], we assume that away from the critical point the truncated l-point functions are integrable, i.e. \( \in L^1(R^{n(l-1)}) \), in the difference variables, \( y_i := x_i - x_{i+1} \). As observables we choose the translates

$$A_R(a_1), \ldots, A_R(a_l), \quad A_R(a) := R^{-n/2} \cdot \int A(x + a)f(x/R)d^n x$$ (8)

(where, for convenience, the labels 1 \ldots l denote also possibly different observables). We then get (for the calculational details see [8], the hat denotes Fourier transform, translation invariance is assumed throughout, the const may change during the calculation but contains only uninteresting numerical factors):

$$\langle A_R(a_1) \cdots A_R(a_l) \rangle_T = \text{const} \cdot R^{n/2} \cdot \int \hat{f}(Rp_1) \cdots \hat{f}(-R[p_1 + \cdots + p_{l-1}]) \cdot \hat{W}_T(p_1, \ldots, p_{l-1}) \cdot e^{-i \sum_{1}^{l-1} p_i a_i} \cdot e^{i a_l \sum_{1}^{l-1} p_i} \prod dp_i$$

$$= \text{const} \cdot R^{n/2} \cdot R^{-(l-1)n} \cdot \int \hat{f}(p'_1) \cdots \hat{f}(-[p'_1 + \cdots + p'_{l-1}]) \cdot \hat{W}_T(p'_1/R, \ldots, p'_{l-1}/R) \cdot e^{-i \sum_{1}^{l-1} (p'_i/R) a_i} \cdot e^{i a_l \sum_{1}^{l-1} p'_i/R} \prod dp'_i$$ (9)
We now scale the $a_i$’s like

$$a_i := R \cdot X_i, \; X_i \text{ fixed}$$

(10)

This yields

$$\langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T =$$

$$\text{const} \cdot R^{(2-l)n/2} \cdot \int e^{-i \sum_{i=1}^{l-1} p'_i \cdot X_i} \cdot e^{i X_l \sum_{i=1}^{l-1} p'_i} \cdot$$

$$\hat{f}(p'_1) \cdots \hat{f}(-[p'_1 + \cdots + p'_{l-1}]) \cdot \hat{W}^T(p'_1/R, \ldots, p'_{l-1}/R) \prod dp'_i$$

(11)

As the $\hat{f}$ are of strong decrease and $\hat{W}^T$ continuous and bounded by assumption, we can perform the limit $R \to \infty$ under the integral and get:

**Case 1** ($l \geq 3$):

$$\lim_{R \to \infty} \langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = 0$$

(12)

**Case 2** ($l = 2$):

$$\lim_{R \to \infty} \langle A_R(R \cdot X_1) A_R(R \cdot X_2) \rangle^T = \text{const} \cdot \int \hat{W}^T(0) \cdot e^{-ip'_1(X_1-X_2)} \cdot \hat{f}(p'_1) \cdot \hat{f}(-p'_1) dp'_1$$

(13)

**Conclusion 2.1** In the normal regime, away from the critical point, where we assumed $L^1$-clustering, all the truncated correlation functions vanish in the limit $R \to \infty$ apart from the 2-point function. We hence have a quasi free theory in the limit as described in [8] or in the work of Verbeure et al (cf. the references in [10])

### 2.3 The Relation to the Heuristic Scaling Hypothesis

In the following sections we develop a rigorous approach to block-spin renormalisation in the realm of quantum statistical mechanics, which tries to implement the physically well-motivated but, nevertheless, to some extent heuristic scaling hypothesis. The analysis will be performed both in coordinate space and Fourier space. In this subsection we restrict our discussion to the two-point correlation function, for which the asymptotic behavior is simpler and more transparent.

Remark: In the rest of the paper we replace the exponent $n/2$ in the definition of $A_R(a)$ by a scaling exponent $\gamma'$, which will usually be fixed during or at the end of a calculation. It plays the role of a critical scaling exponent.
Let us hence study the behavior of

\[
\langle A_R(R \cdot X_1)A_R(R \cdot X_2) \rangle^T = R^{-2\gamma'} \cdot \int W^T((x_1 - x_2) + R(X_1 - X_2)) \cdot f(x_1/R)f(x_2/R)dx_1dx_2
\]

\[
= R^{-2\gamma'+2n} \int W^T(R[(x_1 - x_2) + (X_1 - X_2)]) \cdot f(x_1)f(x_2)dx_1dx_2 \quad (14)
\]

We make the physically well motivated assumption that, in the critical regime, \( W^T \) decays asymptotically like some inverse power, i.e.

\[
W^T(x_1 - x_2) \sim (\text{const} + F(x_1 - x_2)) \cdot |x_1 - x_2|^{-(n-\alpha)} \quad 0 < \alpha < n \; , \; F(x) \in L^1
\]

for \( |x_1 - x_2| \to \infty \), \( F \) bounded and well-behaved.

From the last line of (14) we see that, as \( f \) has compact support, we can replace \( W^T \), for \( (X_1 - X_2) \neq 0 \) and \( R \to \infty \) by its asymptotic expression and get for \( R \) large:

\[
\langle A_R(R \cdot X_1)A_R(R \cdot X_2) \rangle^T \approx \text{const} \cdot R^{-2\gamma'+2n} \cdot R^{-(n-\alpha)} \cdot \int |y + Y|^{-(n-\alpha)} \cdot f * f(y)dy
\]

We choose now

\[
\gamma' = (n + \alpha)/2
\]

and get a limiting behavior (for \( R \to \infty \)) as

\[
\text{const} \cdot \int |y + Y|^{-(n-\alpha)} \cdot f * f(y)dy
\]

with \( y = x_1 - x_2, Y = Y_1 - Y_2 \).

We see that in contrast to the general folklore, the limit correlation functions are not automatically strictly scale invariant but depend in a weak sense on the chosen smearing functions, \( f \). This phenomenon will be discussed in more detail below as it exhibits a quite interesting and a little bit hidden aspect. Central in the renormalisation group idea is that systems on the critical surface (i.e., critical systems) are driven towards a fixed point, representing a scale invariant theory. This idea is usually formulated in an abstract parameter space of, say, Hamiltonians. In our correlation function approach the fixed point shows its existence via the scaling properties of the correlation functions, that is

\[
W_2^T(L \cdot (X - Y); \mu^*) = L^{-2(n-\gamma')}W_2^T(X - Y; \mu^*)
\]

(19)
with $\mu^*$ describing the fixed point in the (usually) infinite dimensional parameter space. We see from the above that this is asymptotically implemented by our limiting correlation functions, as we have (with the choice $\gamma = (n + \alpha)/2$):

$$W_2^T(X - Y; \mu^*) \sim |X - Y|^{-(n-\alpha)}$$

(20)

in the asymptotic regime. That is, the above scaling limit leads to a limit (i.e. fixed point) theory, reproducing the asymptotic behavior of the original (microscopic) theory.

One should however note that in the more general situation of $l$-point correlation functions we have to expect a more complex decay behavior and the existence of various channels as varying clusters of observables move to infinity. These more intricate technical aspects will be discussed in the second part of the paper. We continue with a discussion of a bundle of general properties of the intermediate and scaling limit systems.

### 3 Rigorous Results on the (Quantum) System in the Intermediate Regime and in the Scaling Limit

In this section we assume that the theory exists in the scaling limit provided that the scaling exponents have been appropriately chosen. Under this proviso we investigate its algebraic and dynamical limit structure.

#### 3.1 The Description of the System at Varying Scales

In algebraic statistical mechanics we describe a system with the help of an observable algebra, $\mathcal{A}$, a state, $\omega$, or expectation functional, $\langle \circ \rangle$, a time evolution, $\alpha_t$. Frequently one also employs the GNS-Hilbert space representation of the theory, introduced by Gelfand, Naimark, Segal (see e.g. [13]). We already gave a brief discussion of these points in [8]. But as the approach of the scaling limit is quite subtle both physically and mathematically, we would like to give a more complete discussion of some of the topics in the following.

We begin with fixing the notation and introducing some technical and conceptual tools. Expectations on the underlying observable algebra, $\mathcal{A}$, at scale “0”, are given by

$$\omega(A(1) \cdots A(l)) = \langle A(1) \cdots A(l) \rangle$$

(21)

where, for convenience, different indices may denote different elements, different times etc. The dynamics is denoted by

$$\alpha_t(A) = A(t) \text{ or } A_t, \ t \in \mathbb{R}$$

(22)
space translations by
\[ \alpha_x(A) = A(x) \text{ or } A_x, \quad x \in \mathbb{R}^n \] (23)

\[ \alpha_{t,x}(A) = A(t, x) \] (24)

Given such a structure, we can construct a corresponding Hilbert space representation (for convenience, we use the same symbols for the algebraic elements).

\[ \omega \to \Omega, \quad \omega(A(1) \cdots A(l)) = (\Omega|A(1) \cdots A(l)\Omega)_{\text{GNS}} \] (25)

\[ \alpha_t \to U_t, \text{ with } \alpha_t(A) \to U_t \cdot A \cdot U_{-t} \] (26)

e tc.

The averaged or renormalized observables, \( A \to A_R \), are a subset of elements in the original algebra, \( \mathcal{A} \). We denote the subalgebra, generated by these elements, by \( \mathcal{A}_R \) with \( \mathcal{A}_R \subset \mathcal{A} \). We can decide to forget the finer algebra, \( \mathcal{A} \), and define the algebra on scale \( R \) by:

**Definition 3.1** We define the system on scale \( R \) by

\[ \omega^{(R)}(A^{(R)}) := \omega(A_R) \] (27)

\[ \alpha_t^{(R)}(A^{(R)}) := (\alpha_t(A))^{(R)} \] (28)

\[ \alpha_X^{(R)}(A^{(R)}) := (A(RX))^{(R)} \] (29)

that is, we define the objects on the lhs implicitly (by reconstruction) via the following correspondence

\[ \langle A^{(R)}(t_1, X_1) \cdots A^{(R)}(t_l, X_l) \rangle^{(R)} := \langle A_R(t_1, RX_1) \cdots A_R(t_l, RX_l) \rangle \] (30)

Remark: Note the different treatment of time and space-translations. We will come back to this point (which has remarkable physical consequences) below in connection with critical slowing down.

**Theorem 3.2** From the above we see that on each scale we have a new theory, \( S^{(R)} \), which we get by reconstruction from the above hierarchy of correlation functions, in particular, a new, non-isomorphic algebra, \( \mathcal{A}^{(R)} \), and a corresponding GNS-Hilbert space representation. We emphasize that the coarse-grained dynamics is also physically different (despite the similarities on both sides of the above definitions).
If the scaling limit does exist, we have, by the same token, a scaling limit system denoted by

$$S^\infty = (\omega^\infty, A^\infty, \alpha_t^\infty, \alpha_X^\infty)$$

(31)

with

$$\langle A^\infty(t_1, X_1) \cdots A^\infty(t_l, X_l) \rangle = \lim_{R \to \infty} \langle A_R(t_1, RX_1) \cdots A_R(t_l, RX_l) \rangle$$

(32)

The proof is more or less obvious from what we have said above.

**Corollary 3.3** We generally assume that $\alpha_t$ is strongly continuous on $A$. By the above identification process we can immediately infer that both $\alpha_t^{(R)}$ and $\alpha_t^\infty$ are also strongly continuous on the corresponding algebras, $A^{(R)}, A^\infty$. By the same token, we can infer that $\omega^{(R)}$ and $\omega^\infty$ are KMS-states at the same inverse temperature $\beta$.

Proof: Note that the original time evolution “commutes” with the scale transformation in the sense described above. This yields the mentioned result for all finite $R$. We have in particular that for suitable elements (for the technical details see [17])

$$\langle B^{(R)}(t) \cdot A^{(R)} \rangle_{(R)} = \langle A^{(R)} \cdot B^{(R)}(t + i\beta) \rangle_{(R)}$$

(33)

and there exists an analytic function, $F^{(R)}_{AB}(z)$, in the strip $\{z = t + i\tau, 0 < \tau < \beta\}$ with continuous boundary values at $\tau = 0, \beta$:

$$F^{(R)}_{AB}(t) = \langle A^{(R)} \cdot B^{(R)}(t) \rangle_{(R)} , F^{(R)}_{AB}(t + i\beta) = \langle A^{(R)} \cdot B^{(R)}(t + i\beta) \rangle_{(R)}$$

(34)

This is equivalent to the following equation (cf. [L7]):

$$\int \omega^{(R)}(A^{(R)} \cdot B^{(R)}(t)) \cdot f(t) dt = \int \omega^{(R)}(B^{(R)}(t) \cdot A^{(R)}) \cdot f(t + i\beta) dt$$

(35)

for $f \in \hat{D}$. As $f(t + i\beta)$ is of strong decrease in $t$ the limit $R \to \infty$ can be performed under the integral and we get the same relation in the scaling limit. The above mentioned equivalence of this property with the KMS-condition shows that the limit state is again KMS. This proves the statement.

Remarks: i) Note what we have already said in [8]. One reason for the non-equivalence of the algebras on different scales stems from the observation that, in general,

$$A_R \cdot B_R \neq (A \cdot B)_R$$

(36)

Furthermore, in the scaling limit, many different observables of $A$ converge to the same limit point, for example, all finite translates of a fixed observable.

ii) A corresponding result in a slightly different context was also proved in [H].
3.2 The Scale Invariance of the Limit Theory

We have seen in sect. 2.3 that the scaling limit of the correlation functions for the block spin observables is not fully scale invariant but only asymptotically so (while the short range details of the original microscopic correlations, encoded in the function \( F(x_1 - x_2) \), have been integrated out, there remains an integrated effect of the initial block-function, \( f(x) \)).

This observation runs a little bit contrary to the general folklore, in which the various limit procedures are frequently interchanged and identified without full justification. We will exhibit the true connections between the various expressions in the following.

With \( f(x) \) now being a general test function of e.g. compact support, we have from sect. 2.3, making now the dependence on \( f \) explicit

\[
\lim_{R \to \infty} \langle A_{R,f}(RX_1) \cdot A_{R,f}(RX_2) \rangle = \text{const} \cdot \int |y + Y|^{-(n-\alpha)} \cdot f \ast f(y) dy
\]

with

\[
A_{R,f}(RX) = R^{-(n+\alpha)/2} \cdot \int A(RX + x) \cdot f(x/R) dx
\]

We now rewrite the limit correlation function as

\[
\langle A_f^\infty(X_1) \cdot A_f^\infty(X_2) \rangle = \int \langle \hat{A}^\infty(x_1 + X_1) \cdot \hat{A}^\infty(x_2 + X_2) \rangle \cdot f(x_1)f(x_2) dx_1 dx_2
\]

that is, we identify

\[
A_f^\infty(X) = \int \hat{A}^\infty(x + X) \cdot f(x) dx
\]

with \( \hat{A}^\infty(x) \) now having rather the character of a field or operator valued distribution.

We have that

\[
\langle \hat{A}^\infty(x_1) \cdot \hat{A}^\infty(x_2) \rangle = W^\infty(x_1 - x_2) = \text{const} \cdot |x_1 - x_2|^{-(n-\alpha)}
\]

Corresponding results would hold for the higher correlation functions, that is, we arrive at

**Conclusion 3.4** In contrast to the block observables, \( A_f^\infty \), the field, \( \hat{A}^\infty(x) \), displays the full scale invariance.

The field, \( \hat{A}^\infty(x) \), can, on the other hand, be directly constructed by means of a related limit procedure, which is however not of block variable type. We start instead with the unsmeared observables and take the scaling limit, \( R \to \infty \)

\[
\lim_{R} \langle \hat{A}_R(RX_1) \cdot \hat{A}_R(RX_2) \rangle \quad \text{with} \quad \hat{A}_R(RX) := R^{(n-\gamma)} \cdot A(RX)
\]
and \( n - \gamma = (n - \alpha)/2 \).

Remark: The extra scaling factor, \( R^n \), replaces the missing integration over the test function, the support of which increases like \( R^n \).

Performing the same calculations, we see that the above limit is equal to \( \langle A^\infty(X_1) \cdot A^\infty(X_2) \rangle \). We arrive at the conclusion

**Conclusion 3.5** The fully scale invariant limit theory is achieved by taking the limits

\[
\lim_{R} \langle \hat{A}_R(RX_1) \cdots \hat{A}_R(RX_l) \rangle =: W^\infty(X_1, \ldots, X_l)
\]

The same construction holds of course for the intermediate scales; we define \( \hat{A}^{(R)}(X) \) by the following identification

\[
\langle \hat{A}^{(R)}(X_1) \cdots \hat{A}^{(R)}(X_l) \rangle_{(R)} := R^{(n - \gamma)} \cdot \langle A(RX_1) \cdots A(RX_l) \rangle
\]

and have for the observables, \( A^{(R)}_f \), defined above

\[
A^{(R)}_f(X) = \int \hat{A}^{(R)}(X + x) f(x) dx
\]

(which can e.g. be checked by direct calculation).

### 3.3 The (Non)-Quantum Character in the Scaling Limit

In the present section we have dealt with model independent properties of the system, living on scale \( R \) or \( \infty \). It is clear, that, in principle, the algebras \( A, A^{(R)} \) or \( A^\infty \), may contain classes of observables which have to be scaled with different critical exponents. This depends on the details of the models under discussion and, in particular, on the form of the joint spectrum, \( \hat{W}(\omega, k) \), of the Fourier transforms of e.g. 2-point functions in the vicinity of \( (\omega, k) = (0, 0) \). We think, we have to postpone a more detailed discussion of all the possible different model classes and concentrate, for the time being, in this subsection on some generic aspects.

In subsection B of section 3 of [8], we already discussed the limiting behavior of commutators of scaled observables. In the regime of normal scaling, that is, scale dimension \( \gamma = n/2 \), we found that commutators are non-vanishing in the generic case in the limit. This means that in general the resulting limit theory is non-abelian (but quasi-free!). Perhaps a little bit surprisingly, the situation changes at the critical point, where the scale-dimensions are, typically, greater than \( n/2 \) for at least some observables.

We make the same observation as Sewell in [14], namely, commutators of certain “critical” observables vanish in the scaling limit, i.e., the corresponding limit
observables are loosing (at least) part of their quantum character.

Remark: We think that the observation that fluctuations and critical behavior at the critical point are typically of a thermal and not of a quantal nature, does somehow belong to the general folklore in the field of critical phenomena, but we are not aware at the moment that this fact has been widely discussed in the literature in greater rigor. Corresponding remarks can e.g. be found in connection with so-called (temperature-zero) quantum phase transitions in [18] or Vojta in [15] and further references given there.

On the other hand, related phenomena were observed in the context of spontaneous symmetry breaking in sect. 6 of [8] and for certain models by Verbeure et al in [10]. A careful analysis of the behavior of commutators in a slightly different context can also be found in [10].

The general argument goes as follows. We assume that the scaling exponents for the initial observables, $A, B, \gamma_A, \gamma_B$ obey:

\[ \gamma_A + \gamma_B > n \] (46)

We then have

\[
\|[A_R, B_R]|| \leq R^{-(\gamma_A+\gamma_B)} \cdot \int \|[A(x_1), B(x_2)]|| \cdot f(x_1/R)f(x_2/R)dx_1dx_2 \\
= R^{-(\gamma_A+\gamma_B)} \cdot \int \|[A, B(y)]|| \cdot f(x_1/R)f((x_1+y)/R)dx_1dy \] (47)

We assume that the observables $A, B$ are taken at equal times and are strictly local, that is, it exist finite supports $V_A, V_B \subset \mathbb{R}^n$ so that

\[ [A, B(x)] = 0 \text{ for } V_B + x \cap V_A = \emptyset \] (48)

Remark: The restriction to equal times can be avoided but has then to be replaced by a cluster assumption on the commutator (see below).

From the support assumption we immediately infer that the above double integral is actually a single integral as the commutator on the rhs vanishes outside a strip of finite diameter. We get

\[ \lim_R \|[A_R, B_R]|| \leq const \cdot \lim R^{n-(\gamma_A+\gamma_B)} = 0 \] (49)

as $\gamma_A + \gamma_B > n$ by assumption.

We arrive at the same result if we assume that the above norm of the commutator happens to be in $L^1(\mathbb{R}^n)$, i.e.

\[ \|[A, B(y)]|| =: F(y) \in L^1(\mathbb{R}^n) \] (50)
We have
\[ R^{-\gamma_A + \gamma_B} \int F(y) \cdot f(x/\sqrt{R}) f((x+y)/\sqrt{R}) dx \, dy = R^{-\gamma_A + \gamma_B} \cdot R^{2n} \int \hat{F}(p) \hat{f}(Rp) \cdot \hat{f}(-Rp) dp \]
\[ = R^{-\gamma_A + \gamma_B} \cdot R^n \cdot \int \hat{F}(p/R) \hat{f}(p) \cdot \hat{f}(-p) dp \quad (51) \]

We can again perform the \( R \)-limit under the integral and get the limit expression
\[ R^n \cdot (\hat{F}(0) \cdot \int \hat{f}(p) \cdot \hat{f}(-p) dp \rightarrow 0 \quad (52) \]
for \( R \rightarrow \infty \).

A simple example where different renormalisation exponents naturally arise is the following. Take a limit observable, \( A^{\infty}(X) \), and consider its spatial derivative, \( \partial_X A^{\infty}(X) \). Then we have in a slightly sloppy notation (the limit being taken in the sense, described above):
\[ \partial_X A^{\infty}(X) = \lim_{R \rightarrow \infty} \partial_X (R^{-\gamma_A} \int A(x+RX) \cdot f_{\sqrt{R}}(x) d^n x) \]
\[ = \lim_{R \rightarrow \infty} (R^{-\gamma_A + 1}) \cdot \int (\partial_x A)(x+RX) \cdot f_{\sqrt{R}}(x) d^n x \quad (53) \]
That is, \( \partial_x A = i [P, A] \) has to be scaled with a different scale exponent. Physically, this can be understood as follows. With \( f_{\sqrt{R}}(x) = f(|x|/\sqrt{R}) \) simulating the integration over a ball with radius \( R \), a partial integration in the above formula shifts the \( \partial_x \) to the test function, \( f_{\sqrt{R}}(x) \). As \( \partial_x f_{\sqrt{R}}(x) = 0 \) in the interior of the ball, the averaging goes roughly over the sphere of radius \( R \) instead of the full ball. This has to be compensated by a weaker renormalisation.

Another result in this direction can be found in [8] sect.6, in connection with the canonical Goldstone pair in the context of spontaneous symmetry breaking.

Further possible candidates are the time derivatives of observables as e.g. \( \langle \dot{A} \dot{A} \rangle \). Fourier transformation yields an additional prefactor, \( \omega^2 \) in the spectral weight, \( W_{AA}(\omega, k) \). The \( KMS \)-condition leads to another constraint:
\[ W_{AB}(\omega, k) = (1 - e^{-\beta \omega})^{-1} \cdot W_{[A,B]}(\omega, k) \quad (54) \]
A combination of such properties shows, that in the scaling limit, the vicinity of \( (\omega, k) = (0, 0) \) is important.

From covariance properties (as e.g. in models of relativistic quantum field theory) one may infer certain characteristics about the energy-momentum spectrum. For arbitrary models on non-relativistic many-body theory, the situation is less generic and certainly model dependent. We refrain from going into the technical details at the moment.

Remark: We had several discussions with D. Buchholz about this point, which are gratefully acknowledged. This applies also to the following subsection.
3.4 The Nature of the Limit Time Evolution and the Phenomenon of Critical Slowing Down

We argued above that the appropriate choice of the respective scaling dimensions of the observables under discussion is a subtle point and perhaps, to some extent, even a matter of convenience. After all, one may have some freedom in the choice of the subset of observables which is to survive the renormalisation process.

We will not give a complete analysis of all possibilities in the following but rather emphasize one, as we think, particularly remarkable phenomenon, namely, the phenomenon of \textit{critical slowing down}. As in the preceding discussion, we choose two observables, $A, B$, with $\gamma_A + \gamma_B > n$, implying that the limit commutator vanishes. We assume this also to hold for non-equal times, at least on the level of two-point functions, i.e.

\[
\langle [A^\infty, B^\infty(t)] \rangle_{\infty} = 0 
\]  

(55)

As the limit state is again a $KMS$-state, the vanishing of the above commutator implies that the analytic function, $F_{AB}^\infty(z)$, fulfills

\[
F_{AB}^\infty(t) = F_{AB}^\infty(t + i\beta) 
\]  

(56)

for all $t$. $F_{AB}^\infty(z)$ can hence be analytically continued to the whole plane and is, furthermore, a globally bounded analytic function, hence a constant by standard reasoning. We can conclude:

\textbf{Conclusion 3.6} Under the assumptions being made, we have

\[
\langle A^\infty \cdot B^\infty(t) \rangle_{\infty} = \text{const for all } t \in \mathbb{R} 
\]  

(57)

We see that the subclass of limit observables, which has vanishing limit commutators (see the preceding subsection), has, by the same token, time independent limit correlation functions. As these pair-correlation functions are usually connected with characteristic observable properties of the system (generalized susceptibilities, transport coefficients etc.), this has remarkable physical consequences. The corresponding phenomenon is called \textit{critical slowing down}. For a review see e.g. [19]. In physical terms, the phenomenon can be understood as follows.

In the critical regime, the patches of strongly correlated degrees of freedom become very large and extend practically over all scales. That is, a reorientation of such clusters or a response to external perturbations takes, if viewed on the microscopic time scale, a very long time. In the scaling limit this time scale goes to infinity. If one wants to see observable dynamical effects one must scale the time also and work with a more macroscopic time scale. We have in the limit for the unscaled time:

\[
d/dt \langle A^\infty \cdot B^\infty(t) \rangle_{\infty} = 0
\]  

(58)
This is the same as
\[\langle A^\infty \cdot [H^\infty, B^\infty(t)] \rangle_\infty = \lim_{R} \langle A^{(R)} \cdot [H^{(R)}, B^{(R)}(t)] \rangle_{(R)} = \lim_{R} \frac{d}{dt} \langle A^{(R)} \cdot B^{(R)}(t) \rangle_{(R)} \] (59)

(At this place we suppress the discussion of the technical details connected with the limit processes in order to keep the matter transparent).

What one now has to do is obvious. We have to compensate the vanishing of the above expression in the limit by adding an appropriate scale factor in the time coordinate. Instead of \(B(t)\) we insert \(B(R^\delta \cdot t)\) with \(\delta\) so chosen that the limit expression is non-vanishing. Note that differentiation with respect to \(t\) now adds an explicit prefactor \(R^\delta\). This fixes the macroscopic time scale, \(t_m\), for these processes. We can define
\[\langle A^\infty \cdot B^\infty(t_m) \rangle_\infty = \lim_{R} \langle A^{(R)} \cdot B^{(R)}(R^\delta \cdot t_m) \rangle_{(R)} \] (60)

It is clear that other observables may live on different macroscopic time scales so that the construction of a common macroscopic limit time evolution may not be immediate. Such more detailed questions have to be separately studied for the various model classes.

4 The Scaling Behavior of the Correlation Functions at the Critical Point: Illustration of the Method

In the following two subsections we illustrate, in a first step, the technical methods with the help of the 2-point functions, which have a more transparent cluster behavior. A slightly different analysis can already be found in section 7 of [8]. The general idea is it, to extract and isolate the characteristic singular behavior of the correlation functions; see also section 2.3 of the present paper. The full analysis of the cluster behavior of the \(l\)-point functions is then given in the following section.

4.1 Method One

We assume the existence of a certain exponent, \(\alpha\), so that \((x^2\) denoting the vector-norm squared) we can make the following decomposition.
\[G(x) := W^T(x) \cdot (1 + x^2)^{(\alpha-\alpha)/2} = \text{const} + F(x) \] (61)
with a decaying (non-singular) $F$ which is assumed to be in $L^1$. Fourier transformation then yields:

$$R^{-2\gamma} \cdot \int W_2^T((x_1 - x_2) + R(X_1 - X_2))f(x_1/R)f(x_2/R)dx_1dx_2$$

$$= R^{-2\gamma} \cdot \int G((x_1 - x_2) + R(X_1 - X_2)) \cdot [1 + ((x_1 - x_2) + R(X_1 - X_2))^2]^{-(n-\alpha)/2} \cdot f(x_1/R)f(x_2/R)dx_1dx_2$$

$$= R^{-2\gamma} \cdot R^{2n-(n-\alpha)} \cdot \int dp \hat{G}(p) \cdot e^{-iRp(x_1-x_2)} \cdot \left[ \int e^{-iRp(x_1-x_2)}(R^{-2} + ((x_1 - x_2) + (X_1 - X_2))^2)^{-(n-\alpha)/2} f(x_1)f(x_2)dx_1dx_2 \right]$$

(62)

where we made the substitution $x \to R \cdot x$.

We now assume the support of $f$ to be contained in a sufficiently small ball around zero (or, alternatively, $(X_1 - X_2)$ sufficiently large so that $(x_1 - x_2) + (X_1 - X_2) \neq 0$ for $x_i$ in the support of $f$). With

$$\hat{G}(p) = const \cdot \delta(p) + \hat{F}(p)$$

(63)

the leading part in the scaling limit $R \to \infty$ is the $\delta$-term. Asymptotically we hence get for $R \to \infty$ (setting $y := x_1 - x_2$, $Y := X_1 - X_2$):

$$R^{n+\alpha-2\gamma} \cdot const \cdot \int |y + Y|^{-(n-\alpha)} \cdot f \ast f(y)dy$$

(64)

with

$$f \ast f(y) := \int f(y + x_2) \cdot f(x_2)dx_2$$

(65)

and $y + Y \neq 0$ on $supp(f)$.

The reason why the contribution, coming from $\hat{F}(p)$, can be neglected for $R \to \infty$ is the following: $f$ is assumed to be in $\mathcal{D}$; by assumption the prefactor never vanishes on the support of $f(x_i)$. Hence the whole integrand in the expression in square brackets is again in $\mathcal{D}$ and therefore its Fourier transform, $\hat{g}(p')$, is in $\mathcal{S}$ (with $p' := Rp$), that is, of rapid decrease. We can therefore perform the $R$-limit under the integral and get a rapid vanishing of the corresponding contribution in $R$ for $R \to \infty$.

$$\lim_{R \to \infty} R^{n-\alpha} \cdot \int \hat{F}(p'/R) \cdot e^{-iyp'} \cdot \hat{g}(p')dp' = 0$$

(66)

As $f \ast f$ has again a compact support, we have that, choosing

$$\gamma = (n + \alpha)/2$$

(67)

the limit correlation function behaves as $\sim |X_1 - X_2|^{-(n-\alpha)}$ as in the above heuristic analysis.
4.2 Method Two

As in the case of normal clustering or (8, last section), one can, on the other hand, improve the too weak decay of \( W^T(x_1 - x_2) \) and transform it into an integrable (i.e. \( L^1 \)) function. So, with a similar notation as in the preceding subsection, we choose a suitable exponent \( \alpha \) in
\[
P_\alpha(x_1 - x_2) := (1 + |x_1 - x_2|^2)^{\alpha/2}
\]
so that
\[
G(y) := W^T(y) \cdot P_\alpha^{-1}(y) \in L^1 \quad (y := x_1 - x_2)
\]

In contrast to Method One, there is of course a whole range of such possible exponents, \( \alpha > \alpha_{inf} \), so that
\[
G(y) = \begin{cases} 
\in L^1 & \text{for } \alpha > \alpha_{inf} \\
\notin L^1 & \text{for } \alpha < \alpha_{inf}
\end{cases}
\]

Proceeding as in Method One, we get
\[
R^{-2\gamma} \int W^T(y + R \cdot Y) \cdot f(x_1/R) f(x_2/R) dx_1 dx_2 =
\]
\[
R^{-2\gamma} \cdot R^{n+\alpha} \int dp \hat{G}(p/R) \cdot e^{-ipY} \cdot \left[ \int e^{-ipy} (R^{-2} + (y + Y)^2)^{\alpha/2} \cdot f \ast f(y) dy \right]
\]

Again the obvious strategy seems to be to choose
\[
\gamma = (n + \alpha)/2
\]
and perform the limit \( R \to \infty \). With the same support properties as above, that is, \( y + Y \neq 0 \) for \( x_1, x_2 \in \text{support of } f \), the integrand in square brackets is again infinitely differentiable with respect to \( y \). Hence, its Fourier transform is fast decaying in \( p \).

Remark: Note that for \( \alpha/2 \) non-integer and without the above support restriction, there will show up a singularity in sufficiently high orders of differentiation for vanishing \( R^{-2} \). One can however control these singularities and show that the analysis still goes through in the case where the support condition does not hold. One gets however some mild constraint on the admissible \( \alpha \)'s.

Therefore we can again apply Lebesgue’s theorem of dominated convergence and perform the \( R \)-limit under the integral. This yields the expression
\[
\hat{G}(0) \cdot \int dp e^{-ipY} \cdot \left[ \int e^{-ipy} \cdot |y + Y|^\alpha \cdot f \ast f(y) dy \right] =
\]
\[
\text{const} \cdot \hat{G}(0) \cdot \int \delta(y + Y) \cdot |y + Y|^\alpha \cdot f \ast f(y) dy = \text{const} \cdot \hat{G}(0) \cdot 0
\]
(as a result of the above support condition).
Conclusion 4.1 With \(\alpha\) chosen so that \(G(y) \in L^1\) and \(\gamma = (n + \alpha)/2\), the limit can be carried out under the integral and yields the result zero. This shows a fortiori that there is no \(\alpha_{\min}\) with the property that there is a non-vanishing limit-two-point function. Put differently, we have an \(\alpha_{\inf}\) but no \(\alpha_{\min}\) (cf. (70)).

So, in contrast to Method One, the relevant exponent, \(\alpha_{\inf}\), is of such a peculiar nature that we definitely cannot apply the above method of interchange of taking the limit \(R \to \infty\) and integration. But nevertheless, we will show that

\[
\gamma := \frac{(n + \alpha_{\inf})}{2} \tag{74}
\]

is the correct critical scaling exponent leading to a sensible limit theory and that this \(\alpha_{\inf}\) is exactly the \(\alpha\), we have determined in Method One.

We have learned above that in order to arrive at a non-zero limit correlation function, we are definitely forbidden to exploit Lebesgues’ theorem of dominated convergence in the above expression. The reason for the vanishing of the respective expression was that with

\[
\lim_{R \to \infty} \hat{G}(p/R) = \hat{G}(0) \tag{75}
\]

we have to evaluate \(\int \hat{g}(p)dp\) with

\[
\hat{g}(p) := \int e^{-ip(y+Y)}|y + Y|^\alpha \cdot f \ast f(y)dy \tag{76}
\]

This integral happens to be zero due to the explicit factor, \(|y + Y|^\alpha\) and the assumed support properties.

So, we have to investigate what happens for \(\alpha = \alpha_{\inf}\). As we learned above that there is no \(\alpha_{\min}\), we can conclude

Observation 4.2 For \(\alpha = \alpha_{\inf}\), \(G_\alpha(y)\) is no longer in \(L^1\), with

\[
G_\alpha(y) := W^T(y) \cdot (1 + y^2)^{-\alpha/2} \tag{77}
\]

We know that for \(\alpha < \alpha_{\inf}\) the decay of \(G_\alpha(y)\) is so weak that the Fourier transform develops a power law singularity in \(p = 0\); that is, we can conclude

Lemma 4.3 For \(\alpha_{\inf} - \alpha := \varepsilon\), \(\hat{G}_\alpha\) has a singularity of the form \(|p|^{-\varepsilon}\) near \(p = 0\).

For \(\alpha = \alpha_{\inf}\) the singularity is of logarithmic type near \(p = 0\).

This statement can again be proved by a scaling argument. Let \(G_\alpha\) have a non-integrable tail of the form \(|y|^{-(n-\varepsilon)}\). For the (distributional) Fourier transform we then have

\[
\hat{G}_\alpha(\lambda \cdot p) = \text{const} \cdot \int e^{i\lambda y} \cdot G_\alpha(y)dy = \text{const} \cdot \lambda^{-n} \cdot \int e^{ipy'} \cdot G_\alpha(y'/\lambda)dy' \tag{78}
\]
For \( \lambda \to 0 \) we can, as above, replace \( G_\alpha \) by its asymptotic expression, which goes as \( |y|^{-(n-\varepsilon)} \) and conclude that \( \hat{G}_\alpha(\lambda p) \) contains a leading singular contribution \( \sim \lambda^{-\varepsilon} \) (modulo logarithmic terms). We hence see that

\[
\hat{G}_\alpha(p) \sim |p|^{-\varepsilon}
\]

(79)

near \( p = 0 \) as a distribution (that is, the above reasoning is to be understood modulo the smearing with appropriate test functions; see e.g. [1]). For \( \alpha = \alpha_{\text{inf}} \), the singularity must be weaker than any power, that is, must be of logarithmic type.

By Method One we get a limit correlation function which clusters as \( |Y|^{-(n-\alpha)} \). One may wonder where this behavior is hidden if we use Method Two. Taking only the singular term in \( \hat{G}(p/R) \) into account, we have (with \( \gamma := (\alpha_{\text{inf}} + n)/2 \))

\[
\lim_{R \to \infty} R^{-2\gamma} \langle A_R(RX_1) \cdot B_R(RX_2) \rangle^T \sim \lim_{R \to \infty} \text{const} \cdot \int \ln(|p|/R) \cdot \hat{g}(p) dp
\]

(80)

and \( \hat{g}(p) \) as in equation (76). We can again neglect the term

\[
\int \ln R \cdot \int \hat{g}(p) dp
\]

(81)

as \( \int \hat{g}(p) dp = 0 \).

Assuming at the moment that \( \alpha \) were an integer (we will get the general result by a scaling argument), the prefactor \( |y + Y|^{\alpha} \) can be transformed into corresponding \( p \)-differentions of \( \hat{f} \hat{f}(p) \), which, by partial integration, can then be shifted to corresponding differentiations of \( \ln(|p|) \). This transformation yields an expression of the type \( |p|^{-\alpha} \) times a smooth and decaying function. That means, we essentially end up with an expression like

\[
\int dp |p|^{-\alpha} \cdot e^{-ipY} \cdot \left[ \int e^{-ipy} f \ast f(y) dy \right]
\]

(82)

By the same reasoning as above we conclude that the singularity, \( |p|^{-\alpha} \), goes over, via Fourier transform, into a weak decay proportional to \( |X_1 - X_2|^{-(n-\alpha)} \), that is, we arrive at the same result as in Method One, whereas the reasoning is a little bit more tricky.

For a general non-integer \( \alpha \) the argument could be made precise by analysing the distributional character of an expression like \( r^\beta \) with \( r := |x| \) and its Fourier transform. As the analysis is a little bit tedious, we refer the reader to [11]. On the other hand, one can use a scaling argument as above (with \( Y := \lambda \cdot Y_0, Y_0 \) fixed as \( \lambda \to \infty \)). This yields an asymptotic behavior of the form

\[
\lambda^{-(n-\alpha)} \cdot \int dp \ln(|p|) \cdot \left[ \int e^{-i\lambda p(y+Y_0)} \cdot |y + Y_0|^{\alpha} \cdot f \ast f(\lambda y) dy \right]
\]

(83)
The evaluation of the integral for $\lambda \to \infty$ can be done as follows: As $f * f$ has compact support, the volume of the support of $f * f(\lambda y)$ shrinks proportional to $\lambda^{-n}$. Therefore the expression in square brackets scales as $\sim \lambda^{-n}$. On the other hand (due to an ‘uncertainty principle’ argument), its essential $p$-support increases proportional to $\lambda^n$. That is, the two effects compensate each other and we have again a large-$Y$ behavior $\sim |Y|^{-(n-\alpha)}$ as before.

We conclude that both methods lead to the same asymptotic scaling behavior of the renormalized two-point function.

5 The General Cluster-Analysis at the Critical Point

We now study the general situation of the presence of some long-range correlations in the l-point functions. In contrast to the much simpler situation prevailing in the case of two-point functions, the clustering may be quite complicated, in particular, the dependence on the number, $l$, i.e. the number of observables, occurring in the expressions, may be non-trivial. Therefore, we have to investigate these aspects in more detail.

Remark: One should note that our, at first glance, rather technical analysis serves also the purpose to clarify and isolate the frequently only tacitly made preassumptions concerning the necessary cluster or scaling behavior of the correlation functions. Put differently, the preceding and the following analysis may show which assumptions have actually to be made, in order that the general picture comes out correctly.

From general principles (see e.g. [12]) we know that in a pure phase there is always a certain degree of clustering. We make the slightly stronger assumption that it is in some way of the kind of an inverse power law at infinity (to be specified below). We want to study the scaling limit of

$$\langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T$$

with

$$A_R(a) := R^{-\gamma} \cdot \int A(x + a) f(x/R) d^n x$$

and an, at the moment, unspecified exponent, $\gamma$.

The above expression can be written as

$$\int W^T((x_1 - x_2) + R(X_1 - X_2), \ldots, (x_{l-1} - x_l) + R(X_{l-1} - X_l)) \cdot \prod_{i=1}^{l} f(x_i/R) \prod_{i=1}^{l} dx_i$$

(86)
Fourier transformation yields (with $\hat{W}^T(q_1, \ldots, q_{l-1})$ considered as a distribution on $S(\mathbb{R}^{(l-1)n})$)

$$
const \cdot R^{((n-\gamma) \cdot \int \hat{W}^T(q_1, \ldots, q_{l-1}) \cdot e^{-i \sum_{j=1}^{l-1} Rq_j Y_j}. 
$$

with $Y_j := X_j - X_{j+1}$ and the well-known relation between the $q$-variables and the $p$-variables (see e.g. section 2 or [8]). For calculational or notational convenience we will employ both sets of variables which are linear combinations of each other.

As $f$ is in $D$ by assumption, the Fourier transform of $\prod f(x_i)$ is in $S$ and the function in square brackets is a function of $(Rq_1, \ldots, Rq_{l-1})$, being of rapid decrease in either set of variables. As a consequence, for $R \to \infty$ and at least one $p_j$ being different from zero, the expression approaches zero faster than any inverse power (together with all its derivatives).

From this we see that, as $R \to \infty$, the region of possible singular behavior is located around $(p)_1^{l-1} = 0$ or $(q)_1^{l-1} = 0$, implying also $p_l = -\sum_{j=1}^{l-1} p_j = 0$. We can hence infer that only the singular behavior of $\hat{W}^T$ in $(q) = 0$ will matter in this limit. As a consequence, it will be our strategy to isolate this singular contribution in $\hat{W}^T$ and transform it in a certain explicit scaling behavior in $R$, which can be encoded in some power, $R^{-\alpha}$, in front of the integral.

The singular behavior of $W^T(q)$ at $(q) = 0$ is related to the weak decay of $W^T(y)$ at infinity. The limiting behavior of $W^T(y)$ can, however, not expected to be simple or uniform (at least not in the generic case) as $(y_1, \ldots, y_{l-1})$ or $(x_1, \ldots, x_l)$ can move to infinity in many different ways. We may, for example, have that $(x_i)$ together with all $|x_i - x_j|$ go to infinity or, on the other side, the variables move to infinity in certain fixed clusters of finite diameter. The rate of decay of $W^T(y)$ should of course depend in general on these details. Correspondingly, the singular behavior of $\hat{W}^T(q)$ in the infinitesimal neighborhood of $(q) = 0$ should depend on the direction in which $(q) = 0$ is approached, that is, the limit may be direction-dependent.

In the light of this general situation we must at first decide, in which kind of limit we are mainly interested. Inspecting the expression (84), we actually started from, we choose in a first step our fixed vectors, $(X_i)$, so that

$$
X_i - X_j \neq 0 \text{ for all } i, j \tag{88}
$$

As a consequence, all distances, $|RX_i - RX_j|$, go to infinity for $R \to \infty$. As in the preceding section, we can choose the support of $f$ so small that, with $x_i, x_j \in supp(f)$, we have

$$
|R(X_i - x_i) - R(X_j - x_j)| \to \infty \tag{89}
$$
In this particular case we may expect a relatively uniform limit behavior on physical grounds.

Remark: Similar problems occur in quantum mechanical scattering theory.

Under this proviso the following assumption seems to be reasonable.

**Assumption 5.1** Under the assumption, being made above, we assume the following decomposition of $W_T^T(y)$ to be valid: It exists a function, $(1 + H(y))$, $H(y)$ homogeneous and positive for $y \neq 0$ so that

$$G(y) := (1 + H(y)) \cdot W^T(y) = \text{const} + F(y)$$

(90)

with $F$ sufficiently decaying at infinity in the channel, indicated above, i.e. $\{|y_i| \to \infty \text{ for all } i = 1, \ldots, l-1\}$ and

$$H(Ry) = R^{\alpha_l'} \cdot H(y)$$

(91)

**Remark 5.2** A typical example for $H(y)$ is $(\sum y_i^2)^{\alpha_l'/2}$.

Fourier transforming $G(y)$, we get

$$\hat{G}(q) = \text{const} \cdot \delta(q) + \hat{F}(q)$$

(92)

and expression (87) becomes (compare the related expression in Method One of the preceding section)

$$\text{const} \cdot R^{(\alpha_{l'-\gamma})} \cdot \int \prod_{1}^{l-1} dp_j \hat{G}(q) \cdot e^{-i \sum q_j Y_j} \cdot \left[ \int e^{-i \sum_{1}^{l-1} R_{p_j x_j} \cdot e^{i R q_{l-1} x_l} \cdot (1 + H(R(y + R Y)))^{-1} \cdot \prod_{1}^{l} f(x_j) \cdot \prod_{1}^{l} dx_j \right]$$

(93)

By assumption, $H$ is homogeneous of degree $\alpha_l'$. So we can extract a negative power of $R$, $R^{-\alpha_l'}$, from the expression in square brackets. Furthermore, we observed above that for $R \to \infty$ only the vicinity of $q = 0$ matters. Finally, by assumption, the contribution coming from $\hat{F}(q)$ can be neglected in this limit (compare the corresponding discussion in the subsection 4.1; as a consequence of the assumed support properties, the expression in square brackets is again strongly decreasing). We hence have

$$\lim_{R \to \infty} \langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = \lim_{R \to \infty} \text{const} \cdot R^{(\alpha_{l'-\gamma})} \cdot \int \prod_{1}^{l-1} dq_j \\
\delta(q) e^{-i \sum_{j=1}^{l-1} R_{p_j x_j} \cdot e^{i R q_{l-1} x_l} \cdot \left( R^{-\alpha_l'} + H(y + Y) \right)^{-1} \cdot \prod_{1}^{l} f(x_i) \cdot \prod_{1}^{l} dx_i}$$

(94)
Remark: We see again the reason for the special choice being made above as to the support properties of the functions $f(x_i)$, leading to the result $y_j + Y_j \neq 0$ on the support of $f$. Without this assumption, we see for our above example, $H(y) = (\sum y_i^2)^{\alpha'/2}$, that in the limit, where $R^{-\alpha'}$ vanishes, we would get a singular contribution at points where $y + Y = 0$ in the integrand in square brackets. These terms would make the following discussion much more tedious.

If we now make the choice

$$\gamma := \gamma_l = n - \alpha'_l/l$$

we arrive at a finite limit expression, depending on the coordinates $(X_i)$:

$$\lim_{R \to \infty} \langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = \text{const} \cdot \int H_l(y + Y)^{-1} \cdot \prod_{i=1}^l f(x_i) \prod_{i=1}^l dx_i$$

which is a function of the coarse grained difference coordinates

$$Y_j = X_j - X_{j+1}$$

For the $Y_j$ sufficiently large, it is approximately a function

$$W_{\text{limit}}(Y) \approx \text{const} \cdot H_l(Y)^{-1}$$

That is, the renormalized limit correlation functions reproduce the asymptotic power law behavior of the original microscopic correlation functions modulo the convolution with the original smearing functions as has been discussed already above for the two point functions.

For later use we introduce the new scaling exponent, $\alpha_l$, via

$$\alpha'_l + \alpha_l = (l - 1)n$$

This implies

$$\gamma_l = (n + \alpha_l)/l$$

The underlying reason for this choice is that an asymptotic decay, $\sim R^{-(l-1)n}$, is just the threshold for $W_l^T$ being integrable or non-integrable (with $r := (\sum y_j^2)^{1/2}$).

We have arrived at the following result: We are interested in a scaling-limit theory for $R \to \infty$. In order to get a non-vanishing and finite limit theory, we have to choose the scaling exponent for $l = 2$ as

$$\gamma = \gamma_2 = (n + \alpha_2)/2$$
Furthermore, we have extracted the asymptotic form from the higher truncated \( l \)-point functions, \( W^T_l(y) \), and have absorbed it in an explicit scaling factor, \( R \) to some power. If the limit theory is to be finite, the corresponding scaling exponents for \( l > 2 \) have to be less or equal to zero. This yields unique \( \gamma \)'s as threshold values.

A cornerstone of the philosophy of the renormalisation group is that the scaling exponents of the scaled observables remain the same, irrespectively of the degree of the correlation functions in which they occur. That is, these exponents are fixed by the exponent, \( \gamma_2 \), and we have

\[
\gamma = \gamma_2 \geq \gamma_l
\]

(102)

(the latter exponent being derived from equation (101)), in order that the limit correlation functions remain finite.

**Conclusion 5.3** We have the following alternatives for \( R \to \infty \):

- \( \gamma_2 > \gamma_l \Rightarrow W^T_{l,R} \to 0 \) (103)
- \( \gamma_2 = \gamma_l \Rightarrow \lim_{R \to \infty} W^T_{l,R} \) is finite and non-trivial (104)
- \( \gamma_2 < \gamma_l \Rightarrow W^T_{l,R} \to \infty \) (105)

If \( \gamma_2 > \gamma_l \) for all \( l \geq 3 \), the fixed point is gaussian or trivial. The limit theory is quasi-free. The limit theory is non-trivial if \( \gamma_2 = \gamma_l \) for at least some \( l \geq 3 \). For \( \gamma_2 < \gamma_l \) for some \( l \), the limit theory does not exist.

**Remark 5.4** The corresponding analysis can also be done by employing Method Two (discussed in the preceding section). One can even omit the support conditions assumed above. The treatment then becomes more involved but the end result is the same. We discuss one particular case below.

To complete the scaling and/or cluster analysis of the truncated correlation functions, we have to analyze the other channels and the respective consequences for scaling exponents and cluster assumptions.

We mentioned several times that without the support condition

\[
(X_i - X_j) + (x_i - x_j) \neq 0
\]

(106)

for \( x_{i,j} \in \text{supp}(f) \), the analysis would become more tedious. On the other side, this assumption is violated if the observables move to spatial infinity in certain clusters. The extreme case occurs when all \( X_i \) are chosen to be zero, i.e:

\[
\langle A_R(1) \cdots A_R(l) \rangle^T, \ R \to \infty
\]

(107)

(the indices \( 1, \ldots, l \) denote the different observables). This scenario was already briefly discussed in section 7 of [8] in connection with phase transitions and/or
spontaneous symmetry breaking, which are also typically related to poor spatial clustering.

With the same notations as above we have

\[ \langle A_R(1) \ldots A_R(l) \rangle^T = \text{const} \cdot R^{l(n-\gamma)}. \]

\[ \int \hat{W}_l^T(q_1, \ldots, q_{l-1}) \cdot \left[ \int e^{-i \sum_{1}^{l-1} R_p x_j} \cdot e^{i R_q x_l} \cdot \prod_{i=1}^{l} f(x_i) \prod_{i=1}^{l} dx_i \right] \prod_{1}^{l-1} dp_j \tag{108} \]

Assuming again the existence of a suitable homogeneous function, \( H_l(y) \), in this channel, we get asymptotically two contributions

\[ \text{const} \cdot R^{l(n-\gamma)-\alpha'_i} \cdot \int H_l(y)^{-1} \cdot \prod_{1}^{l} f(x_i) \prod_{1}^{l} dx_i \tag{109} \]

and

\[ \text{const} \cdot R^{l(n-\gamma)-\alpha'_i} \cdot \int \prod_{1}^{l-1} dq_j \hat{F}(q_1, \ldots, q_{l-1}) \cdot \left[ \int e^{-i \sum_{1}^{l-1} R_p x_j} \cdot e^{i R_q x_l} \cdot (H_l(y))^{-1} \cdot \prod_{i=1}^{l} f(x_i) \prod_{i=1}^{l} dx_i \right] \tag{110} \]

The first term has almost the same form as above. But now the function in square brackets in the second contribution is no longer of strong decrease as the integrand (considered as a function of \( (x) \) or \( (y) \)) is no longer in \( D \) as it will have a singularity in \( y = 0 \). We can however provide the following estimate on the degree of this singularity of \( H_l^{-1} \) in \( y = 0 \). We assumed throughout in this section that \( W_l^T \) is not integrable at infinity, that is, the clustering is weak. On the other side, this asymptotic behavior is exactly encoded in \( H_l^{-1} \), as we observed above.

The threshold where integrability goes over into non-integrability for \( H_l^{-1} \) is a behavior

\[ \sim r^{-(l-1)n}, \quad r := \left( \sum_{1}^{l-1} y_j^2 \right)^{1/2} \tag{111} \]

We can therefore conclude that

\[ \alpha'_i \leq (l - 1)n \tag{112} \]

in the above construction if \( W_l^T \) is non-integrable at infinity. If \( \alpha'_i \) is even strictly smaller than \( (l - 1)n \), which is the ordinary case in the critical region, we have
Observation 5.5

\[ \alpha'_l < (l - 1)n \]  

implies that \( H^{-1}_l \) is integrable near \( y = 0 \). Hence

\[ H^{-1}_l(y) \cdot \prod_1^l f(x_i) \in L^1 \]

due to the compact support of \( f \).

From this we infer again that, with

\[ \gamma_l = n - \frac{\alpha'_l}{l} = (n + \frac{\alpha_L}{l}) \]

the contribution (109) is finite in the scaling limit. For the contribution (110) we have by the same reasoning that the function in square brackets is a continuous function of \( (Rq) \), which goes to zero for \( Rq \to \infty \) (due to the Riemann-Lebesgue lemma).

On the other side, we have no precise apriori information about \( F(y) \) and \( \hat{F}(q) \). \( F(y) \) goes to zero at infinity as the asymptotic behavior is contained in \( H^{-1} \), but its rate of vanishing is not clear.

Conclusion 5.6 If the integrand of contribution (110) is lying in some \( L^p \), so that the limit, \( R \to \infty \), can be performed under the integral, the whole expression vanishes in the scaling limit.

In this situation we are left with again with the first term, which is the limit of

\[ \text{const} \cdot \int H_l(y + Y)^{-1} \cdot \prod_1^l f(x_i) \prod_1^l dx_i \]

for \( Y \to 0 \). That is, in this case it holds

\[ W_l^{lim}(0, \ldots, 0) = \lim_{X \to 0} W_l^{lim}(X_1, \ldots, X_l) \]

Theorem 5.7 If the situation is as in the conclusion, \( W_l^{lim}(X_1, \ldots, X_l) \) is continuous and we have in particular

\[ W_l^{lim}(0, \ldots, 0) = \lim_{X \to 0} W_l^{lim}(X_1, \ldots, X_l) \]

We can hence resume our findings as follows: If the assumptions, made above, are fulfilled and if the functions, \( H_l \), can be chosen consistently in all channels, so that the \( \gamma_l \)'s, resulting from the relation

\[ \gamma_l = (n + \alpha_l)/l \]

are smaller than or identical to \( \gamma_2 \), we arrive at a full limit theory, being well-defined in all channels. In this case the renormalization group program works and yields a non-trivial scaling limit.

Acknowledgement: Several discussions with D.Buchholz are gratefully acknowledged (see also the remark at the end of subsection 3.3).
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