SELF-DIFFUSION AND CROSS-DIFFUSION EQUATIONS: \( W^{1,p} \)-ESTIMATES AND GLOBAL EXISTENCE OF SMOOTH SOLUTIONS

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ABSTRACT. We investigate the global time existence of smooth solutions for the Shigesada-Kawasaki-Teramoto system of cross-diffusion equations of two competing species in population dynamics. If there are self-diffusion in one species and no cross-diffusion in the other, we show that the system has a unique smooth solution for all time in bounded domains of any dimension. We obtain this result by deriving global \( W^{1,p} \)-estimates of Calderón-Zygmund type for a class of nonlinear reaction-diffusion equations with self-diffusion. These estimates are achieved by employing Caffarelli-Peral perturbation technique together with a new two-parameter scaling argument.

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1. INTRODUCTION AND MAIN RESULTS

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) with \( n \geq 2 \). We consider the following popular system of reaction-diffusion equations:

\[
\begin{align*}
  u_t &= \Delta [(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1u - c_1v) \quad \text{in} \quad \Omega \times (0, \infty), \\
  v_t &= \Delta [(d_2 + a_{21}u + a_{22}v)v] + v(a_2 - b_2u - c_2v) \quad \text{in} \quad \Omega \times (0, \infty), \\
  \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
  u(x, 0) &= u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 \quad \text{in} \quad \Omega,
\end{align*}
\]  

(1.1)

where the coefficients \( a_k, b_k, c_k, d_k \) are positive constants, while \( a_{ik} \) are non-negative constants, for \( i, k = 1, 2 \). Hereafter, \( \nu(\cdot) \) denotes the unit outward normal vector field on the boundary \( \partial \Omega \).

The system (1.1) was proposed by Shigesada-Kawasaki-Teramoto in [30] to model the spatial segregation of two competing species in the region \( \Omega \). It is usually referred to as the SKT system of cross-diffusion equations. In (1.1), \( u \) and \( v \) are the population densities of the two species. The terms \( d_1 \Delta u, d_2 \Delta v \) are the diffusion ones due to the random movements of individual species with positive diffusion rates \( d_1, d_2 \). Meanwhile, \( \Delta [(a_{11}u + a_{12}v)u] \) and \( \Delta [(a_{21}u + a_{22}v)v] \) come from the directed movements of the individuals toward favorable environments. The considered species
hereby move away from the high population density to avoid the population pressure, hence \( a_{ik} \) are non-negative. The constants \( a_{11}, a_{22} \) are called self-diffusion coefficients, while \( a_{12} \) and \( a_{21} \) are cross-diffusion coefficients. The homogeneous Neumann boundary conditions mean that there are no movements across the boundary. We note that the zero order nonlinearities in (1.1) are reaction terms of the standard Lotka-Volterra competition type or Fisher-Kolmogorov-Petrovskii-Piskunov reaction type. Also the system (1.1) reduces to the well-known Lotka-Volterra system of predator-prey equations when \( a_{ik} = 0 \) for all \( i, k = 1, 2 \).

The system (1.1) has attracted interests of many mathematicians. We particularly refer the interested readers to the survey paper [38] and the books [24, 25, 37]. The local existence of non-negative solutions is established by H. Amann in the seminal papers [1, 2]. This result is summarized in the following theorem.

**Theorem 1.1 ([1, 2, 3]).** Suppose \( n \geq 2 \) and \( \partial \Omega \) is smooth. Let \( p_0 \in (n, \infty) \) and \( u_0, v_0 \) be non-negative functions in \( W^{1,p_0}(\Omega) \). Then there exists a maximal time \( t_{\text{max}} \in (0, \infty] \) such that the system (1.1) has a unique non-negative solution in \( \Omega \times (0, t_{\text{max}}) \) with \( u, v \in C([0, t_{\text{max}}), W^{1,p_0}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0, t_{\text{max}})) \).

Moreover, if \( t_{\text{max}} < \infty \) then

\[
\lim_{t \to t_{\text{max}}} \left[ \|u(\cdot, t)\|_{W^{1,p_0}(\Omega)} + \|v(\cdot, t)\|_{W^{1,p_0}(\Omega)} \right] = \infty. \tag{1.2}
\]

Many efforts have been made to investigate the existence globally in time of solutions for (1.1). In some special cases with very strong restrictions on the spatial dimension \( n \) and the coefficients \( d_k, a_{ik}, i, k = 1, 2 \), the solutions are proved to exist globally in time (see [10, 11, 15, 17, 18, 20, 21, 22, 31, 32, 33, 36]). Despite these achievements, whether this full system possesses global time solutions or finite time blow up solutions remains challenging and vastly open, even for \( n = 2 \).

In this paper, we study the system (1.1) when there are self-diffusion in one species and no cross-diffusion in the other. Specifically, we investigate (1.1) when \( a_{11} > 0 \) and \( a_{21} = 0 \):

\[
\begin{cases}
  u_t = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(a_1 - b_1u - c_1v) & \text{in } \Omega \times (0, \infty), \\
  v_t = \Delta[(d_2 + a_{22}v)v] + v(a_2 - b_2v - c_2v) & \text{in } \Omega \times (0, \infty), \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
  u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0 & \text{in } \Omega.
\end{cases} \tag{1.3}
\]

The system (1.3) was studied in [10, 11, 17, 18, 20, 21, 22, 33, 34] where the global time existence is established either with some restrictive conditions on the coefficients or for small \( n \). For the latter, the result is proved by Lou-Ni-Wu [22] for \( n = 2 \), by Le-Nguyen-Nguyen [21] and Choi-Lui-Yamada [11] for \( n \leq 5 \), and by Phan [34] for \( n \leq 9 \). However, whether the solution of the system (1.3) exists globally in time for every dimension \( n \) is still a well-known open problem. This question is on the list of open problems made by Y. Yamada in [38]. One main purpose of the current paper is to give it an affirmative answer. Precisely, we prove the following result:

**Theorem 1.2.** Suppose \( n \geq 2 \) and \( \partial \Omega \) is smooth. Let \( a_{11} > 0 \) and \( u_0, v_0 \) be non-negative functions in \( W^{1,p_0}(\Omega) \) for some \( p_0 > n \). Then the system (1.3) possesses a unique, non-negative global solution \((u, v)\) with \( u, v \in C([0, \infty), W^{1,p_0}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0, \infty)) \).

Let us discuss the main difficulties and our strategy of proving Theorem 1.2. Thanks to Theorem 1.1, it is sufficient to show that condition (1.2) for finite time blowup does not happen. It is known that this task could be achieved if one can obtain \( L^\infty \)-estimates for the solutions \( u \) and \( v \) in finite time intervals. As there exists the maximum principle for the second equation in (1.3), the central issue is to establish the boundedness for \( u \). For this, the maximum principle is naturally of
our first consideration. Unfortunately, such maximum principle is not available for the system and this presents a serious obstacle.

One possible approach to get around the lack of the maximum principle for the system is to exploit the first equation in (1.3) to get $L^p$-estimates for $u$ for sufficiently large $p$. Since the Laplacian term in this equation can be expressed as $\nabla \cdot [(d_1 + 2a_{11}u + a_{12}v)\nabla u] + a_{12} \nabla \cdot [u\nabla v]$, the approach is only plausible if one is able to show that $\nabla v \in L^p$ for large $p$. However, this type of gradient estimates for $v$ is essentially not known. We would like to stress that the classical Sobolev regularity theory \cite{14,16,23} as well as its very recent developments \cite{4,5,6,9,26} cannot be applied to get $W^{1,p}$-estimates for $v$ due to the nonlinear structure in the second equation in (1.3). In previous studies, many authors tried to avoid dealing with this key issue by using De Giorgi-Nash-Moser techniques to establish $C^\alpha$-regularity for $v$ first. However for nonlinear equations of reaction-diffusion type in (1.3), this also requires the establishment of a priori $L^p$-estimate for $u$ for some $p > (n + 2)/2$. In general, obtaining such $L^p$-estimate for $u$ is not known and challenging unless one assume that $n \leq 9$. This is the main reason that limits the known works such as \cite{11,17,22,21,33,34,36} to small dimension $n$ only.

Our purpose is to tackle directly the problem of obtaining $L^p$-estimates for $\nabla v$ in terms of $L^p$ norms of $u$. We establish new global $W^{1,p}$-estimates of Calderón-Zygmund type that are suitable for the scalar nonlinear diffusion equation appearing in (1.1). This is our second goal of the paper which is also a topic of independent interest in view of recent developments in \cite{4,5,6,9,14,19,16,23,26}. Not only does it help to prove Theorem 1.2, we believe that our result on $W^{1,p}$-estimates also gives some insight into the structure of equations in (1.3) that is not known before. For the scaling and transformation invariant reason that will be explained below, we study equations in more general form than the one in (1.3).

For any fixed $T > 0$, we consider the following class of nonlinear parabolic equations:

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} = \nabla \cdot [(1 + \alpha u)\mathbf{A} \nabla u] + \beta^2 u(1 - \lambda u) - \lambda \theta u & \text{in } \Omega_T := \Omega \times (0, T],
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T),
\end{cases}
\end{equation}

where $\alpha \geq 0$, $\theta, \lambda > 0$ are constants, and $c(x, t)$ is a non-negative measurable function. We also assume that

\begin{equation}
\mathbf{A} = (a_{ij}) : \Omega_T \rightarrow \mathcal{M}^{n \times n} \text{ is symmetric, measurable, and there exists } \Lambda > 0 \text{ such that:}
\end{equation}

\begin{equation}
\Lambda^{-1} |\xi|^2 \leq \xi^T \mathbf{A}(x, t) \xi \leq \Lambda |\xi|^2 \quad \text{for almost every } (x, t) \in \Omega_T \text{ and all } \xi \in \mathbb{R}^n.
\end{equation}

Here $\mathcal{M}^{n \times n}$ is the linear space of $n \times n$ matrices of real numbers. Our goal is to derive global $W^{1,p}$-estimates for weak solution $u$ of (1.4) for a general class of $\mathbf{A}$ and a general domain $\Omega$. To state the result, we need the following definitions.

**Definition 1.3.** Given $R > 0$. Let $\mathbf{A}$ be a function from $\Omega_T$ to $\mathcal{M}^{n \times n}$. We define

$$
[\mathbf{A}]_{BMO(\Omega_T)} = \sup_{0 < \rho \leq R} \sup_{(y,s) \in \Omega_T} \frac{1}{K_\rho(y,s) \cap \Omega_T} \int_{K_\rho(y,s) \cap \Omega_T} |\mathbf{A}(x, t) - \mathbf{A}_B(y,s)\mathcal{O}(t)|^2 \, dx dt,
$$

where $K_\rho(y,s) = B_\rho(y) \times (s - \rho^2, s)$ is a parabolic cube and $\mathbf{A}_B(t) = \int_0^t A(x, t) \, dx$.

**Definition 1.4.** For $\delta, R > 0$, we say that $\Omega$ is $(\delta, R)$-Lipschitz if for every $x_0 \in \partial \Omega$ there exists a Lipschitz continuous function $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ such that - upon relabeling and reorienting the coordinate axes - we have

$$
\Omega \cap B_R(x_0) = \{(x', x_n) \in B_R(x_0) : x_n > \gamma(x')\}
$$
and

\[
\text{Lip}(\gamma) \overset{\text{def}}{=} \sup \left\{ \frac{|\gamma(x') - \gamma(y')|}{|x' - y'|} : (x', \gamma(x')) , (y', \gamma(y')) \in B_R(x_0) , x' \neq y' \right\} \leq \delta.
\]

Our main result on the regularity of solutions to (1.4) is the following theorem:

**Theorem 1.5.** Let \( n \geq 2, T > 0, \alpha \geq 0, \lambda > 0, 0 < \theta \leq 1, \) and \( A \) satisfy (1.5). Assume that \( c \) is a non-negative function in \( L^p(\Omega_T) \) for some \( p > 2 \). There exists a number \( \delta = \delta(p, R, \Lambda, \alpha, n) > 0 \) such that if \( \Omega \) is \((\delta, R)\)-Lipschitz and \( |A|_{BMO(R, \Omega_T)} \leq \delta \), then any weak solution \( u \) of the problem (1.4) with \( 0 \leq u \leq \lambda^{-1} \) in \( \Omega_T \) satisfies

\[
\int_{\Omega \times [\bar{t}, T]} |\nabla u|^p \, dx \, dt \leq C \left\{ \left( \frac{\theta}{\bar{\Lambda}} \vee \|u\|_{L^2(\Omega_T)} \right)^p + \int_{\Omega_T} |c|^p \, dx \, dt \right\}
\]

for every \( \bar{t} \in (0, T) \). Here \( C > 0 \) is a constant depending only on \( \Omega, \bar{t}, p, R, \Lambda, n, \) and \( \alpha \), but independent of \( \theta, \lambda \).

We remark that the condition \( 0 \leq u \leq \lambda^{-1} \) is natural to ensure that the equation is uniformly parabolic, and is not restrictive for applications (see Lemma 3.2). It is also worth mentioning that \( W^{1,p} \)-estimates for linear parabolic equations are obtained in [41, 5, 6].

The proof of Theorem 1.5 is given in Section 2. We employ the perturbation technique introduced by Caffarelli-Peral [9] for equations in divergent form. Similar approach is also used in [41, 5, 6, 26]. This technique is a variation of the method developed by Caffarelli [7] for fully nonlinear uniformly elliptic equations (see also [8]). We note that the second equation in (1.3) is not invariant with respect to the scalings \( u(x, t) \rightarrow s^{-1}u(sx, s^2t) \) and \( u(x, t) \rightarrow r^{-1}u(x, t) \) for \( s, r > 0 \). It is also not invariant with respect to the transformation that flattens the boundary of \( \Omega \). This presents a serious problem in establishing global \( W^{1,p} \)-estimates without assuming any smallness condition on the relevant functions. We handle this by introducing the pair of constants \( \lambda, \theta \) and the coefficient matrix \( A \) into (1.4) to ensure that this class of equations is invariant under the mentioned scalings and transformation. The parameters \( \lambda \) and \( \theta \) play a key role in our approach. On the other hand, this creates technical difficulties in obtaining approximation estimates that are uniformly in both \( \lambda \) and \( \theta \) (Lemmas 2.11 and 2.21). We overcome this by delicate analysis combining compactness argument with energy estimates.

Next, we outline our strategy for proving Theorem 1.2. First note that the equation of \( v \) in (1.3) can be written in the form (1.4). Therefore, if \( u \in L^p(\Omega_T) \), for some \( p > 2 \), we can apply Theorem 1.5 to derive the \( L^p \)-estimates for \( \nabla v \). Using this new information in the equation of \( u \), we establish the \( L^q \)-estimates for \( u \) with some \( q > p \) depending on \( p \). We then repeat the process using the improved estimate \( u \in L^q \) and applying Theorem 1.5 to the equation of \( v \) to gain \( \nabla v \in L^q \), and so on. With such iteration, we are able to obtain \( \nabla v, u \in L^q \) for sufficiently large \( q \in (2, \infty) \). Combining this with the classical regularity and the known results in [17, 33, 34], we derive a contradiction to (1.2). The full proof of Theorem 1.2 will be given in Section 3.

We close the introduction by noting that the partial differential equations in (1.3) can be rewritten in the following divergence form:

\[
\bar{u}_t = \nabla \cdot \left[ J(x, t, \bar{u}) \nabla \bar{u} \right] + f(x, t, \bar{u}),
\]

where

\[
\bar{u} = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad J(x, t, \bar{u}) = \begin{bmatrix} d_1 + 2a_{11}u + a_{12}v & a_{13}v \\ 0 & d_2 + 2a_{22}v \end{bmatrix}.
\]

Equations of the general form (1.7) appear frequently in many areas of physical and biological applications with different types of nonlinearities for \( J \) and \( f \) (see, for examples, [16, 24, 25, 35, 37]). In the simple case when \( J \) is independent of \( \bar{u} \), they become the standard reaction-diffusion equations and have been studied extensively in the theory of parabolic equations (see [16, 24]). In
our case, the dependence of $J$ on $\tilde{u}$ creates mathematical and physical interesting phenomena and great technical complications. Although we focus only on the explicit system (1.3), the method in this paper might be extended to study general systems of form (1.7) with some structural conditions on $J$ and $f$.

2. Regularity of Solutions to Self-Diffusion Equations

2.1. Existence and uniqueness of weak solutions. This subsection proves the existence and uniqueness of solution of (1.4). We first introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be an open bounded Lipschitz domain, $T > 0$ and $\Omega_T = \Omega \times (0, T]$. Let $\Gamma$ be a relatively open connected subset of $\partial \Omega$. Denote

$$\partial_\Gamma \Omega_T = \Gamma \times (0, T) \cup \Omega \times \{0\}, \quad \partial_\Omega \Omega_T = (\partial \Omega \setminus \Gamma) \times (0, T), \quad \partial_\nu \Omega_T = \partial \Omega \times (0, T) \cup \Omega \times \{0\}.$$  

We also denote the following spaces

$${\mathcal{W}}(\Omega_T) = \{u \in L^2(0, T; H^1(\Omega)) : u_t \in L^2(0, T; H^{-1}(\Omega))\},$$

(2.1)

$${\mathcal{W}}(\Omega_T) = \{u \in L^2(0, T; H^1(\Omega)) : u_t \in L^2(0, T; H^{-1}(\hat{\Omega}))\},$$

where $H^{-1} = (H_0^1)^*$ and $\hat{H}^{-1} = (\hat{H}_0^1)^*$. Moreover, the spaces $\mathcal{W}(\Omega_T)$ and $\hat{\mathcal{W}}(\Omega_T)$ are endowed with the following norms:

$$\|u\|_{\mathcal{W}(\Omega_T)} = \|u\|_{L^2(\Omega_T)} + \|\nabla u\|_{L^2(\Omega_T)} + \|u_t\|_{L^2(0, T; H^{-1}(\Omega))},$$

$$\|u\|_{\hat{\mathcal{W}}(\Omega_T)} = \|u\|_{L^2(\Omega_T)} + \|\nabla u\|_{L^2(\Omega_T)} + \|u_t\|_{L^2(0, T; \hat{H}^{-1}(\Omega))}.$$  

Note that $\hat{\mathcal{W}}(\Omega_T) \subset \mathcal{W}(\Omega_T)$, and $\hat{\mathcal{W}}(\Omega_T) = \mathcal{W}(\Omega_T)$ when $\Gamma = \partial \Omega$. It is well-known that the embedding

$$\mathcal{W}(\Omega_T) \hookrightarrow C([0, T]; L^2(\Omega))$$

is continuous, and the embedding

$$(2.2) \quad \hat{\mathcal{W}}(\Omega_T) \hookrightarrow L^2(\Omega_T)$$

is compact. Therefore, if $u \in \mathcal{W}(\Omega_T)$ then $u(\cdot, t)$ is well-defined and in $L^2(\Omega)$ for each $t \in [0, T]$. Since $\hat{\mathcal{W}}(\Omega_T) \subset \mathcal{W}(\Omega_T)$, these statements also hold true for $\hat{\mathcal{W}}(\Omega_T)$ in place of $\mathcal{W}(\Omega_T)$. Finally, for the spaces of test functions, we define

$$\mathcal{E}_0(\Omega_T) = \{\varphi \in L^2(0, T; H^1(\Omega)) : \varphi = 0 \quad \text{on} \quad \partial_\Gamma \Omega_T\},$$

$$\hat{\mathcal{E}}_0(\Omega_T) = \{\varphi \in L^2(0, T; H^1(\Omega)) : \varphi = 0 \quad \text{on} \quad \partial_\Omega \Omega_T\}.$$  

Definition 2.1. Let $g \in \mathcal{W}(\Omega_T)$, $f \in L^2(0, T; \hat{H}^{-1}(\Omega))$ and $\mathbf{A}$ satisfy (1.5). Let $\alpha \geq \theta \geq 0$ and let $c$ be a measurable function on $\Omega_T$.

(a) We say that $u \in \mathcal{W}(\Omega_T)$ is a weak solution of

$$u_t = \nabla \cdot [(1 + \alpha u)\mathbf{A} \nabla u] + \theta u(1 - u) - cu + f \quad \text{in} \quad \Omega_T,$$

if $\alpha u \nabla u$, $\theta u^2$, $cu \in L^2(\Omega_T)$ and

$$\int_0^T \langle u, \varphi \rangle_{{H^{1,0}, H_0^{1,1}}} dt + \int_{\Omega_T} \left\{ (1 + \alpha u)\langle \mathbf{A} \nabla u, \nabla \varphi \rangle - \{\theta u(1 - u) - cu\} \varphi \right\} dx dt - \int_0^T \langle f, \varphi \rangle_{{\hat{H}^{1,1}, \hat{H}_0^{1,1}}} dt = 0,$$

for all $\varphi \in \mathcal{E}_0(\Omega_T)$. 

(b) We say that \( u \in \hat{W}(\Omega_T) \) is a weak solution of

\[
\begin{aligned}
    u_t &= \nabla \cdot [(1 + \alpha u)A \nabla u] + \theta u(1-u) - cu + f \quad \text{in} \quad \Omega_T, \\
    u &= g \quad \text{on} \quad \partial_D \Omega_T, \\
    \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial_N \Omega_T,
\end{aligned}
\]

(2.4)

if \( \alpha u \nabla u, \theta u^2, cu \in L^2(\Omega_T) \), \( u - g \in \hat{E}_0(\Omega_T) \)

and

\[
\int_0^T \langle u_t, \varphi \rangle_{H^{-1}, H^1_0} dt + \int_{\Omega_T} \{(1 + \alpha u)(A \nabla u, \nabla \varphi) - [\theta u(1-u) - cu] \varphi \} dx dt - \int_0^T \langle f, \varphi \rangle_{H^{-1}, H^1_0} dt = 0,
\]

(2.5)

for all \( \varphi \in \hat{E}_0(\Omega_T) \).

In fact, (2.5) is equivalent to the following variational formulation: for any \( v \in H^1_0(\Omega) \) and almost every \( t \in (0, T) \) one has

\[
\langle u_t, v \rangle_{H^{-1}, H^1_0} + ((1 + \alpha u)A \nabla u, \nabla v)_{L^2(\Omega)} = (\theta u(1-u) - cu, v)_{L^2(\Omega)} + \langle f, v \rangle_{H^{-1}, H^1_0}.
\]

Similar equivalence applies to (2.5). From now on, if there is no confusion, we drop the subscripts \( H^{-1}, H^1_0 \) and \( \hat{H}^{-1}, \hat{H}^1_0 \) for the product notation \( \langle \cdot, \cdot \rangle \). In the statements above and calculations below, \( \langle \cdot, \cdot \rangle \) is also used to denote the scalar product in \( \mathbb{R}^n \), but its meaning is clear in the context.

We now can state the main theorem of this subsection.

**Theorem 2.2.** Let \( A \) satisfy (1.5). Suppose the numbers \( \alpha, \theta \) are non-negative, \( c \in L^2(\Omega_T) \) is non-negative, and \( g \in \hat{W}(\Omega_T) \) satisfies \( 0 \leq g \leq 1 \). Then there exists a unique weak solution \( u \in \hat{W}(\Omega_T) \), \( 0 \leq u \leq 1 \) of the equation

\[
\begin{aligned}
    u_t &= \nabla \cdot [(1 + \alpha u)A \nabla u] + \theta u(1-u) - cu \quad \text{in} \quad \Omega_T, \\
    u &= g \quad \text{on} \quad \partial_D \Omega_T, \\
    \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial_N \Omega_T.
\end{aligned}
\]

(2.6)

Moreover, there is a constant \( C = C(T, \Lambda, \alpha, \theta) \) such that

\[
\sup_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) dx + \int_{\Omega_T} |\nabla u|^2 dx dt \leq C \left[ |\Omega| + \| c \|^2_{L^2(\Omega_T)} + \| g \|^2_{\hat{W}(\Omega_T)} \right].
\]

(2.7)

We prove the uniqueness first. This plays a key role in the existence of solutions to (2.6) and in our paper.

**Lemma 2.3.** Let \( A, \alpha, \theta, c, g \) be as in Theorem 2.2. Then (2.6) has at most one weak solution \( u \in \hat{W}(\Omega_T) \) with \( 0 \leq u \leq 1 \).

**Proof.** Suppose \( u_1, u_2 \in \hat{W}(\Omega_T) \) are two weak solutions of (2.6) satisfying \( 0 \leq u_1, u_2 \leq 1 \). Let

\[
w = \left( u_1 + \alpha \frac{u_1^2}{2} \right) - \left( u_2 + \alpha \frac{u_2^2}{2} \right) = \left( u_1 - u_2 \right) \left( 1 + \alpha \frac{u_1 + u_2}{2} \right).
\]

For each \( k \in \mathbb{N} \), we define the Lipschitz approximations to the \( \text{sgn}^+ \) function:

\[
\text{sgn}^+_k(z) = \begin{cases} 
1 & \text{for } z \geq \frac{1}{k}, \\
kz & \text{for } 0 < z < \frac{1}{k}, \\
0 & \text{for } z \leq 0.
\end{cases}
\]

(2.8)

Since the function \( z \mapsto \text{sgn}^+_k(z) \) is Lipschitz continuous, \( w \in L^2(0, T; H^1(\Omega)) \) and \( w = 0 \) on \( \partial_D \Omega_T \), we have \( \text{sgn}^+_k(w) \in L^2(0, T; H^1(\Omega)) \) with \( \text{sgn}^+_k(w) = 0 \) on \( \partial_D \Omega_T \). Hence by using \( \text{sgn}^+_k(w) \) as a test
function in equation $(2.6)$ for $u_1$, $u_2$ and using integration by parts, one gets
\[
\int_{\Omega} (u_1 - u_2) \cdot \text{sgn}_k^+ (w) \, dx = - \int_{\Omega} \langle A(x, t) [(1 + au_1) \nabla u_1 - (1 + au_2) \nabla u_2], \nabla [\text{sgn}_k^+ (w)] \rangle \, dx \\
\quad + \theta \int_{\Omega} (u_1 - u_2)(1 - u_1 - u_2) \text{sgn}_k^+ (w) \, dx - \int_{\Omega} c(u_1 - u_2) \text{sgn}_k^+ (w) \, dx \\
= - \int_{\Omega} \langle A(x, t) \nabla w, \nabla w \rangle (\text{sgn}_k^+)'(w) \, dx + \theta \int_{\Omega} (u_1 - u_2)(1 - u_1 - u_2) \text{sgn}_k^+ (w) \, dx \\
\quad - \int_{\Omega} c(u_1 - u_2) \text{sgn}_k^+ (w) \, dx.
\]
As $A(x, t)$ is non-negative definite and $(\text{sgn}_k^+)' \geq 0$, we deduce that
\[
\int_{\Omega} (u_1 - u_2) \cdot \text{sgn}_k^+ (w) \, dx \leq \theta \int_{\Omega} (u_1 - u_2)(1 - u_1 - u_2) \text{sgn}_k^+ (w) \, dx - \int_{\Omega} c(u_1 - u_2) \text{sgn}_k^+ (w) \, dx.
\]
Letting $k \to \infty$ and observing that $\text{sgn}(w) = \text{sgn}(u_1 - u_2)$, we obtain
\[
\frac{d}{dt} \int_{\Omega} (u_1 - u_2)^+ \, dx \leq \theta \int_{\Omega} (u_1 - u_2)^+ \, dx
\]
yielding
\[
\int_{\Omega} (u_1(x, t) - u_2(x, t))^+ \, dx \leq e^{\theta t} \int_{\Omega} (u_1(x, 0) - u_2(x, 0))^+ \, dx \quad \text{for every } t > 0.
\]
Since $u_1 - u_2 = 0$ on $\Omega \times \{0\}$, it follows that $(u_1 - u_2)^+ = 0$ a.e. on $\Omega_T$, which gives $u_1 \leq u_2$ a.e. on $\Omega_T$. By interchanging the role of $u_1$ and $u_2$, we infer that $u_1 = u_2$ a.e. on $\Omega_T$. \qed

A modification of the proof of Lemma 2.3 gives the following comparison principle:

**Lemma 2.4.** Assume that $A$ satisfies $\text{(1.5)}$. Suppose that $c \in L^2(\Omega_T)$ is non-negative, $g \in W(\Omega_T)$ and $f \in L^2(0, T; H^1(\Omega))$. Let $u_1, u_2 \in W(\Omega_T)$ be respectively weak sub-solution and weak super-solution to the problem

\[
\begin{cases} 
  u_t &= \nabla \cdot [A \nabla u] - cu + f & \text{in } \Omega_T, \\
  u &= g & \text{on } \partial_D \Omega_T, \\
  \frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial_N \Omega_T.
\end{cases}
\]

That is, $(u_1 - g)^+ \in \hat{E}_0(\Omega_T)$, $(g - u_2)^+ \in \hat{E}_0(\Omega_T)$ and for every function $\varphi \in \hat{E}_0(\Omega_T)$ with $\varphi \geq 0$, we have
\[
\int_0^T \langle (u_1), \varphi \rangle \, dt \leq \int_{\Omega_T} \left\{ - \langle A \nabla u_1, \nabla \varphi \rangle + [\theta u_1(1 - u_1) - cu_1] \varphi \right\} \, dx \, dt + \int_0^T \langle f, \varphi \rangle \, dt
\]
and
\[
\int_0^T \langle (u_2), \varphi \rangle \, dt \geq \int_{\Omega_T} \left\{ - \langle A \nabla u_2, \nabla \varphi \rangle + [\theta u_2(1 - u_2) - cu_2] \varphi \right\} \, dx \, dt + \int_0^T \langle f, \varphi \rangle \, dt.
\]
Then $u_1 \leq u_2$ almost everywhere on $\Omega_T$.

**Proof.** Let $w = u_1 - u_2$. Then $w \leq 0$ on $\partial_D \Omega_T$ since $u_1 \leq g \leq u_2$ on $\partial_D \Omega_T$. Hence $\text{sgn}_k^+ (w) \in L^2(0, T; H^1(\Omega))$ with $\text{sgn}_k^+ (w) = 0$ on $\partial_D \Omega_T$, where $\text{sgn}_k^+$ is the function given by $(2.8)$. Therefore, by arguing as in the proof of Lemma 2.3 we obtain
\[
\int_{\Omega} (u_1(x, t) - u_2(x, t))^+ \, dx \leq \int_{\Omega} (u_1(x, 0) - u_2(x, 0))^+ \, dx \quad \text{for every } t > 0.
\]
As $u_1 \leq u_2$ on $\Omega \times \{0\}$, it follows that $(u_1 - u_2)^+ = 0$ a.e. on $\Omega_T$. That is, $u_1 \leq u_2$ a.e. on $\Omega_T$. \qed
Next, we prove the energy estimate (2.7).

**Lemma 2.5.** Let $A, \alpha, \theta, g, c$ be as in Theorem 2.2 and $f \in L^2(0; \mathcal{H}^{-1}(\Omega))$. Suppose $u \in \mathcal{W}(\Omega_T)$ is a weak solution of (2.4) with $0 \leq u \leq 1$. Then there exists $C > 0$ depending only on $T, \Lambda, \alpha$ and $\theta$ such that

$$\sup_{0 \leq t \leq T} \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \leq C \left[ |\Omega| + \|c\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2(0; \mathcal{H}^{-1}(\Omega))} + \|g\|^2_{\mathcal{W}(\Omega_T)} \right].$$

**Proof.** Let $w = u - g$ and use this as the test function for the equation of $u$. We then have

$$\langle w, w \rangle + \langle g, w \rangle = -\int_\Omega (1 + \alpha u) (A \nabla u, \nabla w) \, dx + \int_\Omega \theta u (1 - u) w \, dx - \int_\Omega c u w \, dx + \langle f, w \rangle.$$

This can be rewritten as

$$\langle w_t, w \rangle + \langle g_t, w \rangle = -\int_\Omega (1 + \alpha u) (A \nabla w, \nabla w) \, dx - \int_\Omega (1 + \alpha u) (A \nabla g, \nabla w) \, dx + \int_\Omega \theta (1 - u) w^2 \, dx + \int_\Omega \theta (1 - u) g w \, dx - \int_\Omega c u w \, dx + \langle f, w \rangle.$$

Using this together with (1.5), Cauchy-Schwartz inequality and the assumption $0 \leq u \leq 1$, we get

$$\frac{d}{dt} \int_\Omega w^2 \, dx + \Lambda^{-1} \int_\Omega |\nabla w|^2 \, dx \leq C \left[ \int_\Omega g^2 \, dx + \int_\Omega |\nabla g|^2 \, dx + \int_\Omega c^2 (x, t) \, dx + \left( \|f(\cdot, t)\|_{\mathcal{H}^{-1}(\Omega)} + \|g(\cdot, t)\|_{\mathcal{H}^{-1}(\Omega)} \right) (\|w\|_{L^2} + \|\nabla w\|_{L^2}) + \int_\Omega w^2 \, dx \right].$$

where $C > 0$ depends only on $\Lambda, \alpha$ and $\theta$. Hence

$$\frac{d}{dt} \int_\Omega w^2 \, dx + (2\Lambda)^{-1} \int_\Omega |\nabla w|^2 \, dx \leq C \left[ \|c(\cdot, t)\|^2_{L^2(\Omega)} + \|f(\cdot, t)\|^2_{L^2(\Omega)} + \left( \|g(\cdot, t)\|^2_{L^2(\Omega)} + \|\nabla g(\cdot, t)\|^2_{L^2(\Omega)} + \|g(\cdot, t)\|^2_{L^2(\Omega)} \right) + \int_\Omega w^2 \, dx \right].$$

Note that $\|w(\cdot, 0)\|_{L^2(\Omega)} = 0$. Then applying Gronwall’s inequality yields

$$\sup_{0 \leq t \leq T} \int_\Omega w^2 \, dx + \int_\Omega |\nabla w|^2 \, dx \leq C \left[ \|c\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2(0; \mathcal{H}^{-1}(\Omega))} + \|g\|^2_{\mathcal{W}(\Omega_T)} \right],$$

for some constant $C = C(T, \Lambda, \alpha, \theta)$. Since $0 \leq g \leq 1$, we therefore obtain the estimate (2.9). \qed

We also need the following result for linear parabolic equations with mixed boundary conditions.

**Lemma 2.6.** Let $A$ satisfy (1.3) and $c, f \in L^2(\Omega_T)$ with $0 \leq f \leq c$. Suppose $g \in \mathcal{W}(\Omega_T)$ with $0 \leq g \leq 1$. Then there exists a unique weak solution $w \in \mathcal{W}(\Omega_T)$ of the problem

$$\begin{cases}
  w_t &= \nabla \cdot [A \nabla w] - cw + f & \text{in } \Omega_T, \\
  w &= g & \text{on } \partial \Omega_T, \\
  \frac{\partial w}{\partial y} &= 0 & \text{on } \partial_N \Omega_T.
\end{cases}
$$

Moreover, $0 \leq w \leq 1$.

**Proof.** For each $k \in \mathbb{N}$, let $c_k = \min\{c, k\}$ and $f_k = \min\{f, k\}$. From the standard theory of linear parabolic equations in divergence forms with bounded coefficients (see, for example, [28, Theorem 9.9]), there exists a unique weak solution $w_k \in \mathcal{W}(\Omega_T)$ of the approximation problem

$$\begin{cases}
  \partial_t w_k &= \nabla \cdot [A \nabla w_k] - c_k w_k + f_k & \text{in } \Omega_T, \\
  w_k &= g & \text{on } \partial \Omega_T, \\
  \frac{\partial w_k}{\partial y} &= 0 & \text{on } \partial_N \Omega_T.
\end{cases}
$$
Since \(0 \leq f_k \leq c_k\) and \(0 \leq g \leq 1\), we see that \(u_1 = 0\) is a weak sub-solution and \(u_2 = 1\) is a weak super-solution to the problem (2.11). Therefore, it follows from Lemma 2.4 that

\[
0 \leq w_k \leq 1 \quad \text{in} \quad \Omega_T, \quad \forall k \in \mathbb{N}.
\]

Also by Lemma 2.5, we obtain

\[
\sup_{0 \leq t \leq T} \int_{\Omega} |w_k|^2 dx + \int_{\Omega_T} |\nabla w_k|^2 dxdt \leq C \left[ |\Omega| + \|c_k\|_{L^2(\Omega_T)}^2 + \|f_k\|_{L^2(\Omega_T)}^2 + \|g\|_{W^1(\Omega_T)}^2 \right]
\]

\[
\leq C \left[ |\Omega| + \|c\|_{L^2(\Omega_T)}^2 + \|f\|_{L^2(\Omega_T)}^2 + \|g\|_{W^1(\Omega_T)}^2 \right].
\]

This together with (2.11) and the boundedness of \(w_k\) yield

\[
\|w_k\|_{\dot{W}(\Omega_T)} \leq C, \quad \forall k \in \mathbb{N}.
\]

By the compact embedding (2.2) and the fact that \(\{c_kw_k\}\) is bounded in \(L^2(0, T; \Omega_T)\), there is a subsequence, still denoted by \(\{w_k\}\), and a function \(w \in \dot{W}(\Omega_T)\) such that

\[
\begin{align*}
&w_k \rightarrow w \text{ strongly in } L^2(\Omega_T), \\
&c_kw_k \rightarrow cw \quad \text{and} \quad \nabla w_k \rightharpoonup \nabla w \text{ weakly in } L^2(\Omega_T), \\
&\partial_tw_k \rightarrow \partial_tw \text{ weakly-* in } L^2(0, T; H^{-1}(\Omega_T)).
\end{align*}
\]

Clearly, \(0 \leq w \leq 1\). Now, by taking \(k \rightarrow \infty\), it follows from (2.11) that \(w\) is a weak solution of (2.10). The uniqueness of the solution \(w\) is guaranteed by Lemma 2.4.

Now, we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2** Thanks to Lemma 2.3 and Lemma 2.5, it remains to prove the existence of a weak solution \(u \in \dot{W}(\Omega_T)\) to (2.6) satisfying \(0 \leq u \leq 1\). We give the proof for the case \(\theta > 0\). The case \(\theta = 0\) is similar and, in fact, simpler. Without loss of generality, let us assume \(\theta = 1\). The proof is based on the Schauder fixed point theorem. Alternatively, one can also use the iterative monotone method based on lower and upper solutions (see [29]). Define

\[
E = \{v \in L^2(\Omega_T) : 0 \leq v \leq 1\}, \quad f(s) = 2s - s^2,
\]

and let

\[
L_v[w] = \nabla \cdot [(1 + \alpha v)A \nabla w] - (c + 1)w.
\]

Note that the function \(f(s)\) is increasing on \([0, 1]\). Hence

\[
0 \leq f(v) \leq f(1) = 1 \leq c + 1, \quad \forall v \in E.
\]

Therefore, it follows from Lemma 2.6 that for each \(v \in E\), there exists a unique weak solution \(w \in \dot{W}(\Omega_T)\) with \(0 \leq w \leq 1\) of the problem

\[
\begin{align*}
\frac{dw}{dt} &= L_v[w] + f(v) \quad \text{in} \quad \Omega_T, \\
-w &= g \quad \text{on} \quad \partial_D \Omega_T, \\
\frac{\partial w}{\partial N} &= 0 \quad \text{on} \quad \partial_N \Omega_T.
\end{align*}
\]

Thus, we can define the map \(L : E \rightarrow E \subset L^2(\Omega_T)\) by \(L(v) = w\), for each \(v \in E\), where \(w\) is the solution of (2.12). It is clear that \(E\) is a closed, convex set in \(L^2(\Omega_T)\). We now seek for \(u \in E\) such that \(u = L(u)\). By [14, Corollary 11.2], it suffices to show that \(L\) is completely continuous. Note that from Lemma 2.5, there is \(C = C(T, \Lambda)\) such that

\[
(2.13) \quad \sup_{0 \leq t \leq T} \int_{\Omega} |L(v)|^2 dx + \int_{\Omega_T} |\nabla L(v)|^2 dxdt \leq C \left[ 1 + |\Omega| + \|c\|_{L^2(\Omega_T)}^2 + \|g\|_{W^1(\Omega_T)}^2 \right], \quad \forall v \in E.
\]

By (2.12), the bound (2.13) and the fact \(0 \leq L(v) \leq 1\), we have

\[
(2.14) \quad \|L(v)\|_{\dot{W}(\Omega_T)} \leq C, \quad \forall v \in E.
\]
From this and the compact imbedding (2.2), we conclude that \( \mathcal{L}(E) \) is pre-compact in \( L^2(\Omega_T) \). Therefore, it remains to show that \( \mathcal{L} \) is continuous in \( L^2(\Omega_T) \)-topology. Let \( \{v_k\} \) be a sequence in \( E \) such that \( v_k \to v \) strongly in \( L^2(\Omega_T) \). For each \( k \in \mathbb{N} \), let \( w_k = \mathcal{L}(v_k) \), i.e. \( 0 \leq w_k \leq 1 \), \( w_k \in \dot{W}(\Omega_T) \) and \( w_k \) is a weak solution of

\[
\begin{aligned}
\partial_t w_k &= \mathbb{L}_{v_k}[w_k] + f(v_k) & \text{in } \Omega_T, \\
 w_k &= g & \text{on } \partial_D \Omega_T, \\
 \frac{\partial w_k}{\partial \nu} &= 0 & \text{on } \partial_N \Omega_T.
\end{aligned}
\]  

(2.15)

From (2.14), we have

\[
\|w_k\|_{\dot{W}(\Omega_T)} \leq C, \quad \forall k \in \mathbb{N}.
\]  

(2.16)

Now, let \( w = \mathcal{L}(v) \), i.e. \( 0 \leq w \leq 1 \), \( w \in \dot{W}(\Omega_T) \) and \( w \) is the weak solution of (2.12). We need to prove that

\[
w_k \to w \quad \text{strongly in } L^2(\Omega_T).
\]  

(2.17)

Let \( \{w'_k\}_k \) be any subsequence of \( \{w_k\} \). From (2.16) and (2.2), there exists a subsequence of \( \{k'_m\}_m \) of \( \{k'\} \) and \( \tilde{w} \in \dot{W}(\Omega_T) \) such that as \( m \to \infty \),

\[
\begin{aligned}
w_{k'_m} &\to \tilde{w} \quad \text{strongly in } L^2(\Omega_T), \\
\nabla w_{k'_m} &\to \nabla \tilde{w} \quad \text{and } v_{k'_m} \nabla w_{k'_m} &\to v \nabla \tilde{w} \quad \text{weakly in } L^2(\Omega_T), \\
\partial_t w_{k'_m} - \partial_t \tilde{w} &\to 0 \quad \text{weakly-* in } L^2(0, T; \dot{H}^{-1}(\Omega)).
\end{aligned}
\]

(2.18)

From (2.15), (2.18), the convergence of \( \{v_k\} \), and the uniform boundedness of \( \{v_k\}, \{w_k\} \), we find that \( \tilde{w} = w = \mathcal{L}(v) \). Thus \( w_{k'_m} \to w \) strongly in \( L^2(\Omega_T) \). Therefore, we infer that (2.17) holds and conclude that the map \( \mathcal{L} \) is continuous. The proof is complete. \( \square \)

2.2. Interior \( W^{1,p} \)-estimates. In this subsection we study interior regularity for solutions to (1.4).

We consider the case \( \alpha > 0 \) since the case \( \alpha = 0 \) is much simpler. For the purpose of brevity, we take \( \alpha = 1 \) from now on. We thus consider the following parabolic equation

\[
u_t = \nabla \cdot [(1 + \lambda u)\nabla u] + \theta^2 u(1 - \lambda u) - \lambda \theta cu
\]  

in \( Q_6 \), where \( \lambda, \theta > 0 \) are constants and \( c(x, t) \) is a non-negative measurable function. The coefficient matrix \( A = (a_{ij}) : Q_6 \to M^{n \times n} \) is assumed to be symmetric, measurable and there exists a constant \( \Lambda > 0 \) such that

\[
\Lambda^{-1} \left| \xi \right|^2 \leq \xi^T A(x, t) \xi \leq \Lambda \left| \xi \right|^2 \quad \text{for a.e. } (x, t) \in Q_6 \text{ and for all } \xi \in \mathbb{R}^n.
\]  

(2.19)

Hereafter, \( Q_p(x, t) \overset{\text{def}}{=} B_p(x) \times (t - \rho^2, t + \rho^2) \) is a centered parabolic cube and \( Q_p \overset{\text{def}}{=} Q_p(0, 0) \). Observe that \( u \) is a weak solution of (2.19) in \( Q_6 \) iff the function \( \bar{u} \overset{\text{def}}{=} \lambda u \) is a weak solution of

\[
\bar{\nu}_t = \nabla \cdot [(1 + \bar{u}) \nabla \bar{u}] + \theta^2 \bar{u}(1 - \bar{u}) - \lambda \theta c \bar{u} \quad \text{in } Q_6.
\]  

(2.20)

We are going to derive interior \( W^{1,p} \)-estimates for solutions of (2.19) by freezing its coefficient and comparing it to solutions of the equation

\[
u_t = \nabla \cdot [(1 + \lambda v) \bar{A}_{B_4}(t) \nabla v] + \theta^2 v(1 - \lambda v) \quad \text{in } Q_4,
\]

(2.21)

where \( \bar{A}_{B_4}(t) \) is the average of \( A(\cdot, t) \) over \( B_4 \), that is, \( \bar{A}_{B_4}(t) := \frac{1}{B_4} \int_{B_4} A(x, t) dx \). Notice that \( v \) is a weak solution of (2.22) iff the function \( \bar{v} \overset{\text{def}}{=} \lambda v \) is a weak solution of

\[
\bar{v}_t = \nabla \cdot [(1 + \bar{v}) \bar{A}_{B_4}(t) \nabla \bar{v}] + \theta^2 \bar{v}(1 - \bar{v}) \quad \text{in } Q_4.
\]  

(2.23)

Our main interior regularity result states as follows:
Theorem 2.7. Assume that \( \lambda > 0, 0 < \theta \leq 1 \), \( A \) satisfies (2.20) and \( c \in L^2(Q_6) \). For any \( p > 2 \), there exists a constant \( \delta = \delta(p, \Lambda, n) > 0 \) such that if

\[
\sup_{0 < \rho \leq 4} \sup_{(x,s) \in Q_1} \int_{Q_6(y,s)} |A(x,t) - \bar{A}_{B_0(t)}|^2 \, dx dt \leq \delta,
\]

and \( u \in W(Q_6) \) is a weak solution of (2.19) satisfying \( 0 \leq u \leq \frac{1}{\lambda} \) in \( Q_5 \), then

\[
\int_{Q_6} |\nabla u|^p \, dx dt \leq C \left\{ \left( \frac{\theta}{\lambda} \sqrt{\|u\|_{L^2(Q_6)}} \right)^p + \int_{Q_6} |c|^p \, dx dt \right\}.
\]

Here \( C > 0 \) is a constant depending only on \( p, \Lambda \) and \( n \).

The proof of this theorem will be given at the end of subsection 2.2.3 and will be based on a series of results presented in the next three subsections.

2.2.1. Some fundamental estimates. Our first result is a \( L^2 \)-estimate for \( \nabla u \) in terms of \( L^2 \)-norm of \( u \).

Lemma 2.8. Assume \( \lambda, \theta > 0 \), \( A \) satisfies (2.20) and \( c \) is a non-negative measurable function on \( Q_4 \). Let \( u \in W(Q_4) \) be a non-negative weak solution of (2.19) in \( Q_4 \). Then there exists \( C = C(n, \Lambda) > 0 \) such that

\[
\int_{Q_4} (1 + \lambda u)|\nabla u|^2 \, dx dt \leq C \int_{Q_4} (1 + \lambda u + \theta^2)u^2 \, dx dt.
\]

Proof. Let \( \varphi \) be the standard cut-off function which is 1 on \( Q_2 \) and zero near \( \partial_{\rho}Q_3 \). Then, by multiplying equation (2.19) by \( \varphi^2 u \) and using integration by parts we get

\[
\int_{Q_4} \left[ \left( \varphi^2 \frac{u^2}{2} \right)_t - \varphi \varphi_u^2 \right] \, dx dt = \int_{Q_4} u \varphi^2 u \, dx dt
\]

\[
= - \int_{Q_4} (1 + \lambda u)\langle A \nabla u, \nabla (\varphi^2 u) \rangle \, dx dt + \theta^2 \int_{Q_4} u(1 - \lambda u)\varphi^2 u \, dx dt - \lambda \theta \int_{Q_4} c \varphi^2 u^2 \, dx dt
\]

\[
\leq - \int_{Q_4} (1 + \lambda u)\langle A \nabla u, \nabla \varphi \varphi \rangle \, dx dt - 2 \int_{Q_4} (1 + \lambda u)\langle A \nabla u, \nabla \varphi \varphi \rangle u \, dx dt + \theta^2 \int_{Q_4} \varphi^2 u^2 \, dx dt.
\]

Using the inequality \( |\langle A \nabla u, \nabla \varphi \varphi \rangle| \leq \langle A \nabla u, \nabla \varphi \rangle \langle A \nabla \varphi, \nabla \varphi \rangle \), we deduce from this that

\[
\int_{Q_4} (1 + \lambda u)\langle A \nabla u, \nabla \varphi \varphi \rangle \varphi^2 dx dt + \frac{1}{2} \int_{Q_4} \varphi(x, 16)^2 u(x, 16)^2 \, dx
\]

\[
\leq 2 \int_{Q_4} (1 + \lambda u) \sqrt{\langle A \nabla u, \nabla \varphi \rangle} \sqrt{\langle A \nabla \varphi, \nabla \varphi \rangle} \varphi u \, dx dt + \int_{Q_4} \varphi \varphi_u^2 dx dt + \theta^2 \int_{Q_4} \varphi u^2 dx dt
\]

\[
\leq \frac{1}{2} \int_{Q_4} (1 + \lambda u)\langle A \nabla u, \nabla \varphi \varphi \rangle \varphi^2 dx dt + \int_{Q_4} \left[ 2(1 + \lambda u)\langle A \nabla \varphi, \nabla \varphi \rangle + \varphi \varphi_u^2 + \theta^2 \varphi^2 \right] u^2 \, dx dt.
\]

Hence it follows from condition (2.20) for \( A \) that

\[
\frac{\Lambda^{-1}}{2} \int_{Q_4} (1 + \lambda u)|\nabla u|^2 \varphi^2 dx dt \leq \int_{Q_4} \left[ 2\Lambda(1 + \lambda u)|\nabla \varphi|^2 + \varphi \varphi_u^2 + \theta^2 \varphi^2 \right] u^2 \, dx dt,
\]

which yields the conclusion (2.25). \( \Box \)

We need the following regularity result for equation (2.25) whose proof is given in Appendix A.
Lemma 2.9. Assume $0 < \theta \leq 1$ and $A_0 : (-16, 16) \to \mathcal{M}^{\alpha\eta}$ is measurable such that
\begin{equation}
\Lambda^{-1}|\xi|^2 \leq \xi^T A_0(t) \xi \leq \Lambda |\xi|^2 \quad \text{for a.e. } t \in (-16, 16) \text{ and for all } \xi \in \mathbb{R}^n.
\end{equation}
Let $\bar{v} \in \mathcal{W}(Q_4)$ be a weak solution of
\begin{equation}
\bar{v}_t = \nabla \cdot ((1 + \bar{v}) A_0(t) \nabla \bar{v}) + \theta^2 \bar{v}(1 - \bar{v}) \quad \text{in } Q_4
\end{equation}
satisfying $0 \leq \bar{v} \leq 1$ in $Q_4$. Then there exists $C > 0$ depending only on $n$ and $\Lambda$ such that
\begin{equation}
\|\nabla \bar{v}\|_{L^2(Q_3)} \leq C \int_{Q_4} |\nabla \bar{v}|^2 \ dx dt.
\end{equation}

The next result will be useful for proving the approximation lemma (Lemma 2.11).

Lemma 2.10. Assume $\bar{u} \in \mathcal{W}(Q_4)$ is a non-negative weak solution of (2.21) in $Q_4$. Suppose $\bar{v} \in \mathcal{W}(Q_4)$ is a weak solution of (2.23) with $\bar{v} = \bar{u}$ on $\partial_B Q_4$ and $0 \leq \bar{v} \leq 1$ in $Q_4$. Then
\begin{equation}
\int_{Q_4} |\bar{u} - \bar{v}|^2 \ dx dt + \Lambda^{-1} \int_{Q_4} |\nabla \bar{u} - \nabla \bar{v}|^2 \ dx dt
\leq 33 \left[ 2\Lambda^3 \int_{Q_4} (|\bar{u} - \bar{v}|^2 + 8)|\nabla \bar{u}|^2 \ dx dt + 3\theta^2 \int_{Q_4} |\bar{u} - \bar{v}|^2 \ dx dt + \Lambda^2 \int_{Q_4} \bar{u}^2 c^2 \ dx dt \right].
\end{equation}

Proof. Let $w = \bar{u} - \bar{v}$. Then it is easy to see that $w \in \mathcal{W}(Q_4)$ is a weak solution of
\begin{equation}
w_t = \nabla \cdot [(1 + w) \hat{A}_B(t) \nabla w] + \nabla \cdot \left\{ |w| A + (1 + w)(A - \hat{A}_B(t)) \nabla \bar{u} \right\} + \theta^2 w(1 - \bar{u} - \bar{v}) - \Lambda \theta c \bar{u} \quad \text{in } Q_4,
\end{equation}
with $w = 0$ on $\partial_B Q_4$. Multiplying the above equation by $w$ and integrating by parts we obtain for each $s \in (-16, 16)$ that
\begin{align*}
\int_{B_4} \frac{w(x, s)}{2} \ dx + \int_{-16}^s \int_{B_4} (1 + \bar{v}) \langle \hat{A}_B(t) \nabla w, \nabla w \rangle \ dx dt \\
= - \int_{-16}^s \int_{B_4} w \langle A \nabla \bar{u}, \nabla w \rangle \ dx dt - \int_{-16}^s \int_{B_4} (1 + \bar{v}) \langle (A - \hat{A}_B(t)) \nabla \bar{u}, \nabla w \rangle \ dx dt \\
+ \theta^2 \int_{-16}^s \int_{B_4} w^2 (1 - \bar{u} - \bar{v}) \ dx dt - \Lambda \theta \int_{-16}^s \int_{B_4} c \bar{u} \ dx dt.
\end{align*}
We deduce from this and condition (2.20) for $A$, which also holds for $\hat{A}_B(t)$, and the fact $\bar{u} \geq 0$, $0 \leq \bar{v} \leq 1$ that
\begin{align*}
\frac{1}{2} \int_{B_4} w(x, s)^2 \ dx + \Lambda^{-1} \int_{-16}^s \int_{B_4} |\nabla w|^2 \ dx dt \\
\leq \Lambda \int_{-16}^s \int_{B_4} |w| |\nabla \bar{u}| |\nabla w| \ dx dt + 4\Lambda \int_{-16}^s \int_{B_4} |\nabla \bar{u}| |\nabla w| \ dx dt \\
+ \theta^2 \int_{-16}^s \int_{B_4} w^2 \ dx dt + \Lambda \theta \int_{-16}^s \int_{B_4} c \bar{u} \ dx dt.
\end{align*}
Hence, applying Cauchy’s inequality and collecting like-terms give
\begin{equation}
\frac{1}{2} \int_{B_4} w(x, s)^2 \ dx + \frac{\Lambda^{-1}}{2} \int_{-16}^s \int_{B_4} |\nabla w|^2 \ dx dt \leq \Lambda^3 \int_{-16}^s \int_{B_4} (w^2 + 8)|\nabla \bar{u}|^2 \ dx dt
\end{equation}
\begin{equation}
+ \frac{3\theta^2}{2} \int_{-16}^s \int_{B_4} w^2 \ dx dt + \frac{\Lambda^2}{2} \int_{-16}^s \int_{B_4} \bar{u}^2 c^2 \ dx dt \quad \text{for each } s \in (-16, 16).
\end{equation}
On the one hand, this immediately yields

\[(2.31) \quad \Lambda^{-1} \int_{Q_4} |\nabla w|^2 \, dx \leq 2\Lambda^3 \int_{Q_4} (w^2 + 8)|\nabla u|^2 \, dx dt + 3\theta^2 \int_{Q_4} w^2 \, dx dt + \lambda^2 \int_{Q_4} \bar{u}^2c^2 \, dx dt.\]

On the other hand, we can drop the second term in (2.30) and then integrate in \(s\) to obtain

\[(2.32) \quad \int_{Q_4} w^2 \, dx dt \leq 64\Lambda^3 \int_{Q_4} (w^2 + 8)|\nabla u|^2 \, dx dt + 96\theta^2 \int_{Q_4} w^2 \, dx dt + 32\lambda^2 \int_{Q_4} \bar{u}^2c^2 \, dx dt.\]

By adding (2.31) and (2.32), we get (2.29). \(\square\)

2.2.2. Interior approximation estimates. We begin this subsection with a result allowing us to approximate a weak solution of (2.19) by that of the reference equation.

**Lemma 2.11.** For any \(\varepsilon > 0\), there exists \(\delta > 0\) depending only on \(\varepsilon\), \(\Lambda\) and \(n\) such that: if \(0 < \theta \leq \lambda\),

\[(2.33) \quad \int_{Q_4} \left[ |A(x, t) - \bar{A}_B(t)|^2 + |c(x, t)|^2 \right] \, dx dt \leq \delta,
\]

and \(u \in \mathcal{W}(Q_5)\) is a weak solution of (2.19) in \(Q_5\) satisfying

\[(2.34) \quad 0 \leq u \leq \frac{1}{\lambda} \text{ in } Q_4 \quad \text{and} \quad \int_{Q_4} |\nabla u|^2 \, dx dt \leq 1,
\]

and \(v \in \mathcal{W}(Q_4)\) is the weak solution of (2.22) with \(v = u\) on \(\overline{\partial Q_4}\) and \(0 \leq v \leq \frac{1}{\lambda} \text{ in } Q_4\), then

\[(2.35) \quad \int_{Q_4} |u - v|^2 \, dx dt \leq \varepsilon^2,
\]

and, consequently,

\[(2.36) \quad \int_{Q_4} |\nabla v|^2 \, dx dt \leq 2 + 66\Lambda \left( 18\Lambda^2 + 3\theta^2\varepsilon^2 + \delta \right).
\]

**Proof.** We first prove (2.35) by contradiction. Suppose that estimate (2.35) is not true. Then there exist \(\varepsilon_0\), \(\Lambda\), \(n\), sequences of numbers \(\{\lambda_k\}_{k=1}^{\infty}\) and \(\{\theta_k\}_{k=1}^{\infty}\) with \(0 < \theta_k \leq \lambda_k\), a sequence of coefficient matrices \(\{A_k\}_{k=1}^{\infty}\), and sequences of non-negative functions \(\{c_k\}_{k=1}^{\infty}\) and \(\{u_k\}_{k=1}^{\infty}\) such that

\[(2.37) \quad \int_{Q_4} \left[ |A_k(x, t) - \bar{A}_k(t)|^2 + |c_k(x, t)|^2 \right] \, dx dt \leq \frac{1}{k},
\]

\(u^k \in \mathcal{W}(Q_5)\) is a weak solution of

\[(2.38) \quad u^k_t = \nabla \cdot [(1 + \lambda_k u^k)A_k \nabla u^k] + \theta_k^2 u_k^2(1 - \lambda_k u^k) - \lambda_k \theta_k c_k u^k \quad \text{in } Q_5
\]

with

\[(2.39) \quad 0 \leq u^k \leq \frac{1}{\lambda_k} \quad \text{in } Q_4,
\]

\[(2.40) \quad \int_{Q_4} |\nabla u^k|^2 \, dx dt \leq 1,
\]

\[(2.41) \quad \int_{Q_4} |u^k - v^k|^2 \, dx dt > \varepsilon_0^2 \quad \text{for all } k.
\]

Here \(\bar{A}_k(t) = \int_{B_4} A_k(x, t) \, dx\) and \(v^k \in \mathcal{W}(Q_4)\) is a weak solution of

\[
\begin{align*}
\{ v^k_t &= \nabla \cdot [(1 + \lambda_k v^k)\bar{A}_k(t)\nabla v^k] + \theta_k^2 v_k^2(1 - \lambda_k v^k) \quad \text{in } Q_4, \\
v^k &= u^k \quad \text{on } \overline{\partial Q_4}
\end{align*}
\]
satisfying $0 \leq \psi^k \leq 1/\lambda_k$ in $Q_4$. Since $0 \leq u^k, \psi^k \leq 1/\lambda_k$ in $Q_4$, we infer from (2.41) that

$$\theta_k \leq \lambda_k \leq \frac{|Q_4|^2}{\varepsilon_0},$$

that is, the sequences $\{\lambda_k\}$ and $\{\theta_k\}$ are bounded. Also $\{{\bar{A}}_k\}$ is bounded in $L^\infty(-16, 16; M^{(\infty, \infty)})$ due to condition (2.20) for $A_k$. Then, by taking subsequences if necessary, we can assume that $\lambda_k \to \lambda$, $\theta_k \to \theta$, $A_k \to A_0$ weakly-* in $L^\infty(-16, 16; M^{(\infty, \infty)})$ and $\bar{A}_k \to \bar{A}_0$ weakly in $L^2(Q_4)$ for some constants $\lambda, \theta$ satisfying $0 \leq \theta \leq \lambda < \infty$ and some $A_0 \in L^\infty(-16, 16; M^{(\infty, \infty)})$. For each vector $\xi \in \mathbb{R}^n$, we have

$$\int_{-16}^{16} \xi^T A_0(t) \xi \phi(t) dt = \lim_{k \to \infty} \int_{-16}^{16} \xi^T \bar{A}_k(t) \xi \phi(t) dt$$

for all non-negative functions $\phi \in L^1(-16, 16)$. This together with the denseness of $Q^n$ in $\mathbb{R}^n$ implies that $A_0$ satisfies condition (2.26). We are going to derive a contradiction by proving the following claim.

**Claim.** There are subsequences $\{u_k^m\}$ and $\{\psi_k^m\}$ such that $u_k^m - \psi_k^m \to 0$ in $L^2(Q_4)$ as $m \to \infty$.

Let us consider the case $\lambda > 0$ first. Then, thanks to (2.39), the sequence $\{u_k\}$ is bounded in $Q_4$. This together with (2.37), (2.38), (2.40) and the boundedness of $\{A_k\}$ and $\{\theta_k\}$ implies that the sequence $\{u_k\}$ is bounded in $W(Q_4)$. Next, we apply Lemma 2.10 for $\tilde{u} \sim \lambda_k u^k$ and $\tilde{v} \sim \lambda_k \psi^k$ to obtain

$$\int_{Q_4} \phi \frac{|\nabla u_k - \nabla \psi_k|^2}{dxdt} \leq 33\sqrt{\lambda} \left[ 18\lambda^3 \int_{Q_4} |\nabla u_k|^2 dxdt + 3\theta_k^2 \int_{Q_4} |u_k - \psi_k|^2 dxdt + \lambda_k^2 \int_{Q_4} (u_k^2 \psi_k^2) dxdt \right]$$

$$\leq 33\sqrt{\lambda} \left[ 18\lambda^3 \int_{Q_4} |\nabla u_k|^2 dxdt + 3|Q_4| + \int_{Q_4} \psi_k^2 dxdt \right].$$

Thanks to (2.37) and (2.40), this gives

$$\int_{Q_4} |\nabla \psi_k|^2 dxdt \leq C \quad \text{for all } k.$$  

Thus, by reasoning as in the case of $\{u_k\}$, the sequence $\{\psi_k\}$ is also bounded in $W(Q_4)$. We infer from these facts and the compact embedding (2.2) that there exist subsequences, still denoted by $\{u_k\}$ and $\{\psi_k\}$, and functions $u, v \in W(Q_4)$ such that

\[
\begin{align*}
\{ u^k \to u \text{ strongly in } L^2(Q_4), & \quad \nabla u^k \to \nabla u \text{ weakly in } L^2(Q_4), \\nonumber \\
\partial_t u^k \to \partial_t u \text{ weakly-* in } L^2(0, T; H^{-1}(B_1)), & \nonumber \\
\{ \psi^k \to v \text{ strongly in } L^2(Q_4), & \quad \nabla \psi^k \to \nabla v \text{ weakly in } L^2(Q_4), \\nonumber \\
\partial_t \psi^k \to \partial_t v \text{ weakly-* in } L^2(0, T; H^{-1}(B_1)). & \nonumber 
\end{align*}
\]

Moreover, from the boundedness of $\{u_k\}$ and (2.37), we see that

\[
\begin{align*}
\{ c_k u_k \to 0 \text{ strongly in } L^2(Q_4) & \quad (1 + \lambda_k u^k)(A_k - \bar{A}_k(t)) \nabla u^k \to 0 \text{ weakly in } L^2(Q_4). \nonumber 
\end{align*}
\]
Then, for all $\varphi \in C_0^\infty(Q_4)$, we see that

$$\lim_{k \to \infty} \int_{Q_4} (1 + \lambda_k u^k)(\bar{A}_k(t)\nabla u^k, \nabla \varphi)dxdt$$

$$= - \lim_{k \to \infty} \int_{Q_4} u^k \bar{A}_k(t) \cdot D^2 \varphi dxdt - \lim_{k \to \infty} \frac{\lambda_k}{2} \int_{Q_4} (u^k)^2 \bar{A}_k(t) \cdot D^2 \varphi dxdt$$

$$= - \int_{Q_4} u A_0(t) \cdot D^2 \varphi dxdt - \frac{\lambda}{2} \int_{Q_4} u^2 A_0(t) \cdot D^2 \varphi dxdt$$

$$= \int_{Q_4} (1 + \lambda u)(A_0(t)\nabla u, \nabla \varphi)dxdt.$$  

Thus by passing $k \to \infty$ for the equation (2.38) and using the boundedness of \{\lambda_k\} and \{\theta_k\}, one sees that $u$ is weak solution of the equation

$$u_t = \nabla \cdot [(1 + \lambda u)A_0(t)\nabla u] + \theta^2 u(1 - \lambda u) \quad \text{in} \quad Q_4.$$  

Similarly, $v$ is a weak solution of

$$v_t = \nabla \cdot [(1 + \lambda v)A_0(t)\nabla v] + \theta^2 v(1 - \lambda v) \quad \text{in} \quad Q_4.$$  

In addition, we infer from the strong convergence of $u^k$ and $v^k$ in $L^2(Q_4)$ that

$$0 \leq u, v \leq \frac{1}{\lambda}, \quad \text{and} \quad u = v \quad \text{on} \quad \partial_p Q_4.$$  

Hence by the uniqueness of the solution of equation (2.43) given by Lemma 2.3, we conclude that $u \equiv v$ in $Q_4$. Therefore, $u^k - v^k \to u - v = 0$ strongly in $L^2(Q_4)$.

Now, consider the case $\lambda = 0$, that is, $\lambda_k \to 0$. Due to $\theta_k \leq \lambda_k$, we also have $\theta_k \to 0$. Let $w^k = u^k - v^k$. Then $w^k$ is a bounded weak solution of

$$w_t^k = \nabla \cdot [(1 + \bar{v}^k)\bar{A}_k(t)\nabla w^k] + \nabla \cdot \left\{[\bar{w}^k A_k + (1 + \bar{v}^k)(A_k - \bar{A}_k(t))][\nabla u^k] \right\}$$

$$+ \theta_k^2 w^k(1 - u^k - v^k) - \theta_k c_A \bar{u}^k \quad \text{in} \quad Q_4,$$

$$w^k = 0 \quad \text{on} \quad \partial_p Q_4,$$

where $\bar{u}^k = \lambda_k u^k$, $\bar{v}^k = \lambda_k v^k$ and $\bar{w}^k = \bar{u}^k - \bar{v}^k$.

Note that $0 \leq \bar{u}^k \leq 1$ in $Q_4$ and $\|\nabla \bar{u}^k\|_{L^2(Q_4)} \to 0$ as $k \to \infty$. We have $\bar{u}^k$ is a weak solution of

$$(\bar{u}^k)_t = \nabla \cdot [(1 + \bar{v}^k)\bar{A}_k(t)\nabla \bar{u}^k] + \theta_k^2 \bar{u}^k(1 - \bar{u}^k) - \lambda_k \theta_k c_A \bar{u}^k \quad \text{in} \quad Q_4.$$  

Then $\{\bar{u}^k\}$ is bounded in $W(Q_4)$. Also, $0 \leq \bar{v}^k \leq 1$ in $Q_4$ and $\bar{v}^k$ is a weak solution of

$$(\bar{v}^k)_t = \nabla \cdot [(1 + \bar{v}^k)\bar{A}_k(t)\nabla \bar{v}^k] + \theta_k^2 \bar{v}^k(1 - \bar{v}^k) \quad \text{in} \quad Q_4, \quad \bar{v}^k = \bar{u}^k \quad \text{on} \quad \partial_p Q_4.$$  

By applying Lemma 2.10 for $\bar{u} \hookrightarrow \bar{u}^k$, $\bar{v} \hookrightarrow \bar{v}^k$ and using the fact $\theta_k$ is small for large $k$, we get for all sufficiently large $k$ that

$$(\bar{u}^k)_t = \nabla \cdot \left\{[1 + \bar{v}^k]A_0(t)\nabla \bar{u}^k] + \theta_k^2 \bar{u}^k(1 - \bar{u}^k) - \lambda_k \theta_k c_A \bar{u}^k \right\} = 0 \quad \text{in} \quad Q_4,$$

and

$$0 \leq ||\bar{u}^k||_{L^2(Q_4)} \leq 1 + \lambda_k \theta_k c_A \bar{u}^k \quad \text{in} \quad Q_4,$$

$$0 \leq ||\bar{v}^k||_{L^2(Q_4)} \leq 1 + \lambda_k \theta_k c_A \bar{v}^k \quad \text{in} \quad Q_4,$$

$$0 \leq ||\bar{w}^k||_{L^2(Q_4)} \leq 1 + \lambda_k \theta_k c_A \bar{w}^k \quad \text{in} \quad Q_4.$$  

Therefore, $\{\nabla w^k\}$ is bounded in $L^2(Q_4)$, and consequently, $\{w^k\}$ is bounded in $W'(Q_4)$. Moreover, $\{\nabla v^k\}$ is bounded in $L^2(Q_4)$ since it follows from (2.40) and (2.46) that

$$\int_{Q_4} (1 + \lambda u)\nabla u \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and

$$\int_{Q_4} (1 + \lambda v)\nabla v \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and

$$\int_{Q_4} (1 + \lambda (u + v))\nabla (u + v) \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and

$$\int_{Q_4} (1 + \lambda (u - v))\nabla (u - v) \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and

$$\int_{Q_4} (1 + \lambda (v - u))\nabla (v - u) \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and

$$\int_{Q_4} (1 + \lambda (w - w))\nabla (w - w) \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and

$$\int_{Q_4} (1 + \lambda (w + w))\nabla (w + w) \cdot \nabla \varphi dxdt \leq C \quad \text{for all large} \quad k.$$  

and
Consequently there are subsequences, still denoted by \{w^k\} and \{\bar{v}^k\} and two functions \(w, \bar{v} \in \mathcal{W}(Q_4)\) with \(0 \leq \bar{v} \leq 1\) in \(Q_4\) such that \(w^k \to w\) and \(\bar{v}^k \to \bar{v}\) strongly in \(L^2(Q_4)\), \(\nabla w^k \to \nabla w\) and \(\nabla \bar{v}^k \to \nabla \bar{v}\) weakly in \(L^2(Q_4)\), \(\partial_\nu w^k \to \partial_\nu w\) and \(\partial_\nu \bar{v}^k \to \partial_\nu \bar{v}\) weakly-* in \(L^2(0, T; H^{-1}(B_4))\).

Since \(\nabla \bar{v}^k = \lambda_k \nabla v^k \to 0\) in \(L^2(Q_4)\) thanks to (2.47), we infer further that \(\nabla \bar{v}^k \to \nabla \bar{v} \equiv 0\) strongly in \(L^2(Q_4)\). Also, by passing to the limit in (2.45) we see that \(\bar{v}\) is a weak solution of \(\bar{v}_t = \nabla \cdot [(1 + \bar{v})A_0(t)\nabla \bar{v}]\) in \(Q_4\). Thus, we deduce that \((\nabla \bar{v}, \bar{v}_t) \equiv 0\) in \(Q_4\) and hence \(\bar{v}\) is a constant function. Due to this fact and by arguing as in (2.42), one gets for all \(\varphi \in C_c^\infty(Q_4)\) that

$$
\lim_{k \to \infty} \int_{Q_4} (1 + \bar{v}^k)(\bar{A}_k(t)\nabla w^k, \nabla \varphi) \, dx \, dt = -\lim_{k \to \infty} \int_{Q_4} w^k \bar{A}_k(t) \cdot D^2 \varphi \, dx \, dt \\
+ \lim_{k \to \infty} \int_{Q_4} (\bar{v}^k - \bar{v})(\bar{A}_k(t)\nabla w^k, \nabla \varphi) \, dx \, dt - \lim_{k \to \infty} \int_{Q_4} \bar{v} \bar{A}_k(t) \cdot D^2 \varphi \, dx \, dt \\
= -\int_{Q_4} w A_0(t) \cdot D^2 \varphi \, dx \, dt - \int_{Q_4} \bar{v} A_0(t) \cdot D^2 \varphi \, dx \, dt = \int_{Q_4} (1 + \bar{v})(A_0(t)\nabla w, \nabla \varphi) \, dx \, dt.
$$

Using the aforementioned convergences and similar to the case \(\lambda > 0\), we can pass to the limit in (2.44) to conclude that \(w\) is a weak solution of

\begin{align}
\left\{ \begin{array}{ll}
w_t &= \nabla \cdot [(1 + \bar{v})A_0(t)\nabla w] & \text{in } Q_4, \\
w &= 0 & \text{on } \partial_\nu Q_4.
\end{array} \right.
\end{align}

By the uniqueness of the trivial solution of the linear equation (2.48), we conclude that \(w \equiv 0\) in \(Q_4\). This gives, again, \(u^k - v^k = w^k \to 0\) in \(L^2(Q_4)\) as \(k \to \infty\).

Therefore, we have proved the Claim which contradicts (2.41). Thus the proof of (2.35) is complete. To prove (2.36), we apply Lemma 2.10 for \(\bar{u} \to \lambda u\) and \(\bar{v} \to \lambda v\) to obtain

$$
\int_{Q_4} |\nabla u - \nabla v|^2 \, dx \, dt \leq 33 \Lambda \left[ 18 \Lambda^3 \int_{Q_4} |\nabla u|^2 \, dx \, dt + 3 \varepsilon^2 \int_{Q_4} |u - v|^2 \, dx \, dt + \int_{Q_4} c^2 \, dx \, dt \right].
$$

This, (2.35) and the assumptions (2.33), (2.34) give

$$
\int_{Q_4} |\nabla u - \nabla v|^2 \, dx \, dt \leq 33 \Lambda \left[ 18 \Lambda^3 + 3 \varepsilon^2 + \delta \right].
$$

Since \(\|\nabla v\|_{L^2(Q_4)} \leq \|\nabla u\|_{L^2(Q_4)} + \|\nabla u - \nabla v\|_{L^2(Q_4)} \leq 1 + \|\nabla u - \nabla v\|_{L^2(Q_4)}\), the estimate (2.36) follows immediately from (2.49). \(\square\)

The next lemma is a localized version of Lemma 2.11 together with a comparison between gradients of solutions. It is crucial for establishing interior \(W^{1,p}\)-estimates in Subsection 2.2.3.

**Lemma 2.12.** Assume that \(0 < \theta \leq \lambda, \theta \leq 1\) and \(0 < r \leq 1\). For any \(\varepsilon > 0\), there exists \(\delta > 0\) depending only on \(\varepsilon, \Lambda\) and \(n\) such that if

\begin{align}
\int_{Q_r} \left[ |A(x, t) - A_{B_r}(t)|^2 + |c(x, t)|^2 \right] \, dx \, dt \leq \delta,
\end{align}

then for any weak solution \(u \in \mathcal{W}(Q_{5r})\) of (2.19) in \(Q_{5r}\) satisfying

\(0 \leq u \leq \frac{1}{\lambda}\) in \(Q_{4r}\) and \(\int_{Q_{4r}} |\nabla u|^2 \, dx \, dt \leq 1\),

and a weak solution \(v \in \mathcal{W}(Q_{4r})\) of

\(\left\{ \begin{array}{ll}
v_t &= \nabla \cdot [(1 + \lambda v)A_{B_r}(t)\nabla v] + \theta^2 v(1 - \lambda v) & \text{in } Q_{4r}, \\
v &= u & \text{on } \partial_\nu Q_{4r}
\end{array} \right.\)
satisfying $0 \leq \nu \leq \frac{1}{4}$ in $Q_{4r}$, we have

\begin{equation}
\int_{Q_{4r}} |u - v|^2 \, dx \, dt \leq \varepsilon^2 r^2,
\end{equation}

\begin{equation}
\int_{Q_{4r}} |\nabla v|^2 \, dx \, dt \leq 4^{n+2} 2 \omega_n \left[ 2 + 66 \Lambda \left( 18 \Lambda^2 + 3 \theta^2 \varepsilon^2 + \delta \right) \right],
\end{equation}

\begin{equation}
\int_{Q_{2r}} |\nabla u - \nabla v|^2 \, dx \, dt \leq \varepsilon^2,
\end{equation}

where $\omega_n$ is the volume of the unit ball $B_1$ in $\mathbb{R}^n$.

**Proof.** Estimates (2.51) and (2.52) are a localized version of Lemma 2.11. Define

\[
u'(x, t) = \frac{u(rx, r^2t)}{r}, \quad \nu'(x, t) = \frac{v(rx, r^2t)}{r}, \quad \Lambda'(x, t) = \Lambda(rx, r^2t) \quad \text{and} \quad \epsilon'(x, t) = \epsilon(rx, r^2t).
\]

Let $\lambda' = \lambda r$ and $\theta' = \theta r$. Then $u'$ is a weak solution of

\[
u'(x, t) = \nabla \cdot \left[ (1 + \lambda'u') \Lambda' \nabla u' \right] + \theta^2 u'(1 - \lambda'u') - \lambda' \theta' \epsilon' u' \quad \text{in} \quad Q_5
\]

and $v'$ is a weak solution of

\[
u'(x, t) = \nabla \cdot \left[ (1 + \lambda'v') \Lambda' \nabla v' \right] + \theta^2 v'(1 - \lambda'v') \quad \text{in} \quad Q_4,
\]

\[v' = u' \quad \text{on} \quad \partial_{\nu} Q_4.
\]

We also have $0 \leq u', v' \leq 1/\lambda'$ in $Q_4$, $\Lambda^{-1} |\xi|^2 \leq \xi^T \Lambda'(x) \xi \leq \Lambda |\xi|^2$ and

\[
\int_{Q_4} |\nabla u'(x, t)|^2 \, dx \, dt = 4^{n+2} 2 \omega_n \int_{Q_4} |\nabla u(y, s)|^2 \, dy \, ds \leq 4^{n+2} 2 \omega_n,
\]

\[
\int_{Q_4} \left[ |\Lambda'(x, t) - \Lambda'B_{4r}(t)|^2 + |\epsilon'(x, t)|^2 \right] \, dx \, dt = 4^{n+2} 2 \omega_n \int_{Q_4} \left[ |\Lambda(y, s) - \Lambda'B_{4r}(s)|^2 + |\epsilon(y, s)|^2 \right] \, dy \, ds.
\]

Therefore, given any $\varepsilon > 0$, by Lemma 2.11 there exists a constant $\delta = \delta(\varepsilon, \Lambda, n) > 0$ such that if condition (2.50) for $\Lambda$ and $\epsilon$ is satisfied then we have

\[
\int_{Q_4} |\nabla u'(x, t) - \nabla v'(x, t)|^2 \, dx \, dt \leq 4^{n+2} 2 \omega_n \varepsilon^2.
\]

By changing variables, we obtain the desired estimate (2.51). On the other hand, the estimate (2.52) is a consequence of (2.36) (see also the calculations at the end of the proof of Lemma 2.11).

We now prove (2.53). Define $w = u - v$. Then $w \in \mathcal{W}(Q_{4r})$ is a bounded weak solution of

\begin{equation}
w_t = \nabla \cdot \left[ (1 + \lambda u) \Lambda \nabla w \right] + \nabla \cdot \left[ \left( \epsilon w \Lambda'B_{4r}(t) + (1 + \lambda u) (\Lambda - \Lambda'B_{4r}(t)) \right) \nabla w \right]
\end{equation}

\[+ \theta^2 w(1 - \lambda u - \lambda v) - \lambda \theta c u \quad \text{in} \quad Q_{4r}.
\]

Let $\varphi$ be the standard cut-off function which is $1$ on $Q_{2r}$, $\text{supp}(\varphi) \subset \overline{Q_{3r}}$, $|\nabla \varphi| \leq C_n/r$ and $|\varphi| \leq C_n/r^2$. We multiply equation (2.54) by $\varphi^2 w$ and use integration by parts to obtain

\[
\int_{Q_{4r}} \left[ \frac{(\varphi^2 w)^2}{2} \right]_t - \varphi \varphi_i w \, dx \, dt = \int_{Q_{4r}} w_i \varphi^2 w \, dx \, dt = -\int_{Q_{4r}} (1 + \lambda u) (\Lambda \nabla w, \nabla (\varphi^2 w)) \, dx \, dt
\]

\[-\int_{Q_{4r}} w (\Lambda'B_{4r}(t) \nabla (\lambda v), \nabla (\varphi^2 w)) \, dx \, dt - \int_{Q_{4r}} (1 + \lambda u) ((\Lambda - \Lambda'B_{4r}(t)) \nabla v, \nabla (\varphi^2 w)) \, dx \, dt
\]

\[+ \theta^2 \int_{Q_{4r}} w(1 - \lambda u - \lambda v) \varphi^2 w \, dx \, dt - \theta \int_{Q_{4r}} c(\lambda u) \varphi^2 w \, dx \, dt.
\]
We deduce from this, condition (2.20) for $A$ and the assumption $\theta \leq 1$ that

\[
\int_{Q_t} (1 + \lambda u) \langle A \nabla w, \nabla \varphi \rangle \varphi^2 \, dx \, dt \leq 2 \int_{Q_t} (1 + \lambda u) \sqrt{\langle A \nabla w, \nabla w \rangle} \sqrt{\langle A \nabla \varphi, \nabla \varphi \rangle} \varphi \, |w| \, dx \, dt \\
+ \Lambda \left( \int_{Q_t} |\nabla (\lambda v)| \nabla |\varphi| \varphi^2 \, |w| \, dx \, dt \right) + 2 \int_{Q_t} |\nabla (\lambda v)| \nabla |\varphi| \varphi w \, dx \, dt \right)
+ \int_{Q_t} \varphi \varphi w \, dx \, dt + \int_{Q_t} \varphi^2 w \, dx \, dt + \int_{Q_t} |c| \varphi \, |w| \, dx \, dt.
\]

Using the Cauchy–Schwarz inequality and moving terms around, we get

\begin{equation}
(2.55) \quad \frac{\Lambda^{-1}}{4} \int_{Q_t} (1 + \lambda u) |\nabla w| \, \varphi^2 \, dx \, dt \leq (4\Lambda^2 + 1) \int_{Q_t} |\nabla \varphi| \, w \, dx \, dt \\
+ \Lambda |\nabla (\lambda v)|_{L^\infty(Q_t)} \left( \int_{Q_t} |\nabla w| \varphi^2 \, |w| \, dx \, dt \right) + 2 \int_{Q_t} |\nabla \varphi| \varphi w \, dx \, dt \right)
+ 2(\Lambda + 2) \int_{Q_t} |A - \mathbf{A}_{B_t}(t)|^2 |\nabla v| \varphi^2 \, dx \, dt \right)
+ \int_{Q_t} \varphi \varphi w \, dx \, dt + \int_{Q_t} w \, dx \, dt + \int_{Q_t} |c| \, |w| \, dx \, dt.
\end{equation}

We estimate $|\nabla (\lambda v)|_{L^\infty(Q_t)}$ and $|\nabla v|_{L^\infty(Q_t)}$ next. Let us define $\bar{v}(x, t) = \lambda v(rx, r^2 t)$ for $(x, t) \in Q_4$. Then $0 \leq \bar{v} \leq 1$ in $Q_4$, and $\bar{v}$ is a weak solution of

\[
\bar{v}_t = \nabla \cdot [(1 + \bar{v}) \mathbf{A}'_{B_t}(t) \nabla \bar{v}] + (\theta r)^2 \bar{v}(1 - \bar{v}) \quad \text{in} \quad Q_4.
\]

Thanks to $\theta r \leq \theta \leq 1$, we then can use Lemma [2.9] to get

\begin{equation}
(2.56) \quad |\nabla \bar{v}|_{L^\infty(Q_3)} \leq C(\Lambda, n) \left( \int_{Q_3} |\nabla \bar{v}|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\end{equation}

This together with Lemma [2.8] yields $|\nabla \bar{v}|_{L^\infty(Q_3)} \leq C(\Lambda, n)$. By rescaling back from $Q_3$ to $Q_{3r}$, we obtain

\begin{equation}
(2.57) \quad |\nabla (\lambda v)|_{L^\infty(Q_{3r})} \leq \frac{C(\Lambda, n)}{r}.
\end{equation}

On the other hand, (2.56) also gives

\begin{equation}
(2.58) \quad |\nabla v|_{L^\infty(Q_{3r})} \leq C(\Lambda, n) \left( \int_{Q_3} |\nabla v(ry, r^2 s)|^2 \, dy \, ds \right)^{\frac{1}{2}} = C(\Lambda, n) \left( \int_{Q_3} |\nabla v(y, s)|^2 \, dy \, ds \right)^{\frac{1}{2}}.
\end{equation}

It follows from (2.55), (2.57) and (2.58) that

\[
\frac{\Lambda^{-1}}{4} \int_{Q_t} (1 + \lambda u) \nabla w \, \varphi^2 \, dx \, dt \leq \frac{C}{r} \int_{Q_t} w \, dx \, dt + \frac{C}{r} \int_{Q_t} |\nabla w| \varphi^2 \, |w| \, dx \, dt \\
+ C \left( \int_{Q_t} |\nabla v|^2 \, dx \, dt \right) \left( \int_{Q_t} |A - \mathbf{A}_{B_t}(t)|^2 \, dx \, dt \right) + \int_{Q_t} |c| \, |w| \, dx \, dt.
\]
which together with the Cauchy–Schwarz inequality yields
\begin{equation}
\int_{Q_r} |\nabla w|^2 \varphi^2 \, dx \, dt \leq \frac{C}{r^2} \int_{Q_r} w^2 \, dx \, dt + C \left( \int_{Q_r} |\nabla v|^2 \, dx \, dt \right) \left( \int_{Q_r} |A - \bar{A}_{B_r}(t)|^2 \, dx \, dt \right)
+ \left( \int_{Q_r} c^2 \, dx \, dt \right)^\frac{1}{2} \left( \int_{Q_r} w^2 \, dx \, dt \right)^\frac{1}{2}.
\end{equation}

Next notice that we can assume \( \delta < \varepsilon^2 \). Then by using \((2.51)\) and \((2.52)\), we have
\[
\int_{Q_r} w^2 \, dx \, dt = \int_{Q_r} |u - v|^2 \, dx \, dt \leq \varepsilon^2 r^2
\]
and
\[
\int_{Q_r} |\nabla v|^2 \, dx \, dt \leq 4^{n+2} 2\omega_n \left[ 2 + 66\Lambda \left( 18\Lambda^2 + 4\varepsilon^2 \right) \right] \leq C'(1 + \varepsilon^2).
\]
By combining these with \((2.59)\) and \((2.50)\) we get
\[
\int_{Q_r} |\nabla w|^2 \varphi^2 \, dx \, dt \leq C\varepsilon^2 + CC'(1 + \varepsilon^2)\varepsilon^2 + \varepsilon^2 \leq (C + 2CC' + 1)\varepsilon^2,
\]
where \( C, C' > 0 \) depend only on \( \Lambda \) and \( n \). Thus \( \int_{Q_{2r}} |\nabla w|^2 \, dx \, dt \leq C(\Lambda, n)\varepsilon^2 \) and the proof is complete. \( \square \)

**Remark 2.13.** Since our equations are invariant under the translation \((x, t) \mapsto (x + y, t + s)\), Lemma \[2.12\] still holds true if \( Q_r \) is replaced by \( Q_r(y, s) \).

**2.2.3. Interior density and gradient estimates.** We will derive interior \( W^{1,p} \)-estimates for solution \( u \) of \((2.19)\) by estimating the distribution functions of the maximal function of \( |\nabla u|^2 \). The precise maximal operators will be used are:

**Definition 2.14.** (i) The parabolic-Littlewood maximal function of a locally integrable function \( f \) on \( \mathbb{R}^n \times \mathbb{R} \) is defined by
\[
(\mathcal{M}f)(x, t) = \sup_{\rho > 0} \int_{Q_{\rho}(x, t)} |f(y, s)| \, dy \, ds.
\]
(ii) If \( f \) is defined in a region \( U \subset \mathbb{R}^n \times \mathbb{R} \), then we denote
\[
\mathcal{M}_U f = \mathcal{M}(\chi_U f).
\]

The next result gives a density estimate for the distribution of \( \mathcal{M}_{Q_r}(|\nabla u|^2) \).

**Lemma 2.15.** Assume that \( 0 < \theta \leq \lambda, \theta \leq 1, A \) satisfies \((2.20)\) and \( c \in L^2(Q_0) \). There exists a constant \( N > 1 \) depending only on \( \Lambda \) and \( n \) such that for any \( \varepsilon > 0 \), we can find \( \delta = \delta(\varepsilon, \Lambda, n) > 0 \) satisfying: if
\begin{equation}
\sup_{0 < \rho \leq 4} \sup_{(y, s) \in Q_1} \int_{Q_{\rho}(y, s)} |A(x, t) - \bar{A}_{B_r}(t)|^2 \, dx \, dt \leq \delta,
\end{equation}
then for any weak solution \( u \in \mathcal{W}(Q_0) \) of \((2.19)\) with \( 0 \leq u \leq \frac{1}{\Lambda} \) in \( Q_5 \) and for any \((y, s) \in Q_1, 0 < r \leq 1 \) with
\begin{equation}
Q_r(y, s) \cap Q_1 \cap \{Q_5 : \mathcal{M}_{Q_1}(|\nabla u|^2) \leq 1\} \cap \{Q_5 : \mathcal{M}_{Q_1}(c^2) \leq \delta\} \neq \emptyset,
\end{equation}
we have
\[
\left| \{Q_1 : \mathcal{M}_{Q_1}(|\nabla u|^2) > N\} \cap Q_r(y, s) \right| \leq \varepsilon |Q_r(y, s)|.
\]
Proof. By condition (2.61), there exists a point \((x_0, t_0) \in Q_r(y, s) \cap Q_1\) such that
\[
\mathcal{M}_{Q_2}(\|\nabla u\|^2)(x_0, t_0) \leq 1 \quad \text{and} \quad \mathcal{M}_{Q_2}(c^2)(x_0, t_0) \leq \delta.
\]
Notice that \(Q_{5r}(y, s) \subset Q_6\). Since \(Q_{4r}(y, s) \subset Q_{5r}(x_0, t_0) \cap Q_5\), it follows from (2.62) that
\[
\int_{Q_{4r}(y, s)} |\nabla u|^2 \, dx dt \leq \frac{|Q_{5r}(x_0, t_0)|}{|Q_{4r}(y, s)|} \int_{Q_{5r}(x_0, t_0) \cap Q_5} |\nabla u|^2 \, dx dt \leq \left(\frac{5}{4}\right)^{n+2},
\]
\[
\int_{Q_{4r}(y, s)} c^2 \, dx dt \leq \frac{|Q_{5r}(x_0, t_0)|}{|Q_{4r}(y, s)|} \int_{Q_{5r}(x_0, t_0) \cap Q_5} c^2 \, dx dt \leq \left(\frac{5}{4}\right)^{n+2} \delta.
\]
Also the assumption (2.60) gives
\[
\int_{Q_{4r}(y, s)} |A(x, t) - \bar{A}_{B_{4r}(y)}(t)|^2 \, dx dt \leq \delta.
\]
Therefore, we can use (2.53) and Remark 2.13 to obtain
\[
(2.63) \quad \int_{Q_{2r}(y, s)} |\nabla u - \nabla v|^2 \, dx dt \leq \eta^2,
\]
where \(v \in \mathcal{W}(Q_{4r}(y, s))\) is the unique weak solution of
\[
\begin{cases}
  v_t = \nabla \cdot [(1 + \lambda v)\bar{A}_{B_{4r}(y)}(t)\nabla v] + \theta^2 v(1 - \lambda v) & \text{in } Q_{4r}(y, s), \\
  v = u & \text{on } \partial_{p} Q_{4r}(y, s),
\end{cases}
\]
satisfying \(0 \leq v \leq 1/\lambda\) in \(Q_{4r}(y, s)\), and \(\delta = \delta(\eta, \Lambda, n)\) with \(\eta\) being determined later. We remark that the existence and uniqueness of such weak solution \(v\) is guaranteed by Theorem 2.2.

Let \(\bar{v}(x, t) = \lambda v(rx + y, r^2t + s)\) and \(A'(x, t) = A(rx + y, r^2t + s)\) for \((x, t) \in Q_4\). Then \(0 \leq \bar{v} \leq 1\) in \(Q_4\) and \(\bar{v}\) is a weak solution of
\[
\bar{v}_t = \nabla \cdot [(1 + \bar{v})\bar{A}'_{B_{4r}(y)}(t)\nabla \bar{v}] + (\theta r)^2 \bar{v}(1 - \bar{v}) \quad \text{in } Q_4.
\]
Since \(\theta r \leq \theta \leq 1\), applying Lemma 2.9 we get
\[
\|\nabla \bar{v}\|_{L^\infty(Q_{2r})}^2 \leq C \int_{Q_2} |\nabla \bar{v}|^2 \, dx dt,
\]
which together with (2.63) and (2.62) gives
\[
(2.64) \quad \|\nabla v\|_{L^\infty(Q_{5r}(y, s))}^2 \leq C \int_{Q_{2r}} |\nabla v|^2 \, dx dt \leq 2C \left( \int_{Q_{2r}} |\nabla u - \nabla v|^2 \, dx dt + \int_{Q_{4r}} |\nabla u|^2 \, dx dt \right) \leq C(\Lambda, n)(\eta^2 + 1).
\]
We claim that (2.62), (2.63) and (2.64) yield
\[
(2.65) \quad \{Q_r(y, s) : \mathcal{M}_{Q_{2r}(y, s)}(\|\nabla u - \nabla v\|^2) \leq C(\Lambda, n) \} \subset \{Q_r(y, s) : \mathcal{M}_{Q_2}(\|\nabla u\|^2) \leq N\}
\]
with \(N = \max \{6C(\Lambda, n), 5^{n+2}\}\). Indeed, let \((x, t)\) be a point in the set on the left hand side of (2.65), and consider \(Q_p(x, t)\). If \(\rho \leq r/2\), then \(Q_p(x, t) \subset Q_{3r/2}(y, s) \subset Q_3\) and hence
\[
\frac{1}{|Q_p(x, t)|} \int_{Q_{p}(x, t) \cap Q_5} |\nabla u|^2 \, dx dt \leq \frac{2}{|Q_p(x, t)|} \left( \int_{Q_{p}(x, t) \cap Q_5} |\nabla u - \nabla v|^2 \, dx dt + \int_{Q_{p}(x, t) \cap Q_5} |\nabla v|^2 \, dx dt \right) \leq 2M_{Q_{2r}(y, s)}(\|\nabla u - \nabla v\|^2)(x, t) + 2\|\nabla v\|_{L^\infty(Q_{2r}(y, s))}^2 \leq 2C(\Lambda, n)(\eta^2 + 2) \leq 6C(\Lambda, n).
\]
On the other hand if \( \rho > r/2 \), then \( Q_{\rho}(x,t) \subset Q_{2\rho}(x_0,t_0) \). This and the first inequality in (2.62) imply that

\[
\frac{1}{|Q_{\rho}(x,t)|} \int_{Q_{\rho}(x,t) \cap Q_5} |\nabla u|^2 \, dx dt \leq \frac{5^{n+2}}{|Q_{2\rho}(x_0,t_0)|} \int_{Q_{2\rho}(x_0,t_0) \cap Q_5} |\nabla u|^2 \, dx dt \leq 5^{n+2}.
\]

Therefore, \( M_{Q_5}(|\nabla u|^2)(x,t) \leq N \) and the claim (2.65) is proved. Note that (2.65) is equivalent to

\[
\{Q_r(y,s) : M_{Q_r}(|\nabla u|^2) > N\} \subset \{Q_r(y,s) : M_{Q_2(y,s)}(|\nabla u - \nabla v|^2) > C(\Lambda, n)\}.
\]

It follows from this, the weak type 1–1 estimate and (2.63) that

\[
\left|\{Q_r(y,s) : M_{Q_r}(|\nabla u|^2) > N\}\right| \leq \left|\{Q_r(y,s) : M_{Q_2(y,s)}(|\nabla u - \nabla v|^2) > C(\Lambda, n)\}\right|
\]

\[
\leq C \int_{Q_2(y,s)} |\nabla u - \nabla v|^2 \, dx dt \leq C\eta^2 |Q_r(y,s)|,
\]

where \( C' > 0 \) depends only on \( \Lambda \) and \( n \). By choosing \( \eta = \sqrt{\varepsilon} \), we obtain the desired result. \( \Box \)

In view of Lemma 2.15, we can apply the Vitali covering lemma (see [5, Lemma 2.4]) for \( E = \{Q_1 : M_{Q_5}(|\nabla u|^2) > N\} \) and \( F = \{Q_1 : M_{Q_5}(|\nabla u|^2) > 1\} \cup \{Q_1 : M_{Q_5}(c^2) > \delta\} \) to obtain:

**Lemma 2.16.** Assume that \( 0 < \theta \leq \lambda, \theta \leq 1, A \) satisfies (2.20) and \( c \in L^2(Q_0) \). There exists a constant \( N > 1 \) depending only on \( \Lambda \) and \( n \) such that for any \( \varepsilon > 0 \), we can find \( \delta = \delta(\varepsilon, \Lambda, n) > 0 \) satisfying: if

\[
\sup_{0 < \rho \leq 4} \sup_{(y,s) \in Q_1} \int_{Q_{\rho}(y,s)} |A(x,t) - \tilde{A}_{B_{\rho}(y)}(t)|^2 \, dx dt \leq \delta,
\]

then for any weak solution \( u \in W(Q_0) \) of (2.19) satisfying

\[
0 \leq u \leq \frac{1}{\lambda} \quad \text{in} \quad Q_5 \quad \text{and} \quad \left|\{Q_1 : M_{Q_5}(|\nabla u|^2) > N\}\right| \leq \varepsilon |Q_1|,
\]

we have

\[
\left|\{Q_1 : M_{Q_5}(|\nabla u|^2) > N\}\right| \leq 2(10)^{n+2} \varepsilon \left|\{Q_1 : M_{Q_5}(|\nabla u|^2) > 1\}\right| + \left|\{Q_1 : M_{Q_5}(c^2) > \delta\}\right|.
\]

We are now ready to prove Theorem 2.7

**Proof of Theorem 2.7** Let \( N > 1 \) be as in Lemma 2.16 and let \( q = p/2 > 1 \). We choose \( \varepsilon = \varepsilon(p, \Lambda, n) > 0 \) be such that

\[
\varepsilon_1 \overset{\text{def}}{=} 2(10)^{n+2} \varepsilon = \frac{1}{2N^q},
\]

and let \( \delta = \delta(p, \Lambda, n) \) be the corresponding constant given by Lemma 2.16

Assuming for a moment that \( u \) satisfies

\[
\left|\{Q_1 : M_{Q_5}(|\nabla u|^2) > N\}\right| \leq \varepsilon |Q_1|.
\]

We first consider the case \( \theta \leq \lambda \). Then it follows from Lemma 2.16 that

\[
\left|\{Q_1 : M_{Q_5}(|\nabla u|^2) > N\}\right| \leq \varepsilon_1 \left|\{Q_1 : M_{Q_5}(|\nabla u|^2) > 1\}\right| + \left|\{Q_1 : M_{Q_5}(c^2) > \delta\}\right|.
\]

Let us iterate this estimate by considering

\[
u_1(x,t) = \frac{u(x,t)}{\sqrt{N}}, \quad c_1(x,t) = \frac{c(x,t)}{\sqrt{N}} \quad \text{and} \quad \lambda_1 = \sqrt{N} \lambda \geq \theta.
\]

It is easy to see that \( u_1 \in W(Q_0) \) is a weak solution of

\[
(u_1)_i = \nabla \cdot [(1 + \lambda_1 u_1)A\nabla u_1] + \theta^2 u_1(1 - \lambda_1 u_1) - \lambda_1 \theta c_1 u_1 \quad \text{in} \quad Q_0.
\]
Moreover, thanks to (2.66) we have
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_1|^2) > N| = |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > N| \leq \varepsilon |Q_1|. \]
Therefore, by applying Lemma 2.16 to \( u_1 \) we obtain
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_1|^2) > N| \leq \varepsilon_1 \left( |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_1|^2) > 1| + |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|c_1|^2) > \delta| \right) \]
\[ = \varepsilon_1 \left( |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > N| + |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta N| \right). \]
We infer from this and (2.67) that
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_2|^2) > N| \leq \varepsilon_1^2 |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_2|^2) > 1| \]
\[ + \varepsilon_1^2 |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta| + \varepsilon_1 |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta N|. \]
Next, let
\[ u_2(x, t) = \frac{u(x, t)}{N}, \quad c_2(x, t) = \frac{c(x, t)}{N} \quad \text{ and } \quad \lambda_2 = b \lambda \geq \theta. \]
Then \( u_2 \in \mathcal{W}(Q_6) \) is a weak solution of
\[ (u_2)_t = \nabla \cdot \left( (1 + \lambda_2 u_2) A \nabla u_2 \right) + \theta^2 u_2 (1 - \lambda_2 u_2) - \lambda_2 \theta c_2 u_2 \quad \text{ in } \quad Q_6 \]
and
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_2|^2) > N| = |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > N^3| \leq \varepsilon |Q_1|. \]
Hence by applying Lemma 2.16 to \( u_2 \) we get
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_2|^2) > N| \leq \varepsilon_1 \left( |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u_2|^2) > 1| + |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|c_2|^2) > \delta| \right) \]
\[ = \varepsilon_1 \left( |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > N^2| + |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta N^2| \right). \]
This together with (2.68) gives
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u^2|^2) > N^3| \leq \varepsilon_1^3 |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > 1| + \sum_{i=1}^{3} \varepsilon_i |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta N^{3-i}|. \]
By repeating the iteration, we then conclude that
\[ |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u^2|^2) > N^k| \leq \varepsilon_1^k |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > 1| + \sum_{i=1}^{k} \varepsilon_i |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta N^{k-i}| \]
for all \( k = 1, 2, \ldots \). Since
\[ \int_{Q_1} \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2)^q \, dx \, dt = q \int_0^\infty t^{q-1} \left| \left( Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > t \right) \right| \, dt \]
\[ = q \int_0^N t^{q-1} \left| \left( Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > t \right) \right| \, dt + q \sum_{k=1}^{N} \int_{N^{k}}^{N^{k+1}} t^{q-1} \left| \left( Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > t \right) \right| \, dt \]
\[ \leq N^q |Q_1| + (N^q - 1) \sum_{k=1}^{\infty} N^q k |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > N^k|, \]
we obtain
\[ \int_{Q_1} \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2)^q \, dx \, dt \leq N^q |Q_1| + (N^q - 1) \sum_{k=1}^{\infty} N^q k |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(|\nabla u|^2) > N^k| \]
\[ \leq N^q |Q_1| + (N^q - 1) |Q_1| \sum_{k=1}^{\infty} (\varepsilon_1 N^q)^k + \sum_{k=1}^{\infty} (N^q - 1) N^q k \varepsilon_i |Q_1 : \mathcal{M}_{\mathcal{Q}_5}(c^2) > \delta N^{k-i}|. \]
But we have
\[
\sum_{k=1}^{\infty} \sum_{i=1}^{k} (N^{\theta} - 1)N^{q}\varepsilon_{1i}^j \mathcal{Q}_1 : \mathcal{M}_q(c^2) > \delta N^{k-i-j} \]
\[
= \left( \frac{N}{\delta} \right)^q \sum_{i=1}^{\infty} (\varepsilon_1 N^q)^j \left[ \sum_{k=1}^{\infty} (N^{\theta} - 1)\delta^q N^{q(k-j-1)} \mathcal{Q}_1 : \mathcal{M}_q(c^2) > \delta N^{k-i-j} \right] \]
\[
= \left( \frac{N}{\delta} \right)^q \sum_{i=1}^{\infty} (\varepsilon_1 N^q)^j \left[ \sum_{j=0}^{\infty} (N^{\theta} - 1)\delta^q N^{q(j-1)} \mathcal{Q}_1 : \mathcal{M}_q(c^2) > \delta N^j \right] \]
\[
\leq \left( \frac{N}{\delta} \right)^q \left[ \int_{Q_1} \mathcal{M}_q(c^2)^q \, dx \, dt \right] \sum_{i=1}^{\infty} (\varepsilon_1 N^q)^j, \]

where we have used Remark 2.17 below to get the last inequality. Thus we infer that
\[
\int_{Q_1} \mathcal{M}_q(|\nabla u|^2)^q \, dx \, dt \leq N^q |Q_1| + \left( \frac{N}{\delta} \right)^q \int_{Q_1} \mathcal{M}_q(c^2)^q \, dx \, dt \sum_{k=1}^{\infty} (\varepsilon_1 N^q)^k \]
\[
= N^q |Q_1| + \left( \frac{N}{\delta} \right)^q \int_{Q_1} \mathcal{M}_q(c^2)^q \, dx \, dt \sum_{k=1}^{\infty} 2^{-k} \]
\[
\leq C \left( 1 + \int_{Q_1} \mathcal{M}_q(c^2)^q \, dx \, dt \right) \]

with the constant \(C\) depending only on \(p, \Lambda\) and \(n\). On the other hand, by the Lebesgue differentiation theorem one has
\[
|\nabla u(x, t)|^2 = \lim_{\rho \to 0^+} \int_{Q_{\rho}(x, t)} |\nabla u(y, s)|^2 \, dy \, ds \leq \mathcal{M}_q(|\nabla u|^2)(x, t) \]

for almost every \((x, t) \in Q_1\). Therefore, it follows from the strong type \(q - q\) estimate for the maximal function and the fact \(q = p/2\) that

\[ (2.69) \quad \int_{Q_1} |\nabla u|^p \, dx \, dt \leq C \left( 1 + \int_{Q_3} |c|^p \, dx \, dt \right). \]

The estimate \(2.69\) was derived under the assumption that \(\theta \leq \lambda\). In the case \(\theta > \lambda\), we define \(u' = u/K\), \(c' = c/K\) and \(\lambda' = \lambda K\) where \(K = \theta/\lambda > 1\). Then \(u'\) is a weak solution of
\[
u_i' = \nabla \cdot \left[ (1 + \lambda' u') \mathbf{A} \nabla u' \right] + \theta^2 u'(1 - \lambda' u') - \lambda' \theta c' u' \quad \text{in} \quad Q_6. \]

Since \(\theta \leq \lambda'\) and \(u'\) inherits the property \(2.66\) from that of \(u\), we can employ \(2.69\) to conclude that
\[
\int_{Q_1} |\nabla u'|^p \, dx \, dt \leq C \left( 1 + \int_{Q_3} |c'|^p \, dx \, dt \right). \]

This implies that

\[ (2.70) \quad \int_{Q_1} |\nabla u|^p \, dx \, dt \leq C \left( \left( \frac{\theta}{\lambda} \right)^p + \int_{Q_3} |c|^p \, dx \, dt \right). \]

Combining \(2.69\) and \(2.70\) yields

\[ (2.71) \quad \int_{Q_1} |\nabla u|^p \, dx \, dt \leq C \left( \left( \frac{\theta}{\lambda} \right)^p \vee 1 + \int_{Q_3} |c|^p \, dx \, dt \right). \]
as long as \( \lambda > 0 \) and \( 0 < \theta \leq 1 \). We next remove the extra assumption \((2.66)\) for \( u \). Notice that for any \( M > 0 \), by using the weak type \( 1 - 1 \) estimate for the maximal function and Lemma \((2.8)\) we get

\[
\left|\{ Q_1 : M_{Q_6}(\| \nabla u \|)^2 > NM^2\}\right| \leq \frac{C}{NM^2} \int_{Q_6} \| \nabla u \|^2 \, dx dt \leq \frac{C_n}{M^2} \int_{Q_6} u^2 \, dx dt.
\]

Therefore, if we let

\[
\bar{u}(x,t) = \frac{u(x,t)}{M} \quad \text{with} \quad M^2 = \frac{C_n\|u\|_{L^2(Q_6)}^2}{\varepsilon|Q_1|}
\]

then \( \left|\{ Q_1 : M_{Q_6}(\| \nabla \bar{u} \|)^2 > N\}\right| \leq \varepsilon|Q_1| \). Hence we can apply \((2.71)\) to \( \bar{u} \) with \( c \) and \( \lambda \) being replaced by \( \bar{c} = c/M \) and \( \bar{\lambda} = \lambda M \). By reversing back to the functions \( u \) and \( c \), we obtain \((2.24)\). \(\square\)

**Remark 2.17.** Assume that \( V \subset U \subset \mathbb{R}^n \times \mathbb{R} \), \( c \in L^2(U) \) and \( q > 1 \). Then for any \( \delta > 0 \) and \( N > 1 \), we have

\[
(2.72) \quad \sum_{j=0}^{\infty} (N^q - 1)\delta^q N^{q(j-1)}|\{ V : M_U(c^2) > \delta N^j\}| \leq \int_V M_U(c^2)^q \, dx dt.
\]

Indeed,

\[
\int_V M_U(c^2)^q \, dx dt = \int_0^\infty t^{q-1}|\{ V : M_U(c^2) > t\}| \, dt \geq \sum_{j=0}^{\infty} \int_{\delta N^{j-1}}^{\delta N^j} t^{q-1}|\{ V : M_U(c^2) > t\}| \, dt \geq \sum_{j=0}^{\infty} \int_{\delta N^{j-1}}^{\delta N^j} ((\delta N^j)^q - (\delta N^{j-1})^q)|\{ V : M_U(c^2) > \delta N^j\}|.
\]

Note that our interior gradient estimate for \( u \) in Theorem \((2.7)\) is independent of the boundary values of \( u \) on \( \partial_p Q_6 \). On the contrary, the interior \( W^{1,p}\)-estimates obtained in \((4, 5)\) for linear parabolic equations depend essentially on the boundary values of the solutions.

### 2.3. Boundary \( W^{1,p}\)-estimates on flat domains

We will use the following notation:

\[
B^+_p = \{ x \in B_q : x_n > 0 \}, \quad \partial_c B^+_p = \partial B_q \cap \{ x : x_n > 0 \},
\]

\[
T_p = B_q \cap \{ x : x_n = 0 \}, \quad \bar{T}_p = T_q \times (-\rho^2, \rho^2),
\]

\[
Q^+_p = B_q \times (-\rho^2, \rho^2), \quad Q^+_q = B_q^+ \times (-\rho^2, \rho^2),
\]

\[
\partial_c Q^+_p = \partial_c B^+_q \times (-\rho^2, \rho^2), \quad \partial_b Q^+_q = \partial B^+_q \times \{ -\rho^2 \}.
\]

Our aim is to derive boundary \( W^{1,p}\)-estimates for solutions to the problem

\[
(2.73) \quad \begin{cases} u_t = \nabla \cdot [(1 + \lambda u)A \nabla u] + \partial^2 u(1 - \lambda u) - \lambda \theta cu & \text{in } Q^+_q, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \bar{T}_q, \end{cases}
\]

where \( \theta, \lambda > 0 \) are constants and \( c(x,t) \) is a non-negative measurable function. We assume that \( A : Q^+_q \to \mathcal{M}^{\text{sym}} \) is symmetric, measurable and there exists a constant \( \Lambda > 0 \) such that

\[
(2.74) \quad \Lambda^{-1}|\xi|^2 \leq \xi^T A(x,t) \xi \leq \Lambda |\xi|^2 \quad \text{for a.e. } (x,t) \in Q^+_q \text{ and for all } \xi \in \mathbb{R}^n.
\]

Throughout this subsection, the space \( \tilde{W}(Q^+_q) \) is defined as in \((2.1)\) with \( \Omega = Q^+_q \) and \( \Gamma = \partial_c B^+_q \). Note also that in this case \( \partial_b Q^+_q = \partial_b Q^+_q \cup \partial_c Q^+_q \).
2.3.1. **Boundary approximation estimates.** Let us consider the parabolic equation

\[
(2.75) \quad \begin{cases} 
    u_t - \nabla \cdot ([1 + \lambda u] A \nabla u) + \theta^2 u(1 - \lambda u) - \lambda \theta c u = 0 & \text{in } Q^+_t, \\
    \frac{\partial u}{\partial \nu} = \psi & \text{on } \bar{T}_4, \\
    u = 0 & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{cases}
\]

Observe that \( u \) is a weak solution of (2.75) iff the function \( \bar{u} \overset{\text{def}}{=} \lambda u \) is a weak solution of

\[
(2.76) \quad \begin{cases} 
    \bar{u}_t = \nabla \cdot ([1 + \bar{u}] A \nabla \bar{u}) + \theta^2 \bar{u}(1 - \bar{u}) - \lambda \theta c \bar{u} & \text{in } Q^+_4, \\
    \frac{\partial \bar{u}}{\partial \nu} = 0 & \text{on } \bar{T}_4, \\
    \bar{u} = \bar{\psi} \overset{\text{def}}{=} \lambda \psi & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{cases}
\]

We will establish boundary \( W^{1,p} \) estimates for solutions to (2.75) by freezing its coefficient and comparing it to solutions of the equation

\[
(2.77) \quad \begin{cases} 
    v_t = \nabla \cdot ([1 + \lambda v] \tilde{A}_B(t) \nabla v) + \theta^2 v(1 - \lambda v) & \text{in } Q^+_4, \\
    \frac{\partial v}{\partial \nu} = 0 & \text{on } \bar{T}_4, \\
    v = u & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{cases}
\]

Notice that \( v \) is a weak solution of (2.77) iff the function \( \bar{v} \overset{\text{def}}{=} \lambda \bar{v} \) is a weak solution of

\[
(2.78) \quad \begin{cases} 
    \bar{v}_t = \nabla \cdot ([1 + \bar{v}] \tilde{A}_B(t) \nabla \bar{v}) + \theta^2 \bar{v}(1 - \bar{v}) & \text{in } Q^+_4, \\
    \frac{\partial \bar{v}}{\partial \nu} = 0 & \text{on } \bar{T}_4, \\
    \bar{v} = \bar{u} & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{cases}
\]

By arguing similarly to the proof of Lemma 2.8, we have:

**Lemma 2.18.** Assume that \( \lambda, \theta > 0 \), \( A \) satisfies (2.74) and \( c \) is a non-negative measurable function on \( Q^+_4 \). Let \( u \in \dot{W}(Q^+_4) \) be a non-negative weak solution of (2.73). Then there exists a constant \( C > 0 \) depending only on \( \Lambda \) and \( n \) such that

\[
\int_{Q^+_4} (1 + \lambda u) |\nabla u|^2 \, dx \, dt \leq C \int_{Q^+_4} (1 + \lambda u + \theta^2 u)^2 \, dx \, dt.
\]

We will need the following boundary \( W^{1,\infty} \)-estimate for solutions of the reference equation.

**Lemma 2.19.** Assume that \( 0 < \theta \leq 1 \) and \( A_0 : (-16, 16) \to M^{n \times n} \) is a measurable matrix-valued function satisfying (2.26). Let \( \bar{v} \in \dot{W}(Q^+_4) \) be a weak solution of

\[
\begin{cases} 
    \bar{v}_t = \nabla \cdot ([1 + \bar{v}] A_0(t) \nabla \bar{v}) + \theta^2 \bar{v}(1 - \bar{v}) & \text{in } Q^+_4, \\
    \frac{\partial \bar{v}}{\partial \nu} = 0 & \text{on } \bar{T}_4
\end{cases}
\]

satisfying \( 0 \leq \bar{v} \leq 1 \) in \( Q^+_4 \). Then there exists \( C = C(\Lambda, n) > 0 \) such that

\[
\int_{Q^+_4} |\nabla \bar{v}|^2 \, dx \, dt \leq C \int_{Q^+_4} |\nabla \bar{v}|^2 \, dx \, dt.
\]

**Proof.** From the classical boundary regularity result, we have \( \bar{v} \in C^1(\overline{Q^+_4}) \). Therefore, the reflected function

\[
v'(x', x_n, t) \overset{\text{def}}{=} \begin{cases} 
    \bar{v}(x', x_n, t) & \text{when } x_n \geq 0, \\
    \bar{v}(x', -x_n, t) & \text{when } x_n < 0
\end{cases}
\]

is a weak solution of (2.79).
belongs to the class $C^1(\overline{Q}_4^+)$. Consequently, it is clear that the function $v^*$ is a weak solution of
\[
v^*_t = \nabla \cdot [(1 + v^*)A_0(t)\nabla v^*] + \theta^2 v^*(1 - v^*) \quad \text{in} \quad Q_4^+.
\]
Thus, by applying the interior estimate in Lemma 2.9 we obtain
\[
||\nabla v^*||^2_{L^\infty(Q_n)} \leq C(\Lambda, n) \int_{Q_4^+} |\nabla v^*|^2 \, dx \, dt,
\]
yielding the estimate (2.79).

**Lemma 2.20.** Let $\bar{u} \in \dot{W}(Q_4^+)$ be a non-negative weak solution of (2.76) and $\bar{v} \in \dot{W}(Q_4^+)$ be a weak solution of (2.78) satisfying $0 \leq \bar{v} \leq 1$ in $Q_4^+$. Then
\[
\int_{Q_4^+} |\bar{u} - \bar{v}|^2 \, dx \, dt + \Lambda^{-1} \int_{Q_4^+} |\nabla \bar{u} - \nabla \bar{v}|^2 \, dx \, dt
\leq 33 \left[ 2\Lambda^3 \int_{Q_4^+} (|\bar{u} - \bar{v}|^2 + 8)|\nabla \bar{u}|^2 \, dx \, dt + 3\theta^2 \int_{Q_4^+} |\bar{u} - \bar{v}|^2 \, dx \, dt + \Lambda^2 \int_{Q_4^+} \bar{u}^2 c^2 \, dx \, dt \right].
\]

**Proof.** Let $w = \bar{u} - \bar{v}$. Then $w \in \dot{W}(Q_4^+)$ is a weak solution of
\[
\begin{align*}
\left\{ \begin{array}{ll}
w_t = \nabla \cdot [(1 + \bar{v})\tilde{A}_{B_4^+}(t)\nabla w] + \nabla \cdot [(wA + (1 + \bar{v})(A - \tilde{A}_{B_4^+}(t))\nabla \bar{u}] + \theta^2 w(1 - \bar{u} - \bar{v}) - \lambda \theta c \bar{u} & \quad \text{in} \quad Q_4^+, \\
\frac{\partial w}{\partial \nu} = 0 & \quad \text{on} \quad \bar{T}_4, \\
w = 0 & \quad \text{on} \quad \partial_b Q_4^+ \cup \partial_c Q_4^+.
\end{array} \right.
\end{align*}
\]
Since $\partial \bar{u}/\partial \nu = 0$ on $\bar{T}_4$, by multiplying the above equation by $w$ and integrating by parts we obtain for each $s \in (-16, 16)$
\[
\int_{B_4^+} \frac{w(x, s)^2}{2} \, dx + \int_{-16}^s \int_{B_4^+} (1 + \bar{v})(\tilde{A}_{B_4^+}(t)\nabla w, \nabla w) \, dx \, dt
\leq -\int_{-16}^s \int_{B_4^+} w(\nabla \bar{u}, \nabla w) \, dx \, dt - \int_{-16}^s \int_{B_4^+} (1 + \bar{v})(A - \tilde{A}_{B_4^+}(t))\nabla \bar{u}, \nabla w) \, dx \, dt
+ \theta^2 \int_{-16}^s \int_{B_4^+} w^2(1 - \bar{u} - \bar{v}) \, dx \, dt - \lambda \theta \int_{-16}^s \int_{B_4^+} c \bar{u} \, dx \, dt.
\]
The result then follows by the same arguments as in the proof of Lemma 2.10. \qed

The following approximation result is a global version of Lemma 2.11.

**Lemma 2.21.** Assume that $0 < \theta \leq \lambda$. For any $\varepsilon > 0$, there exists $\delta > 0$ depending only on $\varepsilon$, $\Lambda$ and $n$ such that: if
\[
\int_{Q_4^+} \left[ |A(x, t) - \tilde{A}_{B_4^+}(t)|^2 + |c(x, t)|^2 \right] \, dx \, dt \leq \delta,
\]
and $u \in \dot{W}(Q_4^+)$ is a weak solution of (2.75) satisfying
\[
0 \leq u \leq \frac{1}{\Lambda} \quad \text{in} \quad Q_4^+ \quad \text{and} \quad \int_{Q_4^+} |\nabla u|^2 \, dx \, dt \leq 1,
\]
then
\[
\int_{Q_4^+} |u - v|^2 \, dx \, dt \leq \varepsilon^2,
\]
where \( v \in \hat{W}(Q_4^+) \) is a weak solution of (2.77) with \( 0 \leq v \leq 1/\lambda \) in \( Q_4^+ \). Moreover,

\[
(2.81) \quad \int_{Q_4^+} |\nabla v|^2 \, dx \, dt \leq 2 + 66\Lambda (18\Lambda^2 + 3\theta^2\varepsilon^2 + \delta).
\]

**Proof.** We first prove (2.80) by contradiction. Suppose that estimate (2.80) is not true. Then there exist \( \varepsilon_0, \Lambda, n \), sequences of numbers \( \{\lambda_k\}_{k=1}^\infty \) and \( \{\theta_k\}_{k=1}^\infty \) with \( 0 < \theta_k \leq \lambda_k \), a sequence of coefficient matrices \( \{A_k\}_{k=1}^\infty \), and sequences of non-negative functions \( \{c_k\}_{k=1}^\infty \), \( \{\psi_k\}_{k=1}^\infty \) and \( \{u_k\}_{k=1}^\infty \) such that

\[
(2.82) \quad \int_{Q_4^+} \left[ |A_k(x, t) - \bar{A}_k(t)|^2 + |c_k(x, t)|^2 \right] \, dx \, dt \leq \frac{1}{k},
\]

\( u^k \in \hat{W}(Q_4^+) \) is a weak solution of

\[
(2.83) \quad \begin{cases} 
\frac{\partial u^k}{\partial t} &= \nabla \cdot [(1 + \lambda_k u^k)A_k \nabla u^k] + \theta_k^2 u^k (1 - \lambda_k u^k) - \lambda_k \theta_k c_k u^k & \text{in } Q_4^+, \\
\frac{\partial v^k}{\partial t} &= 0 & \text{on } \bar{T}_4, \\
v^k &= \psi_k & \text{on } \partial_b Q_4^+ \cup \partial_c Q_4^+.
\end{cases}
\]

with \( 0 \leq u^k \leq 1/\lambda_k \) in \( Q_4^+ \),

\[
(2.84) \quad \int_{Q_4^+} |\nabla u^k|^2 \, dx \, dt \leq 1,
\]

and

\[
(2.85) \quad \int_{Q_4^+} |u^k - v^k|^2 \, dx \, dt > \varepsilon_0^2 \quad \text{for all } k.
\]

Here \( \bar{A}_k(t) := \frac{1}{\partial_t A_k(x, t)} \, dx \), \( 0 \leq v^k \leq 1/\lambda_k \) in \( Q_4^+ \) and \( v^k \in \hat{W}(Q_4^+) \) is a weak solution of

\[
\begin{cases} 
\frac{\partial v^k}{\partial t} &= \nabla \cdot [(1 + \lambda_k v^k)\bar{A}_k(t) \nabla v^k] + \theta_k^2 v^k (1 - \lambda_k v^k) & \text{in } Q_4^+, \\
\frac{\partial \psi^k}{\partial t} &= 0 & \text{on } \bar{T}_4, \\
\psi^k &= u^k & \text{on } \partial_b Q_4^+ \cup \partial_c Q_4^+.
\end{cases}
\]

Since \( 0 \leq u^k, v^k \leq 1/\lambda_k \) in \( Q_4^+ \), we infer from (2.85) that

\[
\theta_k \leq \lambda_k \leq \frac{|Q_4^+|^\frac{1}{2}}{\varepsilon_0},
\]

that is, the sequences \( \{\lambda_k\} \) and \( \{\theta_k\} \) are bounded. Also \( \{\bar{A}_k\} \) is bounded in \( L^\infty(-16, 16; M^{\infty}) \) due to condition (2.74) for \( A_k \). Then as in the proof of Lemma 2.11 and by taking subsequence if necessary, we can assume that \( \lambda_k \to \lambda, \theta_k \to \theta \) and \( \bar{A}_k \to A_0 \) weakly in \( L^2(Q_4^+) \) for some constants \( \lambda, \theta \) satisfying \( 0 \leq \theta \leq \lambda < \infty \) and some \( A_0 \in L^\infty(-16, 16; M^{\infty}) \) satisfying (2.26). We are going to derive a contradiction by showing that there are subsequences \( \{u^k_m\} \) and \( \{v^k_m\} \) such that \( u^{k_m} - v^{k_m} \to 0 \) in \( L^2(Q_4^+) \).

Let us first consider the case \( \lambda > 0 \). Then the sequence \( \{u^k\} \) is bounded in \( Q_4^+ \). This together with (2.82), (2.83), (2.84) and the boundedness of \( \{A_k\} \) and \( \{\theta_k\} \) implies that the sequence \( \{u^k\} \) is
bounded in $\hat{W}(Q^+_4)$. Next, we apply Lemma 2.20 for $\bar{u} \rightsquigarrow \lambda_k u^k$ and $\bar{v} \rightsquigarrow \lambda_k v^k$ to obtain
\[
\int_{Q^+_4} |\nabla u^k - \nabla v^k|^2 \, dx \, dt \\
\leq 33\Lambda \left[ 18\Lambda^3 \int_{Q^+_4} |\nabla u|^2 \, dx \, dt + 3\theta_k^2 \int_{Q^+_4} |u - v|^2 \, dx \, dt + \lambda_k^2 \int_{Q^+_4} (u^k)^2 \, dx \, dt \right]
\leq 33\Lambda \left[ 18\Lambda^3 \int_{Q^+_4} |\nabla u|^2 \, dx \, dt + 3|Q^+_4| + \int_{Q^+_4} c_k^2 \, dx \, dt \right].
\]

Thanks to (2.82), (2.84) and the triangle inequality, this gives
\[
\int_{Q^+_4} |\nabla v^k|^2 \, dx \, dt \leq C \text{ for all } k.
\]

Thus, by reasoning as in the case of $\{u^k\}$, the sequence $\{v^k\}$ is also bounded in $\hat{W}(Q^+_4)$. As in the proof of Lemma 2.11, we infer from these facts and the compact embedding (2.2) that there exist subsequences, still denoted by $\{u^k\}$ and $\{v^k\}$, and $u, v \in \hat{W}(Q^+_4)$ such that for $w^k = u^k$ (or $w^k = v^k$) and $w = u$ (or $w = v$) we have
\[
\begin{cases}
  w^k \to w & \text{strongly in } L^2(Q^+_4), \\
  \nabla w^k \to \nabla w & \text{weakly in } L^2(Q^+_4), \\
  \partial_i w^k \to \partial_i w & \text{weakly-* in } L^2(0, T; \dot{H}^{-1}(B^+_4)), \\
  (1 + \lambda_k u^k)(A_k - \bar{A}_k(t))\nabla u^k \to 0 & \text{weakly in } L^2(Q^+_4),
\end{cases}
\]

and furthermore,
\[
\lim_{k \to \infty} \int_{Q^+_4} (1 + \lambda_k w^k)(A(t)\nabla w^k, \nabla \varphi) \, dx \, dt = \int_{Q^+_4} (1 + \lambda w)(A_0(t)\nabla w, \nabla \varphi) \, dx \, dt
\]

for all $\varphi \in C^\infty(\overline{Q^+_4})$ satisfying $\varphi = 0$ on $\partial, Q^+_4$. Since $u^k = v^k$ on $\partial_b Q^+_4 \cup \partial_t Q^+_4$, we also have $u = v$ on $\partial_b Q^+_4 \cup \partial_t Q^+_4$. Thus by passing to limits and using (2.82) together with the boundedness of $\{\lambda_k\}$ and $\{\theta_k\}$, one sees that $u$ and $v$ are weak solutions of the equation
\[
\begin{cases}
  w_t = \nabla \cdot [(1 + \lambda w)A_0(t)\nabla w] + \theta^2 w(1 - \lambda w) & \text{in } Q^+_4, \\
  \frac{\partial w}{\partial \nu} = 0 & \text{on } \tilde{T}_4, \\
  \frac{\partial w^k}{\partial \nu} = u = v & \text{on } \partial_b Q^+_4 \cup \partial_t Q^+_4.
\end{cases}
\]

In addition, we infer from the strong convergence of $u^k$ and $v^k$ in $L^2(Q^+_4)$ and $\lambda_k \to \lambda$ that $0 \leq u, v \leq 1/\lambda$ in $Q^+_4$. By the uniqueness of solutions given by Lemma 2.3, we conclude that $\lambda u = \lambda v$ in $Q^+_4$. Therefore, $u^k - v^k \to u - v = 0$ strongly in $L^2(Q^+_4)$ giving a contradiction to (2.85).

It remains to consider the case $\lambda = 0$, that is, $\lambda_k \to 0$. Due to $\theta_k \leq \lambda_k$, we also have $\theta_k \to 0$. Let $w^k = u^k - v^k$. Then $w^k$ is a weak solution of
\[
\begin{cases}
  w_t = \nabla \cdot [(1 + \lambda_k v^k)\bar{A}_k(t)\nabla w^k] + \nabla \cdot \left[(1 + \lambda_k v^k)(A_k - \bar{A}_k(t))\nabla u^k \right] + \theta_k^2 w^k (1 - \lambda_k u^k - \lambda_k v^k) - \lambda_k \theta_k c_k u^k & \text{in } Q^+_4, \\
  \frac{\partial w^k}{\partial \nu} = 0 & \text{on } \tilde{T}_4, \\
  \frac{\partial w^k}{\partial \nu} = 0 & \text{on } \partial_b Q^+_4 \cup \partial_t Q^+_4.
\end{cases}
\]

On the other hand, by applying Lemma 2.20 for $\bar{u} \rightsquigarrow \lambda_k u^k$, $\bar{v} \rightsquigarrow \lambda_k v^k$ and using the fact $\theta_k$ is small for large $k$, we get for all sufficiently large $k$ that
\[
\int_{Q^+_4} |\nabla w^k|^2 \, dx \, dt \leq 33\Lambda \left[ 18\Lambda^3 \int_{Q^+_4} |\nabla u|^2 \, dx \, dt + \int_{Q^+_4} c_k^2 \, dx \, dt \right] \leq 33\Lambda (18\Lambda^3 + 1)
\]
and
\[
\int_{Q_4^+} |w^k|^2 \, dxdt \leq 66 \left[ 18 \Lambda^3 \int_{Q_4^+} |\nabla u^k|^2 \, dxdt + \int_{Q_4^+} c_k^2 \, dxdt \right] \leq 66(18 \Lambda^3 + 1).
\]
These estimates together with (2.86), (2.84) and the boundedness of \(A_k\) imply that \(\{w^k\}\) is bounded in \(\dot{W}(Q_4^+)\). Hence there exist a subsequence \(\{w^k\}\) and a function \(w \in \dot{W}(Q_4^+)\) such that
\[
\begin{align*}
\text{(2.88)} & \quad \begin{cases} 
 w^k \rightarrow w \quad \text{strongly in } L^2(Q_4^+), \\
 \nabla w^k \rightarrow \nabla w \quad \text{weakly in } L^2(Q_4^+), \\
 \partial_i w^k \rightarrow \partial_i w \quad \text{weakly-* in } L^2(0, T; \dot{H}^{-1}(B_1^+)).
\end{cases}
\end{align*}
\]
Now let \(\bar{v}^k = \lambda_k \bar{v}^k\). Then \(0 \leq \bar{v}^k \leq 1\) in \(Q_4^+\), and \(\bar{v}^k\) is a weak solution of
\[
(\bar{v}^k)_t = \nabla \cdot [(1 + \bar{v}^k)\bar{A}_k(t)\nabla \bar{v}^k] + \partial_i^2 \bar{v}^k(1 - \bar{v}^k) \quad \text{in } Q_4^+.
\]
Moreover, \(\nabla \bar{v}^k = \lambda_k \nabla \bar{v}^k \rightarrow 0\) strongly in \(L^2(Q_4^+)\) since it follows from (2.84) and (2.87) that
\[
\int_{Q_4^+} |\nabla \bar{v}^k|^2 \, dxdt \leq C \quad \text{for all large } k.
\]
Thus \(\{\bar{v}^k\}\) is bounded in \(\dot{W}(Q_4^+)\) and so, up to a subsequence, \(\bar{v}^k \rightharpoonup \bar{v}\) strongly in \(L^2(Q_4^+)\) for some function \(\bar{v} \in \dot{W}(Q_4^+)\) with \(0 \leq \bar{v} \leq 1\) in \(Q_4^+\). By arguing as in the proof of Lemma 2.11, we infer in addition that \(\bar{v}\) is a constant function in \(Q_4^+\), and
\[
\lim_{k \to \infty} \int_{Q_4^+} (1 + \bar{v}^k)\langle \bar{A}_k(t)\nabla w^k, \nabla \varphi \rangle \, dxdt = \int_{Q_4^+} (1 + \bar{v})\langle A_0(t)\nabla w, \nabla \varphi \rangle \, dxdt
\]
for all \(\varphi \in C_c^\infty(Q_4^+)\) satisfying \(\varphi = 0\) on \(\partial_i Q_4^+\). Using these convergences, (2.82) and (2.88), we can pass to the limits in (2.86) to conclude that \(w\) is a weak solution of the equation
\[
\begin{align*}
\text{(2.89)} & \quad \begin{cases} 
 w_t & = \nabla \cdot [(1 + \bar{v})A_0(t)\nabla w] \quad \text{in } Q_4^+,
 \frac{\partial w}{\partial \bar{v}} & = 0 \quad \text{on } \bar{T}_4,
 w & = 0 \quad \text{on } \partial_\nu Q_4^+ \cup \partial_\nu Q_4^+.
\end{cases}
\end{align*}
\]
Thanks to the uniqueness (see Lemma 2.4) of the trivial solution to the linear equation (2.89), we conclude that \(w \equiv 0\) in \(Q_4^+\). This gives
\[
\lim_{k \to \infty} \int_{Q_4^+} |u^k - \bar{v}|^2 \, dxdt = \lim_{k \to \infty} \int_{Q_4^+} |w^k|^2 \, dxdt = \int_{Q_4^+} |w|^2 \, dxdt = 0,
\]
which contradicts to (2.85). Thus the proof of (2.80) is complete and it remains to prove (2.81). For this we apply Lemma 2.20 for \(\bar{u} \sim \lambda u\) and \(\bar{v} \sim \lambda v\) to obtain
\[
\int_{Q_4^+} |\nabla u - \nabla v|^2 \, dxdt \leq 33\Lambda \left[ 18\Lambda^3 \int_{Q_4^+} |\nabla u|^2 \, dxdt + 3\theta^2 \int_{Q_4^+} |u - v|^2 \, dxdt + \int_{Q_4^+} c^2 \, dxdt \right].
\]
This together with (2.80) and the assumptions gives
\[
\int_{Q_4^+} |\nabla u - \nabla v|^2 \, dxdt \leq 33\Lambda \left[ 18\Lambda^3 + 3\theta^2 c^2 + \delta \right].
\]
Since \(||\nabla v||_{L^2(Q_4^+)} \leq 1 + ||\nabla u - \nabla v||_{L^2(Q_4^+)}\), the estimate (2.81) follows immediately from (2.90). \(\square\)

In the next lemma, we establish an approximation of gradients of solutions near the flat boundary. This will play a key role in our derivation of boundary \(W^{1,p}\)-estimates in Subsection 2.3.2.
Lemma 2.22. Assume that $0 < \theta \leq \lambda$, $\theta \leq 1$ and $0 < r \leq 1$. For any $\varepsilon > 0$, there exists $\delta > 0$ small depending only on $\varepsilon$, $\Lambda$ and $n$ such that: if

\[
\int_{Q^+_4} [\|A(x, t) - \bar{A}_{B^+_r}(t)\|^2 + \|c(x, t)\|^2] \, dx dt \leq \delta,
\]

then for any weak solution $u \in \mathring{W}(Q^+_4)$ of (2.75) satisfying

\[
0 \leq u \leq \frac{1}{\lambda} \text{ in } Q^+_4 \text{ and } \int_{Q^+_4} |\nabla u|^2 \, dx dt \leq 1,
\]

and a weak solution $v \in \mathring{W}(Q^+_4)$ of

\[
\begin{cases}
\begin{aligned}
\partial_t v &= \nabla \cdot [(1 + \lambda v)\bar{A}^+_{B^+_r}(t)\nabla v] + \theta^2 v(1 - \lambda v) & \text{in } Q^+_4, \\
\frac{\partial v}{\partial \nu} &= 0 & \text{on } \bar{T}^+_4, \\
v &= u & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{aligned}
\end{cases}
\]

satisfying $0 \leq v \leq 1/\lambda$ in $Q^+_4$, we have

\[
\int_{Q^+_4} |u - v|^2 \, dx dt \leq \varepsilon^2 r^2,
\]

\[
\int_{Q^+_4} |\nabla v|^2 \, dx dt \leq 4^{n+2} \omega_n \left[2 + 66\Lambda \left(18\Lambda^2 + 3\theta^2 - 2\right)\right],
\]

and

\[
\int_{Q^+_4} |\nabla u - \nabla v|^2 \, dx dt \leq \varepsilon^2.
\]

Proof. Define

\[
u'(x, t) = \frac{u(rx, r^2 t)}{r}, \quad v'(x, t) = \frac{v(rx, r^2 t)}{r}, \quad A'(x, t) = A(rx, r^2 t) \text{ and } c'(x, t) = c(rx, r^2 t).
\]

Let $\lambda' = \lambda r$ and $\theta' = \theta r$. Then $u'$ is a weak solution of

\[
\begin{cases}
\begin{aligned}
\partial_t u' &= \nabla \cdot [(1 + \lambda' u')A'\nabla u'] + \theta'^2 u'(1 - \lambda' u') - \lambda' \theta' c' u' & \text{in } Q^+_4, \\
\frac{\partial u'}{\partial \nu} &= 0 & \text{on } \bar{T}^+_4,
\end{aligned}
\end{cases}
\]

and $v'$ is a weak solution of

\[
\begin{cases}
\begin{aligned}
\partial_t v' &= \nabla \cdot [(1 + \lambda' v')\bar{A}'_{B^+_r}(t)\nabla v'] + \theta'^2 v'(1 - \lambda' v') & \text{in } Q^+_4, \\
\frac{\partial v'}{\partial \nu} &= 0 & \text{on } \bar{T}^+_4, \\
v' &= u' & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{aligned}
\end{cases}
\]

Hence by applying Lemma 2.21 for the solutions $u'$, $v'$ and rescaling back, we obtain the estimates (2.98) and (2.99).

In order to prove (2.95), we define $w = u - v$. Then $w \in \mathring{W}(Q^+_4)$ is a weak solution of

\[
\begin{cases}
\begin{aligned}
w_t &= \nabla \cdot [(1 + \lambda u)A\nabla w] + \nabla \cdot \left\{[\lambda w\bar{A}'_{B^+_r}(t) + (1 + \lambda u)(A - \bar{A}'_{B^+_r}(t))]\nabla v\right\} \\
\quad + \theta^2 w(1 - \lambda u - \lambda v) - \lambda \theta^2 c u & \text{in } Q^+_4, \\
\frac{\partial w}{\partial \nu} &= 0 & \text{on } \bar{T}^+_4, \\
w &= 0 & \text{on } \partial_b Q^+_4 \cup \partial_c Q^+_4.
\end{aligned}
\end{cases}
\]
Let \( \varphi \) be the standard cut-off function which is 1 on \( Q_{2r} \), \( \supp(\varphi) \subset \overline{Q_3} \), \( |\nabla \varphi| \leq C_n/r \) and \( |\varphi| \leq C_n/r^2 \). Let us multiply the equation (2.96) by \( \varphi^2w \) and use integration by parts together with the fact \( \partial v/\partial \nu = 0 \) on \( T_{4r} \). Then by arguing as in the proof of Lemma 2.12 we obtain

\[
(2.97) \quad \frac{\Lambda^{-1}}{4} \int_{Q_4^+} (1 + \lambda u)|\nabla w|^2 \varphi^2 dx \leq (4\Lambda^2 + 1) \int_{Q_4^+} |\nabla \varphi|^2 w^2 dx dt \\
+ \Lambda \|\nabla (\lambda v)\|_{L^\infty(Q_{4r})} \left( \int_{Q_4^+} |\nabla w| \varphi^2 w dx dt + 2 \int_{Q_4^+} |\nabla \varphi| \varphi w^2 dx dt \right) \\
+ 2(\Lambda + 2) \int_{Q_4^+} |\Lambda - \Lambda_{B_{4r}}(t)|^2 |\nabla v|^2 \varphi^2 dx dt \\
+ \int_{Q_4^+} \varphi \varphi \varphi w^2 dx dt + \frac{1}{2} \int_{Q_4^+} w^2 dx dt + \int_{Q_4^+} |c||w| dx dt.
\]

We next estimate \( \|\nabla (\lambda v)\|_{L^\infty(Q_{4r})} \) and \( \|\nabla v\|_{L^\infty(Q_{4r})} \). Let us define \( \tilde{v}(x,t) := \lambda v(rx, r^2t) \) for \((x,t) \in Q_4^+ \). Then \( 0 \leq \tilde{v} \leq 1 \) in \( Q_4^+ \) and \( \tilde{v} \) is a weak solution of

\[
\begin{cases}
\tilde{v}_t = \nabla \cdot [(1 + \tilde{v}) \Lambda' B_{4r}(t) \nabla \tilde{v}] + (\theta r)^2 \tilde{v}(1 - \tilde{v}) & \text{in } Q_4^+ , \\
\nabla \tilde{v} \cdot \nu = 0 & \text{on } T_4.
\end{cases}
\]

Thanks to \( \theta r \leq \theta \leq 1 \), we then can use Lemma 2.19 to get

\[
(2.98) \quad \|\nabla \tilde{v}\|_{L^\infty(Q_{4r}^+)} \leq C(\Lambda, n) \left( \int_{Q_{4r}^+} |\nabla \tilde{v}|^2 \right)^{\frac{1}{2}}.
\]

This together with Lemma 2.18 yields \( \|\nabla v\|_{L^\infty(Q_{4r}^+)} \leq C(\Lambda, n) \). By rescaling back, we obtain

\[
(2.99) \quad \|\nabla (\lambda v)\|_{L^\infty(Q_{4r})} \leq \frac{C(\Lambda, n)}{r}.
\]

On the other hand, (2.98) also gives

\[
(2.100) \quad \|\nabla v\|_{L^\infty(Q_{4r}^+)} \leq C(\Lambda, n) \left( \int_{Q_{4r}^+} |\nabla v(rx, r^2t)|^2 dx dt \right)^{\frac{1}{2}} = C(\Lambda, n) \left( \int_{Q_{4r}^+} |\nabla v(y, s)|^2 dy ds \right)^{\frac{1}{2}}.
\]

It follows from (2.97), (2.99), (2.100) and the Cauchy–Schwarz inequality that

\[
\int_{Q_{4r}^+} |\nabla w|^2 \varphi^2 dx dt \leq \frac{C}{r^2} \int_{Q_{4r}^+} w^2 dx dt + C \left( \int_{Q_{4r}^+} |\nabla v|^2 dx dt \right) \left( \int_{Q_{4r}^+} |\Lambda - \Lambda_{B_{4r}}(t)|^2 dx dt \right) \\
+ \left( \int_{Q_{4r}^+} c^2 dx dt \right)^{\frac{1}{2}} \left( \int_{Q_{4r}^+} w^2 dx dt \right)^{\frac{1}{2}}.
\]

This together with (2.93), (2.94) and the assumption (2.91) gives the estimate (2.95). \( \square \)

**Remark 2.23.** Since our equations are invariant under the translation \((x,t) \mapsto (x + y, t + s)\), Lemma 2.22 still holds true if \( Q_4^+ \) is replaced by \( Q_4^+(y,s) \).

2.3.2. Boundary density and gradient estimates on flat domains. We begin this subsection with a density estimate which is the boundary version of Lemma 2.15.

**Lemma 2.24.** Assume that \( 0 < \theta \leq \lambda, \theta \leq 1 \), \( \Lambda \) satisfies (2.74) and \( c \in L^2(Q_4^+) \). There exists a constant \( N > 1 \) depending only on \( \Lambda \) and \( n \) such that for any \( \varepsilon > 0 \), we can find \( \delta = \delta(\varepsilon, \Lambda, n) > 0 \)
satisfying: if

\[
(2.101) \quad \sup_{0 < r \leq \epsilon} \sup_{(y,s) \in \tilde{Q}_t^+} \frac{1}{|Q_r(y,s)|} \int_{Q_r(y,s) \cap Q_t^+} |A(x,t) - \tilde{A}_{B_t}(r)B_t(t)|^2 \, dx \, dt \leq \delta,
\]

then for any weak solution \( u \in \mathring{W}(Q_t^+) \) of (2.75) with \( 0 \leq u \leq \frac{1}{r} \) in \( Q_t^+ \) and for any \( (y,s) \in \tilde{Q}_t^+ \), \( 0 < r \leq 1/12 \) with

\[
Q_s(y,s) \cap Q_t^+ \cap \{ \tilde{Q}_s^+ : M_{\tilde{Q}_s^+}(\nabla u^2) \leq 1 \} \cap \{ \tilde{Q}_s^+ : M_{\tilde{Q}_s^+}(c^2) \leq \delta \} \neq \emptyset,
\]

we have

\[
(2.102) \quad \left| \{ \tilde{Q}_s^+ : M_{\tilde{Q}_s^+}(\nabla u^2) > N \} \cap Q_s(y,s) \right| \leq \varepsilon |Q_s(y,s)|.
\]

**Proof.** Let \( Q_s(\tilde{y}, \tilde{s}) \) be a parabolic cube satisfying \( (\tilde{y}, \tilde{s}) \in \tilde{T}_s^+ \), \( 0 < \tilde{r} \leq 1/2 \) and

\[
(2.103) \quad Q_s(\tilde{y}, \tilde{s}) \cap Q_t^+ \cap \{ \tilde{Q}_s^+ : M_{\tilde{Q}_s^+}(\nabla u^2) \leq 1 \} \cap \{ \tilde{Q}_s^+ : M_{\tilde{Q}_s^+}(c^2) \leq \delta \} \neq \emptyset.
\]

We then claim that there exists \( N > 0 \) depending only on \( \Lambda \) and \( n \) such that

\[
(2.104) \quad \left| \{ \tilde{Q}_s^+ : M_{\tilde{Q}_s^+}(\nabla u^2) > N \} \cap Q_s(\tilde{y}, \tilde{s}) \right| \leq \frac{\varepsilon}{6^{n+2}} |Q_s(\tilde{y}, \tilde{s})|.
\]

Indeed, it follows from (2.103) that

\[
(2.105) \quad M_{\tilde{Q}_s^+}(\nabla u^2)(x_0, t_0) \leq 1 \quad \text{and} \quad M_{\tilde{Q}_s^+}(c^2)(x_0, t_0) \leq \delta
\]

for some point \( (x_0, t_0) \in Q_s(\tilde{y}, \tilde{s}) \cap Q_t^+ \). Since \( Q_{4r}^+(\tilde{y}, \tilde{s}) \subset Q_{5r}(x_0, t_0) \cap \tilde{Q}_t^+ \), we deduce from (2.105) that

\[
\int_{Q_{4r}(\tilde{y}, \tilde{s})} |\nabla u^2| \, dx \, dt \leq \frac{|Q_{5r}(x_0, t_0)|}{|Q_{4r}^+(\tilde{y}, \tilde{s})|} \left( \frac{1}{|Q_{5r}(x_0, t_0)|} \int_{Q_{5r}(x_0, t_0) \cap \tilde{Q}_t^+} |\nabla u^2| \, dx \, dt \right) \leq 2 \left( \frac{5}{4} \right)^{n+2},
\]

\[
\int_{Q_{4r}(\tilde{y}, \tilde{s})} c^2 \, dx \, dt \leq \frac{|Q_{5r}(x_0, t_0)|}{|Q_{4r}^+(\tilde{y}, \tilde{s})|} \left( \frac{1}{|Q_{5r}(x_0, t_0)|} \int_{Q_{5r}(x_0, t_0) \cap \tilde{Q}_t^+} c^2 \, dx \, dt \right) \leq 2 \left( \frac{5}{4} \right)^{n+2} \delta.
\]

Also, as \( B_{4r}(\tilde{y}) \cap B_{3r}^+ = B_{4r}^+(\tilde{y}) \) the assumption (2.101) gives

\[
\int_{Q_{4r}(\tilde{y}, \tilde{s})} |A(x,t) - \tilde{A}_{B_{4r}^+(\tilde{y})}(t)|^2 \, dx \, dt \leq 2\delta.
\]

Therefore, we can use Lemma 2.22 and Remark 2.23 to obtain

\[
(2.106) \quad \int_{Q_{4r}(\tilde{y}, \tilde{s})} |\nabla u - \nabla \tilde{v}|^2 \, dx \, dt \leq \eta^2,
\]

where \( v \in \mathring{W}(Q_{4r}^+(\tilde{y}, \tilde{s})) \) is the weak solution of

\[
\begin{cases}
  \frac{\partial v}{\partial t} = \nabla \cdot [(1 + \lambda v)\tilde{A}_{B_{4r}^+(\tilde{y})}(t)\nabla v] + \theta^2 v(1 - \lambda v) & \text{in} \ Q_{4r}^+(\tilde{y}, \tilde{s}), \\
  \frac{\partial v}{\partial \nu} = 0 & \text{on} \ \tilde{T}_{4r}, \\
  v = u & \text{on} \ \partial_r Q_{4r}^+(\tilde{y}, \tilde{s}) \cup \partial_t Q_{4r}^+(\tilde{y}, \tilde{s})
\end{cases}
\]

satisfying \( 0 \leq v \leq 1/\lambda \) in \( Q_{4r}^+(\tilde{y}, \tilde{s}) \), and \( \delta = \delta(\eta, \Lambda, n) \) with \( \eta \) being determined later. We note that the existence and uniqueness of such weak solution \( v \) is guaranteed by Theorem 2.2.

Let \( \tilde{v}(x,t) = \lambda v(\tilde{r}x + \tilde{y}, \tilde{r}^2t + \tilde{s}) \) and \( A'(x,t) = A(\tilde{r}x + \tilde{y}, \tilde{r}^2t + \tilde{s}) \) for \( (x,t) \in Q_t^+ \). Then \( 0 \leq \tilde{v} \leq 1 \) in \( Q_4^+ \) and \( \tilde{v} \) is a weak solution of

\[
\begin{cases}
  \frac{\partial \tilde{v}}{\partial \tilde{t}} = \nabla \cdot [(1 + \tilde{v})A'(\tilde{r})\nabla \tilde{v}] + \theta(\tilde{r})^2 \tilde{v}(1 - \tilde{v}) & \text{in} \ Q_4^+, \\
  \frac{\partial \tilde{v}}{\partial \tilde{\nu}} = 0 & \text{on} \ \tilde{T}_4.
\end{cases}
\]
Thanks to $\theta \tilde{\theta} \leq \theta \leq 1$, we can apply Lemma 2.19 to get
\[
||\nabla v||_{L^\infty(Q^+_{s_2})} \leq C \int_{Q^+_{s_2}} |\nabla v|^2 \, dx \, dt,
\]
which together with (2.106) and (2.105) gives
\[
(2.107) \quad ||\nabla v||_{L^\infty(Q^+_{s_2}(y,3,3))} \leq C \int_{Q^+_{s_2}(y,3,3)} |\nabla v|^2 \, dx \, dt
\]
\[
\leq 2C \left( \int_{Q^+_{s_2}(y,3,3)} |\nabla u - \nabla v|^2 \, dx \, dt + \int_{Q^+_{s_2}(y,3,3)} |\nabla u|^2 \, dx \, dt \right) \leq C(\Lambda, n)(\eta^2 + 1).
\]
We assert that (2.105), (2.106) and (2.107) yield
\[
(2.108) \quad \{Q_r(\tilde{y}, \tilde{s}) : M_{Q^+_{s_2}(y,3,3)}(|\nabla u - \nabla v|^2) \leq C(\Lambda, n)\} \subset \{Q_r(\tilde{y}, \tilde{s}) : M_{Q^+_{s_2}}(|\nabla u|^2) \leq N\}
\]
with $N = \max \{6C(\Lambda, n), 5^{n+2}\}$. To see this, let $(x, t)$ be a point in the set on the left hand side of (2.108), and consider $Q^+_\rho(x, t)$. If $\rho \leq \tilde{\rho}/2$, then as $\tilde{y}_n = 0$ we have $Q^+_\rho(x, t) \cap Q^+_3 \subset Q^+_{3\tilde{\rho}/2}(\tilde{y}, \tilde{s})$ and hence
\[
1 \int_{Q^+_\rho(x, t) \cap Q^+_3} |\nabla u|^2 \, dx \, dt \leq \frac{2}{|Q^+_\rho(x, t)|} \left( \int_{Q^+_\rho(x, t) \cap Q^+_3} |\nabla u - \nabla v|^2 \, dx \, dt + \int_{Q^+_\rho(x, t) \cap Q^+_3} |\nabla v|^2 \, dx \, dt \right)
\]
\[
\leq 2M_{Q^+_{s_2}(y,3,3)}(|\nabla u - \nabla v|^2)(x, t) + 2|\nabla v|^2_{L^\infty(Q^+_{s_2}(y,3,3))} \leq 2C(\Lambda, n)(\eta^2 + 2) \leq 6C(\Lambda, n).
\]
On the other hand if $\rho > \tilde{\rho}/2$, then $Q^+_\rho(x, t) \subset Q_{5\rho}(x_0, t_0)$. This and the first inequality in (2.105) imply that
\[
1 \int_{Q^+_\rho(x, t) \cap Q^+_3} |\nabla u|^2 \, dx \, dt \leq \frac{5^{n+2}}{|Q_{5\rho}(x_0, t_0)|} \int_{Q_{5\rho}(x_0, t_0) \cap Q^+_3} |\nabla u|^2 \, dx \, dt \leq 5^{n+2}.
\]
Therefore, we conclude that $M_{Q^+_{s_2}}(|\nabla u|^2)(x, t) \leq N$ and (2.108) is proved. Note that (2.108) is equivalent to
\[
\{Q_r(\tilde{y}, \tilde{s}) : M_{Q^+_{s_2}}(|\nabla u|^2) > N\} \subset \{Q_r(\tilde{y}, \tilde{s}) : M_{Q^+_{s_2}(y,3,3)}(|\nabla u - \nabla v|^2) > C(\Lambda, n)\}.
\]
It follows from this, the weak type 1 – 1 estimate and (2.106) that
\[
\left| \left| Q_r(\tilde{y}, \tilde{s}) : M_{Q^+_{s_2}}(|\nabla u|^2) > N \right| \right| \leq \left| \left| Q_r(\tilde{y}, \tilde{s}) : M_{Q^+_{s_2}(y,3,3)}(|\nabla u - \nabla v|^2) > C(\Lambda, n) \right| \right| \\
\leq C \int_{Q^+_{s_2}(y,3,3)} |\nabla u - \nabla v|^2 \, dx \, dt \leq C' \eta^2 |Q_r(\tilde{y}, \tilde{s})|,
\]
where $C' > 0$ depends only on $\Lambda$ and $n$. By choosing $\eta := \sqrt{\frac{c}{\kappa^{n+2}C'}}$, we obtain the claim (2.104).

To proceed with the proof, we consider the following two cases:

**Case 1:** $\text{dist}(y, T_1) > 5r$. Then $B_{4r}(y) \subset B_{3\tilde{\rho}}$, $Q_{3r}(y, s) \subset Q^+_3$ and $\int_{Q_{3r}(y, s)} |A(x, t) - \tilde{A}_{B_{4r}(y)}(t)|^2 \, dx \, dt \leq \delta$ by (2.101). Hence (2.102) follows from the interior estimate in Lemma 2.15 (see the proof of Lemma 2.15).

**Case 2:** $\text{dist}(y, T_1) \leq 5r$. Then there exists $\tilde{y} \in T_1$ such that $B_{r}(y) \subset B_{6r}(\tilde{y})$. Consequently, $Q_{r}(y, s) \subset Q_{6r}(\tilde{y}, s)$ and due to the assumption (2.103) we have
\[
Q_{6r}(\tilde{y}, s) \cap Q^+_1 \cap \{Q^+_3 : M_{Q^+_{s_2}}(|\nabla u|^2) \leq 1\} \cap \{Q^+_3 : M_{Q^+_{s_2}}(|\nabla u|^2) \leq \delta\} \neq \emptyset.
\]
Therefore, it follows from the claim (2.104) that
\[
\left| \left| Q^+_1 : M_{Q^+_{s_2}}(|\nabla u|^2) > N \right| \right| \cap Q_{6r}(\tilde{y}, s) \leq \frac{\varepsilon}{6^{n+2}} |Q_{6r}(\tilde{y}, s)| = \varepsilon |Q_r(y, s)|.
yielding (2.102).

In view of Lemma 2.24, we can apply the Vitali covering lemma (see [4, Theorem 2.6]) for $E = \{Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > N\}$ and $F = \{Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > 1\} \cup \{Q^*_1 : M_{Q^*_1}(c^2) > \delta\}$ to obtain:

**Lemma 2.25.** Assume that $0 < \theta \leq \lambda, \theta \leq 1$, $A$ satisfies (2.74) and $c \in L^2(Q^*_1)$. There exists a constant $N > 1$ depending only on $\Lambda$ and $n$ such that for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, \Lambda, n) > 0$ satisfying: if

$$\sup_{0 < \rho \leq 2} \sup_{(y, s) \in Q^*_1} \frac{1}{|Q^*_1(y, s)|} \int_{Q^*_1(y, s) \cap Q^*_1} |A(x) - \tilde{A}_{B_{y}(y) \cap B_{3}^*(t)}(t)|^2 \, dx \, dt \leq \delta,$$

then for any weak solution $u \in \mathring{W}(Q^*_1)$ of (2.73) satisfying

$$0 \leq u \leq \frac{1}{\lambda} \quad \text{in} \quad Q^*_1 \quad \text{and} \quad |Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > N| \leq \varepsilon |Q^*_1|,$$

we have

$$|Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > N| \leq 2(10)^{n+2} \varepsilon \left[ |Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > 1| + |Q^*_1 : M_{Q^*_1}(c^2) > \delta| \right].$$

We are ready to state and prove the boundary $W^{1,p}$-estimates for flat domains.

**Theorem 2.26.** Assume that $\lambda > 0, 0 < \theta \leq 1$, $A$ satisfies (2.74) and $c \in L^2(Q^*_1)$. Let $u \in \mathring{W}(Q^*_1)$ be a weak solution of (2.73) satisfying $0 \leq u \leq 1/\lambda$ in $Q^*_1$. Then for any $p > 2$, there exists a constant $\delta = \delta(p, \Lambda, n) > 0$ such that if

$$\sup_{0 < \rho \leq 2} \sup_{(y, s) \in Q^*_1} \frac{1}{|Q^*_1(y, s)|} \int_{Q^*_1(y, s) \cap Q^*_1} |A(x, t) - \tilde{A}_{B_{y}(y) \cap B_{3}^*(t)}(t)|^2 \, dx \, dt \leq \delta,$$

we have

$$\int_{Q^*_1} |\nabla u|^p \, dx \, dt \leq C \left( \frac{\theta}{\lambda} + \|u\|_{L^1(Q^*_1)} \right)^p + \int_{Q^*_1} |c|^p \, dx \, dt \right).$$

(2.109)

Here $C > 0$ is a constant depending only on $p, \Lambda$ and $n$.

**Proof.** The arguments follow similar lines as in the proof of Theorem 2.7 using Lemma 2.25 and Lemma 2.18 in place of Lemma 2.16 and Lemma 2.8. Therefore, we will only present the main points.

Let $N > 1$ be as in Lemma 2.25 and let $q := p/2 > 1$. We choose $\varepsilon = \varepsilon(p, \Lambda, n) > 0$ be such that

$$\varepsilon_1 \overset{\text{def}}{=} 2(10)^{n+2} \varepsilon = \frac{1}{2N^q},$$

and let $\delta = \delta(p, \Lambda, n)$ be the corresponding constant given by Lemma 2.25. Assuming for a moment that $u$ satisfies

$$|Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > N| \leq \varepsilon |Q^*_1|.$$  

We first consider the case $\theta \leq \lambda$. Then it follows from Lemma 2.25 that

$$|Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > N| \leq \varepsilon_1 \left[ |Q^*_1 : M_{Q^*_1}(|\nabla u|^2) > 1| + |Q^*_1 : M_{Q^*_1}(c^2) > \delta| \right].$$

Let us iterate this estimate by considering

$$u_1(x, t) = \frac{u(x, t)}{\sqrt{N}}, \quad c_1(x, t) = \frac{c(x, t)}{\sqrt{N}} \quad \text{and} \quad \lambda_1 = \sqrt{N}\lambda \geq \theta.$$
It is easy to see that $u_1 \in \hat{W}(Q^+_1)$ is a weak solution of
\[
\begin{aligned}
(u_1)_t &= \nabla \cdot [(1 + \lambda_1 u_1) \mathbf{A} \nabla u_1] + \theta_1 u_1 (1 - \lambda_1 u_1) - \lambda_1 \theta_c u_1 & \text{in } Q^+_4, \\
\frac{\partial u_1}{\partial \nu} &= 0 & \text{on } \bar{T}_4, \\
u_1 &= \psi / \sqrt{N} & \text{on } \partial_b Q^+_4 \cup \partial Q^+_4.
\end{aligned}
\]
Therefore, by applying Lemma 2.25 for $u_1$ we obtain
\[
\left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > |N| \} \right| \leq \epsilon_1 \left( \left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > 1 \} \right| + \left| \{Q^+_1 : M_{Q^+_1}(|c_1|^2) > \delta \} \right| \right).
\]
We infer from this and (2.111) that
\[
\left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > \epsilon_2^2 \} \right| \leq \epsilon_1 \left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > 1 \} \right| + \epsilon_1 \left| \{Q^+_1 : M_{Q^+_1}(c^2) > \delta \} \right| + \epsilon_1 \left| \{Q^+_1 : M_{Q^+_1}(c^2) > \delta \} \right|.
\]
By repeating the iteration, we then conclude that
\[
\left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > \epsilon_2^k \} \right| \leq \epsilon_1 \left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > 1 \} \right| + \sum_{i=1}^{k} \epsilon_1 \left| \{Q^+_1 : M_{Q^+_1}(c^2) > \delta \} \right| \quad \text{for all } k = 1, 2, \ldots
\]
As a consequence of this and by arguing as in the proof of Theorem 2.7 we obtain
\[
\int_{Q^+_1} |\nabla u|^2 \, dx \, dt \leq \int_{Q^+_1} M_{Q^+_1}(|\nabla u|^2)^q \, dx \, dt \leq C \left( 1 + \int_{Q^+_1} M_{Q^+_1}(c^2)^q \, dx \, dt \right)
\]
with the constant $C$ depending only on $p$, $\Lambda$ and $n$. Therefore, it follows from the strong type $q - q$ estimate for the maximal function and the fact $q = p/2$ that
\[
(2.112) \quad \int_{Q^+_1} |\nabla u|^p \, dx \, dt \leq C \left( 1 + \int_{Q^+_1} |c|^p \, dx \, dt \right).
\]
The estimate (2.112) was derived under the assumption that $\theta \leq \lambda$. But as in the proof of Theorem 2.7 we deduce from (2.112) that
\[
(2.113) \quad \int_{Q^+_1} |\nabla u|^p \, dx \, dt \leq C \left( \frac{\theta^p}{\lambda^p} \vee 1 + \int_{Q^+_1} |c|^p \, dx \, dt \right)
\]
as long as $\lambda > 0$ and $0 < \theta \leq 1$. We next remove the extra assumption (2.110) for $u$. Notice that for any $M > 0$, by using the weak type $1 - 1$ estimate for the maximal function and Lemma 2.18 we get
\[
\left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > NM^2 \} \right| \leq \frac{C}{NM^2} \int_{Q^+_1} |\nabla u|^2 \, dx \, dt \leq \frac{C_n}{M^2} \int_{Q^+_1} u^2 \, dx \, dt.
\]
Thus, if we let
\[
\bar{u}(x, t) = \frac{u(x, t)}{M} \quad \text{with} \quad M^2 = \frac{C_n |\nabla u|_{L^2(Q^+_1)}^2}{\epsilon |Q^+_1|},
\]
then $\left| \{Q^+_1 : M_{Q^+_1}(|\nabla u|^2) > |N| \} \right| \leq \epsilon |Q^+_1|$. Hence we can apply (2.113) for $\bar{u}$ with $c$ and $\lambda$ being replaced by $\tilde{c} = c/M$ and $\tilde{\lambda} = \lambda M$. By reversing back to the functions $u$ and $c$, we obtain (2.109).

\[\square\]

**Remark 2.27.** By inspection we see that the interior and boundary $W^{1, p}$-estimates (i.e. Theorem 2.7 and Theorem 2.26) also hold true if the parabolic cubes $K_p(y, s) = B_p(y) \times (s - r^2, s)$ and $K'_p(y, s) = B'_p(y) \times (s - r^2, s)$ are used in these statements instead of the centered parabolic cubes $Q_p(y, s)$ and $Q'_p(y, s)$.
2.4. **Global $W^{1,p}$-estimates on Lipschitz domains.** In this subsection we consider the Neumann problem (1.4) and derive global $W^{1,p}$-estimates for the solution $u$. To this end, we will flatten the boundary of the domain $\Omega$ and then employ Theorem 2.2.6.

**Proof of Theorem 2.5** For simplicity, we assume that $\alpha = 1$. In order to establish the estimates up to the top boundary of $\Omega_T = \{(\delta, R)\}$-Lipschitz, we may assume upon relabeling and reorienting the coordinate axes if necessary - that

$$\Omega \cap B_R(x_0) = \{(x', x_n) \in B_R(x_0) : x_n > \gamma(x')\}$$

for some Lipschitz continuous function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\text{Lip}(\gamma) \leq \delta$. By translating by a suitable vector, we can assume that $x_0 = 0$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$\Phi(x', x_n) := (x', x_n - \gamma(x'))$$

and let $\Psi(y', y_n) := \Phi^{-1}(y', y_n) = (y', y_n + \gamma(y'))$. We have $\nabla \Phi = (\nabla \Psi)^{-1}$, and

$$\nabla \Phi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\gamma_{x_1} & -\gamma_{x_2} & -\gamma_{x_3} & \cdots & -\gamma_{x_{n-1}} & 1 \end{bmatrix} \quad \text{and} \quad \nabla \Psi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Moreover, $\Phi$ and $\Psi$ are measure-preserving transformations, that is, $\det \nabla \Phi = \det \nabla \Psi = 1$. As a consequence of (2.114), we obtain

\begin{equation}
\|\nabla \Phi\|_{L^\infty(B_R(x_0))}^2 \leq n + \|\nabla \gamma\|_{L^\infty}^2 \leq n + \text{Lip}(\gamma)^2 \leq n + 1 \quad \text{and} \quad \|\nabla \Psi\|_{L^\infty(\Phi(B_R(x_0)))}^2 \leq n + 1.
\end{equation}

Let us choose $\rho \in (0, \bar{\rho})$ small such that $\rho < 2R/\sqrt{n+1}$ and $B^+_{\rho} \subset \Phi(\Omega \cap B_R(x_0))$, and define

$$\hat{u}(y, t) := u(\Psi(y), t) \quad \text{for} \; y \in B^+_{\rho} \quad \text{and} \; t \in [0, T].$$

Then $\hat{u} \in \hat{W}(K^+_\rho(0, t_0))$ is a weak solution of

$$\begin{cases}
\hat{u}_t = \nabla \cdot \left( (1 + \lambda \hat{u}) \hat{A} \nabla \hat{u} \right) + \theta^2 \hat{u}(1 - \lambda \hat{u}) - \lambda \theta \hat{c} \hat{u} & \text{in} \; K^+_\rho(0, t_0), \\
\frac{\partial \hat{u}}{\partial \nu} = 0 & \text{on} \; T_{\rho} \times (t_0 - \rho^2, t_0], \\
\hat{u}(y, t) = u(\Psi(y), t) & \text{on} \; \partial_b K^+_\rho(0, t_0) \cup \partial_c K^+_\rho(0, t_0)
\end{cases}$$

with

$$\hat{A}(y, t) = \nabla \Phi(\Psi(y)) \cdot A(\Psi(y), t) \cdot \nabla \Phi(\Psi(y))^T \quad \text{and} \quad \hat{c}(y, t) = c(\Psi(y), t).$$

Here $\partial_b K^+_\rho(0, t_0) \overset{\text{def}}{=} B^+_{\rho} \times (t_0 - \rho^2)$ and $\partial_c K^+_\rho(0, t_0) \overset{\text{def}}{=} \partial_c B^+_{\rho} \times (t_0 - \rho^2, t_0)$. We would like to apply Theorem 2.2.6 for $\hat{u}$ and so we need to verify conditions in this theorem. Since $\langle \hat{A}(y, t) \cdot \xi, \xi \rangle = \langle A(\Psi(y), t) \cdot [\nabla \Phi(\Psi(y))^T \cdot \xi], [\nabla \Phi(\Psi(y))^T \cdot \xi] \rangle$ for $(y, t) \in K^+_\rho(0, t_0)$ and $\xi \in \mathbb{R}^n$, we have

$$\Lambda^{-1} |\eta|^2 \leq \langle \hat{A}(y, t) \cdot \xi, \xi \rangle \leq \Lambda |\eta|^2,$$

where $\eta := \nabla \Phi(\Psi(y))^T \cdot \xi$. Moreover, by using (2.115) we get $|\eta|^2 \leq |\nabla \Phi|^2 |\xi|^2 \leq (n + 1) |\xi|^2$ and $|\xi|^2 = |\nabla \Psi|^2 |\eta|^2 \leq |\nabla \Phi|^2 |\eta|^2 \leq (n + 1) |\eta|^2$. Thus we conclude that

$$[\Lambda^{-1} |\eta|^2] \leq \langle \hat{A}(y, t) \cdot \xi, \xi \rangle \leq (n + 1) \Lambda |\xi|^2 \quad \text{for a.e.} \; (y, t) \in K^+_\rho(0, t_0) \; \text{and} \; \xi \in \mathbb{R}^n.$$
We next show that the mean oscillation of $\hat{A}$ is small. For this, let us write $A = (a_{ij})$. A direct computation using (2.114) gives

$$
\hat{A}(y, t) = A(\Psi(y), t) + \begin{bmatrix}
    0 & 0 & \cdots & 0 & -\sum_{j=1}^{n-1} a_{1j} \gamma_{y_j} \\
    0 & 0 & \cdots & 0 & -\sum_{j=1}^{n-1} a_{2j} \gamma_{y_j} \\
    \cdot & \cdot & \cdots & \cdot & \cdot \\
    0 & 0 & \cdots & 0 & -\sum_{j=1}^{n-1} a_{(n-1)j} \gamma_{y_j} \\
    -\sum_{i=1}^{n-1} a_{i1} \gamma_{\lambda_i} & -\sum_{i=1}^{n-1} a_{i2} \gamma_{\lambda_i} & \cdots & -\sum_{i=1}^{n-1} a_{i(n-1)} \gamma_{\lambda_i} & -\sum_{i=1}^{n-1} a_{in} \gamma_{y_j} - \sum_{i,j=1}^{n-1} a_{ij} \gamma_{y_i} \gamma_{y_j}
\end{bmatrix}.
$$

Hence for fixed $(z, s) \in K_{\frac{1}{r}}(0, t_0)$ and $r \in (0, \rho/2]$, we have

$$
\tag{2.117}
\frac{1}{|K_r(z, s)|} \left| \int_{K_r(z, s) \cap K_{\frac{1}{r}}(0, t_0)} |\hat{A}(y, t) - \overline{A}_{B_{\sqrt{r}+t}(\Psi(y)) \cap \Omega}(t)|^2 \, dy \, dt \right| 
\leq 2 \frac{1}{|K_r(z, s)|} \left| \int_{K_r(z, s) \cap K_{\frac{1}{r}}(0, t_0)} |A(\Psi(y), t) - \overline{A}_{B_{\sqrt{r}+t}(\Psi(y)) \cap \Omega}(t)|^2 \, dy \, dt \right| + C(\Lambda, n) \text{Lip}(\gamma)^2.
$$

Since $B_r(z) \cap B_{\frac{1}{r}+t} \subset \Phi(\Omega \cap B_R(x_0))$, we infer from the second inequality in (2.115) that $\Psi(B_r(z) \cap B_{\frac{1}{r}+t}) \subset B_{\sqrt{r}+t}(\Psi(z)) \cap \Omega \cap B_R(x_0)$. Therefore,

$$
\frac{1}{|K_r(z, s)|} \left| \int_{K_r(z, s) \cap K_{\frac{1}{r}}(0, t_0)} |A(\Psi(y), t) - \overline{A}_{B_{\sqrt{r}+t}(\Psi(y)) \cap \Omega}(t)|^2 \, dy \, dt \right| 
\leq \frac{(n+1)^2}{|K_{\sqrt{r}+t}(\Psi(z), s)|} \int_{K_{\sqrt{r}+t}(\Psi(z), s) \cap \Omega_T} |A(x, t) - \overline{A}_{B_{\sqrt{r}+t}(\Psi(y)) \cap \Omega}(t)|^2 \, dx \, dt \leq (n+1)^2 [A]_{\text{BMO}(R, \Omega_T)},
$$

where we have used the fact $r \sqrt{n+1} \leq \rho \sqrt{n+1/2} \leq R$ to achieve the last inequality. Plug this into (2.117) we arrive at

$$
\frac{1}{|K_r(z, s)|} \left| \int_{K_r(z, s) \cap K_{\frac{1}{r}}(0, t_0)} |\hat{A}(y, t) - \overline{A}_{B_{\sqrt{r}+t}(\Psi(y)) \cap \Omega}(t)|^2 \, dy \, dt \right| 
\leq 2(n+1)^2 [A]_{\text{BMO}(R, \Omega_T)} + C(\Lambda, n) \text{Lip}(\gamma)^2.
$$

It follows that

$$
\sup_{0 < t \leq T} \frac{1}{|K_r(z, s)|} \left| \int_{K_r(z, s) \cap K_{\frac{1}{r}}(0, t_0)} |\hat{A}(y, t) - \overline{A}_{B_{\sqrt{r}+t}(\Psi(y)) \cap \Omega}(t)|^2 \, dy \, dt \right| 
\leq 2(n+1)^2 [A]_{\text{BMO}(R, \Omega_T)} + C(\Lambda, n) \text{Lip}(\gamma)^2.
$$

Thus we can apply a rescaled version of Theorem 2.26 (see Remark 2.27) to get

$$
\int_{K_{\sqrt{r}}(0, t_0)} |\nabla \hat{u}|^p \, dy \, dt \leq C \left\{ \left( \frac{\theta}{\lambda} \sqrt{\text{L}^2(\nabla \hat{u}(K_{\sqrt{r}}(0, t_0)))} \right)^p + \int_{K_{r}^p(0, t_0)} |\hat{e}|^p \, dy \, dt \right\}.
$$

By changing variables back and using the fact

$$
|\nabla u(\Psi(y), t)| = \left| [\nabla \Phi(\Psi(y))]^T \nabla \Psi(y) \right|^T \nabla u(\Psi(y), t) = \left| [\nabla \Phi(\Psi(y))]^T \nabla \hat{u}(y, t) \right| 
\leq \sqrt{n+1} |\nabla \hat{u}(y, t)|,
$$

we obtain

$$
\int_{\Psi(B_{\sqrt{r}}^p)(x_0) \cap (0, \frac{\rho}{\sqrt{r}+t_0})} |\nabla u|^p \, dx \, dt \leq C(n+1)^\frac{p}{2} \left\{ \left( \frac{\theta}{\lambda} \sqrt{\text{L}^2(\nabla \hat{u}(\Omega_T)))} \right)^p + \int_{\Omega_T} |\hat{e}|^p \, ds \, dt \right\}.
$$
But as $\Omega \cap B_{\frac{c_2 d_2}{2 c_2}}(x_0) \subset \Psi(B^+_{\frac{c_2}{2}})$ by the first inequality in (2.115), we thus conclude that
\begin{equation}
(2.118) \int_{\Omega \cap B_{\frac{c_2 d_2}{2 c_2}}(x_0) \times (t_0 - \frac{c_2 d_2}{2 c_2}, t_0)} |\nabla u|^p \, dx \, dt \leq C \left( \left( \frac{\theta}{\lambda} \sqrt{\|u\|_{L^1(\Omega_T)}} \right)^p + \int_{\Omega_T} |c|^p \, dx \, dt \right)
\end{equation}
for any $x_0 \in \partial \Omega$ and $t_0 \in [\bar{t}, T]$.

The rest of the proof is standard. We cover the region $\Omega \times [\bar{t}, T]$ by a suitable finite family of parabolic cubes $K_{p_i}(x_i, t_i) = B_{p_i}(x_i) \times (t_i - p_i^2, t_i)$ with $(x_i, t_i) \in \Omega \times [\bar{t}, T]$ whose members are either interior cubes (i.e. $B_{p_0}(x_i) \subset \Omega$) or cubes centered at a point on $\partial \Omega \times [\bar{t}, T]$. For each of these cubes, we can either apply a rescaled version of Theorem 2.7 (see also Remark 2.7) or use the estimate (2.118). The desired estimate (1.6) then follows by adding up the resulting inequalities.

3. Global smooth Solutions for the SKT System

We prove Theorem 1.2 in this section. Note that the equations of $u$ and $v$ can be written in the divergence form as
\begin{align*}
u_t &= \nabla \cdot [(d_1 + 2a_1 u + a_1 v) \nabla u + a_1 \nabla v] + u(a_1 - b_1 u - c_1 v), \\
v_t &= \nabla \cdot [(d_2 + 2a_2 v) \nabla v] + v(a_2 - c_2 v) - b_2 u v.
\end{align*}
(3.1)

For given $u_0, v_0, p_0$ as in Theorem 1.2 let $T = t_{\text{max}} \in (0, \infty)$ be the maximal time existence of the solution $u, v$ for (1.3) as in Theorem 1.1. Let $\bar{t} \in (0, T)$ be fixed. Then Theorem 1.1 implies that there is a constant $C(\bar{t}) > 0$ such that
\begin{equation}
(3.2) \sup_{0 < t < \bar{t}} \left[ \int_{\Omega} |\nabla u(x, t)|^{p_0} \, dx + \int_{\Omega} |\nabla v(x, t)|^{p_0} \, dx \right] \leq C(\bar{t}).
\end{equation}
We assume that $T < \infty$ and derive a contradiction to (1.2) by establishing
\begin{equation}
(3.3) \sup_{\bar{t} < t < T} \left[ \|u(\cdot, t)\|_{W^{1, p_0}(\Omega)} + \|v(\cdot, t)\|_{W^{1, p_0}(\Omega)} \right] < \infty.
\end{equation}

In the following estimates, all constants $C$ are positive, continuously dependent on $T$ and may change from line to line. They also can depend on $\bar{t}, \Omega, \|u_0\|_{W^{1, p_0}(\Omega)}, \|v_0\|_{W^{1, p_0}(\Omega)}$ and the coefficients in the system (1.3), but we may not explicitly specify such dependence. We begin with a lemma which is a consequence of the maximum principle.

**Lemma 3.1.** [22, Lemma 2.1] The solution $v$ of (1.3) satisfies
\begin{align*}
0 &\leq v(x, t) \leq M_0 := \max \left\{ \frac{a_2}{c_2}, \max_\Omega v_0 \right\}, \quad \forall (x, t) \in \Omega \times [0, T).
\end{align*}

The next lemma is an important consequence of Theorem 1.5.

**Lemma 3.2.** For each $p \in (2, p_0)$, there exists a constant $C = C(p, T) > 0$ such that
\begin{equation}
(3.4) \|\nabla v\|_{L^p(\Omega_T)} \leq C \left[ 1 + \|u\|_{L^p(\Omega_T)} \right].
\end{equation}

**Proof.** Without loss of generality, we assume that $a_2 = 1$. Let $\lambda = M_0^{-1}$, where $M_0$ is defined in Lemma 3.1. Then we rewrite the equation (3.1) as
\begin{align*}
u_t &= \nabla \cdot [(1 + \alpha \lambda) d_2 \nabla v] + v(1 - \lambda v) - cv,
\end{align*}
where
\begin{align*}
c(x, t) &= [c_2 - \lambda] v(x, t) + b_2 u(x, t) \geq 0, \quad \text{and} \quad \alpha = \frac{2a_2}{\lambda d_2}.
\end{align*}
Since $\|c\|_{L^p(\Omega_T)} \leq C(1 + \|u\|_{L^p(\Omega_T)})$, inequality (3.4) follows from estimate (1.6) of Theorem 1.5 and (3.2). \(\square\)
Our next goal is to derive an $L^l$-estimate for $u$ assuming that $\nabla v \in L^l(\Omega_T)$ for some $l = l(p) > p > 2$. For this, we first recall Lemma 3.2 of [34].

Lemma 3.3. [34] Lemma 3.2 Let $p > 2$ and assume there is a constant $M(p, T) < \infty$ such that

$$\|\nabla v\|_{L^p(\Omega_T)} \leq M(p, T).$$

Then, for each $q > 1$, there is a constant $C = C(q, T, M)$ such that for every $T_1 \in (0, T)$

$$\|u(\cdot, t)\|_{L^q(\Omega)}^q + \|\nabla(u^{q/2})\|_{L^2(\Omega_1)}^2 + \|\nabla(u^{(q+1)/2})\|_{L^2(\Omega_1)}^2 \leq C \left[ 1 + \|u\|_{L^{\frac{q-1}{q+1}}(\Omega_T)}^{q-1} \right], \quad \forall t \in (0, T_1).$$

For each number $a \in \mathbb{R}$, we write $a^+ = \max\{a, 0\}$. The following lemma is one of our main ingredients for the bootstrap argument.

Lemma 3.4. Let $p > 2$ and assume there is a constant $M(p, T) < \infty$ such that

$$\|\nabla v\|_{L^p(\Omega_T)} \leq M(p, T).$$

Then, for every $q \in \left(1, \frac{n(p-1)}{(p+2)p_+}\right]$ and $l \in \left(1, \frac{n(p+1)}{(n+2)p_+}\right]$ with $q, l \neq \infty$, there exists a positive constant $C = C(l, q, T, M)$ such that

$$\sup_{0 \leq t \leq T_1} \int_{\Omega} u(x, t) dx + \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \int_{\Omega} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{2}{\tilde{q}}} dx < \infty,$$

where $\tilde{p} = \frac{2p}{p-2}$. For any number $T_1$ in $(0, T)$, define

$$E(T_1) = \sup_{0 \leq t \leq T_1} \int_{\Omega} u(x, t) dx + \int_{\Omega} |\nabla u|^2 dx = \sup_{0 \leq t \leq T_1} \int_{\Omega} \int_{\Omega} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{2}{\tilde{q}}} dx.$$

It follows from Lemma 3.3 that

$$E(T_1) + \|\nabla(u^{q/2})\|_{L^2(\Omega_1)}^2 \leq C \left[ 1 + \|w\|_{L^{\frac{2(q+1)}{2q}}(\Omega_T)} \right], \quad \text{where} \quad \tilde{q} = 2 + \frac{4q}{n(q+1)}.$$

Now, let $\tilde{q} = 2 + \frac{4q}{n(q+1)}$. By Gagliardo–Nirenberg’s inequality, there exists a constant $C > 0$ depending on $n, \tilde{q}$ and $\Omega$ such that

$$\|w(\cdot, t)\|_{L^\tilde{q}(\Omega)} \leq C \left\{ \|\nabla w(\cdot, t)\|_{L^2(\Omega)}^2 \|w(\cdot, t)\|_{L^{\frac{2(q+1)}{2q}}(\Omega_T)}^{\frac{4q}{4q+12q}} \right\}.$$

By integrating the equation of $u$ and using Gronwall’s inequality, we note that

$$\sup_{0 \leq t \leq T_1} \int_{\Omega} u(x, t) dx \leq C(T).$$

Then the interpolation inequality yields

$$\int_{\Omega} w(x, t) dx \leq \|u(\cdot, t)\|_{L^\tilde{q}(\Omega)} \|w(\cdot, t)\|_{L^\tilde{q}(\Omega)}^{1-\frac{\lambda}{\tilde{q}}} \leq C(T) \|w(\cdot, t)\|_{L^\tilde{q}(\Omega)}^{1-\frac{\lambda}{\tilde{q}}} \quad \text{with} \quad \frac{1-\lambda}{\tilde{q}} + \frac{\lambda}{2q+1} = 1.$$

This, together with (3.7) and Young’s inequality imply that

$$\|w(\cdot, t)\|_{L^\tilde{q}(\Omega_1)} \leq C \left\{ \|\nabla w(\cdot, t)\|_{L^2(\Omega)}^2 \|w(\cdot, t)\|_{L^\tilde{q}(\Omega_T)}^{\frac{4q}{4q+12q}} + 1 \right\}.$$

Therefore,

$$\|w\|_{L^\tilde{q}(\Omega_1)} \leq C \left[ \|\nabla w\|_{L^2(\Omega_1)}^2 \left( \sup_{0 \leq t < T_1} \|w(\cdot, t)\|_{L^\tilde{q}(\Omega)}^{\frac{4q}{4q+12q}} + 1 \right) \right].$$
Also, since \( q \in \left( 1, \frac{n(p-1)}{(n+2-p)}, \right] \) and \( q \neq \infty \), we see that
\[
(n\bar{p} - 2n - 4)q \leq n(\bar{p} + 2) \quad \text{and} \quad \frac{\bar{p}(q-1)}{q+1} \leq \tilde{q}.
\]
From this and (3.8), we obtain
\[
(3.9) \quad \|w\|_{L^\frac{(q+1)n}{2n+q}\left(\Omega_T\right)} \leq C \|w\|_{L^\frac{n}{q}\left(\Omega_T\right)} \leq C\left[1 + E(T)\right]^{\frac{2n+q}{n+4}}.
\]
Hence, it follows from this last inequality and (3.6) that
\[
(3.10) \quad E(T_1) \leq C\left[1 + E(T_1)^\mu\right], \quad \text{with} \quad \mu = \frac{2(q-1)}{\bar{p}(q+1)}\left(\frac{2}{n+1}\right) \quad \text{and} \quad C = C(T).
\]
A simple calculation shows \( \mu < 1 \). Because of this and the fact \( E(T_1) \) is finite, we infer from (3.10) that there exists a constant \( C(T) \) such that
\[
(3.11) \quad E(T_1) \leq C(T), \quad \forall \ T_1 \in (0, T).
\]
By passing \( T_1 \to T^- \), we obtain the first inequality of (3.5). The second inequality of (3.5) follows directly from (3.9) and (3.11), and again, passing \( T_1 \to T^- \). The proof is therefore complete. \( \square \)

To initiate our iteration process, we start with the following \( L^4 \)-estimate for \( \nabla v \).

**Lemma 3.5.** [34] Lemma 3.1] There exists a constant \( C \) depending on \( T \), the coefficients in the system (1.3) and the initial data \( u_0, v_0 \) such that
\[
\|\nabla v\|_{L^4(\Omega_T)} \leq C.
\]

We now can combine the previous results to improve the regularity of \( u \) and \( v \).

**Lemma 3.6.** There exists a constant \( C \) depending on \( T \) such that
\[
(3.12) \quad \sup_{0 < t < T} \int_{\Omega} u(x, t)^{p_0}dx + \int_{\Omega_T} |\nabla v(x, t)|^{p_0}dxdt \leq C.
\]
Moreover, for all finite \( p \in \left( 2, \frac{p_0(n+1)}{n+2-p_0}, \right] \), there exists \( C = C(p, T) \) such that
\[
(3.13) \quad \int_{\Omega_T} u(x, t)^pdxdt \leq C(p, T).
\]

**Proof.** Since (3.13) follows (3.12) and the second inequality of (3.5) in Lemma 3.4 it suffices to prove (3.12) only. Let \( l_1 = 4 \).

**Step 1.** If \( l_1 \geq \min\{p_0, n+2\} \), then (3.12) can be obtained directly from Lemmas 3.2 and 3.4. Now, we consider the case \( l_1 < \min\{p_0, n+2\} \). Note in this case that \( n > 2 \). We then infer from Lemmas 3.4 and 3.5 that
\[
\int_{\Omega_T} |u(x, t)|^{l_2}dxdt \leq C(T) \quad \text{with} \quad l_2 = \frac{l_1(n+1)}{(n+2-l_1)_+} = \frac{l_1(n+1)}{n-2}.
\]

**Step 2.** If \( l_2 \geq \min\{p_0, n+2\} \), we can use Lemmas 3.2 and 3.4 to obtain (3.12) and we then stop. We therefore only consider the case \( l_2 < \min\{p_0, n+2\} \). By Lemma 3.2 we see that
\[
\int_{\Omega_T} |\nabla v(x, t)|^{l_2}dxdt \leq C(T) < \infty.
\]
From this and Lemma 3.4 we have
\[
\int_{\Omega_T} u^{l_3}(x, t)dxdt \leq C(T) < \infty \quad \text{with} \quad l_3 = \frac{l_2(n+1)}{n+2-l_2}.
\]
Observe that
\[
l_3 = \frac{l_2(n + 1)}{n + 2 - l_2} > \frac{l_2(n + 1)}{n - 2} = l_i \left(\frac{n + 1}{n - 2}\right)^i.
\]

We will repeat this procedure. For \( i \geq 3 \), define \( l_{i+1} = \frac{l_i(n + 1)}{n + 2 - l_i} \). Then the sequence \( \{l_i\} \) is strictly increasing and
\[
l_{i+1} \geq l_i \left(\frac{n + 1}{n - 2}\right)^i \quad \forall i \geq 1.
\]
Hence \( \lim_{i \to \infty} l_i = \infty \). Let \( k \) be the smallest integer in \([2, \infty)\) such that \( l_k = \min\{p_0, n + 2\} \).

We repeat Step 2 above with \( l_i \), for \( i = 3, \ldots, k - 1 \), to arrive at Step \((k - 1)\) and obtain
\[
\int_{\Omega_t} |\nabla v(x, t)|^p \, dx \, dt \leq C \quad \forall \, i = 1, \ldots, k - 1, \quad \text{and} \quad \int_{\Omega_t} |u(x, t)|^p \, dx \, dt \leq C.
\]
Since \( l_k = \min\{p_0, n + 2\} \), we, again, obtain (3.12) from Lemmas 3.2 and 3.4. The proof is complete. □

The next estimate for \( \nabla v \) is crucial for obtaining the boundedness of \( u \).

**Lemma 3.7.** There exists \( p_1 > n + 2 \) and a constant \( C(T) > 0 \) such that
\[
\|\nabla v\|_{L^{p_1}(\Omega_0; [\bar{t}, T])} \leq C(T).
\]

**Proof.** If \( n = 2 \), from Lemmas 3.4 and 3.5 we obtain
\[
\|u\|_{L^p(\Omega_T)} \leq C(p, T), \quad \forall \, p \in (2, \infty).
\]
This together with Theorem 1.5 imply that
\[
\|\nabla v\|_{L^p(\Omega_0; [\bar{t}, T])} \leq C \left[ 1 + \|u\|_{L^p(\Omega_T)} \right] \leq C(p, T), \quad \forall \, p \in [2, \infty).
\]
Hence we obtain (3.14) for \( n = 2 \). Now, consider \( n > 2 \). Since \( p_0 > n \) and \( n \geq 3 \), a simple calculation gives
\[
\frac{p_0(n + 1)}{(n + 2 - p_0)n} > n + 2.
\]
Therefore, it follows from Lemma 3.6 that there exists \( p_1 > n + 2 \) such that
\[
\|u\|_{L^{p_1}(\Omega_T)} \leq C(T).
\]
Then applying Theorem 1.5 again, we obtain
\[
\|\nabla v\|_{L^{p_1}(\Omega_0; [\bar{t}, T])} \leq C \left[ 1 + \|u\|_{L^{p_1}(\Omega_T)} \right] \leq C(T),
\]
which proves (3.14). □

We now show that \( u \) is bounded:

**Lemma 3.8.** There exists a constant \( C(T) > 0 \) such that
\[
\|u\|_{L^\infty(\Omega_T)} \leq C(T).
\]

**Proof.** From Theorem 1.1 we have \( u \in C([0, \bar{t}], W^{1,p_0}(\Omega)) \) with \( p_0 > n \). By Morrey’s imbedding theorem, there exists \( \tilde{C}_0 = \tilde{C}_0(\bar{t}) > 0 \) such that
\[
\|u\|_{L^{p_0}(\Omega_0; [0, \bar{t}]}) \leq \tilde{C}_0.
\]
We thus only need to prove that \( u \) is bounded in \( \Omega \times [\bar{t}, T] \). For each \( t_1 \in (\bar{t}, T) \), and each \( k > \tilde{C}_0 \), denote \( W_k(x, t) = \max\{u(x, t) - k, 0\} \) and \( \Omega_{t_1}(k) = \{(x, t) \in \Omega \times [\bar{t}, t_1] : u(x, t) > k\} \).
We write the equation of $u$ as
\begin{equation}
(3.17) \quad u_t = \nabla \cdot [A(x,t) \nabla u] + a_{12} \nabla \cdot [u \nabla v] + f(x,t) \quad \text{in } \Omega_T,
\end{equation}
with homogeneous Neumann boundary condition, where
\begin{equation}
(3.18) \quad A(x,t) = d_1 + 2a_{11}u(x,t) + a_{1,2}v(x,t), \quad f(x,t) = u(x,t)[a_1 - b_1 u(x,t) - c_1 v(x,t)].
\end{equation}
We note that $A(x,t)$ is bounded below by $d_1 > 0$, and not known to be bounded above. However, we can follow the De Giorgi’s iteration technique \cite[Theorem 7.1, p. 181]{[16]} to prove (3.15). By multiplying the equation (3.17) with $W_k$ and using the integration by parts, we obtain
\begin{align*}
\frac{1}{2} \int_{\Omega} W_k^2(x,t)dx + d_1 \int_{\Omega} \nabla W_k^2 dx dt + a_{11} \int_{\Omega_{i}(k)} |\nabla W_k|^2 u dx dt & \\
\leq 2a_{12} \int_{\Omega_{i}(k)} |\nabla| \nabla W_k|u dx dt + 2a_1 \int_{\Omega_{i}(k)(k)} u W_k dx dt & \\
\leq \frac{d_1}{2} \int_{\Omega_{i}(k)} |\nabla W_k|^2 dx dt + C \int_{\Omega_{i}(k)} \left[|\nabla v|^2 + 1\right] |W_k^2 + k^2| dx dt & \forall t \in [\bar{t}, t_1].
\end{align*}
Therefore, there exists a constant $C > 0$ depending only on the coefficients $d_1$ and $a_1, a_{11}, a_{12}$ such that
\begin{equation}
(3.19) \quad \|W_k\|_{L^2}^2 := \sup_{0 < r < r_1} \int_{\Omega} W_k^2(x,t)dx dt + \int_{\Omega \times (\bar{t}, t_1)} |\nabla W_k(x,t)|^2 dx dt \\
\leq C \int_{\Omega_{i}(k)} \left[|\nabla v|^2 + 1\right] |W_k^2 + k^2| dx dt.
\end{equation}
We estimate the right hand side of (3.19). Let $p_1$ be as in Lemma [3.7]. Then by Hölder’s inequality and Lemma [3.7], we have
\begin{equation}
(3.20) \quad \int_{\Omega_{i}(k)} \left[|\nabla v|^2 + 1\right] |W_k^2 + k^2| dx dt \leq \|\nabla v\| + 1\|W_k\|_{L^{p_1}(\Omega_{i}(k))}^2 + k\|W_k\|_{L^{p_1}(\Omega_{i}(k))}^2 \\
\leq C(T) \left[\|W_k\|_{L^{p_1}(\Omega_{i}(k))}^2 + k^2\mu(k)^{\frac{p_1 - 2}{p_1}}\right],
\end{equation}
where $\mu(k) := |\Omega_{i}(k)|$. Observe that $p_1 > n + 2$ implies
\[
\frac{p_1}{p_1 - 2} < \frac{n + 2}{n}, \quad \text{hence} \quad \delta := \frac{p_1 - 2}{p_1} - \frac{n}{n + 2} > 0.
\]
Therefore, we can apply Hölder’s inequality and then the parabolic imbedding theorem (\cite[3.4, p. 75]{[16]}) to infer that
\begin{equation}
(3.21) \quad \|W_k\|_{L^{p_1/(n+2)}(\Omega_{i}(k))}^2 \leq \|W_k\|_{L^{2(n+2)}(\Omega_{i}(k))}^2 \mu(k)^{\delta} \leq C(T) \|W_k\|_{L^2}^2 \mu(k)^{\delta}.
\end{equation}
By (3.19), (3.20) and (3.21), there exists $C_1(T) > 0$ such that
\begin{equation}
(3.22) \quad \|W_k\|_{L^2}^2 \leq C_1(T) \left[\|W_k\|_{L^2}^2 \mu(k)^{\delta} + k^2\mu(k)^{\frac{p_1 - 2}{p_1}}\right].
\end{equation}
Let $t_1 \in (\bar{t}, T]$ satisfy $C_1(T) [\Omega(t_1 - \bar{t})]^{\delta} \leq 1/2$. Then the inequality (3.22) yields
\begin{equation}
(3.23) \quad \|W_k\|_{L^2}^2 \leq 2C_1(T)k^2\mu(k)^{\frac{p_1 - 2}{p_1}}.
\end{equation}
From this and the standard iteration technique (\cite[Theorem 6.1, p. 102]{[16]}), we deduce that
\begin{equation}
(3.24) \quad \sup_{\Omega \times [\bar{t}, t_1]} u(x,t) \leq C_1.
\end{equation}
where $\tilde{C}_1 = \tilde{C}_1(T) > 0$.

Now, partition the interval $[\bar{t}, T]$ evenly with
\[
\bar{t} = t_0 < t_1 < t_2 < \cdots < t_{N+1} = T, \quad t_j = t_0 + jh,
\]
where the time step $h > 0$ is chosen to satisfy $C_1(T)[h |\Omega|]^6 \leq 1/2$. Repeating the proof of (3.24) on each time interval $[t_j, t_{j+1}]$ for $j = 1, 2, \cdots, N$, we obtain
\[
\sup_{\Omega \times [t_j, t_{j+1}]} u(x, t) \leq \tilde{C}_{j+1},
\]
where $\tilde{C}_{j+1} = \tilde{C}_{j+1}(T) > 0$ also depends on the preceding bound $\tilde{C}_j$ on $[t_{j-1}, t_j]$. (In the proof, we replace $\tilde{C}_0$ by $\tilde{C}_j$.) Therefore, we obtain
\[
\sup_{\Omega \times [\bar{t}, T]} u \leq C(T).
\]
The estimate (3.15) then follows from (3.16) and (3.26).

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2** For all $(x, t) \in \Omega_T$, as in the proof of Lemma 3.8 let $A(x, t), f(x, t)$ be defined by (3.18) and let
\[
B(x, t) = d_2 + 2a_{22}v(x, t), \quad g(x, t) = v(x, t)[a_2 - b_2u(x, t) - c_2v(x, t)].
\]
From Lemmas 3.3 and 3.8 there are constants $\Lambda > 0$ and $C > 0$ such that
\[
\Lambda^{-1} \leq A(x, t) \leq \Lambda \quad \text{and} \quad \Lambda^{-1} \leq B(x, t) \leq \Lambda \quad \forall \ (x, t) \in \Omega_T,
\]
\[
\|f\|_{L^\infty(\Omega_T)} + \|g\|_{L^\infty(\Omega_T)} \leq C.
\]
We rewrite the equation (3.1) as
\[
v_t = \nabla \cdot [B(x, t)\nabla v] + g.
\]
From (3.27), (3.28), and the classical H"{o}lder regularity theory ([16] Theorem 10.1, p. 204), ([12] Theorem 1.3, Remark 1.1, p. 43)), there exist $\alpha_1 \in (0, 1)$ and $C(T) > 0$ such that
\[
\|v\|_{C^{\alpha_1, \frac{\alpha_1}{2}}(\Omega_T)} \leq C(T).
\]
Next, we rewrite the equation of $u$ as
\[
\dot{u}_t = \nabla \cdot [A(x, t)\nabla u] + \nabla \cdot \tilde{F}_1 + \nabla \cdot \tilde{F}_2 + f,
\]
where
\[
\tilde{F}_1(x, t) = a_{12} \chi_{[0, \bar{t}]}(t)u(x, t)\nabla v(x, t) \quad \text{and} \quad \tilde{F}_2(x, t) = a_{12} \chi_{[\bar{t}, T]}(t)u(x, t)\nabla v(x, t).
\]
Here, $\chi_I$ denotes the characteristic function of the set $I \subset \mathbb{R}$. From Theorem 1.1 Lemmas 3.7 and 3.8 we see that
\[
\|\tilde{F}_1\|_{L^p([0, \bar{t}]; L^q(\Omega))} \leq C(\bar{t}) \quad \text{and} \quad \|\tilde{F}_2\|_{L^p(\Omega_T)} \leq C(T),
\]
where $p_1 > n + 2$ is given by Lemma 3.7. Therefore, we can again apply the classical H"{o}lder regularity theory ([16] Theorem 10.1, p. 204), ([12] Theorem 1.3, Remark 1.1, p. 43)) to the equation (3.30) to infer that there are $\alpha_2 \in (0, 1)$ and $C(T) > 0$ such that
\[
\|u\|_{C^{\alpha_2, \frac{\alpha_2}{2}}(\Omega_T)} \leq C(T).
\]
Let $w = (d_2 + a_{22}v)v$, then $w$ solves
\[
w_t = B(x, t)\Delta w + B(x, t)g(x, t) \quad \text{in} \quad \Omega_T
with homogeneous Neumann boundary condition. Since all of the coefficients in the equation of \( w \) are Hölder continuous, we apply the Schauder estimate (\cite[Theorem 5.3, p. 320–321]{16}) to obtain
\[
\|w\|_{C^{2,\beta}}(Q_4) \leq C(\bar{t}, T) \quad \text{for some } \beta \in (0, 1).
\]
Using this fact and, again, the Schauder estimate for the equation of \( w_1 := (d_1 + a_{11}u + a_{12}v)u \), we find that
\[
\|w\|_{C^{2,\beta}}(Q_4) \leq C(\bar{t}, T) \quad \text{for some } \mu \in (0, 1).
\]
Thus (A.3) follows and the proof is complete. \( \square \)

**Appendix A. Proof of Lemma 2.9**

Since \( 0 \leq \bar{v} \leq 1 \), the equation (2.27) is uniformly parabolic. From this, the boundedness of the nonlinear term in (2.27), we see that \( \bar{v} \) is locally Hölder continuous (see \cite[Theorem 6.28, p. 130]{23} or \cite[Theorem 1.1, p. 419]{16}). This, \cite[Theorem 3.1, p. 437]{16} and Schauder estimates for linear uniformly parabolic equations further imply that \( \bar{v} \in C^{2,\alpha}_{\text{loc}}(Q_4) \). Therefore, there exists some constants \( \alpha \in (0, 1) \) and \( C > 0 \) depending only on \( n \) such that
\[
\|\bar{v}\|_{C^{2,\alpha}(Q_4^2)} \leq C.
\]
Let \( i = 1, 2, \ldots, n \) and denote \( w = \bar{v}_x \). Taking partial derivative of (2.27) in \( x_i \), we have
\[
w_i = \nabla \cdot [(1 + \bar{v})A_0 \nabla w + wA_0 \nabla \bar{v}] + \theta^2 (1 - 2\bar{v})w.
\]
With (A.1), we can apply the well-known De Giorgi-Nash-Moser iteration technique to this quasilinear uniformly parabolic equation to obtain
\[
\|w\|_{L^{n}(Q_4)} \leq C_n \left( \int_{Q_4} |w|^2 \, dx \right)^{\frac{1}{n}}.
\]
This immediately yields the inequality (2.28). For the sake of completeness, we give the detailed proof for (A.3). (For further references, one can see \cite[Theorem 8.1, p. 192, Theorem 3.1, p. 437]{16}, \cite[Theorem 2.1]{27}, or \cite[Theorem 1.3]{32}.) For an \( n \times n \) matrix \( A \), we denote its operator norm by \( \|A\|_{\text{op}} \) and \( \|A_0\|_{\infty} = \text{ess sup}_{x \in [0, 16]} \|A_0(x)\|_{\text{op}} \). By (2.26), we have \( \|A_0\|_{\infty} \leq \Lambda \), and \( \|A_0^{1/2}\|_{\infty} \|A_0^{-1/2}\|_{\infty} \leq \Lambda^{1/2} \). Let \( M_0 = \|\bar{v}\|_{L^{n}(Q_4^2)} \) and \( M_1 = \|\nabla \bar{v}\|_{L^{n}(Q_4^2)} \). For \( k \geq 0 \), define
\[
w^{(k)} = \max\{w - k, 0\} \quad \text{and} \quad S_k = \{(x, t) : w(x, t) > k\}.
\]
Let \( \zeta(x, t) \) be a cut-off function in \( Q_4 \). Multiplying equation (A.2) by \( w^{(k)} \zeta^2 \), integrating over \( B_4 \) and using integration by parts yield
\[
\frac{1}{2} \frac{d}{dt} \int_{B_4} |w^{(k)}\zeta|^2 \, dx = \int_{B_4} |w^{(k)}|^2 \zeta \zeta_t \, dx - \int_{B_4} [(1 + \bar{v})A_0 \nabla w + wA_0 \nabla \bar{v}] \cdot \nabla (w^{(k)} \zeta^2) \, dx
\]
\[
+ \theta^2 \int_{B_4} (1 - 2\bar{v})w w^{(k)} \zeta^2 \, dx = \int_{B_4} |w^{(k)}|^2 \zeta \zeta_t \, dx - \int_{B_4} [(1 + \bar{v})A_0 \nabla w^{(k)}] \cdot \nabla (w^{(k)} \zeta^2) \, dx
\]
\[
- \int_{B_4} (w^{(k)} + k)A_0 \nabla \bar{v} \cdot \nabla (w^{(k)} \zeta^2) \, dx + \theta^2 \int_{B_4} (1 - 2\bar{v})(w^{(k)} + k)w^{(k)} \zeta^2 \, dx.
\]
We estimate
\[
\frac{1}{2} \frac{d}{dt} \int_{B_4} |w^{(k)}\zeta|^2 \, dx \leq \int_{B_4} |w^{(k)}|^2 \zeta |\zeta_t| \, dx - \int_{B_4} [(1 + \bar{v})A_0 \nabla w^{(k)}] \cdot (\nabla (w^{(k)} \zeta) + w^{(k)} \nabla \zeta) \, dx
\]
\[
+ M_1 \|A_0\|_{\infty} \int_{B_4} |w^{(k)} + k| (|\nabla (w^{(k)} \zeta)| + |w^{(k)}| |\nabla \zeta|) \, dx + \theta^2 \int_{B_4} |w^{(k)} + k|w^{(k)} \zeta^2 \, dx.
\]
Note that
\[
\zeta A_0 \nabla w^{(k)} \cdot (\nabla (w^{(k)} \zeta) + w^{(k)} \nabla \zeta) = A_0^{1/2} (\nabla (w^{(k)} \zeta) - w^{(k)} \nabla \zeta) \cdot A_0^{1/2} (\nabla (w^{(k)} \zeta) + w^{(k)} \nabla \zeta)
= |A_0^{1/2} \nabla (w^{(k)} \zeta)|^2 - |w^{(k)} A_0^{1/2} \nabla \zeta|^2.
\]

Then
\[
\frac{1}{2} \frac{d}{dt} \int_{B_4} |w^{(k)} \zeta|^2 dx \leq \int_{B_4} |w^{(k)} \zeta|^2 |\zeta| dx - \int_{B_4} (1 + \bar{v})(|A_0^{1/2} \nabla (w^{(k)} \zeta)|^2 - |w^{(k)} A_0^{1/2} \nabla \zeta|^2) dx
+ M_1 \|A_0\|_{\infty} \int_{B_4} [w^{(k)} + k] \zeta |A_0^{-1/2} A_0^{1/2} \nabla (w^{(k)} \zeta)| + [w^{(k)} + k] \zeta |\nabla \zeta| dx + \theta^2 \int_{B_4} [w^{(k)} + k] |w^{(k)} \zeta|^2 dx.
\]

By Cauchy’s inequality
\[
\frac{1}{2} \frac{d}{dt} \int_{B_4} |w^{(k)} \zeta|^2 dx \leq \int_{B_4} |w^{(k)} \zeta|^2 |\zeta| dx - \int_{B_4} |A_0^{1/2} \nabla (w^{(k)} \zeta)|^2 dx + (1 + M_0) \int_{B_4} |w^{(k)} A_0^{1/2} \nabla \zeta|^2 dx
+ \left\{ \frac{1}{2} \int_{B_4} |A_0^{1/2} \nabla (w^{(k)} \zeta)|^2 dx + \frac{M_2 \|A_0\|_{\infty} \|A_0^{-1/2} A_0^{1/2} \zeta\|_{L^2}}{2} \int_{B_4} \chi(S_k)[w^{(k)} + k] \zeta^2 \zeta \right\}
+ M_1 \|A_0\|_{\infty} \int_{B_4} \chi(S_k)[w^{(k)} + k] |\nabla \zeta| dx + \theta^2 \int_{B_4} \chi(S_k)[w^{(k)} + k] \zeta^2 dx.
\]

Therefore,
\[
\frac{1}{2} \frac{d}{dt} \int_{B_4} |w^{(k)} \zeta|^2 dx + \frac{1}{2} \int_{B_4} |A_0^{1/2} \nabla (w^{(k)} \zeta)|^2 dx
\leq \int_{B_4} [w^{(k)} + k] \zeta \left[ |\zeta| + (1 + M_0) \Lambda |\nabla \zeta|^2 + \frac{M_2 \Lambda^3}{2} \zeta + M_1 \Lambda |\nabla \zeta| + \theta^2 \zeta^2 \right] dx
\leq \int_{B_4} [w^{(k)} + k] \zeta \left[ |\zeta| + 2(1 + M_0) \Lambda |\nabla \zeta|^2 + \left( \frac{M_2 \Lambda^3}{2} + M_1 \Lambda + \theta^2 \right) \zeta^2 \right] dx.
\]

Integrating in time and taking maximum for \( t \in (-16, 16) \) give
\[
\max_{t \in [-16, 16]} \int_{Q_4} |w^{(k)} \zeta|^2 dx + \Lambda^{-1} \int_{Q_4} |\nabla (w^{(k)} \zeta)|^2 dx dt
\leq 4 \int_{Q_4} \chi(S_k)[w^{(k)} + k] \zeta \left[ |\zeta| + 2(1 + M_0) \Lambda |\nabla \zeta|^2 + (2M_2 \Lambda^3 + \theta^2) \zeta^2 \right] dx dt.
\]

We obtain
\[
(A.4) \quad \max_{t \in [-16, 16]} \int_{B_4} |w^{(k)} \zeta|^2 dx + \int_{Q_4} |\nabla (w^{(k)} \zeta)|^2 dx dt \leq \int_{Q_4} \chi(S_k)[w^{(k)} + k] \zeta^2 P[\zeta] dx dt,
\]
where
\[
P[\zeta] = 8\Lambda \left[ |\zeta| + (1 + M_0) \Lambda |\nabla \zeta|^2 + (M_2 \Lambda^3 + \theta^2) \zeta^2 \right].
\]

Let \( r = (n + 2)/n \). The parabolic Sobolev embedding gives
\[
(A.5) \quad |w^{(k)} \zeta|_{L^r(\Omega_t)} \leq C_0 \left( \max_{t \in [-16, 16]} |w^{(k)} \zeta|_{L^2(B_4)} + |\nabla (w^{(k)} \zeta)|_{L^2(\Omega_t)} \right),
\]
where \( C_0 > 0 \). Therefore, we have from (A.4) that
\[
(A.6) \quad |w^{(k)} \zeta|_{L^r(\Omega_t)} \leq 2C_0 \left( \int_{Q_4} \chi(S_k)[w^{(k)} + k] \zeta^2 P[\zeta] dx dt \right)^{1/2}
\]

Now, we can use standard De Giorgi’s iteration. Let \( K > 0 \) and \( k_j = K(1 - 2^{-j}) \) for \( j \geq 0 \). Then \( k_j \not\to K \) as \( j \not\to \infty \). For \( j \geq 0 \), let \( r_j = 3 + 2^{-j-2} \) and \( t_j = r_j^2 \), then \( 3 < r_j < 7/2 \) and \( r_j \searrow 3 \) as \( j \not\to \infty \).
Let $\zeta_j = \phi_j(t)\varphi_j(x)$, with $0 \leq \phi_j, \varphi_j \leq 1$, $\phi_j = 1$ on $|t| < r_j^2$, $\varphi_j = 0$ on $|t| > r_j^2$, $\varphi_j = 1$ on $|x| < r_j$, $\varphi_j = 0$ on $|x| > r_j^2$. In other words, $\zeta_j = 1$ on $Q_{r_j}$ and spt $\zeta_j \subset Q_{r_j} \subset \bar{Q}_{r_0}$ for $j \geq 1$. Also,

(A.7) \[ |\nabla \zeta_j(x,t)| \leq C_1 2^j, \quad |\zeta_j| \leq C_1 4^j, \quad C_1 \geq 1. \]

Then we have on $Q_4$ that

(A.8) \[ |P[\zeta_j](x,t)| \leq 8\alpha C_1^2 4^j[1 + (1 + M_0)\Lambda + (M_1^2 \Lambda^3 + \theta^3)] \leq C_2^2 4^j, \]

where $C_2 > 0$ is defined by

(A.9) \[ C_2^2 = 8\alpha C_1^2[1 + (1 + M_0 + M_1^2)\Lambda^3 + \theta^2]. \]

Let $E_{j,k} = \{(x,t) \in Q_{r_j} : w(x,t) > k\}$. For $j \geq 0$, applying (A.6) with $k = k_{j+1}$ and $\zeta = \zeta_{j+1}$ and using (A.8) give

\[ \|\psi^{(k_{j+1})}\zeta_{j+1}\|_{L^2(E_{j+1},|x|)} \leq 2C_0\left( \int_{Q_{r_j}} \chi(S_{k_{j+1}})[w^{(k_{j+1})}]^2 |P[\zeta_{j+1}]| dx dt \right)^{1/2} \]
\[ \leq 2^{j+1} C_0 C_2\left( \|w^{(k_{j+1})}\|_{L^2(E_{j+1},|x|)} + K|E_{j+1,0}|^{1/2} \right) \]
\[ \leq 2^{j+1} C_0 C_2\left( \|w^{(k)}\|_{L^2(E_{j+1},|x|)} + K|E_{j+1,0}|^{1/2} \right). \]

On the one hand,

\[ \|w^{(k)}\|_{L^2(E_{j+1},|x|)} \leq \|w^{(k)}\|_{L^2(E_{j+1},|x|)} \leq \|w^{(k)}\|_{L^2(E_{j+1},|x|)} \leq \|w^{(k_{j+1})}\zeta_{j+1}\|_{L^2(E_{j+1},|x|)}|E_{j+1,0}|^{1-2/j} \]
\[ \leq \|w^{(k)}\|_{L^2(E_{j+1},|x|)}|E_{j+1,0}|^{1-2/j} . \]

On the other hand,

\[ \|w^{(k)}\|_{L^2(E_{j+1},|x|)} \geq \|w^{(k)}\|_{L^2(E_{j+1},|x|)} \geq (k_{j+1} - k)|E_{j+1,0}|^{1/2} = K2^{-j-1}|E_{j+1,0}|^{1/2} . \]

Hence $|E_{j+1,0}| \leq 4^{j+1} K^{-2}\|w^{(k)}\|_{L^2(E_{j+1},|x|)}^2$. Combining the above gives

\[ \|w^{(k_{j+1})}\|_{L^2(E_{j+1},|x|)} \leq \|w^{(k_{j+1})}\zeta_{j+1}\|_{L^2(E_{j+1},|x|)}|E_{j+1,0}|^{1-2/j} \]
\[ \leq 2^{j+1} C_0 C_2\left( \|w^{(k)}\|_{L^2(E_{j+1},|x|)} + K|E_{j+1,0}|^{1/2} \right)|E_{j+1,0}|^{1-2/j} \]
\[ \leq 2^{j+1} C_0 C_2\left( \|w^{(k)}\|_{L^2(E_{j+1},|x|)} + 2^{j+1}\|w^{(k)}\|_{L^2(E_{j+1},|x|)} \right)^{4^{j+1} K^{-2}}\|w^{(k)}\|_{L^2(E_{j+1},|x|)}^{1-\frac{1}{2}} . \]

Thus,

\[ \|w^{(k_{j+1})}\|_{L^2(E_{j+1},|x|)} \leq \frac{2 \cdot 16^{j+1} C_0 C_2}{K^\mu}\|w^{(k)}\|_{L^2(E_{j+1},|x|)}^{1+\mu} , \]

where $\mu = 2(1 - \frac{1}{2}) > 0$. Letting $Y_j = \|w^{(k)}\|_{L^2(E_{j+1},|x|)}$, then we have

(A.10) \[ Y_{j+1} \leq AB^j Y_j^{1+\mu}, \quad j \geq 0, \]

where $B = 16$ and $A = 32C_0 C_2 / K^\mu$. The classical result on sequences with fast geometric convergence states that if $Y_0 \leq A^{-1/\mu} B^{-1/\mu^2}$ then

(A.11) \[ \|w^{(K)}\|_{L^2(Q_3)} = \lim_{j \to \infty} Y_j = 0. \]

Since $k_0 = 0$, then $Y_0 \leq \|w\|_{L^2(Q_3)}$, and we choose $K$ such that

\[ \|w\|_{L^2(Q_3)} \leq (32C_0 C_2 / K^\mu)^{-1/\mu} 16^{-1/\mu^2} . \]

Specifically, select $K = \|w\|_{L^2(Q_3)} 16^{1/\mu^2} (32C_0 C_2)^{1/\mu}$. Then from (A.11), $w \leq K$ a.e. in $Q_3$. Replacing $w$ by $-w$ we obtain $|w| \leq K$ a.e. in $Q_3$. This, (A.9), definitions of $M_0$ and $M_1$, and interior estimate (A.1) together imply (A.3). The proof is complete.


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