BRAID GROUP ACTION ON EXTENDED CRYSTALS

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Abstract. In the paper, we prove that there exists a braid group action on the extended crystal $\hat{B}(\infty)$ of finite type. The extended crystal $\hat{B}(\infty)$ and its braid group action are investigated from the viewpoint of crystal similarity. We then interpret the braid group action on $\hat{B}(\infty)$ in the Hernandez-Leclerc category $\mathcal{C}_g^\emptyset$.

Contents

Introduction 2
1. Preliminaries 5
1.1. Crystals 5
1.2. Extended crystals 8
1.3. Categorical crystals of Hernandez-Leclerc categories 10
2. Crystals and PBW bases 13
3. Braid group action on $\hat{B}(\infty)$ 16
4. Similarity 23
5. Hernandez-Leclerc categories 27
References 31

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The notion of extended crystals is introduced in [22] for studying the module category of a quantum affine algebra from the viewpoint of the crystal basis theory. The crystals are one of the most powerful tools for studying quantum groups in a combinatorial way, and they appear naturally in a large number of applications in various research areas (see [12, 13, 14, 15], and see also [7, 28] and the references therein). We denote by $U_q(\mathfrak{g})$ the quantum group associated with a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$ and by $B(\infty)$ the crystal of the negative half $U_q^{-}(\mathfrak{g})$. The extended crystal is defined as

$$\hat{B}(\infty) := \left\{(b_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} B(\infty) \mid b_k = 1 \text{ for all but finitely many } k\right\},$$

where 1 is the highest weight vector of $B(\infty)$. The extended crystal operators $\tilde{F}_{i,k}$ and $\tilde{E}_{i,k}$ ($((i, k) \in \hat{I} := I \times \mathbb{Z})$ are defined in terms of the usual crystal operators $\tilde{f}_i$, $\tilde{e}_i^*$, $\tilde{f}_i^*$ and $\tilde{e}_i$ for the crystal $B(\infty)$, and the weight $\tilde{wt}$ for $\hat{B}(\infty)$ is defined by using usual weight function $wt$ on $B(\infty)$ (see Section 1.2 for details). As a usual crystal, one can define an $\hat{I}$-colored graph structure on the extended crystal $\hat{B}(\infty)$ by using the operators $\tilde{F}_{i,k}$ ($((i, k) \in \hat{I})$. It is shown in [22] that $\hat{B}(\infty)$ is connected as an $\hat{I}$-colored graph and that there exist interesting symmetries $D$ and $\ast$ on $\hat{B}_g(\infty)$ compatible with extended crystal operators.

Let $\mathcal{C}_g$ be the category of finite-dimensional integrable modules over a quantum affine algebra $U'_q(\mathfrak{g})$, where $q$ is an indeterminate. The category $\mathcal{C}_g$ has been studied widely in various research fields including representation theory, geometry and mathematical physics (see [1, 3, 6, 8, 17, 29] for example). The category $\mathcal{C}_g$ has a distinguished subcategory $\mathcal{C}_g^0$, which is called a Hernandez-Leclerc category ([8]). We choose a complete duality datum $\mathcal{D} = \{L_i\}_{i \in I_{hn}}$ for $\mathcal{C}_g^0$ (see Section 1.3). Let $\mathcal{B}(\mathfrak{g})$ be the set of the isomorphism classes of simple modules in $\mathcal{C}_g^0$. It is proved in [22] that $\mathcal{B}(\mathfrak{g})$ has the categorical crystal structure defined by

$$\tilde{F}_{i,k}(M) := (\mathcal{D}^k L_i) \triangledown M, \quad \tilde{E}_{i,k}(M) := M \triangledown (\mathcal{D}^{k+1} L_i)$$
BRAID GROUP ACTION ON EXTENDED CRYSTALS

for $M \in \mathcal{B}(\mathfrak{g})$ and $(i, k) \in \hat{I}_{\text{fin}}$ and that $\mathcal{B}(\mathfrak{g})$ is isomorphic to the extended crystal $\hat{B}_{\text{fin}}(\infty)$ of the crystal $B_{\text{fin}}(\infty)$, i.e.,

$$\Phi_D : \hat{B}_{\text{fin}}(\infty) \xrightarrow{\sim} \mathcal{B}(\mathfrak{g})$$

Here $\mathcal{D}$ is the right dual functor in $\mathcal{C}_0^{\mathfrak{g}}$ and $\mathfrak{g}_{\text{fin}}$ is the simple Lie algebra of simply-laced finite type given in (1.7). Under the isomorphism $\Phi_D$ between $\mathcal{B}(\mathfrak{g})$ and $\hat{B}_{\text{fin}}(\infty)$, the categorical crystal operators $\tilde{F}_{i,k}$ and $\tilde{E}_{i,k}$ correspond to the extended crystal operators $\tilde{F}_{i,k}$ and $\tilde{E}_{i,k}$, and the dual functor $\mathcal{D}$ matches with the operator $D$. This extended crystal isomorphism allows us to study simple modules in $\mathcal{C}_0^{\mathfrak{g}}$ in terms of the extended crystal $\hat{B}_{\text{fin}}(\infty)$, which is a new combinatorial approach to the category $\mathcal{C}_0^{\mathfrak{g}}$ from the viewpoint of crystals.

In this paper, we prove that there exists a braid group action on the extended crystal $\hat{B}(\infty)$ associated with a finite Cartan matrix and study several properties of the braid group action. Let $A = (a_{i,j})_{i,j \in I}$ be a Cartan matrix of finite type, and let $\hat{B}(\infty)$ be the extended crystal of the crystal $B(\infty)$ associated with $A$. We denote by $\mathcal{B}_A$ the generalized braid group (or Artin-Tits group) defined by the generators $r_i$ $(i \in I)$ and the following defining relations:

$$r_i r_j r_i r_j \cdots = r_j r_i r_j r_i \cdots$$

for $i, j \in I$ with $i \neq j$,

where $m(i, j)$ is the integer determined by $A$ (see (3.2) for the precise definition). We simply call $\mathcal{B}_A$ the braid group associated with $A$.

For each $i \in I$, we define bijections

$$R_i : \hat{B}(\infty) \xrightarrow{\sim} \hat{B}(\infty) \quad \text{and} \quad R_i^* : \hat{B}(\infty) \xrightarrow{\sim} \hat{B}(\infty)$$

using usual crystal operators and the Saito crystal reflections ([32]) on $B(\infty)$ (see Section 3). We prove that $R_i$ and $R_i^*$ are inverse to each other and that they satisfy the braid group relations for $\mathcal{B}_A$ (Theorem 3.4). Thus we have the action of $\mathcal{B}_A$ on $\hat{B}(\infty)$ via the bijections $R_i$. In the course of proofs, PBW bases are used crucially. The actions $R_i$ and $R_i^*$ commute with the operator $D$, i.e., $R_i \circ D = D \circ R_i$ and $R_i^* \circ D = D \circ R_i^*$, and they are
compatible with the reflection $s_i$ on the weight lattice via the weight function $\hat{\text{wt}}$, i.e.,

$$\hat{\text{wt}}(R_i(b)) = s_i(\hat{\text{wt}}(b)), \quad \hat{\text{wt}}(R^*_i(b)) = s_i(\hat{\text{wt}}(b))$$

for any $b \in \hat{B}(\infty)$ (see Lemma 3.2). Thus the braid group action $R_i$ can be understood as a natural extension of the Saito crystal reflection on $B(\infty)$ to the extended crystal $\hat{B}(\infty)$. Let $R(w_0)$ be the set of all reduced expression of the longest element $w_0$ (see (1.1)). We show that, for any $i \in R(w_0)$,

$$R_i = D \circ \zeta \quad \text{and} \quad R^*_i = D^{-1} \circ \zeta,$$

where $\zeta$ is the involution defined in (1.4). In particular, unless the Cartan matrix $A$ is of type $A_n$ ($n \in \mathbb{Z}_{>1}$), $D_n$ ($n$ is odd) or $E_6$, we have

$$R_i = D \quad \text{and} \quad R^*_i = D^{-1}.$$

Hence the central elements of the braid group $R_A$ act as $D^t$ on $\hat{B}(\infty)$ for some $t \in \mathbb{Z}$. As a $R_A$-set, $\hat{B}(\infty)$ is not transitive. It is conjectured that $\hat{B}(\infty)$ is faithful as a $R_A$-set for any finite Cartan matrix $A$ (Remark 3.5). We also conjecture that a braid group action with similar properties exists for a generalized Cartan matrix of arbitrary type.

We next investigate the extended crystal $\hat{B}(\infty)$ from the viewpoint of similarity of crystals ([16]). Applying the Dynkin diagram folding ([16, Section 5]) for $B(\infty)$ to extended crystals, we obtain an analogue of [16, Theorem 5.1] for extended crystals. Let $\sigma$ be a Dynkin diagram automorphism given in (4.1). We denote by $\hat{B}_\sigma(\infty)$ the extended crystal associated with the Cartan matrix folded by $\sigma$ and by $\hat{B}(\infty)^\sigma$ the set of fixed points of $\hat{B}(\infty)$ under the action $\sigma$. Proposition 4.3 says that there exists an crystal isomorphism

$$\hat{\Upsilon}_\sigma : \hat{B}_\sigma(\infty) \overset{\sim}{\rightarrow} \hat{B}(\infty)^\sigma,$$

where the extended crystal operator for $\hat{B}(\infty)^\sigma$ is defined as a product of usual extended crystal operators of $\hat{B}(\infty)$ in the same $\sigma$-orbit. Moreover, the isomorphism $\hat{\Upsilon}_\sigma$ is compatible with the braid group actions, i.e.,

$$\hat{\Upsilon}_\sigma (r'_j(b)) = r'_j \left( \hat{\Upsilon}_\sigma (b) \right)$$

for any $j$,
where \( r'_j \) is a generator of the braid group \( B_\sigma \) associated with the Cartan matrix folded by \( \sigma \) and \( r'^{\sigma}_j \) is defined as a product of usual generators of \( B_\Lambda \) in the same \( \sigma \)-orbit (see (4.2)).

We finally interpret the braid group action on \( \hat{B}(\infty) \) in the Hernandez-Leclerc category \( C^0_g \). It is announced in [21] that there is an action of the braid group \( B_{\Lambda_{\text{fin}}} \) on the quantum Grothendieck ring of \( C^0_g \) and that there are monoidal autofunctors on the localization \( T_N \) which give the braid group actions at the Grothendieck ring. We remark that \( T_N \) can be regarded as a graded version of \( C^0_g \) for affine type \( A(1)^{N-1} \). It is conjectured in [21, Section 5] that such functors \( R_i \) (\( i \in I_{\text{fin}} \)) exist for any arbitrary quantum affine algebra (see Conjecture 1). Under the assumption that the conjecture holds, Proposition 5.1 tells us that the braid group action \( R_i \) on \( \hat{B}(\infty) \) is a crystal-theoretic shadow of the conjectural functor \( R_i \), i.e.,

\[
\Phi_D(R_i(b)) = R_i(\Phi_D(b)) \quad \text{for any } b \in \hat{B}_{\text{fin}}(\infty) \text{ and } i \in I_{\text{fin}}.
\]

This paper is organized as follows. In Section 1, we review briefly the notion of extended crystals and Hernandez-Leclerc categories. In Section 2, we recall the Saito crystal reflections and a connection to PBW bases. In Section 3, we define a braid group action on the extended crystal \( \hat{B}(\infty) \) and investigate its properties. We then study the similarity for \( \hat{B}(\infty) \) in Section 4, and interpret the braid group action for \( \hat{B}(\infty) \) in the Hernandez-Leclerc category \( C^0_g \) in Section 5.

1. Preliminaries

1.1. Crystals.

In this subsection, we briefly review the notion of crystals (see [12, 13, 14, 15], and see also [7]). Let \( I \) be a finite index set.

Definition 1.1. A quintuple \( (A, P, \Pi, P^\vee, \Pi^\vee) \) is called a (symmetrizable) Cartan datum if it consists of

\begin{itemize}
  \item[(a)] a generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \),
  \item[(b)] a free abelian group \( P \), called the weight lattice,
\end{itemize}
(c) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of simple roots,
(d) $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$, called the coweight lattice,
(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, called the set of simple coroots,

which satisfy the following properties:
(i) $\langle h_i, \alpha_j \rangle = \alpha_{ij}$ for $i, j \in I$,
(ii) $\Pi$ is linearly independent over $\mathbb{Q}$,
(iii) for each $i \in I$, there exists $\Lambda_i \in P$, called a fundamental weight, such that $\langle h_j, \Lambda_i \rangle = \delta_{j,i}$ for all $j \in I$.
(iv) there is a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $P$ satisfying $\langle \alpha_i, \alpha_i \rangle \in \mathbb{Q}_{>0}$ and $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$.

We denote by $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ and by $\Delta^+$ the set of positive roots. The Weyl group $W$ associated with $A$ is the subgroup of $\text{Aut}(P)$ generated by $s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i$

for any $i \in I$. In this paper, the standard notation for Dynkin diagrams given in [11] is
used except type $E_k$ ($k = 6, 7, 8$). For the case of type $E_k$ ($k = 6, 7, 8$), we follow the
notation for Dynkin diagrams appeared in [19, Appendix A.1]. For $w \in W$, $\ell(w)$ denotes
the length of $w$, and $R(w)$ is the set of all reduced expressions of $w$, i.e.,

\begin{equation}
R(w) := \{(i_1, i_2, \ldots, i_t) \in I^t \mid w = s_{i_1} s_{i_2} \cdots s_{i_t}\}
\end{equation}

where $t = \ell(w)$. We assume that $A$ is of finite type. Let $w_0$ be the longest element of $W$.
One can show that, for any $(i_1, i_2, \ldots, i_\ell) \in R(w_0)$, we have

\begin{equation}
(i_2, i_3, \ldots, i_\ell, i_1^*) \in R(w_0) \quad \text{and} \quad (i_1^*, i_1, i_2, \ldots, i_\ell-1) \in R(w_0),
\end{equation}

where $i^*$ is defined by

\begin{equation}
\alpha_{i^*} = -w_0(\alpha_i)
\end{equation}

Let $U_q(\mathfrak{g})$ be the quantum group associated with $(A, P, P^\vee, \Pi, \Pi^\vee)$, and $U_q^-(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i$ ($i \in I$) (see [7, Chapter 3] for details). Let $A = \mathbb{Z}[q, q^{-1}]$, and $A_0$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions which are regular
at \( q = 0 \). For each \( i \in I \), we denote by \( \tilde{f}_i \) and \( \tilde{e}_i \) the Kashiwara operators on \( U_q^-(\mathfrak{g}) \) ([13, (3.5.1)])

\[
L(\infty) := \sum_{l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} 1 \subset U_q^-(\mathfrak{g}), \quad L(\infty) := \{ x \in U_q^-(\mathfrak{g}) \mid x \in L(\infty) \},
\]

\[
B(\infty) := \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} 1 \mod qL(\infty) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I \} \subset L(\infty)/qL(\infty),
\]

where \( - : U_q^-(\mathfrak{g}) \xrightarrow{\sim} U_q^+(\mathfrak{g}) \) is the \( \mathbb{Q} \)-algebra automorphism defined by \( \tilde{e}_i = e_i, \tilde{f}_i = f_i, q^h = q^{-h} \), and \( \overline{\mathfrak{g}} = \mathfrak{g}^{-1} \). We call \( B(\infty) \) the crystal of \( U_q^-(\mathfrak{g}) \). When we need to emphasize \( B(\infty) \) as a \( U_q^-(\mathfrak{g}) \)-crystal, we write \( B^\infty(\infty) \) instead of \( B(\infty) \).

Let \( G^{\text{low}} \) be the inverse of the \( \mathbb{Q} \)-linear isomorphism \( (\mathbb{Q} \otimes_{\mathbb{Z}} U^\infty_-(\mathfrak{g})) \cap L(\infty) \cap \overline{L(\infty)} \xrightarrow{\sim} L(\infty)/qL(\infty) \). The set

\[
G^{\text{low}}(\infty) := \{ G^{\text{low}}(b) \in U^\infty_-(\mathfrak{g}) \mid b \in B(\infty) \}
\]

is an \( \mathbb{A} \)-basis of \( U^\infty_-(\mathfrak{g}) \), which is called the lower global basis (or canonical basis). Then we have the dual basis with respect to the Kashiwara bilinear form ([13, Proposition 3.4.4])

\[
G^{\text{up}}(\infty) := \{ G^{\text{up}}(b) \mid b \in B(\infty) \},
\]

which is called the upper global basis (or dual canonical basis). The \( \mathbb{Q}(q) \)-antiautomorphism \( \ast \) of \( U_q(\mathfrak{g}) \) given by \( (e_i)^\ast = e_i, (f_i)^\ast = f_i, \) and \( (q^h)^\ast = q^{-h} \) provides another crystal operators \( \tilde{e}_i^\ast \) and \( \tilde{f}_i^\ast \). For any \( b \in B(\infty) \), we set

\[
\tilde{e}_i^{\ast \text{max}}(b) := \tilde{e}_i^{\epsilon_i(b)}(b) \quad \text{and} \quad \tilde{e}_i^{\ast \text{max}}(b) := \tilde{e}_i^{\ast \epsilon_i^\ast(b)}(b).
\]

Here we define \( \epsilon_i(b) := \max\{ k \geq 0 \mid \tilde{e}_i^k(b) \neq 0 \} \) and \( \varphi_i(b) := \epsilon_i(b) + \langle h_i, \text{wt}(b) \rangle \) (resp. \( \epsilon_i^\ast(b) := \max\{ k \geq 0 \mid \tilde{e}_i^{\ast k}(b) \neq 0 \} \) and \( \varphi_i^\ast(b) := \epsilon_i^\ast(b) + \langle h_i, \text{wt}(b) \rangle \)). For more details of crystals, we refer the reader to [12, 13, 14, 15] and [7, Chapter 4].

We define

\[
(1.4) \quad \zeta : I \xrightarrow{\sim} I, \quad i \mapsto i^\ast,
\]

where \( i^\ast \) is given in (1.3). If \( \mathfrak{g} \) is of type \( A_n \) (\( n \in \mathbb{Z}_{>1} \)), \( D_n \) (\( n \) is odd), and \( E_6 \), then \( \zeta \) is given as follows:

(a) (Type \( A_n \)) \( i^\ast = n + 1 - i \),
(b) (Type $D_n$) $i^* = \begin{cases} n - (1 - \xi) & \text{if } n \text{ is odd and } i = n - \xi \ (\xi = 0, 1), \\ i & \text{otherwise}, \end{cases}$

(c) (Type $E_6$) The map $i \mapsto i^*$ is determined by

$$i^* = \begin{cases} 6 & \text{if } i = 1, \\ i & \text{if } i = 2, 4, \\ 5 & \text{if } i = 3. \end{cases}$$

Otherwise, $\zeta$ is the identity. Note that we use the index set of the Dynkin diagram of type $E_6$ given in [19, Appendix A.1]. The involution $\zeta$ induces an involution on the crystal $B(\infty)$, which is also denoted by $\zeta$.

**Example 1.2.** We briefly review the multisegment realization of $B(\infty)$ of type $A_n$ (see [33] and see also [5] and [22, Section 7.1]). This realization will be used as an example in the paper. Let $[a, b]$ be an interval for $1 \leq a \leq b \leq n$. A multisegment is a multiset of $[a, b]$’s, and we denote by $\text{MS}_n$ the set of multisegments. If $a > b$, then we define $[a, b] := \emptyset$. We write $[a] = [a, b]$ if $a = b$. For any multisegment $m = \{m_1, \ldots, m_k\}$, we write $m = m_1 + m_2 + \cdots + m_k$. The crystal structure of $\text{MS}_n$ is described in [22, Section 7.1], which is isomorphic to $B(\infty)$. We follow the description of [22, Section 7.1].

1.2. Extended crystals.

In this subsection, we briefly review the notion of extended crystals introduced in [22]. We keep the notations given in the previous subsection. We set

$$\widehat{B}(\infty) := \left\{(b_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} B(\infty) \mid b_k = 1 \text{ for all but finitely many } k \right\},$$

For any $b = (b_k)_{k \in \mathbb{Z}} \in \widehat{B}(\infty)$, we sometimes write $b = (\ldots, b_2, b_1, \underline{b_0}, b_{-1}, \ldots)$, where the underlined element is located at the 0-position. We define $1 := (1)_{k \in \mathbb{Z}} \in \widehat{B}(\infty)$, which can be viewed as a highest weight vector of $\widehat{B}(\infty)$. Set $\widehat{I} := I \times \mathbb{Z}$, and let $(i, k) \in \widehat{I}$ and $b = (b_k)_{k \in \mathbb{Z}} \in \widehat{B}(\infty)$. Define

$$\widehat{\text{wt}}(b) := \sum_{k \in \mathbb{Z}} \text{wt}_k(b),$$
Lemma 4.2]). It is shown in \([\text{Lemma 4.4}]\), and set
\[
\hat{B}(\infty)_\beta := \{b \in \hat{B}(\infty) | \hat{\text{wt}}(b) = \beta\}
\] for any \(\beta \in \mathbb{Q}\).

We define
\[
\hat{\varepsilon}_{i,k}(b) := \varepsilon_{i,k}(b) - \varepsilon_{i,k+1}(b),
\]
where \(\varepsilon_j(b) := \varepsilon_j(b_t)\) and \(\varepsilon_{j,t}^*(b) := \varepsilon_j^*(b_t)\) for \((j, t) \in \hat{I}\).

The extended crystal operators
\[
\tilde{F}_{i,k} : \hat{B}(\infty) \rightarrow \hat{B}(\infty) \quad \text{and} \quad \tilde{E}_{i,k} : \hat{B}(\infty) \rightarrow \hat{B}(\infty)
\]
are defined by
\[
\tilde{F}_{i,k}(b) := \begin{cases} \cdots, b_{k+2}, b_{k+1}, \tilde{f}_i(b_k), b_{k-1}, \cdots \text{ if } \hat{\varepsilon}_{i,k}(b) \geq 0, \\ \cdots, b_{k+2}, \tilde{e}_i^*(b_{k+1}), b_k, b_{k-1}, \cdots \text{ if } \hat{\varepsilon}_{i,k}(b) < 0, \end{cases}
\]
and
\[
\tilde{E}_{i,k}(b) := \begin{cases} \cdots, b_{k+2}, b_{k+1}, \hat{e}_i(b_k), b_{k-1}, \cdots \text{ if } \hat{\varepsilon}_{i,k}(b) > 0, \\ \cdots, b_{k+2}, \hat{f}_i^*(b_{k+1}), b_k, b_{k-1}, \cdots \text{ if } \hat{\varepsilon}_{i,k}(b) \leq 0, \end{cases}
\]
for any \((i, k) \in \hat{I}\) and \(b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)\). Note that \(\tilde{F}_{i,k}\) is the inverse of \(\tilde{E}_{i,k}\) ([22, Lemma 4.2]). It is shown in [22, Section 4] that the extended crystals \(\hat{B}(\infty)\) enjoy similar properties to usual crystals.

For \(p \in \mathbb{Z}\) and \(b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)\), we define \(D^p(b) = (b'_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)\) by
\[
b'_k = b_{k-p} \quad \text{for any } k \in \mathbb{Z},
\]
which gives a bijection
\[
D^p : \hat{B}(\infty) \rightarrow \hat{B}(\infty).
\]
We remark that \(D^p(\tilde{F}_{i,k}(b)) = \tilde{F}_{i,k+p}(D^p(b))\) for any \(p \in \mathbb{Z}\) and \((i, k) \in \hat{I}\).

The extended crystal \(\hat{B}(\infty)\) has the \(\hat{I}\)-colored graph structure defined by \(\tilde{F}_{i,k}\) for \((i, k) \in \hat{I}\), i.e.,
\[
b \xrightarrow{(i,k)} b' \quad \text{if and only if} \quad b' = \tilde{F}_{i,k}(b).
\]

**Proposition 1.3** ([22, Lemma 4.4]). As an \(\hat{I}\)-colored graph, \(\hat{B}(\infty)\) is connected.
Example 1.4. We keep the notation given in Example 1.2. We define
\[ \widehat{\text{MS}}_n := \left\{ (m_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} \text{MS}_n \mid m_k = \emptyset \text{ for all but finitely many } k \right\} . \]
Since \( \text{MS}_n \) is isomorphic to \( B(\infty) \), \( \widehat{\text{MS}}_n \) has the extended crystal structure induced from the crystal structure of \( \text{MS}_n \).

1.3. Categorical crystals of Hernandez-Leclerc categories.

In this subsection, we briefly review the categorical crystals of Hernandez-Leclerc categories developed in [22]. This subsection will be used only in Section 5.

We denote by \( \mathcal{C}_g \) the category of finite-dimensional integrable modules over \( U_q' (g) \) as-\( s \)ociated with an affine Cartan matrix \( A \). We set \( I_0 := I \setminus \{0\} \). Here we refer the reader to [20, Section 2.3] for the choice of 0. We write \( M \otimes^k := M \otimes \cdots \otimes M \) \( k \)-times for \( k \in \mathbb{Z} \geq 0 \). For any module \( X \) of finite length, \( \text{hd}(X) \) is the head of \( X \) and, for the sake of simplicity, \( M \nabla N \) denotes the head of \( M \otimes N \) for \( M, N \in \mathcal{C}_g \). A simple module \( N \) is real if \( N \otimes N \) is simple.

For any \( M \in \mathcal{C}_g \), \( \mathcal{D}(M) \) denotes the right dual of \( M \), which is extended to \( \mathcal{D}^k(M) \) for all \( k \in \mathbb{Z} \). We denote by \( \mathcal{C}_g^0 \) the Hernandez-Leclerc category, which is a full subcategory of \( \mathcal{C}_g \) with certain conditions (see [8] and see also [22, Section 2.2] for details).

We call \( \mathcal{D} := \{ L_i \}_{i \in J} \) a strong duality datum associated with a simply-laced finite Cartan matrix \( C = (c_{i,j})_{i,j \in J} \) if it satisfies the following:

(a) \( L_i \) is a root module for any \( i \in J \),

(b) \( \mathfrak{d}(L_i, \mathcal{D}^k(L_j)) = -\delta(k = 0) c_{i,j} \) for any \( k \in \mathbb{Z} \) and \( i, j \in J \) with \( i \neq j \),

where a simple module \( N \) is a root module if it is real and \( \mathfrak{d}(N, \mathcal{D}^k N) = \delta(k = \pm 1) \) for any \( k \in \mathbb{Z} \). For a strong duality datum \( \mathcal{D} = \{ L_i \}_{i \in J} \), one can consider the corresponding quantum affine Schur-Weyl duality functor

\[ \mathcal{F}_\mathcal{D} : R_C\text{-gmod} \longrightarrow \mathcal{C}_g^0. \]

Here \( R_C \) is the symmetric quiver Hecke algebra corresponding to \( C \) (see [18, 20] for details). We denote by \( \mathcal{C}_\mathcal{D} \) the smallest full subcategory of \( \mathcal{C}_g^0 \) satisfying

(a) \( \mathcal{C}_\mathcal{D} \) contains \( \mathcal{F}_\mathcal{D}(N) \) for any simple \( R_C \)-module \( N \),

(b) \( \mathcal{C}_\mathcal{D} \) is stable by taking subquotients, extensions, and tensor products.
A strong duality datum $\mathcal{D}$ is said to be complete if for any simple module $M$ in $\mathcal{C}_g^0$, there are simple modules $M_k$ in $\mathcal{C}_D$ ($k \in \mathbb{Z}$) such that

(a) $M_k$ is isomorphic to the trivial module $\mathbf{1}$ for all but finitely many $k$,
(b) $M$ is isomorphic to $\text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M_1 \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots)$.

Let $\mathfrak{g}_C$ be the Lie algebra associated with $C$. If $D$ is complete, then the simple Lie algebra $\mathfrak{g}_C$ has to be of the type $X_g$ given in the table (1.7) (see [20, Proposition 6.2]). In this case, we write $\mathfrak{g}_\text{fin}$, $I_\text{fin}$, $A_\text{fin}$ etc. instead of $\mathfrak{g}_C$, $J$, $C$ etc.

| Type of $\mathfrak{g}$ | $A_n^{(1)}$ | $B_n^{(1)}$ | $C_n^{(1)}$ | $D_n^{(1)}$ | $A_{2n}^{(2)}$ | $A_{2n-1}^{(2)}$ | $D_{n+1}^{(2)}$ |
|------------------------|-------------|-------------|-------------|-------------|----------------|-----------------|----------------|
| Type $X_g$             | $A_n$       | $A_{2n-1}$  | $D_{n+1}$   | $D_n$       | $A_{2n}$       | $A_{2n-1}$     | $D_{n+1}$     |
| Type $E_6$             | $E_6^{(1)}$ | $E_7^{(1)}$ | $E_8^{(1)}$ | $F_4^{(1)}$ | $G_2^{(1)}$    | $E_6^{(2)}$    | $D_4^{(3)}$   |
| Type $X_g$             | $E_6$       | $E_7$       | $E_8$       | $E_6$       | $D_4$          | $E_6$          | $D_4$         |

From now on, we assume that $\mathcal{D} = \{L_i\}_{i \in I_{\text{fin}}}$ is a complete duality datum. Let $\mathcal{B}(g)$ be the set of the isomorphism classes of simple modules in $\mathcal{C}_g^0$. It is proved in [22] that $\mathcal{B}(g)$ has the categorical crystal structure defined by

$$
\tilde{\mathcal{F}}_{i,k}(M) := (\mathcal{D}^k L_i) \nabla M, \quad \tilde{\mathcal{E}}_{i,k}(M) := M \nabla (\mathcal{D}^{k+1} L_i)
$$

for $M \in \mathcal{B}(g)$ and $(i, k) \in \widehat{I}_{\text{fin}}$, and that $\mathcal{B}(g)$ is isomorphic to the extended crystal $\widehat{B}_{g_{\text{fin}}} (\infty)$. We write $\mathcal{B}_D(g)$ for $\mathcal{B}(g)$ when considering $\mathcal{B}(g)$ with $\tilde{\mathcal{F}}_{i,k}$ and $\tilde{\mathcal{E}}_{i,k}$. Let us briefly explain the isomorphism between $\widehat{B}_{g_{\text{fin}}} (\infty)$ and $\mathcal{B}_D(g)$.

Let $B_D$ be the set of the isomorphism classes of simple modules in $\mathcal{C}_D$. It is shown in [26] that the set of the isomorphism classes of simple modules in $R_{g_{\text{fin}}}$-gmod forms a crystal isomorphic to the crystal $B_{g_{\text{fin}}} (\infty)$. As $\mathcal{F}_D$ preserves simple modules, $\mathcal{F}_D$ induces a bijective map

$$
\mathcal{L}_D: B_{g_{\text{fin}}} (\infty) \xrightarrow{\sim} B_D
$$
For any $b = (b_k)_{k \in \mathbb{Z}} \in \widehat{B}_{\text{fin}}(\infty)$, define
\[
\mathcal{L}_D(b) := \text{hd} \left( \bigotimes_{k=+\infty}^{0} D^k L_k \right) = \text{hd}(\cdots \otimes D^2 L_2 \otimes D L_1 \otimes L_0 \otimes D^{-1} L_{-1} \otimes \cdots),
\]
where $L_k = \mathcal{L}_D(b_k)$ for $k \in \mathbb{Z}$. Then the map $\Phi_D: \widehat{B}_{\text{fin}}(\infty) \to B_D(g)$ defined by
\[
\Phi_D(b) := \mathcal{L}_D(b) \quad \text{for any } b \in \widehat{B}_{\text{fin}}(\infty)
\]
is an extended crystal isomorphism, i.e.,
\[
\Phi_D(\tilde{F}_{i,k}(b)) = \tilde{F}_{i,k}(\Phi_D(b)), \quad \Phi_D(\tilde{E}_{i,k}(b)) = \tilde{E}_{i,k}(\Phi_D(b))
\]
for $(i,k) \in \widehat{I}_{\text{fin}}$ and $b \in \widehat{B}_{\text{fin}}(\infty)$ (see [22, Theorem 5.9]).

Example 1.5. Let $U_q'(g)$ be the quantum affine algebra of affine type $A_2^{(1)}$. In this case, $g_{\text{fin}}$ is of type $A_2$ and $I_0 = \{1,2\}$. We review briefly the extended crystal isomorphism between $\widehat{M}_2$ and $B_D(g)$ given in [22, Section 7.4]. Let
\[
\mathcal{P}_2 := (\mathbb{Z}_{\geq 0})^{\otimes \mathcal{S}_2},
\]
where $\mathcal{S}_2 := \{(i,a) \in I_0 \times \mathbb{Z} \mid a - i \equiv 1 \mod 2\}$. We regard $\mathcal{S}_2$ as a subset of $\mathcal{P}_2$. For $(i,a) \in \mathcal{P}_2$ and $k \in \mathbb{Z}$, we set
\[
D^k(i,a) := \begin{cases} 
(i,a+3k) & \text{if } k \text{ is even}, \\
(3-i,a+3k) & \text{if } k \text{ is odd}.
\end{cases}
\]
We define $\gamma_k: M_2 \to \mathcal{P}_2$ by
\[
\gamma_k(a[2] + b[12] + c[1]) = aD^k(1,2) + bD^k(2,1) + cD^k(1,0).
\]
Then we have the bijection
\[
\gamma: \widehat{M}_2 \to \mathcal{P}_2
\]
defined by $\gamma(\widehat{m}) := \sum_k \gamma_k(m_k)$ for any $\widehat{m} = (m_k)_{k \in \mathbb{Z}} \in \widehat{M}_2$.

For any $\lambda = (i_k,a_k) \in \mathcal{P}_2$, we denote by $V(\lambda)$ the simple module in $\mathcal{C}_g$ whose affine highest weight is $\sum_k (i_k,(-q)^a_k)$ (see [22, Theorem 2.2] for affine height weights). Note that $V(i,a) = V(\varpi_i)(-q)^a$ and $D^k(V(i,a)) \simeq V(D^k(i,a))$ for $k \in \mathbb{Z}$.
We now choose a complete duality datum $\mathcal{D} = \{L_1, L_2\}$, where $L_1 := V(\omega_1)$ and $L_2 := V(\omega_1)(-q)^2$, and define

$$\Phi_D : \widehat{MS}_2 \rightarrow B_D(g)$$

by $\Phi_D(\hat{m}) := V(\gamma(\hat{m}))$ for any $\hat{m} \in \widehat{MS}_2$. Then $\Phi_D$ becomes an extended crystal isomorphism (see [22, Section 7.4]).

We remark that $P_2$ has also an extended crystal structure and its crystal operators are described in [22, Section 7.3] in a combinatorial manner.

2. Crystals and PBW bases

In this section, we shall investigate several properties about crystals and PBW bases.

From now on, we assume that $A = (a_{i,j})_{i,j \in I}$ is a Cartan matrix of finite type. Let $B(\infty)$ be the crystal of the negative half of the quantum group associated with $A$. For $i \in I$, we denote Lusztig’s braid symmetries by

$$T_i^{',-1} := T_i^* := T_i^{''},$$

where $T_i^{',-1}$ and $T_i^{''}$ are the symmetries defined in [28, Chapter 37.1]. Let $\ell := \ell(w_0)$ and let $i = (i_1, i_2, \ldots, i_\ell) \in R(w_0)$. For any $\epsilon = \pm 1$, $c \in \mathbb{Z}_{\geq 0}$ and $k = 1, \ldots, \ell$, we set $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_k) \in \Delta^+$ and

$$f_{i,\epsilon}(\beta_k)^{(c)} := \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(f_{i_k}^{(c)}) & \text{if } \epsilon = -1, \\ T_{i_1}^* T_{i_2}^* \cdots T_{i_{k-1}}^*(f_{i_k}^{(c)}) & \text{if } \epsilon = 1. \end{cases}$$

For any $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$, we define

$$f_{i,\epsilon}(a) := \begin{cases} f_{i,\epsilon}(\beta_1)^{(a_1)} f_{i,\epsilon}(\beta_2)^{(a_2-1)} \cdots f_{i,\epsilon}(\beta_\ell)^{(a_\ell)} & \text{if } \epsilon = -1, \\ f_{i,\epsilon}(\beta_1)^{(a_1)} \cdots f_{i,\epsilon}(\beta_{\ell-1})^{(a_{\ell-1})} f_{i,\epsilon}(\beta_\ell)^{(a_\ell)} & \text{if } \epsilon = 1. \end{cases}$$

The set $\{f_{i,\epsilon}(a)\}_{a \in \mathbb{Z}_{\geq 0}^{\ell}}$ is called the $PBW$ basis of $U_q^-(g)$ with respect to $i$ and $\epsilon$.

We now consider the dual version of the PBW basis. For any $a \in \mathbb{Z}_{\geq 0}^{\ell}$, we define

$$f_{i,\epsilon}^{up}(a) := f_{i,\epsilon}(a)/(f_{i,\epsilon}(a), f_{i,\epsilon}(a)).$$
where \((-,-)\) is the Kashiwara bilinear form defined in [13, Proposition 3.4.4]. Then the set \(\{f_{i,\epsilon}^\uparrow(a)\}_{a \in \mathbb{Z}_{\geq0}}\) becomes a basis of \(U_q^-(\mathfrak{g})\), which is called the dual PBW basis with respect to \(i\) and \(\epsilon\). Note that

\[
\begin{align*}
    f_{i,\epsilon}^\uparrow(a) = \begin{cases} 
        f_{i,\epsilon}^\uparrow(\beta_\ell) f_{i,\epsilon}^\uparrow(\beta_{\ell-1}) \cdots f_{i,\epsilon}^\uparrow(\beta_1) & \text{if } \epsilon = -1, \\
        f_{i,\epsilon}^\uparrow(\beta_1) \cdots f_{i,\epsilon}^\uparrow(\beta_{\ell-1}) f_{i,\epsilon}^\uparrow(\beta_\ell) & \text{if } \epsilon = 1,
    \end{cases}
\end{align*}
\]

where we define \(f_{i,\epsilon}^\uparrow(\beta_k)^{\{a\}} := q_{i_k}^{a(a-1)/2} f_{i,\epsilon}^\uparrow(\beta_k)^a\). Note that the dual PBW vectors \(f_{i,\epsilon}^\uparrow(\beta_k)^{\{a\}}\) are contained in \(G^\uparrow(\infty)\).

For any \(i \in I\), we set

\[
\begin{align*}
    i\mathcal{B}(\infty) & := \{b \in B(\infty) \mid \varepsilon_i(b) = 0\}, \\
    B_i(\infty) & := \{b \in B(\infty) \mid \varepsilon_i^*(b) = 0\}.
\end{align*}
\]

The Saito crystal reflections (see [32]) on the crystal \(B(\infty)\) are defined as follows

\[
\begin{align*}
    T_i : i\mathcal{B}(\infty) & \to B_i(\infty), & T_i(b) & := \tilde{f}_i^{\varepsilon_i^*(b)} \varepsilon_i^*(b), \\
    T_i^* : B_i(\infty) & \to i\mathcal{B}(\infty), & T_i^*(b) & := \tilde{f}_i^{\varepsilon_i(b)} \varepsilon_i(b).
\end{align*}
\]

Note that \(T_i \circ T_i^* = \text{id}\) and \(T_i^* \circ T_i = \text{id}\). The crystal reflections \(T_i\) and \(T_i^*\) are the crystal counterparts of the braid symmetries \(T_i\) and \(T_i^*\). Let \(i\pi : B(\infty) \to i\mathcal{B}(\infty)\) and \(\pi_i : B(\infty) \to B_i(\infty)\) be the surjective maps defined by

\[
\begin{align*}
    i\pi(b) & := \varepsilon_i^{\text{max}}(b) & \pi_i(b) & := \varepsilon_i^*(\text{max})(b) \quad \text{for } b \in B(\infty),
\end{align*}
\]

and we set

\[
\begin{align*}
    \tilde{T}_i & := T_i \circ i\pi : B(\infty) \to B_i(\infty) \subset B(\infty), \\
    \tilde{T}_i^* & := T_i^* \circ \pi_i : B(\infty) \to i\mathcal{B}(\infty) \subset B(\infty).
\end{align*}
\]

For any \(j = (j_1, \ldots, j_t) \in \text{R}(w)\), we define

\[
\begin{align*}
    \tilde{T}_j & := \tilde{T}_{j_1} \tilde{T}_{j_2} \cdots \tilde{T}_{j_t} & \tilde{T}_j^* & := \tilde{T}_{j_1}^* \tilde{T}_{j_2}^* \cdots \tilde{T}_{j_t}^*.
\end{align*}
\]
Definition 2.1 (Lusztig datum). For \( b \in B(\infty) \) and \( i = (i_1, \ldots, i_\ell) \in R(w_0) \), we define

\[
\mathcal{L}_{i,\epsilon}(b) := \begin{cases} 
(\varepsilon^*_{i_\ell}(\tilde{T}_{i_{\ell-1}} \cdots \tilde{T}_{i_2} \tilde{T}_{i_1}(b)), \ldots, \varepsilon^*_{i_2}(\tilde{T}_{i_1}(b)), \varepsilon^*_{i_1}(b)) & \text{if } \epsilon = -1, \\
(\varepsilon_{i_1}(b), \varepsilon_{i_2}(\tilde{T}_{i_1}(b)), \ldots, \varepsilon_{i_\ell}(\tilde{T}_{i_{\ell-1}} \cdots \tilde{T}_{i_2} \tilde{T}_{i_1}(b))) & \text{if } \epsilon = 1. 
\end{cases}
\]

We remark that

\[
G_{\uparrow}(b) \equiv f_{i,\epsilon}^\uparrow(a) \mod qL(\infty) \quad \text{for } b \in B(\infty),
\]

where \( a = \mathcal{L}_{i,\epsilon}(b) \) (see [27, 28], and see also [24, Theorem 4.29] and [25, Section 3.2.2]). By the PBW theory, it is easy to see that the map

\[
\mathcal{L}_{i,\epsilon} : B(\infty) \xrightarrow{\sim} \mathbb{Z}_{\geq 0}^\oplus, \quad b \mapsto \mathcal{L}_{i,\epsilon}(b)
\]

is bijective and the inverse of \( \mathcal{L}_{i,\epsilon} \) is given by

\[
G_{\uparrow}^\uparrow(\mathcal{L}_{i,\epsilon}^{-1}(a)) \equiv f_{i,\epsilon}^{\uparrow}(a) \mod qL(\infty).
\]

Lemma 2.2. Let \( i = (i_1, i_2, \ldots, i_\ell) \in R(w_0) \) and set \( i^\vee := (i^*_\ell, \ldots, i^*_2, i^*_1) \). Then, for any \( b \in B(\infty) \), we have

\[
\mathcal{L}_{i,-1}(b) = \mathcal{L}_{i^\vee,1}(b).
\]

Proof. Thanks to (1.2), we have

\[
(i^*_{k+1}, i^*_{k+2}, \ldots, i^*_\ell, i_1, \ldots, i_k) \in R(w_0)
\]

for any \( k = 0, 1, \ldots, \ell \). Since

\[
f_{i_k} = \mathcal{T}_{i_{k+1}}^* \mathcal{T}_{i_{k+2}}^* \cdots \mathcal{T}_{i_{\ell}}^* \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{k-1}}(f_{i_k}),
\]

we have

\[
\mathcal{T}_{i_{\ell}}^* \cdots \mathcal{T}_{i_{k+1}}^* (f_{i_k}) = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{k-1}}(f_{i_k}),
\]

which implies that

\[
f_{i^\vee,1}(\beta^\vee_k) = f_{i,-1}(\beta_k),
\]

where \( \beta^\vee_k := s_{i^*_k} \cdots s_{i^*_{k+1}}(\alpha_k) \) and \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_k) \). Therefore, since both of the PBW vectors are the same, the assertion follows from (2.1). \( \square \)
Lemma 2.3. Let $i = (i_1, i_2, \ldots, i_\ell) \in R(w_0)$ and set $j := (i_2, \ldots, i_\ell, i_1^*)$. Let $b \in B(\infty)$ and $t \in \mathbb{Z}_{\geq 0}$.

(i) If we write $L_{i,1-1}(b) = (m_\ell, m_{\ell-1}, \ldots, m_2, m_1)$, then
\[ \varepsilon_{i_1}^*(b) = m_1, \quad L_{i,1-1} \left( \tilde{f}_{i_1}^* \left( \tilde{T}_{i_1}^* (b) \right) \right) = (t, m_\ell, m_{\ell-1}, \ldots, m_2). \]

(ii) If we write $L_{i,1}(b) = (n_1, n_2, \ldots, n_{\ell-1}, n_\ell)$, then
\[ \varepsilon_{i_1}(b) = n_1, \quad L_{j,1} \left( \tilde{f}_{i_1}^{*t} \left( \tilde{T}_{i_1}^* (b) \right) \right) = (n_2, \ldots, n_{\ell-1}, n_\ell, t). \]

Proof. (i) It follows from Definition 2.1 and the bijection $L_{i,\varepsilon}$ that
\[ \varepsilon_{i_1}^*(b) = m_1 \quad \text{and} \quad L_{i,1-1}(\varepsilon_{i}^{\text{max}} b) = (m_\ell, m_{\ell-1}, \ldots, m_2, 0). \]

Since $\tilde{T}_{i^*} \tilde{T}_{i^*} \cdots \tilde{T}_{i^*} \tilde{T}_{i^*} (b) = 1 \in B(\infty)$, Definition 2.1 says that
\[ L_{j,1-1} \left( \tilde{T}_{i_1}^* (b) \right) = (0, m_\ell, m_{\ell-1}, \ldots, m_2). \]

Since Lemma 2.2 tells us that $L_{j,1-1} \left( \tilde{T}_{i_1}^* (b) \right) = L_{j^*,1} \left( \tilde{T}_{i_1}^* (b) \right)$ and $j^* = (i_1, i_2^*, \ldots, i_2^*)$, we obtain
\[ L_{j,1-1} \left( \tilde{f}_{i_1}^{*t} \left( \tilde{T}_{i_1}^* (b) \right) \right) = (t, m_\ell, m_{\ell-1}, \ldots, m_2). \]

(ii) It can be proved by the same argument. \qed

3. Braid group action on $\hat{B}(\infty)$

We keep the notations given in the previous section. Let $\hat{B}(\infty)$ be the extended crystal of the crystal $B(\infty)$. In this section, we show that there exists a braid group action on $\hat{B}(\infty)$ arising from the Saito crystal reflections.

For $i \in I$ and $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$, we define
\[ R_i(b) = (b'_k)_{k \in \mathbb{Z}} \quad \text{and} \quad R_i^*(b) = (b''_k)_{k \in \mathbb{Z}} \]
by
\[ b'_k := \tilde{f}_{i}^{*c_1(b_{k-1})} \left( \tilde{T}_{i} (b_k) \right) \quad \text{and} \quad b''_k := \tilde{f}_{i}^{*c_{k+1}} \left( \tilde{T}_{i}^* (b_k) \right). \]
for any $k \in \mathbb{Z}$ respectively. Thus we have the maps
\[ R_i : \hat{B}(\infty) \rightarrow \hat{B}(\infty) \quad \text{and} \quad R_i^* : \hat{B}(\infty) \rightarrow \hat{B}(\infty). \]
For any $j = (j_1, \ldots, j_t) \in I^t$, we simply write
\begin{align*}
R_j &:= R_{j_1} R_{j_2} \cdots R_{j_t} \quad \text{and} \quad R_j^* := R_{j_1}^* R_{j_2}^* \cdots R_{j_t}^*.
\end{align*}

**Proposition 3.1.** Let $i = (i_1, i_2, \ldots, i_\ell) \in R(w_0)$ and set $j := (i_2, i_3, \ldots, i_\ell, i_1^*)$. Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$ and write
\begin{align*}
L_{i,1}(b_k) &= (n_{k,1}, n_{k,2}, \ldots, n_{k,\ell-1}, n_{k,\ell}), \\
L_{i,-1}(b_k) &= (m_{k,1}, m_{k,2}, \ldots, m_{k,\ell-1}, m_{k,\ell}),
\end{align*}
for any $k \in \mathbb{Z}$.

(i) If we write $R_{i_1}(b) = (b'_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$, then we have
\[ L_{j,1}(b'_k) = (n_{k,2}, n_{k,3}, \ldots, n_{k,\ell-1}, n_{k,\ell}) \quad \text{for any } k \in \mathbb{Z}. \]

(ii) If we write $R_{i_1}^*(b) = (b''_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$, then we have
\[ L_{j,-1}(b''_k) = (m_{k+1,\ell}, m_{k,1}, m_{k,2}, \ldots, m_{k,\ell-1}) \quad \text{for any } k \in \mathbb{Z}. \]

**Proof.** By Lemma 2.3 (ii), we have $\varepsilon_{i_1}(b_{k-1}) = n_{k-1,1}$ and
\[ L_{j,1}(b'_k) = L_{j,1} \left( f^* \varepsilon_{i_1}(b_{k-1}) \left( \tilde{T}_{i_1}(b_k) \right) \right) = (n_{k,2}, n_{k,3}, \ldots, n_{k,\ell}, n_{k,1}, n_{k,\ell-1}), \]
which gives (i).

By the same argument, one can prove (ii). \qed

**Lemma 3.2.** Let $i \in I$.

(i) $R_i$ and $R_i^*$ are bijective.

(ii) $R_i$ and $R_i^*$ are inverse to each other.

(iii) $R_i \circ D = D \circ R_i$ and $R_i^* \circ D = D \circ R_i^*$.

(iv) For any $b \in \hat{B}(\infty)$, we have
\[ \hat{\text{wt}}(R_i(b)) = s_i(\hat{\text{wt}}(b)) \quad \text{and} \quad \hat{\text{wt}}(R_i^*(b)) = s_i(\hat{\text{wt}}(b)). \]
Proof. We first show that $R_i^* \circ R_i = \text{id}$. We choose $i = (i_1, i_2, \ldots, i_\ell) \in R(w_0)$ with $i = i_1$, and set

$$i^\vee := (i_{\ell}, i_{\ell-1}, \ldots, i_1), \quad j := (i_2, i_3, \ldots, i_\ell, i_1), \quad j^\vee := (i_1, i_\ell, \ldots, i_3, i_2).$$

Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$ and write

$$L_{i,1}(b_k) = (n_{k,1}, n_{k,2}, \ldots, n_{k,\ell-1}, n_{k,\ell})$$

for any $k \in \mathbb{Z}$. We write

$$R_i(b) = (b'_k)_{k \in \mathbb{Z}} \quad \text{and} \quad R_i^* \circ R_i(b) = (b''_k)_{k \in \mathbb{Z}}.$$  

By Proposition 3.1 (i), we have

$$L_{j,1}(b'_k) = (n_{k,2}, n_{k,3}, \ldots, n_{k,\ell}, n_{k,-1,1}) \quad \text{for any } k \in \mathbb{Z}.$$

Since $L_{j^\vee,1}(b'_k) = L_{j,1}(b'_k)$ by Lemma 2.2, Proposition 3.1 (ii) tells us that

$$L_{j^\vee,1}(b''_k) = (n_{k,1}, n_{k,2}, n_{k,3}, \ldots, n_{k,\ell}) \quad \text{for any } k \in \mathbb{Z}.$$

Therefore, Lemma 2.2 says that $b_k = b''_k$ for any $k \in \mathbb{Z}$, i.e., $R_i^* \circ R_i(b) = b$.

In the same manner, one can prove that $R_i \circ R_i^* = \text{id}$. Thus we have (i) and (ii).

(iii) It follows from the definitions of $R_i$ and $R_i^*$.

(iv) Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$ and write $R_i(b) = (b'_k)_{k \in \mathbb{Z}}$. For any $k \in \mathbb{Z}$, we have

$$\text{wt}_k(R_i(b)) = (-1)^k \text{wt}(b'_k) = (-1)^k \text{wt} \left( \tilde{f}_i^{\varepsilon_i(b_{k-1}} \left( \tilde{T}_i(b_k) \right) \right)$$

$$= (-1)^k \text{wt} \left( \tilde{f}_i^{\varepsilon_i(b_{k-1}} \left( T_i \left( \varepsilon_i(b_k) \right) \right) \right)$$

$$= (-1)^k (s_i(\text{wt}(b_k)) + \varepsilon_i(b_k) - \varepsilon_i(b_{k-1}) \alpha_i)$$

$$= s_i(\text{wt}(b_k)) + \left( (-1)^{k-1} \varepsilon_i(b_{k-1}) - (1)^k \varepsilon_i(b_k) \right) \alpha_i,$$

which tells us that

$$\hat{\text{wt}}(R_i(b)) = \sum_{k \in \mathbb{Z}} \text{wt}_k(R_i(b)) = \sum_{k \in \mathbb{Z}} s_i(\text{wt}_k(b)) = s_i(\hat{\text{wt}}(b)).$$

The case for $R_i^*$ can be proved in the same manner. \qed
Remark 3.3. Let $\ast$ be the involution on $\hat{B}(\infty)$ defined in [22, Section 4], which is a counterpart of the involution $\ast$ on $B(\infty)$. Using the fact that $T_i^* = \ast \circ T_i \circ \ast$, one can prove that $R_i^* = \ast \circ R_i \circ \ast$.

Recall that $A = (a_{i,j})_{i,j \in I}$ is a Cartan matrix of finite type. For $i, j \in I$ with $i \neq j$, we set

$$m(i, j) := \begin{cases} 2 & \text{if } a_{i,j}a_{j,i} = 0, \\ 3 & \text{if } a_{i,j}a_{j,i} = 1, \\ 4 & \text{if } a_{i,j}a_{j,i} = 2, \\ 6 & \text{if } a_{i,j}a_{j,i} = 3. \end{cases}$$

(3.2)

Note that $m(i, j) = m(j, i)$ for any $i, j \in I$ with $i \neq j$. We denote by $B_\mathcal{A}$ the generalized braid group (or Artin-Tits group) defined by the generators $r_i$ ($i \in I$) and the following defining relations:

$$r_i r_j r_i r_j \cdots \overset{m(i,j) \text{ factors}}{=} r_j r_i r_j r_i \cdots \quad \text{for } i, j \in I \text{ with } i \neq j.$$ 

We set $B_\mathcal{A}^+$ to be the submonoid of $B_\mathcal{A}$ generated by $r_i$ ($i \in I$). We call $B_\mathcal{A}$ the braid group associated with $A$. We simply write $B$ instead of $B_\mathcal{A}$ if no confusion arises.

Theorem 3.4. The bijections $R_i$ and $R_i^*$ satisfy the braid group relations for $B$, i.e., for $i, j \in I$ with $i \neq j$,

$$R_i R_j R_i R_j \cdots \overset{m(i,j) \text{ factors}}{=} R_j R_i R_j R_i \cdots \quad \text{and} \quad R_i^* R_j^* R_i^* R_j^* \cdots \overset{m(i,j) \text{ factors}}{=} R_j^* R_i^* R_j^* R_i^* \cdots.$$

Proof. Let $m(i, j) := (i, j, i, j, \ldots) \in I^{m(i,j)}$ such that $m(i, j) \ast k \in R(w_0)$, where $A \ast B$ is the concatenation of $A$ and $B$. Note that $m(j, i) \ast k \in R(w_0)$. By (1.2), the sequences $k \ast m(j^*, i^*)$ and $k \ast m(i^*, j^*)$ are also contained in $R(w_0)$. We set

$$i_1 := m(i, j) \ast k, \quad i_2 := m(j, i) \ast k, \quad j_1 := k \ast m(i^*, j^*), \quad j_2 := k \ast m(j^*, i^*).$$
Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$, and for any $k \in \mathbb{Z}$ we write
\begin{align*}
\mathcal{L}_{i_1,1}(b_k) &= (x_{k,1}, x_{k,2}, \ldots, x_{k,\ell-1}, x_{k,\ell}), \\
\mathcal{L}_{i_2,1}(b_k) &= (y_{k,1}, y_{k,2}, \ldots, y_{k,\ell-1}, y_{k,\ell}).
\end{align*}

Let $m = m(i,j)$. By the construction, we have
\begin{equation}
(3.3) \quad x_{k,t} = y_{k,t} \quad \text{for any } k \in \mathbb{Z} \text{ and } t \in \{m + 1, m + 2, \ldots, \ell\}.
\end{equation}

Moreover, since the transition map $T_{i,j}$ between two Lusztig’s data was computed in [2, Section 3] (see also [5, Section 2.3]), the first $m$ components of $(x_{k,1}, x_{k,2}, \ldots, x_{k,\ell})$ and $(y_{k,1}, y_{k,2}, \ldots, y_{k,\ell})$ satisfy
\begin{equation}
(3.4) \quad T_{i,j}(x_{k,1}, x_{k,2}, \ldots, x_{k,m}) = (y_{k,1}, y_{k,2}, \ldots, y_{k,m}) \quad \text{for any } k \in \mathbb{Z}.
\end{equation}

We write $R_{m(i,j)}(b) = (b'_{k})_{k \in \mathbb{Z}}$ and $R_{m(j,i)}(b) = (b''_{k})_{k \in \mathbb{Z}}$. By Proposition 3.1, we have
\begin{align*}
\mathcal{L}_{j_1,1}(b'_k) &= (x_{k,m+1}, x_{k,m+2}, \ldots, x_{k,\ell}, x_{k-1,1}, \ldots, x_{k-1,m}), \\
\mathcal{L}_{j_2,1}(b''_k) &= (y_{k,m+1}, y_{k,m+2}, \ldots, y_{k,\ell}, y_{k-1,1}, \ldots, y_{k-1,m})
\end{align*}
for any $k \in \mathbb{Z}$. Since $a_{i,j} = a_{j,i}^\ast$, the transition map $T_{i,j}$ coincides with $T_{j,i}^\ast$. By (3.3) and (3.4), the transition map $T_{j,i}^\ast$ sends $\mathcal{L}_{j_1,1}(b'_k)$ to $\mathcal{L}_{j_2,1}(b''_k)$. We thus conclude that $b'_{k} = b''_{k}$ for any $k \in \mathbb{Z}$, i.e.,
\begin{equation}
R_{m(i,j)}(b) = R_{m(j,i)}(b).
\end{equation}

The relation for $R_{i}^\ast$ follows from Lemma 3.2. \qed

Thanks to Theorem 3.4, the braid group $\mathcal{B}$ acts on the extended crystal $\hat{B}(\infty)$ as follows:
\begin{equation}
(3.5) \quad r_i \cdot b := R_i(b) \quad \text{and} \quad r_i^{-1} \cdot b := R_i^\ast(b) \quad \text{for } i \in I \text{ and } b \in \hat{B}(\infty).
\end{equation}

For any $w = r_{i_1}^{\epsilon_{i_1}}r_{i_2}^{\epsilon_{i_2}} \cdots r_{i_t}^{\epsilon_{i_t}} \in \mathcal{B}$ with $\epsilon_{k} \in \{-1, 1\}$, we define
\begin{equation}
R_{w} := R_{1}R_{2} \cdots R_{t},
\end{equation}
where $R_{k} := \begin{cases} R_{k} & \text{if } \epsilon_{k} = 1, \\
R_{k}^\ast & \text{if } \epsilon_{k} = -1. \end{cases}$
Remark 3.5. By Lemma 3.2 (iv), the set $\hat{B}(\infty)_0$ of weight 0 is invariant under the action of $\mathcal{B}$. Thus the action of $\mathcal{B}$ on $\hat{B}(\infty)$ is not transitive. In the case where $A$ is of type $A_1$, i.e. $g = \mathfrak{sl}_2$, it is easy to prove that $\hat{B}(\infty)$ is faithful as a $\mathcal{B}$-set. It is conjectured that $\hat{B}(\infty)$ is faithful as a $\mathcal{B}$-set for any Cartan matrix $A$.

Let us recall the involution $\zeta$ defined in (1.4). Since the involution $\zeta$ gives an involution on $B(\infty)$, we have the induced involution on the extended crystal $\hat{B}(\infty)$, i.e.,

$$\zeta (b) := (\zeta(b_k))_{k \in \mathbb{Z}} \quad \text{for any } b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty).$$

Theorem 3.6.

(i) For any $i \in I$, we have $R_i \circ \zeta = \zeta \circ R_{\zeta(i)}$ and $R_i^* \circ \zeta = \zeta \circ R_{\zeta(i)}^*$.

(ii) For any $i \in R(w_0)$, we have

$$R_i = D \circ \zeta \quad \text{and} \quad R_i^* = D^{-1} \circ \zeta.$$

In particular, unless the Cartan matrix $A$ is of type $A_n$ ($n \in \mathbb{Z}_{>1}$), $D_n$ ($n$ is odd) or $E_6$, we have

$$R_i = D \quad \text{and} \quad R_i^* = D^{-1}.$$

Proof. (i) Since

$$f_i (\zeta(b)) = \zeta \left( f_{\zeta(i)}(b) \right) \quad \text{and} \quad e_i (\zeta(b)) = \zeta \left( e_{\zeta(i)}(b) \right)$$

for any $i \in I$ and $b \in B(\infty)$, we have

$$T_i (\zeta(b')) = \zeta \left( T_{\zeta(i)}(b') \right) \quad \text{and} \quad T_i^* (\zeta(b'')) = \zeta \left( T_{\zeta(i)}^*(b'') \right)$$

for any $b' \in \zeta(i)B(\infty)$ and $b'' \in B(\infty)\zeta(i)$. Thus (i) follows from the definitions of $R_i$ and $R_i^*$.

(ii) Let $i = (i_1, i_2, \ldots, i_\ell) \in R(w_0)$ and set

$$j := (i_\ell, i_{\ell-1}, \ldots, i_1), \quad j^* := (i^*_\ell, i^*_{\ell-1}, \ldots, i^*_1).$$

By (3.6) and (3.7), we have

$$f_{j^*,i}(\beta_k^*) = \zeta(f_{j,i}(\beta_k)).$$
where $\beta_k^\ast := s_{i_k} \cdots s_{i_{k+2}}(a_{i_{k+1}}^\ast)$ and $\beta_k := s_{i_k} \cdots s_{i_{k+2}}(a_{i_{k+1}})$. Thus, for any $b \in B(\infty)$, if we write $L_{j,\ell}(b) = (n_1, n_2, \ldots, n_{\ell-1}, n_{\ell})$, then we have

$$L_{j,\ell}(\zeta(b)) = (n_1, n_2, \ldots, n_{\ell-1}, n_{\ell}).$$

(3.8)

Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$, and write $L_{j,1}(b_k) = (n_{k,1}, n_{k,2}, \ldots, n_{k,\ell-1}, n_{k,\ell})$ for $k \in \mathbb{Z}$. Let $R_i(b) = (b'_k)_{k \in \mathbb{Z}}$. By Proposition 3.1 and (3.8), we have

$$L_{j,1}(b'_k) = (n_{k-1,1}, n_{k-1,2}, \ldots, n_{k-1,\ell-1}, n_{k-1,\ell}) = L_{j,1}(\zeta(b_{k-1})),
$$

which implies that $R_i = D \circ \zeta$. In the same manner, one can prove $R_i^\ast = D^{-1} \circ \zeta$.

The last identities follow from the fact that $\zeta$ is the identity unless $A$ is of type $A_n$ ($n \in \mathbb{Z}_{>1}$), $D_n$ ($n$ is odd) or $E_6$. □

**Example 3.7.** We keep the notations given in Example 1.2 and Example 1.4.

Let $n = 2$. In this case, $R(w_0) = \{(1, 2, 1), (2, 1, 2)\}$ and the segments of $\text{MS}_2$ are $[1]$, $[12]$, and $[2]$.

(i) Let $\hat{m} = (\ldots, \emptyset, \emptyset, [2], \emptyset, \ldots) \in \hat{\text{MS}}_2$, where the underlined element is located at the 0-position. Note that $\tilde{F}_{2,0}(1) = \hat{m}$. By direct computations, we have

$$R_1(\hat{m}) = (\ldots, \emptyset, \emptyset, [12], \emptyset, \emptyset, \ldots),$$

$$R_2R_1(\hat{m}) = (\ldots, \emptyset, \emptyset, [1], \emptyset, \emptyset, \ldots),$$

$$R_1R_2R_1(\hat{m}) = (\ldots, \emptyset, [1], \emptyset, \emptyset, \emptyset, \ldots),$$

and

$$R_2(\hat{m}) = (\ldots, \emptyset, [2], \emptyset, \emptyset, \emptyset, \ldots),$$

$$R_1R_2(\hat{m}) = (\ldots, \emptyset, [12], \emptyset, \emptyset, \emptyset, \ldots),$$

$$R_2R_1R_2(\hat{m}) = (\ldots, \emptyset, [1], \emptyset, \emptyset, \emptyset, \ldots).$$

Thus we have

$$R_1R_2R_1(\hat{m}) = R_2R_1R_2(\hat{m}) = D \circ \zeta(\hat{m}).$$
(ii) Let \( \hat{m} = (\ldots, \emptyset, \emptyset, m_1, m_0, m_{-1}, \emptyset, \emptyset, \ldots) \in \hat{\mathcal{MS}}_2 \), where \( m_1 = 2[2] + 3[12] + 4[1], m_0 = [2] + [12] + 2[1] \) and \( m_{-1} = 3[2] + 2[12] + [1] \). Then we have
\[
\varepsilon_1(m_1) = 5, \quad \varepsilon_1(m_0) = 2, \quad \varepsilon_1(m_{-1}) = 2,
\]
and
\[
\varepsilon_1^{\text{max}}(m_1) = 5[2] + 2[1], \quad \varepsilon_1^{\text{max}}(m_0) = 2[2] + [1], \quad \varepsilon_1^{\text{max}}(m_{-1}) = 5[2] + [1].
\]
Thus, we have
\[
\tilde{T}_1(m_1) = 2[2] + 3[12], \quad \tilde{T}_1(m_0) = [2] + [12], \quad \tilde{T}_1(m_{-1}) = [2] + 4[12],
\]
which implies that \( \mathcal{R}_1(\hat{m}) = (\ldots, \emptyset, m'_2, m'_1, m'_0, m'_{-1}, \emptyset, \emptyset, \ldots) \), where
\[
\begin{align*}
m'_2 &= 5[1], \\
m'_1 &= 2[2] + 3[12] + 2[1], \\
m'_0 &= [2] + [12] + 2[1], \\
m'_{-1} &= [2] + 4[12].
\end{align*}
\]

4. Similarity

In this section, we apply the result of \([16]\) to the extended crystals.

Let \( J \) be a finite set and let \( \vartheta : I \rightarrow J \) be a surjective map. We take positive integers \( m_i \) for each \( i \in I \), and set
\[
\tilde{\alpha}_j := \sum_{i \in \vartheta^{-1}(j)} m_i \alpha_i,
\]
where \( \vartheta^{-1}(j) = \{ i \in I \mid \vartheta(i) = j \} \). We define \( \tilde{\mathcal{P}} \) to be the subset of \( \mathcal{P} \) consisting of \( \lambda \in \mathcal{P} \) such that, for any \( j \in J \), the value \( \langle \tilde{h}_j, \lambda \rangle \) is an integer and does not depend on the choice of \( i \in \vartheta^{-1}(j) \). For \( i \in I \), we denote by \( \tilde{h}_i \in \tilde{\mathcal{P}}^\vee := \text{Hom}_Z(\tilde{\mathcal{P}}, \mathbb{Z}) \) the element satisfying
\[
\langle \tilde{h}_j, \lambda \rangle = \langle \tilde{h}_i, \lambda \rangle \quad \text{for} \quad i \in \vartheta^{-1}(j) \quad \text{and} \quad \lambda \in \tilde{\mathcal{P}}.
\]
We assume that
\[
\begin{align*}
(\text{a}) & \quad \langle \tilde{h}_i, \alpha_{i'} \rangle = 0 \quad \text{for} \quad i, i' \in I \quad \text{such that} \quad \vartheta(i) = \vartheta(i') \quad \text{and} \quad i \neq i', \\
(\text{b}) & \quad \tilde{\alpha}_j \quad \text{is contained in} \quad \tilde{\mathcal{P}} \quad \text{for any} \quad j \in J.
\end{align*}
\]
We define the matrix $A = (c_{i,j})_{i,j\in J}$ by

$$c_{i,j} := \langle \tilde{h}_i, \tilde{\alpha}_j \rangle$$

for any $i, j \in J$.

Note that $A$ is a generalized Cartan matrix. Let $B(\infty)$ and $B_\vartheta(\infty)$ be the crystals of the negative halves of the quantum groups associated with $A$ and $A_\vartheta$ respectively. By [16, Section 5] (see [30, Proposition 3.2] and [31, Theorem 2.3.1] for the $*$-operator), we have the following theorem.

**Theorem 4.1** ([16, Section 5]). There exists a unique injective map $\Upsilon_\vartheta : B_\vartheta(\infty) \to B(\infty)$ such that $\Upsilon_\vartheta(1) = 1$ and, for any $j \in J$,

$$\Upsilon_\vartheta(\tilde{e}_j(b)) = \left( \prod_{i \in \vartheta^{-1}(j)} \tilde{e}_i^{m_i} \right) \Upsilon_\vartheta(b), \quad \Upsilon_\vartheta(\tilde{f}_j(b)) = \left( \prod_{i \in \vartheta^{-1}(j)} \tilde{f}_i^{m_i} \right) \Upsilon_\vartheta(b),$$

$$\Upsilon_\vartheta(\tilde{e}_j^*(b)) = \left( \prod_{i \in \vartheta^{-1}(j)} e_i^{*m_i} \right) \Upsilon_\vartheta(b), \quad \Upsilon_\vartheta(\tilde{f}_j^*(b)) = \left( \prod_{i \in \vartheta^{-1}(j)} f_i^{*m_i} \right) \Upsilon_\vartheta(b).$$

The above theorem can be easily extended to the extended crystals.

**Corollary 4.2.** There exists a unique injective map $\hat{\Upsilon}_\vartheta : \hat{B}_\vartheta(\infty) \to \hat{B}(\infty)$ such that $\hat{\Upsilon}_\vartheta(1) = 1$ and, for any $(j, k) \in \hat{J}$,

$$\hat{\Upsilon}_\vartheta(\tilde{E}_{j,k}(b)) = \left( \prod_{i \in \vartheta^{-1}(j)} E_{i,k}^{m_i} \right) \hat{\Upsilon}_\vartheta(b), \quad \hat{\Upsilon}_\vartheta(\tilde{F}_{j,k}(b)) = \left( \prod_{i \in \vartheta^{-1}(j)} F_{i,k}^{m_i} \right) \hat{\Upsilon}_\vartheta(b).$$

An automorphism $\sigma$ of a Dynkin diagram gives a desired example for $\vartheta$ (see [16, Section 5] and [28, Section 14.4]). We illustrate such examples as follows. We take $m_i = 1$ for
any \( i \in I \).

\[
\begin{align*}
A_{2n-1} : & & E_6 : \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad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Let $\sigma$ be an automorphism given in (4.1). Since $\sigma$ acts on $B(\infty)$, we obtain the induced automorphism of $\hat{B}(\infty)$, i.e.,

$$\sigma(b) := (\sigma(b_k))_{k \in \mathbb{Z}} \quad \text{for any } b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty).$$

Let $\mathcal{B} = \langle r_i \mid i \in I \rangle$ be the Braid group associated with $A$ and let $\mathcal{B}_\sigma = \langle r'_j \mid j \in J \rangle$ be the Braid group associated with $A_\sigma$. Then $\mathcal{B}_\sigma$ is embedded in $\mathcal{B}$ via the injection

$$\iota_\sigma : \mathcal{B}_\sigma \hookrightarrow \mathcal{B}$$

defined by $\iota_\sigma(r'_j) = \prod_{i \in \text{orb}_\sigma(j)} r_i$ for $j \in J$ (see [4]).

One can prove the following proposition by using Lusztig’s result ([28, Section 14.4]) and Corollary 4.2.

**Proposition 4.3.** Let $\sigma$ be an automorphism given in (4.1).

(i) Let $\hat{B}(\infty)^\sigma$ be the set of fixed points of $\hat{B}(\infty)$ under the action $\sigma$. For any $(j, k) \in \hat{J}$, we define

$$\tilde{E}_{j,k}^\sigma := \prod_{i \in \text{orb}_\sigma(j)} \tilde{E}_{i,k} \quad \text{and} \quad \tilde{F}_{j,k}^\sigma := \prod_{i \in \text{orb}_\sigma(j)} \tilde{F}_{i,k}. $$

Then the map $\hat{\Upsilon}_\sigma$ given in Corollary 4.2 is compatible with the extended crystal operators, i.e.,

$$\hat{\Upsilon}_\sigma \left( \tilde{E}_{j,k}(b) \right) = \tilde{E}_{j,k}^\sigma \left( \hat{\Upsilon}_\sigma(b) \right) \quad \text{and} \quad \hat{\Upsilon}_\sigma \left( \tilde{F}_{j,k}(b) \right) = \tilde{F}_{j,k}^\sigma \left( \hat{\Upsilon}_\sigma(b) \right)$$

for any $(j, k) \in \hat{J}$ and $b \in \hat{B}_\sigma(\infty)$, which provides an isomorphism of extended crystals

$$\hat{\Upsilon}_\sigma : \hat{B}_\sigma(\infty) \sim \hat{B}(\infty)^\sigma.$$

(ii) Let $\mathcal{B} = \langle r_i \mid i \in I \rangle$ be the Braid group associated with $A$ and let $\mathcal{B}_\sigma = \langle r'_j \mid j \in J \rangle$ be the Braid group associated with $A_\sigma$. For $j \in J$, we define

$$r_j^\sigma := \prod_{i \in \text{orb}_\sigma(j)} r_i \in \mathcal{B}. $$

Then the isomorphism $\hat{\Upsilon}_\sigma$ is compatible with the braid group actions, i.e.,

$$\hat{\Upsilon}_\sigma \left( r'_j(b) \right) = r_j^\sigma \left( \hat{\Upsilon}_\sigma(b) \right) \quad \text{for any } j \in J.$$

5. Hernandez-Leclerc categories

We keep the notation given in Section 1.3. Let $D := \{L_i\}_{i \in I_{\text{fin}}}$ be a complete duality datum of $\mathcal{C}_g^0$ and we write $A_{\text{fin}} = (a_{i,j}^\text{fin})_{i,j \in I_{\text{fin}}}$. It is announced in [21] that

(i) there exists an action of the braid group $B_{A_{\text{fin}}}$ associated with the Cartan matrix $A_{\text{fin}}$ on the quantum Grothendieck ring of the Hernandez-Leclerc category $\mathcal{C}_g^0$,

(ii) there exists a family of monoidal autofunctors $S_i$ on the localization $\mathcal{T}_N$ such that the autofunctors induce the braid group action in (i) at the Grothendieck ring level.

Note that $\mathcal{T}_N$ can be understood as a graded version of the Hernandez-Leclerc category $\mathcal{C}_g^0$ for affine type $A_{N-1}$. It is also conjectured in [21, Section 5] that, for an arbitrary quantum affine algebra $U'_q(\mathfrak{g})$, there exist monoidal autofunctors on the Hernandez-Leclerc category $\mathcal{C}_g^0$ with the same properties as the autofunctors $S_i$ on $\mathcal{T}_N$. In this section, we assume that this conjecture is true. Namely, we assume that the following conjecture holds.

**Conjecture 1.** There exist exact monoidal autofunctors $\{R_i\}_{i \in I_{\text{fin}}}$ on $\mathcal{C}_g^0$ satisfying the following properties: for any $i, j \in I_{\text{fin}},$

(a) $R_i$ sends simple modules to simple modules,

(b) $R_i(L_i) \simeq D(L_i),$

(c) $R_i \circ D \simeq D \circ R_i,$

(d) if $a_{i,j}^\text{fin} = 0$, then $R_i \circ R_j \simeq R_j \circ R_i,$

(e) if $a_{i,j}^\text{fin} = -1$, then $R_i \circ R_j \circ R_i \simeq R_j \circ R_i \circ R_j,$

(f) at the Grothendieck ring level, the following diagram commutes,

$$
\begin{array}{ccc}
K(iR_{\text{fin}} \text{-gmod}) & \stackrel{[F_T]}{\longrightarrow} & K(\mathcal{C}_g^0) \\
| \mathcal{R}_i \downarrow & \downarrow & \downarrow | \mathcal{R}_i \downarrow \\
K(iR_{\text{fin}} \text{-gmod}) & \stackrel{[F_T]}{\longrightarrow} & K(\mathcal{C}_g^0),
\end{array}
$$

where we use the following notations:
• $i^!R_{\text{fin}}$-mod (resp. $i^*R_{\text{fin}}$-mod) is the full subcategory of $R_{\text{fin}}$-mod consisting of graded modules $M$ with $E_1 M = 0$ (resp. $E_1^* M = 0$),

• $\mathcal{R}_i : i^!R_{\text{fin}}$-mod $\to i^*R_{\text{fin}}$-mod is the reflection functor due to S. Kato [23],

• $S_i(D)$ is the duality datum obtained from $D$ by applying the reflection $S_i$ introduced in [20, Section 5.3].

The above properties (a)-(f) of Conjecture 1 come from [21, Theorem 2.4, Theorem 3.1, Proposition 3.2 and Section 5]. Note that $T_i$ is the same as $T_{i-1}$ in [21, Theorem 2.4] and that $\mathcal{R}_i$ is compatible with the Saito crystal reflection $T_i$ (see [23, Theorem 3.6]). Since $T_i$, $R_i$ and $F_D$ send simple modules to simple modules, it follows from (f) that

$$F_D \circ \mathcal{R}_i(M) \simeq R_i \circ F_D(M)$$

for a simple module $M$ in $i^!R_{\text{fin}}$-mod.

In particular, if we write $F_D(M) \simeq L_D(b)$ for some $b \in \hat{B}_{\text{fin}}(\infty)$ (see (1.8) for the definition of $L_D$), then we have

$$(5.1) \quad L_D(T_i(b)) \simeq R_i(L_D(b)),$$

where $T_i$ is the Saito crystal reflection.

**Proposition 5.1.** We assume that Conjecture 1 is true. Then we have

$$\Phi_D(R_i(b)) = R_i(\Phi_D(b)) \quad \text{for any } b \in \hat{B}_{\text{fin}}(\infty) \text{ and } i \in I_{\text{fin}},$$

where $\Phi_D : \hat{B}_{\text{fin}}(\infty) \xrightarrow{\sim} \mathcal{B}_D(g)$ is the extended crystal isomorphism given in Section 1.3, and $R_i$ is the braid group action on $\hat{B}_{\text{fin}}(\infty)$ defined in Section 3.

**Proof.** Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_{\text{fin}}(\infty)$ and let $L := \Phi_D(b)$. By the definition of $\Phi_D$, we have

$$\Phi_D(b) = \text{hd} \left( \bigotimes_{k=-\infty}^{\infty} D^k L_k \right) = \text{hd}(\cdots \otimes D^2 L_2 \otimes D L_1 \otimes L_0 \otimes D^{-1} L_{-1} \otimes \cdots),$$

where $L_k := L_D(b_k) \in \mathcal{C}_D$ for $k \in \mathbb{Z}$ (see [22, Lemma 3.2]). For any $k \in \mathbb{Z}$, let

$$a_k := \varepsilon_i(b_k) \quad \text{and} \quad L'_k := L_D(\varepsilon^a_i(b_k)).$$

We then have $L_k \simeq L_i^{\otimes a_k} \bigtriangledown L'_k$ and, by (5.1),

$$(5.2) \quad \mathcal{R}_i(L'_k) \simeq L_D(T_i(\varepsilon^a_i(b_k))) = L_D(\tilde{T}_i(b_k)).$$
By [22, Lemma 2.6, Lemma 3.2 and Lemma 5.7, (5.2) and the properties of \( \mathcal{R}_i \), we have

\[
\mathcal{R}_i(L) = \mathcal{R}_i \left( \text{hd} \left( \bigotimes_{k=+\infty}^{-\infty} \mathcal{D}^k L_i \otimes \mathcal{D}^k L'_k \right) \right)
\]

\[
\simeq \text{hd} \left( \bigotimes_{k=+\infty}^{-\infty} \mathcal{D}^k (\mathcal{A}_i(L_i) \otimes \mathcal{A}_i(L'_k)) \right)
\]

\[
\simeq \text{hd} \left( \bigotimes_{k=+\infty}^{-\infty} \mathcal{D}^{k+1} (L_i \otimes \mathcal{A}_i(L'_k)) \otimes \mathcal{D}^k (\mathcal{L}_D(\tilde{T}_i(b_k))) \right)
\]

\[
\simeq \text{hd} \left( \bigotimes_{k=+\infty}^{-\infty} \mathcal{D}^k (\mathcal{L}_D(\tilde{T}_i(b_k))) \otimes \mathcal{D}^k (L_i \otimes \mathcal{A}_{k-1}) \right)
\]

\[
\simeq \text{hd} \left( \bigotimes_{k=+\infty}^{-\infty} \mathcal{D}^k (L_i \otimes \mathcal{A}_{k-1}) \nabla L_i \otimes \mathcal{A}_{k-1} \right)
\]

\[
\simeq \text{hd} \left( \bigotimes_{k=+\infty}^{-\infty} \mathcal{D}^k (\mathcal{L}_D(\tilde{T}_i(b_k))) \right)
\]

\[
\simeq \Phi_D(R_i(b)).
\]

\[\square\]

**Example 5.2.** We keep all notations given in Example 1.5 and Example 3.7. Recall the complete duality datum \( \mathcal{D} = \{L_1, L_2\} \), where \( L_1 := V(\varpi_1) \) and \( L_2 := V(\varpi_1(-q)^2) \). We assume that Conjecture 1 is true.

(i) We set

\[
i = (1, 2, 1, 2, 1, 2, \ldots),
\]

and write \( i = (i_k)_{k \in \mathbb{Z}_{>0}} \). Mimicking the construction to make PBW vectors in a quantum group, we define

\[
V_k := \mathcal{R}_{i_1} \mathcal{R}_{i_2} \cdots \mathcal{R}_{i_{k-1}}(L_{i_k}) \quad \text{for any } k \in \mathbb{Z}_{>0}.
\]
By Proposition 5.1, we have
\begin{equation}
P := \{V_k \mid k \in \mathbb{Z}_{>0}\} = \left\{ \Phi_D(R_i R_{i_2} \cdots R_{i_{k-1}}(\tilde{F}_{i_k,0}(1)) \mid k \in \mathbb{Z}_{>0}\right\}.
\end{equation}

Let \(\tilde{m}_2 = (\ldots, \emptyset, [2], \emptyset, \ldots)\), \(\tilde{m}_{12} = (\ldots, \emptyset, [12], \emptyset, \ldots)\), and \(\tilde{m}_1 = (\ldots, \emptyset, [1], \emptyset, \ldots)\). Note that
\[
\Phi_D(\tilde{m}_2) = V(\omega_1)_{(-q)^2}, \quad \Phi_D(\tilde{m}_{12}) = V(\omega_2)_{-q}, \quad \Phi_D(\tilde{m}_1) = V(\omega_1).
\]

By the same argument given in Example 3.7 (i), it follows from (5.3) that
\[
P = \{ \Phi_D(D^t(\tilde{m}_2)), \Phi_D(D^t(\tilde{m}_{12})), \Phi_D(D^t(\tilde{m}_1)) \mid t \in \mathbb{Z}_{>0}\}
= \{ V(\omega_1)_{(-q)^a}, V(\omega_2)_{-q}^b \mid a \in 2\mathbb{Z}_{>0}, b \in 2\mathbb{Z}_{>0} + 1 \}.
\]

We remark that the set \(P\) can be viewed as the set of fundamental modules contained in the Hernandez-Leclerc category \(\mathcal{C}_{\theta}^-\) introduced in [10]. From the viewpoint of the PBW theory given in [20], the set \(P\) is also understood as the set of affine cuspidal modules in \(\mathcal{C}_{\theta}^-\) corresponding to \(D\) and the reduced expression \(s_1 s_2 s_1\).

(ii) We recall the element \(\tilde{m}\) given in Example 3.7 (ii). Let \(\lambda := \gamma(\tilde{m}) \in \mathcal{R}_2\), where \(\gamma\) is given in (1.9). One can write \(\lambda = \lambda_1 + \lambda_0 + \lambda_{-1}\), where
\[
\begin{align*}
\lambda_1 &:= \gamma_1(\tilde{m}_1) = 2(2, 5) + 3(1, 4) + 4(2, 3), \\
\lambda_0 &:= \gamma_0(\tilde{m}_0) = (1, 2) + (2, 1) + 2(1, 0), \\
\lambda_{-1} &:= \gamma_{-1}(\tilde{m}_{-1}) = 3(2, -1) + 2(1, -2) + (2, -3).
\end{align*}
\]

By Proposition 5.1, we have
\[
\mathcal{R}_1(\lambda) \simeq \mathcal{R}_1(\Phi_D(\tilde{m})) \simeq \Phi_D(\mathcal{R}_1(\tilde{m})) \simeq V(\lambda'),
\]
where \(\lambda' = \gamma(\mathcal{R}_1(\tilde{m}))\). Example 3.7 (ii) says that \(\lambda' = \lambda'_2 + \lambda'_1 + \lambda'_0 + \lambda'_{-1}\), where
\[
\begin{align*}
\lambda'_2 &:= \gamma_1(\tilde{m}'_2) = 5(1, 6), \\
\lambda'_1 &:= \gamma_1(\tilde{m}'_1) = 2(2, 5) + 3(1, 4) + 2(2, 3), \\
\lambda'_0 &:= \gamma_0(\tilde{m}'_0) = (1, 2) + (2, 1) + 2(1, 0), \\
\lambda'_{-1} &:= \gamma_{-1}(\tilde{m}'_{-1}) = (2, -1) + 4(1, -2).
\end{align*}
\]
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