Sufficient conditions for convergence of multiple Fourier series with $J_k$-lacunary sequence of rectangular partial sums in terms of Weyl multipliers

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Abstract

We obtain sufficient conditions for convergence (almost everywhere) of multiple trigonometric Fourier series of functions $f$ in $L_2$ in terms of Weyl multipliers. We consider the case where rectangular partial sums of Fourier series $S_n(x; f)$ have indices $n = (n_1, \ldots, n_N) \in \mathbb{Z}^N$, $N \geq 3$, in which $k$ ($1 \leq k \leq N - 2$) components on the places $\{j_1, \ldots, j_k\} = J_k \subset \{1, \ldots, N\} = M$ are elements of (single) lacunary sequences (i.e., we consider the, so called, multiple Fourier series with $J_k$-lacunary sequence of partial sums). We prove that for any sample $J_k \subset M$ the Weyl multiplier for convergence of these series has the form $W(\nu) = \prod_{j=1}^{N-k} \log(|\nu_{\alpha_j}| + 2)$, where $\alpha_j \in M \setminus J_k$, $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}^N$. So, the "one-dimensional" Weyl multiplier $- \log(|\cdot| + 2)$ presents in $W(\nu)$ only on the places of "free" (nonlacunary) components of the vector $\nu$. Earlier, in the case where $N - 1$ components of the index $n$ are elements of lacunary sequences, convergence almost everywhere for multiple Fourier series was obtained in 1977 by M. Kojima in the classes $L_p$, $p > 1$, and by D. K. Sanadze, Sh. V. Kheladze in Orlitz class. Note, that presence of two or more "free" components in the index $n$ (as follows from the results by Ch. Fefferman (1971)) does not guarantee the convergence almost everywhere of $S_n(x; f)$ for $N \geq 3$ even in the class of continuous functions.

Keywords: multiple trigonometric Fourier series, convergence almost everywhere, lacunary sequence, Weyl multipliers.
1. Introduction

1. Consider the $N$-dimensional Euclidean space $\mathbb{R}^N$, whose elements will be denoted as $x = (x_1, \ldots, x_N)$, and set $(nx) = n_1x_1 + \cdots + n_Nx_N$. We introduce $\mathbb{R}^N_\sigma = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq \sigma, j = 1, \ldots, N\}$, $\sigma \in \mathbb{R}^1$, and the set $\mathbb{Z}^N \subset \mathbb{R}^N$ of all vectors with integer coordinates. Set $\mathbb{Z}^N_\sigma = \mathbb{R}^N_\sigma \cap \mathbb{Z}^N$.

Let a $2\pi$-periodic (in each argument) function $f \in L_1(\mathbb{T}^N)$, where $\mathbb{T}^N = \{x \in \mathbb{R}^N : -\pi \leq x_j < \pi, j = 1, \ldots, N\}$, be expanded in a multiple trigonometric Fourier series: $f(x) \sim \sum_{\nu \in \mathbb{Z}^N} c_{\nu} e^{i(\nu \cdot x)}$.

For any vector $n = (n_1, \ldots, n_N) \in \mathbb{Z}_0^N$ consider a rectangular partial sum of these series

$$S_n(x; f) = \sum_{|\nu_1| \leq n_1} \cdots \sum_{|\nu_N| \leq n_N} c_{\nu} e^{i(\nu \cdot x)}. \quad (1.1)$$

The main purpose of our investigation is to study the behavior on $\mathbb{T}^N$ of the partial sum (1.1) as $n \to \infty$ (i.e. $\min_{1 \leq j \leq N} n_j \to \infty$), depending on the restrictions imposed as on the function $f$, so as on the components $n_1, \ldots, n_N$ of the vector $n$ – the index of $S_n(x; f)$.

In 1971 P. Sjolin [14] proved that for any lacunary sequence $\{n_{1}(\lambda_{1})\}$, $n_{1}(\lambda_{1}) \in \mathbb{Z}_{1}^{1}, \lambda_{1} = 1, 2, \ldots$, and for any function $f \in L_p(\mathbb{T}^2)$, $p > 1$,

$$\lim_{\lambda_{1}, n_{2} \to \infty} S_{n_{1}(\lambda_{1}), n_{2}}(x; f) = f(x) \quad \text{almost everywhere (a.e.) on} \quad \mathbb{T}^2. \quad (1.2)$$

In 1977 M. Kojima [8] generalized P. Sjolin’s result by proving that, if a function $f \in L_p(\mathbb{T}^N)$, $p > 1$, $N \geq 2$, and $\{n_{j}(\lambda_{j})\}, n_{j}(\lambda_{j}) \in \mathbb{Z}_{1}^{1}, \lambda_{j} = 1, 2, \ldots, j = 1, \ldots, N - 1$, are lacunary sequences, then

$$\lim_{\lambda_{1}, \ldots, \lambda_{N-1}, n_{N} \to \infty} S_{n_{1}(\lambda_{1}), \ldots, n_{N-1}(\lambda_{N-1}), n_{N}}(x; f) = f(x) \quad \text{a.e. on} \quad \mathbb{T}^N. \quad (1.3)$$

---

1 A sequence $\{n^{(s)}\}, n^{(s)} \in \mathbb{Z}_{1}^{1}$, is called lacunary, if $n^{(1)} = 1$ and $\frac{n^{(s+1)}}{n^{(s)}} \geq q > 1$, $s = 1, 2, \ldots$.

2 In 1970 N. R. Tevzadze [9] obtained the following result: for any given two sequences of integers $\{n_{j}^{(l)}\}, j = 1, 2$, increasing to $\infty$, $n_{j}^{(l)} \in \mathbb{Z}_{1}^{1}, l = 1, 2, \ldots$, $S_{n_{1}^{(l)}, n_{2}^{(l)}}(x; f)$ converges to $f(x)$ a.e. on $\mathbb{T}^2$ for $f \in L_2(\mathbb{T}^2)$.

3 In the same 1977 for $f \in L(\log^{+} L)^{3N-2}(\mathbb{T}^N)$ analogous result was obtained by D. K. Sanadze, Sh. V. Kheladze [13].
However, as soon as we remain "free" two components of the vector \( n = (n_1, \ldots, n_N) \in \mathbb{Z}_0^N \) – the index of \( S_n(x; f) \) (in particular, in the case where they are not elements of any lacunary sequences), even the class of continuous functions \( C(\mathbb{T}^N), N \geq 3 \), does not remain the "class of convergence a.e." of the considered expansions; this can be easily shown using Ch. Fefferman’s function from [3] (see, e.g. [8, Theorem 2]). Nevertheless, some conditions can be imposed on the "nonlacunary" components of the vector \( n \) (in the sequence of indices of partial sums), such that even the classes \( L_p(\mathbb{T}^N), p > 1, N \geq 3 \), remain the classes of convergence a.e. (of the considered expansions), in the case where there are more than one nonlacunary components; moreover, for the certain subsets of \( L_p(\mathbb{T}^N) \) all nonlacunary components can be even "free".

For functions in the classes \( L_p(\mathbb{T}^N), p > 1, N \geq 3 \), I. L. Bloshanskii and D. A. Grafov [2] proved convergence a.e. on \( \mathbb{T}^N \) of the sequence of partial sums of multiple trigonometric Fourier series whose indices \( n \) contain \( k \) lacunary components, \( 1 \leq k \leq N - 2 \), while the rest \( N - k \) nonlacunary components \( n_j \) of the vector \( n \) satisfy restrictions: \( c_1 \leq n_j / n_m \leq c_2 \), where \( c_1, c_2 = const \) (so, along the nonlacunary components, the summation over an extending system of rectangles takes place).

A question naturally arises to find such classes of functions which guarantee convergence a.e. of the sequence of partial sums of multiple Fourier series with indices \( n \) whose \( k \) components are lacunary (\( 1 \leq k \leq N - 2 \)), and at the same time, \( N - k \) nonlacunary components of the vector \( n \) are either "more free" than in the paper [2], or "free at all".

2. In the present paper we give an answer to this question in terms of Weyl multipliers.

**Definition.** A sequence \( W(\nu), \nu \in \mathbb{Z}^N \), is called a Weyl multiplier for rectangular convergence of a multiple trigonometric Fourier series if, first, it satisfies the conditions:

1. \( W(\nu) > 0, \nu \in \mathbb{Z}^N \);
2. \( W(\nu_1, \ldots, \nu_N) = W(|\nu_1|, \ldots, |\nu_N|), \nu \in \mathbb{Z}^N \);
3. \( W(\nu_1, \ldots, \nu_{j-1}, \nu_j + 1, \nu_{j+1}, \ldots, \nu_N) \geq W(\nu_1, \ldots, \nu_{j-1}, \nu_j, \nu_{j+1}, \ldots, \nu_N) \),
\( j = 1, \ldots, N, \quad \nu \in \mathbb{Z}_0^N; \)

and, second, if the convergence of the series \( \sum_{\nu \in \mathbb{Z}^N} |c_\nu(f)|^2W(\nu) < +\infty \) implies that the Fourier series of the function \( f \in L_2(T^N) \) converge over rectangles a.e. on \( T^N \).

From the L. Carleson theorem [3] it follows that in the one-dimensional case the Weyl multiplier \( W(\nu) = 1, \nu \in \mathbb{Z}^1 \). For the \( N \)-multiple Fourier series summed over rectangles the Weyl multiplier is the sequence

\[
W(\nu) = \prod_{j=1}^{N} \log(|\nu_j| + 2), \quad \nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}^N, \quad N \geq 2.
\]

It is difficult to determine the authorship of this result. Since in the multidimensional case it follows from more general (thoroughly proved) results by F. Moricz [10] (1981), so usually F. Moricz is considered to be its author. For \( N = 2 \) it was obtained by S. Kaczmarz [7] (1930); however, proof of several estimates in [7] causes questions – the matter concerns the proofs of Lemma 1 on asymptotic of partial sums (p. 93) and of the Theorem (p. 95). Remarks concerning this see in [10], [1]. Proof of this result for \( N > 2 \) was made in 1977 by M. Kojima [8, Theorem 3], but the lemma on asymptotic of partial sums in his paper is given without the proof with the reference that it can be proved the same as in [7] for \( N = 2 \). Note that this result for multidimensional case was stated in 1973 by L. V. Zhizhiashvili [17, p. 90]; in 1977 J. Chen, N. Shieh [4] actually stated this result once more (without reference to [17]), remaining the basic estimates in their paper without proofs. For \( N = 1 \) all the results listed above are analogs of the classical theorem by A. N. Kolmogorov, G. A. Seliverstov, [9] and A. I. Plessner [12] (1925-1926).

In the case \( N = 2 \) P. Sjolin [14, Theorem 7.2] (1971) proved that the following sequence can be taken as the Weyl multiplier

\[
W(\nu_1, \nu_2) = \log^2\left( \min(|\nu_1|, |\nu_2|) + 2 \right), \quad (1.2)
\]

and E. M. Nikishin [11, Theorem 4] (1972) proved that (1.2) is the exact Weyl multiplier.
3. Let $N \geq 1$, $M = \{1, \ldots, N\}$ and $s \in M$. Denote: $J_s = \{j_1, \ldots, j_s\}, j_q < j_l$ for $q < l$, and (in the case $s < N$) $M \setminus J_s = \{\alpha_1, \ldots, \alpha_{N-s}\}, \alpha_q < \alpha_l$ for $q < l$, these are nonempty subsets of the set $M$. We also consider $J_0 = M \setminus J_N = \emptyset$.

Fix an arbitrary $k, 1 \leq k < N, N \geq 2$, and consider a sample $J_k \subset M$.

Define the vectors

$$\lambda = \lambda[J_k] = (\lambda_{j_1}, \ldots, \lambda_{j_k}) \in \mathbb{Z}_1^k, \quad j_s \in J_k, \quad s = 1, \ldots, k,$$

and

$$m = m[J_k] = (n_{\alpha_1}, \ldots, n_{\alpha_{N-k}}) \in \mathbb{Z}^{N-k}, \quad \alpha_s \in M \setminus J_k, \quad s = 1, \ldots, N - k.$$

We will denote by the symbol

$$n^{(\lambda, m)} = n^{(\lambda, m)}[J_k] = (n_1, \ldots, n_N) \in \mathbb{Z}_1^N$$

such $N$-dimensional vector, whose components $n_j$ with indices $j \in J_k$ are elements of some (single) lacunary sequences, i.e., for $j \in J_k: n_j = n_j^{(\lambda_j)} \in \mathbb{Z}_1^1$, $n_j^{(1)} = 1, \frac{n_j^{(\lambda_j+1)}}{n_j^{(\lambda_j)}} \geq q_j > 1, \lambda_j = 1, 2, \ldots, \text{and } n_j^{(\lambda_j)} \to \infty \text{ as } \lambda_j \to \infty$; we set

$$q = q(J_k) = (q_{j_1}, \ldots, q_{j_k}) \in \mathbb{R}^k, \quad j_s \in J_k, \quad s = 1, \ldots, k. \quad (1.3)$$

In its turn, the components of the vector $m[J_k]$ are free. Further in the paper a sequence of partial sums of the type $S_{n^{(\lambda, m)}[J_k]}(x; f)$ we will call a "$J_k$-lacunary" sequence of partial sums of multiple Fourier series.

Denote

$$W(m[J_k]) = \prod_{j=1}^{N-k} \log(|n_{\alpha_j}| + 2). \quad (1.4)$$

**Theorem 1.** Let $J_k$ be an arbitrary sample from $M, 1 \leq k \leq N - 2, \ N \geq 3$.

For any function $f \in L_2(\mathbb{T}^N)$

$$\left\| \sup_{\lambda_j > 0, j \in J_k, \ n_j > \alpha_j \in M \setminus J_k} \frac{|S_{n^{(\lambda, m)}[J_k]}(x; f)|}{\sqrt{W(m[J_k])}} \right\|_{L_2(\mathbb{T}^N)} \leq C\|f\|_{L_2(\mathbb{T}^N)}, \quad (1.5)$$

where the constant $C$ does not depend on the function $f$, $C = C(J_k, q)^4$, and the quantity $q$ is defined in (1.3).

\[\text{Further we will denote as } C \text{ the constants, which are, generally speaking, different.}\]
Remark 1. In the case $k = N - 1$ (i.e., one component is free and the rest $N - 1$ are lacunary), M. Kojima proved [8, Theorem 2] that for any function $f \in L_p(\mathbb{T}^N), p > 1, N \geq 2$, the following estimate is true:

$$\left\| \sup_{\lambda_j > 0, j \in J_{N-1}, n_j > 0, j \in M \setminus J_{N-1}} |S_{n^{(\lambda, \cdot)}[J_{N-1}]}(x; f)| \right\|_{L_p(\mathbb{T}^N)} \leq C \|f\|_{L_p(\mathbb{T}^N)}. \quad (1.6)$$

**Theorem 2.** Let $J_k$ be an arbitrary sample from $M, 1 \leq k \leq N - 2, N \geq 3$. If the Fourier coefficients $c_n(f), n \in \mathbb{Z}^N$, of the function $f \in L_2(\mathbb{T}^N)$ satisfy condition

$$\Sigma[f, J_k] = \sum_{n \in \mathbb{Z}^N} |c_n(f)|^2 W(m[J_k]) < +\infty, \quad (1.7)$$

then

$$\lim_{\lambda_j \to \infty, n_j \to \infty, j \in M \setminus J_k} S_{n^{(\lambda, \cdot)}[J_k]}(x; f) = f(x) \text{ almost everywhere on } \mathbb{T}^N. \quad (1.8)$$

Moreover, for any $\alpha > 0$ the inequality is true

$$\mu \left\{ x \in \mathbb{T}^N : \sup_{\lambda_j > 0, n_j > 0, j \in M \setminus J_k} |S_{n^{(\lambda, \cdot)}[J_k]}(x; f)| > \alpha \right\} \leq \frac{C}{\alpha^2} \cdot \Sigma[f, J_k], \quad (1.9)$$

where $\mu$ is the $N$-dimensional Lebesgue measure, and the constant $C$ does not depend on the function $f$.

Theorem 2 can be strengthened for $k = N - 2$.

**Theorem 3.** Let $J_{N-2}$ be an arbitrary sample from $M, N \geq 3$. If the Fourier coefficients $c_n(f)$ of the function $f \in L_2(\mathbb{T}^N)$ satisfy condition

$$\Sigma_0[f, J_{N-2}] = \sum_{n \in \mathbb{Z}^N} |c_n(f)|^2 \log^2 \left[ \min_{i, j \in M \setminus J_{N-2}} (|n_i|, |n_j|) + 2 \right] < +\infty, \quad (1.10)$$

then

$$\lim_{\lambda_j \to \infty, n_j \to \infty, j \in M \setminus J_{N-2}} S_{n^{(\lambda, \cdot)}[J_{N-2}]}(x; f) = f(x) \text{ almost everywhere on } \mathbb{T}^N; \quad (1.11)$$

Moreover,

$$\left\| \sup_{\lambda_j > 0, n_j > 0, j \in M \setminus J_{N-2}} |S_{n^{(\lambda, \cdot)}[J_{N-2}]}(x; f)| \right\|_{L_2(\mathbb{T}^N)}^2 \leq C \cdot \Sigma_0[f, J_{N-2}], \quad (1.12)$$

where the constant $C$ does not depend on the function $f$.\]
2. Proof of Theorem 1

Proof of the theorem is based on the ideas represented by us in [2]; furthermore, for simplicity of understanding of the proof of this theorem, we’ll use the structure and notations elaborated by us in the proofs of Lemma 1 and Theorem 1 in [2].

In order to prove the theorem it is necessary to prove the following lemma.

**Lemma 1.** Let \( J_1 = \{ r \} \), \( 1 \leq r \leq N \). Then for any function \( f \in L_2(\mathbb{T}^N) \), \( N \geq 3 \),

\[
\left\| \sup_{\lambda > 0, r \in J_1} \left| S_{n^{(\lambda,m)}[J_1]}(x; f) \right| \right\|_{L_2(\mathbb{T}^N)} \leq C \| f \|_{L_2(\mathbb{T}^N)},
\]

(2.1)

where the constant \( C \) does not depend on the function \( f \), \( C = C(J_1, q) \), and the quantity \( q \) is defined in (1.3).

**Proof of Lemma 1.**

Not to complicate the proof, let us consider \( r = 1 \).

We denote \( \tilde{x} = (x_2, x_3, \ldots, x_N) \in \mathbb{T}^{N-1} \), and consider

\[
\tilde{T}^{N-1} = \{ \tilde{x} \in \mathbb{T}^{N-1} : g(x_1) = f(x_1, \tilde{x}) \in L_2(\mathbb{T}^1) \};
\]

(2.2)

it is obvious,

\[
\mu_{N-1} \tilde{T}^{N-1} = \mu_{N-1} T^{N-1} = (2\pi)^{N-1},
\]

(2.3)

here \( \mu_{N-1} \) is the \((N - 1)\)-dimensional Lebesgue measure.

Fix an arbitrary point \( \tilde{x} \in \tilde{T}^{N-1} \) and expand the function \( g(x_1) \) in the (single) trigonometric Fourier series

\[
g(x_1) \sim \sum_{k \in \mathbb{Z}} c_k e^{ikx_1}.
\]

(2.4)

Consider the partial sums of this series \( S_{n^{(1)}}(x_1; g) \) with the indices \( m = n^{(\lambda_1)}_1 \in \mathbb{Z}_1 \), \( \lambda_1 = 1, 2, \ldots \), where \( \{ n^{(\lambda_1)}_1 \} \) is a lacunary sequence; set \( n^{(0)}_1 = 0 \) and define the difference

\[
\Delta_{\lambda_1}(x_1; g) = \begin{cases} S_0(x_1; g) & \text{for } \lambda_1 = 0, \\ S_{n^{(\lambda_1)}_1}(x_1; g) - S_{n^{(\lambda_1-1)}_1}(x_1; g) & \text{for } \lambda_1 = 1, 2, \ldots. \end{cases}
\]

*In the proof of this lemma, some ideas represented by P.Sjolin [1] and M.Kojima [2] are used.*
Let us split the series in (2.4) into two series:

\[ \sum_{\lambda_1 = 0}^{\infty} \Delta_{2\lambda_1 + 1}(x_1; g), \quad \sum_{\lambda_1 = 0}^{\infty} \Delta_{2\lambda_1}(x_1; g). \]  

(2.5)

From [18, Ch. 15, Theorem (4.11)] it follows that trigonometric series (2.5) are Fourier series of some functions \( g_1(x_1) = f_1(x_1, \tilde{x}) \) and \( g_2(x_1) = f_2(x_1, \tilde{x}) \), \( g_1, g_2 \in L_2(\mathbb{T}^1) \) (here we took account of notation (2.2)), and inequalities are true

\[ \|g_1\|_{L_2(\mathbb{T}^1)} \leq C\|g\|_{L_2(\mathbb{T}^1)}, \quad \|g_2\|_{L_2(\mathbb{T}^1)} \leq C\|g\|_{L_2(\mathbb{T}^1)}. \]  

(2.6)

In its turn, taking into account L. Carleson’s result [3] (for the one-dimensional trigonometric Fourier series), we have:

\[ g_1(x_1) = \sum_{\lambda_1 = 0}^{\infty} \Delta_{2\lambda_1 + 1}(x_1; g), \quad g_2(x_1) = \sum_{\lambda_1 = 0}^{\infty} \Delta_{2\lambda_1}(x_1; g) \quad \text{for a.e.} \ x_1 \in \mathbb{T}^1. \]

Hence, in view of the definition of the functions \( g_1, g_2 \) (as well as notation (2.2)) we obtain

\[ f(x_1, \tilde{x}) = g(x_1) = g_1(x_1) + g_2(x_1) = f_1(x_1, \tilde{x}) + f_2(x_1, \tilde{x}) \quad \text{for a.e.} \ x_1 \in \mathbb{T}^1. \]  

(2.7)

In its turn, taking into account that, according to the assumption of the lemma, \( f \in L_2(\mathbb{T}^N) \), in view of estimates (2.3), (2.6) and arbitrariness of the choice of \( \tilde{x} \in \mathbb{T}^{N-1} \), we obtain the following estimates

\[ \|f_j\|_{L_2(\mathbb{T}^N)} \leq C\|f\|_{L_2(\mathbb{T}^N)}, \quad j = 1, 2. \]  

(2.8)

Now, denoting for convenience

\[ b_m = \{W(m)\}^{-\frac{1}{2}}, \quad m = m[J_1], \]

we define the functions \( G_m(x_1, \tilde{x}), G_m^{(1)}(x_1, \tilde{x}) \) and \( G_m^{(2)}(x_1, \tilde{x}) \), as follows

\[ G_m(x_1, \tilde{x}) = S_m(\tilde{x}; f(x_1, \cdot)) \cdot b_m, \quad G_m^{(j)}(x_1, \tilde{x}) = S_m(\tilde{x}; f_j(x_1, \cdot)) \cdot b_m, \quad j = 1, 2. \]  

(2.9)

From equality (2.7) we get:

\[ S_m(x_1; f) b_m = S_{n_1^{(1)}}(x_1; G_m^{(1)}(\cdot, \tilde{x}))+ S_{n_1^{(2)}}(x_1; G_m^{(2)}(\cdot, \tilde{x})). \]  

(2.10)
Note, that in view of the definition of the functions \( f_j(x_1, \bar{x}) \), \( j = 1, 2 \), in (2.7), for any fixed \( \bar{x} \in \mathbb{T}^N \) the Fourier coefficients of the function \( f_1(x_1, \bar{x}) \) (over the variable \( x_1 \)) \( c_k(f_1) = 0 \) for \( n_1(2\lambda_1 + 1) < |k| \leq n_1(2\lambda_1 + 2) \), and the Fourier coefficients of the function \( f_2(x_1, \bar{x}) \) (over the variable \( x_1 \)) \( c_k(f_2) = 0 \) for \( n_1(2\lambda_1) < |k| \leq n_1(2\lambda_1 + 1) \). In its turn, taking account of the definition of the functions \( G^{(1)}_m(x_1, \bar{x}) \), \( j = 1, 2 \) (see (2.3)), the Fourier coefficients of the function \( G^{(1)}_m(x_1, \bar{x}) \) (over the variable \( x_1 \)) \( c_k(G^{(1)}_m) = 0 \) for \( n_1(2\lambda_1 + 1) < |k| \leq n_1(2\lambda_1 + 2) \), and the Fourier coefficients of the function \( G^{(2)}_m(x_1, \bar{x}) \) (over the variable \( x_1 \)) \( c_k(G^{(2)}_m) = 0 \) for \( n_1(2\lambda_1) < |k| \leq n_1(2\lambda_1 + 1) \).

Hence, both functions \( G^{(j)}_m(x_1, \bar{x}) \), \( j = 1, 2 \) (over the variable \( x_1 \)) satisfy the assumptions of Lemma (1.19) from [18, Ch. 13] (see also [6, Ch. VI, p. 73], [19, Ch. III, p. 79]; for the detailed formulation of this statement, appropriate for understanding of the proof, see [2, Theorem B]). So, in view of this lemma, the following estimates hold true:

\[
\sup_{\lambda_1 > 0} |S^{(j)}_{n_1(\lambda_1), m}(x_1; G^{(j)}_m(\cdot, \bar{x}))| \leq C \sup_{n_1 > 0} |\sigma_{n_1}(x_1; G^{(j)}_m(\cdot, \bar{x}))|, \quad j = 1, 2, \tag{2.11}
\]

where the constant \( C = C(q) \) does not depend on \( G^{(j)}_m, j = 1, 2 \), and \( \sigma_n(t; \varphi) \) are the Cezaro means

\[
\sigma_n(t; \varphi) = \frac{1}{n+1} \sum_{r=0}^{n} S_r(t; \varphi), \quad t \in \mathbb{T}^1. \tag{2.12}
\]

In its turn, for the Cezaro means (2.12) the estimate is true (see [19, Ch. 4, Theorem (7.8))):

\[
\left\| \sup_{n>0} |\sigma_n(t; \varphi)| \right\|_{L_p(\mathbb{T}^1)} \leq C \|\varphi\|_{L_p(\mathbb{T}^1)}, \quad \varphi \in L_p(\mathbb{T}^1), 1 < p < \infty, \quad \tag{2.13}
\]

where the constant \( C \) does not depend on the function \( \varphi \).

By (2.11) and (2.13) we have:

\[
\left\| \sup_{\lambda_1 > 0, n_1 > 0, j \in M \setminus J_1} |S^{(j)}_{n_1(\lambda_1), m}(x_1; G^{(j)}_m(\cdot, \bar{x}))| \right\|_{L_2(\mathbb{T}^N)} \leq C \left\| \sup_{n_1 > 0} |\sigma_{n_1}(x_1; \sup_{n_1 > 0, j \in M \setminus J_1} |S_m(\bar{x}; f_j(x_1, \cdot))| b_m|) \right\|_{L_2(\mathbb{T}^N)}
\]

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Thus, taking account of our assumptions, we prove estimate (2.1).

In view of the result by F. Moricz [10, Theorem 1] for \( \varphi \in L_2(\mathbb{T}^d), \ d \geq 2, \ n \in \mathbb{Z}_d^d \), the estimate is true:

\[
\left\| \sup_{n_j > 0, \ j \in M \setminus J_1} \left| S_n(x; f_j(x_1, \cdot)) \right| \cdot b_m \right\|_{L_2(\mathbb{T}^n)} \leq C_1 \| f \|_{L_2(\mathbb{T}^n)}, \quad j = 1, 2. \tag{2.14}
\]

From (2.14) and (2.15), considering (2.9), we get:

\[
\left\| \sup_{\lambda_1 > 0, n_j > 0, j \in M \setminus J_1} \left| S_{n_{\lambda_1}}(x; G_m^{(j)}(\cdot, \tilde{x})) \right| \right\|_{L_2(\mathbb{T}^n)} \leq C(J_1) \| f \|_{L_2(\mathbb{T}^n)}, \quad j = 1, 2. \tag{2.16}
\]

Further, from equality (2.10) and estimates (2.8) and (2.11) it follows:

\[
\left\| \sup_{\lambda_1 > 0, n_j > 0, j \in M \setminus J_1} \left| S_{n_{\lambda_1}}(x; f) \right| \cdot b_m \right\|_{L_2(\mathbb{T}^n)} \leq C(J_1, q) \| f \|_{L_2(\mathbb{T}^n)}. \tag{2.17}
\]

Thus, taking account of our assumptions, we prove estimate (2.1).

Lemma 1 is proved.

**Proof of Theorem 1.** The proof of estimate (1.3) will be conducted by the induction on \( N, N \geq 3 \).

The first step of induction, i.e. \( N = 3 \); in this case we must prove that for any \( J_1 = \{ r \}, 1 \leq r \leq 3 \), for any function \( f \in L_2(\mathbb{T}^3) \):

\[
\left\| \sup_{\lambda_1 > 0, n_j > 0, j \in M \setminus J_1} \left| S_{n_{\lambda_1}}(x; f) \right| \cdot b_m \right\|_{L_2(\mathbb{T}^3)} \leq C(J_1, q) \| f \|_{L_2(\mathbb{T}^3)}. \tag{2.18}
\]

As we see, the validity of (2.17) follows from the validity of Lemma 1, i.e., from estimate (2.14) for \( N = 3 \).

Further, suppose that (1.3) is true for some \( N = l, l \geq 3 \), i.e., for any \( J_k \) in \( M = \{ 1, \ldots, l \}, 1 \leq k \leq l - 2 \), for any function \( f \in L_2(\mathbb{T}^l) \),

\[
\left\| \sup_{\lambda_j > 0, j \in J_k} \left| S_{n_{\lambda_j}}(x; f) \right| \cdot b_m \right\|_{L_2(\mathbb{T}^l)} \leq C(J_k, q) \| f \|_{L_2(\mathbb{T}^l)}. \tag{2.18}
\]

Let us prove that estimate (1.3) is true for \( N = l + 1, l \geq 3 \), i.e., for any \( J_d \) in \( M = \{ 1, \ldots, l + 1 \}, 1 \leq d \leq l - 1 \), for any function \( f \in L_2(\mathbb{T}^{l+1}) \),

\[
\left\| \sup_{\lambda_j > 0, j \in J_d, n_j > 0, j \in M \setminus J_d} \left| S_{n_{\lambda_j}}(x; f) \right| \cdot b_m \right\|_{L_2(\mathbb{T}^{l+1})} \leq C(J_d, q) \| f \|_{L_2(\mathbb{T}^{l+1})}. \tag{2.19}
\]
If \( d = 1 \), then (2.19) again follows from the result of Lemma 1.

Consider now \( d \geq 2 \), and, to simplify the notation, let us assume that the sample \( J_d \) is of the form: \( J_d = \{1, 2, \ldots, d\} \). In this case,

\[
n^{(\lambda, m)}[J_d] = (n^{(\lambda)}_1, n^{(\lambda)}_2, \ldots, n^{(\lambda)}_d, n_{d+1}, \ldots, n_{t+1}), \quad m[J_d] = (n_{d+1}, \ldots, n_{t+1}).
\]

We denote

\[
\tilde{n}^{(\lambda, m)} = \tilde{n}^{(\lambda, m)}[J_d] = (n^{(\lambda)}_2, \ldots, n^{(\lambda)}_d, n_{d+1}, \ldots, n_{t+1}) \in \mathbb{Z}_0^l.
\]

Let the set \( \tilde{T}^l \) be defined analogously to (2.2), precisely:

\[
\tilde{T}^l = \{ \tilde{x} = (x_2, x_3, \ldots, x_{l+1}) \in T^l : g(x_1) = f(x_1, \tilde{x}) \in L_2(T^l) \}; \quad (2.20)
\]

it is obvious, \( \mu \tilde{T}^l = \mu T^l = (2\pi)^l \) (here \( \mu \) is the \( l \)-dimensional Lebesgue measure). Fixing an arbitrary point \( \tilde{x} \in \tilde{T}^l \), by the same argumentation as in Lemma 1 (see (2.5) – (2.7)), we define two functions \( g_1(x_1) = f_1(x_1, \tilde{x}) \) and \( g_2(x_1) = f_2(x_1, \tilde{x}) \), \( g_1, g_2 \in L_2(T^l) \),

\[
f(x_1, \tilde{x}) = g(x_1) = g_1(x_1) + g_2(x_1) = f_1(x_1, \tilde{x}) + f_2(x_1, \tilde{x}) \quad \text{for a.e.} \quad x_1 \in T^l,
\]

which satisfy (in account of Ch. 15, Theorem (4.11)) the estimates:

\[
\|f_j\|_{L_2(T^{l+1})} \leq C\|f\|_{L_2(T^{l+1})}, \quad j = 1, 2. \quad (2.22)
\]

Further, analogously to (2.9) we define the following functions:

\[
\begin{align*}
G^{(j)}_{n^{(\lambda, m)}}(x_1, \tilde{x}) &= S^{(j)}_{n^{(\lambda, m)}}(\tilde{x}; f(x_1, \cdot)) \cdot b_m, \quad G^{(j)}_{n^{(\lambda, m)}}(x_1, \tilde{x}) = \\
&= S^{(j)}_{n^{(\lambda, m)}}(\tilde{x}; f_j(x_1, \cdot)) \cdot b_m, \quad j = 1, 2, \quad m = m[J_d], \quad b_m = \{W(m[J_d])\}^{-\frac{1}{2}}.
\end{align*}
\]

From equality (2.21) we have:

\[
S^{(\lambda, m)}_{n^{(\lambda, m)}[J_d]}(x; f) \cdot b_m = S^{(\lambda)}_{n^{(\lambda)}_1}(x_1; G^{(j)}_{n^{(\lambda, m)}}(\cdot, \tilde{x}))
\]

\[
= S^{(\lambda)}_{n_1^{(\lambda)}}(x_1; G^{(1)}_{n^{(\lambda, m)}}(\cdot, \tilde{x})) + S^{(\lambda)}_{n_1^{(\lambda)}}(x_1; G^{(2)}_{n^{(\lambda, m)}}(\cdot, \tilde{x})). \quad (2.23)
\]
By the same argumentation as in the proof of (2.11), we obtain:

\[
\sup_{\lambda_1 > 0} |S_{n_1^{(\lambda_1)}}(x_1; G_{n_1}^{(j)}(\cdot, \bar{x}))| \leq C \sup_{n_1 > 0} |\sigma_{n_1}(x_1; G_{n_1}^{(j)}(\cdot, \bar{x}))|, \quad j = 1, 2.
\]

The same as in the proof of Lemma 1, using estimate (2.13), we get:

\[
\left\| \sup_{\lambda_1, \ldots, \lambda_d > 0, n_j > 0, j \in M \setminus J_d} |S_{n_1^{(\lambda_1)}}(x_1; G_{n_1}^{(j)}(\cdot, \bar{x}))| \right\|_{L_2(T^{l+1})} \leq C \left\| \sup_{n_1 > 0} |\sigma_{n_1}(x_1; \sup_{\lambda_2, \ldots, \lambda_d > 0, n_j > 0, j \in M \setminus J_d} |S_{n_2^{(\lambda_2)}}(x_2; \bar{x}; f_j(x_1, \cdot))| \cdot b_m| \right\|_{L_2(T^{l+1})}, \quad j = 1, 2.
\]

(2.24)

Note that \(\{n_j^{(\lambda_j)}\}, n_j^{(\lambda_j)} \in \mathbb{Z}_0^1, \lambda_j = 1, 2, \ldots, j = 2, \ldots, d,\) are lacunary sequences, and also \(1 \leq d - 1 \leq l - 2,\) and the functions \(f_j(x_1, \bar{x}) \in L_2(T^{l+1}), j = 1, 2.\) So, in order to estimate the right part of (2.24), we can use the inductive proposition, i.e., the majorant estimate (2.18), namely:

\[
\left\| \sup_{\lambda_2, \ldots, \lambda_d > 0, n_j > 0, j \in M \setminus J_d} |S_{n_2^{(\lambda_2)}}(x_2; \ldots; n_d^{(\lambda_d)}; n_{d+1}, \ldots, n_{l+1} (\bar{x}; f_j(x_1, \cdot))| \cdot b_m| \right\|_{L_2(T^{l+1})}^2 = \int \left\{ \int_{T^1} \left\{ \sup_{\lambda_2, \ldots, \lambda_d > 0, n_j > 0, j \in M \setminus J_d} |S_{n_2^{(\lambda_2)}}(x_2; \ldots; n_d^{(\lambda_d)}; n_{d+1}, \ldots, n_{l+1} (\bar{x}; f_j(x_1, \cdot))| \cdot b_m| \right)^2 d\bar{x} \right\} dx_1
\]

\[
\leq C \int_{T^1} \left\{ \int_{T^1} |f_j(x_1, \bar{x})|^2 d\bar{x} \right\} dx_1 = C\|f_j\|_{L_2(T^{l+1})}^2, \quad j = 1, 2.
\]

By this and (2.24) we have:

\[
\left\| \sup_{\lambda_1 > 0, n_j > 0, j \in M \setminus J_d} |S_{n_1^{(\lambda_1)}}(x_1; G_{n_1}^{(j)}(\cdot, \bar{x}))| \right\|_{L_2(T^{l+1})} \leq C\|f_j\|_{L_2(T^{l+1})}, \quad j = 1, 2.
\]

(2.25)

Further, from (2.23), (2.22) and (2.25) it follows the validity of estimate (2.19).

In view of the induction method, we get that estimate (1.5) is true for any \(N \geq 3\) and any \(k\) (the number of lacunary components in the vector \(n^{(\lambda, m)}[J_k] \in \mathbb{Z}_0^N, 1 \leq k \leq N - 2.\)
Theorem 1 is proved.

A simple corollary of Theorem 1 is the following statement which will be used in the proof of Theorem 2.

Let $J_k$ be an arbitrary sample from $M$, $1 \leq k \leq N - 2$, $N \geq 3$. Fix an integer $s, 1 \leq s \leq N - k$, and indices $\nu_1, \ldots, \nu_s \in M \setminus J_k$. Denote

$$Q_{n(\lambda, m)[J_k]}^{(p_{\nu_1}, \ldots, p_{\nu_s})}(x; f) = \sum_{n_{\nu_1}=0}^{p_{\nu_1}} \cdots \sum_{n_{\nu_s}=0}^{p_{\nu_s}} S_{n(\lambda, m)[J_k]}(x; f), \quad p_{\nu_1}, \ldots, p_{\nu_s} \in \mathbb{Z}_1^s. \quad (2.26)$$

**Corollary of Theorem 1.** For any function $f \in L_2(T^N)$, for any $s, 1 \leq s \leq N - k$, for any $\nu_1, \ldots, \nu_s \in M \setminus J_k$ and $p_{\nu_1}, \ldots, p_{\nu_s} \in \mathbb{Z}_1^s$, the following estimate is true

$$Q_{n(\lambda, m)[J_k]}^{(p_{\nu_1}, \ldots, p_{\nu_s})}(x; f) = O\left(\prod_{j=1}^{s} p_{\nu_j} \prod_{\substack{l \in M \setminus J_k, \ l \neq \nu_1, \ldots, \nu_s}} \sqrt{\log(|n_l| + 2)}\right) \quad \text{for a.e.} \quad x \in T^N. \quad (2.27)$$

**Proof of Corollary of Theorem 1.** Let us prove estimate (2.27) for $s = 1$ (for $s > 1$ the proof is similar). Let for definiteness $J_k = \{1, \ldots, k\}$ and $\nu_1 = k + 1$, then $n(\lambda, m)[J_k] = (n_1^{(\lambda_1)}, \ldots, n_k^{(\lambda_k)}, n_{k+1}, \ldots, n_N)$. Set $p = p_{k+1}$ and consider

$$Q_{n(\lambda, m)[J_k]}^{(p)}(x; f) = \sum_{n_{k+1}=0}^{p} S_{n_1^{(\lambda_1)}, \ldots, n_k^{(\lambda_k)}, n_{k+1}, \ldots, n_N}(x; f)$$

$$= (p + 1)\pi^{-1} \int_{T^1} K_p(u_{k+1}) \left\{ \pi^{-N+1} \int_{T^N-1} f(u + x) \prod_{j=1}^{k} D_{n_j}^{(\lambda_j)}(u_j) \right\} d\tilde{u}, \quad \tilde{u} = (u_1, \ldots, u_k, u_{k+2}, \ldots, u_N), \quad (2.28)$$

where $D_{n_j}(u_j)$ is the Dirichlet kernel, $K_p(u_{k+1})$ is the Fejér kernel.

Denote as $G(x_1, \ldots, x_k, x_{k+2}, \ldots, x_N; u_{k+1})$ the expression in the braces in (2.28). Considering notation (2.12) $Q_{n(\lambda, m)[J_k]}^{(p)}(x; f) = (p + 1)\sigma_p(x; G)$. Hence, for $\sigma_p(x; G)$ estimate (2.13) is true. Further, note that $G$ is the partial sum of the Fourier series of the function $f$ whose index has $k$ lacunary components, and thus, for $G$ the estimate from Theorem 1 is true. Thus, (2.27) is proved.
Corollary of Theorem 1 is proved.

3. Proof of Theorem 2

In the proof of the theorem ideas from the papers [7] and [8, Theorem 3] are used.

Let us fix a sample $J_k$, $1 \leq k \leq N - 2$. Without loss of generality, let us consider that $J_k = \{N - k + 1, \ldots, N\}$. In this case the vector $m[J_k] = (n_1, \ldots, n_{N - k})$. Consider

$$n^{(\lambda, m)}[J_k] = (n_1, \ldots, n_{N - k}, n^{(\lambda, N - k + 1)}_{N - k + 1}, \ldots, n^{(\lambda, N)}_N) = (n_1, \ldots, n_{N - k}, n'),$$

where

$$n' = (n^{(\lambda, N - k + 1)}_{N - k + 1}, \ldots, n^{(\lambda, N)}_N). \quad (3.1)$$

Let condition (1.7) be satisfied, i.e., in our case

$$\sum_{n \in \mathbb{Z}^N} |c_n(f)|^2 N-k \prod_{\alpha=1}^{N-k} \log(|n_\alpha| + 2) < +\infty. \quad (3.2)$$

Estimate (3.2) permits to assert (see, e.g., [7, Lemma 3]) that there exists a sequence of numbers $\{p_j\}$, $p_j > 0$, $p_j = p_{-j}$, increasing to infinity as slowly as we like as $j \to \infty$, such that

$$\sum_{n \in \mathbb{Z}^N} |c_n(f)|^2 N-k \prod_{\alpha=1}^{N-k} \log(|n_\alpha| + 2) p_{n_\alpha} < +\infty. \quad (3.3)$$

Set

$$b_j = \left\{ \log(|j| + 2) \right\}^{\frac{1}{2}}, \quad j \in \mathbb{Z}^1. \quad (3.4)$$

Note that $\{b_j\}$ (taking into account the choice of the sequence $\{p_j\}$) is a convex sequence, satisfying the following conditions (see [19, Ch. III, p. 93])

$$b_j = b_{-j}, \quad b_j \to 0 \quad \text{and} \quad j\Delta^1 b_j \to 0 \quad \text{as} \quad j \to \infty, \quad \sum_{j=1}^{\infty} j \Delta^2 b_j < \infty, \quad (3.5)$$

where $\Delta^0 b_j = b_j$, $\Delta^1 b_j = b_j - b_{j+1}$, $\Delta^2 b_j = \Delta^1 b_j - \Delta^1 b_{j+1}$, $j \in \mathbb{Z}^1$. 

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Let $M_j \in \mathbb{Z}_1^1$ be such that inequalities are true: $2^{M_j^2} \leq n_j \leq 2^{(M_j+1)^2}$ (i.e., $M_j^2 \sim \log n_j$, $j = 1, \ldots, N - k$). Denote $\alpha_j = 2^{M_j^2}$ and represent the $J_k$-lacunary partial sum $S_{n(\lambda, m)[J_k]}(x; f)$ as follows:

$$S_{n(\lambda, m)[J_k]}(x; f) = \left\{ S_{n_1, n_2, \ldots, n_{N-k}, n'}(x; f) - S_{n_1, n_2, \ldots, n_{N-k}, n'}(x; f) \right\}$$

$$+ \sum_{j=2}^{N-k-1} \left\{ S_{n_1, \ldots, n_{j-1}, n_j, \ldots, n_{N-k}, n'}(x; f) - S_{n_1, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_{N-k}, n'}(x; f) \right\}$$

$$+ S_{n_1, \ldots, n_{N-k-1}, n_{N-k}, n'}(x; f) = \sum_{j=1}^{N-k-1} \Delta S_{n_j, \alpha_j}(x; f) + S_{n_1, \ldots, n_{N-k-1}, n_{N-k}, n'}(x; f).$$

(3.6)

Because the index $(\alpha_1, \ldots, \alpha_{N-k-1}, n_{N-k}, n') \in \mathbb{Z}^N$ of the latter partial sum has $N - 1$ lacunary components (see (3.1)), by Kojima’s result [8, Theorem 2] (see Remark 1 in the Introduction) the equality holds true

$$\lim_{n_1, \ldots, n_{N-k-1} \to \infty; n_{N-k} \to \infty} S_{n_1, \ldots, n_{N-k-1}, n_{N-k}, n'}(x; f) = f(x) \text{ a.e. on } \mathbb{T}^N. \quad (3.7)$$

Thus, theorem will be proved if we prove that each difference in (3.6) tends to zero a.e. on $\mathbb{T}^N$, precisely,

$$\lim_{n_1 \to \infty, \ldots, n_{N-k-1} \to \infty; n_{N-k} \to \infty; n' \to \infty} \Delta S_{n_j, \alpha_j}(x; f) = 0 \text{ a.e. on } \mathbb{T}^N, \quad j = 1, \ldots, N - k - 1. \quad (3.8)$$

Let us prove (3.8) for $j = 1$, for other differences the proof is similar. So, consider

$$\Delta S_{n_1, \alpha_1}(x; f) = S_{n_1, n_2, \ldots, n_{N-k}, n'}(x; f) - S_{n_1, n_2, \ldots, n_{N-k}, n'}(x; f).$$

(3.9)

In view of (3.3) and (3.4)

$$\sum_{n \in \mathbb{Z}^N} |c_n(f)|^2 \prod_{j=1}^{N-k} b_{n_j}^{-2} < +\infty; \quad (3.10)$$

so, by the Riesz-Fischer theorem, there exists a function $g \in L_2(\mathbb{T}^N)$ such that

$$c_n(f) = c_n(g) \prod_{j=1}^{N-k} b_{n_j}. \quad (3.11)$$

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Let us represent the $J_k$-lacunary partial sum of the Fourier series of the function $g$ in the real form and denote: $I = (i_1, \ldots, i_{N-k})$, $I' = (i_{N-k+1}, \ldots, i_N)$, $i_j \in \mathbb{Z}_0^1$, $j = 1, \ldots, N$; we get

$$S_{n(\lambda, m)[J_k]}(x; g) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_{N-k}=0}^{n_{N-k}} \sum_{\lambda_{N-k+1}=0}^{n_{(\lambda_{N-k+1})}} \cdots \sum_{i_N=0}^{n_{\lambda_N}} G_{I,I'}(x; g) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_{N-k}=0}^{n_{N-k}} A_I,$$

where

$$A_I = A_{i_1, \ldots, i_{N-k}} = \sum_{i_{N-k+1}=0}^{n_{(\lambda_{N-k+1})}} \cdots \sum_{i_N=0}^{n_{\lambda_N}} G_{I,I'}(x; g). \tag{3.13}$$

Thus, by (3.12), (3.13), the $J_k$-lacunary partial sum of the Fourier series of the function $f$ in the real form looks as

$$S_{n(\lambda, m)[J_k]}(x; f) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_{N-k}=0}^{n_{N-k}} A_I \cdot b_{i_1} \cdots b_{i_{N-k}}. \tag{3.14}$$

Given (3.14), the difference in (3.9) looks as

$$\Delta S_{n_1, \alpha_1}(x; f) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_{N-k}=0}^{n_{N-k}} A_I \cdot b_{i_1} \cdots b_{i_{N-k}}
- \sum_{i_1=0}^{\alpha_1} \sum_{i_2=0}^{n_2} \cdots \sum_{i_{N-k}=0}^{n_{N-k}} A_I \cdot b_{i_1} \cdots b_{i_{N-k}} = B(n_1) - B(\alpha_1). \tag{3.15}$$

Set $\nu = N - k$ and introduce the following notation.

Let $s, p, q$ be integers, $0 \leq s, p, q \leq \nu$, $s + p + q \leq \nu$. Denote

$$\mathfrak{A}(s, p, q) = \{ L = (l_1, \ldots, l_\nu) \in \mathbb{Z}_0^\nu : 0 = l_0 < l_1 < \cdots < l_\nu \leq \nu; \}
0 = l_0 < l_{s+1} < \cdots < l_{s+p} \leq \nu; \quad 0 = l_0 < l_{s+p+1} < \cdots < l_{s+p+q} \leq \nu;$$

$$1 \leq l_{s+p+q+1} < \cdots < l_\nu \leq \nu; \quad l_{\mu_1} \neq l_{\mu_2} \quad \text{for} \quad \mu_1 \neq \mu_2 \}. \tag{3.16}$$

For any vector $L \in \mathfrak{A}(s, p, q)$ we define the vector

$$R = R(L) = (r_1, \ldots, r_\nu) \in \mathbb{Z}_0^\nu : \quad r_{l_j} = 0 \quad \text{for} \quad j = 1, \ldots, s;
\quad r_{l_j} = 1 \quad \text{for} \quad j = s + 1, \ldots, s + p;
\quad r_{l_j} = 2 \quad \text{for} \quad j = s + p + 1, \ldots, s + p + q;$$

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For any vector \( \kappa = (\kappa_{s+1}, \ldots, \kappa_{\nu}) \in \mathbb{Z}^{\nu-s} \) we define the vector
\[
I_\kappa = (i_1, \ldots, i_\nu) : \quad i_{j} = \kappa_{j} \quad \text{for} \quad j = s + 1, \ldots, \nu, \quad \text{in the case} \quad 0 \leq s < \nu;
\]
and set \( I_0 \equiv I = (i_1, \ldots, i_\nu) \) in the case \( s = \nu \).

Let \( \{A_l\} = \{A_{i_1, \ldots, i_\nu}\}, \quad A_{i_1, \ldots, i_\nu} \in \mathbb{R}, i_l \in \mathbb{Z}^1, \quad l = 1, \ldots, \nu, \quad \nu \geq 1, \)
be an arbitrary sequence, and let \( \{b_{\alpha}\}, \alpha \in \mathbb{Z}^1, \) be a convex sequence of real numbers, satisfying conditions (3.3). Let elements of the sequence \( \{A_{l}^{(R)}\} \) be defined as follows:
\[
A_{l}^{(R)} = \sum_{i_{j}+1}^{\kappa_{j}} \cdots \sum_{i_{j}+p}^{\kappa_{j+p+1}} \sum_{t_{s+p+1}=0}^{t_{j}} \sum_{t_{s+p+1}=0}^{t_{j}} \cdots \sum_{t_{s+p+q}=0}^{t_{j}} \sum_{t_{s+p+q}=0}^{t_{j}} \prod_{i=1}^{\nu} \frac{1}{\Delta^{2}b_{i_{j}}}
\times \sum_{\alpha_{i_{j}+p+q+1}=0}^{\kappa_{j}} \sum_{t_{s+p+q+1}=0}^{t_{j}} \sum_{t_{s+p+q+1}=0}^{t_{j}} \cdots \sum_{t_{s+p+q+1}=0}^{t_{j}} \sum_{t_{s+p+q+1}=0}^{t_{j}} \Delta^{2}b_{\alpha_{i_{j}}} \sum_{i_{j}=0}^{i_{j}} \sum_{i_{j}=0}^{i_{j}} A_{j}.
\]

(3.18)

Here we assume that in (3.18): there are no sums of the type \( \sum_{i_{j}=0}^{i_{j}} \) in the case \( p = 0 \); no sums of the type \( \sum_{t_{j}=0}^{t_{j}} \sum_{i_{j}=0}^{i_{j}} \) in the case \( q = 0 \); no sums of the type
\[
\sum_{\alpha_{i_{j}}=0}^{\kappa_{j}} \sum_{t_{j}=0}^{t_{j}} \sum_{i_{j}=0}^{i_{j}} , \quad j = s + 1, \ldots, \nu, \quad \text{in the case} \quad s + p + q = \nu.
\]

In particular, by (3.18) we have: \( A_{l}^{(0, \ldots, 0)} = A_{l} \),
\[
A_{l}^{(1, \ldots, 1)} = \sum_{i_{1}=0}^{\kappa_{1}} \cdots \sum_{i_{\nu}=0}^{\kappa_{\nu}} A_{1, \ldots, i_{\nu}}, \quad A_{l}^{(2, \ldots, 2)} = \sum_{t_{1}=0}^{t_{1}} \cdots \sum_{t_{\nu}=0}^{t_{\nu}} \sum_{i_{1}=0}^{i_{\nu}} \sum_{i_{1}=0}^{i_{\nu}} A_{1, \ldots, i_{\nu}},
\]
\[
A_{l}^{(3, \ldots, 3)} = \prod_{l=1}^{\nu} \frac{1}{\Delta^{2}b_{\kappa_{l}}} \cdot \sum_{i_{1}=0}^{\kappa_{1}} \cdots \sum_{i_{\nu}=0}^{\kappa_{\nu}} \sum_{i_{1}=0}^{i_{\nu}} \sum_{i_{1}=0}^{i_{\nu}} A_{1, \ldots, i_{\nu}}^{(2, \ldots, 2)} \cdot \Delta^{2}b_{1} \cdots \Delta^{2}b_{\nu}.
\]

(3.19)

**Proposition 1.** For any \( n_{1}, \ldots, n_{\nu} \in \mathbb{Z}^1 \) the following equality holds true
\[
\sum_{i_{1}=0}^{n_{1}} \cdots \sum_{i_{\nu}=0}^{n_{\nu}} A_{l} b_{1} \cdots b_{\nu} = 2 \sum_{r_{1}=0}^{2} \cdots \sum_{r_{\nu}=0}^{2} \Delta^{r_{1}}b_{n_{1}-r_{1}} \cdots \Delta^{r_{\nu}}b_{n_{\nu}-r_{\nu}} A_{l}^{(r_{1}+1, \ldots, r_{\nu}+1)}.
\]

(3.20)
Proof of Proposition 1. First, consider $\nu = 1$. By setting in (3.15), (3.19) $\nu = 1$ and $\nu_1 = n_1$, we get

$$A_{n_1}^{(1)} = \sum_{i_1 = 0}^{n_1} A_{i_1}, \quad A_{n_1}^{(2)} = \sum_{i_1 = 0}^{n_1} \sum_{i_2 = 0}^{i_1} A_{i_1}, \quad A_{n_1}^{(3)} = \frac{1}{\Delta^2 b_{n_1}} \sum_{i_1 = 0}^{n_1} A_{i_1}^{(2)} \Delta^2 b_{i_1}. \quad (3.21)$$

By applying twice the Abel transformation and considering notation (3.21) we obtain:

$$\sum_{i_1 = 0}^{n_1} A_{i_1} b_{i_1} = \sum_{i_1 = 0}^{n_1-1} \sum_{k=0}^{i_1} A_k \Delta b_{i_1} + b_{n_1} \sum_{i_1 = 0}^{n_1} A_{i_1} = \sum_{i_1 = 0}^{n_1-2} \sum_{l=0}^{i_1} \sum_{k=0}^{l} A_k \Delta^2 b_{i_1}$$

$$+ \Delta b_{n_1-1} \sum_{i_1 = 0}^{n_1-1} \sum_{k=0}^{i_1} A_k + b_{n_1} \sum_{i_1 = 0}^{n_1} A_{i_1} = \Delta^2 b_{n_1-2} A_{n_1-2} + \Delta b_{n_1-1} A_{n_1-1} + b_{n_1} A_{n_1}.$$  

Finally, we get that estimate (3.20) is true for $\nu = 1$:

$$\sum_{i_1 = 0}^{n_1} A_{i_1} b_{i_1} = \sum_{r_1 = 0}^{2} \Delta^{r_1} b_{n_1-1-r_1} A_{n_1-1-r_1}^{(r_1+1)}. \quad (3.22)$$

Proposition 1 for $\nu > 1$ is proved by application of formula (3.22) on each index $i_1, l = 1, \ldots, \nu$.

Proposition 1 is proved.

Let us estimate the difference in (3.15) using formula (3.20) with $\nu = N - k$; we have

$$\Delta R(n_1) - B(\alpha_1) = \sum_{r_1 = 0}^{2} \cdots \sum_{r_\nu = 0}^{2} \{ R(n_1) - R(\alpha_1) \}, \quad R = (r_1, \ldots, r_\nu),$$

where

$$Q_R(z) = \Delta^{r_1} b_{z-r_1} \cdot \Delta^{r_2} b_{n_1-1-r_2} \cdots \Delta^{r_\nu} b_{n_\nu-r_\nu} A_{z-r_1, n_1-1-r_2, \ldots, n_\nu-r_\nu}^{(r_1+1, r_2+1, \ldots, r_\nu+1)}, \quad z = n_1, \alpha_1. \quad (3.23)$$

We introduce a set

$$\Omega = \{ R(L) : L \in \bigcup_{0 \leq s, p, q \leq \nu, s+p+q=\nu} \mathcal{A}(s, p, q) \} \setminus \{(2, \ldots, 2)\}, \quad (3.24)$$

and by (3.23), (3.24), write the difference (3.15) as follows

$$\Delta S_{n_1, \alpha_1}(x; f) = \sum_{R \in \Omega} \{ Q_R(n_1) - Q_R(\alpha_1) \} + \{ Q_{(2, \ldots, 2)}(n_1) - Q_{(2, \ldots, 2)}(\alpha_1) \}. \quad (3.25)$$
Consider a vector \( R \in \Omega \). Without loss of generality, we can consider that \( r_1 = \cdots = r_s = 0, r_{s+1} = \cdots = r_{s+p} = 1, r_{s+p+1} = \cdots = r_\nu = 2 \). Denote as \( R + 1 = (r_1 + 1, \ldots, r_\nu + 1) = (\tilde{r}_1, \ldots, \tilde{r}_\nu) \). Taking into account the choice of the vector \( R \), we have: \( \tilde{r}_1 = \cdots = \tilde{r}_s = 1, \tilde{r}_{s+1} = \cdots = \tilde{r}_{s+p} = 2, \tilde{r}_{s+p+1} = \cdots = \tilde{r}_\nu = 3 \). In this case, considering the definition of \( Q_R(n_1) \) by (3.23), we can write

\[
Q_R(n_1) = \prod_{i=1}^{s} b_{n_1} \cdot \prod_{i=s+1}^{s+p} \Delta^1 b_{n_{i-1}} \cdot \prod_{i=s+p+1}^{\nu} \Delta^2 b_{n_i - 2} A_{n_{i-1}-r_1, \ldots, n_{\nu}-r_\nu}^{(R+1)} \tag{3.26}
\]

In its turn, taking account of notation (3.18), we have:

\[
A_{n_1-r_1, \ldots, n_\nu-r_\nu}^{(R+1)} = A_{n_1-1, \ldots, n_{s+p-1}, \ldots, n_{s+p+1}-2, \ldots, n_{\nu}-2}^{(R+1)}
\]

\[
= \left\{ \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} \left\{ \sum_{l_{s+1}=0}^{n_{s+1}-1} \cdots \sum_{l_{s+p+1}=0}^{n_{s+p+1}-2} \sum_{i_{s+p}=0}^{n_{s+p}} \Delta^2 b_{\mu_{s+p+1}} \cdot \frac{1}{\Delta^2 b_{n_i-2}} \sum_{\mu_\nu=0}^{n_\nu-2} \sum_{\mu_{s+p+1}=0}^{n_{s+p+1}-2} \sum_{\mu_{s+p}=0}^{n_{s+p}} \Delta^2 b_{\mu_\nu} \right\} \right\} = \prod_{i=s+p+1}^{\nu} (\Delta^2 b_{n_{i-1}})^{-1} \sum_{\mu_{s+p}+1=0}^{n_{s+p}+1} \Delta^2 b_{\mu_{s+p+1}} \cdots \Delta^2 b_{\mu_s} \cdot \left[ \sum_{l_{i_1}=0}^{n_1-1} \cdots \sum_{l_{i_{s+1}=0}^{n_{s+1}-1}} \sum_{l_{i_{s+p+1}=0}^{n_{s+p+1}-2}} \sum_{l_{i_{s+p}}=0}^{n_{s+p}} \sum_{l_{i_{s}}=0}^{n_s} \Delta^2 b_{\mu_\nu} \cdot \left[ \sum_{l_{i_{s+1}}=0}^{n_s+1} \cdots \sum_{l_{i_{s+p+1}=0}^{n_{s+p+1}-2}} \sum_{l_{i_{s+p}}=0}^{n_{s+p}} \sum_{l_{i_{s}}=0}^{n_s} A_{i_1, \ldots, i_\nu} \right] \right]. \tag{3.27}
\]

Note that the expression in the braces equals \( S_{n_1, \ldots, n_s, l_{s+1}, \ldots, l_{s+p}, n'}(x; g) \), in view of \( \nu = N - k \) (see (3.12)). Denote the expression in the square brackets by \( \Phi = \Phi(\mu_{s+p+1}, \ldots, \mu_\nu) \). According to what was said above, we have

\[
\Phi = \sum_{l_{s+1}=0}^{n_{s+1}-1} \cdots \sum_{l_{s+p}=0}^{n_{s+p}} \sum_{l_{s+p+1}=0}^{n_{s+p+1}-2} \sum_{l_\nu=0}^{n_\nu} S_{n_1, \ldots, n_s, l_{s+1}, \ldots, l_{s+p}, n'}(x; g)
\]

By Corollary of Theorem 1 we have

\[
\Phi = O\left( \prod_{i=1}^{s} \sqrt{\log n_1} \cdot \prod_{i=s+1}^{s+p} (n_i - 1) \right) \cdot \mu_{s+p+1} \cdot \cdots \cdot \mu_\nu. \tag{3.28}
\]
Thus, (3.27) and (3.28) permit us to write

\[ A^{(r_1+1,\ldots, r_\nu+1)}_{n_1-1} = \prod_{i=s+p+1}^{\nu} (\Delta^2 b_{n_1-2}^{i-2})^{-1} \cdot O \left( \prod_{i=1}^{s+p} \sqrt{\log n_i} \prod_{i=s+1}^{n_\nu-1} (n_i - 1) \right) \]

\[ \times \sum_{\mu=0}^{n_\nu-2} \mu_s \Delta^2 b_{\mu_s+1} \cdot \ldots \cdot \sum_{\mu=0}^{n_\nu-2} \mu_\nu \Delta^2 b_{\mu_\nu}. \]  

(3.29)

Using estimate (3.29) in (3.26) we get

\[ Q_R(n_1) = O \left( \prod_{i=1}^{s+p} \sqrt{\log n_i} \cdot b_{n_i} \cdot \prod_{i=s+1}^{n_\nu-1} (n_i - 1) \Delta^1 b_{n_1-1} \cdot \prod_{i=s+p+1}^{n_\nu-1} \sum_{\mu=0}^{n_\nu-2} \mu_i \Delta^2 b_{\mu_i} \right). \]

(3.30)

Note that if \( s + p > 0 \), then, in view of (3.5) and the definition of \( b_i \) (3.4), we obtain from (3.30)

\[ Q_R(n_1) = o(1) \text{ as } n_1, \ldots, n_\nu \to \infty, \text{ if } 0 \leq p, s \leq \nu, 0 < s + p \leq \nu. \]  

(3.31)

Thus,

\[ Q_R(n_1) = o(1) \text{ as } n_1, \ldots, n_\nu \to \infty \text{ for any } R \in \Omega. \]  

(3.32)

The similar estimate is true for \( Q_R(\alpha_1) \) (considering the definition of \( \alpha_1 \)):

\[ Q_R(\alpha_1) = o(1) \text{ as } n_1, \ldots, n_\nu \to \infty \text{ for any } R \in \Omega. \]  

(3.33)

Consider now the case \( s + p = 0 \), i.e., \( s = p = 0 \). It means that \( r_1 = \cdots = r_\nu = 2 \). By setting in (3.27) \( s = p = 0 \), in account of notation (3.12), we have

\[ Q_{(2,\ldots,2)}(n_1) = \prod_{i=1}^{\nu} \Delta^2 b_{n_1-2} A_{n_1-2,\ldots,n_\nu-2}^{(3,\ldots,3)} = \]

\[ = \sum_{\mu=0}^{n_\nu-2} \Delta^2 b_{\mu_1} \sum_{l_1=0}^{\mu_1} \ldots \left\{ \sum_{\mu_\nu=0}^{n_\nu-2} \Delta^2 b_{\mu_\nu} \sum_{l_\nu=0}^{\mu_\nu} S_{l_1,\ldots,l_\nu,n'(x;g)} \right\}. \]

In this case, by (3.19), (3.28) and (3.29), we obtain:

\[ Q_{(2,\ldots,2)}(n_1) - Q_{(2,\ldots,2)}(\alpha_1) = \Delta^2 b_{n_1-2} \prod_{i=2}^{\nu} \Delta^2 b_{n_1-2} A_{n_1-2,n_2-2,\ldots,n_\nu-2}^{(3,\ldots,3)} \]

\[ - \Delta^2 b_{\alpha_1-2} \prod_{i=2}^{\nu} \Delta^2 b_{\alpha_1-2} A_{\alpha_1-2,n_2-2,\ldots,n_\nu-2}^{(3,\ldots,3)}. \]
By (3.5) and the definition of $\alpha_1$ we have:

$$n_1 - 2 \sum_{\mu_1=\alpha_1-1}^{n_1-2} \mu_1 \Delta^2 b_{\mu_1} = \alpha_1 - 1 \Delta^2 b_{\mu_1} = o(1) \text{ as } n_1 \to \infty.$$

Thus, by Corollary of Theorem 1 the same way as above we obtain that, as $n_1, \ldots, n_\nu \to \infty$

$$Q_{(2,\ldots,2)}(n_1) - Q_{(2,\ldots,2)}(\alpha_1) = O\left(\prod_{i=2}^{\nu} \sum_{\mu_i}^{n_i-2} \mu_i \Delta^2 b_{\mu_i} \cdot \sum_{\mu_i=\alpha_1-1}^{n_i-2} \mu_i \Delta^2 b_{\mu_i} \right) = o(1).$$

(3.34)

Thus, in view of (3.23), (3.25), by estimates (3.32), (3.34) we get that (3.3) is true for $j = 1$.

Estimate (1.9) follows from (1.8) and results by E.Stein [15] (moreover, even a slightly more strong estimate can be reduced from (1.8), see, e.g. estimate (22) in [14, p. 347]).

Theorem 2 is proved.

4. Proof of Theorem 3

Proof of Theorem 3. In order to prove the theorem, it is sufficient to prove the validity of estimate (1.12); estimate (1.11) is deduced from it by means of standard argumentation.

Let us fix an arbitrary sample $J_{N-2} \subset M$. Without loss of generality, we consider that $J_{N-2} = \{1, \ldots, N-2\}$, $M \setminus J_{N-2} = \{N-1, N\}$.

We introduce the following notations which permit to carry out the proof with less complexity. Let $t = m_{N-1}$, $q = m_N$ and

$$n' = (n_1^{(\lambda_1)}, \ldots, n_{N-2}^{(\lambda_{N-2})}) \in \mathbb{Z}^{N-2}, \quad m' = (m_1, \ldots, m_{N-2}) \in \mathbb{Z}^{N-2}. \quad (4.1)$$

Thus, the vectors $n^{(\lambda)}[J_{N-2}] \in \mathbb{Z}^N$ and $m \in \mathbb{Z}^N$ can be written in the form:

$$n^{(\lambda)}[J_{N-2}] = (n', n_{N-1}, n_N), \quad m = (m', t, q). \quad (4.2)$$

---

6 In the proof of this theorem, some ideas represented in [14] are used.
Denote also
\[ x' = (x_1, \ldots, x_{N-2}) \in \mathbb{T}^{N-2}, \quad u' = (u_1, \ldots, u_{N-2}) \in \mathbb{T}^{N-2}. \] (4.3)

We represent the partial sum \( S_{n^{(\lambda)}[J_{N-2}]}(x; f) \) in the real form (considering notations (4.1)- (4.3)):
\[ S_{n^{(\lambda)}[J_{N-2}]}(x; f) = \sum_{m'=0}^{n'} \left\{ \sum_{t=0}^{n_{N-1}} \sum_{q=0}^{n_N} G_{(m',t,q)}(x,f) \cdot l(t,q) \right\}. \] (4.4)

Further denote
\[ l(t, q) = \left\{ \log \left[ \min(|t|, |q|) + 2 \right] \right\}^{-1}. \] (4.5)

Thus, the condition (1.10) in Theorem 3 looks as follows:
\[ \sum_{m \in \mathbb{Z}^N} |c_m(f)|^2 l^{-2}(t, q) < +\infty, \]
and hence, according to the Riesz-Fischer theorem, there exists a function \( g \in L^2(\mathbb{T}^N) \) such that the Fourier coefficients of the functions \( f \) and \( g \) are connected by relations
\[ c_m(f) = c_m(g) l(t, q), \quad m \in \mathbb{Z}^N. \] (4.6)

So, the partial sum in (4.4) can be rewritten as follows
\[ S_{n^{(\lambda)}[J_{N-2}]}(x; f) = \sum_{m'=0}^{n'} \left\{ \sum_{t=0}^{n_{N-1}} \sum_{q=0}^{n_N} G_{(m',t,q)}(x,g) \cdot l(t,q) \right\}. \] (4.7)

Denote as \( G \) the sum in the braces in (4.7), i.e.,
\[ G = \sum_{t=0}^{n_{N-1}} \sum_{q=0}^{n_N} G_{(m',t,q)}(x,g) l(t,q). \] (4.8)

Further, let us again introduce ”shorthand” notations. Taking into account that \( l(t, q) = l(q,t) \) (see (4.5)), we set
\[ \Delta_q l(t,q) = l(t,q) - l(t,q+1), \quad \Delta_l(\Delta_q l(t,q)) = \Delta_q l(t,q) - \Delta_q l(t+1,q) \]
\[ = l(t,q) - l(t+1,q) - l(t,q+1) + l(t+1,q+1), \] (4.9)
and as well

\[ G_{t,q} = G_{(m',t,q)}(x,g); \quad U_i(\beta) = \sum_{j=0}^{\beta} G_{i,j}. \] (4.10)

Applying the Abel transformation to the sum over \( q \) in (4.8) and considering notations (4.9), (4.10), we have:

\[
G = \sum_{t=0}^{n_N-1} \left[ \sum_{q=0}^{n_N-1} \Delta_q l(t,q) U_t(q) + l(t,n_N) U_t(n_N) \right]
\]

\[
= \sum_{q=0}^{n_N-1} \sum_{t=0}^{n_N-1} \Delta_q l(t,q) U_t(q) + \sum_{t=0}^{n_N-1} l(t,n_N) U_t(n_N). \] (4.11)

Applying the Abel transformation to each sum over \( t \) in (4.11) and denoting (in account of (4.10))

\[
V(\alpha, \beta) = \alpha \sum_{i=0}^{4} \beta \sum_{j=0}^{4} G_{i,j} = \sum_{i=0}^{4} U_t(\beta), \] (4.12)

we obtain

\[
G = \sum_{q=0}^{n_N-1} \left[ \sum_{t=0}^{n_N-1} \Delta_t \Delta_q l(t,q) V(t,q) \right] + \sum_{q=0}^{n_N-1} \Delta_q l(n_N-1,q) V(n_N-1,q)
\]

\[
+ \sum_{t=0}^{n_N-1} \Delta_t l(t,n_N) V(t,n_N) + l(n_N-1,n_N) V(n_N-1,n_N) = I^{(1)}_{m',n_N-1,n_N}(x,g)
\]

\[
+ I^{(2)}_{m',n_N-1,n_N}(x,g) + I^{(3)}_{m',n_N-1,n_N}(x,g) + I^{(4)}_{m',n_N-1,n_N}(x,g). \] (4.13)

Returning to (4.7) and taking account of (4.8)-(4.13), we have:

\[
S_{n_{\lambda}[J_{N-2}]}(x; f) = \sum_{m'=0}^{n'} \sum_{j=1}^{4} I^{(j)}_{m',n_{N-1},n_N}(x,g) = \sum_{j=1}^{4} J^{(j)}_{n_{\lambda}[J_{N-2}]}(x,g). \] (4.14)

**Lemma 2.** The following estimates are true

\[
\sup_{\lambda_1, \ldots, \lambda_{N-2}, n_{N-1}, n_N > 0} \left\| I^{(j)}_{n_{\lambda}[J_{N-2}]}(x,g) \right\|_{L^2(T^N)} \leq C \| g \|_{L^2(T^N)}, \quad j = 1, 2, 3, 4. \] (4.15)

**Proof of Lemma 2.** Note that in view of the definition of \( l(t,q) \) – (4.5) and the differences \( \Delta_j \) – (4.9), we have

\[
\Delta_t (\Delta_q l(t,q)) = 0 \quad \text{if} \quad t \neq q. \] (4.16)
Denote \( l(s) = l(s, s) \) and \( \Delta l(s) = l(s) - l(s + 1) \).

Let us prove estimate (4.15) for \( j = 1 \). In account of (4.13), as well as notations (4.10), (4.12), we have

\[
I_{n(\lambda_1)J_{n-2}}^{(1)}(x, g) = \sum_{m' = 0}^{n'} \sum_{t = 0}^{n_{n-1}} \sum_{q = 0}^{n_{N-1}} \Delta t(\Delta q l(t, q)) V(t, q)
\]

From this, denoting as

\[
n_0 = \min(n_{N-1}, n_N),
\]

and considering (4.16) and (4.17), we obtain

\[
I_{n(\lambda_1)J_{n-2}}^{(1)}(x, g) = \sum_{m' = 0}^{n'} \sum_{t = 0}^{n_{n-1}} \sum_{i = 0}^{t} \sum_{j = 0}^{t} G_{m', i, j}(x, g) \Delta l(t) = \sum_{t = 0}^{n_0-1} S_{n', t, t}(x; g) \Delta l(t). \tag{4.18}
\]

Repeatedly applying the Abel transformation in (4.18) and taking into account argumentation in [18, Ch. 13, Theorem (1.8)], we obtain:

\[
\left| \sum_{t = 0}^{n_0-1} S_{n', t, t}(x; g) \Delta l(t) \right| \leq C \sup_{t > 0} \left| \frac{t}{t+1} \sum_{i = 0}^{t} S_{n', i, i}(x; g) \right|, \quad x \in \mathbb{T}^N. \tag{4.19}
\]

The following result is a particular case of the theorem proved by us (see [2, Theorem 1]).

**Theorem A.** Let \( 2 \leq k \leq N-1, N \geq 3, \) and the vector \((n_{1}^{(\lambda_1)}, \ldots, n_{N-k}^{(\lambda_1)})\), \(n_0, \ldots, n_0) \in \mathbb{Z}_1^N\), where \( \{n_i^{(\lambda_i)}\}, i = 1, \ldots, N - k, \) are lacunary sequences, and \( n_0 \in \mathbb{Z}_1^1 \). For any function \( \varphi \in L_2(\mathbb{T}^N) \) the estimate is true

\[
\left\| \sup_{\lambda_1, \ldots, \lambda_{N-k}, n_0 > 0} |S_{n_{1}^{(\lambda_1)}, \ldots, n_{N-k}^{(\lambda_1)}, n_0, \ldots, n_0}(x; \varphi)| \right\|_{L_2(\mathbb{T}^N)} \leq C \| \varphi \|_{L_2(\mathbb{T}^N)}, \tag{4.20}
\]

where the constant \( C \) does not depend on the function \( \varphi \).

Applying in the right part of (4.19) estimate (4.20) with \( k = 2 \), we get:

\[
\left\| \sup_{\lambda_1, \ldots, \lambda_{N-2}, n_{N-1}, n_0 > 0} \left| \sum_{t = 0}^{n_0-1} S_{n', t, t}(x; g) \Delta l(t) \right| \right\|_{L_2(\mathbb{T}^N)} \leq C \| g \|_{L_2(\mathbb{T}^N)}. \tag{4.21}
\]

This estimate, in view of (4.18), (4.19), proves estimate (4.15) for \( j = 1 \).
Let us prove estimate (4.15) for $j = 2, 3$. Consider $I_{n(\lambda)|J_{N-2}}^{(2)}(x, g)$. In account of (4.13), (4.14) and notations (4.10), (4.12), we obtain:

$$I_{n(\lambda)|J_{N-2}}^{(2)}(x, g) = \sum_{m'=0}^{n'} \sum_{q=0}^{n_{N-1}} \Delta_q l(n_{N-1}, q)V(n_{N-1}, q)$$

$$= \sum_{m'=0}^{n'} \sum_{q=0}^{n_{N-1}} \Delta_q l(n_{N-1}, q)U_t(q) = \sum_{m'=0}^{n'} \sum_{t=0}^{n_{N-1}} \left\{ \sum_{q=0}^{n_{N-1}} \Delta_q l(n_{N-1}, q)U_t(q) \right\}.$$  

(4.21)

Denote the expression in the braces as $\bar{I}$; given (4.21), we have:

$$\bar{I} = \sum_{q=0}^{n_{N-1}} \Delta_q l(n_{N-1}, q)U_t(q) = \sum_{q=0}^{n_{N-1}} \{l(n_{N-1}, q) - l(n_{N-1}, q+1)\}U_t(q).$$  

(4.22)

Let us "simplify" $\bar{I}$; for this purpose consider two cases: $n_{N-1} > n_N$ and $n_{N-1} \leq n_N$. If $n_{N-1} > n_N$, then in the sum (4.22) $n_{N-1} > q$. Here, in account of the definition of $l(t, q)$ (4.5), we get:

$$\bar{I} = \sum_{q=0}^{n_{N-1} - 1} \{l(q) - l(q + 1)\}U_t(q) = \sum_{q=0}^{n_{N-1} - 1} \Delta l(q)U_t(q).$$

Let now $n_{N-1} \leq n_N$, then

$$\bar{I} = \sum_{q=0}^{n_{N-1} - 1} \{l(n_{N-1}, q) - l(n_{N-1}, q+1)\}U_t(q) + \sum_{q=n_{N-1} + 1}^{n_{N-1}} \{l(n_{N-1}, q) - l(n_{N-1}, q+1)\}U_t(q) \times U_t(q) = \sum_{q=0}^{n_{N-1} - 1} \{l(q) - l(q + 1)\}U_t(q) = \sum_{q=0}^{n_{N-1} - 1} \Delta l(q)U_t(q).$$

In this case, from (4.21), by (4.10) and (4.4), we have:

$$I_{n(\lambda)|J_{N-2}}^{(2)}(x, g) = \sum_{m'=0}^{n'} \sum_{t=0}^{n_{N-1} - 1} \sum_{q=0}^{n_{N-1} - 1} \Delta_l(q)U_t(q)$$

$$= \sum_{q=0}^{n_{N-1} - 1} \sum_{m'=0}^{n'} \sum_{t=0}^{n_{N-1} - 1} \sum_{j=0}^{n_{N-1} - 1} G_{m', t, j}(x, g) \Delta_l(q) = \sum_{q=0}^{n_{N-1} - 1} S_{n', n_{N-1}, q}(x, g) \Delta_l(q).$$  

(4.23)

The same way as above we can obtain

$$I_{n(\lambda)|J_{N-2}}^{(3)}(x, g) = \sum_{t=0}^{n_{N-1} - 1} S_{n', t, n_{N}}(x, g) \Delta_l(t).$$  

(4.24)
The same as for $j = 1$ (see (4.19)), we again apply the Abel transformation in (4.23) and get: for $x \in \mathbb{T}^N$

$$|I_{n^2, l}(x, g)| \leq C \sup_{n_N > 0} \left| \frac{1}{n_N + 1} \sum_{i=0}^{n_N} S_{n', n, n-1, i} (x; g) \right| =$$

considering the form of the Cezaro means

$$= C \sup_{n_N > 0} \left| \int_{\mathbb{T}^N} K_{nN} (x_N - u_N) F(u_N, o) du_N \right| = \sigma_{nN} (x_N; F), \quad (4.25)$$

where $K_{nN}(u)$ is the Fejér kernel and

$$F(u_N, o) = S_{n', n, n-1} (x', x_{N-1}; g; u_N). \quad (4.26)$$

We apply in (4.25) estimate (2.13) and therefore get:

$$\left\| \sup_{\lambda_1, \ldots, \lambda_{N-2} > 0, n_N-1, n_N > 0} |I_{n^2, l}(x, g)| \right\|_{L_2(\mathbb{T}^N)} \leq \left\| \sup_{\lambda_1, \ldots, \lambda_{N-2} > 0, n_N-1 > 0} \left\{ \sup_{n_N > 0} |\sigma_{nN} (x_N; F)| \right\} \right\|_{L_2(\mathbb{T}^N)} \leq \left\| \sup_{\lambda_1, \ldots, \lambda_{N-2} > 0, n_N-1 > 0} |S_{n', n, n-1, l} (x', x_{N-1}; g; u_N)| \right\|_{L_2(\mathbb{T}^N)}.$$

Applying inequality (1.6), we obtain

$$\left\| \sup_{\lambda_1, \ldots, \lambda_{N-2} > 0, n_N-1, n_N > 0} |I_{n^2, l}(x, g)| \right\|_{L_2(\mathbb{T}^N)} \leq C \|g\|_{L_2(\mathbb{T}^N)}.$$

Estimate (4.16) for $j = 2$ is proved. Proof of this estimate for $j = 3$ (see (4.24)) is similar.

And finally, let us prove estimate (4.15) for $j = 4$. From (4.13), taking into account (4.4), (4.10), (4.12), we have:

$$I_{n^4, l}(x, g) = \sum_{m'=0}^{n'} I_{m', nN-1, nN} (x, g) = \sum_{m'=0}^{n'} l(nN-1, nN) \cdot V(nN-1, nN)$$

$$= l(nN-1, nN) \sum_{m'=0}^{n'} \sum_{t=0}^{nN-1} \sum_{q=0}^{nN} G_{m', t, q} (x, g)$$

$$= S_{n', nN-1, nN} (x; g) \cdot l(nN-1, nN) = S_{n^4, l}(x; g) \cdot l(nN-1, nN). \quad (4.27)$$
With the help of the function $F(u_N, \circ)$, defined in (4.20), we represent the partial sum $S_{n(\lambda)[J_{N-2}]}(x; g)$ in the form:

$$S_{n(\lambda)[J_{N-2}]}(x; g) = \frac{1}{\pi} \int_{\mathbb{T}^1} D_n(u_N) F(x_N + u_N, \circ) du_N.$$

(4.28)

Further, using standard argumentation (see [14, p. 84]) and notation (4.3), from (4.28) we obtain for $x = (x', x_{N-1}, x_N) \in \mathbb{T}^N$:

$$\sup_{\lambda_j > 0, j \in J_{N-2}, n_{N-1} > 0, n_N \geq 2} |S_{n(\lambda)[J_{N-2}]}(x; g)| \{\log n_N\}^{-1} \leq C \cdot \mathfrak{M} \left\{ \sup_{\lambda_j > 0, j \in J_{N-2}, n_{N-1} > 0} |S_{n', n_{N-1}}(x', x_{N-1}, g; u_N)| \right\},$$

(4.29)

where $\mathfrak{M}(\circ)$ is the Hardy-Littlewood maximal function. From (4.29), using inequality (1.6), we get:

$$\left\| \sup_{\lambda_j > 0, j \in J_{N-2}, n_{N-1} > 0, n_N \geq 2} |S_{n(\lambda)[J_{N-2}]}(x; g)| \{\log n_N\}^{-1} \right\|_{L_2(\mathbb{T}^N)} \leq C \|g\|_{L_2(\mathbb{T}^N)}.$$

Analogously we can prove

$$\left\| \sup_{\lambda_j > 0, j \in J_{N-2}, n_{N-1} \geq 2, n_N > 0} |S_{n(\lambda)[J_{N-2}]}(x; g)| \{\log n_{N-1}\}^{-1} \right\|_{L_2(\mathbb{T}^N)} \leq C \|g\|_{L_2(\mathbb{T}^N)}.$$

The last two inequalities and estimate (4.27) (in account of the definition of $l(n_{N-1}, n_N)$ (4.5)) prove the validity of estimate (4.15) for $j = 4$.

Lemma 2 is proved.

From estimates (4.14), (4.15), (4.5), (4.6) the validity of estimate (1.10) follows.

Theorem 3 is proved.

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