The Blow up Rate Estimates for a Reaction Diffusion System with Gradient Terms

Maan A. Rasheed and Miroslav Chlebik

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Abstract

We consider the blow-up solutions for a coupled reaction diffusion system with gradient terms. The main purpose is to understand whether the gradient terms effect the blow-up properties. We derive the upper and lower blow-up rate estimates under certain assumptions.

1 Introduction

In this section, we consider the Cauchy (Dirichlet) parabolic problem:

\[\begin{align*}
    u_t &= \Delta u + |\nabla u|^{q_1} + v^{p_1}, & v_t &= \Delta v + |\nabla v|^{q_2} + u^{p_2} & \text{in } \Omega \times (0,T), \\
    u(x,0) &= u_0(x), & v(x,0) &= v_0(x), & \text{in } \Omega,
\end{align*}\]

where \(p_1, p_2, q_1, q_2 \in (1, \infty)\), \(u, v \geq 0\) are nonzero smooth bounded functions on \(\Omega\) (not necessarily radial), \(\Omega = \mathbb{R}^n\) or \(B_R\). Moreover, in case of \(\Omega = B_R\), \(u, v\) are further required to satisfy the condition:

\[u(x,t) = 0, \quad v(x,t) = 0, \quad \text{on } \partial\Omega \times [0,T).\] (1.2)

The problems of semilinear parabolic equations have been studied by many authors, for instance, consider the Cauchy (Dirichlet) problem of the semilinear heat equation:

\[u_t = \Delta u + u^p, \quad \text{in } \Omega \times (0,T),\] (1.3)

where \(p > 1, \Omega = \mathbb{R}^n\) or \(B_R\). It is well known that every positive solution blows up in finite time, if the initial data is nonnegative and suitably large \([6, 9]\). Moreover, it was proved in \([5, 17]\) that the blow-up rate estimate for (1.3) takes the following form

\[u(x,t) \leq c(T-t)^{-\frac{1}{p-1}}, \quad (x,t) \in \Omega \times (0,T).\]
Later, in [10] it has been shown that if we add a positive gradient term to the equation (1.3), namely

\[ u_t = \Delta u + |\nabla u|^q + u^p, \]

(1.4)

then that enhancing blow-up, and the influence of the gradient term becoming more important as the value of \( p \) decreases. In the case \( q = 2 \) for radial positive solutions in \( \mathbb{R}^n \), it was shown in [7, 8] that blow-up solutions behave asymptotically like the self-similar solution of the Hamilton-Jacobi equation without diffusion \( u_t = |\nabla u|^2 + u^p \), which takes the form

\[ u(x, t) = (T - t)^{\frac{1}{p-1}} w\left( \frac{x}{(T - t)^m} \right), \quad m = (2 - p)/2(p - 1), \]

where \( w \in C^2(\mathbb{R}^n) \) is a positive radial decreasing function. On the other hand, the existence of nonnegative global solutions is shown in [15] for small initial data.

In [3, 4], it was considered, the Cauchy (Dirichlet) problem of the following semilinear system:

\[ u_t = \Delta u + v^{p_1}, \quad v_t = \Delta v + u^{p_2}, \quad (x, t) \in \Omega \times (0, T), \]

(1.5)

where \( p_1, p_2 > 1, \Omega = B_R \) or \( \mathbb{R}^n \), with nonzero initial data \( u_0, v_0 \geq 0 \), it was shown that any positive solution of this problem blows up in finite time if the initial data are large enough. Moreover, for the Cauchy problem of (1.5), it is well known [3] that any nontrivial positive solution blows up in finite time, if

\[ \max\{\alpha, \beta\} \geq \frac{n}{2}, \]

(1.6)

where

\[ \alpha = \frac{p_1 + 1}{p_1 p_2 - 1}, \quad \beta = \frac{p_2 + 1}{p_1 p_2 - 1}. \]

(1.7)

The blow-up rate estimates of this system was studied in [1, 2], it was proved that there exist a positive constant \( C \) such that

\[ u(x, t) \leq C(T - t)^{-\alpha}, \quad (x, t) \in \Omega \times (0, T), \]

\[ v(x, t) \leq C(T - t)^{-\beta}, \quad (x, t) \in \Omega \times (0, T). \]

In this paper, for problem (1.1), under some restricted assumptions, we prove that the upper blow-up rate estimates of the positive solutions and their gradient terms, take the following forms:

\[ u(x, t) + |\nabla u(x, t)|^{\frac{2(p_1 + 1)}{p_1 p_2 + 2 p_1 + 1}} \leq C_1(T - t)^{-\alpha}, \quad (x, t) \in \Omega \times (0, T), \]

\[ v(x, t) + |\nabla v(x, t)|^{\frac{2(p_2 + 1)}{p_1 p_2 + 2 p_2 + 1}} \leq C_2(T - t)^{-\beta}, \quad (x, t) \in \Omega \times (0, T), \]

where \( C_1, C_2 > 0. \)
2 Preliminaries

Set
\[ F_1(v, \nabla u) = |\nabla u|^{q_1} + v^{p_1}, \quad F_2(u, \nabla v) = |\nabla v|^{q_2} + u^{p_2}. \]

Since the system (1.1) is uniformly parabolic and \( F_1, F_2 \) are \( C^1([0, \infty) \times \mathbb{R}^n) \), moreover, the growth of the nonlinearities \( F_1 \) and \( F_2 \) with respect to the gradient is sub-quadratic, it follows that, the local existence of the unique nonnegative classical solutions to the Dirichlet (Cauchy) problem of (1.1) is guaranteed, for smooth and bounded initial data \( u_0, v_0 \), by the standard parabolic theory [11] (see also [12]). On the other hand, the positive solutions of problem (1.1) may blow up in finite time, and that due to the known blow-up results of the system (1.5) and the maximum principle [13].

Remark 2.1. Since the growth of the nonlinear terms in problem (1.1) with respect to the gradients is sub-quadratic, the gradient functions \( \nabla u, \nabla v \) are bounded as long as the solution \((u, v)\) is bounded (see [12]).

2.1 Blow-up Rate Estimates

In the next theorem, we establish the upper blow-up rate estimates for the problem (1.1). Furthermore, without comparing the blow-up solutions of this problem with those of problem (1.3), we show that the blow-up can only occur simultaneously.

**Theorem 2.2.** If \( p_1, p_2, q_1 \) and \( q_2 \) satisfy the following conditions

1. \( \max\{\alpha, \beta\} \geq \frac{n}{2} \),
2. \( 1 < q_1 < \frac{2\alpha + 2}{2\alpha + 1}, \quad 1 < q_2 < \frac{2\beta + 2}{2\beta + 1}, \)

where \( \alpha, \beta \) are given in (1.7), then for any positive blow-up solution \((u, v)\) of problem (1.1) there exist positive constants \( C_1, C_2 \) such that

\[ u(x, t) + |\nabla u(x, t)|^{\frac{2(p_1 + 1)}{p_1 p_2 + 2p_1 + 1}} \leq C_1(T - t)^{-\alpha}, \quad (2.1) \]

\[ v(x, t) + |\nabla v(x, t)|^{\frac{2(p_2 + 1)}{p_1 p_2 + 2p_2 + 1}} \leq C_2(T - t)^{-\beta}, \quad (2.2) \]

in \( \Omega \times (0, T) \), where \( T < \infty \) is the blow-up time.

**Proof.** Let

\[ M_u(t) = \sup_{\Omega \times (0, t)} [u(x, t) + |\nabla u(x, t)|^{\frac{2(p_1 + 1)}{p_1 p_2 + 2p_1 + 1}}], \]

\[ M_v(t) = \sup_{\Omega \times (0, t)} [v(x, t) + |\nabla v(x, t)|^{\frac{2(p_2 + 1)}{p_1 p_2 + 2p_2 + 1}}], \]
for \( t \in (0, T) \).

Clearly, \( M_u, M_v \) are positive, continuous and nondecreasing functions on \((0, T)\).

At least one of them diverges as \( t \to T \), due to \((u, v)\) blows up at time \( T \).

We show later that there is \( \delta \in (0, 1) \) such that

\[
\delta \leq M_u^{-\frac{1}{2\alpha}}(t)M_v^{-\frac{1}{2\beta}}(t) \leq \frac{1}{\delta}, \quad t \in (T/2, T).
\]  

(2.3)

So that, consequently, both \( M_u, M_v \) have to diverge as \( t \to T \).

To establish the blow-up rate estimates, we use a scaling argument similar as in [1]. The proof is divided into several steps.

**Step 1: Scaling**

If \( M_u \) diverges as \( t \to T \), the following procedure can be applied.

Given \( t_0 \in (0, T) \), choose \((x^*, t^*) \in \Omega \times (0, t_0]\) such that

\[
u(x^*, t^*) + |\nabla u(x^*, t^*)|^{\frac{2(p_1+1)}{p_1(p_2+2p_1+1)} \geq \frac{1}{2} M_u(t_0).}
\]  

(2.4)

Let \( \gamma = \gamma(t_0) = M_u^{-\frac{1}{2\alpha}}(t_0) \) be a scaling factor. Define the rescaled functions

\[
\varphi_1^\gamma(y, s) = \gamma^{2\alpha} u(\gamma y + x^*, \gamma^2 s + t^*),
\]

(2.5)

\[
\varphi_2^\gamma(y, s) = \gamma^{2\beta} v(\gamma y + x^*, \gamma^2 s + t^*),
\]

(2.6)

for \((y, s) \in \Omega_\gamma \times (-\gamma^{-2}t^*, \gamma^{-2}(T - t^*))\), where

\[
\Omega_\gamma = \{y \in \mathbb{R}^n : \gamma y + x^* \in \Omega\}.
\]

Clearly,

\[
\Omega_\gamma := \begin{cases} \mathbb{R}^n & \text{if } \Omega = \mathbb{R}^n, \\ B_R/\gamma & \text{if } \Omega = B_R. \end{cases}
\]

Next, we aim to show that \((\varphi_1^\gamma, \varphi_2^\gamma)\) is a solution of the following system

\[
\begin{aligned}
\varphi_{1s}^\gamma - \Delta \varphi_1^\gamma &= \gamma^{\mu_1} |\nabla \varphi_1^\gamma|^{q_1} + (\varphi_2^\gamma)^{p_1}, \\
\varphi_{2s}^\gamma - \Delta \varphi_2^\gamma &= \gamma^{\mu_2} |\nabla \varphi_2^\gamma|^{q_2} + (\varphi_1^\gamma)^{p_2},
\end{aligned}
\]  

(2.7)

where \( \mu_1 = 2\alpha + 2 - (2\alpha + 1)q_1 \), \( \mu_2 = 2\beta + 2 - (2\beta + 1)q_2 \).

From the assumption (2), it follows that \( \mu_1, \mu_2 > 0 \).

Clearly,

\[
\varphi_{1s}^\gamma = \gamma^{2\alpha+2} u, \quad \nabla \varphi_1^\gamma = \gamma^{2\alpha+1} \nabla u, \quad \Delta \varphi_1^\gamma = \gamma^{2\alpha+2} \Delta u.
\]  

(2.8)

From [1,1], (2.8), it follows

\[
\frac{1}{\gamma^{2\alpha+2}} \varphi_{1s}^\gamma = \frac{1}{\gamma^{2\alpha+2}} \Delta \varphi_1^\gamma + \frac{1}{\gamma^{2\alpha+2}} |\nabla \varphi_1^\gamma|^{q_1} + \frac{1}{\gamma^{2\alpha+2}} (\varphi_2^\gamma)^{p_1}.
\]  

4
Multiply the last equation by $\gamma^{(2\alpha+2)}$, we get the first equation of (2.7). In the same way we can show that $\varphi_2$ satisfies the second equation in system (2.7).

Restrict $s$ to $s \in (-\gamma^{-2}t^*, 0]$, our aim now is to show that

$$
\varphi^\gamma_1 (y, s) + |\nabla \varphi^\gamma_1 (y, s)|^{\frac{2(p_1+1)}{p_1 p_2 + 2p_1 + 1}} \leq 1,
$$

for $(y, s) \in \Omega_\gamma \times (-\gamma^{-2}t^*, 0]$. From (2.8), we obtain

$$
|\nabla \varphi^\gamma_1 (y, s)|^{\frac{2(p_1+1)}{p_1 p_2 + 2p_1 + 1}} = \gamma^{\frac{2(p_1+1)}{p_1 p_2 - 1} + 1}|\nabla u|^{\frac{2(p_1+1)}{p_1 p_2 + 2p_1 + 1}},
$$

Moreover,

$$
\varphi^\gamma_1 (y, s) + |\nabla \varphi^\gamma_2 (y, s)|^{\frac{2(p_2+1)}{p_1 p_2 + 2p_2 + 1}} \leq M_u^{-\frac{\alpha}{\gamma}} (t_0) M_\nu (t_0),
$$

for $(y, s) \in \Omega_\gamma \times (-\gamma^{-2}t^*, 0]$. From (2.9), we obtain

$$
\varphi^\gamma_1 (0, 0) + |\nabla \varphi^\gamma_1 (0, 0)|^{\frac{2(p_1+1)}{p_1 p_2 + 2p_1 + 1}} \geq \frac{1}{2},
$$

If $M_\nu$ diverges as $t \to T$ we can proceed in the same way by changing the role of $u$ and $v$.

**Step 2: Schauder’s estimates**

We need interior Schauder’s estimates of the functions $\varphi_1, \varphi_2$ on the sets

$$
S_K = \{y \in \Omega_\gamma, |y| \leq K\} \times [-K, KL], \quad K > 0, \quad L = 0, 1.
$$

Assume that $\varphi_1, \varphi_2$ satisfy in $S_{2K}$ the condition

$$
0 \leq \varphi^\gamma_1 + |\nabla \varphi^\gamma_1|^{\frac{2(p_1+1)}{p_1 p_2 + 2p_1 + 1}} \leq B, \quad 0 \leq \varphi^\gamma_2 + |\nabla \varphi^\gamma_2|^{\frac{2(p_2+1)}{p_1 p_2 + 2p_2 + 1}} \leq B.
$$

We claim that for any $K > 0, B > 0$ and $\sigma > 0$ small enough, there is a constant $C = C(K, B, \sigma)$ such that

$$
||\varphi^\gamma_1||_{C^{2+\sigma,1+\frac{\alpha}{2}}(S_{2K})} \leq C, \quad ||\varphi^\gamma_2||_{C^{2+\sigma,1+\frac{\alpha}{2}}(S_{2K})} \leq C.
$$

From (2.14) we deduce that each of $\varphi^\gamma_1, \varphi^\gamma_2, \nabla \varphi^\gamma_1, \nabla \varphi^\gamma_2$, is uniformly bounded function in $S_{2K}$. Therefore, the functions $(\varphi^\gamma_1)^{p_1}, (\varphi^\gamma_2)^{p_2}, |\nabla \varphi^\gamma_1|^{q_1}, |\nabla \varphi^\gamma_2|^{q_2}$ are uniformly bounded in $S_{2K}$. So, the right hand sides of the two equations in (2.7) are
uniformly bounded functions in $S_{2K}$, applying the interior regularity theory (see \cite{[11]}), we obtain (locally) uniform estimates in $C^{1+\sigma,\frac{1+\sigma}{2}}$-norms. Consequently, by Lemma 2.6, we obtain (locally) uniform estimates in Hölder norms $C^{\alpha, \frac{\alpha}{2}}$ on the right hand sides of the both equations in (2.11). Thus the parabolic interior Schauder's estimates imply (2.15) (see [\cite{11}]).

**Step 3: The proof of (2.3)**

Suppose that this lower bound were false. Then there exist a sequence $t_j \to T$. For each $t_j$ in the role of $t_0$ from Step 1, we scale about the corresponding point $(x_j^*, t_j^*)$ for all $j$ such that $t_j^* \leq t_j$, with the scaling factor

$$
\gamma_j = \gamma(t_j) = M_u^{-\frac{1}{3\sigma}}(t_j).
$$

We obtain the corresponding rescaled solution $(\varphi_{y^*_j}, \varphi_{x^*_j})$,

$$
\varphi_{y_j}(y, s) = \gamma_j^{-\sigma} u(\gamma_j y + x_j^*, \gamma_j^2 s + t_j^*), \quad (2.17)
$$

$$
\varphi_{x_j}(y, s) = \gamma_j^{2\sigma} v(\gamma_j y + x_j^*, \gamma_j^2 s + t_j^*). \quad (2.18)
$$

Clearly, $(\varphi_{y_j}, \varphi_{x_j})$ satisfies (as in Step 1) the following problem

$$
\begin{aligned}
\varphi_{y_j}^{\gamma_j} - \Delta \varphi_{y_j}^{\gamma_j} &= \gamma_j^{\rho_1} |\nabla \varphi_{y_j}^{\gamma_j}| q_1 + (\varphi_{x_j}^{\gamma_j})^{\rho_1}, \\
\varphi_{x_j}^{\gamma_j} - \Delta \varphi_{x_j}^{\gamma_j} &= \gamma_j^{\rho_2} |\nabla \varphi_{x_j}^{\gamma_j}| q_2 + (\varphi_{y_j}^{\gamma_j})^{\rho_2},
\end{aligned} \quad (2.19)
$$

with

$$
\begin{aligned}
\varphi_{y_j}^{\gamma_j}(0, 0) + |\nabla \varphi_{y_j}^{\gamma_j}(0, 0)|^{\frac{2(p_1+1)}{p_1 p_2^{(p_1+1)}+1}} &\geq 1/2, \\
0 &\leq \varphi_{y_j}^{\gamma_j} + |\nabla \varphi_{y_j}^{\gamma_j}|^{\frac{2(p_1+1)}{p_1 p_2^{(p_1+1)}+1}} \leq 1, \\
\varphi_{x_j}^{\gamma_j} + |\nabla \varphi_{x_j}^{\gamma_j}|^{\frac{2(p_2+1)}{p_2 p_1^{(p_2+1)}+1}} &\leq M_u^{-\frac{1}{\alpha}}(t_j) M_v(t_j),
\end{aligned} \quad (2.20)
$$

for $(y, s) \in \Omega_{\gamma_j} \times \left(0, -\gamma_j^{-2} t_j^*, 0\right]$, where

$$
\Omega_{\gamma_j} := \begin{cases}
R^n & \text{if } \Omega = R^n, \\
B_{\gamma_j} & \text{if } \Omega = B_R.
\end{cases}
$$

Clearly,

$$
\Omega_{\gamma_j} \to R^n, \quad \text{as } j \to \infty.
$$

From (2.16), (2.20), we see that

$$
\varphi_{x_j}^{\gamma_j} + |\nabla \varphi_{x_j}^{\gamma_j}|^{\frac{2(p_2+1)}{p_2 p_1^{(p_2+1)}+1}} \to 0, \quad \text{as } j \to \infty.
$$

Thus $\varphi_{y_j}^{\gamma_j}, \nabla \varphi_{x_j}^{\gamma_j}$ are bounded in $\Omega_{\gamma_j} \times \left(0, -\gamma_j^{-2} t_j^*, 0\right], \forall j.$
Using the uniform Schauder’s estimate derived in Step 2 to \((\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})\)
\[
||\varphi_1^{\gamma_j}||_{C^{2+\sigma,1+\sigma}}(\{y \in \Omega_{\gamma_j}, |y| \leq K\} \times [-K,0]) \leq C_K;
\]
\[
||\varphi_2^{\gamma_j}||_{C^{2+\sigma,1+\sigma}}(\{y \in \Omega_{\gamma_j}, |y| \leq K\} \times [-K,0]) \leq C_K,
\]
where \(C_K\) is independent of \(j\).

Since \((\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})\) is defined on a compact set, by the Arzela-Ascoli theorem, there exist a convergent subsequence, we still denote it by \((\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})\).

Since \(\mu_1, \mu_2 > 0\) and \(\nabla \varphi_1^{\gamma_j}, \nabla \varphi_2^{\gamma_j}\) are bounded, it follows that, the limit point \((\varphi_1, \varphi_2)\) is a solution of the following system
\[
\varphi_1 = \Delta \varphi_1 + \varphi_2, \quad \varphi_2 = \Delta \varphi_2 + \varphi_1, \quad \text{in} \quad \mathbb{R}^n \times (-\infty, 0]. \quad (2.21)
\]

Since \(\varphi_2^{\gamma_j} \to 0\), as \(j \to \infty\), it follows that \(\varphi_2 \equiv 0\), in \(\mathbb{R}^n \times (-\infty, 0]\).

Consequently, from the second equation in \((2.21)\), we obtain that
\[
\varphi_1 \equiv 0, \quad \text{in} \quad \mathbb{R}^n \times (-\infty, 0].
\]

This means
\[
\varphi_1(0,0) + |\nabla \varphi_1(0,0)|^{\frac{2(p_1+1)}{p_1+p_2+2}} = 0,
\]
which contradicts with \((2.20)\). Thus, the lower bound is held.

To prove the upper bound of \((2.3)\) we proceed similarly as in the proof of lower bound with changing the role of \(u\) and \(v\).

**Step 4: Estimate on doubling of \(M_u\)**

As \(M_u\) is continuous and diverges as \(t \to T\), for any \(t_0 \in (0,T)\) we define \(t_0^+\) by
\[
t_0^+ = \max\{t \in (t_0,T) : M_u(t) = 2M_u(t_0)\}.
\]

Clearly,
\[
u(x,t) + |\nabla u(x,t)|^{\frac{2(p_1+1)}{p_1+p_2+2}} \leq 2M_u(t_0), \quad (x,t) \in \Omega \times (0,t_0^+]. \quad (2.22)
\]

Take \(\gamma = \gamma(t_0) = M_u^{-\frac{1}{2\alpha}}(t_0)\).

We claim that
\[
\gamma^{-2}(t_0)(t_0^+ - t_0) \leq A, \quad t_0 \in \left(\frac{T}{2}, T\right),
\]
where the constant \(A \in (0, \infty)\) is independent of \(t_0\). Suppose that this estimate were false, then there would exist a sequence \(t_j \to T\) such that
\[
\gamma_j^{-2}(t_j)(t_j^+ - t_j) \to \infty,
\]
where
\[
t_j^+ = \max\{t \in (t_j,T) : M_u(t) = 2M_u(t_j)\}. \quad (2.23)
\]
For each $t_j$ we scale about the corresponding point $(x_j^*, t_j^*)$ such that
\[ 0 < t_j^* \leq t_j, \quad \frac{T}{2} < t_j < t_j^+ < T, \quad \forall j \]
with the scaling factor
\[ \gamma_j = \gamma(t_j) = M_u^{-\frac{1}{2m}}(t_j). \]
As in Step 3, we obtain the corresponding rescaled functions $(\varphi_1^\gamma, \varphi_2^\gamma)$, which satisfies (2.19) with the following conditions
\[
\begin{align*}
\varphi_1^\gamma(0, 0) + |\nabla \varphi_1^\gamma(0, 0)|^{2(p_1 + 1)} & \geq 1/2, \\
0 \leq \varphi_1^\gamma + |\nabla \varphi_1^\gamma|^{2(p_1 + 1)} & \leq 2, \\
\varphi_2^\gamma + |\nabla \varphi_2^\gamma|^{2(p_1 + 1)} & \leq M_u^{-\frac{\beta}{2m}}(t_j)M_v(t_j^+),
\end{align*}
\]
for $(y, s) \in \Omega_{\gamma_j} \times (-\gamma_j^{-1}t_j^*, \gamma_j^{-1}(t_j^+ - t_j^*))$.

From (2.23) and (2.24), it follows that
\[
\varphi_2^\gamma + |\nabla \varphi_2^\gamma|^{2(p_2 + 1)} \leq 2\pi M_u^{-\frac{\beta}{2m}}(t_j^+)M_v(t_j^+). \tag{2.25}
\]
From (2.3), we have
\[ M_v(t) \leq \delta^{-2}\beta M_u^\frac{\beta}{2m}(t), \quad t \in \left(\frac{T}{2}, T\right). \]
Therefore, (2.25) becomes
\[ \varphi_2^\gamma + |\nabla \varphi_2^\gamma|^{2(p_2 + 1)} \leq \frac{2\pi}{\delta^{2/3}}. \]
By using the Schauder’s estimates derived in Step 2 for $(\varphi_1^\gamma, \varphi_2^\gamma)$ we get a convergent subsequence in $C^{2+\sigma, 1+\sigma/2}_{loc}(R^n \times R)$ to the solution of system (2.21) in $R^n \times R$. This is a contradiction because all the nontrivial positive solutions of system (2.21), under the assumption (1), blow up in finite time (see [3]).

Thus, there is $A > 0$ such that
\[ \gamma^{-2}(t_0)(t_0^+ - t_0) \leq A, \quad t_0 \in \left(\frac{T}{2}, T\right). \tag{2.26} \]

**Step 5: Rate estimates**
As in Step 4, for any $t_0 \in (T/2, T)$ we define
\[ t_1 = t_0^+ \in (t_0, T) \quad \text{such that} \quad M_u(t_1) = 2M_u(t_0). \]
Due to (2.26),

\[(t_1 - t_0) \leq AM_u^{-\frac{1}{\alpha}}(t_0).\]

We can use \(t_1\) as a new \(t_0\) and obtain \(t_2 \in (t, T)\) such that

\[M_u(t_2) = 2M_u(t_1) = 4M_u(t_0),\]

\[(t_2 - t_1) \leq AM_u^{-\frac{1}{\alpha}}(t_1) = 2^{-\frac{1}{\alpha}} AM_u^{-\frac{1}{\alpha}}(t_0).\]

Continuing this process we obtain a sequence \(t_j \to T\) such that

\[(t_{j+1} - t_j) \leq 2^{-\frac{1}{\alpha}} AM_u^{-\frac{1}{\alpha}}(t_0), \quad j = 0, 1, 2, \ldots\]

If we add these inequalities we get

\[(T - t_0) \leq \sum_{j \geq 0} 2^{-\frac{j}{\alpha}} AM_u^{-\frac{1}{\alpha}}(t_0).\]

Thus

\[(T - t_0) \leq (1 - 2^{-\frac{1}{\alpha}})^{-1} AM_u^{-\frac{1}{\alpha}}(t_0)\]

From using (2.3) we obtain

\[M_v(t_0) \leq \delta^{-2\beta} M_u^\beta (t_0), \quad t_0 \in (T/2, T).\]

Thus

\[M_v(t_0) \leq \delta^{-2\beta} (1 - 2^{-\frac{1}{\alpha}})^{-\beta} A^\beta (T - t_0)^{-\beta}, \quad t_0 \in (T/2, T).\]

From above there exist two constants \(C_1^*, C_2^*\) such that

\[M_u(t_0) \leq C_1^*(T - t_0)^{-\alpha}, \quad t_0 \in \left(\frac{T}{2}, T\right),\]

\[M_v(t_0) \leq C_2^*(T - t_0)^{-\beta}, \quad t_0 \in \left(\frac{T}{2}, T\right).\]

From the last two equations and the definitions of \(M_u, M_v\), it follows that there exist constants \(C_1, C_2\) such that

\[u(x, t) + |\nabla u(x, t)|^{2(p_1+1)}\frac{1}{p_1 p_2 + 2 p_1 + 1} \leq C_1(T - t)^{-\alpha},\]

\[v(x, t) + |\nabla v(x, t)|^{2(p_2+1)}\frac{1}{p_1 p_2 + 2 p_2 + 1} \leq C_2(T - t)^{-\beta},\]

for \((x, t) \in \Omega \times (0, T)\).
Remark 2.3. If \( u_0 \equiv v_0, p = p_1 = p_2, q = q_1 = q_2, \) then problem (1.1) can be reduced to a scalar Dirichlet (Cauchy) problem for (1.4). Moreover, if

\[
1 < p \leq 1 + \frac{2}{n}, \quad 1 < q < \frac{2p}{1 + p},
\]

then in a similar way to the proof of Theorem 2.2 we can show that, for a nontrivial positive blow-up solution \( u, \) there exist \( C > 0 \) such that

\[
u(x, t) + |\nabla u(x, t)|^{\frac{2}{p+1}} \leq C(T-t)^{\frac{1}{p-1}}, \quad \text{in} \quad \Omega \times (0, T),
\]

i.e.

\[
u(x, t) \leq C(T-t)^{\frac{1}{p-1}}, \quad \text{in} \quad \Omega \times (0, T).
\]

As we have mentioned before, the rate estimate (2.29) is also known for the blow-up solutions of equations (1.3). Therefore, if \( p, q \) satisfy (2.27), then the positive gradient terms which appear in equation (1.4), does not affect the blow-up rate estimates of these problems. A similar observation holds for problem (1.1) by Theorem 2.2 which shows that the upper rate estimates of the Cauchy or Dirichlet problem for system (1.1) are the same as those known for the system (1.5). Therefore, under the assumptions of Theorem 2.2, the gradient terms in system (1.1) have no effect on the blow-up rate estimates.

2.2 Blow-up Set

It is well known that for the semilinear system (1.5) defined in a ball, under some restricted assumptions on \( u_0, v_0 \) (nonnegative, radially decreasing functions), that the only blow-up point is the centre of that ball (see [14]), while it is unknown whether this holds for the system (1.1). However, for the radial solutions of the single equation (1.4) defined in \( \Omega, \) in case \( q = 2, \) there is global blow-up, if \( 1 < p < 2, \) \( \Omega = B_R \) or \( \mathbb{R}^n, \) and regional blow-up, if \( p = 2, \) \( \Omega = \mathbb{R}^n, \) while a single blow-up point, if \( p > 2, \) \( \Omega = B_R \) (see [14, 16] and the references therein). The proof relies on the transformation \( v = e^u - 1, \) which converts (1.4) into the semilinear heat equation \( v_t = \Delta v + (1 + v) \log^p(1 + v). \) We note that, these results are much different from those known for equation (1.3) (see [14]), because for any \( p > 1, \) \( \Omega = B_R \) or \( \mathbb{R}^n, \) only a single blow-up point is known to occur for that problem, where the initial date are nonnegative, radially nonincreasing and bounded function.

References

[1] M. Chlebík and M. Fila, From critical exponents to blow-up rates for parabolic problems, Rend. Mat. Appl. (7)19, 449-470, (1999).
[2] K. Deng, *Blow-up rates for parabolic systems*, Z. Angew. Math. Phys. 47,132-143, (1996).

[3] M. Escobedo and M.A. Herrero, *Boundedness and blow up for a semilinear reaction diffusion system*, J. Differ. Equ. 89, 176-202, (1991).

[4] M. Escobedo and M.A. Herrero, *A semilinear parabolic system in a bounded domain*, Annali Matematica pura ed applicata, CLXV(IV), 315-336,(1993).

[5] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. 34, 425-447, (1985).

[6] H. Fujita, *On the blow-up of solutions to the Cauchy problem for ut = ∆u + uα*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math.13, 109-104, (1966).

[7] V. A. Galaktionov and J. L. Vazquez, *Regional blow up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation*, SIAM J. Math. Anal. 24,1254-1276, (1993).

[8] V. A. Galaktionov and J. L. Vazquez, *Blowup for quasilinear heat equations described by means of nonlinear Hamilton-Jacobi equations*, J. Differ. Equ. 127,1-40, (1996).

[9] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. 16, 305 -333, (1963).

[10] B. Kawohl and L. Peletier, *Remarks on blowup and dead cores for nonlinear parabolic equations*, Math. Z. 202, 207-217, (1989).

[11] O. A. Ladyzenskaja, V.A.Solonnikov and N.N.Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, American Mathematical Society, 23, (1968).

[12] J.H. Petersson, *On global existence for semilinear parabolic systems*, Nonlinear Anal., 60 (2), 337347, (2005).

[13] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*, Birkhuser Advanced Texts, Birkhuser, Basel, (2007).

[14] Ph. Souplet, *Single-point blow-up for a semilinear parabolic system*, J. Eur. Math. Soc. 11, 169-188, (2009).

[15] S. Snoussi, S. Tayachi and F.B. Weissler, *Asymptotically self-similar global solutions of a semilinear parabolic equation with a non-linear gradient term*, Proc. R. Soc. Edinb., Sect. A 129, No. 6, 1291-1307, (1999).

[16] Ph. Souplet, *Recent results and open problems on parabolic equations with gradient nonlinearities*, Electron. J. Differential Equations 1-19, (2001).

[17] F. Weissler, *An L∞ blow-up estimate for a nonlinear heat equation*, Comm. Pure Appl. Math. 38, 291-295, (1985).