Gaudin models with $\mathcal{U}_q(\mathfrak{osp}(1|2))$ symmetry

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Abstract

We consider a Gaudin model related to the $q$-deformed superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$. We present an exact solution to that system diagonalizing a complete set of commuting observables, and providing the corresponding eigenvectors and eigenvalues. The approach used in this paper is based on the coalgebra supersymmetry of the model.

1 Introduction

The Gaudin model, introduced by M. Gaudin in 1976, is a quantum mechanical system involving long–range spin interaction [1, 2].

In [3] it was solved in the framework of the algebraic Bethe Ansatz. It was also shown there that the model is governed by a Yang–Baxter algebra, called the Gaudin algebra, with commutation relations linear in the generators and determined by a classical $r$-matrix. It is to be stressed that these features are present in the model despite its quantum mechanical nature. In fact the Gaudin model is one of a large class of models, with such an algebraic nature, so that its study becomes an important issue.

Let us recall that the superalgebra extension of the Gaudin algebra, and of the related $r$–matrix structure, has been worked out in some remarkable papers (see for instance [4, 5]) where the Gaudin model related to orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ has been constructed and solved through a brilliant generalization of the Bethe-Ansatz.

It is known that this algebraic richness and robustness allows one to use it as a testing ground for many ideas such as the Bethe Ansatz and the general procedure of separation of variables.

Among these approaches, the coalgebraic one was introduced in a series of papers [6, 7, 8, 9]. A general and constructive connection between coalgebras and integrability can be stated as follows: given any coalgebra $(\mathfrak{g}, \Delta)$ with Casimir element $C$, each of its representations gives rise to a family of completely integrable Hamiltonians $H^{(m)}$, $m = 1, ..., N$ with an arbitrary number $N$ of degrees of freedom.

Endowing this coalgebra with a suitable additional structure (either a Poisson bracket or a non–commutative product on $\mathfrak{g}$), both classical and quantum mechanical systems can be obtained from the same $(\mathfrak{g}, \Delta)$. It is important to emphasize that the validity of this general procedure by no means depends on the explicit form of $\Delta$ (i.e., on whether the coalgebra $(\mathfrak{g}, \Delta)$ is deformed or not).

In this framework a particular class of coalgebras that can be used to construct systematically integrable systems are the so–called $q$-algebras [10]. The feature of such systems will be that they are integrable deformations of the ones obtained applying the same method to the corresponding non-deformed coalgebra.
We briefly recall here a general construction of completely integrable quantum systems associated with Lie (rank-1) superalgebras based on a coalgebraic approach\footnote{[11]}. Applying the method to higher ranks (super)algebras it is still possible to obtain commuting observables but completeness is by no means guaranteed\footnote{[12]}.

Let us consider a Lie superalgebra $\mathfrak{g}$ with Casimir $C \in \mathcal{U}(\mathfrak{g})$, and a co-associative linear mapping $\Delta : \mathcal{U}(\mathfrak{g}) \mapsto \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ (denoted as coproduct) such that $\Delta$ is a Lie homomorphism:

$$\{\Delta(a), \Delta(b)\} = \Delta(\{a, b\}), \quad \forall a, b \in \mathcal{U}(\mathfrak{g}),$$

where $[\cdot, \cdot]$ denotes the supercommutator. The coassociativity property allows one to construct from $\Delta$ in an unambiguous way subsequent homomorphisms

$$\Delta^{(2)} = \Delta, \quad \Delta^{(3)} : \mathcal{U}(\mathfrak{g}) \mapsto \mathcal{U}(\mathfrak{g})^{\otimes 3}, \quad \ldots, \Delta^{(N)} : \mathcal{U}(\mathfrak{g}) \mapsto \mathcal{U}(\mathfrak{g})^{\otimes N}.$$ 

Thus, we can associate to our superalgebra, (or better co-superalgebra) with $N$ generators and the remaining $N - 1$ integrals of motion are provided by $\Delta^{(m)}(C)$, $m = 2, \ldots, N$.

In\footnote{[13] [14]} it has been shown how to associate to a Lie–Hopf superalgebra a quantum integrable system and how to extend this procedure to $q$-superalgebras. In fact, $q$-superalgebras are obtained by Lie–Hopf superalgebras through a process of deformation that preserve their Lie–Hopf structure. It is therefore possible to associate to $q$-superalgebras integrable systems that are deformed version of the ones associated to the original superalgebra.

In the present paper we will consider an integrable $q$-deformation of the Lie superalgebra $\mathfrak{osp}(1|2)$ in order to obtain a Gaudin model with $\mathcal{U}(\mathfrak{osp}(1|2))$ symmetry.

### 2 A $q$-deformation of $\mathcal{U}(\mathfrak{osp}(1|2))$

The quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$\footnote{[15] [16]} as a deformation of the universal enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1|2)$ is generated by three elements $E, F, H$. The $q$-deformed commutation relations between the generators are:

$$\{E, F\} = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad [H, E] = E, \quad [H, F] = -F$$

(1)

In the following we will also need the operators $F^2$ and $E^2$ fulfilling the commutation relations

$$[F^2, E] = \kappa \left(q^{H+1/2} + q^{-H-1/2}\right) F,$$

$$[E^2, F] = -\kappa \left(q^{-H+1/2} + q^{-H+1/2}\right) E,$$

$$[E^2, F^2] = \kappa^2 - \frac{q^{2H+1/2} - q^{-2H-1/2}}{q - q^{-1}} + (q^H - q^{-H}) EF,$$

$$[H, E^2] = 2E^2,$$

$$[H, F^2] = -2F^2,$$

where

$$\kappa = \frac{1}{q^{1/2} + q^{-1/2}}.$$ 

The center of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ is spanned by the $q$-deformed Casimir element (provided that $q$ is not a root of the unity:\footnote{[17]})

$$C(q) = \left(\frac{q^{H+1/2} + q^{-H+1/2}}{q - q^{-1}}\right)^2 - \kappa^2 E^2 F^2 - (q^{H-1} - q^{-H+1}) EF.$$ 

(2)

We now endow $\mathcal{U}_q(\mathfrak{osp}(1|2))$ with a coalgebra structure. This can be done assigning the following $q$-deformed coproduct:

$$\Delta_q(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta_q(E) = E \otimes q^{H} + q^{-H} \otimes E,$$

$$\Delta_q(F) = F \otimes q^{H} + q^{-H} \otimes F.$$ 

(3)
which establish a superalgebra homomorphism:

\[
\{ \Delta_q(E), \Delta_q(F) \} = \frac{\Delta_q(q^H) - \Delta_q(q^{-H})}{q - q^{-1}},
\]

\[
[\Delta_q(H), \Delta_q(E)] = \Delta_q(E),
\]

\[
[\Delta_q(H), \Delta_q(F)] = -\Delta_q(F).
\]

For the sake of completeness we give the corresponding antipode and counit,

\[
\epsilon(H) = \epsilon(E) = \epsilon(F) = 0, \quad \epsilon(q^{\pm H}) = 1,
\]

\[
\sigma(E) = -q E, \quad \sigma(F) = -q^{-1} F, \quad \sigma(H) = -H, \quad \sigma(q^{\pm H}) = q^{\mp H},
\]

obtaining a Hopf superalgebra.

The coproducts \([3]\) can be extended to the \(N\)-th order by means of the coassociativity property as in the non–deformed case, taking into account that

\[
\Delta_q(q^H) = q^H \otimes q^H.
\]

Explicitly,

\[
\Delta_q^{(N)}(H) = \sum_{i=1}^{N} H_i,
\]

\[
\Delta_q^{(N)}(E) = \sum_{i=1}^{N} E_i q^{\frac{1}{2} \sum_{j=1}^{N} \text{sgn}(i-j) H_j},
\]

\[
\Delta_q^{(N)}(F) = \sum_{i=1}^{N} F_i q^{\frac{1}{2} \sum_{j=1}^{N} \text{sgn}(i-j) H_j}.
\]

**Remark 2.1** In the limit \(q \to 1\) the above deformed supercommutation relations obviously reduce to well-known supercommutation relations of the Lie superalgebra \(\mathfrak{osp}(1|2)\) \([13]\). Let us recall that \(\mathfrak{osp}(1|2)\) has dimension five and rank one; the supercommutation relations between its generators are

\[
\{ E, F \} = H, \quad [H, E] = E, \quad [H, F] = F.
\]

\[
\{ E, E \} = 2E^2, \quad \{ F, F \} = 2F^2,
\]

\[
[E^2, F] = -E, \quad [F^2, E] = F,
\]

\[
[H, E^2] = 2E^2, \quad [H, F^2] = -2F^2, \quad [F^2, E^2] = H.
\]

We see that the operators \(H, E^2, F^2\) generate the Lie algebra \(\mathfrak{sl}(2)\). The above supercommutation relations define \(H, E^2, F^2\) as the bosonic generators, and \(E, F\) as the fermionic ones, i.e. \(\text{deg}(H) = \text{deg}(E^2) = \text{deg}(F^2) = 0\) and \(\text{deg}(F) = \text{deg}(E) = 1\). This gradation can naturally be extended to the deformed enveloping superalgebra \(\mathcal{U}_q(\mathfrak{osp}(1|2))\), since

\[
\text{deg}(a b) = \text{deg}(a) + \text{deg}(b), \quad \text{mod } 2 \quad \forall a, b \in \mathcal{U}_q(\mathfrak{osp}(1|2)), \quad (4)
\]

and \(\text{deg}(q^H) = 1\).

In the same limit \(q \to 1\), definitions \([11]\) also reduce to the non–deformed coproducts for the superalgebra \(\mathfrak{osp}(1|2)\), which we will denote with \(\Delta\).

**Remark 2.2** In order to obtain a superalgebra homomorphism from the coproduct (in the deformed case \(\Delta_q\) as in the non–deformed one) a necessary requirement is that the tensor product be a suitable graded one. The proper definition of multiplication between \(N\) elements tensor products is the following one \([11]\):

\[
(a_1 \otimes \cdots \otimes a_N)(b_1 \otimes \cdots \otimes b_N) = (-1)^{\sum_{i<j=2}^{N} \text{deg}(a_i) \text{deg}(b_j)} (a_1 b_1) \otimes \cdots \otimes (a_N b_N),
\]

for all \(a_j, b_l \in \mathcal{U}_q(\mathfrak{osp}(1|2))\). Notice that \([11]\), together with the definitions \([13]\), assures that

\[
\text{deg}(\Delta_q^{(m)}(a)) = \text{deg}(a), \quad \forall a \in \mathcal{U}_q(\mathfrak{osp}(1|2)), \quad m \in \mathbb{N}.
\]

In other words the deformed coproduct preserves the gradation of the superalgebra.
3 Exact solution of a $U_q(\mathfrak{osp}(1|2))$ Gaudin model

Now we have all we need to construct a Gaudin model with $U_q(\mathfrak{osp}(1|2))$ symmetry in the coalgebra setting.

We consider the $N$ commuting observables $\{C^{(n)}(q)\}_{n=1}^N$:

$$\left[C^{(m)}(q), C^{(n)}(q)\right] = 0, \quad \forall m, n = 1, \ldots, N,$$

where

$$C^{(m)}(q) = \Delta_q^{(m)} [C(q)] =$$

$$= \left[ \frac{\Delta_q^{(m)} (q^{-1/2}) + \Delta_q^{(m)} (q^{1/2})}{q - q^{-1}} \right] - \kappa^2 \Delta_q^{(m)} (E^2) \Delta_q^{(m)} (F^2) +$$

$$- \left[ \Delta_q^{(m)} (q^{1/2}) - \Delta_q^{(m)} (q^{-1/2}) \right] \Delta_q^{(m)} (E) \Delta_q^{(m)} (F).$$

Hereafter we parametrize the deformation parameter with $z \equiv \ln q$.

A “physical” Gaudin Hamiltonian for the $N$-bodies system can be choosen as the $N$-th order deformed coproduct of the Casimir $\Delta_q^{(N)}(C(z))$, namely

$$\mathcal{H}_q = \frac{\sinh^2 \left[ z \left( \frac{\Delta_q^{(N)} (H) - \frac{1}{2}}{2} \right) \right]}{\sinh^2 z} - \kappa^2 \Delta_q^{(N)}(E^2) \Delta_q^{(N)}(F^2) +$$

$$- 2 \cosh \left[ z \left( \frac{\Delta_q^{(N)} (H) - 1}{2} \right) \right] \Delta_q^{(N)}(E) \Delta_q^{(N)}(F). \quad (6)$$

This Hamiltonian can be written in any representation of the deformed superalgebra $U_q(\mathfrak{osp}(1|2))$. While it’s always possible to choose a particular one, we will work in the general case of spin $j$ representation (with integer or half-integer $j$). Further generalization can be obtained by allowing site–dependent representations $(j_1, \ldots, j_N)$. However, for the sake of simplicity, we will not deal with this more general case in the present paper.

A complete set of independent commuting observables is provided by

$$\left\{ \Delta_q^{(N)}(H), C^{(2)}(z), \ldots, C^{(N)}(z) \right\}. \quad (7)$$

We can write the Hamiltonian (6) in the following form:

$$\mathcal{H} = \frac{\sinh^2 \left[ z \left( \sum_{i=1}^N H_i - \frac{1}{2} \right) \right]}{\sinh^2 z} - \kappa^2 \sum_{i,j,k,l=1}^N \eta_i \eta_j \phi_k \phi_l - 2 \cosh \left[ z \left( \sum_{i=1}^N H_i - 1 \right) \right] \sum_{i,j=1}^N \eta_i \phi_j,$$

where

$$\eta_i \equiv E_i q^{1 \sum_{j=1}^N \text{sgn}(i-j)H_j}, \quad \phi_i \equiv F_i q^{1 \sum_{j=1}^N \text{sgn}(i-j)H_j}.$$

Notice that the interaction involves more than two sites: this non–local feature is a peculiar property of $q$–deformed models.

We will show that the common eigenstates of the family of observables (7) take the form

$$\varphi_z(k, m_1, s_{m_1}; \ldots, 0, 0) = \left[ \Delta_q^{(N)}(E) \right]^{k-m_1} \psi_z(m_1, s_{m_1}; \ldots, 0, 0), \quad (8)$$

where $\psi_z(m_1, s_{m_1}; \ldots; 0, 0)$ is an element of the basis spanning the kernel of the lowering operator $\Delta_q^{(s_{m_1})}(F)$. These elements can be obtained through the recursive formula:

$$\psi_z(m_1, s_{m_1}; \ldots; 0, 0) = \sum_{i=0}^{\delta m} \eta_i(z) \left[ \Delta_q^{(s_{m_1}-1)}(E) \right]^{\delta m-i} \psi_z(m_{i-1}, s_{m_1-i}; \ldots; 0, 0), \quad (9)$$
where \( \delta m = m_t - m_{t-1} \) and \( \{ \alpha_i(z) \}_{i=1}^{\delta m} \) denotes a set of suitable coefficients. Since in each representation of \( \mathcal{U}_q(\text{osp}(1|2)) \) we have \( E^j + 1 = 0, j \) being the spin of the chosen representation, the sum in formula (9) will have at most \( 4j + 1 \) terms, so that \( \delta m \leq 4j \). If we consider the pseudo–vacuum state

\[
\psi_z(0,0) = | \downarrow \cdots \downarrow \rangle \in \text{Ker} \left( \Delta_q^{(k)}(F) \right), \quad \forall k = 1, \ldots, N,
\]

we recognize that \( m_t \) stands for the total number of excitations with respect to \( \psi_z(0,0) \) and \( s_{m_t} \) indicates the number of the last excited site (counting from the left).

**Proposition 3.1** The states (9) are annihilated by \( \Delta_q^{(s_{m_l})}(F) \) iff

\[
\frac{\alpha_{i+1}(z)}{\alpha_i(z)} = (-1)^{i+1} e^{\frac{i}{2}(\tau + \delta m - 2)} \frac{(-1)^{\delta m - 1} \sinh \left[ z \left( \tau + \delta m - i - \frac{1}{2} \right) \right] - \sinh \left[ z \left( \tau - \frac{1}{2} \right) \right]}{(-1)^{2j} \sinh \left[ z \left( j + \frac{1}{2} \right) \right] + \sinh \left[ z \left( i - j + \frac{1}{2} \right) \right]},
\]

\( i = 0, \ldots, \delta m - 1 \), where \( \tau \) is the eigenvalue of \( \Delta_q^{(s_{m_l})}(H) \).

**Proof:** A straightforward computation. Notice that it may be useful the following expression

\[
FE^k + (-1)^{k-1}E^{k-1}F = \frac{(-1)^{k-1}E^{k-1}}{2 \sinh z \cosh \frac{z}{2}} \left\{ (-1)^k \cosh \left[ z \left( H + \frac{1}{2} \right) \right] + \cosh \left[ z \left( H - \frac{1}{2} \right) \right] \right\},
\]

holding for all \( k \in \mathbb{N} \). The above formula is a plain consequence of the supercommutation relations (11).

\[ \square \]

Up to a normalization constant, proposition (8) determines all coefficients \( \alpha_i(z) \) with \( i = 1, \ldots, \delta m \).

**Proposition 3.2** The states (8) are eigenvectors of the set (7), namely

\[
C^{(n)}(z) \varphi_z(k, m_t, s_{m_t}, \ldots, 0, 0) = \lambda_n(z) \varphi_z(k, m_t, s_{m_t}, \ldots, 0, 0),
\]

with eigenvalues \( \lambda_n \) given by

\[
\lambda_n = \frac{\sinh^2 \left[ z \left( \rho_z - \frac{1}{2} \right) \right]}{\sinh^4 z},
\]

where \( \rho_z \) is the eigenvalue of \( \Delta_q^{(n)}(H) \) on the state \( \psi_z(i, s_i, \ldots) \), and the value of \( i \leq l \) is selected by the condition

\[
s_{m_l} \leq n < s_{m_{l+1}}, \quad s_{m_{l+1}} = N + 1.
\]

**Proof:** Notice that

\[
C^{(n)}(z) \varphi_z(k, m_t, s_{m_t}, \ldots, 0, 0) = \left[ \Delta_q^{(n)(E)} \right]^{k-m_t} C_h(z) \psi(m_t, s_{m_t}, \ldots, 0, 0),
\]

since \( [C^{(h)}(z), \Delta_q^{(n)(E)}] = 0 \) for all \( n = 1, \ldots, N \). If \( n \geq s_{m_t} \) we readily get \( \psi_z(m_t, s_{m_t}, \ldots, 0, 0) \) is in Ker \( \Delta_q^{(n)(F)} \). Otherwise, if \( n < s_{m_t} \) we can note that

\[
\left[ C^{(n)}(z), \sum_{i=0}^{\delta m} \alpha_i(z) \left[ \Delta_q^{(s_{m_i} - 1)(E)} \right]^{\delta m - i} (E_{s_{m_i}}) \right] = 0,
\]

so that we can act with \( C^{(n)}(z) \) on \( \psi_z(m_{t-1}, s_{m_{t-1}}, \ldots, 0, 0) \). By iteration we will arrive to a value of \( i \) such that condition (14) holds and to a function \( \psi_z(i, s_i, \ldots) \) which fixes the value of \( \rho_z \) and so the eigenvalue (13). This proves the proposition.

\[ \square \]
Remark 3.3 We stress the fact that our approach has a simple algebraic interpretation. Indeed, each eigenstate $\varphi_z(k, m_l, s_m, \ldots, 0, 0)$ has to be a basis vector of the tensor product representation

$$[D^{(j)}]_{\otimes N} = \bigoplus_{l=0}^{N} c_{j,l}^{(N)} D_l,$$

where $D^{(j)}$ denotes the representation of each site and $\{c_{j,l}^{(N)}\}$ is the set of Clebsch–Gordan coefficients. Our method constructs first the lowest weight vectors $\psi_z(m_l, s_m, \ldots, 0, 0)$ for each $D_l$, taking account that $l = N j - m_l$ and then allows us to complete the basis with suitable raising operators.

Thanks to the Schur’s Lemma the eigenvalues of the family (14) are the values taken by the Casimir $\Delta$ on each $D_l$. Furthermore the coefficients $\{c_{j,l}^{(N)}\}$ are related to the spectrum degeneracies; in fact the number of eigenstates of the Hamiltonian (9) that belong to the energy eigenvalue corresponding to the representation $D_l$ is given by

$$g_{j,l}^{(N)} = c_{j,l}^{(N)} (4l + 1),$$

being the factor $4l + 1$ the degeneracy of each $D_l$. This latter term could be removed by an external field, while the first one remains.

Remark 3.4 This graded model shares a remarkable feature with other supersymmetric integrable systems (16). Namely, as we show in Appendix 1, it is possible to assign an arbitrary grading to the pseudo–vacuum state (10). Each choice give rise, through the above construction, to a different family of eigenstates although the spectrum remains the same. In Appendix 2 we explicitly present two families of eigenstates for the $U_q(\mathfrak{osp}(1|2))$ Gaudin model with $j = 1/2$ and $N = 2$.

3.1 The $q \to 1$ limit

We now obtain some known results (11) on the Gaudin model with $\mathfrak{osp}(1|2)$ symmetry considering the limit $q \to 1$.

The family of $N$ commuting observables is $\{C^{(n)}\}_{n=1}^{N}$:

$$\left[ C^{(m)}, C^{(n)} \right] = 0, \quad \forall m, n = 1, \ldots, N,$$

where

$$C^{(m)} = \Delta^{(m)} (C) = \left[ \Delta^{(m)} (H) \right]^2 - 2 \left\{ \Delta^{(m)} (E^2), \Delta^{(m)} (F^2) \right\} - \left[ \Delta^{(m)} (E), \Delta^{(m)} (F) \right].$$

A “physical” non–deformed Gaudin Hamiltonian for the $N$-bodies system can be choosen as the $N$-th order coproduct of the Casimir $\Delta^{(N)} (C)$, namely

$$\mathcal{H} = \sum_{i \neq j} H_i H_j - 2 \left( E_i^2 F_j^2 + E_j^2 F_i^2 \right) - (E_i F_j - F_j E_i).$$

(16)

Up to a term proportional to the identity, (16) corresponds to the limit $z \to 0$ of the Hamiltonian (9), i.e. $\lim_{z \to 0} \mathcal{H}_q = \mathcal{H} + 1/4$. A complete set of independent commuting observables is provided by

$$\left\{ \Delta^{(N)} (H), C^{(2)}, \ldots, C^{(N)} \right\}.$$

(17)

Taking the limit $z \to 0$ in the definition of the states $\psi(m_l, s_m, \ldots, 0, 0)$ (i.e. replacing $\Delta_q$ with $\Delta$) we obtain the following results:

Proposition 3.5 The states $\psi(m_l, s_m, \ldots, 0, 0)$ are annihilated by $\Delta^{(s_m)} (F)$ iff

$$\frac{\alpha_{i+1}}{\alpha_i} = \frac{2 (-1)^{s_m-i} (\tau + \delta m - i - \frac{1}{2}) - 1 - 2\tau}{(-1)^{i+1}(1 + 4j) + 2i + 1 - 4j},$$

(18)

where $\tau$ is the eigenvalue of $\Delta^{(s_m)} (H)$. 
Proposition 3.6 The states $\varphi(k, m_l, s_{m_l}, \ldots, 0, 0)$ are eigenvectors of the set (17), namely
\[ C^{(n)} \varphi(k, m_l, s_{m_l}, \ldots, 0, 0) = \lambda_n \varphi(k, m_l, s_{m_l}, \ldots, 0, 0), \] with eigenvalues $\lambda_n$ given by
\[ \lambda_n = (\rho - i + 1)(\rho - i) + \frac{1}{4}, \] where $\rho$ is the eigenvalue of $\Delta^{(n)}(H)$ on the state $\psi(i, s_i, \ldots)$, and the value of $i \leq l$ is selected by the condition
\[ s_{m_l} \leq n < s_{m_{l+1}}, \quad s_{m_{l+1}} = N + 1. \]

3.2 $\mathcal{U}_q(\mathfrak{osp}(1|2))$ Gaudin model with $j = 1/2$

Here we consider the particular case of the fundamental representation, namely the spin $j = 1/2$ one ($1 \leq \delta m \leq 2$). This case greatly simplify calculations, allowing a meaningful understanding of the results we have presented in the previous section.

Proposition 3.1 becomes the following one.

Proposition 3.7 The states (9) with $\delta m = 1$ are annihilated by $\Delta_q^{(s_{m_l})}(F)$ iff
\[ \alpha_0(z) = 1, \quad \alpha_1(z) = e^{\frac{i}{2}(\tau - 1)}\frac{\sinh(z \tau)}{\sinh z}. \]

The states (9) with $\delta m = 2$ are annihilated by $\Delta_q^{(s_{m_l})}(F)$ iff
\[ \alpha_0(z) = 1, \quad \alpha_1(z) = -e^{\frac{i}{2}}\frac{\cosh[z(\tau + \frac{i}{2})]}{\cosh \frac{i}{2}}, \quad \alpha_2(z) = e^z \frac{\sinh(z \tau)}{\sinh z}, \quad \alpha_3(z) = e^{\frac{i}{2}(\tau - 1)}\frac{\sinh(z \tau)}{\sinh z}, \] where $\tau = m_{l-1} - s_{m_l} + 1$.

On the other hand proposition 3.2 reduces to

Proposition 3.8 The states (k) are eigenvectors of the set (7), namely
\[ C^{(n)} \varphi_z(k, m_l, s_{m_l}, \ldots, 0, 0) = \lambda_n \varphi_z(k, m_l, s_{m_l}, \ldots, 0, 0), \] with eigenvalues $\lambda_n(z)$ given by
\[ \lambda_n(z) = \frac{\sinh^2 \left[ z \left( n - i + \frac{1}{2} \right) \right]}{\sinh^2 z}, \] where the value of $i \leq l$ is selected by the condition
\[ s_{m_l} \leq n < s_{m_{l+1}}, \quad s_{m_{l+1}} = N + 1. \]

In this case it is also possible to determine explicitly the degeneracies of the spectrum. These obviously correspond to those of the spin 1 case of the original $\mathfrak{sl}(2)$ Gaudin model. Namely, (15) now reads:
\[ \left[ D(\frac{1}{2}) \right] \otimes^N = \bigoplus_{l=0}^{N} \sum_{\frac{N}{2},l} \sum_{\frac{N}{2},l} c^{(N)}_{\frac{N}{2},l} D_l, \] and the following result can be proved by means of the character identity.

Proposition 3.9 The total number of eigenstates $\varphi(k, m_l, s_{m_l}, \ldots, 0, 0)$ with $m_l = N/2 - l$ is given by
\[ c^{(N)}_{\frac{N}{2},l} = \sum_{k=l}^{N} \binom{N}{2k-l} \binom{2k-l}{k} - \sum_{k=l}^{N} \binom{N}{2k-l+1} \binom{2k-l+1}{k+1}. \]
4 Concluding remarks

We have constructed a Gaudin model that shares both a deformed coalgebraic structure and a superalgebra symmetry. This has been achieved first deforming the Lie superalgebra $\mathfrak{osp}(1|2)$, endowing it with a Hopf structure and then applying to it a slightly modified version of the algorithm proposed in [3, 7, 11].

We obtained an exhaustive description of the spectrum and eigenstates of a particular Hamiltonian, which reduces to the known one for the Gaudin model associated to $\mathfrak{osp}(1|2)$ [11, 5].

Our approach, thanks to its purely algebraic nature, can be obviously used for any spin of the representation.

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Appendix 1: Some remarks about $\mathfrak{osp}(1|2)$ representations

In this appendix we shall recall the basic concepts of graded vector spaces as modules of superalgebras representations. Graded vector spaces are vector spaces equipped with a notion of odd and even degree, that allows us to treat fermions.

Let $\rho$ be an irreducible finite–dimensional representation of the Lie superalgebra $\mathfrak{osp}(1|2)$, $\rho : \mathfrak{osp}(1|2) \rightarrow \text{End}(V)$, where $V$ is the module of the representation.

The following results hold [15]:

- $\dim(V) = 4j + 1$, where $j$ is a non negative integer or half integer, called the spin of the representation $\rho$;
- $V = V_0 \oplus V_1$ with $\dim V_0 = 2j + 1$ and $\dim V_1 = 2j$. We shall call $v_0 \in V_0$ even (or bosonic) and $v_1 \in V_1$ odd (or fermionic). The subspaces $V_0$ and $V_1$ are called the homogeneous components of $V$. If we choose a basis of homogeneous elements $e_i \in V$, $i = 0, 1, \ldots, 4j + 1$, we can define a grading $G : i \rightarrow \mathbb{Z}_2$:

$$G(i) = \begin{cases} 0 & \text{if } e_i \in V_0, \\ 1 & \text{if } e_i \in V_1. \end{cases}$$

- Elements $[\rho(a)]_{ik}$, $a \in \mathfrak{osp}(1|2)$, $i, k = 1, \ldots, 4j + 1$ of the representation $\rho$ have a grading $\pi : \{1, \ldots, 4j + 1\} \times \{1, \ldots, 4j + 1\} \rightarrow \mathbb{Z}_2$ such that

$$\pi[\rho(a)]_{ik} \equiv G(i) + G(k) \mod 2.$$

The bosonic (resp. fermionic) sector is the set of elements with $\pi = 0$ (resp. $\pi = 1$).

In this way we give a graded structure (that we call “grading”) both to the module $V$ of the representation and to the elements of $\text{End}(V)$, thus reflecting the “gradation” of the elements of $\mathfrak{osp}(1|2)$.

- The bosonic sector of each representation $\rho_j$ is the completely reducible representation of $\mathfrak{sl}(2)$ given by

$$\rho_j = \mathcal{D}_j \oplus \mathcal{D}_{j-1/2}, \quad j \neq 0.$$

- The tensor product of two irreducible representations $\rho_j$ and $\rho_k$ is given by:

$$\rho_j \otimes \rho_k = \bigoplus_{J=|j-k|}^{j+k} \rho_J,$$

where $J$ integer or half-integer.
Let \( v_i \in V \) and \( A_i \in \text{End}(V) \), \( i = 1, \ldots, N \) be respectively \( N \) homogeneous vectors and \( N \) homogeneous endomorphisms. Hence we can construct the operator \( A_1 \otimes \cdots \otimes A_N \in \text{End}(V)^{\otimes N} \) and the vector \( v_1 \otimes \cdots \otimes v_N \in V^{\otimes N} \) using (5).

One can check that it holds the following

**Proposition 4.1** The action of \( A_1 \otimes \cdots \otimes A_N \in \text{End}(V)^{\otimes N} \) on \( v_1 \otimes \cdots \otimes v_N \in V^{\otimes N} \) is associative if

\[
(A_1 \otimes \cdots \otimes A_N)(v_1 \otimes \cdots \otimes v_N) = (-1)^{\sum_{i=2}^{N} \deg(A_i)} \sum_{j=1}^{N} \vartheta(v_j)(A_1 v_1) \otimes \cdots \otimes (A_N v_N).
\]

The above proposition is a key point in applying the algorithm (both in the deformed case and in the non-deformed case) exposed in Section 3.

**Appendix 2: Co-existence of two families of eigenstates**

Let us consider the \( \mathcal{U}_q(\mathfrak{osp}(1|2)) \) Gaudin model with \( j = 1/2 \) and \( N = 2 \) in order to show the occurrence of two complete families of eigenstates, corresponding to the possible choices \( \deg |\downarrow\rangle = 1 \) (fermion–boson–fermion) and \( \deg |\downarrow\rangle = 0 \) (fermion–fermion–boson).

In the first case we obtain the following results:

| \( \psi(0, 0) \) | \( \varphi(k; m_1, s_{m_1}; \cdots, 0, 0) \) | \( C^{(2)} \) |
|------------------|----------------------------------|--------|
| \( \varphi(0; 0, 0) = |\downarrow\downarrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(1; 0, 0) = q^{-1/2} |\downarrow\rangle - q^{1/2} |\downarrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(2; 0, 0) = q^{-1} |\uparrow\downarrow\rangle - (q^{1/2} - q^{-1/2}) |00\rangle + q |\downarrow\uparrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(3; 0, 0) = q^{-3/2} |\uparrow0\rangle + q^{3/2} |0\uparrow\rangle + (q^{1/2} - q^{-1/2}) (|\uparrow0\rangle + |0\uparrow\rangle) \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(4; 0, 0) = |\uparrow\uparrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |

while the second choice leads to

| \( \psi(0, 0) \) | \( \varphi(k; m_1, s_{m_1}; \cdots, 0, 0) \) | \( C^{(2)} \) |
|------------------|----------------------------------|--------|
| \( \varphi(0; 0, 0) = |\downarrow\downarrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(1; 0, 0) = q^{-1/2} |\downarrow\rangle + q^{1/2} |\downarrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(2; 0, 0) = q^{-1} |\uparrow\downarrow\rangle + (q^{1/2} - q^{-1/2}) |00\rangle - |\downarrow\uparrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(3; 0, 0) = q^{-3/2} |\uparrow0\rangle - q^{3/2} |0\uparrow\rangle - (q^{1/2} - q^{-1/2}) (|\uparrow0\rangle - |0\uparrow\rangle) \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |
| \( \varphi(4; 0, 0) = |\uparrow\uparrow\rangle \) | | \( \sinh^2(5z/2) \) \( \sinh^2 z \) |

Let us notice that the eigenvalues of \( C^{(2)} \) and their degeneracies are the same in both cases.
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