THE GALOIS MODULE STRUCTURE OF HOLOMORPHIC POLY-DIFFERENTIALS AND RIEHMANN-ROCH SPACES

FRAUKE M. BLEHER AND ADAM WOOD

ABSTRACT. Suppose $X$ is a smooth projective geometrically irreducible curve over a perfect field $k$ of positive characteristic $p$. Let $G$ be a finite group acting faithfully on $X$ over $k$ such that $G$ has non-trivial, cyclic Sylow $p$-subgroups. If $E$ is a $G$-invariant Weil divisor on $X$ with $\deg(E) > 2g(X) - 2$, we prove that the decomposition of $H^0(X, \mathcal{O}_X(E))$ into a direct sum of indecomposable $kG$-modules is uniquely determined by the class of $E$ modulo $G$-invariant principal divisors, together with the ramification data of the cover $X \to X/G$. The latter is given by the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover. As a consequence, we obtain that if $m > 1$ and $g(X) \geq 2$, then the $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ is uniquely determined by the class of a canonical divisor on $X/G$ modulo principal divisors, together with the ramification data of $X \to X/G$. This extends to arbitrary $m > 1$ the $m = 1$ case treated by the first author with T. Chinburg and A. Kontogeorgis. We discuss applications to the tangent space of the global deformation functor associated to $(X, G)$ and to congruences between prime level cusp forms in characteristic 0. In particular, we complete the description of the precise $k\text{PGL}(2,F)$-module structure of all prime level $\ell$ cusp forms of even weight in characteristic $p = 3$.

1. INTRODUCTION

Let $k$ be a perfect field, let $X$ be a smooth projective geometrically irreducible curve over $k$ of genus $g(X)$, and let $G$ be a finite group acting faithfully on the right on $X$ over $k$. As in [14 §IV.1], we denote by $\Omega_X$ the sheaf of relative differentials of $X$ over $k$. Moreover, for every positive integer $m$, we let $\Omega_X^{\otimes m}$ be the $m$-fold tensor product of $\Omega_X$ with itself over $\mathcal{O}_X$. The group $G$ acts on the left on $\Omega_X^{\otimes m}$ and also on $H^0(X, \Omega_X^{\otimes m})$. We call the latter the space of holomorphic $m$-differentials of $X$. It is a classical problem, first posed by Hecke in [15], to determine how $H^0(X, \Omega_X^{\otimes m})$ decomposes into a direct sum of indecomposable $kG$-modules and to give an explicit description of these indecomposables. When $k$ is algebraically closed and its characteristic does not divide $\#G$, this was solved by Chevalley and Weil in [9] (see also [17]).

For the remainder of the paper, we assume that the characteristic of the perfect field $k$ is a prime number $p$ that divides $\#G$. Several authors have studied the $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ under various assumptions on the group $G$, the parameter $m$, and the ramification of the cover $X \to X/G$. Their research has often focused on cyclic groups or abelian $p$-groups or the case when $m = 1$. See [33, 19, 38, 32, 24, 31, 20, 21, 28, 5] for a sample of previous results.

We would like to point out two of these articles, as they are closely related to our results. In [20], Karanikolopoulos determined the $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ for $m > 1$ when $G$ is a finite cyclic $p$-group. The main tool in [20] was the use of so-called Boseck invariants to find a suitable $k$-basis of $H^0(X, \Omega_X^{\otimes m})$. In [5], the authors took a more geometric approach to determine the $kG$-module structure of $H^0(X, \Omega_X)$ when $G$ has cyclic Sylow $p$-subgroups. It is the latter approach we will use in this paper.

Let $K_{X/G}$ be a canonical divisor on $X/G$, and denote the $G$-cover $X \to X/G$ by $\gamma$ and its ramification divisor by $\text{Ram}_{\gamma}$. Then $K_X = \gamma^* K_{X/G} + \text{Ram}_{\gamma}$ is a $G$-invariant canonical divisor on $X$ (see [14] Prop.

Date: March 18, 2023.

2010 Mathematics Subject Classification. Primary 11G20; Secondary 14H05, 14G17, 20C20.

Both authors were supported in part by NSF Grant No. DMS-1801328. Frauke M. Bleher is the corresponding author.
IV.2.3], and $\Omega_X^{\otimes m} \cong \mathcal{O}_X(mK_X)$ as $\mathcal{O}_X$-$G$-modules. If $m > 1$ and $g(X) \geq 2$ then $\deg(mK_X) > 2g(X) - 2$. Hence, it is natural to consider the more general case of the Riemann-Roch space $H^0(X, \mathcal{O}_X(E))$ associated to a $G$-invariant Weil divisor $E$ on $X$ satisfying $\deg(E) > 2g(X) - 2$.

This leads to our first main result. By the $G$-divisor class of $E$, we mean the equivalence class of all Weil divisors $E'$ on $X$ such that $E - E'$ is the divisor of a $G$-invariant function in $k(X)^*$. 

**Theorem 1.1.** Suppose $G$ has non-trivial cyclic Sylow $p$-subgroups. Let $E$ be a $G$-invariant Weil divisor on $X$ with $\deg(E) > 2g(X) - 2$. Then the $kG$-module structure of $H^0(X, \mathcal{O}_X(E))$ is uniquely determined by the $G$-divisor class of $E$, together with the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover $X \to X/G$.

The $G$-divisor class of the canonical divisor $K_X = \gamma^* K_{X/G} + \text{Ram}_\gamma$ is uniquely determined by the class of $K_{X/G}$ modulo principal divisors on $X/G$. Therefore, we obtain as a consequence of Theorem 1.1 our second main result, which extends [5, Thm. 1.1] from $m = 1$ to arbitrary $m > 1$. Since for all $m \geq 1$, $g(X) = 0$ implies $H^0(X, \Omega_X^{\otimes m}) = 0$ and $g(X) = 1$ implies $H^0(X, \Omega_X^{\otimes m}) = k$ with trivial $G$-action, the assumption $g(X) \geq 2$ is not a real restriction.

**Corollary 1.2.** Suppose $m > 1$ and $g(X) \geq 2$, and that $G$ has non-trivial cyclic Sylow $p$-subgroups. The $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ is uniquely determined by the class of canonical divisors on $X/G$ modulo principal divisors, together with the lower ramification groups and the fundamental characters of the closed points of $X$ that are ramified in the cover $X \to X/G$.

The statement of Corollary 1.2 differs from [5, Thm. 1.1] since if $m = 1$ then the $kG$-module structure of $H^0(X, \Omega_X)$ does not depend on the class of canonical divisors on $X/G$.

When $m = 2$, Corollary 1.2 can be used to determine the dimension of the tangent space of the global deformation functor associated to the pair $(X, G)$ (see [8]). Similarly to the case when $m = 1$, Corollary 1.2 also has implications for the study of classical modular forms. More precisely, for any prime number $\ell$ with $p \neq \ell \geq 7$, let $X(\Gamma_\ell)$ be the modular curve associated to the principal congruence subgroup $\Gamma_\ell$ of level $\ell$, and let $X_p(\ell)$ be the reduction of $X(\Gamma_\ell)$ in characteristic $p$. For all $p \geq 3$, we obtain non-trivial congruences modulo appropriate maximal ideals over $p$ between level $\ell$ cusp forms in characteristic 0 of even weight $2m > 2$ such that these congruences arise from isotypic components with respect to the action of $\text{PSL}(2, \mathbb{F}_\ell)$ on $X(\Gamma_\ell)$ (see Theorems 5.3 and 5.4). To prove this when $p = 3$, we determine for all $m > 1$, the precise $kP\text{SL}(2, \mathbb{F}_\ell)$-module structure of $H^0(X_3(\ell), \Omega_X^{\otimes m}(\ell))$ (see Theorem 5.8 and 40).

In particular, Theorem 5.3 and Propositions 6.5.1 and 6.5.2 together with [5, Thm. 1.4 and Props. 6.4.1 - 6.4.4], give a complete solution to Hecke’s classical problem [15, 9] of determining the precise module structure of all prime level cusp forms of even weight in the case when $k$ has characteristic $p = 3$. Using [19] or [30] for $p > 3$, it follows that the only remaining open case is when $p = 2$.

The proof of Theorem 1.1 follows the main outline used in the proof of [5, Thm. 1.1]. The first two steps are (I) to use the Conlon induction theorem [10, Thm. (80.51)] to reduce to the case when $G$ is $p$-hypo-elementary, and (II) to use the representation theory of $p$-hypo-elementary groups [11, §II.5-II.6] to reduce to the case when $k$ is algebraically closed (see [5, §3] for details on these two reduction steps).

Once these two reduction steps are made, Theorem 1.6 below gives a precise algorithm of how to determine the $kG$-module structure of $H^0(X, \mathcal{O}_X(E))$ when $G$ is $p$-hypo-elementary and $k$ is algebraically closed. Since information about the module structure of Riemann-Roch spaces has been useful in the context of algebraic geometry codes, Theorem 1.6 provides new insights into this subject. Previous work has mostly focused on the case when $kG$ is semisimple, see, for example, [18], [12], [8].

To state Theorem 1.6, we need the following assumptions and notations.
**Assumption 1.3.** Let $k$ be an algebraically closed field of prime characteristic $p$. Suppose $G = P \times \chi C$ is a $p$-hypto-elementary group, where $P = \langle \sigma \rangle$ is a cyclic $p$-group of order $p^n > 1$, $C = \langle \rho \rangle$ is a cyclic $p'$-group of order $c$, and $\chi : C \to \mathbb{F}_p^*$ is a character which we also view as a character of $G$ by inflation. Let $X$ be a smooth projective curve over $k$, and suppose there is a faithful right action of $G$ on $X$ over $k$. Let $E$ be a $G$-invariant Weil divisor on $X$ with $\deg(E) > 2g(X) - 2$ and write $E = \sum_{x \in X} e_xx$.

When $k$ and $G$ are as in Assumption 1.3 then there are precisely $\# G$ isomorphism classes of indecomposable $kG$-modules. Each simple $kG$-module has $k$-dimension one and it is uniquely determined by the action of $C$ on it. Moreover, each indecomposable $kG$-module $U$ has a simple socle, which is the kernel of the action of $(\sigma - 1)$ on $U$, its radical is equal to $(\sigma - 1)U$, and the isomorphism class of $U$ is uniquely determined by its socle and its $k$-dimension (see [11] II.5-II.6 and also [33] Remark 3.4). In other words, $U$ is a uniserial $kG$-module, in the sense that it has a unique composition series. For a discussion of uniserial modules over Artin algebras, see [23] IV.2.

**Notation 1.4.** Let $k$ and $G$ be as in Assumption 1.3. Let $\zeta \in k^*$ be a primitive $c$th root of unity.

(a) For $0 \leq a \leq c - 1$, let $S_a$ be a one-dimensional $k$-vector space on which $\rho$ acts by multiplication by $\zeta^a$, and view $S_a$ as a (simple) $kG$-module by inflation.

(b) For $0 \leq a \leq c - 1$ and $1 \leq b \leq p^n$, let $U_{a,b}$ be an indecomposable $kG$-module with socle $S_a$ and $k$-dimension $b$.

(c) For $i \in \mathbb{Z}$ and $0 \leq a \leq c - 1$, we define $\chi^i.a$ to be the unique element in $\{0, 1, \ldots, c - 1\}$ such that $S_{\chi^i.a} \cong S_{\chi^i} \otimes_k S_a$, where $S_{\chi^i}$ is the simple $kG$-module with character $\chi^i$. By [11] II.5-II.6, $U_{a,b}$ has ascending composition factors $S_a, S_{\chi^{-1}.a}, S_{\chi^{-2}.a}, \ldots, S_{\chi^{-b+1}.a}$.

**Notation 1.5.** Suppose $k$, $G$, and $X$ are as in Assumption 1.3.

(a) Let $I = \langle \tau \rangle$ be the (cyclic) subgroup of $P$ that is the greatest among the Sylow $p$-subgroups of the inertia groups of all closed points of $X$. Write $\# I = p^{n_I}$, where $0 \leq n_I \leq n$.

(b) For any closed point $x$ on $X$ and integer $i \geq 0$, let $I_{x,i}$ denote the $i$th lower ramification subgroup of $I$ at $x$. In other words, $I_{x,i}$ is the group of all elements in $I$ that fix $x$ and act trivially on $\mathcal{O}_{X,x}/m_{x,i}^{I_{x,i}}$.

The inertia group of $x$ in $I$ is $I_x = I_{x,0}$, and we write $\# I_x = p^{n_x}$. Since $I_x$ is cyclic, there are precisely $n_x$ jumps $1 \leq i_1(x) < i_2(x) < \cdots < i_{n_x}(x)$ in the numbering of the lower ramification groups $I_{x,i} \geq 0$ (see [33] Chap. IV.4). We have $I_x = I_{x,0} = I_{x,i_1(x)}$, $I_{x,i_j(x)} > I_{x,i_{j+1}(x)}$ for $1 \leq j \leq n_x - 1$, and $I_{x,i_{n_x}(x)} = I_{x,i_{n_x}(x)+1} = 1$.

(c) Let $Y = X/I$, and let $\pi : X \to Y$ be the corresponding quotient morphism. Define $\overline{G} = G/I$, and let $Z = X/G = Y/\overline{G}$ with corresponding quotient morphism $\lambda : Y \to Z$. Let $Z_{br}$ be the set of closed points in $Z$ that are branch points of the cover $\lambda$.

(d) For $y \in Y$, let $\overline{G}_y$ be the inertia group of $y$ in $\overline{G}$, and let $\theta_y : \overline{G}_y \to k(y)^* = k^*$ be the fundamental character of $\overline{G}_y$. In other words, if $\pi_y$ is a uniformizer and $m_{y,y}$ is the maximal ideal of $\mathcal{O}_{Y,y}$, then $\theta_y(\overline{g}) = \frac{\overline{g} - 1}{\pi_y} \mod m_{y,y}$ for all $\overline{g} \in \overline{G}_y$. We write $\# \overline{G}_y = c_y$. Since $c_y$ divides $c$ and since we can view the modules from Notation 1.4(a) as $kG$-modules, there exists $\varphi(y) \in \{0, 1, \ldots, c_y - 1\}$ such that $\theta_y$ is the character of $\text{Res}_{\overline{G}_y} S_{\varphi(y)}$. For $0 \leq a \leq c - 1$ and $i \in \mathbb{Z}$, define $\mu_{a,i}(y)$ to be $1$ if $a \equiv i \varphi(y) \mod c_y$ and to be $0$ otherwise. In other words, $\mu_{a,i}(y) = 1$ if and only if $\theta_y(\overline{g})$ is the character of $\text{Res}_{\overline{G}_y} S_a$.

(e) Let $K_Z$ be a canonical divisor on $Z$. Let $\text{Ram}_\pi$ (resp. $\text{Ram}_\lambda$) be the ramification divisor of the cover $\pi$ (resp. $\lambda$). By [13] IV.2 and [33] IV.1, $\text{Ram}_\pi = \sum_{x \in X} \sum_{i \geq 0} (\# I_{x,i} - 1)x$ and $\text{Ram}_\lambda = \sum_{y \in Y} (c_y - 1)y$. Define $K_Y = \lambda^* K_Z + \text{Ram}_\lambda$ and $K_X = \pi^* K_Y + \text{Ram}_\pi$. By [13] Prop. IV.2.3, $K_Y$ is a $\overline{G}$-invariant canonical divisor on $Y$ and $K_X$ is a $G$-invariant canonical divisor on $X$. 

3
Theorem 1.6. Suppose Assumption 1.3 holds, and assume Notations 1.4 and 1.5. Fix $0 \leq j \leq p^{n_1} - 1$, and define a $G$-invariant Weil divisor $E_j = \sum_{y \in Y} e_{y,j} y$ on $Y$ by

$$
e_{y,j} = \left[ \frac{c_x(y) - \sum_{\ell=1}^{n_x(y)} a_{\ell,t} y^{n_x(y)} - \ell i(x(y))}{p^{n_x(y)}} \right]$$

where $x(y) \in X$ is a point above $y$, $t \geq 0$ is the unique integer such that $p^{n_1-n_x(y)} t \leq j < p^{n_1-n_x(y)} (t+1)$, and $a_{1,t}, \ldots, a_{n_x(y),t} \in \{0, 1, \ldots, p-1\}$ are given by the $p$-adic expansion of $t$,

$$t = a_{1,t} + a_{2,t} p + \cdots + a_{n_x(y),t} p^{n_x(y)-1}.$$ 

For $z \in Z$, let $y(z) \in Y$ be such that $\lambda(y(z)) = z$. Define $\ell_{y(z),j} \in \{0, 1, \ldots, c_y(z) - 1\}$ by

$$\ell_{y(z),j} \equiv e_{y(z),j} \mod c_y(z),$$

and define

$$n_j = 1 - g(Z) + \sum_{z \in Z} \frac{c_y(z) - \ell_{y(z),j}}{c_y(z)}.$$

For $0 \leq a \leq c-1$, define

$$n(a,j) = \sum_{z \in Z_{br}} \left( \sum_{d=1}^{c_y(z)-1} \mu_{a,d}(y(z)) - \sum_{d=1}^{c_y(z)-1} \frac{d}{c_y(z)} \mu_{a,d}(y(z)) \right) + n_j.$$

Then $n(a,j)$ is a non-negative integer. For $1 \leq b \leq p^n$, the number $n_{a,b}$ of indecomposable direct $kG$-module summands of $H^0(X, O_X(E))$ that are isomorphic to $U_{a,b}$ is given as

$$n_{a,b} = \begin{cases} n(a,j) - n(a,j+1) & \text{if } b = (j+1) p^{n-n_1} \text{ and } 0 \leq j \leq p^{n_1} - 2, \\ n(a,p^{n_1} - 1) & \text{if } b = p^{n_1} p^{n-n_1} = p^n, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 1.7. If $z \in Z - Z_{br}$ then $\ell_{y(z),j} = 0$ for $0 \leq j \leq p^{n_1} - 1$. Moreover, for $0 \leq j \leq p^{n_1} - 2$, we have

$$n(a,j) - n(a,j+1) = \sum_{z \in Z_{br}} \left( \max\{\ell_{y(z),j}, \ell_{y(z),j+1}\} \right) + n_j - n_{j+1}$$

where $\epsilon_{z,j} = 1$ if $\ell_{y(z),j} \geq \ell_{y(z),j+1}$ and $\epsilon_{z,j} = -1$ otherwise.

As a consequence of Theorem 1.6, we obtain the following result concerning holomorphic poly-differentials.

Corollary 1.8. Under the assumptions of Theorem 1.6, suppose $m > 1$, $g(X) \geq 2$ and $E = mK_X$, where $K_X$ is as in Notation 1.5(e), so that $\Omega_X^{\otimes m} \cong O_X(mK_X)$ as $O_X$-G-modules. Then, for $0 \leq j \leq p^{n_1} - 1$, the divisor $E_j$ given by Equation (1.1) satisfies $E_j = mK_X + D_j$, where $D_j = \sum_{y \in Y} d_{y,j} y$ is defined by

$$d_{y,j} = \left[ \frac{m \sum_{z \geq 0} (\# I_{x(y),i} - 1) - \sum_{\ell=1}^{p_x(y)} a_{\ell,t} p^{n_x(y)} - \ell i(x(y))}{p^{n_x(y)}} \right].$$

In particular, $d_{y,j} = 0$ when $y$ is not a branch point of $\pi$. Using these $E_j$, Theorem 1.6 determines the precise $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$.

In Remark 2.1, we will use arguments from the proof of Theorem 1.6 to give a more explicit version of the algorithm provided in [5] Remark 4.4 concerning $H^0(X, \Omega_X)$. The two main differences in the proofs of [5] Thm. 1.1 and Remark 4.4 and of Theorems 1.6 and 1.6 including Corollaries 1.7 and 1.8 are as follows, when assuming $k$ and $G$ are as in Assumption 1.3.
(A) Suppose $I$ is not trivial, i.e. the cover $\pi: X \to Y$ is wildly ramified. When considering the sheaf $\Omega_X \cong \mathcal{O}_X(K_X)$ then $H^1(Y, \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y(D_j))$ does not vanish when $j = \#I - 1$, and neither does $H^1(X, \Omega_X)$. On the other hand, when considering the sheaf $\mathcal{O}_X(E)$ then $H^1(Y, \mathcal{O}_Y(E))$ vanishes for all $0 \leq j \leq \#I - 1$. Since $H^1(X, \mathcal{O}_X(E)) = 0$, this is crucial for a dimension argument needed to compare the filtrations of $H^0(X, \mathcal{O}_X(E))$ and of $\mathcal{O}_X(E)$, given by the actions of the powers of the radical of $kI$ (see Notation 2.1).

(B) When considering the sheaf $\Omega_X \cong \mathcal{O}_X(K_X)$ then one can use Serre duality to avoid having to refer to the class of the canonical divisor $K_Z$ modulo principal divisors on $Z$. On the other hand, when $E = mK_X$ for $m > 1$ then the class of $K_Z$ is essential when analyzing the tamely ramified cover $\lambda: Y \to Z$. For general $E$ as in Assumption 1.3 the $G$-divisor class of $E$ is used to analyze $\lambda$.

The paper is organized as follows. In §2 we prove Theorems 1.1 and 1.6 and deduce Corollary 1.8. Moreover, in Remark 2.4 we give a new version of the algorithm for $H^0(X, \Omega_X)$ from [5, Remark 4.4]. In §3 we show in Proposition 3.1 how Corollaries 1.2 and 1.8 can be used to find the $k$-dimension of the tangent space of the global deformation functor associated to $(X,G)$. In §4 we illustrate Corollaries 1.2 and 1.8 and Proposition 5.1 by considering a particular family of hyperelliptic curves. In §5 we discuss non-trivial congruences modulo appropriate maximal ideals between prime level $\ell$ cusp forms of even weight $2m > 2$; see Theorems 5.3 and 5.4. If $p = 3$, we use Corollaries 1.2 and 1.8 to fully determine the $kPSL(2,\mathbb{F}_3)$-module structure of $H^0(X_3(\ell), \Omega_{X_3(\ell)}^{\otimes m})$; see Theorem 5.5 and Propositions 5.5.1 and 5.5.2, which we prove in §6.

Part of this paper is the Ph.D. thesis of the second author under the supervision of the first (see [36]).

We would like to thank the referee for valuable comments that helped improve the paper.

2. Proof of Theorems 1.1 and 1.6 and Corollary 1.8

By using the Conlon induction theorem [10, Thm. (80.51)] and the representation theory of $p$-hypo-elementary groups [11, II.5-II.6], we can reduce the proof of Theorem 1.1 to the case when $G$ is $p$-hypo-elementary and $k$ is algebraically closed (see [5, Lemma 3.2 and Prop. 3.5] for details). Since $H^0(X, \mathcal{O}_X(E))$ and $H^0(X, \mathcal{O}_X(E'))$ are isomorphic $kG$-modules when $E$ and $E'$ lie in the same $G$-divisor class, Theorem 1.1 then follows from Theorem 1.6. As we have seen in the introduction, Corollary 1.2 is a direct consequence of Theorem 1.1.

For the remainder of this section, we suppose Assumption 1.3 holds, and we assume Notations 1.4 and 1.5. In particular,

$$E = \sum_{x \in X} e_xx$$

is a $G$-invariant Weil divisor on $X$ with $\deg(E) > 2g(X) - 2$. The sheaf $\mathcal{O}_X(E)$ is a coherent $\mathcal{O}_X$-$G$-module that is a locally free $\mathcal{O}_X$-module of rank one, and $H^1(X, \mathcal{O}_X(E)) = 0$.

If $x(y) \in X$ lies above $y \in Y$, then $p^{\pi(y)} = \#I_{x(y)}$ is the ramification index of $x(y)$ over $y$. Under the assumptions of Corollary 1.8 this immediately implies that if $E = mK_X$ then the divisor $E_j$ from Equation (1.1) equals $mK_Y + D_j$ with $D_j$ as in Equation (1.6). Hence Corollary 1.8 follows from Theorem 1.6.

It remains to prove Theorem 1.6. We use the geometric method that was introduced in [5, §4] when considering the holomorphic differentials $H^0(X, \Omega_X)$. The first step is to consider the cover $\pi: X \to Y$, which is wildly ramified if $\#I > 1$, and to study a particular filtration of the $G$-sheaf $\pi_*\mathcal{O}_X(E)$. We need the following notation, which was introduced in [5, §2].

Notation 2.1. Suppose $\mathcal{F}$ is a coherent $\mathcal{O}_X$-$G$-module that is a locally free $\mathcal{O}_X$-module of finite rank, and let $\mathcal{J} = kI(\tau - 1)$ be the Jacobson radical of the group ring $kI$. For $0 \leq j \leq \#I$, we denote by $H^0(X, \mathcal{J}^j)$ the kernel of the action of $\mathcal{J}^j = kI(\tau - 1)^j$ on $H^0(X, \mathcal{F})$, and we denote by $\pi_*\mathcal{F}^{(j)}$ the kernel of the action of $\mathcal{J}^j$ on $\pi_*\mathcal{F}$. 5
Since $I$ is a normal subgroup of $G$, $\mathcal{J}^1$ is taken to itself by the conjugation action of $G$ on $I$. It follows that $H^0(X, \mathcal{O}_X(E))(\mathcal{J}^1)$ is a $kG$-module and that $\pi_*\mathcal{O}_X(E)(\mathcal{J}^1)$ is a quasi-coherent $\mathcal{O}_X-G$-module. Moreover, since $\pi_*\mathcal{O}_X(E)(\mathcal{J}^1)$ is a subsheaf of a locally free coherent $\mathcal{O}_X$-module of finite rank, we obtain that $\pi_*\mathcal{O}_X(E)(\mathcal{J}^1)$ is a locally free coherent $\mathcal{O}_X-G$-module.

We now analyze the filtration $\{\pi_*\mathcal{O}_X(E)(\mathcal{J}^j)\}_{j=0}^6$ of the $G$-sheaf $\pi_*\mathcal{O}_X(E)$.

**Proposition 2.2.** Suppose Assumption [1.3] holds and assume Notations [1.3] and [2.1]. Fix $0 \leq j \leq p^n - 1$. The action of $\mathcal{O}_Y$ and of $G$ on $\pi_*\mathcal{O}_X(E)$ makes the quotient sheaf $\mathcal{L}_j := \pi_*\mathcal{O}_X(E)(\mathcal{J}^{j+1})/\pi_*\mathcal{O}_X(E)(\mathcal{J}^j)$ into a locally free coherent $\mathcal{O}_Y-G$-module. Let $E_j$ be the $G$-invariant Weil divisor on $Y$ given by Equation (1.1) in Theorem (4).

1. There is an isomorphism of locally free coherent $\mathcal{O}_Y-G$-modules between $\mathcal{L}_j$ and $S_{X,Y} \otimes_k \mathcal{O}_Y(E_j)$.

2. If $n \geq 1$, i.e. $\# I > 1$, then $\deg(\mathcal{L}_j) = \deg(E_j) > 2g(Y) - 2$, and $H^1(Y, \mathcal{L}_j) = H^1(Y, \mathcal{O}_Y(E_j)) = 0$.

**Proof.** Let $\mathcal{D}^{-1}_{X/Y}$ be the inverse different of $X$ over $Y$. Then $\mathcal{D}^{-1}_{X/Y}$ is a coherent $\mathcal{O}_X-G$-module that is a subsheaf of the constant sheaf $k(X)$. Moreover, $\mathcal{D}^{-1}_{X/Y}$ can be identified with the line bundle $\mathcal{O}_X(\text{Ram}_\pi)$.

Fix $0 \leq j \leq p^n - 1$. Using the same arguments as in the proof of [5] Prop. 4.1], where we replace $\mathcal{D}^{-1}_{X/Y}$ by $\mathcal{O}_X(E)$, we obtain part (i).

To prove part (ii), we show that $\deg(E_j) > 2g(Y) - 2$ by using our assumption that $2g(X) - 2 < \deg(E) = \sum_{x \in X} e_x$. By the Riemann-Hurwitz formula (see also [5] Eq. (4.15)]) and since $E$ is $G$-invariant and $a_{\ell,t} \leq p - 1$ for all $\ell, t$, we obtain

$$2g(Y) - 2 = \frac{1}{p^n} \left( 2g(X) - 2 - \sum_{x \in X} \sum_{\ell=1}^{n_x} \sum_{y=-1}^{p-1} p^{n_x-\ell}(i_{\ell}(x) + 1) \right)$$

$$\leq \sum_{y \in Y} \left( e_{x(y)} - \sum_{\ell=1}^{n_x} p^{n_x-\ell} a_{\ell,t} \right) \frac{p^{n_x-\ell} i_{\ell}(x(y)) - \sum_{\ell=1}^{n_x} (p - 1) p^{n_x-\ell} i_{\ell}(x(y))}{p^{n_x}}$$

$$\leq \sum_{y \in Y} e_{y,j} = \deg(E_j)$$

where $x(y)$ stands for one particular point on $X$ above $y \in Y$, and we use that $(p - 1) \sum_{\ell=1}^{n_x} p^{n_x-\ell} i_{\ell}(x(y)) = p^{n_x} - 1$. 

The next step is to compare the filtration $\{\pi_*\mathcal{O}_X(E)(\mathcal{J}^j)\}_{j=0}^6$ of the $G$-sheaf $\pi_*\mathcal{O}_X(E)$ to the filtration $\{H^0(X, \mathcal{O}_X(E)(\mathcal{J}^j))\}_{j=0}^6$ of the $kG$-module $H^0(X, \mathcal{O}_X(E))$.

We note that $H^1(Y, -)$ applied to the top filtered piece of $\pi_*\mathcal{O}_X$ does not vanish, and neither does $H^1(X, \mathcal{O}_X)$ (see [5] Lemma 4.2]). On the other hand, if we replace $\mathcal{O}_X$ by $\mathcal{O}_X(E)$ as in Assumption [1.3] then part (ii) of Proposition [2.2] shows that $H^1(Y, \mathcal{O}_Y(E_j)) = 0$ for all $0 \leq j \leq p^n - 1$. Additionally, our assumption that $\deg(E) > 2g(X) - 2$ implies that $H^1(Y, \mathcal{O}_X(E)) = 0$.

These two differences balance each other out so we are able to adapt the dimension argument in the proof of [5] Lemma 4.2] to make it work under Assumption [1.3] More precisely, replacing $\mathcal{O}_X$ by $\mathcal{O}_X(E)$ and using Proposition [2.2] instead of [5] Prop. 4.1, the proof of the following result proceeds using the same main steps as in the proof of [5] Lemma 4.2].
Lemma 2.3. Suppose Assumption 1.3 holds and assume Notations 1.4, 1.5 and 2.1. For all $0 \leq j \leq p^{n_1} - 1$, there are isomorphisms

$$H^0(X, O_X(E))^{(j+1)}/H^0(X, O_X(E))^{(j)} \cong H^0(Y, \pi_*O_X(E)^{(j+1)}/\pi_*O_X(E)^{(j)}) \cong S_{\chi\sim j} \otimes_k H^0(Y, O_Y(E_j))$$

of $kG$-modules, where $E_j$ is the divisor from Theorem 1.6 defined by Equation 1.1.

We are now ready to prove Theorem 1.6 using Proposition 2.2 and Lemma 2.3 together with the results of [30] §3 (see also [24] Thm. 4.5) applied to the tamely ramified cover $\lambda: Y \to Z$.

Proof of Theorem 1.6 Under the assumptions of Theorem 1.5, fix $0 \leq j \leq p^{n_1} - 1$, and let $E_j$, $\ell_{g(z),j}$ and $n_j$ be as in Equations 1.1, 1.2 and 1.3, respectively. By [30] Thm. 2 and [24] Thm. 4.5 and using that $H^1(Y, O_Y(E_j)) = 0$, we obtain that $H^0(Y, O_Y(E_j))$ is a projective $kG$-module whose Brauer character $\beta(j)$ is given as

$$\beta(j) = \sum_{z \in Z_g} \left( \ell_{g(z),j} \prod_{d=1}^{c_{g(z)}} \theta_d^{y(z)} - \sum_{d=1}^{c_{g(z)} - 1} \frac{d}{c_{g(z)}} \prod_{\ell \in \ell_{g(z)}} \theta_d^{y(z)} \right) + n_j \beta(kG)$$

where $\beta(kG)$ is the Brauer character of $kG$.

Let $y \in Y$ and $i \in Z$. Since $c_y = \#G_y$ is not divisible by $p$, $\theta_y^i$ is the character of a projective $kG_y$-module, which implies that $\text{Ind}_{G_y}^{Z} \theta_y^i$ is the character of a projective $kG$-module. Moreover, using Frobenius reciprocity and Notation 1.5(d), we see that the socle of this projective $kG$-module equals

$$\bigoplus_{0 \leq a \leq -1} S_a$$

when we view $S_a$ as a $kG$-module. Therefore, the Brauer character $\beta(j)$ in Equation (2.1) is an explicitly given $Q$-linear combination of Brauer characters of projective indecomposable $kG$-modules, which implies that the rational coefficients must in fact be non-negative integers. It follows that, for all $0 \leq a \leq c - 1$, $n(a,j)$ from Equation (1.4) equals the multiplicity of the projective indecomposable $kG$-module with socle $S_a$ as a direct summand of the projective $kG$-module $H^0(Y, O_Y(E_j))$.

Let $M = H^0(X, O_X(E))$, and let $0 \leq a \leq c - 1$ and $1 \leq b \leq p^n$. Since by Lemma 2.3

$$M^{(j+1)}/M^{(j)} \cong S_{\chi\sim j} \otimes_k H^0(Y, O_Y(E_j))$$

as $kG$-modules, we have that $n(a,j)$ from Equation (1.4) equals the multiplicity of the projective indecomposable $kG$-module with socle $S_{\chi\sim -1,a}$ as a direct summand of the projective $kG$-module $M^{(j+1)}/M^{(j)}$.

Since the $k$-dimension of all projective indecomposable $kG$-modules is equal to $p^{n-n_1}$, it follows that if $b$ is not a multiple of $p^{n-n_1}$, then $U_{a,b}$ is not a direct $kG$-module summand of $M$. On the other hand, if $b = (j+1)p^{n-n_1}$ then the description of the ascending composition factors of $U_{a,b}$ from Notation 1.4(c) shows that $U_{a,b}^{(i+1)}/U_{a,b}^{(i)}$ is a projective $kG$-module with socle $S_{\chi\sim -1,a}$ for $0 \leq i \leq j$, and $U_{a,b}^{(i+1)}/U_{a,b}^{(i)} = 0$ for $j+1 \leq i \leq p^{n_1} - 1$. Hence, $n(a,j)$ equals the number of indecomposable $kG$-module summands of $M = H^0(X, O_X(E))$ whose socle equals $S_a$ and whose $k$-dimension is a multiple of $p^{n-n_1}$ that is greater than or equal to $(j+1)p^{n-n_1}$.

Therefore, for $0 \leq j \leq p^{n_1} - 2$, $n(a,j) - n(a,j+1)$ equals the number of indecomposable $kG$-module summands of $H^0(X, O_X(E))$ that are isomorphic to $U_{a,(j+1)p^{n-n_1}}$, and $n(a,p^{n_1} - 1)$ equals the number of indecomposable $kG$-module summands of $H^0(X, O_X(E))$ that are isomorphic to $U_{a,p^n}$. Since there are no additional indecomposable $kG$-modules occurring as direct summands of $H^0(X, O_X(E))$, it follows that $n_{a,b}$ from Equation (1.5) equals the number of indecomposable direct $kG$-module summands of $H^0(X, O_X(E))$ that are isomorphic to $U_{a,b}$. This completes the proof of Theorem 1.6.
Using some of the arguments from the proof of Theorem 1.6 above, we obtain the following more explicit version of the algorithm provided in [5, Remark 4.4] concerning $H^0(X, \Omega_X)$. To distinguish this case from the case discussed in Theorem 1.6, we add decorations $\Omega$ as needed.

**Remark 2.4.** Suppose $k$, $G$ and $X$ are as in Assumption 1.3 and assume Notations 1.4 and 1.5.

For $0 \leq j < p^{\nu_1} - 1$, define a $G$-invariant Weil divisor $D^\Omega_j = \sum_{y \in Y} d^\Omega_{y,j} y$ by

$$d^\Omega_{y,j} = \left[ \frac{\sum_{t=1}^{\nu_1} p^{\nu_1-\ell} (p-1 + (p-1-a_{\ell,t}) \nu(x(y)))}{p^{\nu_1}} \right],$$

where $x(y) \in X$ is a point above $y$, $t \geq 0$ is the unique integer with $p^{\nu_1-n_y} t \leq j < p^{\nu_1-n_y}(t+1)$, and $t = a_{1,t} + a_{2,t} + \cdots + a_{n_y,t}$. Let $p^{\nu_1-1}$ be its $p$-adic expansion, with $a_{1,t}, \ldots, a_{n_y,t} \in \{0,1,\ldots,p-1\}$. For $z \in Z$, let $y(z) \in Y$ be such that $\lambda(y(z)) = z$. Define $c^\Omega_{y(z),j} \in \{0,1,\ldots,c_{y(z)}-1\}$ by

$$c^\Omega_{y(z),j} \equiv -d^\Omega_{y(z),j} \mod c_{y(z)}.$$ 

and define

$$n^\Omega_j = g(Z) - 1 + \sum_{z \in Z} d^\Omega_{y(z),j} + c^\Omega_{y(z),j}. $$

For $0 \leq a \leq c-1$, define

$$n(a,j)^\Omega = \delta_j, p^{\nu_1}-1 \cdot \delta_{a,0} + \sum_{z \in Z} \left( \frac{(c_{y(z)})^{-1} d}{c_{y(z)}} \mu_{a,d}(y(z)) \right) - \sum_{d=1}^\Omega \mu_{a,-d}(y(z)) + n^\Omega_j,$$

and define $\mu_{a,\chi}$ to be $1$ if $S_a \cong S_\chi$ and to be $0$ otherwise.

Then, for $1 \leq b \leq p^n$, the number $n^\Omega_{a,b}$ of indecomposable direct $kG$-module summands of $H^0(X, \Omega_X)$ that are isomorphic to $U_{a,b}$ is given as

$$n_{a,b}^\Omega = \begin{cases} n(a,j)^\Omega - n(a,j+1)^\Omega & \text{if } b = (j+1) p^{n-1} \text{ and } 0 \leq j \leq p^{\nu_1} - 2, \\ n(a, p^{\nu_1} - 1)^\Omega + (\delta_{a,n_1} - 1) \cdot \mu_{a,\chi} & \text{if } b = (p^{\nu_1} - 1) p^{n-1} + 1 \text{ and } n_1 < n, \\ 0 & \text{otherwise.} \end{cases}$$

As in Remark 1.7 if $z \in Z - Z_{br}$ then $\delta^\Omega_{y(z),j} = 0$ for $0 \leq j < p^{\nu_1} - 1$. Moreover, for $0 \leq j \leq p^{\nu_1} - 2$,

$$n(a,j)^\Omega - n(a,j+1)^\Omega = -\delta_j, p^{\nu_1}-2 \cdot \delta_{a,0} + \sum_{z \in Z} \left( \frac{\max\{d^\Omega_{y(z),j+1} \leq d^\Omega_{y(z),j+1} \leq \min\{d^\Omega_{y(z),j+1} \}}{c^\Omega_{c,z,j}} \right) \mu_{a,-d}(y(z)) \right) + n^\Omega_j,$$

where $c^\Omega_{c,z,j} = 1$ if $d^\Omega_{y(z),j} \leq d^\Omega_{y(z),j+1}$ and $c^\Omega_{c,z,j} = -1$ otherwise.

To prove the formula in Equation 2.4, we note that the definitions in Equations 2.2, 2.3 and 2.4 coincide with the corresponding definitions in [5] Remark 4.4, where we use the Riemann-Hurwitz formula to obtain Equation 2.4. Let $M = H^0(X, \Omega_X)$, and let $0 \leq a \leq c-1$ and $1 \leq b \leq p^n$.

Suppose first that $j < p^{\nu_1} - 1$ or $n_1 = n$. By Step (2) of [5] Remark 4.4, it follows that $M^{(j+1)}/M^{(j)}$ is a projective $k\overline{G}$-module. Moreover, since $S_{c-a}$ is the $k$-dual of $S_a$, we can use Step (2) of [5] Remark 4.4, together with similar arguments as in the proof of Theorem 1.6 above, to see that $n(a,j)^\Omega$ from Equation 2.5 equals the multiplicity of the projective indecomposable $kG$-module with socle $S_{\chi^{-1}-j,a}$ as a direct summand of the projective $k\overline{G}$-module $M^{(j+1)}/M^{(j)}$.

Suppose next that $j = p^{\nu_1} - 1$ and $n_1 < n$. Since the order of the character $\chi$ divides $p-1$, $\chi^{-1} \cdot (p^{\nu_1} - 1)$ is the trivial character. By Step (2) of [5] Remark 4.4, it follows that $M^{(p^{\nu_1})}/M^{(p^{\nu_1} - 1)}$ is a direct sum of $S_{\chi}$ and a projective $k\overline{G}$-module. Moreover, we can use Step (2) of [5] Remark 4.4 and similar arguments as in the proof
of Theorem 1.6 above, to see that \( n(a, p^n - 1)^\Omega - \mu_{a, \chi} \) equals the multiplicity of the projective indecomposable \( kG \)-module with socle \( S_{\chi^{-1}(p^n - 1), a} = S_a \) as a direct summand of \( M(p^n - 1) / M(p^n - 1) \). Moreover, \( n(a, p^n - 1)^\Omega \) equals the multiplicity of \( S_{\chi^{-1}(p^n - 1), a} = S_a \) as a direct summand of the socle of \( M(p^n - 1) / M(p^n - 1) \).

Using similar arguments as in the proof of Theorem 1.6 it follows that \( n_{a,b}^\Omega \) from Equation 2.6 equals the number of indecomposable direct \( kG \)-module summands of \( \text{H}^0 (X, \Omega_X^X) \) that are isomorphic to \( U_{a,b} \).

We end this section with a computational remark, which will be useful in the next sections.

**Remark 2.5.** Suppose Assumption 1.3 holds, and assume Notations 1.4 and 1.5.

(a) By applying the Riemann-Hurwitz formula, \( n_j \) from Equation (1.3) can also be written as

\[
n_j = \frac{\text{deg}(E_j) + 1 - g(Y)}{\#G} + \sum_{x \in \mathbb{Z}_n} \frac{1}{z^{\gamma(z)}} \left( \frac{c(y(z) - 1)}{2} - f(y(z), j) \right)
\]

which in some applications may be easier to use (see, for example, §6.2).

(b) Let \( y \in Y \) be a ramification point of the tame cover \( \lambda : Y \to Z \), and let \( x \in X \) be a point above it. If \( I_x \) (resp. \( G_x \)) is the inertia group of \( x \) inside \( I \) (resp. \( G \)), we obtain \( \mathcal{G}_y \cong G_x / I_x \). The fundamental character \( \theta_x : G_x \to k(x)^* = k^* \) is given by \( \theta_x (g) = \frac{2 \pi_x}{\pi_x} \mod m_{X,x} \) where \( \pi_x \) is a uniformizer and \( m_{X,x} \) is the maximal ideal of \( \mathcal{O}_{X,x} \). Because \( k^* \) has no non-identity elements of \( p \)-power order, \( \theta_x \) factors through \( G_x / I_x \) which we identify with \( \mathcal{G}_y \). Since \( \text{ord}_x (\pi_y) = \#I_x \) and since \( k(y) = k(x) = k \) is algebraically closed, we can identify \( \theta_y = (\theta_x)^{\#I_x} \) when viewing both as characters of \( \mathcal{G}_y = G_x / I_x \).

3. AN APPLICATION TO DEFORMATION THEORY

In this section, we show an application of Corollaries 1.2 and 1.8 to deformation theory and the equivariant deformation problem introduced in [1]. Let \( X \) be a smooth projective curve defined over an algebraically closed field \( k \) of prime characteristic \( p \), and suppose \( G \) is a finite group acting faithfully on the right on \( X \) over \( k \). Suppose \( C \) is the category of commutative local Artinian \( k \)-algebras with residue field \( k \). Let \( D_{gl} : C \to \text{Sets} \) be the global deformation functor associated to the pair \((X, G)\) as introduced in [1] §2, and let \( t_{D_{gl}} \) be its tangent space. Using the arguments in [27] §3 and in [20] §7, we obtain, as in the beginning of the proof of [20] Thm. 7.1], that

\[
\dim_k t_{D_{gl}} = \dim_k \text{H}^0 (X, \Omega_X^{2X})_G
\]

where the subscript \( G \) denotes the \( G \)-coinvariants of a \( kG \)-module.

**Proposition 3.1.** Suppose \( G \) has non-trivial cyclic Sylow \( p \)-subgroups. If \( g(X) \geq 2 \) then the \( k \)-dimension of the tangent space of the global deformation functor \( D_{gl} \) is uniquely determined by the class of canonical divisors on \( X/G \) modulo principal divisors, together with the lower ramification groups and the fundamental characters of the closed points of \( X \) that are ramified in the cover \( X \to X/G \).

Under the hypotheses of Theorem 1.6 we obtain

\[
\dim_k t_{D_{gl}} = \sum_{j=0}^{p^n - 1} n_{\chi^j, 0, (j+1)p^n - n_j}
\]

where \( \chi^j.0 \) is as in Notation 1.4 (c) and \( n_{\chi^j, 0, (j+1)p^n - n_j} \) is given by Equation 1.5 from Theorem 1.6 applied to \( E = 2K_X \).

**Proof.** The first paragraph of the statement follows from Corollary 1.2 using Equation (5.1).

Assume now the hypotheses of Theorem 1.6 and let \( E = 2K_X \). By Equation (1.5) in Theorem 1.6 the number of indecomposable direct \( kG \)-module summands of \( \text{H}^0 (X, \Omega_X^{2X}) \) that are isomorphic to \( U_{a,b} \) (for \( 0 \leq a \leq c - 1 \) and \( 1 \leq b \leq p^n \)) is given by \( n_{a,b} \). Since taking \( G \)-coinvariants defines an additive functor
$\langle - \rangle_G$ from the category of finitely generated $kG$-modules to the category of $k$-vector spaces, it follows from Equation (3.1) that
\[
\dim_k t_{D_{g\ell}} = \sum_{a=0}^{c-1} \sum_{b=1}^{n} n_{a,b} \dim_k (U_{a,b})_G
\]
where $n_{a,b}$ is as in Equation (1.5) from Theorem 1.6. As discussed prior to Notation 1.1, each $U_{a,b}$ is uniserial and its radical is equal to $(\sigma - 1)U_{a,b}$. This implies that $(U_{a,b})_G \neq 0$ if and only if $U_{a,b}/\text{rad}(U_{a,b}) \cong S_0$ is the trivial simple $kG$-module. Using the description of the ascending composition factors of $U_{a,b}$ from Notation 1.3(c), we have
\[
U_{a,b}/\text{rad}(U_{a,b}) \cong S_{\chi^{-b+1},a}
\]
The latter module is isomorphic to $S_0$ if and only if $\chi^{-b+1,a} = 0$, which is equivalent to $a = \chi^{-b+1,0}_G$. We obtain
\[
\dim_k t_{D_{g\ell}} = \sum_{b=0}^{n} \dim_k (U_{-b,0})_G.
\]
By Equation (1.5) from Theorem 1.6, $n_{-b,0} = 0$ unless $b = (j+1)p^{n-t-1} - 1$. Since the order of the character $\chi$ divides $p-1$, it follows that $\chi^{p_n-t-1} = \chi$, which implies $\chi^t = \chi$. Hence the formula in Equation (3.2) follows.

**Remark 3.2.** Proposition 3.1 generalizes [24, Cor. 2.3] from cyclic $p$-groups to groups $G$ with cyclic Sylow $p$-subgroups, and it gives a formula when $G$ is $p$-hypo-elementary. In [25, Thm. 1.1], the $k$-dimension of $t_{D_{g\ell}}$ has been determined for any finite group $G$ satisfying the additional assumption that $\dim_k M_G = \dim_k M_G$ for all finitely generated $kG$-modules $M$. However, for a $p$-hypo-elementary group $G$, this additional assumption does not hold in general.

## 4. A FAMILY OF HYPERELLiptIC CURVES

In this section, we illustrate Corollaries 1.2 and 1.8 and Proposition 3.1 by considering a particular family of hyperelliptic curves. We freely utilize the notation introduced in §1 and §2 without referring to these sections explicitly. We make the following assumptions throughout this section.

**Assumption 4.1.** Let $m > 1$ be an integer, let $p > 3m$ be a prime number, and let $k$ be an algebraically closed field of characteristic $p$. Write the function field of $\mathbb{P}^1_k$ as $k(\mathbb{P}^1_k) = k(t)$, and consider the quadratic cover $X$ of $\mathbb{P}^1_k$ with function field $k(X) = k(t)\sqrt{f(t)}$ where
\[
f(t) = t^p - t - \prod_{\alpha \in \mathbb{F}_p} (t - \alpha).
\]
Let $P = \langle \sigma \rangle$ be a cyclic group of order $p$, and let $C = \langle \rho \rangle$ be a cyclic group of order $2(p-1)$. Fix an isomorphism $\xi : C \xrightarrow{\cong} \mu_{2(p-1)} \subset k^*$ and define $\chi : C \rightarrow \mu_{p-1} = \mathbb{F}_p^* \subset k^*$ to be $\chi = \xi^2$. Define $G$ to be the semidirect product $G = P \rtimes C$, and let the generators $\sigma$ and $\rho$ of $G$ act on $k(X)$ as
\[
\begin{align*}
\sigma(t) &= t + 1, \\
\rho(t) &= \chi^{-1}(t)\rho(t).
\end{align*}
\]

In §4.1, we first determine the data needed to apply Theorem 1.6 when $E = mK_X$. In §4.2, we then use this theorem to determine the precise $kG$-module structure for $H^0(X, \Omega_X^\otimes m)$. In §4.3, we use Proposition 3.1 to determine the $k$-dimension of the tangent space $t_{D_{g\ell}}$. 


4.1. Lower ramification groups, fundamental characters, and canonical divisors. We first determine the lower ramification groups of the action of $G$ on $X$. Define $\nu = \rho^{p-1} \in C$. Then $\nu$ has order 2 and acts on $k(X)$ as $\nu(t) = t$ and $\nu(\sqrt{f}) = -\sqrt{f}$, which implies $k(X)^{(\nu)} = k(t)$. Using that $\sigma(t^p - t) = t^p - t$ and that the character $\chi$ has order $p-1$, it moreover follows that $k(X)^{(\sigma, \nu)} = k(t^p - t)$ and $k(X)^G = k((t^p - t)^{p-1})$. Hence we can identify $X/\langle \nu \rangle$, $X/(\sigma, \nu)$ and $X/G = Z$ with $\mathbb{P}_{k}^1$. We see that the cover $X \rightarrow X/\langle \nu \rangle$ has precisely $p^2 + 1$ branch points given by $A^1(\mathbb{F}_{p^2}) \cup \{\infty\} \subset X/\langle \nu \rangle$, which are all totally ramified. On the other hand, the cover $X/\langle \nu \rangle \rightarrow X/(\sigma, \nu)$ has precisely one branch point, which is totally ramified. This branch point lies below $\infty \in X/\langle \nu \rangle$. Finally, the cover $X/(\sigma, \nu) \rightarrow X/G = Z$ has two branch points, which are totally ramified. These branch points lie below $A^1(\mathbb{F}_{p^2}) \cup \{\infty\} \subset X/\langle \nu \rangle$. Hence, the branch points of the cover $X \rightarrow X/G = Z$ are the points on $Z$ lying below $A^1(\mathbb{F}_{p^2}) \cup \{\infty\} \subset X/\langle \nu \rangle$. We note that $\alpha^p - \alpha = (\Phi - 1)(\alpha)$ for all $\alpha \in k$, where $\Phi$ is the Frobenius automorphism of $k$. Since $\Phi$ restricts to the unique order 2 automorphism of $\mathbb{F}_{p^2}$, it follows that $(\alpha^p - \alpha)^{p-1}$ equals 0 (resp. $-1$) if $\alpha \in \mathbb{F}_p$ (resp. $\alpha \in \mathbb{F}_{p^2} - \mathbb{F}_p$). Therefore, the cover $X \rightarrow X/G = Z$ has 3 branch points given by $z_0 = 0$, $z_1 = -1$ and $z_\infty = \infty$.

For $u \in \{0, 1, \infty\}$, let $x_u \in X$ be above $z_u$. We claim we have the following lower ramification groups

$$
\begin{align*}
G_{x_0} &= G_{x_0,0} = \langle \rho \rangle, \quad G_{x_0,1} = 1, \\
G_{x_1} &= G_{x_1,0} = \langle \nu \rangle, \quad G_{x_1,1} = 1, \\
G_{x_\infty} &= G_{x_\infty,0} = G, \quad G_{x_\infty,1} = \emptyset, \quad G_{x_\infty,2} = P, \quad G_{x_\infty,3} = 1.
\end{align*}
$$

(4.1)

The discussion in the previous paragraph gives us the inertia groups $G_{x_u} = G_{x_u,0}$ for $u \in \{0, 1, \infty\}$. This leads to the first two rows of Equation (4.1) since the inertia groups of $x_0$ and $x_1$ are $\rho'$-groups. For the third row, we use that $f = t^p + 1(t^{-1} - t^{-p^2})$ to obtain that $\pi_\infty = \sqrt{t^{-(p^2+1)/2}}$ is a uniformizer of $O_{X,x_\infty}$. We have

$$
\frac{\sigma, \pi_\infty}{\pi_\infty} = \frac{\sqrt{t}((t + 1) - (t^{-1} - t^{-p^2}))/2}{\sqrt{t} - (-(p^2+1)/2)} = (1 + t^{-1}) - (p^2+1)/2 = 1 - \frac{p^2 + 1}{2} t^{-1} + \cdots
$$

and hence

$$
\sigma, \pi_\infty \equiv \pi_\infty \mod \left(\frac{p^2 + 1}{2} t^{-1} \pi_\infty\right).
$$

Since

$$
\text{ord}_{x_\infty} \left(\frac{p^2 + 1}{2} t^{-1} \pi_\infty\right) = 2 + 1 = 3,
$$

we obtain $P_{x_\infty,i} = P$ for $i \in \{0, 1, 2\}$ and $P_{x_\infty,i} = 1$ for $i \geq 3$, which leads to the third row of Equation (4.1).

Since $x_\infty \in X$ is the only point for which the order of its inertia group is divisible by $p$, it follows that $I = P$ is the greatest among the Sylow $p$-subgroups of the inertia groups of all closed points on $X$. Hence $I = I_{x_\infty} = P = \langle \sigma \rangle$ and $x_\infty$ is the unique ramification point of the cover $\pi : X \rightarrow Y = X/I$.

For $u \in \{0, 1, \infty\}$, let $y_u = \pi(x_u)$. If $G = G/I$, the orbits $\overline{G}, y_0$ and $\overline{G}, y_\infty$ are each singletons, and the orbit $\overline{G}, y_1$ contains $p - 1$ points of $Y$. The points in these orbits are the only ramification points of the cover $\lambda : Y \rightarrow Z = X/G$. Identifying $\overline{G} = \langle \rho \rangle$, we obtain the following inertia groups:

$$
\overline{G}, y_0 = \langle \rho \rangle = \overline{G}, y_\infty \quad \text{and} \quad \overline{G}, y_1 = \langle \nu \rangle.
$$

(4.2)

By the Riemann-Hurwitz formula, it follows that $g(Y) = (p-1)/2$. For $u \in \{0, 1, \infty\}$, we obtain the following uniformizer $\pi_u$ of $O_{X,x_u}$, where $\pi_\infty$ is as above:

$$
\pi_0 = \sqrt{f} = \pi_1 \quad \text{and} \quad \pi_\infty = \sqrt{t^{-(p^2+1)/2}}.
$$

By Remark 2.5(b), for each $u \in \{0, 1, \infty\}$, we can identify the corresponding fundamental character $\theta_{y_u}$ with $\langle \theta_{x_u} \rangle \# I_{x_u}$ on $\overline{G}, y_u = G_{x_u}/I_{x_u}$. Considering the action of $\rho$ on the uniformizer $\pi_u$, this leads to the
fundamental characters

\[
\begin{cases}
\theta_{y_0} = \xi^{-1}, \\
\theta_{y_1} = \text{Res}_{[\nu]}(\xi^{-1}), \\
\theta_{y_\infty} = \left(\xi^{-1}, \chi^\nu(\nu^2+1)/2\right) = \xi^p = \xi^p,
\end{cases}
\]

where we use that the order of \( \xi \) is \( 2p - 2 \) and that \( p^3 \equiv p \mod 2p - 2 \). Using Equations \ref{eq:12} and \ref{eq:13}, we obtain the following values for \( c_{y_0} \) and \( \varphi(y_0) \) from Notation \ref{eq:1,3}(d):

\[
\begin{cases}
\varphi = 0 - 2p - 2, \\
\varphi = 2p - 3, \\
\varphi = 1, \\
\varphi = 2p - 2, \\
\varphi = p.
\end{cases}
\]

Since \( z_\infty = \infty \) on \( Z = \mathbb{P}^1_k \), we have that \( K_2 = -2z_\infty \) is a canonical divisor on \( Z \). Therefore, \( K_Y = \lambda^*(-2z_\infty) + \text{Ram}_\lambda \) is a canonical divisor on \( Y \). Using Equation \ref{eq:1,2} to determine \( \text{Ram}_\lambda \), we obtain

\[
K_Y = (2p - 3)y_0 + \sum_{y \in \mathcal{G}, y_0} y - (2p - 1)y_\infty.
\]

4.2. The \( kG \)-module structure of the holomorphic poly-differentials. We use Corollary \ref{cor:1,8} and in particular Theorem \ref{thm:1,6} applied to \( E = mK_X \), to determine the \( kG \)-module structure for \( H^0(X, \Omega^\otimes_n X) \) for all \( m > 1 \). As seen in \ref{par:4,1} \( I = P = \langle \sigma \rangle \), and hence \( n_I = 1 \). Moreover, \( y_\infty \in Y \) is the only branch point of \( \lambda : X \to Y \), and \( I_{x_\infty,i} = I \) for \( i \in \{0, 1, 2\} \) and \( I_{x_\infty,i} = 1 \) for \( i \geq 3 \). By Corollary \ref{cor:1,8} we have \( p \) divisors \( D_j \) on \( Y \), for \( 0 \leq j \leq p - 1 \), which are given as \( D_j = d_{y_\infty,j} y_\infty \) where

\[
d_{y_\infty,j} = \left\lfloor \frac{3m(p-1)-2j}{p} \right\rfloor = \left\lfloor \frac{3m-3+2j}{p} \right\rfloor.
\]

Since we assume \( p > 3m \), we can rewrite this as follows. Define

\[
\delta_m = \left\{ \begin{array}{ll}
0, & m \text{ even}, \\
1, & m \text{ odd},
\end{array} \right.
\]

and define

\[
\begin{align*}
A_1 &= \left\{ 0, 1, \ldots, p-3m-1 + \frac{\delta_m}{2} \right\}, \\
A_2 &= \left\{ \frac{p-3m-1+\delta_m}{2} + 1, \ldots, 2p-3m-\delta_m \right\}, \\
A_3 &= \left\{ \frac{2p-3m-\delta_m}{2} + 1, \ldots, p-1 \right\}.
\end{align*}
\]

Then \( D_j = (3m - \ell) y_\infty \) if \( j \in A_\ell \) for \( 1 \leq \ell \leq 3 \).

We now apply Theorem \ref{thm:1,6} to \( E = mK_X \). We have \( Z_{\text{br}} = \{ z_0, z_1, z_\infty \} \). Using that \( E_j = mK_Y + D_j \), for \( 0 \leq j \leq p - 1 \), together with Equations \ref{eq:4,4} and \ref{eq:4,5}, we obtain, for \( 0 \leq j \leq p - 1 \),

\[
\ell_{y_\infty,j} = \left\{ \begin{array}{ll}
2p-2-m, & u = 0, \\
\delta_m, & u = 1, \\
2m-\ell, & u = \infty, j \in A_\ell \text{ for } 1 \leq \ell \leq 3.
\end{array} \right.
\]

Since \( \ell_{y,j} = 0 \) for all \( y \in Y \) above \( z \in Z - Z_{\text{br}} \), we have, using Equation (4.5),

\[
n_j = 1 - 0 + \frac{m(2p-3)-(2p-2-m)}{2p-2} + \frac{m-\delta_m}{2} + \frac{-m(2p-1)+(3m-\ell)-(2m-\ell)}{2p-2} = \frac{m-\delta_m}{2}.
\]
Using Equation (1.4) and Remark 1.7, it follows immediately that
\[ n_{a,b} = 0 \quad \text{for} \quad b \not\in \left\{ \frac{p-3m-1-\delta_m}{2} + 1, \frac{2p-3m-\delta_m}{2} + 1, p \right\}. \]
Additionally,
\[ n_{a, \frac{p-3m+1+\delta_m}{2}} = \mu_{a-(2m-1)}(y_{\infty}) \quad \text{and} \quad n_{a, \frac{2p-3m+2+\delta_m}{2}} = \mu_{a-(2m-2)}(y_{\infty}). \]
By Equation (1.3), \( \mu_{a-(2m+1)}(y_{\infty}) = 1 \) if and only if \( a \equiv (-2m+1)p \mod (2p-2) \and \mu_{a-(2m+2)}(y_{\infty}) = 1 \) if and only if \( a \equiv (-2m+2)p \mod (2p-2) \). Since \((2-2m+1)p = -m(2p-2) + 2p - 2m \) and \((2-2m+2)p = -m(2p-2) + 2p - 2m \), the only non-projective indecomposable \( kG \)-modules that are direct summands of \( H^0(X, \Omega_X^2) \) are \( U_{p-2m, \frac{p-3m+1+\delta_m}{2}} \and U_{2p-2m, \frac{p-3m+2+\delta_m}{2}} \) and they each occur with multiplicity one.

It remains to determine the number \( n_{a,p} \) of projective indecomposable direct \( kG \)-module summands of \( H^0(X, \Omega_X^2) \) of the form \( U_{a,p} \), for \( 0 \leq a \leq 2p-3 \). Since \( n_{a,p} = n(a, p-1) \), we obtain, using Equation (1.3),
\[
\begin{align*}
n_{a,p} &= \sum_{1 \leq a < p} \left( \frac{t_{y_1,p-2,m}}{d_{y_1}} \mu_{a-d}(y_1) - \sum_{d=1}^{2p-3} \frac{d}{2p-2} \mu_{a,d}(y_1) \right) + n_{p-1} \\
&= \left( \sum_{d=1}^{2p-3} \frac{d}{2p-2} \mu_{a,d}(y_1) \right) + \left( \delta_m \mu_{a-1}(y_1) - \frac{1}{2} \mu_{a,1}(y_1) \right) + \frac{m - \delta_m}{2}.
\end{align*}
\]
Since \( 2p-3 \equiv -1 \mod (2p-2) \), we obtain, by Equation (1.4), that \( \mu_{a-d}(y_0) = 1 \) if and only if \( a = d \) and \( \mu_{a,d}(y_0) = 1 \) if and only if \( a = 2p - 2 - d \). Moreover, \( \mu_{a-1}(y_1) = \mu_{a,1}(y_1) = 1 \) if and only if \( a \) is odd. To determine, \( \mu_{a-d}(y_1) \) and \( \mu_{a,d}(y_1) \), we use the following congruences:
\[
\begin{align*}
t &= 2p - 2 - t \mod (2p-2), \quad \text{for} \quad 0 \leq t \leq 2p-3, \\
p(2i) &= 2i \mod (2p-2), \quad \text{for} \quad 0 \leq i \leq p-2, \\
p(2i+1) &= 2i + p \mod (2p-2), \quad \text{for} \quad 0 \leq i \leq \frac{p-3}{2}, \\
p(2i+1) &= 2i + 2 - p \mod (2p-2), \quad \text{for} \quad \frac{p-1}{2} \leq i \leq p-2.
\end{align*}
\]
For \( 0 \leq i \leq p-2 \), this implies \( \mu_{a-2i}(y_1) = 1 \) if and only if \( a = 2p - 2 - 2i \) and \( \mu_{a,2i}(y_1) = 1 \) if and only if \( a = 2i \). Similarly, for \( 0 \leq i \leq \frac{p-3}{2} \), we obtain \( \mu_{a-(2i+1)}(y_1) = 1 \) if and only if \( a = p - 2 - 2i \) and \( \mu_{a,2i+1}(y_1) = 1 \) if and only if \( a = 2i + p \). Finally, for \( \frac{p-1}{2} \leq i \leq p-2 \), we have \( \mu_{a-(2i+1)}(y_1) = 1 \) if and only if \( a = 3p - 4 - 2i \) and \( \mu_{a,2i+1}(y_1) = 1 \) if and only if \( a = 2i + 2 - p \).

Suppose first that \( 0 \leq i \leq p - 2 \). Then
\[
n_{2i,p} = \left( \frac{1 - \frac{2p-2-2i}{2p-2}}{2} \right) + (0 - 0) + \left( \frac{1 - \frac{2i}{2p-2}}{2} \right) + \frac{m - \delta_m}{2} = 1 + \frac{m - \delta_m}{2}
\]
provided \( 2i \leq 2p - 2 - m \) and \( 2p - 2 - 2i \leq 2m - 3 \), which is equivalent to \( p - m + 1 \leq i \leq p - 1 - \frac{m + \delta_m}{2} \).

For all other \( i \) with \( 0 \leq i \leq p - 2 \), we have to subtract 1, leading to \( n_{2i,p} = \frac{m - \delta_m}{2} \).

Suppose next that \( 0 \leq i \leq \frac{p-3}{2} \). Since we have assumed that \( p > 3m \), it follows that \( 2i + 1 \leq 2p - 2 - m \). Therefore,
\[
n_{2i+1,p} = \left( \frac{1 - \frac{2p-2-2i-1}{2p-2}}{2} \right) + \left( \delta_m - \frac{1}{2} \right) + \left( \frac{1 - \frac{2i+p}{2p-2}}{2} \right) + \frac{m - \delta_m}{2} = \frac{m + \delta_m}{2}
\]
provided \( p - 2 - 2i \leq 2m - 3 \), which is equivalent to \( \frac{p+1}{2} - m \leq i \). For all other \( i \) with \( 0 \leq i \leq \frac{p-3}{2} \), we have to subtract 1, leading to \( n_{2i+1,p} = \frac{m + \delta_m}{2} - 1 \).
Finally, suppose that \( \frac{p-1}{2} \leq i \leq p-2 \). Since we have assumed that \( p > 3m \), it follows that \( 3p - 4 - 2i > 2m - 3 \). Therefore,

\[
n_{2i+1,p} = \left( 1 - \frac{2p - 2 - 2i - 1}{2p - 2} \right) + \left( \frac{\delta_m - 1}{2} \right) + \left( \frac{0 - 2i + 2 - p}{2p - 2} \right) + \frac{m - \delta_m}{2} = \frac{m + \delta_m}{2}
\]

provided \( 2i + 1 \leq 2p - 2 - m \), which is equivalent to \( i \leq p - 2 - \frac{m - \delta_m}{2} \). For all other \( i \) with \( \frac{p-1}{2} \leq i \leq p-2 \), we have to subtract 1, leading to \( n_{2i+1,p} = \frac{m + \delta_m}{2} - 1 \).

We obtain the following result.

**Proposition 4.2.** Under Assumption 4.1 there is an isomorphism of \( kG \)-modules

\[
H^0(X, \Omega^\otimes_X) \cong \bigoplus_{i \in E_1} \left( \frac{m - \delta_m}{2} + 1 \right) U_{2i,p} \oplus \bigoplus_{i \in E_2} \left( \frac{m - \delta_m}{2} \right) U_{2i,p}
\]

where \( \delta_m \) is as in Equation (4.6) and

\[
E_1 = \{ p - m + 1, p - m + 2, \ldots, p - 1 - \frac{m + \delta_m}{2} \}, \quad E_2 = \{ 0, 1, \ldots, p - 2 \} - E_1,
\]

\[
O_1 = \{ \frac{p+1}{2} - m, \frac{p+1}{2} - m + 1, \ldots, p - 2 - \frac{m - \delta_m}{2} \}, \quad O_2 = \{ 0, 1, \ldots, p - 2 \} - O_1.
\]

### 4.3. The tangent space of the global deformation functor

Under Assumption 4.1, we now use Proposition 4.1 to determine the dimension of the tangent space \( t_{D_{gl}} \) of the global deformation functor \( D_{gl} \) associated to the pair \((X, G)\). We use Proposition 4.2 for \( m = 2 \).

Since \( \chi = \xi^2 \), we have \( \chi^j.0 = 2j \) for \( 0 \leq j \leq p - 2 \) and \( \chi^{p-1}.0 = 0 \). Because \( n = n_I = 1 \), Equation (3.2) becomes

\[
\dim_k t_{D_{gl}} = \sum_{j=0}^{p-2} n_{2j,j+1} + n_{0,p}.
\]

By Proposition 4.2 applied to \( m = 2 \), the coefficient \( n_{0,p} \) of \( U_{0,p} \) equals \( \frac{m - \delta_m}{2} = 1 \). Moreover, the only non-zero coefficients of non-projective direct \( kG \)-module summands of \( H^0(X, \Omega^\otimes_X) \) are

\[
n_{p-4, \frac{p+1}{2}} \quad \text{and} \quad n_{2p-4, p-2}.
\]

Since \( p - 4 \) is odd and since \( 2p - 4 = 2(p - 2) \) but \( p - 2 \neq (p - 2) + 1 \), we obtain the following result.

**Proposition 4.3.** Suppose Assumption 4.1 holds, and let \( P_0 \) be the projective \( kG \)-module cover of the trivial simple \( kG \)-module \( S_0 \). The \( k \)-dimension of the tangent space \( t_{D_{gl}} \) of the global deformation functor \( D_{gl} \) associated to the pair \((X, G)\) is equal to the multiplicity of \( P_0 \) as a direct summand of \( H^0(X, \Omega^\otimes_X) \), which equals one.

### 5. Holomorphic poly-differentials of the modular curves \( X(\Gamma_0) \) modulo \( p \)

In this section and the next, we consider modular curves in prime characteristic \( p \). We show that Corollary 1.2 has applications to congruences between prime level holomorphic cusp forms in characteristic 0 of even weight \( 2m > 2 \) (see Theorems 5.3 and 5.4). To prove these results, we give a solution to Hecke’s classical problem 1.3 of determining the module structure of such cusp forms when the characteristic is \( p = 3 \) (see Theorem 5.8 and Propositions 5.5.1 and 5.5.2). Together with the results in [5], and using [19] or [30] for \( p > 3 \), it follows that the only remaining open case for Hecke’s problem in all characteristics is when \( p = 2 \).

For the remainder of this section, we will use the following assumptions and notations (see [5] §5).
Assumption 5.1. We assume $\ell \geq 3$ is a prime number with $\ell \neq p$. Let $F$ be a number field that is unramified over $p$ and that contains a primitive $\ell$th root of unity $\zeta_\ell$. We assume $A \subset F$ is a Dedekind domain whose fraction field equals $F$ and that contains $\mathbb{Z}[\frac{1}{\ell}, \zeta_\ell]$. Let $\mathcal{V}(F, p)$ be the set of places of $F$ over $p$. For each $v \in \mathcal{V}(F, p)$, let $\mathcal{O}_{F,v}$ be the ring of integers of the completion $F_v$ of $F$ at $v$ and let $\mathfrak{m}_{F,v}$ be its maximal ideal. For all $v \in \mathcal{V}(F, p)$, we assume that $\mathcal{O}_F \cap \mathfrak{m}_{F,v}$ is a proper smooth canonical model $X$ in the sense of [\textit{Cor. 3.2}] for an explicit formula.

Notation 5.2. Suppose Assumption [5.1] holds. Let $\Gamma = \text{SL}(2, \mathbb{Z})$, and let $\Gamma_\ell$ be the principal congruence subgroup of $\Gamma$ of level $\ell$. Let $\Gamma_\ell = \text{SL}(2, \mathbb{F}_\ell) = \Gamma / \Gamma_\ell$, and let $\ell$ be the reduction of $\Gamma_\ell$ to $\Gamma$. Let $\ell = \text{PSL}(2, \mathbb{F}_\ell)$ be the reduction of $\Gamma_\ell$ to $\Gamma$. Let $\mathcal{O}_{F,v} \otimes_A H^0(\mathcal{X}_A(\ell), \Omega_{\mathcal{X}_A(\ell)}^{\otimes m})$ be a projective $B$-module whose local results and congruences. Let $v \in \mathcal{V}(F, p)$, let $k$ be an algebraically closed field containing $k(v)$, and let $X_p(\ell)$ be as in Equation (5.1). We obtain the following result about the structure of the holomorphic poly-differentials of $X_A(\ell)$ viewed as a module for $G$ over the local ring $\mathcal{O}_{F,v}$.

Theorem 5.3. Suppose Assumption [5.1] holds, and let $p \geq 3$ and $m > 1$. The $\mathcal{O}_{F,v}G$-module

$$\mathcal{O}_{F,v} \otimes_A H^0(\mathcal{X}_A(\ell), \Omega_{\mathcal{X}_A(\ell)}^{\otimes m}) = \mathcal{O}_{F,v} \otimes_A S_{2m}(A)$$

is a direct sum over blocks $B$ of $\mathcal{O}_{F,v}G$ of modules of the form $P_B \oplus U_B$, in which $P_B$ is a projective $B$-module and $U_B$ is either the zero module or a single indecomposable non-projective $B$-module. If $p > 3$ then $U_B = \{0\}$. For all $p \geq 3$, $P_B$ is uniquely determined by the lower ramification and the fundamental characters of $X_p(\ell)$ are ramified in the cover $X_p(\ell) \rightarrow X_p(\ell)/G$. Moreover, if $p = 3$ then also the reduction $U_B$ modulo $\mathfrak{m}_{F,v}$ is uniquely determined this way.

Theorem 5.3 extends [5] Thm. 1.2] from $m = 1$ to arbitrary $m > 1$. Since $X_p(\ell)/G \cong \mathbb{P}^1$, we do not need to refer to the class of canonical divisors on $X_p(\ell)/G$.

Similarly to [5], we can use the approach in [31] to define congruences modulo $p$ between modular forms. Namely, let $\mathcal{T}$ be a ring of Hecke operators acting on $S_{2m}(F) := F \otimes_A S_{2m}(A)$. By [31] §6.1-6.2, together with flat base change, it follows that $S_{2m}(F)$ coincides with the space of all weight $2m$ cuspidal forms for $\Gamma_\ell$ whose Fourier expansions with respect to $e^{2\pi iz/\ell}$ have coefficients in $F$. Suppose there exists a decomposition

$$S_{2m}(F) = E_1 \oplus E_2$$
into a direct sum of $T$-stable $F$-subspaces. Let $a$ be an ideal of $A$. Following [31], we define a non-trivial congruence modulo $a$ linking $E_1$ and $E_2$ to be a pair $(f_1, f_2)$ of forms $f_i \in S_{2m}(A) \cap E_i$, for $i \in \{1, 2\}$, such that

$$f_1 \equiv f_2 \mod a \cdot S_{2m}(A) \quad \text{but} \quad f_1 \not\equiv a \cdot S_{2m}(A).$$

Congruences of this kind have played an important role in the development of the theory of modular forms, Galois representations and arithmetic geometry (see, for example [11, 12]).

We call a $T$-stable decomposition into $F$-subspaces of the form given in Equation (5.2) $G$-isotypic if there exist two orthogonal central idempotents of $FG$ such that $1 = e_1 + e_2$ in $FG$ and

$$E_i = e_i S_{2m}(F) \quad \text{for} \quad i \in \{1, 2\}.$$

The following theorem extends [5, Thm. 1.3] from $m = 1$ to arbitrary $m > 1$.

**Theorem 5.4.** With the assumptions of Theorem 5.3, suppose further that $F$ contains a root of unity of order equal to the prime-to-$p$ part of the order of $G$. Let $a$ be the maximal ideal over $v \in \mathcal{V}(F, p)$. A $T$-stable decomposition of $S_{2m}(F)$, as given in Equation (5.2), that is $G$-isotypic, in the sense of Equation (5.3), results in non-trivial congruences modulo $a$, as given in Equation (5.3), if and only if the following is true. There is a block $B$ of $O_{F, \ell}G$ such that when $P_B$ and $U_B$ are as in Theorem 5.3, $M_B = P_B \oplus U_B$ is not equal to the direct sum $(M_B \cap e_1 M_B) \oplus (M_B \cap e_2 M_B)$. For a given $B$, there will be orthogonal idempotents $e_1$ and $e_2$ for which this is true if and only if $B$ has non-trivial defect groups, and either $P_B \neq \{0\}$ or $F_v \otimes_{O_{F, v}} U_B$ has two non-isomorphic irreducible constituents.

We will prove Theorems 5.3 and 5.4 for $p > 3$ in §5.3. The case when $p = 3$ will be discussed in §5.4 and §6 using Corollaries 1.2 and 1.8 together with Theorem 1.6 applied to $E = mK_X$.

### 5.2. Isotypic Hecke stable decompositions of even weight cusp forms

In this subsection, we extend the results of [5, §7] to construct non-trivial $G$-isotypic $T$-stable decompositions of the space of cusp forms of even weight $2m > 2$ when $T$ is the ring of Hecke operators that have index prime to the level $\ell$.

Let $\Delta'_\ell$ be the set of all matrices $\alpha \in \text{Mat}(2, \mathbb{Z})$ such that $\det(\alpha) > 0$ and $\alpha \equiv \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \mod \ell$ for some $x \in (\mathbb{Z}/\ell)^*$. Let $R(\Gamma_\ell, \Delta'_\ell)$ be the free $\mathbb{Z}$-module generated by the double cosets $\Gamma_\ell \alpha \Gamma_\ell$ for $\alpha \in \Delta'_\ell$. By [34, §3.1], $R(\Gamma_\ell, \Delta'_\ell)$ is a ring. For each positive integer $n$ that is relatively prime to $\ell$, let $\psi'_\ell(n)$ be a set of representatives $\alpha \in \Delta'_\ell$ of all distinct double cosets in $\Gamma_\ell \backslash \Delta'_\ell / \Gamma_\ell$ such that $\det(\alpha) = n$, and define $T'(n) := \sum_{\alpha \in \psi'_\ell(n)} \Gamma_\ell \alpha \Gamma_\ell$. By [34, Thm. 3.34], $T := R(\Gamma_\ell, \Delta'_\ell) \otimes \mathbb{Q}$ is the $\mathbb{Q}$-algebra generated by all $T'(n)$ when $n$ ranges over all positive integers that are relatively prime to $\ell$.

Suppose $m > 1$, and let $f \in S_{2m}(F)$. For a matrix $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}(2, \mathbb{Q})$ and $z$ in the complex upper half plane, we define

$$(f|\gamma)(z) := \det(\gamma)^m (cz + d)^{-2m} f \left( \frac{az + b}{cz + d} \right).$$

This leads to well-defined right actions by $\mathbb{T} = \text{SL}(2, F_\ell)$ and $G = \text{PSL}(2, F_\ell)$ on $S_{2m}(F)$, which can be made into left actions by defining the left action of a group element to be the right action of its inverse. Moreover, Equation (5.5) leads to the following right action of $R(\Gamma_\ell, \Delta'_\ell)$, and hence of $T$, on $S_{2m}(F)$. For $\alpha \in \Delta'_\ell$, write $\Gamma_\ell \alpha \Gamma_\ell = \bigcup_i \Gamma_\ell \alpha_i$ as a finite disjoint union of right cosets, and define

$$f|\Gamma_\ell \alpha \Gamma_\ell := \det(\alpha)^{m-1} \sum_i f|\alpha_i.$$

With these definitions, we can use similar arguments as in [5, §7] to obtain the following result.
Proposition 5.5. Let $m > 1$, and suppose $e_1, e_2$ are orthogonal central idempotents of $FG$ such that $1 = e_1 + e_2$ and each $e_i$ is fixed by the conjugation action of $\text{PGL}(2, F)$ on $G$. Then setting $E_i = S_{2m}(F)e_i$ for $i \in \{1, 2\}$ gives a $G$-isotypic $T$-stable decomposition of $S_{2m}(F)$, as defined in Equations (5.22) and (5.24).

5.3. Proofs of Theorems 5.3 and 5.4 when $p > 3$. These proofs follow the same main steps as the proofs of [5] Thms. 1.2 and 1.3, where we replace [5, Lemma 5.2] by Theorem 5.8 below, which gives a description of the direct summands of $H_0$ which implies $H_1$.

Theorem 5.8 from which a description of the isomorphism classes of all indecomposable modules belonging to the principal block, represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $\bar{T}_0$ of $k$-dimension $\ell - 1$. It follows, for example, from [4] that the isomorphism

Lemma 5.6. Suppose $m > 1$, $p > 3$ and $p \neq \ell \geq 7$. Let $v \in V(F, p)$, let $k$ be an algebraically closed field containing $\mathbb{F}_p$, and let $X := X_p(\ell)$ be as in Equation (5.1).

(i) The $kG$-module $H^0(X, \Omega^m_X)$ is projective.

(ii) Let $k_1$ be a perfect field containing $k(v)$, and let $k$ be an algebraic closure of $k_1$. Define $X_1 := k_1 \otimes_{k(v)} (k(v) \otimes_A \mathcal{X}_A(\ell))$. The $k_1G$-module $H^0(X_1, \Omega^m_{X_1})$ is projective.

The $kG$-module structure of $H^0(X, \Omega^m_X)$ as in (i) and the $k_1G$-module structure of $H^0(X_1, \Omega^m_{X_1})$ as in (ii) are both determined by the lower ramification groups and the fundamental characters associated to the cover $X \rightarrow X/G$.

Lemma 5.6 is proved using similar arguments to the ones used in the proof of [5, Lemma 5.2]. The main difference is that since $m > 1$ and $g(X) \geq 3$ by [3, Cor. 3.2], it follows that $\deg(\Omega^m_X) > 2g(X) - 2$, which implies $H^1(X, \Omega^m_X) = 0$. This forces $H^0(X, \Omega^m_X)$ to be a projective $kG$-module by [30, Thm. 2].

5.4. Proofs of Theorems 5.3 and 5.4 when $p = 3$. These proofs follow the same main steps as the proofs of [5] Thms. 1.2 and 1.3, where we replace [5, Thm. 1.1 and Remark 4.4] by Corollaries [4,2] and [4,8] and we replace [5, Thm. 1.4] by Theorem 5.8 below, which gives a description of the $kG$-module structure of the holomorphic $m$-differentials of $X = X_3(\ell)$ when $m > 1$.

The proof of Theorem 5.8 is more complicated than the proof of [5] Thm. 1.4], for the following reasons. Let $m > 1$.

(A) When restricting $H^0(X, \Omega^m_X)$ to 3-hypo-elementary subgroups $\Gamma$ of $G$, we need to determine canonical divisors on $X/\Gamma$.

(B) The indecomposable non-projective $kG$-modules that occur as direct summands of $H^0(X, \Omega^m_X)$ depend on the congruence class of $m$ modulo 6.

(C) The values of the Brauer character of $H^0(X, \Omega^m_X)$ at elements of $G$ of order $\ell$ depend on the congruence class of $m$ modulo $\ell$.

(D) There are indecomposable non-projective $kG$-modules that are not uniserial and that occur as direct summands of $H^0(X, \Omega^m_X)$.

Because of (D), our notation for the isomorphism classes of indecomposable $kG$-modules that occur as direct summands of $H^0(X, \Omega^m_X)$ is also more complicated, see Notation [4,7] below. We refer to the work in [7] from which a description of the isomorphism classes of all indecomposable $kG$-modules follows.

Notation 5.7. Let $\ell \geq 7$ be a prime number, let $G = \text{PSL}(2, F_\ell)$, and let $k$ be an algebraically closed field of characteristic $p = 3$.

(a) If $T$ is a simple $kG$-module, then $U_{T,b}^{(G)}$ denotes a uniserial $kG$-module with $b$ composition factors whose socle is isomorphic to $T$. The isomorphism class of $U_{T,b}^{(G)}$ is uniquely determined by $T$ and $b$ (see, for example, [4]).

(b) If $\ell \equiv -1 \mod 3$ then there exist indecomposable $kG$-modules that are not uniserial. These modules all belong to the principal block of $kG$. There are precisely two isomorphism classes of simple $kG$-modules belonging to the principal block, represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $\bar{T}_0$ of $k$-dimension $\ell - 1$. It follows, for example, from [4] that the isomorphism
classes of the non-uniserial indecomposable $kG$-modules are all uniquely determined by their socles and their tops (i.e. radical quotients), together with the number of their composition factors that are isomorphic to $\overline{T}_0$. There are three different types of such modules. We use the notation $U^G_{T_0, T_0, b}$ (resp. $U^G_{T, T_0, b}$, resp. $U^G_{T, T_0, b}$) to denote an indecomposable $kG$-module that has $b$ composition factors isomorphic to $\overline{T}_0$, whose socle is isomorphic to $T_0 \oplus \overline{T}_0$ (resp. $T_0 \oplus \overline{T}_0$, resp. $\overline{T}_0$) and whose top is isomorphic to $T_0 \oplus \overline{T}_0$ (resp. $T_0 \oplus \overline{T}_0$, resp. $\overline{T}_0$).

**Theorem 5.8.** Suppose $m > 1$ and $\ell \geq 7$. Let $v \in \mathcal{V}(F, 3)$, let $k$ be an algebraically closed field containing $k(v)$, and let $X := X_3(\ell)$ be as in Equation (5.11). Let $\epsilon = \pm 1$ be such that $\ell \equiv \epsilon \mod 3$. Write $\ell = 2 \cdot 3^n \cdot n'$ where 3 does not divide $n'$. For $i \in \{0, 1\}$, define $\delta_i$ to be 1 if $m \equiv i \mod 3$ and to be 0 otherwise. Define $\delta_m \in \{0, 1\}$ by $m \equiv \delta_m \mod 2$.

1. There exists a projective $kG$-module $Q_\ell$, depending on $\ell$, such that the following is true:
   
   (1) Suppose $\ell \equiv 1 \mod 4$ and $\ell \equiv 1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module. For $0 \leq t \leq (n' - 1)/2$, let $T_i$ be representatives of simple $kG$-modules of $k$-dimension $\ell - 1$ such that $T_0$ belongs to the principal block of $kG$. As a $kG$-module,
   $\mathcal{H}_g(X, \Omega^m_X) \cong \delta_0 (1 - \delta_m) U^G_{T_0, T_0, 3^{n-1}+1} \oplus \delta_0 \delta_m U^G_{T_0, 3^{n-1}}$
   $\oplus \delta_1 \delta_m U^G_{T_0, T_0, (3^{n-1}+1)/2} \oplus \delta_1 (1 - \delta_m) U^G_{T_0, T_0, (3^{n-1}+1)/2}$
   $\oplus \bigoplus_{t=1}^{(n' - 1)/2} \delta_0 U^G_{T_1, 2 \cdot 3^{n-1}} \oplus \bigoplus_{t=1}^{(n' - 1)/2} \delta_1 U^G_{T_1, 2 \cdot 3^{n-1}} \oplus Q_\ell$.

2. Suppose $\ell \equiv -1 \mod 4$ and $\ell \equiv 1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module, and let $T_1$ be a simple $kG$-module of $k$-dimension $\ell + 1$. As a $kG$-module,
   $\mathcal{H}_g(X, \Omega^m_X) \cong \delta_0 (1 - \delta_m) U^G_{T_0, T_0, 3^{n-1}} \oplus \delta_0 \delta_m U^G_{T_1, 2 \cdot 3^{n-1}}$
   $\oplus \delta_1 \delta_m U^G_{T_0, T_0, 2 \cdot 3^{n-1}} \oplus \delta_1 (1 - \delta_m) U^G_{T_1, 2 \cdot 3^{n-1}}$
   $\oplus \bigoplus_{t=1}^{(n' - 1)/2} \delta_0 U^G_{T_1, 3^{n-1}} \oplus \bigoplus_{t=1}^{(n' - 1)/2} \delta_1 U^G_{T_1, 3^{n-1}} \oplus Q_\ell$.

3. Suppose $\ell \equiv 1 \mod 4$ and $\ell \equiv 1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module, and let $T_1$ be a simple $kG$-module of $k$-dimension $\ell$. For $1 \leq t \leq (n' - 1)/2$, let $T_i$ be representatives of simple $kG$-modules of $k$-dimension $\ell + 1$. There exist simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ of $k$-dimension $(\ell + 1)/2$ such that, as a $kG$-module,
   $\mathcal{H}_g(X, \Omega^m_X) \cong \delta_0 (1 - \delta_m) \left(U^G_{T_{0,1}, 3^{n-1}} \oplus U^G_{T_{0,1}, 3^{n-1}}\right) \oplus \delta_0 \delta_m \left(U^G_{T_{1,0}, 3^{n-1}} \oplus U^G_{T_{1,0}, 3^{n-1}}\right)$
   $\oplus \delta_1 \delta_m \left(U^G_{T_{0,1}, 2 \cdot 3^{n-1}} \oplus U^G_{T_{0,1}, 2 \cdot 3^{n-1}}\right) \oplus \delta_1 (1 - \delta_m) \left(U^G_{T_{1,0}, 2 \cdot 3^{n-1}} \oplus U^G_{T_{1,0}, 2 \cdot 3^{n-1}}\right)$
   $\oplus \bigoplus_{t=1}^{(n' - 2)/2} \delta_0 U^G_{T_{1,0}, 3^{n-1}} \oplus \bigoplus_{t=1}^{(n' - 2)/2} \delta_1 U^G_{T_{1,0}, 2 \cdot 3^{n-1}} \oplus Q_\ell$.

4. Suppose $\ell \equiv -1 \mod 4$ and $\ell \equiv -1 \mod 3$. Let $T_0$ denote the trivial simple $kG$-module. For $0 \leq t \leq (n'/2 - 1)$, let $T_i$ be representatives of simple $kG$-modules of $k$-dimension $\ell - 1$ such that $T_0$ belongs to the principal block of $kG$. There exist simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ of
Suppose Assumption 6.1 holds. Write Notation 6.2.

The subgroup structure of those are isomorphic to the symmetric group $\Sigma_3$ that are ramified in the cover $X \to X/G$.

Theorem 5.8 extends [5, Thm. 1.4] from $m > 7$ be a prime number, and let $p = 3$. Let $v \in V(F, 3)$, let $k$ be an algebraically closed field containing $k(v)$, and let $X = X_3(\ell)$ be as in Equation (4.21). Let $G = \text{PSL}(2, F_\ell)$.

The goal of this section is to determine the precise $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$. In particular, we will prove Theorem 5.9. We will adapt the strategy used in [31] §6 to prove [31] Thm. 1.4] when $m = 1$ to our situation when $m > 1$. As we pointed out in the previous section, the proof of Theorem 5.9 is more involved than the proof of [5] Thm. 1.4.

By [29] p. 193], the ramification points of the cover $X \to X/G$ that are wildly ramified have inertia groups that are isomorphic to the symmetric group $\Sigma_3$ on three letters. We use that there is precise knowledge of the subgroup structure of $G = \text{PSL}(2, F_\ell)$ (see, for example, [16] §II.8]).

Notation 6.2. Suppose Assumption 6.1 holds. Write

$$m = 3 \cdot m' + i_m \quad \text{where} \quad i_m \in \{0, 1, 2\}.$$
For $i \in \{0, 1\}$, define $\delta_i$ to be 1 when $i = i_m$ and 0 otherwise. Define $\delta_m$ to be 1 if $m$ is odd and 0 otherwise.

Let $\epsilon, \epsilon' \in \{\pm 1\}$ be such that

$$\ell \equiv \epsilon \mod 3 \quad \text{and} \quad \ell \equiv \epsilon' \mod 4.$$ 

Write $\ell - \epsilon = 3^n \cdot 2 \cdot n'$ where 3 does not divide $n'$.

**Remark 6.3.** As in [4, §6], we fix the following 3-hypo-elementary subgroups of $G$:

(a) a cyclic subgroup $V = \langle v \rangle$ of order $(\ell - \epsilon)/2 = 3^n \cdot n'$;
(b) two dihedral groups $\Delta_1 = \langle v', s \rangle$ and $\Delta_2 = \langle v', vs \rangle$ of order $2 \cdot 3^n$, where $v' = v^n \in V$ is an element of order $3^n$ and $s \in \mathcal{N}_G(V) - V$ is an element of order 2;
(c) a cyclic subgroup $W = \langle w \rangle$ of order $(\ell + \epsilon)/2$;
(d) a cyclic subgroup $R$ of order $\ell$.

We let $\tau = (v')^{3^n - 1}$ and $I = \langle \tau \rangle$, so that $I$ is the unique subgroup of order 3 in each of $V, \Delta_1, \Delta_2$. If $\epsilon = -\epsilon'$, i.e. if $\ell \equiv -\epsilon \mod 4$, we also let $\Delta = \Delta_1$. Moreover, we let $P = \langle v' \rangle$, $P_I = I$, and $N_I = \mathcal{N}_G(P_I)$. It follows, for example, from [13, §II.8] that $N_I = \langle v, s \rangle$ is a dihedral group of order $\ell - \epsilon$.

Note that not all 3-hypo-elementary subgroups of $G$ are conjugate to one of the groups in (a)-(d) of Remark [6.3]. Rather than using the Conlon induction theorem, which we used to prove Theorem [11] and Corollary [12.2] in general and which requires all 3-hypo-elementary subgroups of $G$, we will adapt to our situation the approach used in [5, §6] which only needs the 3-hypo-elementary subgroups in (a)-(d) of Remark [6.3]. The main observations that make this approach work are as follows (see [5, §6] for details):

- Every $kN_I$-module is uniquely determined by its restrictions to the subgroups in $\{V, \Delta_1, \Delta_2\}$.
- There is a stable equivalence between the module categories of $kG$ and $kN_I$, i.e. an equivalence of these categories modulo projective modules (see [1, §V.17}). This allows us to use [2, §X.1] to detect the non-projective indecomposable modules for $kG$ and $kN_I$, respectively, that correspond to each other under the Green correspondence [1, Thm. V.17.3].
- The stable $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ and its Brauer character determine the full $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$.

In [6.1], we determine necessary information about the canonical divisors on $X/\Gamma$ and $X/I \cap \Gamma$ when $\Gamma \in \{\Delta_1, \Delta_2, V, W, R\}$. In [6.2], we use Corollary [1.8] to determine the precise $k\Gamma$-module structure of $\text{Res}_{\Gamma}^G \mathcal{H}^0(X, \Omega_X^{\otimes m})$ for $\Gamma \in \{\Delta_1, \Delta_2\}$. We then use this to determine the stable $kN_I$-module structure of $\text{Res}_{N_I}^G \mathcal{H}^0(X, \Omega_X^{\otimes m})$. In [6.3], we use the Green correspondence to determine the stable $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$. In [6.4], we determine the Brauer character of the $kG$-module $H^0(X, \Omega_X^{\otimes m})$. In [6.5], we use [6.3] and [6.4] to determine the full $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ and to complete the proof of Theorem [5.8].

### 6.1. Canonical divisors

We consider the canonical divisors on $X/\Gamma$ and $X/I \cap \Gamma$ when

$$\Gamma \in \{\Delta_1, \Delta_2, V, W, R\},$$

and we determine the information needed to be able to apply Corollary [1.8]. Let $Y = X/I \cap \Gamma$ and $Z = X/\Gamma$, and let $\pi : X \to Y$ and $\lambda : Y \to Z$ be the corresponding Galois covers with Galois groups $I \cap \Gamma$ and $\Gamma = \Gamma/I \cap \Gamma$, respectively. Let $Z_{br}$ be the set of closed points in $Z$ that are branch points of $\lambda$. The cover $X \to X/\Gamma \cong \mathbb{P}_k^1$ factors into the Galois cover $\lambda \circ \pi : X \to Z$ followed by a separable morphism $f : Z = X/\Gamma \to X/\Gamma$ which is not Galois. A canonical divisor on $X/\Gamma \cong \mathbb{P}_k^1$ is given by $K_{X/\Gamma} = -2\infty$, and a canonical divisor on $Z$ is given by

$$K_Z = f^*(-2\infty) + \text{Ram}_f$$
where Ram$_f$ is the ramification divisor of $f$ (see [14 Prop. IV.2.3]). Therefore,

\[(6.1) \quad K_Y = \lambda^* K_Z + \text{Ram}_\lambda = (f \circ \lambda)^*(-2\infty) + \lambda^* \text{Ram}_f + \text{Ram}_\lambda.\]

For each $\Gamma$ as above, we now determine ord$_y(z)(K_Y)$ for all points $y(z) \in Y$ above $z \in Z_{br}$. We give precise details in one of the cases and then list the remaining cases in Table 6.1.

Suppose first that $\epsilon = -\epsilon'$ and that $\Gamma = \Delta = \Delta_1$. Then $I \cap \Gamma = I, Y = X/I$ and $\Gamma = I/I$. Let $z \in Z_{br}$, and let $y(z) \in Y$ and $x(z) \in X$ be points above it. By [29, p. 193], $x(z), y(z)$ and $z$ all lie above $0 \in X/G \cong \mathbb{P}^1_k$ and $G_{x(z), 0} \cong \Sigma_3$. Hence, in Equation (6.1), the coefficient of $y(z)$ in $(f \circ \lambda)^*(-2\infty)$ is zero. If $I_{x(z)}$ (resp. $\Gamma_{x(z)}$) is the inertia group of $x(z)$ inside $I$ (resp. $\Gamma$), then $\Gamma_y(z) \cong \Gamma_{x(z)}/I_{x(z)}$. Because $z \in Z_{br}$, $\Gamma_y(z)$ must be a non-trivial group of order prime to 3, which implies $\#\Gamma_y(z) = 2$. Hence the coefficient of $y(z)$ in $\lambda^* \text{Ram}_f$ is 1. On the other hand, the coefficient of $y(z)$ in $\lambda^* \text{Ram}_f$ equals $e_{y(z)/z} \cdot d_{z/0}$, where $e_{y(z)/z} = 2$ is the ramification index of $y(z)$ over $z$ and $d_{z/0}$ is the different exponent of $z$ over $0 \in X/G \cong \mathbb{P}^1_k$. By [29, p. 193], $d_{x(z)/0} = (6 - 1) + (3 - 1) = 7$. Using the transitivity of the different, we have

\[d_{x(z)/0} = e_{x(z)/z} \cdot d_{z/0} + d_{x(z)/z}.\]

It follows from [3, §6.1.1 and §6.2.1] that there is precisely one point $z_1 \in Z_{br}$ with $e_{x(z_1)/z_1} = 6$ and $d_{x(z_1)/z_1} = 7$, and there are precisely $\frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2} = 1$ points $z_2, \ldots, z_{(e_{x(z)/z} - e_{x(z_1)/z_1})/2} \in Z_{br}$ with $e_{x(z_i)/z_i} = 2$ and $d_{x(z_i)/z_i} = 1$ for $2 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2}$. Therefore, we obtain $d_{z_i/0} = 0$ and $d_{z_i/0} = 3$ for $2 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2}$, which means

\[\text{ord}_{y(z_1)}(K_Y) = \begin{cases} 1, & i = 1, \\ 7, & 2 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2}, \end{cases}\]

if $\epsilon = -\epsilon'$ and $\Gamma = \Delta$.

We can use similar arguments for the remaining $\Gamma$. In Table 6.1 we list for all $z \in Z_{br}$, the values for ord$_y(z)(K_Y)$ together with the orders of the inertia groups $\Gamma_y(z)$ and $\Gamma_{x(z)}$.

**Table 6.1.** Canonical divisor $K_Y$ on $Y = X/I \cap \Gamma$ when $\Gamma \in \{\Delta_1, \Delta_2, V, W, R\}$.

| $\Gamma$ | $I \cap \Gamma$ | all $z \in Z_{br}$ | ord$_y(z)(K_Y)$ | $\#\Gamma_y(z)$ | $\#\Gamma_{x(z)}$ |
|----------|-----------------|---------------------|-----------------|-----------------|-----------------|
| $\Delta = \Delta_1$ | $\epsilon = -\epsilon'$ | $I$ | $z_1$ | 1 | 2 | 6 |
| | | $z_i$ | $(2 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2})$ | 7 | 2 | 2 |
| $\Delta_1$ | $\epsilon = \epsilon'$ | $I$ | $z_i$ | $(i = 1, 2)$ | 1 | 2 | 6 |
| | | $z_i$ | $(3 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2})$ | 7 | 2 | 2 |
| $\Delta_2$ | $\epsilon = \epsilon'$ | $I$ | $z_i$ | $(1 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2})$ | 7 | 2 | 2 |
| $V$ | $\epsilon = -\epsilon'$ | $I$ | $Z_{br} = \emptyset$ | | | |
| $V$ | $\epsilon = \epsilon'$ | $I$ | $z_i$ | $(i = 1, 2)$ | 7 | 2 | 2 |
| $W$ | $\epsilon = -\epsilon'$ | 1 | $z_i$ | $(i = 1, 2)$ | 7 | 2 | 2 |
| $W$ | $\epsilon = \epsilon'$ | 1 | $Z_{br} = \emptyset$ | | | |
| $R$ | $\epsilon = \pm\epsilon'$ | 1 | $z_i$ | $(1 \leq i \leq \frac{e_{x(z)/z} - e_{x(z_1)/z_1}}{2})$ | $-\ell - 1$ | $\ell$ | $\ell$ |

6.2. The stable $kN_1$-module structure of the holomorphic poly-differentials. We use a similar strategy to [3, §6.2]. More precisely, we first calculate the stable $k\Gamma$-module structure of Res$_{k \mathbb{F}_p}^k H^0(X, \Omega^m_X)$ for the subgroups $\Gamma \in \{V, \Delta_1, \Delta_2\}$ of $N_1$. In other words, we find the non-projective indecomposable direct
Moreover, there are precisely $3^{n-1} n'$ closed points $x$ on $X$ such that $3$ divides $\# \Gamma_x$, which is equivalent to $\Gamma_x \geq I$. Using [33, Prop. IV.2 and Cor. 4 of Prop. IV.7], it follows from [29, p. 193] that $I_{x,0} = I_{x,1} = I$ and $I_{x,i} = 1$ for $i \geq 2$ for these points $x$. Let $y_{t,1}, \ldots, y_{t,3^{n-1}}$, for $1 \leq t \leq n'$, be points on $Y = X/I$ that lie below these points on $X$. Then, for $0 \leq j \leq 2$, the divisor $D_j$ from Corollary 1.8 is given as

$$(6.2) \quad D_j = \sum_{t=1}^{n'} \sum_{i=1}^{3^{n-1}} \left[ \frac{4m-j}{3} \right] y_{t,i}.$$ 

Writing $m = 3m' + i_m$ as in Notation 6.2 where $i_m \in \{0, 1, 2\}$, we have

$$(6.3) \quad \left[ \frac{4m-j}{3} \right] = 4m' + \left[ \frac{4i_m-j}{3} \right] = \begin{cases} 4m' - 1, & i_m = 0, j \in \{2, 1\}, \\ 4m', & (i_m = 0, j = 0) \text{ or } (i_m = 1, j = 2), \\ 4m' + 1, & i_m = 1, j \in \{1, 0\}, \\ 4m' + 2, & i_m = 2, j \in \{2, 1, 0\}. \end{cases}$$

This leads to the following equalities, for $j \in \{0, 1\}$:

$$(6.4) \quad \frac{\deg(D_j) - \deg(D_{j+1})}{3^{n-1} n'} = \left[ \frac{4m-j}{3} \right] - \left[ \frac{4m-(j+1)}{3} \right] = \begin{cases} 1, & (i_m, j) \in \{(0, 0), (1, 1)\}, \\ 0, & \text{otherwise}. \end{cases}$$

6.2.1. The stable $kv$-module structure and the values of the Brauer character at 3-regular elements of $V$.

There are $n'$ isomorphism classes of simple $kv$-modules. As in [33, §6.2], we denote representatives of these by $S_{a}^{(V)}$ for $0 \leq a \leq n'-1$. Similarly to Notation 11.21(b), we write $U_{a,b}^{(V)}$ for an indecomposable $kv$-module whose socle is isomorphic to $S_{a}^{(V)}$ and whose $k$-dimension is equal to $b$ with $1 \leq b \leq 3^n$. The Galois cover $\lambda : Y = X/I \to Z = X/V$ is unramified if $\epsilon = -\epsilon'$ (resp. tamely ramified if $\epsilon = \epsilon'$) with Galois group $V = V/I$. We now apply Theorem 1.6 to $E = mK_X$ and the group $V$, where we will write $n_j(V)$ for the integers $n_j$ from Equation (1.3) and $n_{a,b}^{(V)}$ for the integers $n_{a,b}$ from Equation (1.4). Note that $n_1 = 1$ since $\#(I \cap V) = 3$. As in Table 6.1, if $\epsilon = -\epsilon'$ then $Z_{br} = \emptyset$, and if $\epsilon = \epsilon'$ then $Z_{br} = \{z_1, z_2\}$. In the latter case, we use that $E_j = mK_Y + D_j$, together with the information from Table 6.1 to obtain that $\ell_{y(z_i),j} = \delta_m$ for $i = 1, 2$ and $0 \leq j \leq 2$. By Remark 11.7, this immediately implies that for both cases $\epsilon = -\epsilon'$ and $\epsilon = \epsilon'$, we have $n(a,j) - n(a,j+1) = n_j(V) - n_{j+1}(V)$ for $j \in \{0, 1\}$. Using the alternative formula for $n_j(V)$ from Remark 2.9(a), together with Equation (6.4), we obtain, for $j \in \{0, 1\},$

$$(6.5) \quad n_j(V) - n_{j+1}(V) = \frac{\deg(D_j) - \deg(D_{j+1})}{\# V} = \begin{cases} 1, & (i_m, j) \in \{(0, 0), (1, 1)\}, \\ 0, & \text{otherwise}, \end{cases}$$

for all $\epsilon, \epsilon' \in \{\pm 1\}$. By Equation (1.5), $n_{a,3(n+1)n-1}^{(V)} = n_j(V) - n_{j+1}(V)$, for $j \in \{0, 1\}$ and $0 \leq a \leq n'-1$, and these are all possible non-zero $n_{a,b}^{(V)}$ for $1 \leq b \leq 3^n - 1$. Therefore, the non-projective indecomposable
direct $kV$-module summands of $\text{Res}^G_H H^0(\Gamma, \Omega^{\otimes m}_X)$, with their multiplicities, are given by the direct sum

$$
\delta_0 \bigoplus_{a=0}^{n'-1} U_{a,3^n-1}^{(V)} \oplus \delta_1 \bigoplus_{a=0}^{n'-1} U_{a,2.3^n-1}^{(V)}
$$

where $\delta_0$ and $\delta_1$ are as in Notation 6.2.

We now use Equation (6.6) to determine the values of the Brauer character $\beta$ of $H^0(\Gamma, \Omega^{\otimes m}_X)$ at elements of $V$ that are 3-regular, i.e. whose orders are not divisible by 3. We write $V = (\nu)$ as in Remark 6.3 and let $v'' = v^{3^n}$, so that $v''$ has order $n'$. Let $\xi_{n'}$ be a fixed primitive $(n')^{\text{th}}$ root of unity so that, for $0 \leq a \leq n'-1$, $v''$ acts as multiplication by $\xi_{n'}^a$ on $S_a^{(\Gamma)}$. Applying Notation 6.3(c) to the group $V$, it follows that each $U_{a,b}^{(V)}$ has $b$ composition factors, which are all isomorphic to $S_a^{(\Gamma)}$. Therefore, letting $\bar{\beta}(V)$ be the Brauer character of the maximal projective direct $kV$-module summand of $\text{Res}^G_H H^0(\Gamma, \Omega^{\otimes m}_X)$, we obtain that the value at $(v''^i, 1 \leq i < \frac{n'}{2})$, of the Brauer character of the direct sum in Equation (6.6) is equal to

$$
(\beta - \bar{\beta}(V))((v''^i)^i) = \delta_0 \sum_{a=0}^{n'-1} 3^{n-1} \xi_{n'}^a + \delta_1 \sum_{a=0}^{n'-1} 2 \cdot 3^{n-1} \xi_{n'}^a = 0.
$$

Here, the last equality follows since $\xi_{n'}^i$ is an $(n')^{\text{th}}$ root of unity, and hence $\sum_{a=0}^{n'-1} \xi_{n'}^a = 0$ for all $1 \leq i < \frac{n'}{2}$.

To find $\bar{\beta}(V)((v''^i)^i)$, we need to determine $n_{a,3^n}$ for $0 \leq a \leq n'-1$. Since the Brauer character of $kV$ has zero value at all non-identity elements of $V$, it follows from Equations (1.3) and (1.5) that it suffices to determine $n_{a,3^n} - n_{2(V)}$. Using that, for $\epsilon = \epsilon'$ and $i = 1, 2, \mu_a(z_i) = \mu_{a,1}(y(z_i)) = 1$ if and only if $a$ is odd, we get, for $0 \leq a \leq n'-1$,

$$
n_{a,3^n} - n_{2(V)} = \begin{cases} 0, & \epsilon = \epsilon' \text{ and } a \text{ odd}, \\ 2 \cdot (\delta_m - \frac{1}{2}), & \epsilon = -\epsilon' \text{ or } a \text{ even}. \end{cases}
$$

If $\epsilon = \epsilon'$ then $n'$ is even, and hence $|\frac{n'}{2} - 1| = \frac{n'}{2} - 1$ and $\xi_{n'}^2$ is a primitive $(\frac{n'}{2})^{\text{th}}$ root of unity. Therefore, we obtain, for all $\epsilon, \epsilon' \in \{-1, 1\}$ and all $1 \leq i < \frac{n'}{2}$,

$$
\bar{\beta}(V)((v''^i)^i) = \delta_{\epsilon,\epsilon'} (2\delta_m - 1) \sum_{i=0}^{\lfloor \frac{n'}{2} \rfloor - 1} 3^n \xi_{n'}^{i(2t+1)} = \delta_{\epsilon,\epsilon'} (2\delta_m - 1) 3^n \xi_{n'}^i \sum_{t=0}^{\lfloor \frac{n'}{2} \rfloor - 1} (\xi_{n'}^2)^it = 0.
$$

6.2.2. The stable $k\Delta_1$- and $k\Delta_2$-module structures and the values of the Brauer character at elements of order 2. Let $\Gamma \in \{\Delta_1, \Delta_2\}$, where we only consider $\Gamma = \Delta = \Delta_1$ if $\epsilon = -\epsilon'$. There are precisely two isomorphism classes of simple $k\Gamma$-modules. As in [3] §6.2, we denote representatives of these by $S_a^{(\Gamma)}$ for $a \in \{0, 1\}$. Similarly to Notation 6.3(b), we write $U_{a,b}^{(\Gamma)}$ for an indecomposable $k\Gamma$-module whose socle is isomorphic to $S_a^{(\Gamma)}$ and whose $k$-dimension is equal to $b$ with $1 \leq b \leq 3^n$. The Galois cover $\lambda : Y = X/I \rightarrow Z = X/\Gamma$ is tamely ramified with Galois group $\Gamma = \Gamma/I$. We now apply Theorem 1.10 to $E = mK_X$ and the group $\Gamma$, where we will write $n_j(\Gamma)$ for the integers $n_j$ from Equation (1.3) and $n_{a,b}^{(V)}$ for the integers $n_{a,b}$ from Equation (1.5). Note that $n_1 = 1$ since $\#(I \cap \Gamma) = 3$. As in Table 6.1, $Z_{\Gamma} = \{z_1, \ldots, z_\ell, 0, z, z_{\ell-\epsilon'}/2\}$. Define

$$
i_0 = \begin{cases} 1, & \epsilon = -\epsilon', \Gamma = \Delta = \Delta_1, \\ 2, & \epsilon = \epsilon', \Gamma = \Delta_1, \\ 0, & \epsilon = \epsilon', \Gamma = \Delta_2. \end{cases}
$$

By Table 6.1 there are precisely $i_0$ points among the points $y_{t,i}$ ($1 \leq t \leq n', 1 \leq i \leq 3^{n-1}$) on $Y$ occurring in Equation (6.2) that lie above points in $Z_{\Gamma}$. Hence, we obtain by Table 6.1 that

$$\text{ord}_{y_{zt,i}}(mK_Y + D_j) = \begin{cases} m + \lfloor \frac{4m-1}{3} \rfloor, & 1 \leq i \leq i_0, \\ \frac{m}{3}, & i_0 + 1 \leq i \leq (\ell - \epsilon')/2. \end{cases}$$
By Equation (6.3), it follows that

\[
\ell_{y(z_i),j} = \begin{cases} 
1 - \delta_m, & (i_m, j) \in \{(0, 1), (0, 2), (1, 0), (1, 1)\} \\
\delta_m, & \text{otherwise.}
\end{cases}
\]

Using the alternative formula for \(n_j(\Gamma)\) from Remark 2.6(a), together with Equations (6.4) and (6.10), we have, for \(j \in \{0, 1\},\)

\[
n_j(\Gamma) - n_{j+1}(\Gamma) = \frac{\deg(D_j) - \deg(D_{j+1})}{2 \cdot 3^{n-1}} - \sum_{i=1}^{i_0} \ell_{y(z_i),j} - \ell_{y(z_i),j+1}
\]

\[
= \begin{cases} 
\frac{n' + i_0(1 - 2\delta_m)}{2}, & (i_m, j) = (0, 0), \\
\frac{n' - i_0(1 - 2\delta_m)}{2}, & (i_m, j) = (1, 1), \\
0 & \text{otherwise.}
\end{cases}
\]

Using Equation (6.10), and that \(\mu_{0,-1}(y(z_i)) = 0\) and \(\mu_{1,-1}(y(z_i)) = 1\) for \(1 \leq i \leq i_0\), we obtain, for \(j \in \{0, 1\},\)

\[
\sum_{z \in \mathbb{Z}_{nr}} \left( \max \left\{ \ell_{y(z_i),j}, \ell_{y(z_i),j+1} \right\} \right) = \begin{cases} 
-\delta_0(1 - 2\delta_m), & (i_m, j) = (0, 0) \text{ and } a = 1, \\
\delta_0(1 - 2\delta_m), & (i_m, j) = (1, 1) \text{ and } a = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

By Equation (6.11) and Remark 1.5, it therefore follows that the non-projective indecomposable direct \(k\Gamma\)-module summands of \(\text{Res}^G_H\mathbb{H}(X, \Omega_X^m)\), with their multiplicities, are given by the direct sum

\[
(6.11) \quad \delta_0 \left( \frac{n' + i_0(1 - 2\delta_m)}{2} \ell_{0,3^{n-1}}^{(\Gamma)} \right) + \delta_1 \left( \frac{n' - i_0(1 - 2\delta_m)}{2} \ell_{0,2 \cdot 3^{n-1}}^{(\Gamma)} \right) = \delta_0 \left( \frac{n' + i_0(1 - 2\delta_m)}{2} \ell_{0,3^{n-1}}^{(\Gamma)} \right) + \delta_1 \left( \frac{n' - i_0(1 - 2\delta_m)}{2} \ell_{0,2 \cdot 3^{n-1}}^{(\Gamma)} \right)
\]

where \(\delta_0, \delta_1, \delta_m\) are as in Notation (6.2) and \(i_0\) is as in Equation (6.2).

Similarly to Equations (6.7) and (6.8), we now use Equation (6.11) to determine the values of the Brauer character \(\beta\) of \(\mathbb{H}(X, \Omega_X^m)\) at elements of order 2 in \(\Gamma\). Since, by [10] §II.8, all elements in \(G\) of order 2 are conjugate to the element \(s \in \Delta_1\) from Remark 6.3, we only have to consider the element \(s \in \Delta_1\) for \(\beta(\Delta_1)\) as multiplication by 1 (resp. \(-1\)). Applying Notation (1.4) to the group \(\Delta_1\), it follows that, for \(a \in \{0, 1\}\), each \(U_{a,b}^{(\Delta_1)}\) has \([\frac{1 + a}{2}]\) composition factors that are isomorphic to \(S_a^{(\Delta_1)}\) and \([\frac{b}{2}]\) composition factors that are isomorphic to \(S_a^{(\Delta_1)}\). Therefore, letting \(\beta^{(\Delta_1)}\) be the Brauer character of the maximal projective direct \(k\Delta_1\)-module summand of \(\text{Res}^G_H\mathbb{H}(X, \Omega_X^m)\), the value at \(s \in \Delta_1\) of the Brauer character of the direct sum in Equation (6.11) is equal to

\[
(6.12) \quad (\beta - \beta^{(\Delta_1)})(s) = \delta_0 \left( \frac{n' + i_0(1 - 2\delta_m)}{2} - \frac{n' - i_0(1 - 2\delta_m)}{2} \right) + \delta_1(0 + 0) = \delta_0(1 - 2\delta_m).
\]

To find \(\beta^{(\Delta_1)}(s)\), we need to determine \(n_{a,3^n}^{(\Delta_1)}\) for \(a \in \{0, 1\}\). Since the Brauer character of \(k\Delta_1\) has zero value at all non-identity elements of \(\Delta_1\), it follows from Equations (1.2) and (1.4) that it suffices to determine \(n_{a,3^n}^{(\Delta_1)} - n_{2(\Delta_1)}\). Using that, for \(1 \leq i \leq \ell - \epsilon'\), \(\mu_{a,-1}(y(z_i)) = \mu_{a,1}(y(z_i)) = 1\) if and only if \(a = 1\), we get

\[
n_{a,3^n}^{(\Delta_1)} - n_{2(\Delta_1)} = \begin{cases} 
(1 - 2\delta_m) \left( \delta_0 \delta_0 - \frac{\ell - \epsilon'}{4} \right), & a = 1, \\
0, & a = 0.
\end{cases}
\]
Therefore, we get, since $3^n$ is odd,

\[ (6.13) \]

\[ \overline{\beta}(\Delta_1) (s) = - (1 - 2\delta_m) \left( \delta_{0i0} - \frac{\ell - \ell'}{4} \right). \]

6.2.3. The stable $kN_1$-module structure when $\epsilon = -\epsilon'$. We use the notation from [5 §6.2.1] for the isomorphism classes of indecomposable $kN_1$-modules. Since $P$ is a normal Sylow 3-subgroup of $N_1$, all simple $kN_1$-modules are inflated from simple $kN_1/P$-modules. Moreover, $N_1/P$ is isomorphic to a dihedral group of order $2n'$, which is not divisible by 3. Hence we can use ordinary character theory to see the following. There are $2 + \frac{n'-1}{2}$ isomorphism classes of simple $kN_1$-modules, represented by 2 one-dimensional $kN_1$-modules $S_{t_1}^{(N_1)}$, for $i \in \{0, 1\}$, such that $S_{t_1}^{(N_1)}$ restricts to $S_{0}^{(V)}$ and to $S_{0}^{(\Delta)}$, together with $\frac{n'-1}{2}$ two-dimensional simple $kN_1$-modules $\overline{S}_{t}^{(N_1)}$, for $1 \leq t \leq \frac{n'-1}{2}$, such that $\overline{S}_{t}^{(N_1)}$ restricts to $S_{t}^{(V)} \oplus S_{n'-t}^{(V)}$ and to $S_{0}^{(\Delta)} \oplus S_{1}^{(\Delta)}$. For $i \in \{0, 1\}$, we write $U_{i, b}$ for an indecomposable $kN_1$-module whose socle is isomorphic to $S_{i}^{(N_1)}$ and whose $k$-dimension is equal to $b$ with $1 \leq b \leq 3^n$. For $t \in \{1, \ldots, \frac{n'-1}{2}\}$, we write $U_{t, b}$ for an indecomposable $kN_1$-module whose socle is isomorphic to $\overline{S}_{t}^{(N_1)}$ and whose $k$-dimension is equal to $2b$ with $1 \leq b \leq \frac{3^n}{2}$.

By [6.2.1] and by [6.2.2] for $\Gamma = \Delta$, we can write the non-projective indecomposable direct summands of $\text{Res}^{G}_{N_1} H^0(X, \Omega_X^m)$, with their multiplicities, as a direct sum

\[ (6.14) \]

\[ \bigoplus_{i \in \{0, 1\}} \bigoplus_{j \in \{0, 1\}} n_{0, (j+1)3^n-1}^{(N_1)} U_{i, (j+1)3^n-1}^{(N_1)} \oplus \bigoplus_{t \in \{0, 1\}} \bigoplus_{j \in \{0, 1\}} n_{t, (j+1)3^n-1}^{(N_1)} \overline{U}_{t, (j+1)3^n-1}^{(N_1)} \]

where we need to determine the coefficients. Let $j \in \{0, 1\}$. Restricting Equation (6.14) to $V$, Equation (6.6) leads to the conditions

\[ (6.15) \]

\[ \begin{cases} n_{0, (j+1)3^n-1}^{(N_1)} + n_{1, (j+1)3^n-1}^{(N_1)} = \delta_0 \delta_{j, 0} + \delta_1 \delta_{j, 1}, \\ n_{t, (j+1)3^n-1}^{(N_1)} = \delta_0 \delta_{j, 0} + \delta_1 \delta_{j, 1} \quad \text{for } t \in \{1, \ldots, \frac{n'-1}{2}\}. \end{cases} \]

On the other hand, restricting Equation (6.14) to $\Delta$, Equation (6.11), with $i_0 = 1$, leads to the conditions

\[ (6.16) \]

\[ \begin{cases} n_{0, (j+1)3^n-1}^{(N_1)} + \sum_{t=1}^{(n'-1)/2} n_{t, (j+1)3^n-1}^{(N_1)} = \delta_0 \delta_{j, 0} \frac{n' + (1 - 2\delta_m)}{2} + \delta_1 \delta_{j, 1} \frac{n' - (1 - 2\delta_m)}{2}, \\ n_{1, (j+1)3^n-1}^{(N_1)} + \sum_{t=1}^{(n'-1)/2} \overline{n}_{t, (j+1)3^n-1}^{(N_1)} = \delta_0 \delta_{j, 0} \frac{n' - (1 - 2\delta_m)}{2} + \delta_1 \delta_{j, 1} \frac{n' + (1 - 2\delta_m)}{2}. \end{cases} \]

Using Equations (6.15) and (6.16), for $j \in \{0, 1\}$, it follows that the non-projective indecomposable direct summands of $\text{Res}^{G}_{N_1} H^0(X, \Omega_X^m)$, with their multiplicities, are given by the direct sum

\[ (6.17) \]

\[ \delta_0 \left( (1 - \delta_m) U_{0, 3^n-1}^{(N_1)} \oplus \delta_m U_{1, 3^n-1}^{(N_1)} \oplus \bigoplus_{t=1}^{(n'-1)/2} \overline{U}_{t, 3^n-1}^{(N_1)} \right) \]

\[ \oplus \delta_1 \left( \delta_m U_{0, 3^n-1}^{(N_1)} \oplus (1 - \delta_m) U_{1, 3^n-1}^{(N_1)} \oplus \bigoplus_{t=1}^{(n'-1)/2} \overline{U}_{t, 3^n-1}^{(N_1)} \right) \]

where $\delta_0, \delta_1, \delta_m$ are as in Notation (6.2).

6.2.4. The stable $kN_1$-module structure when $\epsilon = \epsilon'$. We use the notation from [5 §6.2.2] for the isomorphism classes of indecomposable $kN_1$-modules. Arguing similarly to [6.2.2], there are $4 + \left( \frac{n'}{2} - 1 \right)$ isomorphism classes of simple $kN_1$-modules, represented by 4 one-dimensional $kN_1$-modules $S_{i_1, i_2}^{(N_1)}$ for $i_1, i_2 \in \{0, 1\}$, such that $S_{i_1, i_2}^{(N_1)}$ restricts to $S_{i_1}^{(\Delta)}$ and to $S_{i_2}^{(\Delta)}$ and to $S_{0}^{(V)}$ if $i_1 = i_2$ and to $S_{n'-t}^{(V)}$ if $i_1 \neq i_2$, together with $\left( \frac{n'}{2} - 1 \right)$ two-dimensional simple $kN_1$-modules $S_{t}^{(N_1)}$, for $1 \leq t \leq \left( \frac{n'}{2} - 1 \right)$, such that $S_{t}^{(N_1)}$ restricts to $S_{t}^{(V)} \oplus S_{n'-t}^{(V)}$.
and to $S_0^{(\Delta_1)} \oplus S_1^{(\Delta_1)}$ and to $S_0^{(\Delta_2)} \oplus S_1^{(\Delta_2)}$. For $i_1, i_2 \in \{0, 1\}$, we write $U_{i_1, i_2}^{(N_1)}$ for an indecomposable $kN_1$-module whose socle is isomorphic to $S_{i_1, i_2}^{(N_1)}$ and whose $k$-dimension is equal to $b$ with $1 \leq b \leq 3^m$. For $t \in \{1, \ldots, (\frac{m}{3}) - 1\}$, we write $\tilde{U}_{tb}^{(N_1)}$ for an indecomposable $kN_1$-module whose socle is isomorphic to $S_t^{(N_1)}$ and whose $k$-dimension is equal to $2b$ with $1 \leq b \leq 3^m$. By §6.2.1 and by §6.2.2 for $\Gamma \in \{\Delta_1, \Delta_2\}$, we obtain, using similar arguments as in Example 6.2.20, that the non-projective indecomposable direct summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^m)$, with their multiplicities, are given by the direct sum

$$(6.18) \quad \delta_0 \left(1 - \delta_m \right) \left( U_{0,0,3^m-1}^{(N_1)} \oplus U_{0,1,3^m-1}^{(N_1)} \right) \oplus \delta_m \left( U_{1,1,3^m-1}^{(N_1)} \oplus U_{1,0,3^m-1}^{(N_1)} \right) \oplus \bigoplus_{t=1}^{(n'/2-1)} \tilde{U}_{t,3^m-1}^{(N_1)} \oplus \delta_1 \left( \delta_m \left( U_{0,0,2,3^m-1}^{(N_1)} \oplus U_{0,1,2,3^m-1}^{(N_1)} \right) \oplus (1 - \delta_m) \left( U_{1,1,2,3^m-1}^{(N_1)} \oplus U_{1,0,2,3^m-1}^{(N_1)} \right) \oplus \bigoplus_{t=1}^{(n'/2-1)} \tilde{U}_{t,2,3^m-1}^{(N_1)} \right),$$

where $\delta_0, \delta_1, \delta_m$ are as in Notation 6.1.

6.3. The stable $kG$-module structure of the holomorphic poly-differentials. We use the same strategy as in [5] §6.4, i.e. we determine the non-projective indecomposable $kG$-modules that are direct summands of $H^0(X, \Omega_X^m)$, together with their multiplicities. These $kG$-modules are precisely the Green correspondents of the non-projective indecomposable direct $kN_1$-module summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^m)$. The two main ingredients we use to find these Green correspondents are §6.2.3 and §6.2.4, together with [7] §III-§VI.

As in [5] §6.4, it is important that there is a stable equivalence between the module categories of $kG$ and $kN_1$, which allows us to use the results from [5] §X.1 on almost split sequences to be able to detect the Green correspondents. More precisely, we first identify the indecomposable $kG$-modules that are the Green correspondents of the simple $kN_1$-modules, since each of these lies at the end of the component of the stable Auslander-Reiten quiver to which it belongs. We then follow the irreducible homomorphisms in the appropriate component until we reach the Green correspondent of each of the non-projective indecomposable direct $kN_1$-module summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^m)$. As in [5] §6.4, we have to consider four cases.

The main difference when $m > 1$ is that the indecomposable non-projective $kG$-modules that occur as direct summands of $H^0(X, \Omega_X^m)$ depend on the congruence class of $m$ modulo 6. Moreover, if $\ell \equiv -1 \mod 3$ then there are indecomposable non-projective $kG$-modules that are not uniserial and that occur as direct summands of $H^0(X, \Omega_X^m)$ (see Notation 5.7(b)).

6.3.1. The stable $kG$-module structure when $\epsilon = -\epsilon'$ and $\epsilon' = 1$. This is the case when $\ell \equiv 1 \mod 4$ and $\ell \equiv -1 \mod 3$. By §6.2.3 the non-projective indecomposable direct summands of $\text{Res}_{N_1}^G H^0(X, \Omega_X^m)$, with their multiplicities, are those in the direct sum in Equation (6.17). As recorded in [5] §6.4.1, it follows from [7] §IV that there are $1 + (n'/2)$ blocks of $kG$ of maximal defect $n$, consisting of the principal block $B_0$ and $(n'-1)/2$ blocks $B_1, \ldots, B_{(n'-1)/2}$, and there are $1 + (\ell-1)/4$ blocks of $kG$ of defect 0. There are precisely two isomorphism classes of simple $kG$-modules that belong to $B_0$, represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $\tilde{T}_0$ of $k$-dimension $\ell - 1$. For each $t \in \{1, \ldots, (n'-1)/2\}$, there is precisely one isomorphism class of simple $kG$-modules belonging to $B_t$, represented by a simple $kG$-module $\tilde{T}_t$ of $k$-dimension $\ell - 1$.

Using Notation 5.7, it follows, as in [5] §6.4.1, that the Green correspondent of $S_0^{(N_1)}$ is $T_0$, the Green correspondent of $S_1^{(N_1)}$ is $U_{\tilde{T}_0,3^{n-1}/2}$, and the Green correspondent of $\tilde{S}_t^{(N_1)}$ is $U_{\tilde{T}_t,3^{n-1}}$, for $1 \leq t \leq (n'-1)/2$. Following the irreducible homomorphisms in the stable Auslander-Reiten quiver of $B_0$ (resp. $B_1$, for $1 \leq t \leq (n'-1)/2$), we obtain the indecomposable $kG$-modules that are the Green correspondents of the indecomposable $kN_1$-modules occurring in Equation (6.17). Because of our assumptions on $\epsilon$ and $\epsilon'$, the Green correspondents of $U_{0,3^{n-1}}^{(N_1)}, U_{0,2,3^{n-1}}^{(N_1)}$ and $U_{1,2,3^{n-1}}^{(N_1)}$ are indecomposable $kG$-modules that are not
uniserial. More precisely, using Notation 6.2.3, the Green correspondent of $U_{1,2}^{(N_{i-1})}$ is $U_{T_0, T_0, (3^{n-1}+1)/2}$, the Green correspondent of $U_{0, 2, 3^{n-1}+1}^{(N_{i-1})}$ is $U_{T_0, T_0, (3^{n-1}+1)/2}$ and the Green correspondent of $U_{0, 3^{n-1}+1}^{(N_{i-1})}$ is $U_{T_0, T_0, (3^{n-1}+1)/2}$. Therefore, we obtain that the non-projective indecomposable direct $kG$-module summands of $H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are as stated in part (i)(1) of Theorem 5.8. It follows that for every possible value of $\delta_0, \delta_1, \delta_m$, the $kG$-module $H^0(X, \Omega_X^{\otimes m})$ has, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as a direct summand.

6.3.2. The stable $kG$-module structure when $\epsilon = -\epsilon'$ and $\epsilon' = -1$. This is the case when $\ell \equiv -1 \mod 4$ and $\ell \equiv 1 \mod 3$. By §6.2.3, the non-projective indecomposable direct summands of $\operatorname{Res}_N^G H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are again those in the direct sum in Equation (6.17). As recorded in [5] §6.4.2, it follows from [7] §V that there are $1 + (n' - 1)/2$ blocks of $kG$ of maximal defect $n$, consisting of the principal block $B_0$ and $(n' - 1)/2$ blocks $B_1, \ldots, B_{(n' - 1)/2}$, and there are $1 + (\ell + 1)/4$ blocks of $kG$ of defect 0. There are precisely two isomorphism classes of simple $kG$-modules that belong to $B_0$, represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $T_1$ of $k$-dimension $\ell$. For each $t \in \{1, \ldots, (n' - 1)/2\}$, there is precisely one isomorphism class of simple $kG$-modules belonging to $B_t$, represented by a simple $kG$-module $T_t$ of $k$-dimension $\ell + 1$.

As in [5] §6.4.2, it follows that the Green correspondent of $S_0^{(N_i)}$ is $T_0$, the Green correspondent of $S_1^{(N_i)}$ is $T_1$, and the Green correspondent of $\overline{S}_1^{(N_i)}$ is $\overline{T}_1$, for $1 \leq t \leq (n' - 1)/2$. Following the irreducible homomorphisms in the stable Auslander-Reiten quiver of $B_0$ (resp. $B_t$, for $1 \leq t \leq (n' - 1)/2$), we obtain the indecomposable $kG$-modules that are the Green correspondents of the indecomposable $kN_1$-modules occurring in Equation (6.17). Because of our assumptions on $\epsilon$ and $\epsilon'$, all indecomposable $kG$-modules are uniserial. Hence we obtain that the non-projective indecomposable direct $kG$-module summands of $H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are as stated in part (i)(2) of Theorem 5.8. It follows that for every possible value of $\delta_0, \delta_1, \delta_m$, the $kG$-module $H^0(X, \Omega_X^{\otimes m})$ has, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as a direct summand.

6.3.3. The stable $kG$-module structure when $\epsilon = \epsilon'$ and $\epsilon' = 1$. This is the case when $\ell \equiv 1 \mod 4$ and $\ell \equiv 1 \mod 3$. By §6.2.4, the non-projective indecomposable direct summands of $\operatorname{Res}_N^G H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are those in the direct sum in Equation (6.18). As recorded in [5] §6.4.3, it follows from [7] §III that there are $1 + (n'/2)$ blocks of $kG$ of maximal defect $n$, consisting of the principal block $B_{00}$, another block $B_{01}$, and $(n'/2 - 1)$ blocks $B_{11}, \ldots, B_{(n'/2 - 1)}$, and there are $(\ell - 1)/4$ blocks of $kG$ of defect 0. There are precisely two isomorphism classes of simple $kG$-modules that belong to $B_{00}$ (resp. $B_{01}$, represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $T_1$ of $k$-dimension $\ell$ (resp. by two simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ of $k$-dimension $(\ell + 1)/2$). For each $t \in \{1, \ldots, (n'/2 - 1)\}$, there is precisely one isomorphism class of simple $kG$-modules belonging to $B_t$, represented by a simple $kG$-module $T_t$ of $k$-dimension $\ell + 1$.

As in [5] §6.4.3, it follows that the Green correspondent of $S_0^{(N_i)}$ is $T_0$, the Green correspondent of $S_1^{(N_i)}$ is $T_1$, and the Green correspondent of $\overline{S}_1^{(N_i)}$ is $\overline{T}_1$, for $1 \leq t \leq (n'/2 - 1)$. On the other hand, the Green correspondent of $S_{0,0}^{(N_i)}$ is one of $T_{0,1}$ or $T_{1,0}$. We relabel the simple $kG$-modules, if necessary, to be able to assume that the Green correspondent of $S_{0,0}^{(N_i)}$ is $T_{0,1}$ and the Green correspondent of $S_{1,0}^{(N_i)}$ is $T_{1,0}$. Because of our assumptions on $\epsilon$ and $\epsilon'$, all indecomposable $kG$-modules are uniserial. Using similar arguments as in §6.3.2 we obtain that the non-projective indecomposable direct $kG$-module summands of $H^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are as stated in part (i)(3) of Theorem 5.8. It follows that for every possible value of $\delta_0, \delta_1, \delta_m$, the $kG$-module $H^0(X, \Omega_X^{\otimes m})$ has, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as a direct summand.
6.3.4. The stable $kG$-module structure when $\epsilon = \epsilon'$ and $\epsilon' = -1$. This is the case when $\ell \equiv -1 \mod 4$ and $\ell \equiv -1 \mod 3$. By the non-projective indecomposable direct summands of $\text{Res}_R^G \mathcal{H}^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are again those in the direct sum in Equation (6.11). As recorded in [4] §6.4.4, it follows from [7] §VI that there are $1 + (n'/2)$ blocks of $kG$ of maximal defect $n$, consisting of the principal block $B_0$, another block $B_01$, and $(n'/2 - 1)$ blocks $B_1, \ldots, B_{n'(2 - 1)}$, and there are $(\ell - 3)/4$ blocks of $kG$ of defect 0. There are precisely two isomorphism classes of simple $kG$-modules that belong to $B_0$ (resp. $B_01$), represented by the trivial simple $kG$-module $T_0$ and a simple $kG$-module $T_01$ of $k$-dimension $\ell - 1$ (resp. by two simple $kG$-modules $T_{0,1}$ and $T_{1,0}$ of $k$-dimension $(\ell - 1)/2$). For each $t \in \{1, \ldots, (n'/2 - 1)\}$, there is precisely one isomorphism class of simple $kG$-modules belonging to $B_t$, represented by a simple $kG$-module $T_t$ of $k$-dimension $\ell - 1$.

Using Notation 5.7, it follows, as in [3] §6.4.4, that the Green correspondent of $S_{0,0}^{(N_1)}$ is $T_0$, the Green correspondent of $S_{1,1}^{(N_1)}$ is $U_{T_0,3n-1}$, and the Green correspondent of $S_{1}^{(N_1)}$ is $U_{T,3n-1}$, for $1 \leq t \leq (n'/2 - 1)$. On the other hand, the Green correspondent of $S_{1}^{(N_1)}$ is a uniserial $kG$-module of length $3n - 1$ whose socle is isomorphic to either $T_{0,1}$ or $T_{1,0}$. We relabel the simple $kG$-modules, if necessary, to be able to assume that the Green correspondent of $S_{0,1}^{(N_1)}$ is $U_{T_0,1,3n-1}$ and the Green correspondent of $S_{1}^{(N_1)}$ is $U_{T_1,3n-1}$. Because of our assumptions on $\epsilon$ and $\epsilon'$, the Green correspondents of $U_{0,0}^{(N_1)}$, $U_{0,0}^{(N_1)}$ and $U_{1,1,2}^{(N_1)}$ are indecomposable $kG$-modules that are not uniserial. Similarly to [3] §6.1 we obtain, using Notation 6.7, that the Green correspondent of $U_{0,0,3n-1}^{(N_1)}$ is $U_{T_0,1,3n-1}^{(G)}$, the Green correspondent of $U_{0,0,3n-1}^{(N_1)}$ is $U_{T_0,1,3n-1}^{(G)}$, and the Green correspondent of $U_{1,1,2}^{(N_1)}$ is $U_{T_0,1,3n-1}^{(G)}$. Using similar arguments as in §6.3.1 we obtain that the non-projective indecomposable direct $kG$-module summands of $\mathcal{H}^0(X, \Omega_X^{\otimes m})$, with their multiplicities, are as stated in part (ii) of Theorem 6.3. It follows that for every possible value of $\delta_0, \delta_1, \delta_m$, the $kG$-module $\mathcal{H}^0(X, \Omega_X^{\otimes m})$ has, for each block $B$ of $kG$, at most one non-projective indecomposable $kG$-module belonging to $B$ as a direct summand.

6.4. The Brauer character of the holomorphic poly-differentials. We use the same strategy as in [4] §6.3 to compute the values of the Brauer character of $\mathcal{H}^0(X, \Omega_X^{\otimes m})$ at all elements $g \in G$ that are 3-regular, i.e. whose order is not divisible by 3.

The main difference when $m > 1$ is that the values of the Brauer character of $\mathcal{H}^0(X, \Omega_X^{\otimes m})$ at elements of $G$ of order $\ell$ depend on the congruence class of $m$ modulo $\ell$.

6.4.1. The values of the Brauer character at elements of order $\ell$. As recorded in [4] §6.3, it follows from [16] §II.8 that the elements of order $\ell$ fall into 2 conjugacy classes. Moreover, we can choose representatives $r_1$ and $r_2$ of these classes in such a way that $R = \langle r_1 \rangle = \langle r_2 \rangle$ and $r_2 = r_1^\ell$ where $\mu \in \{1, 2, \ldots, \ell - 1\}$ with $\ell = 1$. Let $Y = X$ and $Z = X/R$, and let $\lambda : Y = X \rightarrow Z$ be the corresponding tamely ramified Galois cover with Galois group $R$. We apply Theorem 1.3 to $E = mK_X$ and the group $R$. Since $\#(I \cap R) = 1$, $n_I = 0$ and $E_0 = mK_X | Y = X$. As in Table 6.1 we write $Z_{br} = \{z_1, \ldots, z_{\ell - 1}/2\}$. Because $E_0 = mK_X$, the information from this table implies, for $1 \leq i \leq \ell - 1$,

$$\ell x(z_i, 0) = \ell - 1 - m \ell,$$

where $m \ell \in \{0, 1, \ldots, \ell - 1\}$ satisfies $m \ell \equiv m - 1 \mod \ell$.

Since for all $1 \leq i \leq \ell - 1$, $R_{x(z_i)} = R$ and since $Y = X$, Equations 1.4 and 1.5 imply that the Brauer character of $\text{Res}_R^G \mathcal{H}^0(X, \Omega_X^{\otimes m})$ equals

$$\sum_{i=1}^{\ell-2/\ell} \left( \sum_{d=1}^{\ell-1-m-1} \theta_{x(z_i)} - \sum_{d=1}^{\ell-1} \theta_{x(z_i)} \right) + n_0(R) \beta(kR).$$

Since $\beta(kR)$ has zero value at all non-identity elements of $R$, we do not need to determine $n_0(R)$. 

28
By [10] §II.8, the normalizer \( N_G(R) \) is a semidirect product with normal subgroup \( R \) and cyclic quotient group of order \((\ell - 1)/2\). This implies, by [29] p. 193, that there is precisely one point on \( X/N_G(R) \) that lies below the points \( z_1, \ldots, z_{(\ell - 1)/2} \) of \( Z = X/R \). Hence \( N_G(R) \) permutes transitively the points \( x(z_1), \ldots, x(z_{(\ell - 1)/2}) \) of \( X \) lying above them. Let \( \xi \) be a primitive \( \ell \)th root of unity such that \( \theta_{x(z_1)}(r_1) = \xi \).

Since \( \theta_{x(z_1)}(r_1) = \theta_{x(z_1)}(g r g^{-1}) \) for all \( g \in N_G(R) \) and since, by [10] §II.8, the conjugates of \( r_1 \) by \( N_G(R) \) are precisely the elements \( (r_1)^{a^2} \) for \( 1 \leq a \leq (\ell - 1)/2 \), we obtain

\[
(6.20) \quad \{ \theta_{x(z_1)}(r_1) : 1 \leq i \leq (\ell - 1)/2 \} = \{ \xi^a : 1 \leq a \leq (\ell - 1)/2 \}.
\]

Using Gauss sums, we see, similarly to [5, §6.3.1], that there exists a choice of square root of \( \ell', \) say \( \sqrt[\ell]{\ell} \), such that, for all positive integers \( d, \)

\[
(6.21) \quad \sum_{a=1}^{(\ell - 1)/2} (\xi^a)^{d} = -\frac{1 + (d)}{2} \sqrt[\ell]{\ell} \quad \text{and} \quad \sum_{a=1}^{(\ell - 1)/2} (\xi^a)^{d} = -\frac{1 - (d)}{2} \sqrt[\ell]{\ell}
\]

where \( \left(\frac{\cdot}{\ell}\right) \) denotes the Legendre symbol. Using Equations (6.20) and (6.21), we get

\[
(6.22) \quad \sum_{i=1}^{(\ell - 1)/2} \sum_{d=0}^{\ell - 1 - m_\ell} \theta_{x(z_1)}^{d}(r_1) = -\frac{\ell - 1}{2} + m_\ell - \sum_{d=1}^{m_\ell} (\frac{d}{\ell}) \sqrt[\ell]{\ell}.
\]

Letting \( h_\ell = h_{Q(\sqrt[\ell]{\ell})} \) be the class number of \( Q(\sqrt[\ell]{\ell}) \), it follows by [5, §6.3.1] that

\[
(6.23) \quad \left( -\sum_{d=0}^{\ell - 1} \frac{\eta_{x(z_1)}^{d}}{d} \right) (r_1) = \begin{cases} \ell - 1 & \text{if } \epsilon' = 1, \\ \ell - 1 & \text{if } \epsilon' = -1. \end{cases}
\]

Replacing \( \xi \) by \( \xi^d \) results in replacing \( \sqrt[\ell]{\ell} \) by \( -\sqrt[\ell]{\ell} \) in Equation (6.22), and hence also in Equation (6.23). Running the computations in [5, §6.3.1] with \( \xi^d \) instead of \( \xi \), we see that this also results in replacing \( \sqrt[\ell]{\ell} \) by \( -\sqrt[\ell]{\ell} \) in Equation (6.23) when \( \epsilon' = -1 \). Since \( \theta_{x(z_1)}(r_2) = \theta_{x(z_1)}(r_1^{m_\ell}) = \xi_{\ell}^{m_\ell} \), we therefore obtain from Equations (6.20), (6.22) and (6.23) that

\[
(6.24) \quad \beta(H^0(X, \Omega_X^{\otimes m}))(r_\lambda) = \begin{cases} \ell - 1 & \text{if } \epsilon' = 1, \\ \ell - 1 & \text{if } \epsilon' = -1. \end{cases}
\]

for \( b \in \{1, 2\} \), where \( m_\ell \in \{0, 1, \ldots, \ell - 1\} \) satisfies \( m_\ell \equiv m - 1 \mod \ell \).

6.4.2. The values of the Brauer character at all 3-regular elements of \( G \). By Notation 6.2 and Remark 6.3 \( v \) is an element of order \((\ell - \epsilon)/2 = 3^n \cdot n' \), where \( n' \) is not divisible by 3, \( s \) is an element of order 2, and \( w \) is an element of order \((\ell + \epsilon)/2 \). Let \( v'' = v^{3^n} \) be of order \( n' \). By [10] §II.8, a full set of representatives of the conjugacy classes of 3-regular elements of \( G \) is given by Table 6.2

| representative | class length | \( r_b \) | \( s \) | \((v''^i) \) | \( w^j \) |
|---------------|-------------|---------|------|---------|--------|
| \( 1_G \)     | 1           | \( \frac{\ell^2 - 1}{2} \) | \( \frac{\ell(\ell + \epsilon)}{2} \) | \( \ell(\ell + \epsilon) \) | \( \ell(\ell - \epsilon) \) |

where \( b \in \{1, 2\}, 1 \leq i < \frac{n'}{2}, 1 \leq j < \frac{\ell + \epsilon}{2} \).
From Equation (6.24), we know the values of $\beta(H^0(X, \Omega_X^{\otimes m}))$ at $r_b$, for $b \in \{1, 2\}$. The other values of $\beta(H^0(X, \Omega_X^{\otimes m}))$ are as follows:

\[
\beta(H^0(X, \Omega_X^{\otimes m}))(1_G) = (2m - 1) \frac{(\ell^2 - 1)(\ell - 6)}{24},
\]
\[
\beta(H^0(X, \Omega_X^{\otimes m}))(s) = (1 - 2\delta_m) \frac{\ell - \ell'}{4},
\]
\[
\beta(H^0(X, \Omega_X^{\otimes m}))(\xi^i) = 0,
\]
\[
\beta(H^0(X, \Omega_X^{\otimes m}))(\mu^i) = 0.
\]

where $1 \leq i < \frac{w}{2}$ and $1 \leq j < \frac{\ell + \epsilon}{4}$.

Note that we obtain Equation (6.25) by using the Riemann-Roch theorem, together with the formula for the genus $g(X)$ from [3, Cor. 3.2]. Equation (6.26) follows from Equations (6.12) and (6.13), and Equation (6.27) follows from Equations (6.7) and (6.8).

For Equation (6.28), we consider the cyclic subgroup $W = \langle w \rangle$ of $G$. Since $\#W = (\ell + \epsilon)/2$ is not divisible by 3, there are $\#W$ isomorphism classes of indecomposable $kW$-modules, which are all simple of $k$-dimension one. We denote representatives of these by $S_a^{(W)}$, for $0 \leq a \leq \#W - 1$. Let $Y = X$ and $Z = X/W$, and let $\lambda : Y = X \rightarrow Z$ be the corresponding tamely ramified Galois cover with Galois group $W$. We apply Theorem 1.6 to $E = mK_X$ and the group $W$. Since $\#(I \cap W) = 1$, $n_I = 0$ and $E_0 = mK_X$ on $Y = X$. As in Table 6.1, if $\epsilon = -\epsilon'$ then $Z_{br} = \{z_1, z_2\}$, and if $\epsilon = \epsilon'$ then $Z_{br} = \emptyset$. In the former case, we use that $E_0 = mK_X$, together with the information from Table 6.1 to obtain that $\ell(y(z_i), 0) = \delta_m$ for $i = 1, 2$. Let $0 \leq a \leq \#W - 1$. Using that, for $\epsilon = -\epsilon'$ and $i = 1, 2$, $\mu_{a, -1}(y(z_i)) = \mu_{a, 1}(y(z_i)) = 1$ if and only if $a$ is odd, we get from Equations (1.4) and (1.5) that

\[
n_{a, 1}^{(W)} - n_0(W) = \begin{cases} 2(\delta_m - \frac{1}{2}), & \epsilon = -\epsilon' \text{ and } a \text{ odd}, \\ 0, & \epsilon = \epsilon' \text{ or } a \text{ even}. \end{cases}
\]

This implies

\[
\Res_W^G H^0(X, \Omega_X^{\otimes m}) \cong \delta_{\epsilon, -\epsilon'}(2\delta_m - 1) \bigoplus_{t=0}^{\frac{\#W - 1}{2}} S_{2t+1}^{(W)} \oplus n_0(W) kW.
\]

Let $\xi_w$ be a fixed primitive $(\#W)^{th}$ root of unity so that, for $0 \leq a \leq \#W - 1$, $w$ acts as multiplication by $\xi_w^a$ on $S_a^{(W)}$. Then Equation (6.29) implies, for $1 \leq j < \frac{\ell + \epsilon}{4}$,

\[
\beta(H^0(X, \Omega_X^{\otimes m}))(\mu^i) = \delta_{\epsilon, -\epsilon'}(2\delta_m - 1) \sum_{t=0}^{\frac{\#W - 1}{2}} \xi_w^{(2t+1)} = \delta_{\epsilon, -\epsilon'}(2\delta_m - 1) \xi_w^{\frac{\#W - 1}{2}} \sum_{t=0}^{\#W} (\xi_w^2)^{ij} = 0,
\]

where the last equality follows, since, if $\epsilon = -\epsilon'$ then $\#W$ is even, and hence $\frac{\#W}{2} - 1 = \frac{\#W}{2} - 1$ and $\xi_w^2$ is a primitive $(\frac{\#W}{2})^{th}$ root of unity.

6.5. The $kG$-module structure of the holomorphic poly-differentials. We use the same strategy as in [3, §6.4], i.e. we determine the $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$ using $6.3$ and $6.4$. In $6.3.1$ and $6.3.4$, we already determined the non-projective indecomposable $kG$-modules, with their multiplicities, that are direct summands of $H^0(X, \Omega_X^{\otimes m})$ and we showed that these are as stated in parts (i)(1) - (i)(4) of Theorem 5.8. Hence, to determine the full $kG$-module structure of $H^0(X, \Omega_X^{\otimes m})$, it remains to determine the decomposition of the projective $kG$-module $Q_2$ in part (i) of Theorem 5.8 into a direct sum of projective indecomposable $kG$-modules. We will do this in $6.5.1$ and $6.5.2$. The remainder of Theorem 5.8 is then proved based on these results, using similar arguments as in [3, §6.5].
As in [§ 6.4], let \( \tilde{\beta} \) denote the Brauer character of the largest projective direct summand of \( H^0(X, \Omega_X^{\otimes m}) \). In other words, \( \tilde{\beta} \) is the Brauer character of the projective \( kG \)-module \( Q_\ell \) in part (i) of Theorem 5.3. Let \( \text{IBr}(kG) \) denote the set of Brauer characters of simple \( kG \)-modules, and for each \( \phi \in \text{IBr}(kG) \), let \( E(\phi) \) be a simple \( kG \)-module with Brauer character \( \phi \) and let \( P(G, E(\phi)) \) be a projective \( kG \)-module cover of \( E(\phi) \).

The strategy for § 6.5.1 and § 6.5.2 is as follows. We will first determine the Brauer character \( \tilde{\beta} \), by using Equations (6.24) - (6.28), together with the description of the non-projective indecomposable direct \( kG \)-module summands of \( H^0(X, \Omega_X^{\otimes m}) \) from § 6.3.1, § 6.3.4. For all \( \phi \in \text{IBr}(kG) \), we will then calculate

\[
\langle \tilde{\beta}, \phi \rangle = \frac{1}{\#G} \sum_{x \in G'_3} \tilde{\beta}(x)\phi(x^{-1})
\]

where \( G'_3 \) denotes the set of 3-regular elements of \( G \). Since \( \langle \tilde{\beta}, \phi \rangle \) equals the multiplicity of the projective \( kG \)-module \( P(G, E(\phi)) \) as a direct summand of \( Q_\ell \), we obtain

\[
Q_\ell = \bigoplus_{\phi \in \text{IBr}(kG)} \langle \tilde{\beta}, \phi \rangle P(G, E(\phi))
\]

which will lead to the precise \( kG \)-module structure of \( H^0(X, \Omega_X^{\otimes m}) \).

It follows from [§ III–§ VI] that all elements in \( \text{IBr}(kG) \) can be described in terms of ordinary irreducible characters of \( G \). To give these descriptions in § 6.5.1 and § 6.5.2, we list the relevant ordinary irreducible characters, together with their values at all 3-regular conjugacy classes, in Table 6.3. Here \( \xi_\omega \) is a fixed primitive \((n')^{th}\) root of unity and \( \xi_\omega \) is a fixed primitive \((\ell/4)^{th}\) root of unity. Note that Table 6.3 combines all cases of \( \epsilon, \epsilon' \in \{\pm 1\} \) and uses the labels of the ordinary characters from [§ IV], where we replace \( \delta^* \) by \( \tilde{\delta}^* \) to avoid confusion with other uses of \( \delta \).

**Table 6.3.** Restrictions of important ordinary irreducible characters of \( G = \text{PSL}(2, F_\ell) \) to the 3-regular conjugacy classes.

| \( \delta_0^s \) | \( \delta_t^s \) | \( \gamma_a \) | \( \eta_u^G \) | \( 1_G \) | \( r_b \) | \( s \) | \( (v'^r)_i \) | \( w^j \) |
|---|---|---|---|---|---|---|---|---|
| \( \ell + \epsilon \) | \( \ell + \epsilon \) | \( \ell = \epsilon \) | \( \ell = \epsilon \) | \( \epsilon \) | \( \epsilon \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( 2 \epsilon \) | \( 2 \epsilon \) | \( 0 \) | \( 0 \) |
| \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| (1) \) | \( \epsilon |\epsilon' + \epsilon| (1) \) | \( 0 \) | \( 0 \) |
| \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( 1 \) | \( 1 \) |
| \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( \epsilon |\epsilon' + \epsilon| \) | \( 1 \) | \( 1 \) |

where \( a, b \in \{1, 2\} \), \( 1 \leq i \leq t < n/2 \), \( 1 \leq j \leq \ell/4 + u < \ell/4 \).

6.5.1. The largest projective direct \( kG \)-module summand of \( H^0(X, \Omega_X^{\otimes m}) \) when \( \epsilon = \epsilon' \). We use [§ IV and §V] to give a description of \( \text{IBr}(kG) \) using the restrictions of the ordinary irreducible characters in Table 6.3. Let \( \psi_0 \) denote the Brauer character of the trivial simple \( kG \)-module \( T_0 \). If \( \epsilon = 1 \) then \( \delta_0^s \) gives the Brauer character of the simple \( kG \)-module \( T_0 \). If \( \epsilon = 1 \) then the Brauer character of the simple \( kG \)-module \( T_1 \) is given by \( \psi_1 = \delta_0^s - \psi_0 \). In both cases, for \( 1 \leq t \leq n/2 - 1 \), the Brauer character of the simple \( kG \)-module \( T_t \) is given by \( \tilde{\delta}^s_t \). There are \( 1 + \ell/4 \) additional Brauer characters of simple \( kG \)-modules that are also projective, given by \( \gamma_a, a \in \{1, 2\} \), and \( \eta_u^G \), \( 1 \leq u \leq \ell/4 + 1 \). Therefore, if \( \epsilon = \epsilon' \) then

\[
\text{IBr}(kG) = \{ \psi_0, \psi_0', \delta_0^s, \gamma_a, \eta_u^G : 1 \leq t \leq n/2 - 1, 1 \leq a \leq 2, 1 \leq u \leq \ell/4 + 1 \}
\]

where \( \psi_0' = \delta_0^s \) if \( \epsilon = -1 \) and \( \psi_0' = \psi_1 \) if \( \epsilon = 1 \), i.e. \( \psi_0' = \delta_0^s - 1 + \frac{\ell}{4} \psi_0 \).
We determine the Brauer character $\tilde{\beta}$ by using Equations (6.24) - (6.28), together with Table 6.3 applied to the composition factors of the non-projective indecomposable direct summands of $H^0(X, \Omega_X^{\otimes m})$ from (6.3.1) and (6.3.2) which are as in parts (i)(1) and (i)(2) of Theorem 5.8.

Suppose first that $(\epsilon, \epsilon') = (-1, 1)$, so we are in part (i)(1) of Theorem 5.8. Let $\beta$ be the Brauer character of $H^0(X, \Omega_X^{\otimes m})$. Using Notation 6.7 together with the above description of $\text{IBr}(kG)$, it follows that

$$\tilde{\beta}(1_G) = \frac{(2m-1)(\ell^2 - 1)(\ell - 6)}{24} \cdot \frac{\ell + 1}{2}(2\delta_0(1 - \delta_m) + \delta_1) - \frac{\ell^2 - 1}{12}(2\delta_0 + \delta_1),$$

$$\tilde{\beta}(r_b) = \frac{\ell - 1}{4} + \frac{m_\ell + (-1)^b \sum_{i=1}^{m_\ell} \left(\frac{d}{b}\right)}{2} \cdot \frac{\sqrt{\ell}}{2} + \frac{\ell + 1}{12}(2\delta_0 + \delta_1) - \frac{2\delta_0(1 - \delta_m) + \delta_1}{2},$$

$$\tilde{\beta}(s) = (1 - 2\delta_m) \frac{\ell - 1}{4} - (2\delta_0(1 - \delta_m) + \delta_1),$$

$$\tilde{\beta}(w^j) = -2\delta_0(1 - \delta_m) + \delta_1, \quad \text{for } b \in \{1, 2\}, \ 1 \leq i \leq \frac{m_\ell - 1}{2}, \text{ and } 1 \leq j \leq \frac{\sqrt{\ell} - 1}{2}.$$

To determine $\langle \tilde{\beta}, \phi \rangle$ for all $\phi \in \text{IBr}(kG)$ when $(\epsilon, \epsilon') = (-1, 1)$, we use Equation (6.3.10), together with the conjugacy class lengths from Table 6.2. For example, we have, for $0 \leq t \leq \frac{\ell - 1}{2}:

$$\langle \tilde{\beta}, \tilde{\delta}_t \rangle = \frac{1}{#G} \left\{ \tilde{\beta}(1_G)\tilde{\delta}_t(1_G) + \frac{\ell^2 - 1}{2}\tilde{\beta}(r_1)\tilde{\delta}_t(r_1^{-1}) + \frac{\ell^2 - 1}{2}\tilde{\beta}(r_2)\tilde{\delta}_t(r_2^{-1}) \right\} = \frac{2}{\ell(\ell^2 - 1)} \left\{ \frac{(2m-1)(\ell^2 - 1)(\ell - 6)}{24} - \frac{\ell + 1}{2}(2\delta_0(1 - \delta_m) + \delta_1) - \frac{\ell^2 - 1}{12}(2\delta_0 + \delta_1) \right\} (\ell - 1)$$

$$+ \frac{\ell^2 - 1}{2} \left[ - \frac{\ell - 1}{4} + \sum_{i=1}^{m_\ell} \left(\frac{d}{b}\right) \sqrt{\ell} \cdot \frac{\sqrt{\ell}}{2} + \frac{\ell + 1}{12}(2\delta_0 + \delta_1) - \frac{2\delta_0(1 - \delta_m) + \delta_1}{2} \right] (\ell - 1)$$

$$+ \frac{\ell^2 - 1}{2} \left[ - \frac{\ell - 1}{4} + \sum_{i=1}^{m_\ell} \left(\frac{d}{b}\right) \sqrt{\ell} \cdot \frac{\sqrt{\ell}}{2} + \frac{\ell + 1}{12}(2\delta_0 + \delta_1) - \frac{2\delta_0(1 - \delta_m) + \delta_1}{2} \right] (\ell - 1)$$

$$= \frac{(2m-1)(\ell - 7) - 4(2\delta_0 + \delta_1) + 6 + m - 1 - m_\ell}{\ell}.$$
Performing similar computations in the case when \( (\epsilon, \epsilon') = (1, -1) \), we obtain the following result, which gives a detailed description of \( Q_\ell \) in Theorem 5.3.1(1),(2).

**Proposition 6.5.1.** Suppose \( \epsilon = -\epsilon' \) in Notation 6.2. Let \( \tilde{\beta} \) be the Brauer character of the largest projective direct \( kG \)-module summand of \( H^0(X, \Omega_X^{\otimes m}) \), which is denoted by \( Q_\ell \) in Theorem 5.3.1. Then \( Q_\ell \) is as in Equation (6.31), where \( \langle \tilde{\beta}, \phi \rangle \) is given by:

\[
\begin{align*}
\langle \tilde{\beta}, \psi_0 \rangle &= \frac{m - 2 - 2(2\delta_0 + \delta_1) + 3\delta_m(2\delta_0 - 1)}{\ell} - \frac{m - 1 - m_\ell}{\ell}, \\
\langle \tilde{\beta}, \psi_0' \rangle &= \frac{(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{3\epsilon}{2} \delta_0 + \frac{3\epsilon'}{2} \delta_1) - 6\epsilon}{12} - \frac{m - 1 - m_\ell}{\ell} \frac{1 + \epsilon}{2} - \langle \tilde{\beta}, \psi_0 \rangle, \\
\langle \tilde{\beta}, \delta^* \rangle &= \frac{(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{3\epsilon}{2} \delta_0 + \frac{3\epsilon'}{2} \delta_1) - 6\epsilon}{12} - \frac{m - 1 - m_\ell}{\ell}, \\
\langle \tilde{\beta}, \gamma_a \rangle &= \frac{(2m - 1)(\ell - 6 + \epsilon) + 6\epsilon (1 - (1 - 2\delta_m)(-1)^{(\ell - \epsilon')/4})}{24} + \frac{m - 1 - m_\ell}{\ell} + \epsilon \frac{1 - \epsilon}{2} - \frac{m_\ell}{\ell}, \\
\langle \tilde{\beta}, \eta_u^G \rangle &= \frac{(2m - 1)(\ell - 6 + \epsilon) + 6\epsilon (1 - (1 - 2\delta_m)(-1)^{\epsilon})}{12} + \frac{m - 1 - m_\ell}{\ell},
\end{align*}
\]

for \( 1 \leq t \leq \frac{n'_1}{2} \), \( a(1, 2) \), and \( 1 \leq u \leq \frac{\ell + \epsilon}{4} + 1 \).

6.5.2. The largest projective direct \( kG \)-module summand of \( H^0(X, \Omega_X^{\otimes m}) \) when \( \epsilon = \epsilon' \). We use \( \mathbb{1} \) §III and §VI to give a description of \( IBr(kG) \) using the restrictions of the ordinary irreducible characters in Table 6.3.

Let \( \psi_0 \) denote the Brauer character of the trivial simple \( kG \)-module \( T_0 \). If \( \epsilon = 1 \) then the Brauer character of the simple \( kG \)-module \( T_1 \) is given by \( \psi_1 = \delta^*_0 - \psi_0 \). If \( \epsilon = -1 \) then \( \delta^*_0 \) gives the Brauer character of the simple \( kG \)-module \( T_0 \). In both cases, for \( 1 \leq t \leq \frac{n'}{2} - 1 \), the Brauer character of the simple \( kG \)-module \( T_t \) is given by \( \delta^*_t \). Moreover, the Brauer characters of the simple \( kG \)-modules \( T_{0,1} \) and \( T_{1,0} \) are given by \( \gamma_1 \) and \( \gamma_2 \). Note that these characters only differ with respect to their values on the elements of order \( \ell \) in \( G \). Since we have already chosen a square root of \( \epsilon \ell \) to obtain Equations (6.21) and (6.24), we define \( s_{01} \in \{1, 0\} \) such that the Brauer character \( \psi_{0,1} \) of \( T_{0,1} \) satisfies

\[
(6.32) \quad \psi_{0,1}(r_1) = \frac{\ell' + s_{01} \sqrt{\epsilon} \ell}{2}.
\]

There are \( \frac{\ell + \epsilon}{4} \) additional Brauer characters of simple \( kG \)-modules that are also projective, given by \( \eta^G_u \), \( 1 \leq u \leq \frac{\ell + \epsilon}{4} - 1 \). Therefore, if \( \epsilon = \epsilon' \) then

\[
IBr(kG) = \left\{ \psi_0, \psi_0', \psi_{0,1}, \psi_{1,0}, \delta^*_t, \eta^G_u : 1 \leq t \leq \frac{n}{2} - 1, 1 \leq u \leq \frac{\ell + \epsilon}{4} - 1 \right\}
\]

where \( \psi_0' = \psi_1 \) if \( \epsilon = 1 \) and \( \psi_0' = \delta^*_0 \) if \( \epsilon = -1 \), i.e. \( \psi_0' = \delta^*_0 + \frac{1 + \epsilon}{2} \psi_0 \).

Similarly to 6.5.1 we determine the Brauer character \( \tilde{\beta} \) by using Equations (6.22) - (6.28), together with Table 6.3 applied to the composition factors of the non-projective indecomposable direct summands of \( H^0(X, \Omega_X^{\otimes m}) \) from 6.3.3 and 6.3.4, which are as in parts (i)(3) and (i)(4) of Theorem 5.8.

Performing similar computations as in 6.5.1 we obtain the following result, which gives a detailed description of \( Q_\ell \) in Theorem 5.3.1(3),(4).

**Proposition 6.5.2.** Suppose \( \epsilon = \epsilon' \) in Notation 6.2. Let \( \tilde{\beta} \) be the Brauer character of the largest projective direct \( kG \)-module summand of \( H^0(X, \Omega_X^{\otimes m}) \), which is denoted by \( Q_\ell \) in Theorem 5.3.1. Then \( Q_\ell \) is as in
Equation (6.31), where $⟨\tilde{\beta}, \phi⟩$ is given by:

\[
\begin{align*}
⟨\tilde{\beta}, \psi_0⟩ &= \frac{m - 2 - 2(2\delta_0 + \delta_1) + 3\delta_m(2\delta_0 - 1)}{6\ell} - m - 1 - m_\ell, \\
⟨\tilde{\beta}, \psi'_0⟩ &= \frac{(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{3}{2} + \epsilon)\delta_0 + \frac{3}{2} \epsilon \delta_1) - 12\epsilon \delta_m}{12} - \frac{m - 1 - m_\ell}{\ell} - \frac{1 + \epsilon}{2} ⟨\tilde{\beta}, \psi_0⟩, \\
⟨\tilde{\beta}, \psi_{i,1-i}⟩ &= \frac{(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{3}{2} + \epsilon)\delta_0 + \frac{3}{2} \epsilon \delta_1)}{24} \\
&\quad - 6\epsilon \left(1 - (1 - 2\delta_m) \left((-1)^{(1-\epsilon)/4} - (-1)^{(1+\epsilon)(1+\epsilon)}\right)\right) - \frac{m - 1 - m_\ell}{2\ell} \\
&\quad + \frac{(-1)^i s_{\ell}^1}{2} \left(1 - \frac{1}{2} \text{h}_\ell - \sum_{d=1}^{m_\ell}\left(\frac{d}{\ell}\right)\right) \\
⟨\tilde{\beta}, \tilde{\delta}^*_i⟩ &= \frac{(2m - 1)(\ell - 6 + \epsilon) - 4(\frac{3}{2} + \epsilon)\delta_0 + \frac{3}{2} \epsilon \delta_1)}{12} - \frac{m - 1 - m_\ell}{\ell}, \\
⟨\tilde{\beta}, \eta^G_{i,\ell}⟩ &= \frac{(2m - 1)(\ell - 6 - \epsilon) + 6\epsilon}{12} + \frac{m - 1 - m_\ell}{\ell},
\end{align*}
\]

for $i \in \{0, 1\}$, $1 \leq t \leq \frac{m_\ell}{2} - 1$, and $1 \leq u \leq \frac{t + \epsilon - 2}{2}$.

References

[1] J. L. Alperin. Local representation theory. Cambridge Studies in Advanced Mathematics, No. 11. Cambridge University Press, Cambridge, 1986.
[2] M. Auslander, I. Reiten, and S. O. Smalo. Representation theory of Artin algebras, volume 36 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.
[3] P. Bending, A. Camina, and R. Guralnick. Automorphisms of the modular curve. In Progress in Galois theory, volume 12 of Dev. Math., pages 25–37. Springer, New York, 2005.
[4] J. Bertin and A. M{é}zard. D{é}formations formelles des revêtments sauvagement ramifiés de courbes algébriques. Invent. Math., 141(1):195–238, 2000.
[5] F. M. Bleher, T. Chinburg, and A. Kontogeorgis. Galois structure of the holomorphic differentials of curves. J. Number Theory, 216:1–68, 2020.
[6] N. Borne. Cohomology of G-sheaves in positive characteristic. Adv. Math., 201(2):454–515, 2006.
[7] R. Burkhardt. Die Zerlegungsmatrizen der Gruppen PSL(2, p^f). J. Algebra, 40(1):75–96, 1976.
[8] A. Carocca and D. Vásquez. Group actions on Riemann-Roch space. Preprint, arXiv:1904.02748, 2015.
[9] C. Chevalley, A. Weil, and E. Hecke. Über das Verhalten der Integrale 1. Gattung bei Automorphismen des Funktionenkörpers. Abh. Math. Sem. Univ. Hamburg, 10(1):358–361, 1934.
[10] C. W. Curtis and I. Reiner. Methods of representation theory. Vol. II. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1987.
[11] A. Carocca and D. Vásquez. Group actions on Riemann-Roch space. Preprint, arXiv:1904.02748, 2015.
[12] L. V. Dieulefait, J. Jiménez Urrutia, and K. A. Ribet. Modular forms with large coefficient fields via congruences. Res. Number Theory, 1:Art. 2, 14, 2015.
[13] D. Glass, D. Joyner, and A. Ksir. Codes from Riemann-Roch spaces for $y^2 = x^p - x$ over GF(p). Int. J. Inf. Coding Theory, 1(3):298–312, 2010.
[14] R. Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York, 1977.
[15] E. Hecke. Über ein Fundamentalproblem aus der Theorie der elliptischen Modulfunktionen. Abh. Math. Sem. Univ. Hamburg, 6(1):235–257, 1928.
[16] B. Huppert. Endliche Gruppen. I. Springer-Verlag, Berlin, 1967. Grundlehren der Math. Wissenschaften, Band 134.
[17] A. Hurwitz. Über algebraische Gebilde mit eindeutigen Transformationen in sich. Math. Ann., 41(3):403–442, 1892.
[18] D. Joyner and A. Ksir. Decomposition representations of finite groups on Riemann-Roch spaces. Proc. Amer. Math. Soc., 135(11):3465–3476, 2007.
[19] E. Kani. The Galois-module structure of the space of holomorphic differentials of a curve. J. Reine Angew. Math., 367:187–206, 1986.
[20] S. Karanikolopoulos. On holomorphic polydifferentials in positive characteristic. Math. Nachr., 285(7):852–877, 2012.
[21] S. Karanikolopoulos and A. Kontogeorgis. Representation of cyclic groups in positive characteristic and Weierstrass semi-groups. J. Number Theory, 133(1):158–175, 2013.
[22] N. M. Katz. p-adic properties of modular schemes and modular forms. In Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pages 69–190. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.
[23] N. M. Katz and B. Mazur. Arithmetic moduli of elliptic curves. Princeton University Press, Princeton, 1985.
[24] B. Köck. Galois structure of Zariski cohomology for weakly ramified covers of curves. Amer. J. Math., 126(5):1085–1107, 2004.
[25] B. Köck and A. Kontogeorgis. Quadratic differentials and equivariant deformation theory of curves. Ann. Inst. Fourier (Grenoble), 62(3):1015–1043, 2012.
[26] B. Köck and J. Tait. Faithfulness of actions on Riemann-Roch spaces. Canad. J. Math., 67(4):848–869, 2015.
[27] A. Kontogeorgis. Polydifferentials and the deformation functor of curves with automorphisms. J. Pure Appl. Algebra, 210(2):551–558, 2007.
[28] S. Marques and K. Ward. Holomorphic differentials of certain solvable covers of the projective line over a perfect field. Math. Nachr., 291(13):2057–2083, 2018.
[29] C. Moreno. Algebraic Curves Over Finite Fields. Cambridge Tracts in Mathematics. Cambridge University Press, 1993.
[30] S. Nakajima. Galois module structure of cohomology groups for tamely ramified coverings of algebraic varieties. J. Number Theory, 22(1):115–123, 1986.
[31] K. A. Ribet. Congruence relations between modular forms. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), pages 503–514. PWN, Warsaw, 1984.
[32] M. Rzedowski-Calderón, G. Villa-Salvador, and M. L. Madan. Galois module structure of holomorphic differentials in characteristic p. Arch. Math. (Basel), 66(2):150–156, 1996.
[33] J.-P. Serre. Corps locaux. Publications de l’Université de Nancago, No. VIII. Hermann, Paris, 1968. Troisième édition.
[34] G. Shimura. Introduction to the arithmetic theory of automorphic functions, volume 11 of Publications of the Mathematical Society of Japan. Princeton University Press, Princeton, 1994. Reprint of the 1971 original, Kano Memorial Lectures, 1.
[35] R. C. Valentini and M. L. Madan. Automorphisms and holomorphic differentials in characteristic p. J. Number Theory, 13(1):106–115, 1981.
[36] A. Wood. Galois Structure of Holomorphic Polydifferentials in Positive Characteristic. ProQuest LLC, Ann Arbor, MI, 2020. Thesis (Ph.D.)–The University of Iowa.

F.B.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, 14 MACLEAN HALL, IOWA CITY, IA 52242-1419, U.S.A.
Email address: frauke-bleher@uiowa.edu

A.W.: DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, ST. OLAF COLLEGE, 1520 ST. OLAF AVENUE, NORTHFIELD, MN 55057, U.S.A.
Current address: Mathematics Department, Breck School, 123 Ottawa Ave N, Golden Valley, MN 55422, U.S.A.
Email address: acw8794@gmail.com