Equiaffine Isoparametric Functions and Their Regular Level Hypersurfaces

Wenjing Hao and Xingxiao Li

Abstract. In this paper, we aim to introduce and study the (locally strongly convex) equiaffine isoparametric functions on the affine space $A^{n+1}$, making the emphasis on their relation with the affine isoparametric hypersurfaces. Motivated by the case in the Euclidean space $E^{n+1}$, we first introduce the concept of equiaffine parallel hypersurfaces in $A^{n+1}$, and then equivalently re-define the equiaffine isoparametric hypersurfaces to be ones that are among families of equiaffine parallel hypersurfaces in $A^{n+1}$ of constant affine mean curvature. As the main result, we prove that an equiaffine isoparametric hypersurface is nothing but exactly a regular level set of some equiaffine isoparametric function.

Mathematics Subject Classification. Primary 53A15; Secondary 53B25.

Keywords. Affine isoparametric function, affine isoparametric hypersurfaces, affine parallel hypersurfaces, affine principal curvature, affine mean curvature.

1. Introduction

For a long period of time, the study of isoparametric functions (together with the isoparametric hypersurfaces and their focal submanifolds) has been a highly influential field in differential geometry. In the history of this subject, E. Cartan was the pioneer who made a comprehensive study of isoparametric functions/hypersurfaces of the real space forms. Originally, a hypersurface on real space forms was said to be isoparametric if it is of constant principal...
curvature. Due to Cartan [1, 2] and Münzner [19, 20], these isoparametric hypersurfaces, especially, in the standard Euclidean spheres, became fascinating to study and are closely related to a class of smooth functions satisfying certain equations called Cartan–Münzner equations, which people also call isoparametric functions or Cartan polynomial. In fact, any isoparametric hypersurface in real space forms must be a regular level set of an isoparametric function and, conversely, any level set of an isoparametric function is among a family of parallel isoparametric hypersurfaces (cf. [5]). Later, more generally in a Riemannian manifold \((N, \tilde{g})\), an isoparametric hypersurface was naturally defined as the regular level set of an isoparametric function, equivalent to that it is among a (smooth) family of parallel hypersurfaces of constant mean curvature [31]. In case that \((N, \tilde{g})\) is not of constant sectional curvature, the conclusion that an isoparametric hypersurface is of constant principal curvature is not true any more in general. In fact, as is known, an isoparametric hypersurface in the complex projective space can not be of constant principal curvature [30]. Note that there are several nice systematic surveys on isoparametric functions, isoparametric hypersurfaces and their generalizations [3, 6, 32]. See also [4, 7–9, 11] for recent progresses and applications.

Beside the Riemannian case, there are also some other geometric theories of submanifolds in which the concept of isoparametric hypersurfaces appears, being introduced by the constancy of some certain “principal curvatures”. For example, in the Möbius geometry of submanifolds in spheres, the Möbius and Blaschke isoparametric hypersurfaces are systematically studied, and some interesting classification theorems are obtained (see [10, 14–18, 26], and so on). Thus, in a rather broad field of geometry, it is much interesting to find certain relevant isoparametric functions of which the regular level sets are exactly those isoparametric hypersurfaces in geometries other than the Riemannian geometry.

On the other hand, the concept of affine isoparametric hypersurfaces in the affine geometry of hypersurfaces was introduced by R. Niebergall and P. J. Ryan in terms of the characteristic polynomial of the affine shape operator (see [23], and also [21, 22]). This kind of hypersurfaces were also named (affine) Blaschke isoparametric hypersurfaces in [12], where the author gave constructions of Blaschke Dupin and Blaschke isoparametric hypersurfaces in terms of the notion of equiaffine tubes. Later in [25], N. Koike considered a more general class of so-called equiaffine isoparametric hypersurfaces which are not necessarily nondegenerate. As the result, he obtained the relevant Cartan-type identities for affine principal curvatures of an equiaffine isoparametric hypersurface under certain conditions.

In this paper, we aim to introduce, in a natural and reasonable manner, the concept of (equi-)affine isoparametric functions, and establish a close relation between the (equi-)affine isoparametric hypersurfaces and our newly defined affine isoparametric functions. For simplicity, we shall only consider
the locally strongly convex case where all hypersurfaces are of positive definite affine metric.

To obtain our main result (see Theorem 1.1 below), it is much convenient to make a good use of the concept of equiaffine parallel hypersurfaces (Definition 3.2). Using some other equivalent definitions for equiaffine parallel hypersurfaces (Proposition 3.2 and Corollary 3.3), and a few fundamental identities for the basic equiaffine geometric invariants of these parallel hypersurfaces, we can equivalently define (in Definition 3.3) the equiaffine isoparametric hypersurface to be the one that is among a family of equiaffine parallel hypersurfaces of constant affine mean curvature in $A^{n+1}$.

The major part of this paper is contained in Sect. 4 where we shall first introduce the concept of equiaffine isoparametric functions on $A^{n+1}$ (Definition 4.1), and then prove the following main theorems:

**Theorem 1.1.** A non-degenerate hypersurface $x : M^n \to A^{n+1}$ in the affine space $A^{n+1}$ is (locally strongly convex) equiaffine isoparametric if and only if it is a regular level set of an equiaffine isoparametric function defined on an open neighbourhood of $x(M^n)$ in $A^{n+1}$.

**Remark 1.1.** While the definition of affine isoparametric hypersurfaces given by Niebergall and Ryan [23] is purely algebraic one, Definition 3.3 of this paper is purely geometrical one. Alternately, the above main theorem gives an analytic description for the same concept.

It is well-known that affine hyperspheres are among the most important objects to study in the field of affine geometry. As one byproduct of discussions in this paper, we also obtain the following result:

**Corollary 1.2.** (Corollary 4.3) A locally strongly convex hypersurface in the affine space $A^{n+1}$ is an (equi)affine hypersphere if and only if it is among a family of equiaffine parallel affine hyperspheres of the same type.

**Remark 1.2.** The very last corollary may provide a new direction of insight to deal with the classification problem for affine hyperspheres.

### 2. Equiaffine Differential Geometry of Hypersurfaces

In this section, we brief some basic facts for the equiaffine differential geometry of locally strongly convex hypersurfaces, including the necessary notations we shall use in this paper. For more details of this, we refer the readers to [13,24].

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^{n+1}$ be the real vector space of all ordered $(n + 1)$-tuples of real numbers, that is,

$$
\mathbb{R}^{n+1} = \{x = (x^1, \ldots, x^n, x^{n+1}); \ x^1, \ldots, x^n, x^{n+1} \in \mathbb{R}\}.
$$
Then on $\mathbb{R}^{n+1}$ there are a canonical flat connection $d$ defined by the usual component-wise differentiation of $\mathbb{R}^{n+1}$-valued functions, and a canonical volume measure $\operatorname{Vol}$ defined by the determinant function of $(n+1)$-vectors. Endowed with the connection $d$ and the volume measure $\operatorname{Vol}$, $\mathbb{R}^{n+1}$ is taken to be a measured affine space which we denote by $A^{n+1}$.

Note that the group $\text{GL}(n+1)$ of linear transformations on $\mathbb{R}^{n+1}$ together with the additive group $\mathbb{R}^{n+1}$ of translations on $\mathbb{R}^{n+1}$ makes into a semi-direct product group $A(n+1) := \text{GL}(n+1) \ltimes \mathbb{R}^{n+1}$, called the affine transformation group on $A^{n+1}$. An element $T \in A(n+1)$ is called a uni-modular transformation if it preserves the volume measure $\operatorname{Vol}$ of $A^{n+1}$. We denote by $\text{UA}(n+1)$ the subgroup of $A(n+1)$ consisting of all the uni-modular transformations on $A^{n+1}$. Moreover, given an open domain $U \subset A^{n+1}$, we call a Riemannian metric $\tilde{g}$ on $U$ admissible to the volume measure $\operatorname{Vol}$, if the isometry group $\text{Iso}(\tilde{g}) \subset \text{UA}(n+1)$.

Since the tangent bundle $TA^{n+1} = A^{n+1} \times \mathbb{R}^{n+1}$, for a tangent frame field $\{e_1, \ldots, e_{n+1}\}$ on $A^{n+1}$, the volume $\operatorname{Vol}(e_1, \ldots, e_{n+1})$ of $\{e_1, \ldots, e_{n+1}\}$ makes sense. We call $\{e_1, \ldots, e_{n+1}\}$ uni-modular if $\operatorname{Vol}(e_1, \ldots, e_{n+1}) \equiv 1$.

For simplicity, the following ranges of indices are always assumed without other specification in the present paper:

$$1 \leq i, j, k, \ldots \leq n; \quad 1 \leq A, B, C, \ldots \leq n + 1.$$ 

Now let $x : M^n \rightarrow A^{n+1}$ be an $n$-dimensional immersion of manifold $M^n$ into $A^{n+1}$. Then we simply call $x$ or $x(M^n)$ a hypersurface of $A^{n+1}$. A frame field $\{e_1, \ldots, e_{n+1}\}$ around $x(M^n)$ is called uni-modal affine Darboux, or simply, affine Darboux, if it is uni-modal and, when restricted to $x(M^n)$, $e_1, \ldots, e_n$ are tangent to $x(M^n)$. In this case, $\{e_1, \ldots, e_n\}$, by restriction to $x(M^n)$ and then via the tangent map $x_*$, locally defines a frame field on $M^n$ which we still denote as $\{e_1, \ldots, e_n\}$. By fixing a (uni-modal) affine Darboux frame field $\{e_1, \ldots, e_{n+1}\}$, one has the local decomposition of vector bundle:

$$x^*TA^{n+1} = x_*(TM^n) \oplus \mathbb{R} \cdot e_{n+1}.$$ 

So that

$$x_i \equiv x_*(e_i) = e_i(x), \quad x_{ij} \equiv e_j(e_i(x)) = \sum \Gamma^k_{ij} x_k + h_{ij} e_{n+1}, \quad \forall i, j,$$ 

where $\Gamma_{ij}^k$ and $h_{ij}$ are local smooth functions. The (induced) connection on $M^n$ given by the coefficients $\Gamma_{ij}^k$ is called the affine connection of $x$, while (2.1) is called the affine Gauss formula.

When $x$ is non-degenerate, that is, the matrix $(h_{ij})$ is non-singular everywhere on $M^n$, the locally defined function $H = |\det(h_{ij})| > 0$. Define $G_{ij} = H^{-\frac{1}{n+2}} h_{ij}$ for each pair of $i, j$. Then $G = \sum G_{ij} \omega^i \omega^j$ is a well-defined pseudo-Riemannian metric on $M^n$ [13], where $\omega^1, \ldots, \omega^n, \omega^{n+1}$ is the dual frame field of $\{e_1, \ldots, e_{n+1}\}$. In particular, if $x$ is locally strongly convex, or the matrix $(h_{ij})$ is definite everywhere, we can suitably choose the orientation
to make the matrix \((h_{ij})\) and hence the metric \(G\) positive definite. Conventionally the metric \(G\) defined in this way is called the Blaschke metric or affine metric, by which the affine normal vector \(Y\) of \(x\) is defined as

\[Y = \frac{1}{n} \Delta_G x.\] (2.2)

For a given point \(u \in M\), the straight line passing through \(x(u)\) and parallel to \(Y\) is called the affine normal line of \(x\) at \(u\).

Let \(\tilde{\Gamma}^k_{ij}\) be the coefficients of the Levi-Civita connection of \(G\) with respect to \(\{e_1, \ldots, e_n\}\). Then the \((1, 2)\)-tensor \(A\) defined by \(A^k_{ij} = \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}\) is called the difference tensor. The difference tensor is identified, via the metric \(G\), with a symmetric 3-form

\[A = \sum A_{ijk} \omega^i \omega^j \omega^k\] with \(A_{ijk} = \sum G_{kl} A^l_{ij}\),

which is called the Fubini-Pick form, or the cubic form, or the affine third fundamental form. Furthermore, if the one-forms \(\omega^1_{n+1}, \ldots, \omega^{n+1}_{n+1}\) are given by

\[de^{n+1} = \sum \omega^i_{n+1} x_i + \omega^{n+1}_{n+1} e^{n+1},\]

Then we have [13].

**Proposition 2.1.** Let \(\{e_1, \ldots, e_{n+1}\}\) be an arbitrary affine Darboux frame field of the hypersurface \(x\). Then the following three conditions are equivalent to each other:

1. \(e_{n+1}\) is parallel to the affine normal vector \(Y\);
2. \(\text{tr}_G A \equiv \sum G^{ij} A^k_{ij} = 0\) where \((G^{ij}) = (G_{ij})^{-1}\);
3. \(\omega^{n+1}_{n+1} + \frac{1}{n+2} d \log H = 0\).

Moreover, when \(e_{n+1}\) is parallel to \(Y\) we have

\[Y = H^{\frac{1}{n+2}} e_{n+1}, \quad \text{Vol}(x_1, \ldots, x_n, Y) = H^{\frac{1}{n+2}}.\] (2.3)

Thus, by using the affine normal vector \(Y\), the affine Gauss formula (2.1) can be written as

\[x_{ij} = \sum \Gamma^k_{ij} x_k + G_{ij} Y, \quad \forall i, j\] (2.4)

On the other hand, the affine Weingarten map or the affine shape operator \(B = (B^j_i)\) is defined by

\[Y_i = - \sum B^j_i x_j, \quad \forall i,\] (2.5)

and the corresponding 2-form \(B = \sum B_{ij} \omega^i \omega^j\) with \(B_{ij} = \sum G_{ik} B^k_j\) is called the affine (second) fundamental form. Furthermore, the eigenvalues \(\lambda_1, \ldots, \lambda_n\) of the matrix \((B^j_i)\), which are globally well-defined on \(M^n\), are called the affine principal curvatures of \(x\), from which a class of important and interesting hypersurfaces in the affine geometry are defined as follows.
Definition 2.1. A locally strongly convex hypersurface is called an (equi)affine hypersphere if all of its affine principal curvatures are equal to a same constant \( \lambda \). Furthermore, an affine hypersphere is called elliptic (resp. parabolic, hyperbolic) if \( \lambda > 0 \) (resp. \( \lambda = 0 \), \( \lambda < 0 \)).

Definition 2.2. The affine mean curvature of the hypersurface \( x \), denoted by \( L_1 \) is defined as
\[
L_1 = \frac{1}{n} \text{tr} G B = \frac{1}{n} \sum B_i^i = \frac{1}{n} \sum G^{ij} B_{ij} = \frac{1}{n} \sum \lambda_i. \quad (2.6)
\]
Moreover, \( x \) is called affine maximal if \( L_1 \equiv 0 \).

Finally, the affine Gauss equation, relating the Riemannian curvature tensor \( R_{ijkl} \) to the fundamental affine invariants \( G, A \) and \( B \), and the affine Codazzi equations are expressed as follows [13]:
\[
R_{ijkl} = \sum (A_{ik}^m A_{mj} - A_{il}^m A_{mj}) \quad (2.7)
\]
\[
+ \frac{1}{2} (G_{il} B_{jk} + G_{jk} B_{il} - G_{ik} B_{jl} - G_{jl} B_{ik}), \quad (2.8)
\]
\[
A_{ijk,l} - A_{ijl,k} = \frac{1}{2} (G_{ik} B_{jl} + G_{jk} B_{il} - G_{il} B_{jk} - G_{jl} B_{ik}), \quad (2.9)
\]
\[
B_{ij,k} - B_{ik,j} = \sum (A_{ij}^l B_{kl} - A_{ik}^l B_{jl}) \quad (2.10)
\]
for all \( i, j, k, l \).

3. Equiaffine Parallel Hypersurfaces and Equiaffine Isoparametric Hypersurfaces

Let \( x : M^n \to A^{n+1} \) be a locally strongly convex hypersurface with the affine metric \( G \), the Fubini-Pick form \( A \), the affine second fundamental form \( B \) and the affine normal vector \( Y \). For each given \( \mu \in C^\infty(M^n) \), we can define a new hypersurface \( ^\mu x \) as follows:
\[
^\mu x(u) = x(u) + \mu(u) Y(u), \quad u \in M^n. \quad (3.1)
\]

In what follows, we always use a left superscript \( \mu \) to an affine invariant \( \Gamma \), that is, \( ^\mu \Gamma \), to denote the corresponding invariant for the hypersurface \( ^\mu x \). For example, the affine metric, the affine normal vector, the difference tensor or the Fubini-Pick form, the affine Weingarten map or the affine fundamental form and the affine mean curvature are denoted by \( ^\mu G \), \( ^\mu Y \), \( ^\mu A \), \( ^\mu B \) and \( ^\mu L_1 \), accordingly.

Definition 3.1. A hypersurface \( ^\mu x : M^n \to A^{n+1} \) defined by (3.1) is called affine parallel to \( x : M^n \to A^{n+1} \) if

1. \( ^\mu x \) is locally strongly convex;
2. the tangent spaces \( ^\mu x_u(T_u M^n) \) and \( x_u(T_u M^n) \) are parallel in \( A^{n+1} \) for all \( u \in M^n \).
Then the following corollary is direct from the above definition:

**Corollary 3.1.** Let a hypersurface $\mu x$ be given by (3.1). Then $\mu x$ is affine parallel to $x$ if and only if the function $\mu$ is a constant one.

Indeed, from (3.1), it is direct that

$$\mu x_i = x_i + \mu_i Y + \mu_i = \sum (\delta^j_i - \mu B^j_i) x_j + \mu_i Y, \quad \forall i.$$  

Thus $\mu x$ is affine parallel to $x$ if and only if $\mu_i = 0$ for each $i$, that is, $\mu$ is a small enough constant.

Now we are concerned with the equiaffine geometry, so the following definition is more relevant here:

**Definition 3.2.** A hypersurface $\mu x : M^n \rightarrow A^{n+1}$ defined by (3.1) is called equiaffine parallel to $x : M^n \rightarrow A^{n+1}$ if

1. $\mu x$ is affine parallel to $x$;
2. the affine normal line of $\mu x$ is parallel to that of $x$ everywhere on $M^n$.

**Remark 3.1.** (i) It seems natural to seek conditions under which either of the conditions (1) and (2) in Definition 3.2 will imply the other.

(ii) For relatively (affine) parallel hypersurfaces, we refer the readers to [27–29] where, by means of the standard Euclidean metric on the ambient space and support functions, the authors obtain many facts on the relatively parallel hypersurfaces, including some curvature conditions for relatively parallel hypersurfaces to have parallel (equi-)affine normal lines.

Denote by $\mu T = (\mu T^i_j)$ the matrix with $\mu T^i_j = \delta^i_j - \mu B^i_j$. Then, by (3.2), $\mu x$ is affine parallel to $x$ if and only if

$$\mu x_i = \sum \mu T^i_j x_j, \quad \forall i.$$  

**Proposition 3.2.** Let the hypersurface $\mu x : M^n \rightarrow A^{n+1}$ given in (3.1) be affine parallel to $x$. Then $\mu x$ is equiaffine parallel to $x$ if and only if there is a constant $c \in \mathbb{R}$ depending on $\mu$ such that, the affine normal vector $\mu Y(u) = c Y(u)$ for every $u \in M^n$.

**Proof.** The assumption that $\mu x$ is affine parallel to $x$ is, by Corollary 3.1, equivalent to that $\mu$ is constant.

Suppose that $\mu Y = \sum a^i x_i + c Y$ for some functions $a^i, c$ on $M^n$. Then, by (3.3) and the affine Gauss formula, we directly compute

$$\mu x_{ij} = \left( \sum \mu T^k_i x_k \right)_j = \sum \mu T^k_i j x_k + \sum \mu T^k_i x_{kj}$$

$$= \sum \mu T^k_i j x_k + \sum \mu T^k_i \left( \sum \Gamma^k_{ij} x_k + G_{ij} Y \right)$$

$$= \sum \left( \mu T^k_i j + \sum \mu T^k_i \Gamma^k_{ij} \right) x_k + \sum \mu T^k_i G_{kj} Y.$$  

(3.4)
On the other hand,
\[ \mu x_{ij} = \sum \mu \Gamma_{ij}^{k} \mu x_{k} + \mu G_{ij} \mu Y = \sum \mu \Gamma_{ij}^{k} \left( \sum \mu T_{i}^{j} x_{k} \right) + \mu G_{ij} \left( \sum a^{k} x_{k} + cY \right) = \sum \left( a^{k} \mu G_{ij} + \sum \mu \Gamma_{ij}^{l} \mu T_{i}^{k} \right) x_{k} + c\mu G_{ij} Y. \] (3.5)

Comparing the above two equalities we obtain for any \( i, j \)
\[ (\mu T_{i}^{j})_{j} + \sum \mu T_{i}^{l} \Gamma_{i}^{j} = a^{k} \mu G_{ij} + \sum \mu \Gamma_{ij}^{l} \mu T_{i}^{k}, \quad \forall k, \quad \sum \mu T_{i}^{k} G_{kj} = c \mu G_{ij}. \] (3.6)

Moreover, we have
\[ \mu Y_{i} = \left( \sum a^{j} x_{j} + cY \right)_{i} = \sum (e_{i}(a^{j}) x_{j} + a^{j} x_{ji}) + c_{i} Y + cY_{i} = \sum \left( e_{i}(a^{j}) + \sum a^{k} \Gamma_{k}^{j} - cB_{i}^{j} \right) x_{j} + \left( \sum a^{j} G_{ji} + c_{i} \right) Y, \] (3.7)
and
\[ \mu Y_{i} = -\sum \mu B_{i}^{j} \mu x_{j} = -\sum \mu B_{i}^{k} \mu T_{k}^{j} x_{j}. \] (3.8)

So it holds that
\[ e_{i}(a^{j}) + \sum a^{k} \Gamma_{k}^{j} - cB_{i}^{j} + \sum \mu B_{i}^{k} \mu T_{k}^{j} = 0, \quad \sum a^{j} G_{ji} + c_{i} = 0. \] (3.9)

Furthermore, by the second formula of (2.3), it holds that
\[ \mu H^{\frac{1}{n+2}} = \text{Vol}(\mu x_{1}, \ldots, \mu x_{n}, \mu Y) = \text{Vol} \left( \sum \mu T_{i}^{1} x_{i}, \ldots, \sum \mu T_{i}^{n} x_{i}, \sum a^{i} x_{i} + cY \right) = \det(\mu T)cH^{\frac{1}{n+2}}. \] (3.10)

Since \( G_{ij} = H^{-\frac{1}{n+2}} h_{ij} \) and \( \mu G_{ij} = \mu H^{-\frac{1}{n+2}} \mu h_{ij} \), implying
\[ \det G = H^{-\frac{n}{n+2}}, \quad \det \mu G = \mu H^{-\frac{n}{n+2}}, \] (3.11)
it is easily seen from the second equality of (3.6) that \( c^{n} \mu H^{\frac{n}{n+2}} = \det(\mu T)H^{\frac{n}{n+2}} \). This last equality together with (3.10) gives
\[ c^{n+2} \det(\mu T) = 1, \quad c^{n+1} = \left( H \frac{1}{\mu H} \right)^{-\frac{1}{n+2}}, \quad \det(\mu T) = \left( H \frac{1}{\mu H} \right)^{-\frac{1}{n+2}}. \] (3.12)

Apparently, we only need to prove the necessity part of the proposition. Suppose that \( \mu x \) is equiaffine to \( x \). Then \( \mu Y \) is parallel to \( Y \), which is equivalent to that \( a^{i} = 0, \quad \forall i \). This with the second equality of (3.9) shows that \( \mu Y = cY \) with \( c \) being a constant.
Denote by $L_r$ the $r$-th normalized elementary symmetric functions of the affine principal curvatures $\lambda_1, \ldots, \lambda_n$:

$$L_r = \frac{1}{C_n^r} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}, \quad 1 \leq r \leq n.$$ 

Then the following corollary is direct from Proposition 3.2, (3.11) and (3.12):

**Corollary 3.3.** For an hypersurface $^\mu x$ that is affine parallel to $x$, the following four conditions are equivalent to each other:

1. $^\mu x$ is equiaffine parallel to $x$;
2. $\det(^\mu T)$ is constant, which is equivalent to that
   $$nL_1 - C_n^2 \mu L_2 + \cdots + (-1)^n C_n^{n-1} \mu^{n-2} L_{n-1} + (-1)^{n+1} \mu^{n-1} L_n = \text{const};$$
   (3.13)
3. The ratio $\frac{H}{H_\mu}$ of the functions $H$ and $H_\mu$ is constant;
4. The ratio $\frac{\det G}{\det ^\mu G}$ of the squared volumes $\det G$ and $\det ^\mu G$ is constant.

From Proposition 3.2, we clearly have

**Corollary 3.4.** $^\mu x$ is equiaffine parallel to $x$ if and only if $x$ is equiaffine parallel to $^\mu x$.

If it is the case, we shall say that $x$ and $^\mu x$ are equiaffine parallel to each other.

**Proposition 3.5.** If a locally strongly convex hypersurfaces is of constant affine principal curvature, then all of its equiaffine parallel hypersurfaces are also of constant affine principal curvature.

**Proof.** Since $^\mu Y$ is parallel to $Y$ for all $\mu$, the first equality of (3.9) becomes

$$\sum \mu B^k_i \mu T^j_k = cB^j_i. \quad (3.14)$$

This with the second equality of (3.6) gives

$$c^\mu B_{ij} = c\sum \mu B^k_i \mu G_{kj} = \sum \mu B^k_i \mu T^l_k G_{lj} = c\sum B^k_i G_{kj} = cB_{ij}.$$ 

So (3.14) is equivalent to that

$$^\mu B_{ij} = B_{ij}. \quad (3.15)$$

Now choose a suitable frame field $\{e_1, \ldots, e_n\}$ such that $B^j_i = \lambda_i \delta^j_i$. Then

$$^\mu T^j_k = (1 - \mu \lambda_k)\delta^j_k.$$ 

Putting this into (3.14) we obtain

$$^\mu B^j_i (1 - \mu \lambda_j) = \sum \mu B^k_i (1 - \mu \lambda_k)\delta^j_i = c\lambda_i \delta^j_i,$$

implying

$$^\mu B^j_i = \frac{c\lambda_i}{1 - \mu \lambda_i} \delta^j_i.$$
Thus $e_1, \ldots, e_n$ are also the eigen-vectors of the affine Weingarten map $\mu B$ of $\mu x$, and the corresponding affine principal curvatures of $\mu x$ are, respectively,

$$\mu \lambda_i = \frac{c\lambda_i}{1 - \mu \lambda_i}, \forall i. \quad (3.16)$$

The conclusion of the proposition follows easily, since $c$ is constant along $\mu x$. □

Motivated by the isoparametric hypersurfaces in Riemannian manifolds, we naturally introduce the following definition:

**Definition 3.3.** A locally strongly convex hypersurface $x : M \to A^{n+1}$ is called equiaffine isoparametric if it is among a family of equiaffine parallel hypersurfaces that are of constant affine mean curvature.

The following result shows that Definition 3.3 is equivalent to the one given by Niebergall and Ryuan [23]:

**Theorem 3.6.** A locally strongly convex hypersurface in the affine space $A^{n+1}$ is equiaffine isoparametric if and only if it is of constant affine principal curvature.

**Proof.** Due to Proposition 3.5, it suffices to show that all the affine principal curvatures $\lambda_1, \ldots, \lambda_n$ of $x$ are constant if each of its equiaffine parallel hypersurfaces is of constant affine mean curvature. To do this, we use (3.16) to find

$$L_1(\mu) := \mu L_1 = \frac{1}{n} \sum \frac{c\lambda_i}{1 - \mu \lambda_i}, \text{ for } \mu \in (-\delta, \delta),$$

that is,

$$\sum \frac{\lambda_i}{1 - \mu \lambda_i} = \frac{n}{c} L_1(\mu), \mu \in (-\delta, \delta). \quad (3.17)$$

If $\mu L_1$ is constant for all $\mu \in (-\delta, \delta)$, then right hand side of (3.18) is a smooth function of $\mu$. Taking the differentiation of (3.18) by $(n-1)$-times gives

$$\sum \frac{\lambda_i^\iota}{(1 - \mu \lambda_i)^\iota} = \left( \frac{1}{c} L_1(\mu) \right)^\iota, \mu \in (-\delta, \delta), \iota = 2, \ldots, n. \quad (3.19)$$

Evaluating (3.18) and (3.19) at $\mu = 0$ simply gives

$$\sum \lambda_i^\iota = \text{const}, \iota = 1, 2, \ldots, n,$$

which implies that $\lambda_i = \text{const}$ for each $i$, completing the proof of Theorem 3.6. □

Thus we have the following canonical examples of equiaffine isoparametric hypersurfaces:

**Example 3.1.** All equiaffine hyperspheres are equiaffine isoparametric hypersurfaces. In fact, the reason of this is rather simple (see Theorem 3.6).
Example 3.2. All (locally strongly convex) equiaffine homogeneous hypersurfaces are equiaffine isoparametric hypersurfaces.

Remark 3.2. So, two questions may be asked: Firstly, are there any equiaffine isoparametric hypersurfaces in $A^{n+1}$ other than the affine hyperspheres or, more broadly, other than the equiaffine homogeneous hypersurfaces? Secondly, if either of the answers is positive, then the corresponding classification problem arises naturally which seems interesting!

4. Equiaffine Isoparametric Functions and Their Regular Level Sets

First of all, we recall the definition of isoparametric functions on a Riemannian manifold.

Let $(N, \tilde{g})$ be a Riemannian manifold, and $F$ be a non-constant smooth function on $N$ with the gradient $\tilde{\nabla}^g F$ and the Laplacian $\tilde{\Delta}^g F$. Denote $J = F(N) \subset \mathbb{R}$. Then, according to Wang [31], $F$ is called an isoparametric function if there exist smooth functions $a(t)$ and $b(t)$ of one variable $t$, $t \in J$, such that

$$|\tilde{\nabla}^g F|_{\tilde{g}} = a(F), \quad \tilde{\Delta}^g = b(F).$$

(4.1)

Define

$$N_* = \{x \in N; \text{ } dF \neq 0\}, \quad \xi = \frac{1}{a} \tilde{\nabla}^g F.$$  

(4.2)

Then $\xi$ is a globally defined smooth unit vector field on $N_*$, of which the orthogonal complement $\xi^\perp$ is a distribution on $N_*$. Furthermore, we have the following well-known conclusion [2,31]

**Proposition 4.1.** Let $F$ be an isoparametric function on $(N, \tilde{g})$. Then

1. Each integral curves $\gamma_\xi$ of $\xi$ is a unit speed geodesic with the arc-length being its parameter;
2. The distribution $\xi^\perp$ is integrable, generating a foliation $\mathcal{F}$ of $N_*$ by a family of hypersurfaces of constant mean curvature;
3. Each geodesic $\gamma_\xi$ intersects any hypersurface in $\mathcal{F}$ (orthogonally) once and only once;
4. Every two leaf hypersurfaces in $\mathcal{F}$ are of equi-distance, that is, all the geodesics $\gamma_\xi$ restricted between these two hypersurfaces are of the same length. In particular, all leaf hypersurfaces in $\mathcal{F}$ form a family of parallel ones and can be parametrized or labeled by the arc-length parameter.

**Remark 4.1.** Clearly, by Proposition 4.1, the arc-length function $s$ of the geodesics $\gamma_\xi$ well defines a one-form $ds$ globally on $N_*$. Each leaf hypersurface of the foliation $\mathcal{F}$, with $\xi$ being its unit normal, was originally named to be an isoparametric hypersurface of $(N, \tilde{g})$. 


Analysis of the equiaffine parallel hypersurfaces motivates us to introduce the concept of equiaffine isoparametric functions as follow:

**Definition 4.1.** Let $U \subset A^{n+1}$ be a non-empty open domain. A smooth function $F : U \to \mathbb{R}$ is called *equiaffine isoparametric*, if the following conditions are satisfied:

1. There exists a Riemannian metric $\tilde{g}$ on $U$ admissible to the volume measure $\text{Vol}$, such that $F$ is an (Riemannian) isoparametric function on the Riemannian manifold $(U, \tilde{g})$;
2. The Hessian $\text{Hess}_0 F$ of $F$ with respect to the standard flat connection is semi-negative definite with $\mathbb{R}\xi$ being the null space, where $\xi = \frac{1}{a} \tilde{\nabla} F$;
3. $(\tilde{g} + \frac{1}{a} \text{Hess}_0 F) |_{\xi \perp \times \xi \perp} \equiv 0$.
4. $2 \tilde{\nabla} \xi = d\xi - (d \log a) \xi$.

**Remark 4.2.** It is not hard to see that, by using the well-defined one-form $ds$, we can replace conditions (2) and (3) in Definition 4.1 by a new equation

$$\tilde{g} + \frac{1}{a} \text{Hess}_0 F = ds^2. \quad (4.3)$$

**Remark 4.3.** From Definition 4.1 one easily sees that, if $F$ is an equiaffine isoparametric function then, for any constants $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$, $\tilde{F} = c_1 F + c_2$ is also an equiaffine isoparametric function. Moreover, $\tilde{F}$ and $\tilde{F}$ have the same regular level hypersurfaces. In particular, $\tilde{\xi} = \xi$.

Now we are in a position to give a proof for our main theorem (Theorem 1.1) in this paper.

**Proof of Theorem 1.1.** We first prove the necessity part of the theorem.

Suppose that $x : M^n \to A^{n+1}$ is an equiaffine isoparametric hypersurfaces with the affine normal vector $Y$, the Blaschke metric $G$ and the affine second fundamental form $B$. Then, by Definition, $x$ is locally strongly convex, or the same, the metric $G$ is positive definite. Furthermore, from Definition 3.3 and Corollary 3.1, it follows that there must be a $\delta > 0$ such that, for each $\mu \in (-\delta, \delta)$, the hypersurface $^{\mu}x = x + \mu Y$ is equiaffine parallel to $x$ with constant affine mean curvature $^{\mu}L_1$.

Define

$$U = \{^{\mu}x(u) ; \; u \in M^n, \; \mu \in (-\delta, \delta)\}.$$ 

Then $U$ is an open neighbourhood of $x(M^n)$, and a Riemannian metric $\tilde{g}$ on $U$ can be given as follows:

Let $(u^1, \ldots, u^n)$ be a local coordinate system on $M^n$. Then $(u^1, \ldots, u^n, u^{n+1} := \mu)$ is a local coordinate system for $U$. In particular, $\frac{\partial}{\partial \mu} = Y$. By Proposition 3.2, the affine normal vector $^{\mu}Y = cY$ with $c \equiv c(\mu)$, a smooth function of the parameter $\mu \in (-\delta, \delta)$. Using the Blaschke metric $^{\mu}G = \sum ^{\mu}G_{ij} \omega^i \omega^j$ of $^{\mu}x$ and the function $c(\mu)$, a Riemannian metric $\tilde{g}$ can be introduced on $U$ by

$$\tilde{g} = \sum ^{\mu}G_{ij} \omega^i \omega^j + \frac{1}{c^2} d\mu^2.$$
Equivalently speaking, $\tilde{g}$ is the Riemannian metric on $U$ which corresponds to a pseudo-product of $\mu G$ and $d\mu^2$ on $M^n \times (-\delta, \delta)$. For simplicity, we shall always use the index $\mu$ to denote the index $n + 1$, say,

$$\tilde{g}_{\mu\mu} \equiv \tilde{g}_{n+1\ n+1} = \tilde{g}(Y, Y) = \frac{1}{c^2}.$$ 

From the discussion of Sect. 3, we know that for all $i, j$,

$\mu G_{ij} = \frac{1}{c} (G_{ij} - \mu B_{ij})$, where $G_{ij} = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}$,

$B_{ij} = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}$.

Denote by $\tilde{\Gamma}_{ij}^k$ the coefficients of the Levi-Civita connection of $\mu G$. Then we easily find the coefficients $\tilde{\Gamma}_{CAB} \equiv \tilde{\Gamma}_{CAB}$, $\forall A, B, C$ of Levi-Civita connection of $\tilde{g}$ as follows:

$$\tilde{\Gamma}_{ij}^k = \frac{\mu}{c} \tilde{\Gamma}_{ij}^k, \quad \tilde{\Gamma}_{ij}^\mu = \frac{1}{2} c B_{ij}, \quad \forall i, j, k, \quad (4.4)$$

$$\tilde{\Gamma}_{ij}^\mu = \tilde{\Gamma}_{ij}^\mu = -\frac{1}{2} c \sum \mu G^{ij} B_{kj} = -\frac{1}{2} c B_{ij}^{\mu}, \quad \forall i, j, \quad (4.5)$$

$$\tilde{\Gamma}_{ij}^{\mu\mu} = \tilde{\Gamma}_{ij}^{\mu\mu} = 0, \quad \forall i, \quad (4.6)$$

$$\tilde{\Gamma}_{ij}^{\mu\mu} = -\frac{\partial}{\partial \mu} \log c. \quad (4.7)$$

Define a smooth function $F : U \rightarrow \mathbb{R}$ by $F(\mu x(u)) = \mu$. Then $F_i = 0$, $F_\mu = 1$. By using this and (4.4)–(4.7), we easily find that, with respect to the metric $\tilde{g}$ on $U$, the gradient and the Laplacian of $F$ are as follows:

$$\nabla_{\tilde{g}} F = c \mu Y, \quad (4.8)$$

$$\Delta_{\tilde{g}} F = \sum \tilde{g}^{AB} (F_{AB} - F_C \tilde{\Gamma}_{AB}^C) = -\frac{1}{2} c \sum \mu G^{ij} \tilde{\Gamma}_{ij}^\mu - \tilde{g}_{ij} \tilde{\Gamma}_{ij}^{\mu\mu}$$

$$= -\frac{1}{2} c \sum \mu G^{ij} B_{ij} + cc' \equiv -\frac{1}{2} nc \mu L_1 + cc'. \quad (4.9)$$

Since $\mu L_1$ is constant along $\mu x(M^n)$, it depends only on the parameter $\mu$. Therefore, $\Delta_{\tilde{g}} F$ is a function of $F$. So, by the fact that

$$|\nabla_{\tilde{g}} F|_{\tilde{g}}^2 = \tilde{g}(c \mu Y, c \mu Y) = c^2,$$

$F$ is an isoparametric function on the Riemannian manifold $(U, \tilde{g})$ with

$$a(t) \equiv c(t), \quad b(t) \equiv -\frac{1}{2} nc(t) \mu L_1 + c(t)c'(t).$$

In particular,

$$\xi = \frac{1}{a} \nabla_{\tilde{g}} F = \mu Y, \quad s = \int_0^\mu \frac{dt}{c(t)}.$$
Furthermore, the Hessian \( \text{Hess}_0 F(\mu x(u)) \) of \( F \) with respect to the standard flat connection \( \tilde{\Gamma}^C_{AB} \) can be computed as

\[
(\text{Hess}_0 F)_{ij}(\mu x(u)) = F_{ij} - F_{ij}^\mu = -c^\mu G_{ij}(u) = -c \tilde{g}_{ij}(u, \mu), \quad \forall i, j; \tag{4.10}
\]

\[
(\text{Hess}_0 F)_{i\mu}(\mu x(u)) = (\text{Hess}_0 F)_{i\mu}(\mu x(u)) = (\text{Hess}_0 F)_{i\mu}(\mu x(u)) = 0, \quad \forall i. \tag{4.11}
\]

So (4.3) holds since \( a = c \).

On the other hand, since \( \frac{1}{c} \xi = \frac{1}{c} \mu Y = Y \) is a constant vector along \( \mu \)-curves which are both geodesics in \((U, \tilde{g})\), with the arc-length \( s \), and straight lines in \( A^{n+1} \), we find by (4.4)–(4.7) that

\[
2 \tilde{\nabla}_i \xi = 2c \tilde{\nabla}_i Y = c \sum \tilde{\Gamma}^j_{\mu i} \mu x_j = -\mu B^j_i \mu x_j = \mu Y_i = \frac{\partial}{\partial u^i} \xi - \frac{\partial}{\partial u^i} (\log c) \xi, \quad \forall i,
\]

\[
2 \tilde{\nabla}_\mu \xi = 2 \sum \tilde{\Gamma}^i_{\mu i} \mu x_i = 0, \quad \frac{\partial}{\partial \mu} \xi - \frac{\partial}{\partial \mu} (\log c) \xi = c' Y - \frac{1}{c} c' c Y = 0.
\]

So, the condition (4) in Definition 4.1 is also met by \( F \).

Now we have proved that \( F \) is an equiaffine isoparametric function. Moreover, it is clear that each \( \mu x \) including \( x \equiv 0 \) \( x \) is a regular level set of \( F \).

Next we prove the sufficiency part of Theorem 1.1.

Suppose that \( x : M^n \to A^{n+1} \) is a regular level set of an equiaffine isoparametric function \( F \) defined on an open domain \( U \subset A^{n+1} \) where, by Definition 4.1, \( F \) is first an isoparametric function on the Riemannian manifold \((U, \tilde{g})\) with some Riemannian metric \( \tilde{g} \). From Proposition 4.1, we know that \( x \) is one of the parallel hypersurfaces of constant mean curvature in the foliation \( \mathcal{F} \), for which the unit normal \( \xi := \frac{1}{a} \tilde{\nabla} F \) is tangent to and parallel along each of its integral curves \( \gamma_\xi \), that is, these curves \( \gamma_\xi \) are all unit-speed geodesics on \((U, \tilde{g})\) with respect the arc-length parameter \( s \). So by Condition (4) in Definition 4.1, \( Y := \frac{1}{c} \xi \) is constant as \( \mathbb{R}^{n+1} \)-valued function along \( \gamma_\xi \). Thus, all the curves \( \gamma_\xi \) are straight lines in \( A^{n+1} \) and can be parametrized, starting from points on \( x(M^n) \), as \( \gamma_{\xi(u)}(\mu) = x(u) + \mu Y, \mu \in (-\delta, \delta) \) with some \( \delta > 0 \), for each \( u \in M^n \).

Consequently, starting from the given hypersurface \( x \), the above family of parallel hypersurfaces in \( \mathcal{F} \) can be parametrized by the parameter \( \mu \) simply as

\[
\mu x(u) = x(u) + \mu \xi(u), \quad u \in M^n, \tag{4.12}
\]

for \( \mu \in (-\delta, \delta) \).

We shall prove that, for each \( \mu \in (-\delta, \delta) \), \( \mu x \) is a locally strictly convex hypersurface with constant affine mean curvature \( \mu L_1 \). For doing this, we take the following three steps:

1. The induced metric on \( M^n \) by the immersion \( \mu x : M^n \to (U, \tilde{g}) \) is equal to the affine metric \( \mu G \).

In fact, for any tangent frame field \( \{e_1, \ldots, e_n\} \) on \( M \), let \( \lambda = (\det(\mu x^* \tilde{g}_{ij}))^{\frac{1}{2}(n+2)} \) and \( e_{n+1} \) = \( \lambda^{-\frac{1}{n+2}} \xi \). Then by the Lagrange identity
and the fact that \( \tilde{g} \) is admissible to the measure Vol, we find
\[
\text{Vol}(e_1, \ldots, e_n, e_{n+1}) = \text{Vol}_{\tilde{g}}(e_1, \ldots, e_n, e_{n+1}) = (\det(\tilde{g}_{AB}))^{\frac{1}{2}} = (\text{det}(\mu x^* \tilde{g}_{ij}))^{\frac{1}{2}} |e_{n+1}|_{\tilde{g}} = (\text{det}(\mu x^* \tilde{g}_{ij}))^{\frac{1}{2}} \lambda^{-\frac{n+2}{n+2}} |\xi|_{\tilde{g}} = 1.
\]
Thus \( \{e_1, \ldots, e_n, e_{n+1}\} \) is an affine Darboure frame field. Moreover, it holds by the definition of the affine metric that
\[
\mu x_{ij} = \sum \mu \Gamma_{ij}^k \mu x_k + \mu h_{ij} e_{n+1} = \sum \mu \Gamma_{ij}^k \mu x_k + \mu G_{ij} \mu H \lambda^{-\frac{1}{n+2}} \lambda^{-\frac{1}{n+2}} \xi, \quad \forall i,
\]
with \( \mu H = \det(\mu h_{ij}) \).

On the other hand, since the hypersurface \( \mu x(M^n) \) is a regular level set of the function \( F \), we have that \( F(\mu x^1, \ldots, \mu x^{n+1}) = \text{const.} \) Take differentiation of this equation twice, we find that
\[
\sum \frac{\partial^2 F}{\partial x^A \partial x^B} \mu x^A_i \mu x^B_j + \sum \frac{\partial F}{\partial x^A} \mu x^A_{ij} = 0, \quad \forall i, j
\]
or
\[
\mu x^*(\text{Hess}_0 F)_{ij} + \sum F_A \mu x^A_{ij} = 0, \quad \forall i, j \tag{4.14}
\]
for each \( \mu \in (-\delta, \delta) \).

Now, by using (4.13), (4.14) and (4.3), we obtain that, for all \( i, j \),
\[
\mu G_{ij} \mu H \lambda^{-\frac{n}{n+2}} \lambda^{-\frac{n}{n+2}} = \tilde{g}(\mu x_{ij}, \xi) = \tilde{g}(\mu x_{ij}, \frac{1}{a} \nabla^g F) = \frac{1}{a} \sum \tilde{g}_{AB} \mu x^A_{ij} (\nabla^g F)^B
\]
\[
= \frac{1}{a} \sum \mu x^A_{ij} F_A = - \mu x^* \left( \frac{1}{a} \text{Hess}_0 F \right)_{ij} = \mu x^*(\tilde{g} - ds^2)_{ij} = (\mu x^* \tilde{g})_{ij}.
\]
\[\tag{4.15}\]
The equality (4.15) means that \( \mu G \) is positive definite, that is, \( \mu x \) is locally strongly convex for each \( \mu \in (-\delta, \delta) \). Moreover, by the definition of the affine metric \( \mu G \), we have \( \det(\mu G_{ij}) = \mu H \lambda^{-\frac{2}{n+2}} \) [see (3.11)]. Using this we take the determinant of (4.15) and obtain
\[
\mu H \lambda^{-\frac{n}{n+2}} = \det(\mu G_{ij}) \mu H \lambda^{-\frac{n}{n+2}} \lambda^{-\frac{n}{n+2}} = \det(x^* (\tilde{g}))_{ij} = \lambda^{-\frac{2}{n+2}},
\]
implying that \( \lambda = \mu H \). Putting this into (4.15) we have finally proved that
\[
\mu x^* \tilde{g} = \mu G.
\]

2 The affine normal vector \( \mu Y \) of \( \mu x \) at any \( u \in M^n \) is equal to \( \xi(\mu x(u)) \) for each \( \mu \in (-\delta, \delta) \).

In fact, we can choose an affine Darboux frame \( \{e_1, \ldots, e_n, e_{n+1}\} \) such that \( e_{n+1} = \xi \). In this case, by the condition (4) in Definition 4.1, we have along \( \mu x(M^n) \) that
\[
d e_{n+1} = d \xi = 2 \nabla \xi = -2 \sum \mu A^j_i \omega^j \mu x_j, \tag{4.16}
\]
where \( \mu A^j_i \) are the components of the Weingarten map of \( \mu x \). This shows that \( \omega^{n+1} \equiv 0 \). On the other hand, since \( \tilde{g}(\xi, \xi) = 1 \), we can make a choice of
\{e_1, \ldots, e_n\} such that \(\mu H = \det(\mu h_{ij}) = \text{const}\), say, by choosing \(\{e_1, \ldots, e_n\}\) to be orthonormal with respect to the affine metric \(\mu G\) which, as shown in (1), is now exactly the induced metric of \(\tilde{g}\). Consequently, we have

\[
\omega_{n+1} + \frac{1}{n+2} d\log \mu H \equiv 0,
\]

which implies, by Proposition 2.1, that \(e_{n+1}\) or \(\xi\) is parallel to the affine normal vector \(\mu Y\), and

\[
\sum \mu G^{ij} \mu A^k_{ij} \equiv \sum \mu G^{ij} \left( \mu \Gamma^k_{ij} - \mu \tilde{\Gamma}^k_{ij} \right) = 0. \tag{4.17}
\]

In particular, if \(\{e_1, \ldots, e_n\}\) is chosen to be orthonormal w.r.t. \(\mu G\), then we have

\[
\begin{align*}
1 &= \det(\mu G_{ij}) = \mu H_{\frac{2}{n+1}}, \\
\mu Y &= \frac{1}{n} \Delta_{\mu G} x = \frac{1}{n} \sum \mu G^{ij} \left( \mu x_{ij} - \sum \mu x_k \mu \tilde{\Gamma}^k_{ij} \right) \\
&= \frac{1}{n} \left( \sum \mu G^{ij} \left( \mu \Gamma^k_{ij} - \mu \tilde{\Gamma}^k_{ij} \right) \mu x_k + \sum \mu G^{ij} \mu G_{ij} e_{n+1} \right) \\
&= e_{n+1} = \xi.
\end{align*}
\]

By Remark 4.3, we can suitably choose the constant \(c_1\), if necessary, such that \(\alpha(0) = 1\). It then follows that, \(Y = \frac{1}{\alpha(0)} 0 x = 0 Y\) is the affine normal vector of the original hypersurface \(x \equiv 0 x\). So we can conclude that the hypersurface \(x \equiv 0 x\) is among a family (4.12) of equiaffine parallel hypersurfaces in \(A^{n+1}\).

(3) All hypersurfaces \(\mu x\) in (4.12) are of constant affine mean curvature. In fact, we use once again (4.16) and compare it with (2.5) to obtain that

\[
- \sum \mu B^i_j \mu x_j = \mu Y_i = \xi_i = 2 \tilde{\nabla}_e \xi = -2 \sum \mu A^i_j \mu x_j, \quad \forall i,
\]

which proves that \(\mu A^i_j = \frac{1}{2} \mu B^i_j\) for all \(i, j\) and \(\mu\). Take the trace we find that, for each \(\mu\), the mean curvature of the isometric immersion \(\mu x : M^n \to (U, \tilde{g})\) is equal to one half of the affine mean curvature \(\mu L_1\) of the locally strongly convex hypersurface \(\mu x : M^n \to A^{n+1}\). But, since the former \(\mu x\) is the regular level set of the (Riemannian) isoparametric function \(F\) on \((U, \tilde{g})\), being of constant mean curvature, the latter \(\mu x\) must be of constant affine mean curvature, for each \(\mu \in (-\delta, \delta)\).

Summing up conclusions (1), (2) and (3), the sufficiency part of the theorem is proved.

The following corollaries are direct:

**Corollary 4.2.** A locally strongly convex hypersurface in the affine space \(A^{n+1}\) is equiaffine isoparametric if and only if it is among a family of equiaffine parallel hypersurfaces that are of constant affine principal curvature.

In fact, this is a direct consequence of Proposition 3.5 and Theorem 3.6.
Corollary 4.3. A locally strongly convex hypersurface in the affine space $A^{n+1}$ is an (equi)affine hypersphere if and only if it is among a family of equiaffine parallel affine hyperspheres of the same type.

In fact, by (3.16), all the affine principal curvatures $\lambda_i$ of $x$ are equal to each other if and only if those $\lambda_i$ of $x$ are.

Remark 4.4. To end this paper, we would like to give some final remarks: Firstly, Theorem 3.6 can be equivalently and directly obtained by (3.13). In fact, from (3.13) one easily sees that the two conditions for the equiaffine parallel hypersurfaces are rather restrictive. Secondly, it turns out that, in our definition of equiaffine isoparametric hypersurfaces (Definition 3.3), the adjective subordinate clause “that are of constant affine mean curvature” can be really deleted! For example, Corollary 4.2 can be simply restated as

Corollary 4.4. A non-degenerate hypersurface in the affine space $A^{n+1}$ is equiaffine isoparametric if and only if it is among a family of equiaffine parallel hypersurfaces.

Acknowledgements

The second author thanks Professor A.-M. Li for his constant encouragement.

References

[1] Cartan, E.: Families des surface isoparametrique dans les espaces a courbure constante. Ann. Mat. 17, 177–191 (1938)
[2] Cartan, E.: Sur des familles remarquables dhypersurfaces isoparametriques dans les espaces sphriques. Math. Z. 45, 335–367 (1939)
[3] Cecil, T.E.: Isoparametric and Dupin hypersurfaces. Symmetry Integr. Geom. Methods Appl. (SIGMA) 4, 062 (2008). https://doi.org/10.3842/SIGMA.2008.062
[4] Cecil, T.E., Chi, Q.S., Jensen, G.R.: Isoparametric hypersurfaces with four principal curvatures. Ann. Math. 166, 1–76 (2007)
[5] Cecil, T.E., Ryan, P.T.: Tight and Taut Immersions of Manifolds, Research Notes in Mathematics, vol. 107. Pitman, London (1985)
[6] Chi, Q.S.: The Isoparametric Story. National Taiwan University, Taipei City (2012)
[7] Ge, J.Q., Tang, Z.Z.: Isoparametric functions and exotic spheres. J. Reine Angew. Math. 683, 161–180 (2013)
[8] Ge, J.Q., Tang, Z.Z.: Geometry of isoparametric hypersurfaces in Riemannian manifolds. Asian J. Math. 18(1), 117–126 (2014)
[9] Ge, J.Q., Xie, Y.Q.: Gradient map of isoparametric polynomial and its application to Ginzburg-Landau system. J. Funct. Anal. 258, 1682–1691 (2010)
[10] Hu, Z.J., Li, X.X., Jie, S.J.: On the Blaschke isoparametric hypersurfaces in the unit sphere with three distinct Blaschke eigenvalues. Sci. China Math. 54(10), 2171–2194 (2011). https://doi.org/10.1007/s11425-011-4291-9

[11] Immervoll, S.: On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres. Ann. Math. 168, 1011–1024 (2008)

[12] Koike, N.: Blaschke Dupin hypersurfaces and equiaffine tubes. Results Math. 48, 97–108 (2005)

[13] Li, A.-M., Simon, U., Zhao, G.S.: Global Affine Differential Geometry of Hypersurfaces. Walter de Gruyter and Co, Berlin (1993)

[14] Li, H.Z., Liu, H.L., Wang, C.P., Zhao, G.S.: M"obius isoparametric hypersurfaces in $S^{m+1}$ with two distinct principal curvatures. Acta Math. Sin. (Eng. Ser.) 18, 437–446 (2002)

[15] Li, T.Z., Qing, J., Wang, C.P.: M"obius and Laguerre Geometry of Dupin Hypersurfaces. arXiv:1503.02914v1 [math.DG]

[16] Li, T.Z., Wang, C.P.: A note on Blaschke isoparametric hypersurfaces. Int. J. Math. 25, 1450117 (2014). https://doi.org/10.1142/S0129167X14501171

[17] Li, X.X., Zhang, F.Y.: Immersed hypersurfaces in the unit sphere $S^{m+1}$ with constant Blaschke eigenvalues. Acta Math. Sin. Engl. Ser. 23, 533–548 (2007)

[18] Li, X.X., Zhang, F.Y.: On the Blaschke isoparametric hypersurfaces in the unit sphere. Acta Math. Sin. Engl. Ser. 25, 657–678 (2009)

[19] M"unzner, H.F.: Isoparametric hyperfl"achen in sph"aren, I. Math. Ann. 251, 57–71 (1980)

[20] M"unzner, H.F.: Isoparametric hyperfl"achen in sph"aren, II. Math. Ann. 256, 215–232 (1981)

[21] Niebergall, R., Ryan, P.J.: Isoparametric Hypersurfaces-the Affine Case, Geometry and Topology of Submanifolds, V, vol. 201-214. World Scientific Publishing, River Edge (1993)

[22] Niebergall, R., Ryan, P.J.: Focal Sets in Affine Geometry, Geometry and Topology of Submanifolds VI. World Scientific, River Edge (1994)

[23] Niebergall, R., Ryan, P.J.: Affine isoparametric hypersurfaces. Math. Z. 217, 479–485 (1994)

[24] Nomizu, K., Sasaki, T.: Affine Differential Geometry. Cambridge University Press, Cambridge (1994)

[25] Ooguri, M.: On Carlan’s identities of equiaffine isoparametric hypersurfaces. Results Math. 46, 79–90 (2004)

[26] Rodrigues, L.A., Tenenblat, K.: A characterization of Moebius isoparametric hypersurfaces of the sphere. Monatsh. Math. 158, 321–327 (2009)

[27] Stamatakis, S., Kaffas, I., Delivos, I.: Generalization of two Bonnets Theorems to the relative differential geometry of the 3-dimensional Euclidean space. J. Geom. 108, 1073–1082 (2017)

[28] Stamatakis, S., Kaffas, I.: Bonnet’s Type Theorems in the Relative Differential Geometry of the 4-Dimensional Space. arXiv:1707.07549 [math.DG] (2017)
[29] Stamatakis, S., Kaffas, I.: On the Shape Operator of Relatively Parallel Hypersurfaces in the n-Dimensional Relative Differential Geometry. arXiv:1712.10319v1 [math.DG] (2017)

[30] Wang, Q.-M.: Isoparametric hypersurfaces in complex projective spaces. In: Differential Geometry and Differential Equations, Proceedings 1980 Beijing Symposium, vol. 3, pp. 1509–1523 (1982)

[31] Wang, Q.-M.: Isoparametric functions on Riemannian manifolds, I. Math. Ann. 277(277), 639–646 (1987)

[32] Thorbergsson, G.: A Survey on Isoparametric Hypersurfaces and Their Generalizations. In: Handbook of Differential Geometry, vol. I, pp. 963–995. North-Holland, Amsterdam (2000)

Wenjing Hao and Xingxiao Li
School of Mathematics and Information Sciences
Henan Normal University
Xinxiang 453007 Henan
People’s Republic of China
e-mail: haojw0223@163.com;
xxl@henannu.edu.cn

Received: March 18, 2019.
Accepted: June 28, 2019.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.