THREE FLAVORS OF EXTREMAL BETTI TABLES

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ABSTRACT. We discuss extremal Betti tables of resolutions in three different contexts. We begin over the graded polynomial ring, where extremal Betti tables correspond to pure resolutions. We then contrast this behavior with that of extremal Betti tables over regular local rings and over a bigraded ring.

1. Introduction

Classification problems can be often discretized by replacing a collection of complicated objects by numerical invariants. For instance, if we are interested in modules over a local or graded ring, then we can study their Hilbert Polynomial, Betti numbers, Bass numbers, and more. Describing the behavior of these invariants becomes a proxy for understanding the modules; identifying the extremal behavior of an invariant provides structural limitations.

The conjectures of M. Boij and J. Söderberg [BS08], proven by D. Eisenbud and F.O. Schreyer [ES09], link the extremal properties of invariants of free resolutions over the graded polynomial ring $S = k[x_1, \ldots, x_n]$ with the Herzog–Huneke–Srinivasan Multiplicity Conjectures. Here $k$ is any field, $S$ has the standard $\mathbb{Z}$-grading, and we study the graded Betti tables of $S$-modules. The Boij–Söderberg Conjectures state that the extremal rays of the cone of Betti tables are given by Betti tables of Cohen–Macaulay modules with pure resolutions. There exist two excellent introductions to Boij–Söderberg Theory [ES10, Flo11].

In this paper, we explore the notion of an extremal Betti table in three different contexts: in the original setting of a standard graded polynomial ring; over a regular local ring; and over a finely graded polynomial ring.

Previous work has considered the extremal behavior of free resolutions, in a manner unconnected to Boij–Söderberg theory. Each graded Betti number of the Eliahou–Kervaire resolution of a lex-segment ideal is known to be maximal among cyclic modules with the same Hilbert function [Big93, Hul93, Par96]. Also, [Avr96] studies the Betti numbers of modules with extremal homological dimensions, complexity, or curvature. Though we will not discuss these types of results further, the interested reader might consider [Avr98, Pec11, IP99].

Throughout this paper, $S$ will denote a standard graded polynomial ring, $R$ will denote a regular local ring, and $T$ will denote a finely graded polynomial ring. For a graded $S$-module $M$, we define the graded Betti numbers $\beta_{i,j}(M) := \dim_k \Tor^S_i(M, k)$. Betti numbers also have a more concrete interpretation: if $F = [F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow 0]$ is a minimal graded free resolution of $M$, then $\beta_{i,j}(M)$ is the number of minimal generators of $F_i$ of degree $j$. The graded Betti table of $M$, denoted $\beta(M)$, is the vector with coordinates $\beta_{i,j}M$ in the vector space $\mathbb{V} = \bigoplus_{i=0}^n \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}$. 

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For local and multigraded rings, there are analogous definitions. For a regular local ring $R$ with residue field $k$, we define the \textbf{(local) Betti numbers} of an $R$-module as $\beta^R_i(M) = \dim_k \text{Tor}^R_i(M,k)$. Over a $\mathbb{Z}^m$-graded polynomial ring $T$, we define the \textbf{multigraded Betti numbers} of a $T$-module as $\beta^T_{i,\alpha}(M) = \dim_k \text{Tor}^T_i(M,k)_\alpha$, where $\alpha \in \mathbb{Z}^m$. We denote the respective Betti tables by $\beta^R(M)$ and $\beta^T(M)$.

To streamline the exposition, we focus on modules of finite length. With minor adjustments, most results we discuss can be extended to the case of finitely generated modules. See [BS08b, ES10, Flø11] for the standard graded case and [BEKS11b] for the local case.

Let $M$ be a graded $S$-module (or an $R$-module or a multigraded $T$-module) of finite length. We say that $\beta(M)$ is \textbf{extremal} if, for any decomposition of the form $\beta(M) = \beta(M') + \beta(M'')$ with $M'$, $M''$ graded $S$-modules (or $R$-modules or multigraded $T$-modules, respectively), we have that $\beta(M')$ is a scalar multiple of $\beta(M)$. Extremal Betti tables correspond to extremal rays of the cone of Betti tables of finite length. In the case of $S$, this is the cone

$$B^\text{fin}_Q(S) := \mathbb{Q}_{\geq 0} \cdot \{ \beta(M) \mid M \text{ is a graded } S\text{-module of finite length} \} \subseteq V.$$

Boij and Söderberg observed that for graded $S$-modules, there is a natural sufficient condition for extremality.

\textbf{Claim 1.1.} For a graded $S$-module $M$ of finite length, if $M$ has a pure resolution, then $\beta(M)$ is extremal.

Here we say that $M$ has a \textbf{pure resolution} if, for each $i$, $\beta_{i,j}(M) \neq 0$ for at most one $j$. After proving the claim, Boij and Söderberg conjectured that this condition is not only sufficient but also necessary. In fact, after imposing some obvious degree restrictions on the Betti table, they conjecture the existence of pure resolutions of Cohen–Macaulay modules of essentially any combinatorial type. This was later proven by [EFW07] in characteristic 0 and by [ES09] in a characteristic-free manner; see Theorem 3.3.

In §3, we first quickly review why Claim 1.1 provides a sufficient condition for extremality. The remainder of the section is an expository overview of Eisenbud and Schreyer’s construction of modules with pure resolutions.

We then turn our attention to the case of a regular local ring, as considered in [BEKS11b]. In contrast with the graded case, there is no obvious analogue of Claim 1.1. In retrospect this is inevitable, as there are no modules of finite length whose Betti tables are extremal.

In the final section, we move in the opposite direction, refining the grading to a finely graded polynomial ring $T$. One possibility for understanding extremal Betti tables in the multigraded setting is to seek out multigraded lifts of pure resolutions from the standard $\mathbb{Z}$-graded setting. This approach is taken in [Flø10], which considers the linear space of such multigraded Betti tables. Moreover, in the case of $k[x, y]$ with $\mathbb{Z}^2$-grading, [BF11] constructs the entire cone of bigraded Betti tables spanned by such lifted pure resolutions.

Not all extremal Betti tables arise in this way in the multigraded setting, and we provide a sufficient condition for a bigraded Betti table to be extremal, which demonstrates this fact. The extra rigidity induced by the bigrading seems to greatly complicate the picture. We use this condition to show the existence of a zoo of extremal Betti tables.
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2. Preliminaries

Given a ring \( R \) (or a scheme \( X \)) and a complex \( F \) of \( R \)-modules (or \( \mathcal{O}_X \)-modules) with differential \( \partial_i : F_i \to F_{i-1} \), we denote the homology modules of \( F \) by \( H_i(F) = (\ker \partial_i) / (\im \partial_{i+1}) \). The derived category of \( R \)-modules (or of \( \mathcal{O}_X \)-modules) is the category consisting complexes of \( R \)-modules (or \( \mathcal{O}_X \)-modules) modulo the equivalence relation generated by quasi-isomorphisms. We may represent any object in the derived category by a genuine complex of modules.

For a projection of the form \( \pi_1 : X \times \mathbb{P}^m \to X \) of schemes, there are well-defined higher direct image functors \( R^i \pi_{1*} \) that take a sheaf on \( X \times \mathbb{P}^m \) (or a complex of sheaves on \( X \times \mathbb{P}^m \)) to a sheaf on \( X \) (or a complex of sheaves on \( X \)). Further, if we are willing to work with the derived category, then there is a single functor \( R\pi_{1*} \) that combines all of these higher direct image functors: the functor \( R\pi_{1*} \) takes a sheaf \( \mathcal{F} \) on \( X \times \mathbb{P}^m \) (or a complex \( F \) of sheaves) and returns an object in the derived category of \( \mathcal{O}_X \)-modules. The functor \( R\pi_{1*} \) combines the higher direct image functors in the sense that, if \( G \) is any complex that represents \( R\pi_{1*}F \), then \( H_i(G) \cong R^{-i}\pi_{1*}F \) for all \( i \). In the special case where \( X = \text{Spec}(A) \), we will view each \( R^i\pi_{1*}\mathcal{F} \) as an \( A \)-module (instead of writing \( \Gamma(X, R^i\pi_{1*}\mathcal{F}) \)), and similarly for \( R\pi_{1*} \). If \( \mathcal{F} \) is an \( \mathcal{O}_{X \times \mathbb{P}^m} \)-module, then \( R^i\pi_{1*}\mathcal{F} = 0 \) for all \( i < 0 \). Since computing \( R\pi_{1*} \mathcal{F} \) depends only on the quasi-isomorphism class of \( \mathcal{F} \), the same fact holds for any (locally free) resolution \( F \) of an \( \mathcal{O}_{X \times \mathbb{P}^m} \)-module.

Let \( \pi_2 \) be the second projection \( X \times \mathbb{P}^m \to \mathbb{P}^m \). Given a sheaf \( \mathcal{G} \) on \( X \) and a sheaf \( \mathcal{L} \) on \( \mathbb{P}^m \), we set

\[
\mathcal{G} \boxtimes \mathcal{L} := \pi_1^* \mathcal{G} \boxtimes \pi_2^* \mathcal{L}.
\]

If \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^m}(-e) \) is a line bundle on \( \mathbb{P}^m \), then by way of the projection formula [Har77, III, Ex. 8.3], computing \( R\pi_{1*}(\mathcal{G} \boxtimes \mathcal{L}) \) is straightforward, and we will use this computation repeatedly. There are three cases, depending on the value of \( e \).

(i) If \( -e \geq 0 \), then the only nonzero cohomology of \( \mathcal{O}_{\mathbb{P}^m}(-e) \) is \( H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-e)) \), and we have that \( R\pi_{1*}(\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^m}(-e)) \) is the complex consisting of the sheaf \( \mathcal{G} \boxtimes H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-e)) \) in homological degree 0.

(ii) If \( -1 \geq -e \geq -m \), then \( \mathcal{O}_{\mathbb{P}^m}(-e) \) has no cohomology, so \( R\pi_{1*}(\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^m}(-e)) = 0 \).

(iii) If \( -m-1 \geq -e \), then the only nonzero cohomology of \( \mathcal{O}_{\mathbb{P}^m}(-e) \) is \( H^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-e)) \), and we have that \( R\pi_{1*}(\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^m}(-e)) \) is the complex consisting of sheaf \( \mathcal{G} \boxtimes H^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-e)) \) in homological degree \( -m \).
3. Extremal Betti Tables in the Graded case

In this section, we first prove Claim 1.1, providing a sufficient condition for extremality in the graded case. We then focus on the Eisenbud–Schreyer construction of pure resolutions.

We assume throughout this section $\mathbb{k}$ is an infinite field. By [EE12, Lemma 9.6], this assumption will not affect questions related to cones of Betti tables. A strictly increasing sequence of integers $d = (d_0 < d_1 < \cdots < d_n) \in \mathbb{Z}^{n+1}$ is called a degree sequence of $S$. We say a free resolution $F$ is pure of type $d$ if it has the form

$$F : \quad S(-d_0)^{\beta_{0,d_0}} \leftarrow S(-d_1)^{\beta_{1,d_1}} \leftarrow \cdots \leftarrow S(-d_n)^{\beta_{n,d_n}} \leftarrow 0.$$  

Proof of Claim 1.1. Our argument follows [BS08, §2.1], which extends a computation of Herzog and Kühl [HK84]; see also [ES10, Proposition 2.1]. Let $M$ be a finite length module with a pure resolution

$$0 \leftarrow M \leftarrow S(-d_0)^{\beta_{0,d_0}} \leftarrow S(-d_1)^{\beta_{1,d_1}} \leftarrow \cdots \leftarrow S(-d_n)^{\beta_{n,d_n}} \leftarrow 0.$$  

Suppose that $\beta(M) = \beta(M') + \beta(M'')$. Since $M$ has finite length, it follows that $M'$ would also have to be a finite length module (the Hilbert series is determined by the Betti table, and is additive). Thus, by the Auslander–Buchsbaum Theorem, the projective dimension of $M'$ is $n$. It then follows from the decomposition of $\beta(M)$ that $M'$ admits a pure resolution of type $(d_0 < d_1 < \cdots < d_n)$. Thus, if the Betti table of a pure resolution is unique up to scalar multiple, then $\beta(M')$ will be a scalar multiple of $\beta(M)$.

To prove that the $\beta_{i,d_i}$ are determined (up to scalar multiple), we consider the Herzog–Kühl equations for $M$ from [HK84]. Since $M$ has finite length, the following $n$ equations must vanish:

$$\begin{align*}
\sum_{i=0}^{n} (-1)^i \beta_{i,d_i} &= 0; \\
\sum_{i=0}^{n} (-1)^i d_i \beta_{i,d_i} &= 0; \\
&\vdots \\
\sum_{i=0}^{n} (-1)^i d_i^{n-1} \beta_{i,d_i} &= 0.
\end{align*}$$  

Thinking of this as a system of $n$ linear equations in the $(n+1)$-unknowns $\beta_{i,d_i}$, the solutions are given by the kernel of the matrix

$$\begin{pmatrix}
1 & -1 & \cdots & (-1)^n \\
d_0 & -d_1 & \cdots & (-1)^n d_n \\
& \ddots & \ddots & \vdots \\
& & -d_1^{n-1} & \cdots & (-1)^n d_n^{n-1}
\end{pmatrix}.$$  

This is a rank $n$ matrix; in fact, the $n \times n$ minor given by the first $n$ columns is nonzero. To see this, rescale every other column by $-1$ to obtain an $n \times n$ Vandermonde matrix for $(d_0, \ldots, d_{n-1})$. Since the $d_i$ are strictly increasing, this Vandermonde determinant is nonzero. It thus follows that the kernel of this matrix has rank 1, so the $\beta_{i,d_i}$ are uniquely determined, up to scalar multiple. \qed
Remark 3.2. Using Cramer’s rule and the formula for Vandermonde determinants, any solution \((\beta_{0,d_0}, \beta_{1,d_1}, \ldots, \beta_{n,d_n})\) to the system (3.1) is a scalar multiple of
\[
\left( \frac{1}{\prod_{j \neq 0} |d_0 - d_j|}, \frac{1}{\prod_{j \neq 1} |d_1 - d_j|}, \ldots, \frac{1}{\prod_{j \neq n} |d_n - d_j|} \right).
\]

We now show that any degree sequence of \(S\) is realized by a pure resolution. The first two constructions of pure resolutions are due to Eisenbud, Floystad, and Weyman [EFW07]. Their constructions are based on representation theory and Schur functors, and they thus require that \(k\) has characteristic 0. See [Flø11, §3] for an expository treatment of those constructions. The first characteristic-free construction is due to Eisenbud and Schreyer [ES09].

Theorem 3.3 ([ES09, Theorem 5.1]). For any degree sequence \(d = (d_0 < d_1 < \cdots < d_n)\), there exists a finite length graded \(S\)-module whose minimal free resolution is pure of type \(d\).

Of course, it suffices to prove the theorem in the case where \(d_0 = 0\), as we can obtain a pure resolution of type \((d_0 < \cdots < d_n)\) by tensoring a pure resolution of type \((0 < d_1 - d_0 < \cdots < d_n - d_0)\) with \(S(-d_0)\). When Boij and Söderberg conjectured the existence of pure resolutions, there were very few known examples. One family of examples that was known came from the Eagon–Northcott complex, the Buchsbaum–Rim complex, and other related complexes [BE73]. Lascoux had shown that these complexes could be constructed by applying a pushforward construction to a Koszul complex [Las78]. This pushforward construction has the effect of collapsing strands of the Koszul complex, and Eisenbud and Schreyer realized that (with the appropriate setup) this collapsing effect could be iterated. This became the key to their construction of pure resolutions.\(^1\)

Before presenting Eisenbud and Schreyer’s general construction for a pure resolution, we review the original collapsing technique in the following lemma. This produces a pure resolution of type \((0, q + 1, \ldots, q + n)\), which is the Eagon–Northcott complex for an \(n \times (q + 1)\) matrix of linear forms over \(k[x_1, \ldots, x_{n+q}]\). The proof of this lemma contains all of the technical features required for the general case. An example is provided in Figure 1.

Lemma 3.4. Let \(q\) be a positive integer and let \(S' := k[x_1, \ldots, x_{n+q}]\). Let \(f_1, \ldots, f_{n+q}\) be generic bilinear forms on \(\text{Spec}(S') \times \mathbb{P}^q\) and let \(K\) be the Koszul complex of locally free sheaves on \(\text{Spec}(S') \times \mathbb{P}^q\) given by the \(f_i\). Then \(R\pi_1^*(K)\) is represented by a pure resolution \(F\) of type \((0, q + 1, q + 2, \ldots, q + n)\) that resolves a Cohen–Macaulay \(S'\)-module of codimension \(n\).

Proof of Lemma 3.4. Since \(k\) is infinite, we may assume that the \(f_i\) form a regular sequence, and hence they define a \(q\)-dimensional subscheme \(Z \subseteq \mathbb{A}^{n+q} \times \mathbb{P}^q\). The Koszul complex \(K\) is thus a resolution of \(O_Z\). The support of \(\pi_1^*O_Z\) has dimension at most \(q\), and therefore has codimension at least \(n\). In fact, we will later see that the \(S'\)-module \(\pi_1^*O_Z\) is a Cohen–Macaulay of codimension \(n\).

\(^1\)The idea that Eisenbud and Schreyer’s construction of pure resolutions is a higher-dimensional analogue of the Eagon–Northcott and Buchsbaum–Rim complexes is developed explicitly in [BEKS11, §10].
For $0 \leq i \leq n + q$, the $\mathbb{P}^q$-degree of the generators of $K_i$ is $i$. By taking the direct images under the map $\pi_1 : \text{Spec}(S') \times \mathbb{P}^q \rightarrow \text{Spec}(S')$, we will collapse the terms $K_1, K_2, \ldots, K_q$, resulting in the desired pure resolution.

Our first goal is to show that $R^\ell \pi_1^* K \neq 0$ if and only if $\ell = 0$. We do this in two steps. As noted in Section 2, since $K$ is a resolution of $\mathcal{O}_Z$, it follows that $R^\ell \pi_1^* K \neq 0$ only if $\ell \geq 0$.

By computing $R^\ell \pi_1^* (K)$ in a second way, we will now show that $R^\ell \pi_1^* K \neq 0$ only if $\ell \leq 0$. Note that $K_i = S^{(n+q-i)} \otimes \mathcal{O}_{\mathbb{P}^q}(-i)$. For each $i$, let $C_i$ be the Čech resolution of $K_i$ with respect to the standard Čech cover $\{\text{Spec}(S') \times U_0, \ldots, \text{Spec}(S') \times U_q\}$ of $\text{Spec}(S') \times \mathbb{P}^q$.

Since the construction of Čech resolutions is functorial, we obtain a double complex $C_{\bullet, \bullet}$ consisting of $\pi_1^*$-acyclic sheaves on $\text{Spec}(S') \times \mathbb{P}^q$, which has the form:

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & S' \otimes \left( \bigoplus_{k,k'=0}^q \mathcal{O}|_{U_k \cap U_{k'}} \right) & \longrightarrow & S'(-1)^{n+q} \otimes \left( \bigoplus_{k,k'=0}^q \mathcal{O}(-1)|_{U_k \cap U_{k'}} \right) & \longrightarrow & \cdots \\
0 & \longrightarrow & S' \otimes \left( \bigoplus_{k=0}^q \mathcal{O}|_{U_k} \right) & \longrightarrow & S'(-1)^{n+q} \otimes \left( \bigoplus_{k=0}^q \mathcal{O}(-1)|_{U_k} \right) & \longrightarrow & \cdots \\
\end{array}
\]

We may now compute $R^\ell \pi_1^* K$ by applying $\pi_1^*$ to this double complex $C_{\bullet, \bullet}$ and running the vertical spectral sequence for the resulting double complex of $S'$-modules. After taking

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**Table 1.** To construct a pure resolution $F$ of type $(0, 4, 5, 6)$ on $\text{Spec}(S')$, we begin with a Koszul complex $K$ on $\text{Spec}(S') \times \mathbb{P}^3$ and then use a pushforward construction to collapse three of the terms. A term $K_i$ gets collapsed if the second factor is a line bundle on $\mathbb{P}^3$ with no cohomology.

| $K_0$ | $S^1$ | $\otimes \mathcal{O}_{\mathbb{P}^3}$ | $\stackrel{R^2 \pi_2^*}{\longrightarrow}$ |
|-------|-------|-------------------------------|--------------------------|
| $K_1$ | $S(-1)^6$ | $\otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ | |
| $K_2$ | $S(-2)^{15}$ | $\otimes \mathcal{O}_{\mathbb{P}^3}(-2)$ | |
| $K_3$ | $S(-3)^{20}$ | $\otimes \mathcal{O}_{\mathbb{P}^3}(-3)$ | |
| $K_4$ | $S(-4)^{15}$ | $\otimes \mathcal{O}_{\mathbb{P}^3}(-4)$ | |
| $K_5$ | $S(-5)^{6}$ | $\otimes \mathcal{O}_{\mathbb{P}^3}(-5)$ | |
| $K_6$ | $S(-6)^1$ | $\otimes \mathcal{O}_{\mathbb{P}^3}(-6)$ | |

**Figure 1.** The complex $K$ on $\text{Spec}(S') \times \mathbb{P}^3$ and the complex $F$ on $\text{Spec}(S')$.
vertical homology of $C^{\bullet,\bullet}$, we obtain the $vE^{\bullet,\bullet}_1$-page with differential $\partial^{1,\bullet}_1$.

\[
vE^{\bullet,\bullet}_1 : \cdots 0 \leftarrow S' \otimes H^1(\mathbb{P}^q, \mathcal{O}) \xrightarrow{\partial^{1,1}_1} S'(-1)^{n+q} \otimes H^1(\mathbb{P}^q, \mathcal{O}(-1)) \leftarrow \cdots
\]

\[
0 \leftarrow S' \otimes H^0(\mathbb{P}^q, \mathcal{O}) \xrightarrow{\partial^{1,0}_1} S'(-1)^{n+q} \otimes H^0(\mathbb{P}^q, \mathcal{O}(-1)) \leftarrow \cdots
\]

The general entry on the $vE_1$-page is given by

\[
v_{E^{1}}^{-i,j} = S'(-i)^{(n+q)} \otimes H^1(\mathbb{P}^q, \mathcal{O}(-i)).
\]

Since $H^j(\mathbb{P}^q, \mathcal{O}(-i)) = 0$ unless $j = 0$ or $q$, most of these entries of $vE_1$ are equal to 0. In fact, $vE_1$ has a single nonzero entry on row 0, with the only remaining nonzero entries appearing on row $q$, as shown below.

\[
vE^{\bullet,\bullet}_1 : 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots S'(-q-1)^{(n+q)} \otimes H^q(\mathcal{O}(-q-1)) \xrightarrow{\partial^{q-2,q}_1} \cdots
\]

\[
0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots 0 \leftarrow \cdots
\]

Since all of the terms of the $vE_1$ page lie in total degree $-i + j \leq 0$, we see that $R^\ell\pi_1_*K \neq 0$ only if $\ell \leq 0$, as claimed.

Note that after passing the $vE_1$-page, the only other differential exiting or entering a nonzero term will occur on $vE_{q+1}$, from position $(-i, j) = (-q - 1, q)$ to $(-i, j) = (0, 0)$. Since this spectral sequence satisfies $vE^{1-i,j}_1 \Rightarrow R^{i+j}\pi_1_*K$ and our previous computation shows that $R^\ell\pi_1_*K \neq 0$ if only if $\ell = 0$, only positions $(0, 0)$ and $(-q - 1, q)$ may contain nonzero entries on the $vE_2$-page. In addition, since $R^{-1}\pi_1_*K = 0$, we see that $vE_{\infty}^{-q-1,q} = 0$. Hence the differential $\partial_{q+1}^{-q-1,q}$ must be injective.

\[
vE^{\bullet,\bullet}_{q+1} : 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots \coker \partial_1^{-q-2,q} \leftarrow 0
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
0 \leftarrow S' \otimes H^0(\mathcal{O}) \leftarrow \cdots 0 \leftarrow \cdots
\]
The differential $\partial_{q+1}^{-q-1,q}$ lifts to a map $\phi$ of the free modules on the $e_1$ page:

$$S' \otimes H^0(\mathcal{O}) \xrightarrow{\phi} S'(-q-1)^{(n+q)_{q+1}} \otimes H^q(\mathcal{O}(-q-1))$$

We thus conclude that $R^0\pi_1^*\mathbf{K} = \pi_1^*\mathcal{O}_Z$ is represented by a minimal complex of the form

$$S' \xleftarrow{\phi} S'(-q-1)^{(n+q)_{q+1}} \otimes H^q(\mathcal{O}(-q-1)) \xleftarrow{\phi} S'(-q-2)^{(n+q)_{q+2}} \otimes H^q(\mathcal{O}(-q-2)) \xleftarrow{\phi} \cdots$$

Notice this is a pure complex of type $(0, q+1, \ldots, q+n)$. Since it is acyclic, it is actually a resolution of the $S'$-module $\pi_1^*\mathcal{O}_Z$. Hence this module has projective dimension $n$, and since we noted initially that it has codimension at least $n$, it follows that $\pi_1^*\mathcal{O}_Z$ is a Cohen–Macaulay module of codimension $n$. $\square$

The following proposition, due to Eisenbud and Schreyer, provides a more general framework than Lemma 3.4 for collapsing terms from a resolution. The proof is nearly identical. See Figure 2 for an illustration of this result.

**Proposition 3.5** ([ES09, Proposition 5.3]). Let $\mathcal{F}$ be a sheaf on $X \times \mathbb{P}^m$ that has a resolution $\mathbf{G}$ arising from $\mathcal{O}_X$-modules $\mathcal{G}_i$, such that

$$\mathbf{G}_i = \mathcal{G}_i \otimes \mathcal{O}(-e_i) \text{ for } 0 \leq i \leq N$$

and $e_0 < \cdots < e_N$. If this sequence contains the subsequence $(e_{k+1}, \ldots, e_{k+m}) = (1, 2, \ldots, m)$ for some $k \geq -1$, then

$$R^\ell\pi_1^*\mathcal{F} \cong R^\ell\pi_1^*\mathbf{G} = 0 \text{ for } \ell \neq 0,$$

and $\pi_1^*\mathcal{F}$ has a resolution $\mathbf{G}'$, where

$$\mathbf{G}'_i = \begin{cases} 
\mathcal{G}_i \otimes H^0(\mathbb{P}^m, \mathcal{O}(-e_i)) & \text{for } 0 \leq i \leq k, \\
\mathcal{G}_{i+m} \otimes H^m(\mathbb{P}^m, \mathcal{O}(-e_{i+m})) & \text{for } k + 1 \leq i \leq N - m.
\end{cases}$$

**Proof.** We proceed in a matter similar to the proof of Lemma 3.4. Our first goal is to show in two steps that $R^\ell p_*\mathbf{G} \neq 0$ if and only if $\ell = 0$. First, since $\mathbf{G}$ is a resolution of $\mathcal{F}$, it follows that $R^\ell p_*\mathbf{K} \neq 0$ only if $\ell \geq 0$.

We now compute $R^\ell\pi_1^*(\mathbf{G})$ in a second way to show that $R^\ell\pi_1^*\mathbf{G} \neq 0$ only if $\ell \leq 0$. For each $i$, let $\mathbf{C}_{i-\bullet}$ be the Čech resolution of $\mathcal{G}_i$ with respect to the standard Čech cover $\{X \times U_0, \ldots, X \times U_m\}$ of $X \times \mathbb{P}^m$. Since the construction of Čech resolutions is functorial, we obtain a double complex $\mathbf{C}_{\bullet,\bullet}$ consisting of $\pi_1^*$-acyclic sheaves on $X \times \mathbb{P}^m$. To compute $R^\ell\pi_1^*\mathbf{G}$, we apply $\pi_1^*$ to the double complex $\mathbf{C}_{\bullet,\bullet}$ and run the vertical spectral sequence for the resulting double complex of $\mathcal{O}_X$-modules. This yields an $e_1$-page with general entry

$$e_1^{-i,j} = \mathcal{G}_i \otimes H^j(\mathbb{P}^m, \mathcal{O}(-e_i)).$$

Since $H^j(\mathbb{P}^m, \mathcal{O}(-e_i)) = 0$ unless $j = 0$ or $m$, most of these entries are equal to 0. In fact, the resulting $e_1$-page consists of a strand of nonzero entries in row 0, followed by all zeroes.
The complex $G$

| $X$ | $\mathbb{P}^2$ |
|-----|----------------|
| $G_0$ | $G_0 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_0)$ |
| $G_1$ | $G_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_1)$ |
| $G_2$ | $G_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_2)$ |
| $G_3$ | $G_3 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ |
| $G_4$ | $G_4 \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$ |
| $G_5$ | $G_5 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_5)$ |
| $G_6$ | $G_6 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_6)$ |
| $G_7$ | $G_7 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_7)$ |
| $G_8$ | $G_8 \otimes \mathcal{O}_{\mathbb{P}^2}(-e_8)$ |

The complex $G'$

| $X$ |
|-----|
| $G_0'$ | $G_0 \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(-e_0))$ |
| $G_1'$ | $G_1 \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(-e_1))$ |
| $G_2'$ | $G_2 \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(-e_2))$ |
| $G_3'$ | $G_3 \otimes H^2(\mathcal{O}_{\mathbb{P}^2}(-e_5))$ |
| $G_4'$ | $G_4 \otimes H^2(\mathcal{O}_{\mathbb{P}^2}(-e_6))$ |
| $G_5'$ | $G_5 \otimes H^2(\mathcal{O}_{\mathbb{P}^2}(-e_7))$ |
| $G_6'$ | $G_6 \otimes H^2(\mathcal{O}_{\mathbb{P}^2}(-e_8))$ |

Figure 2. Proposition 3.5 uses a pushforward and the vanishing cohomology of line bundles on $\mathbb{P}^m$ to collapse terms from a free resolution. The above illustrates the proposition when $m = k = 2$ and $N = 8$.

In columns $k + 1, \ldots, k + m$, followed by a strand of nonzero entries in row $m$.

$vE_1^{\bullet, \bullet}$: 

\[
\begin{array}{ccccccc}
&-i=k & k+1 & k+m & k+m+1 \\
\ldots & \rightarrow & 0 & \rightarrow & \ldots & \rightarrow & 0 & \rightarrow \mathcal{G}_{k+m+1} \otimes H^m(\mathcal{O}(-e_{k+m+1})) \\
&\rightarrow & 0 & \rightarrow & \ldots & \rightarrow & 0 & \rightarrow \mathcal{G}_{k+m} \otimes H^m(\mathcal{O}(-e_{k+m})) \\
&\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
&\rightarrow & 0 & \rightarrow & \ldots & \rightarrow & 0 & \rightarrow \mathcal{G}_k \otimes H^0(\mathcal{O}(-e_k)) \\
\end{array}
\]

Since all of the nonzero terms of this $vE_1$-page lie in total cohomological degree $-i + j \leq 0$, we see that $R^\ell \pi_1^* G \neq 0$ only if $\ell \leq 0$, as desired.

We have now nearly constructed our complex $G'$. The nonzero entries on the $vE_1$-page are precisely the terms we use in $G'$, and as its differential, we will use $\partial_1$ everywhere except for the map $G'_k \leftarrow G'_{k+1}$:

\[
G'_0 \stackrel{\partial_1^{-1,0}}{\leftarrow} G'_1 \leftarrow \ldots \leftarrow \partial_1^{-k,0} \mathcal{G}_k \leftarrow \ldots G'_{k+1} \leftarrow G'_{k+2} \leftarrow \partial_1^{-N,m} G'_{N-m} \leftarrow 0.
\]

To complete the construction of $G'$ and to check its exactness, we note that after the $vE_2$-page, the only other differential exiting or entering a nonzero term will occur on the $vE_{m+1}$-page, from position $(-i,j) = (-k - m, -1)$ to $(-i,j) = (-k,0)$. Since we have $vE_1^{-i,j} \Rightarrow R^{i+j} \pi_1^* G$ and our previous computation shows that $R^\ell \pi_1^* G \neq 0$ if and only if $\ell = 0$, only positions $(-k,0)$ and $(-k - m, -1)$ may contain nonzero entries on the $vE_2$-page. In particular, although we have not yet fully constructed the differential for $G'$, we already see that our complex is exact in every position except possibly at $G'_k$ or $G'_{k+1}$.
We now examine the differential $\partial_{m+1}^{k-m-1,m}$ on $E_{m+1}$. This differential must be an isomorphism when $k > 0$, as otherwise $R^{k+1}G$ and $R^kG$ would be nonzero. When $k = 0$, it must be injective for the same reason.

This differential $\partial_{m+1}^{k-m-1,m}$ lifts to a map $\phi: G'_{k+1} \rightarrow G'_k$,

$$G'_k = G_k \otimes H^0(\mathcal{O}(-e_k)) \leftarrow \leftarrow \leftarrow \cdots \leftarrow \cdots \leftarrow G'_{k+1} = G_k \otimes H^m(\mathcal{O}(-e_m))$$

completing our construction of $G'$:

$$G'_0 \leftarrow G'_1 \leftarrow \cdots \leftarrow G'_k \leftarrow \cdots \leftarrow G'_{k+1} \leftarrow \cdots \leftarrow G'_{N-m} \leftarrow 0.$$

It follows that $G'$ is exact at $G'_k$ and at $G'_{k+1}$. Since $G'$ is acyclic, it follows that it is a resolution $\pi_1F$, as desired. \hfill \Box

Proposition 3.5 provides a tool to construct a pure free resolution with a prescribed degree sequence. We illustrate this by explaining how to construct a pure free resolution with a prescribed degree sequence. We thus define a Koszul complex $K$ involving 6 multidegree $(1,1,1)$-forms on $\text{Spec}(S') \times \mathbb{P}^2 \times \mathbb{P}^1$, and we set $G := K \otimes_{\mathcal{O}_{\text{Spec}(S') \times \mathbb{P}^2 \times \mathbb{P}^1}} (\mathcal{O}_{\text{Spec}(S')} \otimes_{\mathbb{P}^2} \mathcal{O}_{\text{Spec}(S')} \otimes_{\mathbb{P}^1}(3))$. This twist of the Koszul complex is engineered so that we are able collapse the proper terms, as shown in Figure 3. Put another way, we have attached a line bundle with vanishing cohomology to each of the terms in $G$ that we want to collapse. By applying Proposition 3.5 twice to $G$, we obtain a pure resolution of type $(0,3,5,6)$ on $\text{Spec}(S')$ that resolves a Cohen–Macaulay module of codimension 3. Finally, we mod out by 3 generic linear forms to obtain a pure resolution $F$ of type $(0,3,5,6)$ on $\text{Spec}(S)$ that resolves a module of finite length:

$$F = \left[ S^4 \leftarrow S(-3)^20 \leftarrow S(-5)^{36} \leftarrow S(-6)^{20} \leftarrow 0 \right].$$
The original complex $\mathbf{G}$

| Spec($S'$) | $\mathbb{P}^2$ | $\mathbb{P}^4$ |
|------------|----------------|----------------|
| $G_0$      | $(S')^1$       | $\mathcal{O}_{\mathbb{P}^2}$ | $\mathcal{O}_{\mathbb{P}^1}(3)$ |
| $G_1$      | $S'(-1)^6$     | $\mathcal{O}_{\mathbb{P}^2}(-1)$ | $\mathcal{O}_{\mathbb{P}^1}(2)$ |
| $G_2$      | $S'(-2)^{15}$  | $\mathcal{O}_{\mathbb{P}^3}(-2)$ | $\mathcal{O}_{\mathbb{P}^1}(1)$ |
| $G_3$      | $S'(-3)^{20}$  | $\mathcal{O}_{\mathbb{P}^3}(-3)$ | $\mathcal{O}_{\mathbb{P}^1}$ |
| $G_4$      | $S'(-4)^{15}$  | $\mathcal{O}_{\mathbb{P}^3}(-4)$ | $\mathcal{O}_{\mathbb{P}^1}(-1)$ |
| $G_5$      | $S'(-5)^6$     | $\mathcal{O}_{\mathbb{P}^3}(-5)$ | $\mathcal{O}_{\mathbb{P}^1}(-2)$ |
| $G_6$      | $S'(-6)^1$     | $\mathcal{O}_{\mathbb{P}^3}(-6)$ | $\mathcal{O}_{\mathbb{P}^1}(-3)$ |

$R_{\pi_3^*}$

The complex $\mathbf{G'}$ after one projection

| Spec($S'$) | $\mathbb{P}^1$ |
|------------|----------------|
| $G'_0$     | $(S')^1$       | $\mathcal{O}_{\mathbb{P}^3} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(3))$ |
| $G'_1$     | $S'(-1)^6$     | $\mathcal{O}_{\mathbb{P}^3}(-1) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ |
| $G'_2$     | $S'(-2)^{15}$  | $\mathcal{O}_{\mathbb{P}^3}(-2) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ |
| $G'_3$     | $S'(-3)^{20}$  | $\mathcal{O}_{\mathbb{P}^3}(-3) \otimes H^0(\mathcal{O}_{\mathbb{P}^1})$ |
| $G'_4$     | $S'(-5)^6$     | $\mathcal{O}_{\mathbb{P}^3}(-5) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$ |
| $G'_5$     | $S'(-6)^1$     | $\mathcal{O}_{\mathbb{P}^3}(-6) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-3))$ |

$R_{\pi_2^*}$

The pure resolution $\mathbf{F}$

| Spec($S'$) |
|------------|
| $F_0$      | $(S')^1 \otimes H^0(\mathcal{O}_{\mathbb{P}^2}) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(3))$ |
| $F_1$      | $S'(-3)^{20} \otimes H^2(\mathcal{O}_{\mathbb{P}^3}(-3)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1})$ |
| $F_2$      | $S'(-5)^6 \otimes H^2(\mathcal{O}_{\mathbb{P}^3}(-5)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$ |
| $F_3$      | $S'(-6)^1 \otimes H^2(\mathcal{O}_{\mathbb{P}^3}(-6)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-3))$ |

Figure 3. We iterate Proposition 3.5 to build a pure resolution $\mathbf{F}$ of type $(0,3,5,6)$ over $S'$. Modding out by linear forms yields a resolution over $S$.

Proof of Theorem 3.3. Without loss of generality, we may assume that $d_0 = 0$. We define $S' = \mathbb{k}[y_1, \ldots, y_{d_n}]$. It suffices to construct a Cohen–Macaulay $S'$-module of codimension $n$ with a pure resolution of type $d$, as we may then mod out by generic linear forms to obtain a pure resolution of a finite length $S$-module.

We define an auxiliary space $\mathbb{P}$ which is a product of projective spaces corresponding to the gaps in the degree sequence $d = (d_0 < d_1 < \cdots < d_n)$. To record these gaps, set

$$m_i := d_i - d_{i-1} - 1 \quad \text{for } 1 \leq i \leq n.$$
Set $\mathbb{P} := \mathbb{P}^m_1 \times \cdots \times \mathbb{P}^m_n$, which has dimension $d_n - n$. Choose $d_n$ generic multilinear forms of multidegree $(1, 1, \ldots, 1)$. Since $k$ is an infinite field, these forms give a regular sequence. Let $K$ denote the Koszul complex on these multilinear forms, and define
\[
G := K \otimes (\mathcal{O}_{\text{Spec}(S')} \otimes \mathcal{O}_{\mathbb{P}^m_1} \otimes \mathcal{O}_{\mathbb{P}^m_2}(-d_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^m_n}(-d_{n-1})).
\]
Note that $G$ is an exact complex with
\[
G_i = S'(-i)^{d_i} \otimes \mathcal{O}_{\mathbb{P}^m_1}(-i) \otimes \mathcal{O}_{\mathbb{P}^m_2}(-d_1 - i) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^m_n}(-d_{n-1} - i).
\]
By repeatedly applying Proposition 3.5 (the order in which we pushforward does not matter), all terms from $G$ will eventually be collapsed away with exception of $G_{d_i}$ for $0 \leq i \leq n$. More precisely, when we push away from $\mathbb{P}^m_i$, Proposition 3.5 implies that we will collapse away the terms that originally corresponded to $G_{d_i+1}, \ldots, G_{d_i+1-k}$. This process produces a pure resolution $F$ of graded $S'$-modules, where
\[
F_k = S'(-d_k)^{d_k} \otimes \bigotimes_{i=1}^{k-1} H^0(\mathbb{P}^m_i, \mathcal{O}(-d_{i-1} - k)) \otimes \bigotimes_{i=k}^{n} H^m_i(\mathbb{P}^m_i, \mathcal{O}(-d_{i-1} - k)).
\]
Since $G$ resolves a module of codimension $d_n$ and the fibers of the projection $p : X \times \mathbb{P} \rightarrow X$ have dimension $d_n - n$, it follows that the cokernel of $F$ has support of codimension at least $n$. However, since $F$ is a resolution of projective dimension $n$, we conclude that the cokernel of $F$ is a Cohen–Macaulay $S'$-module of codimension $n$, as desired. \qed

If one works with the base scheme Proj$(S)$ instead of Spec$(S)$, then there is a slightly different argument which eliminates the need to pass to the intermediate ring $S'$, but this requires different steps to check exactness. This was Eisenbud and Schreyer’s original approach in [ES09, §5].

**Remark 3.6.** There is a useful shorthand for reverse-engineering the Eisenbud–Schreyer construction of a pure resolution. For instance, to construct a pure resolution of type $(0, 3, 5, 6)$, begin by considering the table on the left, where we have marked with an asterisk the degrees that we need to collapse. We may then use a copy of $\mathbb{P}^2$ to collapse the first two asterisks and a copy of $\mathbb{P}^1$ to collapse the last asterisk. To do so, line up integers as in the middle table so that the vanishing cohomology degrees of $\mathbb{P}^2$ and $\mathbb{P}^1$ align with the asterisks. Now fill in the remaining entries of the table linearly.

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 0 \\
-1 & -1 & 0 \\
-2 & -2 & -1 \\
-3 & -3 & -2 \\
-4 & -4 & -3 \\
-5 & -5 & -4 \\
-6 & -6 & -5 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]

\[\begin{array}{c|c|c}
\text{Spec}(S') & \mathbb{P}^2 & \mathbb{P}^1 \\
\hline
0 & 0 & 3 \\
-1 & -1 & 2 \\
-2 & -2 & 1 \\
-3 & -3 & 0 \\
-4 & -4 & -1 \\
-5 & -5 & -2 \\
-6 & -6 & -3 \\
\end{array}\]
The last table tells us that we should build a Koszul complex of six \((1,1,1)\)-forms on \(\text{Spec}(S') \times \mathbb{P}^2 \times \mathbb{P}^1\) and then twist by the degrees we see in the top row: \(\mathcal{O}(0,0,3)\). Note that this is precisely the construction from Figure 3.

Although the proof of Theorem 3.3 is constructive, it does not provide an efficient technique for understanding the differentials of the resulting pure resolution \(F\). To obtain explicit formulas for the differentials from the proof, we would have to carry a description of the differential through the spectral sequence.

A more efficient approach to understanding the differentials of these Eisenbud–Schreyer pure resolutions is given in [BEKS11, §4]. That article constructs a generic version of the Eisenbud–Schreyer pure resolution, referred to as a \textbf{balanced tensor complex}, which is defined over a polynomial ring in many more variables. The differentials for the tensor complex can be expressed in terms of explicit multilinear constructions (e.g., (co)multiplication maps on symmetric and exterior products, among others). Since the Eisenbud–Schreyer pure resolutions are obtained as specializations of balanced tensor complexes [BEKS11, Theorem 10.2], this construction provides closed formulas for the various differentials in the Eisenbud–Schreyer pure resolutions.

Example 3.7. There is a Macaulay2 package \texttt{TensorComplexes} that can be used to compute the Eisenbud–Schreyer pure resolutions explicitly [M2]. With \(k = \mathbb{F}_{101}\), the following code computes the first differential for a pure resolution of type \((0,1,3,5)\).

\begin{verbatim}
i1 : loadPackage "TensorComplexes";
i2 : FF = pureResES({0,1,3,5},ZZ/101);
i3 : betti FF
     0      1     2      3
  total: 8 15 10 3
    0: 8 15 ..
    1: . . 10 .
    2: . . . 3

i4 : FF.dd_1
   | x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 0 0 0 |
   | x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 0 0 0 |
   | 0 0 0 0 0 x_1 0 0 0 x_2 0 0 0 0 0 0 |
   | 0 0 0 0 x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 |
   | 0 0 0 0 x_2 x_0 0 -x_2 0 x_1 x_2 0 0 0 0 |
   | 0 0 0 0 x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 |
   | 0 0 0 0 x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 |
   | 0 0 0 0 x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 |
   | 0 0 0 0 x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 |
   | 0 0 0 0 x_0 0 0 0 x_1 0 0 0 x_2 0 0 0 |
\end{verbatim}

4. Extremal Betti Tables in the Local case

Let \(M\) be a finitely generated module over a regular local ring \(R\) of dimension \(n\). From the minimal free resolution of \(M\),

\[
0 \leftarrow M \leftarrow R^\beta_0(M) \leftarrow R^\beta_1(M) \leftarrow \cdots \leftarrow R^\beta_\ell(M) \leftarrow 0,
\]
we obtain the (local) Betti table $\beta^R(M) = (\beta^R_0(M), \ldots, \beta^R_n(M))$. Here we again restrict our attention to the case when $M$ is of finite length.

As in the graded case, we would like to find modules $M$ of finite length where $\beta^R(M)$ is extremal. However, unlike the graded case, there are no natural candidates for such vectors. It turns out that this is because no local Betti table is extremal.

**Claim 4.1.** If $\dim(R) > 1$, then there does not exist any $R$-module $M$ of finite length whose Betti table is extremal.

**Example 4.2.** Let $R = \mathbb{k}[x, y]$. Then the local Betti table of the residue field $\mathbb{k}$ is $\beta^R(\mathbb{k}) = (1, 2, 1)$. If $M = R/(x^2, xy, y^2)$ and $N = \text{Hom}(M, \mathbb{k})$, then we have the decomposition $(1, 2, 1) = \beta^R(\mathbb{k}) + \frac{1}{3} \beta^R(M) + \frac{1}{3} \beta^R(N) = \frac{1}{2}(1, 3, 2) + \frac{1}{2}(2, 3, 1)$.

To understand how this comes to pass, we now assume that $n > 1$ and view each $\beta^R(M) \in \mathbb{Q}^{n+1}$. An extremal local Betti table corresponds to a ray of the cone

$$B^\text{fin}_R := \mathbb{Q}_{>0} \cdot \{ \beta^R(M) \mid M \text{ is an } R\text{-module of finite length} \} \subset \mathbb{Q}^{n+1}.$$  

**Theorem 4.3** ([BEKS11b, Theorem 1.1]). If $R$ is an $n$-dimensional regular local ring with $n > 1$, then $B^\text{fin}_R$ is an open cone that has no extremal rays. More precisely,

$$B^\text{fin}_R = \mathbb{Q}_{>0} \cdot \{ \rho_0, \rho_1, \ldots, \rho_{n-1} \},$$

where $\rho_i = e_i + e_i+1$ is the sum of the $i$th and $(i+1)$st standard basis vectors of $\mathbb{Q}^{n+1}$.

The story for finitely generated modules is similar; see [BEKS11b, §4]. Of course, if $\dim(R) = 1$, then $\rho_0$ is an extremal ray, as it spans the entire cone.

**Proof of Theorem 4.3.** For brevity, set $C := \mathbb{Q}_{>0} \cdot \{ \rho_0, \rho_1, \ldots, \rho_{n-1} \}$. Clearly the cone $C$ lies in the linear subspace of $\mathbb{Q}^{n+1}$ defined by $\sum_{k=0}^n (-1)^k \beta^R_k = 0$. Inside this subspace, an elementary computation confirms that $C$ equals the open cone defined by the inequalities:

$$0 < \sum_{k=i}^n (-1)^{i-k} \beta^R_k \quad \text{for } 1 \leq i \leq n.$$  

When applied to the Betti numbers of a module $M$, the above sum is a partial Euler characteristic (computed from the back of the resolution) that computes the rank of the $k$th syzygy module of $M$. In particular, each such linear functional is strictly nonnegative when evaluated on the Betti table of a finite length module, and hence we have $B^\text{fin}_R \subseteq C$.

The reverse containment $C \subseteq B^\text{fin}_R$ requires a limiting argument. We show that for each $i$, there is a sequence of pairs of positive scalars and modules $\{ (\lambda_{i,j}, M_{i,j}) \}_{j=1}^\infty$ such that

$$\rho_i = \lim_{j \to \infty} \lambda_{i,j} \beta^R(M_{i,j}).$$

The key fact used in the construction of these $R$-modules is that there exist local ring analogues to the $S$-modules with pure resolutions constructed in Theorem 3.3. Thus, given a degree sequence $d \in \mathbb{Z}^{n+1}$, we may construct an $R$-module $M(d)$ whose total Betti numbers are computed (up to scalar multiple) by the Herzog–Kühl equations. The precise existence statement for the $R$-module $M(d)$ is given in Lemma 4.4 below.
As noted in Remark 3.2, the Betti table of $M(d)$ is, up to scalar multiple, given by

$$b(d) = \prod_{\ell \neq i} |d_\ell - d_i| \left( \frac{1}{\prod_{\ell \neq 0} |d_\ell - d_0|}, \frac{1}{\prod_{\ell \neq 1} |d_\ell - d_1|}, \ldots, \frac{1}{\prod_{\ell \neq n} |d_\ell - d_n|} \right)$$

$$= \left( \frac{\prod_{\ell \neq i} |d_\ell - d_i|}{\prod_{\ell \neq 0} |d_\ell - d_0|}, \frac{\prod_{\ell \neq i} |d_\ell - d_i|}{\prod_{\ell \neq 1} |d_\ell - d_1|}, \ldots, \frac{\prod_{\ell \neq i} |d_\ell - d_i|}{\prod_{\ell \neq n} |d_\ell - d_n|} \right) \in \mathbb{Q}^{n+1}.$$  

Note that $b(d)_i = 1$. By carefully choosing degree sequences $d^{i,j}$, we will realize $\rho_i$ as the desired limit using $M_{i,j} = M(d^{i,j})$. To make this choice, set $d^{i,j} := (0, j, 2j, \ldots, ij, ij+1, (i+1)j + 1, \ldots, (n-1)j + 1)$, so that

$$d^{i,j}_k = \begin{cases} 
  k & \text{if } k \leq i, \\
  (k-1)j + 1 & \text{if } k > i.
\end{cases}$$

For these degree sequences, the Herzog–Kühl equations imply that, as $i \to \infty$, the $i$th and $(i+1)$st Betti numbers go to infinity more quickly than the other Betti numbers do. Of course, this limit does not make sense for graded Betti numbers. In the local case, where the Betti numbers are ungraded, we may consider such limits.

We thus set $M_{i,j} := M(d^{i,j})$ and $\lambda_{i,j} := \frac{1}{\beta^R(M(d^{i,j}))}$. This yields

$$\lambda_{i,j} \beta^R(M_{i,j}) = b(d^{i,j}),$$

since they are equal up to scalar multiple and the $i$th entry in both vectors is equal to 1.

We now claim that $\lim_{j \to \infty} b(d^{i,j}) = \rho_i$. By construction, the limit equals 1 in the $i$th position. Also, each element $b(d^{i,j})$ lies in the linear subspace given by $\sum_{k=0}^n (-1)^k \beta_k = 0$. Thus it suffices to show that $\lim_{j \to \infty} b(d^{i,j})_k = 0$ for $k \neq i, i+1$, which we directly compute:

$$\lim_{j \to \infty} b(d^{i,j})_k = \lim_{j \to \infty} \frac{\prod_{\ell \neq i} |d^{i,j}_\ell - d^{i,j}_i|}{\prod_{\ell \neq k} |d^{i,j}_\ell - d^{i,j}_k|} = \lim_{j \to \infty} \frac{O(j^{n-1})}{O(j^n)} = 0.$$

Thus $B^\text{fin}_Q(R)$ contains points that are arbitrarily close to each $\rho_i$. Since $C$ equals the interior of the closed cone spanned by the $\rho_i$, we have shown that $B^\text{fin}_Q(R)$ contains $C$, as desired. \qed

The following lemma is proven in [BEKS11b, Proposition 2.1].

**Lemma 4.4.** Let $R$ be an $n$-dimensional regular local ring, and let $d = (d_0, \ldots, d_n)$ be a degree sequence. If $N$ is the cokernel of the pure resolution of type $d$ constructed in Theorem 3.3, then there exists a finite length $R$-module $M(d)$ where $\beta^R(M(d)) = \beta_{i,\bar{d}_i}(N)$.

**5. Extremal Betti Tables in the Multigraded case**

Whereas in the previous section, we considered regular local rings, we now move in the opposite direction by refining the grading on the polynomial ring. As we will see, this greatly increases the complexity of the situation. The results discussed in this section stem from our original work plus extended discussions with Eisenbud and Schreyer.

We restrict attention to the simplest example of a finely graded polynomial ring, namely $T := \mathbb{k}[x, y]$ with the bigrading $\deg(x) = (1, 0)$ and $\deg(y) = (0, 1)$. We seek $T$-modules $M$ of finite length such that $\beta^T(M)$ is extremal.
Over $S$, extremal was synonymous with having a pure resolution, but over $T$ this is not the case. In fact, there cannot exist a finite length module $M$ with a resolution of $F$ where each $F_i$ is generated in a single bidegree. This is because $T$ is finely graded, so the cokernel of any map $T^a(-\mu_1, -\mu_2) \to T^b(-\lambda_1, -\lambda_2)$ has codimension at most 1.

There are, however, other natural candidates for extremal Betti tables. For instance, in the standard $\mathbb{Z}$-graded case, every pure resolution over $k[x,y]$ can be realized by taking the resolution of a quotient of monomial ideals [BS08, Remark 3.2]. Since each of these modules is naturally bigraded, we might expect that these provide extremal Betti tables in the bigraded sense, as well as in the graded sense. While this is quite often the case (see Example 5.4), there are many other extremal bigraded Betti tables as well.

To describe a sufficient condition for extremality, we introduce the notion of the **matching graph** $\Gamma(M)$ of a bigraded $T$-module of finite length. By imposing rather weak conditions on matching graphs, we produce a wide array of bigraded $T$-modules with extremal Betti tables. This illustrates the additional complexity that arises from refined gradings.

**Claim 5.1.** Let $M$ be a bigraded $T$-module of finite length. If its matching graph $\Gamma(M)$ is $(1,1)$-valent and connected, then $\beta^T(M)$ is extremal.

For a bigraded $T$-module $M$ of finite length, let $F$ be the bigraded minimal free resolution of $M$. The **matching graph** of $M$ is a graph whose vertices have weights in $\mathbb{Z}$ and whose edges are of two types: $x$-edges and $y$-edges. The vertices correspond to the degrees of the generators of the $F_i$; to a vertex $\alpha \in \mathbb{Z}^2$, we assign the weight $\beta^T_{0,\alpha}(M) + \beta^T_{1,\alpha}(M) + \beta^T_{2,\alpha}(M)$. We then include an $x$-edge (or $y$-edge, respectively) between any two vertices with the same $x$-degree (or $y$-degree).

If a vertex of $\Gamma(M)$ meets precisely $a$ of the $x$-edges and precisely $b$ of the $y$-edges, then we say that this vertex has **valency** $(a,b)$. If all of the vertices of $\Gamma(M)$ have valency $(a,b)$, then we say that $\Gamma(M)$ is an $(a,b)$-valent graph. In addition, we say that $\Gamma(M)$ is **connected** if the underlying graph (i.e., the graph on the same vertices whose edges are the union of the $x$-edges and $y$-edges of $\Gamma(M)$) is connected.

**Example 5.2.** Let $M = T/(x^2, xy, y^2)$. The minimal free resolution of $M$ has the form

$$
\begin{align*}
T^1(-2,0) & \oplus \\
T^1(-1,-1) & \oplus \\
T^1(0,-2) & \oplus
\end{align*}
\begin{align*}
T^1(-2,-1) & \to \\
T^1(-1,-2) & \to
\end{align*}
\oplus 0.
$$

Using the natural embedding of the matching graph $\Gamma(M)$ in the first orthant, $\Gamma(M)$ has $x$-edges as shown in the figure on the left.

We omit the weights on the vertices, since all weights are 1. The graph $\Gamma(M)$ appears on the right, and is $(1,1)$-valent and connected. Hence $\beta^T(M)$ is extremal by Claim 5.1.
Example 5.3. Let $M = \langle x, y \rangle / \langle x^2, xy^2, y^3 \rangle$. Then $\Gamma(M)$ fails to be $(1,1)$-valent. In fact, at each vertex of the form $(1,1)$, there are $3$ $x$-edges. In this case, $\beta^T(M)$ equals $\beta^T(\langle x \rangle / \langle x^2, xy^2 \rangle) + \beta^T(\langle y \rangle / \langle xy, y^3 \rangle)$. These last two Betti tables are extremal by Claim 5.1.

Proof of Claim 5.1. For any $\lambda_1 \in \mathbb{Z}$, we can consider the subgraph of $\Gamma(M)$ obtained by restricting to the vertices of $\Gamma(M)$ whose degrees have the form $(\lambda_1,*)$. By definition of the $x$-edges, there will be an $x$-edge between any two vertices of this subgraph. Hence, by the $(1,1)$-valency, we see that $\Gamma(M)$ has at most two vertices of the form $(\lambda_1, *)$.

In fact, for each $\lambda_1 \in \mathbb{Z}$, we claim that $\Gamma(M)$ has either zero or two vertices of the form $(\lambda_1, *)$. The bigraded Hilbert series of $M$ is given by the rational function

$$H_M(s_1, s_2) = \frac{K_M(s_1, s_2)}{(1 - s_1)(1 - s_2)} := \sum_{i=0}^{2} \sum_{\lambda \in \mathbb{Z}} (-1)^{i} \beta^T_{i,\lambda}(M)s_i^\lambda \quad (1 - s_1)(1 - s_2).$$

Since $M$ has finite length, $H_M(s_1, s_2)$ is actually a polynomial. This implies that the $K$-polynomial of $M$, $K_M$, is in $\langle 1 - s_1 \rangle \cap \langle 1 - s_2 \rangle \subseteq \mathbb{Z}[s_1, s_2]$. We thus have

$$K_M(s_1, 1) = \sum_{\lambda \in \mathbb{Z}} \left( \sum_{\lambda_2 \in \mathbb{Z}} \beta^T_{0, 1, \lambda_2} - \beta^T_{1, 1, \lambda_2} + \beta^T_{2, 1, \lambda_2} \right) s_1^{\lambda_1} = 0.$$

Thus, if $\Gamma(M)$ has a vertex of the form $(\lambda_1, *)$, then it has at least two such vertices of this form. Further, one of these vertices must correspond to a generator of $F_1$ and the other must correspond to a generator of either $F_0$ or $F_2$, and the corresponding Betti numbers be equal. By alternately considering Betti numbers with the same $x$-degrees and Betti numbers with the same $y$-degrees, we may show that any two Betti numbers in the same connected component of $\Gamma(M)$ must have the value. The connectedness of $\Gamma(M)$ this implies that each nonzero Betti number of $M$ has the same positive value.

Suppose now that $\beta^T(M) = a' \beta^T(M') + a'' \beta^T(M'')$ for some bigraded modules $M', M''$ of finite length and some $a', a'' \in \mathbb{Q}_{>0}$. We start by considering a bigdegree $(\lambda_1, \lambda_2)$ where $\beta^T_{0, 1, \lambda_2}(M') = r$. Since $K_{M'}(s_1, 1) = 0$ and $\Gamma(M)$ is $(1,1)$-valent, the argument above implies that there is a unique $\mu_2$ such that $\beta^T_{1, 1, \mu_2}(M') \neq 0$, and hence this Betti number must also equal $r$. We then consider $y$-degrees, and a similar argument shows that there is a unique $\mu_1$ such that either (but not both) $\beta^T_{0, 1, \mu_2}(M') \neq 0$ or $\beta^T_{2, 1, \mu_2}(M') \neq 0$. In either case, this Betti number must also equal $r$.

Continuing to alternate between $x$-degrees and $y$-degrees, we eventually form a subcycle of $\Gamma(M)$. However, since $\Gamma(M)$ is $(1,1)$-valent and connected, this cycle must equal $\Gamma(M)$, so we have shown that $\beta^T(M')$ is simply $r$ times $\beta^T(M)$. \hfill \Box

Example 5.4. Quotients of monomial ideals provide many examples of extremal bigraded Betti tables. For instance, let $M = I/J$, where $I = \langle x^4, xy^2, x^2 y, y^4 \rangle$ and $J = \langle x^6, x^3 y^3, y^6 \rangle$. THREE FLAVORS OF EXTREMAL BETTI TABLES 17
Then $\Gamma(M)$ is the graph on the left (each vertex has weight 1), which is extremal by Claim 5.1. The middle and right graphs above correspond to the matching graphs of other quotients of monomial ideals: the middle graph is the matching graph of $\langle x^7, y^7 \rangle / \langle x^{15}, x^{14}y^6, x^8y^8, y^{15} \rangle$. These types of examples may be very far from the pure resolutions we saw in §3. For instance, we can produce an extremal bigraded Betti table given by a resolution $F_i$, where $F_i$ has minimal generators in arbitrarily many different bidegrees.

Claim 5.1 begs the question of which $(1, 1)$-valent, connected graphs can be realized as $\Gamma(M)$ for some $M$. If $\Gamma(M)$ is a $(1, 1)$-valent, connected graph that comes from a quotient of monomial ideals, then it decomposes as the union of two nonintersecting monotonic paths (from the upper left corner to the lower right corner). But the following example illustrates that not all extremal Betti tables arise in this way.

**Example 5.5.** The cokernel of the matrix below induces the following matching graph:

\[
\begin{pmatrix}
3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 \\
1 & 0 & 2 & 1 \\
0 & 1 & 2 & 3 \\
\end{pmatrix}
\begin{pmatrix}
x^2 & xy & y^2 & 0 \\
x^2 & xy & y^2 \\
0 & x^2 & xy & y^2 \\
\end{pmatrix}
\]

However, not every $(1, 1)$-valent connected graph arises as the matching graph of a module.

**Example 5.6.** Suppose that the graph on the left below is the matching graph of a module $M$ of finite length. Then the free resolution of $M$ has the form shown on the right.

\[
\begin{aligned}
T & \oplus T & \oplus T(-2, -1) & \oplus T(-2, 2) \oplus 0 \\
T & \oplus T(-1, -1) & \oplus T(-1, -3) & \oplus T(-3, -3) \\
T & \oplus T(0, -2) & & \\
\end{aligned}
\]

In this case, the matrix $\phi$ would have the form:

\[
\begin{pmatrix}
3 & 2 & 1 & 0 \\
0 & 1 & 2 & 3 \\
1 & 0 & 2 & 1 \\
0 & 1 & 2 & 3 \\
\end{pmatrix}
\begin{pmatrix}
x^3 & ax^2y & a_2xy^2 & y^2 \\
x & -x & y^2 & 0 \\
\end{pmatrix}
\]
for some scalars $a_1, a_2$. After performing an appropriate row operation and column operation, we can assume that $a_1 = a_2 = 0$. However, the kernel of the resulting matrix is generated by $T(-3, -2) \oplus T(-2, -3)$, providing the contradiction.

Though we know of no condition for determining which $(1,1)$-valent, connected graphs arise as $\Gamma(M)$ for some $M$, Claim 5.1 provides a zoo of extremal rays. If we restrict to Betti tables whose support is contained in the square with corners $(0, 0)$ and $(3, 3)$, Claim 5.1 produces 74 extremal rays. They are generated by the tables of quotients of monomial ideals, along with the table in Example 5.5 and its dual. We conclude with a conjecture.

**Conjecture 5.7.** All extremal Betti tables of the cone of bigraded $T$-modules with finite length are generated by Betti tables of modules $M$ that satisfy Claim 5.1.
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