Updating Zigzag Persistence and Maintaining Representatives over Changing Filtrations

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Abstract

Computing persistence over changing filtrations give rise to a stack of 2D persistence diagrams where the birth-death points are connected by the so-called ‘vines’ [5]. We consider computing these vines over changing filtrations for zigzag persistence. We observe that eight atomic operations are sufficient for changing one zigzag filtration to another and provide an update algorithm for each of them. As with the zigzag persistence algorithms for a static filtration, these updates are implemented with the maintenance of representatives. Since finding consistent representatives for zigzag persistence is more involved, the updates for the zigzag case are more costly than their counterparts in the non-zigzag case. As motivations, we identify some potential use of our update algorithms including the case of dynamic point cloud data, where a vineyard of zigzag persistence diagrams captures changing homological features across distance and time.

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1 Introduction

Computation of the persistence diagram (PD) from a given filtration has turned out to be a central task in topological data analysis. Such a filtration usually represents a nested sequence of sublevel sets of a function. In scenarios where the function changes, the filtration and hence the PD may also change. For example, a time-varying function may thus generate a series of PDs viewed as a stack of 2D point sets in 3D. Tracking the birth-death pairs on PDs through this stack generates a set of curves called vines in a vineyard. Cohen-Steiner et al. [5] who coined these terms also provided an efficient algorithm to compute the vineyard from a series of filtrations obtained from a time-varying function. The changes in the function are captured by atomic operations which transpose two consecutive simplex additions in the filtration. Cohen-Steiner et al. showed that the update in the PD due to this atomic operation can be computed in $O(n)$ time if $n$ simplices constitute the filtration. In this paper, we consider the same problem but for zigzag filtrations.

Updating the persistence diagram of a zigzag filtration under atomic operations is more complicated than in the non-zigzag case. First of all, we find eight such operations (see Section 3) instead of a single one that are necessary to go from one zigzag filtration to any other zigzag filtration (Proposition 1). Second, one has to maintain representatives explicitly with more complicated computations (see Section 4 and 5). Computing the PD for a given zigzag filtration already requires maintaining representatives in the algorithms proposed by Maria and Oudot [13, 14], and our updates are built on a renewed understanding of the roles played by the representatives in computing PDs [13, 14]. Four among the eight operations are switches [2, 3, 4, 13, 14, 15] that are equivalents of transpositions [5] in the non-zigzag case. They take linear or quadratic time for updates. The other four operations allow ‘enlarging’ [13, 14] or ‘shrinking’ a filtration locally whose equivalents for non-zigzag filtrations have not been considered. These are more costly which run in cubic time (see Table 1). One may argue that since these updates take cubic time, why not compute the zigzag persistence afresh on the new filtration. In defense, we argue that representatives over changing filtrations cannot be coherently maintained by computing from scratch, which is also one of our goals. We further note that while representatives as maintained in our updates can be computed directly from a zigzag filtration by existing algorithms (e.g., [13]), not only may we lose the connections of representatives for consecutive filtrations by doing so, but we also worsen the time complexity from cubic to quartic.

To motivate our work, we mention below some potential applications of the update operations and algorithms presented in this paper.

**Dynamic point cloud.** Consider a set of points $P$ moving with respect to time [8, 11]. For each point pair in $P$, we can draw its distance-time curve revealing the variation of distance between the points w.r.t. time. For example, Figure 1a draws the curves for a simple $P$ with three points, where $e_1$, $e_2$ and $e_3$ denote edges formed by the three point pairs. Consider the Vietoris-Rips complex of $P$ with $\delta$ as the distance threshold. Since distances of the point pairs may become greater or less than $\delta$ at different time, edges formed by these pairs are added to or deleted from the Rips complex accordingly. This forms a zigzag filtration of Rips complexes, which we denote as $R^\delta$. Letting $\delta$ vary from 0 to $\infty$, and taking the persistence diagram (PD) of $R^\delta$, we obtain a vineyard [5] as a descriptor for the dynamic point cloud. We note that $R^\delta$ changes only at the critical points of the distance-time curves, which are local minima/maxima and intersections (as illustrated by the dots in Figure 1a). To compute the vineyard, one only needs to compute the PD of each $R^\delta$ where $\delta$ is in between distance values of two critical points. For example, $\{\delta_i\}_i$ are the distance values for the critical points in Figure 1a, and $\{d_i\}_i$ are the values in between. Figure 1b lists the zigzag filtration $R^{d_i}$ for each $d_i$, where each horizontal arrow is either an equality, addition of an edge, or deletion...
(a) Distance-time curves of the three point pairs.

(b) Zigzag filtration $\mathcal{R}^{d_i}$ for each $d_i$ is listed horizontally, while vertically each Rips complex is included into the one on the above.

Figure 1: An example of a dynamic point cloud with three points.
of an edge. Each transition from $\mathcal{R}^{d_i}$ to $\mathcal{R}^{d_{i+1}}$ can be realized by a sequence of atomic operations described in this paper, which provides natural associations for the PDs [5]. For example, starting from the top and going down, one needs to perform forward/backward/outward switches, inward contractions, and outward expansions (defined in Section 3). One could also start from the bottom and go up, which requires the reverse operations. In Appendix A, we provide details on how the zigzag filtrations are built for a dynamic point cloud and how the atomic operations can be used to realize the transitions.

**Levelset zigzag for time-varying function.** It is known that the level sets of a function give rise to a special type of zigzag filtrations called levelset zigzag filtrations [4], which are known to capture more information than the non-zigzag sublevel-set filtrations. Thus, even for a time-varying function, computing the vineyard for a levelset zigzag filtration may capture more information than the one by non-zigzag filtrations.

**Other potential applications.** We also hope that our algorithms for maintaining the representatives may be of independent interest. For example, an efficient maintenance of these representatives provided an efficient algorithm for computing zigzag persistence on graphs [6] and also explained why a persistence algorithm proposed by Agarwal et al. [1] for elevation functions works. Hilbert (dimension) function or rank function are among some of the basic features for a multiparameter persistence module. One may use zigzag updates to compute these functions more efficiently as Figure 2a suggests. Thinking forward, we see a potential use of our algorithms for maintaining representatives to compute generalized rank invariants [12, 16] for 2-parameter persistence modules. This may help compute different homological structures as advocated recently [7]; see Figure 2b.

![Figure 2](image-url)

Figure 2: (a) Computing dimensions or rank function on a persistence module with support over a 2D zigzag grid (poset) can be more efficiently computed by considering zigzag persistence on an initial zigzag filtration (indicated by red path) and then updating it with switches, which gives other zigzag paths (indicated by blue and golden paths). Assuming $t$ points in the grid, this will take $O(t^3)$ time with the updates instead of $O(t^{\omega+2})$ with brute force zigzag persistence computation on every path. (b) Recently, it is shown that the generalized rank of an interval in a 2-parameter module can be derived from the zigzag persistence on the boundary as shown with red and blue paths for the grey and pink intervals respectively [7]. We can leverage our update algorithms to compute the zigzag persistence over these two paths and multiple boundaries in general.
2 Preliminaries

A zigzag filtration (or simply filtration) is a sequence of simplicial complexes

$$\mathcal{F} : K_0 \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_m,$$

in which each $K_i \leftrightarrow K_{i+1}$ is either a forward inclusion $K_i \hookrightarrow K_{i+1}$ or a backward inclusion $K_i \hookrightarrow K_{i+1}$. Taking the $p$-th homology $H_p$, we derive a zigzag module

$$H_p(\mathcal{F}) : H_p(K_0) \leftrightarrow H_p(K_1) \leftrightarrow \cdots \leftrightarrow H_p(K_m),$$
in which each $H_p(K_i) \hookrightarrow H_p(K_{i+1})$ is a linear map induced by inclusion. In this paper, we take the coefficient $\mathbb{Z}_2$ for $H_p$, and thereby treat chains or cycles in $\mathbb{C}_p$ and $\mathbb{Z}_p$ as sets of simplices. The zigzag module $H_p(\mathcal{F})$ has a decomposition [2, 10] of the form $H_p(\mathcal{F}) \simeq \bigoplus_{k \in \Lambda} \mathcal{I}^{[b_k,d_k]}$, in which each $\mathcal{I}^{[b_k,d_k]}$ is an interval module over the interval $[b_k,d_k] \subseteq \{0, \ldots, m\}$. The (multi-)set of intervals $\text{Pers}_p(\mathcal{F}) := \{[b_k,d_k] \mid k \in \Lambda\}$ is an invariant of $H_p(\mathcal{F})$ and is called the $p$-th zigzag barcode (or simply barcode) of $\mathcal{F}$. Each interval in $\text{Pers}_p(\mathcal{F})$ is called a $p$-th persistence interval. We usually consider the homology $H_\ast$ in all dimensions and take the zigzag module $H_\ast(\mathcal{F})$, for which we have $\text{Pers}_\ast(\mathcal{F}) = \bigsqcup_{p \geq 0} \text{Pers}_p(\mathcal{F})$. In this paper, sometimes a filtration may have nonconsecutive indices on the complexes (i.e., some indices are skipped); notice that the barcode is still well-defined.

An inclusion in a filtration is called simplex-wise if it is an addition or deletion of a single simplex $\sigma$, which we sometimes denote as, e.g., $K_i \xleftarrow{\sigma} K_{i+1}$. A filtration is called simplex-wise if it contains only simplex-wise inclusions. For computational purposes, we especially focus on simplex-wise filtrations starting and ending with empty complexes; notice that any filtration can be converted into this form by expanding the inclusions and attaching complexes to both ends.

Now let $\mathcal{F}$ in Equation (1) be a simplex-wise filtration starting and ending with empty complexes. Then, each map $H_\ast(K_i) \leftrightarrow H_\ast(K_{i+1})$ in $H_\ast(\mathcal{F})$ is either (i) injective with a one-dimensional cokernel or (ii) surjective with a one-dimensional kernel. The inclusion $K_i \hookrightarrow K_{i+1}$ provides a birth index $i+1$ (start of a persistence interval) if $H_\ast(K_i) \rightarrow H_\ast(K_{i+1})$ is forward and injective, or $H_\ast(K_i) \hookrightarrow H_\ast(K_{i+1})$ is backward and surjective. Symmetrically, the inclusion provides a death index $i$ (end of a persistence interval) if $H_\ast(K_i) \rightarrow H_\ast(K_{i+1})$ is forward and surjective, or $H_\ast(K_i) \hookrightarrow H_\ast(K_{i+1})$ is backward and injective. We denote the set of birth indices of $\mathcal{F}$ as $\mathcal{P}(\mathcal{F})$ and the set of death indices of $\mathcal{F}$ as $\mathcal{N}(\mathcal{F})$.

3 Update operations

We present all the update operations in this section. We always denote the filtration before the update as $\mathcal{F}$ and the filtration after the update as $\mathcal{F}'$, which are both simplex-wise filtrations starting and ending with empty complexes. The eight update operations can be grouped into three types, i.e., switches, expansions, and contractions. A switch is an interchange of two consecutive additions or deletions; an expansion is an insertion of the addition and deletion of a simplex in the middle; a contraction is the reverse of an expansion. The time complexities of the update algorithms for these operations (detailed in Section 5) are listed in Table 1. Note that $m$ is the max length of $\mathcal{F}$ and $\mathcal{F}'$, and $n$ is the number of simplices in the total complex $K$, which is the union of all complexes in $\mathcal{F}$ and $\mathcal{F}'$. At the end of the section, we provide a universality property saying that every two zigzag filtrations can be connected by a sequence of the update operations.

Notice that theorems describing the interval mapping for some operations in this section have already been given in previous works. For these theorems, we provide concrete algorithms in this paper on how to perform the interval mappings computationally. Maria and Oudot [13, 14] presented a theorem on the forward/backward switches (Theorem 2.4, Transposition Diamond Principle [13]);
Table 1: Time complexities of the update algorithms

|               | forward switch | backward switch | outward switch | inward switch |
|---------------|----------------|-----------------|----------------|--------------|
|               | $O(mn)$        | $O(mn)$         | $O(n^2 + m)$   | $O(m)$       |
| outward switch| $O(mn^2)$      | $O(mn^2)$       | $O(mn^2)$      | $O(mn^2)$    |

Carlsson and de Silva [2] presented a theorem on the inward/outward switches (Mayer-Vietoris Diamond Principle [2, 3, 4]); Maria and Oudot [13, 14] presented a theorem on the inward expansion (Theorem 2.3, Surjective Diamond Principle [13]).

Notice that outward expansion and inward/outward contractions have not been considered elsewhere before for which our algorithms implicitly provide theorems on the interval mappings.

**Forward switch** is as follows, where $\sigma \not\subseteq \tau$:

$$
\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xleftarrow{\sigma} K_i \xrightarrow{\tau} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

\[\Downarrow\]

$$
\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xrightarrow{\tau} K_i' \xleftarrow{\sigma} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

Notice that if $\sigma \subseteq \tau$, then adding $\tau$ to $K_{i-1}$ in $\mathcal{F}'$ does not produce a simplicial complex.

**Backward switch** is the symmetric version of forward switch, where $\tau \not\subseteq \sigma$:

$$
\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xleftarrow{\sigma} K_i \xrightarrow{\tau} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

\[\Downarrow\]

$$
\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xrightarrow{\tau} K_i' \xleftarrow{\sigma} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

**Outward switch** is as follows, where $\sigma \neq \tau$:

$$
\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xleftarrow{\sigma} K_i \xrightarrow{\tau} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

\[\Downarrow\]

$$
\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xrightarrow{\tau} K_i' \xleftarrow{\sigma} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

Notice that if $\sigma = \tau$, then we cannot delete $\tau$ from $K_{i-1}$ in $\mathcal{F}'$ because $\tau \not\subseteq K_{i-1}$.

**Inward switch** is the reverse of outward switch, where we also have $\sigma \neq \tau$:

$$
\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xleftarrow{\sigma} K_i \xrightarrow{\tau} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

\[\Downarrow\]

$$
\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \xrightarrow{\tau} K_i' \xleftarrow{\sigma} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
$$

**Outward expansion** is as follows:

$$
\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
$$

\[\Downarrow\]

$$
\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_i' \xleftarrow{\sigma} K_{i-1} \xrightarrow{\sigma} K_i' \leftrightarrow \cdots \leftrightarrow K_m
$$
where $K'_{i-1} = K_i = K'_{i+1}$. Notice that to clearly show the correspondence of complexes in $\mathcal{F}$ and $\mathcal{F}'$, indices for $\mathcal{F}$ are made nonconsecutive in which $i-1$ and $i+1$ are skipped.

**Outward contraction** is the reverse of outward expansion:

$$\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_{i-1} \overset{\sigma}{\leftarrow} K_i \overset{\sigma}{\rightarrow} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m$$

$$\downarrow$$

$$\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K'_{i-1} \overset{\sigma}{\leftarrow} K'_i \overset{\sigma}{\rightarrow} K'_{i+1} \leftrightarrow \cdots \leftrightarrow K_m$$

where $K'_i = K_{i-1} = K_{i+1}$.

**Inward expansion** is similar to outward expansion with the difference that the two inserted arrows now pointing toward each other:

$$\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m$$

$$\downarrow$$

$$\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K'_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m$$

Note that $K'_{i-1} = K_i = K'_{i+1}$.

**Inward contraction** is the reverse of inward expansion:

$$\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_{i-1} \overset{\sigma}{\rightarrow} K_i \overset{\sigma}{\leftarrow} K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m$$

$$\downarrow$$

$$\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K'_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m$$

where $K'_i = K_{i-1} = K_{i+1}$.

**Universality of the operations.** We present the following fact:

**Proposition 1.** Let $\mathcal{F}_1, \mathcal{F}_2$ be any two simplex-wise zigzag filtrations starting and ending with empty complexes. Then $\mathcal{F}_1$ can be transformed into $\mathcal{F}_2$ by a sequence of update operations listed above.

**Proof.** We prove that any simplex-wise zigzag filtration as stated in the proposition can be transformed into an empty filtration by the update operations in this paper. This implies that an empty filtration can be transformed into any simplex-wise filtration by the reverse operations. The proposition is then true.

Let $\mathcal{F} : \emptyset = K_0 \overset{\sigma_0}{\leftarrow} K_1 \overset{\sigma_1}{\rightarrow} \cdots \overset{\sigma_{m-1}}{\leftarrow} K_m = \emptyset$ be a simplex-wise zigzag filtration. We first transform $\mathcal{F}$ into an up-down [4] simplex-wise filtration:

$$\mathcal{U} : \emptyset = L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_n \leftarrow L_{n+1} \leftarrow \cdots \leftarrow L_{2n} = \emptyset.$$ 

Let $K_i \overset{\sigma_i}{\rightarrow} K_{i+1}$ be the first deletion in $\mathcal{F}$ and $K_j \overset{\sigma_j}{\leftarrow} K_{j+1}$ be the first addition after that. That is, $\mathcal{F}$ is of the form

$$\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_i \overset{\sigma_i}{\rightarrow} K_{i+1} \overset{\sigma_{i+1}}{\leftarrow} \cdots \overset{\sigma_{j-2}}{\rightarrow} K_{j-1} \overset{\sigma_{j-1}}{\leftarrow} K_j \overset{\sigma_j}{\rightarrow} K_{j+1} \leftrightarrow \cdots \leftrightarrow K_m.$$ 

If $\sigma_{j-1} \neq \sigma_j$, we perform inward switch on $\overset{\sigma_{j-1}}{\leftarrow} \overset{\sigma_j}{\rightarrow}$ to derive a filtration

$$K_0 \leftrightarrow \cdots \leftrightarrow K_i \overset{\sigma_i}{\leftarrow} K_{i+1} \overset{\sigma_{i+1}}{\rightarrow} \cdots \overset{\sigma_{j-2}}{\leftarrow} K_{j-1} \overset{\sigma_{j-1}}{\rightarrow} K_j \overset{\sigma_j}{\leftarrow} K_{j+1} \leftrightarrow \cdots \leftrightarrow K_m.$$
If $\sigma_{j-1} = \sigma_j$, we perform outward contraction on $\sigma_j \leftarrow \sigma_{j-1}$ to derive a filtration

$$K_0 \leftrightarrow \cdots \leftrightarrow K_i \leftarrow \sigma_i K_{i+1} \leftarrow \cdots \leftarrow \sigma_{j-2} K_j \leftrightarrow \cdots \leftrightarrow K_m.$$  

We can continue the above operations until there are no additions after deletions, so that the filtration becomes an up-down one.

Finally, on the up-down filtration, we perform forward/backward switches and inward contractions to transform it into an empty one.

4 Principles of updating steps

We lay the foundations for the updating algorithms presented in Section 5. Throughout the section, let $\mathcal{F} : \emptyset = K_0 \leftarrow \sigma_0 K_1 \leftarrow \sigma_1 \cdots \leftarrow \sigma_{m-1} K_m = \emptyset$ be a simplex-wise filtration starting and ending with empty complexes.

**Definition 2** (Representative). Let $[b, d] \subseteq \{1, \ldots, m-1\}$ be an interval. A $p$-th representative sequence (sometimes simply called $p$-th representative) for $[b, d]$ consists of a sequence of $p$-cycles $\{z_i \in Z_p(K_i) \mid b \leq i \leq d\}$ and a sequence of $(p+1)$-chains $\{c_i \mid b-1 \leq i \leq d\}$, typically denoted as

$$c_{b-1} \leftarrow \cdots \leftarrow z_b \leftarrow \cdots \leftarrow z_d \rightarrow c_d,$$

such that for each $i$ with $b \leq i < d$:

- if $K_i \leftarrow K_{i+1}$ is forward, then $c_i \in C_{p+1}(K_i)$ and $z_i + z_{i+1} = \partial(c_i)$ in $K_{i+1}$;
- if $K_i \leftarrow K_{i+1}$ is backward, then $c_i \in C_{p+1}(K_i)$ and $z_i + z_{i+1} = \partial(c_i)$ in $K_i$.

Furthermore, the sequence satisfies the additional conditions:

**Birth condition:** If $K_{b-1} \leftarrow \sigma_{b-1} K_b$ is backward, then $z_b = \partial(c_{b-1})$ for $c_{b-1}$ a $(p+1)$-chain in $K_{b-1}$ containing $\sigma_{b-1}$; if $K_{b-1} \leftarrow \sigma_{b-1} K_b$ is forward, then $\sigma_{b-1} \in z_b$ and $c_{b-1}$ is undefined.

**Death condition:** If $K_d \leftarrow \sigma_d K_{d+1}$ is forward, then $z_d = \partial(c_d)$ for $c_d$ a $(p+1)$-chain in $K_{d+1}$ containing $\sigma_d$; if $K_d \leftarrow \sigma_d K_{d+1}$ is backward, then $\sigma_d \in z_d$ and $c_d$ is undefined.

Moreover, we relax the above definition and define a *post-birth* representative sequence for $[b, d]$ by ignoring the death condition. Similarly, we define a *pre-death* representative sequence for $[b, d]$ by ignoring the birth condition.

**Remark 3.** We sometimes ignore the undefined chains (e.g., $c_{b-1}$ or $c_d$) for $[b, d]$ when denoting a representative sequence. Also, the cycle $z_i$ in the above definition is called the representative $p$-cycle at index $i$ for $[b, d]$.

**Remark 4.** A representative sequence is first defined in [13] with only the $p$-cycles $\{z_i\}$ (more accurately the homology classes they represent). We augment the definition in [13] by adding the $(p+1)$-chains $\{c_i\}$ which are essential to the update algorithms in Section 5.

The following proposition from [6] says that as long as one has a pairing of the birth and death indices s.t. each interval induced by the pairing has a representative sequence, one has the barcode.

**Proposition 5.** Let $\pi : P(\mathcal{F}) \rightarrow N(\mathcal{F})$ be a bijection. If every $b \in P(\mathcal{F})$ satisfies that $b \leq \pi(b)$ and the interval $[b, \pi(b)]$ has a representative sequence, then $\text{Pers}_b(\mathcal{F}) = \{[b, \pi(b)] \mid b \in P(\mathcal{F})\}$.
Remark 6. Following Proposition 5, our update algorithms for all operations have a similar procedure: before the update, we assume that the representatives for all intervals are given. Then, based on the given representatives, we compute representatives for a pairing of the birth and death indices for the new filtration, and obtain the updated barcode; see Section 5.

Definition 7 (Birth/death order [13]). Define a total order ‘≺_b’ for the birth indices in \(F\). For two indices \(b_1, b_2 \in \{1, \ldots, m - 1\}\) s.t. \(b_1 \neq b_2\), one has that \(b_1 \prec_b b_2\) iff: (i) \(b_1 < b_2\) and \(K_{b_2-1} \leftarrow K_{b_2}\) is forward, or (ii) \(b_1 > b_2\) and \(K_{b_1-1} \leftarrow K_{b_1}\) is backward. Symmetrically, define a total order ‘≺_d’ for the death indices in \(F\). For two indices \(d_1, d_2 \in \{1, \ldots, m - 1\}\) s.t. \(d_1 \neq d_2\), one has that \(d_1 \prec_d d_2\) iff: (i) \(d_1 > d_2\) and \(K_{d_2} \leftarrow K_{d_2+1}\) is backward, or (ii) \(d_1 < d_2\) and \(K_{d_1} \leftarrow K_{d_1+1}\) is forward.

Remark 8. The motivation behind the orders defined above is as follows: for two intervals \([b_1, i], [b_2, i]\) s.t. \(b_1 ≤ b_2\), a post-birth representative for \([b_1, i]\) can always be ‘added to’ a post-birth representative for \([b_2, i]\) (see Section 4.1). A similar fact holds for the order ‘≺_d’.

Definition 9. Two non-disjoint intervals \([b_1, d_1], [b_2, d_2]\) \(\subseteq \{1, \ldots, m - 1\}\) are called comparable if \(b_1 ≤ b_2\) and \(d_1 ≤ d_2\), or \(b_2 ≤ b_1\) and \(d_2 ≤ d_1\). Also, we use \([b_1, d_1] ≺ [b_2, d_2]\) to denote the situation that \(b_1 ≤ b_2\) and \(d_1 ≤ d_2\).

4.1 Operations on representatives

We present some operations on representative sequences useful for the update algorithms.

Sum for post-birth representatives. For the following \(p\)-th post-birth representatives

\[
\zeta_1 : c_{b_1-1} \leftarrow \cdots \leftarrow c_{b_1} \leftarrow \cdots \leftarrow c_{i}, \\
\zeta_2 : c'_{b_2-1} \leftarrow \cdots \leftarrow c'_{b_2} \leftarrow \cdots \leftarrow c'_{i},
\]

for two intervals \([b_1, i], [b_2, i]\) \(\subseteq \{1, \ldots, m - 1\}\) where \(b_1 ≤ b_2\), we define a sum of \(\zeta_1\) and \(\zeta_2\), denoted \(\zeta_1 \oplus_b \zeta_2\). If \(b_1 < b_2\) (i.e., \(K_{b_2-1} \leftarrow K_{b_2}\) is forward), then \(\zeta_1 \oplus_b \zeta_2\) is defined as:

\[
\zeta_1 \oplus_b \zeta_2 : z_{b_2} \leftarrow \cdots \leftarrow z_{b_2} \leftarrow \cdots \leftarrow z_i \leftarrow \cdots \leftarrow z_i,
\]

if \(b_1 > b_2\) (i.e., \(K_{b_1-1} \leftarrow K_{b_1}\) is backward), then \(\zeta_1 \oplus_b \zeta_2\) is defined as:

\[
\zeta_1 \oplus_b \zeta_2 : z'_{b_2} \leftarrow \cdots \leftarrow z'_{b_2} \leftarrow \cdots \leftarrow z'_{b_2} \leftarrow \cdots \leftarrow z_i \leftarrow \cdots \leftarrow z_i.
\]

It can be verified that \(\zeta_1 \oplus_b \zeta_2\) is a \(p\)-th post-birth representative for \([b_2, i]\). For example, when \(b_1 < b_2\), since \(\sigma_{b_2-1} \notin z_{b_2}\) (Proposition 10) and \(\sigma_{b_2-1} \in z'_{b_2}\), we have that \(\sigma_{b_2-1} \in z_{b_2} + z'_{b_2}\).

Sum for pre-death representatives. Symmetrically, for \(p\)-th pre-death representatives

\[
\zeta_1 : z_i \leftarrow \cdots \leftarrow z_{d_1}, \\
\zeta_2 : z'_i \leftarrow \cdots \leftarrow z'_{d_2},
\]
for intervals \([i, d_1], [i, d_2]\) s.t. \(d_1 < d_2\), we define a sum \(\zeta_1 \oplus_d \zeta_2\) as a \(p\)-th pre-death representative for \([i, d_2]\). If \(d_1 > d_2\) (i.e., \(K_{d_2} \leftarrow K_{d_2+1}\) is backward), then \(\zeta_1 \oplus_d \zeta_2\) is:

\[
\zeta_1 \oplus_d \zeta_2 : z_i + z_i' + c_{i+1} + c_{d_1-1} + c_{d_1} \rightarrow z_{d_1} + z_{d_2}.
\]

if \(d_1 < d_2\) (i.e., \(K_{d_1} \leftarrow K_{d_1+1}\) is forward), then \(\zeta_1 \oplus_d \zeta_2\) is:

\[
\zeta_1 \oplus_d \zeta_2 : z_i + z_i' + c_{i+1} + c_{d_1-1} \rightarrow z_{d_1} + z_{d_2}.
\]

**Concatenation.** Let \(\zeta_1\) be a \(p\)-th post-birth representative for \([b, i]\) and \(\zeta_2\) be a \(p\)-th pre-death representative for \([i, d]\), which are of the forms:

\[
\zeta_1 : c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_i, \quad \zeta_2 : c_i \leftarrow z_i' \leftarrow \cdots \leftarrow c_{d-1} \leftarrow c_d.
\]

If \(z_i\) is homologous to \(z_i'\) in \(K_i\), i.e., \(z_i + z_i' = \partial(A)\) for \(A \in C_{p+1}(K_i)\), then we define a concatenation of \(\zeta_1\) and \(\zeta_2\), denoted \(\zeta_1 \| \zeta_2\), as:

\[
\zeta_1 \| \zeta_2 : c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_i \leftarrow \cdots \leftarrow z_{i-1} \leftarrow z_i' \leftarrow \cdots \leftarrow z_d \leftarrow c_d.
\]

Notice that \(\zeta_1 \| \zeta_2\) is a \(p\)-th representative sequence for \([b, d]\).

**Prefix, suffix, and sum for representatives.** Let

\[
\zeta : c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow c_{d-1} \leftarrow z_d \leftarrow c_d
\]

be a \(p\)-th representative sequence for an interval \([b, d] \subseteq \{1, \ldots, m - 1\}\). For an index \(i \in [b, d]\), define a prefix \(\zeta[:i]\) as a \(p\)-th post-birth representative for \([b, i]\):

\[
\zeta[:i] : c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_i.
\]

Similarly, define a suffix \(\zeta[i:]\) as a \(p\)-th pre-death representative for \([i, d]\):

\[
\zeta[i:] : z_i \leftarrow z_i' \leftarrow \cdots \leftarrow z_{d-1} \leftarrow z_d \leftarrow c_d
\]

Let \([b_1, d_1], [b_2, d_2] \subseteq \{1, \ldots, m - 1\}\) be two intervals containing a common index \(i\) and let \(\zeta_1, \zeta_2\) be \(p\)-th representative sequences for \([b_1, d_1], [b_2, d_2]\) respectively. We define a sum of \(\zeta_1\) and \(\zeta_2\), denoted \(\zeta_1 \oplus \zeta_2\), as a \(p\)-th representative sequence for the interval \([\max_{\prec_d} \{b_1, b_2\}, \max_{\prec_d} \{d_1, d_2\}]\):

\[
\zeta_1 \oplus \zeta_2 := (\zeta_1[i:] \oplus \zeta_2[i:] \| (\zeta_1[i:] \oplus_d \zeta_2[i:])).
\]

Notice that the values of \(\zeta_1 \oplus \zeta_2\) are indeed irrelevant to the choice of \(i\). Specifically, if \([b_1, d_1] < [b_2, d_2]\), then \(\zeta_1 \oplus \zeta_2\) is a \(p\)-th representative for \([b_2, d_2]\).

**Data structures for representatives.** We use a simple data structure to implement a \(p\)-th representative sequence for an interval \([b, d]\). Using an array, each index \(i \in [b, d]\) is associated with a pointer to the \(p\)-cycle at \(i\). Notice that consecutive indices in \([b, d]\) may be associated with the same \(p\)-cycle. In this case, to save memory space, we let the pointers for these indices point to the same memory location. We also do the similar thing for the \((p + 1)\)-chains. Let \(K = \bigcup_{i=0}^{n} K_i\) and \(n\) be the number of simplices in \(K\). Then, the summation of two representative sequences takes \(O(mn)\) time because \([b, d]\) has \(O(m)\) indices and adding two cycles or chains at a index takes \(O(n)\) time.
5 Update algorithms

In this section, we provide algorithms for the update operations. For all the operations, $\mathcal{F}$ and $\mathcal{F}'$ denote the filtration before and after the update respectively. Before the update, we assume that we are given the barcode $\text{Pers}_* (\mathcal{F})$ and the representatives for the intervals in $\text{Pers}_* (\mathcal{F})$. Our goal is to compute $\text{Pers}_* (\mathcal{F}')$ and the representatives for $\text{Pers}_* (\mathcal{F}')$ based on what is given, as stated in Remark 6. Proposition 5 implies that a proposed pairing of birth/death indices indeed forms a valid barcode as long as we can identify representatives for every such pair. Hence, the correctness of the algorithms in this section follows from the correctness of the representatives being updated, which is implicit in our description.

We first present the following proposition useful throughout the section.

**Proposition 10.** For a simplex-wise inclusion $X \hookrightarrow X'$ of two complexes, let $z$ be a cycle in $X'$ homologous to a cycle in $X$. Then, $\sigma \notin z$.

**Proof.** Let $y$ be the cycle in $X$ that $z$ is homologous to. We have $z = y + \partial (A)$ for $A \subseteq X'$. Since $\sigma \notin y$ and $\sigma \notin \partial (A)$ ($\sigma$ has no cofaces in $X'$), we have that $\sigma \notin z$. \qed

5.1 Forward switch

Recall that a forward switch is the following operation:

$$\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \hookrightarrow \cdots \hookrightarrow K_i \hookrightarrow K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m$$

$$\downarrow$$

$$\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \hookrightarrow \cdots \hookrightarrow K'_i \hookrightarrow K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m$$

where $\sigma \notin \tau$. We have the following four cases, and the updating for each case is different:

A. $K_{i-1} \hookrightarrow K_i$ provides a birth index $i$ and $K_i \hookrightarrow K_{i+1}$ provides a birth index $i+1$ in $\mathcal{F}$.

B. $K_{i-1} \hookrightarrow K_i$ provides a death index $i-1$ and $K_i \hookrightarrow K_{i+1}$ provides a death index $i$ in $\mathcal{F}$.

C. $K_{i-1} \hookrightarrow K_i$ provides a birth index $i$ and $K_i \hookrightarrow K_{i+1}$ provides a death index $i$ in $\mathcal{F}$.

D. $K_{i-1} \hookrightarrow K_i$ provides a death index $i-1$ and $K_i \hookrightarrow K_{i+1}$ provides a birth index $i+1$ in $\mathcal{F}$.

5.1.1 Case A

We have the following fact:

**Proposition 11.** By the assumptions of Case A, one has that $K_{i-1} \hookrightarrow K_i$ provides a birth index $i$ and $K_i \hookrightarrow K_{i+1}$ provides a birth index $i+1$ in $\mathcal{F}'$.

**Proof.** Let $x \subseteq K_i$, $x' \subseteq K_{i+1}$ be cycles s.t. $\sigma \in x$, $\tau \in x'$. If $\sigma \notin x'$, then $\tau \in x' \subseteq K'_i$ and $\sigma \in x \subseteq K_{i+1}$, and hence the proposition is true. If $\sigma \in x'$, we can update $x'$ by summing it with $x$. The new $x'$ satisfies that $x' \subseteq K_{i+1}$, $\tau \in x'$, and $\sigma \notin x'$, and hence the proposition is also true. \qed
Step I. An interval $[b, d] \in \text{Pers}_s(F)$ s.t. $b \neq i, i + 1$ is also an interval in $\text{Pers}_s(F')$. For updating its representative, we have the following situations:

$i \notin [b, d]:$ Since $b \neq i, i + 1$, and $i - 1$ is not a death index in $F$, we have that $b > i + 1$ or $d < i - 1$. So the representative for $[b, d]$ stays the same from $F$ to $F'$.

$i \in [b, d]:$ Since $b \neq i$ and $i$ is not a death index in $F$, we have that $b \leq i - 1$ and $d \geq i + 1$. Let

$$ζ : \tilde{c}_{b - 1} \leftarrow \tilde{z}_b \leftarrow \cdots \leftarrow \tilde{c}_{i - 2} \leftarrow \tilde{z}_{i - 1} \leftarrow \tilde{c}_{i - 1} \leftarrow \tilde{z}_i \leftarrow \tilde{c}_i \leftarrow \tilde{z}_{i + 1} \leftarrow \cdots \leftarrow \tilde{c}_{d - 1} \leftarrow \tilde{z}_d \rightarrow \tilde{c}_d$$

be the representative for $[b, d] \in \text{Pers}_s(F)$. If $σ \notin \tilde{c}_{i - 1}$ and $σ \notin \tilde{z}_i$, then $ζ$ is still a representative for $[b, d] \in \text{Pers}_s(F')$. If $σ \in \tilde{c}_{i - 1}$ or $σ \in \tilde{z}_i$, then let the following

$$\tilde{c}_{b - 1} \leftarrow \tilde{z}_b \leftarrow \cdots \leftarrow \tilde{z}_{i - 1} \leftarrow \tilde{z}_i \leftarrow \tilde{c}_i \leftarrow \tilde{z}_{i + 1} \leftarrow \cdots \leftarrow \tilde{z}_d \rightarrow \tilde{c}_d$$

be the representative for $[b, d] \in \text{Pers}_s(F')$, where $\tilde{z}'_i := \tilde{z}_{i - 1}$

Step II. Let $[i, d_1], [i + 1, d_2]$ be the intervals in $\text{Pers}_s(F)$ starting with $i, i + 1$ respectively, which have the following representatives:

$$ζ_1 : \tilde{z}_i \rightarrow \tilde{c}_i \rightarrow \cdots \rightarrow \tilde{c}_{d_1 - 1} \rightarrow \tilde{z}_{d_1} \rightarrow \tilde{c}_{d_1},$$

$$ζ_2 : \tilde{z}'_{i + 1} \rightarrow \tilde{c}'_{i + 1} \rightarrow \cdots \rightarrow \tilde{c}'_{d_2 - 1} \rightarrow \tilde{z}'_{d_2} \rightarrow \tilde{c}'_{d_2}.$$  

Then, in order to obtain $\text{Pers}_s(F')$, we only need to pair the birth indices $i, i + 1$ with the death indices $d_1, d_2$ besides the intervals we inherit directly in Step I. By definition, we have that $σ \in z_i \subseteq K_i$ and $τ \in z'_{i + 1} \subseteq K_{i + 1}$.

If $σ \notin z'_{i + 1}$, then $[i + 1, d_1]$ and $[i, d_2]$ form two intervals in $\text{Pers}_s(F')$ with the following representatives:

$$z_{i + 1} \leftarrow \tilde{c}_{i + 1} \leftarrow \cdots \leftarrow \tilde{c}_{d_1 - 1} \leftarrow \tilde{z}_{d_1} \leftarrow \tilde{c}_{d_1}, \quad (2)$$

$$z'_i \rightarrow z'_{i + 1} \leftarrow \tilde{c}'_{i + 1} \leftarrow \cdots \leftarrow \tilde{c}'_{d_2 - 1} \leftarrow \tilde{z}'_{d_2} \leftarrow \tilde{c}'_{d_2}, \quad (3)$$

where $z'_i := z'_{i + 1}$. It can be verified that the above representatives are valid. For example, $σ \in z_{i + 1}$ because $z_{i + 1} = z_i + δ(c_i)$, $σ \in z_i$, and $σ \notin δ(c_i)$ ($σ$ has no cofaces in $K_{i + 1}$).

If $σ \in z'_{i + 1}$, then we have the following situations (note that $[i, d_1], [i + 1, d_2] \in \text{Pers}_s(F)$ are now of the same dimension):

$d_1 \prec d_2 :$ Since $i \prec b i + 1$, we first update the representative for $[i + 1, d_2] \in \text{Pers}_s(F)$ as $ζ_1 \oplus ζ_2$. Note that $σ \notin z_{i + 1} + z'_{i + 1}$ because $σ \in z_{i + 1}$ as seen previously, where $z_{i + 1} + z'_{i + 1}$ is the cycle in $ζ_1 \oplus ζ_2$ at index $i + 1$. With the updated representative for $[i + 1, d_2] \in \text{Pers}_s(F)$, the rest of the operations are the same as done previously for $σ \notin z'_{i + 1}$.

$d_2 \prec d_1 :$ In this situation, the two intervals $[i, d_1], [i + 1, d_2] \in \text{Pers}_s(F)$ are still intervals for $\text{Pers}_s(F')$. The representative for $[i + 1, d_2] \in \text{Pers}_s(F')$ is set to $ζ_2$ because $σ \in z'_{i + 1}$. The representative for $[i, d_1] \in \text{Pers}_s(F')$ is derived by prepending $z_{i + 1} + z'_{i + 1} \subseteq K_{i + 1}$ to the beginning of $ζ_1 \oplus ζ_2$ (which is defined over $[i + 1, d_1]$), similarly to what is done to $ζ_2$ in Equation (3); note that $τ \in z_{i + 1} + z'_{i + 1}$ because $τ \notin z_{i + 1}$ (by Proposition 10) and $τ \in z'_{i + 1}$.
5.1.2 Case B

We have the following fact:

**Proposition 12.** By the assumptions of Case B, one has that \( K_{i-1} \xrightarrow{\tau} K'_i \) provides a death index \( i-1 \) and \( K'_i \xrightarrow{\sigma} K_{i+1} \) provides a death index \( i \) in \( \mathcal{F}' \).

**Proof.** Since \( \partial(\tau) \) is not a boundary in \( K_i \), \( \partial(\tau) \) must not be a boundary in \( K_{i-1} \), and hence \( K_{i-1} \xrightarrow{\tau} K'_i \) must provide a death index. Now, for contradiction, suppose that \( K'_i \xrightarrow{\sigma} K_{i+1} \) provides a birth index, i.e., there is a cycle \( x \subseteq K_i \) s.t. \( \sigma \in x \). Since \( x \not\subseteq K_i \) (because otherwise \( K_{i-1} \xrightarrow{\sigma} K_i \) would have provided a birth index), we have that \( \tau \in x \), which contradicts the fact that \( K_i \xrightarrow{\tau} K_{i+1} \) provides a death index. \( \square \)

**Step I.** An interval \([b, d] \in \text{Pers}_s(\mathcal{F})\) s.t. \( d \neq -1 \), \( i \) is also an interval in \( \text{Pers}_s(\mathcal{F}') \), and the updating of representative for \([b, d] \in \text{Pers}_s(\mathcal{F})\) is the same as in Step I for Case A described in Section 5.1.1.

**Step II.** Let \([b_1, i-1], [b_2, i] \) be the intervals in \( \text{Pers}_s(\mathcal{F}) \) ending with \( i-1 \), \( i \) respectively, which have the following representatives:

\[
\zeta_1 : c_{b_1-1} \leftarrow z_{b_1} \rightarrow \cdots \rightarrow z_{i-1} \rightarrow c_{i-1},
\]

\[
\zeta_2 : c'_{b_2-1} \leftarrow z'_{b_2} \rightarrow \cdots \rightarrow z'_{i-1} \rightarrow c'_{i-1}.
\]

Then, in order to obtain \( \text{Pers}_s(\mathcal{F}') \), we only need to pair the birth indices \( b_1, b_2 \) with the death indices \( i-1, i \). By definition, we have that \( \sigma \in c_{i-1} \subseteq K_i \) and \( \tau \in c'_i \subseteq K_{i+1} \).

If \( \sigma \not\in c'_{i-1} + c'_i \), then \([b_1, i], [b_2, i-1] \) form two intervals in \( \text{Pers}_s(\mathcal{F}') \) with the following representatives:

\[
c_{b_1-1} \leftarrow z_{b_1} \rightarrow \cdots \rightarrow z_{i-1} \rightarrow z_i \rightarrow c_i \tag{4}
\]

\[
c'_{b_2-1} \leftarrow z'_{b_2} \rightarrow \cdots \rightarrow z'_{i-1} \rightarrow c'_{i-1} + c'_i \tag{5}
\]

where \( z_i := z_{i-1} \) and \( c_i := c_{i-1} \). It can be verified that the above representatives are valid. For example, we have that \( \tau \in c'_{i-1} + c'_i \subseteq K'_i \) because \( \tau \not\in c'_{i-1} \subseteq K_i \), \( \tau \in c'_i \), and \( \sigma \not\in c'_{i-1} + c'_i \subseteq K_{i+1} \).

If \( \sigma \in c'_{i-1} + c'_i \), then we have the following situations (note that \([b_1, i-1], [b_2, i] \in \text{Pers}_s(\mathcal{F}) \) are now of the same dimension):

\[b_1 \prec_b b_2 : \text{Now } [b_1, i], [b_2, i-1] \text{ form two intervals for } \text{Pers}_s(\mathcal{F}'). \text{ The representative for } [b_1, i] \in \text{Pers}_s(\mathcal{F}') \text{ is the same as in Equation } (4). \text{ The representative for } [b_2, i-1] \in \text{Pers}_s(\mathcal{F}') \text{ is derived from } \zeta_1[i-1] \oplus_b \zeta_2[i-1] \text{ by appending the chain } c_{i-1} + c'_{i-1} + c'_i \text{ to the end, where } z_{i-1} + z'_{i-1} = \partial(c_{i-1} + c'_{i-1} + c'_i). \text{ Note that } \tau \in c_{i-1} + c'_{i-1} + c'_i \subseteq K'_i \text{ because } \tau \not\in c_{i-1}, \tau \not\in c'_{i-1}, \text{ and } c_{i-1} + c'_{i-1} + c'_i \subseteq K_{i+1}.\]

\[b_2 \prec_b b_1 : \text{In this situation, the two intervals } [b_1, i-1], [b_2, i] \in \text{Pers}_s(\mathcal{F}) \text{ are still intervals for } \text{Pers}_s(\mathcal{F}'). \text{ The representative for } [b_2, i] \in \text{Pers}_s(\mathcal{F}') \text{ is:}
\]

\[
c'_{b_2-1} \leftarrow z'_{b_2} \rightarrow \cdots \rightarrow z'_{i-1} \rightarrow z''_i \rightarrow c'_{i-1} + c'_i,
\]

where \( z''_i := z'_{i-1} \) and \( \sigma \in c'_{i-1} + c'_i \). The representative for \([b_1, i-1] \in \text{Pers}_s(\mathcal{F}') \) is derived from \( \zeta_1[i-1] \oplus_b \zeta_2[i-1] \) by appending the chain \( c_{i-1} + c'_{i-1} + c'_i \) to the end.
5.1.3 Case C

We have the following fact:

**Proposition 13.** By the assumptions of Case C, one has that \( K_{i-1} \xrightarrow{\tau} K'_i \) provides a death index \( i - 1 \) and \( K'_i \xrightarrow{\sigma} K_{i+1} \) provides a birth index \( i + 1 \) in \( \mathcal{F}' \).

**Proof.** Since \( \partial(\tau) \) is not a boundary in \( K_i \), \( \partial(\tau) \) must not be a boundary in \( K_{i-1} \), and hence \( K_{i-1} \xrightarrow{\tau} K'_i \) must provide a death index. Now, let \( x \subseteq K_i \) be a cycle s.t. \( \sigma \in x \). Then, \( x \) is also in \( K_{i+1} \), and hence \( K'_i \xrightarrow{\sigma} K_{i+1} \) must provide a birth index. \( \square \)

**Step I.** An interval \( [b, d] \in \text{Pers}_*(\mathcal{F}) \) s.t. \( b \neq i \) and \( d \neq i \) is also an interval in \( \text{Pers}_*(\mathcal{F}') \), and the updating of representative for \( [b, d] \in \text{Pers}_*(\mathcal{F}') \) is the same as in Step I for Case A described in Section 5.1.1.

**Step II.** Note that \([i, i]\) cannot form an interval in \( \text{Pers}_*(\mathcal{F}) \). To see this, suppose instead that \([i, i] \in \text{Pers}_*(\mathcal{F}) \). Then the fact that \( \sigma \) is in a boundary in \( K_{i+1} \) (by Definition 2) and \( \sigma \) has no cofaces in \( K_i \) means that \( \sigma \subseteq \tau \), which is a contradiction. Let \([b, i] \) and \([i, d] \) be the intervals in \( \text{Pers}_*(\mathcal{F}) \) ending and starting with \( i \) respectively, which have the following representatives:

\[
\zeta_1 : c_{b-1} \longleftrightarrow z_b \longleftrightarrow \cdots \longleftrightarrow z_{i-1} \longleftrightarrow z_i \longrightarrow c_i,
\]
\[
\zeta_2 : c'_i \longleftrightarrow z'_{i+1} \longleftrightarrow \cdots \longleftrightarrow z'_d \longrightarrow c'_d.
\]

Then, \([b, i - 1] \) and \([i + 1, d] \) form intervals in \( \text{Pers}_*(\mathcal{F}') \). The representative for \([b, i - 1] \in \text{Pers}_*(\mathcal{F}') \) is:

\[
c_{b-1} \longleftrightarrow z_b \longleftrightarrow \cdots \longleftrightarrow z_{i-1} \longrightarrow c'_{i-1},
\]

where \( c'_{i-1} = c_i + z_i \) if \( \sigma \nsubseteq c_i + c_i \) and equals \( c_i + z'_i \) otherwise. The representative for \([i + 1, d] \in \text{Pers}_*(\mathcal{F}') \) is:

\[
z'_{i+1} \longleftrightarrow \cdots \longleftrightarrow z'_d \longrightarrow c'_d,
\]

where the proof for \( \sigma \in z'_{i+1} \) is as done previously.

5.1.4 Case D

We have the following fact:

**Proposition 14.** Given the assumptions of Case D, let \( x \subseteq K_{i+1} \) be a cycle s.t. \( \tau \in x \). If \( \sigma \in x \), then \( K_{i-1} \xrightarrow{\tau} K'_i \) provides a death index \( i - 1 \) and \( K'_i \xrightarrow{\sigma} K_{i+1} \) provides a birth index \( i + 1 \) in \( \mathcal{F}' \); otherwise, \( K_{i-1} \xrightarrow{\tau} K'_i \) provides a birth index \( i \) and \( K'_i \xrightarrow{\sigma} K_{i+1} \) provides a death index \( i \) in \( \mathcal{F}' \).

**Proof.** If \( \sigma \in x \), then \( K'_i \xrightarrow{\sigma} K_{i+1} \) must provide a birth index because \( x \) is a new cycle in \( K_{i+1} \) created by the addition of \( \sigma \). This implies that \( \text{rank } Z_*(K_{i+1}) = \text{rank } Z_*(K'_i) + 1 \). By the assumptions of Case D, we have that

\[
\text{rank } Z_*(K_{i+1}) = \text{rank } Z_*(K_{i-1}) + 1 \quad \text{and} \quad \text{rank } B_*(K_{i+1}) = \text{rank } B_*(K_{i-1}) + 1. \quad (6)
\]

So we must have that \( \text{rank } B_*(K'_i) = \text{rank } B_*(K_{i-1}) + 1 \), implying that \( K_{i-1} \xrightarrow{\tau} K'_i \) provides a death index. If \( \sigma \notin x \), then \( x \subseteq K'_i \) and is created by the addition of \( \tau \), implying that \( K_{i-1} \xrightarrow{\tau} K'_i \) provides a birth index. So for Equation (6) to hold, \( K'_i \xrightarrow{\sigma} K_{i+1} \) must provide a death index. \( \square \)
Step I. An interval \([b, d] \in \text{Pers}_*(\mathcal{F})\) s.t. \(b \neq i + 1\) and \(d \neq i - 1\) is also an interval in \(\text{Pers}_*(\mathcal{F}')\), and the updating of representative for \([b, d] \in \text{Pers}_*(\mathcal{F}')\) is the same as in Step I for Case A described in Section 5.1.1.

Step II. Let \([b, i - 1]\) and \([i + 1, d]\) be the intervals in \(\text{Pers}_*(\mathcal{F})\) ending with \(i - 1\) and starting with \(i + 1\) respectively, which have the following representatives:

\[
\zeta_1: c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow c_{i-2} \leftarrow z_{i-1} \leftarrow c_{i-1},
\]

\[
\zeta_2: z'_{i+1} \leftarrow \cdots \leftarrow z'_{d-1} \leftarrow c'_d.
\]

Note that \(\tau \in z'_{i+1} \subseteq K_{i+1}\). By Proposition 14, the updating is different based on whether \(\sigma \in z'_{i+1}\):

\(\sigma \in z'_{i+1}\): We have that \(K_{i-1} \leftarrow \tau \rightarrow K'_i\) provides a death index \(i - 1\) and \(K'_i \leftarrow \sigma \rightarrow K_{i+1}\) provides a birth index \(i + 1\) in \(\mathcal{F}'\). Since \(\partial(z'_{i+1}) = 0\) and \(\sigma, \tau \in z'_{i+1}\), we have that \(\partial(\sigma) + \partial(\tau) = \partial(z'_i \setminus \{\sigma, \tau\})\), where \(z'_i \setminus \{\sigma, \tau\} \subseteq K_{i-1}\). Hence, \(\partial(\sigma)\) is homologous to \(\partial(\tau)\) in \(K_{i-1}\), and \([b, i - 1]\) is still an interval in \(\text{Pers}_*(\mathcal{F}')\) with the following representative:

\[
c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow c_{i-2} \leftarrow z_{i-1} \leftarrow c_{i-1} + z'_{i+1},
\]

where \(\tau \in c_{i-1} + z'_{i+1} \subseteq K'_i\) and \(\partial(c_{i-1} + z'_{i+1}) = \partial(c_{i-1}) = z_{i-1}\). Also, since \(\sigma \in z'_{i+1}\), \([i + 1, d]\) is still an interval in \(\text{Pers}_*(\mathcal{F}')\) with \(\zeta_2\) as a representative.

\(\sigma \notin z'_{i+1}\): We have that \(K_{i-1} \leftarrow \tau \rightarrow K'_i\) provides a birth index \(i\) and \(K'_i \leftarrow \sigma \rightarrow K_{i+1}\) provides a death index \(i\) in \(\mathcal{F}'\). Now, \([b, i], [i, d]\) form two intervals in \(\text{Pers}_*(\mathcal{F}')\) with the following representatives:

\[
c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow c_{i-2} \leftarrow z_{i-1} \leftarrow 0 \leftarrow z_i \leftarrow c_i,
\]

\[
z'_i \leftarrow z'_{i+1} \leftarrow \cdots \leftarrow c'_{d-1} \leftarrow c'_d
\]

where \(z_i := z_{i-1}\), \(c_i := c_{i-1}\), and \(z'_i := z'_{i+1}\).

5.1.5 Time complexity

Step I of all the cases needs to go over \(O(m)\) intervals. Since the representatives of only \(O(n)\) intervals need to be updated, each of which takes \(O(n)\) time, this step takes \(O(n^2 + m)\) time. The bottleneck of Step II of all the cases is the addition of two representative sequences, which takes \(O(mn)\) time. So the forward switch operation takes \(O(mn)\) time.

5.2 Backward switch

A backward switch is symmetric to a forward switch and hence the algorithm for it is also symmetric. So, we omit the details.

5.3 Outward switch

Recall that an outward switch is the following operation:

\[
\mathcal{F}: K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \leftarrow \sigma \rightarrow K_i \leftarrow \tau \rightarrow K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
\]

\[
\downarrow
\]

\[
\mathcal{F}': K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \leftarrow \tau \rightarrow K'_i \leftarrow \sigma \rightarrow K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
\]
where $\sigma \neq \tau$. By the Mayer-Vietoris Diamond Principle [2, 3, 4], there is an bijection between $\text{Pers}_s(F)$ and $\text{Pers}_s(F')$. Let $[b, d]$ be an interval in $\text{Pers}_s(F)$ with the following representatives:

$$\zeta : c_{b-1} \longrightarrow z_b \longrightarrow \cdots \longrightarrow z_d \longrightarrow c_d.$$ 

We have seven different cases for $[b, d] \in \text{Pers}_s(F)$ (see below). In each case, the form of the corresponding interval in $\text{Pers}_s(F')$ and the updating of representative are different. Note that the seven cases are disjoint and cover all the possibilities of $[b, d]$ because: (i) Case A–C correspond to $b = i$ or $d = i$ (which implies that $i \in [b, d]$); (ii) Case D corresponds to $i \in [b, d]$ but $b \neq i$ and $d \neq i$; (iii) the remaining cases correspond to $i \not\in [b, d]$.

**Case A** $(b = i, d = i)$: Suppose that $[b, d] \in \text{Pers}_s(F)$ is in dimension $p$. The corresponding interval in $\text{Pers}_s(F')$ is also $[b, d]$ but in dimension $p - 1$. The representative for $[b, d] \in \text{Pers}_{p-1}(F')$ is set to $c'_{i-1} \longleftrightarrow z'_i \rightarrow c'_i$, where $z'_i = \partial(\tau)$, $c'_{i-1} = \tau$, and $c'_i = z_i \setminus \{\tau\}$.

**Case B** $(b < i, d = i)$: The corresponding interval in $\text{Pers}_s(F')$ is $[b, i - 1]$ with the following representative:

$$c_{b-1} \longleftrightarrow z_b \longrightarrow \cdots \longrightarrow z_{i-1},$$

where $\tau \in z_{i-1}$ because $z_{i-1} = z_i \cup \partial(c_{i-1})$, $\tau \in z_i$, and $\tau \not\in \partial(c_{i-1})$ ($\tau$ has no cofaces in $K_i$).

**Case C** $(b = i, d > i)$: This case is symmetric to Case B and the details are omitted.

**Case D** $(b < i, d > i)$: See Section 5.3.1.

**Case E** $(b = i + 1)$: The corresponding interval in $\text{Pers}_s(F')$ is $[i, d]$. If $\sigma \not\in c_i$, then $[i, d] \in \text{Pers}_s(F')$ has the following representative:

$$c_{i-1} \longleftrightarrow z_i \longrightarrow z_{i+1} \longrightarrow \cdots \longrightarrow z_d \longrightarrow c_d,$$

where $z_i := z_{i+1}$ and $c_{i-1} := c_i$. Note that $\sigma \not\in \partial(c_i) = z_{i+1}$ because $\sigma$ has no cofaces in $K_i$, and hence $z_{i+1} \subseteq K'_i$. If $\sigma \in c_i$, then $[i, d] \in \text{Pers}_s(F')$ has the following representative:

$$c_{i-1} \longleftrightarrow z_i \longrightarrow z_{i+1} \longrightarrow \cdots \longrightarrow z_d \longrightarrow c_d,$$

where $z_i := z_{i+1} + \partial(\sigma)$ and $c_{i-1} := c_i + \sigma$.

**Case F** $(d = i - 1)$: This case is symmetric to Case E and the details are omitted.

**Case G** $(b > i + 1$ or $d < i - 1)$: The corresponding interval in $\text{Pers}_s(F')$ is $[b, d]$ and the representative stays the same.

**5.3.1 Case D**

In this case, the corresponding interval in $\text{Pers}_s(F')$ is still $[b, d]$. If $\sigma \not\in c_{i-1}$ and $\tau \not\in c_i$, then the representative for $[b, d] \in \text{Pers}_s(F')$ stays the same besides the changes on the arrow directions. For example, $z_{i-1} \longrightarrow z_i$ in $\zeta$ now becomes $z_{i-1} \longleftarrow z_i$. After the switch, where $c_{i-1} \subseteq K_{i-1}$. Note that we always have $z_i \subseteq K'_i$ because $\sigma, \tau \not\in z_i$ by Proposition 10.

If $\sigma \in c_{i-1}$ or $\tau \in c_i$, then we have the following situations:
\( \sigma \not\in c_{t-1} + c_t : \) The representative for \([b, d] \in \text{Pers}_s(F')\) is set to:

\[
c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_{i-2} \leftarrow z_i \leftarrow \cdots \leftarrow z_{i+1} \leftarrow c_{i+1} \leftarrow \cdots \leftarrow z_{d-1} \leftarrow z_d \rightarrow c_d,
\]

where \(z'_i := z_{i+1}, z_{i-1} + z_{i+1} = \partial(c_{i-1} + c_i)\), and \(c_{i-1} + c_i \subseteq K_i - 1\) because \(\sigma \not\in c_{i-1} + c_i\). Note that \(z_{i+1} \subseteq K_i\) because \(z_{i+1}\) as a cycle in \(K_i\) does not contain \(\sigma\) by Proposition 10.

\( \tau \not\in c_{i-1} + c_i : \) Symmetrically, the representative for \([b, d] \in \text{Pers}_s(F')\) is set to:

\[
c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_{i-2} \leftarrow z_i \leftarrow \cdots \leftarrow z_{i+1} \leftarrow c_{i+1} \leftarrow \cdots \leftarrow z_{d-1} \leftarrow z_d \rightarrow c_d,
\]

where \(z'_i := z_{i-1}\).

\( \tau, \sigma \in c_{i-1} + c_i : \) The representative for \([b, d] \in \text{Pers}_s(F')\) is set to:

\[
c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_{i-2} \leftarrow z_i \leftarrow \cdots \leftarrow z_{i+1} \leftarrow c_{i+1} \leftarrow \cdots \leftarrow z_{d-1} \leftarrow z_d \rightarrow c_d,
\]

where \(z'_i := z_{i+1} + \partial(\sigma)\).

### 5.3.2 Time complexity

Going over all the intervals in \(\text{Pers}_s(F)\) takes \(O(m)\) time, and each case takes no more than \(O(n)\) time. Since Case D can be executed for no more than \(n\) times, the time complexity of outward switch operation is \(O(n^2 + m)\).

### 5.4 Inward switch

Recall that an inward switch is the following operation:

\[
F : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \leftrightarrow K_i \leftrightarrow K_{i+1} \leftrightarrow \cdots \leftrightarrow K_m
\]

\[
\downarrow
\]

\[
F' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-1} \leftrightarrow K'_i \leftrightarrow \cdots \leftrightarrow K_m
\]

where \(\sigma \neq \tau\). By the Mayer-Vietoris Diamond Principle [2, 3, 4], there is a bijection between \(\text{Pers}_s(F)\) and \(\text{Pers}_s(F')\). Let \([b, d]\) be an interval in \(\text{Pers}_s(F)\) with the following representative:

\[
\zeta : c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_{d-1} \leftarrow c_d.
\]

As in Section 5.3, we have the following seven cases for \([b, d] \in \text{Pers}_s(F)\):

**Case A** \((b = i, d = i) : \) Suppose that \([b, d] \in \text{Pers}_s(F)\) is in dimension \(p\). The corresponding interval in \(\text{Pers}_s(F')\) is also \([b, d]\) but in dimension \(p + 1\). Since \(\partial(c_{i-1}) = z_i = \partial(c_i)\), we have that \(\partial(c_{i-1} + c_i) = 0\), which means that \(c_{i-1} + c_i\) is a \((p + 1)\)-cycle in \(K_i^t\). Also, since \(\sigma \in c_{i-1}, \tau \in c_i, \tau \not\in c_{i-1}\) (because \(\tau \not\in K_{i-1}\)), and \(\sigma \not\in c_i\) (because \(\sigma \not\in K_{i+1}\)), we have that \(\sigma, \tau \in c_{i-1} + c_i\).

**Case B** \((b < i, d = i) : \) The corresponding interval in \(\text{Pers}_s(F')\) is \([b, i-1]\). We have that \(z_{i-1} + z_i = \partial(c_{i-1})\) and \(z_{i-1} = \partial(c_i)\) for \(c_{i-1} \subseteq K_{i-1} \subseteq K'_i\) and \(c_i \subseteq K_{i+1} \subseteq K'_i\). So \(z_{i-1} = \partial(c_{i-1} + c_i)\). Since \(\tau \in c_i\) and \(\tau \not\in c_{i-1}\) (because \(\tau \not\in K_{i-1}\)), it is true that \(\tau \in c_{i-1} + c_i\). Then, the representative for \([b, i-1] \in \text{Pers}_s(F')\) is:

\[
c_{b-1} \leftarrow z_b \leftarrow \cdots \leftarrow z_{i-2} \leftarrow c_{i-1} \leftarrow c_i.
\]
Case C \((b = i, d > i)\): This case is symmetric to Case B and the details are omitted.

Case D \((b < i, d > i)\): The corresponding interval in \(\text{Pers}_s(F')\) is still \([b, d]\) and the representative stays the same besides the changes on the arrow directions. For example, \(z_{i-1} \leftrightarrow c_{i-1} \leftrightarrow z_i \leftrightarrow c_i \) in \(\zeta\) now becomes \(z_{i-1} \leftrightarrow c_{i-1} \leftrightarrow z_i \) after the switch, where \(c_{i-1} \subseteq K_{i-1} \subseteq K'_i\) and \(z_i \subseteq K_i \subseteq K'_i\).

Case E \((b = i + 1)\): The corresponding interval in \(\text{Pers}_s(F')\) is \([i, d]\) with the following representative:
\[
z_i \leftrightarrow z_{i+1} \leftrightarrow \cdots \leftrightarrow z_d \mapsto c_d, \text{where } z_i := z_{i+1}.
\]

Case F \((d = i - 1)\): This case is symmetric to Case E and the details are omitted.

Case G \((b > i + 1 \text{ or } d < i - 1)\): The corresponding interval in \(\text{Pers}_s(F')\) is \([b, d]\) and the representative stays the same.

**Time complexity.** Traversing the intervals in \(\text{Pers}_s(F)\) takes \(O(m)\) time, and all the cases take no more than \(O(n)\) time with Case G taking constant time. Since Cases A–F execute for only a fixed number of times, the time complexity of inward switch operation is \(O(m)\).

### 5.5 Outward expansion

Recall that an outward expansion is the following operation:

\[
F: K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
\]

\[
\downarrow
\]

\[
F': K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
\]

where \(K'_{i-1} = K_i = K'_{i+1}\). We also assume that \(\sigma\) is a \(p\)-simplex. Notice that indices for \(F\) are nonconsecutive in which \(i - 1\) and \(i + 1\) are skipped.

For the update, we first determine whether the induced map \(H_s(K'_{i-1}) \leftarrow H_s(K'_i)\) is injective or surjective by checking whether \(\sigma\) is in a \(p\)-cycle in \(K'_{i-1}\) (injective) or not (surjective). Let \(\{I_j \mid j \in B\}\) be the set of intervals in \(\text{Pers}_s(F)\) containing \(i\), where \(B\) is an indexing set. Also, let \(z^j_i\) be the representative \(p\)-cycle at index \(i\) for \(I_j\). Note that \(\{z^j_i \mid j \in B\}\) is a basis for \(H_p(K_i) = H_p(K'_i)\). Then, we claim that (i) \(\sigma\) is in a \(p\)-cycle in \(K'_{i-1}\) \(\Leftrightarrow\) (ii) \(\sigma \in z^j_i\) for a \(j \in B\). Hence, to determine the injectivity/surjectivity, we only need to check condition (ii). To prove the claim, let \(z \subseteq K'_{i-1}\) be a \(p\)-cycle containing \(\sigma\). Then, \(z = \sum_{j \in \Lambda} z^j_i + x\), where \(\Lambda \subseteq B\) and \(x\) is a \(p\)-boundary in \(K'_{i-1}\). We have that \(\sigma \notin x\) because \(\sigma\) has no cofaces in \(K'_{i-1}\). Hence, \(\sigma \in \sum_{j \in \Lambda} z^j_i\), which implies condition (ii). This proves the ‘only if’ part of the claim, and the proof for the ‘if’ part is obvious.

#### 5.5.1 \(H_s(K'_{i-1}) \leftarrow H_s(K'_i)\) is surjective

The only difference of \(\text{Pers}_s(F)\) and \(\text{Pers}_s(F')\) in this case is that there is a new interval \([i, i]\) in \(\text{Pers}_s(F')\) with the representative \(c_{i-1} \leftrightarrow z_i \mapsto c_i\), where \(z_i = \partial(\sigma)\) and \(c_{i-1} = c_i = \sigma\). Let \([b, d]\) be an interval in \(\text{Pers}_s(F)\). If \(i \notin [b, d]\), the representative for \([b, d] \in \text{Pers}_s(F)\) can be directly used as a representative for \([b, d] \in \text{Pers}_s(F')\). If \(i \in [b, d]\), let

\[
\zeta: c_{b-1} \leftrightarrow z_b \leftrightarrow \cdots \leftrightarrow z_{i-2} \leftrightarrow z_i \leftrightarrow \cdots \leftrightarrow z_{i+2} \leftrightarrow \cdots \leftrightarrow z_d \mapsto c_d
\]
be the representative for $[b, d] \in \text{Pers}_s(\mathcal{F})$. Then, the representative for $[b, d] \in \text{Pers}_s(\mathcal{F}')$ is updated to the following:

$$
\ldots \leftrightarrow z_{i-2} \leftrightarrow z_i' \leftrightarrow z_{i-1} \leftrightarrow 0 \leftrightarrow z_i \leftrightarrow z_{i+1}' \leftrightarrow z_{i+2} \leftrightarrow \ldots ,
$$

where $z'_{i-1} = z'_{i+1} := z_i$, $c'_{i+1} := c_i$, and the remaining cycles/chains are as in $\zeta$.

### 5.5.2 $H_*(K'_{i-1}) \leftrightarrow H_*(K'_i)$ is injective

In this case, $P(\mathcal{F}') = P(\mathcal{F}) \cup \{i + 1\}$ and $N(\mathcal{F}') = N(\mathcal{F}) \cup \{i - 1\}$. In order to obtain $\text{Pers}_s(\mathcal{F}')$, we need to find ‘pairings’ for the death index $i - 1$ and birth index $i + 1$ in $\mathcal{F}'$. Let $\{I_j \mid j \in \mathcal{B}\}$ be the set of intervals in $\text{Pers}_p(\mathcal{F})$ containing $i$, where $\mathcal{B}$ is an indexing set, and let $z'_i$ be the representative $p$-cycle at index $i$ for $I_j$. Moreover, let $\Lambda := \{j \in \mathcal{B} \mid \sigma \in z'_i\}$, and for each $j \in \Lambda$, let $\zeta_j$ be the $p$-th representative sequence for $I_j$. We do the following:

- Whenever there exist $j, k \in \Lambda$ s.t. $I_j \prec I_k$, update the representative for $I_k$ as $\zeta_j \oplus \zeta_k$, and delete $k$ from $\Lambda$. Note that the $p$-cycle at index $i$ in $\zeta_j \oplus \zeta_k$ does not contain $\sigma$.

After the above operations, we have that no two intervals in $\{I_j \mid j \in \Lambda\}$ are comparable. We then rewrite the intervals in $\{I_j \mid j \in \Lambda\}$ as

$$[b_1, d_1], [b_2, d_2], \ldots, [b_\ell, d_\ell] \text{ s.t. } b_1 \prec_b b_2 \prec_b \ldots \prec_b b_\ell.$$

Also, for each $j$, let $\zeta_j$ be the $p$-th representatives sequence for $[b_j, d_j] \in \text{Pers}_s(\mathcal{F})$.

For $j \leftarrow 1, \ldots, \ell - 1$, we do the following:

- Note that $d_{j+1} \prec_a d_j$ because otherwise $[b_j, d_j]$ and $[b_{j+1}, d_{j+1}]$ would be comparable. Then, let $[b_{j+1}, d_j]$ form an interval in $\text{Pers}_s(\mathcal{F}')$. The representative is set as follows: since $\zeta_j \oplus \zeta_{j+1}$ is a representative for $[b_{j+1}, d_j]$ in $\mathcal{F}$, in which the $p$-cycle at index $i$ does not contain $\sigma$, $\zeta_j \oplus \zeta_{j+1}$ can be ‘expanded’ to become a representative for $[b_{j+1}, d_j] \in \text{Pers}_s(\mathcal{F}')$ as done in Section 5.5.1.

After this, let $[b_1, i - 1]$ and $[i + 1, d_\ell]$ form two intervals in $\text{Pers}_s(\mathcal{F}')$ with representatives $\zeta_1[i:]$ and $\zeta_\ell[i:]$ respectively.

Finally, all the intervals in $\text{Pers}_s(\mathcal{F})$ that are not ‘touched’ in the previous steps are carried into $\text{Pers}_s(\mathcal{F}')$. The updates of representatives for these intervals remain the same as described in Section 5.5.1.

### 5.5.3 Time complexity

Determining injectivity/surjectivity at the beginning takes $O(m + n \log n)$ time. Representative update for each interval containing $i$ in Section 5.5.1 takes $O(m)$ time, and there are no more than $n$ intervals containing $i$, so the total time spent on the surjective case is $O(mn)$. The bottleneck of the injective case is caused by the two loops, both of which take $O(mn^2)$ time. Hence, the outward expansion takes $O(mn^2)$ time.

### 5.6 Outward contraction

Recall that an outward contraction is the following operation:

$$\mathcal{F} : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_{i-1} \overset{\sigma}{\leftarrow} K_i \overset{\sigma}{\leftarrow} K_{i+1} \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m$$

$$\downarrow$$

$$\mathcal{F}' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K'_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m$$
where $K'_i = K_{i-1} = K_{i+1}$. We also assume that $\sigma$ is a $p$-simplex. Notice that the indices for $F'$ are not consecutive, i.e., $i - 1$ and $i + 1$ are skipped.

For the update, we first determine whether the induced map $H_*(K_{i-1}) \leftarrow H_*(K_i)$ is injective or surjective by checking whether $i - 1$ is a death index in $F$ (injective) or $i$ is a birth index in $F$ (surjective).

### 5.6.1 $H_*(K_{i-1}) \leftarrow H_*(K_i)$ is surjective

Since outward contractions are inverses of outward expansions (see Section 5.5), the only difference of $\text{Pers}_s(F)$ and $\text{Pers}_s(F')$ in this case is that $[i, i] \in \text{Pers}_s(F)$ is deleted in $\text{Pers}_s(F')$. Let $[b, d] \neq [i, i]$ be an interval in $\text{Pers}_s(F)$. If $i \notin [b, d]$, i.e., $b > i$ or $d < i$, then since $b \neq i + 1$ and $d \neq i - 1$, we have that $b \geq i + 2$ or $d \leq i - 2$. So the representative for $[b, d] \in \text{Pers}_s(F)$ can be directly used as a representative for $[b, d] \in \text{Pers}_s(F')$. If $i \in [b, d]$, then suppose that

$$c_{b-1} \leftarrow \cdots \leftarrow z_b \leftarrow \cdots \leftarrow c_d$$

is the representative for $[b, d] \in \text{Pers}_s(F)$, which needs to be updated to the following for $[b, d] \in \text{Pers}_s(F')$:

$$c_{b-1} \leftarrow \cdots \leftarrow z_b \leftarrow \cdots \leftarrow c_{d-1} \leftarrow z_d$$

### 5.6.2 $H_*(K_{i-1}) \leftarrow H_*(K_i)$ is injective

**Step I.** In this case, $i - 1 \in \mathbb{N}(F)$, $i + 1 \in P(F)$, $\mathbb{N}(F') = \mathbb{N}(F) \setminus \{i - 1\}$, and $P(F') = P(F) \setminus \{i + 1\}$. Let $[b_k, i - 1]$ and $[i + 1, d_k]$ be the $p$-th intervals in $\text{Pers}_s(F)$ ending/staring with $i - 1$, $i+1$ respectively, which have the following representatives:

$$\zeta^*_i : c^*_b \leftarrow \cdots \leftarrow z^*_b \leftarrow \cdots \leftarrow z^*_{i-1},$$

$$\zeta^*_d : z^*_{i+1} \leftarrow \cdots \leftarrow c^*_{d-1} \leftarrow z^*_{d} \leftarrow \cdots \leftarrow c^*_{d}.$$

Then, let $\{[\beta_j, \delta_j] \mid j \in \mathcal{B}\}$ be the set of intervals in $\text{Pers}_p(F)$ containing $i + 1$, where $\mathcal{B}$ is an indexing set. Notice that $[i + 1, d_k] \in \{[\beta_j, \delta_j] \mid j \in \mathcal{B}\}$. Moreover, for each $j \in \mathcal{B}$, denote the $p$-th representative for $[\beta_j, \delta_j]$ as:

$$\tilde{\zeta}_j : \tilde{c}^j_{\beta_{j-1}} \leftarrow \cdots \leftarrow \tilde{z}^j_{\beta_j} \leftarrow \cdots \leftarrow \tilde{z}^j_{\delta_j} \leftarrow \cdots \leftarrow \tilde{c}^j_{\delta_j}.$$  

Then, the set of homology classes $\{[\tilde{z}^j_{i+1}] \mid j \in \mathcal{B}\}$, which contains $[z^o_{i+1}]$, is a basis for $H_p(K_{i+1})$. Since $z^*_i \subseteq K_{i-1} = K_{i+1}$, we can write $z^*_i$ as the following sum:

$$z^*_i = z^o_{i+1} + \sum_{j \in \Lambda} \tilde{z}^j_{i+1} + C,$$  

(7)

where $\Lambda \subseteq \mathcal{B}$ and $C$ is the boundary of a $(p + 1)$-chain $A$ in $K_{i+1}$. The sum in Equation (7) must contain $z^o_{i+1}$ because: (i) $\sigma \in z^*_i$ and $\sigma \in z^o_{i+1}$ by Definition 2; (ii) no cycle in $\{\tilde{z}^j_{i+1} \mid j \in \mathcal{B}\}$ other than $z^o_{i+1}$ contains $\sigma$ (Proposition 10); (iii) no boundary in $K_{i+1}$ contains $\sigma$ since $\sigma$ has no cofaces in $K_{i+1}$. Equation (7) can be executed by first computing a boundary basis for $K_{i+1}$, which forms a cycle basis for $K_{i+1}$ along with $\{\tilde{z}^j_{i+1} \mid j \in \mathcal{B}\}$, and then performing a Gaussian elimination on the cycle basis.
Step II. Do the following:

- Whenever there is a \( j \in \Lambda \) s.t. \( \beta_j \sim_b b_s \), update the representative for \([b_s, i-1]\) as \( \zeta_* := \zeta_* \oplus \tilde{z}_j \). The update of \( \zeta_* \) is valid because \( \delta_j \sim_d i-1 \) (\( \delta_j > i-1 \) and \( K_{i-1} \leftrightarrow K_i \) is backward), which means that \( [\beta_j, \delta_j] < [b_s, i-1] \). Then, delete \( j \) from \( \Lambda \).

- Similarly, whenever there is a \( j \in \Lambda \) s.t. \( \delta_j \sim_d d_o \), update the representative for \([i+1, d_o]\) as \( \zeta_o := \zeta_o \oplus \tilde{z}_j \) because \( [\beta_j, \delta_j] < [i+1, d_o] \). Then, delete \( j \) from \( \Lambda \).

Note that Equation (7) still holds after the above operations. To see this, suppose that, e.g., there is an \( \ell \in \Lambda \) s.t. \( \beta_\ell \sim_b b_s \). We can rewrite Equation (7) as:

\[
\tilde{z}_{i-1}^{\ell} + \tilde{z}_{i-1}^{\ell} = z_{i+1}^0 + \sum_{j \in \Lambda \setminus \{\ell\}} \tilde{z}_{i+1}^j + C + \partial(\tilde{c}_{i-1}^\ell + \tilde{c}_i^\ell),
\]

in which \( \tilde{z}_{i-1}^{\ell} = \tilde{z}_{i+1}^j + \partial(\tilde{c}_{i-1}^\ell + \tilde{c}_i^\ell) \). Since \( z_{i-1}^* + \tilde{z}_{i-1}^{\ell} \) is the cycle at index \( i-1 \) for the updated \( \zeta_* \) in this iteration, Equation (7) still holds; but we also need to update \( C \) and \( A \) in this case.

After the operations in this step, we have that \( b_s \sim_b \beta_j \) and \( d_o \sim_d \delta_j \) for any \( j \in \Lambda \).

Step III. Rewrite the intervals in \( \{[\beta_j, \delta_j] \mid j \in \Lambda \} \) as

\[
[b_1, d_1], [b_2, d_2], \ldots, [b_\ell, d_\ell] \text{ s.t. } b_1 \sim_b b_2 \sim_b \cdots \sim_b b_\ell.
\]

Also, for each \( j \) s.t. \( 1 \leq j \leq \ell \), denote the \( p \)-th representative for \([b_j, d_j]\) as

\[
\zeta_j : c_{b_j-1}^j \leftarrow \cdots \leftarrow z_{b_j}^j \leftarrow \cdots \leftarrow c_{d_j}^j.
\]

Then, Equation (7) can be rewritten as

\[
z_{i-1}^* = z_{i+1}^0 + \sum_{j=1}^\ell \tilde{z}_{i+1}^j + C. \tag{8}
\]

Next, we pair the birth indices \( b_s, b_1, \ldots, b_\ell \) with the death indices \( d_o, d_1, \ldots, d_\ell \) to form intervals for \( \text{Pers}_*(\mathcal{F}') \). Initially, all these indices are ‘unpaired’. We first pair \( b_s \) with \( d^* = \max_{<d}\{d_1, \ldots, d_\ell\} \) (and hence \( b_s, d^* \) become ‘paired’) to form an interval \([b_s, d^*]\) in \( \text{Pers}_*(\mathcal{F}') \), with the following representative:

\[
\zeta_* \parallel (\zeta_o \oplus_d \zeta_1[i+1:] \oplus_d \cdots \oplus_d \zeta_\ell[i+1:]). \tag{9}
\]

We treat \( \zeta_* \) as a \( p \)-th post-birth representative for \([b_s, i]\) in \( \mathcal{F}' \) and treat \( \zeta_o \oplus_d \zeta_1[i+1:] \oplus_d \cdots \oplus_d \zeta_\ell[i+1:] \) as a \( p \)-th pre-death representative for \([i, d^*]\) in \( \mathcal{F}' \) (because \( d_o \sim_d d^* \) and \( d^* = \max_{<d}\{d_1, \ldots, d_\ell\} \)). The concatenation in Equation (9) is well-defined because (i) \( z_{i-1}^* \) is the \( p \)-cycle at index \( i-1 \) in \( \zeta_* \); (ii) \( z_{i+1}^* + \sum_{j=1}^\ell z_{i+1}^j \) is the \( p \)-cycle at index \( i+1 \) in \( \zeta_o \oplus_d \zeta_1[i+1:] \oplus_d \cdots \oplus_d \zeta_\ell[i+1:] \); (iii) the two \( p \)-cycles are homologous in \( K'_i = K_{i-1} = K_{i+1} \) due to Equation (8).

Similarly, we pair \( b_\ell = \max_{<b}\{b_1, \ldots, b_\ell\} \) with \( d_o \) to form an interval \([b_\ell, d_o]\) in \( \text{Pers}_*(\mathcal{F}') \), with the following representative:

\[
(\zeta_* \oplus_b \zeta_1[:i-1] \oplus_b \cdots \oplus_b \zeta_\ell[:i-1]) \parallel \zeta_o. \tag{10}
\]

Then, we pair the remaining indices \( \{b_1, \ldots, b_{\ell-1}\} \) with \( \{d_1, \ldots, d_\ell\} \setminus \{d^*\} \). Specifically, for \( r := 1, \ldots, \ell - 1 \), pair \( b_r \) with a death index as follows.
• If \( d_r \) is unpaired, then pair \( b_r \) with \( d_r \). The representative for \([b_r, d_r] \in \text{Pers}_s(\mathcal{F}')\) can be updated from the representative for \([b_r, d_r] \in \text{Pers}_s(\mathcal{F})\) as described in Section 5.6.1.

• If \( d_r \) is paired, then \( d_{r-1}, \ldots, d_r \) must be all the paired death indices so far because (i) \( d_1, \ldots, d_{r-1} \) must be paired in previous iterations; (ii) the paired birth indices so far are \( b_*, b_1, \ldots, b_{r-1}, b_r \), which match the cardinality of \( d_{r-1}, d_1, \ldots, d_r \), and so there can be no more paired death indices. Since \( d_{r-1}, \ldots, d_r \) are all unpaired, we pair \( b_r \) with \( \delta = \max_{\delta_{d_{r+1}, \ldots, d_r}} \). The representative for \([b_r, \delta] \in \text{Pers}_s(\mathcal{F})'\) is set as

\[
(\zeta \oplus_b \zeta_1[i-1] \oplus_b \cdots \oplus_b \zeta_r[i-1]) \parallel (\zeta_0 \oplus_d \zeta_{r+1}[i+1] \oplus_d \cdots \oplus_d \zeta[i+1]).
\] (11)

The validity of the above representative follows from: (i) \( b_*, b_1 \prec_b \cdots \prec_b b_r \); (ii) the concatenation is well-defined because by Equation (8), \( z_{i-1}^* + \sum_{j=1}^{r} z_j^i \) is homologous to \( z_{i+1}^0 + \sum_{j=r+1}^{\ell} z_j^i \) in \( K_i' \).

Note that in order to compute the representative in Equation (11) efficiently, we maintain the sum \( \zeta_0 \oplus_b \zeta_1[i-1] \oplus_b \cdots \oplus_b \zeta_r[i-1] \) at each iteration, by adding \( \zeta_r[i-1] \) to the sum for the previous iteration. Similarly, we maintain the sum \( \zeta_0 \oplus_d \zeta_{r+1}[i+1] \oplus_d \cdots \oplus_d \zeta[i+1] \), which is initially \( \zeta_0 \oplus_d \zeta_1[i+1] \oplus_d \cdots \oplus_d \zeta[i+1] \), and add \( \zeta_r[i+1] \) at each iteration. Since each iteration only performs a constant number of sums and concatenations of representatives, which take \( O(mn) \) time, the total time spent on computing Equation (11) is \( O(mn^2) \).

**Step IV.** Every interval in \( \text{Pers}_s(\mathcal{F}) \) that is not ‘touched’ in the previous steps is carried into \( \text{Pers}_s(\mathcal{F}') \). The update of representatives for these intervals are the same as in Section 5.6.1.

5.6.3 Time complexity

By a similar analysis as in Section 5.5.3, the time spent on the surjective case is \( O(mn) \). For the injective case, the complexity of Step I is dominated by the cost of boundary basis computation, which can be accomplished in \( O(n^3) \) time by invoking a persistence algorithm [5]. In Step II, there are at most \( n \) iterations and each iteration takes \( O(mn) \) time. So Step II takes \( O(mn^2) \) time. Step III is dominated by the computation of the representatives in Equation (9)–(11), which takes \( O(mn^2) \) time. The time taken in Step IV is the same as in the surjective case. Hence, the outward contraction takes \( O(mn^2) \) time.

5.7 Inward expansion

Recall that an inward expansion is the following operation:

\[
\mathcal{F}: K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_i \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
\]

\[
\downarrow
\]

\[
\mathcal{F}': K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_{i-1}^s \leftrightarrow K_i^s \leftrightarrow \cdots \leftrightarrow K_{i+1}^s \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
\]

where \( K_{i-1}' = K_i = K_{i+1}' \). We also assume that \( \sigma \) is a \( p \)-simplex. Note that indices for \( \mathcal{F} \) are nonconsecutive in which \( i-1 \) and \( i+1 \) are skipped.

For the update, we first determine whether the induced map \( H_s(K_{i-1}^s) \rightarrow H_s(K_i^s) \) is injective or surjective by checking whether \( \partial(\sigma) \) is a \( (p-1) \)-boundary in \( K_{i-1}^s \) (injective) or not (surjective). The checking can be done by performing a reduction of \( \partial(\sigma) \) on a \( (p-1) \)-boundary basis for \( K_{i-1}^s \) (which can be computed by a persistence algorithm [5]).
5.7.1 \( \mathcal{H}_s(K'_{i-1}) \rightarrow \mathcal{H}_s(K'_i) \) is injective

The only difference of \( \text{Pers}_s(F) \) and \( \text{Pers}_s(F') \) in this case is that there is a new interval \([i, i]\) in \( \text{Pers}_s(F') \). The representative \( p \)-cycle at index \( i \) for \([i, i] \in \text{Pers}_s(F')\) can be any \( p \)-cycle in \( K'_i \) containing \( \sigma \), which can be computed from the reduction done previously on \( \partial(\sigma) \) and the \((p-1)\)-boundary basis for \( K'_{i-1} \). Also, any interval \([b, d] \in \text{Pers}_s(F)\) is an interval in \( \text{Pers}_s(F') \); the update of representative for \([b, d] \in \text{Pers}_s(F')\) is as in Section 5.5.1.

5.7.2 \( \mathcal{H}_s(K'_{i-1}) \rightarrow \mathcal{H}_s(K'_i) \) is surjective

In this case, \( \mathcal{P}(F') = \mathcal{P}(F) \cup \{i + 1\} \) and \( \mathcal{N}(F') = \mathcal{N}(F) \cup \{i - 1\} \). Let \( \{I_j \mid j \in B\} \) be the set of intervals in \( \text{Pers}_{p-1}(F) \) containing \( i \), where \( B \) is an indexing set. Also, let \( z_i^j \) be the representative \((p-1)\)-cycle at index \( i \) for \( I_j \). We have that the homology classes \( \{z_i^j \mid j \in B\} \) form a basis for \( \mathcal{H}_{p-1}(K_i) = \mathcal{H}_{p-1}(K'_{i-1}) \). Denote the map \( \mathcal{H}_s(K'_{i-1}) \rightarrow \mathcal{H}_s(K'_i) \) as \( \rho \); then, there exists a non-empty set \( \Lambda \subseteq B \) s.t. \( \sum_{j \in \Lambda}[z_i^j] \in \ker(\rho) \). The set \( \Lambda \) can be computed by forming a \((p-1)\)-cycle basis for \( K'_{i-1} \) by combining \( \{z_i^j \mid j \in B\} \) with the \((p-1)\)-boundary basis for \( K'_{i-1} \), and then performing a Gaussian elimination and reduction.

We then rewrite the intervals in \( \{I_j \mid j \in \Lambda\} \) as

\[
[b_1, d_1], [b_2, d_2], \ldots, [b_\ell, d_\ell] \text{ s.t. } b_1 <_b b_2 <_b \cdots <_b b_\ell.
\]

For each \( j \) s.t. \( 1 \leq j \leq \ell \), let \( \zeta_j \) denote the representative sequence for \([b_j, d_j] \in \text{Pers}_s(F)\), and let \( z_i^j \) denote the \((p-1)\)-cycle at index \( i \) in \( \zeta_j \). We then pair the birth indices \( i + 1, b_1, \ldots, b_\ell \) with the death indices \( i - 1, d_1, \ldots, d_\ell \) to form intervals for \( \text{Pers}_s(F') \). We first pair \( b_t \) with \( i - 1 \) to form an interval \([b_t, i - 1] \in \text{Pers}_s(F')\), whose representative is derived from \( \zeta_1[i:] \oplus \cdots \oplus \zeta_\ell[i:] \). The representative for \([b_t, i - 1] \in \text{Pers}_s(F')\) is valid because: (i) \( \sum_{j=1}^\ell[z_i^j] \in \ker(\rho) \); (ii) \( b_t = \max_{<_b} \{b_1, \ldots, b_\ell\} \).

Symmetrically, we pair \( i + 1 \) with \( d_\ell = \max_{<_d} \{d_1, \ldots, d_\ell\} \) to form an interval \([i + 1, d_\ell] \in \text{Pers}_s(F')\), whose representative is derived from \( \zeta_1[i:] \oplus \cdots \oplus \zeta_\ell[i:] \).

Then, we pair the remaining indices. Specifically, for \( r := 1, \ldots, \ell - 1 \), pair \( b_r \) with a death index as follows:

- If \( d_r \) is unpaired, then pair \( b_r \) with \( d_r \). The representative for \([b_r, d_r] \in \text{Pers}_s(F')\) can be updated from the representative for \([b_r, d_r] \in \text{Pers}_s(F)\) as described in Section 5.7.1.

- If \( d_r \) is paired, then \( d_1, \ldots, d_r \) must be all the paired death indices among \( d_1, \ldots, d_\ell \) so far. Since \( d_{r+1}, \ldots, d_\ell \) are all unpaired, we pair \( b_r \) with \( \delta = \max_{<_d} \{d_{r+1}, \ldots, d_\ell\} \). We then describe how we obtain the representative for \([b_r, \delta] \in \text{Pers}_s(F)\). For each \( j \) s.t. \( 1 \leq j \leq r \), we define the following representative \( \tilde{\zeta}_j \) for \([b_r, i] \in F'\): first take the representative sequence \( \zeta_j[i:] \) in \( F \) and treat it as a representative sequence for \([b_r, i - 1] \in F'\); then attach a cycle at index \( i \) to \( \zeta_j[i:] \) by copying the cycle at index \( i - 1 \), to derive \( \tilde{\zeta}_j \) (note that \( \zeta_j[i:] \) is treated as a representative in \( F' \) and hence the last index is \( i - 1 \)). Symmetrically, for each \( j \) s.t. \( r < j \leq \ell \), we define the representative \( \tilde{\zeta}_j \) for \([i, d_j] \in F'\), which is derived from \( \zeta_j[i:] \). With the above definitions, the representative for \([b_r, \delta] \in \text{Pers}_s(F')\) is the following:

\[
(\tilde{\zeta}_1 \oplus_b \cdots \oplus_b \tilde{\zeta}_r) \| (\tilde{\zeta}_{r+1} \oplus_d \cdots \oplus_d \tilde{\zeta}_\ell).
\]

The concatenation in the above representative is well-defined because \( \sum_{j=1}^\ell[z_j^1] = 0 \) in \( K'_i \), which means that \( \sum_{j=r}^\ell[z_j^1] = \sum_{j=r-1}^\ell[z_j^1] \).

Finally, all remaining intervals in \( \text{Pers}_s(F) \) are carried into \( \text{Pers}_s(F') \); the update of representatives for these intervals is the same as in Section 5.7.1.
5.7.3 Time complexity

The inward expansion operation takes \(O(mn^2)\) time. The analysis is similar to the analysis for outward contraction in Section 5.6.3 but is easier.

5.8 Inward contraction

Recall that an inward contraction is the following operation:

\[
F : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_{i-1} \overset{\sigma}{\rightarrow} K_i \leftrightarrow K_{i+1} \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
\]

\[
F' : K_0 \leftrightarrow \cdots \leftrightarrow K_{i-2} \leftrightarrow K_{i-1}^o \leftrightarrow K_{i+2} \leftrightarrow \cdots \leftrightarrow K_m
\]

where \(K_i = K_{i-1} = K_{i+1}\). We also assume that \(\sigma\) is a \(p\)-simplex. Note that indices for \(F'\) are not consecutive, i.e., \(i - 1\) and \(i + 1\) are skipped.

For the update, we first determine whether the induced map \(H_*(K_{i-1}) \rightarrow H_*(K_i)\) is injective or surjective by checking whether \(i\) is a birth index in \(F\) (injective) or \(i - 1\) is a death index in \(F\) (surjective).

5.8.1 \(H_*(K_{i-1}) \rightarrow H_*(K_i)\) is injective

Since inward contractions are inverses of inward expansions (see Section 5.7), the only difference of \(\text{Pers}_*(F)\) and \(\text{Pers}_*(F')\) in this case is that \([i, i] \in \text{Pers}_*(F)\) is deleted in \(\text{Pers}_*(F')\).

Let \([b, d] \neq [i, i]\) be an interval in \(\text{Pers}_*(F)\). If \(i \notin [b, d]\), i.e., \(b > i\) or \(d < i\), then since \(b \neq i + 1\) and \(d \neq i - 1\), we have that \(b \geq i + 2\) or \(d \leq i - 2\). So the representative for \([b, d] \in \text{Pers}_*(F)\) can be directly used as a representative for \([b, d] \in \text{Pers}_*(F')\).

If \(i \in [b, d]\), let \(\tilde{z}_i\) be the representative \(p\)-cycle at index \(i\) for \([i, i] \in \text{Pers}_*(F)\), and let

\[
\zeta : c_{b-1} \leftrightarrow z_b \leftrightarrow \cdots \leftrightarrow z_{d-1} \leftrightarrow c_d
\]

be the representative sequence for \([b, d] \in \text{Pers}_*(F)\). Note that \(\sigma \in \tilde{z}_i \subseteq K_i\). Since \(z_{i-1} = \partial(c_{i-1})\) and \(z_i + z_{i+1} = \partial(c_i)\) for \(c_{i-1}, c_i \subseteq K_i\), we have that \(\tilde{z}_{i-1} + \tilde{z}_{i+1} = \partial(c_{i-1} + c_i)\) for \(c_{i-1} + c_i \subseteq K_i\).

If \(\sigma \notin c_{i-1} + c_i\), then \(c_{i-1} + c_i \subseteq K_i^\prime\). If \(\sigma \in c_{i-1} + c_i\), we say that \(\zeta\) is \(\sigma\)-relevant. We have that

\[
\partial(c_{i-1} + c_i + \tilde{z}_i) = \partial(c_{i-1} + c_i) = z_{i-1} + z_{i+1},
\]

where \(c_{i-1} + c_i + \tilde{z}_i\) does not contain \(\sigma\) and hence is in \(K_i^\prime\). So we always have that \(z_{i-1} + z_{i+1} = \partial(\tilde{z})\) for a chain \(\tilde{z} \subseteq K_i^\prime\). Then, the representative for \([b, d] \in \text{Pers}_*(F')\) is set as:

\[
c_{b-1} \leftrightarrow z_b \leftrightarrow \cdots \leftrightarrow c_{i-3} \leftrightarrow z_{i-2} \leftrightarrow c_{i-1} \leftrightarrow \tilde{z}_i \leftrightarrow \cdots \leftrightarrow z_{i+2} \leftrightarrow c_{i+2} \leftrightarrow \cdots \leftrightarrow c_{d-1} \leftrightarrow z_d \leftrightarrow c_d,
\]

where \(z_i^\prime := z_{i-1}\).

5.8.2 \(H_*(K_{i-1}) \rightarrow H_*(K_i)\) is surjective

In this case, \(i - 1 \in \text{N}(F), i + 1 \in \text{P}(F), \text{N}(F') = \text{N}(F) \setminus \{i - 1\}\), and \(\text{P}(F') = \text{P}(F) \setminus \{i + 1\}\). Let \(\{[\beta_j, \delta_j] \mid j \in B\}\) be the set of intervals in \(\text{Pers}_*(F)\) containing \(i\), where \(B\) is an indexing set, and let \(\zeta_j\) be the representative sequence for each \([\beta_j, \delta_j]\). Moreover, define a set \(\Lambda \subseteq B\) as:

\[
\Lambda := \{j \in B \mid \tilde{\zeta}_j\text{ is }\sigma\text{-relevant}\}.
\]

We do the following:
• Whenever there exist \(j, k \in \Lambda\) s.t. \([\beta_j, \delta_j] \prec [\beta_k, \delta_k]\), update the representative for \([\beta_k, \delta_k]\) as \(\tilde{\zeta}_j \oplus \tilde{\zeta}_k\), and delete \(k\) from \(\Lambda\). Note that \(\tilde{\zeta}_j \oplus \tilde{\zeta}_k\) is \(\sigma\)-irrelevant.

After the above operations, we have that no two intervals in \(\{[\beta_j, \delta_j] | j \in \Lambda\}\) are comparable. Let \([b_s, i - 1]\) and \([i + 1, d_o]\) be the \((p - 1)\)-th intervals in \(\text{Pers}_s(F)\) ending/starting with \(i - 1\), \(i + 1\) respectively. Moreover, let \(\zeta_s\) be the representative sequence for \([b_s, i - 1]\), and let \(\zeta_o\) be the representative sequence for \([i + 1, d_o]\). We do the following:

• Whenever there is a \(j \in \Lambda\) s.t. \(b_s \prec_b \beta_j\), update the representative for \([\beta_j, \delta_j]\) as \(\zeta_s \oplus \tilde{\zeta_j}\), and delete \(j\) from \(\Lambda\). Note that \(i - 1 \prec_d \delta_j\) and \(\zeta_s \oplus \tilde{\zeta_j}\) is \(\sigma\)-irrelevant.

• Whenever there is a \(j \in \Lambda\) s.t. \(d_o \prec_d \delta_j\), update the representative for \([\beta_j, \delta_j]\) as \(\zeta_o \oplus \tilde{\zeta_j}\), and delete \(j\) from \(\Lambda\). Note that \(i + 1 \prec_b \beta_j\) and \(\zeta_o \oplus \tilde{\zeta_j}\) is \(\sigma\)-irrelevant.

After the above operations, we have that \(\beta_j \prec_b b_s\) and \(\delta_j \prec_d d_o\) for each \(j \in \Lambda\). If \(\Lambda = \emptyset\), then let \([b, d]\) form an interval in \(\text{Pers}_s(F')\) with a representative \(\zeta_s \parallel \zeta_o\). If \(\Lambda \neq \emptyset\), then rewrite the intervals in \(\{[\beta_j, \delta_j] | j \in \Lambda\}\) as:

\[
[b_1, d_1], [b_2, d_2], \ldots, [b_\ell, d_\ell] \text{ s.t. } b_\ell \prec_b b_1 \prec_b \cdots \prec_b b_\ell.
\]

Also, for each \(j\), let \(\zeta_j\) be the \(p\)-th representative sequence for \([b_j, d_j] \in \text{Pers}_s(F)\).

For \(j = 1, \ldots, \ell - 1\), we do the following:

• Note that \(d_{j+1} \prec_d d_j\) because otherwise \([b_j, d_j]\) and \([b_{j+1}, d_{j+1}]\) would be comparable. Then, let \([b_{j+1}, d_j]\) form an interval in \(\text{Pers}_s(F')\). The representative is set as follows: since \(\zeta_j \oplus \zeta_{j+1}\) is a representative for \([b_{j+1}, d_j]\) in \(F\) which is \(\sigma\)-irrelevant, \(\zeta_j \oplus \zeta_{j+1}\) can be ‘contracted’ to become a representative for \([b_{j+1}, d_j] \in \text{Pers}_s(F')\) as done in Section 5.8.1.

We then do the following:

• Let \([b_s, d_\ell]\) form an interval in \(\text{Pers}_s(F')\) whose representative is derived from \(\zeta_s \oplus \zeta_\ell\) (which is \(\sigma\)-irrelevant); let \([b_1, d_s]\) form an interval in \(\text{Pers}_s(F')\) whose representative is derived from \(\zeta_o \oplus \zeta_1\) (which is \(\sigma\)-irrelevant).

Finally, for each remaining interval \([b, d] \in \text{Pers}_s(F),\) whose representative is \(\sigma\)-irrelevant, \([b, d]\) forms an interval in \(\text{Pers}_s(F')\), whose representative is updated as in Section 5.8.1.

### 5.8.3 Time complexity

By a similar analysis as in Section 5.6.3, the total time spent on the injective case is \(O(mn)\). The bottleneck of the surjective case is the loops, which take \(O(mn^2)\) time. Hence, the inward contraction operation takes \(O(mn^2)\) time.

### 6 Conclusion

We have presented update algorithms to maintain representatives for the barcodes over a changing zigzag filtration. Two main questions ensue from this research: (i) Can we make the updates more efficient? (ii) Are there interesting applications of these algorithms other than the ones mentioned?
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A Details on dynamic point clouds

We first define the following:

Definition 15. Throughout the section, let $D = (P, D_0, D_1, \ldots, D_s)$ denote a dynamic point cloud in which: (i) $P$ is a set of points; (ii) each map $D_i : P \to \mathbb{R}^p$ specifies the positions of points in $P$ at time $i$.

Note that while 1-dimensional persistence [9] with Rips filtration serves as an effective descriptor for a fixed point cloud, it cannot naturally characterize a dynamic point cloud as defined above [11]. In view of this, we build vines and vineyards [5] as descriptors for $D$ using zigzag persistence. We first let the time in $D$ range continuously in $[0, s]$, i.e., the position of each point in $P$ during time $[0, s]$ is linearly interpolated based on the discrete samples given in $D$. For each $t \in [0, s]$, let $P_t$ denote the point cloud which is the point set $P$ with positions at time $t$. Also, for a $\delta \geq 0$, let $R_t^\delta$ denote the Rips complex of $P_t$ with distance $\delta$.

Now fix a $\delta \geq 0$, and consider the continuous sequence $\mathcal{R}^\delta := \{R_t^\delta\}_{t \in [0, s]}$. We claim that $\mathcal{R}^\delta$ is encoded by a zigzag filtration, and hence admits a barcode (persistence diagram) as descriptor. To see this, we note that each $R_t^\delta$ in $\mathcal{R}^\delta$ is completely determined by the vertex pairs in $P$ with distances no greater than $\delta$ at time $t$. Let $\pi$ be a vertex pair whose distance varies with time as illustrated in the red curve in Figure 3a, where the horizontal axis denotes time and the vertical axis denotes distance. For the $\delta$ in Figure 3a, the edge formed by $\pi$ is in $R_t^\delta$ when $t$ falls in the intervals $[0, t_1]$, $[t_2, t_3]$, and $[t_4, t_5]$. Also, in Figure 3b, for three vertex pairs $\pi_1$, $\pi_2$, $\pi_3$, we illustrate respectively the time intervals in which their distances are no greater than $\delta$. With the time varying, the edges formed by the vertex pairs are added to or deleted from the Rips complex. As illustrated in Figure 3b, this naturally defines a zigzag filtration which we denote as $\mathcal{F}^\delta$. For example, $R_{t_2}^\delta$ in Figure 3b is defined by edges formed by $\pi_2$ and $\pi_3$, and $R_{t_5}^\delta$ is defined by edges formed by $\pi_1$ and $\pi_3$.

We then consider the one-parameter family of persistence diagrams $\{B^\delta\}_{\delta \in [0, \infty]}$, with $B^\delta$ being the persistence diagram of $\mathcal{R}^\delta$, which forms a vineyard [5]. Treating each $B^\delta$ as a multi-set of points in $\mathbb{R}^2$, the vineyard $\{B^\delta\}_{\delta \in [0, \infty]}$ contains vines tracking the movement of points in persistence diagrams w.r.t. $\delta$. For computing the vineyard $\{B^\delta\}_{\delta \in [0, \infty]}$, we utilize the update operations and algorithms presented in this paper. As in [5], our atomic update operations help associate points for persistence diagrams in $\{B^\delta\}_{\delta \in [0, \infty]}$ without ambiguity, which is otherwise unavoidable if attempting to associate directly. Let $\overline{\delta}$ be the maximum distance of vertex pairs at all time in $D$. We start with $\mathcal{R}^\delta$. Since $R_t^\delta$ equals a contractible (high-dimensional) simplex at any $t$, $B^\delta$ contains only a 0-th
I. Increasing crossing
II. Decreasing crossing
III. Opposite crossing
IV. Local minimum
V. Local maximum

Figure 4: The events that change the zigzag filtration of $\mathcal{R}^\delta$ as $\delta$ varies. Each (partial) distance-time curve corresponds to a vertex pair, and for some events, edges formed by the vertex pairs are also denoted.

interval $[0, s]$ whose representative sequence is straightforward*. Now consider the distance-time curves of all vertex pairs of $P$ (e.g., Figure 3a illustrates curves of two pairs), which indeed defines a **dynamic metric space** [11]. When decreasing the distance $\delta$, $\mathcal{F}^\delta$ changes only at the following types of points in the plot of all distance-time curves (see Figure 4):

I. **Increasing crossing**: In Figure 4, $e_1$ is deleted first at $t_3$ and then $e_2$ is deleted at $t_4$ in $\mathcal{R}^\delta$. In $\mathcal{R}^{\delta_2}$, the deletions of $e_1, e_2$ are switched. The switch of edge deletions in the zigzag filtrations is realized by a sequence of simplex-wise **backward switches**.

II. **Decreasing crossing**: This is symmetric to the increasing crossing where additions of two edges are switched. It is realized by a sequence of simplex-wise **forward switches**.

III. **Opposite crossing**: In Figure 4, $e_1$ is added first at $t_2$ and then $e_2$ is deleted at $t_3$ in $\mathcal{R}^\delta$. In $\mathcal{R}^{\delta_2}$, the addition of $e_1$ and the deletion of $e_2$ are switched. The simplex-wise version of $\mathcal{F}^{\delta_1}$ contains the following part

$$R_{t_1}^{\delta_1} \leftarrow \sigma_1 \rightarrow \cdots \rightarrow \sigma_4 \rightarrow R_{t_3}^{\delta_1} \leftarrow \tau_1 \rightarrow \cdots \rightarrow \tau_r \rightarrow R_{t_5}^{\delta_1},$$

where $t_1, t_3, t_5$ are as defined in Figure 4. To obtain $\mathcal{F}^{\delta_2}$, we do the following for each $i = 1, \ldots, r$:

- If $\tau_i$ is not equal to any of $\sigma_1, \ldots, \sigma_q$, then use **outward switches** to make $\sigma_1 \sigma_2 \cdots \sigma_q$ appear immediately before the additions of $\sigma_1, \ldots, \sigma_q$. If $\tau_i$ is equal to a $\sigma_j$, first use outward switches to make $\sigma_j \tau_i$ appear immediately after $\sigma_j \tau_i$. Then, apply the **inward contraction** on $\sigma_j \tau_i$. Note that $\sigma_j (= \tau_i)$ which contains both $e_1, e_2$ does not exist in any complex $R_t^{\delta_2}$ for $t$ a time shown in Figure 4 because $e_1, e_2$ do not both exist in these complexes.

IV. **Local minimum**: In Figure 4, an edge $e$ corresponding to the black curve is added at $t_1$ and then deleted at $t_2$ in $\mathcal{R}^\delta$. In $\mathcal{R}^{\delta_2}$, the addition and deletion of $e$ disappear. Correspondingly, simplices containing $e$ are added and then deleted in $\mathcal{F}^\delta$, but in $\mathcal{F}^{\delta_2}$, the addition and deletion of the above mentioned simplices do not exist. Hence, we need to perform **inward contractions**. Note that before this, we may need to perform forward or backward switches to properly order the additions and deletions. (For example, suppose that $\sigma$ is the last simplex added due to the addition of $e$. However, if $\sigma$ is not the first simplex deleted due to the deletion of $e$, we need to perform backward switches to make this true so that we can perform an inward contraction on $\sigma$.)

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*In practice, one may only consider simplices up to a dimension to save time; $\mathcal{F}^\delta$ and the representatives in this case can then be computed from a homology basis for the complex at a time $t$. 

27
V. Local maximum: In Figure 4, an edge $e$ corresponding to the black curve exists in any complex $R_{t_{1}}$ for $t$ a time shown in the figure. However, in $R_{t_{2}}$, $e$ is deleted at $t_{1}$ and then added at $t_{2}$. Accordingly, we need to perform outward expansions on simplices which are deleted and then added.

All the above five types of points appear in Figure 3a with the numbering of types labelled.