Semi-discrete finite element approximation applied to Maxwell’s equations in nonlinear media

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January 9, 2019

In this paper the semi-discrete finite element approximation of initial boundary value problems for Maxwell’s equations in nonlinear media of Kerr-type is investigated. For the case of Nédélec elements from the first family, a priori error estimates are established for the approximation.

Keywords: Semi-discrete finite element method, nonlinear Maxwell’s equations, error estimate
MSC 2010: 35Q61, 65M60, 65M15

1 Introduction

In this paper we investigate the semi-discrete conforming finite element approximation to the solution of Maxwell’s equations for nonlinear media of Kerr-type. As a concrete example, we consider the (meanwhile classical) Nédélec elements from the so-called first family. To the best knowledge of the author, the nonlinear situation is not yet well investigated, most works dealing with nonlinear effects are computational or experimental (see, e.g., [AY18]). Here we derive energy (stability) estimates for the weakly formulated problem and error estimates for the semi-discretized problem.

Let \( Q_T := (0, T) \times \Omega \), where \( \Omega \subset \mathbb{R}^3 \) is a simply connected domain with a sufficiently smooth boundary \( \partial \Omega \) and \( T > 0 \) is the length of the time interval under consideration. Let \( D, B, E, H : Q_T \to \mathbb{R}^3 \) represent the displacement field, the magnetic induction, the electric and magnetic field intensities, respectively. The time-dependent Maxwell’s equations in a nonlinear medium can be written in the form

\[
\begin{align*}
\partial_t D - \nabla \times H &= 0 \quad \text{in } Q_T, \\
\partial_t B + \nabla \times E &= 0 \quad \text{in } Q_T,
\end{align*}
\]

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where the following constitutive relations hold:

\[ B := \mu_0 H, \quad D := \varepsilon_0 E + P(E). \] (3)

Here \( \varepsilon_0 > 0 \) and \( \mu_0 > 0 \) are the vacuum permittivity and the permeability, respectively. Often the constitutive relation for the polarization \( P = P(E) \) is approximated by a truncated Taylor series [Boy03]. In the case of an isotropic material, it takes the form

\[ P(E) := \varepsilon_0 \left( \chi^{(1)} E + \chi^{(3)} |E|^2 E \right), \]

where \( \chi^{(j)} : \Omega \to \mathbb{R} \) are the media susceptibility coefficients, \( j = 1, 3 \). Then from (3) we obtain the representation

\[ D = \varepsilon_0 \varepsilon_s E \quad \text{with} \quad \varepsilon_s = \varepsilon_s(E) := 1 + \chi^{(1)} + \chi^{(3)} |E|^2, \]

and it follows by a simple calculation that

\[ \partial_t D = 2\varepsilon_0 \chi^{(3)} (E \cdot \partial_t E) E + \varepsilon_0 \varepsilon_s \partial_t E = \varepsilon_0 (\varepsilon_s I + \varepsilon_m) \partial_t E \quad \text{with} \quad \varepsilon_m = \varepsilon_m(E) := 2\chi^{(3)} E E^\top, \]

where \( I \) denotes the identity in \( \mathbb{R}^d \). Setting

\[ \varepsilon(E) := \varepsilon_0 (\varepsilon_s(E) I + \varepsilon_m(E)), \]

the system (1)–(3) can be written as

\[ \varepsilon(E) \partial_t E - \nabla \times H = 0 \quad \text{in} \quad Q_T, \] (4)
\[ \mu_0 \partial_t H + \nabla \times E = 0 \quad \text{in} \quad Q_T. \] (5)

Next we state a simple result which in particular implies that the matrix \( \varepsilon(E) \) is regular for all electric field intensities \( E \) under consideration.

**Lemma 1.1.** Let \( \chi^{(1)}, \chi^{(3)} \geq 0 \) a.e. in \( \Omega \). Then the symmetric matrix \( \varepsilon(\Psi) \) is uniformly positive definite a.e. for any \( \Psi \in \mathbb{R}^3 \).

**Proof.** For all \( \Phi \in \mathbb{R}^3 \), it holds that

\[ \varepsilon_0^{-1} \Phi^\top \varepsilon(\Psi) \Phi = \left( 1 + \chi^{(1)} + \chi^{(3)} |\Psi|^2 \right) |\Phi|^2 + 2\chi^{(3)} |\Psi \cdot \Phi|^2 \geq |\Phi|^2. \]

As in [PNTB09], we denote the inverse by

\[ C(E) := (\varepsilon(E))^{-1}. \]

By means of the Sherman-Morrison formula (see, e.g., [GvL96]), the matrix \( C(E) \) can be given explicitly:

\[ C(E) = \frac{1}{\varepsilon_0} C_m(E) \] (6)
with
\[C_m(E) := (\varepsilon_s(E)I + \varepsilon_m(E))^{-1} = \frac{1}{\varepsilon_s(E)} \left( I - \frac{1}{\varepsilon_s(E) + 2\chi^{(3)}|E|^2} \varepsilon_m(E) \right)\]

\[= \frac{1}{\varepsilon_s(E)} \left( I - \frac{1}{1 + \chi^{(1)} + 3\chi^{(3)}|E|^2} \varepsilon_m(E) \right).\]

Therefore, if the formula (6) holds, the system (4)–(5) takes the form
\[\varepsilon_0 \partial_t E - C_m(E) \nabla \times H = 0 \text{ in } Q_T,\]
\[\mu_0 \partial_t H + \nabla \times E = 0 \text{ in } Q_T.\]

This is an appropriate formulation for the development of time-discrete numerical algorithms, see, e.g., [PNTB09].

A perfect conducting boundary condition on \(\Omega\) is assumed so that
\[\nu \times E = 0 \text{ on } (0, T) \times \partial \Omega,\]
where \(\nu\) denotes the outward unit normal on \(\partial \Omega\). In addition, initial conditions have to be specified so that
\[E(0, x) = E_0(x) \text{ and } H(0, x) = H_0(x) \text{ for all } x \in \Omega,\]
where \(E_0 : \Omega \to \mathbb{R}^3\) and \(H_0 : \Omega \to \mathbb{R}^3\) are given functions, and \(H_0\) satisfies
\[\nabla \cdot (\mu_0 H_0) = 0 \text{ in } \Omega, \quad H_0 \cdot \nu = 0 \text{ on } \partial \Omega.\]

The divergence-free condition in (11) together with (5) implies that
\[\nabla \cdot (\mu_0 H) = 0 \text{ in } Q_T,\]

2 Notation

For a real number \(p \in [1, \infty]\), the symbol \(L^p(\Omega)\) denotes the usual Lebesgue spaces equipped with the norm \(\| \cdot \|_{L^p(\Omega)}\). The analogous spaces of vector fields \(u : \Omega \to \mathbb{R}^3\) are denoted by \(L^p(\Omega) := [L^p(\Omega)]^3\) with the norm \(\| \cdot \|_{L^p(\Omega)}\).

In what follows we have to deal with weighted function spaces. Given a weight \(\omega : \Omega \to \mathbb{R}\), where the values of \(\omega\) are positive a.e. on \(\Omega\), we define a weighted inner product and a weighted norm by
\[(u, v)_\omega := \int_\Omega \omega \cdot u \cdot v \, dx \quad \text{and} \quad \|u\|_\omega := \|u\|_{L^2(\Omega)} := \sqrt{(u, u)_\omega}.\]

The space \(L^2_\omega(\Omega)\) consists of vector fields \(u : \Omega \to \mathbb{R}^3\) with Lebesgue-measurable components and such that \(\|u\|_\omega < \infty\).

In the case \(\omega = 1\), the subscript is omitted. An elementary property of weighted spaces, which we will apply at different places without special emphasis, is the monotonicity w.r.t. the weight: If \(\omega_1, \omega_2\) are two weights such that \(\omega_1 \leq \omega_2\) a.e. on \(\Omega\), then
\[\|u\|_{L^2_{\omega_1}(\Omega)} \leq \|u\|_{L^2_{\omega_2}(\Omega)} \quad \text{for all } u \in L^2_{\omega_1}(\Omega) \cap L^2_{\omega_2}(\Omega).\]
As transient problems are addressed, we will work with functions that depend on time and have values in certain Banach spaces. If \( u = u(t,x) \) is a vector field of the space variable \( x \) and the time variable \( t \), it is suitable to separate these variables in such a way that \( u(t) = u(t,\cdot) \) is considered as a function of \( t \) with values in a Banach space, say \( X \). That is, for any \( t \in (0,T) \), the mapping \( x \mapsto u(t,x) \) is interpreted as a parameter-dependent element \( u(t) \) of \( X \). In this sense we will write \( E(t) = E(t,\cdot) \), \( H(t) = H(t,\cdot) \) and so on.

The space \( C^m(0,T;X) \), \( m \in \mathbb{N} \cup \{0\} \), consists of all continuous functions \( u: (0,T) \to X \) that have continuous derivatives up to order \( m \) on \( (0,T) \). It is equipped with the norm \( \| u \|_{C^m(0,T;X)} := \sum_{j=0}^{m} \sup_{t \in (0,T)} \| u^{(j)}(t) \|_X \).

For the sake of consistency in the notation we will write \( C(0,T;X) := C^0(0,T;X) \).

The space \( L^p(0,T;X) \) with \( p \in [1,\infty) \) contains (equivalent classes of) strongly measurable functions \( u: (0,T) \to X \) such that \( \int_0^T \| u(t) \|_X^p dt < \infty \) (for the definition of strongly measurable functions we refer to \([KJF77]\)). The norm on \( L^p(0,T;X) \) is defined by

\[
\| u \|_{L^p(0,T;X)} := \left( \int_0^T \| u(t) \|_X^p dt \right)^{1/p}.
\]

These spaces can be equipped with a weight, too. In particular, we will write

\[
\| u \|_{L^p_d(0,T;L^2_\omega(\Omega))} := \left( \int_0^T \int_\Omega |u(t)|^2 \omega dx dt \right)^{1/2}.
\]

Finally, all the above definitions can be extended to the standard Sobolov spaces of functions with weak spatial derivatives of maximal order \( r \in \mathbb{N} \) in \( L^p(\Omega) \): \( W^{r,p}(\Omega) \) with norm \( \| \cdot \|_{W^{r,p}(\Omega)} \).

If \( p = 2 \), we write \( H^r(\Omega) := W^{r,2}(\Omega) \) and \( \| \cdot \|_{H^r(\Omega)} := \| \cdot \|_{W^{r,2}(\Omega)} \).

The space \( H^1_0(\Omega) \) is defined as the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \| \cdot \|_{H^1(\Omega)} \), where \( C_0^\infty(\Omega) \) denotes the space of all arbitrarily often differentiable functions with compact support on \( \Omega \). It is well known that \( H^1_0(\Omega) \) is a closed subspace of \( H^1(\Omega) \) and consists of elements \( u \) such that \( u = 0 \) on \( \partial \Omega \) in the sense of traces \([AF03]\). As in the case of the \( L^p \)-spaces, we shall write \( W^{r,p}(\Omega) := [W^{r,p}(\Omega)]^3 \) and so on.

Furthermore, we need the following Hilbert spaces that are related to the (weak) rotation and divergence operators:

\[
H(\text{curl},\Omega) := \{ u \in L^2(\Omega) : \nabla \times u \in L^2(\Omega) \},
\]

\[
H_0(\text{curl},\Omega) := \{ u \in H(\text{curl},\Omega) : u \times \nu|_{\partial \Omega} = 0 \},
\]

\[
H(\text{div},\Omega) := \{ u \in L^2(\Omega) : \nabla \cdot u \in L^2(\Omega) \}.
\]
These spaces are equipped with the norms (resp. induced norms)
\[ \|u\|_{H_{(\text{curl},\Omega)}} := \left\{ \|\nabla \times u\|_0^2 + \|\nabla \cdot u\|_0^2 \right\}^{1/2}, \]
\[ \|u\|_{H_{(\text{div},\Omega)}} := \left\{ \|\nabla \cdot u\|_0^2 + \|\nabla \times u\|_0^2 \right\}^{1/2}. \]

We refer to [DL76], [RT77], [GR86] and [Cia02] for details about these spaces.

### 3 Weak formulations

We assume that a solution
\[(E, H) \in (C^1(0, T; L_{2,\varepsilon}(E)(\Omega)) \cap C(0, T; H_0(\text{curl}, \Omega))) \times (C^1(0, T; L_2(\Omega)) \cap C(0, T; H(\text{curl}, \Omega))) \]
of the nonlinear Maxwell’s equations (4)–(5) exists and is unique.

We multiply equation (4) by a test function \( \Psi \in L_2(\Omega) \) and integrate over \( \Omega \). Similarly we multiply (5) by a test function \( \Phi \in H(\text{curl}, \Omega) \), integrate the result over \( \Omega \) and integrate by parts the second term. This shows that it is natural to look for a weak solution \( (E, \Psi) \) such that
\[
(\varepsilon(E)\partial_t E, \Psi) - (\nabla \times H, \Psi) = 0 \quad \forall \Psi \in L_2(\Omega),
\]
\[
(\mu_0 \partial_t H, \Phi) + (E, \nabla \times \Phi) = 0 \quad \forall \Phi \in H(\text{curl}, \Omega).
\]

Alternatively, the use of test functions \( \Psi \in H_0(\text{curl}, \Omega) \) and \( \Phi \in L_2(\Omega) \) and the integration by parts in the equation (4) leads to the notion of a weak solution \((E, H) \in (C^1(0, T; L_{2,\varepsilon}(E)(\Omega)) \cap C(0, T; H_0(\text{curl}, \Omega))) \times (C^1(0, T; L_2(\Omega)) \cap C(0, T; H(\text{curl}, \Omega))) \) of (4)–(5) such that
\[
(\varepsilon(E)\partial_t E, \Psi) - (H, \nabla \times \Psi) = 0 \quad \forall \Psi \in H_0(\text{curl}, \Omega),
\]
\[
(\mu_0 \partial_t H, \Phi) + (\nabla \times E, \Phi) = 0 \quad \forall \Phi \in L_2(\Omega).
\]

In both cases, the initial conditions (10) have to be satisfied at least in the sense of \( C(0, T; L_2(\Omega)) \).

**Remark 1.** As a consequence of the embedding (as sets)
\[ [C_0^\infty(\Omega)]^3 \subset [C^\infty(\Omega) \cap H^1(\Omega)]^3 \subset [H^1(\Omega)]^3 \subset H(\text{div}, \Omega) \subset L_2(\Omega) \]
and of the fact that \( C_0^\infty(\Omega) \) is dense in \( L^2(\Omega) \) we see that \( H(\text{div}, \Omega) \) is a dense subset of \( L_2(\Omega) \) [AF03]. Therefore the test space in (17) can be reduced to \( H(\text{div}, \Omega) \). Also note that \( H \in C(0, T; H(\text{div}, \Omega)) \) due to (12).

**Remark 2.** In the case where \( \mu_0 \) is not a constant but a highly variable function \( \mu = \mu(x) \) it is more convenient to use the magnetic flux density \( B = \mu H \) instead of \( H \) as a dependent variable [MM95]. In such a case, the formulation (16)–(17) is replaced by
\[
(\varepsilon(E)\partial_t E, \Psi) - (\mu^{-1}B, \nabla \times \Psi) = 0 \quad \forall \Psi \in H_0(\text{curl}, \Omega),
\]
\[
(\mu^{-1} \partial_t B, \Phi) + (\nabla \times E, \mu^{-1} \Phi) = 0 \quad \forall \Phi \in H(\text{div}, \Omega),
\]
where
\[(E, B) \in (C^1(0, T; L_{2,\varepsilon}(E)(\Omega)) \cap C(0, T; H_0(\text{curl}, \Omega))) \times (C^1(0, T; L_{2,\varepsilon}(B)(\Omega)) \cap C(0, T; H(\text{div}, \Omega))).\]
Next we will formulate a stability result for the problem (14)–(15). For this, we extend this problem to the case of a nontrivial right-hand side for a moment.

**Theorem 3.1.** Let $J_e \in L_2(0, T; L_{2,\varepsilon_0^{-1}(1+\chi^{(1)}-1)}(\Omega))$, $J_m \in L_2(0, T; L_{2,\mu_0^{-1}}(\Omega))$ – the electric and magnetic current densities, respectively – be given and assume that the system

\begin{align}
(\varepsilon(E)\partial_t E, \Psi) - (\nabla \times H, \Psi) &= -(J_e, \Psi) \quad \forall \Psi \in L_2(\Omega), \quad (20) \\
(\mu_0 \partial_t H, \Phi) + (E, \nabla \times \Phi) &= -(J_m, \Phi) \quad \forall \Phi \in H(\text{curl}, \Omega) \quad (21)
\end{align}

together with the initial conditions (10) has a weak solution

$$(E, H) \in \left(C^1(0, T; L_{2,\varepsilon_0}(\Omega)) \cap C(0, T; H_0(\text{curl}, \Omega)) \right) \times \left(C^1(0, T; L_2(\Omega)) \right).$$

If

$$W(t) := \frac{1}{2} \left[ \|E(t)\|_{\varepsilon_0(1+\chi^{(1)})}^2 + \frac{3}{2} \|E(t)\|_{L_{\varepsilon_0(3)}}^2 + \|H(t)\|_{\mu_0}^2 \right]$$

denotes the nonlinear electromagnetic energy at the time $t$, the following energy law in differential form holds:

$$\frac{dW}{dt} = -(J_e, E) - (J_m, H), \quad t \in (0, T). \quad (22)$$

**Proof.** Taking $\Psi = E$ and $\Phi = H$ in (20)–(21) and adding the result gives

$$(\varepsilon(E)\partial_t E, E) + (\mu_0 \partial_t H, H) = -(J_e, E) - (J_m, H) \quad (23)$$

An elementary calculation shows that

$$\varepsilon(E)\partial_t E = \varepsilon_0 \left( (1 + \chi^{(1)}) + \chi^{(3)}|E|^2 \right) I + 2\chi^{(3)}EE^\top \partial_t E$$

$$= \varepsilon_0 (1 + \chi^{(1)}) + \chi^{(3)}|E|^2 \partial_t E + 2\varepsilon_0 \chi^{(3)} E(E \cdot \partial_t E)$$

$$= \varepsilon_0 (1 + \chi^{(1)}) + \chi^{(3)}|E|^2 \partial_t E + \varepsilon_0 \chi^{(3)} \partial_t |E|^2,$$

hence

$$E \cdot (\varepsilon(E)\partial_t E) = \varepsilon_0 (1 + \chi^{(1)}) + \chi^{(3)}|E|^2 E \cdot \partial_t E + \varepsilon_0 \chi^{(3)} E \cdot \partial_t |E|^2$$

$$= \frac{\varepsilon_0}{2} (1 + \chi^{(1)}) + \chi^{(3)}|E|^2 \partial_t |E|^2 + \frac{3\varepsilon_0}{2} \chi^{(3)} |E|^2 \partial_t |E|^2$$

$$= \frac{\varepsilon_0}{2} (1 + \chi^{(1)}) \partial_t |E|^2 + \frac{3\varepsilon_0}{4} \chi^{(3)} \partial_t |E|^4.$$
Proof. We estimate the right-hand side of (22) by means of the Cauchy-Bunyakovsky-Schwarz’ inequality (twice – in the integral version and in the finite sum version):

\[ ||(J_e, E) + (J_m, H)|| \leq ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} ||E||_{\varepsilon_0(1+\chi(1))} + ||J_m||_{\mu_0^{-1}} ||H||_{\mu_0} \]

\[ \leq \left\{ ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} + ||J_m||_{\mu_0} \right\}^{1/2} \left\{ ||E||_{\varepsilon_0(1+\chi(1))} + ||H||_{2,\mu_0} \right\}^{1/2} \]

\[ \leq \sqrt{2} \left\{ ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} + ||J_m||_{\mu_0} \right\}^{1/2} \sqrt{W(t)}. \]

Since \( \frac{d}{dt}\sqrt{W} = \frac{1}{2\sqrt{W}} \frac{dW}{dt} \) for \( W(t) > 0 \) (the case \( W(t) = 0 \) is trivial), it follows from (22) that

\[ \sqrt{2} \frac{d}{dt}\sqrt{W(t)} \leq \left\{ ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} + ||J_m||_{\mu_0} \right\}^{1/2}. \]

Integrating this inequality w.r.t. \( t \), we get

\[ \sqrt{2}\sqrt{W(t)} \leq \sqrt{2}\sqrt{W(0)} + \int_0^t \left\{ ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} + ||J_m||_{\mu_0} \right\}^{1/2} ds. \]

Then

\[ W(t) \leq 2W(0) + \left( \int_0^t \left\{ ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} + ||J_m||_{\mu_0} \right\}^{1/2} ds \right)^2 \]

\[ \leq 2W(0) + t \int_0^t \left[ ||J_e||_{\varepsilon_0^{-1}(1+\chi(1))^{-1}} + ||J_m||_{\mu_0} \right] ds \]

\[ \leq 2W(0) + t \left[ ||J_e||_{L_2(0,T;L_2,\varepsilon_0^{-1}(1+\chi(1))^{-1}(\Omega))} + ||J_m||_{L_2(0,T;L_2,\mu_0^{-1}(\Omega))} \right]. \]

\[ \square \]

Remark 3. Analogous results as in Thm. 3.1 and Cor. 3.2 can be obtained for the corresponding “non-homogeneous” version of (16)–(17) and for the subsequent semi-discretizations.

4 Spatial discretization

4.1 Semi-discretization of the weak formulations

Let \( W_h \subset L_2(\Omega) \), \( U_h \subset H(curl, \Omega) \), \( U_{0h} \subset H_0(curl, \Omega) \), and \( V_h \subset H(div, \Omega) \) be finite-dimensional subspaces.

The semi-discrete (in space) problem for the system (14)–(15) consists in determining elements \((E_h, H_h) \in C^1(0,T;W_h) \times C^1(0,T;U_h)\) such that

\[ (\varepsilon(E_h) \partial_t E_h, \Psi_h) - (\nabla \times H_h, \Psi_h) = 0 \quad \forall \Psi_h \in W_h, \quad (24) \]

\[ (\mu_0 \partial_t H_h, \Phi_h) + (E_h, \nabla \times \Phi_h) = 0 \quad \forall \Phi_h \in U_h. \quad (25) \]

For the the equations (16)–(17), the semi-discrete problem involves the determination of elements \((E_h, H_h) \in C^1(0,T;U_{0h}) \times C^1(0,T;V_h)\) such that

\[ (\varepsilon(E_h) \partial_t E_h, \Psi_h) - (H_h, \nabla \times \Psi_h) = 0 \quad \forall \Psi_h \in U_{0h}, \quad (26) \]
\begin{equation}
(\mu_0 \partial_t \mathbf{H}_h, \Phi_h) + (\nabla \times \mathbf{E}_h, \Phi_h) = 0 \quad \forall \Phi_h \in \mathbf{V}_h.
\end{equation}

The initial conditions for both problems read formally as

\[ \mathbf{E}_h(0, \mathbf{x}) = \mathbf{E}_{0h}(\mathbf{x}) \quad \text{and} \quad \mathbf{H}_h(0, \mathbf{x}) = \mathbf{H}_{0h}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \]

where the concrete requirements to the particular choice of the discrete initial data \((\mathbf{E}_{0h}, \mathbf{H}_{0h})\) will be seen later (Thm. 5.1).

### 4.2 The choice of the finite element spaces

In the rest of the paper we will choose the so-called first family of Nédélec edge elements, usually denoted by \( \mathbf{D}_k \) and \( \mathbf{R}_k \) for \( k \in \mathbb{N} \), for the construction of the concrete finite element spaces (for details see [Néd80] or [Mon03, Ch. 5]). That is, given an arbitrary member \( \mathcal{T}_h \) of a family of triangulations of \( \Omega \) consisting of open tetrahedra \( K \), we set

\[
\mathbf{U}_h := \{ \mathbf{w} \in \mathbf{H}(\text{curl}; \Omega) : \mathbf{w}|_K \in \mathbf{R}_k \ \forall K \in \mathcal{T}_h \},
\]

\[
\mathbf{V}_h := \{ \mathbf{w} \in \mathbf{H}(\text{div}; \Omega) : \mathbf{w}|_K \in \mathbf{D}_k \ \forall K \in \mathcal{T}_h \},
\]

\[
\mathbf{W}_h := \{ \mathbf{w} \in \mathbf{L}_2(\Omega) : \mathbf{w}|_K \in \mathbf{P}_{k-1} \ \forall K \in \mathcal{T}_h \},
\]

where \( \mathbf{P}_k := [\mathbf{P}_\ell]^3 \) and \( \mathbf{P}_\ell \) is the space of scalar real-valued polynomials in three variables of maximal degree \( \ell \in \mathbb{N} \cup \{0\} \). To deal with the case of the Nédélec formulation (16)–(17), we still have to introduce the space \( \mathbf{U}_{0h} := \mathbf{U}_h \cap \mathbf{H}_0(\text{curl}, \Omega) \). In the subsequent error analysis, we will make use of some projection operators. For the Lee-Madsen formulation (24)–(25), we need projections \( \mathbf{P}_h : \mathbf{L}_2(\Omega) \rightarrow \mathbf{W}_h, \Pi_h : \mathbf{H}(\text{curl}, \Omega) \rightarrow \mathbf{U}_h \).

Let \( \mathbf{P}_h \) be the standard \( \mathbf{L}_2(\Omega) \)-projection operator onto \( \mathbf{W}_h \), i.e. for given \( \mathbf{w} \in \mathbf{L}_2(\Omega) \) the image \( \mathbf{P}_h \mathbf{w} \in \mathbf{W}_h \) is defined by

\begin{equation}
(\mathbf{P}_h \mathbf{w}, \mathbf{p}) = (\mathbf{w}, \mathbf{p}), \quad \forall \mathbf{p} \in \mathbf{W}_h.
\end{equation}

For this operator the following standard error estimate holds: If \( \mathbf{w} \in \mathbf{H}^k(\Omega) \), then

\begin{equation}
\| \mathbf{w} - \mathbf{P}_h \mathbf{w} \| \leq C h^k \| \mathbf{w} \|_{\mathbf{H}^k(\Omega)}.
\end{equation}

Moreover, since \( \nabla \times \mathbf{U}_h \subset \mathbf{W}_h \) (see the beginnings of the proofs of [Mon91, Thm. 3.3] or [Mon03, Lemma 5.40]), it holds that

\begin{equation}
(\mathbf{P}_h \mathbf{w}, \nabla \times \mathbf{P}_h \Phi) = (\mathbf{w}, \nabla \times \Phi), \quad \forall \mathbf{p} \in \mathbf{U}_h.
\end{equation}

Next, for \( \mathbf{v} \in \mathbf{H}(\text{curl}, \Omega) \) we define \( \mathbf{P}_h \mathbf{v} \in \mathbf{U}_h \) by

\begin{equation}
(\nabla \times \mathbf{P}_h \mathbf{v}, \nabla \times \mathbf{P}_h \Phi) = (\nabla \times \mathbf{v}, \nabla \times \Phi), \quad \forall \mathbf{p} \in \mathbf{U}_h,
\end{equation}

\begin{equation}
(\mathbf{P}_h \mathbf{v}, \nabla \mathbf{p}_h) = (\mathbf{v}, \nabla \mathbf{p}_h), \quad \forall \mathbf{p}_h \in \mathbf{S}_h^k.
\end{equation}

where \( \mathbf{S}_h^k \) is defined as

\[
\mathbf{S}_h^k := \{ v \in H_1(\Omega)/\mathbb{R} : v|_K \in \mathbf{P}_k \ \forall K \in \mathcal{T}_h \},
\]

see [Mon91, Subsect. 4.2].

If \( \mathbf{v} \in \mathbf{H}^{k+1}(\Omega) \) such that \( \nabla \cdot \mathbf{v} = 0 \) in \( \Omega \) and \( \nu \cdot \mathbf{v} = 0 \) on \( \partial \Omega \), then there exists a constant \( C > 0 \) such that

\begin{equation}
\| \mathbf{v} - \mathbf{P}_h \mathbf{v} \| \leq C h^k \| \mathbf{v} \|_{\mathbf{H}^{k+1}(\Omega)}
\end{equation}

(see [Mon91, Thm. 4.6]).
5 An error estimate for the semi-discrete problem

In this section we formulate and prove the main result.

**Theorem 5.1** (Semi-discrete error estimate for the Lee-Madsen formulation). Let $k \in \mathbb{N}$, $\chi^{(1)}, \chi^{(3)} \in L_\infty(\Omega)$, $E_0 \in L_\infty(\Omega)$, $H_0 \in L_{2,\nu_0}(\Omega)$ satisfying (11),

$$(E, H) \in (C(0, T; L_\infty(\Omega)) \cap H_0(\text{curl}, \Omega)) \cap E \in C^1(0, T; H^k(\Omega))) \times C^1(0, T; H^{k+1}(\Omega))$$

be the weak solution of the system (14)–(15), and

$$(E_h, H_h) \in C(0, T; W_h) \cap C^1(0, T; L_\infty(\Omega)) \times C(0, T; U_h)$$

be the finite element solution of the system (24)–(25) respectively, where the inclusion is to be understood uniformly w.r.t. the mesh parameter $h$ in the sense that $\|E_h\|_{C^1(0, T; L_\infty(\Omega))}$ is bounded by a constant independent of $h$. Then there exists a constant $C > 0$ independent of $h$ such that the following error estimate holds:

$$\|E_h(T) - E(T)\|_{L_\infty} + \|H_h(T) - H(T)\|_{L_\infty} \leq C \left[ \|P_h E_0 - E_0\|_{L_\infty} + \|\Pi_h H_0 - H_0\|_{L_\infty} + h^k \right]$$

(the detailed structure of the bound is given at the end of the proof).

**Proof.** We set $\Psi := \Psi_h \in W_h$ in (14) and $\Phi := \Phi_h \in U_h$ in (15):

$$(\varepsilon(E) \partial_t E, \Psi_h) - (\nabla \times H, \Psi_h) = 0 \quad \forall \Psi_h \in W_h,$$

$$(\mu_0 \partial_t H, \Phi_h) + (E, \nabla \times \Phi_h) = 0 \quad \forall \Phi_h \in U_h.$$ 

By means of the projection operators $P_h$ and $\Pi_h$ defined in (28) and (31)–(32), resp., from this we get

$$(\varepsilon(E) \partial_t E, \Psi_h) - (\nabla \times \Pi_h H, \Psi_h) = (\nabla \times (H - \Pi_h H), \Psi_h) \quad \forall \Psi_h \in W_h,$$  

$$(\mu_0 \partial_t \Pi_h H, \Phi_h) + (P_h E, \nabla \times \Phi_h) = \mu_0 (\Pi_h \partial_t H - \partial_t H, \Phi_h)$$

$$+ \mu_0 (\partial_t \Pi_h H - \Pi_h \partial_t H, \Phi_h)$$

$$+ (P_h E - E, \nabla \times \Phi_h) \quad \forall \Phi_h \in U_h.$$  

The last term on the right-hand side of (34) vanishes thanks to the properties of $\Pi_h$, see [Mon91, eq. (2.4)] and (31).

The second term on the right-hand side of (35) can be omitted because of the commutation property $\partial_t \Pi_h H = \Pi_h \partial_t H$, which results from the continuity properties of the operator $\Pi_h$.

The last term on the right-hand side vanishes thanks to the property (30) of $P_h$.

Therefore (34)–(35) simplify to

$$(\varepsilon(E) \partial_t E, \Psi_h) - (\nabla \times \Pi_h H, \Psi_h) = 0 \quad \forall \Psi_h \in W_h,$$  

$$(\mu_0 \partial_t \Pi_h H, \Phi_h) + (P_h E, \nabla \times \Phi_h) = \mu_0 (\Pi_h \partial_t H - \partial_t H, \Phi_h) \quad \forall \Phi_h \in U_h.$$  

Now, subtracting (36)–(37) from the system (24)–(25) and taking into consideration that $\mu_0$ is constant, we obtain:

$$(\varepsilon(E_h) \partial_t E_h - \varepsilon(E) \partial_t E, \Psi_h) - (\nabla \times (H_h - \Pi_h H), \Psi_h) = 0 \quad \forall \Psi_h \in W_h,$$  

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Now we will deal with the first term of (38), where we have in mind the choice \( \Psi_h = E_h - P_h E \) in what follows:

\[
\varepsilon_0^{-1} [\varepsilon(E_h) \partial_t E_h - \varepsilon(E) \partial_t E] = (\varepsilon_s(E_h) I + \varepsilon_m(E_h)) \partial_t E_h - (\varepsilon_s(E) I + \varepsilon_m(E)) \partial_t E
\]

\[
= \left((1 + \chi^{(1)} + \chi^{(3)} |E_h|^2) I + \varepsilon_m(E_h)\right) \partial_t E_h - \left((1 + \chi^{(1)} + \chi^{(3)} |E|^2) I + \varepsilon_m(E)\right) \partial_t E
\]

\[
= (1 + \chi^{(1)}) \partial_t E_h - (1 + \chi^{(1)}) \partial_t E + \left(\chi^{(3)} |E_h|^2 I + \varepsilon_m(E_h)\right) \partial_t E_h - \left(\chi^{(3)} |E|^2 I + \varepsilon_m(E)\right) \partial_t E
\]

\[
= (1 + \chi^{(1)}) \partial_t (E_h - E) + \chi^{(3)} [|E_h|^2 \partial_t E_h - |E|^2 \partial_t E] + 2 \chi^{(3)} [E_h E_h^T \partial_t E_h - EE^T \partial_t E]
\]

The treatment of \( \delta_1 \) is quite obvious. With \( E_h - E = \Psi_h + P_h E - E \) we get

\[
\delta_1 = (1 + \chi^{(1)}) \partial_t \Psi_h + (1 + \chi^{(1)}) \partial_t (P_h E - E) =: \delta_{11} + \delta_{12}.
\]

The term \( \delta_2 \) is decomposed as follows:

\[
\delta_2 = \chi^{(3)} \left[|E_h|^2 \partial_t E_h - |E|^2 \partial_t E\right]
\]

\[
= \chi^{(3)} \left[|E_h|^2 - |E|^2\right] \partial_t E_h + \chi^{(3)} |E|^2 \partial_t E_h
\]

\[
= \chi^{(3)} (E_h + E)^T (E_h - E) \partial_t E_h + \chi^{(3)} |E|^2 \partial_t \Psi_h + \chi^{(3)} |E|^2 \partial_t (P_h E - E)
\]

\[
=: \delta_{21} + \delta_{22} + \delta_{23}.
\]

For \( \delta_3 \), we use the following decomposition:

\[
\delta_3 = 2 \chi^{(3)} \left[E_h E_h^T \partial_t E_h - EE^T \partial_t E\right]
\]

\[
= 2 \chi^{(3)} \left[E_h E_h^T - EE^T\right] \partial_t E_h + 2 \chi^{(3)} EE^T \partial_t (E_h - E)
\]

\[
= 2 \chi^{(3)} (E_h - E) E_h^T \partial_t E_h + 2 \chi^{(3)} E(E_h - E)^T \partial_t E_h
\]

\[
+ 2 \chi^{(3)} EE^T \partial_t \Psi_h + 2 \chi^{(3)} EE^T \partial_t (P_h E - E)
\]

\[
=: \delta_{31} + \delta_{32} + \delta_{33} + \delta_{34}.
\]

With these decompositions, equation (38) takes the form

\[
(\varepsilon(E_h) \partial_t E_h - \varepsilon(E) \partial_t E, \Psi_h) - (\nabla \times (H_h - \Pi_h H), \Psi_h)
\]

\[
= \varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^T \Psi_h d\mathbf{x} + \varepsilon_0 \int_\Omega [\delta_{12} + \delta_{21} + \delta_{31} + \delta_{32} + \delta_{34}]^T \Psi_h d\mathbf{x}
\]

\[
- (\nabla \times (H_h - \Pi_h H), \Psi_h) = 0,
\]

or, after some rearrangement,

\[
\varepsilon_0 \int_\Omega [\delta_{11} + \delta_{22} + \delta_{33}]^T \Psi_h d\mathbf{x} - (\nabla \times (H_h - \Pi_h H), \Psi_h)
\]

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\[ \varepsilon_0 \int_{\Omega} \left[ \delta_{11} + \delta_{22} + \delta_{33} \right]^T \Psi_h dx \]

Then:

\[ \varepsilon_0 \int_{\Omega} \left[ \delta_{11} + \delta_{22} + \delta_{33} \right]^T \Psi_h dx \\
= \varepsilon_0 \int_{\Omega} \left[ (1 + \chi^{(1)}) \partial_t \Psi_h^T \Psi_h + \chi^{(3)} |\mathbf{E}|^2 \partial_t \Psi_h^T \Psi_h + 2 \chi^{(3)} (\mathbf{EE}^T \partial_t \Psi_h)^T \Psi_h \right] dx \\
= \frac{\varepsilon_0}{2} \int_{\Omega} \left[ (1 + \chi^{(1)}) \partial_t |\Psi_h|^2 + \chi^{(3)} |\mathbf{E}|^2 \partial_t |\Psi_h|^2 + 4 \chi^{(3)} \mathbf{E}^T \partial_t \Psi_h \mathbf{E}^T \Psi_h \right] dx. \]

Since

\[ |\mathbf{E}|^2 \partial_t |\Psi_h|^2 = \partial_t (|\mathbf{E}|^2 |\Psi_h|^2) - \partial_t (|\mathbf{E}|^2) |\Psi_h|^2 \]

it follows that

\[ \varepsilon_0 \int_{\Omega} \left[ \delta_{11} + \delta_{22} + \delta_{33} \right]^T \Psi_h dx \\
= \frac{\varepsilon_0}{2} \int_{\Omega} \left[ (1 + \chi^{(1)}) \partial_t (|\mathbf{E}|^2 |\Psi_h|^2) - \chi^{(3)} \partial_t (|\mathbf{E}|^2) |\Psi_h|^2 \right. \\
+ \left. 2 \chi^{(3)} \partial_t |\mathbf{E}|^2 |\Psi_h|^2 - 4 \chi^{(3)} \partial_t |\mathbf{E}|^2 \partial_t |\Psi_h|^2 \right] dx \\
= \frac{\varepsilon_0}{2} \int_{\Omega} \left[ (1 + \chi^{(1)}) \partial_t |\Psi_h|^2 + \chi^{(3)} \partial_t (|\mathbf{E}|^2 |\Psi_h|^2) - \chi^{(3)} \partial_t (|\mathbf{E}|^2) |\Psi_h|^2 \\
+ 2 \chi^{(3)} \partial_t |\mathbf{E}|^2 |\Psi_h|^2 - 4 \chi^{(3)} \partial_t |\mathbf{E}|^2 \partial_t |\Psi_h|^2 \right] dx \\
- \frac{\varepsilon_0}{2} \int_{\Omega} \chi^{(3)} \partial_t (|\mathbf{E}|^2) |\Psi_h|^2 dx - 2 \varepsilon_0 \int_{\Omega} \chi^{(3)} \partial_t |\mathbf{E}|^2 \partial_t |\Psi_h|^2 dx. \]

From the estimates

\[ \frac{\varepsilon_0}{2} \int_{\Omega} \chi^{(3)} \partial_t (|\mathbf{E}|^2) |\Psi_h|^2 dx \leq \frac{\varepsilon_0}{2} \int_{\Omega} \chi^{(3)} \partial_t |\mathbf{E}|^2 |\Psi_h|^2 dx \]

and, analogously,

\[ \varepsilon_0 \int_{\Omega} \chi^{(3)} \partial_t |\mathbf{E}|^2 \partial_t |\Psi_h|^2 dx \leq \frac{\varepsilon_0}{2} \int_{\Omega} \chi^{(3)} |\mathbf{E}|^2 \partial_t |\Psi_h|^2 dx \]

we conclude that

\[ \varepsilon_0 \int_{\Omega} \left[ \delta_{11} + \delta_{22} + \delta_{33} \right]^T \Psi_h dx \]

\[ \geq \frac{\varepsilon_0}{2} \int_{\Omega} \left[ (1 + \chi^{(1)}) \partial_t |\Psi_h|^2 + \chi^{(3)} \partial_t (|\mathbf{E}|^2 |\Psi_h|^2) + 2 \chi^{(3)} \partial_t |\mathbf{E}|^2 \partial_t |\Psi_h|^2 \right] dx \]
\[-3\|\chi^{(3)}\|_{L_{\infty}(\Omega)}\|E\|_{C^{1}(0,T;L_{\infty}(\Omega))}\|\Psi_h\|_{e_0}^2\]
\[= \frac{1}{2} \partial_t \|\Psi_h\|_{e_0(1+\chi^{(1)})}^2 + \frac{\varepsilon_0}{2} \int_{\Omega} \chi^{(3)} \left[ |E|^2 |\Psi_h|^2 + 2 |E^\top \Psi_h|^2 \right] dx\]
\[-3\|\chi^{(3)}\|_{L_{\infty}(\Omega)}\|E\|_{C^{1}(0,T;L_{\infty}(\Omega))}\|\Psi_h\|_{e_0}^2.\]

For the right-hand side, we have:

\[-\varepsilon_0 \int_{\Omega} \left[ (1 + \chi^{(1)}) \partial_t (P_h E - E)^\top \Psi_h \right.\]
\[+ \chi^{(3)} (E_h + E)^\top (E_h - E) \partial_t E_h^\top \Psi_h + \chi^{(3)} |E|^2 \partial_t (P_h E - E)^\top \Psi_h \]
\[+ 2 \chi^{(3)} (E_h - E) E_h^\top \partial_t E_h^\top \Psi_h + 2 \chi^{(3)} (E_h - E)^\top \partial_t E_h E^\top \Psi_h \]
\[+ 2 \chi^{(3)} (E^\top E_h \Psi_h) d\Psi_h \right] dx\]
\[\leq \varepsilon_0 \int_{\Omega} \left[ (1 + \chi^{(1)}) |\partial_t (P_h E - E)| |\Psi_h| \right.\]
\[+ \chi^{(3)} |E_h + E| |\partial_t E_h||\Psi_h|^2 + \chi^{(3)} |E_h + E||P_h E - E| |\partial_t E_h||\Psi_h| \]
\[+ 2 \chi^{(3)} E_h^\top \partial_t E_h |\Psi_h|^2 + 2 \chi^{(3)} E_h^\top \partial_t E_h (P_h E - E)^\top \Psi_h \]
\[+ 2 \chi^{(3)} \Psi_h^\top \partial_t E_h E^\top \Psi_h + 2 \chi^{(3)} (P_h E - E)^\top \partial_t E_h E^\top \Psi_h \]
\[+ 2 \chi^{(3)} E^\top \partial_t (P_h E - E)^\top \Psi_h \right] dx\]
\[\leq \varepsilon_0 \int_{\Omega} \left[ (1 + \chi^{(1)} + \chi^{(3)} |E|^2) |\partial_t (P_h E - E)| |\Psi_h| \right.\]
\[+ \chi^{(3)} |E_h||\partial_t E_h||\Psi_h|^2 + \chi^{(3)} |E||\partial_t E_h||\Psi_h|^2 \]
\[+ \chi^{(3)} |E_h||P_h E - E| |\partial_t E_h||\Psi_h| + \chi^{(3)} |E||P_h E - E| |\partial_t E_h||\Psi_h| \]
\[+ 2 \chi^{(3)} |E_h||\partial_t E_h||\Psi_h|^2 + 2 \chi^{(3)} |E_h||\partial_t E_h||P_h E - E||\Psi_h| \]
\[+ 2 \chi^{(3)} |E||\partial_t E_h||\Psi_h|^2 + 2 \chi^{(3)} |E||\partial_t E_h||P_h E - E||\Psi_h| \]
\[+ 2 \chi^{(3)} |E|^2 |\partial_t (P_h E - E)||\Psi_h| \right] dx\]
\[= \varepsilon_0 \int_{\Omega} \left[ (1 + \chi^{(1)} + 3 \chi^{(3)} |E|^2) |\partial_t (P_h E - E)||\Psi_h| \right.\]
\[+ 3 \chi^{(3)} |E_h||\partial_t E_h||P_h E - E||\Psi_h| + 3 \chi^{(3)} |E||\partial_t E_h||P_h E - E||\Psi_h| \]
\[+ 3 \chi^{(3)} |E_h||\partial_t E_h||\Psi_h|^2 + 3 \chi^{(3)} |E||\partial_t E_h||\Psi_h|^2 \right] dx\]
\[= \varepsilon_0 \int_{\Omega} \left[ (1 + \chi^{(1)} + 3 \chi^{(3)} |E|^2) |\partial_t (P_h E - E)||\Psi_h| \right.\]
\[+ 3 \chi^{(3)} (|E_h| + |E|)|\partial_t E_h||P_h E - E||\Psi_h| + 3 \chi^{(3)} (|E_h| + |E|)|\partial_t E_h||\Psi_h|^2 \right] dx\]
\[ \leq \left[ \| 1 + \chi^{(1)} \|_{L^\infty(\Omega)} + 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \| E \|_{C(0,T; L^\infty(\Omega))}^2 \right] \| \partial_t (P_h E - E) \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \\
+ 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \left[ \| E_0 \|_{C(0,T; L^\infty(\Omega))} + \| E \|_{C(0,T; L^\infty(\Omega))} \| \partial_t E_h \|_{C(0,T; L^\infty(\Omega))} \| P_h E - E \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \\
+ 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \left[ \| E_0 \|_{C(0,T; L^\infty(\Omega))} + \| E \|_{C(0,T; L^\infty(\Omega))} \| \partial_t E_h \|_{C(0,T; L^\infty(\Omega))} \| P_h E - E \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \right] \right] \|
\]

\[ =: C_1 \| \partial_t (P_h E - E) \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} + C_2 \| P_h E - E \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} + C_3 \| \Psi_h \|_{\varepsilon_0}^2, \tag{42} \]

where the positive constants \( C_1, C_2, C_3 \) depend on certain norms of \( \chi^{(1)}, \chi^{(3)}, E, \) and \( E_h. \) Combining the estimates (41) and (42) with (40), we get

\[ \frac{1}{2} \partial_t \| \Psi_h \|_{\varepsilon_0(1+\varepsilon_0)}^2 + \frac{\varepsilon_0}{2} \partial_t \int_{\Omega} \chi^{(3)} \left[ \| E \|_{\Psi_h}^2 + 2 \| E \|_{\Psi_h}^2 \right] \| \Psi_h \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \| - (\nabla \times (H_h - \Pi_h H), \Psi_h) \leq \varepsilon_0 \int_{\Omega} [\delta_{11} + \delta_{22} + \delta_{33}] \| \Psi_h \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \| - (\nabla \times (H_h - \Pi_h H), \Psi_h) \leq -\varepsilon_0 \int_{\Omega} [\delta_{11} + \delta_{22} + \delta_{33} + \delta_{31} + \delta_{32} + \delta_{34}] \| \Psi_h \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \leq C_1 \| \partial_t (P_h E - E) \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} + C_2 \| P_h E - E \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} + C_3 \| \Psi_h \|_{\varepsilon_0}^2. \]

This finally leads to

\[ \frac{1}{2} \partial_t \| \Psi_h \|_{\varepsilon_0(1+\varepsilon_0)}^2 + \frac{\varepsilon_0}{2} \partial_t \int_{\Omega} \chi^{(3)} \left[ \| E \|_{\Psi_h}^2 + 2 \| E \|_{\Psi_h}^2 \right] \| \Psi_h \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \| - (\nabla \times (H_h - \Pi_h H), \Psi_h) \leq C_4 := C_3 + 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \| E \|_{C(0,T; L^\infty(\Omega))}^2 \]

where

\[ C_4 := C_3 + 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \| E \|_{C(0,T; L^\infty(\Omega))}^2 \]

Now we consider (39) with \( \Phi_h = H_h - \Pi_h H \) and get

\[ \frac{1}{2} \partial_t \| \Phi_h \|_{\mu_0}^2 + (E_h - P_h E, \nabla \times \Phi_h) = \mu_0 (\partial_t H - \Pi_h \partial_t H, \Phi_h) \leq \| \partial_t H - \Pi_h \partial_t H \|_{\mu_0} \| \Phi_h \|_{\mu_0}. \]

Adding both inequalities and making use of the commutation property of \( P_h, \) we arrive at

\[ \frac{1}{2} \partial_t \| \Psi_h \|_{\varepsilon_0(1+\varepsilon_0)}^2 + \frac{1}{2} \partial_t \| \Phi_h \|_{\mu_0}^2 + \frac{\varepsilon_0}{2} \partial_t \int_{\Omega} \chi^{(3)} \left[ \| E \|_{\Psi_h}^2 + 2 \| E \|_{\Psi_h}^2 \right] \| \Psi_h \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} \| - (\nabla \times (H_h - \Pi_h H), \Psi_h) \leq C_4 \| \Psi_h \|_{\varepsilon_0}^2 + C_2 \| P_h E - E \|_{\varepsilon_0} \| \Psi_h \|_{\varepsilon_0} + \| \partial_t H - \Pi_h \partial_t H \|_{\mu_0} \| \Phi_h \|_{\mu_0} + C_3 \| \Psi_h \|_{\varepsilon_0}^2. \]

The projection errors can be estimated by means of (29) and (33), that is, for \( E, \partial_t E \in H^k(\Omega) \) and \( \partial_t H \in H^{k+1}(\Omega), \) we have that

\[ \| E - P_h E \|_{\varepsilon_0} \leq C \sqrt{\varepsilon_0} h^k \| E \|_{H^k(\Omega)} \leq C \sqrt{\varepsilon_0} h^k \| E \|_{C(0,T; H^k(\Omega))}, \]

\[ \| \partial_t E - P_h \partial_t E \|_{\varepsilon_0} \leq C \sqrt{\varepsilon_0} h^k \| \partial_t E \|_{H^k(\Omega)} \leq C \sqrt{\varepsilon_0} h^k \| \partial_t E \|_{C(0,T; H^k(\Omega))}, \]

\[ \| \partial_t H - \Pi_h \partial_t H \|_{\mu_0} \leq C \sqrt{\mu_0} h^k \| \partial_t H \|_{H^{k+1}(\Omega)} \leq C \sqrt{\mu_0} h^k \| \partial_t H \|_{C(0,T; H^{k+1}(\Omega))}. \]
In this way the above estimate can be written as

\[
\begin{align*}
\frac{1}{2} \partial_t \| \Psi_h \|_{\varepsilon_0(1+\chi(t))}^2 + \frac{1}{2} \partial_t \| \Phi_h \|_{\mu_0}^2 + \frac{\varepsilon_0}{2} \partial_t \int_\Omega \chi^{(3)} \left[ |E|^2 |\Psi_h|^2 + 2 |E^T \Psi_h|^2 \right] d\mathbf{x} \\
\leq C_5 h^k \left[ \| \Psi_h \|_{\varepsilon_0} + \| \Phi_h \|_{\mu_0} \right] + C_4 \| \Psi_h \|_{\varepsilon_0}^2.
\end{align*}
\]

Setting

\[ w_h(t) := \sqrt{\| \Psi_h(t) \|_{\varepsilon_0}^2 + \| \Phi_h(t) \|_{\mu_0}^2}, \]

we get

\[
\begin{align*}
\frac{1}{2} \partial_t \| \Psi_h \|_{\varepsilon_0(1+\chi(t))}^2 + \frac{1}{2} \partial_t \| \Phi_h \|_{\mu_0}^2 + \frac{\varepsilon_0}{2} \partial_t \int_\Omega \chi^{(3)} \left[ |E|^2 |\Psi_h(t)|^2 + 2 |E^T \Psi_h(t)|^2 \right] d\mathbf{x} \\
\leq C_5 \sqrt{2} h^k w_h(t) + C_4 \| \Psi_h \|_{\varepsilon_0}^2 \\
\leq C_5 \sqrt{2} h^k w_h(t) + C_4 w_h^2(t).
\end{align*}
\]

Integrating this inequality, we obtain

\[
\begin{align*}
&\frac{1}{2} \| \Psi_h(t) \|_{\varepsilon_0(1+\chi(t))}^2 + \frac{1}{2} \| \Phi_h(t) \|_{\mu_0}^2 \\
+ &\frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)} \left[ |E(t)|^2 |\Psi_h(t)|^2 + 2 |E(t)^T \Psi_h(t)|^2 \right] d\mathbf{x} \\
\leq &\frac{1}{2} \| \Psi_h(0) \|_{\varepsilon_0(1+\chi(t))}^2 + \frac{1}{2} \| \Phi_h(0) \|_{\mu_0}^2 \\
+ &\frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)} \left[ |E(0)|^2 |\Psi_h(0)|^2 + 2 |E(0)^T \Psi_h(0)|^2 \right] d\mathbf{x} \\
+ &\int_0^t \left[ C_5 \sqrt{2} h^k w_h(s) + C_4 w_h^2(s) \right] ds. \tag{43}
\end{align*}
\]

By the monotonicity of the weighted norms w.r.t. the weight and the nonnegativity of the integral term on the left-hand side, we see that

\[
\frac{1}{2} w_h^2(t) \leq \frac{1}{2} \| \Psi_h(t) \|_{\varepsilon_0(1+\chi(t))}^2 + \frac{1}{2} \| \Phi_h(t) \|_{\mu_0}^2 \\
+ \frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)} \left[ |E(t)|^2 |\Psi_h(t)|^2 + 2 |E(t)^T \Psi_h(t)|^2 \right] d\mathbf{x}. \tag{44}
\]

On the other hand, we have the estimates

\[
\| \Psi_h(0) \|_{\varepsilon_0(1+\chi(t))}^2 \leq \| 1 + \chi^{(1)} \|_{L^\infty(\Omega)} \| \Psi_h(0) \|_{\varepsilon_0}^2 \leq \| 1 + \chi^{(1)} \|_{L^\infty(\Omega)} w_h^2(0) \tag{45}
\]

and

\[
\frac{\varepsilon_0}{2} \int_\Omega \chi^{(3)} \left[ |E(0)|^2 |\Psi_h(0)|^2 + 2 |E(0)^T \Psi_h(0)|^2 \right] d\mathbf{x} \\
\leq 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \| E(0) \|_{L^\infty(\Omega)}^2 \| \Psi_h(0) \|_{\varepsilon_0}^2 \\
\leq 3 \| \chi^{(3)} \|_{L^\infty(\Omega)} \| E(0) \|_{L^\infty(\Omega)}^2 w_h^2(0). \tag{46}
\]
Combining (44), (45), (46) with (43), we get
\[
\frac{1}{2} w_h^2(t) \leq \frac{1}{2} \|1 + \chi(1)\|_{L^\infty(\Omega)} w_h^2(0) + \frac{3}{2} \|\chi(3)\|_{L^\infty(\Omega)} \|E(0)\|_{L^\infty(\Omega)}^2 w_h^2(0) \\
\quad + \int_0^t \left[ C_5 \sqrt{2} h^k w_h(s) + C_4 w_h^2(s) \right] \, ds,
\]
or, equivalently,
\[
w_h^2(t) \leq C_6^2 w_h^2(0) + \int_0^t \left[ 2 C_5 \sqrt{2} h^k w_h(s) + 2 C_4 w_h^2(s) \right] \, ds, \tag{47}
\]
where
\[
C_6^2 := \|1 + \chi(1)\|_{L^\infty(\Omega)} + 3 \|\chi(3)\|_{L^\infty(\Omega)} \|E(0)\|_{L^\infty(\Omega)}^2.
\]

In the paper [Dal79], a Gronwall-type lemma (Lemma 4.1) is specified which gives a bound on the value \(w(T)\) provided an inequality like (47) is satisfied:
\[
w_h(T) \leq C_6 e^{C_4 T} w_h(0) + C_5 \sqrt{2} h^k T e^{C_4 T}.
\]

From this and the triangle inequality in conjunction with (29) and (33) the statement follows. Indeed, since
\[
w(t) \leq \|\Psi_h(t)\|_{\mathcal{V}_0} + \|\Phi_h(t)\|_{\mu_0} \leq \sqrt{2} w(t) \quad \text{for all} \ t \in [0, T],
\]
we get
\[
\|\Psi_h(T)\|_{\mathcal{V}_0} + \|\Phi_h(T)\|_{\mu_0} \leq \sqrt{2} \left[ C_6 w_h(0) + C_5 \sqrt{2} h^k T \right] e^{C_4 T} \\
\quad \leq \sqrt{2} \left[ C_6 \|\Psi_h(0)\|_{\mathcal{V}_0} + C_6 \|\Phi_h(0)\|_{\mu_0} + C_5 \sqrt{2} h^k T \right] e^{C_4 T}.
\]

Then
\[
\|E_h(T) - E(T)\|_{\mathcal{V}_0} + \|H_h(T) - H(T)\|_{\mu_0} \\
\leq \|\Psi_h(T)\|_{\mathcal{V}_0} + \|\Phi_h(T)\|_{\mu_0} + \|\Pi_h E(T) - E(T)\|_{\mathcal{V}_0} + \|\Pi_h H(T) - H(T)\|_{\mu_0} \\
\leq \sqrt{2} \left[ C_6 \|\Psi_h(0)\|_{\mathcal{V}_0} + C_6 \|\Phi_h(0)\|_{\mu_0} + C_5 \sqrt{2} h^k T \right] e^{C_4 T} \\
\quad + C h^k \left[ \sqrt{\|E\|_{C(0,T;H^{k}(\Omega))}} + \sqrt{\mu_0} \|H\|_{C(0,T;H^{k+1}(\Omega))} \right],
\]
which implies the stated estimate. \(\square\)

Remark 4. Note that the constant \(C\) in this estimate behaves as \(Te^{C_4 T}\) for large \(T\).

6 Conclusion

In this paper we have investigated a semi-discrete conforming finite element approximation to the solution of Maxwell’s equations for nonlinear media of Kerr-type using Nédélec elements from the first family. We have demonstrated energy (stability) estimates for the weakly formulated problem and error estimates for the semi-discretized problem. The results can be extended to other conforming finite element methods provided the corresponding projection operators \(P_h\) and \(\Pi_h\) ((28), (31)–(32)) admit analogous properties.
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