Invited Comment

Isovector spin-singlet \((T = 1, S = 0)\) and isoscalar spin-triplet \((T = 0, S = 1)\) pairing interactions and spin-isospin response

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Abstract

We review several experimental and theoretical advances that emphasize common aspects of the study of spin-singlet, \(T = 1\), and spin-triplet, \(T = 0\), pairing correlations in nuclei. We first discuss various empirical evidence of the special role played by the \(T = 1\) pairing interaction. In particular, we show the peculiar features of the nuclear pairing interaction in the low-density regime, and possible outcomes such as the BCS–BEC crossover in nuclear matter and, in an analogous way, in loosely bound nuclei. We then move to the competition between \(T = 1\) and \(T = 0\) pairing correlations. The effect of such competition on the low-lying spectra is studied in \(N = Z\) odd-odd nuclei by using a three-body model; in this case, it is shown that the inversion of the \(J^\pi = 0^+\) and \(J^\pi = 1^+\) states near the ground state, and the strong magnetic dipole transitions between them, can be considered as a clear manifestation of strong \(T = 0\) pairing correlations in these nuclei. The effect of \(T = 0\) pairing correlations is also quite evident if one studies charge-changing transitions. The Gamow–Teller (GT) states in \(N = Z + 2\) nuclei are studied here by using self-consistent Hartree–Fock–Bogoliubov (HFB) plus quasiparticle random-phase approximation calculations in which the \(T = 0\) pairing interaction is taken into account. Strong GT states are found, near the ground state of daughter nuclei; these are compared with available experimental data from charge-exchange reactions, and such comparison can pinpoint the value of the strength of the \(T = 0\) interaction. Pair transfer reactions are eventually discussed. While two-neutron transfer has long been proposed as a tool to measure the \(T = 1\) superfluidity in the nuclear ground states, the study of deuteron transfer is still in its infancy. Despite its potential interest for revealing effects coming from both \(T = 1\) and \(T = 0\) interactions. We also point out that the reaction mechanism may mask the strong pair transfer amplitudes predicted by the HFB calculations, because of the complexity arising from simultaneous and sequential pair transfer processes.

Keywords: nuclear structure, pairing interaction, mean-field and DFT-based methods

(Some figures may appear in colour only in the online journal)
1. Introduction—superfluidity in nuclei

The effective nucleon–nucleon interactions that are employed within self-consistent mean-field approaches have recently reached a high level of sophistication, and have become quite successful in describing many nuclear properties. Though they can be based on different kinds of ansatz, central, spin–orbit, and tensor terms always show up. In open-shell nuclei the pairing interaction or, in fact, its isovector ($T = 1, S = 0$) part, was originally introduced to account for the odd-even binding energy staggering, the gap in the excitation spectrum of even–even and odd-$A$ nuclei [1–3], the moment of inertia of deformed nuclei [2] and also the fission barrier of actinide nuclei (see p 158 of [3]). In the literature, largely only the spin-singlet $T = 1$ pairing has been discussed in nuclear physics, since the large spin–orbit splitting prevents the coupling of a spin-triplet ($T = 0, S = 1$) pair in the ground state [4, 5]. Another reason for this is that the neutron excess along the stability line of the nuclear chart suppresses the proton–neutron pairing for medium-mass and heavy nuclei. The recent availability of radioactive beams has opened up new opportunities to measure structure properties of unstable nuclei along the $N = Z$ line, strongly enhancing the possibility to measure new properties of nuclei such as pairing correlations related to the spin-triplet $T = 0$ pairing [6]. It is thus quite interesting and important to study the competition between the spin-singlet $T = 1$ and the spin-triplet $T = 0$ pairing interactions in $N \approx Z$ nuclei, and seek experimental evidence for their competition in the energy spectra and the transition rates.

One of the widely used mean-field approaches is based on zero-range Skyrme forces: Hartree–Fock (HF) plus Bardin–Cooper–Schrieffer (BCS) equations [7, 8] or Hartree–Fock–Bogoliubov (HFB) equations [8, 9], which include the pairing interaction can be, and have been, solved to study the ground state properties of the open-shell nuclei [10–13]. On top of these ground-state solutions, the self-consistent quasiparticle random-phase-approximation (QRPA) has been adopted by many authors to study the collective excited states [8, 14–19].

The parameters that characterize the effective interactions like the Skyrme ones can be fitted by using empirical properties of uniform nuclear matter, as well as a few ground-state (or sometimes excited state) properties of finite nuclei. However, some channels of the interactions are not well constrained, one of the clearest examples being the pairing interaction between protons and neutrons in the isoscalar spin-triplet ($T = 0, S = 1$) channel. Indeed, there is no consensus on the observables that can be directly related to such a channel, and there are not yet unambiguous signatures of strong neutron–proton particle-particle correlations, despite several efforts [20–26].

It is well known that one of the effects of the bare isoscalar spin-triplet force is to give rise to the deuteron bound state. Several speculations have been made about the relevance of an n-p pairing force in nuclei with $N = Z$, starting from the schematic solution of the pairing Hamiltonian performed in [27]. Many of them are reviewed in [28]. Nonetheless, there is no unambiguous evidence of its effects, let alone evidence of a p-p condensate. In [25], it has been shown that ordinary bound nuclei are dominated by spin–orbit effects, but on the other hand, if such effects can be made less important and one considers either very large nuclei or nuclei with low angular momentum orbitals close to the Fermi surface, then n-p pairing does manifest itself strongly.

Low-lying states of nuclei with a neutron and a proton outside a closed core may be good candidates for studying n-p pairing, as far as the two particles lie in the same orbital. However, collective states are probably better candidates to extract firmer and more general information. In this respect, the effects of isoscalar pairing may be present in charge-exchange excitations and related phenomena. Indeed, in self-consistent Skyrme QRPA calculations it can be shown that the Gamow–Teller Resonance (GTR) is only sensitive to $T = 0$ pairing, while the isobaric analog resonance (IAR) is only sensitive to $T = 1$ pairing [29]. This is related to the zero-range character of Skyrme forces, but it remains to a large extent true when finite-range pairing interactions are adopted [16]—also in the context of relativistic mean field (RMF) calculations [19]. In particular, in [16, 30, 31] it has been shown in self-consistent HF and QRPA calculations that the isoscalar pairing interaction affects some low-energy Gamow–Teller (GT) strength downwards, so that by fitting the n-p pairing strength (at least locally) one can account for the $\beta$-decay half-lives in neutron-rich nuclei. The isoscalar pairing interaction is also important for the double-$\beta$ decay [32]. However, it is not only the isoscalar pairing, but also other terms of the effective interaction which affect the main peak and low-energy part of the GT strength [33–37] in an important way: in particular, this is true for the spin–orbit one-body potential, the spin two-body terms and also tensor terms. One goal of [29] was to point out the necessity of improving Skyrme forces, and this goal was reached in [38].

In short, although it has been put into evidence clearly that both the GTR and the higher order multipole of charge-exchange transitions, such as the spin-dipole and spin-quadrupole transitions, will receive a contribution from both the isoscalar and the isovector pairing forces, firm constraints for the isoscalar pairing have not been extracted until very recently. One of the reasons is that in many of the previous studies the nuclei that have been considered possess neutron excess and are not close to the regions where isoscalar pairing effects are expected to show up. Thus, an important purpose of this contribution is to demonstrate that one is bound to consider specific nuclei and/or specific properties to pin down unambiguous information about isoscalar pairing for specific nuclei and/or specific properties.

Obviously, there could be frameworks other than the self-consistent mean field in which one can study the competition between $T = 1$ and $T = 0$ pairing. Algebraic approaches are discussed in [39] (see the references therein, and in particular [40]). We will not dwell on this topic here, however. Many attempts to extract pairing properties from beyond mean-field approaches, or large-scale shell-model calculations, are also outside the scope of the present paper.
2. Nuclear structure and isovector spin-singlet pairing interaction

2.1. Odd-even mass staggering

The nuclear binding energies are found to show a systematic variation depending on the even or odd values of $Z$ and $N$:

$$\Delta B = \begin{cases} \Delta & \text{for } Z = \text{even and } N = \text{even}, \\ -\Delta & \text{for } A = \text{odd}, \\ \Delta & \text{for } Z = \text{odd and } N = \text{odd}, \end{cases}$$

where $A = N + Z$. To illustrate this point, the separation energy which is the difference of binding energies $B(A)$ of two neighboring nuclei,

$$S_n(A) = B(A) - B(A - 1),$$

is shown in figure 1 in the case of Sn isotopes. The staggering of $S_n(A)$ is certainly due to the extra binding of even-$N$ Sn isotopes. Specific filters can be introduced to prove this. In particular, the 3-point formula for the neutron pairing gap, or pairing index, reads

$$\Delta^{3v}(N) = (-)^{A+1} \frac{B(N + 1) - 2B(N) + B(N - 1)}{2} = (-)^{A+1} \frac{S_n(A + 1) - S_n(A)}{2}.$$  \hspace{1cm} (3)

This pairing index is expected to be proportional to the extra binding energy $\Delta$ in equation (2). This is, of course, to be considered as a reasonable first approximation. Subtle interferences between pairing effects and mean-field effects are discussed in [12, 41, 42]. The 3-point formula centered in the odd-$A$ nucleus can remove the major shell effect on the pairing index [41], while the 4- and 5-point formulas were also considered to avoid a large shell effect in the systematic studies of [12, 42]. Again, to a first approximation, the pairing index can be parametrized as

$$\Delta^{3v}(A) \approx 12/A^{1/2} \text{ MeV},$$

in the broad region of the mass table $16 < A < 250$ [1].

As mentioned in the introduction, HF-BCS or HFB are the most widely used microscopic theories to describe pairing in nuclei. In such frameworks, the quasiparticle energies are expressed as

$$E_k = \sqrt{(\varepsilon_k - \lambda)^2 + \Delta_k^2},$$

where $\varepsilon_k$ and $\Delta_k$ are the single-particle energies and (state-dependent) pairing gaps, respectively, and $\lambda$ is the chemical potential. If we assume that the ground state of the odd nucleus is a quasiparticle state on top of the even core, then the comparison between equations (3) and (5) clearly shows that $E_k \approx \Delta_k \approx \Delta^{3v}$ for $\varepsilon \sim \lambda$.

Only for n-n and p-p pairing do such clear filters exist and point to values of the pairing gap that are consistent with other observables like those mentioned at the start of the introduction. From a theoretical viewpoint, however, the omission of n-p pairing was already found to not be fully justified fifty years ago (see [20] and references therein). The complete generalized isospin pairing theory can be introduced [20, 21]. In the BCS version, the Cooper pairs are not simply formed by two identical nucleons in time-reversed states, $|k\rangle$ and $|\bar{k}\rangle$, but rather one can assume pairs made up in a general way with the four states $|k, \pi\rangle, |k, \nu\rangle, |\bar{k}, \pi\rangle$ and $|\bar{k}, \nu\rangle$. In such generalized theory, nonetheless, the quasiparticle energy keeps the same form as in (5), and the pairing gap reads

$$\Delta^2_k = |\Delta_{k,\pi}|^2 + |\Delta_{k,\nu}|^2 + |\Delta_{\bar{k},\nu}|^2.$$  \hspace{1cm} (6)

In HFB one can even further generalize the wave function, by considering pairs that do not correspond to time-reversed states.

There is a subtle interplay between pairing and deformation: the first calculations [43] showed that different types of pairing arise if the nucleus is spherical, axially symmetric or triaxial, respectively. Only light nuclei with $N \approx Z$ are candidates for n-p pairing. Neutron excess and/or large spin-orbit splittings hinder n-p pairing as discussed in the introduction.

One may wonder whether the results for the dominance of the different kinds of pairing are confirmed within calculations that employ realistic effective interactions. In the work of [25], extensive HFB calculations were performed with the hope of clarifying if and where $T = 0$ pairing can dominate over $T = 1$ pairing. A simple yet realistic choice has been adopted for the mean-field Hamiltonian, that is, a Woods–Saxon potential plus spin–orbit. The pairing forces have been chosen to be zero-range form, namely

$$V^{T=1}(\hat{r}_1, \hat{r}_2) = \hat{P}_n V_0 \delta(\hat{r}_1 - \hat{r}_2),$$

and

$$V^{T=0}(\hat{r}_1, \hat{r}_2) = \hat{P}_s V_0 \delta(\hat{r}_1 - \hat{r}_2),$$

where $\hat{P}_n$ and $\hat{P}_s$ are the projectors onto the spin-singlet and spin-triplet channels, respectively:

$$\hat{P}_n = \frac{1}{4} - \frac{1}{4} \sigma \cdot \sigma, \quad \hat{P}_s = \frac{3}{4} + \frac{1}{4} \sigma \cdot \sigma.$$  \hspace{1cm} (9)

If these forces are adjusted so as to reproduce, at best, the matrix elements of realistic shell-model calculations, the ratio
where $\Delta$ is the magnitude of the spin–orbit splittings. In fact, if the nuclei are large enough so that the spin–orbit effects are weakened, $T = 0$ pairing can take over; such nuclei have mass larger than $A \approx 130–140$ and are mostly prone to be unbound via proton emission.

2.2. Low-energy collective excitations

In figure 2 we display the low part of the excitation spectra in a few Sn isotopes. As is well known, and discussed above, in odd-$N$ isotopes the unpaired neutron does not feel pairing correlations and can occupy several orbitals close to one another, so that there is no gap in the excitation spectra. In even-$N$ isotopes, the lowest state is a $2^+$ state. If we interpret this state as a two quasiparticle ($2qp$) excited state, its excitation energy with respect to the ground state $0qp$ should be written as

$$E_{2qp} - E_{0qp} = 2E_{qp} \approx 2\Delta,$$

(10)

where the last approximation stems from the idea that both excited quasiparticles lie at the Fermi surface. This latter equation also has a very transparent interpretation, namely it shows the value of the minimal energy to create an excited state of $2qp$ type by overcoming the energy gap of each quasiparticle, that is, $\Delta$.

In the tin region, the value of $\Delta$ is expected to be about 1.0 MeV (see equation (4)). However, the excitation energies of the $2^+$ states, as visible in figure 2, are observed to be smaller than 2 MeV because of extra correlation energy. In fact, in QRPA, that is, the standard theory for such vibrational states, the energy of a collective state $h\omega$ can be schematically written as

$$h\omega = \sqrt{(E_{2qp} - E_{0qp})^2 + \langle V \rangle^2},$$

(11)

where $\langle V \rangle$ is an average value of the residual interaction between quasiparticle states (see [8] for a complete account of QRPA with detailed formulas). In a magic nucleus, in which pairing is too weak to manifest itself, the residual interaction would be only of particle-hole (p-h) type, and the p-h channel of the nuclear effective Hamiltonian is, in a sense, uniquely sensitive to a given multipole component (e.g. quadrupole if low-lying $2^+$ states are under study). In open-shell nuclei like the Sn isotopes, particle-particle (p-p) force plays an important role, and low-lying states are sensitive both to its $J = 0$ component (which is usually called ‘pairing force’) and to its multipole component. Last but not least, one should remember that not only energies carry signatures of the p-p force as well as of other components of the nuclear Hamiltonian, but electric quadrupole transition probabilities $B(E2)$ do as well [44]. Approximately, one could say that the transition probabilities are inversely proportional to the excitation energies.

To disentangle such effects is not an easy task. Over the course of the last decade, fully microscopic QRPA calculations have become available and have been extensively used to study low-lying vibrational excitations. In [45], for instance, the global performance of QRPA with Skyrme forces has been tested by basically calculating all nuclei that are experimentally known. Unfortunately, only global conclusions have been extracted, and the specific role of the pairing force has not been pinned down as only one type of force has been used, namely the so-called volume pairing force already defined in equation (7). This is the closest possible thing to a pairing force with constant matrix elements $G$, since it is easy to understand that if all wave functions have similar integral values within the nuclear volume then the radial matrix elements of such a force cannot vary much. Starting from the work [46], it has been suggested that the pairing force would be better if it were density dependent, and may be taken as

$$V_{\text{surface}}(\vec{r}_1, \vec{r}_2) = \hat{V} \delta \left(1 - x \left(\frac{\rho}{\rho_0}\right)^\gamma\right) \delta (\vec{r}_1 - \vec{r}_2),$$

(12)

with $x = 1$ and $\gamma$ often taken as 1 for simplicity but in principle arbitrary. Such an interaction is called the ‘surface pairing force’, because if $\rho_0$ is around the saturation density then the parameters can be chosen so as to enhance pairing at the surface. Several studies have shown that the low-lying states, and in particular their transition probabilities, are very sensitive to the choice of pairing force (see, e.g. [47]). Although some authors have expressed a preference for a surface pairing force in this context [48], no conclusive study exists. We show in figure 3 some results for the energy and reduced transition probability of the low-lying $2^+$ states along the neutron-rich part of the Ni isotope chain. The surface and volume pairing forces have been chosen so that the values of the pairing gaps do not differ by more than 20%. Only in $^{78}$Ni can the solution be either superfluid or normal according to the strength of the pairing force. This mainly affects the quadrupole transition probability.

It would be paramount to improve our understanding of the effects of different type of pairing forces in the low-lying nuclear spectroscopy. $3^-$ or higher multipole states can, of
course, be discussed on an equal footing. This could pave the way for a better understanding of how pairing evolves far from the stability valley, because the properties of low-lying vibrational states of even-even nuclei are among the very first observables that one can collect when producing new, exotic nuclei. At the same time, we stress that we are so far concerned only with the \( T = 1 \) pairing force. The microscopic origin of the density dependence of the pairing force, and/or the contribution to it coming from polarization effects, is outside the scope of the present paper.

### 2.3. Moments of inertia of deformed nuclei

The energy spectrum of a deformed nucleus is characterized by the presence of rotational bands in which levels with angular momentum \( I \) have energies that scale like

\[
E_I = E_0 + \frac{I(I+1)}{2J},
\]

where \( E_0 \) is the energy of the so-called bandhead, that is, the level with a given intrinsic configuration and zero-point rotational motion while \( J \) is the moment of inertia of the nucleus. From an experimental point of view, the measurement of the \( \gamma \)-transitions between the states (13) provides the value of the energy of these states and consequently of the moment of inertia. From a theoretical point of view, it is possible to perform so-called cranking calculations, in which the nuclear Hamiltonian is solved with a constraint \( \omega_f \) which corresponds to a rotation around the \( x \)-axis (we assume here that the nucleus has the \( z \)-axis as a symmetry axis and needs to rotate around a perpendicular axis). In particular, one can obtain the value of the moment of inertia within the framework of second-order perturbation theory (where the cranking term \( \omega_f \) plays the role of the perturbing field). The moment of inertia obtained in this way, the so-called Inglis’s one, is given by

\[
\mathcal{I}_{\text{Inglis}} = 2 \sum_{\rho,h} \left| \langle \rho | J_b | h \rangle \right|^2 \left( \varepsilon_{\rho} - \varepsilon_{h} \right),
\]

where \( \rho \) and \( h \) correspond to particle and hole states, since the perturbing field creates \( p-h \) pairs at lowest order. It is known that the Inglis moment of inertia is equivalent to the moment of inertia of a rigid body. Formally, this is proven within the harmonic oscillator potential approximation (see p 77 of [49]); however, the result can be empirically checked to be the same in more realistic cases.

In the presence of superfluidity, the ground state is represented by the BCS vacuum and the lowest order \( p-h \) excitations are replaced by the two quasiparticle ones. Accordingly, the moment of inertia can be obtained as

\[
\mathcal{I}_{\text{BCS}} = 2 \sum_{\alpha,\beta} \left| \langle \alpha | J_b | \beta \rangle \right|^2 \frac{(u_{\alpha}v_{\beta} - u_{\beta}v_{\alpha})^2}{E_{\alpha} + E_{\beta}},
\]

where \( E \) denotes the quasiparticle energy (equation (5)), and \( u, v \) are the BCS unoccupation and occupation factors with \( u^2 + v^2 = 1 \). The formula (15) generally gives smaller values than equation (14) since the denominator \( E_{\alpha} + E_{\beta} \) is larger than \( \varepsilon_{\rho} - \varepsilon_{h} \), and \( (u_{\alpha}v_{\beta} - u_{\beta}v_{\alpha})^2 \) is smaller than one. The experimental values for the moment of inertia are compared with the calculated ones in figure 4. The quenching with respect to the rigid-body value is part of the evidence of superfluidity in nuclei.

### 2.4. Pairing correlations at low-density and BCS-BEC crossover

The pairing correlations at low nucleon density are of special interest, since the theoretical predictions for low-density uniform matter suggest that the pairing gap may take, at around 1/10 of the normal nuclear density, a value which is considerably larger than that around the normal density, both in the case of the isovector spin-singlet and the isoscalar spin-
triplet channel (see, e.g. [24] and references therein). This feature is expected to have direct relevance for the properties of neutron stars, especially those associated with the inner crust. The strong pairing at low density may also be relevant for finite nuclei, if one considers neutron-rich nuclei near the drip-line as, for example, $^6$He and $^{11}$Li. This is because such nuclei are characterized by low-density distributions of neutrons around the nuclear surface (the so-called neutron skin and/or neutron halo). It is interesting to clarify to what extent the pair correlations in these exotic nuclei are different from those in stable nuclei, reflecting the strong density dependence of the pairing correlations. In fact, the di-neutron correlation in the two-neutron halo nuclei such as $^{11}$Li, in which a spatially correlated pair formed by halo neutrons appears, may be considered as a manifestation of the strong pairing correlations in a low-density regime. A recent theoretical analysis using the HFB method [50] also predicts the presence of similar di-neutron correlations in medium-mass neutron-rich nuclei where more than two weakly bound neutrons contribute to the creation of the neutron skin in the outer part of the nuclear surface.

It has been argued that the BCS superconducting phase will change to a strong coupling regime, or Bose–Einstein condensate (BEC) phase made up of spatially compact bound Fermion pairs if the pairing strength becomes strong enough to drive a phase transition. The schematic solutions of the BCS equations [3] suggest that the relevant parameter in this respect is the pairing strength times the level density. In ordinary nuclei, the level density is, as a rule, too small to have a phase transition. Moreover, it is impossible to design systems where these parameters can be varied. Instead, the BCS-BEC crossover phenomenon was recently observed in ultra-cold atomic gases in a trap, for which the interaction is controllable [51]. In the case of the nuclear pairing, the BCS-BEC crossover has been argued mostly because of the neutron-proton $T = 0$ pairing in the tensor coupled spin-triplet $^{3}$SD$_{1}$ channel, for which the strong deuteron-like spatial correlations may be the driving force, while at the same time uniform systems like matter in neutron stars can realize a situation where the level density is large. Concerning the neutron pairing in the $T = 1$, $S = 0$ channel, we may expect strong coupling regimes in the low-density matter, such as unique di-neutron or di-proton correlations, whereas the transition to BEC phase is more questionable.
The pattern change of the spatial correlations of Cooper pairs in nuclear matter was discussed by M Matsuo in [52]. The square of the wave function of the neutron Cooper pair is shown in figure 6 as a function of the relative distance \( r \) between the two neutrons and at various densities. The solid curves show the results in symmetric nuclear matter obtained with the Gogny D1 force, while the dotted curves are the results in neutron matter obtained with G3RS force, which is a simple representation with three Gaussians of the bare force [53]. At normal nuclear density, the wave function has an oscillatory behavior in the coordinate space and the coherence length \( \xi \) is large (the coherence length \( \xi \) is the measure of the size of the Cooper pair, defined in a similar way to the rms radius of wave function [3]). This is a typical BCS-type wave function in the weak-coupling pairing scheme. On the other hand, at lower density such as in the case in which \( \rho/\rho_0 = 1/8 \), the pairing correlations are strong enough to create a BEC-like wave function which has a short coherence length and is quite compact in coordinate space.

To move to finite, weakly bound nuclei, we show in figure 7 the neutron pair wave function in \(^{11}\)Li calculated by means of a three-body model that assumes an \(^3\)Li core [46, 54]. The two-neutron wave function is calculated in the coordinate system where \( \vec{r}_1 \) and \( \vec{r}_2 \) are the distances of the two neutrons from the center of the core. This wave function is then written in terms of the relative distance \( \vec{r} \) between the two neutrons, and the distance \( \vec{R} \) between the core and the center of the two neutrons. The projected wave function on the total spin \( S = 0 \) state reads

\[
\psi^{(S=0)}(\vec{r}_1, \vec{r}_2) = \sum_L f_L(\vec{r}, \vec{R}) [Y_L(\hat{r})Y_L(\hat{R})]^{(00)} [\chi_{1/2}\chi_{1/2}]^{(00)}. \tag{16}
\]

The density-dependent pairing interaction adopted in the three-body model calculations is strongly attractive near the surface, and rather weak near the center of the nucleus so that, in terms of density-dependence, it is quite similar to the realistic pairing interactions shown in figure 5. The pair wave function in general has two peaks in the two-dimensional plane \( (r, R) \), as shown in the bottom-right corner of figure 7. There is a strong peak at large \( R \) and small \( r \): this configuration is referred to as the ‘di-neutron’ configuration. On the other hand, there is a smaller peak at large \( r \) and small \( R \): this is called the ‘butterfly’ configuration. The total spin (\( S \)) component of the di-neutron configuration is dominantly

![Figure 7](image-url)
$S = 0$, while a large $S = 1$ component is found in the butterfly configuration.

The wave function associated with the di-neutron configuration is cut at different distances $R$ from the center and shown as well in figure 7. Near the center of the core at $R = 0.5$ fm, the pair wave function has an oscillatory behavior similar to the nuclear matter result at the normal density $\rho_0$. When the value of $R$ is increased and the density becomes dilute, the oscillatory behavior gradually disappears and one finds a single peak at $R \approx 4$ fm, where the pairing correlations reach their maximum. This di-neutron wave function in $^{11}$Li shows, therefore, a very similar behavior to the pair wave function in the nuclear matter which has been associated to the BCS-BEC crossover and shown in figure 6.

3. Competition between isoscalar and isovector pairing interaction

3.1. Pairing correlation energy for pf-shell configurations

After having reviewed many general features of pairing in nuclei, we aim in this section to point to some observations that can pin down, quantitatively, the interplay between isovector and isoscalar pairing. Certainly, pairing correlation energies are a fairly direct observable: they represent the energy gain due to the pairing correlation.

In this case we adopt a separable form, namely the spin-singlet $T = 1$ pairing interaction which reads

$$V^{(T=1)}(\vec{r}_1, \vec{r}_2) = -G^{(T=1)} \sum_{i,j} P^{(i,1)}(\vec{r}_1, \vec{r}_2) D^{(i,1)}(\vec{r}_1, \vec{r}_2),$$

(17)

where the pair field operator is defined as

$$P^{(i,5^+)}(\vec{r}_1, \vec{r}_2) = \delta_{i,j} \sqrt{2i + 1} \{a_i^\dagger \psi_i \} \psi_i^\dagger(\vec{r}_2)^\dagger,$$

(18)

in terms of a single-particle wave function $\psi_i(r)$ having quantum numbers $i \equiv \{n_i, l_i, j_i\}$. Here, $a_i^\dagger$ and $a_i$ are the creation and annihilation operators for the single-particle configuration $i$, respectively. The pairing strength $G^{(T=1)}$ can be fitted to the empirical pairing gaps given by equation (4) [3, 49, 56], and in this case turns out to be

$$G^{(T=1)} = \frac{24}{A} \text{MeV}.$$

(19)

Even though the value in equation (19) is a reasonable choice for calculations performed within a model space consisting of one major shell [3, 49, 56], the absolute value of the pairing strength should not be taken seriously since it depends on the model space adopted. It was pointed out in [57] that the separable form of the pairing interaction is quite useful as far as the non-separable realistic Hamiltonians adopted, e.g. in shell-model calculations.

The spin-triplet $T = 0$ pairing can also be given by a similar separable form:

$$V^{(T=0)}(\vec{r}_1, \vec{r}_2) = -fG^{(T=1)} \sum_{i,j} P^{(i,0)}(\vec{r}_1, \vec{r}_2) D^{(i,0)}(\vec{r}_1, \vec{r}_2),$$

(20)

where the scaling factor $f$ is defined in close analogy as we have already done in equations (7) and (8), and as we shall do below. This factor is varied between 1 and 2 in the following study of the pairing correlation energy [5].

We can now discuss the energy gain due to the pairing correlations, that is, the pairing correlation energy. Figure 8 shows these energies for the $p$-orbit ($l = 1$) and the $f$-orbit ($l = 3$) configurations, as a function of the scaling factor $f$ for the $T = 0$ pairing. The energies for both the $J^p = 0^+$ state with the isospin $T = 1$, and the $J^p = 1^+$ state with the isospin $T = 0$ are shown in the figure. In the case of the $T = 0$ pairing, we need to add to the Hamiltonian the spin-orbit splitting parametrized as

$$\Delta \epsilon_{ls} = -V_{ls}(\vec{r} \cdot \vec{s}),$$

(21)

where the spin-orbit coupling strength $V_{ls}$ is taken to be [49]

$$V_{ls} = \frac{24}{A^{1/3}} \text{MeV}.$$

(22)

This spin-orbit potential reproduces well the empirical spin-orbit splitting $\Delta \epsilon = 7.0 \text{MeV}$ between the $1f_{7/2}$ and $1f_{5/2}$ states in $^{41}$Ca [58].

We diagonalize the pairing Hamiltonian separately for the $p$- and $f$-orbit configurations in order to disentangle the roles of the pairing and those of the spin-orbit interactions in a transparent way. It should be noticed that, for the $T = 0$ pairing, the pair configurations are constructed not only with two equal orbitals having $(l_i = l_f, j_i = j_f)$, but also with the spin-orbit partner orbits, namely $(l_i = l_f, j_i = j_f \pm 1)$, as is seen in the two-body $T = 0$ pairing matrix elements discussed in the appendix (see equation (57)). Thus, in the $l = 1$ case, the $2p_{3/2}^2$ and $2p_{1/2}^2$ configurations are available for the $T = 1$, $J^p = 0^+$ state, while the $2p_{3/2}^2 2p_{1/2}$ configuration is also available for the $T = 0$, $J^p = 1^+$ state. In a similar way, the $1f_{7/2}^2$ and $1f_{5/2}^2$ configurations participate in the $J^p = 0^+$ state in the $l = 3$ case, and also the $1f_{7/2} 1f_{5/2}$ configuration is involved in the $J^p = 1^+$ state.

As one can see in figure 8, the lowest energy state with $J^p = 0^+$ for the $l = 3$ case gains more binding energy than the $J^p = 1^+$ state if the factor $f$ is smaller than 1.5. In the strong $T = 0$ pairing case, that is, $f > 1.6$, the $J^p = 1^+$ state obtains more binding energy than the lowest $J^p = 0^+$ state. These results are largely due to the quenching of the $T = 0$ pairing matrix element by the transformation coefficient from the $jj$ to $LS$ coupling schemes [5]. This quenching never happens for the $T = 1$ pairing matrix element, since the mapping of the two-particle wave function between the two coupling schemes is simply implemented by a factor $\sqrt{j + 1/2}$ in equation (55). In the $l = 1$ case, the competition between the $J^p = 0^+$ and the $J^p = 1^+$ states is also seen in figure 8. Because of the smaller spin-orbit splitting in this case, the couplings among the available configurations are
orbits, where the spin-triplet pairing $T = 0$ is favored in the ground state rather than the $T = 0^+$ one, especially when the $p_{3/2}$ orbit is the main configuration for the valence particles. However, the onset of spin-triplet pair condensation is not be guaranteed by the simple inspection of the spin of the ground state, and may need a careful examination of many-body wave functions emerging from HFB or large-scale shell-model calculations [62].

As mentioned above, the shell-model matrix elements are consistent with a factor $f$ in equation (20) in the range of 1.6-1.7, for both $sd$-shell and $pf$-shell configurations [59–61]. In [56], the ratio 1.5 is adopted to analyze the spin-triplet pairing correlations in the $N = Z$ nuclei within shell-model calculations. These adopted values of $f$, together with the results shown in figure 8, suggest that, in the odd-odd $N = Z$ nuclei, the configuration with $J^\pi = 1^+$ is favored in the ground state rather than the $J^\pi = 0^+$ one, especially when the $p_{3/2}$ orbit is the main configuration for the valence particles. However, the onset of spin-triplet pair condensation is not be guaranteed by the simple inspection of the spin of the ground state, and may need a careful examination of many-body wave functions emerging from HFB or large-scale shell-model calculations [62].

3.2. IAR and Gamow–Teller states in normal nuclei

3.2.1. RPA model and formalism. We recall, here, the main features of charge-exchange RPA and QRPA in the standard matrix formulation. The elementary excitations are either proton particle–neutron hole pairs of the type $a_{\alpha,\gamma}^+ a_{\beta,\nu}^-$ or neutron particle–proton hole pairs of the type $a_{\alpha,\pi}^+ a_{\beta,\gamma}^-$. Without pairing correlations, i.e. when the ground state is HF, the index $\alpha$ ($\beta$) denotes the unoccupied (occupied) states in what follows. Within the linear response theory, or RPA, the excited states $|n, \pm\rangle$ result from the action of the operators $\Gamma_{n,\pm}^\dagger$ on the correlated RPA ground state denoted by $|0\rangle$ for a nucleus with proton number $Z$. The RPA states have well-defined $\Delta T_s$, and here the label $\pm$ distinguishes the $T_s$ and $T_{-1}$ modes leading respectively to the $Z + 1$ and $Z - 1$ nuclei. The RPA operators can be expressed as follows:

$$\Gamma_{n,\pm}^\dagger = \sum_{\alpha,\beta} X_{\alpha,\beta}^{(n)} a_{\alpha,\gamma}^+ a_{\beta,\nu}^- - \sum_{\alpha,\beta} Y_{\alpha,\beta}^{(n)} a_{\alpha,\gamma}^+ a_{\beta,\nu}^- ,$$

$$\Gamma_{n,\pm}^\dagger = \sum_{\alpha,\beta} X_{\alpha,\beta}^{(n)} a_{\alpha,\pi}^+ a_{\beta,\gamma}^- - \sum_{\alpha,\beta} Y_{\alpha,\beta}^{(n)} a_{\alpha,\pi}^+ a_{\beta,\gamma}^- .$$

With the additional assumption that the correlated RPA ground state is the vacuum for the RPA operators $\Gamma_{n,\pm}^\dagger$, one can show that the $X_{\alpha,\gamma}^{(n)}$, $Y_{\alpha,\gamma}^{(n)}$ amplitudes define the eigenvectors of the following RPA secular matrix (where the indices $\alpha$ and $\beta$ have been dropped for simplicity):

$$\begin{pmatrix}
A_{\gamma,\gamma'}^{(n)} & 0 & 0 & B_{\gamma,\gamma'}^{(n)} \\
0 & A_{\gamma',\gamma'}^{(n)} & B_{\gamma',\gamma'}^{(n)} & 0 \\
0 & -B_{\gamma,\gamma'}^{(n)} & -A_{\gamma,\gamma'}^{(n)} & 0 \\
-B_{\gamma',\gamma'}^{(n)} & 0 & 0 & -A_{\gamma',\gamma'}^{(n)}
\end{pmatrix} \times \begin{pmatrix}
X_{\gamma}^{(n)} \\
X_{\gamma'}^{(n)} \\
Y_{\gamma}^{(n)} \\
Y_{\gamma'}^{(n)}
\end{pmatrix} = E_n \begin{pmatrix}
X_{\gamma}^{(n)} \\
X_{\gamma'}^{(n)} \\
Y_{\gamma}^{(n)} \\
Y_{\gamma'}^{(n)}
\end{pmatrix} .$$

(24)

The corresponding eigenvalues $E_n$ are the excitation energies of the RPA modes. The expressions of the $A$ and $B$ matrices
are

\[ A_{\alpha\nu,\alpha'\nu'} \equiv A_{\alpha\beta,\beta,\alpha'\beta',\beta'} \]
\[ = (\varepsilon_{\alpha} - \varepsilon_{\beta})_{\alpha\beta,\alpha'\beta'} + \langle \alpha' \beta' | V_{\text{ph}} | \beta \alpha' \rangle, \quad (25a) \]

\[ B_{\alpha\nu,\alpha'\nu'} \equiv B_{\alpha\beta,\beta,\alpha'\beta',\beta'} = \langle \alpha' \beta' | V_{\text{ph}} | \beta \alpha' \rangle, \quad (25b) \]

and similarly for the other cases. In these formulas the single-particle energies \( \varepsilon \) and the matrix elements of the residual particle-hole (p-h) interaction \( V_{\text{ph}} \) appear. At variance with the case of normal RPA, the matrix breaks into two blocks with different dimensions, and the matrix \( B \) is made up of two rectangular blocks. Under the spherical symmetry assumption the RPA excited states have good angular momentum and parity \( J^\pi \) and, therefore, each \( J^\pi \)-mode corresponds to a separate diagonalization in the corresponding \( J^\pi \)-p-h space. In this case, all previous expressions can be cast in angular momentum coupled form: the final result is that the matrix equation (24) retains its structure and the matrix elements are coupled in this way: \( \langle \alpha' \beta' | V_{\text{ph}} | \beta \alpha' \rangle \) the coupling is between \( \langle \alpha \beta | \rangle \) and similarly \( \langle \alpha' \beta' \rangle \).

In open-shell nuclei, pairing correlations must be taken into account and, within the HFB framework the independent particles are replaced by quasiparticles so that RPA becomes QRPA. Quasiparticles are mixtures of particles and holes. If we use the symbol \( \alpha' \) (\( \alpha \)) for the quasiparticle creation (annihilation) operator, the QRPA modes are generated by

\[ \Gamma^\dagger_n = \sum_{\alpha,\pi,\beta,\nu} X^{(n)}_{\alpha,\pi,\beta,\nu} \alpha^\dagger_{\alpha,\pi,\beta,\nu} - Y^{(n)}_{\alpha,\pi,\beta,\nu} \alpha_{\alpha,\pi,\beta,\nu}. \quad (26) \]

In addition, this formula (and the following ones) can be recast in angular momentum coupled form. Similarly as in RPA, in QRPA the states also have well-defined \( \Delta T \). The QRPA matrix equation has the form

\[
\begin{pmatrix}
A_{\alpha\nu,\alpha'\nu'} & B_{\alpha\nu,\alpha'\nu'} \\
-B_{\alpha'\nu',\alpha\nu'} & -A_{\alpha'\nu',\alpha\nu'}
\end{pmatrix}
\begin{pmatrix}
X_{\alpha'\nu'}^{(n)} \\
Y_{\alpha'\nu'}^{(n)}
\end{pmatrix}
= E_n
\begin{pmatrix}
X_{\alpha\nu}^{(n)} \\
Y_{\alpha\nu}^{(n)}
\end{pmatrix},
\]

where once more the indices \( \alpha \) and \( \beta \) have been dropped. The matrix elements appearing in the last formula include both the particle-hole and the particle-particle (p-p) residual interaction. QRPA can be built on top of HFB, or on top of the simpler HF-BCS approximation. The QRPA matrix elements display a quite similar form in the HF-BCS case and in the HFB case, provided one uses the canonical basis. They read

\[ A_{\alpha\nu,\alpha'\nu'} = (E_{\alpha\nu} - E_{\beta\nu})_{\alpha\beta,\alpha'\beta'} + \langle \alpha' \beta' | V_{\text{ph}} | \beta \alpha' \rangle, \quad (28a) \]

\[ B_{\alpha\nu,\alpha'\nu'} = \langle \alpha' \beta' | V_{\text{ph}} | \beta \alpha' \rangle + \langle \alpha' \beta' | V_{\text{ph}} | \beta \alpha' \rangle, \quad (28b) \]

Here \( u \) and \( v \) are the usual unoccupation and occupation factors, respectively, either of the BCS or of the canonical states. \( E_{\alpha'\nu'} \) is either \( E_{\nu} - E_{\nu'} \) (being \( E_{\nu} \) the BCS quasiparticle energy), or the canonical basis matrix element of the HFB Hamiltonian.

Having the (Q)RPA amplitudes \( X^{(n)} \) and \( Y^{(n)} \) at one’s disposal, it is possible to calculate the transition probability of the state \( |\nu\rangle \) associated with a given operator \( \hat{O} \). It is clear that

\[ S(E) = \sum_{\nu} |\langle \nu | \hat{O}_n | \nu \rangle|^2 \delta(E - E_n), \]

and its k-th moment is

\[ m_k = \int dE E^k S(E) = \sum_{\nu} |\langle \nu | \hat{O}_n | \nu \rangle|^2 E_n^k. \]

The operators \( \hat{O}_n \) are written in this form to emphasize that they are proportional to \( T^\pi_n \), where \( T^\pi_n \) is the total isospin, \( \beta = \sum_{i} t(i) \). The Gamow–Teller operators read \( \sum \delta \theta(i) t_{2\,\lambda}(i) \).

Note that quite different ingredients are used even in self-consistent RPA or QRPA by different groups. In the case of Skyrme calculations, the spin-isospin component of the residual interaction in the p-h channel is supplemented by a zero-range p-p force. In the case of QRPA calculations within the RMF framework, in the residual interaction the pion exchange is introduced; since this has no contribution to the ground state, which is treated at the Hartree level, the corresponding coupling constant must be fitted. In the case of relativistic Hartree–Fock (RHF), the pion is introduced and fitted at the ground-state level. However, its role is modest in the residual interaction, and the position of the change-exchange states turn out to be mainly determined by the exchange terms associated with the isoscalar \( \sigma \) and \( \omega \) mesons [63]. Pairing is usually treated non-relativistically, both in RMF and RHF. This is important to keep in mind, because even if our focus is on pairing effects, they cannot be completely decoupled from the underlying mean field.

### 3.2.2. Results

As we have mentioned already, we expect that the IAR is sensitive, in open-shell nuclei, to only or mainly \( T = 1 \) pairing. In the previous section it is clearly shown that the p-n matrix elements enter into the QRPA residual interaction. It could be possible to just fit these matrix elements. Although, generally, one adopts pairing strengths of the order of what is predicted by equation (19), the p-p and n-n empirical matrix elements have often been taken with different strengths: for instance, the values \( G_{2n} = 21/A \) and \( G_{2p} = 26/A \) have been adopted in [64]. However, the Coulomb (anti-)pairing effect should be taken into account and so far few groups have done it intensively [65]. This effect may reduce the pairing gaps by 100–200 keV [66] which means between about 5% and 20%. Therefore, it is legitimate to assume that the pairing force is isospin invariant, as all other components of any bare and/or effective nuclear forces are. In this case, one can say that ground-state 0\(^+\) p-p and n-n matrix elements are equal to the p-n ones.

This is, at least, the attitude that has been adopted in the context of both Skyrme [67] and RMF [68] calculations of the IAR, in which the \( T = 1 \) residual p-n matrix elements have been calculated consistently with the ground-state pairing force and the assumption of isospin invariance. The results for the IAR energies agree very well with experiment (see figure 9), and it has been shown, moreover, that without p-n
pairing in the residual interaction, the IAR is fragmented in more than one peak and does not exhaust the whole $N-Z$ sum rule as it should.

The Gamow–Teller resonance is, instead, sensitive to the largely unknown p-n $T=0$ pairing. Unfortunately, in standard nuclei this sensitivity is not strong enough to allow one to pin down the strength of such pairing force. In [68] it was found that the energy of the main GT peak in $^{118}$Sn changes by, at most, 100 keV when the $T=0$ pairing strength is changed by 50%. The inclusion of $T=0$ pairing reduces the configuration splitting in the region of the main GTR, and has more influence generally speaking on the low-lying strength as can be expected. In fact, it could be said that $T=0$ pairing does play a role in the GT excitation spectra because of the partial occupations that are, in turn, induced by the $T=1$ pairing in the ground state (the latter being, of course, more active around the Fermi surface).

As already discussed, the GT strength function is more sensitive to the spin–orbit splitting and p-h force. Nonetheless, it is interesting to notice that despite the rather different ansatz for such ingredients in the RMF and Skyrme frameworks, some of the conclusions reached in [29] are the same as in [68]. In particular, even in the Skyrme calculations of [29] the reduction of the configuration splitting due to $T=0$ pairing has been observed. This has been specifically attributed to the attractive matrix elements of such force. For instance, in the case of $^{118}$Sn, without $T=0$ pairing the two QRPA states associated with (mainly) the $(\nu g_{9/2}, \pi g_{9/2})$ and $(\nu h_{1/2}, \pi h_{9/2})$ configurations are split considerably, the latter configuration being at higher energy than the former. However, due to the partial occupation of the $l=5$ levels which is caused by the $T=1$ pairing, they are quite sensitive to $T=0$ pairing whose attractive matrix elements push the $(\nu h_{1/2}, \pi h_{9/2})$ downwards; thus, this configuration is finally more admixed in the wave function of the main resonance than would happen without $T=0$ pairing.

Although these effects are of some interest and could be detected experimentally in decay experiments, or other kinds of exclusive experiments, the need is clear for more direct evidence of $T=0$ pairing. This will be the subject of the next subsections.

### 3.3. Gamow–Teller states and $T=0$ pairing in $N \approx Z$ nuclei

As is discussed above, pairing shows up in the QRPA equations through its contribution to the p-p matrix elements of the residual interaction. From the viewpoint of the QRPA model, for IAR states with $J^P = 0^+$, only $T=1$ pairing provides a contribution, while for the GT states with $J^P = 1^+$, only $T=0$ pairing contributes; for other spin-isospin transitions, such as spin-dipole and spin-quadrupole states, both $T=0$ and $T=1$ pairing provide contributions. Therefore, GT states are good candidates for the study of $T=0$ pairing.

Due to selection rules, the GT transitions connect either single-particle states with $j_c = j_p$, or single-particle states with $j_c = j_p \pm 1$, while the IAR is made up only of the former type of transitions. Here, and in what follows, the notation $j_c$ ($j_p$) denotes the spin–orbit partner with $j_c = l - 1/2$ ($j_p = l + 1/2$). Usually, the former type of transitions $\nu j_c \rightarrow \pi j_c$, or $\nu j_c \rightarrow \pi j_p$, is $\approx 3$–7 MeV lower (higher) in energy than the latter type of transitions $\nu j_c \rightarrow \pi j_c$ ($\pi j_c \rightarrow \pi j_p$) due to the spin–orbit splitting. These considerations would completely govern the unperturbed nuclear response. RPA correlations, however, play an important role and we can focus on the matrix elements appearing in the $A$ matrix of equation (28), as they are more important than the matrix elements that fill up matrix $B$, as a rule. In the particle-hole sector the coefficient including occupation factors reads $\alpha, \nu \mu, \nu \nu$, and is non-negligible mainly for configurations at relatively high excitation energy. On the other hand, in the particle-particle sector the relevant configurations are those made up of partially occupied states near the Fermi surface, that is, at relatively low excitation energy. Because of these reasons, the high-lying GT resonance is more sensitive to the p-h spin-isospin interaction, whose strength has often been related to the Landau–Migdal parameter $g_0^2$ [34] as far as the central terms are concerned; tensor terms also play a role, in fact [35, 36]. For the low-energy GT strength, the situation becomes much more complex [16]: in this case, the $T=0$ pairing residual interaction plays a role and its strength can somehow be pinned down; however, central and tensor terms of the p-h residual interaction are also important.

In order to study the GT transitions in $N \approx Z$ nuclei, we have applied the self-consistent Skyrme HFB+QRPA model. In the calculations, zero-range surface $T=0$ pairing was used and its form is the same as shown in equation (12) with a scaling factor $f$, namely

$$ V_{\text{surface}}(\vec{r}_1, \vec{r}_2) = \bar{P}_f V_0 \left(1 - x \left(\frac{j_p}{j_0}\right)^2\right) \delta(\vec{r}_1 - \vec{r}_2). \quad (31) $$

where $\bar{P}_f$ is the projector on spin-triplet states.

#### 3.3.1. Low-energy GT state and $T=0$ pairing in $N \approx Z$ nuclei

We have calculated the GT strength distribution (29) in a series of $N=Z$ pf-shell nuclei, namely $^{48}$Cr, $^{56}$Ni,
and $^{64}$Ge, and the results are shown in figure 10. We have performed these QRPA calculations by assuming different strengths for the $T=0$ pairing or, in other words, the factor $f$ is taken as 0.0, 0.5, 1.0, 1.5, and 1.7 in equation (31). The GT strength distribution of $^{64}$Ge is shown in the panel (c), and it is evident that with the increasing of the $T=0$ pairing strength, a small amount of GT strength is shifted to the low-energy region while the energy of the low-energy peak is shifted downwards, albeit only slightly. In general, only a small amount of GT strength is distributed in this low-energy region, which is about 6 MeV lower than the high-energy peak. This is the normal case of GT strength distributions, as is widely known in standard nuclei with considerable neutron excess. In the case of $^{56}$Ni, the situation changes: when the $T=0$ pairing is not included the GT strength distribution is also like the normal case, i.e. with a main peak in the high-energy region and little strength distributed at low energy (actually, if we do an HF+RPA calculation, without pairing, there is only one peak). With the increase of the $T=0$ pairing, the strength is shifted to the low-energy region and meanwhile the low-energy peak is shifted downward, in a more significant fashion than in the previous case. In particular, when $f=1.5$, there are two peaks with a similar amount of strength, and this is qualitatively consistent with the measured two peak structure [70]. However, as was commented on by the authors of [28], the experimental results are not well reproduced by our present calculation; actually, there are other causes that may affect the peak energies in $^{56}$Ni, such as the particle-vibration coupling (PVC) effect [72]. One might need to use a well-constrained Skyrme force together with a theoretical model that goes beyond QRPA, like QRPA plus PVC or second QRPA, if one wishes to obtain a good reproduction of experimental data [73]. For $^{48}$Cr, the situation becomes quite extreme, that is, when the $T=0$ pairing strength is strong enough (i.e. with $f=1.5$ or 1.7), most GT strength is shifted to the low-energy region and forms the main peak, whereas the original main peak in the high-energy region calculated without $T=0$ pairing disappears. Thus, in this nucleus the $T=0$ pairing might play a much more important role. Therefore, in some $N=Z$ nuclei, the $T=0$ pairing may play an important role to shift a large amount of GT strength to the low-energy region to form a strong GT state there. Interestingly, this happens for values of $f$ that are very consistent with those extracted from shell-model matrix elements and discussed in [25] as well as in the text above.

Figure 10. GT strength distribution (29) in $N=Z$ nuclei $^{48}$Cr, $^{56}$Ni, and $^{64}$Ge calculated by HFB+QRPA with the Skyrme force T21 and with different strengths of $T=0$ pairing. The excitation energies are calculated with respect to the mother nuclei. The experimental data are taken from [70]. The figure is reproduced with permission from [71]. Copyright Elsevier 2013.

3.3.2. Low-energy GT state and $T=0$ pairing in $N=Z+2$ nuclei. GT states in a series of $N=Z+2$ pf-shell nuclei have been measured in [74], and it has been found that, while in some nuclei the usual high-lying GTR is found and has no significant strength at low energy appears, in other cases the main GT peak is detected in the low-energy region. In these cases, the main GT may exhaust more than 80% of the total strength, and it has been called the low-energy super Gamow–Teller (LeSGT) state.

The Skyrme HFB+QRPA results for the GT strength distribution (29) in $N=Z+2$ nuclei from $A=42$ to 58 are shown in figure 11. Here, one can see that in $^{42}$Ca and $^{46}$Ti the $T=0$ pairing (assumed to have a strength similar to that of the corresponding $T=1$ pairing) may shift a large amount of GT strength to the low-energy region to form a strong GT state (LeSGT). For this reason, the high-energy peak disappears. For $^{50}$Cr, $T=0$ pairing is also responsible for the shift of some amount of GT strength to the low-energy region, but its role is not as strong in the case of $^{42}$Ca and $^{46}$Ti. In $^{50}$Fe and $^{58}$Ni, the importance of $T=0$ pairing further decreases.

The GT strength distributions (29) in $^{42}$Ca, calculated by means of HFB+QRPA with the Skyrme force SGII (with or without the tensor force), are shown in figure 12. In the upper (lower) panel, results without (with) the tensor force are
From this figure one can see that the tensor force plays an important role when the $T = 0$ pairing is not strong, i.e. when $f = 0.0$ and 0.5. Instead, when $f$ takes more realistic values like 1.0 or 1.1, the strength distributions with and without tensor force are almost the same. This indicates that the tensor force effects are suppressed by the $T = 0$ pairing with proper strength, and this means that one can extract reliable information on $T = 0$ pairing from such a nucleus.

This conclusion may change if another nucleus is chosen. From figure 13, we can see that the GT strength distributions with and without tensor force are quite different in $^{58}$Ni even with strong $T = 0$ pairing strength. This means that in this case we do not have a unique signature of the effects of the $T = 0$ pairing from the low-energy GT strength.

Therefore, one can use the low-energy GT states in $^{42}$Ca and $^{46}$Ti to extract reliable information on $T = 0$ pairing. Actually, the $T = 0$ pairing strength was suggested to be about 1.0 to 1.05 times the strength of the corresponding

---

**Figure 11.** GT strength distribution (29) in $N = Z + 2$ nuclei with mass number between 42 and 58, calculated by means of HFB + QRPA with the Skyrme force SAMi and different strengths for $T = 0$ pairing. The vertical black line corresponds to the IAR state. The excitation energies are calculated with respect to the mother nuclei. The figure is reproduced with permission from [75]. Copyright American Physical Society 2014.

**Figure 12.** GT strength distribution (29) in $^{42}$Ca calculated by means of HFB+QRPA with the Skyrme force SGII, with or without tensor force. The vertical black line corresponds to the IAR state. The excitation energies are calculated with respect to the mother nuclei. The figure is reproduced with permission from [75]. Copyright American Physical Society 2014.

**Figure 13.** Same as figure 13 in the case of $^{58}$Ni. The figure is reproduced with permission from [75]. Copyright American Physical Society 2014.
In this section we focus on the energy spectra and spin-isospin excitations of odd-odd \(N = Z\) ad- and \(pf\)-shell nuclei. To this end, we apply a three-body model with a density-dependent contact interaction between the valence neutron and proton. The three-body model Hamiltonian for odd-odd \(N = Z\) nuclei, assuming the core + \(p + n\) structure \([6]\), is given by

\[
H = \frac{\hat{p}_p^2}{2m} + \frac{\hat{p}_n^2}{2m} + V_{pc}(\hat{r}_p) + V_{nc}(\hat{r}_n) + V_{pn}(\hat{r}_p, \hat{r}_n) + \left(\frac{\hat{p}_p + \hat{p}_n}{2}\right)^2 2A_c m, \tag{32}
\]

where \(m\) is the nucleon mass, \(A_c\) is the mass number of the core nucleus, and \(V_{pc}\) and \(V_{nc}\) are the mean-field potentials for the valence proton and neutron, respectively, generated by the core nucleus. These are given as

\[
V_{nc}(\hat{r}_n) = V^{(N)}(\hat{r}_n), \quad V_{pc}(\hat{r}_p) = V^{(N)}(\hat{r}_p) + V^{(C)}(\hat{r}_p), \tag{33}
\]

where \(V^{(N)}\) and \(V^{(C)}\) are the nuclear and the Coulomb parts, respectively. In equation (32), \(V_{pn}\) is the pairing interaction between the two valence nucleons. For simplicity, one can safely neglect the recoil kinetic energy of the core nucleus, that is, the last term in equation (32). The nuclear part of the core-valence particle interaction, equation (33), is taken to be

\[
V^{(N)}(r) = \nu_0 f(r) + \nu_1 \frac{1}{r} \frac{df(r)}{dr}(\hat{r} \cdot \hat{s}), \tag{34}
\]

where \(f(r)\) is a Fermi function defined by \(f(r) = 1/(1 + \exp[(r - R)/a])\). For the \(^{18}\)F nucleus, as in \([6]\), we set \(\nu_0 = -49.21\) MeV and \(\nu_1 = 21.6\) MeV fm\(^{-2}\). For the other nuclei, we adjust \(\nu_0\) so as to reproduce the neutron separation energies, while \(\nu_1\) is kept constant for all the nuclei. For more details on the adopted parameters in the one-body potential, the reader can consult \([6]\). We use a contact interaction between the valence neutron and proton, \(V_{np}\), that is the sum of \(T = 0\) and \(T = 1\) pairing forces with surface character as defined above in equations (12) and (31). We write it here with the inclusion of the appropriate projectors for the sake of clarity, namely

\[
V_{np}(\hat{r}_1, \hat{r}_2) = \hat{P}_V V_p \delta(\hat{r}_1 - \hat{r}_2) \left(1 - x_1 \frac{\rho(r)}{\rho_0}\right)^\gamma + \hat{P}_V V_n \delta(\hat{r}_1 - \hat{r}_2) \left(1 - x_2 \frac{\rho(r)}{\rho_0}\right)^\gamma, \tag{35}
\]

where \(\hat{P}_V\) and \(\hat{P}_V\) are the projectors onto the spin-singlet and spin-triplet channels, respectively, defined in equation (9). Notice that the isovector spin-singlet pairing acts on even-\(J\) states \(J^\pi = 0^+, 2^+, \ldots (2J - 1)^+\), while the isoscalar spin-triplet pairing acts on odd-\(J\) states \(J^\pi = 1^+, 3^+, \ldots (2J - 1)^+\) for the configuration \(f^2\). In each channel in equation (35), the first term in brackets corresponds to the interaction in the vacuum while the second term takes into account the medium effects through the density dependence. Here, the core density is assumed to be a Fermi distribution of the same radius and diffuseness as in the core-valence particle interaction (equation (34)). The strength parameters, \(\nu_0\) and \(\nu_1\), are determined from the proton–neutron scattering length as \([76]\)

\[
\nu_0 = \frac{2\pi^2\hbar^2}{m} \frac{a^{(p)}_{pn}}{\pi - 2a^{(p)}_{pn}k_{cut}}, \tag{36}
\]

\[
\nu_1 = \frac{2\pi^2\hbar^2}{m} \frac{a^{(n)}_{pn}}{\pi - 2a^{(n)}_{pn}k_{cut}}, \tag{37}
\]

where \(a^{(p)}_{pn} = -23.749\) fm and \(a^{(n)}_{pn} = 5.424\) fm \([77]\) are the empirical p-n scattering lengths in the spin-singlet and spin-triplet channels, respectively. \(k_{cut}\) is the momentum cutoff that must be introduced when treating the delta function, and is related to the cutoff energy by \(E_{cut} = \hbar^2 k_{cut}^2/m\). In other words, the strengths \(\nu_0\) and \(\nu_1\) determined from the scattering lengths depend on the cutoff energy, \(E_{cut}\). The three parameters \(x_1, x_2,\) and \(\gamma\) in the density-dependent terms in equation (35) are determined so as to reproduce the energies of the ground state \((J^\pi = 1^+)\), the first excited state \((J^\pi = 3^+)\), and the second excited state \((J^\pi = 0^+)\) in \(^{18}\)F (see also \([6]\)). The Hamiltonian (32) is diagonalized in the valence two-particle model space. The basis states for this diagonalization are given by a product of proton and neutron single-particle states having single-particle energies \(\epsilon^{(i)}\). These energies are obtained with the use of the single-particle potential \(V_{np}\) in equation (32) (\(r = p\) or \(n\)). To this end, the single-particle continuum states are discretized in a large box.

We include only those states satisfying \(\epsilon^{(p)} + \epsilon^{(n)} \leq E_{cut}\).

The calculated spectra for \(^{14}\)N, \(^{18}\)F, \(^{30}\)P, \(^{34}\)Cl, \(^{38}\)Sc, and \(^{56}\)Cu nuclei are shown in figure 14 together with the experimental data. The spin-parity for the ground state of the nuclei in figure 14 are \(J^\pi = 1^+\) except for \(^{34}\)Cl and \(^{56}\)Sc. This feature is entirely due to the interplay between the isoscalar spin-triplet...
and the isovector singlet coupling interactions in these $N = Z$ nuclei. In the present calculations, the ratio between the isoscalar and the isovector coupling interactions is $v_{\text{I}}/v_{\text{S}} = 1.9$ for the energy cutoff of the model space $E_{\text{cut}} = 20$ MeV. This ratio is somewhat larger than the value $\approx 1.6$ extracted in [59] from the shell-model matrix elements in p- and sd-shell nuclei and already discussed in the previous sections (see also the value $\approx 1.3$ deduced from HF-BCS plus Lipkin-Nogami calculations reported in [78]). For a larger model space with $E_{\text{cut}} = 30$ MeV, the ratio becomes 1.6, but the agreement between the experimental data and the calculations somewhat worsens qualitatively even though the general features remain the same. It is remarkable that the energy differences $\Delta E = E(0^+_1) - E(1^+_0)$ are well reproduced in $^{34}$Cl and $^{42}$Sc both qualitatively (there is an inversion between the $0^+$ state and the $1^+$ state for the ground state) and quantitatively (if one looks at the absolute value of the energy difference). The model description is somewhat poor in $^{14}$N and $^{30}$P because the cores of these two nuclei are deformed: nevertheless, the ordering of the two lowest levels are correctly reproduced.

The probability of the total spin $S = 0$ and $S = 1$ components for the $0^+$ and the $1^+$ states, respectively, are listed in table 1. The total spin $S = 0$ and $S = 1$ components in two-particle configurations can be calculated with the formula

$$
\langle j_1 j_2 | J \rangle = \sum_{L S} \begin{pmatrix} j_1 & I_1 & L \\ s & s & J_1 \end{pmatrix} \hat{S}_{LJ} \hat{J}_L \langle L S | j_1 j_2 | LS; J \rangle,
$$

where $\hat{L}$, $\hat{S}$, $\hat{J}$ are $\hat{L} = \sqrt{2L + 1}$, $\hat{S} = \sqrt{2S + 1}$, $\hat{J} = \sqrt{2J + 1}$, respectively. For a $j_1 = j_2 = j = l + 1/2$ configuration, the $S = 0$ and $S = 1$ components are given by the factors $(j + 1/2)/2j$ and $(j - 1/2)/2j$, respectively, for $J = 0$. For a $j_1 = j_2 = j = l - 1/2$ configuration, on the other hand, they are $(j + 1/2)/(2j + 2)$ and $(j + 3/2)/(2j + 2)$ for $S = 0$ and $S = 1$, respectively. Notice that the $J_{1/2}$ configuration only has an $S = 0$ component if $J = 0$. Otherwise, all the two-particle states have a large mixture of the $S = 0$ and $S = 1$ components. In general, the $S = 1$ and $S = 0$ components are thus largely mixed in the wave functions of both the ground and the excited states. An exception is $^{30}$P. In this nucleus, the dominant configuration in the $0^+$ state is $(2S_{1/2} \otimes 2S_{1/2})$, which can couple only to the total spin $S = 0$. On the other hand, in the $1^+$ state, the dominant configuration is $(2S_{1/2} \otimes 1D_{3/2})$ $T = 0$ which can couple only to the total spin $S = 1$ with the total angular momentum $L = 2$.

The reduced magnetic dipole transition probability is given by

$$
B(M1; J_i \rightarrow J_f) = \left( \frac{3}{4\pi} \right) \frac{1}{2J_i + 1} \left| \langle J_f | \sum_i (g_i (\nu) \hat{S}_i + g_i (\pi) \hat{L}_i) | J_i \rangle \right|^2,
$$

where the double bar means a reduced matrix element in the spin space. We take the bare g factors $g_\nu (\pi) = 5.58$, $g_\nu (\nu) = -3.82$, $g_\pi (\pi) = 1$, and $g_\nu (\nu) = 0$ for the magnetic transition operators, and the magnetic dipole transitions are given in units of the nuclear magneton $\mu_N = e\hbar/2mc$. The spin-quadrupole transition is defined instead by

$$
B(IVSQ; J_i \rightarrow J_f) = \left( \frac{1}{2J_i + 1} \right) \left| \langle J_f | \sum_i r_i^2 \{ \hat{\sigma} (i) Y_2 (i) \}^{L-1} \tau_z (i) | J_i \rangle \right|^2.
$$
The calculated magnetic moments and magnetic dipole transitions are listed in table 2 together with the spin-quadrupole transitions. The calculated magnetic moment in $^{14}\text{N}$ well reproduces the observed one, while the agreement is worse in $^{56}\text{Cu}$. This is due to the fact that the core of $^{56}\text{Ni}$ might be largely broken and the $f_{7/2}$ hole configuration is mixed in the ground state of $^{58}\text{Cu}$ [80, 81]. The values for $B(M1)$ are also shown in figure 14. Very strong $B(M1)$ values are found both experimentally and theoretically in two of the $N = Z$ nuclei in table 2, that is, in $^{18}\text{F}$ and $^{42}\text{Sc}$.

The $B(M1)$ transition from $0^+ \rightarrow 1^+$ in $^{18}\text{F}$ is the largest one so far observed in the entire region of the nuclear chart. We notice that our three-body calculations provide fine agreement not only for these strong transitions in $^{18}\text{F}$ and $^{42}\text{Sc}$, but also for the quenched transitions in the other $N = Z$ nuclei such as in $^{14}\text{N}$ and $^{34}\text{Cl}$. The shell-model calculation of [82] also shows a large enhancement for the $B(M1)$ transition in $^{18}\text{F}$ which is consistent with both the present study and the experiment.

In the case of $^{18}\text{F}$, the $0^+$ and $1^+$ states are largely dominated by the $S = 0$ and $S = 1$ spin components, respectively, with orbital angular momentum $l = 2$ (see table 1). Therefore, the two states can be considered as members of the SU(4) multiplet in the spin-isospin space [83]. This is the main reason why the $B(M1)$ value is so large in this nucleus, since the spin-isospin operator $g_s V_s \tau \pi$, connects two states in the same SU(4) multiplet, that is, the transition is allowed, and the isovector g-factor is the dominant term in equation (39) with $g_s V_s = (g_s (\nu) - g_s (\pi))/2 = -4.70$. The configurations in $^{42}\text{Sc}$ are also similar to those in $^{18}\text{F}$ in terms of SU(4) multiplets, although they are dominated by $l = 3$ wave functions. For $^{14}\text{N}$ and $^{34}\text{Cl}$, the $B(M1)$ transitions do not acquire any enhancement, since the $S = 0$ component in the $0^+$ state is suppressed due to the $j = l - 1/2$ coupling: both the $0^+$ and $1^+$ states have very large $1p_{3/2}^2 (1d_{5/2})^2$ configurations in $^{14}\text{N}$ and $^{34}\text{Cl}$. These indications for the SU(4) symmetry in $^{18}\text{F}$ and $^{42}\text{Sc}$ are consistent with the results obtained in [84–86].

In the nuclei $^{30}\text{P}$ and $^{58}\text{Cu}$, the $1^+$ state is dominated by $1d_{3/2}^2 2s_{1/2}$ and $2p_{3/2}^2 1f_{5/2}$ configurations, respectively, while the $0^+$ state is governed by the $2s_{1/2}^2 2p_{3/2}^2$ configurations, respectively. Therefore, the isovector spin-quadrupole transitions are largely enhanced in the two nuclei even though the $B(M1)$ is much more quenched. The validity of SU(4) symmetry has been known already for a quite long time in $p$-shell shell nuclei [87]. The two-body matrix element of Cohen-Kurath [88] for the spin-triplet $(J, T) = (1, 0)$ interaction is certainly stronger than that for the spin-singlet $(J, T) = (0, 1)$ pairing interaction. Then, the structure of the two-body wave function will rather be described by the LS coupling scheme than the $jj$ coupling scheme.

### 3.5. Gamow–Teller transitions in $N \approx Z$ nuclei by the three-body model

We have applied the same three-body model as described in subsection 3.4 to calculate the GT strength of $N = Z + 2$ nuclei. The GT transition strength reads

$$B(GT; 0^+ \rightarrow 1^+) = \frac{g_A^2}{4\pi} \left| \langle 1^+ | \sum_i L_{ij}(i) \sigma(i) | 0^+ \rangle \right|^2,$$

where $g_A$ is the axial-vector coupling constant. The results are summarized in table 3. One can again see a strong GT transition between the lowest $0^+$ and $1^+$ states in $A = 18$ and 42 systems, which exhausting a large portion of the GT sum rule value. This can also be interpreted as a manifestation of SU(4) symmetry in the wave functions of these nuclei. We also note that the result obtained in [84] by an analysis of GT transitions also implies a good SU(4) symmetry in the $A = 18$ system. On the other hand, for $^{58}\text{Cu}$, the GT strength is largely fragmented and no state with a strong $B(GT)$ is seen near the ground state. The experimental data are consistent with the calculated results as can be seen in table 3. The ratio of $B(GT)$ values from the ground states to the lowest $1^+$ states, between $A = 18$, 42 and 58, is $1:0.69:0.05$ if one takes the experimental values while the calculated ratio is $1.071:0.04$ and is very close to the experimental one. This agreement suggests the validity of the three-body model wave functions in these nuclei.

Extensive shell-model calculations have been performed in the full $p$-shell, $sd$-shell and $pf$-shell model spaces in the literature (see [82, 88] and [81], respectively). In these studies, the magnetic moments, M1 transitions and GT transitions were studied and the calculated results reproduce the

| $^{18}\text{O} \rightarrow ^{18}\text{F}$ | | |
|---|---|---|
| $E_\alpha$ (MeV) | $B(GT)$ ($g_A^2/4\pi$) | |
| Th. Exp. | Th. Exp. | |
| 0.0 | 0.0 | 2.48 | 3.11 ± 0.03 |
| 4.79 | 0.028 | |
| 6.87 | 0.036 | |

| $^{42}\text{Ca} \rightarrow ^{42}\text{Sc}$ | | |
|---|---|---|
| $E_\alpha$ (MeV) | $B(GT)$ ($g_A^2/4\pi$) | |
| Th. Exp. | Th. Exp. | |
| 0.61 | 0.61 | 1.80 | 2.16 ± 0.15 |
| 1.89 | 0.09 | |
| 3.71 | 3.69 | 0.346 | 0.15 ± 0.03 |

| $^{58}\text{Ni} \rightarrow ^{58}\text{Cu}$ | | |
|---|---|---|
| $E_\alpha$ (MeV) | $B(GT)$ ($g_A^2/4\pi$) | |
| Th. Exp. | Th. Exp. | |
| 0.0 | 0.0 | 0.097 | 0.155 ± 0.01 |
| 1.24 | 1.05 | 0.74 | 0.30 ± 0.04 |
experimental observations well. The validity of SU(4) symmetry in GT decays was also studied in terms of shell-model calculations of p-shell nuclei in [92]. In contrast, our aim in this paper is not to compete with these complete shell-model calculations, but to extract the role of the spin-singlet and the spin-triplet pairing interactions for the ground states and the excited states in the odd-odd $N = Z$ nuclei by the three-body model with one set of input data for the entire mass region, and thus to explore the validity of SU(4) symmetry in the spin-isospin space simple terms. We should note that the present model is quite appropriate for $^{15}$F and $^{42}$Sc since $^{16}$O and $^{46}$Ca are double closed-shell nuclei and can be considered as good cores. On the other hand, the model space of the three-body model is not quite large enough for $^{39}$P, $^{34}$Cl and $^{58}$Cu since excited states of the core might be coupled to the configurations of the present model space.

For $^{14}$N, it might be more appropriate to adopt the two-hole three-body model since $^{16}$O is a better core than $^{12}$C. With the density-dependent, surface-type pairing ($\alpha = \beta = +1$ in equation (35)), the two-hole three-body model gives the energy difference of about 2 MeV between the $0^+$ and $1^+$ states which is close to the experimental observation. However, the magnetic dipole transition becomes much larger than the observed one. It can be pointed out that the mixing of $sd$-shell components will play an important role for the quenching of B(M1) and B(GT) values.

### 3.6. Pair transfer reactions

As is intuitive and well known, two-particle transfer reactions are sensitive to the correlations between those particles, so that two-neutron transfer has, for a long time, been used to pin down fingerprints of $T = 1$ pairing, and recently there is a strong interest in understanding to what extent deuteron transfer can probe $T = 0$ pairing.

There are some basic problems, however, both at the conceptual and experimental level. Let us assume we can restrict ourselves to $L = 0$ states excited in transitions between even-even nuclei. Normal QRPA, or shell-model calculations, can provide the wave functions of such excited states $n$; in particular, one can calculate the strength functions $S(E)$ associated either with the pair addition (ad) or pair removal (rm) operators. These strength functions read

$$S_{\text{ad}} = \sum_{n \in A + 2} |\langle n | \hat{P} | 0 \rangle|^2 \delta (E - E_n),$$  \hspace{1cm} (42a)

$$S_{\text{rm}} = \sum_{n \in A - 2} |\langle n | \hat{P} | 0 \rangle|^2 \delta (E - E_n),$$  \hspace{1cm} (42b)

where the sums run over the appropriate set of final states in the $A + 2$ or $A - 2$ nuclei, and $|0\rangle$ is the ground state of the even-even nucleus under study. The pair-addition operators are

$$\hat{P}_{T=1, S=0} \langle T=0, S=1 \rangle = \frac{1}{2} \sum_{\sigma, \sigma', \tau} \int d^3 r \ \psi^* (\hat{r} \sigma \tau) \langle \sigma | \tau | \sigma' \tau' \rangle \psi (\hat{r} \sigma' \tau'),$$  \hspace{1cm} (43)

where $\psi^* (\hat{r} \sigma \tau)$ is the nucleon field operator, and the time-conjugate state is defined by $\psi^* (\hat{r} \sigma \tau) \equiv (-2 \sigma \hat{r}) (-2 \tau) \psi^* (\hat{r} \sigma \tau)$. We have specifically singled out the $S, T$-dependence. One can consider the neutron–neutron case [93] where only $J^\pi = 0^+$, $T = 1, S = 0$ is possible if $L = 0$, or the neutron–proton case [94], where either $J^\pi = 0^+, T = 1, S = 0$ or $J^\pi = 1^+, T = 0, S = 1$ are possible.

The strength functions (42) are characterized by collective modes like pair vibrations and rotations [3]. A so-called giant pairing vibration (GPV) was predicted long ago [95], and yet so far it has not been unambiguously identified despite many efforts carried out since the 1970s until very recently (see [96] and references therein). One of the reasons advocated for this is that GPVs are expected to lie at high energies, and the semi-classical condition of the optimal $Q$-value favors ground-state to ground-state transitions in reactions with stable beams and targets (see [97] and references therein).

Interest has recently switched to the behavior of such excitations far from stability, and to the issue of whether this can provide information on pairing in neutron-rich nuclei. The strength functions (42) have been calculated for the Sn isotopes in [93]. It has been found that ground-state to ground-state transitions have large pair-addition strength in $^{110}$–$^{130}$Sn. In $^{132}$–$^{140}$Sn, characteristic pair vibrational modes appear whose strength is slightly smaller than that of the ground-state transitions, but still large enough in terms of matrix elements associated with single-particle pair transfer. In these nuclei, which are considerably neutron-rich, such vibrational modes have an extended radial tail, associated with contributions from continuum transitions. A signature of this fact resides in the participation of high angular momentum states with $l > 5$ in these modes. Pairing rotational modes appear instead in more neutron-rich nuclei. These states, and especially their collectivity, are very sensitive to the kind of pairing interaction that is employed like volume pairing, as in (7), or surface pairing, as in (12).

However, the pairing addition or pairing removal strength is not directly observable. The measurable quantity, namely the two-particle transfer cross-section is not simply a factorized product of such matrix-element-time kinematical factors since the reaction process is quite involved. The full microscopic theory of the reaction process is explained in textbooks [98] and review papers [99]. Sophisticated calculations of absolute cross-sections have been published starting...
from the 1960s (see, e.g. [100] for a compact résumé and a survey of references) up to the very recent state-of-the-art scheme of [101]. We do not wish to give a full account of these calculations here, but simply to convey the main points which are also connected to similar difficulties that one may expect in the case of $T = 0$ pairing.

In a distorted-wave Born approximation (DWBA) picture, the reaction cross-section associated with $A + a \rightarrow B + b$ will be proportional to the square of the transition amplitude. At variance with the inelastic case, the nucleons are in different partitions in the initial $Aa$ and final $Bb$ channels. One could define a natural coordinate system in each of these two initial and final channels, and in this way identify the two relative coordinates $\mathbf{r}_a$ and $\mathbf{r}_b$ in such channels (see, e.g. [102, 103] for details and figure 15 for illustration). The projectile-target interaction can be defined either in the initial or final channel (where the words 'ejectile--residual' might be more appropriate), and one refers to the two choices as prior or post representations, respectively. In case of a finite-range interaction $V$ such features bring in many complications associated with non-localities. Neglecting them, that is, approximating $\mathbf{r}_a \approx \bar{\mathbf{r}}_a \approx \bar{\mathbf{R}}$ as in equation (17) of [99], the transition amplitude reads

$$T_{A+a\rightarrow B+b} \approx \int d^3R \chi^{\dagger}_{\mathbf{R}a}(\bar{\mathbf{r}}_a, \bar{\mathbf{K}}_f) F(\bar{\mathbf{R}}) \chi_{Aa}(\mathbf{r}_a, \mathbf{K}_i),$$

(44)

where the $\chi$ are distorted wave functions that carry appropriate momentum labels, while $F$ is the reaction form factor,

$$F(\bar{\mathbf{R}}) \approx \langle \Psi_b | V | \Psi_a \rangle,$$

(45)

which is written in terms of intrinsic wave functions $\Psi$. We remember that the differential cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right) = |T_{A+a\rightarrow B+b}|^2.$$

(46)

What appears clearly, then, is that while the matrix elements that appear in equation (44) involve a full integration over space, in the form factor (45) the effective interaction $V$ is active only in the range allowed by the reaction mechanism and acts as a kind of filter that makes the connection between pairing correlations and reaction cross-sections quite indirect.

In such a complicated situation, the comparison of the latest theoretical calculations of [101] with the experimental data from [104] have nonetheless provided strong evidence that our current understanding of $T = 1$ pairing is confirmed by the analysis of the results of ($p$, $t$) transfer reactions on the stable Sn isotopes. On the other hand, reaction cross-section calculations have not been performed for neutron-rich and weakly bound isotopes, e.g. beyond $^{130}$Sn. At present, such highly neutron-rich nuclei are not yet available in such intensities that allow two-particle transfer reaction experiments, but the topic is certainly of interest for the future.

It is also quite obvious to analyze whether neutron–proton transfer reactions can play a similar role to that discussed so far, to pin down better evidence for $T = 0$ pairing. The issue has been discussed recently in [28]. One clear physics case would be to perform a reaction like $(^3$He, $p)$ on an even-even $N = Z$ nucleus. $N = Z$ nuclei are stable only up to $^{40}$Ca. As the pairing collectivity is expected to be more pronounced for heavier systems, experimental programs for $(^3$He, $p)$ or $(^4$He, $d)$ reactions with unstable beams in inverse kinematics are strongly desired and called for. In such nuclei the $T = 0$ pairing correlations are expected to be enhanced, as already discussed. Moreover, in this case one starts from a $T = 0$, $J^e = 0^+$ state and can probe the isospin invariance of $T = 1$ pairing by looking at the $T = 1, J^e = 0^+$ states in the odd-odd system and investigate $T = 0$ pairing by looking at the $T = 0, J^e = 1^+$ states.

A full theory like the one developed for equal-particle transfer reactions, that is, capable of predicting absolute values of the cross-section (as we have just discussed) still need to be developed. This is one of the important priorities in the nuclear reaction domain. Meanwhile, if experimentally available, relative cross-sections $\sigma(T = 0, J = 1^+) / \sigma(T = 1, J = 0^+)$ can first provide valuable information. It has to be noted that in such cross-sections, strong interaction matrix elements that are
The proton–neutron \( L = 0 \) pair transfer strength in the \( N = Z \) nuclei of \( ^{40}\text{Ca} \) and \( ^{56}\text{Ni} \) has been studied by Yoshida in [94] by using the proton–neutron RPA described in section 3.2.1 with a Skyrme energy functional. The pair-addition strength of \( ^{40}\text{Ca} \rightarrow ^{42}\text{Sc} \) associated with both the \( J^p = 1^+ \) and \( J^p = 0^+ \) states is shown in figure 16. It was found that the collectivity of the lowest \( J^p = 1^+ \) in the neighboring odd-odd nucleus for \( ^{42}\text{Sc} \) is stronger than that of the lowest \( J^p = 0^+ \) state when the IS spin-triplet pairing is taken to be equal to or stronger than the IS spin-singlet pairing.

One can see that the excitation energy and the strength of the \( J^p = 1^+ \) states are strongly affected by the \( T = 0 \) pairing interaction. In the case of \( f = 0 \), without the \( T = 0 \) pairing interaction, the lowest \( 1^+ \) state in \( ^{42}\text{Sc} \) located at \( \omega = 7.5 \text{ MeV} \) is a single-particle excitation \( \pi f_{1/2} \) located on \( \omega = 7.5 \text{ MeV} \). As the pairing interaction is switched on, and the strength is increased, the \( 1^+ \) state is shifted downwards in energy with the enhancement of the transition strength. By increasing pairing strength to \( f = 1.0 \) or 1.3, the lowest \( 1^+ \) state is constructed by many particle-particle excitations involving \( f_{1/2} \) and \( p_{3/2} \) orbitals located above the Fermi level as well as the \( \pi f_{1/2} \) excitation. It is particularly worth noting that the hole-hole excitations from the \( 5\text{d} \)-shell Earth made an appreciable contribution to the generation of this \( T = 0 \) proton–neutron pair-addition vibrational mode, indicating \( ^{40}\text{Ca} \) core-breaking. The strong collectivity is associated by a coherent phase of these configurations of \( pp \) and \( hh \) excitations.

4. Summary and future perspectives

The superfluidity in nuclei was firstly pointed out in a milestone paper by Bohr, Mottelson and Pines in 1958. This historical paper has had a strong impact on the study of nuclear superfluidity, both from experimental and theoretical viewpoints. These studies have established solid evidence of superfluidity in nuclei, such as odd-even staggering in mass systematics, the large energy gaps in the spectra of even-even nuclei, the quenching of the moment of inertia associated with rotational spectra of rare-Earth deformed nuclei, and the enhancement of fission of actinide nuclei. Theoretically, the HF plus BCS and HFB theories have been developed and successfully applied to calculate the effect of pairing interactions on these phenomena. It is well known that one can also go beyond the mean-field HF-BCS or HFB calculations by performing particle number projection.

It has also been pointed out that the pairing correlations will be much stronger in the lower density regime than at normal density. Nuclear matter calculations show the onset of the BCS-BEC crossover phenomenon in the low-density regime, i.e. the spatial correlation of the Cooper pair changes from a BCS-type behavior with a large coherence length at normal density to a BEC-type behavior with a compact coherence length. It was also pointed out that the weakly bound nuclei with halo or skin nature may show strong di-nucleon correlations which give rise to a similar behavior as in the BCS-BEC crossover when one studies the nucleon pairs as a function of the distance between the core and the center of the two nucleons.

So far, most of the studies of the pairing correlations have been concentrated on the isovector \( T = 1 \) spin-singlet pairing interaction. On the other hand, the isoscalar \( T = 0 \) spin-triplet pairing correlations could be much stronger than the \( T = 1 \) pairing ones. The importance of \( T = 0 \) spin-triplet pairing was already pointed out in the 1970s. There have been many discussions on possible signatures of the spin-triplet pairing correlations, such as the Wigner energy term in the mass formula, the enhancement of neutron–proton pair transfer cross-sections, the inversion of \( J = 0^+ \) and \( J = 1^+ \) states, and the large enhancement of GT strength in the low-energy region of \( N = Z \) and \( N = Z + 2 \) nuclei. However, so far no convincing evidence has been found in relation to the spin-triplet superfluidity in nuclei. This may be due to the large spin–orbit splittings in the nuclear mean field and the large neutron excess \( N > Z \) in stable nuclei with \( A > 40 \), which prevent the formation of spin-triplet Cooper pairs in nuclei.

In our paper, we have pointed out that the inversions of \( J = 0^+ \) and \( J = 1^+ \) states in the odd-odd \( N = Z \) nuclei can be considered as a manifestation of strong spin-triplet pairing correlations. These nuclei also display strong magnetic dipole transitions that can be explained by the introduction of \( T = 0 \) pairing. Another kind of spin-dependent transition is the Gamow--Teller excitation from even-even \( N = Z + 2 \) to the neighboring odd-odd ones. We have shown that the strong concentration of GT strength at low energy, close to the ground state of daughter nuclei, can be understood by QRPA calculations only with a strong \( T = 0 \) spin-triplet pairing interaction. In all these cases, the strength of the \( T = 0 \) pairing is larger, or slightly larger, than the strength of the \( T = 1 \) pairing, in agreement with similar conclusions extracted from the analysis of the shell-model matrix elements.

Direct evidence of the strong neutron–proton pairing correlations may appear in the enhancement of the pair transfer cross-sections. Several theoretical studies show clear signs of the strong pairing correlations in the two-nucleon pairing strength. However, the reaction process of two-nucleon transfer reactions is very involved. While some calculations have been performed in the case of two-neutron transfer, and are consistent with the assumptions on \( T = 1 \) pairing coming from other pieces of evidence, the study of proton–neutron transfer reactions is still in its infancy and more effort must be focused on this topic.

The spin-triplet superfluidity is a fascinating subject for theoretical and experimental study. To this end, theories should be capable of performing particle-number projection on top of HF-BCS or HFB including \( T = 0 \) pairing in addition to \( T = 1 \) pairing. Recently, isoscalar and isovector pairing were treated in a model of alpha-like quartets which are made up of two protons and two neutrons with \( J = 0 \) and \( T = 0 \) [105]. It was pointed out that the exact treatment of particle number and isospin projections significantly changes the pairing correlations in the ground states of \( N = Z \) nuclei,
so that they show the coexistence phase of the isoscalar and isovector pairing correlations.

In the future, it would also be highly interesting to build a realistic model able to predict the two-particle cross-sections theoretically, and point to possible spin-triplet pairing modes in the pair transfer reactions such as $^{3}$He, p) or (p, $^{3}$He).

A large-amplitude collective motion such as fission is also very sensitive to whether the nucleus is a viscous fluid or a superfluid. Qualitatively, it is definitely true that superfluidity enhances the fission probability significantly. The fission induced by low-energy excitations which occur in neutron capture reactions are now feasibly studied by microscopic theories like time-dependent HFB. These theoretical studies can provide observables such as the internal energies of fission fragments which can be compared with experimental data. We expect that this may also be instrumental for our understanding of pairing in nuclei.

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Appendix A. Two-body matrix elements of $\delta$–type and separable pairing interactions

A.1. $\delta$–type pairing interaction

We adopt the helicity–Gordon coefficients [49, 54]. In the helicity representation, the single-particle wave function reads

$$\psi_{nljm} = R_{nl}(r) \left( \frac{2j + 1}{16\pi^2} \right)^{1/2} \sum_{h=\pm 1/2} \alpha(jh) D_{nm}^j(r) \chi_h,$$

where $R_{nl}(r)$, $D_{nm}^j$ and $\chi_h$ are the radial wave function, $D$–function and the spinor wave function in the helicity function, respectively, and

$$\alpha(jh) = (-)^{j-1/2}(j-1/2).$$

The two-particle wave function can be written for the contact interaction $\delta(\vec{r}_1 - \vec{r}_2)$ with $\vec{r}_1 = \vec{r}_2$ as

$$\Psi(\alpha\beta)_{JM} = \frac{1}{\sqrt{1 + \delta_{\alpha,\beta}}} \sum_{h_1h_2J} \langle j_h h_{\alpha_1} \alpha_{\beta_1} | J | H \rangle D_{mh_1}^j(\vec{r}) \times R_{\alpha}(r) R_{\beta}(r) \frac{\hat{\delta}_{\alpha_1\beta_1}}{16\pi^2} \alpha(l_{\alpha_1} h_{\alpha_1}) \alpha(l_{\beta_1} h_{\beta_1}) \chi_{h_1} \chi_{h_2},$$

where $R_{\alpha} \equiv R(a_{\alpha} k_{\alpha})$ and $\hat{\delta} = (2j + 1)^{1/2}$. To derive equation (49) we use the addition and the orthogonality relations of the $D$–functions [see (49) for details]. For the total spin $S = 0$ state, the helicity $H$ is restricted to be $H = 0$ only. By using the helicity representation of the two-particle state, the two-body matrix element for the contact pairing for the isovector spin-singlet channel $\nu^{T=0, S=1}(r) = G^{T=0}f(r) \delta(r)$ is calculated to be

$$\langle \Psi(\alpha'\beta')_{JH} | \nu^{T=1, S=0} \rangle = \frac{1}{\sqrt{1 + \delta_{\alpha,\beta}}} \left( \frac{1}{\sqrt{1 + \delta_{\alpha',\beta'}}} \right) \langle j_{\alpha'} h_{\alpha'} h_{\beta'} | J | \hat{\gamma} \rangle \times \langle j_{\alpha} h_{\alpha} h_{\beta} | J | \hat{\gamma} \rangle \times I^{T=1},$$

where $I$ is the radial integral

$$I^{T=1} = \int d\vec{r}^2 R_{\alpha}(r) R_{\beta}(r) G^{T=1}(\rho) f(\rho).$$

Here, the $T = 1$ pairing strength is $G^{T=1}$ and the density-dependent form factor is written as $f(\rho)$. For the anti-symmetrized state $\Psi$, the two-body matrix elements will be

$$\langle \Psi(\alpha'\beta')_{JH} | \nu^{T=1, S=0} \rangle = \langle \Psi(\alpha'\beta')_{JH} | \nu^{T=1, S=0} \rangle = \langle \Psi(\alpha'\beta')_{JH} | \nu^{T=1, S=0} \rangle$$

$$\times \left( \frac{1}{\sqrt{1 + \delta_{\alpha,\beta}}} \right) \times \langle j_{\alpha} h_{\alpha} h_{\beta} | J | \hat{\gamma} \rangle \times \langle j_{\alpha} h_{\alpha} h_{\beta} | J | \hat{\gamma} \rangle \times I^{T=1}.$$

For the spin-triplet case, we have to calculate not only the $H = 0$ state, but also $H = \pm 1$ states. The $H = 0$ state gives the same two-body matrix element as the spin-singlet case (51) except for the $J$ selection term which will be $[1 \pm (-)^{j_{\alpha} + j_{\beta} + J}]$ in the $S = 1$ case. We will present how to calculate the matrix element for the $H = -1$ state in the following. The matrix element for $H = 1$ is essentially the same as that of the $H = -1$ state. The two-particle state for the total helicity $H = -1$ reads

$$\Psi(\alpha\beta)_{JM} = \frac{1}{\sqrt{1 + \delta_{\alpha,\beta}}} \left( \frac{1}{\sqrt{1 + \delta_{\alpha,\beta}}} \right) \langle j_{\alpha} - 1/2 j_{\beta} - 1/2 | J - 1 \rangle \times D_{M=-1}(\vec{r}) R_{\alpha}(r) R_{\beta}(r) \frac{\hat{\delta}_{\alpha_1\beta_1}}{16\pi^2} \chi_{-1/2} \chi_{-1/2},$$

where $\alpha(jh)$ in equation (49) becomes always $+1$ for the $h = 1/2$ state. The two-body matrix element of the $S = 1, H = -1$ state for the isoscalar spin-triplet interaction $\nu^{T=0, S=1}(\rho) = G^{T=0}f(\rho) \delta(\rho)$ will be

$$\langle \Psi(\alpha'\beta')_{H=-1} | \nu^{T=0, S=1} \rangle = \langle \Psi(\alpha'\beta')_{H=-1} | \nu^{T=0, S=1} \rangle$$

$$\times \left( \frac{1}{\sqrt{1 + \delta_{\alpha,\beta}}} \right) \times \langle j_{\alpha} - 1/2 j_{\beta} - 1/2 | J - 1 \rangle \times D_{M=-1}(\vec{r}) R_{\alpha}(r) R_{\beta}(r) \frac{\hat{\delta}_{\alpha_1\beta_1}}{16\pi^2} \chi_{-1/2} \chi_{-1/2},$$

For details [see (49) for details]. For the total spin $S = 0$ state, the helicity $H$ is restricted to be $H = 0$ only. By using the helicity representation of the two-particle state, the two-body matrix element for the contact pairing for the isovector spin-singlet channel $\nu^{T=0, S=1}(\rho) = G^{T=0}f(\rho) \delta(\rho)$
where the radial integral $I^{(T=0)}$ has the pairing strength $G^{(T=0)}$ instead of $G^{(T=1)}$ in equation (50). This formula (53) can be rewritten further by a recursion formula

$$
\langle j_0 - 1/2 | j_b - 1/2 | J - 1 \rangle = \langle j_0 - 1/2 | j_b - 1/2 | J | 0 \rangle
\times (-)^{l+1/2} \langle j_a + 1/2 | j_a + 1/2 | J | 0 \rangle
\times \frac{1}{[J(J+1)]^{1/2}}.
$$

(53)

Then, the two-body matrix element becomes

$$
\langle \Psi(\alpha\beta) | H_{H=1} | \Psi(\alpha\beta) \rangle = \frac{1}{\sqrt{1 + \delta_{\alpha\beta}}} \sqrt{1 + \delta_{\alpha',\beta'}} \langle j_a - 1/2 | j_b - 1/2 | J | 0 \rangle
\times (-)^{l+1/2} \langle j_a + 1/2 | j_a + 1/2 | J | 0 \rangle
\times \frac{1}{[J(J+1)]^{1/2}} \times \frac{1}{16\pi (2J+1)}
\times (-)^{l+1/2} \langle j_b, j_a | J, j_a | 0 \rangle.
$$

(54)

We have to sum up three terms $H = 0$, $H = \pm 1$ for the matrix element of the $S = 1$ state. We are reminded that the $H = +1$ state gives the same matrix element as that of the $H = -1$ state. Eventually, for the anti-symmetrized $S = 1$ state, the two-body matrix element is given by

$$
\langle \hat{\Psi}(\alpha'\beta') | H_{H=0} | \hat{\Psi}(\alpha\beta) \rangle = \frac{1}{\sqrt{1 + \delta_{\alpha\beta}}} \sqrt{1 + \delta_{\alpha',\beta'}} \langle j_a - 1/2 | j_b - 1/2 | J | 0 \rangle
\times \frac{1}{[J(J+1)]^{1/2}} \times \frac{1}{8\pi (2J+1)}
\times (-)^{l+1/2} \langle j_b, j_a | J, j_a | 0 \rangle.
$$

(55)

A.2. Separable pairing interaction

Next, let us discuss the separable pairing interaction, i.e. the spin-singlet $T = 1$ pairing interaction (17) and the spin-triplet $T = 0$ pairing interaction (20). The two-body matrix element for the $T = 1$ pairing is evaluated to be

$$
\langle (j_j j_b) | T = 1, J = 0 \rangle | (j_j j_b) | T = 1, J = 0 \rangle = -\sqrt{J(j_j + 1/2)(j_b + 1/2)} G^{(T=1)} I_{j_j j_b},
$$

(55)

where $I_{j_j j_b}$ is the overlap integral given by

$$
I_{j_j j_b} = \int \psi_j^*(\mathbf{r}) \psi_b(\mathbf{r}) d\mathbf{r}.
$$

(56)

# Table 4. The transformation coefficient $R$ between the $jj$ coupling and the LS coupling for the pair wave functions,

$$
R = \left\{ \begin{array}{cc}
(\frac{l_j}{2} \frac{l_b}{2}) & (l_j + 1/l_j) \frac{(1/2)}{(1/2)} \frac{1}{(l_j + 1)} \frac{1}{(l_b + 1)}
\end{array} \right\}^{T-1} \Omega
$$

\[ \Omega \equiv 3(2l+1)^2. \]

For the $T = 0$ pairing, the two-body matrix element involves the coefficient for the transformation from the $jj$ coupling scheme to the LS coupling scheme, and is given by

$$
\langle (j_j j_b) | T = 0, J = 1 \rangle = \left\{ \begin{array}{cc}
(\frac{l_j}{2} \frac{l_b}{2}) & (l_j + 1/l_j) \frac{(1/2)}{(1/2)} \frac{1}{(l_j + 1)} \frac{1}{(l_b + 1)}
\end{array} \right\}^{T-1} \Omega
$$

References

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