Spin$^T$ structure and Dirac operator on Riemannian manifolds

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Abstract

In this paper, we describe the group Spin$^T(n)$ and give some properties of this group. We construct Spin$^T$ spinor bundle $\mathcal{S}$ by means of the spinor representation of the group Spin$^T(n)$ and define covariant derivative operator and Dirac operator on $\mathcal{S}$. Finally, Schrödinger-Lichnerowicz-type formula is derived by using these operators.

Key Words Spinor bundle, the group Spin$^T(n)$, Dirac operator, Schrödinger-Lichnerowicz-type formula.

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1 Introduction

Spin and Spin$^c$ structures is effective tool to study the geometry and topology of manifolds, especially in dimension four. Spin and Spin$^c$ manifolds have been studied extensively in \cite{2,3,4,5}. For any compact Lie group $G$ the Spin$^G$ structure have been studied in \cite{1}. However, the spinor representation is replaced by a hyperkahler manifold, also called target manifold. In this paper, we define the Lie group Spin$^T(n)$ as a quotient group by taking $G = S^1 \times S^1$. The groups Spin$(n)$ and Spin$^c(n)$ are the subset of Spin$^T(n)$. We define Spin$^T$ structure on any Riemannian manifold. The spinor representation of Spin$^T(n)$ is defined by the help of the spinor representation of Spin$(n)$. By using the spinor representation of Spin$^T(n)$ we construct the Spin$^T$ spinor bundle $\mathcal{S}$. Finally, we give Schrödinger-Lichnerowicz-type formula by using covariant derivative operator and Dirac operator on $\mathcal{S}$.

This paper is organized as follows. We begin with a section introducing the group Spin$^T(n)$. In the following section, we define Spin$^T$ structure on any Riemannian manifold. The final section is dedicated to the construction of
the spinor bundle $S$, the study of the Dirac operator associated to Levi-Civita connection $\nabla$ and Schrödinger-Lichnerowicz-type formula.

## 2 The group $\text{Spin}^T(n)$

**Definition 1** The $\text{Spin}^T$ group is defined as

$$\text{Spin}^T(n) := (\text{Spin}(n) \times S^1 \times S^1)/\{\pm 1\}.$$ 

The elements of $\text{Spin}^T(n)$ are thus classes $[g, z_1, z_2]$ of pairs $(g, z_1, z_2) \in \text{Spin}(n) \times S^1 \times S^1$ under the equivalence relation

$$(g, z_1, z_2) \sim (-g, -z_1, -z_2).$$

We can define the following homomorphisms:

a. The map $\lambda^T : \text{Spin}^T(n) \longrightarrow \text{SO}(n)$ is given by $\lambda^T([g, z_1, z_2]) = \lambda(g)$ where the map $\lambda : \text{Spin}(n) \rightarrow \text{SO}(n)$ is the two-fold covering given by $\lambda(g)(v) = gvg^{-1}$.

b. $i : \text{Spin}(n) \longrightarrow \text{Spin}^T(n)$ is the natural inclusion map $i(g) = [g, 1, 1]$.

c. $j : S^1 \times S^1 \longrightarrow \text{Spin}^T(n)$ is the inclusion map $j(z_1, z_2) = [1, z_1, z_2]$.

d. $l : \text{Spin}^T(n) \longrightarrow S^1 \times S^1$ is given by $l([g, z_1, z_2]) = (z_1^2, z_1z_2)$.

e. $p : \text{Spin}^T(n) \longrightarrow \text{SO}(n) \times S^1 \times S^1$ is given by $p([g, z_1, z_2]) = (\lambda(g), z_1^2, z_1z_2)$.

Hence, $p = \lambda^T \times l$. Here $p$ is a 2-fold covering.

Thus, we obtain the following commutative diagram where the row and the column are exact.

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & \text{Spin}^T(n) & \rightarrow & \text{SO}(n) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 \times S^1 & \rightarrow & \text{Spin}(n) & \rightarrow & S^1 \times S^1 & \rightarrow & 1 \\
\downarrow & & \downarrow_j & & \downarrow_i & & \downarrow_l & & \downarrow_{\lambda^T} \\
1 & \rightarrow & 1 & \rightarrow & \text{Spin}(n) & \rightarrow & S^1 \times S^1 & \rightarrow & 1 \\
\end{array}
\]

Moreover, we have the following exact sequence:

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}^T(n) \rightarrow \text{SO}(n) \times S^1 \times S^1 \rightarrow 1.$$
Theorem 2  The group Spin⁰(n) is isomorphic to Spin⁺(n) × S¹.

Proof We define the map ϕ in the following way:

\[
\begin{array}{ccc}
\text{Spin}(n) \times S¹ \times S¹ & \xrightarrow{\varphi} & \text{Spin}⁺(n) \times S¹ \\
(g, z₁, z₂) & \mapsto & ([g, z₁], z₁z₂)
\end{array}
\]

It can be easily shown that ϕ is a surjective homomorphism and the kernel of ϕ is \{(1, 1, 1), (−1, −1, −1)\}. Thus, the group Spin⁰(n) is isomorphic to Spin⁺(n) × S¹. □

Since Spin(n) is contained in the complex Clifford algebra \(\mathbb{C}l_n\), the spin representation κ of the group Spin(n) extends to a Spin⁰(n)-representation. For an element \([g, z₁, z₂] \in \text{Spin}(n)\) and any spinor \(ψ \in \Delta_n\), the spinor representation \(κ^T\) of Spin⁰(n) is given by

\[
κ^T[g, z₁, z₂]ψ = z₁π₂z₂κ(g)(ψ).
\]

Proposition 3 If \(n = 2k + 1\) is odd, then \(κ^T\) is irreducible.

Proof Assume that \(\{0\} \neq W \neq \Delta_{2k+1}\) is a Spin⁰ invariant subspace. Thus, we have \(κ^T[g, z₁, z₂](W) \subseteq W\). That is, \(z₁π₂z₂κ(g)(W) \subseteq W\). In this case, for every \(w \in W\) there exists a \(w' \in W\) such that \(z₁π₂z₂κ(g)(w) = w'\). As \(κ(g)(w) = \frac{1}{z₁π₂z₂}w' \in W\) and the representation \(κ\) of Spin(n) is irreducible if \(n\) is odd, this is a contradiction. The representation \(κ^T\) of Spin⁰(n) has to be irreducible for \(n = 2k + 1\).

□

Proposition 4 If \(n = 2k\) is even, then the spinor space \(\Delta_{2k}\) decomposes into two subspaces \(\Delta_{2k}^+ \oplus \Delta_{2k}^−\).

Proof We know that the Spin(n) representation \(\Delta_{2k}\) decomposes into two subspaces \(\Delta_{2k}^+ \oplus \Delta_{2k}^−\). Thus, we obtain \(z₁π₂z₂κ(g)(\Delta_{2k}^+) \subseteq \Delta_{2k}^+\) and \(z₁π₂z₂κ(g)(\Delta_{2k}^-) \subseteq \Delta_{2k}^−\). Namely, \(κ^T[g, z₁, z₂](\Delta_{2k}^+) \subseteq \Delta_{2k}^+\) and \(κ^T[g, z₁, z₂](\Delta_{2k}^-) \subseteq \Delta_{2k}^−\). Hence, the Spin⁰(2k) representation \(\Delta_{2k}\) decomposes into two subspaces \(\Delta_{2k}^+ \oplus \Delta_{2k}^−\).

It can be easily seen that the Spin⁰(2k) representation \(\Delta_{2k}^±\) is irreducible. □

The Lie algebra of the group Spin⁰(n) is described by

\[
\text{spin}⁰(n) = \mathfrak{m}_₂ \oplus i\mathbb{R} \oplus i\mathbb{R}.
\]

The differential \(p_* : \text{spin}⁰(n) \rightarrow \mathfrak{so}(n) \oplus i\mathbb{R} \oplus i\mathbb{R}\) is defined by

\[
p_*(e_\alpha e_\beta, λi, μi) = (2E_{αβ}, 2λi, (λ + μ)i)
\]

where \(λ\) and \(μ\) are any real numbers and \(E_{αβ}\) is the \(n \times n\) matrix with entries \((E_{αβ})_{αβ} = −1, (E_{αβ})_{βα} = 1\) and all others are equal to zero. The inverse of the differential \(p_*\) is given by

\[
p_*^{-1}(E_{αβ}, λi, μi) = (\frac{1}{2}e_α e_β, \frac{1}{2}λi, (μ - \frac{1}{2}λ)i).
\]
3 Spin$^T$ structure

**Definition 5** A Spin$^T$ structure on an oriented Riemannian manifold $(M^n, g)$ is a Spin$^T(n)$ principal bundle $P_{\text{Spin}^T(n)}$ together with a smooth map $\Lambda : P_{\text{Spin}^T(n)} \to P_{\text{SO}(n)}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
P_{\text{Spin}^T(n)} \times \text{Spin}^T(n) & \xrightarrow{\Lambda \times \lambda^T} & P_{\text{Spin}^T(n)} \\
\downarrow & & \downarrow \\
P_{\text{SO}(n)} \times \text{SO}(n) & \xrightarrow{\Lambda} & P_{\text{SO}(n)}
\end{array}
\]

From above definition we can construct a two-fold covering map

$$\Pi : P_{\text{Spin}^T(n)} \to P_{\text{SO}(n)} \times P_{S^1 \times S^1}.$$ 

Given a Spin$^T$ structure $(P_{\text{Spin}^T(n)}, \Lambda)$, the map $\lambda^T : \text{Spin}^T(n) \to \text{SO}(n)$ induces an isomorphism

$$P_{\text{Spin}^T(n)} / S^1 \times S^1 \cong P_{\text{SO}(n)}.$$ 

In similar way, $\text{Spin}^T(n) / \text{Spin}(n) \cong S^1 \times S^1$ implies the isomorphism

$$P_{\text{Spin}^T(n)} / \text{Spin}(n) \cong P_{S^1 \times S^1}.$$ 

Note that on account of the inclusion map $i : \text{Spin}(n) \to \text{Spin}^T(n)$, every spin structure on $M$ induces a Spin$^T$ structure. Similarly, since there exists a inclusion map $\text{Spin}^c(n) \to \text{Spin}^T(n)$, every Spin$^c$ structure on $M$ induces a Spin$^T$ structure.

4 Spinor bundle and Dirac operator

Let $(M^n, g)$ be an oriented connected Riemannian manifold and $P_{\text{SO}(n)} \to M$ the $\text{SO}(n)$–principal bundle of positively oriented orthonormal frames. The Levi-Civita connection $\nabla$ on $P_{\text{SO}(n)}$ determine a connection 1–form $\omega$ on the principal bundle $P_{\text{SO}(n)}$ with values in $\mathfrak{so}(n)$, locally given by

$$\omega^e = \sum_{i<j} g(\nabla e_i, e_j) E_{ij}$$

where $e = \{e_1, \ldots, e_n\}$ is a local section of $P_{\text{SO}(n)}$ and $E_{ij}$ is the $n \times n$ matrix with entries $(E_{ij})_{ij} = -1$, $(E_{ij})_{ji} = 1$ and all others are equal to zero.

We fix a connection

$$(A, B) : TP_{S^1 \times S^1} \to i\mathbb{R} \oplus i\mathbb{R}$$

on the principal bundle $P_{S^1 \times S^1}$. The connections $\omega$ and $(A, B)$ induce a connection

$$\omega \times (A, B) : T(P_{\text{SO}(n)} \times P_{S^1 \times S^1}) \to \mathfrak{so}(n) \oplus i\mathbb{R} \oplus i\mathbb{R}$$
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on the fibre product bundle $P_{SO(n)} \times P_{S^1 \times S^1}$. Now we can define a connection
1–form $\omega \times (A, B)$ on the principal bundle $P_{Spin^T(n)}$ such that the following
diagram commutes:

$$
\begin{array}{ccc}
TP_{Spin^T(n)} & \xrightarrow{\omega \times (A, B)} & \text{spin}^T(n) \\
\Pi_* & & p_* \\
T(P_{SO(n)} \times P_{S^1 \times S^1}) & \xrightarrow{\omega \times (A, B)} & \mathfrak{so}(n) \oplus i\mathbb{R} \oplus i\mathbb{R}
\end{array}
$$

That is, the equality

$$
p_* \circ \omega \times (A, B) = \omega \times (A, B) \circ \Pi_*
$$

holds.

**Definition 6** The spinor bundle of a Spin$^T$ manifold is defined as the associated
vector bundle

$$
\mathcal{S} = P_{Spin^T(n)} \times_{\kappa^T} \Delta_n
$$

where $\kappa^T : Spin^T(n) \to GL(\Delta_n)$ is the spinor representation of Spin$^T(n)$. In
the case of $n = 2k$ the spinor bundle splits into the sum of two subbundles $\mathcal{S}^+$ and $\mathcal{S}^-$ such that

$$
\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-, \quad \mathcal{S}^\pm = P_{Spin^T(n)} \times_{\kappa^T} \Delta^\pm_n.
$$

Any spinor field $\psi$ can be identified with the map $\psi : P_{Spin^T(n)} \to \Delta_n$ satisfying
the transformation rule $\psi(pg) = \kappa^T(g^{-1}) \psi(p)$. The absolute differential of a
section $\psi$ with respect to $\omega \times (A, B)$ determines a covariant derivative

$$
\tilde{\nabla} : \Gamma(\mathcal{S}) \to \Gamma(T^*M \otimes \mathcal{S})
$$

given by

$$
\tilde{\nabla} \psi = d\psi + \kappa^T_{*,1}(\omega \times (A, B))\psi
$$

where $\kappa^T_{*,1} : \text{spin}^T(n) \to \text{End}(\Delta_n)$ is the derivative of $\kappa$ at the identity
$1 \in Spin^T(n)$. It can be also shown that

$$
\kappa^T_{*,1}(e_\alpha e_\beta, \lambda i, \mu i) = \kappa(e_\alpha e_\beta) + (2\lambda i + \mu i)Id
$$

where $\lambda$ and $\mu$ are any real numbers and $\kappa$ is the spin representation of the
group Spin($n$).

Now we give the local formulas for connections. Fix a section $s : U \to P_{S^1 \times S^1}$
of the principal bundle $P_{S^1 \times S^1}$. Then, we obtain the local connection form

$$
(A^*, B^*) : TU \to i\mathbb{R} \oplus i\mathbb{R}
$$

where $A^*, B^* : TU \to i\mathbb{R}$. $e \times s : U \to P_{SO(n)} \times P_{S^1 \times S^1}$ is a local section
of the fiber product bundle $P_{SO(n)} \times P_{S^1 \times S^1}$. $e \times s$ is a lift of this section to the
two-fold covering $\Pi : P_{Spin^c(n)} \to P_{SO(n)} \times P_{S^1 \times S^1}$. The local connection form $\omega \times (A, B)$ on the principal bundle $P_{Spin^c(n)}$ is given by the formula

$$\omega \times (A, B) = \left( \frac{1}{2} \sum_{i<j} g(\nabla e_i, e_j) e_i e_j, \frac{1}{2} A^s, B^s - \frac{1}{2} A^s \right)$$

Hence, this connection form induces a connection $\tilde{\nabla}$ on the spinor bundle $S$. We can locally describe $\tilde{\nabla}$ by

$$\tilde{\nabla}_X \psi = d\psi(X) + \frac{1}{2} \sum_{i<j} g(\nabla_X e_i, e_j) e_i e_j \psi + \frac{1}{2} A^s \psi + B^s \psi \quad (1)$$

where $\psi : U \to \Delta_n$ is a section of the spinor bundle $S$.

**Definition 7** The first order differential operator

$$D = \mu \circ \tilde{\nabla} : \Gamma(S) \to \Gamma(T^* M \otimes S) \to \Gamma(S)$$

where $\mu$ denotes Clifford multiplication, is called the Dirac operator.

The Dirac operator $D$ is locally given by

$$D\psi = \sum_{i=1}^n e_i \cdot \tilde{\nabla}_{e_i} \psi \quad (2)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on the manifold $M$.

The Dirac operator has the following property:

**Theorem 8** Let $f$ be a smooth function and $\psi \in \Gamma(S)$ be a spinor field. Then,

$$D(f \cdot \psi) = (\text{grad} f \cdot \psi) + f D\psi.$$

**Proof** By using the definition of the Dirac operator $D$ we can compute $D(f \cdot \psi)$ as follows:

$$D(f \cdot \psi) = \sum_{i=1}^n e_i \cdot \tilde{\nabla}_{e_i} (f \cdot \psi)$$

$$= \sum_{i=1}^n e_i \cdot (e_i(f) \cdot \psi + f \tilde{\nabla}_{e_i} \psi)$$

$$= \sum_{i=1}^n e_i(f) e_i \cdot \psi + f \sum_{i=1}^n e_i \cdot \tilde{\nabla}_{e_i} \psi$$

$$= (\text{grad} f) \cdot \psi + f D\psi$$

□

Now we can define the Laplace operator on the spinor bundle $S$.

**Definition 9** Let $\psi \in \Gamma(S)$ be a spinor field. The Laplace operator $\Delta$ on spinors is defined by

$$\Delta \psi = -\sum_{i=1}^n \left( \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \psi + \text{div}(e_i) \tilde{\nabla}_{e_i} \psi \right). \quad (3)$$
4.1 Schrödinger-Lichnerowicz type formula

The square $D^2$ of the Dirac operator and the Laplace operator $\Delta$ are second order differential operators. We derive Schrödinger-Lichnerowicz type formula by computing their difference $D^2 - \Delta$.

The curvature $R^S$ of the spinor covariant derivative $\tilde{\nabla}$ is an $\text{End}(S)$ valued 2-form by

$$R^S(X,Y)\psi = \tilde{\nabla}_X \tilde{\nabla}_Y \psi - \tilde{\nabla}_Y \tilde{\nabla}_X \psi - \tilde{\nabla}_{[X,Y]} \psi$$

where $\psi \in \Gamma(S)$ and $X,Y \in \Gamma(TM)$. Now we want to describe $R^S$ in terms of the curvature tensor $R$.

Let $\Omega^\omega : TP_{\text{SO}(n)} \times TP_{\text{SO}(n)} \to \mathfrak{so}(n)$ be the curvature form of the Levi-Civita connection with the components

$$\Omega^\omega_{ij} = \sum_{i<j} \Omega_{ij}^* E_{ij}$$

where $\Omega_{ij} : TP_{\text{SO}(n)} \times TP_{\text{SO}(n)} \to \mathbb{R}$. The commutative diagram defining the connection $\omega \times (A,B)$ implies that the curvature form of $\omega \times (A,B)$ is

$$\Omega^{\omega \times (A,B)} = \frac{1}{2} \sum_{i<j} \Pi^* (\Omega_{ij}) e_i e_j + \frac{1}{2} \Pi^* (dA) \oplus \Pi^* (dB).$$

Hence the 2-form $R^S$ with values in the spinor bundle $S$ is obtained by the following formula:

$$R^S(.,.)\psi = \frac{1}{2} \sum_{i<j} \Omega_{ij} e_i e_j \cdot \psi + \frac{1}{2} dA \cdot \psi + dB \cdot \psi.$$

Let $\{e_1, \ldots, e_n\}$ be orthonormal frame field, $\Omega_{ij}(X,Y) = g(R(X,Y)e_i, e_j)$ the components of the curvature form of the Levi-Civita connection,

$$X = \sum_{k=1}^{n} X^k e_k$$

and

$$Y = \sum_{l=1}^{n} Y^l e_l$$

be vector fields on the Riemannian manifold $M$. Then we have

$$\Omega_{ij}(X,Y) = g(R(X,Y)e_i, e_j) = \sum_{k,l=1}^{n} R_{klij} X^k Y^l$$

$$= \sum_{k,l=1}^{n} R_{klij} e^k(X) e^l(Y)$$

$$= \frac{1}{2} \sum_{k,l=1}^{n} R_{klij} (e^k \wedge e^l)(X,Y).$$

where $\{e^1, \ldots, e^n\}$ is the frame dual to $\{e_1, \ldots, e_n\}$. Thus, we obtain the following local formula for the curvature form

$$\Omega^{\omega \times (A,B)} = \frac{1}{4} \sum_{i<j} \sum_{k,l=1}^{n} R_{klij} (e^k \wedge e^l) e_i e_j + \frac{1}{2} dA + dB.$$
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and the 2-form $R^S(\cdot, \cdot)$ is calculated as follows:

$$R^S(\cdot, \cdot) = \frac{1}{4} \sum_{i<j,k,l=1}^n R_{klij} (e^k \wedge e^l) e_i e_j \cdot \psi + \frac{1}{2} dA \cdot \psi + dB \cdot \psi.$$ 

By using the above properties of the curvature form $R^S$ on spinor bundle $S$ we deduce the following result:

**Proposition 10** Let $\text{Ric}$ be the Ricci tensor. Then, the following relation holds:

$$\sum_{\alpha=1}^n e_\alpha \cdot R^S(X, e_\alpha) \psi = -\frac{1}{2} \text{Ric}(X) \cdot \psi + \frac{1}{2} (X \lrcorner dA) \cdot \psi + (X \lrcorner dB) \cdot \psi \quad (4)$$

**Proof** In [2] it is proved the following relation:

$$\sum_{\alpha=1}^n \sum_{i<j,k,l=1}^n R_{klij} (e^k \wedge e^l) e_\alpha e_i e_j \cdot \psi = -2 \text{Ric}(X) \cdot \psi \quad (5)$$

It can be easily seen the following two relations:

$$\sum_{\alpha=1}^n e_\alpha \cdot dA(X, e_\alpha) \cdot \psi = (X \lrcorner dA) \cdot \psi \quad (6)$$

and

$$\sum_{\alpha=1}^n e_\alpha \cdot dB(X, e_\alpha) \cdot \psi = (X \lrcorner dB) \cdot \psi. \quad (7)$$

Then, using (5), (6) and (7), we obtain the claimed equivalence. □

Now, we derive Schrödinger-Lichnerowicz-type formula in the following way:

**Proposition 11** Let $s$ be scalar curvature of the Riemannian manifold and let $dA = \Omega^A$ and $dB = \Omega^B$ be the imaginary-valued 2–forms of the connections $(A, B)$ in the $(S^1 \times S^1)$–bundle associated with $\text{Spin}^7$ structure. Then, we have the following formula:

$$D^2 \psi = \Delta \psi + \frac{s}{4} \psi + \frac{1}{2} dA \cdot \psi + dB \cdot \psi.$$ 

**Proof**

$$D^2 \psi = \sum_{i,j} e_i \cdot \nabla_{e_i} (e_j \cdot \nabla_{e_j} \psi)$$

$$= \sum_{i,j} e_i \cdot \nabla_{e_i} e_j \cdot \nabla_{e_j} \psi + e_i e_j \cdot \nabla_{e_i} \nabla_{e_j} \psi$$

$$= \sum_{i,j,k} g(\nabla_{e_i} e_j, e_k) e_i e_k \cdot \nabla_{e_j} \psi + \sum_{i,j} e_i e_j \cdot \nabla_{e_i} \nabla_{e_j} \psi \quad (8)$$

$$= \Delta \psi + \sum_{j,i \neq k} g(\nabla_{e_i} e_j, e_k) e_i e_k \cdot \nabla_{e_j} \psi + \sum_{i \neq j} e_i e_j \cdot \nabla_{e_i} \nabla_{e_j} \psi$$
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Now we can calculate the following sum:

$$\sum_{i \neq k} g(\nabla_{e_i} e_j, e_k) e_i e_k = -\sum_{i \neq k} g(e_j, \nabla_{e_i} e_k) e_i e_k$$

$$= -\sum_{i < k} g(e_j, \nabla_{e_i} e_k - \nabla_{e_k} e_i) e_i e_k$$

$$= \sum_{i < k} g(e_j, [e_k, e_i]) e_i e_k$$

From (8) we get

$$D^2 \psi = \Delta \psi + \sum_{j, i < k} g([e_j, e_k], e_i e_k) \nabla_{e_j} \nabla_{e_i} \psi + \sum_{i < j} e_i e_j \cdot (\nabla_{e_i} \nabla_{e_j} \psi - \nabla_{e_j} \nabla_{e_i} \psi)$$

$$= \Delta \psi + \sum_{i < j} e_i e_j (\nabla_{e_i} \nabla_{e_j} \psi - \nabla_{e_j} \nabla_{e_i} \psi - \nabla_{[e_i, e_j]} \psi)$$

$$= \Delta \psi + \frac{1}{2} \sum_{i, j} e_i e_j R^S(e_i, e_j) \psi.$$ 

Using the identity (4) and multiplying by $e_i$ we deduce

$$D^2 \psi = \Delta \psi - \frac{1}{4} \sum_i e_i \text{Ric}(e_i) \cdot \psi + \frac{1}{4} \sum_i e_i \cdot (e_i \cup dA) \cdot \psi + \frac{1}{2} \sum_i e_i \cdot (e_i \cup dB) \cdot \psi$$

$$= \Delta \psi + \frac{s}{4} \psi + \frac{1}{2} dA \cdot \psi + dB \cdot \psi.$$

□

References

[1] Thakre, V., Dimensional reduction of non-linear Seiberg-Witten equations, arXiv:1502.01486v1.

[2] Friedrich, T., Dirac operators in Riemannian Geometry, AMS, 2000.

[3] Lawson, H. B., Michelsohn, M.L., Spin Geometry, Princeton Univ., 1989.

[4] Salamon, D.A., Spin Geometry and Seiberg-Witten invariants, in preparation.

[5] Nicolaescu, L. I., Lectures on the Geometry of Manifolds, World Scientific, 2007.