Five-dimensional $N = 4$, $SU(2) \times U(1)$ Gauged Supergravity from Type IIB

H. Lü$\dagger$, C.N. Pope$\ddagger$ and T.A. Tran$\dagger$

$\dagger$Department of Physics and Astronomy  
University of Pennsylvania, Philadelphia, Pennsylvania 19104

$\ddagger$Center for Theoretical Physics  
Texas A&M University, College Station, Texas 77843

ABSTRACT

We construct the complete and explicit non-linear Kaluza-Klein ansatz for deriving the bosonic sector of $N = 4$ $SU(2) \times U(1)$ gauged five-dimensional supergravity from the reduction of type IIB supergravity on $S^5$. This provides the first complete example of such an $S^5$ reduction that includes non-abelian gauge fields, and it allows any bosonic solution of the five-dimensional $N = 4$ gauged theory to be embedded in $D = 10$.

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1 Introduction

Recently, a new duality was conjectered, which relates supergravities in anti-de Sitter backgrounds to superconformal field theories on their boundaries \[1, 2, 3\]. Since the anti-de Sitter backgrounds arise as solutions of lower-dimensional gauged supergravities, the conjectured AdS/CFT correspondence has led to a revival of interest in deriving these supergravities by Kaluza-Klein reduction from the fundamental string theories and M-theory in \(D = 10\) and \(D = 11\).

It has long been believed that many gauged supergravities arise from the Kaluza-Klein reduction of eleven-dimensional supergravity or the type IIA and IIB supergravities on certain spherical internal spaces. At the linearised level, it is well established that the spectra of the massless supermultiplets in the maximal gauged supergravities in \(D = 4\) and \(D = 7\) coincide with those coming from appropriate truncations of the \(S^4\) and \(S^7\) reductions of \(D = 11\) supergravity \[4, 5\]. Likewise, the spectrum of maximal gauged supergravity in \(D = 5\) is known to arise from a truncation of the \(S^5\) reduction from type IIB supergravity \[6, 7\].

Although these linearised results are rather easily established, it is much harder to determine whether the truncations to the massless supermultiplets are consistent as the full non-linear level. The key point of concern here is that once the non-linear interactions are included, the possibility exists that in a full untruncated reduction, source terms built purely from the fields of the massless multiplet might arise in the field equations for the lower-dimensional massive multiplets, making it inconsistent to set the massive fields to zero. Indeed, it is not hard to establish that in a sphere reduction of a generic higher-dimensional theory, there will definitely be such couplings, making a consistent truncation to the massless sector impossible. Thus if the sphere reductions (with truncation to the massless supermultiplet) in supergravities are to be consistent, it must be because of remarkable special properties of these particular theories. General arguments suggesting that reductions to the supergravity multiplet should always be consistent were constructed in \[8\].

Early indications of such properties were found for the \(S^7\) reduction of \(D = 11\) supergravity in \[9\]. Subsequently, a complete demonstration of the consistency in this case was presented in \[10\], although the construction was rather implicit. Recently, a somewhat less implicit construction was presented for the easier case of the reduction ansatz for the \(S^4\) reduction of \(D = 11\) supergravity, to give the maximal gauged supergravity in \(D = 7\) \[12\]. No analogous results have been obtained for the \(S^5\) reduction of type IIB supergravity,
however, and so for now the consistency of this reduction remains conjectural.

Although one would certainly like to know the complete reduction ansätze in all cases, for many purposes it is in practice more useful to have a fully explicit ansatz for a restricted subset of the fields of the entire maximal massless multiplet. For example, the known charged anti-de Sitter black hole solutions of gauged supergravities make use only of gauge fields in the maximal abelian subgroup of the full gauge group. Thus, for example, if one is interested in oxidising these solutions back to the higher dimension of the original parent supergravity, it would be sufficient to have a consistent Kaluza-Klein reduction ansatz for the abelian truncation of the full gauged theory. Kaluza-Klein ansätze of this sort were constructed in [11], for the $S^7$ and $S^4$ reductions of $D = 11$ supergravity, and the $S^5$ reduction of type IIB supergravity. In all cases fully non-linear bosonic ansätze were obtained. In the case of the $S^5$ reduction, the ansatz was fully consistent, while in the reductions from $D = 11$, the ansätze were fully consistent within limited subspaces of solutions, including the charged AdS black holes.

Another way of making the reductions more manageable, while maintaining full consistency, is to truncate from the maximal supermultiplet to a smaller supergravity multiplet. By doing this, the full non-linear ansatz for the $S^4$ reduction from $D = 11$ to $N = 1$ SU(2)-gauged supergravity in $D = 7$ was obtained in [13]. Although in principle subsumed by the $N = 2$ results in [12], in practice the full explicitness of the $N = 1$ results makes them more transparent and usable in those cases where the $N = 1$ truncation is sufficient.

In a similar vein, the bosonic sector of the full $N = 2$ SU(2)-gauged supergravity in $D = 6$ was recently obtained, as an explicit consistent (local) $S^4$ reduction of the massive type IIA theory [14]. A distinction in this case is that there is no gauged supergravity in $D = 6$ with the maximal ($N = 4$) supersymmetry that can occur in ungauged theories, and so the construction in [14] yields the largest gauged pure supergravity that exists in $D = 6$.

As far as the five-dimensional gauged supergravities are concerned, no non-linear reduction ansätze other than the $U(1)^3$ maximal abelian case in [11] have been obtained until recently. Studying the states on the Coulomb branch of $N = 4$ super Yang-Mills theory from the viewpoint of five-dimensional gauged supergravity [15] provides further indications of consistent truncations from type IIB supergravity. For example, in [16] non-linear reduction ansätze involving larger numbers of the scalar fields of the maximal $D = 4, 5$ and 7 theories were presented, although without the inclusion of gauge fields.

In this paper, we shall construct the complete and explicit non-linear Kaluza-Klein reduction ansätze for the bosonic sector of $N = 4, SU(2) \times U(1)$ gauged supergravity in
$D = 5$, obtained from the $S^5$ reduction of type IIB supergravity. The $N = 4$ gauged theory in $D = 5$ was constructed by L. Romans [17]; hereafter, we shall refer to this model as the Romans theory. Our ansatz allows us to re-interpret any bosonic solution of the five-dimensional $N = 4$ theory back in $D = 10$. The ansatz provides the first example of a fully consistent non-linear Kaluza Klein reduction of type IIB supergravity on $S^5$ that includes non-abelian gauge fields.

A new feature of this $S^5$ reduction, which has not been encountered in previous explicit constructions [13, 14], is the presence of higher-rank gauge potentials that transform non-trivially under the gauge group. Specifically, there are two 2-form potentials in the $D = 5$ theory, which form a charged doublet under the $U(1)$ factor of the gauge group.

The paper is organised as follows. In section 2, we establish our notation and conventions, giving Lagrangians and equations of motion for type IIB supergravity and the $D = 5$, $N = 4$ $SU(2) \times U(1)$ gauged supergravity. In section 3, we obtain the reduction ansätze, and discuss some of the interesting features. After the conclusion, in section 4, we include an appendix giving results for the Ricci tensor of the metric reduction ansatz.

2 $D = 10$ IIB supergravity and Romans’ theory in $D = 5$

The bosonic sector of ten dimensional type IIB supergravity comprises the metric, a self-dual 5-form field strength, a scalar, an axion, an R-R 3-form and an NS-NS 3-form field strength. There is no simple covariant Lagrangian for type IIB supergravity, on account of the self-duality constraint for the 5-form. However, one can write a Lagrangian in which the 5-form is unconstrained, which must then be accompanied by a self-duality condition which is imposed by hand at the level of the equations of motion [18]. This type IIB Lagrangian is

$$L_{10}^{IIB} = \hat{R} * 1 - \frac{1}{2} * d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} e^{2\hat{\phi}} * d\hat{\chi} \wedge d\hat{\chi} - \frac{1}{4} * \hat{H}_5 \wedge \hat{H}_5$$

$$- \frac{1}{2} e^{-\hat{\phi}} * \hat{F}_2^2 \wedge \hat{F}_2^2 - \frac{1}{4} e^{\hat{\phi}} * \hat{F}_1^1 \wedge \hat{F}_1^1 - \frac{1}{2} \hat{B}_4 \wedge d\hat{A}_{(2)}^1 \wedge d\hat{A}_{(2)}^2,$$

where

$$\hat{F}_2^2 = d\hat{A}_{(2)}^2, \quad \hat{F}_1^1 = d\hat{A}_{(2)}^1 - \hat{\chi} d\hat{A}_{(2)}^2, \quad \hat{H}_5 = d\hat{B}_4 - \frac{1}{2} \hat{A}_{(2)}^1 \wedge d\hat{A}_{(2)}^2 + \frac{1}{2} \hat{A}_{(2)}^2 \wedge d\hat{A}_{(2)}^1,$$

and we use hats to denote ten-dimensional fields and the ten-dimensional Hodge dual $*$. The equations of motion following from the Lagrangian, together with the self-duality condition, are

$$\hat{R}_{MN} = \frac{1}{2} \partial_M \hat{\phi} \partial_N \hat{\phi} + \frac{1}{2} e^{2\hat{\phi}} \partial_M \hat{\chi} \partial_N \hat{\chi} + \frac{1}{96} \hat{H}^2_{MN}$$

$$+ \frac{1}{4} e^{\hat{\phi}} \left((\hat{F}_1^1)^2_{MN} - \frac{1}{12} (\hat{F}_2^2)^2 g_{MN}\right).$$
We then find that the equations of motion are
\[ 2 \hat{\phi} \left( \hat{F}^2_{(3)} \right)^2 = \frac{1}{4} \left( \hat{F}^2_{(3)} \right)^2 g_{MN}, \]
\[ 2 \hat{\phi} \hat{d} \hat{d} \hat{\phi} = -e^{2 \hat{\phi}} \hat{d} \hat{X} \wedge \hat{d} \hat{X} - \frac{1}{2} e^{\hat{\phi}} \hat{d} \hat{F}^1_{(3)} \wedge \hat{F}^1_{(3)} + \frac{1}{2} e^{-\hat{\phi}} \hat{d} \hat{F}^2_{(3)} \wedge \hat{F}^2_{(3)}, \]
\[ d \left( e^{2 \hat{\phi}} \hat{d} \hat{\phi} \right) = e^{\hat{\phi}} \hat{d} \hat{F}^1_{(3)} \wedge \hat{F}^2_{(3)} , \]
\[ d \left( e^{\hat{\phi}} \hat{d} \hat{F}^1_{(3)} \right) = \hat{H}^2_{(3)}, \]
\[ d \left( e^{\hat{\phi}} \hat{d} \hat{F}^2_{(3)} \right) = -\hat{H}^1_{(3)} \wedge (\hat{F}^1_{(3)} + \hat{F}^2_{(3)}), \]
\[ d \left( \hat{d} \hat{H}^1 \right) = -\hat{F}^1_{(3)} \wedge \hat{F}^2_{(3)}, \quad \hat{H}^1 = \hat{d} \hat{H}^1. \]

The equations (2)-(6) are precisely those which were found in [17].

Turning now to the $N = 4$ gauged supergravity in $D = 5$ [17], it has a bosonic sector comprising the metric, a scalar, the $SU(2)$ Yang-Mills potentials $A^A_{\mu}$, a $U(1)$ gauge potential $B_{\mu}$, and two 2-form potentials $A^\alpha_{\mu
u}$, which transform as a charged doublet under the $U(1)$.

The Lagrangian is given by
\[
\mathcal{L} = R \ast 1 - 3 X^{-2} \ast dX \wedge dX - \frac{1}{2} X^4 \ast G_{(2)} \wedge G_{(2)} - \frac{1}{2} X^{-2} \left( * F^i_{(2)} \wedge F^i_{(2)} + * A^\alpha_{(2)} \wedge A^\alpha_{(2)} \right) + \frac{1}{4 g_1} \epsilon_{\alpha\beta} A^\alpha_{(2)} \wedge A^\beta_{(2)} - \frac{1}{2} A^\alpha_{(2)} \wedge A^\alpha_{(2)} \wedge B_{(3)} - \frac{1}{2} F^i_{(2)} \wedge F^i_{(2)} \wedge B_{(3)} + 2 g_2 (g_2 X^2 + 2 \sqrt{2} g_1 X^{-1}) \ast 1,
\]

where $X$ parameterises the scalar degree of freedom, and can be written in terms of a canonically-normalised dilaton $\phi$ as $X = e^{-\sqrt{2} \phi}$. The 2-form field strengths are given by
\[
F^i_{(2)} = dA^i_{(1)} + \frac{1}{2} g_2 \epsilon^{ijk} A^j_{(1)} \wedge A^k_{(1)} , \quad G_{(2)} = dB_{(1)}. \]

Without loss of generality we may set $g_1 = 1$ and $g_2 = \sqrt{2} g$, since the two independent gauge coupling constants may be recovered by appropriate rescalings. We also find it advantageous to adopt a complex notation for the two 2-form potentials, which form a charged doublet with respect to the gauge field $B_{(1)}$. Thus we define
\[
A_{(2)} \equiv A^1_{(2)} + i A^2_{(2)}. \]

We then find that the equations of motion are
\[
d(X^{-1} \ast dX) = \frac{1}{3} X^4 \ast G_{(2)} \wedge G_{(2)} - \frac{1}{6} X^{-2} \left( * F^i_{(2)} \wedge F^i_{(2)} + * A^\alpha_{(2)} \wedge A^\alpha_{(2)} \right) - \frac{4}{3} g^2 (X^2 - X^{-1}) \ast 1, \]
\[
d(X^4 \ast G_{(2)}) = - \frac{1}{2} F^i_{(2)} \wedge F^i_{(2)} + \frac{1}{2} \hat{A}_{(2)} \wedge A_{(2)}, \]
\[
d(X^{-2} \ast F^i_{(2)}) = \sqrt{2} g \epsilon^{ijk} X^{-2} * F^j_{(2)} \wedge A^k_{(1)} - F^i_{(2)} \wedge G_{(2)} , \]
\[
X^2 \ast F_{(3)} = - i g A_{(2)} , \]
\[
R_{\mu\nu} = 3 X^{-2} \partial_\mu X \partial_\nu X - \frac{4}{3} g^2 (X^2 + 2 X^{-1}) g_{\mu\nu} + \frac{1}{2} X^4 (G_\mu^{\rho} G_\nu^{\rho} - \frac{1}{4} g_{\mu\nu} G_{(2)}^2) + \frac{1}{2} X^{-2} (F^i_{\mu} F^i_{\nu} - \frac{1}{6} g_{\mu\nu} (F^i_{(2)})^2) + \frac{1}{2} X^{-2} (A^{i}_{(\mu} A^{j}_{\rho)} \delta_{\rho}^{\nu} - \frac{1}{6} g_{\mu\nu} |A|_{(2)}^2), \]

(10)
where
\( F_3 = DA_{(2)} \equiv dA_{(2)} - ig B_{(1)} \wedge A_{(2)}. \) (11)

The operator \( D \) defined in this equation is the \( U(1) \) gauge-covariant exterior derivative. Note that the complex field \( A_{(2)} \) satisfies a first-order equation of motion, of the kind referred to as “odd-dimensional self-duality” in [20].

As discussed in [17], there are three inequivalent theories, corresponding to different choices for the gauge couplings in (5): \( N = 4^0 \) in which \( g_2 = 0; N = 4^+ \) in which \( g_2 = \sqrt{2}g_1 \) and \( N = 4^- \) in which \( g_2 = -\sqrt{2}g_1 \) (see also [21]). The \( N = 4^+ \) theory is obtained by truncating the gauged \( SO(6) \), \( N = 8 \) supergravity theory in five-dimensions, while the \( N = 4^0 \) and \( N = 4^- \) theories arise as truncations of non-compact \( N = 8 \) supergravities.

The equations of motion (10) are precisely those of the \( N = 4^+ \) theory. From the Lagrangian of the Romans theory one might conclude that it is not possible to set the \( U(1) \) coupling constant \( g_1 \) to be zero. However, as was pointed out in [22], after appropriate rescalings the limit can be taken.

In the next section, we shall construct the ansätze that give this theory by reducing type IIB supergravity on a 5-sphere.

### 3 Reduction ansätze on the five-sphere

To construct the reduction ansätze, we follow a procedure similar to that used in [13, 14]. We take as our starting point the \( U(1)^3 \) abelian truncation obtained in [11], which involved two independent scalar fields. One can perform a consistent truncation of this, in which two of the three \( U(1) \) potentials are set equal, and at the same time one of the scalar degrees of freedom in eliminated. After doing so, the metric on the internal 5-sphere takes the following form

\[
d\omega_5^2 = X \Delta d\xi^2 + X^2 s^2 (d\tau - gB_{(1)})^2 \\
+ \frac{1}{4} X^{-1} c^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi - \sqrt{2}g A_{(1)})^2 \right),
\]

where \( X \equiv e^{-\frac{1}{\sqrt{2}} \phi} \), and we have defined

\[
c \equiv \cos \xi, \quad s \equiv \sin \xi.
\]

The function \( \Delta \) is given by

\[
\Delta = X^{-2} s^2 + X c^2.
\]
At $X = 1$, in the absence of the $U(1)$ gauge fields, the metric (12) describes a unit 5-sphere. (The gauge field $A_{(1)}$ appearing in (12) is the one that comes from setting two of the original $U(1)$ gauge fields in [11] equal.)

We can now make a non-abelian generalisation of the the 5-sphere metric ansatz, by introducing the three left-invariant 1-forms $\sigma^i$ on the 3-sphere, which satisfy $d\sigma^i = -\frac{1}{2} \varepsilon_{ijk} \sigma^j \wedge \sigma^k$. These can be written in terms of the Euler angles as $\sigma_1 + i \sigma_2 = e^{-i\psi} (d\theta + i \sin \theta \, d\phi)$, $\sigma_3 = d\psi + \cos \theta \, d\phi$. Thus we are naturally led to generalise (12) to

$$d\omega_5^2 = X \Delta d\xi^2 + X^2 s^2 \left( d\tau - gB_{(1)} \right)^2 + \frac{1}{4} X^{-1} c^2 \sum_{i=1}^3 (\sigma^i - \sqrt{2} g A^i_{(1)})^2 .$$

(15)

The abelian limit (12) is recovered by setting the $i = 1$ and $i = 2$ components of the $SU(2)$ gauge potentials to zero.

By proceeding along these lines, we are eventually led to the following ansätze for the ten-dimensional metric, the 5-form field strength $\hat{H}_{(5)} = \hat{G}_{(5)} + \hat{G}^\ast_{(5)}$, and the two 2-form potentials:

$$ds_{10}^2 = \Delta^{1/2} ds_5^2 + g^{-2} X \Delta^{1/2} d\xi^2 + g^{-2} \Delta^{-1/2} X^2 s^2 \left( d\tau - g B_{(1)} \right)^2 + \frac{1}{4} g^{-2} \Delta^{-1/2} X^{-1} c^2 \sum_{i=1}^3 (\sigma^i - \sqrt{2} g A^i_{(1)})^2 ,$$

$$\dot{\hat{G}}_{(5)} = 2gU \varepsilon_5 - \frac{3sc}{g} X^{-1} \ast * dX \wedge d\xi + \frac{c^2}{8 \sqrt{2} g^2} X^{-2} \ast F^i_{(2)} \wedge h^i \wedge h^k \varepsilon_{ijk} - \frac{sc}{2 \sqrt{2} g^2} X^{-2} \ast F^i_{(2)} \wedge h^i \wedge d\xi - \frac{sc}{g^2} X^4 \ast G_{(2)} \wedge d\xi \wedge (d\tau - g B_{(1)}),$$

$$\dot{\hat{A}}_{(2)} = \hat{A}^1_{(2)} + i \hat{A}^2_{(2)} = -\frac{s}{\sqrt{2}g} e^{-i\tau} A_{(2)} ,$$

$$\dot{\hat{\phi}} = 0, \quad \dot{\xi} = 0,$$

(16)

where $h^i \equiv \sigma^i - \sqrt{2} g A^i_{(1)}$, $U \equiv X^2 c^2 + X^{-1} s^2 + X^{-1}$, and $\varepsilon_5$ is the volume form in the five-dimensional spacetime metric $ds_5^2$. Note that we have defined the complex 2-form potential $\hat{A}_{(2)} \equiv \hat{A}^1_{(2)} + i \hat{A}^2_{(2)}$ in the type IIB theory. The ten-dimensional dilaton and the axion are constants, which without loss of generality we have set to zero. The $SU(2)$ Yang-Mills field strengths $F^i_{(2)}$ are given by $F^i_{(2)} = dA^i_{(1)} + \frac{1}{\sqrt{2} g} \epsilon_{ijk} A^j_{(1)} \wedge A^k_{(1)}$. It should be emphasised that all the fields $X$, $A^\alpha_{(2)}$ and $A^i_{(1)}$, and the metric $ds_5^2$, appearing on the right-hand sides of (16), are taken to depend only on the coordinates of the five-dimensional spacetime.\footnote{Note that the metric reduction ansatz in (14) has a fairly simply form, and fits in with the general pattern of Kaluza-Klein metric ansätze, such as have been seen in many previous examples. (See [23, 24, 25] for some earlier examples, and [26, 27, 28, 29] for some more recent ones.) It is the determination of the reduction ansätze for the field strengths that is the more difficult part of the problem, and since the consistency of the reduction depends crucially on conspiracies between the field strength and metric contributions, it is only when the full ansatz is known that the consistency can be established.}
that since the two 2-form potentials of the 5-dimensional supergravity are charged under the U(1) factor in the SU(2) × U(1) gauge group, these fields appear in the reduction ansatz with the appropriate τ dependence for charged U(1) harmonics.

The ten-dimensional Hodge dual of \( \hat{G}_5 \) turns out to be

\[
\hat{G}_5 = \frac{-s c^3}{4 g^4} U \Delta^{-2} d\xi \wedge \omega \wedge \epsilon_3 + \frac{3 s^2 c^4}{8 g^4} X^{-2} \Delta^{-2} dX \wedge \omega \wedge \epsilon_3 \\
- \frac{s^2 c^2}{8 \sqrt{2} g^3} X^{-2} \Delta^{-1} F_{(2)} \wedge h^i \wedge h^k \wedge \omega \epsilon_{ijk} + \frac{8 c}{2 \sqrt{2} g^3} F_{(2)} \wedge h^i \wedge d\xi \wedge \omega \\
- \frac{c^4}{8 g^3} X \Delta^{-1} G_{(2)} \wedge \epsilon_3,
\]

where \( \omega \equiv d\tau - gB_{(1)} \), and \( \epsilon_3 \equiv h^1 \wedge h^2 \wedge h^3 \).

We must now verify that the ansätze (16) do indeed yield a consistent reduction of the type IIB theory to the \( N = 4 \) gauged supergravity in \( D = 5 \). Let us begin by noting that the Bianchi identity for \( \hat{H}_5 \), and the field equations for the NS-NS and R-R 3-forms of the type IIB theory become, in the complex notation,

\[
d\hat{H}_5 = \frac{i}{2} \hat{F}_{(3)} \wedge F_{(3)}, \quad d\hat{F}_{(3)} = -i \hat{H}_5 \wedge \hat{F}_{(3)}. \tag{18}
\]

We now find that the Bianchi identity for \( \hat{H}_5 \) gives rise to the following five-dimensional equations:

\[
d(X^{-1} \ast dX) = \frac{4}{g^2} (X^{-1} - X^2) \ast 1 - \frac{1}{6} X^{-2} (\ast F_{(2)} \wedge F_{(2)} + \ast \bar{A}_{(2)} \wedge A_{(2)}) \\
+ \frac{1}{3} X^4 \ast G_{(2)} \wedge G_{(2)}; \nonumber \\
d(X^{-2} \ast F_{(2)}) = \sqrt{2} g X^{-2} \epsilon_{ijk} \ast F_{(2)}^j \wedge A_{k(1)} - F_{(2)}^i \wedge G_{(2)}; \\
d(X^4 \ast G_{(2)}) = -\frac{1}{2} F_{(2)}^i \wedge F_{(2)}^i - \frac{1}{2} \bar{A}_{(2)} \wedge A_{(2)}. \tag{19}
\]

These are precisely equations of motion of the Romans theory.

Plugging the ansätze (14) into the equation of motion for \( \hat{F}_{(3)} \) in (18) gives rise to the five-dimensional first-order equation for \( A_{(2)} \) in (14), and also the second-order equation

\[
D(X^2 \ast F_{(3)}) = g^2 X^{-2} \ast A_{(2)}, \tag{20}
\]

where the \( U(1) \) gauge-covariant exterior derivative \( D \) is as defined in (11). This second order equation of motion is in fact implied by the first order equation.

With our ansätze, the ten-dimensional equations of motion for the dilaton and axion are automatically satisfied. The final step is to substitute the ansätze into the ten-dimensional Einstein equation (2). There are ten independent types of components of the Ricci tensor, of which eight are non-vanishing, and are given in the appendix. (The components of
the Einstein equation (2) associated with the two vanishing ones, \( \hat{R}_{56} \) and \( \hat{R}_{5i} \), just give identities 0 = 0.) The \( \hat{R}_{\alpha 5} \) and \( \hat{R}_{5i} \) components simply yield identities of the type \( Z = Z \) in (2), while \( \hat{R}_{66} \) gives an equation of motion for the \( U(1) \) field strength \( G_{\omega} \) which coincides with (10). The \( \hat{R}_{55} \), \( \hat{R}_{66} \) and \( \hat{R}_{ij} \) components reproduce the scalar equation of motion in (10). Finally, the \( \hat{R}_{\alpha \beta} \) components yield the five-dimensional Einstein equations, while \( \hat{R}_{\alpha i} \) give the equations of motion for the \( SU(2) \) gauge field strengths. Thus the full consistency of the reduction ansatz (16) is established.

4 Conclusion

In this paper, we have constructed the explicit and complete all-orders ansatz for obtaining the bosonic sector of \( SU(2) \times U(1) \) gauged \( N = 4 \) supergravity in \( D = 5 \), by Kaluza-Klein reduction on \( S^5 \) from type IIB supergravity. This is the largest supersymmetric \( S^5 \) reduction that has been constructed. This is a particularly significant case from the point of view of the AdS/CFT correspondence, since the \( D = 5 \) gauged supergravities coming from type IIB have AdS\(_5\) solutions whose boundary CFT’s are four-dimensional. In this embedding, we showed that the ten-dimensional moduli, parameterised by the dilaton and the axion, are fixed.

Our results for this consistent \( N = 4 \) reduction lend further credence to the conjecture that the \( SO(6) \) gauged \( N = 8 \) supergravity in \( D = 5 \) should also arise as a consistent \( S^5 \) reduction. The \( N = 8 \) reduction can be expected to be extremely complicated, and so our simpler \( N = 4 \) truncation could prove to be very useful in circumstances where only the \( N = 4 \) subset of fields are excited. For example, our reduction ansatz allows any bosonic solution of the \( N = 4 \) gauged theory in five dimensions to be oxidised back to an exact solution in type IIB supergravity. From the standpoint of the AdS/CFT conjecture, the fact that there is a consistent reduction ansatz means that the contributions of the five-dimensional fields in massive supergravity multiplets can be ignored when computing correlation functions in the conformal field theory on the boundary.

5 Appendix

In this appendix we present the vielbeins, spin connection, and Ricci tensor components for the metric (16). The vielbeins are taken to be

\[
\hat{e}^\alpha = \Delta^{1/4} e^\alpha, \quad \hat{e}^5 = \frac{1}{g} \Delta^{1/4} X^{1/2} d\xi,
\]

8
\[ e^6 = \frac{s}{g} \Delta^{-1/4} X \left( d\tau - g B_{(1)} \right), \quad \epsilon^2 = \frac{c}{2g} \Delta^{-1/4} X^{-1/2} h. \]  

(21)

From these we find that the connection 1-form components are given by

\[
\hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} - \frac{1}{4} (c^2 - 2X^{-3}s^2) \Delta^{-5/4} (\partial_\alpha \eta_{\beta\gamma} - \partial_\beta \eta_{\alpha\gamma}) \hat{e}^\gamma \\
+ \frac{s}{2} X \Delta^{-3/4} G_{\alpha\beta} \epsilon^6 + \frac{c}{2\sqrt{2}} X^{-1/2} \Delta^{-3/4} F_{\alpha\beta} \hat{e}^i, \\
\hat{\omega}_{\alpha 5} = \frac{g sc}{2} X^{-1/2} (X^{-2} - X) \Delta^{-5/4} \epsilon^\alpha - \frac{3c^2}{4} \Delta^{-5/4} \partial_\alpha X \hat{e}^5, \\
\hat{\omega}_{\alpha 6} = -\frac{3}{4} \Delta^{-5/4} (c^2 + 2X^{-3}s^2) \partial_\alpha X \epsilon^6 + \frac{s}{2} X \Delta^{-3/4} G_{\alpha\beta} \epsilon^\beta, \\
\hat{\omega}_{\alpha i} = \frac{3c^2}{4} \Delta^{-5/4} \partial_\alpha X \epsilon^i + \frac{c}{2\sqrt{2}} X^{-1/2} \Delta^{-3/4} F_{\alpha\beta} \hat{e}^\beta, \\
\hat{\omega}_{56} = -\frac{gc}{2s} X^{-1/2} (\Delta + X) \Delta^{-5/4} \epsilon^6, \\
\hat{\omega}_{5i} = \frac{gs}{2c} X^{-1/2} (\Delta + X^{-2}) \Delta^{-5/4} \epsilon^i, \\
\hat{\omega}_{ij} = -\frac{g}{c} X^{1/2} \Delta^{1/4} \epsilon^{ijk} \hat{e}^k - \sqrt{2} g \Delta^{-1/4} \epsilon^{ijk} A^k_\alpha \hat{e}^\alpha,
\]

(22)

and the other components are zero. The non-vanishing Ricci tensor components in the vielbein basis are

\[
\hat{R}_{\alpha\beta} = \Delta^{-1/2} R_{\alpha\beta} - \frac{1}{4} (c^2 - 2X^{-3}s^2) \Delta^{-3/2} \square X \eta_{\alpha\beta} \\
+ \frac{1}{4} \Delta^{-5/2} (c^4 - 10X^{-3}s^2 c^2 - 2X^{-6}s^4)(\partial_\alpha X) \eta_{\beta\alpha} \\
- \frac{1}{2}(6c^4 + 3X^{-3}s^2 c^2 + 6X^{-6}s^4)\partial_\alpha X \partial_\beta X \\
+ g^2 \Delta^{-5/2} (X^{-3} - 1)(2X^{-2}s^4 + 2X c^2 s^2 - X^2 c^2) \eta_{\alpha\beta} \\
- \frac{1}{4} s^2 X^2 \Delta^{-3/2} G_{\alpha\beta} \rho - \frac{1}{4} \epsilon^2 X^{-1} \Delta^{-3/2} F^i_{\alpha\beta} F^i_{\beta\rho} \\
\hat{R}_{55} = -3gs c X^{-1/2} \Delta^{-5/2} U \partial_\alpha X, \\
\hat{R}_{66} = -\frac{s}{2} X \Delta^{-1} \nabla^\beta G_{\alpha\beta} - \frac{s}{2} \Delta^{-2} (\delta^\beta \partial_\alpha X) G_{\alpha\beta}, \\
\hat{R}_{\alpha i} = -\frac{1}{2\sqrt{2}} c X^{-1/2} \Delta^{-1} D^\beta F^i_{\alpha\beta} + \frac{1}{2\sqrt{2}} X^{-1/2} \Delta^{-2} (c^2 - X^{-3}s^2) \partial_\alpha X F^i_{\alpha\beta}, \\
\hat{R}_{55} = -\frac{3}{4} c^2 \Delta^{-3/2} \square X + \frac{3}{4} c^2 \Delta^{-5/2} (c^2 - 2X^{-3}s^2)(\partial X)^2 \\
+ g^2 \Delta^{-5/2} (X^{-2} + 3\Delta), \\
\hat{R}_{66} = -\frac{3}{4} (c^2 + 2X^{-3}s^2) \Delta^{-3/2} \square X + \frac{3}{4} s^2 X^2 \Delta^{-3/2} G^2 \\
+ \frac{3}{4} \Delta^{-5/2} (c^4 + 2X^{-6}s^4 + 6X^{-3}s^2 c^2)(\partial X)^2 \\
+ g^2 \Delta^{-5/2} (2X c^4 + X c^2 + 2X^{-2} + 4X^{-2} c^2 s^2 + 2X^{-5}s^4 - X^{-2} c^4), \\
\hat{R}_{6i} = \frac{s c}{4\sqrt{2}} X^{1/2} \Delta^{-3/2} F^i_{\alpha\beta} G^{\alpha\beta}, \\
\hat{R}_{ij} = \frac{3c^2}{4} \Delta^{-3/2} \partial_\alpha \delta_{ij} - \frac{3c^2}{4} \Delta^{-5/2} (c^2 - 2X^{-3}s^2)(\partial X)^2 \delta_{ij} + \frac{1}{8} c^2 \Delta^{-3/2} F^i_{\alpha\beta} F^j_{\alpha\beta} \\
+ g^2 \Delta^{-5/2} (2X^2 \Delta^2 + XU) \delta_{ij},
\]

(23)
where $D^\alpha F^i_{\alpha \beta} \equiv \nabla^\alpha F^i_{\alpha \beta} + \frac{1}{\sqrt{2}} g e^{ijk} F^j_{\alpha \beta} A^{k \alpha}$.

The scalar curvature is given by

$$
\hat{R} = \Delta^{-1/2} R - \frac{1}{2}(c^2 - 2X^{-3} s^2) \Delta^{-3/2} \Box X - \frac{1}{2} (5X^{-1} c^2 + 8X^{-4} s^2) \Delta^{-3/2} (\partial X)^2 \\
+ g^2 \Delta^{-3/2} (14 - 6c^2 + 6X^3 c^2 + 12X^{-3} s^2) \\
- \frac{s^2}{4} X^2 \Delta^{-3/2} G^2_{(2)} - \frac{c^2}{8} X^{-1} \Delta^{-3/2} (F^i_{(2)})^2 
$$

(24)

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