ON A CLASS OF IMMERSIONS OF SPHERES INTO SPACE FORMS OF NONPOSITIVE CURVATURE

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Abstract. Let $M^{n+1}$ ($n \geq 2$) be a simply-connected space form of sectional curvature $-\kappa^2$ for some $\kappa \geq 0$, and $I$ an interval not containing $[-\kappa, \kappa]$ in its interior. It is known that the domain of a closed immersed hypersurface of $M$ whose principal curvatures lie in $I$ must be diffeomorphic to the $n$-sphere $S^n$. These hypersurfaces are thus topologically rigid.

The purpose of this paper is to show that they are also homotopically rigid. More precisely, for fixed $I$, the space $\mathcal{F}$ of all such hypersurfaces is either empty or weakly homotopy equivalent to the group of orientation-preserving diffeomorphisms of $S^n$. An equivalence assigns to each element of $\mathcal{F}$ a suitable modification of its Gauss map. For $M$ not simply-connected, $\mathcal{F}$ is the quotient of the corresponding space of hypersurfaces of the universal cover of $M$ by a natural free proper action of the fundamental group of $M$.

0. Introduction

Convention. Throughout the article, $n \geq 2$ is an integer and all manifolds are assumed to be connected, oriented and smooth, i.e., of class $C^\infty$. Maps between manifolds are also assumed to be smooth, and sets of maps are furnished with the $C^\infty$-topology.

Motivation. In 1958, S. Smale [29] proved that any two immersions of the 2-sphere $S^2$ into euclidean space $E^3$ are regularly homotopic, that is, homotopic through immersions. In particular, the inclusion $\iota: S^2 \to E^3$ is regularly homotopic to the composition $-\iota$ of $\iota$ with the antipodal map of $S^2$. In words, the sphere can be everted, or turned inside out, without developing singularities.

There is an obvious invariant which must be preserved by any regular homotopy, viz., the degree of the Gauss map. However, it is not too hard to prove that for any immersion $S^2 \to E^3$, this degree must be 1, so it is not an obstruction in this situation.

As the Gaussian curvatures of $\pm \iota: S^n \to E^{n+1}$ are both equal to $(-1)^n$, a natural question in this context is whether it is possible to deform $\iota$ into $-\iota$ in such a way that the Gaussian curvature does not vanish during the homotopy. For $n$ even such an eversion does not exist, because the principal curvatures of $\iota$ and $-\iota$ have opposite signs. If $n$ is odd, then such an eversion is possible. In fact, it will be shown that for odd $n$, two immersions of $S^n$ into $E^{n+1}$ having nonvanishing Gaussian curvature are homotopic through immersions of the same type if and only if their Gauss maps are isotopic as diffeomorphisms of $S^n$.

Summary of results. Let $M^{n+1}$ be a space form of sectional curvature $-\kappa^2$ for some $\kappa \geq 0$, and let $S^n$ be equipped with its standard smooth structure, but no pre-assigned Riemannian metric. The purpose of this paper is to study the homotopy type of the space $\mathcal{F}(M; I)$ of immersions $S^n \to M^{n+1}$ whose principal curvatures take on values in an interval $I$ not containing $[-\kappa, \kappa]$.

Let $\tilde{M}$ be the universal cover of $M$. It is shown in (5.7) that $\mathcal{F}(M; I)$ is a quotient of $\mathcal{F}(\tilde{M}; I)$ by a free proper action of the fundamental group of $M$, which is induced by the

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action of the latter on $\tilde{M}$ by deck transformations. This essentially reduces the problem to the case where $M$ is simply-connected.

A partial justification for considering only immersions of $S^n$, as opposed to any other closed manifold $N^n$, is provided by the following previously known facts. Suppose that either $N$ or $M$ is simply-connected. If $I$ is contained in $[-\kappa, \kappa]$, then no immersion $f: N \to M$ with principal curvatures constrained to $I$ exists; in particular, $\mathcal{F}(M; I)$ is empty. In contrast, if $I$ is disjoint from or overlaps (i.e., intersects but neither contains nor is contained in) $[-\kappa, \kappa]$, then such immersions exist only for $N$ diffeomorphic to $S^n$. A diffeomorphism $N \to S^n$ is given by a suitable version of the Gauss map of any lift of $f$ to the universal cover of $M$.

Our definition of this modified Gauss map depends on $\kappa$ and on the position of $I$ relative to $[-\kappa, \kappa]$ (but when $M = \mathbb{E}^{n+1}$, it coincides with the usual Gauss map); see (2.1), (3.1) and (4.1).

If $I$ does not contain $[-\kappa, \kappa]$, $\mathcal{F}(M; I)$ may thus be interpreted as the space of all closed hypersurfaces of $M$ with principal curvatures in $I$. For $M$ not simply-connected, the same interpretation is valid provided that any two hypersurfaces which have a common factor are identified, where $\tilde{f}: \tilde{N} \to M$ is called a factor of $f: N \to M$ if $f$ is the composition of $\tilde{f}$ with a covering map $N \to \tilde{N}$ (see (4.1) of [35] for the details).

Let the space of orientation-preserving (resp. -reversing) diffeomorphisms of $S^n$ be denoted by $\text{Diff}_+(S^n)$ (resp. $\text{Diff}_-(S^n)$). The cornerstone of the paper, which is a combination of (2.5), (3.5) and (4.3), is that if $M$ is simply-connected and $I$ overlaps or is disjoint from $[-\kappa, \kappa]$, then

$$\Phi: \mathcal{F}(M; I) \to \text{Diff}_+(S^n), \quad f \mapsto \tilde{\nu}_f \quad \text{and} \quad \Psi: \text{Diff}_+(S^n) \to \mathcal{F}(M; I), \quad g \mapsto f_0 \circ g$$

are weak homotopy equivalences. Here $\tilde{\nu}_f$ denotes the modified Gauss map of $f$ and $f_0$ is an arbitrary fixed element of $\mathcal{F}(M; I)$; the sign $\pm$ in the range of $\Phi$ depends explicitly on $I$ and $n$.

If $I$ is an open interval, then by invoking well-known general results on infinite-dimensional manifolds, the word “weak” in these assertions can be omitted, and it can even be concluded that $\mathcal{F}(M; I)$ is homeomorphic to $\text{Diff}_+(S^n)$. As a corollary, the space of immersions, or embeddings, of $S^n$ into $M$ having nonvanishing Gaussian curvature is homeomorphic to the full group $\text{Diff}(S^n)$; see (4.4) and (4.5). Our main results can be summarized as follows.

0.1 Theorem. Let $M^{n+1}$ be a space form of sectional curvature $-\kappa^2$, for some $\kappa \geq 0$. Let $I$ be an interval which either overlaps or is disjoint from $[-\kappa, \kappa]$. Then

$$\Psi: \text{Diff}_+(S^n) \to \mathcal{F}(M; I), \quad \Psi(g) = f_0 \circ g \quad (f_0 \in \mathcal{F}(M; I) \text{ arbitrary})$$

induces isomorphisms on $\pi_i$ for all $i \neq 1$. For $i = 1$, there is an exact sequence

$$1 \to \pi_1(\text{Diff}_+(S^n)) \to \pi_1(\mathcal{F}(M; I)) \to \pi_1(\mathcal{F}(M; I)) \to \pi_1(M) \to 1$$

where $\pi: \tilde{M} \to M$ is the universal cover of $M$ and $\Pi: \tilde{f} \mapsto \pi \circ \tilde{f}$.

The condition that a map $N^n \to M^{n+1}$ be an immersion with principal curvatures in an interval may be expressed by stating that its 2-jet extension satisfies a certain (complicated) second-order, underdetermined partial differential relation (see [10] or [13]). Roughly, the preceding theorem states that a compact family of hypersurfaces of $M$ with principal curvatures in $I$ is rigid in that it is uniquely determined, up to homotopy, by the corresponding family of modified Gauss maps of their lifts. We conjecture that the remaining case in which $I$ contains $[-\kappa, \kappa]$ in its interior abides to the h-principle in the sense that the inclusion of $\mathcal{F}(M; I)$ into the space of all immersions $S^n \to M$ is a weak homotopy equivalence. The latter is known by the work of S. Smale and M. Hirsch ([30], [17]) to be weakly homotopy equivalent to the space of bundle monomorphisms $TS^n \to T\mathbb{E}^{n+1}$, or alternatively, to the space of smooth maps $S^n \to SO_{n+1}$. However, the arguments here are fairly direct and no familiarity with the h-principle is assumed.
Related results and problems. To our knowledge, the topology of spaces of immersions with constrained second fundamental form has not been studied previously. However, there is an extensive literature on the geometry of individual hypersurfaces of this kind, especially concerning convexity and embeddedness. J. Hadamard showed in [14] that if $N^2$ is a closed surface and $f: N^2 \to \mathbb{E}^3$ is an immersion whose Gaussian curvature never vanishes, then $f$ is an embedding, $f(N)$ is the boundary of a convex body and $N$ is diffeomorphic to $S^2$. J. Stoker proved in [32] that if $N^2$ is complete, then under the same hypotheses on $f$, $N$ must be diffeomorphic to either $\mathbb{E}^2$ or $S^2$, and $f$ must again be an embedding. The generalization to higher dimensions was carried out by J. van Heijenoort [33] and R. Sacksteder [25]. M. do Carmo and F. Warner [8] proved versions of Hadamard’s theorem for closed immersed hypersurfaces of $S^{n+1}$ and $\mathbb{H}^{n+1}$. S. Alexander [1] extended it to any complete simply-connected ambient manifold of nonpositive sectional curvature. Her results include as immediate corollaries versions of (2.2) and (3.2), for which different proofs are given here, and the fact that a complete hypersurface of $\mathbb{H}^{n+1}$ with principal curvatures in $[-1,1]$ must be diffeomorphic to $\mathbb{R}^n$. Later R. Currier [7] proved an analogue of Stoker’s result for $\mathbb{H}^{n+1}$, under the hypothesis that the principal curvatures are everywhere $\geq 1$. For further results of a similar nature, the reader is referred to [2, 3, 4, 6, 9, 11, 12, 22, 34]. The analogues of the results presented here for $n = 1$ are studied in [26, 27, 28].

0.2 Question. Suppose that $I$ contains $[-1,1]$. Is the inclusion of $\mathcal{I}(\mathbb{H}^{n+1}; I)$ into the space of all immersions $S^n \to \mathbb{H}^{n+1}$ a weak homotopy equivalence?

0.3 Question. Does (0.1) still hold when $M$ is a Hadamard manifold of sectional curvature $\leq -\kappa^2$?

0.4 Question. Let $I = \cot J$ be an interval, where $J \subset (0,\pi)$. It is shown in [35] that $\mathcal{I}(S^{n+1}; I)$ is weakly homotopy equivalent to the twisted product of $SO_{n+2}$ and $\text{Diff}_+(S^n)$ by $SO_{n+1}$ (regarded as a subgroup of the former two) provided that $\text{length}(J) < \frac{\pi}{2}$. Roughly, this covers “half” of the cases, and is another instance of homotopical rigidity. What is the homotopy type of $\mathcal{I}(S^{n+1}; I)$ when $\text{length}(J) > \frac{\pi}{2}$?

Outline of the sections. Section 1 introduces the concepts that appear in the paper; it contains no new results and may be skipped and referred to only as necessary. In §2 we consider closed hypersurfaces of hyperbolic space of curvature $-\kappa^2$ in the case where $I$ is disjoint from $[\kappa, \kappa]$. In §3 the same is done for intervals which overlap $[\kappa, \kappa]$. In §4 the analogues for immersions into $\mathbb{E}^{n+1}$ are established. Section 5 deals with hypersurfaces of space forms of nonpositive curvature. Theorem (0.1) is proved in this section.

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1. Basic terminology

Let $N^n$ be a manifold and $M^{n+1}$ be a Riemannian manifold. As it will be necessary to consider several several immersions $f: N \to M$ having the same domain simultaneously, we adopt the convention that $N$ is furnished with the respective induced metric in each case. The Gauss map $\nu = \nu_f: N \to TN$ of $f$ is uniquely determined by the condition that for all $p \in N$, $(u_1, \ldots, u_n)$ is a positively oriented orthonormal frame in $TN_p$ if and only if

$$\left( df_p(u_1), \ldots, df_p(u_n), \nu(p) \right)$$
is a positively oriented orthonormal frame in $TM_f(p)$.

The shape operator $S = S_f$ of $f$ is the section of $\text{End}(TN)$ defined by:

$$S(u) = S_p(u) = -d f_p^{-1}(\nabla d f_p(u) \nu) \quad \text{for } u \in TN_p.$$  

(1) Here $\nabla$ denotes the Levi-Civita connection of $M$. Being a symmetric linear operator, $S_p$ is diagonalizable by a basis of $TN_p$, consisting of mutually orthogonal eigenvectors. The associated eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$ are the principal curvatures of $f$ at $p$. Its Gaussian or Gauss-Kronecker curvature $K_f$ is the real function on $N$ whose value at $p$ is $\lambda_1(p) \cdots \lambda_n(p)$.

1.1 Convention. The orientation of $\mathbb{R}^{n+1}$ will be the canonical one throughout. The unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ about the origin will be oriented by declaring that $(u_1, \ldots, u_n, p)$ is a positively oriented frame in $T\mathbb{S}^n_p$ if and only if $(u_1, \ldots, u_n, p)$ is positively oriented as a frame in $\mathbb{R}^{n+1}$. When working with the Poincaré half-space or ball model of $\mathbb{H}^{n+1}$, the orientation of the latter is that induced by $\mathbb{R}^{n+1}$.

1.2 Remark. The following geometric interpretation of the principal curvatures when $M = \mathbb{E}^{n+1}$ or $\mathbb{H}^{n+1}$ will be convenient. Given a one-dimensional subspace $\langle u \rangle \subset TN_p$ and a sufficiently small neighborhood $U$ of $p$ in $N$, the intersection of $f(U)$ with the totally geodesic submanifold $P$ tangent to the subspace of $TM_f(p)$ spanned by $df_p(u)$ and $\nu(p)$ is a regular curve $\gamma$ in $P$. Moreover, $P$ is isometric to the euclidean or hyperbolic plane, respectively. The curvature of $\gamma$ as a curve in $P$ at $f(p)$, with sign taken to be positive or negative according to whether $\gamma$ curves towards or away from $\nu(p)$, is the value of the normal curvature of $f$ at $p$ in the direction of $u$. When $0 \neq u_k$ is one of the principal directions at $p$, that is, $S(u_k) = \lambda_k(p) u_k$, then the corresponding normal curvature is exactly the principal curvature $\lambda_k(p)$.

1.3 Remark. Let $f: N^n \to \mathbb{E}^{n+1}$ be an immersion. The canonical trivialization of $T\mathbb{E}^{n+1}$ allows us to regard $\nu_f$ as a map $N \to \mathbb{S}^n$. As subspaces of $\mathbb{R}^{n+1}$, $T\mathbb{S}^n_{\nu_f(p)}$ and $df_p(TN_p)$ are both orthogonal to $\nu(p)$, hence they coincide. If $TN_p$ is identified with $df_p(TN_p)$ via $df_p$, then the derivative of $d\nu_p$ may be considered as a linear operator on $TN_p$. Its negative coincides with the shape operator at $p$. In symbols:

$$S = -df^{-1} \circ d\nu.$$  

(2)

1.4 Definition. Let $M^{n+1}$ be a Riemannian manifold and $I$ be any interval of the real line. The set of all immersions $\mathbb{S}^n \to M^{n+1}$, furnished with the $C^\infty$-topology, will be denoted by $\mathcal{F}(M)$. Its subspace consisting of those immersions whose principal curvatures are constrained to $I$ will be denoted by $\mathcal{F}(M; I)$.

1.5 Remark. The group $\text{Diff}(\mathbb{S}^n)$ acts effectively on the right on the space $\mathcal{F}(M)$ by pre-composition. (It can also be shown that this action is free, though this fact will not be used anywhere.) Given $f \in \mathcal{F}(M)$ and $g \in \text{Diff}(\mathbb{S}^n)$, the Gauss map and shape operator of $f \circ g$ are related to those of $f$ by

$$\nu_{f \circ g} = \pm \nu_f \circ g, \quad d\nu_{f \circ g} = \pm d\nu_f \circ dg \quad \text{and} \quad S_{f \circ g} = \pm dg^{-1} \circ S_f \circ dg;$$  

(3) here the signs $\pm$ are consistent and depend on whether $g$ is orientation-preserving $(\pm)$ or orientation-reversing $(-)$. In other words, the sign is that of $\deg(g)$, the degree of $g$.

It follows that the principal curvatures $\mu_k$ of $f \circ g$ and $\lambda_k$ of $f$ are related by

$$\mu_k = \deg(g)(\lambda_k \circ g) \quad (k = 1, \ldots, n).$$  

(4) The Gaussian curvatures satisfy

$$K_{f \circ g} = \deg(g)^n(K_f \circ g).$$
Similarly, for immersions into \( \mathbb{E}^{n+1} \), the degrees of the Gauss maps (interpreted as maps \( S^n \to S^n \), as in (1.3)) are related by

\[
\deg(\nu_{fg}) = \deg(g)^n \deg(\nu_f).
\]

In particular, pre-composition with \( g \) maps \( \mathcal{F}(M; I) \) homeomorphically onto itself or onto \( \mathcal{F}(M; -I) \), depending on the sign of \( \deg(g) \). Thus it is reasonable to expect the topology of \( \mathcal{F}(M; I) \) to be at least as complicated as that of \( \text{Diff}_+(S^n) \).

1.6 Lemma. Given any interval \( I \), \( \mathcal{F}(M; I) \) and \( \mathcal{F}(M; -I) \) are homeomorphic. \( \square \)

1.7 Examples. The following simple examples should clarify the definitions.

(a) Let \( \iota: S^n \to \mathbb{E}^{n+1} \) denote set inclusion. Then \( \nu_\iota = \text{id}_{S^n} \) has degree 1 and \( S_\iota = -\text{id}_{T^n} \).

The principal curvatures equal \( \kappa > 1 \) everywhere. The Gaussian curvature is \( (-1)^n \).

(b) Let \( \rho: S^n \to S^n \) be a reflection in a hyperplane. Then \( \nu_{\iota \rho} = -\rho \) has degree \( (-1)^n \) and \( S_{\iota \rho} = \text{id}_{T^n} \).

The principal and Gaussian curvatures of \( \iota \circ \rho \) equal 1 everywhere.

(c) Let \( -\iota: S^n \to \mathbb{E}^{n+1} \) be the composition of \( \iota \) with the antipodal map \( -\text{id}_{S^n} \).

By (3):

\[
\nu_{-\iota} = (-1)^n \text{id}_{S^n} \quad \text{and} \quad S_{-\iota} = (-1)^n \text{id}_{T^n}.
\]

Consequently \( \deg(\nu_{-\iota}) = 1 \) regardless of the value of \( n \). The Gaussian and principal curvatures are everywhere equal to \( (-1)^n \).

Remark. A theorem due to H. Hopf states that if \( n \) is even and \( N^n \) is a closed manifold, then the degree of the Gauss map of any immersion \( f: N \to \mathbb{E}^{n+1} \) equals half the Euler characteristic of \( N \). See [20], p. 275 or the original sources [18, 19].

2. Locally convex hypersurfaces of \( \mathbb{H}^{n+1}_{-\kappa^2} \)

Let \( \kappa > 0 \) and \( \mathbb{H}^{n+1}_{-\kappa^2} \) denote hyperbolic \((n + 1)\)-space of sectional curvature \(-\kappa^2\), obtained from the usual hyperbolic space \( \mathbb{H}^{n+1} = \mathbb{H}^{n+1}_{-1} \) by rescaling all distances by the factor \( 1/\kappa \).

2.1 Definition. Regard \( \mathbb{H}^{n+1}_{-\kappa^2} \) as a subset of \( \mathbb{R}^{n+1} \) via the half-space model, and let

\[
f = (f^1, \ldots, f^{n+1}): N^n \to \mathbb{H}^{n+1}_{-\kappa^2} \subset \mathbb{R}^{n+1}
\]

be an immersion. Under the natural trivialization of \( T\mathbb{H}^{n+1}_{-\kappa^2} \) induced by that of \( T\mathbb{R}^{n+1} \), the Gauss map \( \nu_f: N^n \to T\mathbb{H}^{n+1}_{-\kappa^2} \) corresponds to:

\[
\nu_f = (f, \kappa f^{n+1} \tilde{\nu}_f): N^n \to \mathbb{H}^{n+1}_{-\kappa^2} \times \mathbb{R}^{n+1}.
\]

The map \( \tilde{\nu}_f: N^n \to S^n \) so defined will be called the flat Gauss map of \( f \). It is obtained from \( \nu_f \) by ignoring basepoints and normalizing so that \( \tilde{\nu}_f \) has unit euclidean length.

2.2 Lemma. Let \( N^n \) be a closed manifold. Suppose that \( f: N^n \to \mathbb{H}^{n+1}_{-\kappa^2} \) is an immersion whose principal curvatures have absolute value greater than \( \kappa \). Then:

(a) \( \tilde{\nu}_f: N^n \to S^n \) is a diffeomorphism.

(b) The principal curvatures of \( f \) are either all \( > \kappa \) or all \( < -\kappa \).

(c) \( f \) is an embedding.

Proof. To simplify the notation, it will be assumed (without loss of generality) that \( \kappa = 1 \). Part (a) can be proved by a straightforward calculation relating differentiation of \( \tilde{\nu} = \tilde{\nu}_f \) to covariant differentiation of \( \nu = \nu_f \), as follows.

\footnote{Parts (b) and (c) are not new, and neither is the fact (due to S. Alexander, Prop. 1 of [1]) that \( N \) must be diffeomorphic to \( S^n \). However, for our proof of (2.5), it is necessary to establish that \( \tilde{\nu}_f \) is a diffeomorphism.}
Let \( u \in TN_p \) and let \( \gamma : \mathbb{R} \to N \) be a smooth curve with \( \gamma(0) = p \) and \( \dot{\gamma}(0) = u \). We may regard \( \bar{\nu} \) as a map of \( N^n \) into \( \mathbb{S}^n \) but also as a vector field along \( f \), through the trivialization of \( T\mathbb{H}^{n+1} \). Then:

\[
\begin{align*}
\bar{\nu}_p(u) &= \frac{d}{dt}(\bar{\nu} \circ \gamma)(0) = \frac{d}{dt}((\bar{\nu} \circ f^{-1}) \circ (f \circ \gamma))(0) - \sum_{i,j,k=1}^{n+1} \Gamma^k_{ij}(f(p)) \frac{d}{dt}(f \circ \gamma)^i(0)\bar{\nu}^j(p)e_k \\
&= \nabla d\nu_p(u) - \sum_{i,j,k=1}^{n+1} \Gamma^k_{ij}(f(p)) \frac{d}{dt}(f \circ \gamma)^i(0)\bar{\nu}^j(p)e_k \\
&= \frac{1}{f^{n+1}(p)} \nabla d\nu_p(u) - \frac{d}{dt}(f \circ \gamma)^{n+1}(0)\nu(p) - \sum_{i,j,k=1}^{n+1} \Gamma^k_{ij}(f(p)) \frac{d}{dt}(f \circ \gamma)^i(0)\bar{\nu}^j(p)e_k.
\end{align*}
\]

(5)

Recall that the Christoffel symbols in the half-space model are given by:

\[
\Gamma^r_{r(n+1)}(q) = \Gamma^r_{(n+1)r}(q) = \Gamma^{n+1}_{(n+1)(n+1)}(q) = -\frac{1}{q^{n+1}} = -\Gamma^{n+1}_{rr}(q) \quad (1 \leq r \leq n).
\]

The second term in (5) cancels those terms in the summation involving \( \Gamma^k_{(n+1)k} \) for \( k = 1, \ldots, n+1 \). Moreover,

\[
\begin{align*}
\sum_{k=1}^{n+1} \Gamma^k_{k(n+1)}(f(p)) \frac{d}{dt}(f \circ \gamma)^k(0)\bar{\nu}^{n+1}(p)e_k &= -\frac{\bar{\nu}^{n+1}(p)}{f^{n+1}(p)} d\nu_p(u) \quad \text{and} \\
\sum_{r=1}^{n} \Gamma_{rr}^{n+1}(f(p)) \frac{d}{dt}(f \circ \gamma)^r(0)\nu^r(p)e_{n+1} - \Gamma_{(n+1)(n+1)}^{n+1}(f(p)) \frac{d}{dt}(f \circ \gamma)^{n+1}(0)\nu^{n+1}(p)e_{n+1} &= (d\nu_p(u), \nu(p)) e_{n+1} = 0,
\end{align*}
\]

where \( \langle , \rangle \) denotes the hyperbolic Riemannian metric. Now assume that \( u \) is a principal direction for \( f \), associated to the principal curvature \( \lambda \). Adding the preceding equations and substituting the result in (5), we deduce that:

\[
\begin{align*}
d\bar{\nu}_p(u) &= \left( \frac{\bar{\nu}^{n+1}(p) - \lambda}{f^{n+1}(p)} \right) d\nu_p(u) \in \mathbb{R}^{n+1}.
\end{align*}
\]

This is a nonzero multiple of \( d\nu_p(u) \) because by hypothesis \(| \lambda | > 1 \) and \( \bar{\nu} \) is a unit vector in the euclidean sense. Since \( TN_p \) admits a basis consisting of principal directions for each \( p \in N \), \( \bar{\nu} \) is a local diffeomorphism. As \( N \) is compact by hypothesis, \( \bar{\nu} \) must be a covering map and hence a global diffeomorphism, since \( \mathbb{S}^n \) is simply-connected. This proves (a).

Now let \( o \in \mathbb{H}^{n+1} \) be any point not in the image of \( f(N) \). Let \( p \) be a point where the function \( q \mapsto d(f(q), o) \) defined on \( N \) attains its maximum value \( r \). By comparison with the metric sphere of radius \( r \) centered at \( o \), one deduces that all principal curvatures of \( f \) at \( p \) must have the same sign, which depends on whether \( \nu(p) \) points towards or away from \( o \) along the geodesic \( op \). But then by connectedness, the principal curvatures have the same sign everywhere. This establishes (b).

The following proof of (c) is a straightforward adaptation of the proof of the analogous result for immersions into \( \mathbb{E}^{n+1} \) given in [24], p. 96. It suffices to show injectivity of \( f \). Let \( p \in N \) be arbitrary. Composing \( f \) with an isometry of \( \mathbb{H}^{n+1} \) if necessary, it may be assumed that \( \bar{\nu}(p) = e_{n+1} \). The smooth function \( \delta : q \mapsto f^{n+1}(q) - f^{n+1}(p) \) must attain its global extrema on \( N \) by compactness, hence there exist at least two critical points. At any critical point, \( \bar{\nu} = \pm e_{n+1} \). By injectivity of \( \bar{\nu} \), established in (a), there are thus exactly two critical points, and one of them must be \( p \); let \( q \) denote the other one. If \( \delta(q) = \delta(p) = 0 \), then \( f^{n+1}(N) = f^{n+1}(p) \), that is, the image of \( f \) is contained in a horosphere, which has principal
curvatures identically equal to $\pm 1$; this is impossible. Therefore, say, $f^{n+1}(q) > f^{n+1}(p)$. If $f(p') = f(p)$, then in particular $p'$ is a global minimum, hence $p' = p$. Thus $f$ is injective. \(\square\)

2.3 Remark. The argument used to prove (2.2) (b) actually shows that if $N^n$ is closed and $f: N^n \to \mathbb{H}^{n+1}_{-\kappa^2}$ is an immersion whose Gaussian curvature never vanishes, then its principal curvatures are everywhere positive or everywhere negative.

In the sequel, the notation $I > \kappa$ indicates that all elements of $I$ are greater than $\kappa$.

2.4 Corollary. Let $I$ be an interval disjoint from $[-\kappa, \kappa]$ and $f \in \mathcal{F}(\mathbb{H}^{n+1}_{-\kappa^2}; I)$. Then $\bar{\nu}_f \in \text{Diff}_+ (\mathbb{S}^n)$ unless $n$ is odd and $I > \kappa$, in which case $\bar{\nu}_f \in \text{Diff}_- (\mathbb{S}^n)$.

Proof. By (2.2) (a), $\bar{\nu} = \bar{\nu}_f \in \text{Diff}(\mathbb{S}^n)$. Let $u$ be a principal direction for $f$ at $p$. As in (7), $d\bar{\nu}_p (u)$ is a positive or negative multiple of $df_p (u)$ according to whether $I < -\kappa$ or $I > \kappa$, respectively. The assertion is now a straightforward consequence of the definition of $\nu, \bar{\nu}$ and of the choice (1.1) of orientation of $\mathbb{H}^{n+1}$.

2.5 Proposition. Let $I$ be disjoint from $[-\kappa, \kappa]$ and $f_0 \in \mathcal{F}(\mathbb{H}^{n+1}_{-\kappa^2}; I)$ be arbitrary. Then

$$\Phi: \mathcal{F}(\mathbb{H}^{n+1}_{-\kappa^2}; I) \to \text{Diff}_\pm (\mathbb{S}^n), \quad f \mapsto \bar{\nu}_f \quad \text{and} \quad \Psi: \text{Diff}_+ (\mathbb{S}^n) \to \mathcal{F}(\mathbb{H}^{n+1}_{-\kappa^2}; I), \quad g \mapsto f_0 \circ g$$

are homotopy equivalences. The sign in the range of $\Phi$ is positive unless $I > \kappa$ and $n$ is odd.

Remark. A classical theorem due to S. Smale [31] states that $\text{Diff}(\mathbb{S}^2)$ has $O_2$ as a deformation retract, and a theorem of A. Hatcher [15] states that the inclusion of $O_4$ in $\text{Diff}(\mathbb{S}^3)$ is a homotopy equivalence. This allows one to replace $\text{Diff}_+ (\mathbb{S}^n)$ by $SO_n$ in the statement when $n = 2, 3$. For $n \geq 5, \text{Diff}_+ (\mathbb{S}^n)$ is not homotopy equivalent to $SO_{n+1}$; in fact, it is even disconnected whenever there exist exotic spheres of dimension $n + 1$ (see [21], [5] and the references therein).

Proof. The last assertion is just (2.4). It suffices to consider the case where $\kappa = 1$. By (1.6), no generality is lost in assuming that $I$ lies to the left of $[-1, 1]$. We shall work with the half-space model. Let $\mu \in I$ be arbitrary and

$$\sigma: \mathbb{S}^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1}, \quad \sigma: p \mapsto c + rp \quad (p \in \mathbb{S}^n),$$

where

$$r = \frac{-1}{\mu + 1} > 0 \quad \text{and} \quad c = \frac{\mu}{\mu + 1} \rho_{n+1} \in \mathbb{H}^{n+1}.$$ 

This is an embedding whose image is a sphere in $\mathbb{H}^{n+1}$ of hyperbolic radius $\text{arccoth}(-\mu)$. Its relevant properties are the following:

* At each point $p \in \mathbb{S}^n$, $d\sigma_p = rtp$, where $tp: T_{S^p} \hookrightarrow \mathbb{R}^{n+1}$ denotes set inclusion.

* $\bar{\nu}_\sigma = \text{id}_{\mathbb{S}^n}$, by the conformality of the hyperbolic and euclidean metrics in the half-space model.

* All principal curvatures of $\sigma$ are equal to $\mu$. For $\text{Iso}(\mathbb{H}^{n+1})$ contains a copy of $SO_n$ preserving $\sigma(\mathbb{S}^n)$ (this is more easily visualized in the ball model), hence all principal curvatures are equal; and a circle of hyperbolic radius $\rho$ has curvature $\pm \text{coth}(\rho)$.

The sign of the principal curvatures can be gleaned immediately from (1.2).

Define

$$\Psi_\sigma: \text{Diff}_+ (\mathbb{S}^n) \to \mathcal{F}(\mathbb{H}^{n+1}; I) \quad \text{by} \quad \Psi_\sigma (g) = \sigma \circ g.$$ 

It will be proven that $\Phi$ and $\Psi_\sigma$ are homotopy inverses, where $\Phi$ is as in the statement. By (3), $\nu_{\sigma \circ g} = \nu_\sigma \circ g$ for all $g \in \text{Diff}_+ (\mathbb{S}^n)$, hence

$$\Phi \circ \Psi_\sigma (g) = \nu_{\sigma \circ g} = \nu_\sigma \circ g = g.$$ 

Thus $\Phi \circ \Psi_\sigma = \text{id}_{\text{Diff}_+ (\mathbb{S}^n)}$. To show that $\Psi_\sigma \circ \Phi \simeq \text{id}_{\mathcal{F}(\mathbb{H}^{n+1}; I)}$, consider the homotopy

$$(s, f) \mapsto f_s = (1 - s)f + sh, \quad \text{where} \ s \in [0, 1], \ f \in \mathcal{F}(\mathbb{H}^{n+1}; I) \ \text{and} \ h = \sigma \circ \bar{\nu}_f.$$
This is clearly continuous. Moreover, \( f_0 = f \) and \( f_1 = \Psi_\sigma \circ \Phi(f) \). We claim that for each \( s \in [0, 1] \):

(i) \( f_s \) is indeed an immersion;

(ii) \( \tilde{\nu}_f = \nu_f \colon S^n \to S^n \);

(iii) the principal curvatures of \( f_s \) lie in \( I \).

Fix an arbitrary \( p \in S^n \) and let \( u \in T_{f_s} S^n \) be a principal direction for \( f \) at \( p \), corresponding to the principal curvature \( \lambda < -1 \). By (7),

\[
dh p(u) = d \sigma_{\nu_f(p)} \circ d(\tilde{\nu}_f)_p(u) = \frac{r}{f_n+1(p)}(\tilde{\nu}_f^{n+1}(p) - \lambda df_p(u)
\]

is a positive multiple of \( df_p(u) \). Thus, so is \( d(f_s)_p(u) \) for all \( s \in [0, 1] \). This in turn implies (i), since \( f \) is an immersion. It also implies that \( \tilde{\nu}_{f_s}(p) = \pm \nu_f(p) \) for each \( s \in [0, 1] \). But then (ii) holds by continuity with respect to \( s \).

It remains to prove (iii). Notice first that

\[
\nu_{f_s} = f_s^{n+1} \tilde{\nu}_{f_s} = f_s^{n+1} \nu_f.
\]

To keep track of basepoints, let \( v_p \) denote the element of \( T^{n+1}_{f(p)} S^n \) corresponding to \( v \in \mathbb{R}^{n+1} \) under the trivialization of \( T^{n+1} S^n \) yielded by the half-space model. Then:

\[
\nabla d(f_s)_p(u)(\nu_{f_s}) = f_s^{n+1}(p) \nabla d(f_s)_p(u)(\tilde{\nu}_{f_s}) + d(f_s)_p(u) \tilde{\nu}_{f_s}(p)
\]

\[
= f_s^{n+1}(p)[(1-s)\nabla [df_p(u)]_{f_s(p)}(\tilde{\nu}_{f_s}) + s\nabla [df_p(u)]_{f_s(p)}(\nu_{f_s})] + T_1
\]

where \( T_i \) (\( i = 1, 2, 3 \)) will represent terms which are proportional to \( \nu_f \), and hence can be ignored, because \( \nabla d(f_s)_p(u)(\nu_{f_s}) \) has no normal component. Recalling the special form (6) of the Christoffel symbols and the fact that \( \tilde{\nu}_{f_s} = \nu_f \) as maps from \( S^n \) to \( S^n \),

\[
\nabla [df_p(u)]_{f_s(p)}(\nu_{f_s}) = d(\nu_f)_p(u) + \sum_{i,j,k=1}^{n+1} \Gamma_{ijk}^k (f_s(p)) df^i_p(u) \nu_f^j(p) e_k
\]

\[
= d(\nu_f)_p(u) + \frac{f^{n+1}(p)}{f_s^{n+1}(p)} \sum_{i,j,k=1}^{n+1} \Gamma_{ijk}^k (f(p)) df^i_p(u) \nu_f^j(p) e_k
\]

\[
= \left(1 - \frac{f^{n+1}(p)}{f_s^{n+1}(p)}\right) d(\nu_f)_p(u) + \frac{f^{n+1}(p)}{f_s^{n+1}(p)} \nabla df_p(u)(\nu_f)
\]

\[
= \left(1 - \frac{f^{n+1}(p)}{f_s^{n+1}(p)}\right) d(\nu_f)_p(u) + \frac{1}{f_s^{n+1}(p)} \nabla df_p(u)(\nu_f) + T_2
\]

\[
= \left(1 - \frac{f^{n+1}(p)}{f_s^{n+1}(p)}\right) d(\nu_f)_p(u) - \frac{\lambda}{f_s^{n+1}(p)} df_p(u) + T_2
\]

To obtain (11), (6) was used. And in passing to (12), the hypothesis that \( u \in T^n S^n \) is a principal direction for \( f \) associated to \( \lambda \) was used. Since any tangent vector to \( S^n \) is a principal direction of \( h \) associated to \( \mu \), we obtain similarly:

\[
\nabla [d\theta_p(u)]_{f_s(p)}(\nu_{f_s}) = \left(1 - \frac{h^{n+1}(p)}{f_s^{n+1}(p)}\right) d(\nu_f)_p(u) - \frac{\mu}{f_s^{n+1}(p)} df_p(u) + T_3.
\]

Recall from (9) that \( dh_p(u) = \rho df_p(u) \) for some \( \rho > 0 \). Substituting (12) and (13) into (10), we conclude that:

\[
\nabla d(f_s)_p(u)(\nu_{f_s}) = -[(1-s)\lambda + s \rho \mu] df_p(u).
\]

Therefore

\[
S_{f_s}(u) = -(d(f_s))^{-1}(\nabla d(f_s)_p(u) \nu_{f_s}) = \frac{(1-s)\lambda + s \rho \mu}{(1-s) + s \rho} u.
\]
Immersions of spheres into space forms of nonpositive curvature

The factor multiplying \( u \) in the preceding expression is some convex combination of \( \lambda \) and \( \mu \), hence it belongs to \( I \). This completes the proof of claim (iii).

Observe that in our definition (8) of \( \Psi_\sigma \), a particular immersion \( \sigma \in \mathcal{F}(\mathbb{H}^{n+1}; I) \) was chosen, instead of an arbitrary one \( f_0 \) as in the definition of \( \Psi = \Psi_{f_0} \) in the statement. Let \( g_0 = \hat{\nu}_f \). Then \( f_0 \) is homotopic to \( \sigma \circ g_0 \), as they have the same image under the homotopy equivalence \( \Phi \). Therefore \( \Psi_{f_0} \simeq \Psi_{\sigma \circ g_0} = \Psi_\sigma \circ L_{g_0} \), where \( L_{g_0} \) denotes left multiplication by \( g_0 \) in the group \( \text{Diff}_+(\mathbb{S}^n) \), which is a homeomorphism. Since \( \Psi_\sigma \) is a homotopy equivalence, so is \( \Psi_{f_0} \).

\( \square \)

3. Hypersurfaces of \( \mathbb{H}^{n+1} \) locally supported by horospheres

For concreteness, it will be assumed in the sequel that \( I < \kappa \); the case where \( I > -\kappa \) is analogous by (1.6).

3.1 Definition. Let \( f : N^n \to \mathbb{H}^{n+1} \) be an immersion. The visual Gauss map \( \hat{\nu}_f : N^n \to S_\infty^n \) is the map which assigns to each \( p \in N \) the endpoint in the ideal boundary \( S_\infty^n = \partial \mathbb{H}^{n+1} \) of the geodesic ray which issues from \( \nu_f(p) \) (or the asymptotic class of this ray).

3.2 Lemma. Let \( N^n \) be a manifold and \( I < \kappa \) be any interval (not necessarily one that overlaps \([ -\kappa, \kappa] \)). Suppose that \( f : N^n \to \mathbb{H}^{n+1}_{-\kappa^2} \) is an immersion whose principal curvatures take on values in \( I \). Then \( \hat{\nu}_f : N^n \to S_\infty^n \) is a local diffeomorphism.

Proof. Let \( p \in S^n \) be arbitrary and \( u \in T_{p}S^n \) be a principal direction. Let \( P \) be the 2-plane tangent to \( u \) and \( \nu_f(p) \), and let \( \hat{u} \) be a nonzero vector in \( T(S_\infty^n)_{\nu_f(p)} \) tangent to \( P \). Finally, let \( \gamma \) be the normal section to \( f \) at \( p \) in the direction of \( \hat{u} \), oriented so that its unit normal at \( p \) agrees with \( \nu_f(p) \). If \( \eta \) is the constant-curvature curve which osculates \( \gamma \) at \( p \), then \( \eta \) may be a circle, hypercycle or horocycle, but in any case its curvature is smaller than \( \kappa \), by (1.2) and the hypothesis on \( I \). Let \( \nu_\eta \) and \( \nu_\gamma \) denote the unit normals to \( \eta \) and \( \gamma \), respectively, and \( \hat{\nu}_\eta \) and \( \hat{\nu}_\gamma \) be the corresponding visual normals (that is, the maps into \( S_\infty^n \) which assign the endpoints of the geodesic rays issuing from \( \nu_\gamma \), \( \nu_\gamma \)). Then \( \hat{\nu}_\eta \) and \( \hat{\nu}_\gamma \) coincide up to first order at \( f(p) \), by construction. Furthermore, direct calculation shows that \( \hat{\nu}_\eta \) is immersive everywhere, so that \( \hat{\nu}_\gamma \) is immersive at \( f(p) \). In turn, this implies that \( d(\hat{\nu}_f)_p(u) \) is some nonzero multiple of \( \hat{u} \). Since \( T_{S^n}S^n \) admits a basis consisting of principal directions, we conclude that \( \hat{\nu}_f \) is a local diffeomorphism.

\( \square \)

3.3 Corollary. Let \( N^n \) be a closed manifold and \( I < \kappa \). If \( f : N^n \to \mathbb{H}^{n+1}_{-\kappa^2} \) is an immersion whose principal curvatures take on values in \( I \), then \( \hat{\nu}_f : N^n \to S_\infty^n \) is a diffeomorphism. 2

Remark. In the situation of (3.3), \( f \) need not be an embedding, but this does hold if the Gaussian curvature of \( f \) never vanishes. See Rmk. 1 and Thm. 1 in [1].

Remark. Given an immersion \( f : N^n \to \mathbb{H}^{n+1}_{-\kappa^2} \), let \( \hat{\nu}_f : N^n \to S_\infty^n \) denote the map which assigns to each \( p \in N \) the endpoint of the geodesic ray issuing from \( -\nu_f(p) \). It is not necessarily true that \( \hat{\nu}_f \) is a local diffeomorphism, even if \( I < \kappa \). On the other hand, if \( I > -\kappa \), then, in analogy with (3.2), \( \hat{\nu}_f \) is a local diffeomorphism (while \( \hat{\nu}_f \) may not be).

We will now study the topology of \( \mathcal{F}(\mathbb{H}^{n+1}; I) \) when \( I \) overlaps \([ -\kappa, \kappa] \), i.e., when \( I \) intersects but neither contains nor is contained in \([ -\kappa, \kappa] \).

3.4 Lemma. Let \( I < \kappa \) overlap \([ -\kappa, \kappa] \) and let \( f \in \mathcal{F}(\mathbb{H}^{n+1}_{-\kappa^2}; I) \). Given \( r \geq 0 \), define

\[
\begin{align*}
f_r : S^n &\to \mathbb{H}^{n+1}_{-\kappa^2}, \\
f_r : p &\mapsto \exp_{f(p)}(r\nu_f(p)),
\end{align*}
\]

where \( \exp \) denotes the exponential map of \( \mathbb{H}^{n+1}_{-\kappa^2} \). Then:

\( \underline{2} \)Again, that \( N \) is diffeomorphic to \( S^n \) is a corollary of Prop. 1 of [1], but for the proof of (3.5), we need the fact that \( \hat{\nu}_f \) is a diffeomorphism.
(a) \( f_r \in \mathcal{F}(\mathbb{H}^{n+1}_1; I) \) for all \( r \geq 0 \).
(b) \( \hat{\nu}_f = \nu_f \).
(c) The principal curvatures of \( f_r \) approach \(-\kappa\) monotonically and uniformly over \( \mathbb{S}^n \) as \( r \to +\infty \).

**Proof.** Again, it suffices to consider the case where \( \kappa = 1 \). We will work in the hyperboloid model of \( \mathbb{H}^{n+1}_1 \). Then \( f_r \) may be expressed as

\[
    f_r : p \mapsto \cosh r f(p) + \sinh r \nu_f(p) \quad (p \in \mathbb{S}^n).
\]

Let \( u \in T_{\hat{\nu}_f} \mathbb{S}^n_p \) be a principal direction for \( f \), associated to the principal curvature \( \lambda \in I \). The Christoffel symbols of the Lorentz metric on the ambient space \( \mathbb{R}^{n+1,1} \subset \mathbb{H}^{n+1}_1 \) are identically equal to 0. Therefore, if \( \nu_f \) is regarded as a map into \( \mathbb{R}^{n+1,1} \) as well as a vector field along \( f \), then

\[
    \nabla_{df_r(u)} \nu_f = d(\nu_f)_p(u) = -\lambda df_p(u),
\]

whence

\[
    d(f_r)_p(u) = \cosh r df_p(u) + \sinh r d(\nu_f)_p(u) = (\cosh r - \lambda \sinh r) df_p(u).
\]

Now \( \lambda < 1 \) by hypothesis, hence the factor multiplying \( df_p(u) \) in (14) is positive for all \( r \geq 0 \). This implies that \( f_r \) is an immersion. Furthermore, it can be verified directly that

\[
    \nu_f(p) = \sinh r f(p) + \cosh r \nu_f(p).
\]

The geodesic ray issuing from \( \nu_f(p) \) is thus parametrized by

\[
    t \mapsto \cosh(r + t)f(p) + \sinh(r + t) \nu_f(p) \quad (t \geq 0),
\]

so that its image is contained in that of the geodesic ray which issues from \( \nu_f(p) \). In particular, \( \hat{\nu}_f = \nu_f \), which proves (b). Finally, combining (14) and (15), one deduces that

\[
    d(\nu_f)_p(u) = \frac{\tanh r - \lambda}{1 - \lambda \tanh r} d(f_r)_p(u).
\]

It follows that \( u \) is a principal direction for \( f_r \) at \( p \) associated to the principal curvature

\[
    \lambda(r) = \frac{\lambda - \tanh r}{1 - \lambda \tanh r}.
\]

This has the following behavior with respect to \( r \): If \( \lambda = -1 \), then \( \lambda(r) = -1 \) for all \( r \geq 0 \); if \( \lambda < -1 \), say \( \lambda = -\coth(\ell) \) for some \( \ell > 0 \), then \( \lambda(r) = -\coth(\ell + r) \); and if \( \lambda \in (-1, 1) \), say \( \lambda = \tanh(\ell) \) for some \( \ell \in \mathbb{R} \), then \( \lambda(r) = \tanh(\ell - r) \). In any case, \( \lambda(r) \) is monotone with respect to \( r \) and approaches \(-1\) as \( r \to +\infty \). Moreover, the convergence is uniform over \( \mathbb{S}^n \) by compactness. \( \square \)

3.5 Proposition. Let \( I \) overlap \([-\kappa, \kappa]\) and let \( f_0 \in \mathcal{F}(\mathbb{H}^{n+1}_{1-\kappa^2}; I) \) be arbitrary. Then

\[
    \Psi : \text{Diff}_+(\mathbb{S}^n) \to \mathcal{F}(\mathbb{H}^{n+1}_{1-\kappa^2}; I), \quad g \mapsto f_0 \circ g \quad \text{and}
\]

\[
    \Phi : \mathcal{F}(\mathbb{H}^{n+1}_{1-\kappa^2}; I) \to \text{Diff}_+(\mathbb{S}^n), \quad \begin{cases} f \mapsto \hat{\nu}_f & \text{if } I < \kappa \\ f \mapsto \nu_f & \text{if } I > -\kappa \end{cases}
\]

are weak homotopy equivalences.

**Proof.** As before, no generality is lost in assuming that \( \kappa = 1 \) and \( I < 1 \). We will work in the Poincaré ball model; \( \mathbb{S}^n_\infty = \partial \mathbb{H}^{n+1} \) will thereby be identified with the unit sphere \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \) about the origin. Let \( \mu < -1 \) be an arbitrary element of \( I \), and let

\[
    \sigma : \mathbb{S}^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1}, \quad p \mapsto -\mu^{-1} p.
\]

...
The image of \( \sigma \) is a sphere of hyperbolic radius \( \text{arctanh} \left( -\mu^{-1} \right) \), and its principal curvatures are everywhere equal to \( \mu \). Because in the ball model the geodesics through the origin are radial segments,

\[
\hat{\nu}_{\sigma g} = \hat{\nu}_e \circ g = \text{id}_{S^n} \circ g = g \quad \text{for any } g \in \text{Diff}_+(S^n).
\]

Define

\[
\Psi_\sigma : (\text{Diff}_+(S^n), \text{id}_{S^n}) \to (\mathcal{F}(\mathbb{H}^{n+1}; I), \sigma) \quad \text{by} \quad \Psi_\sigma : g \mapsto \sigma \circ g
\]

and let \( \Phi : (\mathcal{F}(\mathbb{H}^{n+1}; I), \sigma) \to (\text{Diff}_+(S^n), \text{id}_{S^n}) \) be as in the statement. By (16), \( \Phi \circ \Psi_\sigma = \text{id}_{\text{Diff}_+(S^n)} \). Let \( k \geq 0 \) be an arbitrary integer and

\[
F : (S^k, -e_{k+1}) \to \mathcal{F}(\mathbb{H}^{n+1}; I), \sigma) \].

We will construct a basepoint-preserving homotopy between \( \Psi_\sigma \circ \Phi \circ F \) and \( F \).

Denote by \( f^z \) the immersion \( F(z) \in \mathcal{F}(\mathbb{H}^{n+1}; I) \) \( (z \in S^k) \). Define \( r : [0, 1] \to [0, +\infty] \) by \( r(s) = \tan(\frac{s}{r_1}) \). By (3.4)(a), the principal curvatures of \( f^z_{r(s)} \) take on values inside \( I \) for all \( s < 1 \). However, \( f^z_{r(1)} \) is not a valid immersion into \( \mathbb{H}^{n+1} \), for it coincides with \( \hat{\nu}_f : S^n \to S^n \).

This can be corrected as follows. For fixed \( \tau \in \left[ \frac{1}{2}, 1 \right] \), let \( \tau(s) f^z_{r(s)} \) denote the composition of \( f^z_{r(s)} \) with an euclidean homothety, centered at 0, by

\[
\tau(s) = 1 + s(\tau - 1) \quad (s \in [0, 1]).
\]

Then \( F_s : z \mapsto \tau(s) f^z_{r(s)} \) defines a homotopy in \( \mathcal{F}(\mathbb{H}^{n+1}) \) from \( F_0 = F \) to

\[
F_1 = -\tau \mu(\Psi_\sigma \circ \Phi \circ F).
\]

By (3.4)(c) and compactness of \( S^k \), the principal curvatures of \( f^z_{r(s)} \) converge uniformly to \(-1\) over \( S^k \times S^n \) as \( s \to 1 \), hence it is possible to choose \( \tau \) close enough to 1 so that this homotopy takes place inside \( \mathcal{F}(\mathbb{H}^{n+1}; I) \). Since \( I \) is convex, the homotopy can then be extended using homotheties to \( s \in [0, 2] \) so that \( F_2 = \Psi_\sigma \circ \Phi \circ F \). Finally, the loop \( s \mapsto F_s(-e_{k+1}) \) described by the basepoint in \( \mathcal{F}(\mathbb{H}^{n+1}; I) \) is null-homotopic, since it is of the form \( s \mapsto \rho(s) \sigma \) for some continuous \( \rho : [0, 2] \to [0, +\infty) \) with \( \rho(0) = \rho(2) = 1 \). Consequently, the homotopy can be further modified to become basepoint-preserving.

As \( \Psi_\sigma(g) = \sigma \circ g \) and \( \Phi(\sigma \circ g) = g \) for any \( g \in \text{Diff}_+(S^n) \), we conclude that \( \Psi_\sigma \) and \( \Phi \) induce the identity on \( \pi_0 \) and isomorphisms on the respective \( k \)-th homotopy groups based at \( g \) and \( \sigma \circ g \) for all \( g \). That \( \Psi_\sigma \) is an isomorphism on homotopy groups based at an arbitrary basepoint \( f_0 \) now follows from the fact that

\[
f_0 \simeq \mu \hat{\nu}_{f_0} \quad \text{for all } f_0 \in \mathcal{F}(\mathbb{H}^{n+1}; I).
\]

The same relation implies that if \( \Psi = \Psi_{f_0} \) is defined as in the statement, then \( \Psi \) is a weak homotopy equivalence, as it is homotopic to the composition \( \Psi_{\sigma \circ \hat{\nu}_{f_0}} = \Psi_\sigma \circ L_{\hat{\nu}_{f_0}} \) of a weak homotopy equivalence with a homeomorphism (where \( L_g \) denotes left multiplication by \( g \) in \( \text{Diff}_+(S^n) \)).

\[ \Box \]

\section{4. Locally convex hypersurfaces of \( \mathbb{E}^{n+1} \)}

In this section the analogues of the results of §2 for euclidean space will be stated. The proofs are also analogous, but easier since covariant differentiation is simpler. The “modified” Gauss map of an immersion \( f : S^n \to \mathbb{E}^{n+1} \) will be \( \nu_f \) itself, but regarded as a map \( S^n \to S^n \) instead of \( S^n \to T\mathbb{E}^{n+1} \); cf. (1.3).

\subsection{4.1 Lemma} Let \( N^n \) be a closed manifold. Suppose that \( f : N^n \to \mathbb{E}^{n+1} \) is an immersion whose Gaussian curvature never vanishes. Then:

(a) \( \nu_f : N^n \to S^n \) is a diffeomorphism.

(b) The principal curvatures of \( f \) are either all positive or all negative.
(c) $f$ is an embedding.

Proof. For proofs of (a) and (c), see [24], p. 96. The proof of (b) will be left to the reader (compare the proof of (2.2) (b)).  

4.2 Lemma. Let $I$ be an interval not containing 0 and $f \in \mathcal{F}(\mathbb{E}^{n+1}; I)$. Then $\nu_f \in \text{Diff}_+(\mathbb{S}^n)$ unless $n$ is odd and $I > 0$, in which case $\nu_f \in \text{Diff}_-(\mathbb{S}^n)$; compare (1.7) (a) and (b).

Proof. If $I < 0$, then all principal curvatures of $f$ are negative, so by (2) $d\nu_f$ is orientation-preserving, i.e., $\text{deg}(\nu_f) = 1$. If $I > 0$, then $\text{deg}(\nu_f) = (-1)^n$ by (2) again.  

4.3 Proposition. Let $I$ be an interval not containing 0 and $f_0 \in \mathcal{F}(\mathbb{E}^{n+1}; I)$ be arbitrary. Then

$$\Psi : \text{Diff}_+(\mathbb{S}^n) \to \mathcal{F}(\mathbb{E}^{n+1}; I), \quad g \mapsto f_0 \circ g$$

is a homotopy equivalence. In the other direction, $\Phi : f \mapsto \nu_f$ is a homotopy equivalence between $\mathcal{F}(\mathbb{E}^{n+1}; I)$ and $\text{Diff}_+(\mathbb{S}^n)$, the sign being positive unless $I > 0$ and $n$ is odd.

Proof. By (1.6), it can be assumed that $I < 0$. Let $\iota : \mathbb{S}^n \to \mathbb{E}^{n+1}$ denote set inclusion. Let $\mu \in I$ be arbitrary (in particular, $\mu < 0$). Then

$$\sigma = -\mu^{-1} \iota : \mathbb{S}^n \to \mathbb{E}^{n+1},$$

that is, $\iota$ followed by a homothety of ratio $-\mu^{-1}$, has principal curvatures identically equal to $\mu$; see (1.7) (a). Define

$$\Psi_\sigma : \text{Diff}_+(\mathbb{S}^n) \to \mathcal{F}(M; I) \quad \text{by} \quad \Psi_\sigma(g) = \sigma \circ g.$$  

Then $\Phi \circ \Psi_\sigma = \text{id}_{\text{Diff}_+(\mathbb{S}^n)}$ by (3). To show that $\Psi_\sigma \circ \Phi \simeq \text{id}_{\mathcal{F}(M; I)}$, consider the homotopy

$$(s, f) \mapsto f_s = (1 - s)f + s(\sigma \circ \nu_f), \quad \text{where } s \in [0, 1], \ f \in \mathcal{F}(M; I).$$

Then $f_0 = f$ and $f_1 = \Psi_\sigma \circ \Phi(f)$. Moreover, for all $s \in [0, 1]$:

(i) $f_s$ is an immersion;
(ii) $\nu_{f_s} = \nu_f$;
(iii) the principal curvatures of $f_s$ lie in $I$.

To prove these claims, fix $p \in M$ and let $u \in T_{\iota(p)} \mathbb{S}^n$ be a principal direction for $f$, corresponding to the principal curvature $\lambda < 0$. Then

$$d(\sigma \circ \nu_f)_p(u) = -\mu^{-1} d(\nu_f)_p(u) = \mu^{-1} \lambda d_f p(u).$$

Consequently,

$$(df_s)_p(u) = [(1 - s) + s\mu^{-1}\lambda] d_f p(u) = \mu^{-1}[(1 - s)\mu + s\lambda] d_f p(u)$$

is a positive multiple of $d_f p(u)$, as $I < 0$ by hypothesis. Since $f$ is an immersion, so is $f_s$. This proves (i), and (ii) is also an immediate consequence of (17).

Let $S_{f_s}$ be the shape operator of $f_s$. Then:

$$S_{f_s}(u) = -(df_s)^{-1}_p \circ (d\nu_{f_s})_p(u) = -(df_s)^{-1}_p \circ (d\nu_f)_p(u) = \lambda(d_{f_s})^{-1}_p(d_f p(u))$$

$$= [(1 - s)\lambda^{-1} + s\mu^{-1}]^{-1} u.$$  

The set $I^{-1} = \{ t^{-1} : t \in I \}$ is an interval, hence convex. Thus the factor multiplying $u$ lies in $(I^{-1})^{-1} = I$. This establishes (iii), showing that $\Phi$ and $\Psi_\sigma$ are homotopy inverses.

The same argument as in the last paragraph of the proof of (2.5) now implies that the map $\Psi$ described in the statement is also a homotopy equivalence.  

4.4 Remark. If $I$ is an open interval, then $\mathcal{F}(M; I)$ is a metrizable Fréchet manifold, as is $\text{Diff}(\mathbb{S}^n)$. It is known since the 1960’s [16, 23] that a weak homotopy equivalence between (infinite-dimensional) manifolds of this type is actually a homotopy equivalence, and that homotopy equivalence implies homeomorphism within this class.
4.5 Corollary. Let $M^{n+1}$ be a simply-connected space form of nonpositive curvature and let $N$ denote the space of all closed immersed (resp. embedded) hypersurfaces of $M$ whose Gaussian curvature never vanishes. Then

$$\Psi: \text{Diff}(S^n) \rightarrow N, \quad g \mapsto f \circ g \quad (f \in N \text{ arbitrary})$$

is a homotopy equivalence. In particular, $N$ is homeomorphic to $\text{Diff}(S^n)$ by (4.4).

Proof. That “immersed” and “embedded” are interchangeable in this situation follows from Thm. 1 in [1]. By (2.3) and (4.1)(b), a closed hypersurface of $M$ has nonvanishing Gaussian curvature if and only if its principal curvatures are all positive or all negative. Thus, if $I_- = (-\infty, 0)$ and $I_+ = (0, +\infty)$, then (2.2) and (4.1) imply that

$$N = \mathcal{F}(M; I_-) \sqcup \mathcal{F}(M; I_+)$$

By (4), (3.5) and (4.3), $\Psi$ is a (weak) homotopy equivalence onto one of these subspaces when restricted to each of $\text{Diff}_+(S^n)$. As it also induces a bijection on $\pi_0$, it is a homotopy equivalence. \qed

5. HYPERSURFACES OF SPACE FORMS OF NONPOSITIVE CURVATURE

5.1 Definition. Let $M^{n+1}$ be a Riemannian manifold and fix $q \in M$. We denote by $\mathcal{F}_s(M)$ and $\mathcal{F}_s(M; I)$ the subspaces of $\mathcal{F}(M)$ and $\mathcal{F}(M; I)$, respectively, consisting of those immersions mapping $-e_{n+1} \in S^n$ to $q \in M$.

Let $\text{Iso}_+(M)$ denote the group of orientation-preserving isometries of $M$.

5.2 Lemma. If $\text{Iso}_+(M)$ acts transitively on $M$, then $\mathcal{F}_s(M; I)$ is independent of the choice of basepoint used to define it; that is, different choices yield homeomorphic spaces. \qed

5.3 Lemma. Suppose that $\Gamma$ is a subgroup of $\text{Iso}_+(M)$ which acts simply transitively on $M$. Assume moreover that the map $\Gamma \rightarrow M$, $\gamma \mapsto \gamma p$ is open for some (and hence all) $p \in M$. Then $\mathcal{F}(M; I)$ is homeomorphic to $\Gamma \times \mathcal{F}_s(M; I)$ for any interval $I$.

Proof. Define $\phi: \Gamma \times \mathcal{F}_s(M; I) \rightarrow \mathcal{F}(M; I)$ by $\phi(\gamma, f) = \gamma \circ f$. It is easily checked that $\phi$ is a homeomorphism. \qed

5.4 Corollary. If $M^{n+1}$ is a simply-connected space form of nonpositive curvature, then $\mathcal{F}(M; I)$ is homeomorphic to $M \times \mathcal{F}_s(M; I)$.

Proof. If $M = \mathbb{E}^{n+1}$, then take $\Gamma = \mathbb{R}^{n+1}$ acting by translations on $\mathbb{E}^{n+1}$. If $M = \mathbb{H}^{n+1}_{-\kappa^2}$, then take $\Gamma$ to be the image of the monomorphism $\mathbb{R}^n \times \mathbb{R}^+ \rightarrow \text{Iso}_+(M)$, $(a, t) \mapsto \gamma_{(a, t)}$, where, in the half-space model,

$$\gamma_{(a,t)}: p \mapsto tp + (a, 0) \quad (a \in \mathbb{R}^n, \ t \in \mathbb{R}^+, \ p \in \mathbb{H}^{n+1}_{-\kappa^2} \subset \mathbb{R}^{n+1}).$$ \qed

In contrast, unless $n = 0$, 1 or 3, there exists no simply-transitive subgroup of $\text{Iso}_+(S^n)$, since $S^n$ cannot be given a Lie group structure.

In what follows let $M^{n+1}$ be any Riemannian manifold and $\tilde{M}$ be a covering space of $M$, with the induced smooth structure, orientation and Riemannian metric.

5.5 Lemma. A covering map $\pi: \tilde{M} \rightarrow M$ induces homeomorphisms $\mathcal{F}_s(\tilde{M}) \rightarrow \mathcal{F}_s(M)$ which restrict to homeomorphisms $\mathcal{F}_s(M; I) \rightarrow \mathcal{F}_s(M; I)$ for all $I$.

Proof. Let $q \in M$ be the basepoint used to define $\mathcal{F}_s(M)$ and let $\tilde{q} \in \pi^{-1}(q)$ be arbitrary. Then we have an induced map $\mathcal{F}_s(M) \rightarrow \mathcal{F}_s(M)$ given by post-composition with $\pi$, whose inverse is given by lifting elements of $\mathcal{F}_s(M)$ to $(\tilde{M}, \tilde{q})$. Because $\pi$ is an orientation-preserving local isometry, it preserves principal curvatures. \qed
5.6 Corollary. Let $M$ be a space form. Then different choices of basepoints for the definition of $\mathcal{F}_n(M; I)$ yield homeomorphic spaces.

Proof. Immediate from (5.2) and (5.5), since the group of orientation-preserving isometries of the universal cover of $M$ is transitive on points. □

5.7 Lemma. Let $M$ be any Riemannian manifold and $\pi: \tilde{M} \to M$ be a covering map. Then

\begin{equation}
\Pi: \mathcal{F}(\tilde{M}) \to \mathcal{F}(M), \quad \tilde{f} \mapsto \pi \circ \tilde{f}
\end{equation}

restricts to covering maps $\mathcal{F}(\tilde{M}; I) \to \mathcal{F}(M; I)$ for all intervals $I$. If $\pi$ is regular, then so is $\Pi$, and the corresponding groups of deck transformations are naturally isomorphic.

Proof. Fix $f \in \mathcal{F}(M)$ and let $p = f(-e_{n+1})$. Let $U$ be an evenly covered neighborhood of $p$, with $\pi^{-1}(U) = \bigcup_{\alpha \in I} \tilde{U}_\alpha$. Define a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{F}(M)$ by

$$\mathcal{U} = \{ h \in \mathcal{F}(M) : h(-e_{n+1}) \in U \}.$$ 

Then

$$\Pi^{-1}(\mathcal{U}) = \bigcup_{\alpha \in I} \tilde{\mathcal{U}}_\alpha, \quad \text{where} \quad \tilde{\mathcal{U}}_\alpha = \{ \tilde{h} \in \mathcal{F}(\tilde{M}) : \tilde{h}(-e_{n+1}) \in \tilde{U}_\alpha \}.$$ 

Moreover, $\Pi|_{\tilde{\mathcal{U}}_\alpha}: \tilde{\mathcal{U}}_\alpha \to \mathcal{U}$ is a homeomorphism for each $\alpha \in I$. Its inverse maps an arbitrary immersion $h \in \mathcal{U}$ to its lift

$$\tilde{h}_\alpha: (\mathbb{S}^n, -e_{n+1}) \to \left( \tilde{M}, (\pi|_{\tilde{U}_\alpha})^{-1}(h(-e_{n+1})) \right) \quad (\alpha \in I).$$

In addition, $\Pi$ restricts to covering maps $\mathcal{F}(\tilde{M}; I) \to \mathcal{F}(M; I)$ since it preserves principal curvatures.

Now suppose that $\pi$ is regular, i.e., that the deck transformation group $\text{Aut}(\pi)$ acts simply transitively on each fiber $\pi^{-1}(p)$ ($p \in M$). Define

$$\phi: \text{Aut}(\pi) \to \text{Aut}(\Pi), \quad \gamma \mapsto \gamma_*, \quad \text{where} \quad \gamma_*: \mathcal{F}(\tilde{M}) \to \mathcal{F}(\tilde{M}), \quad \tilde{f} \mapsto \gamma \circ \tilde{f}.$$ 

Then $\phi$ is a group monomorphism. Given two lifts $\tilde{f}, \tilde{f}'$ of $f \in \mathcal{F}(M)$, there exists $\gamma \in \text{Aut}(\pi)$ such that $\tilde{f}'(-e_{n+1}) = \gamma \circ \tilde{f}(-e_{n+1})$, so that $\tilde{f}' = \gamma_*(\tilde{f})$ by uniqueness of lifts. This implies that $\phi$ is surjective, hence an isomorphism. □

We are finally ready to prove the main theorem (0.1).

Proof of (0.1). Let $\tilde{f}: \mathbb{S}^n \to \tilde{M}^{n+1}$ be any lift of $f$. Let

$$\tilde{\Psi}: \text{Diff}_+((\mathbb{S}^n)) \to \mathcal{F}(\tilde{M}; I), \quad \tilde{\Psi}(g) = \tilde{f} \circ g.$$ 

Then in the following commutative diagram:

$$\begin{array}{ccc}
\text{Diff}_+((\mathbb{S}^n)) & \xrightarrow{\tilde{\Psi}} & \mathcal{F}(\tilde{M}; I) \\
& \downarrow{\Psi} & \downarrow{\Pi} \\
& \mathcal{F}(M; I) & \\
\end{array}$$

the map $\tilde{\Psi}$ is a weak homotopy equivalence by (2.5), (3.5) and (4.3), while $\Pi$ is a regular covering map with covering group isomorphic to $\pi_1(M)$ by (5.7). □

The assertion about $\pi_0$ can be paraphrased as follows when $I = (0, +\infty)$ or $I = (0, +\infty)$

5.8 Corollary. Let $M^{n+1}$ be space form of nonpositive curvature and $f, g: \mathbb{S}^n \to M$ be immersions whose principal curvatures are either both positive or both negative everywhere. Then $f$ and $g$ are homotopic through immersions of nonvanishing Gaussian curvature if and only if the (visual) Gauss maps of their lifts to $\tilde{M}$ are isotopic as diffeomorphisms of $\mathbb{S}^n$. □
In the situation of (5.8), note that if \( n \) is odd, then the sign of the principal curvatures of \( f \) is the same as that of \( K_f \). However, if \( n \) is even, then it can occur that \( f, g: S^n \to M^{n+1} \) are not homotopic through immersions of nonvanishing Gaussian curvature even though \( K_f = K_g \neq 0 \) is a constant and \( \nu_f, \nu_g \) (or \( \tilde
u_f, \tilde
u_g \)) are isotopic, because their principal curvatures have opposite signs. An example is obtained by taking \( f \) to be the inclusion \( S^n \hookrightarrow \mathbb{E}^{n+1} \) and \( g \) as its composition with the antipodal map (compare (1.7)).

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