Controlling nonlinear photon-photon interaction 
via a two-level system*

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Abstract

The problem of photon-photon interaction controlled by a two-level system is studied in this paper. Specifically, we have proposed two scenarios: Case 1, how a two-level system changes the pulse shapes of two initially uncorrelated input photons in a single input channel; and Case 2, how a two-level system entangles two counter-propagating photons, one in each input channel. The steady-state output field states for both cases are derived explicitly. For Case 1, the Wigner spectrum is used to exhibit the interesting properties of the output field state. For Case 2, the nonlinear property of the interaction between the two-level system and the two input photons has been revealed by the probabilities of observing photons in the output channels. In addition, two-photon interference, similar to Hong-Ou-Mandel effect, occurs when the two input photons are chosen with the same pulse shape.

Keywords: quantum control; two-level system; steady-state output field state; Wigner spectrum.

1 Introduction

Two-level systems (e.g., two-level atoms) are the simplest nonlinear quantum systems yet with rich nonlinear dynamics. The study of the interaction between photons and two-level systems is of great importance for quantum information processing, as photon-photon interaction controlled by a two-level system has many applications. For example, the strong nonlinear interaction between photons and optical emitters can be used to engineer a single-photon transistor [4]. The operation principle of the single-photon transistor is to use either zero or one photon in the storage step, then the subsequent transmitted or reflected photons are controlled by the conditional flip of the “gate” pulse. Another single-photon transistor is introduced in [27] to setup a circuit quantum electrodynamical (circuit QED) model, which consists of two two-level systems. Although no photons are exchanged between the two transmission lines in this circuit, one photon can completely block or enable the propagation of the other by the interaction between the two two-level systems. Recently, the realization of an optical transistor is given in [5], which consists of a four-level system and a stored photon to control the transmission of source photons.

The interaction between two-level systems and single-photon states has been well studied, see e.g., [37, 36]. In particular, the transmission and reflection probabilities in terms of the stationary output photon state are discussed. In [52], an analytical expression of the output field state has been derived for a class of quantum finite-level systems driven by single-photon input states. Interestingly, it is shown that linear systems theory [55] can be adopted to derive the pulse shapes of the output single-photon states.

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In addition to single-photon input states, two-photon scattering by a two-level system has been studied, see e.g., [38, 56, 22, 29, 53]. Based on the analysis of photon pulse shapes, scattering dynamics for co-propagating and counter-propagating photons are investigated in [29]. By combining the input-output formalism and the scattering matrix (S-matrix) method [38], one- and two-photon transport behavior has been investigated in [10, 55, 46]. The above studies were generalized to the N-photon scattering in [46], where an N-photon S-matrix is derived. For control engineering, it would be very useful to have the explicit form of the output filed state to enable the cascade systems analysis [14]. Unfortunately, no exact analytical expressions for the pulse shape of the output two-photon field state have been given in the literature.

Recently, a quantum stochastic differential equation (QSDE) approach has been presented in [31], the response of quantum systems (including optical cavities, optomechanical systems and two-level atoms) to a continuous-mode two-photon state has been analyzed. The theory proposed there is for general passive quantum systems, see in particular Section 3 in [31]. An analytic expression of the output field state of a two-level system driven by two photons is not given; instead, a simulation with two specific pulse shapes of the input photons is done in Section 6.2. Unfortunately, the form of the output field state [31, Eq. (56)] is not accurate; as a consequence, the subsequent simulation results are not convincing. In fact, the probability of observing at least one photon in the first output (namely, the right-going) channel [31, Fig. 6] cannot be less than 0.5. Indeed, it has been shown that the probability of observing two photons in the second output (namely, left-going) channel is less than 0.25 when the temporal decay rate of the incident photon pulse is bigger than the atom’s spontaneous emission rate, see \( \alpha \geq 1 \) in [36, Fig. 2(a)]. In other words, the probability of observing at least one photon in the first output (namely, the right-going) channel should be at least 0.75. Our simulation has also confirmed this, see Fig. 4 in section 5.

Motivated by the above discussions, in this paper we give explicit expressions of the output field states of a two-level system driven by two input photons. Two cases are studied. In Case 1, there is one input channel which contains two photons. The analytic form of the output two-photon state is derived, from which one can see that the pulse shape contains 16 terms (see Eq. (3.51) for detail). Interestingly, this is consistent with two-photon filtering for a two-level system driven by a two-photon state, in which a system of 16 ordinary differential equations are needed to derive the two-photon quantum filter [40]. We also investigate the Wigner spectrum of the output two-photon state. The Wigner spectrum can be used to study continuous-mode photon states in the time domain and frequency domain simultaneously. Moreover, it could provide more information than the normal ordering where the Dirac delta function in the correlation function is discarded [44, 9]. Here, we compare the Wigner spectra for the input and the output two-photon states, see Fig. 1. The comparison reveals several interesting properties of the output two-photon state. By the best knowledge of the authors, these properties have not yet been reported in the literature. In Case 2, there are two input channels, each of which having one photon. After deriving the analytic form of the output two-photon state, a rather detailed simulation is performed, which reveals the Hong-Ou-Mandel effect [18] and simulated emission phenomenon [36] in quantum optics.

In recent years, the problem of analysis and control of two-level systems has been investigated in the quantum control community. For example, control of closed two-level systems via Hamiltonian modulation has been studied in, e.g., [6, 17, 23, 20, 8, 50, 31, 7]. The systems studied in these papers are closed systems in the sense that there are no fluctuating external fields and the control appears in the form of modulated system Hamiltonian; therefore, quantum noise was not considered in these studies. On the other hand, control of open two-level systems has been studied in, e.g., [12, 26, 19]. In these studies, the input signal is a probe laser. Single- and multi-photon states have found promising applications in quantum information and quantum computation [28, 30, 44, 11, 27, 5, 51, 55, 52, 48]. The problem of single- and/or multi-photon filtering of two-level systems has been studied in, e.g., [10, 15, 40, 12]. The explicit expression of the output field state of a two-level system driven by a single-photon state is given in [52].

The paper is organized as follows. In section 2, some preliminary results are reviewed, including quantum system and field, two-level system, single-photon and two-photon states. The explicit form of the output field state for a two-level system driven by a two-photon input state in a single input channel is discussed in section 3. The scenario of a two-level system driven by two counter-propagating photons is considered in section 4. Two examples are given in section 5. Section 6 concludes this paper.
2 Preliminary

Notation $|0\rangle$ denotes the vacuum state of a free field, $|g\rangle$ and $|e\rangle$ stand for the ground and excited states of a two-level system respectively. The symbol $^\dagger$ stands for the complex conjugate of a complex number or the adjoint of a Hilbert space operator. $\sigma_- = |g\rangle \langle e|$, $\sigma_+ = |e\rangle \langle g| = (\sigma_-)^\dagger$, $\sigma_z = 2\sigma_+\sigma_- - I$, where $I$ is the identity operator. $\delta(t)$ is the Dirac delta function. $i = \sqrt{-1}$. The commutator between two operators $A$ and $B$ is $[A,B] = AB - BA$.

2.1 System and field

In this subsection, quantum systems and fields are briefly introduced, more details can be found in, e.g., [19, 33, 13, 3, 2, 43, 45, 48].

The $(S, L, H)$ formalism [14, 41, 54] is very convenient for describing Markovian quantum systems and networks. Here, $S$ is a unitary scattering operator, the operator $L$ determines the coupling between the system and the environment (which in this paper is a light field), and the self-adjoint operator $H$ is the initial system Hamiltonian. $S$, $L$, and $H$ are all system operators defined on a Hilbert space $\mathcal{H}_S$ in which system states reside. For clarity of presentation, in this paper we assume that $S = I$, namely, an identity operator. The light field has a bosonic annihilation operator $b(t)$ and a creation operator $b^\dagger(t)$ (the adjoint operator of $b(t)$); these are operators on a Fock space $\mathcal{H}_F$ (an infinite-dimensional Hilbert space). These field operators have the following properties

$$b(t)|0\rangle = 0, \quad [b(t), b^\dagger(r)] = [b^\dagger(t), b(r)] = 0, \quad [b(t), b^\dagger(r)] = \delta(t - r), \quad \forall t, r \in \mathbb{R}.$$  

Define integrated annihilation and creation field operators $B(t) \triangleq \int_{t_0}^t b(r)dr$ and $B^\dagger(t) \triangleq \int_{t_0}^t b^\dagger(r)dr$, where $t_0$ is the initial time, i.e., the time when the system starts its interaction with the field.

The dynamics of the compound (system plus field) system can be described by a unitary operator $U(t,t_0)$ on the tensor product Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_F$, which is the solution to the following quantum stochastic differential equation (QSDE) in Itô form

$$dU(t,t_0) = \left\{ -\frac{1}{2}L^\dagger L + iH \right\} dt + LdB^\dagger(t) - L^\dagger dB(t) \right\} U(t,t_0), \quad t \geq t_0$$  

with the initial condition $U(t_0,t_0) = I$. In Heisenberg picture, a system operator $X$ at time $t \geq t_0$ is

$$X(t) \equiv j_t(X) \triangleq U(t,t_0)^\dagger(X \otimes I)U(t,t_0),$$  

which is an operator on $\mathcal{H}_S \otimes \mathcal{H}_F$ and solves the following QSDE

$$dj_t(X) = j_t(\mathcal{L}_{00}(X))dt + j_t(\mathcal{L}_{01}(X))dB(t) + j_t(\mathcal{L}_{10}(X))dB^\dagger(t), \quad t \geq t_0$$  

with the initial condition $j_{t_0}(X) = X \otimes I$, where the Evans-Parthasarathy superoperators are [40],

$$\mathcal{L}_{00}(X) \triangleq \frac{1}{2}L^\dagger [X, L] + \frac{1}{2}[L^\dagger, X]L - i[X, H],$$  

$$\mathcal{L}_{01}(X) \triangleq [L^\dagger, X],$$  

$$\mathcal{L}_{10}(X) \triangleq [X, L].$$  

After interaction, the quantum output field

$$B_{\text{out}}(t) \triangleq U(t,t_0)^\dagger (I \otimes B(t))U(t,t_0)$$

is generated, which is also an operator on $\mathcal{H}_S \otimes \mathcal{H}_F$ and whose dynamics are given by the following QSDE

$$dB_{\text{out}}(t) = j_t(L)dt + dB(t).$$  


In this paper, instead of integrated quantum processes $B(t)$ and $B_{\text{out}}(t)$, we find it more convenient to work directly with the quantum processes $b(t)$ and

$$b_{\text{out}}(t) \doteq U(t,t_0)^\dagger b(t)U(t,t_0), \ t \geq t_0. \quad (2.9)$$

Moreover, the output field annihilation operator $b_{\text{out}}(t)$ enjoys the following property, see, e.g. [2] Section 5.2,

$$b_{\text{out}}(t) = U(\tau,t_0)^\dagger b(t)U(\tau,t_0), \ \forall \tau \geq t \geq t_0. \quad (2.10)$$

Finally, let $\tau = \max\{t_1,t_2\}$. Then by Eq. $(2.10)$, we have

$$[b_{\text{out}}(t_1),b_{\text{out}}(t_2)] = [U(\tau,t_0)^\dagger b(t_1)U(\tau,t_0),U(\tau,t_0)^\dagger b(t_2)U(\tau,t_0)] = U(\tau,t_0)^\dagger \{b(t_1),b(t_2)\}U(\tau,t_0).$$

However, noticing Eq. $(2.11)$, we conclude that

$$[b_{\text{out}}(t_1),b_{\text{out}}(t_2)] = 0, \ \forall t_1,t_2 \geq t_0. \quad (2.11)$$

Eq. $(2.11)$ is the so-called self-nondemolition feature of quantum light fields, [2].

### 2.2 Two-level system

In the $(S, L, H)$ formalism introduced above, the two-level system studied in this paper has the following system parameters

$$S = 1, \ L = \sqrt{\kappa} \sigma_z, \ H = \frac{\omega_d}{2} \sigma_z,$$

where $\kappa > 0$ determines the coupling strength between the system and the field, and $\omega_d \in \mathbb{R}$ is the frequency detuning (between the carrier frequency of the input field and the transition frequency of the two-level system). With these parameters, by Eqs. $(2.1)$ and $(2.5)$ we have the following QSDEs

$$d\sigma_-(t) = -\left(\frac{\kappa}{2} + i\omega_d\right)\sigma_-(t)dt + \sqrt{\kappa} \sigma_z(t)dB(t), \quad (2.12)$$

$$dB_{\text{out}}(t) = \sqrt{\kappa} \sigma_-(t)dt + dB(t), \quad t \geq t_0. \quad (2.13)$$

Alternatively, we may write the system $(2.12)-(2.13)$ in the following form

$$\dot{\sigma}_-(t) = -\left(\frac{\kappa}{2} + i\omega_d\right)\sigma_-(t) + \sqrt{\kappa} \sigma_z(t)b(t), \quad (2.14)$$

$$b_{\text{out}}(t) = \sqrt{\kappa} \sigma_-(t) + b(t), \quad t \geq t_0. \quad (2.15)$$

Next, we study several properties of the system $(2.14)-(2.15)$. Firstly, notice

$$L|g\rangle = 0, \ H|g\rangle = -\frac{\omega_d}{2}|g\rangle. \quad (2.16)$$

That is, the coupling operator $L$ does not generate photons and the initial system Hamiltonian $H$ does not excite the two-level system. As a result, when this system is initialized in the vacuum state $|g\rangle$ and is driven by a two-photon state (to be discussed in the sequel), at any time instant $t$ the joint system may have either two photons in the field, or one photon in the field and one excited atomic state. That is, the number of excitations is a conserved quantity at all times. (Here the word “excitation” stands for a photon or an excited two-level system.)

Secondly, by Eq. $(2.15)$, it can be shown that [32] Lemma 3], up to a global phase factor, the solution $U(t,t_0)$ to the QSDE $(2.14)$ satisfies

$$U(t,t_0)|0g\rangle = |0g\rangle. \quad (2.17)$$

Multiplying both sides of Eq. $(2.17)$ by $U(t,t_0)^\dagger$ yields

$$|0g\rangle = U(t,t_0)^\dagger |0g\rangle. \quad (2.18)$$

Conjugating both sides of Eq. $(2.18)$ we get

$$\langle 0g|U(t,t_0) = \langle 0g|. \quad (2.19)$$
Furthermore, by Eqs. 2.17 and 2.18 we have
\[
\sigma_z(t)|0g\rangle = U(t, t_0)\sigma_z(t_0)U(t, t_0)|0g\rangle = U(t, t_0)^\dagger\sigma_z(t_0)|0g\rangle = -U(t, t_0)^\dagger|0g\rangle = -|0g\rangle.
\] (2.20)
That is,
\[
\sigma_z(t)|0g\rangle = -|0g\rangle, \quad \langle 0g|\sigma_z(t) = -\langle 0g|.
\] (2.21)
Thirdly, it is worth mentioning that the quantum causality conditions [16]
\[
[X(t), b(\tau)] = [X(t), b^\dagger(\tau)] = 0, \quad t \leq \tau,
\] (2.22)
\[
[X(t), b_{out}(\tau)] = [X(t), b_{out}^\dagger(\tau)] = 0, \quad t \geq \tau.
\] (2.23)
Eq. (2.22) indicates that system operators \(X(t)\) is influenced by the past input field \(b(r)\), \(t_0 \leq r < t\). On the other hand, it is clear that \([X(t), b_{out}(\tau)] = 0\) for \(t > \tau\). When \(t = \tau\),
\[
[X(t), b_{out}(\tau)] = U(t, t_0)[X \otimes I, I \otimes b(t)]U(t, t_0)^\dagger = 0,
\]
\[
[X(t), b_{out}^\dagger(\tau)] = U(t, t_0)[X \otimes I, I \otimes b^\dagger(t)]U(t, t_0)^\dagger = 0.
\]
Consequently, Eq. (2.23) holds.
Finally, because of Eq. (2.22), \(\sigma_z(t)b(t)\) in Eq. (2.14) is equal to \(b(t)\sigma_z(t)\). That is \([\sigma_z(t), b(t)] = 0\).

2.3 Two-photon states

Given a function \(\xi \in L_2(\mathbb{R}, \mathbb{C})\), define an operator
\[
B(\xi) \triangleq \int_{-\infty}^\infty \xi(t)b(t)dt,
\] (2.24)
whose adjoint operator is
\[
B^\dagger(\xi) = \int_{-\infty}^\infty \xi(t)b^\dagger(t)dt.
\] (2.25)
A continuous-mode single-photon state can be defined to be
\[
|1_\xi\rangle \triangleq B^\dagger(\xi)|0\rangle,
\] (2.26)
where \(\|\xi\| = 1\) for normalization. For example, if the pulse shape is
\[
\xi(t) = -\sqrt{\gamma/\pi}e^{\gamma t^2/2}(1 - u(t)),
\] (2.27)
where \(u(t)\) is the Heaviside function
\[
u(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}
\]
then in the frequency domain we have
\[
f[\omega] \triangleq \int_{-\infty}^\infty e^{-i\omega t}\xi(t)dt = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\gamma}}{\omega - \gamma/2}.
\]
Clearly, \(f[\omega]\) describes a Lorentzian spectrum with full width at half maximum (FWHM) \(\gamma\) [23].

In the calculation of various few-photon states, the notation
\[
|1_t\rangle \triangleq b^\dagger(t)|0\rangle, \quad \forall t \in \mathbb{R}
\] (2.28)
turns out to be very useful. By Eq. (2.11) we have
\[
\langle 1_t|1_r \rangle = \langle 0|b(t)b^\dagger(r)|0\rangle = \delta(t - r).
\]
Moreover, by Eqs. (2.25) and (2.28), the single-photon state \(|1_\xi\rangle\) can be re-written as
\[
|1_\xi\rangle = \int_{-\infty}^\infty \xi^\dagger(t)|1_t\rangle.
\]
Therefore, the single-photon state $|1_t\rangle$ is in the form of continuum superposition of $|1_t\rangle$. Consequently, $\{1_t : t \in \mathbb{R}\}$ is a complete single-photon basis. Similarly

$$\int_{-\infty}^{\infty} dl \ |1_g\rangle \langle 1_g| + |0_c\rangle \langle 0_c|$$

(2.29)

is an identity in the one-excitation case.

In what follows, we introduce two-photon states. Given two functions $\xi_1, \xi_2 \in L_2(\mathbb{R}, \mathbb{C})$ satisfying $\|\xi_1\| = \|\xi_2\| = 1$, we define an uncorrelated two-photon state by

$$|2_{\xi_1, \xi_2}\rangle \equiv \frac{1}{\sqrt{N_2}} B^\dagger (\xi_1) B^\dagger (\xi_2)|0\rangle,$$

(2.30)

where $N_2 = 1 + |\langle \xi_1 | \xi_2 \rangle|^2$ is the normalization coefficient. If $\xi_1 \equiv \xi_2$, then $|2_{\xi_1, \xi_2}\rangle$ is a continuous-mode two-photon Fock state \cite{11,10}. More generally, an arbitrary continuous-mode two-photon state can be written as

$$\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ f(p_1, p_2) b^\dagger (p_1)b^\dagger (p_2)|0\rangle = \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ f(p_1, p_2)|1_{p_1, p_2}\rangle,$$

(2.31)

where $f(p_1, p_2)$ is an ordinary function of two time variables $p_1$ and $p_2$, satisfying the symmetry property $f(p_1, p_2) = f(p_2, p_1)$. It can be easily checked that

$$\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ f(p_1, p_2) b^\dagger (p_1)b^\dagger (p_2)|0\rangle = \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ f(p_1, p_2)|1_{p_1, p_2}\rangle,$$

(3.37)

Therefore,

$$\left\{ \frac{1}{\sqrt{2}} |1_{p_1, p_2}\rangle : p_1, p_2 \in \mathbb{R} \right\}$$

(2.32)

is a complete orthonormal basis of continuous-mode two-photon pure states. As a result,

$$\frac{1}{2} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ |1_{p_1, p_2}\rangle \langle 1_{p_1, p_2}| + \int_{-\infty}^{\infty} dp |1_p e\rangle \langle 1_p e|$$

(2.33)

is an identity in the 2-excitation system.

3 One-channel case

In this section, we consider a two-level system which is driven by a two-photon state $|2_{\xi_1, \xi_2}\rangle$. The main result is an explicit expression of the steady-state output field state.

Integrating (2.13)-(2.15) from $t_0$ to $t$ gives

$$\sigma_-(t) = e^{-\left(\frac{t-t_0}{\hbar}\right)\sigma_-(t_0)} + \sqrt{\kappa} \int_{t_0}^{t} dr \ e^{-\left(\frac{t-r}{\hbar}\right)\sigma_+(r)} b(r),$$

(3.34)

$$b_{out}(t) = \sqrt{\kappa} \sigma_-(t) + b(t)$$

(3.35)

and

$$\int_{t_0}^{t} dr \ e^{-\left(\frac{t-r}{\hbar}\right)\sigma_-(r)} b(r) + b(t).$$

(3.36)

Assume that the two-level system is initialized in the ground state $|g\rangle$ and the input field is in the two-photon state $|2_{\xi_1, \xi_2}\rangle$, as is defined in Eq. (2.30) above. Then the initial joint system-field state is

$$|\Psi(t_0)\rangle = |2_{\xi_1, \xi_2} g\rangle = \frac{1}{\sqrt{N_2}} B^\dagger (\xi_1) B^\dagger (\xi_2) |0 g\rangle.$$
By the Schrödinger equation, the joint system-field state at time $t \geq t_0$ is
\begin{equation}
|\Psi(t)\rangle = U(t, t_0) |\Psi(t_0)\rangle. \tag{3.38}
\end{equation}

In this paper we are interested in the steady-state output field state, that is, we assume that the interaction starts in the remote past ($t_0 = -\infty$) and terminates in the far future ($t = \infty$). The term $\eqref{3.42}$ is established.

The aim of this section is to derive analytic expressions of $|\Psi_{\text{out}}\rangle$.

We first state a result for the single-photon case. Define a time-domain function
\begin{equation}
g_C(t) \equiv \left\{ \begin{array}{ll}
\delta(t) - \kappa e^{-\nu t}, & t \geq 0, \\
0, & t < 0.
\end{array} \right. \tag{3.40}
\end{equation}

And define another function $\eta(t)$ via convolution
\begin{equation}
\eta(t) \equiv g_C * \xi(t) \equiv \int_{-\infty}^{\infty} g_C(t-r) \xi(r) dr. \tag{3.41}
\end{equation}

It has been proven in \cite{32} that the steady-state output field state of the two-level system driven by a single-photon state is another single-photon state. More specifically,

\textbf{Lemma 3.1} \textit{(\cite{32} Theorem 4)} If the two-level system \textit{\textup{2.14} - \textup{2.15})} is initialized in the ground state $|g\rangle$ and is driven by a single-photon state $|\xi\rangle$ defined in Eq. \textit{\textup{2.26}}, then the steady-state output field state is another single-photon state $|\xi\rangle$ with the pulse shape $\eta(t)$ given in Eq. \textit{\textup{3.44}}.

The following lemma is important in the proof of the subsequent lemmas.

\textbf{Lemma 3.2} If the system \textit{\textup{2.14} - \textup{2.15)} is initialized in the ground state $|g\rangle$ and driven by a two-photon state $|2\xi_1, \xi_2\rangle$, then
\begin{equation}
\langle g | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle = \frac{1}{2} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ |1_{p_1, p_2}g | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle, \quad t \geq t_0. \tag{3.42}
\end{equation}

\textbf{Proof.} As discussed above, the system \textit{\textup{2.14} - \textup{2.15)} satisfies the conditions \textit{\textup{2.16}}. As a result, if the system is initialized in the ground state $|g\rangle$ and driven by a two-photon state $|2\xi_1, \xi_2\rangle$, the number of excitations of the joint system is always 2 for all times. Consequently,
\begin{equation}
\left\{ \frac{1}{\sqrt{2}} |1_{p_1, p_2}g \rangle, \ |\rho\xi\rangle : p_1, p_2, \rho \in \mathbb{R} \right\} \tag{3.43}
\end{equation}
is a complete orthonormal basis of this 2-excitation system. We have
\begin{equation}
\langle g | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle = \frac{1}{2} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \ |1_{p_1, p_2}g | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle + \int_{-\infty}^{\infty} dp \ |\rho\xi\rangle \langle \rho\xi | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle \tag{3.44}
\end{equation}

Eq. \textit{\textup{3.42}} is established. $\square$

\textbf{Remark 3.1} The term $\langle g | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle$ in Eq. \textit{\textup{3.42}} is an (unnormalized) two-photon field state after tracing out the system. Eq. \textit{\textup{3.44}} expresses this field state in terms of a coherent superposition of a complete two-photon basis \textit{\textup{\frac{1}{\sqrt{2}} |1_{p_1, p_2}g \rangle : p_1, p_2 \in \mathbb{R} \}} with weights $\frac{1}{\sqrt{2}} |1_{p_1, p_2}g | U(t, t_0) b^\dagger(t_1) b^\dagger(t_2) | 0g \rangle$.

With the help of Lemma \textit{\textup{3.2}}, the following result expresses the steady-state output field state $|\Psi_{\text{out}}\rangle$ in a form that facilitates the derivation of its final analytic expression.
Lemma 3.3 The steady-state output field state $|\Psi_{\text{out}}\rangle$ in Eq. (3.46) can be re-written as

$$
|\Psi_{\text{out}}\rangle = \frac{1}{2\sqrt{N_2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \left| 1_{p_1}1_{p_2} \right|_{t_0 \to -\infty, t \to -\infty} \int_{t_0}^{t} dt_1 \xi_1(t_1) \int_{t_0}^{t} dt_2 \xi_2(t_2) \tag{3.45}
$$

$$
\times \left[ \sqrt{\kappa} \left\langle 0g|\sigma_- (p_1) b_{\text{out}}(p_2) b^\dagger (t_1) b^\dagger (t_2)|0g \right\rangle + \left\langle 0g|b(p_1) b_{\text{out}}(p_2) b^\dagger (t_1) b^\dagger (t_2)|0g \right\rangle \right].
$$

The proof of Lemma 3.3 is given in Appendix A.

Lemma 3.4 tells us that to derive the steady-state output field state $|\Psi_{\text{out}}\rangle$, we have to calculate the two terms $\left\langle 0g|\sigma_- (p_1) b_{\text{out}}(p_2) b^\dagger (t_1) b^\dagger (t_2)|0g \right\rangle$ and $\left\langle 0g|b(p_1) b_{\text{out}}(p_2) b^\dagger (t_1) b^\dagger (t_2)|0g \right\rangle$ on the right-hand side of Eq. (3.45). Such calculations are provided in Lemma 3.4 and Lemma 3.5 below. As the proofs of Lemmas 3.4 and 3.5 are straightforward (although tedious), thus are omitted.

Lemma 3.4 We have

$$
\left\langle 0g|\sigma_- (p_1) b_{\text{out}}(p_2) b^\dagger (t_1) b^\dagger (t_2)|0g \right\rangle = -2\kappa^{3/2} e^{-(\frac{\kappa}{2} + i\omega_d)(p_1 - t_0)} \int_{t_0}^{P_2} dr e^{-(\frac{\kappa}{2} + i\omega_d)(p_2 - r)} e^{-(\frac{\kappa}{2} + i\omega_d)(r - t_0)} \times \int_{t_0}^{P_1} dr e^{-(\frac{\kappa}{2} + i\omega_d)(p_1 - r)} [\delta(r - t_1) \delta(r - t_2) + \delta(r - t_2) \delta(r - t_1)]
$$

$$
- \sqrt{\kappa} \int_{t_0}^{P_1} dr e^{-(\frac{\kappa}{2} + i\omega_d)(p_1 - r)} [\delta(p_2 - t_1) \delta(r - t_2) + \delta(r - t_1) \delta(p_2 - t_2)]
$$

$$
+ \kappa^{3/2} \int_{t_0}^{P_1} dr \int_{t_0}^{P_2} d\eta e^{-(\frac{\kappa}{2} + i\omega_d)(p_1 + p_2 - r - \eta)} [\delta(r - t_1) \delta(n - t_2) + \delta(n - t_1) \delta(r - t_2)]
$$

$$
- 2\kappa^{3/2} \int_{t_0}^{P_1} dr \int_{t_0}^{P_2} d\eta e^{-(\frac{\kappa}{2} + i\omega_d)(p_1 + p_2 - r - \eta)} \int_{t_0}^{\eta} d\tau_1 e^{-(\frac{\kappa}{2} + i\omega_d)(n - \tau_1)} \delta(\tau_1 - r)
$$

$$
\times \int_{t_0}^{\eta} d\tau_2 e^{-(\frac{\kappa}{2} + i\omega_d)(n - \tau_2)} [\delta(\tau_2 - t_1) \delta(n - t_2) + \delta(n - t_1) \delta(\tau_2 - t_2)].
$$

Notice that, due to the presence of the factor $e^{-(\frac{\kappa}{2} + i\omega_d)(p_1 - t_0)}$, the first term in Eq. (3.46) vanishes in the limit $t_0 \to -\infty$.

Lemma 3.5 We have

$$
\left\langle 1_{p_1} g|b_{\text{out}}(p_2) b^\dagger (t_1) b^\dagger (t_2)|0g \right\rangle
$$

$$
= \delta(p_2 - t_1) \delta(p_1 - t_2) + \delta(p_1 - t_1) \delta(p_2 - t_2)
$$

$$
- \kappa \int_{t_0}^{P_2} dr e^{-(\frac{\kappa}{2} + i\omega_d)(p_2 - r)} [\delta(p_1 - t_1) \delta(r - t_2) + \delta(r - t_1) \delta(p_1 - t_2)]
$$

$$
+ 2\kappa^2 \int_{t_0}^{P_2} dr e^{-(\frac{\kappa}{2} + i\omega_d)(p_2 - r)} \int_{t_0}^{r} d\tau_1 e^{-(\frac{\kappa}{2} + i\omega_d)(r - \tau_1)} \delta(\tau_1 - p_1)
$$

$$
\times \int_{t_0}^{r} d\tau_2 e^{-(\frac{\kappa}{2} + i\omega_d)(r - \tau_2)} [\delta(\tau_2 - t_1) \delta(r - t_2) + \delta(r - t_1) \delta(\tau_2 - t_2)].
$$

By Lemmas 3.4 and 3.5 we have the following result.

Lemma 3.6 The steady-state output field state is

$$
|\Psi_{\text{out}}\rangle = \frac{1}{2\sqrt{N_2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \eta_1(p_1, p_2) b^\dagger (p_1) b^\dagger (p_2)|0\rangle.
$$

(3.48)
where

\[ \eta_1(p_1, p_2) \triangleq \xi_1(p_1)\xi_2(p_2) + \xi_1(p_2)\xi_2(p_1) \]

\[- \kappa \int_{-\infty}^{p_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2 - \tau)} [\xi_1(p_1)\xi_2(\tau) + \xi_2(p_2)\xi_1(\tau)] \]

\[- \kappa \int_{-\infty}^{p_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1 - \tau)} [\xi_1(p_2)\xi_2(\tau) + \xi_2(p_1)\xi_1(\tau)] \]

\[ + \kappa^2 \int_{-\infty}^{p_1} d\tau \int_{-\infty}^{p_2} dr e^{-\left(\frac{r}{\kappa}\right)(p_1 + p_2 - \tau - r)} [\xi_1(\tau)\xi_2(\tau) + \xi_2(\tau)\xi_1(\tau)] \]

\[ + 2\kappa^2 \int_{-\infty}^{p_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2 - \tau)} \int_{-\infty}^{\tau} d\tau_1 e^{-\left(\frac{\tau_1}{\kappa}\right)(\tau - \tau_1)} \times \delta(\tau_1 - p_1) \int_{-\infty}^{\tau} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(\tau - \tau_2)} [\xi_2(\tau_2)\xi_1(\tau_2) + \xi_1(\tau_2)\xi_2(\tau_2)] \]

\[ + \delta(\tau - \tau_1) \int_{-\infty}^{\tau} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(\tau - \tau_2)} [\xi_2(\tau_2)\xi_1(\tau_2) + \xi_1(\tau_2)\xi_2(\tau_2)]. \]  

(3.49)

**Proof.** Substituting Eqs. (3.46) and (3.47) into Eq. (3.45) yields Eq. (3.49). \( \square \)

**Remark 3.2** By the explicit expression of \( \eta_1(p_1, p_2) \) given in Eq. (3.49), it is hard to see that \( \eta_1(p_1, p_2) \) is symmetric. In the following, we derive a function \( \eta(p_1, p_2) \), which, in contrast to \( \eta_1(p_1, p_2) \) defined in Eq. (3.49), exhibits the symmetry \( \eta(p_1, p_2) = \eta(p_2, p_1) \) more clearly.

**Lemma 3.7** The steady-state output field state is

\[ |\Psi_{\text{out}}\rangle = \frac{1}{2\sqrt{N_2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \eta(p_1, p_2) b^\dagger(p_1)b^\dagger(p_2) |0\rangle, \]  

(3.50)

where

\[ \eta(p_1, p_2) \triangleq \xi_1(p_2)\xi_2(p_1) + \xi_1(p_1)\xi_2(p_2) \]

\[- \kappa \int_{-\infty}^{p_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1 - \tau)} [\xi_1(p_2)\xi_2(\tau) + \xi_2(p_2)\xi_1(\tau)] \]

\[- \kappa \int_{-\infty}^{p_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2 - \tau)} [\xi_1(p_1)\xi_2(\tau) + \xi_2(p_1)\xi_1(\tau)] \]

\[ + \kappa^2 \int_{-\infty}^{p_2} d\tau \int_{-\infty}^{p_1} dr e^{-\left(\frac{r}{\kappa}\right)(p_1 + p_2 - \tau - r)} [\xi_1(\tau)\xi_2(\tau) + \xi_2(\tau)\xi_1(\tau)] \]

\[ + \kappa^2 \left[ \int_{-\infty}^{p_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1 - \tau)} \int_{-\infty}^{\tau} d\tau_1 \delta(\tau_1 - p_1) + \int_{-\infty}^{p_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2 - \tau)} \int_{-\infty}^{\tau} d\tau_1 \delta(\tau_1 - p_1) \right] \times e^{-\left(\frac{\tau}{\kappa}\right)(\tau - \tau_1)} \int_{-\infty}^{\tau} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(\tau - \tau_2)} [\xi_1(\tau_2)\xi_2(\tau) + \xi_2(\tau_2)\xi_1(\tau)] \]

\[ - \kappa^3 \int_{-\infty}^{p_1} d\tau \int_{-\infty}^{p_2} dr e^{-\left(\frac{\tau}{\kappa}\right)(p_1 + p_2 - \tau - r)} \times \left\{ \int_{-\infty}^{\tau} d\tau_1 e^{-\left(\frac{\tau_1}{\kappa}\right)(\tau_1 - \tau)} \int_{-\infty}^{\tau} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(\tau_2 - \tau)} [\xi_2(\tau_2)\xi_1(\tau_2) + \xi_1(\tau_2)\xi_2(\tau_2)] + \int_{-\infty}^{\tau} d\tau_1 e^{-\left(\frac{\tau_1}{\kappa}\right)(\tau_1 - \tau)} \int_{-\infty}^{\tau} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(\tau_2 - \tau)} [\xi_2(\tau_2)\xi_1(\tau_2) + \xi_1(\tau_2)\xi_2(\tau_2)] \right\}. \]

(3.51)

**Proof.** By Eq. (3.11) we have

\[ \langle 0| b_{\text{out}}(p_1)b_{\text{out}}(p_2)b^\dagger(t_1)b^\dagger(t_2)|0\rangle = \frac{1}{2} \langle 0| b_{\text{out}}(p_1)b_{\text{out}}(p_2)b^\dagger(t_1)b^\dagger(t_2)|0\rangle + \frac{1}{2} \langle 0| b_{\text{out}}(p_2)b_{\text{out}}(p_1)b^\dagger(t_1)b^\dagger(t_2)|0\rangle, \]  

(3.52)
In analog to Eq. (3.49), the term
\[ \langle 0g|b_{\text{out}}(p_2)b_{\text{out}}(p_1)b^\dagger(t_1)b^\dagger(t_2)|0g \rangle \]
produces
\[
\eta_2(p_2, p_1) = \xi_1(p_2)\xi_2(p_1) + \xi_1(p_1)\xi_2(p_2) \\
- \kappa \int_{-\infty}^{p_1} d\tau e^{-i(\tau + i\omega d)(p_1 - r)} [\xi_1(p_2)\xi_2(\tau) + \xi_2(p_2)\xi_1(\tau)] \\
- \kappa \int_{-\infty}^{p_2} d\tau e^{-i(\tau + i\omega d)(p_2 - r)} [\xi_1(p_1)\xi_2(\tau) + \xi_2(p_1)\xi_1(\tau)] \\
+ \kappa^2 \int_{-\infty}^{p_2} d\tau \int_{-\infty}^{p_1} d\tau e^{-i(\tau + i\omega d)(p_1 + p_2 - r - \tau)} [\xi_1(\tau)\xi_2(\tau) + \xi_2(\tau)\xi_1(\tau)] \\
+ 2\kappa^2 \int_{-\infty}^{p_2} d\tau \int_{-\infty}^{p_1} d\tau e^{-i(\tau + i\omega d)(p_1 - \tau)} \int_{-\infty}^{r} d\tau_1 \delta(\tau_1 - p_2) \\
\times e^{-i(\tau - \tau_1)(r - \tau_1)} \int_{-\infty}^{r} d\tau_2 e^{-i(\tau + i\omega d)(\tau - \tau_2)} [\xi_1(\tau_2)\xi_2(\tau) + \xi_2(\tau_2)\xi_1(\tau)] \\
- 2\kappa^3 \int_{-\infty}^{p_2} d\tau \int_{-\infty}^{p_1} d\tau e^{-i(\tau + i\omega d)(p_1 + p_2 - r - \tau)} \int_{-\infty}^{r} d\tau_1 \delta(\tau_1 - \tau) \\
\times e^{-i(\tau - \tau_1)(r - \tau_1)} \int_{-\infty}^{r} d\tau_2 e^{-i(\tau + i\omega d)(\tau - \tau_2)} [\xi_2(\tau_2)\xi_1(\tau_2) + \xi_1(\tau_2)\xi_2(\tau_2)].
\]

Define the overall pulse shape
\[
\eta(p_1, p_2) = \frac{\eta_1(p_1, p_2) + \eta_2(p_2, p_1)}{2}.
\]
Then by Eqs. (3.49) and (3.53) we have Eq. (3.51). Thus, the steady-state output state is that in Eq. (3.50). The proof is completed. □

**Remark 3.3** From Eq. (3.51) one can see that the output pulse shape contains 16 terms. Interestingly, in the study of quantum filtering of a two-level system driven by the two-photon state \( |2\xi_1, \xi_2 \rangle \), a system of 16 ordinary differential equations are needed to represent the two-photon filter or the master equation [10]. That is, there is consistency between output two-photon field state and two-photon quantum filtering.

The expression of the steady-state output field state in Eq. (3.51) has 16 terms, which look rather complicated. In what follows we present the main result of this paper, which gives a much compact form of the steady-state output field state \( |\Psi_{\text{out}} \rangle \).

**Theorem 3.1** The steady-state output field state is
\[ |\Psi_{\text{out}} \rangle = \frac{1}{2\sqrt{N^2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 [\nu_1(p_1)\nu_2(p_2) + \zeta(p_1, p_2) + \nu_1(p_2)\nu_2(p_1) + \zeta(p_2, p_1)] b^\dagger(p_1)b^\dagger(p_2)|0 \rangle, \tag{3.54} \]
where
\[ \nu_j \triangleq g_G \ast \xi_j, \quad j = 1, 2, \tag{3.55} \]
and
\[ \zeta(p_1, p_2) \triangleq 2\kappa e^{-\xi_1(p_1 - p_2) - \omega d(p_1 + p_2)} \int_{p_2}^{p_1} d\tau e^{2i\omega d\tau} \left[ \xi_1(\tau)\xi_2(\tau) - \frac{\xi_1(\tau)\nu_2(\tau) + \nu_1(\tau)\xi_2(\tau)}{2} \right]. \tag{3.56} \]
In particular, if \( \omega d = 0 \),
\[ \zeta(p_1, p_2) = 2\kappa e^{-\xi_1(p_1 - p_2)} \int_{p_2}^{p_1} d\tau \left[ \xi_1(\tau)\xi_2(\tau) - \frac{\xi_1(\tau)\nu_2(\tau) + \nu_1(\tau)\xi_2(\tau)}{2} \right]. \]
If further \( \xi_1 = \xi_2 = \xi \) (an input two-photon Fock state), then \( \nu_1 = \nu_2 = \nu \), and
\[ \zeta(p_1, p_2) = 2\kappa e^{-\frac{\xi}{2}(p_1 - p_2)} \int_{p_2}^{p_1} d\tau \xi(\tau)[\xi(\tau) - \nu(\tau)]. \]

The proof of Theorem 3.1 is given in the Appendix [13].

**Remark 3.4** By the explicit form of \( \zeta(p_1, p_2) \) which is given in Eq. (3.56), we can see that the interaction between the two-level system and the two input photons is not a linear transformation.
4 Two-channel case

In this section, we consider the two-level system which is driven by two input channels, each containing one photon. Assume there is no detuning (namely \( \omega_d = 0 \)), the system model is

\[
\dot{\sigma}_- = -\frac{\kappa_1 + \kappa_2}{2} \sigma_+ + \sqrt{\kappa_1} \sigma_+ (t) b_1(t) + \sqrt{\kappa_2} \sigma_+ (t) b_2(t),
\]

\[
b_{\text{out},1}(t) = \sqrt{\kappa_1} \sigma_-(t) + b_1(t),
\]

\[
b_{\text{out},2}(t) = \sqrt{\kappa_2} \sigma_-(t) + b_2(t), \quad t \geq t_0.
\]

Define a function in the time domain

\[
g_c(t) \triangleq \begin{cases} 
\delta(t) I - \left[ \frac{\sqrt{\kappa_1}}{\sqrt{\kappa_2}} \right] e^{-\frac{\kappa_1 + \kappa_2}{2} t} \left[ \begin{array}{cc} \sqrt{\kappa_1} & \sqrt{\kappa_2} \\ \sqrt{\kappa_2} & \sqrt{\kappa_1} \end{array} \right], & t \geq 0, \\
0, & t < 0.
\end{cases}
\]

The initial joint state is

\[
|\Psi(t_0)\rangle = B_1^\dagger (\xi_1) B_2^\dagger (\xi_2) |0g\rangle,
\]

where \( ||\xi_1|| = ||\xi_2|| = 1 \). At time \( t \), the joint system-field state is

\[
|\Psi(t)\rangle = U(t, t_0) B_1^\dagger (\xi_1) B_2^\dagger (\xi_2) |0g\rangle.
\]

The steady-state output field state is

\[
|\Psi_{\text{out}}\rangle = \lim_{t_0 \to -\infty, t \to \infty} \langle g | \Psi(t) \rangle = \lim_{t_0 \to -\infty, t \to \infty} \langle g | U(t, t_0) B_1^\dagger (\xi_1) B_2^\dagger (\xi_2) |0g\rangle.
\]

Notice that

\[
\langle g | U(t, t_0) B_1^\dagger (\xi_1) B_2^\dagger (\xi_2) |0g\rangle = \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \xi_1(t_1)\xi_2(t_2) \langle g | U(t, t_0) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle,
\]

and

\[
\lim_{t_0 \to -\infty, t \to \infty} \langle g | U(t, t_0) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle = \frac{1}{2} \int dp_1 \int dp_2 |1_{1p_1} 1_{2p_2}\rangle \langle 0g | b_{\text{out},1}(p_1) b_{\text{out},1}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle + \int dp_1 \int dp_2 |1_{1p_1} 1_{2p_2}\rangle \langle 0g | b_{\text{out},2}(p_1) b_{\text{out},2}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle + \frac{1}{2} \int dp_1 \int dp_2 |2_{1p_1} 2_{2p_2}\rangle \langle 0g | b_{\text{out},2}(p_1) b_{\text{out},2}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle,
\]

where the two-photon basis

\[
\left\{ \frac{1}{2} \int dp_1 \int dp_2 |1_{1p_1} 1_{2p_2}\rangle \langle 1_{1p_1} 1_{2p_2} |, \int dp_1 \int dp_2 |1_{1p_1} 1_{2p_2}\rangle \langle 1_{1p_1} 1_{2p_2} |, \frac{1}{2} \int dp_1 \int dp_2 |1_{1p_1} 1_{2p_2}\rangle \langle 1_{2p_1} 1_{2p_2} |, \frac{1}{2} \int dp_1 \int dp_2 |1_{2p_1} 1_{2p_2}\rangle \langle 1_{2p_1} 1_{2p_2} | \right\}
\]

has been used. Eq. (4.61) becomes

\[
|\Psi_{\text{out}}\rangle = \frac{1}{2} \int dp_1 \int dp_2 |1_{1p_1} 1_{2p_2}\rangle \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \xi_1(t_1)\xi_2(t_2)
\]

\[
\times \sum_{i,j=1}^{2} \langle 0g | b_{\text{out},i}(p_1) b_{\text{out},j}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle,
\]

So we need to calculate the following quantities:

\[
\langle 0g | b_{\text{out},1}(p_1) b_{\text{out},1}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle,
\]

\[
\langle 0g | b_{\text{out},1}(p_1) b_{\text{out},2}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle,
\]

\[
\langle 0g | b_{\text{out},2}(p_1) b_{\text{out},1}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle,
\]

\[
\langle 0g | b_{\text{out},2}(p_1) b_{\text{out},2}(p_2) b_1^\dagger (t_1) b_2^\dagger (t_2) |0g\rangle.
\]
Notice that
\[
\langle 0g | b_{\text{out},1}(p_1)b_{\text{out},2}(p_2)b_1^i(t_1)b_2^j(t_2) | 0g \rangle = \langle 0g | b_{\text{out},2}(p_2)b_{\text{out},1}(p_1)b_1^i(t_1)b_2^j(t_2) | 0g \rangle .
\] (4.67)

The following lemmas simply present the calculation for the quantities given by Eqs. (4.69)-(4.70), which can be easily verified.

**Lemma 4.1** We have
\[
\langle 0g | \sigma_-(r)b_1^i(q) | 0g \rangle = -\sqrt{\kappa_k} \left[ \delta_{1k} + \delta_{2k} \right] \int_{t_0}^{r} d\tau e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau)} \delta(\tau - q),
\] (4.68)

and
\[
\langle 0g | b_1(q)\sigma_+(r) | 0g \rangle = \langle 0g | \sigma_-(r)b_1^i(q) | 0g \rangle, \quad k = 1, 2.
\] (4.69)

By Lemma 4.1 we have

**Lemma 4.2** We have
\[
\langle 0g | b_1(t)\sigma_+(r)b_1^i(t) | 0g \rangle = 2\sqrt{\kappa_{jk}} \kappa_k \int_{t_0}^{r} d\tau_1 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_1)} \delta(\tau_1 - l) \int_{t_0}^{r} d\tau_2 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_2)} \delta(\tau_2 - t) - \delta_{jk} \delta(l - t), \quad j, k = 1, 2.
\] (4.70)

By Lemma 4.2 we have

**Lemma 4.3** We have
\[
\langle 0g | b_1(t)\sigma_+(r)b_2(t)b_1^i(t_1)b_2^j(t_2) | 0g \rangle = 2\sqrt{\kappa_{1\kappa}} \delta(r - t_1) \int_{t_0}^{r} d\tau_1 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_1)} \delta(\tau_1 - l) \int_{t_0}^{r} d\tau_2 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_2)} \delta(\tau_2 - t_2),
\]

\[
\langle 0g | b_2(t)\sigma_+(r)b_2(t)b_1^i(t_1)b_2^j(t_2) | 0g \rangle = 2\sqrt{\kappa_{1\kappa}} \delta(r - t_2) \int_{t_0}^{r} d\tau_1 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_1)} \delta(\tau_1 - l) \int_{t_0}^{r} d\tau_2 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_2)} \delta(\tau_2 - t_1),
\]

\[
\langle 0g | b_1(t)\sigma_+(r)b_2(t)b_1^i(t_1)b_2^j(t_2) | 0g \rangle = 2\sqrt{\kappa_{1\kappa}} \delta(r - t_1) \int_{t_0}^{r} d\tau_1 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_1)} \delta(\tau_1 - l) \int_{t_0}^{r} d\tau_2 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_2)} \delta(\tau_2 - t_2) - \delta_{jk} \delta(l - t_1) \delta(r - t_2),
\]

\[
\langle 0g | b_2(t)\sigma_+(r)b_1(t)b_1^i(t_1)b_2^j(t_2) | 0g \rangle = 2\sqrt{\kappa_{1\kappa}} \delta(r - t_1) \int_{t_0}^{r} d\tau_1 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_1)} \delta(\tau_1 - l) \int_{t_0}^{r} d\tau_2 e^{-\frac{\kappa_{1\kappa}}{2}(r-\tau_2)} \delta(\tau_2 - t_2) - \delta(l - t_1) \delta(r - t_2).
\]

By Lemma 4.3 we have the following result.

**Lemma 4.4** We have
\[
\langle 0g | b_{\text{out},i}(p_1)b_{\text{out},j}(p_2)b_1^i(t_1)b_2^j(t_2) | 0g \rangle = \kappa_j \int_{-\infty}^{p_2} dr e^{-\frac{\kappa_{1\kappa}}{2}(p_2-r)} \langle 0g | g_{1i} | b_{\text{out},i}(p_1)\sigma_+(r)b_1^i(t_1)b_2^j(t_2) | 0g \rangle
\]
\[
+ \sqrt{\kappa_{1\kappa}} \int_{-\infty}^{p_2} dr e^{-\frac{\kappa_{1\kappa}}{2}(p_2-r)} \langle 0g | g_{1i} | b_{\text{out},i}(p_1)\sigma_+(r)b_1^i(t_1)b_2^j(t_2) | 0g \rangle
\]
\[
- k_j \sqrt{\kappa_{1\kappa}} \int_{-\infty}^{p_2} dr e^{-\frac{\kappa_{1\kappa}}{2}(p_2-r)} \int_{-\infty}^{p_1} d\tau e^{-\frac{\kappa_{1\kappa}}{2}(p_1-\tau)} \langle 0g | g_{2i} | \sigma_+(r)b_1^i(t_1)b_2^j(t_2) | 0g \rangle
\]
\[
- k_j \sqrt{\kappa_{1\kappa}} \int_{-\infty}^{p_2} dr e^{-\frac{\kappa_{1\kappa}}{2}(p_2-r)} \int_{-\infty}^{p_1} d\tau e^{-\frac{\kappa_{1\kappa}}{2}(p_1-\tau)} \langle 0g | g_{2i} | \sigma_+(r)b_1^i(t_1)b_2^j(t_2) | 0g \rangle
\]
\[
- \sqrt{\kappa_{1\kappa}} \delta(p_2 - t_j) \int_{-\infty}^{p_1} d\tau e^{-\frac{\kappa_{1\kappa}}{2}(p_1-\tau)} \delta(\tau - t_j) + (1 - \delta_{ij}) \delta(p_2 - t_j) \delta(p_1 - t_i), \quad i, j = 1, 2,
\]
where \(g_{1i,j}(t)\) is the \((i,j)\) entry of \(g(t)\) defined in Eq. (4.58).
The resulting steady-state output field state can be re-written as

\[ |\Psi_{\text{out}}\rangle = \frac{1}{2} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 |\psi_1, 1_{p_1} 1_{p_2}\rangle \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \xi_1(t_1) \xi_2(t_2) \]

\[ \times \langle 0|b_{\text{out}, i}(p_1)b_{\text{out}, j}(p_2) b_1^\dagger(t_1)b_2^\dagger(t_2)|0\rangle, \]

where \( \langle 0|b_{\text{out}, i}(p_1)b_{\text{out}, j}(p_2) b_1^\dagger(t_1)b_2^\dagger(t_2)|0\rangle \) is given by Lemma 1.4.

The following is the main result of this section, which follows Lemma 4.4 and Eq. 4.72.

**Theorem 4.1** The steady-state output field state is

\[ |\Psi_{\text{out}}\rangle = \frac{1}{2} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \eta_{ij}(p_1, p_2) \times b_1^\dagger(p_1)b_2^\dagger(p_2)|0\rangle, \]  

(4.73)

where

\[ \eta_{ij}(p_1, p_2) = \left[ g_{C_{ij}} \ast \xi_1(p_1) \right] \times \left[ g_{C_{ji}} \ast \xi_2(p_2) \right] + \left[ g_{C_{ij}} \ast \xi_1(p_2) \right] \times \left[ g_{C_{ji}} \ast \xi_2(p_1) \right] \]

\[ - 2 \int_{-\infty}^{p_2} dr e^{-\frac{\kappa_1 + \kappa_2}{2}(p_2-r)} \xi_i(r) \left[ g_{C_{ji}} \ast \xi_2(r) \right] \int_{-\infty}^{r} d\tau_1 e^{-\frac{\kappa_1 + \kappa_2}{2}(r-\tau_1)} \]

\[ \times \left[ \kappa_j g_{C_{ij}} \ast \delta(p_1 - \tau_1) + \sqrt{\kappa_1 \kappa_2} g_{C_{ij}} \ast \delta(p_1 - \tau_1) \right], \]

\[ - 2 \int_{-\infty}^{p_2} dr e^{-\frac{\kappa_1 + \kappa_2}{2}(p_2-r)} \xi_2(r) \left[ g_{C_{ji}} \ast \xi_1(r) \right] \int_{-\infty}^{r} d\tau_1 e^{-\frac{\kappa_1 + \kappa_2}{2}(r-\tau_1)} \]

\[ \times \left[ \kappa_i g_{C_{ij}} \ast \delta(p_1 - \tau_1) + \sqrt{\kappa_1 \kappa_2} g_{C_{ij}} \ast \delta(p_1 - \tau_1) \right], \]

(4.74)

In particular, if \( \kappa_1 = \kappa_2 = \kappa, \xi_1 = \xi_2 = \xi, \) then

\[ \eta_{11}(p_1, p_2) = \left[ g_{C_{11}} \ast \xi_1(p_1) \right] \times \left[ g_{C_{12}} \ast \xi(p_2) \right] + \left[ g_{C_{11}} \ast \xi(p_2) \right] \times \left[ g_{C_{12}} \ast \xi(p_1) \right] \]

\[ - 4\kappa \int_{-\infty}^{p_2} dr e^{-\kappa(p_2-r)} \xi(r) \times \left[ g_{C_{12}} \ast \xi(r) \right] \]

\[ \times \int_{-\infty}^{r} d\tau_1 e^{-\kappa(r-\tau_1)} \left[ g_{C_{11}} \ast \delta(p_1 - \tau_1) \right] \ + \left[ g_{C_{12}} \ast \delta(p_1 - \tau_1) \right] \]

\[ = \eta_{22}(p_1, p_2), \]

\[ \eta_{12}(p_1, p_2) = \left[ g_{C_{11}} \ast \xi_1(p_1) \right] \times \left[ g_{C_{12}} \ast \xi(p_2) \right] + \left[ g_{C_{12}} \ast \xi_2(p_2) \right] \times \left[ g_{C_{12}} \ast \xi(p_1) \right] \]

\[ - 4\kappa \int_{-\infty}^{p_2} dr e^{-\kappa(p_2-r)} \xi(r) \times \left[ g_{C_{12}} \ast \xi(r) \right] \]

\[ \times \int_{-\infty}^{r} d\tau_1 e^{-\kappa(r-\tau_1)} \left[ g_{C_{11}} \ast \delta(p_1 - \tau_1) \right] \ + \left[ g_{C_{12}} \ast \delta(p_1 - \tau_1) \right] \]

\[ = \eta_{21}(p_1, p_2). \]

The resulting steady-state output field state is

\[ |\Psi_{\text{out}}\rangle = \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \left[ \frac{1}{2} \eta_{11}(p_1, p_2) b_1^\dagger(p_1)b_1^\dagger(p_2) + \eta_{12}(p_1, p_2) b_1^\dagger(p_1)b_2^\dagger(p_2) + \frac{1}{2} \eta_{11}(p_1, p_2) b_2^\dagger(p_1)b_2^\dagger(p_2) \right]|0\rangle. \]

(4.77)

## 5 Numerical studies

In this section, Example 1 is used to demonstrate the results in section 3 (the one-channel case) and Example 2 is for section 4 (the two-channel case).

**Example 1.** In this example, we study the Wigner spectra of input and output two-photon states. The two-time correlation function of a two-photon state \( |\xi_1, \xi_2\rangle \) is given by

\[ r(t, \tau) = \text{E}_{\xi_1, \xi_2} [b(t)b^\dagger(\tau)] , \]

(5.78)
where the subscripts \( \xi_1 \) and \( \xi_2 \) indicate that expectation is taken with respect to the two-photon state \( |2\xi_1,\xi_2\rangle \). Applying the Fourier transform to the correlation function \( r(t, \tau) \), yields the Wigner spectrum

\[
W(t, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r(t, \tau) e^{-i\omega \tau} d\tau. \tag{5.79}
\]

Inserting the input state Eq. (2.30) and the output state Eq. (3.54) into Eq. (5.78), we can get the input and output covariance functions \( r_{in}(t, \tau) \) and \( r_{out}(t, \tau) \), respectively. Then by the definition of Wigner spectrum in Eq. (5.79), the input and output spectra can be derived, which are shown in Fig. 1.

In Fig. 1, we assume that

\[
\xi_1(t) = \xi_2(t) = -\sqrt{\gamma} e^{\gamma t^2} (1 - u(t)), \tag{5.80}
\]

where \( \gamma \) is the pulse shape parameter and \( u(t) \) is the Heaviside function:

\[
u(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{5.81}
\]

That is, the input state \( |2\xi_1,\xi_2\rangle \) is a two-photon Fock state with decaying exponential pulse shapes. Notice that the input Wigner spectrum, shown in Fig. 1(a), is monotonically increasing when \( t < 0 \), while it falls to zero suddenly at \( t = 0 \) (This can be clearly seen by Eq. (5.80)). In contrast, the output Wigner spectrum, shown in Fig. 1(b), exhibits complex behavior. Specifically, the spectrum remains very small when \( t < 0 \). This indicates that the energy is mostly shared by the input field and the two-level system. At \( t = 0 \), there is a sharp increase in the output Wigner spectrum, which implies that partial energy leaks out at \( t \geq 0 \) suddenly. Interestingly, the output Wigner spectrum decreases monotonically in the time period \( t = 0 \) and \( t = 0.4 \); after that it increases monotonically. On the other hand, if we read in the frequency perspective (along the \( \omega \) axis), we can see that, as \( t \) increases, the output Wigner spectrum becomes very large near the line \( \omega = 0 \). This is consistent with the Wigner spectrum of Wiener process, [11] Fig. 2.

**Example 2.** The system could be depicted as in Fig. 2. In this scheme, the first output channel \( b_{out,1} \) can be regarded as the right-going direction, while the second output channel \( b_{out,2} \) indicates the left-going direction. The pulse shapes of the two input photons are given respectively by

\[
\xi_i(t_i) = -\sqrt{\gamma_i} e^{\gamma_i t_i^2} (1 - u(t_i)), \quad i = 1, 2. \tag{5.82}
\]

where \( \gamma_i \) are the pulse shape parameters, \( u(t_i) \) are the Heaviside function defined in Eq. (5.81). The photon \( i \) is coupled to the two-level system with the coupling strength \( \kappa_i (i = 1, 2) \). Clearly, the input field state is a product state \( B_{1}^{\xi_1}(\xi_1) |0\rangle_1 \otimes B_{2}^{\xi_2}(\xi_2) |0\rangle_2 \). According to Theorem 4.1, the two photons in the output channels are entangled by the two-level system.

Two cases of simulation have been performed:

**Case 1** the two photons have the same pulse shape, namely, \( \gamma_1 = \gamma_2 \);

**Case 2** the two photons are equally coupled to the two-level system, namely, \( \kappa_1 = \kappa_2 \).
Figure 2: (Color online) Two counter-propagating photons coupled to a two-level system initialized in the ground state. The nonlinear interaction between the two initially separated photons controlled by the two-level system is able to exhibit the Hong-Ou-Mandel (HOM) phenomenon (see Case 1) and stimulated emission (see Case 2).

Figure 3: (Color online) The three subfigures correspond to the probability that the two photons are: (a) in the first output channel, (b) in the second output channel, and (c) one in each output channel. We choose the pulse shape parameters \( \gamma_1 = \gamma_2 = 1.0 \).

We look at Case 1 first. The simulation results are summarized in Fig. 3. In this figure, we have considered the following three scenarios:

(a) the probability \( P_{11} \) of observing two photons in the first output channel, see Fig. 3(a);

(b) the probability \( P_{22} \) of observing two photons in the second output channel, see Fig. 3(b);

(c) the probability \( P_{12} \) of observing one photon in each output channel, see Fig. 3(c).

By Theorem 4.1 we have

\[
P_{11} = \frac{1}{4} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \left[ |\eta_{11}(p_1, p_2)|^2 + |\eta_{12}(p_2, p_1)|^2 \right], \tag{5.83}
\]

\[
P_{22} = \frac{1}{4} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \left[ |\eta_{22}(p_1, p_2)|^2 + |\eta_{21}(p_2, p_1)|^2 \right], \tag{5.84}
\]

and

\[
P_{12} = \frac{1}{4} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \left[ |\eta_{12}(p_1, p_2)|^2 + |\eta_{21}(p_1, p_2)|^2 \right]. \tag{5.85}
\]

For ease of presentation, we fix \( \gamma_1 = \gamma_2 = 1 \) in our simulation. The probabilities under different coupling strengths \( \kappa_1, \kappa_2 \) are shown in Fig. 3. We have the following observations.

- \( P_{11} \) is large when \( \kappa_1 \) is small and \( \kappa_2 \) is big (the up-left corner), see Fig. 3(a). It agrees with the waveguide QED scheme proposed in [56, Fig. 4(d)]. This phenomenon can be interpreted in the
Figure 4: (Color online) The probability $P$ of observing at least one photon in the first output channel.

following way. In the strong-coupling limit with respect to a big $\kappa_2$, the two-level system acts as a mirror for the second input photon, and this photon is nearly reflected. While in the first channel, the photon is perfectly transmitted due to the weak-coupling (small $\kappa_1$) with the two-level system. Thus, high probability values for observing two photons in the first output (in other words, right-going) channel can be attained.

- Similar as above, by symmetry, $P_{22}$ is large when $\kappa_2$ is small and $\kappa_1$ is big (the bottom-right corner), see Fig. 3(b).

- $P_{12}$ decreases as coupling strengths $\kappa_1$ and $\kappa_2$ increase, see Fig. 3(c). In particular, $P_{12}$ is close to zero when $\kappa_1 = \kappa_2 \geq 4$. That is, both photons are either in the first output channel or in the second output channel. Before interaction with the two-level system, there is one photon in each channel. Thus, strong nonlinear interaction (large $\kappa_1, \kappa_2$) between the two photons, controlled by the two-level system, occurred that makes the two photons exit from the same channel. This can be regarded as a continuous-variable version of the famous Hong-Ou-Mandel (HOM) effect [18, 23, 55].

Next, we look at **Case 2**. The probability $P$ of observing at least one photon in the first output channel (namely, $P_{11} + P_{12}$) is simulated and the results are shown in Fig. 4. Notice that $P \geq 0.7$ for all the cases in Fig. 4 in contrast to Fig. 6 in [31]. It can be seen that high probability values are concentrated in the area of small $\gamma_2$ and big $\gamma_1$ (the bottom-right corner). If we fix $\gamma_2 = \kappa$ and let $\gamma_1 \geq \gamma_2$, we may get the black lines in these subfigures. In this case, the probability $P$ increases as $\gamma_1$ increases. Particularly, if $\gamma_1$ is very big as compared to $\gamma_2$, it can be observed that $P$ is bigger than 0.9, equivalently $P_{22}$ is less than 0.1. In the limit, $P_{22} \to 0$, i.e., two photons hardly exit from the second output (left-going) channel. This can be interpreted as follows. When $\gamma_2 = \kappa$, the two-level system can be fully excited by the input photon in the second input channel [32]. Thus, when $\gamma_1 \gg \gamma_2$, the process is similar to the one where the photon in the first input (right-going) channel interacts with an *excited* two-level system. Therefore, our scheme can be regarded as a process of simulated emission, see, e.g., [36, Fig. 2(a)].
6 Conclusion

In this paper, the nonlinear photon-photon interaction controlled by a two-level system has been investigated. The output two-photon states have been explicitly derived when the input state is either a two-photon state or a tensor product of two counter-propagating single-photon states. For both cases, simulation results have demonstrated rich and interesting properties of output two-photon states induced by the control of the two-level system on the two input photons.

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Proof of Lemma 3.3

Substituting Eqs. (3.37) and (3.38) into Eq. (3.39) yields

\[ |\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{N_2}} \lim_{t_0 \to -\infty, t \to \infty} \langle g | U(t, t_0) \int_{-\infty}^{\infty} dt_1 \xi_1(t_1) \int_{-\infty}^{\infty} dt_2 \xi_2(t_2) b^\dagger(t_1)b^\dagger(t_2) |0g\rangle \]

\[ = \frac{1}{\sqrt{N_2}} \lim_{t_0 \to -\infty, t \to \infty} \langle g | U(t, t_0) \int_{t_0}^{t} dt_1 \xi_1(t_1) \int_{t_0}^{t} dt_2 \xi_2(t_2) b^\dagger(t_1)b^\dagger(t_2) |0g\rangle \]

\[ = \frac{1}{\sqrt{N_2}} \lim_{t_0 \to -\infty, t \to \infty} \langle t_0 | U(t_0, t) \int_{t_0}^{t} dt_1 \xi_1(t_1) \int_{t_0}^{t} dt_2 \xi_2(t_2) \langle g | U(t, t_0)b^\dagger(t_1)b^\dagger(t_2) |0g\rangle . \]
The substitution of Eq. (3.29) into Eq. (1.86) produces

\[
|\Psi_{\text{out}}\rangle = \frac{1}{2\sqrt{N_2}} \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt_1 \xi_1(t_1) \int_{t_0}^{t} dt_2 \xi_2(t_2)
\]

\[
\times \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 |1_{p_1, 1_{p_2}}\rangle \langle 1_{p_1, 1_{p_2}}| U(t, t_0) b^\dagger(t_1) b(t_2) |0_g\rangle
\]

\[
= \frac{1}{2\sqrt{N_2}} \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt_1 \xi_1(t_1) \int_{t_0}^{t} dt_2 \xi_2(t_2) \langle 1_{p_1, 1_{p_2}}| U(t, t_0) b^\dagger(t_1) b(t_2) |0_g\rangle.
\]

Furthermore, by Eqs. (2.10) and (2.19), for \( t \geq \max\{p_1, p_2\} \geq t_0 \) (this can always be guaranteed because we are interested in the steady state case \((t_0 \to -\infty \text{ and } t \to \infty)\), we have

\[
\langle 1_{p_1, 1_{p_2}}| U(t, t_0) b^\dagger(t_1) b(t_2) |0_g\rangle
\]

\[
= \langle 0_g| b(p_1) b(p_2) U(t, t_0) b^\dagger(t_1) b(t_2) |0_g\rangle
\]

\[
= \langle 0_g| U(t, t_0) U(t, t_0)^\dagger b(p_1) U(t, t_0)b^\dagger(t_1) b(t_2) |0_g\rangle
\]

\[
= \langle 0_g| b_{\text{out}}(p_1) b_{\text{out}}(p_2) b^\dagger(t_1) b(t_2) |0_g\rangle.
\]

Substituting Eq. (1.88) into Eq. (1.87) we get

\[
|\Psi_{\text{out}}\rangle = \frac{1}{2\sqrt{N_2}} \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt_1 \xi_1(t_1) \int_{t_0}^{t} dt_2 \xi_2(t_2) \langle 0_g| b_{\text{out}}(p_1) b_{\text{out}}(p_2) b^\dagger(t_1) b(t_2) |0_g\rangle.
\]

Finally, by Eq. (3.30) we have

\[
\langle 0_g| b_{\text{out}}(p_1) b_{\text{out}}(p_2) b^\dagger(t_1) b(t_2) |0_g\rangle
\]

\[
= \sqrt{\kappa} \langle 0_g| \sigma_- (p_1) b_{\text{out}}(p_2) b^\dagger(t_1) b(t_2) |0_g\rangle + \langle 0_g| b(p_1) b_{\text{out}}(p_2) b^\dagger(t_1) b(t_2) |0_g\rangle.
\]

Substituting Eq. (1.90) into Eq. (1.89) yields Eq. (3.45). \( \square \)

**B  Proof of Theorem 3.1**

It can be readily shown that

\[
\xi_1(p_2) \xi_2(p_1) + \xi_1(p_1) \xi_2(p_2) - \kappa \int_{-\infty}^{01} d\tau \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_1 - \tau)} [\xi_2(\tau) \xi_1(p_2) + \xi_1(\tau) \xi_2(p_2)]
\]

\[
= g_C \ast \xi_1(p_1) \times \xi_2(p_2) + g_C \ast \xi_2(p_1) \times \xi_1(p_2).
\]

Similarly,

\[
- \kappa \int_{-\infty}^{02} d\tau \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_2 - \tau)} \left[\xi_1(p_1) \xi_2(\tau) + \xi_2(p_1) \xi_1(\tau)\right]
\]

\[
+ \kappa^2 \int_{-\infty}^{02} d\tau \int_{-\infty}^{\tau} d\tau' \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_1 + p_2 - \tau - \tau')} \left[\xi_1(\tau') \xi_2(\tau) + \xi_2(\tau') \xi_1(\tau)\right]
\]

\[
= - \kappa \int_{-\infty}^{02} d\tau \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_2 - \tau)} \left[\xi_1(p_1) \xi_2(\tau) + \xi_2(p_1) \xi_1(\tau)\right]
\]

\[
- \kappa \int_{-\infty}^{\tau} d\tau' \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_1 - \tau')} \left[\xi_1(\tau') \xi_2(\tau) + \xi_2(\tau') \xi_1(\tau)\right]
\]

\[
= - \kappa g_C \ast \xi_1(p_1) \times \int_{-\infty}^{02} d\tau \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_2 - \tau)} \xi_2(\tau)
\]

\[
- \kappa g_C \ast \xi_2(p_1) \times \int_{-\infty}^{02} d\tau \ e^{-\left(\frac{i}{\omega + \omega_d}\right) (p_2 - \tau)} \xi_1(\tau),
\]
where Eq. (2.91) is used in the last step. Because

\[ g_G \ast \xi_1(p_1) \times \xi_2(p_2) - g_G \ast \xi_1(p_1) \times \kappa \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \xi_2(\tau) = g_G \ast \xi_1(p_1) \times g_G \ast \xi_2(p_2), \]

and

\[ g_G \ast \xi_2(p_1) \times \xi_1(p_2) - g_G \ast \xi_2(p_1) \times \kappa \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \xi_1(\tau) = g_G \ast \xi_2(p_1) \times g_G \ast \xi_1(p_2), \]

adding (2.91) and (2.92), the first 8 terms of \( \eta(p_1, p_2) \) becomes

\[ g_G \ast \xi_1(p_1) \times g_G \ast \xi_2(p_2) + g_G \ast \xi_2(p_1) \times g_G \ast \xi_1(p_2). \]  

(2.93)

Notice that the remaining 8 items of \( \eta(p_1, p_2) \) (ignoring the common coefficient \( \kappa^2 \)) can be written as a sum of

\[
\int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1-\tau)} \int_{-\infty}^{r} d\tau_1 \delta(\tau_1 - p_1) e^{-\left(\frac{\tau_1}{\kappa}\right)(r-\tau_1)} \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(r-\tau_2)} [\xi_1(\tau_2) \xi_2(\tau) + \xi_2(\tau_2) \xi_1(\tau)]
\]

\[ - \kappa \int_{-\infty}^{P_1} d\tau \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1+P_2-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau_1}{\kappa}\right)(r-\tau_1)} \delta(\tau_1 - r) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(r-\tau_2)} [\xi_2(\tau_2) \xi_1(\tau_2) + \xi_1(\tau_2) \xi_2(\tau_2)]
\]

\[ = \int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_1)} \left[ \delta(\tau_1 - p_2) - \kappa \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \delta(\tau - r) \right]
\]

\[ \times \int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} [\xi_2(\tau) \xi_1(\tau_2) + \xi_2(\tau_2) \xi_1(\tau)]
\]

\[ = \int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_1)} g_G \ast \delta(p_2 - \tau_1) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} [\xi_2(\tau) \xi_1(\tau_2) + \xi_1(\tau) \xi_2(\tau_2)]
\]

\[ = \int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_1)} g_G \ast \delta(p_2 - \tau_1) \times \left[ \xi_1(\tau) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} \xi_2(\tau_2) + \xi_2(\tau) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} \xi_1(\tau_2) \right]
\]

(2.94)

and

\[
\int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \int_{-\infty}^{r} d\tau_1 \delta(\tau_1 - p_1) e^{-\left(\frac{\tau_1}{\kappa}\right)(r-\tau_1)} \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(r-\tau_2)} [\xi_2(\tau_2) \xi_1(\tau) + \xi_2(\tau) \xi_1(\tau)]
\]

\[ - \kappa \int_{-\infty}^{P_2} d\tau \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2+P_2-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau_1}{\kappa}\right)(r-\tau_1)} \delta(\tau_1 - \tau) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau_2}{\kappa}\right)(r-\tau_2)} [\xi_1(\tau_2) \xi_2(\tau) + \xi_2(\tau_2) \xi_1(\tau)]
\]

\[ = \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_1)} \left[ \delta(\tau_1 - \tau_1) - \kappa \int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_1-\tau)} \delta(\tau - r) \right]
\]

\[ \times \int_{-\infty}^{P_1} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} [\xi_2(\tau_2) \xi_1(\tau) + \xi_2(\tau) \xi_1(\tau)]
\]

\[ = \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_1)} g_G \ast \delta(p_1 - \tau_1) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} [\xi_2(\tau_2) \xi_1(\tau) + \xi_2(\tau_2) \xi_1(\tau)]
\]

\[ = \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-\tau)} \int_{-\infty}^{r} d\tau_1 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_1)} g_G \ast \delta(p_1 - \tau_1) \times \left[ \xi_1(\tau) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} \xi_2(\tau_2) + \xi_2(\tau) \int_{-\infty}^{r} d\tau_2 e^{-\left(\frac{\tau}{\kappa}\right)(r-\tau_2)} \xi_1(\tau_2) \right]
\]

(2.95)

where the fact

\[ \delta(p_2 - \tau_1) - \kappa \int_{-\infty}^{P_2} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-s)} \delta(\tau_1 - s)
\]

\[ = \delta(p_2 - \tau_1) - \kappa \int_{-\infty}^{P_2-r} d\tau e^{-\left(\frac{\tau}{\kappa}\right)(p_2-r)} \delta(\tau)
\]

(2.96)

\[ = g_G \ast \delta(p_2 - \tau_1) \]
is used in the derivation. Adding (2.93), (2.94) and (2.95), then \( \eta(p_1, p_2) \) in (3.51) becomes

\[
\eta(p_1, p_2) = gG \ast \xi_1(p_1) \times gG \ast \xi_2(p_2) + gG \ast \xi_2(p_1) \times gG \ast \xi_1(p_2) \\
+ \kappa^2 \left[ e^{-\left(\frac{\tau_1}{\tau_2} + i\omega_d\right)} \int_{p_1}^{p_1} \int_{\tau_1}^{\tau_1} \int_{\tau_1}^{\tau_1} \eta \left( p_1, p_2 \right) b^\dagger(p_1) b^\dagger(p_2) \right] \\
\times e^{-\left(\frac{\tau_1}{\tau_2} - i\omega_d\right)(\tau - \tau_1)} \left[ \xi_1(\tau) \int_{-\infty}^{\tau} d\tau_2 e^{\left(\frac{\tau_1}{\tau_2} + i\omega_d\right)\tau_2} \xi_2(\tau_2) + \xi_2(\tau) \int_{-\infty}^{\tau} d\tau_2 e^{\left(\frac{\tau_1}{\tau_2} + i\omega_d\right)\tau_2} \xi_1(\tau_2) \right].
\]

(2.97)

Then

\[
\eta(p_1, p_2) = \nu_1(p_1)\nu_2(p_2) + \nu_2(p_1)\nu_1(p_2) + \zeta(p_1, p_2) + \zeta(p_2, p_1).
\]

where functions \( \nu_1, \nu_2, \) and \( \zeta \) are given in Eqs. (3.55) and (3.56) respectively. As a result, the steady-state output state is

\[
|\Psi_{out}\rangle = \frac{1}{2\sqrt{N_2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \eta(p_1, p_2) b^\dagger(p_1) b^\dagger(p_2) |0\rangle
\]

\[
= \frac{1}{2\sqrt{N_2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \left[ \nu_1(p_1)\nu_2(p_2) + \nu_2(p_1)\nu_1(p_2) + \zeta(p_1, p_2) + \zeta(p_2, p_1) \right] b^\dagger(p_1) b^\dagger(p_2) |0\rangle.
\]

Eq. (3.54) is established. \( \square \)