Octonionic instantons in eight dimensions

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Abstract

In this paper, we study the construction of the instanton moduli space for eight-dimensional octonionic instantons. We find a general $N$-instanton solution which depends on $8N + 1$ free parameters and we investigate the cases $N = 2$ and $N = 3$ in detail.

1 Introduction

The discovery of regular solutions to the Yang-Mills field equations in the four-dimensional Euclidean space, which correspond to absolute minimum of the action, has led to an intensive study of such theory and the search multidimensional generalizations the self-duality equations. In refs. [1,2], such equations were found and classified. These were first-order equations satisfy the Yang-Mills field equations as a consequence of the Bianchi identity. Later, solutions to these equations were found and then used to construct classical solitonic solutions of the low energy effective theory of the heterotic string [3–13].

An alternative approach to the construction of self-duality equations proposed in [14] was to consider self-duality relations between higher order terms of the field strength. An explicit example instantons satisfying such self-duality relations was obtained on the eight dimensional sphere in ref. [15]. As shown in ref. [16], these instantons play a role in smoothing out the singularity of heterotic string soliton solutions by incorporating one-loop corrections. In ref. [17–22] these exotic solutions were used to construct various

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string and membrane solutions. In ref. [23,24] they were used to construct the higher dimensional quantum Hall effect.

In this paper, we study the ADHM construction of the self-dual instantons in eight dimensions and we find a multi-instanton solution generalizing the solution that was found in ref [15].

2 Preliminaries

In this section, we give a brief summary of the Clifford algebra and the octonion algebra. We list the features of the mathematical structure as far as they are of relevance to our work.

We recall that the Clifford algebra \(\text{Cl}_{0,7}(\mathbb{R})\) is a real associative algebra generated by the elements \(\Gamma_1, \Gamma_2, \ldots, \Gamma_7\) and defined by the relations

\[
\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2\delta_{ij}.
\]

The element \(\Omega = \Gamma_1 \Gamma_2 \ldots \Gamma_7\) commutes with all other elements of the algebra, and its square \(\Omega^2 = 1\). Therefore the pair \(\Gamma^\pm = \frac{1}{2}(1 \pm \Omega)\) forms a complete system of mutually orthogonal central idempotents, and hence the algebra \(\text{Cl}_{0,7}(\mathbb{R})\) decomposes into the direct sum of two ideals. It can be shown that these ideals are isomorphic to the algebra \(M_8(\mathbb{R})\) of all real matrices of size \(8 \times 8\). The latter, in turn, is the algebra of endomorphisms of the octonion algebra. Let us describe these relations in a little more detail.

The algebra of octonions \(\mathbb{O}\) is a real linear algebra with the canonical basis \(1, e_1, \ldots, e_7\) such that

\[
e_i e_j = -\delta_{ij} + c_{ijk} e_k,
\]

where the structure constants \(c_{ijk}\) are completely antisymmetric and nonzero as \(c_{123} = c_{145} = c_{167} = c_{246} = c_{275} = c_{374} = c_{365} = 1\). The algebra of octonions is not associative but alternative, i.e. the associator \((x, y, z) = (xy)z - x(yz)\) is totally antisymmetric in \(x, y, z\). Consequently, any two elements of \(\mathbb{O}\) generate an associative subalgebra. The algebra of octonions satisfies the identity \(((zx)y)x = z(xy)x\) which is called the right Moufang identity. The algebra \(\mathbb{O}\) permits the involution (i.e. an anti-automorphism of period two) \(x \rightarrow \bar{x}\) such that the elements \(t(x) = x + \bar{x}\) and \(n(x) = \bar{x}x\) are in \(\mathbb{R}\). In the canonical basis, this involution is defined by \(\bar{e}_i = -e_i\). It follows that the bilinear form \((x, y) = \frac{1}{2}(\bar{x}y + \bar{y}x)\) is positive definite and defines an inner
product on \( \mathbb{O} \). It is easy to prove that the quadratic form \( n(x) \) permits the composition \( n(xy) = n(x)n(y) \). Since this form is positive definite, it follows that \( \mathbb{O} \) is a division algebra.

Denote by \( R_x \) the operator of right multiplication on \( x \) in the octonion algebra, i.e. \( yR_x = yx \) for all \( y \in \mathbb{O} \). The set of all such operators generates a subalgebra in the algebra \( \text{End} \mathbb{O} \) of endomorphisms of the linear space \( \mathbb{O} \). Using the multiplication law \([2]\) and antisymmetry of the associator \((e_i, e_j, e_k)\), we prove the equalities

\[
R_{e_i}R_{e_j} + R_{e_j}R_{e_i} = -2\delta_{ij}1_8,
\]

where \( 1_8 \) is the identity \( 8 \times 8 \) matrix. Comparing \((3)\) with \((1)\), we see that the correspondence \( \Gamma_i \to R_{e_i} \) can be extended to the homomorphism \( Cl_{0,7}(\mathbb{R}) \to \text{End} \mathbb{O} \). Since the algebra \( M_8(\mathbb{R}) \) is simple, it follows that this mapping is surjective and therefore \( \text{End} \mathbb{O} \simeq \mathbb{R}(8) \). Note also that the product

\[
R_{e_1}R_{e_2} \ldots R_{e_7} = 1_8.
\]

This equality follows from simplicity of \( M_8(\mathbb{R}) \) and the fact that the element \( \Omega \) lies in the center of \( Cl_{0,7}(\mathbb{R}) \).

Now we redenote the unit of \( \mathbb{O} \) by the symbol \( e_0 \) so that \( R_{e_0} = 1_8 \) and consider the combinations

\[
R_{\mu\nu} = R_{e_\mu} \bar{R}_{e_\nu} - R_{e_\nu} \bar{R}_{e_\mu},
\]

where \( \bar{R}_{e_\mu} \equiv R_{e_\mu} \) and the Greek index takes values from 0 to 7. Then it easily follows from the identity \((4)\) that

\[
R_{[\mu\nu}R_{\lambda\rho]} = \epsilon_{\mu\nu\lambda\rho\alpha\beta\gamma\sigma} R_{(\alpha\beta} R_{\gamma\sigma)}.
\]

Hence, the tensor \( R_{[\mu\nu}R_{\lambda\rho]} \) satisfies the self-duality equations.

### 3 ADHM construction

Let \( M_{m,n}(\mathbb{R}) \) be the set of all real \( m \times n \) matrices, and let \( K = \{ R_a \mid a \in \mathbb{O} \} \). Obviously, \( K \) is a linear subspace of \( \text{End} \mathbb{O} \). Further, let \( M \) be a real \( 8m \times 8n \) matrix. We call \( M \) the \( K \)-matrix of size \( m \times n \) if it is representable as a matrix with elements from \( K \), i.e. if \( M \in M_{m,n}(\mathbb{R}) \otimes K \). We say that this matrix is real over \( K \) if \( M \in M_{m,n}(\mathbb{R}) \otimes \mathbb{R}1_8 \). In the case when \( m = n \) and
\( M \in \mathbb{R}^{1_n \otimes K} \), where \( 1_n \) is the identity \( n \times n \) matrix, we say that the matrix \( M \) is diagonal over \( K \).

Now we choose two \( K \)-matrices \( C \) and \( D \) of size \((n + N) \times N\) in such a way that for any \( x \in K \) the matrix

\[
M(x) = Cx + D
\]

satisfies the following conditions:

i) the matrix \( \bar{M}^tM \) is real over \( K \);

ii) the matrix \( \bar{C}^tM(\bar{M}^tM)^{-1}\bar{M}^tC \) is real over \( K \);

iii) the matrix \( M \) has the maximal rank \( 8N \).

Condition iii) means that the columns of \( M(x) \) generate in \( \mathbb{R}^{n+8N} \otimes K \) a subspace \( W_x \) of dimension \( 8N \). Let \( W_x^\perp \) be the orthogonal complement of \( W_x \) with respect to the Euclidean metric in \( \mathbb{R}^{n+8N} \otimes K \). We choose a \( K \)-matrix \( U(x) \) of size \((n + N) \times n\), whose columns form an orthonormal basis of \( W_x^\perp \), and define the linear potential

\[
A_\mu(x) = \bar{U}^t \partial_\mu U.
\]

Then the corresponding completely antisymmetric 4-tensor \( F_{[\mu\nu}F_{\lambda\rho]} \) will have the form

\[
F_{[\mu\nu}F_{\lambda\rho]} = \bar{U}^t C R_{[\mu\nu} R_{\lambda\rho]} (\bar{M}^tM)^{-1}\bar{C}^t \bar{U} \bar{U}^t C (\bar{M}^tM)^{-1}\bar{C}^t U.
\]

To prove this equality, it suffices to use the orthonormality of the chosen basis, i.e. the conditions

\[
\bar{U}^t M = 0, \quad \bar{U}^t U = 1_n,
\]

as well as the above conditions i) and ii). Obviously, the tensor (9) satisfies the self-duality equations.

Note that the subspace \( W_x \) can be considered as fibers of a vector bundle over \( S^8 = K \cup \{\infty\} \) with half the second Pontryagin number equal to \( N \). We also note that the use of the octonion algebra in this construction is not necessary. Instead of the algebra \( \text{End} \mathfrak{O} \), we can use the matrix algebra \( M_8(\mathbb{R}) \) which is isomorphic to \( \text{End} \mathfrak{O} \). In this case, it suffices to construct the homomorphism \( Cl_{0,7}(\mathbb{R}) \to M_8(\mathbb{R}) \) and find the images of generators of the Clifford algebra in \( M_8(\mathbb{R}) \). This idea was made in ref. [25], where an expression for the tensor \( F_{[\mu\nu}F_{\lambda\rho]} \) coinciding in form with (9) was obtained.
It is clear that when replacing $U$ with $UT$, where $T$ is the orthogonal $n \times n$ matrix with elements from $Spin(7) \times \mathbb{R}$, the potential (8) undergoes a gauge transformation, and the $N$-instanton (9) does not change. Similarly, the multiplication of $K$-matrices $C$ and $D$ on the right by a non-degenerate real over $K$ matrix of size $N \times N$ leads to a replacement of the basis of the space $W_x$, and multiplication of $C$ and $D$ to the left by a non-degenerate orthogonal $(n + N) \times (n + N)$ matrix with elements from $Spin(7) \times \mathbb{R}$ leads to a replacement of the basis in the space $\mathbb{R}^{n+N} \otimes K$. Since the group $Spin(7)$ acts transitively on a subset of elements of the norm 1 in $K$, by such transformations the matrix $C^t$ can be reduced to the form $(0, -1_N)$.

Therefore, without loss of generality, we can assume that the matrix (7) has the form
\[
M(x) = \begin{pmatrix}
\Lambda & -x1_N \\
B & x1_N
\end{pmatrix},
\]
where $\Lambda$ and $B$ are constant $K$-matrices and $x \in K$. It follows from the condition i) that the matrices $\bar{B}^tB + \bar{\Lambda}^t\Lambda$ and $\bar{B}^t x + \bar{x}B$ must be real over $K$ for any $x \in K$. Then it follows from the properties of the octonion algebra that the second condition is satisfied if and only if the matrix $B$ is symmetric. Hence, the matrix $\bar{B}^tB + \bar{\Lambda}^t\Lambda$ is also symmetric.

Generally speaking, the same $N$-instanton can be obtained from different matrices $\Lambda$ and $B$. Indeed, let $T$ be the orthogonal matrix defined above, and let $S$ be an orthogonal $K$-matrix of size $N \times N$ that is real over $K$. Then the transformation $\Lambda \to TAS$ and $B \to S^t BS$ leads only to a replacement of bases and therefore leaves the $N$-instanton unchanged. Therefore, by virtue of the theorem on reduction to principal axes, the matrix $\bar{B}^tB + \bar{\Lambda}^t\Lambda$ can be considered diagonal over $K$. Modifying the condition ii) in a similar way, we finally obtain the following result:

i) the matrix $B$ is symmetric, and the matrix $\bar{B}^tB + \bar{\Lambda}^t\Lambda$ is real and diagonal over $K$;

ii) the matrix $B(\bar{M}^tM)^{-1}\bar{B}^t$ is real over $K$;

iii) the matrix $M$ has the maximal rank $8N$.

4 Multi-instanton solutions

Let us consider in more detail the case $n = 1$, when the potential (8) takes values in the Lie algebra $so(8)$. To construct an $N$-instanton satisfying the
conditions (10), we will search for the matrix \( U = U(x) \) in the following form

\[
U = k \left( \frac{-1}{V} \right),
\]

(12)

where the column vector \( V = (v_1, \ldots, v_N)^t, v_i \in K \) and the real \( k > 0 \). Substituting this expression into the formula (8) and using the conditions (10), we get the expressions

\[
A_\mu = \frac{1}{2} \bar{V}^t \partial_\mu V - \partial_\mu \bar{V}^t V
\]

where \( \bar{V} = \Lambda (B - x 1_s)^{-1} \) and \( 1 + \bar{V}^t V = k^{-2} \).

Obviously, the conditions i)’, ii)’, and iii)’ are automatically satisfied in the 1-instanton case. Let \( N = 2 \). Then the diagonal elements of the \( K \)-matrix \( \bar{B}^t B + \bar{\Lambda}^t \Lambda \) have the form

\[
k_i = \bar{b}_{11} b_{1i} + \bar{b}_{12} b_{2i} + \bar{\lambda}_i \lambda_i,
\]

(14)

and obviously they are real over \( K \) as \( b_{21} = b_{12} \). Suppose \( b_{12} \neq 0 \). Then the condition i)’ reduces to the equation

\[
\bar{b}_{11} = -\bar{b}_{12} b_{22} b_{12}^{-1} - \bar{\lambda}_1 \lambda_2 b_{12}^{-1},
\]

(15)

which expresses the vanishing of off-diagonal elements of the \( K \)-matrix \( \bar{B}^t B + \bar{\Lambda}^t \Lambda \). We show that this equation has a solution in the space \( K \). Indeed, using identities

\[
\bar{R}_a = |a|R_a^{-1}, \quad R_a^{-1} = R_a^{-1}, \quad R_a R_b R_a = R_{aba},
\]

(16)

we prove that the first term on the right-hand side (15) belongs to \( K \). To prove the same for the second term, we use the transformation \( \Lambda \to T \Lambda \), where \( T \in Spin(7) \). Since the group \( Spin(7) \) acts transitively on a subset of elements of the norm 1 in \( K \), we can assume that \( \lambda_1 = k' \lambda_2 \) or \( \lambda_1 = k' b_{12} \) for some \( k' \in \mathbb{R} \). Under these conditions, the second term on the right-hand side (15) belongs to \( K \). Hence, \( \bar{b}_{11} \in K \), as it should be.

Now we turn to condition ii)’. Suppose that the matrix \( B \) is non-degenerate. Then the condition ii)’ is equivalent to the requirement that the matrix \( (B^t)^{-1} M^t MB^{-1} \) is real over \( K \). Substituting the expression (11) in it, we obtain the conditions

\[
B (\bar{B}^t B + \bar{\Lambda}^t \Lambda)^{-1} \bar{B}^t, \quad x B^{-1} + (\bar{B}^t)^{-1} \bar{x}
\]

(17)
for the matrix to be real over $K$. Since the inverse matrix of the symmetric matrix is itself symmetric, the second matrix in (17) is real over $K$. The reality condition for the first matrix over $K$ is equivalent to the equality

$$k_1^{-1}b_{12}\bar{b}_{11} + k_2^{-1}b_{22}\bar{b}_{12} = k_{18},$$

(18)

where $k \in \mathbb{R}$. Eliminating the element $\bar{b}_{11}$ from (15) and (18), we get the equation

$$(k_2^{-1} - k_1^{-1})\bar{b}_{12}b_{22} = k_1^{-1}\tilde{\lambda}_1\lambda_2 + k.$$  

(19)

If $k_1 = k_2$, then choosing $\lambda_1 = k'\lambda_2$, we get the set $\lambda_2 > 0$, $b_{22}$ and $b_{12} \neq 0$ as independent variables. If $k_1 \neq k_2$, then choosing $\lambda_1 = k'b_{12}$, we get $b_{22}$ as a linear combination of the independent variables $\lambda_2$ and $b_{12} \neq 0$.

Finally, if $b_{12} = 0$, then $\lambda_1 = 0$ and $\lambda_2 > 0$, and the elements $b_{11}$ and $b_{22}$ are independent variables. Since the matrix $B$ is non-degenerate, the condition $\text{iii)$'$}$ is satisfied in all cases. Thus, the ansatz (13) defines a 2-instanton in the following cases:

1) the elements $\lambda_2 > 0$, $b_{22}$ and $b_{12} \neq 0$ are independent variables satisfying the condition $k_1 = k_2$, $\lambda_1 = k'\lambda_2$, and $b_{11}$ is defined by the formula (15);

2) the elements $\lambda_2$ and $b_{12} \neq 0$ are independent variables as $k_1 \neq k_2$, $\lambda_1 = k'b_{12}$, and $b_{11}$ and $b_{22}$ are determined by the formulas (15) and (19), respectively.

3) the elements $\lambda_2 > 0$, $b_{11}$ and $b_{22}$ are independent variables, and $b_{12} = \lambda_1 = 0$.

Obviously, in all cases there are 17 free real parameters.

The case $N = 3$ is investigated in a similar way. Suppose $B$ is a non-degenerate symmetric matrix of size $3 \times 3$ over $K$. Then condition $\text{i)$'$} reduces to solving the system of equations

$$\bar{b}_{11}b_{12} + \bar{b}_{12}b_{22} + \bar{b}_{13}b_{23} + \tilde{\lambda}_1\lambda_2 = 0,$$

$$\bar{b}_{11}b_{13} + \bar{b}_{12}b_{23} + \bar{b}_{13}b_{33} + \tilde{\lambda}_1\lambda_3 = 0,$$

$$\bar{b}_{12}b_{13} + \bar{b}_{22}b_{23} + \bar{b}_{23}b_{33} + \tilde{\lambda}_2\lambda_3 = 0.$$  

(20)

Consider the following four possibilities.

1) Let $b_{12} = b_{13} = b_{23} = 0$. Setting $\lambda_1 > 0$, we have $\lambda_2 = \lambda_3 = 0$. The conditions $\text{ii)$'$}$ and $\text{iii)$'$}$ are satisfied automatically. Thus, we have 25 free real parameters in total.
2) Let $b_{12} \neq 0$ and $b_{13} = b_{23} = 0$. Setting $\lambda_3 = 0$, we reduce the system under consideration to one equation equivalent to (15). Arguing as above, we prove that the elements $b_{12} \neq 0$, $b_{22}$, $b_{33}$ and $\lambda_2 > 0$ are independent variables satisfying the condition $k_1 = k_2$. Thus, we again have a total of 25 free real parameters.

3) Let $b_{12} \neq 0$, $b_{13} \neq 0$ and $b_{23} = 0$. We put $\lambda_3 = k\lambda_2 = k'\lambda_1$ and choose the elements $b_{12} \neq 0$, $b_{22}$ and $\lambda_1 > 0$ as independent variables. Through them, $b_{11}$, $b_{13}$ and $b_{33}$ are easily expressed. To satisfy condition ii)', put $k_1 = k_2 = k_3$. As a result, we again obtain a solution with 17 free real parameters.

4) Let $b_{12} \neq 0$, $b_{13} \neq 0$ and $b_{23} \neq 0$. We put $b_{23} = kb_{13}$ and $\lambda_2 = k'\lambda_1$. Then the first equation in (20) has a solution $b_{11}$ in the space $K$. We rewrite the other two equations in the form of a system with respect to the unknown $b_{33}$ and $\lambda_3$. This system will have a unique solution in the space $K$ if $b_{13} = \lambda_1$ and $k \neq k'$. Hence, the independent variables are $b_{12} \neq 0$, $b_{22}$ and $\lambda_1 > 0$. To satisfy condition ii)', we again set $k_1 = k_2 = k_3$. As a result, we again obtain a solution with 17 free real parameters.

We see that the number of free parameters does not increase with an increase in the number of nonzero elements of the system (20). This is due to the need to have solutions in the space $K$. This is a very strong condition. It can be shown that a similar situation takes place in the case of arbitrary $N$. It follows, in particular, that the $N$-instanton in eight dimensions cannot have more than $8N + 1$ free parameters.

A solution containing exactly $8N + 1$ free parameters is easy to construct explicitly. Suppose the $K$-matrix $B$ is non-degenerate and diagonal over $K$, the $K$-matrix $\Lambda$ is real over $K$, and the elements of the latter satisfy the condition $\text{Tr} \lambda_i > 0$ for all $1 \leq i \leq N$. Then the conditions i') and iii') are satisfied automatically, and the condition ii') is satisfied in view of the obvious realness of the matrices (17) over $K$. Substituting the values of $\Lambda$ and $B$ in (13), we find the potential

$$A_\mu = \frac{R_{\nu\mu}}{\rho} \sum_{i=1}^{N} \frac{\lambda_i^2 (b_i - x)_\nu}{|b_i - x|^4}, \quad \rho = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{|b_i - x|^2}. \quad (21)$$

Obviously, we got an eight-dimensional analogue of the 't Hooft instanton in dimension four. In particular, for $N = 1$, the obtained instanton is gauge equivalent to the 1-instanton that was found in ref. [15].
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