Category of Quantizations and Inverse Problem

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Abstract

We introduce a category composed of all quantizations of all Poisson algebras. By the category, we can treat in a unified way the various quantizations for all Poisson algebras and develop a new classical limit formulation. This formulation proposes a new method for the inverse problem, that is, the problem of finding the classical limit from a quantized space. Equivalence of quantizations is defined by using this category, and the conditions under which the two quantizations are equivalent are investigated. Two types of the classical limits are defined as the limits in the context of category theory and they are determined by giving a sequence of objects. Using these classical limits, we discussed the inverse problem of determining the classical limit from some noncommutative Lie algebra. From a Lie algebra, we construct a sequence of quantized spaces, from which we determine a Poisson algebra. We also present a method to obtain this sequence of quantizations from the principle of least action by using matrix regularization. Apart from the above category of quantizations of all Poisson algebras, we also introduce a category of quantizations of a fixed single Poisson algebra. In this category, the other classical limit is defined, and it is automatically determined for the category.

1 Introduction

In M-theory and string theory, it has been proposed for a long time that matrix models give their constructive formulation [5, 22]. Noncommutative manifolds are obtained as classical solutions of such matrix models. (See for example [17, 45] and references therein.) Assuming that the universe as a noncommutative manifold is realized, the space we inhabit is perceived as a smooth manifold, at least to an approximation. Therefore, it is important to obtain smooth manifolds as classical limits of noncommutative manifolds.

In many cases, noncommutative geometry is some kind of a noncommutative deformation of a smooth manifold. In other words, it is often made as a quantization of a smooth manifold. So, it is often called an inverse problem to construct a usual commutative manifold or commutative algebra as a classical limit of a noncommutative manifold.

Taking the classical limit of a noncommutative manifold is often fraught with difficulty. The images of the quantization maps have information of the classical geometry of manifolds, but the algebras generated from them lose information of the original geometry. For example, as a well-known phenomenon, matrix regularization generates the same matrix algebra whether a two-dimensional torus is transformed into a
fuzzy torus or a two-dimensional sphere into a fuzzy sphere. (See Appendix C and [18, 3, 21, 33, 34].) Put another way, when reading a classical geometry (Poisson algebra) from a matrix algebra, the Poisson algebra obtained depends on what classical limit is taken [12, 15, 16]. Various approaches have been taken to the inverse problem of how to extract geometric properties from matrix algebras [44, 11, 41, 24, 25, 4]. Therefore, a framework including any Poisson algebra and its quantizations is important in considering such issues to be investigated in a unified manner.

Another important lesson implied by the difficulty described above is that the commutative limit cannot be identified by information in the algebra obtained as the target of a quantization map. Rather, information about the modules or vector spaces that are images of quantization maps before their generating algebras is important to identify the classical limits. This tendency can be read from, for example, fuzzy spheres and fuzzy torus in Appendix C. Note that an image of a quantization map is not an algebra but a module in general since a quantization map is not an algebra homomorphism but a module homomorphism (linear map). Furthermore, the classical limit depends on what classical limit is taken as already mentioned. Accordingly, it is suggested that we should prepare a sequence of modules to determine a Poisson algebra as a classical limit, not just one of them.

In light of the above, the first purpose of this study is to construct a category composed of all quantizations of all Poisson algebras. This is actually performed in Section 3. We denote this category by $QW$ (Quantum World), and its strict definition is given in Section 3. Quantization here refers to a noncommutative deformation in the sense of approximate canonical quantization of a Poisson algebra. The definition and properties of this quantization are given in Section 2. Quantizations in this paper are not necessarily identified with them in the sense of quantum mechanics, nor in the sense of spectral discretization, but in the sense of many noncommutative geometries. The morphisms that define this $QW$ consist of Poisson morphisms, quantization maps, and algebra homomorphisms whose domain is restricted. Their compositions are constructed in such a way that they satisfy the axioms of the category without contradiction. Furthermore, in this context, it is also important to show that quantization can be properly classified. To this end, we construct a functor from $QW$ to some comma category and show that the isomorphism of the object of the comma category gives the isomorphism pair of the Poisson algebra and the algebra generated by its quantized space. $QW$ allows any quantization of any Poisson algebra to be treated as a single framework.

The second purpose is to define the classical limit within the framework of this category of all quantizations $QW$ in a non-traditional way. Of course, the question of what classical limit is best within the $QW$ framework depends on what classical limit is needed. In this paper, we propose three types of classical limits and investigate their properties by considering concrete examples in Section 5. We define the classical limit using the limit in the category theory or a slight modified limit. The first two types are defined by using sequence of objects (modules) in $QW$ and morphisms between them. One is a naive adaptation of the category theory limit to $QW$. When that category theory limit corresponds to a Poisson algebra, it is defined to be the classical limit. To define another classical limit, it is necessary to modify the definition of the limit of
category theory. We impose universality within the framework that the vertex of the cone is a Poisson algebra. The classical limit so defined is called the weak classical limit. These limits depends on how to choose the sequence of objects, and the existence of the limits are not guaranteed. As concrete examples, we investigate the classical limits for a sequence of deformation quantizations of a polynomial ring and for a sequence of fuzzy spheres. By considering the all quantizations as a single category, we can make a structure that a sequence of modules as a sequence of objects in the category determines a Poisson algebra as the classical limit from all Poisson algebras. To make the sequence of objects, we use matrix regularization as an example. Furthermore, developing this argument we study how to obtain the classical limit by using principle of least action in Section 6.

The third type of the classical limit is made by focusing a single Poisson algebra. When we choose a Poisson algebra, this classical limit is determined automatically. It is constructed in the framework of a category made from $QW$ with a fixed Poisson algebra whose objects are quantization maps from the Poisson algebra. We prove that the classical limit coincides with the fixed Poisson algebra. We call this classical limit the strong classical limit.

We would like to make a few comments on recent previous studies that discuss quantization as noncommutative deformation of a Poisson algebra by using category theory approach. The construction of a quantization category was done by [19, 20]. The quantization category in [19, 20] was pioneering in the sense that it was a category-theoretic version of quantization, but it contained two problems. The first problem is that the category in [19, 20] is constructed as a whole of quantization with respect to a single fixed Poisson algebra. As already mentioned, this is a weakness for the inverse problem because it is important to treat the all quantizations for the all Poisson algebras, in a unified manner. The second problem is that the quantization maps treated in [19, 20] are maps from some Poisson algebra to Lie algebras. Target spaces of the maps do not necessarily reflect the reality of the image of the mapping. A Lie algebra that is a target space of some quantization map may have lost some of their original information of the image of quantization map, as already mentioned. So, to classify quantization maps, we need more complex structure in the formulation. On the other hand, the category constructed in this article conquers these two problems.

This paper is organized as follows. In Section 2 we study quantization maps. Using the properties of quantization maps, we construct a category of the all quantizations of all Poisson algebras $QW$ in Section 3. In Section 4 a way to classify the quantization maps is established by using the context of $QW$. In Section 5 we discuss how to find the classical limit in $QW$. Three types of classical limits are discussed. Examples of these classical limits are studied. In Section 6 the invers problem is discussed in the context of the classical limit defined in Section 5. Especially, a way to obtain the classical limit by the principle of least action with a sequence of actions is proposed. We make a summary of this article in Section 7. A symbol like Landau notation, used throughout this paper, is defined in Appendix A. Many new definitions appear in this paper, and a list of the symbols is available in Appendix B. We often use a fuzzy sphere or a fuzzy torus to make examples. So, we make a short introduction to them in Appendix C.
2 Quantization Maps

We define the quantization maps in this section. To formulate them, we use $R$-modules and $R$-algebras. $R$ denote a fixed commutative ring over $\mathbb{C}$, in this paper. Only finitely generated $R$-modules and $R$-algebras are considered. As the simplest case, we can employ $\mathbb{C}$ as $R$ in all the discussions in this paper.

2.1 Definition of Weak Quantization Maps

We define a category of Poisson algebras whose morphisms are restricted into surjective Poisson morphisms as follows.

**Definition 2.1.** Let $(A, \cdot_A, \{ , \}_A)$ and $(B, \cdot_B, \{ , \}_B)$ be Poisson algebras over $R$. When a linear $\phi_{A,B}: A \to B$ satisfies

$$\phi_{A,B}(f \cdot_A g) = \phi_{A,B}(f) \cdot_B \phi_{A,B}(g) \quad (2.1)$$

$$\phi_{A,B}(\{f, g\}_A) = \{\phi_{A,B}(f), \phi_{A,B}(g)\}_B, \quad (2.2)$$

for any $f, g \in A$, $\phi_{A,B}$ is called a Poisson morphism. $\mathcal{Poisss}$ is a category whose objects are Poisson algebras over $R$ and its morphisms are “surjective” Poisson morphisms.

The reason why the morphisms were restricted to surjective maps will become clear later. Often the subscripts of Poisson brackets and the symbol for the product $\cdot_A$ and so on are omitted in the following. When we put $R = \mathbb{C}$, Poisson algebras are usual ones. All examples appearing in this paper are the cases that $R$ is fixed to $\mathbb{C}$.

**Definition 2.2** (Weak Quantization map $wkQ$). Let $A$ be a Poisson algebra over $R$, and let $T_i$ be an $R$-module that is given by a subset of some $R$-algebra $(M, *_M)$. If an $R$-module homomorphism (linear map) $t_{Ai} \in \text{Hom}_{R,\text{-mod}}(A, T_i)$ equips a constant $h(t_{Ai}) \in \mathbb{C}$ and satisfies

$$[t_{Ai}(f), t_{Ai}(g)]_M = \sqrt{-1}h(t_{Ai})t_{Ai}(\{f, g\}) + \tilde{O}(h^{1+\epsilon}(t_{Ai})) \quad (\epsilon > 0) \quad (2.3)$$

for arbitrary $f, g \in A$, where $[a, b]_M := a *_M b - b *_M a$, we call $t_i$ a weak quantization map. For the case that $[t_{Ai}(f), t_{Ai}(g)]_M = 0$ and $t_{Ai}(\{f, g\}) = 0$ for $\forall f, g \in A$, we put $h(t_{Ai}) = 0$. $\tilde{O}$ is defined in the Appendix A. We denote the set of all weak quantization maps by $wkQ$:

$$wkQ := \bigsqcup_{A \in \mathcal{Poisss}} \{t_{Ai} \in \text{Hom}_{R,\text{-mod}}(A, T_i) \mid t_{Ai} \text{ satisfies } (2.3)\}$$

Note that the target of $t_{Ai}$ is not an algebra in general, but it is a subset of an algebra $M$, so the left-hand side of (2.3) is defined through the product in $M$. We call $h(t_{Ai}) \in \mathbb{C}$ a noncommutativity parameter. As we will see in Proposition 2.13 the $h$ can take any value of $\mathbb{C}^\times$ for fixed $A$ and $T_i$ if $h(t_{Ai}) \neq 0$. In the following, $*_M$ is omitted as appropriate. For simplicity, $[a, b]_M$ is abbreviated to $[a, b]$ when there is no risk of confusion. For the case $R = \mathbb{C}$, $t_{Ai} \in \text{Hom}_{\mathbb{C}}(A, T_i)$ is a linear map between $\mathbb{C}$-vector spaces with the condition (2.3).

**Definition 2.2** includes many kinds of quantizations. For example, matrix regularization [21, 3], fuzzy spaces [33], and Berezin-Toeplitz quantization [12, 15, 42] which
have original ideas of the matrix regularization, satisfy Definition 2.2. In fact, the fuzzy sphere appears frequently as an example in this paper. In addition, the strict deformation quantization introduced by Rieffel [38, 39, 28], the prequantization [8, 30, 48], and Poisson enveloping algebras [35, 36, 47, 49] are also in wkQ. These facts are derived immediately from these definitions. (In [19, 20], we can see organized discussions about these quantization maps. The conditions for wkQ is a part of the definition of pre-Ω in [19, 20].)

2.2 Properties for Weak Quantization Maps

For the later part, we summarize basic properties of R-algebra homomorphisms used in this paper, and we derive some propositions of the weak quantization maps. In order to make this article a self-contained article, proofs of basic matters are not omitted.

**Proposition 2.3.** Let \((M, *_M)\) and \((N, *_N)\) be R-algebras, and let \(T\) be a generating set of \(M\), i.e. \(\langle T \rangle = M\). For any R-algebra homomorphisms \(h_{MN}, h'_{MN}\) from \(M\) to \(N\),

\[
h_{MN}|_T = h'_{MN}|_T \iff h_{MN} = h'_{MN},
\]

where \(h_{NM}|_T\) and \(h'_ {NM}|_T\) are \(h_{MN}\) and \(h'_ {MN}\) whose domains are restricted to \(T\), respectively.

**Proof.** For a multi index \(I_k = (i_1, \cdots, i_k)\), we denote \(a_i \ast_M \cdots \ast_M a_i\) by \(a_{I_k}\), where \(a_{ij} \in T\). Since \(T\) generates \(M\), any \(b \in M\) can be written as a polynomial

\[
b = \sum_k \sum_{I_k} b_{I_k} a_{I_k} \quad (b_{I_k} \in R).
\]

If \(h_{MN}|_T = h'_{MN}|_T\), \(h_{NM}(a_{ij}) = h'_{MN}(a_{ij})\) for each \(a_{ij}\), and

\[
h_{NM}(b) = \sum_k \sum_{I_k} b_{I_k} h_{NM}(a_{I_k})
= \sum_k \sum_{I_k} b_{I_k} h'_{MN}(a_{I_k}) = h'_{NM}(b).
\]

The proof for the opposite direction is trivial. \(\square\)

**Proposition 2.4.** Let \(M\) and \((T)\) be R-algebras, where \(T\) is a subset of \(\langle T \rangle\) that generates algebra \((T)\). For any R-algebra homomorphism \(h : \langle T \rangle \to M\)

\[
\langle h(T) \rangle = h(\langle T \rangle).
\]

**Proof.** For any \(m \in \langle h(T) \rangle\) can be written as a polynomial

\[
m = \sum_k \sum_{I_k} m_{I_k} e_{I_k} \quad (m_{I_k} \in R),
\]

where \(e_{I_k} = e_{i_1} \ast \cdots \ast e_{i_k}\) with \(e_{ij} \in h(T)\). There exists \(e'_{ij} \in T\) such that \(e_{ij} = h(e'_{ij})\) for each \(e_{ij}\). Then,

\[
m = \sum_k \sum_{I_k} m_{I_k} h(e'_{i_1}) \ast \cdots \ast h(e'_{i_k})
= h(\sum_k \sum_{I_k} m_{I_k} e'_{i_1} \ast \cdots \ast e'_{i_k}).
\]

\(\square\)
Proposition 2.5. Let $A$ be a Poisson algebra. Let $t_i : A \to M_i$ be a weak quantization map whose image $t_i(A)$ generates an $R$-algebra $M_i$, i.e. $\langle t_i(A) \rangle = M_i$. For an arbitrary $R$-algebra $M_j$ and any $R$-algebra homomorphism $h_{ij} : M_i \to M_j$, an $R$-module homomorphism $t_j : A \to t_j(A) \subset M_j$ defined by

$$t_j = h_{ij} \circ t_i,$$

satisfies

$$[t_j(f), t_j(g)]_{M_j} = \sqrt{-1}h(t_i)t_j(\{f, g\}) + \tilde{O}(h^{1+\epsilon}(t_i)) \quad (\forall f, g \in A).$$

Proof. For any $f, g \in A$, from (2.6) and the fact that $h_{ij}$ is an $R$-algebra homomorphism,

$$[t_j(f), t_j(g)]_{M_j} = (h_{ij}|_{t_i(A)} \circ t_i)(f) *_{M_j} (h_{ij}|_{t_i(A)} \circ t_i)(g) - (h_{ij}|_{t_i(A)} \circ t_i)(g) *_{M_j} (h_{ij}|_{t_i(A)} \circ t_i)(f)$$

$$= h_{ij}(t_i(f) *_{M_i} t_i(g) - t_i(f) *_{M_i} t_i(g))$$

$$= h_{ij}([t_i(f), t_i(g)]_{M_i}).$$

Since $t_i$ is in $wkQ$, the above equation is written as

$$[t_j(f), t_j(g)]_{M_j} = h_{ij}(\sqrt{-1}h(t_i)t_i(\{f, g\}) + \tilde{O}(h^{1+\epsilon}(t_i)))$$

$$= \sqrt{-1}h(t_i)t_j(\{f, g\}) + \tilde{O}(h^{1+\epsilon}(t_i)).$$

Here we use Proposition [A.2] in Appendix [A]. \qed

Note that (2.7) does not mean that $t_j$ is a weak quantization map with $h(t_j) = h(t_i)$. For example, $h(t_i) \neq 0$ case, if $t_j(A)$ generates commutative algebra, (2.7) means $\{f, g\}$ is in $\text{Ker } t_j$. In this case $h(t_j) = 0$ by the definition of the weak quantization.

Corollary 2.6. Let $A$ be a nontrivial Poisson algebra, i.e. $\exists f, g \in A$ such that $\{f, g\} \neq 0$. Let $t_i, t_j, h_{ij}, M_i, M_j$ be them in Proposition 2.5. If $\langle t_i(A) \rangle$ is noncommutative and if $\langle t_j(A) \rangle$ is commutative and $\langle t_i(A) \rangle \subset M_j$ is commutative, then

$$\{A, A\} := \{ \{f, g\} \in A \mid f, g \in A \} \subset \text{Ker } t_j,$$

is not $\{0\}$. $t_i(\{A, A\})$ is not $\{0\}$, too, and

$$t_i(\{A, A\}) \subset \text{Ker } h_{ij}.$$ (2.9)

Proof. When $\langle t_j(A) \rangle$ is a commutative algebra, for any $f, g \in A$,

$$0 = [t_j(f), t_j(g)] = ih(t_i)t_j(\{f, g\}) + \tilde{O}(h^{1+\epsilon}).$$ (2.10)

Since $(t_i(A))$ is noncommutative, existence of $f, g \in A$ such that $[t_i(f), t_i(g)] \neq 0$ derives $h(t_i) \neq 0$. Then we get $t_j(\{f, g\}) = h_{ij} \circ t_i(\{f, g\}) = 0$ for $\forall f, g \in A$. $h(t_i) \neq 0$ also means that there exist $f, g \in A$ such that $t_i(\{f, g\}) \neq 0$, and $t_i(\{A, A\}) \subset \text{Ker } h_{ij}$ is also obtained. \qed
Proposition 2.7. Suppose that a noncommutativity parameter of a weak quantization map \( t_i : A \to t_i(A) \subset M_i \) is non-zero, i.e. \( h(t_i) \neq 0 \), where \( M_i \) is some noncommutative \( R \)-algebra. For an \( R \)-algebra homomorphism \( h_{ij} : M_i \to M_j \) and \( t_j := h_{ij} \circ t_i \), \( \langle t_j(A) \rangle \) is a commutative algebra if and only if

\[
\{ [t_i(f), t_i(g)] \mid f, g \in A \} \subset \text{Ker } h_{ij}.
\]

Proof. For any \( f, g \in A \) with \( [t_i(f), t_i(g)] \neq 0 \),

\[
h_{ij}([t_i(f), t_i(g)]) = [h_{ij} \circ t_i(f), h_{ij} \circ t_i(g)] = [t_j(f), t_j(g)].
\]

Therefore, if \( \langle t_j(A) \rangle \) is commutative, \( \forall [t_i(f), t_i(g)] \in \text{Ker } h_{ij} \).

Conversely, suppose that \( [t_i(f), t_i(g)] \) is in \( \text{Ker } h_{ij} \) for \( \forall f, g \in A \). \( \forall m_t \in \langle t_j(A) \rangle \) is expressed as \( m_t = \sum_k \sum_{I_k} m_{I_k} a_{I_k} \), where \( a_{I_k} = a_{i_1} \cdots a_{i_k}, a_{i_p} \in \text{Im } h_{ij} \) and \( m_{I_k} \in R \). There exists \( e_{i_p} \in t_i(A) \) such that \( a_{i_p} = h_{ij}(e_{i_p}) \). Any commutator \( [m_t, m_p] \) is written as

\[
[m_t, m_p] = \sum_k \sum_{I_k} m_{I_k} m_{pJ_q}[a_{I_k}, a_{J_q}],
\]

and \( [a_{I_k}, a_{J_q}] = \sum_{1 \leq s \leq k} \sum_{1 \leq n \leq q} (\cdots [a_{i_s}, a_{j_n}] \cdots) \). So,

\[
[m_t, m_p] = \sum_k \sum_{I_k, J_q} m_{I_k} m_{pJ_q} \sum_{1 \leq s \leq k} \sum_{1 \leq n \leq q} (\cdots [a_{i_s}, a_{j_n}] \cdots).
\]

\[
= \sum_k \sum_{I_k, J_q} m_{I_k} m_{pJ_q} \sum_{1 \leq s \leq k} \sum_{1 \leq n \leq q} (\cdots h_{ij}(e_{i_s}, e_{j_n}) \cdots).
\]

Since \( [e_{i_s}, e_{j_n}] \in \text{Ker } h_{ij} \), we find \( [m_t, m_p] = 0 \).

To avoid commutative algebras from quantum theories, we will restrict \( R \)-algebra homomorphisms to injective ones later.

We found that \( t_j = h_{ij} \circ t_i \) in Proposition 2.5 is a weak quantization map. \( \langle t_j(A) \rangle = \langle h_{ij} \circ t_i \rangle \subset M_j, \text{Im } h_{ij} \), and \( M_j \) are not always equal. So, “when \( h_{ij} \circ t_i(A) \) generates \( M_j \)” is a natural question. The following discussion derives a useful tool to judge this problem.

Proposition 2.8. Let \( A, B, C \in \text{ob}(\text{R-Mod}) \) be \( R \)-modules satisfying \( \langle A \rangle = M_i, \langle B \rangle = \langle C \rangle = M_j \), where \( M_i \) and \( M_j \) are \( R \)-algebras. If an \( R \)-algebra homomorphism \( h_{ji} : M_j \to M_i \) satisfies \( h_{ji}|_{B}(B) = A \), where \( h_{ji}|_{B} \) is defined by restriction of its domain to \( B \), then the image of \( C \) by \( h_{ji} \) generates \( M_j \), i.e. \( \langle h_{ji}|_{C}(C) \rangle = M_i \).

Proof. From \( \langle B \rangle = \langle C \rangle = M_j \), an arbitrary \( b \in B \) is expressed by elements of \( C \). By \( h_{ji}|_{B}(B) = A \) and \( \langle A \rangle = M_i, h_{ji}|_{B}(B) \) generates \( M_i \). Then \( \langle h_{ji}|_{C}(C) \rangle = M_i \) is given.
We introduce a sequence of weak quantization maps \( t^{(l)}_{\mu j} : A^\mu \rightarrow t^{(l)}_{\mu j}(A^\mu) \subset M_j, (l \in I_j, \mu = 1, 2, \ldots) \) that generate an \( R \)-algebra \( M_j \)

\[
M_j = (t^{(l)}_{\mu j}(A^\mu)) = \cdots = (t^{(n)}_{\mu j}(A^\mu)) = \cdots \]

\[
= (t^{(l)}_{\mu j}(A^\nu)) = \cdots = (t^{(n)}_{\mu j}(A^\nu)) = \cdots \]

\[
: \cdots , \tag{2.11}
\]

where \( A^\mu, A^\nu, \cdots \in \text{ob}(\mathcal{Poisss}) \) and \( I_j \) is an index set.

**Proposition 2.9.** Let \( A^\mu, A^\tau \) be Poisson algebras. Let \( t^{(k)}_{\mu j} : A^\mu \rightarrow t^{(k)}_{\mu j}(A^\mu) \) and \( t^{(l)}_{\mu i} : A^\mu \rightarrow t^{(l)}_{\mu i}(A^\mu) \) be weak quantization maps such that \( (t^{(k)}_{\mu j}(A^\mu)) = M_j, (t^{(l)}_{\mu i}(A^\mu)) = M_i. \)

If an \( R \)-algebra homomorphism \( m_{ji} : M_j \rightarrow M_i \) satisfies commutativity \( m_{ji} \circ t^{(k)}_{\mu j} = t^{(l)}_{\mu i}, \)

then \( \forall t^{(n)}_{\tau j} \in \text{wkQ} \) such that \( (t^{(n)}_{\tau j}(A^\tau)) = M_j, t_i := m_{ji} \circ t^{(n)}_{\tau j} : A^\tau \rightarrow M_i \) satisfies

\[
[t_i(f), t_i(g)]_{M_i} = \sqrt{-1}h(t^{(n)}_{\tau j})t_i(\{f, g\}) + \hat{O}(h^{1+\epsilon}(t^{(n)}_{\tau j})), (f, g \in A^\tau), \tag{2.12}
\]

and

\[
\langle t_i(A^\tau) \rangle = M_i. \tag{2.13}
\]

**Proof.** This proposition follows from Proposition 2.8 and Proposition 2.8.

\[ \square \]

**Remark.** Note that the existence of Poisson maps between \( A^\mu \) and \( A^\tau \) are not requested in Proposition 2.9

**Example 2.10.** As an example, let us consider fuzzy spheres and fuzzy tori. These are obtained by matrix regularization for \( S^2 \) and \( T^2 \), and their processes are summarized in Appendix C. \( A \) and \( B \) denote the Poisson algebras defined as a set of functions on \( S^2 \) and \( T^2 \), respectively. \( t_k : A \rightarrow T_k := t_k(A) \subset M_k = \text{Mat}_k(\mathbb{C}) \) and \( q_k : B \rightarrow Y_k := q_k(B) \subset M_k = \text{Mat}_k(\mathbb{C}) \) are defined as weak quantization maps. The details are given in Appendix C. It is known that both \( t_k(A) \) and \( q_k(B) \) generate the same matrix algebra \( \text{Mat}_k \).
Let introduce a map \( t_{k\oplus k} : A \to T_k \oplus T_k \subset M_k \oplus M_k \) as
\[
t_{k\oplus k}(f) = \begin{pmatrix} t_k(f) & 0 \\ 0 & t_k(f) \end{pmatrix} \quad (f \in A),
\]
(2.14)
This satisfies \([t_{k\oplus k}(f), t_{k\oplus k}(g)]_M = \sqrt{-1}h(t_k)t_{k\oplus k}(\{f, g\}) + \tilde{O}(h^{1+\epsilon}(t_k))\). In other words, \( t_{k\oplus k} \) is a weak quantization map. We define \( q_{k\oplus k} : B \to Y_k \oplus Y_k \) as a weak quantization map, in the same way. Let \( \pi : M_k \oplus M_k \to M_k \) be a projection map to one \( M_k \). From Proposition 2.9, by using a projection \( \pi \) and commutativity \( \pi \circ t_{k\oplus k} = t_k \), we obtain \( q_k = \pi \circ q_{k\oplus k} \) as a weak quantization map and \( \langle q_k(B) \rangle = M_k \). This result reproduces known facts.

To prevent later confusion, a note is made here. From the fact that this projection \( \pi \) is an \( R \)-algebra homomorphism but not injective one, this \( \pi \) is removed from the set of \( R \)-algebra homomorphisms we will consider in the following parts of this paper. As we will see in Section 3 only injective \( R \)-algebra homomorphisms are in \( \text{Mor}(QW) \).

### 2.3 Quantization Maps

The parameter of noncommutativity \( h \) is determined by the choice of weak quantization map i.e. \( h : wkQ \to \mathbb{C} \). The set of the weak quantization maps \( wkQ \) is the disjoint union as
\[
wkQ = \bigsqcup_{x \in \mathbb{R}_{\geq 0}} Qh_x,
\]
where
\[
Qh_x := \{ q \in wkQ \mid |h(q)| = x \in \mathbb{R}_{\geq 0} \}.
\]
We denote the set of all targets of \( Qh_x \) by
\[
Qh_x(\mathcal{Poisss}) := \{ t(q) \mid q \in Qh_x \},
\]
(2.15)
where \( t(q) \) is a target of \( q \). Particularly, \( Qh_0 := \{ q \in wkQ \mid |h(q)| = 0 \} \) is a set of maps to commutative algebras including the identity \( \text{Id}_A \) for \( A \in U_p(\mathcal{Poisss}) \), where \( U_p \) is a forgetful functor from \( \mathcal{Poisss} \) to \( QW \) defined in Section 3.
**Definition 2.11.** The set of quantization maps $Q$ is defined by

$$Q := wkQ \setminus Qh_0 = \bigcup_{x \neq 0} Qh_x,$$

and the set of all targets of the quantization maps is defined by

$$Q(Poisss) := \{ t(q) \mid q \in Q \}$$

(2.17)

We call $q \in Q$ a quantization map.

At first, we see the following proposition.

**Proposition 2.12.** Let $A^\mu, A^\nu$ be Poisson algebras and let $T_i$ be an $R$-module as a subset of an $R$-algebra $M_i$. For a weak quantization map $t_{\nu i} : A^\nu \to T_i$ and a Poisson morphism $\phi : A^\mu \to A^\nu$, $t_{\nu i} \circ \phi : A^\mu \to T_i$ is also a weak quantization map.

**Proof.**

$$\sqrt{-1} h(t_{\nu i}) \circ \phi(\{ f, g \}) + \tilde{O}(h^{1+\epsilon}) = \sqrt{-1} h(t_{\nu i}) (\phi(f), \phi(g)) + \tilde{O}(h^{1+\epsilon})$$

$$= [t_{\nu i}(\phi(f)), t_{\nu i}(\phi(g))] + \tilde{O}(h^{1+\epsilon}).$$

Note that Proposition 2.12 does not show that $t_{\nu i} \circ \phi$ is always in $Q$ even if $t_{\nu i} \in Q$. For example, suppose that $C \subset T_i \subset M_i$ is a commutative subalgebra of $M_i$, and $\text{Im} \phi \subset t^{-1}_{\nu i}(C)$. Then the image of $t_{\nu i} \circ \phi$ is in $C$ and $h(t_{\nu i} \circ \phi) = 0$, i.e. $t_{\nu i} \circ \phi \notin Q$. On the other hand, if $\phi$ is surjective $t_{\nu i} \circ \phi(A^\mu) = t_{\nu i}(A^\nu)$. This fact and Proposition 2.12 derives the following.

**Proposition 2.13.** Let $A^\mu, A^\nu$ be Poisson algebras. Let $t_{\nu i} \in Q$ be a quantization map whose source is $A^\nu$. If a Poisson morphism $\phi : A^\mu \to A^\nu$ is surjective, then $t_{\nu i} \circ \phi \in Q$.

This property is important to make the category $QW$, so an ancillary condition that the morphisms are surjective is added to the conditions for $\text{Mor}(Poisss)$.

Next, we make sure that we can treat $h$ as a continuous parameter in $Q$.

**Proposition 2.14.** Let $A$ be a Poisson algebra. Let $T_i$ be an $R$-module. For $\forall q_i \in Q : A \to T_i$ and $\forall x \in C^x$,

$$q_i^x := \frac{x}{h(q_i)} q_i$$

(2.18)

is a quantization map from $A$ to $T_i$, such that $h(q_i^x) = x$, $q_i(A) = q_i^x(A)$.

**Proof.** From $\{q_i(f), q_i(g)\} = \sqrt{-1} h(q_i) q_i(\{ f, g \}) + \tilde{O}(h^{1+\epsilon})$, we obtain

$$[q_i^x(f), q_i^x(g)] = i x q_i^x(\{ f, g \}) + \frac{x^2}{h^2(q_i)} \tilde{O}(h^{1+\epsilon}).$$

Because $\frac{x^2}{h^2(q_i)} \tilde{O}(h^{1+\epsilon}) = \tilde{O}(x^{1+\epsilon})$, the proof is finished.

This proposition teaches us that there are uncountable infinite number quantizations for each target $T_i$.  

10
3 Category of Quantization

In this section, we construct a category $QW$ that describe whole space of the all quantizations of all Poisson algebras.

We denote the target and the source of a morphism $q \in Q$ by $t(q)$ and $s(q)$, respectively. Recall that $t(q)$ is a subspace of some $R$-algebra for each $q$.

At the first step, we introduce a category of target spaces $\{t(q)| q \in Q\}$.

**Definition 3.1.** $QP$ is a subcategory of $R$-Mod defined as follows.

- $\text{ob}(QP) := Q(\mathcal{Poisss}) = \{t(q)| q \in Q\}$.
- Suppose that two objects $t(q_i), t(q_j)$ are subsets of $R$-algebras $M_i, M_j$, respectively. The morphism $\text{Mor}(t(q_i), t(q_j))$ is the set of all $h_{ij}|_{t(q_i)} : t(q_i) \to t(q_j)$. Here, $h_{ij}|_{t(q_i)}$ is given as an injective $R$-algebra homomorphism $h_{ij} : M_i \to M_j$ whose source is restricted into $t(q_i)$, and $\text{Im}(h_{ij}|_{t(q_i)}) \subset t(q_j)$:

$$\text{Mor}(t(q_i), t(q_j)) := \{h_{ij}|_{t(q_i)} : t(q_i) \to t(q_j) | h_{ij} : M_i \to M_j \text{ is an injective } R \text{-algebra homomorphism }\}.$$  

The composition of $m_{ij} = h_{ij}|_{t(q_i)} \in \text{Mor}(t(q_i), t(q_j))$ and $m_{jk} = h_{jk}|_{t(q_j)} \in \text{Mor}(t(q_j), t(q_k))$ is well-defined by

$$m_{jk} \circ m_{ij} = h_{jk}|_{t(q_j)} \circ h_{ij}|_{t(q_i)} = (h_{jk} \circ h_{ij})|_{t(q_i)}.$$  

Next, we consider a category $\mathcal{Poisss} \sqcup QP$ defined by

- $\text{ob}(\mathcal{Poisss} \sqcup QP) = \text{ob}(\mathcal{Poisss}) \sqcup \text{ob}(QP)$
- $\text{Mor}(\mathcal{Poisss} \sqcup QP) = \text{Mor}(\mathcal{Poisss}) \sqcup \text{Mor}(QP)$.

Since $\text{ob}(\mathcal{Poisss}) \cap \text{ob}(QP) = \emptyset$ and $\text{Mor}(\mathcal{Poisss}) \cap \text{Mor}(QP) = \emptyset$, it is trivial that $\mathcal{Poisss} \sqcup QP$ is a category.

The final step to introduce a category $QW$ that describe whole space of the all quantizations of all Poisson algebras is to define its morphism by

$$\text{Mor}(QW) := \text{Mor}(\mathcal{Poisss}) \sqcup \text{Mor}(QP) \sqcup Q.$$  \hspace{1cm} (3.1)

**Proposition 3.2.** If $\forall f, g \in \text{Mor}(QW)$ satisfy $s(f) = t(g)$, then $f \circ g \in \text{Mor}(QW)$.

**Proof.** For the case $f, g \in \text{Mor}(\mathcal{Poisss})$ or for the case $f, g \in \text{Mor}(QP)$, this statement is trivial. For $\forall f, g \in Q$, $f \circ g$ never exists, because $s(f) \in \text{ob}(\mathcal{Poisss})$ and $t(g) \in \text{ob}(QP)$. So, we have to show this statement in the case i) $g \in \text{Mor}(\mathcal{Poisss})$, $f \in Q$ and the case ii) $g \in Q$, $f \in \text{Mor}(QP)$. For the case i), Proposition 2.13 shows that $f \circ g \in Q$. For the case ii), Proposition 2.5 Corollary 2.6 and Proposition 2.7 show that $f \circ g \in Q$, since every morphism in $QP$ is injective. Therefore, this proposition is proved.  

Proposition 3.2 guarantees the following definition of a category of whole space of the quantizations makes sense.

**Definition 3.3** (Quantization World $QW$). The category of quantization $QW$ is a subcategory of $R$-Mod whose object is defined by

$$\text{ob}(QW) := \text{ob}(\mathcal{Poisss}) \cup \text{ob}(QP)$$  \hspace{1cm} (3.2)

and its morphism is defined by (3.1).
A schematic diagram of this category of quantization $QW$ is shown below.

\[
\begin{array}{cccccc}
\cdots & M_1 & h_{12} & M_2 & h_{23} & M_3 & \cdots \\
\cdots & \langle q_1(A) \rangle & \langle q_1(A) \rangle & \langle q_2(A) \rangle & \langle q_2(A) \rangle & \langle q_3(B) \rangle & \cdots \\
\cdots & q_1(A) & q_2(A) & q_3(B) & q_3(B) & q_3(B) & \cdots \\
A & q_1 & q_2 & q_3 & q_3 & B & \cdots \\
\end{array}
\]

Here $M_i$ is an $R$-algebra, and $\langle q_i(A) \rangle, \cdots$ are subalgebras of $M_i, \cdots$ that generated by $q_i(A), \cdots$, respectively. $A, B, \cdots$ are Poisson algebras, and $q_i$ is a quantization map. $\phi_{AB}$ is a surjective Poisson morphism, and $h_{ij}$ is an injective $R$-algebra homomorphism. This figure (3.3) is not strict but just for help to understand relations between objects in $QW$.

It would not be waste to emphasize the importance of injectivity of $\text{Mor}(t(q_j), t(q_i))$ here, even though this is an overlap with what was previously stated around Proposition 2.5, Corollary 2.6 and Proposition 2.7. Consider some $q_j : A \rightarrow T_j \in Q$ i.e. $h(q_j) \neq 0$, where $\langle T_j \rangle = M_i$ is a noncommutative $R$-algebra. Suppose there exists an $R$-algebra homomorphism $h_{ji} : M_j \rightarrow M_i$ such that its image $\text{Im} h_{ji} \subset M_i$ is a commutative subalgebra. For $\forall f, g \in A$,

\[
0 = [h_{ji} \circ q_j(f), h_{ji} \circ q_j(g)] = h_{ji}([q_j(f), q_j(g)]) = h(q_j) h_{ji} \circ q_j(\{f, g\}) + \hat{O}(h^{1+\epsilon}(q_j)),
\]

since $\text{Im} h_{ji}$ is commutative. From $h(q_j) \neq 0$,

\[
h_{ji} \circ q_j(\{f, g\}) = 0, \quad \hat{O}(h^{1+\epsilon}(q_j)) = 0.
\]

Furthermore, we find that there exists a pair of $f, g$ such that $q_j(\{f, g\}) \neq 0$ since $h(q_j) \neq 0$. Then we obtain that

\[
\text{Ker} h_{ji} |_{T_j} \supset \{q_j(\{f, g\}) \mid f, g \in A\} \neq \{0\}.
\]

The presence of this kernel prevents the construction of a category $QW$ using $Q$. To avoid this, injectivity is added to the definition for $\text{Mor}(t(q_j), t(q_i))$.

## 4 Classification of Quantizations

As the classical limits of matrix regularization and so on teach us, what characterizes quantum geometry is not only the algebra generated by the target space of the quantization map, but also the information about the map of quantization itself. Even if we confine our considerations to algebra, $\langle q_1(A) \rangle, \langle q_2(A) \rangle, \cdots$ are more important than $M_1, M_2, \cdots$ in the diagram (3.3) to characterize the quantization maps. In this section, we introduce a framework that classifies quantizations.
4.1 Equivalence of Quantization Maps

Recall that Poisson category $\mathcal{Poiss}$ is a category whose objects are Poisson algebras, and morphisms are surjective Poisson homomorphisms.

Let us introduce $U_P$ as a forgetful functor from $\mathcal{Poiss}$ to $QW$ forgetting multiplication and Poisson structure;

$$U_P : \mathcal{Poiss} \to QW.$$ 

For simplicity, we abbreviate $U_P(A)$ and $U_P(\phi)$ as $A$ and $\phi$ for $A \in \text{ob}(\mathcal{Poiss})$, $\phi \in \text{Mor}(\mathcal{Poiss})$, respectively. Next, let $I_{QP}$ be a functor from $QP$ to $QW$ such that all objects and morphisms are identically embedded in $QW$ as $I_{QP}(QP) = QP \subset QW$ i.e. $I_{QP}(T_i) = T_i \in \text{ob}(QW)$ for any object $T_i \in \text{ob}(QP)$ and $I_{QP}(h_{ij}) = h_{ij} \in \text{Mor}(QW)$ for any morphism $h_{ij} \in \text{Mor}(QP)$. As similar to the case of $U_P$, we abbreviate $I_{QP}(T_i)$ and $I_{QP}(h)$ as $T_i$ and $h$ for $T_i \in \text{ob}(QP)$, $h \in \text{Mor}(QP)$, respectively.

Using these functors a comma category is defined as follows.

\textbf{Definition 4.1 (Comma category $(U_P \downarrow I_{QP})$).} $(U_P \downarrow I_{QP})$ is a comma category whose object is a triple

$$\text{ob}(U_P \downarrow I_{QP}) := \{(A, q_{Ai}, T_i) \mid A \in \text{ob}(\mathcal{Poiss}), T_i \in \text{ob}(QP), q_{Ai} : A \to T_i \in \text{Mor}(QW)\}$$

(4.1)

and its morphism $(\phi_{AB}, t_{ij}) : (A, q_{Ai}, T_i) \to (B, q_{Bi}, T_j)$ is a pair of a Poisson map $\phi_{AB} : A \to B \in \text{Mor}(\mathcal{Poiss})$ and an $R$-module homomorphism $t_{ij} : T_i \to T_j \in \text{Mor}(QP)$ such that

$$t_{ij} \circ q_{Ai} = q_{Bi} \circ \phi_{AB}. \quad (4.2)$$

$$\begin{array}{ccc}
   U_P(A) = A & \xrightarrow{\phi_{AB}} & B = U_P(B) \\
   \downarrow q_{Ai} & & \downarrow q_{Bi} \\
   I_{QP}(T_i) = T_i & \xrightarrow{t_{ij}} & T_j = I_{QP}(T_j)
\end{array}$$

Using this comma category, quantization maps are classified. For every $q_{Ai} : A \to T_i \in Q$ there is a unique map $q'_{Ai} : A \to q_{Ai}(A) \in Q$ defined by $q'_{Ai}(f) := q_{Ai}(f)$ for any $f \in A$. For simplicity, $q'_{Ai}$ will simply be abbreviated as $q_{Ai}$ in the following.

\textbf{Definition 4.2.} When $q_{Ai} : A \to T_i \in Q$ and $q_{Bi} : B \to T_j \in Q$ satisfy

$$(A, q_{Ai}, q_{Ai}(A)) \simeq (B, q_{Bi}, q_{Bi}(B))$$

in $(U_P \downarrow I_{QP})$, that is, there exists an isomorphism $(\phi_{AB}, t_{ij}) : (A, q_{Ai}, q_{Ai}(A)) \to (B, q_{Bi}, q_{Bi}(B)))$, then we say that the quantization $q_{Ai}$ and $q_{Bi}$ are equivalent and we denote $q_{Ai} \simeq q_{Bi}$.

This is natural definition for equivalence of quantizations. However, the relation between algebras generating from $q_{Ai}(A)$ and $q_{Bi}(B)$ are not mentioned in this definition. Note that $q_{Ai}(A)$ and $q_{Bi}(B)$ are not algebras in general but just $R$-modules, subsets of some $R$-algebras. So, we investigate this relation, next.

Let us introduce the following functor.
Definition 4.3. A functor $F : (U_P \downarrow I_{QP}) \to \mathcal{Poisss} \times R$-alg is defined by the map between these objects:

$$(A, q_{A_i}, T_i) \mapsto (A, \langle q_{A_i}(A) \rangle),$$

and the map between these morphisms:

$$(\phi_{AB}, t_{ij}) \mapsto (\phi_{AB}, h_{ij}|_{\langle q_{A_i}(A) \rangle}),$$

where $h_{ij}$ is an $R$-algebra homomorphism from $\langle T_i \rangle \to \langle T_j \rangle$ such that $h_{ij}|_{T_i} = t_{ij}$.

Let us make sure that this $F$ is correctly defined as a functor. We have to check three points: 1) $h_{ij}|_{\langle q_{A_i}(A) \rangle}$ is uniquely determined from $t_{ij}$, 2) $h_{ij}(\langle q_{A_i}(A) \rangle) \subset \langle q_{B_j}(B) \rangle$, and 3) compositions are defined in a consistent manner.

1) At first, we check that $h_{ij}|_{\langle q_{A_i}(A) \rangle}$ is uniquely determined from $t_{ij}$.

From $h_{ij}|_{T_i} = t_{ij}$ and $q_{A_i}(A) \subset T_i$, $h_{ij}|_{\langle q_{A_i}(A) \rangle}$ uniquely exists. Therefore, we found $h_{ij}|_{\langle q_{A_i}(A) \rangle}$ is uniquely determined from Proposition 2.3.

2) Next, we would like to show $h_{ij}(\langle q_{A_i}(A) \rangle) \subset \langle q_{B_j}(B) \rangle$, because if the following proposition does not hold, then the composition of morphisms is not defined.

Proposition 4.4. Let $A, B$ be Poisson algebras with a surjective Poisson morphism $\phi_{AB} : A \rightarrow B$. Let $q_{A_i} : A \rightarrow T_i$ and $q_{B_j} : B \rightarrow T_j$ be quantization maps. Suppose that there exists $t_{ij} : T_i \rightarrow T_j$ defined by $t_{ij} = h_{ij}|_{T_i}$, where $h_{ij}$ is some $R$-algebra homomorphism $h_{ij} : \langle T_i \rangle \rightarrow \langle T_j \rangle$. If they satisfy the commutativity (4.3), then,

$$h_{ij}(\langle q_{A_i}(A) \rangle) = \langle q_{B_j}(B) \rangle.$$  

Proof. Because $\phi_{AB}$ is surjective,

$$\langle q_{B_j}(B) \rangle = \langle q_{B_j} \circ \phi_{AB}(A) \rangle.$$  

From the commutativity (4.2),

$$q_{B_j} \circ \phi_{AB} = t_{ij} \circ q_{A_i} = h_{ij}|_{\langle q_{A_i}(A) \rangle} \circ q_{A_i}.$$  

Proposition 2.4 shows that

$$h_{ij}(\langle q_{A_i}(A) \rangle) = \langle h_{ij}|_{\langle q_{A_i}(A) \rangle}(q_{A_i}(A)) \rangle.$$  

From (4.5), (4.6), and (4.7), we obtain the conclusion that we want.

By this proposition, $h_{ij}(\langle q_{A_i}(A) \rangle) \subset \langle q_{B_j}(B) \rangle$ is shown.

3) Finally, we make sure that $F$ is defined as a functor. Consider the compositions of morphisms $(\phi_{AB}, t_{ij}) : (A, q_{A_i}, T_i) \rightarrow (B, q_{B_j}, T_j)$ and $(\phi_{BC}, t_{jk}) : (B, q_{B_j}, T_j) \rightarrow (C, q_{C_k}, T_k)$ in $(U_P \downarrow I_{QP})$. From the results of 1) and 2)

$$F(\phi_{BC}, t_{jk}) \circ F(\phi_{AB}, t_{ij}) = (\phi_{BC}, h_{jk}|_{\langle q_{B_j}(B) \rangle}) \circ (\phi_{AB}, h_{ij}|_{\langle q_{A_i}(A) \rangle})$$

$$= (\phi_{BC} \circ \phi_{AB}, h_{jk} \circ h_{ij}|_{\langle q_{A_i}(A) \rangle}) = F((\phi_{BC}, t_{jk}) \circ (\phi_{AB}, t_{ij})).$$
From the above, it is shown that $F$ is a functor.

$F(\mathcal{U}_P \downarrow \mathcal{I}_{QP})$ is a subcategory of $\mathcal{Poisss} \times R$-alg. We call $F(\mathcal{U}_P \downarrow \mathcal{I}_{QP}) \mathcal{Poisss} - Q(\mathcal{Poisss})$ pair category. In the following, this is abbreviated as $P - QP := F(\mathcal{U}_P \downarrow \mathcal{I}_{QP})$. The object of $P - QP$ is a pair of a Poisson algebra and a generated algebra from an image of a quantization map, so we call the object of $P - QP$ quantization pair. Using $P - QP$, equivalence of quantization is expressed as follows.

**Theorem 4.5.** Let $q_{A_i} : A \to T_i$ and $q_{B_j} : B \to T_j$ be quantization maps in $Q$. In $P - QP := F(\mathcal{U}_P \downarrow \mathcal{I}_{QP})$, if and only if there exists an isomorphism of $\mathcal{Poisss} \times R$-alg $(\phi_{AB}, h_{ij})$ from $(A, \langle q_{A_i}(A) \rangle)$ to $(B, \langle q_{B_j}(B) \rangle)$, $q_{A_i}$ and $q_{B_j}$ are equivalent quantization:

$$\left( A, \langle q_{A_i}(A) \rangle \right) \simeq \left( B, \langle q_{B_j}(B) \rangle \right) \iff q_{A_i} \simeq q_{B_j}. \quad (4.8)$$

**Proof.** When $(\phi_{AB}, h_{ij}) : (A, \langle q_{A_i}(A) \rangle) \to (B, \langle q_{B_j}(B) \rangle)$ is an isomorphism in $\mathcal{Poisss} \times R$-alg, we put $t_{ij} = h_{ij}|_{q_{A_i}(A)}$ and $t_{ji} := h_{ij}^{-1}|_{q_{B_j}(B)}$. Then for $\forall a \in q_{A_i}(A) \subset \langle q_{A_i}(A) \rangle$

$$t_{ji} \circ t_{ij}(a) = h_{ij}^{-1} \circ h_{ij}(a) = a,$$

i.e. $t_{ji} \circ t_{ij} = id_{q_{A_i}(A)}$. $t_{ij} \circ t_{ji} = id_{q_{B_j}(B)}$ is obtained similarly. These show that $q_{A_i} \simeq q_{B_j}$.

Conversely, if $q_{A_i} \simeq q_{B_j}$, there exists an isomorphism $(\phi_{AB}, t_{ij}) : (A, q_{A_i}, q_{A_i}(A)) \to (B, q_{B_j}, q_{B_j}(B))$ and some $R$-alg homomorphism $h_{ij} : \langle q_{A_i}(A) \rangle \to \langle q_{B_j}(B) \rangle$ such that $h_{ij}|_{q_{A_i}(A)} = t_{ij}$. Similarly, there exists $t_{ji} = t_{ij}^{-1} : q_{B_j}(B) \to q_{A_i}(A)$ given as some restriction of an $R$-alg homomorphism $h_{ij}$. $\forall a \in \langle q_{A_i}(A) \rangle$ is expressed as $a = \sum k \sum I_k a_k e_{i_1} \cdots e_{i_k}$ by using $e_{i_k} \in q_{A_i}(A)$.

$$h_{ij} \circ h_{ij}(a) = \sum k \sum I_k a_k h_{ij} \circ h_{ij}(e_{i_1}) \cdots h_{ij} \circ h_{ij}(e_{i_k})$$

$$= \sum k \sum I_k a_k t_{ji} \circ t_{ij}(e_{i_1}) \cdots t_{ji} \circ t_{ij}(e_{i_k})$$

$$= \sum k \sum I_k a_k e_{i_1} \cdots e_{i_k} = a.$$

So, we find that $h_{ij}|_{q_{B_j}(B)} \circ h_{ij}|_{q_{A_i}(A)} = id|_{q_{A_i}(A)}$. $h_{ij}|_{q_{A_i}(A)} \circ h_{ij}|_{q_{B_j}(B)} = id|_{q_{B_j}(B)}$ is shown in the same way. Therefore, we obtain $(A, \langle q_{A_i}(A) \rangle) \simeq (B, \langle q_{B_j}(B) \rangle)$. 

**Remark.** We can introduce a functor $\langle - \rangle : QP \to R$-alg, by $T_i \in ob(QP) \mapsto \langle T_i \rangle \in ob(R$-alg) and $t_{ij} \in Mor(T_i, T_j) \mapsto h_{ij} \in Mor(\langle T_i \rangle , \langle T_j \rangle)$ where $t_{ij} = h_{ij}|_{T_i}$. Proposition 2.3 guarantees the existence of this functor. We denote the image of this functor by $\langle QP \rangle$. There are natural projection functors $\pi_p : P - QP \to R$-alg and $\pi_QP : P - QP \to \langle QP \rangle$.

### 4.2 Restriction to a Single Poisson Algebra $A$

Until now, we have considered whole space of all Poisson algebras and their all quantizations. However, it is also important to focus on quantizations on a single Poisson algebra. In this subsection, we will discuss about it.
Let $\mathbf{1}$ be the trivial category with the only one object $*$ and the only one morphism $Id_*$. For a Poisson algebra $A$, we define a functor $A : \mathbf{1} \to \QP$ by $A(*) = A \in \ob(\QP)$ and $A(Id_*) = Id_A \in \mor(\QP)$. Let us consider the comma category associated with $A$ and $\QP$.

**Definition 4.6** ($\QP$). Let $A$ be a fixed Poisson algebra. $\QP$ is a category whose objects and morphisms are given by

\[
\ob(\QP) = \{ (q_i, T_i) := (*, q_i, T_i) \mid q_i : A \to T_i \in Q, \ T_i \in \ob(\QP) \}, \quad (4.9)
\]

\[
\mor((q_i, T_i), (q_j, T_j)) = \{ t_{ij} \in \mor_{\QP}(T_i, T_j) \mid t_{ij} \circ q_i = q_j \}. \quad (4.10)
\]

Roughly speaking, $\QP$ is a category of all quantizations of $A$. Note that $\QP$ is similar to a co-slice category but different. In the same way as $\P - \QP$, we define the following functor.

**Definition 4.7** ($F_A$). The functor $F_A : \QP \to R\text{-alg}$ is defined by

\[
F_A((q_i, T_i)) = \langle q_i(A) \rangle
\]

for $\forall (q_i, T_i) \in \ob(\QP)$, and for $t_{ij} \in \mor_{\QP}((q_i, T_i), (q_j, T_j))$

\[
F_A(t_{ij}) = h_{ij}|_{q_i(A)},
\]

where $h_{ij}$ is an $R$-algebra homomorphism satisfying $h_{ij}|_{T_i} = t_{ij}$.

From Theorem 4.5 with $A = B$, we obtain the following.

**Corollary 4.8.** Let $q_i$ and $q_j$ be quantization maps whose source is a Poisson algebra $A$. $q_i$ and $q_j$ are equivalent quantization of $A$ in $F_A(A \downarrow \QP)$ if and only if object $\langle q_i(A) \rangle \in R\text{-alg}$ is isomorphic to $\langle q_j(A) \rangle \in R\text{-alg}$ in $R\text{-alg}$. From this corollary, we found that $F_A(A \downarrow \QP)$ classifies quantization of $A$.

**Example 4.9.** Let $\P(\mathbb{C}^2)$ be the algebra of polynomials of a coordinate $(x, y) \in \mathbb{C}^2$. By introducing a Poisson bracket by

\[
\{f, g\} = \partial_x f \partial_y g - \partial_y f \partial_x g,
\]

for any $f, g \in \P(\mathbb{C}^2)$, $(\P(\mathbb{C}^2), \cdot, \{, \})$ is regarded as a Poisson algebra.

We use the notation $x^1 := x$, $x^2 = y$, $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$. Let $H = (H_{ij})$ be a $2 \times 2$ complex matrix. We introduce a bi-differential operator $\overrightarrow{H}$ as

\[
f \overrightarrow{H} g = H_{ij} \partial_i f \partial_j g, \quad (f, g \in \P(\mathbb{C}^2)). \quad (4.11)
\]

Here $H_{ij} \partial_i f \partial_j g$ is an abbreviation for $\sum_{i,j=1}^n H_{ij} \partial_i f \partial_j g$. Moyal product $\ast_H$ is defined by

\[
f \ast_H g = f(\exp \nu \overrightarrow{H})g = \sum_{l=0}^\infty \frac{\nu^l}{l!} H_{i_1j_1} \cdots H_{i_kj_k} (\partial_{i_1} \cdots \partial_{i_k} f) \cdot (\partial_{j_1} \cdots \partial_{j_k} g)
\]
where $\nu$ is a complex number. Then, by the Moyal product $*_{H}$, the strict deformation quantization $(\mathcal{P}(\mathbb{C}^{2}),*_{H})$ is a $\mathbb{C}$-algebra \([37]\). We define $q_{H} : (\mathcal{P}(\mathbb{C}^{2}),\cdot,\{\cdot,\cdot\}) \to (\mathcal{P}(\mathbb{C}^{2}),*_{H})$ by the inclusion map i.e. $q_{H}(\sum_{ij}a_{ij}x^{i}y^{j}) = \sum_{ij}a_{ij}x^{i}y^{j}$. It is clear that $q_{H}$ is a quantization map.

Let us decompose $H = (H_{ij})$ into a symmetric matrix and a skew symmetric matrix as

$$H = J + K, \quad J = (J_{ij}), \quad K = (K_{ij}), \text{where } J_{ij} = -J_{ji}, \quad K_{ij} = K_{ji}.$$ 

Here we introduce a differential operator

$$T = \exp(-\frac{1}{2}\nu K_{ij}\partial_{i}\partial_{j}) = \sum_{l=0}^{\infty} \frac{1}{l!}(-\frac{1}{2}\nu K_{ij}\partial_{i}\partial_{j})^{l}$$

(4.12)

that act a polynomial $f$ as

$$T(f) = f + \sum_{l=1}^{\infty} \nu^{l} \frac{1}{l!}(-\frac{1}{2}K_{mj}\partial_{m}\partial_{j})^{l}f.$$  

(4.13)

To compare two algebra $(\mathcal{P}(\mathbb{C}^{2}),*_{J})$ $(\mathcal{P}(\mathbb{C}^{2}),*_{H})$, the following theorem is useful.

**Theorem 4.10** (Tomihisa-Yoshioka \([46]\)). $(\mathcal{P}(\mathbb{C}^{2}),*_{H})$ and $(\mathcal{P}(\mathbb{C}^{2}),*_{J})$ are $\mathbb{C}$-algebra isomorphic. $T$ defined by (4.12) satisfies

1. $T : (\mathcal{P}(\mathbb{C}^{2}),*_{H}) \to (\mathcal{P}(\mathbb{C}^{2}),*_{J})$ is linear and bijective.
2. $T(1) = 1$
3. For $f,g \in (\mathcal{P}(\mathbb{C}^{2}),*_{H})$, $T(\tilde{f}*_{H}\tilde{g}) = T(\tilde{f})*_{J}T(\tilde{g})$.

In other words, $T$ is an $R$-algebra isomorphism.

We define $q_{J} : (\mathcal{P}(\mathbb{C}^{2}),\cdot,\{\cdot,\cdot\}) \to (\mathcal{P}(\mathbb{C}^{2}),*_{J})$ as an inclusion map i.e. $q_{J}(\sum_{ij}a_{ij}x^{i}y^{j}) = \sum_{ij}a_{ij}x^{i}y^{j}$, too. Then $(q_{H}(\mathcal{P}(\mathbb{C}^{2}))) = (\mathcal{P}(\mathbb{C}^{2}),*_{H}) \simeq (q_{J}(\mathcal{P}(\mathbb{C}^{2}))) = (\mathcal{P}(\mathbb{C}^{2}),*_{J})$.

From Corollary 4.8, $q_{H}$ and $q_{J}$ are equivalent quantizations.

## 5 Classical Limit

One of the most important problem is : how can we find the classical limit from the spaces given by the quantizations of Poisson algebras? To make the point of this problem easier to understand, let us consider matrix regularization of $S^{2}$ and $T^{2}$. (In Appendix \([\text{A}]\) we summarize the matrix regularization of $S^{2}$ and $T^{2}$.) As an well-known fact, both images of quantization maps from $A$ and $B$ generate the same matrix algebras. This example shows that the classical limit is not determined by observation of the algebra obtained from a single quantization. This shows that the Poisson algebra cannot be determined as the classical limit from the algebra obtained by quantization. But this example also teaches that the classical limit may be identified from the sequence of modules before generating algebras.

Since a new formulation of quantization using category theory was introduced in Section \([\text{A}]\) we can define various classical limits in the framework. The purpose of this
section is to define some classical limits, study their properties by examining them with concrete examples, and look for useful definitions of classical limits for approaching inverse problems. For this purpose, we pay particular attention to the way the sequence of $R$-modules obtained by the quantization maps determines the Poisson algebra.

In the following $\lim_{\leftarrow I} D$ means the categorical limit in this article. For example we can see its definition in Chapter V in [32] or Chapter 5 in [29].

5.1 Naive Classical Limit

For a start, the classical limit is defined as follows using a naive definition of the limit (in the sense of category theory).

**Definition 5.1** (Naive classical limit of $D(I)$). Let $\mathcal{I}$ be an index category, and let $D : \mathcal{I} \to QW$ be a diagram of shape $\mathcal{I}$. Let $A$ be a Poisson algebra. Suppose that $t_A$ is a set of $t_i : A \to D(i) \in Q \ (i \in \mathcal{I})$ such that $t_i$ and $t_j$ in $t_A$ satisfy $t_j = D(u_{ij}) \circ t_i$ for $\forall u_{ij} \in Mor_{\mathcal{I}}(i, j)$, that is $(A, t_A)$ is a cone over $D$. When $\mathcal{I}$ and $D$ have a limit $(A, t_A)$:

$$\lim_{\leftarrow I} D = (A \xrightarrow{t_i} D(i))_{i \in \mathcal{I}}, \quad (5.1)$$

and $\lim_{\leftarrow I} D$ is called the naive classical limit of $D(I)$.

This definition may lead some readers to think that the naive classical limit is unrelated to the conventional classical limit. However, the following theorem shows that the naive classical limit is a generalization that includes the classical limit when only one type of quantization for one Poisson algebra is considered conventionally.

**Theorem 5.2.** Let $\mathcal{I}$ be an index category, and let $D : \mathcal{I} \to QW$ be a diagram of shape $\mathcal{I}$ such that there is only one Poisson algebra $A$ in $D(I)$. If $(A, t_A)$ is a cone over $D$ such that $Id_A \in t_A$, then $(A, t_A)$ is the naive classical limit.

**Proof.** In the following, we denote $(A, \cdot, \{ , \})$ by $A$ for simplicity, in contexts where there is no risk of confusion. The cone $(A, t)$ is expressed as:

\[
(A, \{ , \}) \xrightarrow{Id_A} (T_1, *_1) \xrightarrow{t_{12}} (T_2, *_2) \xrightarrow{t_{ij}} \cdots
\]

Here each $t_i : A \to T_i$ is a quantization map, and $t_{ij} = D(u_{ij}) : T_i \to T_j$ is in $Mor_{QP}(T_i, T_j)$. $D(I)$ is expressed as

\[
(A, \{ , \}) \xrightarrow{t_{12}} (T_1, *_1) \xrightarrow{t_{ij}} (T_2, *_2) \xrightarrow{t_{ij}} \cdots
\]

If there is another cone of $(B, q')$, $B$ is a Poisson algebra, since $A$ is a Poisson algebra, and only Poisson algebras have possibility to be sources of morphisms in $QW$ which
target is $A$. We denote this Poisson morphisms by $\phi : B \to A$. $q' = \{\phi, q'^{B_1}, q'^{B_2}, \cdots \}$ is a set of morphisms that satisfy the following commutative diagram:

```
\begin{align*}
B & \xrightarrow{\phi} q'_B \\
\downarrow & \quad \downarrow \\
(A, \{, \}) & \xrightarrow{t_1} (T_1, *) \xrightarrow{t_2} (T_2, *) \xrightarrow{\cdots}
\end{align*}
```

Suppose two $\phi, \phi' : B \to A$ satisfy the following commutative diagram:

```
\begin{align*}
(A, \{, \}) & \xrightarrow{\phi} B \\
\downarrow & \quad \downarrow \\
(A, \{, \}) & \xrightarrow{\phi'} (A, *, p) \xrightarrow{t_p} (A, *, r) \xrightarrow{\cdots}
\end{align*}
```

From the condition that $(A, \{, \})$ is an object in $D(\mathcal{I})$, $Id_A$ is also a morphism in $D(\mathcal{I})$ because $D$ is a functor. From the commutativity $\phi = Id_A \circ \phi' = \phi'$, we obtain the uniqueness of $\phi$. \hfill $\square$

In practice, the condition $Id_A \in t_A$ in this theory can be relaxed. Theorem 5.2 holds even if $Id_A$ is replaced by an arbitrary automorphism.

In the following subsections, we will actually consider two examples of quantization families corresponding to only one type of quantization for a single Poisson algebra and check that their naive limits are in fact what is expected.

### 5.2 Example of Naive Classical Limit $(\mathcal{P}(\mathbb{C}^2))$

We consider $\mathcal{P}(\mathbb{C}^2)$, the set of all polynomials of $x$ and $y$, and its quantization by deformation quantization, again. This is an example of fixing the commutative ring $R$ to $\mathbb{C}$. In the following, we denote $(\mathcal{P}(\mathbb{C}^2), \{, \})$ by $\mathcal{P}(\mathbb{C}^2)$ for simplicity, in contexts where there is no risk of confusion. (See Example 4.9.)

Let introduce sequence of Moyal products with $p (p \in \mathbb{Q})$ by

$$f \ast_p g = f \exp \left( \frac{-i}{2} \frac{\partial}{\partial x} p \frac{\partial}{\partial y} - \frac{i}{2} \frac{\partial}{\partial y} p \frac{\partial}{\partial x} \right) g,$$

where $f, g \in \mathcal{P}(\mathbb{C}^2)$. We define $q_h : (\mathcal{P}(\mathbb{C}^2), \{, \}) \to (\mathcal{P}(\mathbb{C}^2), \ast_h)$ by inclusion map i.e. $q_h(f) = f$, as similar to Example 4.9.

Let $\mathcal{I}_\text{moyal}$ be an index category defined by $ob(\mathcal{I}_\text{moyal}) = \mathbb{Q}$ and

$$
\begin{align*}
Mor(p, r) := \{(p \mapsto r =: +r - p) \} & \text{ for } r \neq 0, p \in \mathbb{Q}, \\
Mor(p, 0) := \emptyset.
\end{align*}
$$

Note that $+p - p$ is an identity for $p$.

Let $D_\text{moyal} : \mathcal{I}_\text{moyal} \to QW$ be a diagram of shape $\mathcal{I}_\text{moyal}$ defined by

$$
D_\text{moyal}(p) = (\mathcal{P}(\mathbb{C}^2), \ast_p) \text{ for } \forall p \neq 0 \in \mathbb{Q} \\
D_\text{moyal}(0) = (\mathcal{P}(\mathbb{C}^2), \{, \}) \text{ for } 0 \in \mathbb{Q}
$$

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and
\[
D_{\text{moyal}}(+r - p) = t_{p,r} \\
D_{\text{moyal}}(+r - 0) = q_r.
\]

Here \( t_{p,r} \) is an intertwiner between \((\mathcal{P}(\mathbb{C}^2), \ast_p)\) and \((\mathcal{P}(\mathbb{C}^2), \ast_r)\). The intertwiner is given as a \( \mathbb{C} \)-algebra isomorphism. The existence of such intertwiner is shown in [9, 10, 13, 14].

The existence of this \( t_{p,r} \) is not essential to the following discussion. We choose here each isomorphism \( t_{p,r} \) satisfying \( t_{p,r} \circ q_p = q_r \).

The diagram \( D_{\text{moyal}}(\mathcal{I}_{\text{moyal}}) \) is given as:
\[
(\mathcal{P}(\mathbb{C}^2), \{, \}) \xrightarrow{q_p} (\mathcal{P}(\mathbb{C}^2), \ast_p) \xrightarrow{t_{pr}} (\mathcal{P}(\mathbb{C}^2), \ast_r) \xrightarrow{id} \cdots.
\]

Then, \( (\mathcal{P}(\mathbb{C}^2), q) \) is a cone:
\[
(\mathcal{P}(\mathbb{C}^2), \{, \}) \xrightarrow{Id_{\mathcal{P}(\mathbb{C}^2)}} (\mathcal{P}(\mathbb{C}^2), \{, \}) \xrightarrow{q_p} (\mathcal{P}(\mathbb{C}^2), \ast_p) \xrightarrow{t_{pr}} (\mathcal{P}(\mathbb{C}^2), \ast_r) \xrightarrow{id} \cdots.
\]

From Theorem 5.2 ((\( \mathcal{P}(\mathbb{C}^2), \cdot, \{, , \} \)), \( q \)) is the naive classical limit.

**Corollary 5.3.** Let \( q \) be a set of the above quantization \( q_p : (\mathcal{P}(\mathbb{C}^2), \cdot, \{, \}) \rightarrow (\mathcal{P}(\mathbb{C}^2), \ast_p) \) (\( p \in \mathbb{Q} \)) and identity \( Id_{\mathcal{P}(\mathbb{C}^2)} \) for Poisson algebra \((\mathcal{P}(\mathbb{C}^2), \cdot, \{, \})\) i.e. \( q = \{q_p \in Q \mid p \in \mathbb{Q}\} \cup \{Id_{\mathcal{P}(\mathbb{C}^2)}\} \). For the above \( \mathcal{I}_{\text{moyal}}, D_{\text{moyal}}, ((\mathcal{P}(\mathbb{C}^2), \cdot, \{, \}), q) \) is the naive classical limit.

### 5.3 Example of Naive Classical Limit (Fuzzy sphere)

In this subsection, let us consider the naive classical limit of fuzzy spheres. Basic knowledge for the fuzzy sphere and the notations used in this subsection are summarized in the Appendix C.

Let \( P[x^a] \) be the algebra of polynomial generated by \( x^a \). For an ideal \( I \) which is generated by the relation \((C.1)\), we consider a Poisson algebra \( \mathcal{A} = P[x^a]/I \) with Poisson bracket \( \{x^a, x^b\} = \varepsilon^{abc}x^c \). For arbitrary \( f \in \mathcal{A} \) is given as
\[
f = f_0 + f_ad^a + \frac{1}{2}f_{ab}x^ax^b + \cdots
\]

where \( f_{a \cdot \cdot a_i} \in \mathbb{C} \) is completely symmetric and trace-free. In the case of \( k \geq 2 \), quantization maps \( t_k : \mathcal{A} \rightarrow T_k := t_k(\mathcal{A}) \subset M_k = \text{Mat}_k \) are defined by
\[
t_k(f) := f_01_k + f_{a_1}t_k(x^{a_1}) + \cdots + \frac{1}{(k-1)!}f_{a_1 \cdot \cdot a_{k-1}}t_k(x^{a_1} \cdot \cdot x^{a_{k-1}})
\]
\[
t_k(x^a) := h_kJ^a_k = X^a_k,
\]

where \( J_a (a = 1, 2, 3) \) satisfy \( [J^a_k, J^b_k] = i\varepsilon^{abc}J^c_k \); that is \( J_a \) are generators of \( k \)-dimensional irreducible representation of \( \mathfrak{su}(2) \). Its Casimir relation \( J^a_kJ_a = \frac{k^2}{4}(k^2 - 1)1_k \) and (C.1)
derive the relation \( \frac{4}{k^2 - 1} = \hbar^2_k \). The commutation relation of \( J_k^a \) or (C.4) shows that \( t_k \in Q \) \((k \geq 2)\).

Let \( \mathcal{I}_{fzS 2} \) be an index category defined by \( ob(\mathcal{I}_{fzS 2}) = \mathbb{Z}_{>0} \) and

\[
\begin{align*}
Mor(1, k) &= \{(1 \mapsto k) = \hat{k}\} \text{ for } k \in \mathbb{Z}_{>0}, \\
Mor(p, q) &= \emptyset \text{ for integers } p, q > 1, p \neq q, \\
Mor(p, p) &= Id_p.
\end{align*}
\]

The absence of a morphism between \( p \) and \( q \) \((p, q > 1 \text{ and } p \neq q)\) in the index category is due to the fact that there is no morphism between \( T_p \) and \( T_q \). Since \( \ker t_p = \{f_{a_1 \cdots a_p} x^{a_1} \cdots x^{a_p} + f_{a_1 \cdots a_{p+1}} x^{a_1} \cdots x^{a_{p+1}} + \cdots \} \), there is no injective morphism between \( T_p \) and \( T_q \) that satisfies the commutative diagram with \( t_p \) and \( t_q \).

Let \( D_{fzS 2} : \mathcal{I}_{fzS 2} \to QW \) be a diagram of shape \( \mathcal{I}_{fzS 2} \) defined by

\[
\begin{align*}
D_{fzS 2}(1) &= \mathcal{A} \text{ for } 1, \\
D_{fzS 2}(k) &= T_k \text{ for } \forall k \neq 1 \in \mathbb{Z}_{>0},
\end{align*}
\]

and

\[
\begin{align*}
D_{fzS 2}(\hat{1}) &= Id_A, \\
D_{fzS 2}(\hat{k}) &= t_k \text{ for } \forall k > 1, \\
D_{fzS 2}(Id_k) &= Id_{T_k} \text{ for } \forall k > 1,
\end{align*}
\]

For the \( \mathcal{I}_{fzS 2} \) and \( D_{fzS 2} \), the naive classical limit is determined.

**Corollary 5.4.** Let \( t \) be a set of the above quantizations \( t = \{t_k \in Q \mid k > 1\} \cup \{Id_A\} \). For the above \( \mathcal{I}_{fzS 2}, D_{fzS 2}, (\mathcal{A}, t) \) is the naive classical limit.

This follows trivially from Theorem 5.2.

### 5.4 Weak Classical Limit

As we have seen, in the naive classical limit, the limit is trivially determined for some kinds of diagrams \( D(\mathcal{I}) \). From the viewpoint of discussing inverse problems, the range of the naive classical limit is too wide. In this subsection, we give a definition of the other classical limit with inverse problems in mind. We introduce it by restricting the diagram to \( QP \) and by using a limit that differs from the limit of category theory.

Let us introduce a quantization family and a kind of classical limit.
Note that $p$ with $p$ as the same one in Subsection 5.2. D \index category, and let $I$ be a Poisson algebra. Suppose that $t_A$ is a set of $t_i : A \rightarrow D(i) \in Q (i \in I)$ such that $t_i$ and $t_j$ in $t_A$ satisfy $t_j = D(u_{ij}) \circ t_i$ for $\forall u_{ij} \in Mor_I(i,j)$, that is, $(A, t_A)$ is a cone over $D$. $D(I)$ is called a quantization family of $A$, and $(A, t_A)$ is called the weak classical limit of $D(I)$, when the cone $(A, t_A)$ satisfies the following condition:

- For any other cone $(B, q_B)$ over $D$, where $B \in ob(\mathcal{P}oisss) \subset ob(QW)$, a unique map $\phi_{BA} : B \rightarrow A \in Mor_{QW}(B, A)$ such that $q_i : B \rightarrow D(i) \in q_B$ satisfies

$$q_i = t_i \circ \phi_{BA}$$

for all $i \in I$.

![Diagram](image)

: $\mathcal{P}oisss(= U_p(\mathcal{P}oisss)) \subset QW$

: $QP(= I_{QP}(QP)) \subset QW$

5.5 **Examples of Candidates of Weak Classical limits**

We consider $\mathcal{P}(\mathbb{C}^2)$, again. As in Subsection 5.2, the sequence of Moyal products with $p (p \in \mathbb{Q}^\times)$ by $f \ast_p g = f \exp \left( \sum \frac{\partial_x g}{\partial_y} \frac{\partial_y f}{\partial_x} - \frac{\partial_y g}{\partial_x} \right) g$, where $f, g \in \mathcal{P}(\mathbb{C}^2)$, and $q_h : (\mathcal{P}(\mathbb{C}^2), \ast, \{, \}) \rightarrow (\mathcal{P}(\mathbb{C}^2), \ast_h)$ is defined by $q_h(f) = f$.

$\mathcal{T}'_{moyal}$ differs from the one in Subsection 5.2 in that 0 is excluded from the objects in $\mathcal{T}_{moyal}$. Let $\mathcal{T}'_{moyal}$ be an index category defined by $ob(\mathcal{T}'_{moyal}) = \mathbb{Q}^\times$ and

$$Mor(p, r) := \{(p \mapsto r =: +r - p)\} \text{ for } r, p \in \mathbb{Q}^\times.$$

Note that $+p - p$ is an identity for $p$.

Let $D'_{moyal} : \mathcal{T}'_{moyal} \rightarrow QP \subset QW$ be a diagram of shape $\mathcal{T}'_{moyal}$ defined by $D'_{moyal}(p) = (\mathcal{P}(\mathbb{C}^2), \ast_p)$ for $\forall p \in \mathbb{Q}^\times$ and $D'_{moyal}(+r - p) = t_{p,r}$. Here $t_{p,r}$ is an intertwiner as the same one in Subsection 5.2.

The diagram $D'_{moyal}(\mathcal{T}'_{moyal})$ is given as:

$$(\mathcal{P}(\mathbb{C}^2), \ast_p) \xrightarrow{t_{p,r}} (\mathcal{P}(\mathbb{C}^2), \ast_r) \xrightarrow{\cdots}.$$

Let $q'$ be a set of the above quantization $q_p : (\mathcal{P}(\mathbb{C}^2), \ast, \{, \}) \rightarrow (\mathcal{P}(\mathbb{C}^2), \ast_p) (p \in \mathbb{Q}^\times)$. Even with these change, $(\mathcal{P}(\mathbb{C}^2), q')$ is still a cone:

$$(\mathcal{P}(\mathbb{C}^2), \{, \}) \quad q_p \quad \cdots \quad q_{\cdots}$$

$$(\mathcal{P}(\mathbb{C}^2), \ast_p) \xrightarrow{t_{p,r}} (\mathcal{P}(\mathbb{C}^2), \ast_r) \xrightarrow{\cdots}.$$

At this case, the following proposition holds.
Proposition 5.6. If there exist the weak classical limit for the above \( I'_{\text{moyal}}, D'_{\text{moyal}} \), the weak classical limit is isomorphic to \(((\mathcal{P}(\mathbb{C}^2), \cdot, \{ , \}), q')\).

Proof. Let \((L, q_{\text{lim}})\) be the weak classical limit. Then there exists a unique Poisson morphism \( \phi_{PL} : \mathcal{P}(\mathbb{C}^2) \to L \) such that \( q_p = q_{LP} \circ \phi_{PL} \) for any \( q_{LP} : L \to (\mathcal{P}(\mathbb{C}^2), *_p) \in q_{\text{lim}} \) and any \( q_p \in q' \). Recall that this \( \phi_{PL} \) is a surjective Poisson morphism. By definition of \( q_p \), \( f = q_p(f) = q_{LP} \circ \phi_{PL}(f) \) for \( \forall f \in \mathcal{P}(\mathbb{C}^2) \). Then \( \phi_{PL} \) is injective. \( \square \)

In fact, if there is another cone, from which there is a morphism to \( \mathcal{P}(\mathbb{C}^2) \) satisfying commutativity, then it is the unique one.

Proposition 5.7. Let \(((\mathcal{P}(\mathbb{C}^2), \cdot, \{ , \}), q')\) be a cone defined above. Let \((B, t)\) be the other cone where \( B \) is arbitrary object in \( \mathbb{Q} \). If there is \( \phi : B \to \mathcal{P}(\mathbb{C}^2) \in \text{Mor}(\mathbb{Q}) \) satisfies the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{P}(\mathbb{C}^2) & \xrightarrow{\phi} & B \\
q_p \downarrow & & \downarrow t_p \\
(\mathcal{P}(\mathbb{C}^2), *_p) & \xrightarrow{t_{pr}} & (\mathcal{P}(\mathbb{C}^2), *_r) \\
\end{array}
\]

then \( \phi \) is unique.

Proof. Suppose two different \( \phi, \phi' : B \to \mathcal{P}(\mathbb{C}^2) \) satisfy the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{P}(\mathbb{C}^2) & \xrightarrow{\phi, \phi'} & B \\
q_p \downarrow & & \downarrow t_p \\
(\mathcal{P}(\mathbb{C}^2), *_p) & \xrightarrow{t_{pr}} & (\mathcal{P}(\mathbb{C}^2), *_r) \\
\end{array}
\]

There are \( b \in B \) such that \( \phi(b) \neq \phi'(b) \). From the commutativity \( t_p = q_p \circ \phi = q_p \circ \phi' \). By definition of each \( q_p, \phi(b) = q_p \circ \phi(b) = q_p \circ \phi'(b) = \phi'(b) \). This is a contradiction. \( \square \)

Unfortunately, we do not know if the weak classical limit actually exists for this \( I'_{\text{moyal}}, D'_{\text{moyal}} \).

Next, as similar in Subsection[5.3] let us consider the example of fuzzy sphere, again. Let \( (\mathcal{A} = P [x^a]/I, \{ , \}) \) be a Poisson algebra, where \( \{ x^a, x^b \} = \epsilon^{abc} x^c \) \((a, b, c = 1, 2, 3) \). \( f \in \mathcal{A} \) is given as \( f = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + \cdots \) where \( f_{a_1 \cdots a_i} \in \mathbb{C} \) is completely symmetric and trace-free. Quantization maps \( t_k : \mathcal{A} \to T_k := t_k(\mathcal{A}) \subset M_k = \text{Mat}_k \) are defined by \( t_k(f) := f_0 1_k + f_a X^a_k + \cdots + \frac{1}{(k-1)!} f_{a_1 \cdots a_{k-1}} X^a_k \cdots X^a_k \).

Let \( I_{\mathbb{Z}S_2} \) be an index category defined by \( \text{ob}(I_{\mathbb{Z}S_2}) = \{ 2, 3, 4, \cdots \} \) and

\[
\text{Mor}(p, q) := \emptyset \text{ for integers } p, q > 1, p \neq q,
\]

\[
\text{Mor}(p, p) := Id_p.
\]
In other words, $T'_{fzS2}$ is a discrete category. Let $D'_{fzS2} : T'_{fzS2} \to QP \subset QW$ be a diagram of shape $T'_{fzS2}$ defined by

$$D'_{fzS2}(k) = T_k \text{ for } \forall k \in \{2, 3, 4, \cdots \},$$

and

$$D'_{fzS2}(Id_k) = Id_{T_k} \text{ for } \forall k \in \{2, 3, 4, \cdots \}.$$  

Of course, $(A,t)$ is a cone, where $t = \{t_k \in Q \mid k > 1\}$.

For the $T'_{fzS2}$ and $D'_{fzS2}$, the following proposition is derived.

**Proposition 5.8.** If there exists the weak classical limit for the $T'_{fzS2}$ and $D'_{fzS2}$, then the weak classical limit is isomorphic to $(A,t)$.

**Proof.** Let $(L, q_{lim})$ be the weak classical limit. Then there exists a unique Poisson morphism $\phi_{AL} : A \to L$ such that $t_k = q_{Lk} \circ \phi_{AL}$ for any $q_{Lk} : L \to T_k \in q_{lim}$ and any $t_k \in t$. Recall that this $\phi_{AL}$ is a surjective Poisson morphism. Let us show that this $\phi_{AL}$ is injective by contradiction. Suppose that $\phi_{AL}$ is not injective. There are $f, g \in A$ such that $f \neq g$ and $\phi_{AL}(f) = \phi_{AL}(g)$. For any $k$, $t_k(f) = t_k(g)$ since $t_k = q_{Lk} \circ \phi_{AL}$. For sufficiently large $N$, this is contradiction, and $\phi_{AL}$ is injective. 

As a matter of fact, there is stronger uniqueness of the Poisson morphism in the case of fuzzy spheres.

**Proposition 5.9.** Let $t$ and $q$ be sets of the quantizations $t = \{t_k : A \to T_k \in Q \mid k > 1\}$ and $q = \{q_k : B \to T_k \in Q \mid k > 1\}$, where $B$ is arbitrary object in $QW$. If there is $\phi : B \to A \in Mor(QW)$ satisfies the following commutative diagram:

\[
\begin{array}{c}
A \\
\downarrow \phi \\
B
\end{array} \xymatrix{ & \cdots \\
T_2 & T_3 & T_4 & \cdots}
\]

then $\phi$ is unique.

**Proof.** $B$ is a Poisson algebra, since $A$ is a Poisson algebra, and only Poisson algebras have possibility to be sources of morphisms in $QW$ which target is $A$. Suppose that there exist two morphisms $\phi, \phi' : B \to A$ such that $t_k \circ \phi = t_k \circ \phi' = q_k$.

If $\phi \neq \phi'$, there exist $b \in B$ such that $\phi(b) \neq \phi'(b)$ and $t_n(\phi(b)) = t_n(\phi'(b))$ for $\forall n \in \mathbb{Z}_{>0}$. We denote the coimage of $t_N$ by $H_N := \{a_0 + a_1 x^a + \cdots + a_{N-1} x^a \cdots x^a \}$. For sufficiently large $N$, $\phi(b), \phi'(b) \in H_N$, and $t_N(\phi(b)) \neq t_N(\phi'(b))$. This is a contradiction, so the uniqueness of the map $\phi$ is shown. 


Proposition 5.10. For the above $\mathcal{I}_{f_{zS2}}$ and $D'_{f_{zS2}}$, the weak classical limit does not exist.

Proof. Suppose that there exists the weak classical limit. From Proposition 5.8 the weak classical limit is isomorphic to $(\mathcal{A}, t)$. Let us introduce quantization maps $\hat{t}_k : \mathcal{A} \to T_k$ $(k = 2, 3, \cdots)$ by $\hat{t}_k(f) := f_0 1_k + f_{a_1} X_k^{a_1} \cdot \phi(f)$ for $f = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + \cdots \in \mathcal{A}$. $(\mathcal{A}, \hat{t})$ is a cone too, where $\hat{t} = \{ \hat{t}_k \in Q \, | \, k > 1 \}$. Then there exist the unique surjective $\phi : (\mathcal{A}, \hat{t}) \to (\mathcal{A}, t)$ such that $t_k \circ \phi = \hat{t}_k$. Consider $k \geq 3$ case. Because $t_k \circ \phi(f) = \hat{t}_k(f) = f_0 1_k + f_{a_1} X_k^{a_1}$, $\phi(f) = f_0 + f_a x^a + g$, where $g$ is a polynomial of degree $\phi$ or higher given as $\sum_{j \geq k} \frac{1}{j!} g_{a_1 \cdots a_j} x^{a_1} \cdots x^{a_j}$, for any $f$. For example, $x^1 x^2 + x^2 x^1 \notin \Im \phi$, and this contradicts that the $\phi$ is surjective. □

If $\mathcal{I}$ and $D$ are chosen to satisfy Theorem 5.2 the naive classical limit is obviously determined, as shown in the examples above. On the other hand, to get the weak classical limits we have to choose appropriate pairs of an index category and a diagram. If we can not find an appropriate pair of them, the weak classical limit is not determined in general. Moreover, since any Poisson algebra is a candidate for this weak classical limit, proving the existence of the weak classical limit may be difficult in general.

5.6 Strong Classical Limit through $(A \downarrow I_{QP})$

Apart from the weak classical limit that we have examined in the previous subsections, it is also natural to assume that the classical limit should be determined from the all quantization spaces corresponding to a single Poisson algebra. In particular, it is expected that the fixed Poisson algebra is a candidate for the classical limit. In this subsection, we investigate such classical limits.

To prepare for this, we first introduce a way to construct an index category from a category with total order.

Definition 5.11. Let $\mathcal{C}$ be a category with total order denoted by $c_i \leq c_j$ $(c_i, c_j \in \text{ob}(\mathcal{C}))$. $J^\bullet$ is a functor like correspondent (but not functor) between $\mathcal{C}$ and an index category $\mathcal{I} := J^\bullet(\mathcal{C})$ defined by the following conditions:

1. Between objects of the two categories,

$$J^\bullet : \text{ob}(\mathcal{C}) \xrightarrow{\sim} \text{ob}(\mathcal{I}) = \text{ob}(J^\bullet(\mathcal{C}))$$

$$c \mapsto i_c := J^\bullet(c)$$

is a one to one correspondence.

2. $\forall t_{ij} \in \text{Mor}_C(c_i, c_j)$ if $c_i \leq c_j$ then there exists a $f_{ij} = J^\bullet(t_{ij}) \in \text{Mor}_T(i_c, i_{c_j})$.

3. $J^\bullet(Id_{i_c}) = Id_{i_c}$.

4. $\forall f_{kl} \in \text{Mor}_T(i_{c_k}, i_{c_l})$, there exists a unique $t_{kl} \in \text{Mor}_C(c_k, c_l)$ such that $c_k \leq c_l$ and $f_{kl} = J^\bullet(t_{kl})$.

5. For $c_i \leq c_j \leq c_k$, if $t_{ij} \in \text{Mor}_C(c_i, c_j)$, $t_{jk} \in \text{Mor}_C(c_j, c_k)$, and $t_{ik} \in \text{Mor}_C(c_i, c_k)$ satisfy $t_{jk} \circ t_{ij} = t_{ik}$, then $J^\bullet(t_{ik}) = J^\bullet(t_{jk}) \circ J^\bullet(t_{ij})$.

Note that if $c_k > c_l$ then $\text{Mor}_C(i_{c_k}, i_{c_l}) = \emptyset$ even when $\text{Mor}_C(c_k, c_l) \neq \emptyset$. In short, $J^\bullet$ acts to remove morphisms from bigger object to smaller object in $\mathcal{C}$. This is the reason that $J^\bullet$ is not a functor.

Therefore, we have to show the following.
**Proposition 5.12.** \( \mathcal{I} := J^\bullet(\mathcal{C}) \) is a category.

**Proof.** The only condition that needs to be checked is consistency regarding \( \text{Mor}_\mathcal{I}(i_{c_k}, i_{c_l}) = \emptyset \) when \( \text{Mor}_\mathcal{C}(c_k, c_l) \neq \emptyset \) with \( c_k > c_l \). For any two \( f : X \to Y \) and \( g : Y \to Z \) their composition \( g \circ f : X \to Z \) have to be a morphism of \( \text{Mor}(X, Z) \). In our situation, if the morphism corresponding to \( g \circ f \) is removed without removing the morphism corresponding to \( g \) or the morphism corresponding to \( f \), then it is impossible for \( \mathcal{I} := J^\bullet(\mathcal{C}) \) to be a category. Let \( c_X, c_Y, c_Z \) be objects of \( \mathcal{C} \) and \( i_X = J^\bullet(c_X), i_Y = J^\bullet(c_Y), i_Z = J^\bullet(c_Z) \) be objects of \( \mathcal{I} \). For the case \( c_X > c_Z \), \( \text{Mor}_\mathcal{I}(i_X, i_Z) \) becomes \( \emptyset \), so if \( \text{Mor}_\mathcal{I}(i_Y, i_Z) \neq \emptyset \), then it is inconsistent for \( \mathcal{I} \) to be a category. However, such a situation cannot arise. The reasons are as follows. Suppose that \( \text{Mor}_\mathcal{I}(i_X, i_Z) \) becomes \( \emptyset \) by the condition \( c_X > c_Z \). For the case \( c_Y \geq c_X, c_Y > c_Z \) also holds, then \( \text{Mor}_\mathcal{I}(i_Y, i_Z) = \emptyset \). For the case \( c_X > c_Y \), \( \text{Mor}_\mathcal{I}(i_X, i_Y) = \emptyset \). Thus, it is shown that the consistency for compositions of morphisms is guaranteed. \( \square \)

We introduce a total order to \((A \downarrow \text{IQ}_P)\), as follows.

**Definition 5.13** (total order of \((A \downarrow \text{IQ}_P)\)). For \((q_i, T_i), (q_j, T_j) \in \text{ob}(A \downarrow \text{IQ}_P)\), we define \((q_i, T_i) \leq (q_j, T_j)\) by \(|h(q_i)|^2 \leq |h(q_j)|^2\).

By this total order, an index category \( \mathcal{I}_h(A) := J^\bullet(A \downarrow \text{IQ}_P) \) is determined from \((A \downarrow \text{IQ}_P)\).

\[
\begin{pmatrix}
A \\
\downarrow q_1 \downarrow q_2 \downarrow q_3 \cdots \\
T_1 \rightarrow t_{12} T_2 \rightarrow t_{23} T_3 \rightarrow \cdots
\end{pmatrix} \rightarrow \begin{pmatrix}
i_1 \xrightarrow{f_{12}} i_2 \xrightarrow{f_{23}} i_3 \cdots
\end{pmatrix} = \mathcal{I}_h(A).
\]

Here \( J^\bullet(q_k, T_k) = i_k \) and \( J^\bullet(t_{kl}) = f_{kl} \) when \(|h(q_k)|^2 \leq |h(q_l)|^2\). Next, the diagram \( D_h : \mathcal{I}_h(A) \to (A \downarrow \text{IQ}_P) \) is defined by \( D_h(i_k) = (q_k, T_k) \) and \( D_h(f_{kl}) = t_{kl} \).

\[
\begin{pmatrix}
i_1 \xrightarrow{f_{12}} i_2 \xrightarrow{f_{23}} i_3 \cdots
\end{pmatrix} \xrightarrow{D_h} \begin{pmatrix}
(q_1, T_1) \xrightarrow{t_{12}} (q_2, T_2) \xrightarrow{t_{23}} (q_3, T_3) \cdots
\end{pmatrix}.
\]

Notice that it is possible to put a natural initial object \((\text{Id}_A, A)\) into this \( D_h(\mathcal{I}_h(A)) \).

**Definition 5.14.** \( A/\text{QW}_h \) is a category defined by

\[
\text{ob}(A/\text{QW}_h) := \text{ob}(D_h(\mathcal{I}_h(A))) \cup \{(\text{Id}_A, A)\}
\]

and

\[
\text{Mor}_{A/\text{QW}_h}((q_i, T_i), (q_j, T_j)) = \text{Mor}_{D_h(\mathcal{I}_h(A))}((q_i, T_i), (q_j, T_j))
\]

for \( \forall (q_i, T_i), (q_j, T_j) \in \text{ob}(D_h(\mathcal{I}_h(A))) \),

\[
\text{Mor}_{A/\text{QW}_h}((\text{Id}_A, A), (q_i, T_i)) = \{q_i\}
\]

for \( \forall (q_i, T_i) \in \text{ob}(D_h(\mathcal{I}_h(A))) \).

\[
\begin{pmatrix}
\downarrow q_1 \downarrow q_2 \downarrow q_3 \cdots \\
q_1, T_1 \rightarrow t_{12} T_2 \rightarrow t_{23} T_3 \rightarrow \cdots
\end{pmatrix}.
\]

Note that the index \( \mathcal{I}_h(A) \) and the diagram \( D_h \) are still valid in \( A/\text{QW}_h \).
Proposition 5.15. $A/QW_\hbar$ has a limit $((Id_A, A), q)$ for the index category $\mathcal{I}_h(A)$ with the diagram $D_h$. Here, $((Id_A, A), q)$ is a cone of $D_h(\mathcal{I}_h(A))$ with $q = \{ q_i \in Q \mid s(q_i) = A \}$.

Proof. $(Id_A, A)$ is the initial object in $A/QW_\hbar$. By definition, $((Id_A, A), q)$ is a cone of $D_h(\mathcal{I}_h(A))$. Because every morphism $t_{ij} : (q_i, T_i) \rightarrow (q_j, T_j)$ satisfies commutativity $q_j = t_{ij} \circ q_i$ by the definition of $(A \downarrow \mathcal{I}_Q^P)$. In $A/QW_\hbar$, there is at most one morphism between any two objects. So, $((Id_A, A), q)$ is one of the candidates of the limit. Next, we prove that there are no other candidates for the limit other than $((Id_A, A), q)$, by contradiction. If the other candidate of the limit of the diagram $D_h$ exists, it is given by a cone with the form $((q_k, T_k), t_{k-})$, where $q_k : A \rightarrow T_k$ is in $Q$ and $t_{k-} = \{ t_{kl} : q_k \rightarrow q_l = t_{kl} \circ q_k \mid t_{kl} \in Mor_{QP}(T_k, T_l) \}$. From Proposition 2.14, there exist $q_x : A \rightarrow T_x \in Q$ such that $|q_x|^2 < |q_k|^2$. $(q_x, T_x)$ is an object in $D_h(\mathcal{I}_h(A))$, and $(q_k, T_k) > (q_x, T_x)$. Then $Mor_{A/QW_\hbar}((q_k, T_k), (q_x, T_x)) = \emptyset$, and this contradicts $((q_k, T_k), t_{k-})$ is a cone. Therefore, we find that $((Id_A, A), q)$ is the limit.

From this proposition, we found that for any Poisson algebra $A$ in $QW$ the limit for the index category $\mathcal{I}_h(A)$ with the diagram $D_h$ is determined without additional information. We call this limit $((Id_A, A), q)$ “strong classical limit of $A$”.

This Proposition 5.15 shows that the strong classical limit satisfies at least the properties that the limit should have that we wanted in this subsection.

5.7 Example of Strong Classical Limit

Let us consider $\mathcal{A}$ we studied in the fuzzy sphere in Appendix C. The index $\mathcal{I}_h(A)$, the diagram $D_h(\mathcal{I}_h(A))$ and $A/QW_\hbar$ are determined without extra information. We obtain the following fact.

Example 5.16. Let $\mathcal{A}$ be a Poisson algebra studied in Appendix C. The strong classical limit of $\mathcal{A}$ is $((Id_A, A), q)$, where $q = \{ q_i \in Q \mid s(q_i) = A \}$.

From Proposition 5.15 there is no need to prove this Example 5.16. However, this claim can be shown just from the properties of some quantization maps of matrix regularization. This is so suggestive that we will also provide the proof below.

Proof. As we saw in the proof of Proposition 5.15, $((Id_A, A), q)$ is a cone for the diagram $D_h$ in $A/QW_\hbar$. We show that there are no other cones.

Suppose that there is another cone. Then, the cone is given as some $((q_{cone}, T_{cone}), t_{cone,-})$ where $q_{cone} \in Q$ with $s(q_{cone}) = A$ and $t_{cone,-} = \{ t_{cone,k} : (q_{cone}, T_{cone}) \rightarrow (q_k, T_k) \mid q_k \in Q, s(q_k) = A, q_k = t_{cone,k} \circ q_{cone} \}$. Consider $t_{cone,2} \in Mor((q_{cone}, T_{cone}), (t_2, T_2))$, where $t_2 \in Q$ is defined in Appendix C. Since $t_{cone,2}$ is an injection and $T_2$ is a 4-dimensional vector space, $\dim T_{cone} \leq 4$ is derived. For $t_3 : A \rightarrow T_3$ defined in Appendix C $\dim \text{Im } t_3 > 4$. On the other hand, $\dim \text{Im } t_{cone,3} \leq 4$ for $t_{cone,3} : T_{cone} \rightarrow T_3$. Therefore $t_3 \neq t_{cone,3} \circ q_{cone}$. This is contradiction to the assumption that $((q_{cone}, T_{cone}), t_{cone,-})$ is a cone.

Thus, it is shown that $((Id_A, A), q)$ is the only cone and the limit for the diagram $D_h$ in $A/QW_\hbar$.  

[27]
6 One Attempt to Address the Inverse Problem

As mentioned at the beginning of this article, the inverse problems of quantizations are important for physics related to noncommutative geometries. The inverse problem of quantizations is the problem of how to determine the classical limits (classical manifolds) based on the information of the spaces appearing in the quantizations. In general, classical limits are not uniquely determined from algebras as target spaces of quantization maps in a naive way, so we need to experiment with various subtle techniques. This problem is closely related to the question of how we can identify manifolds in the membrane theory, thus, it was often investigated in that context. For example, in [12, 15, 16] it is discussed how to construct the classical limit in geometric quantization. In [44, 11, 41], how membrane topology is distinguished in the context of matrix regularization. In [24, 25, 4], inverse problems of Berezin-Toeplitz quantizations are discussed.

Unlike in the past, the purpose of this section is to study this inverse problem as a problem of the classical limit discussed in the previous section. Until now, we have studied whole world of the quantizations of all Poisson algebras. In particular, the naive classical limit and the weak classical limits introduced in Subsection 5.1 and Subsection 5.4 are chosen from among all Poisson algebras in $QW$. To determine the naive classical limit or weak classical limit from the spaces appearing in the quantizations, we are going to examine how to construct a sequence of objects in $QW$ and are going to discuss how to give the sequence by a process found in physics.

6.1 Matrix Regularization and Inverse Problem

The goal of this section is making an example of a method to obtain a Poisson algebra as a classical limit from a noncommutative associative algebra. Here we make this example using a fuzzy sphere as a role model. In this section, the commutative ring $R$ is fixed to $\mathbb{C}$, again.

At first, let $\mathfrak{g}$ be a Lie algebra as a subset of a noncommutative algebra over $\mathbb{C}$ whose commutator is defined by its associative product i.e. $[a, b] := ab - ba$. Only semisimple Lie algebras are treated in this section. This $\mathfrak{g}$ is the origin to get the sequence of objects in $QW$ that may provide a classical limit. Such sequence of objects in $QW$ is named the quantization family for the weak classical limit in Subsection 5.4. Let $e = \{e_1, e_2, \cdots, e_d\}$ be a fixed base of $\mathfrak{g}$ satisfying commutation relations $[e_i, e_j] = f_{ij}^k e_k$, where $f_{ij}^k$ are structure constants of $\mathfrak{g}$. For this Lie algebra $\mathfrak{g}$ we introduce a sequence of representation $\rho^\mu : \mathfrak{g} \rightarrow gl(V_\mu)$ and a sequence of numbers $h(\mu) \neq 0$, $(\mu = 1, 2, 3, \cdots)$. Here $V_\mu$ is a finite dimensional vector space chosen as appropriate. We denote the corresponding basis of $e$ by

$$e^{(\mu)} = \{h(\mu)\rho^\mu(e_1), h(\mu)\rho^\mu(e_2), \cdots, h(\mu)\rho^\mu(e_d)\} = \{e_1^{(\mu)}, e_2^{(\mu)}, \cdots, e_d^{(\mu)}\}.$$ (6.1)

Then they satisfy

$$[e_i^{(\mu)}, e_j^{(\mu)}] = h(\mu) f_{ij}^k e_k^{(\mu)}.$$ (6.2)

The Lie algebra $\rho^\mu(\mathfrak{g})$ or $(e^{(\mu)})$ are constructed by this basis.
For each \( \mu \), we have relations that are originated in Casimir invariants. To characterize a classical limit, we can use these relations. For the later convenience, we denote them by

\[
f^\mu_i (c(\mu), \nu(\mu), \hbar(\mu)), \ldots, f^\mu_{N\mu} (c(\mu), \nu(\mu), \hbar(\mu)),
\]

where \( \nu(\mu) = \{\nu_1(\mu), \nu_2(\mu), \ldots\} \) is a set of parameters, and \( N_\mu \leq \text{rank}g \). We will use these relations to induce corresponding relations in the Poisson algebra, later. We will be back to this subject, at the end of this subsection.

Next, let \( T_\mu \) be the vector space that is \( \langle e(\mu) \rangle \) forgetting multiplication structure. We choose a basis of \( T_\mu := \langle e(\mu) \rangle, E_1, E_2, \ldots, E_D \), as polynomials of \( e(\mu) \). The highest degree is denoted by \( n_\mu \), i.e. \( n_\mu = \text{max}\{\text{deg}E_1, \ldots, \text{deg}E_D\} \). For later use, let us consider symmetrized polynomial

\[
c_{i_1 \ldots i_k} e_{(i_1 \ldots i_k)}^{(\mu)} \ldots e_{(i_1 \ldots i_k)}^{(\mu)} = c_{i_1 \ldots i_k} e_{(i_1 \ldots i_k)}^{(\mu)} \quad (k = 0, 1, 2, \ldots, n_\mu),
\]

where the fixed coefficients \( c_{i_1 \ldots i_k} \in \mathbb{C} \) are complete symmetric, and

\[
\epsilon_{(i_1 \ldots i_k)}^{(\mu)} := \frac{1}{k!} \sum_{\sigma \in \text{Sym}(k)} \epsilon_{\sigma(i_1)}^{(\mu)} \ldots \epsilon_{\sigma(k)}^{(\mu)}.
\]

Here \( \text{Sym}(k) \) is a symmetric group. Obviously, each \( e_{(i_1 \ldots i_k)}^{(\mu)} \) is given as a linear combination of \( E_1, E_2, \ldots, E_D \) and its degree is smaller than or equal to \( n_\mu \). For \( m > n_\mu \) \( e_{(i_1 \ldots i_m)}^{(\mu)} \) is written by some linear combination of lower degree polynomial, then \( h(\mu) \) appear because of the definition (6.1). From the set of representation \( \rho(\mu) \) and relations \( f^\mu_i \), we obtain sets of \( T_\mu \). We denote a set of all \( T_\mu \) by \( \{T_\mu\} \). To show that \( \{T_\mu\} \) is a sequence of objects in \( QW \), we have to find one Poisson algebra and to make quantization maps from the Poisson algebra to \( \{T_\mu\} \).

The next step we introduce a way to obtain one Poisson algebra from a Lie algebra \( g \) as a candidate of a weak classical limit. There is a well-known way known as Kirillov-Kostant Poisson bracket, that is the way constructing a Poisson algebra. (See also 31 [43 I] ). We focus the following fact (See 21 27), here.

**Theorem 6.1.** Let \( g \) be a \( d \)-dimensional Lie algebra. Let \( e = \{e_1, e_2, \ldots, e_d\} \) be a basis of \( g \) satisfying commutation relations \( [e_i, e_j] = f^k_{ij} e_k \). Let \( x = (x^1, x^2, \ldots, x^d) \) be commutative variables. We obtain a Poisson algebra \( \mathbb{C}[x], \{, \} \) by

\[
\{f, g\} := f \omega g := f \partial_i \omega_{ij} j g := (\partial_i f) \omega_{ij} (\partial_j g),
\]

where \( \partial_i = \frac{\partial}{\partial x^i} \) and \( \omega_{ij} = f^k_{ij} x^k \).

The proof is given in 21 but for readers convenience we give the proof here.

**Proof.** \( \{f, g\} = -\{g, f\} \) is followed from \( f^k_{ij} = -f^k_{ji} \). The Leibniz's rule and the bilinearity are trivially satisfied by the definition. By direct calculations, the Jacobi identity is obtained as follows.

\[
\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}
\]

\[
= (\partial_i f_1)(\partial_j f_2)(\partial_k f_3)(\partial_l k \omega_{ij})\omega_{kl} + (\partial_i \omega_{ij})\omega_{kl} + (\partial_i \omega_{ij})\omega_{kl}.
\]
By \( \omega_{ij} = f_{ij}^k x^k \):
\[
(\partial_k \omega_{ij}) \omega_{kl} + (\partial_l \omega_{ij}) \omega_{ki} + (\partial_i \omega_{kl}) \omega_{kj} = x^m (f_{ij}^k f_{kl}^m + f_{ji}^k f_{kl}^m + f_{kj}^k f_{il}^m) = 0
\]
The last equality follows from the Jacobi identity of the Lie bracket \([e_i, e_j] = f_{ij}^k e_k\).  

We denote this \((\mathbb{C}[x], \cdot, \{, \})\) by \(A_\theta\). From the above discussions, we found that we can obtain a sequence of Lie algebras and a Poisson algebra from a fixed single Lie algebra.

Next, let us construct quantization maps from the Poisson algebra \(A_\theta\) we have just obtained in Theorem 6.1 to the vector spaces in \(\{T_\mu\}\) that generate original algebras. We define linear function \(q_\mu : A_\theta \to T_\mu\) by
\[
\sum_k f_{i_1, \ldots, i_k} x^{i_1} \cdots x^{i_k} \mapsto \sum_k f_{i_1, \ldots, i_k} e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)} = \sum_k f_{i_1, \ldots, i_k} e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)},
\]
(6.5)
where \(f_{i_1, \ldots, i_k} \in \mathbb{C}\) is completely symmetric, and we assume that the multiplicative identity of \(A_\theta\) maps to the unit matrix in \(T_\mu\). Then this correspondence \(q_\mu\) satisfies the following.

**Theorem 6.2.** Let \(q_\mu : A_\theta \to T_\mu = \langle e^{(\mu)} \rangle\) be a linear function defined as (6.5). Then it satisfies
\[
[q_\mu(f), q_\mu(g)] = h(\mu) q_\mu([f, g]) + \hat{O}(h(\mu))
\]
for \(f, g \in A_\theta \). In other words, \(q_\mu \in Q\).

**Proof.** We put \(f = \sum_k f_{i_1, \ldots, i_k} x^{i_1} \cdots x^{i_k}\) and \(g = \sum_l g_{j_1, \ldots, j_l} x^{j_1} \cdots x^{j_l}\).

\[
[q_\mu(f), q_\mu(g)] = \sum_{k,l} f_{i_1, \ldots, i_k} g_{j_1, \ldots, j_l} [e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)}, e_{j_1}^{(\mu)} \cdots e_{j_l}^{(\mu)}]
\]
\[
= \sum_{k,l} f_{i_1, \ldots, i_k} g_{j_1, \ldots, j_l} \sum_{n,m} e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)} \cdots e_{j_1}^{(\mu)} \cdots e_{j_l}^{(\mu)}.
\]

Here using \([e_{i_n}^{(\mu)}, e_{j_m}^{(\mu)}] = h(\mu) f_{i_n, j_m}^{p} e_{p}^{(\mu)}\),
\[
[q_\mu(f), q_\mu(g)] = h(\mu) \sum_{k,l=1}^{n} f_{i_1, \ldots, i_k} g_{j_1, \ldots, j_l} \sum_{n,m} f_{i_n, j_m}^{p} e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)} \cdots e_{j_1}^{(\mu)} \cdots e_{j_l}^{(\mu)}
\]
\[
= h(\mu) \sum_{k,l=1}^{n} f_{i_1, \ldots, i_k} g_{j_1, \ldots, j_l} \sum_{1 \leq n \leq k \leq m \leq l} f_{i_n, j_m}^{p} e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)} \cdots e_{j_1}^{(\mu)} \cdots e_{j_l}^{(\mu)} + \hat{O}(h^{1+\epsilon}(\mu)).
\]
(6.6)

Here the index \(\{i_1, \ldots, i_n, \ldots, i_m, \ldots, i_l, p\}\) means \(\{i_1, \ldots, i_l, j_1, \ldots, j_l, p\} - \{i_n, j_m\}\), so \(e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)} \cdots e_{i_l}^{(\mu)} \cdots e_{j_1}^{(\mu)} \cdots e_{j_l}^{(\mu)}\) is a degree \(k + l - 1\) polynomial. \(\hat{O}(h^{1+\epsilon}(\mu))\) appeared when \(e_{i_1}^{(\mu)} \cdots e_{i_k}^{(\mu)} \cdots e_{i_l}^{(\mu)} \cdots e_{j_1}^{(\mu)} \cdots e_{j_l}^{(\mu)}\) were sorted in the symmetric order. On the other hand,
\[
\{f, g\} = \sum_{k,l=1}^{n} f_{i_1, \ldots, i_k} g_{j_1, \ldots, j_l} \sum_{1 \leq n \leq k \leq m \leq l} f_{i_n, j_m}^{p} x^{i_1} \cdots x^{i_k} \cdots x^{j_1} \cdots x^{j_l}.
\]

Note that the degree of $x^{i_1} \cdots x^{p} \cdots x^{i_l}$ is $k + l - 1$. Then

$$q_\mu(\{f, g\}) = \sum_{1 \leq n \leq k, m \leq l} f_{i_1, \ldots, i_n} g_{j_1, \ldots, j_l} \sum_{1 \leq n \leq k} f_{i_{n+1}} e^{(\mu)}_{(i_1, \ldots, \hat{i}_n, \ldots, i_{n+1}, \ldots, i_l, j_1, \ldots, j_l)}.$$  

(6.7)

Let us subtract $h$ times (6.7) from (6.6):

$$[q_\mu(f), q_\mu(g)] - h(\mu)q_\mu(\{f, g\}) = h(\mu) \sum_{2 \leq k, l \leq n_\mu} f_{i_1, \ldots, i_k} g_{j_1, \ldots, j_l} \sum_{1 \leq n \leq k} f_{i_{n+1}} e^{(\mu)}_{(i_1, \ldots, \hat{i}_n, \ldots, i_{n+1}, \ldots, i_l, j_1, \ldots, j_l)} + \tilde{O}(h^{1+\epsilon}(\mu)).$$

Note that the degree of $e^{(\mu)}_{(i_1, \ldots, \hat{i}_n, \ldots, \hat{i}_m, \ldots, j_1, \ldots, j_l)}$ in the right hand side is bigger than or equal to $n_\mu + 1$. Recall that $n_\mu$ is determined as the highest degree of polynomials of $E_i(i = 1, \cdots, D)$ that constitute basis of $T_\mu$. So, any degree $m$ polynomial of $e^{(\mu)}$ with $m > n_\mu$ is represented by some linear combination of polynomials whose degree is smaller than or equal to $n_\mu$. From the definition (6.1), such linear combinations are $\tilde{O}(h(\mu))$. Then we obtain

$$[q_\mu(f), q_\mu(g)] = h(\mu)q_\mu(\{f, g\}) + \tilde{O}(h(\mu)).$$

Thus, a sequence of quantization maps $\{q_\mu\} \subset Q$ is obtained, and we find $\{T_\mu\}$ is a sequence of objects in $QW$.

Other candidates for Poisson algebras exist besides $A_\mathfrak{g}$. Using $I$ as an ideal of $A_\mathfrak{g}$ generated by relations that is invariants under acting Poisson brackets, we can introduce $A_\mathfrak{g}/I$. Even in this case, we expect to be able to construct $\{T_\mu\}$ and $\{q_\mu\}$ in the same way as in the $A_\mathfrak{g}$ case. In fact, $q_\mu$ is realized as $t_k$ for a fuzzy sphere case in Appendix C.

Example 6.3. Let $e = \{e_1, e_2, e_3\}$ be a fixed base of $\mathfrak{su}(2)$ satisfying commutation relations $[e_i, e_j] = i\epsilon_{ijk} e_k$. Consider a sequence of spin $j$ representation $\rho^{(j)} : \mathfrak{g} \to gl(V_{\mu_j})$ $(\dim V_{\mu_j} = 2j + 1)$. Then $e^{(\mu_j)}$ satisfies $[e^{(\mu_j)}_i, e^{(\mu_j)}_j] = ih(\mu_j)\epsilon_{ijk} e^{(\mu_j)}_k$, and a Casimir relation

$$f^{(\mu_j)}_{(\mu_j)}(R, h(\mu_j)) = (e^{(\mu_j)}_1)^2 + (e^{(\mu_j)}_2)^2 + (e^{(\mu_j)}_3)^2 - R^2 Id$$

is imposed for each $\rho^{(j)}$. Here $R/h(\mu_j) = \sqrt{j(j + 1)}$. From this sequence we obtain the sequence of $T_\mu$ in Subsection 5.3 or in Subsection 5.3. The corresponding Poisson algebra of $\mathfrak{su}(2)$ determined by Theorem 6.2 is differ from $A$ because of the existence of the relation. However, quantization maps $t_k$ in Subsection 5.3 or Subsection 5.3 is constructed in the same manner with $q_\mu$ in Theorem 6.2. Furthermore, as we saw in Subsection 5.3, this sequence gives the naive classical limit $A$, when its diagram includes $A$. 

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As above, starting from a Lie algebra we obtain a sequence of corresponding objects \( \{ T_\mu \} \) in \( QW \) and some corresponding Poisson algebras. We already know that the diagram \( \{ T_\mu \} \cup A_{g} \) gives at least a naive classical limit. This indicates that the above procedure may provide one new approach to the inverse problem.

Is it possible to derive a quantization family, a sequence like \( \{ T_\mu \} \) that gives a naive classical limit or a weak classical limit, from the principle of least action in terms of physics? Here we mention only one attempt to give such a quantization family. Let us consider a sequence of matrix models. Using \( N \times N \) matrices \( X_\mu^N (\mu = 1, \cdots, D, \ N \in \mathbb{N}) \) and a mass \( h(N) \), we consider the action

\[
S_N(h^2(N)) = \text{tr} \left( \frac{1}{4}[X_\mu^N, X_\nu^N]^2 + h^2(N) \frac{1}{2}X_\mu^N \frac{1}{2}X_\mu^N \right) + \sum_i \langle \lambda_i^N, f_i^N(X_\mu^N, \nu) \rangle, \quad (6.8)
\]

where we take contraction by the Killing metric with the structure constants \( f^\mu_{\rho\sigma} \) of \( g \), i.e. \( g_{\mu\nu} = f^\kappa_{\mu\rho}f^\rho_{\kappa\sigma} \), and \( \langle -,- \rangle \) represents the inner product with respect to this Killing metric. \( f_i^N(X_\mu^N, \nu) \) is a solution of the equation of motion of the action \( S \). (In the previous works by Ishii et al. [23] and by Kim et al. [26], actions that derive similar equation of motion are investigated.) We denote these generators by \( e^{(N)} \). A vector space \( V_N \) spanned by the solutions \( e^{(N)} \) is determined. A subset of natural number \( SN \subset \mathbb{N} \) determine a sequence of actions \( \{ S_N \} := \{ S_N \mid N \in SN \subset \mathbb{N} \} \). Then we obtain sequence of \( V_N \) from this \( \{ S_N \} \) similarly. The sequence \( \{ \langle V_N \rangle \} \) is a sequence of objects \( QW \).

The way to get one Poisson algebra \( A_g \) from generators \( e^{(N)} \) is given in Theorem 6.1. For the case with relations \( \{ f_i^N(X_\mu^N, \nu^{(N)}) \} \), a Poisson algebra with relations \( \{ f_i^N(x, \nu^{(N)}) \} \) is also expected. (Recall that \( f_i^N \) does not depend on \( N \).) We assume the existence of the Poisson algebra and denote the Poisson algebra by \( A_g/I \), where \( I = \langle f_1^N(x, \nu), \cdots, f_k^N(x, \nu) \rangle \). To obtain the classical limit, we need a sequence of quantization maps from the Poisson algebra to objects.

While a detailed examination of the following discussion is left for future work, an outline of the methods expected to be accomplished is provided here. Let us find the
way to obtain the set of these quantization maps. From Theorem 6.2 quantization $q_N \in Q$ from $A_\theta/I$ to $\langle V_N \rangle$ is defined similar to (6.5). Using $\{q_N \mid N \in SN\}$, we define a new quantization sequence $\{q'_N\} = \{q'_N = q_N, q'_J = q_J, q'_K = q_K, \ldots\} \subset Q$ as appropriate such that $\ker q'_N \supseteq \ker q'_J$ if $N < J$. (The similar condition appears in Fuzzy sphere case.)

Finally, using the objects $A_\theta/I$ and $q'_N(A_\theta/I)$ and the morphisms $\{q'_N\}$ in $QW$, as in Section 5.3 we can construct the index category and its diagram. We denote the diagram by $D_\theta$. The constructing process is parallel to the one in Section 5.3. Then, the diagram $D_\theta$ is obtained by (6.9) and the naive classical limit is given by $A_\theta/I$.

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots \\
N & J & K & \cdots \\
\end{pmatrix}
\xrightarrow{D_\theta}
\begin{pmatrix}
A_\theta/I & q'_N(A_\theta/I) & q'_J(A_\theta/I) & q'_K(A_\theta/I) & \cdots \\
q'_N(A_\theta/I) & q'_J(A_\theta/I) & q'_K(A_\theta/I) & \cdots \\
\end{pmatrix}
\tag{6.9}
\]

Let us summarize the process to obtain the classical limit.

1. Prepare the sequence of actions $\{S_N\}$.
2. A sequence of the solution $e^{(N)} = \{e_1^{(N)}, \ldots, e_d^{(N)}\}$ and the sequence of vector spaces $\{V_N\}$, where $e^{(N)}$ is a basis of $V_N$ are obtained.
3. When $A_\theta/I$ is given as a Poisson algebra from $\{e^{(N)}\}$, a sequence of quantization $\{q'_N\} \subset Q$, and $q'_N(A_\theta/I)$ are constructed.
4. Objects $A_\theta/I$, $q'_N(A_\theta/I)$ and morphisms $q'_N$ induce an index category and its diagram $D_\theta$.
5. The naive classical limit is determined as $A_\theta/I$.

It should be noted that, in contrast to conventional classical mechanics, the classical solution gives a basis for the vector space, and the set of basis (interpreted as the set of vector spaces) derives the sets of objects and morphisms in $QW$ that may determine the classical limit.

The process discussed here is an example of approaching an inverse problem using the quantization world $QW$ through a physics method. Since the method is based on a generalization of the fuzzy sphere, it works well for the case of fuzzy spheres, but it is not known at this time whether it will work for other manifolds. It is another future task to investigate this point.

7 Summary

We constructed a category $QW$ that is composed of the all quantizations of all Poisson algebras. The characteristic of this category is that quantization is treated as a linear map not to an algebra, but to a module, which is a subset of some algebra. We defined what is required in this formulation and saw next that the definition works well. First, equivalence of quantization was defined. And it was shown that iff there is a pair of equivalent quantizations by the definition then there exists a pair of isomorphisms of
Poisson algebras and the algebras generated by the images of the quantization maps. We also considered the category of quantizations for a fixed single Poisson algebra and discussed its classification of quantizations. Next, we defined the classical limits. Three types of classical limits were introduced: naive classical limits, weak classical limits and strong classical limits. The naive classical limit was defined in the context of category theory. It is a limit in category $\mathcal{QW}$ whose vertex of the cone is in Poisson algebras. As concrete examples, we introduced the naive classical limits for deformation quantizations of polynomials with the Moyal products and for matrix regularization of spheres. The weak classical limit was defined by modifying the definition of the limit in the category theory. The diagrams for the limit were restricted them in $\mathcal{QP} \subset \mathcal{QW}$, and the cones to define the limit were restricted to only those with vertices in the Poisson algebra. Like the limit of ordinary number sequences, the naive classical limit and the weak classical limit may not have a limit, depending on how the sequence of objects is chosen. On the other hand, the strong classical limit was defined for quantizations when a Poisson algebra is fixed. In contrast to the other classical limit, there is no flexibility in the choice of the classical limit, which is automatically determined when the Poisson algebra is chosen. Also, we found the fixed Poisson algebra is always the strong classical limit. Finally, we discussed the inverse problem of determining the classical limit from some noncommutative Lie algebra. From a fixed Lie algebra, we constructed a sequence of representations of the Lie algebra with relations, from which we constructed a Poisson algebra. Next, by constructing a sequence of quantization maps from the Poisson algebra, we obtain the sequence of representations of the Lie algebra as a sequence of $\mathcal{QW}$ objects. A method to obtain this series of procedures from the principle of least action was proposed. The proposed method is just an example of how to approach to the inverse problem in the framework of $\mathcal{QW}$, which is a generalization of the fuzzy sphere case. A more precise discussions on how generalizations should be made are needed in the future. The challenge of obtaining classical manifolds from solutions of noncommutative manifolds of matrix models that actually describe string theory or M-theory is a future problem, too.

As for $\mathcal{QW}$, it is important to investigate its properties from a purely mathematical point of view as well, since it is a noncommutative geometric object that naturally follows from the overall formulation of the quantization of all Poisson algebras. There are many interesting problems around $\mathcal{QW}$ to understand the whole picture of noncommutative geometry. These are issues to be clarified in the future.

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A Definition of $\tilde{O}(z^{1+\epsilon})$

Since we have not defined a norm for objects of category $QW$ in this paper, Landau symbol $O$ does not make sense. So, we define an order $\tilde{O}$ by $x \in \mathbb{R}$ with the Euclidean norm.

Definition A.1. Let $\mathcal{M}$ be an $R$-module for a commutative algebra $R$ over $\mathbb{C}$. Let $f_i$ be a complex valued continuous function such that

$$\lim_{x \to 0} \frac{f_i(xz)}{x} = 0,$$

where $x \in \mathbb{R}$ and $z \in \mathbb{C}$. For $a_i \in \mathcal{M}$ which is independent of $z \in \mathbb{C}$, we denote the element described as $\sum_i f_i(z)a_i \in \mathcal{M}$ by $\tilde{O}(z^{1+\epsilon})$.

From this definition, the term of $\tilde{O}(h^{1+\epsilon})$ in (2.3) is also 0 when $h = 0$. Note that $h$ itself is not necessarily continuous.

We use the following fact:

**Proposition A.2.** Let $t_i : A \to M_i$ be a weak quantization, and let $h_{ij} : M_i \to M_j$ be an $R$-algebra homomorphism. Then

$$h_{ij}(\tilde{O}(h^{1+\epsilon}(t_i))) = \tilde{O}(h^{1+\epsilon}(t_i)) \in M_j.$$ 

**Proof.** For $\tilde{O}(h^{1+\epsilon}(t_i)) = \sum_i f_i(z)a_i \in M_i$

$$h_{ij}(\tilde{O}(h^{1+\epsilon}(t_i))) = h_{ij}(\sum_i f_i(z)a_i) = \sum_i f_i(h)h_{ij}(a_i) = \tilde{O}(h^{1+\epsilon}(t_i)) \in M_j.$$ 

\[\square\]
B List of Symbols

In this paper, many symbols including new symbols are used. So, the list of them is useful to read through.

| Symbol | Description |
|--------|-------------|
| $R$    | commutative ring over $\mathbb{C}$ |
| $R$-Mod | category of $R$-module |
| $R$-alg | category of $R$-algebra |
| $\mathcal{P}_{\text{oisss}}$ | category of Poisson algebras whose morphisms are surjective Poisson morphism |
| $wkQ$ | set of weak quantization maps (including maps with $\hbar = 0$) |
| $Q_{hx}$ | set of weak quantization maps with $|\hbar| = x$ |
| $Q$ | set of weak quantization maps with $\hbar \neq 0$ |
| $Q(\mathcal{P}_{\text{oisss}})$ | \{t(q) | q $\in Q$\} |
| $QP$ | subcategory of $R$-Mod : $ob(QP) := Q(\mathcal{P}_{\text{oisss}})$, $Mor(t(q_i), t(q_j))$ is a set of $R$-algebra homomorphisms restricted their domain into $t(q_i)$ |
| $\mathcal{P}_{\text{oisss}} \sqcup QP$ | subcategory of $R$-Mod : $ob(\mathcal{P}_{\text{oisss}} \sqcup QP) = ob(\mathcal{P}_{\text{oisss}}) \sqcup ob(QP)$, $Mor(\mathcal{P}_{\text{oisss}} \sqcup QP) = Mor(\mathcal{P}_{\text{oisss}}) \sqcup Mor(QP)$ |
| $QW$ | subcategory of $R$-Mod : $ob(QW) := ob(\mathcal{P}_{\text{oisss}}) \cup ob(QP)$, $Mor(QW) := Mor(\mathcal{P}_{\text{oisss}}) \cup Mor(QP) \cup Q$ |
| $U_P$ | forgetful functor from $\mathcal{P}_{\text{oisss}}$ to $QW$ |
| $I_{QP}$ | identically immersion functor from $QP$ to $QW$ |
| $\mathcal{P}_{\text{oisss}} \downarrow I_{QP}$ | comma category of $U_P$ and $I_{QP}$ |
| $F$ | functor from $(U_P \downarrow I_{QP})$ to $\mathcal{P}_{\text{oisss}} \times R$-alg |
| $P - QP$ | $P - QP := F(U_P \downarrow I_{QP}) : \mathcal{P}_{\text{oisss}} - Q(\mathcal{P}_{\text{oisss}})$ quantization pair category |
| $\mathcal{P}_{\mathcal{I}}$ | category of all quantizations of $A$ |
| $F_A$ | functor from $(A \downarrow I_{QP})$ to $R$-alg |
| $D(\mathcal{I})$ | Quantization Family with an index category $\mathcal{I}$ |
| $J^*$ | map from a category with total order to an index category |
| $\mathcal{I}_h(A)$ | $J^*(A \downarrow I_{QP})$ |
| $D_h$ | diagram $\mathcal{I}_h(A) \rightarrow (A \downarrow I_{QP})$ |
| $A/QW_h$ | category consists of $D_h(\mathcal{I}_h(A))$ and $\{(Id_A, A)\}$ with quantization maps whose source is $A$ |

C Fuzzy Sphere and Fuzzy Torus

The fuzzy sphere is considered in [21, 33]. See [21, 33, 2, 7] for details. In [40], more general and mathematically precise statements are given. Let $x^a (1 \leq a \leq 3)$ be coordinates of three-dimensional Euclidean space and $S^2$ be the two-sphere given by

$$\delta_{ab}x^a x^b = 1. \quad (C.1)$$
Let $P[x^a]$ be the algebra of polynomials generated by $x^a$. For an ideal $I$ which is generated by the relation $(C.1)$, we consider a Poisson algebra $\mathcal{A} = P[x^a]/I$ with Poisson bracket

$$\{x^a, x^b\} = \epsilon^{abc}x^c. \tag{C.2}$$

For arbitrary $f \in \mathcal{A}$ is given as

$$f = f_0 + f_ax^a + \frac{1}{2}f_{ab}x^ax^b + \cdots$$

where $f_{a_1\cdots a_i} \in \mathbb{C}$ is completely symmetric and trace-free. The morphism $t_1$ from $\mathcal{A}$ to $T_1 = \mathbb{C}$ is defined by $t_1(f) := f_0$ and the morphism $t_2$ from $\mathcal{A}$ to a subspace $T_2 := t_2(\mathcal{A})$ of a matrix algebra $M_2 = \text{Mat}_2$ is defined by

$$t_2(f) := f_01_2 + f_at_2(x^a), \quad t_2(x^a) := \frac{\hbar}{2}\sigma^a$$

where $\sigma^a$ is Pauli matrix and $1_2$ is a $k \times k$ unit matrix. $t_2$ means that the morphism gives a map from a polynomial to a $2 \times 2$ matrix. In the case of $k \geq 2$, morphisms $t_k : \mathcal{A} \to T_k := t_k(\mathcal{A}) \subset M_k = \text{Mat}_k$ are defined by

$$t_k(f) := f_01_k + f_a t_k(x^a) + \cdots + \frac{1}{(k-1)!}f_{a_1\cdots a_{k-1}} t_k(x^{a_1} \cdots x^{a_{k-1}})$$

$$t_k(x^a) := h_k J^a_k = X^a_k$$

where $J^a_k$ are generators for the $k$-dimensional irreducible representation of $\mathfrak{su}(2)$, and for $i \geq 2$ each $t_i(x^{a_1} \cdots x^{a_{i-1}})$, which is fixed to be well-defined, is generated by $J^a_i$. The morphism $t_k$ gives a map from a polynomial to a $k \times k$ matrix. From the Casimir relation

$$J^a_k J_{ka} = \frac{1}{4}(k^2 - 1)1_k$$

and $(C.1)$, the relationship between $k$ and $\hbar$ is given as

$$\frac{4}{k^2 - 1} = \hbar^2. \tag{C.3}$$

From the commutation relation of $J^a_k$

$$[J^a_k, J^b_k] = i\epsilon^{abc}J^c_k, \quad [t_k(x^a), t_k(x^b)] = \frac{\hbar^2}{\epsilon^{abc}}J^c_k = \frac{\hbar^2}{\epsilon^{abc}}t_k(x^c). \tag{C.4}$$

Using $(C.4)$ we can show that $t_k \in Q \ (k \geq 2)$. Note that $(T_k) = \text{Mat}_k =: M_k.$

Next, we consider a fuzzy torus in a similar way. (See for example [3].) Let $(\theta_1, \theta_2)$ be a coordinate of $T^2 = S^1 \times S^1$, where $\theta_i \in [0, 2\pi)$. A algebra $\mathcal{B} = \{f(\theta_1, \theta_2) = \sum_{l_1, l_2} f_{l_1, l_2} y_{l_1, l_2}\}$, where $y_{l_1, l_2} := \exp(\sum_{i}^{2} \sqrt{-1}l_i \theta_i)$ ($l_i \in \mathbb{Z}_{\geq 0}$) are functions on $T^2$. Poisson bracket is defined by

$$\{y_{l_1, l_2}, y_{m_1, m_2}\} = -\pi(l_1 m_2 - l_2 m_1)y_{l_1+m_1, l_2+m_2}$$

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The algebra of Fuzzy torus in $M_k = \text{Mat}_k$ is generated by the clock matrix $U$ and shift matrix $V$ of size $k$ determined by a complex number $q = e^{2\pi i/k}$, where

$$U = \begin{pmatrix} q^0 & q^1 & \cdots & q^{k-1} \\ q^1 & q^0 & \cdots & q^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ q^{k-1} & q^{k-2} & \cdots & q^0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$ 

$U, V$ satisfy $U^k = V^k = I_{dk}$ and $V U = q UV$. By using these $U, V$, we introduce $Y_{m,n} := U^m V^n$.

Then they satisfy

$$[Y_{l_1,l_2}, Y_{m_1,m_2}] = (q^{l_2 m_1} - q^{l_1 m_2}) Y_{l_1+m_1,l_2+m_2}.$$ 

Note that $q^{l_2 m_1} - q^{l_1 m_2} = -\sqrt{-1} \frac{2\pi}{k} (l_1 m_2 - l_2 m_1) + O(1/k^2)$. The quantization $q_k : B \to Y_k := q_k(B) \subset M_k$ is defined as the same manner with the fuzzy sphere:

$$q_k\left(\sum_{l_1,l_2} f_{l_1,l_2} y_{l_1,l_2}\right) = \sum_{l_1,l_2} f_{l_1,l_2} Y_{l_1,l_2}.$$ 

Note that $\langle Y_k \rangle = M_k$.

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