The kaon electromagnetic form factor

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Abstract

We use recent data on $K^+ \to \pi^+ e^+ e^-$, together with known values for the pion form factor, to derive experimental values for the kaon electromagnetic form factor for $0 < q^2 < 0.125 \ (GeV/c)^2$. The results are then compared with predictions of the Vector-Meson-Dominance model, which gives a good fit to the experimental results.

Key words: kaons, form factors

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1 Introduction

The pion electromagnetic form factor, $F_\pi(q^2)$, is well studied experimentally. Many measurements, for both positive and negative $q^2$, have been reported in the literature [1]. In addition to direct measurements of $F_\pi(q^2)$, further information comes from the pion charge radius, which is related to the slope of the form factor at $q^2 = 0$. By contrast, much less is known about the kaon form factor, $F_K(q^2)$. There are some measurements [1] for negative $q^2$. For positive $q^2$, the only measurements [2] are for $q^2 > 1 \ (GeV/c)^2$, leaving the region $0 < q^2 < 1 \ (GeV/c)^2$ unexplored.

Information on $F_K$ can be deduced from experimental data on the decay $K^+ \to \pi^+ e^+ e^-$, such as that provided by the recent high-statistics Brookhaven experiment E865 [3]. The amplitude for this decay was measured for $q^2$ up to 0.125 $\ (GeV/c)^2$, the maximum allowed by the kinematics of this kaon decay. This amplitude does not give $F_K$ directly, but rather the difference $F_K(q^2) - F_\pi(q^2)$. Since $F_\pi(q^2)$ is relatively well known, then, this decay is a source of information on $F_K$. 
Fig. 1. Graphs for $K^+ \to \pi^+ e^+ e^-$. (a) and (b) are long-distance graphs, (c) is a short-distance graph and (d) is the pion loop term. In each graph, the blob denotes the weak (strangeness-changing) vertex. The (off-shell) photon converts to $e^+ e^-$. 

In the present paper, we extract $|F_{K}(q^2)|^2$ from the E865 results for $K^+ \to \pi^+ e^+ e^-$ and compare the results with the predictions of the Vector-Meson-Dominance (VMD) model [4].

2 The decay $K^+ \to \pi^+ e^+ e^-$ and the kaon form factor

The decay $K^+ \to \pi^+ e^+ e^-$ has been studied theoretically for many years. Already in 1985 it became clear [5] that the process is dominated by the “long-distance” (LD) terms, in which a virtual photon is radiated by either the pion or the kaon. However, it was not until the detailed data of experiment E865 [3] became available that a convincing description of both the scale and the $q^2$ dependence of the amplitude was found [6].

Burkhardt et al. [6] considered four contributions to the amplitude, depicted in Fig. 1. The LD terms are those in Fig. 1(a) and (b). These two graphs are related to the pion and kaon form factors as shown below. Fig. 1(c) represents all short-distance (SD) terms. These were already known in ref. [5] to be small, and subsequent work [7] has shown them to be still smaller than previously believed. Therefore here, as in ref. [6], we neglect the SD contribution from Fig. 1(c). Fig. 1(d) is a “pion loop” term, first discussed by Ecker et al. [8]. Its contribution is small, but it gives a characteristic shape to the $q^2$ dependence of the amplitude. The data of ref. [3] and the analysis of ref. [6] each show the “kink” at $q^2 \sim (2m_\pi)^2$ resulting from this pion loop term, providing convincing evidence for the presence of the term. As in ref. [6], we take this term directly from [8].

The pion and kaon form factors enter via the graphs of Fig. 1(a) and (b), which give the LD amplitude [6]

$$| A_{LD}(q^2) | = e^2 \left| \frac{\langle \pi^+ | H_W | K^+ \rangle}{m_{K^+}^2 - m_{\pi^+}^2} \right| \left| \frac{F_{\pi^+}(q^2) - F_{K^+}(q^2)}{q^2} \right|. \quad (1)$$
In the numerical calculations below, we take the value of the weak matrix element $\langle \pi^+ | H_W | K^+ \rangle$ from ref. [9], following from the average over eleven different measures of weak-decay matrix elements:

$$| \langle \pi^+ | H_W | K^+ \rangle | = (3.59 \pm 0.05) \times 10^{-8} \text{ GeV}^2. \quad (2)$$

Adding the pion loop amplitude, Fig. 1(d), from Ecker et al. [8], we obtain

$$A(q^2) = A_{LD}(q^2) + A_{\pi \text{loop}}(q^2) = e^2 \left| \frac{\langle \pi^+ | H_W | K^+ \rangle}{m_{K^+}^2 - m_{\pi^+}^2} \right| \left| \frac{F_{\pi^+}(q^2) - F_{K^+}(q^2)}{q^2} \right| + A_{\pi \text{loop}}(q^2) \quad (3)$$

from which

$$| F_K - F_{\pi} | = \frac{q^2(m_{K^+}^2 - m_{\pi^+}^2)}{e^2 \left| \langle \pi^+ | H_W | K^+ \rangle \right| \left| A(q^2) - A_{\pi \text{loop}}(q^2) \right|}. \quad (4)$$

To apply Eq. (4), we need experimental values for $A(q^2)$ and $F_\pi(q^2)$. For the former, we use data from Brookhaven E865 [3]. Their amplitude $f(q^2)$ is related to our $A(q^2)$ by

$$| A(q^2) | = f(q^2) \frac{G_F \alpha}{4\pi} \quad (5)$$

where $G_F$ is the Fermi constant and $\alpha$ is the fine structure constant.

The experimental values of $F_\pi(q^2)$ from ref. [1] are in general not measured at precisely the required values of $q^2$. However, the data, which are plotted in Fig. 2, are well described for the region of $q^2$ of interest by the VMD model using a rho-meson pole:

$$F_\pi^{VMD}(q^2) = \frac{m_\rho^2}{m_\rho^2 - q^2} \quad (6)$$

where [10] $m_\rho = 775.8$ MeV. This is shown by the solid line in Fig. 2. A quantitative test of the agreement between the VMD and the experimental data is provided by the value of $\chi^2$, defined by
\[ \chi^2 = \sum \left[ \frac{|F_{\pi}^{\exp}|^2 - |F_{\pi}^{\text{VMD}}|^2}{\sigma} \right]^2 \]

where the sum runs over the 14 points for \( q^2 > 0 \), \( |F_{\pi}^{\exp}|^2 \) and \( |F_{\pi}^{\text{VMD}}|^2 \) are respectively the experimental values and the VMD predictions, and \( \sigma \) is the experimental error on \( |F_{\pi}^{\exp}|^2 \). There are no variable parameters to search on, so the number of degrees of freedom is 14. The result is \( \chi^2 / \text{degree of freedom} = 0.89 \), which shows that the VMD curve reproduces the data satisfactorily in the relevant region of \( q^2 \). Therefore we use Eq. (6) to calculate the required values of \( F_{\pi} \). We emphasise that the values we calculate this way are essentially experimental; the VMD prediction is used only as an interpolating function between the measured experimental points.

![Figure 2](image-url)

Fig. 2. Pion electromagnetic form factors squared. The points are the experimental data and the solid line is the VMD prediction.

The relative sign of \( A_{LD} \) and \( A_{\pi\text{loop}} \) was already established in refs. [3] and [6]. To derive \( F_K \) from Eq. (4), we need also to determine the sign of \( F_K - F_{\pi} \). To do so, we observe that in the VMD model, and in other models such as all those discussed by Scadron et al. [11], as well as in all the available data for \( q^2 < 0 \), \( F_K \) differs less from unity than does \( F_{\pi} \), i.e.

\[ |F_K - 1| < |F_{\pi} - 1|. \]

We assume that this inequality holds also for \( 0 < q^2 < 0.125 \) (GeV/c)^2. This defines the required sign, giving
The extracted values of $|F_K|^2$ are listed in Tab. 1 and plotted in Fig. 3 which also shows previous data, from ref. [1], for $q^2 < 0$. The errors in our values of $F_K$ arise from experimental errors in the amplitude $A(q^2)$ for $K^+ \rightarrow \pi^+ e^+ e^-$ and the error in the weak matrix element $\langle \pi^+ | H_W | K^+ \rangle$, but dominantly from the errors on the experimental values of $F_\pi$. The VMD model, Eq. (6), gives a reasonably unambiguous value for $F_\pi$. However, our input values for $F_\pi(q^2)$ are basically experimental, and the use of VMD as an interpolating function is only justified to the extent that it agrees with the experimental points in Fig. 2. The weighted RMS deviation of the positive-$q^2$ experimental points in Fig. 2 from the VMD prediction is 0.0837, and we take this as the error in $|F_\pi|^2$.

3 Vector meson dominance

Since the VMD picture gives a good description of the pion form factor, as shown in Fig. 2, it is of interest to test it also for the kaon form factor. As discussed in sect. 2 above, the pion form factor is dominated in this model by the $\rho$ pole, Eq. (6). For the kaon form factor, there are contributions from the $\rho$, $\omega$ and $\phi$ poles:
Fig. 3. Kaon electromagnetic form factors squared. The solid points are the experimental data from the present analysis and the circles show the previously existing data. The solid line is the VMD prediction.

\[
F_{K}^\text{VMD}(q^2) = N\left(\frac{1}{2} \frac{g_{\rho ee}}{m_{\rho}^2 - q^2} + \frac{1}{2} \frac{g_{\omega ee}}{m_{\omega}^2 - q^2} + \sqrt{\frac{1}{2} \frac{g_{\phi ee}}{m_{\phi}^2 - q^2}}\right).
\]  

(8)

where \( g_{\rho ee} = 4.97 \), \( g_{\omega ee} = 17.06 \) and \( g_{\phi ee} = 13.38 \), derived from the decay widths. The masses in Eq. (8) are taken from ref. [10]. The \( \rho^0 K^+ K^- \), \( \omega K^+ K^- \) and \( \phi K^+ K^- \) SU(3) coefficients are \( 1/2 \), \( 1/2 \) and \( 1/\sqrt{2} \) respectively. The requirement that \( F(0) = 1 \) gives the normalisation coefficient as \( N = 0.03682 \text{ GeV}^2 \).

The prediction of Eq. (8) is plotted with the data for \( |F_K|^2 \) in Fig. 3. As for \( |F_\pi|^2 \), the VMD gives excellent agreement with data, both for the previously available data for \( q^2 < 0 \) and for the new data derived in the present paper.

A further check on the VMD model is provided by the charge radius, \( r \), which is related to the form factor by [11]

\[
r \equiv \sqrt{\langle r^2 \rangle} = \frac{\hbar c}{6} \frac{\sqrt{dF(q^2)}}{dq^2}_{q^2=0}.
\]

(9)

The quantity \( dF(q^2)/dq^2 \) is straightforwardly obtained from Eq. (8) giving
\[ r_{K}^{VMD} = \hbar c \sqrt{6N \left( \frac{1}{2} \frac{g_{\rho ee}}{m_{\rho}^2} + \frac{1}{2} \frac{g_{\omega ee}}{m_{\omega}^2} + \sqrt{\frac{1}{2} \frac{g_{\phi ee}}{m_{\phi}^2}} \right)} \] (10)

with \( \hbar c \approx 197.3 \text{ MeV fm} \). The result is \( r_{K}^{VMD} = 0.574 \text{ fm} \), which is in good agreement with the experimental value \( r_{K}^{exp} = (0.56 \pm 0.03) \text{ fm} \).

4 Vector and axial-vector form factors

Returning to meson form factors generated from \( \pi^- , K^- \rightarrow e^+ e^- \) decays, p. 498 of the PDG tables [10] gives the vector and axial vector charged pion form factors as \( F_V(0) = 0.017 \pm 0.008 \), \( F_A(0) = 0.0116 \pm 0.0016 \). However, the ratio of axial vector to vector form factors at \( q^2 = 0 \) is [12] \( \gamma^{LsM} = 1 - 1/3 = 2/3 \) due to the Linear \( \sigma \) Model quark plus meson loops for \( \pi \rightarrow e^+ e^- \) decay. Then, using \( f_\pi \approx 93 \text{ MeV} \), the theoretical CVC estimates [13] of these pion form factors are, for charged pion mass 139.57 MeV, \( F_V(0) = m_\pi / 8\pi^2 f_\pi = 0.0190 \), \( F_A(0) = m_\pi / 12\pi^2 f_\pi = 0.0127 \), in reasonable agreement with the pion form-factor data above.

As for the \( K \rightarrow e^+ e^- \) decay, p. 621 of the PDG tables [10] gives the form factor sum as \( |F_A + F_V| = 0.148 \pm 0.010 \), whereas refs. [12,14] find in the linear sigma model 0.109+0.044=0.153, compatible with the data above. This sum is for the kaon form factor, which is the main subject of this paper. For the kaon charge radius, the SU(3) analogue of the VMD SU(2) value \( r_\pi = \sqrt{6}/m_\rho \) is then \( r_K \approx \sqrt{6}/m_{K^*} \approx 0.54 \text{ fm} \). However the exact VMD kaon charge radius of 0.574 fm can only be found from Eq. (10).

5 Summary

We have derived new values for \( |F_K|^2 \) for positive \( q^2 \) from the experimental amplitude for the decay \( K^+ \rightarrow \pi^+ e^+ e^- \). Both the new values of \( |F_K|^2 \), as well as the previous results for \( q^2 < 0 \), agree well with the VMD prediction, as does the experimental value of the kaon charge radius. This picture is consistent with the conclusions of Ivanov et al. [2] who measured \( F_K \) at \( q^2 > 1 \text{ GeV}^2 \), substantially higher than in the present work. They found that VMD gives a good fit to their data, at least at the lower end of their range of \( q^2 \).
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