1. Introduction

In September of 1959, at the conference on Infinitistic Methods in Warsaw, Ernst Specker presented a joint paper with Robert MacDowell [27]. MacDowell and Specker were studying the additive structure of nonstandard models of arithmetic. For a brief description of their main result see Hillary Putnam’s review [36]. At the end of the review Putnam writes:

The proof of this theorem requires the result, which is of independent interest, that every model $M$ possesses a proper elementary (“arithmetic”) extension $M'$ such that the positive elements of $M' - M$ are greater than the elements of $M$. The proof of this result in turn depends upon an extension of a method due to Skolem. In essence, the method is to form the ultrapower $M^M/\mathcal{P}$, where $M^M$ is the ring of all (first-order) definable functions from $M$ into $M$ and $\mathcal{P}$ is a suitable ultrafilter. An ultrafilter with the required properties (the authors speak of a “finitely additive measure with values 0 and 1”, but this is equivalent to the ultrafilter terminology) is obtained by an ingenious construction. Little previous acquaintance with model theory is required.

That result of independent interest is known now as the MacDowell-Specker Theorem. It can be stated simply as follows.

Theorem 1 (MS). Every model of Peano Arithmetic has an elementary end extension.

MacDowell and Specker proved their theorem for any set of axioms that contains the Peano axioms (PA) for addition and multiplication of integers (positive and negative) in the language with constant symbols for 0, 1 and possibly countably many other symbols, with the induction schema for all formulas of the extended language.

In the 60 years that followed, MS has been a constant inspiration for many developments in the model theory of arithmetic. The most striking feature of the theorem is that the same proof covers both the countable and the uncountable cases. It is a rarity in model theory. Jim Schmerl has said that every model of PA thinks it is countable. Thanks to MS, one can form arbitrarily long chains of elementary end extensions, and one can study unions of such chains that exhibit interesting properties. Such models are of independent interest not only in the model theory of arithmetic. For examples, see [17, 28, 43].

\footnote{In this note, elementary extension will always mean proper elementary extension.}
MS made the model theory of PA a special discipline. This becomes evident when one tries to generalize it to other theories such as fragments of PA or axiomatic systems of set theory, just to realize that it cannot be done, or at least cannot be done in full generality. Ali Enayat’s [7] is a relevant survey of corresponding results for models of ZF.

My aim in this note is to discuss briefly the proof of the theorem, and to give an account of its most important extensions and generalizations. Then, I will describe the ongoing research and mention some of the main open problems.

2. The Theorem

Using the Omitting Types Theorem, one can show that every countable model of PA has an elementary end extension. In fact more is true. If a countable structure M for a countable language is linearly ordered, has no last element and satisfies the regularity schema consisting of the universal closures of the axioms of the form:

\[ [\forall x > v \exists y < z \varphi(x, y)] \rightarrow [\exists y < z \forall v \exists x > v \varphi(x, y)] \]

then M has an elementary end extension. This was proved by H. Jerome Keisler in [20, 21]. A proof for countable models of PA is included in [22, Theorem 6.17]. For linearly ordered structures with no last element, the regularity schema is equivalent to the collection principle consisting of the universal closures of the axioms of the form:

\[ [\forall x < z \exists y_1 \ldots \exists y_n \varphi(x, \bar{y})] \rightarrow \exists \bar{v} [\forall x < z \exists y_1 < v \ldots \exists y_n < v \varphi(x, \bar{y})] \]

For models, M, N, . . . , their domains will be denoted M, N, . . . . For a cardinal number κ, a linearly ordered model M is κ-like if |M| = κ and for each a in M, |{x : M \models x < a}| < κ.

In [10], Enayat and Shahram Mohsenipour studied the model theory of the regularity schema. They formulate Keisler’s theorem as follows.

Theorem 2. The following are equivalent for a complete countable first-order theory T.

1. Some model of T has an elementary end extension.
2. T proves the regularity schema.
3. Every countable model of T has an elementary end extension.
4. Every countable model of T has an \( \omega_1 \)-like elementary end extension.
5. T has a κ-like model for some regular cardinal κ.

Because of the limitations on the cardinality of the language in the Omitting Types Theorem, proofs of MS must follow different routes. MacDowell and Specker modified a construction of Thoralf Skolem. By the compactness theorem, the standard model of arithmetic \( \mathbb{N} = (\omega, +, \times, 0, 1) \) has an elementary end extension. Skolem proved it directly using what is today called the definable ultrapower construction [57]. His proof works for any countable model of PA.

Here is a version of Skolem’s argument. Let M be a countable model of PA, and let \( X_n, n \in \omega \), be an enumeration of all sets that are definable in M. By induction on n, we can

2In this note, definability means definability with parameters.
define a sequence of unbounded definable sets \( Y_0 \supseteq Y_1 \supseteq \cdots \) such that for each \( n \), either \( Y_n \subseteq X_n \) or \( Y_n \subseteq M \setminus X_n \). Let \( p(x) \) be the set of formulas \( \varphi(x) \) with parameters from \( M \), such that for some \( n \), \( Y_n \) is contained in the set defined by \( \varphi(x) \) in \( M \). It is easy to check that \( p(x) \) is a complete nonprincipal type of \( \text{Th}(M, a) \) for some \( a \in M \).

Using the regularity schema in \( M \), we can arrange that for each definable \( f : M \to M \), if there is an \( n \) such that \( f(Y_n) \) is bounded in \( M \), then \( f \) restricted to \( Y_m \) is constant for some \( m \geq n \). It is not difficult to check that the model generated over \( M \) by an element realizing \( p(x) \) is an elementary end extension of \( M \).

Using induction in \( M \), we can thin down the \( Y_n \)'s further, so that for every definable \( f : M \to M \), there is an \( n \) such that the restriction of \( f \) to \( Y_n \) is either one-to-one or constant. In this case, the model generated by \( M \) and an element realizing \( p(x) \) is a minimal elementary end extension of \( M \), i.e., an elementary end extension \( N \) such that \( N \) has no proper elementary submodels properly containing \( M \).

The proof outlined above begins with fixing an enumeration of all formulas of the language of \( \text{PA} \) with all parameters of a given countable model. The ingenuity of MacDowell and Specker’s proof was in their observation that one can avoid having to deal with uncountably many parameters by (almost) forgetting the parameters altogether. Their argument is based on two inductions. One external that applies to a enumeration \( \varphi_n(x,y), n \in \omega \), of all parameter-free formulas of the language of \( \text{PA} \), and one internal, performed in \( M \) for each \( \varphi_n(x,y) \) and all instances of \( \varphi_n(x,b) \), for \( b \in M \). For a proof with all technical details see [18].

Here is an abridged proof from [40]. Let \( \varphi_n(x,y), n \in \omega \), be a list of all formulas in two variables. Using the fact that induction holds, we can find formulas \( \psi_n(y) \) such that for each \( n \), the following is a theorem of \( \text{PA} \):

\[
\forall w, z \exists x > z \forall y < w [ \bigwedge_{i<n} (\varphi_i(x,y) \iff \psi_i(y))].
\]

Define the type \( p(x) \) of \( \text{Th}(M, a)_{a \in M} \) by

\[
\varphi_n(x,b) \in p(x) \text{ iff } M \models \psi_n(b).
\]

Again, it is easy to check that the model that is generated over \( M \) by an element realizing \( p(x) \) is an elementary end extension of \( M \). For details see [26] Chapter 3.

Once we know that every model of \( \text{PA} \) has an elementary end extension, there are still many questions that can be asked. An obvious one is: Why is it interesting? I will try to answer it to some extent, but there is also a more technical issue. If we agree, that there is something special about \( \text{MS} \), then we can still ask whether what MacDowell and Specker proved is formulated in the sufficient generality, or, in other words, what is the right \( \text{MS} \). One way this can be made more precise is as follows.

Let \( N \) be an elementary end extension, of \( M \). What can we say about the isomorphism type of the pair \( (N, M) \)? There are two main cases to consider. The first one is when \( N \) is generated over \( M \) by a single element, and the second, when it is not the case, and in particular when \( N \) is the union of a chain of elementary end extensions. There is much we
could discuss here, but it in this note we will concentrate on the first case. For interesting applications of long chains of MacDowell-Specker extensions see [39, 42].

3. Conservativity

The definition (⋆) of the type \( p(x) \) in the previous section turns out to be crucial. A type \( p(x) \) of a completion \( T \) of \( \mathsf{PA} \) is definable if for each parameter-free formula \( \varphi(x, y) \) there is a formula \( \psi(y) \) such that for all constant Skolem terms \( t \),

\[
\varphi(x, t) \in p(x) \text{ iff } T \vdash \psi(t).
\]

So (⋆) is an instance of (**) for \( T = \mathsf{Th}(\mathcal{M}, a)_{a \in M} \).

At this point more notation will be useful. Let \( p(x) \) be a complete type of \( \mathsf{Th}(\mathcal{M}, a)_{a \in M} \), and let \( b \) realize \( p(x) \) in \( \mathcal{N} \). Then, let \( \mathcal{M}(p) \) be the smallest elementary submodel of \( \mathcal{N} \) containing \( M \cup \{b\} \). Such a model always exists because \( \mathsf{PA} \) has definable Skolem functions, and it is unique up to isomorphism.

If \( p(x) \) is a definable type of \( \mathsf{Th}(\mathcal{M}, a)_{a \in M} \), then it follows that for each \( X \) that is definable in \( \mathcal{M}(p) \), \( X \cap M \) is definable in \( \mathcal{M} \). Extensions with that property are called conservative. If \( \mathcal{N} \) is a conservative extension of \( \mathcal{M} \), then for every \( c \in \mathcal{N} \setminus M \), \( \{x \in M : \mathcal{N} \models x < c\} \) is definable in \( \mathcal{M} \). This implies that every conservative extension is an end extension. MacDowell and Specker did not mention conservativity, but a careful reading of their proof reveals that every model of \( \mathsf{PA} \) has a conservative elementary end extension. Robert Phillips defined conservativity in [34], and gave an explicit proof that every model of \( \mathsf{PA} \) has a conservative elementary end extension.

The standard system, \( \mathsf{SSy}(\mathcal{M}) \), of a nonstandard model \( \mathcal{M} \) is the set of the sets of the form \( \omega \cap X \), for all \( X \) that are definable in \( \mathcal{M} \). By compactness, for every set of natural numbers \( X \), the standard model \( \mathbb{N} \) has an elementary extension \( \mathcal{M} \) such that \( X \) is in \( \mathsf{SSy}(\mathcal{M}) \). Hence, \( \mathbb{N} \) has nonconservative extensions.

Phillips proved in [35] that the standard model has nonconservative minimal elementary end extensions. This can be generalized to all countable models of \( \mathsf{PA} \) as follows. By a result of Stephen Simpson [55], every countable model of \( \mathsf{PA} \) has undefinable subsets \( X \) such that \( (\mathcal{M}, X) \models \mathsf{PA}^* \), i.e., \( (\mathcal{M}, X) \) satisfies the induction schema in the language with an additional unary predicate symbol interpreted as \( X \). By \( \mathsf{MS} \), \( (\mathcal{M}, X) \) has an elementary end extension \( (\mathcal{N}, Y) \). By induction on \( a \), one can show that for each \( a \) in \( \mathcal{N} \), the set \( Y \cap \{x : N \models x < a\} \) is coded by an element of \( N \), hence it is definable in \( \mathcal{N} \). For each \( a \) in \( \mathcal{N} \setminus M \), \( X = M \cap (Y \cap \{x : N \models x < a\}) \); hence \( \mathcal{N} \) is not a conservative extension of \( \mathcal{M} \). If \( \mathcal{N} \) is an elementary extension of \( \mathcal{M} \), then a subset \( X \) of \( M \) is coded in \( \mathcal{N} \), if for some \( Y \) that is definable in \( \mathcal{N} \), \( X = M \cap Y \). It is shown in [25] that every subset of a countable model of \( \mathsf{PA} \) that can be coded in an elementary end extension, can be coded in a minimal elementary end extension.

The set of subsets of a model \( \mathcal{M} \) that are coded in an elementary end extension \( \mathcal{N} \) is denoted by \( \mathsf{Cod}(\mathcal{N}/\mathcal{M}) \). In Section 8 we will discuss in detail the question: Which sets subsets of a model \( \mathcal{M} \) are of the form \( \mathsf{Cod}(\mathcal{N}/\mathcal{M}) \) for some elementary end extension of \( \mathcal{M} \)?
A subset $X$ of a model $M$ of PA is a class if for each $a$ in $M$, \{a : M \models x < a\} \cap X$ is definable in $M$. Each definable set is a class, and so is each set $X$ such the $(M, X) \models PA^*$. Also, if $N$ is an elementary end extension of $M$, then all sets in $\text{Cod}(N/M)$ are classes of $M$. Schmerl took the term “rather classless” from the title of [17] and defined in [42] a model to be rather classless if all of its classes are definable.

Conservativity in MS is not an incidental property. By MS, every model of PA has arbitrarily long chains of conservative elementary end extensions. Schmerl showed that if $\text{cf}(\kappa) > \aleph_0$, and $N$ is $\kappa$-like model of PA that is the union of a $\kappa$-chain of conservative elementary end extensions, then $N$ is rather classless [40, Theorem 1.6]. All elementary end extension of rather classless models are conservative. This suggests that every proof of MS must, explicitly or implicitly, yield the existence of conservative extensions.

4. Elementarity

In [27], MacDowell and Specker used MS to show that for every model $M$ of PA and every cardinal $\kappa \geq |M|$, there is a model of PA, $N$ such that $|N| = \kappa$ and $\text{SSy}(M) = \text{SSy}(N)$. This is a simple consequence of the fact that, by MS, every model of PA has arbitrarily long chains of elementary end extensions.

To prove that every model of PA has an end extension, one can apply the Arithmetized Completeness Theorem (ACT). ACT implies that every model of PA has a conservative end extension. The construction can be iterated along $\omega$, but then it breaks down. The union of a chain of end extensions of models of PA may not be a model of PA. Kenneth McAloon constructed $\omega$-chains of end extensions of models of PA such that in their unions $\omega$ is definable by a $\Delta^0_1$ formula [29].

Attempts to generalize MS to fragments of PA have led to interesting problems. If a model of $I\Delta^0_0$ has an elementary end extension, then it is a model of PA. This was shown by Laurence Kirby and Jeff Paris in [33], where they also proved the following theorem.

**Theorem 3.** For every $n \geq 2$, every countable model of $B\Sigma^0_n$ has a $\Sigma^0_n$-elementary end extension.

Peter Clote tried to show that Theorem 3 holds for uncountable models as well, but only succeeded to do it with the stronger assumption that $M \models I\Sigma^0_n$ [2]. The problem whether the result holds for $B\Sigma^0_n$ is open. For a discussion of the end extension problem for fragments of PA, see [5], where, among other results, Costas Dimitracopoulos gives an alternative proof of Clote’s theorem using ACT.

If $\kappa$ is a regular cardinal and $M$ is a $\kappa$-like model of $I\Delta^0_0$, then, by the results of Kirby and Paris, $M$ is a model of PA. For singular $\kappa$ the situation is more complex. Richard Kaye proved in [19] that for each $n \geq 1$ and each singular $\kappa$, there are $\kappa$-like models of $B\Sigma^0_n + \exp + \neg I\Sigma^0_n$, and he asked whether $B\Sigma^0_n + \exp + \{I\Sigma^0_n \rightarrow B\Sigma^0_{n+1} : n \in \omega\}$ is an axiomatization of the theory of all $\kappa$-like models for singular strong limit cardinals $\kappa$. This was answered in the negative by lan Robert Haken and Theodore Slaman, who constructed a $\kappa$-like model of $I\Sigma^0_1 + \neg B\Sigma^0_2$ for a singular strong limit $\kappa$, [15, Chapter 3].

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3$I\Sigma_n$ and $B\Sigma_{n+1}$ are finitely axiomatized for all $n \geq 1$. 
5. Canonicity

The study of elementary extensions of models of PA was raised to another level by Gaifman’s seminal paper [14].

Let \( T \) be a completion of \( \text{PA} \). A type \( p(x) \) is **unbounded** if it contains all formulas \( (t < x) \), where \( t \) is a constant Skolem term; \( p(x) \) is **end-extensional** if it is unbounded, and for every model \( M \) of \( T \), if \( \mathcal{N} \) is generated over \( M \) by an element \( b \) realizing \( p(x) \), and \( a < b \) for all \( a \) in \( M \), then \( \mathcal{N} \) is an elementary end extension of \( M \). If \( p(x) \) is unbounded and each \( \mathcal{N} \) as above is a minimal elementary end extension of \( M \), then \( p(x) \) is **minimal**.

By the definition, each minimal type is end-extensional, and Gaifman shows that each end-extensional type is definable, and that none of these inclusions reverses. Moreover, the set of end-extensional types is dense in the space of complete unbounded 1-types of \( T \).

There are several equivalent definitions of minimal types (see [26, Theorem 3.2.10]). In particular, each minimal type \( p(x) \) is strongly indiscernible, i.e., for every model \( M \) of \( \text{PA} \), for every set of elements \( I \) realizing \( p(x) \) in \( M \), and every \( a \) in \( I \), for all \( c_0, c_1, \ldots, c_k \leq a \), \( \{ x \in I : a < x \} \) is a set of indiscernibles in the structure \( (M, c_0, c_1, \ldots, c_k) \).

If \( p(x) \) is an end-extensional type of \( T \) and \( M \) and \( N \) are isomorphic models of \( T \), then \( M(p) \) and \( N(p) \) are isomorphic as well. Moreover, in the standard topological space of countable models, the operation \( M \mapsto M(p) \) is Borel.

We can also define a canonical operation \( M \mapsto M(I) \), where \( (I, <) \) is a linearly ordered set, and \( M(I) \) is generated over \( M \) by a set of indiscernibles \( \{ a_i : i \in I \} \), realizing a fixed minimal type and such that \( M(I) \models a_i < a_j \) iff \( i < j \). Gaifman proved that for all \( M \) and \( (I, <) \), the group of automorphisms of \( M(I) \) that fix \( M \) pointwise is isomorphic to \( \text{Aut}(I, <) \) [14, Theorem 4.11]. This is improved by the following theorem of Jim Schmerl.

**Theorem 4** ([47]). For every linearly ordered structure \( \mathfrak{A} \), every model of \( \text{PA} \) has an elementary end extension \( N \) such that \( \text{Aut}(N) \cong \text{Aut}(\mathfrak{A}) \).

The operation \( M \mapsto M(I) \), reduces the isomorphism relation for countable linear orders to the isomorphism relation of countable models of \( T \). This shows that the latter relation is Borel complete. Along with the MacDowell-Specker-Gaifman techniques, there are other ways of constructing elementary end extensions with special properties. Canonicity of those constructions has not been systematically studied, but there is one interesting example which we will discuss next.

We say that \( N \) is a **superminimal** extension of \( M \) if for every \( a \in N \setminus M \), the only elementary submodel of \( N \) containing \( a \) is \( N \). Every countable model of \( \text{PA} \) has a conservative superminimal elementary end extension. The proof given in [26, Theorem 2.1.12] begins with enumerating \( M \), and it is not a good prognosis for canonicity of the operation

\[ M \mapsto \text{a superminimal elementary end extension of } M. \]

It is shown in [4] that in fact there is no operation that would in a Borel way map isomorphic models of \( \text{PA} \) to their isomorphic superminimal elementary end extensions.

If \( \mathcal{N} \) is the union of a chain \( M_\alpha, \alpha < \omega_1 \), of models such that for each \( \alpha, M_{\alpha+1} \) is a superminimal elementary end extension of \( M_\alpha \), then \( \mathcal{N} \) is Jónsson, i.e., \( \mathcal{N} \) has no proper
elementary submodel of cardinality \( \aleph_1 \). By a theorem of Andrzej Ehrenfeucht [6], \( N \) realizes \( \aleph_1 \) complete types.

**Problem 5.** Is there an uncountable Jónsson model of \( \text{PA} \) that realizes only \( \aleph_0 \) complete types?

Gaifman defines a dependence relation on types as follows. Let \( p(x) \) and \( q(x) \) be complete types of a completion of \( \text{PA} \). Then, \( q(x) \) depends on \( p(x) \) if there is a Skolem term \( t(x) \) such that for all \( \varphi(x) \), \( \varphi(x) \in q(x) \) iff \( \varphi(t(x)) \in p(x) \). Gaifman showed that for every completion \( T \), there are \( 2^{\aleph_0} \) independent minimal types of \( T \) [14, Theorem 4.13]. It follows that every countable model of \( \text{PA} \) has \( 2^{\aleph_0} \) conservative minimal elementary end extensions.

Schmerl has shown that there is a family of minimal types \( \{ p_\sigma(x) : \sigma \in 2^\omega \} \), such that for any \( \sigma, \tau \in 2^\omega \), \( p_\sigma(x) \) and \( p_\tau(x) \) are dependent if and only if there is a \( k \) such that for all \( n \geq k \), \( \sigma(n) = \tau(n) \) [4, Lemma 3.6]. This result was used in [4] to show that for any completion \( T \), the isomorphism relation for finitely generated models of \( T \) is not Borel reducible to the identity relation on \( 2^\omega \).

In [52], Saharon Shelah looked for possible extensions of Gaifman’s result for theories other than \( \text{PA} \). He wrote

Now Gaifman [12, 13, 14], following MacDowell and Specker [27], proved that any theory \( T = \text{Th}(\omega, +, \times, <, \ldots) \) has definable end extensions, minimal ones, rigid ones, etc. He uses the fact that definitions by induction are allowable. We show that for some of the results proof by induction only is sufficient, so \( T = \text{Th}(\omega, <, \ldots) \) is sufficient (for almost minimal end-extension types). This seems maximum we can get, but we do not have counterexamples.

Shelah called a model \( N \) an almost minimal elementary end extension extension of \( M \) if all elementary submodels of \( N \) that contain \( M \) are cofinal in \( N \), and he defined almost minimal end-extensional types. Shelah proved that every theory \( T \) in a countable language that proves the induction schema and the sentence \( \forall x \exists y (x < y) \), has almost minimal end-extensional types.

In [8], Enayat introduced a general framework for constructions based on iterations of what he calls Skolem-Gaifman ultrapowers, and other types of restricted ultrapowers over models of \( \text{PA} \), and he obtains several interesting results on automorphism of countable recursively saturated models of \( \text{PA} \).

### 6. Uncountable Languages

\( \text{MS} \) and Gaifman’s results hold for all theories that include Peano’s axioms in countable languages extending the language of \( \text{PA} \). Therefore, many later results on models of \( \text{PA} \) extend without modifications to such languages. To indicate that, they are formulated for \( \text{PA}^* \), without specifying the language.

Gaifman asked if the main results of [14] can be proved for extensions of \( \text{PA} \) in uncountable languages. That quickly turned out not to be the case. George Mills used a version of arithmetic forcing to show that each countable model \( M \) of \( \text{PA} \) can be expanded by a
set of functions $f_\alpha : M \rightarrow M$, $\alpha < \omega_1$, so that the model $(M, f_\alpha)_{\alpha < \omega_1}$ satisfies Peano’s axioms, and it has no elementary end extension [30]. Nevertheless, some positive results can be proved as well.

In [40], Schmerl defined generic sets of subsets of a model of PA, and he proved the following generalization of MS (proofs are included in [26, Chapter 6]):

**Theorem 6** ([38, 40]). Let $\mathcal{M}$ be a model of PA.

1. For every generic $X$, $(\mathcal{M}, A)_{A \in X}$ satisfies the induction schema.
2. For every generic $X$, $(\mathcal{M}, A)_{A \in X}$ has a conservative elementary end extension.
3. For any infinite cardinal $\kappa$, there are models of PA of cardinality $\kappa$ with generic sets of subsets of cardinality $\kappa^+$.

In [1], Andreas Blass extended some of Gaifman’s results to models of PA in the full language of arithmetic, i.e., the language with function and relation symbols for all functions and relations of the standard model. Blass characterized end-extensional and minimal types of this theory in terms of ultrafilters on $\omega$.

Every elementary extension of the standard model is an end extension, but even in this case MS gives additional information: the standard model has a conservative elementary end extension.

Every extension of $(\mathbb{N}, A)_{A \subseteq \omega}$ is conservative, but there are expansions of the standard model $\mathbb{N}$ without elementary conservative extensions. This was shown by Enayat in [9].

The problem, listed as Question 7’ in [26, Chapter 12], whether there is an expansion $\mathbb{N}$ of the standard model such that some nonstandard model of Th($\mathbb{N}$) has no (conservative) elementary end extension, was settled in the positive direction by Shelah in [54].

7. The Lattice Problem

For a model $\mathcal{M}$ of PA, the set of elementary submodels of $\mathcal{M}$ forms a lattice $\text{Lt}(\mathcal{M})$, with the join of two models being the smallest elementary submodel of $\mathcal{M}$ containing both of them, and the meet being their intersection. The general lattice problem is to characterize those lattices that can be represented as $\text{Lt}(\mathcal{M})$, for some $\mathcal{M}$. While much is known about the problem, it is still open whether every finite lattice can be represented this way. For a given $\mathcal{M}$, one can also ask about lattices of interstructures $\text{Lt}(\mathcal{N}/\mathcal{M})$, where $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, and $\text{Lt}(\mathcal{N}/\mathcal{M})$ is the set of elementary submodels of $\mathcal{N}$ that contain $\mathcal{M}$. In particular, we can ask about lattices that can be realized as $\text{Lt}(\mathcal{N}/\mathcal{M})$, where $\mathcal{N}$ is an elementary end extension of $\mathcal{M}$. Here Gaifman’s technique comes to play.

Let $T$ be a completion of PA and let $p(x)$ be an end-extensional type of $T$. By a theorem of Gaifman [14], for every model $\mathcal{M}$ of $T$, $p(x)$ can be (uniquely) extended to a complete definable type $p'(x)$ of $\text{Th}(\mathcal{M}, a)_{a \in M}$, such that $a < x$ is in $p'(x)$ for all $a$ in $M$. Since $p'(x)$ is uniquely determined by $p(x)$, we will write $\mathcal{M}(p)$ instead of $\mathcal{M}(p')$.

After Mills [30], we will say a type of $T$ produces a lattice $L$, if for every model $\mathcal{M}$ of $T$, $\text{Lt}(\mathcal{M}(p)/\mathcal{M})$ is isomorphic to $L$. Thus, every minimal $p(x)$ produces the two-element Boolean algebra $\{0, 1\}$. Gaifman proved several results on lattices representable as substructure and interstructure lattices, and conjectured that for every finite distributive
lattice $L$ with exactly one atom there is an end-extensional type that produces $L$ \cite{14}. The conjecture was confirmed by Schmerl \cite{11}, and extended further by Mills \cite{31}. Mills extended Gaifman’s technique of end-extensional 1-types to types with arbitrary (finite or infinite) sets of variables, and he characterized completely distributive lattices (of any cardinality) that can be produced by end-extensional types.

The case of the distributive lattices is closed, but there is still much mystery about the nondistributive ones. While, as shown in \cite{45}, the lattice $M_3$ (Figure 1A) can be represented as $\text{Lt}(N/M)$, where $N$ is a cofinal extension of $M$, $M_3$ has no interstructure lattice representation in which $N$ is an elementary end extension of $M$ \cite{14, 32}. Alex Wilkie proved in \cite{59} that every countable model $M$ has an elementary end extension such that $\text{Lt}(N/M)$ is isomorphic to $N_5$ (Figure 1B).

It is not easy to say why the cases of $M_3$, and $N_5$ are so different, and what makes them both different from the distributive lattices case. All constructions of elementary end extensions require some Ramsey style combinatorics, nontrivial results in lattice representation theory, and proofs involve some rather intricate details. Much work on the lattice problem after the 1980’s was done by Schmerl who has developed special techniques for constructing models with prescribed properties. An account of those developments can be found in \cite[Chapter 4]{26}. One of the results which is directly related to the theme of this note that is not covered there is the following theorem.

**Theorem 7** (\cite{18}). Let $T$ be a completion of $\text{PA}$, and let $L$ be a finite lattice.

1. If some countable model $M$ of $T$ has an almost minimal elementary end extension $N$ such that $\text{Lt}(N/M) \cong L$, then every countable model $M$ of $T$ has an almost minimal elementary end extension $N$ such that $\text{Lt}(N/M) \cong L$.

2. If some countable model $M$ of $\text{Th}(\mathbb{N})$ has an almost minimal elementary end extension $N$ such that $\text{Lt}(N/M) \cong L$, then every model $M$ of $\text{Th}(\mathbb{N})$ has an almost minimal elementary end extension $N$ such that $\text{Lt}(N/M) \cong L$.

The next problem is listed as Question 4 in \cite[Chapter 12]{26}.
**Problem 8.** What finite lattices $L$ are such that every $M$ has an elementary end extension such that $\text{Lt}(N/M) \cong L$? What finite lattices $L$ are such that every countable $M$ has an elementary end extension such that $\text{Lt}(N/M) \cong L$? What finite lattices $L$ are such that some countable $M$ has an elementary end extension such that $\text{Lt}(N/M) \cong L$?

**Problem 9.** Let $L$ be a finite lattice, and suppose that each model $M$ of PA, has an elementary end extension $N$ such that $\text{Lt}(N/M) \cong L$. Is there an end-extensional type $p(x)$ such that for every $M$, $\text{Lt}(M(p)/M) \cong L$?

### 8. Coded sets

Recall that if $N$ is an elementary end extension of $M$, then $\text{Cod}(N/M)$ denotes the set of subsets of $M$ that are coded in $N$. With this notation, we can rephrase MS as follows:

**Theorem 10.** Every model $\mathcal{M}$ of PA* has an elementary end extension $N$ such that $\text{Cod}(N/M)$ is the collection of all definable subsets of $M$.

Let $(N, \mathcal{X})$ be a model of ACA$^0$, and let $(M, \mathcal{A})_{A \in \mathcal{X}}$ be a conservative elementary end extension of $(N, A)_{A \in \mathcal{X}}$. Then $\mathcal{X} = \text{SSy}(\mathcal{M})$. Hence, for each countable model $(N, \mathcal{X})$ of ACA$^0$ there is an elementary end extension $\mathcal{M}$ of $N$ such that $\mathcal{X} = \text{SSy}(\mathcal{M})$. This is a special case of Scott’s theorem that characterizes the countable $\omega$-models of WKL$^0$ as standard systems of countable models of PA, [51]. The same argument gives the following result.

**Theorem 11.** If $\mathcal{M}$ and $\mathcal{X}$ are countable and $(\mathcal{M}, \mathcal{X})$ is a model of ACA$^0$, then $\mathcal{M}$ has an elementary end extension such that $\text{Cod}(N/M) = \mathcal{X}$.

Theorem 11 was generalized by Schmerl in the next theorem which I am tempted to call the ultimate MS. It is not difficult to show that if $N$ is an elementary end extension of $\mathcal{M}$, then $\text{Cod}(N/M)$ contains all definable subsets of $\mathcal{M}$, and that $(\mathcal{M}, \text{Cod}(N/M))$ is a model of WKL$^0$, which is the usual WKL$^0$ of second-order arithmetic in which $I\Sigma^0_1$ is replaced with $I\Delta^0_1 + \exp$. A set $\mathcal{X}$ of subsets of $\mathcal{M}$ is countably generated if there is a countable $\mathcal{X}_0$ such that every set in $\mathcal{X}$ is $\Delta^0_1$-definable in $(\mathcal{M}, \mathcal{X}_0)$.

**Theorem 12 ([49]).** Let $\mathcal{X}$ be a set of subsets of a model $\mathcal{M}$ of PA. Then the following are equivalent:

1. $\mathcal{X}$ is countably generated, it contains all definable subsets of $\mathcal{M}$, and $(\mathcal{M}, \mathcal{X})$ is a model of WKL$^0$.
2. $\mathcal{M}$ has a finitely generated elementary end extension $N$ of $\mathcal{M}$ such that $\mathcal{X} = \text{Cod}(N/M)$.
3. $\mathcal{M}$ has a countably generated elementary end extension $N$ of $\mathcal{M}$ such that $\mathcal{X} = \text{Cod}(N/M)$.

The question whether this Theorem 12 can be strengthened further by requiring that $N$ is a minimal extension was also settled by Schmerl, with an intriguing twist:

**Theorem 13 ([50]).** Let $\mathcal{X}$ be a set of subsets of a model $\mathcal{M}$ of PA. Then the following are equivalent:
(1) $X$ is countably generated, it contains all definable subsets of $\mathcal{M}$, $(\mathcal{M}, X)$ is a model of WKL$^*_0$, and every set that is $\Pi^0_1$-definable in $(\mathcal{M}, X)$ is the union of countably many $\Sigma^0_1$-definable sets.

(2) $\mathcal{M}$ has a minimal elementary end extension $\mathcal{N}$ such that $X = \text{Cod}(\mathcal{N}/\mathcal{M})$.

Proofs of both theorems are very finely tuned versions of the original proof of MacDowell and Specker.

It is not clear what the rather exotic condition in Theorem 13 (1) really means, but, by the next theorem, we know that it is necessary. Notice that if $(\mathcal{M}, X)$ $\models$ ACA$^*_0$, then $X$ trivially satisfies the condition.

**Theorem 14** ([50]). For every model $\mathcal{M}_0$ of PA, there are $\mathcal{M}$ and $X$ such that $\text{Th}(\mathcal{M}_0) = \text{Th}(\mathcal{M})$, $(\mathcal{M}, X) \models \text{WKL}_0$, $X$ is countably generated and contains all definable subsets of $\mathcal{M}$, and there is a set that is $\Pi^0_1$-definable in $(\mathcal{M}, X)$ that is not the union of countably many $\Sigma^0_1$-definable sets.

Athar Abdul-Quader has observed (unpublished) that Theorem 13 can be extended by adding the third condition: For each finite distributive lattice $D$, $\mathcal{M}$ has an elementary end extension $\mathcal{N}$ such that $X = \text{Cod}(\mathcal{N}/\mathcal{M})$ and $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong D$. Schmerl gave a short proof of this result using arguments from [26, Chapter 4].

Another interesting development was triggered by an attempt to generalize Wilkie’s theorem about $\mathcal{N}_5$ to the uncountable case. It turns out that this can’t be done in general. If $\text{Lt}(\mathcal{N}/\mathcal{M})$ is isomorphic to $\mathcal{N}_5$, then $\mathcal{N}$ is not a conservative extension. An interesting proof, due to Schmerl, is included in [26, Section 4.6]. It follows that if $\mathcal{M}$ is rather classless and $\mathcal{N}$ is an elementary end extension, then $\text{Lt}(\mathcal{N}/\mathcal{M})$ is not isomorphic to $\mathcal{N}_5$. This implies that, $\mathcal{N}_5$ cannot be produced by an end-extensional type.

Matt Kaufmann has observed that $\text{MS}$ can be proved using the Arithmetized Completeness Theorem. An outline of his proof is given in [46]. In [11], Enayat and Tin Lok Wong use a refined version of this argument to give an alternative proof of a slightly weaker version of Theorem 12. It is not clear if their method yields finitely generated extensions.

**Problem 15.** Let $\mathcal{M}$ be a model of PA.

(1) If $\text{Lt}(\mathcal{N}/\mathcal{M}) \cong \mathcal{N}_5$, what can we say about $\text{Cod}(\mathcal{N}/\mathcal{M})$?

(2) If $\mathcal{M}$ is countable, is it true that for every undefinable subset $A$ of $\mathcal{M}$ there is an elementary end extension $\mathcal{N}$ such that $\mathcal{N}/\mathcal{M}$ $\cong \mathcal{N}_5$ and $A$ is not in $\text{Cod}(\mathcal{N}/\mathcal{M})$?

In the recent paper [56], Simpson and Tin Lok Wong combine several results from the literature to give the following characterizations of WKL$^*_0$ and ACA$^*_0$. The extension $(\mathcal{N}, Y)$ of $(\mathcal{M}, X)$ is conservative if for all $Y \in Y$, $Y \cap M \in X$.

**Theorem 16.** Let $(\mathcal{M}, X)$ be a countable model of RCA$^*_0$. Then the following are equivalent:

(1) $(\mathcal{M}, X) \models \text{WKL}_0$.

(2) $(\mathcal{M}, X) \models \text{I} \Sigma^0_1$ and it has a proper conservative extension that satisfies RCA$^*_0$.

**Theorem 17.** Let $(\mathcal{M}, X)$ be a countable model of RCA$^*_0$. Then the following are equivalent:

(1) $(\mathcal{M}, X) \models \text{ACA}_0$.
(2) \((M, \mathcal{X})\) has a proper \(\Sigma^1_1\)-elementary conservative extension that satisfies ACA\(_0\).

In [44], Schmerl proved that every countable model of ACA\(_0\) has a \(\Sigma^1_1\)-elementary exclusive extension which satisfies ACA\(_0\). Simpson and Tin Lok Wong give an alternative proof of Schmerl’s result, and they characterized countable models of ATR\(_0\) and \(\Pi^1_1\)-CA\(_0\) in terms of exclusive extensions.

Kirby proved that a countable model of RCA\(_0\) admits a nonprincipal definable type if and only if it satisfies ACA\(_0\) [23]. Simpson and Tin Lok Wong pose the following problem.

**Problem 18.** Can one characterize countable models of ATR\(_0\) and \(\Pi^1_1\)-CA\(_0\) in terms of some variants of definable types, without requiring the defining scheme to be in any specific form?

### 9. Recursive saturation

A systematic study of the model theory of countable recursively saturated models of PA was initiated by Craig Smoryński in [58]. Smoryński proved, among many other results, that every countable recursively saturated model \(M\) of PA has an elementary end extension that is isomorphic to \(M\). Prior to that, John Schlipf [37] proved that resplendent model of PA has a resplendent elementary end extension. So there is a version of MS for resplendent models, but it does not extend to all recursively saturated models. There are recursively saturated models of PA that have no recursively saturated elementary end extension. The prime example is the \(\omega_1\)-like rather classless recursively saturated model constructed by Kaufmann in [17]. It is not difficult to show that if \(N\) is a recursively saturated elementary end extension of \(M\), then the extension is not conservative. This implies that Kaufmann’s model has no recursively saturated elementary end extensions. Kaufman’s proof uses Jensen’s diamond principle which can be eliminated by Shelah’s Absoluteness Theorem [53]. Still, the following problem, posed by Wilfrid Hodges in [16], is open.

**Problem 19.** Prove the existence of rather classless recursively saturated models of PA in cardinality \(\aleph_1\) without assuming diamond at any stage of the proof.

Here is the key lemma in Kaufman’s proof:

**Lemma 20.** Let \(M\) be a countable recursively saturated model of PA. If \(A\) is an undefinable subset of \(M\), then \(M\) has a recursively saturated elementary end extension \(N\) such that \(A \notin \text{Cod}(N/M)\).

In [42], Schmerl studied rather classless recursively saturated model of PA in cardinalities higher than \(\aleph_1\) by means of special types of MacDowell-Specker extensions. Here is one of his main results.

**Theorem 21.** Assume \(V = L\). Let \(\kappa\) be an infinite cardinal and let \(T\) be a completion of PA. Then the following are equivalent:

1. there is a \(\kappa\)-like, recursively saturated model of \(T\);
2. \(\text{cf}(\kappa) > \aleph_0\) and \(\kappa\) is not weakly compact.

See [41] or [56] for a definition.
Lemma 20 says that each undefinable subset of $M$ can be omitted from $\text{Cod}(N/M)$ in some recursively saturated elementary end extension of $M$. The following problem asks if in a similar manner we can also omit the complete theory of any undefinable subset of $M$. It is listed as Problem 17 in [26, Chapter 12], but is badly marred by typos. Here is the correct version.

**Problem 22.** Let $A$ be an undefinable subset of a countable recursively saturated model $M$ of $\text{PA}$. Does $M$ have a recursively saturated elementary end extension $N$ such that for every $B \in \text{Cod}(N/M)$, $\text{Th}(M, A) \neq \text{Th}(M, B)$?

If $\text{Th}(M, A)$ is not in $\text{SSy}(M)$, then the answer to the above question is positive. By chronic resplendency, $M$ has a recursively saturated elementary end extension $N$ such that $(N, M)$ is recursively saturated. In this case, for each $B \in \text{Cod}(N/M)$, $\text{Th}(M, B) \in \text{SSy}(M)$.

There is an open problem concerning elementary end extension of recursively saturated models that is similar in flavor to Problem 22. It was first posed in [24], and despite some effort, it remains open.

**Problem 23.** Let $M$ be countable and recursively saturated and let $K$ be a proper cofinal submodel of $M$. Does $M$ have a recursively saturated elementary end extension $N$ such that for every recursively saturated $N'$ such that $K \preceq N' \preceq N$, if $N' \cap M = K$, then $N' = K$.

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