THE RADIUS OF COMPARISON OF THE TENSOR PRODUCT
OF A C*-ALGEBRA WITH C(X)

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ABSTRACT. Let $X$ be a compact metric space, let $A$ be a unital AH algebra with large matrix sizes, and let $B$ be a stably finite unital C*-algebra. Then we give a lower bound for the radius of comparison of $C(X) \otimes B$ and prove that the dimension-rank ratio satisfies $drr(A) = drr(C(X) \otimes A)$. We also give a class of unital AH algebras $A$ with $\text{rc}(C(X) \otimes A) = \text{rc}(A)$. We further give a class of stably finite exact $\mathcal{Z}$-stable unital C*-algebras with nonzero radius of comparison.

1. Introduction

The radius of comparison of a C*-algebra, based on the Cuntz semigroup, and the dimension-rank ratio of an AH algebra are numerical invariants which were introduced in [29] to study exotic examples of simple amenable C*-algebras that are not $\mathcal{Z}$-stable. Sometimes there is a tight relationship between them. For example, it was shown in Corollary 4.3 of [29] that $\text{rc}(A) = 0$ if and only if $\text{drr}(A) = 0$ whenever $A$ is a simple infinite-dimensional real rank zero unital AH algebra.

The Cuntz semigroup plays a crucial role in the Elliott program for the classification of C*-algebras [7, 12, 30]. See [1, 8, 28] for many aspects of the Cuntz semigroup. It is generally complicated and large. For simple nuclear C*-algebras, the classifiable ones are those whose Cuntz semigroups are easily understood (Section 5 of [1]). With the near completion of the Elliott program, nonclassifiable C*-algebras receive more attention (see [2, 15, 21]) and the Cuntz semigroup is the main additional available invariant.

Let $X$ be a compact metric space. The covering dimension of $X$ is denoted by $\dim(X)$ and the cohomological dimension with rational coefficients is denoted by $\dim_{\mathbb{Q}}(X)$. For many results about the Cuntz semigroup of $C(X)$ when $\dim(X) \leq 3$, we refer to the work of Robert and Tikusis [22]. In the commutative setting, it is well known that the radius of comparison of $C(X)$ is dominated by $\frac{1}{2} \dim(X)$ [5, 11]. For this reason, comparison theory can be viewed as a non-commutative dimension theory. This fact and the main result of [27] are part of our motivation to wonder about an upper bound for the radius of comparison of C*-algebras of the form $C(X, A)$, where $A$ is a unital C*-algebra. The following conjecture is proposed by N. Christopher Phillips.

**Conjecture 1.1.** Let $A$ be a stably finite unital C*-algebra and let $X$ be a compact metric space. Then

$$
\text{rc}(A) \leq \text{rc}(C(X) \otimes A) \leq \frac{1}{2} \dim(X) + \text{rc}(A) + 1.
$$

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The purpose of this paper is to give some preliminary results related to Conjecture 1.1.  

In Section 3, we prove that the left-hand side of (1.1) is true for any stably finite unital C*-algebra $A$ and any compact metric space $X$. (See Proposition 3.4.) The right-hand side of (1.1) is also true for the case $A = M_n(C(Y))$ for $n \in \mathbb{Z}_{>0}$ with technical hypotheses on $X$ and $Y$. (See Proposition 3.5.) But there are some difficulties to prove it for a general C*-algebra $A$ and a metric space $X$. So we decide to show that Conjecture 1.1 is valid at least for some choices of $A$ and $X$.

As an important special case, if $A$ is a residually stably finite $\mathcal{Z}$-stable unital C*-algebra, then it is shown in Proposition 3.9 that $\text{rc}(C(X) \otimes A) = 0$. To prove this, we apply Proposition 3.2.4(ii) of [5]. Therefore, finiteness and residual stable finiteness of $C(X,A)$ is one of the starting points.

Another special case comes when $\dim(X) = 0$. (See Corollary 3.8 and the discussion after it.) To deal with this, we apply Proposition 3.6 which states a connection between Cuntz comparison over $C(X,A)$ and $\dim(X)$.

It is definitely known that $\text{rc}(A) = 0$ for a stably finite exact simple $\mathcal{Z}$-stable unital C*-algebra $A$. (See Corollary 4.6 of [24].) One might naively expect that $\text{rc}(A) = 0$ for all stably finite exact (not necessarily simple) $\mathcal{Z}$-stable unital C*-algebras. To show that simplicity of $A$ is necessary, we exhibit a class of stably finite exact $\mathcal{Z}$-stable unital C*-algebras with nonzero radius of comparison. (See Remark 4.5.)

In Section 5, we prove that if $X$ is a compact metric space and $A$ is a unital AH algebra with large matrix sizes, then the dimension-rank ratios of $A$ and $C(X,A)$ are related by $\text{drr}(A) = \text{drr}(C(X) \otimes A)$. (See Proposition 5.3.) There is an example in which $A$ doesn’t have large matrix sizes and the equality fails. We further give a class of AH algebras $A$ with $\text{rc}(C(X) \otimes A) = \text{rc}(A)$. So, (1.1) is also true if $A$ is chosen from this class of AH algebras.

Along the way, we show the radius of comparison of a unital AH algebra is dominated by one half of its dimension-rank ratio. (See Lemma 5.1.) This result is known, but we have not found it in the literature.

At the end, we give many examples of infinite-dimensional stably finite simple unital C*-algebras $A$ for which the radii of comparison of $A$ and $C(X) \otimes A$ are the same.

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2. Preliminaries

In this section, we gather for easy reference some information on the Cuntz semigroup, the dimension-rank ratio of AH algebras, and the radius of comparison.
**Notation 2.1.** We use the following standard notation. If $A$ is a C*-algebra, or if $A = M_{\infty}(B)$ for a C*-algebra $B$, we write $A_+$ for the set of positive elements of $A$. The unitization of a C*-algebra $A$ is denoted by $A^+$. (We add a new unit even if $A$ is already unital.) We let $\mathcal{K}$ denote the algebra of compact operators on a separable and infinite-dimensional Hilbert space $\mathcal{H}$.

2.1. **The Cuntz semigroup.** Let $A$ be a C*-algebra. For $a, b \in M_{\infty}(A)_+$, we say that $a$ is Cuntz subequivalent to $b$ in $A$, written $a \preceq_A b$, if there is a sequence $(c_n)_{n=1}^\infty$ in $M_{\infty}(A)$ such that $\lim_{n \to \infty} c_n b c_n^* = a$. We say that $a$ and $b$ are Cuntz equivalent in $A$, written $a \simeq_A b$, if $a \preceq_A b$ and $b \preceq_A a$. This relation is an equivalence relation, and we write $[a]_A$ for the equivalence class of $a$. We define $W(A) = M_{\infty}(A)_+/\simeq_A$, together with the commutative semigroup operation $[a]_A + [b]_A = [a \oplus b]_A$ and the partial order $[a]_A \leq [b]_A$ if $a \preceq_A b$. We write 0 for $[0]_A$. We also define $\Cu(A) = W(A) / \mathcal{K}$.

The common notation for Cuntz subequivalence is $a \preceq b$ and it is originally from [9]. We include $A$ in the notation because we want notation for the Cuntz subequivalence with respect to C*-algebras.

Part (1) of the following is taken from Proposition 2.4 of [23]. Part (2) is Lemma 2.5(ii) of [16]. Part (3) is Lemma 1.5 of [20].

**Lemma 2.2.** Let $A$ be a C*-algebra.

1. Let $a, b \in A_+$. Then the following are equivalent:
   a. $a \preceq_A b$.
   b. $(a - \varepsilon)_+ \preceq_A b$ for all $\varepsilon > 0$.
   c. For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a - \varepsilon)_+ \preceq_A (b - \delta)_+$.

2. Let $a, b \in A_+$ and let $\varepsilon > 0$. If $\|a - b\| < \varepsilon$, then $(a - \varepsilon)_+ \preceq_A b$.

3. Let $a, b \in A_+$ and let $\varepsilon_1, \varepsilon_2 \geq 0$. Then
   $((a + b) - (\varepsilon_1 + \varepsilon_2))_+ \sim_A (a - \varepsilon_1)_+ + (b - \varepsilon_2)_+ \preceq_A (a - \varepsilon_1)_+ \oplus (b - \varepsilon_2)_+$.

2.2. **The dimension-rank ratio of an AH algebra.** Part (1) and Part (2) of the following definition are taken from Definition 1.1 of [29]. Part (3) is defined by us to simplify notation. What we call strictly homogeneous here was called homogeneous in [10, 29]. We want to avoid conflict with Definition IV.1.4.1 of [3].

**Definition 2.3.** A C*-algebra is said to be strictly homogeneous if it is isomorphic to $p(C(X) \otimes \mathcal{K})p$ for a compact Hausdorff space $X$ and a projection of constant rank $p \in C(X) \otimes \mathcal{K}$. A strictly semihomogeneous C*-algebra is a finite direct sum of strictly homogeneous C*-algebras.

1. An approximately homogeneous (AH) C*-algebra is a direct limit
   $$A = \lim_{j \to \infty} (A_j, \psi_j)$$
   where $A_j$ is strictly semihomogeneous for each $j$.

2. Let $A = \lim_{j \to \infty} (A_j, \psi_j)$ be a unital (i.e., both $A_j$ and $\psi_j : A_j \to A_{j+1}$ are unital for every $j \in \mathbb{Z}_{>0}$) AH algebra, where
   $$A_j = \bigoplus_{l=1}^{m_j} p_{j,l} (C(X_{j,l}) \otimes \mathcal{K}) p_{j,l}$$
   for compact Hausdorff spaces $X_{j,l}$, projections $p_{j,l} \in C(X_{j,l}) \otimes \mathcal{K}$, and natural numbers $m_j$. Define $\psi_{j,k} = \psi_{j-1} \circ \psi_{j-2} \circ \cdots \circ \psi_k$, and write $\psi_{j,\infty} : A_j \to A$ for the canonical map. We consider this collection of objects and maps as...
a decomposition for $A$. If $\psi_j$ is injective for every $j$, then we describe this collection as an injective decomposition.

(3) If a unital AH algebra $A$ admits a decomposition as in (2) for which

$$\lim_{j \to \infty} \min_{1 \leq i \leq m_j} \left( \text{rank}(p_{j,i}) \right) = \infty,$$

then we say that $A$ has large matrix sizes.

**Definition 2.4** (Definition 2.1 of [29]). Let $A$ be a unital AH algebra. The **dimension-rank ratio** of $A$, denoted $\text{drr}(A)$, is the infimum of the set of strictly positive reals $r$ such that $A$ has a decomposition which satisfies

$$\limsup_{j \to \infty} \max_{1 \leq i \leq m_j} \frac{\dim(X_{j,i})}{\text{rank}(p_{j,i})} = r,$$

whenever this set is not empty, and $\infty$ otherwise.

Part (1), Part (2), and Part (3) of the following lemma are parts of Proposition 2.2 of [29] and Part (4) is Corollary 2.4 of [29].

**Lemma 2.5.** Let $A$ and $B$ be unital AH algebras and let $I$ a closed ideal of $A$. Then:

1. $\text{drr}(A/I) \leq \text{drr}(A)$.
2. $\text{drr}(A \oplus B) = \max \left( \text{drr}(A), \text{drr}(B) \right)$.
3. $\text{drr}(A \otimes M_k) \leq \frac{1}{k} \text{drr}(A)$ for $k \in \mathbb{Z}_{>0}$.
4. Let $A = \lim_{\omega} (A_j, \varphi_j)$ be a unital AH algebra where each $A_j$ is semihomogeneous. Then $\text{drr}(A) \leq \liminf_{j \to \infty} \text{drr}(A_j)$.

### 2.3. Radius of comparison.

The set of normalized 2-quasitraces on a unital C*-algebra $A$ is denoted by $\text{QT}(A)$. By the discussion after Proposition II.4.6 of [4], we have $\text{QT}(A) \neq \emptyset$ for every stably finite unital C*-algebra $A$. See [4] for more details about 2-quasitraces.

**Definition 2.6** (Definition 12.1.7 of [13]). Let $A$ be a unital C*-algebra, and let $\tau \in \text{QT}(A)$. Define $d_{\tau} : M_\infty(A)_+ \to [0, \infty)$ by $d_{\tau}(a) = \lim_{n \to \infty} \tau(a^{1/n})$. We also use the same notation for the corresponding functions on $\text{Cu}(A)$ and $W(A)$.

The following is Definition 6.1 of [29], except that we allow $r = 0$ in (1). This change makes no difference.

**Definition 2.7.** Let $A$ be a stably finite unital C*-algebra.

1. Let $r \in [0, \infty)$. We say that $A$ has $r$-**comparison** if whenever $a, b \in M_\infty(A)_+$ satisfy $d_{\tau}(a) + r < d_{\tau}(b)$ for all $\tau \in \text{QT}(A)$, then $a \prec_A b$.
2. The **radius of comparison** of $A$, denoted $\text{rc}(A)$, is
   
   $$\text{rc}(A) = \inf \left\{ r \in [0, \infty) : A \text{ has } r\text{-comparison} \right\}$$

   if it exists, and $\infty$ otherwise.

If $A$ is simple, then the infimum in Definition 2.7(2) is attained, that is, $A$ has $\text{rc}(A)$-comparison. (See Proposition 6.3 of [29].)

**Theorem 2.8** (Corollary 1.2 of [11]). Let $X$ be a compact metrizable space with $\dim_\mathbb{Q}(X) = \dim(X)$. Then:

1. If $\dim(X)$ is even, then $\frac{\dim(X)}{2} - 2 \leq \text{rc}(C(X)) \leq \max \left( 0, \frac{\dim(X)}{2} - 1 \right)$.
2. If $\dim(X)$ is odd, then $\text{rc}(C(X)) = \max \left( 0, \frac{\dim(X) - 1}{2} - 1 \right)$. 

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2.4. **Functionals.** Let $\Omega$ be a semigroup in the category $\text{Cu}$ which is given in Definition 4.1 of [1]. A functional on $\Omega$ is a map $\lambda : \Omega \to [0, \infty]$ which is order preserving, additive, preserves suprema of increasing sequences and satisfies $\lambda(0) = 0$. We denote the set of functionals on $\Omega$ by $F(\Omega)$. Let $A$ be a unital $C^*$-algebra. The radius of comparison of $(\text{Cu}(A), \langle 1_A \rangle)$, denoted $r_A$, is the infimum of the set of real numbers $r$ in $[0, \infty)$ such that, if whenever $a, b \in M_\infty(A)_+$ satisfy $\lambda(\langle a \rangle A) + r\lambda(\langle 1_A \rangle) \leq \lambda(\langle b \rangle A)$ for all $\lambda \in F(\text{Cu}(A))$, then $a \preceq_A b$.

Since we allow functionals on $\text{Cu}(A)$ which take the value $\infty$ at the unit, it follows that $r_A \leq r(C)$. For sufficiently finite $C^*$-algebras, we have $r_A = r(C)$.

**Definition 2.9.** A unital $C^*$-algebra is called residually stably finite if all of its quotients are stably finite.

**Proposition 2.10** (Proposition 3.2.3 of [5]). Let $A$ be a residually stably finite unital $C^*$-algebra. Then $r(C) = r_A$.

**Proposition 2.11.** Let $A$ be a residually stably finite unital $C^*$-algebra.

1. For every closed ideal $I$ of $A$, we have $r(C/I) \leq r(C)$.
2. If $A = \lim_{\to} A_j$ where the homomorphisms of the inductive system are unital, then $r(C) \leq \liminf_{j \to \infty} r(A_j)$.

**Proof.** Part (1) is essentially immediate from residual stable finiteness of $A/I$ and Proposition 3.2.3(i) of [5]. Also Part (2) is immediate from residual stable finiteness of $A$, Proposition 2.10, and Proposition 3.2.3(iii) of [5].

The following remark is taken from Theorem 2.3 of [29] and Theorem 5.1 of [31].

**Remark 2.12.** For every compact Hausdorff space $X$ and projection $p \in C(X) \otimes K$, we have

$$r(C(X) \otimes K)_p \leq \max \left(0, \frac{\dim(X) - 1}{2 \text{ rank}(p)} \right) \text{ and } dr(C(X) \otimes K)_p = \frac{\dim(X)}{\text{ rank}(p)}.$$  

3. **AN APPROACH TO THE RADIUS OF COMPARISON OF $C(X) \otimes A$**

In this section, we obtain a lower bound for the radius of comparison of $C(X) \otimes A$ when $A$ is any unital stably finite $C^*$-algebra and $X$ is any compact metric space.

We further discuss the radius of comparison of $C(X) \otimes A$ when we have stronger assumptions on $A$ or $X$.

Let $A$ be a $C^*$-algebra. We denote by $\text{Prim}(A)$ the set of primitive ideals of $A$. See Section II.6.5 of [3], Section 5.4 of [18], and [19] for more details about primitive ideals of a $C^*$-algebra.

**Lemma 3.1.** Let $X$ be a compact metric space, let $A$ be a $C^*$-algebra, and let $J$ be a proper closed ideal of $C(X, A)$. Then there is a closed subset $F \subseteq X \times \text{Prim}(A)$ such that $J = \{ f \in C(X, A) : f(x) \in I \text{ for all } (x, I) \in F \}$.

**Proof.** The result is essentially immediate from Theorem 5.4.3 of [18], Proposition 2.16 of [6] and Theorem II.2.2.4 of [3].
The first part of the following lemma is needed to apply Definition 2.7 and the second part is needed to prepare for Proposition 3.9.

Lemma 3.2. Let $X$ be a compact metric space and let $A$ be a unital $C^*$-algebra. Then:

1. If $A$ is stably finite, then $C(X, A)$ is also stably finite.
2. If $A$ is residually stably finite, then $C(X, A)$ is also residually stably finite.

Proof. We prove (1). Since $C(X, A)$ is unital, we apply Lemma 5.1.2 of [25]. Let $f \in C(X, A)$ satisfy $f^*f = 1_{C(X, A)}$. So, $f(x)^*f(x) = 1_A$ for all $x \in A$. Since $A$ is finite, it follows that $f^*f = 1_{C(X, A)}$. A similar argument works if $f \in M_n(C(X, A))$ for $n \in \mathbb{Z}_{>0}$.

To prove (2), we must show that all quotients of $C(X, A)$ are stably finite. Let $J$ be a proper closed ideal of $C(X, A)$. Then, by Lemma 3.1, there is a closed subset $F \subseteq X \times \text{Prim}(A)$ such that

$$J = \{ f \in C(X, A) : f(x) \in I \text{ for all } (x, I) \in F \}.$$

Since $M_n(C(X, A)/J) \cong C(X, M_n(A))/M_n(J)$ for all $n \in \mathbb{Z}_{>0}$, it suffices to show that $C(X, A)/J$ is finite. Let $f \in C(X, A)$ satisfy $f^*f + J = 1_{C(X, A)} + J$. Then $f^*f - 1_{C(X, A)} \in J$. Thus $f(x)^*f(x) - 1_A \in I$ for all $(x, I) \in F$. Since $A/I$ is finite for all $(x, I) \in F$, it follows that $f(x)^*f(x) - 1_A \in F$ for all $(x, I) \in F$. This relation implies that $f^*f - 1_{C(X, A)} \in J$ and therefore $f^*f + J = 1_{C(X, A)} + J$. □

By Lemma 3.2(1) and the discussion after Proposition II.4.6 of [4], we have $\text{QT}(C(X) \otimes A) \neq \emptyset$ for every compact space $X$ and stably finite unital $C^*$-algebra $A$.

Lemma 3.3. Let $X$ be a compact metric space and $A$ be a unital $C^*$-algebra. Let also $l \in \mathbb{Z}_{>0}$ and let $a, b \in M_l(A)_+$. Then $1_{C(X)} \otimes a \prec_{C(X) \otimes A} 1_{C(X)} \otimes b$ if and only if $a \prec_A b$.

Proof. Without loss of generality, assume $l = 1$. Set $B = C(X) \otimes A$.

To show the backward implication, assume $a \prec_A b$. Define a homomorphism $\varphi : A \to B$ by $\varphi(c) = 1_{C(X)} \otimes c$. Using this and $a \prec_A b$, we get $1_{C(X)} \otimes a \prec_B 1_{C(X)} \otimes b$.

To show the forward implication, assume $a \otimes 1_{C(X)} \prec_{B} b \otimes 1_{C(X)}$. It suffices to find a sequence $(v_n)_{n=1}^{\infty}$ in $A$ with $\lim_{n \to \infty} \|v_n b v_n^* - a\| = 0$.

Identifying $C(X) \otimes A$ with $C(X, A)$ in the standard way, we get $a \prec_{C(X, A)} b$. This relation implies that there exists a sequence $(w_n)_{n=1}^{\infty}$ in $C(X, A)$ such that

$$\lim_{n \to \infty} \|w_n b v_n^* - a\| = 0.$$

Now fix some $x_0 \in X$ and define $v_n = w_n(x_0)$ for $n \in \mathbb{Z}_{>0}$. □

The following proposition is essentially immediate from Proposition 2.11(1) as soon as we assume that $A$ is residually stably finite.

Proposition 3.4. Let $X$ be a compact metric space and let $A$ be a stably finite unital $C^*$-algebra. Then $\text{rc}(A) \leq \text{rc}(C(X) \otimes A)$.

Proof. Set $B = C(X) \otimes A$. We may clearly assume $\text{rc}(B) < \infty$. Let $r \in [0, \infty)$ and suppose that $B$ has $r$-comparison. Let $l \in \mathbb{Z}_{>0}$. Let $a, b \in M_l(A)_+$ satisfy

$$d_r(a) + r < d_r(b)$$

(3.1)
for all $\tau \in \text{QT}(A)$. Let $\theta \in \text{QT}(B)$. Define $\tau_\theta : A \to \mathbb{C}$ by $\tau_\theta(a) = \theta(1_{C(X)} \otimes a)$ for all $a \in A$. Then $\tau_\theta \in \text{QT}(A)$ and $d_{\tau_\theta}(c) = d_\theta(1_{C(X)} \otimes c)$ for all $c \in M_l(A)_+$. Using this and (3.1), we get
\[d_\theta(1_{C(X)} \otimes a) + r < d_\theta(1_{C(X)} \otimes b)\]
for all $\theta \in \text{QT}(B)$. Since $B$ has $r$-comparison, it follows that $1_{C(X)} \otimes a \not\preccurlyeq_B 1_{C(X)} \otimes b$. Then, by Lemma 3.3, $a \not\preccurlyeq_A b$. Therefore $r_c(A) \leq r$. Taking the infimum over $r \in [0, \infty)$ such that $B$ has $r$-comparison, we get $r_c(A) \leq r_c(B)$. \hfill $\Box$

**Proposition 3.5.** Let $X$ and $Y$ be compact metric spaces with $\dim_Q(X) = \dim(X)$, $\dim_Q(Y) = \dim(Y)$, and $\dim_Q(X \times Y) = \dim(X \times Y)$. Then Conjecture 1.1 is true for $A = M_n(C(Y))$ for $n \in \mathbb{Z}_{>0}$.

**Proof.** By Proposition 3.4, it suffices to show that the right-hand side of (1.1) holds. To prove it, we may assume $\dim(X \times Y) \geq 5$ and $\dim(Y) \geq 4$. Then, by Theorem 2.8, we have the following:

1. If $\dim(Y)$ is odd and $\dim(X \times Y)$ is even, then
\[r_c(C(X) \otimes A) - 1 \leq \frac{1}{n} \left( \frac{\dim(X \times Y) - 1}{2} - 1 \right) - 1 \leq \frac{1}{2} \dim(X) + \frac{1}{n} \left( \frac{\dim(Y) - 1}{2} - 1 \right) = \frac{1}{2} \dim(X) + r_c(A).\]

2. If $\dim(Y)$ is even and $\dim(X \times Y)$ is odd, then
\[r_c(C(X) \otimes A) - 1 = \frac{1}{n} \left( \frac{\dim(X \times Y)}{2} - 1 \right) - 1 \leq \frac{1}{2} \dim(X) + \frac{1}{n} \left( \frac{\dim(Y)}{2} - 2 \right) = \frac{1}{2} \dim(X) + r_c(A).\]

The argument above is similar for other choices of $\dim(Y)$ and $\dim(X \times Y)$. \hfill $\Box$

The following proposition provides a relationship between pointwise Cuntz comparison over $C(X, A)$ and the topological dimension of $X$. Although we only apply the proposition to the case $\dim(X) = 0$ in this paper, the more general setting of the proposition might be helpful when dealing with pointwise Cuntz comparison over $C(X, A)$.

**Proposition 3.6.** Let $X$ be a compact metric space with $m = \dim(X) < \infty$. Let $A$ be a unital C*-algebra, let $l \in \mathbb{Z}_{>0}$, and let $a, b \in M_l(C(X, A))_+$. If $a(x) \not\preccurlyeq_A b(x)$ for all $x \in X$, then $a \not\preccurlyeq_{C(X, A)} 1_{M_{m+1}} \otimes b$.

**Proof.** Set $B = C(X, A)$, $b = 1_{M_{m+1}} \otimes b$, and $\Gamma_m = \{0, 1, \ldots, m\}$. We must show that $a \not\preccurlyeq_B b$. By Lemma 2.2(1b), it suffices to show that for every $\varepsilon > 0$ we have $(a - \varepsilon)_+ \not\preccurlyeq_B b$. So, let $\varepsilon > 0$ and let $x \in X$ be arbitrary. We may assume without loss of generality that $l = 1$.

For $x \in X$, since $a(x) \not\preccurlyeq_A b(x)$, there exists $v_x \in A$ such that
\[\|v_x b(x) v_x^* - a(x)\| < \frac{\varepsilon}{m + 1}.\]

Since $a, b \in B$, the map $\xi_x : X \to [0, \infty)$, given by
\[z \mapsto \|v_x b(z) v_x^* - a(z)\|,
Since a, b ∈ B, the map ξx : X → [0, ∞), given by
is continuous. Using this, (3.2), and \( \xi_\varepsilon(x) < \frac{\varepsilon}{m+1} \), we get an open neighborhood \( N(x) \) of \( x \) such that, for all \( z \in N(x) \),

\[
\| v_x b(z) v_x^* - a(z) \| < \frac{\varepsilon}{m+1}.
\]

Since \( X \) is compact and \( X = \bigcup_{x \in X} N(x) \), we can choose \( x_0, x_1, \ldots, x_n \in X \) such that \( X = \bigcup_{l=0}^{n} N(x_l) \) for some \( n \in \mathbb{Z}_{>0} \). By Proposition 1.5 of [17], there exists an \( m \)-decomposable finite open refinement of \( \{N(x_l) : l = 0, \ldots, n\} \) of the form

\[
\{U_{0,j}\}_{j \in J_0} \cup \{U_{1,j}\}_{j \in J_1} \cup \ldots \cup \{U_{m,j}\}_{j \in J_m}
\]

with \( U_{k,j} \cap U_{k,j'} = \emptyset \) for \( j \neq j' \) and \( k \in \Gamma_m \).

Choose a partition of unity subordinate to this cover, say \( f_{k,j} : X \to [0,1] \) such that, for all \( x \in X \),

\[
\text{supp}(f_{k,j}) \subseteq U_{k,j} \quad \text{and} \quad \sum_{k \in \Gamma_m} \sum_{j \in J_k} f_{k,j}(x) = 1.
\]

Also, for all \( k \in \Gamma_m \) and \( j \in J_k \), we choose \( l(k,j) \in \{0,1,\ldots,n\} \) such that \( U_{k,j} \subseteq N(x_{l(k,j)}) \).

Now set

\[
v = \text{diag} \left( \sum_{j \in J_0} f_{0,j}^{1/2} v_{x_{l(0,j)}}, \sum_{j \in J_1} f_{1,j}^{1/2} v_{x_{l(1,j)}}, \ldots, \sum_{j \in J_m} f_{m,j}^{1/2} v_{x_{l(m,j)}} \right).
\]

Clearly \( v \in M_{m+1}(B) \). Now we claim

\[
\left\| \text{diag} \left( \sum_{j \in J_0} f_{0,j} a, \ldots, \sum_{j \in J_m} f_{m,j} a \right) - \bar{v} \bar{b}^* \right\| < \frac{\varepsilon}{m+1}.
\]

To prove the claim, for every \( z \in X \), define \( \Lambda_z = \{(s,t) \in \Gamma_m \times \left( \bigcup_{k=0}^{m} J_k \right) : z \in U_{s,t} \} \). Using (3.4) and (3.5) at the last step, we compute

\[
v \bar{b}^* = \text{diag} \left( \sum_{j \in J_0} f_{0,j}^{1/2} f_{0,t}^{1/2} v_{x_{l(0,j)}} b v_{x_{l(0,j)}}, \sum_{j \in J_1} f_{1,j}^{1/2} f_{1,t}^{1/2} v_{x_{l(1,j)}} b v_{x_{l(1,j)}}, \ldots, \sum_{j \in J_m} f_{m,j}^{1/2} f_{m,t}^{1/2} v_{x_{l(m,j)}} b v_{x_{l(m,j)}} \right)
\]

\[
= \text{diag} \left( \sum_{j \in J_0} f_{0,j} v_{x_{l(0,j)}} b v_{x_{l(0,j)}}, \sum_{j \in J_1} f_{1,j} v_{x_{l(1,j)}} b v_{x_{l(1,j)}}, \ldots, \sum_{j \in J_m} f_{m,j} v_{x_{l(m,j)}} b v_{x_{l(m,j)}} \right).
\]
Therefore, using this at the first step, using (3.3) at the third step, and using the second part (3.5) at the last step, for all \( z \in X \),
\[
\left\| \text{diag} \left( \sum_{j \in J_0} f_{0,j} a, \ldots, \sum_{j \in J_m} f_{m,j} a \right) - v b^* \right\| (z) \\
= \left\| \text{diag} \left( \sum_{j \in J_0} f_{0,j} (a - v x_{i(0,j)} b v_{x_{i(0,j)}}^*), \ldots, \sum_{j \in J_m} f_{m,j} (a - v x_{i(m,j)} b v_{x_{i(m,j)}}^*) \right) \right\| (z) \\
\leq \sum_{k=0}^{m} \sum_{j \in J_k} f_{k,j}(z) \left\| a(z) - v x_{i(k,j)}(z) b(z) v_{x_{i(k,j)}}^* (z) \right\| \\
< \sum_{(k,j) \in A} f_{k,j}(z) \cdot \frac{\varepsilon}{m+1} \\
\leq \sum_{k \in \Gamma_m} \sum_{j \in J_k} f_{k,j}(z) \cdot \frac{\varepsilon}{m+1} = \frac{\varepsilon}{m+1}.
\]
This completes the proof of the claim. Using the claim and Lemma 2.2(2), we get
\[
(3.6) \quad \left( \text{diag} \left( \sum_{j \in J_0} f_{0,j} a, \ldots, \sum_{j \in J_m} f_{m,j} a \right) - \frac{\varepsilon}{m+1} \right) \lesssim_B v b^* \lesssim_B b.
\]
Therefore, using the second part of (3.5) at the first step, using Lemma 2.2(3) at the second step, and using (3.6) at the last step,
\[
(a - \varepsilon)_+ = \left( \left( \sum_{k \in \Gamma_m} \sum_{j \in J_k} f_{k,j} \right) a - \varepsilon \right)_+ \\
\lesssim_B \left( \sum_{j \in J_0} f_{0,j} a - \frac{\varepsilon}{m+1} \right)_+ \oplus \ldots \oplus \left( \sum_{j \in J_m} f_{m,j} a - \frac{\varepsilon}{m+1} \right)_+ \\
= \left( \text{diag} \left( \sum_{j \in J_0} f_{0,j} a, \ldots, \sum_{j \in J_m} f_{m,j} a \right) - \frac{\varepsilon}{m+1} \right)_+ \lesssim_B b.
\]
This completes the proof. \( \square \)

The following corollary is essentially immediate from Proposition 3.6.

**Corollary 3.7.** Let \( X \) be a compact metric space with \( \dim(X) = 0 \). Let \( A \) be a unital \( C^* \)-algebra, let \( l \in \mathbb{Z}_{\geq 0} \), and let \( a, b \in M_l(C(X, A))_+ \). Then \( a \lesssim_{C(X,A)} b \) if and only if \( a(x) \lesssim_A b(x) \) for all \( x \in X \).

As an application of Corollary 3.7, we obtain the following corollary.

**Corollary 3.8.** Let \( X \) be a compact metric space with \( \dim(X) = 0 \) and let \( A \) be a stably finite unital \( C^* \)-algebra. Then \( \text{rc}(C(X) \otimes A) = \text{rc}(A) \).

**Proof.** Set \( B = C(X, A) \). By Proposition 3.4, it suffices to show \( \text{rc}(B) \leq \text{rc}(A) \). We may assume \( \text{rc}(A) < \infty \). Let \( r \in [0, \infty) \) and suppose that \( A \) has \( r \)-comparison. Let \( l \in \mathbb{Z}_{\geq 0} \) and let \( a, b \in M_l(B)_+ \). Assume
\[
(3.7) \quad d_r(a) + r < d_r(b)
\]
for all $r \in \text{QT}(B)$. Let $\rho \in \text{QT}(A)$. Then $\rho \circ \text{ev}_x \in \text{QT}(B)$ for all $x \in X$. Therefore $d_{\rho \circ \text{ev}_x}(c) = d_{\rho}(c(x))$ for all $x \in X$, all $\rho \in \text{QT}(A)$, and all $c \in B$. Using this and (3.7), we get
\[ d_{\rho}(a(x)) + r < d_{\rho}(b(x)) \]
for all $x \in X$ and all $\rho \in \text{QT}(A)$. Since $A$ has $r$-comparison, it follows that $a(x) \preceq_A b(x)$ for all $x \in X$. Then, by Corollary 3.7, $a \preceq_B b$. Therefore $\text{rc}(B) \leq r$. Taking the infimum over $r \in [0, \infty)$ such that $A$ has $r$-comparison, we get $\text{rc}(B) \leq \text{rc}(A)$. \hfill \Box

Note that if the $C^*$-algebra $A$ in Corollary 3.8 is also residually stably finite, then the result is immediate from Proposition 3.2.4(i) of [5], Proposition 3.4, and the fact that $X = \lim X_j$ with $X_j$ a finite set for all $j$.

Corollary 3.8 fails if $\dim(X) > 0$. For example, it is enough to take $A = \mathbb{C}$ and $X = [0, 1]^5$.

The following proposition is a consequence of Proposition 3.2.4(ii) of [5].

**Proposition 3.9.** Let $X$ be a compact metric space and let $A$ be a residually stably finite $\mathcal{Z}$-stable unital $C^*$-algebra. Then $\text{rc}(C(X) \otimes A) = 0$.

**Proof.** First of all, although simplicity of $A$ was assumed in Proposition 3.2.4(ii) of [5], it is not necessary for the backward implication. That is, almost unperforation implies $r_A = 0$ even when $A$ is not simple.

On the other hand, $\text{Cu}(C(X, A))$ is almost unperforated by Theorem 4.5 of [24]. Thus, $r_{C(X, A)} = 0$.

Since $A$ is residually stably finite, it follows from Lemma 3.2(2) that $C(X, A)$ is also residually stably finite. Therefore, using Proposition 2.10, we get $\text{rc}(C(X, A)) = r_{C(X, A)} = 0$. \hfill \Box

Combining Proposition 3.4 and Proposition 3.9, we get the following corollary which is a generalization of Corollary 4.6 of [24].

**Corollary 3.10.** Let $A$ be a residually stably finite $\mathcal{Z}$-stable unital $C^*$-algebra. Then $\text{rc}(A) = 0$.

In light of Proposition 3.9 and Corollary 3.10, it is an important question whether there exists a stably finite $\mathcal{Z}$-stable unital $C^*$-algebra with nonzero radius of comparison. Such a $C^*$-algebra can’t be residually stably finite. We will give examples in the next section.

4. A CLASS OF STABLY FINITE EXACT $\mathcal{Z}$-STABLE UNITAL $C^*$-ALGEBRA WITH NONZERO RADIUS OF COMPARISON

It is a result of Rørdam [24] that all stably finite exact simple $\mathcal{Z}$-stable unital $C^*$-algebras have strict comparison of positive elements. To see that not all $\mathcal{Z}$-stable unital $C^*$-algebras have strict comparison of positive elements, we exhibit a class of stably finite exact $\mathcal{Z}$-stable unital $C^*$-algebras with nonzero radius of comparison.

4.1. Cones over $C^*$-algebras. Let $A$ be a $C^*$-algebra. The cone over $A$, denoted $\text{CA}$, is the set of continuous functions $f: [0, 1] \to A$ with $f(0) = 0$. Clearly $\text{CA} \cong C_0((0, 1]) \otimes A$. If $A$ is unital, $(\text{CA})^+$ is isomorphic to the set of continuous functions $f: [0, 1] \to A$ such that $f(0) \in \text{CA}_1$. For a nonunital $C^*$-algebra $A$, we say that $A$ is stably finite if $A^+$ is stably finite. Then the cone over any $C^*$-algebra is stably finite but often fails to be residually stably finite. In general, we do not know whether the tensor product of two (non-exact) stably finite simple $C^*$-algebras is stably finite.
or not. But we have the following nice lemma for the case that one tensor factor is the unitization of the cone over a unital C*-algebra.

**Lemma 4.1.** Let $A$ be a unital C*-algebra and let $B$ be a stably finite unital C*-algebra. Then $(CA)^+ \otimes B$ is stably finite for any choice of tensor product.

**Proof.** Let $k \in \mathbb{Z}_{>0}$ and let $f \in (CA)^+ \otimes M_k(B).$ Suppose $f^* f = 1_{(CA)^+} \otimes 1_{M_k(B)}.$ We must show $f^* f = 1_{(CA)^+} \otimes 1_{M_k(B)}.$ We may identify $f$ as a continuous function $f: [0,1] \rightarrow A \otimes M_k(B)$ with $f(0) = 1_A \otimes b$ for some $b \in M_k(B).$ So, it suffices to show that $f(x) f(x)^* = 1_A \otimes 1_{M_k(B)}$ for all $x \in [0,1].$

We note that $1_A \otimes (b^* b) = f(0)^* f(0) = 1_A \otimes 1_{M_k(B)}.$

This relation implies that $b^* b = 1_{M_k(B)}.$ Since $B$ is stably finite, it follows that $bb^* = 1_{M_k(B)}.$ Therefore

$$f(0) f(0)^* = 1_A \otimes (bb^*) = 1_A \otimes 1_{M_k(B)}.$$ 

Now define a uniformly continuous map $\Delta: [0,1] \rightarrow A \otimes M_k(B)$ by $\Delta(x) = f(x) f(x)^*.$ Clearly $\Delta(0) = 1_A \otimes 1_{M_k(B)}$ and $\Delta(x) \in \text{Proj}(A \otimes M_k(B))$ for all $x \in [0,1].$

Choose $\delta > 0$ such that $\|\Delta(t) - \Delta(s)\| < 1$ whenever $|t - s| < \delta$ for $s,t \in [0,1].$

Choose $n \in \mathbb{Z}_{>0}$ and the partition $P = \{t_0 = 0, t_1, \ldots, t_n = 1\}$ of $[0,1]$ with $\max_{1 \leq j \leq n} |t_j - t_{j-1}| < \delta.$ Then $\|\Delta(t_j) - \Delta(t_{j-1})\| < 1$ for $j = 1, \ldots, n.$ Using this, $\Delta(t_j)$ is projection for $j = 0, 1, \ldots, n,$ and Lemma 10.1.7 of [13], we get $\Delta(x) = 1_A \otimes 1_{M_k(B)}$ for all $x \in [0,1]$ and therefore $f^*(x) f(x) = 1_A \otimes 1_{M_k(B)}.$

We let $T(A)$ denote the set of normalized traces on a unital C*-algebra. To prepare the next proposition, we need the following remark.

**Remark 4.2.** Let $A$ be unital C*-algebra with $\tau|_{CA} = 0$ for all $\tau \in T((CA)^+).$ We may identify $(CA)^+$ as the set of continuous functions $f: [0,1] \rightarrow A$ such that $f(0) \in \mathbb{C}1_A.$ Define $\rho: (CA)^+ \rightarrow \mathbb{C}$ by $\rho(f) = \lambda_f$ where $f(0) = \lambda f 1_A.$ Clearly $\rho \in T((CA)^+).$ Now we claim that $\rho$ is the only tracial state on $(CA)^+.$

To prove the claim, let $\tau: (CA)^+ \rightarrow \mathbb{C}$ be a tracial state and $f \in (CA)^+.$ Since $\tau|_{CA} = 0,$ we have

$$\tau(f) = \tau(f - f(0) 1_A) + \tau(f(0) 1_A) = \rho(f).$$

Hence, $\rho$ is the unique trace on $(CA)^+.$ Moreover, let $f \in M_l((CA)^+)$ for some $l \in \mathbb{Z}_{>0}.$ Choose $f_{jk} \in (CA)^+$ for $j,k = 1, \ldots, l$ with $f = (f_{jk})_{j,l}.$ Since $f_{jk}(0) \in \mathbb{C}1_A$ for $j,k = 1, \ldots, l,$ it follows that $f(0) = (\lambda_{jk} 1_A)_{j,l}.$ Set $\lambda_f = (\lambda_{jk})_{j,l}.$ Now we have

$$(4.1) \quad d_{\text{Tr}_l \otimes \rho}(f) = \lim_{n \rightarrow \infty} (\text{Tr}_l \otimes \rho)(f^{1/n}) = \lim_{n \rightarrow \infty} \text{Tr}_l(\lambda_f^{1/n}) = \text{rank}(\lambda_f).$$

It is known (see Theorem 5.11 of [14]) that all 2-quasitraces on a unital exact C*-algebra are traces.

**Proposition 4.3.** Let $A$ be a exact unital C*-algebra such that $\tau|_{CA}$ is zero for all $\tau \in T((CA)^+).$ Let $B$ be a stably finite exact unital C*-algebra. Then

$$\text{rc}((CA)^+) = \text{rc}((CA)^+ \otimes_{\text{min}} B) = \infty.$$ 

**Proof.** First we prove $\text{rc}((CA)^+) = \infty.$ For every $r \in [0,\infty),$ we must find $f,g \in M_\infty((CA)^+)$ such that $d_\rho(f) + r < d_\rho(g)$ and $f \not\preceq g$ where $\rho$ is as in Remark 4.2.
The largest integer which is less than or equal to \( r \) is denoted by \( \lfloor r \rfloor \). Set \( n = \lfloor r \rfloor + 2 \). Define \( f_0, g_0 : [0, 1] \to [0, \infty) \) by

\[
f_0(t) = 1 \quad \text{and} \quad g_0(t) = \begin{cases} -4t + 2 & 0 \leq t \leq 1/2 \\ 0 & 1/2 \leq t \leq 1. \end{cases}
\]

We let \( e_{i,k} \in M_n \) be the standard matrix units. Then set \( f(t) = f_0(t)1_A \otimes e_{1,1} \) and \( g(t) = g_0(t)1_A \otimes 1_{M_n} \) for \( t \in [0, 1] \). Clearly \( f, g \in M_n((CA)^+) \). By (4.1), we have \( d_\rho(f) = 1 \) and \( d_\rho(g) = n \). Therefore

\[
d_\rho(f) + r = 1 + r < \lfloor r \rfloor + 2 = n = d_\rho(g).
\]

Now assume \( f \preceq_{(CA)^+} g \). Choose \( v \in M_n((CA)^+) \) such that \( \| f - vgv^* \| < \frac{1}{2} \). Then

\[
1 = \| f(1) - v(1)g(1)v^*(1) \| \leq \| f - vgv^* \| < \frac{1}{2}.
\]

This is a contradiction.

Now we show \( \text{rc}((CA)^+ \otimes_{\min} B) = \infty \). The proof is similar to the first part, except that we now find \( \tilde{f} \) and \( \tilde{g} \) instead of \( f \) and \( g \). Since \( (CA)^+ \) has a unique tracial state \( \rho \), it is easy to check that

(4.2) \[
T((CA)^+ \otimes_{\min} B) = \{ \rho \otimes \tau : \tau \in T(B) \}.
\]

Set \( \tilde{f} = f \otimes 1_B \) and \( \tilde{g} = g \otimes 1_B \). Clearly \( \tilde{f}, \tilde{g} \in (M_n((CA)^+) \otimes_{\min} B)^+ \). Using (4.1) and (4.2), we get, for all \( \theta \in T((CA)^+ \otimes_{\min} B) \),

\[
d_\theta(\tilde{f}) = d_\rho(f) = 1 \quad \text{and} \quad d_\theta(\tilde{g}) = d_\rho(g) = n.
\]

Then a similar calculation gives us a similar contradiction as in the first part.  

We refer to Section 4 of [16] for the definition of a purely infinite C*-algebra and its properties.

**Corollary 4.4.** Let \( A \) be a purely infinite exact simple unital C*-algebra and \( B \) be a stably finite exact unital C*-algebra. Then

\[
\text{rc}((CA)^+) = \text{rc}((CA)^+ \otimes_{\min} B) = \infty.
\]

**Remark 4.5.** Applying Corollary 4.4 with \( \mathcal{Z} \) in place of \( B \), we get \( \text{rc}((CA)^+ \otimes \mathcal{Z}) = \infty \). More generally, if the C*-algebra \( B \) in Corollary 4.4 is also \( \mathcal{Z} \)-stable, then we can get a class of stably finite exact \( \mathcal{Z} \)-stable unital C*-algebras with nonzero radius of comparison.

5. A CLASS OF AH ALGEBRAS WITH PARTICULAR RADIUS OF COMPARISON

In this section, we give a class of AH algebras satisfying Conjecture 1.1. As preparation, we prove that \( drr(A) = drr(C(X) \otimes A) \) for any unital AH algebra \( A \) with large matrix sizes.

The dimension-rank ratio of an AH algebra is related to its radius of comparison by the following lemma.

**Lemma 5.1.** Let \( A \) be a unital AH algebra. Then \( \text{rc}(A) \leq \frac{1}{4}drr(A) \).

**Proof.** We may assume \( drr(A) < \infty \). Suppose \( \lim A_k \) is an arbitrary decomposition for \( A \), where

\[
A_k = \bigoplus_{l=1}^{m_k} p_{k,l}(C(X_{k,l}) \otimes K)p_{k,l}
\]

with

\[
drr(A) = \text{dim} \bigoplus_{l=1}^{m_k} C(X_{k,l}) \otimes K \leq 4 \text{dim} C(X_{k,l}) \otimes K.
\]

Therefore

\[
\text{rc}(A) = \frac{1}{\text{dim} A} \leq \frac{1}{4}drr(A).
\]
for compact Hausdorff spaces $X_{k,l}$, projections $p_{k,l} \in C(X_{k,l}) \otimes \mathcal{K}$, and $m_k \in \mathbb{Z}_{>0}$. Set

\begin{equation}
(5.1) \quad r = \limsup_{k \to \infty} \max_{1 \leq l \leq m_k} \left( \frac{\dim(X_{k,l})}{\text{rank}(p_{k,l})} \right)
\end{equation}

which is a nonnegative real number or $\infty$. Using Proposition 6.2(i) of [29] at the first step and Remark 2.12 at the second step, we get, for every $k$,

\begin{equation}
(5.2) \quad \text{rc}(A_k) = \max_{1 \leq l \leq m_k} \text{rc}(p_{k,l}(C(X_{k,l}) \otimes \mathcal{K})p_{k,l}) \leq \max_{1 \leq l \leq m_k} \left( \frac{\dim(X_{k,l})}{2 \text{rank}(p_{k,l})} \right).
\end{equation}

It follows easily from the discussion after Definition V.2.1.9 of [26] that AH algebras are residually stably finite. Therefore, using Proposition 2.11(2) at the first step, using (5.2) at the second step, and using (5.1) at the last step,

\[ \text{rc}(A) \leq \liminf_{j \to \infty} \text{rc}(A_j) \leq \liminf_{j \to \infty} \max_{1 \leq l \leq m_j} \left( \frac{\dim(X_{j,l})}{2 \text{rank}(p_{j,l})} \right) \leq \frac{1}{2} \limsup_{j \to \infty} \max_{1 \leq l \leq m_j} \left( \frac{\dim(X_{j,l})}{\text{rank}(p_{j,l})} \right) = \frac{r}{2}. \]

Using this and taking infimum over all $r$ with $r = \limsup_{k \to \infty} \max_{1 \leq l \leq m_k} \left( \frac{\dim(X_{k,l})}{\text{rank}(p_{k,l})} \right)$, we get $\text{rc}(A) \leq \frac{1}{2} \text{drr}(A)$. \hfill $\Box$

There is an example in which $A$ doesn’t have large matrix sizes and the equality $\text{rc}(A) = \frac{1}{2} \text{drr}(A)$ fails. For example, it is enough to take $A = C([0,1]^6)$.

We now give a more precise statement of Proposition 6.8 of [29].

**Corollary 5.2.** For every $r \in [0, \infty)$, there exists a simple unital AH algebra such that $\text{rc}(A) = \frac{1}{2} \text{drr}(A) = r$.

**Proof.** Combine Proposition 6.8 of [29] and Lemma 5.1. \hfill $\Box$

**Proposition 5.3.** Let $A$ be a unital AH algebra with large matrix sizes and let $X$ be a compact metric space. Then $\text{drr}(A) = \text{drr}(C(X) \otimes A)$.

**Proof.** First we show $\text{drr}(C(X) \otimes A) \leq \text{drr}(A)$. We may assume $\text{drr}(A) < \infty$ and $X = \lim_{j \to \infty} X_j$ with $\dim(X_j) < \infty$ for all $j$ (by Corollary 5.2.6 of [26]). Suppose $\lim_k A_k$ is an arbitrary decomposition for $A$, where

\[ A_k = \bigoplus_{l=1}^{m_k} p_{k,l}(C(Y_{k,l}) \otimes \mathcal{K})p_{k,l} \]

for compact Hausdorff spaces $Y_{k,l}$, projections $p_{k,l} \in C(Y_{k,l}) \otimes \mathcal{K}$, and $m_k \in \mathbb{Z}_{>0}$. Set

\begin{equation}
(5.3) \quad r = \limsup_{k \to \infty} \max_{1 \leq l \leq m_k} \left( \frac{\dim(Y_{k,l})}{\text{rank}(p_{k,l})} \right)
\end{equation}

which is a nonnegative real number or $\infty$. Choose a strictly increasing sequence $r(j)$ such that, for all $j$,

\begin{equation}
(5.4) \quad \min_{1 \leq l \leq m_{r(j)}} \left( \text{rank}(p_{r(j),l}) \right) \geq j \dim(X_j).
\end{equation}

Clearly, $C(X) \otimes A \cong \lim_{j \to \infty} C(X_j) \otimes A_{r(j)}$. Using Lemma 2.5(2) at the second step, using Remark 2.12 at the third step, and using

\[ \dim(X_j \times Y_{r(j),l}) \leq \dim(Y_{r(j),l}) + \dim(X_j), \]

...
the fourth step, we get, for every $j$,
\[
\text{drr} \left( C(X_j) \otimes A_{r(j)} \right) \\
= \text{drr} \left( \bigoplus_{l=1}^{m_{r(j)}} (1_C(X_j) \otimes p_{r(j),l}) (C(X_j \times Y_{r(j),l}) \otimes \mathcal{K}) (1_C(X_j) \otimes p_{r(j),l}) \right) \\
\leq \max_{1 \leq l \leq m_{r(j)}} \left( \text{drr} \left( (1_C(X_j) \otimes p_{r(j),l}) (C(X_j \times Y_{r(j),l}) \otimes \mathcal{K}) (1_C(X_j) \otimes p_{r(j),l}) \right) \right) \\
\leq \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(X_j \times Y_{r(j),l})}{\text{rank}(p_{r(j),l})} \right) \\
\leq \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(X_j) + \dim(Y_{r(j),l})}{\text{rank}(p_{r(j),l})} \right) \\
\leq \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(X_j)}{\text{rank}(p_{r(j),l})} \right) + \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(Y_{r(j),l})}{\text{rank}(p_{r(j),l})} \right).
\]

Therefore, using this relation at the second step, using Lemma 2.5(4) at the first step, and (5.4) at the fourth step,
\[
\text{drr}(C(X) \otimes A) \\
\leq \liminf_{j \to \infty} \text{drr} \left( C(X_j) \otimes A_{r(j)} \right) \\
\leq \liminf_{j \to \infty} \left( \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(X_j)}{\text{rank}(p_{r(j),l})} \right) + \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(Y_{r(j),l})}{\text{rank}(p_{r(j),l})} \right) \right) \\
\leq \limsup_{j \to \infty} \left( \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(X_j)}{\text{rank}(p_{r(j),l})} \right) \right) + \limsup_{j \to \infty} \left( \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(Y_{r(j),l})}{\text{rank}(p_{r(j),l})} \right) \right) \\
\leq \lim_{j \to \infty} \frac{1}{j} + \limsup_{j \to \infty} \left( \max_{1 \leq l \leq m_{r(j)}} \left( \frac{\dim(Y_{r(j),l})}{\text{rank}(p_{r(j),l})} \right) \right) = r.
\]

Using this and taking the infimum over all $r$ with $r = \limsup_{k \to \infty} \max_{1 \leq l \leq m_k} \left( \frac{\dim(Y_{r(l),k})}{\text{rank}(p_{r(l),k})} \right)$, we get
\[
\text{drr}(C(X) \otimes A) \leq \text{drr}(A).
\]

To show $\text{drr}(A) \leq \text{drr}(C(X) \otimes A)$, let $x_0 \in X$. Define a surjective homomorphism $\varphi : C(X, A) \to A$ by $\varphi(f) = f(x_0)$. Using this at the first step and using Lemma 2.5(1) at the second step, we get
\[
\text{drr}(A) = \text{drr} \left( C(X, A)/\ker \varphi \right) \leq \text{drr} \left( C(X) \otimes A \right).
\]

This completes the proof.

There is an example in which $A$ doesn’t have large matrix sizes and the equality $\text{drr}(A) = \text{drr}(C(X) \otimes A)$ fails. For example, it is enough to take $A = C([0, 1])$ and $X = [0, 1]$. As we promised, we introduce a class of AH algebras $A$ with $\text{rc}(C(X) \otimes A) = \text{rc}(A)$ in the following theorem.

**Theorem 5.4.** Let $A$ be a unital AH algebra with large matrix sizes and let $X$ be a compact metric space. Suppose $\text{rc}(A) = \frac{1}{2} \text{drr}(A)$. Then:

1. $\text{rc}(C(X) \otimes A) = \frac{1}{2} \text{drr}(C(X) \otimes A)$.
2. $\text{rc}(C(X) \otimes A) = \text{rc}(A)$. 
Proof. We prove (1). Using Proposition 5.3 at the first step, using Proposition 3.4 at the third step, and using Lemma 5.1 at the last step, we get
\[ \frac{1}{2} \text{drr}(C(X) \otimes A) = \frac{1}{2} \text{drr}(A) = \text{rc}(A) \leq \text{rc}(C(X) \otimes A) \leq \frac{1}{2} \text{drr}(C(X) \otimes A). \]

This relation implies \( \text{rc}(C(X) \otimes A) = \frac{1}{2} \text{drr}(C(X) \otimes A) \).

We prove (2). Using Part (1) at the first step and using Proposition 5.3 at the second step, we get
\[ \text{rc}(C(X) \otimes A) = \frac{1}{2} \text{drr}(C(X) \otimes A) = \frac{1}{2} \text{drr}(A) = \text{rc}(A). \]

This completes the proof. \( \Box \)

One should ask whether there exists an AH algebra satisfying the hypotheses of Theorem 5.4. To see that there is such an AH algebra, we give a simple unital AH algebra \( A \) with large matrix sizes, stable rank one, nonzero radius of comparison, and \( \text{rc}(A) = \frac{1}{2} \text{drr}(A) \). The example was recently constructed in Section 6 of [2].

Example 5.5. Let \( X \) be a compact metric space and let \( A = \lim_{\rightarrow} A_n \) be the AH algebra which was constructed in Section 6 of [2]. Adopt the assumptions and notation of Section 6 of [2]. Then:
\[ \text{rc}(A) = \frac{1}{2} \text{drr}(A), \quad \text{rc}(C(X) \otimes A) = \frac{1}{2} \text{drr}(C(X) \otimes A), \quad \text{and} \quad \text{rc}(C(X) \otimes A) = \text{rc}(A). \]

By Lemma 5.1 and Theorem 5.4, it suffices to show \( \text{drr}(A) \leq 2 \text{rc}(A) \). Using Lemma 2.5(4) at the first step, using Lemma 2.5(2) and Remark 2.12 at the third step, and using Theorem 6.15 of [2] at the last step, we get
\[ \text{drr}(A) \leq \lim_{n \rightarrow \infty} \inf \text{drr}(A_n) = \lim_{n \rightarrow \infty} \inf \text{drr} \left( M_{r(n)} (C(X_n) \oplus C(X_n)) \right) \]
\[ \leq \lim_{n \rightarrow \infty} \inf \frac{\dim(X_n)}{r(n)} = \lim_{n \rightarrow \infty} \frac{2s(n)}{r(n)} = 2 \text{rc}(A), \]

as desired.

Corollary 5.2 provides more examples to feed into Theorem 5.4.

Corollary 5.6. For every \( r \in [0, \infty) \) and for every compact metric space \( X \), there exists a simple unital AH algebra such that \( \text{rc}(C(X) \otimes A) = \text{rc}(A) = r \).

6. Open problem

The most obvious problem is whether the right-hand side of inequality (1.1) holds for any stably finite unital C*-algebra \( A \) and any compact metric space \( X \).

Question 6.1. Let \( A \) be a unital stably finite C*-algebra and let \( X \) be a compact metric space. Does it follow that
\[ \text{rc} (C(X) \otimes A) \leq \text{rc}(A) + \frac{1}{2} \dim(X) + 1? \]
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