SYMPLECTIC 4-MANIFOLDS WITH ARBITRARY FUNDAMENTAL GROUP NEAR THE BOGOMOLOV-MIYAOKA-YAU LINE

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ABSTRACT. In this paper we construct a family of symplectic 4–manifolds with positive signature for any given fundamental group \( G \) that approaches the BMY line. The family is used to show that one cannot hope to do better than than the BMY inequality in finding a lower bound for the function \( f = \chi + b_2 \sigma \) on the class of all minimal symplectic 4-manifolds with a given fundamental group.

1. Introduction

Let \( \chi(S) \) and \( \sigma(S) \) denote the Euler characteristic and signature of a 4-manifold respectively. Minimal complex surfaces \( S \) of general type satisfy \( c_2^1(S) > 0, \chi(S) > 0 \) and

\[
2\chi_h(S) - 6 \leq c_2^1(S) \leq 9\chi_h(S)
\]

where \( c_2^1(S) = 2\chi(S) + 3\sigma(S) \) and \( \chi_h(S) = \frac{1}{4}(\chi(S) + \sigma(S)) \). The second inequality is usually referred to as the Bogomolov-Miyaoka-Yau inequality. Finding symplectic (or Kähler) 4-manifolds on or near the BMY line has a long and interesting history (c.f. [8], [2], [3], [9], [5], [10], [11]). In particular this means looking for symplectic 4-manifolds with positive signature.

All known examples of symplectic 4–manifolds on the BMY line have large fundamental groups. In fact, if \( S \) is a complex surface differing from \( \mathbb{CP}^2 \) (the only known simply-connected example on the BMY line), the equality \( c_2^1(S) = 9\chi_h(S) \) holds if and only if the unit disk \( D^4 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 \leq 1 \} \) covers \( S \) [6], [13], [7]. This implies for such \( S \) that \( |\pi_1(S)| = \infty \). The goal has generally been to produce examples that fill in the geography with respect to \( (c_2^1, \chi_h) \). In this paper we are interested what can be said for a given fundamental group.

Stipsicz constructed simply connected symplectic 4–manifolds \( C_n \) for which \( c_2^1(C_n)/\chi_h(C_n) \to 9 \) as \( n \to \infty \) in [11]. Our main theorem generalizes this result to any fundamental group.

**Theorem 1.** Let \( G \) have a presentation with \( g \) generators \( x_1, \cdots, x_g \) and \( r \) relations \( w_1, \cdots, w_r \). For each integer \( n > 1 \), there exists a symplectic
4–manifold \(M(G, n)\) with fundamental group \(G\) with Euler characteristic
\[
\chi(M(G, n)) = 75n^2 + 256n + 130 + 12(g + r + 1),
\]
and signature
\[
\sigma(M(G, n)) = 25n^2 - 68n - 78 - 8(g + r + 1).
\]

Our interest in this question developed while investigating pairs \((a, b)\) \(\in \mathbb{R}^2\) for which the function \(f = a\chi + b\sigma\) has a lower bound on the class of symplectic manifolds with a given fundamental group (see [1]). In that article we considered the following.

Fix a finitely presented group \(G\) and let \(\mathfrak{M}\) denote either the class \(\mathfrak{M}(G)\) of closed symplectic 4-manifolds with fundamental group \(G\) or the class \(\mathfrak{M}^{\text{min}}(G)\) of minimal, closed symplectic 4-manifolds with fundamental group \(G\).

For \(b \in \mathbb{R}\), define \(f_{\mathfrak{M}}(b) \in \mathbb{R} \cup \{-\infty\}\) to be the infimum
\[
f_{\mathfrak{M}}(b) = \inf_{M \in \mathfrak{M}} \{\chi(M) + b\sigma(M)\}.
\]
(In [1] we considered the infimum \(f_{\mathfrak{M}}(a, b)\) of \(a\chi + b\sigma\) on \(\mathfrak{M}\) and showed that if \(a \leq 0\) the infimum is \(-\infty\). Thus we restrict to \(f_{\mathfrak{M}}(1, b)\), which we more compactly denote by \(f_{\mathfrak{M}}(b)\) in the present article.)

We showed in [1] that the set
\[
D_{\mathfrak{M}} = \{b \mid f_{\mathfrak{M}}(b) \neq -\infty\}
\]
(the domain of \(f_{\mathfrak{M}}\)) is an interval satisfying
\[
[-1, 1] \subset D_{\mathfrak{M}(G)} \subset (-\infty, 1] \text{ and } [-1, \frac{3}{2}] \subset D_{\mathfrak{M}^{\text{min}}(G)} \subset (-\infty, \frac{3}{2}].
\]
The upper bounds are sharp; in fact \(1 \in D_{\mathfrak{M}(G)}\) and \(\frac{3}{2} \in D_{\mathfrak{M}^{\text{min}}(G)}\).

We are interested in the value of the left endpoint \(e_G\) of \(D_{\mathfrak{M}(G)}\), which is an intriguing invariant of a group \(G\). (It may or may not be contained in \(D_{\mathfrak{M}(G)}\).) Since \(e_G \leq -1\), a straightforward argument shows that \(e_G\) is also the left endpoint of \(D_{\mathfrak{M}^{\text{min}}(G)}\).

In [1] we observed that the results of Stipcitz gives a better lower bound (than \(-\infty\)) when \(G\) is the trivial group, and so a consequence of the result of this article is an extension to all \(G\). In fact Theorem 1 easily implies the following corollary.

**Corollary 2.** For any finitely presented group \(G\),
\[
D_{\mathfrak{M}(G)} \subset [-3, 1] \text{ and } D_{\mathfrak{M}^{\text{min}}(G)} \subset [-3, \frac{3}{2}].
\]

The BMY inequality \(c_1^2 \leq 9\chi_h\) is equivalent to \(f_{\mathfrak{M}^{\text{min}}(G)}(-3) \geq 0\) provided \(G\) is not a surface group. Hence the BMY conjecture and Corollary 2 together imply that \(e_G = 3\). Thus, a weaker form of the BMY conjecture could be stated as follows.

**Conjecture 3** (Weak BMY Conjecture). For each finitely presented group \(G\), \(e_G = -3\).
2. The construction

We use the following notation. If $X$ and $Y$ are symplectic 4-manifolds containing symplectic surfaces $F_X \subset X$ and $F_Y \subset Y$ such that $F_X^2 + F_Y^2 = 0$, then the symplectic sum ([4]) of $X$ and $Y$ along $F_X$ and $F_Y$ will be denoted by

$$X \#_{F_X, F_Y} Y.$$ 

Recall that topologically $X \#_{F_X, F_Y} Y$ is obtained by removing tubular neighborhoods of $F_X$ and $F_Y$ and identifying the resulting boundaries in a fiber-preserving way. Moreover, if $E_X \subset X$ (resp. $E_Y \subset Y$) is a symplectic surface intersecting $F_X$ once transversally (resp. intersecting $F_Y$ transversally), then the fiber sum can be constructed so that (the connected sum) $E_X \# E_Y$ is a symplectic surface in $X \#_{F_X, F_Y} Y$.

2.1. The first piece: symplectic manifolds with given fundamental group. The following theorem was proven in [1].

**Theorem 4.** Let $G$ have a presentation with $g$ generators $x_1, \cdots, x_g$ and $r$ relations $w_1, \cdots, w_r$. Then there exists a minimal symplectic 4-manifold $M(G)$ with $\pi_1(M(G)) \cong G$, Euler characteristic $\chi(M(G)) = 12(g + r + 1)$, and signature $\sigma(M(G)) = -8(g + r + 1)$.

We will use the following observation. The manifold $M(G)$ constructed in Theorem 4 is obtained by taking fiber sums of a certain base manifold with $g + r + 1$ copies of the basic elliptic surface $E(1)$. Since $E(1)$ admits a singular fibration with symplectic generic fibers and 6 cusp fibers (which are simply connected), so does $E(1) - F$, where $F$ denotes the generic fiber in $E(1)$ along which the fiber sum giving $M$ is constructed. Thus each $M(G)$ contains a symplectic torus $T_0$ such that the induced homomorphism $\pi_1(T_0) \to \pi_1(M(G))$ is trivial.

2.2. The second piece: symplectic manifolds near the BMY line. In [11], Stipsicz proved the following theorem.

**Proposition 5** (Stipsicz). For each non-negative integer $n$, there exists a symplectic 4–manifold $X(n)$ which admits a genus-$15n+1$ Lefschetz fibration with a section $T_{n+2}$ of genus $(n+2)$ and self-intersection $-(n+1)$. Furthermore, $X(n)$ can be equipped with a symplectic structure such that $T_{n+2}$ is a symplectic submanifold. The projection map $X(n) \to T_{n+2}$ induces an isomorphism on fundamental groups. The Euler characteristic of $X(n)$ is $\chi(X(n)) = 75n^2 + 180n + 12$ and the signature is $\sigma(X(n)) = 25n^2 - 60n - 8$.

Denote by $F_{15n+1} \subset X(n)$ a fixed generic fiber of $X(n)$. This is a symplectic surface with trivial normal bundle.

2.3. The third piece: a simply connected manifold. Gompf constructs a symplectic 4-manifold $S_{1,1}$ in [4, Lemma 5.5] which contains a disjoint pair $T, F$ of symplectically embedded surfaces $T$ of genus one and $F$ of genus two,
with trivial normal bundles such that $S_{1,1} - (T \cup F)$ is simply connected. Thus the symplectic sum $A$ of two copies $S_{1,1}$ along the genus two surfaces

$$A = S_{1,1} \#_{F,F} S_{1,1}$$

contains a pair of disjointly embedded symplectic tori $T_1 \cup T_2 \subset A$ with trivial normal bundles so that the complement $A - (T_1 \cup T_2)$ is simply connected. Since $S_{1,1}$ has Euler characteristic 23 and signature $-15$, $\chi(A) = 50$ and $\sigma(A) = -30$.

The manifold $A$ has a useful property, whose proof is a simple application of the Seifert-Van Kampen theorem.

**Proposition 6.** Suppose $B$ and $C$ are symplectic 4-manifolds containing symplectic tori $i_B : T_B \subset B$ and $i_C : T_C \subset C$ with trivial normal bundles.

Let $D = B \#_{T_B,T_1} A \#_{T_2,T_C} C$ be the fiber sum of $B$, $A$, and $C$. Then

$$\pi_1(D) = (\pi_1(B) / N((i_B)_*(\pi_1(T_B)))) \ast (\pi_1(C) / N((i_C)_*(\pi_1(T_C))))$$

where $\ast$ denotes free product and $N(H)$ denotes the normal closure of a subgroup $H$. \hfill $\square$

2.4. The fourth piece: “Elbows”. Let $T$ be a torus and $\{a, b\}$ a pair of smoothly embedded loops forming a symplectic basis of $\pi_1 T$. Let $\varphi : T \to T$ be the Dehn twist around $a$. The mapping torus $Y_\varphi$ fibers over $S^1$ with fiber $T$. Let $t_1 : S^1 \to Y_\varphi$ denote a section. Taking a product of $Y_\varphi$ with $S^1$ yields a symplectic 4-manifold $Y_\varphi \times S^1$ (this is just Thurston’s manifold from [12]) which fibers over a torus with symplectic torus fibers. Moreover, the symplectic structure can be chosen so that the section $t_1 \times \mathrm{id} : S^1 \times S^1 \to Y_\varphi \times S^1$ is symplectic. Denote by $s_1 : S^1 \to \{p\} \times S^1 \subset Y_\varphi \times S^1$ the loop representing the second factor.

Note that $Y_\varphi \times S^1$ contains a torus $T' = b \times s_1$, where $b$ is the curve described above in the fiber of $Y_\varphi$. The torus $T'$ is homologically non-trivial by the Kunneth theorem, since $b$ is non-trivial in $H_1(Y_\varphi)$, and is Lagrangian with respect to the symplectic structure on $Y_\varphi \times S^1$. Thus the symplectic structure on $Y_\varphi \times S^1$ can be perturbed slightly to make $T'$ symplectic. Note moreover that $T'$ is disjoint from the section $t_1 \times s_1 : S^1 \times S^1 \to Y_\varphi \times S^1$ since we can assume that $t_1$ intersects the fiber containing $b$ in a point which does not lie on $b$. The tubular neighborhood of $T'$ in $Y_\varphi \times S^1$ is trivial since $b$ can isotoped off itself in a fiber of $Y_\varphi \to S^1$. Similarly the tubular neighborhood of the section $t_1 \times s_1$ is trivial since $t_1$ can be pushed off itself in $Y_\varphi$.

Define $Elb(n)$ to be the fiber sum $Elb(n) = (Y_\varphi \times S^1) \#_{T,T^2} (T^2 \times \Sigma_{n-1})$. The fiber sum can be carried out so that the sections of $Y_\varphi \times S^1 \to S^1 \times S^1$ and $T^2 \times \Sigma_{n-1} \to \Sigma_{n-1}$ yield a symplectic section of the resulting fibration $Elb(n) \to \Sigma_n$. Thus $Elb(n)$ contains a disjoint pair of symplectic surfaces with trivial normal bundles, a torus $T' = b \times s_1$ and a genus $n$ surface, the image of the section, which we denote by $D_n$. 
Letting $t_2, s_2, \cdots, t_n, s_n$ denote the generators of $\pi_1(\Sigma_{n-1})$, one computes
\[ \pi_1(Elb(n)) = \langle a, b, t_1, s_1, \ldots, t_n, s_n \mid a \text{ central}, [b, t_1] = a, [b, t_i] = 1 \text{ for } i > 1, [b, s_i] = 1 \text{ for all } i, \prod_{i=1}^{n}[t_i, s_i] = 1 \rangle. \]

The inclusion of $T'$ into $Elb(n)$ takes the generators of $\pi_1T'$ to $b$ and $s_1$, and the inclusion of $D_n$ takes the standard basis to $t_1, s_1, \ldots, t_n, s_n$. The Euler characteristic and signature of $Elb(n)$ both vanish.

The manifold $Elb(n) - D_n$ is a punctured torus fibration over $\Sigma_n$, and hence has a presentation with the same generators and all the same relations except that one no longer has $a$ commuting with $b$, i.e., $a$ commutes with all generators except $b$.

2.5. The fifth piece: an elliptic surface. We find a symplectically embedded surface $J$ of genus $n + 3$ and self-intersection $n + 1$ in the elliptic surface $E(n + 5)$ such that $E(n + 5) - J$ is simply connected as follows. Consider $n + 3$ copies of the generic fiber and one copy of the section in a fibration $E(n + 5) \rightarrow \mathbb{C}P^1$ with $6(n + 5)$ cusp fibers. The section and fibers are symplectic with regards to the symplectic structure on the elliptic fibration $E(n + 5)$. Resolve the $n + 3$ transverse double points (4) to get a symplectically embedded surface $J$ of genus $n + 3$ and self-intersection $n + 1$ (the fiber hits the section once and that section has self-intersection $-(n + 5)$). The complement $E(n + 5) - J$ is simply connected because $E(n + 5)$ has a simply connected fiber which intersects $J$ in one point: the normal circle of a tubular neighborhood of $J$ is nullhomotopic in $E(n + 5) - J$.

2.6. Putting the pieces together. We begin by a modification of Stipsicz’s construction. Let $Z(n)$ be the fiber sum of $Elb(15n + 1)$ and $X(n)$ along $D_{15n+1} \subset Elb(15n + 1)$ and the fiber $F_{15n+1}$ of the Lefschetz fibration $X(n) \rightarrow \Sigma_{n+2}$
\[ Z(n) = Elb(15n + 1)\#D_{15n+1}, F_{15n+1}, X(n). \]

The fiber sum can be constructed so that the fiber $T \subset Elb(15n + 1)$ and the section $T_{n+2} \subset X(n)$ add to yield a symplectic surface of genus $n + 3$, $K_{n+3} = T\#T_{n+2} \subset Z(n)$ [4]. The important property of $Z(n)$ is that it contains a symplectic torus $T'$, since $D_{15n+1}$ and $T'$ are disjoint.

The fundamental group of $Z(n)$ is easily computed, since $Elb(15n + 1) - D_{15n+1}$ is a fiber bundle with punctured torus fibers and $X(n) - F_{15n+1}$ is a Lefschetz fibration over a punctured genus $n + 2$ surface with at least one simply connected fiber. Using the Seifert-Van Kampen theorem and Novikov additivity one obtains the following.

Lemma 7. The fundamental group of $Z(n)$ is the free product of $\mathbb{Z}$ with generator $b$, and a genus $n + 2$ surface group generated by $x_i, y_i$:
\[ \pi_1(Z(n)) = \mathbb{Z}b \ast \langle x_i, y_i, i = 1, \cdots, n + 2 \mid \prod(x_i, y_i) = 1 \rangle. \]
Moreover, \( Z(n) \) contains a disjoint pair of symplectic surfaces, \( T' \cup K_{n+3} \subset Z(n) \) satisfying \([T']^2 = 0\), and \([K_{n+3}]^2 = -n - 1\). The induced homomorphism \( \pi_1(T') \rightarrow \pi_1(Z(n)) \) is the map
\[
\langle a, s_1 | [a, s_1] \rangle \rightarrow \pi_1(Z(n)) \quad a \mapsto a, s_1 \mapsto 1.
\]
The induced homomorphism \( \pi_1(K_{n+3}) \rightarrow \pi_1(Z(n)) \) is the map
\[
\langle a, b, x_1, y_1, \ldots, x_{n+2}, y_{n+2} | [a, b] \prod [x_i, y_i] = 1 \rangle \rightarrow \pi_1(Z(n))
\]
\[
a \mapsto a, b \mapsto 1, x_i \mapsto x_i, y_i \mapsto y_i.
\]
Moreover, \( \chi(Z(n)) = 75n^2 + 240n + 12 \) and \( \sigma(Z(n)) = 25n^2 - 60n - 8. \)

The symplectic sum of \( Z(n) \) with \( E(n+5) \) along \( J \), \( Z(n) \# K_{n+3}, J \cdot E(n+5) \) is a simply connected symplectic 4-manifold containing a torus \( T_1 \) with trivial normal bundle and appropriate Euler characteristic and signature. We take symplectic sum of this manifold with \( A \) to obtain an example with a torus whose complement is simply connected.

Define \( W(n) \) to be the symplectic sum
\[
W(n) = A \#_{T_1, T'} Z(n) \#_{K_{n+3}, J} E(n+5).
\]
Then since \( \pi_1(A - (T_2 \cup T_2)) = 1 \), the following proposition follows straightforwardly.

**Proposition 8.** The symplectic manifold \( W(n) \) is simply connected and contains a symplectic torus \( T_2 \subset W(n) \) with trivial normal bundle so that \( \pi_1(W(n) - T_2) = 1 \). It has Euler characteristic \( \chi(W(n)) = 75n^2 + 256n + 130 \) and signature \( \sigma(W(n)) = 25n^2 - 68n - 78. \)

We can now prove Theorem 1.

**Proof of Theorem 1.** The symplectic sum
\[
M(G, n) = M(G) \#_{T_0, T_2} W(n)
\]
has fundamental group \( G \) by Proposition 6. The calculations of \( \chi(M(G, n)) \) and \( \sigma(M(G, n)) \) are routine.

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