W-ALGEBRAS RELATED TO PARAFERMION ALGEBRAS

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Abstract. We study a W-algebra of central charge \(2(k - 1)/(k + 2)\), \(k = 2, 3, \ldots\) contained in the commutant of a Heisenberg algebra in a simple affine vertex operator algebra \(L(k, 0)\) of type \(A_1^{(1)}\) with level \(k\). We calculate the operator product expansions of the W-algebra. We also calculate some singular vectors in the case \(k \leq 6\) and determine the irreducible modules and Zhu’s algebra. Furthermore, the rationality and the \(C_2\)-cofiniteness are verified for such \(k\).

1. Introduction

A Virasoro field and its finitely many primary fields generate a W-algebra of various kinds. Those W-algebras are important in the study of vertex operator algebras, for they provide many interesting examples. In [15], a possible structure of a W-algebra \(W(2, 3, 4, 5)\) with primary fields of conformal weight 3, 4 and 5 was discussed. Such an algebra was constructed in the commutant of a Heisenberg algebra in a Weyl module \(V(k, 0)\) for an affine Lie algebra \(\widehat{sl}_2\) of type \(A_1^{(1)}\) with level \(k\) in [1]. Our main concern is the commutant \(K_0\) of a Heisenberg algebra in a simple quotient \(L(k, 0)\) of \(V(k, 0)\), where \(k\) is an integer greater than 1. The commutant \(K_0\), including the characters of its irreducible modules, was studied in [2] (see [14] also). In this paper, we study \(K_0\) from a point of view of vertex operator algebra. The central charge of \(K_0\) is \(2(k - 1)/(k + 2)\), which coincides with the central charge of the parafermion algebra. We refer the reader to [7] for the relationship between \(K_0\) and the parafermion algebra.

It is also known that \(K_0\) appears as the commutant of a certain subalgebra in the vertex operator algebra \(V_{\sqrt{2}A_{k-1}}\) associated with \(\sqrt{2}A_{k-1}\), where \(\sqrt{2}A_{k-1}\) denotes \(\sqrt{2}\) times an ordinary root lattice of type \(A_{k-1}\) [18]. Such a realization of \(K_0\) leads to a natural study of \(V_{\sqrt{2}A_{k-1}}\) as a module for \(K_0\). The \(K_0\)-module structure of \(V_{\sqrt{2}A_{k-1}}\) for some special \(k\) is expected to play an important role in a better understanding of the moonshine vertex operator algebra \(V^\natural\) [12].

It is widely believed that \(K_0\) is a rational and \(C_2\)-cofinite vertex operator algebra. It is also anticipated that \(K_0\) has exactly \(k(k + 1)/2\) inequivalent irreducible modules (see Conjecture [4,6]). In this paper, we treat these subjects. The key of our arguments here is a detailed analysis of some singular vectors. Unfortunately, we do not succeed in describing those singular vectors explicitly for a general \(k\). Therefore, we restrict ourselves to the case \(k \leq 6\). We determine Zhu’s algebra and classify the irreducible modules of \(K_0\) for \(k \leq 6\). Moreover, we show that \(K_0\) is rational and \(C_2\)-cofinite for such \(k\). In the case \(k \geq 3\), we show that \(K_0\) is generated by a primary field of weight 3.

The organization of the paper is as follows. In Section 2, we introduce the conformal vector \(\omega\) of central charge \(2(k - 1)/(k + 2)\) and Virasoro primary vectors \(W^3, W^4\) and

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$W^5$ of weight 3, 4 and 5, respectively in the commutant $N_0$ of a Heisenberg algebra in the Weyl module $V(k, 0)$ for $\mathfrak{sl}_2$ with level $k$, where $k$ is an integer greater than 1. Such a vector $W^i$, $i = 3, 4, 5$ is unique up to a scalar multiple. Let $W^i_\gamma$ be a component operator, that is, the coefficient of $x^{-n-1}$ in the vertex operator associated with $W^i$. The vectors $W^i_\gamma W^j$, $3 \leq i \leq j \leq 5$, $0 \leq n \leq i+j-1$ are known in [1]. We compute these vectors in the Weyl module $V(k, 0)$ and express them as linear combinations of vectors of normal form (see (2.12)). The computation has been done by a computer algebra system Risa/Asir. The results can be found in Appendix B. In the computation of the vectors $W^i_\gamma W^j$, we do not assume that $k$ is an integer greater than 1. Thus in Appendix B we can think of the parameter $k$ as a formal variable. Using the explicit expression of $W^i_\gamma W^j$ as a linear combination of vectors of normal form, we study a subalgebra $\tilde{W}$ of the commutant $N_0$ generated by $\omega$, $W^3$, $W^4$ and $W^5$. It turns out that $\tilde{W}$ is in fact generated by $W^3$ if $k \geq 3$. As a consequence, the automorphism group of $\tilde{W}$ is of order 2 and it is generated by an automorphism which maps $W^3$ to its negative if $k \geq 3$. We also show that Zhu’s algebra of $\tilde{W}$ is commutative. It is known that $\tilde{W}$ has two (resp. four) linearly independent singular vectors of weight 8 (resp. 9) [1]. We use these singular vectors to determine Zhu’s algebra of $K_0$ for $k = 5, 6$ in Section 5. In addition to them, a weight 10 singular vector is necessary to establish the $C_2$-cofiniteness for $k = 5, 6$ in Section 5.

Since $k$ is an integer greater than 1, the vertex operator algebra $V(k, 0)$ possesses a unique maximal ideal $J$, which is generated by $e(-1)^{k+1}1$. In Section 3 we study the commutant $K_0$ of a Heisenberg algebra in the quotient vertex operator algebra $L(k, 0) = V(k, 0)/J$. We denote the image of $\tilde{W}$ in $L(k, 0)$ by $\mathcal{W}$. Then $\mathcal{W}$ is a subalgebra of $K_0$. The ideal $J$ is not contained in the commutant $N_0$. It is expected that a unique maximal ideal $J \cap N_0$ of $N_0$ is generated by a weight $k + 1$ vector $u^0 = f(0)^{k+1}e(-1)^{k+1}1$ (see Lemma 3.1, Conjecture 3.3).

In Section 4, we embed $L(k, 0)$ into a vertex operator algebra $V_L$ associated with a lattice $L$ of type $A_{2k}^{\pm}$. This is accomplished by the use of level-rank duality [7, Chapter 14]. Let $V^\text{aff}$ be a subalgebra of $V_L$ obtained by the embedding. Then $V^\text{aff} \cong L(k, 0)$. There is a sublattice $L'$ of $L$ isomorphic to $\sqrt{2}A_{k-1}$ such that the vertex operator algebra $V_{L'}$ associated with $L'$ is the commutant of the vertex operator algebra $V_{Z\gamma}$ associated with a rank one lattice $Z\gamma$ in $V_L$. We have $V_L \supset V^\text{aff} \supset V_{Z\gamma}$ and $K_0 \cong V^\text{aff} \cap V_{L'}$. That is, $K_0$ is isomorphic to the commutant of $V_{Z\gamma}$ in $V^\text{aff}$. This consideration has some advantages. For instance, using the representation theory of the vertex operator algebra $V_{Z\gamma}$, we construct a certain family of irreducible $K_0$-modules inside $V_{L^\perp}$ and study their properties, where $L^\perp$ is the dual lattice of $L$.

The singular vectors of weight at most 10 in $\mathcal{W}$ are calculated explicitly for any $k$. However, this is not the case for $u^0$. We can describe $u^0$ as a linear combination of vectors of normal form only for a given small $k$. For this reason, we deal with only the case $k \leq 6$ in Section 5. If $k = 2$, 3 or 4, then $\mathcal{W}$ is degenerate. In fact, it turns out that $u^0$ is a scalar multiple of $W^{k+1}$ for $k = 2, 3, 4$. In such a case, $\mathcal{W}$ is isomorphic to a well-known vertex operator algebra. Thus the main part of Section 5 is devoted to the case $k = 5, 6$. We show that $\mathcal{W} = K_0$ and classify its irreducible modules. Moreover, we show that $K_0$ is rational and $C_2$-cofinite. We note that $K_0$ is related to a 2A, 3A, 4A, 5A or 6A element of the Monster simple group according as $k = 2, 3, 4, 5$ or 6 (see [19].
Section 3, Appendix B], [21, Section 4]). In fact, this is part of the motivation of our work.

The argument heavily depends on singular vectors \( v^0, v^1 \) and \( v^2 \) of weight 8, 9 and 10, respectively in \( \tilde{W} \) and on singular vectors \( u^r = (W^3)^r u^0 \) of weight \( k + 1 + r \), \( r = 0, 1, 2, 3 \) in \( W \). It seems that we can take \( W^3 v^0 \) and \( (W^3)^2 v^0 \) in place of \( v^1 \) and \( v^2 \), respectively. However, we do not verify it. The importance of \( u^0 \) is clear from the degenerate case, namely, the case \( k = 2, 3, 4 \), for \( u^0 \) is a scalar multiple of \( W^3, W^4 \) or \( W^5 \) in such a case. It would be difficult to express \( u^r \), \( r = 0, 1, 2, 3 \) in terms of \( \omega, W^3, W^4 \) and \( W^5 \) for an arbitrary \( k \). We should take a different approach for a general case.

Our notation is fairly standard [12, 20]. Let \( V \) be a vertex operator algebra and \((M, Y_M)\) be its module. Then \( Y_M(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \) is the vertex operator associated with \( v \in V \). The linear operator \( v_n \) on \( M \) is called a component operator. For a subalgebra \( U \) of \( V \) and a subset \( S \) of \( M \), let \( U \cdot S = \text{span}\{u_n w \mid u \in U, w \in S, n \in \mathbb{Z}\} \), which is the \( U \)-submodule of \( M \) generated by \( S \).

Part of the results in this paper was announced in [6]. We remark that \( N_0 \) (resp. \( K_0 \)) is denoted by \( \tilde{W} \) (resp. \( W \)) in [6]. In this paper, we distinguish \( N_0 \) and \( \tilde{W} \) (resp. \( K_0 \) and \( W \)) clearly to avoid confusion.

### 2. \( \tilde{W} \) and its singular vectors

Let \( \{h, e, f\} \) be a standard Chevalley basis of \( sl_2 \). Thus \([h, e] = 2e, [h, f] = -2f, [e, f] = h\) for the bracket and \( \langle h, h \rangle = 2, \langle e, f \rangle = 1, \langle h, e \rangle = \langle h, f \rangle = \langle e, e \rangle = \langle e, f \rangle = 0 \) for the normalized Killing form. We fix an integer \( k \geq 2 \). Let \( V(k, 0) = V_{sl_2}(k, 0) \) be a Weyl module for the affine Lie algebra \( \hat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \) with level \( k \), that is, a Verma module for \( \tilde{sl}_2 \) with level \( k \) and highest weight 0. Let \( \mathbb{1} \) be its canonical highest weight vector, which is called the vacuum vector. Then \( sl_2 \otimes \mathbb{C}[t] \) acts as 0 and \( C \) acts as \( k \) on \( \mathbb{1} \). We denote by \( h(n), e(n) \) and \( f(n) \) the operators on \( V(k, 0) \) induced by the action of \( h \otimes t^n, e \otimes t^n \) and \( f \otimes t^n \), respectively. Thus \( h(n) \mathbb{1} = e(n) \mathbb{1} = f(n) \mathbb{1} = 0 \) for \( n \geq 0 \) and

\[
[a(m), b(n)] = [a, b](m + n) + m(a, b)\delta_{m+n,0}k
\]

for \( a, b \in \{h, e, f\} \). The elements

\[
h(-i_1) \cdots h(-i_p) e(-j_1) \cdots e(-j_q) f(-m_1) \cdots f(-m_r) \mathbb{1}, \quad \quad (2.2)
\]

\( i_1 \geq \cdots \geq i_p \geq 1, j_1 \geq \cdots \geq j_q \geq 1, m_1 \geq \cdots \geq m_r \geq 1 \) form a basis of \( V(k, 0) \).

Let \( a(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1} \) for \( a \in \{h, e, f\} \) and

\[
a(x)_n b(x) = \text{Res}_{x_1} \left( (x_1 - x)^n a(x_1) b(x) - (-x + x_1)^n b(x) a(x_1) \right).
\]

Then the vertex operator \( Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \in (\text{End} V(k, 0))[\![x, x^{-1}]\!] \) associated with \( v = a^1(n_1) \cdots a^r(n_r) \mathbb{1} \) is given by

\[
Y(a^1(n_1) \cdots a^r(n_r) \mathbb{1}, x) = a^1(x)_{n_1} \cdots a^r(x)_{n_r} \quad \quad (2.3)
\]
for $a^i \in \{h, e, f\}$ and $n_i \in \mathbb{Z}$, where 1 denotes the identity operator. Set
\[
\omega_{\text{aff}} = \frac{1}{2(k+2)} \left( \frac{1}{2} h(-1)^2 \mathbb{1} + e(-1)f(-1)\mathbb{1} + f(-1)e(-1)\mathbb{1} \right)
\]
\[
= \frac{1}{2(k+2)} \left( -h(-2)\mathbb{1} + \frac{1}{2} h(-1)^2 \mathbb{1} + 2e(-1)f(-1)\mathbb{1} \right).
\]
Then $(V(k,0), Y, \mathbb{1}, \omega_{\text{aff}})$ is a vertex operator algebra with the conformal vector $\omega_{\text{aff}}$, whose central charge is $3k/(k+2)$ [13] (see [20], Section 6.2) also. The vector of the form (2.2) is an eigenvector for $(\omega_{\text{aff}})_i$ with eigenvalue $i_1 + \cdots + i_p + j_1 + \cdots + j_q + m_1 + \cdots + m_r$. The eigenvalue is called the weight of the vector in $V(k,0)$. We denote the weight of $v$ by $\text{wt} v$.

We consider two subalgebras $\tilde{\mathfrak{h}} = \mathbb{C} h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} C$ and $\tilde{\mathfrak{h}}_s = (\oplus_{n \neq 0} \mathbb{C} h \otimes t^n) \oplus \mathbb{C} C$ of the Lie algebra $\tilde{s}_2$. Let $V_{\tilde{\mathfrak{h}}}(k,0)$ be the subalgebra of $V_{\tilde{s}_2}(k,0)$ with basis $h(-i_1) \cdots h(-i_p)\mathbb{1}$, $i_1 \geq \cdots \geq i_p \geq 1$. That is, $V_{\tilde{\mathfrak{h}}}(k,0)$ is a vertex operator algebra associated with the Heisenberg algebra $\tilde{\mathfrak{h}}_s$ of level $k$. The conformal vector of $V_{\tilde{\mathfrak{h}}}(k,0)$ is given by
\[
\omega_{\gamma} = \frac{1}{4k} h(-1)^2 \mathbb{1}.
\]
Its central charge is 1.

Now, $V(k,0)$ is completely reducible as a $V_{\tilde{\mathfrak{h}}}(k,0)$-module. More precisely,
\[
V(k,0) = \bigoplus_{\lambda} M_{\tilde{\mathfrak{h}}}(k, \lambda) \otimes N_{\lambda}.
\]
Here $M_{\tilde{\mathfrak{h}}}(k, \lambda)$ denotes an irreducible highest weight module for $\tilde{\mathfrak{h}}$ with a highest weight vector $v_{\lambda}$ such that $h(0)v_{\lambda} = \lambda v_{\lambda}$ and
\[
N_{\lambda} = \{v \in V(k,0) \mid h(m)v = \lambda \delta_{m,0}v \text{ for } m \geq 0\}.
\]
The index $\lambda$ runs over all even integers, since the eigenvalues of $h(0)$ in $V(k,0)$ are even integers. In fact, $h(0)$ acts as $2(q-r)$ on the vector of the form (2.2).

In the case $\lambda = 0$, $M_{\tilde{\mathfrak{h}}}(k,0)$ is identical with $V_{\tilde{\mathfrak{h}}}(k,0)$ and $N_0$ is the commutant [13, Theorem 5.1] of $V_{\tilde{\mathfrak{h}}}(k,0)$ in $V(k,0)$. The commutant $N_0$ is a vertex operator algebra with the conformal vector $\omega = \omega_{\text{aff}} - \omega_{\gamma}$;
\[
\omega = \frac{1}{2k(k+2)} \left( -kh(-2)\mathbb{1} - h(-1)^2 \mathbb{1} + 2ke(-1)f(-1)\mathbb{1} \right),
\]
whose central charge is $3k/(k+2) - 1 = 2(k-1)/(k+2)$. Since the conformal vector of $V_{\tilde{\mathfrak{h}}}(k,0)$ is $\omega_{\gamma}$, we have $N_0 = \{v \in V(k,0) \mid (\omega_{\gamma})_0v = 0\}$ by [13, Theorem 5.2]. It is also the commutant of $\text{Vir}(\omega_{\gamma})$ in $V(k,0)$, where $\text{Vir}(\omega_{\gamma})$ is the subalgebra of $V(k,0)$ generated by $\omega_{\gamma}$. Since $\omega_1v = (\omega_{\text{aff}})_1v$ for $v \in N_0$, the weight of $v$ in $N_0$ agrees with that in $V(k,0)$.

By a direct computation, we see that the dimension of the weight $i$ subspace $(N_0)_i$ of $N_0$ is 2, 4 and 6 for $i = 3, 4$ and 5, respectively. Furthermore, we can verify that there is up to a scalar multiple, a unique Virasoro primary vector $W^i$ in $(N_0)_i$ for $i = 3, 4, 5$. Here a Virasoro primary vector of weight $i$ means that $\omega_3 W^i = \omega_3 W^i = 0$ and $\omega_1 W^i = iW^i$. In this paper, we take
\[
W^3 = k^2 h(-3)\mathbb{1} + 3kh(-2)h(-1)\mathbb{1} + 2h(-1)^3 \mathbb{1} - 6kh(-1)e(-1)f(-1)\mathbb{1}
+ 3k^2 e(-2)f(-1)\mathbb{1} - 3k^2 e(-1)f(-2)\mathbb{1},
\]
As to $W^4$ and $W^5$, see Appendix A. We denote by $\tilde{W}$ the subalgebra of $N_0$ generated by $\omega$, $W^3$, $W^4$ and $W^5$. Actually, $\tilde{W}$ coincides with $W(2, 3, 4, 5)$ of [1].

**Remark 2.1.** Our $W^3$, $W^4$ and $W^5$ are scalar multiples of $W^3$, $W^4$ and $W^5$ in the notation of [1] Appendix A. In fact,

$$W^3 = \frac{1}{2}W_3,$$

$$W^4 = \frac{16k + 17}{144k(2k + 3)}W_4,$$

$$W^5 = -\frac{64k + 107}{3456k^2(2k + 3)(3k + 4)}W_5.$$

Notice that $h$, $e$, $f$ and $\omega$ are denoted by $J^0$, $J^+$, $J^-$ and $L$, respectively in [1].

Recall that $W^i_n$ is a component operator, that is, the coefficient of $x^{-n-1}$ in the vertex operator associated with $W^i$. The computation of $W^i_nW^j$, $3 \leq i \leq j \leq 5$, $0 \leq n \leq i + j - 1$ has been done in [1]. In this paper, we compute the commutation relation (2.1) of the operators $W^i_n$ associated with $W$ in these four elements. We also notice that $\omega$ is treated as a formal variable. That is, we do not assume that $k$ is an integer greater than 1 in the computation. Hence in Appendix B, $k$ can be considered as a formal variable.

Our computation of $W^i_nW^j$’s in $V(k, 0)$ has been done by a computer algebra system Risa/Asir. During the computation, we only use the condition $a(n)\mathbb{1} = 0$ for $a \in \{h, e, f\}$, $n \geq 0$, the commutation relation (2.1) and the definition (2.3) of vertex operators on $V(k, 0)$. The parameter $k$ is treated as a formal variable. That is, $\tilde{W}$ is closed within these four elements. We also notice that

$$W^5W^3 = 12k^3(k - 2)(k - 1)(3k + 4)\mathbb{1},$$

$$W^4W^3 = 0,$$

$$W^3W^3 = 36k^3(k - 2)(k + 2)(3k + 4)\omega.$$
Remark 2.3. From $W_3^3W_3^3$, $W_3^3W_3^3$ and $W_3^3W_3^4$, we see that $\tilde{W}$ is generated by a single element $W_3^3$ if $k \geq 3$. It turns out that $W_3^3$, $W_4^4$ or $W_5^5$ is contained in a maximal ideal of $\tilde{W}$ if $k = 2$, 3 or 4 (see Section 5 for detail). These are the degenerate cases.

Let $\text{Vir}(\omega)$ be the subalgebra of $\tilde{W}$ generated by $\omega$. Each $W_i^iW_j^j$, $3 \leq i \leq j \leq 5$, $0 \leq n \leq i + j - 1$ is a linear combination of elements in $\text{Vir}(\omega)$, $\text{Vir}(\omega) \cdot W_p^p\mathbb{1}$ with $p < i + j$ and $m \leq -1$, and $\text{Vir}(\omega) \cdot W_r^rW_s^s\mathbb{1}$ with $r + s < i + j$, $r \leq s$ and $k, m, r \leq -1$, where $p, r, s \in \{3, 4, 5\}$ (see Appendix B). Note also that $\omega_iW_i^i = 0$ if $n \geq 2$, $\omega_iW_i^i = iW_i^i$ and $\omega_0W_i^i = W_i^i\mathbb{1}$ for $i = 3, 4, 5$. Hence, using basic formulas for a vertex operator algebra [20], (3.1.9), (3.1.12)

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (uv)_{m+n-i},$$

$$\sum_{i \geq 0} (-1)^i \left( \binom{m}{i} (u_{m-i}v_{n+i} - (-1)^m v_{m+n-i} u_i) \right).$$
we obtain the following lemma by induction.

Lemma 2.4. $\tilde{W}$ is spanned by the elements

$$\omega_{-i_1} \cdots \omega_{-i_s} W_3^3 \cdots W_3^3 W_3^4 \cdots W_3^4 W_5^5 \cdots W_5^5 W_5^5 \mathbb{1}$$

with $i_1 \geq \cdots \geq i_p \geq 1$, $j_1 \geq \cdots \geq j_q \geq 1$, $m_1 \geq \cdots \geq m_r \geq 1$ and $n_1 \geq \cdots \geq n_s \geq 1$.

An element of the form (2.12) is said to be of normal form. Another notation is more convenient on some occasion. Set $L(n) = \omega_{n+1}$, $W_3^3(n) = W_3^3_{n+2}$, $W_4^4(n) = W_4^4_{n+3}$ and $W_5^5(n) = W_5^5_{n+4}$. All of these operators are of weight $-n$. The spanning set of $\tilde{W}$ can also be described by

$$L(-i_1) \cdots L(-i_p) W_3^3(-j_1) \cdots W_3^3(-j_q) \cdot W_4^4(-m_1) \cdots W_4^4(-m_r) W_5^5(-n_1) \cdots W_5^5(-n_s) \mathbb{1}$$

with $i_1 \geq \cdots \geq i_p \geq 2$, $j_1 \geq \cdots \geq j_q \geq 3$, $m_1 \geq \cdots \geq m_r \geq 4$ and $n_1 \geq \cdots \geq n_s \geq 5$.

The weight of the vector (2.13) is

$$i_1 + \cdots + i_p + j_1 + \cdots + j_q + m_1 + \cdots + m_r + n_1 + \cdots + n_s.$$

A vector $u$ of a $\tilde{W}$-module is called a highest weight vector for $\tilde{W}$ with highest weight $(a_2, a_3, a_4, a_5)$ if $L(n)u = W_3^3(n)u = 0$ for $n \geq 1$, $L(0)u = a_2u$ and $W_3^3(0)u = a_3u$ for $i = 3, 4, 5$. By a similar argument as above, we see that the vectors

$$L(-i_1) \cdots L(-i_p) W_3^3(-j_1) \cdots W_3^3(-j_q) \cdot W_4^4(-m_1) \cdots W_4^4(-m_r) W_5^5(-n_1) \cdots W_5^5(-n_s)u$$

with $i_1 \geq \cdots \geq i_p \geq 1$, $j_1 \geq \cdots \geq j_q \geq 1$, $m_1 \geq \cdots \geq m_r \geq 1$ and $n_1 \geq \cdots \geq n_s \geq 1$ span the $\tilde{W}$-submodule $\tilde{W} \cdot u$ generated by such a highest weight vector $u$.

An automorphism of the Lie algebra $sl_2$ given by $h \mapsto -h$, $e \mapsto f$, $f \mapsto e$ lifts to an automorphism $\theta$ of the vertex operator algebra $V(k, 0)$ of order 2. The Virasoro element $\omega_7$ is invariant under $\theta$ by (2.4). Hence $\theta$ induces an automorphism of $N_0$. In fact, $\theta \omega = \omega$ and $\theta W^3 = -W^3$ by (2.6) and (2.7), respectively, and so $\theta W^4 = W^4$ and $\theta W^5 = -W^5$ by
Let $\theta v = (−1)^{q+\bar{s}}v$ for an element $v$ of the form (2.12). Let $W^\pm = \{ v \in W | \theta v = \pm v \}$.

**Theorem 2.5.** If $k \geq 3$, then the automorphism group $\text{Aut} \tilde{W}$ of $\tilde{W}$ is $\langle \theta \rangle$, a group of order 2 generated by $\theta$.

**Proof.** Let $\sigma \in \text{Aut} \tilde{W}$. Since any Virasoro primary vector of weight 3 in $N_0$ is a scalar multiple of $W^3$, we have $\sigma W^3 = \xi W^3$ for some scalar $\xi \neq 0$. Then $\sigma(W^3 W^3) = \xi^2 W^3 W^3$. Now, $W^3 W^3$ is a nonzero scalar multiple of the vacuum vector $1$ since we are assuming that $k \geq 3$. Thus $\xi^2 = 1$ and the assertion holds, for $\tilde{W}$ is generated by $W^3$.

We will consider Zhu’s algebra $A(\tilde{W})$ of $\tilde{W}$. Zhu’s algebra $A(V)$ of a vertex operator algebra $(V, Y, \mathbb{1}, \omega)$ introduced by Zhu [23] is very powerful for the study of irreducible $V$-modules. For $u, v \in V$ with $u$ being homogeneous, let

$$u * v = \sum_{i \geq 0} \binom{\text{wt}(u) + m}{i} u_{i-1} v,$$

$$u \circ v = \sum_{i \geq 0} \binom{\text{wt}(u)}{i} u_{i-2} v.$$  

(2.15)

We extend these two binary operations to arbitrary $u, v \in V$ by linearity. The subspace $O(V)$ spanned by all $u \circ v$ with $u, v \in V$ is a two-sided ideal with respect to $*$. We denote by $[v]$ the image of $v \in V$ in the quotient space $A(V) = V/O(V)$. Then $A(V)$ is an associative algebra with the identity $[1]$ by the binary operation $[u] * [v] = [u * v]$. We write $u \sim v$ if $[u] = [v]$.

We need some formulas. By [23] Lemma 2.1.2,

$$\sum_{i \geq 0} \binom{\text{wt}(u) + m}{i} u_{i-n-2} v \in O(V) \text{ for } n \geq m \geq 0.$$  

(2.16)

By (2.15), $u_{-1} v$ can be written as a linear combination of $u * v$ and $u_o v$, $n \geq 0$. Note that $\text{wt}(u_o v) = \text{wt} u + \text{wt} v - n - 1$, $n > 0$ is strictly smaller than $\text{wt}(u_{-1} v)$ for homogeneous elements $u, v \in V$. For $n \geq 0$, (2.16) implies that $[u_{-n-2} v]$ is a linear combination of $[u_{i-n-2} v]$, $i \geq 1$. As before, note that $\text{wt}(u_{i-n-2} v) < \text{wt}(u_{i-n-2} v)$ for $i \geq 1$.

Another useful formula [23] Lemma 2.1.3 is

$$u * v - v * u \sim \sum_{j \geq 0} \binom{\text{wt}(u) - 1}{j} u_j v.$$  

(2.17)

It follows from [22] (4.2), (4.3) that

$$[L(-n)v] = (-1)^n((n - 1) [\omega] * [v] + [L(0)v]) \text{ for } n \geq 1.$$  

(2.18)

If $v$ is homogeneous, then $[L(-1)v] = -(\text{wt} v)[v]$ and so [23] (1.2.17) implies that

$$[v_{-m-1} \mathbb{1}] = \frac{(-1)^m}{m!} \left( \prod_{i=0}^{m-1} (\text{wt}(v) + i) \right) [v] \text{ for } m \geq 0.$$  

(2.19)

Let $o(v) = v_{\text{wt}(v)-1}$ for a homogeneous element $v \in V$ and extend it to an arbitrary element by linearity. If $U = \oplus_{n=0}^{\infty} U(n)$ is an admissible $V$-module as in [9] with $U(0) \neq 0$, then $o(v)$ acts on its top level $U(0)$. Zhu’s theory [23] Theorems 2.1.2 and 2.2.2 can be summarized as follows.
(1) \( o(u)o(v) = o(u \ast v) \) as operators on \( U(0) \) and \( o(v) \) acts as 0 if \( v \in O(V) \). Hence \( U(0) \) is an \( A(V) \)-module, where \([v]\) acts as \( o(v) \) on \( U(0) \).

(2) \( U \mapsto U(0) \) is a bijection between the set of equivalence classes of irreducible admissible \( V \)-modules and the set of equivalence classes of irreducible \( A(V) \)-modules.

We now study Zhu’s algebra \( A(\tilde{W}) = \tilde{W}/O(\tilde{W}) \) of \( \tilde{W} \). First, we show that \( A(\tilde{W}) \) is generated by \([\omega] \), \([W^3] \), \([W^4] \), and \([W^5] \). For this purpose, take an element \( v \) of the form \((2.12)\). It is sufficient to confirm that \([v]\) is a linear combination of monomials in \([\omega] \), \([W^3] \), \([W^4] \), and \([W^5] \). We proceed by induction on weight. By \((2.18)\), we may assume that \([v]\) is a linear combination of the images in \([\omega] \), \([W^3] \), \([W^4] \), and \([W^5] \) as required. Actually, \((2.19)\) is useful in computing \([v]\).

Next, we show that \( A(\tilde{W}) \) is commutative. By a property of Zhu’s algebra, \([\omega] \) lies in the center of \( A(\tilde{W}) \) \[23\], Theorem 2.1.1\]. It follows from \((2.17)\) that

\[ W^3 \ast W^4 - W^4 \ast W^3 = W_0^3 W^4 + 2W_1^3 W^4 + W_2^3 W^4. \]

Using an explicit expression of \( W_n^3 W^4 \), \( n = 0, 1, 2 \) as a linear combination of vectors of normal form \((2.12)\) given in Appendix \[12\] we can describe its image \([W_n^3 W^4]\) in \( A(\tilde{W}) \) as a polynomial in \([\omega] \), \([W^3] \) and \([W^5] \). In fact, we can verify that

\[ [W^3] \ast [W^4] - [W^4] \ast [W^3] = [W_0^3 W^4] + 2[W_1^3 W^4] + [W_2^3 W^4] = 0. \]

Likewise, we have

\[ [W^3] \ast [W^5] - [W^5] \ast [W^3] = [W_0^3 W^5] + 2[W_1^3 W^5] + [W_2^3 W^5] = 0, \]

\[ [W^4] \ast [W^5] - [W^5] \ast [W^4] = [W_0^4 W^5] + 3[W_1^4 W^5] + 3[W_2^4 W^5] + [W_3^4 W^5] = 0. \]

Thus \( A(\tilde{W}) \) is commutative. We have obtained the following lemma.

**Lemma 2.6.** \( A(\tilde{W}) \) is commutative and it is generated by \([\omega] \), \([W^3] \), \([W^4] \) and \([W^5] \).

The above lemma implies that \( w_2 \mapsto [\omega] \), \( w_3 \mapsto [W^3] \), \( w_4 \mapsto [W^4] \), \( w_5 \mapsto [W^5] \) define a homomorphism \( \varphi \) of associative algebras from a polynomial algebra \( \mathbb{C}[w_2, w_3, w_4, w_5] \) of four variables \( w_2, w_3, w_4, w_5 \) onto \( A(\tilde{W}) \). In particular, \( A(\tilde{W}) \) is spanned by

\[ [\omega]^{rp} [W^3]^{rq} [W^4]^{rs} [W^5]^{ss}, \quad p, q, r, s \geq 0, \quad (2.20) \]

where \([u]^{rp}\) is a product of \( p \) copies of \([u]\) in \( A(\tilde{W}) \).

We will study linear relations among vectors of normal form \((2.12)\) of small weight. The generating function of the number of vectors of normal form with respect to weight is

\[ \frac{(1 - q)^4(1 - q^2)^3(1 - q^3)^2(1 - q^4)}{\prod_{n \geq 1}(1 - q^n)^4}. \]
The sum of its first several terms are
\[ 1 + q^2 + 2q^3 + 4q^4 + 6q^5 + 11q^6 + 16q^7 + 29q^8 + 44q^9 + 72q^{10} + \cdots. \]

We express all vectors of normal form (2.12) of weight at most 10 as linear combinations of the basis (2.2) of \( V(k, 0) \). By a direct calculation, we can verify that those vectors of normal form of weight at most 7 are all linearly independent. However, this is not the case if the weight is greater than 7 \([2, (2.1.9)]\). There are 29 vectors of normal form of weight 8, which span a subspace of dimension 27. Thus the dimension of the weight 8 subspace of \( \widetilde{W} \) is 27. If we eliminate \((W^3_2)^21\) and \(W_{-1}^3W_{-2}^41\), then the remaining 27 vectors form a basis of the weight 8 subspace of \( \widetilde{W} \). That is, there are two nontrivial linear relations in the weight 8 subspace of \( \widetilde{W} \). One involves \((W^3_2)^21\) and the other involves \(W_{-1}^3W_{-2}^41\). Such a nontrivial linear relation is called a null field \([1, 15]\). Recall that \( \widetilde{W}^\pm = \{v \in \widetilde{W} | \theta v = \pm v\} \). Among 29 vectors of normal form (2.12) of weight 8, we see that 17 are contained in \( \widetilde{W}^+ \) and the remaining 12 are contained in \( \widetilde{W}^- \). Note that \((W^3_2)^21 \in \widetilde{W}^+ \) and \(W_{-1}^3W_{-2}^41 \in \widetilde{W}^- \). Thus there is up to a scalar multiple, a unique nontrivial linear relation in each weight 8 subspace of \( \widetilde{W}^+ \) and \( \widetilde{W}^- \). Let \( v^0 \) be a nontrivial linear relation in the weight 8 subspace of \( \widetilde{W}^+ \). It is a linear combination of the 17 vectors of normal form of weight 8 in \( \widetilde{W}^+ \). An explicit form of such a linear combination can be found in Appendix 2.4. It is obtained by describing each of those 17 vectors as a linear combination of the basis (2.2). Actually, \( v^0 = 0 \) in \( V(k, 0) \).

We can express the image of each of those 17 vectors of normal form in Zhu’s algebra \( A(\widetilde{W}) \) as a linear combination of the elements of the form (2.20) by using a similar argument as in the proof of Lemma 2.6. Then the image \([v^0]\) of \( v^0 \) in \( A(\widetilde{W}) \) becomes a linear combination of the elements of the form (2.20). Replace \([\omega], [W^3], [W^4] \) and \([W^5]\) with \( w_2, w_3, w_4 \) and \( w_5 \) respectively in \([v^0]\). Then we obtain
\[
Q_0 = -8k^4(k + 2)^2(3k + 4)(4k - 1)(64k + 107)(k^2 + k + 1)w_2^2 \\
+ 4k^4(k + 2)^3(3k + 4)(64k + 107)(80k^2 + 30k + 61)w_3^2 \\
- 112k^4(k + 2)^4(3k + 4)(6k - 5)(64k + 107)w_4^2 \\
+ 2k(16k + 17)^2(2k^2 + k + 5)w_3 \\
+ k(k + 2)(16k + 17)^2(26k + 83)w_2w_3 \\
+ 2k^2(k + 2)(64k + 107)(8k^2 + 9k - 8)w_2w_4 \\
- 4k^2(k + 2)^2(36k + 61)(64k + 107)w_2^2w_4 \\
+ 2(64k + 107)w_3^2 + (16k + 17)^2w_3w_5.
\]

Since \( v^0 = 0 \), the above polynomial lies in the kernel of \( \bar{\phi} \). One may discuss the image in Zhu’s algebra of a nontrivial linear relation in the weight 8 subspace of \( \widetilde{W}^- \). However, the polynomial obtained in this manner becomes \( 0 \). Thus the null field of weight 8 in \( \widetilde{W}^- \) gives no information on \( A(\widetilde{W}) \).

Next, we study null fields of weight 9. There are 44 vectors of normal form (2.12) of weight 9. Among them, 22 vectors are contained in \( \widetilde{W}^+ \) and the other 22 vectors are contained in \( \widetilde{W}^- \). We eliminate \( W_{-3}^3W_{-2}^41, W_{-3}^3W_{-2}^41, W_{-1}^3W_{-3}^11, W_{-1}^3W_{-2}^51 \) and \( W_{-1}^3W_{-2}^51 \). Then
the remaining 40 vectors form a basis of the weight 9 subspace of $\tilde{W}$. We take a nontrivial linear relation $v^1$ in $\tilde{W}^-$ which involves $W_3^1W_4^1\mathbb{1}$ (see Appendix C.2). We calculate $[v^1]$ by a similar method as above and obtain the following polynomial.

$$Q_1 = -16k^3(k + 2)(2k + 1)(13k^3 + 24k^2 + 7k + 10)w_2w_3$$
$$+ 4k^3(k + 2)^2(1040k^3 + 2232k^2 + 1213k + 1116)w_2^2w_3$$
$$- 16k^3(k + 2)^3(674k^2 + 637k - 1100)w_2^3w_3$$
$$+ (16k + 17)(64k + 107)w_3^2 + 2k(68k^2 + 119k + 20)w_3w_4$$
$$- 4(k + 2)(358k + 559)w_2w_3w_4 + 4k^2(k + 2)(3k + 4)(4k - 1)w_2w_5$$
$$- 112k^2(k + 2)^2(3k + 4)w_2^2w_5 + 4w_4w_5.$$

As before, $v_1^1 = 0$ in $V(k, 0)$ and the polynomial $Q_1$ lies in the kernel of $\tilde{\rho}$. No further relation in $A(\tilde{W})$ is obtained from the null fields of weight 9.

Let $C_2(\tilde{W})$ be the subspace of $\tilde{W}$ spanned by the elements $u_{-2}v$ with $u, v \in \tilde{W}$. The quotient space $\tilde{W}/C_2(\tilde{W})$ has a commutative Poisson algebra structure [23, Section 4.4]. Since $(L(-1)v)_n = -nv_{n-1}$, we have $u_{-m}v \in C_2(\tilde{W})$ for $m \geq 2$. By Lemma 2.4, we see that $x_2 \mapsto \omega + C_2(\tilde{W})$, $x_i \mapsto W^i + C_2(\tilde{W})$, $i = 3, 4, 5$ define a homomorphism $\tilde{\rho}$ of associative algebras from a polynomial algebra $\mathbb{C}[x_2, x_3, x_4, x_5]$ of four variables $x_2, x_3, x_4, x_5$ onto $\tilde{W}/C_2(\tilde{W})$.

We consider the images of some null fields in $\tilde{W}/C_2(\tilde{W})$. The nontrivial linear relation involving $W_3^1W_4^1\mathbb{1}$ is written by vectors of normal form contained in $C_2(\tilde{W})$ (see Appendix C.1). Hence its image in $\tilde{W}/C_2(\tilde{W})$ is trivial. On the other hand, $(W_3^2)^2\mathbb{1}$ is a linear combination of vectors of normal form not all of which lie in $C_2(\tilde{W})$ (see Appendix C.1). Therefore, its image in $\tilde{W}/C_2(\tilde{W})$ is a nonzero polynomial in $\omega + C_2(\tilde{W})$, $W^i + C_2(\tilde{W})$, $i = 3, 4, 5$. Replace $\omega + C_2(\tilde{W})$ and $W^i + C_2(\tilde{W})$ with $x_2$ and $x_i$, $i = 3, 4, 5$, respectively in the polynomial and multiply it by $(17/9)k(k + 1)(16k + 17)^2(64k + 107)$.

Let $B_0$ be the polynomial obtained in this manner. Then

$$B_0 = -112k^4(k + 2)^4(3k + 4)(6k - 5)(64k + 107)x_2^4$$
$$+ k(k + 2)(16k + 17)^2(26k + 83)x_2x_3^2$$
$$- 4k^2(k + 2)^2(36k + 61)(64k + 107)x_2^2x_4$$
$$+ 2(64k + 107)x_2^3 + (16k + 17)^2x_3x_5.$$

Since $(W_3^2)^2\mathbb{1} \in C_2(\tilde{W})$, $B_0$ lies in the kernel of $\tilde{\rho}$. Likewise, we consider the images of the four nontrivial linear relations among the vectors of normal form of weight 9 in $\tilde{W}/C_2(\tilde{W})$. We obtain only one nonzero polynomial $B_1$ up to a scalar multiple in this manner, namely,

$$B_1 = 16k^3(k + 2)^3(674k^2 + 637k - 1100)x_2^3x_3 - (16k + 17)(64k + 107)x_3^3$$
$$+ 4k(k + 2)(358k + 559)x_2x_3x_4 + 112k^2(k + 2)^2(3k + 4)x_2^2x_5 - 4x_4x_5.$$

In fact, the polynomial $B_1$ comes from the null field $v^1$. 

The weight 10 subspace of $\tilde{W}^+$ is of dimension 35, while there are 40 vectors of normal form in the subspace. We eliminate $\omega - 1(W^3_2)^2 1, (W^3_3)^2 1, (W^4_2)^2 1, W^3_2 W^5_2 1,$ and $W^3_2 W^5_3 1.$ Then the remaining 35 vectors form a basis of the weight 10 subspace of $\tilde{W}^+.$ For instance, $(W^3_2)^2 1$ can be written uniquely as a linear combination of those 35 vectors just as in the case for weight 8 or 9. We denote by $W_1^k$ the nontrivial linear relation in $\tilde{W}$ involving $(W^3_2)^2 1.$ Take the image of $\tilde{W}$ in $\tilde{W}/C_2(\tilde{W})$ and replace $\omega + C_2(\tilde{W})$ and $W^1 + C_2(\tilde{W})$ with $x_2$ and $x_1,$ $i = 3, 4, 5,$ respectively. Then we obtain a polynomial in $x_2, x_3, x_4, x_5.$ Its suitable constant multiple $B_2$ is as follows.

$$B_2 = 16k^5(k + 2)^5(6k - 5)(64k + 107)$$

$$- k^2(k + 2)^2(16k + 17)(29745920k^5 + 282149936k^4$$

$$+ 715730704k^3 + 459700602k^2 - 375262083k - 379918040)x_2^5x_3^2$$

$$- 20k^3(k + 2)^3(64k + 107)$$

$$\cdot (81056k^4 - 691736k^3 - 2503316k^2 - 1005811k + 1451208)x_2^3x_4$$

$$+ 17(k + 1)(16k + 17)(64k + 107)^3x_2x_4$$

$$- 2k(k + 2)(64k + 107)(979064k^3 + 3791032k^2 + 4574059k + 1616792)x_2x_4$$

$$- k(k + 2)(16k + 17)^2(256632k^3 + 825008k^2 + 598779k - 114896)x_2x_3x_5$$

$$- 34(k + 1)(16k + 17)^2(64k + 107)x_5.$$

We note that $B_0, B_1$ and $B_2$ lie in the kernel of $\tilde{\rho}.$ These polynomials will be used in Section 5.

**Remark 2.7.** We eliminate $(W^3_2)^2 1$ and $W^3_2 W^3_2 1$ from the vectors of normal form $\omega - k + 1$ vector $e(-1)^k 1.$ The quotient algebra $L(k, 0) = V(k, 0)/\mathcal{J}$ is the simple vertex operator algebra associated with an affine Lie algebra $s\hat{\mathfrak{l}}_2$ of type $A_1^{(1)}$ with level $k.$ Since the Heisenberg vertex operator algebra $V_{\mathfrak{h}}(k, 0)$ is a simple subalgebra of $V(k, 0),$ it follows that $\mathcal{J} \cap V_{\mathfrak{h}}(k, 0) = 0.$ Thus $V_{\mathfrak{h}}(k, 0)$ can be considered as a subalgebra of $L(k, 0).$ Just as in (2.5), we have that $L(k, 0)$ is a completely reducible $V_{\mathfrak{h}}(k, 0)$-module and we can write

$$L(k, 0) = \bigoplus \lambda M_{\mathfrak{h}}(k, \lambda) \otimes K_{\lambda},$$

where

$$K_{\lambda} = \{v \in L(k, 0) \mid h(m)v = \lambda \delta_{m,0}v \text{ for } m \geq 0\}.$$
Note that $K_0$ is the commutant of $V_b^\gamma(k,0)$ in $L(k,0)$. Similarly, $\mathcal{J}$ is completely reducible as a $V_b^\gamma(k,0)$-module. Hence by [2.5],

$$\mathcal{J} = \oplus \lambda M_b^\lambda(k,\lambda) \otimes (\mathcal{J} \cap N_\lambda).$$

In particular, $\mathcal{I} = \mathcal{J} \cap N_0$ is an ideal of $N_0$ and $K_0 \cong N_0/\mathcal{I}$.

**Lemma 3.1.** $\mathcal{I}$ is a unique maximal ideal of $N_0$.

**Proof.** The top level of the subalgebra $V_b^\gamma(k,0) \otimes N_0$ of $V(k,0)$ is $C\mathbb{1}$, and so it is the top level of $N_0$ also. Hence $N_0$ has a unique maximal ideal, say $S$. Since $\mathcal{J}$ does not contain $\mathbb{1}$, we have $\mathcal{I} \subset S$. The subspace $V(k,0) \cdot S$ spanned by $u_\gamma v$, $u \in V(k,0)$, $v \in S$, $\gamma \in \mathbb{Z}$ is an ideal of $V(k,0)$. Let $u \in M_b^\gamma(k,\lambda) \otimes N_\lambda$ and $v \in S$. Since $S \subset V_b^\gamma(k,0) \otimes N_0$ and $M_b^\gamma(k,\lambda) \otimes N_\lambda$ is a $V_b^\gamma(k,0) \otimes N_0$-module, the skew symmetry [20, (3.1.30)] in the vertex operator algebra $V(k,0)$ implies that $u_\gamma v$ lies in $M_b^\gamma(k,\lambda) \otimes N_\lambda$. Then we see from [2.5] that the intersection of $V(k,0) \cdot S$ and $V_b^\gamma(k,0) \otimes N_0$ is $V_b^\gamma(k,0) \otimes S$. Thus $V(k,0) \cdot S$ is a proper ideal of $V(k,0)$ and so it is contained in the unique maximal ideal $\mathcal{J}$. Therefore, $\mathcal{I} = S$ as required.

The vector $e(-1)^{k+1}\mathbb{1}$ is not contained in $N_0$. For $r \geq 1$ and an integer $n$, we calculate that

$$h(n)f(0)^r e(-1)^r \mathbb{1} = (-2rf(0)^{r-1}f(n) + f(0)^r h(n)) e(-1)^r \mathbb{1}$$

$$= -2rf(0)^{r-1}f(n)e(-1)^r \mathbb{1} + 2rf(0)^r e(-1)^r e(n-1) \mathbb{1}$$

$$+ f(0)^r e(-1)^r h(n) \mathbb{1}$$

in $V(k,0)$ by using (2.10). Moreover,

$$f(n)e(-1)^r \mathbb{1} = \begin{cases} r(k + r + 1)e(-1)^{r-1} \mathbb{1} & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Hence, $h(n)f(0)^r e(-1)^r \mathbb{1} = 0$ if $n = 0$ or $n \geq 2$, and

$$h(1)f(0)^r e(-1)^r \mathbb{1} = -2rf(0)^{r-1}f(1)e(-1)^r \mathbb{1}$$

$$= -2r^2(k - r + 1)f(0)^{r-1}e(-1)^{r-1} \mathbb{1}.$$ 

In particular, $h(n)f(0)^{k+1} e(-1)^{k+1} \mathbb{1} = 0$ for $n \geq 0$. This proves the next theorem.

**Theorem 3.2.** $f(0)^{k+1} e(-1)^{k+1} \mathbb{1} \in \mathcal{I}$.

It seems natural to expect the following properties of $f(0)^{k+1} e(-1)^{k+1} \mathbb{1}$ (see Section 5).

**Conjecture 3.3.** (1) The unique maximal ideal $\mathcal{I}$ of $N_0$ is generated by a weight $k + 1$ vector $f(0)^{k+1} e(-1)^{k+1} \mathbb{1}$.

(2) The automorphism $\theta$ acts as $(-1)^{k+1}$ on $f(0)^{k+1} e(-1)^{k+1} \mathbb{1}$.

We have considered the elements $\omega_{\text{aff}}$, $\omega_\gamma$, $\omega$, $W^3$, $W^4$ and $W^5$ of $V(k,0)$. For simplicity of notation, we use the same symbols to denote their images in $L(k,0) = V(k,0)/\mathcal{J}$. Then $\omega$ is the conformal vector of $K_0$. Moreover, $K_0 = \{ v \in L(k,0) \mid (\omega_\gamma)_0 v = 0 \}$ by [13, Theorem 5.2]. It is also the commutant of $\text{Vir}(\omega_\gamma)$ in $L(k,0)$. The automorphism $\theta$ of $V(k,0)$ induces an automorphism of $L(k,0)$. We denote it by the same symbol $\theta$. Let
\( \mathcal{W} \) be a subalgebra of \( K_0 \) generated by \( \omega, W^3, W^4 \) and \( W^5 \). Thus \( \mathcal{W} \) is a homomorphic image of \( \hat{\mathcal{W}}. \) We are interested in \( \mathcal{W} \). The following theorem is a direct consequence of Remark 2.3 and Theorem 2.5.

**Theorem 3.4.** If \( k \geq 3 \), then the vertex operator algebra \( \mathcal{W} \) is generated by \( W^3 \) and its automorphism group \( \text{Aut} \mathcal{W} = \langle \theta \rangle \) is of order 2.

4. **Irreducible Modules for \( K_0 \)**

In the preceding section, \( K_0 \) is defined to be the commutant of \( V_0^z(k, 0) \) in \( L(k, 0) \). We will follow the argument in [7] to realize \( K_0 \) in a vertex operator algebra associated with a lattice and construct some of its irreducible modules. Those irreducible \( K_0 \)-modules will be denoted by \( M^{i,j}, 0 \leq i \leq k, 0 \leq j \leq k - 1 \). Actually, \( M^{0,0} = K_0. \)

Let \( L = Z\alpha_1 + \cdots + Z\alpha_k \) with \( \langle \alpha_p, \alpha_q \rangle = 2\delta_{pq} \). Thus \( L \) is an orthogonal sum of \( k \) copies of a root lattice of type \( A_1 \). Its dual lattice is \( L^\perp = \frac{1}{2}L \). The commutator map \( c_0(\cdot, \cdot) \) defined in [20] Remark 6.4.12 is trivial on \( L^\perp \) so that the twisted group algebras \( C\{L \} \) and \( C\{L^\perp \} \) considered in [20] Section 6.4 are isomorphic to ordinary group algebras \( C[L] \) and \( C[L^\perp] \) of additive groups \( L \) and \( L^\perp \), respectively. Denote a standard basis of \( C[L^\perp] \) by \( e^\alpha, \alpha \in L^\perp \) with multiplication \( e^\alpha e^\beta = e^{\alpha+\beta} \). We consider the vertex operator algebra \( V_L = M(1) \otimes C[L] \) associated with the lattice \( L \) and its module \( V_{L^\perp} = M(1) \otimes C[L^\perp] \). Here, \( M(1) = [\alpha_p(-n); 1 \leq p \leq k, n \geq 1] \) as a vector space. The vertex operator algebra \( V_L \) is spanned by \( \beta_1(-n_1) \cdots \beta_r(-n_r) e^\alpha, \beta_i \in \{\alpha_1, \ldots, \alpha_k\}, \alpha \in L, n_1 \geq \cdots \geq n_r \geq 1 \). Let \( \gamma = \alpha_1 + \cdots + \alpha_k. \) Thus \( \langle \gamma, \gamma \rangle = 2k. \) Set

\[
H = \gamma(-1)1, \quad E = e^{\alpha_1} + \cdots + e^{\alpha_k}, \quad F = e^{-\alpha_1} + \cdots + e^{-\alpha_k},
\]

where \( 1 = 1 \otimes 1 \) is the vacuum vector.

Then \( H_0H = 0, H_1H = 2k1, H_0E = 2E, H_1E = 0, H_0F = -2F, H_1F = 0, E_0F = H, E_1F = k1, E_0E = E_1E = F_0F = F_1F = 0 \) and \( A_nB = 0 \) for \( A, B \in \{H, E, F\}, n \geq 2 \) in the vertex operator algebra \( V_L. \) Therefore, the component operators \( H_n, E_n \) and \( F_n, n \in Z, \) which are endomorphisms of \( V_L \) or \( V_{L^\perp} \), give a representation of \( sl_2 \) with level \( k \) under the correspondence

\[
H_n \leftrightarrow h(n), \quad E_n \leftrightarrow e(n), \quad F_n \leftrightarrow f(n).
\]

We have

\[
(E_{-1})^j1 = j! \sum_{I \subset \{1, 2, \ldots, k\} \mid |I| = j} e^{\alpha_I}, \quad (F_{-1})^j1 = j! \sum_{I \subset \{1, 2, \ldots, k\} \mid |I| = j} e^{-\alpha_I},
\]

and \( (E_{-1})^{k+1}1 = (F_{-1})^{k+1}1 = 0, \) where \( \alpha_I = \sum_{p \in I} \alpha_p \) for a subset \( I \) of \( \{1, 2, \ldots, k\}. \) In particular, \( (E_{-1})^k1 = k! e^\gamma \) and \( (F_{-1})^k1 = k! e^{-\gamma}. \) Moreover,

\[
e^{-\gamma} \cdot (E_{-1})^j1 = \frac{j!}{(k-j)!} (F_{-1})^{k-j}1, \quad e^\gamma \cdot (F_{-1})^j1 = \frac{j!}{(k-j)!} (E_{-1})^{k-j}1
\]

in the group algebra \( C[L]. \)

Let \( V^{\text{aff}} \) (resp. \( V^{\gamma} \)) be the subalgebra of \( V_L \) generated by \( H, E \) and \( F \) (resp. \( e^\gamma \) and \( e^{-\gamma} \)). Then \( V_L \supset V^{\text{aff}} \supset V^{\gamma} \) with \( V^{\text{aff}} \cong L(k, 0) \) and \( V^{\gamma} \cong V_{Z\gamma}, \) where \( V_{Z\gamma} \) is the vertex operator algebra associated with a rank one lattice \( Z\gamma. \) Note that \( V^{\gamma} \) contains \( (e^\gamma)^{2k-2}e^{-\gamma} = H. \) The \(-1 \) isometry of the lattice \( L^\perp \) lifts to a linear isomorphism \( \theta \) of
We have Proposition 4.1. Moreover, \(V\) trivial. Its restriction to \(V\) is an automorphism of the vertex operator algebra \(V_L\). We have \(\theta H = -H, \theta E = F\) and \(\theta F = E\). Thus the restriction of the automorphism \(\theta\) here to \(V_{\text{aff}}\) agrees with the automorphism of \(L(k, 0) = V(k, 0)/J\) induced by the automorphism \(\theta\) of \(V(k, 0)\) discussed in Section 2.

For simplicity of notation, we use the same symbols \(\omega_{\text{aff}}, \omega, W^3, W^4\) and \(W^5\) to denote their images in \(V_{\text{aff}} \cong L(k, 0)\) under the correspondence (4.1). Then

\[
\omega_{\text{aff}} = \frac{1}{2(k + 2)} \left( \frac{1}{2} H_{-1} H + E_{-1} F + F_{-1} E \right),
\]

\[
\omega_\gamma = \frac{1}{4k} H_{-1} H,
\]

\[
\omega = \frac{1}{2k(k + 2)} \left( -H_{-1} H + k(E_{-1} F + F_{-1} E) \right),
\]

\[
W^3 = k^2 H_{-3} \mathbb{I} + 3k H_{-2} H + 2H_{-1} H_{-1} H - 6k H_{-1} E_{-1} F + 3k^2 (E_{-2} F - E_{-1} F_{-2} \mathbb{I}).
\]

Let \(L' = \bigoplus_{p=1}^{k-1} \mathbb{Z}(\alpha_p - \alpha_{p+1})\), which is \(\sqrt{2}\) times a root lattice of type \(A_{k-1}\). We have

\[
L' = \{ \alpha \in L \mid \langle \alpha, \gamma \rangle = 0 \}.
\]

Moreover,

\[
L = \bigcup_{i=0}^{k-1} (i\alpha_1 + L' \oplus \mathbb{Z}\gamma)
\]

is a coset decomposition of \(L\) by \(L' \oplus \mathbb{Z}\gamma\) (4.1).

Let \(V_{L'} = M(1)' \otimes \mathbb{C}[L']\) be the vertex operator algebra associated with \(L'\), where \(M(1)' = \{ (\alpha_p - \alpha_q)(-n) \mid 1 \leq p < q \leq k, n \geq 1 \}\) as a vector space and \(\mathbb{C}[L']\) is an ordinary group algebra of the additive group \(L'\). Then \(V_{L'} \cong V_{\mathbb{Z}A_{k-1}}\) is a subalgebra of \(V_L\). By (4.2) and (4.3), the following proposition holds (see [13, Theorem 5.2] also).

**Proposition 4.1.** We have \(V_{L'} = \{ v \in V_L \mid \langle \omega_\gamma \rangle v = 0 \text{ for } n \geq 0 \}\). In particular, \(K_0 \cong V_{\text{aff}} \cap V_{L'}\).

We will describe \(\omega\) and \(W^3\) explicitly as elements of \(V_{L'}\) (see [13, Lemma 4.1]). In the vertex operator algebra \(V_L\) we have

\[
H_{-1} H = \sum_{1 \leq p \leq k} \alpha_p(-1)^2 \mathbb{I} + \sum_{1 \leq p, q \leq k} \alpha_p(-1)\alpha_q(-1) \mathbb{I},
\]

\[
E_{-1} F + F_{-1} E = \sum_{1 \leq p \leq k} \alpha_p(-1)^2 \mathbb{I} + 2 \sum_{1 \leq p, q \leq k} \alpha_p(-1)\alpha_q(-1) \mathbb{I}.
\]

Now, \((\alpha_p - \alpha_q)(-1)^2 = \alpha_p(-1)^2 + \alpha_q(-1)^2 - 2\alpha_p(-1)\alpha_q(-1)\) and so

\[
\frac{1}{2} \sum_{1 \leq p, q \leq k} (\alpha_p - \alpha_q)(-1)^2 = (k - 1) \sum_{1 \leq p \leq k} \alpha_p(-1)^2 - \sum_{1 \leq p, q \leq k} \alpha_p(-1)\alpha_q(-1).
\]
Hence we obtain (see [18, Lemma 4.1])

\[
\omega = \frac{1}{4k(k+2)} \sum_{1 \leq p,q \leq k, p \neq q} (\alpha_p - \alpha_q)(-1)^2 \mathbb{1} + \frac{1}{k+2} \sum_{1 \leq p,q \leq k, p \neq q} e^{\alpha_p - \alpha_q}. \tag{4.4}
\]

Moreover, we calculate that

\[
H_{-3} \mathbb{1} = \sum_{1 \leq p \leq k} \alpha_p(-3) \mathbb{1},
\]

\[
H_{-2} H = \sum_{1 \leq p,q \leq k} \alpha_p(-2) \alpha_q(-1) \mathbb{1},
\]

\[
H_{-1} H_{-1} = \sum_{1 \leq p,q,r \leq k} \alpha_p(-1) \alpha_q(-1) \alpha_r(-1) \mathbb{1},
\]

\[
H_{-1} E_{-1} F = \left( \sum_{1 \leq p \leq k} \alpha_p(-1) \right) \left( \sum_{1 \leq q \leq k} \frac{1}{2} (\alpha_q(-2) \mathbb{1} + \alpha_q(-1)^2 \mathbb{1}) + \sum_{1 \leq q,r \leq k, q \neq r} e^{\alpha_q - \alpha_r} \right),
\]

\[
E_{-2} F - E_{-1} F \mathbb{1} = \frac{1}{3} \sum_{1 \leq p \leq k} (-\alpha_p(-3) \mathbb{1} + \alpha_p(-1)^3 \mathbb{1})
\]

\[
+ \sum_{1 \leq p,q \leq k, p \neq q} (\alpha_p(-1) + \alpha_q(-1)) e^{\alpha_p - \alpha_q}.
\]

Using these equations, we have

\[
W^3 = \sum_{1 \leq p,q,r \leq k} (\alpha_p - \alpha_q)(-1)^2(\alpha_p - \alpha_r)(-1) \mathbb{1}
\]

\[
- 3k \sum_{1 \leq q,r \leq k, q \neq r} \left( \sum_{1 \leq p \leq k} (\alpha_p - \alpha_q)(-1) + \sum_{1 \leq p \leq k, p \neq r} (\alpha_p - \alpha_r)(-1) \right) e^{\alpha_q - \alpha_r}. \tag{4.5}
\]

Equation (4.5) in particular implies that \(W^3 = 0\) in \(V_L\), if \(k = 2\) (see Section 5.1). We identify \(V^\text{aff}\) with \(L(k,0)\) and \(V^\gamma\) with \(V_{\mathbb{Z}^\gamma}\) from now on.

It is well known that the vertex operator algebra associated with a positive definite even lattice is rational [3]. We study a decomposition of \(V^\text{aff}\) into a direct sum of irreducible modules for \(V^\gamma = V_{\mathbb{Z}^\gamma}\). Any irreducible module for \(V_{\mathbb{Z}^\gamma}\) is isomorphic to one of \(V_{\mathbb{Z}^\gamma+n\gamma/2k}^\gamma\), \(0 \leq n \leq 2k-1\) [3]. Let \(V^\gamma \cdot v\) be the \(V^\gamma\)-submodule of \(V_{\mathbb{Z}^\gamma}^\gamma\) generated by an element \(v\) of \(V_{\mathbb{Z}^\gamma}^\gamma\). Then \(V^\gamma \cdot (E_{-1})^j \mathbb{1}\) (resp. \(V^\gamma \cdot (F_{-1})^j \mathbb{1}\)) is isomorphic to \(V_{\mathbb{Z}^\gamma+j\gamma/k}^\gamma\) (resp. \(V_{\mathbb{Z}^\gamma-j\gamma/k}^\gamma\)).

Now, the action of \(H_0 = \gamma(0)\) on \(e^{n\gamma/2k}\) is given by \(H_0 e^{n\gamma/2k} = ne^{n\gamma/2k}\). Moreover, the eigenvalues of \(H_0\) on \(V^\text{aff}\) are even integers, since \(h(0)u = 2(q-r)u\) for a vector \(u\) of the form (2.2) in \(V(k,0)\). Hence \(V_{\mathbb{Z}^\gamma+n\gamma/2k}^\gamma\) does not appear as a direct summand in \(V^\text{aff}\) unless \(n\) is even. Let

\[
M^0_j = \{ v \in V^\text{aff} \mid H_m v = -2j \delta_{m,0} v \text{ for } m \geq 0 \}
\]

for \(0 \leq j \leq k-1\). Then \(V^\text{aff} = \bigoplus_{j=0}^{k-1} (V^\gamma \cdot (F_{-1})^j \mathbb{1}) \otimes M^0_j\) as \(V^\gamma\)-modules. That is, the following lemma holds.

**Lemma 4.2.** \(L(k,0) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}^\gamma-j\gamma/k}^\gamma \otimes M^0_j\) as \(V_{\mathbb{Z}^\gamma}^\gamma\)-modules.
In the case \( j = 0 \), \( M^{0,0} \) coincides with the commutant \( K_0 \) of \( V_b(k,0) \) in \( L(k,0) \). The restriction of \( \theta \) to \( M^{0,0} \), which we denote by the same symbol \( \theta \), is an automorphism of \( M^{0,0} \) of order 2.

In order to describe irreducible \( M^{0,0} \)-modules contained in \( V_{L^+} \), let

\[
v^i = \sum_{I \subseteq \{1,2,\ldots,k\}} e^{\alpha_i/2}, \quad v^{i,j} = \frac{1}{j!}(F_0)^j v^i
\]

for \( 0 \leq i \leq k, 0 \leq j \leq i \). Note that \( v^0 \) is the vacuum vector 1 and \( v^{i,0} = v^i \). In fact\(^2\),

\[
v^{i,j} = \sum_{I \subseteq \{1,2,\ldots,k\}} \sum_{|J| = j} e^{\alpha_I/2} v^{i,J}
\]

From this explicit form of \( v^{i,j} \), we see that \( \theta v^{i,j} = v^{i-j,j} \), \( 0 \leq i \leq k, 0 \leq j \leq i \).

We have \( H_0 v^{i,j} = (i-2j)v^{i,j} \), \( E_0 v^{0,i} = 0 \), \( E_0 v^{i,j} = (i-j+1)v^{i-1,j} \) for \( 1 \leq j \leq i \), \( F_0 v^{i,j} = (j+1)v^{i+1,j} \) for \( 0 \leq j \leq i-1 \), \( F_0 v^{i,j} = 0 \), and \( H_n v^{i,j} = E_n v^{i,j} = F_n v^{i,j} = 0 \) for \( n \geq 1 \). Hence the subspace \( U^i \) of \( V_{L^+} \) spanned by \( v^{i,j} \), \( 0 \leq j \leq i \) is an \( i+1 \) dimensional irreducible module for \( \text{span}\{H_0,E_0,F_0\} \cong sl_2 \). Furthermore, the \( V^\text{aff} \)-submodule \( V^\text{aff} \cdot v^i \) of \( V_{L^+} \) generated by \( v^i \) is isomorphic to an irreducible \( L(k,0) \)-module \( L(k, U^i) \) with top level \( U^i \)\(^{13} \). If \( i = 0 \), then \( U^0 = \mathbb{C}1 \) and \( L(k, U^0) \) coincides with \( L(k,0) \). We denote \( L(k, U^i) \) by \( L(k,i) \) for a general \( i \) also and identify it with \( V^\text{aff} \cdot v^i \). Let

\[
M^{i,j} = \{ v \in V^\text{aff} \cdot v^i \mid H_m v = (i-2j)\delta_{m,0} v \text{ for } m \geq 0 \}
\]

for \( 0 \leq i \leq k, 0 \leq j \leq k-1 \). Each \( M^{i,j} \) is an \( M^{0,0} \)-module. We decompose \( V^\text{aff} \cdot v^i \) into a direct sum of irreducible \( V^\gamma \)-modules and obtain the following lemma, which is a generalization of Lemma\(^4,2 \).

**Lemma 4.3.** \( L(k,i) = \bigoplus_{j=0}^{k-1} V_{Z\gamma+(i-2j)\gamma/2k} \otimes M^{i,j} \) as \( V_{Z\gamma} \)-modules.

Next, we consider an isomorphism \( \sigma = \exp(2\pi\sqrt{-1}\gamma(0)/2k) \) induced by the action of \( \gamma(0) \) on \( V_{L^+} \). Recall that \( h(0)u = 2(q-r)u \) for \( u \in V(k,0) \) being as in \( (2.2) \) and that \( H_0 v^{i,j} = (i-2j)v^{i,j} \). Thus there are exactly \( k \) distinct eigenvalues \( i-2j \), \( 0 \leq j \leq k-1 \) modulo \( 2k \) of the action of \( \gamma(0) = H_0 \) on \( L(k,i) \). Hence \( \sigma \) has \( k \) distinct eigenvalues \( \exp(2\pi\sqrt{-1}(i-2j)/2k) \), \( 0 \leq j \leq k-1 \) on \( L(k,i) \). Let \( L(k,i) = \bigoplus_{j=0}^{k-1} L(k,i)^j \) be the eigenspace decomposition, where \( L(k,i)^j \) is the eigenspace for \( \sigma \) on \( L(k,i) \) with eigenvalue \( \exp(2\pi\sqrt{-1}(i-2j)/2k) \). Since \( \langle \gamma, (i-2j)\gamma/2k \rangle = i-2j \), Lemma\(^4,3 \) implies that

\[
L(k,i)^j = V_{Z\gamma+(i-2j)\gamma/2k} \otimes M^{i,j}. \tag{4.6}
\]

For convenience, we understand the second parameter \( j \) of \( M^{i,j} \) to be modulo \( k \). We study some basic properties of \( M^{0,0} \)-modules \( M^{i,j} \), \( 0 \leq i \leq k, 0 \leq j \leq k-1 \).

**Theorem 4.4.** (1) \( M^{i,j} \) is an irreducible \( M^{0,0} \)-module.

(2) \( M^{i,j} \cong M^{k-i,k-i+j} \) as \( M^{0,0} \)-modules.

(3) The automorphism \( \theta \) of \( M^{0,0} \) induces a permutation \( M^{i,j} \mapsto M^{i,j} \) on these irreducible \( M^{0,0} \)-modules.

(4) The top level of \( M^{i,j} \) is a one dimensional space \( \mathbb{C}v^{i,j} \), \( 0 \leq i \leq k, 0 \leq j \leq \min\{i,k-1\} \).

\(^2\)The corresponding equation on page 31 of [6] is incorrect.
Proof. We first show the assertion (1). By Lemma 4.2, \( L(k, 0) = \oplus_{j=0}^{k-1} V_{Z \gamma - j \gamma/k} \otimes M^{0,j} \). Suppose \( M^{0,j} \) is not an irreducible \( M^{0,0} \)-module and let \( U \) be a proper submodule of \( M^{0,j} \). Take \( 0 \neq u \in U \). Let \( S = L(k, 0) \cdot u \) be the subspace of \( L(k, 0) \) spanned by \( v_n u \), \( v \in L(k, 0) \), \( n \in \mathbb{Z} \), which is an ideal of \( L(k, 0) \). Actually, \( L(k, 0) \) is equal to \( S \), since it is a simple vertex operator algebra. Now, let \( v \in V_{Z \gamma - r \gamma/k} \otimes M^{0,r} \). Then the fusion rules \( V_{Z \gamma + a} \times V_{Z \gamma + b} = V_{Z \gamma + a+b} \) of irreducible \( V_{Z \gamma} \)-modules [7, Chapter 12] imply that \( v_n u \) lies in \( V_{Z \gamma - (r+j) \gamma/k} \otimes M^{0,r+j} \). Hence \( S \cap (V_{Z \gamma - j \gamma/k} \otimes M^{0,j}) = V_{Z \gamma - j \gamma/k} \otimes U \), which contradicts the fact that \( L(k, 0) = S \). Thus \( M^{0,j} \) is an irreducible \( M^{0,0} \)-module. We apply a similar argument to the irreducible \( L(k,0) \)-module \( L(k,i) \) and use the decomposition in Lemma 4.3. Then we obtain (1).

Next, we show the assertion (2). A multiplication by \( e^{-\gamma/2} \) on the group algebra \( \mathbb{C}[L^1] \) induces a linear isomorphism \( \psi : u \otimes e^\beta \mapsto u \otimes e^{\beta - \gamma/2}, u \in M(1), \beta \in L^1 \) of \( V_L \) onto itself. Since \( M^{0,0} \) is contained in \( V_L \) by Proposition 4.1, it follows from (4.2) and the definition of \( V_{V_i} \) that \( \psi \) commutes with the vertex operator \( V_{V_i} \) associated with any \( v \in M^{0,0} \). Thus \( \psi \) is an isomorphism of \( M^{0,0} \)-modules. We have \( \psi(V_{Z \gamma + (i-2) \gamma/2k}) = V_{Z \gamma + (i-k-2j)(i-j) \gamma/2k} \). Hence (2) holds.

We show the assertion (3). Since \( \theta^j \psi^j = v^{i,j} \), \( U^i \) is invariant under \( \theta \). Moreover, \( \theta(H_m v) = -H_m \theta(v) \) for \( m \in \mathbb{Z} \) and \( v \in V^{aff} \cdot v^i \), since \( \theta H = -H \). Thus (3) holds. Actually, \( \theta \) maps \( V_{Z \gamma + (i-2j) \gamma/k} \) onto \( V_{Z \gamma + (i-2j) \gamma/k} \).

Finally, we show the assertion (4). The top level of the irreducible \( L(k,0) \)-module \( L(k, i) \) is \( U^i = \oplus_{j=0}^{k} \mathbb{C} v^{i,j} \). Moreover, \( \gamma(0)v^{i,j} = (i-2j)v^{i,j} \) and so \( v^{i,j} \in L(k, i)^j \), \( 0 \leq j \leq \min\{i, k-1\} \). That is, the eigenvalues of \( \sigma \) on the eigenvectors \( v^{i,j} \), \( 0 \leq j \leq \min\{i, k-1\} \) are all different. The only exception is the case \( i = k \) and \( j = 0, k \). Hence the top level of \( L(k,i)^j \) is a one dimensional space \( \mathbb{C} v^{i,j} \) unless \( i = k \) and \( j = 0 \). In such a case the top level of \( M^{i,j} \) is also \( \mathbb{C} v^{i,j} \) by (4.6). As to the top level of \( M^{k,0} \), recall that \( v^{k,0} = e^{\gamma/2} \) and \( v^{k,k} = e^{-\gamma/2} \), on which \( \sigma \) acts as \(-1 \). Thus the top level of \( L(k,k)^0 \) is a two dimensional space \( \mathbb{C} v^{k,0} + \mathbb{C} v^{k,k} \), which coincides with the top level of \( V_{Z \gamma + \gamma/2} \). Therefore, the top level of \( M^{k,0} \) is one dimensional by (4.6). This proves (4). Note that \( M^{k,0} \cong M^{0,0} \) as \( M^{0,0} \)-modules by the assertion (2). Indeed, the isomorphism \( \psi \) of \( M^{0,0} \)-modules maps \( v^{k,0} \) to the vacuum vector \( \mathbb{1} \) and \( \mathbb{1} \) to \( v^{k,k} \).

One may prove the irreducibility of \( M^{i,j} \) in a different manner. Indeed, each irreducible \( L(k, 0) \)-module \( L(k, i) \) is \( \sigma \)-stable, for \( \sigma \) is an inner automorphism. We see from (4.6) that the space \( L(k, 0)^{(\sigma)} \) consisting of the elements of \( L(k, 0) \) fixed by \( \sigma \) is \( V_{Z \gamma} \otimes M^{0,0} \). By [11, Theorem 5.4], the eigenspace \( L(k, i)^j \) for \( \sigma \) is an irreducible \( L(k, 0)^{(\sigma)} \)-module. Hence (4.6) implies that \( M^{i,j} \) is irreducible as a module for \( M^{0,0} \). In the proof of Theorem 1.4 (1), we have shown that \( v_n u \in V_{Z \gamma - (r+j) \gamma/k} \otimes M^{0,r+j} \) by using the fusion rules of \( V_{Z \gamma} \). This fact comes from (4.6) also. The isomorphism of \( M^{0,0} \)-modules discussed in Theorem 1.4 (2) is a special case of [8, Corollary 5.7] (see [8, Remark 5.8] also). That is, it is a kind of spectrum flow.

The character of \( M^{i,j} \) is known [2, 14]. In fact,

\[
\chi M^{i,j} = \eta(\tau) c_{i-2j}(\tau)
\]

by [14, (3.34)]. Note that \( k, l \) and \( m \) of [14] are \( k, i \) and \( i - 2j \), respectively in our notation. Then [14, (3.36), (3.40)] mean \( c_{i-2j} = c_{i-2(j-k)} \), \( c_{i-2j} = c_{2j-i} \), and \( c_{i-2j} = c_{k+i-2j} \). As to
the first equation, recall that our $j$ of $M^{i,j}$ is considered to be modulo $k$. The remaining
two equations are compatible with Theorem 4.4 (3) and (2), respectively.

**Proposition 4.5.** The weight 0 operators $o(\omega) = \omega_1$, $o(W^p) = W^p_{p-1}$, $p = 3, 4, 5$ act on
$v^{i,j}$, $0 \leq i \leq k$, $0 \leq j \leq i$ as follows.

$$
o(\omega) v^{i,j} = \frac{1}{2(k + 2)} \left( k(i - 2j) - (i - 2j)^2 + 2k(i - j + 1)j \right) v^{i,j},$$

$$
o(W^3) v^{i,j} = \left( k^2(i - 2j) - 3k(i - 2j)^2 + 2(i - 2j)^3 - 6k(i - 2j)(i - j + 1)j \right) v^{i,j},$$

$$
o(W^4) v^{i,j} = \left( 2k^2(k^2 + k + 1)(i - 2j) - k(13k^2 + 8k + 2)(i - 2j)^2 + 2k(11k + 6)(i - 2j)^3 - (11k + 6)(i - 2j)^4 + 4k^2(k - 3)(k - 2)(i - j + 1)j - 4k^2(6k - 5)(i - 2j)(i - j + 1)j + 4k(11k + 6)(i - 2j)^2(i - j + 1)j - 2k^2(6k - 5)(i - j + 1)(i - j + 2)(j - 1)j \right) v^{i,j},$$

$$
o(W^5) v^{i,j} = \left( -2k^3(k^2 + 3k + 5)(i - 2j) + 5k^2(5k^2 + 6k + 6)(i - 2j)^2 - 20k(4k^2 + 3k + 1)(i - 2j)^3 + 5k(19k + 12)(i - 2j)^4 - 2(19k + 12)(i - 2j)^5 + 10k^2(5k^2 - 14k + 20)(i - 2j)(i - j + 1)j - 20k^2(10k - 7)(i - 2j)^2(i - j + 1)j + 10k(19k + 12)(i - 2j)^3(i - j + 1)j - 10k^2(10k - 7)(i - 2j)(i - j + 1)(i - j + 2)(j - 1)j \right) v^{i,j}.$$

**Proof.** We use the representation of $\hat{sl}_2$ given by the correspondence (4.1), namely

(i) $h(0) v^{i,j} = (i - 2j)v^{i,j}$,

(ii) $e(0) v^{i,0} = 0$, $e(0) v^{i,j} = (i - j + 1)v^{i-1,j}$ for $1 \leq j \leq i$,

(iii) $f(0) v^{i,j} = 0$, $f(0) v^{i,j} = (j + 1)v^{j+1,j}$ for $0 \leq j \leq i - 1$,

(iv) $a(n) v^{i,j} = 0$ for $a \in \{h, e, f\}$, $n \geq 1$.

Recall the vertex operator $Y(u, x)$ of the $L(k, 0)$-module $L(k, i)$. For $a \in \{h, e, f\}$ and

$$m \geq 1,$$

$$Y(a(-m) 1, x) = \frac{1}{(m - 1)!} \left( \frac{d}{dx} \right)^{m-1} a(x).$$

Hence the coefficient $o(a(-m))$ of $x^{-m}$ in $Y(a(-m) 1, x)$ is $(-1)^{m-1}a(0)$. Let $a^1, \ldots, a^r \in \{h, e, f\}$ and $m_1, \ldots, m_r \geq 1$. The vertex operator $Y(u, x)$ associated with the vector

$$u = a^1(-m_1) \cdots a^r(-m_r) 1$$

is given recursively by

$$Y(u, x) = \frac{1}{(m_1 - 1)!} \left( \frac{d}{dx} \right)^{m_1-1} a^1(x)^- Y(a^2(-m_2) \cdots a^r(-m_r) 1, x)$$

$$+ Y(a^2(-m_2) \cdots a^r(-m_r) 1, x) \left( \frac{1}{(m_1 - 1)!} \left( \frac{d}{dx} \right)^{m_1-1} a^1(x)^+ \right).$$
where \( a(x)^- = \sum_{n<0} a(n)x^{-n-1} \) and \( a(x)^+ = \sum_{n\geq0} a(n)x^{-n-1} \). The operator \( o(u) \) is the coefficient of \( x^{-m_1-\cdots-m_r} \) in \( Y(u, x) \). Since \( a(n)v^{i,j} = 0 \) for \( n \geq 1 \), we have

\[
o(a^1(-m_1)\cdots a^r(-m_r)1)v^{i,j} = (-1)^{m_1+\cdots+m_r-r}a^r(0)\cdots a^1(0)v^{i,j}.
\]

For instance, \( o(h(-2)) \), \( o(h(-1)^2) \) and \( o(e(-1))f(-1)) \) act on \( v^{i,j} \) as \(-(i-2j)\), \((i-2j)^2\) and \((i-j+1)j\), respectively. Therefore, we obtain \( o(\omega)v^{i,j} \). The action of \( o(W^3) \), \( o(W^4) \) and \( o(W^5) \) on \( v^{i,j} \) can be calculated similarly. \( \square \)

Let \( a(i, j) \) and \( b(i, j) \) be the eigenvalues of the operators \( o(\omega) \) and \( o(W^3) \) on \( v^{i,j} \) given in Proposition 4.5, respectively. We can verify that \( a(i, i-j) = a(i, j) \) and \( b(i, i-j) = -b(i, j) \). These relations are compatible with Theorem 4.4 (3), since \( \theta \) fixes \( \omega \) and maps \( W^3 \) to its negative. Similar relations hold for the eigenvalues of \( o(W^4) \) and \( o(W^5) \) on \( v^{i,j} \).

We close this section with the following conjecture.

**Conjecture 4.6.**

1. \( M^{i,j}, 0 \leq i \leq k, 0 \leq j \leq i \) are not isomorphic each other except the isomorphisms \( M^{i,j} \cong M^{k-i,0} \). There are \( k(k+1)/2 \) inequivalent irreducible \( M^{0,0} \)-modules among those \( M^{i,j} \)'s.

2. \( M^{0,0} = W \) and there are exactly \( k(k+1)/2 \) inequivalent irreducible \( M^{0,0} \)-modules, which are represented by \( M^{i,j}, 0 \leq i \leq k, 0 \leq j \leq i-1 \). Furthermore, Zhu’s algebra \( A(M^{0,0}) \) of \( M^{0,0} \) is generated by \( [\omega] \) and \( [W^3] \) if \( k \geq 3 \).

3. The vertex operator algebra \( M^{0,0} \) is rational and \( C_2 \)-cofinite.

5. Case \( k \leq 6 \)

In this section we will show that Conjecture 4.6 is true for \( k \leq 6 \). Indeed, one may verify that there are \( k(k+1)/2 \) different pairs of eigenvalues of the operators \( o(\omega) \) and \( o(W^3) \) on the top level \( C_0v^{i,j} \) of \( M^{i,j}, 0 \leq i \leq k, 0 \leq j \leq i-1 \) once \( k \) is given (see Proposition 4.5). That is, \( o(\omega) \) and \( o(W^3) \) are expected to be sufficient to distinguish inequivalence of those irreducible \( M^{0,0} \)-modules \( M^{i,j} \)'s. This is the case if \( k \) is a small positive integer, say \( k \leq 6 \). In this way we see that the assertion (1) of Conjecture 4.6 is true for \( k \leq 6 \).

The singular vector discussed in Section 3 plays a crucial role in the proof of the remaining assertions of Conjecture 4.6. Let \( u^0 = f(0)^{k+1}e(-1)^{k+1}1 \). By Theorem 3.2, \( u^0 \in \tilde{I} \) and so \( u^0 = 0 \) in \( K_0 = M^{0,0} \). Our argument is based on a detailed analysis of the vector \( u^0 \). First of all, we express \( u^0 \) as a linear combination of the basis (2.2) of \( V(k, 0) \). The expression enables us to write \( u^0 \) as a linear combination of the vectors of normal form (2.12) of weight \( k+1 \). This in particular implies that \( \tilde{W} \) contains \( u^0 \). Unfortunately, we do not succeed in handling this process for a general \( k \). It seems difficult even to show that \( u^0 \in \tilde{W} \). Therefore, we discuss only the case \( k \leq 6 \) in this section. Actually, \( u^0 \) is a scalar multiple of \( W^3, W^4 \) or \( W^5 \) in the case \( k = 2, 3 \) or 4. In such a degenerate case, \( W \) is isomorphic to a well-known vertex operator algebra (see below for details). Thus we concentrate on the cases \( k = 5 \) and 6.

We study Zhu’s algebra \( A(W) \) of \( W \) for the classification of irreducible \( W \)-modules. It turns out that the null fields \( v^0 \) and \( v^1 \) considered in Section 2 and \( u^r = (W^3)^ru^0 \), \( r = 0, 1, 2, 3 \) are sufficient to determine \( A(W) \) in the case \( k = 5, 6 \). Once all irreducible \( W^r \)-modules are known, we can show that \( W = M^{0,0} \). One more null field \( v^2 \) is necessary
for the proof of the $C_2$-cofiniteness of $\mathcal{W}$. Finally, we use [4, Proposition 5.11] to establish the rationality of the vertex operator algebra $\mathcal{W}$.

5.1. Case $k = 2$. In this case $u^0$ is a scalar multiple of $W^3$. In fact, we have $u^0 = -3W^3$. Thus $W^3 \in \tilde{I}$. Now, $W_1^3W^3 = (72/7)W^4$ by (2.8), for we are assuming that $k = 2$. Hence $W^4 \in \tilde{I}$. Then (2.9) implies that $W^5 \in \tilde{I}$. Therefore, $W^3, W^4$ and $W^5$ become 0 in $M^0$. The vertex operator algebra $\mathcal{W}$ is generated by the conformal vector $\omega$ and it is isomorphic to a simple Virasoro vertex operator algebra $L(1/2, 0)$ of central charge $1/2$. It is well known that $L(1/2, 0)$ has exactly three irreducible modules $L(1/2, h), h = 0, 1/2, 1/16$, where $L(c, h)$ denotes an irreducible highest weight module with highest weight $h$ for a Virasoro algebra of central charge $c$. Actually, $M^0, M^2$ and $M^4$ are isomorphic to those irreducible modules, respectively. Moreover, we have $\mathcal{W} = M^0$. Thus Conjecture 4.6 is true for $k = 2$.

5.2. Case $k = 3$. In this case $u^0 = -(8/13)W^4$ is a scalar multiple of $W^4$. Thus $W^4 \in \tilde{I}$. Then (2.9) with $k = 3$ implies that $W^5 \in \tilde{I}$. Hence $W^4$ and $W^5$ become 0 in $M^0$. The vertex operator algebra $\mathcal{W}$ is isomorphic to a three state Potts model $L(4/5, 0) \oplus L(4/5, 3)$. It is known that a three state Potts model has exactly six irreducible modules [17]. Moreover, we have $\mathcal{W} = M^0$. The results in [17] agree with the assertions of Conjecture 4.6. The vertex operator algebra $V_{L'} \cong V_{\sqrt{2}A_2}$ was studied in detail [17]. For the relationship between $M^0$ and a three state Potts model, see [2, Section 5], [14, Appendix B].

5.3. Case $k = 4$. In this case $u^0 = (15/22)W^5$ is a scalar multiple of $W^5$ and so $W^5$ becomes 0 in $M^0$. The vertex operator algebra $\mathcal{W}$ is isomorphic to $V_{z_\beta}^+$ with $\langle \beta, \beta \rangle = 6$, which has exactly ten irreducible modules [10]. Moreover, we have $\mathcal{W} = M^0$. The results in [10] agree with the assertions of Conjecture 4.6. The vertex operator algebra $V_{L'} \cong V_{\sqrt{2}A_2}$ was studied in detail [5].

5.4. Case $k = 5$. Let $v^0$ and $v^1$ be as in Section 2. In addition to these two null fields of $\mathcal{W}$, we consider the image of $u^0$ under the operator $W_3^3$ successively, that is, $u^r = (W_3^3)^ru^0, r = 1, 2, 3$. The weight of $u^r$ is $k + 1 + r$. We first express $u^r$ as a linear combination of the basis (2.2) of $V(k, 0)$, and then express it as a linear combination of the vectors of normal form (2.12). For instance,

$$
u^0 = -(56260915200/97)\omega_{-5}11 - (47822745600/97)\omega_{-3}\omega_{-1}11
+ (43180603200/97)(\omega_{-2})^211 + (33230937600/97)(\omega_{-1})^311
- (4032/5)(W_{-1}^3)^211 + (550368/97)\omega_{-1}W_{-1}^411 + (340704/97)W_{-3}^411.$$

This equation is obtained from the expression of the vectors $v^0, \omega_{-5}11, \omega_{-3}\omega_{-1}11, (\omega_{-2})^211, (\omega_{-1})^311, (W_{-1}^3)^211, \omega_{-1}W_{-1}^411$ and $W_{-3}^411$ as linear combinations of the basis (2.2) of $V(k, 0)$. The above expression of $u^0$ implies that $u^0 \in \tilde{W}^+$. As to $u^1, u^2$ and $u^3$, see Appendix D)

Next, take the image in Zhu’s algebra $A(\tilde{W}) = \tilde{W}/O(\tilde{W})$ of the right hand side of the expression of $u^r$ as a linear combination of the vectors of normal form. Then we can express the image $[u^r]$ of $u^r$ in $A(\tilde{W})$ as a linear combination of elements of the form

$\tilde{\nu} = \frac{k^2}{4} - \frac{k}{4}$.

\[\text{In the table of [10] page 186] the action of } J \text{ on the top level of } V_{L+\alpha/2}^\pm \text{ should read } k^2/4 - k/4.\]
(2.20). We then replace $[\omega], [W^3], [W^4]$ and $[W^5]$ with $w_2, w_3, w_4$ and $w_5$, respectively in the expression of $[u^r]$. Let $P_r$, $r = 0, 1, 2, 3$ be the polynomial in $w_2, w_3, w_4, w_5$ obtained from $u^r$ in this manner. Actually, we multiply it by a suitable integer. Then we have

\[
\begin{align*}
P_0 &= 82418000w_2^3 - 36225000w_2^2 + (1365w_4 + 2530000)w_2 - 194w_3^2 - 130w_4, \\
P_1 &= (5116834800w_2^3 - 3289532400w_2 - 49959w_4 + 190779600)w_3 \\
&- 3354260w_2w_5 + 479180w_5, \\
P_2 &= -51997017891021000w_4^2 + 301201024956142500w_2^2 \\
&+ (-8403180446500w_4 - 4771869340518000)w_2^2 \\
&+ (123120505775w_2^2 + 1890038332025w_4 + 2220670158630000)w_2 \\
&- 180544972860w_3^2 + 33957081w_3w_5 + 1437404w_2^2 - 102193394550w_4, \\
P_3 &= -46312512741411w_3^3 \\
&+ (8531538341629506000w_2^3 - 737916475955662500w_2^2 \\
&+ (433503066092460w_4 - 286997147877132000)w_2 \\
&- 37952176698930w_4 + 27372745589112000)w_3 \\
&+ 6990074602966000w_2^2w_5 - 1318615129549900w_2w_5 \\
&- 3246519796w_4w_5 + 53681912466000w_5.
\end{align*}
\]

Since $W$ is a homomorphic image of $\tilde{W}$, Zhu’s algebra $A(W)$ of $W$ is a homomorphic image of $A(\tilde{W})$. Take the composition with the surjective homomorphism $\tilde{\varphi} : \mathbb{C}[w_2, w_3, w_4, w_5] \rightarrow A(\tilde{W})$ of associative algebra considered in Section 2. Then we obtain a surjective homomorphism $\varphi : \mathbb{C}[w_2, w_3, w_4, w_5] \rightarrow A(W)$. The kernel of $\varphi$ contains the polynomials $Q_0$ and $Q_1$ studied in Section 2 with $k = 5$, for $\tilde{v}^0 = \tilde{v}^1 = 0$ in $V(k, 0)$. The kernel also contains the above four polynomials $P_0, P_1, P_2$ and $P_3$, for $u^0, u^1, u^2$ and $u^3$ lie in $\tilde{I}$. We can verify that a Gröbner basis of the ideal $P$ of $\mathbb{C}[w_2, w_3, w_4, w_5]$ generated by $P_0, P_1, P_2, P_3, Q_0$ and $Q_1$ with $k = 5$ consists of the five polynomials

\[
\begin{align*}
R_1 &= w_2(5w_2 - 6)(5w_2 - 4)(7w_2 - 6)(7w_2 - 2) \\
&\quad \cdot (35w_2 - 23)(35w_2 - 17)(35w_2 - 3)(35w_2 - 2), \\
R_2 &= w_3(5w_2 - 6)(5w_2 - 4)(35w_2 - 23)(35w_2 - 17)(35w_2 - 3)(35w_2 - 2), \\
R_3 &= p(w_2) + 564841728w_3^2, \\
R_4 &= q(w_2) + 14685884928w_4, \\
R_5 &= r(w_2)w_3 + 5575284w_5,
\end{align*}
\]

where $p(w_2), q(w_2)$ and $r(w_2)$ are polynomials in $w_2$ of degree 8, 8 and 5, respectively. The common factor of $R_1$ and $p(w_2)$ is $w_2(7w_2 - 6)(7w_2 - 2)$ and that of $R_1$ and $q(w_2)$ is $w_2$, while $r(w_2)$ has no common factor with $R_1$. The Gröbner basis implies that $\mathbb{C}[w_2, w_3, w_4, w_5]/P$ is a 15 dimensional space with basis $w_2^m + P$, $0 \leq m \leq 8$, $w_2^nw_3 + P$, $0 \leq n \leq 5$. In particular, $\mathbb{C}[w_2, w_3, w_4, w_5]/P$ is generated by $w_2 + P$ and $w_3 + P$.

We do not show that $W = M^{0,0}$ so far. Hence the 15 inequivalent irreducible $M^{0,0}$-modules $M^{i,j}$ constructed in Section 4 may not be irreducible nor inequivalent as $W$-modules. Let $N^{i,j}$ be the $W$-submodule of $M^{i,j}$ generated by $v^{i,j}$, so that the top level
of \( N^{i,j} \) is a one dimensional space \( \mathbb{C}v^{i,j} \). Then \( N^{i,j} \) has a unique maximal submodule, possibly 0. Consider the quotient module \( U^{i,j} \) of \( N^{i,j} \) by its unique maximal submodule. It is an irreducible \( \mathcal{W} \)-module with top level \( \mathbb{C}v^{i,j} \). By Proposition \ref{4.2.3} we know how \( o(\omega) \), \( o(W^3) \), \( o(W^4) \) and \( o(W^5) \) act on \( \mathbb{C}v^{i,j} \). We can verify that the 15 quartets of the eigenvalues of these four operators on \( \mathbb{C}v^{i,j} \) are all different and that they agree with the solutions \((w_2, w_3, w_4, w_5)\) of a system of equations

\[ R_1 = R_2 = R_3 = R_4 = R_5 = 0. \tag{5.1} \]

By \cite[Theorem 2.2.2]{23}, we conclude that \( A(\mathcal{W}) \cong \mathbb{C}[w_2, w_3, w_4, w_5]/\mathcal{P} \) and any irreducible \( \mathcal{W} \)-module is isomorphic to one of \( U^{i,j} \)'s. Furthermore, \( \mathbb{C}[w_2, w_3, w_4, w_5]/\mathcal{P} \) is semisimple, for the system of equations \((5.1)\) has no multiple root. Hence \( A(\mathcal{W}) \) is semisimple. Note that \( A(\mathcal{W}) \) is generated by \([\omega]\) and \([W^3]\). This is consistent with the fact that the 15 pairs of the eigenvalues of \( o(\omega) \) and \( o(W^3) \) on \( \mathbb{C}v^{i,j} \) are all different. That is, \( o(\omega) \) and \( o(W^3) \) are sufficient to distinguish \( U^{i,j} \)'s.

Now, suppose \( \mathcal{W} \neq M^{0,0} \) and consider the quotient \( \mathcal{W} \)-module \( M^{0,0}/\mathcal{W} \). It has integral weights. We can easily verify that the weight \( n \) subspace of \( \mathcal{W} \) coincides with that of \( M^{0,0} \) for a small \( n \), say \( n = 0, 1, 2 \). Therefore, the weight of any irreducible quotient of \( M^{0,0}/\mathcal{W} \) is greater than 2. This is a contradiction since the weight of the top level of \( U^{i,j} \) is at most 6/5 and 0 is the only integral one. Thus \( \mathcal{W} = M^{0,0} \) and the assertion (2) of Conjecture \ref{4.6} is true for \( k = 5 \).

It remains to prove the \( C_2 \)-cofiniteness and the rationality of \( \mathcal{W} \). We have studied \( u^r \), \( r = 0, 1, 2, 3 \) and the null fields \( v^0 \) and \( v^1 \) modulo \( O(\tilde{\mathcal{W}}) \) for the determination of Zhu’s algebra of \( \mathcal{W} \). As to the proof of the \( C_2 \)-cofiniteness, we consider \( u^r \), \( r = 0, 1, 2, 3 \) and the null fields \( v^0 \), \( v^1 \) and \( v^2 \) modulo \( C_2(\tilde{\mathcal{W}}) \). In fact, \( u^r \), \( r = 0, 1, 2, 3 \), \( v^0 \) and \( v^1 \) are not sufficient to show the \( C_2 \)-cofiniteness. Take the image in \( \tilde{\mathcal{W}}/C_2(\tilde{\mathcal{W}}) \) of the right hand side of the expression of \( u^r \) as a linear combination of vectors of normal form given in Appendix \ref{D}. It is a polynomial in \( \omega + C_2(\tilde{\mathcal{W}}) \) and \( W^i + C_2(\tilde{\mathcal{W}}) \), \( i = 3, 4, 5 \). Replace \( \omega + C_2(\tilde{\mathcal{W}}) \) and \( W^i + C_2(\tilde{\mathcal{W}}) \) with \( x_2 \) and \( x_i \), \( i = 3, 4, 5 \), respectively in the polynomial and multiply it by a suitable integer. Let \( A_r \), \( r = 0, 1, 2, 3 \) be the polynomial obtained from \( u^r \) in this manner. Then

\[
A_0 = 82418000x_2^3 + 1365x_2x_3^2 - 194x_3^2,
\]
\[
A_1 = 730976400x_3x_2^2 - 479180x_2x_3^2 - 7137x_4x_3,
\]
\[
A_2 = 51997017891021000x_2^4 + 8403180446500x_2x_3^2 - 1231205050775x_2^2x_3
\quad - 33957081x_2x_3^2 - 1437404x_4,
\]
\[
A_3 = 8531538341629506000x_3x_2^3 + 6990074602966000x_2x_3^2
\quad + 433503066092460x_2x_3^2 - 46312512741411x_3^3 - 3246519796x_3x_4.
\]

Let \( C_2(\mathcal{W}) \) be the subspace of \( \mathcal{W} \) spanned by \( u_{-2}v \) with \( u, v \in \mathcal{W} \). Since \( \mathcal{W} \) is a homomorphic image of \( \tilde{\mathcal{W}} \), there is a homomorphism from \( \tilde{\mathcal{W}}/C_2(\tilde{\mathcal{W}}) \) onto \( \mathcal{W}/C_2(\mathcal{W}) \). Its composition \( \rho \) with the surjective homomorphism \( \tilde{\rho} : \mathbb{C}[x_2, x_3, x_4, x_5] \to \tilde{\mathcal{W}}/C_2(\tilde{\mathcal{W}}) \) discussed in Section \ref{2} is a homomorphism from \( \mathbb{C}[x_2, x_3, x_4, x_5] \) onto \( \mathcal{W}/C_2(\mathcal{W}) \). The kernel of \( \rho \) contains the polynomials \( B_0, B_1 \) and \( B_2 \) studied in Section \ref{2} with \( k = 5 \). It also contains the above four polynomials \( A_r \), \( r = 0, 1, 2, 3 \), for \( u^r \) is 0 in \( \mathcal{W} \). Let \( A \) be the
ideal of \( \mathbb{C}[x_2, x_3, x_4, x_5] \) generated by \( A_r, r = 0, 1, 2, 3 \) and \( B_s, s = 0, 1, 2 \) with \( k = 5 \). We can verify that a Gröbner basis of \( \mathcal{A} \) consists of the eleven polynomials

\[
\begin{align*}
S_1 &= x_2^6, \\
S_2 &= x_3x_4^4, \\
S_3 &= 2780750x_2^5 - 29x_3^2x_2^2, \\
S_4 &= 378000x_3x_2^3 - x_3^3, \\
S_5 &= 82418000x_2^2 + 1365x_4x_2 - 194x_3^2, \\
S_6 &= 33674025000x_2^5 + 377x_4x_3^2, \\
S_7 &= 804763750000x_2^4 + 61327280x_3x_2^2 - 2379x_4^2, \\
S_8 &= 730976400x_3x_2^2 - 479180x_5x_2 - 7137x_4x_3, \\
S_9 &= 4633930000x_2^4 - 28315x_3^2x_2 - 13x_5x_3, \\
S_{10} &= 2018093000x_3x_2^3 + 13x_5x_4, \\
S_{11} &= 173625253725000x_2^5 - 377x_5^2.
\end{align*}
\]

From \( S_1, S_4, S_7 \) and \( S_{11} \), we see that \( \mathbb{C}[x_2, x_3, x_4, x_5]/\mathcal{A} \) is finite dimensional. This establishes the \( C_2 \)-cofiniteness of \( \mathcal{W} \) for \( k = 5 \).

The set of eigenvalues of the action of \( o(\omega) \) on \( \mathbb{C}v^{i,j}, 0 \leq i \leq 5, 0 \leq j \leq i - 1 \) is

\[
\mathcal{E} = \{0, 2/35, 3/35, 2/7, 17/35, 23/35, 6/7, 4/5, 6/5\}.
\]

The difference of any two rational numbers in the set \( \mathcal{E} \) is not an integer. Thus by a similar argument as in the proof of [4, Lemma 5.13], we have that any \( \mathcal{W} \)-module generated by an irreducible \( A(\mathcal{W}) \)-module is irreducible. Hence the vertex operator algebra \( \mathcal{W} \) is rational by [4, Proposition 5.11]. Thus the assertion (3) of Conjecture [4.6] is true for \( k = 5 \).

5.5. Case \( k = 6 \). The argument is essentially the same as in the case \( k = 5 \). The six singular vectors \( u^0, u^1, u^2, u^3, v^0 \) and \( v^1 \) are sufficient for the determination of Zhu’s algebra \( A(\mathcal{W}) \), while we need one more singular vector \( v^2 \) for the proof of the \( C_2 \)-cofiniteness of \( \mathcal{W} \). Indeed, we have

\[
u^0 = -((1420529376000/55483)\omega_3 W_3 \mathbb{1} + (1356106752000/55483)\omega_{-1}^3 W_3 \mathbb{1}) + (19141808000/491)\omega_2 W_2 \mathbb{1} - (2212337344000/55483)\omega_{-1} W_3 \mathbb{1} + (2043429304000/55483)W_3^2 \mathbb{1} - (33950/339)W_3^2 W_4 \mathbb{1} - (4632320/491)W_{-5} \mathbb{1} - (1995840/491)W_{-3}^5 \mathbb{1}
\]

in the case \( k = 6 \). In particular, \( v^0 \in \mathcal{W}^- \).

Set \( u^r = (W_3)^r u^0, r = 1, 2, 3 \) as in Section 5.4. We consider the polynomial \( P_r \in \mathbb{C}[w_2, w_3, w_4, w_5] \) obtained from \( u^r \) in a similar manner as in the case \( k = 5 \). Let \( \mathcal{P} \) be the ideal of \( \mathbb{C}[w_2, w_3, w_4, w_5] \) generated by the six polynomials \( P_r, r = 0, 1, 2, 3, Q_0 \) and \( Q_1 \) with \( k = 6 \). We calculate that a Gröbner basis of \( \mathcal{P} \) consists of the following five
polynomials.

\[
R_1 = w_2(2w_2 - 3)(3w_2 - 4)(4w_2 - 3)(4w_2 - 1)(6w_2 - 5)(12w_2 - 7)(12w_2 - 1) \\
\quad \cdot (32w_2 - 23)(32w_2 - 3)(96w_2 - 101)(96w_2 - 41)(96w_2 - 5),
\]

\[
R_2 = w_3(3w_2 - 4)(6w_2 - 5)(12w_2 - 7)(12w_2 - 1) \\
\quad \cdot (32w_2 - 23)(96w_2 - 101)(96w_2 - 41)(96w_2 - 5),
\]

\[
R_3 = p(w_2) + 2399941984319748410712448453175w_5^2,
\]

\[
R_4 = q(w_2) + 5999854960799371026781121329375w_4,
\]

\[
R_5 = r(w_2)w_3 + 171818959801082568975w_5,
\]

where \(p(w_2), q(w_2)\) and \(r(w_2)\) are polynomials in \(w_2\) of degree 12, 12 and 7, respectively. The common factor of \(R_1\) and \(p(w_2)\) is \(w_2(2w_2 - 3)(4w_2 - 3)(4w_2 - 1)(32w_2 - 3)\) and that of \(R_1\) and \(q(w_2)\) is \(w_2\), while \(r(w_2)\) has no common factor with \(R_1\). The Gröbner basis implies that \(\mathbb{C}[w_2, w_3, w_4, w_5]/\mathcal{P}\) is a 21 dimensional space with basis \(w_2^m + \mathcal{P}, 0 \leq m \leq 12, w_2^3w_3 + \mathcal{P}, 0 \leq n \leq 7\).

In the case \(k = 6\), we have 21 inequivalent irreducible \(M^{0,0}\)-modules \(M^{i,j}\). Consider an irreducible subquotient \(U^{i,j}\) of \(M^{i,j}\) as in the case \(k = 5\). The 21 quartets of the eigenvalues of \(o(\omega)\), \(o(W^3)\), \(o(W^4)\) and \(o(W^5)\) on \(\mathbb{C}v^{i,j}\) are all different and they agree with the solutions \((w_2, w_3, w_4, w_5)\) of a system of equations \(R_1 = R_2 = R_3 = R_4 = R_5 = 0\). Thus \(A(W)\) is isomorphic to \(\mathbb{C}[w_2, w_3, w_4, w_5]/\mathcal{P}\). Moreover, it is semisimple and generated by \([\omega]\) and \([W^3]\).

The set of eigenvalues of \(o(\omega)\) on \(\mathbb{C}v^{i,j}\), \(0 \leq i \leq 6, 0 \leq j \leq i - 1\) is

\[
\mathcal{E} = \{0, 5/96, 1/12, 1/4, 3/32, 41/96, 7/12, 3/4, 23/32, 101/96, 5/6, 4/3, 3/2\}.
\]

Thus the weight of \(v^{i,j}\) is at most 3/2 and 0 is the only integral one. By a similar argument as in the case \(k = 5\), we have that \(W = M^{0,0}\) and the assertion (2) of Conjecture 4.6 holds.

For the proof of the \(C_2\)-cofiniteness of \(W\), we use \(u^r, r = 0, 1, 2, 3\) and \(v^s, s = 0, 1, 2, 3\). Consider seven polynomials \(A_r, r = 0, 1, 2, 3\) and \(B_s, s = 0, 1, 2\) obtained in a similar manner as in the case \(k = 5\). We can verify that the ideal \(\mathcal{A}\) generated by these seven polynomials is of finite codimension in \(\mathbb{C}[x_2, x_3, x_4, x_5]\). Thus \(W\) is \(C_2\)-cofinite.

Let \(U\) be an irreducible \(A(W)\)-module. Then \(U = \mathbb{C}u\) is one dimensional and \(L(0)u = \lambda u\) for some \(\lambda \in \mathcal{E}\). We want to show that any \(W\)-module \(M\) generated by \(U\) is irreducible (see [4, Lemma 5.13]). If \(\lambda \neq 5/96\), then \(\mathcal{E} \cap (\lambda + \mathbb{Z}) = \{\lambda\}\) and so there is no singular vector for \(W\) of weight greater than \(\lambda\) in \(M\). By a similar argument as in the proof of [4, Lemma 5.13], we obtain that \(M\) is an irreducible \(W\)-module.

Suppose \(\lambda = 5/96\). We can assume that \(u = v^{1,0}\) or \(u = v^{5,0}\). The \(W\)-module \(M\) is spanned by the elements of the form \((2.14)\). Thus the weight \(\lambda + 1\) subspace of \(M\) is spanned by \(L(-1)u, W^p(-1)u, p = 3, 4, 5\). Let

\[
v = c_1L(-1)u + c_2W^3(-1)u + c_3W^4(-1)u + c_4W^5(-1)u
\]

be an element of \(M\) of weight \(\lambda + 1\).

We first consider the case \(u = v^{1,0}\). Then \(L(0), W^3(0), W^4(0)\) and \(W^5(0)\) act on \(u\) as \(5/96, 20, 780\) and \(-1560\), respectively. We study \(L(1)v\) and \(W^p(1)v, p = 3, 4, 5\). Each of these vectors is a scalar multiple of \(u\). Let \(L(1)v = \eta_1u, W^p(1)v = \eta_{p-1}u, p = 3, 4, 5\).
Using the expression of $W^i_W^j$, $3 \leq i \leq j \leq 5$, $0 \leq n \leq i+j-1$ as a linear combination of vectors of normal form given in Appendix B together with basic formulas (2.10) and (2.11), we can determine the constant $\eta_p$, $p = 1, 2, 3, 4$. In fact, a suitable constant multiple $F_p$ of $\eta_p$ is as follows.

\[
F_1 = c_1 + 576c_2 + 29952c_3 - 74880c_4,
\]
\[
F_2 = 113c_1 + 65088c_2 + 3384576c_3 + 400721629860c_4,
\]
\[
F_3 = 13c_1 + 7488c_2 + 13498935756c_3 - 973440c_4.
\]
\[
F_4 = 6217083815033c_1 + 16960079666412680418c_2 + \ldots
\]
\[
+ 18621409427868416c_3 - 62241257449122360326409060c_4.
\]

We can verify that a system of equations $F_1 = F_2 = F_3 = F_4 = 0$ has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. That is, there is no nonzero vector $v$ of weight $\lambda + 1$ in $M$ such that $L(1)v = 0$ and $W^p(1)v = 0$, $p = 3, 4, 5$. Since $\mathcal{E} \cap (\lambda + \mathbb{Z}) = \{\lambda, \lambda + 1\}$, this implies that $M$ has no singular vector of weight greater than $\lambda$. Hence $M$ is irreducible.

Next, we deal with the case $u = v^{5,0}$. The operators $L(0)$, $W^3(0)$, $W^4(0)$ and $W^5(0)$ act on $u$ as $5/96$, $-20$, $780$ and $1560$, respectively. We only need to replace $c_3$ with $-c_3$ and $c_5$ with $-c_5$ in $F_1$, $F_2$, $F_3$ and $F_4$, and we obtain that $M$ is irreducible.

Now, we can apply [4, Proposition 3.11] to conclude that the vertex operator algebra $W$ is rational. Thus the assertion (3) of Conjecture 4.6 is true for $k = 6$.

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**Appendix A. Virasoro primary vectors $W^3$, $W^4$ and $W^5$**

$W^3 = k^2h(-3)1 + 3kh(-2)h(-1)1 + 2h(-1)^31 - 6kh(-1)e(-1)f(-1)1$
$+ 3k^2e(-2)f(-1)1 - 3k^2e(-1)f(-2)1,$

$W^4 = -2k^2(k^2 + k + 1)h(-4)1 - 8k(k^2 + k + 1)h(-3)h(-1)1 - k(5k^2 - 6)h(-2)^21$
$- 2k(11k + 6)h(-2)h(-1)^21 - (11k + 6)h(-1)^41 + 4k^2(6k - 5)h(-2)e(-1)f(-1)1$
$+ 4k(11k + 6)e(-1)f(-1)1 - 4k^2(5k + 11)h(-1)e(-2)f(-1)1$
$+ 4k^2(5k + 11)h(-1)e(-1)f(-2)1 + 8k^2(k - 3)(k - 2)e(-3)f(-1)1$
$- 4k^2(3k^2 - 3k + 8)e(-2)f(-2)1 - 2k^2(6k - 5)e(-1)^2f(-1)1$
$+ 8k^2(k^2 + k + 1)e(-1)f(-3)1,$
\[ W^5 = -2k^3(k^2 + 3k + 5)h(-5)I - 10k^2(k^2 + 3k + 5)h(-4)h(-1)I \]
\[ - 5k^2(3k^2 - 4)h(-3)h(-2)I - 5k(7k^2 + 12k + 16)h(-3)h(-1)I \]
\[ - 15k(3k^2 - 4)h(-2)h(-1)I - 5k(19k + 12)h(-2)h(-1)^3I - 2(19k + 12)h(-1)^5I \]
\[ + 10k^2(4k^2 - 7k + 8)h(-3)e(-1)f(-1)I + 20k^2(10k - 7)h(-2)h(-1)e(-1)I \]
\[ + 10k(19k + 12)h(-1)^3e(-1)f(-1)I - 5k^2(11k^2 - 14k + 12)h(-2)e(-2)f(-1)I \]
\[ - 5k^2(17k + 64)h(-1)^2e(-2)f(-1)I + 15k^2(3k^2 - 4)h(-2)e(-1)f(-2)I \]
\[ + 5k^2(17k + 64)h(-1)^2e(-1)f(-2)I + 30k^2(k - 4)(k - 3)h(-1)e(-3)f(-1)I \]
\[ - 40k^2(k^2 + 3k + 5)h(-1)e(-2)f(-2)I - 10k^2(10k - 7)h(-1)e(-1)^2f(-1)I \]
\[ + 10k^2(3k^2 + 19k + 8)h(-1)e(-1)f(-3)I - 10k^3(k - 4)(k - 3)e(-4)f(-1)I \]
\[ + 20k^3(k - 4)(k - 3)e(-3)f(-2)I + 5k^3(10k - 7)e(-2)e(-1)f(-1)I \]
\[ - 10k^3(2k^2 - 4k + 17)e(-2)f(-3)I - 5k^3(10k - 7)e(-1)^2f(-2)f(-1)I \]
\[ + 10k^3(k^2 + 3k + 5)e(-1)f(-4)I. \]

**APPENDIX B.** \( W^i_nW^j, 3 \leq i \leq j \leq 5, 0 \leq n \leq i + j - 1 \)

**B.1.** \( W^3_nW^3, 0 \leq n \leq 5. \)

\[ W^3_5W^3 = 12k^3(k - 2)(k - 1)(3k + 4)I, \]
\[ W^3_4W^3 = 0, \]
\[ W^3_3W^3 = 36k^3(k - 2)(k + 2)(3k + 4)\omega_{-1}I, \]
\[ W^3_2W^3 = 18k^3(k - 2)(k + 2)(3k + 4)\omega_{-1}I, \]
\[ W^3_1W^3 = -(162k^3(k - 2)(k + 2)(3k + 4)/(16k + 17))\omega_{-3}I \]
\[ + (288k^3(k - 2)(k + 2)^2(3k + 4)/(16k + 17))\omega_{-1}\omega_{-1}I \]
\[ + (36k(2k + 3)/(16k + 17))W^4_1I, \]
\[ W^3_0W^3 = -(108k^3(k - 2)(k + 2)(3k + 4)/(16k + 17))\omega_{-4}I \]
\[ + (288k^3(k - 2)(k + 2)^2(3k + 4)/(16k + 17))\omega_{-2}\omega_{-1}I \]
\[ + (18k(2k + 3)/(16k + 17))W^4_2I. \]

**B.2.** \( W^3_nW^4, 0 \leq n \leq 6. \)

\[ W^3_5W^4 = W^3_4W^4 = W^3_3W^4 = W^3_2W^4 = 0, \]
\[ W^3_1W^4 = 48k^2(k - 3)(2k + 1)(2k + 3)W^3_1I, \]
\[ W^3_0W^4 = 16k^2(k - 3)(2k + 1)(2k + 3)W^3_2I, \]
\[ W^3_1W^4 = (1248k^2(k - 3)(k + 2)(2k + 1)(2k + 3)/(64k + 107))\omega_{-1}W^3_1I \]
\[ - (48k^2(k - 3)(2k + 1)(2k + 3)(2k + 7)/(64k + 107))W^3_3I \]
\[ - (12k(3k + 4)(16k + 17)/(64k + 107))W^5_1I, \]
\[ W^3_0W^4 = (120k^2(k - 3)(k + 2)(2k + 3)(16k + 17)/(64k + 107))\omega_{-2}W^3_1I \]
\[ + (48k^2(k - 3)(k + 2)(2k + 3)(8k - 11)/(64k + 107))\omega_{-1}W^3_2I \]
\[ - (12k^2(k - 3)(2k + 3)(32k^2 + 47k - 52)/(64k + 107))W^3_4I \]
\[ - (24k(3k + 4)(16k + 17)/(5(64k + 107)))W^5_2I. \]
B.3. $W_n^3 W^5$, $0 \leq n \leq 7$.

\[
W_2^3 W^5 = W_6^3 W^5 = W_3^3 W^5 = W_4^3 W^5 = 0,
\]
\[
W_2^3 W^5 = -(15/2)k^2(k - 4)(5k + 8)W^4_1 \mathbb{I},
\]
\[
W_3^3 W^5 = -(6480k^4(k + 2)(k + 3)(2k + 3)\omega_5 \mathbb{I}
- (360k^4(k + 2)^2(2k + 3)(3k + 4)(32k^2 + 797k + 863)/(16k + 17))\omega_3 \omega_1 \mathbb{I}
+ (45k^4(k + 2)^2(2k + 3)(3k + 4)(1408k^2 + 1315k - 977)/(16k + 17))\omega_2 \omega_2 \mathbb{I}
+ (240k^4(k + 2)^3(2k + 3)(3k + 4)(202k - 169)/(16k + 17))\omega_1 \omega_1 \omega_1 \mathbb{I}
- 15k(2k + 3)(41k + 61)W^3_1 W^3_1 \mathbb{I}
+ (60k^2(k + 2)(404k^2 + 1170k + 835)/(16k + 17))\omega_1 W^4_1 \mathbb{I}
+ (15k^2(2176k^3 + 9481k^2 + 13792k + 6708)/(2(16k + 17)))W^3_3 \mathbb{I},
\]
\[
W_4^3 W^5 = -(3240k^4(k + 2)(k + 3)(2k + 3)(3k + 4)(12k^2 + 8k - 17)/(16k + 17))\omega_6 \mathbb{I}
- (120k^4(k + 2)^2(2k + 3)(3k + 4)(184k^2 + 1669k + 1801)/(16k + 17))\omega_4 \omega_1 \mathbb{I}
+ (2700k^4(k + 2)^2(2k + 3)(3k + 4)(8k^2 + 5k + 5)/(16k + 17))\omega_3 \omega_2 \mathbb{I}
+ (240k^4(k + 2)^3(2k + 3)(3k + 4)(202k - 169)/(16k + 17))\omega_2 \omega_1 \omega_1 \mathbb{I}
- 10k(2k + 3)(41k + 61)W^3_2 W^3_1 \mathbb{I}
+ (60k^2(k + 2)(2k + 3)(64k + 107)/(16k + 17))\omega_2 W^4_1 \mathbb{I}
+ (60k^2(k + 2)(138k^2 + 382k + 257)/(16k + 17))\omega_1 W^4_2 \mathbb{I}
+ (15k^2(104k^3 + 317k^2 + 308k + 108)/(16k + 17))W^4_4 \mathbb{I}.
\]

B.4. $W_n^4 W^4$, $0 \leq n \leq 7$.

\[
W_2^4 W^4 = 16k^4(k - 3)(k - 2)(k - 1)(2k + 1)(3k + 4)(16k + 17)\mathbb{I},
\]
\[
W_6^4 W^4 = 0,
\]
\[
W_3^4 W^4 = 64k^4(k - 3)(k - 2)(k + 2)(2k + 1)(3k + 4)(16k + 17)\omega_1 \mathbb{I},
\]
\[
W_4^4 W^4 = 32k^4(k - 3)(k - 2)(k + 2)(2k + 1)(3k + 4)(16k + 17)\omega_2 \mathbb{I},
\]
\[
W_5^4 W^4 = -96k^4(k - 3)(k - 2)(k + 2)(k + 5)(2k + 1)(3k + 4)\omega_3 \mathbb{I}
+ 672k^4(k - 3)(k - 2)(k + 2)^2(2k + 1)(3k + 4)\omega_1 \omega_1 \mathbb{I}
+ 72k^4(4k^3 - 15k^2 - 33k - 4)W^4_1 \mathbb{I},
\]
\[
W_7^4 W^4 = -64k^4(k - 3)(k - 2)(k + 2)(k + 5)(2k + 1)(3k + 4)\omega_4 \mathbb{I}
+ 672k^4(k - 3)(k - 2)(k + 2)^2(2k + 1)(3k + 4)\omega_2 \omega_1 \mathbb{I}
+ 36k^2(4k^3 - 15k^2 - 33k - 4)W^4_2 \mathbb{I},
\]
\[ W_1^4 W^4 = 1920k^4(k + 2)(2k + 3)(3k + 4)(4k^3 + 12k^2 - 4k - 9)\omega \cdot 5 \mathbb{1} \]
\[ + 32k^5(2k + 1)(3k + 4)(6k^2 + 180k + 53k + 13)\omega \cdot 1 \]
\[ - 120(k + 2)^2(3k + 4)(5k + 4)(6k + 5)\omega \cdot 2 \]
\[ + 8k(k + 1)(16k + 17)^2 W^3_1 W^3_1 \]
\[ - 48k^2(k + 2)(52k^2 + 109k + 50)\omega \cdot 1 W^3_1 \]
\[ - 4k^2(142k^3 + 503k^2 + 1753k + 640)W^4_1, \]
\[ W_0^4 W^4 = 1440k^4(k + 2)(2k + 3)(3k + 4)(4k^3 + 12k^2 - 4k - 9)\omega \cdot 6 \]
\[ + 96k^4(k + 2)^2(3k + 4)(11k + 13)(6k + 55k + 41)\omega \cdot 1 \]
\[ - 48k^4(k + 2)^2(3k + 4)(136k^3 + 180k^2 + 161k + 75)\omega \cdot 2 \]
\[ - 480k^4(k + 2)^3(3k + 4)(5k + 4)(6k + 5)\omega \cdot 1 \]
\[ + 8k(k + 1)(16k + 17)^2 W^3_1 W^3_1 \]
\[ - 24k^2(k + 2)(52k^2 + 109k + 50)\omega \cdot 2 W^4_1 \]
\[ - 24k^2(k + 2)(52k^2 + 109k + 50)\omega \cdot 1 W^4_1 \]
\[ - 12k^2(20k^2 + 45k + 29k + 8)W^4_1. \]

B.5: \( W_1^4 W^5 \), \( 0 \leq n \leq 8 \)
\[ W_8^4 W^5 = W_7^4 W^5 = W_6^4 W^5 = 0, \]
\[ W_5^4 W^5 = -(40k^3(k - 2)(k - 3)(2k + 1)(5k + 8)(16k + 17)W^3_1 \mathbb{1}, \]
\[ W_4^4 W^5 = -(40/3)k^3(k - 2)(k - 3)(2k + 1)(5k + 8)(16k + 17)W^3_2 \mathbb{1}, \]
\[ W_3^4 W^5 = -(1320k^3(k - 4)(k - 3)(2k + 1)(5k + 8)(16k + 17)/(64k + 107))\omega \cdot 1 W^3_1 \mathbb{1} \]
\[ + (40k^3(k - 4)(k - 3)(2k + 1)(5k + 8)(16k + 17)/(64k + 107))W^3_2 \mathbb{1} \]
\[ + (180k^2(3k + 4)(32k^3 - 236k^2 - 535k - 125)/(64k + 107))W^4_1 \mathbb{1}, \]
\[ W_2^4 W^5 = -(160k^3(k - 4)(k - 3)(2k + 1)(5k + 8)(16k + 17)/(64k + 107))\omega \cdot 2 W^3_1 \mathbb{1} \]
\[ - (80/3)k^3(k - 4)(k - 3)(2k + 1)(5k + 8)(14k - 23)(16k + 17)/(64k + 107))\omega \cdot 2 W^3_2 \mathbb{1} \]
\[ + (60k^3(k - 4)(k - 3)(16k + 17)(8k^2 + 12k - 11)/(64k + 107))W^4_1 \mathbb{1} \]
\[ + (72k^2(3k + 4)(32k^3 - 236k^2 - 535k - 125)/(64k + 107))W^5_1 \mathbb{1}, \]
\[ W_1^4 W^5 = (20k^3(k + 2)(5k + 8)(2624k^4 - 83108k^3 + 341706k^2 - 433511k + 177319)/(64k + 107))\omega \cdot 3 W^3_1 \mathbb{1} \]
\[ + (120k^3(k + 2)^2(2k + 1)(5k + 8)(16k - 9)(75k + 74)/(64k + 107))\omega \cdot 1 W^3_1 \mathbb{1} \]
\[ + ((10/3)k^3(k + 2)(5k + 8)(16k + 17)(7960k^3 + 18296k^2 + 6119k - 4457)/(64k + 107))\omega \cdot 2 W^3_2 \mathbb{1} \]
\[ - ((40/3)k^3(k + 2)(5k + 8)(28800k^4 + 128704k^3 + 133404k^2 - 43341k - 76171) \]
\[ / (64k + 107))\omega \cdot 1 W^3_2 \mathbb{1} \]
\[ + ((20/3)k^3(k + 8)(5k + 8)(45440k^5 + 358008k^4 + 884944k^3 + 619369k^2 - 360351k - 404824) \]
\[ / (64k + 107))W^5_1 \mathbb{1} \]
\[ - 10k(5k + 8)(20k + 19)W^3_1 W^4_1 \mathbb{1}, \]
\[ - (40k^2(k + 2)(3k + 4)(116k^2 + 55k + 1171)/(64k + 107))\omega \cdot 1 W^5_1 \mathbb{1} \]
\[ - (120k^3(2k + 3)(3k + 4)(88k^2 + 202k + 97)/(64k + 107))W^5_1 \mathbb{1}. \]
\[ W_6^4 W^5 = - (60k^3(k + 2)(5k + 8)(16k + 17)(236k^3 + 2566k^2 + 5577k + 3207)/(64k + 107)) \omega^{-4} W_1^3 1 + (240k^3(k + 1)(k + 2)^2(5k + 8)(10k - 7)(16k + 17)/(64k + 107)) \omega^{-2} W_1^3 1 + (40k^3(k + 2)(5k + 8)(6976k^4 + 26048k^3 + 28428k^2 + 2426k - 6881)/(64k + 107)) \omega^{-1} W_1^3 1 \]

B.6. \( W_n^5 W^5 \), 0 \( \leq n \leq 9 \).

\[ W_0^5 W^5 = 40k^5(k - 4)(k - 3)(k - 2)(k - 1)(2k + 1)(5k + 8)(64k + 107) 1, \]
\[ W_1^5 W^5 = 0, \]
\[ W_2^5 W^5 = 200k^5(k - 4)(k - 3)(k - 2)(2k + 1)(5k + 8)(64k + 107) \omega^{-1} 1, \]
\[ W_3^5 W^5 = 100k^5(k - 4)(k - 3)(k - 2)(2k + 1)(5k + 8)(64k + 107) \omega^{-2} 1, \]
\[ W_4^5 W^5 = -(300k^5(k - 4)(k - 3)(k - 2)(2k + 1)(2k + 7)(5k + 8)(64k + 107)/(16k + 17)) \omega^{-1} W_1^3 1 + (2600k^5(k - 4)(k - 3)(k - 2)(k + 2)^2(2k + 1)(5k + 8)(64k + 107)/(16k + 17)) \omega^{-1} W_1^3 1 \]
\[ + (450k^5(k - 4)(5k + 8)(32k^3 - 236k^2 - 535k - 125)/(16k + 17)) W_1^4 1, \]
\[ W_5^5 W^5 = -(200k^5(k - 4)(k - 3)(k - 2)(2k + 1)(2k + 7)(5k + 8)(64k + 107)/(16k + 17)) \omega^{-4} W_1^3 1 + (2600k^5(k - 4)(k - 3)(k - 2)(k + 2)^2(2k + 1)(5k + 8)(64k + 107)/(16k + 17)) \omega^{-1} W_1^3 1 \]
\[ + (225k^5(k - 4)(5k + 8)(32k^3 - 236k^2 - 535k - 125)/(16k + 17)) W_1^4 1, \]
\[ W_6^5 W^5 = (400k^5(k + 2)(9728k^7 - 345370k^6 - 2884229k^5 - 7339690k^4 - 5652707k^3 + 3682145k^2 + 6580220k + 1862400)/(16k + 17)) \omega^{-5} 1 \]
\[ - (600k^5(k + 2)^2(256k^6 + 790k^5 + 1054568k^4 + 4865734k^3 + 8044197k^2 + 5415116k + 1171136)/(16k + 17)) \omega^{-3} W_1^3 1 \]
\[ - (25k^5(k + 2)^2(137728k^6 - 4923496k^5 - 24095252k^4 - 35522641k^3 - 12391265k^2 + 8406500k + 3657600)/(16k + 17)) \omega^{-2} W_1^3 1 \]
\[ - (200k^5(k + 2)^3(12288k^5 - 487882k^4 - 1447853k^3 - 491400k^2 + 1135840k + 463040)/(16k + 17)) \omega^{-1} W_1^3 1 \]
\[ + (150k^5(k + 2)(1632k^4 - 54468k^3 - 209305k^2 - 225706k - 59200)/(16k + 17)) W_1^3 1 + 25k^5(544k^4 - 16660k^3 - 65657k^2 - 72453k - 19600) W_1^3 1 + (150k^5(k + 2)(1632k^4 - 54468k^3 - 209305k^2 - 225706k - 59200)/(16k + 17)) \omega^{-1} W_1^3 1 \]
\[ - (25k^5(6816k^5 - 206652k^4 - 1172123k^3 - 2196873k^2 - 1637466k - 371200)/(16k + 17)) W_1^3 1, \]
\[ W_2^5 W^5 = \frac{(300k^5(k + 2)(9728k^7 - 345370k^6 - 2884229k^5 - 7339690k^4 - 5652707k^3 + 3682145k^2
+ 6580220k + 1862400)/(16k + 17))\omega_{-6}\mathbb{I}}{\omega_{-6} - 6}\] 
\[ + \frac{(100k^5(k + 2)^2(11264k^6 - 476132k^5 - 8238118k^4 - 32234405k^3 - 50927083k^2
- 34091060k - 7406592)/(16k + 17))\omega_{-4}\mathbb{I}}{\omega_{-4} - \omega_{-1}\mathbb{I}} - \frac{(150k^5(k + 2)^2(12800k^6 - 428732k^5 - 1910710k^4 - 3040001k^3 - 2661901k^2
- 1600364k - 379776)/(16k + 17))\omega_{-2}\mathbb{I}}{\omega_{-2} - \omega_{-1}\mathbb{I}} - \frac{(300k^5(k + 2)^3(12288k^5 - 487882k^4 - 1447853k^3 - 491400k^2 + 1135840k + 463040)/(16k + 17))\omega_{-1}\mathbb{I}}{\omega_{-1} - \omega_{-1}\mathbb{I}} + \frac{25k^2(544k^4 - 16660k^3 - 65657k^2 - 72453k - 19600)W_{-2}^3W_{-1}\mathbb{I}}{\omega_{-2}W_{-1}\mathbb{I}} - \frac{(75k^3(k + 2)(1632k^4 - 54468k^3 - 209305k^2 - 225706k - 59200)/(16k + 17))\omega_{-2}W_{-1}\mathbb{I}}{\omega_{-2}W_{-1}\mathbb{I}} - \frac{(75k^3(k + 2)(1632k^4 - 54468k^3 - 209305k^2 - 225706k - 59200)/(16k + 17))\omega_{-1}W_{-2}\mathbb{I}}{\omega_{-2}W_{-1}\mathbb{I}} - \frac{(75k^3(384k^5 - 10608k^4 - 43480k^3 - 52785k^2 - 21126k - 3200)/(16k + 17))W_{-1}\mathbb{I}}{\omega_{-1}W_{-2}\mathbb{I}}. \]
\[ W_1^5 W_5 = -(25k^5(k + 2)(170027057152k^{10} + 2356580095488k^9 + 13676829114720k^8 \\
+ 42735238046312k^7 + 7479365683474k^6 + 6082864002771k^5 \\
- 186300342678k^4 - 94115224713312k^3 - 92379208458276k^2 \\
- 41524853935184k - 7347787324608)/(17(k + 1)(16k + 17)^2(64k + 107)) \omega_7 \mathbb{I} \\
- (600k^5(k + 2)^2(3950979072k^9 + 64351956480k^8 + 426934964416k^7 + 1559551526014k^6 \\
+ 3508072702889k^5 + 580429991599k^4 + 475848076095k^3 + 2785329492007k^2 \\
+ 924445415740k + 132238676112)/(17(k + 1)(16k + 17)^2(64k + 107)) \omega_5 \omega_1 \mathbb{I} \\
+ (150k^5(k + 2)^2(1929743840k^9 + 207821377280k^8 + 951110388112k^7 + 2382556615296k^6 \\
+ 3466010214237k^5 + 2750850253947k^4 + 729192953742k^3 - 542736421532k^2 \\
- 472337350908k - 105657243168)/(17(k + 1)(16k + 17)^2(64k + 107)) \omega_4 \omega_2 \mathbb{I} \\
- (600k^5(k + 2)^2(9399296k^9 + 115780640k^8 + 833628608k^7 + 3394583580k^6 \\
+ 6845449350k + 353888811k^4 - 10856512820k^3 - 2151965045k^2 \\
- 14915524340k + 3556147692)/(17(k + 1)(16k + 17)^2(64k + 107)) \omega_3 \omega_3 \mathbb{I} \\
- (100k^5(k + 2)^2(49197952k^7 + 1316567424k^6 + 7996912590k^5 + 22351947279k^4 + 34102766376k^3 \\
+ 29430436515k^2 + 13488718172k + 2525554272)/(17(k + 1)(16k + 17)^2(64k + 107)) \omega_3 \omega_1 \omega_1 \mathbb{I} \\
+ ((25/2)^5(k + 2)^3(1808895488k^7 + 12570420576k^6 + 34774013520k^5 \\
+ 47177699946k^4 + 28725570387k^3 + 203645169k^2 - 8060040140k \\
- 264690336)/(17(k + 1)(16k + 17)^2) \omega_2 \omega_2 \omega_1 \mathbb{I} \\
+ (200k^5(k + 2)^4(87914016k^6 + 45846908k^5 + 817692930k^4 + 409776935k^3 \\
- 39461175k^2 - 507021564k - 148662512)/(17(k + 1)(16k + 17)^2) \omega_1 \omega_1 \omega_1 \omega_1 \mathbb{I} \\
- ((25/2)^5(k + 2)(23714656k^5 + 162824160k^4 + 438179214k^3 + 573852691k^2 \\
+ 361829501k + 86225720)/(17(k + 1)(64k + 107)) \omega_1 W_1^2 W_2^1 \mathbb{I} \\
- (25k^2(2088128k^6 + 10843456k^5 + 10746960k^4 - 34653451k^3 - 92307847k^2 \\
- 7711088k - 20771320)/(17(k + 1)(64k + 107)) W_3^3 W_3^1 \mathbb{I} \\
- (300k^3(k + 2)(504864k^6 + 454764k^5 - 10972571k^4 - 41935547k^3 - 63119109k^2 \\
- 4286543k - 10561420)/(17(k + 1)(16k + 17)^2) \omega_3 W_1^2 \mathbb{I} \\
+ (150k^3(k + 2)^2(9637952k^5 + 58430080k^4 + 139113500k^3 + 162223837k^2 \\
+ 92206985k + 20184520)/(17(k + 1)(16k + 17)^2) \omega_1 W_1^4 \mathbb{I} \\
- ((7/8)^5(k + 2)^2(10958592k^6 + 71925552k^5 + 180045944k^4 + 196657979k^3 \\
+ 56586147k^2 - 47214608k - 24536960)/(17(k + 1)(16k + 17)^2) \omega_2 W_4^2 \mathbb{I} \\
+ ((25/2)^4(k + 2)^2(81593088k^6 + 609224400k^5 + 1877915632k^4 + 3065560093k^3 \\
+ 279794176k^2 + 1347940946k + 262697360)/(17(k + 1)(16k + 17)^2) \omega_1 W_4^3 \mathbb{I} \\
- (150^5(18189312k^6 + 131109696k^5 + 275959096k^4 - 155647282k^5 - 1459360618k^4 - 2182319517k^3 \\
- 1238636927k^2 - 142994954k + 43729880)/(17(k + 1)(16k + 17)^2(64k + 107)) W_4^5 \mathbb{I} \\
- (75k^2(21712k^4 + 134672k^3 + 395445k^2 + 266275k + 78990)/(17(k + 1)(16k + 17)^2) W_4^1 W_4^1 \mathbb{I} \\
- ((7/2)(47824k^4 + 265368k^3 + 534533k^2 + 455349k + 133560)/(17(k + 1)(64k + 107))) W_4^1 W_4^5 \mathbb{I},
\]
\[ W_0^5 W^5 = -(200 k^5 (k + 2) (5244455936 k^{10} + 7019685904 k^9 + 3981532469 k^8 + 1233587976582 k^7 + 220101621153 k^6 + 2027732203871 k^5 + 165492882072 k^4 - 1736353418536 k^3 - 1889476600504 k^2 - 869314211744 k - 154555766032) \)/((16 k + 17) (131 k^2 + 351 k + 229)) \omega \omega_8 \mathbb{I} \\
- (400 k^5 (k + 2)^2 (724906768 k^9 + 13032033712 k^8 + 92438438140 k^7 + 355708801440 k^6 + 835873623678 k^5 + 1257476731455 k^4 + 1218124586462 k^3 + 734755072085 k^2 + 250569685256 k + 36757401408) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega \omega_6 \omega_1 \mathbb{I} \\
- (400 k^5 (k + 2)^2 (214796800 k^9 + 2717542336 k^8 + 13231630738 k^7 + 30829431549 k^6 + 28628760321 k^5 - 20502956550 k^4 - 80533588567 k^3 - 85066432663 k^2 - 41491861540 k - 7925033520) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega \omega_5 \omega_2 \mathbb{I} \\
+ (200 k^5 (k + 2)^2 (268100608 k^9 + 2728543168 k^8 + 12034185208 k^7 + 30531316020 k^6 + 50376301665 k^5 + 58543428453 k^4 + 50148887738 k^3 + 30806117372 k^2 + 1175806152 k + 1966237968) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega \omega_4 \omega_3 \mathbb{I} \\
+ (1200 k^5 (k + 2)^3 (57147904 k^8 + 134150272 k^7 - 1594361818 k^6 - 10000010232 k^5 - 25615972350 k^4 - 35776722093 k^3 - 28439133457 k^2 - 1204226386 k^1 - 2090245832) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega \omega_4 \omega_1 \omega_1 \mathbb{I} \\
+ (200 k^5 (k + 2)^3 (10186188 k^8 - 2776465367 k^7 - 28779796440 k^6 - 112345101336 k^5 - 234716374893 k^4 - 28779482491 k^3 - 207844230254 k^2 - 8132486600 k^1 - 13505087256) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega \omega_3 \omega_2 \omega_1 \mathbb{I} \\
+ (25 k^5 (k + 2)^3 (410452736 k^6 + 3102590048 k^6 + 9337359325 k^5 + 13766811166 k^4 + 9179515875 k^3 + 4416256912 k^2 - 2501518260 k^1 - 881437904) \)/((16 k + 17) (131 k^2 + 351 k + 229)) \omega \omega_2 \omega_2 \omega_1 \mathbb{I} \\
+ (400 k^5 (k + 2)^4 (632017152 k^7 + 4297675600 k^6 + 11097398636 k^5 + 1223816128 k^5 + 1797240891 k^4 - 8140638059 k^3 - 681774300 k^2 - 1683752944) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega \omega_2 \omega_1 \omega_1 \omega_1 \mathbb{I} \\
- (25 k^2 (k + 2) (1346512 k^5 + 9136620 k^4 + 24342198 k^3 + 31659247 k^2 + 19927547 k + 4787240) \)/((131 k^2 + 351 k + 229)) \omega_2 W_3^2 W_1^3 \mathbb{I} \\
- (100 k^2 (k + 2) (227056 k^5 + 1523004 k^4 + 3994369 k^3 + 5084081 k^2 + 3107416 k + 716920) \)/((131 k^2 + 351 k + 229)) \omega_1 W_2^2 W_2^3 \mathbb{I} \\
+ (25 k^2 (218864 k^6 + 1387084 k^5 + 3127878 k^4 + 2581115 k^3 - 382393 k^2 - 1483740 k - 448840) \)/((131 k^2 + 351 k + 229)) W_3^2 W_2^3 \mathbb{I} \\
+ (300 k^3 (k + 2) (13304576 k^7 + 146690032 k^6 + 668880716 k^5 + 1643048125 k^4 + 2351854427 k^3 + 1958853655 k^2 + 874329498 k + 159392280) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega_4 W_2^1 \mathbb{I} \\
+ (300 k^3 (k + 2)^2 (44903552 k^6 + 347191688 k^5 + 1102896188 k^4 + 1838425452 k^3 + 1691025439 k^2 + 810350893 k + 156998080) \)/((16 k + 17)^2 (31 k^2 + 351 k + 229)) \omega_2 W_1 W_2^1 \mathbb{I} \\
- (75 k^3 (k + 2) (12201216 k^7 + 95826768 k^6 + 305102656 k^5 + 494975652 k^4 + 412040321 k^3 + 138210893 k^2 - 14086422 k - 13469960) \)/((16 k + 17)^2 (131 k^2 + 351 k + 229)) \omega_3 W_2^1 \mathbb{I}
\begin{align*}
&+ (300k^3(k + 2)^2(12518976k^6 + 93551364k^5 + 286503394k^4 + 459301951k^3 + 405337557k^2 \\
&+ 185853764k + 34297340)/(16 + 17)^2(131k^2 + 351k + 229))\omega_{-1}W_{4,2}^1 \mathbb{1} \\
&+ (50k^3(k + 2)(8988192k^6 + 73184988k^5 + 245480957k^4 + 433108373k^3 + 422458956k^2 \\
&+ 214916428k + 44202700)/(16 + 17)^2(131k^2 + 351k + 229))\omega_{-2}W_{4,3}^1 \mathbb{1} \\
&+ (150k^3(k + 2)(2k + 3)(2310272k^6 + 14302664k^5 + 3597052k^4 + 47185363k^3 + 34385905k^2 \\
&+ 12495594k + 1193880)/(16 + 17)^2(131k^2 + 351k + 229))\omega_{-1}W_{4,1}^4 \mathbb{1} \\
&- (50k^3(3095808k^8 + 22633488k^7 + 54967200k^6 + 17421355k^5 - 157516472k^4 - 327442197k^3 - 312185014k^2 - 170931184k - 48119360)/(16 + 17)^2(131k^2 + 351k + 229))W_{4,6}^1 \mathbb{1} \\
&- (75k(2k + 3)(64k + 107)(368k^3 + 1897k^2 + 2671k + 950)/(16 + 17)^2(131k^2 + 351k + 229))W_{4,1}^2 \mathbb{1} \\
&- (50k(2k + 3)(5k + 8)(388k^2 + 863k + 405)/(131k^2 + 351k + 229))W_{3,2}^2 W_{5,1}^5 \mathbb{1}.
\end{align*}
C.1. Linear relations in the weight 8 subspace. We express \((W^3_2)^2 \mathbb{1}\) and \(W^3_1 W^4_2 \mathbb{1}\) as linear combinations of the remaining 27 vectors of normal form of weight 8.

\[
(W^3_2)^2 \mathbb{1} = -(18k^3(k + 2)(3k + 4)(217088k^5 + 132352k^4 + 1864570k^3 - 459533k^2 - 145384k - 18520)/(17(k + 1)(16k + 17)^2(64k + 107)))\omega_{-7} \mathbb{1}
\]

\[
- (288k^3 + 2)(3k + 4)(6976k^4 + 112048k^3 + 316803k^2 + 301883k + 91892)/(17(k + 1)(16k + 17)^2(64k + 107)))\omega_{-5} \omega_{-1} \mathbb{1}
\]

\[
+ (54k^3(k + 2)^2(3k + 4)(56320k^4 - 6240k^3 - 136598k^2 - 541975k - 173763)/(17(k + 1)(16k + 17)^2(64k + 107)))\omega_{-4} \omega_{-2} \mathbb{1}
\]

\[
+ (972k^3(k + 2)^2(3k + 4)(1792k^4 + 5096k^3 + 6100k^2 + 4783k + 229)/(17(k + 1)(16k + 17)^2(64k + 107)))\omega_{-3} \mathbb{1}
\]

\[
+ (72k^3(k + 2)^3(3k + 4)(920k^2 + 1598k + 187)/(17(k + 1)(16k + 17)^2)\omega_{-3}(\omega_{-1})^2 \mathbb{1}
\]

\[
+ (9k^3(k + 2)^3(3k + 4)(5792k^3 - 1566k - 3425)/(17(k + 1)(16k + 17)^2)\omega_{-2}^2 \omega_{-1} \mathbb{1}
\]

\[
- (1008k^3(k + 2)^3(3k + 4)(6k - 5)/(17(k + 1)(16k + 17)^2)\omega_{-1}^4 \mathbb{1}
\]

\[
+ (9(k + 2)(26k + 83)/(17(k + 1)(64k + 107)))\omega_{-1}(W^3_2)^2 \mathbb{1}
\]

\[
- (6(380k^2 + 822k + 301)/(17(k + 1)(64k + 107)))W_{-3}^3 W_{-1}^3 \mathbb{1}
\]

\[
+ (54k^3(k + 2)^2(120k^2 + 141k - 34)/(17(k + 1)(16k + 17)^2)\omega_{-3}W_{-1}^4 \mathbb{1}
\]

\[
- (36k^2(k + 2)^2(36k + 61)/(17(k + 1)(16k + 17)^2)\omega_{-3}W_{-1}^4 \mathbb{1}
\]

\[
+ (27k^2(k + 2)(78k^2 + 1565k + 664)/(68(k + 1)(16k + 17)^2)\omega_{-2}W_{-1}^4 \mathbb{1}
\]

\[
+ (9(k + 2)(496k^2 + 889k + 214)/(17(k + 1)(16k + 17)^2)\omega_{-1}W_{-1}^4 \mathbb{1}
\]

\[
+ (54k(2688k^4 + 9208k^3 + 9951k^2 + 5793k + 3788)/(17(k + 1)(16k + 17)^2(64k + 107)))W_{-1}^5 \mathbb{1}
\]

\[
+ (18/(17k(k + 1)(16k + 17)^2)W_{-1}^4 \mathbb{1}
\]

\[
+ (9/(17k(k + 1)(64k + 107)))W_{-1}^3 W_{-1}^5 \mathbb{1}
\]

\[
W_{-1}^3 W_{-1}^5 \mathbb{1} = (8k^2(k + 2)(2k + 3)(320k^2 - 155k - 621)/(64k + 107))\omega_{-4}W_{-1}^3 \mathbb{1}
\]

\[
- (8k^2(k + 2)^2(2k + 3)(136k - 109)/(64k + 107))\omega_{-2}W_{-1}^3 \mathbb{1}
\]

\[
+ (16k^2(k + 2)(2k + 3)(40k^2 - 310k - 327)/(64k + 107))\omega_{-3}W_{-1}^5 \mathbb{1}
\]

\[
+ (16k^2(k + 2)^2(2k + 3)(136k - 109)/(3(64k + 107)))\omega_{-3}W_{-1}^5 \mathbb{1}
\]

\[
+ (8k^2(k + 2)(2k + 3)(168k^2 - 16k + 31)/(64k + 107))\omega_{-2}W_{-1}^5 \mathbb{1}
\]

\[
- (20k^2(k + 2)(2k + 3)(64k^2 + 167k - 180)/(64k + 107))\omega_{-1}W_{-1}^5 \mathbb{1}
\]

\[
+ (24k^2(2k + 3)(32k^3 + 344k^2 + 347k - 516)/(64k + 107))W_{-1}^6 \mathbb{1}
\]

\[
+ (4/3)W_{-1}^3 W_{-1}^5 \mathbb{1}
\]

\[
+ (4k(k + 2)(16k + 17)/(64k + 107))\omega_{-2}W_{-1}^5 \mathbb{1}
\]

\[
- (9(k + 2)(16k + 17)/(5(64k + 107)))\omega_{-1}W_{-1}^5 \mathbb{1}
\]

\[
- (8(7k + 9)(16k + 17)/(5(64k + 107)))W_{-1}^5 \mathbb{1}
\]

C.2. Linear relations in the weight 9 subspace. We can express \(W_{-3}^3 W_{-3}^2 \mathbb{1}\), \(W_{-2}^3 W_{-4}^2 \mathbb{1}\), \(W_{-1}^3 W_{-3}^3 \mathbb{1}\), and \(W_{-1}^3 W_{-5}^2 \mathbb{1}\) as linear combinations of the remaining 40 vectors of normal form.
form of weight 9. For instance, $W_{-3}^3 W_{-3}^3$ is expressed as follows. We omit the expression of $W_{-3}^3 W_{-3}^3$, $W_{-2}^3 W_{-2}^4$ and $W_{-1}^3 W_{-1}^5$, for they are not used in our argument.

$$W_{-3}^3 W_{-3}^3 = (16k^2(k + 2)(1283648k^5 + 3440448k^4 - 3245504k^3 - 18055627k^2 - 18583431k - 5789692)$$

$$/(64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (8k^2(k + 2)^2(455808k^4 + 2175980k^3 + 3327583k^2 + 1752535k + 133700)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (k^2(k + 2)^2(8192k^3 - 432k^2 - 30515k - 15420)/(1424k^2 + 3241k + 1542))(W_{-2}^3 W_{-2}^3$$

$$+ (16k^2(k + 2)^3(674k^2 + 63k - 1100)/(1424k^2 + 3241k + 1542))(W_{-1}^3 W_{-1}^3$$

$$+ (8k^2(k + 2)(5k + 8)(16k + 17)(58944k^3 - 96692k^2 - 505205k - 340649)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$+ (4k^2(k + 2)^2(5k + 8)(16k + 17)(11248k^2 - 3953k - 6251)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$+ (8k^2(k + 2)^2(1209472k^3 + 1405772k^2 - 12112961k^3 - 3415532k^2 - 32424710k - 10585768)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$+ (32k^2(k + 2)^2(627872k^4 + 2346827k^3 + 2091732k^2 - 1239437k - 1843412)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$+ (3k^2(k + 2)(4445184k^5 + 20157312k^4 + 34479058k^3 + 27362195k^2 + 8585804k - 569072)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (8k^2(k + 2)^2(6894208k^5 + 54479264k^4 + 15313147k^3 + 19292401k^2 + 10564731k + 33600368k + 9480832)/(64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (16k + 17)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$+ (4k^2(k + 2)(358k + 559)/(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$+ (8k^2k^2 + 1392k + 1075)/(1424k^2 + 3241k + 1542)W_{-3}^3 W_{-3}^3$$

$$+ (112k^2 + 236 + 31)/(1424k^2 + 3241k + 1542)W_{-3}^3 W_{-3}^3$$

$$+ (112k^2 + 236 + 31)/(1424k^2 + 3241k + 1542)W_{-3}^3 W_{-3}^3$$

$$- (2k(k + 2)(192k^2 + 229k - 400)/(5(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (16k^2(k + 2)(16k + 17)(168k^2 + 3815k + 1786)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (96k(k + 17)(2687k^3 + 10929k^2 + 12927k + 5342)$$

$$/((64k + 107)(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3$$

$$- (4/(k(1424k^2 + 3241k + 1542))W_{-3}^3 W_{-3}^3. $$

**Appendix D. $u^0, u^1, u^2$ and $u^3$ in Case $k = 5$**

We express $u^0 = f(0)^{k+1} (-1)^{k+1} u$ and $u^r = (W^3)^r u^0$, $r = 1, 2, 3$ as linear combinations of the vectors of normal form in the case $k = 5$. 
\[ u^0 = -\left(56260915200/97\right)\omega_{-5} \mathbb{I} - \left(47822745600/97\right)\omega_{-3}\omega_{-1} \mathbb{I} + \left(43180603200/97\right)(\omega_{-2})^2 \mathbb{I} + \left(33230937600/97\right)(\omega_{-1})^3 \mathbb{I} - \left(4032/5\right)(W_{-2}^3)^2 \mathbb{I} + \left(550368/97\right)\omega_{-1}W_{-1}^4 \mathbb{I} + \left(340704/97\right)W_{-3}^2 \mathbb{I}, \]

\[ u^1 = -\left(17721088761600/5917\right)\omega_{-3}W_{-1}^2 \mathbb{I} + \left(13262835501600/5917\right)(\omega_{-1})^2 W_{-3}^2 \mathbb{I} + \left(221863017600/61\right)\omega_{-2}W_{-2}^4 \mathbb{I} - \left(365470963200/97\right)\omega_{-1}W_{-3}^3 \mathbb{I} + \left(21001925203200/5917\right)W_{-3}^3 \mathbb{I} - \left(2122848/97\right)W_{-3}^4 \mathbb{I} - \left(389631360/61\right)\omega_{-1}W_{-5}^3 \mathbb{I} - \left(3401438440/61\right)W_{-5}^3 \mathbb{I}, \]

\[ u^2 = \left(8181452462686782123600000/9757133\right)\omega_{-5} \mathbb{I} + \left(8868381288151420627200000/9757133\right)\omega_{-4}\omega_{-1} \mathbb{I} - \left(5147471345450314255200000/9757133\right)\omega_{-4}\omega_{-2} \mathbb{I} + \left(23321410696693972800000/9757133\right)(\omega_{-3})^2 \mathbb{I} + \left(47380877265410942400000/159953\right)\omega_{-3}(\omega_{-1})^2 \mathbb{I} - \left(41194303644229799800000/159953\right)(\omega_{-2})^2 \mathbb{I} - \left(32478871712964566400000/159953\right)(\omega_{-1})^2 \mathbb{I} + \left(4985850497040000/1037\right)\omega_{-1}(W_{-1}^3 \mathbb{I}) - \left(42034377168000/1037\right)W_{-3}^2 W_{-1}^2 \mathbb{I} - \left(8566112126376000/159953\right)\omega_{-3}W_{-2}^4 \mathbb{I} - \left(524887446958560000/159953\right)(\omega_{-1})^2 W_{-3}^4 \mathbb{I} + \left(3017834596261800/159953\right)\omega_{-2}W_{-2}^4 \mathbb{I} - \left(340143285584592000/159953\right)\omega_{-1}W_{-4}^3 \mathbb{I} + \left(357554169263088000/9757133\right)W_{-4}^3 \mathbb{I} + \left(89784495360/159953\right)(W_{-1}^4 \mathbb{I})^2 + \left(13751156160/1037\right)W_{-3}^2 W_{-1}^2 \mathbb{I}, \]

\[ u^3 = \left(633349572703577384045440000/45093457\right)\omega_{-5}W_{-1}^3 \mathbb{I} - \left(304333657131610010822400000/45093457\right)\omega_{-4}\omega_{-1}W_{-1}^3 \mathbb{I} + \left(1141592140148275607400000/464881\right)\omega_{-2}W_{-3}^3 \mathbb{I} + \left(69658304251736590588800000/45093457\right)(\omega_{-1})^3 W_{-3}^3 \mathbb{I} + \left(175753574599043599200000/464881\right)\omega_{-4}W_{-2}^3 \mathbb{I} - \left(620956662666585739200000/464881\right)\omega_{-2}W_{-1}W_{-2}^3 \mathbb{I} - \left(5158511194620039076800000/45093457\right)\omega_{-3}W_{-3}^3 \mathbb{I} + \left(820496583738354986582400000/45093457\right)(\omega_{-1})^2 W_{-3}^3 \mathbb{I} + \left(15050173408252695423000000/464881\right)\omega_{-2}W_{-2}^3 \mathbb{I} - \left(2704800801881903228784000000/45093457\right)\omega_{-1}W_{-3}^3 \mathbb{I} + \left(314127287318119508442720000000/45093457\right)W_{-3}^3 \mathbb{I} - \left(63906101745998400/7621\right)(W_{-1}^3 \mathbb{I})^3 \mathbb{I} + \left(5804030066903728000/739237\right)\omega_{-1}W_{-1}^3 W_{-4}^2 \mathbb{I} + \left(70446353688003384000/739237\right)W_{-3}^2 W_{-1}^3 \mathbb{I} + \left(29475630099262095840000/45093457\right)\omega_{-3}W_{-2}^3 \mathbb{I} + \left(570725611182967968000000/45093457\right)(\omega_{-1})^2 W_{-2}^3 \mathbb{I} - \left(586292703314111956800/45093457\right)\omega_{-2}W_{-2}^5 \mathbb{I} - \left(82578847924067040000/464881\right)\omega_{-1}W_{-5}^3 \mathbb{I} - \left(2748731861102520384000/464881\right)W_{-5}^5 \mathbb{I} - \left(434544013612800/739237\right)W_{-4}^2 W_{-2}^3 \mathbb{I}. \]

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