A Study of the Dirac-Sidharth Equation

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Abstract

The Dirac-Sidharth Equation has been constructed from the Sid-
dharth hamiltonian by quantization of the energy and momentum in Pauli
algebra. We have solved this equation by using tensor product of matrices.

Keywords: Dirac-Sidharth equation, Dirac equation.

1 Introduction

In the special relativity of Einstein (A.Einstein., 1905), from the energy-momentum
relation
\[ E^2 = c^2 p^2 + m^2 c^4 \] (1)
we can deduce the Klein-Gordon equation and the Dirac equation. This theory
use the concept of continuous spacetime.

Quantized spacetime was introduced at the first time by Snyder (H.S. Sny-
der., Phys Rev. 1947)[2, 3], which known as Snyder noncommutative geometry.
That is because the commutation relations are modified and becomes [2, 3]
\[
[x^{\mu}, x^{\nu}] = i \alpha \frac{\ell^2 c^2}{\hbar} (x^{\mu} p^{\nu} - x^{\nu} p^{\mu}),
\] (2)
\[
[x^{\mu}, p_{\nu}] = i \hbar \left( \delta^{\mu}_{\nu} + i \alpha \frac{\ell^2 c^2}{\hbar^2} p^{\mu} p_{\nu} \right),
\] (3)
\[
[p_{\mu}, p_{\nu}] = 0
\] (4)

\[ \epsilon = \frac{\hbar c}{\sqrt{\alpha \ell}} \] (5)
is the energy due to the length scale $\ell$, where $\alpha$ a dimensionless constant. As consequence the energy momentum relation gets modified and becomes

$$E^2 = c^2 p^2 + m^2 c^4 + \alpha \left( \frac{c}{h} \right)^2 \ell^2 p^4$$

(6)

$$\ell = \ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} \text{ cm}, \text{ Planck scale, the fundamental length scale,}$$

where $G$ is the gravitational constant.

$$\ell_c = \frac{\hbar}{e m_e c} \propto 10^{-12} \text{ cm}, \text{ Compton scale, where } e \text{ is the electron charge and } m_e \text{ the electron mass.}$$

$$\ell_{LHC} \approx 2 \times 10^{-18} \text{ cm.}$$

So,

$$\ell_p < \ell_{LHC} < \ell_c$$

(7)

From the above energy- momentum relation, Sidharth has deduced the so-called Dirac-Sidharth Equation \[4, 5\], a modified Dirac equation.

In the Section 2, we will derive the Dirac-Sidharth Equation by quantizing energy and momentum. In the Section 3, we will solve the Dirac-Sidharth Equation by using tensor product of matrices.

We think that using different mathematical tools in physics will make to appear different hidden mathematical or physical properties.

## 2 A derivation of the Dirac-Sidharth equation

For deriving the Dirac-Sidharth equation we use the method used by J.J. Sakurai \[6\] for deriving the Dirac equation.

The wave function of a spin-$\frac{1}{2}$ particle is two components. So, for quantizing the energy-momentum relation in order to have the modified Klein-Gordon equation \[4, 5\], or Klein-Gordon-Sidharth equation, of the spin-$\frac{1}{2}$ particle, the operators which take part in the quantization should be $2 \times 2$ matrices. So, let us take as quantization rules

$$E \rightarrow i \hbar \sigma^0 \frac{\partial}{\partial t} = i \hbar \frac{\partial}{\partial t}$$

$$\vec{p} \rightarrow -i \hbar \sigma^1 \frac{\partial}{\partial x^1} - i \hbar \sigma^2 \frac{\partial}{\partial x^2} - i \hbar \sigma^3 \frac{\partial}{\partial x^3} = -i \hbar \vec{\sigma} \vec{\nabla} = \hat{p}_1 \sigma^1 + \hat{p}_2 \sigma^2 + \hat{p}_3 \sigma^3$$

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. Then we have, at first the Klein-Gordon-Sidharth equation

$$c^2 \hbar^2 \left( \frac{\partial^2}{\partial t^2} - \Delta - m^2 c^2 - \alpha \ell^2 \frac{\nabla^4}{\hbar^2} \right) \phi = 0$$

(8)

$$\left( i \hbar \frac{\partial}{\partial t} + i c \hbar \vec{\sigma} \vec{\nabla} \right) \frac{1}{mc^2} \left\{ \sum_{k=0}^{+\infty} (-1)^k \frac{ic}{mc \hbar} \ell \left( -i \hbar \vec{\sigma} \vec{\nabla} \right)^2 \right\} \left( i \hbar \frac{\partial}{\partial t} - i c \hbar \vec{\sigma} \vec{\nabla} \right) \phi$$

$$= \left[ mc^2 + i \sqrt{\alpha \frac{c}{\hbar} \ell \left( -i \hbar \vec{\sigma} \vec{\nabla} \right)^2} \right] \phi$$

(9)
with application of the operator to two components wave function $\phi$, which is solution of the Klein-Gordon-Sidharth equation. Let

$$\chi = \frac{1}{mc^2} \left( \sum_{k=0}^{+\infty} (-1)^k \left[ \frac{i\sqrt{\alpha}}{mch} \left( -i\hbar\nabla \right)^2 \right]^k \right) \times \left( i \frac{\partial}{\partial t} - i\hbar \nabla \right) \phi$$  \hspace{1cm} (10)

then, we have the following system of partial differential equations

$$\begin{cases}
    i\hbar \frac{\partial}{\partial t} \chi + i\hbar \nabla \chi = mc\phi + i\sqrt{\alpha} \ell \left( i\hbar \nabla \right)^2 \phi \\
    i\hbar \frac{\partial}{\partial t} \phi - i\hbar \nabla \phi = mc\chi - i\sqrt{\alpha} \ell \left( i\hbar \nabla \right)^2 \chi
\end{cases}$$  \hspace{1cm} (11)

In additioning and in subtracting these equations, and in transforming the obtained equation under matricial form, we have the Dirac-Sidharth equation

$$i\hbar \gamma^\mu D \partial_{\mu} \psi_D - mc\psi_D - i\sqrt{\alpha} \ell \gamma^5 D \Delta \psi_D = 0$$  \hspace{1cm} (12)

in the Dirac (or "Standard") representation of the $\gamma$-matrices, where

$$\begin{align*}
\gamma^0_D &= \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} = \sigma^3 \otimes \sigma^0, \\
\gamma^j_D &= \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = i\sigma^2 \otimes \sigma^j, \quad j = 1, 2, 3, \\
\gamma^5_D &= i\gamma^0_D \gamma^2_D \gamma^1_D = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} = \sigma^1 \otimes \sigma^0, \quad \psi_D = \begin{pmatrix} \chi + \phi \\ \chi - \phi \end{pmatrix}.
\end{align*}$$

We know that (J.D. Bjorken and S.D. Drell., 1964)[7]

$$P \gamma^5 = -\gamma^5 P$$  \hspace{1cm} (13)

It follows that the Dirac-Sidharth equation is not invariant under reflections (B.G. Sidharth., Mass of the Neutrinos, 2009). The equation

$$i\hbar \gamma^\mu W D \partial_{\mu} \psi_W - mc\psi_W - i\sqrt{\alpha} \ell \gamma^5 W \Delta \psi_W = 0$$  \hspace{1cm} (14)

is the Dirac-Sidharth equation in the Weyl(or "chiral") representation, where

$$\psi_W = \begin{pmatrix} \chi \\ \phi \end{pmatrix}.$$  \hspace{1cm}

So, $\chi$ is the left-handed two components spinor and $\phi$ the right-handed one. This method makes to appear that the right-handed two components spinor is solution of the Klein-Gordon-Sidharth equation.

3 Resolution of the Dirac-Sidharth equation

In this section we will use the tensor product of matrices for solving the Dirac-Sidharth equation. We had used this method, suggested by Raoelina Andriambololona for solving the Dirac equation (C. Rakotonirina., Thesis, 2003). Let us look for a solution of the form

$$\psi_D = U(p)e^{i\left(\vec{p}\cdot\vec{x} - Et\right)}$$  \hspace{1cm} (15)
Let $\Psi$ a four components spinor which is eigenstate both of $\hat{p}_j = -i\hbar \frac{\partial}{\partial x_j}$ and $\hat{E} = i\hbar \frac{\partial}{\partial t}$, and $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$, and $\vec{n} = \frac{\vec{p}}{p}$.

The Dirac-Sidharth equation becomes

$$\begin{align*}
\sigma^0 \otimes \sigma^0 U(p) - \frac{2}{\hbar} cp \sigma^1 \otimes \left( \frac{\hbar}{2} \vec{\sigma} \vec{n} \right) U(p) - mc^2 \sigma^3 \otimes \sigma^0 U(p) + c\sqrt{\alpha p^2} \frac{\vec{E}}{\hbar} \sigma^2 \otimes \sigma^0 U(p) = 0
\end{align*}$$

Let us take $U(p)$ of the form

$$U(p) = \varphi \otimes u$$

where $u$ is the eigenvector of the spin operator $\left( \frac{\hbar}{2} \vec{\sigma} \vec{n} \right)$. $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}$ and $u$ are two components.

Since $u \neq 0$, so

$$\left( \epsilon cp \sigma^1 - c\sqrt{\alpha p^2} \frac{\vec{E}}{\hbar} \sigma^2 + mc^2 \right) \varphi = E\varphi$$

with $\epsilon = \begin{cases} +1 \text{ spin up} \\ -1 \text{ spin down} \end{cases}$

Solving this equation with respect to $\varphi^1$ and $\varphi^2$, we have

$$\begin{align*}
\Psi_+ &= \sqrt{\frac{E + mc^2}{2E}} \left( \frac{1}{mc^2 + E} \right) \otimes se^{\frac{1}{2}(\vec{p} \vec{x} - E t)}
\end{align*}$$

the solution with positive energy, where $s = \frac{1}{\sqrt{2(1+n^3)}} \left( -n^1 + in^2 \right)$ spin up,

$$s = \frac{1}{\sqrt{2(1+n^3)}} \left( \frac{1+n^3}{n^1 + in^2} \right)$$

spin down.

This method makes to appear the $2 \times 2$ matrix $h = \epsilon cp \sigma^1 - c\sqrt{\alpha p^2} \frac{\vec{E}}{\hbar} \sigma^2 + mc^2 \sigma^3$ whose eigenvalues are the positive and the negative energies. $h$ is like a vector in Pauli algebra. So, energy of the spin-$\frac{1}{2}$ particle can be associated to a vector in Pauli algebra, whose length or intensity is given by the energy-momentum relation.

$$h^2 = E^2$$

References

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