13. Abelian extensions of absolutely unramified complete discrete valuation fields

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In this section we discuss results of [K]. We assume that \( p \) is an odd prime and \( K \) is an absolutely unramified complete discrete valuation field of mixed characteristics \((0, p)\), so \( p \) is a prime element of the valuation ring \( \mathcal{O}_K \). We denote by \( F \) the residue field of \( K \).

13.1. The Milnor \( K \)-groups and differential forms

For \( q > 0 \) we consider the Milnor \( K \)-group \( K_q(K) \), and its \( p \)-adic completion \( \widehat{K}_q(K) \) as in section 9. Let \( U_1\widehat{K}_q(K) \) be the subgroup generated by \( \{1 + p\mathcal{O}_K, K^*, \ldots, K^*\} \).

Then we have:

**Theorem.** Let \( K \) be as above. Then the exponential map \( \exp_p \) for the element \( p \), defined in section 9, induces an isomorphism

\[
\exp_p: \widehat{\Omega}_{\mathcal{O}_K}^{q-1}/pd\widehat{\Omega}_{\mathcal{O}_K}^{q-2} \rightarrow U_1\widehat{K}_q(K).
\]

The group \( \widehat{K}_q(K) \) carries arithmetic information of \( K \), and the essential part of \( \widehat{K}_q(K) \) is \( U_1\widehat{K}_q(K) \). Since the left hand side \( \widehat{\Omega}_{\mathcal{O}_K}^{q-1}/pd\widehat{\Omega}_{\mathcal{O}_K}^{q-2} \) can be described explicitly (for example, if \( F \) has a finite \( p \)-base \( I \), \( \widehat{\Omega}_{\mathcal{O}_K}^{1} \) is a free \( \mathcal{O}_K \)-module generated by \( \{dt_i\} \) where \( \{t_i\} \) are a lifting of elements of \( I \)), we know the structure of \( U_1\widehat{K}_q(K) \) completely from the theorem.

In particular, for subquotients of \( \widehat{K}_q(K) \) we have:

**Corollary.** The map \( \rho_m: \Omega_F^{q-1} \oplus \Omega_F^{q-2} \rightarrow \text{gr}_mK_q(K) \) defined in section 4 induces an isomorphism

\[
\Omega_F^{q-1}/B_{m-1}\Omega_F^{q-1} \rightarrow \text{gr}_mK_q(K).
\]
where $B_{m-1}\Omega_F^{q-1}$ is the subgroup of $\Omega_F^{q-1}$ generated by the elements $a^p d \log a \wedge d \log b_1 \wedge \cdots \wedge d \log b_{q-2}$ with $0 \leq j \leq m-1$ and $a, b_i \in F^\ast$.

### 13.2. Cyclic $p$-extensions of $K$

As in section 12, using some class field theoretic argument we get arithmetic information from the structure of the Milnor $K$-groups.

**Theorem.** Let $W_n(F)$ be the ring of Witt vectors of length $n$ over $F$. Then there exists a homomorphism

$$\Phi_n: H^1(K, \mathbb{Z}/p^n) = \text{Hom}_{cont}(\text{Gal}(\overline{K}/K), \mathbb{Z}/p^n) \rightarrow W_n(F)$$

for any $n \geq 1$ such that:

1. The sequence
   $$0 \rightarrow H^1(K_{ur}/K, \mathbb{Z}/p^n) \rightarrow H^1(K, \mathbb{Z}/p^n) \xrightarrow{\Phi_n} W_n(F) \rightarrow 0$$
   is exact where $K_{ur}$ is the maximal unramified extension of $K$.

2. The diagram
   $$
   \begin{array}{ccc}
   H^1(K, \mathbb{Z}/p^{n+1}) & \xrightarrow{p} & H^1(K, \mathbb{Z}/p^n) \\
   \downarrow{\Phi_{n+1}} & & \downarrow{\Phi_n} \\
   W_{n+1}(F) & \xrightarrow{F} & W_n(F)
   \end{array}
   $$
   is commutative where $F$ is the Frobenius map.

3. The diagram
   $$
   \begin{array}{ccc}
   H^1(K, \mathbb{Z}/p^n) & \xrightarrow{\Phi_n} & H^1(K, \mathbb{Z}/p^{n+1}) \\
   \downarrow{\Phi_{n+1}} & & \downarrow{\Phi_n} \\
   W_n(F) & \xrightarrow{V} & W_{n+1}(F)
   \end{array}
   $$
   is commutative where $V((a_0, \ldots, a_{n-1})) = (0, a_0, \ldots, a_{n-1})$ is the Verschiebung map.

4. Let $E$ be the fraction field of the completion of the localization $O_K[T]_{(p)}$ (so the residue field of $E$ is $F(T)$). Let
   $$\lambda: W_n(F) \times W_n(F(T)) \xrightarrow{\Phi_n} p^n \text{Br}(F(T)) \oplus H^1(F(T), \mathbb{Z}/p^n)$$
   be the map defined by $\lambda(w, w') = (i_2(p^{n-1}wdw'), i_1(w'w'))$ where $p^n \text{Br}(F(T))$ is the $p^n$-torsion of the Brauer group of $F(T)$, and we consider $p^{n-1}wdw'$ as an element of $W_n\Omega^1_{F(T)}$ ($W_n\Omega^1_{F(T)}$ is the de Rham Witt complex). Let
   $$i_1: W_n(F(T)) \rightarrow H^1(F(T), \mathbb{Z}/p^n)$$

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be the map defined by Artin–Schreier–Witt theory, and let
\[ i_2: W_n \Omega^1_{F(T)} \to p^n \text{Br}(F(T)) \]
be the map obtained by taking Galois cohomology from an exact sequence
\[ 0 \to (F(T)_{\text{sep}})^* / ((F(T)_{\text{sep}})^*)^p \to W_n \Omega^1_{F(T)_{\text{sep}}} \to W_n \Omega^1_{F(T)_{\text{sep}}} \to 0. \]

Then we have a commutative diagram
\[
\begin{array}{ccc}
H^1(K, \mathbb{Z}/p^n) \times E^*/(E^*)^p & \xrightarrow{\cup} & \text{Br}(E) \\
\Phi_n \downarrow & & \uparrow i \\
W_n(F) \times W_n(F(T)) & \xrightarrow{\lambda} & p^n \text{Br}(F(T)) \oplus H^1(F(T), \mathbb{Z}/p^n)
\end{array}
\]
where \( i \) is the map in subsection 5.1, and
\[
\psi_n((a_0, \ldots, a_{n-1})) = \exp \left( \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} p^{i+j} \tilde{a}_i p^{n-i-j} \right)
\]
(\( \tilde{a}_i \) is a lifting of \( a_i \) to \( \mathcal{O}_K \)).

(5) Suppose that \( n = 1 \) and \( F \) is separably closed. Then we have an isomorphism
\[ \Phi_1: H^1(K, \mathbb{Z}/p) \simeq F. \]
Suppose that \( \Phi_1(\chi) = a \). Then the extension \( L/K \) which corresponds to the character \( \chi \) can be described as follows. Let \( \tilde{a} \) be a lifting of \( a \) to \( \mathcal{O}_K \). Then \( L = K(x) \) where \( x \) is a solution of the equation
\[ X^p - X = \tilde{a}/p. \]

The property (4) characterizes \( \Phi_n \).

Corollary (Miki). Let \( L = K(x) \) where \( x^p - x = a/p \) with some \( a \in \mathcal{O}_K \). \( L \) is contained in a cyclic extension of \( K \) of degree \( p^n \) if and only if
\[ a \mod p \in F_{p^{n-1}}. \]

This follows from parts (2) and (5) of the theorem. More generally:

Corollary. Let \( \chi \) be a character corresponding to the extension \( L/K \) of degree \( p^n \), and \( \Phi_n(\chi) = (a_0, \ldots, a_{n-1}) \). Then for \( m > n \), \( L \) is contained in a cyclic extension of \( K \) of degree \( p^m \) if and only if \( a_i \in F_{p^{m-n}} \) for all \( i \) such that \( 0 \leq i \leq n - 1 \).

Remarks.

(1) Fesenko gave a new and simple proof of this theorem from his general theory on totally ramified extensions (cf. subsection 16.4).
For any \( q > 0 \) we can construct a homomorphism
\[
\Phi_n : H^q(K, \mathbb{Z}/p^n(q-1)) \to W_n^q \Omega^{q-1}_F
\]
by the same method. By using this homomorphism, we can study the Brauer group of \( K \), for example.

Problems.
1. Let \( \chi_K \) be the character of the extension constructed in 14.1. Calculate \( \Phi_n(\chi_K) \).
2. Assume that \( F \) is separably closed. Then we have an isomorphism
\[
\Phi_n : H^1(K, \mathbb{Z}/p^n) \simeq W_n(F).
\]
This isomorphism is reminiscent of the isomorphism of Artin–Schreier–Witt theory. For \( w = (a_0, \ldots, a_{n-1}) \in W_n(F) \), can one give an explicit equation of the corresponding extension \( L/K \) using \( a_0, \ldots, a_{n-1} \) for \( n \geq 2 \) (where \( L/K \) corresponds to the character \( \chi \) such that \( \Phi_n(\chi) = w \)?)

References

[K] M. Kurihara, Abelian extensions of an absolutely unramified local field with general residue field, Invent. math., 93 (1988), 451–480.