Non-axisymmetric oscillations of differentially rotating relativistic stars

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Non-axisymmetric oscillations of differentially rotating stars are studied using both slow rotation and Cowling approximation. The equilibrium stellar models are relativistic polytropes where differential rotation is described by the relativistic j-constant rotation law. The oscillation spectrum is studied versus three main parameters: the stellar compactness $M/R$, the degree of differential rotation $A$ and the number of maximum couplings $\ell_{\text{max}}$. It is shown that the rotational splitting of the non-axisymmetric modes is strongly enhanced by increasing the compactness of the star and the degree of differential rotation. Finally, we investigate the relation between the fundamental quadrupole mode and the corotation band of differentially rotating stars.

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I. INTRODUCTION

Differential rotation is believed to play an important role in nascent neutron stars as well as in binary neutron stars mergers. Until viscosity, turbulent motion and/or magnetic fields force the star to uniform rotation, dynamical or secular instabilities can be developed due to differential rotation.

The study of the differential rotation phase of a neutron star life re-gained attention lately. Recent numerical simulations in Newtonian hydrodynamics show that in stars with high differentially rotating, non-axisymmetric dynamical instabilities can develop even for low values of $\beta = T/|W| \simeq 0.01 - 0.08$, where $T$ is the rotational kinetic energy and $W$ the gravitational binding energy. The so-called “low $T/|W|$” instability can drive either one-armed spiral or bar-mode instabilities. The gravitational wave signal emitted via these instabilities might be detectable with the advanced generation of ground based detectors. If this instability develops in supermassive stars, it may produce gravitational waves detectable even by the interferometric space detector LISA. As suggested in, the low $T/|W|$ instability might be due to the shear instability that the corotating $f$ mode develops when it enters the corotation band. A Newtonian study of this instability, which is based on the analysis of the canonical angular momentum, confirms the presence of corotation modes.

A key feature of the oscillation spectrum of a uniformly or differentially rotating star is the splitting of the eigen-modes, like the Zeeman effect in atomic physics. In Newtonian theory, the splitting of the non-axisymmetric oscillations of differentially rotating stars have been studied with perturbative techniques in. A general relativistic description of the perturbations of differentially rotating neutron stars may alter, at least quantitatively, the Newtonian results. In general relativity, the perturbations of uniformly rotating stars are mainly studied in the so called slow rotation approximation. This approximation is practically valid for the study of all known pulsars even for those with rotational periods of 1.5-2ms. The slow rotation approximation fails to treat perturbations of neutron stars with periods below 1.5ms or $\Omega/\Omega_{\text{Kepler}} > 0.25 - 0.3$. Actually, there are no known pulsars with such small periods but it is not at all impossible that newly born neutron stars may have rotational periods below 1ms.

The lack of perturbative studies for differentially rotating relativistic stars, the absence of any results for non-axisymmetric perturbations and the issue of low $T/|W|$ instability are the main motivations of this work. There only a few studies of fast rotating neutron stars using a perturbative approach. While there is significant progress in the study of neutron star oscillations using evolutions of the non-linear equations. Still the non-axisymmetric perturbations of differentially rotating relativistic stars have not been treated by any method yet, this paper is the first attempt to address the problem. Actually, in an earlier paper we derived, in the perturbative framework of general relativity, the equations describing the oscillations of a slowly and differentially rotating neutron star.

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star in the Cowling approximation [19] (from now on Paper I). Here we study the effect of differential rotation on the oscillation spectrum. Moreover, by using the perturbative approach developed here we examine the relation between the low $T$/W instability with the existence of corotating modes. The main results of our study can be summarized in the following sentence: the rotational splitting of the non-axisymmetric modes is enhanced by stellar compactness and the degree of differential rotation.

The structure of this paper is as follows. In section II we briefly describe the perturbative framework (more details are given in Paper I). In Section III we address the boundary value problem and introduce the numerical techniques for solving it. In Section IV we present and discuss our numerical results derived for different stellar models. Section V is dedicated to the conclusions and to the possible extensions of this work. Finally, in the Appendix A we describe the structure of the eigenvalue equations, while the definition of the linear angular operators is given in Appendix B.

As is common, throughout the paper we use geometrical units i.e. $c = G = 1$. Prime (′) denotes derivatives with respect to the radial coordinate $r$ and overdot (¨) denotes derivatives with respect to the time coordinate $t$.

II. THE PERTURBATIVE FRAMEWORK

Equilibrium configurations of differentially rotating relativistic stars have been already studied in the early ’70s by Hartle [20] and Will [21]. In the approach that we follow here, the background spacetime of a slowly and differentially rotating star assumed to be axially symmetric and can be described, at first order with respect to the angular velocity of the star, in Schwarzschild coordinates, by the following line element:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 - 2\omega r^2\sin^2\theta dt d\phi + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

The scalar functions $\nu$, $\lambda$ depend only on the radial coordinate $r$ and are determined by solving the Tolman-Oppenheimer-Volkoff (TOV) equations for a given equation of state (EoS). The metric function $\omega = \omega(r, \theta)$ describes the dragging of inertial frames due to stellar rotation and obeys the following ODE [22]:

$$\omega'' - \left[4\pi(e + p) r e^{2\lambda} - \frac{4}{r}\right] \omega' - \left[16\pi(e + p) + \frac{\ell(\ell + 1) - 2}{r^2}\right] e^{2\lambda} \omega = -16\pi(e + p) e^{2\lambda} \Omega,$$

where $e$ and $p$ are the total energy density and the pressure respectively, $\ell$ the harmonic index and $\Omega = \Omega(r, \theta)$ is the angular velocity of the star as measured by an observer at infinity.

A differentially rotating stellar model can be constructed in the framework of the slow rotation approximation in two steps. First, one constructs the non-rotating stellar model by specifying the central energy density and the EoS and then solving the TOV equations. Afterwards, a law describing the differential rotation is specified, that is one assumes a specific functional form for $\Omega(r, \theta)$ and then the metric function $\omega(r, \theta)$ is determined by solving ODE (2).

Here, we use the perturbative version of the relativistic j-constant rotation law [19, 23]:

$$\Omega(r, \theta) = \frac{A^2\Omega_c + e^{-2\nu}\omega(r, \theta)r^2\sin^2\theta}{A^2 + e^{-2\nu}r^2\sin^2\theta},$$

where $\Omega_c$ denotes the angular velocity on the rotation axis, while the parameter $A$ specifies the degree of differential rotation of the star. For $A \to \infty$, the j-constant rotation law leads to uniformly rotating configurations, i.e. $\Omega \to \Omega_c$. More details about the procedure of constructing equilibrium configurations are given in Paper I, where we have adopted a harmonic expansion of the variables $\omega$ and $\Omega$ up to $\ell = 3$.

The perturbation equations describing non-barotropic, non-axisymmetric oscillations of slowly and differentially rotating relativistic stars have already been derived in Paper I. We have actually used the so called Cowling approximation, i.e. the equations are derived by perturbing only the fluid variables in the energy momentum equations $\delta (T_{\mu\nu}) = 0$. The oscillations of the fluid are then described by five functions, namely the enthalpy $H^{\ell m}$, the three perturbed velocity components $u_1^{\ell m}$, $u_2^{\ell m}$ (polar), $u_3^{\ell m}$ (axial) and the radial component of Lagrangian displacement vector $\xi^{\ell m}$. The perturbation equations read:

$$\dot{H}^{\ell m} + \text{im} (\Omega_1 + 6\Omega_3) H^{\ell m} = \left\{ \left(\frac{2}{r} - \lambda' + 2\nu'\right) c_s^2 - \nu' \right\} u_1^{\ell m} + c_s^2(u_1^{\ell m})' e^{2(\nu - \lambda)} - c_s^2 A \frac{2\nu}{r^2} u_2^{\ell m} + \text{im} \left\{ 2c_s^2 (\omega_1 + 6\omega_3) - \frac{15}{2} (2c_s^2 \omega_3 - \Omega_3) L_1^{\ell 2} \right\} H^{\ell m},$$

where $\lambda' = 0$ and are determined by solving ODE (2).
\[ u_{1m}^{\ell m} + \text{im} (\Omega_1 + 6\Omega_3) u_{1m}^{\ell m} = (H^{\ell m})' + \nu' e^{-2\lambda} \left( 2 \left( \frac{1}{r} - \nu' \right) \varpi_1 \omega_1' + 2 \left( \frac{1}{r} - \nu' \right) \varpi_3 \varpi_3' \right) \left( 6 - \frac{15}{2} \mathcal{L}_1^{\pm 2} \right) u_{1m}^{\ell m} \]
\[ + \text{im} \left\{ \left[ 2 \left( \frac{1}{r} - \nu' \right) \varpi_3 - \omega_1' \right] \left( 6 - \frac{15}{2} \mathcal{L}_1^{\pm 2} \right) \right\} u_{1m}^{\ell m} \]
\[ + \left\{ \left[ 2 \left( \frac{1}{r} - \nu' \right) \varpi_1 + 6\varpi_3 \right] \mathcal{L}_1^{\pm 1} \left( 6 - \frac{15}{2} \mathcal{L}_1^{\pm 2} \right) \right\} u_{1m}^{\ell m}, \]
\[ (5) \]
\[ u_{2m}^{\ell m} + \text{im} (\Omega_1 + 6\Omega_3) u_{2m}^{\ell m} = H_{m}^{\ell m} \]
\[ + \frac{\text{im}}{\Lambda} r e^{-2\lambda} \left\{ (2 - 2r\nu') (\varpi_1 + 6\varpi_3) + r (\varpi_1' + 6\varpi_3') \right\} u_{1m}^{\ell m} \]
\[ - \frac{15}{2} \left\{ (2 - 2r\nu') \varpi_3 + r \varpi_3'I \mathcal{L}_4^{\pm 2} \right\} u_{1m}^{\ell m} \]
\[ + \frac{\text{im}}{\Lambda} \left\{ (2(\varpi_1 + 6\varpi_3) - 30\varpi_3 \mathcal{L}_4^{\pm 2} - 15(\Omega_3 - 2\omega_3) \mathcal{L}_4^{\pm 2} + \frac{15}{2} \Omega_3 \mathcal{L}_4^{\pm 2}) \right\} u_{2m}^{\ell m} \]
\[ + \frac{1}{\Lambda} \left\{ (2(\varpi_1 + 6\varpi_3) \mathcal{L}_4^{\pm 1} - 15m^2(2\varpi_3 + \Omega_3) \mathcal{L}_4^{\pm 1} - 15(\Omega_3 - 2\omega_3) \mathcal{L}_4^{\pm 3}) \right\} u_{3m}^{\ell m}, \]
\[ (6) \]
\[ u_{3m}^{\ell m} + \text{im} (\Omega_1 + 6\Omega_3) u_{3m}^{\ell m} = \frac{\text{im}}{\Lambda} \left\{ (2(\varpi_1 + 6\varpi_3) - 30\varpi_3 \mathcal{L}_4^{\pm 2} - 15(\Omega_3 - 2\omega_3) \mathcal{L}_4^{\pm 2} + \frac{15}{2} \Omega_3 \mathcal{L}_4^{\pm 2}) \right\} u_{3m}^{\ell m} \]
\[ - \frac{1}{\Lambda} \left\{ (2(\varpi_1 + 6\varpi_3) \mathcal{L}_4^{\pm 1} - 30m^2\varpi_3 \mathcal{L}_4^{\pm 1} - 30\varpi_3 \mathcal{L}_4^{\pm 3}) \right\} u_{2m}^{\ell m} \]
\[ - \frac{r}{\Lambda} e^{-2\lambda} \left\{ (2 - 2r\nu') (\varpi_1 + 6\varpi_3) + r (\varpi_1' + 6\varpi_3') \right\} \mathcal{L}_4^{\pm 1} \]
\[ - \frac{15}{2} \left\{ (2 - 2r\nu') \varpi_3 + r \varpi_3'I \mathcal{L}_4^{\pm 3} \right\} u_{1m}^{\ell m}, \]
\[ (7) \]
\[ \xi_{m}^{\ell m} + \text{im} (\Omega_1 + 6\Omega_3) \xi_{m}^{\ell m} = e^{2\nu - 2\lambda} \left( \frac{\Gamma_1 - 1}{\Gamma} \right) \nu' u_{1m}^{\ell m} + \frac{15}{2} \text{im} \Omega_3 \mathcal{L}_1^{\pm 2} \xi_{m}^{\ell m}, \]
\[ (8) \]
where \( \varpi = \Omega - \omega \). The explicit form of the linear angular operators \( \mathcal{L}_1^{\pm j} \) are given in Appendix B.

As in the case of uniformly rotating stars [23, 24], Eqs. (4) and (5) form an infinitely coupled system of equations. In particular, differential rotation introduces extra couplings with respect to the uniformly rotating case. In the limit of uniform stellar rotation, \( \Omega_3 \) and \( \omega_3 \) vanish and Eqs. (4, 5) reduce to the perturbative equations presented in [24]. In addition, like in the uniformly rotating case, these equations can be devided into two independent subsystems, the so called axial-led and polar-led [26]. The polar-led system is the one that includes polar perturbations with \( \ell = |m| + 2k \), and axial perturbations with \( \ell = k + 1 \), where \( k \) is an integer. Instead, the axial-led system is the one that includes axial perturbations with \( \ell = |m| + 2k \), and polar perturbations with \( \ell = |m| + 2k + 1 \). The overall parity of each system is preserved and is polar for the first and axial for the second. In the rest of the paper we will focus mainly on the polar-led perturbations and leave axial-led for another study. Finally, for barotropic oscillations, where the background adiabatic index \( \Gamma_1 \) is equal to the adiabatic index of the perturbations \( \Gamma_1 \), the last equation is obsolete and the system reduces to four evolution equations.

The study of the oscillations described by Eqs. (4, 5) can be done either in the time domain as an initial value problem or in the frequency domain as an eigenvalue problem (Sec. III). Our study was based on the last method although we have also tried numerical evolutions of the system. In this case, we implemented a numerical code based on the two step Lax Wendroff scheme [27]. For some of the stellar models, the simulations show some numerical instabilities after an evolution time of \( \approx 20 \) – \( 30 \) ms. Before the development of the instability, we were able to determine with a Fast Fourier Transformation (FFT) the mode frequencies which agree better than \( \approx 3 \) – \( 5 \% \) to the ones derived by an eigenvalue problem.
III. PERTURBATION EQUATIONS IN FREQUENCY DOMAIN

The perturbation equations (11) can be studied in the frequency domain by assuming a harmonic time dependence for the perturbation functions $e^{-i\omega t}$, with $\omega$ being the oscillation frequency. By replacing the time derivatives in Eqs. (11S) by $-i\omega$ (e.g. $H \rightarrow -i\omega H$) and transforming $\mathcal{H}_{\ell m} \rightarrow i\mathcal{H}_{\ell m}$, $u_{1_{\ell m}} \rightarrow iu_{1_{\ell m}}$ and $\xi_{\ell m} \rightarrow i\xi_{\ell m}$ one obtains a purely real eigenvalue problem, which is formed by two ODEs for $\mathcal{H}_{\ell m}$ and $u_{1_{\ell m}}$ and three algebraic equations for $u_{2_{\ell m}}$, $u_{3_{\ell m}}$ and $\xi_{\ell m}$.

In order to prescribe the eigenvalue problem in a more compact form we first define the following three vectors:

$$ u_{\ell m} \equiv (H_{\ell m}, u_{1_{\ell m}})^T, \quad s_{\ell m} \equiv (u_{2_{\ell m}}, u_{3_{\ell m}})^T, \quad v_{\ell m} \equiv (\xi_{\ell m}, 0)^T, \quad (9) $$

and then three infinite dimensional vectors:

$$ U \equiv (\ldots, u_{\ell-1 m}, u_{\ell m}, u_{\ell+1 m}, \ldots)^T, \quad S \equiv (\ldots, s_{\ell-1 m}, s_{\ell m}, s_{\ell+1 m}, \ldots)^T, \quad V \equiv (\ldots, v_{\ell-1 m}, v_{\ell m}, v_{\ell+1 m}, \ldots)^T. \quad (10) $$

By using these definitions, the perturbative equations can be written in an operatorial form as follows:

$$ \frac{dU}{dr} = \mathcal{A}_{\sigma}^{\text{tot}} \cdot U + \mathcal{C} \cdot S + \mathcal{D} \cdot V, \quad (11) $$

$$ \mathcal{S}_{\sigma} \cdot S = \mathcal{M} \cdot U, \quad (12) $$

$$ \mathcal{Q}_{\sigma}^{\text{tot}} \cdot V = \mathcal{N} \cdot U, \quad (13) $$

where $\mathcal{A}_{\sigma}^{\text{tot}}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{S}_{\sigma}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{Q}_{\sigma}^{\text{tot}}$, are infinite dimensional linear operators which are defined in Appendix A. In $\mathcal{A}_{\sigma}^{\text{tot}}$, $\mathcal{S}_{\sigma}$, $\mathcal{Q}_{\sigma}^{\text{tot}}$ the subscript denotes the dependence of these matrices on the oscillation frequency $\omega$. These operators, as already mentioned above, couple perturbations with different harmonic indices $\ell$, leading to an infinite system of coupled equations. For a given azimuthal index $m$, the number of couplings in the system of equations (11, 12, 13) can be controlled by the parameter $\ell_{\text{max}}$, which is the harmonic index $\ell$ where the tensor harmonic expansion of perturbations is truncated. In this way, the number of couplings is $n_c = \ell_{\text{max}} - |m| + 1$ as $\ell$ runs $|m|$ to $\ell_{\text{max}}$ i.e. $|m| \leq \ell \leq \ell_{\text{max}}$. The truncation of the couplings up to a certain $\ell_{\text{max}}$ has been done for practical reasons since it is impossible to deal with infinity many couplings which do not contribute significantly in the final result. This approximation has been tested by studying the variation of the eigenfrequencies with respect to a varying value of $\ell_{\text{max}}$. In general, we found that even by keeping only a small number of couplings the eigenfrequencies were converging (Sec. IV B).

One can form a boundary value problem by using Eqs. (11, 12, 13) together with an appropriate set of boundary conditions at the center and the surface of the star. In the parameter domain where the inverse of the operators $\mathcal{S}_{\sigma}$ and $\mathcal{Q}_{\sigma}^{\text{tot}}$ exist, both Eqs. (12) and (13) can be solved for $S$ and $V$ respectively:

$$ S = \mathcal{S}_{\sigma}^{-1} \cdot \mathcal{M} \cdot U, \quad V = (\mathcal{Q}_{\sigma}^{\text{tot}})^{-1} \cdot \mathcal{N} \cdot U, \quad (14) $$

and inserted into Eq. (11). Then the final matrix equation, together with the appropriate boundary conditions, can be used for the calculation of the eigenfrequency $\omega$.

In the domain of frequencies where the two operators $\mathcal{S}_{\sigma}$ and $\mathcal{Q}_{\sigma}^{\text{tot}}$ are singular, Eqs. (12) and (13) cannot be solved for $S$ and $V$. This issue will be discussed in more detail in Sec. III C.

A. Boundary Conditions

The perturbation equations (11, 12, 13) can be solved as an eigenvalue problem for the variable $\omega$ by fixing the boundary conditions at the center and the surface of the star. Regularity at the center of the star suggests that the various perturbation functions have the following behavior:

$$ H_{\ell m} \sim r^\ell, \quad u_{1_{\ell m}} \sim r^{\ell-1}, \quad u_{2_{\ell m}} \sim r^\ell, \quad u_{3_{\ell m}} \sim r^{\ell+1}. \quad (15) $$

For any combination of $\ell$ and $m$ there is only one independent solution at the center, as $H_{\ell m}$ and $u_{1_{\ell m}}$ are related via the relation:

$$ [\ell H_{\ell m} - (-\sigma + m\Omega_1) u_{1_{\ell m}}]|_{r=0} = 0. \quad (16) $$
Moreover, since at the center of the star \( H^{\ell m} \) and \( H^{\ell' m} \) are independent, there exist \( \ell_{\text{max}} - |m| + 1 \) independent boundary conditions.

The second boundary condition comes from the vanishing of the pressure's Lagrangian perturbation on the surface. This condition is satisfied when the perturbations obey the following \( \ell_{\text{max}} - |m| + 1 \) number of equations:

\[
\left[ -\sigma + m (\Omega_1 + 6\Omega_3) - \frac{15}{2} m \Omega_3 \ell^{\pm 2}_1 \right] e^{2\lambda - 2\nu} H^{\ell m} - \nu' u^{\ell m}_1 = 0, \quad m \leq \ell \leq \ell_{\text{max}}. \tag{17}
\]

Notice that in the previous equation, the operator acting on the enthalpy perturbation \( H^{\ell m} \) comes from the total derivative:

\[
\frac{d}{dt} u^\alpha \nabla_\alpha = e^{-\nu} \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) \rightarrow ic^{-\nu} \left[ -\sigma + m (\Omega_1 + 6\Omega_3) - \frac{15}{2} m \sin^2 \theta \Omega_3 \right]. \tag{18}
\]

In fact, the last term of Eq. (18) becomes the operator \( \mathcal{L}_1^{\pm 2} \) of Eq. (17) after the angular integration of the perturbation equations.

### B. Numerical Method

The numerical technique that is used here has been already described earlier in [24]. By selecting a certain value for \( \ell_{\text{max}} \) we allow \( n_c = \ell_{\text{max}} - |m| + 1 \) couplings between various harmonics where \( |m| \leq \ell \leq \ell_{\text{max}} \). Then one must specify \( n_c \) independent conditions at the origin for \( H^{\ell m} \) that can be denoted by a vector

\[
\mathcal{H}_k \equiv (H^{\ell_1 m_1}, \ldots, H^{\ell_{\text{max}} m_{\text{max}}}) , \quad \text{where} \quad k = 1, \ldots, n_c, \tag{19}
\]

and here we assume that \( \mathcal{H}_1 = (1,0,\ldots,0) \), \( \mathcal{H}_2 = (0,1,0,\ldots,0) \), \ldots, \( \mathcal{H}_{n_c} = (0,\ldots,0,1) \). Then the corresponding values for \( u^{\ell m}_1 \) are estimated via Eq. (18). By specifying a trial eigensolution \( \sigma \), we integrate for each \( \mathcal{H}_k \) Eqs. (14) and we calculate the value of the Lagrangian pressure perturbation \( \Delta p^\ell_k \) on the surface. This quantity in general does not vanish and after \( n_c \) integrations, we construct an \( n_c \times n_c \) “pressure matrix” \( \mathbf{P} \). An eigenmode \( \sigma \) is a solution of Eqs. (14) that simultaneously satisfy \( \Delta p^\ell = 0 \) for any \( \ell \). Since the \( n_c \) initial values for \( \mathcal{H}_k \) are independent, one can find appropriate coefficients \( a_k \) such that

\[
\sum_{k=m}^{\ell_{\text{max}}} a_k \Delta p^\ell_k = 0, \quad \text{for every} \quad \ell = m \ldots \ell_{\text{max}}. \tag{20}
\]

This is equivalent with the requirement that \( \det \mathbf{P} = 0 \). After integrating Eqs. (14) for a wide range of frequencies we isolate the zeros of \( \det \mathbf{P} = 0 \) by a “root finding algorithm”. These zeros are the eigenmodes \( \sigma \) of the problem.

The eigenfunctions of the enthalpy and velocity perturbations can be constructed as a linear combinations of the \( n_c \) solutions that correspond to the \( n_c \) independent initial conditions \( \mathcal{H}_k \). The coefficients \( a_k \) of these linear combinations are determined by solving the homogeneous system of linear equations described by Eq. (20).

### C. Continuous Spectrum

For the range of frequencies \( \sigma \) that the inverse operators of \( S_\sigma \) and \( Q^{\text{tot}}_\sigma \) do not exist, Eqs. (14) are singular and consequently the eigenmodes cannot be determined using the frequency domain code. This is the same singularity that generates the continuous spectrum as in the r-mode studies [28, 29, 30, 31, 32], which have been carried out in slow rotation approximation. The continuous spectrum (CS) practically can be determined by solving the determinant of the \( 2n_c \times 2n_c \) matrices that represent the operators \( S_\sigma \) and \( Q^{\text{tot}}_\sigma \), for the mode frequency \( \sigma \). Let us first consider the operator \( S_\sigma \) for the case \( \ell_{\text{max}} = |m| \). The determinant is a second order polynomial in \( \sigma \) which has a double root of the form:

\[
\sigma = m (\Omega_1 + 6\Omega_3) - \frac{2m}{\ell (\ell + 1)} (\omega_1 + 6\omega_3) - \frac{m}{\ell (\ell + 1)} \Psi (\ell, m, \Omega_3, \omega_3), \tag{21}
\]

where the function \( \Psi \) is defined as follows:

\[
\Psi \equiv -15 (3\Omega_3 - 4\omega_3) [\ell Q^2_{\ell+1,m} - (\ell + 1) Q^2_{\ell,m}] + \frac{15}{2} \Omega_3 \left[ \ell (\ell + 1) - (\ell + 1)(\ell - 2) Q^2_{\ell,m} - \ell (\ell + 3) Q^2_{\ell+1,m} \right], \tag{22}
\]
and the coefficient $Q_{\ell,m}^\sigma$ is given in Eq. (B12). The interval of the CS is then $\sigma_{R}^\sigma \leq \sigma \leq \sigma_{c}^\sigma$, where $\sigma_{R}^\sigma$ and $\sigma_{c}^\sigma$ are for the operator $S_{\sigma}$ the values of $\sigma$ given by Eq. (21) at stellar surface and centre respectively. In general, for $\ell_{\text{max}} \geq m$, there are $\ell_{\text{max}} - |m| + 1$ branches of the CS, which might as well overlap widening the band of it. Similarly to the uniformly rotating star [24], for $\ell_{\text{max}} \to \infty$ the CS will cover all the spectrum.

Similar behavior can be observed for the operator $Q_{\ell,m}^\text{tot}$, where one can calculate a singular patch for $\ell_{\text{max}} = m$ via the following expression:

$$\sigma = m (\Omega_1 + 6\Omega_3) - \frac{15}{2} m \Omega_3 \left[ 1 - Q_{\ell,m}^2 - Q_{\ell+1,m}^2 \right].$$  \hfill (23)

Again in this case, the width of the CS is defined by the values of $\sigma$ given by Eq. (23) at the stellar center and surface i.e. $\sigma_{R}^Q \leq \sigma \leq \sigma_{c}^Q$ and the number of continuous spectrum patches when more coupling terms are included is again $\ell_{\text{max}} - |m| + 1$.

It is obvious from the above analysis, that the operators $S_{\sigma}$ and $Q_{\ell,m}^\text{tot}$ generate different bands of CS. For a barotropic EoS, Eq. (13) is trivial and the continuous spectrum is only generated from operator $S_{\sigma}$. Furthermore, when expression (23) is satisfied the output of the angular integration of the total time derivative given by Eq. (18) vanishes and the CS of the operator $Q_{\ell,m}^\text{tot}$ lies in the corotation band of oscillation modes.

It should be noted that the existence of a continuous spectrum for uniformly rotating relativistic stars is highly debatable and it might be a result of the slow rotation approximation [32]. On the other hand, differential rotation is known to be associated with the presence of corotation points and existence of a continuous spectrum but this is a technical problem that may not be solvable within the limits of the slow rotation approximation. Thus we will not address the issues related to the presence of CS in the rest of this paper.

| Model | $\rho_c$ ($\times 10^{-3}\text{km}^{-2}$) | $M$ ($M_\odot$) | $R$ (km) | $M/R$ | $\Omega_K$ ($\times 10^{-2}\text{km}^{-1}$) |
|-------|-----------------------------------|----------------|---------|--------|----------------------------------|
| B0    | 0.662                             | 1.400          | 14.151  | 0.146  | 2.700                            |
| C0    | 1.484                             | 1.637          | 11.218  | 0.216  | 4.168                            |

TABLE I: The parameters of the background non-rotating stellar models of the B and C sequence. Here $\rho_c$ is the central rest mass density, $M$ the mass, $R$ the radius and $\Omega_K$ denotes the mass sheeding limit (Kepler frequency).

| $a_k$ ($\times 10^{-3}$) | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|-------------------------|-------|-------|-------|-------|-------|
| B                       | 0.726 | -1.272| 3.972 | -0.853| 3.970 |
| C                       | -2.239| 1.811 | 5.388 | 2.233 | 5.386 |

TABLE II: The coefficients $a_k$ of the Padé approximation of Eq (27) for both the B and C sequences of stellar modes.

| Model | $T_c$ (ms) | $\Omega_c$ ($\times 10^{-2}\text{km}^{-1}$) | $\Omega_e$ ($\times 10^{-2}\text{km}^{-1}$) | $\varepsilon_e$ | $J$ (km$^2$) |
|-------|------------|-------------------------------------------|-------------------------------------------|-----------------|---------------|
| B1    | 1.719      | 1.218                                     | 0.435                                     | 0.161           | 0.935         |
| B3    | 0.970      | 2.160                                     | 0.771                                     | 0.286           | 1.657         |
| B6    | 0.657      | 3.189                                     | 1.139                                     | 0.422           | 2.447         |
| C1    | 1.719      | 1.218                                     | 0.535                                     | 0.129           | 0.832         |
| C3    | 0.970      | 2.160                                     | 0.949                                     | 0.229           | 1.474         |
| C6    | 0.657      | 3.189                                     | 1.401                                     | 0.338           | 2.176         |

TABLE III: Parameters describing rotating models of the B and C sequences, see also Table I. Here $T_c$ and $\Omega_c$ are respectively the period and the angular velocity on the rotational axis, while $\Omega_e$ and $\varepsilon_e$ represent the angular velocity and the dimensionless parameter, described in Eq. (28), at the equator. The angular momentum of the star is denoted with $J$. 

\[\]
FIG. 1: This figure displays the dependence of $\phi_e = \Omega_e/\Omega_c$ on the parameter $A$ for the B (solid line) and C (dashed line) models.

| Model BJ1 | $T_c$ (ms) | $\Omega_c$ ($\times 10^{-2}$ km$^{-1}$) | $\Omega_e$ ($\times 10^{-2}$ km$^{-1}$) | $\epsilon_e$ | $J$ (km$^2$) |
|-----------|-----------|------------------------------------|------------------------------------|-------------|-------------|
| A (km)    |           |                                    |                                    |             |             |
| 5         | 0.667     | 3.143                              | 0.268                              | 0.099       | 0.935       |
| 15        | 1.989     | 1.054                              | 0.491                              | 0.182       | 0.935       |
| 50        | 2.746     | 0.763                              | 0.692                              | 0.256       | 0.935       |
| 100       | 2.834     | 0.739                              | 0.725                              | 0.268       | 0.935       |

TABLE IV: This table displays the main properties of the differentially rotating models BJ1 for four typical values of $A$.

IV. NUMERICAL RESULTS

In this section, we solve Eqs. (11-13) and investigate several aspects of the oscillation spectrum of non-axisymmetric oscillations of slowly and differentially rotating stars. In the first subsection we describe the procedure in constructing the background stellar models while in the second subsection the spectrum of non-axisymmetric oscillations is discussed and we provide tables with frequencies for several background configurations.

A. Stellar Models

In the slow rotation approximation the stellar models maintain their spherical shape and the rotation is treated as a small perturbation. This approximation is practically applicable to all known neutron stars even to those with rotation frequency of a few hundreds Hz. Here we specify a specific value for the central density for a given EoS and we generated a sequence of rotating models by varying the angular velocity of the star. For simplicity, we adopted the relativistic barotropic EoS:

$$p = K \rho^\Gamma, \quad \epsilon = \rho + \frac{p}{\Gamma - 1},$$

where $\rho$ is the rest mass density and $\epsilon$ the total energy density, while $K$ and $\Gamma$ are the polytropic parameters. We have chosen the so called B-models [18, 19], which represent differentially rotating polytropic stars with central rest mass density $\rho_c = 7.91 \times 10^{14}$ g cm$^{-3}$ and polytropic parameters $\Gamma = 2$ and $K = 217.86$ km$^2$. The nonrotating
TABLE V: Frequencies ($\sigma/2\pi$), in kHz, of the fundamental ($^2f$) and the first pressure mode ($^2p_1$) for the B1 stellar model with $A = 12$ km and for $\ell_{\text{min}} \leq \ell_{\text{max}} \leq \ell_{\text{min}} + 3$.

| $m$ | $\ell_{\text{max}} = 2$ | $\ell_{\text{max}} = 3$ | $\ell_{\text{max}} = 4$ | $\ell_{\text{max}} = 5$ |
|-----|----------------|----------------|----------------|----------------|
| $^2f$ | 1.4793 | 1.4766 | 1.4768 | 1.4769 |
| -2 | 1.7037 | 1.7016 | 1.7016 | |
| -1 | 2.1271 | 2.1252 | 2.1246 | |
| 1 | 2.2932 | 2.2901 | 2.2913 | 2.2913 |

| $^2p_1$ | 3.5798 | 3.5811 | 3.5815 | 3.5815 |
| -2 | 3.8156 | 3.8205 | 3.8206 | |
| -1 | 4.4221 | 4.4278 | 4.4273 | |
| 1 | 4.6467 | 4.6485 | 4.6488 | 4.6488 |

FIG. 2: Frequencies ($\sigma/2\pi$), in kHz, of the $^2f$ and $^2p_1$ nonradial modes with $m = 1$ for the B sequence of models having $A = 12$ km. The solid line corresponds to the purely first order slow rotation approximation, while the dashed line includes some second order corrections due to the component $u^m_\ell$ of the perturbed fluid velocity.

In a differentially rotating star, the angular velocity on the rotation axis $\Omega_c$ and the parameter $A$, which describes the degree of differential rotation, are the other two free parameters which need to be specified in constructing a sequence of rotating stellar model by using Eq. (2). The angular velocity at surface, $\Omega_s$, is then related to the one at the axis by the following relation:

$$\Omega_s = \Omega_c \phi(A, \theta),$$

where $\phi = \phi(A, \theta)$ is a scalar function that depends on the law that describes the differential rotation. For the relativistic $j$-constant rotation law given by Eq. (3) this function reads:

$$\phi = \frac{1}{A^2 + e^{-2\nu r^2} \sin^2 \theta \left( A^2 + e^{-2\nu r^2} \sin^2 \theta \frac{\omega(r, \theta)}{\Omega_c} \right)}. \quad (26)$$
The form of \( \phi_e \) on the equator is drawn in Fig. 1 for both B and C sequence of stellar models. These two curves, for practical reasons, can be Padé approximated by the following rational function:

\[
\phi_{e}^{fit} = \frac{a_0 + a_1 A + a_2 A^2}{1 + a_3 A + a_4 A^2},
\]

where the coefficients \( a_k \) are listed in Table III.

The parameter that is commonly used in describing differentially rotating stars is the ratio \( T/W \), where \( T \) is the rotational kinetic energy and \( W \) the gravitational potential energy respectively. Another possible rotational parameter, which have been used in [34], is a function of the total angular momentum, the total mass and the two parameters defining polytrope. In the relativistic slow rotation approximation, the gravitational potential energy \( W \) can be accurately determined only at order \( O(\Omega^2) \), where the monopole and quadrupole corrections of the gravitational mass and internal energy can be defined. Therefore, we define as dimensionless rotation parameter the following ratio:

\[
\varepsilon_e \equiv \frac{\Omega_e}{\Omega_K} = \frac{\Omega_e}{\Omega_K} \phi(A, \pi/2) = \varepsilon_e \phi(A, \pi/2),
\]

where \( \Omega_e \) represents the angular rotation at the stellar equator and \( \Omega_K \) is the Kepler angular velocity that defines the mass shedding limit of a rotating star. In slowly rotating neutron stars (first order in \( \Omega \)), \( \Omega_K \) can be approximately described by the angular velocity of a particle in a stable circular Keplerian orbit at the equator of a nonrotating star \( \Omega_K = \sqrt{\mu/R^3} \). Finally, the quantity \( \varepsilon_e \equiv \Omega_e/\Omega_K \) is the value of the dimensionless rotation parameter at the axis. The parameters of a few selected models, belonging to the B and C sequences, are shown in Table III for \( A = 12 \text{ km} \). Differential rotation is described by a number of parameters i.e. \( \Omega_e \), \( J \) and \( A \). The effect of these parameters on the oscillation spectrum will be studied for two families of background stellar models. In each family we fix one of the first two parameters and vary \( A \). The first family of stellar models is the so called B-sequence in which we fix \( \Omega_e \) and we vary the parameter \( A \). The properties of the various members of this family, e.g. their angular velocity \( \Omega_e \) at the equator and of the dimensionless parameter \( \varepsilon_e \), can be estimated from those of the \( A = 12 \text{ km} \) model (Table III) by using the Padé expression (27). The second sequence is constructed by keeping \( \Omega_e \) fixed at the value \( \Omega_e = 12 \text{ km} \) model (Table III) by using the Padé expression (27). The models of this new family will be named BJ models. The second family of background stellar models is the so called B-sequence in which we fix \( \Omega_e \) and \( A \) and vary the parameter \( J \). The models of this new family will be named BJ models.

B. Spectral Properties

Any nonradial mode of a rotating star is characterized by its harmonic indices \( (\ell, m) \), where \( -\ell \leq m \leq \ell \). The axisymmetric modes \( m = 0 \) have been already discussed in Paper I, so here we focus on the non-axisymmetric oscillations \( m \neq 0 \).

The non-axisymmetric modes of a rotating star split in corotating and counterrotating branches, whose pattern speed \( \sigma_p = \sigma/m \) is respectively positive and negative. Note that the modes are assumed to behave as \( e^{-i(\sigma t - m\phi)} \). In the slow rotation approximation, the eigenfrequencies of any \( (\ell, m) \) non-axisymmetric mode are linear functions of the
|               | m   | B0   | B1   | B3   | B6   |
|---------------|-----|------|------|------|------|
| $^2f$         | -2  | 1.883| 1.479| 1.171| 0.839|
|               | -1  | 1.883| 1.672| 1.515| 1.349|
|               | 1   | 1.883| 2.102| 2.276| 2.473|
|               | 2   | 1.883| 2.293| 2.614| 2.970|
| $^2p_1$      | -2  | 4.107| 3.579| 3.177| 2.744|
|               | -1  | 4.107| 3.808| 3.579| 3.334|
|               | 1   | 4.107| 4.416| 4.657| 4.925|
|               | 2   | 4.107| 4.647| 5.068| 5.534|

TABLE VI: Frequencies (σ/2π), in kHz, of the $^2f$ and $^2p_1$ nonradial modes for selected B stellar models with $A = 12$ km.

Any ($\ell, m$) nonradial mode, Eqs. (11-13) can be integrated by fixing the parameters $\ell_{\text{min}}$ and $\ell_{\text{max}}$. The value of $\ell_{\text{min}}$ is fixed by the selection of index $m$ ($\ell_{\text{min}} = |m|$), whereas the value of $\ell_{\text{max}}$ is chosen according to the maximum rotation parameter $\varepsilon_c$ or $\varepsilon_e$. Note that $\varepsilon_c$ and $\varepsilon_e$ are related through Eq. (28). For a given stellar model the splitting of the modes, due to differential rotation, can be described by the following relation:

$$\sigma^{\ell m} = \sigma^{\ell m}_0 \pm \alpha(\ell, |m|, A) \varepsilon_c = \sigma^{\ell m}_0 \pm \tilde{\alpha}(\ell, |m|, A) \varepsilon_e,$$

where $\alpha$ and $\tilde{\alpha}$ are scalar functions depending on the harmonic indices and the differential parameter $A$ while $\sigma_0$ is the oscillation frequency of the nonrotating configuration with respect to an inertial observer. For rotational laws that are independent from the $\theta$ coordinate (e.g. uniform rotation), relation (32) is simplified and becomes linear with respect to the azimuthal index $m$,

$$\sigma^{\ell m} = \sigma^{\ell m}_0 + m \alpha_{\text{uni}} \varepsilon_c,$$

as it is known from the Newtonian theory [35].

1. Dependence on $\ell_{\text{max}}$

For any ($\ell, m$) nonradial mode, Eqs. (11-13) can be integrated by fixing the parameters $\ell_{\text{min}}$ and $\ell_{\text{max}}$. The value of $\ell_{\text{min}}$ is fixed by the selection of index $m$ ($\ell_{\text{min}} = |m|$), whereas the value of $\ell_{\text{max}}$ is chosen according to the maximum
number of desired couplings \( n_c \) and it’s upper limit is \( \infty \). Before proceeding in deriving any result it is important to test the dependence of the oscillation frequencies for a given value of \( \ell \) on \( \ell_{\text{max}} \). In Table \( \text{V} \) we present the results of such a study, we actually show how the values of the fundamental \( ^2 f \) and pressure \( ^2 p_1 \) modes of the B1 model depend on \( \ell_{\text{max}} \). The variation of \( \ell_{\text{max}} \) has not affected the actual values of the modes by more than \( 0.2\%-0.3\% \). The same order of dependence of the mode frequencies on \( \ell_{\text{max}} \) has been also observed for the other models of the B and C-sequences. Therefore, \( \ell = 2 \) mode frequencies can be accurately estimated by ignoring couplings with higher \( \ell \)’s i.e. by keeping \( \ell_{\text{max}} = 2 \).

In paper I, we discussed the role of the coupling between the polar and axial perturbation functions in the axisymmetric case. Actually, the component of the velocity perturbation \( u_3^{\ell_m} \) is intrinsically of \( O(\Omega) \) order, (see Eq. (7)). This implies that there are “hidden” second order in \( \Omega \) terms which induce some extra couplings into Eqs. (5,6) that can influence the results. The same type of problem is also present for the non-axisymmetric perturbation, and will especially influence specific modes as for example the \( (\ell, m) = (2, 1) \). Thus we studied the \( (\ell, m) = (2, 1) \) eigenfrequencies either by using the full coupled system of equations, i.e. \( (\ell_{\text{min}}, \ell_{\text{max}}) = (1, 2) \), or by removing the “hidden” second order couplings, i.e. \( (\ell_{\text{min}}, \ell_{\text{max}}) = (2, 2) \). The results are shown in Fig. 2 and the similarities to the axisymmetric results of Paper I are obvious. That is, as we expected the dependence of the mode frequencies on the rotation rate is not any more linear due to the presence of these implicit second order rotational terms. This effect is actually more pronounced for the \( ^2 f \) mode.

We have chosen to neglect all of these “hidden” second order terms in order to get the expected linear relation between the oscillation and the rotational frequencies.

### 2. Eigenfrequencies

Here, we study how the mode frequencies depend on the rotation parameter \( \varepsilon_e \), the compactness \( M/R \) of the star and the degree of differential rotation \( A \).

Let us start by choosing the B sequence of stellar models for a fixed value of \( A = 12 \) km, these are the equilibrium configurations already used in [17, 18]. Due to rotation, these modes are split in two symmetrical branches characterized by the azimuthal number \( m \). In Fig. 3 the splitting induced by rotation for the \( \ell = 2 \) fundamental and first pressure modes is shown while the actual values of the eigenfrequencies for these sequence of models are given in Table \( \text{VI} \).

Stellar compactness affects significantly the mode splitting, but one should be careful on how to quantify this effect. In fact, the angular velocity profile of the stellar model depends on the compactness and it is inconsistent to create a sequence of models by just keeping \( A \) constant. Instead, we have chosen to compare stellar models according to the ratio between the angular velocity at the equator and at the rotational axis, i.e. by keeping \( \phi_e = \Omega_e/\Omega_c \) constant. We then consider the B sequence of stellar models with \( A = 12 \) km as reference for generating other polytropic stars.
described by the Eq. (24). All these models have \( \phi_e = 0.3572 \), which for the C models corresponds to \( A = 10.08 \) km. In Figure 3, we show the dependence of the “splitting” coefficient \( \hat{\alpha} \) on the stellar compactness for the \( 2f \) and \( 2p_1 \) modes. It is obvious that the splitting is enhanced by the compactness and it can be easily be twice as large for very compact neutron stars. In principle, within the gravitational wave asteroseismology [36], this effect of rotation can be used, to infer the rotation and compactness of the oscillating neutron star.

Finally, we investigate the dependence of the non-axisymmetric modes on the parameter \( A \) describing the degree of differential rotation. We consider the two sequences of stellar models described earlier in Sec. IV A. These are, the B sequence of stellar models with \( \Omega_c \) constant and the BJ sequence with \( J \) constant. In Table VIII we report the frequencies of the \( 2f \) and \( 2p_1 \) modes for some of these models. In Fig. 4, it is shown as the value of \( A \) affects the splitting of the \( 2f \) and \( 2p_1 \) modes. The splitting of the eigenfrequencies for stellar models with high degree of differential rotation depends strongly on the rotation parameter \( \varepsilon_c \). Furthermore, it is worth noticing that the B and BJ models show the same dependence of \( \hat{\alpha} \) with respect to \( A \). This behavior is expected since the differential rotation is described in both stellar sequences by the relativistic j-constant rotation law [3] and the no-rotating model of the sequences is the same.

As it has been mentioned in the introduction, differential rotation plays an important role on the onset of rotational instabilities. Non-axisymmetric pulsations of rotating stars can become dynamical or secular unstable and enhance gravitational wave emission, e.g. see [31, 38] and references there in. These instabilities appear when the rotational parameter \( \beta = T/|W| \), reaches some critical value \( \beta_c \). The onset of secular and dynamical bar mode instabilities in Newtonian incompressible and uniformly rotating bodies is at \( \beta_c = 0.14 \) and \( \beta_c = 0.27 \) respectively. The secular instabilities are driven by dissipative processes, such as gravitational radiation via the Chandrasekhar-Friedman-Schutz (CFS) mechanism and viscosity.

Actually, the so called “low \( T/W \)” dynamical instability appears in stellar models with high degree of differential rotation. Several studies carried out with nonlinear hydrodynamical codes [4, 7] and perturbative methods [9, 10] suggest a strong correlation between the onset of the low \( T/W \) instability and the presence of corotation modes. By definition, these modes have their pattern speed equal to the local angular velocity of the star, i.e. \( \sigma/m = \Omega(r, \theta) \).

For a differentially rotating star the possible corotation band of the spectrum is given by the interval \( \Omega \leq \sigma/m \leq \Omega_c \), which is obviously larger for highly differentially rotating stars, i.e. with smaller \( A \). For the B sequence of stellar models, we estimated the required differential and rotation parameters for having corotation modes. Initially, we show in Fig. 6 the cases \( A = 5 \) km and \( A = 12 \) km. As expected, for \( A = 5 \) km the \( \ell = m = 2 \) fundamental mode “corotates” for smaller rotation rate \( \varepsilon_c \) than the \( A = 12 \) km configuration. Fig. 7 shows the variation of the \( \ell = m = 2 \) fundamental mode with respect to the parameter \( A \). All these models with the exception of the B1 and B2 present a corotation \( 2f \) mode. The upper limit of the parameter \( A \) for having corotation is indicated with \( A_c \) and it is illustrated with a circle in Fig. 7. Therefore, the star has corotation modes when \( A \leq A_c \). In Table VIII, the critical values \( A_c \) and the corresponding eigenfrequencies are listed for various models of the B sequence. When non-axisymmetric modes are into the corotation band the eigenvalue problem is mathematically singular. Still, for some stellar models it was possible to isolate the eigenmode frequencies by carefully studying their eigenfunctions into the continuous spectrum [10].

V. CONCLUSIONS AND DISCUSSION

We presented a first comprehensive study of non-axisymmetric oscillations of slowly and differentially rotating neutron stars in the perturbative framework of General Relativity. By using Cowling approximation, we examined the spectral properties of polytropic stars and investigated their dependence on four main parameters: stellar compactness \( M/R \), stellar rotation rate at the equator \( \varepsilon_e \), degree of differential rotation \( A \) and the maximum number of perturbative couplings \( \ell_{max} \). In accordance with the first order slow rotation approximation, the non-axisymmetric modes exhibit a linear splitting with respect to the rotational parameter \( \varepsilon_c \). However, in some of the eigenmode patterns appears a quadratic deviation with respect to the expected linear behavior. This is due to the presence of “implicit” second order perturbative terms in the perturbation equations. We have identified and then neglected these terms in order to be consistent with the order of approximation adopted for the background spacetime. Moreover, we show that the non-axisymmetric spectrum can be described by including only a small number of couplings between perturbation functions. For instance, we determined the quadrupolar spectrum with an accuracy to better than 1% by setting \( \ell_{max} = 2 \), which corresponds to the lowest possible number of coupling terms in the perturbation equations.

We found that both differential rotation and stellar compactness affect the non-axisymmetric spectrum. In fact, the rotational splitting of the non-axisymmetric modes is enhanced by stellar compactness and the degree of differential rotation.

For the study of the “low \( T/W \)” instability we calculated the corotation band of some polytropic models and the necessary rotational configuration for having a corotating quadrupolar fundamental mode. Moreover, we verified
| Modes | m | A (km) | B1 (kHz) | B3 (kHz) | B6 (kHz) | BJ1 (kHz) | BJ3 (kHz) | BJ6 (kHz) |
|-------|---|--------|---------|---------|---------|---------|---------|---------|
|       | 1 | 5      | 1.965   | 2.029   | 2.101   | 2.097   | 2.269   |
|       | 1 | 12     | 2.102   | 2.276   | 2.473   | 2.102   | 2.276   | 2.473   |
|       | 1 | 50     | 2.244   | 2.535   | 2.864   | 2.107   | 2.285   | 2.484   |
|       | 1 | 100    | 2.256   | 2.101   | 2.896   | 2.107   | 2.285   | 2.483   |
|       | 2 | 5      | 2.027   | 2.138   | 2.627   | 2.274   |
|       | 2 | 12     | 2.293   | 2.614   | 2.970   | 2.293   | 2.614   |
|       | 2 | 50     | 2.615   | 3.220   | 3.917   | 2.334   | 2.698   | 3.113   |
|       | 2 | 100    | 2.647   | 3.283   | 4.018   | 2.338   | 2.707   | 3.128   |
|       | 1 | 5      | 4.226   | 4.317   | 4.416   | 4.412   | 4.648   | 4.908   |
|       | 1 | 12     | 4.416   | 4.657   | 4.925   | 4.416   | 4.657   | 4.925   |
|       | 1 | 50     | 4.609   | 5.011   | 5.465   | 4.419   | 4.665   | 4.940   |
|       | 1 | 100    | 4.624   | 5.038   | 5.508   | 4.418   | 4.664   | 4.938   |
|       | 2 | 5      | 4.295   | 4.438   | 4.593   | 4.586   | 4.949   | 5.486   |
|       | 2 | 12     | 4.647   | 5.068   | 5.534   | 4.647   | 5.068   | 5.534   |
|       | 2 | 50     | 5.114   | 5.947   | 6.912   | 4.728   | 5.229   | 5.799   |
|       | 2 | 100    | 5.164   | 6.044   | 7.067   | 4.737   | 5.247   | 5.829   |

TABLE VII: The mode frequencies ($\sigma/2\pi$) of the fundamental ($^2f$) and the first pressure mode ($^2p_1$) in kHz, for the B and BJ sequences of stellar models. The B6 models for $A > 38$ km are rotating faster than the mass sheeding limit, their eigenmodes are labeled with a star. In the BJ3 and BJ6 models, some of the eigenfrequencies cannot be determined as they are inside the continuous spectrum. Therefore, we leave a blank space in the table.

![FIG. 5: The dependence of the splitting factor $\tilde{\alpha}$ on the differential rotation parameter $A$ for the B and BJ models is drawn for the $^2f$ and $^2p_1$ modes. The solid and the dashes lines represents respectively the $^2f$ and $^2p_1$ modes of the B models, whereas the open circles and triangles the $^2f$ and $^2p_1$ modes of the BJ models.](image)

the results of Newtonian studies, i.e. that the value of the rotational parameter $\epsilon_\alpha$, which is required for having a corotating fundamental mode, is inversely proportional to the degree of differential rotation. The onset and the details of the low $T/|W|$ dynamical instability of differentially rotating stars will be the subject of a future investigation.
FIG. 6: The upper panel displays the $\ell = m = 2$ f-mode (solid line) for the B sequence of stellar models with a high degree of differential rotation $A = 5$ km. The limiting lines of the corotation band are represented in dashed lines. The lower panel is similar but for $A = 12$ km.

| Models | $\varepsilon_e$ | $A_e$ (km) | $\sigma/2\pi$ (kHz) |
|--------|----------------|------------|---------------------|
| B3     | 0.042          | 3.898      | 2.0615              |
| B4     | 0.162          | 7.422      | 2.4105              |
| B5     | 0.302          | 10.182     | 2.7336              |
| B6     | 0.446          | 12.534     | 3.0439              |
| B7     | 0.610          | 15.095     | 3.3454              |
| B8     | 0.778          | 17.625     | 3.6707              |

TABLE VIII: Critical values of the differential rotational parameters for having corotation modes. These values correspond to the $\ell = m = 2$ f-mode.

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APPENDIX A: COEFFICIENT MATRICES

In Sec. III we wrote the fluid perturbations as three infinite dimensional vectors $U$, $S$ and $V$, see Eq. (10). These vectors can be written in terms of the 2-vectors $u^{\ell m}$, $s^{\ell m}$ and $v^{\ell m}$ that are defined in Eq. (9). They contain the fluid perturbations with harmonic indices $(\ell, m)$. These definitions enable us to describe the perturbation equations in a more compact form by introducing a set of infinite dimensional linear matrix operators, i.e. $A_{\sigma}^{\text{tot}}, \mathcal{C}, \mathcal{D}, S_\sigma, M, N, Q_\sigma^{\text{tot}}$. In this Appendix, we provide the transformation laws of these operators. Actually, it is much more convenient to
FIG. 7: For B stellar models, the figure displays the variation of the \( \ell = m = 2 \) fundamental mode with respect to the differential parameter \( A \). In any sequence, the circles denotes when the fundamental mode goes in corotation. For \( A \) lower then these values, the fundamental mode is always in corotation. The dashed line denotes instead the stellar models that are rotating faster than the mass shedding limit.

define how these operators act on the single vectors i.e. the ones defined in Eq. (9), rather than dealing with the infinite dimensional vectors \( U, S \) and \( V \) given by Eq. (10). Let us start with the operator \( \mathcal{A}^{\text{tot}}_{\sigma} \) used in Eq. (11), which can be written as the sum of two operators \( \mathcal{A}_{\sigma} \) and \( \mathcal{B}^{\pm 2} \) defined as follows:

\[
\mathcal{A}_{\sigma} \equiv \hat{A} I, \quad \mathcal{B}^{\pm 2} \equiv \hat{B} L^{\pm 2}_1,
\]

where \( \hat{A} \) is the following \( 2 \times 2 \) matrix:

\[
\hat{A} = \begin{pmatrix}
\left( \frac{\Gamma_1}{\Gamma} - 1 \right) \nu' c_s^{-2} & -\sigma + m (\Omega_1 + 6\Omega_3) \\
\left[ (\sigma - m (\Omega_1 + 6\Omega_3)) \frac{1}{c_s^2} + 2m (\varpi_1 + 6\varpi_3) \right] e^{2(\lambda - \nu)} & -\frac{2}{r} + \lambda' - 2\nu' + \frac{\nu'}{\varpi_1}
\end{pmatrix},
\]

where \( I \) is the identity matrix and \( L^{\pm 2}_1 \) is an operator defined in Eq. (B5). The action of \( \mathcal{A}_{\sigma} \) on \( u^{\ell m} \) is given by:

\[
\mathcal{A}_{\sigma} u^{\ell m} \longrightarrow \hat{A} u^{\ell m},
\]

whereas by the definition of \( L^{\pm 2}_1 \), the operator \( \mathcal{B}^{\pm 2} \) can transform \( u^{\ell m} \) in three ways:

\[
\mathcal{B}^{\pm 2} u^{\ell m} \longrightarrow \hat{B} \left[ -Q_{\ell-1m} Q_{\ell m} u^{\ell-2m} + (1 - Q_{\ell m}^2 - Q_{\ell+1m}^2) u^{\ell m} - Q_{\ell m} Q_{\ell+1m} u^{\ell+2m} \right],
\]

where the coefficient \( Q_{\ell m} \) is defined in Eq. (B12) and \( \hat{B} \) is the following \( 2 \times 2 \) matrix:

\[
\hat{B} = \begin{pmatrix}
0 & -\frac{13}{2} m (\Omega_3) \\
\frac{15}{2} m (\Omega_3) - 2\varpi_3) e^{2(\lambda - \nu)} & 0
\end{pmatrix}.
\]

Note that the operator \( \mathcal{B}^{\pm 2} \) couple perturbations with different index \( \ell \) while \( \mathcal{A}_{\sigma} \) does not.

The other two operators \( \mathcal{C} \) and \( \mathcal{D} \) of Eq. (11) are defined as follows:

\[
\mathcal{C} \equiv \hat{C}_0 I \oplus \hat{C}_1 L^{\pm 1}_1 \oplus \hat{C}_2 L^{\pm 2}_1 \oplus \hat{C}_3 L^{\pm 3}_1,
\]

\[
\mathcal{D} \equiv \hat{D}_0 I,
\]
where $\mathcal{L}_{1}^{\pm 1}$, $\mathcal{L}_{1}^{\pm 2}$ and $\mathcal{L}_{1}^{\pm 3}$ are given in Sec. [ ] and the matrices $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{D}_0$ have the following expressions:

$$\mathbf{C}_0 = \begin{pmatrix} -m (f_1 + 6 f_3) & 0 \\ \ell (\ell + 1) e^{2 \lambda r^{-2}} & 0 \end{pmatrix}, \quad \mathbf{C}_1 = \begin{pmatrix} 0 & -(f_1 + 6 f_3) \\ 0 & 0 \end{pmatrix}, \quad \tag{A7}$$

$$\mathbf{C}_2 = \begin{pmatrix} \frac{15}{2} m f_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{C}_3 = \begin{pmatrix} 0 & \frac{15}{2} f_3 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \nu' e^{-2} & 0 \\ 0 & 0 \end{pmatrix}. \quad \tag{A8}$$

and

$$f_1 = 2 \left( \frac{1}{r} - \nu' \right) \omega_1 - \omega'_1, \quad f_3 = 2 \left( \frac{1}{r} - \nu' \right) \omega_3 - \omega'_3. \quad \tag{A9}$$

The operation of $\mathcal{L}_{1}^{\pm j}$ for $j = 1, 2, 3$ on $u^{em}$ follows the previous description as for the operator $\mathcal{B}^{\pm 2}$.

In Eq. [ ] the operators $\mathcal{S}_\sigma$ and $\mathcal{M}$ are defined as follows:

$$\mathcal{S}_\sigma = \mathbf{K}_d \mathcal{I} \oplus \mathbf{K}_1 \mathcal{L}_3^{\pm 1} \oplus \mathbf{K}_2 \mathcal{L}_3^{\pm 2} \oplus \mathbf{K}_1 \mathcal{L}_3^{\pm 2} \oplus \mathbf{J}_2 \mathcal{L}_3^{\pm 2} \oplus \mathbf{W} \mathcal{L}_3^{\pm 3}, \quad \tag{A10}$$

$$\mathcal{M} = \mathbf{M}_d \mathcal{I} \oplus \mathbf{M}_1 \mathcal{L}_2^{\pm 1} \oplus \mathbf{M}_2 \mathcal{L}_2^{\pm 2} \oplus \mathbf{M}_3 \mathcal{L}_3^{\pm 3}, \quad \tag{A11}$$

where the quantities with a hat are matrices. Those associated with the operator $\mathcal{S}_\sigma$ have the following form:

$$\mathbf{\hat{S}}_\sigma = \begin{pmatrix} -\sigma + m (\Omega_1 + 6 \Omega_3) - \frac{2m}{\Lambda} (\omega_1 + 6 \omega_3) \end{pmatrix} \mathbf{I}_{2 \times 2}, \quad \tag{A12}$$

$$\mathbf{\hat{K}}_1 = \begin{pmatrix} 2 (\omega_1 + 6 \omega_3) & 0 \\ \ell (\ell + 1) & 1 \end{pmatrix}, \quad \mathbf{\hat{K}}_2 = \begin{pmatrix} \frac{15m^2}{\ell (\ell + 1)} & 0 \\ 2 \omega_3 + \Omega_3 \\ 0 \end{pmatrix}, \quad \tag{A13}$$

$$\mathbf{\hat{J}}_1 = \begin{pmatrix} \frac{15m}{\ell (\ell + 1)} & 0 \\ \Omega_3 - 2 \omega_3 \\ 0 \\ 2 \omega_3 \end{pmatrix}, \quad \mathbf{\hat{J}}_2 = \begin{pmatrix} \frac{15m}{\ell (\ell + 1)} & 0 \\ 2 \omega_3 \\ 0 \\ \Omega_3 - 2 \omega_3 \end{pmatrix}, \quad \tag{A14}$$

$$\mathbf{\hat{J}}_3 = \begin{pmatrix} -m & \Omega_3 - 2 \omega_3 \end{pmatrix} \mathbf{I}_{2 \times 2}, \quad \mathbf{\hat{W}} = \begin{pmatrix} \frac{15}{\ell (\ell + 1)} & 0 \\ 0 \\ 2 \omega_3 \\ 0 \end{pmatrix}, \quad \tag{A15}$$

here $\mathbf{I}_{2 \times 2}$ is the identity matrix of rank 2. While the matrices associated with the operator $\mathcal{M}$ are:

$$\mathbf{\hat{M}}_0 = \frac{1}{\ell (\ell + 1)} \begin{pmatrix} \ell (\ell + 1) & 0 \\ 0 & m^2 e^{-2\lambda} (g_1 + 6g_3) \end{pmatrix}, \quad \mathbf{\hat{M}}_1 = \frac{1}{\ell (\ell + 1)} \begin{pmatrix} 0 & 0 \\ 0 & r^2 e^{-2\lambda} (g_1 + 6g_3) \end{pmatrix}, \quad \tag{A16}$$

$$\mathbf{\hat{M}}_2 = \frac{m}{\ell (\ell + 1)} \begin{pmatrix} 0 & -\frac{15}{2} r^2 e^{-2\lambda} g_3 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{\hat{M}}_3 = \frac{1}{\ell (\ell + 1)} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{15}{2} r^2 e^{-2\lambda} g_3 \end{pmatrix}, \quad \tag{A17}$$

where

$$g_1 = 2 \left( \frac{1}{r} - \nu' \right) \omega_1 + \omega'_1, \quad g_3 = 2 \left( \frac{1}{r} - \nu' \right) \omega_3 + \omega'_3. \quad \tag{A18}$$

Finally, we consider the operators $\mathcal{Q}^{\text{tot}}_\sigma$ and $\mathcal{N}$ of Eq. [ ], which exists only in the non-barotropic case:

$$\mathcal{Q}^{\text{tot}}_\sigma = \mathbf{\hat{Q}}_\sigma \mathcal{I} \oplus \mathbf{\hat{Q}}_1 \mathcal{L}_1^{\pm 2}, \quad \tag{A19}$$

$$\mathcal{N} = \mathbf{\hat{N}}_0 \mathcal{I}, \quad \tag{A20}$$

where the matrices $\mathbf{\hat{Q}}_\sigma$, $\mathbf{\hat{Q}}_1$ and $\mathbf{\hat{N}}_0$ have the form:

$$\mathbf{\hat{Q}}_\sigma = \begin{pmatrix} -\sigma + m (\Omega_1 + 6 \Omega_3) & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{\hat{Q}}_1 = \begin{pmatrix} -\frac{15}{2} m \Omega_3 \\ 0 \end{pmatrix}, \quad \mathbf{\hat{N}}_0 = \begin{pmatrix} 0 \\ 0 \\ e^{2\nu - 2\lambda} (1 - \frac{\Gamma_1}{\Gamma}) \nu' \end{pmatrix}. \quad \tag{A21}$$

However, the definition of the vector $v^{em}$ in [ ] and the values of the matrix coefficients in [ ] lead to simpler equations:

$$\begin{pmatrix} -\sigma + m (\Omega_1 + 6 \Omega_3) - \frac{15}{2} m \Omega_3 \mathcal{L}_1^{\pm 2} \end{pmatrix} \xi^{em} = e^{2\nu - 2\lambda} \left( 1 - \frac{\Gamma_1}{\Gamma} \right) \nu' u_1^{em}. \quad \tag{A22}$$
APPENDIX B: ANGULAR OPERATORS

In this Appendix, we present the definition of the linear angular operators $L_i^{\pm j}$ with $i, j \in \mathbb{N}$, which are used in the angular integration of the perturbation equations derived in [19] and couple the various perturbation functions. For convenience we don’t write down the detailed expression of each operator, but rather the final part that shows the couplings that each one of them introduces on a generic harmonic component perturbation $A_{\ell m}$.

The operators that introduce couplings of the form $\ell \pm 1$ are:

$$
L_1^{\pm 1} A_{\ell m} = (\ell - 1) Q_{\ell m} A_{\ell - 1 m} - (\ell + 2) Q_{\ell + 1 m} A_{\ell + 1 m},
$$

$$
L_2^{\pm 1} A_{\ell m} = - (\ell + 1) Q_{\ell m} A_{\ell - 1 m} + \ell Q_{\ell + 1 m} A_{\ell + 1 m},
$$

$$
L_3^{\pm 1} A_{\ell m} = (\ell - 1) (\ell + 1) Q_{\ell m} A_{\ell - 1 m} + \ell (\ell + 2) Q_{\ell + 1 m} A_{\ell + 1 m},
$$

$$
L_4^{\pm 1} A_{\ell m} = Q_{\ell m} A_{\ell - 1 m} + Q_{\ell + 1 m} A_{\ell + 1 m},
$$

while the following operators introduce couplings of the form $\ell \pm 2$,

$$
L_1^{\pm 2} A_{\ell m} = - Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 2 m} + (1 - Q_{\ell m}^2 - Q_{\ell + 1 m}^2) A_{\ell m} - Q_{\ell m} Q_{\ell + 1 m} A_{\ell + 2 m},
$$

$$
L_2^{\pm 2} A_{\ell m} = - (\ell + 1) Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 2 m} + \left[\ell Q_{\ell + 1 m}^2 - (\ell + 1) Q_{\ell m}^2\right] A_{\ell m} + \ell Q_{\ell + 1 m} Q_{\ell + 2 m} A_{\ell + 2 m},
$$

$$
L_3^{\pm 2} A_{\ell m} = (\ell - 2) Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 2 m} + \left[\ell Q_{\ell + 1 m}^2 - (\ell + 1) Q_{\ell m}^2\right] A_{\ell m} - (\ell + 3) Q_{\ell + 1 m} Q_{\ell + 2 m} A_{\ell + 2 m},
$$

$$
L_4^{\pm 2} A_{\ell m} = - (\ell - 2) (\ell + 1) Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 2 m} - \ell (\ell + 3) Q_{\ell + 1 m} Q_{\ell + 2 m} A_{\ell + 2 m} + \left[\ell (\ell + 1) - (\ell + 1) (\ell - 2) Q_{\ell m}^2 - \ell (\ell + 3) Q_{\ell + 1 m}^2\right] A_{\ell m}.
$$

Finally, the following operators introduce couplings of the form $\ell \pm 3$,

$$
L_1^{\pm 3} A_{\ell m} = - (\ell - 3) Q_{\ell - 2 m} Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 3 m} + (\ell + 4) Q_{\ell + 1 m} Q_{\ell + 2 m} Q_{\ell + 3 m} A_{\ell + 3 m} + Q_{\ell - 1 m} \left[\ell Q_{\ell - 1 m}^2 + (\ell - 1) (1 - Q_{\ell m}^2 - Q_{\ell + 1 m}^2)\right] A_{\ell - 1 m} - Q_{\ell + 1 m} \left[\ell (\ell + 1) Q_{\ell + 1 m}^2 + (\ell + 2) (1 - Q_{\ell m}^2 - Q_{\ell + 1 m}^2)\right] A_{\ell + 1 m},
$$

$$
L_2^{\pm 3} A_{\ell m} = - (\ell - 3) (\ell + 1) Q_{\ell - 2 m} Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 3 m} - (\ell + 1) Q_{\ell + 1 m} Q_{\ell + 2 m} Q_{\ell + 3 m} A_{\ell + 3 m} + Q_{\ell - 1 m} \left[\ell (\ell + 1) Q_{\ell - 1 m}^2 + (\ell - 1) (\ell + 1) Q_{\ell m}^2 + (\ell (\ell - 1) Q_{\ell + 1 m}^2\right] A_{\ell - 1 m} - Q_{\ell + 1 m} \left[\ell (\ell + 1) Q_{\ell + 1 m}^2 - (\ell + 2) Q_{\ell m}^2 - (\ell + 1) Q_{\ell + 1 m}^2\right] A_{\ell + 1 m},
$$

$$
L_3^{\pm 3} A_{\ell m} = (\ell + 1) Q_{\ell - 2 m} Q_{\ell - 1 m} Q_{\ell m} A_{\ell - 3 m} - (\ell + 1) Q_{\ell + 1 m} Q_{\ell + 2 m} Q_{\ell + 3 m} A_{\ell + 3 m} - Q_{\ell - 1 m} \left[\ell (\ell + 1) Q_{\ell - 1 m}^2 + (\ell + 1) (Q_{\ell m}^2 - Q_{\ell + 1 m}^2)\right] A_{\ell - 1 m} - Q_{\ell + 1 m} \left[\ell + (\ell + 1) Q_{\ell m}^2 - (Q_{\ell + 1 m}^2 + Q_{\ell + 2 m}^2)\right] A_{\ell + 1 m},
$$

where $Q_{\ell m}$ is defined as

$$
Q_{\ell m} = \sqrt{\frac{(\ell - m)(\ell + m)}{(2\ell - 1)(2\ell + 1)}}.
$$

A useful relation between operators is the following:

$$
L_4^{\pm 1} = -\frac{1}{2} (L_1^{\pm 1} + L_2^{\pm 1}).
$$
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