Fast is hot: energetics of information erasure and the overhead to Landauer’s bound.

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Information processing in the physical world comes with an energetic cost: erasing a 1-bit memory at temperature \( T \) requires at least \( k_B T_0 \ln 2 \) of work, as demonstrated theoretically\(^1\) and experimentally\(^2\)-\(^10\), with \( k_B \) Boltzmann’s constant. Practical implementations require an overhead to Landauer’s bound, observed to scale as \( k_B T_0 \times B/\tau \), with \( \tau \) the protocol duration and \( B \) close to the system relaxation time\(^1\). Most experiments use overdamped systems, for which minimizing the overhead means minimizing the dissipation. Therefore, underdamped systems seem a good way to reduce this energetic cost. Here we show experimentally and theoretically that, for such systems, fast erasures induce a heating of the memory: the work influx is not instantaneously compensated by the inefficient heat transfer to the thermostat. This temperature rise results in a kinetic energy contribution superseding the viscous dissipation term. Our model covering all damping regimes paves the way to new optimization strategies, based on the thorough understanding of the energy exchanges.

The system under scrutiny is illustrated in Fig. 1: an underdamped micro-mechanical oscillator confined in a double-well potential \( U_1(x, x_1) = \frac{1}{2} k (|x| - x_1)^2 \), with \( k \) its stiffness and \( x_1 \) the user-controlled parameter tuning the barrier height\(^10\). The 1-bit information is encoded in the mean position of the oscillator: using a large barrier \((x_1 = X_1 \gg \sigma_0 = \sqrt{k_B T_0/k})\), the system can be at equilibrium either in the state 0 (in the left-hand well centered in \(-X_1\)) or in the state 1 (in the right-hand well centered in \(+X_1\)). The erasure process consists in lowering the barrier and merging the wells (stage 1: decreasing \( x_1 \) from \( X_1 \) to 0 in a time \( \tau \)), then translating the single well \( U_2(x, x_1) = \frac{1}{2} k (x + x_1)^2 \) to position \(-X_1\) (stage 2: increasing \( x_1 \) from 0 to \( X_1 \) in a time \( \tau \)), before recreating the second well centered in \(+X_1\) to recover the initial potential \( U_1 \). \(^10\) The system is allowed to equilibrate after each ramp in \( x_1 \). The experimental probability distribution function (PDF) evolution in grey on Fig. 1c points out the 100% success rate of the operation on the \( N = 2000 \) erasures performed: this protocol drives the system in state 0 independently of its initial state.

Along a trajectory, the total energy of the system consists in the sum of the kinetic energy \( K = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m v^2 \) (with \( m \) the oscillator mass, and dotted quantities their time derivative) and of the potential energy \( U: E = K + U \). This quantity equilibrates with the stochastic work \( \mathcal{W} \) and heat \( \mathcal{Q} \) through the energy balance equation:

\[
\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = \frac{d\mathcal{W}}{dt} - \frac{d\mathcal{Q}}{dt}. \tag{1}
\]

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The data plotted in Fig. 1c (x and z along a trajectory) contains all we need to compute the quantities involved in Eq. (1). Indeed, applying to the underdamped regime the generic computations of stochastic energy exchanges\textsuperscript{10,12-17}, we have:

\[
dV = \frac{\partial U}{\partial x_1} \ddot{x}_1, \quad \frac{dQ}{dt} = -\frac{\partial U}{\partial x} \ddot{x} - \frac{dK}{dt}.
\] (2)

In Ref. 10 we measure the mean work and dissipated heat of erasure processes at different ramp speeds. The duration range under study starts from the quasi-static regime with $\tau_{f_0} = 250$, to very fast erasures reaching $\tau_{f_0} \approx 6$ such as the one presented in Fig. 1c. As stated in the introduction, reducing the operation time requires an overhead to Landauer’s bound (LB): the mean work and heat on an erasure cycle are, to a first approximation, $(\mathcal{W}) \sim k_B T_0 (\ln 2 + B/\tau)$. In this Letter, we propose a thorough study of all energy exchanges during the procedure, eventually allowing for the first time to predict quantitatively the approach to LB.

For this purpose, we measure the mean kinetic and potential energy during either a quasi-static erasure (Fig. 2a-b) and a fast one (Fig. 2c-d). When we proceed in a quasi-static fashion, the mean kinetic energy is expected to stay at its equilibrium value $\frac{1}{2} k_B T_0$. One might be surprised by the equilibrium evolution of the potential energy during step 1, which nevertheless complies with equipartition for the bi-quadratic shape of $U_1$, as detailed below. On the other hand, for fast operations, the energy profiles are completely different: in particular the kinetic energy strongly increases during stage 1, before relaxing during the equilibration step. $K$ can be decomposed into $\langle K \rangle = \frac{1}{2} m \langle v^2 \rangle + \sigma_v^2$, summing the contribution of the velocity mean value $\langle v \rangle$ that reflects the response to the well motion, and the velocity variance $\sigma_v^2$ which defines the kinetic temperature of the system $T = m \sigma_v^2 / k_B$. The first term is responsible for the transient oscillations at the beginning of step 1 and during step 2, but the energy rise during step 1 comes from the thermic term, and therefore demonstrates a temperature rise.

This warming and its consequences on the operation cost can be interpreted using a simple analogy: during step 1, the system behaves similarly to a single-particle gas\textsuperscript{18} at pressure $p$, compressed so that the available volume $V$ is divided by 2. The infinitesimal work required for the compression is $dW^C = -pdV = -k_B T d\ln V$. If the transformation is quasi-static, the temperature remains constant at $T = T_0$ and the work simplifies into $W^C = k_B T_0 \ln 2$. On the contrary, if the process is too fast to allow heat exchanges with the surrounding thermostat, the transformation is adiabatic, and the temperature $T$ of the gas increases during the compression. Hence, the compression work for fast operations writes $W^C = k_B T_0 \int_0^{T_0} T dT \ln V = k_B T_{dB} \ln 2$ with $T_{dB} \geq T_0$. The heat exchanges after the adiabatic compression will then allow the system to thermalize at $T_0$.

By analogy, we will also call "compression" the reduction of the phase space volume when the bi-stable potential progressively shrinks until a single well during step 1. This analogy highlights the fact that the warming during the compression is specific to the underdamped system, and would not exist if a strong coupling to the bath allowed efficient heat exchanges. The objective of the following sections is to build a model which describes both the compression and translational motion as observed in experiment.

We first proceed with the mean dissipated heat. Starting from the underdamped Langevin equation and using the Ito discretization, we show that (see methods):

\[
\frac{d\langle Q \rangle}{dt} = \frac{\omega_0}{Q} m \langle v^2 \rangle^2 + k_B (T - T_0).
\] (4)

This expression is completely general and doesn’t depend on the energy potential shape or actual transformations occurring in the system. It also highlights that the heat exchanges with the thermal bath are reduced at high quality factors $Q$.

To compute the other energetic terms ($\langle W \rangle$, $\langle K \rangle$ and $\langle U \rangle$), we rely on the PDF of position $x$ and speed $v$. Let us introduce this PDF during the compression stage, supposing that the system is at equilibrium: it is governed
by the Boltzmann distribution
\[ P^c(x, v) = \frac{1}{Z} e^{-\frac{1}{2} \beta m v^2} e^{-\frac{1}{2} \beta k (|x| - x_1)^2} \]  
(5a)
\[ Z^c(\beta, x_1) = \frac{2\pi}{\sqrt{\kappa m \beta}} V, \quad V = 1 + \text{erf} \left( \sqrt{\frac{k\beta}{2x_1}} \right). \]  
(5b)

with \( \beta = 1/(k_B T) \), \( Z^c \) the partition function, and \( V \) a volume-like function that shrinks by a factor 2 when \( x_1 \) decreases from \( X_1 \) to 0. We can directly apply this PDF to the slow erasures, which are quasi-static and thus in equilibrium at temperature \( T_0 \) at all time. We extend the use of this PDF to the case of fast erasures as well, under the hypothesis that the cantilever oscillates several times in the double-well before its shape changes significantly (\( |x_1| \ll \omega_0 \sigma_0 \)), so that the phase space is adequately sampled and a Boltzmann-like distribution still holds. In this case, however, we let the temperature \( T \) as a parameter free to evolve due to a possible heating. Note that the PDF \( P^c(x, v) \) only describes the volume compression and does not include any transients. The main transient, due to the translational motion of the wells, is addressed in the next paragraph.

During stage 2, or at the beginning of stage 1 before the oscillator crosses the barrier, the dynamics is ruled by a linear Langevin equation: the potential energy is quadratic (no switching). \( x(t) \) is therefore the sum of the stochastic response to the thermal fluctuations, and of the deterministic response \( \pm x_D(t) \) to the driving force \( F_D(t) = \pm k x_1(t) \) (the sign depending of which well is considered). \( x_D(t) \) can be easily computed for our simple \( x(t) \) ramps, and the PDF \( P^t(x, v) \) which determines the translational motion is then described by:

\[ P^t(x, v) = \frac{1}{Z} e^{-\frac{1}{2} \beta m (v - \dot{x}_D)^2} e^{-\frac{1}{2} \beta k (x - x_D)^2} \]  
(6a)
\[ Z^t = \frac{2\pi}{\sqrt{\kappa m \beta}} V, \quad V = 1. \]  
(6b)

We easily retrieve \( \langle x \rangle = x_D \) and \( \langle v \rangle = \dot{x}_D \).

In complement to Eq. (4) for the mean heat, the knowledge of the PDF allows the computation of all mean energetic quantities. During compression for example, the mean energy is \( \langle E^c \rangle = -\partial \ln Z^c / \partial \beta \), while the mean work derivative is \( \langle W^c \rangle = \langle \partial U / \partial x_1 \rangle \dot{x}_1 = -\dot{x}_1 / \beta \partial \ln Z^c / \partial x_1 \). In Methods, we derive the following expressions, valid for all stages:

\[ \frac{d\langle Q \rangle}{dt} = \frac{\omega_0}{Q} (2K_D + k_B T - k_B T_0) \]  
(7a)
\[ \frac{d\langle W \rangle}{dt} = \frac{dW_D}{dt} - k_B T \frac{\partial \ln V}{\partial x_1} \dot{x}_1 \]  
(7b)
\[ \langle K \rangle = K_D + \frac{1}{2} k_B T \]  
(7c)
\[ \langle U \rangle = U_D + \frac{1}{2} k_B T + k_B T^2 \frac{\partial \ln V}{\partial T} \]  
(7d)

where \( W_D, K_D \) and \( U_D \) are respectively the deterministic work, kinetic and potential energy (see Methods) which vanish in the quasi-static regime. With Eq. (7b) for a quasi-static compression in equilibrium at \( T_0 \), we recover the gas analogy \( dW = -k_B T_0 d\ln V \), hence LB: \( \langle W^c \rangle = k_B T_0 \ln 2 \).

Using Eqs. (1) and (7), we derive a differential equation governing the time evolution of the temperature: the deterministic terms cancels out, since they comply to the energy balance as well, and we’re left with

\[ \frac{d\langle E \rangle}{dt} = \frac{\partial \langle E \rangle}{\partial T} T + \frac{\partial \langle E \rangle}{\partial x_1} \dot{x}_1 \]  
\[ = -k_B T \frac{\partial \ln V}{\partial x_1} \dot{x}_1 + \frac{k_B \omega_0}{Q} (T - T_0). \]  
(8)

Explicit formulas for \( \partial \langle E \rangle / \partial T \) and \( \partial \langle E \rangle / \partial x_1 \) are readily computed from Eqs. (7c-7d). When we proceed in quasistatic fashion \( (\dot{x}_1 \sim 0) \), or when the volume is constant \( (\partial / \partial x_1 = 0) \), we observe no heating: \( T = T_0 \). For fast compressions, this equation can be solved numerically and leads to the evolution of the kinetic temperature \( T(t) \).

Thanks to the knowledge of \( T(t) \), our model describes the time evolution of all energetic quantities in Eqs. (7), at all times of the erasure process. We plot those quantities in Fig. 3. For slow erasures, the mean kinetic energy is almost constant at \( k_B T_0/2 \), as expected from equipartition. The mean potential energy displays a surprising drop, going below \( k_B T_0/2 \) during compression. Since \( U_1 \) is not quadratic in \( x \), this shape remains however in agreement with equipartition. For fast erasures, we obtain a high rise in temperature during step 1, visible both on the kinetic (a) and potential (b) energy profile, and also visible by the reduction of the potential energy dip present for the quasi-static process. The system then thermalizes, before responding to the translational motion of step 2 with transient oscillations. Those theoretical results superposed on Fig. 2 in red lines are in very good agreement with the experimental observations for both slow and fast erasures, with no adjustable parameters. The very small difference between the model prediction and the experimental results may be explained by the calibration drift during the experiment (in terms of initial distance \( X_1 \)), by the limited number of trajectories averaged \( (N = 2000) \), by the experimental imperfection when building the double-well potential\(^{20} \), or by the shortcomings of the model. For those reasons, we supplemented the model validation by numerical simulation data (see Methods for the simulation implementation). The \( N_{\text{sim}} = 5 \times 10^6 \) simulated trajectories for step 1 give the black curve on Fig. 2c-d that closely matches the model, except for tiny ripples that correspond to transients unaccounted for by the model during the thermalization. Additionally, the model predicts that a fast erasure cycle will cause a mean power evolution that displays transient oscillations and a rise during compression, both of which are consistent with the experimental data of Fig. 3c, and perfectly matches the simulation results.

All in all, we propose an efficient theoretical framework to predict the energy exchanges and explore the
fast information erasure cost. The model only requires the system parameters \((f_0\) and \(Q\)) and the protocol ones \((X_1\) and \(\tau\)) to provide the erasure cost of a 1-bit information stored in an under-damped memory. As a further illustration of the model reliability, we compare in Fig. 4 the experimental mean work and heat, measured for in-

FIG. 3. Model prediction: energy and stochastic work profiles (a) Time evolution of the mean kinetic energy (in \(k_B T_0\) units) for different duration \(\tau\) of the erasure steps computed from Eq. (7c). For small \(\tau\), step 1 (red background) causes a strong rise in temperature that impacts the kinetic energy \(\langle K \rangle\) combined to an oscillating transient due to the dragging. The same dragging transient appears during step 2 (green background). (b) Same plot for the potential energy \(\langle U \rangle\) from Eq. (7d). The dip due to the bistable shape during quasi-static step 1 disappears when \(\tau\) gets smaller, as the temperature increases. (c) Time evolution of the mean power over 2000 trajectories, following the fast protocol \((\tau = 5\, \text{ms})\) corresponding to Fig. 1 (blue). The red line is computed using Eq. (7b) and closely matches the experimental results. To get rid of the experimental calibration drift and to average over more samples, we also compare to a numerical simulation in black dashed line which corresponds to \(5 \times 10^6\) trajectories. The simulation result matches the model so well that we cannot distinguish the curves.

As a conclusion, this Letter proposes an unprecedented insight on the approach to LB for fast erasures. Indeed, the underdamped framework, never developed in previous studies of LB, opens up new possibilities in information processing: the operation times are several orders of magnitude smaller than the ones encountered in the overdamped regime, as well is the cost required to move the system in the bath. Nevertheless, the price to pay to get rid of the viscous slowdown hides in the low cou-
pling to the bath, allowing the system temperature to strongly rise for fast drivings. That is why we developed a full theoretical description of an erasure cycle, in order to predict the energetic exchanges. The success of the model, whose results were verified by a wide panel of high-resolution experimental measurements and complementary numerical simulations, culminates in the panel of high-resolution experimental measurements and future approaches to the information processing optimisation.

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DATA AVAILABILITY

The data that support the findings of this study are openly available in Zenodo\textsuperscript{13}.

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METHODS

M1. EXPERIMENT

The experimental setup is described in Ref. 10, and we summarise in this paragraph the main characteristics. The underdamped oscillator is a conductive cantilever (Doped silicon cantilever OCTO 1000S from Micromotiv Mikrotechnik, 1 mm long, 90 μm large, 1 μm thick) in air at room temperature. Its deflection is measured with a differential interferometer, which features a high stability and high resolution (~1 pm for a 10 kHz bandwidth). To modify the energy potential seen by the oscillator, we create an electrostatic force by applying a voltage $V$ to a nearby electrode. The value of $V$ is set by a feedback loop on the position $x$ measured by the interferometer. $V = V_0 + S(x - x_0)V_1$, where $S$ is the sign function. The threshold $x_0$ is chosen midway between the two equilibrium positions to create the symmetric double-well, or chosen far above (or below) to force one single well. The amplitude of $V_1$ tunes the distance between the wells, thus $x_1$ in our protocols. The sign function is implemented with a high speed analog comparator (response time below 1 μs). To avoid multiple fast switching of the comparator around the threshold because of measurement noise, we apply a short temporal lockup: once triggered, the comparator cannot switch back to the preceding state during the next quarter period $f_0^{-1}/4 \sim 200 \mu s$. During this lockup period, the oscillator is expected to have reached a position far enough from the threshold before rearming the trigger. This strategy allows the feedback potential experienced by the oscillator to be equivalent to the prescribed double-well 10,20.

M2. NUMERICAL SIMULATION

The experimental results are supplemented by a numerical simulation providing a large number of trajectories ($N_{\text{sim}} = 5 \times 10^6$) without any calibration drift in the initial position $X_1$. The simulation is meant to mimic the experimental system during step 1, and therefore uses the experimental parameters $\omega_0, m, Q, X_1$ and $\tau$. The simulation code consists in integrating the Langevin equation that rules the cantilever position:

$$\ddot{x} + \frac{\omega_0}{Q} \dot{x} + \frac{\omega_0^2}{m} x = \frac{F_{th}}{m} + S(x)\omega_0^2(X_1 - \frac{X_1}{\tau} t), \quad (M1)$$

where $F_{th}$ the stochastic forcing from the bath, satisfying $\langle F_{th}(t)F_{th}(0)\rangle = \delta(t)2k_B T_0\omega_0/Q$. $F_{th}$ is implemented as a random number normally distributed around 0, with a standard deviation $\sqrt{2k_B T_0\omega_0/(Q\Delta t)}$, with $\Delta t$ the simulation time step. We choose the symplectic Euler method, better suited to stochastic differential equation than the Runge-Kutta one 24, to solve numerically Eq. (M1) and output the position and speed of the cantilever at every time step. The initial position and speed are distributed according to the Boltzmann equilibrium distribution, corresponding to Eq. (5) with $\beta = 1/k_B T_0$ and $x_1 = X_1$.

M3. ENERGETIC QUANTITIES COMPUTATION

A. Mean heat

The computation of the mean dissipated heat requires writing the general Langevin equation of an underdamped system in a potential $U$:

$$m\ddot{x} = -\frac{\partial U}{\partial x} - \frac{m\omega_0}{Q} \dot{x} + F_{th} \quad (M2)$$

Multiplying Eq. (M2) by $\dot{x}$ leads to the dissipated heat defined by Eq. (3):

$$\frac{dQ}{dt} = m\ddot{x} \dot{x} - \frac{dK}{dt} + \frac{m\omega_0}{Q} \dot{x}^2 - F_{th} \dot{x} \quad (M3)$$

Some caution is required before taking the mean value of the above expression, because it involves products of stochastic quantities: in that respect, the Ito discretization prescribes for a stochastic function $f(v)$,

$$\frac{df}{dt} = \frac{\partial f}{\partial v} \dot{v} + \frac{1}{2} \frac{\partial^2 f}{\partial v^2} \dot{v}^2 dt \quad (M4)$$

If we combine Eq. (M2) and Eq. (M4) applied to $K = 1/2mv^2$, before taking the mean value and making $dt$ tend to 0, most terms simplify out. Indeed, only remain the terms that involve the thermal noise scaling in $1/\sqrt{\Delta t}$, some of which are cancelled by the Ito prescription: $\langle F_{th}(v)\rangle = \langle F_{th}\dot{x}\rangle = 0$. Finally, we obtain the relation: $d\langle K\rangle/dt = m\langle \ddot{x}\dot{x}\rangle + k_B T_0\omega_0/Q$. Eq. (M3) then simplifies into Eq. (4).

B. Mean work, potential and kinetic energy

Let us split the PDF $P_c(x,v)$ describing compression and its associated partition function $Z_K$ of Eqs. (5) into their kinetic and potential contribution:

$$P_c^K(v) = \frac{e^{-\beta K^c}}{Z_K^c}, \quad Z_K^c = \int_{-\infty}^{+\infty} e^{-\beta K^c} dv = \sqrt{\frac{2\pi}{m\beta}} \quad (M5a)$$

$$P_c^U(v) = \frac{e^{-\beta U^c}}{Z_U^c}, \quad Z_U^c = \int_{-\infty}^{+\infty} e^{-\beta U^c} dx = \sqrt{\frac{2\pi}{k_B V}} \quad (M5b)$$
We now easily compute the mean values of $K$ and $U$:

$$(K^c) = \int_{-\infty}^{+\infty} K^c e^{-\beta K^c} d\nu = -\frac{\partial \ln Z_K^c}{\partial \beta} = \frac{1}{2\beta} \quad (M6a)$$

$$(U^c) = \int_{-\infty}^{+\infty} U^c e^{-\beta U^c} d\nu = -\frac{\partial \ln Z_U^c}{\partial \beta} = \frac{1}{2\beta} - \frac{\partial \ln V}{\partial \beta} \quad (M6b)$$

Because there are no deterministic terms ($\langle x \rangle = 0$ and $\langle v \rangle = 0$) in the compression framework, those equations equivalently arise from Eqs. (7c-7d). The mean work time derivative is computed from Eq. 2:

$$\langle \frac{dW^c}{dt} \rangle = \int_{-\infty}^{+\infty} \frac{\partial U^c}{\partial x_1} e^{-\beta U^c} dx_1 = -\frac{1}{\beta} \frac{\partial \ln Z_U^c}{\partial x_1} \frac{dx_1}{\beta}$$

which is again equivalent to Eq. (7b) in the absence of deterministic work.

The case of translation follows a similar pattern, except that we now need to include deterministic terms, since here $\langle x \rangle = x_D$ and $\langle v \rangle = \dot{x}_D$. We then rewrite the energies as:

$$\langle K^t \rangle = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m (v - \dot{x}_D)^2 + m \dot{x}_D (v - \frac{1}{2} \dot{x}_D))$$

$$= \frac{1}{2} m (v - \dot{x}_D)^2 + \frac{1}{2} m \dot{x}_D^2 \quad (M8)$$

$$\langle U^t \rangle = \frac{1}{2} k (x - x_1)^2$$

$$= \frac{1}{2} k (x - x_D)^2 + \frac{1}{2} k (x_D - x_1)^2$$

$$= \frac{1}{2} k (x - x_D)^2 + \frac{1}{2} k (x_D - x_1)^2 \quad (M9)$$

The mean values of the energies are thus the sum of a deterministic and a stochastic term. The expressions of latter and of the PDF $P^t(x,v)$ in Eq. (6a) are those of an harmonic oscillator in the referential centered in $x_D$, which directly lead to the equipartition. Since $V = 1$ during a translation, we recover the mean values anticipated by Eqs. (7c-7d). The mean work time derivative is again computed from Eq. 2:

$$\langle \frac{dW^t}{dt} \rangle = \langle \frac{\partial U^t}{\partial x_1} \dot{x}_1 = -k (x_D - x_1) \dot{x}_1 \rangle$$

In this case, the mean work is purely deterministic, as expected in Eq. (7b) with $V = 1$. The ansatz for the PDF in the compression or translation stages thus lead to the Eqs. (7) describing all the energetic terms in any situation.

C. Deterministic terms

The trajectory $x(t)$ in a moving well decomposes into the stochastic response to the thermal fluctuations, which vanishes on average, and the response to the driving force ramp which is the solution of the following deterministic equation:

$$\ddot{x}_D + \frac{\omega_0}{Q} \dot{x}_D + \omega_D^2 x_D = \omega_D^2 x_1(t) \quad (M11)$$

with $x_1(t) = X_1 (1-t/\tau)$ decreasing from $X_1$ to 0 during step 1. We solve the above equation of motion, introducing $\Omega = \omega_0 \sqrt{1-1/4Q^2}$, and obtain the deterministic trajectory $\pm x_D(t)$ (the sign depending of which well is considered):

$$x_D(t) = x_1(t) + \frac{X_1}{\tau} \left[ 1 - \frac{1}{Q\omega_0} \left( 1 - e^{\frac{t}{\tau}} \cos \Omega t \right) \right]$$

$$- \frac{1}{2}\omega_0^2 \frac{X_1}{\tau} \sin \Omega t \quad (M12)$$

Those results can be applied for the translational motion during step 2, and at the beginning of step 1. Indeed, as long as the cantilever hasn’t left its initial well, the above description holds during step 1. After the first commutation, the cantilever switches frequently between the symmetric wells so that the deterministic terms can be neglected. We therefore introduce $\Pi(t)$ the probability that the cantilever remains in its initial well until time $t$ (see next section for its expression), in order to modulate the deterministic contribution accordingly. The deterministic work, kinetic and potential energies are then given by:

$$\frac{dW_D}{dt} = -k(x_D - x_1) \dot{x}_1 \times \Pi(t) \quad (M13a)$$

$$K_D(t) = \frac{1}{2} m \dot{x}_D^2 \times \Pi(t) \quad (M13b)$$

$$U_D(t) = \frac{1}{2} k (x_D - x_1)^2 \times \Pi(t) \quad (M13c)$$

During step 2, $x_D(t)$ is still described by Eq. (M12) with $x_1(t) = -X_1 t/\tau$, and the energetic terms correspond at all time to Eqs. (M13) with $\Pi(t) = 1$ as the cantilever remains in the single well allowed.

D. First commutation time probability distribution

To compute $\Pi(t)$ during step 1, we tackle the first commutation time probability distribution of the cantilever. In a good approximation, the Kramer’s theory prescribes the escape rate $\Gamma(t) = \frac{\omega_0}{2\pi} e^{-\Delta U/t\kappa T_0}$, with the potential barrier at time $t$ being worth $\Delta U(t) = k x_1(t)^2/2$. The probability that a trajectory hasn’t commuted at time $t$ derives from it:

$$\Pi(t) = e^{-\int_0^t \Gamma(u) du} \quad (M14)$$

The integral in Eq. (M14) can be analytically expressed:

$$\int_0^t \Gamma(u) du = \frac{\omega_0 \tau}{2\pi \sqrt{2\pi} x_1^2} \left[ \text{erf} \left( \frac{X_1}{\sqrt{2}\sigma_0} \right) - \text{erf} \left( \frac{\tau - t}{\sqrt{2}\sigma_0} \right) \right] \quad (M15)$$

This estimation is consistent with the experimental data.
Fast is hot: energetics of information erasure and the overhead to Landauer’s bound
Salambô Dago and Ludovic Bellon
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SUPPLEMENTARY MATERIALS

FIG. S1. Comparison of the PDFs $P_{\text{model}}(x,v,t)$ from our ansatz and $P_{\text{sim}}(x,v,t)$ from a numerical simulation. This frame is captured from the movie PDF_erasure_process.mov in the supplementary materials, corresponding to $t = 4.5f_0^{-1}$. (a) Mean kinetic energy $\langle K \rangle$ vs time from the simulation and the model. The quasistatic case is shown for comparison. (b) Average absolute value of the position $|x|$ vs time (c) 2D representation of the PDF in $P_{\text{sim}}(x,v,4.5f_0^{-1})$ with the color scale on top. (d) PDFs in position (obtained by integrating over all speeds the 2D PDF) from the simulation and the model. The quasistatic case is shown for comparison. (e) PDFs in speed (obtained by integrating over all positions the 2D PDF). (f) Ratio of the PDFs $P_{\text{sim}}(x,v,4.5f_0^{-1})/P_{\text{model}}(x,v,4.5f_0^{-1})$ in a 2D representation using the color scale on top.

S1. VALIDATION OF THE PDF ANSATZ

Using the PDFs of Eqs. (5) and (6) is based on some approximations: the dragging effect is assumed to vanish after the first commutation of the system. We unavoidably leave aside some transients mixing position and speed during compression. In order to investigate on the validity of this approach we compare the numerical simulation data to the PDF models, using the following ansatz

$$P_{\text{model}}(x,v,t) = \Pi(t) P^t(x,v) + [1 - \Pi(t)] P^c(x,v)$$  \hspace{1cm} (S1)

$$P^c(x,v) = \frac{1}{Z} e^{-\frac{1}{2} \beta m v^2} e^{-\frac{1}{2} \beta k (|x| - x_1)^2}$$  \hspace{1cm} (S2)

$$P^t(x,v) = \frac{1}{Z} e^{-\frac{1}{2} \beta m (v - S(x) \dot{x})^2} e^{-\frac{1}{2} \beta k (|x| - x_D)^2}$$  \hspace{1cm} (S3)

with $S(x)$ the sign of $x$. The PDF $P_{\text{sim}}(x,v,t)$ is computed from $5 \times 10^6$ trajectories, half of them starting from each well, with $X_1 = 5\sigma_0$ and $\tau = 6f_0^{-1}$. We can then study the relevance of our model by comparing the PDFs. This is done in the movie PDF_erasure_process.mov in the supplementary materials, a frame of which is shown as an example in Fig. S1 corresponding to $t = 4.5f_0^{-1}$. It demonstrates how good is the model to estimate the position and velocity distribution: the oscillations due to the dragging force are replaced by the temperature rise predicted by the compression model. The relaxation in temperature after step 1 predicted by the model also matches the simulation data, except for the transient relaxation oscillations which are not included in the model after the 1st commutation of
the system during step 1. The comparison of the PDFs by their ratio in panel (f) also demonstrates that for statistically relevant portion of the phase space, the agreement between the two is better than 20%. The main deviations occur in the middle of step 1 in the bottom left and top right corner areas. These areas corresponds to trajectories where the system has switched once and presents a mean velocity component from motion of the well it has switched to. The deviation is therefore explained by the fact that the model PDF doesn’t includes the mean driving velocity after the first commutation. But because the error made is symmetrical with the initial state, it doesn’t impact the computation of average values such as the velocity variance.

S2. \( T_{\text{eff}} \) APPROXIMATION

To retrieve the gas analogy, we apply Eq. (7b) to the step 1 compression (no deterministic work), and reframe it to identify the volume total derivative:

\[
\langle \frac{dW_{c}^{1}}{dt} \rangle = -k_{B}T \frac{\partial \ln V}{\partial x_{1}} \dot{x}_{1}
\]

\[
= -k_{B}T \frac{d\ln V}{dt} \left(1 + \frac{d\ln T}{d\ln(x_{1}^{2}/T)}\right)
\]

(S4) (S5)

The second term can be evaluated from our model once the time evolution of the temperature has been numerically computed. After integration, it represents at most 10% of the final result (upper limit reached for the highest temperature rise in the adiabatic limit). The work required for a fast compression can therefore be approximated by \( \langle W_{c}^{1} \rangle \sim k_{B}T_{\text{eff}} \ln 2 \), and meet the gas analogy with the effective temperature being worth

\[
T_{\text{eff}} = \frac{1}{\ln 2} \int T d\ln V.
\]

(S6)

For the erasure cycle in Fig. 1, we derive \( T_{\text{eff}} = 1.35T_{0} \) which gives the compression work with a 6% error.

S3. ADIABATIC LIMIT OF THE COMPRESSION WORK

For large quality factors, heat exchanges with the bath are cancelled: \( d\langle Q \rangle = 0 \). Such compressions, called adiabatic compressions, correspond to the highest temperature rise because the kinetic temperature of the system cannot dissipate in the bath. Let us remind that the entropy variation during an adiabatic compression (assumed reversible) vanishes: \( dS = d\langle Q \rangle/T = 0 \). Consequently, we have by definition of the entropy:

\[
\Delta(k_{B} \ln Z^{c} + \frac{\langle E^{c} \rangle}{T}) = 0
\]

As for \( X_{1} \gg 1 \) the system starts and ends in the same quadratic potential, the energy in the initial and final states satisfies \( \langle E^{c}_{i}\rangle/T_{0} = \langle E^{c}_{f}\rangle/T_{f} = k_{B} \) (derived from equipartition, or equivalently from \( \langle E^{c}\rangle = \partial \ln Z^{c}/\partial \beta \)).

Therefore, only remains in Eq. (S7) the variation of the compression partition function written in Eq. (5b):

\[
\Delta(k_{B} \ln Z^{c}) = 0 \rightarrow \Delta(TV) = 0
\]

(S8)

As the volume is divided by two, the temperatures doubles during the adiabatic compression: \( T_{f} = 2T_{0} \). The corresponding work is given by the first law of thermodynamics (with \( \langle Q \rangle = 0 \)):

\[
\langle W_{c}^{c} \rangle = \Delta \langle E^{c} \rangle = k_{B}T_{f} - k_{B}T_{0} = k_{B}T_{0}
\]

(S9) (S10) (S11)

As a conclusion, an adiabatic compression results in doubling the system temperature and requires on average \( k_{B}T_{0} \) of work.