A note on edge colorings and trees

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We point out some connections between existence of homogenous sets for certain edge colorings and existence of branches in certain trees. As a consequence, we get that any locally additive coloring (a notion introduced in the paper) of a cardinal \( \kappa \) has a homogeneous set of size \( \kappa \) provided that the number of colors \( \mu \) satisfies \( \mu^+ < \kappa \). Another result is that an uncountable cardinal \( \kappa \) is weakly compact if and only if \( \kappa \) is regular, has the tree property, and for each \( \lambda, \mu < \kappa \) there exists \( \kappa^* < \kappa \) such that every tree of height \( \mu \) with \( \lambda \) nodes has less than \( \kappa^* \) branches.

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1 Introduction

In [3] Shelah studied additive colorings as a tool in the study of monadic theories. Here, we focus on combinatorics only. On the one hand, we study Ramsey theorems and the connection between them and the existence of branches in some trees. On the other hand, we find a new characterization of weakly compact cardinals.

In § 2 we introduce the basic definitions and notations about the colorings we deal with in this paper. In § 3 we point out a connection between additive colorings and locally-additive colorings. In § 4 we assign to each coloring \( C \) a tree \( T(C) \) and to each tree \( T \) with an enumeration \( e \), a locally additive coloring \( C(T, e) \). The function \( C \mapsto T(C) \) is not injective on the class of locally additive colorings. On the other hand, we prove that if \( e \) covers \( T \) in some sense, then the tree \( T(C(T, e)) \) is isomorphic to the history tree of \( T \). In § 5, we prove the following lemma: for any uncountable regular cardinal \( \kappa \) and \( \mu < \kappa \), the existence of a homogeneous set of size \( \kappa \) for every locally additive coloring from \( [\kappa]^2 \) to \( \mu \) is equivalent to the property that every tree of height \( \kappa \) whose levels have size \( \leq \mu \) admits a \( \kappa \)-sized branch. As a consequence, we get a characterization of the successor cardinals satisfying the tree property. In § 6 we find an equivalent definition for weakly compact cardinals using the tree power. In § 7 we prove theorems about existence of large homogeneous sets. On the one hand, we slightly improve a theorem of Shelah about *-locally-additive colorings. On the other hand, we derive a theorem on locally-additive colorings from the lemma in § 5.

2 Basic definitions

We commence by defining several classes of colorings: additive colorings, locally-additive colorings and *-locally-additive colorings. We generalize the well-known Erdős and Rado arrow notation.

Definition 2.1 A coloring (or edge coloring) of a set \( A \) with \( \mu \) colors is a function \( C : [A]^2 \to D \) for some set \( D \) of cardinality \( \mu \). A coloring of a set \( A \) is a coloring of a set \( A \) with \( \mu \) colors for some \( \mu \). A coloring is a coloring of some set \( A \). An element in \( D \) is called a color.

If \( A \) is a linearly ordered set and \( C \) is a coloring of \( A \), then we write \( C(a, b) \) instead of \( C([a, b]) \) when \( a < b \) are in \( A \).

Definition 2.2 Let \( C \) be a coloring of a set \( A \). A subset \( B \) of \( A \) is a called homogeneous for \( C \) if \( C([B]^2) \) is constant.

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Definition 2.3 ([3, § 1]) Let \((A, <)\) be a linear ordering and let \(D\) be a set. A function \(C : [A]^2 \to D\) is said to be an additive coloring if there is a function \(s : D \times D \to D\) such that if \(a, b, c\) are in \(A\) and \(a < b < c\) then
\[
C(a, c) = s(C(a,b), C(b,c)).
\]

Definition 2.4 Let \((A, <)\) be a linear ordering and let \(D\) be a set. A function \(C : [A]^2 \to D\) is said to be a locally-additive coloring if for all \(a < b < c < d\) we have
\[
C(b, c) = C(b, d) \implies C(a, c) = C(a, d).
\]

Notation 2.5 Let \((A, <)\) be a linear ordering, \(D\) be a set, and \(C : [A]^2 \to D\) be a locally additive coloring. For \(b\) in \(A\), let \(D_b\) be the set of colors \(t \in D\) such that \(C(b, c) = t\) for some \(c > b\). For every \(a < b\) in \(A\) we define the following function:
\[
\begin{align*}
ua_{a,b} : & D_b \to D, \\
ua_{a,b}(C(b, c)) &= C(a, c).
\end{align*}
\]

Note that \(ua_{a,b}\) is well-defined for each \(a < b\) if and only if the coloring \(C\) is locally-additive.

Definition 2.6 Let \(\kappa\) be a cardinal. A coloring \(C\) of \(\kappa\) is said to be a locally-additive coloring if it is locally-additive with respect to the \(\varepsilon\)-order of \(\kappa\). \(C\) is a \(*\)-locally additive coloring if \(C\) is locally-additive with respect to the reverse order of \(\kappa\).

Remark 2.7 Every additive coloring is locally-additive.

Claim 2.8 For every locally additive coloring \(C\) on a set \(A\), for all \(a < b < c\) in \(A\), we have \(ua_{a,b} \circ ub_{b,c} = ua_{a,c}\).

Proof. Assume \(C : [A]^2 \to D\) and let \(a < b < c\). Each side of the equality is a function from
\[
D_c = \{ t \in D : t = C(c, d) \text{ for some } d > c \}
\]
into \(D\). Let \(t \in D_c\), namely, \(C(c, d) = t\) for some \(d > c\). On the one hand,
\[
ua_{a,c}(t) = ua_{a,c}(C(c, d)) = C(a, d).
\]
But on the other hand,
\[
(ua_{a,b} \circ ub_{b,c})(t) = ua_{a,b}(ub_{b,c}(C(c, d))) = ua_{a,b}(C(b, d)) = C(a, d).
\]
□

Notation 2.9 Let \(\mu\) be a cardinal. We introduce the following notations:
1. \(add\) = the class of additive colorings.
2. \(add_\mu\) = the class of additive colorings with \(\mu\) colors.
3. \(add_{\prec\mu}\) = the class of additive colorings with less than \(\mu\) colors.
4. \(la\) = the class of locally-additive colorings.
5. \(*\text{-}la\) = the class of \(*\)-locally additive colorings on a regular cardinal.
6. Similarly, \(la_\mu, la_{\prec\mu}, *\text{-}la_\mu\) and \(*\text{-}la_{\prec\mu}\).

We generalize the well-known Erdős and Rado arrow notation:

Definition 2.10 Let \(F_1, F_2\) be families of subsets of a fixed linearly ordered set. Let \(\mathcal{C}\) be a class of colorings. \(F_1 \rightarrow (F_2)_\kappa\) is the following principle: For every set \(D \in F_1\) and every coloring \(C \in \mathcal{C}\) of \(D\), there is a subset \(E\) of \(D\) such that \(E \in F_2\) and \(C|[E]\) is constant. We put \(\kappa\) for \(F_1\) when \(F_1\) is the family of subsets of \(\kappa\) of cardinality \(\kappa\) and similarly for \(F_2\). We put \(\mu\) for \(\mathcal{C}\) when \(\mathcal{C}\) is the class of all colorings with \(\mu\) colors.

Special cases of \(\mathcal{C}\) may be \(add, la, *\text{-}la, add_{\prec\kappa}, la_\kappa\), and more.
3 Replacing additivity by local additivity

In this section we prove that every Ramsey theorem related to additive colorings can be translated to the context of locally additive colorings at one price: the number of colors is decreased (in the infinite case, it is decreased to its logarithm). As an application, we get a direct generalization of a theorem of Shelah on additive colorings of dense linear orderings.

Notation 3.1 For a cardinal $\mu$ we define:

$$c(\mu) = \begin{cases} 2\mu & \mu \text{ is infinite}, \\ (1 + \mu)^\mu & \mu \text{ is finite}. \end{cases}$$

Theorem 3.2 Let $\kappa, \mu$ be cardinals such that $c(\mu) \leq \kappa$ and let $F$ be a family of subsets of a linearly ordered set. Assume that $F \rightarrow (F)_{add,\kappa}$. Then $F \rightarrow (F)_{la,\mu}$. (Recall that $la,\mu$ is the class of locally additive colorings with $\mu$ colors.)

Proof. Let $A \in F$ and $C : [A]^2 \rightarrow \mu$ be a locally additive coloring. For each $a < b$ in $A$ let $u_{a,b}$ be the function that is defined in Notation 2.5. Define

$$D^* = \bigcup_{D^- \subseteq \mu} D^-$$

(the set of partial functions from $\mu$ into $\mu$). We define a new coloring

$$C^* : [A]^2 \rightarrow D^*,$$

$$C^*(a, b) = u_{a,b}, \text{ where } a < b.$$

By Claim 2.8, $C^*$ is an additive coloring. Since $|D^*| \leq \kappa$ (for finite $\mu$, $|D^*| = \sum_{i \leq \mu} (i\mu)^i = (1 + \mu)^\mu = c(\mu) \leq \kappa$), we can apply the assumption, $F \rightarrow (F)_{add,\kappa}$ on the additive coloring $C^*$. So there is a subset $B$ of $A$ so that $B \in F$ and $C^*([B]^2)$ is constant. Let $f$ be the function such that $C^*(a, b)$ equals $f$, whenever $a < b$ are in $B$.

Now for all $a < b < c$ in $B$ the following holds:

$$C(a, c) = C^*(a, b)(C(b, c)) = f(C(b, c)).$$

So $C([B]^2)$ is additive. We apply again the assumption, $F \rightarrow (F)_{add,\kappa}$ to get a subset $E$ of $B$ such that $E \in F$ and $C([E]^2)$ is constant. \hfill $\square$

Corollary 3.3 Let $\mu$ be a strong limit cardinal and let $F$ be a family of subsets of a linearly ordered set. Assume that $F \rightarrow (F)_{add,\mu}$. Then $F \rightarrow (F)_{la,\mu}$. 

Corollary 3.4 Let $F$ be a family of subsets of a linearly ordered set. Assume that $F \rightarrow (F)_{add,\kappa,0}$. Then $F \rightarrow (F)_{la,\kappa,0}$. 

In our notation, Shelah proved the following fact:

Fact 3.5 ([3, Theorem 1.3]) Let $I$ be a dense linear ordering and let $F$ be the family of subsets of $I$ that are dense in some interval of $I$. Then $F \rightarrow (F)_{add,\kappa,0}$. 

Theorem 3.2 & Fact 3.5 yield:

Corollary 3.6 Let $I$ be a dense linear ordering and let $F$ be the family of subsets of $I$ that are dense in some interval of $I$. Then $F \rightarrow (F)_{la,\kappa,0}$. 

Proof. By Fact 3.5 & Corollary 3.4. \hfill $\square$

4 Trees and colorings

In this section we study two functions: one assigns to each coloring $C$ a tree $T(C)$, and the other assigns to each tree $T$ with an enumeration $\{t^e_i : i < \mu_e\}$ for each level of $T$, a locally additive coloring $C(T, e)$. The function
Consider the induced fiber map \( \text{inducedfibermap} \). The coloring \( \text{branch} \) is said to be isomorphic to its history tree, so it does not depend on \( \text{branch} \).

Let \( T \) be a tree and \( t \in T \), let \( f_t : ht(t, T) \rightarrow T \) be the function such that \( f_t(\alpha) \) is the unique \( t' <_T t \) in \( \text{Lev}_\alpha(T) \). The history tree of \( T \) is the tree \( FT \) = \((f_t : t \in T), \subseteq\).

For a tree \( T \) and an ordinal \( \alpha \), let \( \text{Lev}_\alpha(T) \) denote the \( \alpha \)th level of \( T \) and let \( \text{Lev}_{<\alpha}(T) \) denote the subtree of \( T \), consisting of the nodes of height smaller than \( \alpha \). We denote the height of \( t \in T \) in \( T \) by \( ht(t, T) \), and \( ht(T) \) is the first \( \alpha \) so that \( \text{Lev}_\alpha(T) = \emptyset \). A tree \( T \) is said to be well-pruned if \( \text{Lev}_0(T) \) has only one element and we have

\[ \forall \alpha \in T \backslash \alpha [ht(x, T) < \alpha < ht(T) \rightarrow \exists \gamma \in \text{Lev}_\alpha(T)(x <_T \gamma)]. \]

An outline of a tree is a maximal element in the tree. Clearly, every well-pruned tree of limit ordinal height has no leaves.

### 4.1 Reconstructing a tree from its history

**Definition 4.1** For a tree \( T \) and \( t \in T \), let \( f_t : ht(t, T) \rightarrow T \) be the function such that \( f_t(\alpha) \) is the unique \( t' <_T t \) in \( \text{Lev}_\alpha(T) \). The history tree of \( T \) is the tree \( FT \) = \((f_t : t \in T), \subseteq\).

**Definition 4.2** For a tree \( T \), let \( T^{\text{succ}} \) be the tree consisting of the set \( \{t \in T : ht(t, T) \text{ is a successor ordinal}\} \) equipped with the suborder induced by the order of \( T \).

Clearly, \( (FT)^{\text{succ}} = \{(f_t : t \in T^{\text{succ}}), \subseteq\} \) (as \( ht(f_t, FT) = ht(t, T) \) for every \( t \in T \)).

**Remark 4.3** Let \( T, S \) be trees. If \( g : FT \rightarrow FS \) is an isomorphism then \( g|T^{\text{succ}} \) is an isomorphism from \( (FT)^{\text{succ}} \) to \( (FS)^{\text{succ}} \).

**Proposition 4.4** Let \( T \) be a tree without leaves. Then \( T \) is isomorphic to the tree \( (FT)^{\text{succ}} \).

**Proof.** Define a function \( h : T \rightarrow (FT)^{\text{succ}} \) as follows: \( h(t) = f_t^{-1}\{\} \). Clearly, \( h(t) \in (FT)^{\text{succ}} \). Indeed, since \( T \) has no leaves, for each \( t \in T \) we can find an immediate successor \( t^* \) of \( t \) in \( T \). By the definition of a tree, \( h(t) = f_t^{-1}\{\} \) for each such \( t^* \). Clearly, \( h \) is an isomorphism.

**Corollary 4.5** Let \( T, S \) be two trees without leaves. Then the trees \( T, S \) are isomorphic if and only if the history trees \( FT, FS \) are isomorphic.

**Proof.** For the non-trivial direction, assume that \( FT \cong FS \). Then by Remark 4.3, \( (FT)^{\text{succ}} \cong (FS)^{\text{succ}} \), and hence by Proposition 4.4, \( T \cong S \).

### 4.2 The tree of a coloring

The following definition assigns to each coloring a tree.

**Definition 4.6** (\([2, \S 1.3]\)) Let \( \kappa \) and \( \mu \) be cardinals and let \( C : [\kappa]^2 \rightarrow \mu \) be a coloring. For every \( \delta < \kappa \), consider the induced fiber map \( C(\cdot, \delta) : \delta \rightarrow \mu \), which satisfies \( C(\cdot, \delta)(\alpha) = C(\alpha, \delta) \) for all \( \alpha < \delta \). Then consider the following downward-closed subtree of \( (^\kappa \mu, \subseteq) \):

\[ T(C) := \{C(\cdot, \delta) | \beta \mid \beta \leq \delta < \kappa\}. \]

**Remark 4.7** For each \( f : \kappa \rightarrow 2 \), define a coloring

\[ C_f : [\kappa]^2 \rightarrow 2, \]

\[ C_f(\alpha, \beta) = f(\alpha). \]

The coloring \( C_f \) is both locally-additive and \( * \)-locally-additive and \( T(C_f) \) is isomorphic to \( (\kappa, \in) \).

**Remark 4.8** For each \( f : \kappa \rightarrow 2 \), define a coloring

\[ C_f : [\kappa]^2 \rightarrow 2, \]

\[ C_f(\alpha, \beta) = f(\beta). \]
If \( f \) is not eventually constant, then \( C_f \) is both locally-additive and \( * \)-locally-additive and we have 
\[
T(C_f) = \{0\} \cup \{1\}.
\]

**Problem 4.9** Find a characterization of the equivalence classes of locally additive colorings induced by the map \( C \mapsto T(C) \).

### 4.3 The locally additive coloring of an enumerated tree

**Definition 4.10** Let \( T \) be a \( \kappa \)-tree. For each \( \alpha < \kappa \), let \( \langle t^i_\alpha : i < \mu_\alpha \rangle \) be a bijective enumeration of \( \text{Lev}_\alpha(T) \). A sequence \( e = \langle \langle t^i_\alpha : i < \mu_\alpha \rangle : \alpha < \kappa \rangle \) of enumerations of the levels of \( T \) is called an enumeration of \( T \).

**Definition 4.11** Let \( T \) be a \( \kappa \)-tree and let \( e \) be an enumeration of \( T \) as given by Definition 4.10. Let \( \mu \) be the supremum of \( \{ \mu_\alpha : \alpha < \kappa \} \). Let \( C(T, e) : [\kappa]^2 \to \mu \) be the coloring defined by \( C(T, e)(\alpha, \delta) = i \) if and only if \( \alpha < \delta \) and \( i < \mu \) is the unique ordinal such that \( t^i_\alpha \leq_T t^0_\delta \).

**Claim 4.12** For each \( \kappa \)-tree \( T \) and an enumeration \( e \) of \( T \), the coloring \( C(T, e) \) is locally additive.

**Proof.** Denote \( C = C(T, e) \). Let \( \alpha < \beta < \gamma_1, \gamma_2 < \kappa \). We have to show 
\[
C(\beta, \gamma_1) = C(\beta, \gamma_2) \implies C(\alpha, \gamma_1) = C(\alpha, \gamma_2).
\]
Let \( i = C(\beta, \gamma_1) = C(\beta, \gamma_2) \). So
\[
\begin{align*}
t^{\beta}_{\gamma_1} &\leq_T t^0_{\gamma_1}, \\
t^{\alpha}_{\gamma_1} &\leq_T t^0_{\gamma_1},
\end{align*}
\]
\[
\begin{align*}
t^{\alpha}_{\gamma_1} &\leq_T t^0_{\gamma_1},
\end{align*}
\]
Since \( \alpha < \beta \), we have 
\[
t^{\alpha}_{\gamma_1} \leq t^{i}_{\beta}.
\]
Since \( t^{\alpha}_{\gamma_1}, t^{\alpha}_{\gamma_2} \) have the same height and comparable, we have 
\[
t^{\alpha}_{\gamma_1} = t^{\alpha}_{\gamma_2}.
\]
By the injectivity of the enumeration of each level, we have 
\[
C(\alpha, \gamma_1) = C(\alpha, \gamma_2).
\]

**Definition 4.13** Let \( e \) be an enumeration of a \( \kappa \)-tree \( T \). We say that \( e \) covers \( T \) if for every \( t \in T \) for some \( \delta < \kappa \), we have \( t <_T t^0_\delta \).

**Remark 4.14** For every regular cardinal \( \kappa \) and a well-pruned \( \kappa \)-tree \( T \), there is an enumeration \( e \) of \( T \) that covers \( T \).

**Proof.** Fix a bijection \( f : \kappa \to T \). We define \( \delta_\gamma < \kappa \) and choose \( t^0_\gamma \) by induction on \( \gamma \) in the following way:
\[
\begin{align*}
\delta_\gamma &= \max(\text{ht}(f(\gamma), T), \sup\{\delta_\beta : \beta < \gamma\}) + 1, \\
f(\gamma) < t^0_\delta \quad \text{and} \quad \text{ht}(t^0_\delta, T) = \delta_\gamma.
\end{align*}
\]
Since \( \kappa \) is regular, \( \delta_\gamma < \kappa \). Note that the function \( \gamma \mapsto \delta_\gamma \) is increasing and so injective. For each \( \delta \not\in \{ \delta_\gamma : \gamma < \kappa \} \) we choose \( t^0_\delta \) arbitrarily. It remains to choose an enumeration \( \langle t^i_\alpha : 0 < i < \mu_\alpha \rangle \) of \( \text{Lev}_\alpha(T) \setminus \{ t^0_\alpha \} \) for each \( \alpha < \kappa \). Clearly, the enumeration \( e \) covers \( T \).

**Theorem 4.15** Let \( T \) be a \( \kappa \)-tree and let \( e = \langle \langle t^i_\alpha : i < \mu_\alpha \rangle : \alpha < \kappa \rangle \) be an enumeration of \( T \) as given by Definition 4.10. If \( e \) covers \( T \), then \( T(C(T, e)) \) is isomorphic to \( \tilde{T} \), the history tree of \( T \).
Proof. In order to simplify the notation, we write $C$ for $C(T, e)$. For $\delta < \kappa$ we stipulate $C(\delta, \delta) = 0$. We shall prove that the following function is a well-defined isomorphism:

$$h : T(C) \to F_T,$$

$$h(C(\cdot, \delta)|\beta) = f_{C(\cdot, \delta)}^\beta$$

for every $\beta \leq \delta < \kappa$.

We separate the proof into several claims. First note that the height of $t_{C(\cdot, \delta)}^\beta$ is $\beta$, so the domain of the function

$$f_{C(\cdot, \delta)}^\beta$$

is $\beta$.

Claim 4.16 Let $\alpha < \beta \leq \delta < \kappa$. Let $i < \mu$ and let $s = t_i^\alpha$. Then

$$(C(\cdot, \delta)|\beta)(\alpha) = i \iff f_{C(\cdot, \delta)}^\beta(\alpha) = t_i^\alpha.$$ 

Proof. We prove that the following clauses are equivalent:

1. $(C(\cdot, \delta)|\beta)(\alpha) = i$;
2. $C(\alpha, \delta) = i$;
3. $t_i^\alpha <_T t_0^\beta$;
4. $s < t_0^\beta$;
5. $s < t_{C(\cdot, \delta)}^\beta$;
6. $f_{C(\cdot, \delta)}^\beta(\alpha) = t_i^\alpha$.

The equivalence between every pair of consecutive clauses holds immediately by the definitions, except the equivalence between clauses (4) and (5). Note that $t_{C(\cdot, \delta)}^\beta \leq T t_0^\beta$. Assume that clause (5) holds. Then by the transitivity of $\leq_T$, clause (4) holds. Conversely, assume that Clause (4) holds. So $s$ and $t_{C(\cdot, \delta)}^\beta$ are comparable. As the height of $s$ is $\alpha$ and $\alpha < \beta$, clause (5) holds.

Claim 4.17 Let $\beta_1 \leq \delta_1$ and $\beta_2 \leq \delta_2$. The following are equivalent:

1. $C(\cdot, \delta_1)|\beta_1 = C(\cdot, \delta_2)|\beta_2$ (so $\beta_1 = \beta_2$).
2. $f_{C(\cdot, \delta_1)}^{\beta_1} = f_{C(\cdot, \delta_2)}^{\beta_2}$.

Hence, the function $h$ is well-defined and injective.

Proof. The equivalence holds by Claim 4.16 and the injectivity of each enumeration in $e$. So by its definition, $h$ is well-defined and injective.

Claim 4.18 The function $h$ is an order-homomorphism.

Proof. For the first direction, suppose

$$C(\cdot, \delta_1)|\beta_1 \subseteq C(\cdot, \delta_2)|\beta_2.$$ 

We should prove

$$f_{C(\cdot, \delta_1)}^{\beta_1} \subseteq f_{C(\cdot, \delta_2)}^{\beta_2}.$$ 

Let $\alpha < \beta_1$. Then

$$(C(\cdot, \delta_1)|\beta_1)(\alpha) = (C(\cdot, \delta_2)|\beta_2)(\alpha).$$ 

So Claim 4.16 yields

$$f_{C(\cdot, \delta_1)}^{\beta_1}(\alpha) = f_{C(\cdot, \delta_2)}^{\beta_2}(\alpha).$$ 

The converse is similar, using the injectivity of each enumeration in $e$. 

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Claim 4.19 The function $h$ is surjective.

Proof. Let $t \in T$. Since $e$ covers $T$, we can find $\delta$ such that $t <_{T} t_0^{\delta}$. Let $\alpha$ be the height of $t$. Then $t = t_{\alpha}^{(\alpha, \delta)}$. So $h(C(\alpha, \delta) | \alpha) = f_{\alpha}^{(\alpha, \delta)} = f_\alpha$.

This completes the proof of Theorem 4.15. □

Corollary 4.20 Let $T_1, T_2$ be two $\kappa$-trees without leaves and for $i = 1, 2$ let $e_i$ be an enumeration of $T_i$ that covers $T_i$. Then the trees $T_1, T_2$ are isomorphic if and only if the trees $T(C(T_1, e_1))$ and $T(C(T_2, e_2))$ are isomorphic.

Proof. The following conditions are equivalent:
1. The trees $T_1$ and $T_2$ are isomorphic.
2. The history trees of $T_1$ and $T_2$ are isomorphic.
3. The trees $T(C(T_1, e_1))$ and $T(C(T_2, e_2))$ are isomorphic.

The equivalence between (2) and (3) is satisfied by Theorem 4.15, and the equivalence between (1) and (2) is satisfied by Corollary 4.5, as the trees have no leaves. □

5 The tree property at successor cardinals

In this section, we establish a connection between two properties:
1. existence of $\kappa$-sized branches in certain trees, and
2. existence of $\kappa$-sized homogeneous sets for locally additive colorings.

As a result, we get a characterization of the tree property at a successor cardinal. Recall two equivalent definitions of a weakly compact cardinal:

Fact 5.1 For every strongly inaccessible cardinal $\kappa$, the following are equivalent:
1. There exists no $\kappa$-Aronszajn tree.
2. For every $\lambda < \kappa$, every coloring $C : [\kappa]^2 \to \lambda$ admits a homogeneous set of size $\kappa$.

Here, we shall get a similar characterization for successor cardinals.

Theorem 5.2 For every infinite cardinal $\mu$, the following are equivalent:
1. There exists no $\mu^+$-Aronszajn tree.
2. $\mu^+ \to (\mu^+)^{\kappa}_{\mu^+}$, namely: every locally additive coloring $C : [\mu^+]^2 \to \mu$ admits a homogeneous set of size $\mu^+$.

The preceding theorem is a particular case of the following lemma (where $\mu^+$ stands for $\kappa$).

Lemma 5.3 Suppose that $\kappa$ is a regular uncountable cardinal and $\mu < \kappa$. The following are equivalent:
1. Every tree of height $\kappa$ whose levels have size $\leq \mu$ admits a $\kappa$-sized branch.
2. $\kappa \to (\kappa)^{\mu}_{\mu^+}$, namely: every locally additive coloring $C : [\kappa]^2 \to \mu$ admits a homogeneous set of size $\kappa$.

Proof. (1) $\implies$ (2): Let $C : [\kappa]^2 \to \mu$ be an arbitrary locally additive coloring.

Claim 5.4 All the levels of $T(C)$ have size $\leq \mu$.

Proof. Suppose not. Pick $\beta < \kappa$ such that $|T(C) \cap \beta| \mu > \mu$. Let $\langle \delta_i \mid i < \mu^+ \rangle$ be a strictly increasing sequence of ordinals $< \kappa$ for which $i \mapsto C(\cdot, \delta_i) | \beta$ is one-to-one over $\mu^+$. Without loss of generality, $\delta_0 > \beta$. Pick
i < j < \mu^+ such that C(\beta, \delta_i) = C(\beta, \delta_j). As C(\cdot, \delta_i) | \beta \neq C(\cdot, \delta_j) | \beta, pick \alpha < \beta such that C(\alpha, \delta_i) \neq C(\alpha, \delta_j).

Then we arrive at the following contradiction:

\[ C(\alpha, \delta_i) = u_{\alpha, \beta}(C(\beta, \delta_i)) = u_{\alpha, \beta}(C(\beta, \delta_j)) = C(\alpha, \delta_j). \]

By (1), \( T(C) \) must admit a cofinal branch. Pick \( b : \kappa \to \mu \) such that \( \{ b | \beta | \beta < \kappa \} \subseteq T(C) \). Fix a strictly increasing and continuous function \( f : \kappa \to \kappa \) such that \( f(0) = 0 \) and \( b(\{ f(i) + 1 \} \subseteq C(\cdot, f(i + 1)) \) for all \( i < \kappa \). Pick \( H \subseteq \{ f(i + 1) | i < \kappa \} \) of size \( \kappa \) on which \( b \) is constant. Then \( H \) is a \( C \)-homogeneous set of size \( \kappa \).

\((\neg 1) \implies (\neg 2)\): Suppose that \( (T, <_T) \) is a tree of height \( \kappa \) whose levels have size \( \leq \mu \), and \( T \) admits no \( \kappa \)-sized branch. We fix an enumeration \( \epsilon \) of \( T \) and consider the locally additive coloring \( C(T, e) \) (given by Claim 4.12).

Towards a contradiction, suppose that \( H \) is a \( \kappa \)-sized homogeneous set for \( C(T, e) \), with value, say, \( i \). So for all \( \alpha < \beta < \gamma \) from \( H \) we have \( t^i_{\alpha} \leq_T t^i_{\beta} \) and \( t^i_{\beta} \leq_T t^i_{\gamma} \), and hence \( t^i_{\alpha} \leq_T t^i_{\beta} \). So \( \{ t^i_{\alpha} | \alpha \in H \} \) is a \( \kappa \)-sized chain in \( T \). This is a contradiction. \( \square \)

## 6 Strongly inaccessibles in the tree sense

In this section, we prove that a regular cardinal that satisfies the tree property is strongly inaccessible if and only if it is strongly limit in the sense of the tree exponent.

**Definition 6.1** [\([4]\)] For infinite cardinals \( \mu \leq \lambda \), let \( \mathfrak{T}_{\lambda, \mu} \) denote the set of trees \( T \) of height \( \mu \) with \( \leq \lambda \) nodes. Let \( \text{Lim}(T) \) denote the set of cofinal-branches of \( T \). We define the tree exponent \( \lambda^{\mu, tr} \) as follows:

\[ \lambda^{\mu, tr} = \sup |\text{Lim}(T)| : T \in \mathfrak{T}_{\lambda, \mu} \].

**Remark 6.2** For every two infinite cardinals \( \lambda \) and \( \mu \) we have \( \lambda^{\mu, tr} \leq \lambda^{\mu} \).

Recall, that a weakly compact cardinal is a strongly inaccessible cardinal that satisfies the tree property.

**Fact 6.3** Let \( \kappa \) be an uncountable cardinal. The principle \( \kappa \to (\kappa)_2 \) holds if and only if \( \kappa \) is weakly compact.

By the following theorem, a cardinal satisfying the tree property is strongly inaccessible in the tree sense if and only if it is strongly inaccessible.

**Theorem 6.4** Let \( \kappa \) be an uncountable regular cardinal that satisfies the tree property. The following are equivalent:

1. \( \lambda^{\mu, tr} < \kappa \) for each \( \lambda, \mu < \kappa \).
2. \( \kappa \) is strongly inaccessible.

**Proof.** On the one hand, by Remark 6.2, clause (2) implies clause (1). Conversely, assume that clause (1) holds. By Fact 6.3, it is enough to prove that \( \kappa \to (\kappa)_2 \) holds.

Take a coloring \( F : [\kappa]^2 \to 2 \). We construct a tree \( T = \{ t_\alpha : \alpha < \kappa \} \) of functions of ordinals into 2: We define \( t_\alpha \) by induction on \( \alpha < \kappa \). We define \( t_\alpha | \xi \) by induction on \( \xi \). If \( t_\alpha | \xi \) is not in \( \{ t_\beta | \beta < \alpha \} \) then we define \( t_\alpha = t_\alpha | \xi \).

Otherwise \( t_\alpha | \xi = t_\beta \) for some \( \beta < \alpha \). In this case, we define \( t_\alpha(\xi) = F(\beta, \alpha) \). Clearly, the domain of each \( t_\alpha \) is an ordinal smaller than \( \kappa \) and \( T = \{ t_\alpha : \alpha < \kappa \} \) is a tree. Note that \( t_\alpha \neq t_\beta \) for all \( \alpha \neq \beta \) and therefore if \( t_\alpha <_T t_\beta \), then \( \alpha < \beta \).

We prove by induction on \( \alpha < \kappa \) that the cardinality of \( \text{Lev}_\alpha(T) \) is smaller than \( \kappa \). By the induction hypothesis and the regularity of \( \kappa \), \( |\text{Lev}_\alpha(T)| < \kappa \). Therefore by assumption (1), we have \( |\text{Lim}(\text{Lev}_\alpha(T))| < \kappa \). But the function on \( \text{Lev}_\alpha(T) \) into \( \text{Lim}(\text{Lev}_\alpha(T)) \) assigning \( t_\beta \) to \( \{ t_\gamma \in T : t_\gamma <_T t_\beta \} \) is injective. So \( |\text{Lev}_\alpha(T)| < \kappa \).

\( T \) is a \( \kappa \)-tree. So by assumption, \( T \) has a \( \kappa \)-branch, \( B \). If \( t_\alpha <_T t_\beta <_T t_\gamma \) are in \( B \) then \( t_\beta | \xi = t_\gamma | \xi = t_\alpha \) where \( \xi = \beta t_\alpha(T) \). So \( F(\alpha, \beta) = t_\beta(\xi) = t_\alpha(\xi) = F(\alpha, \gamma) \). For \( i < 2 \), define

\[ H_i = \{ \alpha \in \kappa : t_\alpha \in B \text{ and } \forall \beta > \alpha \{ t_\beta \in B \implies F(\alpha, \beta) = i \} \} \].

Each ordinal in \( \{ \alpha \in \kappa : t_\alpha \in B \} \) is in \( H_0 \) or in \( H_1 \). So one of \( H_0 \) and \( H_1 \) has cardinality \( \kappa \). But each \( H_i \) is homogeneous, because \( F(\alpha, \beta) = i \) holds whenever \( \alpha < \beta \) are in \( H_i \). \( \square \)
Corollary 6.5 Let $\kappa$ be an uncountable cardinal. Then $\kappa$ is weakly compact if and only if $\kappa$ satisfies the following properties:

1. $\kappa$ is regular,
2. $\kappa$ satisfies the tree property; and
3. $\lambda^{<\lambda} < \kappa$ for each $\lambda, \mu < \kappa$.

Proof. Assume that $\kappa$ is weakly compact. Clearly $\kappa$ satisfies properties (1) & (2) and is strongly inaccessible. By Remark 6.2 $\kappa$ satisfies property (3).

Conversely, if $\kappa$ satisfies properties (1)-(3), then by Theorem 6.4, $\kappa$ is strongly inaccessible, hence weakly compact.

7 Ramsey theorems for locally-additive colorings

We present two theorems relating colorings of $\kappa$ by $\mu$ colors where $\mu^+ < \kappa = cf(\kappa)$; while in Theorem 7.4 the coloring is assumed to be locally-additive, in Theorem 7.6 the coloring is assumed to be $\ast$-locally-additive. We observe that the result for $\ast$-locally additive colorings cannot be generalized to the case $\mu^+ = \kappa$. On the other hand, the result for locally additive colorings in the case $\mu^+ = \kappa$ is equivalent by Theorem 5.2 to the non-existence of a $\kappa$-Aronszajn tree. Here is a summary of the results of this section:

$$
\kappa \to (\kappa)_{\kappa-\text{la}_{\mu}} \quad \text{(Theorem 7.4)},
$$
$$
\kappa \to (\kappa)_{\text{la}_{\mu}} \quad \text{(Theorem 7.6)},
$$
$$
\mu^+ \to (\mu^+)_{\kappa-\text{la}_{\mu}} \quad \text{(Proposition 7.5)}.
$$

7.1 $\ast$-Locally-additive colorings

In this subsection we verify that Shelah’s proof of ([3, Theorem 1.1]) works for $\ast$-locally additive colorings (rather than additive colorings) with few colors (rather than finite). For the rest of this subsection, we assume

$$
\mu < \kappa = cf(\kappa) \text{ and } C : [\kappa]^2 \to \mu \text{ is a } \ast\text{-locally additive coloring.}
$$

Definition 7.1 For $\beta, \gamma < \kappa$, we write $\beta \sim \gamma$ if for some $\alpha < \kappa$ with $\beta, \gamma < \alpha$, we have $C(\beta, \alpha) = C(\gamma, \alpha)$. In this case, we say that $\alpha$ witnesses that $\beta \sim \gamma$.

Remark 7.2 By $\ast$-local additivity, if $\beta \neq \gamma, \alpha_1$ witnesses that $\beta \sim \gamma$ and $\alpha_1 < \alpha_2$, then $\alpha_2$ witnesses that $\beta \sim \gamma$ as well. Hence, $\sim$ is an equivalence relation.

Lemma 7.3 If $\mu^+ < \kappa$, then the number of $\sim$-equivalence classes is $\mu$ at most.

Proof. Assume towards contradiction that there are $\mu^+$ equivalence classes at least. Let $\{\alpha_i : i < \mu^+\}$ be a system of representatives. Since $\mu^+ < \kappa$, we can find an ordinal $\alpha^* < \kappa$ which is bigger than each $\alpha_i$. The set $\{C(\alpha_i, \alpha^*) : i < \mu^+\}$ is included in $\mu$. So for some distinct $i, j$ we have $C(\alpha_i, \alpha^*) = C(\alpha_j, \alpha^*)$, so $\alpha_i \sim \alpha_j$, a contradiction.

Theorem 7.4 $\kappa \to (\kappa)_{\kappa-\text{la}_{\mu}}$ holds whenever $\mu^+ < \kappa = cf(\kappa)$.

Proof. By Lemma 7.3, some $\sim$-equivalence class, $A$, is unbounded. We construct by induction a strictly increasing sequence $\langle y_\alpha : \alpha \in \kappa \rangle$ of elements of $A$, such that whenever $\beta < \gamma < \alpha$ we have $C(y_\beta, y_\alpha) = C(y_\gamma, y_\alpha)$. Given $y_\beta$ for all $\beta < \alpha$ we choose $y_\alpha$ as follows: for every $\beta < \gamma$ there is $\varepsilon_{\beta, \gamma} < \kappa$ such that $\varepsilon_{\beta, \gamma} > y_\beta, y_\gamma$ and $C(y_\beta, \varepsilon_{\beta, \gamma}) = C(y_\gamma, \varepsilon_{\beta, \gamma})$. By Remark 7.2, we may choose $y_\alpha$ to be any element of $A$ greater than $\sup \{\varepsilon_{\beta, \gamma} : \beta < \gamma < \alpha\}$.

It follows that for every $\beta < \gamma < \alpha$ in $\kappa$ we have $C(y_\beta, y_\alpha) = C(y_\gamma, y_\alpha)$. Consider the function $f : \kappa \setminus \{0\} \to \mu$ defined by $f(\alpha) = C(y_0, y_\alpha)$. As $\mu < \kappa = cf(\kappa)$, there is an unbounded subset $B$ of $\kappa \setminus \{0\}$ and $t \in \mu$ such that...
Proposition 7.5 \( \kappa \to (\kappa)_{\kappa^{+}} \) does not hold if \( \mu^{+} = \kappa \).

Proof. We present a coloring witnessing the negation of this principle. For each \( \alpha < \kappa \), let \( f_{\alpha} : \alpha \to \mu \) be an injection. For \( \beta < \alpha < \kappa \), we define \( C(\beta, \alpha) = f_{\alpha}(\beta) \). The coloring \( C \) is \( \ast \)-locally additive vacuously, because for any \( \beta < \gamma < \alpha < \kappa \), we have \( C(\beta, \alpha) \neq C(\gamma, \alpha) \). By the same reason, there is no homogeneous set of cardinality 3.

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\[ \Box \]

7.2 Locally-additive colorings

Theorem 7.6 Let \( \kappa \) be a regular cardinal and let \( \mu \) be a cardinal with \( \mu^{+} < \kappa \). The following principle holds:

\[ \kappa \to (\kappa)_{\mu^{+}}. \]

Proof. Define \( \lambda = \mu^{+} \). By Claim 7.7, every tree of height \( \kappa \) whose levels have size \( \leq \mu \) is not \( \kappa \)-Aronszajn. So by Lemma 5.3, the principle \( \kappa \to (\kappa)_{\mu^{+}} \) holds.

Recall the following claim [1, Proposition 7.9].

Claim 7.7 Let \( \kappa \) and \( \lambda \) be cardinals such that \( \kappa \) is regular and \( \lambda < \kappa \). If \( (T, <) \) is a \( \kappa \)-tree whose levels have size \( \leq \lambda \), then \( (T, <) \) is not \( \kappa \)-Aronszajn.

For completeness, we prove Claim 7.7.

Proof. Case A: \( \lambda \) is regular. Without loss of generality \( (T, <) \) is well-pruned (it is a well-known fact that every \( \kappa \)-tree has a well-pruned subtree which is a \( \kappa \)-tree, and so there is no harm in assuming that \( (T, <) \) is well-pruned).

Claim 7.8 For every ordinal \( \alpha < \kappa \) with \( \text{cf}(\alpha) = \lambda \), there is an ordinal \( q(\alpha) < \alpha \), such that for every ordinal \( \beta \) with \( q(\alpha) < \beta < \alpha \), and any element \( a \in \text{Lev}_{q(\alpha)}(T) \) there are no two distinct elements \( b, c \in \text{Lev}_{\beta}(T) \) such that \( a < b < c \).

Proof. Towards a contradiction assume that there exists an \( \alpha \) with \( \text{cf}(\alpha) = \lambda \), such that for every \( q(\alpha) < \alpha \) there are \( \beta, a, b, c \) such that \( q(\alpha) < \beta < \alpha \), \( a \in \text{Lev}_{q(\alpha)}(T) \), \( b, c \in \text{Lev}_{\beta}(T) \), \( a < b \) and \( a < c \). By the induction hypothesis, there is at most one ordinal \( j < i \) such that \( d_{i} | \text{Lev}_{\beta}(T) = a \). We choose \( d_{i} \in \text{Lev}_{\alpha}(T) \) such that \( d_{i} | \text{Lev}_{\alpha}(T) = a \), and for some \( j < i \), \( d_{j} | \text{Lev}_{\alpha}(T) = a \), then \( d_{j} | \text{Lev}_{\alpha}(T) = b \), then \( d_{j} | \text{Lev}_{\alpha}(T) = c \) and vice versa. Hence, we can carry out the induction. The \( d_{j} \)'s are pairwise distinct. So \( \text{Lev}_{\alpha}(T) \) is \( \lambda \) at least, a contradiction. Claim 7.8 is proved.

We now can apply Fodor’s Lemma with the function from \( \kappa \) to \( \alpha \mapsto q(\alpha) \). So for some stationary subset \( S \) of \( \kappa \) and some ordinal \( \alpha^{*} < \kappa \), \( q(\alpha) = \alpha^{*} \) for each \( \alpha \in S \). Fix \( a \in \text{Lev}_{\alpha}(T) \). The set \( \{ b \in \text{Lev}_{\geq \alpha}(T) : a < b \} \) is a chain of size \( \kappa \) since \( (T, <) \) is well-pruned. So \( (T, <) \) is not \( \kappa \)-Aronszajn.

Case B: \( \lambda \) is singular, so, in particular, \( \lambda \) is a limit cardinal. For the sake of a contradiction, assume that \( (T, <) \) is \( \kappa \)-Aronszajn. By case A, for each \( \mu < \lambda \) there is a level of size \( \mu^{+} \) at least. Since \( \lambda < \text{cf}(\kappa) = \kappa \), it follows that there is a level of cardinality \( \lambda \) at least, contradicting our assumption. Claim 7.7 is proved.

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