Classical Spacetime from Quantum Gravity

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Abstract

We show how classical spacetime emerges from quantum gravity through the study of a quantum FRW cosmological model coupled to a free massive scalar field using a new asymptotic expansion method of the Wheeler-DeWitt equation. It is shown that the coherent states of the nonadiabatic basis of a particular generalized invariant give rise to the quantum back reaction of matter field proportional to classical energy and the Einstein-Hamilton-Jacobi with matter becomes equivalent to the classical Einstein equation.

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In recent years, apart from the attempt to endow canonical gravity with a quantum gravity theory by overcoming many conceptual and technical problems, quantum cosmology has also been used to derive semiclassical gravity in which one has the quantum field theory in a curved spacetime and the classical Einstein equation with the expectation value of quantum matter field.

If canonical quantum gravity in the form of the Wheeler-DeWitt (WD) equation is to have successful and useful applications in cosmology, there should be a consistent limiting procedure from a supposed quantum gravity written formally as $\hat{G}_{\mu\nu} = 8\pi\kappa\hat{T}_{\mu\nu}$, in which both the gravity and matter field are quantized at the same time and $\kappa$ is the gravitation constant, first to the semiclassical gravity written also formally as $G_{\mu\nu} = 8\pi\kappa\langle\hat{T}_{\mu\nu}\rangle$, in which the spacetime emerges as a classical background in some sense but the matter field maintains the status of quantum variables, and finally down to the classical gravity $G_{\mu\nu} = 8\pi\kappa T_{\mu\nu}$, in which both gravity and matter field are treated as classical variables.

In the conventional semiclassical approach [1], the WKB approximation of the WD equation for gravity coupled to a matter field results in the vacuous Einstein equation in the form of the Einstein-Hamilton-Jacobi (EHJ) equation together with a time-dependent functional Schrödinger equation for the matter field. In order to include quantum back reaction of the matter field into the EHJ equation one should keep the expectation value of the quantum matter field by hand in the Born-Oppenheimer expansion. However, there are still gaps in this approach to the semiclassical gravity [2]. Firstly, there is an arbitrariness in the separation of the WD equation into the gravitational field equation which should reduce to the EHJ equation and the matter field equation which should describe the Schrödinger equation for the matter field in a curved spacetime with some gravitational quantum corrections. Secondly, even after the cosmological time is introduced from the EHJ equation and the Schrödinger equation is derived, the form of quantum states of the matter field should still have the WKB form to give rise to classical gravity equated with the matter field in the form of the EHJ equation.

In this letter we show how a classical spacetime emerges from a quantum Friedmann-
Robertson-Walker (FRW) cosmological model coupled to a free massive scalar field in the semiclassical gravity obtained from a new asymptotic expansion of the WD equation [3]. In the new asymptotic expansion method, the matter field obeys purely a quantum equation equivalent to the time-dependent functional Schrödinger equation with higher order gravitational quantum corrections, and semiclassical gravity is described by the EHJ equation equated with the quantum back reaction of the scalar field. In the nonadiabatic basis of a particular generalized invariant [4], it is shown that the quantum back reaction of the scalar field has exactly the same form as the time-time component of the energy-momentum tensor in the classical Einstein equation by matching appropriately quantum numbers to the amplitudes of classical field. However, there is a noticeable distinction between the conventional approach such as the WKB approximation or the Born-Oppenheimer expansion to semiclassical gravity and the new asymptotic expansion in that in the former one should use the WKB form of wave function at some stage of its derivation to obtain the classical Einstein equation, whereas in the latter one keeps the quantum status without assuming the WKB wave function for the scalar field and the quantum back reaction just yields the time-time component of the energy-momentum tensor.

The FRW cosmology has a homogeneous and isotropic spacetime manifold with the metric

\[ ds^2 = -N^2(t)dt^2 + R^2(t)d\Omega_3^2, \]

where \( N \) is the lapse function and \( R(t) \) is the scale factor depending only on \( t \). The time will be scaled in unit of \( c = 1 \), and the Planck mass squared will thus denote \( m_P^2 = \frac{\hbar}{8\pi\kappa} \).

The action for the FRW cosmology coupled to a free massive scalar field, which is also homogeneous and isotropic, i.e., depends only on time \( t \), takes the form

\[ S = \int dt \left[ -\frac{m_P^2}{\hbar} R^3 \left( \frac{1}{2N} \left( \frac{\dot{R}}{R} \right)^2 + N \left( -\frac{k}{2R^2} + \frac{\Lambda}{6} \right) \right) + R^3 \left( \frac{1}{2N} \dot{\phi}^2 - \frac{Nm^2}{2} \phi^2 \right) \right]. \]

(2)

The classical equations of motion are obtained by taking variation with respect to \( N \):

\[ -\frac{m_P^2}{\hbar} \left( \frac{R\dot{R}^2}{2} + \frac{kR}{2} - \frac{\Lambda R^3}{6} \right) + R^3 \left( \frac{\dot{\phi}^2}{2} + \frac{m^2}{2} \phi^2 \right) = 0. \]

(3)
by taking variation with respect to $R$:

$$\frac{m_p^2}{\hbar} \left( \frac{d}{dt}(R\dot{R}) - \frac{\dot{R}^2}{2} + \frac{k}{2} - \frac{\Lambda R^2}{2} R^2 \right) + 3R^2 \left( \frac{\dot{\phi}^2}{2} - m^2 \phi^2 \right) = 0,$$  \hspace{1cm} (4)

and by taking variation with respect to $\phi$:

$$\frac{d}{dt} \left( R^3 \dot{\phi} \right) + m^2 R^3 \phi = 0.$$  \hspace{1cm} (5)

We rewrite the classical equation (3) in the form

$$\left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = 16\pi\kappa \frac{1}{R^3} T_{00},$$  \hspace{1cm} (6)

where $T_{00}$ is the time-time component of the energy-momentum stress tensor

$$T_{00} = \frac{1}{2} R^3 \dot{\phi}^2 + \frac{1}{2} m^2 R^3 \phi^2.$$  \hspace{1cm} (7)

Up to some operator ordering ambiguity, we quantize the Hamiltonian a la Dirac quantization to obtain the WD equation

$$\left[ \frac{\hbar^3}{2m^2 \rho R} \frac{\partial^2}{\partial R^2} + \frac{m_p^2}{\hbar} \left( - \frac{kR}{2} + \frac{\Lambda R^3}{6} \right) - \frac{\hbar^2}{2R^3} \frac{\partial^2}{\partial \phi^2} + \frac{m^2 R^3}{2} \phi^2 \right] \Psi(R, \phi) = 0.$$  \hspace{1cm} (8)

The key tool of the new asymptotic expansion method is to search for the wave function of the form

$$\Psi(R, \phi) = \psi(R) \Phi(\phi, R).$$  \hspace{1cm} (9)

Here $\psi$ and $\Phi$ are still quantum states of gravity and the scalar field, respectively. The classical spacetime will emerge later only after finding the wave function for gravity in the WKB form with the quantum back reaction of scalar field. The wave function (9) has the form obtained from the adiabatic expansion or superadiabatic expansion of the WD equation (8). Any quantum state of the scalar field can be expanded by the basis of some Hermitian operator relevant to the matter field Hamiltonian:

$$\Phi(\phi, R) = \sum_k c_k(R) \left| \Phi_k(\phi, R) \right>.$$  \hspace{1cm} (10)
The main result is the separation of the WD equation into the gravitational field equation

\[
\left[ \frac{h^3}{2m_P^2 R} \frac{\partial^2}{\partial R^2} + \frac{m_P^2}{\hbar} \left( -\frac{kR}{2} + \frac{\Lambda R^3}{6} \right) + : H_{\cdot n} (R) \right] \psi(R) = 0, \tag{11}
\]

and in the limit \( \frac{h^2}{m_P} \to 0 \) the asymptotic matter field equation

\[
i\hbar \frac{\partial}{\partial t} c_n + \left( \Omega_{nn}^{(1)} - H_{nn} + : H_{\cdot n} \right) c_n + \sum_{k \neq n} \left( \Omega_{nk}^{(1)} - H_{nk} \right) c_k = 0, \tag{12}
\]

where

\[
\Omega_{nk}^{(1)} = i\hbar \left\langle \Phi_n(\phi, R) \frac{\partial}{\partial t} \Phi_k(\phi, R) \right\rangle, \quad H_{nk} = \left\langle \Phi_n(\phi, R) | \hat{H}_m | \Phi_k(\phi, R) \right\rangle, \tag{13}
\]

and the expectation value of normal ordered Hamiltonian

\[
: H_{\cdot n} = \left\langle \Phi_n(\phi, R) | : \hat{H}_m : | \Phi_n(\phi, R) \right\rangle, \tag{14}
\]

through the introduction of cosmological time

\[
\frac{\partial}{\partial t} = \text{Im} \left( \frac{h^2}{m_P^2 R \psi \partial R} \frac{\partial}{\partial R} \right). \tag{15}
\]

For the gravitational wave function in the WKB form

\[
\psi(R) = f(R) \exp \left[ \frac{m_P^2}{\hbar^2} S(R) \right] \tag{16}
\]

the cosmological time becomes

\[
\frac{\partial}{\partial t} = \frac{1}{R} \frac{\partial S(R)}{\partial R} \frac{\partial}{\partial R}. \tag{17}
\]

From the gravitational wave equation we obtain the EHJ equation

\[
\frac{1}{2R} \left( \frac{\partial S}{\partial R} \right)^2 + \frac{kR}{2} - \frac{\Lambda R^3}{6} = 8\pi \kappa : H_{\cdot n}(R), \tag{18}
\]

which can be rewritten as

\[
\left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = 8\pi \kappa \frac{1}{R^3} : H_{\cdot n}(R). \tag{19}
\]
On the other hand, the matter field Hamiltonian is a time-dependent harmonic oscillator of the form

\[ \hat{H}_m = T_{00} = \frac{1}{2R^3} \pi_\phi^2 + \frac{1}{2} m^2 R^3 \phi^2. \]  

(20)

In the nonadiabatic bases

\[ \hat{I}(\pi_\phi, \phi, R) |\Phi_n(\phi, R)\rangle = \lambda_n |\Phi_n(\phi, R)\rangle. \]  

(21)

of the generalized invariants, which obey the invariant equation

\[ \frac{\partial}{\partial t} \hat{I} - i \frac{\hbar}{\hbar} [\hat{I}, \hat{H}_m] = 0, \]  

(22)

there is a well-known decoupling theorem [5]

\[ H_{nk}(R) = \Omega^{(1)}_{nk}, \]  

(23)

for \( n \neq k \). Then the asymptotic matter field equation becomes a diagonal equation whose solution is given by

\[ c_n(t) = c_n(t_0) \exp \left[ \frac{i}{\hbar} \int \left( \Omega^{(1)}_{nn} - H_{nn} + : H :_{nn} \right) dt \right]. \]  

(24)

It should be noted that the asymptotic quantum state is the exact quantum state of the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \Phi(\phi, R) = \hat{H}_m(\pi_\phi, \phi, R)\Phi(\phi, R) \]  

(25)

except for the time-dependent phase factor \( \exp \left[ \frac{i}{\hbar} \int : H :_{nn} dt \right] \).

First we find the particular second order generalized invariant of the form [4]

\[ \hat{I} = \frac{1}{2} \left( \hat{I}_+ \hat{I}_- + \hat{I}_- \hat{I}_+ \right), \]  

(26)

where

\[ \hat{I}_+ = \phi^*(t) \hat{\pi}_\phi - R^3(t) \dot{\phi}^*(t) \hat{\phi}, \]  

\[ \hat{I}_- = \phi(t) \hat{\pi}_\phi - R^3(t) \dot{\phi}(t) \hat{\phi}, \]  

(27)
in terms of one of classical solutions of Eq. (5) such that

\[ R^3(t) \left( \dot{\phi}^*(t)\dot{\phi}(t) - \phi(t)\dot{\phi}^*(t) \right) = i, \]

\[ \text{Im} \left( \frac{\dot{\phi}(t)}{\phi(t)} \right) < 0. \] (28)

Then \( \hat{I}_+ \) acts as the creation operator \( \hat{A}^\dagger(t) \) and \( \hat{I}_- \) as the annihilation operator \( \hat{A}(t) \). The ground state quantum state is given by

\[ \langle \phi | \Phi_0(\phi, R) \rangle = \frac{1}{\sqrt{2\pi\hbar}}} \exp \left[ i \frac{R^3 \dot{\phi}(t)\phi^2}{2\hbar\phi(t)} \right], \] (29)

and the \( n \)th quantum state by

\[ \langle \phi | \Phi_n(\phi, R) \rangle = \frac{1}{\sqrt{2\pi\hbar}}} \frac{1}{\sqrt{2\pi\hbar}}} \left( \frac{\phi^*(t)}{\phi(t)} \right)^n H_n \left( \frac{\phi}{\sqrt{2\pi\hbar}}} \right) \exp \left[ i \frac{R^3 \dot{\phi}(t)\phi^2}{2\hbar\phi(t)} \right], \] (30)

where \( H_n \) is the \( n \)th Hermite polynomial. From the quantum back reaction of the scalar field

\[ :H :_{\phi}(t) = n\hbar R^3(t) \left( \dot{\phi}(t)\dot{\phi}^*(t) + m^2\phi(t)\phi^*(t) \right) \] (31)

we obtain the EHJ equation with the quantum back reaction in the nonadiabatic basis

\[ \left( \frac{\dot{R}}{\dot{R}} \right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = 8\pi n\hbar \kappa \left( \dot{\phi}(t)\dot{\phi}^*(t) + m^2\phi(t)\phi^*(t) \right). \] (32)

The ground state \( (n = 0) \) of the scalar field leads to the Einstein vacuum equation. Furthermore, one can show that the semiclassical gravity reduces classical gravity by identifying the amplitude of classical field \( \phi_c = \phi_0\phi_q \) where \( \phi_0 = \sqrt{n\hbar} \) and \( \phi_q \) satisfies the condition (28). In particular, the classical field energy is proportional to the field squared and the quantum energy to \( n\hbar \), and therefore for a large quantum number \( n \) one may expect the correspondence \( \phi_0 = \sqrt{n\hbar} \).

To summarize, we showed how classical spacetime obeying the Einstein-Hamilton-Jacobi equation with the back reaction of classical matter field, which is equivalent to classical Einstein equation equated with the matter field, can emerge through the investigation of the quantum FRW cosmological model minimally coupled to a free massive scalar field. It
differs somewhat conceptually and technically from the conventional approach. The quan-
tum status of matter field has been still kept through the asymptotic limiting procedure,
a Born-Oppenheimer expansion, for the gravitational field equation. There needs no re-
striction on the form of the wave function of matter field as far as the cosmological time
is appropriately defined. The remaining open question is to see how the method used in
this letter can be applied successfully to the quantum FRW cosmological model minimally
coupled to a scalar field with an arbitrary potential and whether classical spacetime emerges
really from quantum gravity. Even though to find the generalized invariant is not so simple
beyond a quadratic potential, the extension to a general potential is worthy to be worked
out.

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