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Norm-Controlled Inversion of Banach algebras of infinite matrices

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Abstract. In this paper we provide a polynomial norm-controlled inversion of Baskakov–Gohberg–Sjöstrand Banach algebra in a Banach algebra $B(\ell^q)$, $1 \leq q \leq \infty$, which is not a symmetric $*$-Banach algebra.

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1. Introduction

N. Wiener in [19] proved that if a periodic function with absolutely convergent Fourier series never vanishes, then it also has an absolutely convergent Fourier series.

A Banach subalgebra $\mathcal{A}$ of a Banach algebra $B$ having a common identity is called inverse-closed in $B$ if $A \in \mathcal{A}$ with $A^{-1} \in B$ implies $A^{-1} \in \mathcal{A}$. For a Banach subalgebra $\mathcal{A}$ which is inverse-closed in $B$, we say that $\mathcal{A}$ admits a norm-controlled inversion in $B$ if there exists a function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|A^{-1}\|_{\mathcal{A}} \leq h(\|A^{-1}\|_B, \|A\|_{\mathcal{A}})$$

for all $A \in \mathcal{A}$ with an inverse $A^{-1}$ in $B$, where $\| \cdot \|_{\mathcal{A}}$ and $\| \cdot \|_B$ are norms on $\mathcal{A}$ and $B$, respectively.

N. Nikolski in [9] showed that the algebra of absolutely convergent Fourier series does not admit norm-controlled inversion in the algebra of continuous periodic functions.

Let a discrete set $\Lambda \subset \mathbb{R}^d$ be relatively-separated, that is,

$$R(\Lambda) = \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} \chi_{\lambda + [0, 1)^d}(x) < \infty. \quad (1)$$
The set \( \Lambda \) may not form a group. Our prime models are paraboloids \( \{(x, y, z) : z = ax^2 + by^2, x, y \in \mathbb{Z} \} \), and elliptical hyperboloids \( \{(x, y, z) : z^2 = ax^2 + by^2, x, y \in \mathbb{Z} \} \), where \( a, b > 0 \), and the set \( \{k + \delta_k : k \in \mathbb{Z}^d, \delta_k \in [0, 1)^d \} \).

For \( 1 \leq p \leq \infty \) and \( r \geq 0 \), define the Baskakov–Gohberg–Sjöstrand (BGS) class \( \mathcal{E}_{p, r}(\Lambda) \) by

\[
\mathcal{E}_{p, r}(\Lambda) = \{ A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} : \| A \|_{\mathcal{E}_{p, r}(\Lambda)} < \infty \}
\]

where for \( 1 \leq p < \infty \),

\[
\| A \|_{\mathcal{E}_{p, r}(\Lambda)} = \left( \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')|^p (1 + |\lambda - \lambda'|)^{pr} \chi_{k+\{0,1\}^d}(\lambda - \lambda') \right)^{1/p},
\]

and for \( p = \infty \),

\[
\| A \|_{\mathcal{E}_{\infty, r}(\Lambda)} = \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')|(1 + |\lambda - \lambda'|)^r,
\]

where for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d, |x| = \max(|x_1|, \ldots, |x_d|) \). The above classes of infinite matrices form Banach algebras. In particular, when \( p = \infty \), \( \mathcal{E}_{\infty, r}(\Lambda) \) is called a Jaffard algebra and written as \( \mathcal{J}_r(\Lambda) \) with the norm \( \| \cdot \|_{\mathcal{J}_r(\Lambda)} \).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach \( * \)-algebras with common identity and involution. If \( \mathcal{B} \) is a symmetric algebra (see [4]) and \( \mathcal{A} \) is a differential subalgebra of \( \mathcal{B} \), then \( \mathcal{A} \) admits norm-controlled inversion in \( \mathcal{B} \) (see [6, 7, 12]). Several algebras of infinite matrices with certain off-diagonal decay including Gröchenig–Schur algebra, Baskakov–Gohberg–Sjöstrand algebra and Jaffard algebra are shown to be differential \( * \)-subalgebra of \( \mathcal{B}(\ell^2(\mathbb{Z}^d)) \) (see [3, 5–8, 10–13, 15–18]), where for \( 1 \leq q \leq \infty \), \( \mathcal{B}(\ell^q(\mathcal{V})) \) denotes the space of all bounded linear operators on \( \ell^q(\mathcal{V}) \) with the norm \( \| \cdot \|_{\mathcal{B}(\ell^q(\mathcal{V}))} \) and \( \ell^q(\mathcal{V}) \) is the set of all \( q \)-summable sequences on \( \mathcal{V} \) with the norm \( \| \cdot \|_q \).

Using the commutator trick and the partition of the identity, J. Sjöstrand in [14] showed Wiener’s lemma for \( \mathcal{E}_{1,0}(\mathbb{Z}^d) \). The polynomial norm-controlled inversion is studied in [6] for a differential subalgebra of a symmetric Banach algebra and in [7] for matrices in Besov algebras, Bessel algebras, Banales–Davie algebras, Baskakov–Gohberg–Sjöstrand algebras and Jaffard algebras. A. G. Baskakov in [1, 2] depending on Bochner–Phillips theorem proved that Jaffard algebras and Baskakov–Gohberg–Sjöstrand algebras with \( p = 1 \) admit norm-controlled inversion in \( B(\ell^2) \).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach \( * \)-algebras with common identity and involution. If \( \mathcal{B} \) is a symmetric algebra (see [4]) and \( \mathcal{A} \) is a differential subalgebra of \( \mathcal{B} \), then \( \mathcal{A} \) admits norm-controlled inversion in \( \mathcal{B} \) (see [6, 7, 12]). Several algebras of infinite matrices with certain off-diagonal decay including Gröchenig–Schur algebra, Baskakov–Gohberg–Sjöstrand algebra and Jaffard algebra are shown to be differential \( * \)-subalgebra of \( \mathcal{B}(\ell^2(\mathbb{Z}^d)) \) (see [3, 5–8, 10–13, 15–18]), where for \( 1 \leq q \leq \infty \), \( \mathcal{B}(\ell^q(\mathcal{V})) \) denotes the space of all bounded linear operators on \( \ell^q(\mathcal{V}) \) with the norm \( \| \cdot \|_{\mathcal{B}(\ell^q(\mathcal{V}))} \) and \( \ell^q(\mathcal{V}) \) is the set of all \( q \)-summable sequences on \( \mathcal{V} \) with the norm \( \| \cdot \|_q \).

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In some applications in the field of mathematics and engineering, widespread-used algebras \( \mathcal{B} \) of infinite matrices are Banach algebras \( \mathcal{B}(\ell^p) \) for \( p \in [1, \infty) \), while they are symmetric only when \( p = 2 \). The results in [1, 2, 6, 7, 11, 13] deal with the norm-controlled inversions in symmetric algebras, on the other hand, we provide the norm-controlled inversion in a nonsymmetric algebra. In this paper, for \( 1 \leq p, q \leq \infty \), \( r > d(1-1/p) \) and a relatively-separated subset \( \Lambda \) of \( \mathbb{R}^d \), we give the simple proof for the norm-controlled inversion of the Baskakov–Gohberg–Sjöstrand subalgebra \( \mathcal{E}_{p, r}(\Lambda) \) of a nonsymmetric Banach algebra \( \mathcal{B}(\ell^d(\Lambda)) \). We expect that the method in this paper can be applied to algebras of infinite matrices having off-diagonal decay with different weights from polynomial functions. The proof of the main theorem is based on commutator trick and the partition of the identity in [14].

For \( a = (a_1, \ldots, a_d) \in \mathbb{R} \), we write \( |a| = (|a_1|, \ldots, |a_d|) \), where for \( b \in \mathbb{R} \), \( |b| \) denotes the largest preceding integer of \( b \).
2. Norm-Controlled Inversion

To state our result on norm-controlled inversion for localized infinite matrices, we recall some concepts. For a relatively-separated subset $\Lambda$ of $\mathbb{R}^d$ satisfying (1), we define Schur norm of an infinite matrix $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}$ by

$$\|A\|_{\mathcal{S}(\Lambda)} = \max \left( \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |a(\lambda, \lambda')|, \sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |a(\lambda, \lambda')| \right).$$

(4)

For any $1 \leq q \leq \infty$, one can show that the Schur class $\mathcal{S}(\Lambda)$ is a subalgebra of the Banach algebra $\mathcal{B}(\ell^q(\Lambda))$ and

$$\|A\|_{\mathcal{B}(\ell^q(\Lambda))} \leq \|A\|_{\mathcal{S}(\Lambda)}.$$  

(5)

Let $A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda}$ be an infinite matrix in a BGS algebra, we define its approximation

$$A_N := \left( a(\lambda, \lambda') \chi_{[0,1]}(|\lambda - \lambda'|/N) \right)_{\lambda, \lambda' \in \Lambda}.$$  

(6)

We have the following properties of the algebra $\mathcal{E}_{p,r}(\Lambda)$ for $1 \leq p \leq \infty$ and $r > 0$.

**Proposition 1.** Let $1 \leq p, q \leq \infty$ and $r > d(1 - 1/p)$, and let $\Lambda$ be a relatively-separated subset of $\mathbb{R}^d$ satisfying (1). Then the following statements hold.

1. The BGS algebra $\mathcal{E}_{1,0}(\Lambda)$ is a subalgebra of Schur algebra $\mathcal{S}(\Lambda)$, and

$$\|A\|_{\mathcal{S}(\Lambda)} \leq 2R(\Lambda)\|A\|_{\mathcal{E}_{1,0}(\Lambda)} \text{ for all } A \in \mathcal{E}_{1,0}(\Lambda).$$  

(7)

2. The BGS algebra $\mathcal{E}_{1,0}(\Lambda)$ is a subalgebra of the Banach algebra $\mathcal{B}(\ell^q(\Lambda))$, and

$$\|Ac\|_{\ell^q(\Lambda)} \leq 2R(\Lambda)\|A\|_{\mathcal{E}_{1,0}(\Lambda)}\|c\|_{\ell^q(\Lambda)} \text{ for all } A \in \mathcal{E}_{1,0}(\Lambda) \text{ and } c \in \ell^q(\Lambda).$$  

(8)

3. The BGS algebra $\mathcal{E}_{p,r}(\Lambda)$ is a subalgebra of the algebra $\mathcal{E}_{1,0}(\Lambda)$, and

$$\|A\|_{\mathcal{E}_{1,0}(\Lambda)} \leq \left( \frac{3^d r}{r - d(1 - 1/p)} \right)^{1 - 1/p} \|A\|_{\mathcal{E}_{p,r}(\Lambda)} \text{ for all } A \in \mathcal{E}_{p,r}(\Lambda).$$  

(9)

4. The BGS algebra $\mathcal{E}_{p,r}(\Lambda)$ is a Banach algebra, and there exists a positive constant $C_1$ such that

$$\|AB\|_{\mathcal{E}_{p,r}(\Lambda)} \leq C_1 \|A\|_{\mathcal{E}_{p,r}(\Lambda)}\|B\|_{\mathcal{E}_{p,r}(\Lambda)} \text{ for all } A, B \in \mathcal{E}_{p,r}(\Lambda).$$  

(10)

5. A matrix $A$ in $\mathcal{E}_{p,r}(\Lambda)$ is well approximated by its truncated matrix $A_N$, $N \geq 1$, in the norm $\| \cdot \|_{\mathcal{E}_{1,0}(\Lambda)}$, and

$$\|A - A_N\|_{\mathcal{E}_{1,0}(\Lambda)} \leq \|A\|_{\mathcal{E}_{p,r}(\Lambda)} \times \left( \frac{d(1-1/p)}{r-d(1-1/p)} \right)^{1-1/p} N^{-r+d(1-1/p)}$$  

if $p \neq 1,$

$$N^{-r}$$  

if $p = 1.$  

(11)

**Proof.**

(i) and (ii). Observing that for $\lambda \in \Lambda$,

$$\sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |a(\lambda, \lambda')| \leq R(\Lambda) \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')| \chi_{k+[0,1]^d}(\lambda - \lambda')$$  

(12)

and

$$\sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |a(\lambda, \lambda')| \leq 2R(\Lambda) \sum_{k \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')| \chi_{k+[0,1]^d}(\lambda - \lambda'),$$  

(13)

which imply (7) in (i). Combining (5) and (7), one can get (8) in (ii).

(iii) and (v). By direct computation, we obtain (iii) and (v).
(iv). Let \(1 \leq p < \infty\) and \(r > d(1 - 1/p)\), and take the matrices \(A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} \in \mathcal{C}_{p, r}(\Lambda)\) and \(B = (b(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} \in \mathcal{C}_{r, p}(\Lambda)\). Then by the fact that \(|a + b|^r \leq 2^r(|a|^r + |b|^r)\) for \(a, b \in \mathbb{R}\), we have

\[
\|AB\|_{\mathcal{C}_{p, r}(\Lambda)} \leq 2^r \left( \sum_{\ell \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} \left\{ \sum_{\lambda'' \in \Lambda} |a(\lambda, \lambda'')(1 + |\lambda - \lambda''|)\|b(\lambda'', \lambda')\| \right\}^p \chi_{k+0,1}^d(\lambda - \lambda') \right)^{1/p} \\
+ 2^r \left( \sum_{\ell \in \mathbb{Z}^d} \sup_{\lambda, \lambda' \in \Lambda} \left\{ \sum_{\lambda'' \in \Lambda} |a(\lambda, \lambda'')(1 + |\lambda - \lambda''|)\|b(\lambda'', \lambda')\| \right\}^p \chi_{k+0,1}^d(\lambda - \lambda') \right)^{1/p} \\
=: J_1 + J_2. \tag{14}
\]

Observing from (12) that

\[
J_1/2^r \leq \left( \sum_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')(1 + |\lambda - \lambda|)\|b(\lambda', \lambda)\| \right)^{1/p} \chi_{k+\ell, -1}^d(\lambda - \lambda) \cdot |b(\lambda', \lambda')\|^{1/p} \chi_{k+\ell, 1}^d(\lambda - \lambda') \\
\leq R(\Lambda) \| A \|_{\mathcal{C}_{p, r}(\Lambda)} \| B \|_{\mathcal{C}_{r, p}(\Lambda)},
\]

and similarly \(J_2/2^r \leq R(\Lambda) \| A \|_{\mathcal{C}_{p, r}(\Lambda)} \| B \|_{\mathcal{C}_{r, p}(\Lambda)},\) these together with (14) and (9) in (iii) imply (10) with \(C_1 = 2^{r+1} R(\Lambda) \left( \frac{d^q \cdot \rho}{r - d(1 - 1/p)} \right)^{1-1/p} \) for \(1 \leq p < \infty\) and \(r > d(1 - 1/p).\)

For \(p = \infty\), we have

\[
\|AB\|_{\mathcal{C}_{\infty, r}(\Lambda)} \leq 2^r \left( \sum_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')(1 + |\lambda - \lambda'|)\|b(\lambda', \lambda)\| \right) \\
+ 2^r \left( \sum_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')(1 + |\lambda - \lambda'|)\|b(\lambda', \lambda)\| \right) \\
\leq 2^r \left( \| B \|_{\mathcal{B}(\ell^q)} \| A \|_{\mathcal{C}_{\infty, r}(\Lambda)} + \| A \|_{\mathcal{B}(\ell^q)} \| B \|_{\mathcal{C}_{\infty, r}(\Lambda)} \right). \tag{15}
\]

The desired result (10) follows from (7) and (15) for \(p = \infty.\) \(\square\)

Let \(h(t) := \min(\max(2 - |t|, 0), 1)\) be the trapezoidal-shaped function. The function \(h\) is Lipschitz continuous.

For \(1 \leq q \leq \infty,\) a positive integer \(N\) and \(A \in \mathcal{B}(\ell^q(\Lambda)),\) define localization operators \(\Psi_i^N, \chi_i^N\) and commutators \([\Psi_i^N, A], i \in \mathbb{Z}^d,\) by

\[
\Psi_i^N c := (h(\lambda/N - i) c(\lambda))_{\lambda \in \Lambda} \tag{16}
\]

\[
\chi_i^N c := (\chi_{(0,1)}(|i - \lambda|/N) c(\lambda))_{\lambda \in \Lambda} \tag{17}
\]

and

\[
[\Psi_i^N, A] c = \Psi_i^N Ac - A \Psi_i^N c \quad \text{for} \quad c := (c(\lambda))_{\lambda \in \Lambda} \in \ell^q(\Lambda),
\]

where for a set \(I, \chi_I(\cdot)\) denotes the characteristic function on \(I.\)

In the next theorem, we show the norm-controlled inversion of a Banach algebra \(\| A \|_{\mathcal{C}_{p, r}(\Lambda)} \) in \(\mathcal{B}(\ell^q(\Lambda))\) which is not a symmetric \(*\)-Banach algebra.

**Theorem 2.** Let \(1 \leq p, q < \infty, r > d(1 - 1/p),\) \(\Lambda\) be a relatively-separated subset of \(\mathbb{R}^d\) satisfying (1), and let \(A \in \mathcal{C}_{p, r}(\Lambda)\) be invertible in \(\mathcal{B}(\ell^q(\Lambda)).\) Then there exists an absolute constant \(C,\) independent of \(A,\) such that

\[
\|A^{-1}\|_{\mathcal{C}_{p, r}(\Lambda)} \leq C \|A^{-1}\|_{\mathcal{B}(\ell^q)} \| A^{-1}\|_{\mathcal{B}(\ell^q)} \| A \|_{\mathcal{C}_{p, r}(\Lambda)} \left( \frac{d^q \cdot \rho}{r - d(1 - 1/p)} \right)^{(d/p + r)/\min(1, r - d(1 - 1/p))} \\
\times \begin{cases} 
1 & \text{if } r \neq d(1 - 1/p) + 1, \\
(\ln(\|A^{-1}\|_{\mathcal{B}(\ell^q)} \| A \|_{\mathcal{C}_{p, r}(\Lambda)}))^{(d/p + r)(1-1/p)} & \text{if } r = d(1 - 1/p) + 1.
\end{cases} \tag{18}
\]
**Proof.** We follow the arguments in [18]. Let $1 \leq q < \infty$ and $1 < p < \infty$. When $q = \infty$, $p = 1$ or $p = \infty$, we can follow the same proof. Write $p' = p/(p-1)$. Define the linear operator $\Phi_N$ on $\ell^q(\Lambda)$ by

$$
\Phi_N c := \left( H(\lambda/N) c(\lambda) \right)_{\lambda \in \Lambda}
$$

for $c := (c(\lambda))_{\lambda \in \Lambda} \in \ell^q(\Lambda)$, where $H(t) = (\sum_{i \in \mathbb{Z}^d} h(t-i) \mathbf{1}_1, t \in \mathbb{R}^d$. By the invertibility of $A$, we have that

$$
\|A^{-1}\|_{B(\ell^q)} \|\Psi_N^i c\|_q \leq \|\Psi_N^i Ac\|_q + \|\Psi_N^i, A\|\|c\|_q
$$

\begin{align*}
& \leq \|\Psi_N^i Ac\|_q + \|\chi_{\Lambda_N}^N [\Psi_N^i, A]\|\|c\|_q + (I - \chi_{\Lambda_N}^N)A \chi_{\Lambda_N}^N \Psi_N^i c\|_q \\
& \leq \|\Psi_N^i Ac\|_q + \sum_{j \in \mathbb{Z}^d} \|\chi_{\Lambda_N}^N [\Psi_N^i, A]\| \|\Phi_N \Psi_N^j c\|_q + \|A - A_N\|\|c\|_q \|\Psi_N^i c\|_q.
\end{align*}

Choose $N$ so large that

$$
N \geq \left( \frac{2}{r p'/d} \right)^{1/p'} \left( \left( \frac{1}{r} \|\Psi_N^i, A\|\right)^{1/(r-d/p')}
$$

It follows from (11), (19) and (20) that

$$
\|A^{-1}\|_{B(\ell^q)} \|\Psi_N^i c\|_q \leq 2 \|\Psi_N^i Ac\|_q + 2 \sum_{j \in \mathbb{Z}^d} \|\chi_{\Lambda_N}^N [\Psi_N^i, A]\| \|\Phi_N \Psi_N^j c\|_q.
$$

For $i, j \in \mathbb{Z}^d$ with $|i - j| \leq 10$, we obtain from Lipschitz property of $h$ that

$$
\|\chi_{\Lambda_N}^N [\Psi_N^i, A]\| \|\Phi_N \Psi_N^j c\|_q \leq \left( \sum_{\lambda \in \Lambda} \chi_{\Lambda_N}^N(\lambda) a(\lambda, \lambda') |h(\lambda/N - i) - h(\lambda'/N - i)| |c(\lambda')| \right)^{1/q}
$$

\begin{align*}
& \leq \left( \sum_{\lambda \in \Lambda} \chi_{\Lambda_N}^N(\lambda) \min(|\lambda - \lambda'|/N, 1) a(\lambda, \lambda') |h(\lambda'/N - j)| |c(\lambda')| \right)^{1/q} \\
& \leq R(\Lambda) \left( \sum_{|k| \leq 15N} \min((|k| + 1)/N, 1) A_k \right) \|\Psi_N^j c\|_q.
\end{align*}

For $i, j \in \mathbb{Z}^d$ with $|i - j| > 10$, we have that

$$
\|\chi_{\Lambda_N}^N [\Psi_N^i, A]\| \|\Phi_N \Psi_N^j c\|_q \leq \left( \sum_{\lambda \in \Lambda} \chi_{\Lambda_N}^N(\lambda) a(\lambda, \lambda') |h(\lambda'/N - j)| |c(\lambda')| \right)^{1/q}
$$

\begin{align*}
& \leq \left( \sum_{\lambda \in \Lambda} \chi_{\Lambda_N}^N(\lambda) \sum_{|\lambda - \lambda'| \leq |i - j| + 5N} a(\lambda, \lambda') |h(\lambda'/N - j)| |c(\lambda')| \right)^{1/q} \\
& \leq 2R(\Lambda) \left( \sum_{|k| \leq 15N} A_k \right) \|\Psi_N^j c\|_q.
\end{align*}

where

$$
A_k = \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')| \chi_{k+[0,1]^d}(\lambda - \lambda').
$$

We define a function

$$
\overline{V}_{A,N}(i) = 2R(\Lambda) \times \left\{ \begin{array}{ll}
\sum_{|k| \leq 15N} \min((|k| + 1)/N, 1) A_k & \text{if } |i| \leq 10, \\
\sum_{|k| \leq 15N} A_k & \text{if } |i| > 10.
\end{array} \right.
$$

We have from (21), (22), (23) and (24) that

$$
\|\Psi_N^i c\|_q \leq 2 \|A^{-1}\|_{B(\ell^q)} \|\Psi_N^i Ac\|_q + \sum_{j \in \mathbb{Z}^d} \overline{V}_{A,N}(i, j) \|\Psi_N^j c\|_q.
$$
where \( V_{A,N} := (V_{A,N}(i,j))_{i,j \in \mathbb{Z}^d} \) and \( V_{A,N}(i,j) = 2 \| A^{-1} \|_{\mathcal{B}(\ell^q)} \| \tilde{V}_{A,N}(i-j) \|. \) We write
\[ V_{A,N} = ((V_{A,N})^\ell (i,j))_{i,j \in \mathbb{Z}^d}. \] (26)
We apply (25) repeatedly to have that
\[ \| \Psi_N c \|_q \leq 2 \| A^{-1} \|_{\mathcal{B}(\ell^q)} \| \Psi_N A c \|_q + 2 \| A^{-1} \|_{\mathcal{B}(\ell^q)} \sum_{\ell=1}^{n-1} \sum_{j \in \mathbb{Z}^d} \| \tilde{V}_{A,N}(i,j) \| \| \Psi_N A c \|_q \]
\[ + \sum_{j \in \mathbb{Z}^d} \| \tilde{V}_{A,N}(i,j) \| \| \Psi_N c \|_q. \] (27)
Note that
\[ \| A \|_{\mathcal{E}_{p,r}(\Lambda)} \leq 2^r \left( \sum_{k \in \mathbb{Z}^d} A_k^p (1 + |k|)^q \right)^{1/p} \] (28)
and
\[ \left( \sum_{k \in \mathbb{Z}^d} A_k^p (1 + |k|)^q \right)^{1/p} \leq 2^r \| A \|_{\mathcal{E}_{p,r}(\Lambda)}. \] (29)
Observing from (29) that
\[ \left( \sum_{|i| > 10} (\tilde{V}_{A,N}(i))^p (1 + |i|)^q \right)^{1/p} \leq 2 R(\Lambda)(10N)^{(d-1)/p} A N^{-r} \left( \sum_{|i| > 10} \sum_{|j| \leq 5N} A_k^p (1 + |k|)^q \right)^{1/p} \leq 10^d 2^{3r+1} R(\Lambda) A N^{-r+d/p} \| A \|_{\mathcal{E}_{p,r}(\Lambda)} \] (30)
and
\[ \left( \sum_{|i| \leq 10} (\tilde{V}_{A,N}(i))^p (1 + |i|)^q \right)^{1/p} \leq 2 R(\Lambda) \left( \sum_{|i| \leq 10} (1 + |i|)^q \left( \sum_{|k| \leq 15N} \min((|k|+1)/N,1) A_k^p \right)^{1/p} \right)^{1/p} \leq 2 R(\Lambda) 11^{r+d/p} N^{-1} \sum_{|k| \leq N-1} \sum_{N \leq |k| \leq 15N} A_k \]
\[ \leq R(\Lambda) 2^{r+2} 11^{r+d/p} \| A \|_{\mathcal{E}_{p,r}(\Lambda)} \]
\[ \times \left\{ \left( \frac{d}{rp^d} \right)^{1/p} + \left( \frac{2d+rp^d-d}{r^d-p^d} \right)^{1/p} \right\} N^{-\min(1,r-d/p')} \]
\[ \left( \frac{2d}{r^d} \right)^{1/p} N^{-1}(\ln(N+1))^{1/p'} \]
\[ \text{if } r \neq d/p' + 1, \]
\[ \text{if } r = d/p' + 1, \] (31)
we have that
\[ \| V_{A,N} \|_{\mathcal{E}_{p,r}((\mathbb{Z}^d))} \leq D_{d,p,r} \| A^{-1} \|_{\mathcal{B}(\ell^q)} \| A \|_{\mathcal{E}_{p,r}(\Lambda)} \times \left\{ \begin{array}{ll} N^{-\min(1,r-d/p')} \text{ if } r \neq d/p' + 1, \\ N^{-1}(\ln(N+1))^{1/p'} \text{ if } r = d/p' + 1, \end{array} \right. \] (32)
where
\[ D_{d,p,r} = 2^{3r+1} 11^{d+r} R(\Lambda) \times \left\{ \begin{array}{ll} \left( \frac{d}{r^d-p^d} \right)^{1/p} + \left( \frac{2d+rp^d-d}{r^d-p^d} \right)^{1/p} \text{ if } r \neq d/p' + 1, \\ \left( \frac{2d}{r^d} \right)^{1/p} \text{ if } r = d/p' + 1. \end{array} \right. \] (33)
where \( C_1 \) is the constant in (10). Then \( N_0 \) satisfies (20), so (21) and (27) hold. From (33) there exist absolute constants \( C_2, C_3 \) such that for \( r \neq d/p' + 1, \)
\[
N_0 \leq C_2(\| A^{-1} \|_{\mathcal{B}(\ell^p)} \| A \|_{\mathcal{B}(\ell^q)})(1/r-d/p')^{1/\min(1,r-d/p')}
\]
and for \( r = d/p' + 1 \)
\[
N_0 \leq C_3(\| A^{-1} \|_{\mathcal{B}(\ell^p)} \| A \|_{\mathcal{B}(\ell^q)})(\ln(\| A^{-1} \|_{\mathcal{B}(\ell^p)} \| A \|_{\mathcal{B}(\ell^q)}))^{1/p'}.
\]
It follows from (10), (32) and (33) that for \( n \in \mathbb{N}, \)
\[
\| V^N_{A,N_0} \|_{e_{p,r}(\mathbb{Z}^d)} \leq 2^{-n}.
\]
This combining with Proposition 1 (ii) and (iii) implies that
\[
\lim_{n \to \infty} \left( \sum_{j \in \mathbb{Z}^d} V^N_{A,N_0}(i,j) \| \Psi^N_{j} c \|_q \right)_{i \in \mathbb{Z}^d} = 0,
\]
so taking \( n \to \infty \) in (27), we have that
\[
\| \Psi^N_i c \|_q \leq \| A^{-1} \|_{\mathcal{B}(\ell^q)} \sum_{j \in \mathbb{Z}^d} W_{A,N_0}(i,j) \| \Psi^N_j A c \|_q,
\]
where \( W_{A,N_0}(i,j) = 2I + 2\sum_{\ell=1}^{\infty} V^\ell_{A,N_0}(i,j) \). It follows from (36) that
\[
\| W_{A,N_0} \|_{e_{p,r}(\mathbb{Z}^d)} \leq 4.
\]
We define an infinite matrix \( H_{A,N_0} := (H_{A,N_0}(\lambda,\lambda'))_{\lambda,\lambda' \in \Lambda}, \) where
\[
H_{A,N_0}(\lambda,\lambda') = \sum_{|mN_0-\lambda| \leq N_0} \sum_{|nN_0-\lambda'| \leq 2N_0} W_{A,N_0}(m,n).
\]
Since for \( k \in \mathbb{Z}^d, \)
\[
\sup_{\lambda,\lambda' \in \Lambda} \chi_{k+\{0,1\}^d}(\lambda-\lambda') \sum_{|mN_0-\lambda| \leq N_0} \sum_{|nN_0-\lambda'| \leq 2N_0} W_{A,N_0}(m,n) \leq \sum_{\epsilon=-4,-3,\ldots,4} \sum_{m-n=[k/N_0]+\epsilon} W_{A,N_0}(m,n),
\]
we have from (38) that
\[
\| H_{A,N_0} \|_{e_{p,r}(\Lambda)} \leq 10^{d/p+r+d+1} N_0^{d/p+r}.
\]
Let \( A^{-1} := (a^{-1}(\lambda,\lambda'))_{\lambda,\lambda' \in \Lambda}, a^{-1}_\lambda := (a^{-1}(\lambda,\lambda'))_{\lambda \in \Lambda} \) and \( a^{-1}_\lambda(\lambda) = a^{-1}(\lambda,\lambda'). \) Replace \( c \) by \( a^{-1}_\lambda \) in (37) to get that for \( \lambda \in \Lambda \) and \( m \in \mathbb{Z}^d \) with \( |mN_0-\lambda| \leq N_0 \)
\[
|a^{-1}(\lambda,\lambda')| \leq \| \Psi^N_m a^{-1}_\lambda \|_q \leq \| A^{-1} \|_{\mathcal{B}(\ell^q)} \sum_{|mN_0-\lambda| \leq N_0} \sum_{|nN_0-\lambda'| \leq 2N_0} W_{A,N_0}(m,n) \| \Psi^N_n A a^{-1}_\lambda \|_q \leq \| A^{-1} \|_{\mathcal{B}(\ell^q)} H_{A,N_0}(\lambda,\lambda').
\]
It follows from (40) and (41) that
\[
\| A^{-1} \|_{e_{p,r}(\Lambda)} \leq \| A^{-1} \|_{\mathcal{B}(\ell^q)} \| H_{A,N_0} \|_{e_{p,r}(\mathbb{Z}^d)} \leq 10^{d/p+r+d+1} N_0^{d/p+r} \| A^{-1} \|_{\mathcal{B}(\ell^q)}.
\]
From (34), (35) and (42), (18) holds.

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References

[1] A. G. Baskakov, “Wiener’s theorem and asymptotic estimates for elements of inverse matrices”, Funkts. Anal. Prilozh. 24 (1990), no. 3, p. 64-65, translation in Funkt. Anal. Appl. 24 (1990), no. 3, p. 222-224.
[2] ———, “Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis”, Sib. Mat. Zh. 38 (1997), no. 1, p. 14-28, translation in Sib. Math. J. 38 (1997), no. 1, p. 10-22.
[3] L. H. Brandenburg, “On identifying the maximal ideals in Banach Algebras”, J. Math. Anal. Appl. 50 (1975), p. 489-510.
[4] K. Gröchenig, “Wiener’s lemma: theme and variations, an introduction to spectral invariance and its applications”, in Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis, Applied and Numerical Harmonic Analysis, Birkhäuser, 2010.
[5] K. Gröchenig, A. Klotz, “Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices”, Constr. Approx. 32 (2010), no. 3, p. 429-466.
[6] ———, “Norm-controlled inversion in smooth Banach algebras. I”, J. Lond. Math. Soc. 88 (2013), no. 1, p. 49-64.
[7] ———, “Norm-controlled inversion in smooth Banach algebras. II”, Math. Nachr. 287 (2014), no. 8-9, p. 917-937.
[8] S. Jaffard, “Propriétés des matrices “bien localisées” près de leur diagonale et quelques applications”, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 7 (1990), no. 5, p. 461-476.
[9] N. Nikolski, “In search of the invisible spectrum”, Ann. Inst. Fourier 49 (1999), no. 6, p. 1925-1966.
[10] K. S. Rim, C. E. Shin, Q. Sun, “Stability of localized integral operators on weighted $L^p$ spaces”, Numer. Funct. Anal. Optim. 33 (2012), no. 7-9, p. 1166-1193.
[11] E. Samei, V. Shepelska, “Norm-controlled inversion in weighted convolution algebras”, J. Fourier Anal. Appl. 25 (2019), no. 6, p. 3018-3044.
[12] C. E. Shin, Q. Sun, “Differential subalgebras and norm-controlled inversion”, submitted, https://arxiv.org/abs/1911.08679, 2019.
[13] ———, “Polynomial control on stability”, J. Funct. Anal. 276 (2019), no. 1, p. 148-182.
[14] J. Sjöstrand, “Wiener type algebra of pseudodifferential operators”, Sémin. Équ. Dériv. Partielles 1994-1995 (1995), article ID 4 (19 pages).
[15] Q. Sun, “Wiener’s lemma for infinite matrices with polynomial off-diagonal decay”, C. R. Math. Acad. Sci. Paris 340 (2005), p. 567-570.
[16] ———, “Wiener’s lemma for infinite matrices”, Trans. Am. Math. Soc. 359 (2007), no. 7, p. 3099-3123.
[17] ———, “Wiener’s lemma for localized integral operators”, Appl. Comput. Harmon. Anal. 25 (2008), no. 2, p. 148-167.
[18] ———, “Wiener’s lemma for infinite matrices. II.”, Constr. Approx. 34 (2011), no. 2, p. 209-235.
[19] N. Wiener, “Tauberian theorem”, Ann. Math. 33 (1932), p. 1-100.