Triplicate Relativistic Fermions and Parity Anomaly in 2+1 Dimensions

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Attention is drawn to the triplicate relativistic fermion (TRF) and the (2 + 1) dimensional parity anomaly. The Hall conductivity of TRFs in continuum model and lattice model have been studied respectively. We find that each single TRF gives a contribution of quantized Hall conductance \( e^2/h \), and then obtain the effective Chern-Simons theory of TRFs interacting with an (auxiliary) \( U(1) \) gauge field. We also find that the Nielsen-Ninomiya theorem for Dirac fermions is broken down for TRFs, so there may exist single TRF in a lattice model. In addition, our results indicate the quantum anomaly in odd-dimensional spinor space.

I. INTRODUCTION

Dirac and Weyl fermions are massless relativistic particles which were predicted firstly in the context of high-energy physics. In the standard model of particle physics, they describe the matter fields. In condensed matter physics, Dirac and Weyl fermions have attracted extensive attention because the low-energy excitations around some band degenerate point display a relativistic dispersion relation and thus are described by Dirac or Weyl equation.

Graphene, topological states of matter and d-wave superconductor are three typical systems for realizing relativistic fermions in condensed matter physics. Graphene is a semi-metal at half filling, in which the conduction band and valence band cross linearly at the Dirac points, hallmarks of relativistic particles. The topological insulator has gapped bulk spectrum but hosts an odd number of gapless Dirac fermions on surface. Under the protection by the bulk topology, these gapless surface states are robust. The low-energy excitations in d-wave superconductor are node Dirac fermions. Besides, another important class of topological phases are topological semi-metals. In Dirac semi-metal phase, the Dirac fermions are characterized by four-fold degenerate linear band crossing points; while in Weyl semi-metal phase, the Weyl fermions are characterized by non-degenerate and separated band-touching nodes. Importantly, the relativistic Dirac and Weyl fermions in solid materials are responsible for most of novel phenomena such as quantum anomalous Hall effect, unusual transport features, i.e., chiral magnetic effect, chiral anomaly and axion electrodynamics.

Unlike the relativistic fermions in quantum field theory, the low-energy excitations in solid refers to pseudospin that depends on the geometry of lattice, they are called pseudospin fermions. The fermions in solid preserve space group rather than Poincare group, so rich fermions may exist in solid universe. Started from the discovery of the celebrated Dirac fermions in graphene, the unconventional quasiparticle that have no analog in quantum field theory have stimulated an enormous interest. Recently, a new type of fermion in three dimensions was predicted in topological semi-metal called unconventional fermions with highly degenerate points, for example the triply degenerate fermion in molybdenum phosphide material. The highly degenerate fermions have also been studied in many families, i.e., pseudospin-one fermions in optical lattice, dice lattice or Monolayer MgC\(_2\)-type lattice, and photonic crystal materials. The unconventional fermions show some extraordinary properties, such as super-Klein tunneling effect and bosonic topological phases. More, the pairing between higher spin fermions was suggested to explain the unconventional superconducting of half-Heusler compounds. For example, the strong spin-orbit coupled half-Heusler compounds, in particular YPtBi and LuPtBi, are described by 3/2 spin-vector-coupling, the Cooper pair between spin-3/2 fermions yields the pairing term for superconductor.

It’s well known that the theory of a relativistic Dirac and Weyl fermions interacting with a fluctuating gauge field must break some symmetries, i.e., parity, time-reversal and chirality, this is known as quantum anomaly. Anomaly is a fundamental phenomenon in quantum field theory and it leads to some important physical consequences. The (3 + 1)-D axial (chiral) anomaly closely relates to the topological magnetoelectric effect of topological insulator. On the contrary, it relates to chiral magnetic effect and strong CP problem in quantum chromodynamics. The (2 + 1)-D parity anomaly has close relation with quantum Hall effect. The (1+1)-D chiral anomaly will lead to fractional charged soliton. All these anomaly arise from the breaking of the symmetry by quantum effect and the background topology of gauge field.

The triply degenerate relativistic fermions (here we call it triplicate relativistic fermion (TRF)), show some distinctive features with respect to graphene fermions. Many literatures have pointed out the quantum Hall effect of TRFs in two-dimensional materials or in cold-atom systems, which yields a Hall conductivity of \( e^2/h \). By now, to our knowledge, wether the quantum theory of TRFs interacting with a gauge field exhibits anomaly.
or not remains unclear. The quantum anomaly, especially the parity anomaly, relates closely to the quantum anomalous Hall effect. In this paper, we pay our attention to the Hall conductivity of the TRFs and the 

The remainder of this paper is organized as follows: in Sec.II we describe the effective relativistic model of TRFs and then study the Landau quantization for massless TRFs and massive TRFs. In Sec.III, we calculate the Hall conductivity of single TRF and the Chern invariant based on general lattice model of TRF. The relationship between the Chern number and the winding number is discussed in this section. In Sec.IV, we present two typical lattice models of TRF as examples to illustrate the exotic properties of TRFs. In Sec.V, we derive the effective theory of TRFs and study the parity of the theory with TRF interacting with electromagnetic field (Abelian U(1) gauge field). Finally, we draw our conclusions in Sec.VI.

II. LANDAU LEVELS FOR TRIPlicate RELATIVISTIC FERMIONS

The two-dimensional electrons subjected to a magnetic field exhibit some fantastic properties, i.e., integer quantum Hall effect and parity anomaly. These fundamental phenomenon take place even in the noninteracting limit. In non-relativistic case, the energy levels of electrons depend linearly on the magnetic field, these levels are called Landau levels (LLs). In this section, we shall solve the LLs for both the massless and massive TRFs at two spatial dimensions, and we will show the special zeroth LL in the same manner as relativistic Dirac fermion.

The gapless Dirac cone can appear in some system bulk (i.e., graphene) or on the surface of topological insulator. So, the low energy effect theory is described by Dirac equation with specific chirality, with the Hamiltonian can be expressed as

\[ H = \sigma \cdot \mathbf{k}, \]

where \( \mathbf{k} \) is the quasimomentum and \( \sigma \) is the Pauli matrices. Further, the Hamiltonian for massive Dirac fermions can be generally written as

\[ H = \sigma \cdot \mathbf{k} + \gamma \mathbf{M}. \]

Similar to the relativistic Dirac fermions, the TRFs can be described by the model

\[ H = v_F \mathbf{S} \cdot \mathbf{k} + S_z M \]

where \( S = (S_x, S_y, S_z) \) is the matrix representation of \( S = 1 \). Here, \( M = 0 \) describes massless TRF and \( M \neq 0 \) describes massive one.

In this paper, we focus on following two types of models. For the first type of model is characterized by the following three matrices,

\[
\begin{align*}
\text{type-I TRF} : & \quad S_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
S_y & = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix}.
\end{align*}
\]

These matrices form the specific foundation representation of SU(3) group, named Gell-Mann matrices. Such type of model have been proposed to realize boson and fermion topological states in cold atom and photon quantum Hall effect. The dispersive energy spectrum is easily obtained as \( E = \sqrt{k^2 + M^2}. \) As a result, the system is gapped for non-vanishing \( M \), so we call it (the coefficient of \( S_z \)) mass term. For the second type of model is characterized by the following three matrices,

\[
\begin{align*}
\text{type-II TRF} : & \quad S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
S_y & = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{align*}
\]

The type-II matrices are standard matrix representation of spin \( S = 1 \).

Note that both types satisfy the SU(2) algebra relation but does not form a Clifford algebra. The type-I model supports only single TRF in the lattice (or see Appendix B), as Qi-Wu-Zhang model of Chern insulator. Whereas, type-II model supports two TRFs, as Haldane model of anomalous quantum Hall insulator. These results relate to Nielsen-Ninomiya theorem and important for understanding TRFs. The detailed differences between the type-I and type-II TRF will be shown in Sec.IV.

A. LLs for massless TRF

We have shown the general form of TRF. Now we add a perpendicular magnetic field \( B = \hat{z} \), which can be implemented by replacing the canonical momentum with a gauge-invariant quantity,

\[ p \rightarrow \Pi = p - \epsilon \mathbf{A}, \]

where \( \mathbf{A} = (-By, 0, 0) \). The procedure for a uniform magnetic field lead to Peierls substitution on the lattice. One can prove that \( [\Pi_x, \Pi_y] = -i\hbar^2/2B \) with \( l_B = \sqrt{\hbar/eB} \) to be the magnetic length. To solve the energy level, it’s convenient to introduce a pair of ladder operators \( a = \frac{i}{\sqrt{2}B} (\Pi_x - i\Pi_y) \) and \( a^\dagger = \frac{i}{\sqrt{2}B} (\Pi_x + i\Pi_y) \). For type-I TRF, the Hamiltonian becomes

\[ H = \frac{\hbar v_F}{\sqrt{2}B} (a^\dagger S_- + a S_+) + MS_z, \]
where $S_+ = S_x + iS_y$, $S_- = S_x - iS_y$. Without $M$, the LLs are found to be

$$E_{n\pm} = \pm \sqrt{2n + 1}v_F/l_B, \; n \geq 1,$$

(6)

that is in contrast to the case of relativistic Dirac fermion $E_{n\pm} \sim \sqrt{2n}$. The corresponding wave functions are given by

$$\psi_n^\pm = \left( \begin{array}{c} \pm \sqrt{2n + 1}\phi_n \\ \sqrt{\frac{n+1}{2}}\phi_{n+1} + i\sqrt{\frac{n}{2}}\phi_{n-1} \\ i\sqrt{\frac{n+1}{2}}\phi_{n+1} - i\sqrt{\frac{n}{2}}\phi_{n-1} \end{array} \right),$$

(7)

where $\phi_n$ denotes the wave function of harmonic oscillator with frequency $\omega_c = v_F/l_B$. Besides, there is a highly degenerate flat band lying exactly at $E_n = 0$, and the wave functions read

$$\psi_n^0 = \left( \begin{array}{c} 0 \\ \sqrt{\frac{n+1}{2}}\phi_{n+1} - \frac{1}{\sqrt{2}}\phi_{n-1} \\ i\sqrt{\frac{n+1}{2}}\phi_{n+1} + \frac{1}{\sqrt{2}}\phi_{n-1} \end{array} \right).$$

(8)

This is right the zero energy flat band under zero magnetic field and does not contribute the Hall conductance. In addition, another zero energy level becomes interest, of which the wave function is $\psi_n = (0, \phi_n, -i\phi_n)^T$.

By using similar approach, the Hamiltonian for type-II TRF becomes

$$H = \frac{\hbar v_F}{\sqrt{2}l_B}(a^+S_- + aS_+) + MS_z.$$  

(9)

Similar solution can be found in Ref. 38, here we present it for completeness. For $M = 0$, the level spectrum are also given by Eq.(6) and the wave functions read

$$\psi_n^\pm = \left( \begin{array}{c} \pm \sqrt{2n+1}\phi_n \\ \sqrt{n+1}\phi_{n+1} \end{array} \right), \; n \geq 1.$$  

(10)

The zero energy flat band that is independent of magnetic field is given by

$$\psi_n^0 = \left( \begin{array}{c} 0 \\ \sqrt{n+1}\phi_{n+1} \\ -\sqrt{n}\phi_{n+1} \end{array} \right), \; n \geq 1.$$  

(11)

Finally, the zero-th LLs is given by $\psi_0 = (0, 0, \phi_0)^T$.

### B. LLs for massive TRF

In this subsection, we solve the LLs for massive TRFs. Let’s consider a momentum dependence mass term $M \rightarrow M(k) = M - (\Pi_x^2 + \Pi_y^2)$, in terms of ladder operators, it reads

$$M(k) = M - 2\hbar^2/l_B^2(a^+a + 1/2).$$  

(12)

![FIG. 1: (Color online) The Landau-level of TRFs versus magnetic field $B$. (a) for massless fermion and (b) for massive fermions. In the massive case (b), the red level shows the zeroth Landau level. In this plot, we set the units $\hbar = e = 1$. The LLs can be obtained in the same manner as massless case. After some straightforward algebra, we find that the energy spectrum for $n \geq 1$ come into pairs, as depicted in Fig 1(b). Special care need to be taken for $n = 0$ in the massive case, which yields

$$E_{n=0} = -M + \hbar^2/l_B^2.$$  

(13)

This result is valid for both type-I and type-II TRFs. Because the zero-th Landau level depends on the magnetic field and increases as the magnetic field, it crosses Fermi energy. Therefore, the zero-th Landau level provides an one-dimensional chiral mode, which is hallmark of the nontrivial topological edge states. Taking the continuum limit, the one-dimensional chiral mode can be described by the effective Lagrangian of free fermion:

$$\mathcal{L} = \frac{i}{2} \int dx \psi (\partial_t + \partial_x)\psi,$$  

(14)

where $\psi$ denotes single fermion field. Under parity transformation, the chiral mode lead to violation of parity. So, the TRFs are expected to have parity anomaly, and we will study this explicitly in Sec.V.

### III. TRANSPORT AND TOPOLOGICAL PROPERTIES

#### A. Hall conductivity for single TRF

The parity anomaly relates closely to the quantum anomalous Hall effect. In this section, we consider the transport properties and quantum Hall response of a single free relativistic fermions. The low-energy theory is described by a single cone, which allows us to study it analytically. In general, the dc response to external field
can be obtained in terms of standard Kubo formula:\(^9\):

\[
\sigma_{ij}(\omega) = \lim_{\omega \to 0} \frac{i}{\omega} Q_{ij}(i\omega_n \to \omega + i0^+), \tag{15}
\]

\[
Q_{ij}(i\nu_n) = \frac{1}{V\beta} \sum_{k,m} \text{Tr}[J_i(k) \times G(k,i(\omega_n + \nu_n))]\tag{16}
\]

where \(\text{Tr}\) means trace over spinor indices, \(\beta = 1/T\) is the inverse temperature and \(V\) is the area of the system. The dc current is \(J_i(k) = \partial h(k)/\partial \mathbf{k}_i\), with \(i = x, y\). The single-body Green function is given by \(G^{-1}(k,i\omega_n) = i\omega_n - H(k)\), and \(\omega_n = (2m + 1)\pi/\beta\) is a Matsubara frequency.

\[
Q_{xy}(i\nu_n) = \frac{i}{V\beta} \sum_{k,m} \frac{i\omega_n(i\nu_n)^2 M + (i\omega_n)^2 i\nu_n M + i\nu_n k^2 M + i\nu_n M^3}{i\omega_n + i\nu_n - (k^2 + M^2)\left[(i\omega_n)^2 - (k^2 + M^2)\right]}, \tag{18}
\]

where the identity \(\text{Tr}(S_x S_y S_z) = i\) has been used. Only the antisymmetric part survive and all terms that are symmetric for \(x \leftrightarrow y\) had been neglected to reach this result. The Hall conductivities is obtained in the dc limit, say \(\nu \sim 0\). After analytic continuation \(i\nu_n \to \nu + i0^+\), then taking the limit \(\nu \to 0\), we obtain the final result

\[
\sigma_{xy} = \frac{i}{\nu \to 0} \frac{i}{\nu} Q_{ij}(\nu + i0^+) =\]

\[
\lim_{\nu \to 0} \frac{i}{\nu} \frac{i}{\nu} Q_{ij}(\nu + i0^+) = \frac{i}{\nu} \frac{i}{\nu} Q_{ij}(\nu + i0^+) =
\]

\[
i \int d\omega d^2k \frac{\omega^2 M + (k^2 + M^2)M}{(2\pi)^3 \omega^2 - (k^2 + M^2)^2}
\]

\[
= \frac{1}{2\pi} \frac{M}{|M|}. \tag{19}
\]

More details about this calculation are presented in Appendix A. Putting back the standard conductivity units, we have \(\sigma_{xy} = -\text{sgn}(M)\times e^2/h\), which differs from the result calculated from single Dirac cone. Here and below in this paper, we will set the units \(e = \hbar = 1\) such that \(e^2/h = 1/2\pi\).

\[\text{B. Consequence of flat band and Chern number}\]

In this section, we shall derive some relations for the Hall conductivity of TRFs on general lattice model. In order to make the results clearer, we begin with the review of Chern insulator. In Chern insulator, the Hall conductivity is always an integer (in units of \(e^2/h\)), named TKNN number or the first Chern number.\(^5\) Mathematically, the Chern number is an integral of Berry curvature corresponding to all filled bands. For a two-band model of general Chern insulator\(^9,53\), the first Chern number is right the winding number which measure the mapping from two-dimensional Brillouin zone (BZ) \(T^2\) to two-sphere \(S^2\). The equality between these values provide us an easy way to reach the topological invariant. However, we will see the Chern number may not equal to the winding number in topological insulator which supports TRFs.

A simple lattice Hamiltonian with TRFs can be generally written in momentum space as

\[
h(k) = \tilde{d}_i(k) \cdot \tilde{S}_i,
\]

where the vector \(\tilde{d}_i(k) = (d_x(k), d_y(k), d_z(k))\), \(\tilde{S}_i\) are three matrices given in Eq.\((2)\) and \((3)\). The Hamiltonian have two dispersive bands and a zero-energy flat band, which are given as \(E_{\pm} = \pm |\tilde{d}_i(k)|\), \(E_0 = 0\). If \(E_{\pm}\) is non-vanishing in the whole BZ, the system is an insulator when the fermi energy lies in the gap. To evaluate the Hall conductivity, we use the TKNN expression of Kubo formula\(^57\):

\[
\sigma^m_H = \frac{ie^2}{\hbar} \sum_{k} \sum_{E_m < E_F < E_n} \frac{J^x_{mn} J^y_{nm} - J^y_{mn} J^x_{nm}}{(E_m - E_n)^2}, \tag{21}
\]

where \(J^x_{mn} = \langle m|\partial h(k)/\partial \mathbf{k}_i|n\rangle\) is the current operator. The Hall response of the lowest occupied band is of interest. For type-I TRFs model, the normalized eigenstates are given by \(\phi^0_\pm = (id_x, -d_y, d_z)/d\), and

\[
\phi_\pm = \frac{1}{d \sqrt{(d^2 - d_z^2)}} \begin{pmatrix} \pm dd_x + id_yd_z \\ d_x^2 + d_z^2 \\ d_xd_y + id_zd_x \end{pmatrix}. \tag{22}
\]

where \(d \equiv |\tilde{d}_i(k)|\). Plugging the eigenstates into Eq.\((21)\), we find the highest band with \(E_n = E_+\) makes no contribution to the Hall conductivity when \(E_+ < E_F < 0\).
and the total Hall conductivity comes only from the zero-energy flat band. We obtain
\begin{equation}
\sigma_{xy} = \frac{e^2}{h} \int \frac{d^3k}{2\pi} \epsilon^{abc} \hat{d}_a \partial_k \hat{d}_b \partial_k \hat{d}_c, \tag{23}
\end{equation}
where \( \epsilon^{abc} \) is an antisymmetric tensor, with \( \hat{d}_a = d_a / d \). This result is astonishing, because it differs from the winding number from \( T^2 \) to \( S^2 \) which is given as
\begin{equation}
W[\vec{d}] = \int \frac{d^3k}{4\pi} \epsilon^{abc} \hat{d}_a \partial_k \hat{d}_b \partial_k \hat{d}_c. \tag{24}
\end{equation}
Since the dispersive bands are all gapped from the flat band, one can derive Eq.(23) from the method of differential geometry. The Berry gauge potential is given by \( A = -i(\partial_k \phi_k, \partial_k \phi_b) \). Here, the exterior differential operator has been used for the sake of brevity. Accordingly, the first Chern number can be expressed as
\begin{equation}
C_1 = \frac{1}{2\pi} \int_{BZ} dk_x dk_y F, \tag{25}
\end{equation}
where \( F = dA = -i(\partial_k \phi_k, \partial_k \phi_b) \) is the Berry curvature. To see its explicit expression, one can substitute the eigenstates into it, we find
\begin{equation}
F = dA = \frac{1}{2d^3} \epsilon^{abc} \partial_{db} \partial_{dc}, \tag{26}
\end{equation}
Finally, we obtain the Chern number for the lowest occupied band:
\begin{equation}
C_1 = \frac{1}{2\pi} \int_{BZ} dk_x dk_y F
= \frac{1}{4\pi} \int_{BZ} \frac{1}{|d(k)|^3} \epsilon^{abc} \partial_{db} \partial_{dc}
= \frac{1}{2\pi} \int_{BZ} \frac{dk_x dk_y}{|d(k)|^3} \epsilon^{abc} \partial_{db} \partial_{dc}, \tag{27}
\end{equation}
Again, this result agrees with Eq.(23) but differs from the general winding number. In the same way, repeating above calculations for type-II TRFs model we reproduces the same result expression. Therefore, these results are universal for TRFs.

So far we only consider the TRFs, and then generalize procedure to higher degenerate fermions is possible. The general Hamiltonian for higher degenerate fermions can be written as
\begin{equation}
h(k) = d_x(k)J_x + d_y(k)J_y + d_z(k)J_z, \tag{28}
\end{equation}
where \( J_i \) stands for the \((2j + 1)\)-dimensional standard representation of angular momentum \( \hat{J} \). It’s convenient to define the rotation operator
\begin{equation}
P_{x,\theta} P_{y,\theta} = e^{-i\theta \hat{J}_x} e^{-i\theta \hat{J}_y}. \tag{29}
\end{equation}
Then, the \( i \)-th eigenstates can be given by the coherent states \( u_i = P_{x,\theta} P_{y,\theta} u_{i0} \), where \( u_{i0} \) denotes the normalized eigenstates with non-vanning \( i \)-th component. Furthermore, we obtain
\begin{equation}
A = u_i^T [P_{x,\theta}^{-1} J_x \phi P_{y,\theta} + J_y \phi \theta] u_i
= u_i^T [J_x \cos \theta \phi - J_y \sin \theta \phi + J_y \phi \theta] u_i. \tag{30}
\end{equation}
To reach the last line, we have used the formula
\[ e^{A B e^{-A}} = \sum_n \frac{1}{n!} [A^{(n)}, B]. \]
The Berry curvature is determined by \( \mathbf{F} = d \mathbf{A} + i \mathbf{A} \times \mathbf{A} \), explicitly, we have
\begin{equation}
\mathbf{F} = u_i^T [-J_x \sin \theta \phi \frac{\partial}{\partial \phi} - J_y \cos \theta \phi \frac{\partial}{\partial \phi} + J_y \phi \theta] u_i. \tag{31}
\end{equation}
In spherical coordinates, the curvature is expressed as
\[ \mathbf{F} = F_\theta \phi (|d(k)| \frac{\partial}{\partial \phi}), \]
\begin{equation}
F_\theta \phi = -\langle J_x \rangle_n \frac{\cos \theta}{|d(k)|^2 \sin \theta} - \langle J_x \rangle_n \frac{\partial h}{|d(k)|}. \tag{32}
\end{equation}
Therefore, the Chern number for the \( i \)-th band can be obtained as
\begin{equation}
C_1 = -u_i^T J_x u_i \frac{1}{2\pi} \int \sin \theta \frac{\partial}{\partial \phi} + \frac{\partial h}{\partial \phi} = -2 j_z (i) \tag{33}
\end{equation}
where \( j_z (i) \) is the \( i \)-th eigenvalue for the system. In summary, we have the conclusion: For the general degenerate fermion with the form of Eq.(28), if the system is gapped in the whole BZ and there exists nontrivial topology, the Chern number for each band coincides with the eigenvalue.

As an universal conclusion, all the results obtained in this section should be checked in the lattice model. In next section, we will present two lattice models of TRFs for verification.

IV. LATTICE MODEL FOR TRFS

In section III, we have shown some general results of Hall conductivity and Chern number for the TRFs. All these results can be reached in the cases of lattice model, we now present two lattice models which serves as the realization of the TRFs.

A. TRF on Lieb lattice

The Lieb lattice (Fig. 2(a)) provides a natural realization of three-component excitations, of which the Hamiltonian reads
\begin{equation}
H_0 = -t \sum_{\langle ij \rangle} c_i^\dagger c_j + h.c., \tag{34}
\end{equation}
where \( \langle ij \rangle \) denotes sum over all NN pairs, \( c_i^\dagger / c_i \) are creation/annihilation operators for particle at site \( i \). In terms of the basis \( \psi_k^T = (c_{1k}, c_{2k}, c_{3k}) \), the Hamiltonian in momentum space is
\[ h(k) = -2t \cos k_x S_x - 2t \cos k_y S_y, \]
where \( S_x \) and \( S_y \) are given in Eq.(2). The single particle
We observe that the model is given by dependent next nearest-neighbor (NNN) hopping. Now, the single gapless TRF can be gapped by adding a time-reversal or translational breaking term. The physical significance of the symmetry protected degenerate point implies that the single triply degenerate point in the Brillouin zone and which is protected by the imposed symmetries on the lattice. More details about the single triply degenerate point are presented in Appendix B. The physical significance of the symmetry protected degenerate point implies that the single gapless TRF can be gapped by adding a time-reversal or translational breaking term.

In analogy with the Haldane model, the Lieb lattice exhibits quantum Hall effect with an additional phase reversal or translational breaking term. One can prove that the degenerate fermion point is stabilized by some space groups imposed on the system. For Lieb lattice, there exists one single triply degenerate point in the Brillouin zone which is protected by the imposed symmetries on the lattice. The quantized Chern number is the integral of the Berry curvature in the Brillouin zone, 

$$C_1 = \frac{1}{2\pi} \int d^2k \mathbf{F}_{xy} = 1. \quad (40)$$

Now, a few remarks need to be addressed in order: (i) Each massive Dirac fermion in two-dimensions contributes a one-half Hall conductance. So the integer Hall conductivity requires additional fermion. The reason lies in the Nielsen-Ninomiya theorem: the continuum Dirac fermions need to be regularized properly (for example putting it on a lattice). (ii) The Hall conductance calculated from Lieb lattice model is equal to the value from single TRF. This result implies that the single triply degenerate cone can exist on the lattice. Furthermore, the fact that the highest band makes no contribution to the Hall conductance of lowest occupied band also verifies the existence of single triply degenerate fermions.

An important property of the topological nontrivial states is the presence of spatially localized edge states. In order to visualize the edge states, a standard method is to diagonalize the lattice model with periodic boundary conditions imposed along one of the spatial directions, say, $x$-direction. The energy spectrum as a function of momentum are shown in Fig. 2 (b). We observe that there is only one chiral edge state, and another branch of twisted edge state is interesting. The single chiral reflects the bulk-edge correspondence of topological states. Further, the chiral edge state can be studied analytically, this has been done in Ref. 58.

### B. TRFs on dice lattice

Another lattice realization of TRFs is the model on the so-called dice or AB-stacked bilayer graphene-like lattice. Fig. 3(a) shows its crystal structure that contains three inequivalent sites per unit cell and the point group is $D_{3d}$. The primitive lattice vectors are $\mathbf{a}_1 = 3\mathbf{x}/2 - \sqrt{3}\mathbf{y}/2$ and $\mathbf{a}_2 = 3\mathbf{x}/2 + \sqrt{3}\mathbf{y}/2$. Here, we choose the lattice spacing $a = 1$. This lattice can be viewed as two honeycomb lattices shifted by unit vector $\mathbf{x} = (1, 0)$. Similar
to graphene lattice, the system has two triply degenerate relativistic cones at two corners of its first Brillouin zone. The NN tight-binding model on dice lattice have two particle-hole symmetric dispersive bands and a zero-energy flat band. After expanding the model near the two corners, we derive the type-II TRFs model.

The magnetic phases dependent NNN hoppings break time reversal without net flux per plaquette but keep the translational symmetry. Noting that the second neighbor hopping for the blue sites canceled each other. In momentum space, the Hamiltonian is given by $h(k) = \sum_i d_i(k)S_i$, with

$$d_x(k) = -t(\cos k \cdot \delta_1 + \cos k \cdot \delta_2 + \cos k \cdot \delta_3),$$

$$d_y(k) = -t(\sin k \cdot \delta_1 + \sin k \cdot \delta_2 + \sin k \cdot \delta_3),$$

$$d_z(k) = 2\lambda(\sin k \cdot a_1 - \sin k \cdot a_2 - \sin k \cdot (a_1 - a_2)),$$

where $t$ ($\lambda$) are NN (NNN) hopping amplitude, $\delta_i$ as three NNN vectors of red circle. In the same way as Haldane model, near the degenerate point $\xi = \pm k$, we have

$$h_{\xi}(q) = -\frac{3t}{2} (q Sax - \xi qSy) - \xi M_{-\xi}(q) Sz,$$

where $q = k - K_\xi$, $M_{-\xi}(q) = 3\sqrt{3}\lambda + O(q^2)$. We found the system is a topological insulating phase when $\lambda > 0$, with the Hall conductivity given by

$$\sigma_{xy} = \frac{e^2}{\hbar}.$$

This result verifies the general formula Eq. (33). Moreover, form Fig. 3(b), there exist two chiral edge states and a branch of twisted edge state.

**V. PARITY ANOMALY AND CHERN-SIMONS ACTION**

In some gauge theory, i.e. the Abelian QED and the non-Abelian Yang-Mills theory, the invariant of action in the classical field theory does not survive when the theory is quantized, this phenomena is called quantum anomaly. An important consequence of anomaly is the associated topological term in the ground-state current. In (2 + 1)-dimensions QED, parity (or time-reversal) anomaly is great importance because it may be relevant to the quantized Hall effects. In this section, we shall derive the parity anomaly for massless TRFs coupled to U(1) gauge fields (a special QED theory). This work provide the first derivation of quantum anomaly from a Hamiltonian in odd-dimensional spinor space in (2 + 1)-dimensions.

**A. Invariance of the action**

To demonstrate the parity anomaly more clearer, we begin by briefly reviewing the anomaly in usual QED. The parity anomaly has been firstly discussed by Redlich\cite{51} that arises from the derivation of effective action upon integrating the fermion fields. The action for massless fermions coupled to U(1) gauge fields in three time-space is invariant under gauge transformations and parity reflections. The gauge invariance and parity invariance of effective action, however, depend on the procedure used to regulate the ultraviolet divergences. There are two ways to regulate the theory, we can regulate in a way that maintains parity well but the effective action is not gauge invariant. Alternatively, we can use a gauge invariant regularization at the expense of parity invariant (as similar as Pauli-Villars scheme). The Pauli-Villars regulator field includes a heavy mass which is not parity invariant in (2 + 1)-dimensions.

If we use Pauli-Villars regularization by introducing a parity-violating mass $M$, the regulated effective action is

$$I_{\text{eff}}^R[A] = I_{\text{eff}}[A, m = 0] - I_{\text{eff}}[A, M \to \infty].$$

The second term produces a finite parity-violating topological action $\pm \pi W[A]$, that is the Chern-Simons (CS) secondary characteristic class\cite{51}. The gauge non-invariance of Chern-Simons action cancels the gauge noninvariance of $I_{\text{eff}}[A]$ under a homotopically non-trivial gauge transformation with winding number $n$, which yields a gauge invariant but parity-violating action $I_{\text{eff}}^R[A]$.

In the same way, we are going to show the TRFs coupled to U(1) gauge fields exhibits anomaly. The Lagrangian corresponding to the TRFs is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - \zeta_\alpha \beta^\alpha m)\psi$$

where $\psi$ are three-component spinors. Here, we use Minkowski Space metric

$$\eta^{\mu\nu} = \text{diag}(1, -1, -1),$$

with $\mu, \nu = 0, 1, 2$. We have introduced the matrix $\zeta_\alpha$ such that the adjoint spinor $\bar{\psi} = \psi^\dagger \zeta_1$, $\zeta = \text{diag}(1, -1, 1)$ for type-I triplet Dirac fermions and $\zeta_\alpha = \text{diag}(1, -1, 1)$ for type-II triplet Dirac fermions. The gamma matrices are define as $\gamma^1 = \zeta_\alpha S_{x}$, $\gamma^2 = \zeta_\alpha S_{y}$, $\gamma^0 = \zeta_\alpha$, $\beta^0 = S_{z}$. One can check that such Lagrangian recovers the Hamiltonian Eq.(1) for TRFs. Some remarks need to be made in order, the Lagrangian seem strange compared with usual Dirac theory, which arises from the absence of three-dimensional matrix representation of anticommutative Clifford algebra.

The discrete parity transformation in (2 + 1)-dimensions is unusual. In two-dimensional plane, the parity transformation act as $(x, y) \rightarrow (-x, -y)$ rather than $(x, y) \rightarrow (-x, y)$. The latter is equivalent to Lorentz rotation with $\text{det}(\Lambda) = 1$. Under parity $P$, we see that the type-II triplet Dirac fermion fields transform as

$$P\psi P^{-1} = \begin{pmatrix}
0 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix} \psi \equiv \Lambda_P \psi.$$
where the parity satisfies \( \det(A_P) = -1 \). The gauge fields transform as \( PAP^{-1} = (A_0, -A_x, A_y) \). Therefore, the massless QED theory

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu)\psi
\]

(49)
is parity invariant. We observe that, however, the mass term \( m\bar{\psi}\beta^0\psi \) and the CS term \( \epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda \) are not parity invariant. Instead, they change sign under parity transformation. Finally, one can show that the special massless QED theory is time-reversal invariant but the mass term and Chern-Simons term are not invariant.

B. Perturbation calculations of Chern-Simons term

We have shown the invariance of the massless QED theory. In this section, we will demonstrate that the Chern-Simons action will arise in the effective action of gauge fields after regulating the theory in a gauge invariant way. To this end, we use Pauli-Villars regularization and regulate the theory according to Eq. (45).

Now, consider the massive fermions, the effective action can be derived by integrating over the fermion fields in the functional integral:

\[
Z = \int d\bar{\psi}d\psi dA \exp \left(i \int d^4x \mathcal{L}\right),
\]

(50)

\[
\mathcal{L} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - \gamma^\lambda \partial_\lambda \gamma^\mu m)\psi,
\]

(51)

where \( D_\mu = \partial_\mu + ieA_\mu \) is the covariant derivative, \( F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the usual notation of field strength. The effective action, upon integrating the fermion fields, is given by

\[
Z = \int dA \exp \left(i \int d^3x \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + I_{\text{eff}}[A, m]\right),
\]

(52)

where

\[
I_{\text{eff}}[A, m] = -iN_f \ln(i\gamma^\mu D_\mu - \gamma^\lambda \beta^0 m).
\]

(53)

We have introduced the fermion flavors \( N_f \), which will be important for the gauge invariance of the action. Taking the large \( m \) limit, this allows us to do perturbative expansion at one-loop order, all other higher-order graphs are order \( O(1/m) \) and thus vanish when \( m \to \infty \). As shown before, in the Abelian theory, only the vacuum polarization shown in Fig. 4 is ultraviolet divergent, therefore require regularization. In the non-Abelian theory, the triangle diagram is also ultraviolet divergent.

To seek the parity anomaly term, to order \( e^2 \), we have

\[
I_{\text{eff}}[A, m] = -i\frac{N_f}{2} \int \frac{d^3q}{(2\pi)^3} A_\mu(q)\Pi^{\mu\nu}(q)A_\nu(-q),
\]

(54)

where the polarization tensor is

\[
\Pi^{\mu\nu}(q) = e^2 \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \gamma^\mu S(p + q)\gamma^\nu S(p) \right],
\]

(55)

with \( S(p) = i/(\gamma p - \gamma^\lambda \beta^0 m) \). The polarization tensor is superficially linearly divergent as the cut-off \( \Lambda \to \infty \).

Since our gamma matrices defined in odd-dimensions representation do not have nice trace and anti-commutative properties as in even-dimensional representation, the possible nonzero tensor should be calculated one by one.

We restrict our attention to \( \Pi^{01}(q) \), after performing matrix inversion and trace operation, we obtain the antisymmetric part:

\[
\Pi^{01}_A(q, m) = \frac{i}{8\pi} \frac{m}{|m|} \left(\frac{m^2 - p_0^2}{(p + q)^2 - m^2} - \frac{m^2 - p_0^2}{|p - m|^2}\right),
\]

(56)

\[
\Pi_{A1} = \int \frac{d^3p}{(2\pi)^3} \frac{m}{(p + q)^2 - m^2|p - m|^2} = \frac{i}{8\pi} \frac{m}{|m|},
\]

(57)

\[
\Pi_{A2} = \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{p_0^2|p + q|^2 - m^2|p - m|^2} = \frac{3i}{8\pi} \frac{m}{|m|}.
\]

(58)

To reach the last result, we have (i) introduced Feynman parameter to rewrite the denominator, (ii) shifted integration variable and taken the limit \( q_2 \to 0 \), (iii) carried out the integration by parts. The calculations of polarization can be referred to Appendix A. For \( m \to \infty \), we find

\[
\Pi^{01}_A(m \to \infty) = \frac{m}{|m|} \frac{e^2}{2\pi} \epsilon^{012} q_2.
\]

(59)
Repeating the calculation for other nonzero tensors, the results are displayed in Table I. Putting these results together, we observe
\[ \Pi^\mu_A(m \to \infty) = \frac{m}{|m|} \frac{e^2}{2\pi} \varepsilon^{\mu\nu\lambda} q_\lambda. \] (60)

By Lorentz invariance, the symmetric part is linearly divergent. In general, it must be of the form
\[ \Pi^\mu_S(m \to \infty) = \text{constant} \times e^2 \Lambda f^{\mu\nu}, \] (61)
where \( \Lambda \) is a momentum cutoff. Inserting these results into Eq. (54) yields
\[ I_{\text{eff}}[A, m \to \infty] = -i \frac{N_f}{2} \text{constant} \times e^2 \Lambda A^2 \]
\[ - \frac{im}{|m|} \frac{e^2}{2\pi} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \] (62)

The divergent term in \( I_{\text{eff}}[A, m = 0] \) cancels the divergent term in \( I_{\text{eff}}[A, m \to \infty] \). We obtain the regularized and finite effective action as
\[ I_{\text{eff}}^R[A] = \text{finite} \pm \frac{N_f}{4\pi} e^2 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \] (63)
where the symbol \( \pm \) depends on the sign of \( m \). The finite part conserves parity, and the parity violation comes from the Chern-Simons term. In summary, we have shown that the parity is spontaneously broken to maintain the gauge invariance of the regulated effective action.

C. Physical implication

In (2 + 1)-dimensional quantum field theory, the Chern-Simons term produce a locally gauge theory. It’s helpful to consider a non-Abelian theory
\[ \mathcal{L} = \frac{k}{4\pi} \varepsilon^{\mu\nu\lambda} \text{tr}(A_\mu \partial_\nu A_\lambda - \frac{2}{3} A_\mu A_\nu A_\lambda), \] (64)
where the notation \( A_\mu = A_\mu^a T_a \) as usual Yang-Mills theory. It’s proven that under a homotopically non-trivial gauge transformation, the non-Abelian term is invariant on a closed space-time manifold only if the parameter \( k \) is quantized as \( k \in \mathbb{Z} \).

The usual QED with massless relativistic Dirac fermions predicts the coefficient \( k = 1/2 \). It is puzzling because the Chern-Simons term requires \( k \) should be an integer. The understanding of the puzzle lies in the Nielsen-Ninomiya theorem which implies that there are at least two local fermions on the lattice. Three typical models are of great help for us: the Haldane model for quantum anomalous effect\(^{53}\), the Qi-Wu-Zhang model for Chern insulator, the axion topological insulator\(^9\). When the time-reversal or parity is spontaneously broken in Haldane model, there are two Dirac fermions due to the Nielsen-Ninomiya theorem\(^{55}\), each of which contributes a half quantized Hall conductivity such that the total value equals to be an integer. In Qi-Wu-Zhang model, there exists a single fermion in the bulk. In this case, we need consider a compact Brillouin zone in lattice model and then the Nielsen-Ninomiya theorem plays the role of Pauli-Villars regulator with mass \(-m\). As a result, the linearly divergent has canceled each other and the Chern-Simons term is doubled, yields integer Chern number that depends on the sign of \( m \). In axion topological insulator, the bulk states preserve time-reversal but the boundary state is time-reversal broken. So, there is a single massive fermion on each boundary. The Nielsen-Ninomiya theorem is avoided due to the massive fermion lives on different boundary. Therefore, the axion topological insulator supports surface Hall effect with half Hall conductivity.

As shown in above, the CS term for QED with TRFs is
\[ J_{\text{eff}}^R[A] = \frac{N_f}{4\pi} e^2 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \] (65)
Here we take the units \( e = \hbar = 1 \). The minimum value \( N_f = 1 \) satisfies the quantized condition, which implies that there exists single local triplicate relativistic cone in discrete lattice. That is, the Nielsen-Ninomiya theorem for Dirac fermions breaks down for TRFs. Furthermore, the Chern-Simons term predicts the Hall current
\[ J^\nu = \frac{\delta I_{\text{eff}}^R[A]}{\delta A_\nu} = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda, \] (66)
which agrees with the Hall conductivity in the Lieb lattice: \( \sigma_{xy} = 1 \) in units of \( e^2/\hbar \). This number, known as Chern number, indicates the non-trivial bands topology of the lattice model.

VI. CONCLUSIONS

In summary, we have studied the Hall conductivity and the parity anomaly of TRFs, which is the extension of usual Dirac fermions. The low-energy excitations of TRF are described by Dirac-type equation with three-component spinor. Each TRF in continuum model contributes an unit Hall conductance, and the Hall conductivity of TRF on general lattice is determined by Eq. (23) in the main text rather than the general winding number from \( T^2 \) to two-sphere \( S^2 \). Based on TKNN expression of Hall conductivity, we find the zero-energy flat band is nontrivial for the integer Hall conductance of the lowest occupied band.

In the same way as usual QED theory, we find the theory for TRFs coupled with \( U(1) \) gauge field exhibits parity anomaly. We have derived the anomaly-induced Chern-Simons action, as shown in Eq. (65), which breaks the parity and time-reversal spontaneously. From the Chern-Simons term, we obtain the integer quantum Hall current of the ground states, yielding a conductivity of
$e^2/h$. The quantized condition requires that the coefficient $N_f$ must be an integer. Further, we find that the Hall conductivity of a single TRF meets the prediction of Chern-Simons term. As a result, there exists single TRF in discrete lattice. The existence of single TRF in discrete lattice implies that the Nielsen-Ninomiya theorem for Dirac fermions is broken down for TRFs in two $(2 + 1)$ dimensions. More importantly, our finding imply anomaly also takes place in odd-dimensional spinor space.

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**Appendix A: Derivation of Hall conductivity for single TRF**

In this Appendix, we want to calculate the Hall conductance for single TRF. Starting from the TKNN expression of Kubo formula, we need to calculate the explicit expression for the Green function, since the gamma matrices in odd dimensions do not have "nice" trace property as in even dimensions. The Green function is given by

$$ G = \frac{1}{i \omega_n - H} = \left( \begin{array}{ccc} i \omega_n & -k_x & -k_y \\ -k_x & i \omega_n & i \star M \\ -k_y & i \star M & i \omega_n \end{array} \right)^{-1} \quad (A1) $$

Inserting this into Eq.(17), and then considering the dc limit yields Eq.(18). After analytic continuation $\nu_n \rightarrow \nu + i 0^+$, we have

$$ \sigma_{xy} = \lim_{\nu \rightarrow 0} \frac{i}{\nu} Q_{ij}(\nu + i 0^+) $$

$$ = \frac{1}{i} \int \frac{d\omega d^2k}{(2\pi)^3} \frac{M}{[\omega^2 - (k^2 + M^2)]^2} $$

$$ + \frac{1}{i} \int \frac{d\omega d^2k}{(2\pi)^3} \frac{(k^2 + M^2)M}{[\omega^2 - (k^2 + M^2)]^2}. \quad (A2) $$

From Wick rotation $\omega = i \omega_E$, the integral in the second line is elementary:

$$ i \int \frac{d\omega d^2k}{(2\pi)^3} \frac{M}{[\omega^2 - (k^2 + M^2)]^2} $$

$$ = i \int \frac{d^3k E}{(2\pi)^3} \frac{M}{[k^2 + M^2]^2} $$

$$ = -\frac{M}{8\pi |M|}. \quad (A3) $$

In the same way, for the integral in the third line, we have

$$ i \int \frac{d\omega d^2k}{(2\pi)^3} \frac{(k^2 + M^2)M}{[\omega^2 - (k^2 + M^2)]^2} $$

$$ = \frac{1}{i} \int \frac{d\omega d^2k}{(2\pi)^3} \frac{(k^2 + M^2)M}{[\omega^2 + (k^2 + M^2)]^2} $$

$$ = -4 \int \frac{d\omega d^2k}{(2\pi)^3} \frac{(k^2 + M^2)M}{[\omega^2 + (k^2 + M^2)]^3/2} \quad (A4) $$

In order to reach the last line, we have to integrate over $\omega_E$ by parts. Then the integral over $\omega_E$ and $|k|$ is elementary, and yields

$$ -4 \int \frac{d\omega d^2k}{(2\pi)^3} \frac{(k^2 + M^2)M}{[\omega^2 + (k^2 + M^2)]^3/2} $$

$$ = -6M \int \frac{d^3k}{(2\pi)^3} \frac{\pi}{4(k^2 + M^2)^3/2} $$

$$ = -3M \frac{8\pi |M|}{|M|}. \quad (A5) $$

In conclusion, we have obtained

$$ \sigma_{xy} = -\frac{M}{8\pi |M|} - \frac{3M}{8\pi |M|} = -\frac{M}{2\pi |M|}. \quad (A6) $$

**Appendix B: The stability of TRF on Lieb lattice**

In this Appendix, we show the TRF on Lieb lattice is stabilized by some imposed symmetries. The time-reversal symmetry is defined by $\hat{T}h(k)\hat{T}^{-1} = h^*(-k)$, where $\hat{T} = \eta \hat{T}$, $\eta = 1$ for integer spin while $\eta = -1$ for half-odd-integer spin. The spatial inversion or parity is defined by $\hat{P}h(k)\hat{P}^{-1} = h(-k)$. The NN tight-binding model on Lieb lattice exhibits chiral symmetry, which is defined by $\hat{C}h(k)\hat{C}^{-1} = -h^*(k)$. The mirror symmetry is defined by $\hat{M}h(k)\hat{M}^{-1} = h(\hat{R}k)$, where $\hat{M}$ is the mirror operator and $\hat{R}$ is the mirror reflection with respect to reflection plane. From the general definition, the mirror symmetry acting on the local basis in given by

$$ KM_{xy} \begin{pmatrix} c_1(k_x, k_y) \\ c_2(k_x, k_y) \\ c_3(k_x, k_y) \end{pmatrix} (KM_{xy})^{-1} = \begin{pmatrix} c_1(-k_y, -k_x) \\ c_2(-k_y, -k_x) \\ c_3(-k_y, -k_x) \end{pmatrix}. \quad (B1) $$

where $K$ is the complex conjugate operator, $M_{xy}$ is the mirror reflection respect to $x = y$ line on a plane. This symmetry requires

$$ KMh(k_x, k_y)(KM)^{-1} = h^*(-k_y, -k_x). \quad (B2) $$

Now, let’s define the symmetry:

$$ \Gamma = KM_{xy}T_{\hat{x}}, \quad (B3) $$
where $T_x$ represents translation along the horizontal direction that globally moves the lattice by $\hat{x}$. The $\Gamma$-symmetry can be represented as $[H, \Gamma] = 0$. According to Bloch’s theorem, the wave function on lattice can be written as $\psi_k(\mathbf{r}) = e^{ik\mathbf{r}}u_k(\mathbf{r})$, where $u_k(\mathbf{r})$ is the Bloch wavefunction satisfying the periodic condition $u_k(\mathbf{r}) = u_k(\mathbf{r} + \mathbf{R})$, the lattice vectors $\mathbf{R} = n_1a_1 + n_2a_2$, where $n_1, n_2$ are integers. Given a Hamiltonian with the lattice vectors basis $\psi_k(\mathbf{R})$, the Bloch Hamiltonian can be expressed as 

$$h(k) = e^{-ik\mathbf{r}}H_0e^{ik\mathbf{r}}$$

where $h(k)$ satisfies $h(k) = h(k + \mathbf{G}_{h})$, with $\mathbf{G}_{h}$ is the reciprocal lattice vector. It is convenient to chose the Bloch function

$$\Psi_k(\mathbf{R}, \mathbf{r}) = \begin{pmatrix} c_1(\mathbf{r}, k_x, k_y) \\ c_2(\mathbf{r}, k_x, k_y) \\ c_3(\mathbf{r}, k_x, k_y) \end{pmatrix} e^{ik\mathbf{r}}, \quad (B4)$$

then the $\Gamma$ operator acts on it as follows:

$$\Gamma \Psi_k(\mathbf{r}) = \begin{pmatrix} c_1(\mathbf{r} - \hat{x}, -k_y, -k_x) \\ c_2(\mathbf{r} - \hat{x}, -k_y, -k_x) \\ c_3(\mathbf{r} - \hat{x}, -k_y, -k_x) \end{pmatrix} e^{ik\mathbf{r}} e^{-ikx_1 - ikx_2 - ikx_3}, \quad (B5)$$

$$\equiv \Psi_{k'}(\mathbf{r}).$$

Owing to the symmetry $[\Gamma, H] = 0$, we have $\Gamma h(k)\Gamma^{-1} = h(k')$, so $\Psi_k(\mathbf{r})$ and $\Gamma \Psi_k(\mathbf{r})$ are two degenerate states corresponding to a same energy. Therefore, we have $k' = -k$. As is shown in topological insulator\textsuperscript{60,61}, if $k' = k + \mathbf{G}_{h}$ in the Brillouin zone, we say that $k$ is a symmetry invariant points. These special symmetry invariant points can be indexed by half a “mod2 reciprocal lattice vector”

$$k_{n_1n_2} = \frac{1}{2}(n_1b_1 + n_2b_2), \quad (B6)$$

where $b_i$ are primitive reciprocal lattice basis. Base on above discussion, there are four $\Gamma$-invariant points: $k_{00}$, $k_{01}$, $k_{10}$, $k_{11}$. From the definition of $\Gamma$ symmetry,

$$\langle \psi_\phi, \Gamma \psi_\phi \rangle = \langle M_{xy}T_\phi \psi_\phi^*, M_{xy}T_\phi \psi_\phi^* \rangle = \langle \phi, \phi \rangle, \quad (B7)$$

so $\Gamma$ is an antiunitary operator. Specially, we have the following inner product

$$\langle \psi_k(\mathbf{r})|\Gamma \psi_k(\mathbf{r}) \rangle = \langle \Gamma^2 \psi_k(\mathbf{r})|\Gamma \psi_k(\mathbf{r}) \rangle. \quad (B8)$$

From Eq.(B5), we have $\Gamma \psi_k(\mathbf{r}) = e^{2ikz} \psi_k(\mathbf{r})$, thus

$$\langle \psi_k(\mathbf{r})|\Gamma \psi_k(\mathbf{r}) \rangle = e^{-2ikz} \langle \psi_k(\mathbf{r})|\Gamma \psi_k(\mathbf{r}) \rangle. \quad (B9)$$

At $k_{00}$ and $k_{01}$, Eq.(B9) gives a trivial result. At $k_{10}$, $k_{11}$, we have

$$\langle \psi_k(\mathbf{r})|\Gamma \psi_k(\mathbf{r}) \rangle = 0. \quad (B10)$$

Therefore, the system must be degenerate at $k_{10}$, $k_{11}$. In the same way, one can show that $k_{10}$ is excluded by $\Gamma' = K M_{xy}T_\gamma$. In conclusion, we have reached the conclusion that if the time-reversal, mirror reflection and translation are all satisfied, there exists only one single stable triply degenerate point on Lieb lattice.

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