Spin contribution to the perihelion advance in binary systems like OJ 287: higher order corrections

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Abstract
Higher order corrections are obtained for the perihelion precession in binary systems like OJ 287, Sagittarius A*-S2 and H1821+643 using both the Schwarzschild metric and the Kerr metric to take into account the spin effect. The corrections are performed considering the third root of the motion equation and developing the expansion in terms of parameters related to each other and depending on orbit variables. The results are compared with those obtained in a previous paper.

Keywords Perihelion advance · Kerr · Spin · Binary systems · Orbits

1 Introduction
In a previous paper (Marín and Poveda 2018), higher order corrections (up to n-th order) were obtained for the perihelion precession (see Fig. 1) in binary systems like OJ 287, Sagittarius A*-S2 and H1821+643 using the Schwarzschild metric and complex integration. The corrections were performed considering the third root of the motion equation and developing the expansion in terms of $\epsilon \equiv r_s / (a(1 - e^2))$, where $r_s$ is the Schwarzschild radius, $a$ is the semi-major axis and $e$ is the eccentricity. The results were compared with other expansions that appear in the literature giving corrections to second and third order (Fokas and Vayegas 2016; Lemmon and Mondragon 2009; Rosales and Castro-Quilantán 1983; Biesel 2008; D’Eliseo 2011; Scharf 2011; Do-Nhat 1998). In this paper, we will consider both Schwarzschild and Kerr metrics to take into account the spin effect (to first order) and developing the expansion in terms of both $\epsilon \equiv r_s / (a(1 - e^2))$ and $\epsilon^* \equiv \left(1 - \frac{2\alpha E}{cJ}\right)\epsilon$, where $E'$ and $J$ are the energy and angular momentum per unit mass and $\alpha$ is a factor proportional to the black hole spin.

Fig. 1 Perihelion precession. $\omega$ is the initial inclination of the orbit and $\delta\omega$ is the angle of precession

Kerr’s black holes are very interesting because most of the black holes in the universe probably have a rotational movement (spin) and one of the most important consequences of that spin is that spacetime is dragged around the rotating black hole leading to an effect that is known as “frame dragging”. In the case of a Schwarzschild black hole, a precession occurs which is the rotation of the elliptical orbit in the fixed plane of that orbit. For Kerr’s solution, the drag of the reference frame introduces an additional precession of the plane of the orbit around its axis of rotation in the same direction of the black hole rotation. To facilitate the calculations, it was assumed that the axis of rotation of the Kerr black hole is perpendicular to the plane of the orbit.
2 Spin contribution to the perihelion precession

The Schwarzschild metric describes a body with spherical symmetry, but without electric charge and rotational movement. Taking into account that black holes found rotate on their own axis, such as the OJ 287 system, one might think that this rotation should influence the advance of the perihelion of the elliptical orbits. The rotation around its own axis, such as the OJ 287 system, one might think that this rotation should influence the advance of the perihelion of the elliptical orbits. The rotation around its own axis is given by the angular momentum of spin $S_c$ of the massive body $M$.

To include the spin in the calculation, the Kerr metric is at our disposal. The Kerr metric is an exact solution to the Einstein field equations in vacuum for a body of mass $M$ that rotates on its own axis with an angular momentum $S_c$. The Kerr metric is given by (Misner et al. 1973; Ryder 2009; Hobson et al. 2006; Chandrasekhar 1992; ’t Hooft 2001; Ludvigsen 1999; Marín 2019):

$$ds^2 = c^2 (dt)^2 - \frac{\rho^2}{\Delta} (dr)^2 - \rho^2 (d\theta)^2 - \frac{r_s r}{\rho^2} \left( c \, dt - \alpha \sin^2 \theta d\phi \right)^2,$$

where $\alpha = \frac{S_c}{Mc}$, $\Delta = r^2 - r_s r + \alpha^2$ and $\rho^2 = r^2 + \alpha^2 \cos^2 \theta$, with coordinates $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius. We will take $\theta = \pi/2$ (equatorial plane). With this value of $\theta$, the last equation can be written as

$$(ds)^2 = c^2 (d\tau)^2 - \frac{r_s^2}{\Delta} (dr)^2 - r^2 \left( 1 + \frac{\alpha^2}{r^2} + \frac{r_s \alpha^2}{r^4} \right) (d\phi)^2 + \frac{2rs \alpha c}{r} c dt d\phi,$$

where $\tau$ is the proper time and $\gamma = 1 - \frac{\gamma}{c}$. In this paper, we only intend to observe how the spin could contribute to the perihelion precession and if it is relevant to the calculation, for which only first order terms in $\frac{\Delta}{\rho}$ will be taken into account. By doing this, the metric is reduced to

$$(ds)^2 = c^2 (d\tau)^2 - \frac{1}{\gamma} (dr)^2 - r^2 (d\phi)^2 + \frac{2rs \alpha c}{r} c dt d\phi. \tag{3}$$

The arc length $ds$ satisfies the relation $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, then, the covariant metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & \frac{r_s \alpha c}{r} \\ 0 & -\gamma^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s \alpha c}{r} & 0 & 0 & -r^2 \end{pmatrix}. \tag{4}$$

The geodesic equation can be written in an alternative form using the Lagrangian

$$L \left( x^\mu, \frac{dx^\mu}{d\sigma} \right) = -g_{\alpha\beta} \left( x^\mu \right) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}, \tag{5}$$

where $\sigma$ is a parameter of the trajectory of the particle, which is usually taken to be the proper time, $\tau$ for a massive particle. Using the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\sigma} \left( \frac{\partial L}{\partial (\frac{dx^\mu}{d\sigma})} \right) = 0,$$

one obtains the geodesic equation for the particle

$$\frac{du^\mu}{d\sigma} = \frac{1}{2} \left( \partial_{\mu} g_{\alpha\beta} \right) u^\alpha u^\beta, \tag{7}$$

where $u^\mu = \frac{dx^\mu}{d\sigma}$.

With $\sigma = \tau$, for the coordinates $x^\mu (\mu = 0)$ and $\phi (\mu = 3)$, the geodesic equation (7) gives us, respectively

$$\frac{d}{d\tau} \left( \gamma c^2 \left( \frac{dt}{d\tau} \right) + \frac{r_s \alpha c}{r} \left( \frac{d\phi}{d\tau} \right) \right) = 0,$$

$$\frac{d}{d\tau} \left( r^2 \left( \frac{d\phi}{d\tau} \right) - \frac{r_s \alpha c}{r} \left( \frac{dt}{d\tau} \right) \right) = 0. \tag{9}$$

Both of these equations define the following constants along the trajectory of the particle around the massive object

$$\gamma c^2 \left( \frac{dt}{d\tau} \right) + \frac{r_s \alpha c}{r} \left( \frac{d\phi}{d\tau} \right) = E', \tag{10}$$

$$r^2 \left( \frac{d\phi}{d\tau} \right) - \frac{r_s \alpha c}{r} \left( \frac{dt}{d\tau} \right) = J. \tag{11}$$

where $E'$ has units of energy per unit mass and $J$ of angular momentum per unit mass.

From these equations one has

$$\frac{d\phi}{d\tau} = \frac{1}{\gamma cr^2} \left( \frac{r_s \alpha c}{r} E' + \gamma c J \right), \tag{12}$$

$$\frac{dt}{d\tau} = \frac{1}{\gamma cr^2} \left( \frac{r^2}{c} E' - \frac{r_s \alpha c}{r} J \right). \tag{13}$$

Equation (3) can be written as

$$c^2 = \gamma c^2 \left( \frac{dt}{d\tau} \right)^2 - \frac{1}{\gamma} \left( \frac{dr}{d\tau} \right)^2 - r^2 \left( \frac{d\phi}{d\tau} \right)^2 + \frac{2rs \alpha c}{r} \frac{dt}{d\tau} \frac{d\phi}{d\tau}, \tag{14}$$

and replacing the values of $\frac{dt}{d\tau}$ and $\frac{d\phi}{d\tau}$

$$\gamma^2 r^4 c^4 = \gamma c^2 \left( \frac{r^2}{c} E' - \frac{r_s \alpha c}{r} J \right)^2$$
\[-\gamma c^2 r^4 \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{rs\alpha}{r} E' + \gamma c J\right)^2 + 2rs\alpha \left(\frac{r^2 E' - rs\alpha}{r} J c J'\right) = 0,\]

Simplifying, and taking only first order terms in \(\frac{\alpha}{r}\),

\[
y r^4 c^4 = c^2 \left(\frac{r^2}{c^2} E^2 - \frac{2r rs\alpha}{c} J E'\right) - c^2 r^4 \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(2c \frac{rs\alpha}{r} J E' + \gamma c^2 J^2\right) + 2rs\alpha \left(\frac{r^2 c J E'}{c r^3}\right).
\]

Introducing the value of \(\gamma\) in the left side of the last equation one has

\[
c^2 - \frac{rs^2 c^2}{r} = \frac{1}{c^2} E^2 - \frac{2r rs\alpha}{cr^3} J E' - \left(\frac{dr}{d\tau}\right)^2 - \gamma \frac{J^2}{r^2}.
\]

Finally, an energy conservation equation is obtained, similar to the one obtained in the case of the Schwarzschild metric, but with an additional crossed term proportional to \(J E'\)

\[
\frac{E'^2}{c^2} - c^2 = \left(\frac{dr}{d\tau}\right)^2 + \gamma \frac{J^2}{r^2} - \frac{rs^2 c^2}{r} + \frac{2rs\alpha}{cr^3} J E'.
\]

This allows us to define an effective potential

\[
\tilde{V} = \gamma \frac{J^2}{r^2} - \frac{rs^2 c^2}{r} + \frac{2rs\alpha}{cr^3} J E'.
\]

From equation (15), one obtains the expression for the radial kinetic energy per unit mass

\[
\left(\frac{dr}{d\tau}\right)^2 = A + \frac{rs^2 c^2}{r} - \frac{J^2 rs}{r^2} + \frac{J^2 rs}{r^3} - \frac{2rs\alpha}{cr^3} J E',
\]

where \(A = \frac{E'^2}{c^2} - c^2\).

Recall now that since the orbit is an ellipse, there are two points in which the temporal derivative becomes zero, and they are aphelion and perihelion. For these two points one has

\[
A + \frac{rs^2 c^2}{R_a} - \frac{J^2}{R_a^2} + \frac{J^2 rs}{R_a^3} - \frac{2rs\alpha}{c R_a^3} J E' = 0,
\]

\[
A + \frac{rs^2 c^2}{R_p} - \frac{J^2}{R_p^2} + \frac{J^2 rs}{R_p^3} - \frac{2rs\alpha}{c R_p^3} J E' = 0,
\]

where \(R_a = a (1 + e)\) and \(R_p = a (1 - e)\).

For an ellipse, the equation of motion has three real and positive roots. The roots are \(R_a, R_p\) and \(R'_o\). To obtain \(R'_o\), it is necessary to use equation (17) and write

\[
\frac{A}{J^2} r^3 + \frac{rs^2 c^2}{J^2} \left(\frac{r^2}{r} E' - \frac{rs\alpha}{c} J\right)^2 - r + r_s - \frac{2rs\alpha}{c J} E' = 0.
\]

The last equation can be expressed as

\[
\frac{r_s c^2}{J^2} r^2 - r + \left(\frac{rs}{c J} E'\right) = \frac{|A|}{J^2} \left(\frac{R'_o}{R_o} + \frac{R_p}{R_p}\right)^2 - \frac{|A|}{J^2} (R_p R'_o + R_o R_p + R_a R'_o) r + \frac{|A|}{J^2} R_a R_p R'_o.
\]

It is important to recall that \(\gamma\) is negative for elliptic orbits, i.e., \(\gamma = -|\gamma|\).

Comparing the coefficients of \(r^3, r\) and \(r^2\) of both sides of the last equation, and replacing the values of \(R_a\) and \(R_p\), the following relations are obtained

\[
\frac{r_s c^2}{J^2} = \frac{|A|}{J^2} (R'_o + 2a),
\]

\[
1 = \frac{|A|}{J^2} \left(2a R'_o + a^2 (1 - e^2)\right),
\]

\[
R'_o = \frac{J^2 r_s}{|A| (1 - e^2) a^2} \left(1 - \frac{2a}{c J} E'\right).
\]

The value of \(R'_o\) given in (22) is the same result that the one obtained in a previous paper (Marín and Poveda 2018), but with an extra factor \(1 - \frac{2a}{c J} E'\). Using equations (21) and (22) we can finally write

\[
R'_o = \frac{a (1 - e^2) r_s \left(1 - \frac{2a E'}{c J}\right)}{\left(a (1 - e^2) - 2r_s \left(1 - \frac{2a E'}{c J}\right)\right)},
\]

and

\[
\frac{J}{|A|^\frac{3}{2}} = \frac{a^3 (1 - e^2)}{\left(a (1 - e^2) - 2r_s \left(1 - \frac{2a E'}{c J}\right)\right)^\frac{3}{2}}.
\]

Going back to equation (17), and because \(\frac{dr}{d\tau} = \left(\frac{dr}{d\phi}\right) \times \left(\frac{d\phi}{d\tau}\right)\), replacing the expression of \(\frac{d\phi}{d\tau}\) given by (12), one obtains

\[
\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{c^2 r^4} \left(\frac{rs\alpha}{r} E' + \gamma c J\right)^2 = A + \frac{rs^2 c^2}{r} - \frac{J^2}{r^2} + \frac{J^2 rs}{r^3} - \frac{2rs\alpha}{c r^3} J E'.
\]
Leaving only first order terms in \( \frac{a}{r} \), it is easy to arrive at the following equation

\[
\left( \frac{dr}{d\phi} \right)^2 \left( 1 + \frac{2r_s \alpha E'}{\sqrt{r c J}} \right) = \frac{A}{J^2} r^4 + \frac{r_s c^2}{J^2} r^3 - r^2 + \left( r_s - \frac{2r_s \alpha E'}{c J} \right) r, \tag{26}
\]

that also can be written as

\[
\left( \frac{dr}{d\phi} \right)^2 \left( 1 + \frac{2r_s \alpha E'}{\sqrt{r c J}} \right) = \frac{|A|}{J^2} \left( R_a - r \right) \left( r - R_p \right) \left( r - R'_o \right) r. \tag{27}
\]

Then, the advance of the perihelion will be given by the integral

\[
\Delta \phi_{kerr} = \frac{2J}{|A|^{1/2}} \int_{R_p}^{R_o} \left( 1 + \frac{2r_s \alpha E'}{\sqrt{r c J}} \right)^{1/2} \frac{dr}{\left[ \left( R_a - r \right) \left( r - R_p \right) \left( r - R'_o \right) r \right]^{1/2}}, \tag{28}
\]

and keeping only terms of up to first order in \( \frac{a}{r} \) one has

\[
\Delta \phi_{kerr} = \frac{2J}{|A|^{1/2}} \int_{R_p}^{R_o} \frac{dr}{\left[ \left( R_a - r \right) \left( r - R_p \right) \left( r - R'_o \right) r \right]^{1/2}} + \frac{2r_s \alpha E'}{c |A|^{1/2}} \int_{R_p}^{R_o} \frac{dr}{r y \left[ \left( R_a - r \right) \left( r - R_p \right) \left( r - R'_o \right) r \right]^{1/2}}, \tag{29}
\]

that can be written as

\[
\Delta \phi_{kerr} = \Delta \phi_{sch}(R'_o) + \Delta \phi_2(R'_o), \tag{30}
\]

where

\[
\Delta \phi_{sch}(R'_o) = \frac{2J}{|A|^{1/2}} \int_{R_p}^{R_o} \frac{dr}{\left[ \left( R_a - r \right) \left( r - R_p \right) \left( r - R'_o \right) r \right]^{1/2}}, \tag{31}
\]

and

\[
\Delta \phi_2(R'_o) = \frac{2r_s \alpha E'}{c |A|^{1/2}} \int_{R_p}^{R_o} \frac{dr}{r \left[ \left( R_a - r \right) \left( r - R_p \right) \left( r - R'_o \right) r \right]^{1/2}}. \tag{32}
\]

The first term in equation (30) is the same that was obtained in a previous paper (Marín and Poveda 2018) using the Schwarzschild metric, but replacing \( R_o \) by \( R'_o \).

The parameter \( \alpha \) is related to the spin of the black hole \( s \), also known as the Kerr parameter through the relationship

\[
\alpha = \frac{GM}{c^2} s,
\]

in such a way that the angular momentum of spin is

\[
S_z = \frac{GM^2}{c} s, \tag{33}
\]

and \( s \) is a dimensionless parameter that can take values between 0 and 1. If \( s \) were greater than 1, there would be no event horizons and the singularity of \( r = 0 \) would be naked, which is not allowed (Ryder 2009; Wald 1984).

### 3 Calculation of \( \Delta \phi_{sch}(R'_o) \)

The first term in equation (30) also can be expressed as

\[
\Delta \phi_{sch}(R'_o) = \frac{2J}{|A|^{1/2}} \int_{R_p}^{R_o} \frac{r^{-\frac{1}{2}} \left( 1 - \frac{R'_o}{r} \right)^{-\frac{1}{2}} dr}{\left[ \left( R_a - r \right) \left( r - R_p \right) r \right]^{1/2}}, \tag{34}
\]

and because \( R'_o \ll R_p < R_a \), one can expand around \( R'_o \)

\[
\left( 1 - \frac{R'_o}{r} \right)^{-\frac{1}{2}} = \sum_{n=1}^{\infty} \left( -\frac{1}{n-1} \right) (-1)^{n-1} \left( \frac{R'_o}{r} \right)^{n-1}. \tag{35}
\]

Then,

\[
\Delta \phi_{sch}(R'_o) = \frac{2J}{|A|^{1/2}} \sum_{n=1}^{\infty} \left( -\frac{1}{n-1} \right) (-1)^{n-1} \left( \frac{R'_o}{r} \right)^{n-1} I_n, \tag{36}
\]

where

\[
I_n = \int_{R_p}^{R_o} \frac{dr}{r^n \left[ \left( R_a - r \right) \left( r - R_p \right) r \right]^{1/2}}. \tag{37}
\]

The value of \( I_n \) was calculated using complex integration in reference (Marín and Poveda 2018), where we obtained

\[
I_n = \frac{\pi (-1)^{n+1}}{a^n 2^{n-1} (1 - e^2)^{n-1/2}} \sum_{k=0}^{n-1} \binom{n}{k} \left( \frac{1}{2} \right)^{n-1-k} \varepsilon^{n-1}, \tag{38}
\]

where

\[
\varepsilon = \frac{\left( 1 + \sqrt{1 - e^2} \right)}{2}, \tag{39}
\]
and

\[ z_2 = -\frac{1 - \sqrt{1 - e^2}}{e}. \]  

(40)

Defining the functions \( Q_{n-1} (z_1, z_2) \)

\[ Q_n (z_1, z_2) = \sum_{k=0}^{n} \binom{n}{k} \frac{2}{k} \frac{z_1^{n-k} z_2^k}{e^n}, \]  

(41)

equation (36) can be expressed as

\[ \Delta \phi_{sch}(R'_o) = \frac{2\pi}{|A|^{1/2}} \sum_{n=0}^{\infty} \left( \frac{-\frac{1}{n}}{n} \right)^2 \frac{(R'_o)^n}{a^{n+1/2}} Q_n(z_1, z_2), \]  

(42)

In Table 1 it is shown the first five functions \( Q_n \).

| Function | Expression |
|----------|-------------|
| \( Q_0 \) | 1 |
| \( Q_1 \) | \(-2\) |
| \( Q_2 \) | \((4 + 2e^2)\) |
| \( Q_3 \) | \(-8 + 12e^2\) |
| \( Q_4 \) | \((16 + 48e^2 + 6e^4)\) |
| \( Q_5 \) | \(-32 + 160e^2 + 60e^4\) |

Let us call: \( a_1 = 2r_s + R'_o, a_2 = 2R'_o r_s + r_s^2, \) and \( a_3 = r_s^2 R'_o \). Performing the expansion

\[ \left( 1 - \frac{a_1}{r} + \frac{a_2}{r^2} - \frac{a_3}{r^3} \right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left( -\frac{\frac{1}{2}}{n} \right)^n \left( -\frac{a_1}{r} + \frac{a_2}{r^2} - \frac{a_3}{r^3} \right)^n = \sum_{n=0}^{\infty} c_n \frac{r^n}{r^n}. \]  

(47)

one can check for example that

\[ c_0 = 1, \]
\[ c_1 = \frac{a_1}{2}, \]
\[ c_2 = \frac{1}{2} \left( \frac{3a_1^2 - a_2}{4} \right), \]
\[ c_3 = \frac{5}{16}a_1^3 - \frac{3}{4}a_1 a_2 + \frac{1}{2}a_3, \]
\[ c_4 = \frac{35}{128}a_1^4 + \frac{3}{4}a_1 a_3 - \frac{15}{16}a_1^2 a_2 + \frac{3}{8}a_2^2, \] etc...

In general, the \( c_n \) satisfy the following recurrence relationship

\[ c_{n+3} = \frac{1}{6 + 2n} (a_1 c_{n+2} (5 + 2n) - 2a_2 c_n (2 + n) + a_3 c_n (3 + 2n)). \]  

(53)

In terms of the \( c_n \), it is possible to write (46) as

\[ \Delta \phi_2(R'_o) = \frac{2r_s a E'}{c |A|^{1/2}} \sum_{n=0}^{R_0} \frac{c_n dR}{R_p^{n+2} \left( [R_d - r] (r - R_p) \right)^{1/2}} \]
\[ = \frac{2r_s a E'}{c |A|^{1/2}} \sum_{n=0}^{\infty} c_n I_{n+2}. \]  

(54)

Replacing the value of \( I_{n+2} \) given by (38) and the value of \( B_{n+2} \) given by (24) one obtains

\[ \Delta \phi_2(R'_o) = \frac{\pi \epsilon}{2 \left( 1 - 2 \epsilon \left( 1 - \frac{2a E'}{cJ} \right) \right)^{1/2}} \left( \frac{2a E'}{cJ} \right) \times \sum_{n=0}^{\infty} (-1)^{n+1} c_n Q_{n+1}(z_1, z_2) e^n. \]  

(55)

### 4 Calculation of \( \Delta \phi_2(R'_o) \)

The integral \( \Delta \phi_2(R'_o) \) also can be written as

\[ \Delta \phi_2(R'_o) = \frac{2r_s a E'}{c |A|^{1/2}} \]
5 Expansion of $\Delta \phi_{sch}(R_o')$ in terms of $\epsilon^*\equiv \left(1 - \frac{2\alpha E'}{cJ}\right)\epsilon$

In terms of $\epsilon^*\equiv \left(1 - \frac{2\alpha E'}{cJ}\right)\epsilon$ (43) is

$$\Delta \phi_{sch}(R_o') = \frac{2\pi}{\left(1 - 2\epsilon^*\right)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \left(-\frac{1}{n}\right) \frac{(\epsilon^*)^n}{2^n (1 - 2\epsilon^*)^n} Q_n(z_1, z_2).$$

(56)

We will expand $\Delta \phi_{sch}(R_o')$ in terms of $\epsilon^*$. Let us compute the first four terms of (56) to recover the expansion until third order in $\epsilon^*$.

$$\Delta \phi_{sch}(R_o')^{(3)} = \frac{2\pi}{\left(1 - 2\epsilon^*\right)^{\frac{1}{2}}} \left( Q_0(z_1, z_2) - \frac{1}{2} Q_1(z_1, z_2) \epsilon^* \right. $$

$$+ \left( \frac{3}{8} \right) \frac{Q_2(z_1, z_2)}{2^2 (1 - 2\epsilon^*)^2} (\epsilon^*)^2 $$

$$- \left( \frac{5}{16} \right) \frac{Q_3(z_1, z_2)}{2^3 (1 - 2\epsilon^*)^3} (\epsilon^*)^3 + \cdots \right).$$

(57)

Replacing the values of the $Q_i$ ($i = 0, 1, 2, 3$) give us

$$\Delta \phi_{sch}(R_o')^{(3)} = \frac{2\pi}{\left(1 - 2\epsilon^*\right)^{\frac{1}{2}}} \left( 1 + \frac{\epsilon^*}{2 (1 - 2\epsilon^*)} \right.$$

$$+ \left( \frac{3}{16} \right) \frac{2 + e^2}{(1 - 2\epsilon^*)^2} (\epsilon^*)^2 $$

$$+ \left( \frac{5}{32} \right) \frac{2 + 3e^2}{(1 - 2\epsilon^*)^3} (\epsilon^*)^3 + \cdots \right).$$

(58)

which in third order in $\epsilon^*$ is reduced to

$$\Delta \phi_{sch}(R_o')^{(3)} = 2\pi \left( 1 + \frac{3}{2} \epsilon^* + \frac{54 + 3e^2}{16} \right) (\epsilon^*)^2 $$

$$+ \left( \frac{135}{16} + \frac{45}{32} e^2 \right) (\epsilon^*)^3 + \cdots \right).$$

(59)

In terms of $\epsilon$ it can be written as

$$\Delta \phi_{sch}(R_o')^{(3)} = 2\pi \left( 1 + \frac{3}{2} \left( 1 - \frac{2\alpha E'}{cJ} \right) \epsilon \right.$$

$$+ \frac{3 (18 + e^2)}{16} \left( 1 - \frac{2\alpha E'}{cJ} \right)^2 \epsilon^2 $$

$$+ \frac{45}{32} \left( 6 + e^2 \right) \left( 1 - \frac{2\alpha E'}{cJ} \right)^3 \epsilon^3 + \cdots \right).$$

(60)

In general to order $m$ in $\epsilon^*$, one has

$$\Delta \phi_{sch}(R_o')^{(m)} = \frac{2\pi}{\left(1 - 2\epsilon^*\right)^{\frac{1}{2}}} \times m \left( -\frac{1}{n}\right) \frac{(\epsilon^*)^n}{2^n (1 - 2\epsilon^*)^n} Q_n(z_1, z_2).$$

(61)

6 Expansion of $\Delta \phi_2(R_o')$ in terms of $\epsilon$

To evaluate $\Delta \phi_2(R_o')$ to third order in $\epsilon$, it is sufficient to consider the first three terms in the sum given by equation (55)

$$\Delta \phi_2(R_o')^{(3)} = \frac{\pi \epsilon}{2 \left( 1 - 2\epsilon \left( 1 - \frac{2\alpha E'}{cJ} \right) \right)^{\frac{1}{2}}} \left( \frac{2\alpha E'}{cJ} \right)$$

$$\times \left( -c_0 Q_1 + \frac{c_1 Q_2 \epsilon}{2r_s} + \frac{c_2 Q_3 \epsilon^2}{2^2 r_s^2} \right).$$

(62)

where

$$c_0 = 1,$$

$$c_1 = \frac{2r_s + R_o'}{2},$$

$$c_2 = \frac{3}{8} \left( R_o' \right)^2 + \frac{1}{2} r_s R_o' + r_s^2.$$

(63, 64, 65)

Introducing the values of the $Q_i$ in (62) we have

$$\Delta \phi_2(R_o')^{(3)} = \frac{\pi \epsilon}{\left( 1 - 2\epsilon \left( 1 - \frac{2\alpha E'}{cJ} \right) \right)^{\frac{1}{2}}} \left( \frac{2\alpha E'}{cJ} \right)$$

$$\times \left( 1 + \frac{2r_s + R_o'}{2} \right) \epsilon$$

$$+ \left( 3 \left( R_o' \right)^2 + 4r_s R_o' + 8r_s^2 \right) \left( 2 + 3\epsilon^2 \right) \epsilon^2$$

$$+ \left( 16r_s^2 \right).$$

(66)

where

$$\frac{(2r_s + R_o')}{r_s} = \frac{2 + (1 - 4\epsilon) \left( 1 - \frac{2\alpha E'}{cJ} \right)}{2 \left( 1 - 2\epsilon \left( 1 - \frac{2\alpha E'}{cJ} \right) \right)},$$

(67)

and

$$\frac{3 \left( R_o' \right)^2 + 4r_s R_o' + 8r_s^2}{r_s^2}.$$
Using (67) and (68) in (66), and employing the Taylor series (around \( x = 0 \)): \((1 - x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots \), one has to third order in \( \epsilon \)

\[
\Delta \phi_2(R_e^{(3)}) = \pi \frac{2\alpha E'}{cJ} \epsilon + \frac{\pi}{4} \left( 10 + 3\epsilon^2 - \left( 6 + \epsilon^2 \right) \frac{2\alpha E'}{cJ} + \frac{3}{2} \frac{2\alpha E'}{cJ} \epsilon^2 \right) \\
+ \frac{\pi}{16} \left( 94 + 65\epsilon^2 - \left( 132 + 62\epsilon^2 \right) \frac{2\alpha E'}{cJ} + \left( 82 + 62\epsilon^2 \right) \frac{2\alpha E'}{cJ} \epsilon^2 \right) \\
+ \left( 54 + 21\epsilon^2 \right) \left( \frac{2\alpha E'}{cJ} \right)^2 \frac{2\alpha E'}{cJ} \epsilon^3. \tag{69}
\]

### 7 Perihelion precession to third order in \( \epsilon \)

As the perihelion precession in a cycle is \( \chi = \Delta \phi_{kerr} - 2\pi \), we have until third order in \( \epsilon \)

\[
\chi^{(3)} = \Delta \phi_{sch}(R_e^{(3)}) + \Delta \phi_2(R_e^{(3)}) - 2\pi, \tag{70}
\]

that give us

\[
\chi^{(3)} = 3\pi \left( 1 - \frac{2}{3} \left( \frac{2\alpha E'}{cJ} \right) \right) \epsilon \\
+ \left( \frac{54 + 3\epsilon^2}{8} - 11 \left( \frac{2\alpha E'}{cJ} \right) \right) \pi \epsilon^2 \\
+ \frac{1}{16} \left( 45 \left( 6 + \epsilon^2 \right) - \left( 716 + 70\epsilon^2 \right) \left( \frac{2\alpha E'}{cJ} \right) \right) \\
+ \left( 678 + 73\epsilon^2 \right) \left( \frac{2\alpha E'}{cJ} \right)^2 \\
- \left( 216 + 24\epsilon^2 \right) \left( \frac{2\alpha E'}{cJ} \right)^3 \pi \epsilon^3. \tag{71}
\]

### 8 Applications

To analyze how the spin of the central black hole changes the perihelion precession, it was calculated until third order (using equation (71)) for three different binary systems. In Table 2 are shown the results.

The term \( \chi(\epsilon^i) \) is the one that depends only on the \( i \)-th power of \( \epsilon \), since \( \chi^{(i)} = \chi(\epsilon^1) + \chi(\epsilon^2) + \cdots + \chi(\epsilon^i) \).

The expansion was done in terms of \( \alpha \), that is a parameter with units of length, but the experimental measurements are done in terms of the spin \( s \). The spin \( s \) of the black hole is a dimensionless parameter that is related to \( \alpha \) by

\[
\alpha = \frac{GM}{c^2} = \frac{r_s}{2} s. \tag{72}
\]

The spin of a central black hole in a binary system is very difficult to measure. Nevertheless, some groups have measured it for the binary system OJ 287. These measurements can be seen in articles by Pihajoki (2014), Valtonen (2016), Dey et al. (2018). The problem is that the values obtained form those measurements are different. For the numerical calculations we have used some values of the spin including the one presented in the paper by Dey et al. (2018). Also, we made the calculations for \( s = 0 \).

For the calculations, the values of the energy per unit mass and the angular momentum per unit mass are necessary. For this, both values can be calculated numerically using equations (18) and (19). These are the equations of motion evaluated for the critical points \( R_q \) (aphelion) and \( R_p \) (perihelion). As the equations are quadratic, there are two solutions for each parameter. We took the negative value for the energy (that is necessary for the orbit to be elliptical), and the positive value for the angular momentum, because a counterclockwise rotation was considered.

In Table 2, it can be seen the values for \( s = 0 \) for the systems Sagittarius A*-S2, OJ 287, H1821+643. These values agree with those calculated in a previous paper (Marín and Poveda 2018) using the Schwarzschild metric. These results are in concordance with the fact that the Kerr spacetime transforms in Schwarzschild spacetime when \( s = 0 \).

Then, for other values of \( s \), the perihelion precession increases with \( s \). Also, it is important to recall that we considered that \( s \) is positive, i.e., the central black hole rotates in the same direction of the movement of the secondary massive object. The values of the spin of OJ287 were used with all the systems to see how the perihelion precession changes with it because of the lack of information about the values for those systems.

For OJ 287, the experimental accepted value for the perihelion precession is 38.62° per period (Dey et al. 2018). The theoretical calculation (for \( s = 0.381 \)) predicts a higher value of 42.741°. Without considering the spin, the theoretical value is 38.713°. It was expected that the introduction of the spin would reduce the error with respect to the experimental value, but instead it exceeded this value. Nevertheless, there is another correction that is expected to reduce the value of the perihelion precession. This correction consist in taking into account the lost of energy in the gravitational radiation. That correction will be done in further works.
Table 2  Perihelion advance in degrees per period for some binary systems, and for different values of the spin

| System                  | Sagittarius A*-S2 | OJ 287     | H1821+643 |
|-------------------------|--------------------|------------|-----------|
| $M(\times M_\odot)$     | $4.310 \times 10^6$| $1.830 \times 10^{10}$ | $3.000 \times 10^{10}$ |
| $r_s$(AU)               | 0.085              | 360.847    | 591.553   |
| $\alpha$(AU)            | 923.077            | 11500      | 40000     |
| $e$                     | 0.870              | 0.700      | 0.900     |
| $\epsilon$              | $3.787 \times 10^{-4}$ | $6.153 \times 10^{-2}$ | $7.784 \times 10^{-2}$ |
| $s$                     | 0                  | 0          | 0         |
| $\chi(\epsilon)$       | 0.205°             | 33.223°    | 42.031°   |
| $\chi(\epsilon^2)$     | $(1.816 \times 10^{-4})^o$ | 4.724°    | 7.692°    |
| $\chi(\epsilon^3)$     | $(1.858 \times 10^{-7})^o$ | 0.765°    | 1.625°    |
| $\chi^{(3)}$            | 0.205°             | 38.713°    | 51.349°   |
| $s$                     | 0.280              | 0.280      | 0.280     |
| $\chi(\epsilon)$       | 0.206°             | 35.253°    | 46.232°   |
| $\chi(\epsilon^2)$     | $(1.838 \times 10^{-4})^o$ | 5.442°    | 9.622°    |
| $\chi(\epsilon^3)$     | $(1.895 \times 10^{-7})^o$ | 0.962°    | 2.332°    |
| $\chi^{(3)}$            | 0.206°             | 41.657°    | 58.186°   |
| $s$                     | 0.381              | 0.381      | 0.381     |
| $\chi(\epsilon)$       | 0.206°             | 35.986°    | 47.747°   |
| $\chi(\epsilon^2)$     | $(1.846 \times 10^{-4})^o$ | 5.715°    | 10.382°   |
| $\chi(\epsilon^3)$     | $(1.908 \times 10^{-7})^o$ | 1.041°    | 2.635°    |
| $\chi^{(3)}$            | 0.206°             | 42.741°    | 60.764°   |
| $s$                     | 0.500              | 0.500      | 0.500     |
| $\chi(\epsilon)$       | 0.206°             | 36.848°    | 49.533°   |
| $\chi(\epsilon^2)$     | $(1.855 \times 10^{-4})^o$ | 6.048°    | 11.322°   |
| $\chi(\epsilon^3)$     | $(1.924 \times 10^{-7})^o$ | 1.140°    | 3.027°    |
| $\chi^{(3)}$            | 0.207°             | 44.035°    | 63.882°   |

9 Conclusions

In this paper, we have considered both Schwarzschild and Kerr metrics to take into account the spin effect in the calculation of the perihelion precession in binary systems like OJ 287, Sagittarius A*-S2, and H1821+643. The expansion was developed in terms of both $\epsilon \equiv r_s/(a(1-e^2))$ and $\epsilon^* \equiv \left(1 - \frac{2aE'}{cJ}\right)\epsilon$, where $E'$ and $J$ are the energy and angular momentum per unit mass and $a$ is a factor proportional to the black hole spin. Here, the notations $\chi(\epsilon^n)$ for the contribution of the $n$th term and $\chi^{(n)}$ for the complete expansion until $n$th term, were used.

Now, to see if the spin correction is relevant for massive objects, the perihelion advance for three binary systems, Sagittarius A*-S2, OJ 287 and H1821+643, was calculated. In Table 2 are shown these calculations performed using equation (71) for different orders.

Sagittarius A* is a bright and very compact radio source located at the center of the Milky Way. There is great evidence that Sagittarius A* is a supermassive black hole with a mass approximately equal to $4 \times 10^8 M_\odot$. The star S2 has been taken, because is the one that presents a very peculiar orbit. As it can be seen in Table 2, if the value of $\chi^{(3)}$ for $s = 0$ that is 0.205° is compared with the corresponding values for $s = 0.280$ and $s = 0.381$ that is 0.206°, one can see that the difference is not significant. It is worth mentioning that the values of $\chi(\epsilon^n)$ and $\chi^{(n)}$ corresponding to $s = 0$ are in agreement with those calculated in an earlier paper (Marín and Poveda 2018) using the Schwarzschild metric.

OJ 287 is a binary system of black holes located 3.500 million light years from Earth having a total mass of around
1.845 × 10^{10} M_\odot. It can be seen that between the first and the second order terms (\(\chi(e)\) and \(\chi(e^2)\)) there is on average a difference of approximately 5.5° for spin values less than 0.5. For higher orders, the difference is less than or on the order of 1° for the same values of spin. For higher values of spin, such as \(s = 0.55\), the differences increase.

At the third order the contribution to the perihelion precession in a cycle is 0.962° in our expansion for \(s = 0.280\) compared with the value of 0.765° obtained in a earlier paper (Marín and Poveda 2018) taking \(s = 0\). Then, the total perihelion precession is 41.657°. For \(s = 0.381\) the perihelion precession is slightly different (approximately 42.7°). Something interesting about our expansion is that it begins to stabilize taking into account higher order corrections around 43°. Then, equation (71) gives an important correction to the perihelion precession. It is important to mention that the gravitational radiation of the system (that affects directly to the values of the period, the radial distance and the eccentricity of the orbit (Banerjee et al. 2017)), could introduce important modifications in the perihelion advance equation.

For H1821+643 system, that is a very massive black hole with a mass of \(3 \times 10^{10} M_\odot\), the orbital parameters of the gravitational companion are not known, so we used random parameters. In this system, the corrections are very important. For example, taking \(s = 0.381\), the second order contribution \(\chi(e^2)\) is 10.382°, while the third order contribution \(\chi(e^3)\) is 2.635°.

Taking into account all these results, we can conclude that this is an excellent approach to calculate the perihelion precession for binary systems as it can be calculated until any order in the third root of the motion equation \(R_o\) (see equation (23)). With other methods as perturbation theory, the calculations are more complicated and due to the approximations that are made it is easy to make mistakes.

Declarations

Conflict of Interest The authors declare no conflict of interest.

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