EXISTENCE AND STABILITY OF TRAVELING WAVES FOR
LESLIE-GOWER PREDATOR-PREY SYSTEM
WITH NONLOCAL DIFFUSION

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Abstract. This paper will mainly study the information about the existence
and stability of the invasion traveling waves for the nonlocal Leslie-Gower
predator-prey model. By using an invariant cone in a bounded domain with
initial function being defined on and applying the Schauder’s fixed point theo-
rem, we can obtain the existence of traveling waves. Here, the compactness of
the support set of dispersal kernel is needed when passing to an unbounded do-
main in the proof. Then we use the weighted energy to prove that the invasion
traveling waves are exponentially stable as perturbation in some exponentially
as $x \to -\infty$. Finally, by defining the bilateral Laplace transform, we can obtain
the nonexistence of the traveling waves.

1. Introduction. The dynamic relationship between predators and their prey has
been one of the dominant themes in ecology due to its universal existence and
importance. The interactions of prey and predator species are predicted by the
behavior of some mathematical models. According to the different functional re-
response to predation, researchers have constructed many various models. The classi-
cal Lotka-Volterra model and its modified models have been studied for the stability
of equilibria and the existence of traveling waves, see [3, 12, 15, 18, 20, 30, 31, 34]
et al. The properties of the model with the Leslie-Gower functional response and
its modified models have been discussed in [1, 17, 20, 37]. For the Holling-Tanner
predator-prey model, Tanner in [32] has considered the stability of the underlying
ordinary differential system

$$\begin{align*}
\frac{du(t)}{dt} &= u(t)[1 - u(t)] - \Pi(u(t))v(t), \\
\frac{dv(t)}{dt} &= rv(t) \left(1 - \frac{v(t)}{u(t)}\right),
\end{align*}$$

(1)

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where \( r \) denotes the growth rate of predator and \( \Pi(u) = \frac{mu}{A + u} \) (\( m > 0, A > 0 \)) denotes the functional response to predation suggested by Holling in [13]. The second equation of (1) means that the intrinsic population growth rate \( r \) affects not only the potential increase of the population but also its decrease. If \( v \) is greater than \( u \), the population will decline, and the speed of its decline is directly proportional to the intrinsic growth rate. It seems to be a contradiction, but it is realistic since species of small body size and early maturity have high intrinsic growth rates and also have low survival rates and short lives. This equation is the same as that used by Leslie and Gower in [20]. In the last decades, the model (1) has attracted some attention of many researchers. For instance, one can refer to May [22], Hsu and Huang [14] and Murray [25]. Typical example of the functional response \( \Pi \) is given by Holling type functional response. Let us also mention the case where \( \Pi(u) \equiv \alpha u \) with constant \( \alpha > 0 \), that corresponds to the classical Lotka-Volterra functional response. A. Ducrot in [11] has studied the reaction-diffusion system with the Laplacian diffusion

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= d\Delta u(x,t) + u(x,t)[1 - u(x,t)] - \Pi(u(x,t))v(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= \Delta v(x,t) + rv(x,t) \left( 1 - \frac{v(x,t)}{u(x,t)} \right),
\end{align*}
\]

where \( d > 0 \) describes the diffusivity of prey, \( r \) denotes the growth rate of predator and \( \Pi(u) = u\pi(u), \pi : [0, \infty) \rightarrow [0, \infty) \) is a function of the class \( C^1 \) such that \( \pi(u) > 0 \) for all \( u \in (0, 1] \). He has proved that the system has a generalized transition wave with some determined global mean speed of propagation.

As we all know, the standard Laplacian operator corresponds to expected values for individuals moving under a Brownian process. But the movement of individuals which cannot be limited in a small area is often free and random. Recently, various integral operators have been widely used to describe the nonlocal diffusion phenomena. For example, an operator of the form

\[
K[u](x) = \int_{\mathbb{R}} k(x, y)[u(y) - u(x)]dy
\]

appears in the theory of phase transition, ecology, genetics and neurology, see [2, 16, 17, 33]. Meanwhile, many researchers give more attention on the study of traveling waves for the nonlocal reaction diffusion equations. For instance, many authors have obtained the properties of the solution for the reaction-diffusion systems with nonlocal diffusion term, see [3, 4, 7, 8, 9, 10, 28, 30]. They consider the models under the condition of monotone or quasi-monotone, for instance in [35, 38, 39]. The important condition can permit the applications of the powerful monotone dynamical systems and comparison arguments. But some models cannot satisfy the monotone condition. In [27], the authors have obtained the existence of the traveling waves of the model without monotone condition. In [5], we have considered the nonexistence of the general wave solution for the model [2] with \( \Pi(u) = \alpha u \) and the fractional diffusion term \( \Delta^\alpha (\alpha \in (0, 1)) \).

Inspired by these results, we consider the Leslie-Gower predator-prey model with the nonlocal diffusion term, that is
the below assumptions of the kernel function \( J \),

Linearizing the second equation of the system (4) at (1 2.

Existence of the traveling waves.

Laplace transform.

Finally, we obtain the nonexistence of the invasion traveling waves by the bilateral

we get the stability of the invasion traveling wave solution by the weighted energy.

speed \( c \) that for \( J \in C^1(\mathbb{R}) \), \( J(y) = J(-y) \geq 0 \), \( \int_{\mathbb{R}} J(y)dy = 1 \).

Assumption 1.1. (J1) The function \( J \) is a smooth function in \( \mathbb{R} \) and satisfies

\( J(y) = J(-y) \geq 0 \), \( \int_{\mathbb{R}} J(y)dy = 1 \).

(J2) There exists \( \lambda_0 \in (0, +\infty) \) such that \( \int_{\mathbb{R}} J(y)e^{-\lambda y}dy < +\infty \) for any \( \lambda \in [0, \lambda_0) \), and \( \int_{\mathbb{R}} J(y)e^{-\lambda y}dy \to +\infty \) as \( \lambda \to \lambda_0 - 0 \).

In this work, we mainly consider the existence and stability of the invasion traveling wave solution which connects the predator free state (1, 0) with the coexistence state \( \left( \frac{1}{1+\beta}, \frac{1}{1+\beta} \right) \) of the system (3). We will obtain that there exists \( c^* > 0 \) such that for \( c > c^* \), system (3) admits an invasion traveling wave solution with wave speed \( c \); for \( 0 < c < c^* \), system (3) has no invasion traveling waves with wave speed \( c \). Further, we can get the stability of the invasion traveling wave by the weighted energy. Due to the nonlocal diffusion effect, it is more hard to obtain the uniform boundness of solutions. To overcome the difficulties, we construct an invariant cone in a large bounded domain with initial functions being defined on, then pass to the unbounded domain by limiting argument.

This paper is organized as follows. In the next section, we present the proof of the existence of the invasion traveling waves by Schauder’s fixed point theorem under the assumption of the compactly supported for the kernel function \( J \). In Section 3, we get the stability of the invasion traveling wave solution by the weighted energy. Finally, we obtain the nonexistence of the invasion traveling waves by the bilateral Laplace transform.

2. Existence of the traveling waves. In this section, we will prove the existence of the invasion traveling wave solution for the system (3). The traveling wave solution means a solution of the form \( (u(x+ct), v(x+ct)) \). Let \( \xi = x + ct \), then \((u(\xi), v(\xi))\) satisfies

\[
\begin{cases}
    cu'(\xi) = \int_{\mathbb{R}} J(y)(u(\xi - y) - u(\xi))dy + u(\xi)(1 - u(\xi)) \quad - \beta u(\xi)v(\xi), \\
    cv'(\xi) = d \int_{\mathbb{R}} J(y)(v(\xi - y) - v(\xi))dy + rv(\xi) \left( 1 - \frac{v(\xi)}{u(\xi)} \right).
\end{cases}
\]

(4)

Linearizing the second equation of the system (4) at \((1, 0)\), we have

\[
cv'(\xi) = d \int_{\mathbb{R}} J(y)(v(\xi - y) - v(\xi))dy + rv(\xi).
\]

Then we can get a characteristic equation

\[
\Delta(\lambda, c) := d \int_{\mathbb{R}} J(y)(e^{-\lambda y} - 1)dy - c\lambda + r.
\]

(5)
By easy calculations, we can obtain
\[ \Delta(0, c) = r > 0, \quad \frac{\partial \Delta(\lambda, c)}{\partial \lambda} \bigg|_{\lambda=0} = -c < 0, \]
\[ \frac{\partial \Delta(\lambda, c)}{\partial \xi} = -\lambda < 0 \quad \text{for all } \lambda > 0, \]
\[ \frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} = d \int_{\mathbb{R}} J(y) y^2 e^{-\lambda y} dy > 0. \]

In view of the above properties of the function \( \Delta(\lambda, c) \), we can get the following lemma.

**Lemma 2.1.** Assume that (\( J_1 \)) and (\( J_2 \)) hold, then there exists \( c^* > 0 \) and \( \lambda^* > 0 \) such that
\[ \frac{\partial \Delta(\lambda, c)}{\partial \lambda} \bigg|_{(\lambda^*, c^*)} = 0 \quad \text{and} \quad \Delta(\lambda^*, c^*) = 0. \]

Furthermore,
(i) if \( 0 < c < c^* \), then \( \Delta(\lambda, c) > 0 \) for all \( \lambda \in (0, \lambda_0) \);
(ii) if \( c > c^* \), then the equation \( \Delta(\lambda, c) = 0 \) has two positive real roots \( \lambda_1(c), \lambda_2(c) \) with \( 0 < \lambda_1(c) < \lambda^* < \lambda_2(c) < \lambda_0 \), such that \( \Delta(\lambda, c) < 0 \) in \( (\lambda_1(c), \lambda_2(c)) \) and \( \Delta(\lambda, c) > 0 \) in \( (0, \lambda_1(c)) \cup (\lambda_2(c), \lambda_0) \).

In the sequel, we always assume \( c > c^* \) and simply denote \( \lambda_i(c) \) by \( \lambda_i \) for \( i = 1, 2 \), respectively.

**Lemma 2.2.** The function \( v^+(\xi) = e^{\lambda_1 \xi} \) satisfies the following equation
\[ cu'(\xi) = d \int_{\mathbb{R}} J(y)(v^-(\xi - y) - v(\xi)) dy + rv(\xi). \]

**Lemma 2.3.** Let \( 0 < \alpha < \lambda_1 \) be sufficiently small and \( \sigma > \beta \) be large enough. Then the function \( u^-(\xi) := \max\{1 - \sigma e^{\alpha \xi}, 1 - \beta\} \) satisfies
\[ cu'(\xi) \leq \int_{\mathbb{R}} J(y)(u^-(\xi - y) - u(\xi)) dy + u(\xi)(1 - u(\xi)) - \beta u(\xi)V^+(\xi), \quad (6) \]
where \( V^+(\xi) = \min\{e^{\lambda_1 \xi}, 1\} \).

**Proof.** If \( \xi > 0 \), we can get that \( u^-(\xi) = 1 - \beta \) and \( V^+(\xi) = 1 \) which implies that (6) holds.

For \( \frac{1}{\alpha} \ln \frac{\beta}{\sigma} \leq \xi \leq 0 \), it is easy to show \( u^-(\xi) = 1 - \beta \) and \( V^+(\xi) = e^{\lambda_1 \xi} \). By easy calculation, we can get that (6) holds.

When \( \xi < \frac{1}{\alpha} \ln \frac{\beta}{\sigma} \), we can easily obtain \( u^-(\xi) = 1 - \sigma e^{\alpha \xi} \) and \( V^+(\xi) = e^{\lambda_1 \xi} \).

Then we need to prove
\[ -c\sigma \alpha + \sigma \int_{\mathbb{R}} J(y)(e^{-\alpha y} - 1) dy + \beta e^{(\lambda_1 - \alpha)\xi} - \beta \sigma e^{\lambda_1 \xi} \leq 0. \]

Since \( \xi < \frac{1}{\alpha} \ln \frac{\beta}{\sigma} \), we just need to prove
\[ -c\sigma \alpha + \sigma \int_{\mathbb{R}} J(y) \left( e^{-\alpha y} - 1 \right) dy + \beta e^{\frac{\lambda_1 - \alpha}{\alpha}} \ln \frac{\beta}{\sigma} \leq 0. \]

That is,
\[ -c\sigma \alpha + \sigma \int_{\mathbb{R}} J(y)(e^{-\alpha y} - 1) dy + \beta \left( \frac{\beta}{\sigma} \right)^{\frac{\lambda_1 - \alpha}{\alpha}} \leq 0. \quad (7) \]
Keeping $\sigma \alpha = 1$ and letting $\sigma \to \infty$, $\alpha \to 0$, for some $\sigma > 0$ large enough and $0 < \alpha < \lambda_1$ small enough, we can get that holds. This completes the proof of the lemma.

**Lemma 2.4.** Let $\eta \in (0, \min\{\lambda_1, \lambda_2 - \lambda_1\})$ be sufficiently small and some well-chosen $M > 1$ satisfy $\frac{r}{1 - \beta} \leq -M \frac{r}{1 - \beta} \Delta(\lambda_1 + \eta, c)$. Then the function $\nu^{-}(\xi) = e^{\lambda_1 \xi}(1 - Me^{\eta \xi})$ satisfies

$$cv'(\xi) \leq d \int_{\mathbb{R}} J(y)[v(\xi - y) - v(\xi)]dy + rv(\xi) - \frac{r}{1 - \beta}(V^+ (\xi))^2$$

for $\xi < \frac{1}{\eta} \ln \frac{1}{M}.

**Proof.** By the condition, we can get

$$L[v^{-}(\xi)] := cv^{-}(\xi) - d \int_{\mathbb{R}} J(y)[v^{-}(\xi - y) - v^{-}(\xi)]dy - rv^{-}(\xi) + \frac{r}{1 - \beta}(V^+ (\xi))^2$$

$$= c \left[ \lambda_1 e^{\lambda_1 \xi}(1 - Me^{\eta \xi}) - M\eta e^{(\lambda_1 + \eta) \xi} \right]$$

$$-d \int_{\mathbb{R}} J(y) \left[ e^{\lambda_1 (\xi - y)}(1 - Me^{\eta (\xi - y)}) - e^{\lambda_1 \xi}(1 - Me^{\eta \xi}) \right] dy$$

$$-re^{\lambda_1 \xi}(1 - Me^{\eta \xi}) + \frac{r}{1 - \beta} e^{2\lambda_1 \xi}$$

$$= e^{\lambda_1 \xi} \left( c\lambda_1 - d \int_{\mathbb{R}} J(y)(e^{\lambda_1 y} - 1)dy - r \right) + \frac{r}{1 - \beta} e^{2\lambda_1 \xi}$$

$$-Me^{(\eta + \lambda_1) \xi} \left( c(\eta + \lambda_1) - d \int_{\mathbb{R}} J(y)(e^{-(\eta + \lambda_1) y} - 1)dy - r \right).$$

By the definition of $\Delta(\lambda_1, c)$, Lemma 2.1 and $\frac{r}{1 - \beta} \leq -M \frac{r}{1 - \beta} \Delta(\lambda_1 + \eta, c)$, we have

$$L[v^{-}(\xi)] = \Delta(\lambda_1 + \eta, c)Me^{(\lambda_1 + \eta) \xi} + \frac{r}{1 - \beta} e^{2\lambda_1 \xi}$$

$$\leq e^{2\lambda_1 \xi} \Delta(\lambda_1 + \eta, c) \left( Me^{(\eta - \lambda_1) \xi} - M \frac{\lambda_1}{\eta} \right).$$

Since $0 < \eta < \lambda_2 - \lambda_1$, we can know $\Delta(\lambda_1 + \eta, c) < 0$. Then by $\xi < \frac{1}{\eta} \ln \frac{1}{M}$ and $0 < \eta < \lambda_1$, we can get

$$L[v^{-}(\xi)] \leq e^{2\lambda_1 \xi} \Delta(\lambda_1 + \eta, c) \left( Me^{\frac{2\lambda_1}{\eta} \ln \frac{1}{M} - M \frac{\lambda_1}{\eta}} \right) = 0.$$

So we have that the function $v^{-}(\xi)$ satisfies the inequality (8). This completes the proof of the lemma.

Let $\delta \in (0, \min\{1 - \beta, \frac{\beta}{\alpha} c^{\alpha}, e^{\lambda_1 \alpha}(1 - Me^{\eta \alpha})\})$ be sufficiently small, where $a = \frac{1}{\eta} \ln \frac{1}{M} + \frac{1}{\alpha} \ln \frac{\beta}{\alpha}$. Then we define a function

$$V^{-}(\xi) = \begin{cases} v^{-}(\xi), & \xi \leq \frac{1}{\eta} \ln \frac{1}{M}, \\ \delta, & \xi \geq \frac{1}{\eta} \ln \frac{1}{M}, \end{cases}$$

and a function set

$$\Gamma_A = \left\{ (\phi(\cdot), \varphi(\cdot)) \in C([-A, A], \mathbb{R}^2) : \begin{array}{l} \phi(-A) = u_-(A), \varphi(-A) = V^{-}(-A), \\ u^{-}(\xi) \leq \phi(\xi) \leq 1 - \beta \delta, \\ V^{-}(\xi) \leq \varphi(\xi) \leq V^{+}(\xi) \\ \text{for any } \xi \in [-A, A] \end{array} \right\},$$
where $A > \max\{\frac{1}{n} \ln M, \frac{1}{n} \ln \frac{2}{\pi}\}$. For any $(\phi(\cdot), \varphi(\cdot)) \in C([-A, A], \mathbb{R}^2)$, we define

$$
\hat{\phi}(\xi) = \begin{cases} 
\phi(A), & \xi > A, \\
\phi(\xi), & |\xi| \leq A, \\
u^-(\xi), & \xi < -A,
\end{cases} \quad \hat{\varphi}(\xi) = \begin{cases} 
\varphi(A), & \xi > A, \\
\varphi(\xi), & |\xi| \leq A, \\
V^-(\xi), & \xi < -A,
\end{cases}
$$

and consider the following initial value problems

\begin{align}
\frac{cu'}{\beta u} &= \int \limits_{\mathbb{R}} J(y)(\hat{\phi}(\xi - y) - u(\xi))dy + \phi(\xi)(1 - u(\xi)) - \beta u(\xi)\varphi(\xi), \quad (9) \\
\frac{cv'}{\beta v} &= d \int \limits_{\mathbb{R}} J(y)(\hat{\varphi}(\xi - y) - v(\xi))dy + \varphi(\xi)\left(1 - \frac{v(\xi)}{\phi(\xi)}\right), \quad (10)
\end{align}

with

$$u(-A) = u^-(\xi), \quad v(-A) = V^-(\xi). \quad (11)$$

Obviously, the problems (9)-(11) admit a unique solution $(u_A(\cdot), v_A(\cdot))$ satisfying $u_A(\cdot) \in C^1([-A, A])$ and $v_A(\cdot) \in C^1([-A, A])$. Then, we can define an operator $F = (F_1, F_2) : \Gamma_A \to C([-A, A])$ by $F_1[\phi, \varphi](\xi) = u_A(\xi)$ and $F_2[\phi, \varphi](\xi) = v_A(\xi)$ for $\xi \in [-A, A]$.

**Lemma 2.5.** The operator $F$ maps $\Gamma_A$ into $\Gamma_A$.

**Proof.** For $\forall (\phi(\cdot), \varphi(\cdot)) \in \Gamma_A$, we should prove that

$$F_1[\phi, \varphi](-A) = u^-((-A)), \quad F_2[\phi, \varphi](-A) = V^-(\xi),$$

and

$$u^-(\xi) \leq F_1[\phi, \varphi](\xi) \leq 1 - \beta\delta, \quad V^-(\xi) \leq F_2[\phi, \varphi](\xi) \leq V^+(\xi)$$

for any $\xi \in [-A, A]$. By the definition of the operator $F$, it is obvious to see

$$F_1[\phi, \varphi](-A) = u_A(-A) = u^-((-A)) \quad \text{and} \quad F_2[\phi, \varphi](-A) = v_A(-A) = V^-(\xi).$$

For $\xi \in [-A, A]$, we first consider $F_1[\phi, \varphi](\xi)$. Using the definition of the operator $F$, it is sufficient to show $u^-(\xi) \leq u_A(\xi) \leq 1 - \beta\delta$. According to the definition of $\hat{\phi}(\xi)$ and the choosing of the constant $\delta$, we can know that

$$\int \limits_{\mathbb{R}} J(y)[\hat{\phi}(\xi - y) - (1 - \beta\delta)]dy + \beta \delta \varphi(\xi) - \beta(1 - \beta\delta)\varphi(\xi) \leq 0,$$

which implies that $1 - \beta\delta$ is a super-solution of (9). Thus we can obtain that $u_A(\xi) \leq 1 - \beta\delta$ for $\xi \in [-A, A]$. By the definition of $\varphi(\xi)$ and Lemma 2.3, we know that

\begin{align}
&cu^-(\xi) - \int \limits_{\mathbb{R}} J(y)(\hat{\phi}(\xi - y) - u^-(\xi))dy - \phi(\xi)(1 - u^-(\xi)) + \beta u^-(\xi)\varphi(\xi) \\
&\leq cu^-(\xi) - \int \limits_{\mathbb{R}} J(y)(u^-(\xi - y) - u^-(\xi))dy - u^-(\xi)(1 - u^-(\xi)) + \beta u^-(\xi)V^+(\xi) \\
&\leq 0
\end{align}

for any $\xi \in (-A, A)$. Since $u_A(-A) = u^-((-A))$, the comparison principle implies that $u^-(\xi) \leq u_A(\xi)$ for $\xi \in [-A, A]$. So we get that $u^-(\xi) \leq u_A(\xi) \leq 1 - \beta\delta$ for all $\xi \in [-A, A]$.

By a similar argument and using Lemmas 2.2 and 2.4 it is easy to show that

$$V^-(\xi) \leq v_A(\xi) \leq V^+(\xi)$$

for $\xi \in [-A, A]$.

This ends the proof. 

**Lemma 2.6.** The operator $F : \Gamma_A \to \Gamma_A$ is completely continuous.
Proof. By the definition of the operator $\mathcal{F}[\phi, \varphi](\xi)$, we have that $\mathcal{F}_1[\phi, \varphi](\xi) = u_A(\xi)$ and $\mathcal{F}_2[\phi, \varphi](\xi) = v_A(\xi)$ for any $\xi \in [-A, A]$. We first show that $\mathcal{F}$ is continuous. By a direct calculation, we have that

$$u_A(\xi) = u^-(A) \exp \left\{ -\frac{1}{c} \int_{-A}^{\xi} [1 + \phi(s) + \beta \varphi(s)] ds \right\}$$

$$+ \frac{1}{c} \int_{-A}^{\xi} \exp \left\{ -\frac{1}{c} \int_{-A}^{\xi} [1 + \phi(s) + \beta \varphi(s)] ds \right\} [f_\phi(\eta) + \phi(\eta)] d\eta,$$

and

$$v_A(\xi) = V^-(A) \exp \left\{ -\frac{1}{c} \int_{-A}^{\xi} \left[ d + \frac{\varphi(s)}{\phi(s)} \right] ds \right\}$$

$$+ \frac{1}{c} \int_{-A}^{\xi} \exp \left\{ -\frac{1}{c} \int_{-A}^{\xi} \left[ d + \frac{\varphi(s)}{\phi(s)} \right] ds \right\} [g_\varphi(\eta) + r \varphi(\eta)] d\eta,$$

where

$$f_\phi(\eta) = \int_{-\infty}^{-A} J(\eta-y) u^-(y) dy + \int_{-A}^{A} J(\eta-y) \phi(y) dy + \int_{A}^{+\infty} J(\eta-y) \phi(A) dy,$$

and

$$g_\varphi(\eta) = \int_{-\infty}^{-A} J(\eta-y) V^-(y) dy + \int_{-A}^{A} J(\eta-y) \varphi(y) dy + \int_{A}^{+\infty} J(\eta-y) \varphi(A) dy.$$

For $\forall (\phi_1(\cdot), \varphi_1(\cdot)), (\phi_2(\cdot), \varphi_2(\cdot)) \in \Gamma_A$, we have that

$$|f_{\phi_1}(\eta) - f_{\phi_2}(\eta)| \leq \int_{-A}^{A} |J(\eta-y)| |\phi_1(y) - \phi_2(y)| dy$$

$$+ \int_{A}^{+\infty} |J(\eta-y)| |\phi_1(A) - \phi_2(A)| dy$$

$$\leq 2 \max_{y \in [-A, A]} |\phi_1(y) - \phi_2(y)|,$$

and

$$|g_{\varphi_1}(\eta) - g_{\varphi_2}(\eta)| \leq 2 \max_{y \in [-A, A]} |\varphi_1(y) - \varphi_2(y)|.$$

Combining with the continuity of the compound function, we can obtain that $\mathcal{F}$ is continuous from equations (12) and (13).

Next we show that $\mathcal{F}$ is compacted, that is, we should prove that for any bounded subset $\Omega \subseteq \Gamma_A$, $\mathcal{F}(\Omega)$ is precompact. By the definition of $\mathcal{F}$, we have that for all $(u_A, v_A) \in \mathcal{F}(\Omega)$, there exists $(\phi, \varphi) \in \Omega$ such that

$$\mathcal{F}[\phi, \varphi](\xi) = (u_A, v_A)(\xi), \forall \xi \in [-A, A].$$

Since $(\phi, \varphi) \in \Omega$, (12) and (13), we have that there exists a constant $M_1 > 0$ such that

$$|u_A(\xi)| \leq M_1 \text{ and } |v_A(\xi)| \leq M_1, \forall \xi \in [-A, A].$$

That is, $\mathcal{F}(\Omega)$ is uniformly bounded. Further, according to the equations (9), (10) and the above inequality, then there exists some constant $M_2 > 0$ such that

$$|u_A'(\xi)| \leq M_2 \text{ and } |v_A'(\xi)| \leq M_2, \forall \xi \in [-A, A].$$

So we can get that $\mathcal{F}(\Omega)$ is equicontinuous. By Arzela-Ascoli Theorem, we have that $\mathcal{F}(\Omega)$ is precompact. Then we get that $\mathcal{F} : \Gamma_A \to \Gamma_A$ is completely continuous with respect to the maximum norm. \qed
Proof. By the definition of $\Gamma_A$, it is easy to see that $\Gamma_A$ is closed and convex. Thus, according to Lemma 2.6 and using the Schauder’s fixed point theorem, there exists $(u_A^*(\cdot), v_A^*(\cdot)) \in \Gamma_A$ such that

$$(u_A^*(\xi), v_A^*(\xi)) = F[u_A^*, v_A^*](\xi), \ \forall \xi \in [-A, A].$$

To obtain the existence of solutions for system (4), we need some estimates about $(u_A^*(\cdot), v_A^*(\cdot))$. For the sake of convenience, we use $(u_A(\cdot), v_A(\cdot))$ instead of $(u_A^*(\cdot), v_A^*(\cdot))$. In order to get the estimates, we make another assumption on the kernel function $J$.

Assumption 2.1. $(J_3)$ The kernel function $J$ is compactly supported.

Lemma 2.8. Assume that $(J_1)$ and $(J_3)$ hold, then there exists some constant $C > 0$ such that

$$\|u_A\|_{C^1([-B,B])} < C \quad \text{and} \quad \|v_A\|_{C^1([-B,B])} < C$$

for any $B > 0$ satisfying $B < A$ where $A > \max \left\{ \frac{1}{\alpha} \ln M, \frac{1}{\alpha} \ln \frac{\sigma}{\beta} \right\}$.

Proof. By Theorem 2.7 we have that $(u_A(\cdot), v_A(\cdot))$ satisfies

$$cu_A'(\xi) = \int_{\mathbb{R}} J(y)[\dot{u}_A(\xi - y) - u_A(\xi)]dy + u_A(\xi)[1 - u_A(\xi)] - \beta u_A(\xi)v_A(\xi), \quad (14)$$

and

$$cv_A'(\xi) = d \int_{\mathbb{R}} J(y)[\dot{v}_A(\xi - y) - v_A(\xi)]dy + rv_A(\xi) \left( 1 - \frac{v_A(\xi)}{u_A(\xi)} \right), \quad (15)$$

where

$$\dot{u}_A(\xi) = \begin{cases} u_A(\xi), & \xi > A, \\ u_A(\xi), & |\xi| \leq A, \\ u^-(\xi), & \xi < -A, \end{cases} \quad \dot{v}_A(\xi) = \begin{cases} v_A(\xi), & \xi > A, \\ v_A(\xi), & |\xi| \leq A, \\ V^-(\xi), & \xi < -A. \end{cases}$$

Following that $1 - \beta \leq u_A(\xi) \leq 1 - \beta \delta < 1$ and $0 < \delta \leq v_A(\xi) \leq 1$ for $\xi \in [-B, B]$, we have

$$\frac{|v_A'(\xi)|}{c} \leq \frac{d}{c} \left| \int_{\mathbb{R}} J(y)\dot{v}_A(\xi - y)dy \right| + \frac{r}{c} |v_A(\xi)| + \frac{1}{c} |u_A(\xi)| \left| 1 - \frac{v_A(\xi)}{u_A(\xi)} \right| \leq \frac{2d + r/4}{c},$$

and

$$\frac{|v_A'(\xi)|}{c} \leq \frac{d}{c} \left| \int_{\mathbb{R}} J(y)\dot{v}_A(\xi - y)dy \right| + \frac{r}{c} |v_A(\xi)| + \frac{1}{c} |u_A(\xi)| \left| 1 - \frac{v_A(\xi)}{u_A(\xi)} \right| \leq \frac{2d + r/4}{c}.$$
for any $\xi, \eta \in [-B, B]$. In view of (14), we have

$$c|u'_A(\xi) - u'_A(\eta)| \leq \left| \int_{-R}^{+R} J(y) [\dot{u}_A(\xi - y) - \dot{u}_A(\eta - y)] dy \right| + |u_A(\xi) - u_A(\eta)|$$

$$+ |u_A(\xi)[1 - u_A(\xi)] - u_A(\eta)[1 - u_A(\eta)]|$$

$$+ |\beta| u_A(\xi)v_A(\xi) - u_A(\eta)v_A(\eta)|$$

$$:= U_1 + U_2 + U_3 + \beta U_4. \tag{17}$$

By the conditions $(J_1)$ and $(J_2)$, we can assume that $L$ is its Lipschitz constant and $R$ is the radius of supp $J$. Then, we have

$$U_1 = \left| \int_{-R}^{+R} J(y) \dot{u}_A(\xi - y) dy - \int_{-R}^{+R} J(y) \dot{u}_A(\eta - y) dy \right|$$

$$= \left| \int_{\xi - R}^{\xi + R} J(\xi - y) \dot{u}_A(y) dy - \int_{\eta - R}^{\eta + R} J(\eta - y) \dot{u}_A(y) dy \right|$$

$$\leq \left| \int_{\eta + R}^{\xi + R} J(\xi - y) \dot{u}_A(y) dy + \int_{\xi - R}^{\eta - R} J(\xi - y) \dot{u}_A(y) dy \right|$$

$$+ \int_{\eta - R}^{\eta + R} \left| J(\xi - y) - J(\eta - y) \right| \dot{u}_A(y) dy$$

$$\leq 2 \left( \| J \|_{L^\infty} + RL \right) |\xi - \eta|,$$

$$U_3 = |u_A(\xi) - u_A^2(\xi) - u_A(\eta) + u_A^2(\eta)|$$

$$\leq |u_A(\xi) - u_A(\eta)| + |u_A^2(\xi) - u_A^2(\eta)|$$

$$\leq 3|u_A(\xi) - u_A(\eta)|,$$

and

$$U_4 = |u_A(\xi)v_A(\xi) - u_A(\eta)v_A(\eta)|$$

$$\leq |u_A(\xi) - u_A(\eta)||v_A(\xi)| + |u_A(\eta)||v_A(\xi) - v_A(\eta)|$$

$$\leq |u_A(\xi) - u_A(\eta)| + |v_A(\xi) - v_A(\eta)|.$$

Combining (17) and (16), we obtain that there exists some constant $L_2 > 0$ such that

$$|u'_A(\xi) - u'_A(\eta)| \leq L_2 |\xi - \eta|.$$

Then applying with (15), we also have

$$c|v'_A(\xi) - v'_A(\eta)| \leq \left| \int_{-R}^{+R} J(y) [\dot{v}_A(\xi - y) - \dot{v}_A(\eta - y)] dy \right| + |v_A(\xi) - v_A(\eta)|$$

$$+ r |v_A(\xi) - v_A(\eta)| \left( 1 - \frac{v_A(\xi)}{u_A(\xi)} \right) - v_A(\eta) \left( 1 - \frac{v_A(\eta)}{u_A(\eta)} \right)$$

$$:= dV_1 + dV_2 + rV_3. \tag{18}$$

Similar to the same argument of $U_1$, we have that

$$V_1 = \left| \int_{-R}^{+R} J(y) \dot{v}_A(\xi - y) dy - \int_{-R}^{+R} J(y) \dot{v}_A(\eta - y) dy \right|$$

$$\leq 2 \left( \| J \|_{L^\infty} + RL \right) |\xi - \eta|.$$

Then, by (16), (18) with

$$V_3 \leq \left| v_A(\xi) - v_A(\eta) \right| + \left| \frac{v_A^2(\xi)u_A(\eta) - v_A^2(\eta)u_A(\xi)}{u_A(\xi)u_A(\eta)} \right|$$

$$\leq \left| v_A(\xi) - v_A(\eta) \right| + \left| \frac{v_A^2(\xi) - v_A^2(\eta)}{u_A(\xi)} \right| + \left| v_A^2(\eta)u_A(\eta) - u_A(\xi) \right| \left| u_A(\xi)u_A(\eta) \right|$$
Proof. Choosing an increasing sequence 

Assume that 

we can get that 

for any \( \xi, \eta \in [-B, B] \). So we have obtained that there exists a constant \( C > 0 \) for any \( B \) satisfying \( B < A \) independent of \( A > \max \left\{ \frac{1}{\eta} \ln M, \frac{1}{\eta} \ln \frac{\sigma}{\eta} \right\} \) such that 

\[
\|u_A\|_{C^{1,1}([-B, B])} < C \quad \text{and} \quad \|v_A\|_{C^{1,1}([-B, B])} < C.
\]

\( \square \)

**Theorem 2.9.** Assume that \((J_1)\) and \((J_3)\) hold. For any \( c > c^* \), there exists a pair function \((\tilde{u}(\xi), \tilde{v}(\xi))\) satisfying \((4)\), \((\tilde{u}(-\infty), \tilde{v}(-\infty)) = (1, 0)\) and \((\tilde{u}(+\infty), \tilde{v}(+\infty)) = \left(1 + \frac{1}{1 + \beta} \right)\).

Proof. Choosing an increasing sequence \( \{A_n\}_{n=1}^{+\infty} \) such that \( A_n > \max \left\{ \frac{1}{\eta} \ln M, \frac{1}{\eta} \ln \frac{\sigma}{\eta} \right\} \) for each \( n \) and \( \lim_{n \to +\infty} A_n = +\infty \). For every \( c > c^* \), there exists \((u_{A_n}, v_{A_n}) \in \Gamma_{A_n}\) which satisfies Lemma 2.8 and equations \((14), (15)\) in \( \xi \in [-A_n, A_n] \). According to the estimates for the sequence \( \{(u_{A_n}, v_{A_n})\} \) in Lemma 2.8, we can extract a subsequence by a standard diagonal extract argument, denoted by \( \{(u_{A_n}, v_{A_n})\}_{k \in \mathbb{N}} \), tending towards \((\tilde{u}, \tilde{v}) \in C^1(\mathbb{R})\) in the following topologies

\[
u_{A_{n_k}} \to \tilde{u} \quad \text{and} \quad v_{A_{n_k}} \to \tilde{v} \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}) \quad \text{as} \quad k \to +\infty.
\]

By the assumption of the kernel function \( J(y) \) and applying the dominated convergence theorem, we can get that

\[
\lim_{k \to +\infty} \int_{\mathbb{R}} J(y)\tilde{u}_{A_{n_k}}(\xi - y)dy = \int_{\mathbb{R}} J(y)\tilde{u}(\xi - y)dy
\]

and

\[
\lim_{k \to +\infty} \int_{\mathbb{R}} J(y)\tilde{v}_{A_{n_k}}(\xi - y)dy = \int_{\mathbb{R}} J(y)\tilde{v}(\xi - y)dy
\]

for any \( \xi \in \mathbb{R} \). Then, it is easy to show that \((\tilde{u}, \tilde{v})\) satisfies system \((4)\) and

\[
\begin{split}
\tilde{u}^-(\xi) &\leq \tilde{u}(\xi) \leq 1, \quad \tilde{V}^-(-\infty) \leq \tilde{v}(\xi) \leq \tilde{V}^+(\xi).
\end{split}
\]

By the above inequality, we can further get that \((\tilde{u}, \tilde{v})\) satisfies \((\tilde{u}(-\infty), \tilde{v}(-\infty)) = (1, 0)\), and

\[
1 - \beta \leq \liminf_{\xi \to +\infty} \tilde{u}(\xi) \leq \limsup_{\xi \to +\infty} \tilde{u}(\xi) \leq 1 - \beta \delta, \quad \delta \leq \liminf_{\xi \to +\infty} \tilde{v}(\xi) \leq \limsup_{\xi \to +\infty} \tilde{v}(\xi) \leq 1.
\]

Next we give the proof of the asymptotic behavior of the solution \((\tilde{u}(\xi), \tilde{v}(\xi))\) at \(+\infty\). Since \(\limsup_{\xi \to +\infty} \tilde{u}(\xi) \leq 1 - \beta \delta\) and \(\liminf_{\xi \to +\infty} \tilde{v}(\xi) \geq \delta\), there exist \(\xi > \xi_0\) and \(\delta_0 \in (0, \delta)\) such that \(\tilde{u}(\xi) \leq 1 - \beta \delta_0\) and \(\tilde{v}(\xi) \geq \delta_0\) for any \(\xi > \xi_0\). Then by \((19)\), we have that

\[
\delta_0 < \tilde{u}(\xi) \leq 1 - \beta \delta_0 \quad \text{and} \quad \delta_0 \leq \tilde{v}(\xi) \leq 1
\]

for any \(\xi > \xi_0\). Now, we introduce a sequence \(\{\gamma_n\}_{n \geq 0}\) defined by

\[
\gamma_0 = 1, \quad \gamma_1 = \delta_0, \quad \frac{1 - \gamma_{n+1}}{\beta} = \gamma_n, \quad n \geq 1.
\]
By the above definition, we can get that the sequences \( \{ \gamma_{2n} \}_{n \geq 0} \) and \( \{ \gamma_{2n+1} \}_{n \geq 0} \) are adjacent. They converge to \( \frac{1}{1+\beta} \) and satisfy for each \( n \geq 0 \),

\[
\gamma_1 < \gamma_3 < \cdots < \gamma_{2n+1} < \cdots < \frac{1}{1+\beta} < \cdots < \gamma_{2n} < \cdots < \gamma_2 < \gamma_0.
\]

Next we prove that

\[
\gamma_{2n+1} \leq \tilde{u}(\xi) \leq \gamma_{2n+2} \quad \text{and} \quad \gamma_{2n+1} \leq \tilde{v}(\xi) \leq \gamma_{2n}
\]

for all \( n \geq 0 \) and \( \xi > \xi_0 \). According to the inequality \( (20) \), this inequality \( (21) \) holds true for \( n = 0 \). Let us now argue by induction on \( n \). Assume that \( (21) \) hold true for all \( n \geq 0 \) and let us prove that \( (21) \) holds true for \( n + 1 \). Since \( \tilde{u}(\xi) \leq \gamma_{2n+2} \), then \( \tilde{v}(\xi) \)

satisfies

\[
c_P'(\xi) - d \int_{\mathbb{R}} J(y)[\tilde{v}(\xi - y) - \tilde{v}(\xi)]dy - r \tilde{v}(\xi)
\]

\[
\left( 1 - \frac{\tilde{v}(\xi)}{\gamma_{2n+2}} \right) \leq 0 \quad \text{for} \quad \xi \geq \xi_0.
\]

That is to say that \( \tilde{v}(\xi) \) is the subsolution of the equation

\[
c_P'(\xi) - d \int_{\mathbb{R}} J(y)[\tilde{v}(\xi - y) - \tilde{v}(\xi)]dy - r \tilde{v}(\xi)
\]

\[
\left( 1 - \frac{\tilde{v}(\xi)}{\gamma_{2n+2}} \right) = 0 \quad \text{for} \quad \xi \geq \xi_0. \quad (22)
\]

Since \( \gamma_{2n+2} \) is a solution of the equation \( (22) \), we can get that \( \tilde{v}(\xi) \leq \gamma_{2n+2} \) for all \( \xi \geq \xi_0 \). Then we can get that \( \tilde{u}(\xi) \) satisfies

\[
c_P'(\xi) - \int_{\mathbb{R}} J(y)[\tilde{u}(\xi - y) - \tilde{u}(\xi)]dy - \tilde{u}(\xi)[1 - \tilde{u}(\xi)] + \beta \gamma_{2n+2} \tilde{u}(\xi) \geq 0 \quad \text{for} \quad \xi \geq \xi_0.
\]

That is, \( \tilde{u}(\xi) \) is the supersolution of the equation

\[
c_P'(\xi) - \int_{\mathbb{R}} J(y)[\tilde{u}(\xi - y) - \tilde{u}(\xi)]dy - \tilde{u}(\xi)[1 - \tilde{u}(\xi)] + \beta \gamma_{2n+2} \tilde{u}(\xi) = 0 \quad \text{for} \quad \xi \geq \xi_0.
\]

Using the fact that \( 1 - \gamma_{2n+3} = \beta \gamma_{2n+2} \) for \( n \geq 0 \), we can get that \( \tilde{u}(\xi) \geq \gamma_{2n+3} \) for \( \xi \geq \xi_0 \). By the same arguments as before and \( \tilde{u}(\xi) \geq \gamma_{2n+3} \), one can easily conclude that \( \tilde{v}(\xi) \geq \gamma_{2n+3} \) for \( \xi \geq \xi_0 \). Then we can use the result to get that \( \tilde{u}(\xi) \leq \gamma_{2n+4} \) for \( \xi \geq \xi_0 \). Thus \( (21) \) holds true for \( n + 1 \). Letting \( n \to +\infty \) of \( (21) \), we can obtain that

\[
\tilde{u}(\xi) \equiv \frac{1}{1+\beta}, \quad \tilde{v}(\xi) \equiv \frac{1}{1+\beta}
\]

for \( \xi \geq \xi_0 \). So we can get \( \tilde{u}(+\infty) = \tilde{v}(+\infty) = \frac{1}{1+\beta} \). This completes the proof of the theorem.

\( \Box \)

3. Stability of traveling waves . In this section, we will give the proof of the stability for the invasion traveling waves. Before giving our stability result, we need to show an assumption on parameters. \( (J_0) \) The parameters \( \beta, r \) and \( d \) satisfy

\[
r < \frac{\beta^2 + \beta^2 - 7\beta - 2}{4(1+\beta)} \quad \text{and} \quad d < \frac{4r(1-\beta) + 2\beta}{1+\beta} - \frac{4r + 2}{(1-\beta)^2}.
\]

Define two functions on \( \eta \) as follows

\[
f_1(\eta) = \frac{2 + \beta}{1+\beta} - \int_{-\infty}^{0} J(y)e^{-\eta y}dy - \frac{2r + 1}{(1-\beta)^2},
\]

and

\[
f_2(\eta) = \frac{2r(1-\beta) + \beta}{1+\beta} - d \int_{-\infty}^{0} J(y)e^{-\eta y}dy - \frac{2r + 1}{(1-\beta)^2}.
\]
By the assumptions \((J_1), (J_2)\) and the continuity of \(f_i(\eta)\) \((i = 1, 2)\), it is easily show
that there exists \(\eta_0\) such that \(f_i(\eta_0) > 0\) \((i = 1, 2)\).

Next, we define two functions on \(\xi\) as follows

\[
g_1(\xi) = (4 + \beta)\tilde{u} - 2 + 2\beta \tilde{v} - \int_{-\infty}^{0} J(y)e^{-\rho_0y}dy - \frac{2r + 1}{(1 - \beta)^2},
\]

and

\[
g_2(\xi) = 4r\tilde{v} - 2r + \beta \tilde{u} - d \int_{-\infty}^{0} J(y)e^{-\rho_0y}dy - \frac{2r + 1}{(1 - \beta)^2},
\]

where \((\tilde{u}, \tilde{v})\) is a traveling wave solution given in Theorem 2.9. It is easy to see that

\[
\lim_{\xi \to +\infty} g_1(\xi) = f_1(\eta_0) > 0 \quad \text{and} \quad \lim_{\xi \to +\infty} g_2(\xi) = f_2(\eta_0) > 0,
\]

which implies that there exists \(\xi_0\) large enough such that

\[
g_i(\xi) > 0 \quad \text{for} \quad \xi > \xi_0 \quad (i = 1, 2). \tag{23}
\]

Following we define a weight function by

\[
w(\xi) = \begin{cases} e^{-\rho_0(\xi - \xi_0)}, & \xi \leq \xi_0, \\ 1, & \xi > \xi_0. \end{cases} \tag{24}
\]

Here we state the stability of the traveling wave solution for the system \([3]\) with
the following initial data

\[
u(x, 0) = u_0(x) \quad \text{and} \quad v(x, 0) = v_0(x). \tag{25}
\]

**Theorem 3.1.** Assume that \((J_1), (J_3)\) and \((J_4)\) hold and \((\tilde{u}(x + ct), \tilde{v}(x + ct))\) is a
traveling wave solution with the wave speed \(c > \max\{c^*, \frac{1}{\rho_0}\}\) \(c_1, c_2\} \}\) given in
Theorem 2.9 where

\[
c_1 = -\frac{5}{2} + 3\beta + \beta^2 + \int_{\mathbb{R}} J(y)e^{-\rho_0y}dy + \frac{2r + 1}{(1 - \beta)^2}
\]

and

\[
c_2 = 2r - \frac{d}{2} - \frac{\beta(1 - \beta)}{2} + d \int_{\mathbb{R}} J(y)e^{-\rho_0y}dy + \frac{2r + 1}{(1 - \beta)^2}.
\]

If the initial data satisfies

\[(1 - \beta, 0) \leq (u_0(x), v_0(x)) \leq (1, 1), \quad x \in \mathbb{R},
\]

and the initial perturbations satisfy \((u_0(x) - \tilde{u}(x)) \in H^1_{w}(\mathbb{R})\) \(v_0(x) - \tilde{v}(x) \in H^1_{w}(\mathbb{R})\),
then the nonnegative solution of the Cauchy problem \([3]\) and \([25]\) uniquely
exists and satisfies

\[(1 - \beta, 0) \leq (u(x, t), v(x, t)) \leq (1, 1), \quad x \in \mathbb{R}, t > 0
\]

and

\[(u(x, t) - \tilde{u}(x + ct)), (v(x, t) - \tilde{v}(x + ct)) \in C([0, +\infty); H^1_{w}(\mathbb{R})) \cap L^2([0, +\infty); H^1_{w}(\mathbb{R})),
\]

where \(w(x)\) is defined by \([24]\). Further, they satisfy

\[\sup_{x \in \mathbb{R}} |u(x, t) - \tilde{u}(x + ct)| \leq Ce^{-\mu t} \quad \text{and} \quad \sup_{x \in \mathbb{R}} |v(x, t) - \tilde{v}(x + ct)| \leq Ce^{-\mu t}
\]

for all \(t > 0\), where \(C\) and \(\mu\) are some positive constants. Namely, \((u(x, t), v(x, t))\)
converges to the traveling wave solution \((\tilde{u}(x + ct), \tilde{v}(x + ct))\) exponentially in time \(t\).
By the standard energy method and continuity extension method (see [23, 24]) or the theory of abstract functional differential equations in [21], we can obtain the existence and uniqueness of the solution for (3) and (25) in Theorem 3.1. Here we omit the details and mainly show the proof of the stability.

Define the functions $u_0^-(x) = \min\{u_0(x), \tilde{u}(x)\}$, $u_0^+(x) = \max\{u_0(x), \tilde{u}(x)\}$, $v_0^-(x) = \min\{v_0(x), \tilde{v}(x)\}$ and $v_0^+(x) = \max\{v_0(x), \tilde{v}(x)\}$ for $x \in \mathbb{R}$, it is easily obtained that

$$
\begin{align*}
1 - \beta & \leq u_0^-(x) \leq u_0(x) \leq u_0^+(x) \leq 1, \ x \in \mathbb{R}, \\
1 - \beta & \leq u_0^-(x) \leq \tilde{u}(x) \leq u_0^+(x) \leq 1, \ x \in \mathbb{R}, \\
0 & \leq v_0^-(x) \leq v_0(x) \leq v_0^+(x) \leq 1, \ x \in \mathbb{R}, \\
0 & \leq v_0^-(x) \leq \tilde{v}(x) \leq v_0^+(x) \leq 1, \ x \in \mathbb{R}.
\end{align*}
$$

(26)

From the comparison principle, we can get that

$$
\begin{align*}
1 - \beta & \leq u^-(x, t) \leq u(x, t) \leq u^+(x, t) \leq 1, \ (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
1 - \beta & \leq u^-(x, t) \leq \tilde{u}(x + ct) \leq u^+(x, t) \leq 1, \ (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
0 & \leq v^-(x, t) \leq v(x, t) \leq v^+(x, t) \leq 1, \ (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
0 & \leq v^-(x, t) \leq \tilde{v}(x + ct) \leq v^+(x, t) \leq 1, \ (x, t) \in \mathbb{R} \times \mathbb{R}^+.
\end{align*}
$$

(27)

Setting

$$
U_0(\xi) = u_0^+(x) - \tilde{u}(x), \ U(\xi, t) = u^+(x, t) - \tilde{u}(x + ct)
$$

and

$$
V_0(\xi) = v_0^+(x) - \tilde{v}(x), \ V(\xi, t) = v^+(x, t) - \tilde{v}(x + ct),
$$

where $\xi = x + ct$, it follows from (26) and (27) that

$$(1 - \beta, 0) \leq (U_0(\xi), V_0(\xi)) \leq (1, 1) \quad \text{and} \quad (1 - \beta, 0) \leq (U(\xi, t), V(\xi, t)) \leq (1, 1).$$

First of all, we will establish some key inequalities to get the basic estimates. Define $G_w^i(\xi, t)$ ($i = 1, \cdots, 4$) as follows

$$
\begin{align*}
G_w^1(\xi, t) &= -c \frac{w'(\xi)}{w(\xi)} + 4\tilde{u}(\xi) - 1 + 2\beta[V(\xi, t) + \tilde{v}(\xi)] + 2U(\xi, t) + \beta \tilde{u}(\xi) \\
&\quad - \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} \, dy - \frac{r\tilde{v}^2}{\tilde{u}^2}, \\
G_w^2(\xi, t) &= -c \frac{w'(\xi)}{w(\xi)} + d + 4r\tilde{v}(\xi) - 2r + 2rV(\xi) + \beta \tilde{u}(\xi) \\
&\quad - d \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} \, dy - \frac{r\tilde{v}^2}{\tilde{u}^2}(\xi), \\
G_w^3(\xi, t) &= -c \frac{w'(\xi)}{w(\xi)} + 4\tilde{u}(\xi) - 1 + 2\beta[V(\xi, t) + \tilde{v}(\xi)] + 4U(\xi, t) \\
&\quad - \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} \, dy + \beta[\tilde{u}(\xi) + U(\xi, t)] - \frac{2r\tilde{v}(\xi)V(\xi, t)}{(1 - \beta)^2} \\
&\quad - rV^2(\xi, t) + \tilde{v}(\xi) \frac{1}{2(1 - \beta)^2} + \frac{r\tilde{v}^2(\xi)U(\xi, t)}{2}, \\
R(\xi, t) &= -c \frac{w'(\xi)}{w(\xi)} + d + 4r\tilde{v}(\xi) + V(\xi, t) - 2r + \beta[U(\xi, t) + \tilde{u}(\xi)] \\
&\quad - \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} \, dy - \frac{2r\tilde{v}(\xi)V(\xi, t)}{(1 - \beta)^2} - \frac{r[V^2(\xi, t) + \tilde{v}(\xi)]}{2(1 - \beta)^2}.
\end{align*}
$$

(28)

(29)

(30)

(31)
Lemma 3.2. Assume that \((J_1), (J_2)\) and \((J_3)\) hold and \(w(\xi)\) is defined by (24). For any \(c > \max \left\{ c^*, \frac{1}{\eta_0} \max \{c_1, c_2\} \right\}\), there exists some positive constants \(C_i\) such that
\[
G^i_w(\xi, t) \geq C_i
\]
for all \(\xi \in \mathbb{R}, t > 0\) and \(i = 1, 2, 3, 4\), respectively.

Proof. We only show the proof of \(G^1_w(\xi, t) \geq C_1\) for some positive constant \(C_1\), since the other inequalities can be proven in a similar way. By the condition, we have
\[
c\eta_0 > c_1, \text{ that is } \quad c\eta_0 + \frac{5}{2} - 3\beta - \beta^2 - \int_{\mathbb{R}} J(y)e^{-\eta_0y}dy - \frac{2r + 1}{(1 - \beta)^2} > 0.
\]
When \(\xi \leq \xi_0\), it is easy to show that \(w(\xi) = e^{-\eta_0(\xi - \xi_0)}\) and \(w(\xi)\) is non-increasing. Then we can obtain
\[
G^1_w(\xi, t) = -e^{w'(\xi)}w(\xi) + 4\bar{u}(\xi) - 1 + 2\beta[V(\xi, t) + \bar{v}(\xi)] + 2U(\xi, t) + \beta u(\xi)
\]
\[
- \int_{\mathbb{R}} J(y)w(\xi + y)w(\xi)dy - \frac{r\bar{u}^2}{u^2},
\]
\[
\geq c\eta_0 + 3 - 3\beta - \beta^2 - \int_{\xi_0 - \xi}^{\xi_0} J(y)e^{-\eta_0y}dy
\]
\[
- \int_{\mathbb{R}} J(y)e^{\eta_0(\xi - \xi_0)}dy - \frac{r}{(1 - \beta)^2}
\]
\[
\geq c\eta_0 + 3 - 3\beta - \beta^2 - \int_{\mathbb{R}} J(y)e^{-\eta_0y}dy
\]
\[
- \int_{\mathbb{R}} J(y)e^{\eta_0(\xi - \xi_0)}dy - \frac{r}{(1 - \beta)^2}
\]
\[
\geq c\eta_0 + 3 - 3\beta - \beta^2 - \int_{\mathbb{R}} J(y)e^{-\eta_0y}dy - \frac{r}{(1 - \beta)^2}
\]
\[
> \frac{r + 1}{(1 - \beta)^2} > 0.
\]

For \(\xi > \xi_0\), it follows from the definition (24) that \(w(\xi) = 1\) and \(w'(\xi) = 0\). Thus by (23), we have
\[
G^1_w(\xi, t) \geq 4\bar{u}(\xi) - 1 + 2\beta \bar{v}(\xi) + \beta u(\xi) - \int_{\xi_0 - \xi}^{\xi_0 - \xi} J(y)e^{-\eta_0(y + \xi - \xi_0)}dy
\]
\[
- \int_{\mathbb{R}} J(y)dy - \frac{r}{(1 - \beta)^2}
\]
\[
\geq (4 + \beta)\bar{u}(\xi) - 2 + 2\beta \bar{v}(\xi) - \int_{\mathbb{R}} J(y)e^{-\eta_0y}dy - \frac{r}{(1 - \beta)^2}
\]
\[
> \frac{r + 1}{(1 - \beta)^2} > 0.
\]
Setting \(C_1 = \frac{r + 1}{(1 - \beta)^2}\), we can obtain \(G^1_w(\xi) \geq C_1\). 

According to the above argument, we can easily obtain the following result.

Lemma 3.3. Assume that \((J_1), (J_2)\) and \((J_3)\) hold and \(w(\xi)\) is defined by (24). For any \(c > \max \left\{ c^*, \frac{1}{\eta_0} \max \{c_1, c_2\} \right\}\), there exists some positive constant \(C\) such
Lemma 3.4. Assume that \( H \) that for all \( t > 0 \) and \( 0 < \mu < \min_{i=1,2,3,4} \left\{ \frac{C_i}{2} \right\}. \)

Here we begin to give the priori estimates about \( U(\xi,t) \) and \( V(\xi,t) \) in the weighted Sobolev space \( H^1_{\mu}(\mathbb{R}) \).

**Lemma 3.4.** Assume that \( (J_1), (J_3) \) and \( (J_4) \) hold and \( w(\xi) \) is defined by (24). For any \( c > \max \left\{ e^r, \frac{1}{r_0} \max \{c_1, c_2, c_3\} \right\} \), there exists some positive constant \( C \) such that

\[
\|U(\cdot,t)\|_{L^2_\mu}^2 + \|V(\cdot,t)\|_{L^2_\mu}^2 + \int_0^t e^{-2\mu(t-s)} \left( \|U(\cdot,s)\|_{L^2_\mu}^2 + \|V(\cdot,s)\|_{L^2_\mu}^2 \right) ds \\
\leq Ce^{-2\mu t} \left[ \|U_0(\cdot)\|_{L^2_\mu}^2 + \|V_0(\cdot)\|_{L^2_\mu}^2 \right]
\]

for all \( t > 0 \).

**Proof.** By the equations (3), (4) and the definition of \( (U(\xi,t), V(\xi,t)) \), we can obtain that \( (U(\xi,t), V(\xi,t)) \) satisfies

\[
\begin{align*}
U_t + cU_\xi &= \int_{\mathbb{R}} J U dy - (2\hat{u} + \beta V + \beta \hat{v})U - U^2 - \beta \hat{u}U, \\
V_t + cV_\xi &= d \int_{\mathbb{R}} J V dy + \left( r - d - \frac{2r\hat{v}}{U + \hat{u}} \right) V - \frac{rV^2}{U + \hat{u}} + \frac{r\hat{v}^2 U}{U + \hat{u}},
\end{align*}
\]

with the initial data \( (U(\xi,0), V(\xi,0)) = (U_0(\xi), V_0(\xi)) \). Multiplying two sides of the equation (32) by \( e^{2\mu t} w(\xi) U(\xi,t) \) and \( e^{2\mu t} w(\xi) V(\xi,t) \), respectively, where \( \mu \) is defined in Lemma 3.3, we can obtain

\[
\begin{align*}
\left( \frac{1}{2} e^{2\mu t} U^2 \right)_t + \left( \frac{c}{2} e^{2\mu t} U^2 \right)_\xi + e^{2\mu t} U \int_{\mathbb{R}} J(y) U(\xi - y, t) dy \\
+ \left( \frac{c w'}{2w} - \mu + 2\hat{u} + \beta V + \beta \hat{v} \right) e^{2\mu t} U^2 \\
= -e^{2\mu t} U^3 - \beta \hat{u} e^{2\mu t} UV
\end{align*}
\]

and

\[
\begin{align*}
\left( \frac{1}{2} e^{2\mu t} V^2 \right)_t + \left( \frac{c}{2} e^{2\mu t} V^2 \right)_\xi - de^{2\mu t} V \int_{\mathbb{R}} J(y) V(\xi - y, t) dy \\
+ \left( \frac{c w'}{2w} - \mu + d - r - \frac{2r\hat{v}}{U + \hat{u}} \right) e^{2\mu t} V^2 \\
= -e^{2\mu t} V^3 + \frac{r e^{2\mu t} \hat{v}^2 U^2 V}{U + \hat{u}}.
\end{align*}
\]

Since \( U \in H^1_{\mu} \) and \( V \in H^2_{\mu} \), then we have

\[
\left\{ \begin{array}{l}
\frac{c}{2} e^{2\mu t} U^2 \bigg|_{t=0} = 0 \\
\frac{c}{2} e^{2\mu t} V^2 \bigg|_{t=0} = 0
\end{array} \right\} \quad \text{if} \quad \xi = \pm \infty.
\]

Therefore, integrating (33) and (34) on \( \mathbb{R} \times [0, t] \) with respect to \( \xi \) and \( t \), we have

\[
e^{2\mu t} \|U(\cdot,t)\|_{L^2_\mu}^2 - 2 \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) U(\xi, s) J(y) U(\xi - y, s) dy d\xi ds \\
+ \int_0^t \int_{\mathbb{R}} \left( \frac{c w'(\xi)}{w(\xi)} - 2\mu + 4\hat{u}(\xi) + 2\beta V(\xi, s) + 2\beta \hat{v}(\xi) \right) e^{2\mu s} w(\xi) U^2(\xi, s) d\xi ds
\]
Then, by substituting (37)-(40) into (35) and (36), we have

\[
= \|U_0(\cdot)\|^2_{L^2_w} - 2 \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} U^3(\xi, s) d\xi ds \\
- 2\beta \int_0^t \int_{\mathbb{R}} \tilde{u}(\xi) e^{2\mu_s w(\xi)} U(\xi, s) V(\xi, s) d\xi ds
\tag{35}
\]

and

\[
e^{2\mu t} \|V(\cdot, t)\|^2_{L^2_w} - 2 \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} V(\xi, s) \int_{\mathbb{R}} J(y) V(\xi - y, s) dy d\xi ds \\
+ \int_0^t \int_{\mathbb{R}} \left( - \frac{cw'(\xi)}{w(\xi)} - 2\mu + 2d - 2r + 4r\tilde{v}(\xi) \right) e^{2\mu_s w(\xi)} V^2(\xi, s) d\xi ds \\
\leq \|V_0(\cdot)\|^2_{L^2_w} - 2 \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} V^3(\xi, s) d\xi ds \\
+ 2 \int_0^t \int_{\mathbb{R}} \frac{re^{2\mu_s w(\xi)} \tilde{v}^2(\xi)}{\tilde{u}^2(\xi)} U(\xi, s) V(\xi, s) d\xi ds.
\tag{36}
\]

By using the Cauchy-Schwarz inequality, it is easy to show

\[
2 \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} U(\xi, s) \int_{\mathbb{R}} J(y) U(\xi - y, s) dy d\xi ds \\
= 2 \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} \int_{\mathbb{R}} J(y) U(\xi, s) U(\xi - y, s) dy d\xi ds \\
\leq \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} \int_{\mathbb{R}} J(y) U^2(\xi, s) + U^2(\xi - y, s) dy d\xi ds \\
= \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} U^2(\xi, s) d\xi ds + \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} \int_{\mathbb{R}} J(y) U^2(\xi - y, s) dy d\xi ds \\
= \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} U^2(\xi, s) d\xi ds + \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} U^2(\xi, s) \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} dy d\xi ds.
\tag{37}
\]

Similarly, we can get

\[
2d \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} V(\xi, s) \int_{\mathbb{R}} J(y) V(\xi - y, s) dy d\xi ds \\
\leq d \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} V^2(\xi, s) d\xi ds \\
+ d \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} V^2(\xi, s) \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} dy d\xi ds,
\tag{38}
\]

\[
2\beta \int_0^t \int_{\mathbb{R}} \tilde{u}(\xi) e^{2\mu_s w(\xi)} U(\xi, s) V(\xi, s) d\xi ds \\
\leq \beta \int_0^t \int_{\mathbb{R}} \tilde{u}(\xi) e^{2\mu_s w(\xi)} U^2(\xi, s) d\xi ds + \beta \int_0^t \int_{\mathbb{R}} \tilde{u}(\xi) e^{2\mu_s w(\xi)} V^2(\xi, s) d\xi ds
\tag{39}
\]

and

\[
2 \int_0^t \int_{\mathbb{R}} \frac{re^{2\mu_s w(\xi)} \tilde{v}^2(\xi)}{\tilde{u}^2(\xi)} U(\xi, s) V(\xi, s) d\xi ds \\
\leq \int_0^t \int_{\mathbb{R}} \frac{re^{2\mu_s w(\xi)} \tilde{v}^2(\xi)}{\tilde{u}^2(\xi)} U^2(\xi, s) d\xi ds + \int_0^t \int_{\mathbb{R}} \frac{re^{2\mu_s w(\xi)} \tilde{v}^2(\xi)}{\tilde{u}^2(\xi)} V^2(\xi, s) d\xi ds.
\tag{40}
\]

Then, by substituting (37)-(40) into (35) and (36), we have

\[
e^{2\mu t} \|U(\cdot, t)\|^2_{L^2_w} + \int_0^t \int_{\mathbb{R}} \left( - \frac{cw'(\xi)}{w(\xi)} - 2\mu + 4\tilde{u}(\xi) - 1 + 2\beta V(\xi, s) + 2\beta \tilde{v}(\xi)
\]

\[
\leq \|U_0(\cdot)\|^2_{L^2_w} - 2 \int_0^t \int_{\mathbb{R}} e^{2\mu_s w(\xi)} U^3(\xi, s) d\xi ds \\
- 2\beta \int_0^t \int_{\mathbb{R}} \tilde{u}(\xi) e^{2\mu_s w(\xi)} U(\xi, s) V(\xi, s) d\xi ds
\]
Furthermore, we can obtain

\[- \int_{\mathbb{R}} J(y) \left( \frac{w(\xi + y)}{w(\xi)} \right) dy + 2U(\xi, s) + \beta \bar{u}(\xi) \right) e^{2\mu_s w(\xi)} U^2(\xi, s) d\xi ds \]

\[= \|U_0(\cdot)\|^2_{L^2_w} - \beta \int_{\mathbb{R}} ^{\xi} \int_{\mathbb{R}} U(\xi) e^{2\mu_s w(\xi)} V^2(\xi, s) d\xi ds \]

and

\[e^{2\mu t} \|V(\cdot, t)\|^2_{L^2_w} + \beta \int_{\mathbb{R}} ^{\xi} \int_{\mathbb{R}} \left( - \frac{cw(\xi)}{w(\xi)} - 2\mu + d - 2r + 4r \bar{v}(\xi) + 2r V(\xi, s) \right) \]

\[- \frac{rv^2(\xi)}{\bar{u}^2(\xi)} - d \int_{\mathbb{R}} J(y) \left( \frac{w(\xi + y)}{w(\xi)} \right) dy \right) e^{2\mu_s w(\xi)} V^2(\xi, s) d\xi ds \]

\[\leq \|V_0(\cdot)\|^2_{L^2_w} + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{r e^{2\mu_s w(\xi)} v^2(\xi)}{\bar{u}^2(\xi)} U^2(\xi, s) d\xi ds. \]

Furthermore, we can obtain

\[e^{2\mu t} \left( \|U(\cdot, t)\|^2_{L^2_w} + \|V(\cdot, t)\|^2_{L^2_w} \right) \]

\[+ \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\mu_s w(\xi)} [\left( G_u^1 - 2\mu w(\xi) U^2(\xi, s) + (G_w^2 - 2\mu w(\xi) V^2(\xi, s) \right] d\xi ds \]

\[\leq \|U_0(\cdot)\|^2_{L^2_w} + \|V_0(\cdot)\|^2_{L^2_w}, \]

where \(G_u^1\) and \(G_w^2\) are defined in (28) and (29). By Lemma 3.3, it is easy to show that

\[\|U(\cdot, t)\|^2_{L^2_w} + \|V(\cdot, t)\|^2_{L^2_w} + \int_{\mathbb{R}} ^{\xi} \int_{\mathbb{R}} e^{-2\mu(t-s)} \left( \|U(\cdot, s)\|^2_{L^2_w} + \|V(\cdot, s)\|^2_{L^2_w} \right) ds \]

\[\leq Ce^{-2\mu t} \left[ \|U_0(\cdot)\|^2_{L^2_w} + \|V_0(\cdot)\|^2_{L^2_w} \right] \]

for some positive constant \(C\). This completes the proof. \( \Box \)

The following estimates can be proved by modifying the arguments in the above lemma. So we omit the details of the proof for the following lemma.

**Lemma 3.5.** Assume that \((J_1), (J_2),\) and \((J_4)\) hold and \(w(\xi)\) is defined by (24). For any \(c > \max \left\{ c^*, \frac{1}{m_0} \max \{ c_1, c_2 \} \right\} \), there exists some positive constant \(C\) such that

\[\|U(\cdot, t)\|^2_{H^2_w} + \|V(\cdot, t)\|^2_{H^2_w} + \int_{\mathbb{R}} ^{\xi} \int_{\mathbb{R}} e^{-2\mu(t-s)} \left( \|U(\cdot, s)\|^2_{H^2_w} + \|V(\cdot, s)\|^2_{H^2_w} \right) ds \]

\[\leq Ce^{-2\mu t} \left[ \|U_0(\cdot)\|^2_{H^2_w} + \|V_0(\cdot)\|^2_{H^2_w} \right] \]

for all \(t > 0\).

Then we can easily have a prior estimate by Lemmas 3.4 and 3.5.

**Lemma 3.6.** Assume that \((J_1), (J_2),\) and \((J_4)\) hold and \(w(\xi)\) is defined by (24). For any \(c > \max \left\{ c^*, \frac{1}{m_0} \max \{ c_1, c_2 \} \right\} \), there exists some positive constant \(C\) such that

\[\|U(\cdot, t)\|_{H^2_w} \leq Ce^{-\mu t} \left[ \|U_0(\cdot)\|^2_{H^2_w} + \|V_0(\cdot)\|^2_{H^2_w} \right]^{\frac{1}{2}} \]

and

\[\|V(\cdot, t)\|_{H^2_w} \leq Ce^{-\mu t} \left[ \|U_0(\cdot)\|^2_{H^2_w} + \|V_0(\cdot)\|^2_{H^2_w} \right]^{\frac{1}{2}} \]

for all \(t > 0\).
Proof of Theorem 3.1. According to the standard Sobolev embedding inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, and the embedding inequality $H^1(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ since $w \geq 1$, we can obtain

$$\sup_{x \in \mathbb{R}} |U(x, t)| \leq C \|U(\cdot, t)\|_{H^1} \leq C \|U(\cdot, t)\|_{H_w^1}$$

and

$$\sup_{x \in \mathbb{R}} |V(x, t)| \leq C \|V(\cdot, t)\|_{H^1} \leq C \|V(\cdot, t)\|_{H_w^1}.$$

By Lemma 3.6, it is easily show

$$\sup_{x \in \mathbb{R}} |u^+(x, t) - \tilde{v}(x + ct)| = \sup_{\xi \in \mathbb{R}} |U(\xi, t)| \leq Ce^{-\mu t},$$

$$\sup_{x \in \mathbb{R}} |v^+(x, t) - \tilde{v}(x + ct)| = \sup_{\xi \in \mathbb{R}} |V(\xi, t)| \leq Ce^{-\mu t}.$$

By the same argument, we can also obtain

$$\sup_{x \in \mathbb{R}} |u^-(x, t) - \tilde{v}(x + ct)| = \sup_{\xi \in \mathbb{R}} |U(\xi, t)| \leq Ce^{-\mu t},$$

$$\sup_{x \in \mathbb{R}} |v^-(x, t) - \tilde{v}(x + ct)| = \sup_{\xi \in \mathbb{R}} |V(\xi, t)| \leq Ce^{-\mu t}.$$

Thus, using the squeezing technique, we can easily get

$$\sup_{x \in \mathbb{R}} |u(x, t) - \tilde{v}(x + ct)| \leq Ce^{-\mu t} \text{ and } \sup_{x \in \mathbb{R}} |v(x, t) - \tilde{v}(x + ct)| \leq Ce^{-\mu t}.$$

\[ \square \]

4. Nonexistence of traveling waves . In this section, we will give the proof of the nonexistence of traveling waves for (4) when $0 < c < c^*$.

Theorem 4.1. Assume that (J1) and (J2) hold. For any speed $0 < c < c^*$, there exist no traveling waves $(u(\xi), v(\xi))$ of system (4) satisfying $u(-\infty) = 1$, $v(-\infty) = 0$ and $\sup_{\xi \in \mathbb{R}} u(\xi) < 1$, $\inf_{\xi \in \mathbb{R}} v(\xi) > 0$.

Proof. We prove this theorem by the way of contradiction. Here, we assume that there exists a traveling wave solution $(u(\xi), v(\xi))$ of system (4) satisfying the limit behavior at infinity. Since $u(-\infty) = 1$ and $v(-\infty) = 0$, there exist $\hat{\xi} < 0$ and $\epsilon_1 > 0$ such that $u(\xi) \geq 1 - \epsilon_1$ and $v(\xi) \leq \epsilon_1$ for any $\xi \leq \hat{\xi}$. Therefore, we have that

$$cv'(\xi) = d \int_{\mathbb{R}} J(y) [v(\xi - y) - v(\xi)] dy + rv(\xi) \left(1 - \frac{v(\xi)}{u(\xi)}\right)$$

$$\geq d \int_{\mathbb{R}} J(y) [v(\xi - y) - v(\xi)] dy + \frac{r(1-2\epsilon_1)}{1-\epsilon_1} v(\xi) \quad (44)$$

for any $\xi \leq \hat{\xi}$. Letting $V(\xi) = \int_{-\infty}^{\xi} v(\eta) d\eta$ for any $\xi \in \mathbb{R}$ and integrating two sides of inequality (44) from $-\infty$ to $\xi$ with $\xi \leq \hat{\xi}$, we can obtain

$$cv(\xi) \geq d \left( \int_{-\infty}^{\xi} \int_{\mathbb{R}} J(y) v(\eta - y) dy d\eta - V(\xi) \right) + \frac{r(1-2\epsilon_1)}{1-\epsilon_1} V(\xi). \quad (45)$$

By Fubini theorem, we have

$$\int_{-\infty}^{\xi} \int_{\mathbb{R}} J(y) v(\eta - y) dy d\eta = \int_{\mathbb{R}} J(y) \int_{-\infty}^{\xi} v(\eta - y) dy d\eta$$

$$= \int_{\mathbb{R}} J(y) \int_{-\infty}^{\xi-y} v(\eta) dy d\eta \triangleq J * V(\xi).$$
So we have
\[ cv(\xi) \geq d[J \ast V(\xi) - V(\xi)] + \frac{r(1 - 2\epsilon_1)}{1 - \epsilon_1} V(\xi). \] (46)

In view of
\[
\int_{-\infty}^{\xi} [J \ast V(\eta) - V(\eta)]d\eta = \int_{-\infty}^{\xi} \int_{\mathbb{R}} J(y)[V(\eta - y) - V(\eta)]dyd\eta \\
= \int_{-\infty}^{\xi} \int_{\mathbb{R}} (-y) J(y) \int_{0}^{1} V'(\eta - y\theta)d\theta dyd\eta \\
= \int_{0}^{1} \int_{\mathbb{R}} (-y) J(y) \int_{-\infty}^{\xi} V'(\eta - y\theta)d\eta dyd\theta \\
= \int_{\mathbb{R}} (-y) J(y) \int_{0}^{1} V(\xi - y\theta)d\theta dy,
\]
we have that \([J \ast V(\xi) - V(\xi)]\) is integrable on \((-\infty, \xi]\) for any \(\xi \leq \hat{\xi}\). Hence, from equation \([46]\), we have that \(V(\xi)\) is integrable on \((-\infty, \xi]\) for any \(\xi \leq \hat{\xi}\). Now integrating two sides of inequality \([46]\) from \(-\infty\) to \(\xi\) with \(\xi \leq \hat{\xi}\), we can obtain
\[
\frac{r(1 - 2\epsilon_1)}{1 - \epsilon_1} \int_{-\infty}^{\xi} V(\eta)d\eta \leq c V(\xi) + d \int_{\mathbb{R}} y J(y) \int_{0}^{1} V(\xi - y\theta)d\theta dy.
\]
Due to the fact that \(y V(\xi - \theta y)\) is non-increasing for \(\theta \in [0, 1]\) and the symmetry of the kernel function \(J\), we have
\[
\frac{r(1 - 2\epsilon_1)}{1 - \epsilon_1} \int_{-\infty}^{\xi} V(\eta)d\eta \leq \left\{ c + d \int_{\mathbb{R}} y J(y)dy \right\} V(\xi) = c V(\xi).
\]
Since \(V(\xi)\) is increasing with respect to \(\xi\), there exists some \(\tau > 0\) such that
\[
\frac{r(1 - 2\epsilon_1)}{1 - \epsilon_1} \tau V(\xi - \tau) \leq c V(\xi).
\]
Hence, there exists a constant \(\tau_0 > 0\) sufficiently large and some \(\nu \in (0, 1)\) such that \(V(\xi - \tau_0) \leq \nu V(\xi)\) for each \(\xi \leq \hat{\xi}\). Setting \(W(x) = V(x)e^{-\mu_0 x}\) and \(\mu_0 = \frac{1}{\tau_0} \ln(\frac{1}{\nu})\), then
\[
W(\xi - \tau_0) = V(\xi - \tau_0)e^{-\mu_0(\xi - \tau_0)} \leq \nu V(\xi) e^{-\mu_0(\xi - \tau_0)} = W(\xi).
\]
Thus, there exists \(W_0 > 0\) such that \(W(\xi) \leq W_0\) for any \(\xi \leq \hat{\xi}\), which implies that
\[ V(\xi) \leq W_0 e^{\mu_0 \xi} \text{ for any } \xi \leq \hat{\xi}. \]
Since
\[ cv'(\xi) \leq d \int_{\mathbb{R}} J(y)[v(\xi - y) - v(\xi)]dy + rv(\xi), \]
we have that there exists \(W_1 > 0\) such that \(v(\xi) \leq W_1 e^{\mu_0 \xi}\) for any \(\xi \leq \hat{\xi}\). Since \(v(\xi)\) is bounded, we can get
\[
\sup_{\xi \in \mathbb{R}} \{ v(\xi) e^{-\mu_0 \xi} \} < \infty \text{ and } \sup_{\xi \in \mathbb{R}} \{ v' (\xi) e^{-\mu_0 \xi} \} < \infty. \quad (47)
\]
For \(\lambda \in \mathbb{C}\) with \(0 < Re \lambda < \mu_0\), we can define a bilateral Laplace transform of \(v(\cdot)\) by
\[
\mathcal{L}(\lambda) = \int_{\mathbb{R}} e^{-\lambda \xi} v(\xi)d\xi.
\]
Then, multiplying two sides of the second equation of (4) by $e^{-\lambda \xi}$ and integrating two sides on $\mathbb{R}$, we can obtain
\[
d \int_{\mathbb{R}} e^{-\lambda \xi} \int_{\mathbb{R}} J(y)[v(\xi - y) - v(\xi)]dyd\xi - e \int_{\mathbb{R}} e^{-\lambda \xi} v'(\xi)d\xi + r \int_{\mathbb{R}} e^{-\lambda \xi} v(\xi)d\xi = r \int_{\mathbb{R}} e^{-\lambda \xi} \frac{v(\xi)^2}{u(\xi)}d\xi.
\]
In view of
\[
\int_{\mathbb{R}} e^{-\lambda \xi} \int_{\mathbb{R}} J(y)[v(\xi - y) - v(\xi)]dyd\xi = \int_{\mathbb{R}} J(y)(e^{-\lambda y} - 1)dyL(\lambda)
\]
and
\[
\int_{\mathbb{R}} e^{-\lambda \xi} v'(\xi)d\xi = \lambda L(\lambda),
\]
we have that
\[
\Delta(\lambda, c) L(\lambda) = r \int_{\mathbb{R}} e^{-\lambda \xi} \frac{v(\xi)^2}{u(\xi)}d\xi
\]
for $\lambda \in \mathbb{C}$ with $0 < Re\lambda < \mu_0$, where $\Delta(\lambda, c)$ is defined by equation (5). Due to $1 - \beta \leq u(\xi) \leq 1$ and (47), we can get that the right hand side of equation (48) is well defined for $\lambda \in \mathbb{C}$ with $0 < Re\lambda < 2\mu_0$. By the way of recursion, we can get that the right hand side of (48) is well defined for $\lambda \in \mathbb{C}$ with $0 < Re\lambda < \mu_0$ ($n \geq 2$). Letting $n \to +\infty$, we can obtain that the right hand side of (48) is well defined for $\lambda \in \mathbb{C}$ with $0 < Re\lambda < \infty$. Then we can know that $L(\lambda)$ is well defined with $Re\lambda > 0$. But (48) can be re-written as
\[
\int_{\mathbb{R}} e^{-\lambda \xi} \left\{ \Delta(\lambda, c) - r \frac{v(\xi)}{u(\xi)} \right\} v(\xi)d\xi = 0,
\]
it follows from the definition of $\Delta(\lambda, c)$ and Lemma 2.1 that $\Delta(\lambda, c) \to +\infty$ as $\lambda \to \lambda_0$. This contradicts with the equation (49) and we complete our proof. \qed

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