On generalized ModMax model of nonlinear electrodynamics

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Abstract

A new generalized ModMax model of nonlinear electrodynamics with four parameters is proposed. The ModMax model and Born–Infeld-type electrodynamics are particular cases of the present model. It is shown that a singularity of the electric field at the center of point-like charged particles is absent. We found corrections to Coulomb’s law at \( r \to \infty \) and obtained the total electrostatic and magnetic energies of point-like charges. Free electric and magnetic charges and their densities are obtained.

Recently, a new model, named ModMax model, of nonlinear electrodynamics (NED) which is duality and conformal invariant, as well as Maxwell electrodynamics, was proposed in [1]. Some aspects of this model and its applications were studies in [2, 3, 4, 5, 6, 7]. Earlier proposals of conformal invariant NED was in [8, 9, 10, 11].

The duality-invariant conformal electrodynamics was introduced in [1] and it is described by the Lagrangian density

\[
L = -\mathcal{F} \cosh(\gamma) + \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \sinh(\gamma),
\]

where

\[
\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (B^2 - E^2), \quad \mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = B \cdot E,
\]

are Lorentz invariants with \( B, E \) being the magnetic induction and electric fields correspondingly, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( \tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \) is the dual electromagnetic field, and \( \gamma \) is the dimensionless parameter. This model as well as Maxwell electrodynamics with the Lagrangian density \( \mathcal{L}_M = -\mathcal{F} \)

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possess singularities in the centre of point-like charges. In addition, the
electromagnetic energy of charges is infinite. To smooth singularities we
propose the generalized ModMax model with the Lagrangian density
\[ L = \frac{1}{\beta} \left( 1 - \left( 1 - \frac{\beta L}{\sigma} - \frac{\beta \lambda G^2}{2\sigma} \right)^\sigma \right), \] (3)
where \( L \) is given by Eq. (1), \( \beta \) and \( \lambda \) have the dimensions of \((\text{length})^4\) and
\( \sigma \) is the dimensionless parameter. At \( \sigma = 1, \lambda = 0 \) we come to ModMax
model, when \( \gamma = 0 \) one has the Born–Infeld-type model [12], [13], at \( \sigma = 1/2, \gamma = 0 \)
we arrive at the generalized Born–Infeld model [14], and at \( \sigma = 1/2, \gamma = 0, \lambda = \beta \) one comes to Born–Infeld model \( (\beta = 1/b^2) \) [15]. At \( \sigma \to \infty \) the Lagrangian density (3) becomes
\[ L_{\exp} = \frac{1}{\beta} \left( 1 - \exp \left( -\beta L - \beta \lambda G^2 / 2 \right) \right). \] (4)
Some exponential NED models were considered in [16], [17]. Thus, our model
(3) allows us to consider different NED by fixing the parameters introduced.
The Born–Infeld model is of interest because at the low energy D-brain
dynamics is governed by Born–Infeld-type action [18], [19]. Making use of the
Taylor series, at \( \beta L \ll 1, \beta \lambda G^2 \ll 1 \), the Lagrangian density (3) becomes
\[ L = L + \frac{\beta (1 - \sigma) L^2}{2\sigma} + \frac{\lambda}{2} G^2 + O(LG^2) + O(L^3). \] (5)
As a result, in the weak-field limit and small \( \gamma \) when the condition \( \beta L \ll 1 \)
is satisfied, the Lagrangian density (3) approaches to the ModMax model.
At \( \gamma = 0 \) Eq. (5) corresponds to the Heisenberg–Euler-type electrodynamics
[20].
Adding to Eq. (3) the source term \( A_\mu j^\mu \) and varying the action we obtain
the Euler–Lagrange equations
\[ \partial_\mu \left( L_\mathcal{F} F^{\mu\nu} + L_\mathcal{G} \tilde{F}^{\mu\nu} \right) = j^\nu, \] (6)
where
\[ L_\mathcal{F} = \frac{\partial L}{\partial F} = \left( 1 - \frac{\beta L}{\sigma} - \frac{\beta \lambda G^2}{2\sigma} \right)^{\sigma-1} L_F, \quad L_F = - \cosh(\gamma) + \frac{F \sinh(\gamma)}{\sqrt{F^2 + G^2}}, \]
\[ L_\mathcal{G} = \frac{\partial L}{\partial G} = \left( 1 - \frac{\beta L}{\sigma} - \frac{\beta \lambda G^2}{2\sigma} \right)^{\sigma-1} (L_\mathcal{G} + \lambda G), \quad L_\mathcal{G} = \frac{G \sinh(\gamma)}{\sqrt{F^2 + G^2}}. \] (7)
Field equations (6) can be represented as Maxwell equations making use of
definitions of the electric displacement and magnetic fields

\[ D = \frac{\partial L}{\partial E} = \varepsilon E + \nu B, \quad \varepsilon = -L_F, \quad \nu = L_G, \]
\[ H = -\frac{\partial L}{\partial B} = \mu^{-1} B - \nu E, \quad \mu = \varepsilon^{-1}. \] (8)

With help of Eq. (8) Euler–Lagrange equations (6) can be represented as
Maxwell equations in Gaussian quantities

\[ \nabla \cdot D = 4\pi \rho, \quad \frac{\partial D}{\partial t} - \nabla \times H = 4\pi j. \] (9)

Second pair of Maxwell equations follows from the Bianchi identity \( \partial_\mu \tilde{F}^{\mu\nu} = 0, \)
\[ \nabla \cdot B = 0, \quad \frac{\partial B}{\partial t} + \nabla \times E = 0. \] (10)

From Eq. (8) we obtain the relation
\[ D \cdot H = (\varepsilon^2 - \nu^2) G + 2\varepsilon \nu \mathcal{F}. \] (11)

According to the criterion of [21] the dual symmetry takes place if \( D \cdot H = E \cdot B. \) One can verify from Eq. (11) that the dual symmetry holds in two cases: \( \sigma = 1, \lambda = 0 \) which corresponds to ModMax model or for Born–Infeld-type model with \( \sigma = 1/2, \lambda = \beta. \) It is worth noting that the two-parametric
generalized Born–Infeld model \( (\sigma = 1/2, \lambda = \beta) \) was considered and shown
to be duality invariant in [4].

From Eq. (9), for the source of the point-like charged particle with the
electric charge \( q, \) in Gaussian units, we obtain the equation as follows

\[ \nabla \cdot D = 4\pi q \delta(r), \] (12)
with the solution
\[ D = \frac{qr}{r^3}. \] (13)

From Eqs. (8) and (13) one finds
\[ E \exp(\gamma) \left( 1 - \frac{\beta E^2}{2\sigma} \exp(\gamma) \right)^{-1} = \frac{q}{r^2}. \] (14)
Introducing the dimensionless variables

\[ v = \sqrt{\beta} E \frac{\exp \left( \frac{\gamma}{2} \right)}{\sqrt{2} \sigma}, \quad u = r \frac{\sqrt{2} \sigma}{\sqrt{2} \beta \sqrt{q}} \exp \left( \frac{\gamma}{4} \right), \]  

(15)

Eq. (14) takes the form

\[ v \left( 1 - v^2 \right)^{\sigma-1} = \frac{1}{u^2}. \]  

(16)

It follows from Eq. (16) that at \( u \to 0 \), \( v(0) = 1 \) for \( \sigma < 1 \), or in terms of electric fields

\[ E(0) = \sqrt{\frac{2 \sigma}{\beta}} \exp \left( -\frac{\gamma}{2} \right). \]  

(17)

Thus, the electric field of the point-like charged particle in the center is finite and possesses the maximum value for \( \sigma < 1 \). When \( \sigma \) increases the maximum value of electric fields increases, but if \( \gamma \) increases the electric field in the center decreases. For the cases \( \sigma = 1/2 \) (Born–Infeld-type model) and \( \sigma = 3/4 \) exact solutions to Eq. (16) and their asymptotic as \( u \to 0 \) are

\[ v = \frac{1}{\sqrt{1 + u^8}}; \quad v = 1 - \frac{u^4}{2} + \frac{3u^8}{8} + \mathcal{O}(u^{12}) \quad \sigma = \frac{1}{2}, \]

\[ v = \frac{\sqrt{\sqrt{1 + 4u^8} - 1}}{\sqrt{2}u^4}; \quad v = 1 - \frac{u^8}{2} + \mathcal{O}(u^{16}) \quad \sigma = \frac{3}{4}. \]  

(18)

The plots of function \( v(u) \) for \( \sigma = 1/4 \), \( \sigma = 1/2 \) and \( \sigma = 3/4 \) are given in Fig. (1). In the general case for \( \sigma < 1 \), the functions \( v(u) \) as \( u \to 0 \) (\( r \to 0 \)) and \( u \to \infty \) (\( r \to \infty \)) become

\[ v = 1 - \frac{u^{2/(1-\sigma)}}{2} + \mathcal{O}(u^{4/(1-\sigma)}) \quad u \to 0, \]

\[ v = \frac{1}{u^2} - \frac{1 - \sigma}{u^6} + \mathcal{O}(u^{-8}) \quad u \to \infty. \]  

(19)

Making use of Eqs. (15) and (19) we obtain the asymptotic value of the electric field as \( r \to 0 \) and \( r \to \infty \)

\[ E = \sqrt{\frac{2 \sigma}{\beta}} \exp \left( -\frac{\gamma}{2} \right) - \frac{(2\sigma)^{(2-\sigma)/(2(1-\sigma))}r^{2/(1-\sigma)} \sigma}{2\beta(2-\sigma)/(2(1-\sigma))q^{1/(1-\sigma)}} \exp \left( \frac{\gamma \sigma}{(2(1-\sigma))} \right). \]
Equation (20) gives the correction to Coulomb’s law as $r \to \infty$. We have damping of the electric field because of parameter $\gamma$. Corrections to Coulomb’s law for the case of Born–Infeld-type electrodynamics ($\sigma = 1/2$) and exponential-like electrodynamics (4) ($\sigma = \infty$) are similar but with the opposite sign. At $\sigma = 1$, $\lambda = 0$, $\gamma = 0$, one has Maxwell’s electrodynamics and we come to the Coulomb law $E = q/r^2$ as $r \to \infty$, but the electric field at the origin is infinite.

The energy-momentum tensor is given by

$$T_{\mu\nu} = F_{\mu\alpha} \left( \mathcal{L}_F F^\alpha_{\nu} + \mathcal{L}_g \tilde{F}^\alpha_{\nu} \right) - g_{\mu\nu} \mathcal{L}. \quad (21)$$
From Eq. (21) we obtain the energy density

\[ T_0^0 = -E^2 \mathcal{L}_E - G \mathcal{L}_G - \mathcal{L}. \]  
(22)

In the case of pure electric field \((B = 0)\) the energy density (22) becomes

\[ T_0^0 = E^2 e^\gamma \left( 1 - \frac{\beta E^2}{2\sigma e^\gamma} \right)^{\sigma-1} - \frac{1}{\beta} \left( 1 - \left( 1 - \frac{\beta E^2}{2\sigma e^\gamma} \right)^\sigma \right). \]  
(23)

For the Born–Infeld-type electrodynamics with \(\sigma = 1/2\) we can calculate the total electrostatics energy of point-like charged particles. Introducing the dimensionless variables

\[ y = \beta E^2 \exp(\gamma), \quad x = \frac{r^2 \exp(\gamma/2)}{q\sqrt{\beta}}, \]  
(24)

we obtain from Eq. (14) equation

\[ y = \frac{1}{1 + x^2}. \]  
(25)

Then, making use of Eqs. (23), (24) and (25), the total electrostatics energy of point-like charged particles (for \(\sigma = 1/2\)) is given by

\[ \mathcal{E} = 4\pi \int_0^\infty T_0^0 r^2 dr = \frac{2\pi q^{3/2} \exp(-3\gamma/4)}{\beta^{1/4}} I, \]

(26)

\[ I = \int_0^\infty \left( \frac{\sqrt{1 + x^2}}{\sqrt{x}} - \sqrt{x} \right) dx = -\frac{2}{3} \sqrt{x} \left( x - 3_2F_1 \left( -\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; -x^2 \right) \right)_0^\infty. \]

Taking into account the asymptotic of hypergeometric function \(_2F_1(-1/2, 1/4; 5/4; -x^2)\) we obtain the total electrostatics energy of point-like charged particles

\[ \mathcal{E} \approx 15.5327 \frac{q^{3/2}}{\beta^{1/4}} \exp \left( -\frac{3\gamma}{4} \right). \]  
(27)

For Born–Infeld electrodynamics \((\beta = 1/b^2)\) at \(\gamma = 0\) we come to the result obtained in [15].

To study the conformal invariance of the proposed model (3), we calculate the trace of the energy-momentum tensor \(T^\mu_\mu\). According to [22], the
conformal invariance takes place if $T_{\mu}^{\mu} = 0$. From Eqs. (7) and (27) we obtain

$$T_{\mu}^{\mu} = 4 \left( F \mathcal{L}_F + G \mathcal{L}_G - \mathcal{L} \right)$$

$$= 4 \left( 1 - \frac{\beta L}{\sigma} - \frac{\beta \lambda G^2}{2\sigma} \right) ^{\sigma^{-1}} \left( \frac{(\sigma - 1)L}{\sigma} + \frac{(2\sigma - 1)\lambda G^2}{2\sigma} + \frac{1}{\beta} \right) - \frac{4}{\beta}. \quad (28)$$

Equation (28) shows that the conformal invariance ($T_{\mu}^{\mu} = 0$) holds only for $\sigma = 1, \lambda = 0$ corresponding to the ModMax model.

In accordance with [15] we introduce the “free charge density” $\rho_{free}$ by the equation

$$\nabla \cdot E = \frac{1}{r^2} \frac{d}{dr} (r^2 E) = 4\pi \rho_{free}, \quad (29)$$

where $E$ obeys Eq. (14). To calculate the distribution of the free charge one has to obtain $\rho_{free}$ from Eq. (29). We have exact solutions to Eq. (14) for $\sigma = 1/2$ and $\sigma = 3/4$. Making use of Eq. (15), from Eq. (18) we find

$$E = \frac{q \exp(-\gamma/2)}{r_0^2 \sqrt{1 + (r/r_0)^4 \exp(\gamma)}} \quad \sigma = \frac{1}{2},$$

$$E = \frac{qr_0^2 \exp(-3\gamma/2) \sqrt{1 + (9r^8/r_0^8) \exp(2\gamma)} - 1}{\sqrt{3r^4}} \quad \sigma = \frac{3}{4}, \quad (30)$$

where $r_0 = \sqrt[3]{\beta} \sqrt{q}$. When $q$ is the charge of the electron, $r_0$ is the electron radius [15]. It follows from Eq. (30) that for $r \to \infty$ the electric field $E \to (q/r^2) \exp(-\gamma)$ and for $r \to 0$ we have

$$E \to \frac{1}{\sqrt{\beta}} \exp \left( -\frac{\gamma}{2} \right) \quad \sigma = \frac{1}{2},$$

$$E \to \sqrt{\frac{3}{2\beta}} \exp \left( -\frac{\gamma}{2} \right) \quad \sigma = \frac{3}{4}, \quad (31)$$

according to Eq. (20). From Eqs. (29) and (30) we obtain “free charge density”

$$\rho_{free} = \frac{q \exp(-\gamma/2)}{2\pi r_0^2 r (1 + (r/r_0)^4 \exp(\gamma))^{3/2}} \quad \sigma = \frac{1}{2},$$

$$\rho_{free} = \frac{qr_0^2 \exp(-3\gamma/2) \sqrt{1 + (9r^8/r_0^8) \exp(2\gamma)} - 1}{\sqrt{3r^4} \sqrt{1 + (9r^8/r_0^8) \exp(2\gamma)}} \quad \sigma = \frac{3}{4}, \quad (32)$$
For Born–Infeld electrodynamics, in the case $\sigma = 1/2$, $\gamma = 0$, one comes from Eq. (32) to the result found in [15]. Now we can calculate free charges

$$q_{\text{free}} = 4\pi \int_0^\infty r^2 \rho_{\text{free}} dr = \left(r^2 E^2\right)_0.$$  \hspace{1cm} (33)

Making use of Eqs. (20) and (33) we find the free charge for any parameter $\sigma$

$$q_{\text{free}} = q \exp(-\gamma). \hspace{1cm} (34)$$

Equation (34) shows that the free charge is less from $q$ by the factor $\exp(-\gamma)$. From Eq. (30) we obtain the free electric charges, in the cases $\sigma = 1/2$ and $\sigma = 3/4$, inside the sphere $r < r_0$

$$q_{\text{free}}(r_0) = 4\pi \int_0^{r_0} r^2 \rho_{\text{free}} dr = (r^2 E)_0^{r_0} = q_{\text{free}} \frac{\exp(\gamma/2)}{\sqrt{1 + \exp(\gamma)}} \quad \sigma = 1/2,$$

$$q_{\text{free}}(r_0) = q_{\text{free}} \frac{\sqrt{\sqrt{(1 + 9 \exp(2\gamma) - 1)/3} \exp \left(-\frac{\gamma}{2}\right)}}{\sigma = 3/4} \hspace{1cm} (35)$$

The plots of the functions $q_{\text{free}}(r_0)/q_{\text{free}}$ versus $\gamma$ is presented in Fig. 2. It follows from Eq. (35) and the Fig. 2 that at $\gamma = 0$ $q_{\text{free}}(r_0)/q_{\text{free}} \approx 0.71$ for $\sigma = 1/2$ and $q_{\text{free}}(r_0)/q_{\text{free}} \approx 0.85$ for $\sigma = 3/4$. Therefore, 71% of the electron charge $q_{\text{free}}$ is contained in the electron radius sphere for $\sigma = 1/2$ and 85% of the electron charge is concentrated within the electron radius for $\sigma = 3/4$. When parameter $\gamma$ increases more charge is inside the electron radius sphere.

For the magnetic monopole we have equation

$$\nabla \cdot \mathbf{B} = 4\pi Q \delta(r), \hspace{1cm} (36)$$

where $Q$ is the magnetic charge. For a magnetic monopole we have equation

$$B = \frac{Q}{r^2}. \hspace{1cm} (37)$$

From Eq. (22) we obtain the magnetic energy density ($E = 0$) of the magnetic monopole

$$T_0^0 = -\mathcal{L} = \frac{1}{\beta} \left(1 + \frac{\beta Q^2}{2\sigma r^4 e^{-\gamma}}\right)^\sigma - 1 \hspace{1cm}. \hspace{1cm} (38)$$
Figure 2: The plot of the functions $q_{\text{free}}(r_0)/q_{\text{free}}$ versus $\gamma$ for $\sigma = 1/2$ and $\sigma = 3/4$.

The total magnetic energy is given by

$$E_m = 4\pi \int_0^\infty T_0 r^2 dr = -\frac{4\pi}{\beta} \int_0^\infty r^2 \left(1 - \left(1 + \frac{\beta Q^2}{2\sigma r^4 e^{-\gamma}}\right)^\sigma\right) dr$$  \hspace{1cm} (39)

Integral (39) converges for $0 < \sigma < 1$ and the energy of the magnetic monopole is finite within our model. In Table 1 we present the approximate values of the dimensionless energy $\bar{E}_m = \beta^{1/4} Q^{-3/2} \exp(3\gamma/4) E_m$ for $\sigma = 0.1, 0.2, ..., 0.7$. Table 1 shows that with increasing parameter $\sigma$ the magnetic energy of the monopole increasing. When parameter $\gamma$ increases the magnetic energy $E_m$ decreases.

From Eq. (8) we obtain the magnetic field of the monopole

$$H = \frac{Q}{r^2} e^{-\gamma} \left(1 + \frac{\beta Q^2}{2\sigma r^4 e^{-\gamma}}\right)^{\sigma-1}. \hspace{1cm} (40)$$
Table 1: The dimensionless energy $\bar{E}_m$

| $\sigma$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |
|----------|-----|-----|-----|-----|-----|-----|-----|
| $\bar{E}_m$ | 6.58 | 8.38 | 10.13 | 12.28 | 15.45 | 22.29 | 58.78 |

The “free magnetic charge density” $\eta_{\text{free}}$ is defined by the equation

$$\nabla \cdot H = \frac{1}{r^2} \frac{d}{dr} (r^2 H) = 4\pi \eta_{\text{free}}.$$  

(41)

Making use of Eq. (41) one finds from Eq. (40) the “free magnetic charge density”

$$\eta_{\text{free}} = -\frac{\beta Q^3 (\sigma - 1)}{2\pi \sigma r^7} e^{-2\gamma} \left(1 + \frac{\beta Q^2}{2\sigma r^4} e^{-\gamma}\right)^{\sigma-2}.$$  

(42)

Similar to Eq. (33) we obtain, by using Eq. (40), the free magnetic charge

$$Q_{\text{free}} = \frac{1}{4\pi} \int \nabla \cdot H d^3 x = \int_0^\infty \frac{d}{dr} (r^2 H) dr = (r^2 H^2)_0^\infty = Q \exp(-\gamma).$$  

(43)

Thus, the equation for the free magnetic charge is similar to the free electric charge (34). From Eq. (40), one finds the free magnetic charge inside the sphere of the radius $r_m = \beta^{1/4} \sqrt{Q}$

$$Q_{\text{free}}(r_m) = (r^2 H)_0^r = Q_{\text{free}} \left(1 + \frac{1}{2\sigma} e^{-\gamma}\right)^{\sigma-1}.$$  

(44)

The plots of the functions $Q_{\text{free}}(r_m)/Q_{\text{free}}$ versus $\gamma$ for $\sigma = 1/8$, $\sigma = 1/2$ and $\sigma = 3/4$ is presented in Fig. 3. According to Eq. (44) $(1 + e^{-\gamma}/(2\sigma))^{\sigma-1}$ of the magnetic charge $Q_{\text{free}}$ is inside the sphere of the radius $r_m$. The plot of the function $Q_{\text{free}}(r_m)/Q_{\text{free}}$ is in Fig. 4. Figure 4 shows that the ratio $Q_{\text{free}}(r_m)/Q_{\text{free}}$ increases with increasing parameters $\sigma$ and $\gamma$. As a result, more magnetic charge is concentrated in the sphere of the radius $r_m$ with greater values of $\sigma$ and $\gamma$.

The generalised ModMax model proposed as compared to ModMax model (1) possesses the attractive features as follows.

- At some parameters $\beta$, $\lambda$, $\gamma$ and $\sigma$ we come to different models (including the Born–Infeld electrodynamics) discussed in the literature.
Figure 3: The plot of the functions $Q_{\text{free}}(r_m)/Q_{\text{free}}$ versus $\gamma$ for $\sigma = 1/8$, $\sigma = 1/2$ and $\sigma = 3/4$.

- In the weak-field limit and $\gamma = 0$, the Lagrangian density leads to the Heisenberg–Euler electrodynamics.
- The electric field of the point-like charged particle in the origin is finite and possesses the maximum value (for $\sigma < 1$).
- The electric and magnetic energies of point-like charged particles is finite for some parameters $\sigma$.

Such properties take place also for the two-parametric duality invariant generalization of Born–Infeld electrodynamics found in [1], [4].

In addition, we calculated the free electric and magnetic densities in our model that allow us to study the distributes of the electric and magnetic charges in the space.
Figure 4: The plot of the functions $Q_{\text{free}}(r_m)/Q_{\text{free}}$ versus $\sigma$, for $\gamma = 0, 1, 2, 3$.

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