Beyond Lie algebras and group representations: combinatorics

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Abstract. While acknowledging the extraordinary influence and profound implications of symmetry methods in physics, we argue that these methods have a broader setting in enumerative combinatorics. This is illustrated in several ways by explicit results.

1. Introduction and Viewpoint
The role of symmetry in physics is still succinctly summarized by Wigner’s [1] remark in the Preface that he had come to agree with M. von Laue’s remark “that the recognition that almost all the rules of spectroscopy follow from the symmetry of the problem is the most remarkable result” contained in that volume. One now knows that this result extends to the term spectroscopy in the broadest sense: atomic, nuclear, particle, etc. The algebra of the operators corresponding to observables in quantum theory leads naturally to Lie algebra and group symmetry in physical law with a strong focus on representation theory and invariants, which arises naturally because of the action of operators in the Hilbert space of state vectors that describe physical systems, and the universality of the invariant subspaces that encode the patterns of symmetry.

Our friend and colleague, the late Brian Wybourne, recognized the importance of symmetry early in his career, and spent much of his time in developing the tools for implementing such techniques into complex atomic spectroscopy, as well as in other areas (Wybourne [2,3]), as we have heard in the opening remarks by Ron King. He was particular fascinated with the role of Schur functions and their plethysms. This paper is dedicated to his extraordinary enthusiasm for science and the hope that his contributions, insightful writings, and viewpoints will inspire others.

The role of symmetry in governing physical law cannot be overstated, and the viewpoints expressed in this paper are evolutions of that role to what I believe is the more general setting of combinatorics, much in the spirit of Brian's focus on Schur functions. Much of what I shall report has all been said before by my collaborators and me (see Refs.[4,5,11,12]). The purpose of this paper is to pull this all together in a compendium format that justifies the title of this presentation.

We contrast the present approach to the traditional methods of the representation theory of symmetry where the group and its Lie algebra are realized by their action in an abstract Hilbert space with suitable properties. An alternative approach is to use special Hilbert spaces that realize all the properties of the abstract postulates and perform calculations within that
framework. The framework must be sufficiently rich in structure so as to apply to a manifold of physical situations. It is an approach that is particularly useful for revealing the combinatorial foundations of quantum angular momentum theory, and its generalizations to the general linear and unitary groups.

We shall introduce the subject in the simplest context through the concept of $SU(2)$ solid harmonics, which are the unitary group analogues of the familiar solid harmonics, or spherical harmonics, when restricted to the unit sphere of physical Cartesian $\mathbb{R}^3$–space, that we meet in elementary quantum theory.

2. $SO(3,\mathbb{R})$ and $SU(2)$ Solid Harmonics
It is often the case that when a mathematical entity is polynomial in, say, some constrained variables that define it, a useful procedure is to replace those variables by arbitrary indeterminates with no constraints, and then consider the properties of the new object. A simple example of this is provided by the binomial coefficients defined by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

which is polynomial in the nonnegative integer $n$. This suggests we consider the general polynomial $(x^k)$ for an arbitrary variable $x$ defined by

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!},$$

which is then a polynomial of degree $k$ with roots $x = 0, 1, \ldots, k-1$, which reduces at $x = n \geq k$ to the binomial coefficient. These polynomials then have a host of properties generalizing those of the numerical binomial coefficients such as the sum rule

$$\binom{x+y}{k} = \sum_{k_1+k_2=k} \binom{x}{k_1} \binom{y}{k_2}.$$

The idea for obtaining the $SU(2)$ solid harmonics extends this simple example for binomial polynomials in the obvious way. The unitary irreducible representations of the unitary unimodular group $SU(2)$ can be written in terms of the elements of the 2 unitary matrix $U = (u_{ij})_{1\leq i,j\leq 2}, \det U = 1$, by the familiar expression

$$D^j_{m,m'}(U) = \sqrt{(j+m)!(j-m)!(j+m')!(j-m')!} \times \sum_k \frac{u_{11}^ku_{12}^{j+m-k}u_{21}^{j+m'-k}u_{22}^{k-m-m'}}{k!(j+m-k)!(j+m'-k)!(k-m-m')!},$$

(2.1)

where the angular momentum quantum number $j$ is integral or half-odd integral, and each of the projection quantum numbers $m$ and $m'$ can assume values $j, j-1, \ldots, -j$. (See Biedenharn and Louck [5, p. 53] for this form).

Following the above advisement, we define the $SU(2)$ solid harmonics to be the polynomials homogeneous of degree $2j$ in four commuting indeterminates $Z = (z_{ij})_{1\leq i,j\leq 2}$ given by

$$D^j_{m,m'}(Z) = \sqrt{\alpha_1!\beta_1!} \sum_{A\in M_2(\alpha,\beta)} Z^A A!/A!,$$

(2.2)

in which

$$\alpha_1 = j+m, \alpha_2 = j-m, \beta_1 = j+m', \beta_2 = j-m',$$

(2.3)
and the nonnegative exponents \( A = (a_{11}, a_{21}, a_{12}, a_{22}) \) are subject to the constraints

\[
M_2(\alpha, \beta) = \left\{ a_{11}, a_{21}, a_{12}, a_{22} \mid a_{11} + a_{12} = \alpha_1, a_{21} + a_{22} = \alpha_2, a_{11} + a_{21} = \beta_1, a_{12} + a_{22} = \beta_2 \right\}.
\]

(2.4)

These constraints on the exponents are imposed to preserve the homogeneity properties of (2.1); namely, they are homogenous of degree \( j + m \) in the variable pair \((z_{11}, z_{12})\), degree \( j - m \) in the variable pair \((z_{21}, z_{22})\), degree \( j + m' \) in the variable pair \((z_{11}, z_{21})\), and degree \( j - m' \) in the variable pair \((z_{12}, z_{22})\), which gives polynomials that are homogenous of total degree \( 2j \). We also use the space saving standard notations

\[
Z^A = \prod_{i,j=1}^n z_{ij}^{a_{ij}}, \quad A! = \prod_{i,j=1}^n (a_{ij})!, \quad \alpha! = \prod_{i=1}^n \alpha_i!, \quad \beta! = \prod_{i=1}^n \beta_i!,
\]

(2.5)

where \( n = 2 \) in (2.2).

The nomenclature \( SU(2) \) solid harmonics for the polynomials defined by (2.2) is by analogy with the term \( SO(3, \mathbb{R}) \) solid harmonics. These latter solid harmonics are defined over all points of Cartesian 3-space \( \mathbb{R}^3 \) by

\[
\mathcal{Y}_{l,m}(x_1,x_2,x_3) = \left[ \frac{2l+1}{4\pi} (l+m)!(l-m)! \right]^{1/2} \times \sum_{k \geq 0} \frac{(-ix_1 - ix_2)^m (x_1 - ix_2)^k x_3^{l-m-2k}}{2^{m+2k}(m+k)!k!(l-m-2k)!}. \quad (2.6)
\]

The components \((L_1, L_2, L_3)\) of the orbital angular momentum operator \( \mathbf{L} = -i\mathbf{x} \times \nabla \) have the standard action on the solid harmonics, which are homogeneous polynomial solutions of Laplace’s equation in \( \mathbb{R}^3 \):

\[
L^2 \mathcal{Y}_{l,m}(x) = l(l+1)\mathcal{Y}_{l,m}(x), \quad L_3 \mathcal{Y}_{l,m}(x) = m \mathcal{Y}_{l,m}(x),
\]

\[
L_{\pm} \mathcal{Y}_{l,m}(x) = \sqrt{(l+m)(l \pm m+1)} \mathcal{Y}_{l \pm 1,m}(x),
\]

(2.7)

\[
l \in \{0,1,2,\ldots\}, \quad m = l,l-1,\ldots,-l.
\]

The eigenvalue relation \( L^2 \mathcal{Y}_{l,m}(x) = l(l+1)\mathcal{Y}_{l,m}(x) \) is a consequence of the fact that the polynomials \( \mathcal{Y}_{l,m}(x) \) are homogeneous of degree \( l \) that solve Laplace’s equation, since \( L^2 \) can be written as

\[
L^2 = -(\mathbf{x} \cdot \mathbf{x}) \nabla^2 + (\mathbf{x} \cdot \nabla)^2 + (\mathbf{x} \cdot \nabla),
\]

(2.8)

which is a sum of three commuting operators, each of which is invariant under orthogonal frame transformations.

These solid harmonics in \( \mathbb{R}^3 \) are normalized to unity over the unit sphere, and the components \( L_i \) are Hermitian with respect to the inner product of square integral functions over the unit sphere. That only integral values of the angular momentum quantum number \( l \) can occur is a consequence of the Hermitian property and the fact that both \( L_+ \mathcal{Y}_{l1} = 0 \) and \( L_- \mathcal{Y}_{l,-l} = 0 \) must be satisfied. (See Ref [5, p. 319] for more discussion of this result.)

The polynomials \( D^l_{m,m'}(Z), \quad z = (z_{11}, z_{21}, z_{12}, z_{22}) \in \mathbb{C}^4 \) are homogeneous of degree \( 2j \). The angular momentum operator \( \mathbf{J}^2 \), with \( \mathbf{J} \) components \((J_1, J_2, J_3)\) is given by

\[
\mathbf{J}^2 = -(\det Z)(\det \frac{\partial}{\partial Z}) + J_0(J_0 + 1), \quad J_0 = (\mathbf{z} \cdot \mathbf{\partial})/2,
\]

(2.9)

\[
\mathbf{\partial} = (\partial/\partial z_{11}, \partial/\partial z_{21}, \partial/\partial z_{12}, \partial/\partial z_{22}),
\]
which is a sum of two commuting operators $-(\det Z)(\det \frac{\partial}{\partial Z})$ and $J_0(J_0 + 1)$, each of which is invariant under $SU(2)$ transformations. The $SU(2)$ solid harmonics are homogeneous polynomials of degree $2j$ such that

$$\det \left( \frac{\partial}{\partial Z} \right) D^j_{m m'}(Z) = 0, \quad J^2 D^j_{m m'}(Z) = j(j + 1)D^j_{m m'}(Z). \quad (2.10)$$

The components $(J_1, J_2, J_3)$ of the angular momentum operators $\mathbf{J}$ corresponding to left transformations of $Z$ and the components $(J'_1, J'_2, J'_3)$ corresponding to right transformations have the standard actions on these polynomials:

$$J_{\pm} D^j_{m m'}(Z) = \sqrt{j \mp m}(j \pm m + 1)D^j_{m \pm 1 m'}(Z),$$

$$J'_{\pm} D^j_{m m'}(Z) = \sqrt{j \mp m'}(j \pm m' + 1)D^j_{m m' \pm 1}(Z). \quad (2.11)$$

The component $J_3$ and $J'_3$ are diagonal with eigenvalues $m$ and $m'$. Under either left or right $SU(2)$ transformations these polynomials give the standard unitary irreducible representations (2.1) of the group $SU(2)$.

The $SU(2)$ solid harmonics are among the most important functions in angular momentum theory. Not only do they unify the irreducible representations of $SU(2)$ in any parametrization by the appropriate definition of the indeterminates in terms of generalized coordinates, they also include the popular boson calculus realization of state vectors for quantum mechanical systems, as well as the state vectors for the symmetric rigid rotator. The role of the inner product is essential. Physical theory demands an inner product that is given in terms of integrations of wave functions over the variables of the theory, as required by the probabilistic interpretation of wave functions. It is the requirement that realizations of angular momentum operators be Hermitian with respect to the inner product for the spaces being used that assures orthogonality of functions. Thus, one can use different inner products having this property with assurance that the results for the properties of angular momentum operators will be identical, except for normalization of state vectors. For polynomials in arbitrary indeterminates it is convenient to use an inner product discussed in detail in Refs. [6,7], which we denote here by $\langle , \rangle$. With respect to this inner product, the $SU(2)$ solid harmonics satisfy the normalization rule:

$$\langle D^j_{m m'}, D^j_{m'' m'''} \rangle = \delta_{m m''} \delta_{m', m'''}(2j)! \quad (2.12)$$

Often, in combinatorial arguments, the inner product plays no direct role.

We next introduce natural generalizations of the $SU(2)$ solid harmonics, called $SU(n)$ solid harmonics. We will give many properties of these polynomials, which subsume for $n = 2$ properties of the $SU(2)$ solid harmonics.

### 3. $SU(n)$ Solid Harmonics

Generating functions codify the content of many mathematical entities in a unifying, comprehensive way. The are very popular in combinatorics, and Schwinger [8] used them extensively in his fundamental treatment of angular momentum theory. In this subsection, we present a natural generalization of the $SU(2)$ solid harmonics to a class of polynomials that are homogeneous in $n^2$ indeterminates. While these polynomials are of interest in their own right (see Gelfand and Graev [9]), it is their fundamental role in the addition of $n$ kinematically independent angular momenta that motivates their introduction here. They bring an unexpected unity and coherence to angular momentum coupling and recoupling theory through their relationship to MacMahon’s [10] master theorem, which was rediscovered by Schwinger [8].
The natural generalization of definition (2.2) to \( n^2 \) indeterminates \( Z = (z_{ij})_{1 \leq i,j \leq n} \) is given by

\[
D^p_{\alpha \beta}(Z) = \sqrt{\alpha! \beta!} \sum_{A \in \mathcal{M}^p_{\alpha \beta}} Z^A A!,
\]

where \( \alpha \) and \( \beta \) are sequences \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) of \( n \) nonnegative integers that sum to \( p \), and are called compositions of \( p \) into \( n \) nonnegative parts, denoted by \( \alpha \vdash p; \beta \vdash p \). For each pair \((\alpha, \beta)\) of such compositions \( \alpha \vdash p \) and \( \beta \vdash p \), the notation \( \mathcal{M}^p_{\alpha \beta} \) denotes the set of all matrices \( A \) such that the entries in row \( i \) sum to \( \alpha_i \) and those in column \( j \) to \( \beta_j \), for all \( i, j = 1, 2, \ldots, n \). The sequences \( \alpha \) and \( \beta \) are referred to as the row sum and column sum vectors of \( A \). The significance of the row sum vector \( \alpha \) is that \( \alpha_i \) is the degree of the polynomial \( D^p_{\alpha \beta}(Z) \) in the variables \( (z_{11}, z_{12}, \ldots, z_{1n}) \) in row \( i \) of \( Z \), while the column sum vector \( \beta \) has the significance that \( \beta_j \) is the degree of the polynomial \( D^p_{\alpha \beta}(Z) \) in the variables \( (z_{j1}, z_{j2}, \ldots, z_{jn}) \) in column \( j \) of \( Z \).

The following compendium summarizes some of their important properties:

(i) Matrix of the \( D^p_{\alpha \beta}(Z) \) polynomials: The number of compositions of the integer \( k \) into \( n \) nonnegative parts is given by \( \binom{n+p-1}{p} \). The compositions in this set may be linearly ordered by the lexicographical rule \( \alpha > \beta \), if the first nonzero part of \( \alpha - \beta \) is positive. The polynomial \( D^k_{\alpha \beta}(Z) \) is then the entry in row \( \alpha \) and column \( \beta \) in the matrix \( D^p(Z) \) of dimension

\[
\dim D^p(Z) = \binom{n+p-1}{p},
\]

where, following the convention for \( SU(2) \), the rows are labeled from top to bottom by the greatest to the least sequences, and the columns are labeled in the same manner as read from left to right.

(ii) Multiplication property: Chen and Louck [11] have given a purely combinatorial proof that these polynomials satisfy the following multiplication rule for arbitrary matrices \( X \) and \( Y \), singular and nonsingular:

\[
D^p(X)D^p(Y) = D^p(XY).
\]

(iii) Orthogonality in the inner product \( \langle \cdot, \cdot \rangle \):

\[
\langle D^p_{\alpha \beta}, D^p_{\alpha' \beta'} \rangle = \delta_{p,p'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} p!.
\]

(iv) Reduction to spinor harmonics: Specializing the matrix \( Z \) of indeterminates gives some classical results

\[
D^p_{\alpha \beta}(0, \ldots, 0, z^{(j)}, 0, \ldots, 0) = \left( \prod_{i \neq j}^{n} \delta_{0,\beta_i} \right) \frac{p!}{\alpha!} z^\alpha,
\]

in which all columns of \( Z \) are \( \text{col}(0, 0, \ldots, 0) \), except column \( j \) which is \( z^{(j)} = \text{col}(z_1, z_2, \ldots, z_n) \), \( \alpha \vdash p \), and \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \). Also, we have the relation

\[
D^p_{\alpha \beta}(\text{diag}(z_1, z_2, \ldots, z_n)) = \delta_{\alpha,\beta} z^\alpha.
\]

(v) Diagonal matrix: The value at the diagonal matrix is a diagonal matrix:

\[
D^p(I_n) = I_{\binom{n+p-1}{p}}.
\]
(vi) Transposition property: Transposition is a basic symmetry:
\[ D^p(Z^T) = (D^p(Z))^T. \] (3.8)

(vii) Special irreducible unitary representations of \( SU(n) \):
\[ D^p(U)D^p(V) = D^p(UV), \text{ all } U, V \in SU(n). \] (3.9)

(viii) MacMahon and Schwinger master theorems. Classical relations:
(a) Schwinger’s master theorem: For any two matrices \( X \) and \( Y \) of order \( n \), the following identities hold:
\[ e^{(\partial_x : X : \partial_y)}e^{(x : Y : y)}|_{x=y=0} = \sum_{p=0}^{\infty} \sum_{\alpha, \beta \vdash p} D^p_{\alpha \beta}(X)D^p_{\beta \alpha}(Y) = \frac{1}{\det(I - XY)}, \] (3.10)
\[ (x : Z : y) = xZy^T = \sum_{i,j=1}^{n} z_{ij}x_iy_j. \] (3.11)

(b) MacMahon’s master theorem: Let \( X \) be the diagonal matrix \( X = \text{diag}(x_1, x_2, \ldots, x_n) \) and \( Y \) a matrix of order \( n \). Then, the coefficient of \( x^\alpha \) in the product \( y^\alpha \), where \( y_i = \sum_{j=1}^{n} y_{ij}x_j \); that is,
\[ \frac{1}{\det(I - XY)} = \sum_{p=0}^{\infty} \sum_{\alpha \vdash p} D^p_{\alpha \alpha}(Y) x^\alpha. \] (3.12)

(c) Basic master theorem: Let \( Z \) be a matrix of order \( n \), then
\[ \frac{1}{\det(I - tZ)} = \sum_{p=0}^{\infty} t^p \sum_{\alpha \vdash p} D^p_{\alpha \alpha}(Z). \] (3.13)

Schwinger’s relation (3.10) follows from this relation by setting \( Z = XY \) and using the multiplication property (3.3); MacMahon’s relation follows from Schwinger’s result by setting \( X = \text{diag}(x_1, x_2, \ldots, x_n) \). Of course, MacMahon’s master theorem preceeding Schwinger’s result by many years. The unification into the single form by using properties of the \( D^k_{\alpha \beta}(Z) \) polynomials was observed by Louck [12]. More surprisingly, the right-hand side of relation (3.10) was already discovered for the general linear group in 1897 by Molien [13]. The properties of this relation for groups are developed extensively in Michel and Zhilinski [14].

4. Coupling of Two Angular Momenta
The coupling of two angular momenta in abstract Hilbert space is realized explicitly by \( SU(2) \) solid harmonics and spinor harmonics by the following relationship:
\[ \psi_{(j_1, j_2)j m}(Z) = \sum_{m_1 + m_2 = m} C^{j_1 j_2 j}_{m_1 m_2 m} \times \psi_{j_1 m_1}(z_{11}, z_{21})\psi_{j_2 m_2}(z_{12}, z_{22}), \] (4.1)
where \( \psi_{j_1 m_1}(z_{11}, z_{21}) \) and \( \psi_{j_2 m_2}(z_{12}, z_{22}) \) are obtained from the spinor harmonics defined by
\[ \psi_{j m}(z_1, z_2) = \frac{z_1^{j+m} z_2^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \] (4.2)
and \( \psi_{(j_1,j_2)} m(Z) \) is defined by

\[
\psi_{(j_1,j_2)} m(Z) = \sqrt{\frac{2j + 1}{(j_1 + j_2 - j)!(j_1 + j_2 + j + 1)!}} \times (\det Z)^{j_1 + j_2 - j} D_{m,j_1-j_2}^j(Z).
\]  

(4.3)

Explicit knowledge of the Clebsch-Gordan coefficients (also called Wigner-Clebsch-Gordan or WCG coefficients) is not needed to prove these relationships. Their explicit form can be obtained by a purely combinatorial technique in the umbral calculus known as an \textit{evaluation}, which goes as follows (see Roman and Rota [15]):

(i) Explicit WCG coefficients as evaluations:

(a) Rota evaluations: The evaluation at \( y \) of a divided power \( x^k/k! \) of a single indeterminate \( x \) to a nonnegative integral power \( k \) is defined by

\[
eval y \frac{x^k}{k!} = \frac{(y)_{k}}{k!} = \frac{y(y-1)\cdots(y-k+1)}{k!} = \binom{y}{k},
\]

where \((y)_k\) is the falling factorial. This definition is extended to products by

\[
eval(y_1,y_2,\ldots,y_n) \prod_{i=1}^{n} \frac{x_{k_i}}{k_i!} = \prod_{i=1}^{n} \frac{y_i}{k_i!} = \prod_{i=1}^{n} \frac{(y_i)_{k_i}}{k_i!}.
\]

(4.5)

It is also extended by linearity to sums of such divided powers, multiplied by arbitrary numbers.

(b) A special case of Rota evaluation: The basic relation underlying relation (4.3) is

\[
\frac{(\det Z)^n}{n!} \sum_{A \in M_2(\alpha,\alpha')} \frac{Z^A}{A!} = \sum_{B \in M_2(\beta,\beta')} \left( \frac{\det Z}{n!} \right)^n \frac{X^B}{B!},
\]

(4.6)

in which the “line sums” \( \beta \) and \( \beta' \) of \( B \) are given in terms of the “line sums” \( \alpha \) and \( \alpha' \) of \( A \) by

\[
\beta = (\alpha_1 + n, \alpha_2 + n), \quad \beta' = (\alpha'_1 + n, \alpha'_2 + n).
\]

(4.7)

The evaluation operation is given by

\[
eval_{B} \frac{(\det Z)^n}{n!} = \sum_{k_1+k_2=n} (-1)^{k_2} k_! k_2! \binom{b_{11}}{k_1} \binom{b_{12}}{k_2} \binom{b_{21}}{k_1} \binom{b_{22}}{k_2}.
\]

(4.8)

This identity is a purely combinatorial, algebraic relation for arbitrary indeterminates and arbitrary row and column sum constraints on the array \( A \) as specified by \( \alpha = (\alpha_1, \alpha_2) \) and \( \alpha' = (\alpha'_1, \alpha'_2) \).

(c) Van der Waerden form of WCG coefficients as an evaluation: We now apply relations (4.6)-(4.8) to the case at hand, where we have \( n = j_1 + j_2 - j, \alpha = (j + m, j - m), \alpha' = (j + j_1 - j_2, j - j_1 + j_2), \beta = (j_1 + j_2 + m, j_1 + j_2 - m), \beta' = (2j_1, 2j_2) \). This gives the following result:

\[
C_{m_1 m_2}^{j_1 j_2 j} = \sqrt{\frac{(2j + 1)(j + m)!(j - m)!}{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!}} \times (\det Z)^{2j_1 + 2j_2 - j} \frac{\det Z}{(j_1 + j_2 - j)!} \times \eval_{A} \frac{(\det Z)^{j_1 + j_2 - j}}{(j_1 + j_2 - j)!},
\]

(4.9)
\[
eval_A \frac{(\det Z)^{j_1 + j_2 - j}}{(j_1 + j_2 - j)!} = \sum_{k_1 + k_2 = j_1 + j_2 - j} (-1)^{k_2} k_1! k_2! \binom{j_1 + m_1}{k_1} \binom{j_2 + m_2}{k_2} \times \binom{j_1 - m_1}{k_1} \binom{j_2 - m_2}{k_2}.
\]

(4.10)

In summary, we have: Up to multiplicative square-root factors, a WCG coefficient is the integer obtained from the evaluation at the point \(B = \left( \begin{array}{c} j_1 + m_1 j_2 + m_2 \\ j_1 - m_1 j_2 - m_2 \end{array} \right)\) of the divided power \(\frac{\det Z^{j_1 + j_2 - j}}{(j_1 + j_2 - j)!}\) of a \(2 \times 2\) determinant.

5. Magic Squares and the Coupling of Two Angular Momenta

One of the most fascinating non-Lie algebraic origins of the coupling of two angular momenta is contained in Regge's [16] observation that the triangle rule on the three angular momenta \(J_1, J_2, j\) given by \(j = J_1 + J_2, J_1 + J_2 - 1, \ldots, |J_1 - J_2|\), and the sum rule on their three projections \((m_1, m_2, m)\) given by \(m_1 + m_2 = m\), are realized by a \(3 \times 3\) magic square with fixed line-sum \(J\), which is given in terms of angular momentum quantum numbers by \(J = J_1 + J_2 + j\):

\[
A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} j_1 + m_1 & j_2 + m_2 & j - m \\ j_1 - m_1 & j_2 - m_2 & j + m \\ j_2 - j_1 + j & j_1 - j_2 + j & j_1 + j_2 - j \end{pmatrix}.
\]

(5.1)

This relation gives a bijection between magic squares \(A\) of fixed line-sum \(J\) and the angular momentum quantum numbers \((J_1, m_1; J_2, m_2; j, m)\). The abstract relationship between coupled and uncoupled state vectors is completely codified in the properties of magic squares and may be formulated as

\[
\left| (a_{11} + a_{21})/2, (a_{12} + a_{22})/2; (a_{13} + a_{23})/2, (a_{23} - a_{13})/2 \right|
= \sum_{A_2} W_J(A) \left| (a_{11} + a_{21})/2, (a_{11} - a_{21})/2 \right| \otimes \left| (a_{12} + a_{22})/2, (a_{12} - a_{22})/2 \right|
\]

(5.2)

where the summation is over all matrix arrays of nonnegative integers \(A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\) of order 2 with row and column sums given by \(a_1 = J_2 - a_{13}, a_2 = J_2 - a_{23}, \beta_1 = J_2 - a_{31}, \beta_2 = J_2 - a_{32}\), and the elements \((a_{31}, a_{32}, a_{33})\) in row 3 and the elements \((a_{13}, a_{23}, a_{33})\) in column 3 are held fixed. The coefficients \(W_J(A)\) in relation (5.2) are not determined by the magic square structure, but all domains of definition of the entries in the ket-vectors are determined. The coefficients \(W_J(A)\) themselves are WCG coefficients, as given in terms of the elements of the magic square \(A\) by

\[
W_J(A) = C^{(a_{11} + a_{21})/2, (a_{12} + a_{22})/2, (a_{13} + a_{23})/2, (a_{13} - a_{23})/2, (a_{23} - a_{13})/2}.
\]

(5.3)

It is a quite interesting result that the number of magic squares of order 3 and line-sum \(J\) has an easy derivation as given by Stanley [17, p.92], which leads to the following formula::

Define \(\Delta_J = \{\text{all triangles}(j_1, j_2, j) | j_1 + j_2 + j = J\}\) and \(M(j_1, j_2, j) = \{(m_1, m_2) | -j_1 \leq m_1 \leq j_1; -j_2 \leq m_2 \leq j_2; -j \leq m_1 + m_2 \leq j\}\). Then, we have the following identity:

\[
\sum_{(j_1, j_2, j) \in \Delta_J} |M(j_1, j_2, j)| = \binom{J + 5}{5} - \binom{J + 2}{5}.
\]

(5.4)
It is nontrivial to effect the summation on the left-hand side of this relation to obtain the right-hand side.

Thus, to each magic square $A$ of order 3 there corresponds a unique WCG coefficient $W_j(A)$, and conversely. Thus, the full abstract structure of the addition of angular momentum is one-to-one with the structure of magic squares of order 3. These rich combinatorial footings of angular momentum theory are completed by the observation that the WCG coefficients themselves are obtained by the Regge [16] generating function for the expansion of a determinant of order 3, again there being no reference to Lie algebraic structures. Addition of angular momentum in quantum theory can be presented in terms of magic squares and the Regge generating function for WCG coefficients without reference to Lie algebras.

6. Kronecker Products

From a physical viewpoint, Kronecker products of representations of groups underly the theory of composite systems. Irreducible symmetry subspaces of a composite system are built from those of its individual parts by reducing the Kronecker product of the irreducible representations of the parts. The Kronecker product extends to the polynomial forms on arbitrary commuting indeterminates, so that one has the general rule for $SU(2)$ solid harmonics given by

$$
(C^{(j_1,j_2)})^T (D^{j_1}(Z) \otimes D^{j_2}(Z)) C^{(j_1,j_2)} = \sum_j \bigoplus (\det Z)^{j_1+j_2-j} D^j(Z),
$$

(6.1)

in which $\oplus$ denotes the direct sum of matrices, $T$ denotes matrix transposition, the summation is over all $j$ that satisfy the triangle rule $j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$, and $C^{(j_1,j_2)}$ is a matrix of WCG coefficients and 0's of dimension $(2j_1+1)(2j_2+1)$, appropriately defined. The validity of this result follows from the fact that the implied double WCG coefficient coupling of the solid harmonics $D^{j_1}_{m_1,m_1'}(Z)$ and $D^{j_2}_{m_2,m_2'}(Z)$ is insensitive to the detailed properties of the polynomials: It depends only on the angular momentum properties of the coupling of angular momenta. The presence of $(\det Z)^{j_1+j_2-j}$ is required by the homogeneity properties of the solid harmonics. There are three additional forms of this result, obtained by using the fact that $C^{(j_1,j_2)}$ is an orthogonal matrix, so that it can be moved in three ways to the other side of (6.1).

Relation (6.1) applies to an arbitrary matrix $Z$, including singular ones. These relations can be specialized in many ways to obtain results for special functions (see Chen and Louck [11]) and algebras. In particular, they hold for $Z$ an element of the groups $SU(2), U(2), GL(2, \mathbb{C})$.

7. Pfaffians and Double Pfaffians

The basic mathematical objects in the theory of recoupling coefficients are Pfaffians and double Pfaffians. Schwinger [8] observed that the calculation of $3n - j$ coefficients involves taking the square root $\sqrt{(T - AB)}$, where $A$ and $B$ are skew symmetric (antisymmetric) matrices of order $n$, but the procedure is rather obscure. The appropriate concepts for taking the square root is that of a Pfaffian and a double Pfaffian, denoted, respectively, by $Pf(A)$ and $Pf(A,B)$. The definitions require the concept of a matching of the set of integers $\{1, 2, \ldots, n\}$. A matching of $\{1, 2, \ldots, n\}$ is an unordered set of disjoint subsets $\{i, j\}$ containing two elements. For example, the matchings of 1, 2, 3 are $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. The Pfaffian and double Pfaffian of skew symmetric matrices $A = (a_{ij})$ and $B = (b_{ij})$ of order $n$ are defined by

$$
Pf(A) = \sum_{\text{all matchings}} \varepsilon(i_1i_2\cdots i_n)a_{i_1,i_2}a_{i_3,i_4}\cdots a_{i_{n-1},i_n},
$$

(7.1)

$$
Pf(A, B) = 1 + \sum_{k \geq 1} \sum_{\text{all double matchings}} \varepsilon(i_1i_2\cdots i_{2k})\varepsilon(j_1j_2\cdots j_{2k}) \times a_{i_1,i_2}a_{i_3,i_4}\cdots a_{i_{2k-1},i_{2k}}b_{j_1,j_2}b_{j_3,j_4}\cdots b_{j_{2k-1},j_{2k}},
$$

(7.2)
where $\varepsilon(i_1, i_2 \cdots i_n)$ is the sign of the permutation (number of inversions), and the double matching in the double Pfaffian are the 2-subsets of matchings of a subset of $\{1, 2, \ldots, n\}$ of even length.

The relations of skew symmetric matrices $A, B$ to Pfaffians are

$$
\sqrt{\det A} = Pf(A); \quad \sqrt{\det(I - AB)} = Pf(A, B).
$$

(7.3)

It is possible to identify a skew symmetric matrix with each binary coupling scheme in the addition of $n$ angular momenta, and the recoupling coefficients between each pair of such coupling schemes are generated by the reciprocal of $(Pf(A, B))^2$. (See the Fifth SSCPM under Ref. [4].)

8. The General $D^\lambda$–Polynomials

The $D^\lambda$–polynomials are invertible real linear transformations of the $Z^\lambda$ polynomials that preserve their homogeneity properties in the variables appearing in the rows and columns of $Z$; they have the form:

$$
D \begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z) = \sum_{A \in M^p_{n \times n}(\alpha, \alpha')} C \begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix} (A) \frac{Z^A}{A!}.
$$

(8.1)

The symbol $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition having $n$ parts, including parts that are zero, so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, $(\lambda_m)$ and $(\lambda_m')$ are Gelfand-Tsetlin patterns of the same shape $\lambda$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and, similarly, $\alpha'$ are compositions of $p = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ into $n$ nonnegative parts. These compositions are the weights of the respective Gelfand-Tsetlin patterns: $\alpha = W^{\lambda_m}(\lambda)$, $\alpha' = W^{\lambda_m'}(\lambda')$. The symbol $M^p_{n \times n}(\alpha, \alpha')$ denotes the set of all $n \times n$ matrix arrays of nonnegative integers having row and columns sums given by the parts of compositions $\alpha$ and $\alpha'$. This implies that the $D^\lambda$–polynomials are homogeneous of degree

$$
\alpha_i = \sum_{j=1}^{n} a_{ij} \text{ in the variables } z_i = (z_{i1}, z_{i2}, \ldots, z_{in}) \text{ in row } i \text{ of } Z,
$$

$$
\alpha'_j = \sum_{i=1}^{n} a_{ij} \text{ in the variables } z^j = (z_{1j}, z_{2j}, \ldots, z_{nj}) \text{ in column } j \text{ of } Z.
$$

The inversion of the transformation (8.1) is given by

$$
\frac{Z^A}{A!} = \sum_{\lambda \vdash p \atop m, m' \in G_{\lambda}(\alpha, \alpha')} \frac{1}{M(\lambda) A!} C \begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix} (A) D \begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix}(Z),
$$

(8.2)

where $G_{\lambda}(\alpha, \alpha')$ is the set of double Gelfand-Tsetlin patterns of weight $(\alpha, \alpha')$ and shape $\lambda$, and the quantity $M(\lambda)$ is the invariant normalizing factor defined in terms of Weyl’s [18] dimension formula, $\text{Dim}\lambda$, by

$$
M(\lambda) = \frac{\prod_{i=1}^{n} (\lambda_i + n - i)!}{\text{1!2!} \cdots (n-1)! \text{Dim}\lambda},
$$

$$
\text{1!2!} \cdots (n-1)! \text{Dim}\lambda = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i).
$$
The orthogonality and normalization of the \( D^\lambda \)-polynomials is expressed by

\[
\langle D \left( \begin{array}{c} m' \\ \lambda \\ m \end{array} \right) (Z) , D \left( \begin{array}{c} m'' \\ \lambda' \\ m'' \end{array} \right) (Z) \rangle = \delta_{m,m''} \delta_{m',m''} \delta_{\lambda,\lambda'} M(\lambda). \tag{8.3}
\]

The normalization of the \( D^\lambda \)-polynomials to the factor \( M(\lambda) \) is made so that the \( D^\lambda \) - matrices will have the property \( D^\lambda (I_n) = I_{\text{Dim}\lambda} \).

There are many orthogonal transformations between the polynomials \( Z^A/A! \) and a set of \( D^\lambda \)-polynomials that might be considered. But the polynomials we have in mind here are uniquely defined by the \( C \)-coefficients that occur in (8.1). These coefficients are determined by the action of a set of fundamental shift operators acting on a Gelfand-Tsetlin pattern. Shift operators are basic objects in combinatorial theory. They usually cannot be realized by differential operators, which is the case we have at hand. Nonetheless, the Lie algebra associated with the \( D^\lambda \)-polynomials can be expressed in terms of these still more basic objects, the fundamental shift operators. We cannot get into these details here. Our main point in mentioning the general \( D^\lambda \)-polynomials is to list some of their principal properties to show the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier. What we have called the generalization of all that we have said earlier.

A short compendium of some of the more important formulas, repeated from the Fourth SSPCM in Ref. [4] for the \( D^\lambda \)-polynomials, which generalize the earlier list for the \( D^\mu \)-polynomials is the following:

(i) Multiplication property:

\[
D^\lambda (X) D^\lambda (Y) = D^\lambda (XY), \text{ for arbitrary } X,Y.
\]

This property is fundamental, and has a combinatorial proof.

(ii) Transpositional symmetry:

\[
\left( D^\lambda (Z) \right)^T = D^\lambda (Z^T).
\]

(iii) Diagonal property: For \( Z \) a diagonal matrix, \( Z = \text{diag}(x_1,x_2,\ldots,x_n) \), the following identity holds:

\[
D^\lambda (\text{diag}(x_1,x_2,\ldots,x_n)) = \sum_{\alpha \in W_\lambda} \oplus x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} I_{K(\lambda,\alpha)},
\]

where \( I_{K(\lambda,\alpha)} \) is the identity matrix of dimension \( K(\lambda,\alpha) \), the Kostka number, and \( W_\lambda \) is the set of all weights. This result, in turn, gives \( D^\lambda (I_n) = I_{\text{Dim}\lambda} \).

(iv) Reduction property: Let \( Z_{n-1} \) denote the principal submatrix of \( Z \) obtained by striking row \( n \) and column \( n \). Then, the following relation holds between \( D^\lambda \)-polynomials in the \( n^2 \) variables of \( Z \) for each partition \( \lambda = (\lambda_1,\ldots,\lambda_n) \) and \( D^\mu \)-polynomials in the \( (n-1)^2 \) variables in \( Z_{n-1} \) for each partition \( \mu = (\mu_1,\ldots,\mu_{n-1}) \) :

\[
D \left( \begin{array}{c} m' \\ \mu' \\ \lambda \\ \mu \\ m \end{array} \right) \left( \begin{array}{c} Z_{n-1} \end{array} \begin{array}{c} 0 \\ t \end{array} \right) = \delta(\mu,\mu') t^{\lambda-|\mu|} D \left( \begin{array}{c} m' \\ \mu \\ m \end{array} \right) (Z_{n-1}),
\]

where \( \binom{\mu}{m} \) is the Gelfand-Tsetlin pattern consisting of rows 1 through \( n-1 \) of the \( n \)-rowed pattern, and similarly for \( \binom{\mu'}{m'} \).
(v) Kronecker product: the following identity holds:

\[ C_{\mu\nu}^T(D^\mu(Z) \otimes D^\nu(Z))C_{\mu\nu} = \sum_\lambda \oplus c_{\mu\nu}^{\lambda} D^\lambda(Z), \]

where \( C_{\mu\nu}^{\lambda} \) is a real orthogonal matrix of dimension

\[ \text{Dim} C_{\mu\nu}^{\lambda} = \text{Dim} \mu \text{ Dim} \nu = \sum_\lambda c_{\mu\nu}^{\lambda} \text{Dim} \lambda \]

that effects the reduction of the Kronecker product into a direct sum of \( D^\lambda \)–polynomials. We leave the summation over \( \lambda \) unspecified, but note that is it fully determined by the properties of the Littlewood-Richardson numbers.

(vi) Trace functions: Define \( T_\lambda(Z) = \text{trace} D^\lambda(Z) \). Then, these functions satisfy, in consequence of the Kronecker product relation, the identity:

\[ T_\mu(Z)T_\nu(Z) = \sum_\lambda c_{\mu\nu}^{\lambda} T_\lambda(Z). \]

This result is a consequence of the fact that Kronecker product is brought to a direct sum by a similarity transformation.

(vii) Relation to Schur functions: The polynomials \( T_\lambda(Z) \) are not symmetric functions in the variables \( z_{ij} \), but if \( Z \) is specialized to the diagonal matrix, \( Z = (\text{diag}(x_1, x_2, \ldots, x_n)) \), then, by the diagonal property in Item (iii), they become symmetric functions in these variables, in which case, we have the identity:

\[ s_\lambda(x) = T_\lambda(\text{diag}(x_1, x_2, \ldots, x_n)). \]

The Kronecker product relation in Item (vi) then gives

\[ s_\mu(x)s_\nu(x) = \sum_\lambda c_{\mu\nu}^{\lambda} s_\lambda(x). \]

9. Concluding Remarks
The \( D^\lambda \)–polynomials, and there specialization to \( U(n) \) and \( SU(2) \) solid harmonics can be defined and all their properties deduced by the use of combinatorial methods alone, without reference to Lie algebras and groups. If the indeterminate matrices \( Z \) are specialized to be the elements of a group or algebra, then representations of that group or algebra are obtained. In the case of \( U(n) \), when powers of \( \text{det} Z \) are adjoined, all irreducible unitary representations result, with similar results for the integral representations of the group \( GL(n, \mathbb{C}) \). It is in this sense that the representations of Lie algebras and groups have a deeper setting in combinatorial objects, thus justifying the title of our presentation.

We conclude by noting that the \( D^\lambda \)–polynomials can be considered to be matrix generalizations of the Schur functions. Is anyone bold enough to take on the generalization of Brian Wybourne’s plethysms for Schur functions for the \( D^\lambda \)–polynomials?

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