Affine connections with $W = 0$

Francis Burstall

F.E.Burstall@maths.bath.ac.uk

Mathematical Sciences
University of Bath
Bath BA2 7AY
United Kingdom

John Rawnsley

J.Rawnsley@warwick.ac.uk

Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom

26 February 2007

Abstract

If $\nabla$ is a torsionless connection on the tangent bundle of a manifold $M$ the Weyl curvature $W^\nabla$ is the part of the curvature in kernel of the Ricci contraction. We give a coordinate free proof of Weyl’s result that $W^\nabla$ vanishes if and only if $(M, \nabla)$ is (locally) diffeomorphic to $\mathbb{R}P^n$ with $\nabla$, when transported to $\mathbb{R}P^n$, in the projective class of $\nabla_{LC}$, the Levi-Civita connection of the Fubini–Study metric on $\mathbb{R}P^n$.

If $M$ is even dimensional and $J(M)$ denotes the bundle of all endomorphisms $j$ of the tangent spaces of $M$, a connection $\nabla$ determines an almost complex structure $J^\nabla$ on $J(M)$ [9]. We show that $J^\nabla$ is a projective invariant, that an integrable $J^\nabla$ can be obtained from a torsionless connection and that we must then have $W^\nabla = 0$. We also show for torsionless connections $\nabla$, $\nabla'$ that $J^\nabla = J^{\nabla'}$ if and only if $\nabla$ and $\nabla'$ are projectively equivalent.
1 Introduction

In Riemannian geometry the Ricci tensor splits into two irreducible pieces under the orthogonal group, the scalar curvature which is its trace and the traceless Ricci tensor. The part of the curvature tensor lying in the kernel of the Ricci contraction is also irreducible and known as the Weyl tensor. If the manifold is oriented, this decomposition is still irreducible under the special orthogonal group except in dimension 4 where the Weyl tensor decomposes into its self- and anti-self-dual parts. This decomposition was obtained by Singer and Thorpe [12]. It is shown in Eisenhart [7] that the Weyl tensor is the obstruction to the Riemannian manifold being conformally flat.

The bundle of Hermitean structures on the tangent spaces of an even dimensional Riemannian manifold carries a natural almost complex structure whose integrability condition is the vanishing of the Weyl tensor [3, 6, 9]. In dimension 4, if only Hermitean structures compatible with an orientation are used then the integrability condition is the vanishing of the self-dual part of the Weyl tensor which leads to the celebrated Riemannian analogue of Penrose’s twistor theory developed by Atiyah, Hitchin and Singer [1].

Other authors have considered decompositions of the curvature tensor into irreducible components and corresponding integrability conditions in a number of contexts, in particular for unitary [9, 13], quaternionic [11] and symplectic structures [14, 15].

In the present paper we look at the case with the least restriction on the structure group, the case of a linear connection on a manifold and show that this leads naturally to projective geometry.

A torsionless connection $\nabla$ on the tangent bundle of an $n$-dimensional manifold $M$ defines a projective structure: the family of connections sharing the same geodesics as $\nabla$. The Weyl curvature tensor $W_\nabla$ of $\nabla$ is the part of the curvature in kernel of the Ricci contraction. When $n = 2$, $W_\nabla \equiv 0$ but there is an analogue $C_\nabla$ of the Cotton–York tensor of 3-dimensional conformal geometry. We give a low-technology, coordinate-free proof of Weyl’s theorem [18] that $W_\nabla$ and, for $n = 2$, $C_\nabla$, are projectively invariant and vanish if and only if $(M, \nabla)$ is (locally) isomorphic to $(\mathbb{R}P^n, \nabla_{LC})$ where $\nabla_{LC}$ is the Levi-Civita connection.

If $J(M)$ denotes the bundle of all endomorphisms $j$ of the tangent spaces of $M$, a connection $\nabla$ determines an almost complex structure $J_\nabla$ on $M$. We show that $J_\nabla$ is a projective invariant, that an integrable $J_\nabla$ can always be obtained from a torsionless connection and, in that torsionless case, $J_\nabla$ is integrable if and only if $W_\nabla = 0$. In particular, when $n = 2$, $J_\nabla$ is always integrable even when $C_\nabla$ is non-zero. We also
show that for torsionless connections $\nabla$ and $\nabla'$ we have $J^\nabla = J^{\nabla'}$ if and only if $\nabla$ and $\nabla'$ are projectively equivalent.

The paper is structured as follows:

In section 2 we set the scene and establish notation.

In section 3 we describe the decompositions of torsion and curvature tensors and show that the Weyl component of the curvature is projectively invariant.

In section 4 we look at the twistor theory and show that the Weyl component of the curvature is the obstruction to integrability of the twistor almost complex structure.

Theorem 4.2 The almost complex structure $J^\nabla$ on $J(M)$ defined by a connection $\nabla$ on $M$ only depends on the projective class of $\nabla$. If $\nabla$ and $\nabla'$ both have zero torsion then $J^\nabla = J^{\nabla'}$ if and only if $\nabla$ and $\nabla'$ are projectively equivalent.

Theorem 4.3 If $J^\nabla$ is integrable then there is another connection $\nabla'$ defining the same almost complex structure on $J(M)$ and with zero torsion.

Theorem 4.4 Let $\nabla$ be a torsion-free connection then $J^\nabla$ is integrable if and only if $W^\nabla = 0$.

In section 5 we prove

Theorem 5.6 ([18]) Let $M$ be an $n$-dimensional manifold with torsion-free connection $\nabla$. Suppose that either $n \geq 3$ and $W^\nabla = 0$ or $n = 2$ and $C^\nabla = 0$. Then there are local affine diffeomorphisms between $M$ and $\mathbb{RP}^n$ equipped with a connection in the projective class of the symmetric connection.

2 Preliminaries

2.1 Connections, curvature and torsion

Let $M$ be a manifold and $\nabla$ a connection in $TM$. Its torsion $T^\nabla$ and curvature $R^\nabla$ are given by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R^\nabla(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$  

$T^\nabla$ is a $TM$-valued 2-form and $R^\nabla$ an $\text{End } TM$-valued 2-form. As forms they have covariant exterior derivatives computed using $\nabla$. It is easy to check that we have the First Bianchi Identity

$$(d^\nabla T^\nabla)(X,Y,Z) = R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y$$  \hspace{1cm} (1)
and the Second Bianchi Identity

\[ 0 = (d^\nabla R^\nabla)(X, Y, Z) \]

\[ = (\nabla_X R^\nabla)(Y, Z) + (\nabla_Y R^\nabla)(Z, X) + (\nabla_Z R^\nabla)(X, Y) \]

\[ + R(T(X, Y), Z) + R(T(Y, Z), X) + R(T(Z, X), Y). \]  \hspace{1cm} (2)

The Ricci curvature, \( r^\nabla \), of a linear connection is given by

\[ r^\nabla(X, Y) = \text{Tr} \left( Z \mapsto R^\nabla(X, Z)Y \right). \]

There is a second trace we could take

\[ s^\nabla(X, Y) = \text{Tr}(R^\nabla(X, Y)) \]

which gives a 2-form. If \( X_i \) is a local frame field and \( \alpha^i \) the dual frame field so that \( \alpha^i(X_j) = \delta^i_j \) then, using the First Bianchi Identity,

\[ s^\nabla(X, Y) = \sum_i \alpha^i(R(X, Y)X_i) \]

\[ = \sum_i \alpha^i \left( (d^\nabla T^\nabla)(X, Y, X_i) - R(Y, X_i)X - R(X_i, X)Y \right) \]

\[ = r^\nabla(X, Y) - r^\nabla(Y, X) + \sum_i \alpha^i \left( (d^\nabla T^\nabla)(X, Y, X_i) \right). \]

In particular, the Ricci tensor is symmetric when \( s^\nabla = 0 \) and \( \nabla \) is torsion-free, but not in general.

When \( \nabla \) is torsion free, the second trace is determined by the antisymmetric part of the Ricci tensor.

### 2.2 A bundle of Lie algebras

The bundle \( TM \oplus \text{End} \, TM \oplus T^*M \) carries the structure of a bundle of Lie algebras that will be useful to us. For this, declare \( TM \) and \( T^*M \) to be abelian subalgebras, give \( \text{End} \, TM \) the usual commutator bracket and then, for \((X, A, \alpha) \in TM \oplus \text{End} \, TM \oplus T^*M\), set

\[ [A, X] = AX \]

\[ [\alpha, A] = \alpha \circ A \]

and define \([X, \alpha] \in \text{End} \, TM\) by

\[ [X, \alpha]Y = \alpha(X)Y + \alpha(Y)X. \]
It is straightforward to check that this bracket does indeed satisfy the Bianchi identity but a more conceptual explanation is available: fix a line bundle \( \Lambda \) and set \( V = \Lambda \oplus T \Lambda \) where here and below we use juxtaposition to denote tensor product with a line bundle. Contemplate the bundle \( \mathfrak{sl}(V) \) of trace-free endomorphisms of \( V \). This bundle of Lie algebras decomposes:

\[
\mathfrak{sl}(V) = \text{Hom}(\Lambda, T \Lambda) \oplus \mathfrak{sl}(\text{End}(\Lambda) \oplus \text{End}(T \Lambda)) \oplus \text{Hom}(T \Lambda, \Lambda).
\]

The first and last summands are canonically isomorphic to \( T \Lambda \) and \( T^* \Lambda \) respectively, while the adjoint action provides an isomorphism

\[
\mathfrak{sl}(\text{End}(\Lambda) \oplus \text{End}(T \Lambda)) \cong \text{End}(\text{Hom}(\Lambda, T \Lambda)) \cong \text{End} T \Lambda.
\]

Putting all this together, we arrive at a bundle isomorphism

\[
\mathfrak{sl}(V) \cong T \Lambda \oplus \text{End} T \Lambda \oplus T^* \Lambda
\]

which is readily shown to be an isomorphism of Lie algebras.

### 2.3 G-structures and Representations

Let \( G \) be a Lie group and \( M \) a manifold of dimension \( n \) then a \( G \)-structure on \( M \) is a principle \( G \)-bundle \( \pi: P \to M \) together with an \( n \)-dimensional representation \( V \) of \( G \) such that \( T \Lambda \) is isomorphic to the associated bundle \( P \times_G V \). More precisely, we consider the \( V \)-frame bundle \( Fr(M) \) consisting all the linear isomorphisms \( b: V \to T_x M \) of \( V \) with the tangent spaces of \( M \) and the obvious projection map \( \pi_M \), then a \( G \)-structure is a morphism \( P \to Fr(M) \) covering the homomorphism \( G \to GL(V) \) given by the representation. The isomorphism of \( P \times_G V \) with \( T \Lambda \) is then induced by the identification \( T \Lambda = Fr(M) \times_{GL(V)} V \).

This leads to further identifications of space of tensors on \( M \) with associated bundles of \( P \). For instance, torsion tensors are sections of the bundle associated to the representation \( \Lambda^2 V^* \otimes V \) and curvature tensors to \( \Lambda^2 V^* \otimes \mathfrak{gl}(V) = \Lambda^2 V^* \otimes V^* \otimes V \). The \( G \)-structure in this case is the standard one with \( G = GL(V) \) and \( P = Fr(M) \). We adopt the more general language simply because the questions we examine here make sense in the more general context.

Even when the initial representation of \( G \) on \( V \) is irreducible, the representations on tensor spaces may not be. For example \( \Lambda^k V \) and \( \Lambda^k V^* \) are irreducible under \( GL(V) \) but \( \Lambda^2 V^* \otimes V \) is not. If \( W \) is some representation of \( G \) and \( W = \bigoplus_k W_k \) a decomposition
into irreducible subrepresentations (we assume here that we are dealing with groups for which such a decomposition is always possible and unique up to multiplicities) then this splitting induces a splitting of the corresponding associated bundles and we can project $P \times_G W$ into the subbundles $P \times_G W_k$. We refer to the projections of a section of $P \times_G W$ into the $P \times_G W_k$ as its irreducible components.

In the next section we determine the irreducible components of tensors of torsion and curvature type for $G = \text{GL}(V)$.

3 Decomposition of torsion and curvature

On a manifold $M$ with a particular structure group $G$ we can break the spaces of tensors on $M$ of a particular kind into those taking values in irreducible subspaces under $G$. For example, on a Riemannian manifold (of dimension at least 4) the Singer–Thorpe Theorem \cite{12} says that the curvature of the Levi-Civita connection breaks into 3 pieces under the orthogonal group, the scalar curvature, the traceless Ricci tensor and the Weyl curvature. On oriented 4-manifolds there is a further decomposition of the Weyl curvature under the special orthogonal group into self-dual and anti-self-dual parts. We are interested here in the case where $G$ is the general linear group. Irreducibility is determined by the semisimple part, and only when counting multiplicities do we need to look at how the centre acts on summands where highest weight of the semisimple part is the same.

In the following we suppose that $V$ is a complex vector space and we handle real vector spaces by replacing them by their complexifications. Denote by $S^p V$ and $\Lambda^p V$ the $p$-th symmetric and exterior powers of $V$, respectively, and similarly for the dual space $V^*$. We denote the symmetric product by juxtaposition and exterior multiplication by a wedge. The Lie algebra $\mathfrak{gl}(V)$ is isomorphic to $V^* \otimes V$.

As noted in the previous section, torsion tensors are sections of a bundle whose fibre is associated to the representation of $\text{GL}(V)$ on $\Lambda^2 V^* \otimes V$ and curvature tensors the representation on $\Lambda^2 V^* \otimes \mathfrak{gl}(V) = \Lambda^2 V^* \otimes V^* \otimes V$.

Let us summarise the highest weight theory that we need to decompose these representations. If $\dim V = n$ then $\mathfrak{sl}(V)$ has rank $n - 1$. There are $n - 1$ fundamental representations with highest weights $\omega_i$ and we number them so that $\omega_i$ is the highest weight of $\Lambda^i V$. As representations of $\text{SL}(V)$, $\Lambda^i V^* = \Lambda^{n-i} V$. The weights of $V$ are \{$\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_2, \ldots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}$\} and the weights of $V^*$ are \{-$\omega_1, \omega_1 - \omega_2, \omega_2 - \omega_3, \ldots, \omega_{n-2} - \omega_{n-1}, \omega_{n-1}$\}.
We denote by $V(m_1, \ldots, m_{n-1})$ the irreducible representation of $SL(V)$ whose highest weight is $m_1 \omega_1 + \cdots + m_{n-1} \omega_{n-1}$. Thus $V = V(1, 0, \ldots, 0)$, $V^* = V(0, \ldots, 0, 1)$ $\Lambda^2 V^* = V(0, \ldots, 0, 1, 0)$ and so on.

The highest weights occurring in the decomposition of the tensor product of two irreducibles can be found amongst the highest weight of one factor plus an arbitrary weight of the other. When the second factor has weights of multiplicity 1 then it follows easily from Littelmann’s method \[8\] that the set of irreducible components is in bijection with the dominant elements of this set and these are the highest weights of the components. Fortunately, both $V$ and $V^*$ have all weights of multiplicity 1.

### 3.1 Torsion

Torsion tensors live in $\Lambda^2 V^* \otimes V$ and so the highest weights of the irreducible factors of this space will be the dominant elements of the set $\omega_{n-2} + \{\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_2, \ldots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\}$. Dominant weights have all coefficients non-negative and so, by inspection, these are just $\{\omega_{n-2} + \omega_1, \omega_{n-1}\}$. The second of these is the highest weight of $V^*$. Thus we have $\Lambda^2 V^* \otimes V = T_1 \oplus T_2$ with $T_1 = V(1, 0, \ldots, 0, 1, 0)$ and $T_2 = V(0, \ldots, 0, 1) = V^*$.

We have a map $\Lambda^2 V^* \otimes V \rightarrow V^*$ given by taking a basis $e_i$ for $V$ and dual basis $\delta_i$ for $V^*$ and setting

$$\hat{T}(X) = \sum_i \delta^i(T(X, e_i))$$

which defines an element $\hat{T}$ for each element $T$ of $\Lambda^2 V^* \otimes V$. Conversely, given $\alpha \in V^*$ we can obtain $\bar{\alpha} \in \Lambda^2 V^* \otimes V$ by setting

$$\bar{\alpha}(X, Y) = \alpha(X)Y - \alpha(Y)X.$$ 

Then

$$\hat{\alpha}(X) = \sum_i \delta^i(\alpha(X)e_i - \alpha(e_i)X) = (n-1)\alpha(X).$$

It follows that the component of $T$ in $T_2$ is

$$T_2(X, Y) = \frac{1}{n-1} \sum_i \delta^i(T(X, e_i))Y - \delta^i(T(Y, e_i))X$$

and the component of $T$ in $T_1$ is $T_2 = T - T_1$. $T = T_1 + T_2$ is then the decomposition of the torsion into irreducible components under $GL(V)$. 

6
3.2 Curvature

We decompose curvature by first looking at $\Lambda^2 V^* \otimes V^*$. This has highest weights the dominant elements amongst $\omega_{n-2} + \{-\omega_1, \omega_1 - \omega_2, \omega_2 - \omega_3, \ldots, \omega_{n-2} - \omega_{n-1}, \omega_{n-1}\}$ which are $\{\omega_{n-3}, \omega_{n-2} + \omega_{n-1}\}$. The first of these corresponds with $\Lambda^3 V^* = V(0, \ldots, 0, 1, 0, 0)$ and the second is a representation we call $B_0 = V(0, \ldots, 0, 1, 1)$.

Curvatures live in the space $\Lambda^2 V^* \otimes V^* \otimes V$ which thus has a partial decomposition $\Lambda^3 V^* \otimes V \oplus B$ where $B = B_0 \otimes V$. To decompose this further we can apply the method again. The first piece decomposes as the irreducibles which have a highest weight the dominant elements of $\omega_{n-3} + \{\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_2, \ldots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\}$, and these are $\{\omega_{n-3} + \omega_1, \omega_{n-2}\}$. $B$ decomposes as the irreducibles which have a highest weight the dominant elements of $\omega_{n-2} + \omega_{n-1} + \{\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_2, \ldots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\}$, and these are $\{\omega_{n-2} + \omega_{n-1} + \omega_1, 2\omega_{n-1}, \omega_{n-2}\}$. The last two are highest weights of $S^2 V^*$ and $\Lambda^2 V^*$ respectively.

3.3 Zero Torsion

We look at the case where the torsion vanishes. In this case (11) implies that the curvature satisfies the first Bianchi Identity

$$R^\nabla(X, Y)Z + R^\nabla(Y, Z)X + R^\nabla(Z, X)Y = 0.$$ 

But, when we have an element $\alpha \in \Lambda^2 V^* \otimes V^*$, then the combination $1/3(\alpha(X, Y)Z + \alpha(Y, Z)X + \alpha(Z, X)Y)$ is alternating and is precisely its projection into $\Lambda^3 V^*$. Thus curvatures of torsion zero connections lie in the subspace $B$ which we call the Bianchi tensors.

On the space $B$ the Ricci contraction produces an element $\tau^\nabla$ in $V^* \otimes V^* = \Lambda^2 V^* \oplus S^2 V^*$. We want to go in the reverse direction and build a Bianchi tensor from an element of $V^* \otimes V^*$. For this, we use the bracket of section 2.2 view $Q \in V^* \otimes V^*$ as a $V^*$-valued 1-form and the identity map Id as a $V$-valued 1-form then $[Q \wedge \text{Id}] \in \Lambda^2 V^* \otimes \text{End}(V)$ defined by

$$[Q \wedge \text{Id}](X, Y) = [Q(X), Y] - [Q(Y), X]$$

takes values in $B$. Moreover, the Ricci contraction of $[Q \wedge \text{Id}]$ is

$$-(n + 1)Q_- - (n - 1)Q_+$$

where $Q_+, Q_-$ are the symmetric and skew parts of $Q$. Thus the decomposition of $B$ into irreducibles reads

$$B = W \oplus [S^2 V^* \wedge \text{Id}] \oplus [\Lambda^2 V^* \wedge \text{Id}],$$
with $\mathcal{W}$ the kernel of the Ricci contraction, and the corresponding decomposition of a curvature tensor $R^\nabla$ is

$$R^\nabla = W^\nabla - \frac{1}{n-1}[r_+^\nabla \wedge \text{Id}] - \frac{1}{n+1}[r_-^\nabla \wedge \text{Id}].$$

By analogy with the Riemannian case, we call $W^\nabla$ so defined, the \textit{Weyl component} of the curvature.

### 3.4 Projective Invariance of the Weyl Tensor

The projective class of a torsion free connection $\nabla$ consists of all (necessarily) torsion-free connections of the form

$$\nabla^\alpha = \nabla - [\alpha, \text{Id}],$$

for a 1-form $\alpha$. Thus

$$\nabla^\alpha_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X.$$ 

A standard computation gives

$$R^{\nabla^\alpha} = R^\nabla - d^\nabla[\alpha, \text{Id}] + \frac{1}{2}[[\alpha, \text{Id}] \wedge [\alpha, \text{Id}]]$$

$$= R^\nabla - [d^\nabla \alpha \wedge \text{Id}] + \frac{1}{2}[[\alpha, \text{Id}] \wedge [\alpha, \text{Id}]]$$

since $d^\nabla \text{Id} = T^\nabla = 0$. As for the zero-order term, since $TM$ is abelian, $[\text{Id} \wedge \text{Id}] = 0$ and the Jacobi identity (for the superalgebra of Lie algebra valued forms) then gives

$$[\text{Id} \wedge [\alpha \wedge \text{Id}]] = [[\text{Id} \wedge \alpha] \wedge \text{Id}]$$

$$= -[\text{Id} \wedge [\alpha \wedge \text{Id}]]$$

so that $[\text{Id} \wedge [\alpha \wedge \text{Id}]] = 0$. Bracketing this last with $\alpha$ gives

$$0 = [\alpha, [\text{Id} \wedge [\alpha \wedge \text{Id}]]] = [[\alpha, \text{Id}] \wedge [\alpha, \text{Id}]] + [\text{Id} \wedge [\alpha, [\alpha \wedge \text{Id}]]]$$

and we conclude that

$$R^{\nabla^\alpha} = R^\nabla - [(d^\nabla \alpha + \frac{1}{2}[\alpha, [\alpha \wedge \text{Id}]]) \wedge \text{Id}].$$

Thus $R^{\nabla^\alpha} - R^\nabla$ lies entirely in $[T^* M \otimes T^* M \wedge \text{Id}]$ and in particular

$$W^{\nabla^\alpha} = W^\nabla.$$

Thus the Weyl curvature is a projective invariant.
4 Application to Twistor Theory

Additional details on the structure of twistor spaces needed to prove the results in this section can be found in \[9, 10\].

If \(M\) is a manifold we denote by \(J(M)\) the bundle over \(M\) whose fibre at \(x \in M\) consists of all endomorphisms \(j\) of the tangent space at \(x\) with \(j^2 = -1\), and we let \(\pi: J(M) \to M\) be the bundle projection. For this to make sense, the dimension \(n\) of \(M\) must be even, say \(n = 2m\). The differential \(d\pi\) of \(\pi\) is a surjective bundle morphism from \(TJ(M)\) to the pull-back \(E\) of \(TM\) to \(J(M)\). The kernel of \(d\pi\) is the vertical tangent bundle \(\mathcal{V}\). At \(j \in J(M)\) the vertical space \(\mathcal{V}_j\) can be identified with endomorphisms of \(E_j\) which anticommute with \(j\). \(\text{End}(E)\) has a canonical section \(\mathcal{J}\) given by \(\mathcal{J}_j = j\).

If \(\nabla\) is a connection in \(TM\) it induces a pull-back connection \(\pi^*\nabla\) in \(E\) and the covariant differential \(\pi^*\nabla\mathcal{J}\) is an \(\text{End}(E)\)-valued 1-form on \(J(M)\) whose values anticommute with \(\mathcal{J}\). In fact, if we identify \(\mathcal{V}\) with such endomorphisms, then \(\pi^*\nabla\mathcal{J}: TJ(M) \to \mathcal{V}\) is surjective, so the kernel is a subbundle \(\mathcal{H}\) of \(TJ(M)\) which is mapped isomorphically onto \(d\pi\). \(\mathcal{J}\) gives \(E\) a complex structure and left multiplication by \(\mathcal{J}\) gives \(\mathcal{V}\) a complex structure. There is then a unique almost complex structure \(\nabla\mathcal{J}\) on \(J(M)\) such that

\[d\pi(J^\nabla X) = \mathcal{J}d\pi(X)\quad \text{and} \quad \pi^*\nabla_{J^\nabla X} \mathcal{J} = \mathcal{J}\pi^*\nabla_X \mathcal{J}.
\]

The \((1, 0)\) tangent spaces of \(J^\nabla\) on \(J(M)\) are spanned by vectors of the form \((J^\nabla + i)X\) and the \((0, 1)\) tangent spaces by \((J^\nabla - i)X\). Thus the \((1, 0)\) forms are spanned by components of \(d\pi \circ (J^\nabla + i) = (\mathcal{J} + i)d\pi X\) and \(X \mapsto \pi^*\nabla_{(J^\nabla + i)X} \mathcal{J} = (\mathcal{J} + i)\pi^*\nabla_X \mathcal{J}\).

If we change the connection from \(\nabla\) to \(\nabla' = \nabla + A\) then \(\pi^*\nabla' = \pi^*\nabla + \pi^*A\). Thus

\[\pi^*\nabla' \mathcal{J} = \pi^*\nabla \mathcal{J} + [\pi^*A, \mathcal{J}]
\]

and

\[(\mathcal{J} + i)\pi^*\nabla' \mathcal{J} = (\mathcal{J} + i)\pi^*\nabla \mathcal{J} + (\mathcal{J} + i)[\pi^*A, \mathcal{J}].\]

\(\nabla'\) will define the same almost complex structure as \(\nabla\) provided \((\mathcal{J} + i)[\pi^*A, \mathcal{J}]\) is a \((1, 0)\)-form. This will be the case if and only if

\[(\mathcal{J} + i)[\pi^*A((J^\nabla - i)X), \mathcal{J}] = (\mathcal{J} + i)A((\mathcal{J} - i)d\pi X)((\mathcal{J} - i) = 0.
\]

As an endomorphism-valued 1-form, \((j + i)A_x((j - i)X)(j - i)\) is the projection of \(A_x\) into the \(3i\) eigenspace of \(j\) on \(T^*_x M \otimes T^*_x M \otimes T_x M\).
Proposition 4.1 Two connections $\nabla$ and $\nabla' = \nabla + A$ define the same almost complex structure on $J(M)$ if and only if no irreducible component of $A$ has values in an irreducible subspace of $V^* \otimes V^* \otimes V$ where $j_0$ has a $3i$ eigenvalue.

Proof Each $j$ is obtained from one fixed $j_0$ by conjugation by an element $g$ of $GL(V)$. So a statement about all $j$ is equivalent to a statement about a single $j_0$ provided we apply it to whole irreducible components with respect to $GL(V)$ at a time. The result now follows from the preceding calculation.

Theorem 4.2 The almost complex structure $J^\nabla$ on $J(M)$ defined by a connection $\nabla$ on $M$ only depends on the projective class of $\nabla$. If $\nabla$ and $\nabla'$ both have zero torsion then $J^\nabla = J^{\nabla'}$ if and only if $\nabla$ and $\nabla'$ are projectively equivalent.

Proof If $\nabla$ and $\nabla'$ are projectively equivalent then $A \in [T^*M \wedge \text{Id}]$ and so has values in the irreducible components where only $\pm i$ eigenvalues occur, so $J^\nabla = J^{\nabla'}$ by Proposition 4.1.

What remains is to show that for torsion zero connections $J^\nabla = J^{\nabla'}$ implies that $\nabla$ and $\nabla'$ are projectively equivalent. But when $\nabla$ and $\nabla' = \nabla + A$ both have zero torsion, then $A$ has values in the bundle associated to $S^2V^* \otimes V$. The highest weights of irreducible components will be the dominant elements of the set $2\omega_{n-1} + \{\omega_1, \omega_2 - \omega_1, \omega_3 - \omega_2, \ldots, \omega_{n-1} - \omega_{n-2}, -\omega_{n-1}\}$ and these are $\{\omega_{n-1}, 2\omega_{n-1} + \omega_1\}$. The first is $V^*$ which has no $3i$ eigenvalue and the second is $V(1, 0, \ldots, 0, 2)$ which therefore does have a $3i$ eigenvalue. It follows from Proposition 4.1 that $J^\nabla = J^{\nabla'}$ implies that $A$ has values in the irreducible component of $S^2V^* \otimes V$ corresponding to $V^*$. This is embedded via $\alpha \in V^* \mapsto \alpha(X)Y + \alpha(Y)X$. Hence $\nabla$ and $\nabla'$ are projectively equivalent.

When is the almost complex structure $J^\nabla$ integrable? This question was considered in various cases in [9]. The condition discovered there is the analogue for the Nijenhuis tensor of $J^\nabla$ of Proposition 4.1 and tells us that the torsion and curvature of $\nabla$ should have irreducible components only in subrepresentations of $\Lambda^2V^* \otimes V$ and $\Lambda^2V^* \otimes \mathfrak{gl}(V)$ where $j_0$, viewed as an element of the Lie algebra $\mathfrak{gl}(V)$ acting in these representations has no $3i$ or $4i$ eigenvalue, respectively. Let us look at the torsion condition first, as it leads to a simplification of the curvature case.

Theorem 4.3 If $J^\nabla$ is integrable then there is another connection $\nabla'$ defining the same almost complex structure on $J(M)$ and with zero torsion.

Proof The eigenvalues of $j_0$ are imaginary and so we really need to work with the
complexification of $V$, but to keep the notation simple we shall still refer to this as $V$. The eigenvalues of $j_0$ on $V$ and $V^*$ are $\pm i$, and on a $k$-fold tensor product of these we get eigenvalues which are a sum of $k$ of these. Thus possible eigenvalues on torsion tensors are $\pm 3i, \pm i$, and the $3i$ does occur on the whole space as it is the value of the highest weight on $j_0$. It cannot occur on $T_2$ since this is isomorphic to $V^*$ and so the $3i$ eigenvalue occurs on $T_1$. By the result of [9] it follows that when $J^\nabla$ is integrable, $T^\nabla$ must lie in the $T_2$ subspace. If we set

$$\alpha(X) = \frac{1}{n-1} \sum_i e^i(T(X, e_i)),$$

in the notation of Section 3.1, then we have

$$T^\nabla(X, Y) = \alpha(X)Y - \alpha(Y)X.$$

Consider

$$\nabla'_X Y = \nabla_X Y - \frac{1}{2}(\alpha(X)Y - \alpha(Y)X).$$

$\nabla'$ clearly has torsion zero and differs from $\nabla$ by an endomorphism-valued 1-form with no component having eigenvalue $3i$. So by Proposition 4.1 $J^\nabla = J^{\nabla'}$.  

Theorem 4.3 allows us to assume, without loss of generality, that the connection $\nabla$ defining the almost complex structure has torsion zero, which we do from now on.

**Theorem 4.4** Let $\nabla$ be a torsion-free connection then $J^\nabla$ is integrable if and only if $W^\nabla = 0$.

**Proof** The curvature argument is similar to the torsion case with the appropriate changes of representation and eigenvalue. Integrability forces any irreducible component of the curvature to vanish if the $4i$ eigenvalue of $j_0$ occurs on the corresponding irreducible component of $B$. It is easy to see that (i) the $4i$ eigenvalue does occur on $B$, that (ii) the only eigenvalues on $V^* \otimes V^*$ are $0$ and $\pm 2i$ and hence that the $4i$ eigenvalue must actually occur on the space of Weyl tensors. Thus the integrability condition for $J^\nabla$ is that the Weyl component, $W^\nabla$, must vanish.

**Remark 4.5** When dim $M = 2$, any $W^\nabla = 0$ since then the space of curvature tensors is just $[T^*M \otimes T^*M \wedge \text{Id}]$. Thus, in this case, any $J^\nabla$ is integrable.
5 Projectively Flat Connections

Definition 5.1 Say that a connection $\nabla$ on a manifold $M$ is *projectively flat* or of *Ricci* type if $W\nabla = 0$.

We are going to show that when $\dim M = n \geq 3$, a projectively flat connection $\nabla$ induces a local diffeomorphism between $M$ and $\mathbb{R}P^n$ which intertwines the projective class of $\nabla$ with the projective class of the Levi-Civita connection on $\mathbb{R}P^n$.

We begin by describing the relevant geometry of $\mathbb{R}P^n$.

Denote by $V$ the trivial bundle $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ and let $\Lambda \to \mathbb{R}P^n$ be the tautological subbundle of $V$ whose fibre at $\ell \in \mathbb{R}P^n$ is $\ell \subset V$. The flat connection $d$ on $V$ induces a canonical isomorphism $\beta: T\mathbb{R}P^n \to \text{Hom}(\Lambda, V/\Lambda)$ such that, for $\sigma$ a section of $\Lambda$,

$$\beta(X)\sigma = d_X\sigma \mod \Lambda.$$  

There is a dual isomorphism (also called $\beta$) from $T^*\mathbb{R}P^n$ to $\text{Hom}(V/\Lambda, \Lambda)$ determined by

$$\beta(\alpha)\beta(X)\sigma = \alpha(X)\sigma$$

or, equivalently,

$$\beta(\alpha)d_X\sigma = \alpha(X)\sigma.$$  

The connections in the projective class we wish to consider are in bijective correspondence with complements to $\Lambda$ in $V$. Indeed, if $U$ is such a complement so that $V = \Lambda \oplus U$, then $d$ followed by projection along $\Lambda$ or $U$ gives connections $\nabla$ on $\Lambda$ and $U \cong V/\Lambda$ and so connections on $\text{Hom}(\Lambda, V/\Lambda)$ and so, via $\beta$, on $T\mathbb{R}P^n$.

To compute the curvature and torsion of such a $\nabla$, first note that $\beta$ induces an isomorphism $T\mathbb{R}P^n\Lambda \cong U$ via

$$X \otimes \sigma \mapsto \beta(X)\sigma \in V/\Lambda \cong U$$

so that we have a connection-preserving isomorphism $V \cong \Lambda \oplus T\mathbb{R}P^n\Lambda$ with respect to which the flat derivative decomposes as

$$d = \begin{pmatrix} \nabla & Q \\ \text{Id} & \nabla \end{pmatrix}$$

for some $Q$ a 1-form with values in $T^*\mathbb{R}P^n$. Explicitly,

$$d_X(\sigma, Y \otimes \tau) = (\nabla_X\sigma + Q(X, Y)\tau, \nabla_X(Y \otimes \tau) + X \otimes \sigma).$$
We compute the curvature of $d$:

$$0 = R^d = \begin{pmatrix} R^\nabla + [Q \wedge \text{Id}] & d^\nabla Q \\
\d^\nabla \text{Id} & R^\nabla + [Q \wedge \text{Id}] \end{pmatrix}.$$ 

In particular, $d^\nabla \text{Id} = T^\nabla = 0$ so that $\nabla$ is torsion-free. Further, $R^\nabla + [Q \wedge \text{Id}] = 0$ on both $\Lambda$ and $T\mathbb{R}P^n \Lambda$ and so on $T\mathbb{R}P^n$ also (we are in the situation of section 2.2 so there is no ambiguity in the definition of our brackets). In particular, $R^\nabla$ has no component in $\mathcal{W}$ so that $\nabla$ is projectively flat.

To see how $\nabla$ varies with $U$, we need an explicit formula for $\nabla$. For this, let $\pi : V \to \Lambda$ be the projection along $U$. Then

$$\beta(\nabla_X Y)\sigma = (\nabla_X \beta(Y))\sigma$$

$$= \nabla_X (\beta(Y)\sigma) - \beta(Y)\nabla_X \sigma$$

$$= d_X ((1 - \pi)d_Y \sigma) - d_Y (\pi d_X \sigma) \mod \Lambda$$

$$= d_X d_Y \sigma - d_X (\pi d_Y \sigma) - d_Y (\pi d_X \sigma) \mod \Lambda.$$

The projection along any other complement differs from $\pi$ by a section of $\text{Hom}(V/\Lambda, \Lambda)$ and so is of the form $\pi - \beta(\alpha)$ for some 1-form $\alpha$. If $\nabla^{\alpha}$ is the corresponding connection then we have

$$\beta(\nabla^{\alpha}_X Y - \nabla_X Y)\sigma = d_X (\beta(\alpha) d_Y \sigma) + d_Y (\beta(\alpha) d_X \sigma) \mod \Lambda$$

$$= \beta(X)(\alpha(Y)\sigma) + \beta(X)(\alpha(Y)\sigma)$$

$$= \beta([X, \alpha]Y)\sigma.$$

Thus $\nabla^{\alpha} = \nabla - [\alpha, \text{Id}]$.

We have therefore constructed a projective class of projectively flat torsion-free connections on $\mathbb{R}P^n$.

**Remark 5.2** Equip $V$ with a flat metric and take $U = \Lambda^\perp$. The metric induced on $\text{Hom}(\Lambda, U)$ gives, via $\beta$, an $\text{SO}(n + 1)$-invariant Riemannian structure on $\mathbb{R}P^n$ which is that for which $\mathbb{R}P^n$ is a Riemannian symmetric space. Moreover, the corresponding connection $\nabla$, being torsion-free and clearly metric, is the Levi-Civita connection for the symmetric metric.

Suppose now that $M$ is an $n$-manifold, $n \geq 2$, with torsion-free connection $\nabla$ and let $\Lambda$ be a bundle of $-n/(n + 1)$-densities. Then $\nabla$ induces a connection on $\Lambda$, also called $\nabla$, with curvature $F^{\nabla}$ given by

$$F^{\nabla} = -\frac{1}{n + 1} \text{Tr} R^{\nabla} = -\frac{1}{n + 1} s^{\nabla} = -\frac{2}{n + 1} r^{\nabla}.$$
Moreover, set
\[ Q^\nabla = \frac{1}{n-1}r^+ + \frac{1}{n+1}r^-, \]
so that, on \( TM \), \( R^\nabla + [Q^\nabla \wedge \text{Id}] = W^\nabla \).

Set \( V = \Lambda \oplus T\Lambda \Lambda \) and, as before, define a connection \( \mathcal{D} \) on \( V \) by
\[ \mathcal{D} = \begin{pmatrix} \nabla & Q^\nabla \\ \text{Id} & \nabla \end{pmatrix}. \]

Then
\[ R^\mathcal{D} = \begin{pmatrix} F^\nabla + [Q^\nabla \wedge \text{Id}] & d^\nabla Q^\nabla \\ d^\nabla \text{Id} & (R^\nabla + [Q^\nabla \wedge \text{Id}]) \otimes \text{Id} + \text{Id} \otimes (F^\nabla + [Q^\nabla \wedge \text{Id}]) \end{pmatrix}. \]

Once more, \( d^\nabla \text{Id} = T^\nabla = 0 \) while, for \( \sigma \) a section of \( \Lambda \),
\[ [Q^\nabla \wedge \text{Id}](X, Y)\sigma = Q^\nabla_X(Y \otimes \sigma) - Q^\nabla_Y(X \otimes \sigma) = (Q^\nabla(X, Y) - Q^\nabla(Y, X))\sigma \]
so that \( [Q^\nabla \wedge \text{Id}] = 2Q^\nabla = -F^\nabla \) on \( \Lambda \). Therefore,
\[ R^\mathcal{D} = \begin{pmatrix} 0 & d^\nabla Q^\nabla \\ 0 & W^\nabla \otimes \text{Id} \end{pmatrix}. \]

**Remark 5.3** Although it is not at first apparent, our construction of \((V, \mathcal{D})\) depends only on the projective class of \( \nabla \). From a more invariant viewpoint, \( V^* \) is the bundle \( J_1(\Lambda^*) \) of 1-jets of sections of \( \Lambda \) and \( \mathcal{D} \) is the normal Cartan connection thereon, c.f. [2, 3].

**Proposition 5.4** For \( n = \dim M \geq 3 \), \( \mathcal{D} \) is flat if and only if \( W^\nabla = 0 \)

**Proof** The only issue is to show that when \( W^\nabla = 0 \) then \( d^\nabla Q^\nabla \) vanishes also. Now the second Bianchi identity together with \( T^\nabla = 0 \) gives
\[ d^\nabla W^\nabla = d^\nabla [Q^\nabla \wedge \text{Id}] = [d^\nabla Q^\nabla \wedge \text{Id}]. \]

Now, for any 2-form \( \omega \) with values in \( T^*M \), one readily computes that
\[ \alpha^i([\omega \wedge \text{Id}](X, Y, X_i)Z) = -((n-2)\omega_+ + (n+1)\omega_-)(X, Y, Z) \]
where \( \omega_+ \) and \( \omega_- \) are, respectively, the components of \( \omega \) in \( \mathcal{B}_0 \) and \( \Lambda^3 T^*M \). Thus, when \( n \geq 3 \) and \( W^\nabla \) vanishes, both \( (d^\nabla Q^\nabla)_\pm \) vanish whence \( d^\nabla Q^\nabla = 0 \) and \( \mathcal{D} \) is flat. \[ \blacksquare \]
Remark 5.5 When $n = 2$, any connection is projectively flat so that $C^\nabla = d^\nabla Q^\nabla$ is the only obstruction to flatness of $\nabla$. In this case, $C^\nabla$ is an invariant of the projective class of $\nabla$: it is the projective covariant of Veblen–Thomas [17]. This is the projective analogue of the Cotton–York tensor of conformal geometry.

Suppose now that $\nabla$ is flat. On a simply connected open subset $\Omega$ of $M$, we can trivialise the pair $(V, \nabla)$: that is, there is a bundle isomorphism $\Phi : V|_\Omega \cong M \times \mathbb{R}^{n+1}$ such that $\Phi \circ \nabla = d \circ \Phi$. We now have a map $\phi : \Omega \to \mathbb{R}P^n$ given by $\phi(x) = \Phi^\Lambda_x \subset \mathbb{R}^{n+1}$ for which

$$\phi^{-1} \Lambda_{RP^n} = \Phi \Lambda.$$

Set $U = \Phi(TM\Lambda)$: a complement to $\Phi \Lambda$. Then we may view $\phi^* \beta$ as taking values in $\text{Hom}(\Phi \Lambda, U) \cong \phi^{-1} T\mathbb{R}P^n$. We have

$$d_X(\Phi \sigma) = \Phi(\nabla_X \sigma)$$

and taking the component in $U$ yields

$$\phi^* \beta(X) \Phi \sigma = \Phi(X \otimes \sigma).$$

We conclude that $\phi^* \beta$ is an isomorphism so that $\phi$ is a local diffeomorphism. Moreover, $\phi^* \beta$ and hence $d\phi$ intertwines $\nabla$ with the connection on $\mathbb{R}P^n$ induced by $U$.

To summarise:

Theorem 5.6 ([18]) Let $M$ be an $n$-dimensional manifold with torsion-free connection $\nabla$. Suppose that either $n \geq 3$ and $W^\nabla = 0$ or $n = 2$ and $C^\nabla = 0$. Then there are local affine diffeomorphisms between $M$ and $\mathbb{R}P^n$ equipped with a connection in the projective class of the symmetric connection.

In particular, in this case, the projective class of $\nabla$ contains a locally symmetric connection and the twistor space of $M$ is locally biholomorphic to that of $\mathbb{R}P^n$.

Acknowledgements

The first author thanks David Calderbank for helpful conversations on Cartan Geometry.

The second author thanks Roger Carter for explaining the use of Littelmann’s method [5, 8] for decomposing the tensor product of irreducible representations, and Christian Duval and the CPT, Marseille-Luminy for its hospitality during part of this work.

We thank Izu Vaisman for bringing his work on transversal twistor spaces of foliations [16] to our attention which has a substantial overlap with the results of Section 4. The
work for our present paper was completed in November 2000, but publication delayed
due to the pressures of other work.

We also thank Thomas Mettler for bringing the paper of Veblen–Thomas [17] to our
attention.

References

[1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional
Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), no. 1711, 425–
461. MR MR506229 (80d:53023)

[2] R. J. Baston, *Almost Hermitian symmetric manifolds. I. Local twistor theory*, Duke
Math. J. **63** (1991), no. 1, 81–112. MR MR1106939 (93d:53064)

[3] L. Bérard-Bergery and T. Ochiai, *On some generalizations of the construction of
twistor spaces*, Global Riemannian geometry (Durham, 1983), Ellis Horwood Ser.
Math. Appl., Horwood, Chichester, 1984, pp. 52–59. MR MR757205 (86h:53028)

[4] E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France **52**
(1924), 205–241. MR MR1504846

[5] R. W. Carter, *Representations of simple Lie algebras: modern variations on a clas-
sical theme*, Algebraic groups and their representations (Cambridge, 1997), NATO
Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 517, Kluwer Acad. Publ., Dordrecht,
1998, pp. 151–173. MR MR1670769 (2000b:17014)

[6] Michel Dubois-Violette, *Structures complexes au-dessus des variétés, applications,
Mathematics and physics (Paris, 1979/1982)*, Progr. Math., vol. 37, Birkhäuser
Boston, Boston, MA, 1983, pp. 1–42. MR MR728412 (85h:53053a)

[7] Luther Pfahler Eisenhart, *Riemannian geometry*, Princeton Landmarks in Mathe-
matics, Princeton University Press, Princeton, NJ, 1997, Eighth printing, Princeton
Paperbacks. MR MR1487892 (98h:53001)

[8] Peter Littelmann, *Paths and root operators in representation theory*, Ann. of Math.
(2) **142** (1995), no. 3, 499–525. MR MR1356780 (96m:17011)

[9] N. R. O’Brian and J. H. Rawnsley, *Twistor spaces*, Ann. Global Anal. Geom. **3**
(1985), no. 1, 29–58. MR MR812312 (87d:32054)
[10] John H. Rawnsley, *f*-structures, *f*-twistor spaces and harmonic maps, Geometry seminar “Luigi Bianchi” II—1984, Lecture Notes in Math., vol. 1164, Springer, Berlin, 1985, pp. 85–159. MR MR829229 (87h:58048)

[11] S. M. Salamon, Quaternionic structures and twistor spaces, Global Riemannian geometry (Durham, 1983), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1984, pp. 65–74. MR MR757207

[12] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 355–365. MR MR0256303 (41 #959)

[13] Franco Tricerri and Lieven Vanhecke, Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc. 267 (1981), no. 2, 365–397. MR MR626479 (82j:53071)

[14] Izu Vaisman, Symplectic curvature tensors, Monatsh. Math. 100 (1985), no. 4, 299–327. MR MR814206 (87d:53077)

[15] ———, Variations on the theme of twistor spaces, Balkan J. Geom. Appl. 3 (1998), no. 2, 135–156. MR MR1746886 (2001a:53076)

[16] ———, Transversal twistor spaces of foliations, Ann. Global Anal. Geom. 19 (2001), no. 3, 209–234. MR MR1828080 (2002d:53061)

[17] Oswald Veblen and Joseph Miller Thomas, Projective invariants of affine geometry of paths, Ann. of Math. (2) 27 (1926), no. 3, 279–296. MR MR1502733

[18] Hermann Weyl, Zur infinitesimalgeometrie: Einordnung der projektiven und der konformen auffassung, Göt. Nachr. (1921), 99–112.