Cyclotomic FFT of Length 2047 Based on a Novel 11-point Cyclic Convolution

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Abstract

In this manuscript, we propose a novel 11-point cyclic convolution algorithm based on alternate Fourier transform. With the proposed bilinear form, we construct a length-2047 cyclotomic FFT.

I. INTRODUCTION

Discrete Fourier transforms (DFTs) over finite fields have widespread applications in error correction coding [1]. For Reed-Solomon (RS) codes, all syndrome-based bounded distance decoding methods involve DFTs over finite fields [1]: syndrome computation and the Chien search are both evaluations of polynomials and hence can be viewed as DFTs; inverse DFTs are used to recover transmitted codewords in transform-domain decoders. Thus efficient DFT algorithms can be used to reduce the complexity of RS decoders. For example, using the prime-factor fast Fourier transform (FFT) in [2], Truong et al. proposed [3] an inverse-free transform-domain RS decoder with substantially lower complexity than time-domain decoders; FFT techniques are used to compute syndromes for time-domain decoders in [4].

Cyclotomic FFT was proposed recently in [5] and two variations were subsequently considered in [6], [7]. Compared with other FFT techniques [2], [8], CFFTs in [5]–[7] achieve significantly lower multiplicative complexities, which makes them very attractive. But their additive complexities (numbers of additions required) are very high if implemented directly. A common subexpression elimination (CSE) algorithm was proposed to significantly reduce the additive complexities of CFFTs in [9]. Along with those full CFFTs, reduced-complexity partial and dual partial CFFTs were used to design low complexity RS decoders in [10]. The lengths of CFFTs in [9] are only up to 1023 while longer CFFTs are required to decode long RS codes. To pursue a length-2047 CFFT, 11-point cyclic convolution over characteristic-2 fields is necessary, which is not readily available in the literature.

In this manuscript, we first propose a novel 11-point cyclic convolution for characteristic-2 fields in Section II. Based on this cyclic convolution, a length-2047 CFFT is presented in Section III. Using the same approach, CFFTs of any lengths that divide 2047 can also be constructed.
II. 11-POINT CYCLIC CONVOLUTION OVER CHARACTERISTIC-2 FIELDS

We first derive a fast cyclic convolution of 11 points over the real field. Denote the cyclic convolution of 11 point sequences \( x \) and \( y \) by the sequence \( z \). The sequence \( z \) may be computed by Fourier transforming \( x \) and \( y \), multiplying the transforms point-by-point and finally, inverse Fourier transforming the product sequence. Let \( X \), \( Y \) and \( Z \) denote the Fourier transforms of \( x \), \( y \) and \( z \) respectively. As defined by the Fourier transform,

\[
X_0 = \sum_{i=0}^{10} x_i \quad Y_0 = \sum_{i=0}^{10} y_i. \tag{1}
\]

We express the rest components, \( X' \) and \( Y' \) (over reals) using the basis \( \langle 1, W, W^2, \ldots, W^9 \rangle \) where \( W \) denotes the 11th primitive root of unity. This basis is sufficient because \( W^{11} - 1 = 0 \) yields \( W^{10} = -1 - W - W^2 - \cdots - W^9 \). Thus, \( X' = \sum_{i=0}^{9} X'_i W^i \) and \( Y' = \sum_{i=0}^{9} Y'_i W^i \), in which

\[
X'_i = (x_i - x_{10}) \quad Y'_i = (y_i - y_{10}). \tag{2}
\]

We will call the vector \( (X_0, X'_0, X'_1, \ldots, X'_9) \) as the Alternate Fourier transform (AFT) of sequence \( x \). Note that AFT is simply the DFT components \( X_0 \) and \( X' \) in their special bases. From (1) and (2) it is obvious that the AFT computation may be described as a multiplication with a \( 11 \times 11 \) matrix \( B \) with structure

\[
B = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
& & & -1 & \\
& I_{10} & & -1 & \\
& & \vdots & & \\
& & & \vdots & \\
& & & & -1
\end{bmatrix}
\]

where \( I_{10} \) is a \( 10 \times 10 \) identity matrix. Alternately, given the AFT of \( x \), one can determine \( x \) by using matrix \( B^{-1} \) given by

\[
B^{-1} = \frac{1}{11} \begin{bmatrix}
1 & A_1 \\
A_2 & A_3
\end{bmatrix}
\]

where length-10 row \( A_1 = (10, -1, -1, \ldots, -1) \), length-10 column \( A_2 = (1, 1, \ldots, 1)^T \) and \( 10 \times 10 \) submatrix \( A_3 \) has 10 on the first upper diagonal and -1 everywhere else.

Now consider the product of \( B^{-1} \) and an AFT vector:

\[
B^{-1} \begin{bmatrix} U_0 \\ U' \end{bmatrix} = \frac{1}{11} \begin{bmatrix}
1 & A_1 \\
A_2 & A_3
\end{bmatrix} \begin{bmatrix} U_0 \\ U' \end{bmatrix} = \begin{bmatrix} V_0 \\ V' \end{bmatrix}
\]

\(^1\)In this manuscript, vectors and matrices are represented by boldface letters, and scalars by normal letters.
where $U_0$, $U'$, and $V_0$, $V'$ are appropriate partitions of the AFT and the signal vectors. Values of $V_0$ and $V'$ can be computed as $V_0 = (1/11)U_0 + (1/11)A_1U'$ and $V' = (1/11)A_2U_0 + (1/11)A_3U'$. Note that $A_1$ and $A_3$ are related as $A_1 = -(1,1\ldots,1)A_3$. This implies that the sum of the components of $(1/11)A_3U'$ gives $-(1/11)A_1U'$. Furthermore, $A_2$ contains only 1’s. Thus the computation of $V_0$ and $V'$ reduces to

$$V_0 = (1/11)U_0 - (1/11)\sum (A_3U')$$

$$V' = (1/11)[U_0,U_0,\ldots,U_0]^T + (1/11)A_3U'.$$

Relation (4) shows that the inverse of an AFT only needs an evaluation of $(1/11)A_3U'$.

To compute cyclic convolution of $x$ and $y$, one should multiply the Fourier transforms of $x$ and $y$ and then take the inverse Fourier transform of the product. We use AFT instead of classical Fourier transform. Multiplying $X_0$ and $Y_0$ is simple, but since $X'$ and $Y'$ are expressed in a basis with 10 elements, their product may be difficult. Similarly inverse AFT requires multiplication by matrix $A_3$ which may be complicated. However, we now show that both these two difficult computation stages are equivalent to only a Toeplitz product (i.e., product of a Toeplitz matrix and a vector) [11].

The pointwise multiplication results are $Z_0 = X_0Y_0$ and $Z'$ defined as

$$\left(\sum_{i=0}^{9} X'_iW^i\right)\left(\sum_{i=0}^{9} Y'_iW^i\right) = \sum_{i=0}^{9} Z'_iW^i.$$  \hspace{1cm} (5)

Vector $Z'$ can be computed through the matrix product $(Z'_0, Z'_1, \ldots, Z'_9)^T = M(X'_0, X'_1, \ldots, X'_9)^T$ where the elements of matrix $M$ are

$$M_{k,j} = Y'_{k-j} + Y'_{k-j+11} - Y'_{10-j}.$$  \hspace{1cm} (6)

Note that in (6), $Y'_i$ are considered as zero outside its valid range, i.e., $Y'_i = 0$ if $i < 0$ or $i > 9$. The terms in (6) are easy to deduce from (5). Matrix element $M_{k,j}$ sums up those terms in $Y'$ that after multiplication with $X'_jW^j$ result in $W^k$ terms. For example, since product of $X'_jW^j$ and $Y'_{k-j}W^{k-j}$ results in $X'_jY'_{k-j}W^k$, we get the first term in (6) as given. Second term of (6) can be similarly argued. The third term is due to the product $(X'_jW^j)(Y'_{10-j}W^{10-j}) = X'_jY'_{10-j}W^{10} = -X'_jY'_{10-j}\sum_{i=0}^{9} W^i$.

Computing inverse DFT of $Z$ requires one to multiply $A_3$ and vector $(Z'_0, Z'_1, \ldots, Z'_9)^T$ where $A_3$ is the matrix defined in (3). Thus one has to compute $R(X_1(0), X_1(1), \ldots, X_9)^T$ where the 10 x 10 matrix $R = (1/11)A_3M$. 

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We now show by direct computation that \( R \) is a Toeplitz matrix. From the structure of \( A_3 \), we have

\[
R_{i,j} = \frac{1}{11} \sum_{k=0, k \neq i+1}^{9} -M_{k,j} + \frac{10}{11} M_{i+1,j}
\]

(7)

\[
= -\frac{1}{11} \sum_{k=0}^{9} M_{k,j} + M_{i+1,j}.
\]

From (6), using the appropriate ranges for the three terms we now get

\[
\sum_{k=0}^{9} M_{k,j} = -\sum_{k=0}^{9} Y'_{10-j} + \sum_{k=0}^{j-2} Y'_{k-j+p} + \sum_{k=j}^{9} Y'_{k-j}
\]

\[
= -10Y'_{10-j} + \sum_{s=11-j}^{9} Y'_s + \sum_{s=0}^{9-j} Y'_s
\]

(8)

\[
= \sum_{s=0}^{9} Y'_s - 11Y'_{10-j}
\]

Finally, combining (6), (7) and (8) gives

\[
R_{i,j} = Y'_{i-j+1} + Y'_{i-j+12} - \frac{1}{11} \sum_{s=0}^{9} Y'_s.
\]

(9)

Since \( R_{i,j} \) is a function of only \( i - j \), \( R \) is a Toeplitz matrix. Thus \( Z' = RX' \) is computed as

\[
\begin{bmatrix}
Z'_0 \\
Z'_1 \\
\vdots \\
Z'_9
\end{bmatrix} =
\begin{bmatrix}
Y'_{1} & Y'_{0} & 0 & \ldots & Y'_{3} \\
Y'_{2} & Y'_{1} & Y'_{0} & \ldots & Y'_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & Y'_{9} & Y'_{8} & \ldots & Y'_{1}
\end{bmatrix}
\begin{bmatrix}
X'_{0} \\
X'_{1} \\
\vdots \\
X'_{9}
\end{bmatrix} +
\begin{bmatrix}
\sum_{i=0}^{9} X'_i \sum_{i=0}^{9} Y'_i
\end{bmatrix}.
\]

Recall that \( Y'_i \) is assumed zero if its index is outside the valid range from 0 to 9. Thus in (9), exactly one of the first two terms is valid for any combination of \( i \) and \( j \). Fig. 1 illustrates the bilinear cyclic convolution algorithm of length 11 based on this discussion.

The multiplication of the 10 \times 10 Toeplitz matrix \( R \) with a vector can be obtained using the Toeplitz product algorithms of lengths 2 and 5. The matrix \( R \) can be split into four 5 \times 5 submatrices and the vector \( X' \) can be split into two length-5 vectors. By the definition of Toeplitz matrices, the Toeplitz product \( RX' \) can be computed as

\[
RX' = \begin{bmatrix}
R_0 & R_1 \\
R_2 & R_0
\end{bmatrix}
\begin{bmatrix}
X'_0 \\
X'_1
\end{bmatrix} =
\begin{bmatrix}
R_0(X'_0 + X'_1) + (R_1 - R_0)X'_1 \\
R_0(X'_0 + X'_1) + (R_2 - R_0)X'_1
\end{bmatrix}.
\]

Although the cyclic convolution is derived over the real field, it can be easily converted to characteristic-2 fields. Based on the method in [12], we multiply both sides of all equations above by 11 modulus 2. In
the converted form, $X = T x$, $Y = T y$, and $z = S Z$. Thus we obtain 11-point cyclic convolution over characteristic-2 fields. To find its bilinear form, we need the bilinear form of Toeplitz product of length 10. The bilinear form of length-5 Toeplitz product over characteristic-2 fields $v = Q^{(T5)} (R^{(T5)} r \cdot P^{(T5)} u)$ is given in Appendix II where $\cdot$ stands for pointwise multiplication.

Based on the length-10 Toeplitz product, the bilinear form of 11-point cyclic convolution over $GF(2^m)$ is given by

$$z = Q^{(11)} (R^{(11)} y \cdot P^{(11)} x) = S \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Q^{(T5)} & 0 & 0 \\ 0 & 0 & Q^{(T5)} & 0 \\ 0 & R^{(T5)} \Pi_0 & R^{(T5)} \Pi_1 & R^{(T5)} \Pi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R^{(T5)} \Pi_0 \\ 0 & R^{(T5)} \Pi_1 \\ 0 & R^{(T5)} \Pi_2 \end{bmatrix} T y \cdot \begin{bmatrix} 1 & 0 \\ 0 & P^{(T5)} \Pi_3 \\ 0 & P^{(T5)} \Pi_4 \\ 0 & P^{(T5)} \Pi_5 \end{bmatrix} T x.$$ 

Details of matrices $S, T, \Pi_0, \ldots, \Pi_5$ are given in Appendix II.

The proposed length-11 cyclic convolution needs only 43 multiplications. We compare it with cyclic convolutions of other lengths from [1], [13], [14] in Table II.

### III. CYCLOTOMIC FFT OVER GF(2^{11})

Based on the derived 11-point cyclic convolution over $GF(2^m)$, we can construct a length-2047 cyclotomic FFT over $GF(2^{11})$. In this manuscript, we focus on direct CFFT as in [5] since it was
shown in [9] all variants of CFFTs have the same multiplicative complexity and they have the same additive complexity under direct implementation.

Given a primitive element \( \alpha \in \text{GF}(2^m) \), the DFT of a vector \( f = (f_0, f_1, \ldots, f_{n-1})^T \) is defined as

\[ F \triangleq (f(\alpha^0), f(\alpha^1), \ldots, f(\alpha^{n-1}))^T, \]

where \( f(x) \triangleq \sum_{i=0}^{n-1} f_i x^i \in \text{GF}(2^m)[x] \).

We choose the field generated by the polynomial \( x^{11} + x^2 + 1 \). In this field, there are one size-1 coset and 186 size-11 cosets. We permute the input \( f \) to \( f' \) such that \( f' = (f_0, f_1, f_2, \ldots, f_{186}) \) and each size-11 vector \( f_i \) contains the components of \( f \) whose indices are in the same coset \((k_i, k_i^2, \ldots, k_i 2^{m_i-1}) \mod 2047 \) where \( m_i \mid 11 \) is the coset size. Thus the polynomial \( f(x) \) is divided into parts, each one is \( L_i(x^{k_i}) = \sum_{j=0}^{m_i-1} f_{k_i 2^j} x^{k_i 2^j} \). Hence \( L_i(x) \)’s are linearized polynomials. Each element \( \alpha^{k_i} \) can be decomposed with respect to a basis \( \beta_i = (\beta_{i,0}, \beta_{i,1}, \ldots, \beta_{i,m_i-1}) \) such that \( \alpha^{k_i} = \sum_{s=0}^{10} a_{i,j,s} \beta_{i,s}, a_{i,j,s} \in \text{GF}(2) \). So each component of DFT is factored into

\[ f(\alpha^j) = \sum_{i=0}^{186} L_i(\alpha^{j k_i}) = \sum_{i=0}^{186} \sum_{s=0}^{m_i} a_{i,j,s} L_i(\beta_{i,s}) = \sum_{i=0}^{186} \sum_{s=0}^{10} a_{i,j,s} \left( \sum_{p=0}^{10} \beta_{i,s}^{2^p} f_{k_i 2^p} \right). \]

In matrix form, it is \( F = aL f \), in which \( L \) is a block diagonal matrix with each diagonal block being

\[
L_i = \begin{bmatrix}
\beta_{i,0} & \beta_{i,0}^2 & \ldots & \beta_{i,0}^{2^{m_i-1}} \\
\beta_{i,1} & \beta_{i,1}^2 & \ldots & \beta_{i,1}^{2^{m_i-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{i,m_i-1} & \beta_{i,m_i-1}^2 & \ldots & \beta_{i,m_i-1}^{2^{m_i-1}}
\end{bmatrix}
\]

Using a normal basis as \( \beta_i \), the matrix \( L_i \) becomes a cyclic matrix and \( L_i f_i \) becomes a size-\( m_i \) cyclic convolution. For length-2047 CFFT, \( m_i \) is 1 or 11. Thus we obtain a length-2047 CFFT using the bilinear form of 11-point cyclic convolution as

\[
F = a \begin{bmatrix}
1 & Q^{(11)} & \cdots & Q^{(11)} \\
Q^{(11)} & 1 & \cdots & \cdots \\
\cdots & \cdots & 1 & R^{(11)} \\
\cdots & \cdots & R^{(11)} & 1
\end{bmatrix}
\begin{bmatrix}
1 & \cdots & \cdots & \cdots \\
1 & \beta_1 & \cdots & \cdots \\
1 & \cdots & \beta_{186} & \cdots \\
1 & \cdots & \cdots & \beta_{186}
\end{bmatrix}
\begin{bmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{186}
\end{bmatrix}.
\]
It requires 7812 multiplications to compute the constructed length-2047 CFFT. Under direct implementation, it requires 2154428 additions. With incomplete optimization using the CSE algorithm [9], its additive complexity can be reduced to 529720. We compare its complexity with those of shorter CFFTs in Table II. In Table II our numbers of additions for CFFTs of lengths 7, 15, · · · , 1023 are reproduced from [9].

### Table II

**Complexity of Full Cyclotomic FFT**

| n  | Mult. | Additions |
|----|-------|-----------|
|    |       | Ours | [5] | Direct |
| 7  | 6     | 24   | 25  | 34     |
| 15 | 16    | 74   | 77  | 154    |
| 31 | 54    | 299  | 315 | 570    |
| 63 | 97    | 759  | 805 | 2527   |
| 127| 216   | 2576 | 2780| 9684   |
| 255| 586   | 6736 | 7919| 37279  |
| 511| 1014  | 23130| 26643| 141710 |
| 1023| 2827 | 75360| -   | 536093 |
| 2047| 7812 | 529720| -   | 2154428 |

### Appendix I

**Toeplitz Product of Length 5**

Toeplitz product of length 5 as

\[
\begin{bmatrix}
  v_0 \\
  v_1 \\
  v_2 \\
  v_3 \\
  v_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
  r_4 & r_5 & r_6 & r_7 & r_8 \\
  r_3 & r_4 & r_5 & r_6 & r_7 \\
  r_2 & r_3 & r_4 & r_5 & r_6 \\
  r_1 & r_2 & r_3 & r_4 & r_5 \\
  r_0 & r_1 & r_2 & r_3 & r_4 \\
\end{bmatrix}
\begin{bmatrix}
  u_0 \\
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
\end{bmatrix}
\]

can be done in bilinear form as \( v = Q^{(T5)} (R^{(T5)} r \cdot P^{(T5)} u) \).
\[
R^{(T5)} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
P^{(T5)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1
\end{bmatrix}
\]
\[
Q^{(T5)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

**APPENDIX II**

**BILINEAR FORM OF 11-POINT CYCLIC CONVOLUTION OVER CHARACTERISTIC-2 FIELDS**

\[
z = S \begin{bmatrix}
1 \\
Q^{(T5)} & Q^{(T5)} & 0 \\
Q^{(T5)} & 0 & Q^{(T5)}
\end{bmatrix}
\begin{bmatrix}
1 \\
R^{(T5)} \Pi_0 \\
R^{(T5)} \Pi_1 \\
R^{(T5)} \Pi_2
\end{bmatrix}
T_y
\begin{bmatrix}
1 \\
P^{(T5)} \Pi_3 \\
P^{(T5)} \Pi_4 \\
P^{(T5)} \Pi_5
\end{bmatrix}
T_x
\]
\[ T = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

\[ \Pi_0 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix} \]

\[ \Pi_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \]

\[ \Pi_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

\[ \Pi_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]
$$\Pi_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

$$\Pi_5 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

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