Vortex-Model for the Inverse Cascade of 2D-Turbulence

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(Dated: November 28, 2011)

We generalize Kirchhoff’s point vortex model of two-dimensional fluid motion to a rotor model which exhibits an inverse cascade by the formation of rotor clusters. A rotor is composed of two vortices with like-signed circulations glued together by a nonelastic bond. The model is motivated by a treatment of the vorticity equation representing the vorticity field as a superposition of vortices with elliptic Gaussian shapes of variable widths, augmented by a suitable forcing mechanism. The rotor model opens up the way to discuss the energy transport in the inverse cascade on the basis of dynamical systems theory.

PACS numbers: 05.40.Fb, 05.10.Gg, 52.65.Ff

The theoretical treatment of the longstanding problem of turbulent flows [1]-[3] has to relate dynamical systems theory with nonequilibrium statistical physics [5]. The central notion of physical turbulence theory is the concept of the energy cascade, highlighting the fact that turbulent flows are essentially transport processes of quantities like energy or enstrophy in scale. Although well-established theories due to Richardson, Kolmogorov, Onsager, Heisenberg and others (for reviews we refer the reader to [1]-[3]) can capture gross features of the cascade process in a phenomenological way, the dynamical aspects are by far less understood, and usually are investigated by direct numerical simulations of the basic fluid dynamic equations. An exception, in some sense, are inviscid fluid flows in two dimensions. Based on the work of Helmholtz [6] Kirchhoff [7] pointed out that the partial differential equation can be reduced to a Hamiltonian system for the locations of point vortices, provided one considers initial conditions where the vorticity is a superposition of delta-distributions (we refer the reader to the works of Aref [8] as well as the monographs [9], [10]). Due to Onsager [11] (for a discussion we refer the reader to [12]), a statistical treatment of point vortex dynamics is possible for equilibrium situations because of the Hamiltonian character of the dynamics, provided the ergodic hypothesis holds. Extensions to nonequilibrium situations based on kinetic equations have been pursued by Joyce and Montgomery [13], Lundgren and Pointin [14], as well as more recently by Chavanis [15].

The purpose of the present Letter is to generalize Kirchhoff’s point vortex dynamics to a rotor model which exhibits an inverse cascade, i.e. the spontaneous formation of large scale velocity fields from an initially random distribution of vortices. Kraichnan [16] has emphasized that such an inverse cascade can exist in two-dimensional flows. The predicted scaling of the energy spectrum has been experimentally verified by Paret and Tabeling [17]. Recently, it has been pointed out by Chen et al. [18] that a vortex thinning mechanism is an important feature of the inverse cascade. A similar mechanism is contained in the rotor model presented in this Letter. We will explicitly demonstrate that the large scale fields are generated by a clustering of roters, induced by a change of the rotor size due to a large scale straining field, i.e. a kind of vortex thinning process. In a second part of the Letter we shall discuss how the rotor model is related to the vorticity equation of dissipative fluid dynamics with forcing.

The Model. Instead of single vortices located at position $\mathbf{R}_i$ with circulations $\Gamma_i$, we consider vortex couples, where each vortex of the couple $i$ has circulation $\Gamma_i$, located at the positions $\mathbf{x}_i$ and $\mathbf{y}_i$. In the following we denote these objects as roters. We assume that the positions $\mathbf{x}_i$ and $\mathbf{y}_i$ are determined by the evolution equations

\[
\begin{align*}
\dot{x}_i &= \frac{\gamma}{2}(D_0 - |x_i - y_i|)\mathbf{e}_i + \Gamma_i \mathbf{u}(x_i - y_i) \\
\dot{y}_i &= -\frac{\gamma}{2}(D_0 - |y_i - x_i|)\mathbf{e}_i + \Gamma_i \mathbf{u}(y_i - x_i) \\
\end{align*}
\]

We have defined the unit vector $\mathbf{e}_i = \frac{\mathbf{x}_i - \mathbf{y}_i}{|\mathbf{x}_i - \mathbf{y}_i|}$. The velocity field $\mathbf{u}(r)$ is the velocity field of a point vortex centered at the origin, $\mathbf{u}(r) = c_2 \times \frac{\mathbf{r}}{2\pi r^2}$. For roters moving inside a closed regime, the velocity field has to be changed based on the introduction of mirror vortices, as is described e.g. in [9], [10]. The size of a single rotor relaxes with relaxation time $2/\gamma$ to $D_0$.

In our rotor model, the vortex couple is glued together by an inelastic coupling. The roters generate a far field similar to the one of an elliptical vortex. Furthermore, due to the inelastic coupling it possesses an internal degree of freedom, which is sensitive with respect to a shear velocity field, as is easily seen from the multipole expansion of the evolution equation (1):

\[
\dot{r}_i = \gamma(D_0 - r_i)\frac{r_i}{r_i^3} + 2\Gamma_i \mathbf{u}(r_i) + \sum_j \Gamma_j \mathbf{r}_j \cdot \nabla \mathbf{u}(\mathbf{R}_{ij})
\]
Here, we have introduced the center \( \mathbf{R}_i = \frac{\sum_j r_j}{N} \) of a rotor, the distance vector \( \mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j \) between two rotors, as well as the distance vector \( \mathbf{r}_i = x_i - y_i \), which is a measure of the elliptical character of the velocity field of the rotor. We have included only the leading terms in the multipole expansion of the velocity field with respect to \( |\mathbf{r}_i|/|\mathbf{R}| \). As is seen from eq. (2) the elliptical deformation of the velocity field, i.e. the size and the direction of \( \mathbf{r}_i \), depends on the shear induced by the other rotors. The inelastic coupling between the vortices of the couple can be considered to model viscous damping as well as forcing.

**Numerical results** We have numerically solved the dynamical system (1). The temporal evolution of 200 rotors with an equal number of positive and negative circulations starting from a random initial condition exhibits an inverse cascade. Large scale vortical motion is generated by a clustering of rotors. A typical time series is exhibited in figs. 1, a,b, for the parameter values \((\gamma = 5, D_0 = .5, \Gamma = \pm 1, \tilde{R} = 20)\). The temporal evolution of a number of rotors with the same circulations is exhibited in figs. 2, a,b. Starting from random initial positions of the rotors a fluctuating lattice of rotor clusters appear.

The fact that the rotor vortex system exhibits a pronounced inverse cascade already for moderate numbers of rotors (the figures contain 100 rotors) allows us to investigate the inverse cascade using methods of nonlinear dynamics.

Furthermore, the rotor model incorporates the aspect of vortex thinning, due to a possible change of the ellipticity of the rotor in much the same way as identified in the experiments of Chen et al. [18]. Hence, it is a minimal dynamical model containing the mechanisms of the inverse cascade. In the following we shall discuss the origin of the formation of clusters of rotors with like-signed circulations.

**Interaction of two rotors.** It is straightforward to show that the center of vorticity \( \frac{(\mathbf{r}_1 \cdot \mathbf{R} + \mathbf{r}_2 \cdot \mathbf{R})}{|\mathbf{R}|^2} \) is a conserved quantity. The distance vector \( \mathbf{R} \) between the two rotors obeys the evolution equation (3), written explicitly

\[
\dot{\mathbf{R}} = 2\frac{(\Gamma_1 + \Gamma_2)}{2\pi} \mathbf{e}_z \times \frac{|\mathbf{R}|}{|\mathbf{R}|^2} - \frac{1}{8} \left[ \frac{2}{|\mathbf{R}|} |\mathbf{r}_1|^2 - \frac{R}{|\mathbf{R}|^2} \mathbf{r}_1 \cdot \mathbf{R} + \frac{R}{|\mathbf{R}|^6} (\mathbf{R} \cdot \mathbf{r}_1)^2 \right] + \frac{1}{8} \left[ \frac{2}{|\mathbf{R}|} |\mathbf{r}_2|^2 - \frac{R}{|\mathbf{R}|^2} \mathbf{r}_2 \cdot \mathbf{R} + \frac{R}{|\mathbf{R}|^6} (\mathbf{R} \cdot \mathbf{r}_2)^2 \right] \]

FIG. 1: Inverse cascade in the rotor model with \( N = 200 \) rotors: a) An initial stage starting from random initial conditions.

b) Evolved state showing the formation of rotor clusters.

FIG. 2: Inverse cascade in the rotor model (\( N = 200 \)) with like signed circulations: a) Initial stage starting from random initial conditions. b) Evolved state showing the formation of rotor clusters generating large scale flow.

\[
\dot{\mathbf{R}} = \frac{(\Gamma_1 + \Gamma_2)}{2\pi} \frac{1}{R^3} \left[ |\mathbf{r}_1|^2 h(\mathbf{e}_1, \mathbf{r}_1) + r_2^2 h(\mathbf{e}_2, \mathbf{r}_2) \right] (6)
\]

with the unit vectors \( \mathbf{e}_i = \frac{\mathbf{r}_i}{|\mathbf{r}_i|} \), \( \mathbf{e}_R = \frac{\mathbf{R}}{|\mathbf{R}|} \). We obtain the equation for the relative distance

\[
\dot{\mathbf{r}} = \frac{(\Gamma_1 + \Gamma_2)}{2\pi} \frac{1}{R^3} \left[ |\mathbf{r}_1|^2 h(\mathbf{e}_1, \mathbf{r}_1) + r_2^2 h(\mathbf{e}_2, \mathbf{r}_2) \right] (6)
\]

The evolution equation for the relative distance of a rotor reads

\[
\dot{\mathbf{r}}_1 = \frac{\gamma (D_0 - r_1)}{r_1} + \frac{\Gamma_1}{2\pi} \mathbf{e}_z \times \frac{\mathbf{r}_1}{r_1} + \frac{\Gamma_2}{2\pi} \mathbf{e}_z \times \left[ \frac{1}{|\mathbf{r}_1|^2} - \frac{2}{|\mathbf{R}|^4} (\mathbf{r}_1 \cdot \mathbf{R}) \right] (7)
\]

We have to determine the quantities \( r_1^2, r_2^2 \), which are determined by the evolution equations

\[
\dot{\mathbf{r}} = \frac{\gamma (D_0 - r_1)}{R^2} - 2\frac{\Gamma_2}{2\pi} \mathbf{r}_1 \mathbf{r}_2 \cdot h(\mathbf{e}_1, \mathbf{e}_R) \]

We can solve iteratively for small deviations of \( r_i \) from \( D_0 \):

\[
r_1 = D_0 - 2\frac{\Gamma_2}{2\pi} \frac{D_0}{R^2} \int_{-\infty}^{t} e^{-(t-t') \gamma} h(\mathbf{e}_1(t'), \mathbf{e}_R(t')) (9)
\]

A similar treatment applies to \( r_2 \). Assuming that the damping constant \( \gamma \) is large compared to the rotation
frequency of the rotor, adiabatic approximation of the integral yields to lowest order in \( \gamma^{-1} \)

\[
    r_1^2 = D_0^2 [1 - 4 \frac{\Gamma_2}{2 \pi \gamma R^2} h(e_1, e_R)]
\]  

(10)

and a similar equation for \( r_2^2 \). Here, the last terms on the right hand side arise due to the change of the size or the rotors, connected with a change of the far field, induced by the mutually generated shear. It mimics the mechanism of vortex thinning, identified in [18].

The relative motion of the rotors obeys the evolution equation

\[
    \dot{R} = \frac{(\Gamma_1 + \Gamma_2) D_0}{2 \pi} \frac{\gamma R^3}{R^3} \left[ \frac{\gamma R^2}{\Gamma_1} h(e_1, e_R) + h(e_2, e_R) \right] 
\]  

(11)

We now average the evolution equation with respect to the rotations of the vectors \( e_i(t), i = 1, 2 \), taking into account that the average \( \langle h(e_i, e_R) \rangle = 0 \) vanishes. Furthermore, the average \( \langle h(e_i, e_R)^2 \rangle = a \) is positive. As a consequence, the relative distance behaves according to

\[
    \dot{R} = -4 \frac{(\Gamma_1 + \Gamma_2)^2 D_0^2}{(2 \pi)^2} \frac{\gamma R^3}{R^3} a
\]  

(12)

Two rotors approach each other, except for \( \Gamma_1 = -\Gamma_2 \). It is important to stress that this attractive relative motion arises only if we include the irreversible effect of the strain induced stretching of the rotors.

Relation to vorticity equation. In the following we shall discuss the relationship of our point vortex model to the vorticity equation of fluid dynamics. Thereby, we replace the point vortices by vortices with elliptic Gaussian profiles, along similar lines as described by Melander, Styčzek, and Zabusky [19].

We consider the two dimensional vorticity equation in Fourier-space

\[
    \omega(k, t) - ik \cdot \int dk' u(k') \omega(k - k', t) \omega(k', t) = -\nu k^2 \omega(k, t)
\]  

(13)

with \( u(k) = \frac{1}{4\pi^2} [e_z \times \frac{k}{k}] \). In order to deal with vortex dynamics, we perform the ansatz

\[
    \omega(k, t) = \sum_j \Gamma_j e^{i k \cdot x_j(t) + W_j(k, t)}
\]  

(14)

For \( W_j(k, t) = 0 \) we obtain the field of point vortices

\[
    \omega(x, t) = \sum_j \Gamma_j \delta(x - x_j(t)).
\]

14 We now insert our ansatz into the vorticity equation and obtain

\[
    \sum_j \Gamma_j e^{i k \cdot x_j(t) + W_j(k, t)} \left[ i k \cdot x_j(t) + \dot{W}_j(k, t) + \nu k^2 \right] = i k \cdot \sum_{j,l} \Gamma_j \Gamma_l \int dk'' u(k'') e^{i k' - k'' \cdot x_j + i k'' \cdot x_l} e^{W_j(k'' + W_l(k'' + W_l(k', t))}
\]  

(15)

This equation contains sweeping of the vortices, encoded in the location of the centers \( x_j(t) \), as well as a change of the shapes of the vortices, contained in \( W_j(k, t) \).

A separation of these two processes becomes possible by defining sweeping via the terms in the evolution equation (13) which are proportional to \( k \): \[
    \dot{x}_j = \sum_l \Gamma_l U_{jl} (x_j - x_l)
\]  

(16)

We have defined the velocity fields

\[
    U_{jl}(r) = \int dk' u(k') e^{-ik' \cdot r + W_j(-k', t) + W_l(k', t)}
\]  

(17)

This is the extension of the set of evolution equations for the \( \delta \)-point vortices. Let us now consider the resulting equation for the shapes, which read

\[
    \dot{W}_j(k, t) = -\nu k^2 + ik \cdot \sum_l \Gamma_l \int dk'' u(k'') e^{-ik' \cdot (x_j - x_l)} e^{W_j(-k', t) + W_l(k', t)}
\]

(18)

\[
    \times \left[ \left( e^{W_j(-k', t) + W_l(k', t)} - 1 \right) \right]
\]

Here, the sum includes also the terms with \( i = l \). We now invoke the approximation

\[
    W_i(k, t) \approx -\frac{1}{2} k C_i(t) k
\]  

(19)

introducing the symmetric matrix \( C_i \). Approximating the last term of the right-hand side of eq. (18) by \( \frac{1}{2} [k C_i k' + k' C_i k] \) we arrive at

\[
    \dot{C}_i = \nu E + \Gamma_i [S_d C_i + C_i S_d^T]
\]

(20)

\[
    + \sum_{l \neq i} \Gamma_i [S_d (x_l - x_i) C_i + C_i S_d (x_l - x_i)^T]
\]

Here, we explicitly have introduced the matrix \( S_d = [\nabla U_{jl}(x_j - x_l)] \) and have singled out the term with \( l = i \).

In order to make contact with the previously introduced rotor model we determine the velocity field eq. (17) up to the first order in \( C_i + \dot{C}_i \) valid for widely separated vortices

\[
    U_{jl}(r) = \int dk' u(k') e^{-ik' \cdot r + \frac{1}{2} k' (C_j + C_l) k'}
\]

(21)

\[
    \approx e_z \times \frac{1}{2} \nabla r [C_j + C_l] \nabla r' \frac{r}{2 \pi |k|^2}
\]

The evolution equation for the vortex centers then read

\[
    \dot{x}_j = \sum_l \Gamma_l e_z \times \frac{x_l - x_i}{2 \pi |x_i - x_l|^2}
\]

(22)

\[
    + \sum_l \Gamma_l \nabla r [C_j + C_l] \nabla r' e_z \times \frac{x_l - x_i}{4 \pi |x_i - x_l|^2}
\]

(23)

This equation is identical to the equation (3) for the vortex centers of our point vortex model, provided we represent the matrix as the direct product \( C_i = r_i(t) r_i(t) \).
This corresponds to an infinitely thin elliptical vortex, oriented in \( r_i \)-direction. In this case, the evolution equation for \( r_i \), obtained by inserting \( C_i = r_i(t) \) into eq. (20), is identical to the model equation eq. (2) of the rotor model with \( \gamma = 0 \).

As we have seen the forcing term \( \gamma(D_0 - r_i) \) is crucial for the emergence of the inverse cascade in the rotor model. It is tempting to include forcing and damping terms for the emergence of the inverse cascade in the rotor model, where the centers \( C_i \) are oriented in the \( x \)-direction. This corresponds to an infinitely thin elliptical vortex, where the centers \( C_i \) are oriented in the \( x \)-direction.

\[
\begin{align*}
\dot{\Gamma}_i &= -a \Gamma_i + f_i \\
\dot{x}_i &= \sum_l \Gamma_i U_{il}(x_i - x_l) + U_i(t) \\
\dot{C}_i &= \nu E + \gamma(C_0 - C_i) + \Gamma_i [S_{il} C_l + C_i S_{il}] \\
&+ \sum_l \Gamma_i [S_{il}(x_i - x_l) C_l + C_i S_{il}(x_i - x_l)^T] 
\end{align*}
\]

Such type of forcing may be obtained from the vorticity equation by just adding a linear damping term, \(-a \omega(k,t)\) as well as the forcing term \( F(k,t) \),

\[
F(k,t) = \sum_j f_j e^{ik \cdot x_j(t)} + \sum_j e^{ik \cdot x_j(t)} W_j \\
\times \left[ f_j + i \Gamma_j \mathbf{k} \cdot U_j(t) - \frac{1}{2} \Gamma_j \mathbf{k} \gamma(C_0(t) - C_j(t)) \right]
\]

where the centers \( x_j = x_j(t) + \frac{\Gamma_i}{\Gamma_j} U_j(t) \) as well as the shapes \( W_j = W_j - \frac{1}{2} \Gamma_j \gamma(C_0(t) - C_j(t)) \mathbf{k} \) are close to the centers and the shapes of the elliptical vortices. The first contribution in eq. (23) leads to a modulation of the circulation, the second term describes a shift of the rotor center and the third one corresponds to a modification of the width of the Gaussian vortex shape counteracting the viscous broadening of the vortex.

**Conclusions** We have presented a generalized point vortex model, a rotor model, exhibiting an inverse cascade based on clustering of rotors. We have discussed how this rotor model can be derived from the vorticity equation by an expansion of the vorticity field into a set of elliptical vortices at locations \( x_i(t) \) and shapes \( C_i(t) \). An important point has been the inclusion of a forcing term, which prevents the elliptical far field of the rotors from diffusing away. The added forcing term breaks the symmetry \( \Gamma_i \to -\Gamma_i, t \to -t \). This symmetry breaking lies at the origin of cluster formation and the inverse cascade, as can be seen from the two-rotor interaction inducing in average a relative motion proportional to \( \frac{D_0^2}{R^2} \).

The presented rotor model can be investigated by applying methods from dynamical systems theory like the evaluation of finite time Ljapunov exponents and Ljapunov vectors. The model system may also be studied as a stochastic system by considering the velocity \( U_i(t) \) to be a white noise force. The corresponding Fokker-Planck equation allows one to draw analogies with quantum mechanical many body problems. Furthermore, we emphasize that a continuum version of the model equations leads to a subgrid model exhibiting analogies with the work of Eyink.

It will be a task for the future to investigate the cluster formation from a statistical point of view, based on the formulation of kinetic equations, along the lines as has been performed for fully developed turbulence and Rayleigh-Bénard convection. In this respect we hope to find a relation to the kinetic equation for the two-point vorticity statistics recently derived on the basis of the Monin-Lundgren-Novikov hierarchy, taking conditional averages from direct numerical simulations.

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