L-functions of symmetric powers of the generalized Airy family of
exponential sums

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Abstract

This paper looks at the L-function of the k-th symmetric power of the \( \mathbb{Q}_l \)-sheaf \( \text{Ai}_f \) over the affine line \( \mathbb{A}^1_{\mathbb{Q}_l} \) associated to the generalized Airy family of exponential sums. Using \( \ell \)-adic techniques, we compute the degree of this rational function and local factors at infinity.

1 Introduction

In this paper we study the L-function attached to the \( k \)-th symmetric power of the \( \mathbb{Q}_l \)-sheaf \( \text{Ai}_f \) associated to the generalized Airy family of exponential sums. Symmetric powers appear in the proofs of many arithmetic problems. For instance, Deligne’s proof \([5]\) of the Ramanujan-Petersson conjecture relies on the construction of a Galois module coming from the \( k \)-th symmetric power of a certain \( \ell \)-adic sheaf. The Sato-Tate conjecture \([1]\) \([2]\) \([25]\) relies on the analytic continuation of the \( L \)-function attached to the \( k \)-th symmetric power of an \( \ell \)-adic representation coming from an elliptic curve. Another equidistribution result concerning Kloosterman angles was proven by Adolphson \([3]\) using results of Robba’s \([21]\) on the Kloosterman sheaf \( K_2 \). Symmetric powers also arise in the proof of Dwork’s conjecture \([27]\) \([28]\) \([29]\). To begin, let us recall the general setup of an \( L \)-function of an \( \ell \)-adic representation.

Let \( \mathbb{F}_q \) be the finite field of \( q \) elements and characteristic \( p \). Let \( Y \) be a smooth, geometrically connected, open variety defined over \( \mathbb{F}_q \); for instance, take \( Y \) to be affine s-space \( \mathbb{A}^d_{\mathbb{F}_q} \) or the torus \( \mathbb{G}_m^d \). Denote its function field by \( K \), and its corresponding absolute Galois group by \( G_K := \text{Gal}(K^{\text{sep}}/K) \). Let \( V \) be a finite dimensional vector space over a finite extension field of \( \mathbb{Q}_l \), where \( l \neq p \). Let \( \rho : G_K \to GL(V) \) be a continuous \( \ell \)-adic representation unramified on \( Y \), and let \( \mathcal{F} \) be the corresponding lisse sheaf on \( Y \). Define the \( L \)-function of \( \rho \) on \( Y \) by

\[
L(Y, \rho, T) := \prod_{x \in Y|} \frac{1}{\det(1 - \rho(\text{Frob}_x)T^{\deg(x)})}.
\]

By the Lefschetz trace formula, this is a rational function whose zeros and poles may be described using étale cohomology with compact support:

\[
L(Y, \rho, T) = \prod_{i=0}^{2\dim(Y)} \det(1 - \text{Frob}_q T | H^i_c(Y \otimes \mathbb{F}_q, \mathcal{F}))^{(-1)^{i+1}}
\]

Given such a representation, we may construct new \( L \)-functions via operations such as tensor, symmetric, or exterior products. Natural questions about these new \( L \)-functions concern the determination of their degrees (Euler characteristic) and describing various properties about their zeros and poles. In this paper, we will focus on the symmetric powers of a particular family of exponential sums called the generalized Airy family. Other families whose symmetric powers have been investigated are the Legendre family of elliptic curves \([2]\) \([8]\) and the hyperKloosterman family \([10]\) \([11]\) \([21]\). We note that the former seems to have been motivated by Dwork’s \( p \)-adic interest in the Ramanujan-Petersson conjecture.

The generalized Airy family is defined as follows. Let \( f \) be a polynomial over \( \mathbb{F}_q \) of degree \( d \) with \( p \nmid d \). Let \( \psi \) be a nontrivial additive character on \( \mathbb{F}_q \). For each \( \ell \in \mathbb{F}_q^\times \) define its degree by \( \deg(\ell) := [\mathbb{F}_q(\ell) : \mathbb{F}_q] \). It is well-known that the associated \( L \)-function of the sequence of exponential sums

\[
S_m(\ell) := \sum_{x \in \mathbb{F}_q^m \text{deg}(\ell)} \psi \circ \text{Tr}_{\mathbb{F}_q^m/\mathbb{F}_q}(f(x) + \ell x) \quad \text{for } m = 1, 2, 3, \ldots
\]

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is a polynomial of degree $d - 1$:

$$L(f, A^1, \ell; T) := \exp \left( \sum_{m=1}^{\infty} S_m(\ell) \frac{T^m}{m} \right) = (1 - \pi_1(\ell)T) \cdots (1 - \pi_{d-1}(\ell)T).$$

As we will describe later, the relative cohomology of this family may be represented $\ell$-adically as a lisse sheaf of rank $d - 1$ over $A^1$ via Fourier transform. Let us denote this sheaf by $A_f$. The $L$-function of the $k$-th symmetric power of $A_f$ takes the form:

$$M_k(f, T) := L(A^1, \Sym^k(A_f), T) := \prod_{\ell \in [A^1]} \prod_{a_1 + \cdots + a_{d-1} = k} (1 - \pi_1(t)^{a_1} \cdots \pi_{d-1}(t)^{a_{d-1}} T^{\deg(t)})^{-1},$$

where $[A^1]$ denotes the set of closed points on $A^1$. By the Lefschetz trace formula, $M_k(f, T)$ is a rational function.

The $\ell$-adic sheaf $A_f$ was extensively studied by N. Katz in [16], where its monodromy group is determined and, as a consequence, an equidistribution result is obtained for the exponential sums in the family ([16, Corollary 20]). From these results it follows that, for $p > 2d - 1$, $M_k(f, T)$ is in fact a polynomial. For $d = 3$, a study of the monodromy group may be avoided using Adolphson’s method [3].

Our first result is the computation of the degree of $M_k(f, T)$ for $p > d$. The degree of the rational function $M_k(f, T)$ equals the $k$-th coefficient of a generating series which is explicitly given in Corollary 5.3. Simplified formulas are given in section 5 for some particularly nice values of $f$ and $p$.

As an example of this theorem, consider the family generated by $f(x) = x^d$. Then the degree of $M_k(x^d, T)$ may be described as follows. Let $\zeta$ be a primitive $(d-1)$-th root of unity in $\mathbb{F}_q$. Denote by $N_{d-1,k}$ the number of $(d-1)$-tuples $(a_0, a_1, \ldots, a_{d-2})$ of nonnegative integers such that $a_0 + a_1 + \cdots + a_{d-2} = k$ and $a_0 + a_1 \zeta + \cdots + a_{d-2} \zeta^{d-2} = 0$ in $\mathbb{F}_q$.

**Theorem 1.1.** With the notation defined above, we have

$$\deg M_k(x^d, T) = \frac{1}{d-1} \left( \binom{k + d - 2}{d - 2} - dN_{d-1,k} \right).$$

It was conjectured in [13] that $M_k(x^3, T)$ is a polynomial for all $p > 3$ since it was shown, in that paper, that $M_k(x^3, T)$ is a polynomial for every odd integer $k$, and also for every $k$ even with $k < 2p$. Surprisingly, for $p = 5$, $M_k(x^3, T)$ is not a polynomial for infinitely many $k$. This was communicated to the first author by N. Katz and is a consequence of the geometric monodromy group of $A_{x^3}$ being finite.

**Theorem 1.2.** Suppose $p > 2d - 1$. Then $M_k(f, T)$ is a polynomial which may be factored into a product $Q_k(f, T)P_k(f, T)$, where $P_k(f, T)$ satisfies the functional equation

$$P_k(f, T) = cT^{\deg(P_k)}P_k(f, 1/q^{k+1}T)$$

with $|c| = q^{\deg(P_k)(k+1)/2}$ and $Q_k(f, T)$ has reciprocal roots of weight $\leq k$. Furthermore, writing $f(x) = \sum_{i=0}^{d} c_i x^i$, if we assume $\mathbb{F}_q$ contains the $2(d-1)$-th roots of $-d c_d$ then an explicit description of $Q_k(f, T)$ may be given, see Corollary 4.3.

Describing the $p$-adic behavior of the reciprocal roots of $M_k(f, T)$ is also of interest. Motivation for such a study comes from Wan’s reciprocity theorem [26] of the Gouvêa-Mazur conjecture [12] on the slopes of modular forms; see [2] for the connection between symmetric powers of the Legendre crystal with Hecke polynomials. At present we are able to prove the following improvement to [13]. Assume $p \geq 7$, $k$ is odd and $k < p$. Write $M_k(x^3, T) = 1 + c_1 T + \cdots + c_{k+1/2} T^{(k+1)/2}$. Then the $q$-adic Newton polygon lies on or above the quadratic function $\frac{1}{2}(m^2 + m + km)$ for $m = 0, 1, 2, \ldots, \frac{k-1}{2}$. Furthermore, as a consequence of the functional equation, the endpoints of the $q$-adic Newton polygon of $M_k(x^3, T)$ coincide with this lower bound. We will prove this in a separate paper.

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## 2 Cohomological interpretation of $M_k(f, T)$

In this section we will study the generalised Airy family of exponential sums from the point of view of $\ell$-adic cohomology. We will do so by studying the sheaf $A_f$ that represents this family on the affine line $A^1$ over the given finite field $\mathbb{F}_q$. We begin by observing that the map $\mathbb{F}_q \to \mathbb{C}$ given by $t \mapsto \sum_{x \in \mathbb{F}_q} \psi(f(x) + \ell x)$ is the Fourier transform with respect to $\psi$, in the classical sense, of the map $t \mapsto \psi(f(t))$. This will translate, in the cohomological sense, to the fact that $A_f$ is the Fourier transform, in the sheaf-theoretical sense, of the $\mathcal{O}_f$-sheaf.
that represents the latter map, which is just the pull-back of the Artin-Schreier sheaf associated to ψ via the map given by f. Let us be more precise.

The polynomial f naturally defines a morphism, also denoted by f : \( \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q} \). Let \( \mathcal{L}_\psi \) be the Artin-Schreier sheaf on \( \mathbb{A}^1_{\mathbb{F}_q} \) associated to ψ (cf. [6, 1.7]). For every finite extension \( \mathbb{F}_q^m \) of \( \mathbb{F}_q \), every \( t \in \mathbb{A}^1(\mathbb{F}_q^m) = \mathbb{F}_q^m \) and every geometric point \( t \) over \( t \), we have \( \text{Trace}(\text{Frob}_t|_{\mathcal{L}_\psi}) = \psi(\text{Trace}_{\mathbb{F}_q^m/\mathbb{F}_q}(t)) \), where \( \text{Frob}_t \) denotes a geometric Frobenius element at \( t \). Consider the pullback \( \mathcal{L}_{\psi(f)} := f^*\mathcal{L}_\psi \).

By [16, Theorem 17], for \( d \geq 2 \) the Fourier transform with respect to \( \psi \) of \( \mathcal{L}_{\psi(f)} \) (which, in principle, is an element of the derived category \( \mathcal{D}_c^b(\mathbb{A}^1, \mathcal{M}_f) \)) is in fact a (shifted) lisse sheaf on \( \mathbb{A}^1 \), of rank \( d-1 \) and with \( d/(d-1) \) as its single slope at infinity. Its Swan conductor is therefore \( \sum_{x \in \mathbb{F}_q} \psi(f(x) + tx) \) (denoting this sheaf by \( \mathcal{L}_{\psi(f)} \)).

The characteristic polynomial of the action of a geometric Frobenius element \( \text{Frob}_t \) at \( t \) on the stalk of \( \mathcal{L}_{\psi(f)} \) at a geometric point \( t \) is the geometric monodromy \( \text{Frob}_t \) of \( \mathcal{L}_{\psi(f)} \) at \( t \). Consider the pullback \( \mathcal{L}_{\psi(f)} := f^*\mathcal{L}_{\psi(f)} \).

By [16, Theorem 17], for \( d \geq 2 \) the Fourier transform with respect to \( \psi \) of \( \mathcal{L}_{\psi(f)} \) (which, in principle, is an element of the derived category \( \mathcal{D}_c^b(\mathbb{A}^1, \mathcal{M}_f) \)) is in fact a (shifted) lisse sheaf on \( \mathbb{A}^1 \), of rank \( d-1 \) and with \( d/(d-1) \) as its single slope at infinity. Its Swan conductor is therefore \( \sum_{x \in \mathbb{F}_q} \psi(f(x) + tx) \) (denoting this sheaf by \( \mathcal{L}_{\psi(f)} \)).

The characteristic polynomial of the action of a geometric Frobenius element \( \text{Frob}_t \) at \( t \) on the stalk of \( \mathcal{L}_{\psi(f)} \) at a geometric point \( t \) has the form

\[
L(A_{ij}, t, T) = (1 - \pi_1(t)T) \cdots (1 - \pi_{d-1}(t)T)
\]

where \( \pi_i(t) \) is a Weil algebraic number of weight 1 (i.e., all its complex conjugates have absolute value \( q^{1/2} \)) and \( \sum_{x \in \mathbb{F}_q} \psi_m(f(x) + tx) = -\sum_i \pi_i(t)m \) for all \( m \geq 1 \). Its \( k \)-th “symmetric power” is given by

\[
L(k; A_{ij}, t, T) := \prod_{a_1 + \cdots + a_d-k = k} (1 - \pi_1(t)^{a_1} \cdots \pi_{d-1}(t)^{a_d}T).
\]

These are the local factors of the \( L \)-function of the \( k \)-th symmetric power of \( A_{ij} \), which is given by the infinite product

\[
M_k(f, T) := \prod_{t \in \mathbb{A}^1} L(k; A_{ij}, t, T^{\deg(t)})^{-1}.
\]

The Lefschetz trace formula demonstrates that the zeros and poles of \( M_k(f, T) \) may be described in terms of cohomology:

\[
M_k(f, T) = \prod_{i=0}^{2} \det(1 - \text{Frob}_T|\mathcal{H}^i_c(\mathbb{A}^1_{\mathbb{F}_q}, \text{Sym}^k A_{ij}))^{(-1)^{i+1}}.
\]

Since \( \text{Sym}^k A_{ij} \) is a lisse sheaf on the affine line, we have \( \mathcal{H}^0_c(\mathbb{A}^1_{\mathbb{F}_q}, \text{Sym}^k A_{ij}) = 0 \), and the previous formula simplifies to

\[
M_k(f, T) = \frac{\det(1 - \text{Frob}_T|\mathcal{H}^1_c(A_{ij}, \mathcal{M}_f)^{-1})}{\det(1 - \text{Frob}_T|\mathcal{H}^2_c(A_{ij}, \mathcal{M}_f))}.
\]

On the other hand, \( \mathcal{H}^2_c(\mathbb{A}^1_{\mathbb{F}_q}, \text{Sym}^k A_{ij}) \) is just the space of co-invariants of the sheaf \( \text{Sym}^k A_{ij} \), regarded as a representation of the fundamental group \( \pi_1(\mathbb{A}^1_{\mathbb{F}_q}) \), which is the \( k \)-th symmetric power of \( A_{ij} \) regarded as a representation of the same group. This is the same as the space of co-invariants for its monodromy group, which is defined to be the Zariski closure of its image in the group of automorphisms of the generic stalk of \( A_{ij} \), isomorphic to \( \text{GL}(d-1) \). By [19, Theorem 19], for \( p > 2d-1 \) the geometric monodromy group of \( A_{ij} \) is either \( \text{SL}(d-1) \) for \( d \) even, or \( \text{Sp}(d-1) \) for \( d \) odd if \( c_{d-1} = 0 \) and \( \mu_p \cdot \text{SL}(d-1) \) for \( d \) odd or \( \mu_p \cdot \text{Sp}(d-1) \) for \( d \) odd if \( c_{d-1} \neq 0 \), where \( f(x) = \sum_{i=0}^{d} c_i x^i \). In either case, its \( k \)-th symmetric power is still an irreducible representation of rank \( (d+k-2)/(d-2) \) of the monodromy group (because it is an irreducible representation of its subgroup \( \text{SL}(d-1) \) or \( \text{Sp}(d-1) \)), and in particular the space of co-invariants vanishes. More generally, it was proven by O. Such ([24, Proposition 1.6]) that, for \( p > 2 \), either \( A_{ij} \) has finite monodromy or its monodromy group contains \( \text{SL}(d-1) \) or \( \text{Sp}(d-1) \). In order to rule out the finite monodromy case for \( p \leq 2d-1 \) one may use for instance [18, Proposition 8.14.3], which implies that \( A_{ij} \) has finite monodromy if and only if for every element \( t \in \mathbb{F}_q \) the Newton polygon of the \( L \)-function associated to the exponential sum \( \sum \psi(f(x) + tx) \) has a single slope.

Consequently, we have the following:

**Theorem 2.1.** If \( A_{ij} \) does not have finite monodromy (e.g. if \( p > 2d-1 \)), the \( L \)-function of the \( k \)-th symmetric power of \( A_{ij} \) is a polynomial:

\[
M_k(f, T) = \det(1 - \text{Frob}_T|\mathcal{H}^1_c(A_{ij}, \mathcal{M}_f)^{-1})
\]
While it is tempting to believe that $M_k(f, T)$ is always a polynomial, this is not true, as mentioned in the introduction. In fact, the monodromy group can be finite in certain cases; for instance when $p = 5$ and $f(x) = x^3$, as proven in [19]. In such cases, $H^t_c(\mathcal{H}_{q_F}^1, \text{Sym}^k \mathcal{A}_f)$ will be non-trivial for infinitely many values of $k$, and consequently $M_k(f, T)$ will have a denominator.

**Remark 2.2.** Arithmetic difficulties often arise when the characteristic $p$ is small compared to $d$, as demonstrated above by the link between the finiteness of the monodromy group when $p \leq 2d - 1$ and the Newton polygons of the fibres of the family. By the functional equation, if we denote by $NP_1(t)$ the slope of the first line segment of the Newton polygon of the fibre $t$ then $NP_1(t) \leq 1/2$ with equality if and only if the Newton polygon is a single line segment. If $p \equiv 1$ modulo $d$, and in particular when $p = d + 1$, then by [23, Theorem 3.11] the Newton polygon of every fibre equals the $q$-adic Newton polygon of the polynomial $\prod_{x=1}^{d-1}(1 - q^i/dT)$. Thus, $NP_1(t) = 1/d$ and so the monodromy group is infinite when $p = d + 1 > 3$.

Let $[f(x)]_{x,N}$ denote the coefficient of $x^N$ in $f(x)$. Suppose $\frac{3}{2} + 1 < p \leq 2d - 1$ and $f$ has coefficients over $F_p$. By [22, Theorem 2.1], if $[(f(x) + tx)^{\frac{p-1}{d}}]_{x,N-1} \neq 0$ modulo $p$ for some $0 \leq t \leq p - 1$, then $NP_1(t) \leq \left[ \frac{d-1}{p-1} \right]$ for those $t$. Hence, the monodromy group is infinite when such a $t$ exists and $d \geq 3$ and $p \geq 7$. Their argument can be extended to show the following. Let $(h(x))_s = h(x)(h(x) - 1) \cdots (h(x) - s + 1)$. Define the linear operator $U : F_p[x] \to F_p$ by sending $x^n$ to 0 if $(p - 1) \nmid n$ and 1 otherwise. Let $c_1 = \cdots = c_{k-1} = 0$ modulo $p$ and $c_k \neq 0$ mod $p$ for some $t$, then $NP_1(t) \leq \frac{k}{p-1}$. Hence, if this happens for some $k < (p - 1)/2$ then the monodromy group is infinite. For example, for $d > p - 1$ and $f(x) = x^d + x^{p-1}$ then $c_1 = 1$ and hence the monodromy group is infinite for $p \geq 5$.

Lastly, we mention the case when $d = 4$, $p = 7$ and $f \in F_q[x]$ is not of the form $(x + a)^4 + bx + c$, then by [13, Theorem 4.6] the monodromy of $A_i f$ is not finite.

### 3. Computation of the degree of the $L$-function

We will now study the degree of $M_k(f, T)$ when $p > d$. From the formula above we have

$$\deg(M_k(f, T)) = \dim(H^1_c(\mathcal{H}_{q_F}^1, \text{Sym}^k \mathcal{A}_f)) - \dim(H^1_c(\mathcal{H}_{q_F}^1, \text{Sym}^k \mathcal{A}_f)) = -\chi_c(\mathcal{H}_{q_F}^1, \text{Sym}^k \mathcal{A}_f),$$

where $\chi_c$ denotes the Euler characteristic with compact supports. Using the Grothendieck-Néron-Ogg-Shafarevich formula, we have then

$$\deg(M_k(f, T)) = \text{Swan}_\infty(\text{Sym}^k \mathcal{A}_f) - \text{rank}(\text{Sym}^k \mathcal{A}_f)$$

$$= \text{Swan}_\infty(\text{Sym}^k \mathcal{A}_f) - \left(\frac{k + d - 2}{d - 2}\right).$$

In order to compute the Swan conductor of $\text{Sym}^k \mathcal{A}_f$ we have to study the sheaf $\mathcal{A}_f$ as a representation of the inertia group $I_\infty$ of $\mathcal{H}_{q_F}^1$ at infinity. Since $\mathcal{L}_{\psi(f)}$ is lisse on $\mathcal{H}_1$, as a representation of the decomposition group at infinity we have $\mathcal{A}_f \cong \mathcal{F}_{\infty}(\mathcal{L}_{\psi(f)}(t^i))$, where $\mathcal{F}_{\infty}$ is the local Fourier transform as defined in [20].

Recently, Fu [9] and, independently, Abbes and Saito [11] have given an explicit description of the different local Fourier transforms for a wide class of $\ell$-adic sheaves. We will mainly be using the description given in [11], which works over an arbitrary (not necessarily algebraically closed) perfect base field, and therefore gives an explicit formula for $\mathcal{A}_f$ as a representation of the decomposition group $D_\infty$.

If $S(\infty)$ is the henselization of the local ring of $\mathcal{F}_q$ at infinity with uniformizer $1/t$, the triple $(\mathcal{L}_{\psi(f(t))}, t, -f'(t))$ is a Legendre triple in the sense of [11, Definition 2.16]. Therefore by [11, Theorem 3.9] we conclude that, as a representation of $D_\infty$, $\mathcal{A}_f$ is isomorphic to

$$(-f')^* \mathcal{L}_{\psi(f(t))} \otimes \mathcal{L}_{\psi(-t'/t)} \otimes \mathcal{L}_\rho(f'(t)) \otimes \mathbb{Q} = (-f')^* \mathcal{L}_{\psi(f(t)-t'/t)} \otimes \mathcal{L}_\rho(f'(t)) \otimes \mathbb{Q}$$

where $\rho$ is the unique character $I_\infty \to \overline{\mathbb{Q}}_{\ell}$ of order 2, $\mathcal{L}_\rho$ the corresponding Kummer sheaf and $\mathbb{Q}$ is the pull-back of the character $\text{Gal}(\mathbb{F}_q/\mathbb{Q}) \to \overline{\mathbb{Q}}_{\ell}$ mapping the geometric Frobenius to the quadratic Gauss sum $g(\psi, \rho) := -\sum_{t \in \mathbb{F}_q} \psi(t) \rho(t)$.

Write $f(t) = \sum_{i=0}^d c_i t^i$. For simplicity, from now on we will assume that $\mathbb{F}_q$ contains the $2(d - 1)$-th roots of $-dc_d$ (which can always be achieved by a finite extension of the base field). Following [9, Proposition 3.1] we can find an invertible power series $\sum_{i \geq 0} r_i t^{-i} \in \mathbb{F}_q[[t^{-1}]]$ with $r_0^{d-1} = -dc_d$ such that $u(t) := t \sum_{i \geq 0} r_i t^{-i}$ is a solution to $f'(t) + u(t)^{d-1} = 0$ (the other solutions being $\xi u(t)$ for every $(d - 1)$-th root of unity $\xi$). The map $\phi : 1/t \mapsto 1/u(t)$ defines an automorphism $S(\infty) \to S(\infty)$, and by construction $-f' = [d - 1] \circ \phi$, where $[d - 1]$ is
the \((d - 1)\)-th power map. So \(A_{if}\) is isomorphic to
\[
[d - 1]_* \phi_* (L_{\psi(f(t) - t I'(t))} \otimes L_{\rho(\frac{1}{2} f''(t))} \otimes \mathcal{Q}) = [d - 1]_* (\phi^{-1})^* (L_{\psi(f(t) - t I'(t))} \otimes L_{\rho(\frac{1}{2} f''(t))} \otimes \mathcal{Q})
\]
\[
= [d - 1]_* L_{\psi(f(t) + \psi(t)(t - 1)^{d - 1})} \otimes L_{\rho(\frac{1}{2} f''(t))} \otimes \mathcal{Q}
\]
\[
= [d - 1]_* L_{\psi(f(t) + \psi(t)(t - 1)^{d - 1})} \otimes L_{\rho(\frac{1}{2} f''(t))} \otimes \mathcal{Q}
\]
since \([d - 1]^* \mathcal{Q} = \mathcal{Q}\), where \(v(t) := \phi^{-1}(t) = t \sum_{i \geq 0} s_i t^{-i}\).

Let \(g(t)\) be the polynomial of degree \(d\) obtained from \(f(v(t)) + v(t)^{d - 1}\) by removing the terms with negative powers of \(t\). It is important to notice that the coefficients of \(g\) are polynomials in the coefficients of \(f\). More precisely, if we write \(g(t) = \sum b_i t^i\), the coefficient \(b_i\) is a polynomial in the coefficients \(a_i, a_{i+1}, \ldots, a_d\) of \(f\). Since \(L_{\psi(h(t))}\) is trivial as a representation of \(D_{\infty}\) for any \(h(t) \in t^{-1} F_q[[t^{-1}]]\), we have an isomorphism \(L_{\psi(f(t)) + \psi(t)(t - 1)^{d - 1}} \cong L_{\psi(g(t))}\) as representations of \(D_{\infty}\).

On the other hand, from \(f'(v(t)) + t^{d - 1} = 0\) we get \(f''(v(t)) v'(t) + (d - 1) t^{d - 2} = 0\), so \(L_{\rho(\frac{1}{2} f''(t))} = L_{\rho(-\frac{1}{2} v'(t) t^{d - 2})}\). Since \(v'(t) = \sum_{i \geq 0} (1 - i) s_i t^{-i} = s_0 (1 + \sum_{i \geq 2} (1 - i) \frac{s_0}{s_0} t^{-i})\) and \(1 + \sum_{i \geq 2} (1 - i) \frac{s_0}{s_0} t^{-i}\) is a square in \(F_q[[t^{-1}]]\), we have \(L_{\rho(-\frac{1}{2} v'(t) t^{d - 2})} = L_{\rho(-\frac{1}{2} v'(t) t^{d - 2})} = L_{\rho(d - d^2)}(\ell_{s_0} t^{d - 2})\) (since \(s_0^{d - 1} = -1/d c_d\)). So we finally get
\[
A_{if} \cong [d - 1]_* (L_{\psi(g(t))} \otimes L_{\rho(s_0 t^{d - 2})} \otimes L_{\rho(d - d^2/2)} \otimes \mathcal{Q}).
\]

We can now easily compute the Swan conductor at infinity of its symmetric powers. By [17, 1.13.1],
\[
\text{Swan}_{\infty} \text{Sym}^k A_{if} = \frac{1}{d - 1} \text{Swan}_{\infty} [d - 1]^* \text{Sym}^k A_{if} = \frac{1}{d - 1} \text{Swan}_{\infty} \text{Sym}^k [d - 1]^* A_{if}
\]

**Lemma 3.1.** Let \(\zeta\) be a primitive \((d - 1)\)-th root of unity if \(\mathbb{F}_q, I_{\infty}^{d - 1}\) the unique closed subgroup of \(I_{\infty}\) of index \(d - 1\). As a representation of \(I_{\infty}^{d - 1}\), the restriction \([d - 1]^* A_{if}\) of \(A_{if}\) is isomorphic to the direct sum
\[
\bigoplus_{i=0}^{d - 2} L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)} \cong \bigoplus_{i=0}^{d - 2} L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(t)}
\].

**Proof.** Since \((\zeta^i)^* L_{\psi(s)} = L_{\psi(g(\zeta^i t))}\), \((\zeta^i)^* L_{\rho(s_0 t)} = L_{\rho^t(s_0 \zeta^i t)}\) and \([d - 1] \circ \zeta^i = [d - 1]\) for every \(i\), we have \([d - 1]^* (L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)}) = [d - 1]_* (L_{\psi(g(t))} \otimes L_{\rho^t(s_0 t)}),\) and therefore by Frobenius reciprocity \(\text{Hom}_{I_{\infty}^{d - 1}}([d - 1]^* A_{if}, L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)}) = \text{Hom}_{I_{\infty}}(A_{if}, [d - 1]_* (L_{\psi(g(t))} \otimes L_{\rho^t(s_0 t)}) = \text{Hom}_{I_{\infty}}(A_{if}, A_{if}) \cong \mathbb{Q}_l\) since the latter is an irreducible representation of \(I_{\infty}\). So for every \(i\), \(L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)}\) is a subrepresentation of \([d - 1]^* A_{if}\).

Now \(L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)}\) and \(L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)}\) are isomorphic if and only if \(L_{\psi(g(\zeta^i t))}\) and \(L_{\psi(g(\zeta^i t))}\) are, if and only if \(g(\zeta^i t) - g(\zeta^i t) = h^p - h\) for some \(h \in \mathbb{F}_q[t]\). Since \(p > d\), this can only happen if \(g(\zeta^i t) = g(\zeta^i t)\). Comparing the highest degree coefficients we conclude that \(\zeta^i\) and \(\zeta^j\) must be equal. Therefore the direct sum of the \(L_{\psi(g(\zeta^i t))} \otimes L_{\rho^t(s_0 \zeta^i t)}\) for \(i = 0, \ldots, d - 2\) injects into \([d - 1]^* A_{if}\) and we conclude that it must be isomorphic to it, since they have the same rank.

Consequently, we have an isomorphism of \(\mathbb{Q}_l[I_{\infty}]\)-modules
\[
\text{Sym}^k [d - 1]^* A_{if} \cong \bigoplus_{a_0 + a_1 + \ldots + a_d - 2 = k} L_{\psi(s_0 \zeta^i t)} \otimes L_{\rho^t(t)}
\].

For every finite subset \(I \subset \mathbb{Z}\) and every integer \(k \geq 0\) define
\[
S_{d - 1}(k, I) := \{(a_0, a_1, \ldots, a_d) \in \mathbb{Z}_{\geq 0}^{d - 1} | a_0 + a_1 + \ldots + a_d - 2 = k, a_0 + a_1 \zeta^i + \ldots + a_d \zeta^{d - 2} = 0\}
\]

It is clear from the definition that \(S_{d - 1}(k, I) = S_{d - 1}(k, I')\) if \(\phi(I) = \phi(I')\), where \(\phi : \mathbb{Z} \to \mathbb{Z}/(d - 1)\mathbb{Z}\) is reduction modulo \(d - 1\). Also, \(S_{d - 1}(k, I) = \emptyset\) if \(p\) does not divide \(k\) and \(I \cap (d - 1) \mathbb{Z} \neq \emptyset\). The number of elements in \(S_{d - 1}(k, I)\) can be conveniently expressed in terms of a generating function:

**Lemma 3.2.** Let \(F_{d - 1}(I; T) := \sum_{k=0}^{\infty} \# S_{d - 1}(k, I) T^k\). Then
\[
F_{d - 1}(I; T) = \frac{1}{q^{|I|}} \sum_{\gamma \in (\mathbb{F}_q[t])^I} \prod_{j=0}^{d - 2} (1 - \psi(\sum_{i \in I} \gamma_i \zeta^j t)) T^{-1}
\]
where \(\psi\) is any non-trivial additive character of \(\mathbb{F}_q\).
Proof. From the definition,
\[
F_{d^{-1}}(I; T) = \sum_{(a_0, \ldots, a_d) \in \mathbb{Z}_{\geq 0}^d} \prod_{i \in I} \delta(a_0 + a_1 \zeta_i + \cdots + a_d \zeta_i^{(d-2)}) T_{a_0 + a_1 + \cdots + a_d - 2}
\]
where \(\delta(a) = 1\) if \(a = 0\), 0 otherwise. Equivalently, \(\delta(a) = \frac{1}{q} \sum_{\gamma \in \mathbb{F}_q} \psi(\gamma a)\). So we get
\[
F_{d^{-1}}(I; T) = \sum_{(a_0, \ldots, a_d) \in \mathbb{Z}_{\geq 0}^d} \prod_{i \in I} \frac{1}{q^I} \sum_{\gamma_i \in \mathbb{F}_q} \psi(\gamma_i(a_0 + a_1 \zeta_i + \cdots + a_d \zeta_i^{(d-2)})) T_{a_0 + a_1 + \cdots + a_d - 2}
\]
\[
= \frac{1}{q^I} \sum_{\gamma \in (\mathbb{F}_q)^I} \prod_{i \in I} \psi(\gamma_i a_0) \prod_{i \in I} \psi(\gamma_i a_1 \zeta_i) T_{a_0} \prod_{i \in I} \psi(\gamma_i a_d \zeta_i^{(d-2)}) T_{a_d - 2}
\]
\[
= \frac{1}{q^I} \sum_{\gamma \in (\mathbb{F}_q)^I} \psi(\sum_{i \in I} \gamma_i a_0 T_{a_0}) \psi(\sum_{i \in I} \gamma_i a_1 \zeta_i T_{a_1}) \cdots \psi(\sum_{i \in I} \gamma_i a_d \zeta_i^{(d-2)} T_{a_d - 2})
\]
\[
= \frac{1}{q^I} \prod_{j=0}^{d-2} (1 - \psi(\sum_{i \in I} \gamma_i \zeta_i^{j}))^{-1}.
\]

Write \(g(t) = \sum_{j=0}^d b_j t^j\), and let \(J = \{1 \leq j \leq d | b_j \neq 0\}\) and \(J_{> j} := J \cap \{j, j+1, \ldots, d\}\) for every \(j \in \{1, \ldots, d+1\}\). We have
\[
\text{Swan}_\infty \text{Sym}^k[d-1] \text{Ai}_f = \sum_{a_0 + a_1 + \cdots + a_{d-2} = k} \text{Swan}_\infty \mathcal{L}_{\psi(\sum_{i=0}^{d-2} a_i g(\zeta_i^t))} \otimes \mathcal{L}_{\rho^k(t)}
\]
\[
= \sum_{a_0 + a_1 + \cdots + a_{d-2} = k} \text{deg}(\sum_{i=0}^{d-2} a_i g(\zeta_i^t))
\]
and
\[
\sum_{i=0}^{d-2} a_i g(\zeta_i^t) = \sum_{i=0}^{d-2} a_i \sum_{j=0}^d b_j \zeta_i^{ij} t^j = \sum_{j=0}^d (b_j \sum_{i=0}^{d-2} \zeta_i^{ij}) t^j
\]
so its degree is the greatest \(j\) such that \(b_j \sum_{i=0}^{d-2} \zeta_i^{ij} \neq 0\). Therefore we get
\[
(d-1)\text{Swan}_\infty \text{Sym}^k[d-1] \text{Ai}_f = \text{Swan}_\infty \text{Sym}^k[d-1] \text{Ai}_f = \sum_{j \in J} j \cdot (\#S_{d-1}(k, J_{>j+1}) - \#S_{d-1}(k, J_{>j}))
\]
\[
d = \binom{k + d - 2}{d - 2} - \sum_{j \in J} h(j) \cdot \#S_{d-1}(k, J_{>j})
\]
where \(h(j) := j - \text{sup}(J_{>j+1})\) is the “gap” between the \(t^j\) term and the next lower degree term in \(g(t)\). Taking the corresponding generating function we get the formula.

Corollary 3.3. Let \(G(f; T) := \sum_{k=0}^\infty (\text{Swan}_\infty \text{Sym}^k \text{Ai}_f) T^k\), then
\[
G(f; T) = \frac{d}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \sum_{j \in J} h(j) \cdot F_{d^{-1}}(J_{>j}; T)
\]
Using the previous formula for the degree, we deduce

Corollary 3.4. The degree of \(M_k(f; T)\) is the \(k\)-th coefficient of the power series expansion of
\[
\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \sum_{j \in J} h(j) \cdot F_{d^{-1}}(J_{>j}; T).
\]

Corollary 3.5. For every \(J \subset \{1, \ldots, d-1\}\), let \(\mathcal{P}_d(J)\) be the subspace of the affine space \(\mathcal{P}_d\) of polynomials of degree \(d\) over \(k\) such that \(b_j = 0\) if and only if \(j \in J\). The sets \(\{\mathcal{P}_d(J) | J \subset \{1, \ldots, d-1\}\}\) define a stratification of \(\mathcal{P}_d\) such that the degree of \(M_k(f; T)\) is constant in each stratum.
4 The trivial factor

Suppose $p > d$ and the monodromy of $\Lambda_f$ is not finite. We will now study the weights of the (reciprocal) roots of the polynomial $M_k(f, T)$. Let us first consider the easier case where $d$ is even, and therefore $\Lambda_f$ is isomorphic to $[d-1,1]\Lambda_{(d-1)\cdot e(d-2)} \otimes \mathbb{Q}$ as a representation of $D_{\infty}$. Let $D_{d-1}^{d-1} = \text{Gal}(\mathbb{F}_q((1/t))/\mathbb{F}_q((1/t^{(d-1)})))$, denote by $\alpha : D_{d-1}^{d-1} \to \mathbb{Q}^\ast$, the character corresponding to the sheaf $\mathcal{L}_f$, and let $b \in I_{\infty}$ be a generator of the cyclic group $D_{d-1}/D_{d-1}^{d-1} \cong I_{\infty}/I_{\infty}^{d-1}$. By the explicit description of induced representations, there is a basis $\{v_0, \ldots, v_{d-2}\}$ of the underlying vector space $V$ such that $a \cdot v_0 = (a_0)v_0$ for every $a \in I_{\infty}^{d-1}$ and $b \cdot v_i = v_{i+1}$ for $i = 0, \ldots, d-3$. Then $b \cdot v_{d-2} = v_{d-1}$, $v_0 = \alpha(b^{-1})v_0$. Replacing $b$ by $a^{-1}b$, we get an element such that $\alpha(a)^{d-1} = \alpha(b^{-1})$ (which is always possible since the values of $\alpha$ are the $p$-th roots of unity and $d-1$ is prime to $p$ since $p > d$) we may assume without loss of generality that $\alpha(b^{-1}) = 1$.

Furthermore, for any $a \in I_{\infty}^{d-1}$, we have $a \cdot v_i = (a_0b^{-1})(a^{-1})v_i$ for every $a_0 \in I_{\infty}$ and $a^{-1}b \cdot v_i = v_{i+1}$ for $i = 0, \ldots, d-3$. Then $b \cdot v_{d-2} = b^{d-1}$, $v_0 = \alpha(b^{-1}a_{d-1})v_0$. Replacing $b$ by $a^{-1}b$, we get an element such that $\alpha(a)^{d-1} = \alpha(b^{-1})$ (which is always possible since the values of $\alpha$ are the $p$-th roots of unity and $d-1$ is prime to $p$ since $p > d$) we may assume without loss of generality that $\alpha(b^{-1}) = 1$.

We turn now to the case $d$ odd. Let $\chi$ be a multiplicative character of $\mathbb{F}_q$ of order $2(d-1)$ (which exists, since we are assuming that $\mathbb{F}_q$ contains the $2(d-1)$-th roots of unity). Then by the projection formula $\Lambda_f$ is isomorphic to $[d-1,1] \mathcal{L}_{\mathbb{Q}}(d-1) \otimes \mathbb{Q}$ as a representation of $D_{\infty}$ and $D_{d-1}/D_{d-1}^{d-1}$ is a basis $\{v_0, \ldots, v_{d-2}\}$ of $V$ such that $a \cdot v_i = \alpha(a\beta)(a^{-1})v_i$ for $a \in D_{d-1}$ and $b \cdot v_i = \beta(b)v_{i+1}$ for $i = 0, \ldots, d-3$, $b \cdot v_{d-2} = \beta(b)v_0$ in this case. $\prod_{i=0}^{d-2} \alpha_i$ is trivial on $D_{d-1}$ if and only if $\sum a_i g(\zeta^j t)$ is a constant in $\mathbb{F}_q[t]$, that is, if and only if $\sum a_i c^{ij} = 0$ for every $j \in J$.

We now compute the dimension of the invariant subspace of the action of $I_{\infty}$ on $\text{Sym}^k \Lambda_f$, in very much the same way it is done for the Kloosterman sheaf in [10, Lemma 2.1]. Its underlying vector space is $\text{Sym}^k V$. An element $w$ is given by a linear combination

$$w = \sum_{a_0 + \cdots + a_{d-2} = k} c_{a_0 \cdots a_{d-2}} v_0^{a_0} \cdots v_{d-2}^{a_{d-2}}.$$

In the $d$ even case we have

$$a \cdot \sum_{a_0 + \cdots + a_{d-2} = k} c_{a_0 \cdots a_{d-2}} v_0^{a_0} \cdots v_{d-2}^{a_{d-2}} = \sum_{a_0 + \cdots + a_{d-2} = k} c_{a_0 \cdots a_{d-2}} (a_0 a_1 \cdots a_{d-2}) (a_0 v_0 ^{a_0} \cdots v_{d-2}^{a_{d-2}})$$

for $a \in I_{\infty}^{d-1}$ and

$$b \cdot \sum_{a_0 + \cdots + a_{d-2} = k} c_{a_0 \cdots a_{d-2}} v_0^{a_0} \cdots v_{d-2}^{a_{d-2}} = \sum_{a_0 + \cdots + a_{d-2} = k} c_{a_0 \cdots a_{d-2}} v_0^0 v_1^{a_1} \cdots v_{d-2}^{a_{d-2}}.$$

So $w$ is fixed by $I_{\infty}$ if and only if the character $c_{a_0 \cdots a_{d-2}}$ is trivial whenever $c_{a_0 \cdots a_{d-2}} \neq 0$ and $c_{a_0 \cdots a_{d-2}} = c_{a_0 \cdots a_{d-2}}$ for all $a_0, \ldots, a_{d-2}$. A basis for the invariant subspace is thus given by all distinct sums of the form (setting $v_{d-1+t} := v_t$ for all $l \geq 0$):

$$\sum_{j=0}^{d-2} v_0^{a_0} v_1^{a_1} \cdots v_j^{a_j}.$$
order 2, $\alpha_0 \cdot \alpha_d \cdot \beta$ is trivial if and only if both $\alpha_0 \cdot \alpha_d \cdot \beta$ and $\beta$ are trivial as characters of $T^d$, that is, if and only if $\sum a_i \zeta^j = 0$ in $\mathbb{F}_q$ for every $j \in J$ and $k$ is even. In particular, there are no non-zero invariants for $I^d$ if $k$ is odd. If $k$ is even, a generating set for the invariant subspace is given by all distinct sums of the form

$$
\sum_{j=0}^{d-2} \beta(h)^{j} v_{i}^a v_{j+1} \cdot \cdot \cdot v_{j+d-2}
$$

for all $a_0, \ldots, a_{d-2}$ such that $\sum a_i \zeta^j = 0$ in $\mathbb{F}_q$ for every $j \in J$. Let $r$ be the size of the orbit of $(a_0, \ldots, a_{d-2})$ under the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ by cyclic permutations. If $r \neq d-1$, we can write

$$
\sum_{j=0}^{d-2} \beta(h)^{j} v_{i}^a v_{j+1} \cdot \cdot \cdot v_{j+d-2} = \sum_{j=0}^{r-1} \beta(h)^{j} (1 + \beta(h)^r)^k + \cdots + \beta(h)^{\frac{(d-1)}{r}} v_{i}^a v_{j+1} \cdot \cdot \cdot v_{j+d-2}.
$$

Notice that $k$ must be a multiple of $\frac{d-1}{r}$, since $k = \sum_{i=0}^{d-2} a_i = \frac{d-1}{r} \sum_{i=0}^{r-1} a_i$. If $\frac{d-1}{r}$ is odd we have

$$
1 + \beta(h)^r + \cdots + \beta(h)^{\frac{(d-1)}{r}} = \frac{1 - \beta(h)^{d-1}}{1 - \beta(h)} = 0,
$$

so the above sum vanishes. On the other hand, if $\frac{d-1}{r}$ is even it is clear that the element

$$
\sum_{j=0}^{d-2} \beta(h)^{j} v_{i}^a v_{j+1} \cdot \cdot \cdot v_{j+d-2} = \frac{d-1}{r} \sum_{j=0}^{r-1} \beta(h)^{j} v_{i}^a v_{j+1} \cdot \cdot \cdot v_{j+d-2}
$$

is non-zero, and to different orbits correspond different elements. To summarize, we have

**Proposition 4.1.** Let $T_{d-1}(k,J)$ be the set of orbits of the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ on the set $S_{d-1}(k,J)$ by cyclic permutations, and let $U_{d-1}(k,J)$ be the subset of orbits such that $\frac{k}{d-1}$ is even, where $r$ is their cardinality. If $d$ is even, the invariant subspace of the representation $\text{Sym}^k \text{Sym}^d$ of $I^d$ has dimension $\#T_{d-1}(k,J)$. If $d$ is odd and $k$ is even, it has dimension $\#U_{d-1}(k,J)$. If $d$ and $k$ are odd, the representation has no non-zero invariants.

The sequences $\#T_{d-1}(k,J)$ and $\#U_{d-1}(k,J)$ can also be described by means of generating functions. By Burnside’s lemma, the dimension of the invariant subspace for $d$ even is given by

$$
\#T_{d-1}(k,J) = \frac{1}{d-1} \sum_{r=1}^{d-1} \# \{(a_0, a_1, \ldots, a_{d-2}) | a_i = a_{i+r \mod d-1} \} = \frac{1}{d-1} \sum_{r=d-1}^{d-1} \phi(d-1) \#S_r(\frac{kr}{d-1}, J)
$$

where $S_r(k,J) = \emptyset$ if $k$ is not an integer and $\phi$ is Euler’s totient function. So the generating function for the sequence $\{\#T_{d-1}(k,J) | k \geq 0 \}$ is

$$
G_{d-1}(J; T) = \sum_{k=0}^{\infty} \#T_{d-1}(k,J) T^k
$$

Next, suppose that $d$ is odd, and let $(a_0, \ldots, a_{d-2}) \in S_{d-1}(k,J)$. Let $r$ be the number of elements in its orbit. Then $\sum_{i=0}^{r-1} a_i = k \frac{r}{d-1}$. We want to count the number of orbits such that this value is even. Since $k = \frac{kr}{d-1} \cdot \frac{d-1}{r}$, if the largest power of 2 that divides $d-1$ is smaller than the largest power of 2 dividing $k$, $\frac{kr}{d-1}$ must always be even. Suppose that the largest power of 2 that divides $k$, $2^{\alpha(k)}$, divides $d-1$. Then $\frac{kr}{d-1}$ is odd if and only if $2^{\alpha(k)}$ is odd.
divides $\frac{d-1}{r}$, if and only if $r$ divides $\frac{d-1}{2m(r)}$. Therefore $\#U_{d-1}(k, J) = \#T_{d-1}(k, J)$ if $2^m(k)$ does not divide $d - 1$ and $\#T_{d-1}(k, J) - \#T_{d-1}(k, J)$ if it does. The generating function is then

$$
\sum_{k=0}^{\infty} \#U_{d-1}(k, J) T^k = \sum_{k=0}^{\infty} \#T_{d-1}(k, J) T^k - \sum_{j \geq 1/2 \mid d-1 \text{ odd}} \#T_{d-1}(l, J) T^{2l} = G_{d-1}(J; T) - \sum_{j \geq 1/2 \mid d-1} H_{d-1}(J; T^{2j})
$$

where

$$H_r(J; T) := \frac{1}{2}(G_r(J; T) - G_r(J; -T)).$$

Let $F \in D^{d-1}_\infty \subset D_\infty$ be a geometric Frobenius element, and $w = \sum_{j=0}^{d-2} j_0^{a_0} j_1^{a_1} \cdots j_{d-2}^{a_{d-2}}$ (resp. $w = \sum_{j=0}^{d-2} \beta(h)^2 v_j v_j^{a_0} v_j^{a_1} \cdots v_j^{a_{d-2}}$) a generator of the $I_\infty$-invariant subspace of $\text{Sym}^k V$. $F$ acts on $V_\infty$ via the character corresponding to $L_\infty$. Since $2d-1 \not\mid d-1$, we have $L_\infty$ acts on $V_\infty$ via the character corresponding to $L_\infty$. Additionally, if $d$ is odd and $k$ even, we have $L_\infty$ acts on $V_\infty$ via the character corresponding to $L_\infty$. We conclude:

**Proposition 4.2.** A Frobenius geometric element at infinity acts on the $I_\infty$-invariant subspace of $\text{Sym}^k \text{Ai}_f$ by multiplication by $\psi(kh_0)\rho(d(d-1)c_{d/2})g(\psi, \rho)^k$.

As an immediate consequence we get

**Corollary 4.3.** The local L-function of $\text{Sym}^k \text{Ai}_f$ at infinity $\det(1 - \text{Frob} T) \mid (\text{Sym}^k \text{Ai}_f)^{I_\infty}$ is given by $(1 - \psi(kh_0) \rho(d(d-1)c_{d/2})g(\psi, \rho)^k T)^{\#T_{d-1}(k, J)}$ if $d$ is even, $(1 - \psi(kh_0) \rho(d(d-1)c_{d/2})g(\psi, \rho)^k T)^{\#U_{d-1}(k, J)}$ if $d$ is odd and $k$ is even, and 1 if $d$ and $k$ are odd.

**Theorem 4.4.** The polynomial $M_k(f, T)$ decomposes as a product $P_k(f, T) Q_k(f, T)$, where $Q_k(f, T)$ is given by the formula in Corollary 4.3 and $P_k(d, T)$ satisfies a functional equation

$$P(T) = cT^n P(1/q^{k+1}T)$$

where $|c| = q^{r(k+1)/2}$ and $r$ is its degree.

**Proof.** Let $j : A^1 \to P^1$ be the inclusion. From the exact sequence

$$0 \to \text{Sym}^k \text{Ai}_f \to j_* \text{Sym}^k \text{Ai}_f \to (j_* \text{Sym}^k \text{Ai}_f)_\infty \to 0$$

we get an exact sequence of Gal($\overline{F}_q / F_q$)-modules

$$0 \to (j_* \text{Sym}^k \text{Ai}_f)^{I_\infty} \to H^1(A^1, \text{Sym}^k \text{Ai}_f) \to H^1(P^1, j_* \text{Sym}^k \text{Ai}_f) \to 0$$

and therefore a decomposition

$$M_k(f, T) = \det(1 - \text{Frob} T | H^1(A^1, \text{Sym}^k \text{Ai}_f))$$

$$= \det(1 - \text{Frob} T | (j_* \text{Sym}^k \text{Ai}_f)^{I_\infty}) \det(1 - \text{Frob} T | H^1(P^1, j_* \text{Sym}^k \text{Ai}_f)).$$

The first factor is described by the previous corollary. On the other hand, by [Théorème 1.3] we have a perfect pairing

$$H^1(P^1, j_* \text{Sym}^k \text{Ai}_f) \times H^1(P^1, j_* \text{Sym}^k \text{Ai}_f) \to \overline{\mathbb{Q}}(-k - 1)$$

where $\text{Ai}_f^*$ is the dual of $\text{Ai}_f$, which is constructed in the same way as $\text{Ai}_f$ using the complex conjugate character $\bar{\psi}$ instead of $\psi$. If the eigenvalues of the action of Frobenius on $H^1(P^1, j_* \text{Sym}^k \text{Ai}_f)$ are $\alpha_1, \cdots, \alpha_r$, so that $P_k(f, T) = \prod (1 - \alpha_i T)$, it follows that $P_k(f, T) = \prod (1 - (q^{k+1}/\alpha_i) T)$ and therefore the functional equation holds. Applying the functional equation twice we get $|c| = q^{r(k+1)/2}$.  

\end{document}
5 Some special cases

We will now see how the previous results apply to some special values of $f$. First, consider the case $f(t) = t^d$. In this case the equation $f'(t) + u(t) t^{d-1} = 0$ gives $u(t) = r_0 t$, where $r_0^{d-1} = -d$. Then $v(t) = t/r_0$, and $g(t) = f(v(t)) + v(t) t^{d-1} = t^d(1/r_0^d + 1/r_0) = 1/r_0^d t^d$. By corollary 3.4, we get that the degree of $M_k(f; T)$ is the $k$-th coefficient in the power series expansion of

$$ \frac{1}{d-1} \left( \frac{1}{1-T} - dF_{d-1}(\{1\}; T) \right) $$

where

$$ F_{d-1}(\{1\}; T) = \frac{1}{q} \sum_{\gamma \in \mathbb{F}_q} (1-\gamma^T)^{-1} $$

Explicitly,

$$ \deg M_k(f; T) = \frac{1}{d-1} \left( \frac{k+d-2}{d-2} - d \cdot \#S_{d-1}(k, \{1\}) \right). $$

In particular, for $d = 3$

$$ F_2(\{1\}; T) = \frac{1}{q} \sum_{\gamma \in \mathbb{F}_q} (1-\gamma^T)^{-1} (1-\gamma^{-1} T)^{-1} = \frac{1}{p} \sum_{m=0}^{p-1} \left( \frac{2 \pi i m}{p} \right)^{-1} \left( \frac{2 \pi i m}{p} \right)^{-1}. $$

It is easily checked that $S_2(k, \{1\}) := \{(a,b)|a+b=k, a \equiv b \pmod{p}\}$ has $\frac{k}{p} + \delta$ elements, where $\delta = 0$ (resp. $\delta = 1$) if $k - \frac{k}{p}$ is odd (resp. even). So in this case we get an explicit formula for the degree:

$$ \deg M_k(f(t) = t^3; T) = \frac{1}{2} \left( k + 1 - 3 \left( \frac{k}{p} + \delta \right) \right). $$

If $p > k$ this gives $(k+1)/2$ for $k$ odd and $(k-2)/2$ for $k$ even.

Corollary 3.3 states for $f(t) = t^d$ that the local $L$-function of $\text{Sym}^k \mathfrak{A}_f$ at infinity is $(1-\rho(d(d-1)/2)k g(\psi, \rho)^k T)^{\#T_{d-1}(k, J)}$ if $d$ is even, $(1-\rho(d(d-1)/2)k g(\psi, \rho)^k T)^{\#U_{d-1}(k, J)}$ if $d$ is odd and $k$ is even and 1 if $d$ and $k$ are odd. For $d = 3$, we can again provide a more explicit expression.

Since $3$ is odd, the local $L$-function is 1 for $k$ odd. For $k$ even, we can write $\#S_2(k, \{1\}) = \lfloor \frac{k}{2} \rfloor + \delta = 2 \lfloor \frac{k}{2p} \rfloor + 1$. Every orbit of $\mathbb{Z}/2\mathbb{Z}$ acting on $S_2(k, \{1\})$ has two elements except for $\{(k/2, k/2)\}$, so $\#T_2(k, \{1\}) = \lfloor \frac{k}{2p} \rfloor + 1$. $U_2(k, \{1\})$ contains the orbits such that $rk$ is a multiple of 4. If $k \equiv 0(\pmod{4})$ this includes all orbits. If $k \equiv 2(\pmod{4})$ the orbit $\{(\frac{k}{2}, \frac{k}{2})\}$ must be excluded. So the trivial factor for $k$ even is

$$ (1-g(\psi, \rho)^k T)^{\lfloor \frac{k}{2} \rfloor} \quad \text{for } k \equiv 2(\pmod{4}) $$

$$ (1-g(\psi, \rho)^k T)^{\lfloor \frac{k}{2} \rfloor + 1} \quad \text{for } k \equiv 0(\pmod{4}) $$

In particular, for $p > \frac{k}{2}$ the trivial factor of $M_k(t^3; T)$ is 1 if $k \equiv 2(\pmod{4})$ and $(1-g(\psi, \rho)^k T)$ if $k \equiv 0(\pmod{4})$.

We will now consider the case where $g(t) = \sum b_i t^i$ has $b_i \neq 0$ for $i = 1, \ldots, d-2$. This includes the generic case where all coefficients of $g(t)$ are non-zero as a special case. Suppose first that $b_{d-1} = 0$ (or, equivalently, that $c_{d-1} = 0$). $S_{d-1}(k, J)$ is the set of all $(a_0, \ldots, a_{d-2}) \in \mathbb{Z}_{\geq 0}^{d-1}$ such that $\sum a_i = k$ and $\sum a_i \zeta^{ji} = 0$ for all $j = 1, \ldots, d-2$. The system of equations $\{ \sum_i \zeta^{ji} x_i = 0 \}$ has rank $d-2$ (since the $(d-2) \times (d-2)$ minors are Vandermonde determinants) and has $(1,1,\ldots,1)$ as a solution, so all solutions must be of the form $(a,a,\ldots,a)$ modulo $p$ for some $a$. Therefore

$$ F_{d-1}(J; T) := \sum_{k=0}^{\infty} \#S_{d-1}(k, J) T^k $$

$$ = \sum_{r=0}^{p-1} \sum_{a_0, \ldots, a_{d-2} = 0} \sum_{a_0, \ldots, a_{d-2} = 0} T^{r+s_0 p + \cdots + (r+s_{d-2} p)} $$

$$ = \sum_{r=0}^{p-1} T^{(d-1)r} \sum_{a_0, \ldots, a_{d-2} = 0} T^{p(a_0 + \cdots + a_{d-2})} $$

$$ = \left( \frac{1}{1-Tp} \right)^{d-1} \left( \frac{1}{1-T^d} \right) $$

$$ = \left( \frac{1}{1-T} \right)^{d-1} \left( \frac{1}{1-T^d} \right) $$

$$ = \left( \frac{1}{1-T} \right)^{d-1} \left( \frac{1}{1-T^d} \right) $$

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Suppose now that $b_{d-1} \neq 0$ (or, equivalently, that $c_{d-1} \neq 0$). Making the change of variable $\hat{f}(t) = f(t - \frac{a_{d-1}}{c_{d-1}})$ we eliminate the degree $d-1$ term. Moreover, $\text{Ai}_f = R_1 \pi t \ell \psi_{f(x(t)+t+ \frac{a_{d-1}}{c_{d-1}})} = \text{Ai}_f \otimes \psi_{\left(-\frac{a_{d-1}}{c_{d-1}}t\right)}$ and thus $\text{Sym}^k \text{Ai}_f = (\text{Sym}^k \text{Ai}_f) \otimes \ell^\otimes k \psi_{\left(-\frac{a_{d-1}}{c_{d-1}}t\right)}$. As a representation of $D_\infty$, we have then $\text{Ai}_f = [d-1], (\text{Sym}^k \text{Ai}_f) \otimes \mathcal{L}_\rho(\text{not}) \otimes \mathcal{L}_\rho(\text{not}) = [d-1], (\text{Sym}^k \text{Ai}_f) \otimes \mathcal{L}_\rho(\text{not}) \otimes \mathcal{L}_\rho(\text{not}) \otimes \mathcal{L}_\rho(\text{not})$. In other words, $g(t) = \hat{g}(t) - \frac{a_{d-1}}{c_{d-1}} t^{d-1}$.

If $p$ divides $k$, the condition $\sum_j a_j \zeta^{ij}$ for $j = d-1$ is void, so both the dimension of $M_k(f; T)$ and the trivial factor at infinity behave as in the $b_{d-1} = 0$ case. If $p$ does not divide $k$, the condition $\sum_j a_j \zeta^{ij}$ does never hold for $j = d-1$, so $S_{d-1}(k, J_j) = 0$ for $j = 1, \ldots, d-1$. In particular, the trivial factor of $M_k(f; T)$ is 1. Furthermore, applying the formula for the degree, we get

$$\text{deg } M_k(f, T) = \frac{1}{d-1} \left( \binom{k+d-2}{d-2} - \# S_{d-1}(k, \{1\}) \right).$$

As a final example, suppose that $d-1$ is prime and $p$ is a multiplicative generator of $F_{d-1}$. In this case, all non-trivial $(d-1)$-th roots of unity are conjugate over $\mathbb{F}_p$, so $a_0 + a_1 \zeta + \cdots + a_{d-2} \zeta^{d-2} = 0$ if and only if $a_0 + a_1 \zeta^j + \cdots + a_{d-2} \zeta^{(d-2)j} = 0$ for any $j = 1, 2, \ldots, d-2$. Therefore $S_{d-1}(k, \{1\}) = S_{d-1}(k, J)$ for every $J \subseteq \mathbb{Z}$ such that $J \cap (d-1) \mathbb{Z} = \emptyset$. As in the previous example, we conclude that, if $c_{d-1} = 0$,

$$F_{d-1}(J \geq j; T) = \frac{1 - T^{(d-1)p}}{(1 - T^{(d-1)^p})^{d-1}(1 - T^{d-1})}$$

for every $j \in J$. By corollary [3.4] the degree of $M_k(f; T)$ is the $k$-th coefficient of the power series expansion of

$$\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \frac{1 - T^{(d-1)p}}{(1 - T^{(d-1)p})^{d-1}(1 - T^{d-1})} \sum_{j \in J} h(j) = \frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \frac{1 - T^{(d-1)p}}{(1 - T^{(d-1)p})^{d-1}(1 - T^{d-1})}.$$

If $c_{d-1} \neq 0$ we have, as in the previous example, the same formula for the degree if $k$ is a multiple of $p$, and the $k$-th coefficient in the power series expansion of

$$\frac{1}{(d-1)(1-T)^{d-1}} - \frac{1}{d-1} \frac{1 - T^{(d-1)p}}{(1 - T^{(d-1)p})^{d-1}(1 - T^{d-1})}$$

if $k$ is prime to $p$.

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