Non-equilibrium two-fluid plasmas can generate magnetic fields and flows simultaneously

Hamid Saleem
National Centre for Physics (NCP),
Quaid-i-Azam University Campus, Islamabad,
Pakistan.
(Dated: 16, July 2010)

A new analytical solution of the set of highly nonlinear two-fluid equations is presented to explain the mechanism for the generation of "seed" magnetic field and plasma flow by assuming the density \( n \) to have a profile like an exponential in xy-plane and temperature profiles of electrons (ions) to be linear in yz-plane. It is shown that the baroclinic vectors - \( \nabla \Psi \times \nabla T_j \) (where \( \Psi = \ln \bar{n}, \bar{n} \) is normalized density, and \( T_j \) denote the temperatures of electrons and ions for \( j = e, i \)) can generate not only the magnetic field but the plasma flow as well. It is also pointed out that the electron magnetohydrodynamics (EMHD) model has inconsistencies because it does not take into account the ion dynamics while the magnetic field is produced on slow time scale. The estimate of the magnitude of the magnetic field in a classical laser plasma using this model is in agreement with the experimental observations.

PACS numbers: 52.38F8, 52.35Mw, 95.30.Qd
I. INTRODUCTION

The presence of large scale magnetic fields in galaxies, galaxy clusters and in intergalactic space [1] is a mystery and several theoretical models have been presented to explain the origin of these fields [2-5]. Most of these works deal with the dynamo theory of single fluid magnetohydrodynamics (MHD). But the set of MHD equations assumes that some magnetic field is already present in the system. Therefore, these models actually investigate the amplification of the existing weak magnetic field and can not explain the generation of the 'seed' field in true sense.

Long ago [6], Biermann presented a mechanism for the generation of stellar magnetic fields which is not based on MHD. He proposed that the electrons faster motion compared to ions can produce electric field in a rotating star which is not curl free due to non-parallel density and temperature gradients and hence the magnetic field is produced. The ions were assumed to be stationary in this work. The Biermann battery and electron diffusion processes were also investigated to explain the generation of 'seed' magnetic fields in galaxies [7].

It is very interesting that the large magnetic fields of the order of kilo and mega Gauss were observed in classical laser-induced plasmas many decades ago [8-9]. These observations indicate that the dynamics of initially unmagnetized nonuniform plasmas can generate magnetic fields. The idea of magnetic field generation by plasma dynamics is very attractive and a huge amount of research work in this direction has already been appeared in literature.

Based on Biermann battery effect, a single fluid plasma model called electron magneto-hydrodynamics (EMHD) was presented to explain the magnetic field generation in laser plasmas [10, 11]. But it does not require the rotation of the system to produce magnetic fields. In EMHD, ions are assumed to be stationary and electrons are treated to be inertialess. Both the fluctuating [12, 13] and steadily growing magnetic fields [8, 9] have been theoretically produced using EMHD models. The so called magnetic electron drift (MEDV) mode was discovered [12] using EMHD equations. The MEDV mode is believed to be a pure transverse low frequency wave of an unmagnetized inhomogeneous plasma. But a critical analysis of MEDV mode shows that it should contain a contribution from electron density perturbation as well.

Therefore, a new mode which is partially transverse and partially longitudinal has been proposed to be a normal mode of pure electron plasmas which can exist only in a very
narrow range of parameters. This mode can couple with the ion acoustic wave which also becomes electromagnetic under certain conditions in a non-uniform plasma [14]. Similarly the model equation widely used for the generation of steadily growing magnetic field [12, 13] is not flaw-less. The field grows on ion time scale while ions are assumed to be stationary. Recently [15], the same model equation containing electron baroclinic term has been used to estimate the magnetic field produced in a laser plasma.

Some weaknesses and contradictions in the approximations and assumptions used in EMHD models for magnetic field generation have already been pointed out [16]. The advantage of EMHD model is that it is very simple. Since it is still being used by many authors, therefore it seems important to discuss at least the two cases of fluctuating and steadily growing magnetic fields which are believed to be generated by EMHD. The EMHD models for the generation of magnetic fields are critically discussed in the next section.

The magnitudes of magnetic fields on galactic scale [7] as well as on laser-plasma scale [8, 9] were estimated by assuming the non-parallel electron density and temperature gradients to be constant using EMHD equations. In these models the gradients of electron temperature and density were assumed to be one-dimensional and the produced magnetic fields had only one component.

A few years ago [17], a theoretical model was presented to show the generation of three-dimensional magnetic field by baroclinic vectors of electrons and ions. However, in this investigation some constant magnetic field was assumed to be present already. A stationary solution of these equations was presented by Mahajan and Yoshida [18] in the form of double Beltrami field.

Later, the model presented in Ref. [17] was modified to explain the creation of all the three components of ‘seed’ magnetic field vector $\mathbf{B}$ from $t=0$ due to externally given forms of baroclinic vectors [19]. Here one does not need to assume some static magnetic field to be present in the system. However, the form of the solution was sinusoidal along one axis which is not physical in general.

The EMHD may have some applications in other areas but for the magnetic field generation on slow time scale, the ion dynamics can not be neglected. Therefore, it is necessary to study the ‘seed’ magnetic field generation by using two fluid model. On the other hand the generated magnetic field vector should not have necessarily only one component for the
sake of generality.

Our aim is to find out an analytical solution of the set of two fluid equations such that the cross products of plasma density and temperature gradients of electrons and ions become the source terms in the electron and ion equations of motion. We assume that the plasma has been produced in a non-equilibrium state and it evolves with time generating the "seed" magnetic field and flow.

The present investigation is very different from the previous work [19] because the density gradient scale length is assumed to have an exponential form in xy-plane. In most of the analytical studies, the density is assumed to have exponential form along one axis. But we want to obtain a two-dimensional solution, therefore the density is assumed to be a function of (x,y) coordinates.

We have chosen special profiles of gradients of density (\(\nabla n\)) electron temperature (\(\nabla T_e\)) and ion temperature (\(\nabla T_i\)) to obtain an analytical solution. Different profiles of density and temperatures can be considered but then the numerical simulation will be needed. In our formulism all the nonlinear terms vanish and ultimately we obtain the two linear equations where the terms \(\nabla \psi \times \nabla T_j\) (i = e, i) with \(\psi = ln\bar{n}\) (where \(\bar{n}\) is normalized density) become the source terms for magnetic field.

The details of the model are discussed in section III. Since the present model contains a very complex system of highly nonlinear equations, therefore to find out an analytical solution one has to use some assumptions and approximations. The main focus is to justify the physical idea that the two fluid plasma with density gradient like an exponential function in xy-plane and constant gradients of electron and ion temperatures along y and z-axes can generate the 'seed' magnetic fields and flow. This model can explain the magnetic fields produced in laser-induced plasmas. In our opinion, the numerical simulation of two-fluid equations is very important to study the 'seed' field generation. In simulation one can use many different profiles of density and temperatures.

II. CONTRADICTORY RESULTS OF EMHD

Here we briefly point out the contradictory results of EMHD models used for magnetic field generation. First we discuss a theoretical model based on EMHD for the generation of fluctuating magnetic fields proposed several years ago [12]. The mode-discovered through
EMHD theory was named as the magnetic electron drift vortex (MEDV) mode. A great deal of research work on this mode has been carried out. The critical analysis of MEDV mode and some new theoretical results have been published recently on the fluctuating magnetic fields [14]. There seems to be a need to briefly clarify here the physical situation to lay down the basis of our theoretical model presented in the next section. In the theory of MEDV mode, the ions are assumed to be stationary but the electron inertial effects are included. The linear description of the MEDV mode is presented very briefly as follows.

Electron equation of motion is,

\[ m_e n_0 \partial_t v_{e1} = -e n_0 E_1 - \nabla p_{e1} \] (1)

where subscripts one (1) and naught (0) denote the linearly perturbed and equilibrium quantities, respectively. In the limit \( \omega_{pi} << \omega << \omega_{pe} \) (where \( \omega \) is the frequency of the wave and \( \omega_{pj} = \left( \frac{4\pi n_0 e^2}{m_j} \right)^{1/2} \) is the plasma oscillation frequency of the jth species while \( j = e \) here), the displacement current is ignored and Maxwell’s equation yields,

\[ \nabla \times B_1 = 4\pi c (J_1) = 4\pi c (-e n_0 v_{e1}) \] (2)

Since \( \nabla \cdot J_1 = 0 \), therefore according to (2), the density perturbation is neglected and we find \( p_{e1} = n_0 T_{e1} \). For \( T_{e1} \), the electron energy equation becomes,

\[ \frac{3}{2} n_0 \partial_t T_{e1} + \frac{3}{2} n_0 (v_{e1} \cdot \nabla) T_{e0} = -p_0 \nabla \cdot v_{e1} \] (3)

Assuming, \( \nabla n_0 = \frac{\dot{x}}{dx} \frac{dn_0}{dx} \), \( \kappa_n = \left| \frac{1}{n_0} \frac{dn_0}{dx} \right| \), \( E_1 = E_1 \hat{x} \), \( k = k_y \hat{y} \) and \( B_1 = B_1 \hat{z} \) the linear dispersion relation for MEDV mode turns out to be

\[ \omega^2 = \frac{2}{3} C_0 \frac{\kappa_n}{k_y^2} \nu_{Te}^2 k_y^2 \] (4)

where \( C_0 = \frac{\lambda_e^2 k_y^2}{1 + \lambda_e^2 k_y^2} \), \( \lambda_e = \frac{c}{\omega_{pe}} \) is the electron collision-less skin depth and \( \nu_{Te} = \left( \frac{T_e}{m_e} \right)^{1/2} \) is the electron thermal speed. If temperature gradient is assumed to be anti-parallel to the density gradient in laser plasma with \( \nabla T_0 = \frac{\dot{x}}{dx} \frac{dT_0}{dx} \) and \( \kappa_T = \left| \frac{1}{T_0} \frac{dT_0}{dx} \right| \), then (4) is modified as,

\[ \omega^2 = C_0 \frac{\kappa_n}{k_y^2} \left[ \left( \frac{2}{3} \kappa_n - \kappa_T \right) \nu_{Te}^2 k_y^2 \right] \] (5)
and these magnetic perturbations become unstable if the condition
\[ \frac{2}{3} \kappa_n < \kappa_T \] (6)
holds. Note that the local approximation requires \( \kappa_n, \kappa_T << k_y \). It has been assumed that \( \nabla \cdot E_1 = 0 \) and \( \nabla \cdot v_{e1} \neq 0 \) in the description of MEDV mode along with \( \omega_{pi} << \omega \).

Equation (1) indicates that the term \( \nabla p_e = \nabla (n_0 T_e) = T_e \nabla n_0 + n_0 \nabla T_e \) will produce a linear term with \( \nabla \) replaced by \( \mathbf{k} \) and hence \( E_1 \) can have a longitudinal component with \( \nabla \cdot E_1 \neq 0 \). Thus the mode can not be a pure transverse mode.

Moreover, the linear theory has been applied under the local approximation therefore the term, \( \left( \frac{\kappa_n}{k_y} \right)^2 v_{Te}^2 k_y^2 \) can be closer to \( c_s^2 k_y^2 \) where \( c_s = (T_e/m_i)^{1/2} \) is the ion acoustic speed while \( C_0 < 1 \) always holds. Using the laser plasma parameters [9] \( T_e = 100eV \) and \( n_0 \sim 10^{20} cm^{-3} \), one obtains \( \omega < \omega_{pi} \) contrary to initial assumption of stationary ions for \( \omega_{pi} << \omega \).

Recently [14], it has been shown that if compressibility effects are also taken into account then one obtains a new partially transverse and partially longitudinal normal mode of a nonuniform pure electron plasma as,
\[ \omega^2 = \frac{2}{3H_0} \lambda_e^2 k_y^2 (v_{Te}^2 \kappa_n^2) \left( 1 - \frac{3 \kappa_T}{2 \kappa_n} \right) \] (7)
where \( H_0 = \left[ \left\{ 1 + \frac{5}{3} \lambda_D e k_y^2 \right\} a - \kappa_n^2 / k_y^2 \right] \) and \( a = (1 + \lambda_D e k_y^2) \).

It is important to note that ions can be assumed to be stationary in the limit \( \frac{m_e}{m_i} \rightarrow 0 \). Since for hydrogen plasma \( \frac{m_e}{m_i} \sim 10^{-3} \), therefore (7) is valid for \( m_e/m_i < \lambda_D e k_y^2, \kappa_n^2 / k_y^2 \) and \( \omega^2 << \omega_{pe}^2 \).

But the important point is that the term \( \nu_{Te}^2 \kappa_n^2 \) can be closer to \( c_s^2 k_y^2 \) and lesser than \( \omega_{pi}^2 \) therefore the dynamics of ions should not be ignored.

It is also interesting to mention here that ion acoustic wave (IAW) has always been treated as a low frequency electrostatic mode. The reason is that in the limit \( \frac{m_e}{m_i} \rightarrow 0 \), the inertia-less electrons are assumed to follow the Boltzmann density distribution in the electrostatic field \( E = -\nabla \varphi \) as,
\[ \frac{n_e}{n_0} \sim e^{-e\varphi/T_e} \] (8)
In an inhomogeneous plasma we may have \( \frac{m_e}{m_i} < \kappa_n^2 / k_y^2 \) and in this case electron inertia should not be neglected. Then for \( \frac{m_e}{m_i} < \lambda_D e k_y^2 \), the IAW follows the dispersion relation [24],
\[ \omega^2 = c_s^2 k_y^2 \left( \frac{a - \kappa_n^2 / k_y^2}{ab - \kappa_n^2 / k_y^2} \right) \] (9)
where \( b = (1 + \lambda_D^2 k_y^2) \). Hence inhomogeneous plasmas can have a low frequency electromagnetic wave on ion time scale.

When the electron temperature perturbation effect is taken into account, then modes described in (7) and (9) will couple to produce a partially longitudinal and partially transverse wave with the dispersion relation [24],

\[
\omega^2 = \frac{5}{3H_0} \left[ (\lambda_e^2 k_y^2) v_{Te}^2 \kappa_n^2 \left( \frac{2}{3} - \frac{\kappa_T}{\kappa_n} \right) + \left( a - \frac{\kappa_n^2}{k_y^2} \right) \right] c_s^2 k_y^2 \left\{ \frac{5}{3} - \left( \frac{k_T^2}{k_y^2} + \frac{\kappa_T \kappa_n}{k_y^2} \right) \right\} \tag{10}
\]

If \( \nabla p_{e0} = 0 \) is used as the steady state condition the above equation yields a basic low frequency electromagnetic wave of inhomogeneous unmagnetized plasmas with the dispersion relation,

\[
\omega^2 = \frac{5}{3H_0} \left[ (\lambda_e^2 k_y^2) v_{Te}^2 \kappa_n^2 + \left( a - \frac{\kappa_n^2}{k_y^2} \right) c_s^2 k_y^2 \right] \tag{11}
\]

This wave has not been studied in plasmas so far. In our opinion it can play very important role in the generation of magnetic fluctuations in unmagnetized plasmas due to several linear and nonlinear mechanisms. In a pure electron plasma where ions are assumed to be stationary, (10) reduces to

\[
\omega^2 = \frac{5}{3H_0} (\lambda_e^2 k_y^2) v_{Te}^2 \kappa_n^2 \tag{12}
\]

But in our point of view the electron plasma wave frequency in (12) is near ion acoustic wave frequency \( c_s^2 k_y \) and hence it couples with it.

Now we look at EMHD theory for the generation of ‘seed’ magnetic field which is steadily growing. Again ions are assumed to be stationary in the time scale \( \tau \ll \omega_{pi}^{-1} \). In addition to this the electron inertia is also neglected assuming \( \omega_{pe}^{-1} \ll \tau \). Then electron equation of motion becomes,

\[
0 \simeq -eE - \frac{\nabla p_e}{n} \tag{13}
\]

The faraday law is,

\[
\partial_t \mathbf{B} = -c \nabla \times \mathbf{E} \tag{14}
\]

If it is assumed that \( \nabla n_0 = \hat{x} \left| \frac{dn_{e0}}{dx} \right| \) and \( \nabla T_0 = \hat{y} \left| \frac{dT}{dx} \right| \), then (13) and (14) yield,

\[
\partial_t \mathbf{B} = -\frac{c}{e} \left( \frac{T_e}{L_n L_T} \right) \hat{z} \tag{15}
\]
where $L_n = \kappa_n^{-1}$ and $L_T = \kappa_T^{-1}$ are constants.

Equation (15) is integrated from $\tau = 0$ to $\tau = \frac{L_n}{c_s}$ to have [9 - 11],

$$
B = \left\{ \frac{c}{e} \left( \frac{T_e}{L_n L_T} \right) \tau \right\} \hat{z}
$$

(16)

This is a well-known equation in laser-plasma literature. Assuming $T_e \sim 100eV$, $n_0 \sim 10^{20}cm^{-3}$, $L_n \sim L_T \sim 0.005cm$, one obtains $c_s \sim 3 \times 10^7 cm/Sec$ and hence $|B| \sim 0.6 \times 10^6$ Gauss [9]. Note that $\tau = \frac{L_n}{c_s} \simeq 1.66 \times 10^{-10}$ Sec and $\omega_{pi} \sim 1.3 \times 10^{13} rad/Sec$ while the laser pulse duration is of the order of a nano second. Thus we have $\omega_{pi}^{-1} << \tau$ contrary to initial assumption of stationary ions for $\tau << \omega_{pi}^{-1}$. An equation similar to (16) has also been used to estimate the 'seed' magnetic field generated by ionized clump of a galactic cloud [7]. The density gradient of the cloud has been assumed to have exponential form.

The brief overview of EMHD models shows clearly that the theoretical models for both the fluctuating and steadily growing magnetic fields suffer from serious contradictions. Since the EMHD is still being used [15] for estimating magnetic fields produced in laser plasmas, therefore the weaknesses and contradictions have been elaborated here again.

Since the plasmas have generally exponential density profiles, therefore there is a need to find out an exact 2-D solution of the set of two fluid equations assuming exponential type density structure in a plane. It may also be mentioned here that in many tokamak plasmas, the density falls almost exponentially near the walls which gives rise to drift waves.

### III. EXACT SOLUTION OF 2-FLUID EQS.

Our aim is to search for an exact analytical solution of the set of highly nonlinear partial differential equations of electron-ion plasma to show how the system from a non-equilibrium state can evolve in time generating the 'seed’ magnetic field.

In our opinion the same physical mechanism is applicable at both astrophysical and laboratory scales. Biermann [6] gave the pioneering idea that the electron baroclinic vector $(\nabla n_e \times \nabla T_e)$ can generate magnetic fields in rotating stars. We just modify it a little by proposing that the 'seed’ magnetic field is a macroscopic phenomenon and it is generated on longer spatial and temporal scales. Hence the ion dynamics can not be neglected.
Therefore, the ion baro-clinic vector \( (\nabla n_i \times \nabla T_i) \) is also crucial to be considered. However, the quasi-neutrality approximation in slow time scale is a valid approximation, therefore we use \( n_i \sim n_e = n \). In thermal equilibrium, the source of magnetic field generation disappears.

For an analytical solution of a complex set of equations, we have to use some assumptions and approximations. The exact solution presented here contains exponential type density profile in xy-plane which is the main deviation from the previous models [17, 19]. It is important to note that the nonlinear terms do not vanish if we assume exponential density fall or rise along both the axes x and y as has been discussed in section II.

Therefore, we have to choose a very special exponential function in xy-plane for density which reduces the nonlinear equations into two linear equations. The detailed mathematical model is presented here.

The electrons are assumed to be inertialess in the limit \( |\partial t| \ll \omega_{pe}, c|\nabla| \) where \( \omega_{pe} = \left( \frac{4\pi n_0 e^2}{m_e} \right)^{\frac{1}{2}} \) is the electron plasma frequency and c is the speed of light. We define four scalar fields \( \varphi, u, \chi \) and h such that the ion velocity \( v_i \) and magnetic field are defined, respectively, as [17, 19],

\[
v_i = (\nabla \varphi \times \hat{z} + u\hat{z}) f(t) = (\partial_y \varphi, -\partial_x \varphi, u) f \tag{17}
\]

\[
B = (\nabla \chi \times \hat{z} + h\hat{z}) f(t) = (\partial_y \chi, -\partial_x \chi, h) f \tag{18}
\]

All these scalar fields are functions of (x, y) coordinates and f is a function of time.

We further assume \( \partial_t n_j = 0 \) and \( \nabla \cdot \mathbf{v}_j = 0 \) which requires

\[
\nabla \psi \cdot \mathbf{v}_j = \{\varphi, \psi\} = 0 \tag{19}
\]

where \( \psi = \ln \bar{n}, \ n_e \simeq n_i = n \) and \( \{\varphi, \psi\} = \partial_y \varphi \partial_x \psi - \partial_x \varphi \partial_y \psi \). Here \( \bar{n} = \frac{n(x,y)}{N_0} \) and \( N_0 \) is an arbitrary number used to normalize \( n \).

The displacement current is neglected and hence we obtain,

\[
\mathbf{v}_e = \left( \mathbf{v}_i - \frac{c}{4\pi e} \frac{\nabla \times B}{n} \right) \tag{20}
\]

Let \( \mathbf{E} = -\nabla \Phi - \frac{1}{c} \partial_t \mathbf{A} \) where \( \Phi \) is electrostatic potential different from \( \varphi \). Since \( \mathbf{B} = 0 \) at \( t=0 \), therefore we do not normalize the equations. This point has been explained in detail in Ref. [19]. The curls of momentum equations of electrons and ions yield, respectively,

\[
\partial_t \mathbf{B} + \nabla \times \left[ \mathbf{B} \times \left( \mathbf{v}_i - \frac{c}{4\pi e} \frac{\nabla \times B}{n} \right) \right] = -\frac{c}{e} (\nabla \psi \times \nabla T_e) \tag{21}
\]
and
\[ \partial_t (a \mathbf{B} + \nabla \times \mathbf{v}_i) - \nabla \times [a(\mathbf{v}_i \times \mathbf{B}) + \mathbf{v}_i \times (\nabla \times \mathbf{v}_i)] \]
\[ = \frac{1}{m_i} (\nabla \psi \times T_i) \]  \hspace{1cm} (22)
where \( a = \frac{e}{m_i c} \). If the conditions
\[ \{ \varphi, \chi \} = \{ \varphi, u \} = \{ h, \varphi \} = 0 \]  \hspace{1cm} (23)
satisfy along with
\[ \{ \nabla^2 \varphi, \varphi \} = 0 \]  \hspace{1cm} (24)
then all the nonlinear terms of (21) and (22) vanish and they reduce, respectively, to simpler equations
\[ \partial_t \mathbf{B} = -\frac{e}{e} (\nabla \psi \times \nabla T_e) \]  \hspace{1cm} (25)
and
\[ \partial_t (a \mathbf{B} + \nabla \times \mathbf{v}_i) = \frac{1}{m_i} (\nabla \psi \times \nabla T_i) \]  \hspace{1cm} (26)
where \( T_e \neq T_i \) and right hand sides of (25) and (26) are the source terms for generating magnetic field and plasma flow. Let us assume that \( \mathbf{B} \) is related with plasma vorticity through the following equation,
\[ \mathbf{B} = \alpha (\nabla \times \mathbf{v}_i) \]  \hspace{1cm} (27)
where \( \alpha \) is a constant. Then (26) becomes
\[ (a + \alpha^{-1}) \partial_t \mathbf{B} = \frac{1}{m_i} (\nabla \psi \times \nabla T_i) \] \hspace{1cm} (28)
Now we discuss an important point of the present theoretical model. In the previous works it was assumed that the field \( \varphi \) satisfies the Poisson equation,
\[ \nabla^2 \varphi = -\lambda \varphi \] \hspace{1cm} (29)
where \( \lambda \) is a constant and \( 0 < \lambda \) holds. The forms of \( \psi \) and \( T_j \) were chosen as \( \psi = \psi_0 e^{\mu_1 x} \cos \mu_2 y \) and \( T_j = \{ T_{00j} + T'_{0j} (y - z) \} f(t) \) where \( \psi_0, \mu_1, \mu_2, T_{00j} \) and \( T'_{0j} \) were constants. We, here, want to find out a 2-D solution in the exponential form without assuming the density gradient to be constant. For this purpose the assumption (29) is modified as
\[ \nabla^2 \varphi = \lambda \varphi \] \hspace{1cm} (30)
and we assume $0 < \lambda$. The form of $\psi_{(x,y)}$ can be chosen like,

$$
\psi_{(x,y)} = A_1 e^{(\mu x + \nu y)} + A_2 e^{(\mu x - \nu y)} = \psi_1 + \psi_2 = \ln \bar{n}
$$

(31)

where $\bar{n} = \frac{n_{(x,y)}}{N_0}$ is dimensionless and $N_0$ is some constant density. Here $A_1$, $\mu$, $\nu$, $A_2$ are constants and

$$
\lambda = \mu^2 + \nu^2
$$

(32)

We may choose $0 < \mu, \nu$ for simplicity. Let the temperatures be only functions of space in yz-plane as,

$$
T_{0j}(y, z) = T_{00j} + T'_{0j}(y - z)
$$

(33)

Then the baroclinic vectors in (25) and (26) become constant with respect to time. These equations can be integrated from $t=0$ to $\tau$ and one obtains,

$$
B = - \frac{c}{e} (\nabla \psi \times \nabla T) \tau
$$

(34)

and

$$
B = \frac{1}{m_1 (a + \alpha^{-1})} (\nabla \psi \times \nabla T_i) \tau
$$

(35)

These equations relate $T'_{e0}$ and $T'_{i0}$ as,

$$
T'_{e0} = \frac{a}{(a + \alpha^{-1})} T'_{i0}
$$

(36)

Equations (31) and (33) yield

$$
(\nabla \psi \times \nabla T_j) = -T'_{0j} (\partial_{yj} \psi, -\partial_{xj} \psi, -\partial_{xj} \psi)
$$

(37)

where $\partial_{yj} \psi = \nu (\psi_1 - \psi_2)$ and $\partial_{xj} \psi = \mu \psi$.

We are free to use any of the equations, (34) or (35) to estimate $B$. Let us choose (34) and use (18) for $f=1$ to find out following relations,

$$
\chi = (\chi_0 \psi) \tau
$$

(38)

and

$$
h = -(h_0 \psi) \tau
$$

(39)

where $\chi_0 = \left( \frac{e_{0i}^{T_{e0}}}{e} \right)$ and $h_0 = \mu \chi_0$. Equation (17) for $f=1$ along with (30) gives,

$$
\nabla \times \mathbf{v}_i = (\partial_{y} u, -\partial_{x} u, -\lambda \phi)
$$

(40)
Then (27) yields,

\[ u = (u_0 \psi) \tau \]  

(41)

and

\[ \varphi = (\varphi_0 \psi) \tau \]  

(42)

where \( u_0 = \frac{\chi_0}{\alpha} \) and \( \varphi_0 = \left( \frac{\mu}{\alpha \lambda} \right) \chi_0. \)

Three dimensional magnetic field \( \mathbf{B} \) and \( \mathbf{v}_i \) can be expressed explicitly for \( A_1 \neq A_2 \) as,

\[
\mathbf{B} = \begin{bmatrix}
\chi_0 \nu (A_1 e^{(\mu x + \nu y)} + A_2 e^{(\mu x - \nu y)}) \\
-\mu \chi_0 (A_1 e^{(\mu x + \nu y)} + A_2 e^{(\mu x - \nu y)}) \\
-h_0 (A_1 e^{(\mu x + \nu y)} + A_2 e^{(\mu x - \nu y)})
\end{bmatrix} \psi_0
\]  

(43)

and

\[
\mathbf{v}_i = \begin{bmatrix}
\nu \varphi_0 (A_1 e^{(\mu x + \nu y)} - A_2 e^{(\mu x - \nu y)}) \\
-\mu \varphi_0 (A_1 e^{(\mu x + \nu y)} + A_2 e^{(\mu x - \nu y)}) \\
-u_0 (A_1 e^{(\mu x + \nu y)} + A_2 e^{(\mu x - \nu y)})
\end{bmatrix} \psi_0
\]  

(44)

Hence all the scalar fields \( \chi, h, u \) and \( \varphi \) become functions of \((x,y)\) through \( \psi \) which is externally given. The complicated nonlinear terms of two-fluid equations vanish and we obtain two simple and beautiful linear equations (34) and (35). These equations show that the terms of electrons and ions \((\nabla \psi \times \nabla T_j)\) (for \( j = e, i \)) become the source for the 'seed' magnetic field and plasma flow \( \mathbf{v}_i \).

**IV. GENERAL APPLICATIONS**

It has been shown that the forms of \( \psi(x,y) \) and \( T_{0j}(y,z) \) given in equations (31) and (33), respectively, reduce the set of nonlinear two fluid partial differential equations into two simpler linear equations (34) and (35) under certain conditions mentioned in the previous section. This theoretical model shows that the non-parallel density and temperature gradients can create magnetic fields and flows in initially unmagnetized plasmas.
Our aim is to apply this model to a system which has exponential-type of density profile as for example in the case of a laser plasma. But the chosen form of $\psi(x, y)$ in (31) with the definition $\psi = \ln \bar{n}$ gives,

$$\bar{n} = \frac{n(x,y)}{N_0} = \exp \left[ A_0 \left\{ e^{(\mu x + \nu y)} + e^{(\mu x - \nu y)} \right\} \right]$$

(45)

where $A_0 = A_1 = A_2$ has been assumed.

It looks as if density $n(x,y)$ has a profile like double exponential in (x,y) plane. Such a steep density variation is not interesting for physical applications, in general. We show here that the density variation can become very similar to exponential form in (x,y) plane by choosing suitable values of the constants $N_0, \mu$ and $\nu$ along with $A_0$. Let us consider an inhomogeneous plasma rectangle in (x,y) plane with four corner points $(0,0)$, $(x_m,0)$, $(0,y_m)$ and $(x_m,y_m)$ where $x_m$ and $y_m$ are the maximum lengths of the system along x and y axis, respectively. Then choose the constants in such a way that the density $n(x,y)$ at $(x_m,y_m)$ will be almost $e$-times (or a little larger) than the value at $(0,0)$, while the density at $(x_m,0)$ and $(0,y_m)$ will be somewhat lesser than $e$-times the density at $(0,0)$.

Such a density function is acceptable physically. For example, in previous EMHD model, the density was chosen to be an exponential function only along x-axis as $n(x) = n_0 e^{x/L_n}$ where $L_n$ is the density scale length and $n_0$ is the magnitude of density at $x=0$ [9]. In our case, $n$ depends upon two coordinates x and y. It’s profile depends upon the values of the constants. Therefore this theoretical model can be applicable to many inhomogeneous plasma systems.

Note that $\psi = \ln \bar{n}$ and if $0 < A_0 < 1$, $0 \leq \mu x + \nu y < 1$ and $0 \leq \mu x - \nu y < 1$, then $\psi$ can have values which give $e^\psi$ of the order of $e^{(1)}$, but not exactly $e^{(1)} \simeq 2.7$ at all points because $\psi$ changes with x and y.

In the next section we shall apply the model to laser plasmas as an example and our point of view will become clearer. The values of the constants will be chosen to show how it works for relatively smooth density profiles in (x,y)-plane. It seems important to point out that any one of $\psi_1$ and $\psi_2$ in (31) should not be much smaller than the other while choosing the constants. If one of them is negligibly small then the solution becomes one-dimensional.

**V. LASER PLASMA**

Here we apply our theoretical model to estimate magnetic field $B$ and the plasma flow $v_i$ in a non-uniform classical laser plasma. Consider a finite plasma rectangle with four corner
points (0, 0), (x_m, 0), (0, y_m) and (x_m, y_m) in (x, y) plane as has been mentioned in previous section. We may assume \( \mu x \) and \( \nu y \) to vary in this finite plasma as,

\[
\mu x : 0 \rightarrow 0.5 = \mu x_m \quad (46a)
\]

and

\[
\nu y : 0 \rightarrow 0.7 = y_m \quad (46b)
\]

Then at (0, 0), we have, \( \overline{n}(0,0) = \frac{n(0,0)}{N_0} \) and \( N_0 \) is chosen such that \( \overline{n} \neq 1 \) or \( \overline{n} \neq 1 \) because density \( n(x,y) \) should be neither zero nor negative. Therefore, we choose \( \frac{n(0,0)}{N_0} = 3 \) which gives, \( \Psi(0,0) \simeq 1.1 \) and due to (45) we find \( A_0 \simeq 0.55 \). If density is of the order of \( 10^{20} cm^{-3} \), then we may assume \( N_0 = 10^{20} cm^{-3} \) and hence \( n(0,0) = 3 \times 10^{20} cm^{-3} \). Or we may assume \( N_0 = 10^{19} cm^{-3} \) and hence \( n(0,0) = 3 \times 10^{19} cm^{-3} \) while \( n(x_m,y_m) \) will turn out to be nearly \( 10^{20} cm^{-3} \). In laser-plasmas, \( L_n = 50 \times 10^{-6} m = L_T \) was assumed in estimating \( |B| \) [8, 9] (where \( L_n \) and \( L_T \) are scale lengths of density and electron temperature along x and y co-ordinates, respectively). In these studies, the \( \nabla n \) was along x-axis and \( \nabla T_e \) was along y-axis only. Then \( L_n \simeq L_T \) was also assumed. For the sake of generality we do not assume \( \mu = \nu \). Instead let \( \nu = 1.5\mu \). In this case we estimate \( \overline{n}, \Psi \) and \( B \) at 4-corner points of the plasma rectangle as follows:

\[
\overline{n}(0,0) = 3, \Psi(0,0) \simeq 1.1
\]

\[
B_{(0,0)} = (0, -1.1, -1.1)9.9 \times 10^5 Gauss \quad (47a)
\]

\[n(x_m,0) = 6 : \Psi(x_m,0) \simeq 1.79\]

\[
B_{(x_m,0)} = (0, -1.79, -1.79)9.9 \times 10^5 Gauss \quad (47b)
\]

\[
\overline{n}(0,y_m) = 3.98; \Psi(y_m,0) = 1.38
\]

\[
B_{(0,y_m)} = (1.24, -1.38, -1.38)9.9 \times 10^5 Gauss \quad (47c)
\]

\[n(x_m,y_m) \simeq 9.48; \Psi(x_m,y_m) \simeq 2.25\]

\[
B_{(x_m,y_m)} \simeq (2.73, -2.25, -2.25)9.9 \times 10^5 Gauss \quad (47d)
\]

These values of \( |B| \) are almost in agreement with the observation [9]. Then we can express,

\[
v_{i(x,y)} = 6 \times 10^7(-0.46, 0.3, -1)\Psi_{(x,y)}cm/sec \quad (48)
\]
If we look at the values of density, we notice that density \( n(x,y) \) at \((x_m, y_m)\) is

\[
n(x_m, y_m) \simeq \{n(0,0)\}\ (3.16)
\]

which is little larger than \( e \)-times the density at \((0, 0)\). Therefore, the density has not been chosen as a double exponential function. We are mainly interested in the order of magnitudes of \(|B|\) and \(|v_i|\) which mainly depends upon our choice of constants.

V. DISCUSSION

It is important to note that the ’seed’ magnetic field generation can not be explained on the basis of magnetohydrodynamics (MHD). There must be a source like thermal energy which converts into magnetic energy. Biermann [6] proposed that the electron baroclinic vector \((\nabla n_e \times \nabla T_e)\) can generate the magnetic fields in rotating stars. Then based on this idea, a very simple model; the electron magnetohydrodynamics (EMHD) was presented to explain the generation of magnetic field in laser-induced plasmas. Later on, the magnetic electron drift vortex (MEDV) mode was discovered using EMHD. This was believed to be a low frequency pure transverse normal mode of electron plasma [12, 13]. The EMHD is also not a convincing theoretical model for the generation of ’seed’ magnetic field. The MEDV mode description also suffers from contradictions.

On the other hand, the two-fluid model is too complicated. The numerical simulation of these equations is a complex problem. But the analytical 3-D solution is also not straightforward. However, a two-dimensional solution has been presented using physical and consistent assumptions and approximations. It is important to note that the present model is different from the previous works because it shown that

1. ion dynamics play a crucial role
2. the baroclinic vectors generate not only the magnetic field but plasma flow as well

An exact 2-D solution of the two-fluid equations was also found a few years ago [19], but it had a serious weakness. The density gradient was assumed to follow sinusoidal behavior contrary to common observations. Since, it was the first effort to get an exact solution of the two fluid equations, therefore it was presented for the interest of researchers working in the field.
The present 2-D exact solution of two-fluid equations is in the form of exponential function of x and y coordinates. This structure of density gradient is more physical. Since all fields become linear function of $\psi$, therefore all fields have the similar spatial structure. This solution is applicable to both astrophysical and laser-induced plasmas in our opinion. The present investigation suggests that instead of EMHD, the numerical simulation of two-fluid equations will be very useful for understanding the mechanism for the generation of ‘seed’ magnetic field in different systems with different profiles of density and temperatures. For analytical solution, we have to choose very special forms of density and temperature gradients. This theoretical model can be very useful for further studies in astrophysical and laser plasmas.

[1] L. M. Widrow, Rev. Mod. Phys. **74**, 775 (2002).
[2] L. Mestel and K. Subramanian, Mon. Not. R. Astron. Soc. **265**, 649 (1993).
[3] E. N. Parker, Cosmical Magnetic Fields (Clarendon, Oxford 1979)
[4] E. G. Blackman, Astrophysical J. **529**, 138 (2000).
[5] A. Brandenburg and K. Subramanian, Astrophysical Magnetic Fields and Nonlinear Dynamo Theory, Physics Reports 417, 1-209 (2005).
[6] L. Biermann, Z. Naturforsch. **5A**, 65 (1950).
[7] A. Lazarian, Astron. Astrophys. **264**, 326 (1992).
[8] J. A. Stamper, K. Papadopoulos, R. N. Sudan, S. O. Dean, E. A. Mclean, and J. W. Dawson, Phys. Rev. Lett. **26**, 1012 (1971).
[9] K. A. Brueckner and S. Jorna, Rev. Mod. Phys. **46**, 325 (1974).
[10] A. A. Kingsseps, K. V. Chukbar, V. V. Yan’Kov, in Reviews of Plasma Physics, edited by B. B. Kadomtsev (Cosultants Bureau, New York 1990), Vol. 16, p. 243.
[11] L. A. Bol’shov, A. M. Dykhne, N. G. Kowalski, and A. I. Yudin, in Handbook of Plasma Physics, edited by M. N. Rosenbluth, and R. Z. Sagdeev (Elsevier Science, New York 1991), Vol. 3, p.519.
[12] R. D. Jones, Phys. Rev. Lett. **51**, 1269 (1963).
[13] M. Y. Yu and Xiao Chijin, Phys. Fluids **30**, 3631 (1987).
[14] H. Saleem, Phys. Plasmas 16, 082102 (2009); H. Saleem in New Developments in Nonlinear Plasma Physics, Editors B. Eliasson and P.K. Shukla, Proc. ICTP Summer College on Plasma Physics and International Symposium on Cutting Edge Plasma Physics 10-28 August 2009, Trieste, Italy.

[15] C. A. Ceccetti, M. Borghesi, J. Fuchs, G. Schurtz, S. Kar, A. Macchi, L. Romagnani, P.A. Wilson, P. Antici, R. Jung, J. Osterholtz, C.A. Pipahl, O. Willi, A. Schiavi, M. Notley and D. Neely. 16, 043102 (2009).

[16] H. Saleem, Phys. Rev. E 54, 4469 (1996); H. Saleem, Phys. Rev. E 59, 6196 (1999).

[17] H. Saleem and Z. Yoshida, Phys. Plasmas 11, 4865 (2004).

[18] S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett. 81, 4863 (1998).

[19] H. Saleem, Phys. Plasmas 14, 072105 (2007).

Acknowledgement
The author is grateful to Professor Zensho Yoshida of Tokyo University for several useful discussions on this work at Abdus Salam-International Centre for Theoretical Physics (AS-ICTP), Trieste, Italy during the Summer College on Plasma Physics 10-28 August 2009.