Almost Kenmotsu manifolds admitting certain vector fields

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Abstract: In the present paper, we characterize almost Kenmotsu manifolds admitting holomorphically planar conformal vector (HPCV) fields. We have shown that if an almost Kenmotsu manifold $\mathcal{M}^{2n+1}$ admits a non-zero HPCV field $V$ such that $\phi V = 0$, then $\mathcal{M}^{2n+1}$ is locally a warped product of an almost Kaehler manifold and an open interval. As a corollary of this we obtain few classifications of an almost Kenmotsu manifold to be a Kenmotsu manifold and also prove that the integral manifolds of $\mathcal{D}$ are totally umbilical submanifolds of $\mathcal{M}^{2n+1}$. Further, we prove that if an almost Kenmotsu manifold with positive constant $\xi$-sectional curvature admits a non-zero HPCV field $V$, then either $\mathcal{M}^{2n+1}$ is locally a warped product of an almost Kaehler manifold and an open interval or isometric to a sphere. Moreover, a $(k, \mu)^{1}$-almost Kenmotsu manifold admitting a HPCV field $V$ such that $\phi V \neq 0$ is either locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^{n}$ or $V$ is an eigenvector of $h'$. Finally, an example is presented.

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1. Introduction

In the present time, the study of existence of Killing vector fields in Riemannian manifolds is a very interesting topic as they preserves a given metric and determine the degree of symmetry of the manifold. Conformal vector fields whose flow preserves a conformal class of metrics are very important in the study of several kind of almost contact metric manifolds.

A smooth vector field $V$ on a Riemannian manifold $(\mathcal{M}, g)$ is said to be conformal vector field if there exist a smooth function $f$ on $\mathcal{M}$ such that

$$\mathcal{L}_V g = 2fg,$$

where $\mathcal{L}_V g$ is the Lie derivative of $g$ with respect to $V$. The vector field $V$ is called homothetic or Killing accordingly as $f$ is constant or zero. Moreover, $V$ is said to be closed conformal vector field if the metrically equivalent 1-form of $V$ is closed. If the conformal vector field $V$ is gradient of some smooth function $\lambda$, then $V$ is called gradient conformal vector field. The geometry of conformal vector fields have been investigated in ([5], [6]).

A vector field $V$ on a contact metric manifold $\mathcal{M}^{2n+1}(\phi, \xi, \eta, g)$ is said to be holomorphically planar conformal vector field if it satisfies

$$\nabla_X V = aX + b\phi X$$

(1.2)
for any vector field $X$, where $a, b$ are smooth functions on $M$. As a generalization of closed conformal vector fields Sharma [16] introduced the notion of holomorphically planar conformal vector (in short, HPCV) fields on almost Hermitian manifold. In [12], Ghosh and Sharma characterize an almost Hermitian manifolds admitting a HPCV field. They shows that if $V$ is strictly non-geodesic non-vanishing HPCV field on an almost Hermitian manifold, then $V$ is homothetic and almost analytic. Further Sharma [17] shows that among all complete and simply connected $K$-contact manifolds only the unit sphere admits a non-Killing HPCV field and a $(k, \mu)$-contact manifold admitting a non-zero HPCV field is either Sasakian or locally isometric to $E^3$ or $E^{n+1} \times S^n(4)$. In [11], Ghosh studied HPCV fields in the framework of contact metric manifolds under certain conditions and proved that a contact metric manifold with pointwise constant $\xi$-sectional curvature admitting a non-closed HPCV field $V$ is either $K$-contact or $V$ is homothetic.

Motivated by the above studies we consider HPCV fields in the framework of a special type of almost contact metric manifolds, called almost Kenmotsu manifolds. The paper is organized as follows:

In section 2, we present some preliminary notions on almost Kenmotsu manifolds existing in the literature. Section 3 deals with HPCV fields on almost Kenmotsu manifolds and section 4 is associated to the study of HPCV fields on $(k, \mu)'$-almost Kenmotsu manifolds.

2. Preliminaries

An almost contact structure on a $(2n + 1)$-dimensional smooth manifold $M^{2n+1}$ is a triplet $(\phi, \xi, \eta)$, where $\phi$ is a $(1,1)$-tensor, $\xi$ is a global vector field and $\eta$ is a 1-form satisfying ([1], [2]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where $I$ denote the identity endomorphism. Here also $\phi \xi = 0$ and $\eta \circ \phi = 0$ hold; both can be derived from (2.1) easily.

If a manifold $M$ with a $(\phi, \xi, \eta)$-structure admits a Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X, Y$ on $M^{2n+1}$, then $M^{2n+1}$ is said to be an almost contact metric manifold. The fundamental 2-form $\Phi$ on an almost contact metric manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for any vector fields $X, Y$ on $M^{2n+1}$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor $N_\phi$, defined by

$$N_\phi = [\phi, \phi] + 2d\eta \otimes \xi,$$

where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ [1]. Recently in ([8], [9], [10]), almost contact metric manifold such that $\eta$ is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. For more details on almost Kenmotsu manifolds we refer the reader to go through the references ([4], [9], [10]). Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$
for any vector fields $X, Y$. Let the distribution orthogonal to $\xi$ is denoted by $\mathcal{D}$, then $\mathcal{D} = Im(\phi) = Ker(\eta)$. Since $\eta$ is closed, $\mathcal{D}$ is an integrable distribution.

The study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of $k$-nullity distribution was introduced by Gray [13] and Tanno [18] in the study of Riemannian manifolds. Blair, Koufogiorgos and Papantonio [3] introduced the generalized notion of the $k$-nullity distribution, named the $(k, \mu)$-nullity distribution on a contact metric manifold. In [8], Dileo and Pastore introduce the notion of $(k, \mu)'$-nullity distribution, another generalized notion of the $k$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu)' = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)h'X - g(X, Z)h'Y]\},$$

(2.2)

where $h' = h \circ \phi$.

Let $M^{2n+1}$ be an almost Kenmotsu manifold with structure $(\phi, \xi, \eta, g)$. The Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \phi = 0$. We denote by $h = \frac{1}{2}\xi \phi$ and $l = R(\cdot, \xi)\xi$ on $M^{2n+1}$. The tensor fields $l$ and $h$ are symmetric operators and satisfy the following relations [15]:

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0,$$

(2.3)

We also have the following formulas given in (8 - 10)

$$\nabla_X \xi = X - \eta(X)\xi - \phi hX,$$

(2.4)

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,$$

(2.5)

$$(\nabla_X \phi)Y - (\nabla_Y \phi)\phi Y = -\eta(Y)\phi X - 2g(X, \phi Y)\xi - \eta(Y)hX,$$

(2.6)

for any $X, Y$ on $M^{2n+1}$. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with $\phi$ and $h'\xi = 0$. Also it is clear that $(8, 20)$

$$h = 0 \iff h' = 0, \ h^2 = (k + 1)\phi^2 \iff h^2 = (k + 1)\phi^2.$$

(2.7)

3. HPCV fields on almost Kenmotsu manifolds

In this section we characterize almost Kenmotsu manifolds admitting a holomorphically planar conformal vector field $V$. Before proving our main theorems we first state and prove the following lemma.

**Lemma 3.1.** Let $M^{2n+1}$ be an almost Kenmotsu manifold admitting a HPCV field $V$. Then the following relation

$$\phi V a = 4nb\eta(V) + (\xi b)\eta(V) - Vb$$

holds on $M^{2n+1}$.

**Proof:** Differentiating (1.2) covariantly along any vector field $Y$, we have

$$\nabla_Y \nabla_X V = a(\nabla_Y X) + (Ya)X + b(\nabla_Y \phi X) + (Yb)\phi X.$$ 

(3.1)

Interchanging $X$ and $Y$ in the above equation, we get

$$\nabla_X \nabla_Y V = a(\nabla_X Y) + (Xa)Y + b(\nabla_X \phi Y) + (Xb)\phi Y.$$ 

(3.2)

Replacing $X$ by $[X, Y]$ in (1.2) yields

$$\nabla_{[X,Y]} V = a(\nabla_X Y) - a(\nabla_Y X) + b(\nabla_X \phi Y) - b(\nabla_Y X).$$ 

(3.3)
Now using $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ gives
\begin{equation}
R(X,Y)V = (Xa)Y - (Ya)X + (Xb)\phi Y - (Yb)\phi X \\
+ b[(\nabla_X^\phi)Y - (\nabla_Y^\phi)X].
\end{equation}
(3.4)

Putting $X = \phi X$ and $Y = \phi Y$ in (3.4) we get
\begin{align*}
R(\phi X, \phi Y)V &= (\phi Xa)\phi Y - (\phi Ya)\phi X + (\phi Xb)[-Y + \eta(Y)\xi] \\
&\quad - (\phi Yb)[-X + \eta(X)\xi] + b[(\nabla_{\phi X}^\phi)\phi Y - (\nabla_{\phi Y}^\phi)\phi X].
\end{align*}
(3.5)

Now adding equations (3.4) and (3.5) and using (2.6) we have
\begin{align*}
R(X,Y)V + R(\phi X, \phi Y)V &= (Xa)Y - (Ya)X + (Xb)\phi Y - (Yb)\phi X \\
&\quad + (\phi Xa)\phi Y - (\phi Ya)\phi X - (\phi Xb)Y \\
&\quad + (\phi Xb)\eta(Y)\xi + (\phi Yb)X - (\phi Yb)\eta(X)\xi \\
&\quad + b[-\eta(Y)\phi X - 2g(X,\phi Y)\xi - \eta(Y)hX \\
&\quad + \eta(X)\phi Y + 2g(\phi X,\phi Y)\xi + \eta(X)hY].
\end{align*}
(3.6)

Taking inner product of (3.6) with $V$ and then substituting $X = \phi X$ and $Y = \phi Y$ yields
\begin{align*}
(\phi Xa)g(\phi Y, V) - (\phi Ya)g(\phi X, V) + (\phi Xb)\eta(Y)\eta(V) \\
- (\phi Yb)\eta(Y)\eta(V) \\
- [\eta(Y)\xi(a)[-g(X, \phi Y) + \eta(X)\eta(V)] \\
- [\eta(Y)\xi(b)[4g(\phi X, V) - 4b\phi Y, \phi Y)\eta(V) = 0.
\end{align*}
(3.7)

Now replacing $Y$ by $\phi Y$ in the foregoing equation we obtain
\begin{align*}
-g(\phi Da, X)[-g(Y, V) + \eta(Y)\eta(V)] + g(Da, Y)g(\phi X, V) - \eta(Y)(\xi a)g(\phi X, V) \\
+ g(\phi Db, X)g(Y, V) + g(Db, Y)[-g(X, V) + \eta(X)\eta(V)] - \eta(Y)(\xi b)[-g(X, V) \\
+ \eta(X)\eta(V)] + g(Db, Y)[-g(Y, V) + \eta(Y)\eta(V)] - \eta(X)(\xi b)[-g(Y, V) \\
+ \eta(Y)\eta(V)] - g(Db, Y)g(\phi X, V) + 4b\eta(X, \phi Y)\eta(V) - 4b\eta(X, \phi Y)\eta(V) = 0.
\end{align*}
(3.8)

Contracting $X$ and $Y$ in (3.8) we have
\[-2\phi V a - 2V b + 2(\xi b)\eta(V) + 8nb\eta(V) = 0,
\]
which implies
\[\phi V a = 4nb\eta(V) + (\xi b)\eta(V) - V b. \quad (3.9)\]
This completes the proof.

**Theorem 3.2.** If an almost Kenmotsu manifold $M^{2n+1}$ admits a non-zero HPCV field $V$ such that $\phi V = 0$, then $M^{2n+1}$ is locally a warped product of an almost Kaehler manifold and an open interval.

**Proof.** Let $M^{2n+1}$ be an almost Kenmotsu manifold admitting a non-zero HPCV field $V$ such that $\phi V = 0$. Operating $\phi$ on it we get
\[V = \eta(V)\xi. \quad (3.10)\]
Now using (3.10) and $\phi V = 0$ in Lemma 3.1 we have $4nb\eta(V) = 0$, which implies either $b = 0$ or $\eta(V) = 0$. If $\eta(V) = 0$, then from (3.10) we have $V = 0$, which is a contradiction to our hypothesis. Thus we get $b = 0$. 

Differentiating (3.10) covariantly along any vector field \( X \) and using \( b = 0, \phi V = 0, (1.2) \) and (2.4) we obtain
\[
a X = a \eta(X) \xi + g(X, V) \xi - 2\eta(X) \eta(V) \xi + \eta(V) X - \eta(V) \phi h X. \tag{3.11}
\]
Contracting \( X \) and using (2.3) in (3.11) yields \( a = \eta(V) \). Substituting the value of \( a \) in (3.11) we get
\[
g(X, V) \xi - \eta(X) \eta(V) \xi - \eta(V) \phi h X = 0. \tag{3.12}
\]
Replacing \( X \) by \( \phi X \) in the above equation and using the hypothesis \( \phi V = 0 \) and \( \eta(V) \neq 0 \) we infer that \( h X = 0 \) for any vector field \( X \) on \( M^{2n+1} \). The rest of the proof follows from Theorem 2 of [10].

Proposition 1 of [10] says that "In an almost Kenmotsu manifold \( M^{2n+1} \), the integral manifolds of \( D \) are totally umbilical submanifolds of \( M^{2n+1} \) if and only if \( h \) vanishes ". Hence, we can state the following:

**Corollary 3.3.** Let \( M^{2n+1} \) be an almost Kenmotsu manifold admitting a non-zero HPCV field \( V \) such that \( \phi V = 0 \). Then the integral manifolds of \( D \) are totally umbilical submanifolds of \( M^{2n+1} \).

**Corollary 3.4.** If a locally symmetric almost Kenmotsu manifold \( M^{2n+1} \) admits a non-zero HPCV field \( V \) such that \( \phi V = 0 \), then \( M^{2n+1} \) is a Kenmotsu manifold.

The above Corollary follows directly from Theorem 3 of [10].

Proposition 2.1 of [19] states that "Any 3-dimensional almost Kenmotsu manifold is Kenmotsu if and only if \( h \) vanishes ". Thus we arrive to the following:

**Corollary 3.5.** A 3-dimensional almost Kenmotsu manifold \( M^{2n+1} \) admitting a non-zero HPCV field \( V \) such that \( \phi V = 0 \) is a Kenmotsu manifold.

**Theorem 3.6.** Let \( M^{2n+1} \) be a complete almost Kenmotsu manifold admitting a non-zero HPCV field \( V \). If \( M^{2n+1} \) has positive constant \( \xi \)-sectional curvature, then either \( M^{2n+1} \) is locally a warped product of an almost Kaehler manifold and an open interval or isometric to a sphere.

**Proof.** If the sectional curvature \( K(\xi, X) = c \) of an almost Kenmotsu manifold is a positive constant, then we can easily obtain the following:
\[
R(\xi, X)\xi = -c[X - \eta(X)\xi]. \tag{3.13}
\]
Now putting \( X = \xi \) in (3.4) we have
\[
R(\xi, Y)V = (\xi a)Y - (Ya)\xi + (\xi b)\phi Y + b\phi Y + bh Y. \tag{3.14}
\]
Taking inner product of (3.14) with \( \xi \) we get
\[
g(R(\xi, Y)V, \xi) = (\xi a)\eta(Y) - (Ya). \tag{3.15}
\]
Again using (3.13) we have
\[
g(R(\xi, Y)V, \xi) = -g(R(\xi, Y)\xi, V) = c[g(Y, V) - \eta(Y)\eta(V)]. \tag{3.16}
\]
Hence from (3.15) and (3.16) we obtain
\[
Da - (\xi a)\xi + cV - c\eta(V)\xi = 0. \tag{3.17}
\]
Taking inner product of (3.14) with \( V \) we get
\[
(\xi a)V - \eta(V)(Da) - (\xi b)\phi V - b\phi V + bh V = 0. \tag{3.18}
\]
Eliminating $Da$ from (3.17) and (3.18) we have
\[-(\xi a)\phi^2 V - c\eta(V)\phi^2 V - (\xi b)\phi V - b\phi V + bh V = 0. \tag{3.19}\]
Now differentiating (3.17) covariantly along any vector field $X$ and then taking inner product of the resulting equation with $Y$ we infer
\[g(\nabla_X Da, Y) - (\xi a)[g(X, Y) - \eta(X)\eta(Y) - g(\phi hX, Y)] - (X(\xi a))\eta(Y) + c[a\phi g(X, Y) + b\phi g(X, Y)] - c\eta(V)[g(X, V) - \eta(X)\eta(V)] + a\eta(X)]
\[= -c\eta(V)[g(X, V) - \eta(X)\eta(V)] - g(\phi hX, Y)] = 0. \tag{3.20}\]
Antisymmetrizing the above equation and using the symmetry of the Hessian operator, that is, $\text{Hess}_a(X, Y) = g(\nabla_X Da, Y) = g(\nabla_Y Da, X)$ we obtain
\[(Y(\xi a))\eta(X) - (X(\xi a))\eta(Y) + 2bcg(\phi X, Y)
- c\eta(Y)g(X, V) + c\eta(X)g(Y, V) = 0. \tag{3.21}\]
Replacing $X$ by $\phi X$ and $Y$ by $\phi Y$ in (3.21) we get $2bcg(\phi X, Y) = 0$, which implies $b = 0$ as $c$ is non-zero constant by hypothesis. Then from (3.18) we have
\[(\xi a)V = (Da)\eta(V). \tag{3.22}\]
Also from (3.19) we obtain
\[[c\eta(V)\phi^2 V = 0, \tag{3.23}\]
which implies either $\phi^2 V = 0$ or $(\xi a) = -c\eta(V)$.

Case 1: If $\phi^2 V = 0$, then we have $V = \eta(V)\xi$ and this implies $\phi V = 0$. Thus from Theorem 3.2 we infer that $M^{2n+1}$ is locally a warped product of an almost Kaehler manifold and an open interval.

Case 2: If $(\xi a) = -c\eta(V)$, then from (3.22) we have
\[(Da + cV)\eta(V) = 0. \tag{3.24}\]
Now if $\eta(V) = 0$, then from (3.22) we have $\xi a = 0$ as $V$ is non-zero. Hence from (3.17) we get $Da = -cV$. Thus in either cases we obtain $Da = -cV$. Differentiating this covariantly along any vector field $X$ and using (1.2) we have $\nabla_X Da = -caX$. We are now in a position to apply Obata’s theorem [14]: ” In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a non-constant function $\lambda$ with $\nabla_X D\lambda = -c^2\lambda X$ for any vector $X$, it is necessary and sufficient that the manifold is isometric with a sphere $S^n(c)$ of radius $\frac{1}{\sqrt{c}}$ ” to conclude that the manifold is isometric to the sphere $S^{2n+1}(\sqrt{c})$ of radius $\frac{1}{\sqrt{c}}$.

**4. HPCV fields on a class of almost Kenmotsu manifolds**

In this section, we study HPCV fields on almost Kenmotsu manifolds with $\xi$ belonging to the $(k, \mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigen vector of $h'$ corresponding to the eigen value $\lambda$. Then from (2.7) it is clear that $\lambda^2 = -(k + 1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm \sqrt{-k - 1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigenvalue $\lambda$ and $-\lambda$ of $h'$, respectively. Before proving our main theorem in this section we recall some results:

**Lemma 4.1.** (Prop. 4.1 of [8]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $\xi$ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and $\text{Spec}(h') = \{0, \lambda, -\lambda\}$, with $0$ as simple eigen value and $\lambda = \sqrt{-k - 1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The
Proof.\ Substituting $V$ field $V$ manifold or Theorem 4.4.\ A (Lemma 4.2.\ X Equations $(8.3)$ and $g$ Taking inner product of $(8.4)$, $\lambda \in \mathbb{R}$, $X, Y \in \mathcal{D}$, $\lambda \in [-\lambda']$, $\lambda$ such that $\xi \in [\lambda']$ and $X, Y \in [-\lambda']$.\ $M^{2n+1}$ has constant negative scalar curvature $r = 2n(k - 2n)$.

Lemma 4.2. (Lemma 4.1 of [8]) Let $M^{2n+1}, \phi, \xi, \eta, \gamma$ be an almost Kenmotsu manifold with $h' \neq 0$ and $\xi$ belongs to the $(k, -2)'$-nullity distribution. Then, for any $X, Y \in \chi(M^{2n+1})$,

$$$(4.1) (\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X)$$$

Lemma 4.3. (Prop. 4.2 of [8]) Let $M^{2n+1}, \phi, \xi, \eta, \gamma$ be an almost Kenmotsu manifold such that $h' \neq 0$ and $\xi$ belongs to the $(k, -2)'$-nullity distribution. Then for any $X, Y, Z \in [\lambda']$ and $X, Y, Z \in [-\lambda']$, the Riemann curvature tensor satisfies:

$$R(X, Y)Z - \lambda = 0,$$

$$R(X, Y)Z = 0,$$

$$R(X, Y)Z - \lambda = (k + 2)g(X, Z)Y - \lambda,$$

$$R(X, Y)Z + \lambda = -(k + 2)g(Y, Z)X + \lambda,$$

$$R(X, Y)Z - \lambda = (k - 2\lambda)[g(Y, Z)X - g(X, Z)Y],$$

$$R(X, Y)Z + \lambda = (k + 2\lambda)[g(Y, Z)X - g(X, Z)Y].$$

From (2.2) we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(Y)h'Y], \tag{4.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (4.2)

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \tag{4.3}$$

Theorem 4.4. A $(k, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ admitting a HPCV field $V$ such that $\phi V \neq 0$ is either locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature $-4$ and a flat $n$-dimensional manifold or $V$ is an eigenvector of $h'$.

Proof. Substituting $X = \xi$ in (3.4) we have

$$R(\xi, Y)V = (\xi a)V - (Y a)\xi + (\xi b)\phi Y + b\phi Y + bh Y. \tag{4.4}$$

Taking inner product of (4.4) with $\xi$ we obtain

$$g(R(\xi, Y)V, \xi) = (\xi a)\eta(Y) - (Y a). \tag{4.5}$$

Making use of (4.2) we get

$$g(R(\xi, Y)V, \xi) = -g(R(\xi, Y)\xi, V) = -k\eta(Y)\eta(V) + kg(Y, V) - 2g(h'Y, V). \tag{4.6}$$

Equations (4.5) and (4.6) together implies

$$-k\eta(Y)\eta(V) + kg(Y, V) - 2g(h'Y, V) = (\xi a)\eta(Y) - (Y a), \tag{4.7}$$
which implies
\[-k\eta(V)\xi + kV - 2h'V = (\xi a)\xi - Da. \quad (4.8)\]

Now taking inner product of (4.4) with \( V \) gives
\[(\xi a)g(Y, V) - (Y a)\eta(V) + (\xi b)g(\phi Y, V) + bg(\phi Y, V) + bg(hY, V) = 0,
\]
which implies
\[(\xi a)V - (Da)\eta(V) - (\xi b)\phi V - b\phi V + bhV = 0. \quad (4.9)\]

Eliminating \( Da \) from (4.8) and (4.9) we have
\[-(\xi a)\phi^2V - k\eta(V)\phi^2V - 2\eta(V)h'V - (\xi b)\phi V - b\phi V + bhV = 0. \quad (4.10)\]

Differentiating (4.8) covariantly along any vector field \( X \) and using (1.2), (2.4), Lemma 4.2 and the value of \( \mu \) from Lemma 4.1 we infer
\[-k[g(X - \eta(X))\xi - \phi hX, V] + g(\xi, aX + b\phi X)]\xi - k\eta(V)[X - \eta(X)\xi - \phi hX] \nonumber
\]
\[+ k[aX + b\phi X] - 2[-g(h'X + h^2X, V)\xi - \eta(V)(h'X + h^2X) + h'(aX + b\phi X)] \nonumber
\]
\[= (\xi a)[X - \eta(X)\xi - \phi hX] + (X(\xi a))\xi - \nabla X Da. \nonumber\]

Taking inner product of the foregoing equation with \( Y \) we obtain
\[-k[g(X, V) - \eta(X)\eta(V) - g(\phi hX, V) + a\eta(X)]\eta(Y) - k\eta(V)[g(X, Y)] \nonumber
\]
\[+ \eta(X)\eta(Y) - g(\phi hX, Y)] + k[a g(X, Y) + bg(\phi X, Y)] - 2[-g(h'X + h^2X, V)\eta(Y) + g(\phi hX, Y)] \nonumber
\]
\[= (\xi a)[g(X, Y) - \eta(Y)\eta(X)] + (X(\xi a))\eta(Y) - g(\nabla X Da, Y(\xi a)). \quad (4.11)\]

Antisymmetrizing the above equation and using the symmetry of the Hessian operator, that is, \( \text{Hess}_a(X, Y) = g(\nabla X Da, Y) = g(\nabla Y Da, X) \) we obtain
\[-k[g(X, V)\eta(Y) - g(Y, V)\eta(X) - g(\phi hX, V)\eta(Y) + g(\phi hY, V)\eta(X)] \nonumber
\]
\[+ 2kbg(\phi X, Y) - 2[-g(h'X + h^2X, V)\eta(Y) + g(h'Y + h^2Y, V)\eta(X)] \nonumber
\]
\[= (X(\xi a))\eta(Y) - (Y(\xi a))\eta(X). \quad (4.12)\]

Putting \( X = \phi X \) and \( Y = \phi Y \) in the previous equation we infer that \( 2kbg(\phi X, Y) = 0, \) which implies \( b = 0 \) as \( k < -1. \) Hence from (4.9) we have
\[(\xi a)V = (Da)\eta(V). \quad (4.13)\]

Now letting \( Y \in [\lambda]' \) in (4.7) yields
\[(k - 2\lambda)g(Y, V) = -(Ya), \]
which implies
\[Da = (2\lambda - k)V \quad \text{and} \quad (\xi a) = (2\lambda - k)\eta(V). \quad (4.14)\]

Now using \( b = 0 \) and the value of \( (\xi a) \) from (4.14) in (4.10) we have
\[2(\lambda + 1)\eta(V)(h'V + \phi^2V) = 0, \quad (4.15)\]
which implies either \( \lambda = -1 \) or \( \eta(V) = 0 \) or \( h'V = -\phi^2V. \)

Case 1: If \( \lambda = -1, \) then from \( \lambda^2 = -(k - 1) \) we obtain \( k = -2. \) Now letting \( X, Y, Z \in [\lambda]' \) and noticing that \( k = -2, \lambda = -1, \) from Lemma 4.3 we have
\[R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0, \]
and
\[R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = -4[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}], \]
for any $X_{\lambda}, Y_{\lambda}, Z_{\lambda} \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$ it follows from Lemma 4.1 that $K(X, \xi) = -4$ for any $X \in [-\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [\lambda]'$. Again from Lemma 4.1 we see that $K(X, Y) = -4$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X, Y \in [\lambda]'$. As is shown in [8] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1 - \lambda)\xi$, where $H$ is the mean curvature tensor field for the leaves of $[-\lambda]'$ immersed in $M^{2n+1}$. Here $\lambda = -1$, then the two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in $M^{2n+1}$. Then we can say that $M^{2n+1}$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. 

Case 2: If $\eta(V) = 0$, then from (4.14) we have $(\xi a) = 0$. Then from (4.8) we have $Da = 2h'V - kV$. Now equating the value of $Da$ from this and (4.14) we get $h'V = \lambda V$. This shows that $V$ is an eigenvector of $h'$.

Case 3: If $h'V = -\phi V = V - \eta(V)\xi$, then applying $h'$ on both side of it we have $h^2V = h'V$. Hence using (2.7) we obtain $-(k + 2)(V - \eta(V)\xi) = 0$. Now $V - \eta(V)\xi \neq 0$ as $\phi V \neq 0$ by hypothesis. Therefore, we have $k = -2$. Now from $\lambda^2 = -k - 1$, we obtain $\lambda^2 = 1$. Without loss of generality we assume that $\lambda = -1$. Then by the same argument as in Case 1 we get $M^{2n+1}$ is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof. 

Example. In [7], the author present an example of a 5-dimensional $(k, \mu)'$-almost Kenmotsu manifold with $k = -2$ and $\mu = -2$. Then by the same argument as in Case 1 of Theorem 4.4, $M^5$ is locally isometric to $\mathbb{H}^{3}(-4) \times \mathbb{R}^2$.

Let $X = \alpha_1(\xi_1 + \alpha_2e_2 + \alpha_3e_3 + \alpha_4e_4 + \alpha_5e_5)$ be any vector field on $M^5$ and let $V = e_4$. Then, $\nabla_XV = 0 = \alpha e_X + b\phi X$, where $a = b = 0$. Hence, $V = e_4$ is an example of a HPCV field, where $\phi V = \phi e_4 = -e_2 \neq 0$. Hence, Theorem 4.4 is verified.

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