Gerbes, 2-gerbes and symplectic fibrations.

1. Introduction.

A symplectic fibration $P \to N$ is a differentiable fibration whose typical fiber is the closed connected symplectic manifold $(F, \omega)$, and such that there exists a trivialization $(U_i, g_{ij})$, such that $g_{ij}(u)$ is a symplectic automorphism of the fiber over $u$, endowed with a symplectic structure $\omega_u$, symplectomorphic to $(F, \omega)$. We suppose that the cohomology class $[\omega_u]$ of $\omega_u$ is fixed. The theory of symplectic bundles have been studied by different authors (see [8], [9], [13], [14]). One purpose of the paper [14] is to determine whether the structural group of the symplectic bundle can be reduced to the Hamiltonian group of $(F, \omega)$, that is whether there exists a symplectic bundle $P' \to N$ isomorphic to $P$, whose coordinate changes $g'_{ij}(u)$ are Hamiltonian automorphisms of the fiber above $u$; such a reduction will be called a Hamiltonian structure, or a Ham-reduction. In [14], it is shown that the existence of such Hamiltonian reductions on a finite cover of $N$ is equivalent to the following two conditions:

(i) There exists a closed 2-form $\Omega$ defined on $P$ whose cohomology class $[\Omega]$ extends $[\omega]$. This means that the restriction to the fiber above $u$ of the cohomology class $[\Omega]$, is the cohomology class $[\omega]$. Following McDuff, we will call the form $\Omega$ a closed connection form.

(ii) Let $\text{Symp}(F, \omega)_0$ be the connected component of the group of symplectomorphisms $\text{Symp}(F, \omega)$, of $(F, \omega)$. The symplectic bundle is isomorphic to a symplectic bundle whose coordinate changes take their values in $\text{Symp}(F, \omega)_0$.

In [14] it was necessary to impose condition (ii) because the Hamiltonian subgroup is connected. In [14], McDuff has defined a disconnected subgroup $\text{Ham}^s$ of the group $\text{Symp}(F, \omega)$, and has shown that the existence of a $\text{Ham}^s$-reduction of a symplectic bundle is equivalent to the existence of a closed connection form.

One purpose of this paper is to study the problem of the existence of Hamiltonian and $\text{Ham}^s$-reductions of a symplectic bundle using gerbes, and 2-gerbes. The theory of gerbes has been defined by Giraud [6] with the purpose of giving geometric interpretations of cohomology classes. These classes represent the obstruction to globally extending locally defined bundles, as it is the case for Hamiltonian bundles. Lawrence Breen [2] has also defined a theory of 2-gerbes. A 2-gerbe represents geometrically the obstruction for a 2-geometric type structure to be defined globally. This theory will be also involved here. For $n \geq 2$, such a geometric obstruction theory has been defined by Tsemo [20].
Let $\omega$ be a 2-closed form defined on the manifold $F$, and $T^1$ the circle. It has been shown by Kostant and Weil, that the cohomology class $[\omega]$ of $\omega$ is integral, if and only if $[\omega]$ is the Chern class of a $T^1$-bundle. When the class is not integral, we define a flat gerbe $C'(\omega)$ bounded by the sheaf of locally constant $\mathbb{R}$-functions defined on $F$ which represents the obstruction of $[\omega]$ to be zero. We can construct from this gerbe, another gerbe $C(\omega)$ bounded by the sheaf of locally constant $T^1$-functions defined on $F$, which represents the obstruction of $[\omega]$ to be integral (see 2.4). These gerbes are used to study the extension of $[\omega]$. We have:

**Theorem 2.5.4.**

Let $p : P \to N$ be a symplectic bundle whose typical fiber is $(F, \omega)$, there exists a gerbe $C^1_F(\omega)$ whose classifying cocycle $c^1_F(\omega)$ represents the obstruction of the symplectic bundle $p$ to have a Hamiltonian reduction.

To show an analogous theorem for $\text{Ham}^s$-reductions, one has to show first as in [13], that the automorphisms group of a $\text{Ham}^s$-reduction of a symplectic bundle is independent of the chosen $\text{Ham}^s$-reduction, in order to define the band of the classifying gerbe. We prove also the following result:

**Theorem 8.2, 8.2.2.**

There exists a 2-gerbe $C^2_F(\omega)$ whose classifying cocycle $c^2_F(\omega)$ represents the obstruction of the class $[\omega]$ to be extended to $P$. The class $[c^2_F(\omega)]$ can be deduced from $[c^1_F(\omega)]$ as follows: Let $L_1$ and $L_0$ be the respective bands of $C^1_F(\omega)$ and $C^2_F(\omega)$. There exists an exact sequence of sheaves $1 \to L_0 \to L'_1 \to L_1 \to 1$, such that the class $[c^1_F(\omega)]$, is the image of the class $[c^1_F(\omega)]$ by the connecting morphism $H^2(N, L_1) \to H^3(N, L_0)$ of the last exact sequence. This shows that the existence of a Hamiltonian reduction implies that the form $\omega$ can be extended to $P$.

In [14], McDuff defines a discrete subgroup $H^1(F, P_\omega)$ of $H^1(F, \mathbb{R})$ and a class in $H^2(N, H^1(F, P_\omega))$ which is the obstruction to have a $\text{Ham}^s$-reduction, that is to obtain a closed connection form. We show that this last class and $[c^2_F(\omega)]$ are the image of the Chern class of a $H^1(F, \mathbb{R})/H^1(F, P_\omega)$-principal bundle by connecting morphisms related to exact sequences of sheaves see 8.3.

The holonomy of a connective structure defined on a gerbe, is the analogous of the holonomy of a connection. It is used to represent the action in string theory. We relate the holonomy of the gerbe $C(\omega)$ to the flux see 4.

We generalize the methods applied here to solve other geometric problems, as for example to find a $H$-reduction of a $G$-bundle such that $G/H$ is a $K(\pi, 1)$ space. For this problem, we define also a gerbe $C_H$ which represents the geometric obstruction to solve it: More precisely we have:

**Theorem 2.6.3.**
Let \( f : P \rightarrow N \) be a \( G \)-bundle defined on \( N \), and \( H \) a subgroup of \( G \) such that the right quotient of \( G \) by \( H \), \( G/H \) is a K(\( \pi, 1 \)) space. Suppose that:

(i) either the coordinate changes take their values in \( \text{Nor}(H) \), the normalizer of \( H \) in \( G \), this condition is satisfied for symplectic bundles whose coordinate changes take their values in the connected component \( \text{Symp}(F, \omega)_0 \) of the group of symplectic automorphisms \( \text{Symp}(F, \omega) \). We consider \( G \) to be \( \text{Symp}(F, \omega)_0 \), and \( H \) to be \( \text{Ham}(F, \omega) \) the group of Hamiltonian diffeomorphisms.

(ii) or \( H \) intersects every connected component of \( G \), and there exists a commutative group \( L \), a continue and surjective cocycle \( F : G \rightarrow L \), for a representation \( \rho : G \rightarrow L \), such that \( \rho(G_0) \) the image of the connected component \( G_0 \) of \( G \) is the identity of \( L \), and the kernel of \( F \) is \( H \). Here \( L \) is a quotient of a vector space by a discrete subgroup. This condition is satisfied if \( H \) is the subgroup \( \text{Ham}^s \), and \( G \) is \( \text{Symp}(F, \omega) \)

Then there exists a gerbe \( C_H \), whose classifying cocycle represents the obstruction to reduce \( G \) to \( H \).

When the gerbe \( C_H \) is defined by a cocycle \( F \), the classifying cocycle of this gerbe is the Chern class of a \( G/H = L \)-bundle.

Analogues of the gerbe which appears in the last theorem can be constructed in more abstract situations: we generalize this construction to the case of topos (elementary topos). This will perhaps suggest applications to algebraic geometry and arithmetic.

The fact that the pull-back of a \( \text{Symp}(F, \omega)_0 \)-bundle endowed with a closed connection form to a finite cover of the base space has Hamiltonian reductions, suggests that the natural category for the study of Hamiltonian reductions is the etale topos of the base (see 3).

The last part of the paper is devoted to geometric quantization. We give an extension of the Kostant-Souriau quantization whenever the class \( \omega \) is not supposed to be integral, using the gerbe \( C(\omega) \). In particular we obtain the following:

**Theorem.**

Let \((M, \omega)\) be a symplectic manifold, and \((C^\infty(M), \{,\})\) the poisson algebra of \((M, \omega)\). There exists a preHilbert space \( H \), and a representation \((C^\infty(M), \{,\}) \rightarrow (\text{Aut}(H), [\cdot, \cdot])\) Where \((\text{Aut}(H), [\cdot, \cdot])\) is the algebra of operators of \( H \) endowed with the commutator bracket.

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2. Gerbes theory.

The notion of gerbe has been defined by Giraud [6] to give a geometric interpretation of 2-Cech cohomology classes, and to find obstructions to solve gluing problems. The basic example of a gerbe is defined as follows: consider a $G$-principal bundle defined on the manifold $N$, and $1 \rightarrow H \rightarrow G' \rightarrow G \rightarrow 1$ a central extension. The geometric obstruction to the existence of a $G'$-principal bundle over $N$, whose quotient by $H$ is the original $G$-bundle is defined by the classifying fibre of a gerbe. Gerbe theory also has a lot of applications in algebraic geometry. In theoretical physics, a notion of holonomy of gerbe allows us to represent geometrically the action in string theory. In this part, we summarize the results of gerbe theory used here. We prefer the point of view of sheaf of categories rather to the one of descent.

Definition 2.1.

Let $N$ be a category. A sieve $T$ is a subclass of objects of $N$, such that if $u$ is an element of $T$, and $v \rightarrow u$ an arrow of $N$, then $v$ is an element of $T$.

Recall that the category $N_u$ is the category whose objects are objects $v$ of $N$ such that there exists an arrow $h_v : v \rightarrow u$, a morphism between two objects $v$ and $v'$ of $N_u$ is an arrow $h : v \rightarrow v'$ such that $h_v \circ h = h_{v'}$.

A topology on the category $N$ is defined as follows: for each object $u$ of $N$, there is a family of sieves $J_u$ of $N_u$, such that:

(i) If $h : v \rightarrow u$ is an arrow, and $T$ an element of $J_u$, then $T^h = \{ v' \in Ob(N) : v' \in T, \exists h' : v' \rightarrow v \}$ is an element of $J(v)$. 

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(ii) Suppose that \( T \) is a sieve of the sub-category \( N_u \) above \( u \), if for each map \( h : v \to u \), \( T^h \) is an element of \( J(v) \), then \( T \) is an element of \( J(u) \).

For example, one can define a topology \( J \) on the category \( \text{Top}(N) \), whose objects are open sets of a topological manifold \( N \), and morphisms are canonical inclusions as follows: For each open set \( U \) of \( N \), an element of \( J(U) \) is a sieve of the category above \( U \), which contains a family of open subsets \( (U_i)_{i \in I} \) of \( U \) whose union is \( U \).

We will suppose in the sequel that our category is a topos, reader unfamiliar to this notion make the stronger assumptions that the category is stable by finite sums, and products, there exists final and initial objects, and the limits exist and are universal.

We will also suppose that the topology is generated by a covering family \( (U_i)_{i \in I} \), where \( U_i \) is an object of \( N \). This means that: for each object \( u \), there exists a subset \( I_u \) contained in \( I \), such that for each \( i \in I_u \), there exists a map \( U_i \to u \) of \( N \), the subcategory \( u(U_i)_{i \in I_u} \) whose objects are objects \( v \) of \( N \) such that there exists a map \( v \to U_i \), \( i \in I_u \) is an element of \( J(u) \). An generating family \( (U_i)_{i \in I} \) of a topological space \( N \) generates the topology of the category \( \text{Top}(N) \).

**Definition 2.2.**

Let \( (N, J) \) be a category \( N \) endowed with a topology \( J \). A sheaf of categories defined on \( (N, J) \) is a correspondence \( C: \)

\[
U \to C(U)
\]

where \( C(U) \) is a category, and \( U \) an object of \( N \), which verifies the following properties:

(i) For each map \( U \to V \), there exists a restriction map \( r_{U,V} : C(V) \to C(U) \) such that

\[
r_{U_1,U_2} \circ r_{U_2,U_3} = r_{U_1,U_3}
\]

In fact, while the previous equality is verified in many examples, only an isomorphism between \( r_{U_1,U_2} \circ r_{U_2,U_3} \) and \( r_{U_1,U_3} \) is needed. The last relation defined the notion of presheaf of categories.

The following properties needed to be verified to complete the notion of sheaf of categories.

(ii) Gluing properties for objects.

Let \( (U_i)_{i \in I} \) be a covering family of the object \( U \) of \( N \), and \( e_i \) an object of \( C(U_i) \). We denote abusively by \( N \) the final object of \( N \). Suppose there are morphisms

\[
g_{ij} : r_{U_i \times U_j, U_i}(e_j) \to r_{U_i \times U_j, U_i}(e_i)
\]
such that on $U_{i1} \times_N U_{i2} \times_N U_{i3}$, the restrictions of the morphisms $g_{i1i2}g_{i2i3}$, and $g_{i1i3}$ between the respective restrictions of $e_{i3}$ and $e_{i1}$ to $U_{i1} \times_N U_{i2} \times_N U_{i3}$ are equal. Then there exists an object $e_U$ of $U$ such that $r_{U, U}(e_U) = e_i$.

(iii) **Gluing conditions for maps.**

For each objects $e, e'$ of $C(U)$, the correspondence defined on the category above $U$ by

$$V \longrightarrow Hom(r_{U, V}(e), r_{U, V}(e'))$$

is a sheaf of sets.

A correspondence $C$ which satisfies properties (i), (ii) and (iii) is a sheaf of categories. A **gerbe** is a sheaf of categories which satisfies the following conditions:

(iv) There exists a covering family $(U_i)_{i \in I}$ of $N$ such that $C(U_i)$ is not empty for each $i$.

(v) **Local connectivity.**

For each object $U$ of $N$, there exists a covering family $(U_i)_{i \in I}$ of $U$ such that, for each pair of elements $e$ and $e'$ of $C(U)$, $r_{U, U}(e)$ and $r_{U, U}(e')$ are isomorphic.

(vi) There exists a sheaf $L$ on $N$ such that for each object $e_U$ of $C(U)$, $Hom(e_U, e_U) = L(U)$, and this identification commutes with restrictions an arrows. The sheaf $L$ is called **the band** of the gerbe $C$, or we say that the gerbe $C$ is **bounded** by $L$.

**The classifying cocycle of a gerbe.**

Let $(U_i)_{i \in I}$ be a covering family of $N$ such that for each $i$, $C(U_i)$ is not empty, and $e_i$ is an object of $C(U_i)$. Choose maps $g_{ij} : r_{U_i \times_N U_j, U_i}(e_j) \longrightarrow r_{U_i \times_N U_j, U_i}(e_i)$ for all $i, j$. Denote by $g_{ii}^{-1}$ the restriction of $g_{i1i2}$ between the restrictions of $e_{i2}$ and $e_{i1}$ to $U_{i1} \times_N U_{i2} \times_N U_{i3}$. Then the map

$$c_{i1i2i3} = g_{i1i2}^{-1}g_{i2i3}^{-1}g_{i3i1}^{-1}$$

is an automorphism of $r_{U_{i1} \times_N U_{i2} \times_N U_{i3}, U_{i1}}(e_1)$ which may be thought of as an element of $L(U_{i1} \times_N U_{i2} \times_N U_{i3})$. The assignment $U_{i1} \times_N U_{i2} \times_N U_{i3} \rightarrow c_{i1i2i3}$ is called the classifying cocycle of the gerbe. If the band is commutative, it is a Čech-cocycle in the classical sense. It has been shown by Giraud [6] that the isomorphism classes of gerbes bounded by the sheaf $L$ is one to one with the Čech cohomology group $H^2(N, L)$, when $L$ is commutative. If the band is not commutative $H^2(N, L)$ is defined to be set the of equivalence classes of gerbes bounded by $L$. The trivial gerbe is a gerbe such that $C(N)$ is not empty. The elements of $C(N)$ are called global sections. They are one to one with $H^1(N, L)$.

**2.2 Notations.**
Let $U_{i_1}, \ldots, U_{i_p}$ be objects of a topos $N$, and $C$ a presheaf defined on $N$. We will denote by $U_{i_1 \ldots i_p}$ the fiber product of $U_{i_1}, \ldots, U_{i_p}$ on the final object. If $e_{i_1}$ is an object of $C(U_{i_1})$, $e_{i_1 \ldots i_p}$ will be the restriction of $e_{i_1}$ to $U_{i_1 \ldots i_p}$. For a map $h : e \rightarrow e'$ between two objects of $C(U_{i_1 \ldots i_p})$, we denote by $h_{i_1 \ldots i_p}$ the restriction of $h$ to a morphism between $e_{i_1 \ldots i_p} \rightarrow e'_{i_1 \ldots i_p}$.

Now we provide details on the classic example of sheaf of categories given at the beginning. Consider an extension:

$$1 \rightarrow H \rightarrow G' \rightarrow G \rightarrow 1$$

such that $H$ is a central group in $G'$, and the map $G' \rightarrow G$ has local sections. Supposed defined a $G$-principal bundle $p_G$, over $N$. The obstruction of the existence of a $G'$-principal bundle over $N$ whose quotient by $H$ is $p_G$, is the cohomology class of the classifying cocycle of the following gerbe $C_H$ defined on the categories of open subsets of $N$ as follows: for each subset $U$ of $N$, we define $C_H(U)$ to be the category whose objects are principal $G'$-bundles over $U$ whose quotient by $H$ is the restriction of $p_G$ to $U$. To explicit the classifying cocycle $c_H$ of this gerbe, consider an open covering $(U_i)_{i \in I}$ of $N$, who trivializes the bundle $p_G$. We denote by $g_{ij} : U_i \cap U_j \rightarrow G$ the transition functions. Since the projection $G' \rightarrow G$ has local sections, we can suppose that we can lift each map $g_{ij}$ to a map $\hat{g}_{ij} : U_i \cap U_j \rightarrow G'$. The classifying cocycle of $C_H$ is defined by:

$$c_{i_1 i_2 i_3} = \hat{g}_{i_1 i_2} \hat{g}_{i_2 i_3} \hat{g}_{i_3 i_1}$$

This situation applies to the case where $H$ is $\mathbb{Z}/2$, $G'$ the spin group, and $G$ the orthogonal group $O(n)$. The $O(n)$-bundle is the orthogonal reduction of the bundle of linear frames of the $n$-dimensional manifold $N$, defined by a riemannian metric. The gerbe represents the geometric obstruction of the existence of a spin structure on $N$. The cocycle in this case is the second Stiefel-Whitney class.

### 2.3 Connective structures on gerbes.

The notion of a connective structure on a gerbe has been defined by Deligne see [3]. It is the analogous to the notion of a connection on a principal bundle.

**Definition 2.3.1.**

Consider a gerbe $C$ defined on a manifold whose band is $L$. A **connective structure** on $C$, is a correspondence which associates to each object $e_U$ of $C(U)$ a torsor $Co(e_U)$, called the torsor of connections, that is an affine space whose underlying vector space is a subset of the set of 1-forms defined on $U$. The following properties are supposed to be satisfied by this assignment:

(i) The correspondence $e_U \rightarrow Co(e_U)$ is funtional with respect to restrictions to smaller subsets.
(ii)- For every isomorphism \( h : e_U \rightarrow e'_U \) between objects of \( C(U) \), there exists an isomorphism of torsors \( h^* : Co(e_U) \rightarrow Co(e'_U) \) compatible with the composition of morphisms of \( C(U) \), and the restrictions to smaller subsets.

Suppose now that the band of the gerbe is a \( T^1 \)-sheaf, where \( T^1 \) is the circle. Then for each morphism \( g \) of the object \( e_U \) of \( C \), and \( \nabla_{e_U} \) a connection of \( Co(e_U) \),

\[
g^* \nabla_{e_U} = \nabla_{e_U} + g^{-1} dg
\]

A **curving** of a connective structure \( Co \) is an assignment to each object \( e_U \), and each element \( \nabla \) of \( Co(e_U) \), a 2-form \( D(e_U, \nabla) \) defined on \( U \) such that for each morphism \( h : e'_U \rightarrow e_U \), \( D(e'_U, h^* \nabla) = D(e'_U, \nabla) \).

If \( \alpha \) is a 1-form on \( U \) such that \( \nabla + \alpha \) is an element of \( Co(e_U) \), then

\[
D(e_U, \nabla + \alpha) = D(e_U, \nabla) + d\alpha
\]

The assignment \( e_U \rightarrow D(e_U, \nabla) \) is compatible with the restrictions to smaller subsets.

**The curvature of the curving** is the form whose restriction to each open subset such that \( C(U) \) is not empty is \( dD(e_U, \nabla) \), where \( e_U \) is an object of \( C(U) \), and \( \nabla \) an element of \( Co(e_U) \).

**2.4 The gerbe associated to a closed 2-form.**

Let \((N, \omega)\) be a manifold \( N \), endowed with a closed 2-form \( \omega \); \((N, \omega)\) is often called a **Dirac manifold**. There exists a Cech-Weil isomorphism between the De Rham cohomology groups of \( N \), and the Cech-cohomology groups of the sheaf of locally constant \( \mathbb{R} \)-functions defined on \( N \). Thus using the theorem of Giraud [6], we deduce that the cohomology class \([\omega]\) of \( \omega \) classifies a gerbe \( C'(\omega) \) defined on \( N \) and bounded by the sheaf of locally constant \( \mathbb{R} \)-functions.

In this part, we present the construction of the classifying cocycle of this gerbe. This is in fact the classic explanation of the Cech-Weil isomorphism. This gerbe is the fundamental gerbe used to define many of the geometric obstructions involved in this paper.

Let \( N \) be a manifold, \( \omega \) a closed 2-form defined on \( N \), and \((U_i)_{i \in I}\) a cover of \( N \) by contractible open subsets. Without loss of generality, we can suppose that \( U_i \cap U_j \) is connected. The Poincare Lemma implies the existence of a family of 1-forms \((\alpha_i)_{i \in I}\) such that

\[
d(\alpha_i) = \omega|_{U_i},
\]

where \( \omega|_{U_i} \) is the restriction of \( \omega \) to \( U_i \). Let \( \alpha_j^i \) and \( \alpha_i^j \) be the respective restrictions of \( \alpha_j \) and \( \alpha_i \) to \( U_i \cap U_j \). Denote by \( \alpha_{ij} \), the form \( \alpha_j^i - \alpha_i^j \) on \( U_i \cap U_j \). The form \( \alpha_{ij} \) is closed. By applying the Poincare lemma to \( \alpha_{ij} \), we obtain a family of real valued functions \( u_{ij} \) defined on \( U_i \cap U_j \) such that
\[ d(u_{i}) = \alpha_{ij}. \]

On \( U_{i1i2i3} \), the differential of \( c_{i1i2i3} = u_{i2i3} - u_{i1i3} + u_{i1i2} \) is zero. This implies that it is a constant map. The family of functions \( c_{i1i2i3} \) is a 2-Cech cocycle for the sheaf of locally constant \( \mathcal{R} \)-functions.

If \( c_{i1i2i3} \in \mathbb{Z} \), the functions \( h_{ij} = exp(2i\pi u_{ij}) \) defines a line bundle over \( N \). This bundle is the well-known Kostant-Weil construction. In this case, the cohomology class \([\omega]\) of \( \omega \) is an element of \( H^{2}(N, \mathbb{Z}) \).

Suppose that \([\omega]\) is not necessarily an element of \( H^{2}(N, \mathbb{Z}) \). Using Giraud’s theorem concerning the classification of gerbes, we can associate to \( \omega \) a gerbe \( C'(\omega) \) bounded by the sheaf of locally constant \( \mathcal{R} \)-functions, whose classifying cohomology class is the image of \([\omega]\) by the De Rham Cech isomorphism. This gerbe represents the obstruction of the class \([\omega]\) to be zero. The objects of \( C'(\omega)(U) \) when it is not empty, can be represented by flat \( \mathcal{R} \)-bundles by using the reconstruction theorem of Giraud presented in Brylinsky [3]. We denote by \( c'_{\omega} \) the classifying cocycle of \( C'(\omega) \).

The following proposition describes a gerbe bounded by \( T^{1} \) which will play a fundamental role in this paper.

**Definition-Proposition 2.4.1.**

Let \( U \) be an open subset of \( N \), and denote by \( C(\omega)(U) \) the category whose objects are circle bundles over \( U \), endowed with a connection whose curvature is \( \omega_{i} \mid U \) the restriction of \( \omega \) to \( U \). We will denote by \((e_{U}, \nabla_{e_{U}})\) an object of \( C(\omega)(U) \); \( e_{U} \) represents a \( T^{1} \)-bundle, and \( \nabla_{e_{U}} \) the connection on \( e_{U} \) whose curvature is the restriction of \( \omega \) to \( U \). The set of morphisms between two objects \((e_{U}, \nabla_{e_{U}})\) and \((e'_{U}, \nabla_{e'_{U}})\) is the set of morphisms of differential bundles over the identity \( f : e_{U} \to e'_{U} \) such that \( f^{*}(\nabla_{e'_{U}}) = \nabla_{e_{U}} \). The correspondence \( U \to C(\omega)(U) \) is a gerbe bounded by the sheaf of locally constant \( T^{1} \)-valued functions. The class of its classifying cocycle is the obstruction of \([\omega]\) to be integral.

**Proof.**

First, we show that \( C(\omega) \) is a sheaf of categories.

Gluing conditions for objects:

Let \((U_{i})_{i \in I}\) be an open cover of an open set \( U \) of \( N \), \((e_{i}, \nabla_{e_{i}})\) an object of \( C(\omega)(U_{i}) \), and \( g_{ij} : e_{j}^{i} \to e_{j}^{i} \) a morphism such that on \( U_{i1i2i3} \), \( g_{i1i2}g_{i2i3}^{i} = g_{i1i3}^{i} \). Since the elements of the family \((e_{i})_{i \in I}\) are bundles, we deduce that there exists a bundle \( e \) over \( U \) whose restriction to \( U_{i} \) is \( e_{i} \). The bundle \( e \) is endowed with a connection whose curvature is the restriction of \( \omega \) to \( U \) since the restriction of this curvature to \( U_{i} \) is the restriction of \( \omega \) to \( U_{i} \).

Gluing conditions for arrows:

Let \( e, e' \) be a pair of elements of \( C(\omega)(U) \), the correspondence defined on the category of open subsets of \( U \) by \( V \to Hom(r_{U,V}(e), r_{U,V}(e')) \) is a sheaf of sets, since it is the sheaf of morphisms between two bundles.
It remains to verified that the sheaf of categories is a gerbe.

Let \((U_i)_{i \in I}\) be an open covering of \(N\) by contractible open subsets. For each pair of objects \((e, \nabla_e)\) and \((e', \nabla_{e'})\) of \(C(\omega)(U)\) we have to show that these objects are locally isomorphic.

To show this, consider two objects \((e_i, \nabla_{e_i})\) and \((e'_i, \nabla_{e'_i})\) of \(C(\omega)(U_i)\). The bundle \(e_i\) and \(e'_i\) are isomorphic to the trivial bundle \(U_i \times T^1\). Let \(d\) be the differential, \(\nabla_{e_i} = d + \alpha_i\), and \(\nabla_{e'_i} = d + \alpha'_i\). For each section \(u : U_i \rightarrow i\mathbb{R}\) of the Lie algebra bundle associated to this bundle, and each automorphism \(g\) defined by a differentiable map \(U_i \rightarrow T^1\), we have:

\[
g^*(d + \alpha_i)(u) = g^{-1}(d + \alpha_i)(gu) = (g^{-1}dg + d + \alpha_i)(u) \tag{1}
\]

Since the connections \(\nabla_{e_i}\) and \(\nabla_{e'_i}\) have the same curvature, there exists a function \(v_i\) such that \(\alpha'_i = \alpha_i + dv_i\). We can suppose (or shrinking \(U_i\) if needed) that the logarithm is defined on \(U_i\), thus \(g^{-1}dg = d\log(g)\). If we take \(g = \exp(v_i)\), where \(\alpha'_i = \alpha_i + dv_i\), then \(g^*(d + \alpha_i) = d + \alpha'_i\). We obtain that the respective restrictions \((e_i, \nabla_{e_i})\) and \((e'_i, \nabla_{e'_i})\) of \((e, \nabla_e)\) and \((e', \nabla_{e'})\) to \(U_i\) are isomorphic.

The automorphism group of the object \((e, \nabla_e)\) of \(C(\omega)(U)\) is the group of gauge transformations which preserve the connection \(\nabla_e\). These gauge transformations are necessarily constant maps, as is shown by (1).

Now we have to interpret geometrically the vanishing of the cohomology class \([c_\omega]\), of the classifying cocycle \(c_\omega\) of \(C(\omega)\). The theorem of Giraud [6] implies that this is equivalent to the existence of a global object of the gerbe, that is a \(T^1\)-bundle over \(N\) whose curvature is \(\omega\). The Kostant-Weil construction implies that this is equivalent to the fact that the class \([\omega]\) of \(\omega\) is integral \(\bullet\).

Now we establish the relation between the gerbes \(C'(\omega)\) and \(C(\omega)\).

**Proposition 2.4.2.**

Consider the exact sequence of sheaves of locally constant functions:

\[
1 \rightarrow \mathcal{L} \rightarrow \mathbb{R} \rightarrow T^1 \rightarrow 1 \tag{1}
\]

where the map \(\mathcal{L} \rightarrow \mathbb{R}\) is the canonical injection, and \(\mathbb{R} \rightarrow T^1\) is the exponential map of the Lie group \(T^1\), that is the composition of the multiplication by \(2\pi i\) and the usual exponential. We obtain the following exact sequence in cohomology:

\[
H^1(N, T^1) \rightarrow H^2(N, \mathcal{L}) \rightarrow H^2(N, \mathbb{R}) \rightarrow H^2(N, T^1)\ldots
\]

The class \([c_\omega]\) is the image of the class \([c'_\omega]\), by the map \(H^2(N, \mathbb{R}) \rightarrow H^2(N, T^1)\).

**Proof.**

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Consider an open covering \((U_i)_{i \in I}\) of \(N\), such that for each \(i\), \(U_i\) is contractible and \(U_{i_1, \ldots, i_p}\) is connected (using a theorem of Weil, we can suppose \(U_{i_1, \ldots, i_p}\) to be connected). Let \(c_{i_1i_2i_3}\) be the classifying cocycle of the gerbe \(C\). The image of \([e']\) by the map \(H^2(N, \mathbb{R}) \to H^2(N, T^1)\) is represented by the cocycle \(\exp(2i\pi c_{i_1i_2i_3})\). Recall that to construct the cocycle \(c_{i_1i_2i_3}\), we have considered the restriction \(\omega_{|U_i}\) of \(\omega\) to \(U_i\). There exists a form \(\alpha_i\) such that \(d\alpha_i = \omega_{|U_i}\). We can define the object \(\epsilon_i = (U_i \times T^1, d + \alpha_i)\) of \(C(\omega)(U_i)\). Let \(\alpha_{ij}^1\) be the restriction of \(\alpha_j\) to \(U_{ij}\), then there exists a function \(u_{ij}\) such that \(d(u_{ij}) = \alpha_{ij}^1 - \alpha_{ij}^0\).

The functions \(\exp(2i\pi u_{ij})\) defines a morphism between \(e_i^j\) and \(e_i^j\). The classifying cocycle of \(C'\) is \(c_{i_1i_2i_3} = u_{i_1i_3} - u_{i_1i_2} + u_{i_2i_3}\), and the classifying cocycle of \(C(\omega)\) is \(\exp(2i\pi u_{i_1i_3})\exp(-2i\pi u_{i_1i_2})\exp(2i\pi u_{i_2i_3}) = \exp(2i\pi c_{i_1i_2i_3})\).

Now, we are going to endow the gerbe \(C(\omega)\) with a connective structure.

**Proposition 2.4.3.**

For each open set \(U\) of \(N\), and the object \(e_U\) of \(C(\omega)(U)\), the set \(\text{Co}(\omega)(e_U)\) of connections defined on \(e_U\) whose curvature is the restriction of \(\omega\) to \(U\) defines a connective structure on \(C(\omega)\). The restriction \(\omega_U\) of \(\omega\) to \(U\), is the curvature of each object \(e_U\) of \(C(\omega)(U)\). The curvature of this curving is zero.

**Proof.**

Let \(\alpha\) and \(\alpha'\) be two elements of \(\text{Co}(e_U)\), and \((U_i)_{i \in I}\) a contractible open cover of \(U\). It is a well-known fact that there exists a 1-form \(v\) such that \(\alpha' = \alpha + v\). The restriction of \(e_U\) to \(U_i\) is diffeomorphic to the trivial \(T^1\)-bundle. This implies that under this identification, the respective restrictions \(\alpha_i\) and \(\alpha'_i\) of the connections \(\alpha\) and \(\alpha'\) to \(U_i\), have the form \(d + u_i\) and \(d + u_i + v|_{U_i}\), where \(u_i\) is a 1-form defined on \(U_i\), and \(v|_{U_i}\) is the restriction of \(v\) to \(U_i\). The respective curvatures of \(d + u_i\) and \(d + u_i + v|_{U_i}\) are the 2-forms \(du_i\) and \(d(u_i + v)\).

Since they coincide with the restriction of \(\omega\) to \(U_i\), we deduce that \(dv = 0\), thus \(\text{Co}(\omega)(e_U)\) is an affine space whose underlying vector space is the vector space of closed 1-forms. We deduce that it is a torsor.

The fact that for each automorphism \(g\) of \(e_U\), \(g^*\nabla_{e_U} = \nabla_{e_U} + g^{-1}dg\) results from the fact that \(\nabla_{e_U}\) is a connection.

For each map \(h : e_U \to e'_U\), we define the map \(h^* : \text{Co}(\omega)(e_U) \to \text{Co}(\omega)(e'_U)\), to be the pull-back of connections by \(h^{-1}\). This implies that \(h^*\) behave naturally in respect to restrictions to smaller subsets and compositions.

Let \(\nabla_{e_U}\) be an element of \(\text{Co}(e_U)\), the curvature of \(\nabla_{e_U}\) is the restriction of \(\omega\) to \(U\), \(\omega|_{U}\). It is also the curvature of \(h^{-1}(\nabla_{e_U})\). This can be shown using (1). This implies that \(\omega\) defines a curving for this connective structure. The fact that the curvature of this connective structure is zero follows from the fact that the form \(\omega\) is closed.

At the end of this paper, we will present a quantization of symplectic manifolds using the gerbe \(C(\omega)\). This gerbe thus appears to be fundamental in symplectic geometry.
2.5 Symplectic fibrations and gerbes.

Let \( p : P \to N \) be a symplectic fibration, whose fiber \( F \) is the closed symplectic manifold \((F, \omega)\). We study the following problem: extend \([\omega]\) to a class \([\Omega]\) defined on \( P \), that is, find a cohomology class \([\Omega] \in H^2(P, \mathbb{R})\) such that for every \( u \in N \), consider the canonical embedding \( i_u : F \to F_u \to P \), \( i_u^*(\[\Omega]\) = [\omega] \).

A result of Thurston [16] implies that in this situation there exists a form \( \Omega \) such that \( i_u^*\Omega = \omega_u \) for all \( u \in N \).

To use the theory of gerbes, we must suppose that the class \([\omega]\) of the symplectic form \( \omega \) is integral. Thus it is the Chern class of a circle bundle \( \mathbb{R}^\text{dim}N \times F \), where \( N \) is an open subset of \( N \), and its differential structure is modelled on \( \mathbb{R}^n \times F \), where \( n \) is the dimension of \( N \). The cohomology class of the classifying cocycle of this gerbe is the obstruction to extend \([\omega]\).

**Proposition. 2.5.1.**

Suppose that \([\omega]\) is integral, and consider for each open set \( U \) of \( N \) the category \( C_F(\omega)(p^{-1}(U)) \) of circle bundles over \( p^{-1}(U) \) whose Chern class is \([\Omega_U]\), an element of \( H^2(p^{-1}(U), \mathbb{R}) \) which extends \([\omega]\). The correspondence defined on the category of open subsets of \( P \) by \( p^{-1}(U) \to C_F(p^{-1}(U)) \), defines a gerbe on \( P \), where \( P \) is endowed with the topology structure generated by \( p^{-1}(U) \), where \( U \) is an open subset of \( N \), and its differential structure is modelled on \( \mathbb{R}^n \times F \), where \( n \) is the dimension of \( N \). The cohomology class of the classifying cocycle of this gerbe is the obstruction to extend \([\omega]\).

**Proof.**

Gluing conditions for objects.

Recall that for every objects \( e_U \) and \( e'_U \) of \( C_F(\omega)(U) \), \( \text{Hom}(e_U, e'_U) \) are morphisms of circle bundles which project to the identity. Let \((U_i)_{i \in I}\) be an open covering of the open set \( U \) of \( N \) by open subsets, \( e_i \) an object of \( C_F(\omega)(p^{-1}(U_i)) \), and \( u_{ij} : e'_j \to e'_i \) a morphism which verifies \( u_{ij1}u_{ij2} = u_{ij3} \). Then there exists a bundle \( e_U \) on \( p^{-1}(U) \) whose restriction to each \( U_i \) is \( e_i \). This is deduced from the classical definition of a \( T^1 \)-bundle \( e_U \) over \( p^{-1}(U) \). Consider a 2-closed form \( \Omega \) which represents the Chern class of \( e_U \), since the restriction of \( e_U \) to \( p^{-1}(U_i) \) is \( e_i \), its Chern class which is the restriction of the class \([\Omega]\) of \( \Omega \) to \( p^{-1}(U_i) \) is the Chern class of \( e_i \). This implies that \([\Omega]\) extends to \( p^{-1}(U) \) the class \([\omega]\).

Gluing condition for arrows

The correspondence defined on the category of open subsets of \( U \) by \( V \to \text{Hom}(e_U|_V, e'_U|_V) \) defines a sheaf on this category, since it is a sheaf of morphisms between two bundles.

This shows that \( C_F(\omega) \) is a sheaf of categories. It remains to prove that it is a gerbe.
Let \((U_i)_{i \in I}\) be a cover of \(N\) by contractible open subsets, \(p^{-1}(U_i) = U_i \times F\). This implies that \(H^*(p^{-1}(U_i)) = H^*(F)\), thus there exists a class \([\Omega_i]\) on \(p^{-1}(U_i)\) which extends \([\omega]\), and which is integral. Thus \(C_F(\omega)(p^{-1}(U_i))\) is not empty.

We deduce from (1) that the group of automorphisms of the objects of \(C_F(\omega)(p^{-1}(U))\) are sections of the sheaf circle valued functions defined on \(p^{-1}(U)\).

Connectivity.

Let \(e_U\) and \(e'_U\) be a pair of objects of \(C_F(\omega)(U)\). Denote respectively by \(e_i\) and \(e'_i\) the respective restrictions of \(e_U\) and \(e'_U\) to \(p^{-1}(U_i)\), where \((U_i)_{i \in I}\) is an open cover of \(U\) by contractible open subsets. Since \(U_i\) is contractible, the Chern class of the differentiable bundle \(e_i\) and \(e'_i\) are mapped to \([\omega]\) by the isomorphism \(H^2(U_i \times F, \mathbb{R}) \to H^2(F, \mathbb{R})\). This implies they are isomorphic since they have the same Chern class.

If the classifying cocycle of the gerbe \(C_F(\omega)\) has a trivial cohomology class, then by a theorem of Giraud [6], the gerbe \(C_F(\omega)\) has a global section \(e\). Let \(u\) be an element of the contractible open subset \(U_i\) of \(N\). The restriction of \(e\) to \(p^{-1}(U_i)\) is an element \(e_i\) of \(C_F(\omega)(U_i)\), by definition, its restriction to \(F_u\) has Chern class \([\omega]\) •

Remark.

Denote the classifying cocycle of the gerbe \(C_F(\omega)\) by \(c_F(\omega)\). Its cohomology class is an element of the sheaf cohomology group of differentiable functions \(H^2(P, T_1)\). We can consider the exact sequence of sheaves of differentiable functions:

\[
1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow T^1 \longrightarrow 1.
\]

We deduce an isomorphism between \(H^2(P, T^1)\) and \(H^3(P, \mathbb{Z})\), since \(H^*(P, \mathbb{R})\) the cohomology of the sheaf of \(\mathbb{R}\)-differentiable functions is zero, because there exist partitions of unity. Thus the gerbe \(C_F(\omega)\) is classified by an element of \(H^3(P, \mathbb{Z})\).

In [3] Brylinski has studied the following problem: Suppose defined on \(P\) a 2-form \(\Omega\) whose restriction to each fiber is closed, integral and symplectic, find obstructions to build a closed 2-form whose restriction to a fiber \(F_u\) above \(u\) coincides with the restriction of \(\Omega\) on the fiber \(F_u\). If \(H^1(F, \mathbb{R}) = 0\), the obstruction to find such a class is a gerbe \(C_p(\omega)\) defined on \(N\).

Recall the construction of \(C_p(\omega)\). For every open set \(U\), \(C_p(\omega)(U)\) is the category whose objects are \(T^1\)-bundles over \(p^{-1}(U)\), endowed with a connection such that the restriction of its curvature to a fiber \(F_u\) above \(u\) coincides with the restriction of \(\Omega\) to \(F_u\). A morphism between two objects \((e_u, \nabla_{e_u})\) and \((e'_u, \nabla_{e'_u})\) is a morphism \(f : e_u \to e'_u\) of \(T^1\)-bundles such that \(f^*(\nabla_{e'_u}) = \nabla_{e_u}\). The group of automorphisms of \((e_U, \nabla_U)\) is the set of \(T^1\)-differentiable functions defined on \(U\). This gerbe is trivial, since as remarked McDuff in [14], in this
case the Guillemin-Lerman-Sternberg method allows to construct a closed form which extends \([\omega]\) if \(H^1(F, \mathbb{R}) = 0\).

**Remark**

Suppose that the symplectic bundle \(p : P \to N\) has a Hamiltonian reduction. Then there exists an extension \(\Omega\) of \(\omega\) (see [13]) which defines the distribution \(D^\Omega\) on \(P\) as follows: let \(u\) be an element of \(P\), then \(T_uP\) and \(TF_{p(u)}\) the respective tangent spaces of \(P\) at \(u\) and at the fiber of \(p(u)\).

\[
D^\Omega_u = \{v \in T_uP : \Omega_u(v, y) = 0, y \in TF_{p(u)}\}.
\]

When the bundle is Hamiltonian, we can suppose that the holonomy of the closed connection form is Hamiltonian. And using a standard process, we can reduce the structural group of this connection to its holonomy, and obtain thus the Hamiltonian reduction.

**Proposition 2.5.2.**

Suppose that there exists an extension \([\Omega]\) of the class \(\omega\). Let \(\Omega\) be a fixed representative. Then the set of cohomology classes of closed 2-forms \(\Omega'\) whose restriction to any fiber \(F_u\) coincides with the restriction of \(\Omega\) to the fiber \(F_u\) and such that \(D^\Omega = D^{\Omega'}\) is isomorphic to \(H^2(N, \mathbb{R})\).

**Proof.**

Remark that while this proposition is very similar to the problem of the Brylinski’s book [3] mentioned above, we cannot apply the result obtained by Brylinski since we do not suppose that the class \([\omega]\) is integral and \(H^1(F, \mathbb{R})\) may not vanish.

Let \(\Omega'\) be a representative of a cohomology class whose restriction to the fiber \(F_u\) of \(p : P \to N\) coincide with the restriction of \(\Omega\) to \(F_u\) and such that \(D^\Omega = D^{\Omega'}\). Then the form \(\Omega - \Omega'\) projects to a closed 2-form \(p(\Omega - \Omega')\) defined on the base, we have thus defined a map between the set of extensions of \([\omega]\) whose have a representant whose restriction to a fiber \(F_u\) coincides with the restriction of \(\Omega\) to \(F_u\) and defines also the distribution \(D^\Omega\), and \(H^2(N, \mathbb{R})\) by assigning to the class of \(\Omega'\) the class of \(\Omega - \Omega'\). We have to show that this map is an isomorphism.

Suppose that the class of \(p(\Omega - \Omega')\) is trivial. Then there exists a 1-form \(\alpha\) on \(N\) such that \(d(\alpha) = p(\Omega - \Omega')\). We denote by \(p^*(\alpha)\) the pulls-back of \(\alpha\) to \(P\). This implies that \(\Omega' = \Omega + d(p^*(\alpha))\), thus the class of \(\Omega\) and \(\Omega'\) coincide. This shows that the map \([\Omega'] \to [p(\Omega - \Omega')]\) is injective.

To show that this map is surjective, consider a closed 2-form \(v\) of \(N\), \(p(\Omega - (\Omega - p^*(v))) = v\).

The initial problem studied by Mc Duff was to find a Hamiltonian reduction of the bundle \(p : P \to N\), that is a symplectic bundle isomorphic to \(p\), whose transition functions take their values in the Hamiltonian group of \((F, \omega)\). This problem can be studied by a sheaf of categories. The definition of this sheaf of
category use the following result of Lalonde-McDuff [13]: which allows to define its band:

**Definition-Proposition 2.5.3.**

Let \( p : P \to N \) be a symplectic bundle. Suppose that there exists a Hamiltonian reduction of \( p \). Then there exists an extension \( \Omega \) of \( \omega \), such that the Hamiltonian reduction is defined by the holonomy of the closed connection form \( \Omega \). A Hamiltonian automorphism of the bundle \( p : P \to N \) is a diffeomorphism \( \phi \) of \( P \) which covers the identity, such that the restriction of \( \phi \) to the fiber over \( n \in N \) is an Hamiltonian automorphism of \( (F, \omega_n) \), and such that \( \phi^*(\Omega) = \Omega \). We denote by \( \text{Aut}(P, \Omega) \) the group of Hamiltonian automorphisms of the Hamiltonian reduction \( (P, \Omega) \). The group \( \text{Aut}(P, \Omega) \) does not depend of the Hamiltonian reduction.

**Remark**

In fact a more general result is shown in Lalonde-McDuff [13] that is: the group of diffeomorphisms \( G(P, \omega) \) which cover the identity and such that the restriction of each of its element \( \phi \) to a fiber \( F_u \), belongs to the connected component of the group of symplectic diffeomorphisms of \( (F, \omega_u) \), and which preserves the symplectic class which defines the Hamiltonian reduction does not depend of the chosen Hamiltonian reduction. This result implies the one stated in the proposition above since this group \( G(P, \omega) \) contains \( \text{Aut}(P, \Omega) \). The elements of \( \text{Aut}(P, \Omega) \) are the elements of \( G(P, \omega) \) which when restricted to \( (F_u, \omega_u) \) are Hamiltonian. We see that this last condition is independent of the chosen Hamiltonian connection \( \Omega \) which defines any Hamiltonian reduction of \( p : P \to N \). A morphism \( f : P \to P' \) between the Hamiltonian bundles \( P \) and \( P' \) is a morphism of fiber bundles \( f \) such that \( f^*(\Omega') = \Omega \), where \( \Omega \) and \( \Omega' \) are the closed connections forms whose holonomy define respectively the Hamiltonian reduction of \( P \) and \( P' \).

Now we can show the following:

**Proposition 2.5.4.**

Let \( p : P \to N \) be a symplectic fibration. For any open set \( U \) of \( N \), we define \( C^1_F(\omega)(U) \) to be the category whose objects are Hamiltonian structures on the symplectic bundle \( p^{-1}(U) \to U \). A morphism between the objects \( (e_U, \Omega_U) \) and \( (e'_U, \Omega'_U) \) of \( C^1_F(\omega)(U) \) is a morphism of bundles \( f : e_U \to e'_U \) such that \( f^*(\Omega'_U) = \Omega_U \). The correspondence defined on the category of open subsets of \( N \) by \( U \to C^1_F(\omega)(U) \) is a gerbe whose band \( L \) is the sheaf induced by the presheaf of Hamiltonian automorphisms such that for each open set \( U \) of \( N \), and each \( e_U \) of \( C^1_F(\omega)(U) \), \( L(U) \) is the group of Hamiltonian automorphisms of \( e_U \) see Definition-Proposition 2.5.3. The cohomology class of the classifying cocycle of \( C^1_F(\omega) \) is the obstruction for the existence of a Hamiltonian reduction of \( p : P \to N \).

**Proof.**
Gluing conditions of objects.

Consider \((U_i)_{i \in I}\) an open cover of the open subset \(U\) of \(N\), such that \(C^1_F(\omega)(U_i)\) is not empty, and \((e_i, \Omega_i)\) an object of \(C^1_F(\omega)(U_i)\). Suppose that there exists a family of morphisms \(u_{ij} : e^i_j \to e^i_3\) such that \(u_{ij}^{-1} u_{ij} = u_{ji}\). Then there exists a \(F\)-bundle \(e\) over \(U\) whose restriction to \(U_i\) is \(e_i\). We have to show that this bundle is Hamiltonian. Since \(u_{ij}^* (\Omega_i) = \Omega_j\), the forms \(\Omega_i\) glue together to define on \(e\) an extension \(\Omega\) of \(\omega\). Consider a path \(c : [0, 1] \to N\), we can suppose that \([0, 1]\) is a union of intervals \(I_l\) such that \(I_l\) is contained in \(U_i\), an open set of the above cover. The holonomy of the connection form \(\Omega\) along \(I_l\) is the holonomy of \(\Omega\) along \(I\). We conclude that the holonomy of \(\Omega\) along \(I\) is Hamiltonian, since each closed form \(\Omega\) define an Hamiltonian reduction on \(e_i\).

Gluing conditions of arrows.

Let \(e_U\) and \(e'_U\) be a pair of objects of \(C^1_F(\omega)(U)\). The correspondence defined on the category of open subsets of \(U\) which associates to \(V\) the set of Hamiltonian morphisms \(\text{Ham}(e_U, e'_U)\) is a sheaf since it is the subsheaf of the sheaf of morphisms between two bundles.

Connectivity.

Let \(e_U\) and \(e'_U\) be a pair of objects of \(C^1_F(\omega)(U)\). We can suppose that the open cover \((U_i)_{i \in I}\) of \(U\) is a Hamiltonian trivialization of the both bundles \(e_U\) and \(e'_U\). This implies that the restrictions of \(e_U\) and \(e'_U\) to \(U_i\) are isomorphic as Hamiltonian bundles to the trivial Hamiltonian bundle \(U_i \times (F, \omega)\). We deduce that these Hamiltonian bundles are locally isomorphic.

Let \((U_i)_{i \in I}\) be a symplectic trivialization of \(p : P \to N\). The trivial symplectic bundle \(U_i \times (F, \omega)\) is an element of \(C^1_F(\omega)(U_i)\), which is not empty.

The result of Lalonde and McDuff [13] recalled above shows that the group \(\text{Aut}(e_U, \Omega_U)\) of Hamiltonian automorphisms of the Hamiltonian reduction of the restriction of \(p\) to \(p^{-1}(U)\) does not depend of the chosen object in \(C^1_F(\omega)(U)\). This implies that the correspondence defined on the category of open subsets of \(N\) by \(U \to \text{Aut}(e_U, \Omega_U)\) defines a presheaf \(L\) on \(U\). We denote by \(L\) the sheaf associated to this presheaf. Remark that if \(C^1_F(\omega)(U)\) is not empty, then \(L(U) = \text{Aut}(e_U, \Omega_U)\) for each object \(e_U\) of \(C^1_F(\omega)(U)\). This implies that the gerbe is bounded by \(L\).

2.6 The McDuff construction of \(\text{Ham}^s\), and closed connection forms.

The existence of a closed connection form \(\Omega\) on the symplectic bundle \(p : P \to N\) does not insure the existence of a Hamiltonian reduction of this bundle. This has motivated McDuff to introduce the group denoted \(\text{Ham}^s\), such that the existence of a closed connection form is equivalent to the existence of a \(\text{Ham}^s\)-reduction. We will now present the construction of the group \(\text{Ham}^s\), and show using gerbe theory that a \(\text{Ham}^s\)-reduction implies the existence of a closed connection form on a symplectic bundle.
**Definitions McDuff 2.6.1.**

Let $H_1(F, \omega, \mathbb{Z})$ be the first homology group of $F$ with integral coefficients, we define $SH_1(F, \omega, \mathbb{Z})$ to be the quotient of the integral 1-cycles of $F$ by the image under the boundary of 2-cycles with zero symplectic area. We denote $SH_1(F, \omega, \mathbb{Q})$ to be the tensor product $SH_1(F, \omega, \mathbb{Z}) \otimes \mathbb{Q}$. Often we will respectively denote $SH_1(F, \omega, \mathbb{Z})$ by $SH_1(F, \omega, \mathbb{Z})$ and $SH_1(F, \omega, \mathbb{Q})$ by $SH_1(F, \omega, \mathbb{Q})$.

Let $P_\omega$ be the values of $\omega$ on rational cycles. We have the exact sequence:

$$0 \to \mathbb{R}/P_\omega \to SH_1(F, \mathbb{Q}) \to H_1(F, \mathbb{Q}) \to 0$$

Consider a section $s$ of $H_1(F, \mathbb{Q}) \to SH_1(F, \mathbb{Q})$. Then we can define on $\text{Symp}(F, \omega)$ the group of symplectomorphisms of $(F, \omega)$, the map $F_s : \text{Symp}(F, \omega) \to H_1(F, \mathbb{R}/P_\omega) = H_1(F, \mathbb{R})/H_1(F, P_\omega)$ by:

$$F_s(g)(u) = g(su) - s(gu)$$

Recall that the group $\text{Symp}(F, \omega)$ acts canonically on $SH_1(F, \mathbb{Q})$ and $H_1(F, \mathbb{Q})$.

McDuff has shown that the application $F_s$ is a 1-cocycle for the canonical representation defined on $\text{Symp}(F, \omega)$ which takes its values in the group of linear automorphisms of $H_1(F, \mathbb{R})/H_1(F, P_\omega)$, and has defined $\text{Ham}^s$ to be the kernel of this cocycle $F_s$.

**Theorem McDuff 2.6.2.**

A symplectic bundle $p : P \to N$ has a $\text{Ham}^s$-reduction if and only if there exists a closed connection form. Moreover, the group $\text{Ham}^s$ intersects every connected component of $\text{Symp}(F, \omega)$.

We will present now a proof of the first part of this theorem using gerbe theory. In fact this problem can be reformulated in a more general situation: Let $G$ be a Lie group whose dimension can be infinite, and $H$ a subgroup of $G$, we suppose that $G/H$ is a $K(\pi, 1)$ space, that is its universal cover is contractible and its fundamental group is $\pi$. We are looking for conditions which insure the existence of a $H$-reduction. This problem can be formulated using gerbe theory. We have:

**Theorem 2.6.3.**

Let $p : P \to N$ be a $G$-principal bundle defined on $N$. Suppose either:

(i) the transitions functions $u_{ij} : U_i \cap U_j \to G$ take their values in the normalizer $\text{Nor}(H)$ of $H$ in $G$, where $H$ is a subgroup of $G$, and $G/H$ is a $K(\pi, 1)$ space.

(ii) or there exists a continuous representation $h : G \to L$ where $L$ is an abelian group isomorphic to the quotient of a vector space $V$ by a discrete subgroup $\pi$ such that the restriction of $h$ to the connected component of the identity $G_0$ of $G$ is trivial, a continuous surjective 1-cocycle for this representation whose kernel $H$ intersects every connected component of $G$.
Then there exists a gerbe $C_H$ defined on $N$, bounded by the locally constant sheaf defined on $N$ by $\pi$ which represents the obstruction of the bundle $p : P \to N$ to have a $H$-reduction.

**Proof.**

The proof is a corollary of the following lemmas:

**Lemma 2.6.4.**

Suppose first that there exists a subgroup $H$ of $G$, such that the transitions functions $u_{ij}$ of $p : P \to N$ are contained in the normalizer $\text{Nor}(H)$ of $H$ in $G$, then the right quotient fiber by fiber of the bundle $p$ by $H$, defines a $G/H$-bundle $\bar{p} : \bar{P} \to N$. Let $G/H$ be the universal cover of $G/H$. For each open subset $U$ of $N$, define the category $C_H(U)$ to be the category whose objects are $G/H$-bundles whose quotient fiber by fiber by $\pi$ (recall that $\pi$ is the fundamental group of $G/H$) is the restriction of $\bar{p}$ to $U$, a morphism $f : e_U \to e'_U$ between two objects $e_U$ and $e'_U$ of $C_H(U)$ is a morphism of $G/H$-bundles which projects to the identity on their quotient by $\pi$. Then the correspondence defined on the category of open subsets of $N$, $U \to C_H(U)$ defines a gerbe, whose classifying cocycle is the obstruction for reduce the structural group $G$ of the bundle $p$ to $H$.

**Proof.**

We have first to show the existence of the bundle $\bar{p}$. Let $(U_i, u_{ij})$ be a trivialization of the bundle $p$. Since $u_{ij}$ take their values in $\text{Nor}(H)$, for each element $x$ of $U_i \cap U_j$, the right multiplication by $u_{ij}(x)$ of $G$ gives rise to a $G/H$-action of $u_{ij}(x)$ on $G/H$. We denote by $\bar{u}_{ij}(x)$ this induced action. The map $\bar{u}_{ij} : U_i \cap U_j \to G$, $x \mapsto \bar{u}_{ij}(x)$ verified the Chasle relation, thus defines a $G/H$-bundle $\bar{p}$ over $N$. Now we show that the correspondence $U \to C_H(U)$ is a gerbe.

Gluing condition for objects.

Let $U$ be an open set of $N$, $(U_i)_{i \in I}$ an open cover of $U$, and $e_i$ an object of $C_H(U_i)$. We suppose that there exists maps $g_{ij} : e_j \to e_i$ such that $g_{ii}^{-1}g_{ij}g_{ij}^{-1} = g_{ij}$. Since $e_i$ is a bundle, there exists a bundle $e$ over $U$ whose restriction to $U_i$ is $e_i$. Since the restriction to $U_i$ of the quotient fiber by fiber, of $e$ by $\pi$ is the quotient fiber by fiber of $e_i$ by $\pi$, we deduce that $e$ is an element of $C_H(U)$.

Gluing condition for arrows.

For each pair of objects $e$ and $e'$, of $C_H(U)$ the correspondence defined on the category of open subsets of $U$ by $V \to \text{Hom}(e_V, e'_V)$ where $e_V$ and $e'_V$ are the respective restrictions of $e$ and $e'$ to $V$ defines a sheaf, since it is the sheaf of morphisms between two bundles.
This shows that the correspondence defined on the category of open subsets of $N$ by $U \to C_H(U)$ is a sheaf of categories. Now we show that it is a gerbe.

Let $(U_i)_{i \in I}$ be a trivialization of the bundle $\bar{\rho}$, then we can lift the restriction of $\bar{\rho}$ to $U_i$ to a bundle $U_i \times G/H$. This shows that $C_H(U_i)$ is not empty.

Let $U$ be an open set of $N$. Consider two objects $e_U$, and $e_U'$ of $C_H(U)$. The restriction of $e_U$ and $e_U'$ to $U_i \cap U$ are isomorphic to $U_i \cap U \times G/H$ this implies that the connectivity property holds.

The definition of $\text{Hom}(e_U, e_U')$, the group of automorphisms of an object $e_U$ shows that its elements coincide with the action of $\pi$, which thus defines a locally constant sheaf on $N$, which is the band of $C_H$.

It remains to show that the triviality of the gerbe $C_H$ is equivalent to the existence of a $H$-reduction of $G$. Let $\tilde{G}$ and $\tilde{H}$ be respectively the universal cover of $G$ and $H$. The homotopy sequence applied to the fibration $H \to \tilde{G} \to \tilde{G}/H$ implies that $\tilde{G}/H$ is simply connected. The map $\tilde{G}/H \to G/H$ is a covering map, thus $G/H$ is the universal cover of $G/H$. Suppose that the gerbe $C_H$ is trivial, then a global object of this gerbe is a $\tilde{G}/H$-bundle. Since $\tilde{G}/H$ is contractible, we deduce that this bundle is trivial, and thus have a global section. This section projects to a section of $\bar{\rho}$. This implies that the bundle $\rho$ has a $H$-reduction $\bullet$

**Lemma 2.6.5.**

Suppose that there exists a continuous representation $h : G \to L$ (where $L$ is a quotient of a vector space $V$ by a discrete subgroup $\pi$), whose restriction to the connected component $G_0$ of $G$ is trivial. Suppose also the existence of a continuous cocycle $F$, surjective, for this representation whose kernel $H$ intersects every connected component of $G$, then for every principal $G$-bundle $p : P \to N$, there exists a gerbe $C_H$, which represents the geometric obstruction for the bundle $p$ to have a $H$-reduction.

**Proof.**

For each elements $g \in G$, and $h \in H$, we have $F(gh) = F(g) + gF(h) = F(g)$. This implies that the cocycle $F$ defines a map $\tilde{F} : G/H \to L$. The map $\tilde{F}$ is surjective since $F$ is surjective. Let $[g]$, and $[g']$ be two elements of $G/H$, suppose that $\tilde{F}([g]) = \tilde{F}([g'])$. Since $H$ intersects every connected component of $G$, we can choose two elements $g$ and $g'$ in $G_0$, and respectively in the class $[g]$ and $[g']$ such that $F(g) = F(g')$.

$$F(gg'^{-1}) = F(g) + h(g)F(g'^{-1}) = F(g) + F(g'^{-1}) = 0$$

since $g \in G_0$, and the restriction of $h$ to $G_0$ is trivial. We deduce that $F(g) = F(g')$, thus $\tilde{F}$ is a diffeomorphism.

Remark that $H \cap G_0 = H_0$ is a normal subgroup of $G_0$. Since $\tilde{F}$ is a diffeomorphism, we deduce that $G/H = G_0/H_0$ is diffeomorphic to $L$. 

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Consider now a $G$-bundle $p : P \to N$, defined by the trivialization $u_{ij} : U_i \cap U_j \to G$. Then $F(u_{ij})$ defines a $L = G/H$-bundle $\bar{p}$ over $N$. The action of $F(u_{ij}(x))$ on an element $[g]$ of $G/H$ is defined by $F(gu_{ij}(x))$ where $g$ is an element of $[g]$ in $G_m$. For each open subset $U$ of $N$, we define $C_H(U)$ to be the category whose objects are $V$-bundles over $U$, whose quotient fiber by fiber by $\pi$ is the restriction of $\bar{p}$ to $U$. Recall that $L$ is the quotient of $V$ by $\pi$. The set of morphisms $\text{Hom}(e, e')$ between two objects $e$ and $e'$ of $C_H(U)$ is the set of morphisms of $L$-bundles which project to the identity on the restriction of $\bar{p}$ to $U$. We are going to show that the correspondence defined on the category of open subsets of $N$, by $U \to C_H(U)$ is a gerbe who represents the geometric obstruction to reduce $G$ to $H$.

Gluing property for objects.

Consider an open subset $U$ of $N$, and an open covering $(U_i)_{i \in I}$ of $U$. Let $e_i$ be an element of $C_H(U_i)$. Consider a morphism $g_{ij} : e_j' \to e_i'$ such that

$$g_{ij}^{-1} g_{ij}^{-1} = g_{ij}^{-1} g_{ij}^{-1}.$$ 

Since $e_i$ are bundles, there exists a bundle $e$ over $U$ whose restriction to $U_i$ is $e_i$. Since the restriction to $U_i$ of the quotient fiber by fiber of $e$ by $\pi$ is the quotient fiber by fiber of $e_i$ by $\pi$, we deduce that $e$ is an element of $C_H(U)$.

Gluing condition for arrows.

For each pair of objects $e$ and $e'$, the correspondence defined on the category of open subsets of $U$ by $V \to \text{Hom}(e_V, e'_V)$, where $e_V$ and $e'_V$ are the respective restrictions of $e$ and $e'$ to $V$ defines a sheaf, since it is the sheaf of morphisms between two bundles.

Let $(U_i)_{i \in I}$ be a trivialization of the bundle $\bar{p}$, then we can lift the restriction of $\bar{p}$ to $U_i$, to a bundle $U_i \times G/H$. This shows that $C_H(U_i)$ is not empty.

Consider two objects $e_U$ and $e'_U$ of $C_H(U)$. The restrictions of $e_U$ and $e'_U$ to $U_i \cap U$ are isomorphic to $U_i \cap U \times G/H$ this implies that the connectivity property holds.

The definition of $\text{Hom}(e_U, e'_U)$ the group of automorphisms of the bundle $e_U$ shows that it can be identified with $\pi$, which thus defines a locally constant sheaf on $N$ which is the band of $C_H$.

It remains to show that the triviality of the classifying cocycle of the gerbe $c_H$ implies the existence of an $H$-reduction. Let $\hat{G}$ and $\hat{H}$ be respectively the universal cover of $G$ and $H$. The homotopy sequence applied to the fibration $\hat{H} \to \hat{G} \to \hat{G}/\hat{H}$ implies that $\hat{G}/\hat{H}$ is simply connected. The map $\hat{G}/\hat{H} \to G/H$ is a covering map, thus $\hat{G}/\hat{H}$ is the universal cover of $G/H$. Suppose that the gerbe $C_H$ is trivial, then a global object of this gerbe is a $\hat{G}/\hat{H}$-bundle. Since $\hat{G}/\hat{H}$ is contractible, we deduce that this bundle is trivial, and thus have a global section. This section projects to a section of $\bar{p}$. This implies that the bundle $p$ has a $H$-reduction.
Remark.
In the case of the lemma 2.6.5, above, the cohomology class of the classifying cocycle $c_H$ is the obstruction for the bundle $\bar{p}$ to be flat. This implies that it is the Chern class of this bundle.

We are going to apply the above result to study the problem of the existence of $Ham^s$-reductions.

**Theorem 2.6.6.**
Let $p: P \to N$ be a symplectic bundle whose typical fiber is $(F, \omega)$, then there exists a gerbe $[c_{Ham^s}]$ such that the cohomology class $[c_{Ham^s}] \in H^2(N, H^1(F, \mathbb{R})/H^1(F, P_\omega))$ of its classifying cocycle $c_{Ham^s}$ is the obstruction to reduce the structural group of the bundle to $Ham^s$. If the coordinate changes of the bundle take their values in the connected component $\text{Symp}(F, \omega)_0$ of $\text{Symp}(F, \omega)$, then there exists a gerbe $C_{Ham}$ whose classifying cocycle is the obstruction for reduce the structural group to $Ham(F, \omega)$.

**Proof.**
The group $Ham^s$ is the kernel of the continuous surjective 1-cocycle $F_s$, and it intersects every connected component of $\text{Symp}(F, \omega)$. The quotient of $\text{Symp}(F, \omega)$ by $Ham^s$ is $H^1(F, \mathbb{R})/H^1(F, P_\omega)$. We can apply theorem 2.6.4.

Suppose that the coordinate changes take their values in $\text{Symp}(F, \omega)_0$, since $Ham(F, \omega)$ is a normal subgroup of $\text{Symp}(F, \omega)_0$, and the flux homomorphism allow us to identify $\text{Symp}(F, \omega)_0/\text{Ham}(F, \omega)$ with $H^1(F, \Gamma)/H^1(F, \Gamma)$, where $\Gamma$ is the flux group, we can apply theorem 2.6.4

**Remark.**
In differential geometry, like in the theory of $G$-structures the question of finding reductions of a $G$-bundle is intensively studied. Let $H$ be a subgroup of $G$, if the left quotient $H/G$ is a $K(\pi, 1)$ space, it is possible to write a similar theorem to the one above, and obtain an obstruction cocycle whose cohomology class decide of the existence of a $H$-reduction. This can be for example applied to the existence of a riemannian structure on a manifold, and also to solve differential equations defined on jet-bundles, since in many cases the existence of solutions is equivalent to the existence of reductions of jets bundles.

We will give now another proof of the theorem of McDuff mentioned above which says that the existence of a closed connection form on a symplectic bundle $p: P \to N$ implies the existence of a $Ham^s$-reduction.

**Theorem McDuff 2.6.7. (see [14]).**
Let $p: P \to N$ be a symplectic bundle endowed with a closed connection form, then there exists on $P$ a $Ham^s$-reduction.

**Other proof.**
Suppose the existence of a closed connection form defined on the bundle $p: P \to N$. We have to show that the cohomology class $[c_{Ham^s}]$ is trivial.
It has been shown by McDuff-Lalonde [13], that the holonomy around a contractible loop is Hamiltonian. Consider the reduction of the symplectic bundle to the holonomy of the closed connection form. Since the Hamiltonian group is the connected component of $\text{Ham}^s$, we deduce that the composition of the transitions functions $u_{ij}$ and of $F_s$, $F_s(u_{ij})$ is constant, if needed, we shrink the open set $U_i$ such that $u_{ij}(U_i \cap U_j)$ is contained in the same connected component of $\text{Symp}(F, \omega)$. This implies that the bundle $\tilde{p}$ (defined in the proof of Lemma 2.6.5) is flat. Thus its Chern class is a torsion class. Since the lattice $\pi$ in this case is a $\mathbb{Q}$-vector space, we deduce that the Chern class of this bundle is zero.

\section*{Sketch of the proof of McDuff [14].}

McDuff defines for each symplectic bundle $p : P \to N$ of fiber $(F, \omega)$, a cohomology 2-class in $H^2(N, H^1(F, \mathbb{P}))$ (in fact it is the class of 2.6.5) as follows: The bundle $p$ is defined by a classifying map $p' : N \to B\text{Symp}(F, \omega)$. The map $F_s$ induces a map $F'_s : B\text{Symp}(F, \omega) \to B\mathbb{H}^1(F, \mathbb{R})/\mathbb{H}^1(F, \mathbb{P})$. There exists a $\text{Ham}^s$-reduction if and only if the composition $F'_s \circ p'$ is null homotopic, since we have an exact sequence

$$1 \longrightarrow \text{Ham}^s \longrightarrow \text{Symp}(F, \omega) \longrightarrow \mathbb{H}^1(F, \mathbb{R})/\mathbb{H}^1(F, \mathbb{P}) \longrightarrow 1.$$ 

The space $B\mathbb{H}^1(F, \mathbb{R})/\mathbb{H}^1(F, \mathbb{P})$ is a $K(\mathbb{H}^1(F, \mathbb{P}), 2)$-space, and the set of homotopy classes of maps $N \to K(\mathbb{H}^1(F, \mathbb{P}), 2)$ is one to one with $H^2(N, \mathbb{P})$. The obstruction class of McDuff is defined to be the homotopy class of $F'_s \circ p'$.

The proof of McDuff of the previous result, is done by showing that the previous class vanishes on the 2-sub-complex of the $CW$-complex $N$. On this purpose, she shows that it is the image by a null connecting homomorphism related to an exact sequence of a one class.

\section*{2.7 The universal obstruction of McDuff.}

In this part, we will show how the universal class defined by McDuff can be defined using gerbe theory.

Let $ES\text{Symp}(F, \omega) \to B\text{Symp}(F, \omega)$ be the universal bundle of the group $\text{Symp}(F, \omega)$. The 1-cocycle $F_s : \text{Symp}(F, \omega) \to H^1(F, \mathbb{R})/\mathbb{H}^1(F, \mathbb{P})$ defined by McDuff, induces a $H^1(F, \mathbb{R})/\mathbb{H}^1(F, \mathbb{P})$-bundle on $B\text{Symp}(F, \omega)$. See lemma 2.6.5. The Chern class $U_F$ of this bundle is the universal class $U_M$ defined by McDuff, it can be viewed as the cohomology class of the classifying cocycle of the gerbe which represents geometrically the obstruction for the previous $H^1(F, \mathbb{R})/\mathbb{H}^1(F, \mathbb{P})$-bundle to be trivial. Since each $(F, \omega)$-symplectic bundle $p : P \to N$, is classified by a classifying map $f : N \to B\text{Symp}(F, \omega)$, the obstruction class to obtain a $\text{Ham}^s$-reduction is $f^*(U_F)$. This is the class defined in the sketch of the proof of McDuff in 2.6.

\section*{2.8 Generalizations to topoi.}
The previous construction applied to symplectic bundles can be generalized to other situations; algebraic geometry, arithmetic,... On this purpose we will adapt this result to topos.

**Definitions 2.8.1.**

Let $G$ be a group endowed with a topology, the topology can be the Zariski, etale, the Lie topology, etc... A continuous right $G$-action of $G$ on the topos $(P, J_P)$, is a continuous functor $d_G : P \times G \to P$, such that if $u$ is the multiplication of $G$ by $g$, $d_G \circ (I_P \times u) = d_G(d_G \times I_G)$.

A $G$-torsor defined on a topos $N$ is a continuous functor $p : (P, J_P) \to (N, J_N)$ such that:

(i) $(P, J_P)$ is endowed with an action of $G$, $p$ commutes with the action of $G$ that is, the composition $P \times G \to P \to N$, (where $P \times G \to P$ is the canonical projection, and $P \to N$ is $p$) and $P \times G \to P \to N$ (where $P \times G \to P$ is the multiplication $d_G$ and $P \to N$ is $p$) coincide.

(ii) The canonical map $P \times G \to P \times P \times G \to P \times P$ which is the composition of the canonical embedding $P \times G \to P \times P \times G$, and the product of the identity on the first factor, and the multiplication $d_G$ on the second and third factor is an isomorphism. We suppose that the quotient of $P$ by $G$ is $N$. Recall that the quotient of $P$ by $G$ is the initial element in the category of maps $p' : P \to N'$ such that $p'$ commutes with the action of $G$.

We will assume that the torsor is locally trivial. This means that there exists a covering family of $N$, $(U_i)_{i \in I}$ such that:

There exists an isomorphism $u_i : P_{|U_i} \to U_i \times G$ between the restriction $P_{|U_i}$ of $P$ to $U_i$ and $U_i \times G$. We can thus define $u_{ij} = u_i \circ u_j^{-1} : U_i \times N U_j \times G : U_i \times N U_j \times G \to U_i \times N U_j \times G$. Let $e' : G \to G, g \to e$, where $e$ is the neutral of $G$ and $e_1 : U_i \times N U_j \times G \to G$ the canonical projection. We can define $u_{ij} : U_i \times N U_j \to G$ by $e_1 \circ u_{ij} \circ (I_{U_i \times N U_j} \times e')$. We have $u_{i_1i_2}^{-1}u_{i_2i_3}^{-1}u_{i_1i_3} = u_{i_1i_2}^{-1}$. $P$ is obtained by gluing the family of $(U_i \times G)_{i \in I}$ using $u_{ij}$.

Let $H$ be a subgroup of $G$, we say that the torsor $P \to N$ has a $H$-reduction if and only if it is isomorphic to a torsor whose transition functions $u_{ij}$ take their values in $H$.

**Theorem 2.8.2.**

Let $p : P \to N$ be a $G$-torsor. Suppose that either,

1. there exists a subgroup $H$ of $G$ such that, $G/H$ is a $K(\pi, 1)$ space, and the torsor has a $\text{Nor}(H)$-reduction.

2. or there exists a 1-cocycle surjective and continue $F : G \to L$ for a representation $h$ of $G$, where $L$ is the quotient of a vector space by a discrete subgroup $\pi$, such that the restriction of $h$ to the connected component $G_0$ of $G$ is trivial, and the kernel $H$ of $F$ intersects every connected component of $G$.

3. or the left quotient $H/G$ is a $K(\pi, 1)$-space,

then there exists a gerbe $C_H$ defined on $N$ such that the cohomology class of its classifying cocycle is the obstruction for reduce $G$ to $H$.

**Proof.**
We will only give the proof in the first case. The fact that the torsor has a \( \text{Nor}(H) \)-reduction implies the existence of a \( G/H \)-torsor \( \bar{P} \), which is the right quotient of \( P \) by \( H \).

For each object \( U \) of \( N \), we define \( C_H(U) \) to be the category whose objects are \( G/H \)-torsors whose quotient by \( \pi \) is the restriction of \( \bar{P} \) to \( U \). A morphism between two objects of \( C_H(U) \), is a morphism of torsors which projects to the identity on the restriction of \( \bar{P} \) to \( U \).

Now we show that the correspondence \( U \to C_H(U) \) is a gerbe.

Gluing condition for objects.

Let \( U \) be an object of \( N \), \((U_i)_{i \in I} \) a covering family of \( U \), \( e_i \) an object of \( C_H(U_i) \). We suppose that there exists maps \( g_{ij} : e_i \to e_j \) such that \( g_{ij} g_{ij} = g_{ij} \). Since \( e_i \) are torsors, there exists a torsor \( e \) over \( U \) whose restriction to \( U_i \) is \( e_i \). Since the restriction to \( U_i \) of the quotient of \( e \) by \( \pi \) is the quotient of \( e_i \) by \( \pi \), we deduce that \( e \) is an element of \( C_H(U) \).

Gluing condition for arrows.

For each objects \( e \) and \( e' \), the set of morphisms defined on the sub-topos over \( U \), by \( V \to \text{Hom}(e_V, e'_V) \), where \( e_V \) and \( e'_V \) are the respective restrictions of \( e \) and \( e' \) to \( V \) defines a sheaf of sets, since it is the sheaf of morphisms between two torsors.

Let \((U_i)_{i \in I} \) be a trivialization of the torsor \( \bar{P} \), we can lift the restriction of \( \bar{P} \) to \( U_i \) to the torsor \( U_i \times G/H \). This shows that \( C_H(U_i) \) is not empty.

Consider two objects \( e_U \) and \( e'_U \) of \( C_H(U) \). The restrictions of \( e_U \) and \( e'_U \) to \( U_i \times_N U \) are isomorphic to \( U_i \times_N U \times G/H \) this implies that the connectivity property holds.

The group \( \text{Hom}(e_U, e_U) \) is the the group of automorphisms of the torsor \( e_U \) which project to the identity isomorphism of the restriction of \( \bar{p} \) to \( U \). This group is identified to \( \pi \), which thus defines a locally constant sheaf on \( N \) which is the band of \( C_H \).

**Remark.**

The triviality of the gerbe \( C_H \) does not necessarily implies the existence of a \( H \)-reduction, if \( N \) is not a manifold. Since for other categories, homotopy is not well-understood, there are no precise definitions of null-homotopic maps.

3. The group \( \text{Ham}^* \) and the etale topos of a manifold.

The group \( \text{Ham}^* \) introduced by McDuff allows to characterize symplectic bundles whose have a closed connection form to be the symplectic bundles endowed with a \( \text{Ham}^* \)-reduction. In [14] it is shown that a \( \text{Symp}(F, \omega) \)-bundle \( p : P \to N \) is endowed with a closed connection form if and only if there exists
a finite cover $\hat{N}$ of $N$, such that the pull-back of $p$ to $\hat{N}$ has a Hamiltonian reduction. This motivates to define $\text{Symp}(F,\omega)$-bundles on the etale topos of $N$. The motivation is due to this historical remark: In algebraic geometry, algebraic principal bundles are locally trivial up to a finite etale cover. This has motivated the definition of the etale topology.

**Definitions 3.1.**

The **etale topos** of a manifold $N$ is the category whose objects are differentiable maps $c : U \rightarrow N$ which are finite covering maps onto their images. A morphism between two objects is a covering map.

A covering family of the etale topos, $Et(N)$ of $N$, is a family $(U_i)_{i \in I}$ such that the arrow $u_i : U_i \rightarrow N$ is a finite etale cover, and the union of $(u_i(U_i))_{i \in I}$ is $N$.

A symplectic bundle $p : P \rightarrow Et(N)$ whose typical fiber is the symplectic manifold $(F,\omega)$ is defined by a covering family $(U_i)_{i \in I}$ of $Et(N)$ for the etale topology. The transition functions are symplectic bundle isomorphisms of the trivial symplectic bundle $U_i \times_N U_j \times \text{Symp}(F,\omega)$, defined by $u_{ij} : U_i \times_N U_j \rightarrow \text{Symp}(F,\omega)$ such that $u_{ij}^{-1} u_{i1} u_{1j} = u_{i1} u_{j2}$.

A closed connection form on the symplectic bundle is defined by a family of closed connections forms $\Omega_i$ of the bundle $c : U_i \times (F,\omega)$ (recall that $\Omega_i$ is a 2-form which extends $\omega$), such that on $U_i \times_N U_j$, we have: $u_{ij}^* (\Omega_i|_{U_i \times_N U_j}) = \Omega_j|_{U_i \times_N U_j}$, where $\Omega_i|_{U_i \times_N U_j}$ and $\Omega_j|_{U_i \times_N U_j}$ are the respective restrictions of $\Omega_i$ and $\Omega_j$ to $U_i \times_N U_j$.

A symplectic bundle defined on $N$ induces canonically a symplectic bundle on $Et(N)$, since an open covering of $N$ defines an etale covering of $N$. 

**Proposition 3.2.**

Let $P$ be a symplectic bundle defined on the etale topos of a manifold $N$, then there exists a symplectic bundle $\hat{P}$ defined on a covering space $\hat{N}$ of $N$, such that the symplectic bundle induced by $\hat{P}$ on $Et(\hat{N})$, is the pull-back of $P$ by the covering map $\hat{N} \rightarrow N$. If $N$ is compact we can suppose that $\hat{N}$ is a finite cover.

**Proof.**

Let $(d_i : U_i \rightarrow N)_{i \in I}$ be the etale covering family of $N$ which defines the symplectic bundle. Then we can define a manifold $\hat{N}$ as follows: $\hat{N}$ is the quotient of the union of $U_i$ by identifying the elements $u_i \in U_i$, and $u_j \in U_j$ such that $d_i(u_i) = d_j(u_j)$. We denote by $\hat{l}_i : U_i \rightarrow \hat{N}$ the canonical map. The manifold $\hat{N}$ is a cover of $N$ since the restriction of the canonical projection $\hat{N} \rightarrow N$ to $\hat{l}_i(U_i)$ is $d_i^{-1}l_i$.

There exists a diffeomorphism $l_{ij} : l_i(U_i) \cap l_j(U_j) \rightarrow U_i \times_N U_j$, such that on $l_{ij}(U_i) \cap l_{ij}(U_j) \cap l_{ij}(U_k)$, $l_{ij}^{-1}l_{ij} = \text{Id}_{l_{ij}(U_i) \cap l_{ij}(U_j) \cap l_{ij}(U_k)}$, thus we can define the symplectic bundle $\hat{P}$ on $\hat{N}$ by gluing $l_i(U_i) \times (F,\omega)$ using $u_{ij}^* = u_{ij} \circ l_{ij}$, where $u_{ij}$ are the transition functions of $P$. 

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The construction of $\hat{P}$ shows that the induced bundle on $Et(\hat{N})$, by $\hat{P}$, is the pull-back of $P$ by the canonical map $\hat{N} \to N$. If $N$ is compact, then we can suppose that there exists a finite number of $U_i$, this implies that $\hat{N}$ is compact, therefore is a finite cover of $N$. 

We can rewrite the theorem of McDuff [14] as follows:

**Theorem 3.3.**

Let $p : \hat{P} \to Et(N)$ be a $\text{Sym}(F,\omega)_0$-bundle defined on the etale topos of a compact manifold $N$, then $\hat{P}$ has a closed connection form if and only if it has a Hamiltonian reduction.

**Proof.**

The previous proposition shows that there exists a finite cover $\hat{N}$ of $N$, and an induced symplectic bundle $\hat{P}$ over $\hat{N}$. Suppose that the closed connection form is defined on $\hat{P}$ by the family of 2-forms $\Omega_i$ defined on the etale cover $(l_i : U_i \to \hat{N})_{i \in I}$. As in the previous proposition, we can show that there exists a finite cover $N'$ of $N$ such that the pull-back $P'$ of $\hat{P}$ to $N'$ is endowed with a closed connection form, such that the closed connection form induced on its etale cover is defined on $U'_i = P' \times_N U_i$ by the pull-back of $\Omega_i$ by $P' \times_N U_i \to U_i$. We can apply the result of McDuff and obtain a Hamiltonian reduction $P'' \to N''$ on the pull-back of $P'$ to a finite cover $N''$ of $N'$. We denote by $l''_i$, the canonical map $l''_i : U''_i = U'_i \times N', P'' \to N''$. There exists a family of maps $u''_i : l''_i(U''_i) \to \text{Sym}(F,\omega)$ such that $u''_i u''_i u''_i u''_i \in \text{Ham}(F,\omega)$, where $u''_i$ are the coordinate changes of $P''$, thus $u''_i u''_i u''_i u''_i \in \text{Ham}(F,\omega)$ defined a Hamiltonian reduction of $\hat{P}$. Since the family $(U''_i \to N'')_{i \in I}$ is an etale cover of $N''$, $(U''_i \to N'' \to N)_{i \in I}$ is also an etale cover of $N$.

4. Flux, and holonomy of gerbes.

In this part, we will relate the flux of a symplectic manifold $(F,\omega)$ to the holonomy of the gerbe $C(\omega)$ defined in 2.4.

Let $E$ be a $T^1$-gerbe defined on $P$, that is a gerbe such that for each open set $U$ of $P$, $E(U)$ is a category of $T^1$-bundles defined on $U$. Consider an open covering $(U_i)_{i \in I}$ of $P$, such that $U_i$ is contractible. Let $c^i_j$ be the restriction of an object $c_i$ of $E(U_i)$ to $U_i \cap U_j$, there exists a morphism $u_{ij} : c^j_i \to c^i_j$. We denote by $c_{ijl}$, the automorphism $u_{ij} u_{ij} u_{ijl}$ of the restriction of $c_{ijl}$ to $U_i \cap U_j \cap U_l$. It is defined by a $T^1$-differentiable function. Since $c_{ijl}$ is the classifying 2-cocycle of $E$, there exists a 1-chain $h_{ij}$ of 1-forms such that:

$$h_{ij} - h_{il} + h_{ij} = -\frac{i}{2\pi} d(\text{Log}(c_{ijl}))$$

since $d(h_{ij})$ is a 1-cocycle, there exists a 0-chain of 2-forms $L_i$ such that

$$L_j - L_i = d(h_{ij})$$
Definitions 4.1.

The family of forms $h_{ij}$ is called a connection of the gerbe, and the family of forms $(L_i)_{i \in I}$ is the curving of the gerbe, this means that there exists a related connective structure $Co$ defined on the gerbe, and elements $\alpha_i$ of $Co(e_i)$, such that $h_{ij} = \alpha_j - u_{ij}^* \alpha_i$. The 3-form whose restriction to $U_i$ is $dL_i$ is the curving of the connective structure. Suppose that the curvature is zero, then $L_i = d(L'_i)$, $h_{ij} = L'_j - L'_i + d(h'_{ij})$, we denote by $c'_{i_1 i_2 i_3} = \frac{i}{2\pi} \log(c_{i_1 i_2 i_3}) + h'_{i_2 i_3} - h'_{i_1 i_3} + h'_{i_1 i_2}$ to be the holonomy of the connection, $c'_{i_1 i_2 i_3}$ is constant and is a 2-cocycle.

Definition 4.2.

For each map $l : N_2 \to P$, where $N_2$ is a surface without a boundary, the pull-back of the gerbe, and its connective structure to $N_2$, by $l$ has a vanishing curving. Using the Cech-De Rham isomorphism, we can identify the holonomy cocycle of this gerbe with a 2-form $Hol(h_{ij}, N_2)$. The holonomy of the connection on $N_2$ is

$$\int_{N_2} Hol(h_{ij}, N_2)$$

Let $(F, \omega)$ be a symplectic manifold, and $C_F(\omega)$ the $T^1$-gerbe representing the obstruction of $[\omega]$ to be integral. If the band of this gerbe is extended to the sheaf of differentiable $T^1$-functions, it becomes trivial and flat.

For each open set $U$ of $F$, the set of connections defined on an object $e_U$ of $C_F(\omega)(U)$ which curvature is the restriction of $\omega$ to $U$ defines a connective structure, the curving of the connective structure is the restriction of $\omega$ to $U$. The cocycle representing the holonomy of this connective structure is the image of $\omega$ by the Cech-De Rham isomorphism. This can be deduced from 2.4.3.

Let $l : N_2 \to F$ be a differentiable map defined on the surface $N_2$, the holonomy of this connective structure around $N_2$ is:

$$\int_{N_2} l^*(\omega)$$

This definition is related to the definition of the flux, since for each path $\gamma = c_t$ of $N$, and each path $\phi_t$ of the connected component of $\text{Symp}(F, \omega)$, $\phi_t(\gamma)$ is a map from $h : I_2 \to F$, the flux of $\phi_t(\gamma)$ is nothing but the half of the holonomy around the sphere $S^2$ obtained by gluing two copies of $I_2$ along their boundaries. The map $f : S^2 \to N$ is obtained by restricting $h$ to each copy of $I_2$. The holonomy of $f$ is defined to be the limit of the holonomy of a sequence of differentiable maps which converges towards $f$.

5. A geometric interpretation of a section of $H_1(M, \mathbb{R}) \to SH_1(M, \mathbb{R})$. 27
In [14], McDuff gives a geometric interpretation of a section \( p : H_1(M, \mathbb{R}) \to SH_1(M, \mathbb{R}) \), when the cohomology class \([\omega]\) is integral. In this section we generalize this interpretation when \([\omega]\) is not necessarily integral. We denote by \( \pi : SH_1(M, \mathbb{R}) \to H_1(M, \mathbb{R}) \) the projection map. Suppose that the class \([\omega]\), of the symplectic manifold \((F, \omega)\) is not necessarily integral. Consider a cycle \([\gamma]\) represented by the chain \( h : T^1 \to F \), where \( T^1 \) is the circle, the pull-back by \( h \), of the gerbe \( C_F(\omega) \), to \( T^1 \) is trivial.

**Proposition 5.1.**
Consider an object \( e \) of \( h^*(C_F(\omega)) \) which is the pull-back of an object \( e' \) of a tubular neighborhood of \( h(T^1) \). Let \( L \) be a connection in \( h^*Co(e') \). Denote by \( h'_L(\gamma) \) the holonomy around \( \gamma \) of \( L \). It does not depend of the element chosen in \( h^*(Co(e')) \).

**Proof.**
To show this, consider another connection \( L' \) in \( h^*(Co(e')) \). We can suppose that \( h(T^1) \) is covered by \( (U_i)_{i \in I} \), the union of \( U_i \) is a tubular neighborhood of \( h(T^1) \), and \( C_F(\omega)(U_i) \) is not empty. The fact that the union of \( U_i \) is a tubular neighborhood of \( h(T_i) \) implies that the restrictions of \( L \) and \( L' \) to \( I_i = h^{-1}(h(T^1) \cap U_i) \) can be supposed to be the pull-back of elements \( d + \alpha_i \) and \( d + \alpha'_i \) of \( Co(h_i) \), where \( d \) is the differential \( e_i \) is an object of \( C_F(\omega)(U_i) \) and \( \alpha_i \) and \( \alpha'_i \) are 1-forms defined on \( U_i \). We have \( \alpha'_i = \alpha_i + df_i \) where \( f_i \) is a function defined on \( U_i \) since the curving of the gerbe is the closed form \( \omega \). Denote by \( L_i \) and \( L'_i \) the restrictions of \( L \) and \( L' \) to \( I_i \). On \( I_i \), \( L_i = d + du_i \), the coordinate changes \( v_{ij} \) of the bundle \( e \) are defined by \( du_j - du_i = -i \frac{1}{2\pi} d Log(v_{ij}) \), the holonomy cocycle of \( L \) is given by \(-i \frac{1}{2\pi} d Log(v_{ij}^{-1}) - u_j + u_i \). Since \( L'_i = d + d(u_i + f_i) \), \( f_i \) is the pull-back of \( f'_i \) by \( h \). We deduce that the holonomy cocycle of \( L \) and \( L' \) coincide up to a boundary. Thus their cocycle have the same cohomology class \( \bullet \).

We can define \( h_L(\gamma) \) the image of the holonomy of this connection in \( \mathbb{R}/P_\omega \). Let \( [\gamma] \in H_1(F, \mathbb{R}) \), defines the section \( p([\gamma]) \) to be the class of elements \( \gamma' \) in \( \pi^{-1}([\gamma]) \) such that the holonomy around \( \gamma' \) is in \( P_\omega \).

**6. Existence of symplectic bundles and gerbes.**

Let \( F \) be the flux, and \( \Gamma_\omega \) be the flux group. The flux conjecture has been shown recently by Ono, thus \( \Gamma_\omega \) is a discrete subgroup of \( H^1(F, \mathbb{R}) \). There exists an exact sequence

\[ 1 \to Ham(F, \omega) \to Symp_0(F, \omega) \to H^1(F, \mathbb{R})/\Gamma_\omega \to 1. \]

Let \( p : P \to N \) be a symplectic bundle defined by the coordinate changes \( g_{ij} : U_i \cap U_j \to Symp_0(F, \omega) \) on the trivialization \((U_i)_{i \in I}\). We can project the cocycle \( g_{ij} \) to maps \( F(g_{ij}) = g'_{ij} : U_i \cap U_j \to H^1(F, \mathbb{R})/\Gamma_\omega \), and obtain a \( H^1(F, \mathbb{R})/\Gamma_\omega \)-bundle as in 2.6.5. A natural question is the following: given a \( H^1(F, \mathbb{R})/\Gamma_\omega \)-bundle \( \tilde{p} \) is there a symplectic bundle which gives rise to \( \tilde{p} \).
This problem is an example of the basic examples which have motivated the definition of gerbes theory. Consider an open set $U$ of $N$, and $C(U)$ a category of symplectic bundles whose transition functions take their values in $\text{Symp}(F,\omega)_0$, and which induces the restriction of $\bar{p}$ to $U$. Suppose that $\bar{p}$ is defined by the transition functions $g'_{ij}$, and there exists elements $g_{ij}$ over $g'_{ij}$ such that the conjugation by $g_{ij}$ in $\text{Ham}(F,\omega)$ defined a bundle over $N$ whose typical fiber is $\text{Ham}(F,\omega)$. We suppose also that the automorphisms group of an object $e_U$ of $C(U)$ are the sections of the previous $\text{Ham}(F,\omega)$-bundle. we denote $L_1$ the sheaf of those sections. The correspondence $U \to C(U)$ is a gerbe bounded by $L_1$.

Denote by $l$ the rank of the group $\Gamma_{\omega}$, the torus $T^l$ is the maximal compact subgroup of $H^1(M,\mathbb{R})/\Gamma_{\omega}$. The bundles defined over $N$, which fiber is $T^l$ are classified by their first Chern class. This can enable to construct symplectic bundles which does not admit Hamiltonian reductions if the Chern class is not zero.

7. 2-gerbes, 2-gerbed towers.

The notion of 2-gerbe has been defined by Lawrence Breen [2], [3] it allows to represent geometrically 3-cohomology classes. In the preprint [20], Tsemo has defined the notion of gerbed towers, this is a recursive definition of geometric representations of cohomology classes. We will present now the notion of 2-gerbes, and 2-gerbed towers, which enable us to cope with the extension problem when $[\omega]$ is not necessarily an integral class. An alternative discussion has been presented above using the group $\text{Ham}^*$, the construction given in this section allows to show the existence of a connection on a bundle which has a Hamiltonian reduction without using the Guillemin-Lerman-Sternberg construction. The definition of sheaf of 2-categories uses the definition of 2-categories or bicategories which has been defined by Benabou.

Definition.

A bicategory $C$ is defined by a class of objects $C$, for each pair of objects $u$ and $v$ of $C$, a category $\text{Hom}(u,v)$. The objects of $\text{Hom}(u,v)$ are called the 1-arrows, and the arrows of $\text{Hom}(u,v)$ are the 2-arrows, there exists a composition map:

$$\text{Hom}(u_2,u_3) \times \text{Hom}(u_1,u_2) \to \text{Hom}(u_1,u_3)$$

For each quadruple $(u_1,u_2,u_3,u_4)$, there exists an isomorphism between the functors

$$(\text{Hom}(u_3,u_4) \times \text{Hom}(u_2,u_3)) \times \text{Hom}(u_1,u_2) \to \text{Hom}(u_1,u_4)$$

and
which satisfies more compatibility axioms which can be found in Breen \[2\].

**Definitions.**

Let \( N \) be a manifold, a **sheaf of 2-categories** is a correspondence \( C \) defined on the category of open subsets of \( N \) by:

\[
U \rightarrow C(U)
\]

where \( C(U) \) is a 2-category, which verifies the following properties:

- For each embedding map \( U \rightarrow V \), there exists a restriction functor \( r_{U,V} : C(V) \rightarrow C(U) \), such that

\[
r_{U,V} \circ r_{U',V} = r_{U,V'}.
\]

**Gluing properties for objects.**

Let \( (U_i)_{i \in I} \) be a covering family of an open set \( U \) of \( N \), \( e_i \) an object of \( C(U_i) \), and a 1-arrow \( g_{ij} : r_{U_i \cap U_j} (e_i) \rightarrow r_{U_i \cap U_j} (e_j) \), suppose there exists a 2-arrow \( h_{i_1i_2i_3} : g_{i_1i_2} \circ g_{i_2i_3} \rightarrow g_{i_1i_3} \) which satisfies:

\[
h_{i_1i_2i_3} (Id \circ h_{i_2i_3i_4} ) = h_{i_1i_3i_4} (h_{i_1i_2i_3} \circ Id)
\]

then there exists an object \( e \) of \( C(U) \) whose restriction to \( U_i \) is \( e_i \).

**Gluing conditions for arrows.**

For each pair of objects \( e \) and \( e' \) of \( U \), the correspondence defined on the category of open sets contained in \( U \) by \( V \rightarrow Hom(r_{U,V}(e), r_{U,V}(e')) \) defines a sheaf of categories.

A **2-gerbe** is a sheaf of bicategories which satisfies the following:

1. The bicategory \( C(U) \) is a 2-groupoid, this means that 1-arrows are invertible up to 2-arrows, and 2-arrows are invertible.
2. For every point \( x \) of \( N \), there exists a neighborhood \( U_x \) of \( x \), such that \( C(U_x) \) is not empty.
3. Any pair of objects \( e \) and \( e' \) of \( C(U) \) are locally isomorphic. This means that there exists an open covering \( (U_i)_{i \in I} \) of \( U \) such that the restrictions \( e_i \) and \( e'_i \) of respectively \( e \) and \( e' \) to \( U_i \) are isomorphic.

We say that a 2-gerbed is bounded by the sheaf of abelian groups \( L \), if the following two conditions are satisfied:

4. Any pair of 1-arrows can be joined by a 2-arrow.
5. Let \( e_U \) and \( e'_U \) be a pair of objects of \( C(U) \). For any 1-arrow \( h : e_U \rightarrow e'_U \), there is a specified isomorphism \( L(U) \rightarrow Aut(h) \), compatible with compositions.
and with restrictions \( \bullet \). We say that the sheaf \( L \) is the band of the 2-gerbe, or that the gerbe is bounded by \( L \).

### 7.2. Classifying cocycle of a 2-gerbe.

Let \( (U_i)_{i \in I} \) be an open covering of \( N \) such that \( C(U_i) \) is not empty. Consider an object \( e_i \) of \( C(U_i) \), and \( g_{ij} : r_{U_{ij}}(e_j) \to r_{U_{ij}}(e_i) \), there exists a 2-arrow \( h_{i\{i\}j} : g_{i\{i\}j}^2 \to g_{i\{i\}j}^1 \), and on \( U_{i\{i\}j} \) a 2-arrow \( u_{i\{i\}j} \) which verifies:

\[
h_{i\{i\}j}^3 (Id \circ h_{i\{i\}j}^1) = u_{i\{i\}j} (h_{i\{i\}j}^2 \circ Id).\]

The family \( u_{i\{i\}j} \) is the classifying 2-cocycle of \( C \), if the sheaf \( L \) is commutative, it is a Cech cocycle in the classical sense, and the set of isomorphic classes of 2-gerbes bounded by \( L \) is isomorphic to \( H^3(N,L) \). If \( L \) is not commutative, we define \( H^3(N,L) \) to be the set of isomorphic classes of 2-gerbes bounded by \( L \).

In [20] we have given a simplified version of 2-gerbes, that we have named 2-gerbed towers.

**Definition 7.2.1.**

A 2-gerbed tower defined on \( N \), is defined by a gerbe \( C \) on \( N \) and for each object \( e_U \) of \( C(U) \), a gerbe \( C_1(e_U) \) defined on \( U \) such that the following conditions are satisfied:

(i) For each embedding map \( U \to V \), there exists a restriction functor \( r_{U,V}^1 : C_1(e_U) \to C_1(r_{U,V}(e_U)) \) such that \( r_{V,U}^1 \circ r_{U,V}^1 = r_{U,V}^1 \), where \( r \) is the restriction functor of the gerbe \( C \).

(iii) There exits a commutative sheaf \( L_1 \) defined on \( N \), such that for each object \( e_U \) of \( C(U) \), the band of \( C_1(e_U) \) is the restriction of \( L_1 \) to \( U \).

(ii) For each morphism \( h : e_U \to e'_U \) of objects of \( C(U) \), there exists a functor \( h^* : C_1(e_U) \to C_1(e'_U) \) which is compatible with restrictions, such that for a morphism \( h' : e'_U \to e''_U \), there exists a natural transformations between the functor \( (h'h)^* \) and \( h'^* h^* \). We suppose also the functors \( (h'h)^* \) and \( h'^* h^* \) coincide on objects. This implies the existence of an element \( l_{h',h} \) of \( L_1(U) \) such that \( (h'h)^* = l_{h',h} \circ h'^* h^* \).

### 7.3. The classifying cocycle of a 2-gerbed tower.

We can associate to a 2-gerbed tower, a 3-Cech cocycle defined as follows: Consider an object \( e_i \) of \( C(U_i) \) and a morphism \( g_{ij} : r_{U_{ij}}(e_j) \to r_{U_{ij}}(e_i) \). The arrow \( c_{i\{i\}j} = g_{i\{i\}j}^2 g_{i\{i\}j}^1 \) is the Cech classifying cocycle of the gerbe \( C \). It can be identified to an element of the band of \( C \).
The classifying cocycle of the 2-gerbed tower is defined by considering the family of automorphisms

\[ c_{i1i2i3i4} = (c_{i2i3i4}^{-1})^* (c_{i3i1i4}^{-1})^* (c_{i1i2i4}^{-1})^* (c_{i1i2i3}^{-1})^* \]

Property (iii) implies that \( c_{i1i2i3i4} \) is an element of \( L_1(U_{i1} \ldots i4) \). Contrary to the case of 2-gerbes, it is after having defined the classifying cocycle, that we set the axiom concerning the gluing property for objects:

**Gluing property for objects.**

Suppose that the cohomology class of the classifying cocycle of a 2-gerbed tower is zero. Let \( (U_i)_{i \in I} \) be the open covering of \( N \) used to construct the cocycle. Then there exists a gerbe \( C_0 \), such that for each open subset \( U \) of \( N \), the restriction of \( C_0 \) to \( U \cap U_i \) is \( C_1(e_iU) \), where \( e_iU \) is the restriction of \( e_i \) to \( U_i \cap U \), and \( e_i \) is the object of \( C(U_i) \) used to construct the 2-cocycle.

**Proposition 7.3.1.**

Let \( (C, C_1) \) be a 2-gerbed tower defined on \( N \), the correspondence defined on the category of open subsets of \( N \) as follows:

To each open set \( U \) of \( N \), \( C'(U) \) is the bicategory whose objects are gerbes \( E_U \) such that for every open covering \( (U_i)_{i \in I} \) of \( U \) such that \( C(U_i) \) is not empty, the restriction of \( E_U \) to \( U_i \) is isomorphic to a gerbe \( C_1(e_i) \) where \( e_i \) is an object of \( C(U_i) \). A 1-arrow \( h : E_U = C_1(e_iU) \to E'_U = C_1(e'_iU) \) between two objects of \( C'(U) \) is a functor \( h^* \), where \( h : e_U \to e'_U \) is an arrow. A 2-arrow is a natural transformation \( l_U \) between two 1-arrows which coincide on objects.

**Proof.**

Gluing property for objects.

Consider an open covering family \( (U_i)_{i \in I} \) of \( N \). Let \( E_i = C_1(e_i) \) be an object of \( C'(U_i) \), a morphism between \( g_{ij} : E'_j \to E'_i \) is a functor \( h_{ij} : C_1(e'_j) \to C_1(e'_i) \), where \( h_{ij} : e'_j \to e'_i \) is an arrow. A 2-arrow between \( h_{i1i2}^{-1} h_{i2i3}^{-1} \) and \( h_{i1i3}^{-1} \), is a natural transformation

\[ c_{i1i2i3} : h_{i1i2}^{-1} h_{i2i3}^{-1} \to h_{i1i3}^{-1} \]

defined by an element of \( L_1(U_{i1i2i3}) \). The fact that:

\[ c_{i1i3i4} \circ (c_{i1i2i3}^{-1} \circ Id) = (Id \circ c_{i2i3i4}^{-1}) \]

is equivalent to the gluing property of objects of a 2-gerbed tower. This implies by definition the existence of an object \( E_U \) whose restriction to \( U_i \) is \( E_i \).
Let $E_U$ and $E'_U$ be two respective objects of $C'(U)$. The correspondence defined on the category of open subsets of $U$ by $V \rightarrow Hom(E_U|_V, E'_U|_V)$, where $E_U|_V$ and $E'_U|_V$ are the respective restrictions of $E_U$ and $E'_U$ to $V$ is a sheaf of categories since it is the sheaf of morphisms between two gerbes.

Let $U$ be an open subset of $N$, the objects $E_U$, and $E'_U$ of $C'(U)$ are locally isomorphic, since the restrictions of $E_U$ and $E'_U$ to an open cover of $(U_i)_{i \in I}$ of $U$ such that the objects of $C(U_i)$ are isomorphic are isomorphic. If we replace $U$ by $N$, and choose a covering family such that $C(U_i)$ is not empty, we obtain that $C'(U_i)$ is not empty.

The set of automorphisms of a 1-arrow is isomorphic to $L_1(U)$ by definition.

The notion of 2-gerbed tower is easier to understand than the one of 2-gerbe, principally because, we do not need the notion of bicategory to define it. In practice, many the examples of 2-gerbed are defined using the notion of 2-gerbed tower, another advantage of this notion is the fact that the classifying cocycle of a 2-gerbed tower $(C, C_1)$ is the image of a 2-cocycle, that is, the classifying cocycle of $C$ by a connecting morphism in cohomology.

8. The general case.

We will now describe the 2-gerbe and 2-gerbed towers bounded by the sheaf of locally constant $\mathcal{R}$-functions which represent the geometric obstruction to extend $\omega$ to $P$ when the cohomology class $[\omega]$ of $\omega$ is not necessarily integral.

Let $U$ be an open subset of $N$, and $[\Omega_U]$ an extension of $[\omega]$ to $p^{-1}(U)$. We cannot define a $T^1$-bundle over $p^{-1}(U)$ (as in the integral case) whose Chern class is $[\Omega_U]$.

Definitions 8.1.

We denote by $C'_p(\Omega, p^{-1}(U))$ the gerbe defined on $p^{-1}(U)$ which is the obstruction of the class $[\Omega_U]$ to be trivial. see 2.4

Let $U$ be an open subset of $N$, we define the bicategory $C^2_p(\omega)(p^{-1}(U))$ to be the class whose elements are categories $C^2_p(\Omega, p^{-1}(U))$. Let $e_1$ and $e_2$ be two objects of $C^2_p(p^{-1}(U))$, a 1-arrow $f : e_1 = C'_p(\Omega_1, p^{-1}(U)) \rightarrow e_2 = C'_p(\Omega_2, p^{-1}(U))$ is an isomorphism of gerbes between $e_1$ and $e_2$, and a 2-arrow is a natural transformation between those functors. More precisely, on a contractible cover $(U'_i)_{i \in I}$ of $p^{-1}(U)$, the restrictions of the objects of $e_1$ are torsors whose objects are isomorphic to trivial $\mathcal{R}$-bundles $U'_i \times \mathcal{R}$, a 1-arrow $f$ is defined by the respective objects $e_1^i$ and $e_2^i$ of the respective restrictions of $e_1$ to $U'_i$, and of $e_2$ to $U'_i$, and an isomorphism $f_i$ between $e_1^i$ and $e_2^i$. Due to the natural properties of $f$, we can use these morphisms to rebuild completely $f$. This implies that these datas satisfy the following properties:

The identification of $e_1^i$ and $e_2^i$ to $U'_i \times \mathcal{R}$ allows to represents $f_i$ has a morphism of the trivial torsor $U'_i \times \mathcal{R}$, the fact that $f$ behave naturally in respect
with restrictions implies the existence of a morphism \( u_{ij} \) of \( U'_{ij} \times R \) such that 
\[ f_i = u_{ij} f_j. \]
We have \( u_{i_1 i_2 i_3} u_{i_2 i_3} = u_{i_1 i_2} \). The map \( u_{ij} \) is a translation by an element of \( R \). The family \((u_{ij})_{i,j \in I} \) defines a 1-cocycle, thus a closed 1-form on \( p^{-1}(U) \). Conversely, a 1-cocycle of the sheaf of locally constant \( R \)-maps defines a torsor, and a 1-arrow between \( e^1 \) and \( e^2 \) by using the previous identification of \( e^1 \) and \( e^2 \) to \( U'_i \times R \).

Using the identification above, a 2-arrow is defined locally by a chain of elements of \( C \) defined on the category of open subsets of \( U \). Gluing conditions for objects. Consider an open covering \((U_i)_{i \in I} \) of an open subset \( U \) of \( N \), and \((e_i, [\Omega_i])\) an object of \( C^2_F(\omega)(p^{-1}(U_i)) \) where \([\Omega_i]\) is a cohomology class defined on \( p^{-1}(U_i) \) which extends \([\omega]\). Suppose that there exists 1-arrows \( h_{ij} : e^1_j \to e^1_i \), a 2-arrow \( d_{i_1 i_2 i_3} : h_{i_2 i_3} h_{i_1 i_3}^{-1} \to h_{i_1 i_2}^{-1} \) such that \( d_{i_1 i_2 i_3} (d_{i_1 i_2 i_3}^{-1} \circ 1d) = d_{i_1 i_2 i_3} (1d \circ d_{i_2 i_3 i_1}) \). The maps \( d_{i_1 i_2 i_3} \) can be identified with a 2-Cech cocycle of the sheaf of locally constant \( R \)-functions defined on \( p^{-1}(U) \). We can identify it using the De Rham Weil isomorphism with an element \([\Omega_U]\) of \( H^2(p^{-1}(U), R) \). The class \([\Omega_U]\) is the classifying cocycle of a gerbe \( e_U \) defined on \( U \) bounded by the sheaf of locally constant \( R \)-functions. We have to show now that this gerbe is an element of \( C^2_F(p^{-1}(U)) \).

The fact that the family \( d_{i_1 i_2 i_3} \) is a 2-Cech cocycle, implies that there exists a gerbe bounded by the sheaf of locally constant functions whose restriction to \( U_i \) is \( e_i \). (See the proof of the classifying theorem for gerbes presented in the book of Breen [2]) this gerbe is isomorphic to \( e_U \). This implies that the restriction of \([\Omega_U]\) to \( p^{-1}(U_i) \) is the classifying cocycle of \( e_i \), and that \([\Omega_U]\) extends \([\omega]\), since the restriction of \([\Omega_U]\) to \( U_i \) is \([\Omega_i]\). We deduce that it is the cohomology class of the classifying cocycle of an element of \( C^2_F(\omega)(p^{-1}(U)) \) whose restriction to \( p^{-1}(U_i) \) is \( e_i \).

Gluing conditions for arrows.

Let \( e \) and \( e' \) be a pair of objects of \( C^2_F(\omega)(p^{-1}(U)) \), the correspondence defined on the category of open subsets of \( U \) by \( U' \to Hom(e_{U'}, e'_{U'}) \) is a sheaf of categories, since it is the sheaf of categories of morphisms between two gerbes. This shows that \( C^2_F(\omega) \) is a sheaf of 2-categories.
Consider an open covering \((U_i)_{i \in I}\) of \(N\) by contractible open sets, since 
\(H^*(\hat{U}_i \times F) = H^*(F)\), we can extend \([\omega]\) to 
\(p^{-1}(U_i) = U_i \times F\), and two such 
extension classes are equal to the class \([\omega]\) as shows the identification 
\(H^*(\hat{U}_i \times F) = H^*(F)\). This implies that 
\(C^2_{\mathbb{R}}(\omega)(p^{-1}(U_i))\) is not empty, an its objects are isomorphic.

The sheaf of 2-categories \(C^2_{\mathbb{R}}(\omega)\) is bounded by the sheaf of \(\mathbb{R}\)-locally constant functions defined on \(P\). This is shown in the paragraph above this theorem.

Suppose that the class of the classifying cocycle of \(C^2_{\mathbb{R}}(\omega)\) vanishes, then 
the 2-gerbe has a global section \(e\), its restriction to 
\(p^{-1}(U_i)\) is an element of 
\(C^2_{\mathbb{R}}(\omega)(p^{-1}(U_i))\) whose classifying cocycle extends \([\omega]\). This implies that the 
classifying cocycle of \(e\) extends \([\omega]\)

The cocycle defined by McDuff, and the classifying cocycle \(c^2_F\) of the 2-gerbe 
\(C^2_{\mathbb{R}}(\omega)\) solve the same geometric problem: decide if the class \([\omega]\) can be extended to the total space of the symplectic bundle 
\((F, \omega)\). We will show now that they are related by an isomorphism of cohomology groups.

Suppose that the family \((U_i)_{i \in I}, g_{ij}\) defines the coordinate changes of \(P\). 
Let \(\tilde{\text{Symp}}(F, \omega)\) be the universal cover of \(\text{Symp}(F, \omega)\). Consider an element \(h_{ij}\) 
of \(\text{Ham}^\ast\) such that \(g_{ij}(x)h_{ij} = g'_{ij}(x)\) is contained in 
\(\tilde{\text{Symp}}(F, \omega)_0\), and a lift: 
\(g'_{ij} : U_i \cap U_j \to \tilde{\text{Symp}}(F, \omega)\) of the functions \(g'_{ij}\). Remark that an element \(g'_{ij}(x)\) 
is an equivalence class of a path in \(c : [0, 1] \to \text{Symp}(F, \omega)\). We choose a path 
\(u_{ij}\) which represents it and set:

\[
\int_0^1 \omega(\frac{d}{dt} u_{ij}(x), \ldots) = g''_{ij}(x)
\]

**Proposition. 8.3**

The chain \(c_{i_1i_2i_3} = g''_{i_2i_3}^{i_1} - g''_{i_1i_3}^{i_2} + g''_{i_1i_2}^{i_3}\) is a 2-Cech cocycle whose cohomology class is identified using the Cech-Weil isomorphism to the McDuff obstruction class.

**Proof.**

The element \(g''_{ij}(x)\) is a lift of \(F_s(g''_{ij})\) in \(H^1(F, \mathbb{R})\), since the restriction of 
\(F_s\) to \(\text{Symp}(F, \omega)_0\) factors by the flux. This implies that \(g''_{ij}\) represents the classifying 
cocycle of the \(H^1(F, \mathbb{R})/H^1(F, P_0)\)-bundle (see 2.6.5) whose coordinate changes are the functions 
\(F_s(g''_{ij})\)

We can use the Cech-Weil isomorphism to identify \(c_{i_1i_2i_3}\) to a closed 2-form 
\(\Omega'\) defined on \(N\) which take values in the vector bundle \(p_\omega\) of closed \(P_0\) 1-forms 
defined on \(F\) induced by \(g_{ij}\). Let \(\Omega(F, P_\omega)\) be the vector space of closed \(P_0\) 1-forms defined on \(F\). The bundle \(p_\omega\) is the quotient of the union of \(U_i \times \Omega(F, P_\omega)\) 
by the following transitions functions:

\[
(x, \alpha) \to (x, g_{ij}(x)^*\alpha)
\]
where \(g_{ij}(x)^*(\alpha)(y)\) is defined by:

\[
g_{ij}(x)^*(\alpha)(y) = \alpha(d(g_{ij}(x)^{-1})(y))
\]

The identification of \(c_{i_{1}j_{1}k_{1}l_{1}}\) to \(\Omega'\) defines a 3-form \(\Omega\) on \(P\) by \(\Omega(x, y, z) = \Omega'(x, y)(z)\) where \(x, y\) are elements of \(T_nN\) the tangent space of \(N\) at \(n\), and \(z\) is an element of the tangent space to the fiber at \(n\).

Consider the Leray-Serre spectral sequence related to the fibration \(p : P \to N\) the McDuff obstruction class is an element of \(E_2^{2,1}\) which converges to \([c_F^2(\omega)]\).

**Theorem 8.3.**

Under the notation just above, the cohomology class of \(\Omega\) is the obstruction to lift \([\omega]\) to \(P\). Its cohomology class can identified to the class of the classifying cocycle of \(C_F^2(\omega)\).

**Proof.**

Let \(e_i\) be the gerbe defined on \(U_i \times F\) whose classifying cohomology class is the image of the class of the 2-form \(\Omega_i\), which is the product of 0 and \(\omega\) by the Cech-Weil isomorphism. The gerbe \(e_i\) is an object of \(C_F^2(\omega)(U_i)\). The morphism \(g^{*}_{ij}\) defined at the proposition above is a morphism between \(e_i^1\) and \(e_j^1\). This implies that \(c_{i_{1}j_{1}k_{1}l_{1}}\) represents also the classifying cocycle of \(C_F^2(\omega)\).

**8.2 Hamiltonian reduction and closed connection forms.**

We have given a gerbe formulation to the problem of the existence of a Hamiltonian reduction, by defining the gerbe \(C_F^1(\omega)\), now we are going to show how the classifying cocycle of \(C_F^1(\omega)\) and \(C_F^2(\omega)\) are related.

The link between the classifying cocycles of \(C_F^1(\omega)\) and \(C_F^2(\omega)\) appears clearly by considering the 2-gerbed towers defined as follows:

**Definition 8.2.1.**

Consider \(U\), an open set of \(N\), \(e_U\) an object of \(C_F^1(\omega)(U)\), it is a Hamiltonian structure defined on the restriction of the symplectic fibration \(p : P \to N\) to \(U\).

We deduce that there exists an extension \([\Omega_U]\) of \([\omega]\) to \(p^{-1}(U)\) whose holonomy defines the Hamiltonian reduction of \(e_U\). Denote by \(C_F^2(e_U)\) the gerbe which represents the obstruction of \([\Omega_U]\) to be trivial. We have just defined a 2-gerbed tower \((C_F^1(\omega), C_F^2)\).

Let \(L\) be the band of the gerbe \(C_F^1(\omega)\), and \(L_0\) the sheaf of locally constant \(\mathcal{R}\)-functions defined on \(P\). We define the following sheaf \(L'\) on \(P\): suppose that \(e_U\) is an object of \(C_F^1(\omega)(U)\), \(V\) an open subset of \(p^{-1}(U)\), and \(e_V\) an object of \(C_F^2(e_U)(V)\). An automorphism \(g\) of \(e_U\) map \(e_U\) to the object \(g^{-1*}(e_V)\) of \(C_F^2(e_U)(g(V))\), given \(c \in \mathcal{R}\), for each morphism \(h : e_V \to e'_{V}\) between objects of \(C_F^2(e_U)(V)\) we consider the morphism between \(g^{-1*}(e_V) \to g^{-1*}(e'_{V})\) defined by composing \(g^{-1*}(h)\) by the translation by \(c\) fiber by fiber. The sheaf generated
by the set of actions on the gerbe $C_2(e_U)$ that we have just defined is $L'$. (It does not depend of $e_U$). We can suppose that $L$ is defined on $P$ by setting $L(U) = L(p(U)), U \subset P$. We have the exact sequence:

$$1 \rightarrow L_0 \rightarrow L' \rightarrow L \rightarrow 1$$

This gives rise to the following exact sequence in cohomology:

$$H^2(P, L_0) \rightarrow H^2(P, L') \rightarrow H^2(P, L) \rightarrow H^3(P, L_0)$$

Here if $E$ is a sheaf defined on $P$, the space $H^2(P, E)$ is the space of isomorphism classes of gerbes bounded by $E$ defined on $P$. The space $H^3(P, E)$ is the space of isomorphism classes of 2-gerbes bounded by $E$. See Breen [2].

The next result show that the class of the classifying cocycle of the 2-gerbe tower $(C^2_1(\omega), C_2)$ is the image of the class of the classifying cocycle of $C^2_P(\omega)$ by the map $H^2(P, L) \rightarrow H^3(P, L_0)$.

**Proposition 8.2.2.**

The class of the classifying cocycle $c^2_P(\omega)$ of $C^2_P(\omega)$, is the image of the class of the classifying cocycle $c^2_P(\omega)$ of $C^2_P(\omega)$, by the map $H^2(P, L) \rightarrow H^3(P, L_0)$. Suppose that there exists a Hamiltonian reduction of the bundle $P \rightarrow N$, then we can extend $[\omega]$ to $P$.

**Proof.**

The classifying cocycle of this 2-gerbed tower is defined as follows, consider an object $e_i$ of $C^2_P(\omega)(U_i)$, and a map $u_{ij} : e_i \rightarrow e_i'$, the map $c_{i_1 i_2 i_3} = u_{i_1 i_2 i_3} : e_i \rightarrow e_i'$ is an automorphism of $e_{i_1 i_2 i_3}$, we can lift it to a map $e_{i_1 i_2 i_3}$ of $C^2_P(e_{i_1 i_2 i_3}(\omega))$, the Cech boundary $c_{i_1 i_2 i_3 i_4}$ of the chain $c_{i_1 i_2 i_3}$ is the classifying cocycle of the 2-gerbed tower. It appears that $c_{i_1 i_2 i_3 i_4}$ is the image of $c_{i_1 i_2 i_3}$ by the connecting map $H^2(P, L) \rightarrow H^3(P, L)$. Considered as a 2-gerbe the 2-gerbed tower involved here is a subgerbe of $C^2_P(\omega)$, since for each object $e_U$ of $C^2_P(\omega)(U)$, the gerbe $C^2_P(e_U)$ is an object of $C^2_P(\omega)(U)$. This shows that if $[c^2_P(\omega)]$ vanishes, then $[c^2_P(\omega)]$ also vanishes.

This result is shown by McDuff in [14] by using the Guillemin-Lerman-Sternberg construction.

9. Quantization of the symplectic gerbe.

Let $(F, \omega)$ be a symplectic manifold, when the class $[\omega]$ is integral, there exists a line bundle $L$ over $F$ whose chern class is $[\omega]$. This line bundle is endowed with a hermitian metric. The hermitian space of sections $L^2(F) = \{ u : F \rightarrow L : \int_F |u|^2 < +\infty \}$ is the quantization of the manifold. The elements of this Banach space are used in theoretical physic, to describe evolution of particles.

The goal of this part is to associate to any symplectic form, a hermitian space endowed with a Hermitian form, which is a candidate to represents the phase space in quantum mechanic.
Let $C(\omega)$ be the symplectic gerbe defined on $F$, which represents the obstruction of $[\omega]$ to be integral see 2.4. Consider an open covering $(U_i)_{i \in I}$ of $F$, and $e_i$ an object of $C(\omega)(U_i)$. We can define the gerbe $L(\omega)$ on $F$, such that $L(\omega)(U)$ is the category, whose objects are $(e_U, e'_U)$ where $e_U$ is an object of $C(\omega)(U)$, and $e'$ the $C$-line vector bundle over $U$, whose transition functions are the transition functions of $e_U$. The object of $L(\omega)(U)$ are endowed with a canonical connective structure $Co$ see 2.4. An element of $Co((e_U, e'_U))$ is a connection on $e_U$ whose curvature is the restriction of $\omega$ to $U$. A morphism between two objects $(e_U, e'_U)$ and $(e'_U, e''_U)$ of $L(\omega)(U)$ is a morphism $e_U \to e'_U$. The correspondence defined on the category of open subsets of $F$ by $U \to L(\omega)(U)$ is a gerbe.

To perform the quantization we need to define the notion of section $s$. We will propose this definition of sections of vectorial gerbes.

**Definition 9.1.**
Let $(U_i)_{i \in I}$ be an open covering family of $F$, such that $L(\omega)(U_i)$ is not empty, and $(e_i, e'_i)$ an object $L(\omega)(U_i)$. A section $u$ of $(e'_i)_{i \in I}$ is a family of sections $u_i : U_i \to e'_i$ such that the union of supports of $u_i$ is compact.

We denote by $V((e'_i)_{i \in I})$ the vector space generated by those sections of $(e'_i)_{i \in I}$. This vector space is endowed with a Hermitian metric defined by

$$< u, v > = \sum_{i \in I} \int_{e'_i} < u_i, v_i >_{e'_i}$$

where $<, >_{e'_i}$ is the Hermitian metric of $e'_i$.

For each function $f$, and each section $(u_i)_{i \in I}$. We can define

$$L_f(u_i) = \nabla_{e'_i, X_f} u_i + 2i\pi f u_i,$$

where $X_f$ is the Hamiltonian of $f$, and $\nabla_{e'_i}$ a connection defined on $e'_i$ whose curvature is the restriction of $\omega$ to $U_i$. The vector field $X_f$ is the vector field such that $\omega(X_f, .) = -df$.

**Proposition 9.2.**
The family of $L_f(u_i)$ defined a section $L_f(u)$. The map

$$f \to L_f$$

verifies

$$[L_f, L_g] = L_{\{f,g\}}$$

**Proof.**
We have to show that \( L_f(u_i) \) has a compact support, and that the union of support of the family \( (L_f(u_i))_{i \in I} \) is compact. The support of \( f u_i \) and \( \nabla e_i X_j(u_i) \) are contained in the support of \( u_i \). The fact that \( [L_f, L_g] = L_{\{f,g\}} \) is classical.

We have obtained a Souriau-Kostant quantization.

We can define using the classifying theorem of Giraud [6] the gerbe \( L'(\omega) \) on \( F \), such that \( L'(\omega)(U) \) is a set of flat \( \mathcal{E} \)-bundles defined on \( U \), and the cohomology class of the classifying cocycle of \( L'(\omega) \) is the obstruction of the class \( [\omega] \) to be integral. This construction of this gerbe using [3] shows that this gerbe is flat, the objects of \( L'(\omega)(U) \) are locally flat \( \mathcal{E} \)-bundles, and morphisms are morphisms of locally flat \( \mathcal{E} \)-bundles.

For \( L'(\omega) \), we can also define the following space of sections. Consider an open covering \( (U_i)_{i \in I} \) of \( F \) by contractible subsets, \( e_i : \mathbb{R}^2 \to \mathbb{R}^2 \) a family of isomorphisms. A section \( u = (u_i)_{i \in I} \) is a family of sections \( u_i : U_i \to e_i \) such that \( u_i = g_{ij}(u_j) \). We denote by \( V(e_i, g_{ij}) \) the set of those sections. It is a vector space which can be endowed with the following scalar product.

Consider a partition of the unity \( p_i \) subordinate to \( (U_i)_{i \in I} \). Let \( u = (u_i)_{i \in I} \), and \( u' = (u'_i)_{i \in I} \) be sections of \( V(e_i, g_{ij}) \). We set

\[
< u, v > = \sum_{i \in I} \int p_i u_i, p_i u'_i >
\]

For each differentiable function \( f \) defined on \( F \) we can define the operator \( L_f \) which acts on the section \( u = (u_i)_{i \in I} \) by:

\[
L_f(u_i) = \nabla X_i u_i + 2i\pi fu_i
\]

The operator \( L_f \) is well defined. Since on \( U_i \cap U_j \), we have \( L_f(u_i) = u_{ij} L_f(u_j) \) since the gerbe \( C(\omega) \) is flat, and the map \( u_{ij} \) are identified using a trivialization with the multiplication by an element of \( T^1 \) in the trivial bundle \( U_i \cap U_j \times \mathbb{R} \).

Quantization of other structures.

The methods of quantization of Kostant-Souriau have been extended in many directions. Here we present a quantization described in [19]

Consider a manifold \( M \), such that the ring \( C^\infty(M) \) of differentiable functions of \( M \) is endowed with a bracket:

\[
\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)
\]

such that \( (C^\infty(M), \{,\}) \) is a Lie algebra and there is a \( \mathbb{R} \)-linear map:
\[ H : C^\infty(M) \rightarrow \chi(M) \]

\[ f \rightarrow X_f \]

where \( \chi(M) \) is the space of vector fields of \( M \), such that \( X_{\{f,g\}} = [X_f, X_g] \)

The map

\[ C^\infty(M) \rightarrow \text{End}(C^\infty(M)) \]

\[ f \rightarrow (g \rightarrow X_f(g)) \]

is a representation of the Lie algebra \( C^\infty(M) \). We denote \( H^*_C(M) \) the cohomology of this representation. The correspondence:

\[ C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) \]

\[ \Lambda_M(f,g) \rightarrow X_f(g) - X_g(f) - \{f,g\} \]

is a 2-cocycle of this representation.

There is a canonical map \( C' : H^*_{DeRham}(M) \rightarrow H^*_C(M) \). Defined on a chain by \( C'(h)(f_1, ..., f_p) = h(X_{f_1}, ..., X_{f_p}) \). In [19], it is shown that if there exists a line bundle \( L \rightarrow M \) such that \( C'(\Omega) = \Lambda_M \), then the structure is quantizable: that is there exists a representation:

\[ P : C^\infty(M) \rightarrow \text{End}(L^2(L)) \]

which verifies

\[ P(\{f,g\}) = [P_f, P_g] \]

\[ P(f) = \nabla X_f + 2i\pi f \]

where \( \nabla \) is the hermitian connection of the bundle.

Let \( (U_i)_{i \in I} \) be a contractible open covering of \( M \) by charts. We can restrict the bracket \( \{\cdot,\cdot\} \) to \( U_i \). Suppose that on 2-chains, the map \( C \) restricted to \( U_i \) is surjective on closed forms. that is there exists a 2-closed form \( \Omega_{U_i} \) on \( U_i \) such that \( C(\Omega_{U_i}) = \Lambda_{U_i} \). The form \( \Omega_{U_i} \) is the Chern class of a connection defined on \( U_i \times E \).

We can define on \( M \) the gerbe \( D \), such that for each open set \( U \) of \( M \), \( D(U) \) is the category of line bundles over \( U \) endowed with a connection whose curvature \( \Omega_U \) verifies:
\[ C(\Omega_U) = \Lambda_U \]

Let \( e_i \) be an object of \( D(U_i) \), we consider the family \( (u_i)_{i \in I} \), where \( u_i : U_i \to e_i \) is a section of \( e_i \), whose support is compact, and the union of support of \( u_i \) is compact. The family of \( (u_i)_{i \in I} \) is a Hermitian space. On \( e_i \) we consider the connection \( \nabla_{e_i} \), whose curvature is the restriction of \( \Omega_{U_i} \).

The representation

\[ f \to \nabla_{e_i} X_f + 2i\pi f \]

defines a quantization of \( M \).

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