Efficiency of Regression (Un)-Adjusted Rosenbaum’s Rank-based Estimator in Randomized Experiments

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Abstract

A completely randomized experiment allows us to estimate the causal effect by the difference in the averages of the outcome under the treatment and control. But, difference-in-means type estimators behave poorly if the potential outcomes are heavy-tailed, or contain a few outliers. We study an alternative estimator by Rosenbaum that estimates the causal effect by inverting a randomization test using ranks. By calculating the asymptotic breakdown point of this estimator, we show that it is provably more robust than the difference-in-means estimator. We obtain the limiting distribution of this estimator and develop a framework to compare the efficiencies of different estimators of the treatment effect in the setting of randomized experiments. In particular, we show that the asymptotic variance of Rosenbaum’s estimator is, in the worst case, about 1.16 times the variance of the difference-in-means estimator, and can be much smaller when the potential outcomes are not light-tailed. Further, we propose a regression adjusted version of Rosenbaum’s estimator to incorporate additional covariate information in randomization inference. We prove gain in efficiency by this regression adjustment method under a linear regression model. Finally, we illustrate through synthetic and real data that these rank-based estimators, regression adjusted or unadjusted, are efficient and robust against heavy-tailed distributions, contamination, and model misspecification.

Keywords: Breakdown point, causal inference, covariate adjustment, Hodges-Lehmann 0.864 lower bound, local asymptotic normality, randomization inference, Wilcoxon rank-sum test statistic.

1 Introduction

When a treatment is randomly assigned to the units in a study, Fisher (1935) showed that the randomization procedure can be used to provide valid statistical inference about the treatment effect without making strong assumptions regarding the outcome-generating model. In particular, we can calculate confidence intervals for the treatment effect based on an estimator using its randomization distribution.

Suppose the treatment is assigned to the units by a completely randomized assignment. The standard estimator for the average treatment effect is the difference-in-means estimator which takes

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the difference of the averages of outcomes for the treated and control units. Conveniently for the practitioner, the usual large sample confidence interval based on two independent samples from infinite populations for the difference in two population means with unequal variances gives an asymptotically valid, conservative confidence interval under the randomization distribution (Splawa-Neyman, 1990; Freedman et al., 2007). Freedman (2008a) establishes its validity under a finite limiting fourth moment assumption on the sequence of potential outcomes, and Li and Ding (2017) relax the assumption to any finite limiting moment of more than two; also see Lin (2013) and Aronow et al. (2014) for further developments.

Despite these advances, this confidence interval shares the same weaknesses as the classical two sample confidence interval for the difference of two means due to its operational similarity. In particular, the confidence intervals can be very wide, with volatile standard errors, when the potential outcomes are: (i) heavy-tailed, or (ii) have a few extreme values, or (iii) contaminated by outliers. Athey et al. (2021) note that in many modern randomized experiments, including in experiments in the digital space, heavy-tailed or extreme outcomes are often likely.

An important class of estimators used in practice in such situations use the ranks of the potential outcomes as opposed to their exact values (see e.g., Imbens and Rubin (2015, Section 5.5.4)). In particular, Rosenbaum (1993) proposed to use a Hodges-Lehmann type point estimator (Hodges and Lehmann, 1963) based on the Wilcoxon test statistic (Wilcoxon, 1945) for estimating an additive treatment effect under randomization inference. We illustrate the robustness of this estimator against extreme potential outcomes or contamination using the concept of breakdown point (see Proposition 2.2). Rosenbaum (2002b, see Sections 4.6.7, 4.6.9) provided a numerical method to construct a confidence interval based on the above estimator: Under randomization, the upper and lower limits of the $100(1 - \alpha)$% two-sided confidence interval are calculated by solving for the hypothesized treatment effects that equate the corresponding standardized Wilcoxon test statistic value to the $\alpha/2$ and $(1 - \alpha/2)$ quantiles of the standard normal distribution. Unfortunately, the above procedure does not shed light on the efficiency gain/loss of this rank-based approach compared to the usual differences-in-means estimator, thereby precluding the practitioner from making an informed decision as to when such a rank-based method should be preferred.

In this paper, we aim to resolve these deficiencies of Rosenbaum’s estimator (aka the Hodges-Lehmann type point estimate based on the Wilcoxon rank-sum test statistic) by undertaking a systematic and rigorous study of its asymptotic properties under randomization and a constant treatment effect assumption. Our asymptotic results have important implications. First, we prove that the asymptotic variance of Rosenbaum’s estimator is always at most $125/108 \approx 1.16$ times the variance of the difference-in-means estimator, and is often much smaller. Even when the potential outcome sequence is drawn from a normal distribution (where the difference-in-means estimator is the most efficient), Rosenbaum’s estimator has about 95% asymptotic relative efficiency compared to the difference-in-means estimator. For potential outcome distributions that are not light-tailed, the efficiency gain is often substantial, e.g., for the exponential distribution, Rosenbaum’s estimator is 200% more efficient than the difference-in-means estimator, while for the Pareto($\alpha$) distribution with $\alpha > 2$, this efficiency tends to infinity as $\alpha$ tends to 2 (see Table 1). In fact, this result is parallel to the lower bound result on the asymptotic relative efficiency between the Wilcoxon rank-sum test and Student’s $t$-test by Hodges and Lehmann (1956) under an infinite population model. Second, we provide a consistent estimator of the asymptotic variance of Rosenbaum’s estimator, which can be calculated analytically, and readily yields a computable confidence interval for the treatment ef-
fect. Thus, we avoid the computational challenges associated with the existing proposal of using this estimator. Third, we highlight the much weaker assumptions that are required for inference using this rank-based estimator compared to the difference-in-means estimator. This is also evident in our simulation studies where the rank-based method gives valid confidence intervals that are much shorter than those obtained from the difference-in-means estimator when the distribution of the potential outcomes has thicker tails.

Next we consider randomization inference for the treatment effect with regression adjustment. In modern randomized experiments it is increasingly common to collect additional data on covariates. This covariate information is typically used with an ANCOVA model using the treatment indicator and covariates aiming to improve the unadjusted difference-in-means type estimator (Fisher, 1935; Cox and Reid, 2000; Freedman, 2008a,b). Lin (2013) proposed a regression adjusted estimator by modifying the ANCOVA model, to include the interaction of the treatment and covariates, which is asymptotically at least as efficient as the unadjusted difference-in-means estimator. Recently, regression adjusted or model assisted estimators have been developed for different study designs (Fogarty, 2018; Li and Ding, 2020; Liu and Yang, 2020; Su and Ding, 2021; Zhao and Ding, 2021). But, these regression adjustment methods based on linear models which typically use robust standard errors (Huber, 1967; White, 1980), are still sensitive to heavy-tailed distributions, extreme values or outliers in the potential outcomes (MacKinnon and White, 1985; Young, 2018).

We propose a regression adjustment method for Rosenbaum’s estimator for randomization inference with covariate information. A similar estimator was suggested by Rosenbaum (2002a) for observational studies. Our proposed regression adjusted estimator is calculated in three steps. First, we calculate the control potential outcomes under the hypothesis of a specified additive treatment effect and regress it on the covariates using least squares. Then, we calculate the Wilcoxon rank-sum statistic between the treatment and control groups using the residuals of this regression. Finally, we invert this test statistic to compute the estimator. Similar to the previous regression adjustment methods, this method is agnostic to the fitted regression model used in the first step; that is, the estimator is consistent for the treatment effect even if the regression model is not correctly specified, under mild conditions. Unlike some of the regression adjustment methods cited above, we do not use the treatment indicator in the regression model; an approach advocated by Tukey (1993) (also see Gail et al. (1988)).

We study the asymptotic properties of this rank-based regression adjusted estimator and derive its limiting distribution. Similar to the unadjusted case, we require much weaker assumptions on the sequence of potential outcomes than the typically assumed finite limiting moment of some order greater than two (see e.g., Li and Ding, 2020), often four (Lin, 2013; Zhao and Ding, 2022), for our asymptotic results. Consequently, these results are valid even for many heavy-tailed distributions. One important result we show is that, under a linear model assumption, this regression adjusted estimator has lower limiting variance than the unadjusted Rosenbaum’s estimator, a finding similar in spirit to that in Lin (2013). Further, we provide a consistent plug-in estimator of the asymptotic variance for this rank-based regression adjusted estimator that readily yields an asymptotically valid confidence interval for the treatment effect. Our simulation studies illustrate notable gains in efficiency for this regression adjustment method over its unadjusted counterpart when the potential outcomes are not light-tailed.

We also provide two applications of the methods, one to data on the electoral impact of Progresa, Mexico’s conditional cash transfer program (De La O, 2013; Imai, 2018), and one to data on housing prices (Linden and Rockoff, 2008). In both cases, the confidence intervals obtained from
rank-based estimators are much shorter than those obtained from various ANOVA-based estimators. Also, the confidence intervals corresponding to the regression-adjusted estimators are shorter than the unadjusted ones, while maintaining the desired significance level.

1.1 Summary of our contributions

We summarize here our main theoretical results and highlight their importance and implications. We consider estimation of the treatment effect under the assumption of a constant additive treatment effect model following Fisher (1935), Rosenbaum (2002b) and Rosenbaum (2020). A constant treatment effect is typically implied by common infinite population models (see Rosenbaum, 2002a, page 323) and is often a convenient starting point in answering a causal question (Rosenbaum, 2002a; Ho and Imai, 2006; Athey et al., 2021). Further, in many situations, identification of a constant treatment effect has immediate practical use (Rosenbaum, 2002b, Section 2.4.5), whereas when the treatment effect is heterogeneous across the units, additional methods and/or studies are needed to understand how and to whom the treatment is more beneficial.

All the theoretical results in this paper are derived under the finite population asymptotics framework where the only source of randomization stems from the completely randomized assignment of the treatment; the potential outcomes are considered fixed. We formally define Rosenbaum’s rank-based estimator without covariate adjustment in (2.11), and conduct a methodical study of the asymptotic properties of this estimator in Section 2. In particular, Theorem 2.2 gives the asymptotic distribution of this rank-based estimator under a completely randomized design as the sample size \( N \) grows to infinity, under a mild regularity condition, namely Assumption 1. We further show that Assumption 1 holds whenever the control potential outcomes are realizations from an absolutely continuous distribution with density \( f_0(\cdot) \) that satisfies \( \int f_0^2(x) \, dx < \infty \) (see Remark 2.3) — a condition which is satisfied by most of the distributions used in practice, including many with heavy-tails, e.g., Cauchy. Theorem 2.2 immediately yields a readily computable and asymptotically valid confidence interval for the treatment effect; see Section 4 for a detailed discussion.

Our proof of Theorem 2.2 is novel and does not follow from existing results in the literature. As Rosenbaum’s rank-based estimator is computed by (essentially) equating the Wilcoxon rank-sum statistic for the adjusted responses to its expected null value under randomization (see (2.9)), following Hodges and Lehmann (1963), we require the asymptotic distribution of this statistic under a sequence of local alternatives to study the asymptotic properties of this solution. Our analysis is complicated by the following facts. First, under a completely randomized assignment, the observed outcomes of different units are neither independent nor identically distributed which precludes the use of standard limit theorems developed for the i.i.d. setting. Second, to study the behavior of the Wilcoxon rank-sum statistic under local alternatives under randomization inference, the classical results from Le Cam’s local asymptotic normality theory (see van der Vaart (1998, Chapter 7) for details) do not apply. Recently, Li and Ding (2017) have established a few useful asymptotic finite population central limit theorems, including the null distribution of the Wilcoxon statistic; see Proposition 2.1 and the accompanying discussion. Some of the results in Li and Ding (2017) have been generalized and used by Li et al. (2018), Lei and Ding (2021) and Wu and Ding (2021). However, these papers only consider statistics that are linear in the observations, and their results do not apply to rank-based methods (which are highly non-linear). Thus, our first technical innovation in Section 2 is deriving the local asymptotic normality of the Wilcoxon statistic; see Theorem 2.1.

Next, Theorem 2.4 establishes that the relative efficiency of Rosenbaum’s estimator compared to
the difference-in-means estimator is lower bounded by $0.864 \equiv 108/125$ with probability one if the control potential outcomes are realizations from a probability distribution. As far as we are aware, this is the first such result which provides a principled approach to comparing the performance of different estimators in randomized experiments.

In Section 3, we define our regression adjusted rank-based estimator in (3.4) and study its asymptotic properties. In particular, Theorem 3.3 gives the asymptotic distribution of our regression adjusted rank-based estimator. Previously, the asymptotic distributions of regression adjusted difference-in-means type estimators were derived under different randomization strategies (Fogarty, 2018; Li and Ding, 2020; Liu and Yang, 2020; Su and Ding, 2021; Zhao and Ding, 2021). These results build on the results of Li and Ding (2017) and do not apply to rank-based estimators, as already discussed above. We prove Theorem 3.3 by deriving the local asymptotic behavior of the regression adjusted rank statistic, which is provided in Theorem 3.2. The proof of this theorem is more involved than the proof of the analogous result in the unadjusted case (i.e., Theorem 2.2) as the regression adjusted estimator involves solving an equation that has a more complex implicit dependence on the parameter through the ranks of the residuals of the least squares fit. We refer the reader to Appendix B for a summary of our proof techniques.

In Section 4, we propose consistent estimators of the standard errors for our rank-based estimators that yield analytical formulas for confidence intervals based on those estimators. Finally, we evaluate our estimators via simulation studies in Section 5. Our simulation results illustrate that the performance of Rosenbaum’s estimator is never substantially worse than the difference-in-means estimator and is often considerably better when the potential outcomes are not light-tailed. This is as expected from our efficiency result, Theorem 2.4. Moreover, in most cases the regression adjusted rank-based estimator provides more powerful inference than the unadjusted Rosenbaum’s estimator. In fact, Theorem 3.4 guarantees that the adjusted rank-based estimator has a higher efficiency over its unadjusted counterpart under a linear model assumption; cf. Lin (2013) for a similar result for the (un)-adjusted regression-based estimator. Our proposed methods have been implemented in the software R, and the codes are available from https://github.com/ghoshadi/RARE2021. The proofs of our main results, additional technical results and discussions are relegated to Appendices C–E.

2 Inference without regression adjustment

Assume that there are $N$ subjects in a randomized trial, and the treatment group is formed by choosing $m = m(N)$ indices by simple random sampling without replacement (SRSWOR) from $\{1, 2, \ldots, N\}$. We work in the Neyman-Rubin potential outcomes framework (Splawa-Neyman, 1990; Rubin, 1974, 1977; Imbens and Rubin, 2015). Let $a_i$ and $b_i$ denote the potential outcomes for the $i$-th subject under the treatment and control, respectively. The observed response $Y_i$ for the $i$-th subject is given by

$$Y_i = Z_ia_i + (1 - Z_i)b_i,$$

where $Z_i$ equals 1 if the $i$-th subject is assigned to the treatment group, and 0 otherwise. In our model $Z_i$ is the only source of randomness in $Y_i$; we will assume throughout this paper that $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ are fixed constants. We also make the assumption of a constant additive treatment effect (Rosenbaum, 1993, 2002a; Athey et al., 2021), i.e.,

$$a_i - b_i = \tau, \quad \text{for all } 1 \leq i \leq N,$$
for some unknown real number $\tau$. We are interested in point estimation and confidence interval for the constant treatment effect $\tau$. In the sequel we will denote vectors by boldface letters, e.g., $Z := (Z_1, \ldots, Z_N)^\top$, $Y := (Y_1, \ldots, Y_N)^\top$, $b := (b_1, \ldots, b_N)^\top$ and so forth. To formally state our asymptotic results as the sample size $N$ grows to infinity, we will use an additional index $N$ in our notation for the later sections. For example, the potential outcomes will be denoted by $\{(a_{N,j}, b_{N,j}) : 1 \leq j \leq N\}$, the vector of treatment indicators by $Z_N = (Z_{N,1}, \ldots, Z_{N,N})^\top$, and so on.

Our strategy for estimating $\tau$ involves inverting a testing problem. Towards that, consider first the problem of testing

$$H_0 : \tau = \tau_0 \quad \text{versus} \quad H_1 : \tau \neq \tau_0. \tag{2.2}$$

When $H_0$ is true, the vector $Y - \tau_0 Z$, called the vector of adjusted responses, equals $b$, which is non-random. Hence any statistic $t(Z, Y - \tau_0 Z)$, which is a function of the treatment indicators and the adjusted responses, can be used to make randomization inference about $\tau_0$, since the null distribution of $t(Z, Y - \tau_0 Z) = t(Z, b)$ is completely specified by the randomization distribution of $Z$. For example, one can consider the difference-in-means test statistic:

$$t_D(Z, \nu) := \frac{1}{m} \sum_{i:Z_i = 1} v_i - \frac{1}{N - m} \sum_{i:Z_i = 0} v_i, \tag{2.3}$$

where $\nu$ is a vector of observations of length $N$, and $m = \sum_{i=1}^N Z_i$.

Next, consider the problem of constructing valid confidence sets for $\tau$ based on the test statistic $t(Z, Y - \tau_0 Z)$. Using randomization, one can construct an exact, finite sample confidence interval for the treatment effect by appealing to the duality of testing and confidence intervals (Lehmann and Romano, 2005, Section 3.5). Note that, we can simulate the randomization null distribution, under $H_0 : \tau = \tau_0$, of the test statistic $t(Z', Y - \tau_0 Z')$ by evaluating the test statistic at each $Z' = (Z_1', \ldots, Z_N') \in \{0, 1\}^N$ such that $\sum_{i=1}^N Z_i' = m$. Then, the $100(1 - \alpha)$ confidence interval for $\tau$ is the set of hypothesized values of the parameter not rejected by the level $\alpha$ test using these null distributions. But, this method is impractical when $N$ and $m$ are moderately large as the calculations of the exact null distributions for each hypothesized treatment effect impose a huge computational burden.

In practice, one of the two following methods are used to approximate the null distribution and thus calculate an approximate $100(1 - \alpha)$ confidence interval for $\tau$. The first method approximates the null distribution by evaluating the test statistic at a randomly selected set of $Z'$s. The second method uses the asymptotic null distribution of the test statistic from the infinite population literature (Conover, 1999; Lehmann, 1975). Although some authors have advocated for the first method (Bind and Rubin, 2020; Luo et al., 2021), the second method is easier to implement and thus more popular. But, to ensure that this method is approximately valid, it is important to understand if and when the asymptotic null distribution of the test statistic derived under the infinite population assumption also holds under randomization inference. This indeed holds for the difference-in-means statistic defined in (2.3) (see Freedman (2008a) for details). However, the confidence intervals based on the difference-in-means statistic have the disadvantage that they can be very wide with volatile standard errors when the potential outcomes are: (i) heavy-tailed, or (ii) have a few extreme values, or (iii) contaminated by outliers. On the other hand, test statistics that use the ranks of the potential outcomes, as opposed to their exact values, are in general less sensitive to thick-tailed or skewed distributions and hence can lead to more powerful tests (see, e.g., Imbens and Rubin (2015, Section 5.5.4)). There are various popular choices for the rank-based statistic $t(\cdot, \cdot); Rosenbaum (1993)$ recommended using the Wilcoxon
rank-sum statistic (Wilcoxon, 1945), defined as
\[ t(Z, Y - \tau Z) := Z^\top q^{(\tau)} = \sum_{j:Z_j=1} q_j^{(\tau)}, \]
(2.4)
where \( q_j^{(\tau)} \) is the rank of \( Y_j - \tau Z_j \) among \( \{Y_i - \tau Z_i : 1 \leq i \leq N\} \). Considering the possibility of ties in the data, we take the following definition for the ranks, known as up-ranks:
\[ q_j^{(\tau)} := \sum_{i=1}^N 1\{Y_i - \tau Z_i \leq Y_j - \tau Z_j\}, \quad \text{for } 1 \leq j \leq N, \]
(2.5)
where \( 1\{\cdot\} \) denotes the indicator function. Note, under \( H_0 : \tau = \tau_0 \), the ranks \( \{q_j^{(\tau_0)} : 1 \leq j \leq N\} \) can equivalently be written as
\[ q_j^{(\tau_0)} = \sum_{i=1}^N 1\{b_i \leq b_j\}, \quad 1 \leq j \leq N. \]
(2.6)
We present the asymptotic null distribution of the Wilcoxon rank-sum statistic in the following result (see Appendix C.1 for a proof).

**Proposition 2.1** (Asymptotic null distribution of \( t_N \)). Let \( t_N := t(Z_N, Y_N - \tau_0 Z_N) \) be the Wilcoxon rank-sum statistic for a sample of size \( N \) in the testing problem (2.2), with \( t(\cdot, \cdot) \) as defined in (2.4). Assume that the random treatment assignment is based on a SRSWOR sample of size \( m \) from the \( N \) subjects, where \( m/N \to \lambda \in (0,1) \) as \( N \to \infty \). Assume further that the ranks \( \{q_j^{(\tau_0)} : 1 \leq j \leq N\} \) in (2.6) satisfy the following as \( N \to \infty \):
\[ \frac{1}{N} \sum_{j=1}^N (q_j^{(\tau_0)} - \bar{q}_N^{(\tau_0)})^2 = \frac{N^2 - 1}{12} + o(N^2), \]
(2.7)
where \( \bar{q}_N^{(\tau_0)} := N^{-1} \sum_{j=1}^N q_j^{(\tau_0)} \). Then, under \( \tau = \tau_0 \),
\[ N^{-3/2} \left( t_N - m\bar{q}_N^{(\tau_0)} \right) \xrightarrow{d} N\left(0, \frac{\lambda(1 - \lambda)}{12}\right). \]
(2.8)

**Remark 2.1** (Null distribution in presence of ties). It is noteworthy that the asymptotic null distribution of the Wilcoxon rank-sum statistic defined through up-ranks (see (2.5)) stated in Proposition 2.1 holds even when there are ties, provided (2.7) holds. If, instead of using up-ranks we break ties by comparing indices or by randomization, then the ranks, being merely a permutation of \( \{1,2,\ldots,N\} \), satisfy (2.7) automatically. As a consequence, (2.8) holds without any assumption on the potential outcomes. Moreover, if ties are broken using average ranks, then also (2.8) holds provided that the blocks of the ties are not large; see Lemma F.1 in Appendix F for a precise result in this direction.

**Remark 2.2** (Comparison with Li and Ding (2017)). Li and Ding (2017) gives a finite population CLT for linear rank statistics, under the null hypothesis of no treatment effect, and assuming that there are no ties (see also Lehmann (1975, Appendix 4)). Moreover, they show that the conditions they need in the finite population framework is satisfied almost surely when the responses are realizations from a distribution having finite moments of any order more than two. In contrast, we derive the null distribution of \( t_N \) in (2.8) under milder conditions (see (2.7)) that also allow ties. Also, we do not require any moment assumption when the potential outcomes are realizations from a continuous distribution.
The limiting distribution in (2.8) is indeed the asymptotic null distribution of the Wilcoxon rank-sum statistic under the infinite population assumption (Lehmann, 1975, Section 1.3). Proposition 2.1 therefore justifies Rosenbaum’s proposal of calculating the lower and upper limit of a confidence interval for \( \tau \) by numerically solving for the hypothesized treatment effects that equate the standardized Wilcoxon rank-sum statistic to the \( \alpha/2 \) and \( (1 - \alpha/2) \) quantiles of the standard normal distribution (see Rosenbaum, 2002b, Section 4.6).

However, a drawback of this approach for constructing confidence sets is that, it does not provide much insight regarding the efficiency of this rank-based approach over the difference-in-means based confidence interval. In the subsequent sections, we develop the theory for a point estimator \( \hat{\tau} \) based on the Wilcoxon rank-sum statistic, derive its asymptotic distribution, and find consistent estimators of its asymptotic variance. This provides, in analytic form, an asymptotically valid confidence interval for \( \tau \) based on the rank-based estimator, which obviates the need for the extensive computation required to implement the aforementioned approach of Rosenbaum. We also establish the robustness and efficiency properties of this estimator, which is reflected in the confidence intervals constructed based on them in our synthetic and real data examples in Section 5.

### 2.1 Rosenbaum’s estimator based on the Wilcoxon rank-sum statistic

For finding a point estimator for \( \tau \), Rosenbaum (2002a) suggested that we invert the testing problem (2.2); in the sense that, we equate \( t(Z, Y - \tau Z) \) to its expectation, under the randomization distribution, and solve for \( \tau \). Thus, the estimator for \( \tau \) advocated in Rosenbaum (2002a) is a solution to the equation

\[
\hat{\tau} = \sum_{i=1}^{N} Z_i Y_i - \frac{1}{N - m} \sum_{i=1}^{N} (1 - Z_i) Y_i.
\]

Note that \( \hat{\tau} \) is same as \( E_t(Z, b) \), which does not depend on \( \tau \). We provide below a concrete example to illustrate this estimation strategy.

**Example 1.** Consider the difference-in-means test statistic defined in (2.3). Since the random treatment assignment is based on a SRSWOR sample of size \( m \) from the \( N \) subjects, we have \( EZ_i = m/N \) for each \( 1 \leq i \leq N \). Observing that \( Z_i Y_i = Z_i a_i \) and \((1 - Z_i) Y_i = (1 - Z_i)b_i\), we deduce that \( E_t(D(Z, Y - \tau Z) = 0 \), and consequently, the solution to (2.9) for \( \tau \) is given by

\[
\hat{\tau}_{\text{dm}} := \frac{1}{m} \sum_{i=1}^{N} Z_i Y_i - \frac{1}{N - m} \sum_{i=1}^{N} (1 - Z_i) Y_i.
\]

Thus, Rosenbaum’s suggestion for \( \hat{\tau} \) coincides with the difference-in-means estimator, for the above choice of the test statistic \( t_D(\cdot, \cdot) \).

Next we implement Rosenbaum’s suggestion in (2.9) with the Wilcoxon rank-sum statistic \( t(\cdot, \cdot) \) defined in (2.4). Assume for the moment that ties are not present in the ranks. We then have

\[
E_t(Z, Y - \tau Z) = E_t Z^\top q(\tau) = \frac{m}{N} \sum_{j=1}^{N} q_j(\tau) \frac{N(N+1)}{2} = \frac{m(N+1)}{2}.
\]

Notice that the equation \( t(Z, Y - \tau Z) = m(N + 1)/2 \) might not always admit a solution for \( \tau \), for instance, when \( m(N + 1) \) is odd. To circumvent this technical issue, we slightly modify the definition of this estimator, following the footsteps of Hodges and Lehmann (1963). We set

\[
\hat{\tau} := \sup \{ \tau : t(Z, Y - \tau Z) > \mu \}, \quad \text{and} \quad \hat{\tau}^* := \inf \{ \tau : t(Z, Y - \tau Z) < \mu \}.
\]
where $\mu := \mathbb{E}_{\tau_0} t(Z, Y - \tau_0 Z)$, and define

$$\hat{\tau}_R := \frac{\hat{\tau}^* + \hat{\tau}^{**}}{2}. \quad (2.11)$$

We first show that $\hat{\tau}_R$ is robust to outliers, as opposed to the difference-in-means estimator $\hat{\tau}_{dm}$ in (2.10). In classical nonparametrics, a natural way to quantify the robustness of an estimator is via its (asymptotic) breakdown point (see, e.g., Hettmansperger and McKean (2011)). As in our setting the indices corresponding to the treatment and control groups are not deterministic but are rather chosen by randomization, we define the (asymptotic) breakdown point of an estimator of the constant treatment effect $\tau$ in the following fashion.

**Definition 1 (Breakdown point).** Consider any estimator $\hat{\tau} = \hat{\tau}(Y_1, \ldots, Y_N; Z_1, \ldots, Z_N)$ of the constant treatment effect $\tau$. Assume that the random treatment assignment is based on a SRSWOR sample of size $m$ from the $N$ subjects. The finite sample breakdown point of $\hat{\tau}$ is defined as:

$$BP_N(\hat{\tau}) := \frac{1}{N} \min \left\{ 1 \leq k \leq N : \forall z_1, \ldots, z_N \in \{0, 1\}, \sum_{i=1}^N z_i = m, \min_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} \sup_{y_{i_1}, \ldots, y_{i_k} \in \mathbb{R}} |\hat{\tau}(y_1, \ldots, y_N; z_1, \ldots, z_N)| = \infty \right\}. \quad (2.12)$$

Further, the asymptotic breakdown point of $\hat{\tau}$ is defined as:

$$ABP(\hat{\tau}) := \lim_{N \to \infty} BP_N(\hat{\tau}).$$

Intuitively, the above notion of breakdown point formalizes the following question: “What is the minimum proportion of responses that, if replaced with arbitrarily extreme values, will cause the estimator to be arbitrarily large (in absolute value), for all configurations of treatment assignments?” Note the emphasis on the phrase “for all configurations of treatment assignments”, which is reflected in the minimum taken over $i_1, i_2, \ldots, i_k$ in (2.12). Another alternative would have been to replace the minimum by a maximum in (2.12) which would have led to a different and more adversarial notion of breakdown point. Also note that when we are taking the supremum in (2.12) over the responses, we are essentially changing both the potential outcomes, keeping the assumption of constant additive treatment effect intact.

It is easy to see that changing a single response arbitrarily can lead to $\hat{\tau}_{dm} = \pm \infty$ irrespective of the treatment assignment, implying that $BP_N(\hat{\tau}_{dm}) = 1/N$ and consequently $ABP(\hat{\tau}_{dm}) = 0$. In contrast, the following result (see Appendix C.2 for a proof) illustrates that the asymptotic breakdown point of Rosenbaum’s (unadjusted) estimator $\hat{\tau}_R$ is strictly positive for all values of the proportion $\lambda \approx m/N$.

**Proposition 2.2 (Asymptotic breakdown point of $\hat{\tau}_R$).** Let $\hat{\tau}_R$ be the Rosenbaum’s estimator based on the Wilcoxon rank-sum statistic, as defined in (2.11). Assume that $m/N \to \lambda \in [0, 1]$, where $m$ is the size of the treatment group. Then the asymptotic breakdown point of $\hat{\tau}_R$ is given by

$$ABP(\hat{\tau}_R) = \begin{cases} 
(1 - \lambda)/2 & \text{if } \lambda < 1/3 \\
1 - \sqrt{2\lambda(1-\lambda)} & \text{if } 1/3 \leq \lambda \leq 2/3 \\
\lambda/2 & \text{if } \lambda > 2/3.
\end{cases}$$
It is well-known (see, for instance, Hettmansperger and McKean (2011, Section 6.5.3)) that the asymptotic breakdown point of the one sample Hodges-Lehmann location estimator (Hodges and Lehmann (1963)) is $1 - 2^{-1/2} \approx 0.29$. Proposition 2.2 illustrates that the asymptotic breakdown point of Rosenbaum’s estimator based on the Wilcoxon rank-sum statistic is never less than $1 - 2^{-1/2}$ (equal when $\lambda = 1/2$). This means that, if half of the units were to receive the treatment, then at least 29% of the responses need to be altered, in order to compel $\hat{\tau}_R$ to be arbitrarily large for any treatment assignment.

Having established the robustness property of Rosenbaum’s estimator $\hat{\tau}_R$ through its high breakdown point, we derive its asymptotic distribution in the following section. This, in particular, will help us understand the asymptotic efficiency of $\hat{\tau}_R$, and obtain confidence intervals for the constant treatment effect $\tau$.

### 2.2 Asymptotic distribution of Rosenbaum’s estimator $\hat{\tau}_R$

In this section, we establish the asymptotic distribution of Rosenbaum’s estimator $\hat{\tau}_R$ defined in (2.11) based on the Wilcoxon rank-sum statistic introduced in (2.4). For notational clarity, we index vectors and matrices by subscript $N$ (the total sample size), which is allowed to grow to infinity. Although the unadjusted estimator $\hat{\tau}_R$ can be expressed as the median of the pairwise differences between the responses from the treatment and the control pool, we follow the strategy of Hodges and Lehmann (1963) that works in more generality (in particular, for the regression adjustment case that we will study later). Before presenting our main results, we discuss this method for obtaining the asymptotic distribution of $\hat{\tau}_R$ from that of the test statistic $t_N := t(Z_N, Y_N - \tau_0 Z_N)$. We prove a result, namely Lemma D.1 in Appendix D, that connects the distribution of $\hat{\tau}_R$ to the distribution of $t_N$ under a sequence of local alternatives $\tau_N := \tau_0 - hN^{-1/2}$ for some fixed $h \in \mathbb{R}$. In particular, we show that

$$
\lim_{N \to \infty} \mathbb{P}_{\tau_N} \left( N^{-3/2}(t_N - \mu_N) \leq x \right) = \Phi \left( \frac{x + hB}{A} \right), \quad \text{for every } x \in \mathbb{R},
$$

where $\mu_N := \mathbb{E}_{\tau_N} t_N$, $\Phi(\cdot)$ is the standard normal distribution function, and $A, B$ are positive constants, then

$$
\lim_{N \to \infty} \mathbb{P}_{\tau_0} \left( \sqrt{N}(\hat{\tau}_R - \tau_0) \leq x \right) = \Phi \left( \frac{xB}{A} \right), \quad \text{for every } x \in \mathbb{R}.
$$

In view of the above, it suffices to establish (2.13) for the Wilcoxon rank-sum statistic $t_N := t(Z_N, Y_N - \tau_0 Z_N)$, with $t(\cdot, \cdot)$ as defined in (2.4). Although Li and Ding (2017, Corollary 1) give the asymptotic null distribution of $t_N$, their proof technique does not work for finding the asymptotic distribution of $t_N$ under local alternatives $\tau_N := \tau_0 - hN^{-1/2}$. This is because their proof hinges on the fact that ranks under the null are deterministic and hence do not depend on the random treatment assignment; this is not the case under local alternatives. Moreover, the classical results from Le Cam’s local asymptotic normality theory (see van der Vaart (1998, Chapter 7) for details) do not apply in our fixed design setting.

We present the asymptotic distribution of $t_N$ under a sequence of local alternatives in Theorem 2.1 below, and this is indeed one of our major contributions. To concisely state an assumption we need here, we introduce the following notation. Define, for any $h, x \in \mathbb{R},$

$$
I_{h,N}(x) := \begin{cases} 
1 \{0 \leq x < hN^{-1/2} \} & \text{if } h \geq 0, \\
-1 \{hN^{-1/2} \leq x < 0 \} & \text{if } h < 0.
\end{cases}
$$

(2.15)
We now make the following assumption on the potential control outcomes.

**Assumption 1.** Let \( \{b_{N,j} : 1 \leq j \leq N\} \) be the potential control outcomes. We assume that, for \( I_{h,N} \) as in (2.15), the following holds:

\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N}(b_{N,j} - b_{N,i}) = h\mathcal{I}_b, \tag{2.16}
\]

for some fixed constant \( \mathcal{I}_b \in (0, \infty) \) and for every \( h \in \mathbb{R} \).

**Remark 2.3** (On Assumption 1). Intuitively, Assumption 1 roughly says that the proportion of the pairwise differences \( (b_{N,j} - b_{N,i}) \) falling into small intervals (shrinking at the rate of \( N^{-1/2} \) scales with the Lebesgue measure of those intervals. Indeed, if \( b_{N,j} \)'s are realizations from a distribution with density \( f_b(\cdot) \) that satisfies \( \int_{\mathbb{R}} f_b^2(x)dx < \infty \), then Assumption 1 holds almost surely, with \( \mathcal{I}_b = \int_{\mathbb{R}} f_b(x)dx \) (see Lemma D.4 in Appendix D for a formal statement). In fact, this condition is satisfied by many heavy-tailed distributions for which the moment assumptions imposed in Li and Ding (2017) do not hold (e.g., Cauchy). Moreover, Lemma D.5 in Appendix D shows that Assumption 1 implies (2.7), and consequently the asymptotic null distribution of \( t_N \) in (2.8) holds.

The following theorem is pivotal; it provides the local asymptotic normality for \( t_N \) (see Appendix C.3 for a proof), which will aid us to study the limiting behavior of \( \tilde{\tau}^R \).

**Theorem 2.1** (Local asymptotic normality of \( t_N \)). Let \( t_N := t(Z_N, Y_N - \tau_0 Z_N) \) be the Wilcoxon rank-sum test statistic, with \( t(\cdot, \cdot) \) as in (2.4). Assume that the random treatment assignment is based on a SRSWOR sample of size \( m \) from the \( N \) subjects, where \( m/N \to \lambda \in (0, 1) \) as \( N \to \infty \), and suppose that Assumption 1 holds. Fix \( h \in \mathbb{R} \) and let \( \tau_N = \tau_0 - hN^{-1/2} \). Then it follows that under \( \tau = \tau_N \),

\[
N^{-3/2} (t_N - \frac{m(N + 1)}{2}) \overset{d}{\to} \mathcal{N} \left( -h\lambda(1 - \lambda)\mathcal{I}_b, \frac{\lambda(1 - \lambda)}{12} \right),
\]

where \( \mathcal{I}_b \) is defined in Assumption 1.

In particular, Proposition 2.1 follows from Theorem 2.1 by plugging in \( h = 0 \). Having established the local asymptotic normality of the statistic \( t_N \), we are now ready to present the asymptotic distribution of the estimator \( \tilde{\tau}^R \). This is the content of the following theorem (see Appendix C.4 for a proof).

**Theorem 2.2** (CLT for the estimator \( \tilde{\tau}^R \)). Let \( \tilde{\tau}^R \) be as in (2.11), with \( t(\cdot, \cdot) \) as in (2.4). Assume that the random treatment assignment is based on a SRSWOR sample of size \( m \) from the \( N \) subjects, where \( m/N \to \lambda \in (0, 1) \) as \( N \to \infty \), and suppose that Assumption 1 holds. If \( \tau_0 \) is the true value of \( \tau \), then

\[
\sqrt{N} (\tilde{\tau}^R - \tau_0) \overset{d}{\to} \mathcal{N} \left( 0, (12\lambda(1 - \lambda)\mathcal{I}_b^2)^{-1} \right),
\]

where \( \mathcal{I}_b \) is defined in Assumption 1.

**Remark 2.4** (No moment assumptions). A parallel result of Freedman (2008a, Theorem 3) on the asymptotic distribution of the difference-in-means estimator (see (2.10)) relies crucially upon moment assumptions, such as, that the fourth order moments of the potential outcomes \( \{a_{N,i}\} \) and \( \{b_{N,i}\} \) are uniformly bounded (uniform over the sample size) and certain mixed moments converge to deterministic
limits. Also, the related results of Li and Ding (2017) require finiteness of moments of some order greater than two. In contrast, our Theorem 2.2, which yields the asymptotic distribution of the rank-based estimator $\hat{\tau}_R$, does not require any assumption on the moments of the sequence of potential outcomes. Thus, our result holds even when the potential outcomes are realizations from a Pareto($\alpha$) distribution for some $\alpha > 0$.

Theorem 2.2 provides a natural way to construct confidence intervals for $\tau$. It also tells us that the problem of estimating the large sample variance of $\hat{\tau}_R$ essentially reduces to the problem of estimating the limiting quantity $I_b$, which is unknown in general. Naturally, our next target is to estimate $I_b$ based on the observed responses $\{Y_{N,j} : 1 \leq j \leq N\}$. We propose an estimator of $I_b$ and prove its consistency under the same set of assumptions as in Theorem 2.2. This is stated in the following theorem (see Appendix C.5 for a proof).

Theorem 2.3 (Consistent estimation of $I_b$). Suppose that Assumption 1 holds and let $m/N \to \lambda \in (0,1)$ as $N \to \infty$. Define

$$\hat{I}_N := \left(1 - \frac{m}{N}\right)^{-2} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} (1 - Z_{N,i})(1 - Z_{N,j}) 1\{0 \leq Y_{N,j} - Y_{N,i} < N^{-1/2}\}.$$  \hspace{1cm} (2.17)

Then $\hat{I}_N \xrightarrow{P} I_b$ as $N \to \infty$.

The above theorem in conjunction with Theorem 2.2 readily yields an asymptotically valid confidence interval for $\tau_0$ (the true value of $\tau$). This is formally stated below.

Corollary 2.1 (Confidence interval for $\tau_0$). Under the assumptions of Theorem 2.2, an approximate $100(1 - \alpha)$% confidence interval for $\tau_0$ is given by

$$\hat{\tau}_R \pm \frac{z_{\alpha/2}}{\sqrt{N}} \left(12 \frac{m}{N} \left(1 - \frac{m}{N}\right) \hat{I}_N^2\right)^{-1/2},$$

where $\hat{I}_N$ is defined in (2.17) and $z_\alpha$ denotes the upper $\alpha$-th quantile for the standard normal distribution.

The estimator of the asymptotic variance of $\hat{\tau}_R$ proposed above only uses the data on the controls. We will propose another consistent estimator of $I_b$ later in Section 4 that uses all the observed samples.

2.3 Efficiency of Rosenbaum’s estimator

In this section we aim to assess the asymptotic relative efficiency of the rank-based estimator $\hat{\tau}_R$ with respect to the difference-in-means estimator $\hat{\tau}_{dm}$. In classical statistics, an asymptotic framework for comparing two consistent, asymptotically normal estimators is through the ratio of their asymptotic variances. A precise definition is given below.

Definition 2 (Asymptotic relative efficiency). Let $\hat{\tau}_{N,1}$ and $\hat{\tau}_{N,2}$ be two asymptotically normal estimators of $\tau$, in the sense that there exist positive sequences $\sigma_{N,1}^2$ and $\sigma_{N,2}^2$ such that as $N \to \infty$,

$$\frac{\hat{\tau}_{N,1} - \tau}{\sigma_{N,1}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{and} \quad \frac{\hat{\tau}_{N,2} - \tau}{\sigma_{N,2}} \xrightarrow{d} \mathcal{N}(0,1).$$

Then the asymptotic relative efficiency of $\hat{\tau}_{N,1}$ with respect to $\hat{\tau}_{N,2}$ is defined as

$$\text{eff}(\hat{\tau}_{N,1}, \hat{\tau}_{N,2}) := \lim_{N \to \infty} \frac{\sigma_{N,2}^2}{\sigma_{N,1}^2}.$$
Our next theorem provides a lower bound on the asymptotic relative efficiency of \( \hat{\tau}^R \) with respect to \( \hat{\tau}_{dm} \) when the potential control outcomes are drawn from a density (see Appendix C.6 for a proof).

**Theorem 2.4 (Efficiency lower bound).** Assume that the potential control outcomes \( \{b_{N,j} : 1 \leq j \leq N\} \) are drawn i.i.d. from a distribution with density function \( f_b \) satisfying \( \int_R f_b^2(x)dx < \infty \). Then,

\[
\sqrt{N} (\hat{\tau}^R - \tau_0) \mid b_{N,1}, \ldots, b_{N,N} \xrightarrow{d} \mathcal{N} \left( 0, \left( 12\lambda(1 - \lambda) \left( \int_R f_b^2(x)dx \right)^2 \right)^{-1} \right),
\]

for almost every sample path \( \{b_{N,j}\}_{j \geq 1} \). When the density \( f_b(\cdot) \) admits a finite variance \( \sigma_b^2 \), the asymptotic efficiency of \( \hat{\tau}^R \) relative to \( \hat{\tau}_{dm} \) is given by

\[
\text{eff}(\hat{\tau}^R, \hat{\tau}_{dm}) = 12\sigma_b^2 \left( \int_R f_b^2(x)dx \right)^2.
\]

Further, if \( \mathcal{F} \) is the family of all probability densities on \( \mathbb{R} \), then

\[
\inf_{\mathcal{F}} \text{eff}(\hat{\tau}^R, \hat{\tau}_{dm}) \geq 0.864.
\]

The above result shows that, conditional on \( \{b_{N,j} : 1 \leq j \leq N\} \), the rank-based estimator \( \hat{\tau}^R \) has a centered limiting normal distribution where the limiting variance, for the worst possible distribution, is just around 15\% more than that for the difference-in-means estimator.

Table 1 records the values of the relative efficiency \( \text{eff}(\hat{\tau}^R, \hat{\tau}_{dm}) \) for some standard probability distributions. As can be seen from the table, even when the \( \{b_{N,j}\} \) are assumed to be realizations from a normal distribution, the rank-based estimator only suffers a 5\% loss of efficiency. Whereas, if \( f_b \) is assumed to be the exponential distribution, the rank-based estimator gives a 2-fold gain in efficiency.

It is easy to construct examples where the rank-based estimator will exhibit infinite gains over the difference-in-means estimator. For instance, in the Pareto(\( \alpha \)) example in Table 1, it is easy to see that the efficiency of \( \hat{\tau}^R_{adj} \) with respect to \( \hat{\tau}_{dm} \) approaches \( \infty \) as \( \alpha \) approaches 2. We also refer the reader to our simulation results in Section 5 which highlights how these efficiencies manifest themselves in moderate sample sizes.

**Table 1:** Asymptotic efficiency of \( \hat{\tau}^R \) relative to \( \hat{\tau}_{dm} \) for some common probability distributions

| distribution  | density \( (f_b) \)                                                                 | \( \text{eff}(\hat{\tau}^R, \hat{\tau}_{dm}) \) |
|---------------|----------------------------------------------------------------------------------|-----------------------------------------------|
| Normal        | \( (2\pi)^{-1/2} \exp(-x^2/2) \)                                               | \( 3/\pi \approx 0.955 \)                     |
| Uniform       | \( 1 \{0 \leq x \leq 1\} \)                                                   | 1                                             |
| Laplace       | \( 2^{-1} \exp(-|x|) \)                                                        | \( 3/2 \)                                     |
| \( t_3 \)     | \( c \left( x^2/3 + 1 \right)^{-2} \)                                         | \( 75/(4\pi^2) \approx 1.9 \)                |
| Exponential   | \( \exp(-x)1\{x \geq 0\} \)                                                   | 3                                             |
| Pareto(\( \alpha \)) | \( \alpha x^{-(\alpha+1)}1\{x \geq 1\} \) | \( \begin{cases} \frac{\alpha^5}{(\alpha-1)^2(2\alpha+1)^2(\alpha-2)} & \text{if } \alpha > 2 \\ +\infty & \text{if } \alpha \in (0, 2] \end{cases} \) |
Remark 2.5 (The 0.864 lower bound). The efficiency lower bound given in Theorem 2.4 coincides with a celebrated efficiency lower bound due to Hodges and Lehmann (1956) in the context of two sample testing under location shift alternatives. They showed that the Pitman efficiency (see, e.g., van der Vaart (1998, Section 14.3)) of the Wilcoxon rank-sum test relative to Student’s t-test never falls below 0.864. Here, \( \hat{\tau} \) is the estimator that inverts the Wilcoxon rank-sum test, and thus, in a way, our Theorem 2.4 mimics the efficiency result of Hodges and Lehmann (1956).

Definition 2 above provides a principled framework for comparing estimators under a randomization inference. From a theoretical perspective, this framework can be used to compare the worst case relative performance between two estimators, as is done above between the difference-in-means and Rosenbaum’s estimators. More importantly, in practice, this framework allows us to use a pilot sample to choose an estimator that is expected to be more efficient in the final study. Pilot sampling is a frequently used tool to plan an empirical study (Wittes and Brittain, 1990). In this context, for example, using a pilot sample, we can estimate \( f_b \), which may then be used to choose between the difference-in-means and Rosenbaum’s estimator by estimating their relative efficiency using (2.18).

3 Inference with regression adjustment

We now study a method advocated by Rosenbaum (2002a) for drawing randomization inference on the treatment effect with regression adjustment for pre-treatment covariates, in the setup of completely randomized experiments as in Section 2. Assume that along with the responses \( \{Y_i : 1 \leq i \leq N\} \) (as in (2.1)) we also collect data on \( p \) many pre-treatment covariates, denoted by \( x_i \) for the \( i \)-th subject. Also denote by \( X \) the \( N \times p \) matrix of covariates. To avoid notational clutter, we suppress the use of the index \( N \) in the notation until Section 3.1.

To motivate our estimation strategy, consider again the testing problem (2.2). The method suggested by Rosenbaum is to apply a randomization test on the residuals, as follows. First we calculate the adjusted responses \( Y - \tau_0 Z \) and regress it on the covariates \( X \) using the least squares criterion. Define the residuals obtained from this fit as:

\[
e_0 := \hat{\varepsilon}(Y - \tau_0 Z, X) = (I - P_X)(Y - \tau_0 Z),
\]

where \( I \) is the identity matrix of order \( N \times N \), and \( P_X \) is the projection matrix onto the column space of \( X \). Now we let the residuals \( e_0 \) play the role of the adjusted responses \( Y - \tau_0 Z \) while performing the randomization test. Define the Wilcoxon rank-sum test statistic based on the residuals \( e_0 \) as \( t_{\text{adj}} := t(Z, e_0) = q^\top Z \), with \( t(\cdot, \cdot) \) as in (2.4) and the up-ranks \( q \) in (2.5) being calculated on the residuals instead of the adjusted responses. That is,

\[
t_{\text{adj}} := t(Z, e_0) = \sum_{i=1}^{N} Z_i \sum_{j=1}^{N} 1\{e_{0,j} \leq e_{0,i}\}.
\]

To find a point estimator for \( \tau \) based on the above test procedure, Rosenbaum (2002a) suggested calculating residuals \( e_\tau \) as

\[
e_\tau := \hat{\varepsilon}(Y - \tau Z, X) = (I - P_X)(Y - \tau Z).
\]

We then equate \( t(Z, e_\tau) \) with its expectation under the randomization distribution of \( Z \) and solve for \( \tau \). Thus, the regression adjusted estimator advocated by Rosenbaum (2002a) is a solution to the
Note that the right hand side of the above equation is free of \( \tau \), because \( \mathbb{E}_t(Z, e_\tau) = \mathbb{E}_0 t(Z, e_0) \) and under \( \tau = \tau_0 \) we have \( e_0 = (I - P_X)(Y - \tau_0 Z) = (I - P_X)b \), whose distribution does not depend on \( \tau \).

Again, due to the discreteness of the ranks, (3.3) might not always have a solution. So we modify the estimator using the idea of Hodges and Lehmann (1963), similarly as we did in the previous section for the unadjusted estimator. We denote the modified Rosenbaum's estimator as \( \hat{\tau}^R_{\mathrm{adj}} \), which is defined using (2.11) in the same manner as \( \hat{\tau}^R \) (our rank-based estimator in the unadjusted case), only with the difference that here \( t(Z, e_\tau) \) plays the role of \( t(Z, Y - \tau Z) \). We set

\[
\hat{\tau}^*_{\mathrm{adj}} := \sup \{ \tau : t(Z, e_\tau) > \mathbb{E}_t t(Z, e_\tau) \}, \quad \text{and} \quad \hat{\tau}^{**}_{\mathrm{adj}} := \inf \{ \tau : t(Z, e_\tau) < \mathbb{E}_t t(Z, e_\tau) \},
\]

and define

\[ \hat{\tau}^R_{\mathrm{adj}} := \frac{\hat{\tau}^*_{\mathrm{adj}} + \hat{\tau}^{**}_{\mathrm{adj}}}{2}. \tag{3.4} \]

In the unadjusted case, the Wilcoxon rank-sum statistic \( t(Z, Y - \tau Z) \), with \( t(\cdot, \cdot) \) as defined in (2.4), can easily be seen to be a monotonic function of \( \tau \). This is not immediate in the regression adjustment case. However, we will assume throughout that \( t(Z, e_\tau) \) is a monotonic function of \( \tau \) (cf. Hodges and Lehmann (1963, p. 599)), so that the above definition of \( \hat{\tau}^R_{\mathrm{adj}} \) makes sense. In our simulation studies in Section 5 and applications to real data in Section 6, we found that this monotonicity assumption is always satisfied.

Analyzing \( \hat{\tau}^R_{\mathrm{adj}} \) presents significant new challenges compared to \( \hat{\tau}^R \). The chief reason being that \( t(Z, e_\tau) \), which determines \( \hat{\tau}^R_{\mathrm{adj}} \), involves indicators of the form \( 1(e_{\tau,j} \leq e_{\tau,i}) \), and each of these indicators depends on the entire random treatment assignment vector \( Z \). This is not the situation in the unadjusted case, where the corresponding indicator only depends on \( Z_i \) and \( Z_j \). We relegate the discussion on how we were able to resolve this technical challenge to Appendix B so as not to impede the flow of the paper.

### 3.1 Asymptotic distribution of \( \hat{\tau}^R_{\mathrm{adj}} \)

Consider now a sequence of completely randomized experiments as in Section 2.2. Denote the residuals obtained from the least squares fitting of \( Y_N - \tau_0 Z_N \) on the covariates \( X_N \) by

\[ e_N := \hat{\varepsilon}(Y_N - \tau_0 Z_N, X_N) \]

with \( \hat{\varepsilon}(\cdot, \cdot) \) as in (3.1), and define the regression adjusted Wilcoxon rank-sum statistic as

\[ t_{N,\mathrm{adj}} := t(Z_N, e_N) \tag{3.5} \]

with \( t(\cdot, \cdot) \) as in (3.2). Finally, define \( \hat{\tau}^R_{\mathrm{adj}} \) as in (3.4) using the statistic \( t_{N,\mathrm{adj}} \).

As in the without regression adjustment case (see (2.13) and (2.14)), we invoke Lemma D.1 to reduce the problem of deriving the asymptotic distribution of the estimator \( \hat{\tau}^R_{\mathrm{adj}} \) to the problem of finding the asymptotic distribution of the test statistic \( t_{N,\mathrm{adj}} \) under the sequence of local alternatives \( \tau_N = \tau_0 - hN^{-1/2} \) for a fixed \( h \in \mathbb{R} \). As a precursor to our result on the local asymptotic normality
of $t_{N, \text{adj}}$, we first present its limiting null distribution. Recall the notation \( \{b_{N,j} : 1 \leq j \leq N\} \) for the potential control outcomes. Define the regression adjusted potential control outcomes as

\[
\tilde{b}_{N,j} := b_{N,j} - p_{N,j}^\top b_N, \quad 1 \leq j \leq N,
\]

where \( p_{N,j} \) is the \( j \)-th column of the projection matrix \( P_{X_N} \) that projects onto the column space of \( X_N \). Note that under \( \tau = \tau_0 \), these \( \tilde{b}_{N,j} \)'s are identical to the residuals obtained by regressing \( Y_N - \tau_0 Z_N \) on \( X_N \). The following assumption mimics Assumption 1 that we used in the without regression adjustment case.

**Assumption 2.** Let \( \{\tilde{b}_{N,j} : 1 \leq j \leq N\} \) be the regression adjusted potential control outcomes as defined in (3.6). We assume that, for \( I_{h,N} \) as in (2.15), the following holds:

\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^N \sum_{i=1}^N I_{h,N}(\tilde{b}_{N,j} - \tilde{b}_{N,i}) = h J_b,
\]

for some fixed constant \( J_b \in (0, \infty) \) and for every \( h \in \mathbb{R} \).

We show in Lemma D.8 in Appendix D that Assumption 2 holds (in probability) when the sequence of potential control outcomes are sampled according to the regression model \( b_N = X_N \beta_N + \varepsilon_N \), where \( \varepsilon_{N,1}, \ldots, \varepsilon_{N,N} \) are i.i.d. from \( N(0, \sigma^2) \). In fact, the limiting quantity \( J_b \) has a neat formula in this special case, namely, \( J_b = (2\sqrt{\pi} \sigma)^{-1} \).

Our next theorem provides the limiting null distribution of the test statistic \( t_{N, \text{adj}} \) under Assumption 2 (see Appendix C.7 for a proof).

**Theorem 3.1** (Asymptotic null distribution of \( t_{N, \text{adj}} \)). Denote by \( t_{N, \text{adj}} \) the Wilcoxon rank-sum statistic based on the residuals obtained from the least squares fitting of \( Y_N - \tau_0 Z_N \) on \( X_N \), as defined in (3.5). Assume that the random treatment assignment is based on a SRSWOR sample of size \( m \) from the \( N \) subjects, where \( m/N \to \lambda \in (0, 1) \) as \( N \to \infty \), and suppose that Assumption 2 holds. Then, under \( \tau = \tau_0 \),

\[
N^{-3/2} \left( t_{N, \text{adj}} - \frac{m(N + 1)}{2} \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\lambda(1 - \lambda)}{12} \right).
\]

(3.7)

It is noteworthy that the limiting null distribution of the regression adjusted statistic \( t_{N, \text{adj}} \) given in (3.7) and that of the unadjusted statistic \( t_N \) given in (2.8) are identical. However, this is expected, since both \( b_N \) and \( \varepsilon_N \) are deterministic vectors under the null, the ranks always take values in \( \{1, 2, \ldots, N\} \), and \( Z_N \) is the only source of randomness.

We next focus on deriving the asymptotic normality of the statistic \( t_{N, \text{adj}} \) under the sequence of local alternatives \( \tau_N = \tau_0 - h N^{-1/2} \) as \( N \to \infty \). This is addressed in Theorem 3.2 below (see Appendix C.8 for a proof), and it immediately yields a CLT for \( \hat{\tau}_{\text{adj}}^R \), which is presented in Theorem 3.3. Although these results are parallel to their unadjusted counterparts (Theorems 2.1 and 2.2, respectively), their proofs are substantially different, and more involved. A more detailed description of the technical challenges is provided in Appendix B.

**Theorem 3.2** (Local asymptotic normality of \( t_{N, \text{adj}} \)). Assume the setting of Theorem 3.1. Fix \( h \in \mathbb{R} \) and let \( \tau_N = \tau_0 - h N^{-1/2} \). Then, under \( \tau = \tau_N \),

\[
N^{-3/2} \left( t_{N, \text{adj}} - \frac{m(N + 1)}{2} \right) \xrightarrow{d} \mathcal{N} \left( -h \lambda(1 - \lambda) J_b, \frac{\lambda(1 - \lambda)}{12} \right),
\]

where \( J_b \) is defined in Assumption 2.
Now that we have established a local asymptotic normality of the statistic \( t_{N,\text{adj}} \), it readily yields a central limit theorem for the estimator \( \hat{\tau}_{\text{adj}}^R \), as stated in the following theorem (see Appendix C.9 for a proof).

**Theorem 3.3** (CLT for the estimator \( \hat{\tau}_{\text{adj}}^R \)). Assume that the random treatment assignment is based on a SRSWOR sample of size \( m \) from the \( N \) subjects, where \( m/N \rightarrow \lambda \in (0,1) \) as \( N \rightarrow \infty \), and suppose that Assumption 2 holds. If \( \tau_0 \) be the true value of \( \tau \), then

\[
\sqrt{N} \left( \hat{\tau}_{\text{adj}}^R - \tau_0 \right) \xrightarrow{d} \mathcal{N} \left( 0, (12\lambda(1-\lambda)J_b)^2\right),
\]

where \( J_b \) is defined in Assumption 2.

The problem of estimating the large sample variance of our estimator \( \hat{\tau}_{\text{adj}}^R \) essentially reduces to the problem of estimating the limiting quantity \( J_b \). However, the estimation of \( J_b \) is more involved than that of \( J_b \) in the unadjusted case. This is deferred until Section 4, where we propose a consistent estimator of \( J_b \) that readily yields asymptotically valid confidence intervals for \( \tau \) (see Corollary 4.2).

### 3.2 Efficiency gain by regression adjustment

As discussed in Section 2, appropriate randomization inference can be drawn ignoring the information available on the covariates. However, it is a popular belief (Rosenbaum, 2002a) that regression adjustment may increase the efficiency of the inference based on the rank-based statistics, although any formal result supporting this belief was missing. We provide such a result in Theorem 3.4 below.

In view of Theorems 2.2 and 3.3, a comparison between the asymptotic variances of the adjusted estimator \( \hat{\tau}_{\text{adj}}^R \) and the unadjusted estimator \( \hat{\tau}^R \) is essentially a comparison of the quantities \( J_b \) and \( I_b \). In the following theorem (see Appendix C.10 for a proof) we show that \( \text{eff}(\hat{\tau}_{\text{adj}}^R, \hat{\tau}^R) \geq 1 \) under the assumption that the regression model is correctly specified (recall the notion of asymptotic relative efficiency from Section 2.3). Thus, in this setting, the regression adjustment provides more efficient estimates than in the unadjusted case.

**Theorem 3.4** (Efficiency gain by regression adjustment). Assume that the potential control outcome sequence \( b_N \) are realizations from the regression model \( b_N = X_N\beta_N + \varepsilon_N \), where \( \varepsilon_{N,i}'s \) are i.i.d. from \( \mathcal{N}(0,\sigma^2) \). Then Assumption 2 holds, with \( J_b = (2\sqrt{\pi}\sigma)^{-1} \). Further, denote \( v_N := X_N\beta_N \) and assume that \( \lim_{N \rightarrow \infty} N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} \exp\left(-\frac{(v_{N,j} - v_{N,i})^2}{4\sigma^2}\right) = \ell \). Then, Assumption 1 holds with \( I_b = \ell J_b \), and consequently,

\[
J_b \geq I_b, \quad \text{i.e.,} \quad \text{eff}(\hat{\tau}_{\text{adj}}^R, \hat{\tau}^R) \geq 1.
\]

Moreover, if it holds that \( \lim_{N \rightarrow \infty} N^{-1} \sum_{j=1}^{N} (v_{N,j} - \bar{v}_N)^2 > 0 \), where \( \bar{v}_N := N^{-1} \sum_{i=1}^{N} v_{N,i} \), then \( J_b > I_b \), and consequently \( \text{eff}(\hat{\tau}_{\text{adj}}^R, \hat{\tau}^R) \) is strictly greater than 1.

Note that Theorem 3.4 is similar in spirit to a result by Lin (2013, Corollary 1.1) on the efficiency gain by regression adjustment for ANOVA-based estimators. In Sections 5 and 6, we illustrate using both simulated and real datasets that the regression adjusted estimators lead to shorter confidence intervals than the unadjusted estimators, while not compromising on the desired significance level. We also observe in our simulations that, even when the regression model is not correctly specified, the regression adjusted estimator \( \hat{\tau}_{\text{adj}}^R \) is at least as efficient as the unadjusted estimator \( \hat{\tau}^R \), suggesting that Theorem 3.4 might hold in a wider generality.
4 Consistent plug-in estimators of standard errors

We now shift our attention in this section to the next natural question: “How do we establish asymptotically honest confidence intervals for \( \tau_0 \) (the true value of \( \tau \))?" To answer this question, note that, in Theorems 2.2 and 3.3, we have established \( \sqrt{N}(\hat{\tau}_R - \tau_0) \) and \( \sqrt{N}(\hat{\tau}_{R,adj} - \tau_0) \) both have centered normal limits with appropriate variances. Both these limiting variances depend on certain properties of the unobserved (fixed) potential outcomes through the quantities \( I_b \) and \( J_b \) (see Assumptions 1 and 2) and hence, are unknown. The main challenge towards obtaining confidence intervals therefore, would be to estimate these unknown variances consistently.

4.1 Consistent estimation of the standard error of \( \hat{\tau}_R \)

In (2.17), we proposed a natural estimator of \( I_b \) that leads to a consistent estimator of the standard error of \( \hat{\tau}_R \). However this estimator has the drawbacks that it is only based on the control outcomes, and we could not generalize it for the regression adjusted case. In this section, we propose a plug-in estimator of \( I_b \) that uses all the observed data and not just the control outcomes. Using the relation \( b_{N,j} = Y_{N,j} - \tau_0 Z_{N,j} \), we define a plug-in estimator of \( b_{N,j} \) as

\[
\hat{b}_{N,j} := Y_{N,j} - \hat{\tau}_N Z_{N,j} = b_{N,j} - (\hat{\tau}_N - \tau_0) Z_{N,j},
\]

(4.1)

where \( \hat{\tau}_N \) is an estimator of \( \tau \). At this point, the next intuitive step would be to replace \((b_{N,1}, \ldots, b_{N,N})\) with \((\hat{b}_{N,1}, \ldots, \hat{b}_{N,N})\) in Assumption 1, i.e., replace the term \( I_{h,N}(b_{N,j} - b_{N,i}) \) on the left hand side of (2.16), with \( I_{h,N,\nu}(\hat{b}_{N,j} - \hat{b}_{N,i}) \) to construct the estimator. However this leads to some technical problems in proving consistency as \( I_{h,N,\nu}(\cdot) \) is a discontinuous function which itself changes with \( N \). This requires us to refine our estimation strategy further and impose a slightly stronger condition than that in Assumption 1. Towards this, for \( \nu > 0 \) and \( N \geq 1 \), define

\[
I_{h,N,\nu}(x) := \begin{cases} 
1 \{0 \leq x < hN^{-\nu}\} & \text{if } h \geq 0, \\
-1 \{hN^{-\nu} \leq x < 0\} & \text{if } h < 0.
\end{cases}
\]

(4.2)

**Assumption 3.** For \( h \in \mathbb{R} \) let \( I_{h,N,\nu} \) be defined in (4.2). We assume that there exists \( 0 < \nu < 1/2 \), the following holds for every \( u \in [\nu, 1/2] \) that

\[
N^{-(2-u)} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N,u}(b_{N,j} - b_{N,i}) \to hI_b
\]

for some constant \( I_b \in (0, \infty) \), which is the same \( I_b \) as in Assumption 1.

**Remark 4.1** (On Assumption 3). Note that if we were to set \( \nu = 1/2 \) in Assumption 3, then \( I_{h,N,1/2}(\cdot) \equiv I_{h,N,1/2}(\cdot) \) is exactly the same function as \( I_{h,N}(\cdot) \) from (2.15), and consequently Assumption 3 would then imply Assumption 1. If we define

\[
T_{h,N,u} := N^{-(2-u)} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N,u}(b_{N,j} - b_{N,i}),
\]

(4.3)

then Assumption 1 requires \( T_{h,N,1/2} \to hI_b \) whereas Assumption 3 requires the mildly stronger condition \( T_{h,N,u} \to hI_b \) for \( u \) varying in any non-degenerate interval with right endpoint at 1/2. We firmly believe
that this is a reasonable assumption. For instance, one of the ways we justified Assumption 1 was by showing that it is satisfied when \( b_{N,i} \)'s are randomly sampled from an absolutely continuous distribution (see Remark 2.3). The same is also true for Assumption 3, see Lemma D.16 in the supplementary material for a formal result.

Finding a consistent estimator of \( \mathcal{I}_b \) is now quite intuitive. With \( T_{h,N,u} \) defined as in (4.3), Assumption 3 implies that \( T_{1,N,\nu} \) converges to \( \mathcal{I}_b \), where \( \nu \) is as in Assumption 3. Consequently, we construct an estimator \( \hat{\mathcal{V}}_N \) from \( T_{1,N,\nu} \) using the plug-in principle as described earlier in this section, by replacing \( b_{N,i} \)'s in (4.3) with \( \hat{b}_{N,i} \)'s (see (4.1)). The following result makes it precise.

**Theorem 4.1** (Consistent estimation of \( \mathcal{I}_b \)). Define \( \hat{b}_{N,j} \) as in (4.1), with \( \hat{\tau}_N \equiv \hat{\tau}^R \). Suppose that Assumption 3 holds for some \( \nu \in (0, 1/2) \), and define

\[
\hat{\mathcal{V}}_N := N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \{ 0 \leq \hat{b}_{N,j} - \hat{b}_{N,i} < N^{-\nu} \}. 
\]

(4.4)

Then \( \hat{\mathcal{V}}_N \xrightarrow{P} \mathcal{I}_b \) as \( N \to \infty \).

Combining Theorems 2.2 and 4.1 we get an asymptotically valid confidence interval for \( \tau_0 \). This is formally stated in the following corollary.

**Corollary 4.1** (Confidence interval for \( \tau_0 \) based on \( \hat{\tau}^R \)). Under Assumptions 1 and 3, an approximate 100(1 - \( \alpha \))% confidence interval for \( \tau_0 \) is given by

\[
\hat{\tau}^R \pm \frac{z_{\alpha/2}}{\sqrt{\hat{\mathcal{V}}_N}} \left( \frac{12}{N} \frac{m}{N} \left( 1 - \frac{m}{N} \right) \hat{\mathcal{V}}_N^2 \right)^{-1/2},
\]

(4.5)

where \( \hat{\mathcal{V}}_N \) is defined in (4.4).

We refer the reader to Remark 4.2 for a discussion on the choice of \( \nu \) in the above result.

### 4.2 Consistent estimation of the standard error of \( \hat{\tau}^R_{\text{adj}} \)

For estimating \( \tau_0 \) (the true value of \( \tau \)) under regression adjustment, we follow the same road map as in the regression unadjusted case in Section 4.1. In this case, by Theorem 3.3, consistently estimating the asymptotic variance of \( \sqrt{N}(\hat{\tau}^R_{\text{adj}} - \tau_0) \) is equivalent to estimating the quantity \( J_b \) (see Assumption 2 for its definition). Once again, due to technical reasons (same as in the regression unadjusted setting above), we require a mildly stronger condition than Assumption 2 to come up with a consistent estimator for \( J_b \). To lay the groundwork, define

\[
\tilde{T}_{h,N,u} := N^{-(2-u)} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N,u}(\tilde{b}_{N,j} - \tilde{b}_{N,i}),
\]

(4.6)

where \( \tilde{b}_{N,j} \)'s are defined as in (3.6) and \( I_{h,N,u}(\cdot) \) is defined as in (4.2). Observe the direct correspondence between \( \tilde{T}_{h,N,u} \) and \( T_{h,N,u} \) from (4.3). The only difference is that the potential control outcomes \( b_{N,j} \)'s are replaced by the regression adjusted potential control outcomes \( \tilde{b}_{N,j} \)'s. Note also that Assumption 2 can be restated as \( \tilde{T}_{h,N,1/2} \to hJ_b \). With this observation, in the same vein as Assumption 3, we now state our mildly stronger condition (compared to Assumption 2):
Assumption 4. There exists \( 0 < \nu < 1/2 \), such that for every \( u \in [\nu, 1/2] \) and \( h \in \mathbb{R} \), \( \bar{T}_{hN,u} \to hJ_b \), for some constant \( J_b \in (0, \infty) \), which is the same \( J_b \) as in Assumption 2.

One can show that Assumption 4 holds under the same regression model assumption as in Theorem 3.4 (see the arguments used in Lemma D.8 in Appendix D).

We are now in the position to construct our consistent estimator for \( J_b \). Towards this direction, recall the definition of \( \hat{\theta}_{N,j} \) from (4.1) and set \( \hat{\theta}_N := (\hat{\theta}_{N,1}, \ldots, \hat{\theta}_{N,N}) \). To carry out the plug-in approach, we define the empirical analogues of \( \hat{\theta}_{N,j} \)’s, as follows:

\[
\hat{\theta}_{N,j} := \hat{\theta}_{N,j} - \hat{p}_{N,j}^\top \theta_N = Y_{N,j} - \hat{\tau}_{N,j}^{R}\hat{Z}_{N,j} - \hat{p}_{N,j}^\top (Y - \hat{\tau}_{N,j}^{R}\hat{Z}), \quad j = 1, 2, \ldots, N. \tag{4.7}
\]

The plug-in (consistent) estimator for \( J_b \) is now constructed in the same manner as we did for \( I_b \), only with the modification that here \( \hat{\theta}_{N,j} \)’s will play the role of \( \hat{\theta}_{N,j} \)’s. This is the content of the following theorem.

**Theorem 4.2** (Consistent estimation of \( J_b \)). Define \( \hat{W}_N \) as in (4.8), and suppose that Assumption 4 holds for some \( \nu \in (0, 1/2) \). Define

\[
\hat{W}_N := N^{-(2-\nu)} \sum_{j=1}^N \sum_{i=1}^N 1 \left\{ 0 \leq \hat{\theta}_{N,j} - \hat{\theta}_{N,i} < N^{-\nu} \right\}. \tag{4.8}
\]

Then \( \hat{W}_N \to J_b \) as \( N \to \infty \).

Theorem 4.2 in conjunction with Theorem 3.3 readily yields an asymptotically valid confidence interval for \( \tau_0 \) based on the regression adjusted estimator \( \hat{\tau}_{R adj} \), which is formally stated below.

**Corollary 4.2** (Confidence interval for \( \tau_0 \) based on \( \hat{\tau}_{R adj} \)). Under Assumptions 2 and 4, an approximate 100(1 - \( \alpha \))% confidence interval for \( \tau_0 \) is given by

\[
\hat{\tau}_{R adj} \pm \frac{z_{\alpha/2}}{\sqrt{N}} \left( \frac{12}{N} \left( 1 - \frac{m}{N} \right) \hat{W}_N \right)^{-1/2}, \tag{4.9}
\]

where \( \hat{W}_N \) is defined in (4.8).

**Remark 4.2** (Choice of \( \nu \)). Note that (4.4) and (4.8) both require a choice of \( \nu \in (0, 1/2) \). We have observed through extensive simulations that the confidence intervals are not very sensitive to the choice of \( \nu \). A more detailed exposition is provided in Remark A.1 in the supplementary material. In fact, our simulations suggest that the choice \( \nu = 1/2 \) also works in practice, although our proof of Theorem 4.1 precludes that choice.

The following sections illustrate the empirical performance of the confidence intervals given in Corollaries 4.1 and 4.2 on both simulated and real datasets.

## 5 Simulations

In this section we illustrate, through numerical experiments, the performance of \( \hat{\tau}_{R} \) and \( \hat{\tau}_{R adj} \). We also compare their performance with those of some conventional estimators, namely, the difference-in-means estimator \( \hat{\tau}_{dm} \) (see (2.10)), the simple regression adjusted estimator \( \hat{\tau}_{adj} \) (Freedman, 2008a,b), and the estimator \( \hat{\tau}_{interact} \) proposed by Lin (2013).
We take \( N = 1000 \) (the total sample size), out of which we choose \( m \) subjects by SRSWOR and assign them to the treatment group and the rest to the control group. We experimented with various values of the proportion \( m/N \), here we only report the results for \( m/N = 0.75, 0.50, \) and \( 0.25 \). Consider the settings listed in Table 2. In each of these settings, the errors \( \varepsilon_i \)'s are drawn i.i.d. from: (a) the standard normal distribution, (b) the \( t_1 \) distribution, (c) the \( t_3 \) distribution; independent of everything else. Thus, all together we have 9 simulation settings. We will refer to the sub-settings as, for instance, Setting 1b, 2a, etc. We repeat the above randomized experiment 1000 times for each of the sub-settings, to calculate the average coverage and length of the confidence intervals. The potential control outcomes are generated as \( b_i = a_i - \tau_0 \), where \( \tau_0 = 2 \) is the true value of the constant treatment effect \( \tau \).

### Table 2: Simulation settings

| Setting 1 | \( x_i \sim \text{Unif}(-4, 4) \) | \( a_i = v_i + \varepsilon_i, \ v_i \sim \text{Exp}(1/10) \) |
| Setting 2 | \( x_i \sim \text{Unif}(-4, 4) \) | \( a_i = 3x_i + \varepsilon_i \) |
| Setting 3 | \( x_i = e^{u_i}, u_i \sim \text{Unif}(-4, 4) \) | \( a_i = \frac{1}{4}(x_i + \sqrt{x_i^2}) + \varepsilon_i \) |

Our motivation for considering the settings in Table 2 are as follows. Setting 1 is an example where the potential outcomes are independent of the covariates. Setting 2 is an example where the regression model is correctly specified. Finally, Setting 3 is a slight modification of an example of Lin (2013, Setting 4.2.3), and helps us illustrate the performance of the estimators under misspecification of the regression model.

We report the coverage and lengths of the confidence intervals obtained from the aforementioned estimators in the above settings. As noted in Remark 4.2, the choice of \( \nu \) is inconsequential for finding the confidence intervals based on the plug-in estimators of the asymptotic variances of \( \hat{\tau}_R \) and \( \hat{\tau}_{R, \text{adj}} \); hence we report the simulation results for only a particular value of \( \nu \), namely, \( \nu = 1/3 \). We also report the results on the oracle confidence intervals obtained by plugging in the true asymptotic variances. Recall that in Corollary 2.1 we proposed another confidence interval based on \( \hat{\tau}_R \). In each of our simulation settings, the confidence intervals constructed using Corollary 2.1 are almost identical, in terms of lengths and coverages, to the confidence intervals constructed using the plug-in estimator of the asymptotic variance of \( \hat{\tau}_R \) (see Corollary 4.1); hence the former are not reported here. The results are summarized in Tables 3–5, and some observations are listed below.

- **Efficiency of \( \hat{\tau}_R \) w.r.t. \( \hat{\tau}_{dm} \):** In all the settings except Setting 2c, the confidence intervals constructed using the unadjusted estimator \( \hat{\tau}_R \) have shorter lengths than those constructed using \( \hat{\tau}_{dm} \), yet maintaining the desired significance level. Even for Setting 2c the differences in the lengths are within a margin dictated by Theorem 2.4.

- **Regression adjustment improves precision:** For Setting 1, where the potential outcomes are drawn independently of the covariates, our adjusted confidence intervals and unadjusted confidence intervals have the same lengths, as one expects. For Setting 2a, where the model is correctly specified and the errors are Gaussian, Table 4 shows that our adjusted confidence intervals have shorter lengths than our unadjusted confidence intervals, yet have the same coverage, which validates our Theorem 3.4. This reinforces the popular belief (see, for instance, Rosenbaum (2002a)) that regression adjustment generally improves the precision over the unadjusted case.
• **Comparison of \( \hat{\tau}_{R \text{adj}} \) with \( \hat{\tau}_{\text{adj}} \) and \( \hat{\tau}_{\text{interact}} \):** In all of the settings, the lengths of our confidence intervals constructed using \( \hat{\tau}_{R \text{adj}} \) are either smaller than or (almost) equal to the lengths of the confidence intervals constructed using \( \hat{\tau}_{\text{adj}} \) or \( \hat{\tau}_{\text{interact}} \), without compromising on coverage.

• **Robustness against heavy-tails:** In each of the settings, when the errors are drawn from the \( t_1 \) distribution, the confidence intervals based on \( \hat{\tau}_{\text{dm}}, \hat{\tau}_{\text{adj}} \) or \( \hat{\tau}_{\text{interact}} \) are far too wide. In fact, in such situations the regression adjustment provides negligible improvement over the unadjusted estimator even when the regression model is correctly specified (see the results for Setting 2b). In contrast, our proposed confidence intervals based on \( \hat{\tau}^R \) and \( \hat{\tau}^R_{\text{adj}} \) have reasonably shorter lengths, and provide coverage close to the nominal level.

• **Robustness against model misspecification:** In Setting 3, the regression model is wrongly specified. Yet, the performance of our rank-based confidence intervals are quite satisfactory, illustrating that the estimators \( \hat{\tau}^R \) and \( \hat{\tau}^R_{\text{adj}} \) are robust against model misspecification. Note also that in this setting, despite the regression model being misspecified, the confidence intervals based on \( \hat{\tau}^R_{\text{adj}} \) are shorter than those constructed using \( \hat{\tau}^R \), suggesting that Theorem 3.4 holds in more generality than what is assumed in the statement of the result.

**Table 3:** Results for simulation Setting 1 (potential outcomes independent of the covariates)

| Estimator            | (a) Gaussian errors | (b) \( t_1 \) errors | (c) \( t_3 \) errors |
|---------------------|---------------------|-----------------------|-----------------------|
|                     | coverage            | length                | coverage              | length                | coverage              | length                | coverage              | length                |
| \( \hat{\tau}^R \)  | 0.94 0.96 0.95      | 1.85 1.60 1.85        | 0.95 0.96 0.96        | 2.29 1.98 2.29        | 0.94 0.95 0.95        | 1.93 1.67 1.93        |
| \( \hat{\tau}^R \)  (oracle)| 0.94 0.96 0.95      | 1.85 1.60 1.85        | 0.95 0.96 0.96        | 2.29 1.98 2.29        | 0.94 0.95 0.95        | 1.93 1.67 1.93        |
| \( \hat{\tau}_{\text{dm}} \)| 0.95 0.95 0.96      | 2.88 2.49 2.88        | 0.97 0.98 0.97        | 57.62 46.47 43.10     | 0.95 0.96 0.94        | 2.91 2.52 2.91        |
| \( \hat{\tau}^R_{\text{adj}} \)| 0.94 0.96 0.95      | 1.86 1.61 1.86        | 0.94 0.96 0.96        | 3.35 3.03 3.38        | 0.95 0.95 0.95        | 1.94 1.68 1.94        |
| \( \hat{\tau}^R_{\text{adj}} \) (oracle)| 0.94 0.96 0.95      | 1.86 1.61 1.86        | 0.95 0.96 0.96        | 3.48 3.01 3.48        | 0.95 0.95 0.95        | 1.94 1.68 1.94        |
| \( \hat{\tau}_{\text{adj}} \)| 0.95 0.95 0.96      | 2.88 2.49 2.87        | 0.97 0.98 0.97        | 57.05 45.88 43.04     | 0.95 0.96 0.94        | 2.90 2.52 2.91        |
| \( \hat{\tau}_{\text{interact}} \)| 0.94 0.95 0.96      | 2.87 2.49 2.87        | 0.97 0.98 0.97        | 56.41 45.85 42.90     | 0.95 0.96 0.94        | 2.90 2.52 2.90        |
### Table 4: Results for simulation Setting 2 (regression model correctly specified)

| Estimator         | (a) Gaussian errors | (b) $t_1$ errors | (c) $t_3$ errors |
|-------------------|---------------------|------------------|------------------|
|                   | coverage            | length           | coverage         | length           | coverage         | length           |
| $\hat{\tau}_R$   | 0.95 0.50 0.25      | 0.95 0.50 0.25   | 0.95 0.50 0.25   | 0.95 0.94 0.95   | 2.08 1.81 2.08   | 2.43 2.11 2.43   |
| $\hat{\tau}_R$ (oracle) | 0.95 0.50 0.25 | 0.95 0.50 0.25 | 0.95 0.94 0.95 | 2.09 1.81 2.09 | 2.43 2.11 2.43 |
| $\hat{\tau}_{dm}$ | 0.95 0.50 0.25      | 0.98 0.97 0.98   | 0.95 0.94 0.96   | 2.01 1.73 2.00   | 63.52 70.98 70.32 | 2.04 1.77 2.04 |
| $\hat{\tau}_{adj}$ | 0.95 0.50 0.25 | 0.94 0.94 0.95 | 0.96 0.94 0.95 | 0.29 0.25 0.29 | 0.36 0.31 0.36 |
| $\hat{\tau}_{adj}$ (oracle) | 0.95 0.50 0.25 | 0.94 0.94 0.95 | 0.96 0.95 0.95 | 0.29 0.25 0.29 | 0.36 0.31 0.36 |
| $\hat{\tau}_{interact}$ | 0.95 0.50 0.25 | 0.98 0.98 0.98 | 0.95 0.93 0.95 | 0.29 0.25 0.29 | 0.48 0.42 0.49 |

### Table 5: Results for simulation Setting 3 (model misspecification)

| Estimator         | (a) Gaussian errors | (b) $t_1$ errors | (c) $t_3$ errors |
|-------------------|---------------------|------------------|------------------|
|                   | coverage            | length           | coverage         | length           | coverage         | length           |
| $\hat{\tau}_R$   | 0.95 0.48 0.56      | 0.95 0.77 0.89   | 0.96 0.95 0.96   | 0.65 0.56 0.65   | 0.56 0.48 0.56   | 0.89 0.77 0.89   |
| $\hat{\tau}_R$ (oracle) | 0.95 0.48 0.56 | 0.95 0.77 0.89 | 0.95 0.95 0.95 | 0.65 0.56 0.65 | 0.48 0.32 0.36 |
| $\hat{\tau}_{dm}$ | 0.95 0.94 0.94      | 0.99 0.98 0.98   | 0.95 0.95 0.95   | 1.02 0.88 1.02   | 37.74 37.56 42.99 | 1.09 0.95 1.09 |
| $\hat{\tau}_{adj}$ | 0.95 0.96 0.95     | 0.94 0.95 0.95 | 0.95 0.95 0.95 | 0.30 0.26 0.30 | 0.71 0.61 0.71 | 0.36 0.31 0.36 |
| $\hat{\tau}_{adj}$ (oracle) | 0.95 0.96 0.95 | 0.94 0.95 0.95 | 0.95 0.95 0.95 | 0.30 0.26 0.30 | 0.71 0.61 0.71 | 0.36 0.32 0.36 |
| $\hat{\tau}_{adj}$ | 0.95 0.95 0.95     | 0.98 0.98 0.98   | 0.95 0.95 0.94 | 0.29 0.25 0.29 | 37.86 37.66 43.07 | 0.49 0.43 0.49 |
| $\hat{\tau}_{interact}$ | 0.95 0.95 0.96     | 0.98 0.97 0.98   | 0.95 0.95 0.95 | 0.29 0.25 0.29 | 37.82 37.62 43.03 | 0.49 0.43 0.49 |

### 6 Data analysis

We demonstrate the empirical performance of the rank-based estimators $\hat{\tau}_R$ and $\hat{\tau}_{R \text{ adj}}$, and compare it with that of the regression-based estimators $\hat{\tau}_{dm}$, $\hat{\tau}_{adj}$ and $\hat{\tau}_{interact}$, on two real datasets, namely: (i) Mexico’s conditional cash transfer program (Progresa) data (De La O, 2013; Imai, 2018), and (ii) the house price data from the replication files of Linden and Rockoff (2008).
6.1 Progresa data

We analyze the data from a randomized trial that aims to study the electoral impact of Progresa, Mexico’s conditional cash transfer program (CCT program) (De La O, 2013; Imai, 2018). In this experiment, eligible villages were randomly assigned to receive the program either 21 months (early Progresa, “treated”) or 6 months (late Progresa, “control”) before the 2000 Mexican presidential election. The data contains 417 observations, each representing a precinct, and for each precinct we have information about its treatment status, outcomes of interest, socioeconomic indicators, and other precinct characteristics.

Following De La O (2013), our outcome is the support rates for the incumbent party (PRI) as shares of the eligible voting population in the 2000 election (\(\text{pri2000s}\)), and our regression models include the following covariates: the average poverty level in a precinct (\(\text{avgpoverty}\)), the total precinct population in 1994 (\(\text{pobtot1994}\)), the total number of voters who turned out in the previous election (\(\text{votos1994}\)), and the total number of votes cast for each of the three main competing parties in the previous election (\(\text{pri1994}\) for PRI, \(\text{pan1994}\) for Partido Acción Nacional or PAN, and \(\text{prd1994}\) for Partido de la Revolución Democrática or PRD), and also include villages as factors. Table 6 gives the point estimates and approximate 95% confidence intervals for the five methods.

It is noteworthy that the standard errors of Rosenbaum’s estimators are substantially smaller than that of \(\hat{\tau}_{\text{dm}}\), \(\hat{\tau}_{\text{adj}}\), or \(\hat{\tau}_{\text{interact}}\). Further, the standard error for \(\hat{\tau}_{\text{adj}}\) is slightly less than that of \(\hat{\tau}_{\text{R}}\), just as one can expect in light of Theorem 3.4. Each of the confidence intervals suggests that the CCT program led to a significant positive increase in support for the incumbent party.

Table 6: Different estimates of the effect of early Progresa on PRI support rates with the corresponding standard errors, 95% approximate confidence intervals and their lengths.

| method  | estimate | std.error | 95% confidence interval | length |
|---------|----------|-----------|------------------------|--------|
| \(\hat{\tau}_{\text{R}}\) | 1.834 | 0.446 | [0.960, 2.707] | 1.747 |
| \(\hat{\tau}_{\text{dm}}\) | 3.622 | 1.728 | [0.235, 7.010] | 6.774 |
| \(\hat{\tau}_{\text{Radj}}\) | 2.185 | 0.411 | [1.380, 2.989] | 1.610 |
| \(\hat{\tau}_{\text{adj}}\) | 3.671 | 1.510 | [0.712, 6.630] | 5.917 |
| \(\hat{\tau}_{\text{interact}}\) | 4.214 | 1.462 | [1.348, 7.079] | 5.731 |

6.2 House price data

Our second example is the house price data from the replication files of Linden and Rockoff (2008), which contain data on property sales for Mecklenburg County (North Carolina) between January 1994 and December 2004. Dropping the sales below $5,000 and above $1,000,000, the dataset comes with 170,239 observations. Exploratory data analysis shows that even after taking logarithm of house prices, the distribution is heavily skewed on the right side with kurtosis equal to 5.1, which makes us anticipate that the rank-based estimators will perform better (in terms of shorter confidence intervals) than the regression-based estimators for this dataset.

We create synthetic data sets from this data set to compare the different estimators. Following Athey et al. (2021), we draw subsamples from the dataset and randomly assign exactly half of each sample to the treatment group and the remaining half to the control group. Thus, we know apriori that the treatment effect is zero, and hence can assess the performance of the five estimators in a
different way than the previous example. In particular, we repeat this experiment and evaluate the performance of the estimators based on the coverage and average lengths of the confidence intervals.

In our simulations, we draw subsamples of size $n = 1000$ in each iteration, take the logarithm of the house prices as the outcome variable, and use several characteristics of the houses (e.g., sales year, age of the house (in years), number of bedrooms) as covariates. The fit of these models are found to be quite satisfactory, with adjusted $R^2$ values being near to 0.7. The estimates, along with their standard errors and approximate 95% confidence intervals obtained from a single simulation are shown in Table 7. Repeating this experiment $B = 1000$ times, we report the coverage and average lengths of the confidence intervals in Table 8.

Observe that the confidence intervals obtained using $\hat{\tau}_{adj}^R$ are the shortest among the five, having about half the length than those obtained using $\hat{\tau}_{adj}$ or $\hat{\tau}_{interact}$, without any compromise in the coverage. Also, the confidence intervals obtained using $\hat{\tau}^R$ are shorter than those obtained using $\hat{\tau}_{dm}$, and are wider than those obtained using $\hat{\tau}_{adj}^R$. These observations are completely aligned with our main theoretical contributions.

Table 7: Results from a single simulation from the house price data

|           | estimate | std.error | 95% confidence interval | length |
|-----------|----------|-----------|-------------------------|--------|
| $\hat{\tau}^R$ | -0.04    | 0.03      | [-0.09, 0.02]           | 0.12   |
| $\hat{\tau}_{dm}$ | -0.05    | 0.04      | [-0.12, 0.03]           | 0.15   |
| $\hat{\tau}_R^{adj}$ | -0.01    | 0.01      | [-0.03, 0.02]           | 0.04   |
| $\hat{\tau}_{adj}$ | -0.02    | 0.02      | [-0.06, 0.03]           | 0.08   |
| $\hat{\tau}_{interact}$ | -0.02    | 0.02      | [-0.06, 0.03]           | 0.08   |

Table 8: Coverage and average lengths of the approximate 95% confidence intervals obtained from different estimators by repeated simulations from the house price data

|           | $\hat{\tau}^R$ | $\hat{\tau}_{dm}$ | $\hat{\tau}_R^{adj}$ | $\hat{\tau}_{adj}$ | $\hat{\tau}_{interact}$ |
|-----------|----------------|-------------------|---------------------|--------------------|-------------------------|
| coverage  | 0.944          | 0.959             | 0.954               | 0.952              | 0.949                   |
| average length | 0.121         | 0.143             | 0.041               | 0.073              | 0.073                   |

7 Discussion

Randomization remains the gold standard method for causal inference as it balances both the observed and unobserved covariates across treatment groups. Technological advances have made randomization experiments more feasible and cheaper to conduct. As a result, large randomization experiments are becoming increasingly popular, see recent discussions in Banerjee et al. (2016) and Deaton and Cartwright (2018). Another remarkable feature of randomization is that it can, by itself, be used as the ‘reasoned basis’ for inference — we do not need to make (generally unverifiable) model assumptions about the outcome (Fisher (1935)).

In this paper, under randomization inference, we study statistical inferential strategies using rank-based methods for the treatment effect. We derive easily computable formulas for asymptotic level $100(1 - \alpha)$ confidence intervals using Rosenbaum’s unadjusted and regression adjusted rank-based estimators (see Rosenbaum (2002a)). Our results highlight many benefits of these estimators in terms of their robustness properties and efficiency relative to the standard (un)-adjusted difference-in-means.
estimator. In particular, we show that the asymptotic relative efficiency of Rosenbaum’s unadjusted estimator is in the worst case only $\sim 15\%$ lower than the unadjusted difference-in-means estimator, but often the efficiency is much higher. We also show that regression adjustment gives provable efficiency gain for Rosenbaum’s estimator under a linear outcome model. Additionally, our detailed simulation study illustrates the superior performance of these estimators for normal and heavy-tailed errors, and for linear and non-linear models.

This paper studies Rosenbaum’s rank-based estimator (see Rosenbaum (2002a)) under the constant treatment effect model — it would be interesting to investigate the properties of the estimator when this model assumption is violated. We may consider relaxing this model assumption in the following ways. First, it may happen that our data or our existing knowledge indicates that the treatment interacts with a covariate. In that case, we can incorporate this information in our inference by: (a) changing the design (e.g., using blocked randomization that blocks on the interacting covariate instead of complete randomization as studied in this paper), or (b) changing the model for the adjusted responses (see Rosenbaum, 2021, Section 1.8 for general adjustment). Our results may be extended in these situations to construct confidence intervals for the treatment effect after conditioning on covariates. Second, Caughey et al. (2021) provide methods to construct confidence intervals for quantiles of individual treatment effects by inverting randomization based tests. Following their approach, it will be interesting to generalize our methods to target the quantiles of individual treatment effects.

Finally, the average treatment effect across the units is a popular choice as a causal estimand. But, this estimand may be undesirable, for example, when only a fraction of the units have a large positive treatment effect and others have no effect or negative effect; see, e.g., Athey et al. (2021) and Rosenbaum (2021, Section 1.3.4). In future, we plan to study a more appropriate estimand under such situations and develop rank-based statistical inference procedures targeting this estimand.

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Appendices A–F

This supplementary material begins with a discussion in Appendix A on some practical considerations for implementing our method of plug-in estimation of variance in Section 4 of the main paper. In Appendix B we briefly outline the main ideas of the proofs of our major results. The subsequent sections (Appendices C–E) contain the proofs of all results in the main paper, other auxiliary results (with their proofs) and further discussions. Finally, Appendix F talks about the analogous results that hold for average ranks instead of the up-ranks we used in the main paper.

If \( \{x_n\}_{n \geq 1} \) and \( \{y_n\}_{n \geq 1} \) are two sequence of positive real numbers, we write \( x_n \sim y_n \) to denote that \( \lim_{n \to \infty} x_n/y_n = 1 \), and \( x_n \preceq y_n \) to denote that \( x_n \leq Cy_n \) holds for all sufficiently large \( n \), for some constant \( C > 0 \).

### A Some practical considerations

The following remark highlights that the choice of the tuning parameter \( \nu \) is inconsequential for both the regression unadjusted and regression adjusted confidence intervals that we proposed in Section 4 of the main paper.

**Remark A.1 (Choice of \( \nu \)).** Our plug-in estimators proposed in (4.4) and (4.8) of the main paper involve a tuning parameter \( \nu \). In our simulations we observed that the choice of \( \nu \) is not crucial, as long as we are only concerned about consistent estimation of the asymptotic variances so as to find confidence intervals for \( \tau_0 \). In fact, \( \nu = 1/2 \) also seems to work in the simulations, even though our proofs for the consistency of these estimators require \( \nu < 1/2 \).

For an illustration, consider our simulation setting 3a described in Section 5 (see Table 2). We calculate the plug-in estimator of the asymptotic variances of \( \tilde{\tau}_R \) and \( \tilde{\tau}_R^{\text{adj}} \) for different values of \( \nu \), ranging from 1/4 to 1/2. Then fixing one particular value of \( \nu \) as a reference, say \( \nu = 1/3 \), we calculate the ratios of these estimates for \( \nu = 1/3 \) to those for the other values of \( \nu \). Repeating this 100 times, we draw the boxplots of the ratios thus obtained, which is provided in Fig. A.1.

The plots suggest that the estimates of the asymptotic variance do not differ much with the choice of \( \nu \). We observed similar phenomenon for the other simulation settings as well. Furthermore, in our simulations we also varied \( \nu \) while calculating the confidence intervals for \( \tau_0 \) proposed above, and found that the results do not vary considerably with the choice of \( \nu \). Thus the choice of \( \nu \) seems to be unimportant, and in practice one may use any number in \( (0,1/2) \) as the value of \( \nu \); in our simulation studies in Section 5 we use \( \nu = 1/3 \).
without reg. adj.

\[ \begin{array}{c|c|c|c|c|c}
\text{ratio} & 0.90 & 1.00 & 1.10 & 0.25 & 0.30 & 0.35 & 0.40 & 0.45 & 0.50
\end{array} \]

with reg. adj.

\[ \begin{array}{c|c|c|c|c|c}
\text{ratio} & 0.90 & 1.00 & 1.10 & 0.25 & 0.30 & 0.35 & 0.40 & 0.45 & 0.50
\end{array} \]

Figure A.1: Boxplots of the ratios of the plug-in estimators of asymptotic variances of \( \hat{\tau}^R \) (left panel) and \( \hat{\tau}^R_{\text{adj}} \) (right panel) for \( \nu = 1/3 \) and various other values of \( \nu \), in simulation setting 4a (see Section 5 for details on the simulation setup).

Remark A.2 (Scaling the outcome variable). Since our method involves counts of differences in the outcome variable (or the fitted residuals) in shrinking intervals, we might not get a reasonable estimate if the responses are typically large in magnitude. This is why we suggest to scale (or modify) the outcome variable appropriately while using our variance estimators in practice. In Section 6.2 of the main paper, where we apply our method on a house price data, we use the logarithm of the house prices as the outcome variable since the house prices are typically large in magnitude.

B Main technical tools

In this section, we discuss the main tools used in the proofs of our main results in Sections 2–4 of the main paper. We begin with Proposition 2.1 where \( \tau = \tau_0 \). In this case, it is easy to check that

\[ t_N \equiv t_N^0 = m + \sum_{j=1}^{N} Z_{N,j} \sum_{i=1,i\neq j}^{N} 1\{b_{N,i} \leq b_{N,j}\}. \]

Note that \( t_N^0 \) is a weighted linear combination of \( Z_{N,j}'s \), where the weights are non-random. We therefore use the classical Hoeffding’s combinatorial CLT (see Hoeffding (1951); also see Appendix C.1 for more details) to complete the proof of Proposition 2.1. The case of local alternatives, i.e., where \( \tau = \tau_N := \tau_0 - h/\sqrt{N} \), is more challenging. This is the subject of Theorem 2.1 of the main paper. Note that under \( \tau = \tau_N \), we have:

\[ t_N \equiv t_N^h = m + \sum_{j=1}^{N} Z_{N,j} \sum_{i=1,i\neq j}^{N} 1\{b_{N,i} - hN^{-1/2}Z_{N,i} \leq b_{N,j} - hN^{-1/2}Z_{N,j}\}. \]

When \( h \neq 0 \), \( t_N^h \) is a weighted linear combination of \( Z_{N,j}'s \) with random weights depending on the \( Z_{N,j}'s \) themselves (unlike what we observed for \( t_N^0 \)). Therefore classical combinatorial CLTs do not apply. Further, as we are in the fixed design setting, the models under \( \tau = \tau_0 \) and \( \tau = \tau_N \) lack the desirable forms for us to use the standard contiguity arguments due to Le Cam for deriving local asymptotic normality (see van der Vaart (1998, Chapter 7) for details). We tackle these challenges with the aid of a special decomposition which is the subject of the following proposition (see Appendix C.13 for a proof).
Proposition B.1. Let \( t_N = t_N(Z_N, Y_N - \tau_0 Z_N) \) be the Wilcoxon rank-sum statistic based on any random treatment assignment, where \( t(\cdot, \cdot) \) is as in (2.4) of the main paper. Fix \( h \in \mathbb{R} \) and let \( \tau_N = \tau_0 - hN^{-1/2} \). It holds under \( \tau = \tau_N \) that

\[
t_N \equiv t_N^h \overset{d}{=} t_N^0 - \gamma_N^h, \quad \text{where} \quad \gamma_N^h := \sum_{j=1}^N \sum_{i=1, i \neq j}^N (1 - Z_{N,i})Z_{N,j}I_{h,N}(b_{N,j} - b_{N,i})
\]

where \( I_{h,N} \) is defined in (2.15) of the main paper.

Based on Proposition B.1, the proof of Theorem 2.1 can be completed via Slutsky’s Theorem if we show the following:

\[
N^{-3/2} \left( t_N^0 - \frac{m(N+1)}{2} \right) \overset{d}{\to} \mathcal{N} \left( 0, \frac{\lambda(1-\lambda)}{12} \right) \quad \text{and} \quad \gamma_N^h \overset{P}{\to} -h(1-\lambda)\mathcal{I}_b,
\]

where \( \mathcal{I}_b \) is defined in Assumption 1 of the main paper. The asymptotic fluctuation of \( t_N^0 \), in the above display, follows directly from Proposition 2.1 as already discussed above. To prove the limit of \( \gamma_N^h \), we use the standard Markov’s inequality coupled with Assumption 1 (see Appendix C.3 for more details). Once the asymptotic distribution of \( t_N \equiv t_N^h \) is obtained, the same for \( \hat{\tau}^R \) as stated in Theorem 2.2, follows as a direct consequence of a classical Hodges and Lehmann argument (see Hodges and Lehmann (1963); also see Theorem 2.1 and Lemma D.1).

The next important question is obtaining asymptotically honest confidence intervals for \( \tau_0 \) (as presented in Theorem 4.1 of the main paper). Based on Theorem 2.2, this reduces to obtaining a consistent estimator for \( \mathcal{I}_b \), which by Assumption 3, is the limit of \( u_N \) where

\[
u_N := N^{-3/2} \sum_{j=1}^N \sum_{i=1}^N \left( 1(Y_{N,j} - Y_{N,i} - \tau_0(Z_{N,j} - Z_{N,i}) \geq 0 \right) - 1(Y_{N,j} - Y_{N,i} - \tau_0(Z_{N,j} - Z_{N,i}) \geq N^{-\nu}) \right).
\]

The natural plug-in estimator can then be constructed by replacing \( \tau_0 \) by \( \hat{\tau}^R \) above, to get a plug-in analogue of \( u_N \), say \( \hat{u}^R_N \). While \( \hat{u}^R_N \) should intuitively be consistent, there are two technical issues: (a) indicators are not continuous functions, and (b) the sets in the indicators used above are not fixed, but effectively shrinking with \( N \) (they are of the form \([0, N^{-\nu}]\)). These technical issues preclude the possibility of using a continuous mapping type argument to establish consistency. We circumvent this by approximating the indicators in \( u_N \) with Gaussian mollifiers with an appropriately chosen variance parameter. In particular, setting \( \Phi(\cdot) \) as the standard Gaussian distribution function and \( \sigma_{N,\nu} := \log N/N^\nu \), define

\[
\hat{u}^G_N := \frac{1}{N^{3/2}} \sum_{i,j=1}^N \left( \Phi(\sigma^{-1}_{N,\nu}(Y_{N,j} - Y_{N,i} - \hat{\tau}^R(Z_{N,j} - Z_{N,i})) \right.
\]

\[
- \Phi(\sigma^{-1}_{N,\nu}(Y_{N,j} - Y_{N,i} - \hat{\tau}^R(Z_{N,j} - Z_{N,i} - N^{-\nu}))) \right).
\]

Using some careful truncation arguments, we then show that \( \hat{u}^G_N - \hat{u}^R_N \overset{P}{\to} 0 \) and \( \hat{u}^G_N - u_N \overset{P}{\to} 0 \), which when combined, yield \( \hat{u}^R_N \overset{P}{\to} \mathcal{I}_b \).

Our next important result in the regression unadjusted setting is the efficiency lower bound for \( \hat{\tau}^R \) when compared with \( \hat{\tau}_{\text{dm}} \), as presented in Theorem 2.4. The main challenge in its proof lies in showing
the following: If \( b_{N,1}, \ldots, b_{N,N} \) are realizations from a random sample from a probability density \( f_b(\cdot) \), then Assumption 1 holds with \( I_b = \int f_b^2(x) \, dx \). In other words, by (2.16), it suffices to show that

\[
\mathcal{T}_{N,h} := N^{-3/2} \sum_{i,j=1}^{N} I_{h,N}(b_{N,j} - b_{N,i}) \xrightarrow{a.s.} h \int f_b^2(x) \, dx.
\]

By an application of the Borel-Cantelli Lemma, it suffices to show that, given any \( \varepsilon > 0 \), the following holds:

\[
\sum_{N \geq 1} \mathbb{P}(|\mathcal{T}_{N,h} - \mathbb{E}(\mathcal{T}_{N,h})| \geq \varepsilon) < \infty \quad \text{and} \quad \mathbb{E}(\mathcal{T}_{N,h}) \to h \int f_b^2(x) \, dx.
\]

We prove the first conclusion in the above display by appealing to a generalized version of the Efron-Stein inequality (see Boucheron et al. (2005, Theorem 2); also see Proposition E.1), while the second conclusion is proved by a careful application of the mean value theorem.

We now move on to our results from Section 3 of the main paper. The major technical difference between the adjusted and unadjusted cases is perhaps reflected in the proof of Theorem 3.2, which requires substantially different techniques compared to its unadjusted counterpart in Theorem 2.1. To understand this difference, recall from (3.5) of the main paper that under \( \tau = \tau_0 - hN - 1/2 \), the regression adjusted Wilcoxon rank-sum statistic is given by:

\[
t_{N,\text{adj}} \equiv t_{N,\text{adj}}^h = \sum_{i,j=1}^{N} Z_{N,j} \mathbf{1}(hN^{-1/2}(p_{N,i} - p_{N,j})^\top Z_N \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} - hN^{-1/2}),
\]

where \( p_{N,j} \) is the \( j \)-th column of the projection matrix \( P_{X_N} \) that projects onto the column space of \( X_N \), and \( \tilde{b}_{N,i} := Y_{N,i} - \tau_0 Z_{N,i} - P_{X_N}^\top b_N \). Therefore, for every pair \((i,j)\), the indicators in \( t_{N,\text{adj}} \) depend on the entire random vector \((Z_{N,1}, \ldots, Z_{N,N})\). This is in sharp contrast to \( t_N^h \) above, where for every pair \((i,j)\), the indicators depend only on \((Z_{N,i}, Z_{N,j})\). As a result, even studying the mean and variance of \( t_{N,\text{adj}}^h \) raises significantly more technical challenges than those of \( t_N^h \). We circumvent this by leveraging the properties of the projection matrix and showing that the term on the left hand side of the indicators in the above display can be replaced by 0, in a suitable asymptotic sense. To be specific, define

\[
\tilde{A}_{N,\text{adj}}^h := \sum_{i,j=1}^{N} Z_{N,j} \mathbf{1}(0 \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} - hN^{-1/2}).
\]

Our main contribution lies in showing that

\[
N^{-3/2}|t_{N,\text{adj}}^h - \tilde{A}_{N,\text{adj}}^h| = o_p(1).
\]

The proof of this step is technical and we defer the reader to Appendix D for details of these computations. However once the above display is established, it suffices to obtain the asymptotic distribution of \( N^{-3/2}\tilde{A}_{N,\text{adj}}^h \). Note that the indicators in \( \tilde{A}_{N,\text{adj}}^h \) are deterministic and do not depend on \( Z_N \). This puts us in a similar situation to the analysis of \( t_N^h \), which we have already discussed above.

Once Theorem 3.2 is established, the proofs of Theorem 3.3 and Corollary 4.2 follow using ideas similar to the proofs of Theorem 2.2 and Theorem 4.1 respectively.
C Proofs of the main results

C.1 Proof of Proposition 2.1

Proof. We begin by recalling that \( t_N := \hat{q}_N^T Z_N \), where

\[
\hat{q}_{N,j} := \sum_{i=1}^{N} 1\{Y_{N,i} - \tau_0 Z_{N,i} \leq Y_{N,j} - \tau_0 Z_{N,j}\}, \quad 1 \leq j \leq N.
\]

It follows under \( \tau = \tau_0 \) that

\[
t_N = \frac{d}{d} q_N^T Z_N, \quad q_{N,j} = \sum_{i=1}^{N} 1\{b_{N,i} \leq b_{N,j}\}.
\]

Further, under \( \tau = \tau_0 \),

\[
q_N^T Z_N = \sum_{i=1}^{m} q_{N,i} \Pi_N(i) = \sum_{i=1}^{N} c_{N,i} \cdot q_{N,i} \Pi_N(i)
\]

where \( c_{N,i} = 1\{i \leq m\} \) and \( \Pi_N \) is a random permutation of \{1, 2, \ldots, N\}. Now, \( \bar{c}_N := N^{-1} \sum_{i=1}^{N} c_{N,i} = m/N \), and \( \max_{1 \leq i \leq N} (q_{N,i} - \bar{q}_N)^2 = O(N^2) \). Hence

\[
\lim_{N \to \infty} N \cdot \max_{1 \leq i \leq N} (c_{N,i} - \bar{c}_N)^2 / \sum_{i=1}^{N} (c_{N,i} - \bar{c}_N)^2 = \lim_{N \to \infty} \frac{\max \{\bar{c}_N^2, (1 - \bar{c}_N)^2\}}{\frac{m}{N} (1 - \bar{c}_N)^2 + (1 - \frac{m}{N}) \bar{c}_N^2} \cdot \frac{O(N^2)}{N(N^2 - 1) / 12 + o(N^3)}
\]

\[
= \max \{\lambda^2, (1 - \lambda)^2\} \cdot \frac{O(N^2)}{\lambda(1 - \lambda)} \cdot \frac{N(N^2 - 1) / 12 + o(N^3)}{N \to \infty}
\]

\[
= 0.
\]

In view of the above observation, we can apply Hoeffding’s combinatorial CLT Hoeffding (1951, Theorem 4) to conclude that under \( \tau = \tau_0 \),

\[
t_N - \mathbb{E}(t_N) \sim \sqrt{\text{Var}(t_N)} \quad \text{as} \quad N \to \infty.
\]

Now invoking Lemma D.6, we can complete the proof.

C.2 Proof of Proposition 2.2

Proof. It follows from the algebraic manipulations provided in Hodges and Lehmann (1963, Section 4) that our modified estimator \( \hat{\tau}_R \) in (2.11), with \( t(\cdot, \cdot) \) as in (2.4) of the main paper, is given by

\[
\hat{\tau}_R = \text{median}\{Y_i - Y_j : Z_i = 1, Z_j = 0, 1 \leq i, j \leq N\}.
\]

Now for any \( 1 \leq k \leq N \), fix any \( k \) indices \( 1 \leq i_1 < i_2 < \cdots < i_k \leq N \), and any treatment assignment \((Z_1, \ldots, Z_N) = (z_1, \ldots, z_N)\). Set \( n_1 := |\{1 \leq j \leq k : z_{i_j} = 1\}| \). Note that changing the responses \( y_{i_1}, \ldots, y_{i_k} \) arbitrarily can make the pairwise differences \( \{y_i - y_j : z_i = 1, z_j = 0\} \) blow up to \( +\infty \) except for \((m - n_1)(N - m - (k - n_1))\) many pairs \((i, j)\). As long as this count is
greater than or equal to half of the total number of pairwise differences, i.e., \( m(N - m)/2 \), the median cannot be made arbitrarily large. Note, this needs to be satisfied for every treatment assignment \( z_1, \ldots, z_N \in \{0, 1\} \) such that \( \sum_{i=1}^{N} z_i = m \). By symmetry, we can assume without loss of generality that \((i_1, i_2, \ldots, i_k) = (1, 2, \ldots, k)\). Thus, we can make \( \tau_R(y_1, \ldots, y_N; z_1, \ldots, z_N) \) arbitrarily large by altering \( k \) responses only if \( k \) satisfies the following:

\[ \forall z_1, \ldots, z_N \in \{0, 1\}, \ (m - n_1)(N - m - (k - n_1)) < \frac{m(N - m)}{2}, \ \text{where} \ n_1 = \sum_{j=1}^{k} z_j. \quad (C.2) \]

Moreover, if \( k \) satisfies this condition, then for any \( z_1, \ldots, z_N \in \{0, 1\} \) such that \( \sum_{i=1}^{N} z_i = m \), and any choice of indices \( 1 \leq i_1 < i_2 < \cdots < i_k \leq N \), we can let \( y_{i_j} \to \infty \) if \( z_{i_j} = 1 \) and \( y_{i_j} \to -\infty \) if \( z_{i_j} = 0 \), which results in \( \tau_R(y_1, \ldots, y_N; z_1, \ldots, z_N) \to \infty \). Therefore,

\[ BP_N(\tau_R) = \frac{1}{N} \min \{1 \leq k \leq N : (C.2) \ \text{holds}\}. \]

It is elementary to observe that \( f(x) = (m - x)(N - m - k + x) \) has a global maximum at \( x = m - (N - k)/2 \), and the maximum value is \( (N - k)^2/4 \). However, we want to maximize \( f(x) \) over \([0, k]\) only. Consider the following cases:

- **Case 1:** \( m > N/3 \). Note, \( 0 \leq m - (N - k)/2 \leq k \iff k \geq 2m - N \). Thus, for \( 2m - N \leq k \leq N \),

\[ (C.2) \ \text{holds} \iff \frac{(N - k)^2}{4} < \frac{m(N - m)}{2} \iff k > N\left(1 - \sqrt{\frac{2m}{N} \left(1 - \frac{m}{N}\right)}\right) \geq 2m - N. \]

In other words, for \( 2m - N \leq k \leq N \), the condition \( (C.2) \) is automatic. On the other hand, for \( 1 \leq k < 2m - N \), the maximum of the function \( f(x) = (m - x)(N - m - k + x) \) is attained at \( x = k \). So, for \( 1 \leq k < 2m - N \),

\[ (C.2) \ \text{holds} \iff (m - k)(N - m) < \frac{m(N - m)}{2} \iff k > \frac{m}{2}. \]

Hence we can write

\[ \{1 \leq k \leq N : (C.2) \ \text{holds}\} = \{1 \leq k < 2m - N : (C.2) \ \text{holds}\} \cup \{2m - N \leq k < N : (C.2) \ \text{holds}\} = \{1 \leq k < 2m - N : k > m/2\} \cup \{2m - N \leq k < N\}. \]

Since in this case \( m/2 < 2m - N \), it follows that for \( m > N/2 \),

\[ BP_N(\tau_R) = \frac{1}{N} \left\lceil \frac{m}{2} \right\rceil. \]

- **Case 2:** \( m < N/3 \). This case is similar to the previous case. In this case we get

\[ BP_N(\tau_R) = \frac{1}{N} \left\lceil \frac{N - m}{2} \right\rceil. \]

- **Case 3:** \( N/3 \leq m \leq 2N/3 \). First, consider \( N/2 \leq m \leq 2N/3 \). We note that

\[ \{1 \leq k \leq N : (C.2) \ \text{holds}\} \]
Now a little algebra shows that \( m/N < N - \sqrt{2m(N-m)} \). Hence in this case we get

\[
\BP_N(\tilde{\tau}^R) = \frac{1}{N} \left[ N \left( 1 - \sqrt{2 \frac{m}{N} \left( 1 - \frac{m}{N} \right)} \right) \right].
\]

Similarly one can deal with the other sub-case, namely, \( N/3 \leq m \leq N/2 \).

Since \( m/N \to \lambda \in [0,1] \), the desired conclusion follows.

C.3 Proof of Theorem 2.1

**Proof.** We start with the decomposition stated in Proposition B.1. Under \( \tau = \tau_N \), it holds that

\[
t_N \stackrel{d}{=} m + \sum_{j=1}^{N} Z_{N,j} \sum_{i=1, i \neq j}^{N} 1\{b_{N,i} \leq b_{N,j}\} - \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} (1 - Z_{N,i})Z_{N,j}I_{h,N}(b_{N,j} - b_{N,i}),
\]

call this \( W_N \)

\[
\text{call this } S_N
\]

where \( I_{h,N}(\cdot) \) is as in (2.15) of the main paper. We now observe that the randomization distribution of \( W_N \) is identical to the null distribution of \( t_N \). Thus, invoking Proposition 2.1 and Lemma D.5, we deduce that

\[
N^{-3/2} \left( W_N - \frac{m(N+1)}{2} \right) \overset{d}{\to} \mathcal{N} \left( 0, \frac{\lambda(1-\lambda)}{12} \right).
\]

On the other hand, Lemma D.7 tells us that

\[
N^{-3/2} S_N \overset{P}{\to} h\lambda(1-\lambda)I_b.
\]

Combining these using Slutsky’s theorem we conclude that under \( \tau = \tau_N \),

\[
N^{-3/2} \left( t_N - \frac{m(N+1)}{2} \right) \overset{d}{\to} N^{-3/2} \left( W_N - \frac{m(N+1)}{2} \right) - N^{-3/2} S_N
\]

\[
\overset{d}{\to} \mathcal{N} \left( -h\lambda(1-\lambda)I_b, \frac{\lambda(1-\lambda)}{12} \right), \tag{C.3}
\]

which completes the proof.

C.4 Proof of Theorem 2.2

**Proof.** Recall from Theorem 2.1 that under \( \tau = \tau_0 + hN^{-1/2} \) we have

\[
N^{-3/2} \left( t_N - \frac{m(N+1)}{2} \right) \overset{d}{\to} \mathcal{N} \left( -h\lambda(1-\lambda)I_b, \frac{\lambda(1-\lambda)}{12} \right),
\]

where \( I_b \) is defined in Assumption 1. Now we can invoke Lemma D.1 to complete the proof.
C.5 Proof of Theorem 2.3

Proof. First note that under any value of $\tau$,

$$B_N = \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} (1 - Z_{N,i})(1 - Z_{N,j})I_{1,N}(b_{N,j} - b_{N,i}).$$

Invoking Assumption 1, we get

$$\mathbb{E} N^{-3/2} B_N = \frac{(N-m)(N-m-1)}{N(N-1)} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} I_{1,N}(b_{N,j} - b_{N,i}) \to (1 - \lambda)^2 I_b.$$

Consequently, $\mathbb{E} \left( (1 - m/N)^{-2} N^{-3/2} B_N \right) \xrightarrow{P} I_b$. Therefore it suffices to show that $N^{-3} \text{Var}(B_N) \to 0$. For brevity, we denote by $I_N(i, j)$ the indicator $I_{1,N}(b_{N,j} - b_{N,i})$ in the rest of the proof. Note that

$$\text{Var}(B_N) = \sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} \sum_{k=1, i \neq k}^{N} \sum_{l=1, i \neq l}^{N} \text{Cov} \left( (1 - Z_{N,i})(1 - Z_{N,j}), (1 - Z_{N,k})(1 - Z_{N,l}) \right) I_N(i, j)I_N(k, l).$$

Since $2 \leq |\{i, j, k, l\}| \leq 4$, we consider the following cases.

(a) $|\{i, j, k, l\}| = 2$, i.e., $(i, j) = (k, l)$ or $(l, k)$. Since $\text{Var}((1 - Z_{N,i})(1 - Z_{N,j})) = p_N(1 - p_N)$ where $p_N = P(Z_{N,j} = 0, Z_{N,i} = 0) = \frac{(N-m)(N-m-1)}{N(N-1)} \sim (1 - \lambda)^2$, the contribution of these terms in $\text{Var}(B_N)$ is given by

$$\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} 2p_N(1 - p_N) I_N^2(i, j) \lesssim 2(1 - \lambda)^2 \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} I_N(i, j) \lesssim N^{3/2}.$$

(b) $|\{i, j, k, l\}| = 4$, i.e., all 4 indices are distinct. Note that in this case

$$\text{Cov} \left( (1 - Z_{N,i})(1 - Z_{N,j}), (1 - Z_{N,k})(1 - Z_{N,l}) \right) = p_N \left( \frac{(N-m-2)(N-m-3)}{(N-2)(N-3)} - \frac{(N-m)(N-m-1)}{N(N-1)} \right)$$

$$= p_N \frac{2m(2N^2 - 2mN - 6N + 3 + 3m)}{N(N-1)(N-2)(N-3)} \sim -4\lambda(1 - \lambda)^3 N^{-1}.$$

Hence the contribution $u_N$ of these terms in $\text{Var}(B_N)$ satisfies the following.

$$u_N \lesssim N^{-1} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \sum_{k=1, i \neq k}^{N} \sum_{l=1, i \neq l}^{N} I_N(i, j)I_N(k, l) \lesssim N^{-1+3/2+3/2} = N^2.$$

(c) $|\{i, j, k, l\}| = 3$. Here we have 4 sub-cases: $i = k, j \neq l; i \neq k, j = l; i = l, j \neq k; i \neq l, j = k$. In each of the subcases,

$$\text{Cov} \left( (1 - Z_{N,i})(1 - Z_{N,j}), (1 - Z_{N,k})(1 - Z_{N,l}) \right) = p_N(Z_{N,1} = Z_{N,2} = Z_{N,3} = 0) - p_N^2$$

$$\sim (1 - \lambda)^3 - (1 - \lambda)^4 = \lambda(1 - \lambda)^3.$$
Hence, if $v_n$ be the contribution of these terms in $\text{Var}(B_N)$, then

\[ v_N \lesssim \sum_{i,j,l \text{ distinct}} I_N(i,j)I_N(i,l) + \sum_{i,j,k \text{ distinct}} I_N(i,j)I_N(k,i) + \sum_{i,j,l \text{ distinct}} I_N(i,j)I_N(j,l) \]

\[ \leq 4N \sum_{i,j \neq i} I_N(i,j) \lesssim N^{1+3/2} = N^{5/2}. \]

Combining the above cases, we conclude that $N^{-3}\text{Var}(B_N) \lesssim N^{-3}(N^{3/2} + N^2 + N^{5/2}) = o(1)$, which completes the proof.

C.6 Proof of Theorem 2.4

Proof. When the \( \{b_{N,j} : 1 \leq j \leq N\} \) are assumed to be sampled from a distribution with density \( f_b(\cdot) \), Lemma D.4 tells us that

\[ I_{b_N} = \int_{\mathbb{R}} f_b^2(x)dx. \]

So in this case the asymptotic distribution of \( \sqrt{N}(\hat{\tau}_R - \tau_0) \) under \( \tau = \tau_0 \) is \( N(0, \sigma_{\text{HL}}^2) \), where

\[ \sigma_{\text{HL}}^2 := \left(12\lambda(1-\lambda) \left(\int_{\mathbb{R}} f_b^2(x)dx\right)^2\right)^{-1}. \]

On the other hand, we deduce from Freedman (2008a, Theorem 3) (and invoking the law of large numbers) that in this setup the asymptotic distribution of \( \sqrt{N}(\hat{\tau}_{\text{dm}} - \tau_0) \) under \( \tau = \tau_0 \) is \( N(0, \sigma_{\text{dm}}^2) \), where

\[ \sigma_{\text{dm}}^2 := (\lambda(1-\lambda))^{-1} \sigma_b^2, \]

where \( \sigma_b^2 = \text{Var}_b(X) \). Therefore, the asymptotic efficiency of \( \hat{\tau}_R \) relative to \( \hat{\tau}_{\text{dm}} \) (see Section 2.3 of the main paper for definition) is given by

\[ \text{eff}(\hat{\tau}_R, \hat{\tau}_{\text{dm}}) = \frac{\sigma_{\text{dm}}^2}{\sigma_{\text{HL}}^2} = 12\sigma_b^2 \left(\int_{\mathbb{R}} f_b^2(x)dx\right)^2. \]

The desired lower bound then follows from Hodges and Lehmann (1956, Theorem 1).

C.7 Proof of Theorem 3.1

Proof. Denote by \( p_{N,i} \) the \( i \)-th row of \( P_X \). Observe that

\[ \epsilon_{N,i} \leq \epsilon_{N,j} \iff Y_{N,i} - \tau_0 Z_{N,i} - p_{N,i}^\top (Y_N - \tau_0 Z_N) \leq Y_{N,j} - \tau_0 Z_{N,j} - p_{N,j}^\top (Y_N - \tau_0 Z_N). \]

So under \( \tau = \tau_0 \), we have \( t_{N,\text{adj}} = \frac{d}{N} \sum_{j=1}^{N} q_{N,j} Z_{N,j} \), where

\[ q_{N,j} := \sum_{i=1}^{N} 1\left\{ b_{N,i} - p_{N,i}^\top b_N \leq b_{N,j} - p_{N,j}^\top b_N \right\}, j = 1,2,\ldots,N. \]

Since the ranks \( q_{N,j} \)'s are deterministic, the asymptotic normality of \( \sum_{j=1}^{N} q_{N,j} Z_{N,j} \) can be derived in the same way as for the without regression adjustment case. A closer look at the proofs of Lemma D.5 and Proposition 2.1 in the without regression adjustment case reveals that the following results hold in this case as well, since Assumption 2 plays the role of Assumption 1.
(a) As \( N \to \infty \), \( \sum_{j=1}^{N} (q_{N,j} - \bar{q}_N)^2 = \frac{1}{12} N(N^2 - 1) + o(N^3) \).

(b) \( \lim_{N \to \infty} \max_{1 \leq j \leq N} (q_{N,j} - \bar{q}_N)^2 / \sum_{j=1}^{N} (q_{N,j} - \bar{q}_N)^2 = 0 \).

(c) Under \( \tau = \tau_0 \), \( \text{Var}(t_{N,\text{adj}}) \sim \frac{1}{12} \lambda(1 - \lambda)N^3 \) as \( N \to \infty \).

Equipped with (b) above, we apply Hoeffding’s combinatorial CLT (Hoeffding, 1951, Theorem 4) to say that under \( \tau = \tau_0 \),
\[
\frac{t_{N,\text{adj}} - \mathbb{E}(t_{N,\text{adj}})}{\sqrt{\text{Var}(t_{N,\text{adj}})}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } N \to \infty.
\]
This, in conjunction with (c) above, completes the proof.

C.8 Proof of Theorem 3.2

Proof. Recall the notation \( \tilde{b}_{N,i} = b_{N,i} - p_{N,i}^\top b_N \) \( (1 \leq i \leq N) \) from (3.6) of the main paper. Under \( \tau = \tau_N \), we have
\[
Y_N - \tau_0 Z_N \xrightarrow{d} b_N + (\tau_N - \tau_0)Z_N = b_N - hN^{-1/2}Z_N,
\]
which implies that
\[
t_{N,\text{adj}} \xrightarrow{d} \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} 1 \left\{ \tilde{b}_{N,i} - hN^{-1/2}Z_{N,i} + hN^{-1/2}p_{N,i}^\top Z_N \leq \tilde{b}_{N,j} - hN^{-1/2}Z_{N,j} + hN^{-1/2}p_{N,j}^\top Z_N \right\}
\]
\[
= \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} (1 - Z_{N,i}) 1 \left\{ \tilde{b}_{N,i} + hN^{-1/2}p_{N,i}^\top Z_N \leq \tilde{b}_{N,j} + hN^{-1/2}p_{N,j}^\top Z_N - hN^{-1/2} \right\} + C_N,
\]
where
\[
C_N := \sum_{j=1}^{N} \sum_{i=1}^{N} Z_{N,j} Z_{N,i} 1 \left\{ \tilde{b}_{N,i} + hN^{-1/2}p_{N,i}^\top Z_N \leq \tilde{b}_{N,j} + hN^{-1/2}p_{N,j}^\top Z_N \right\}.
\]
Thus under \( \tau = \tau_N \) we have \( t_{N,\text{adj}} \xrightarrow{d} I_N - II_N + C_N \), where
\[
I_N := \sum_{j=1}^{N} \sum_{i=1}^{N} Z_{N,j} \xi_{N,i,j}, \quad \text{and} \quad II_N := \sum_{j=1}^{N} \sum_{i=1}^{N} Z_{N,j} Z_{N,i} \xi_{N,i,j},
\]
and
\[
\xi_{N,i,j} := 1 \left\{ b_{N,i} - p_{N,i}^\top (b_N - hN^{-1/2}Z_N) \leq b_{N,j} - p_{N,j}^\top (b_N - hN^{-1/2}Z_N) - hN^{-1/2} \right\}
\]
\[
= 1 \left\{ hN^{-1/2}(p_{N,i} - p_{N,j})^\top Z_N \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} - hN^{-1/2} \right\}.
\]
The indicators \( \xi_{N,i,j} \) are quite complicated to handle, since it depends on the entire random vector \((Z_{N,1}, \ldots, Z_{N,N})\), for every pair \((i, j)\). To circumvent this technical hurdle, we replace \( \xi_{N,i,j} \) with \( \tilde{\xi}_{N,i,j} \), where
\[
\tilde{\xi}_{N,i,j} := 1 \left\{ 0 \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} - hN^{-1/2} \right\}.
\]
Define
\[
\tilde{I}_N := \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} \tilde{\xi}_{N,i,j}, \quad \text{and} \quad \tilde{II}_N := \sum_{j=1}^{N} \sum_{i=1}^{N} Z_{N,j} Z_{N,i} \tilde{\xi}_{N,i,j}.
\]

Thus we decompose $t_{N, \text{adj}}$ under $\tau = \tau_N$ as
\[
t_{N, \text{adj}} = \frac{d}{\tilde{A}_{N, \text{adj}}} \left[ \mathbf{I}_N - \mathbf{P}_N + C_N \right] + \left( \mathbf{I}_N - \mathbf{I}_N \right) - \left( \mathbf{P}_N - \mathbf{P}_N \right). \tag{C.7}
\]

We first focus on $\tilde{A}_{N, \text{adj}}$. Note that in (C.4) we are summing up the ranks of the $m$ numbers
\[
\{\tilde{b}_{N,j} + h N^{-1/2} \mathbf{p}_N^\top \mathbf{Z}_N : 1 \leq j \leq N, Z_{N,j} = 1\}
\]
within this set. Unfortunately, due to possibility of ties, we cannot directly equate it with $m(m + 1)/2$. However, invoking Assumption 2 we can show that $N^{-3/2}(C_N - m(m + 1)/2) = o_P(1)$ as $N \to \infty$. Towards that, observe that for any $\delta > 0$,
\[
\left| C_N - \frac{m(m + 1)}{2} \right| = \sum_{j=1}^N \sum_{i=1}^N 1 \left\{ \tilde{b}_{N,j} - \tilde{b}_{N,i} + h N^{-1/2} (\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N = 0 \right\}
\leq \sum_{j=1}^N \sum_{i=1}^N 1 \left\{ 0 \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} + h N^{-1/2} (\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N < \delta N^{-1/2} \right\}
\leq \sum_{(i,j): |(\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N| < \delta} 1 \left\{ 0 \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} + h N^{-1/2} (\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N < \delta N^{-1/2} \right\}
\leq \sum_{(i,j): |(\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N| < \delta} 1 \left\{ -h \delta N^{-1/2} \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} < (h + 1) \delta N^{-1/2} \right\}
\leq \sum_{(i,j): |(\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N| < \delta} 1 \left\{ |\tilde{b}_{N,j} - \tilde{b}_{N,i}| \leq (h + 1) \delta N^{-1/2} \right\} + \delta^{-2} \sum_{j=1}^N \sum_{i=1}^N \left( (\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N \right)^2. \tag{C.8}
\]

Now we invoke Lemma E.1 to say that
\[
N^{-3/2} \sum_{j=1}^N \sum_{i=1}^N \mathbb{E} \left[ \left( (\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N \right)^2 \right] = O(N^{-1/2}) = o(1),
\]
for every fixed $\delta > 0$. This, in conjunction with (C.8) and Markov inequality, tells us that for any fixed $\delta > 0$,
\[
N^{-3/2} \left\{ \left\{ (i,j) : |(\mathbf{p}_{N,j} - \mathbf{p}_{N,i})^\top \mathbf{Z}_N| \geq \delta \right\} \right\} = o_P(1).
\]

On the other hand, we can use Lemma D.9 to conclude that under Assumption 2,
\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^N \sum_{i=1}^N 1 \left\{ |\tilde{b}_{N,j} - \tilde{b}_{N,i}| \leq (h + 1) \delta N^{-1/2} \right\} = 2(h + 1) \delta.
\]
Appealing to (C.8) we can now conclude, by letting $N \to \infty$ first, and then $\delta \to 0$, that
\[
N^{-3/2}(C_N - m(m + 1)/2) = o_P(1), \text{ as } N \to \infty.
\]
This allows us to write

\[ \tilde{A}_{N,\text{adj}}^h = \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} 1 \left\{ 0 \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} - hN^{-1/2} \right\} + \frac{m(m+1)}{2} + o_P(1). \]

In view of the above display, the asymptotic normality of \( \tilde{A}_{N,\text{adj}}^h \) under \( \tau = \tau_N \) can be derived in the same manner as we proved the local asymptotic normality of \( t_N \) in the without regression adjustment case. To be precise, it follows by mimicking the proof of Theorem 2.1 (in the same manner as we proved Theorem 3.1 by mimicking the proof of Proposition 2.1) that under \( \tau = \tau_N \),

\[ N^{-3/2}(\tilde{A}_{N,\text{adj}}^h - \frac{m(N+1)}{2}) \overset{d}{\rightarrow} N \left( -h\lambda(1-\lambda)\mathcal{J}_b, \frac{\lambda(1-\lambda)}{12} \right), \quad (C.9) \]

where \( \mathcal{J}_b \) is defined in Assumption 2 of the main paper. The proof of the fact that \( D_N \) and \( Q_N \) defined in (C.7) are asymptotically negligible, is split into a couple of lemmas in Appendix D. Lemmas D.11 and D.12 give upper bounds on the second moments of \( D_N \) and \( Q_N \), respectively. Then Lemma D.13 shows that under Assumption 2,

\[ N^{-3/2}D_N = o_P(1) \text{ and } N^{-3/2}Q_N = o_P(1) \text{ as } N \rightarrow \infty. \]

This, in conjunction with (C.9) completes the proof of Theorem 3.2.

C.9 Proof of Theorem 3.3

Proof. Recall from Theorem 3.2 that under \( \tau = \tau_0 + hN^{-1/2} \), we have

\[ N^{-3/2} \left( t_{N,\text{adj}} - \frac{m(N+1)}{2} \right) \overset{d}{\rightarrow} N \left( -h\lambda(1-\lambda)\mathcal{J}_b, \frac{\lambda(1-\lambda)}{12} \right), \]

where \( \mathcal{J}_b \) is defined in Assumption 2. Now we invoke Lemma D.1 to complete the proof.

C.10 Proof of Theorem 3.4

Proof. Under the assumptions of this theorem, Lemma D.8 tells us that Assumption 2 holds in probability, with \( \mathcal{J}_b = (2\sqrt{\pi}\sigma)^{-1} \). Next, fix any \( h \in \mathbb{R} \) and define

\[ I_{h,N} := N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N}(b_j - b_i), \]

where \( I_{h,N} \) is defined in (2.15) of the main paper. Without loss of generality, we take \( h > 0 \) (the other case will be similar). For simplicity in notation, we shall omit the index \( N \) in this proof for \( b_N, X_N \) etc. From \( b = X\beta + \varepsilon \) we write \( b_j - b_i = \varepsilon_j - \varepsilon_i + v_j - v_i \) where \( v_i \) denotes the \( i \)-th coordinate of \( v = X\beta \). It then follows that

\[ \mathbb{E}I_{h,N} = N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} P(0 \leq \varepsilon_j - \varepsilon_i + v_j - v_i < hN^{-1/2}) \]

\[ = N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left[ \Phi \left( \frac{v_i - v_j + hN^{-1/2}}{\sqrt{2}\sigma} \right) - \Phi \left( \frac{v_i - v_j}{\sqrt{2}\sigma} \right) \right] \]

\[ = N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left[ \frac{h}{\sqrt{2}\sigma \sqrt{N}} \phi \left( \frac{v_i - v_j}{\sqrt{2}\sigma} \right) \right. + \left. \frac{h^2}{4N\sigma^2} \phi' \left( \xi_{i,j} \right) \right] \]

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\[
\frac{h}{2\sqrt{\pi}\sigma} N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} e^{-(v_j-v_i)^2/4\sigma^2} + (4\sigma^2)^{-1} h^2 N^{-5/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \phi'(\xi_{i,j}),
\]

where for each \((i, j)\), \(\xi_{i,j}\) is a number between \(v_i - v_j\) and \(v_i - v_j + N^{-1/2}\). Since \(\phi'\) is bounded, we can show (by proceeding in the same manner as in the proof of Lemma D.8) that the second sum in the above display is asymptotically negligible. Hence

\[
\lim_{N \to \infty} \left( \mathbb{E} \mathcal{I}_{h,N} - \frac{h}{2\sqrt{\pi}\sigma} N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} e^{-(v_j-v_i)^2/4\sigma^2} \right) = 0. \tag{C.10}
\]

Since \(0 \leq N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} e^{-(v_j-v_i)^2/4\sigma^2} \leq 1\), it follows that for any \(h > 0\),

\[
\limsup_{N \to \infty} \mathbb{E} \mathcal{I}_{h,N}/h \leq (2\sqrt{\pi}\sigma)^{-1} = J_b.
\]

Next we show that \(\text{Var}(\mathcal{I}_{h,N}) \to 0\). Towards that, we first write

\[
\mathbb{E} \mathcal{I}_{h,N}^2 = N^{-3} \sum_{i,j,k,l} P(0 \leq b_j - b_l < hN^{-1/2}, 0 \leq b_i - b_k < hN^{-1/2}).
\]

Now, as mentioned earlier, Assumption 2 holds in this setting (cf. Lemma D.8); hence appealing to arguments similar to those given in Appendix C.5, we argue that the contribution from the terms with repeated indices are negligible. So we are left with terms with distinct indices \(i, j, k, l\). For such indices, note that we have \(P(0 \leq b_j - b_l < hN^{-1/2}, 0 \leq b_i - b_k < hN^{-1/2}) = P(0 \leq b_j - b_l < hN^{-1/2})P(0 \leq b_i - b_k < hN^{-1/2})\), since the errors \(\varepsilon_i\)'s are independent and \(v_i\)'s non-stochastic. We can thus write

\[
\mathbb{E} \mathcal{I}_{h,N}^2 = \left( N^{-3/2} \sum_{i,j \text{ distinct}} P \left( 0 \leq b_j - b_l < hN^{-1/2} \right) \right)^2 + o(1) = (\mathbb{E} \mathcal{I}_{h,N})^2 + o(1),
\]

from which the conclusion follows. Moreover, it follows from (C.10) that if the following limit exists:

\[
\lim_{N \to \infty} N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} e^{-(v_j-v_i)^2/4\sigma^2} = \ell,
\]

then Assumption 1 holds, with \(\lim_{N \to \infty} \mathbb{E} \mathcal{I}_{h,N}/h = \ell(2\sqrt{\pi}\sigma)^{-1} = \ell J_b\). Finally, for any \(x \in \mathbb{R}\),

\[
e^{-x^2} \leq \frac{1}{1 + x^2} \leq \max \left\{ 1 - \frac{x^2}{2}, \frac{1}{2} \right\}. \tag{C.11}
\]

Hence

\[
N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} e^{-(v_j-v_i)^2/4\sigma^2} \leq \max \left\{ 1 - \frac{1}{2N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} (v_j-v_i)^2, \frac{1}{2} \right\}
\]

\[
= \max \left\{ 1 - \frac{1}{N} \sum_{j=1}^{N} (v_j-\overline{v})^2, \frac{1}{2} \right\}.
\]
Therefore, if \( \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} (v_j - \nu)^2 > 0 \), then

\[
\limsup_{N \to \infty} \frac{1}{2} \frac{1}{\sqrt{\pi \sigma}} \limsup_{N \to \infty} \max \left\{ 1 - \frac{1}{N} \sum_{j=1}^{N} (v_j - \nu)^2, \frac{1}{2} \right\} \\
\leq \frac{1}{2} \max \left\{ 1 - \liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} (v_j - \nu)^2, \frac{1}{2} \right\} \\
< \frac{1}{2} = J_b.
\]

Hence the proof is complete. \( \square \)

### C.11 Proof of Theorem 4.1

**Proof.** Recall the notations from Section 4.1. In this proof, we replace \( \tau^R \) by \( \hat{\tau}_N \), where \( \hat{\tau}_N \) is any \( \sqrt{N} \)-consistent estimator of \( \tau_0 \). For brevity, we shall skip the index \( N \) for \( b_{N,i} \)'s and \( Z_{N,i} \)'s in this proof. Define

\[
V_N := N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1}\{0 \leq b_j - b_i < N^{-\nu}\},
\]

and note that Assumption 3 yields that \( V_N \to \mathcal{J}_b \) as \( N \to \infty \). So, it suffices to show that \( \hat{V}_N - V_N \overset{P}{\to} 0 \) as \( N \to \infty \). We write

\[
\hat{V}_N - V_N = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1}\{0 \leq \hat{b}_j - \hat{b}_i\} - \mathbf{1}\{0 \leq b_j - b_i\} \right) \\
- N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1}\{N^{-\nu} \leq \hat{b}_j - \hat{b}_i\} - \mathbf{1}\{N^{-\nu} \leq b_j - b_i\} \right). \tag{C.12}
\]

Since \( \hat{b}_j - \hat{b}_i = b_j - b_i - (\hat{\tau}_N - \tau_0)(Z_j - Z_i) \), we can write

\[
\Delta_{1,N} := N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1}\{0 \leq \hat{b}_j - \hat{b}_i\} - \mathbf{1}\{0 \leq b_j - b_i\} \right) \\
= N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1}\{(\hat{\tau}_N - \tau_0)(Z_j - Z_i) \leq b_j - b_i\} - \mathbf{1}\{0 \leq b_j - b_i\} \right) \\
= N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1}\{(\hat{\tau}_N - \tau_0)(Z_j - Z_i) \leq b_j - b_i < 0\} - \mathbf{1}\{0 \leq b_j - b_i < (\hat{\tau}_N - \tau_0)(Z_j - Z_i)\} \right).
\]

Now pick \( \nu < \nu' < 1/2 \) and let \( E_N = \{ |\hat{\tau}_N - \tau_0| > N^{-\nu'} \} \). Since \( \hat{\tau}_N - \tau_0 = O_P(N^{-1/2}) \) and \( \nu' < 1/2 \), we get \( \hat{\tau}_N - \tau_0 = o_P(N^{-\nu'}) \), implying that \( P(E_N) \to 0 \) as \( N \to \infty \). Therefore for any \( \varepsilon > 0 \),

\[
P(|\Delta_{1,N}I_{E_N}| > \varepsilon) \leq P(1_{E_N} = 1) = P(E_N) \to 0.
\]

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On the other hand, we have \(|(\widehat{\tau}_N - \tau_0)(Z_j - Z_i)| \leq N^{-\nu'}\) on \(E_N^c\), hence

\[
N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{ (\widehat{\tau}_N - \tau_0)(Z_j - Z_i) \leq b_j - b_i < 0 \} \mathbb{1}_{E_N^c}
\]

\[
\leq N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{ -N^{-\nu'} \leq b_j - b_i < 0 \} \mathbb{1}_{E_N^c}
\]

\[
\leq N^{-(\nu' - \nu)} \left( N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{ -N^{-\nu'} \leq b_j - b_i < 0 \} \right).
\]

Assumption 3 tells us that the term in the above parentheses converges to \(I_0\). Since \(\nu' > \nu\), we conclude that

\[
N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{ (\widehat{\tau}_N - \tau_0)(Z_j - Z_i) \leq b_j - b_i < 0 \} \mathbb{1}_{E_N^c} \overset{P}{\longrightarrow} 0.
\]

Similarly,

\[
N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{ 0 \leq b_j - b_i < (\widehat{\tau}_N - \tau_0)(Z_j - Z_i) \} \mathbb{1}_{E_N^c} \overset{P}{\longrightarrow} 0.
\]

Thus, \(\Delta_{1,N} \mathbb{1}_{E_N^c} \overset{P}{\longrightarrow} 0\) and consequently \(\Delta_{1,N} \overset{P}{\longrightarrow} 0\), as \(N \to \infty\).

We next focus on the second summand in the RHS of (C.12). The key idea to deal with this sum is to replace the indicators with smooth functions, such as a Gaussian CDF. Define \(\sigma_N := N^{-\nu}(\log N)^{-1}\), \(r_N(i, j) := b_j - b_i - N^{-\nu}\), and \(\widehat{r}_N(i, j) := \widehat{b}_j - \widehat{b}_i - N^{-\nu}\). Then

\[
\widehat{V}_N - V_N - \Delta_{1,N} = N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\{ N^{-\nu} \leq \widehat{b}_j - \widehat{b}_i \} - 1\{ N^{-\nu} \leq b_j - b_i \} \right)
= \Delta_{3,N} - \Delta_{2,N} + \Delta_{4,N} + \Delta_{5,N},
\]

where

\[
\Delta_{2,N} := N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\{ r_N(i, j) > 0 \} - \Phi \left( \frac{r_N(i, j)}{\sigma_N} \right) \right),
\]

\[
\Delta_{3,N} := N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\{ \widehat{r}_N(i, j) > 0 \} - \Phi \left( \frac{\widehat{r}_N(i, j)}{\sigma_N} \right) \right),
\]

\[
\Delta_{4,N} := N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \Phi \left( \frac{\widehat{r}_N(i, j)}{\sigma_N} \right) - \Phi \left( \frac{r_N(i, j)}{\sigma_N} \right) \right),
\]

and

\[
\Delta_{5,N} := N^{-(2\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\{ \widehat{r}_N(i, j) = 0 \} - 1\{ r_N(i, j) = 0 \} \right).
\]

Showing \(\Delta_{2,N} \to 0\), \(\Delta_{k,N} \overset{P}{\longrightarrow} 0\) for \(k = 3, 4, 5\), will complete the proof.
(a) First we deal with $\Delta_{5,N}$. Note that
\[
N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{r_N(i, j) = 0\} \leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{N^{-\nu} \leq b_j - b_i < (1 + \delta)N^{-\nu}\} \to \delta \mathbb{I}_b.
\]
Now let $\delta \to 0$ to get
\[
\lim_{N \to \infty} N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{r_N(i, j) = 0\} = 0.
\]

For $\hat{\tau}_N(i, j)$ we proceed just as in the proof of $\Delta_{1,N} \to 0$. Define $E_N = \{|\hat{\tau}_N - \tau_0| > N^{-\nu'}\}$ for some $\nu < \nu' < 1/2$. Then
\[
N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{\hat{\tau}_N(i, j) = 0\} 1_{E_N}
\]
\[
\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{N^{-\nu} \leq \hat{b}_j - \hat{b}_i < (1 + \delta)N^{-\nu}\} 1_{E_N}
\]
\[
= N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{N^{-\nu} \leq b_j - b_i - (\hat{\tau}_N - \tau_0)(Z_j - Z_i) < (1 + \delta)N^{-\nu}\} 1_{E_N}
\]
\[
\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{-|\hat{\tau}_N - \tau_0| + N^{-\nu} \leq b_j - b_i < (1 + \delta)N^{-\nu} + |\hat{\tau}_N - \tau_0|\} 1_{E_N}
\]
\[
\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{-N^{-\nu'} + N^{-\nu} \leq b_j - b_i < (1 + \delta)N^{-\nu} + N^{-\nu'}\}.
\]

Since $\nu' > \nu$, it holds for all sufficiently large $N$ that $N^{-\nu'} \leq \delta N^{-\nu}$, and consequently
\[
\limsup_{N \to \infty} N^{-(2-\nu)} \mathbb{E} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{\hat{\tau}_N(i, j) = 0\} 1_{E_N}
\]
\[
\leq \limsup_{N \to \infty} N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{(1 - \delta)N^{-\nu} \leq b_j - b_i < (1 + 2\delta)N^{-\nu}\} = 3\delta \mathbb{I}_b.
\]

Letting $\delta \to 0$ here, and using $P(E_N) \to 0$, we conclude that
\[
N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{\hat{\tau}_N(i, j) = 0\} \overset{P}{\to} 0, \text{ as } N \to \infty.
\]

(b) Next we deal with $\Delta_{2,N}$, for which the bound provided by Lemma E.2 is crucial. Recall that
\[
\Delta_{2,N} = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left(1\{r_N(i, j) > 0\} - \Phi\left(\frac{r_N(i, j)}{\sigma_N}\right)\right),
\]
where $\sigma_N = N^{-\nu}(\log N)^{-1}$ and $r_N(i, j) = b_j - b_i - N^{-\nu}$. The key idea is to use the bound in Lemma E.2 only for those $i, j$ for which $r_N(i, j)$ is at least as large as $\delta N^{-\nu}$. Towards that, fix $\delta > 0$ and define
\[
S_{N,\delta} = \{(i, j) : |r_N(i, j)| \leq \frac{\delta}{N^{\nu}}, 1 \leq i, j \leq N\}.
\]
Note that
\[(i, j) \in S_{N, \delta} \implies \delta N^{-\nu} \geq |r_N(i, j)| \geq |b_j - b_i| - N^{-\nu} \]
\[\implies |b_j - b_i| \in [(1 - \delta)N^{-\nu}, (1 + \delta)N^{-\nu}] \, .\]

Therefore Lemma D.14 implies that
\[
\lim_{\delta \downarrow 0} \limsup_{N \to \infty} N^{-(2 - \nu)} |S_{N, \delta}| = 0. \tag{C.13}
\]

On the other hand, \((i, j) \not\in S_{N, \delta} \implies |r_N(i, j)| > \delta N^{-\nu}\), and then Lemma E.2 yields that
\[
\left| 1 \{r_N(i, j) > 0\} - \Phi \left( \frac{r_N(i, j)}{\sigma_N} \right) \right| \leq \sigma_N |r_N(i, j)|^{-1} \exp(-r_N^2(i, j)/2\sigma_N^2)
\leq \sigma_N \delta^{-1} N^\nu \exp(-\delta^2 N^{-2\nu}/2\sigma_N^2)
= (\log N)^{-1} \delta^{-1} \exp(-\delta^2 (\log N)^2/2)
\]

Combining the above bounds, the following chain of inequalities emerges.
\[
|\Delta_{2,N}| \leq N^{-(2 - \nu)} \sum_{j=1}^N \sum_{i=1}^N \left| 1 \{r_N(i, j) > 0\} - \Phi \left( \frac{r_N(i, j)}{\sigma_N} \right) \right|
\leq N^{-(2 - \nu)} \sum_{(i, j) \in S_{N, \delta}} 2 + N^{-(2 - \nu)} \sum_{(i, j) \not\in S_{N, \delta}} \left| 1 \{r_N(i, j) > 0\} - \Phi \left( \frac{r_N(i, j)}{\sigma_N} \right) \right|
\leq 2N^{-(2 - \nu)} |S_{N, \delta}| + N^{-(2 - \nu)} N^2 (\log N)^{-1} \delta^{-1} \exp(-\delta^2 (\log N)^2/2)
= 2N^{-(2 - \nu)} |S_{N, \delta}| + (\log N)^{-1} \delta^{-1} \exp(\nu \log N - \delta^2 (\log N)^2/2) .
\]

Therefore,
\[
\limsup_{N \to \infty} |\Delta_{2,N}| \leq 2 \limsup_{N \to \infty} N^{-(2 - \nu)} |S_{N, \delta}| .
\]

Letting \(\delta \downarrow 0\) and invoking \((C.13)\) we get the desired conclusion.

(c) Recall that
\[
\Delta_{3,N} = N^{-(2 - \nu)} \sum_{j=1}^N \sum_{i=1}^N \left( 1 \{\hat{r}_N(i, j) > 0\} - \Phi \left( \frac{\hat{r}_N(i, j)}{\sigma_N} \right) \right) .
\]

We can proceed just as in the previous proof. First we argue that for any fixed \(\delta > 0\),
\[
N^{-(2 - \nu)} \sum_{(i,j):|\hat{r}_N(i,j)| > \delta N^{-\nu}} \left| 1 \{\hat{r}_N(i, j) > 0\} - \Phi \left( \frac{\hat{r}_N(i, j)}{\sigma_N} \right) \right| \xrightarrow{P} 0, \text{ as } N \to \infty .
\]

Proof of the above part is exactly same as what we did in part (b). For the other part, it suffices to show that for any \(\varepsilon > 0\),
\[
\lim\limsup_{\delta \downarrow 0} \limsup_{N \to \infty} P \left( N^{-(2 - \nu)} \sum_{j=1}^N \sum_{i=1}^N \left| \hat{b}_j - \hat{b}_i \right| \in [(1 - \delta)N^{-\nu}, (1 + \delta)N^{-\nu}] \right) > \varepsilon \right) = 0 \quad \text{(C.14)}
\]
because the above implies that
\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} P \left( N^{-(2-\nu)} \left\{ (i,j) : |\tilde{\tau}_N(i,j) - \tau_0| \leq \delta N^{-\nu} \right\} \right) \geq \varepsilon \] 
which further implies that
\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} P (|\Delta_{3,N}| > \varepsilon) = 0. \]

To prove (C.14), we once again use the event \( \tilde{E}_{N,K} = \{ |\tilde{\tau}_N - \tau_0| > N^{-\nu'} \} \) where \( \nu < \nu' < 1/2 \).
\[
N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in \left[ (1-\delta)N^{-\nu}, (1+\delta)N^{-\nu} \right] \right\} \mathbb{I}_{\tilde{E}_N} \\
\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ (1-\delta)N^{-\nu} \leq \tilde{b}_j - \tilde{b}_i < (1+\delta)N^{-\nu} \right\} \mathbb{I}_{\tilde{E}_N} \\
= N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ (1-\delta)N^{-\nu} \leq b_j - b_i - (\tilde{\tau}_N - \tau_0)(Z_j - Z_i) < (1+\delta)N^{-\nu} \right\} \mathbb{I}_{\tilde{E}_N} \\
\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ -|\tilde{\tau}_N - \tau_0| + (1-\delta)N^{-\nu} \leq b_j - b_i < (1+\delta)N^{-\nu} + |\tilde{\tau}_N - \tau_0| \right\} \mathbb{I}_{\tilde{E}_N} \\
\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ -N^{-\nu'} + (1-\delta)N^{-\nu} \leq b_j - b_i < (1+\delta)N^{-\nu} + N^{-\nu'} \right\} .
\]

Since \( \nu' > \nu \), it holds for all sufficiently large \( N \) that \( N^{-\nu'} \leq \delta N^{-\nu} \), and consequently
\[
\lim_{N \to \infty} \sup N^{-(2-\nu)} \mathbb{E} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in \left[ (1-\delta)N^{-\nu}, (1+\delta)N^{-\nu} \right] \right\} \mathbb{I}_{\tilde{E}_N} \\
\leq \lim_{N \to \infty} \sup N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ -2N^{-\nu} \leq b_j - b_i < 2N^{-\nu} \right\} = 4\delta \mathbb{I}_{b}.
\]

Letting \( \delta \to 0 \) here, we conclude that
\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} N^{-(2-\nu)} \mathbb{E} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in \left[ (1-\delta)N^{-\nu}, (1+\delta)N^{-\nu} \right] \right\} \mathbb{I}_{\tilde{E}_N} = 0. \]  \hspace{1cm} (C.15)

Finally, for any \( \varepsilon > 0 \),
\[
P \left( N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in \left[ (1-\delta)N^{-\nu}, (1+\delta)N^{-\nu} \right] \right\} \geq \varepsilon \right) \\
\leq P \left( N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in \left[ (1-\delta)N^{-\nu}, (1+\delta)N^{-\nu} \right] \right\} \mathbb{I}_{\tilde{E}_N} > \varepsilon / 2 \right) + P(E_N) \\
\leq (\varepsilon / 2)^{-1} N^{-(2-\nu)} \mathbb{E} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in \left[ (1-\delta)N^{-\nu}, (1+\delta)N^{-\nu} \right] \right\} \mathbb{I}_{\tilde{E}_N} + P(E_N).
\]

Invoking (C.15) and \( \lim_{N \to \infty} P(E_N) = 0 \), we complete the proof of (C.14).
(d) Next let us focus on $\Delta_{4,N}$. We first write

$$|\Delta_{4,N}| \leq N^{-(2-\nu)} \sum_{(i,j) \in T_{N,\delta}} \frac{|\Phi(\hat{\tau}_N(i,j)) - \Phi(r_N(i,j))|}{\sigma_N}.$$

Now we break the last summation into three parts: (i) $(i, j)$ such that $|r_N(i,j)| \leq \delta N^{-\nu}$, i.e., $(i, j) \in S_{N,\delta}$; (ii) $(i, j)$ such that $|\hat{\tau}_N(i,j)| \leq \delta N^{-\nu}$; and (iii) remaining $(i, j)$'s, for which $|r_N(i,j)| \land |\hat{\tau}_N(i,j)| > \delta N^{-\nu}$. For indices $(i, j)$ in the cases (i) and (ii), we can use the crude bound $|\Phi(x) - \Phi(y)| \leq 2$, since equations (C.13) and (C.14) tell us that these contributions will be asymptotically negligible. For the case (iii), we use the following bound: for any $x, y \in \mathbb{R}$,

$$|\Phi(x) - \Phi(y)| \leq |x - y| \sup_{z \in [x \land y, x \lor y]} \phi(z) \leq |x - y| \exp \left( -\frac{x^2 \lor y^2}{2} \right).$$

Thus, if $T_{N,\delta}$ denote the (random) set of indices in case (iii), we have

$$N^{-(2-\nu)} \sum_{(i,j) \in T_{N,\delta}} \frac{|\Phi(\hat{\tau}_N(i,j)) - \Phi(r_N(i,j))|}{\sigma_N} \leq \sigma_N^{-1} N^{-(2-\nu)} \sum_{(i,j) \in T_{N,\delta}} \frac{|\hat{\tau}_N(i,j) - r_N(i,j)| \exp \left( -\frac{r_N(i,j)^2 \land \hat{\tau}_N(i,j)^2}{2 \sigma_N^2} \right)}{\sigma_N}$$

$$\leq \sigma_N^{-1} N^{-(2-\nu)} \sum_{(i,j) \in T_{N,\delta}} |\hat{\tau}_N(i,j) - r_N(i,j)| \exp \left( -\frac{\delta^2 N^{-2\nu}}{2 \sigma_N^2} \right)$$

$$\leq \sigma_N^{-1} N^{-(2-\nu)} \exp \left( -\frac{\delta^2 N^{-2\nu}}{2 \sigma_N^2} \right) \sum_{j=1}^{N} \sum_{i=1}^{N} \left| \hat{b}_j - \hat{b}_i \right|$$

$$= \sigma_N^{-1} N^{-(2-\nu)} \exp \left( -\frac{\delta^2}{2} (\log N)^2 \right) \sum_{j=1}^{N} \sum_{i=1}^{N} |(\hat{\tau}_N - \tau_0)(Z_j - Z_i)|$$

$$\leq \sigma_N^{-1} N^{\nu} |\hat{\tau}_N - \tau_0| \exp \left( -\frac{\delta^2}{2} (\log N)^2 \right)$$

$$= N^{2\nu - 1/2} \log N \cdot \sqrt{N} |\hat{\tau}_N - \tau_0| \exp \left( -\frac{\delta^2}{2} (\log N)^2 \right).$$

Since $\hat{\tau}_N - \tau_0 = O_P(N^{-1/2})$, we get the desired conclusion that $\Delta_{4,N} \overset{P}{\to} 0$ as $N \to \infty$.

Combining the above steps, the proof is now complete. □

C.12 Proof of Theorem 4.2

Proof. Recall the notations from Section 4.2. In this proof, we replace $\hat{\tau}_{adj}^{R_i}$ by $\hat{\tau}_N$, where $\hat{\tau}_N$ is any $\sqrt{N}$-consistent estimator of $\tau_0$. For brevity, we shall skip the index $N$ for $b_{N,i}$, $Z_{N,i}$, $X_N$, and $p_{N,i}$ in this proof. To start with, we define

$$W_N := N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ 0 \leq \hat{b}_j - \hat{b}_i < N^{-\nu} \right\},$$

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and note that $W_N \rightarrow \mathcal{J}_0$, by Assumption 4. We are to show that $\hat{N}_N - W_N \xrightarrow{P} 0$. Let $I$ and $J$ be two random indices chosen with replacement from \{1, 2, \ldots, N\}. We replace the event $E_N = \{|\hat{N}_N - \tau_0| > N^{-\nu'}\}$ in Appendix C.11 with the following event:

$$E_{N,K} = \{|\hat{N}_N - \tau_0| > N^{-\nu'}\} \cup \{\sqrt{N\|p_j - p_I\|} > K\},$$

where $\nu < \nu' < 1/2$, and $K > 0$ is a fixed positive real number. Observe that

$$P(\sqrt{N\|p_j - p_I\|} > K) \leq K^{-2}N \cdot \mathbb{E}_{J,I}\|p_j - p_I\|^2$$

$$= K^{-2}N^{-1} \sum_{j=1}^{N} \sum_{i=1}^{N} \|p_j - p_i\|^2$$

$$= 2(\text{rank}(X) - 1)K^{-2},$$

where the last step follows from the fact that $p_j$'s are columns of an idempotent matrix (see (E.4) for a proof). Thus,

$$P(E_{N,K}) \leq P(\sqrt{N\|p_j - p_I\|} > K) \lesssim 2(p - 1)K^{-2},$$

where $p$ is the number of covariates (i.e., number of columns of $X$). Also note that,

$$\hat{\tau}_j - \hat{\tau}_N Z_j - p_j^\top(Y - \hat{\tau}_N Z) = \hat{\tau}_j - \hat{\tau}_N \tau_0 \left( Z_j - p_j^\top Z \right).$$

Thus,

$$\hat{\tau}_j - \hat{\tau}_N \tau_0 \left( Z_j - p_j^\top Z \right).$$

Now write

$$\hat{W}_N - W_N = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\left\{ 0 \leq \hat{\tau}_j - \hat{\tau}_N \tau_0 \right\} - 1\left\{ 0 \leq \hat{\tau}_j - \tau_0 \right\} \right)$$

$$- N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\left\{ N^{-\nu} \leq \hat{\tau}_j - \tau_0 \right\} - 1\left\{ N^{-\nu} \leq \hat{\tau}_j - \tau_0 \right\} \right).$$

For the first part, we imitate the proof of $\Delta_{1,N} \xrightarrow{P} 0$, as follows.

$$\tilde{\Delta}_{1,N} = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\left\{ 0 \leq \hat{\tau}_j - \tau_0 \right\} \cdot u_N(i,j) \right)$$

$$= N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\left\{ \hat{\tau}_j - \tau_0 \right\} \cdot u_N(i,j) \right)$$

$$\leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1\left\{ \hat{\tau}_j - \tau_0 \right\} \cdot u_N(i,j) \right).$$

Now fix any $\varepsilon > 0$ and note that on $E_{N,K}$ we have $|\tau_0 - \tau_0| \leq N^{-\nu'}$, and

$$|u_N(i,j)| \leq |Z_j - Z_i| + \left| (p_j - p_i)^\top Z \right| \leq 1 + \|p_j - p_i\| \cdot \|Z\| \leq 1 + \sqrt{N} \|p_j - p_i\| \leq 1 + K.$$
Hence

\[ N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ (\tilde{\tau}_N - \tau_0)u_N(i,j) \leq \tilde{b}_j - \tilde{b}_i < 0 \right\} \mathbb{1}_{E_{N,K}^c} \]

\[ \leq N^{\nu'} \cdot N^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ -N^{-\nu'}(1 + K) \leq \tilde{b}_j - \tilde{b}_i < 0 \right\} \mathbb{1}_{E_{N,K}^c} \]

\[ \leq N^{-(\nu' - \nu)} \left( N^{-2(\nu')} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ -N^{-\nu'}(1 + K) \leq \tilde{b}_j - \tilde{b}_i < 0 \right\} \right). \]

Now Assumption 4 tells us that the term in the above parentheses converges to $(1 + K)J_b$. Since $\nu' > \nu$, we conclude that

\[ N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ (\tilde{\tau}_N - \tau_0)u_N(i,j) \leq \tilde{b}_j - \tilde{b}_i < 0 \right\} \mathbb{1}_{E_{N,K}^c} \xrightarrow{P} 0. \]

Similarly,

\[ N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ 0 \leq \tilde{b}_j - \tilde{b}_i < (\tilde{\tau}_N - \tau_0)u_N(i,j) \right\} \mathbb{1}_{E_{N,K}^c} \xrightarrow{P} 0. \]

Thus $\bar{\Delta}_{1,N}\mathbb{1}_{E_{N,K}^c} \xrightarrow{P} 0$. Hence

\[ \limsup_{N \to \infty} P \left( \left| \bar{\Delta}_{1,N} \right| > \varepsilon \right) \leq \limsup_{N \to \infty} P \left( \left| \bar{\Delta}_{1,N}\mathbb{1}_{E_{N,K}^c} \right| > \varepsilon/2 \right) \]

\[ \leq \limsup_{N \to \infty} P \left( \mathbb{1}_{E_{N,K}^c} = 1 \right) \]

\[ = \limsup_{N \to \infty} P(\bar{E}_{N,K}) \leq 2(p - 1)K^{-2}. \]

Letting $K \to \infty$ here, we conclude that $\bar{\Delta}_{1,N} \xrightarrow{P} 0$, as $N \to \infty$. Let us next focus on the second part of $\tilde{W}_N - W_N$, namely,

\[ N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \left\{ N^{-\nu} \leq \tilde{\tau}_N - \tilde{\tau}_N \right\}, -1 \left\{ N^{-\nu} \leq \tilde{\tau}_N - \tilde{\tau}_N \right\} \right) \]

Once again we approximate the indicators in the above display using the CDF of $\mathcal{N}(0, \sigma_N^2)$ where $\sigma_N := N^{-\nu}(\log N)^{-1}$. Define

\[ \tilde{\tau}_N(i,j) := \tilde{b}_j - \tilde{b}_i - N^{-\nu}, \quad \text{and} \quad \tilde{\tau}_N(i,j) := \tilde{b}_j - \tilde{b}_i - N^{-\nu}. \]

The rest of the proof is essentially same as the proof of Theorem 4.1. We write

\[ N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \left\{ N^{-\nu} \leq \tilde{\tau}_N(i,j) \right\} - 1 \left\{ N^{-\nu} \leq \tilde{\tau}_N(i,j) \right\} \right) = \bar{\Delta}_{3,N} - \bar{\Delta}_{2,N} + \bar{\Delta}_{4,N} + \bar{\Delta}_{5,N}, \]

where

\[ \bar{\Delta}_{2,N} = N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \{ \tilde{\tau}_N(i,j) > 0 \} - \Phi \left( \frac{\tilde{\tau}_N(i,j)}{\sigma_N} \right) \right), \]

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\[ \tilde{\Delta}_{3,N} = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \{ \tilde{r}_N(i,j) > 0 \} - \Phi \left( \frac{\tilde{r}_N(i,j)}{\sigma_N} \right) \right), \]
\[ \tilde{\Delta}_{4,N} = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \Phi \left( \frac{\tilde{r}_N(i,j)}{\sigma_N} \right) - \Phi \left( \frac{\tilde{r}_N(i,j)}{\sigma_N} \right) \right), \]

and
\[ \tilde{\Delta}_{5,N} = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \{ \tilde{r}_N(i,j) = 0 \} - 1 \{ \tilde{r}_N(i,j) = 0 \} \right). \]

The proof of \( \tilde{\Delta}_{2,N} \to 0 \) is exactly same as the proof of \( \Delta_{2,N} \to 0 \) given in Appendix C.11, hence omitted. Next, in order to prove \( \tilde{\Delta}_{3,N} \overset{P}{\to} 0 \), we first argue that for any fixed \( \delta > 0 \),
\[ N^{-(2-\nu)} \sum_{(i,j) \in \mathbb{R}^2} \left| 1 \{ \tilde{r}_N(i,j) > 0 \} - \Phi \left( \frac{\tilde{r}_N(i,j)}{\sigma_N} \right) \right| \overset{P}{\to} 0, \text{ as } N \to \infty. \]

Proof of the above display is omitted, for being completely analogous to the proof of part (b) in the proof of Theorem 4.1. For the other part of \( \tilde{\Delta}_{3,N} \), it suffices to show that for any \( \varepsilon > 0 \),
\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} P \left( N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \tilde{b}_j - \tilde{b}_i \right) \in [(1-\delta)N^{-\nu}, (1+\delta)N^{-\nu}] \right) > \varepsilon \right) = 0 \quad (C.16) \]

because the above implies that
\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} P \left( \left| \tilde{r}_N(i,j) \right| \leq \delta N^{-\nu} \right) > \varepsilon \right) = 0, \]

which further implies that
\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} P \left( \left| \tilde{\Delta}_{3,N} \right| > \varepsilon \right) = 0. \]

In order to prove (C.16), we once again use the event \( \bar{E}_{N,K} \) and the trick with \( I \) and \( J \). For brevity, we use the notation
\[ \kappa_{j,i} := (1 + \sqrt{N}) \| \mathbf{p}_j - \mathbf{p}_i \| \left| \tilde{r}_N - \tau_0 \right|. \]

Observe that,
\[ N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left\{ \tilde{b}_j - \tilde{b}_i \right\} \in [(1-\delta)N^{-\nu}, (1+\delta)N^{-\nu}] \right\} 1_{\bar{E}_{N,K}} \]
\[ \leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left\{ (1-\delta)N^{-\nu} \leq \tilde{b}_j - \tilde{b}_i < (1+\delta)N^{-\nu} \right\} 1_{\bar{E}_{N,K}} \]
\[ = N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left\{ (1-\delta)N^{-\nu} \leq \tilde{b}_j - \tilde{b}_i - (\tilde{r}_N - \tau_0)u_N(i,j) < (1+\delta)N^{-\nu} \right\} 1_{\bar{E}_{N,K}} \]
\[ \leq N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} \left\{ -\kappa_{j,i} + (1-\delta)N^{-\nu} \leq \tilde{b}_j - \tilde{b}_i < (1+\delta)N^{-\nu} + \kappa_{j,i} \right\} 1_{\bar{E}_{N,K}} \]
Proof. The statistic $\tilde{C}_{1.3}$ Proof of Proposition B.1

Since $\nu > \nu^\prime$, it holds for all sufficiently large $N$ that $(1 + K)N^{-\nu^\prime} \leq \delta N^{-\nu}$, and consequently

\[
\limsup_{N \to \infty} N^{-\nu} \mathbb{E} \left( \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in [(1 - \delta)N^{-\nu}, (1 + \delta)N^{-\nu}] \right\} \mathbb{I}_{E_{\tilde{N},K}} \right) = 0. \tag{C.17}
\]

Finally, for any $\varepsilon > 0$,

\[
P \left( N^{-\nu} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in [(1 - \delta)N^{-\nu}, (1 + \delta)N^{-\nu}] \right\} > \varepsilon \right)
\leq P \left( N^{-\nu} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in [(1 - \delta)N^{-\nu}, (1 + \delta)N^{-\nu}] \right\} \mathbb{I}_{E_{\tilde{N},K}} > \varepsilon / 2 \right) + P(\tilde{E}_{N,K})
\leq (\varepsilon / 2)^{-1} N^{-\nu} \mathbb{E} \left( \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ \left| \tilde{b}_j - \tilde{b}_i \right| \in [(1 - \delta)N^{-\nu}, (1 + \delta)N^{-\nu}] \right\} \mathbb{I}_{E_{\tilde{N},K}} \right) + P(\tilde{E}_{N,K}).
\]

Invoking (C.17) and using $\lim_{K \to \infty} \limsup_{N \to \infty} P(\tilde{E}_{N,K}) = 0$, we complete the proof of (C.16), which, in turn, proves that $\tilde{\Delta}_{3,N} \to 0$.

For $k = 4$ or 5, the proof of $\tilde{\Delta}_{k,N} \to 0$ follows by making similar changes in the proof of $\Delta_{k,N} \to 0$ in Appendix C.11, just as we did for $k = 1$ or 3.

C.13 Proof of Proposition B.1

Proof. The statistic $t_N$ is defined as

\[
t_N = \tilde{q}_N^\top Z_N, \quad \tilde{q}_N(j) = \sum_{i=1}^{N} 1 \{ Y_{N,i} - \tau_0 Z_{N,i} \leq Y_{N,j} - \tau_0 Z_{N,j} \}, \quad 1 \leq j \leq N.
\]
Note, the distribution of \( t_N \) under \( \tau = \tau_N \) depends on \( \tau_N \) only through the vector \( \tilde{q}_N \). Under \( \tau = \tau_N \) we can write

\[
(Y_{N,1} - \tau_0 Z_{N,1}, \ldots, Y_{N,N} - \tau_0 Z_{N,N}) \overset{d}{=} (b_{N,1} + (\tau_N - \tau_0)Z_{N,1}, \ldots, b_{N,N} + (\tau_N - \tau_0)Z_{N,N})
\]

and hence the distribution of \( \tilde{q}_N^\top Z_N \) under \( \tau = \tau_N \) is same as the randomization distribution of \( \tilde{q}_N^\top Z_N \) where

\[
q_{N,j} = \sum_{i=1}^{N} 1\{b_{N,i} + (\tau_N - \tau_0)Z_{N,i} \leq b_{N,j} + (\tau_N - \tau_0)Z_{N,j}\}, \ 1 \leq j \leq N.
\]

In order words, we have \( t_N \overset{d}{=} t_N^* \) under \( \tau = \tau_N \), where

\[
t_N^* = \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} 1\{b_{N,i} + (\tau_N - \tau_0)Z_{N,i} \leq b_{N,j} + (\tau_N - \tau_0)Z_{N,j}\}
\]

\[
= \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} \left( Z_{N,i} 1\{b_{N,i} \leq b_{N,j}\} + (1 - Z_{N,i}) 1\{hN^{-1/2} \leq b_{N,j} - b_{N,i}\}\right)
\]

\[
= \sum_{j=1}^{N} Z_{N,j} + \sum_{j=1}^{N} \sum_{i=1,i\neq j}^{N} Z_{N,j} Z_{N,i} 1\{b_{N,i} \leq b_{N,j}\}
\]

\[
+ \sum_{j=1}^{N} \sum_{i=1,i\neq j}^{N} Z_{N,j} (1 - Z_{N,i}) 1\{hN^{-1/2} \leq b_{N,j} - b_{N,i}\}.
\]

Now when \( h \geq 0 \),

\[
1\{hN^{-1/2} \leq b_{N,j} - b_{N,i}\} = 1\{0 \leq b_{N,j} - b_{N,i}\} - 1\{0 \leq b_{N,j} - b_{N,i} < hN^{-1/2}\}
\]

and thus

\[
t_N^* = m + \sum_{j=1}^{N} \sum_{i=1,i\neq j}^{N} Z_{N,j} 1\{b_{N,i} \leq b_{N,j}\} - \sum_{j=1}^{N} \sum_{i=1,i\neq j}^{N} (1 - Z_{N,i}) Z_{N,j} 1\{0 \leq b_{N,j} - b_{N,i} < hN^{-1/2}\}.
\]

Similarly, for \( h < 0 \),

\[
1\{hN^{-1/2} \leq b_{N,j} - b_{N,i}\} = 1\{0 \leq b_{N,j} - b_{N,i}\} + 1\{hN^{-1/2} \leq b_{N,j} - b_{N,i} < 0\}
\]

which gives us the desired expression.

\[\square\]

**D Some Technical Lemmas**

**Lemma D.1.** Let \( t(Z, Y - \tau_0 Z) \) be a test statistic for testing \( H_0 : \tau = \tau_0 \) and \( \hat{\tau}^R \) be the estimator of \( \tau \) based on \( t(\cdot, \cdot) \), as defined in (2.11) of the main paper. Assume that for all values of \( y \) and \( z \), \( t(z, y - \tau z) \) is a non-increasing function of \( \tau \). Let \( \{c_N\}_{N \geq 1} \) and \( \{d_N\}_{N \geq 1} \) be sequences of positive real numbers and \( \mu_N \) be the mean of \( t_N = t(Z, Y_N - \tau_0 Z_N) \) under \( \tau = \tau_0 \). Fix \( h \in \mathbb{R} \) and define \( \tau_N := \tau_0 - h/c_N \). Suppose that

\[
\lim_{N \to \infty} \mathbb{P}_{\tau_N} (d_N(t_N - \mu_N) \leq x) = G((x + hB)/A) \tag{D.1}
\]
holds for every \( x \in \mathbb{R} \), where \( G \) is a distribution function of a continuous random variable with mean 0 and variance 1, and \( A, B > 0 \) are constants. Then it holds that
\[
\lim_{N \to \infty} \mathbb{P}_{\tau_0} \left( c_N (\hat{\tau}_R - \tau_0) \leq h \right) = G(hB/A). \tag{D.2}
\]

**Proof.** Denote \( \tau'_N = \tau_0 + h/c_N \). We use Lemma D.2 to write the following:
\[
\mathbb{P}_{\tau_0} \left( t_N(\mathbf{Z}, \mathbf{Y} - \tau'_N \mathbf{Z}) < \mu_N \right) \leq \mathbb{P}_{\tau_0} \left( \hat{\tau}^* \leq \tau'_N \right)
\leq \mathbb{P}_{\tau_0} \left( \tau^R \leq \tau'_N \right)
\leq \mathbb{P}_{\tau_0} \left( \tau^{**} \leq \tau'_N \right) \leq \mathbb{P}_{\tau_0} \left( t_N(\mathbf{Z}, \mathbf{Y} - \tau'_N \mathbf{Z}) \leq \mu_N \right).
\tag{D.3}
\]

In view of Lemma D.3, the distribution of \( t_N(\mathbf{Z}, \mathbf{Y} - \tau'_N \mathbf{Z}) \) under \( \tau = \tau_0 \) is same as the distribution of \( t_N = t_N(\mathbf{Z}, \mathbf{Y} - \tau_0 \mathbf{Z}) \) under \( \tau = \tau_N \), which we denote by \( G_N \). Then equation (D.1) tells us that \( G_N(x/d_N + \mu_N) \to G((x + hB)/A) \) as \( n \to \infty \), for every \( x \in \mathbb{R} \). In particular, for \( x = 0 \) we can say that \( G_N(\mu_N) \to G(hB/A) \) and that \( G_N(\mu_N -) \to G(hB/A-) \) which is also equal to \( G(hB/A) \) since \( G \) is continuous. Therefore both the extreme sides of (D.3) converge to \( G(hB/A) \) and thus by sandwich principle, equation (D.2) holds.

\[ \square \]

**Lemma D.2.** Let \( \ell(\cdot, \cdot) \) be a test statistic such that for all values of \( y \) and \( z \), \( \ell(z, y - \tau z) \) is a non-increasing function of \( \tau \). Then for any \( h \in \mathbb{R} \),
\[
\mathbb{P}_\tau \left( \ell(\mathbf{Z}, \mathbf{Y} - a \mathbf{Z}) < \mu \right) \leq \mathbb{P}_\tau \left( \hat{\tau}^* \leq a \right) \leq \mathbb{P}_\tau \left( \tau^R \leq a \right)
\leq \mathbb{P}_\tau \left( \tau^{**} \leq a \right) \leq \mathbb{P}_\tau \left( \ell(\mathbf{Z}, \mathbf{Y} - a \mathbf{Z}) \leq \mu \right).
\]

**Proof.** The proof is straightforward from the definitions of \( \hat{\tau}^* \) and \( \tau^{**} \). First observe that \( \tau^* \leq \tau^R \leq \tau^{**} \) holds almost surely. Now, if \( a \) is such that \( \ell(\mathbf{Z}, \mathbf{Y} - a \mathbf{Z}) < \mu \), then
\[
h \geq \sup \{ \tau : \ell(\mathbf{Z}, \mathbf{Y} - \tau \mathbf{Z}) < \mu \} = \hat{\tau}^{**} \implies h \geq \hat{\tau}^*,
\]
so the left-most inequality follows. Next,
\[
\hat{\tau}^* \leq a \implies \tau^R \leq a \implies \tau^{**} \leq a.
\]

Finally, suppose that \( a \) is such that \( \ell(\mathbf{Z}, \mathbf{Y} - a \mathbf{Z}) > \mu \). We split it into two cases:

(a) when \( \hat{\tau}^* < \tau^{**} \), we have
\[
t(\mathbf{Z}, \mathbf{Y} - a \mathbf{Z}) > \mu \implies a \leq \sup \{ \tau : t(\mathbf{Z}, \mathbf{Y} - \tau \mathbf{Z}) > \mu \} = \hat{\tau}^* < \hat{\tau}^{**}.
\]

(b) When \( \hat{\tau}^* = \tau^{**} \), we have
\[
t(\mathbf{Z}, \mathbf{Y} - a \mathbf{Z}) > \mu \implies h < \sup \{ \tau : t(\mathbf{Z}, \mathbf{Y} - \tau \mathbf{Z}) > \mu \} = \hat{\tau}^* = \hat{\tau}^{**}.
\]

This finishes the proof. \[ \square \]

**Lemma D.3.** The distribution of \( t(\mathbf{Z}, \mathbf{Y} - (\tau + \delta) \mathbf{Z}) \) under \( \tau = \tau_0 \) is identical to the distribution of \( t(\mathbf{Z}, \mathbf{Y} - \tau \mathbf{Z}) \) under \( \tau = \tau_0 - \delta \).
Proof. For any $x \in \mathbb{R}$,
\[
\mathbb{P}_{\tau_0}(t(Z, Y - (\tau + \delta)Z) \leq x) = \mathbb{P}_{\tau_0}(t(Z, \tau_0Z + b - (\tau + \delta)Z) \leq x)
= \mathbb{P}(t(Z, (\tau_0 - \delta)Z + b - \tau Z) \leq x)
= \mathbb{P}_{\tau_0 - \delta}(t(Z, Y - \tau Z) \leq x),
\]
where the second step is due to the fact that the randomization distribution of $Z$ is free of $\tau$. \hfill \Box

**Lemma D.4.** If $\{b_{N,j} : 1 \leq j \leq N\}$ be i.i.d. from a distribution with density $f_b(\cdot)$ satisfying $\int_{\mathbb{R}} f_b^2(x) \, dx < \infty$, and $I_{h,N}$ be as in (2.15) of the main paper, then

\[
N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N}(b_{N,j} - b_{N,i}) \xrightarrow{a.s.} h \int_{\mathbb{R}} f_b^2(x) \, dx.
\]

**Proof.** Let us denote $S_N = S_N(b_{N,1}, \ldots, b_{N,N}) = \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N}(b_{N,j} - b_{N,i})$. Observe that for any fixed $i \neq j$, and $h > 0$,

\[
\mathbb{E}(I_{h,N}(b_{N,j} - b_{N,i})) = P(0 \leq b_{N,2} - b_{N,1} < hN^{-1/2}) = g(hN^{-1/2}) - g(0)
\]

where

\[
g(x) = P(b_{N,2} - b_{N,1} \leq x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_b(u + t)f_b(u) \, du \, dt, \ x \in \mathbb{R}.
\]

Using the DCT for integrals, we argue that $g'(x) = \int_{\mathbb{R}} f_b(u + x)f_b(u) \, du$. Hence

\[
\lim_{N \to \infty} N^{-3/2}\mathbb{E}(S_N) = \lim_{N \to \infty} N^{-3/2}N^2(g(hN^{-1/2}) - g(0))
= \lim_{N \to \infty} h \cdot \frac{g(hN^{-1/2}) - g(0)}{hN^{-1/2}}
= hg'(0)
= h \int_{\mathbb{R}} f_b(u)^2 \, du.
\]

Now we bound $\mathbb{E}(S_N - \mathbb{E}S_N)^2$ using the Efron-Stein inequality (Efron and Stein, 1981). For each $1 \leq k \leq N$, let $b_{N,k}'$ be an i.i.d. copy of $b_{N,k}$, independent of everything else, and define

\[
S_N^{(k)} = S_N(b_{N,1}, \ldots, b_{N,k-1}, b_{N,k}', b_{N,k+1}, \ldots, b_{N,N}), \ 1 \leq k \leq N.
\]

Note that

\[
S_N - S_N^{(k)} = \sum_{j=1}^{N} \left( I_{h,N}(b_{N,j} - b_{N,k}) - I_{h,N}(b_{N,j} - b_{N,k}') \right)
+ \sum_{j=1}^{N} \left( I_{h,N}(b_{N,k} - b_{N,j}) - I_{h,N}(b_{N,k}' - b_{N,j}) \right).
\]

An application of the Cauchy-Schwarz inequality yields the following.

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{N} I_{h,N}(b_{N,j} - b_{N,k}) - I_{h,N}(b_{N,j} - b_{N,k}') \right)^2 \mid b_{N,1}, \ldots, b_{N,N} \right]
\]

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would complete the proof, with the aid of Borel-Cantelli lemma. We start with

\[
\sum_{j=1}^{N} \left( I_{h,N}(b_{N,j} - b_{N,k}) - I_{h,N}(b_{N,j} - b'_{N,k}) \right)^2 \mid b_{N,1}, \ldots, b_{N,N}
\]

\[
= 2N \sum_{j=1}^{N} \left( I_{h,N}(b_{N,j} - b_{N,k}) + \mathbb{E} \left[ I_{h,N}(b_{N,j} - b_{N,k}) \mid b_{N,j} \right] \right).
\]

Applying the same argument to the second part yields

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{N} I_{h,N}(b_{N,k} - b_{N,j}) - I_{h,N}(b'_{N,k} - b_{N,j}) \right)^2 \mid b_{N,1}, \ldots, b_{N,N} \right]
\]

\[
\leq 2N \sum_{j=1}^{N} \left( I_{h,N}(b_{N,k} - b_{N,j}) + \mathbb{E} \left[ I_{h,N}(b_{N,k} - b_{N,j}) \mid b_{N,j} \right] \right).
\]

Using the above bounds and a generalized Efron-Stein inequality (see Proposition E.1),

\[
\mathbb{E}(N^{-3/2}(S_N - \mathbb{E}S_N))^6
\]

\[
\lesssim N^{-9} \mathbb{E} \left[ \left( S_N - S^{(k)}_N \right)^2 \mid b_{N,1}, \ldots, b_{N,N} \right]^3
\]

\[
\lesssim N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} (I_{h,N}(b_{N,j} - b_{N,k}) + \mathbb{E} \left[ I_{h,N}(b_{N,j} - b_{N,k}) \mid b_{N,j} \right]) \right]^3
\]

\[
+ N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} (I_{h,N}(b_{N,k} - b_{N,j}) + \mathbb{E} \left[ I_{h,N}(b_{N,k} - b_{N,j}) \mid b_{N,j} \right]) \right]^3
\]

\[
\lesssim N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} I_{h,N}(b_{N,j} - b_{N,k}) \right]^3 + N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left[ I_{h,N}(b_{N,j} - b_{N,k}) \mid b_{N,j} \right] \right]^3
\]

\[
+ N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left[ I_{h,N}(b_{N,k} - b_{N,j}) \mid b_{N,j} \right] \right]^3.
\]

We now show that the contribution of each of last three sums is at most of the order of \( N^{-3/2} \), which would complete the proof, with the aid of Borel-Cantelli lemma. We start with

\[
N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} I_{h,N}(b_{N,j} - b_{N,k}) \right]^3
\]

\[
= N^{-6} \sum_{1 \leq i_1, j_1, i_2, j_2, i_3, j_3 \leq N} \mathbb{E} \left[ I_{h,N}(b_{N,j_1} - b_{N,i_1}) I_{h,N}(b_{N,j_2} - b_{N,i_2}) I_{h,N}(b_{N,j_3} - b_{N,i_3}) \right]
\]
When $|\{i_1, j_1, i_2, j_2, i_3, j_3\}| \leq 4$, the number of ways to choose such a set of indices would be $O(N^4)$, and the summands being bounded above by 1, the contributions from these terms is $O(N^{-2})$. If $|\{i_1, j_1, i_2, j_2, i_3, j_3\}| = 6$, i.e., the indices are all distinct, we use the independence of the $b_{N,j_i}$’s to split the joint probability as the product of marginal probabilities, and thus the contribution from such terms becomes

$$N^{-6} \sum_{i_1,j_1,i_2,j_2,i_3,j_3 \text{ distinct}} \mathbb{E} I_{h,N}(b_{N,j_1} - b_{N,i_1}) \mathbb{E} I_{h,N}(b_{N,j_2} - b_{N,i_2}) \mathbb{E} I_{h,N}(b_{N,j_3} - b_{N,i_3})$$

$$\leq N^{-6} \left( \sum_{1 \leq i \leq N} \mathbb{E} I_{h,N}(b_{N,j} - b_{N,i}) \right)^3 = N^{-3/2} \left( \mathbb{E} N^{-3/2} S_N \right)^3 = O(N^{-3/2}).$$

Finally, consider the case where $|\{i_1, j_1, i_2, j_2, i_3, j_3\}| = 5$, i.e., exactly one index is repeated. Then at least two of the sets $\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}$ will make four distinct indices altogether. If the set left-out is $\{i_3, j_3\}$, we can just upper bound $I_{h,N}(b_{N,j_3} - b_{N,i_3})$ by 1. Also, the number of ways of choosing the indices $\{i_3, j_3\}$ will be $O(N)$, since one of them is repeated within $\{i_1, j_1, i_2, j_2\}$. Using this idea,

$$N^{-6} \sum_{|\{i_1, j_1, i_2, j_2, i_3, j_3\}|=5} \mathbb{E} I_{h,N}(b_{N,j_1} - b_{N,i_1}) I_{h,N}(b_{N,j_2} - b_{N,i_2}) I_{h,N}(b_{N,j_3} - b_{N,i_3})$$

$$\lesssim N^{-5} \sum_{i_1,j_1,i_2,j_2 \text{ distinct}} \mathbb{E} I_{h,N}(b_{N,j_1} - b_{N,i_1}) \mathbb{E} I_{h,N}(b_{N,j_2} - b_{N,i_2})$$

$$\leq N^{-5} \left( \sum_{1 \leq i \leq N} \mathbb{E} I_{h,N}(b_{N,j} - b_{N,i}) \right)^2 = N^{-2} \left( \mathbb{E} N^{-3/2} S_N \right)^2 = O(N^{-2}).$$

Combining the above cases, we conclude that $N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} I_{h,N}(b_{N,j} - b_{N,k}) \right]^3 = O(N^{-3/2})$. We next look at

$$N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbb{E} \left[ I_{h,N}(b_{N,j} - b_{N,k}) \mid b_{N,j} \right] \right]^3$$

$$= N^{-6} \sum_{1 \leq i_1,j_1,i_2,j_2,i_3,j_3 \leq N} \mathbb{E} \left[ I^*(b_{N,j_1}, b_{N,i_1}) I^*(b_{N,j_2}, b_{N,i_2}) I^*(b_{N,j_3}, b_{N,i_3}) \right]$$

where $I^*(b_{N,j}, b_{N,i}) := \mathbb{E} \left[ I_{h,N}(b_{N,j} - b_{N,i}) \mid b_{N,j} \right]$, which is really a function of $b_{N,j}$. Thus, when all the 6 indices are distinct, the joint probability is split into marginals, and once again we can see that the contribution of these terms is $O(N^{-3/2})$. When $|\{i_1, j_1, i_2, j_2, i_3, j_3\}| \leq 4$, the same argument applies as we gave earlier, and tells us that contribution from these terms would be $O(N^{-2})$. Finally, when $|\{i_1, j_1, i_2, j_2, i_3, j_3\}| = 5$ we play the same trick applied in the previous case to conclude that contribution from these terms is

$$N^{-6} \sum_{|\{i_1, j_1, i_2, j_2, i_3, j_3\}|=5} \mathbb{E} \left[ I^*(b_{N,j_1}, b_{N,i_1}) I^*(b_{N,j_2}, b_{N,i_2}) I^*(b_{N,j_3}, b_{N,i_3}) \right]$$

$$\lesssim N^{-5} \sum_{i_1,j_1,i_2,j_2 \text{ distinct}} \mathbb{E} \left[ I^*(b_{N,j_1}, b_{N,i_1}) I^*(b_{N,j_2}, b_{N,i_2}) \right]$$
\[ = N^{-5} \sum_{i_1,j_1,i_2,j_2 \text{ distinct}} \mathbb{E} [I_{h,N}(b_{N,j_1} - b_{N,i_1})] \mathbb{E} [I_{h,N}(b_{N,j_2} - b_{N,i_2})]. \]

\[ \leq N^{-5} \left( \sum_{1 \leq i,j \leq N} \mathbb{E} I_{h,N}(b_i - b_j) \right)^2 = N^{-2} \left( \mathbb{E} N^{-3/2} S_N \right)^2 = O(N^{-2}). \]

Thus, \( N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbb{E} [I_{h,N}(b_{N,j} - b_{N,k}) | b_{N,j}] \right]^3 = O(N^{-3/2}). \) In a similar manner one can show that \( N^{-6} \mathbb{E} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} \mathbb{E} [I_{h,N}(b_{N,k} - b_{N,j}) | b_{N,j}] \right]^3 = O(N^{-3/2}). \) Hence the proof.

\[ \square \]

**Lemma D.5.** Suppose that the potential control outcomes \( \{b_{N,j} : 1 \leq j \leq N, N \geq 1\} \) satisfy Assumption 1. Define

\[ q_{N,j} = \sum_{i=1}^{N} 1\{b_{N,i} \leq b_{N,j}\}, \quad 1 \leq j \leq N. \]

Then, as \( N \to \infty, \)

\[ \bar{q}_N := \frac{1}{N} \sum_{j=1}^{N} q_{N,j} = \frac{N + 1}{2} + o(N^{1/2}), \quad \text{and} \quad \frac{1}{N} \sum_{j=1}^{N} (q_{N,j} - \bar{q}_N)^2 = \frac{N^2 - 1}{12} + o(N^2). \]

**Proof.** We first break the ties in an arbitrary manner, and obtain \( b_{N(1)} \leq b_{N(2)} \leq \cdots \leq b_{N(N)}. \) Define \( r_{N,j} = \sum_{i=1}^{N} 1\{b_{N,i} \leq b_{N(j)}\}. \) Note that for each \( 1 \leq j \leq N, \)

\[ |r_{N,j} - j| \leq \sum_{i=1}^{N} 1\{b_{N,i} = b_{N(j)}\}. \]

The last equation in conjunction with Assumption 1 yields the following bounds.

\[ \sum_{j=1}^{N} |r_{N,j} - j| \leq \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{b_{N,i} = b_{N,j}\} \]

\[ \leq \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{0 \leq b_{N,j} - b_{N,i} < N^{-1/2}\} \approx N^{3/2}, \]

and

\[ \sum_{j=1}^{N} (r_{N,j} - j)^2 \leq \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{b_{N,i} = b_{N,j}\} \]

\[ \leq N \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{0 \leq b_{N,j} - b_{N,i} < N^{-1/2}\} \approx N^{5/2}. \]

Now,

\[ \sum_{i=1}^{N} r_{N,i}^2 - \sum_{i=1}^{N} i^2 = \sum_{i=1}^{N} (r_{N,i} - i)^2 + \sum_{i=1}^{N} 2i(r_{N,i} - i) \]

\[ \leq \sum_{i=1}^{N} (r_{N,i} - i)^2 + 2 \left( \sum_{i=1}^{N} i^2 \sum_{i=1}^{N} (r_{N,i} - i)^2 \right)^{1/2} \]

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\[ \leq N^{5/2} + N^{11/4} = o(N^3). \]

On the other hand,
\[
N \left| \frac{r_N^2}{N} - ((N + 1)/2)^2 \right| = \frac{1}{N} \left| \sum_{i=1}^{N} (r_{N,i} - i) \sum_{i=1}^{N} (r_{N,i} + i) \right|
\leq \frac{1}{N} \sum_{i=1}^{N} |r_{N,i} - i| \sum_{i=1}^{N} (|r_{N,i} - i| + 2i)
\lesssim N^{-1} N^{3/2} \left( N^{3/2} + N^2 \right) = o(N^3).
\]

Combining the above with the fact that \( \{r_{N,1}, \ldots, r_{N,N}\} \) is same as \( \{q_{N,1}, \ldots, q_{N,N}\} \),
\[
\sum_{i=1}^{N} (q_{N,i} - \bar{q}_N)^2 = \sum_{i=1}^{N} r_{N,i}^2 - N r_N^2 = \sum_{i=1}^{N} \left( i - \frac{N + 1}{2} \right)^2 + o(N^3) = \frac{N(N^2 - 1)}{12} + o(N^3).
\]

This finishes the proof of the second assertion. To prove the first assertion, we fix any \( \delta > 0 \) and note that
\[
\left| \sum_{j=1}^{N} q_{N,j} - \frac{N(N + 1)}{2} \right| = \left| \sum_{j=1}^{N} (r_{N,j} - j) \right|
\leq \sum_{j=1}^{N} |r_{N,j} - j|
\leq \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{b_{N,i} = b_{N,j}\}
\leq \sum_{j=1}^{N} \sum_{i=1}^{N} 1\{0 \leq b_{N,j} - b_{N,i} < \delta N^{-1/2}\}.
\]

Invoking Assumption 1 we can say that
\[
\limsup_{N \to \infty} \left| \sum_{j=1}^{N} q_{N,j} - \frac{N(N + 1)}{2} \right| \leq \delta I_b.
\]

Now letting \( \delta \to 0 \) completes the argument.

\[\square\]

**Lemma D.6.** Suppose that ranks \( \{q_{N,j}\} \) satisfy
\[
\sum_{j=1}^{N} (q_{N,j} - \bar{q}_N)^2 = \frac{N(N^2 - 1)}{12} + o(N^3).
\]

Let \( t_N \) be the Wilcoxon rank-sum statistic when the treatments are assigned by an \( m \)-out-of-\( N \) SR-SWOR sample where \( m/N \to \lambda \in (0,1) \) as \( N \to \infty \). Then, under \( \tau = \tau_0 \),
\[
\text{Var}(t_N) \sim \frac{\lambda(1 - \lambda)}{12} N^3 \text{ as } N \to \infty.
\]

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Proof. Recall from (C.1) that under \( \tau = \tau_0 \), we can write \( t_N \overset{d}{=} \sum_{j=1}^{N} q_{N,j} Z_{N,j} \), where \( q_{N,j} = \sum_{i=1}^{N} 1 \{ b_{N,i} \leq b_{N,j} \} \). Note that each \( Z_{N,j} \) is a Bernoulli\((m/N)\) random variable, and for \( i \neq j \) we have

\[
\text{Cov}(Z_{N,i}, Z_{N,j}) = P(Z_{N,i} = 1, Z_{N,j} = 1) - P(Z_{N,i} = 1)P(Z_{N,j} = 1) = -\frac{1}{N - 1} \frac{m}{N} \left( 1 - \frac{m}{N} \right).
\]

Hence we deduce that, under \( \tau = \tau_0 \),

\[
\text{Var}(t_N) = \sum_{j=1}^{N} \text{Var}(q_{N,j} Z_{N,j}) + 2 \sum_{1 \leq i < j \leq N} \text{Cov}(q_{N,i} Z_{N,i}, q_{N,j} Z_{N,j})
\]

\[
= \sum_{j=1}^{N} q_{N,j} \frac{m}{N} \left( 1 - \frac{m}{N} \right) - 2 \sum_{1 \leq i < j \leq N} q_{N,i} q_{N,j} \frac{m}{N} \left( 1 - \frac{m}{N} \right) \frac{1}{N - 1}
\]

\[
= \frac{m}{N} \left( 1 - \frac{m}{N} \right) \frac{N}{N - 1} \sum_{j=1}^{N} \left( q_{N,j} - \overline{q}_N \right)^2.
\]

In light of the given condition on the ranks, the above implies that

\[
\text{Var}(t_N) = \frac{m}{N} \left( 1 - \frac{m}{N} \right) \frac{N}{N - 1} \left( \frac{N(N^2 - 1)}{12} + o(N^3) \right) \sim \lambda(1 - \lambda) \frac{N^3}{12},
\]

which completes the proof. \( \Box \)

**Lemma D.7.** Let \( Z_N \) be the vector of treatment indicators when the treatments are assigned by \( m \)-out-of-\( N \) SRSWOR where \( m(N)/N \to \lambda \in (0, 1) \) as \( N \to \infty \). Define

\[
S_N = \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} (1 - Z_{N,i}) Z_{N,j} I_{h,N}(b_{N,j} - b_{N,i})
\]

where \( I_{h,N} \) is defined in (2.15) of the main paper. Then under Assumption 1 it holds that

\[
N^{-3/2} S_N \xrightarrow{P} h \lambda(1 - \lambda) I_b.
\]

**Proof.** We imitate the proof of Theorem 2.3. The first step is to note that

\[
N^{-3/2} \mathbb{E} S_N = \frac{m}{N} \left( 1 - \frac{m - 1}{N - 1} \right) N^{-3/2} \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} I_{h,N}(b_{N,j} - b_{N,i}) \to h \lambda(1 - \lambda) I_b.
\]

It therefore remains to show that \( N^{-3} \text{Var}(S_N) \to 0 \). For brevity, let us abuse the notation \( I_{h,N}(b_{N,j} - b_{N,i}) \) and write \( I_{h,N}(i, j) \) instead. We have

\[
\text{Var}(S_N) = \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \sum_{k=1}^{N} \sum_{l=1, l \neq k}^{N} \text{Cov}((1 - Z_{N,i}) Z_{N,j}, (1 - Z_{N,k}) Z_{N,l}) I_{h,N}(i, j) I_{h,N}(k, l).
\]

Note, \( 2 \leq |\{i, j, k, l\}| \leq 4 \). Consider the following cases.
(a) \(|\{i, j, k, l\}| = 2\), i.e., \((i, j) = (k, l)\). Note that \(\text{Var}((1 - Z_{N,i})Z_{N,j}) = p_N(1 - p_N)\) where \(p_N = P(Z_{N,i} = 0, Z_{N,j} = 1) = \frac{m N - m}{N - 1} \sim \lambda(1 - \lambda)\). Hence the contribution of these terms in \(\text{Var}(S_N)\) is given by

\[
\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} p_N(1 - p_N) I_{h,N}(i, j) \lesssim \lambda(1 - \lambda) \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} I_{h,N}(i, j) \lesssim N^{3/2}.
\]

(b) \(|\{i, j, k, l\}| = 4\), i.e., all 4 indices are distinct. Note that in this case

\[
\text{Cov} ((1 - Z_{N,i})Z_{N,j}, (1 - Z_{N,k})Z_{N,l}) = P(Z_{N,i} = 0, Z_{N,j} = 1, Z_{N,k} = 0, Z_{N,l} = 1) - p_N^2
\]

\[
= \binom{N}{m-2} - \left(\frac{m N - m}{N - 1}\right)^2
\]

\[
= \frac{m^2 N - m (m - 1)(N - m - 1) - m N - m}{N(N - 1)}
\]

\[
= \frac{m N - m}{N - 1} \left(\frac{(m - 1)(N - m - 1)}{(N - 2)(N - 3)} - \frac{m N - m}{N - 1}\right)
\]

\[
= \frac{m N - m}{N - 1} \left(\frac{(m - 1)(N - m - 1)}{(N - 2)(N - 3)} - \frac{m N - m}{N - 1}\right)
\]

\[
= -\lambda(1 - \lambda)(1 - 2\lambda)^2 N^{-1}.
\]

Hence the contribution \(u_N\) of these terms in \(\text{Var}(S_N)\) satisfies the following.

\[
u_N \lesssim N^{-1} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \sum_{k=1}^{N} \sum_{l=1, l \neq k}^{N} I_{h,N}(i, j)I_{h,N}(k, l) \lesssim N^{-1 + 3/2 + 3/2} = N^2.
\]

(c) \(|\{i, j, k, l\}| = 3\). Here we have 4 sub-cases:

| sub-case | \(\mathbb{E}(1 - Z_{N,i})Z_{N,j}(1 - Z_{N,k})Z_{N,l}\) | \(\text{Cov} ((1 - Z_{N,i})Z_{N,j}, (1 - Z_{N,k})Z_{N,l})\) |
|----------|-------------------------------------------------|--------------------------------------------------|
| \(i = k, j \neq l\) | \(P(Z_{N,i} = 0, Z_{N,j} = 1, Z_{N,l} = 1)\) | \(\frac{m(m - 1)(N - m - 1)}{N(N - 1)(N - 2)} - p_N^2 \sim \lambda^3(1 - \lambda)\) |
| \(i \neq k, j = l\) | \(P(Z_{N,i} = 0, Z_{N,j} = 1, Z_{N,k} = 0)\) | \(\frac{m(m - 1)(N - m - 1)}{N(N - 1)(N - 2)} - p_N^2 \sim \lambda^3(1 - \lambda)^3\) |
| \(i = l, j \neq k\) | \(0\) | \(-p_N^2 \sim -\lambda^2(1 - \lambda)^2\) |
| \(i \neq l, j = k\) | \(0\) | \(-p_N^2 \sim -\lambda^2(1 - \lambda)^2\) |

Hence, if \(v_n\) be the contribution these terms in \(\text{Var}(S_N)\), then

\[
v_n \lesssim \sum_{i, j, l \text{ distinct}} I_{h,N}(i, j)I_{h,N}(i, l) + \sum_{i, j, k \text{ distinct}} I_{h,N}(i, j)I_{h,N}(k, j)
\]

\[
+ \sum_{i, j, k \text{ distinct}} I_{h,N}(i, j)I_{h,N}(k, i) + \sum_{i, j, l \text{ distinct}} I_{h,N}(i, j)I_{h,N}(j, l)
\]

\[
\leq 4N \sum_{i, j \neq i} I_{h,N}(i, j) \lesssim N^{1 + 3/2} = N^{5/2}.
\]

Combining the three cases, we can say that \(N^{-3}\text{Var}(S_N) \lesssim N^{-3}(N^{3/2} + N^2 + N^{5/2}) = o(1)\).

\(\square\)
**Lemma D.8.** Suppose that $b_N = X_N\beta_N + \varepsilon_N$, where $\varepsilon_{N,1}, \ldots, \varepsilon_{N,N}$ are i.i.d. from $\mathcal{N}(0,\sigma^2)$. Define, for any fixed $h$,

$$J_N := N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} I_h,N_i(b_{N,j} - \tilde{b}_{N,i}),$$

(D.5)

where $\tilde{b}_{N,j}$ is defined in (3.6), and $I_h,N_i$ is defined in (2.15) of the main paper. Then $\mathbb{E} J_N \rightarrow h(2\sqrt{\pi\sigma})^{-1}$, and $\text{Var}(J_N) \rightarrow 0$, implying that Assumption 2 holds in probability, with $J_0 = (2\sqrt{\pi\sigma})^{-1}$.

**Proof.** We do the proof for $h > 0$, the other case will be similar. For simplicity in notation, we omit the index $N$ in $\tilde{b}_{N,j}, Z_{N,j}, p_{N,j}$, etc. throughout this proof. Observe that under the linear model, we have $\tilde{b} = (I - P_X)b = (I - P_X)\varepsilon$. Hence for any pair $(i,j)$ of distinct indices, $\tilde{b}_j - \tilde{b}_i \sim \mathcal{N}(0,\sigma_{ij}^2)$, where $\sigma_{ij}^2 := \sigma^2(2 - \|p_j - p_i\|^2)$ (this follows from the fact that $P_X$ is a projection matrix). Thus

$$\mathbb{E} J_N = N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} P \left( 0 \leq \tilde{b}_j - \tilde{b}_i < hN^{-1/2} \right) = N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \Phi \left( \sigma_{ij}^{-1}hN^{-1/2} \right) - \Phi(0) \right).$$

For any $\delta > 0$ and $N \in \mathbb{N}$, define

$$S_{N,\delta} := \{(i,j) : 1 \leq i, j \leq N, \|p_j - p_i\| \leq \delta \}.$$

(D.6)

Invoking Lemma E.1, we get

$$|S_{N,\delta}^c| = \left| \{(i,j) : 1 \leq i, j \leq N, \|p_j - p_i\| > \delta \} \right| \leq \delta^{-2} \sum_{j=1}^{N} \sum_{i=1}^{N} \|p_j - p_i\|^2 = O(N).$$

Consequently,

$$N^{-3/2} \sum_{(i,j) \in S_{N,\delta}^c} \left( \Phi \left( \sigma_{ij}^{-1}hN^{-1/2} \right) - \Phi(0) \right) = O(N^{-1/2}).$$

For $(i,j) \in S_{N,\delta}$ we apply the Taylor theorem to conclude that there exists $\psi_{i,j} \in (0,\sigma_{ij}^{-1}hN^{-1/2})$ such that

$$\Phi \left( \sigma_{ij}^{-1}hN^{-1/2} \right) - \Phi(0) = \sigma_{ij}^{-1}hN^{-1/2}\phi(0) + \sigma_{ij}^{-2}h^2N^{-1}\phi'(\psi_{i,j}).$$

Since $\phi'$ is bounded on $\mathbb{R}$, we can say that

$$\Delta_N := \left| \mathbb{E} J_N - N^{-3/2} \sum_{(i,j) \in S_{N,\delta}} \sigma_{ij}^{-1}hN^{-1/2}\phi(0) \right| \lesssim N^{-1/2} + N^{-3/2} \sum_{(i,j) \in S_{N,\delta}} \sigma_{ij}^{-2}h^2N^{-1} \lesssim N^{-1/2} + N^{-5/2} \sum_{(i,j) \in S_{N,\delta}} (2 - \|p_j - p_i\|^2)^{-1} \lesssim N^{-1/2} + N^{-5/2} \sum_{(i,j) \in S_{N,\delta}} (2 - \delta^2)^{-1} \lesssim N^{-1/2}.$$

Hence $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$, and thus

$$\limsup_{N \rightarrow \infty} \mathbb{E} J_N = \limsup_{N \rightarrow \infty} N^{-3/2} \sum_{(i,j) \in S_{N,\delta}} \sigma_{ij}^{-1}hN^{-1/2}\phi(0) \leq h\sigma^{-1}\phi(0)(2 - \delta^2)^{-1/2}.$$
Since the above holds for every \( \delta > 0 \), we can say that \( \limsup_{N \to \infty} \mathbb{E} J_N \leq \frac{h}{2\sqrt{2} \pi \sigma} \). On the other hand, the fact that \( \Delta_N \to 0 \) also yields the following:

\[
\liminf_{N \to \infty} \mathbb{E} J_N = \liminf_{N \to \infty} N^{-3/2} \sum_{(i,j) \in S_{N,\delta}} (2 - \|p_j - p_i\|)^{-1/2} h \sigma^{-1} N^{-1/2} \phi(0) \\
\geq \liminf_{N \to \infty} N^{-2} |S_{N,\delta}| \cdot h \sigma^{-1} \phi(0) \cdot 2^{-1/2} = h \sigma^{-1} \phi(0) \cdot 2^{-1/2},
\]

where in the last step we again used the fact that \( |S_{N,\delta}| = O(N) \). We thus conclude that

\[
\lim_{N \to \infty} \mathbb{E} J_N \geq \frac{h}{2\sqrt{2} \pi \sigma}.
\]

Next we show that \( \text{Var}(J_N) \to 0 \) as \( N \to \infty \). First,

\[
\mathbb{E} J_N^2 = N^{-3} \mathbb{E} \sum_{i,j,k,l} 1 \left\{ \frac{b_j - b_i}{b_l - b_k} \leq \frac{h}{h N^{-1/2}}, 0 \leq b_l - b_k < h N^{-1/2} \right\}.
\]

Now, as in the proof of Theorem 2.3, observe that the contribution of the terms with repeated indices in the above summation is negligible (since Assumption 2 holds and we already have shown that \( \mathbb{E} J_N \) converges). To analyze the terms with distinct indices, note that for any 4 distinct indices \( i, j, k, l \),

\[
\left( \frac{b_j - b_i}{b_l - b_k} \right) \sim N_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2 \begin{pmatrix} 2 - \|p_j - p_i\|^2 & -(p_j - p_i)^\top (p_l - p_k) \\ -(p_j - p_i)^\top (p_l - p_k) & 2 - \|p_l - p_k\|^2 \end{pmatrix} \right).
\]

Let \( \rho_{i,j,k,l} := \text{corr}(b_j - b_i, b_l - b_k) \). Now we again play the trick of splitting the sum into two groups, such that in one group \( |\rho_{i,j,k,l}| \) is small, whereas for the other group the number of summands is small. Fix any \( \delta \in (0,1) \), and consider the set \( S_{N,\delta} \) defined in (D.6). If \( (i,j) \) or \( (k,l) \) does not belong to \( S_{N,\delta} \), then that brings down the count for such summands. To be precise,

\[
N^{-3} \sum_{(i,j) \in S_{N,\delta} \text{ or } (k,l) \in S_{N,\delta}} P(0 \leq b_j - b_i \leq h N^{-1/2}, 0 \leq b_l - b_k \leq h N^{-1/2}) \\
\leq 2N^{-3} \sum_{1 \leq j,k \leq N} \sum_{(k,l) \in S_{N,\delta}} P(0 \leq b_j - b_i \leq h N^{-1/2}) \\
\leq N^{-2} \sum_{1 \leq j,k \leq N} P(0 \leq b_j - b_k \leq h N^{-1/2}) = N^{-1/2} \mathbb{E} J_N.
\]

Since \( \mathbb{E} J_N \) converges, the above shows that the contribution from such indices are also negligible. Finally, for \( (i,j), (k,l) \in S_{N,\delta} \), we have

\[
|\rho_{i,j,k,l}| = \frac{|(p_j - p_i)^\top (p_l - p_k)|}{\sqrt{2 - \|p_j - p_i\|^2} \sqrt{2 - \|p_l - p_k\|^2}} \leq \frac{\delta}{2 - \delta^2} \leq \delta,
\]

and hence Lemma E.3 tells us that

\[
|P(0 \leq b_j - b_i \leq h N^{-1/2}, 0 \leq b_l - b_k \leq h N^{-1/2}) \\
- P(0 \leq b_j - b_i \leq h N^{-1/2}) P(0 \leq b_l - b_k \leq h N^{-1/2})| \\
\leq |\rho_{i,j,k,l}| \left( (1 - \rho_{i,j,k,l}^2)^{-1/2} + (1 - |\rho_{i,j,k,l}|)^{-2} \right) h^2 N^{-2} \sigma_{i,j}^{-1} \sigma_{k,l}^{-1}
\]

in the work.
\[ \leq h^2 N^{-2} \sigma^{-2} \delta \left( (1 - \delta^2)^{-1/2} + (1 - \delta)^{-2} \right). \]

Thus we have shown that
\[
\left| \mathbb{E} J_N^2 - \left( N^{-3/2} \sum_{(i,j) \in S_{N,\delta}} P \left( 0 \leq \tilde{b}_{j} - \tilde{b}_{i} \leq hN^{-1/2} \right) \right)^2 \right| \leq h^2 \sigma^{-2} \delta \left( (1 - \delta^2)^{-1/2} + (1 - \delta)^{-2} \right) + O(N^{-1/2}).
\]

Now for every \( \delta \in (0,1) \), we have \( |S_{N,\delta}^c| = O(N) \), so it follows that
\[
\limsup_{N \to \infty} \text{Var}(J_N) \leq h^2 \sigma^{-2} \delta \left( (1 - \delta^2)^{-1/2} + (1 - \delta)^{-2} \right).
\]

Since \( \delta \in (0,1) \) is arbitrary here, letting \( \delta \to 0 \) completes the proof.

**Lemma D.9.** Suppose that Assumption 2 holds. Then for any \( h \geq \delta > 0 \),
\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| b_{N,j} - b_{N,i} \right| \leq (h - \delta) N^{-1/2}, (h + \delta) N^{-1/2} \right\} = 4 \delta J_6
\]

where \( J_6 \) is defined in Assumption 2 of the main paper.

**Proof.** First we show that for any \( h \in \mathbb{R} \),
\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ b_{N,j} - b_{N,i} = hN^{-1/2} \right\} = 0. \tag{D.7}
\]

To show this, fix \( h \geq 0 \). Note that for any \( \delta > 0 \),
\[
0 \leq N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ b_{N,j} - b_{N,i} = hN^{-1/2} \right\} \leq N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ hN^{-1/2} \leq b_{N,j} - b_{N,i} < \frac{h + \delta}{\sqrt{N}} \right\}
\]
\[
\leq N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ 0 \leq b_{N,j} - b_{N,i} < \frac{h + \delta}{\sqrt{N}} \right\} - N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ 0 \leq b_{N,j} - b_{N,i} < hN^{-1/2} \right\}.
\]

As \( N \to \infty \), the above RHS converges to \( \delta J_6 \). Then letting \( \delta \to 0 \) we finish the proof of (D.7) for \( h \geq 0 \). The case \( h < 0 \) is similar.

Next, fix any \( h \geq \delta > 0 \). We write
\[
N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbf{1} \left\{ \left| b_{N,j} - b_{N,i} \right| \leq \frac{h - \delta}{\sqrt{N}}, \frac{h + \delta}{\sqrt{N}} \right\}
\]
\[
= N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1} \left\{ b_{N,j} - b_{N,i} \leq \frac{h + \delta}{\sqrt{N}} \right\} - \mathbf{1} \left\{ b_{N,j} - b_{N,i} \leq \frac{h - \delta}{\sqrt{N}} \right\} \right)
\]
\[
= N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \mathbf{1} \left\{ 0 \leq b_{N,j} - b_{N,i} < \frac{h + \delta}{\sqrt{N}} \right\} + \mathbf{1} \left\{ \frac{h + \delta}{\sqrt{N}} \leq b_{N,j} - b_{N,i} < 0 \right\} \right)
\]

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\[-N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \left\{ 0 \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} < \frac{h - \delta}{\sqrt{N}} \right\} + 1 \left\{ -\frac{h - \delta}{\sqrt{N}} \leq \tilde{b}_{N,j} - \tilde{b}_{N,i} < 0 \right\} \right) \]

\[+ N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \left( 1 \left\{ \tilde{b}_{N,j} - \tilde{b}_{N,i} = \frac{h + \delta}{\sqrt{N}} \right\} - 1 \left\{ \tilde{b}_{N,j} - \tilde{b}_{N,i} = \frac{h - \delta}{\sqrt{N}} \right\} \right). \]

Appealing to Assumption 2 and (D.7), we conclude that as \(N \to \infty\), the above display converges to \(2(h + \delta)J_b - 2(h - \delta)J_b = 4\delta J_b\).

Lemma D.10. Suppose that Assumption 1 holds. Then for any \(h \geq \delta > 0\),

\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ |b_{N,j} - b_{N,i}| \in \left[ \frac{h - \delta}{\sqrt{N}}, \frac{h + \delta}{\sqrt{N}} \right] \right\} = 4\delta I_C
\]

where \(I_C\) is defined in Assumption 1 of the main paper.

Proof. The proof is essentially same as the proof of Lemma D.9, hence omitted.

Lemma D.11. Define \(D_N\) as in (C.7), and let

\[
\theta_{N,i,j} := \mathbb{E} \left[ |\xi_{N,i,j} - \bar{\xi}_{N,i,j}| \mid Z_{N,j} = 1 \right]. \quad (D.8)
\]

Then the following holds:

\[
\sqrt{\mathbb{E}(N^{-3/2}D_N^2)} \leq N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\theta_{N,i,j}}.
\]

Proof. Recall the notations \(I_N\) and \(\bar{I}_N\) from (C.5) and (C.6), respectively. We use the simple result \(\sqrt{\mathbb{E}(\sum_{j=1}^{N} V_j)^2} \leq \sum_{j=1}^{N} \sqrt{\mathbb{E}(V_j^2)}\) to derive the following.

\[
\sqrt{\mathbb{E}(D_N)^2} = \sqrt{\mathbb{E} \left( I_N - \bar{I}_N \right)^2} = \left[ \mathbb{E} \left( \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \right]^{1/2}
\]

\[
\leq \sum_{j=1}^{N} \left[ \mathbb{E} \left( Z_{N,j} \sum_{i=1}^{N} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \right]^{1/2}
\]

\[
= \sum_{j=1}^{N} \left[ \mathbb{E} \left( Z_{N,j} \mathbb{E} \left( \left( \sum_{i=1}^{N} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \mid Z_{N,j} = 1 \right) \right) \right]^{1/2}
\]

\[
= \sum_{j=1}^{N} \left[ \mathbb{E} \left( Z_{N,j} \mathbb{E} \left( \left( \sum_{i=1}^{N} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \mid Z_{N,j} = 1 \right) \right) \right]^{1/2}
\]

\[
= \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \left[ \mathbb{E} \left( \left( \sum_{i=1}^{N} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \mid Z_{N,j} = 1 \right) \right]^{1/2}
\]

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≤ \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \sum_{i=1}^{N} \left[ \mathbb{E} \left( (\xi_{N,i,j} - \bar{\xi}_{N,i,j})^2 \bigg| Z_{N,j} = 1 \right) \right]^{1/2}

= \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \sum_{i=1}^{N} \left[ \mathbb{E} \left( |\xi_{N,i,j} - \bar{\xi}_{N,i,j}| \bigg| Z_{N,j} = 1 \right) \right]^{1/2}

≤ \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\theta_{N,i,j}}.

This completes the proof. \(\square\)

**Lemma D.12.** Define \(Q_N\) as in (C.7), and let

\[ \gamma_{N,i,j} := \mathbb{E} \left[ |\xi_{N,i,j} - \bar{\xi}_{N,i,j}| \bigg| Z_{N,i} = 1, Z_{N,j} = 1 \right] \tag{D.9} \]

Then it holds that,

\[ \sqrt{\mathbb{E}(N^{-3/2}Q_N)^2} \leq N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\gamma_{N,i,j}}. \]

**Proof.** This proof mimics the proof of Lemma D.11. Recall the notations \(\Pi_N\) and \(\bar{\Pi}_N\) from (C.5) and (C.6), respectively. Note that,

\[ \sqrt{\mathbb{E}(Q_N^2)} = \sqrt{\mathbb{E} \left( \Pi_N - \bar{\Pi}_N \right)^2} \]

\[ = \left[ \mathbb{E} \left( \sum_{j=1}^{N} Z_{N,j} \sum_{i=1}^{N} Z_{N,i} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \right]^{1/2} \]

\[ \leq \sum_{j=1}^{N} \left[ \mathbb{E} \left( Z_{N,j} \sum_{i=1}^{N} Z_{N,i} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \right]^{1/2} \]

\[ = \sum_{j=1}^{N} \left[ \mathbb{E} \left( Z_{N,j} \mathbb{E} \left( \left( \sum_{i=1}^{N} Z_{N,i} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \bigg| Z_{N,j} \right) \right) \right]^{1/2} \]

\[ = \sum_{j=1}^{N} \left[ \mathbb{E} \left( Z_{N,j} \mathbb{E} \left( \left( \sum_{i=1}^{N} Z_{N,i} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \bigg| Z_{N,j} = 1 \right) \right) \right]^{1/2} \]

\[ = \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \left[ \mathbb{E} \left( \left( \sum_{i=1}^{N} Z_{N,i} (\xi_{N,i,j} - \bar{\xi}_{N,i,j}) \right)^2 \bigg| Z_{N,j} = 1 \right) \right]^{1/2} \]

\[ \leq \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \sum_{i=1}^{N} \left[ \mathbb{E} \left( Z_{N,i} (\xi_{N,i,j} - \bar{\xi}_{N,i,j})^2 \bigg| Z_{N,i}, Z_{N,j} = 1 \right) \right]^{1/2} \]

\[ = \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \sum_{i=1}^{N} \left[ \mathbb{E} \left( Z_{N,i} \mathbb{E} \left( (\xi_{N,i,j} - \bar{\xi}_{N,i,j})^2 \bigg| Z_{N,i}, Z_{N,j} = 1 \right) \right) \right]^{1/2} \]
\[= \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \sum_{i=1}^{N} \left[ \mathbb{E}(Z_{N,i} \mathbb{E} \left( (\xi_{N,i,j} - \tilde{\xi}_{N,i,j})^2 \mid Z_{N,i} = 1, Z_{N,j} = 1 \right) \right]^{1/2} = \sum_{j=1}^{N} \sqrt{\mathbb{E}(Z_{N,j})} \sum_{i=1}^{N} \sqrt{\mathbb{E}(Z_{N,i} \mid Z_{N,j} = 1)} \left[ \mathbb{E} \left( (\xi_{N,i,j} - \tilde{\xi}_{N,i,j}) \mid Z_{N,i} = 1, Z_{N,j} = 1 \right) \right]^{1/2}\]
\[\leq \sum_{j=1}^{N} \sum_{i=1}^{N} \left[ \mathbb{E} \left( (\xi_{N,i,j} - \tilde{\xi}_{N,i,j}) \mid Z_{N,i} = 1, Z_{N,j} = 1 \right) \right]^{1/2}\]
\[= \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\gamma_{N,i,j}}.\]

This completes the proof. \[\square\]

**Lemma D.13.** Define \(D_N\) and \(Q_N\) as in (C.7). It holds under Assumption 2 that \(N^{-3/2}D_N = o_P(1)\) and \(N^{-3/2}Q_N = o_P(1)\) as \(N \to \infty\).

**Proof.** We first focus on \(D_N\). In view of Lemma D.11, it suffices to show that
\[
\lim_{N \to \infty} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\theta_{N,i,j}} = 0,
\]
where \(\theta_{N,i,j}\) is defined in (D.8). Fix \(h\) for the moment, and introduce the notation
\[
r_N(i, j) := b_{N,i,j} - b_{N,i} - hN^{-1/2}, \quad 1 \leq i, j \leq N.
\]
Note,
\[
\xi_{N,i,j} - \tilde{\xi}_{N,i,j} = 1 \left\{ hN^{-1/2} (p_{N,i} - p_{N,j})^\top Z_N \leq r_N(i, j) \right\} - 1 \left\{ 0 \leq r_N(i, j) \right\} = 1 \left\{ hN^{-1/2} (p_{N,i} - p_{N,j})^\top Z_N \leq r_N(i, j) < 0 \right\} - 1 \left\{ hN^{-1/2} (p_{N,i} - p_{N,j})^\top Z_N > r_N(i, j) \geq 0 \right\} = 1 \left\{ hN^{-1/2} \left( (p_{N,i} - p_{N,j})^\top Z_N \right) \geq |r_N(i, j)| \right\} (1 \{r_N(i, j) < 0\} - 1 \{r_N(i, j) \geq 0\}).
\]
Now applying the Markov inequality, we get
\[
\sqrt{\theta_{N,i,j}} = \mathbb{E}^{1/2} \left[ |\xi_{N,i,j} - \tilde{\xi}_{N,i,j}| \mid Z_{N,j} = 1 \right] = \mathbb{E} \left( hN^{-1/2} \left( (p_{N,i} - p_{N,j})^\top Z_N \right) \geq |r_N(i, j)| \mid Z_{N,j} = 1 \right)^{1/2} \leq h^2 N^{-1} r_N(i, j)^{-2} \mathbb{E}^{1/2} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^4 \mid Z_{N,j} = 1 \right].
\]
To upper bound the above RHS, we use Lemma E.1. The key idea is to use this bound only for those \(i, j\) for which \(r_N(i, j)\) is at least as large as \(\delta N^{-1/2}\). Towards that, fix \(\delta > 0\) and define
\[\mathcal{S}_{N,\delta} = \{(i, j) : |r_N(i, j)| \leq \delta N^{-1/2}, 1 \leq i, j \leq N\}.
\]
Then
\[(i, j) \notin S_{N, \delta} \implies \sqrt{\theta_{N, i, j}} \leq a^2 \delta^{-2} \mathbb{E}^{1/2} \left( (p_{N, i} - p_{N, j})^T Z_N \right)^4 \mid Z_{N, j} = 1 \].

Therefore
\[
N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\theta_{N, i, j}} \\
\leq N^{-3/2} \sum_{(i, j) \in S_{N, \delta}} \sqrt{\theta_{N, i, j}} + N^{-3/2} \sum_{(i, j) \notin S_{N, \delta}} \sqrt{\theta_{N, i, j}} \\
\leq N^{-3/2} \sum_{(i, j) \in S_{N, \delta}} 1 + N^{-3/2} \sum_{(i, j) \notin S_{N, \delta}} a^2 \delta^{-2} \mathbb{E}^{1/2} \left( (p_{N, i} - p_{N, j})^T Z_N \right)^4 \mid Z_{N, j} = 1 \\
\leq N^{-3/2} |S_{N, \delta}| + a^2 \delta^{-2} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \mathbb{E}^{1/2} \left( (p_{N, i} - p_{N, j})^T Z_N \right)^4 \mid Z_{N, j} = 1 \\
= N^{-3/2} |S_{N, \delta}| + a^2 \delta^{-2} \text{rank}(X_N) - 1 \cdot O(N^{-1/2}).
\]

In the last step we used Lemma E.1. Letting \(N \to \infty\) and then \(\delta \downarrow 0\), it follows that
\[
\limsup_{N \to \infty} N^{-3/2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sqrt{\theta_{N, i, j}} \leq \limsup_{\delta \downarrow 0} \lim_{N \to \infty} N^{-3/2} |S_{N, \delta}|.
\]

Now
\[(i, j) \in S_{N, \delta} \implies \frac{\delta}{\sqrt{N}} \geq |r_N(i, j)| \geq \left| (b_{N, j} - p_{N,j}^T b_N) - (b_{N,i} - p_{N,i}^T b_N) - \frac{|a|}{\sqrt{N}} \right| \\
\implies \left| (b_{N,j} - p_{N,j}^T b_N) - (b_{N,i} - p_{N,i}^T b_N) \right| \in \left[ \frac{|a| - \delta}{\sqrt{N}}, \frac{|a| + \delta}{\sqrt{N}} \right].
\]

We now invoke Lemma D.9 to arrive at
\[
\limsup_{\delta \downarrow 0} \lim_{N \to \infty} N^{-3/2} |S_{N, \delta}| = 0,
\]
which finishes the proof for \(D_N\). The proof for \(Q_N\) can be done in the same manner, using Lemma D.12, Markov inequality, and Lemma E.1.

\[\square\]

**Lemma D.14.** Suppose that Assumption 3 holds. Then for any \(h \geq \delta > 0\),
\[
\lim_{N \to \infty} N^{-(2-\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ |b_{N,j} - b_{N,i}| \in \left[ \frac{h - \delta}{N^\nu}, \frac{h + \delta}{N^\nu} \right] \right\} = 4 \delta I_b
\]
where \(I_b\) is defined in Assumption 1 of the main paper.

**Proof.** The proof is analogous to those of Lemmas D.9 and D.10, and hence omitted. \[\square\]
Lemma D.15. Suppose that Assumption 4 holds. Then for any $\delta > 0$,

$$
\lim_{N \to \infty} N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} 1 \left\{ \left| \tilde{b}_{N,j} - \tilde{b}_{N,i} \right| \in \left[ \frac{h - \delta}{N^\nu}, \frac{h + \delta}{N^\nu} \right] \right\} = 4\delta J_b
$$

where $J_b$ is defined in Assumption 2 of the main paper.

Proof. The proof is analogous to those of Lemmas D.9 and D.10, and hence omitted. \qed

Lemma D.16. For each $N \geq 1$, let $b_{N,1}, \ldots, b_{N,N}$ be i.i.d. from an unknown distribution $F_b$ with density $f_b(\cdot)$, and $I_{h,N,\nu}$ be defined in (4.2) of the main paper, then for any $0 < \nu \leq 1/2$,

$$
N^{-2(\nu)} \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N,\nu}(b_{N,j} - b_{N,i}) \overset{P}{\to} h \int f_b^2(x) dx.
$$

Proof. We proved the above for $\nu = 1/2$ in Lemma D.4, so assume now that $\nu \in (0, 1/2)$. Define

$$
S_N := S_N(b_{N,1}, \ldots, b_{N,N}) := \sum_{j=1}^{N} \sum_{i=1}^{N} I_{h,N,\nu}(b_{N,j} - b_{N,i}).
$$

Observe that for any fixed $i \neq j$, and $h > 0$,

$$
\mathbb{E}(I_{h,N,\nu}(b_{N,j} - b_{N,i})) = P(0 \leq b_{N,2} - b_{N,1} < hN^{-\nu}) = g(hN^{-\nu}) - g(0)
$$

where

$$
g(x) = P(b_{N,2} - b_{N,1} \leq x) = \int_{-\infty}^{x} \int_{\mathbb{R}} f_b(u + t) f_b(u) du dt, \quad x \in \mathbb{R}.
$$

Using the DCT for integrals, we argue that

$$
g'(x) = \int_{\mathbb{R}} f_b(u + x) f_b(u) du.
$$

Hence

$$
\lim_{N \to \infty} N^{-2(\nu)} \mathbb{E}(S_N) = \lim_{N \to \infty} N^{-2(\nu)} N^2(g(hN^{-\nu}) - g(0)) \quad \text{(D.10)}
$$

$$
= \lim_{N \to \infty} N^2 g'(0)
$$

$$
= h \int_{\mathbb{R}} f_b(u)^2 du.
$$

The case $h < 0$ can be handled similarly, and the case $h = 0$ is straight-forward. Next we bound $\mathbb{E}(S_N - \mathbb{E}S_N)^2$ using the Efron-Stein inequality (Efron and Stein, 1981). For each $1 \leq k \leq N$, let $b_{N,k}'$ be an i.i.d. copy of $b_{N,k}$, independent of everything else, and define

$$
S_{N}^{(k)} = S_N(b_{N,1}, \ldots, b_{N,k-1}, b_{N,k}', b_{N,k+1}, \ldots, b_{N,N}), \quad 1 \leq k \leq N.
$$

Note,

$$
S_N - S_N^{(k)} = \sum_{j=1}^{N} \left( I_{h,N,\nu}(b_{N,j} - b_{N,k}) - I_{h,N,\nu}(b_{N,j} - b_{N,k}') \right)
$$
\[ + \sum_{j=1}^{N} (I_{h,N,v}(b_{N,k} - b_{N,j}) - I_{h,N,v}(b'_{N,k} - b_{N,j})) \].

Hence \(|S_N - S_N^{(k)}| \leq 4N\) almost surely. The Efron-Stein inequality tells us that
\[
\mathbb{E}(N^{-(2-\nu)}(S_N - \mathbb{E}S_N))^2 \leq N^{-(4-2\nu)}\mathbb{E} \left[ \sum_{k=1}^{N} \mathbb{E} \left[ (S_N - S_N^{(k)})^2 \mid b_{N,1}, \ldots, b_{N,N} \right] \right]
\leq N^{-(4-2\nu)} \cdot 16N^3 = 16N^{-(1-2\nu)}.
\]

Since \(\nu < 1/2\), we can now invoke (D.10) to get the desired conclusion. \(\square\)

### E Auxiliary results

**Lemma E.1.** As \(N \to \infty\), the following results hold, for \(r = 1, 2\).

\[
\sum_{j=1}^{N} \sum_{i=1}^{N} \mathbb{E}^{1/r} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^{2r} \right] = (\text{rank}(X_N) - 1) \cdot O(N), \quad (E.1)
\]

\[
\sum_{j=1}^{N} \sum_{i=1}^{N} \mathbb{E}^{1/r} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^{2r} \mid Z_{N,j} = 1 \right] = (\text{rank}(X_N) - 1) \cdot O(N), \quad (E.2)
\]

\[
\sum_{j=1}^{N} \sum_{i=1}^{N} \mathbb{E}^{1/r} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^{2r} \mid Z_{N,i} = 1, Z_{N,j} = 1 \right] = (\text{rank}(X_N) - 1) \cdot O(N). \quad (E.3)
\]

**Proof.** Using the fact that \(P_{X_N}\) is idempotent, we derive the following identity.

\[
\sum_{j=1}^{N} \sum_{i=1}^{N} \|p_{N,i} - p_{N,j}\|^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} (P_{X_N}(i,i) + P_{X_N}(j,j) - 2P_{X_N}(i,j))
\]
\[
= 2N \sum_{i=1}^{N} P_{X_N}(i,i) - 2 \sum_{i=1}^{N} P_{X_N}^\top 1
\]
\[
= 2N (\text{Tr} (P_{X_N}) - 1) = 2N (\text{rank}(X_N) - 1). \quad (E.4)
\]

Now \(1\) belongs to the column space of \(X_N\), so \(\mathbb{E} \left[ (p_{N,i} - p_{N,j})^\top Z_N \right] = 0\) for each \(i, j\). Also note that \(\text{Var}(Z_N)\) is of the form \(\alpha I + \beta 11^\top\), where \(\alpha = \frac{m}{N} (1 - \frac{m}{N})\). Consequently,

\[
\text{Var} \left[ (p_{N,i} - p_{N,j})^\top Z_N \right] = (p_{N,i} - p_{N,j})^\top (\alpha I + \beta 11^\top) (p_{N,i} - p_{N,j}) = \frac{m}{N} \left( 1 - \frac{m}{N} \right) \|p_{N,i} - p_{N,j}\|^2.
\]

Hence

\[
\sum_{j=1}^{N} \sum_{i=1}^{N} \mathbb{E} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^2 \right] \leq \sum_{j=1}^{N} \sum_{i=1}^{N} \|p_{N,i} - p_{N,j}\|^2.
\]

Thus, (E.1) follows for \(r = 1\).

To prove it for \(r = 2\), fix \(i\) and \(j\) for the moment and write \((p_{N,i} - p_{N,j})^\top Z_N = \sum_{k=1}^{N} v_k Z_k\). Then

\[
\mathbb{E} \left( \sum_{k=1}^{N} v_k Z_k \right)^4 = \sum_{k_1=1}^{N} v_{k_1}^4 \mathbb{E} Z_1 + \sum_{k_1, k_2 \text{ distinct}} (3v_{k_1}^2 v_{k_2}^2 + 4v_{k_1}^3 v_{k_2}) \mathbb{E} Z_1 Z_2
\]

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\[
\sum_{k_1,k_2,k_3} 6v_{k_1}^2 v_{k_2} v_{k_3} E Z_1 Z_2 Z_3 + \sum_{k_1,k_2,k_3,k_4} v_{k_1} v_{k_2} v_{k_3} v_{k_4} E Z_1 Z_2 Z_3 Z_4.
\]

Now \( \sum_{k=1}^N v_k = (p_{N,i} - p_{N,j})^T \mathbf{1} = 0 \), so we obtain
\[
\sum_{k_1,k_2,k_3,k_4} v_{k_1} v_{k_2} v_{k_3} v_{k_4} = \sum_{k_1,k_2,k_3} v_{k_1} v_{k_2} v_{k_3} (-v_{k_1} - v_{k_2} - v_{k_3})
\]
\[
= -3 \sum_{k_1,k_2,k_3} v_{k_1}^2 v_{k_2} v_{k_3},
\]
\[
\sum_{k_1,k_2,k_3} v_{k_1} v_{k_2} v_{k_3} = \sum_{k_1,k_2} v_{k_1} v_{k_2} (-v_{k_1} - v_{k_2})
\]
\[
= -3 \sum_{k_1,k_2} v_{k_1}^2 v_{k_2} - \sum_{k_1,k_2} v_{k_1}^2 v_{k_2},
\]
\[
\sum_{k_1,k_2} v_{k_1} v_{k_2} = \sum_{k_1=1}^N v_{k_1} (-v_{k_1}) = -\sum_{k_1=1}^N v_{k_1}^2.
\]

Since \( E |Z_1 \cdots Z_v| \leq 1 \) for \( 1 \leq v \leq 4 \), it follows from the above identities that
\[
E^{1/2} \left( (p_{N,i} - p_{N,j})^T Z_N \right)^4 \leq \sqrt{D'} \left( \sum_{k=1}^N v_k^4 + \sum_{k \neq l} v_k^2 v_l^2 \right)^{1/2}
\]
\[
\leq \sqrt{2D'} \sum_{k=1}^N v_k^2 = \sqrt{2D'}\|p_{N,i} - p_{N,j}\|^2
\]
for some constant \( D' > 0 \) (which is free of \( i, j \) and \( N \)). The rest follows again from (E.4).

Next, we deal with the identities where we condition upon \( Z_{N,j} = 1 \). Observe that
\[
E \left[ \left( (p_{N,i} - p_{N,j})^T Z_N \right)^2 \mid Z_{N,j} = 1 \right] = (p_{N,i} - p_{N,j})^T \text{Var} (Z_N \mid Z_{N,j} = 1) (p_{N,i} - p_{N,j}).
\]

For any \( k \neq l \), \( \text{Var}(Z_{N,k} \mid Z_{N,j} = 1) = (\alpha_1 - \alpha_1^2) 1 \{ k \neq j \} \), and \( \text{Cov}(Z_{N,k}, Z_{N,l} \mid Z_{N,j} = 1) = (\alpha_2 - \alpha_1^2) 1 \{ k \neq j \} \), where \( \alpha_1 = \frac{m-1}{N-1} \), \( \alpha_2 = \frac{m-2}{N-2} \). We thus obtain
\[
E \left[ \left( (p_{N,i} - p_{N,j})^T Z_N \right)^2 \mid Z_{N,j} = 1 \right] = \left( \alpha_1 (1 - \alpha_1) \|p_{N,i} - p_{N,j}\|^2 \right.
\]
\[
- \left. (\alpha_1 - 2\alpha_2 + \alpha_1^2) (P_{X_N}(i,j) - P_{X_N}(j,j))^2 \right].
\]

But \( (P_{X_N}(i,j) - P_{X_N}(j,j))^2 \leq \|p_{N,i} - p_{N,j}\|^2 \). So it follows that
\[
E \left[ \left( (p_{N,i} - p_{N,j})^T Z_N \right)^2 \mid Z_{N,j} = 1 \right] \leq D\|p_{N,i} - p_{N,j}\|^2
\]
for some \( D > 0 \) which is free of \( i, j \) and \( N \). Invoking (E.4) again, we finish the proof of (E.2) for \( r = 1 \).

To prove (E.2) for \( r = 2 \), note the following.
\[
\sqrt{E \left[ \left( (p_{N,i} - p_{N,j})^T Z_N \right)^4 \mid Z_{N,j} = 1 \right]}\]
\[ \leq \sqrt{\text{Var} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^2 \bigg| Z_{N,j} = 1 \right]} + \mathbb{E} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^2 \bigg| Z_{N,j} = 1 \right] \]
\[ \leq \frac{N}{m} \sqrt{\mathbb{E} \left[ \text{Var} \left[ (p_{N,i} - p_{N,j})^\top Z_N \right)^2 \bigg| Z_{N,j} \right]} + \mathbb{E} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^2 \bigg| Z_{N,j} = 1 \right] \]
\[ \leq \frac{N}{m} \sqrt{\mathbb{E} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^4 \right] + \mathbb{E} \left[ \left( (p_{N,i} - p_{N,j})^\top Z_N \right)^2 \bigg| Z_{N,j} = 1 \right]}. \]

Since \( m/N \to \lambda \in (0, 1) \), the desired conclusion follows from (E.1) (with \( r = 2 \)) and (E.2) (with \( r = 1 \)). The proof of (E.3) is completely analogous to the proofs of (E.1) and (E.2), hence omitted.

**Lemma E.2.** For \( x \neq 0 \) it holds that \( |\Phi(x/\sigma) - 1\{x \geq 0\}| \leq \sigma |x|^{-1} \exp(-x^2/2\sigma^2) \), where \( \Phi \) is the standard Normal CDF.

**Proof.** Let \( \phi \) be the density of standard Normal. For \( x > 0 \), we have
\[ |\Phi(x/\sigma) - 1\{x \geq 0\}| = 1 - \Phi(x/\sigma) \leq \sigma \cdot x^{-1} \phi(x/\sigma), \]
using a standard inequality. For \( x < 0 \), we can write
\[ |\Phi(x/\sigma) - 1\{x \geq 0\}| = \Phi(x/\sigma) = 1 - \Phi(-x/\sigma) \leq \sigma \cdot (-x)^{-1} \phi(-x/\sigma). \]
Hence the result.

**Lemma E.3.** Suppose that \((X,Y)\) follows the bivariate Normal distribution with \( \mathbb{E}X = \mathbb{E}Y = 0 \), \( \text{Var}(X) = \text{Var}(Y) = 1 \), and \( \text{corr}(X,Y) = \rho \in (-1, 1) \). Then for any \( h_1, h_2 \in (0, 1) \), the following bound holds:
\[ |P(0 \leq X \leq h_1, 0 \leq Y \leq h_2) - P(0 \leq X \leq h_1)P(0 \leq Y \leq h_2)| \leq C_\rho h_1 h_2, \]
where \( C_\rho = |\rho| \left( (1 - \rho^2)^{-1/2} + (1 - |\rho|)^{-2} \right) \).

**Proof.** Define
\[ g_{x,y}(\rho) := \exp \left( -\frac{1}{2(1 - \rho^2)} \left( x^2 + y^2 - 2\rho xy \right) \right). \]
Note that for any fixed \( x, y \in [0, 1] \),
\[ \left| \frac{\partial g_{x,y}(\rho)}{\partial \rho} \right| = g_{x,y}(\rho) \left| \frac{xy}{1 - \rho^2} - \frac{\rho}{1 - \rho^2} \frac{x^2 + y^2 - 2\rho xy}{1 - \rho^2} \right| \leq \frac{1}{(1 - |\rho|)^2}. \]
Hence
\[ |P(0 \leq X \leq h_1, 0 \leq Y \leq h_2) - P(0 \leq X \leq h_1)P(0 \leq Y \leq h_2)| \]
\[ = \left| \int_{[0, h_1] \times [0, h_2]} \left( (2\pi \sqrt{1 - \rho^2})^{-1} g_{x,y}(\rho) - (2\pi)^{-1} g_{x,y}(0) \right) \, dx \, dy \right|. \]
\begin{align*}
\leq (2\pi)^{-1} & \left|(1 - \rho^2)^{-1/2} - 1 \right| \int_{[0, h_1] \times [0, h_2]} g_{x,y}(\rho) \, dx \, dy \\
& + (2\pi)^{-1} \int_{[0, h_1] \times [0, h_2]} |g_{x,y}(\rho) - g_{x,y}(0)| \, dx \, dy \\
\leq & \rho^2 (1 - \rho^2)^{-1/2} h_1 h_2 + \int_{[0, h_1] \times [0, h_2]} |\rho| \int_0^1 \left| \frac{\partial g_{x,y}(t\rho)}{\partial \rho} \right| \, dt \, dx \, dy \\
\leq & |\rho| \left( (1 - \rho^2)^{-1/2} + (1 - |\rho|)^{-2} \right) h_1 h_2.
\end{align*}

This completes the proof. \hfill \Box

**Proposition E.1** (A generalized Efron-Stein inequality). Given a sequence of independent real-valued random variables $W_1, W_2, \ldots, W_n$ and $F : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. For each $1 \leq i \leq n$, let $W_i'$ be an independent copy of $W_i$, independent of the other $W_j$’s. Define $S := F(W_1, \ldots, W_n)$ and $S_i := F(W_1, \ldots, W_{i-1}, W_i', W_{i+1}, \ldots, W_n)$. Then for all integers $q \geq 2$, there exists a constant $c_q$ (depending only on $q$) such that

\[ E \left| S - E S \right|^q \leq c_q E \left[ \sum_{i=1}^n (S - S_i)^2 \mid (W_1, W_2, \ldots, W_n) \right]^{q/2}. \]

**Proof.** See Boucheron et al. (2005, Theorem 2) for a proof. \hfill \Box

The special case $q = 2$ yields the Efron-Stein inequality (see Efron and Stein (1981)).

**F Average ranks**

In the main paper, we used up-ranks to break ties in the responses (see (2.5) in the main paper for a definition). We discuss in this section the analogous results when ties are broken using average ranks. Suppose that while sorting $\{b_{N,j} : 1 \leq j \leq N\}$ in ascending order, the first $t_1$ many are equal, then the next $t_2$ many are equal, and so on. Assume that we get $k$ such blocks of sizes $t_1, \ldots, t_k$, where the $t_i$’s can equal to 1 as well. Set $s_0 = 0$ and $s_j = \sum_{i=1}^j t_i$. Then for each $1 \leq j \leq k$, define

\[ q_N^{\text{avg}}(s_{j-1} + 1) = q_N^{\text{avg}}(s_{j-1} + 2) = \cdots = q_N^{\text{avg}}(s_j) := \frac{s_{j-1} + 1 + s_j}{2} = s_{j-1} + \frac{1 + t_j}{2}, \]

where the last quantity is nothing but the average of the ranks $\{s_{j-1} + 1, \ldots, s_{j-1} + t_j\}$. With the above notion of average ranks, we define the Wilcoxon rank-sum statistic as

\[ t_N^{\text{avg}} := \sum_{j=1}^N q_N^{\text{avg}}(j) Z_{N,j}. \]

We make the following assumption on the block-sizes, which essentially tells us that the blocks formed by the ties are not too large.

**Assumption 5.** Let $t_1, \ldots, t_k$ denote the block sizes, as defined above. Then assume that $\max_{1 \leq i \leq k} t_i = o(N)$, as $N \to \infty$. 

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The following lemma shows that if Assumption 5 holds, then the set of the average ranks do not differ much from \(\{1, 2, \ldots, N\}\). This implies, in view of Lemma D.5 and Proposition 2.1, that \(t^\text{avg}_N\) has the same asymptotic null distribution as the statistic \(t_N\) that we studied in Section 2 of the main paper.

**Lemma F.1.** It holds under Assumption 5 that

\[
\sum_{i=1}^{N} (q_N(i) - \bar{q}_N)^2 = \frac{N (N^2 - 1)}{12} + o(N^3),
\]

as \(N \to \infty\), where \(\bar{q}_N = N^{-1} \sum_{i=1}^{N} q_N(i)\).

*Proof.* Since for each \(j\), \(\sum_{i=s_{j-1}+1}^{s_j} (q_N(i) - i) = 0\), and thus \(\bar{q}_N = N^{-1} \sum_{i=1}^{N} i = (N + 1)/2\) we obtain the following.

\[
\sum_{i=s_{j-1}+1}^{s_j} \left( (q_N(i) - \bar{q}_N)^2 - (i - \bar{q}_N)^2 \right) = \sum_{i=s_{j-1}+1}^{s_j} (q_N(i)^2 - 2\bar{q}_N (q_N(i) - i) - i^2)
\]

\[
= - \sum_{i=s_{j-1}+1}^{s_j} \left( i^2 - \left( \frac{s_{j-1} + 1 + s_j}{2} \right)^2 \right)
\]

\[
= - \sum_{i=s_{j-1}+1}^{s_j} \left( i - \frac{s_{j-1} + 1 + s_j}{2} \right)^2
\]

\[
= - t_j \left( \frac{t_{2j}^2 - 1}{12} \right).
\]

Therefore

\[
\sum_{i=1}^{N} (q_N(i) - \bar{q}_N)^2 = \sum_{i=1}^{N} (i - \bar{q}_N)^2 - \sum_{j=1}^{k} t_j \left( \frac{t_{2j}^2 - 1}{12} \right)
\]

\[
= \frac{N (N^2 - 1)}{12} - \sum_{j=1}^{k} t_j \left( \frac{t_{2j}^2 - 1}{12} \right).
\]

Notice that

\[
0 \leq \sum_{j=1}^{k} \frac{t_j \left( t_{2j}^2 - 1 \right)}{12} \leq \sum_{i=1}^{k} \frac{t_i \left( t_{2i}^2 - 1 \right)}{12} = \frac{N}{12} \max_{1 \leq i \leq k} t_i^2.
\]

Hence Assumption 5 completes the proof.

Our next lemma extends the last result, by showing that if Assumption 5 is satisfied, then the statistic \(t^\text{avg}_N\) has the same asymptotic distribution as the statistic \(t_N\), for any value of the parameter \(\tau\). In particular, this implies that Theorem 2.1 also holds if we replace \(t_N\) by \(t^\text{avg}_N\). Consequently, Rosenbaum’s estimator for \(\tau\) constructed using \(t^\text{avg}_N\) instead of \(t_N\) must also satisfy Theorem 2.2.

**Lemma F.2.** It holds under Assumption 5 that \(N^{-3/2}(t^\text{avg}_N - t_N) = o_P(1)\) as \(N \to \infty\). Consequently, \(t^\text{avg}_N\) has the same asymptotic distribution as \(t_N\).
Proof. Recall that \( t_N = \sum_{j=1}^{N} q_N(j) Z_{N,j} \) where \( q_N(j) = \sum_{i=1}^{N} 1\{b_{N,i} \leq b_{N,j}\} \). Observe that when \( s_{j-1} < j \leq s_j \),

\[
0 \leq q_N(j) - q_N^{avg}(j) \leq t_j - \frac{t_j + 1}{2} = \frac{t_j - 1}{2}.
\]

Hence \( t_N \geq \bar{t}_N^{avg} \) a.s., and therefore it suffices to show that the following converges to 0, as \( N \to \infty \).

\[
\mathbb{E} N^{-3/2} (t_N - \bar{t}_N^{avg}) = \frac{m}{N} N^{-3/2} \sum_{j=1}^{N} (q_N(j) - q_N^{avg}(j)) = \frac{m}{N} N^{-3/2} \sum_{i=1}^{k} \frac{t_i(t_i - 1)}{2}.
\]

We split the above sum into two parts, as follows. Fix any \( \varepsilon > 0 \) and define \( S_{\varepsilon} = \{ 1 \leq j \leq k : t_j - 1 \geq \varepsilon N^{1/2} \} \). Then

\[
\frac{m}{N} N^{-3/2} \sum_{i \in S_{\varepsilon}} \frac{t_i(t_i - 1)}{2} \leq \varepsilon N^{-1} \sum_{i \in S_{\varepsilon}} t_i \leq \varepsilon.
\]

On the other hand, for \( i \in S_{\varepsilon} \), and \( s_{i-1} < j \leq s_i \) we have \( q_N(j) - q_N^{avg}(j) = t_i(t_i - 1)/2 \geq \varepsilon^2 N/2 \). Hence, if \( J \) be an index chosen uniformly at random from \( \{1, 2, \ldots, N\} \) (independent of everything else), then

\[
N^{-3/2} \sum_{i \in S_{\varepsilon}} \frac{t_i(t_i - 1)}{2} = N^{-3/2} \sum_{i \in S_{\varepsilon}} \sum_{s_{i-1}+1}^{s_i} (q_N(j) - q_N^{avg}(j)) = N^{-1/2} P(q_N(J) - q_N^{avg}(J) \geq \varepsilon^2 N/2)
\]

\[
\leq \frac{\varepsilon^2}{2} N^{-1/2} N^{-2} \mathbb{E}(q_N(J) - q_N^{avg}(J))^2
\]

\[
= \frac{\varepsilon^2}{2} N^{-7/2} \sum_{j=1}^{N} (q_N(j) - q_N^{avg}(j))^2.
\]

Therefore it suffices to show that as \( N \to \infty \),

\[
\sum_{j=1}^{N} (q_N^{avg}(j) - q_N(j))^2 = O(N^3).
\] (F.1)

Note that for each \( 1 \leq j \leq k, \sum_{i=s_{j-1}+1}^{s_j} (q_N^{avg}(i) - i) = 0. \) Thus \( q_N^{avg} = N^{-1} \sum_{i=1}^{N} i = (N + 1)/2 \), and we obtain the following.

\[
\sum_{i=s_{j-1}+1}^{s_j} \left( (q_N^{avg}(i) - q_N^{avg})^2 - (i - q_N^{avg})^2 \right) = \sum_{i=s_{j-1}+1}^{s_j} (q_N^{avg}(i)^2 - 2q_N^{avg}(q_N^{avg}(i) - i) - i^2)
\]

\[
= - \sum_{i=s_{j-1}+1}^{s_j} \left( i^2 - \left( \frac{s_{j-1} + 1 + s_j}{2} \right)^2 \right)
\]

\[
= - \sum_{i=s_{j-1}+1}^{s_j} \left( i - \frac{s_{j-1} + 1 + s_j}{2} \right)^2
\]

\[
= - \frac{1}{12} t_j (t_j^2 - 1).
\]

Therefore

\[
\sum_{i=1}^{N} (q_N^{avg}(i) - q_N^{avg})^2 = \sum_{i=1}^{N} (i - q_N^{avg})^2 - \frac{1}{12} \sum_{j=1}^{k} t_j (t_j^2 - 1).
\]
Notice that
\[ 0 \leq \frac{1}{12} \sum_{j=1}^{k} t_j (t_j^2 - 1) \leq \frac{1}{12} \left( \sum_{i=1}^{k} t_i \right) \max_{1 \leq i \leq k} t_i^2 = \frac{N}{12} \max_{1 \leq i \leq k} t_i^2 \]
and therefore Assumption 5 tells us that
\[ \sum_{i=1}^{N} (\bar{q}_N(i) - \bar{q}_N)^2 = \frac{N(N^2 - 1)}{12} + o(N^3). \]
On the other hand, it follows from the proof of Lemma D.5 that
\[ \sum_{i=1}^{N} \left( q_N(i) - \frac{N + 1}{2} \right)^2 = \frac{N(N^2 - 1)}{12} + o(N^3) \]
Combining these and doing some algebraic manipulations, \((F.1)\) follows.