Electromagnetic field quantization in an anisotropic and inhomogeneous magnetodielectric

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Abstract

The electromagnetic field in an anisotropic and inhomogeneous magnetodielectric is quantized by modelling the medium with two independent quantum fields. Some coupling tensors coupling the electromagnetic field with the medium are introduced. Electric and magnetic polarizations are obtained in terms of the ladder operators of the medium and the coupling tensors explicitly. Using a minimal coupling scheme for electric and magnetic interactions, the Maxwell equations and the constitutive equations of the medium are obtained. The electric and magnetic susceptibility tensors of the medium are calculated in terms of the coupling tensors. Finally the efficiency of the approach is elucidated by some examples.

Keywords: Field quantization, Magnetodielectric, Anisotropic, Inhomogeneous, Coupling tensor, E-M Quantum fields

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1 Introduction

The quantization of electromagnetic field in an absorptive dielectric, represents one of the most and interesting problems in quantum optics, because it gives a rigorous test of our understanding of the interaction of light with matter. One of the important methods to quantize the electromagnetic field in the presence of an absorptive medium is known as Green function method [1]-[7]. In this method by adding the noise electric and magnetic polarization densities to classical constitutive equations of the medium, these equations are considered as definitions of electric and magnetic polarization operators. The noise polarizations are related to two independent sets of bosonic operators. Combination of the Maxwell equations and the constitutive equations in frequency domain, give the electromagnetic field operators in terms of the noise polarizations and classical Green tensor. Suitable commutation relations are imposed on the bosonic operators such that the commutation relations between electromagnetic field operators become identical with those in free space.

An interesting quantization scheme of electromagnetic field in the presence of an absorptive dielectric medium is based on the Hopfield model of a dielectric [8], where the polarization of the dielectric is represented by a damped quantum field [9]. Huttner and Barnett [10] for a homogeneous medium and after Sutterp and Wubs [11] for an inhomogeneous medium in the framework of the damped polarization model have presented a canonical quantization for the electromagnetic field inside an absorptive dielectric. This scheme is based on a microscopic model in which the medium is represented by a collection of interacting matter fields. The absorptive character of the medium is modelled through the interaction of the matter fields with a reservoir consisting of a continuum of the Klein-Gordon fields. In this model, eigen-operators for the coupled systems are calculated and the electromagnetic field operators have been expressed in terms of these eigen-operators. Also, the dielectric function is derived and it is shown that it satisfies the Kramers-Kronig relations [12].

Another approach to quantizing a dissipative system is by considering the dissipation as a result of interaction between the system and a heat bath consisting of a set of harmonic oscillators [13]-[25]. In this method the whole system is composed of two parts, the main system and a heat bath which interacts with the main system and causes the dissipation of energy on it.

In a recent approach to electromagnetic field quantization the present
authors have quantized the electromagnetic field in an isotropic magnetodielectric [26]. In this approach: (i) the electromagnetic field is taken as the main quantum system and the medium as a heat bath. (ii) The polarizability of the medium is defined in terms of dynamical variables of the medium. (iii) The polarizability and absorptivity of the medium are not independent of each other, as expected, this is contrary to the damped polarization model where polarizability and absorptivity are treated independently [10, 11]. (iv) If the medium is both magnetizable and polarizable, one must models the medium with two independent collections of harmonic oscillators, where one collection describes electric properties and the other one describes magnetic properties of the medium. This scheme leads to a consistent quantization of the electromagnetic field in the presence of an absorptive magnetodielectric [26].

In the present article, the idea introduced in the previous work [26] is generalized to the case of an anisotropic and inhomogeneous magnetodielectric.

2 Quantum dynamics

Electromagnetic field quantization can be achieved in an anisotropic magnetodielectric by modelling the medium with two independent quantum fields. Let us call these fields E and M quantum fields, describing the polarizability and magnetizability of the medium respectively. These quantum fields couple the medium with electromagnetic field through some coupling tensors. The electric and magnetic polarization densities of the medium are defined as linear expansions in terms of the ladder operators of the E and M fields. The coefficients of these expansions are real valued coupling tensors. We will see that the electric and magnetic susceptibility tensors can be obtained in terms of the coupling tensors. In the following we use the Coloumb gauge and assume the periodic boundary conditions with no loss of generality of the approach.

The electromagnetic vector potential \( \vec{A} \) inside a box with volume \( V = L_1 L_2 L_3 \) can be expanded in terms of plane waves as

\[
\vec{A}(\vec{r}, t) = \sum_{\vec{n}} \sum_{\lambda=1}^{2} \sqrt{\frac{\hbar}{2 \varepsilon_0 V \omega_{\vec{n}}}} \left[ a_{\vec{n}\lambda}(t) e^{i\vec{k}_{\vec{n}} \cdot \vec{r}} + a_{\vec{n}\lambda}^\dagger(t) e^{-i\vec{k}_{\vec{n}} \cdot \vec{r}} \right] \vec{e}_{\vec{n}\lambda},
\]

where \( \omega_{\vec{n}} = c|\vec{k}_{\vec{n}}| \) is the frequency corresponding to the mode \( \vec{n} \), the vector
$\vec{n}$ is a triplet of integer numbers $(n_1, n_2, n_3)$, $\sum_{\vec{n}}$ means $\sum_{n_1,n_2,n_3=-\infty}^{+\infty}$, $\varepsilon_0$ is the permittivity of the vacuum, $\vec{k}_{\vec{n}} = \frac{2n_1\pi}{L_1} \hat{i} + \frac{2n_2\pi}{L_2} \hat{j} + \frac{2n_3\pi}{L_3} \hat{k}$ is the wave vector and $\vec{e}_{\vec{n}\lambda}$ are polarization unit vectors for each $\vec{n}$, satisfying

\[ \vec{e}_{\vec{n}\lambda}, \vec{e}_{\vec{n}\lambda'} = \delta_{\lambda\lambda'}, \]
\[ \vec{e}_{\vec{n}\lambda} \cdot \vec{k}_{\vec{n}} = 0. \] (2)

The operators $a_{\vec{n}\lambda}(t)$ and $a_{\vec{n}\lambda}^\dagger(t)$ are annihilation and creation operators of the electromagnetic field and satisfy the following equal time commutation rules

\[ [a_{\vec{n}\lambda}(t), a_{\vec{m}\lambda'}^\dagger(t)] = \delta_{\vec{n},\vec{m}} \delta_{\lambda\lambda'}. \] (3)

Quantization in Coloumb gauge usually needs resolution of a vector field in its transverse and longitudinal parts. Any vector field $\vec{F}(\vec{r})$ can be resolved in two components, transverse and longitudinal component which are denoted by $\vec{F}^\perp$ and $\vec{F}^\parallel$ respectively. The transverse part satisfy the coloumb condition $\nabla \cdot \vec{F}^\perp = 0$ and the longitudinal component is a conservative field $\nabla \times \vec{F}^\parallel = 0$.

For a periodic boundary condition these two parts are defined as

\[ \vec{F}^\perp(\vec{r}, t) = \vec{F}(\vec{r}, t) + \int_V d^3r' \nabla' \cdot \vec{F}(\vec{r}', t) \vec{\nabla} G(\vec{r}, \vec{r}'), \] (4)
\[ \vec{F}^\parallel(\vec{r}, t) = - \int_V d^3r' \nabla' \cdot \vec{F}(\vec{r}', t) \vec{\nabla} G(\vec{r}, \vec{r}'), \] (5)

where

\[ G(\vec{r}, \vec{r}') = \sum_{\vec{n}} \frac{1}{|\vec{k}_{\vec{n}}|^2} e^{i\vec{k}_{\vec{n}}(\vec{r} - \vec{r}')}, \] (6)

is the Green function and satisfies the Poison equation

\[ \nabla^2 G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}'). \] (7)

In absence of external charges the displacement field is purely transverse, and we can expand it in terms of the plane waves

\[ \vec{D}(\vec{r}, t) = -i\varepsilon_0 \sum_{\vec{n}} \sum_{\lambda=1}^2 \sqrt{\frac{\hbar \omega_{\vec{n}}}{2\varepsilon_0 V}} \left[ a_{\vec{n}\lambda}^\dagger(t) e^{-i\vec{k}_{\vec{n}} \cdot \vec{r}} - a_{\vec{n}\lambda}(t) e^{i\vec{k}_{\vec{n}} \cdot \vec{r}} \right] \vec{e}_{\vec{n}\lambda}. \] (8)
The commutation relations (3) lead to the following commutation relations between the components of the vector potential $\vec{A}$ and the displacement operator $\vec{D}$

$$[A_l(\vec{r}, t), -D_j(\vec{r}, t)] = i\hbar \delta^\perp_{lj}(\vec{r} - \vec{r}'),$$

where

$$\delta^\perp_{lj}(\vec{r} - \vec{r}') = \frac{1}{V} \sum_{\vec{n}} (\delta_{lj} - \frac{k_{\vec{n}}l_k}{|\vec{k}_{\vec{n}}|^2}) e^{i\vec{k}_{\vec{n}} \cdot (\vec{r} - \vec{r}')} ,$$

is the transverse delta function. From (10), we see that $-\vec{D}$ plays the role of the momentum density of electromagnetic field. The Hamiltonian of the electromagnetic field inside the box is given by

$$H_F(t) = \int_V d^3r \left[ \frac{\vec{D}^2}{2\varepsilon_0} + \frac{(\nabla \times \vec{A})^2}{2\mu_0} \right] = \sum_{\vec{n}} \sum_{\lambda=1}^2 \hbar \omega_{\vec{n}\lambda} a^\dagger_{\vec{n}\lambda}(t)a_{\vec{n}\lambda}(t).$$

(11)

where $\mu_0$ is the magnetic permittivity of the vacuum and we have used normal ordering for $a^\dagger_{\vec{n}\lambda}(t)$ and $a_{\vec{n}\lambda}(t)$.

Now we include the medium in the process of quantization. For this purpose let the Hamiltonian corresponding to E and M quantum fields be denoted by $H_e$ and $H_m$ respectively. Then the medium Hamiltonian can be written as

$$H_d = H_e + H_m,$$

$$H_e(t) = \sum_{\vec{n}} \sum_{\nu=1}^3 \int_{-\infty}^{+\infty} d^3q \hbar \omega_{\vec{n}\nu} d^\dagger_{\vec{n}\nu}(\vec{q}, t)d_{\vec{n}\nu}(\vec{q}, t),$$

$$H_m(t) = \sum_{\vec{n}} \sum_{\nu=1}^3 \int_{-\infty}^{+\infty} d^3q \hbar \omega_{\vec{n}\nu} b^\dagger_{\vec{n}\nu}(\vec{q}, t)b_{\vec{n}\nu}(\vec{q}, t),$$

(12)

where the annihilation and creation operators $d_{\vec{n}\nu}(\vec{q}, t)$, $d^\dagger_{\vec{n}\nu}(\vec{q}, t)$, $b_{\vec{n}\nu}(\vec{q}, t)$ and $b^\dagger_{\vec{n}\nu}(\vec{q}, t)$ satisfy the following equal-time commutation relations

$$[d_{\vec{n}\nu}(\vec{q}, t), d^\dagger_{\vec{m}\nu'}(\vec{q}', t)] = \delta_{\vec{n},\vec{m}} \delta_{\nu\nu'} \delta(\vec{q} - \vec{q}'),$$

$$[b_{\vec{n}\nu}(\vec{q}, t), b^\dagger_{\vec{m}\nu'}(\vec{q}', t)] = \delta_{\vec{n},\vec{m}} \delta_{\nu\nu'} \delta(\vec{q} - \vec{q}').$$

(13)
In relations (12), $\omega_{\vec{q}}$ is the dispersion relation of the magnetodielectric. It is remarkable to note that, although the medium is anisotropic in its electric and magnetic properties, we do not need to take the dispersion relation as a tensor. As discussed in [26], we can assume a linear dispersion relation $\omega_{\vec{q}} = c|\vec{q}|$ with no loss of generality, but taking a linear dispersion relation simplifies the formulas considerably. Therefore from now on we choose the dispersion relation as $\omega_{\vec{q}} = c|\vec{q}|$ where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is the proportionality constant.

The basic idea in this quantization method is that the electric and magnetic properties of an anisotropic magnetodielectric can be described by $E$ and $M$ quantum fields. This means that we can define the electric and magnetic polarization densities of a linear but anisotropic medium as linear combinations of the ladder operators of the $E$ and $M$ quantum fields, respectively. Therefore

$$P_i(\vec{r}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^3\vec{q} f_{ij}(\omega_{\vec{q}}, \vec{r}) \left[ d_{\vec{n}\nu}(\vec{q}, t) e^{i\vec{k}_\nu \cdot \vec{r}} + h.c. \right] v_{\vec{n}\nu}^j,$$

(14)

$$M_i(\vec{r}, t) = \frac{i}{\sqrt{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^3\vec{q} g_{ij}(\omega_{\vec{q}}, \vec{r}) \left[ b_{\vec{n}\nu}(\vec{q}, t) e^{-i\vec{k}_\nu \cdot \vec{r}} - h.c. \right] s_{\vec{n}\nu}^j,$$

(15)

where $\vec{P}$ and $\vec{M}$ are electric and magnetic polarization densities of the medium and

$$\vec{v}_{\vec{n}\nu} = \vec{c}_{\vec{n}\nu}, \quad \nu = 1, 2$$

$$\vec{s}_{\vec{n}\nu} = \vec{k}_{\vec{n}} \times \vec{c}_{\vec{n}\nu}, \quad \nu = 1, 2$$

$$\vec{v}_{\vec{n}3} = \vec{s}_{\vec{n}3} = \vec{k}_{\vec{n}}$$

$$\vec{k}_{\vec{n}} = \frac{\vec{k}_{\vec{n}}}{|\vec{k}_{\vec{n}}|}.$$

(16)

In definitions of polarization densities (14) and (15), the real valued tensors $f_{ij}(\omega_{\vec{q}}, \vec{r})$ and $g_{ij}(\omega_{\vec{q}}, \vec{r})$, are called the coupling tensors of the electromagnetic field and the medium which are dependent (independent) on position $\vec{r}$ for inhomogeneous (homogeneous) magnetodielectrics. The coupling tensors
play the key role in this method and are a measure for the strength of the polarizability and magnetizability of the medium macroscopically. We will see that the imaginary parts of the electric and magnetic susceptibility in frequency domain can be obtained in terms of these coupling tensors. Also, explicit forms for the noise polarization densities can be obtained in terms of the coupling tensors and the ladder operators of the medium. The coupling tensors are common factors in the noise densities and the electric and magnetic susceptibilities, and so the strength of the noise densities are dependent on the strength of the electric and magnetic susceptibility. It can be shown that for a non absorptive medium, the noise densities tend to zero as expected and this quantization scheme reduces to the usual quantization in such media.

A consistent quantization scheme must lead to the correct equations of motion of the system and the medium. These equations are macroscopic Maxwell and constitutive equations of the medium and we will see that these equations can be obtained from the Heisenberg equations using the total Hamiltonian defined by

$$\tilde{H}(t) = \int d^3r \left\{ \frac{[\tilde{D}(\vec{r}, t) - \tilde{P}(\vec{r}, t)]^2}{2\varepsilon_0} + \frac{(\nabla \times \vec{A})^2(\vec{r}, t)}{2\mu_0} - \nabla \times \vec{A}(\vec{r}, t) \cdot \vec{M}(\vec{r}, t) \right\} + H_e + H_m.$$  \hspace{1cm} (17)

### 2.1 Maxwell equations

Using the commutation relations (9) the Heisenberg equations for the vector potential $\vec{A}$ and the displacement field $\vec{D}$ are

$$\frac{\partial \vec{A}(\vec{r}, t)}{\partial t} = \frac{i}{\hbar} [\tilde{H}, \vec{A}(\vec{r}, t)] = -\frac{\tilde{D}(\vec{r}, t) - \tilde{P}^\perp(\vec{r}, t)}{\varepsilon_0},$$  \hspace{1cm} (18)

$$\frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \frac{i}{\hbar} [\tilde{H}, \vec{D}(\vec{r}, t)] = \frac{\nabla \times \nabla \times \vec{A}(\vec{r}, t)}{\mu_0} - \nabla \times \vec{M}(\vec{r}, t),$$  \hspace{1cm} (19)

where $\tilde{P}^\perp$ is the transverse component of $\tilde{P}$. The transverse electrical field $\tilde{E}^\perp$, magnetic induction $\tilde{B}$ and magnetic field $\tilde{H}$ are defined by

$$\tilde{E}^\perp = -\frac{\partial \vec{A}}{\partial t}, \quad \tilde{B} = \nabla \times \vec{A}, \quad \tilde{H} = \frac{\tilde{B}}{\mu_0} - \vec{M}.$$  \hspace{1cm} (20)
Using these recent relations, (18) and (19) can be rewritten as

\[ \vec{D} = \varepsilon_0 \vec{E}^\perp + \vec{P}^\perp, \]  

(21)

\[ \frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H}, \]  

(22)

which are the definitions of the displacement field and the macroscopic Maxwell equation, in the presence of a magnetodielectric, respectively.

2.2 Constitutive equations of the medium

A magnetodielectric subjected to electromagnetic field can be polarized and magnetized in consequence of interaction of the medium with the field. The macroscopic electric and magnetic polarizations is related to electric and magnetic fields, respectively by the constitutive equations of the medium. Therefore a quantization scheme must be able to give the constitutive equations in the Heisenberg picture. In this section by applying the Heisenberg equations to the ladder operators of the medium we find the correct constitutive equations of the medium.

The time evolution of the operators \( d_{\vec{n}\nu}(\vec{q}, t) \) and \( b_{\vec{n}\nu}(\vec{q}, t) \) can be obtained from the commutation relations (13) and the Hamiltonian (17) as follows

\[ \dot{d}_{\vec{n}\nu}(\vec{q}, t) = \frac{i}{\hbar} [\vec{H}, d_{\vec{n}\nu}(\vec{q}, t)] = \]

\[ -\omega_\vec{q} d_{\vec{n}\nu}(\vec{q}, t) + \frac{i}{\hbar V} \int_V d^3 r' e^{-i\vec{k}_\vec{q} \cdot \vec{r}'} f_{ij}(\omega_\vec{q}, \vec{r}') E^i(\vec{r}', t) v^j_{\vec{n}\nu}, \]  

(23)

\[ \dot{b}_{\vec{n}\nu}(\vec{q}, t) = \frac{i}{\hbar} [\vec{H}, b_{\vec{n}\nu}(\vec{q}, t)] = \]

\[ -i\omega_\vec{q} b_{\vec{n}\nu}(\vec{q}, t) + \frac{1}{\hbar V} \int_V d^3 r' e^{-i\vec{k}_\vec{q} \cdot \vec{r}'} g_{ij}(\omega_\vec{q}, \vec{r}') B^i(\vec{r}', t) s^j_{\vec{n}\nu}. \]  

(24)

It is easy to show that these equations have the following formal solutions

\[ d_{\vec{n}\nu}(\vec{q}, t) = d_{\vec{n}\nu}(\vec{q}, 0) e^{-i\omega_\vec{q} t} + \]

\[ \frac{i}{\hbar V} \int_0^t dt' e^{-i\omega_\vec{q} (t-t')} \int_V d^3 r' e^{-i\vec{k}_\vec{q} \cdot \vec{r}'} f_{ij}(\omega_\vec{q}, \vec{r}') E^i(\vec{r}', t') v^j_{\vec{n}\nu}, \]  

(25)
Substituting (25) in (14) and (26) in (15) we obtain the macroscopic constitutive equations of the anisotropic polarizable and magnetizable medium,

\[
\vec{P}(\vec{r}, t) = \vec{P}_N(\vec{r}, t) + \varepsilon_0 \int_{t}^{t'} dt' \chi_e(\vec{r}, |t| - t') \vec{E}(\vec{r}, \pm t'),
\]

\[
\vec{M}(\vec{r}, t) = \vec{M}_N(\vec{r}, t) + \frac{1}{\mu_0} \int_{t}^{t'} dt' \chi_m(\vec{r}, |t| - t') \vec{B}(\vec{r}, \pm t'),
\]

where the upper (lower) sign corresponds to \( t > 0 \) (\( t < 0 \)) and \( \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \frac{\vec{P}}{\varepsilon_0} \) is the total electrical field.

The memory tensors

\[
\chi^e(\vec{r}, t) = \begin{cases} \frac{8\pi}{\hbar c^2 \varepsilon_0} \int_0^\infty d\omega \omega^2 (ff^t)(\omega, \vec{r}) \sin \omega t & t > 0, \\ 0 & t \leq 0, \end{cases}
\]

\[
\chi^m(\vec{r}, t) = \begin{cases} \frac{8\pi}{\hbar c^2 \mu_0} \int_0^\infty d\omega \omega^2 (gg^t)(\omega, \vec{r}) \sin \omega t & t > 0, \\ 0 & t \leq 0, \end{cases}
\]

are called the electric and magnetic susceptibility tensors of the magnetodielectric, respectively and \( f^t, g^t \) denote the transpose of the coupling tensors \( f, g \). If we are given a definite pair of tensors \( \chi^e(\vec{r}, t), \chi^m(\vec{r}, t) \) which are zero for \( t \leq 0 \), then we can inverse (29) and (30) and obtain the corresponding tensors \((ff^t)\) and \((gg^t)\) as,

\[
(ff^t)(\omega, \vec{r}) = \begin{cases} \frac{\hbar c^3 \varepsilon_0}{4\pi^2 \omega^2} \int_0^\infty dt \chi^e(\vec{r}, t) \sin \omega t = \frac{\hbar c^3 \varepsilon_0}{4\pi^2 \omega^2} \text{Im} \left[ \chi^e(\vec{r}, \omega) \right] & \omega > 0, \\ 0 & \omega = 0, \end{cases}
\]
\[(gg^\dagger)(\omega, \vec{r}) = \begin{cases} \frac{\hbar c^3}{4 \pi^2 \mu_0 \omega^2} \int_0^\infty dt \chi^m(\vec{r}, t) \sin \omega t = \frac{\hbar c^3}{4 \pi^2 \mu_0 \omega^2} \text{Im} \left[ \chi^m(\vec{r}, \omega) \right] & \omega > 0, \\ 0 & \omega = 0, \end{cases} (32)\]

where \(\chi^e(\vec{r}, \omega)\) and \(\chi^m(\vec{r}, \omega)\) are the susceptibility tensors in the frequency domain. The operators \(\vec{P}_N\) and \(\vec{M}_N\) in (27) and (28) are the noise electric and magnetic polarization densities

\[P_{Ni}(\vec{r}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{n}} \sum_{\nu=1}^3 \int d^3 q f_{ij}(\omega \vec{q}, \vec{r}) \left[ d_{\vec{n}\nu}(\vec{q}, 0) e^{-i\omega \vec{q} \cdot \vec{r} + \vec{k}_{\vec{n}} \cdot \vec{r}} + \text{h.c.} \right] v^i_{\vec{n}\nu}, (33)\]

\[M_{Ni}(\vec{r}, t) = \frac{i}{\sqrt{V}} \sum_{\vec{n}} \sum_{\nu=1}^3 \int d^3 q g_{ij}(\omega \vec{q}, \vec{r}) \left[ b_{\vec{n}\nu}(\vec{q}, 0) e^{-i\omega \vec{q} \cdot \vec{r} - \vec{k}_{\vec{n}} \cdot \vec{r}} - \text{h.c.} \right] s^j_{\vec{n}\nu}, (34)\]

These noises are necessary for a consistent quantization of the electromagnetic field in the presence of an absorptive medium.

From (31) and (32) it is clear that for a given pair of the susceptibility tensors \(\chi^e\) and \(\chi^m\) there are infinite number of coupling tensors \(f\) and \(g\) satisfying the equations (31) and (32). In fact for a given pair of \(\chi^e\) and \(\chi^m\) if the tensors \(f\) and \(g\) satisfy equations (31) and (32), then the coupling tensors \(fA\) and \(gA\), for any orthogonal matrix \((AA^t = 1)\), are also a solution. Certainly this affect the space-time dependence of the noise polarizations and therefore the space-time dependence of the electromagnetic field operators, but all of these choices are equivalent. This means that the various choices of the coupling tensors \(f\) and \(g\) satisfying (31) and (32), for a given pair of the susceptibilities \(\chi^e\) and \(\chi^m\), do not affect the commutation relations between the field operators and hence the physical observables. This becomes more clear if we compute the commutation relations between the components of the Fourier transform of the noise polarizations.
\[ [\mathcal{P}_{Ni}(\vec{r}, \omega), \mathcal{P}_{Nj}^\dagger(\vec{r}', \omega')] = \frac{\hbar \varepsilon_0}{\pi} \text{Im} \left[ \chi_{ij}(\vec{r}, \omega) \right] \delta(\vec{r} - \vec{r}')\delta(\omega - \omega'), \]
\[ [\mathcal{M}_{Ni}(\vec{r}, \omega), \mathcal{M}_{Nj}^\dagger(\vec{r}', \omega')] = \frac{\hbar}{\mu_0 \pi} \text{Im} \left[ \chi_{mij}(\vec{r}, \omega) \right] \delta(\vec{r} - \vec{r}')\delta(\omega - \omega'). \]

(35)

These relations are generalization of those in reference [27]. For a given pair of \( \chi^e \) and \( \chi^m \), various choices of the coupling tensors \( f \) and \( g \) satisfying the relations (31) and (32), do not affect these commutation relations and accordingly the commutation relations between the electromagnetic field operators. Hence, all of the field operators which are obtained by using a definite pair of the susceptibilities \( \chi^e \) and \( \chi^m \), with different coupling tensors, satisfying (31) and (32), are equivalent.

3 Solution of Heisenberg equations

The Maxwell and constitutive equations of the medium constitute a set of coupled equations. In this section we solve them in terms of their initial conditions using the Laplace transformation technique. For any time-dependent operator \( g(t) \) the forward and backward Laplace transformation of \( g(t) \) are defined by

\[ g^f(s) = \int_0^\infty dt g(t)e^{-st}, \]
\[ g^b(s) = \int_0^\infty dt g(-t)e^{-st}, \]

(36)

respectively. Carrying out the forward and backward Laplace transformation of the Maxwell equation (22) and the constitutive equations (21), (27) and (28) we find

\[ \nabla \times \nabla \times \vec{E}^f, b(\vec{r}, s) + \mu_0 \varepsilon_0 s^2 \vec{\varepsilon}(\vec{r}, s) \vec{E}^f, b(\vec{r}, s) - \nabla \times \bar{\chi}^m(\vec{r}, s) \nabla \times \vec{E}^f, b(\vec{r}, s) = \vec{J}^f, b(\vec{r}, s), \]

(37)

where \( \vec{\varepsilon}(\vec{r}, s) = 1 + \bar{\chi}^e(\vec{r}, s) \) and \( \bar{\chi}^m(\vec{r}, s) \) are the Laplace transformations of the electric permeability tensor and magnetic susceptibility tensor of the medium, respectively and
\( \vec{J}^{f,b}(\vec{r}, s) = \pm \nabla \times \vec{B}(\vec{r}, 0) - \mu_0 s^2 \vec{H}_N^{f,b}(\vec{r}, s) \mp \mu_0 s \nabla \times \vec{M}_N^{f,b}(\vec{r}, s) - \nabla \times \tilde{\chi}^m(\vec{r}, s) \vec{B}(\vec{r}, 0) + \mu_0 s \vec{D}(\vec{r}, 0), \) (38)

is the forward and backward Laplace transformation of the noise current where upper(lower) sign corresponds to \( \vec{J}^f(\vec{r}, s) \) \((\vec{J}^b(\vec{r}, s))\). The wave equation (37) can be solved using the Green tensor method [27]. To see the space-time dependence of electric field more explicitly, let us consider a homogeneous but anisotropic bulk medium. In this case by expanding \( \vec{E}^{f,b}(\vec{r}, s) \) and \( \vec{J}^{f,b}(\vec{r}, s) \) in terms of plane waves as

\[
\vec{E}^{f,b}(\vec{k}, s) = \frac{1}{\sqrt{V}} \sum_{\vec{n}} \hat{\vec{E}}^{f,b}(\vec{k}, s) e^{i\vec{k} \cdot \vec{n}},
\]

\[
\vec{J}^{f,b}(\vec{k}, s) = \frac{1}{\sqrt{V}} \sum_{\vec{n}} \hat{\vec{J}}^{f,b}(\vec{k}, s) e^{i\vec{k} \cdot \vec{n}},
\]

and inserting this expansions in the wave equation (37) we obtain

\[
\Lambda(\vec{k}, s) \hat{\vec{E}}^{f,b}(\vec{k}, s) = \hat{\vec{J}}^{f,b}(\vec{k}, s),
\]

\[
\Lambda_{ij}(\vec{k}, s) = -\varepsilon_{i\mu\nu} \varepsilon_{\alpha\beta j} [\delta_{\nu\alpha} - \tilde{\chi}^m(\vec{k}, s)] k^\mu k^\beta + \mu_0 \varepsilon_0 s^2 \tilde{\varepsilon}_{ij}(s), \quad (40)
\]

where \( \varepsilon_{i\mu\nu} \) is the Levi-Civita symbol. we can use the expansions (1), (8), (33) and (34) to obtain the operator \( \hat{\vec{J}}^{f,b}(\vec{k}, s) \) in terms of the ladder operators of the electromagnetic field and the magnetodielectric medium. Finally using (39) and (40) after some elaborated calculations we obtain the space-time dependence of the electric field in terms of the ladder operators of the field and medium as

\[
E_i(\vec{r}, t) = \sum_n \sum_{\lambda}^2 \sqrt{\frac{h \omega_n \mu_0}{2cV}} \left[ \eta_{ij}^+(\vec{k}, t) a_{\alpha\lambda}(0) e^{i\vec{k} \cdot \vec{r}} + h.c. \right] e^j_{\alpha\lambda},
\]

\[
-\frac{\mu_0}{\sqrt{V}} \sum_n \sum_{\lambda}^3 \int d^3 q \left[ \xi_{ij}^+(\omega, \vec{k}, \vec{q}, t) d_{\alpha\lambda}(\vec{q}) e^{i\vec{k} \cdot \vec{r}} + h.c. \right] v^j_{\alpha\lambda},
\]

\[
\pm \frac{\mu_0}{\sqrt{V}} \sum_n \sum_{\lambda}^3 \int d^3 q \left[ \xi_{ij}^+(\omega, \vec{k}, \vec{q}, t) b_{\alpha\lambda}(\vec{q}) e^{i\vec{k} \cdot \vec{r}} + h.c. \right] s^j_{\alpha\lambda}, \quad (41)
\]
where the upper (lower) sign corresponds to $t > 0$ $(t < 0)$ and $\eta_{ij}^{\pm}(\vec{k}, \mp t)$, $\xi_{ij}^{\pm}(\vec{q}, \vec{k}, \pm t)$, $\zeta_{ij}^{\pm}(\vec{q}, \vec{k}, \mp t)$, $\xi_{ij}^{\pm}(\vec{q}, \vec{k}, \pm t)$ and $\zeta_{ij}^{\pm}(\vec{q}, \vec{k}, \mp t)$ for $t > 0$ are given by

$$
\eta_{ij}^{\pm}(\vec{k}, \pm t) = L^{-1}\left\{\Lambda^{-1}_{il}(\vec{k}, s) \left[ (ts \pm \omega) \delta_{ij} \pm \omega \bar{\varepsilon}_{lmu} \varepsilon_{\alpha \beta j} \hat{k}^\alpha \hat{k}^\beta \chi_{\mu \nu} (s) \right] \right\},
$$

$$
\xi_{ij}^{\pm}(\vec{k}, \pm t) = L^{-1}\left\{\Lambda^{-1}_{il}(\vec{k}, s) \frac{s^2}{s \pm \omega q} \right\} f_{lj}(\omega q),
$$

$$
\zeta_{ij}^{\pm}(\vec{k}, \pm t) = L^{-1}\left\{\Lambda^{-1}_{il}(\vec{k}, s) \frac{s}{s \pm \omega q} \right\} \varepsilon_{\alpha \beta j} k^\alpha k^\beta g_{lj}(\omega q),
$$

(42)

and $L^{-1}\{h(s)\}$ denotes the inverse Laplace transformation of $h(s)$ and $\Lambda^{-1}$ is the inverse of the matrix $\Lambda$.

**Example 1:**
In the first example we show that in the absence of any medium this quantization scheme reduces to the usual quantization in the vacuum. In free space the electric and magnetic susceptibility tensors are zero and from (31) and (32) we deduce that the coupling tensors $f$ and $g$ are also zero. Therefore from (40), (41) and (42) one finds

$$
\vec{E}(\vec{r}, t) = i \sum_{\vec{n}} \sum_{\lambda=1}^{2} \sum_{\lambda=0}^{\triangle} \sqrt{\frac{\hbar \omega_{\vec{n}}}{2\varepsilon_0 V}} \left[ a_{\vec{n}\lambda}(0)e^{-\omega_{\vec{n}}t+i\vec{k}_{\vec{n}} \cdot \vec{r}} - h.c. \right] \vec{e}_{\vec{n}\lambda},
$$

which is the electric field in the free space. So in this case, quantization of electromagnetic field is reduced to the usual quantization in the vacuum as expected.

**Example 2:**
Take the susceptibility tensors $\chi^e$ and $\chi^m$ as follows

$$
\chi^e(\vec{r}, t) = \chi^e_0(\vec{r}) \times \left\{ \begin{array}{ll}
\frac{1}{\triangle} & 0 < t < \triangle, \\
0 & \text{otherwise},
\end{array} \right.
$$

$$
\chi^m(\vec{r}, t) = \chi^m_0(\vec{r}) \times \left\{ \begin{array}{ll}
\frac{1}{\triangle} & 0 < t < \triangle, \\
0 & \text{otherwise},
\end{array} \right.
$$

13
where \( \chi_0^e(\vec{r}) \) and \( \chi_0^m(\vec{r}) \) are some time independent but position dependent tensors and \( \Delta \) is a real positive constant, using (31) and (32) we find

\[
(f f')(\omega, \vec{r}) = \frac{\hbar c^3}{4\pi^2\omega^2} \frac{\sin^2 \frac{\omega \Delta}{2}}{\omega \Delta} \chi_0^e(\vec{r}),
\]

\[
(g g')(\omega, \vec{r}) = \frac{\hbar c^3}{4\pi^2\omega^2 \mu_0} \frac{\sin^2 \frac{\omega \Delta}{2}}{\omega \Delta} \chi_0^m(\vec{r}),
\]

(44)

and from (27) and (28)

\[
\vec{P}(\vec{r}, t) = \vec{P}_N(\vec{r}, t) + \chi_0^e(\vec{r}) \frac{\varepsilon_0}{\Delta} \int_{|t| - \Delta}^{|t|} dt' \vec{E}(\vec{r}, \pm t'),
\]

\[
\vec{M}(\vec{r}, t) = \vec{M}_N(\vec{r}, t) + \chi_0^m(\vec{r}) \frac{1}{\mu_0 \Delta} \int_{|t| - \Delta}^{|t|} dt' \vec{B}(\vec{r}, \pm t'),
\]

(45)

where \( \vec{P}_N(\vec{r}, t), \vec{M}_N(\vec{r}, t) \) are the noise polarization densities correspond to a pair coupling tensors \( f \) and \( g \) satisfying (14). In the limit \( \Delta \to 0 \), from (14) we deduce that the coupling tensors and therefore the noise polarization densities defined by (33) and (34) tend to zero. In this case the constitutive equations (45) are

\[
\vec{P}(\vec{r}, t) = \varepsilon_0 \chi_0^e(\vec{r}) \vec{E}(\vec{r}, t),
\]

\[
\vec{M}(\vec{r}, t) = \frac{1}{\mu_0} \chi_0^m(\vec{r}) \vec{B}(\vec{r}, t),
\]

(46)

and the wave equation (37) becomes

\[
\nabla \times \nabla \times \vec{E}^{f,b}(\vec{r}, s) + \mu_0 \varepsilon_0 s^2 [1 + \chi_0^e(\vec{r})] \vec{E}^{f,b}(\vec{r}, s) - \mu_0 \varepsilon_0 \vec{H}^{f,b}(\vec{r}, s) = \vec{J}^{f,b}(\vec{r}, s),
\]

(47)

where the noise current density (38) is
\[ J^{f,b}(\vec{r}, s) = \pm \nabla \times \vec{B}(\vec{r}, 0) \mp \nabla \times \chi_0^m(\vec{r}) \vec{B}(\vec{r}, 0) + \mu_0 s \vec{D}(\vec{r}, 0). \] (48)

We see that the noise operators have vanished. This is because in the limit \( \Delta \to 0 \), the absorption coefficients tend to zero. The solution of the wave equation (47) can be expressed in terms of the Green tensor as

\[ \vec{E}^{f,b}(\vec{r}, s) = \int d^3 r' G(\vec{r}, \vec{r}', s) \vec{J}^{f,b}(\vec{r}', s), \] (49)

where the Green tensor satisfies the equation

\[ \nabla \times \nabla \times G(\vec{r}, \vec{r}', s) + \mu_0 \varepsilon_0 s^2 [1 + \chi_0^e(\vec{r})] G(\vec{r}, \vec{r}', s) - \nabla \times \chi_0^m(\vec{r}) \nabla \times G(\vec{r}, \vec{r}', s) = \delta(\vec{r} - \vec{r}'), \] (50)

together with some boundary conditions. These boundary conditions guarantee the continuity of the tangential component of electric field and the normal component of magnetic field at some surfaces where the susceptibilities of the medium become discontinuous. For an anisotropic homogeneous medium using (40), (41) and (42) we can write the electric field as follows

\[ E_i(\vec{r}, t) = \sum_{\vec{n}} \sum_{\lambda=1}^{2} \sqrt{\frac{\hbar \omega_{\vec{n}} \mu_0}{2 c V}} \left[ \eta_{ij}(\vec{k}_{\vec{n}}, t) a_{\vec{n}\lambda}(0) e^{i \vec{k}_{\vec{n}} \cdot \vec{r}} + h.c. \right] e^{j}, \] (51)

where

\[ \eta_{ij}(\vec{k}_{\vec{n}}, \pm t) = L^{-1} \left\{ \Lambda^{-1}_{il}(\vec{k}_{\vec{n}}, s) \left[ \delta_{ij} \pm \omega_{\vec{n}} \delta_{ij} \pm \omega_{\vec{n}} \varepsilon_{ij\mu} \varepsilon_{\alpha\beta\lambda} k^\alpha_{\vec{n}} k^\beta_{\vec{n}} (\chi_0^m)_{\nu\alpha} \right] \right\}, \]
\[ \Lambda_{il}(\vec{k}_{\vec{n}}, s) = -\varepsilon_{ij\mu} \varepsilon_{\alpha\beta\lambda} [\delta_{\nu\alpha} - (\chi_0^m)_{\nu\alpha}] k^\mu_{\vec{n}} k^\beta_{\vec{n}} + \mu_0 \varepsilon_0 s^2 (1 + (\chi_0^e)_{il}). \] (52)

This example shows that the present quantization scheme is reduced to the usual quantization in a nonabsorptive medium.

**Example 4: A simple model for electric susceptibility tensor**

If we neglect the difference between local and macroscopic electric field for substances with a low density, then the classical equation of a bound atomic electron in an external electric field for small oscillation can be written as
\[ 
\ddot{x}_i + 2\gamma \dot{x}_i + K_{ij} x_j = -\frac{e}{m} \vec{E}_i(t), \quad i = 1, 2, 3, 
\] (53)

where \( \vec{E}(t) \) is the electric field in the place of the atom and the magnetic force has been neglected in comparison with the electric one and \( \gamma \) is a damping coefficient. We have assumed that for sufficiently small oscillations the ith component of the force exerted on the bound electron by nucleus, can be expressed as a linear combination of the coordinates of the electron with constant coefficients \( K_{ij} \). Therefore in this simple model the motion of the bound electron is as a forced anisotropic harmonic oscillator. Let \( \vec{E}(\omega) \) and \( \vec{r}(\omega) \) be Fourier transforms of the electrical field \( \vec{E}(t) \) and position \( \vec{r}(t) \) respectively. From (53) we find

\[ 
\vec{r}(\omega) = \frac{-e}{m} \left[ (-\omega^2 + 2i\gamma\omega)1 + K \right]^{-1} \vec{E}(\omega). 
\] (54)

Now let there be \( N \) molecules per unit volume of the medium with \( z \) electrons per molecule. We assume that the damping coefficient (\( \gamma \)) and the tensor \( K \) are identical for each electron. Then for the Fourier transform of the electric polarization density we find

\[ 
\vec{P}(\vec{r}, \omega) = \frac{Ne^2}{m} \left[ (-\omega^2 + 2i\gamma\omega)1 + K \right]^{-1} \vec{E}(\vec{r}, \omega). 
\] (55)

From (55) we find the electric susceptibility tensor of the medium in frequency domain

\[ 
\chi^e(\omega) = \frac{Ne^2}{m\varepsilon_0} \left[ (-\omega^2 + 2i\gamma\omega)1 + K \right]^{-1}. 
\] (56)

From (31) we have

\[ 
(f f^t)(\omega) = \left\{ \begin{array}{ll} 
\frac{\hbar e^2}{4\pi^2\omega^2} 2\gamma\omega \left[ (-\omega^2 1 + K)^2 + 4\gamma^2\omega^2 1 \right]^{-1} & \omega \neq 0, \\
0 & \omega = 0
\end{array} \right. 
\] (57)

In the special case \( \gamma = 0 \), the anisotropic dielectric substance is a nonabsorptive one and this relation for \( \omega \neq 0 \) becomes

\[ 
(f f^t)(\omega) = \frac{\hbar e^2\varepsilon_0}{4\pi\omega^2} \sum_{i=1}^{3} \delta(\omega - \omega_i) \left( \begin{array}{ccc} 
x_i^2 & x_iy_i & x_iz_i \\
y_i^2 & y_i^2 & y_iz_i \\
z_i^2 & z_1y_i & z_iz_i
\end{array} \right). 
\] (58)
where \( \omega_i \), are eigenvalues of the tensor \( K \) corresponding to eigenvectors \( R_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \ (i = 1, 2, 3) \). In this case the coupling tensor \( f \) is nonzero only for frequencies \( \omega_1, \omega_2, \omega_3 .. \). These frequencies are the resonance frequencies of the equation (53). Therefore when \( \gamma = 0 \), that is for a non absorptive medium, the coupling tensor \( f \) and therefore the noise electrical polarization density is equal to zero except in resonance frequencies, where the energy of electromagnetic field is absorbed by the oscillator. This example explicitly shows that this quantization scheme is also applicable to anisotropic dispersive but non absorptive media.

4 Summary

The electromagnetic field quantization in the presence of an anisotropic magnetodielectric is investigated consistently by modelling the magnetodielectric with two independent quantum fields namely E and M quantum fields. For a given pair of the electric and magnetic susceptibility tensors \( \chi^e \) and \( \chi^m \), we have found the corresponding coupling tensors \( f \) and \( g \), which couple electromagnetic field to E and M quantum fields respectively. The explicit space-time dependence of the noise polarizations are obtained in terms of the ladder operators of the medium and the coupling tensors as a consequence of Heisenberg equations. Maxwell and constitutive equations are obtained directly from Heisenberg equations. In the limiting case, i.e., when there is no medium, the approach tends to the usual method of quantization of the electromagnetic field in vacuum. Also when the medium is a non absorptive one, the noise polarizations tend to zero and in this case the approach is equivalent to the previous methods, as expected.

References

[1] T. Grunner, D. G. Welsch, Phys. Rev. A 51, 3246 (1995)
[2] R. Matloob, R. Loudon, S. M. Barnett, Phys. Rev. A 52, 4823 (1995)
[3] T. Grunner, D. G. Welsch, Phys. Rev. A 53, 1818 (1996)
[4] T. Grunner, D. G. Welsch, Phys. Rev. A 54, 1661 (1996)
[5] R. Matloob, R. Loudon, Phys. Rev. A 53, 4567 (1996)
[6] H. T. Dung, L. Knoll, D. G. Welsch, Phys. Rev. A 57, 3931 (1998)
[7] S. Scheel, L. Knoll, D. G. Welsch, Phys. Rev. A 58, 700 (1998)
[8] J. J. Hopfield, Phys. Rev. 112, 1555 (1958)
[9] U. Fano, Phys. Rev. 103, 1202 (1956)
[10] B. Huttner, S. M. Barnett, Phys. Rev. A 46, 4306 (1992)
[11] L. G. Suttorp, M. Wubs, Phys. Rev. A 70, 013816 (2004)
[12] L. D. Landau, E. M. Lifshitz, Electrodynamics of Continuous Media, Pergamon, Oxford (1977)
[13] G. W. Ford, J. T. Lewis, R. F. O’Connell, Phys. Rev. A 37, 4419 (1988)
[14] G. W. Ford, J. T. Lewis, R. F. O’Connell, Phys. Rev. Lett. 55, 2273 (1985)
[15] A. O. Caldeira, A. J. Leggett, Phys. Rev. Lett. 46, 211 (1981)
[16] A. O. Caldeira, A. J. Leggett, Ann. Phys. (N.Y.) 149, 374 (1983)
[17] A. H. Castro Neto, A. o. Caldeira, Phys. Rev. Lett. 67, 1960 (1991)
[18] B. L. Hu, J. P. Paz, Y. Zhang, Phys. Rev. D 45, 2843 (1992)
[19] Jie-Lou Liao, E. Pollak, Chem. Phys. 268, 295 (2001)
[20] A. O. Caldeira, A. J. Leggett, Physica (Amsterdam) 121A, 585 (1983)
130A, 374(E) (1985)
[21] A. O. Caldeira, A. J. Leggett, Ann. Phys. (N.Y.) 153, 445 (1984)
[22] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Grag, W. Zwerger, Rev. Mod. Phys. 59, 1 (1987)
[23] H. G. Shuster, V. R. Veria, Phys. Rev. B 34, 189 (1986)
[24] G. W. Ford, M. Kac, P. Mazur, J. Math. Phys. 6, 504 (1964)
[25] F. Kheirandish, M. Amooshahi, Mod. Phys. Lett. A 20, 39 (2005)

[26] F. Kheirandish, M. Amooshahi, Phys. Rev. A 74, 1 (2006)

[27] H. T. Dung, S. Y. Buhmann, L. Knö, D. G. Welsh, Phys. Rev. A 68, 043816 (2003)