On determinism and well-posedness in multiple time dimensions

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We study the initial value problem for the wave equation and the ultrahyperbolic equation for data posed on initial hypersurfaces surface of arbitrary space–time signature. We show that, under a non-local constraint, the initial value problem posed on codimension-one hypersurfaces—the Cauchy problem—has global unique solutions in the Sobolev spaces $H^m$. Thus, it is well-posed. However, we show that the initial value problem on higher codimension hypersurfaces is ill-posed due to failure of uniqueness, at least when specifying a finite number of derivatives of the data. This failure is in contrast to a uniqueness result for data given in an arbitrary neighbourhood of such initial hypersurfaces, which Courant deduces from Asgeirsson’s mean value theorem. We give a generalization of Courant’s theorem that extends to a broader class of equations. The proofs use Fourier synthesis and the Holmgren–John uniqueness theorem.

Keywords: ultrahyperbolic equation; non-local constraint; multiple time dimensions

1. Introduction

The wave equation

$$\Delta u - \partial_y^2 u = 0$$

in (3 + 1)-dimensional space–time is of central physical importance, describing the dynamical evolution of many of the physical quantities of classical and quantum field theories, including the components of the electromagnetic field. Its generalization to a world with two or more time dimensions is an ultrahyperbolic equation, and thus the study of the properties of ultrahyperbolic equations provides a useful window onto the mathematical status and physical viability of theories involving multiple times. Of paramount importance is the insight it provides into the extent to which the ordinary concepts of causality and determinism survive the transition to multiple time dimensions.

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Consideration of theories with multiple times has been relatively rare because it is widely believed that they are inherently unstable and thus are not deterministic in a physically meaningful sense. Certain significant developments in theoretical physics, notably string theory, require additional dimensions. In most work to date, exceptions include the work of Tegmark (1997), Hull (1999), Hull & Khuri (2000) and Bars (2001), the signature for the extra dimensions is spatial, reflecting in part this concern with instability. Furthermore, Dorling (1970) points out a connection between the stability of relativistic particles and the one-dimensionality of time. Motivated by this, the purpose of this paper is to reconsider the questions of stability, uniqueness and determinism of the initial value problem in the presence of multiple time dimensions. As noted, we take the model field equations in this setting to be the simple generalizations of the wave equation to multiple times, the ultrahyperbolic equations. We find that the issue of stability and uniqueness for the Cauchy problem can be addressed by imposing non-local constraints that arise naturally from the field equations.

It may be thought reasonable to go beyond the traditional Cauchy problem and give initial data on hypersurfaces of higher codimension. We show that, under the above constraints, one can preserve stability in this setting, but uniqueness is lost, and thus determinism. Indeed, one may specify an arbitrary finite number of normal derivatives of the solution on the higher codimension hypersurface and insist upon smooth solutions, yet still fail to achieve uniqueness. In contrast to this, we conclude with a result that essentially recovers and generalizes a theorem of Courant, which shows that the values of a solution in an arbitrarily small neighbourhood of the initial hypersurface are sufficient to determine the solution uniquely. In related work, Woodhouse (1992) studied the case of two space and two time dimensions with initial data on a space-like hypersurface (thus of codimension two), using the Penrose twistor transform in the real case.\(^1\) He also recovered the uniqueness result of Courant and its implicit constraints on well-posed initial data for the Cauchy problem. Our work provides a rigorous analytic alternative for his solution method, which is not restricted to this choice of space and time dimensions.

We remark that none of our results rely upon properties of analyticity of the data or the solution. Furthermore, our results on the Cauchy problem and the higher codimensional initial value problem can be extended in a straightforward way to field equations with non-zero mass terms, as can our generalization of Courant’s theorem.

To fix our notation, the wave equation in \(d_1\)-many space dimensions and one time dimension is

\[
\Delta_x u - \partial_y^2 u := \sum_{j=1}^{d_1} \partial_{x_j}^2 u - \partial_y^2 u = 0. 
\]

The standard Cauchy problem is posed on \(N = \{(x, y) \in \mathbb{R}^{d_1}_x \times \mathbb{R}^1_y : y = 0\}\), a space-like codimension-one linear hypersurface, for initial data

\[
u(x, 0) = f(x) \quad \text{and} \quad \partial_y u(x, 0) = g(x).
\]

\(^1\)Sparling (2007) also considers multiple time dimensions in the context of the twistor program.
A non-standard Cauchy problem is posed for a linear hypersurface of mixed signature \( N = \{(x, y) : x_1 = 0\} \subseteq \mathbb{R}^d_x \times \mathbb{R}^1_y \), namely

\[
 u(0, x', y) = f(x', y) \quad \text{and} \quad \partial_{x_1} u(0, x', y) = g(x', y),
\]

where the notation is that \( x = (x_1, x') \in \mathbb{R}^d \). Courant (1962) calls this the non-space-like Cauchy problem, but to avoid confusion with the non-characteristic Cauchy problem, we call it a Cauchy problem of mixed signature.

An ultrahyperbolic equation has the form

\[
 \Delta_x u - \Delta_y u := \sum_{j=1}^{d_1} \partial^2_{x_j} u - \sum_{j=1}^{d_2} \partial^2_{y_j} u = 0, \quad (1.2)
\]

where \( x \in \mathbb{R}^{d_1} \) are considered to be the space-like variables and \( y \in \mathbb{R}^{d_2} \) are time-like variables. The Cauchy problem considers initial data posed on a linear hypersurface of codimension one. Choosing \( y_1 \) as the direction of evolution, Cauchy data consist of

\[
 u(x, 0, y') = f(x, y') \quad \text{and} \quad \partial_{y_1} u(x, 0, y') = g(x, y')
\]

on the hypersurface \( N = \{(x, y) \in \mathbb{R}^d_x \times \mathbb{R}^d_y : y_1 = 0\} \).

The initial value problem on a higher codimension hypersurface \( M \) could take various forms. A natural problem from the perspective of theories with multiple times is to consider the space-like hypersurface \( M = \{(x, y) \in \mathbb{R}^d_x \times \mathbb{R}^d_y : y_1 = 0\} \) of codimension \( d_2 \). Alternatively, one may consider more general \( M = \{(x, y) \in \mathbb{R}^d_x \times \mathbb{R}^d_y : x_{p_1+1} = \cdots = x_{d_1} = 0, y_{p_2+1} = \cdots = y_{d_2-1} = 0\} \), where \( 0 \leq p_1 \leq d_1 \) and \( 0 \leq p_2 \leq d_2 - 1 \). There is, in either case, a question as to how much data, and which constraints, are to be considered on \( M \). Some of the options are to (i) give the zeroth and first normal derivatives of \( u \) on \( M \), (ii) give some finite number of derivatives of \( u \) on \( M \) that are compatible with the constraint imposed by the ultrahyperbolic equation, or (iii) specify infinitely many derivatives of \( u \) on \( M \).

In this paper, we consider the first two of these cases.

An outline of the results of this paper is as follows. In \( \S 2 \), we use Fourier methods to show that the Cauchy problem for the ultrahyperbolic equation (1.2) is ill-posed in general, but well-posed on Sobolev spaces \( H^m \) if an explicit non-local constraint is imposed upon the Cauchy data. This also applies to the wave equation with Cauchy data on a mixed signature hypersurface. In \( \S 3 \), we consider the initial value problem for data given on higher codimension hypersurfaces and we find that solutions are highly non-unique for initial value problems of type (i) and (ii) above, even among \( H^m \) smooth solutions and with the imposition of the constraint given in \( \S 2 \). In particular, for theories with multiple times that can be transformed to the form of equation (1.2), data posed on the hypersurface \( M = \{y = 0\} \) do not uniquely determine the solution at any other point in time \( y \in \mathbb{R}^{d_2} \setminus \{0\} \). The extension problem for higher numbers of derivatives is treated by the same method as case (i) of zeroth and first normal derivatives. Regarding case (iii), in which one specifies infinitely many derivatives on the initial hypersurface \( M \), we do not have an answer. We do show in \( \S 4 \) that among smooth solutions, data in an arbitrarily small ellipsoidal neighbourhood of a disk in \( M \) uniquely determine the data in the envelope of its light cones. This is analogous to a result in Courant (1962) that is derived from Asgeirsson’s mean value theorem.
2. The Cauchy problem

Let \( x \in \mathbb{R}^{d_1} \) and \( y \in \mathbb{R}^{d_2} \) be the Cartesian coordinates of space–time, denote \( y = (y_1, y') \) and consider the Cauchy problem of evolution in the coordinate \( y_1 \). The Cauchy problem of mixed signature that we address is posed as

\[
\partial_{y_1}^2 u = \Delta_x u - \Delta_{y'} u, \tag{2.1}
\]

with Cauchy data \( u(x, 0, y') = u_0(x, y') \) and \( \partial_{y_1} u(x, 0, y') = u_1(x, y') \). The standard Sobolev spaces \( H^m \) of functions of the variables \((x, y')\) are defined as closures of \( C_0^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2-1}) \) with respect to the norms

\[
\|f\|_m^2 = \sum_{|\alpha|+|\beta| \leq m} \int |\partial_x^\alpha \partial_{y'}^\beta f(x, y')|^2 \, dx \, dy'.
\]

Additionally, there is an energy functional, of indefinite sign, which is associated with equation (2.1), namely

\[
E(u) := \frac{1}{2} \int |\partial_{y_1} u|^2 + |\nabla_x u|^2 - |\nabla_{y'} u|^2 \, dx \, dy'.
\]

**Theorem 2.1.** Suppose that the evolution mapping \( y_1 \mapsto (u(x, y_1, y')) \) is in \( C^1(\mathbb{R}_{y_1} : H^1 \times H^0) \). Then, the energy is conserved along a solution \( u(\cdot, y_1, \cdot) \),

\[
E(u(\cdot, y_1, \cdot)) = E(u(\cdot, 0, \cdot)).
\]

**Proof.** Given \( (u(x, y_1, y')) \) in \( C^1 \), the following calculation is justified:

\[
\partial_{y_1} E(u) = \int (\partial_{y_1} u \cdot \partial_{y_1}^2 u + \nabla_x u \cdot \nabla_x \partial_{y_1} u - \nabla_{y'} u \cdot \nabla_{y'} \partial_{y_1} u) \, dx \, dy' = \int \partial_{y_1} u (\partial_{y_1}^2 u + \Delta_x u + \Delta_{y'} u) \, dx \, dy' = 0.
\]

The key issue is that the Cauchy problem for equation (2.1) is ill-posed for \( d_2 \geq 2 \) and the solutions are not in general in \( C^1(\mathbb{R}_{y_1} : H^1 \times H^0) \). The energy is indefinite and in particular not bounded below; hence, it does not, in general, define an energy norm with which to control the Sobolev norms of solutions of the evolution equations.

To move to the next level of analysis, we give a Fourier synthesis of the evolution operator for the Cauchy problem of mixed signature. Given \( (u_0, u_1) \) in \( H^{m+1} \times H^m \), consider the Fourier space variables \((x, y') \mapsto (\xi, \eta')\) and define the Fourier transform in the standard way,

\[
\begin{pmatrix}
\hat{u}_0(\xi, \eta') \\
\hat{u}_1(\xi, \eta')
\end{pmatrix} = \frac{1}{\sqrt{2\pi}^d} \int e^{-i\xi x} e^{-i\eta' y'} \begin{pmatrix}
u_0(x, y') \\
u_1(x, y')
\end{pmatrix} \, dx \, dy'.
\]
where \( d = (d_1 + d_2 - 1) \). On a formal level, equation (2.1) under Fourier transform will read

\[
\hat{\partial}_{y_1}^2 \hat{u} = (-|\xi|^2 + |\eta'|^2) \hat{u},
\]

giving rise to the expression for the propagator, \( \exp(y_1 \sqrt{\Delta_x - \Delta_y}) \). The solution thus reads

\[
\hat{u}(\xi, y_1, \eta') = \cos \left( \sqrt{|\xi|^2 - |\eta'|^2} y_1 \right) \hat{u}_0(\xi, \eta') + \frac{\sin \left( \sqrt{|\xi|^2 - |\eta'|^2} y_1 \right)}{\sqrt{|\xi|^2 - |\eta'|^2}} \hat{u}_1(\xi, \eta'),
\]

for \( |\eta'| \leq |\xi| \), while

\[
\hat{u}(\xi, y_1, \eta') = \cosh \left( \sqrt{|\eta'|^2 - |\xi|^2} y_1 \right) \hat{u}_0(\xi, \eta') + \frac{\sinh \left( \sqrt{|\eta'|^2 - |\xi|^2} y_1 \right)}{\sqrt{|\eta'|^2 - |\xi|^2}} \hat{u}_1(\xi, \eta'),
\]

for \( |\xi| < |\eta'| \). That is, the dispersion relation

\[
\omega(\xi, \eta') = \sqrt{|\xi|^2 - |\eta'|^2}
\]

holds in the Fourier space region \( \{|\eta'| \leq |\xi|\} \), while in the complementary region, the evolution of a Fourier mode is described by the Lyapunov exponent

\[
\lambda(\xi, \eta') = \sqrt{|\eta'|^2 - |\xi|^2}.
\]

When the propagator is applied to data \((u_0/v_0)\) that is analytic, this solution exists for at least a short time; for analytic data of exponential type, the solution is global. However, it is clear that general initial data in \( H^{m+1} \times H^m \) do not even give rise to solutions which are tempered distributions for any non-zero \( y_1 \).

On the other hand, imposing a constraint on the initial data, the solution process is well defined. The fact that some constraint is necessary is indeed evident from the Asgeirsson mean value theorem and its consequences, as discussed in Courant (1962). The form of this non-local constraint is evident from the Fourier synthesis, as we shall now see.

Define a phase space \( X \) using an energy norm adapted to the propagator of equation (2.1). Using the definition of the dispersion relation (2.2) and the Lyapunov exponent (2.3), and the Plancherel identity, set \( v = \left( \frac{u_0}{v_0} \right) \)

\[
\|v\|^2_X := \int_{\{|\eta'| < |\xi|\}} \omega^2(\xi, \eta') |\hat{v}_0(\xi, \eta')|^2 \, d\xi \, d\eta' + \int_{\{|\xi| \leq |\eta'|\}} \lambda^2(\xi, \eta') |\hat{v}_0(\xi, \eta')|^2 \, d\xi \, d\eta' + \int |\hat{v}_1(\xi, \eta')|^2 \, d\xi \, d\eta'.
\]

This is a norm, unlike the actual energy associated with equation (2.1) and can be used to control solutions when the propagator is restricted to the appropriate stable and/or unstable subspaces of \( X \). Define

\[
X^S = \left\{ v = \left( \frac{u_0}{v_1} \right) \in X : \frac{1}{2} \left( \hat{v}_0(\xi, \eta') + \frac{\hat{v}_1(\xi, \eta')}{\lambda(\xi, \eta')} \right) = 0, \quad \text{for} \quad |\xi| < |\eta'| \right\},
\]

\[
X^U = \left\{ v \in X : \frac{1}{2} \left( \hat{v}_0(\xi, \eta') - \frac{\hat{v}_1(\xi, \eta')}{\lambda(\xi, \eta')} \right) = 0, \quad \text{for} \quad |\xi| < |\eta'| \right\}.
\]
and

\[ X^C = \left\{ v \in X : \text{supp}(\hat{v}_0, \eta') (\xi, \eta') \subseteq \{ |\xi| \geq |\eta'| \} \right\} = X^S \cap X^U. \]

The subspace \( X^S \) corresponds to the centre stable subspace for evolution in \( y_1 \in \mathbb{R}^+ \), the subspace \( X^U \) corresponds to the centre unstable subspace and \( X^C \) is the centre subspace. This nomenclature is supported by the following theorem.

**Theorem 2.2.** For \((u_0, u_1) \in X^S\), the Cauchy problem of mixed signature for equation (2.1) has a unique solution in \( X \) for all \( y_1 \in \mathbb{R}^+ \). For \((u_0, u_1) \in X^U\), the problem has a unique solution for all \( y_1 \in \mathbb{R}^- \), and whenever \((u_0, u_1) \in X^C\), the solution exists globally in \( y_1 \in \mathbb{R} \). In each of these cases, the map \( y_1 \mapsto u(x, y_1, y') \) is \( C^1 \).

Denote the propagators by \( \Phi^S \), \( \Phi^U \) and \( \Phi^C \) for data in the respective subspaces. These solutions are continuous with respect to their Cauchy data taken in the respective subspaces. This is the result of the following theorem.

**Theorem 2.3.** Given two phase space points \( u = (u_0, u_1) \) and \( v = (v_0, v_1) \in X^S \), then for \( y_1 \geq 0 \),

\[ \| \Phi^S_{y_1} (u) - \Phi^S_{y_1} (v) \|_X^2 \leq \| u - v \|_X^2. \] (2.6)

The analogous estimate holds for \( u, v \in X^U \), for \( y_1 \leq 0 \),

\[ \| \Phi^U_{y_1} (u) - \Phi^U_{y_1} (v) \|_X^2 \leq \| u - v \|_X^2. \] (2.7)

For \( u - v \in X^C \), \( \Phi^C_{y_1} = \Phi^S_{y_1} \) for \( y_1 \geq 0 \) and \( \Phi^C_{y_1} = \Phi^U_{y_1} \) for \( y_1 \leq 0 \), and equality holds in both equations (2.6) and (2.7).

**Proof.** It suffices in theorem 2.3 to prove the first statement. In \( X^S \), the solution has two components, distinguished by their Fourier support. Consider first \((u_0, u_1)\) such that \( \text{supp}(\hat{u}_0, \hat{u}_1) \subseteq \{ |\xi| \geq |\eta'| \} = R_1 \), which gives the centre component of the evolution. The propagator is expressed as

\[
\mathcal{F} \Phi^S_{y_1} (u_0, u_1) = \begin{pmatrix}
\cos(\omega y_1) & \frac{\sin(\omega y_1)}{\omega} \\
-\omega \sin(\omega y_1) & \cos(\omega y_1)
\end{pmatrix}
\begin{pmatrix}
\hat{u}_0 \\
\hat{u}_1
\end{pmatrix},
\]

where \( \omega = \omega(\xi, \eta') \) is the dispersion relation (2.2). Evaluating this in the energy norm, we obtain

\[
\| \Phi^S_{y_1} (u_0, u_1) \|_X^2 = \left\| \cos(\omega y_1) \hat{u}_0 + \frac{\sin(\omega y_1)}{\omega} \hat{u}_1 \right\|_X^2 \omega^2 \\
+ \left| -\omega \sin(\omega y_1) \hat{u}_0 + \cos(\omega y_1) \hat{u}_1 \right|^2 d\xi d\eta' \\
= \int (|\hat{u}_0|^2 \omega^2 + |\hat{u}_1|^2) d\xi d\eta' = \left\| (u_0, u_1) \right\|_X^2. \] (2.8)

The propagator on the complementary space is more sensitive. Let us suppose that \( \text{supp}(\hat{u}_0, \hat{u}_1) \subseteq \{ |\eta'| > |\xi| \} \), then \( \lambda(\xi, \eta') > 0 \), and we express the propagator in
terms of its Fourier transform as
\[
\mathcal{F} \Phi_{y_1}^S(u_0, u_1) = \begin{pmatrix} \cosh(\lambda y_1) & \frac{\sinh(\lambda y_1)}{\lambda} \\ \lambda \sinh(\lambda y_1) & \cosh(\lambda y_1) \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} = \frac{e^{\lambda y_1}}{2} \begin{pmatrix} 1 & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} + \frac{e^{-\lambda y_1}}{2} \begin{pmatrix} 1 & -\frac{1}{\lambda} \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix}.
\]

The subspace \( X^S \) consists precisely of those data that lie in the null space of the first term; this is the expression of the constraint
\[
\lambda \hat{u}_0(\xi, \eta') + \hat{u}_1(\xi, \eta') = 0. \tag{2.9}
\]

Measuring the remaining term in energy norm, we find
\[
\| \Phi_{y_1}^S(u_0, u_1) \|^2_X = \int \int e^{-2\lambda y_1} \left[ \hat{u}_0 - \frac{\hat{u}_1}{\lambda} \right]^2 d\xi d\eta' \leq \int \int e^{-2\lambda y_1} \left( |\hat{u}_0|^2 \lambda^2 + |\hat{u}_1|^2 \right) d\xi d\eta'.
\]

Since we consider the propagator \( \Phi_{y_1}^S \) for \( y_1 \geq 0 \), the exponent \( -\lambda y_1 \) is negative, and therefore
\[
\| \Phi_{y_1}^S(u_0, u_1) \|^2_X \leq \| u \|^2_X.
\]

for \( u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X^S \). For general data in \( X^S \), one decomposes it into its components with support in \( \{ |\xi| \geq |\eta'| \} \), for which we use equation (2.8), and its components supported in \( \{ |\eta'| > |\xi| \} \), which in addition satisfies the constraint (2.9). Therefore, on \( X^S \),
\[
\| \Phi_{y_1}^S(u) \|^2_X \leq \| u \|^2_X.
\]

Bounded operators on \( X^S \) are continuous with respect to \( u \in X^S \), and it is easy to see that the solution also behaves continuously with respect to \( y_1 \geq 0 \). The case for the subspace \( X^U \) is proved by the same arguments, after reversing time \( y_1 \rightarrow -y_1 \). This proves theorems 2.2 and 2.3. We remark that, on the centre subspace \( X^C \), which yields global solutions, both constraints are imposed,
\[
\lambda \hat{u}_0(\xi, \eta') \pm \hat{u}_1(\xi, \eta') = 0, \tag{2.10}
\]

implying that \( \hat{u}_0(\xi, \eta') = 0 = \hat{u}_1(\xi, \eta') \) for all \( \{ |\xi| \leq |\eta'| \} \).

The proof extends to the initial value problem posed in higher energy spaces, defined by
\[
\| v \|^2_{X^m} := \sum_{|\alpha|+|\beta| \leq m} \| \partial_x^\alpha \partial_y^\beta v \|^2_X.
\]

We then have the following corollary.
Corollary 2.4. The higher energy space $X^m$ decomposes into three subspaces, $X^{m,S}, X^{m,U}$ and $X^{m,C} = X^{m,S} \cap X^{m,U}$, such that for $u, v \in X^{m,S}$ and $y_1 \geq 0$

$$\| \phi_{y_1}^S (u) - \phi_{y_1}^S (v) \|_{X^m}^2 \leq \| u - v \|_{X^m}^2,$$

while for $y_1 \leq 0$ and $u, v \in X^{m,U}$,

$$\| \phi_{y_1}^U (u) - \phi_{y_1}^U (v) \|_{X^m}^2 \leq \| u - v \|_{X^m}^2.$$ 

For $u, v \in X^{m,C}$, both estimates hold, and a global solution exists that has properties of higher Sobolev regularity. When $m > (d_1 + (d_2 - 1))/2 + 2$, then such solutions are known to be classical $C^2$ solutions by the Sobolev embedding theorem.

It is natural to estimate solutions with respect to the energy norm; indeed, it is the energy when restricted to the centre subspace $X^C$. Thus, the Cauchy problem is well-posed in the following sense: data in $X^S$ continuously propagate to all $y_1 \in \mathbb{R}^+$, data in $X^U$ continuously propagate to all times $y_1 \in \mathbb{R}^-$ and data in $X^C$, which constitute an infinite-dimensional Hilbert space, are defined globally in time. In the case of the ordinary wave equation ($d_1 = 1$), solutions in $X^C$ correspond to the full energy space $H^1 \times L^2$.

3. The initial value problem in higher codimensions

In the presence of multiple time dimensions, space-like hypersurfaces are necessarily of higher codimension. Therefore, one might consider the initial value problem with data posed on a hypersurface of codimension greater than or equal to two. Such problems are generally ill-posed. Indeed, solutions can be singular for standard classes of data. Moreover, even imposing the constraint discussed in §2, which is the requirement of global existence, smooth solutions are highly non-unique. The purpose of this section is to study the extension problem of data posed on a non-degenerate higher codimension hypersurface $M$ to Cauchy data on a codimension-one hypersurface $N$. There is a lot of freedom in choosing this extension, even under the constraint equation (2.9) on the resulting Cauchy data. Other extensions can be chosen that fail to satisfy the constraint. Thus, the initial value problem fails to be well-posed in several ways; resulting solutions may be singular, or the constraint could be imposed, whereupon they will be regular for all $y_1 \in \mathbb{R}$, however they will not be unique.

(a) Codimension two to codimension one in $\mathbb{R}^3$

Our analysis is illustrated in the example case of $M = \{y_1 = y_2 = 0\}$ and $N = \{y_1 = 0\}$ subspaces of $\mathbb{R}^3$. We suppose that initial data for a solution $u(x, y)$ is given on $M$ in the form

$$w(x_1) = (w_0(x_1), w_{10}(x_1), w_{01}(x_1)),$$

where $w_0(x_1) = u(x_1, 0)$, $w_{10}(x_1) = \partial_{y_1} u(x_1, 0)$ and $w_{01}(x_1) = \partial_{y_2} u(x_1, 0)$, corresponding to the values of the solution and its normal derivatives on $M$. 

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The objective is to extend $w(x_1)$ to Cauchy data $(w_0(x_1, y_2), u_1(x_1, y_2))$ on $N$ that satisfy

$$u_0(x_1, 0) = w_0(x_1),$$
$$u_1(x_1, 0) = w_{10}(x_1)$$

and the compatibility condition

$$\partial_{y_2} u_0(x_1, 0) = w_{01}(x_1).$$

We give extensions that satisfy the constraint (2.9), therefore giving rise to global solutions in $y_1 \in \mathbb{R}$. Such extensions are non-unique. Additionally, there are extensions that fail to satisfy the constraint, lying in $X^S \setminus X^C$ or $X^U \setminus X^C$ or neither.

**Definition 3.1.** The extension operator is given by

$$E(w)(x_1, y_2) := \frac{1}{2\pi} \int \int e^{i(\xi x_1 + \eta' y_2)} \hat{w}(\xi) \chi(\xi, \eta') \, d\eta' \, d\xi,$$

where the kernel function $\chi(\xi, \eta')$ is chosen such that, for all $\xi$,

$$\frac{1}{\sqrt{2\pi}} \int \chi(\xi, \eta') \, d\eta' = 1.$$

To satisfy the constraint, we ask additionally that $\text{supp}(\chi(\xi, \eta')) \subseteq \{ |\eta'| < |\xi| \}$. A reasonable choice is to take

$$\chi(\xi, \eta') = \psi(\eta' / |\xi|) \frac{1}{|\xi|}$$

for $\psi(\theta) \in C_0^\infty([-1, 1])$, $\psi(\theta) \geq 0$ even, and

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \psi(\theta) \, d\theta = 1. \quad (3.1)$$

**Theorem 3.2.** The extension operator $E$ is a bounded operator on the following space of functions:

$$E : \dot{H}^{-1/2}(M) \to L^2(N)$$

and

$$E : (\dot{H}^{-1/2} \cap H^{m-1/2})(M) \to H^m(N).$$

In addition, when $w \in \dot{H}^{-3/2}(M)$ then $y_2 E(w) \in L^2(N)$ and furthermore

$$y_2 E : \dot{H}^{m-3/2}(M) \to H^m(N).$$

Using the extension operator, we generate constraint-satisfying Cauchy data on $N$ from initial data on $M$ as follows:

$$u_0(x_1, y_2) := E(w_0)(x_1, y_2) + y_2 E(w_{01})(x_1, y_2)$$

and

$$u_1(x_1, y_2) := E(w_{10})(x_1, y_2).$$
Checking that this is a legitimate choice, we have
\[
u_1(x_1,0) = \frac{1}{2\pi} \int e^{i\xi x_1} \hat{\nu}_{10}(\xi) \chi(\xi, \eta') \, d\xi \, d\eta'
\]
\[
= \frac{1}{2\pi} \int e^{i\xi x_1} \hat{\nu}_{10}(\xi) \left[ \frac{1}{\sqrt{2\pi}} \int \psi(\eta' / |\xi|) \frac{1}{|\xi|} \, d\eta' \right] \, d\xi
\]
\[
= \frac{1}{2\pi} \int e^{i\xi x_1} \hat{\nu}_{10}(\xi) \left[ \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \psi(\theta) \, d\theta \right] \, d\xi = w_{10}(x_1)
\]
because of the normalization equation (3.1) of \(\psi\). Similarly,
\[
u_0(x_1,0) = E(\nu_0)(x_1,0) = \nu_0(x_1).
\]
The compatibility condition is satisfied since
\[
\partial_{y_2} \nu_0(x_1,0) = \partial_{y_2} E(\nu_0)(x_1,0) + E(\nu_{01})(x_1,0) = \partial_{y_2} E(\nu_0)(x_1,0) + \nu_{01}(x_1).
\]
The first term of the right-hand side vanishes because
\[
\partial_{y_2} E(\nu_0)(x_1,0) = \frac{1}{2\pi} \int e^{i\xi x_1} i\eta' \hat{\nu}_{0}(\xi) \chi(\xi, \eta') \, d\xi \, d\eta'
\]
\[
= \frac{1}{2\pi} \int e^{i\xi x_1} \hat{\nu}_{0}(\xi) \left[ \frac{1}{\sqrt{2\pi}} \int \eta' \psi(\eta' / |\xi|) \frac{1}{|\xi|} \, d\eta' \right] \, d\xi = 0,
\]
where we use the fact that \(\int \theta \psi(\theta) \, d\theta = 0\) because \(\psi(\cdot)\) has been chosen to be even.

The pair of functions \((\nu_0(x_1, y_2), \nu_1(x_1, y_2))\) gives Cauchy data for the codimension-one problem that is discussed in §2. Because of the properties of the extension, it satisfies the constraint conditions of \(X^c\) for solutions that are globally defined in \(y_1\). To apply the existence theorem, the energy norm of this Cauchy data must be finite.

**Theorem 3.3.** Suppose that \(\nu_0 \in \dot{H}^{1,2}(M)\), \(\nu_{01} \in \dot{H}^{-1,2}\) and \(w_{10} \in \dot{H}^{-1,2}\). Then, the energy norm of the extension \(\nu_0 = E(\nu_0)(x_1, y_2) + y_2 E(\nu_{01})(x_1, y_2)\) and \(\nu_1(x_1, y_2) = E(\nu_{10})(x_1, y_2)\) is finite,
\[
\| (\nu_0, \nu_1) \|_{X^c}^2 \leq C \left( \| \nu_0 \|_{\dot{H}^{1,2}}^2 + \| \nu_{01} \|_{\dot{H}^{-1,2}}^2 + \| w_{10} \|_{\dot{H}^{-1,2}}^2 \right).
\]
Additionally, the higher energy norms with which one defines the \(X^m\) topology for \((\nu_0, \nu_1)\) are also bounded by this extension process, namely
\[
\| (\nu_0, \nu_1) \|_{X^{m,c}}^2 \leq C_m \left( \| \nu_0 \|_{\dot{H}^{m+1,2}}^2 + \| \nu_{01} \|_{\dot{H}^{m-1,2}}^2 + \| w_{10} \|_{\dot{H}^{m-1,2}}^2 \right).
\]

**Proof of theorem 3.2.** Using the Plancherel identity, the \(L^2(N)\) norm of \(E(\nu)\) is
\[
\| E(\nu) \|_{L^2(N)}^2 = \int |\hat{\nu}(\xi)|^2 \psi^2(\eta' / |\xi|) \frac{1}{|\xi|^2} \, d\eta' \, d\xi
\]
\[
= \int \frac{1}{|\xi|^2} |\hat{\nu}(\xi)|^2 \left( \int \psi^2(\eta' / |\xi|) \frac{1}{|\xi|} \, d\eta' \right) \, d\xi = \| \psi \|_{L^2([-1,1])}^2 \| \nu \|_{\dot{H}^{-1/2}(M)}^2,
\]
as $\theta = \eta'/|\xi|$ and
\[
\int \psi^2(\eta'/|\xi|) \frac{1}{|\xi|} \, d\eta' = \int_{-1}^{1} \psi^2(\theta) \, d\theta.
\]

The identity extends to the Sobolev space $H^m(N)$; it suffices to calculate $\|\partial_x^m E(w)\|_{L^2}$ and $\|\partial_y^m E(w)\|_{L^2}$,
\[
\|\partial_x^m E(w)\|_{L^2(N)} = \left\| \left| \hat{w}(\xi) \right|^2 |\xi|^{2m} \psi^2(\eta'/|\xi|) \frac{1}{|\xi|^2} \right\|_{L^2} \, d\eta' \, d\xi
\]
\[
= \int |\hat{w}(\xi)|^2 |\xi|^{2m-1} \left( \int \psi^2(\eta'/|\xi|) \frac{1}{|\xi|} \, d\eta' \right) \, d\xi
\]
\[
= \|\psi\|_{L^2[-1,1]}^2 \|w\|_{H^{m-1/2}(M)}^2.
\]
The second quantity is similar,
\[
\|\partial_y^m E(w)\|_{L^2(N)} = \left\| \left| \hat{w}(\xi) \right|^2 |\xi|^{2m} \psi^2(\eta'/|\xi|) \frac{1}{|\xi|^2} |\eta'|^2 \right\|_{L^2} \, d\eta' \, d\xi
\]
\[
= \int |\hat{w}(\xi)|^2 |\xi|^{2m-1} \left( \int \psi^2(\eta'/|\xi|) \frac{1}{|\xi|} |\eta'| \frac{1}{|\xi|} \, d\eta' \right) \, d\xi
\]
\[
= C_m \|w\|_{H^{m-1/2}(M)}^2,
\]
where $C_m = \int_{-1}^{1} \theta^{2m} \psi^2(\theta) \, d\theta$. The third and fourth estimates of the theorem involve $y_2 E(w)$, whose Fourier transform is
\[
\hat{w}(\xi) \frac{1}{i} \partial_{\eta'} \chi(\eta'/|\xi|).
\]

Measuring the $L^2$ norm of $y_2 E(w),$
\[
\|y_2 E(w)\|_{L^2(N)}^2 = \left\| \left| \hat{w}(\xi) \right|^2 \left( \frac{1}{i} \partial_{\theta} \psi(\eta'/|\xi|) \frac{1}{|\xi|^2} \right) \right\|_{L^2} \, d\eta' \, d\xi
\]
\[
= \int |\hat{w}(\xi)|^2 \left( \int |\partial_{\theta} \psi(\eta'/|\xi|)|^2 \frac{1}{|\xi|} \, d\eta' \right) \, d\xi
\]
\[
= \int |\hat{w}(\xi)|^2 \left( \int_{-1}^{1} |\partial_{\theta} \psi|^2 \, d\theta \right) = C \|w\|_{H^{3/2}(M)}^2,
\]
with $C = \int_{-1}^{1} |\partial_{\theta} \psi|^2 \, d\theta$. The calculations of the $H^m$ norms of $y_2 E(w)$ are similar.

Proof of theorem 3.3. Given initial data $(u_0, u_{01}, u_{10})(x_1)$, we are to give conditions under which the energy norm of the extension $(u_0, u_1)$ is finite. First of all, the contribution to the energy given by $u_1$ is simply $\frac{1}{2} \|u_1\|_{L^2}^2$, hence, by theorem 3.2, it is bounded by $C \|u_{10}\|_{H^{-1/2}}^2$. There are two contributions from $u_0,$
which can be expressed using the Plancherel identity
\[
\int \int \omega^2(\xi, \eta') |\hat{w}_0(\xi)|^2 \chi^2(\xi, \eta') \, d\eta' \, d\xi + \int \int \omega^2(\xi, \eta') |\hat{w}_{01}(\xi)|^2 |\partial_\eta' \chi^2(\xi, \eta')|^2 \, d\eta' \, d\xi.
\]
Using \(\chi(\xi, \eta') = \psi(\eta'/|\xi|)/|\xi|\), we estimate these two integrals as
\[
\int_{|\eta'|<|\xi|} (|\xi|^2 - |\eta'|^2) |\hat{w}_0(\xi)|^2 \psi^2(\eta'/|\xi|) \frac{1}{|\xi|^2} \, d\eta' \, d\xi = \int |\hat{w}_0(\xi)|^2 \left[ \int_{|\eta'|<|\xi|} \left( |\xi| - \frac{|\eta'|^2}{|\xi|^2} \right) \psi^2(\eta'/|\xi|) \frac{1}{|\xi|} \, d\eta' \right] \, d\xi \leq C \|w_0\|^2_{H^{1/2}}
\]
and
\[
\int_{|\eta'|<|\xi|} (|\xi|^2 - |\eta'|^2) |\hat{w}_{01}(\xi)|^2 \psi^2(\eta'/|\xi|) \frac{1}{|\xi|^2} \, d\eta' \, d\xi = \int |\hat{w}_{01}(\xi)|^2 \left[ \int_{|\eta'|<|\xi|} \left( \frac{1}{|\xi|} - \frac{|\eta'|^2}{|\xi|^3} \right) \psi^2(\eta'/|\xi|) \frac{1}{|\xi|} \, d\eta' \right] \, d\xi \leq C \|w_{01}\|^2_{H^{-1/2}}.
\]

(b) The extension problem for general space-like data

We now consider the general problem of initial data given on a maximal space-like hypersurface of dimension \(d_1\), extending it to Cauchy data on a codimension-one hypersurface. That is, for \((x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\),
\[
M = \{ y = 0 \} \subseteq N = \{ y_1 = 0 \}.
\]
Initial data on \(M\) take the form \(w(x) = (w_0(x), w_{\alpha}(x))\), where a solution \(u(x, y)\) of the field equation (1.2) is asked to satisfy
\[
u(x, y) = w_0(x),
\]
with its first derivatives normal to \(M\) satisfying
\[
\partial_y^\alpha u(x, 0) = w_{\alpha}(x),
\]
where \(\alpha \in \mathbb{N}^{d_2}\) is the multi-index \(\alpha = (\alpha_1, \ldots, \alpha_{d_2})\), \(|\alpha| = 1\), such that only one \(\alpha_j = 1\) and the rest are zero. The objective is to extend \(w(x)\) to Cauchy data on \(N\) while satisfying the constraints (2.9) to be in \(X^C\). These Cauchy data satisfy
\[
w_0(x, 0) = w_0(x) \quad \text{and} \quad u_{\alpha'}(x, 0) = w_{\alpha}(x),
\]
for \(\alpha' = (\alpha_2, \ldots, \alpha_{d_2})\), and the first derivatives normal to \(N\) satisfy
\[
\partial_y^{\alpha_1} u(x, 0) = w_{10}(x).
\]
Following the construction given in §3a, define an extension operator
\[
E(w)(x, y') := \frac{1}{\sqrt{2\pi}} \int \frac{d\xi}{d_1 + d_2 - 1} \int \hat{w}(\xi) \chi(\xi, \eta') e^{i\xi x} e^{i\eta' y'} \, d\xi \, d\eta',
\]
where the kernel function is even in $\eta$ and satisfies
\[
\frac{1}{\sqrt{2\pi}^{d_2-1}} \int \chi(\xi, \eta') \, d\eta' = 1.
\]
To satisfy the constraint that $E(w) \in X^C$, we ask that $\text{supp}(\chi(\xi, \eta')) \subseteq \{(\xi, \eta') : |\eta'| < |\xi|\}$. Such kernel functions are readily constructed (they are far from being uniquely determined). For example, a variant of our construction of §3a is based on the choice of a $C^\infty_0$ function $\psi(\theta) \geq 0$, with $\text{supp}(\psi) \subseteq B_1(0)$, the ball of radius one. Then, define
\[
\chi(\xi, \eta') = \psi(\eta' / |\xi|) \frac{1}{|\xi|^{d_2-1}}.
\]
We note that $\chi$ is even in $\eta$ if $\psi(\theta)$ is even and that
\[
\frac{1}{\sqrt{2\pi}^{d_2-1}} \int \chi(\xi, \eta') \, d\eta' = \frac{1}{\sqrt{2\pi}^{d_2-1}} \int \psi(\eta' / |\xi|) \frac{1}{|\xi|^{d_2-1}} \, d\eta' = \frac{1}{\sqrt{2\pi}^{d_2-1}} \int \psi(\theta) \, d\theta.
\]
This is normalized to one by choice of $\psi$.

**Theorem 3.4.** The extension operator $E$ is bounded on the following function spaces:
\[
E : \dot{H}^{(1-d_2)/2}(M) \to L^2(N)
\]
and
\[
E : \dot{H}^{(1-d_2)/2}(M) \cap H^{m+(1-d_2)/2}(M) \to H^m(N),
\]
where $m$ is the exponent of Sobolev regularity,
\[
y' E : \dot{H}^{-(1+d_2)/2}(M) \to L^2(N)
\]
and
\[
y' E : \dot{H}^{-(1+d_2)/2}(M) \cap H^{-m-(1+d_2)/2}(M) \to H^m(N).
\]
Using the extension operator $E$, the vector function $w(x) = (w_0(x), w_\alpha(x))$ extends to Cauchy data on $N$ as
\[
u_0(x, y') := E(w_0)(x, y') + \sum_{|\alpha'|=1} y'^\alpha' E(w_0\alpha')(x, y')
\]
and
\[
u_1(x, y') := E(w_10)(x, y').
\]
This is seen to extend the initial data $w(x)$ in the required way, and, in addition, it satisfies the constraint that $(\nu_0, \nu_1) \in X^C$. However, measuring the functions $(\nu_0, \nu_1)$ in the energy norm is more appropriate for the Cauchy problem, hence, we also state estimates in this setting.
Theorem 3.5. Given \( w_0 \in \dot{H}(M) \) and \( w_\alpha \in \dot{H}(M) \), the energy norm of the extension

\[
(u_0, u_1) = (E(w_0) + y' E(w_{(0,\alpha')}), E(w_{10}))
\]

is finite; indeed

\[
\| (u_0, u_1) \|_{\mathcal{H}C}^2 \leq C (\| w_0 \|_{\dot{H}(-d/2)}^2 + \| w_{0\alpha'} \|_{\dot{H}(1-d/2)}^2 + \| w_{10} \|_{\dot{H}(1-d/2)}^2).
\]

The proofs of theorems 3.4 and 3.5 are similar to the proofs of theorems 3.2 and 3.3, to which we refer the reader.

(c) The extension problem for mixed space-like and time-like data

As a final case, we consider the extension problem for initial data on a lower dimensional hypersurface \( M \) of mixed signature. Given zeroth and first normal derivatives of a solution \( u(x, y) \) on \( M \), the object is to extend this data to the codimension-one hypersurface \( N = \{ y_1 = 0 \} \) in such a way that the constraint for well-posedness is satisfied. This is not always possible for arbitrary data \( w = (w_0, w_\alpha) \) posed on \( M \), due to analogous lower dimensional constraints on \( M \). But it is possible, along with attendant Sobolev bounds on the extended functions, in most cases. This situation will be explained below.

To set the notation, we consider space-like and time-like coordinates on \( M \) to be \((\tilde{x}, \tilde{y}) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}\), with their Fourier transform variables denoted \((\tilde{\xi}, \tilde{\eta}) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}\). The complementary variables will be denoted \((x'', y'') \in \mathbb{R}^{d_1-p_1} \times \mathbb{R}^{d_2-p_2-1}\) and \((\epsilon'', \eta'') \in \mathbb{R}^{d_1-p_1} \times \mathbb{R}^{d_2-p_2-1}\), so that coordinates on \( N \) are \((x, y') = (\tilde{x}, x'', \tilde{y}, y'')\). The evolution variable remains \( y_1 \).

Initial data for a solution \( u(x, y) \) is given on \( N \), which is expressed in the form \((u, \partial_{x''} u, \partial_{y''} u, \partial_{\beta''} u)(\tilde{x}, \tilde{y}, 0, 0) = (w_0, w_{\alpha''}, w_{\beta''}, w_{\alpha''}\beta'')(\tilde{x}, \tilde{y})\), where \( \alpha'' = (\alpha_{p_1+1}, \ldots, \alpha_{d_1}) \) and \( \beta'' = (\beta_{p_2+1}, \ldots, \beta_{d_2}) \) are multi-indices such that \(|\alpha''| + |\beta''| + |\beta_1| = 1\). The idea is the same as in §3.a,b, namely to extend \((u_0, w_{\alpha''}, w_{\beta''}, w_{\alpha''}\beta'')\) to constraint-satisfying Cauchy data on \( N \) in such a way that a solution \( u(x, y) = u(\tilde{x}, x'', \tilde{y}, y'')\) to the field equation (1.2) satisfies

\[
u(\tilde{x}, 0, 0, \tilde{y}, 0) = w_0(\tilde{x}, \tilde{y})
\]

and

\[
\partial_{y_1} u(\tilde{x}, 0, 0, \tilde{y}, 0) = w_{0\beta_1}(\tilde{x}, \tilde{y}),
\]

as well as the compatibility conditions

\[
\partial_{x''} u(\tilde{x}, 0, 0, \tilde{y}, 0) = w_{(\alpha'', 0)}(\tilde{x}, \tilde{y})
\]

and

\[
\partial_{y''} u(\tilde{x}, 0, 0, \tilde{y}, 0) = w_{(0, \beta'')}(\tilde{x}, \tilde{y}).
\]

The existence of such an extension follows as in theorems 3.4 and 3.5 from the construction of an extension operator \( E \) with certain boundedness properties on appropriate Sobolev spaces. We will focus our analysis therefore on the extension operators.
Again following §3a, define an extension operator

$$E(w)(x, y') = \frac{1}{\sqrt{2\pi^{d_1+d_2-1}}} \int \chi(\tilde{\xi}, \xi'', \tilde{\eta}, \eta'') \, d\xi'' \, d\eta'' = 1.$$ 

Furthermore, to satisfy the constraint that $E(w) \in X^C$ for arbitrary data $w$, we ask that

$$\text{supp}(\chi(\xi, \eta')) \subseteq \{(\xi, \eta') : |\eta'|^2 < |\xi|^2\} := R_1.$$ 

These two conditions are always satisfiable, except in the case $\xi'' = \{0\}$, meaning that $d_1 = p_1$ and the extension subspace $\{(\xi'', \eta'')\}$ is purely time-like.

It is to be expected that the constraint induces a restriction on the data $w(\tilde{\xi}, \tilde{\eta})$ in the vicinity of the ‘light cone’ $|\tilde{\xi}| = |\tilde{\eta}| \subseteq \hat{M}$. Subdividing $\hat{M}$ into two sets, we obtain

$$\tilde{R}_1 := \{(\tilde{\xi}, \tilde{\eta}) \in \hat{M} : |\tilde{\eta}| \leq |\tilde{\xi}|\}$$

and

$$\tilde{R}_2 := \{(\tilde{\xi}, \tilde{\eta}) \in \hat{M} : |\tilde{\eta}| > |\tilde{\xi}|\}.$$ 

The orthogonal projections onto functions supported in $\tilde{R}_1$ and $\tilde{R}_2$, respectively, are denoted $\pi_1$ and $\pi_2$. We use standard Sobolev spaces to quantify data supported in $\tilde{R}_1$, namely

$$H^r = \left\{ w(\tilde{x}, \tilde{y}) \in \text{range}(\pi_1) : \|w\|_{H^r}^2 = \int_{\tilde{R}_1} |\hat{w}(\tilde{\xi}, \tilde{\eta})|^2 (|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^r \, d\xi \, d\eta < +\infty \right\}.$$ 

Over $\tilde{R}_2$, we use a modified form of the Sobolev norm that is given by

$$K^r = \left\{ w(\tilde{x}, \tilde{y}) \in \text{range}(\pi_2) : \|w\|_{K^r}^2 = \int_{\tilde{R}_2} |\hat{w}(\tilde{\xi}, \tilde{\eta})|^2 \frac{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^r}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{e_0/2}} \, d\xi \, d\eta < +\infty \right\},$$

where

$$e_0 := d_1 + d_2 - (p_1 + p_2) - 1.$$ 

We note that, in the case where $d_1 = p_1$, we have $\tilde{R}_1 = \{0\}$ and $K^r = H^{r-(r/2)(d_2-p_2-1)}$. More generally, define

$$K^r_s = \left\{ w(\tilde{x}, \tilde{y}) \in \text{range}(\pi_2) : \|w\|_{K^r_s}^2 = \int_{\tilde{R}_2} |\hat{w}(\tilde{\xi}, \tilde{\eta})|^2 \frac{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^r}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{(1/2)\alpha+s}} \, d\xi \, d\eta < +\infty \right\}.$$ 

Decompose an arbitrary function $w = \pi_1 w + \pi_2 w$, so that its components possess Fourier support in $\tilde{R}_1$ and $\tilde{R}_2$, respectively.
Theorem 3.6. If $d_1 > p_1$, then there is a choice of kernel $\chi$ (indeed there are many such choices) such that $u = E(w)$ satisfies

$$\|u\|_{L^2}^2 \leq C \left( \|\pi_1 w\|_{H^{-(1/2)}e_0}^2 + \|\pi_2 w\|_{K^0}^2 \right).$$

Higher Sobolev norms of $u = E(w)$ are bounded as follows:

$$\|u\|_{H^r}^2 \leq C_r \left( \|\pi_1 w\|_{H^{r-(1/2)}e_0}^2 + \|\pi_2 w\|_{K^r}^2 \right).$$

In case $d_1 = p_1$, it is not possible to extend arbitrary data to a function $u = E(w)$ that satisfies the constraint $\text{supp}(\hat{u}(\xi, \eta')) \subseteq R_1$. However, if initially $\text{supp}(\hat{w}(\xi, \eta')) \subseteq \tilde{R}_1$ (i.e. $w = \pi_2 w$), then such an extension is possible, and we have, for $u = E(w)$,

$$\|u\|_{L^2}^2 \leq C \|w\|_{K^0}^2$$

and

$$\|u\|_{H^r}^2 \leq C_r \|w\|_{K^r}^2.$$

Proof. The proof of theorem 3.6 depends upon the construction of a kernel $\chi(\tilde{\xi}, \tilde{\eta}, \tilde{\xi}'', \tilde{\eta}'')$ with satisfactory properties. This construction is slightly different in the two different regions of Fourier space

$$\tilde{R}_1 := \{(\tilde{\xi}, \tilde{\eta}) : |\tilde{\eta}| \leq |\tilde{\xi}|\} \quad \text{and} \quad \tilde{R}_2 := \{(\tilde{\xi}, \tilde{\eta}) : |\tilde{\eta}| > |\tilde{\xi}|\},$$

where we note that the region $\tilde{R}_2$ contains the data that would lead to an ill-posed initial value problem if $M$ were considered itself as a codimension-one hypersurface.

To extend data posed on region $\tilde{R}_1$, define

$$\chi_1(\tilde{\xi}, \tilde{\eta}, \tilde{\xi}'', \tilde{\eta}'') := \psi_1 \left( \frac{\tilde{\xi}''}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2}}, \frac{\tilde{\eta}''}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2}} \right) \frac{1}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2} e_0},$$

where $\psi_1(\theta_1, \theta_2)$ is a $C^\infty$ function of $(d_1 - p_1) \times (d_2 - p_2 - 1)$ variables, respectively, with support in the set $|\theta_2| < |\theta_1|$. Therefore, $\chi_1$ has support in the set

$$\frac{\tilde{\xi}''}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2}} \geq \frac{\tilde{\eta}''}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2}},$$

implying that

$$|\eta'|^2 = |\tilde{\eta}|^2 + |\tilde{\eta}'|^2 < |\tilde{\xi}|^2 + |\tilde{\xi}'|^2 = |\xi|^2.$$
This is the appropriate region of support from functions \( v = E(w) \) to lie in the constraint-satisfying subspace of \( L^2(N) \). In order that \( E \) be an extension operator, we furthermore require that

\[
\sqrt{2\pi}^{d_1+d_2-1} = \int \chi_1(\tilde{\xi}, \tilde{\eta}, \xi'', \eta'') \, d\xi'' \, d\eta'' = \int \psi_1 \left( \frac{\xi''}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2}}, \frac{\eta''}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{1/2}} \right) \frac{1}{(|\tilde{\xi}|^2 + |\tilde{\eta}|^2)^{(1/2)e_0}} \, d\xi'' \, d\eta''
\]

\[
= \int \psi_1(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2.
\]

Asking that this latter integral equals the normalizing constant \( \sqrt{2\pi}^{d_1+d_2-1} \), and asking for \( \psi_1 \) to be even in its variables \((\theta_1, \theta_2)\) gives an acceptable kernel for the extension operator. We note again that this choice of kernel is highly non-unique.

On the region \( \tilde{R}_2 = \{ (\tilde{\xi}, \tilde{\eta}) \in \tilde{M} : |\tilde{\eta}| > |\tilde{\xi}| \} \), we can also attempt a construction of our extension operator. By itself, this region would give rise to data in \( L^2(M) \) for which the Cauchy problem of mixed type is ill-posed. The extension operator will nonetheless come up with data \( u = E(w) \) for which the well-posedness constraint is satisfied, if this is possible. That is, as long as \( d_1 > p_1 \), so that \( \{ \xi'' \} \) is not restricted to a zero-dimensional vector space, extensions can be found in a way that the defect, in satisfying the constraint caused by the fact that \( |\tilde{\xi}| < |\tilde{\eta}| \), can be compensated by a choice of large \( |\xi''| \). In practice, we will build \( \chi_2(\tilde{\xi}, \tilde{\eta}, \xi'', \eta'') \) so that its support is in the regions

\[ \{ |\tilde{\eta}| > |\tilde{\xi}| \} = \tilde{R}_2, \]

as well as

\[ \{ |\tilde{\eta}|^2 + |\eta''|^2 < |\tilde{\xi}|^2 + |\xi''|^2 \}, \]

implying that \( 0 \leq (|\tilde{\eta}|^2 - |\tilde{\xi}|^2) + (|\eta''|^2 + |\xi''|^2) \). Thus, we require \( d_1 > p_1 \). Following the above examples, assume that \( d_1 > p_1 \) and set

\[
\chi_2(\tilde{\xi}, \tilde{\eta}, \xi'', \eta'') := \psi_2 \left( \frac{\xi''}{(|\tilde{\xi}|^2 - |\tilde{\xi}|^2)^{1/2}}, \frac{\eta''}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{1/2}} \right) \frac{1}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{(1/2)e_0}},
\]

for \((\tilde{\xi}, \tilde{\eta}) \in \tilde{R}_2\). Let \( \psi_2(\theta_1, \theta_2) \) be a \( C^\infty \) function of \( e_0 = d_1 + d_2 - (p_1 + p_2) - 1 \) many variables as before and require that

\[
\int \psi_2(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2 = \int \psi_2 \left( \frac{\xi''}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{1/2}}, \frac{\eta''}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{1/2}} \right) \times \frac{1}{(|\tilde{\eta}|^2 - |\tilde{\xi}|^2)^{(1/2)e_0}} \, d\xi'' \, d\eta'' = (2\pi)^{e_0/2}.
\]

Furthermore, we require that \( \psi(\theta_1, \theta_2) \) be even in \((\theta_1, \theta_2)\). Finally, we also require that the support of \( \psi(\theta_1, \theta_2) \) be in the set

\[ \{ (\theta_1, \theta_2) : \theta_1^2 - \theta_2^2 > 1 \}. \]
Such requirements are satisfied by many possible choices of $\psi$. Imposing these requirements gives a satisfactory kernel of an extension operator $E$ with the property that all functions $u = E(w)$ in its range have Fourier support satisfying $\text{supp}(\hat{u}) \subseteq \{ |\eta|^2 < |\xi|^2 \}$. The singularities introduced at the boundaries of the light cone $\{ |\tilde{\eta}| = |\tilde{\xi}| \} \subseteq \tilde{M}$ by the kernel $\chi_2$ impose more severe constraints on the functions $w$ that are permitted in the domain of the operator $E$; this is the origin of the somewhat unusual requirements on functions $w(\tilde{x}, \tilde{y})$ from which we can reasonably draw our data. The Sobolev estimates of the proof are similar to those of theorems 3.4 and 3.5, and we leave the details to the reader. \hfill \blacksquare

Finally, we show that a sufficiently large class of data $(w_0, w_{(\alpha'', 0)}, w_{(0, \beta''}, w_{\beta_1})$ on $M$ extends to Cauchy data on the hypersurface $N$ that is both of finite energy and which satisfies the constraint. This extension is given by

$$
\begin{align*}
    u_0(x, y') &= E(w_0)(x, y') + \sum_{|\alpha''|=1} x^{n\alpha''} E(w_{(\alpha'', 0)})(x, y') + \sum_{|\beta''|=1} y^{n\beta''} E(w_{(0, \beta''}))(x, y') \\
    u_1(x, y') &= E(w_{(0, \beta_1)})(x, y').
\end{align*}
$$

(3.2)

By design, this Cauchy data satisfies the constraint, i.e. $(u_0, u_1) \in X^C$, the centre manifold. As before, its restriction to $M$ reduces to the data $(w_0, w_{(\alpha'', 0)}, w_{(0, \beta''}, \tilde{x}, \tilde{y}, 0)$.

The only remaining task is to show that its energy norm is finite. Recall that, in this context, the energy norm is

$$
H(u_0, u_1) = \frac{1}{2} \int_N |u_1|^2 + |\nabla_x u_0|^2 - |\nabla_y u_0|^2 \, dx \, dy
= \frac{1}{2} \int_N |\hat{u}_1(\xi, \eta')|^2 + (|\xi|^2 - |\eta'|^2) |\hat{u}_0(\xi, \eta')|^2 \, d\xi \, d\eta'.
$$

To show that this energy is finite for the extension (3.2), we use the results of theorem 3.6.

**Theorem 3.7.** Given data $(w_0, w_{(\alpha'', 0)}, w_{(0, \beta''}, w_{\beta_1})$ on $M$ with $|\alpha''| = |\beta''| = 1$, suppose that

$$
\| \pi_1 w_0 \|_{H^{q_0+1}} + \sum_{|\alpha''|=1} \| \pi_1 w_{(\alpha'', 0)} \|_{H^{q_0+1}} + \sum_{|\beta''|=1} \| \pi_1 w_{(0, \beta''}) \|_{H^{q_0+1}} < +\infty,
\quad (3.3)
$$

$$
\| \pi_2 w_0 \|_{K^1} + \sum_{|\alpha''|=1} \| \pi_2 w_{(\alpha'', 0)} \|_{K^1} + \sum_{|\beta''|=1} \| \pi_2 w_{(0, \beta''}) \|_{K^1} < +\infty
\quad (3.4)
$$

and

$$
\| \pi_1 w_{\beta_1} \|_{H^{q_0}} + \| \pi_2 w_{\beta_1} \|_{K^0} < +\infty.
$$

Then, the extension $(u_0, u_1)$, given by expression (3.2), has finite energy and lies in the centre subspace $X^C$. If $d_1 = p_1$, then we have to ask that $\pi_2 w_1 = 0$ in the above statement, for all multi-indices $\gamma$ in question.
Proof. Estimates on the contributions of $u_0$ to $u_0$ follow immediately from theorem 3.6, as do the estimates for $u_1 = E(u_{0}, y)$. Therefore, we have to consider only contributions in one of the two possible forms,

$$x^{\alpha''} E(w(\alpha'', 0)), \quad |\alpha''| = 1,$$

or

$$y^{\beta''} E(w(0, \beta'')), \quad |\beta''| = 1.$$ 

The energy norm includes the quantities $\|x^{\alpha''} E(w(\alpha'', 0))\|_{H^1}$ and $\|y^{\beta''} E(w(0, \beta''))\|_{H^1}$; as the estimates are similar, we will give a sketch of one of them,

$$\|x^{\alpha''} E(w(\alpha'', 0))\|_{H^1}^2 = \frac{1}{2} \left[ \partial_{\xi''} \psi_1 \left( \frac{\xi''}{(|\xi''|^2 + |\eta''|^2)^{1/2}}, \frac{\eta''}{(|\xi''|^2 + |\eta''|^2)^{1/2}} \right) \right] \times \left[ \int_{|\xi''| = |\eta''|} \left( \int_{|\xi''|^2 - |\xi''|^2} \frac{1}{(|\xi''|^2 - |\eta''|^2)^{1/2}} \frac{\eta''}{(|\xi''|^2 - |\eta''|^2)^{1/2}} \right) \right] \int_{\xi''}^{\eta''} \int_{\eta''}^{\eta''} \psi_2 \left( \frac{\xi'}{(|\xi'|^2 - |\eta'|^2)^{1/2}}, \frac{\eta'}{(|\xi'|^2 - |\eta'|^2)^{1/2}} \right) \times \left[ \int_{|\eta'| = |\xi'|} \left( \int_{|\eta'|^2}^{\xi'} \frac{1}{(|\eta'|^2 - |\xi'|^2)^{1/2}} \frac{\eta''}{(|\eta'|^2 - |\xi'|^2)^{1/2}} \right) \right] \int_{\xi'}^{\eta'} \int_{\eta'}^{\eta'} d\xi' d\eta' \int_{\xi'}^{\eta'} d\eta'.

The $\xi''$-derivative introduces one extra factor of $(|\xi''|^2 + |\eta''|^2)$, respectively $(|\eta''|^2 - |\xi''|^2)$, into the denominator. The integral over $(\xi'', \eta'')$ gives a constant, depending upon $\psi_1$ and $\psi_2$, as a bound, while the resulting integral over the variable $(\tilde{\xi}, \tilde{\eta})$ is bounded by the $H^{\alpha_{1}+1}$ norm (respectively, the $K_{1}^{1}$ norm) of $w(\alpha'', 0)$. This completes the proof.

4. Failure of uniqueness in higher codimension

The question addressed in this section is the uniqueness of solutions with prescribed initial data on a hypersurface $M$ of codimension greater than one. This is a non-trivial issue if one requires that solutions exist globally in space-time, which has been the focus of the analysis in the preceding sections. In §3, we showed that initial data, consisting of the values of the solution $u(x, y)$ and its first normal derivatives on $M$, through a procedure of extension, give rise to constraint-satisfying Cauchy data on a codimension-one hypersurface $N$. These extensions are highly non-unique, and so are the resulting global solutions.

We now raise the question whether prescribing an arbitrarily large, but finite, number of normal derivatives on $M$, as well as insisting upon global solutions, would remedy the non-uniqueness. This data should satisfy the compatibility conditions implied by the commuting of mixed partial derivatives and by equation (1.2). Given the classic result of Courant (1962) in the case of purely time-like $M$, that data given in any $\varepsilon$-tubular neighbourhood of $M$ within
N determine solutions uniquely in the $C^2$ category, one might think that specifying additional data for $u(x, y)$ on $M$ would suffice. In fact, if one specifies any finite number of derivatives of $u$ on $M$, it does not.

**Theorem 4.1.** Given $k$, there exist constraint-satisfying data $u_0$ and $u_1$ on $N$ that vanish to order $k$ on $M$.

Therefore, there exists a globally defined solution $u(x, y)$ having initial data $u(x, y) = u_0, \partial_x u(x, y) = u_1$ on $N$, which vanishes to order $k$ on $M$. Hence, any other solution $v(x, y)$ that takes on specified data on $M$ up to $k$-many derivatives may be changed by adding this solution $u$ to it, without changing its initial data.

**Proof.** We follow a construction that was used for the extension operators of §3.

Let $\chi_3(ξ, η')$ be a Schwartz class function with support in the set $\{ |η'|^2 < |ξ|^2 \} \subseteq \hat{N}$. Its Fourier restriction to $\hat{M}$, given by

$$\int \int \chi_3(\tilde{ξ}, \tilde{η}, \xi'', \eta'') \, dξ'' \, dη'' = \mu(\tilde{ξ}, \tilde{η}),$$

is in Schwartz class in $\hat{M}$. Because of the support of $\chi_3$,$$v(x, y') = (F^{-1} \chi_3)(x, y')$$satisfies the constraint. While $v$ may be non-zero on $M$, as may its derivatives, it is the case that, for homogeneous polynomials $p_{k+1}(x'', y'')$ of degree $k + 1$, the function $p_{k+1}(x'', y'')v(x, y')$ on $N$ vanishes on $M$ to at least order $k$. Furthermore, $p_{k+1}v$ satisfies the constraint. Indeed,$$(Fx_{k+1}v)(ξ', η') = p_{k+1} \left( \frac{1}{i} \partial_ξ'', \frac{1}{i} \partial_η'' \right) \chi_3(ξ', η'),$$and differential operators do not affect the support. Set data $u_0(x, y') = (p_{k+1}v)(x, y')$ and $u_1 = 0$, and solve equation (1.2). Because this data satisfies the constraint, the solution $u(x, y)$ is global. Because of the properties of the initial data, all $x$ and $y'$ derivatives of $u(x, y)$ vanish on $M$. Because $u_1 = 0$ and $u$ itself satisfies equation (1.2), all $y_1$ derivatives up to order $k$, as well as any mixed derivatives, also vanish.

**5. A variant of a uniqueness theorem of Courant**

Courant (1962) gives a uniqueness result for the ultrahyperbolic equation with data posed on a hypersurface of mixed signature, which, in our notation, states that, among $C^2$ solutions, initial values of $u(x, 0, y')$ and $\partial_{y_1} u(x, 0, y')$ prescribed in the set in the Cauchy hypersurface $M$ given by

$$\sum_{\ell=1}^{d_1} (x_\ell - x_\ell^0)^2 \leq a^2 \quad \text{and} \quad \sum_{\ell=2}^{d_2} (y_\ell - y_\ell^0)^2 \leq \varepsilon^2 \quad (5.1)$$

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will determine \textit{a priori} the values of the data on the larger set
\[
\left\{(x, y) \in M : \sqrt{\sum_{\ell=1}^{d_1} (x_\ell - x_\ell^0)^2 + \sum_{\ell=2}^{d_2} (y_\ell - y_\ell^0)^2} \leq a \right\}. \tag{5.2}
\]
Furthermore, the solution is determined uniquely in the space–time region
\[
\left\{(x, y) \in \mathbb{R}^{d_1+d_2} : \sqrt{\sum_{\ell=1}^{d_1} (x_\ell - x_\ell^0)^2 + \sum_{\ell=1}^{d_2} (y_\ell - y_\ell^0)^2} \leq a \right\}. \tag{5.3}
\]
Courant’s proof of this fact uses the Asgeirsson mean value theorem in a fundamental way.

The key implication from our point of view is that data on an arbitrarily small cylindrical subset of $M$, plus the stipulation of $C^2$ regularity, determine the data and indeed the solution on much larger sets of $M$ and of space–time, respectively. In turn, knowledge of the data in a small cylinder determines the values of all of its derivatives on $N = \{(x, y) : y = 0\}$ (if the data are smooth). This contrasts with the case discussed in §4, in which it is shown that specification of a possibly large, but finite, number of derivatives does not lead to unique solutions, even when the constraint is imposed and the resulting solutions are globally defined and smooth.

In this section, we give a version of the above theorem of Courant, for data posed in ellipsoidal domains in the Cauchy hypersurface $M$, which are localized near the $\{y' = 0\}$ coordinate axis (or any translate thereof). Our proof of this result is based on the Holmgren–John theorem (John 1982) and therefore remains true under perturbations to the equation. Thus, it is a robust generalization of the Courant result, which being based on Asgeirsson’s theorem is true only for the ultrahyperbolic equation.

**Theorem 5.1.** Let $\varepsilon > 0$ and define the ellipsoid $Z_\varepsilon \subseteq M$ by
\[
Z_\varepsilon = \left\{(x, y) : y_1 = 0, |x|^2 + \frac{|y'|^2}{\varepsilon^2} < 1 \right\}, \quad 0 < \varepsilon \leq 1. \tag{5.4}
\]
A $C^2$ solution to equation (1.2) whose Cauchy data vanish on $Z_\varepsilon$ must necessarily vanish on the set
\[
D = \{(x, y) \in \mathbb{R}^{d_1+d_2} : |x| + |y| < 1\}
\]
and, in particular, its Cauchy data, along with all derivatives, must vanish on the subset of the Cauchy hypersurface given by $\{(x, y') \in M : |x| + |y'| < 1\}$.

**Proof.** Define $R_\varepsilon(w)$ to be the cone over $Z_\varepsilon$ with vertex $v = (0, w_1, w') \in \{(x, y) : x = 0\}$. We will show that, for any $w = (w_1, w')$ with $|w| \leq 1$ (namely the unit sphere in $\mathbb{R}^{d_2}$), the region between the cone $R_\varepsilon(w)$ and the ellipsoid $Z_\varepsilon$ is a region of determinacy for the ultrahyperbolic equation. The closure of the envelope of such ellipsoidal cones includes the region $D$; in fact, it is slightly larger. The result will follow accordingly.

Courant works with sections of cylinders that are $\varepsilon$-tubular neighbourhoods of the Cauchy hypersurface $M$. 

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For a given $R_ε(w)$, the Holmgren–John theorem is based on the construction of an analytic family of non-characteristic hypersurfaces $S_λ$ with which to sweep the region between $Z_ε$ and $R_ε(w)$. Taking the case of the vertex $v = (0, w)$ with $w = e_1 := (1, 0)$, define

$$S_λ := \left\{ (x, y) : (1 - y_1)^2 - \left( |x|^2 + \frac{|y'|^2}{ε^2} \right) = -λ \right\},$$

with $-1 ≤ λ ≤ 0$. The normal to $S_λ$ is $N_λ = -2(x, (1 - y_1), y'/ε^2)^T$, so that the characteristic form calculated on $N_λ$ is

$$\frac{1}{4} N_λ^T \left( -I_{d_1 × d_1} \ 0 \ 0 \ I_{d_2 × d_2} \right) N_λ = \frac{1}{4} \left( -|x|^2 + (1 - y_1)^2 + \frac{|y'|^2}{ε^2} \right).$$

Taking into account that $(x, y_1, y') ∈ S_λ$ and solving for $(1 - y_1)^2$,

$$\frac{1}{4} N_λ^T \left( -I_{d_1 × d_1} \ 0 \ 0 \ I_{d_2 × d_2} \right) N_λ = \frac{1}{4} \left( \left( \frac{1 + ε^2}{ε^4} \right) |y'|^2 - λ \right).$$

Recalling that $λ < 0$ (except in the limiting case $S_λ → R_ε$), observe that this family of hyperboloids constitutes a non-characteristic analytic family that sweeps the region between $Z_ε$ and $R_ε(e_1)$. Thus, the Holmgren–John uniqueness theorem applies, and this region is a region of determinacy for the ultrahyperbolic equation (1.2).

We have already achieved the analogue of the statement (5.3) of Courant. Namely, given the values of a $C^2$ solution $u(x, y)$ to equation (1.2) in the space–time ellipsoid

$$W_ε := \left\{ (x, y) : |x|^2 + \frac{|y|^2}{ε^2} < 1 \right\},$$

we may slice it with a hyperplane that contains the $x$-coordinate axes, but which is otherwise arbitrarily oriented in $y$, to determine a possible $Z_ε$, which in turn determines the solution over the larger conical region $R_ε$ with base $Z_ε$. All of these regions have been shown to be domains of determinacy. Their union contains the set $D = \{(x, y) : |x| + |y| < 1\}$. Therefore, if a solution vanishes in $W_ε$, it must also vanish in $D$, implying that $D$ is contained in the domain of determinacy of $W_ε$.

Returning to the problem of the domain of determinacy of the set $Z_ε ⊆ M$, we generalize the above construction to any $w ∈ \mathbb{R}^{d_2}$ with $|w| = 1$. Let $w = Re_1$ for $e_1 = (1, 0, \ldots)$, where $R$ is an orthogonal matrix. Changing variables to $z = Ry$ and using a symmetric matrix $Q$ of signature $(-, +, +, \ldots)$, an analytic family of hyperboloids is given by

$$S_λ(w) := \{(x, z) : |x|^2 + \langle (z - e_1), Q(z - e_1) \rangle = λ\},$$

where the Euclidean inner product is given by $\langle \cdot, \cdot \rangle$. The matrix $Q$ is to be chosen so that the intersections of the hyperboloids $S_λ(w)$ with the hypersurface $M$ lie in $Z_ε$ and sweeps it as $λ$ is varied.

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At this point, we may assume, without loss of generality, that \( w = (w_1, w') = (w_1, w_2, 0 \ldots) \), whereupon \( Q \) may be chosen such that

\[
Q = \begin{pmatrix} Q_2 & 0 \\ 0 & \frac{1}{\varepsilon^2} I'' \end{pmatrix}
\]

and \( Q_2^T = Q_2 \),

for \( Q_2 \) a \( 2 \times 2 \) symmetric matrix with signature \((- , +)\). Furthermore, the above rotation is then set to be

\[
R = \begin{pmatrix} R_2 & 0 \\ 0 & I'' \end{pmatrix}
\]

and

\[
R_2 = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.
\]

In \( y \)-coordinates, the hyperboloid family is expressed as

\[
S_\lambda(w) := \{(x, y) : |x|^2 + (y - w), R^T Q R(y - w) = \lambda \},
\]

and the stipulation is that \( S_\lambda(w) \) should intersect the hypersurface \( M \) in the original ellipsoid \( Z_\varepsilon \). This imposes the condition that

\[
|x|^2 + ((x, 0, y'), R^T Q R(x, 0, y')) := |x|^2 + ((x, 0, y'), B(x, 0, y')) = |x|^2 + \frac{1}{\varepsilon^2} |y'|^2,
\]

where \( B_2 \) is the upper left-hand \( 2 \times 2 \) block of the matrix \( B \). Therefore, one finds the matrix elements of \( B_2 \)

\[
b_{11} = -\frac{\varepsilon^2 - \sin^2(\theta)}{\varepsilon \cos^2(\theta)}, \quad b_{12} = -\frac{\tan(\theta)}{\varepsilon^2} \quad \text{and} \quad b_{22} = \frac{1}{\varepsilon^2},
\]

and furthermore, the \( 2 \times 2 \) matrix \( Q_2 \) is

\[
Q_2 = \begin{pmatrix} -1 & \tan(\theta) \\ \tan(\theta) & \frac{1}{\varepsilon^2} a \end{pmatrix},
\]

\[ \quad \text{(5.5)} \]

where \( a = a(\varepsilon, \theta) = (1 + (1 - \varepsilon^2) \tan^2(\theta)) \). Calculating the characteristic form on the hyperboloids \( S_\lambda(w) \), we compute the normal \( N_\lambda(w) \) as

\[
-\frac{1}{2} N_\lambda(w) = (x, Q(z - e_1))^T.
\]

Noting that the characteristic form is invariant under rotations \( R \) as above, which leave the coordinate subspaces \( \mathbb{R}^d_{x} \) and \( \mathbb{R}^d_{y} \) invariant, we find that

\[
\frac{1}{4} N_\lambda(w)^T \begin{pmatrix} -I_{d_1 \times d_1} & 0 \\ 0 & I_{d_2 \times d_2} \end{pmatrix} N_\lambda(w) = -|x|^2 + \langle (z - e_1), Q^2 (z - e_1) \rangle.
\]

This is evaluated on the hyperboloid \( S_\lambda(w) \), on which

\[
|x|^2 + ((z - e_1), Q(z - e_1)) = \lambda.
\]

Solving for \( |x|^2 \), we find

\[
\frac{1}{4} N_\lambda(w)^T \begin{pmatrix} -I_{d_1 \times d_1} & 0 \\ 0 & I_{d_2 \times d_2} \end{pmatrix} N_\lambda(w) = \langle (z - e_1), [Q^2 + Q](z - e_1) \rangle - \lambda.
\]
Specifically, the matrix \([Q^2 + Q]\) is
\[
[Q^2 + Q] = \begin{pmatrix}
Q_2^2 + Q_2 & 0 \\
0 & \left(1 + \frac{\epsilon^2}{\epsilon^4}\right) I''
\end{pmatrix}.
\]

Using the form (5.5) for \(Q_2\), one calculates
\[
[Q_2^2 + Q_2] = \begin{pmatrix}
\tan^2(\theta) & \frac{a}{\epsilon^2} \tan(\theta) \\
\frac{a}{\epsilon^2} \tan(\theta) & \frac{a^2}{\epsilon^4} + \tan^2(\theta)
\end{pmatrix}.
\]

It is easily verified that this is positive definite. Recalling that \(\lambda \leq 0\) in the definition of the analytic families of hyperboloids, it follows that \(S_\lambda(w)\) are all non-characteristic and hence the Holmgren–John theorem applies, thus completing the argument.

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