Deformation Quantization: From Quantum Mechanics to Quantum Field Theory

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September 30, 2018

Abstract

The aim of this paper is to give a basic overview of Deformation Quantization (DQ) to physicists. A summary is given here of some of the key developments over the past thirty years in the context of physics, from quantum mechanics to quantum field theory. Also, we discuss some of the conceptual advantages of DQ and how DQ may be related to algebraic quantum field theory. Additionally, our previous results are summarized which includes the construction of the Fedosov star-product on dS/AdS. One of the goals of these results was to verify that DQ gave the same results as previous analyses of these spaces. Another was to verify that the formal series used in the conventional treatment converged by obtaining exact and nonperturbative results for these spaces.

1 Introduction

There are three standard approaches to quantum theories: the operator formalism, the path integral formalism, and deformation quantization (DQ). The aim of this proceedings is to inform the reader of the state of the DQ in terms of issues relevant to physicists: from quantum mechanics to quantum field theory.

The main problem in DQ is the issue of convergence of all perturbative series which remain unknown. To address this issue, I will summarize some of the results of [TiSp1] which includes the computation of the Fedosov star-product (the fundamental object used in DQ) exactly for the dS and AdS space-times. Another goal of the results were to reproduce previous results for the Klein-Gordon (KG) equation on dS and AdS.[Fro1-4]

The question is: Why bother with DQ at all? The reason we do is that DQ provides some distinct advantages over canonical quantization and the path integral methods. One example is that it is not only coordinate invariant but also independent of the choice of dynamics (e.g. Lagrangian). The associativity of the star-product plays a fundamental role in understanding how DQ deviates from canonical quantization as can be seen in [Til,GoRe].

In this paper we will discuss other advantages. For instance, the observables in DQ are functions on phase-space just as they are classically. Therefore, the conceptual break with classical mechanics is less severe than with, for example,
operator methods which map observables to fundamentally different objects. Most of the tools used for functions, whether they are geometric or algebraic, extend much more naturally into DQ than in other quantization methods.

Furthermore, it is argued in [DüFr] that perturbative algebraic quantum field theory (AQFT) can be understood in terms of DQ. It is argued that this is the likely scenario because of the many similarities between the two approaches. Conventional treatments of DQ rely heavily on perturbative expansions in the formal parameter $\hbar$ which makes it easy to obtain perturbative results for various physical quantities. However, besides the perturbative nature of DQ, AQFT focuses on the algebra of observables just as DQ does. Other similarities to AQFT are noticed like the way in which the topology of ordinary phase-space functions induces observable topology in DQ to form the "net of observables" in AQFT.[Haag,Buc]

2 An Introduction to Deformation Quantization

In 1927 Herman Weyl wrote his quantization rule $\mathcal{W}$ that maps every phase-space function $f$ on a flat space-time to a unique observable in the space of linear operators acting on the appropriate Hilbert space. Shortly afterwards, Eugene Wigner wrote the inverse map $\mathcal{W}^{-1}$. Groenewold in 1946 [Gro] (and later Moyal in 1949 [Moy]) investigated the formula $\mathcal{W}^{-1}(\mathcal{W}(f)\mathcal{W}(g))$ and found a remarkable result:

\[
\mathcal{W}^{-1}(\mathcal{W}(f)\mathcal{W}(g)) = f \exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial p_\nu} - \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial x^\nu} \right) \right] g
\]

(1)

where the operator inside the exponential is the Poisson Bracket and the arrows over the derivatives explain the direction in which they act.

Moreover, Groenewold (and again later Moyal) realized that this operator is an associative, noncommutative product of the two phase-space functions $f$ and $g$ defined by $f * g := \mathcal{W}^{-1}(\mathcal{W}(f)\mathcal{W}(g))$ which has the familiar commutators:

\[
[x_\mu, p_\nu]_* = i\hbar \delta^\mu_\nu \quad , \quad [x_\mu, x_\nu]_* = 0 = [p_\mu, p_\nu]_*
\]

In a coordinate independent formulation we have:

\[
f * g = f \exp \left[ \left( \overset{\leftarrow}{\overset{\rightarrow}{\mathcal{P}}} \right) \right] g
\]

(2)

\[
\overset{\leftarrow}{\overset{\rightarrow}{\mathcal{P}}} := \frac{\partial}{\partial A} i\hbar \omega_{AB} \overset{\leftarrow}{\overset{\rightarrow}{\partial}}_B
\]

where $\overset{\leftarrow}{\overset{\rightarrow}{\mathcal{P}}}$ is the Poisson bracket and $\partial_A$ is a (flat) torsion-free phase-space connection ($\partial \otimes \omega = 0$).

In summary, they obtained another equivalent formulation of the quantum theory on phase-space.[HWW,HiHe]

Despite all of this, it wasn’t until 1978 when Bayen F., Flato M., Frønsdal C., Lichnerowicz A., Sternheimer D. proposed an alternative formulation of quantum theory on the phase-space of an arbitrary curved space-time that we now know as DQ. The new formulation (and a quite radical one) can aptly be summarized by a quote from their paper:

"We suggest that quantization be understood as a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of the observables"—Bayen et al. (1978)

The view here is that quantum mechanics is a theory on a classical phase-space by the replacement of the pointwise product by a new associative but noncommutative star-product. This star-product is a pseudodifferential operator i.e. an operator of the form:
\[
f \ast g = \sum_{A,B,j,l,m}^{\infty} (\frac{i\hbar}{2})^{j} G_{j,l,m}^{A_1 \cdots A_l B_1 \cdots B_m} (D_{A_1} \cdots D_{A_l} f) (D_{B_1} \cdots D_{B_m} g)
\]

for any two phase-space functions \( f \) and \( g \), where \( D \) is a phase-space connection and \( G_{j,l,m}^{A_1 \cdots A_l B_1 \cdots B_m} \) is an \( l + m \) index tensor for each \( j, l, \) and \( m \). See [HiHe] for conditions on these coefficients due to Gerstenhaber.

A star-product is a very complicated object because of the infinite order of derivatives. In fact, it is this “infinite orderness” which is the source of most of the trouble in working with star-products and why we need such fancy tools in their construction and classification. Despite the difficult nature of these objects, in the following thirty years, huge advances have been achieved. One is the classification of star-products into equivalence classes due to the contributions of many people (see [DiSt] for a brief history):

**Thm.** All star-products on a symplectic manifold (a generalized phase-space) fall into equivalence classes which are parametrized by a formal series in \( \hbar \),

\[
B = B_0 + \hbar B_1 + \hbar^2 B_2 + \cdots
\]

with coefficients \( B_i \) in the second de Rham cohomology group \( H^2_{dR}[[\hbar]] \).

In each equivalence class, whether we describe our system with \( \ast_1 \) or \( \ast_2 \), all physical quantities (like expectation values) will be identical. Two equivalent products \( \ast_1 \) and \( \ast_2 \) are related by some invertible operator \( T \):

\[
T(f) = \sum_{A,j,l}^{\infty} (i\hbar/2)^j T_{j,l}^{A_1 \cdots A_l} (D_{A_1} \cdots D_{A_l} f)
\]

by the formula:

\[
f \ast_2 g = T^{-1} (T(f) \ast_1 T(g))
\]

In simplified terms, this parametrization is a two-form \( B \) that is closed (\( dB = 0 \)), but not necessarily exact (\( B = dA \)). It was shown in [Bord1] interpret \( B \) as a magnetic field in our space-time. The integral over any closed two-surface is the amount of magnetic monopole charge sitting inside which is directly correlated to the fact that \( B \) is closed but not exact. For example, if the magnetic monopole charge in our space-time is zero then all star-products are equivalent.

Another huge advance was developed by Fedosov in the construction of his Fedosov star-product on an arbitrary finite-dimensional symplectic manifold using geometric approaches. Fedosov, using insights from the index theorems of Atiyah and others, perturbatively constructed a star-product via an iterative process.[GdT,Fed] The convergence behavior of the Fedosov star is still unknown in the general case, but some specific exact formulas have been found.[TiSp1,TiSp2] With the convergence issues aside, the classification theorem above means that all star-products are equivalent to a Fedosov star. The properties of the Fedosov star are [Fed,TiSp2]:

1. It is coordinate invariant.
2. It can be constructed on all symplectic manifolds (including all phase-spaces) perturbatively in powers of \( \hbar \).
3. It assumes no dynamics (e.g. Hamiltonian or Lagrangian), symmetries, or even a metric.
4. The limit \( \hbar \to 0 \) yields classical mechanics.
5. It is equivalent to an operator formalism by a Weyl-like quantization map.[Fed,TiSp2]

Another key development was the application of DQ to quantum field theory (QFT) by Dito. In
Dito has successfully constructed nonperturbative star-products for both a free covariant field and a covariant field with a class of interaction terms which include polynomial ones. The interacting case is done by using a cohomological method of renormalization, called cohomological renormalization in which he uses a linearization program. This involves identifying the diverging terms as singular cocycles or coboundaries of the Hochschild cohomology in the star-product which can be removed by changing to an equivalent star-product.

The main problem in DQ, as I see it, is related to the standard treatments of deformation products which rely heavily on series expansions in a formal parameter \( \hbar \). Therefore, convergence of these series need to be addressed which is a purpose of the results given here. Additionally, the existence of a large enough set of states to describe physical systems (which includes a notion of vacuum) needs to be addressed especially for QFT’s. Moreover, for a star-product on a Maxwell-Dirac field to be constructed in a similar manner to that of Dito, the star-product of the asymptotic fields must be constructed (see [HHS,HiHe2] for work on fermions and bosons). Therefore, much work still needs to be done in this area.

Additional important developments of DQ include the application to statistical quantum mechanics where the KMS condition (a condition for a state to be in thermodynamic equilibrium at a defined temperature, see [BrRo,Haag]) was given in [Bas]. A formal definition of a KMS state of finitely many degrees of freedom was defined in [Bord2]. Also, a formal GNS construction of a Hilbert space associated to any star-product has been formulated in [BoWa] and yields the correct results in the standard representations such as the Bargmann and Schrödinger representations. Finally, Kontsevich in [Kon] has formulated a star-product, called the Kontsevich star, on an arbitrary finite dimensional Poisson manifold perturbatively in powers of \( \hbar \).

*Note: The difficulty of constructing star-products is exemplified by Maxim Kontsevich’s Fields Medal in 1998, won in part because of his brilliant construction of a star-product on arbitrary finite-dimensional Poisson manifold called the Kontsevich star.[Kon] Furthermore, this was the very first solution to a long-standing problem in mathematics: showing that any finite dimensional Poisson manifold admits a formal quantization.*

3 Quantum Mechanics on Phase-Space: A New Perspective of Quantum Theory

The important question is: What advantage does deformation quantization (DQ) have over the other standard formulations of quantum physics? Part of the answer to this lies in the radically different framework: DQ is a theory of quantum mechanics where observables are still phase-space functions. Therefore, the conceptual break with classical mechanics is less severe than the other two standard approaches.

Herein lies the advantage of DQ: Most techniques (geometrical or algebraic) one uses with ordinary functions are valid in, or can be adapted in a natural way into this framework. Therefore, DQ greatly expands our toolbox including tools used in basic algebra and geometry. For instance, coordinate invariance of quantum theory is easily manifest because the star product is coordinate invariant. Also, different operator orderings in quantization can be organized nicely into equivalence classes of star-products in DQ parametrized by the formal series \( B \). We note that there are examples of equivalent orderings in [Til,HiHe].

Finally, building a theory restricted to a compact region in our configuration space-time gives an
innately local quantum theory. This can be done by a simple restriction of set of all observables (which are just functions in a formal series of $\hbar$) to a region of compact support. In this sense one can build a local theory of quantum physics in an analogous to AQFT. [TiSp1, TiSp2, Til] Simply stated, the topology of observable algebra is directly and naturally induced by the topology of the algebra of functions on any given phase-space. Moreover, visualizing these things require much less mental work because the physical observables are the same functions they were in the classical theory.

This is contrary to the Hilbert space operator formulation in which phase-space functions get mapped to fundamentally different objects. So much on this set of operators is awkward and many other things are a lot easier. Therefore, if the hard work involved in DQ is the construction of star-products, but once you have star-product and in AQFT using [Haag, BrRo]. The bulk of the technical details, we obtain the exact commutators for the Fedosov star-product:

$$[x^\mu, x^{\nu}]_s = 0 \quad [x_\mu, M_{\nu\rho}]_s = i\hbar x_\nu \eta_\rho \eta_\mu$$

Omitting the technical details, we obtain the exact commutators for the Fedosov star-product:

$$[x^\mu, x^{\nu}]_s = 0 \quad [x_\mu, M_{\nu\rho}]_s = i\hbar x_\nu \eta_\rho \eta_\mu$$

$$M_{\mu\nu}, M_{\rho\sigma} = M_{\sigma\mu} \eta_\nu \eta_\rho - M_{\sigma\nu} \eta_\rho \eta_\mu$$

indices run from 0 to 4, $M_{\mu\nu} = x_\mu * p_\nu$, $x_\mu = \eta_{\mu\nu} x^{\nu}$.

The conditions of the embedding $x^\mu x_\mu$, $x^\mu p_\mu = x^\mu * p_\mu$ become the Casimir invariants of the algebra in group theoretic language.

We now summarize our two key observations:

1. $M$'s generate $SO(1,4)$ and $SO(2,3)$ for dS and AdS respectively.
2. $M$'s and $x$'s generate $SO(2,4)$ for both dS and AdS.

By calculating $R = -4C$ and $p_\mu * p^\mu$ in terms of $M$ and $x$, the Hamiltonian [3] is:

$$H = 2CM_{\mu\nu} * M^{\mu\nu} + (A - 4\hbar) AC - 4\xi C$$

where $M_{\mu\nu} * M^{\mu\nu}$ is a Casimir invariant of the subgroup $SO(1,4)$ or $SO(2,3)$ for dS or AdS respectively.

In the more familiar form of Hilbert space language the KG equation takes the form:

$$Tr_s (\rho_m) = 1 \quad \rho_m = \rho_m$$

where $*$ is the Fedosov star-product, $R(x)$ is the Ricci scalar, $p^\mu := g^{\mu\nu} p_\nu$, and $\xi \in \mathbb{C}$ is an arbitrary constant. [Fed,GdT] Also, $\rho_m$ is called a Wigner function and is defined by:

$$\rho_m := \sigma (|\phi_m\rangle \langle \phi_m|)$$

In the case of the dS and AdS space-times given by the embedding:

$$\eta_{\mu\nu} x^\mu x^\nu = 1/C \quad \text{and} \quad x^\mu p_\mu = A$$

The Klein-Gordon Equation

It has been shown in [TiSp1, TiSp2, Til] that the Klein-Gordon (KG) equation in an arbitrary (possibly curved) space-time may formulated using Fedosov’s Weyl quantization map $\sigma^{-1}$ (i.e. the analogue of $\mathcal{W}$) and its inverse $\sigma$ as:

$$H \star \rho_m = \rho_m \star H = m^2 \rho_m$$

$$H = p_\mu * p^\mu + \xi R(x)$$

$$(2C M_{\mu\nu} M^{\mu\nu} + \chi C) |\phi_m\rangle = m^2 |\phi_m\rangle$$
where \( \langle \phi_m | \phi_m \rangle = 1 \), \( \mathbb{C} \ni \chi = (A - 4i) A - 4 \xi \) is an arbitrary constant, and we regard all groups to be in a standard irreducible representation on the set of linear Hilbert space operators.

*Note: This result is confirmed by [Fro1-4] as well as others.

5 Covariant Free Field Quantization and the Dito Star-Product

According to Feynman, in QFT positive frequency (energy) solutions to the KG equation correspond to particles that moving forward in time while negative ones correspond to particles moving backwards in time. Particles moving backwards in time correspond to anti-particles moving forward in time. This is why Fourier modes are most suitable for QFT (see [HiHe] for more details).

Given the free KG equation:

\[
(\partial_\mu \partial^\mu - m^2 / \hbar^2) \phi (x, t) = 0
\]

on Minkowski space. First we decompose initial data \( \Phi (x, 0) := (\phi (x, 0), \pi (x, 0)) \) into Fourier modes of definite energy:

\[
\phi (x, 0) = \int_\Sigma \frac{d^3k}{2 (2\pi)^{3/2}} \omega (k) \left( \tilde{a}_k e^{-ik \cdot x} + a_k e^{ik \cdot x} \right)
\]

\[
\pi (x, 0) = i \int_\Sigma \frac{d^3k}{2 (2\pi)^{3/2}} \left( \tilde{a}_k e^{-ik \cdot x} - a_k e^{ik \cdot x} \right)
\]

where \( \pi (x) = \partial_t \phi (x) \) and \( \omega (k) := (k^2 + m^2)^{1/2} \).

The set of solutions \( \Phi (x, t) := (\phi (x, t), \pi (x, t)) \) to the KG equation is a (infinite-dimensional) Poisson manifold with Poisson structure:

\[
[\Phi, \Psi]_P = \frac{2}{i} \int_\Sigma d^3k (D_{\tilde{a}_k} (\Phi) D_{\tilde{a}_k} (\Psi) - D_{\tilde{a}_k} (\Phi) D_{\tilde{a}_k} (\Psi))
\]

\[
D_{\tilde{a}_k} = \sqrt{(2\omega (k))} \frac{\delta}{\delta \tilde{a}_k}, \quad D_{a_k} = \sqrt{(2\omega (k))} \frac{\delta}{\delta a_k}
\]

where \( \Phi = \Phi (x, t) \) and \( \Psi (x, t) \) are any two solutions of the KG equation and the Poisson bracket is invariant under choice of hypersurface \( \Sigma \).[HaEl]

Just as in the case of ordinary quantum mechanics, this Poisson structure induces a star-product on the phase-space. Because different star-products correspond to different operator orderings we must be careful with the star-product we choose to prevent large numbers of divergences.

The choice in Dito’s star product for a free field is normal ordering because normal ordering plays a special role in QFT by annihilating the vacuum state. This choice eliminates the artificial infinity in the vacuum energy of the free field. The normal star-product is defined by:

\[
\Phi \ast_n \Psi = \Phi \Psi + \sum_{n=1}^{\infty} \hbar^n C_n^N (\Phi, \Psi)
\]

where:

\[
C_n^N (\Phi, \Psi) := \frac{1}{n!} \int d^3k_1 \cdots d^3k_n \left( D_{\tilde{a}_{k_1}} \cdots D_{\tilde{a}_{k_n}} (\Phi) D_{\tilde{a}_{k_1}} \cdots D_{\tilde{a}_{k_n}} (\Psi) \right)
\]
For an interacting scalar field on Minkowski space with, for example, a polynomial interaction term $V(\phi)$ the KG equation is
\[ (\partial_\mu \partial^\mu - m^2/\hbar^2 + V'(\phi)) \phi(x,t) = 0. \]
Through a linearization program, Dito constructed star-product $\ast^+$ at the asymptotic future free field and $\ast^-$ at the asymptotic past free field which is free of a large number of divergences by construction.

By investigating infinite dimensional star-products associated to interacting fields Dito saw how divergences form and how renormalization should be performed to remove them. The diverging terms are singular cocycles or coboundaries of the Hochschild cohomology in the star-product which can be removed by changing to an equivalent star-product which is achieved by the linearization program.

For a star-product on a Maxwell-Dirac field to be constructed in a similar manner, the star-product of the asymptotic fields must be constructed. Recently in [HiHe2] a star-product was constructed for a fermionic system. This is based on the works of Berezin and Marinov in which they begin with a pseudoclassical system based on Grassmann algebras. Hirshfeld and Hensler proposed a fermionic star-product as a deformation quantization of this pseudoclassical system. However, work still needs to be done to understand how to properly construct the asymptotic fields of QED in DQ.

6 Connections to the Algebraic Approach to Quantum Field Theory

The Algebraic Approach to quantum field theory (AQFT), invented by Rudolf Haag and Daniel Kastler, formulates QFT from a sufficiently rigorous axiomatic framework which is consistent with all the basic principles of QFT.[Buc,Haag,BrRo] This is contrast to the alternative approach of Constructive QFT which builds on existing mathematical methods for the treatment of physical models.

The relationship between conventional approaches to QFT’s such as Lagrangian/path integral formulations and canonical quantization to AQFT may be compared to the relationship between the coordinate dependent approach to differential geometry and the coordinate independent one.[Buc] To calculate certain quantities it is natural (and sometimes necessary) to describe the situation with coordinates, e.g. writing down Christoffel symbols, etc. However, if you are more interested in a general abstract analysis of the manifold you need coordinate independent quantities like a connection, fiber and vector bundles, etc. It is only through the use of the two complementary approaches can a full understanding of differential geometry be yielded. In this way, a rigorous axiomatic framework for QFT is necessary to compliment the other two methods. AQFT attempts to achieve this more rigorous framework.

The advances obtained through AQFT include the clarification of the roles of locality and covariance in QFT. Also, in AQFT, equivalent QFT’s have the same abstract AQFT structure. However, the issue of the existence of a suitable set of states (which includes a notion of a vacuum) is a difficult and ongoing debate except for the class of maximally symmetric spaces which has been solved. Additionally, there have been numerous successes on an arbitrary globally hyperbolic space-time in perturbative AQFT using renormalization techniques such as configuration space and micro-local techniques. The main new insight here is the complete disentanglement of the ultraviolet and infrared problems in the perturbative expansion. Also, insights into holography of the AdS/CFT correspondence (triggered by string theory) have been made in AQFT. Holography is the notion that a QFT in the bulk of a manifold uniquely determines the QFT on the boundary.
As is pointed out in [DuFr], on the level of concrete models AQFT was less successful. It is in this regard that we believe that DQ can come to the rescue by defining a concrete and practical phase-space description of an abstract net of observables (the standard set used in AQFT). It was conjectured in [DuFr] that the relationship between perturbative AQFT may be understood in terms of DQ (also see [HiHe1]). A reason is because the DQ observables are ordinary functions; therefore, a local theory can be built out of functions of compact support in a way that is entirely natural. This set of observables is the "net of observables" described in AQFT. As long as the set of states that are well-defined in DQ is large enough to describe the theory as well as the convergence of all relevant quantities, DQ should be very fruitful. In addition, the conventional treatment of DQ in terms of a formal series in \( \hbar \) is extremely convenient for perturbative calculations.

7 Conclusion

What we have in DQ is a coordinate and dynamics independent formulation of quantum theory formulated on phase-space. [Til,GoRe] DQ's conceptual advantages come from the fact that the observables are phase-space functions just as they are in classical mechanics. Therefore, many geometric and algebraic tools can be extended very naturally into DQ. It seems that most of the hard work in DQ is in the construction of star-products and issues relating to their convergence. However, once you have a star-product many things like topology, coordinate transformations, etc. are much easier to understand on a fundamentally conceptual level.

Advances have been made in applying DQ to QFT’s. The star-products of Dito on scalar fields, including those with polynomial interaction terms are some of the significant advances. [Dito] In addition, their seems to be a strong connection between DQ and AQFT because of the similarities between the two. [DuFr] One includes their focus on the algebra of observables. Another is the "net of observables" in AQFT can be constructed in DQ by the simple restriction of the observables which are functions on the phase-space. The reason we should care about this connection is because AQFT has provided many advances in the understanding of QFT’s. These include the role of locality, the disentanglement of the ultraviolet and infrared problems on an arbitrary globally hyperbolic space-time, and insights to the AdS/CFT correspondence. We hope that DQ may be able to provide the concrete models that AQFT lacks.

By applying DQ to QFT’s we may hopefully yield a better understanding of quantum theory.

Acknowledgements

I would like to thank my advisor George Sparling for very helpful discussions.

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