An overview of simply pulsed qubits

A. Chalastaras, L. Kaplan, Kh. Kh. Shakov, M. Smith, and J. H. McGuire

Physics Dept., Tulane University, New Orleans, LA 70118, USA

April 1, 2022

Abstract: The behavior of simply pulsed qubits (quantum systems with two linearly independent states) may be characterized by the energy difference $\Delta E$ between the two states of the qubit and by an external stimulating potential $V(t)$ that causes transitions between them. Thus, the operation of such quantum mechanical systems may be categorized in various regions that explicitly depend on $\Delta E$ and $V(t)$. Limiting cases of degenerate, perturbative, and adiabatic regions are discussed. A comprehensive and illustrative map for simply pulsed qubits is presented that can be used as a visual tool for students. Furthermore, analytic solutions may be obtained when the interaction $V(t)$ is proportional to $\delta(t - t_k)$, namely when a fast interaction, called a kick, is used.
1 Introduction

Just as classical information and computation are built upon on-off bits, quantum information and computation may be constructed on the basis of two-state (on-off) quantum systems known as qubits \[1, 2\]. These quantum building blocks are used to describe and to control atomic and molecular reactions \[3, 4\], electron dynamics \[5\], and time ordering in quantum systems \[6\]. However, properties of dynamic two-state quantum systems, such as transition rates from one state to the other, are usually found numerically since analytic solutions are relatively rare. In this paper we describe some simply pulsed qubits where the occupation probability of each of the two states, i.e., the likelihood of being in the on state or the off state, can be easily controlled.

The difference between classical and quantum bits is that the classical occupation probability of the on state must be either 1 or 0 (definitely on or definitely off) at all times, while for a qubit the probability \( P(t) \) of being on may be any number between 0 and 1 until an observation is made. That is, before an observation the qubit may be in a combination or superposition of on and off states, similar to a superposition of classical waves. After a qubit is observed, \( P(t) \) collapses to 1 or 0 for that qubit. If \( N \) qubits (where \( N \) is large), initially all in the off state, are switched by the same external potential and then observed at time \( t \), \( N \cdot P(t) \) of them will be found to be in the on state and \( N \cdot (1 - P(t)) \) will remain in the off state. It is this feature, namely that before an observation a qubit can be simultaneously on and off, that distinguishes the qubit from the classical bit that simply switches between on or off (never being in a superposition of on and off states). In this regard, the mathematical rules (i.e., logic) differ for qubits and bits. The more sophisticated qubits can in principle perform more complex operations \[1, 7\].

2 Qubits

Since bits can only be on or off (e.g., corresponding to small magnets that can be magnetized in one of two directions), they are simpler than qubits. Before the measurement, a qubit can be in both of its two states at the same time. Reliably manipulating this coherent linear superposition of states, e.g., with external fields, is essential for quantum computation \[1\] and quantum control \[3\], but often difficult to achieve. The phenomenon
of quantum parallelism [7] allows a quantum processor to execute actions in parallel (performing many classical computations at the same time). The classical computer, no matter how fast, can only execute commands in series, which limits its computational power. Hence, a quantum computer is more suitable for solving certain computational problems such as factorization of large numbers and database search. Indeed, computer algorithms specifically designed to take advantage of the phenomenon of quantum parallelism in qubits have been developed [8, 9]. Various physical systems that are good candidates for qubits are listed at the end of Section 2.

2.1 Conceptual Description

Here we describe a conceptually simple way of controlling a qubit, that retains the key features of manipulating more general qubits. The simplicity of the following description helps to clarify the concepts without losing the essence of more complicated quantum systems.

We consider a two-state system with possible energies $E_1$ and $E_2$, where $\hbar \omega = \Delta E = E_2 - E_1$ is the energy splitting between these two states. Since the zero point of overall energy is arbitrary, the two eigenvalues may be written as $+\Delta E/2$ and $-\Delta E/2$ without loss of generality. The unperturbed Hamiltonian $\hat{H}_0$ does not couple the two eigenstates, i.e., $\hat{H}_0$ does not cause transition of population between the two states of the qubit. Such a system by itself is not very interesting since the observable properties do not evolve with time. For example, if the system is initially populated with any population number $N_1 = N \cdot P_1$ in state 1 and population $N_2 = N \cdot P_2$ in state 2, then without an external switching potential $N_1$ and $N_2$ do not change with time.

What makes a qubit interesting and useful is the ability to change the populations of its states in a controlled way. This is desirable, for example, in a device such as a computer, where one wishes to switch the state of a storage location back and forth from on to off at will. In classical computers, the logic of these binary operations is simple, and is described in terms of Boolean algebra [10], which is a formal way of working with states that may be described as either on or off. In such a classical device, the on or off state is usually implemented physically by an on or off magnetic field at a given storage location.

Quantum information is similar, except that the system is generally in a linear combination (linear superposition) of on and off states. In both classical bits and qubits, the switching between on and off states is done with an
external field \( V \). Since the state is to be switched from on to off at various times, \( V \) is some function of time \( t \), i.e., \( V = V(t) \). It is this external switching field \( V(t) \) that enables a computer to manipulate information, performing logic operations dynamically. In our model we choose a \( V(t) \) that simply couples the two states without changing the eigenenergies \( +\Delta E/2 \) and \( -\Delta E/2 \) of the unperturbed Hamiltonian \( \hat{H}_0 \). While there are more complex ways to manipulate a two-level system, this is sufficient to illustrate how a simple qubit works.

### 2.2 Mathematical Description

In quantum mechanics, the state of a system at any time \( t \) is fully described by its wave function \( \psi(t) \). If there are two linearly independent states of the system, labeled by \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) (e.g., corresponding to on) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), (e.g., corresponding to off), then in the most general case \( \psi(t) \) is a linear superposition of these two states. Mathematically this corresponds to

\[
\psi(t) = a_1(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix}.
\] (1)

Here \( a_1(t) \) and \( a_2(t) \) are the probability (wave-like) amplitudes such that the probability of finding \( \psi(t) \) in the on state is \( P_1(t) = |a_1(t)|^2 \), and the probability that \( \psi(t) \) is in the off state is \( P_2(t) = |a_2(t)|^2 \). This is analogous to linear superpositions of electromagnetic fields in classical wave mechanics, where the observed intensity is proportional to the square of either the electric or the magnetic field. Normalization condition in quantum mechanics requires the constraint \( P_1(t) + P_2(t) = 1 \) (conservation of probability), meaning that no dissipation\(^1\) has occurred since the system was initially formed.

A two-state system can be coupled (i.e., the particle population can be transferred between the two states) by applying an external potential \( V(t) \). In this paper, the external potential has the shape of a single pulse that can be sensibly characterized by a single time duration, \( \tau \). A simple pulse could be for example a Gaussian, an instantaneous kick (delta function),

\(^1\)Dissipation corresponds to leakage of some population out of the two-state system. While dissipation can be a significant problem in some practical applications, in this paper we assume it is negligible.
or a rectangular pulse. The energy of the system is described by the full Hamiltonian $\hat{H}(t)$, which includes the energy difference term $\Delta E$ and a time-dependent external potential term $V(t)$. Here we take $\Delta E$ to be constant in time, corresponding to most applications. Hence all the time dependence in the system’s Hamiltonian $\hat{H}(t)$ comes from $V(t)$. This full Hamiltonian can be written in terms of the Pauli spin matrices (problem 1), namely

$$\hat{H}(t) = \hat{H}_0 + \hat{V}(t) = \begin{bmatrix} -\Delta E/2 & 0 \\ 0 & +\Delta E/2 \end{bmatrix} + \begin{bmatrix} 0 & V(t) \\ V(t) & 0 \end{bmatrix}$$ (2)

where the widely used Pauli spin matrices $\sigma$ are defined as

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ (3)

Inserting the Hamiltonian $\hat{H}(t)$ from expression (2) and the wave function $\psi(t)$ from expression (1) into the time-dependent Schrödinger equation,

$$i\hbar \frac{d\psi(t)}{dt} = \hat{H}(t)\psi(t),$$ (4)

we obtain (problem 2) a differential equation for the on and off probability amplitudes, $a_1(t)$ and $a_2(t)$, as functions of the energy splitting and the external potential:

$$i\hbar \begin{bmatrix} \dot{a}_1(t) \\ \dot{a}_2(t) \end{bmatrix} = \begin{bmatrix} -\Delta E/2 & V(t) \\ V(t) & +\Delta E/2 \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix}.$$ (5)

Separating the equations, we can also write,

$$i\hbar \frac{da_1(t)}{dt} = i\hbar \dot{a}_1(t) = -\frac{\Delta E}{2}a_1(t) + V(t)a_2(t)$$

$$i\hbar \frac{da_2(t)}{dt} = i\hbar \dot{a}_2(t) = V(t)a_1(t) + \frac{\Delta E}{2}a_2(t).$$ (6)
Conceptually and mathematically, it is convenient to isolate the time
dependence of the system by defining the evolution operator \( \hat{U}(t, t_0) \) that connects the initial state \( \psi(t_0) \) with the final state \( \psi(t) \),

\[
\psi(t) = \hat{U}(t, t_0) \psi(t_0) .
\]  

(7)

All the time dependence of the system is contained in the time evolution
operator (or Green’s function) \( \hat{U}(t, t_0) \), while the initial conditions are specified in \( \psi(t_0) \). Throughout the rest of this paper, we take \( t_0 = 0 \) and write
the evolution operator as \( \hat{U}(t) = \hat{U}(t, t_0) \). Then, the evolution operator in
equation (7), has the following matrix representation

\[
\hat{U}(t) = \begin{bmatrix}
U_{11}(t, t_0) & U_{12}(t, t_0) \\
U_{21}(t, t_0) & U_{22}(t, t_0)
\end{bmatrix} = \begin{bmatrix}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{bmatrix}.
\]  

(8)

It is then straightforward to express the time-dependent probability amplitudes, \( a_1(t) \) and \( a_2(t) \), using the evolution operator \( \hat{U}(t) \), namely

\[
\begin{bmatrix}
a_1(t) \\
a_2(t)
\end{bmatrix} = \hat{U}(t) \begin{bmatrix}
a_1(0) \\
a_2(0)
\end{bmatrix} = \begin{bmatrix}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{bmatrix} \begin{bmatrix}
a_1(0) \\
a_2(0)
\end{bmatrix}.
\]  

(9)

The evolution operator \( \hat{U}(t) \) may be obtained by solving a differential equation very similar to (5):

\[
 i\hbar \begin{bmatrix}
\dot{U}_{11}(t) & \dot{U}_{12}(t) \\
\dot{U}_{21}(t) & \dot{U}_{22}(t)
\end{bmatrix} = \begin{bmatrix}
-\Delta E/2 & V(t) \\
V(t) & +\Delta E/2
\end{bmatrix} \begin{bmatrix}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{bmatrix},
\]  

(10)

and we require the time evolution operator to start out as the identity matrix
\( I \),

\[
\hat{U}(0) = \begin{bmatrix}
U_{11}(0) & U_{12}(0) \\
U_{21}(0) & U_{22}(0)
\end{bmatrix} = I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]  

(11)

so that \( \psi(t) = \psi(0) \) at the initial time \( t = t_0 = 0 \).

Formally, the solution to equation (10) is given by

\[
\hat{U}(t) = \begin{bmatrix}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{bmatrix} = Te^{-i\int_0^t \hat{H}(t')dt'/\hbar} = Te^{-i\int_0^t [\hat{H}_0 + \hat{V}(t')]dt'/\hbar},
\]  

(12)
where $T$ is the Dyson time ordering symbol \cite{13, 14} that enforces $\hat{H}(t_1)$ acting on the system prior to $\hat{H}(t_2)$ if $t_1 < t_2$.

Equations (1), (2), (4), (7), (9), (10), and (12) form the basic equations for a qubit. These equations are governed by the energy splitting $\Delta E$ and the external potential $V(t)$. Depending on the complexity of the time evolution operator, these equations may or may not have analytic solutions. In the latter case, numerical solutions of the coupled differential equations (6) may obscure information about these quantum systems. Analytic solutions, when possible, are more convenient and easy to analyze.

2.3 Practical Considerations for a System of Qubits

Up to this point, we have treated the qubit as an abstract mathematical structure. However, in applications some practical issues have to be addressed. For a system of qubits to be capable of performing quantum computation \cite{11, 6, 7}, control of population transfer between the two states of the qubit is not sufficient. The following list, dubbed the “DiVincenzo checklist” \cite{15} after its developer in 1997, addresses some practical issues for a quantum computer:

- A large number of coupled qubits need to be reliably controlled.

- It should be possible to prepare the qubits in a desired initial state. Commonly, in the initial state all qubits are on or all qubits are off.

- Decoherence should be minimized. In terms of the energy splitting $\Delta E$ between the two states, the decoherence time $t_d$ over which quantum phase information is lost must satisfy $t_d \gg 2\pi \hbar / \Delta E$ \cite{2}.

- Quantum gates need to be designed that will control the operation of the qubits (these gates play a role similar to classical gates).

- The information contained in the qubits must be extractable at the end of the computation if the outcome of the qubit operations is to be used in a productive manner.

A number of quantum systems satisfy these requirements. Optical photons \cite{16}, nuclear spins \cite{17, 18, 19}, ion traps \cite{20}, nuclear magnetic resonance (NMR) \cite{21}, and electrons on superfluid helium \cite{22} are some commonly discussed examples.
The QIST Quantum Computation Roadmap addresses the key issues in quantum computation and the latest advances in the field. In the last few years, considerable progress has been made in overcoming the problem of decoherence, i.e., loss of phase control. Decoherence is an important but complicated issue and is discussed elsewhere.

Next, we discuss different physical limits for simply pulsed qubits and the graphical qubit map is introduced.

3 Map

3.1 Significance of the Qubit Map

A qubit map, such as that shown in Figure 1, is a tool that enables one to visualize how the behavior of simply pulsed qubits may depend on the variables $\Delta E$ and $V(t)$. In order to make the qubit map a helpful visual tool, the map coordinates are taken to be $\Delta E \tau / 2 \hbar$ (where $\tau$ is the time duration of $V(t)$) and $\int_0^\infty V(t') dt'/\hbar$. These two variables are dimensionless and range in absolute value from 0 to $\infty$. The variable $\Delta E \tau / 2 \hbar$ determines the effect of the unperturbed Hamiltonian $\hat{H}_0$ on the evolution operator $\hat{U}(t)$, while $\int_0^\infty V(t') dt'/\hbar$ measures the influence of the external potential on $\hat{U}(t)$. It is useful to think of these variables as independent phase angles or action-like integrals.

It should be noted here that the boundaries separating the qubit map regions are not precisely defined. The regions overlap each other (see curly lines in the map) when either of the two phases $\Delta E \tau / 2 \hbar$ or $\int_0^\infty V(t') dt'/\hbar$ is of the order of $2\pi$. This behavior should not be surprising since the definitions of the degenerate, perturbative, and adiabatic limits are of an approximate nature. The map helps to identify the regions where qubit behavior may be described analytically, and indicate ballpark values of the parameters for which various analytic solutions are applicable. For instance, if we wish to investigate slow weakly perturbed qubits, we look for large phases $\Delta E \tau / 2 \hbar$ and small $\int_0^\infty V(t') dt'/\hbar$. For strongly perturbed degenerate qubits, we look for small $\Delta E \tau / 2 \hbar$ but large values of $\int_0^\infty V(t') dt'/\hbar$.

The arrow pointing toward the lower left corner of the Figure indicates the "fast" or short-time limit $\tau \to 0$. Here the time $\tau$ of the pulse is too short to allow for significant phase accumulation due either to the external potential $V(t)$ or to the energy splitting $\Delta E$, i.e., both $\Delta E \tau / 2 \hbar$ and $\int_0^\infty V(t') dt'/\hbar$ are
small. The opposite $\tau \to \infty$ or “slow” limit indicated by the arrow pointing toward the upper right corner is where the pulse is long enough for both the energy splitting and the external potential to have a large effect over the duration of the pulse.

### 3.2 Regions of the Map

It is sensible to separate our map into four regions including the interesting central region and to examine each one individually. Analytic solutions exist only for some regions of the map. It should be emphasized that this map is valid only for singly pulsed qubits. Multiply pulsed qubits [24], including periodically pulsed qubits [28], are not discussed here since an additional time parameter, e.g., the typical time between pulses or the period of oscillation of the potential, would require a more complicated map.

#### 3.2.1 Degenerate Region

When the energy splitting between the two states of the qubits is much less than the strength of the external interaction (i.e., $\Delta E \ll V(t)$), we are in the degenerate region of the map. The diagonal extending from the lower left (“fast”) corner to the upper right (“slow”) corner of the Figure forms the boundary of the degenerate region: the behavior of the system is dominated by the external interaction $V(t)$ whenever we are to the right of or below this diagonal line. The evolution of the system is then given by simple sine and cosine functions, and the mathematical complexity is greatly reduced. It can easily be shown (problems 3 and 4) that as $\Delta E \to 0$ in equation (12), the evolution matrix of the degenerate qubit or dit$^3$ [27] becomes

$$\hat{U}(t) \to \hat{U}_{\text{dit}}(t) = \begin{bmatrix} \cos \alpha & -i \sin \alpha \\ -i \sin \alpha & \cos \alpha \end{bmatrix},$$

(13)

where

$$\alpha = \frac{\int_0^t V(t')dt'}{\hbar}$$

(14)

$^2$While solutions of two-state systems with rapidly changing external potentials were already examined about 40 years ago [4], the focus has now shifted to quantum control.

$^3$Our usage of the term “dit” is not universal.
energy splitting is small, $\Delta E$ approximation generally applies. 

If both phases are large, then the adiabatic limit may be thought of as action-like phase accumulations due respectively to the energy difference $\Delta E$ between the two possible qubit states and to the external potential $V(t)$. Everywhere outside the central region, approximate analytic solutions may be obtained. When the total phase associated with the external potential is small, i.e. $\int_0^\infty V(t')dt'/\hbar \ll 2\pi$, then the expression for the time evolution operator may be expanded in powers of $V$ using $e^{-iV(t')dt'/\hbar} \approx 1 - iV(t')dt'/\hbar$ and only the first few terms retained. This corresponds to standard perturbation theory for either quickly perturbed qubits where $\tau$ is small or slowly perturbed qubits where $\tau$ is large. Similarly, if the phase associated with the energy splitting is small, $\Delta E\tau/2\hbar \ll 2\pi$, then we can treat the qubit as degenerate, and the solution can be expanded in powers of $\Delta E$. If both phases are large, then the adiabatic approximation generally applies.

Figure 1: Qubit map for qubits interacting with simply pulsed finite external potentials. Here $\tau$ is the duration of the pulse, $\Delta E$ is the energy difference between the two states of the qubit, and $V(t)$ is the external potential. Note that the single pulse $V(t)$ is neither harmonic nor periodic. The two axes are dimensionless. These axes may be thought of as action-like phase accumulations due respectively to the energy difference $\Delta E$ between the two possible qubit states and to the external potential $V(t)$. Everywhere outside the central region, approximate analytic solutions may be obtained. When the total phase associated with the external potential is small, i.e. $\int_0^\infty V(t')dt'/\hbar \ll 2\pi$, then the expression for the time evolution operator may be expanded in powers of $V$ using $e^{-iV(t')dt'/\hbar} \approx 1 - iV(t')dt'/\hbar$ and only the first few terms retained. This corresponds to standard perturbation theory for either quickly perturbed qubits where $\tau$ is small or slowly perturbed qubits where $\tau$ is large. Similarly, if the phase associated with the energy splitting is small, $\Delta E\tau/2\hbar \ll 2\pi$, then we can treat the qubit as degenerate, and the solution can be expanded in powers of $\Delta E$. If both phases are large, then the adiabatic approximation generally applies.
is the action integral. When $\alpha$ is an integer multiple of $\pi/2$, the degenerate bit becomes a classical bit where the occupation probabilities simply get switched between 1 and 0.

The analytic expression (13) is only valid when two conditions $\Delta E \tau / 2\hbar \ll \int_0^t V(t')dt'/\hbar$ and $(\Delta E \tau / 2\hbar)^2 \ll \int_0^t V(t')dt'/\hbar$ are simultaneously satisfied. The need for the second condition may be removed, thereby extending the solution to the entire degenerate region to the right of the diagonal, by replacing $\alpha$ with

$$\theta = \frac{\int_0^t \Omega(t')dt'}{\hbar},$$

where $\Omega(t') = \sqrt{V^2(t') + (\Delta E/2)^2}$. We notice that $\Omega(t') = V(t')$ and thus $\theta = \alpha$ for $\Delta E \to 0$.

We will see below that the degenerate region overlaps other regions of the map, such as the perturbative, adiabatic, and kicked regions. In these areas of overlap, two or more analytic solutions become mathematically equivalent.

### 3.2.2 Perturbative Region

When the external potential $V(t)$ is small, we are in the perturbative region. The perturbative region includes both upper and lower left quadrants of the qubit map. In this particular region $\int_0^\infty V(t')dt'/\hbar \ll 2\pi$. The external interaction $V(t)$ is sufficiently weak that the qubit remains essentially in its initial on or off state, acquiring only an overall phase associated with the energy $\pm \Delta E/2$. It can be shown (problem 5a) that the evolution operator in equation (12) becomes

$$\hat{U}_{\text{pert}}(t) = \begin{bmatrix} e^{it(\Delta E/2\hbar)} & -i \int_0^\infty e^{i(t-2t')\Delta E / 2\hbar} V(t')dt'/\hbar \\ -i \int_0^\infty e^{-i(t-2t')\Delta E / 2\hbar} V(t')dt'/\hbar & e^{-it(\Delta E/2\hbar)} \end{bmatrix}. \quad (16)$$

For the extreme special case where $V(t) = 0$ at all times (problem 5b), the evolution operator in equation (16) becomes

$$\hat{U}(t) \to \hat{U}_{\text{zero}}(t) = \begin{bmatrix} e^{it(\Delta E/2\hbar)} & 0 \\ 0 & e^{-it(\Delta E/2\hbar)} \end{bmatrix}. \quad (17)$$

Setting $V(t) = 0$ uncouples the differential equations in expression (6) making the math trivial. We obtain $a_1(t) = a_1(0)e^{it\Delta E/2\hbar}$ and $a_2(t) = a_2(0)e^{-it\Delta E/2\hbar}$, i.e., the amplitudes of being in the on or off state acquire phases as time
evolves but the probabilities $|a_1(t)|^2$ and $|a_2(t)|^2$ remain what they were at the initial time $t = 0$.

The perturbative region is of little interest in our case since the system does not change appreciably from its original state. In the case of zero external potential, the qubit does not change its state at all.

### 3.2.3 Adiabatic Region

The adiabatic region covers the upper right, upper left, and lower right quadrants of the qubit map, excluding only part of the lower left (“fast”) quadrant and the central region. The explicit condition for adiabaticity is 

$$\hbar \dot{V}(t) \Delta E \ll [V^2(t) + (\Delta E/2)^2]^{3/2},$$  

which can be further simplified in the degenerate and nondegenerate limits. In the degenerate adiabatic case $\Delta E \ll V(t)$, so \[18\] yields the condition $\hbar \dot{V}(t) \Delta E \ll V^3(t)$ or $\hbar \Delta E/\tau \ll V^2(t)$. The time derivative of the external potential is of the order $V(t)/\tau$, i.e., $\dot{V} \sim V/\tau$, which is the typical slope of the single-pulse potential. In the nondegenerate adiabatic case $\Delta E \gg V(t)$, and the validity condition \[18\] becomes $\hbar \dot{V}(t) \ll (\Delta E)^2$ or $\hbar V(t)/\tau \ll (\Delta E)^2$.

The meaning of the adiabatic region is that the external potential changes slowly, so that the quantum system is able continuously to adjust to the new Hamiltonian. Consider for example a classical system of two pendulums connected by a spring. The system can oscillate in one of two normal modes: a lower-frequency swinging mode where the two pendulums move left and then right in unison, and a higher-frequency vibrational mode where the spring is alternately stretched and compressed. If we move the pivot point of one or both pendulums slowly and carefully enough, the pendulums continue to oscillate in the same (lower or higher) mode they started in, because not enough energy is provided by the external perturbation to switch the motion from the lower-frequency oscillation to the higher-frequency oscillation or vice versa. Similarly, an adiabatically driven qubit that is initialized in the lower energy ($-\Delta E/2$ or off) state before the pulse is turned on will at any future time remain in the lower energy state of the full Hamiltonian $\hat{H}(t)$. After the pulse is over and the Hamiltonian returns to the unperturbed Hamiltonian $\hat{H}_0$, the qubit must return to the off state with energy $-\Delta E/2$. Thus, the adiabatic region is of no great interest in the case of pulsed qubits since we wish to completely transfer the population of a qubit from one state to the other.
Mathematically, the evolution operator for an adiabatically driven qubit is

$$
\hat{U}_{\text{adiab}}(t) = \begin{pmatrix}
\cos \theta(t) + i \frac{\Delta E}{2\Omega(t)} \sin \theta(t) & -i \frac{V(t)}{\Omega(t)} \sin \theta(t) \\
-i \frac{V(t)}{\Omega(t)} \sin \theta(t) & \cos \theta(t) - i \frac{\Delta E}{2\Omega(t)} \sin \theta(t)
\end{pmatrix}, \quad (19)
$$

where $\theta(t)$ and $\Omega(t)$ are defined as in Section 3.2.1, and for simplicity we assume $V(t) = V(0)$, i.e., the external potential returns to its initial value at the end of the pulse. Equation (19) can be reduced (problem 6) to equations we presented above for the degenerate and zero external potential cases.

### 3.2.4 Central Region

None of the limits discussed previously overlap with the central region of the map. Here the two independent variables $\Delta E \tau / 2\hbar$ and $\int_0^\infty V(t')dt'/\hbar$ are both of order $2\pi$, and there is no large or small parameter to simplify the solution of the problem. If the pulse has a particularly simple (e.g., rectangular) shape, an analytic expression is possible [24] even in this region, but for a general pulse shape no analytic solution exists and the evolution must be computed numerically.

### 4 Kicked Qubit Approximation

There is a useful approximation for simply pulsed qubits, namely the sharp, narrow pulse or “kick” limit in which the width $t$ of the pulse goes to zero, while the integrated strength or the area under the external potential curve

$$
\alpha_k = \int_0^t V(t')dt'/\hbar
$$

remains fixed. Formally, the shape of a very narrow pulse of finite total strength $\alpha$ may be expressed by a delta function: $V(t') = \alpha_k \hbar \delta(t - t_k)$, where $t_k$ is the time at which the pulse is applied. The kicked region corresponds to the lower half of the qubit map in Figure 1. Here the duration of the pulse is so short that $\Delta E \tau / 2\hbar \ll 2\pi$, i.e., there is not enough time for the energy splitting $\Delta E$ to have a significant effect while the pulse is active. The integrated strength of the pulse, $\alpha_k$, may be either large or small in this region. If $\alpha_k$ is large, we are in the lower right quadrant of the map,
where the kicked region overlaps with the adiabatic region. If $\alpha_k$ is small, we are in the lower left quadrant, where the kicked region overlaps with the perturbative region.

In the kicked region, the evolution of the qubit can be described as follows: (i) before the kick, the qubit evolves in accordance with the unperturbed Hamiltonian $\hat{H}_0$, i.e. the on and off probability amplitudes $a_1(t)$ and $a_2(t)$ acquire phases proportional to the energy splitting $\Delta E$ (as in (17)), while the on and off probabilities, given by $|a_1(t)|^2$ and $|a_2(t)|^2$, remain unchanged. (ii) For the duration of the kick, the energy splitting associated with the unperturbed Hamiltonian $\hat{H}_0$ may be ignored, and a transfer of population between on and off states may occur, depending only on the integrated strength $\alpha_k$ of the kick. (iii) Finally, free evolution governed by $\hat{H}_0$ resumes after the kick is complete, and no further population transfer occurs. This sequence of events is very similar to the collision approximation studied in introductory physics, where (i) a particle initially moves freely with constant velocity, (ii) then undergoes an instantaneous collision with a wall, during which the velocity is changed but there is not enough time for the particle to move, and (iii) finally, the particle once again resumes free flight with a (new) constant velocity.

For such kicks, the integration over time in equation (12) is straightforward (problem 7) and the time evolution matrix becomes a product of three parts,

$$
\hat{U}_{\text{kicked}}(t) = e^{i\frac{\Delta E(t-t_k)}{2\hbar}\sigma_z}e^{-i\int_{t_k-\epsilon}^{t_k+\epsilon} V(t')dt'\sigma_z/\hbar}e^{i\frac{\Delta E}{2\hbar}t_k\sigma_z} = \begin{bmatrix}
  e^{i\Delta E t/2\hbar} \cos \alpha_k & -ie^{i\Delta E(t-2t_k)/2\hbar} \sin \alpha_k \\
  -ie^{-i\Delta E(t-2t_k)/2\hbar} \sin \alpha_k & e^{-i\Delta E t/2\hbar} \cos \alpha_k
\end{bmatrix}. \tag{21}
$$

Here, we may set the Dyson time ordering symbol $T \to 1$ for the duration of the pulse ($t_k - \epsilon < t < t_k + \epsilon$). It can also be shown (problem 8) that equation (21) will reduce to other limiting cases discussed previously.

Moreover, the occupation probabilities for a kicked qubit initially in state 1 are, from equation (9),

$$
P_1(t) = |a_1(t)|^2 = |U_{11 \text{kicked}}(t)|^2 = \cos^2 \alpha_k$$

$$
P_2(t) = |a_2(t)|^2 = |U_{21 \text{kicked}}(t)|^2 = \sin^2 \alpha_k. \tag{22}
$$

\[4\]For a single kick setting $T \to 1$ corresponds to the limit of no time ordering, which is beyond the scope of this paper. For some useful discussions on time ordering see reference [6].
Since \( \sin^2 \alpha_k + \cos^2 \alpha_k = 1 \) for this closed two-state system, the conservation of population holds, i.e., there is no dissipation. We notice also that the transfer probability \( P_2(t) \) does not depend on the energy splitting \( \Delta E \). This simple instructional example may be extended to a series of kicks \[24\]. It is one of the few cases in which analytic solutions may be obtained for qubits controlled by external potentials.

5 Summary

A primary motivation of this paper has been to give an overview of simply pulsed qubits. We have developed a simple, yet comprehensive and instructive, two-dimensional map for such qubits. Analytic solutions with limited applicability are found and their corresponding regions on the map are identified. One promising class of analytic solutions for dynamic qubits are the kicked solutions, valid for qubits subject to fast interactions. These provide clear and useful examples to students and offer an alternative to the well-established rotating wave approximation (RWA) \[28, 29, 30\]. The RWA method assumes a sinusoidal external perturbation with frequency chosen to be in resonance with the desired transition between two states. The solutions presented in this paper are, in principle, valid for a wide spectrum of pulse widths \( \tau \) and energy splittings \( \Delta E \).

Qubits are building blocks for quantum computation and quantum information. Their behavior and interaction with each other and the environment have to be addressed before a quantum computer becomes a reality. Ultimately, N-qubit systems and their interactions will have to be successfully controlled. The simply pulsed qubits discussed in this paper may form a basic building block for more complex interconnected N-qubit systems.
6 Appendix
Problems for Students

- **Problem 1**: (a) Show that a 2x2 matrix that corresponds to a two-state quantum mechanical observable can be written in terms of the Pauli matrices. [Hint: Write down a generic Hermitian 2x2 matrix and try to expand it as a sum of the Pauli matrices with appropriate prefactors]. (b) If the qubit is implemented as a spin-1/2 particle, \[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
represents a state where the spin points in the +z direction, and \[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
represents a state where the spin points in the −z direction, then what do operators \(\sigma_x\), \(\sigma_y\), and \(\sigma_z\) correspond to physically?

- **Problem 2**: Using equations (1), (2), and (4), verify equation (5). Rules for simple matrix operations are available in many elementary math books and on the web.

- **Problem 3**: (a) Solve the evolution of the degenerate qubit or dit by plugging \(\Delta E = 0\) into equation (5), thus proving that the probability amplitudes, \(a_1(t)\) and \(a_2(t)\), oscillate sinusoidally with \(\alpha = \int_0^t V(t')dt'/\hbar\). (b) Make plots of the probabilities \(P_1 = |a_1(t)|^2\) and \(P_2 = |a_2(t)|^2\). What can you deduce from the plots about dissipation in the system? (c) Find the condition for complete transfer from state 1 to state 2.

- **Problem 4**: (a) Solve the evolution of the degenerate qubit or dit by using the evolution operator in equation (12) and then plugging \(\hat{U}(t)\) into equation (9). [Hint: Since \(\Delta E = 0\), \(\hat{H}_0\) vanishes and time ordering effects disappear \((T = 1)\). Then \(\hat{U}(t) = e^{-i\int_0^t V(t')\sigma_x/\hbar}\), and the following identities may be useful: \(e^{i\theta\sigma_z} = I\cos(\theta) + i\sigma_z\sin(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\) and \(e^{i\theta\sigma_x} = I\cos(\theta) + i\sigma_x\sin(\theta)\).] (b) Do you get the same probability amplitudes as in problem (3b)? (c) Check to see that the evolution operator is unitary, that is \(\hat{U}(t)^\dagger\hat{U}(t) = I\).

- **Problem 5**: (a) Show that in the perturbative limit, the evolution operator in equation (16) satisfies equation (10). (b) Do the same for
the case of zero external potential. That is, prove equation (17). You may use the result from part (a).

- **Problem 6**: (a) Show that the adiabatic equation (19) for the evolution operator reduces to the degenerate equation (13) in the limit $\Delta E \to 0$. (b) Show that the adiabatic equation (19) reduces to the zero potential equation (17) in the limit $V(t) \to 0$.

- **Problem 7**: Prove equation (21) by integrating and then multiplying out the exponentials in the first line, thus obtaining the matrix in the second line of equation (21) [Hint: You may find useful the identities given in problem (4)].

- **Problem 8**: (a) Show that equation (21) for a kicked qubit is equivalent to the perturbative equation (16) when the kick strength is small, i.e., $\alpha_k \ll 1$ [Hint: When $\alpha_k \ll 1$, $\cos \alpha_k \approx 1$ and $\sin \alpha_k \approx \alpha_k$. In equation (16), replace $t'$ with $t_k$ since we are in the kicked region in this problem]. (b) Show that the equation (21) for a kicked qubit reduces to the degenerate equation (13) in the limit $\Delta E \to 0$. 

17
References

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).

[2] R. Waser (editor), Nanoelectronics and Information Technology: Advanced Electronic Materials and Novel Devices (Wiley, New York, 2003).

[3] Proceedings of Ann Arbor Conference on “Building Computational Devices using Coherent Control,” June 7-9, 2004, edited by V. Malinovsky (2004); C. Rangan, A. M. Bloch, C. Monroe, and P. H. Bucksbaum, Phys. Rev. Lett. 92, 113004 (2004); D. J. Wineland et al., Chem. Phys. 267, 1 (2001).

[4] F. T. Smith, Phys. Rev. 179, 111 (1969).

[5] J. H. McGuire, Electron Correlation Dynamics in Atomic Scattering, (Cambridge University Press, 1997).

[6] J. H. McGuire, A. L. Godunov, Kh. Kh. Shakov, Kh. Yu. Rakhimov, and A. Chalastaras, in Progress in Quantum Physics Research, edited by V. Krasnoholovets (Nova Science, NY, 2004).

[7] R. Hughes et al., A Quantum Information Science and Technology Roadmap, A.R.D.A, Los Alamos National Laboratory, http://qist.lanl.gov LA-UR-04-1778.

[8] P. W. Shor, Proc. of the 35th Annual Symposium on the Foundations of Computer Science, ed. S. Goldwasser, (IEEE Computer Society Press, Los Alamitos, CA, 1994).

[9] L. K. Grover, In Proceedings 28th Annual ACM Symposium on the Theory of Computing, (1996).

[10] J. D. Monk and R. Bonnet (editors), Handbook of Boolean Algebras, 3 Volumes (North-Holland, Amsterdam, 1989).

[11] G. B. Arfken and H. J. Weber, Mathematical Methods for Physicists, (Academic Press, New York, 1995).

[12] J. J. Sakurai, Modern Quantum Mechanics (Revised Edition), (Addison-Wesley Publishing Company, New York, 1994), pp. 69.
[13] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1967).

[14] E. Merzbacher, *Quantum Mechanics (3rd Edition)*, (Wiley, New York, 1997).

[15] D. P. DiVincenzo, *Topics in Quantum Computers, in Mesoscopic Electron Transport*, edited by L. Kowenhoven, G. Schon, and L. Sohn, NATO ASI Series E (Kluwer Ac. Publ., Dordrecht, 1997).

[16] A. Imamoglou and Y. Yamamoto, *Phys. Rev. Lett.* 72, 210 (1994).

[17] D. P. DiVincenzo, *Science* 270, 255 (1995).

[18] D. G. Cory, A. F. Fahmy, and T. F. Havel, *Proc. Nat. Acad. Sci.* 94, 1634 (1997).

[19] N. Gershenfeld and I. L. Chuang, *Science* 275, 350 (1997).

[20] J. I. Cirac and P. Zoller, *Phys. Rev. Lett.* 74, 4091 (1995).

[21] L. J. Schulman and U. Vazirani, *Scalable NMR quantum computation*, in *Proceedings of the 31st Annual ACM Symposium on the Theory of Computation (STOC)*, El Paso, Texas (ACM Press, New York, 1998).

[22] P. M. Platzman and M. I. Dykman, *Science* 284, 1967 (1999).

[23] A. R. P. Rau and R. A. Wendell, *Phys. Rev. Lett.* 89, 220405 (2002).

[24] L. Kaplan, Kh. Kh. Shakov, A. Chalastaras, M. Maggio, A. L. Burin, and J. H. McGuire, *Phys. Rev. A* 70, 063401 (2004).

[25] Albert Messiah, *Quantum Mechanics Volume I* (Wiley, New York, 1961), p. 41.

[26] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1959), pp. 228, 293.

[27] Kh. Kh. Shakov and J. H. McGuire, *Phys. Rev. A* 67, 033405 (2003).

[28] J. Allen and J. H. Eberly, *Optical Resonance in Two-Level Atoms* (Dover, New York, 1987).
[29] B. W. Shore, *Theory of Coherent Atomic Excitation* (Wiley, New York, 1990).

[30] P. Milonni and J. H. Eberly, *Lasers* (Wiley, New York, 1985).