Variational principle for theories with dissipation from analytic continuation

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[based on JHEP09 (2016) 099]
Dissipation in quantum field theory

- Dissipation is **generation** of entropy
- Unitary evolution **conserves** entropy

- in practice often only incomplete information available
- expectation values of fundamental quantum fields and some composite operators
- quantum states with minimal information given some constraints
- use truncation of 1 PI effective action
Example 1

- consider muon with decay $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$
- full electroweak theory is unitary
- consider now effective action where fields for $e^-$, $\bar{\nu}_e$ and $\nu_\mu$ have been integrated out
- effective action for $\mu^-$ contains decay width: appears as dissipative term

Example 2

- consider electromagnetic field $A_\mu$
- field strength above Schwinger threshold: electron-positron pair production
- electron / positron field can be integrated out: dissipative term for electromagnetic field
Double time path formalism

- formalism for general, far-from-equilibrium situations: Schwinger-Keldysh double time path
- can be formulated with two fields $\Phi = \frac{1}{2}(\phi_+ + \phi_-)$, $\chi = \phi_+ - \phi_-$
- in principle for arbitrary initial density matrices, in praxis mainly Gaussian initial states
- allows to treat also dissipation
- useful also to treat initial state fluctuations or forced noise in classical statistical theories
- difficult to recover thermal equilibrium, in particular non-perturbatively
- formalism algebraically somewhat involved
Close-to-equilibrium situations

- Out-of-equilibrium situations
- Close-to-equilibrium: description by field expectation values and thermodynamic fields
- more complete description achieved by following more fields explicitly
- example: Viscous fluid dynamics plus additional fields
- usually discussed in terms of
  - phenomenological constitutive relations
  - as a limit of kinetic theory
  - in AdS/CFT
- want non-perturbative formulation in terms of QFT concepts
- Analytic continuation as an alternative to Schwinger-Keldysh
- direct generalization of equilibrium formalism
Local equilibrium states

- Dissipation: energy and momentum get transferred to a heat bath
- Even if one starts with pure state $T = 0$ initially, dissipation will generate nonzero temperature
- Close-to-equilibrium situations: dissipation is local
- Convenient to use general coordinates with metric

$$g_{\mu \nu}(x)$$

- Need approximate local equilibrium description with temperature $T(x)$ and fluid velocity $u^\mu(x)$, will appear in combination

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

- **Global** thermal equilibrium corresponds to $\beta^\mu$ Killing vector

$$\nabla_\mu \beta_\nu(x) + \nabla_\nu \beta_\mu(x) = 0$$
**Local equilibrium**

- Use similarity between local density matrix and translation operator
  \[ e^{\beta x}(x) \mathcal{P}_\mu \iff e^{i\Delta x(x)} \mathcal{P}_\mu \]

  to represent partition function as functional integral with periodicity in imaginary direction such that
  \[ \phi(x^\mu - i\beta(x)) = \pm \phi(x^\mu) \]

- Partition function \( Z[J] \), Schwinger functional \( W[J] \) in Euclidean domain
  \[ Z[J] = e^{W_E[J]} = \int D\phi e^{-S_E[\phi] + \int_x J\phi} \]

- First defined on **Euclidean signature manifold** \( \Sigma \times M \) at constant time
- Approximate local equilibrium at all times: Hypersurface \( \Sigma \) can be shifted
Quantum effective action (1 PI effective action)

- Defined in euclidean domain by Legendre transform

\[
\Gamma_E[\Phi] = \int_x J_a(x)\Phi_a(x) - W_E[J]
\]

with expectation values

\[
\Phi_a(x) = \frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta J_a(x)} W_E[J]
\]

- Euclidean field equation

\[
\frac{\delta}{\delta \Phi_a(x)} \Gamma_E[\Phi] = \sqrt{g(x)} \, J_a(x)
\]

resembles classical equation of motion for \( J = 0 \).

- Need analytic continuation to obtain a viable equation of motion
Two-point functions

- Consider homogeneous background fields and global equilibrium
  \[ \beta^\mu = \left( \frac{1}{T}, 0, 0, 0 \right) \]

- Propagator and inverse propagator
  \[ \frac{\delta^2}{\delta J_a(-p)\delta J_b(q)} W_E[J] = G_{ab}(i\omega_n, p) \delta(p - q) \]
  \[ \frac{\delta^2}{\delta \Phi_a(-p)\delta \Phi_b(q)} \Gamma_E[\Phi] = P_{ab}(i\omega_n, p) \delta(p - q) \]

- From definition of effective action
  \[ \sum_b G_{ab}(p) P_{bc}(p) = \delta_{ac} \]
Spectral representation

- Källen-Lehmann spectral representation

\[ G_{ab} (\omega, p) = \int_{-\infty}^{\infty} dz \frac{\rho_{ab}(z^2 - p^2, z)}{z - \omega} \]

with \( \rho_{ab} \in \mathbb{R} \)

- correlation functions can be analytically continued in \( \omega = -u^\mu p_\mu \)
- branch cut or poles on real frequency axis \( \omega \in \mathbb{R} \) but nowhere else
- different propagators follow by evaluation of \( G_{ab} \) in different regions

\[ \Delta_M^{MMS} (p) = G_{ab} (i\omega_n, p) \]
\[ \Delta_R^{R} (p) = G_{ab} (p^0 + i\epsilon, p) \]
\[ \Delta_A^{A} (p) = G_{ab} (p^0 - i\epsilon, p) \]
\[ \Delta_F^{F} (p) = G_{ab} (p^0 + i\epsilon \text{sign} (p^0), p) \]
**Inverse propagator**

- Spectral representation for $G_{ab}$ implies that inverse propagator $P_{ab}(\omega, p)$
  - Can have zero-crossings for $\omega = p^0 \in \mathbb{R}$
  - Has in general branch-cut for $\omega = p^0 \in \mathbb{R}$
- So far reference frame with $u^\mu = (1, 0, 0, 0)$
- More general: analytic continuation with respect to
  \[
  \omega = -u^\mu p_\mu
  \]

- Use decomposition
  \[
  P_{ab}(p) = P_{1,ab}(p) - i s_1(-u^\mu p_\mu) P_{2,ab}(p)
  \]
  with sign function
  \[
  s_1(\omega) = \text{sign}(\text{Im} \ \omega)
  \]
- Both functions $P_{1,ab}(p)$ and $P_{2,ab}(p)$ are regular (no discontinuities)
Sign operator in position space

- In position space, **sign function** becomes **operator**

\[
s_\text{I} (-u^\mu p_\mu) = \text{sign}\left(\text{Im}\left(-u^\mu p_\mu\right)\right)
\]
\[
\rightarrow \text{sign}\left(\text{Im}\left(iu^\mu \frac{\partial}{\partial x^\mu}\right)\right) = \text{sign}\left(\text{Re}\left(u^\mu \frac{\partial}{\partial x^\mu}\right)\right) = s_\text{R}\left(u^\mu \frac{\partial}{\partial x^\mu}\right)
\]

- Geometric representation in terms of Lie derivative

\[
s_\text{R}(\mathcal{L}_u) \quad \text{or} \quad s_\text{R}(\mathcal{L}_\beta)
\]

- **Sign operator** appears also in analytically continued quantum effective action \(\Gamma[\Phi]\)
Analytically continued 1 PI effective action

- Analytically continued quantum effective action defined by analytic continuation of correlation functions

- Quadratic part

$$\Gamma_2[\Phi] = \frac{1}{2} \int_{x,y} \Phi_a(x) \left[ P_{1,ab}(x - y) + P_{2,ab}(x - y) s_R \left( u^\mu \frac{\partial}{\partial y^\mu} \right) \right] \Phi_b(y)$$

- Higher orders correlation functions less understood: no spectral representation

- Use inverse Hubbard-Stratonovich trick: terms quadratic in auxiliary field can be integrated out

- Allows to understand analytic structures of higher order terms

[Floerchinger, JHEP09 (2016) 099]
Can one obtain causal and real renormalized equations of motion from the 1 PI effective action?

naively: time-ordered action / Feynman $i\epsilon$ prescription:

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_{\text{time ordered}}[\Phi] = \sqrt{g} J_a(x)$$

This does not lead to causal and real equations of motion!

[e.g. Calzetta & Hu: *Non-equilibrium Quantum Field Theory* (2008)]
Retarded functional derivative

[Floerchinger, JHEP09 (2016) 099]

- **Real** and **causal dissipative field equations** follow from analytically continued effective action

\[
\left. \frac{\delta \Gamma[\Phi]}{\delta \Phi_a(x)} \right|_{\text{ret}} = \sqrt{g} J(x)
\]

- to calculate retarded variational derivative determine

\[
\delta \Gamma[\Phi]
\]

by varying the fields \(\delta \Phi(x)\) including dissipative terms

- set signs according to

\[
s_R(u^\mu \partial_\mu) \delta \Phi(x) \to -\delta \Phi(x), \quad \delta \Phi(x) \ s_R(u^\mu \partial_\mu) \to +\delta \Phi(x)
\]

- proceed as usual

- opposite choice of sign: field equations for backward time evolution
Causality

- Consider derivative of field equation (in flat space with $\sqrt{g} = 1$)

$$\left. \frac{\delta}{\delta \Phi_b(y)} \frac{\delta \Gamma}{\delta \Phi_a(x)} \right|_{\text{ret}} = \frac{\delta}{\delta \Phi_b(y)} J_a(x)$$

- Inverting this equation gives retarded Green’s function

$$\frac{\delta}{\delta J_b(y)} \Phi_a(x) = \Delta^R_{ab}(x, y)$$

- Only non-zero for $x$ future or null to $y$

- **Causality**: Field expectation value $\Phi_a(x)$ can only be influenced by the source $J_b(y)$ in or on the past light cone √
Damped harmonic oscillator

- Equation of motion
  \[ m\ddot{x} + c\dot{x} + kx = 0 \]
  or
  \[ \ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \]
  with \( \omega_0 = \sqrt{k/m} \) and \( \zeta = c/\sqrt{4mk} \)

- What is action for damped oscillator? This does not work:
  \[ \int \frac{d\omega}{2\pi} \frac{m}{2} x^*(\omega) \left[ \omega^2 + 2i\omega\zeta\omega_0 - \omega_0^2 \right] x(\omega) \]

- Consider inverse propagator
  \[ \omega^2 + 2i s_1(\omega) \omega \zeta\omega_0 - \omega_0^2 \]
  with
  \[ s_1(\omega) = \text{sign} \left( \text{Im} \omega \right) \]
  zero crossings (poles in the eff. propagator) are broadened to branch cut
Take for effective action

$$\Gamma[x] = \int \frac{d\omega}{2\pi} \frac{m}{2} x^*(\omega) \left[ -\omega^2 - 2i s_1(\omega) \omega \zeta \omega_0 + \omega_0^2 \right] x(\omega)$$

$$= \int dt \left\{ -\frac{1}{2} m \dddot{x}^2 + \frac{1}{2} c x s_R(\dot{t}) \dot{x} + \frac{1}{2} k x^2 \right\}$$

where the second line uses

$$s_1(\omega) = \text{sign}(\text{Im} \omega) \rightarrow \text{sign}(\text{Im} i \partial_t) = \text{sign}(\text{Re} \partial_t) = s_R(\partial_t)$$

Variation gives up to boundary terms

$$\delta \Gamma = \int dt \left\{ m \dddot{x} \delta x + \frac{1}{2} c \delta x s_R(\partial_t) \dot{x} - \frac{1}{2} c \dot{x} s_R(\partial_t) \delta x + k x \delta x \right\}$$

Set now $s_R(\partial_t) \delta x \rightarrow -\delta x$ and $\delta x s_R(\partial_t) \rightarrow \delta x$. Defines $\frac{\delta \Gamma}{\delta x}|_{\text{ret}}$.

Equation of motion for forward time evolution

$$\frac{\delta \Gamma}{\delta x}|_{\text{ret}} = m \dddot{x} + c \dot{x} + k x = 0$$
Scalar field with $O(N)$ symmetry

Consider effective action (with $\rho = \frac{1}{2} \varphi_j \varphi_j$)

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U(\rho, T) \right.$$  
$$\left. + \frac{1}{2} C(\rho, T) \left[ \varphi_j, s_R(u^\mu \partial_\mu) \right] \beta^{\nu} \partial_\nu \varphi_j \right\}$$

Variation at fixed metric $g_{\mu\nu}$ and $\beta^\mu$ gives

$$\delta \Gamma = \int d^d x \sqrt{g} \left\{ Z(\rho, T) g^{\mu\nu} \partial_\mu \delta \varphi_j \partial_\nu \varphi_j + \frac{1}{2} Z'(\rho, T) \varphi_m \delta \varphi_m g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j$$  
$$+ U'(\rho, T) \varphi_m \delta \varphi_m$$  
$$+ \frac{1}{2} C(\rho, T) \left[ \delta \varphi_j, s_R(u^\mu \partial_\mu) \right] \beta^{\nu} \partial_\nu \varphi_j$$  
$$+ \frac{1}{2} C(\rho, T) \left[ \varphi_j, s_R(u^\mu \partial_\mu) \right] \beta^{\nu} \partial_\nu \delta \varphi_j$$  
$$+ \frac{1}{2} C''(\rho, T) \varphi_m \delta \varphi_m \left[ \varphi_j, s_R(u^\mu \partial_\mu) \right] \beta^{\nu} \partial_\nu \varphi_j \right\}$$

set now $\delta \varphi_j \ s_R(u^\mu \partial_\mu) \rightarrow \delta \varphi_j$ and $s_R(u^\mu \partial_\mu) \delta \varphi_j \rightarrow -\delta \varphi_j$
Scalar field with $O(N)$ symmetry

- Field equation becomes

$$-\nabla_\mu [Z(\rho, T)\partial_\mu \varphi_j] + \frac{1}{2} Z'(\rho, T)\varphi_j \partial_\mu \varphi_m \partial^\mu \varphi_m$$

$$+ U'(\rho, T)\varphi_j + C(\rho, T)\beta^\mu \partial_\mu \varphi_j = 0$$

- Generalized Klein-Gordon equation with additional damping term
Where do energy & momentum go?

- Modified variational principle leads to equations of motion with dissipation.
- But what happens to the dissipated energy and momentum?
- And other conserved quantum numbers?
- What about entropy production?
Energy-momentum tensor expectation value

- Analogous to field equation, obtain by retarded variation
  \[
  \left. \frac{\delta \Gamma[\Phi, g_{\mu\nu}, \beta^\mu]}{\delta g_{\mu\nu}(x)} \right|_{\text{ret}} = -\frac{1}{2} \sqrt{g} \langle T^{\mu\nu}(x) \rangle
  \]

- Leads to Einstein’s field equation when \( \Gamma[\Phi, g_{\mu\nu}, \beta^\mu] \) contains Einstein-Hilbert term

- Useful to decompose
  \[
  \Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu] + \Gamma_D[\Phi, g_{\mu\nu}, \beta^\mu]
  \]
  where reduced action \( \Gamma_R \) contains no dissipative / discontinuous terms and \( \Gamma_D \) only dissipative terms

- Energy-momentum tensor has two parts
  \[
  \langle T^{\mu\nu} \rangle = (\bar{T}_R)^{\mu\nu} + (\bar{T}_D)^{\mu\nu}
  \]
General covariance

- Infinitesimal general coordinate transformations as a “gauge transformation” of the metric

\[
\delta g^G_{\mu\nu}(x) = g_{\mu\lambda}(x) \frac{\partial \epsilon^\lambda(x)}{\partial x^\nu} + g_{\nu\lambda}(x) \frac{\partial \epsilon^\lambda(x)}{\partial x^\mu} + \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} \epsilon^\lambda(x)
\]

- Temperature / fluid velocity field transforms as vector

\[
\delta \beta^G_\mu(x) = -\beta^\nu(x) \frac{\partial \epsilon^\mu(x)}{\partial x^\nu} + \frac{\partial \beta^\mu(x)}{\partial x^\nu} \epsilon^\nu(x)
\]

- Also fields \( \Phi_a \) transform in some representation, e. g. as scalars

\[
\delta \Phi^G_a(x) = \epsilon^\lambda(x) \frac{\partial}{\partial x^\lambda} \Phi_a(x)
\]

- Reduced action is invariant

\[
\Gamma_R[\Phi + \delta \Phi^G, g_{\mu\nu} + \delta g^G_{\mu\nu}, \beta^\mu + \beta^G_\mu] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu]
\]
Consider first situation **without dissipation** $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}]$.

Field equation implies (for $J = 0$)

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_R[\Phi, g_{\mu\nu}] = 0$$

Gauge variation of the metric

$$\delta \Gamma_R = \int d^d x \sqrt{g} \epsilon^\lambda (x) \nabla_\mu \langle T^\mu_\lambda (x) \rangle$$

General covariance $\delta \Gamma_R = 0$ and field equations imply covariant energy-momentum conservation

$$\nabla_\mu \langle T^\mu_\lambda (x) \rangle = 0$$
Situation with dissipation

Consider now situation with dissipation. General covariance of $\Gamma_R$:

$$\delta \Gamma_R = \int d^d x \left\{ \frac{\delta \Gamma_R}{\delta \Phi_a} \delta \Phi_a + \sqrt{g} \epsilon^\lambda \nabla_\mu (\bar{T}_R)^\mu_\lambda + \frac{\delta \Gamma_R}{\delta \beta^\mu} \delta \beta^\mu \right\} = 0$$

Reduced action not stationary with respect to field variations

$$\frac{\delta \Gamma_R}{\delta \Phi_a(x)} = - \frac{\delta \Gamma_D}{\delta \Phi_a(x)} \bigg|_{\text{ret}} =: - \sqrt{g(x)} M_a(x)$$

Reduced energy-momentum tensor not conserved

$$\nabla_\mu (\bar{T}_R)^\mu_\lambda(x) = - \nabla_\mu (\bar{T}_D)^\mu_\lambda(x)$$

Dependence on $\beta^\mu(x)$ cannot be dropped

$$\frac{\delta \Gamma_R}{\delta \beta^\mu(x)} =: \sqrt{g(x)} K_\mu(x)$$

General covariance implies four additional differential equations that determine $\beta^\mu$

$$M_a \partial_\lambda \Phi_a + \nabla_\mu (\bar{T}_D)^\mu_\lambda = \nabla_\mu [\beta^\mu K_\lambda] + K_\mu \nabla_\lambda \beta^\mu$$
Entropy production

- Contraction of previous equation with $\beta^\lambda$ gives

$$M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu_\lambda = \nabla_\mu \left[ \beta^\mu \beta^\lambda K_\lambda \right]$$

- Consider special case

$$\sqrt{g} K_\mu (x) = \frac{\delta \Gamma_R}{\delta \beta^\mu (x)} = \frac{\delta}{\delta \beta^\mu (x)} \int d^d x \sqrt{g} U(T)$$

with grand canonical potential density $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu} \beta^\mu \beta^\nu}}$$

- Using $s = \partial p/\partial T$ gives entropy current

$$\beta^\mu \beta^\lambda K_\lambda = s^\mu = s u^\mu$$

- Local form of second law of thermodynamics

$$\nabla_\mu s^\mu = M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu_\lambda \geq 0$$
Energy-momentum tensor for scalar field

- Analytic action

\[ \Gamma[\phi, g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \phi_j \partial_\nu \phi_j + U(\rho, T) + \frac{1}{2} C(\rho, T) [\phi_j, s_R(u^\mu \partial_\mu)] \beta^\nu \partial_\nu \phi_j \right\} \]

- Energy-momentum tensor

\[ \langle T^{\mu\nu}(x) \rangle = Z(\rho, T) \partial^\mu \phi_j \partial^\nu \phi_j \]

\[ - \left( g^{\mu\nu} + u^\mu u^\nu T \frac{\partial}{\partial T} \right) \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \phi_j \partial_\nu \phi_j + U(\rho, T) \right\} \]

- Generalizes \( T^{\mu\nu} \) for scalar field and \( T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + g^{\mu\nu} p \) for ideal fluid with pressure \( p = -U \) and enthalpy density \( \epsilon + p = sT = -T \frac{\partial}{\partial T} U \).

- General covariance and covariant conservation law imply

\[ \nabla_\mu \langle T^{\mu\nu}(x) \rangle = 0 \implies \text{Differential eqs. for } \beta^\mu(x) \]
Entropy production for scalar field

- Entropy current

\[ s^\mu = \beta^\mu \beta^\lambda K_\lambda = -\beta^\mu T \frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\} \]

- Generalized entropy density

\[ s_G = -\frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\} \]

- Entropy generation positive semi-definite for \( C(\rho, T) \geq 0 \)

\[ \nabla_\mu s^\mu = C(\rho, T) (\beta^\mu \partial_\mu \varphi_j) (\beta^\nu \partial_\nu \varphi_j) \geq 0 \]

- For fluid at rest \( u^\mu = (1, 0, 0, 0) \)

\[ \nabla_\mu s^\mu = \dot{s}_G = \frac{C(\rho, T)}{T^2} \dot{\varphi}_j \dot{\varphi}_j \]

entropy increases when \( \varphi_j \) oscillates. For example reheating after inflation.
Ideal fluid

- Consider effective action

\[ \Gamma[g_{\mu\nu}, \beta^\mu] = \Gamma_R[g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} U(T) \]

with effective potential \( U(T) = -p(T) \) and temperature

\[ T = \frac{1}{\sqrt{-g_{\mu\nu} \beta^\mu \beta^\nu}} \]

- Variation of \( g_{\mu\nu} \) at fixed \( \beta^\mu \) leads to

\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + pg^{\mu\nu} \]

where \( \epsilon + p = Ts = T \frac{\partial}{\partial T} p \) is the enthalpy density

- Describes ideal fluid. General covariance of covariant conservation \( \nabla_\mu T^{\mu\nu} = 0 \) leads to ideal fluid equations

\[ u^\mu \partial_\mu \epsilon + (\epsilon + p) \nabla_\mu u^\mu = 0 \]

\[ (\epsilon + p) u^\mu \nabla_\mu u^\nu + \Delta^{\nu\mu} \partial_\mu p = 0 \]
Viscous fluid

- Analytic action

\[
\Gamma[g_{\mu\nu}, \beta^\mu] = \int_x \left\{ U(T) + \frac{1}{4} [g_{\mu\nu}, s_R(L_u)] (2\eta(T)\sigma^{\mu\nu} + \zeta(T)\Delta^{\mu\nu}\nabla_\rho u^\rho) \right\}
\]

with projector

\[
\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}
\]

and

\[
\sigma^{\mu\nu} = \left( \frac{1}{2} \Delta^{\mu\alpha} \Delta^{-\alpha\beta} + \frac{1}{2} \Delta^{\mu\beta} \Delta^{-\mu\alpha} - \frac{1}{d-1} \Delta^{\mu\nu} \Delta^{-\alpha\beta} \right) \nabla_\alpha u_\beta
\]

leads to

\[
\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta \Gamma[g_{\mu\nu}, \beta^\mu]}{\delta g_{\mu\nu}} \bigg|_{\text{ret}} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\rho u^\rho
\]

- Describes viscous fluid with shear viscosity \(\eta(T)\) and bulk viscosity \(\zeta(T)\)

- Entropy production

\[
\nabla_\mu s^\mu = \frac{1}{T} \left[ 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2 \right]
\]
Conclusions

- A variational principle for theories with dissipation can be based on analytic continuation.
- Needs a local equilibrium setup: Generalized Gibbs ensemble with $T(x)$ and $u^\mu(x)$.
- Works at least for close-to-equilibrium situations, e.g. fluid dynamics coupled to additional fields.
- General covariance and energy-momentum conservation lead to equations for fluid velocity and entropy production.
- Local form of second law of thermodynamics is implemented on the level of the effective action $\Gamma[\Phi]$.
- Fluid dynamics equations of motion can follow from quantum effective action $\Gamma[\phi]$.
- Higher order functional derivatives contain information about hydrodynamic fluctuations.
- Different methods of QFT can be used to determine $\Gamma[\phi]$ from more microscopic calculations.
Equations of motion from the Feynman action?

Consider damped harmonic oscillator as example. Time-ordered or Feynman action is obtained from analytic action by replacing $s_1(\omega) \rightarrow \text{sign}(\omega)$

$$\Gamma_{\text{time ordered}}[x] = \int \frac{d\omega}{2\pi} \frac{m}{2} x^*(\omega) \left[ -\omega^2 - 2i|\omega|\zeta\omega_0 + \omega_0^2 \right] x(\omega)$$

Field equation $\frac{\delta}{\delta x(t)} \Gamma_{\text{time ordered}}[x] = J(t)$ would give

$$\left[ -\omega^2 - 2i|\omega|\zeta\omega_0 + \omega_0^2 \right] x(\omega) = J(\omega)$$

Violates reality constraint $x^*(\omega) = x(-\omega)$ for $J^*(\omega) = J(-\omega)$

Solution not causal

$$x(t) = \int_{t'} \Delta_F(t - t')J(t')$$

because Feynman propagator $\Delta_F(t - t')$ not causal.

In contrast, retarded variation of analytically continued action leads to real and causal equation of motion
Tree-like structures

- Discontinuous terms in analytic action could be of the form

\[
\Gamma_{\text{Disc}}[\Phi] = \int d^d x \sqrt{g} \left\{ f[\Phi](x) \ s_R(u^\mu(x) \frac{\partial}{\partial x^\mu}) \ g[\Phi](x) \right\}
\]

- More general, tree-like structure are possible such as

\[
\Gamma_{\text{Disc}}[\Phi] = \int_{x,y} \left\{ f[\Phi](x) \ s_R(u^\mu(x) \frac{\partial}{\partial x^\mu}) \ g[\Phi](x,y) \ s_R(u^\mu(y) \frac{\partial}{\partial y^\mu}) \ h[\Phi](y) \right\}
\]
or

\[
\Gamma_{\text{Disc}}[\Phi] = \int_{x,y,z} \left\{ f[\Phi](x) \ s_R(u^\mu(x) \frac{\partial}{\partial x^\mu}) \ g[\Phi](x,y,z) \ s_R(u^\mu(y) \frac{\partial}{\partial y^\mu}) \ h[\Phi](y) \right. \\
\left. \times s_R(u^\mu(z) \frac{\partial}{\partial z^\mu}) \ j[\Phi](z) \right\}
\]

- For retarded variation calculate \( \delta \Gamma \) and set \( s_R(u^\mu \partial_\mu) \to -1 \) if derivative operator points towards node that is varied and \( s_R(u^\mu \partial_\mu) \to 1 \) if derivative operator points in opposite direction
Analytic continuation of FRG equations
[Floerchinger, JHEP 1205 (2012) 021]

- Consider a point \( p_0^2 - \vec{p}^2 = m^2 \) where \( P_1(m^2) = 0 \).
- One can expand around this point

\[
P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \cdots \\
P_2 = Z\gamma^2 + \cdots
\]

- Leads to Breit-Wigner form of propagator (with \( \gamma^2 = m\Gamma \))

\[
G(p) = \frac{1}{Z} \frac{-p_0^2 + \vec{p}^2 + m^2 + i s(p_0) m\Gamma}{(-p_0^2 + \vec{p}^2 + m^2)^2 + m^2\Gamma^2}.
\]

- A few flowing parameters describe efficiently the singular structure of the propagator.

\[
\frac{\gamma_1^2}{\Lambda^2}
\]

black solid line: evaluation at \( p_0 = m_1 \)
red dashed line: evaluation at \( p_0 = 0 \)
Truncation for relativistic scalar $O(N)$ theory

\[ \Gamma_k = \int_{t, \vec{x}} \left\{ \sum_{j=1}^{N} \frac{1}{2} \bar{\phi}_j \bar{P}_\phi(i\partial_t, -i\vec{\nabla}) \phi_j \right. 
\left. + \frac{1}{4} \bar{\rho} \bar{P}_\rho(i\partial_t, -i\vec{\nabla}) \rho + \bar{U}_k(\rho) \right\} \]

with \( \bar{\rho} = \frac{1}{2} \sum_{j=1}^{N} \phi_j^2 \).

- Goldstone propagator massless, expanded around \( p_0 - \bar{p}^2 = 0 \)
  \[ \bar{P}_\phi(p_0, \bar{p}) \approx \bar{Z}_\phi \left(-p_0^2 + \bar{p}^2\right) \]

- Radial mode is massive, expanded around \( p_0^2 - \bar{p}^2 = m_1^2 \)
  \[ \bar{P}_\phi(p_0, \bar{p}) + \bar{\rho}_0 \bar{P}_\rho(p_0, \bar{p}) + \bar{U}'_k + 2\bar{\rho} \bar{U}''_k \]
  \[ \approx \bar{Z}_\phi Z_1 \left[ (-p_0^2 + \bar{p}^2 + m_1^2) - is(p_0) \gamma_1^2 \right] \]
Flow of the effective potential

\[ \partial_t U_k(\rho) \big|_{\bar{\rho}} = \frac{1}{2} \int_{p_0=\omega_n, \bar{p}} \left\{ \frac{(N - 1)}{\bar{p}^2 - p_0^2 + U' + \frac{1}{Z_\phi} R_k} + \frac{1}{Z_1 \left[ (\bar{p}^2 - p_0^2) - i s(p_0) \gamma_1^2 \right] + U' + 2\rho U'' + \frac{1}{Z_\phi} R_k} \right\} \frac{1}{Z_\phi} \partial_t R_k. \]

- Summation over Matsubara frequencies \( p_0 = i2\pi T n \) can be done using contour integrals.
- Radial mode has non-zero decay width since it can decay into Goldstone excitations.
- Use Taylor expansion for numerical calculations

\[ U_k(\rho) = U_k(\rho_{0,k}) + m_k^2 (\rho - \rho_{0,k}) + \frac{1}{2} \lambda_k (\rho - \rho_{0,k})^2 \]