Federated Learning With Quantized Global Model Updates

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Abstract

We study federated learning (FL), which enables mobile devices to utilize their local datasets to collaboratively train a global model with the help of a central server, while keeping data localized. At each iteration, the server broadcasts the current global model to the devices for local training, and aggregates the local model updates from the devices to update the global model. Previous work on the communication efficiency of FL has mainly focused on the aggregation of model updates from the devices, assuming perfect broadcasting of the global model. In this paper, we instead consider broadcasting a compressed version of the global model. This is to further reduce the communication cost of FL, which can be particularly limited when the global model is to be transmitted over a wireless medium. We introduce a lossy FL (LFL) algorithm, in which both the global model and the local model updates are quantized before being transmitted. We analyze the convergence behavior of the proposed LFL algorithm assuming the availability of accurate local model updates at the server. Numerical experiments show that the quantization of the global model can actually improve the performance for non-iid data distributions. This observation is corroborated with analytical convergence results.

1 Introduction

Federated learning (FL) enables wireless devices to collaboratively train a global model by utilizing locally available data and computational capabilities under the coordination of a central parameter server (PS) while the local data never leaves the devices \cite{mcmahan2017communication}.

In FL with $M$ devices the goal is to minimize a loss function $F(\mathbf{\theta}) = \sum_{m=1}^{M} \frac{B_m}{B} F_m(\mathbf{\theta})$ with respect to the global model $\mathbf{\theta} \in \mathbb{R}^d$, where $F_m(\mathbf{\theta})$ is the loss function at device $m$, given by $F_m(\mathbf{\theta}) = \frac{1}{B_m} \sum_{u \in B_m} f(\mathbf{\theta}, u)$, with $B_m$ representing device $m$'s local dataset of size $B_m$, $B \triangleq \sum_{m=1}^{M} B_m$, and $f(\cdot, \cdot)$ is an empirical loss function defined by the learning task. Having access to the global model $\mathbf{\theta}$, device $m$ utilizes its local dataset and performs multiple iterations of stochastic gradient descent (SGD) in order to minimize the local loss function $F_m(\mathbf{\theta})$. It then sends the local model update to the server, which aggregates the updates from all the devices to update the global model.

FL mainly targets mobile applications at the network edge, and the wireless communication links connecting these devices to the network are typically limited in bandwidth and power, and suffer from various channel impairments such as fading, shadowing, or interference; hence the need to develop an FL framework with limited communication requirements becomes more vital. While communication-efficient FL has been widely studied, prior works mainly
focused on the devices-to-PS links, assuming perfect broadcasting of the global model to the
devices at each iteration. In this paper, we design an FL algorithm aiming to reduce the
cost of both PS-to-devices and devices-to-PS communications.

Related work There is a fast-growing body of literature on the communication efficiency
of FL targeting restricted bandwidth devices. Several studies address this issue by considering
communications with rate limitations, and propose different compression and quantization
techniques [2–9], as well as performing local updates to reduce the frequency of commun-
ications from the devices to the PS [10, 11]. Statistical challenges arise in FL since the
data samples may not be independent and identically distributed (iid) across devices. The
common sources of the dependence or bias in data distribution are the participating devices
being located in a particular geographic region, and/or at a particular time window [12].
Different approaches have been studied to mitigate the effect of non-iid data in FL [3, 13, 17].
Also, FL suffers from a significant variability in the system, which is mainly due to the
hardware, network connectivity, and available power associated with different devices [18].
Active device selection schemes have been introduced to alleviate significant variability in
FL systems, where a subset of devices share the resources and participate at each iteration
of training [19–23]. There have also been efforts in developing convergence guarantees for
FL under various scenarios, considering iid data across the devices [11, 24–27], non-iid data
27, 24, 28, 29, participation of all the devices [30, 33]. or only a subset of devices
at each iteration [34, 35, 29, 37], and FL under limited communication constraints [37, 40].
Furthermore, FL with compressed global model transmission has been studied recently in
[41] aiming to alleviate the communication footprint from the PS to the devices. Since
the global model parameters are relatively skewed/diverse, with the scheme in [41] at each
iteration the PS employs a linear transform before quantization, and the devices apply the
inverse linear transform to estimate the global model.

Our contributions With the exception of [41], the literature on FL considers perfect
broadcasting of the global model from the PS to the devices. With this assumption, no
matter what type of local update or device-to-PS communication strategy is used, all the
devices are synchronized with the same global model at each iteration. In this paper, we
instead consider broadcasting a quantized version of the global model update by the PS,
which provides the devices with a lossy estimate of the global model (rather than its accurate
estimate) with which to perform local training. This further reduces the communication cost
of FL, which can be particularly limited for transmission over a wireless medium while serving
a massive number of devices. Also, it is interesting to investigate the impact of various
hyperparameters on the performance of FL with lossy broadcasting of the global model since
FL involves transmission over wireless networks with limited bandwidth. We introduce a
lossy FL (LFL) algorithm, where at each iteration the PS broadcasts a compressed version
of the global model update to all the devices through quantization. The devices recover
an estimate of the current global model by combining the received quantized global model
update with their previous estimate, and perform local training using their estimate, and
return the local model updates, again employing quantization. The PS updates the global
model after receiving the quantized local model updates from the devices. We provide
convergence analysis of the LFL algorithm investigating the impact of lossy broadcasting
on the performance of FL, where for ease of analysis we assume the availability of accurate
local model updates from the devices at the PS. Numerical experiments on the MNIST
dataset illustrate the efficiency of the proposed LFL algorithm for both iid and non-iid data
scenarios across the devices. We observe that, in addition to a significant communication
cost saving with the LFL algorithm, the availability of a compressed global model at the
devices can even improve the performance compared to the accurate global model in non-iid
data scenarios. This observation is corroborated by the analytical convergence result.

We highlight that the proposed LFL algorithm differs from the approach introduced in
[41], where the PS sends a quantized version of the current global model to a subset of
devices that will participate in the learning process at that iteration. The efficiency of
quantization diminishes significantly when the peak-to-average ratio of the parameters is
large. To overcome this, in [41] the PS first employs a linear transform in order to spread the
information of the global model vector more evenly among its dimensions, and broadcasts a
We consider a lossy PS-to-devices transmission, in which the PS shares a compressed
value of ϕ over a constrained bandwidth medium. We denote the estimate of the global model
of communication from the PS to the devices, and can be particularly beneficial when the
size of the model parameters. Furthermore, the performance evaluation in [41] is limited
such scenarios by sending the global model, rather than the model update, every time the
subset of participating devices changes. Note also that, compared to the LFL algorithm, the approach
rather than a randomly chosen subset, would introduce limited additional communication
cost as broadcasting is typically more efficient than sending independent information to
devices. Moreover, in practice, the subset of participating devices remain the same for a
number of iterations, until a device leaves or joins. Our algorithm can easily be adopted to
such iterations, until a device leaves or joins. Our algorithm can easily be adopted to

\begin{align}
\hat{\theta}_t &\triangleq \theta_t - \frac{\theta_{t-1}}{q},
\end{align}

for an integer \( q \geq 1 \), we have

\begin{align}
Q(x_i, q) &\triangleq \text{sign}(x_i) \cdot \left( x_{\min} + (x_{\max} - x_{\min}) \cdot \varphi \left( \frac{|x_i| - x_{\min}}{x_{\max} - x_{\min}}, q \right) \right), \quad i \in [d],
\end{align}

where \( \varphi(\cdot, \cdot) \) is a quantization function defined in the following. For \( 0 \leq x \leq 1 \) and \( q \geq 1 \), let \( l \in \{0, \ldots, q-1\} \) be an integer such that \( x \in [l/q, (l+1)/q) \). We then define

\begin{align}
\varphi(x, q) &\triangleq \begin{cases} 
{l/q,} & \text{with probability } 1 - (xq - l), \\
(l+1)/q, & \text{with probability } xq - l.
\end{cases}
\end{align}

2 Lossy Federated Learning (LFL) Algorithm

We consider a lossy PS-to-devices transmission, in which the PS shares a compressed
information about the global model with the devices at each iteration. This reduces the cost
communication from the PS to the devices, and can be particularly beneficial when the
PS resources, such as power and bandwidth, are limited, and/or communication takes place
over a constrained bandwidth medium. We denote the estimate of the global model \( \theta(t) \) at
the devices by \( \hat{\theta}(t) \), where \( t \) represents the global iteration count. Having recovered \( \hat{\theta}(t) \), the
devices perform a \( \tau \)-step SGD with respect to their local datasets, and transmit their local
model updates to the PS.

2.1 Global Model Broadcasting

In the proposed LFL algorithm, the PS performs stochastic quantization similarly to the
QSGD algorithm introduced in [42] with a slight modification, to broadcast the information
about the global model to the devices. In particular, at global iteration \( t \), the PS aims to
broadcast the global model update \( \theta(t) - \hat{\theta}(t-1) \) to the devices. In the following, we present
the stochastic quantization technique we use, denoted by \( Q(\cdot, \cdot) \).

Stochastic quantization Given \( x \in \mathbb{R}^d \), with the \( i \)-th entry denoted by \( x_i \), we define

\begin{align}
x_{\max} &\triangleq \max \{|x_1|, \ldots, |x_d|\}, \\
x_{\min} &\triangleq \min \{|x_1|, \ldots, |x_d|\}.
\end{align}

Given a quantization level \( q \geq 1 \), we have

\begin{align}
Q(x_i, q) &\triangleq \text{sign}(x_i) \cdot \left( x_{\min} + (x_{\max} - x_{\min}) \cdot \varphi \left( \frac{|x_i| - x_{\min}}{x_{\max} - x_{\min}}, q \right) \right), \quad i \in [d],
\end{align}

where \( \varphi(\cdot, \cdot) \) is a quantization function defined in the following. For \( 0 \leq x \leq 1 \) and \( q \geq 1 \), let
\( l \in \{0, \ldots, q-1\} \) be an integer such that \( x \in [l/q, (l+1)/q) \). We then define

\begin{align}
\varphi(x, q) &\triangleq \begin{cases} 
l/q, & \text{with probability } 1 - (xq - l), \\
(l+1)/q, & \text{with probability } xq - l.
\end{cases}
\end{align}
We define $Q(x, q) \triangleq [Q(x_1, q), \ldots, Q(x_d, q)]^T$, and we highlight that it is represented by
\[ R_Q = 64 + d(1 + \log_2(q + 1)) \text{ bits,} \quad (3) \]
where 64 bits are used to represent $x_{\text{max}}$ and $x_{\text{min}}$. $d$ bits are used for $\text{sign}(x_i)$, $\forall i \in [d]$, and $d\log_2(q + 1)$ bits represent $\varphi ((||x_i| - x_{\text{min}}|)/(x_{\text{max}} - x_{\text{min}}), q)$, $\forall i \in [d]$. We note that we have modified the QSGD scheme proposed in [42] by normalizing the entries of vector $x$ with $x_{\text{max}} - x_{\text{min}}$ rather than $||x||_2$.

**Lemma 1.** For the quantization function $\varphi(x, q)$ given in (2b), we have
\[ E_\varphi [\varphi(x, q)] = x, \quad (4a) \]
\[ E_\varphi [\varphi^2(x, q)] \leq x^2 + \frac{1}{4q^2}, \quad (4b) \]
where $E_\varphi$ represents expectation with respect to the randomness of the quantization function $\varphi(\cdot, \cdot)$.

**Proof.** Given $\varphi(x, q)$ in (2b), we have
\[ E_\varphi [\varphi(x, q)] = \left(\frac{l}{q}\right) (1 + l - qx) + \left(\frac{l+1}{q}\right) (qx - l) = x. \quad (5) \]
Also, we have
\[ E_\varphi [\varphi^2(x, q)] = \left(\frac{l}{q}\right)^2 (1 + l - qx) + \left(\frac{l+1}{q}\right)^2 (qx - l) = \frac{1}{q^2} (-l^2 + 2lxq + qx - l) \]
\[ = x^2 + \frac{1}{q^2} (xq - l)(1 - xq + l) \leq x^2 + \frac{1}{4q^2}, \quad (6) \]
where (a) follows since $(xq - l)(1 - xq + l) \leq 1/4$. \qed

According to Lemma 1, it follows that
\[ E_\varphi [Q(x, q)] = x, \quad (7a) \]
\[ E_\varphi [||Q(x, q)||_2^2] = \sum_{i=1}^d E_\varphi [||Q(x_i, q)||_2^2] = (x_{\text{max}} - x_{\text{min}})^2 \sum_{i=1}^d E_\varphi [\varphi^2(x_i - x_{\text{min}}/x_{\text{max}} - x_{\text{min}}, q)] \]
\[ + dx_{\text{min}}^2 + 2x_{\text{min}}(x_{\text{max}} - x_{\text{min}}) \sum_{i=1}^d E_\varphi [\varphi(x_i - x_{\text{min}}/x_{\text{max}} - x_{\text{min}}, q)] \]
\[ \leq (x_{\text{max}} - x_{\text{min}})^2 \sum_{i=1}^d \left(\frac{|x_i| - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}}^2 + \frac{1}{4q^2}\right) + dx_{\text{min}}^2 + 2x_{\text{min}} \sum_{i=1}^d (|x_i| - x_{\text{min}}) \]
\[ = ||x||_2^2 + d(\frac{x_{\text{max}} - x_{\text{min}}}{4q^2})^2 + \epsilon \frac{||x||_2^2}{4q^2}, \quad (7b) \]
where (b) follows from Lemma 1 and in (c) we define $0 \leq \epsilon \leq 1$ as $\epsilon \triangleq (x_{\text{max}} - x_{\text{min}})^2/||x||_2^2$.

We highlight that the value of $\epsilon$ depends on the skewness of the magnitudes of the entries of $x$, where it increases for a more skewed entries with a higher variance. In one extreme case, we have $\epsilon = 0$, if and only if all the entries of $x$ have the same magnitude. In the other extreme case, we have $\epsilon = 1$, if and only if $x$ has only one non-zero entry.

Given a quantization level $q_1$, the PS broadcasts $Q(\theta(t) - \hat{\theta}(t - 1), q_1)$ to the devices at global iteration $t$. Then the devices obtain the following estimate of $\theta(t)$:
\[ \hat{\theta}(t) = \hat{\theta}(t - 1) + Q(\theta(t) - \hat{\theta}(t - 1), q_1), \quad (8) \]
which is equivalent to
\[ \hat{\theta}(t) = \theta(0) + \sum_{i=1}^t Q(\theta(i) - \hat{\theta}(i - 1), q_1), \quad (9) \]
where we assumed that $\hat{\theta}(0) = \theta(0)$. We note that, having the knowledge of the compressed vector $Q(\theta(i) - \hat{\theta}(i - 1), q_1)$, $\forall i \in [t]$, the PS can also track $\hat{\theta}(t)$ at each iteration.
Algorithm 1 LFL

1: Initialize $\theta(0)$
2: for $t = 0, \ldots, T - 1$ do
   3: Global model broadcasting
      4: PS broadcasts $Q(\theta(t) - \hat{\theta}(t-1), q_1)$
      5: $\hat{\theta}(t) = \hat{\theta}(t-1) + Q(\theta(t) - \hat{\theta}(t-1), q_1)$
   6: Local update aggregation
      7: for $m = 1, \ldots, M$ in parallel do
         8: Device $m$ transmits $Q(\Delta \theta_m(t), q_2) = Q(\theta_{m}^{r+1}(t) - \hat{\theta}(t), q_2)$
      9: end for
   10: $\theta(t + 1) = \hat{\theta}(t) + \sum_{m=1}^{M} \frac{B_m}{B} Q(\Delta \theta_m(t), q_2)$
end for

2.2 Local Update Aggregation

After recovering $\hat{\theta}(t)$, device $m$ performs a $\tau$-step local SGD, where the $i$-th step corresponds to the following update:

$$
\theta_{m}^{i+1}(t) = \theta_{m}^{i}(t) - \eta_{m}(t) \nabla F_{m} \left( \theta_{m}^{i}(t), \xi_{m}(t) \right), \quad i \in [\tau],
$$

where $\theta_{m}^{i}(t) = \hat{\theta}(t)$, and $\xi_{m}(t)$ denotes the local mini-batch chosen uniformly at random from the local dataset $B_{m}$. It then transmits the quantized local model update $\Delta \theta_{m}(t) = \theta_{m}^{r+1}(t) - \hat{\theta}(t)$ using a quantization level $q_2$, i.e., $Q(\Delta \theta_{m}(t), q_2)$. Having received $Q(\Delta \theta_{m}(t), q_2)$ from device $m$, $\forall m \in [M]$, the PS updates the global model as

$$
\theta(t + 1) = \hat{\theta}(t) + \sum_{m=1}^{M} \frac{B_m}{B} Q(\Delta \theta_m(t), q_2).
$$

Algorithm 1 summarizes the proposed LFL algorithm.

3 Convergence Analysis of LFL Algorithm

Here we analyze the convergence behaviour of the LFL algorithm, where for the simplicity of the analysis, we assume that the devices can transmit their local updates, $\Delta \theta_{m}(t)$, $\forall m$, accurately/in a lossless fashion to the PS, and focus on the impact of lossy broadcasting on the convergence performance.

3.1 Preliminaries

We denote the optimal solution minimizing loss function $F(\theta)$ by $\theta^*$, and the minimum loss as $F^*$, i.e., $\theta^* \triangleq \arg \min_{\theta} F(\theta)$, and $F^* \triangleq F(\theta^*)$. We also denote the minimum value of the local loss function at device $m$ by $F_m^*$, for $m \in [M]$. We further define $\Gamma \triangleq F^* - \sum_{m=1}^{M} \frac{B_m}{B} F_m^*$, where $\Gamma \geq 0$, and its magnitude indicates the bias in the data distribution across devices.

For ease of analysis, we set $\eta_{m}(t) = \eta(t)$. Thus, the $i$-th step SGD at device $m$ is given by

$$
\theta_{m}^{i+1}(t) = \theta_{m}^{i}(t) - \eta(t) \nabla F_{m} \left( \theta_{m}^{i}(t), \xi_{m}(t) \right), \quad i \in [\tau], m \in [M],
$$

where $\theta_{m}^{i}(t) = \hat{\theta}(t)$, given in [8]. Device $m$ transmits the local model update

$$
\Delta \theta_{m}(t) = \theta_{m}^{r+1}(t) - \hat{\theta}(t) = -\eta(t) \sum_{i=1}^{\tau} \nabla F_{m} \left( \theta_{m}^{i}(t), \xi_{m}(t) \right), \quad m \in [M],
$$

and the PS updates the global model as

$$
\theta(t + 1) = \hat{\theta}(t) + \sum_{m=1}^{M} \frac{B_m}{B} \Delta \theta_{m}(t) = \hat{\theta}(t) - \eta(t) \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} \nabla F_{m} \left( \theta_{m}^{i}(t), \xi_{m}(t) \right).
$$
Assumption 1. The loss functions $F_1, \ldots, F_M$ are all $L$-smooth; that is, $\forall v, w \in \mathbb{R}^d$,
\[
F_m(v) - F_m(w) \leq \langle v - w, \nabla F_m(w) \rangle + \frac{L}{2} \|v - w\|^2, \quad \forall m \in [M].
\] (15)

Assumption 2. The loss functions $F_1, \ldots, F_M$ are all $\mu$-strongly convex; that is, $\forall v, w \in \mathbb{R}^d$,
\[
F_m(v) - F_m(w) \geq \langle v - w, \nabla F_m(w) \rangle + \frac{\mu}{2} \|v - w\|^2, \quad \forall m \in [M].
\] (16)

Assumption 3. The expected squared $l_2$-norm of the stochastic gradients are bounded; that is
\[
\mathbb{E}_x \left[ \|\nabla F_m \left( \theta^i_m(t), \xi_m^i(t) \right) \|_2^2 \right] \leq G^2, \quad \forall i \in [\tau], \forall m \in [M], \ \forall t.
\] (17)

3.2 Convergence Rate

In the following theorem, whose proof is provided in Appendix A, we present the convergence rate of the LFL algorithm assuming that the devices can send their local updates accurately.

Theorem 1. Let $0 < \eta(t) \leq \min \left\{ 1, \frac{1}{\mu \tau} \right\}$, $\forall t$. We have
\[
\mathbb{E} \left[ \|\theta(t) - \theta^*\|_2^2 \right] \leq \left( \prod_{i=0}^{t-1} A(i) \right) \|\theta(0) - \theta^*\|_2^2 + \sum_{j=0}^{t-1} B(j) \prod_{i=j+1}^{t-1} A(i),
\] (18a)
where
\[
A(i) \triangleq 1 - \mu \eta(i) (\tau - \eta(i)(\tau - 1)),
\] (18b)
\[
B(i) \triangleq (1 - \mu \eta(i) (\tau - \eta(i)(\tau - 1))) \left( \frac{\eta(i - 1) \tau G}{2q_1} \right)^2 \varepsilon d + \eta^2(i)(\tau^2 + \tau - 1) G^2 + (1 + \mu(1 - \eta(i))) \eta^2(i) G^2 \frac{(\tau - 1)(2\tau - 1)}{6},
\] (18c)
for some $0 \leq \varepsilon \leq 1$, and the expectation is with respect to the stochastic gradient function and stochastic quantization.

Choice of $\varepsilon$ We highlight that $\varepsilon$ appears in the convergence analysis of the LFL algorithm in inequality (19), in which we have
\[
\mathbb{E} \left[ \max \left\{ \left\| \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} \nabla F_m \left( \theta^i_m(t - 1), \xi_m^i(t - 1) \right) \right\| \right] - \min \left\{ \left\| \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} \nabla F_m \left( \theta^i_m(t - 1), \xi_m^i(t - 1) \right) \right\| \right\}^2 \right] \leq \varepsilon \mathbb{E} \left[ \left\| \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} \nabla F_m \left( \theta^i_m(t - 1), \xi_m^i(t - 1) \right) \right\|_2^2 \right],
\] (19)
which follows from (18b), where we note that
\[
\theta(t) - \tilde{\theta}(t - 1) = -\eta(t - 1) \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} \nabla F_m \left( \theta^i_m(t - 1), \xi_m^i(t - 1) \right).
\] (20)

On average the entries of $\theta(t) - \tilde{\theta}(t - 1)$, given in (20), are not expected to have very diverse magnitudes. Thus, the inequality in (19) should hold for a relatively small value of $\varepsilon$ using the LFL algorithm. We have observed through numerical experiments that $\varepsilon \approx 10^{-3}$ satisfies the inequality (19) for the LFL algorithm.

Impact of lossy broadcasting The first term in $B(i)$ is due to the imperfect broadcasting of the global model update at the PS, which decreases with the quantization level $q_1$ and increases linearly with $\varepsilon$. This term is a complicated function of the number of local iterations $\tau$ depending on other setting variables. As we will observe in the experiments, a smaller $\tau$ provides the best performance for relatively large and small values of $q_1$ compared to a medium $q_1$ value for non-iid data.

For a decreasing learning rate over time, such that $\lim_{t \to \infty} \eta(t) = 0$, and given small enough $\varepsilon$, it is easy to verify that $\lim_{T \to \infty} \mathbb{E} \left[ F(\theta(T)) \right] - F^* = 0$. 
Figure 1: Test accuracy of LFL and lossless broadcasting for different $q_1$ values and $q_2 = 3$.

4 Numerical Experiments

Here we investigate the performance of the proposed LFL algorithm for image classification on the MNIST dataset [43] utilizing ADAM optimizer [44]. For the experiments, we consider $M = 40$ devices in the system, and we set the size of the local mini-batch sample for each local iteration to $|\mathcal{C}_{m,t}| = 500$. We measure the performance as the accuracy with respect to the test samples, called test accuracy, versus the global iteration count, $t$.

**Network architecture** We train a convolutional neural network (CNN) with 6 layers including two $5 \times 5$ convolutional layers with ReLu activation and the same padding, where the first and the second layers have 32 and 64 channels, respectively, and each has stride 1 and followed by a $2 \times 2$ max pooling layer with stride 2. It also has a fully connected layer with 1024 units and ReLu activation with dropout 0.8 followed by a softmax output layer.

**Data distribution** We consider two data distribution scenarios. In the iid scenario, we randomly split the training data samples to $M$ disjoint subsets, and assign each subset to a distinct device. In the non-iid scenario, we split the training data samples with the same label (from the same class) to $M/10$ disjoint subsets (assume that $M$ is divisible by 10). We then assign each subset of data samples, selected at random, to a different device.

**Convergence variables** For the analytical results on the convergence rate of the LFL algorithm, we set $\eta(t) = \min\{1/(1/(\mu t))\}/(t+1), \forall t$, and consider $M = 40$ devices. We assume that $\mu = 1$, $L = 10$, $\|\Theta(0) - \Theta^*\|_2 = 5 \times 10^3$, and $\varepsilon = 10^{-3}$. We also model the iid and non-iid scenarios by setting $(G^2, \Gamma) = (10, 5)$ and $(G^2, \Gamma) = (100, 50)$, respectively, where we note that the non-iid scenario results in higher $G$ and $\Gamma$ values.

For numerical evaluation, we consider the performance of the lossless broadcasting scenario, where we assume that the devices receive the current global model accurately. We highlight that this approach requires transmission of $R_{LL} = 33d$ bits, where we assume that each entry of the global model is represented by 33 bits, 32 bits to represent the absolute value and 1 bit representing the sign. Thus, the saving ratio in the communication bits of broadcasting from the PS using the LFL algorithm versus the lossless broadcasting approach is

$$\frac{R_{LL}}{R_Q} = \frac{33d}{64 + d(1 + \log_2(q_1 + 1))} \approx \frac{33}{1 + \log_2(q_1 + 1)},$$

where (a) follows assuming that $d \gg 1$.

In Figure 1 we illustrate the performance of LFL for different broadcasting quantization levels $q_1 \in \{5, 7, 30\}$ for iid and non-iid scenarios. We have also included the performance of the lossless broadcasting approach, where we set $q_2 = 3$ for both LFL and the lossless broadcasting approach. For each experimental setting, we have found the number of local iterations $\tau$ providing the best performance taking into consideration the convergence rate, performance stability, and the final accuracy level. As expected, the best $\tau$ value for the iid scenario is larger than its non-iid counterpart, which is due to the heterogeneity of non-iid data, in which case an excessively large $\tau$ value results in biased local gradients. As can be seen for the iid scenario, the performance loss due to using the LFL algorithm for any of the quantization levels $q_1 \in \{5, 7, 30\}$ compared to the lossless approach is negligible, despite a
factor of 9.2 savings in the number of bits that need to be broadcast with LFL for \( q_1 = 5 \). This illustrates the efficiency of the LFL algorithm for the iid scenario providing significant communication cost savings without any visible performance degradation. On the other hand, for the non-iid scenario, the best \( \tau \) is larger for a medium value of \( q_1 = 7 \). When the devices have a relatively good estimate of the global model, i.e., for large \( q_1 \), increasing \( \tau \) excessively results in diverging local model updates, which is due to the bias in the local data, leading to performance degradation/instability. On the other hand, for a very small \( q_1 \) value, the devices have a relatively poor global model estimate, and using a large \( \tau \) might not lead to the best local gradient directions. We further observe that the LFL algorithm for both quantization levels \( q_1 \in \{5, 50\} \) performs very close to the lossless broadcasting approach for the non-iid scenario. Similar to the iid scenario, it can be seen that the gap between the performance of LFL with a small quantization level \( q_1 = 5 \) and that of the lossless broadcasting scheme for non-iid data is marginal despite a significant communication load saving. The advantage of the LFL algorithm is more pronounced in the non-iid scenario, where it is shown that having a lossy estimate of the global model at the devices with a medium quantization level \( q_1 = 7 \) can improve the performance of FL even beyond the lossless broadcasting approach, which requires \( \times 8.25 \) higher communication load. Lossy broadcasting provides the devices with a perturbed global model, which can be beneficial for a medium quantization level, in which case the perturbation is neither too high nor too small, through using a larger \( \tau \) value without any significant performance degradations/instability for the non-iid scenario when local data is biased. As another advantage of the LFL algorithm, it is evident that the performance degradation due to introducing the bias in the local data is relatively small, particularly for a medium quantization level \( q_1 \). The results in Figure 2 for iid and non-iid scenarios are corroborated with the analytical convergence result shown in Figure 3.

5 Conclusion

FL is demanding in terms of bandwidth, particularly when deep networks with huge numbers of parameters are trained across a large number of devices. Communication is typically the major bottleneck, since it involves iterative transmission over a bandwidth-limited wireless medium between the PS and a massive number of devices at the edge. With the goal of reducing the communication cost, we have studied FL with lossy broadcasting, where, in contrast to most of the existing work in the literature, the PS broadcasts a compressed version of the global model to the devices. We have considered broadcasting quantized global model updates from the PS, which can be used to estimate the current global model at the devices for local SGD iterations. The PS aggregates the quantized local model updates from the devices, according to which it updates the global model. We have derived convergence guarantees for the LFL algorithm to analyze the impact of lossy broadcasting on the FL performance assuming accurate local model updates at the PS. Numerical experiments have shown the efficiency of the proposed LFL algorithm despite the significant reduction in the communication load. It performs as good as the lossless broadcasting approach for iid data even for a relatively small quantization level for broadcasting. On the other hand, it can outperform the lossless broadcasting approach for non-iid data thanks to the small perturbations introduced in the global model.
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A Proof of Theorem \[1\]

We have
\[
E \left[ \| \theta(t+1) - \theta^* \|_2^2 \right] = E \left[ \| \hat{\theta}(t) - \theta^* \|_2^2 \right] + E \left[ \left\| \sum_{m=1}^M \frac{B_m}{B} \Delta \theta_m(t) \right\|_2^2 \right] + 2E \left[ (\hat{\theta}(t) - \theta^*, \sum_{m=1}^M \frac{B_m}{B} \Delta \theta_m(t)) \right].
\]
(22)

In the following, we bound the last two terms on the right hand side (RHS) of (22). From the convexity of \( \| \cdot \|_2^2 \), it follows that
\[
E \left[ \left\| \sum_{m=1}^M \frac{B_m}{B} \Delta \theta_m(t) \right\|_2^2 \right] \leq \sum_{m=1}^M \frac{B_m}{B} E \left[ \| \Delta \theta_m(t) \|_2^2 \right]
\]
\[
= \eta^2(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\| \sum_{i=1}^\tau \nabla F_m (\theta^i_m(t), \xi^i_m(t)) \right\|_2^2 \right]
\]
\[
\leq \eta^2(t) \sum_{m=1}^M \frac{B_m}{B} \sum_{i=1}^\tau E \left[ \| \nabla F_m (\theta^i_m(t), \xi^i_m(t)) \|_2^2 \right] \leq \eta^2(t) \tau^2 G^2,
\]
(23)

where (a) follows from Assumption 3.

We rewrite the third term on the RHS of (22) as follows:
\[
2E \left[ (\hat{\theta}(t) - \theta^*, \sum_{m=1}^M \frac{B_m}{B} \Delta \theta_m(t)) \right]
\]
\[
= 2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\langle \theta^* - \hat{\theta}(t), \sum_{i=1}^\tau \nabla F_m (\theta^i_m(t), \xi^i_m(t)) \right\rangle \right]
\]
\[
= 2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\langle \theta^* - \hat{\theta}(t), \nabla F_m (\hat{\theta}(t), \xi^1_m(t)) \right\rangle \right]
\]
\[
+ 2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\langle \theta^* - \hat{\theta}(t), \nabla F_m (\theta^i_m(t), \xi^i_m(t)) \right\rangle \right].
\]
(24)

We have
\[
2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\langle \theta^* - \hat{\theta}(t), \nabla F_m (\hat{\theta}(t), \xi^1_m(t)) \right\rangle \right]
\]
\[
\stackrel{(a)}{=} 2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\langle \theta^* - \hat{\theta}(t), \nabla F_m (\hat{\theta}(t)) \right\rangle \right]
\]
\[
\leq 2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\| F_m (\theta^*) - F_m (\hat{\theta}(t)) \right\| - \frac{\mu}{2} \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \right]
\]
\[
= 2\eta(t) \left( F^* - E \left[ F (\hat{\theta}(t)) \right] \right) - \frac{\mu}{2} \left( \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \right)
\]
\[
\leq -\mu \eta(t) E \left[ \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \right],
\]
(25)

where (a) follows since \( E_\xi \left[ \nabla F_m (\theta^i_m(t), \xi^i_m(t)) \right] = \nabla F_m (\theta^i_m(t)), \forall i, m \), (b) follows from Assumption 2 and (c) follows since \( F^* \leq F (\hat{\theta}(t)) \), \forall t.

Lemma 2. For \( 0 < \eta(t) \leq 1 \), we have
\[
2\eta(t) \sum_{m=1}^M \frac{B_m}{B} E \left[ \left\langle \theta^* - \hat{\theta}(t), \sum_{i=2}^\tau \nabla F_m (\theta^i_m(t), \xi^i_m(t)) \right\rangle \right]
\]
\[
\leq -\mu \eta(t) (1 - \eta(t)) (\tau - 1) E \left[ \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \right] + \eta^2(t) (\tau - 1) G^2
\]
\[
+ (1 + \mu (1 - \eta(t))) \eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} + 2\eta(t)(\tau - 1) \Gamma.
\]
(26)

Proof. See Appendix \[\]
By substituting (25) and (26) in (24), it follows that

\[
2 \mathbb{E} \left[ \| \hat{\theta}(t) - \theta^* \|^2 \right. \sum_{m=1}^{M} \frac{B_m}{B} \Delta \theta_m(t) \left. \right] \\
\leq -\mu \eta(t) (\tau - \eta(t)(\tau - 1)) \mathbb{E} \left[ \| \hat{\theta}(t) - \theta^* \|^2 \right] + \eta^2(t) (\tau - 1) G^2 \\
+ (1 + \mu(1 - \eta(t))) \eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} + 2\eta(t)(\tau - 1) \Gamma, 
\]

(27)

which, together with the inequality in (23), leads to the following upper bound on \( \mathbb{E} \left[ \| \theta(t+1) - \theta^* \|^2 \right] \), when substituted into (22):

\[
\mathbb{E} \left[ \| \theta(t+1) - \theta^* \|^2 \right] \leq (1 - \mu \eta(t) (\tau - \eta(t)(\tau - 1))) \mathbb{E} \left[ \| \hat{\theta}(t) - \theta^* \|^2 \right] + \eta^2(t) (\tau^2 + \tau - 1) G^2 \\
+ (1 + \mu(1 - \eta(t))) \eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} + 2\eta(t)(\tau - 1) \Gamma. 
\]

(28)

**Lemma 3.** For \( \hat{\theta}(t) \) given in (5), we have

\[
\mathbb{E} \left[ \| \hat{\theta}(t) - \theta^* \|^2 \right] \leq \mathbb{E} \left[ \| \theta(t) - \theta^* \|^2 \right] + \left( \frac{\eta(t - 1) \tau G}{2q_1(t)} \right)^2 \varepsilon d. 
\]

(29)

for some \( 0 \leq \varepsilon \leq 1 \).

**Proof.** See Appendix C. \( \square \)

According to Lemma 3, the inequality in (29) can be rewritten as follows:

\[
\mathbb{E} \left[ \| \theta(t+1) - \theta^* \|^2 \right] \leq (1 - \mu \eta(t) (\tau - \eta(t)(\tau - 1))) \mathbb{E} \left[ \| \theta(t) - \theta^* \|^2 \right] \\
+ (1 - \mu \eta(t) (\tau - \eta(t)(\tau - 1))) \left( \frac{\eta(t - 1) \tau G}{2q_1(t)} \right)^2 \varepsilon d + \eta^2(t) (\tau^2 + \tau - 1) G^2 \\
+ (1 + \mu(1 - \eta(t))) \eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} + 2\eta(t)(\tau - 1) \Gamma. 
\]

(30)

Theorem 1 follows from the inequality in (30) having \( 0 < \eta(t) \leq \min \left\{ 1, \frac{1}{\mu \tau} \right\} \), \( \forall t \).

**B Proof of Lemma 2**

We have

\[
2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \langle \theta^* - \hat{\theta}(t), \nabla F_m (\theta^*_m(t), \xi^*_m(t)) \rangle \right] \\
= 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \langle \theta^*_m(t) - \hat{\theta}(t), \nabla F_m (\theta^*_m(t), \xi^*_m(t)) \rangle \right] \\
+ 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \langle \theta^* - \theta^*_m(t), \nabla F_m (\theta^*_m(t), \xi^*_m(t)) \rangle \right]. 
\]

(31)

We first bound the first term on the RHS of (31). We have

\[
2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \langle \theta^*_m(t) - \hat{\theta}(t), \nabla F_m (\theta^*_m(t), \xi^*_m(t)) \rangle \right] \\
\leq \eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \frac{1}{\eta(t)} \| \theta^*_m(t) - \hat{\theta}(t) \|^2 + \eta(t) \| \nabla F_m (\theta^*_m(t), \xi^*_m(t)) \|^2 \right] \\
\leq \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \| \theta^*_m(t) - \hat{\theta}(t) \|^2 \right] + \eta^2(t) (\tau - 1) G^2, 
\]

(32)
where (a) follows from Assumption 3. We have
\[ \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \| \theta_m^i(t) - \hat{\theta}(t) \|^2 \right] = \eta^2(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left\| \sum_{j=1}^{i} \nabla F_m \left( \theta_m^j(t), \xi_m^j(t) \right) \right\|^2 \right] \leq \eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} \tag{33} \]

where (b) follows from the convexity of \( \| \cdot \|_2^2 \) and Assumption 3. Plugging (33) into (32) yields
\[ 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left( \theta_m^i(t) - \hat{\theta}(t), \nabla F_m \left( \theta_m^i(t), \xi_m^i(t) \right) \right) \right] \leq \eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} + \eta^2(t) (\tau - 1) G^2 \tag{34} \]

For the second term on the RHS of (31), we have
\[ 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left( \theta^* - \theta_m^i(t), \nabla F_m \left( \theta^i_m(t), \xi_m^i(t) \right) \right) \right] \]
\[ = 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left( \theta^* - \theta_m^i(t), \nabla F_m \left( \theta_m^i(t) \right) \right) \right] \]
\[ \leq 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ F_m(\theta^*) - F_m(\theta_m^i(t)) - \frac{\mu}{2} \left\| \theta_m^i(t) - \theta^* \right\|_2^2 \right] \]
\[ = 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ F_m(\theta^*) - F_m^* + F_m - F_m(\theta_m^i(t)) - \frac{\mu}{2} \left\| \theta_m^i(t) - \theta^* \right\|_2^2 \right] \]
\[ = 2\eta(t)(\tau - 1) \Gamma + 2\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \left( F_m^* - \mathbb{E} \left[ F_m(\theta_m^i(t)) \right] \right) \]
\[ \quad - \mu\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left\| \theta_m^i(t) - \theta^* \right\|_2^2 \right] \]
\[ \leq 2\eta(t)(\tau - 1) \Gamma - \mu\eta(t) \sum_{m=1}^{M} \frac{B_m}{B} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left\| \theta_m^i(t) - \theta^* \right\|_2^2 \right], \tag{35} \]

where (a) follows since \( \mathbb{E} \left[ \nabla F_m \left( \theta(t), \xi_m(t) \right) \right] = \nabla F_m \left( \theta(t) \right), \forall \theta, \xi_m, t \); (b) follows from Assumption 2 and (c) follows since \( F_m^* \leq F_m(\theta_m^i(t)), \forall \theta, \xi_m, t \). We have
\[ - \left\| \theta_m^i(t) - \theta^* \right\|_2^2 = - \left\| \theta_m^i(t) - \hat{\theta}(t) \right\|_2^2 - \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 - 2(\theta_m^i(t) - \hat{\theta}(t), \hat{\theta}(t) - \theta^*) \]
\[ \leq - \left\| \theta_m^i(t) - \hat{\theta}(t) \right\|_2^2 - \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 + \frac{1}{\eta(t)} \left\| \theta_m^i(t) - \hat{\theta}(t) \right\|_2^2 + \eta(t) \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \]
\[ = -(1 - \eta(t)) \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 + \frac{1}{\eta(t)(\tau - 1)} \left\| \theta_m^i(t) - \hat{\theta}(t) \right\|_2^2, \tag{36} \]

where (a) follows from Cauchy-Schwarz inequality. Plugging (36) into (33) yields
\[ \frac{2\eta(t)}{M} \sum_{m=1}^{M} \sum_{i=2}^{\tau} \mathbb{E} \left[ \left( \theta^* - \theta_m^i(t), \nabla F_m \left( \theta_m^i(t), \xi_m^i(t) \right) \right) \right] \]
\[ \leq -\mu\eta(t)(1 - \eta(t))(\tau - 1) \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 + \mu(1 - \eta(t))\eta^2(t) G^2 \frac{\tau(\tau - 1)(2\tau - 1)}{6} + 2\eta(t)(\tau - 1) \Gamma, \tag{37} \]

where we used the inequality in (33) and \( \eta(t) \leq 1 \). Plugging (34) and (37) into (31) completes the proof of Lemma 2.

C Proof of Lemma 3

We have
\[ \mathbb{E} \left[ \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \right] = \mathbb{E} \left[ \left\| \hat{\theta}(t) \right\|_2^2 \right] + \mathbb{E} \left[ \left\| \theta^* \right\|_2^2 \right] - 2\mathbb{E} \left[ \left( \hat{\theta}(t), \theta^* \right) \right] \]
\[ \overset{(a)}{=} \mathbb{E} \left[ \left\| \hat{\theta}(t) \right\|_2^2 \right] + \mathbb{E} \left[ \left\| \theta^* \right\|_2^2 \right] - 2\mathbb{E} \left[ \left( \hat{\theta}(t), \theta^* \right) \right], \tag{38} \]
where (a) follows since
\[
E \left[ \hat{\theta}(t) \right] = E \left[ \hat{\theta}(t-1) \right] + E \left[ Q(\theta(t) - \hat{\theta}(t-1), q_1(t)) \right] = E[\theta(t)],
\]  
where the last equality follows from (7a). In the following, we upper bound \(E \left[ \left\| \hat{\theta}(t) \right\|_2^2 \right].\) We have
\[
E \left[ \left\| \hat{\theta}(t) \right\|_2^2 \right] = E \left[ \left\| \hat{\theta}(t-1) \right\|_2^2 \right] + E \left[ \left\| Q(\theta(t) - \hat{\theta}(t-1), q_1(t)) \right\|_2^2 \right]
\]
\[
+ 2E \left[ \left\| (\hat{\theta}(t-1), Q(\theta(t) - \hat{\theta}(t-1), q_1(t))) \right\|_2 \right]
\]
\[
\leq E \left[ \left\| \hat{\theta}(t-1) \right\|_2^2 \right] + E \left[ \left\| \theta(t) - \hat{\theta}(t-1) \right\|_2^2 \right] + \frac{\epsilon(t) d}{4q_1^2(t)} E \left[ \left\| \theta(t) - \hat{\theta}(t-1) \right\|_2^2 \right],
\]  (40)
where (a) follows from (7) for some \(0 \leq \epsilon(t) \leq 1\) defined as
\[
\epsilon(t) \triangleq \frac{\left( \max \left\{ \left\| \theta(t) - \hat{\theta}(t-1) \right\| \right\} - \min \left\{ \left\| \theta(t) - \hat{\theta}(t-1) \right\| \right\} \right)^2}{E \left[ \left\| \theta(t) - \hat{\theta}(t-1) \right\|_2^2 \right]},
\]  (41)
noting that
\[
\theta(t) - \hat{\theta}(t-1) = -\eta(t-1) \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} \nabla F_m \left( \theta^i_m(t-1), \xi^i_m(t-1) \right),
\]  (42)
and in (b) we define \(\epsilon \triangleq \max_t \{\epsilon(t)\}.\) According to (42), from the convexity of \(\|\cdot\|_2^2\), it follows that
\[
E \left[ \left\| \theta(t) - \hat{\theta}(t-1) \right\|_2^2 \right] \leq \eta^2(t-1) \sum_{m=1}^{M} \sum_{i=1}^{\tau} \frac{B_m}{B} E \left[ \left\| \nabla F_m \left( \theta^i_m(t-1), \xi^i_m(t-1) \right) \right\|_2^2 \right]
\]
\[
\leq \eta^2(t-1) \tau^2 G^2,
\]  (43)
where (a) follows from Assumption 3. Accordingly, (40) reduces to
\[
E \left[ \left\| \hat{\theta}(t) \right\|_2^2 \right] \leq E \left[ \left\| \theta(t) \right\|_2^2 \right] + \left( \frac{\eta(t-1) \tau G}{2q_1(t)} \right)^2 \epsilon d.
\]  (44)
Substituting the above inequality into (38) yields
\[
E \left[ \left\| \hat{\theta}(t) - \theta^* \right\|_2^2 \right] \leq E \left[ \left\| \theta(t) \right\|_2^2 \right] + E \left[ \left\| \theta^* \right\|_2^2 \right] - 2E \left[ \left\langle \theta(t), \theta^* \right\rangle \right] + \left( \frac{\eta(t-1) \tau G}{2q_1(t)} \right)^2 \epsilon d
\]
\[
= E \left[ \left\| \theta(t) - \theta^* \right\|_2^2 \right] + \left( \frac{\eta(t-1) \tau G}{2q_1(t)} \right)^2 \epsilon d.
\]  (45)