NOISE AND STABILITY IN REACTION-DIFFUSION EQUATIONS

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ABSTRACT. We study the stability of reaction-diffusion equations in presence of noise. The relationship of stability of solutions between the stochastic ordinary different equations and the corresponding stochastic reaction-diffusion equation is firstly established. Then, by using the Lyapunov method, sufficient conditions for mean square and stochastic stability are given. The results show that the multiplicative noise can make the solution stable, but the additive noise will be not.

1. Introduction. The stability of solutions is an important issue in the theory of partial differential equations (PDEs), which has been studied by many authors [27]. There are a lot of sufficient conditions to assure that the solutions are stable or unstable. We note that noise always exists in the real world. The reasons may be that the parameter is obtained by different measurement, and we can not get the real value, so the ordinary (partial) differential equations with noise perturbation is available. In other words, in microscopic world, to describe the particle moving law must be stochastic ordinary (partial) differential equations. In macroscopic world, we often consider the case that the coefficient in equations is random or stochastic. Usually, if we consider the role of noise, we have two cases. First case: the noise is regarded as a small perturbation. In this case, the structure of solutions will not be changed and the biggest possible change is long-time behavior of the solutions, that is to say, the conditions of stability or un-stability may be different from the deterministic case. Second case: the noise is strong, such as $u^\gamma dW_t$, where $\gamma > 1$,
is a unknown function and $W_t$ is the noise. In this case, the structure of solutions will be changed, and the noise may induce the solutions blowup in a finite time.

When the noise appears in a deterministic PDE, the impact of noise on solutions will be the first thing to be considered. For example, noise can make solutions unique [11], can induce singularity [6, 16, 19], can prevent singularities [10], and can impact the regularity of solutions [18]. And in the present paper, we aim to study the impact of noise on stability of solutions, see the important work [1].

Another important topic about stochastic evolution equations is stochastic control problem, see [2, 3, 4]. Recently, Flandoli-Luo [12] considered the 3D stochastic Euler equations and obtained the impact of noise on Euler equations.

There has been a lot of works in this direction, but the main issue is that the noise can stabilize the solution of ordinary differential equations, see the book [1, 14, 21]. Meanwhile, the stochastic stability of functional differential equations is also considered by Mackey-Nechaeva [20]. In the book [8], the long time behavior of solutions was considered in Chapter 11 and sufficient condition of mean square stable is given, see Theorem 11.14. More precisely, Da Prato-Zabczyk studied the following equation

$$\begin{cases}
  dX = AX dt + B(X) dW_t, \\
  X(0) = x \in H,
\end{cases} \tag{1.1}$$

where $H$ is a Hilbert space, $A$ generates a $C_0$ semigroup and $B \in L(H;L^0_0)$ (see p309 of [8] for more details). They proved that the following statements

(i) There exists $M > 0, \gamma > 0$ such that
$$E|X(t,x)|^2 \leq Me^{-\gamma t}|x|^2, \quad t \geq 0.$$  

(ii) For any $x \in H$ we have
$$E \int_0^\infty |X(t,x)|^2 dt < \infty,$$

are equivalent. It is remarked that the nonlinear term is not considered in the book [8]. Moreover, the example in [8] is the stochastic reaction-diffusion equation on a bounded domain, also see [7]. Liu-Mao [15] also considered the stability of trivial solution 0 on a bounded domain. Stability of a pure random delay system and regime-switching diffusion systems are considered in [22] and [26], respectively. In [25], Wang et al. established the stability in distribution of stochastic functional differential equations. In [5], Chow also gave an abstract result to study the stability of null solution (see page 233), the concrete form was not given. Different from these mentioned research works, in the present paper, we consider the concrete model and generalized the classical results. What’s more, we will establish some difference between stochastic partial differential equations (SPDEs) and stochastic differential equations (SDEs).

We discuss the impact of different kinds of noise on stability. Some interesting results are obtained: the additive noise will have a “bad” effect and some multiplicative noise has a “good” effect. In other words, the multiplicative noise can make the solution stable, but the additive noise will be not, which is new for SPDEs. This is different from SDEs, see Remark 4.1. Moreover, we obtain a new result for the stability theory of SDEs, see Theorem 4.5.

This paper is arranged as follows. In Sections 2, we will present some preliminaries. Sections 3 and 4 are concerned with the $L^\infty$ and $L^2$ solutions, respectively. In the present paper, we always assume $W_t$ is a one-dimensional standard Wiener
process and $W_t(x)$ is a Wiener random field which are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The Wiener random field can be chosen to have the following properties: $\mathbb{E}W_t(x) = 0$ and its covariance function $q(x, y)$ is given by

$$
\mathbb{E}W_t(x)W_s(y) = (t \wedge s)q(x, y), \quad x, y \in \mathbb{R}^d,
$$

where $t \wedge s = \min\{t, s\}$ for $t, s \geq 0$. Moreover, we use $W(x, t)$ to denote the space-time white noise.

2. Preliminaries. Firstly, we recall the definitions of stochastic stability. We only consider the stability of constant equilibrium of stochastic reaction-diffusion equations. Consider the following stochastic reaction-diffusion equation

$$
\begin{cases}
\frac{du}{dt} = (\Delta u + f(u))dt + \sigma(u)dW_t, & t > 0, \quad x \in \mathbb{R}^d, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^d.
\end{cases}
$$

(2.1)

We call $u$ a solution to (2.1) if there exists some $p \in [1, \infty]$ such that for every $T > 0$, $u \in C([0, T]; L^p(\mathbb{R}^d; L^2(\Omega)))$ and for every $t \in [0, T]$,

$$
\begin{align*}
&u(x, t) = \int_0^t \int_{\mathbb{R}^d} K(x - y, t - s)f(u(y, s))dyds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} K(x - y, t - s)\sigma(u(y, s))dydW_s \\
&\quad + \int_{\mathbb{R}^d} K(x - y, t)u_0(y)dy
\end{align*}
$$

(2.2)

holds almost surely, where $K$ is the heat kernel associated with the Laplacian operator and $u_0 \in L^p(\mathbb{R}^d)$ ($p \in [1, \infty]$). On the other hand, if $\mathbb{R}^d$ is replaced by a bounded domain $D$ and we call $u$ a mild solution to (2.1) with boundary condition if the above equality holds with the kernel $K$ replaced by the Green function $G$, see Page 79 of [5].

Without loss of generality, we suppose that $f(0) = \sigma(0) = 0$, that is to say, 0 is a trivial solution to the first equation of problem (2.1). Due to that we will use different norms, we only denote $\| \cdot \|$ as some norm with respect to the spatial variable.

**Definition 2.1.** The trivial solution 0 is called mean square stable if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any initial data $u_0$,

$$
\|u_0\| \leq \delta \quad \text{implies} \quad \mathbb{E}\|u(\cdot, t)\|^2 < \varepsilon, \quad \forall \ t \geq 0,
$$

and exponentially mean square stable, if there exists two positive constants $c_1$ and $c_2$ such that

$$
\mathbb{E}\|u(\cdot, t)\|^2 < c_1 e^{-c_2 t}\|u_0\|^2, \quad \forall \ t \geq 0.
$$

**Definition 2.2.** The trivial solution 0 is called stochastically stable if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exists $\delta = \delta(\varepsilon_1, \varepsilon_2) > 0$ such that for $t > 0$ the solution $u$ satisfies

$$
\mathbb{P}\left\{\sup_{t>0} \|u(\cdot, t)\| \leq \varepsilon_1 \right\} \geq 1 - \varepsilon_2 \quad \text{for} \quad \|u_0\| \leq \delta.
$$

Before ending this section, we introduce a notion for SDEs. We will establish the relationship of the stability of solutions between SDEs and stochastic reaction-diffusion equations.
Assume that \( u = 0 \) is a trivial solution to the first equation of problem (2.1), then \( u = 0 \) will be a trivial solution to the following equation
\[
du = f(u)dt + \sigma(u)dW_t.
\]
(2.3)

Problem (2.3) means equation (2.3) with the initial data \( u_0 \) (independent of \( x \)). Initially, we introduce a definition for (2.3).

**Definition 2.3.** The trivial solution 0 is called mean square stable for problem (2.3) if for any \( \varepsilon > 0 \), there exists \( \delta = \varepsilon(\varepsilon) > 0 \) such that for any initial data \( u_0 \),
\[
|u_0| \leq \delta \quad \text{implies} \quad \mathbb{E}|u(\cdot, t)|^2 < \varepsilon, \quad \forall t \geq 0,
\]
and exponentially mean square stable, if there exists two positive constants \( c_1 \) and \( c_2 \) such that
\[
\mathbb{E}|u(\cdot, t)|^2 < c_1 e^{-c_2 t}|u_0|^2, \quad \forall t \geq 0,
\]
and stochastically stable if for any \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \), there exists \( \delta = \delta(\varepsilon_1, \varepsilon_2) > 0 \) such that for \( t > 0 \) the solution \( u \) satisfies
\[
\mathbb{P}\left\{ \sup_{t > 0}|u(\cdot, t)| \leq \varepsilon_1 \right\} \geq 1 - \varepsilon_2 \quad \text{for} \quad |u_0| \leq \delta.
\]

3. \( L^\infty \)-theory. In this section we plan to establish the relationship between problems (2.1) and (2.3), we need the following lemma. Let \( \eta(r) = r^- \) denote the negative part of \( r \) for \( r \in \mathbb{R} \). Set
\[
k(r) = \eta^2(r),
\]
so that \( k(r) = 0 \) for \( r \geq 0 \) and \( k(r) = r^2 \) for \( r < 0 \). For \( \varepsilon > 0 \), let \( k_\varepsilon(r) \) be a \( C^2 \)-regularization of \( k(r) \) defined by
\[
k_\varepsilon(r) = \begin{cases} r^2 - \frac{\varepsilon^2}{6}, & r < -\varepsilon, \\
-r^3 \left( \frac{r}{2\varepsilon} + \frac{4}{3} \right), & -\varepsilon \leq r < 0, \\
0, & r \geq 0.
\end{cases}
\]
Then one can check that \( k_\varepsilon(r) \) has the following properties.

**Lemma 3.1.** [16, Lemma 3.1] The first two derivatives \( k_\varepsilon' \), \( k_\varepsilon'' \) of \( k_\varepsilon \) are continuous and satisfy the conditions: \( k_\varepsilon'(r) = 0 \) for \( r \geq 0 \); \( k_\varepsilon' \leq 0 \) and \( k_\varepsilon'' \geq 0 \) for any \( r \in \mathbb{R} \). Moreover, as \( \varepsilon \to 0 \), we have
\[
k_\varepsilon(r) \to k(r), \quad k_\varepsilon'(r) \to -2\eta(r) \quad \text{and} \quad k_\varepsilon''(r)r^2 \to 2\eta(r)
\]
and the convergence is uniform for \( r \in \mathbb{R} \).

It is noted that most of existing research results for solutions of (2.1) are in \( L^2(\Omega; \mathcal{C}([0, T]; L^2(\mathbb{R}^d))) \) or \( \mathcal{C}([0, T]; L^2(\mathbb{R}^d \times \mathbb{R}^d)) \) (see [5, 8]). However, in this section, we consider \( L^2(\Omega; L^\infty([0, T] \times \mathbb{R}^d)) \cap \mathcal{C}([0, T]; L^\infty(\mathbb{R}^d; L^2(\Omega))) \) as the space on which solutions are posed in order to compare the stability of solutions to (2.1), which is new in the stability theory. For completeness, we first consider the existence of solutions in \( L^2(\Omega; L^\infty([0, T] \times \mathbb{R}^d)) \cap \mathcal{C}([0, T]; L^\infty(\mathbb{R}^d; L^2(\Omega))) \) to (2.3). Denote \( B_R = B_R(0) \) by the ball with radius \( R \) and centred in 0, and \( 1_{B_R}(x) \) by the indicator function. Moreover, we use \( UC(\mathbb{R}^d) \) to denote the set consisted of uniformly continuous and bounded function on \( \mathbb{R}^d \).
Lemma 3.2. Assume $u_0$ is in $UC(\mathbb{R}^d)$, and satisfies
\[ \|u_0\|_{L^\infty} \to 0, \quad R \to \infty. \] (3.1)

If $f$ and $\sigma$ are global Lipschitz continuous, then there is a unique $u$ which is in $C([0,T];L^\infty(\mathbb{R}^d;L^2(\Omega)))$ solving (2.1). Moreover, $u$ also lies in $L^2(\Omega;L^\infty([0,T] \times \mathbb{R}^d))$.

Proof. We divide the proof into two steps.

Step 1. We assume that $u_0 \in UC(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In this case, it is clear to see that
\[ \int_{\mathbb{R}^d} K(\cdot, y)u_0(y)dy \in C([0,T]; L^2 \cap L^\infty(\mathbb{R}^d)). \] (3.2)

By (3.2) and noting that $f$ and $\sigma$ are Lipschitz continuous, the contractive mapping principle yields that problem (2.1) admits a unique solution $u \in C([0,T]; L^2(\Omega \times \mathbb{R}^d)) \cap C([0,T]; L^\infty(\mathbb{R}^d;L^2(\Omega)))$. Since $u \in C([0,T]; L^\infty(\mathbb{R}^d;L^2(\Omega)))$, then (2.2) has an equivalent form (similar proof can be seen [28, Proposition 3.5]): for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and every $t \in [0,T]$,
\[ \int_{\mathbb{R}^d} u(x,t)\varphi(x)dx = \int_0^t \int_{\mathbb{R}^d} u(x,s)\Delta \varphi(x)dxds + \int_0^t \int_{\mathbb{R}^d} f(u(x,s))\varphi(x)dxds 
+ \int_0^t \int_{\mathbb{R}^d} \sigma(u(x,s))\varphi(x)dxds 
+ \int_{\mathbb{R}^d} u_0(x)\varphi(x)dx. \] (3.3)

From (3.3), by virtue of the energy method, then $u \in L^2(\Omega \times [0,T]; H^1(\mathbb{R}^d))$. Now let us prove that
\[ u \in L^2(\Omega; L^\infty([0,T] \times \mathbb{R}^d)). \] (3.4)

Let $u_-(t)$ be the unique strong solution of (2.3) with initial data $-\|u_0\|_{L^\infty}$. For any fixed $T > 0$, we will prove that
\[ u(x,t) \geq u_-(t), \quad \forall (x,t) \in \mathbb{R}^d \times [0,T], \text{ a.s.} \] (3.5)

In order to prove the inequality (3.5), we let $v = u(x,t) - u_-(t)$, then $v$ satisfies
\[ \left\{ \begin{array}{ll}
         dv = (\Delta v + f(v + u_-) - f(u_-))dt 
+ (\sigma(v + u_-) - \sigma(u_-))dW_t, & t > 0, \quad x \in \mathbb{R}^d, \\
         v(x,0) = u_0(x) + \|u_0\|_{L^\infty} \geq 0, & x \in \mathbb{R}^d.
       \end{array} \right. \] (3.6)

Let $R > 0$ be a given real number and let $\phi_1$ be the eigenvalue function of Laplacian operator on $B_R$ with respect to the first eigenvalue $\lambda_1$, i.e.,
\[ \left\{ \begin{array}{ll}
         -\Delta \phi_1 = \lambda_1 \phi_1, & \text{in } B_R, \\
         \phi_1 = 0, & \text{on } \partial B_R.
       \end{array} \right. \]

Set $\psi = \phi_1 1_{B_R}$, then $\psi \in C^2_c(\mathbb{R}^d)$.

Define
\[ \Phi(v(t)) = (\psi, k_\epsilon(v(t))) = \int_{\mathbb{R}^d} \psi(x)k_\epsilon(v(x,t))dx. \]
By Itô’s formula, we have

\[
\Phi_x(v(t)) = \Phi_x(v(0)) + \int_0^t \int_{\mathbb{R}^d} \psi(x) k'_x(v(x,s)) \Delta v(x,s) dx ds + \int_0^t \int_{\mathbb{R}^d} \psi(x) k'_x(v(x,s)) (f(v(x,s) + u_-(s)) - f(u_-(s))) dx ds + \int_0^t \int_{\mathbb{R}^d} \psi(x) k''_x(v(x,s)) (\sigma(v(x,s) + u_-(s)) - \sigma(u_-(s))) dxdW_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \psi(x) k''_x(v_n(x,s)) (\sigma(v(x,s) + u_-(s)) - \sigma(u_-(s)))^2 dxds.
\]

By using the facts \(k''_x \geq 0\), \(\psi(x) \geq 0\), we have

\[
\int_{\mathbb{R}^d} \psi(x) k''_x(v(x,s)) \Delta v(x,s) dx = - \int_{\mathbb{R}^d} \psi(x) k''_x(v(x,s)) \| \nabla v(x,s) \|^2 dx - \int_{\mathbb{R}^d} k'_x(v(x,s)) \nabla \psi(x) \cdot \nabla v(x,s) dx \leq - \int_{\mathbb{R}^d} \nabla k_x(v(x,s)) \cdot \nabla \psi(x) dx = \int_{\mathbb{R}^d} k_x(v(x,s)) \Delta \psi(x) dx = - \lambda_1 \int_{\mathbb{R}^d} k_x(v(x,s)) \psi(x) dx \leq 0.
\]

Consequently,

\[
\Phi_x(v(t)) \leq \Phi_x(v(0)) + \int_0^t \int_{\mathbb{R}^d} \psi(x) k''_x(v(x,s)) \left( \frac{1}{2} (\sigma(v(x,s) + u_-(s)) - \sigma(u_-(s)))^2 \right) dx ds + \int_0^t \int_{\mathbb{R}^d} \psi(x) k'_x(v(x,s)) (f(v(x,s) + u_-(s)) - f(u_-(s))) dx ds + \int_0^t \int_{\mathbb{R}^d} \psi(x) k'_x(v(x,s)) (\sigma(v(x,s) + u_-(s)) - \sigma(u_-(s))) dxdW_s.
\]

Taking expectation over the above equality, we get

\[
\mathbb{E}[\Phi_x(v(t))] = \mathbb{E}[\Phi_x(v(0))] + \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \psi(x) k''_x(v(x,s)) \left( \frac{1}{2} (\sigma(v(x,s) + u_-(s)) - \sigma(u_-(s)))^2 \right) dx ds \right] + \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \psi(x) k'_x(v_n(x,s)) (f(v(x,s) + u_-(s)) - f(u_-(s))) dx ds \right] \leq \mathbb{E}[\Phi_x(v(0))] + \frac{L_0}{2} \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \psi(x) k''_x(v(x,s)) v(x,s)^2 dx ds + L_f \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \psi(x) |k'_x(v(x,s))| |v(x,s)| dx ds.
\]
Note that \( \lim_{\varepsilon \to 0} E(\Phi_\varepsilon(v(t))) = E(\eta(v(t))^2, \psi) \), by taking the limits termwise as \( \varepsilon \to 0 \) and using Lemma 3.1, we have

\[
E(\eta(v(t))^2, \psi) \leq (L_\sigma + 2L_f) \int_0^t E(\eta(v(s))^2, \psi) ds,
\]

which, by means of Gronwall’s inequality, implies that

\[
E(\eta(v(t))^2, \psi) = 0, \quad \forall t \in [0, T].
\]

Note that for any \( R > 0 \), the above inequality always holds. It follows from \( \psi \geq 0 \) that \( \eta(v(t)) = v^{-}(x, t) = 0 \) a.e. \( x \in \mathbb{R}^d \), which implies that \( v^{-} = 0 \) a.e. for a.e. \( x \in \mathbb{R}^d \), and for any \( t \in [0, T] \), i.e. (3.5) holds.

Similarly, if one lets \( u_+ (t) \) be the unique strong solution of (2.3) with initial data \( \|u_0\|_{L^\infty} \), and sets \( v = u_+ - u(x, t) \), then it can be proved that \( v^{-} = 0 \) a.e. for a.e. \( x \in \mathbb{R}^d \), and for any \( t \in [0, T] \). Therefore, (3.4) is true.

Step 2. \( u_0 \in UC(\mathbb{R}^d) \). For any \( n > 0 \), consider the following Cauchy problem

\[
\begin{aligned}
&dv_n = (\Delta v_n + f(v_n)) dt + \sigma(v_n) dW_t, \quad t > 0, \quad x \in \mathbb{R}^d, \\
v_n(x, 0) = (u_0 1_{B_n}) \ast g_n(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

where \( g_n(x) = n^d g(nx) \) and

\[
0 \leq g \in C^\infty(\mathbb{R}^d), \quad \text{support}(g) \subset B_1, \quad \int_{\mathbb{R}^d} g(x) dx = 1.
\]

Then it follows from Step 1 that there is a unique \( v_n \in C([0, T]; L^\infty(\mathbb{R}^d; L^2(\Omega))) \cap L^2(\Omega; L^\infty([0, T] \times \mathbb{R}^d)) \) solving the Cauchy problem (3.7). Moreover, from the calculations in Step 1, we have

\[
u_n \in L^2(\Omega; L^\infty([0, T] \times \mathbb{R}^d)) \leq C(\|u_0\|_{L^\infty(\mathbb{R}^d)}, T).
\]

Let \( 0 < n < m \in \mathbb{N} \). Set \( w_{m,n} = v_m - v_n \), then \( w_{m,n} \) yields that

\[
w_{m,n}(x, t) = \int_0^t \int_{\mathbb{R}^d} K(x - y, t - s)[f(v_m(y, s)) - f(v_n(y, s))] dy ds
+ \int_0^t \int_{\mathbb{R}^d} K(x - y, t - s)[\sigma(v_m(y, s)) - \sigma(v_n(y, s))] dy dW_s
+ \int_{\mathbb{R}^d} K(x - y, t)[(u_0 1_{B_m}) \ast g_m(y) - (u_0 1_{B_n}) \ast g_n(y)] dy.
\]

By taking the second moment for \( w_{m,n} \) and noting that \( f \) and \( \sigma \) are Lipschitz continuous, from (3.10), we conclude that

\[
E|w_{m,n}(x, t)|^2 \leq 3(L_f + L_\sigma) \int_0^t \int_{\mathbb{R}^d} K(x - y, t - s) E|w_{m,n}(y, s)|^2 dy ds
+ \int_{\mathbb{R}^d} K(x - y, t)[(u_0 1_{B_m}) \ast g_m(y) - (u_0 1_{B_n}) \ast g_n(y)]^2 dy.
\]
By Gronwall’s inequality, the solution of (2.1) which also belongs to $C([0, T]; R^d)$ repeating the above calculations, we gain also lies in $C([0, T]; R^d)$ by Theorem 3.1.

The stochastically stable trivial solution of (2.3), then $0$ is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.1) (with the $C$ norm in spatial variable), and $u_0$ also lies in $L^2(\Omega, \mathbb{R}^d)$. Besides, by (3.9), $u$ also lies in $L^2(\Omega, L^\infty([0, T]; \mathbb{R}^d))$.

On the other, if $u_1$ and $u_2$ are solutions of (2.1) with the same initial data, then repeating the above calculations, we gain

$$
\mathbb{E}[(u_1 - u_2)(x, t)]^2 \leq C \int_0^t \|u_1 - u_2\|_{L^\infty([0, T]; \mathbb{R}^d)}(s)ds.
$$

By Gronwall’s inequality, $u_1 = u_2$, so we prove the uniqueness. \hfill \Box

Before giving the main result in this section, we introduce a subset of $UC(\mathbb{R}^d)$, we denote it by $UC(\mathbb{R}^d)$ (UC for short):

$$
UC(\mathbb{R}^d) = \{ \xi \in UC(\mathbb{R}^d) \mid \xi \text{ satisfies (3.1)} \}.
$$

We call the trivial solution of (2.1) mean square stable, exponentially mean square stable or stochastically stable in class of $UC$ if we replace $u_0 \in L^\infty(\mathbb{R}^d)$ in Definitions 2.1 and 2.2 by $u_0 \in UC$.

Theorem 3.1. Assume $f$ and $\sigma$ satisfy the global Lipschitz condition, and $f(0) = \sigma(0) = 0$.

(i) If $0$ is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.1) (with the $L^\infty$ norm in spatial variable), then $0$ is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.3).

(ii) If $0$ is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.3), then $0$ is a stable (mean square stable,
exponentially mean square stable or stochastically stable) trivial solution in class of \( \mathcal{U} \mathcal{C} \) for (2.1).

**Proof.** (i) Clearly, if \( 0 \) is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.1), then \( 0 \) is also a stable trivial solution of (2.3) since we can choose the initial data \( u_0 \) which is spatial variable independent.

(ii) We will check if \( 0 \) is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.3), then \( 0 \) is a stable trivial solution in class of \( \mathcal{U} \mathcal{C} \) for (2.1).

Assume \( 0 \) is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (2.3), then for any \( \varepsilon > 0 \), there exists constant \( \delta = \delta(\varepsilon) > 0 \) such that

\[
E|u_\pm(t)|^2 < \varepsilon,
\]

where \( u_\pm(t) \) is the solution of (2.3) with initial data \( \pm \delta \).

Consider the Cauchy problem (2.1) with the initial \( u_0 \) which is of class \( \mathcal{UC} \) and bounded by \( \delta \). By Lemma 3.2, there is a unique solution \( u \in C([0, T]; L^\infty(\mathbb{R}^d)) \cap L^2(\Omega; L^\infty([0, T] \times \mathbb{R}^d)) \) solving (2.1). Moreover, for any fixed \( T > 0 \), (3.8) is true. So we complete the proof.

**Remark 3.1.** It is remarked that in Theorem 3.1 the norm of solution of problem (2.1) is maximum in \( \mathbb{R}^d \). The advantage of this norm is that the estimates we obtained hold point-wise. Under this norm, Theorem 3.1 also holds if the equation (2.1) is replaced by the equation (3.13). But it is also remarked that the maximum is not unique norm to study the relationship, and we will give another norm to consider the problem (3.13).

Now, we consider a special case on a bounded domain \( D \subset \mathbb{R}^d \) \((d \geq 1)\)

\[
\begin{align*}
\begin{cases}
    du = (\Delta u + f(u))dt + udW_t, & t > 0, \quad x \in D, \\
    u(x, 0) = u_0(x), & x \in D, \\
    u(x, t) = 0, & t > 0, \quad x \in \partial D.
\end{cases}
\end{align*}
\]

(3.13)

We will establish the relationship between (3.13) and the following SDE

\[
\begin{align*}
\begin{cases}
    dX_t = f(X_t)dt + X_t dW_t, & t > 0, \\
    X(0) = u_0.
\end{cases}
\end{align*}
\]

(3.14)

In order to do that, we consider the eigenvalue problem for the elliptic equation

\[
\begin{align*}
\begin{cases}
    -\Delta \phi = \lambda \phi, & \text{in } D, \\
    \phi = 0, & \text{on } \partial D.
\end{cases}
\end{align*}
\]

Then, all the eigenvalues are strictly positive, increasing and the eigenfunction \( \phi_1 \) corresponding to the smallest eigenvalue \( \lambda_1 \) does not change sign in domain \( D \), as shown in [13]. Therefore, we normalize it in such a way that

\[
\phi_1(x) \geq 0, \quad \int_D \phi_1(x) dx = 1.
\]

**Theorem 3.2.** Assume that

\[
(f(u), \phi_1) \leq f((u, \phi_1)).
\]

If \( 0 \) is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (3.14), then \( 0 \) is also a stable (mean square stable,
exponentially mean square sable or stochastically stable) trivial solution of (3.13), where we take the spatial norm as 
\[ \| u \|_{\phi_1} = (u, \phi_1). \]

**Proof.** It follows from [6, Theorem 2.1] and [16] that the solutions of (3.13) keep non-negative almost surely, i.e., \( u(x, t) \geq 0 \), a.s. for almost every \( x \in D \) and for all \( t \in [0, T] \). Moreover, the solutions exist globally. Let
\[ v(t) = (u, \phi_1) = \int_D u(x, t)\phi_1(x)dx. \]

Then \( v \) satisfies
\[ \left\{ \begin{array}{l}
  dv(t) \leq [-\lambda_1 v(t) + f(v(t))]dt + v(t)dW_t, \quad t > 0, \quad x \in D,
  v(0) = v_0 = (u(\cdot, 0), \phi_1).
\end{array} \right. \quad (3.15) \]

It is easy to prove that \( v(t) \) is a sub-solution of the following problem
\[ \left\{ \begin{array}{l}
  dY_t = [-\lambda_1 Y_t + f(Y_t)]dt + Y_t dW_t, \quad t > 0,
  Y(0) = v_0.
\end{array} \right. \quad (3.16) \]

Since the solutions of (3.16) keep non-negative, we obtain that \( v(t) \) is also a sub-solution of (3.15). Set \( Z_t = Y_t - v(t) \), similar to the proof of Theorem 3.1, we can prove that \( Z_t \geq 0 \) almost surely. Indeed, one can first prove that \( Y_t \geq 0 \) almost surely by using the same method to the proof of Theorem 3.1. Then it follows from the definition of \( v \) that \( v(t) = (u, \phi_1) > 0 \) almost surely. Lastly, noting that
\[ -(Y_t^r - v(t)^r) = -r \xi^{-1} Z_t \geq 0, \quad \text{when } Z_t \leq 0, \]
one can use the same method to the proof of Theorem 3.1 to get \( Z_t \geq 0 \) almost surely. Similarly, one can prove \( Y_t \leq X_t \) almost surely. Therefore, if \( 0 \) is a stable (mean square stable, exponentially mean square stable or stochastically stable) trivial solution of (3.14), then \( 0 \) is also a stable trivial solution of (3.13). The proof is complete.

**Example** Let \( f(u) = au - ku^r \), where \( a \in \mathbb{R}, k \geq 0 \). \( r \) is a real number greater than or equal to 1 such that \( -u^r \geq 0 \) for \( u \leq 0 \). It follows from [6, Theorem 2.1] and [16] that the solutions of (3.13) with \( f(u) = au - ku^r \) keep non-negative almost surely. By using the Hölder inequality, we have
\[ (u, \phi_1)^r = \left( \int_D u(x, t)\phi_1(x)dx \right)^r \leq \int_D u^r(x, t)\phi_1(x)dx = (u^r, \phi_1). \]

Therefore, all the assumptions of Proposition 3.2 are satisfied.

**Remark 3.2.** In particular, if \( f(u) = u \), then the stability of the trivial solution \( 0 \) for (3.13) and the following problem
\[ \left\{ \begin{array}{l}
  dY_t = (1 - \lambda_1) Y_t dt + Y_t dW_t, \quad t > 0, \quad x \in D,
  Y(0) = (u_0, \phi_1),
\end{array} \right. \]

are equivalent. The above problem is different from (3.14) because of the Laplacian operator.
4. $L^2$-theory. In this section, we consider the stability results for solutions in $L^2$ space. Since the discussions on $\mathbb{R}^d$ is similar to $D \subset \mathbb{R}^d (|D| < \infty)$. We only study the stability results on a bounded domain. We focus on the conditions which induce the solution stable. Meanwhile, we are interested in the difference between the stochastic partial differential equations and partial differential equations. We will consider the impact of different noise. Throughout this section, $\| \cdot \|$ means the norm of $L^2(D)$.

We first consider the following initial boundary problem

$$
\begin{align*}
\begin{cases}
\frac{du}{dt} = \mu \Delta u + \sigma dW_t, & t > 0, \quad x \in D, \\
u(x,0) = u_0(x), & x \in D, \\
\nu(x,t) = 0, & t > 0, \quad x \in \partial D,
\end{cases}
\end{align*}
$$

(4.1)

where $\mu$ and $\sigma$ are positive constants. When $\sigma = 0$, the point 0 is a trivial solution of (4.1), and if $\sigma > 0$, we call 0 as a point. Taking the Lyapunov function as $\|u\|^2$, one can prove that if $\sigma^2|D| < 2\mu \lambda_1 |\mathbb{E}\|u_0\|^2$, then 0 is mean square stable (The proof is classical and we leave it to readers).

It is easy to see that if $\sigma = 0$, then the trivial solution 0 is stable; and if $\mu = 0$, then the point 0 will be unstable; and when $\sigma^2 > 0$, the point 0 will be a stable point under some more assumptions. Possibly one can say that the additive noise will have a “bad” effect on the stability of trivial solution.

By using the Chebyshev inequality, one can easily prove the point 0 is stochastic stable without any more assumption, see the next theorem for the proof.

We study the impact of additive noise. Consider the following equation

$$
\begin{align*}
\begin{cases}
\frac{du}{dt} = (\Delta_p u + f(u))dt + \sigma dW_t, & t > 0, \quad x \in D, \\
u(x,0) = u_0(x), & x \in D, \\
\nu(x,t) = 0, & t > 0, \quad x \in \partial D,
\end{cases}
\end{align*}
$$

(4.2)

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$. Because we only consider the stability of solutions, throughout this paper we will assume the problem we consider admits a unique global solution. Let $C_\infty$ be the Sobolev embedding constant satisfying

$$
\|u\|_{L^\infty(D)} \leq C_\infty \|u\|_{W^{1,p}(D)}, \quad p > d.
$$

(4.3)

It is noted that the solution $u$ of (4.2) satisfies that $\|u\|_{W^{1,p}(D)} = \|\nabla u\|_{L^p(D)}$.

**Theorem 4.1.** Assume the nonlinear term satisfies

$$
uf(u) \leq au^2 + bu^{2m}, \quad m \geq 1,
$$

(4.4)

where $a, b \in \mathbb{R}$. Assume further that $p > \max\{2m, d\}$. If

$$
a + \frac{2 - \gamma}{2} \left(\frac{\gamma |D| |b| C_\infty}{2}\right)^{\frac{\gamma}{p}} + \frac{\sigma^2 |D|}{2\mathbb{E}\|u_0\|^2} < 0,
$$

(4.5)

where $\gamma \in (0, 2)$ satisfies

$$
(2m - 2 + \gamma) \cdot \frac{2}{\gamma} = p.
$$

Then the point 0 is mean square stable.
Proof. We remark that when \( m = 1 \), Theorem 4.1 will become easier, see [15] for similar results. We pick a Lyapunov function \( V(u) = \|u\|^2 \). By Itô’s formula, taking expectation and integrating with respect to \( t \), we have

\[
\frac{d}{dt} \mathbb{E}\|\cdot(t)\|^2 = 2 \mathbb{E} \int_D u(\Delta_p u(x, t) + f(u)) dx + \sigma^2 |D|
\]

\[
\leq -2 \mathbb{E} \int_D |\nabla u(x, t)|^p dx + 2 \mathbb{E} \int_D (au^2 + bu^{2m}) dx + \sigma^2 |D|. \quad (4.6)
\]

By the Sobolev embedding inequality (4.3), we have

\[
\|u\|_{L^{2m}}^2 = \int_D |u|^{2m}(x, t) dx
\]

\[
\leq \|u\|_{L^\infty}^{2m-2} \int_D u^2(x, t) dx
\]

\[
= \|u\|_{L^\infty}^{2m-2} \|u\|_{L^2}^2
\]

\[
\leq |D|^2 \|u\|_{L^\infty}^{2m-2+\gamma} \|u\|_{L^2}^{2-\gamma}
\]

\[
\leq C_{\infty}^{2m-2+\gamma} |D|^2 \|u\|_{W^{1,p}}^{2m-2+\gamma} \|u\|_{L^2}^{2-\gamma}
\]

\[
\leq \frac{2 - \gamma}{2} (C_{\infty})^{\frac{2(2m-2+\gamma)}{2-\gamma}} \left( \frac{\gamma |D| |b|}{2} \right)^{\frac{\gamma}{2-\gamma}} \|u\|_{L^2}^2 + \frac{\|u\|_{W^{1,p}}^{2m-2+\gamma}}{|b|}. \quad (4.7)
\]

Noting that \( p > \max\{2m, d\} \), there exists a constant \( \gamma \in (0, 2) \) such that

\[
(2m - 2 + \gamma) \cdot \frac{2}{\gamma} = p.
\]

Submitting (4.7) into (4.6), we have

\[
\frac{d}{dt} \mathbb{E}\|\cdot(t)\|^2 \leq 2 \left( a + \hat{C} \right) \mathbb{E}\|u\|_{L^2}^2 + \sigma^2 |D|, \quad (4.8)
\]

where

\[
\hat{C} = \frac{2 - \gamma}{2} \left( \frac{\gamma |D| |b| C_{\infty}}{2} \right)^{\frac{\gamma}{2-\gamma}}.
\]

Solving the differential inequality (4.8) gives

\[
\mathbb{E}\|\cdot(t)\|^2 \leq \left( \mathbb{E}\|u_0\|^2 + \frac{\sigma^2 |D|}{2(a + \hat{C})} \right) e^{2(a + \hat{C})t} - \frac{\sigma^2 |D|}{2(a + \hat{C})}. \quad (4.9)
\]

Note that \( a + \hat{C} < 0 \) implies that \( e^{2(a + \hat{C})t} < 1 \) for all \( t > 0 \). Furthermore, the assumption

\[
\mathbb{E}\|u_0\|^2 + \frac{\sigma^2 |D|}{2(a + \hat{C})} > 0
\]

yields that

\[
\mathbb{E}\|\cdot(t)\|^2 \leq \mathbb{E}\|u_0\|^2,
\]

which completes the proof.

\( \square \)

Example Consider

\[
\begin{cases}
\begin{aligned}
du &= (\Delta_p u - u + \frac{1}{C_{\infty}} u^2) dt + \sigma dW_t, & t > 0, & x \in (0, 1), \\
u(x, 0) &= u_0(x), & x \in (0, 1), \\
u(x, t) &= 0, & t > 0, & x \in \partial(0, 1).
\end{aligned}
\end{cases}
\]
It is easy to check that $\gamma = 1$ satisfies $(2m - 2 + \gamma) \cdot \frac{2}{\gamma} = p$, where $m = 3/2, p = 4$. Then if the initial data satisfies
\[ \frac{\sigma^2}{2E\|u_0\|^2} < \frac{3}{4}, \]
then Theorem 4.1 shows that the point 0 will be mean square stable.

**Remark 4.1.** (1) In Theorem 4.1, we assume the constant $b$ satisfies (4.5). Note that the constant $C_\infty$ depends on the domain $D$, the dimension $d$ and the constant $p$, and thus it is hard to give a concrete constant in an example. The reason is that we used the embedding inequality (4.3). On the other hand, we can use the following embedding inequality replaced (4.3):
\[ W^{1,p}(D) \hookrightarrow W^{2m}(D), \quad p > 2m. \]
Let $C_{2m,2m}$ be the Sobolev embedding constant, i.e., $\|u\|_{L^2m(D)} \leq C_{2m,2m} \|u\|_{W^{1,2m}(D)}$. Then under the assumptions that $b \leq \frac{2}{C_{2m,2m}}$ and
\[ 2aE\|u_0\|^2 + \sigma^2|D| < 0, \]
the point 0 is mean square stable for equation (4.2).

(2) We now explain why we did not get the results of stochastic stability. Like the case of stochastic differential equations, we try to use a Lyapunov function $V(u) = \|u\|^{2r}$ with $0 < r < 1$ to prove the stochastic stability. For additive noise, we can not prove that $\|u(\cdot, t)\|^2 > 0$ for all $t > 0$. In order to use the Itô formula, we consider the following Lyapunov functional $V(u) = (\|u\| + \kappa)^{2r}$ with $0 < r < 1$ and $0 < \kappa \ll 1$. This leads to the expression
\[
\frac{d}{dt}E(\|u\|^2 + \kappa)^r = 2rE \left[ (\|u\|^2 + \kappa)^{r-2} \int_D u(\Delta_p u(x, t) + f(u))dx \right] \\
+ r\sigma^2D|E| \left[ (\|u\|^2 + \kappa)^{r-1} \right] \\
+ 2\sigma^2r(r-1)E(\|u\|^2 + \kappa)^{r-2} \left( \int_D udx \right)^2 \\
\leq E \left[ r(\|u\|^2 + \kappa)^{r-1}(a + \hat{C})\|u(\cdot, t)\|^2 \right. \\
\left. + r\sigma^2(\|u\|^2 + \kappa)^{r-1} \left( |D| + 2(r - 1) \frac{\int_D udx}{\|u\| + \kappa} \right) \right] (4.10) 
\]
Due to the difference $(\int_D udx)^2$ and $(\int_D |u|dx)^2$, we can not get any help to control the term $|D|$. Note that
\[ \int_D udx = 0 \]
maybe happen, so we can not use this term. Even though the term $(\int_D udx)^2$ is replaced by $(\int_D |u|dx)^2$, we can not get the desired result. The reason is the followings. The Hölder inequality implies that
\[ \int_D |u|dx \leq |D|^\frac{1}{2} \left( \int_D |u|^2dx \right)^{\frac{1}{2}}. \]
Consequently,
\[ \frac{\|u\|^2_{L^1}}{\|u\|^2_{L^2}} \leq |D|. \]
Hence we can not use the above inequality in (4.10). The aim of the above discussion is to show the last two terms of right-side hand of (4.10) are in the same level, which are different from the first term for SPDEs.

But for SDEs, there will be another case. In this case, let $|D| = 1$, then we have

$$|D| + 2(r - 1) \left( \frac{\int_D |u|^2 dx}{\|u\|^2 + \kappa} \right)^2 = 1 + \frac{2(r - 1)}{1 + \kappa}.$$ 

Taking $0 < r < 1/2$ such that $2r + \kappa < 1$, we get

$$\frac{d}{dt} \mathbb{E}(\|u\|^2 + \kappa)^r \leq 0.$$ 

Letting $\kappa \to 0$, and using the Chebyshev inequality, we obtain the stochastic stability. In all, we find there is a significant difference between SPDEs and SDEs in the stability theory.

We remark that the noise can be easily generalized the cylindrical Wiener process. We first generalize the classical results of deterministic reaction-diffusion equation [27, Theorem 4.2.1, p 166] to the following equation

$$\begin{cases}
  d\frac{du}{dt}(x,t) = (\Delta u(x,t) + f(x,t,u))dt + udW_t, & t > 0, \quad x \in D, \\
  u(x,0) = u_0(x), & x \in D, \\
  u(x,t) = 0, & t > 0, \quad x \in \partial D.
\end{cases} \quad (4.11)$$

**Theorem 4.2.** Assume that $f(x,t,0) = 0$, $f \in C^1(D \times [0, \infty) \times (-\infty, \infty))$.

(i) If there exists a constant $\alpha > 0$ such that for all $(x,t) \in D \times [0, \infty)$, $\eta \in \mathbb{R}$, we have

$$f(x,t,\eta) \leq (\lambda_1 - \alpha)\eta,$$

then for the initial data $u_0$ satisfying $0 \leq u_0(x) \leq \rho\phi_1(x)$ with $\rho > 0$, problem (4.11) admits a unique positive solution $u(x,t)$ and the following estimate holds almost surely

$$0 \leq u(x,t) \leq \rho e^{-(\lambda_1 + \frac{\alpha^2}{2})t + \sigma W_t}\phi_1(x), \quad (x,t) \in D \times [0, \infty).$$

Consequently,

$$\mathbb{E}u(x,t) \leq e^{\alpha t} \mathbb{E}\phi_1(x). \quad (4.12)$$

Assume further that the initial data $u_0$ is a deterministic function, we have

$$\mathbb{P}\left\{ \int_D u(x,t)dx \leq \rho e^{-(\alpha + \frac{\alpha^2}{2})t} \right\} \geq \frac{1}{2}. \quad (4.13)$$

(ii) If there exists a constant $\alpha > 0$ such that for all $(x,t) \in D \times [0, \infty)$, $\eta \geq 0$, we have

$$f(x,t,\eta) \geq (\lambda_1 + \alpha)\eta,$$

then for every $\delta > 0$, when $u_0(x) \geq \delta \phi_1(x)$, problem (4.11) admits a unique positive solution $u(x,t)$, which exists globally or finite time blowup. On the lifespan, the following estimate holds almost surely

$$u(x,t) \geq \delta e^{(\lambda_1 - \frac{\alpha^2}{2})t + \sigma W_t}\phi_1(x), \quad (x,t) \in D \times [0, \infty).$$

Consequently, $\mathbb{E}\|u(t)\|^2 \geq \delta e^{\alpha t} \mathbb{E}\|u_0\|^2$. 

Proof. We first change the stochastic reaction-diffusion equation into random reaction-diffusion, then by using comparison principle, the desired results are obtained. More precisely, let \( v(x,t) = e^{-\sigma W_t} u(x,t) \), then \( v(x,t) \) satisfies

\[
\begin{cases}
\frac{\partial}{\partial t}v(x,t) = \Delta v(x,t) - \frac{\sigma^2}{2} v(x,t) + e^{-\sigma W_t} f(x,t, e^{\sigma W_t} v), & t > 0, \ x \in D, \\
v(x,0) = u_0(x), & x \in D, \\
v(x,t) = 0, & t > 0, \ x \in \partial D.
\end{cases}
\] (4.14)

By using the assumptions, we get

\[
\frac{\partial}{\partial t}v(x,t) \leq \Delta v(x,t) - \frac{\sigma^2}{2} v(x,t) + (\lambda_1 - \alpha) v(x,t).
\]

It is easy to check that \( \bar{v} = \rho e^{-(\alpha + \frac{\sigma^2}{2})t} \phi_1(x) \) is an upper solution of (4.14) and 0 is a lower solution to (4.14), which implies that for \((x,t) \in D \times [0,\infty)\)

\[
0 \leq v(x,t) \leq \rho e^{-(\alpha + \frac{\sigma^2}{2})t} \phi_1(x) \iff 0 \leq u(x,t) \leq \rho e^{-(\alpha + \frac{\sigma^2}{2})t + \sigma W_t} \phi_1(x), \ a.s.,
\]

which implies that

\[
0 \leq \int_D u(x,t) dx \leq \rho e^{-(\alpha + \frac{\sigma^2}{2})t + \sigma W_t}, \ a.s..
\]

By using \( \mathbb{E}[e^{\sigma W_t}] = e^{\frac{\sigma^2}{2}t} \), we have

\[
\mathbb{E}u(x,t) \leq e^{(\alpha - \lambda_1) t} \phi_1(x).
\]

Note that

\[
\left\{ \int_D u(x,t) dx \leq \rho e^{-(\alpha + \frac{\sigma^2}{2})t} \right\} \iff \{ e^{\sigma W_t} \leq 1 \} \supset \{ W_t \leq 0 \},
\]

thus we have

\[
\mathbb{P}\left\{ \int_D u(x,t) dx \leq \rho e^{-(\alpha + \frac{\sigma^2}{2})t} \right\} \geq \mathbb{P}\{ W_t \leq 0 \} = \frac{1}{2},
\]

which proves (4.13).

Next, we prove (ii). Note that

\[
\frac{\partial}{\partial t}v(x,t) \geq \Delta v(x,t) - \frac{\sigma^2}{2} v(x,t) + (\lambda_1 + \alpha) v(x,t).
\]

It follows from that \( v = \delta e^{(\alpha - \frac{\sigma^2}{2})t} \) is a lower solution of (4.14), thus we have the desired inequality. The proof is complete. \( \square \)

Remark 4.2. Following Theorem 4.2, it is easy to see that in mean square sense, the solution of (4.11) keeps the same properties as the deterministic case, which is different from the additive noise. Of course, the big difference between the stochastic and deterministic cases is that there exists an event such that whose probability is large than 0, where the event is that the solution of stochastic case maybe have exponentially decay. In other words, in (4.13), if \( -\frac{\sigma^2}{2} < \alpha < 0 \), then the solution \( u \) of (4.11) satisfies \( \| u(t) \|^2 \leq \| u_0 \|^2 e^{(\alpha - \lambda_1 - \frac{\sigma^2}{2})t} \) with probability \( \frac{1}{2} \). Maybe from here we can say the noise can stabilize the solutions.

The method we used in Theorem 4.2 is comparison principle, which is different from the Lyapunov functional method. The inequality (4.12) holds pointwise, which is different from the earlier results. What’s more, the index \( \alpha - \lambda_1 \) is different from
that obtained by Lyapunov method, see the next theorem. In part (ii) of Theorem 4.2 implies the unstable condition of the trivial solution 0, which is new in this field.

Comparing with the stochastic ordinary different equations, the role of the Laplacian operator in stochastic reaction-diffusion equations gives a help with $\lambda_1$ in the stability of trivial solutions. Indeed, the reason is the Poincare inequality.

The impact of multiplicative noise in Theorem 4.2 is not satisfied. We give the next result.

**Theorem 4.3.** Assume that $f(x,t,0) = 0$ and there exists a constant $K > 0$ such that for all $(x,t) \in D \times [0,\infty)$, $uf(x,t,u) \leq Ku^2$. If $K - \lambda_1 + \sigma^2/2 \leq 0$, then the trivial solution 0 is mean square stable and if

$$K - \lambda_1 - \sigma^2/2 < 0,$$

(4.15)

then the trivial solution 0 is stochastically stable.

**Proof.** Taking Lyapunov function $V(u) = \|u\|^2$, we have

$$\frac{d}{dt}E\|u(t)\|^2 = -\int_D |
abla u(x,t)|^2dx + \frac{\sigma^2}{2}E\|u(t)\|^2 + \mathbb{E} \int_D uf(x,t,u)dx$$

$$\leq \left(K - \lambda_1 + \frac{\sigma^2}{2}\right)E\|u(t)\|^2,$$

which yields that

$$E\|u(t)\|^2 \leq E\|u_0\|^2 e^{\left(K - \lambda_1 + \frac{\sigma^2}{2}\right)t}.$$

Next, we use a Lyapunov function $V(u) = \|u\|^{2r}$ with $0 < r < 1$ to prove the stochastic stability. Note that in Theorem 4.2, we proved that the solution $u \geq 0$ almost surely and thus we can choose $\|u\|^{2r}$ as Lyapunov functional. This leads to the expression

$$\frac{d}{dt}E\|u(\cdot,t)\|^{2r} = 2rE \left[\|u(\cdot,t)\|^{2r-2} \int_D u(\Delta u(x,t) + f(x,t,u))dx\right]$$

$$+ r\sigma^2E\|u(\cdot,t)\|^{2r} + 2\sigma^2 r(r-1)E\|u(\cdot,t)\|^{2r}$$

$$\leq E \left[2r\|u(\cdot,t)\|^{2r} \left(K - \lambda_1 + \sigma^2 r - \frac{\sigma^2}{2}\right)\right].$$

(4.16)

If $K - \lambda_1 - \frac{\sigma^2}{2} < 0$, we can choose $0 < r < \frac{1}{2}$ such that

$$K - \lambda_1 + \sigma^2 r - \frac{\sigma^2}{2} \leq 0,$$

then from the Chebyshev inequality, stochastic stability for the solution of (4.2) follows from (4.16). The proof is complete.

**Remark 4.3.** It follows from Theorem 4.3 that the multiplicative noise can make solution stable in sense of stochastically stable. Comparing Theorem 4.3 with Theorem 4.2, we can take $K = \lambda_1 + \sigma^2/2 - \varepsilon > \lambda_1 - \alpha$, where $0 < \varepsilon \ll 1$.

In the above theorems, we assume that the noise term satisfies the global Lipschitz condition. In the following theorem, we will see that the assumption can be
weaken as local Lipschitz condition. In order to do this, we consider the following equation
\[
\begin{cases}
    du = (\Delta u - k_1 u^r)dt + k_2 u^m dW_t(x), & t > 0, \ x \in D, \\
    u(x, 0) = u_0(x), & x \in D, \\
    u(x, t) = 0, & t > 0, \ x \in \partial D,
\end{cases}
\] (4.17)
where \(k_1, k_2, r\) and \(m\) are positive constants, the covariance function of \(W_t(x)\) is \(q\). In [16], Lv-Duan proved the existence of global solution of (4.17) under the assumptions of Theorem 4.4. Moreover, we proved the solutions keep non-negative almost surely.

**Theorem 4.4.** Assume that \(r\) is an odd number and \(1 < m < \frac{1+r}{2}\). Assume further that there exists a positive constant \(q_0\) such that the covariance function \(q(x,y)\) satisfies the condition \(\sup_{x,y \in D} q(x,y) \leq q_0\). Then if \(\lambda < \lambda_1\), then the trivial solution \(0\) is exponentially mean square stable with the index \(\lambda - \lambda_1\), where
\[
\lambda_1 := \frac{r - 3}{r - 1} \left( \frac{k_1(r - 1)}{2} \right)^{-\frac{2m-2}{r-m}} (q_0 k_2)^{-\frac{-1}{r-m}}.
\]
In particular, when \(m = 2\) and \(r > 3\), we assume further that there exists a positive constant \(q_1\) such that the covariance function \(q(x,y)\) satisfies the condition \(\sup_{x,y \in D} q(x,y) \geq q_1\). If
\[
\lambda_1 > \frac{r - 3}{r - 1} \left( \frac{k_1(r - 1)}{2} \right)^{-\frac{2m-2}{r-m}} (q_0 k_2)^{-\frac{-1}{r-m}} - 2k_2^2 q_1,
\]
then the trivial solution \(0\) is stochastic stable.

**Proof.** The proof is similar to that of Theorem 4.3 and we give outline of the proof for completeness. Taking Lyapunov function \(V(u) = \|u\|^2\), we have the following inequality
\[
\frac{d}{dt} \mathbb{E}\|u(t)\|^2 \\
= -2 \int_D |\nabla u(t)|^2 dx + k_2^2 \mathbb{E} \int_D u^{2m} q(x,x) dx - 2k_1 \mathbb{E} \int_D u^{r+1} dx \\
\leq -2\lambda_1 \mathbb{E}\|u(t)\|^2 - 2k_1 \mathbb{E}\|u(t)\|^{1+r} + k_2^2 q_0 \mathbb{E}\|u(t)\|^{2m}.
\]
By using the interpolation inequality
\[
\|u\|_{L^r} \leq \|u\|_{L^\theta}^{\frac{r}{\theta}} \|u\|_{L^\infty}^{1-\frac{r}{\theta}},
\]
with \(r = 2m\), \(p = 2\) and \(q = 4\), we have
\[
q_0 \|u\|_{L^{2m}}^2 \leq q_0 \|u\|_{L^2}^{2m} \|u\|_{L^{2m}}^{1-\theta} \\
\leq k_1 \|u\|_{L^{1+r}}^{2m(1-\theta)} + m\theta \left( \frac{k_1}{1-m\theta} \right)^{-\frac{1-m\theta}{m\theta}} (q_0 k_2)^{-\frac{1}{m\theta}} \|u\|_{L^2}^2 \\
= k_1 \|u\|_{L^{1+r}} + \lambda \|u\|_{L^2}^2,
\]
where
\[
\theta = \frac{r + 1 - 2m}{mr - m}, \quad \lambda := \frac{r + 1 - 2m}{r - 1} \left( \frac{k_1(r - 1)}{2m - 2} \right)^{-\frac{2m-2}{r+m}} (q_0 k_2)^{-\frac{-1}{r+m}}.
\]
Consequently, we have
\[ E\|u(t)\|^2 \leq E\|u_0\|^2 e^{-(\lambda_1-\bar{\lambda})t}, \]
where implies that the trivial solution 0 is exponentially mean square stable.

Next, we use a Lyapunov function \( V(u) = \|u\|^{2\gamma} \) with \( 0 < \gamma < 1 \) to prove the stochastic stability (note that the solutions of (4.17) is non-negative function, see [16]). This leads to the expression
\[
\frac{d}{dt} E\|u(\cdot, t)\|^{2\gamma} = \gamma E \left[ \|u(\cdot, t)\|^{2\gamma-2} \int_D u(\Delta u(x, t) - k_1 u^r(x, t)) dx \right] \\
+ \gamma k_2^2 E\|u(\cdot, t)\|^{2\gamma-2} \int_D u^{2m} dx \\
+ 2k_2^2 \gamma (\gamma - 1) E\|u(\cdot, t)\|^{2\gamma-4} \int_D \int_D q(x, y) u^m(x, t) u^m(y, t) dx dy \\
\leq \gamma E \left[ \|u(\cdot, t)\|^{2\gamma-2} \int_D u(\Delta u(x, t) - k_1 u^r(x, t)) dx \right] \\
+ \gamma k_2^2 E\|u(\cdot, t)\|^{2\gamma-2} \int_D u^{2m} dx + 2k_2^2 C_0 \gamma (\gamma - 1) E\|u(\cdot, t)\|^{2\gamma}.
\]
Then by using similar method in proving mean square stable, we can choose \( 0 < r < 1 \) such that
\[
\frac{d}{dt} E\|u(\cdot, t)\|^{2\gamma} \leq 0.
\]
From the Chebyshev inequality, stochastic stability for the solution of (4.17) is obtained. The proof is complete. \( \square \)

In paper [23], the authors considered the following problem
\[
\begin{align*}
    dX_t &= X_t(b(X_t) + k_1 X_t^{m-1}) dt + k_2 X_t^{m+1} \phi(X_t) dW_t, \quad t > 0, \\
    X_0 &= x > 0,
\end{align*}
\]
where \( k_1, k_2 \in \mathbb{R}, \ m \geq 1. \) In [17], we considered the competition between the nonlinear term and noise term. The result of [23] generalized the results of [17]. Now we first recall the main results of [23].

**Proposition 4.1.** [23, Theorem 1.1] Let \( k_1 \) be a real number which is not zero. Assume \( rb(r) \in C^1(\mathbb{R}+) \) and there exist two positive numbers \( c_0 \) and \( m_0 < m \) such that
\[
|b(r)| \leq c_0 (1 + r^{m_0-1}), \quad r \in \mathbb{R}^+.
\]
Assume in addition that \( r^{m+1} \phi(r) \in C^1(\mathbb{R}^+) \). Let \( \beta \in (0, 1) \) and suppose there is a \( r_0 > 0 \) such that
\[
\inf_{r \leq r_0} \phi(r) > \frac{2|k_1|}{(1-\beta)k_2^2}.
\]
There is a unique solution \( X_t(x) \) for (4.18) on \( t \geq 0 \) and the solution is positive for all \( t \geq 0 \) almost surely. Moreover, for every \( T > 0 \)
\[
\sup_{0 \leq t \leq T} E X_t^\beta(x) < +\infty.
\]
Moreover, [23, Theorem 1.2] shows that the result in proposition is sharp. More precisely, if there exists $\gamma \in (\beta, 1)$ such that
\[
\sup_{r \geq r_0} \phi(r) < \sqrt{\frac{2|k_1|}{(1-\gamma)k_2^2}},
\]
then there is a real number $T_0 > 0$ such that
\[
\sup_{0 \leq t \leq T} \mathbb{E}X_t^\gamma(x) = +\infty.
\]
The above results imply that the trivial solution 0 is not mean square stable. But the stochastic stability would be possible. In the following result, we will give a positive answer.

**Theorem 4.5.** Let all the assumptions of Proposition 4.1 hold. Assume further that
\[
rb(r) \leq c_1 r + c_2 r^{m_0}, \quad 1 < m_0 < m, \quad r \in \mathbb{R}_+, \quad c_1 < 0 < c_2.
\]
(i) If $\inf_{r \geq 0} \phi(r) \geq 1$ and
\[
c_1 + \frac{[p(k_1 - \frac{k_2^2}{2})] - p/q}{q} < 0,
\]
where
\[
p = \frac{m-1}{m_0-1}, \quad q = \frac{m-1}{m-m_0}.
\]
Then the trivial solution 0 is stochastic stable.
(ii) If $\phi(r) = r^2$ with $\alpha \geq 0$ and
\[
\frac{m-1}{\alpha + m - 1} \left(-\frac{c_1(\alpha + m - 1)}{2\alpha}\right)^{-\frac{\alpha + m - 1}{k_1}} k_1^{-\frac{\alpha + m - 1}{m - 1}}
\]
\[
+ \frac{m_0-1}{\alpha + m - 1} \left(-\frac{c_4(\alpha + m - 1)}{2(\alpha + m - m_0)}\right)^{-\frac{\alpha + m - m_0}{m_0-1}} c_2^{\alpha + m - m_0} < \frac{k_2^2}{2}.
\]
Then the trivial solution 0 is stochastic stable.

**Proof.** (i) We use a Lyapunov function $V(X) = |X|^\beta$ with $0 < \beta < 1$ being fixed later to prove the stochastic stability (noting that the solutions is positive almost surely, or one can use $(|X| + \kappa)\beta$ to replace $|X|^\beta$ and then let $\kappa \to 0$). This leads to the expression
\[
\frac{d}{dt}\mathbb{E}|X|^\beta = \beta \mathbb{E}|X|^\beta (b(X_t) + k_1 X_t^{m-1}) + \frac{1}{2} k_2^2 \beta(\beta - 1) \mathbb{E}|X|^\beta + \phi(X_t)
\]
\[
\leq \beta \mathbb{E}|X|^\beta \left[ c_1 + c_2 X_t^{m_0-1} + \left(k_1 + (\beta - 1) \frac{k_2^2}{2}\right) |X_t|^{m-1} \right].
\]
Set $0 < \beta \ll 1$ such that
\[
c_1 + \frac{[p(k_1 - \frac{(1-\beta)k_2^2}{2})] - p/q}{q} \leq 0.
\]
By using the $\varepsilon$-Young inequality, we have
\[
\frac{d}{dt}\mathbb{E}|X|^\beta \leq 0.
\]
From the Chebyshev inequality, stochastic stability for the solution of (4.18) is obtained.

(ii) In this case: \( \phi(r) = r^{\frac{\alpha}{2}} \) with \( \alpha \geq 0 \). For every \( \beta \in (0, 1) \), if we take 
\[ r_0 = \left( \frac{3|k_1|}{k_2(1-\beta)} \right)^2, \]
then 
\[ \inf_{r \geq r_0} \phi(r) = \inf_{r \geq r_0} |r|^{\alpha/2} = \frac{3|k_1|}{k_2(1-\beta)} > \frac{2|k_1|}{(1-\beta)k_2^2}. \]
Hence Proposition 4.1 holds for this case.

Similar to case (i), we use a Lyapunov function 
\[ V(X) = |X|^\beta \]
with \( 0 < \beta < \frac{1}{2} \). This leads to the expression
\[
\frac{d}{dt} E|X|^\beta = \beta E|X|^\beta (b(X_t) + k_1 X_t^{m-1}) + \frac{1}{2} k_2^2 \beta (\beta - 1) E|X_t|^\beta + \alpha + m - 1
\leq \beta E|X|^\beta \left[ c_1 + c_2 X_t^{m_0-1} + k_1 |X_t|^{m-1} + (\beta - 1) \frac{k_2^2}{2} |X_t|^\alpha + m - 1 \right].
\]
Set \( 0 < \beta < 1 \) such that
\[
\frac{m - 1}{\alpha + m - 1} \left( \frac{-c_1(\alpha + m - 1)}{2\alpha} \right) \left( \frac{c_2}{m_0 - \frac{\alpha + m - 1}{\alpha + m - m_0}} \right) \leq \frac{k_2^2}{2} (1 - \beta).
\]
By using the \( \varepsilon \)-Young inequality, we have
\[
\frac{d}{dt} E|X|^\beta \leq 0.
\]
From the Chebyshev inequality, stochastic stability for the solution of (4.18) is obtained.

**Remark 4.4.** Theorem 4.5 is new for SDEs. When \( k_2 = 0 \), the solution of (4.18) will blow up in finite time, and thus Theorem 4.5 implies that the multiplicative noise can make the solution stable.

Unfortunately, for SPDEs, we can not get the similar result to Theorem 4.5. Before we end this section, we give the reason. For simplicity, we consider the following problem
\[
\begin{cases}
    du = (\Delta u + k_1 u^r)dt + k_2 u^m dW_t, & t > 0, \ x \in D, \\
    u(x, 0) = u_0(x), & x \in D, \\
    u(x, t) = 0, & t > 0, \ x \in \partial D,
\end{cases}
\]
where \( k_1, k_2, r, m \in \mathbb{R} \). Under the condition \( r < m \), the existence of global solution was established by [19]. In the following, we set forth the reason why we can not get that the trivial solution 0 is stochastic stable.

We use a Lyapunov function \( V(u) = \|u\|^{2\gamma} \) with \( 0 < \gamma < 1/2 \) to prove the stochastic stability (if we worry about \( \|u\| = 0 \), we can use \( (\|u\|^2 + \kappa)^\gamma \) instead and
then let \( \kappa \to 0 \). This leads to the expression
\[
\frac{d}{dt} \mathbb{E}[|u(\cdot, t)|^{2\gamma}] = \gamma \mathbb{E} \left[ |u(\cdot, t)|^{2\gamma - 2} \int_D u(\Delta u(x, t) + k_1 u^\rho(x, t)) dx \right] \\
+ \gamma \mathbb{E} \left[ |u(\cdot, t)|^{2\gamma - 2} \int_D q(x, x) u^{2m} dx \right] \\
+ 2k_2^2 \gamma (\gamma - 1) \mathbb{E}[|u(\cdot, t)|^{2\gamma - 4}] \int_D \int_D q(x, y) u^{m+1}(x, t) u^{m+1}(y, t) dxdy \\
\leq -\gamma \lambda_1 \mathbb{E}[u(\cdot, t)]^{2\gamma} + k_1 \gamma \mathbb{E} \left[ |u(\cdot, t)|^{2\gamma - 2} \|u(\cdot, t)\|_{L^{m+1}}^{2(m+1)} \right] \\
+ \gamma \mathbb{E} \left[ |u(\cdot, t)|^{2\gamma - 2} \left( q_0 \|u\|_{L^m}^{2m} + 2q_1 k_2^2 (\gamma - 1) \frac{\|u(\cdot, t)\|_{L^{m+1}}^{2(m+1)}}{\|u(\cdot, t)\|^2} \right) \right]. 
\]

Hölder inequality implies that
\[
\int_D |u|^{m+1}(x, t) dx \leq \left( \int_D |u|^{2m}(x, t) dx \right)^{\frac{1}{2}} \left( \int_D |w|^2(x, t) dx \right)^{\frac{1}{2}}.
\]

For the last term of right-side hand of (4.20), we want to get
\[
\frac{\|u(\cdot, t)\|_{L^m}^{2m}}{\|u(\cdot, t)\|^2} \geq \|u\|_{L^{2m}}^{2m},
\]
which is a contradiction with respect to the above Hölder inequality. This shows no difference from Remark 4.1.

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