Divisionally free arrangements of hyperplanes

Takuro Abe *

April 2, 2015

Abstract

Let \((A, A', A^H)\) be the triple of hyperplane arrangements. We show that the freeness of \(A^H\) and the division of \(\chi(A; t)\) by \(\chi(A^H; t)\) imply the freeness of \(A\). This “division theorem” improves the famous addition-deletion theorem, and it has several applications, which include a definition of “divisionally free arrangements”. It is strictly larger than the classical important class of inductively free arrangements. Also, in the class of divisionally free arrangements, Terao’s conjecture is true. Moreover, we show that Terao’s conjecture holds true for a lot of recursively free arrangements to which almost all known free arrangements are belonging.

1 Main results

Let \(V\) be an \(\ell\)-dimensional vector space over an arbitrary field \(K\) with \(\ell \geq 1\), \(S = \text{Sym}(V^*) = K[x_1, \ldots, x_\ell]\) its coordinate ring and \(\text{Der} S := \oplus_{i=1}^{\ell} S \partial_{x_i}\) the module of \(K\)-linear \(S\)-derivations. A hyperplane arrangement \(A\) is a finite set of hyperplanes in \(V\). We say that \(A\) is central if every hyperplane is linear. In this article every arrangement is central unless otherwise specified. In the central cases, we fix a linear form \(\alpha_H \in V^*\) such that \(\ker(\alpha_H) = H\) for each \(H \in A\). An \(\ell\)-arrangement is an arrangement in an \(\ell\)-dimensional vector space. Let \(L(A) := \{\cap_{H \in B} H \mid B \subset A\}\) be an intersection lattice. \(L(A)\) has a partial order by reverse inclusion, which equips \(L(A)\) with a poset structure. For \(X \in L(A)\), define the localization \(A_X\) of \(A\) at \(X\) by \(A_X := \{H \in A \mid H \supset X\}\), which is a subarrangement of \(A\). Let \(L_i(A) := \{X \in L(A) \mid \text{codim}_V X = i\}\). Also, we use some notations in §2.

\*Department of Mechanical Engineering and Science, Kyoto University, Kyoto 606-8501, Japan. email:abe.takuro.4c@kyoto-u.ac.jp. MSC primary: 32S22, 52S35.
In the study of hyperplane arrangements, its algebraic structure $D(\mathcal{A})$ is well-studied. The logarithmic derivation module $D(\mathcal{A})$ is defined by

$$D(\mathcal{A}) := \{ \theta \in \text{Der } S \mid \theta(\alpha_H) \in S \cdot \alpha_H \ (\forall H \in \mathcal{A}) \}.$$  

We say that $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ if $D(\mathcal{A})$ is generated as an $S$-module by $S$-independent homogeneous generators $\theta_1, \ldots, \theta_\ell$ with $\deg \theta_i = d_i$ ($i = 1, \ldots, \ell$). The study of free arrangements was initiated by Terao in [19], and has been playing the central role in this area. Recently, there have been several studies to determine when $\mathcal{A}$ is free, e.g., [2], [9], [21], [22] and so on. However, it is still very difficult to determine the freeness.

Freeness of arrangements implies several interesting geometric and combinatorial properties of $\mathcal{A}$. For example, see [1], [2] and [20]. In particular, the most important result among them is Terao's factorization theorem (Theorem 2.5) in [20], which asserts that if $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, then its characteristic polynomial $\chi(\mathcal{A}; t)$ (essentially this is the same as the topological Poincaré polynomial $\pi(\mathcal{A}; t)$ of the complement $M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H$ of $\mathcal{A}$ in $V$ when $K = \mathbb{C}$) factorizes into $\chi(\mathcal{A}; t) = \prod_{i=1}^\ell (t - d_i)$. When $\mathcal{A} \neq \emptyset$, it is known that $(t - 1)$ divides $\chi(\mathcal{A}; t)$. Let $\chi_0(\mathcal{A}; t) := \chi(\mathcal{A}; t)/(t - 1)$ when $\mathcal{A} \neq \emptyset$.

The most useful method to construct free arrangements is the addition-deletion theorems (Theorem 2.3) by Terao in [19]. Let us recall it. For a central arrangement $\mathcal{A}$ and $H \in \mathcal{A}$, define

$$\mathcal{A}' := A \setminus \{ H \}, \quad \mathcal{A}^H := \{ H \cap L \mid L \in \mathcal{A}' \}.$$  

We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$ the triple. Note that $\mathcal{A}^H$ is an arrangement in $H$. The addition-deletion theorem enables us to determine the freeness of all these three when we know the freeness of any two of them with some information on exponents. For example, the addition theorem asserts that if both $\mathcal{A}'$ and $\mathcal{A}^H$ are free with $\exp(\mathcal{A}') \supset \exp(\mathcal{A}^H)$, then $\mathcal{A}$ is also free with certain exponents. By the factorization theorem above, in this case, it follows that $\chi(\mathcal{A}^H; t)$ divides $\chi(\mathcal{A}'; t)$ (the division of polynomials is often denoted by $\chi(\mathcal{A}^H; t) \mid \chi(\mathcal{A}'; t)$). Since there is a famous deletion-restriction formula (see Theorem 2.2)

$$\chi(\mathcal{A}; t) = \chi(\mathcal{A}'; t) - \chi(\mathcal{A}^H; t),$$  

it holds that $\chi(\mathcal{A}^H; t)$ also divides $\chi(\mathcal{A}; t)$ in this case. So the addition-deletion theorem contains a statement about divisions of polynomials, but these divisions have not been studied so much.

The aim of this article is to give a consideration on this aspect, i.e., the division of characteristic polynomials of these triples. The main result in this article is as follows.
Theorem 1.1 (Division theorem)
Let \( A \) be a central \( \ell \)-arrangement. Assume that there is a hyperplane \( H \in A \) such that \( \chi(A^H; t) \) divides \( \chi(A; t) \) and that \( A^H \) is free. Then \( A \) is free.

Theorem 1.1 has several advantages compared to previous results in both theoretical and practical senses. First, let us show a practical advantage of Theorem 1.1 in the following example.

Example 1.2
Let us consider the arrangement \( B \) in \( \mathbb{R}^4 \) defined by

\[
B = \left( \prod_{i=1}^{4} x_i \right) \prod_{a_2, a_3, a_4 \in \{\pm 1\}} (x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0.
\]

\( B \) is a restriction of the famous counterexample to Orlik’s famous conjecture by Edelman and Reiner in [12] onto a coordinate hyperplane. \( B \) is known to be free. Here we can check it soon by Theorem 1.1 combinatorially. Let

\[
C := B^{x_4=0} = \left( \prod_{i=1}^{3} x_i \right) \prod_{a_2, a_3 \in \{\pm 1\}} (x_1 + a_2 x_2 + a_3 x_3) = 0.
\]

Then it is easy to check that

\[
\chi(B; t) = (t - 1)(t - 3)^2(t - 5),
\]

\[
\chi(C; t) = (t - 1)(t - 3)^2,
\]

\[
\chi(C^{x_3=0}; t) = (t - 1)(t - 3).
\]

Since every 2-central arrangement is free (e.g., see Lemma 3.1), applying Theorem 1.1 twice shows that \( B \) is free. In particular, since we only used the combinatorial computations, we can also see that the freeness of \( B \) depends only on its combinatorics \( L(B) \).

There are several ways to understand Theorem 1.1. If we emphasize the aspect of a generalization of the addition-deletion theorem 2.3, Theorem 1.1 can be formulated as Theorems 3.10 and 3.11. Also, it can be regarded as a modified converse of Orlik’s conjecture, see Remark 3.8. Here we focus on the relation between the freeness and combinatorics.

As in Example 1.2, Theorem 1.1 allows us to check the freeness of \( A \) by constructing a divisional “tower” of characteristic polynomials as in (1.1). In other words, we obtain the following theorem on freeness which gives a completely combinatorial algorithm:
Theorem 1.3
An \( \ell \)-arrangement \( \mathcal{A} \) is free if \( \mathcal{A} \) has a filtration
\[
\mathcal{A} = \mathcal{A}_\ell \supset \mathcal{A}_{\ell-1} \supset \cdots \supset \mathcal{A}_2
\]
such that there is \( H_i \in \mathcal{A}_i \) such that \( \mathcal{A}_i^{H_i} = \mathcal{A}_{i-1} \) and that \( \chi(\mathcal{A}_{i-1}; t) \mid \chi(\mathcal{A}_i; t) \) \((i = 3, \ldots, \ell)\). Let us call this filtration a \textit{divisional filtration}.

Apparently, the freeness of \( \mathcal{A} \) satisfying the condition in Theorem 1.3 is combinatorially determined. Hence Theorem 1.3 is very useful to check that the freeness of some arrangement depends only on its combinatorics, which is a theoretical advantage. One of these theoretical applications is the dependency of the freeness of famous Shi arrangements only on combinatorics (see §7 for the notation):

Theorem 1.4
The freeness of the extended Shi arrangement depends only on the combinatorics.

To determine the relation between freeness and combinatorics has been an important but very hard problem. However, in this article, we can show the similar result as in Theorem 1.4 due to Theorem 1.1 (e.g., Example 5.6 and Theorem 6.6), which is very useful for that purpose. To make this framework systematically, we may introduce a new class of free arrangements, called the class of \textit{divisionally free arrangements}.

Definition 1.5 (Divisionally free arrangements)
We say that an \( \ell \)-arrangement \( \mathcal{A} \) is \textit{divisionally free} if \( \mathcal{A} \) has a divisional filtration in Theorem 1.3. The set \( \mathcal{DF}_\ell \) consists of all the divisionally free \( \ell \)-arrangements, and \( \mathcal{DF} := \bigcup_{\ell \geq 1} \mathcal{DF}_\ell \).

Then it is easy to show that \( \mathcal{A} \) is free if \( \mathcal{A} \in \mathcal{DF} \), and the freeness of \( \mathcal{A} \in \mathcal{DF} \) depends only on its combinatorics, see Theorem 5.4 for details. The most famous and important class of free arrangements with the same properties is the class of inductively free arrangements introduced by Terao in [19] (see Definition 5.2). Let \( \mathcal{IF} \) denote the class of inductively free arrangements. In fact, the class of divisionally free arrangements is strictly larger than that of inductively free arrangements.

Theorem 1.6
\( \mathcal{IF} \subset \subset \mathcal{DF} \).

To prove Theorem 1.6, recent developments on the freeness of unitary reflection arrangements (e.g., [10], [13]) play the key roles. Hence Theorem 1.1 develops the theory of free arrangements from the viewpoint of
Terao’s conjecture (Conjecture 5.1) by strictly enlarging the inductively free arrangements. Also, we can show that Terao’s conjecture is true for a lot of recursively free arrangements in Theorem 6.5. Moreover, as in Theorem 1.4, applications to those related to root systems are also given. Hence Theorem 1.1 and the divisionally free arrangements give rise to an essential progress to the theory of free arrangements and its combinatorics.

To prove Theorem 1.1, the key ingredient is the following fact that, the division of characteristic polynomial commutes with localizations along the hyperplane in codimension-three:

**Theorem 1.7 (Localization and Remainder Theorem)**

Let \( A \) be a central \( \ell \)-arrangement and \( H \in A \). Let us consider the polynomial division

\[
\chi_0(A; t) = (t - (|A| - |AH|))\chi_0(A_H; t) + r(t)
\]

with the remainder \( r(t) = \sum_{i=0}^{\ell-3} (-1)^i r_i t^{\ell-3-i} \). Then \( r_0 \geq 0 \). Moreover, if \( r_0 = 0 \), then \( \chi(A_H; t) \) divides \( \chi(A_X; t) \) for all \( X \in L_2(A_H) \), and \( A \) is locally free along \( H \) in codimension three. In particular, if \( \chi(A_H; t) \) divides \( \chi(A; t) \) for a hyperplane \( H \in A \), then the same statement holds.

The statement in Theorem 1.7 is non-trivial. For example, even when \( A \) is free, that statement does not hold in general, e.g., see Example 3.7. Also, a certain converse of Theorem 1.7 holds, see Theorem 3.13.

If \( \chi(A_H; t) \) divides \( \chi(A; t) \), then clearly \( r_0 = 0 \). Hence the first part of Theorem 1.7 implies the second one. The meaning of \( r_0 \geq 0 \) in Theorem 1.7 can be seen in Remark 2.8.

**Remark 1.8**

As in Theorem 1.7, the condition \( \chi(A_H; t) \mid \chi(A; t) \) in Theorem 1.1 can be replaced by \( r_0 = 0 \) in terms of Theorem 1.7.

Though Theorem 1.1 is a generalization of the addition-deletion theorems, the proofs of Theorems 1.1 and 1.7 deeply depend on the new development of the theory of multiarrangements, i.e., we use the definition of multiarrangements by Ziegler in [23], Yoshinaga’s criterion in [21] and its refinement in [9], development of basic tools to treat them in [7] and [8], and the same statement as Theorem 1.1 when \( \ell = 3 \) in [2]. It is interesting to see that in the statement of Theorem 1.1 there are no multiarrangements.

The organization of this article is as follows. In §2 we introduce several results used for the proof of results in §1. In §3 we prove our main theorems in §1. In §4 we give several applications of our result related to multiarrangements. The main applications here are Theorems 4.4 and 4.6 which assert the commutativity of the Euler and Ziegler restrictions. In §5
we investigate the most important application; the definition of divisionally
free arrangements in Definitions 1.5 and 5.3. We show that divisionally free
arrangements contain all the inductively free arrangements, and Terao’s con-
jecture holds true in divisionally free arrangements. Moreover, there is a
divisionally free arrangement which is not inductively free. In §6 we consider
the relation between divisionally and recursively free arrangements to which
almost all known free arrangements belong. In §7 we give applications to the
arrangements related to root systems. In particular, we prove that the Shi
arrangements are divisionally free.

2 Preliminaries

Let us review several definitions and results used in the rest of this article.
We use the notation and definitions appeared in §1. We use [15] as a general
reference in this section. Define the Möbius function $\mu : L(A) \to \mathbb{Z}$ by
$\mu(X) = 1$ if $X = V$ and $\mu(X) = -\sum_{X \subseteq Y \in L(A)} \mu(Y)$ if $X \neq V$. Then the
characteristic polynomial $\chi(A; t)$ and the Poincaré polynomial $\pi(A; t)$
are defined by

$$
\chi(A; t) = \sum_{X \in L(A)} \mu(X) t^{\dim X},
$$

$$
\pi(A; t) = \sum_{X \in L(A)} \mu(X) (-t)^{\codim X}.
$$

As mentioned in §1, $\pi(A; t)$ equals to the topological Poincaré polynomial of
$M(A)$ when $\mathbb{K} = \mathbb{C}$. Also, it is easy to see that $\chi(A)$ is divisible by $(t - 1)$
if $A \neq \emptyset$. Write and define

$$
\chi(A; t) = \sum_{i=0}^{\ell} b_i(A) (-1)^i t^{\ell-i},
$$

$$
\chi_0(A; t) = \chi(A; t) / (t - 1) = \sum_{i=0}^{\ell-1} b_i(dA) (-1)^i t^{\ell-1-i}.
$$

Remark 2.1

The reason why we use the terminology $b_i(dA)$ is as follows. Fix a hyperplane
$H_0 \in A$. Then we may consider the operation named deconing $dA$ of $A$
with respect to $H_0$. Namely, $dA$ is a set of affine hyperplanes obtained as
intersections of $\alpha_{H_0} = 1$ with all hyperplanes $L \in A \setminus \{H_0\}$. Then it is
known that $\chi(dA; t) = \chi_0(A; t)$. See [15] for example. This is the reason of
the notation above.
Let $\theta_E := \sum_{i=1}^\ell x_i \partial x_i$ be the Euler derivation. Let us recall several results on these polynomials and freeness:

**Theorem 2.2 (Deletion-restriction formula, e.g., [15], Corollary 2.57)**

Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$ be a triple. Then it holds that

$$
\chi(\mathcal{A}; t) = \chi(\mathcal{A}'; t) - \chi(\mathcal{A}^H; t),
$$

$$
\chi_0(\mathcal{A}; t) = \chi_0(\mathcal{A}'; t) - \chi_0(\mathcal{A}^H; t).
$$

**Theorem 2.3 (Addition-deletion theorem, [19])**

Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$ be a triple. Then any two of the following three imply the third:

1. $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.
2. $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.
3. $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$.

**Theorem 2.4 (Restriction Theorem, [19])**

In the notation of Theorem 2.3, assume that both $\mathcal{A}$ and $\mathcal{A}'$ are free. Then all the statements (1), (2), and (3) in Theorem 2.3 hold. Moreover, we may choose a basis $\theta_1, \ldots, \theta_\ell$ for $D(\mathcal{A})$ such that $\alpha_H$ divides $\theta_\ell$ and $\pi(\theta_1), \ldots, \pi(\theta_{\ell-1})$ form a basis for $D(\mathcal{A}^H)$, where $\pi$ is a residue map $D(\mathcal{A}) \to D(\mathcal{A}^H)$.

**Theorem 2.5 (Factorization theorem, [20], Main Theorem)**

If $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, then $\chi(\mathcal{A}; t) = \prod_{i=1}^\ell (t - d_i)$.

When $\ell = 3$, the addition-deletion theorem [2, Theorem 1.1 (3)] has the following simple form.

**Theorem 2.6 ([2], Theorem 1.1 (3))**

Let $\mathcal{A}$ be a central arrangement in $V = \mathbb{K}^3$. Then two of the following three imply all the statements (1), (2) and (3) in Theorem 2.3:

(a) $\chi(\mathcal{A}; t) = (t - 1)(t - d_1)(t - d_2)$.

(b) $\chi(\mathcal{A}'; t) = (t - 1)(t - d_1)(t - (d_2 - 1))$.

(c) $|\mathcal{A}^H| = d_1 + 1$. 

7
Among the equivalent conditions in Theorem 2.6, we can find several divisions of characteristic polynomials, and they imply the freeness of each member in the triple $(A, A', A^H)$. From this point of view, Theorem 1.1 is a generalization of Theorem 2.6. Also, if we regard these freeness as a local freeness in codimension three in $K^3$, then Theorem 1.7 is also a generalization of Theorem 2.6. The following is also in [2] with a different formulation.

**Theorem 2.7 ([2], Theorem 1.1)**

Assume that $\ell = 3$ and let us consider the division

$$\chi_0(A; t) = (t - (|A^H| - 1)) (t - (|A| - |A^H|)) + a.$$  

Then $a = \chi_0(A; |A^H| - 1) \geq 0$. Moreover, $a = 0$ implies that $A$ is free.

**Remark 2.8**

Theorem 2.7 says that the integer $a$ can be regarded as the remainder of the polynomial division of $\chi_0(A; t)$ by $t - (|A^H| - 1) = \chi_0(A^H; t)$. The non-negativity of $a$ in the above was generalized in [2], Theorem 7.1 for an arbitrary $\ell \geq 3$ just as an inequality. However, from the viewpoint of polynomial divisions, we may understand the non-negativity as that of the leading term of the remainder of a characteristic polynomial division naturally. Hence the non-negativity $r_0 \geq 0$ in Theorem 1.7 can be regarded as a generalization of Theorem 2.7.

Now let us explain the freeness criterion by using multiarrangements. A **multiarrangement** $(A, m)$ is a pair consisting of an $\ell$-arrangement $A$ and a function $m : A \to \mathbb{Z}_{>0}$. Let $|m| := \sum_{H \in A} m(H)$. For $L \in A$, let $\delta_L : A \to \{0, 1\}$ be a **characteristic multiplicity** of $L$ defined by $\delta_L(H) = 1$ only when $H = L$ (hence $\delta_L(H) = 0$ for all $A \ni H \neq L$). For $(A, m)$, we can define the **logarithmic derivation module** $D(A, m)$ by

$$D(A, m) := \{ \theta \in \text{Der } S \mid \theta(\alpha_H) \in S \alpha_H^{m(H)} (\forall H \in A) \}.$$  

Also, its **freeness** and **exponents** can be defined in the same manner as for $A$ and $D(A)$. When $(A, m)$ is free with $\exp(A, m) = (d_1, \ldots, d_\ell)$, it holds that $\sum_{i=1}^\ell d_i = |m|$. For two multiplicities $m_1 : A \to \mathbb{Z}_{>0} (i = 1, 2)$, $m_1 \leq m_2$ denotes $m_1(H) \leq m_2(H)$ for all $H \in A$. Then the following is the most basic method to determine the freeness.

**Theorem 2.9 (Saito’s criterion, [16])**

Let $\theta_1, \ldots, \theta_\ell \in D(A, m)$. Then $(A, m)$ is free if and only if $\theta_1, \ldots, \theta_\ell$ are $S$-independent and $\sum_{i=1}^\ell \deg \theta_i = |m|$.
Not only the freeness, but also a characteristic polynomial \( \chi(A, m; t) \) for a multiarrangement \((A, m)\) can be defined too. Let us write

\[
\chi(A, m; t) = \sum_{i=0}^{\ell} b_i(A, m)(-1)^i t^{\ell-i}.
\]

Also, there is the factorization theorem for free multiarrangements as for the arrangement cases, i.e., if \((A, m)\) is free with \(\exp(A, m) = (d_1, \ldots, d_\ell)\), then

\[
\chi(A, m; t) = \prod_{i=1}^{\ell} (t - d_i).
\]

For details, see Definition 2.1 and Theorem 4.1 in \cite{7}. Contrary to \(\chi(A; t)\), to compute \(\chi(A, m; t)\) when \(m\) is not identically 1 is very difficult. In fact, it is not combinatorial (see \cite{7}). However, it is easy to check that \(b_1(A, m) = |m|\). Moreover, \(b_2(A, m)\) can be computed by using the following local-global formula.

**Theorem 2.10 (Local-global formula, \cite{7}, Theorem 3.3)**

Let \((A, m)\) be a multiarrangement. For \(X \in L_2(A)\), denote \(\exp(A_X, m|_{A_X}) = (d_X^1, d_X^2, 0, \ldots, 0)\). Then

\[
b_2(A, m) = \sum_{X \in L_2(A)} d_X^1 d_X^2 = \sum_{X \in L_2(A)} b_2(A_X, m|_{A_X}).
\]

From a central arrangement \(A\) and a hyperplane \(H \in A\), we may define an induced multiarrangement \((A^H, m^H)\) (called the Ziegler restriction of \(A\) onto \(H\)) by

\[
m^H(K) := |\{L \in A' \mid L \cap H = K\}|.
\]

The multiplicity \(m^H\) is called the Ziegler multiplicity. By definition, \(|m^H| = |A'| = |A| - 1\). Then Ziegler showed the following fundamental fact.

**Theorem 2.11 (\cite{23})**

Assume that \(A\) is free with \(\exp(A) = (1, d_2, \ldots, d_\ell)\). Then the Ziegler restriction \((A^H, m^H)\) onto an arbitrary \(H \in A\) is free with \(\exp(A^H, m^H) = (d_2, \ldots, d_\ell)\).

We say that \(A\) is locally free in codimension \(i\) along \(H \in A\) if \(A_X\) is free for all \(X \in L_i(A)\) with \(X \subset H\) (equivalently, for all \(X \in L_{i-1}(A^H)\)). By using this notion, as a converse of Theorem 2.11, the following freeness criterion is known.
Theorem 2.12 ([9], Theorem 5.1)

Let $\mathcal{A} \neq \emptyset$ and $H \in \mathcal{A}$. Then $b_2(d\mathcal{A}) \geq b_2(\mathcal{A}^H, m^H)$, and the equality holds if and only if $\mathcal{A}$ is locally free in codimension three along $H$. Moreover, if the Ziegler restriction $(\mathcal{A}^H, m^H)$ is free, then the following conditions are equivalent:

1. $\mathcal{A}$ is free.
2. $b_2(d\mathcal{A}) = b_2(\mathcal{A}^H, m^H)$.
3. $\mathcal{A}_X$ is free for any $X \in L_2(\mathcal{A}^H)$, i.e., $\mathcal{A}$ is locally free in codimension three along $H$.

Also, let us recall the addition-deletion theorem for multiarrangements $(\mathcal{A}, m)$ from [8]. For $H \in \mathcal{A}$, we may define a restriction $(\mathcal{A}^H, m^*)$ of the multiarrangement $(\mathcal{A}, m)$ onto $H$, which is called the Euler restriction of $(\mathcal{A}, m)$ onto $H$. Also, the new multiplicity $m^*$ is called the Euler multiplicity. Let us explain the definition of $m^*$ since it will be used for the proofs of main results.

For $X \in L_2(\mathcal{A})$ with $X \subset H$, let $m_X$ denote the restriction of $m$ onto $\mathcal{A}_X$. Then the multiarrangement $(\mathcal{A}_X, m_X)$ is a direct product of a 2-multiarrangement and the $(\ell - 2)$-empty arrangement. It is well-known that every 2-arrangement is free. Hence we may define $\exp(\mathcal{A}_X, m_X) = (d_1^X, d_2^X, 0, \ldots, 0)$. Therefore, there is a homogeneous free basis $\theta_1, \theta_2, \varphi_1, \ldots, \varphi_{\ell-2}$ for $D(\mathcal{A}_X, m_X)$ such that $\deg \theta_i = d_i$ ($i = 1, 2$) and $\deg \varphi_j = 0$ ($j = 1, \ldots, \ell - 2$). By the same reason, there are free basis $\theta'_1, \theta'_2, \varphi'_1, \ldots, \varphi'_{\ell-2}$ for $D(\mathcal{A}_X, m_X - \delta_H)$ such that $\deg \theta'_i = d'_i$ ($i = 1, 2$) and $\deg \varphi'_j = 0$ ($j = 1, \ldots, \ell - 2$), here $\exp(\mathcal{A}_X, m_X - \delta_H) = (d'_1, d'_2, 0, \ldots, 0)$. It was shown in [8] that we may assume $d_1 = d'_1 + 1$ and $d_2 = d'_2$. Moreover, we may choose bases in such a way that

$$\theta_1 = \alpha_H \theta'_1, \quad \theta_2 = \theta'_2, \quad \varphi_j = \varphi'_j \quad (j = 1, \ldots, \ell - 2).$$

Recall that $X \in L_2(\mathcal{A})$ with $X \subset H \iff X \in \mathcal{A}^H$. Now define $m^*(X) := \deg \theta_2$. For more details of these definitions, see [8]. By using these definitions, we have the addition-deletion theorem for multiarrangements as follows.

Theorem 2.13 ([8], Theorem 0.8)

Let $(\mathcal{A}, m)$ be a multiarrangement, $H \in \mathcal{A}$ and $(\mathcal{A}^H, m^*)$ the Euler restriction of $(\mathcal{A}, m)$ onto $H$. Then any two of the following three imply the third:

1. $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, \ldots, d_{\ell-1}, d_{\ell})$. 

10
(2) \((A, m - \delta_H)\) is free with \(\exp(A, m - \delta_H) = (d_1, \ldots, d_{\ell-1}, d_{\ell} - 1)\).

(3) \((A^H, m^*)\) is free with \(\exp(A^H, m^*) = (d_1, \ldots, d_{\ell-1}).\)

Moreover, if both \((A, m)\) and \((A, m - \delta_H)\) are free, then all the three hold true.

3 Proofs of results

Let us start the proof of main results in §1. Let us add one notation. For an \(\ell\)-multiarrangement \((A, m)\) and \(X \in L_1(A)\), assume that \((A_X, m_X)\) is free with \(\exp(A_X, m_X) = (d_1, \ldots, d_\ell - 1, 0, \ldots, 0)\). Then the set \(\exp^*(A_X, m_X) := \{d_1, \ldots, d_\ell\}\).

First, we introduce a lemma which is used without referring in the rest of this article.

Lemma 3.1
A 2-multiarrangement \((A, m)\) is free. Let \(\exp(A, m) = (d_1, d_2)\). Then for \(H \in A\), it holds that \(\exp(A, m - \delta_H) = (d_1 - 1, d_2)\) or \((d_1, d_2 - 1)\).

Proof. For example, see Lemma 2.7 in [2]. □

Second we need the following lemmas.

Lemma 3.2
Let \(A\) be a 2-arrangement. Then \(\exp(A, m) = (|m| - |A| + 1, |A| - 1)\) when \(|m| \leq 2|A| - 1\). If \(|m| \geq 2|A|\), then \(d_i \geq |A| - 1\) \((i = 1, 2)\) for \(\exp(A, m) = (d_1, d_2)\) with \(d_1 \leq d_2\).

Proof. This result is well-known. Here we give a short proof. For a complete proof, see Lemma 2.10 in [2] for example.

First consider the case \(|m| \leq 2|A| - 1\). Then clearly

\[\theta_1 := (\prod_{H \in A} \alpha_H^{m(H) - 1}) \theta_E \in D(A, m).\]

Note that \(\deg \theta_1 = |m| - |A| + 1\). Also the definition of \(D(A, m)\) shows that there are no \(\theta \in D(A, m)\) such that \(\theta | \theta_1\). Since \(d_1 + d_2 = |m|\) by Theorem 2.9, it has to hold that \(|m| - |A| + 1 = d_1\), or \(|m| - |A| + 1 = d_2\), or \(d_2 < |m| - |A| + 1\). The third case contradicts Saito’s criterion 2.9.

Second assume that \(|m| \geq 2|A|\) and take multiplicities \(m' \leq m\) such that \(|m'| = 2|A| - 1\). Then the first assertion shows that \(\exp(A, m') = (|A| - 1, |A|)\). Since \(D(A, m') \supset D(A, m)\), it holds that \(|A| - 1 \leq d_1 \leq d_2\). □
Lemma 3.3
Let \((A, m)\) be a multiarrangement and fix \(H \in A\) with \(m(H) \geq 2\). Then

1. if \(A\) is a 2-arrangement, then \(b_2(A, m) - b_2(A, m - \delta_H) = \exp^*(A, m) \cap \exp^*(A, m - \delta_H)\), and
2. \(b_2(A, m) - b_2(A, m - \delta_H) \geq |A| - 1\).

Proof.
(1) Immediate by the definition of \(b_2(A, m)\) and Lemmas 3.1 and 3.2.

(2) Let us compute \(b_2(A, m) - b_2(A, m - \delta_H)\) by using the local-global formula 2.10. Since \((A_X, m_X) = (A_X, (m - \delta_H)_X)\) for \(L_2(A) \ni X \not\subset H\), it holds that

\[
(3.1) \quad b_2(A, m) - b_2(A, m - \delta_H) = \sum_{X \in A^H} (b_2(A_X, m_X) - b_2(A_X, (m - \delta_H)_X)).
\]

Now apply (1) and Lemma 3.2 to show that

\[
(3.1) \geq \sum_{X \in A^H} (|A_X| - 1) = |A| - 1,
\]

which completes the proof. \(\square\)

Lemma 3.4
Let \((A, m)\) be a multiarrangement and fix \(H \in A\) with \(m(H) \geq 2\). Let \(m^*\) be the Euler multiplicity of \((A, m)\) onto \(H\). Then

1. \(m^*(X) \geq |A_X| - 1\) for all \(X \in A^H\), \(m^*(X) = \exp^*(A_X, m_X) \cap \exp^*(A_X, m_X - \delta_H)\) and
2. if \(m^*(X) \neq |A_X| - 1\) for some \(X \in A^H\), then \(b_2(A, m) - b_2(A, m - \delta_H) > |A| - 1\).

Proof.
(1) When \(|m_X| \leq 2|A_X| - 1\), the explicit construction of \(\theta_1\) in the proof of Lemma 3.2 shows that \(m^*(X) = |A_X| - 1\). If \(|m_X| \geq 2|A_X|\), then Lemma 3.2 shows that \(d \geq |A_X| - 1\) for all \(d \in \exp^*(A_X, m_X)\). Hence the definition of \(m^*\) shows that \(m^*(X) \geq |A_X| - 1\). The second statement follows immediately by the same proof as Lemma 3.3 (1).

(2) By the assumption, (1) and Lemma 3.3 (1), \(m^*(Y) = b_2(A_Y, m_Y) - b_2(A_Y, (m - \delta_H)_Y) > |A_Y| - 1\) for some \(Y \in A^H\). Then apply this and Lemma 3.3 (2) to the local-global formula 2.10 to obtain

\[
b_2(A, m) - b_2(A, m - \delta_H) = \sum_{X \in A^H} (b_2(A_X, m_X) - b_2(A_X, (m - \delta_H)_X)) \geq \sum_{X \in A^H} (|A_X| - 1) = |A| - 1.
\]

\(\square\)
Lemma 3.5
Let \((A, m)\) be an \(\ell\)-multiarrangement, \(H \in A\) with \(m(H) > 1\), \(X \in L_2(A^H)\) and \((A^H, m^*)\) the Euler restriction of \(A\) onto \(H\). Then
\[
b_2(A^H, m_X) - b_2(A^H, (m - \delta_H)_X) = m^*(X).
\]

Proof. Immediate from Lemmas 3.3 (1) and 3.4 (1). \(\Box\)

Proposition 3.6
In the setup of Theorem 1.7, it holds that
\[
b_2(A^H) + (|A^H| - 1)(|A| - |A^H| - 1) + r_0 \quad \geq \quad b_2(A^H, m_H^*)
\]
In particular, when \(r_0 = 0\), for a multiplicity \(m\) on \(A^H\) such that \(1 \leq m \leq m_H^*\), it holds that
\[
b_2(A^H, m) - b_2(A^H, m - \delta_L) \quad = \quad |A^H| - 1,
\]
\[
b_2(A^H, m_X) - b_2(A^H, (m - \delta_L)_X) \quad = \quad |A^H| - 1
\]
for \(L \in A^H\), \(X \in L_2(A^H)\) with \(X \subset L\) and \(m(L) > 1\).

Proof. Recall the division of characteristic polynomials:
\[
\chi_0(A; t) = \chi_0(A^H; t)(t - (|A| - |A^H|)) + \sum_{i=0}^{\ell-3} r_i t^{\ell-3-i}.
\]
Now let us introduce three (in)equalities:

First inequality. \(b_2(dA) \geq b_2(A^H, m_H)\). This is a part of Theorem 2.12.

Second inequality. \(b_2(A^H, m_H^*) \geq b_2(A^H) + (|A^H| - 1)(|A| - |A^H| - 1)\).

First note that \(|m_H - |A^H| = |A| - |A^H| - 1\). By definition, there are a sequence of hyperplanes \(L_1, \ldots, L_{|A| - |A^H| - 1} \in A^H\) such that \(m_H = 1 + \delta_{L_1} + \cdots + \delta_{L_{|A| - |A^H| - 1}}\). Hence Lemma 3.3 (2) implies that
\[
b_2(A^H, m_H) = \sum_{i=1}^{\frac{|A| - |A^H| - 1}{\ell} - 1} (b_2(A^H, 1 + \delta_{L_1} + \cdots + \delta_{L_i})
\quad - b_2(A^H, 1 + \delta_{L_1} + \cdots + \delta_{L_{i-1}})) + b_2(A^H)
\quad \geq \sum_{i=1}^{\frac{|A| - |A^H| - 1}{\ell} - 1} (|A^H| - 1) + b_2(A^H)
\quad = b_2(A^H) + (|A| - |A^H| - 1)(|A^H| - 1).
\]
Third equality. $b_2(d_A) = b_2(A^H) + (|A^H| - 1)(|A| - |A^H| - 1) + r_0$.

First, note that $\chi_0(A; t) = (t - (|A| - |A^H|))\chi_0(A^H; t) + \sum_{t=0}^{[t-3]} r_t t^{\ell-3}$. Hence comparing the coefficients of $t^{\ell-3}$ implies

$$b_2(d_A) = (|A| - |A^H|)(|A^H| - 1) + b_2(dA^H) + r_0.$$ 

Since $\chi(A^H; t) = (t-1)\chi_0(A^H; t)$, it holds that $b_2(A^H) = b_2(dA^H) + |A^H| - 1$. Hence

$$b_2(dA) = (|A| - |A^H| - 1)(|A^H| - 1) + b_2(A^H) + r_0.$$ 

Now we have three (in)equalities. Combine these three to obtain

$$b_2(A^H) + (|A^H| - 1)(|A| - |A^H| - 1) + r_0$$

$$= b_2(dA) \geq b_2(A^H, m^H)$$

$$\geq b_2(A^H) + (|A^H| - 1)(|A| - |A^H| - 1),$$

which is the first statement in this proposition.

Now assume that $r_0 = 0$. Then the inequalities above are all equalities. Hence

$$b_2(dA) = b_2(A^H, m^H)$$

$$(3.7) b_2(dA) = b_2(A^H, m^H)$$

$$(3.8) = b_2(A^H) + (|A| - |A^H| - 1)(|A^H| - 1).$$

Let us show $b_2(A^H, m) - b_2(A^H, m - \delta_L) = |A^H| - 1$ for a multiplicity $1 \leq m \leq m^H$ and $L \in A^H$ with $m(L) > 1$. We know that $b_2(A^H, m) - b_2(A^H, m - \delta_L) \geq |A^H| - 1$ by Lemma 3.3 (2). If this inequality is strict, then again Lemma 3.3 (2) and (3.6) show that the equality (3.8) cannot hold, which is a contradiction.

Next let us show that $b_2(A^H, m) - b_2(A^H, (m - \delta_L)_X) = |A^H| - 1$ for $X \in L_2(A^H)$ with $X \subset L$. By the above, we know that $b_2(A^H, m) - b_2(A^H, m - \delta_L) = |A^H| - 1$. So the local-global formula 2.10 shows that

$$b_2(A^H, m) - b_2(A^H, m - \delta_L) = \sum_{X \in (A^H)_L} (b_2(A^H_X, m_X) - b_2(A^H_X, (m - \delta_L)_X))$$

$$= |A^H| - 1 = \sum_{X \in (A^H)_L} (|A^H_X| - 1).$$

Since $b_2(A^H_X, m_X) - b_2(A^H_X, (m_X - \delta_L)_X) \geq |A^H_X| - 1$ by Lemma 3.3 (2), this inequality has to be an equality, which completes the proof.

Now let us prove Theorem 1.7.

Proof of Theorem 1.7. The nonnegativity $r_0 \geq 0$ holds immediately from the inequality in Proposition 3.6. Assume that $r_0 = 0$. Then we
have the equation (3.7). Thus Theorem 2.12 shows that \( \mathcal{A} \) is locally free in codimension three along \( H \). Hence to complete the proof by applying Theorem 2.12, it suffices to show that \( \chi(\mathcal{A}_X^H; t) \) divides \( \chi(\mathcal{A}_X; t) \) for all \( X \in L_3(\mathcal{A}) \) with \( X \subset H \).

We show that \( \exp(\mathcal{A}_X^H, (m^H)_X) = (|\mathcal{A}_X^H| - 1, |(m^H)_X| - |\mathcal{A}_X^H| + 1) \) by applying Theorem 2.13. First, note that \((\mathcal{A}^H)_X, (m^H)_X \) coincides with the Ziegler restriction of \( \mathcal{A}_X \) onto \( H \supset X \) by definition. Hence let us express this multiarrangement by \((\mathcal{A}_X^H, m_X^H)\). Now we need to compute the Euler multiplicity \( m^* \), which can be obtained as the Euler restriction of \( \mathcal{A}_X \) onto \( H \). Recall that we have the equation (3.5). Hence Lemma 3.5 shows that

\[(3.9) \quad m^*(X) = b_2(\mathcal{A}_X^H, m_X) - b_2(\mathcal{A}_X^H, (m - \delta_L)_X) = |\mathcal{A}_X^H| - 1\]

for \( L \in \mathcal{A}^H \) and \( X \in (\mathcal{A}^H)^L \) as in the notation of Proposition 3.6.

Since \( \exp^*(\mathcal{A}_X^H) = (1, |\mathcal{A}_X^H| - 1) \) by Lemma 3.2, Theorem 2.13 shows that \( \exp(\mathcal{A}_X^H, (m^H)_X) = (|\mathcal{A}_X^H| - 1, |\mathcal{A}_X^H| - 1) \). Because \( \mathcal{A} \) is locally free in codimension three along \( H \) by (3.7), Theorems 2.5 and 2.11 show that

\[\begin{align*}
\chi(\mathcal{A}_X; t) &= (t-1)(t-(|\mathcal{A}_X^H| - 1))(t-(|\mathcal{A}_X| - |\mathcal{A}_X^H|)), \\
\chi(\mathcal{A}_X^H; t) &= (t-1)(t-(|\mathcal{A}_X^H| - 1))
\end{align*}\]

which completes the proof. \( \square \)

**Example 3.7**

Let \( \mathcal{A} \) be a plane arrangement consisting of the cone of all edges and diagonal lines of a regular pentagon. Hence \( |\mathcal{A}| = 11 \). It is well-known that (for example, see [13], Example 4.54) that \( \mathcal{A} \) is free with \( \exp(\mathcal{A}) = (1, 5, 5) \) and \( |\mathcal{A}^H| = 5 \) for any \( H \in \mathcal{A} \). Let \( \mathcal{B} \) be the coning of \( \mathcal{A} \), hence free with \( \exp(\mathcal{A}) = (1, 1, 5, 5) \). Let \( H_0 \in \mathcal{B} \) be the infinite line in \( \mathcal{A} \) and let \( X \in L_2(\mathcal{A}^{H_0}) \) be a flat which is contained in all the cone of planes belonging to \( \mathcal{A} \). Then it is easy to see that \( \mathcal{B}_X = \mathcal{A} \times \emptyset \) and \( \mathcal{B}_X^{H_0} \) consists of five linear lines in a plane. Hence

\[\chi(\mathcal{B}_X; t) = (t-1)(t-5)^2, \quad \chi(\mathcal{B}_X^{H_0}; t) = (t-1)(t-4).\]

So clearly \( \chi(\mathcal{B}_X^{H_0}; t) \uparrow \chi(\mathcal{B}_X; t) \).

**Proof of Theorem 1.1**

By Theorems 1.7 and 2.12 it suffices to show that the Ziegler restriction \((\mathcal{A}^H, m^H)\) of \( \mathcal{A} \) onto \( H \) is free. By the assumption and Terao’s factorization theorem 2.5 it holds that \( \chi(\mathcal{A}^H; t) = \prod_{i=1}^{t-1}(t - d_i) \), where \( d_1 = 1 \) and \( \exp(\mathcal{A}^H) = (d_1, \ldots, d_{t-1}) \). Since \( \chi(\mathcal{A}^H; t) \uparrow \chi(\mathcal{A}; t) \), we may write \( \chi(\mathcal{A}; t) = (t - d_t)\chi(\mathcal{A}^H; t) = \prod_{i=1}^{t-1}(t - d_i) \) for \( d_t = |\mathcal{A}| - |\mathcal{A}^H| \in \mathbb{Z}_{\geq 0} \). In fact, we show that, for every multiplicity \( m \) with \( 1 \leq m \leq m^H \),
the multiarrangement \((\mathcal{A}^H, m)\) is free with \(\exp(\mathcal{A}^H, m) = (|m| - |\mathcal{A}^H| + 1, d_2, \ldots, d_{\ell-1})\).

We show by induction on \(|m|\). If \(m\) is a constant multiplicity 1, then the statement holds true by the assumption. Assume that the statement holds true for all \(m\) with \(|m| - |\mathcal{A}^H| < n\) \((n \in \mathbb{Z}_{\geq 1})\). Take an arbitrary \(m\) satisfying the condition above and \(|m| - |\mathcal{A}^H| = n\). We may choose a hyperplane \(L \in \mathcal{A}^H\) such that \(m(L) \geq 2\). By the induction hypothesis, the multiarrangement \((\mathcal{A}^H, m - \delta_L)\) is free with \(\exp(\mathcal{A}^H, m - \delta_L) = (n, d_2, \ldots, d_{\ell-1})\). Let us apply the addition theorem for multiarrangements (Theorem 2.13) to confirm that \((\mathcal{A}^H, m)\) is free with \(\exp(\mathcal{A}^H, m) = (n+1, d_2, \ldots, d_{\ell-1})\).

For that purpose, we need to compute the Euler restriction \(((\mathcal{A}^H)^L, m^*)\) of \((\mathcal{A}^H, m)\) onto \(L\). By the assumption, Proposition 3.6, the equation (3.5) and Lemma 3.5, the equation (3.9) also follows here, i.e.,

\[
b_2(\mathcal{A}^H_X, m_X) - b_2(\mathcal{A}^H_X, (m - \delta_L)_X) = m^*(X) = |\mathcal{A}^H_X| - 1
\]

for all \(X \in (\mathcal{A}^H)^L\). On the other hand, by definition, \(m^L (X) = |\mathcal{A}^H_X| - 1\) for \(X \in (\mathcal{A}^H)^L\), where \(((\mathcal{A}^H)^L, m^L)\) is the Ziegler restriction of \(\mathcal{A}^H\) onto \(L\). Hence in this case, these two restrictions coincide; \(((\mathcal{A}^H)^L, m^L) = ((\mathcal{A}^H)^L, m^*)\).

Since \(\mathcal{A}^H\) is free with \(\exp(\mathcal{A}^H) = (1, d_2, \ldots, d_{\ell-1})\), Theorem 2.11 shows that \(((\mathcal{A}^H)^L, m^L) = ((\mathcal{A}^H)^L, m^*)\) is free with exponents \((d_2, \ldots, d_{\ell-1})\). Hence the assumption on the freeness and exponents on \((\mathcal{A}^H, m - \delta_L)\), and Theorem 2.13 show that \((\mathcal{A}^H, m)\) is also free with \(\exp(\mathcal{A}^H, m) = (n+1, d_2, \ldots, d_{\ell-1})\).

Now apply this argument to \((\mathcal{A}^H, m^H)\). Then we obtain that \((\mathcal{A}^H, m^H)\) is free with

\[
\exp(\mathcal{A}^H, m^H) = (|m^H| - |\mathcal{A}^H| + 1, d_2, \ldots, d_{\ell-1}) = (|A| - |\mathcal{A}^H|, d_2, \ldots, d_{\ell-1}) = (d_2, \ldots, d_{\ell-1}, d_{\ell}),
\]

which completes the proof. When the condition \(\chi(\mathcal{A}^H, t) \mid \chi(A; t)\) is replaced by \(r_0 = 0\) as mentioned in Remark 1.8, the proof is totally the same, so we left it to the reader. \(\square\)

**Proof of Theorem 1.3** Apply Theorem 1.1 repeatedly with Lemma 3.1. \(\square\)

**Remark 3.8**

Theorem 1.1 can be regarded as a modified converse of the famous Orlik’s conjecture, which asserted that \(\mathcal{A}^H\) is free if \(A\) is free. To this conjecture, a counter example was found by Edelman and Reiner in [12]. This conjecture was asserting that global freeness implies restricted freeness. Though this
is not true, Theorem 1.1 asserts that the modified converse is true, i.e.,
restricted freeness with a combinatorial condition implies global freeness.

As a corollary of the proof above, the following is immediate.

**Corollary 3.9**

1. Assume that $\chi(A^H; t) | \chi(A; t)$. Then $(A^H, m^H)$ is free if $A^H$ is free.
2. Assume that there is $H \in A$ such that the Ziegler restriction $(A^H, m^H)$
is free and $\chi(A^H; t) | \chi(A; t)$. Then $A$ is free.
3. Assume that $A^H$ is free. If $r_0 = 0$ in the division

$$
\chi_0(A; t) = (t - (|A| - |A^H|))\chi_0(A^H; t) + \sum_{i=0}^{\ell-3} r_i t^{\ell-3-i},
$$

then $r_1 = \cdots = r_{\ell-3} = 0$. Equivalently, $\chi(A^H; t)$ divides $\chi(A; t)$.

If we emphasize the aspect as the addition-deletion theorems, we may
have the following formulations too.

**Theorem 3.10**

Let $A$ be an $\ell$-central arrangement. Assume that there is a hyperplane $L \notin A$
such that $\chi((A \cup \{L\})^L; t)$ divides $\chi(A; t)$ and that $(A \cup \{L\})^L$ is free. Then $A$ is free.

**Proof.** Apply the same proof as that of Theorem 1.1 to $A \cup \{L\}$ to obtain
that $A \cup \{L\}$ is free since $\chi((A \cup \{L\})^L; t) | \chi(A \cup \{L\}; t)$ by Theorem 2.2.
Now apply Theorem 2.3 to complete the proof. \qed

**Theorem 3.11**

Let $(A, A', A^H)$ be the triple with respect to $H \in A$. Assume that $A^H$ is
free. Then the following conditions are equivalent:

1. $A$ is free and $\exp(A) \supset \exp(A^H)$.
2. $A'$ is free and $\exp(A') \supset \exp(A^H)$.
3. All the three of $A, A'$ and $A^H$ are free.
4. $\chi(A^H; t)$ divides $\chi(A; t)$.
5. $\chi(A^H; t)$ divides $\chi(A'; t)$.
6. $\chi(A; t)$ and $\chi(A'; t)$ have a GCD of degree $\ell - 1$.
7. In the division $\chi_0(A; t) = \chi_0(A^H; t)(t - (|A| - |A^H|)) + \sum_{i=0}^{\ell-3} r_i t^{\ell-3-i}$,
it holds that $r_0 = 0$. 

17
In the division \( \chi_0(\mathcal{A}'; t) = \chi_0(\mathcal{A}^H; t)(t - (|\mathcal{A}'| - |\mathcal{A}^H|)) + \sum_{i=0}^{\ell-3} r'_i t^{\ell-3-i}, \) it holds that \( r'_0 = 0. \)

**Proof.** The equivalences among (1), (2) and (3) follow immediately from the addition-deletion theorem 2.3. Those among (4), (5) and (6) follow from the deletion-restriction theorem 2.2. Since \( \chi_0(\mathcal{A}; t) = \chi_0(\mathcal{A}'; t) - \chi_0(\mathcal{A}^H; t), \) that between (7) and (8) is easy. Both (1) \( \iff (4) \) and (1) \( \iff (7) \) follow by Theorem 1.1 and Remark 1.8, which completes the proof. □

The conditions (1), (2) and (3) are the addition-deletion theorems that contain freeness conditions. However, the others are just combinatorial ones. Also, the freeness is assumed for an \((\ell - 1)\)-arrangement \( \mathcal{A}^H \) to check the freeness of \( \ell \)-arrangement \( \mathcal{A} \), which enables us an inductive argument.

The following result is a corollary of the proofs of Theorems 1.7 and 1.1.

**Corollary 3.12**
Let \( \mathcal{A} \) be a free \( \ell \)-arrangement with \( \exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell) \) and let \( m : \mathcal{A} \to \mathbb{Z}_{>0} \). Let us consider the division

\[
\chi(\mathcal{A}, m; t) = (t - (|m| - |\mathcal{A}| + 1))\chi_0(\mathcal{A}; t) + \sum_{i=0}^{\ell-2} s_i t^{\ell-2-i}.
\]

If \( s_0 = 0 \), then \( (\mathcal{A}, m) \) is free with \( \exp(\mathcal{A}, m) = (|m| - |\mathcal{A}| + 1, d_2, \ldots, d_\ell) \).

The following is a certain converse of Theorem 1.7.

**Corollary 3.13**
Let \( \mathcal{A} \) be an \( \ell \)-arrangement.

1. Assume that there is \( H \in \mathcal{A} \) such that \( \chi(\mathcal{A}^H; t) \) divides \( \chi(\mathcal{A}_X; t) \) for all \( X \in L_2(\mathcal{A}^H) \), and \( \mathcal{A}^H \) is free. Then \( \mathcal{A} \) is free.

2. If there is a filtration

\[
\mathcal{A} = \mathcal{A}_\ell \supset \mathcal{A}_{\ell-1} \supset \cdots \supset \mathcal{A}_3 \supset \mathcal{A}_2
\]

such that, \( \mathcal{A}_{i-1} := \mathcal{A}_{H_i}^H \) for some \( H_i \in \mathcal{A}_i \), and \( \chi((\mathcal{A}_i)_{X}; t) \) divides \( \chi((\mathcal{A}_i)_{X}; t) \) for all \( X \in L_2(\mathcal{A}_{i-1}) \) and \( i = \ell, \ldots, 3 \). Then \( \mathcal{A} \) is free.

**Proof.** (1) By the same argument as in the proof of Theorem 1.1 it holds that

\[
b_2(\mathcal{A}, m) - b_2(\mathcal{A}, m - \delta_L) = |\mathcal{A}^H| - 1
\]
and
\[ b_2(\mathcal{A}^H_X, m_X) - b_2(\mathcal{A}^H_X, (m - \delta_L)_X) = m^*(X) = |\mathcal{A}^H_X| - 1 \]
for all \( m : \mathcal{A}^H \rightarrow \mathbb{Z} \) with \( 1 \leq m \leq m^H \), all \( L \in \mathcal{A}^H \) with \( m(L) > 1 \) and \( X \in (\mathcal{A}^H)_L \). Hence Theorems 2.11 and 2.13 complete the proof. (2) is an immediate consequence of (1).

Without the freeness of \( \mathcal{A}^H \), Theorem 3.13 (1) does not hold. See the following example.

**Example 3.14**
Let \( \mathcal{A} := \{ xyzw(x + y + z + w) = 0 \} \). Then it is easy to check that \( \chi(\mathcal{A}_X; t) = (t - 1)^3 \) and \( \chi(\mathcal{A}^H_X; t) = (t - 1)^2 \) for all \( X \in L_2(\mathcal{A}^{w=0}) \). Also, it is easy to compute
\[ \chi(\mathcal{A}; t) = (t - 1)(t^3 - 4t^2 + 6t - 4), \quad \chi(\mathcal{A}^{w=0}; t) = (t - 1)(t^2 - 3t + 3). \]
Hence \( \mathcal{A}^H \) is not free and \( \chi(\mathcal{A}^{w=0}; t) \nmid \chi(\mathcal{A}; t) \), and Theorem 3.13 (1) does not hold.

## 4 Applications to multiarrangements

In this section we give several applications of main results to, mainly multiarrangements. Theorem 1.1 implies that, if an arrangement is a free arrangement, and has a free restriction, then several strong requirements exist. The following is one of such requirements.

**Corollary 4.1**
Let \( \mathcal{A} \) be a central \( \ell \)-arrangement. Assume that there is a hyperplane \( H \in \mathcal{A} \) such that \( \chi(\mathcal{A}^H; t) \nmid \chi(\mathcal{A}; t) \). Then for any flat \( X \in L_2(\mathcal{A}^H) \) with \( |\mathcal{A}^H_X| = 2 \), it does not hold that \( m^H(X) \geq 2 \) and \( m^H(Y) \geq 2 \) for \( X \in (\mathcal{A}^H)_Y \). In particular, if there is such a flat \( X \in L_2(\mathcal{A}) \) with \( X \subset H \), then \( \mathcal{A} \) is not free.

**Proof.** If so, then Lemma 3.3 shows that there is some multiplicity \( m : \mathcal{A}^H \rightarrow \mathbb{Z}_{>0} \) with \( 1 \leq m \leq m^H \) and \( L \in \mathcal{A}^H \) with \( m^H(L) \geq 2 \) such that \( b_2(\mathcal{A}^H_X, m_X) - b_2(\mathcal{A}^H_X, (m - \delta_L)_X) > |A^H_X| - 1 \) for some \( X \in (\mathcal{A}^H)_L \) in the notation of the proof of Proposition 3.6. Hence the argument in the proof of Theorem 1.7 shows that \( b_2(\mathcal{A}, m) - b_2(\mathcal{A}, m - \delta_L) > |\mathcal{A}^H| - 1 \), which implies that \( b_2(\mathcal{A}) < b_2(\mathcal{A}^H, m^H) \) and a contradiction.

The following is immediate from Theorems 2.2 and 1.7.

**Corollary 4.2**
Let \( \mathcal{A} \supset H \) be as the above and \( \mathcal{A}' := \mathcal{A} \setminus \{H\} \). Then \( \chi(\mathcal{A}^H_X; t) \mid \chi(\mathcal{A}_X; t) \) for all \( X \in L_2(\mathcal{A}^H) \) if one of the following three conditions holds;
(1) $\chi(A^H; t) \mid \chi(A; t)$.

(2) $\chi(A^H; t) \mid \chi(A'; t)$.

(3) $\chi(A; t)$ and $\chi(A'; t)$ have a common factor of degree $\ell - 1$.

Also, the division theorem[1.1] gives the following combinatorial statement
for the pair of freeness as in [2].

Corollary 4.3
Assume that an $\ell$-arrangement $A$ contains a hyperplane $H \in A$ such that the freeness of an $(\ell - 1)$-arrangement $A^H$ depends only on $L(A^H)$. Then whether both of $A$ and $A' := A \setminus \{H\}$ are free or not is determined only by $L(A)$.

Proof. Assume that $A^H$ is not free, and it is determined only by $L(A^H)$ for some $H \in A$. If both $A$ and $A'$ are free, then Theorem [2.3] implies that $A^H$ is free. Hence in this case not both free. Assume that $A^H$ is free and it is determined only by $L(A^H)$. By Theorem [3.1] both $A$ and $A'$ are free if and only if $\chi(A^H; t)$ divides $\chi(A; t)$, which depends only on $L(A)$. □

For example, inductively, or divisionally free arrangements satisfy the condition on $A^H$ in Corollary 4.3. For the definition of them, see the next section.

In the proof of Theorem 1.1, two restrictions played key roles. One is the Ziegler restriction, and the other is the Euler restriction. They are different, and the multiplicity obtained by combining them are different in general. However, if the division holds, then they commute in the following sense.

Theorem 4.4
Let $A$ be an $\ell$-arrangement. Assume that there is $H \in A$ such that $\chi(A^H; t)$ divides $\chi(A; t)$. Then the following two restrictions $(m^H)^*$ and $m^*$ coincide;

(a) First take the Ziegler restriction $(A^H, m^H)$ of $A$ onto $H$. Second take the Euler restriction $((A^H)^L, (m^H)^*)$ of $(A^H, m^H)$ onto $L \in A^H$ such that $m_H(L) > 1$.

(b) First take a usual restriction $A^H$ which is in fact the Euler restriction of $A$ onto $H$. Second take the Ziegler restriction $((A^H)^L, m^*)$ of $A^H$ onto $L \in A^H$ such that $m_H(L) > 1$.

Proof. Let us fix $L \in A^H$ as in the statement. Then it suffices to show that $(m^H)^*(X) = m^*(X)$ for $X \in (A^H)^L$. Since codim$_V X = 3$, we may assume that $A$ is a free 3-arrangement and $\chi(A^H; t) \mid \chi(A; t)$.
First compute $m^*(X)$. By definition, it follows immediately that $m^*(X) = |A^H_X| - 1$. Compute $(m^H)^*(X)$. By Theorems 1.7 and 2.6 it follows that $\exp(A) = (1, |A| - |A^H|, |A^H| - 1)$ and $\exp(A^H) = (1, |A^H| - 1)$. By the proof of Theorems 1.1 and 1.7 and (3.6), it holds that $(m^H)^*(X) = |A^H_X| - 1$, which completes the proof. □

Example 4.5
Let us again consider the pentagon example $A$ in Example 3.7. Let $H \in \mathcal{A}$ be the infinite plane. Then it is known that the Ziegler restriction of $A$ onto $H$ is $(A^H, 2)$, where $A^H$ consists of five linear lines and 2 is the constant multiplicity two. It is well-known that $\exp(A^H, 2) = (5, 5)$. Hence any $L \in A^H$ satisfies the condition in Theorem 4.4. By Lemma 3.4 (2), it follows that $(2)^*(L) = 5$. On the other hand, the Ziegler restriction of $A^H$ onto $L$ has the Ziegler multiplicity 4 at the origin $(A^H)L$. Hence they are not equal even when $A$ is free.

In fact, the commutativity in Theorem 4.4 is closely related to the division of $\chi(A; t)$ by $\chi(A^H; t)$ when $A^H$ is free.

Theorem 4.6
Let $A$ be an $\ell$-arrangement. Assume that there is $H \in \mathcal{A}$ such that $A^H$ is free. Then the following conditions are equivalent:

(1) Two restrictions (a) and (b) in Theorem 4.4 coincide.

(2) $\chi(A^H; t)$ divides $\chi(A; t)$.

(3) $A$ is free and $\exp(A) \supset \exp(A^H)$.

Proof. The equivalence of (2) and (3) follows from Theorem 1.1. Assume (2). Then Theorem 4.4 implies (1). So it suffices to show (1) implies (2). The coincidence of two multiplicities and the explicit computation of $m^*$ in Theorem 4.4 shows that

$$m^*(X) = (m^H)^* = |A^H_X| - 1$$

for $X \in L_2((A^H)L)$, where $L \in A^H$ is an arbitrary hyperplane. Since $\exp^*(A^H_X) = (1, |A^H_X| - 1)$, Lemmas 3.1 and 3.2 show that $\exp^*(A^H_X, k) \supset |A^H_X| - 1$ for any multiplicity $k$ with $1 \leq k \leq m^H$. Hence Lemma 3.5 shows that $k^*(X) = |A^H_X| - 1$. Therefore, $((A^H)L, (m^H)^*) = ((A^H)L, k^*) = ((A^H)L, m_L)$ is the Ziegler restriction of $A^H$ onto $L$. Hence Theorems 2.11 and 2.13 shows that $A$ is free with $\exp(A) \supset \exp(A^H)$, which implies (2). □
5 Divisionally free arrangements

In this section we give the main application of main results in this article. Before that, let us recall Terao’s conjecture.

Conjecture 5.1 (Terao)
Let $\mathcal{A}, \mathcal{B}$ be two $\ell$-arrangements such that $L(\mathcal{A}) \simeq L(\mathcal{B})$ as posets and that $\mathcal{A}$ is free. Then $\mathcal{B}$ is also free.

Conjecture 5.1 is still open even when $\ell = 3$. Terao also introduced a class of inductively free arrangements by using Theorem 2.3 in which Conjecture 5.1 holds true. First recall the definition of the inductively free arrangements.

Definition 5.2 ([19])
The set $\mathcal{IF}_\ell$ of $\ell$-arrangements consists of the following:

1. If $\ell = 1, 2$, then all arrangements are in $\mathcal{IF}_\ell$.

2. Assume that $\ell \geq 3$. Then $\mathcal{A} \in \mathcal{IF}_\ell$ if $\mathcal{A} = \emptyset_\ell$, or there is $H \in \mathcal{A}$ such that $\mathcal{A}' := \mathcal{A} \setminus \{H\} \in \mathcal{IF}_\ell$, $\mathcal{A}^H \in \mathcal{IF}_{\ell-1}$ and $\chi(\mathcal{A}^H; t) | \chi(\mathcal{A}'; t)$.

By Theorem 2.3, $\mathcal{A} \in \mathcal{IF}_\ell$ is free. An arrangement $\mathcal{A}$ is called **inductively free (IF)** if $\mathcal{A} \in \mathcal{IF} := \cup_{\ell \geq 1} \mathcal{IF}_\ell$.

Though the class of divisionally free arrangements has been already defined in Definition 1.5, let us give an another equivalent definition of it similar to Definition 5.2.

Theorem-Definition 5.3
The set $\mathcal{DF}_\ell$ of $\ell$-arrangements can be also defined as follows:

1. When $\ell = 1$ and 2, all arrangements are in $\mathcal{DF}_\ell$.

2. Assume that $\ell \geq 3$. Then $\mathcal{A} \in \mathcal{DF}_\ell$ if $\mathcal{A} = \emptyset_\ell$, or there is $H \in \mathcal{A}$ such that $\chi(\mathcal{A}^H; t) | \chi(\mathcal{A}; t)$ and $\mathcal{A}^H \in \mathcal{DF}_{\ell-1}$.

An arrangement $\mathcal{A}$ is called **divisionally free (DF)** if $\mathcal{A} \in \mathcal{DF} := \cup_{\ell \geq 1} \mathcal{DF}_\ell$.

Comparing Definition 5.2 with Definition 5.3, it can be seen that the divisionally free arrangement is an analogy of the inductively free arrangement by replacing the addition-deletion theorem 2.3 by Theorem 1.1.

By the Definitions 1.5, 5.3, Theorems 2.3, 1.1 and 1.3 the following is clear.
Theorem 5.4
(1) If $A \in DF$, then $A$ is free.
(2) $IF \subset DF$.
(3) Let $A_1, A_2$ be $\ell$-arrangements with the same combinatorics $L(A_1) \simeq L(A_2)$. If $A_1 \in DF$, then $A_2$ is free. In particular, Terao’s conjecture is true in $DF$.

Every inductively free arrangement has to be constructed from the empty arrangement by the addition theorem. On the other hand, a divisionally free arrangement need not. In fact, the class $DF$ is strictly larger than $IF$. Let us show it as Theorem 1.6.

Proof of Theorem 1.6 It suffices to find an arrangement $A \in DF \setminus IF$.
We can find such an example among the reflecting hyperplanes of a unitary reflection group.

Let $G_{31}$ be a finite unitary reflection group acting on $\mathbb{C}^4$, where we use the labeling of such groups due to Shephard and Todd in [17]. Let $A$ be the unitary reflection arrangement in $V = \mathbb{C}^4$ corresponding to $G_{31}$. Then it is shown that $A$ is free with $\exp(A) = (1, 13, 17, 29)$ (see [13], Table C.12 for example), but $A$ is not inductively free ([13], Theorem 1.1).

However, Lemma 3.5 in [13] showed that there is $H \in A$ such that $A^H$ is free with $\exp(A^H) = (1, 13, 17)$. Also, Lemma 4.1 in [10] shows that there is $L \in A^H$ such that $|(A^H)^L| = 14$. Hence Theorem 2.5 shows that $\chi(A^H; t) | \chi(A; t)$ and $\chi((A^H)^L; t) | \chi(A^H; t)$, which implies that $A$ is divisionally free by Theorem 1.3. Therefore, $IF \subsetneq DF$.

Also, the intermediate arrangements $A^k_\ell(r) \in DF \setminus IF$ if $k \neq 0$, $\ell \geq 3$. See Theorem 6.6 for details.

To check whether the freeness of some arrangement depends only on its lattice, $DF$ can be used as follows.

Example 5.5
It is known that all the freeness of Weyl arrangements depends only on its combinatorics. Here we show the same statement, for certain cases, by using divisionally free arrangements.

Let $B_\ell$ be the Weyl arrangement of the type $B_\ell$ defined by

$$\prod_{i=1}^\ell x_i \prod_{1 \leq i < j \leq \ell} (x_i^2 - x_j^2) = 0.$$ 

Then immediately $B^{x_i=0}_{\ell-1} = B_{\ell-1}$. Also, we may compute

$$\chi(B_\ell; t) = (t-1)(t-3)\cdots(t-(2\ell-1)).$$
Hence it follows that

\[ \chi(B_{i-1}; t) \mid \chi(B_i; t) \ (i = 3, \ldots, \ell). \]

Therefore, \( B_\ell \in \mathcal{DF} \), which shows that the freeness of \( B_\ell \) depends only on its combinatorics.

The same proof works for the root systems of the types \( A \) and \( C \).

**Example 5.6**

By the proof of Theorem 1.6, \( A(G_{31}) \in \mathcal{DF} \). Hence if a 4-arrangement \( B \) has the same intersection as that of the unitary reflection arrangement \( A \) corresponding to \( G_{31} \), then \( B \) is also free by Theorem 5.4.

For the finite unitary reflection arrangements, inductive freeness and divisional freeness almost coincide:

**Corollary 5.7**

Let \( W \) be a finite irreducible complex reflection group and \( A = A(W) \) its corresponding reflection arrangement. Then \( A \in \mathcal{IF} \) or \( W = G_{31} \) if and only if \( A \in \mathcal{DF} \).

**Proof.** Let \( W \neq G_{31} \). Then Corollary 1.3 in [13] shows that \( A = A(W) \in \mathcal{IF} \) if and only if \( \exp(A^H) \subset \exp(A) \) for any \( H \in A \). Since \( A \) is free, this is also equivalent to say that \( A \in \mathcal{DF} \). Hence the proof of Theorem 1.6 completes the proof.

It seems interesting to study the (non-)divisional freeness of several non-inductively free arrangements around complex reflection arrangements which appeared in the recent development. See [10], [13] for example.

Recalling the fact that there have been several developments on how to check the inductively freeness of several arrangements due to [10], [13] and so on, practically, the following formulation of Theorem 1.3 is also useful.

**Theorem 5.8**

\( A \in \mathcal{DF} \) if and only if \( A \) has a filtration

\[ A = A_\ell \supset A_{\ell-1} \supset \cdots \supset A_2 \]

such that there is \( H_i \in A_i \) such that \( A_i^{H_i} = A_{i-1}, \chi(A_{i-1}; t) \) divides \( \chi(A_i \setminus \{H_i\}; t) \) for \( i = 3, \ldots, \ell \) and that \( A \setminus \{ H_\ell \} \) is free.

**Proof.** Follows immediately from Theorems 1.1, 1.3, 2.3 and Definitions 1.5 and 5.2
Corollary 5.9
\( \mathcal{A} \in \mathcal{DF} \) if and only if \( \mathcal{A} \) has a filtration
\[
\mathcal{A} = \mathcal{A}_\ell \supset \mathcal{A}_{\ell - 1} \supset \cdots \supset \mathcal{A}_2
\]
such that there is \( H_i \in \mathcal{A}_i \) such that \( \mathcal{A}_{i}^{H_i} = \mathcal{A}_{i - 1} \) are all free, \( \exp(\mathcal{A}_{i - 1}) \subset \exp(\mathcal{A}_i) \) for \( i = 3, \ldots, \ell - 1 \) and that \( \mathcal{A} \setminus \{H_\ell\} \) is free with \( \exp(\mathcal{A} \setminus \{H_\ell\}) \supset \exp(\mathcal{A}_{\ell - 1}) \).

**Proof.** Apply the same proof as that of Theorem 5.8 □

The points in Theorem 5.8 and Corollary 5.9 are, though we have to check the freeness of some arrangements, we do not have to check anything else on them, and they enable us to determine the combinatorially determined freeness. Corollary 5.9 is useful when we have a list of free arrangements.

### 6 Recursively and divisionally free arrangements

First let us recall the definition of the recursively free arrangements.

**Definition 6.1 (Recursively free arrangements)**

A set of \( \ell \)-arrangements \( \mathcal{RF}_\ell \) is defined by,

1. all arrangements are in \( \mathcal{RF}_\ell \) when \( \ell = 1, 2 \), and
2. for \( \ell \geq 3 \), \( \mathcal{A} \in \mathcal{RF}_\ell \) if \( \mathcal{A} = \emptyset \), or there is a hyperplane \( H \in \mathcal{A} \) such that \( \mathcal{A}_H \in \mathcal{RF}_{\ell - 1} \), \( \mathcal{A} \setminus \{H\} \in \mathcal{RF}_\ell \) and \( \chi(\mathcal{A}_H; t) \mid \chi(\mathcal{A}; t) \), or there is a hyperplane \( L \not\in \mathcal{A} \) such that \( (\mathcal{A} \cup \{L\}) \in \mathcal{RF}_\ell \), \( (\mathcal{A} \cup \{L\})^L \in \mathcal{RF}_{\ell - 1} \) and \( \chi(\mathcal{A} \cup \{L\}^L; t) \mid \chi(\mathcal{A}; t) \).

By Theorem 2.3, \( \mathcal{A} \) is free if \( \mathcal{A} \in \mathcal{RF} \). We say that an arrangement \( \mathcal{A} \) is recursively free if \( \mathcal{A} \in \mathcal{RF} := \bigcup_{\ell \geq 1} \mathcal{RF}_\ell \).

**Remark 6.2**
The definition of recursively free arrangements may seem to be artificial. However, this contains almost all of known free arrangements. There are only three arrangements known which are free but non-recursively free. The first one is found by Cuntz and Hoge in [11], and the other two are in [4]. Though some free arrangements (like \( \mathcal{A}(G_{31}) \) in the previous section) are not known whether it is recursively free or not (see [10], Corollary 4.3), by these known results on \( \mathcal{RF} \), it is worth consider Conjecture 5.1 for recursively free arrangements.
Contrary to the inductively free arrangements, it is not yet known whether the freeness of recursively free arrangements depends only on the combinatorics.

**Problem 6.3**

Does the freeness of recursively free arrangements depend only on its combinatorics?

We will show that, by using divisionally free arrangements, Terao’s conjecture 5.1 holds true for a lot of recursively free arrangements. For that purpose, let us introduce a definition.

**Definition 6.4**

Let $\mathcal{RF} = \bigcup_{\ell \geq 1} \mathcal{RF}_\ell$ denote the class of recursively free arrangements. Define $\mathcal{RF}_1^* := \mathcal{RF}_1$, $\mathcal{RF}_2^* := \mathcal{RF}_2$, and

$$
\mathcal{RF}_\ell^* := \{ A \in \mathcal{RF}_\ell \mid \exists H \in A \text{ such that } \chi(A^H; t) \mid \chi(A; t) \text{ and } A^H \in \mathcal{RF}_{\ell-1}^* \} \quad (\ell \geq 3),
$$

and put $\mathcal{RF}^* := \bigcup_{\ell \geq 1} \mathcal{RF}_\ell^*$. We call $\mathcal{RF}^*$ the class of **recursively free arrangements at the hill**. If $A \in \mathcal{RF} \setminus \mathcal{RF}^*$, then we say that $A$ is a **recursively free arrangement at the valley**.

As far as we know, there have been no answer to Problem 6.3. Here we give the first partial answer to Problem 6.3 asserting that Conjecture 5.1 holds true for recursively free arrangements not at the valley.

**Theorem 6.5**

$\mathcal{RF}^* \subset \mathcal{DF}$. In particular, Terao’s conjecture holds true for $A \in \mathcal{RF}^*$.

**Proof.** We show by induction on $\ell$. When $\ell = 1, 2$, then there is nothing to show. Assume that $\ell \geq 3$ and $A \in \mathcal{RF}_\ell^*$. Then there is $H \in A$ such that $\chi(A^H; t) \mid \chi(A; t)$ and $A^H \in \mathcal{RF}_{\ell-1}^* \subset \mathcal{DF}_{\ell-1}$ by the induction hypothesis. Hence Definition 5.3 shows that $A \in \mathcal{DF}_\ell$. $\square$

An example of non-inductively free $A \in \mathcal{RF}^*$ is intermediate arrangements. Let $\mathcal{A}_k^r(\ell)$ ($\ell \geq 2$, $0 \leq k \leq \ell$) be an $\ell$-arrangement defined by

$$
\prod_{i=1}^{k} x_i \prod_{1 \leq i < j \leq \ell, 0 \leq n < r} (x_i - \zeta^n x_j) = 0,
$$

where $\zeta$ is a primitive $r$-th root of unity. They are called **intermediate arrangements**, and first studied by Orlik and Solomon in ![14](14). They interpolate between the reflection arrangements corresponding to monomial
groups, and they are all free arrangements (Propositions 2.11 and 2.13, [14]). The inductive freeness of intermediate arrangements is studied in [10], and showed that $A_k^\ell(r)$ is not inductively free if and only if $r \geq 3$ and $0 \leq k \leq \ell - 3$ (Theorem 3.6, [10]). On their divisional freeness, we have the following.

**Theorem 6.6**

Let $\ell \geq 3$. Then $A_k^\ell(r) \in DF$ if and only if $k \neq 0$. In particular, when $k \neq 0$, the freeness depends only on the intersection lattice.

**Proof.** Let $H \in A_k^\ell(r)$ be an arbitrary coordinate hyperplane. When $k = 0$, by Corollary 1.3 in [13], there are no $H \in A$ such that $A^H$ is free and $\chi(A^H; t) \mid \chi(A; t)$. Hence $A_k^\ell(r)$ is not divisionally free. Assume that $k \neq 0$. Then for a coordinate hyperplane $H \in A$, by the result in [14] (see Proposition 3.1 in [10] too), it holds that $(A_k^\ell(r))^H \simeq A_{\ell-1}^{\ell-1}(r)$, and

$$\exp(A_k^\ell(r)) \supset \exp(A_k^\ell(r)^H), \quad \exp(A_k^\ell(r)) \supset \exp(A_{\ell-1}^{\ell-1}(r)) \quad (j = 3, \ldots, \ell).$$

Hence Theorem 1.3 shows that $A_k^\ell(r) \in DF$. $\square$

As shown in Theorem 3.6 in [10], $A_k^\ell(r)$ ($0 < k < \ell - 2, \ell \geq 3$) are not inductively free but recursively free. By Theorem 6.6, the conditions in Theorem 6.5 are satisfied. Hence $A_k^\ell(r) \in RF^\ast$. So only the arrangement $A_k^\ell(r)$ which is at the the valley of free paths of arrangements $A_k^\ell(r)$ ($k \geq 0$) is not divisionally free.

Also, following the definition based on the addition-deletion theorems, we can define the following class of free arrangements.

**Definition 6.7**

(1) A set of $\ell$-arrangements $\mathcal{MF}_\ell$ is defined by,

(a) all arrangements are in $\mathcal{MF}_\ell$ when $\ell = 1, 2,$ and

(b) for $\ell \geq 3$, $A \in \mathcal{MF}_\ell$ if there is a hyperplane $H \in A$ such that $A^H \in \mathcal{MF}_{\ell-1}$ and $\chi(A^H; t) \mid \chi(A; t)$, or there is a hyperplane $L \not\in A$ such that $(A \cup \{L\})^L \in \mathcal{MF}_{\ell-1}$ and $\chi((A \cup \{L\})^L; t) \mid \chi(A; t)$.

We say that an arrangement $A$ is **multiplicatively free** if $A \in \mathcal{MF} := \cup_{\ell \geq 1} \mathcal{MF}_\ell$.

(2) An $\ell$-arrangement is **hereditarily divisionally free** if $A^X$ is divisionally free for all $X \in L(A)$.

Recall that the definition of hereditarily inductively free, where the definition of hereditarily divisionally free generalizes it. An arrangement $A$ is **hereditarily inductively free** if $A^X$ is inductively free for all $X \in L(A)$. Clearly, hereditarily inductively free arrangements are hereditarily divisionally free.
Proposition 6.8
There is an arrangement $\mathcal{A}$ which is not hereditarily inductively free but hereditarily divisionally free.

Proof. Let us again use a divisionally free, but not inductively free unitary reflection arrangement $\mathcal{A} = \mathcal{A}(G_{31})$ in the proof of Theorem 1.6. By Theorem 1.2 in [10], every $\mathcal{A}^X$ is inductively free for $X \in L_3(\mathcal{A})$. Since $\mathcal{A}$ is a 4-arrangement and $\mathcal{IF} \subseteq \mathcal{DF}$ by Theorem 5.4, the proof is completed.

Remark 6.9
Apparently multiplicatively free arrangements are generalizations of recursively free arrangements. The former contains the latter. By the result in [11], there is a free arrangement which is not multiplicatively free. Hence there is a free arrangement which is not either divisionally nor multiplicatively free.

By Theorem 6.5 if there exists a counterexample to Conjecture 5.1 in $\mathcal{RF}$, then that has to be at the valley. So we may pose a problem.

Problem 6.10
Does the freeness of a recursively free arrangement at the valley depend only on its combinatorics?

Even if Problem 6.10 is settled positively, Conjecture 5.1 seems to be still difficult. However, Problem 6.10 is a natural one based on the results in this article. Based on several examples, we pose the following conjecture:

Conjecture 6.11
(1) $\mathcal{RF}^* \subsetneq \mathcal{DF}$.
(2) $\mathcal{RF} \subsetneq \mathcal{MF}$.

To settle Conjecture 6.11 (1) positively, for example, recalling the proof of Theorem 1.6 it suffices to show that $\mathcal{A}(G_{31})$ is not recursively free, which is not yet known (see [10], Corollary 4.3). It seems natural to believe that Conjecture 6.11 (1) is true. On Conjecture 6.11 (2), it seems still difficult to check since there are few known examples of a free arrangement which is non-recursively free (see [11], [4]).

7 Application to Weyl arrangements

In this section assume that $\mathbb{K} = \mathbb{R}$ and consider a real irreducible crystallographic reflection arrangements and its deformations.
First let us recall a notation on root systems and related Weyl arrangements. Let $\Phi$ be a real irreducible crystallographic root system of rank $\ell$ with the Coxeter number $h$. Fix a positive system $\Phi^+$ of $\Phi$ and let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Phi^+$ the associated simple system. Let $W$ be the corresponding Weyl group and $H_{\alpha}$ the reflecting hyperplane with respect to the root $\alpha \in \Phi^+$ in $V = \mathbb{R}^\ell$. Then the Weyl arrangement $A_W$ of $\Phi^+$ is defined by

$$A_W := \{H_{\alpha} \mid \alpha \in \Phi^+\}.$$ 

Also, the $k$-extended Shi arrangement $\text{Shi}^k$ is defined by the equation

$$\text{Shi}^k := \{z = 0\} \bigcup \{H_{\alpha_j} \mid \alpha_j \in \Phi^+, -k+1 \leq j \leq k\},$$

where $z$ is a new coordinate added to the vector space spanned by $\alpha_1, \ldots, \alpha_\ell$, and $H_{\alpha_j} := \{\alpha = jz\}$ for $\alpha \in \Phi^+$ and $j \in \mathbb{Z}$. The Shi arrangement was introduced by J.-Y Shi in [18], and has been well-studied by several mathematicians. See [6] and [21] for example. It is proved by Yoshinaga in [21] that $\text{Shi}^k$ is free with exponents $(1, kh, \ldots, kh)$. We can show that Treao’s conjecture 5.1 holds true for $\text{Shi}^k$. Here we show that in fact $\text{Shi}^k$ is divisionally free, which implies Theorem 1.4 by Theorem 5.4.

**Theorem 7.1**

$\text{Shi}^k \in \mathcal{DF}.$

To prove Theorem 7.1 we need a theorem.

**Theorem 7.2**

Let $\mathbb{K}$ be an arbitrary field, $V = \mathbb{K}^\ell$ and $\mathcal{A}$ be an $\ell$-arrangement in $V$. Assume that there are distinct hyperplanes $H_1, \ldots, H_{\ell-1} \in \mathcal{A}$ such that,

1. $\mathcal{A}'_i := \mathcal{A} \setminus \{H_i\}$ is free with $\exp(\mathcal{A}'_i) = (1, d_1, d_2, \ldots, d_{i-1}, d_i-1, d_{i+1}, \ldots, d_{\ell-1})$ for $i = 1, \ldots, \ell - 1$.
2. $\mathcal{A}' := \mathcal{A} \setminus \{H_1, \ldots, H_{\ell-1}\}$ is free with $\exp(\mathcal{A}') = (1, d_1-1, d_2-1, \ldots, d_{\ell-1} - 1)$.

Then $\mathcal{A} \in \mathcal{DF}$. In particular, the freeness of $\mathcal{A}$ depends only on the combinatorics $L(\mathcal{A})$.

**Proof.** Let $\mathcal{A}_\ell := \mathcal{A}$ and define $\mathcal{A}_i := \mathcal{A}_{i+1}^{H_i}$ ($i = 2, \ldots, \ell - 1$). We show that $\mathcal{A}_\ell \supset \mathcal{A}_{\ell-1} \supset \cdots \supset \mathcal{A}_2$ is a divisional filtration. Let $\alpha_j := \alpha_{H_j}$ for $j = 1, \ldots, \ell - 1$.

For that purpose, at first, we show that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_1, d_2, \ldots, d_{\ell-1})$ with a special basis. By the assumption, there is a derivation $\theta_i \in D(\mathcal{A}'_i)$ ($i =$
1, . . . , ℓ − 1) of degree d, − 1 such that αiθi ∈ D(A) and that θE, θj are a part of a basis for D(A′) for j = 1, . . . , ℓ − 1. We show that θE, θ1, . . . , θℓ−1 form a basis for D(A′). We may assume that 1 ≤ deg θ1 ≤ · · · ≤ deg θℓ−1. Note that 1 + \sum_{i=1}^{\ell-1} \deg θi = |A′| and deg θj = d, − 1 for j = 1, . . . , ℓ − 1. Hence by Proposition 4.42 in [15], it suffices to show that \langle θE, θ1, . . . , θℓ−1\rangle S ̸∋ θi for i = 1, . . . , ℓ − 1. If θ1 ∈ SθE, then it contradicts the fact that θE, θ1 form a part of basis for D(A′). Now assume that θi = fθE + \sum_{j=1}^{i-1} f_jθ_j (f, f_j ∈ S).

By the choice of θi, θk(αj) ∈ Sαj if and only if k ̸= j. Since θE(α) = α for any α ∈ V ∗, it holds that (fθE + \sum_{j=1}^{i-1} f_jθ_j)(αi) ∈ Sαi, but θi(αi) ̸∈ Sαi, which is a contradiction.

Hence D(A′) has a basis θE, θ1, . . . , θℓ−1 with the properties above. Moreover, the construction implies that θE, α1θ1, . . . , αℓ−1θℓ−1 form a basis for D(A). By using these bases, we show that Aℓ ⊃ · · · ⊃ A2 is a divisional filtration.

First, we show that Aℓ−1 is free with exp(Aℓ−1) = (1, d1, d2, . . . , dℓ−1). Let π : D(Aℓ) → D(Aℓ−1) be the restriction map defined by π(θ)(f) := θ(f) modulo αℓ−1. By Terao’s restriction theorem [2.4] the derivations

π(θE), π(α1θ1), . . . , π(αℓ−2θℓ−2)

form a free basis for D(Aℓ−1), hence exp(Aℓ−1) = (1, d1, d2, . . . , dℓ−2).

Second, the arrangement Aℓ−1 in Hℓ−1 = Rℓ−1, hyperplanes H1∩Hℓ−1, . . . , Hℓ−2∩Hℓ−1 ∈ Aℓ−1 and derivations π(θE), π(α1θ1), . . . , π(αℓ−2θℓ−2) satisfy the same assumptions as for Aℓ, H1, H2, . . . , Hℓ−1, θE, α1θ1, . . . , αℓ−1θℓ−1. For f ∈ S, let \overline{f} denote the image of f by the canonical surjection S → S/(αℓ−1). Since αi | θi, clearly \overline{αi} | π(θi) for i = 1, . . . , ℓ − 2. Also, we can show that αi = 0 (i = 1, . . . , ℓ − 2) define distinct hyperplanes in Aℓ. Assume not. Then we may assume that \overline{α2} = \overline{α3}. Then for Bℓ−1 := Aℓ−1 \ {H2∩Hℓ−1 = H3∩Hℓ−1}, the derivations

π(θE), π(α1θ1), π(θ2), π(θ3), π(α4θ4), . . . , π(αℓ−2θℓ−2)

form a basis for D(Bℓ−1). Since the sum of degrees of the derivations above is |Aℓ−1| − 2 = |Bℓ−1| − 1, this contradicts Saito’s criterion [2.9]. Hence H1 ∩ Hℓ−1, . . . , Hℓ−2∩Hℓ−1 are distinct. Hence these satisfy the same assumption in the previous paragraph.

Now apply the same arguments inductively to Aℓ−1, Aℓ−2, . . . , A3 with Theorem 2.5 to show that χ(A;i) | χ(Ai+1;i) for i = 2, . . . , ℓ − 1. Hence Theorem 1.3 shows that A ∈ DDF.

Proof of Theorem 7.1. First recall the result on simple-root basis in [5]. It asserts that, there are derivations φ1, . . . , φℓ ∈ D(Shi) of degree kh such
that \( \theta_E, \phi_1, \ldots, \phi_\ell \) form a homogeneous basis for \( D(\text{Shi}^k) \) and \( (\alpha_i - kz) \) | \( \phi_i \) \((i = 1, \ldots, \ell)\). We call this basis a simple-root basis minus (see [5] for details).

By the property of simple root basis, the Shi arrangement and hyperplanes \( \alpha_i - kz = 0 \) \((i = 1, \ldots, \ell)\) clearly satisfy the assumptions in Theorem 7.2, which implies that \( \text{Shi}^k \in \mathcal{DF} \).

If we use some specific basis for the derivation module, then we can show the following corollary similar to Theorem 7.2.

**Corollary 7.3**

Let \( \mathbb{K} \) be an arbitrary field, \( V = \mathbb{K}^\ell \) and \( \mathcal{A} \) be a free \( \ell \)-arrangement in \( V \) with \( \exp(\mathcal{A}) = (1, d, d_2, \ldots, d_{\ell - 1}) \). Assume that there are distinct hyperplanes \( H_2, \ldots, H_{\ell - 1} \in \mathcal{A} \) and derivations \( \theta_E, \varphi, \theta_2, \ldots, \theta_{\ell - 1} \) for \( D(\mathcal{A}) \) such that \( \deg \varphi = d \), \( \deg \theta_i = d_i \) \((i = 2, \ldots, \ell - 1)\) and \( \alpha_i := \alpha_{H_i} \) divides \( \theta_i \) for \( i = 2, \ldots, \ell - 1 \). Then \( \mathcal{A} \in \mathcal{DF} \). In particular, the freeness of \( \mathcal{A} \) depends only on the combinatorics \( L(\mathcal{A}) \).

**Proof.** The proof is the same as that of Theorem 7.2. \( \square \)

**Acknowledgements.** The author is partially supported by by JSPS Grants-in-Aid for Young Scientists (B) No. 24740012.

**References**

[1] T. Abe, Chambers of 2-affine arrangements and freeness of 3-arrangements. *J. Alg. Combin.*, 38 (2013), no. 1, 65–78.

[2] T. Abe, Roots of characteristic polynomials and and intersection points of line arrangements. *J. Singularities*, 8 (2014), 100–117.

[3] T. Abe, M. Barakat, M. Cuntz, T. Hoge, H. Terao, The freeness of ideal subarrangements of Weyl arrangements. *arXiv:1304.8033*. *J. Eur. Math. Soc.*, to appear.

[4] T. Abe, M. Cuntz, H. Kawaioue and T. Nozawa, Non-recursive freeness and non-rigidity of plane arrangements. *arXiv:1411.3351*.

[5] T. Abe and H. Terao, Simple-root basis for Shi arrangements. *J. Algebra*, 422 (2015), 89-104

[6] T. Abe and H. Terao, The freeness of ideal-Shi arrangements and free paths in affine Weyl arrangements. *arXiv:1405.6379*
[7] T. Abe, H. Terao and M. Wakefield, The characteristic polynomial of a multiarrangement. *Adv. in Math.*, 215 (2007), 825–838.

[8] T. Abe, H. Terao and M. Wakefield, The Euler multiplicity and addition-deletion theorems for multiarrangements. *J. London Math. Soc.*, 77 (2008), no. 2, 335–348.

[9] T. Abe and M. Yoshinaga, Free arrangements and coefficients of characteristic polynomials. *Math. Z.*, 275 (2013), Issue 3, 911-919.

[10] N. Amend, T. Hoge and G. Röhrle, On inductively free restrictions of reflection arrangements. *J. Algebra*, 418 (2014), 197–212.

[11] M. Cuntz and T. Hoge, Free but not recursively free arrangements. *Proc. Amer. Math. Soc.*, 143 (2015), 35–40.

[12] P. H. Edelman and V. Reiner, A counterexample to Orlik’s conjecture. *Proc. Amer. Math. Soc.*, 118 (1993), 927–929.

[13] T. Hoge and G. Röhrle, On inductively free reflection arrangements. *J. Reine Angew. Math.*, to appear. [arXiv:1208.3131](https://arxiv.org/abs/1208.3131).

[14] P. Orlik and L. Solomon, Arrangements defined by unitary reflection groups. *Math. Ann.*, 261 (1982), 339–357.

[15] P. Orlik and H. Terao, *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften, 300. Springer-Verlag, Berlin, 1992.

[16] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo* 27 (1980), 265–291.

[17] G.C. Shephard and J.A. Todd, Finite unitary reflection groups, *Canadian J. Math.*, 6 (1954), 274–304.

[18] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups. Lecture Notes in Math., 1179, Springer-Verlag, 1986.

[19] H. Terao, Arrangements of hyperplanes and their freeness I, II. *J. Fac. Sci. Univ. Tokyo* 27 (1980), 293–320.

[20] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula. *Invent. math.* 63 (1981), 159–179.
[21] M. Yoshinaga, Characterization of a free arrangement and conjecture of Edelman and Reiner. *Invent. Math.* **157** (2004), no. 2, 449–454.

[22] M. Yoshinaga, On the freeness of 3-arrangements. *Bull. London Math. Soc.* **37** (2005), no. 1, 126–134.

[23] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. Singularities (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.