ON THE YUDOVICH’S TYPE SOLUTIONS FOR THE 2D BOUSSINESQ SYSTEM WITH THERMAL DIFFUSIVITY

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(Communicated by Pierre Germain)

Abstract. The goal of this paper is to study the two-dimensional inviscid Boussinesq equations with temperature-dependent thermal diffusivity. Firstly we establish the global existence theory and regularity estimates for this system with Yudovich’s type initial data. Then we investigate the vortex patch problem, and proving that the patch remains in Hölder class $C^{1+s} (0 < s < 1)$ for all the time.

1. Introduction. The general 2D incompressible Boussinesq system describes the influence of the convection-diffusion phenomenon in a viscous fluid as follows:

$$
\begin{align*}
\partial_t u + u \cdot \nabla u - \nabla \cdot (\nu(\theta) \nabla u) &= -\nabla p + \theta e_2, & x \in \mathbb{R}^2, & t > 0, \\
\partial_t \theta + u \cdot \nabla \theta - \nabla \cdot (\kappa(\theta) \nabla \theta) &= 0, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x), \theta(0, x) = \theta_0(x),
\end{align*}
$$

(1)

where $u$ denote the velocity field, $p, \theta$ are two scalar representing pressure and temperature respectively. $e_2 = (0, 1)$ denotes the vertical unit vector field and the forcing term of the first equation $\theta e_2$ indicating the buoyancy force due to the gravity. $\nu(\theta)$ and $\kappa(\theta)$ are the viscosity and thermal diffusivity depending on the temperature.

The Boussinesq system describe the influence of convection phenomenon in the dynamics of the ocean or of the atmosphere (see e.g. [33]). Mathematically, the global well-posedness for system (1) in the case $\nu, \kappa$ are positive constant has been solved in [4, 20]. But it the case $\nu = \kappa = 0$, it is still an unsolved problem that whether we can construct global unique solutions for some non-trivial $\theta_0$. So
this system has been extensively studied in the last few years due to the physical background and mathematical challenging.

For the constant viscosity case \( \nu(\theta) = \nu > 0, \kappa = 0 \), Chae in [5] and Hou, Li in [26] obtained the global well-posedness result for regular initial data. Later, Abidi and Hmidi studied this system in the Besov space in [1]. For lower regularity initial data, the global weak solution with finite energy has been constructed in [23] and has been proved to be unique later in [13]. On the other hand, for the constant thermal diffusive case \( \kappa(\theta) = \kappa > 0, \nu = 0 \), the global well-posed for regular initial data has been obtained by Chae in [5]. Later, Hmidi and Keraani extended this result to rough initial data in some Besov space in [24]. Danchin and the first author studied this system in [14] with Yudovich’s type data.

For the temperature dependent viscosity, system (1) has been studied in [36], and they obtained the global well-posedness result for smooth data with De Giorgi method. Later, Li and Xu studied the case \( \nu = 0, \kappa(\theta) > 0 \) in [28] with smooth initial data. In contrast, the case \( \nu(\theta) > 0, \kappa(\theta) = 0 \) still unsolved even for smooth initial data. Other interesting results corresponding to this model can be found in [27, 29, 35, 37].

In this paper, we investigate the 2D Boussinesq equations with only temperature-dependent thermal diffusion under the Yudovich’s type initial data, the system reads:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\nabla p + \theta e_2, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\partial_t \theta + u \cdot \nabla \theta - \nabla \cdot (\kappa(\theta) \nabla \theta) &= 0, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x), \quad \theta(0, x) = \theta_0(x).
\end{align*}
\]

(2)

Through out this paper, we assume that \( \kappa(\theta) \) satisfies

\[
C_0^{-1} \leq \kappa(\theta) \leq C_0, \quad \kappa'(\theta) \leq C_0,
\]

for some constant \( C_0 > 0 \).

Here we want to introduce an important quantity \( \omega \triangleq \partial_1 u^2 - \partial_2 u^1 \) called vorticity which measures how fast the fluid rotates and its control plays an important role in the literature we mentioned above. The Yudovich’s type initial data is \((u_0, \theta_0)\) in \( L^2 \) with bounded vorticity \( \omega_0 \triangleq \partial_1 u_0^2 - \partial_2 u_0^1 \). Taking \( \text{curl} \) operator to the first equation of (2) we can obtain the corresponding vorticity equation

\[
\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta.
\]

(3)

Let us denote by \( \psi(t, \cdot) \) the flow associated with the vector field \( u \), that is

\[
\begin{align*}
\frac{d}{dt} \psi(t, x) &= u(t, \psi(t, x)), \\
\psi(0, x) &= x.
\end{align*}
\]

(4)

The classical vortex patch problem is concerned about the following 2D incompressible Euler equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\nabla p, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x).
\end{align*}
\]

(5)

The associate vorticity \( \omega \) satisfies the following transport equation

\[
\partial_t \omega + u \cdot \nabla \omega = 0.
\]

(6)
If the initial vorticity
\[
\omega_0 = \partial_1 u_0^2 - \partial_2 u_0^1 = \chi_{D_0}(x) \triangleq \begin{cases} 1 & x \in D_0, \\ 0 & x \notin D_0, \end{cases} \tag{7}
\]
where \( D_0 \) is a connected bounded domain, \( \chi_{D_0} \) is the standard characteristic function of \( D_0 \). Then according to the properties of the flow \( \psi \), we have \( \omega(t) = \chi_{D_t} \) with \( D_t = \chi(D_0, t) \). A natural problem is that whether the regularity of the boundary \( \partial D_t \) preserving through the evolution of the flow. It has been proved by Chemin (see e.g. [6, 7]) that the regularity of the boundary can be persisted for all the time. Later, Gamblin and Saint-Raymond studied the vortex patch problem for 3D Euler equations in [18]. As for Boussinesq system, the vortex patch problem for inviscid Boussinesq equations has been discussed by Hassainia and Hmidi in [21]. Then Danchin and Zhang in [15], Gancedo and García-Juárez in [19] considered the temperature patch problem associate to the Boussinesq system with full Laplacian dissipation in velocity and no diffusion in temperature. Then for the stratified Euler system, which is system (2) with constant temperature diffusion, Hmidi and Zerguine studied the vortex patch problem in [25]. Many similar studies have been subsequently implemented by numerous authors for homogeneous (inhomogeneous) Navier-Stokes and other viscous (inviscid) flows, see for instance [3, 8, 9, 10, 11, 12, 16, 17, 22, 30, 31, 32] and the references therein.

In order to understand the striated regularity clearly, we need first to introduce some notations and definitions which will be used to describe the boundary regularity. Let \( X_0 \) be a vector field defined on \( D_0 \), \( X \) is the evolution of \( X_0 \) along the flow \( \psi \) defining as follows,
\[
X(t, x) \triangleq \partial X_0 \psi(t, \psi^{-1}(t, x)), \tag{8}
\]
where \( \partial X_0 f \triangleq X_0 \cdot \nabla f \) denoting the standard directional derivative.

Taking time derivative of (8), one can obtain \( X \) satisfies the following transport equation,
\[
\begin{aligned}
\partial_t X + u \cdot \nabla X &= \partial_X u, \\
X(0, x) &= X_0(x).
\end{aligned} \tag{9}
\]
It is not hard to check that \( \partial_X \) satisfies,
\[
[\partial_X, \partial_t + u \cdot \nabla] = 0, \tag{10}
\]
where \( [A, B] \triangleq AB - BA \) represents the standard commutator.

Applying \( \text{div} \) operator to (9) and combining with the divergence-free condition of \( u \), we obtain in addition
\[
\begin{aligned}
\partial_t \text{div} X + u \cdot \nabla \text{div} X &= 0, \\
\text{div} X(0, x) &= \text{div} X_0(x).
\end{aligned} \tag{11}
\]
Therefore, the divergence-free property can be preserved through the evolution.

The following definition of \( I(x) \) is needed in order to estimate the striated regularity and state our result, which can be found in [2, 7].

**Definition 1.1.** A family \( (X_\lambda)_{\lambda \in \Lambda} \) of vector fields over \( \mathbb{R}^2 \) is said to be non-degenerate whenever
\[
I(X) \triangleq \inf_{x \in \mathbb{R}^2} \sup_{\lambda \in \Lambda} |X_\lambda(x)| > 0.
\]
Let \( r \in (0,1) \) and \((X_\lambda)_{\lambda \in \Lambda}\) be a non-degenerate family of \(C^r\) vector fields over \(\mathbb{R}^2\). A bounded function \( f \) is said to be in the function space \(C^r_X\) if it satisfies
\[
\|f\|_{C^r_X} \triangleq \sup_{\lambda \in \Lambda} \left( \|f\|_{L^\infty} \|X_\lambda\|_{C^r} + \|\nabla \cdot (X_\lambda f)\|_{C^{r-1}} \right) < \infty.
\]

Then we give the definition about how to describe a boundary curve in \(C^s\) class.

**Definition 1.2.** Let \(0 < s < 1\) and \(\Omega\) be a bounded domain in \(\mathbb{R}^2\). We say that \(\Omega\) is of class \(C^{1+s}\) if there exists a compactly supported function \(f \in C^{1+s}(\mathbb{R}^2)\) and a neighborhood \(V\) of \(\partial \Omega\) such that
\[
\partial \Omega = f^{-1}(\{0\}) \cap V \quad \text{and} \quad \nabla f(x) \neq 0 \quad \forall \; x \in V.
\]

The main result of our paper can be stated as follows.

**Theorem 1.3.** Assume \(u_0 \in L^2\) be a divergence-free vector field, the corresponding vorticity \(\omega_0 \triangleq \partial_t u_0^2 - \partial_x u_1^2 \in L^2 \cap L^\infty\), \(\theta_0 \in L^2 \cap B^2_{p,r}\) with \(2 < p < \infty\), \(r > 1\) and \(\frac{1}{p} + \frac{1}{r} < 2\). Then there exists a small enough positive constant \(\varepsilon_0\) such that if
\[
\|\kappa(\cdot) - 1\|_{L^\infty(\mathbb{R})} \leq \varepsilon_0,
\]
the system (2) has a global solution \((u, \theta)\) satisfies
\[
u \in L^\infty([0,T];H^1), \quad \omega \in L^\infty([0,T];L^\infty), \quad \theta \in L^\infty([0,T];L^2) \cap L^\sigma([0,T];W^{2,p}),
\]
for any \(T > 0\) and some \(\sigma > 1\). If \(p\) and \(r\) satisfy \(\frac{1}{p} + \frac{1}{r} \leq 1\), then the solution is unique.

Furthermore, for any non-degenerate divergence-free vector field \(X_0 \in C^s\) such that \(\partial X_0 \omega_0 \in L^p\), there exists a unique global solution \(X \in L^\infty([0,T];C^s)\) to equation (9) and we have
\[
\partial X \omega \in L^\infty([0,T];L^p), \quad \nabla u \in L^1([0,T];L^\infty),
\]
Moreover, if we assume \(X_0 \in W^{1,p}\) additionally, then
\[
X \in L^\infty([0,T];W^{1,p}).
\]

**Remark 1.** Noticing that with Yudovich’s type data, \(\partial X \omega\) is defined as
\[
\partial X \omega \triangleq \text{div}(\omega X) - \omega \text{div} X,
\]
in the sense of distribution. For the sake of simplicity, we assume \(\text{div} X_0 = 0\) in Theorem 1.3, which can be preserved for all the time. As for the general case \((\text{div} X_0 \neq 0)\), we just need to propagate the regularity of \(\text{div} X\), which is easy to get because \(\text{div} X\) satisfies a transport type equation (11). We will explain the details in Remark 3.

**Remark 2.** In the process of proving \(\nabla u\) in \(L^1([0,T];L^\infty)\), we only need \(\partial X_0 \omega_0 \in C^{s-1}\), which leads to the estimate of \(\partial X \omega\) in \(C^{s-1}\). This is enough to obtain the Lipschitz bound of the velocity \(u\). The reason we choose \(L^p\) space here is because \(L^p \hookrightarrow C^{s-1}\) and the \(L^p\) norm of \(\partial X \omega\) will be used in the proof of \(X \in L^\infty([0,T];W^{1,p})\).

Theorem 1.3 can be used to deal with the following vortex patch problem directly. Let \(\omega_0\) defined as (7), the solution \(\omega(t,x) = \omega^1(t,x) + \omega^2(t,x)\) where \(\omega^1\) is the solution of the system
\[
\begin{align*}
\partial_t \omega^1 + u \cdot \nabla \omega^1 &= 0, \\
\omega^1(0,x) &= \omega_0(x),
\end{align*}
\]
and $\omega^2$ is the solution of the system
\[
\begin{aligned}
\partial_t \omega^2 + u \cdot \nabla \omega^2 &= \partial_1 \theta, \\
\omega^2(0, x) &= 0.
\end{aligned}
\] (14)

Then the main result can be stated as follows.

**Corollary 1.** Assume $u_0$ be a divergence free vector field with vorticity $\omega_0$ defined as in (7) and $D_0$ be a connected bounded domain with its boundary $\partial D_0$ in Hölder class $C^{1+s}$ ($0 < s < 1$), $\theta_0$ defined as Theorem 1.3. Then system (2) exists a unique global solution satisfies the properties shows in Theorem 1.3. Moreover, the solution of systems (13) and (14) satisfying
\[
\omega^1 = \chi_{D_t}, \quad \partial_X \omega^2 \in L^\infty([0, T]; L^p),
\]
with $D_t \triangleq \psi(D_0, t)$ and the boundary of the domain remains in the class $C^{1+s}$.

The rest of this paper is divided into four sections and an appendix. Some tool lemmas will be given in section 2. In the third section, we prove the well-posedness result for system (2) which shows the proof of results of the first part in Theorem 1.3. Section 4 is devoted to give some estimates of the striated regularity which complete the proof the Theorem 1.3. The last section presents the proof of Corollary 1, which solves the corresponding vortex patch problem for system (2). In the appendix, we provide the description of the Littlewood-Paley decomposition, the Besov space and some related facts used in the previous sections.

### 2. Preparations

In this section, we will give some lemmas which will be used in the next several sections. Throughout this paper, $C$ stands for some real positive constant which may be different in each occurrence. $C(t)$ is also a constant which depending on $t$ and the initial data.

Noticing that if $u$ is a divergence-free vector field in $\mathbb{R}^2$, then there exists a stream function $\psi$ such that $u = \nabla^\perp \psi$. Then we can obtain that the velocity $u$ can be recovered from the corresponding vorticity $\omega$ by means of the following Biot-Savart law
\[
u = \nabla^\perp \Delta^{-1} \omega.
\] (15)

Combining the classical Calderón-Zygmund estimate with (15), it can lead to the following lemma (see [7] for details).

**Lemma 2.1.** For any smooth divergence-free vector field $u$ with its vorticity $\omega \in L^p$ and $p \in (1, \infty)$, there exists a constant $C$ such that
\[
\|\nabla u\|_{L^p} \leq C \frac{p^2}{p-1} \|\omega\|_{L^p}.
\] (16)

Then we present the maximum regularity estimate for heat semi-group, which play an important role in the regularity estimate for $\theta$ of system (2). (see e.g. Lem. 7.3 in [34] for the proof.)

**Lemma 2.2.** Let $1 < p, r < \infty$, $f(t, x) \in L^r((0, t); L^p(\mathbb{R}^d))$ and $A$ be an operator satisfies
\[
A(f) = \int_0^t \nabla^2 e^{(t-\tau)\Delta} f(\tau, \cdot) \ d\tau.
\]

Then we have
\[
\|A(f)\|_{L^r((0, t); L^p(\mathbb{R}^d))} \leq C \|f\|_{L^r((0, t); L^p(\mathbb{R}^d))}.
\]
The next lemma shows the Hölder estimate for transport equation, which is useful in the estimate of the striated regularity. The proof can be found in [7].

**Lemma 2.3.** Let \( u \) be a smooth divergence-free vector field, \( r \in (-1, 1) \). Consider two functions \( f \in L^\infty_\text{loc}(\mathbb{R}; C^r) \) and \( g \in L^1_\text{loc}(\mathbb{R}; C^r) \) satisfy the transport equation

\[
\partial_t f + u \cdot \nabla f = g.
\]

Then we have

\[
\|f(t)\|_{C^r} \leq C\|f(0)\|_{C^r} e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} + C\int_0^t \|g(\tau)\|_{C^r} e^{\int_\tau^t \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau,
\]

and the constant \( C \) depends only on \( r \).

Next we give lemma which showing a logarithmic inequality which can be found in [2, 7]. This inequality plays an important role in the proof of the vortex patch problem for Euler equations.

**Lemma 2.4.** Let \( r \in (0, 1) \) and \( (X_\lambda)_{\lambda \in \Lambda} \) be a non-degenerate family of \( C^r \) vector fields over \( \mathbb{R}^2 \). Let \( u \) be a divergence-free vector field over \( \mathbb{R}^2 \) with vorticity \( \omega \in C^r_X \). Assume in addition that \( u \in L^q \) for some \( q \in [1, +\infty] \) or that \( \nabla u \in L^p \) for some finite \( p \). Then there exists a constant \( C \) depending on \( p \) and \( r \) such that

\[
\|\nabla u\|_{L^\infty} \leq C \left( \min(\|u\|_{L^p}, \|\omega\|_{L^p}) + \|\omega\|_{L^\infty} \log \left( e + \frac{\|\nabla \omega\|_{L^\infty}}{\|\omega\|_{L^\infty}} \right) \right).
\]  

Finally we give a commutator estimate for tangential derivatives and Riesz transform, which will be used in the regularity estimate for \( X \). The proof can be found in the Appendix of [32].

**Lemma 2.5.** Let \( p, r \in (1, \infty) \) and \( q \in (1, \infty) \) satisfying \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \). Let \( X \in \dot{W}^{1,q}(\mathbb{R}^2) \) with \( \nabla \cdot X = 0 \), \( g \in L^q(\mathbb{R}^2) \), \( R_i = \frac{\partial}{\partial x_i} (-\Delta)^{\frac{\gamma}{2}} \) be the Riesz transform. Then one has

\[
\|\partial X, R_i R_j g\|_{L^r} \leq C \|\nabla X\|_{L^p} \|g\|_{L^q}.
\]

3. **Well-posedness and global regularity estimate for System (2).** In this section, we will give some a priori estimates for \((u, \theta)\), then obtain the global existence and uniqueness results for system (2).

Firstly, basic \( L^2 \) energy estimate for \( \theta \) shows that

\[
\|\theta(t)\|_{L^2}^2 + 2C_0^{-1} \int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 \, d\tau \leq \|\theta_0\|_{L^2}^2, \quad \text{for all } t \in \mathbb{R}_+.
\]  

Then using the estimate (18) and the divergence free condition of \( u \), one can obtain the \( L^2 \) estimate for \( u \) and \( \omega \) that,

\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2} \, d\tau \leq \|u_0\|_{L^2} + t \|\theta_0\|_{L^2},
\]

and

\[
\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} + \int_0^t \|\nabla \theta(\tau)\|_{L^2} \, d\tau \leq \|\omega_0\|_{L^2} + C(t) \|\theta_0\|_{L^2}.
\]

Similarly, \( L^q \) estimate of \( \omega \) can be obtained by the same way that,

\[
\|\omega(t)\|_{L^q} \leq \|\omega_0\|_{L^q} + \int_0^t \|\nabla \theta(\tau)\|_{L^q} \, d\tau.
\]

So we need to do the \( L^q \) estimate for \( \nabla \theta \) firstly. The following proposition alerting the \( W^{2,p} \) estimate for \( \theta \).
Proposition 1. Let \((u_0, \theta_0)\) satisfies the assumptions in Theorem 1.3, then for some \(\sigma \geq 1\), we have
\[
\theta \in L^\sigma((0, t); W^{2,p}) \quad \text{for any } t > 0.
\]

Proof. Multiplying the second equation of system (2) by \(|\theta|^{q-2}\theta\) \((2 < q < \infty)\) and integrating over \(\mathbb{R}^2\) with respect to \(x\), according to the divergence free condition of \(u\), we have
\[
\frac{1}{q} \frac{d}{dt} \|\theta(t)\|^q_{L^q} + C_0^{-1}(q - 1) \int_{\mathbb{R}^2} |\nabla \theta|^2 |\theta|^{q-2} \, dx = 0.
\]
Then integrating for time variable, we obtain
\[
\|\theta(t)\|_{L^q} \leq \|\theta_0\|_{L^q}, \quad \text{for any } t > 0. \tag{22}
\]
In order to obtain the first and second order derivative estimates for \(\theta\), we need first rewrite the second equation of system (2) as
\[
\partial_t \theta - \Delta \theta = -\nabla \cdot (u \theta) + \nabla \cdot ((\kappa(\theta) - 1)\nabla \theta) = -u \cdot \nabla \theta + \kappa'(\theta)(\nabla \theta)^2 + (\kappa(\theta) - 1)\Delta \theta. \tag{23}
\]
Then \(\theta\) can be represented by
\[
\theta(t) = e^{t\Delta} \theta_0 + \int_0^t e^{(t - \tau)\Delta} (-\nabla \cdot (u \theta) + \nabla \cdot ((\kappa(\theta) - 1)\nabla \theta)) \, d\tau, \tag{24}
\]
where \((e^{t\Delta})_{t > 0}\) stands for the heat semi-group. Applying \(\nabla\) to (24) and taking \(L^p_t(L^p_x)\) \((1 \leq p)\) norm, one can deduce
\[
\|\nabla \theta\|_{L^p_t(L^p_x)} \leq \|\nabla e^{t\Delta} \theta_0\|_{L^p_t(L^p_x)} + \left\| \int_0^t \nabla e^{(t - \tau)\Delta} \nabla \cdot (u \theta) \, d\tau \right\|_{L^p_t(L^p_x)}
\]
\[
+ \left\| \int_0^t \nabla e^{(t - \tau)\Delta} \nabla \cdot ((\kappa(\theta) - 1)\nabla \theta) \, d\tau \right\|_{L^p_t(L^p_x)} \triangleq F_0 + F_1 + F_2.
\]
For \(F_0\), according to Lemma A.2, we have
\[
F_0 = \|\nabla e^{t\Delta} \theta_0\|_{L^p_t(L^p_x)} \leq C\|\nabla \theta_0\|_{B^\sigma_{p,2}} \leq C\|\theta_0\|_{B^{1-\frac{1}{2}\sigma}_{p,2}}. \tag{26}
\]
For the case \(\frac{1}{p} + \frac{1}{r} \leq 1\), by taking \(\sigma = r\) and making use of Proposition 4, we obtain
\[
\|\theta_0\|_{B^{1-\frac{1}{2}\sigma}_{p,2}} \leq C\|\theta_0\|_{B^{1-\frac{1}{2}\sigma}_{p,2}} \leq C\|\theta_0\|_{B^{2-\frac{1}{2}}_{p,r}}.
\]
For the case \(\frac{1}{p} + \frac{1}{r} > 1\). Taking \(\frac{1}{\sigma} = \frac{1}{p} + \frac{2}{r} - 1\) (which implies \(\sigma > 1\)) and by Proposition 4,
\[
\|\theta_0\|_{B^{1-\frac{1}{2}\sigma}_{p,2}} = \|\theta_0\|_{B^{2-\frac{1}{2}}_{p,r}} \leq C\|\theta_0\|_{B^{2-\frac{1}{2}}_{p,r}}.
\]
Then we estimate \(F_1\). Making use of Lemma 2.2 and Hölder inequality, we can bound \(F_1\) by
\[
F_1 \leq \left\| \int_0^t \nabla^2 e^{(t - \tau)\Delta} (u \theta) \, d\tau \right\|_{L^p_t(L^p_x)} \leq \|u\|_{L^p_t(L^p_x)} \|\theta\|_{L^p_t(L^p_x)}.
\]
For \( \|u\|_{L^p_t(L^\infty_x)} \), by interpolation and Lemma 15,
\[
\|u\|_{L^p_t(L^\infty_x)} \leq C \|u\|_{L^2_t(L^\infty_x)}^{\frac{p-1}{p}} \|\nabla u\|_{L^p_t(L^2_x)}^{\frac{1}{p}} \leq C \|u\|_{L^2_t(L^\infty_x)}^{\frac{p-1}{p}} \|\omega\|_{L^p_t(L^2_x)}^{\frac{1}{p}} \text{ for any } 2 < p < \infty.
\]
Combining with (19) and (21), we obtain
\[
\|u\|_{L^p_t(L^\infty_x)} \leq C(t) \left( \|\omega_0\|_{L^p_t(L^2_x)}^{\frac{p}{p-1}} + \|\partial_t \theta\|_{L^1_t(L^2_x)}^{\frac{p}{p-1}} \right). \tag{28}
\]
Inserting (28) into (27), we deduce that
\[
F_1 \leq C(t)(1 + \|\nabla \theta\|_{L^2_t(L^\infty_x)}^{p-1}) \|\theta\|_{L^p_t(L^\infty_x)}^p. \tag{29}
\]
Because \( \theta_0 \in B_{p,r}^{\frac{p}{r}-2} \), by Proposition 4 in the Appendix, it is not hard to see \( \theta_0 \in L^{2p} \). Combining with the estimate (22), we have \( \|\theta\|_{L^p_t(L^\infty_x)} \) is bounded. Then making use of Young’s inequality, we get \( F_1 \) is bounded by
\[
F_1 \leq C(t) + \frac{1}{4} \|\nabla \theta\|_{L^2_t(L^p_x)}. \tag{30}
\]
For \( F_2 \), also by Lemma 2.2,
\[
F_2 \leq \left\| \int_0^t \nabla^2 e(t-\tau) \Delta ((\kappa(\theta) - 1) \nabla \theta)) d\tau \right\|_{L^2_t(L^p_x)} \leq C \|((\kappa(\theta) - 1) \nabla \theta) \|_{L^2_t(L^2_x)} \leq C \|\kappa(\theta) - 1\|_{L^\infty} \|\nabla \theta\|_{L^2_t(L^2_x)}. \tag{31}
\]
According to (12), we obtain
\[
F_2 \leq \frac{1}{4} \|\nabla \theta\|_{L^2_t(L^p_x)}. \tag{32}
\]
Inserting the estimates (26), (30) and (32) into (25), one can deduce
\[
\|\nabla \theta\|_{L^2_t(L^p_x)} \leq C(t). \tag{33}
\]
Then we give the estimate of \( \nabla^2 \theta \). According to (23),
\[
\theta(t) = e^{t \Delta} \theta_0 + \int_0^t e^{(t-\tau) \Delta} \left( -u \cdot \nabla \theta + \kappa'(\theta)(\nabla \theta)^2 + (\kappa(\theta) - 1) \Delta \theta \right) d\tau. \tag{34}
\]
Then we have
\[
\|\nabla^2 \theta\|_{L^p_t(L^p_x)} \leq \|\nabla^2 e^{t \Delta} \theta_0\|_{L^p_t(L^p_x)} + \left\| \int_0^t \nabla^2 e^{(t-\tau) \Delta} (u \cdot \nabla \theta) \ d\tau \right\|_{L^p_t(L^p_x)} \tag{35}
\]
\[
\left. + \left\| \int_0^t \nabla^2 e^{(t-\tau) \Delta} \left( (\kappa'(\theta)(\nabla \theta)^2) \right) d\tau \right\|_{L^p_t(L^p_x)} \right. \]
\[
\left. + \left\| \int_0^t \nabla^2 e^{(t-\tau) \Delta} \left( (\kappa(\theta) - 1) \Delta \theta \right) d\tau \right\|_{L^p_t(L^p_x)} \right. \]
\[
\triangleq G_0 + G_1 + G_2 + G_3.
\]
According to Lemma A.2 in the Appendix,
\[
G_0 = \|\nabla^2 e^{t \Delta} \theta_0\|_{L^p_t(L^p_x)} \leq C \|\nabla^2 \theta_0\|_{B_{p,s}^{\frac{3}{2}}} \leq C \|\theta_0\|_{B_{p,s}^{\frac{2}{p} - \frac{3}{2}}}. \tag{36}
\]
In the case \( \frac{1}{p} + \frac{1}{r} < 1 \), we take \( \sigma = r \), so we have
\[
G_0 \leq C \|\theta_0\|_{B_{p,r}^{\frac{2}{p} - \frac{3}{2}}}. \]
For $\frac{1}{p} + \frac{1}{q} > 1$. Noticing that $\frac{1}{p} = \frac{1}{p} + \frac{1}{q} - 1$, that is $1 < \sigma < r$. Then by Proposition 4,

$$G_0 \leq C\|\theta_0\|_{B_{p,r}^{\frac{1}{2}}} \leq C\|\theta_0\|_{B_{p,r}^{\frac{1}{2}}}.$$

Next we estimate $G_1$, by Lemma 2.2 and Hölder inequality and estimate

$$G_1 = \int_0^t \|\nabla e^{\Delta t} (u \cdot \nabla \theta)\|_{L^\infty_t(L^s_x)} \leq \|u \cdot \nabla \theta\|_{L^\infty_t(L^s_x)} \leq C\|u\|_{L^s_t(L^2_x)}\|\nabla \theta\|_{L^s_t(L^2_x)}.$$

(37)

By interpolation

$$\|u\|_{L^2_t(L^2_x)} \leq C\|u\|_{L^2_t(L^2_x)}\|\nabla u\|_{L^\infty_t(L^2_x)} \leq C\|u\|_{L^2_t(L^2_x)}\|\omega\|_{L^1_t(L^2_x)},$$

and combining with the estimate (19), (20), (33) and (37), we can deduce

$$G_1 \leq C\|u\|_{L^2_t(L^2_x)}\|\omega\|_{L^\infty_t(L^2_x)} \|\nabla \theta\|_{L^2_t(L^2_x)} \leq C(t).$$

(38)

Similarly,

$$G_2 = \int_0^t \|\nabla e^{\Delta t} (\kappa'(|\theta|) (\nabla \theta)^2)\|_{L^\infty_t(L^s_x)} \leq C\|\kappa'(|\theta|)\|_{L^\infty} \|\nabla \theta\|_{L^s_t(L^2_x)} \leq C(t).$$

(39)

$$G_3 = \int_0^t \|\nabla e^{\Delta t} (\kappa(\theta) - 1) \Delta \theta\|_{L^\infty_t(L^s_x)} \leq C\|\kappa(\theta) - 1\|_{L^\infty} \|\nabla^2 \theta\|_{L^s_t(L^2_x)}.$$

(40)

Inserting the estimates (36)-(40) to (35), we deduce that

$$\|\nabla^2 \theta\|_{L^s_t(L^s_x)} \leq C(t),$$

which complete the proof of this proposition.

Then according to Sobolev embedding and estimate (21), we can obtain

$$\nabla \theta \in L^1((0, t); L^\infty), \quad \omega \in L^\infty((0, t); L^\infty).$$

(41)

With these regularity estimates of $(u, \theta)$, it is enough to derive the existence and uniqueness results for system (2). For the proof of the existence, we make use of the Friedrichs method. First we define the spectral cut-off as follows:

$$\hat{J}_N f(\xi) = \chi_{B(0, N)}(\xi) \hat{f}(\xi),$$

where $N > 0$, $B(0, N) = \{\xi \in \mathbb{R}^2 : |\xi| \leq N\}$, $\chi_{B(0, N)}$ is the characteristic function on $B(0, N)$. We define

$$L^2_N = \{f \in L^2(\mathbb{R}^2) : \text{supp } f \subset B(0, N)\}.$$

Let $\mathcal{P}$ denote the Leray projector over divergence free vector fields. Now we consider the following approximate system

$$\begin{cases}
\partial_t u^N + \mathcal{P} J_N (\mathcal{P} J_N u^N \cdot \nabla) \mathcal{P} J_N u^N = \mathcal{P} (J_N \theta^N e_2),

\partial_t \theta^N + \mathcal{P} J_N (\mathcal{P} J_N u^N \cdot \nabla \theta^N) - J_N \nabla \cdot (\kappa(J_N \theta^N) \nabla J_N \theta^N) = 0.
\end{cases}$$

(42)

with smooth initial data

$$u^N(0, x) = J_N u_0(x), \quad \theta^N(0, x) = J_N \theta_0(x).$$

From the Cauchy-Lipschitz Theorem, we can get a unique smooth solution $(u^N, \theta^N)$ in $C^1([0, T^*); L^2_N)$. Due to $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{J}_N^2 = \mathcal{J}_N$ and $\mathcal{P} \mathcal{J}_N = \mathcal{J}_N \mathcal{P}$, we can discover
that \((\mathcal{P}u^N, \theta^N)\) and \((\mathcal{J}_Nu^N, \mathcal{J}_N\theta^N)\) are also solutions to the approximate system (42) with the same initial condition. Thanks to the uniqueness, we deduce

\[
\mathcal{P}u^N = u^N, \quad \mathcal{J}_Nu^N = u^N, \quad \mathcal{J}_N\theta^N = \theta^N.
\]

Thus the approximate system (42) reduces to

\[
\begin{cases}
\partial_t u^N + \mathcal{P}(\mathcal{J}_N(u^N \cdot \nabla u^N)) = \mathcal{P}(\theta^N e_2), \\
\partial_t \theta^N + \mathcal{J}_N(u^N \cdot \nabla \theta^N) - \mathcal{J}_N \nabla \cdot (\kappa(\theta^N) \nabla \theta^N) = 0.
\end{cases}
\tag{43}
\]

According to the previous a priori estimates, we have for any \(T > 0\),

- \(\{u^N\}_{N \in \mathbb{N}} \subset L^\infty([0, T]; L^2)\),
- \(\{\omega^N\}_{N \in \mathbb{N}} \subset L^\infty([0, T]; L^2 \cap L^\infty)\),
- \(\{\theta^N\}_{N \in \mathbb{N}} \subset L^\infty([0, T]; L^2(\mathcal{J}_1^1) \cap L^\infty([0, T]; W^{2, p}) \cap L^1([0, T]; W^{1, \infty})\),

and the bounds are uniformly in \(N\). According to the first equation of (43), it is not hard to verify that \(\partial_t u^N \in L^\infty([0, T]; L^2)\) uniformly in \(N\). Similarly, by the second equation of (43), \(\partial_t \theta^N \in L^2([0, T]; H^{-1})\).

Because \(L^2\) is (locally) compactly embedded in \(H^{-1}\). Then by the standard Aubin-Lions theorem and Cantor diagonal process, we can prove there exists a subsequence of \((u^N, \theta^N)_{N \in \mathbb{N}}\) strong convergence to its limit \((u, \theta)\) in \(L^\infty([0, T]; H^{-1})\). Moreover,

\[
u \in L^\infty([0, T]; L^2), \quad \omega \in L^\infty([0, T]; L^2 \cap L^\infty),
\]

\[
\theta \in L^\infty([0, T]; L^2) \cap L^\infty([0, T]; \dot{H}^1) \cap L^\infty([0, T]; W^{2, p}) \cap L^1([0, T]; W^{1, \infty}).
\]

Then with these results, it is enough to pass the limit in (43) to obtain the existence result for (2).

For the proof of uniqueness part, we make use of the Yudovich method. Let \((u_1, \theta_1, p_1)\) and \((u_2, \theta_2, p_2)\) are two solutions to system (2) with the same initial data. Denote \(\delta u \triangleq u_2 - u_1\), \(\delta \theta \triangleq \theta_2 - \theta_1\) and \(\delta p \triangleq p_2 - p_1\), then we can get the system for \((\delta u, \delta \theta)\) that

\[
\begin{cases}
\partial_t \delta u + u_2 \cdot \nabla \delta u = -\nabla \delta p - \delta u \cdot \nabla u_1 + \delta \theta e_2, \\
\partial_t \delta \theta + u_2 \cdot \nabla \delta \theta - \nabla \cdot (\kappa(\theta_2) \nabla \delta \theta) = -\delta u \cdot \nabla \theta_1 + \nabla \cdot ((\kappa(\theta_2) - \kappa(\theta_1)) \nabla \theta_1).
\end{cases}
\tag{44}
\]

Standard \(L^2\) estimate combined with Hölder inequality yields for all \(q \in [2, \infty)\),

\[
\frac{1}{2} \|\delta u(t)\|_{L^2}^2 \leq \|\nabla u_1\|_{L^q} \|\delta u\|_{L^{2q'}}^2 + \|\delta u\|_{L^2} \|\delta \theta\|_{L^2} \quad \text{with} \quad q' = \frac{q}{q-1}.
\]

Then by interpolation, this inequality can be written as:

\[
\frac{1}{2} \frac{d}{dt} \|\delta u(t)\|_{L^2}^2 \leq q \|\nabla u_1\|_{L^q} \|\delta u\|_{L^2} \|\delta u\|_{L^2} + \|\delta u\|_{L^2} \|\delta \theta\|_{L^2} \|\delta u\|_{L^2},
\tag{45}
\]

with

\[
\|\nabla u_1\|_{L^q} := \sup_{2 \leq q < \infty} \frac{\|\nabla u_1\|_{L^q}}{q}.
\]

Noticing that in (40) and (41), we have \(\omega_1 \in L^\infty([0, T]; L^2 \cap L^\infty)\), so the term \(\|\nabla u_1(t)\|_{L^q}\) is locally bounded. Of course, combining the fact that \(u_i \in L^\infty([0, T]; L^2)\) and \(\omega_i \in L^\infty([0, T]; L^\infty)\) for \(i = 1, 2\), implies that \(\delta u \in L^\infty([0, T]; L^\infty)\).
Next we deal with $\delta\theta$. Also from $L^2$ estimate,
\[
\frac{1}{2} \frac{d}{dt} \|\delta\theta(t)\|_{L^2}^2 + C_0^{-1} \|\nabla\delta\theta\|_{L^2}^2
\leq \|\nabla \theta_1\|_{L^\infty} \|\delta\theta\|_{L^2} \|\nabla u\|_{L^2} + \|\kappa(\theta_2) - \kappa\theta_1\|_{L^{2p'}} \|\nabla \theta_1\|_{L^2} \|\nabla\delta\theta\|_{L^2}
\leq \|\nabla \theta_1\|_{L^\infty} \|\delta\theta\|_{L^2} \|\nabla u\|_{L^2} + C \|\delta\theta\|_{L^{2p'}} \|\nabla \theta_1\|_{L^2} \|\nabla\delta\theta\|_{L^2},
\]
with $\frac{1}{p} + \frac{1}{p'} = 1$. Then by interpolation and Young’s inequality, we get
\[
\frac{1}{2} \frac{d}{dt} \|\delta\theta(t)\|_{L^2}^2 + C_0^{-1} \|\nabla\delta\theta\|_{L^2}^2
\leq \|\nabla \theta_1\|_{L^\infty} \|\delta\theta\|_{L^2} \|\nabla u\|_{L^2} + C \|\delta\theta\|_{L^{2p'}} \|\nabla \theta_1\|_{L^2} \|\nabla\delta\theta\|_{L^2}^{1 + \frac{1}{p}}
\leq \|\nabla \theta_1\|_{L^\infty} \|\delta\theta\|_{L^2} \|\nabla u\|_{L^2} + C \|\delta\theta\|_{L^2}^2 \|\nabla \theta_1\|_{L^2}^2 + \frac{C_0^{-1}}{2} \|\nabla\delta\theta\|_{L^2}^2.
\]
Thus we have
\[
\frac{d}{dt} \|\delta\theta(t)\|_{L^2}^2 + C_0^{-1} \|\nabla\delta\theta\|_{L^2}^2 \leq \|\nabla \theta_1\|_{L^\infty} \|\delta\theta\|_{L^2} \|\nabla u\|_{L^2} + C \|\delta\theta\|_{L^2}^2 \|\nabla \theta_1\|_{L^2}^2 + \frac{2\kappa}{2}. \tag{46}
\]
Let $\varepsilon$ be a small parameter (tend to 0). Denote
\[
X_\varepsilon(t) := \sqrt{\|\delta\theta(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \varepsilon^2}.
\]
Putting inequalities (45) and (46) together gives
\[
\frac{d}{dt} X_\varepsilon \leq q \|\nabla u_1\|_L \|\delta u\|_{L^\infty}^\frac{1}{2} X_\varepsilon^{1 - \frac{1}{q}} + C \left(1 + \|\nabla \theta_1\|_{L^\infty} + \|\nabla \theta_1\|_{L^{2p'}}^\frac{2\kappa}{2} \right) X_\varepsilon.
\]
Let $\gamma(t) := C \left(1 + \|\nabla \theta_1(t)\|_{L^\infty} + \|\nabla \theta_1\|_{L^{2p'}}^\frac{2\kappa}{2} \right)$. Recalling that from (41), $\|\nabla \theta_1(t)\|_{L^\infty}$ belongs to $L^1[0, T]$, and according to Proposition 1, $\nabla \theta_1 \in L^{2r}([0, T]; L^{2p})$ with $\frac{1}{p} + \frac{1}{r} \leq 1$. Noticing that $\frac{2p}{p-1} \leq 2r$, so these ensure that the function $\gamma$ is in $L^1[0, T]w$. Therefore, setting $Y_\varepsilon := e^{-\int_0^t \gamma(\tau)d\tau} X_\varepsilon$, the previous inequality is rewritten:
\[
\frac{2}{q} Y_\varepsilon^{\frac{q-1}{q}} \frac{d}{dt} Y_\varepsilon \leq C \|\nabla u_1\|_L \|\delta u\|_{L^\infty}^\frac{1}{q} e^{-\frac{1}{q} \int_0^t \gamma(\tau)d\tau}.
\]
Performing a time integration yields
\[
Y_\varepsilon(t) \leq \left(\varepsilon^\frac{q}{2} + C \int_0^t \|\nabla u_1\|_L \|\delta u\|_{L^\infty}^\frac{1}{2} d\tau \right)^\frac{2}{q}.
\]
Having $\varepsilon$ tend to 0, we end up with
\[
\|\delta\theta(t)\|_{L^2}^2 + \|\delta u(t)\|_{L^2}^2 \leq \|\delta u\|_{L^{2r}(L^\infty)}^2 \left(\int_0^t \|\nabla u_1\|_L d\tau \right)^q, \tag{47}
\]
for all $t > 0$.

Because $\|\nabla u_1(t)\|_L$ is locally bounded. Hence one may find a positive time $T^*$ such that
\[
\int_0^{T^*} \|\nabla u_1\|_L d\tau < \frac{1}{2}.
\]
Taking $q$ tend to infinity in (47) entails that $(\delta\theta, \delta u) \equiv 0$ on $[0, T^*)$. Then by standard connectivity argument, we can conclude the uniqueness for $[0, T]$ with all $T > 0$. 

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4. A priori estimates for the striated regularity. In this subsection, we will
give the estimates of tangential derivatives of ω and regularity estimates of X. The
first lemma gives $L^p \ (p \in [1, \infty])$ estimate of X.

**Lemma 4.1.** Let $r \in [1, \infty]$, $X_0 \in L^r$. Then the solution X of equation (9) satisfies
\[
\|X_0\|_{L^r} e^{-\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau} \leq \|X(t)\|_{L^r} \leq \|X_0\|_{L^r} e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau},
\]
for any $t > 0$.

**Proof.** Multiplying (9) by $|X|^{p-2}X$ (1 ≤ r < ∞) and integrating over $\mathbb{R}^2$ with
respect to x, by Hölder inequality, we obtain
\[
\frac{1}{r} \frac{d}{dt}\|X(t)\|_{L^r} \leq C\|\nabla u\|_{L^\infty}\|X\|_{L^r}.
\]
Then by Grönwall’s Lemma, we deduce the second inequality of (48). The first
equality can be obtained by the time reversibility. And taking $r \to \infty$ to obtain the
result for the case $r = \infty$.

Then we give a proposition which alerting the $C^s$ (0 < s < 1) estimate for X.
As a by-product, we can also obtain the Lipschitz information for the velocity u.

**Proposition 2.** Assume $(u_0, \theta_0)$ and $X_0$ satisfy the assumptions in Theorem 1.3,
then for any $t > 0$, we have the velocity u satisfies
\[
\nabla u \in L^1([0, t]; L^\infty).
\]
Moreover,
\[
X \in L^\infty([0, t]; C^s), \ \partial_X \omega \in L^\infty([0, t]; L^p).
\]

Before we prove this proposition, we need firstly give the $L^p$ estimate of $\partial_X \omega$.
Applying $\partial_X$ to the vorticity equation (3), according to (10), we get $\partial_X \omega$ satisfies
the following equation
\[
\partial_t \partial_X \omega + u \cdot \nabla \partial_X \omega = \partial_X (\partial_t \theta) = X \cdot \nabla \partial_t \theta.
\]

Multiplying the equation (51) by $|\partial_X \omega|^{p-2}\partial_X \omega$ (2 ≤ p < ∞), and integrating over
$\mathbb{R}^2$ with respect to x, because u satisfies the divergence-free condition, by Hölder
inequality,
\[
\frac{1}{p} \frac{d}{dt}\|\partial_X \omega(t)\|_{L^p}^p \leq \|X\|_{L^\infty}\|\partial_\theta \nabla \theta\|_{L^p}\|\partial_X \omega\|_{L^p}^{p-1}.
\]

Then integrating in time and combining with the results of Proposition 1 and
Lemma 4.1,
\[
\|\partial_X \omega(t)\|_{L^p} \leq \|\partial_X \omega_0\|_{L^p} + \int_0^t \|X(\tau)\|_{L^\infty}\|\nabla^2 \theta(\tau)\|_{L^p} d\tau
\]
\[
\leq \|\partial_X \omega_0\|_{L^p} + \|X\|_{L^\infty(L^p)} \int_0^t \|\nabla^2 \theta(\tau)\|_{L^p} d\tau + C(t) e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau},
\]
Then we give the proof of Proposition 2.
Proof of Proposition 2. Firstly, making use of Lemma 2.3 to (9), we can get the Hölder norm of $X$ satisfies that
\[
\|X(t)\|_{C^s} \leq C\|X_0\|_{C^s}e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} + C\int_0^t \|\partial_X u(\tau)\|_{C^s}e^{\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau
\]
\[
\leq C e^{\tilde{C} t} \|\nabla u(t)\|_{L^\infty} \, d\tau (\|X_0\|_{C^s} + \int_0^t \|\partial_X u(\tau)\|_{C^s}e^{\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau), \tag{53}
\]
where we can choose $\tilde{C} > 2$. In order to estimate Hölder norm of $\partial_X u$, we need the following estimate which proof can be found in [7, 2],
\[
\|\partial_X u\|_{C^s} \leq C(\|\nabla u\|_{L^\infty} \|X\|_{C^s} + \|\partial_X \omega\|_{C^{s-1}}) \tag{54}
\]
By Sobolev embedding $L^p \hookrightarrow C^{s-1} (1 - s = \frac{2}{p})$ and estimate (52), we obtain
\[
\|\partial_X \omega\|_{C^{s-1}} \leq C\|\partial_X \omega\|_{L^p} \leq C\|\partial_X \omega\|_{L^p} + C(t)e^{2 \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{55}
\]
Inserting (54) and (55) into (53), one can deduce that
\[
\|X(t)\|_{C^s} \leq C e^{\tilde{C} t} f_0 \|\nabla u(t)\|_{L^\infty} \, d\tau \left(\|X_0\|_{C^s} + \int_0^t (C(t)
\right.
\]
\[
\left.\ + \|\nabla u(\tau)\|_{L^\infty} \|X(\tau)\|_{C^s}e^{-\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau \right).
\]
Denoting
\[
F(t) \triangleq \|X(t)\|_{C^s}e^{-\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}.
\]
Then according to the above estimates, we obtain
\[
F(t) \leq CF(0) + \int_0^t C(t) \left(\|\nabla u(\tau)\|_{L^\infty} + 1\right)(F(\tau) + 1) \, d\tau.
\]
By Grönwall’s Lemma,
\[
F(t) \leq C(F(0) + 1)e^{\int_0^t C(t)(\|\nabla u(\tau)\|_{L^\infty} + 1) \, d\tau}.
\]
According to the definition of $F(t)$, we obtain the Hölder estimate of $X$ that,
\[
\|X(t)\|_{C^s} \leq C(t)e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{56}
\]
Recalling the logarithmic inequality in Lemma 2.4 that
\[
\|\nabla u\|_{L^\infty} \leq C \left(\|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log \left( e + \|\omega\|_{C^s_X} \right) \right), \tag{57}
\]
where $\|\omega\|_{C^s_X}$ is defined in Definition 1.1. Noticing that $\|\omega\|_{L^2(L^\infty)}$ has been proven to be boundedness in section 3. Then inserting the estimates (55), (56) into (57), it follows that
\[
\|\nabla u\|_{L^\infty} \leq C \left(1 + \log \left( e + C(t)e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty}} \right) \right) \leq C \left(1 + \int_0^t C(t)(1 + \|\nabla u(\tau)\|_{L^\infty}) \, d\tau \right).
\]
Making use of Grönwall’s Lemma, one can deduce
\[
\|\nabla u(t)\|_{L^\infty} \leq C(t), \forall t > 0. \tag{58}
\]
Combining the estimates (56) and (58), we can obtain the desired Hölder norm of $X$. Then inserting the estimate (58) into (52), we can conclude $\|\partial_t \omega\|_{L^p(L^r_X)}$ is bounded, which completes the proof of this proposition.

Remark 3. For the case $\text{div} X_0 \neq 0$, which implies $\text{div} X \neq 0$. (54) should be changed to

$$\|\partial_t u\|_{C^s} \leq C(\|\nabla u\|_{L^\infty} \|X\|_{C^s} + \|\text{div} (X \omega)\|_{C^{s-1}}).$$

The equation that $\text{div} (X \omega)$ satisfies is

$$\partial_t \text{div} (X \omega) + u \cdot \nabla \text{div} (X \omega) = \text{div}(X \partial_t \theta) = X \cdot \nabla \partial_t \theta + \text{div} X \partial_t \theta. \quad (59)$$

So compared with (51), we need to estimate $\text{div} \omega$ additionally. Because $\text{div} X$ satisfies a transport equation (11), using Lemma 2.3, we have

$$\|\text{div} X\|_{C^s} \leq C |\text{div} X_0|_{C^s} e^{C \int_0^t \|\nabla u(t)\|_{L^\infty}} \, dt.$$

Then insert this estimate in (52), one can deduce

$$\|\text{div} (X \omega)(t)\|_{L^p} \leq \|\text{div} (X_0 \omega_0)\|_{L^p} + C(t) e^{C \int_0^t \|\nabla u(t)\|_{L^\infty}} \, dt,$$

which share a similar structure with (52), so the other estimates are the same as Proposition 2.

Then we give a proposition about the $\dot{W}^{1,p}$ estimate for $X$.

Proposition 3. Let $(u_0, \theta_0)$ satisfies the assumptions in Theorem 1.3, $X_0 \in \dot{W}^{1,p}$, then we have

$$X \in L^\infty([0,t] ; \dot{W}^{1,p}) \text{ for any } t > 0. \quad (60)$$

Proof. Applying $\partial_i$ ($i = 1, 2$) to (9), we can obtain $\partial_i X$ satisfies the following equation,

$$\partial_t \partial_i X + u \cdot \nabla \partial_i X = -\partial_t u \cdot \nabla X + \partial_i X \cdot \nabla u + \partial_X (\partial_i u). \quad (61)$$

Multiplying by $|\partial_i X|^{p-2} \partial_i X$ and integrating over $\mathbb{R}^2$ with respect to $x$, according to the divergence-free condition of $u$ and Hölder inequality,

$$\frac{1}{p} \frac{d}{dt} \|\partial_i X(t)\|_{L^p}^p \leq \|\nabla u\|_{L^\infty} \|\nabla X\|_{L^p}^p + \int_{\mathbb{R}^2} \partial_X (\partial_i u) |\partial_i X|^{p-2} \partial_i X \, dx. \quad (62)$$

Noticing that by equation (15),

$$\partial_X (\partial_i u) = \partial_X (\partial_i \nabla^\perp \Delta^{-1} \omega) = |\partial_X, R_i R^\perp| \omega + R_i R^\perp (\partial_X \omega).$$

Then we can write

$$\int_{\mathbb{R}^2} \partial_X (\partial_i u) |\partial_i X|^{p-2} \partial_i X \, dx = \int_{\mathbb{R}^2} [\partial_X, R_i R^\perp] \omega |\partial_i X|^{p-2} \partial_i X \, dx + \int_{\mathbb{R}^2} R_i R^\perp (\partial_X \omega) |\partial_i X|^{p-2} \partial_i X \, dx \triangleq M_1 + M_2.$$

For $M_1$, by Hölder inequality,

$$M_1 \leq \|\partial_X, R_i R^\perp\|_{L^p} \|\nabla X\|_{L^p}^{p-1}.$$
Thus we have $M_1$ can be bounded by
\[ M_1 \leq \|\omega\|_{L^\infty} \|\nabla X\|_{L^p}^p. \]  
(63)

For $M_2$, by Hölder inequality and the boundedness of Riesz transform in $L^p$ ($1 < p < \infty$),
\[ M_2 \leq \|\partial X\omega\|_{L^p} \|\partial_i X\|_{L^p}^{p-1}. \]  
(64)

Inserting estimates (63) and (64) into (62),
\[ \frac{d}{dt} \|\nabla X(t)\|_{L^p} \leq (\|\nabla u\|_{L^\infty} + \|\omega\|_{L^\infty}) \|\nabla X\|_{L^p} + \|\partial X\omega\|_{L^p}. \]

According to the estimate of (41), Proposition 2 and then making use of Grönwall’s Lemma, we obtain
\[ \|\nabla X\|_{L^p} \leq C \text{ for any } t > 0, \ p \in [2, \infty), \]
which completes the proof of this proposition.

5. The vortex patch problem. In this section, we devote to prove Corollary 1, which solving the vortex patch problem. Because
\[ \omega_0 = \chi_{D_0}(x) \triangleq \begin{cases} 1 & x \in D_0, \\ 0 & x \notin D_0, \end{cases} \]
where $D_0$ is a connected bounded domain with $\partial D_0 \in C^{1+s}$ for $0 < s < 1$. Then according to Definition 1.2, there exist a real function $f_0 \in C^{1+s}$ and a neighborhood $V_0$ such that $\partial D_0 = V_0 \cap f^{-1}(0)$ and $\nabla f_0 \neq 0$ on $V_0$. Noticing that at time $t$, the boundary $\partial D_t = \psi(t, D_0)$ is the level set of the function $f(t, \cdot) = f_0(\psi^{-1}(t, \cdot))$ with $f$ being transported by the flow $\psi$:
\[ \begin{cases} \partial_t f + u \cdot \nabla f = 0, \\ f(0, x) = f_0(x). \end{cases} \]  
(65)

Setting the vector field $X \triangleq \nabla \perp f$ with initial data $X_0 \triangleq \nabla \perp f_0$, it is not hard to verifies that $X$ satisfying (8) and the corresponding system (9). Then we can parametrize $\partial D_0$ as
\[ \gamma_0 : \mathbb{S}^1 \to \partial D_0, \text{ via } \sigma \mapsto \gamma_0(\sigma), \]
with
\[ \begin{cases} \partial_\sigma \gamma_0 = X_0(\gamma_0(\sigma)), \ \forall \ \sigma \in \mathbb{S}^1, \\ \gamma_0(0) = x_0 \in \partial D_0. \end{cases} \]  
(66)

In order to conclude the proof of Corollary 1, we observe that a parametrization for $\partial D_t$ is given by $\gamma_t(\sigma) \triangleq \psi(\gamma_0(\sigma), t)$ and by differentiating with respect to the parameter $\sigma$, we get
\[ \begin{cases} \partial_\sigma \gamma_t(\sigma) = X(\gamma_t(\sigma)), \ \forall \ \sigma \in \mathbb{S}^1, \\ \gamma_t(0) = \psi(t, x_0) \in \partial D_t. \end{cases} \]  
(67)

According to Theorem 1.3, $X \in L^\infty([0, T]; C^s)$, thus $\gamma_t \in C^{1+s}(\mathbb{S}^1)$ for all $t \geq 0$. This completes the proof of Corollary 1.
A. **Appendix.** This appendix provides the definitions of Besov space and some related facts are used in the previous sections. Firstly we present the classical Littlewood-Paley theory in \( \mathbb{R}^d \) which plays an important role in the proof of our result. Let \( \chi \) be a smooth function support on the ball \( B \triangleq \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{3}{4} \} \) and \( \varphi \) be a smooth function support on the ring \( C \triangleq \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \) such that
\[
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.
\]
Then for every \( u \in S' \) (tempered distributions), we define the non-homogeneous Littlewood-Paley operators as follows,
\[
\Delta_q u = 0 \text{ for } q \leq -2, \quad \Delta_{-1} u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\hat{u}(\xi)),
\]
\[
\Delta_q u = \varphi(2^{-q}D)u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\hat{u}(\xi)), \quad \forall \; q \geq 0.
\]
Here \( \mathcal{F}(\cdot), (\cdot) \) represent the Fourier transform and \( \mathcal{F}^{-1}(\cdot) \) the inverse Fourier transform.

Meanwhile, we define the homogeneous dyadic blocks as
\[
\hat{\Delta}_q u = \varphi(2^{-q}D)u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\hat{u}(\xi)), \quad \forall \; q \in \mathbb{Z}.
\]

Next we state the definition of homogeneous and non-homogeneous Besov spaces through the dyadic decomposition, more details can be found in [2].

**Definition A.1.** For \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), the homogeneous Besov space \( \dot{B}^s_{p,r} \) and non-homogeneous Besov space \( B^s_{p,r} \) are defined by
\[
\dot{B}^s_{p,r} = \{ f \in S'_p \triangleq S' \setminus \mathcal{P} ; \| f \|_{\dot{B}^s_{p,r}} < \infty \}, \quad B^s_{p,r} = \{ f \in S' ; \| f \|_{B^s_{p,r}} < \infty \},
\]
where \( \mathcal{P} \) is the set of polynomials,
\[
\| f \|_{\dot{B}^s_{p,r}} = \begin{cases} 
\sum_{q \geq -1} (2^{qs} \| \Delta_q f \|_{L^p})^{\frac{1}{r}} & \text{for } r < \infty, \\
\sup_{q \in \mathbb{Z}} (2^{qs} \| \Delta_q f \|_{L^p}) & \text{for } r = \infty,
\end{cases}
\]
and
\[
\| f \|_{B^s_{p,r}} = \begin{cases} 
\sum_{q \geq -1} (2^{qs} \| \Delta_q f \|_{L^p})^{\frac{1}{r}} & \text{for } r < \infty, \\
\sup_{q \geq -1} (2^{qs} \| \Delta_q f \|_{L^p}) & \text{for } r = \infty.
\end{cases}
\]

The following proposition lists some useful equivalence and embedding relations.

**Proposition 4.** For any \( s \in \mathbb{R} \),
\[
B^s_{2,2}(\mathbb{R}^d) \simeq H^s(\mathbb{R}^d).
\]

For any \( s \in \mathbb{R} \), \( 1 < p < \infty \),
\[
B^s_{p,\min(p,2)} \hookrightarrow W^{s,p} \hookrightarrow B^s_{p,\max(p,2)}.
\]

For any \( s \in \mathbb{R} \), \( 1 \leq p_1 \leq p_2 \leq \infty \), \( 1 \leq r_1 \leq r_2 \leq \infty \),
\[
B^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow B^{s-d(\frac{1}{p_1} - \frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^d).
\]

The above properties also valid for homogeneous Besov space.

For Besov space with negative index, we have the following equivalent definition.
Lemma A.2. Let $s > 0$, $(p, r) \in [1, \infty]^2$. A constant $C$ exists such that
\[
C^{-1} \|f\|_{\dot{B}^{-s}_{p,r}} \leq \left\| \int t^{-s} \mathcal{L} f(t) \|_{L^r(\mathbb{R}^2, dt)} \right\|_{L^p(\mathbb{R}^2, dt)} \leq C \|f\|_{\dot{B}^{-s}_{p,r}}.
\]

Acknowledgments. The authors would like to express their great gratitude to the referee for the insightful comments and suggestions. M. Paicu is partially supported by the Agence Nationale de la Recherche, Project IFSMACS, grant ANR-15-CE40-0010. N. Zhu was partially supported by NSFC (No. 11771045, No. 11771043). Part of this work was done when N. Zhu was visiting Université de Bordeaux, he would like to appreciate the warm hospitality of Institut de Mathématiques de Bordeaux.

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Received April 2019; 1st revision December 2019; 2nd revision February 2020.

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