A GEOMETRIC CATEGORIFICATION OF TENSOR PRODUCTS OF $U_q(sl_2)$-MODULES

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Abstract. We give a purely geometric categorification of tensor products of finite-dimensional simple $U_q(sl_2)$-modules and $R$-matrices on them. The work is developed in the framework of category of perverse sheaves and the categorification theorems are understood as consequences of Deligne’s theory of weights.

1. Introduction

The term categorification in mathematics refers to the process of lifting set-theoretic concepts to the level of categories. For example, categorification of a module $M$ over an algebra $A$ means lifting the module $M$ to an additive or abelian category $C$ and, accordingly, lifting the algebra $A$ to a collection of endofunctors of $C$ and functor isomorphisms among them; the lifts are done in such a way that the Grothendieck group of $C$ recovers the module $M$ and the endofunctors and the isomorphisms among them recover the module structure of $M$ and the algebra structure of $A$.

Categorified theories have such advantages as reflecting explicitly the integrity and positivity of the algebraic structures involved and, more importantly, usually providing new insights into the background theory.

Among various known categorifications till now (cf. the review [KMS07]), algebraic approaches are playing the dominant role, partly because there are still lacking of systematic tools for geometric treatment. We will demonstrate here how the profound result in modern algebraic geometry, Deligne’s theory of weights [De80], may enter to change the situation.

In the present paper, we categorify tensor products of $U_q(sl_2)$-modules, as well as $R$-matrices on them. The former task is accomplished in Section 3.3 by using the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [BBD82], which is known to be one of the remarkable consequences of Deligne’s theory of weights. As another consequence of the weight theory, we introduce in Section 4.2 the notion of pure resolution of mixed complexes and establish a uniqueness theorem, then use them in Section 4.3 and 4.4 to categorify $R$-matrices. Thanks to these powerful tools, our categorification is able to be fulfilled in a very simple and elegant way.

Supported in part by the National Natural Science Foundation of China (NSFC).
The main part of the paper consists of Section 3 and Section 4. Further remarks on the motivations and expositions of this work will be given in the beginning of them.

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2. Preliminaries

The references for Section 2.1 are [Kas95], [Lu93] and the references for Section 2.2 are [BBD82], [Bor84], [KS90].

2.1. The algebra $U_A$. Throughout this paper, $A = \mathbb{Z}[q, q^{-1}]$ denotes the Laurent polynomial ring and we set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad \left[ \begin{array}{c} n \\ r \end{array} \right]_q = \prod_{t=1}^r \frac{[n-r+t]_q}{[t]_q}.$$
The quantum enveloping algebra $U = U_q(sl_2)$ is the $\mathbb{Q}(q)$-algebra defined by the generators $K, K^{-1}, E, F$ and the relations
\begin{align}
KK^{-1} &= K^{-1}K = 1,
KE &= q^2EK, \quadKF = q^{-2}FK, \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. 
\end{align}
(2.1)

It is a Hopf algebra with comultiplication
\begin{align}
\Delta K &= K \otimes K, \\
\Delta E &= E \otimes 1 + K \otimes E, \\
\Delta F &= F \otimes K^{-1} + 1 \otimes F, 
\end{align}
(2.2)
counit
\begin{align}
\varepsilon(K) &= 1, \quad \varepsilon(E) = \varepsilon(F) = 0, 
\end{align}
(2.3)
and antipode
\begin{align}
S(K) &= K^{-1}, \quad S(E) = -K^{-1}E, \quad S(F) = -FK. 
\end{align}
(2.4)

To emphasize the integrity of the finite-dimensional representations of $U$, we work on an alternative algebra $U_A$ which is defined as the $A$-subalgebra of $U$ generated by $K, K^{-1}, E^{(n)}, F^{(n)}, n \geq 0$ where
\begin{align}
E^{(n)} &= \frac{E^n}{[n]_q!}, \quad F^{(n)} = \frac{F^n}{[n]_q!}.
\end{align}
(2.5)

For every integer $d \geq 0$, there is a simple $U_A$-module
\begin{align}
\Lambda_d &= U_A/(U_A \cap I_d)
\end{align}
(2.6)
where $I_d$ is the left ideal of $U$ generated by $E, K - q^d$ and $F^{d+1}$. Tensoring with $\mathbb{Q}(q)$, they recover the finite-dimensional simple $U$-modules. The elements
\begin{align}
v_r &= F^{(r)}, \quad r = 0, 1, \ldots, d
\end{align}
(2.7)
form a basis of $\Lambda_d$ and (we define $v_{-1} = v_{d+1} = 0$)
\begin{align}
Kv_r &= q^{d-2r}v_r, \\
Ev_r &= [d - r + 1]_q v_{r-1}, \\
Fv_r &= [r + 1]_q v_{r+1}.
\end{align}
(2.8)

More generally, for a composition $d = (d_1, d_2, \ldots, d_l)$ of $d$ (i.e. a sequence of nonnegative integers summing up to $d$), let
\begin{align}
\Lambda_d &= \Lambda_{d_1} \otimes \Lambda_{d_1} \otimes \cdots \otimes \Lambda_{d_l}
\end{align}
(2.9)
be the tensor product of $U_A$-modules. It has a standard basis
\begin{align}
v_r &= v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_l}
\end{align}
(2.10)
with \( r = (r_1, r_2, \ldots, r_l) \) running over the compositions satisfying \( r_k \leq d_k, k = 1, 2, \ldots, l \).

Let \( \varrho : U_A \to U_A^{op} \) be the \( A \)-algebra isomorphism defined on the generators by
\[
\varrho(K) = K, \quad \varrho(E) = qKF, \quad \varrho(F) = qK^{-1}E.
\] (2.11)

By an inner product of a \( U_A \)-module \( M \) we mean a non-degenerate symmetric bilinear form
\[
(\cdot, \cdot) : M \times M \to A
\]
satisfying
\[
(xu, w) = (u, \varrho(x)w) \quad \text{for } x \in U_A, u, w \in M.
\] (2.12)

Since \( \varrho \) is compatible with the comultiplication of \( U_A \):
\[
(\varrho \otimes \varrho)\Delta(x) = \Delta \varrho(x) \quad \text{for } x \in U_A,
\]
inner products of \( U_A \)-modules \( M_1, M_2 \) automatically give rise to an inner product of the tensor product module \( M_1 \otimes M_2 \) such that
\[
(u_1 \otimes u_2, w_1 \otimes w_2) = (u_1, w_1)(u_2, w_2) \quad \text{for } u_1, w_1 \in M_1, u_2, w_2 \in M_2.
\]

The simple \( U_A \)-module \( \Lambda_d \) has a unique inner product up to a constant, which we will normalize as
\[
(v_r, v_r) = \delta_{rr} \left[ \begin{array}{c} d \\ r \end{array} \right] q^{-r(d-r)}.
\] (2.13)

They automatically extend to inner products of the tensor product modules \( \Lambda_d \).

2.2. Perverse sheaves. Let \( X \) be a complex algebraic variety. We denote by \( \mathcal{D}(X) = \mathcal{D}^b_c(X) \) the bounded derived category of constructible \( \mathcal{C} \)-sheaves on \( X \) and denote by \( \mathcal{M}(X) \) the full subcategory consisting of perverse sheaves. An object of \( \mathcal{D}(X) \) is also referred to as a complex. Given a connected algebraic group \( G \) acting on \( X \), let \( \mathcal{M}_G(X) \) denote the full subcategory of \( \mathcal{M}(X) \) whose objects are the \( G \)-equivariant perverse sheaves on \( X \).

We denote by \( D : \mathcal{D}(X) \to \mathcal{D}(X)^\circ \) the Verdier duality functor. For an integer \( n \), let \([n] : \mathcal{D}(X) \to \mathcal{D}(X)\) denote the shift functor and let \( \mathcal{p}H^n : \mathcal{D}(X) \to \mathcal{M}(X) \) denote the \( n \)-th perverse cohomology functor. There are functor isomorphisms
\[
D^2 = \text{Id}, \quad \mathcal{p}H^n[j] = \mathcal{p}H^{n+j}, \quad D[n] = [-n]D.
\]

A complex \( C \in \mathcal{D}(X) \) is said to be semisimple if \( C \cong \oplus_n \mathcal{p}H^n(C)[-n] \) and if \( \mathcal{p}H^n(C) \in \mathcal{M}(X) \) is semisimple for all \( n \). A semisimple complex \( C \in \mathcal{D}(X) \) is called \( G \)-equivariant if \( \mathcal{p}H^n(C) \in \mathcal{M}_G(X) \) for all \( n \).

The Ext groups of \( C, C' \in \mathcal{D}(X) \) are the \( \mathbb{C} \)-linear spaces
\[
\text{Ext}_{\mathcal{D}(X)}^n(C, C') = \text{Hom}_{\mathcal{D}(X)}(C, C'[n]) = \mathbb{H}^n\mathcal{R}\text{Hom}(C, C');
\]
they satisfy

(1) \( \text{Ext}_{\mathcal{D}(X)}^n(C[n], C'[n']) = \text{Ext}_{\mathcal{D}(X)}^{n+n'}(C, C') \).
(2) \( \text{Ext}_{D(X)}^\bullet(C, DC') = \text{Ext}_{D(X)}^\bullet(C', DC) = H^\bullet D(C \otimes C') \).
(3) \( \text{Ext}_{D(X)}^n(C, C') = 0 \) for \( C, C' \in M(X) \) and \( n < 0 \).
(4) For simple perverse sheaves \( C, C' \in M(X) \), \( \text{Ext}_{D(X)}^0(C, C') \) is isomorphic to \( \mathbb{C} \) if \( C \cong C' \) and vanishes otherwise.

Let \( f : X \to Y \) be a morphism of algebraic varieties. There are induced functors \( f_!, f_* : D(X) \to D(Y) \) and \( f^!, f^* : D(Y) \to D(X) \). For reader’s convenience we list some properties of these functors as follows.

(5) \( Df^* = f^! D \) and \( Df_! = f_* D \).
(6) \( f^* \) is left adjoint to \( f_* \) and \( f_! \) is left adjoint to \( f^! \).
(7) If \( f \) is proper, then \( f_! = f_* \).
(8) If \( f \) is smooth with connected nonempty fibers of dimension \( d \), then \( f^*[d] = f^![-d] \) which induces a fully faithful functor \( M(Y) \to M(X) \) and sends simple perverse sheaves to simple perverse sheaves.
(9) There are natural isomorphisms for \( C, C' \in D(Y), C'' \in D(X) \)

\[
\begin{align*}
  f^!(C \otimes C') &= f^! C \otimes f^! C', \\
  f^\!* \mathcal{R}\!\mathcal{H}\!\mathcal{O}\!\!\!\mathcal{M}(C, C') &= \mathcal{R}\!\mathcal{H}\!\mathcal{O}\!\!\!\mathcal{M}(f^* C, f^! C') , \\
  f_! C'' \otimes C &= f_! (C'' \otimes f^* C) , \\
  \mathcal{R}\!\mathcal{H}\!\mathcal{O}\!\!\!\mathcal{M}(f_! C'', C) &= f_* \mathcal{R}\!\mathcal{H}\!\mathcal{O}\!\!\!\mathcal{M}(C'', f^! C) .
\end{align*}
\]

In particular,
\[
\begin{align*}
  f_! f^* C &= f_! (C_X \otimes f^* C) = f_! C_X \otimes C , \\
  f_* f^! C &= f_* \mathcal{R}\!\mathcal{H}\!\mathcal{O}\!\!\!\mathcal{M}(C_X, f^! C) = \mathcal{R}\!\mathcal{H}\!\mathcal{O}\!\!\!\mathcal{M}(f_! C_X, C) .
\end{align*}
\]

(10) Assume \( f : X \to Y \) is a \( G \)-equivariant morphism. If \( C \in \mathcal{M}_G(X) \), then \( p^nH^n(f_! C) \in \mathcal{M}_G(Y) \) for all \( n \). If \( C' \in \mathcal{M}_G(Y) \), then \( p^nH^n(f^* C') \in \mathcal{M}_G(X) \) for all \( n \).
(11) Assume \( f : X \to Y \) is a (locally trivial) principal \( G \)-bundle. The functor \( f^*[\dim G] \) and the functor \( f_! = p^{-\dim G} f_* \) define an equivalence of the categories \( \mathcal{M}_G(X) \), \( \mathcal{M}(Y) \).
(12) \( (fg)^* = g^* f^* \) and \( (fg)_! = f_! g_! \) for morphisms \( f : X \to Y, g : Y \to Z \).
(13) (Proper base change) \( f^* g_! = g_! f^\!* \) holds for the cartesian square

\[
\begin{array}{ccc}
  X \times_Y Y' & \overset{g'}{\longrightarrow} & X \\
  f' \downarrow & & f' \downarrow \\
  Y' & \overset{g}{\longrightarrow} & Y
\end{array}
\]

For a subvariety \( S \subset X \) and a complex \( C \in D(X) \) we also write \( C|_S \) instead of \( j_S^* C \) where \( j_S : S \to X \) is the inclusion.
For a locally closed irreducible smooth subvariety $S \subset X$, we denote by $IC(S) \in M(X)$ the simple perverse sheaf (the intersection complex) defined as the intermediate extension of the shifted constant sheaf $\mathcal{C}_S[\dim S]$. Below is a rather deep result on the interplay between proper morphisms and perverse sheaves.

(14) (Decomposition theorem) If $f : X \to Y$ is a proper morphism, then for every locally closed irreducible smooth subvariety $S \subset X$, $f_!IC(S) \in D(Y)$ is a semisimple complex.

The following implications of the decomposition theorem will be used in this paper.
(15) If $f : X \to Y$ is a proper morphism with $X$ smooth, then $f_!\mathcal{C}_X \in D(Y)$ is a semisimple complex.
(16) Assume a connected algebraic group $G$ acts on a variety $X$, having finitely many orbits. Then the $G$-equivariant simple perverse sheaves on $X$ are exactly those $IC(S)$ for various $G$-orbits $S$. Therefore, by the decomposition theorem, if $f : X \to Y$ is a proper morphism then $f_!$ sends $G$-equivariant semisimple complexes to semisimple complexes.

2.3. Partial flag varieties. Let $G \supset P \supset B$ be a connected reductive algebraic group, a parabolic subgroup and a Borel subgroup of it, respectively. We have a partial flag variety $X = G/P$. Let $W = N_G(T)/T$ be the Weyl group with respect to a fixed maximal torus $T \subset B$, and for every element $w \in W$ we fix a representative $\dot{w} \in N_G(T)$. We denote by $W_P \subset W$ the subgroup corresponding to $P$ and denote by $W^P$ the set of shortest representatives of the cosets $W/W_P$. The $B$-orbits partition $X$ into a finite number of affine cells (Bruhat decomposition)

$$X = \bigcup_{w \in W^P} X_w$$

(2.14)
where $X_w = B\dot{w}P/P$. The subvarieties $X_w$ are referred to as Schubert cells, and their closures are called Schubert varieties.

It follows that, up to isomorphism, the $B$-equivariant semisimple complexes on $X$ are finite direct sums of $IC(S)[j]$ for various Schubert cells $S$ and integers $j$.

The main concern of this paper is the case that $G = GL(W)$ is a general linear group, where $W$ is a complex linear space of dimension $d$, and that $B$ is the Borel subgroup preserving a fixed complete flag
$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_d = W.$$

Given an ascending sequence of integers $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq d$, there is a partial flag variety
$$X = G/P = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset W \mid \dim V_i = r_i, \ i = 1, 2, \ldots, n\},$$
where $P$ is the parabolic subgroup preserving the subspaces $W_{r_i}, i = 1, 2, \ldots, n$. 
The following lemma is proved by Kazhdan-Lusztig [KL80].

**Lemma 2.3.1.** For each pair of Schubert cells \( X_w, X_v \) of \( X \), \( pH^n(\mathcal{IC}(\overline{X}_w)|_{X_v}) = 0 \) unless \( n \equiv \dim X_w + \dim X_v \) (mod 2).

The proof of the following lemma is borrowed from [BGS96, 3.4].

**Lemma 2.3.2.** Let \( S \subset X \) be a subvariety consisting of Schubert cells \( X_1 \sqcup X_2 \sqcup \cdots \sqcup X_k \). Then, for \( B \)-equivariant semisimple complexes \( C, C' \in \mathcal{D}(X) \), we have

\[
\text{Ext}^*_\mathcal{D}(S)(C|_S, D(C'|_S)) \cong \bigoplus_{i=1}^k \text{Ext}^*_\mathcal{D}(X_i)(C|_{X_i}, D(C'|_{X_i})).
\]

*Proof.* We may assume \( C = \mathcal{IC}(\overline{X}_w), C' = \mathcal{IC}(\overline{X}_v) \) where \( X_w, X_v \) are Schubert cells. Setting

\[
S_p = \sqcup_{\dim X_i = \dim s_p X_i},
\]

we get a filtration of closed subvarieties

\[
S \supset S \setminus S_0 \supset S \setminus (S_0 \sqcup S_1) \supset \cdots \supset \emptyset.
\]

Then \( \text{Ext}^*_\mathcal{D}(S)(C|_S, D(C'|_S)) = \mathbb{H}^* D(C \otimes C'|_S) \) is the limit of a spectral sequence with \( E_1 \)-term

\[
E_1^{pq} = \mathbb{H}^{p+q} D(C \otimes C'|_{S_p}) = \bigoplus_{\dim X_i = \dim s_p} \mathbb{H}^{p+q} D(\mathcal{IC}(\overline{X}_w) \otimes \mathcal{IC}(\overline{X}_v)|_{X_i}).
\]

Since both \( \mathcal{IC}(\overline{X}_w)|_{X_i}, \mathcal{IC}(\overline{X}_v)|_{X_i} \) are direct sums of shifted constant sheaves, by Lemma 2.3.1 \( \mathbb{H}^n D(\mathcal{IC}(\overline{X}_w) \otimes \mathcal{IC}(\overline{X}_v)|_{X_i}) = 0 \) unless \( n \equiv \dim X_w + \dim X_v \) (mod 2). The \( E_1 \)-term therefore “vanishes like a chess-board”. It follows that the spectral sequence degenerates at the \( E_1 \)-term and we deduce that

\[
\text{Ext}^n\mathcal{D}(S)(C|_S, D(C'|_S)) \cong \bigoplus_{p+q=n} E_1^{pq} = \bigoplus_{i=1}^k \text{Ext}^n\mathcal{D}(X_i)(C|_{X_i}, D(C'|_{X_i})). \quad \square
\]

**Corollary 2.3.3.** Let \( S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_k \subset X \) be a subvariety with each \( S_i \) being a union of Schubert cells. Then, for \( B \)-equivariant semisimple complexes \( C, C' \in \mathcal{D}(X) \), we have

\[
\text{Ext}^*_\mathcal{D}(S)(C|_S, D(C'|_S)) \cong \bigoplus_{i=1}^k \text{Ext}^*_\mathcal{D}(S_i)(C|_{S_i}, D(C'|_{S_i})).
\]

Next, we recall the Bott-Samelson resolution of a given Schubert variety \( \overline{X}_w \). Fix a reduced word \( w = s_{i_1}s_{i_2} \cdots s_{i_t} \) and set

\[
Z = B^{s_{i_1}}B \times B^{s_{i_2}}B \times \cdots \times B^{s_{i_t}}B / B^t
\]

in which \( B^t \) acts by the equation

\[
(x_1, x_2, \ldots, x_t) \cdot (g_1, g_2, \ldots, g_t) = (x_1 g_1, g_1^{-1} x_2 g_2, \ldots, g_1^{-1} x_t g_t).
\]

The projection \( \pi : Z \to \overline{X}_w, \ [x_1, x_2, \ldots, x_t] \mapsto [x_1 x_2 \cdots x_t] \) then gives rise to a resolution of singularities. Indeed, \( Z \) is an iterated \( \mathbb{P}^1 \)-bundle.

The following variation of Bott-Samelson resolution for Grassmannians will be used in Section 3.7.
Lemma 2.3.4. Given a Grassmannian variety
\[ X = GL(W)/P = \{ V \subset W \mid \dim V = r \}, \]
for each Schubert variety \( \overline{X}_w \subset X \) and for each integer \( 0 \leq d' \leq d \), there exists a resolution of singularities \( \pi : Z \to \overline{X}_w \) such that the preimage of each
\[ Y_{r'} = \{ V \in \overline{X}_w \mid \dim(V \cap W_{d'}) = r' \} \]
is a smooth subvariety of \( Z \).

Proof. Choose a decomposition \( w = s_{i_1}s_{i_2}\cdots s_{i_t}w' \) such that \( \ell(w) = t + \ell(w') \), \( s_{i_j} \) are simple reflections satisfying \( s_{i_j}W_{d'} = W_{d'} \) and \( \ell(w') \) is minimal in possible. It is straightforward to check that
\[ \overline{X}_{w'} = Bw'P/P \]
is a Grassmannian variety and has each
\[ Y_{r'}' = \{ V \in \overline{X}_{w'} \mid \dim(V \cap W_{d'}) = r' \} \]
as a smooth subvariety. Following the spirit of Bott-Samelson resolution, we can form a resolution of singularities \( \pi : Z \to \overline{X}_w \) with
\[ Z = B\overline{s}_{i_1}B \times B\overline{s}_{i_2}B \times \cdots \times B\overline{s}_{i_t}B \times Bw'P/B' \times P. \]
Moreover, each
\[ \pi^{-1}(Y_{r'}) = \{ [x_1, x_2, \ldots, x_t, x'] \in Z \mid \dim(x'W_r \cap W_{d'}) = r' \} \]
is a \( Y_{r'}' \)-bundle over an iterated \( \mathbb{P}^1 \)-bundle, hence is a smooth subvariety of \( Z \). This completes the proof. \( \square \)

3. Categorification of \( U_A \)-modules

Categorification of representations of quantum groups is a fairly new topic. For the simplest cases, the irreducible representations of \( U_q(sl_2) \) and the tensor products of the fundamental representation of \( U_q(sl_2) \), the picture has been fairly clear; categorifications are implemented via both algebraic and geometric approaches (cf. for instance [BFK99], [CR04]).

The next development along this direction is the very recent work by Frenkel-Khovanov-Stroppel [FKS05], in which the authors succeeded in categorifying tensor products of general \( U_q(sl_2) \)-modules. The work used at full length many deep results on representations of Lie algebras.

In this section, we do the same as [FKS05], but in quite a different way. The categorification is fulfilled in the framework of perverse sheaves on Grassmannians. Moreover, it is tailored to set up an initial stage for the categorification of representations of general quantum groups via the geometry of Nakajima’s quiver varieties [Na94].
Quiver varieties are very natural and successful tools in the study of representations of Kac-Moody algebras \([Na01]\) and their quantum analogues \([Ln91]\), \([KSa97]\). Naturally the same is expected for categorification. As will be justified below, microlocal perverse sheaves \([KS90]\) \([Wa04]\) \([GMV05]\) on them (rather than homology groups or perverse sheaves as usually treated) turn out to provide the appropriate setting for this goal.

Notice that the tensor product varieties \([Na01]\) \([Ma03]\) associated to tensor products of \(U_q(sl_2)\)-modules are conic Lagrangian subvarieties of the cotangent bundles of Grassmannians. A standard result then states the categories of perverse sheaves we use in this section are equivalent via microlocalization functor to the categories of microlocal perverse sheaves supported on these varieties. That being said, our categorification is a priori able to be achieved alternatively in the framework of microlocal perverse sheaves.

Further examination by examples reveals that microlocal perverse sheaves on Nakajima’s quiver varieties do carry the right information necessary for extending the present work to general quantum groups. Actually, it was this observation that motivated the present paper.

Nevertheless, carrying the full plan out needs, that will be our next concern, substantial developments on many aspects of the theory of microlocal perverse sheaves. A version for schemes, which is still vacant from the literature, is especially welcome.

The section is organized as follows. We establish the categorification theorem in the first three subsections. The construction is straightforward and elementary. The only nontrivial tool used is the decomposition theorem.

In Section 3.4 and 3.5 we realize inner product of \(U_A\)-modules and the bar involution of \(U_A\) via certain functors.

In Section 3.6 we translate the categorification into an abelian version. This abelian version exhibits many resemblances with the work \([FKS05]\), but at this moment we have no proof to their equivalence.

The last three subsections are devoted to identify the standard \(U_A\)-modules with what we have categorified. The task can be done in more elementary ways, but we stick to our treatment because of its advantage of being less dependent on the algebraic knowledge of \(U_A\)-modules. This is important when we are confronting with other quantum groups.

Notice the resemblance of this work with Lusztig’s treatment \([Ln91]\)\([Ln93]\) for canonical basis of quantum groups, for example, canonical bases being constructed explicitly from simple perverse sheaves, and the usage of inner product.

### 3.1. The category \(Q_d\)

Let \(W\) be a complex linear space of dimension \(d\) and fix a complete flag

\[
0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_d = W.
\]  

(3.1)
We have for each integer $0 \leq r \leq d$ a Grassmannian variety
\[ X_d^r = \{ V \subset W \mid \dim V = r \}. \] (3.2)

It is convenient to set $X_d^r = \emptyset$ for $r < 0$ or $r > d$.

Given a composition $d = (d_1, d_2, \ldots, d_l)$ of $d$, let $P_d$ denote the parabolic subgroup of $G = GL(W)$ preserving the subspaces $W_{d_1+d_2+\cdots+d_k}$, $k = 1, 2, \ldots, l$. We denote the set of the $P_d$-orbits of $X_d^r$ as $\mathcal{S}_d^r$. It is indexed by the compositions $r = (r_1, r_2, \ldots, r_l)$ of $r$ satisfying $r_k \leq d_k$, $k = 1, 2, \ldots, l$; associated to $r = (r_1, r_2, \ldots, r_l)$ is the orbit
\[ X_r = \{ V \in X_d^r \mid \dim(V \cap W_{d_1+\cdots+d_k}) = r_1 + \cdots + r_k, \ k = 1, 2, \ldots, l \}. \] (3.3)

In the specific case $d = (1, 1, \ldots, 1)$, $P_d$ is the Borel subgroup $B \subset G$ preserving the complete flag (3.1), hence the $P_d$-orbits coincide with the Schubert cells of $X_d^r$.

**Remark 3.1.1.** The conormal variety to the $P_d$-orbits of $\sqcup_r X_d^r$ is precisely the tensor product varieties $\mathcal{IC}(X_d^r)$ associated to the $U$-module $\mathcal{Q}(q) \times \Lambda_d$.

Let $\mathcal{Q}_d^r$ be the full subcategory of $\mathcal{D}(X_d^r)$ consisting of the $P_d$-equivariant semisimple complexes. Up to isomorphism the objects from $\mathcal{Q}_d^r$ are finite direct sums of $\mathcal{IC}(X_r)[j]$ for various $P_d$-orbits $X_r$ and integers $j$.

The categories $\mathcal{Q}_d^r$ are additive (but neither abelian nor triangulated in general). We set
\[ \mathcal{Q}_d = \bigoplus_r \mathcal{Q}_d^r. \] (3.4)

By the Grothendieck group $Q_d$ of $\mathcal{Q}_d$ we mean the free $\mathcal{A}$-module defined by the generators each for an isomorphism class of objects from $\mathcal{Q}_d$ and the relations

(i) $[C \oplus C'] = [C] + [C']$, for $C, C' \in \mathcal{Q}_d$;
(ii) $[C[1]] = q^{-1}[C]$, for $C \in \mathcal{Q}_d$.

It has a canonical basis
\[ b_r = [\mathcal{IC}(X_r)], \quad X_r \in \sqcup_r \mathcal{S}_d^r. \] (3.5)

**Example 3.1.2.** For $d = (d)$ and $0 \leq r \leq d$, $X_d^r$ consists of a single $P_d$-orbit: $\mathcal{S}_d^r = \{ X(r) \} = X_d^r$. Therefore, $\mathcal{Q}_d$ has a canonical basis
\[ b(r) = [\mathcal{IC}(X_d^r)], \quad 0 \leq r \leq d. \] (3.6)

Thus $\mathcal{Q}_d \cong \oplus_{0 \leq r \leq d} \mathcal{A}$.

**Example 3.1.3.** For $d = (2, 2)$ and $r = 2$, we have $\mathcal{S}_d^r = \{ X_{(2,0)}, X_{(1,1)}, X_{(0,2)} \}$ where $X_{(2,0)}$ is a point, $X_{(1,1)}$ consists of four Schubert cells and $X_{(0,2)} \cong \mathbb{C}^4$ is the top Schubert cell. We have met a singular Schubert variety $\overline{X}_{(1,1)} = X_{(1,1)} \sqcup X_{(2,0)}$. 


3.2. **The functors** $\mathcal{K}, \mathcal{E}^{(n)}, \mathcal{F}^{(n)}$. For every integer $n \geq 0$, we have a diagram

$$X_d^{r+n} \xrightarrow{p} X_d^{r,r+n} \xrightarrow{p'} X_d^{r+n}$$

where

$$X_d^{r,r+n} = \{ V \subset V' \subset W \mid \dim V = r, \dim V' = r + n \},$$

and $p(V,V') = V$, $p'(V,V') = V'$. Note that $p$ (resp. $p'$) is an $X_{n}^{r}$-bundle (resp. $X_{d-n}^{r}$-bundle) and that

$$\dim X_d^{r,r+n} - \dim X_d^{r+n} = nr,$$

$$\dim X_d^{r,r+n} - \dim X_d^{r} = n(d - n - r).$$

Define functors

$$\mathcal{K}_r = [2r - d] : \mathcal{D}(X_d^r) \to \mathcal{D}(X_d^r),$$

$$\mathcal{E}^{(n)}_{r+n} = p p' [nr] : \mathcal{D}(X_d^{r+n}) \to \mathcal{D}(X_d^r),$$

$$\mathcal{F}^{(n)}_{r+n} = p' p' [n(d - n - r)] : \mathcal{D}(X_d^r) \to \mathcal{D}(X_d^{r+n}),$$

and assemble them into endofunctors of $\mathcal{Q}_d$.

$$\mathcal{K} = \bigoplus_r \mathcal{K}_r, \quad \mathcal{E}^{(n)} = \bigoplus_r \mathcal{E}_{r}^{(n)}, \quad \mathcal{F}^{(n)} = \bigoplus_r \mathcal{F}_{r}^{(n)}.$$

(3.10)

We also abbreviate $\mathcal{E}_{r}^{(1)}, \mathcal{F}_{r}^{(1)}, \mathcal{E}, \mathcal{F}$ to $\mathcal{E}_r, \mathcal{F}_r, \mathcal{E}, \mathcal{F}$, respectively.

The following proposition states that these functors induce endofunctors of $\mathcal{Q}_d$.

**Proposition 3.2.1.** We have $\mathcal{K} C, \mathcal{E}_{r}^{(n)} C, \mathcal{F}_{r}^{(n)} C \in \mathcal{Q}_{d}$ for $C \in \mathcal{Q}_{d}$.

**Proof.** The statement for $\mathcal{K}$ is trivial. We prove the proposition for $\mathcal{F}$ and a similar argument applies to $\mathcal{E}$. Since $p$ is a Grassmannian bundle and is $P_d$-equivariant, $p^* C$ is a $P_d$-equivariant semisimple complex. Since $p'$ is proper and is also $P_d$-equivariant, by the decomposition theorem $p' p^* C$, and therefore $\mathcal{F}_{r}^{(n)} C$, is a $P_d$-equivariant semisimple complex. \(\square\)

**Example 3.2.2.** For $d = (d)$, we have

$$\mathcal{F}^{(r)} IC(X_d^0) = IC(X_d^r)$$

and

$$\mathcal{K} IC(X_d^r) = IC(X_d^r)[2r - d],$$

$$\mathcal{E} IC(X_d^r) = \bigoplus_{j=0}^{d-r} IC(X_d^{r-1})[d - r - 2j],$$

$$\mathcal{F} IC(X_d^r) = \bigoplus_{j=0}^{r} IC(X_d^{r+1})[r - 2j],$$

in agreement with (2.7), (2.8).
3.3. Categorification theorem. We shall show that the endofunctors $\mathcal{K}$, $\mathcal{E}^{(n)}$, $\mathcal{F}^{(n)}$ categorify the generators $K, E^{(n)}, F^{(n)}$ of $U_A$. Our first two propositions are obvious.

**Proposition 3.3.1.** The functor $\mathcal{K}$ is an autoequivalence.

**Proposition 3.3.2.** We have functor isomorphisms

$$\mathcal{K} \mathcal{E} = \mathcal{E}[\mathcal{K}[-2]], \quad \mathcal{K} \mathcal{F} = \mathcal{F} \mathcal{K}[2].$$

**Proposition 3.3.3.** We have functor isomorphisms

$$\mathcal{E}^{(n-1)} \mathcal{E} \cong \bigoplus_{j=0}^{n-1} \mathcal{E}^{(n)}[n - 1 - 2j],$$

$$\mathcal{F}^{(n-1)} \mathcal{F} \cong \bigoplus_{j=0}^{n-1} \mathcal{F}^{(n)}[n - 1 - 2j].$$

**Proof.** We prove the second isomorphism. Consider the commutative diagram

\[
\begin{array}{ccc}
X_{d}^{r,r+n} & \xrightarrow{p'_1} & X_{d}^{r+n} \\
\downarrow p_3 & & \downarrow p_2 \\
Y & \xrightarrow{p_{23}} & X_{d}^{r+1,r+n} \\
\downarrow p_{12} & & \downarrow p_2 \\
X_{d}^r & \xleftarrow{p_1} & X_{d}^{r+1} \\
\end{array}
\]

where

$$Y = \{ V_1 \subset V_2 \subset V_3 \subset W \mid \dim V_1 = r, \ \dim V_2 = r + 1, \ \dim V_3 = r + n \},$$

$p_{ij}(V_1, V_2, V_3) = (V_i, V_j)$ and $p_i, p'_i$, $i = 1, 2, 3$ are given as (3.7). We have

$$\mathcal{F}^{(n-1)} \mathcal{F}_r = (p'_2)_! (p_2)^* (p'_1)_! (p_1)^* [k] = (p'_2)_! (p_{23})_! (p_{12})^* (p_1)^* [k]$$

$$= (p'_3)_! (p_{13})_! (p_3)^* [k]$$

in which we set

$$k = (d - r - 1) + (n - 1)(d - n - r)$$

and the second equality is by proper base change. Since $p_{13}$ is a $\mathbb{P}^{n-1}$-bundle,

$$(p_{13})_! \mathcal{C}_Y \cong \bigoplus_{j=0}^{n-1} \mathcal{C}_{X_{r,r+n}}[-2j]$$

by the decomposition theorem. Therefore,

$$(p_{13})_! (p_{13})^* \cong (p_{13})_! \mathcal{C}_Y \otimes \cong \bigoplus_{j=0}^{n-1} [-2j].$$
It follows that
\[ \mathcal{F}_{r+1}^{-1} \mathcal{F}_r \cong \bigoplus_{j=0}^{n-1} (p_3^j \circ (p_3^*)^*[k - 2j]) = \bigoplus_{j=0}^{n-1} \mathcal{F}_r^{(n)}[n - 1 - 2j]. \]

Assembling the isomorphism for various \( r \), we prove the proposition. \( \square \)

**Proposition 3.3.4.** There is a functor isomorphism
\[
\mathcal{E}_{r+1} \mathcal{F}_r \oplus \bigoplus_{0 \leq j < 2r - d} \text{Id}[(2r - d) - 1 - 2j] \\
\cong \mathcal{F}_{r-1} \mathcal{E}_r \oplus \bigoplus_{0 \leq j < d - 2r} \text{Id}[(d - 2r) - 1 - 2j].
\]

**Proof.** We start with the commutative diagrams

\[
\begin{align*}
Y & \xrightarrow{p'_3} X^r_d & Y & \xrightarrow{p'_3} X^r_d \\
\downarrow^{p_2} & \downarrow^{p_1} & \downarrow^{p_1} & \downarrow^{p_2} \\
X^r_{d,r+1} & \xrightarrow{p'_1} X^r_{d,r+1} & X^r_{d,r+1} & \xrightarrow{p'_1} X^r_{d,r+1}
\end{align*}
\]

where
\[
Y = \{ V, V' \in X^r_d \mid \dim(V + V') \leq r + 1 \}
= \{ V, V' \in X^r_d \mid \dim(V \cap V') \geq r - 1 \},
Y' = \{ V_1, V_2 \subset V_3 \subset W \mid \dim V_1 = \dim V_2 = r, \dim V_3 = r + 1 \},
Y'' = \{ V_1 \subset V_2, V_3 \subset W \mid \dim V_1 = r - 1, \dim V_2 = \dim V_3 = r \},
\]

and the morphisms are the obvious ones as before. The bottom right corners of the diagrams are cartesian squares. As in the previous proposition, we have
\[
\begin{align*}
\mathcal{E}_{r+1} \mathcal{F}_r &= (p'_3) \circ (p'_1) \circ (p_3^*)^*[d - 1], \\
\mathcal{F}_{r-1} \mathcal{E}_r &= (p'_3) \circ (p''_3) \circ (p_3^*)^*[d - 1].
\end{align*}
\]

Let \( i : \Delta \to Y \) be the inclusion of the diagonal. We have functor isomorphisms
\[
\begin{align*}
\text{Id} &= (p'_3) \circ i \circ (p_3^*)^* = p'_3(i ; C_\Delta \otimes p_3^* -), \\
\mathcal{E}_{r+1} \mathcal{F}_r &= p'_3(i ; C_{Y'}[d - 1] \otimes p_3^* -), \\
\mathcal{F}_{r-1} \mathcal{E}_r &= p'_3(p''_3 \circ C_{Y''}[d - 1] \otimes p_3^* -). \tag{3.11}
\end{align*}
\]

Note that the varieties \( Y', Y'' \) are smooth, but \( Y \) may be singular at the diagonal \( \Delta \). Indeed, the morphisms \( p'_1, \ p''_2 \) are isomorphisms away from \( \Delta \) and are \( \mathbb{P}^{d-r-1} \).
respectively $\mathbb{P}^{r-1}$ fibrations over $\Delta$. By proper base change we have

$$p'_{12!}[C_{Y'}|_{Y \setminus \Delta}] \cong p''_{23!}[C_{Y''}|_{Y \setminus \Delta}] \cong C_{Y \setminus \Delta},$$

and

$$p'_{12!}[C_{Y'}]|_{\Delta} \cong \bigoplus_{j=0}^{d-r-1} \mathbb{C}_{\Delta}[-2j], \quad p''_{23!}[C_{Y''}|_{\Delta}] \cong \bigoplus_{j=0}^{r-1} \mathbb{C}_{\Delta}[-2j].$$

On the other hand, $p'_{12!}[C_{Y'}]$ and $p''_{23!}[C_{Y''}]$ are both semisimple complexes by the decomposition theorem. Therefore, by the classification of simple perverse sheaves, both has $\mathcal{IC}(Y)[\dim Y]$ as a direct summand and

$$p'_{12!}[C_{Y'}] \oplus \bigoplus_{d-r\leq j<r} i_{\Delta}^! \mathbb{C}_{\Delta}[-2j] \cong p''_{23!}[C_{Y''}] \oplus \bigoplus_{r<j<d-r} i_{\Delta}^! \mathbb{C}_{\Delta}[-2j]. \quad (3.12)$$

Combining isomorphisms (3.11) and (3.12), we prove the proposition. □

**Example 3.3.5.** Revisit the case $d = (2, 2)$ and $r = 2$. We may derive the isomorphism $\mathcal{E}_{r+1} \mathcal{F} \mathcal{IC}(\mathcal{X}(2,0)) \cong \mathcal{IC}(\mathcal{X}(1,1))$ as follows. $X(2,0)$ is a point and $p_3^{-1}(X(2,0))$ is isomorphic to the singular Schubert variety $\mathcal{X}(1,1)$. Therefore, $(p_3)^* \mathcal{IC}(\mathcal{X}(2,0))[3]$ is a shifted constant sheaf supported on $p_3^{-1}(X(2,0))$ thus of course not semisimple. But $p'_{12}$ resolves the singularity of $p_3^{-1}(X(2,0))$. Thus $(p'_{12})_!(p'_{12})^* (p_3)^* \mathcal{IC}(\mathcal{X}(2,0))[3]$ is the simple perverse sheaf supported on $p_3^{-1}(X(2,0))$, sent by $(p'_{12})_!$ to $\mathcal{IC}(\mathcal{X}(1,1))$. Indeed, in this case $p'_{12!}[C_{Y'}] \cong p''_{23!}[C_{Y''}] \cong \mathcal{IC}(Y)[\dim Y]$.

Comparing the above propositions with the defining relations (2.1), (2.15) of $U_A$, we obtain the categorification theorem.

**Theorem 3.3.6.** The endofunctors $\mathcal{K}$, $\mathcal{E}^{(n)}$, $\mathcal{F}^{(n)}$ categorify the generators $K$, $E^{(n)}$, $F^{(n)}$ of $U_A$, thus endow the Grothendieck group $Q_d$ with a $U_A$-module structure. More precisely, the followings hold for $C \in Q_d$.

$$[\mathcal{K} \mathcal{E} C] = q^2 [\mathcal{E} C], \quad [\mathcal{K} \mathcal{F} C] = q^{-2} [\mathcal{F} C],$$

$$[\mathcal{F} \mathcal{F} C] - [\mathcal{F} \mathcal{E} C] = \frac{[\mathcal{K} C] - [\mathcal{K}^{-1} C]}{q - q^{-1}},$$

$$[\mathcal{E}^{(n)} C] = \left[ \frac{\mathcal{E}^n C}{[n]_q^n} \right], \quad [\mathcal{F}^{(n)} C] = \left[ \frac{\mathcal{F}^n C}{[n]_q^n} \right].$$

**Remark 3.3.7.** Our categorification described above is a rather rough one, but has the advantage of being clear and simple. Indeed this has been enough if we are only concerned with the representations of $U_q(sl_2)$. To give a more rigorous treatment, one may consider instead the algebra $\hat{U}_A$ (cf. [Lu93]) which is a free $A$-module generated by the symbols

$$F^{(n)} E^{(m)} 1_r, \quad m, n \geq 0, \quad r \in \mathbb{Z}.$$
and subjects to the multiplication
\[
F^{(n)}E^{(m)}_{1_r} \cdot F^{(k)}E^{(l)}_{1_s} = \delta_{r,s+2l-2k} \sum_{0 \leq t \leq m,k} \left[ \begin{array}{c} 2l + s \\ t \end{array} \right]_q \left[ \begin{array}{c} n + k - t \\ n \end{array} \right]_q \left[ \begin{array}{c} m + l - t \\ l \end{array} \right]_q F^{(n+k-t)}E^{(m+l-t)}_{1_s}.
\]

What follows then is straightforward: associate to \( F^{(n)}E^{(m)}_{1_r} \) the functor \( F^{(n)}E^{(m)}_{d-r} \) and use the decomposition theorem to establish functor isomorphisms mimicking the above multiplication, for example, as we have done for the following fundamental cases

\[
F^{(0)}E^{(n-1)}_{1_{r+2}} \cdot F^{(0)}E^{(1)}_{1_r} = [n]_q F^{(0)}E^{(n)}_{1_r},
\]

\[
F^{(n-1)}E^{(0)}_{1_{r-2}} \cdot F^{(1)}E^{(0)}_{1_r} = [n]_q F^{(n)}E^{(0)}_{1_r},
\]

\[
F^{(0)}E^{(1)}_{1_{r-2}} \cdot F^{(1)}E^{(0)}_{1_r} = F^{(1)}E^{(1)}_{1_r} + [r]_q F^{(0)}E^{(0)}_{1_r}.
\]

3.4. Inner product. In this subsection we endow the \( U_A \)-module \( Q_d \) with an inner product by using the bifunctor \( \text{Ext}^*_D(-, -) \) where

\[
\mathcal{D} = \mathcal{D}(\bigcup_r X_d^r) = \bigoplus_r \mathcal{D}(X_d^r).
\]  

By \ref{3.2}(1)(2), there exists a unique symmetric bilinear form

\[
(\cdot, \cdot) : Q_d \times Q_d \to A 
\]

such that

\[
([C], [C']) = \sum_k \dim \text{Ext}^k_D(C, D C') \cdot q^{-k}
\]

for \( C, C' \in Q_d \).

**Remark 3.4.1.** Note the identities

\[
([C], [C']) = \sum_k \dim H^k D(C \otimes C') \cdot q^{-k} = \sum_k \dim H^k_C(C \otimes C') \cdot q^k.
\]

**Proposition 3.4.2.** For \( b_r = [\mathcal{I}(X_r)], b_s = [\mathcal{I}(X_s)], X_r, X_s \in \bigcup_r X_d^r \) we have

\[
(b_r, b_s) \in \delta_{r,s} + q^{-1}Z_{\geq 0}[q^{-1}].
\]

In particular, the bilinear form \((\cdot, \cdot)\) on \( Q_d \) is non-degenerate.

**Proof.** Since \( \mathcal{I}(X_r), \mathcal{I}(X_s) \) are self dual simple perverse sheaves, by \ref{2.2}(3)(4) \( \text{Ext}^k_D(\mathcal{I}(X_r), D \mathcal{I}(X_s)) \) vanishes for \( k < 0 \) and has dimension \( \delta_{r,s} \) for \( k = 0 \). This proves the main claim of the proposition. The non-degeneracy follows from the observation that the bilinear form on the canonical basis \( \ref{3.5} \) produces a unit matrix modulo \( q^{-1} \).

**Corollary 3.4.3.** The following conditions are equivalent for \( C, C' \in Q_d \).

(i) \( C \cong C' \).

(ii) \( ([C], [C'']) = ([C'], [C'']) \) for all \( C'' \in Q_d \).
Proposition 3.4.4. There are bifunctor isomorphisms

\[ \text{Ext}^\bullet_D(\mathcal{K}, D-) = \text{Ext}^\bullet_D(-, DK-), \]
\[ \text{Ext}^\bullet_D(\mathcal{E}^{(n)} -, D-) = \text{Ext}^\bullet_D(-, DK^n\mathcal{F}^{-n2}[-n2]-), \]
\[ \text{Ext}^\bullet_D(\mathcal{F}^{(n)} -, D-) = \text{Ext}^\bullet_D(-, DK^{-n}\mathcal{E}^{(n)}[-n2]-). \]

Proof. The first isomorphism is obvious and the proofs of the next two are similar. The third one follows from the natural isomorphisms for \( C \in \mathcal{D}(X_d^r), C' \in \mathcal{D}(X_d^r+1) \)

\[ \text{Ext}^\bullet_D(X_d^r, D) \]
\[ = \text{Ext}^\bullet_D(X_d^r, \mathcal{D}p^*|n(d-n-r)|C, DC') \]
\[ = \text{Ext}^\bullet_D(X_d^r, C, Dp|n(d-n-r)|C') \]
\[ = \text{Ext}^\bullet_D(X_d^r, C, DK^{-n}\mathcal{E}^{(n)}[-n2]C') \]

Summarizing, we obtain

Theorem 3.4.5. The bilinear form \((,\) is an inner product of the \(U_A\)-module \(Q_d\).

Example 3.4.6. For \(d = (d)\), we have

\[ ([\mathcal{I}C(X_d^r)], [\mathcal{I}C(X_d^r)]) = \sum_k \dim \text{Ext}^k_D(\mathcal{I}C(X_d^r), \mathcal{I}C(X_d^r)) \cdot q^{-k} \]
\[ = \sum_k \dim H^k(X_d^r, \mathbb{C}) \cdot q^{-k} = \left[ \begin{array}{c} d \\ r \end{array} \right] q^{-r(d-r)}, \]

which agrees with (2.13).

3.5. Verdier duality and bar involution. Let \(^-: U_A \rightarrow U_A\) denote the \(\mathbb{Z}\)-algebra isomorphism determined by

\[ \bar{q} = q^{-1}, \quad \bar{K} = K^{-1}, \quad \bar{E} = E, \quad \bar{F} = F. \]  

(3.17)

The following proposition shows that the Verdier duality functor categorifies the bar involution of \(U_A\).

Proposition 3.5.1. We have functor isomorphisms

\[ D[-1] = [1]D, \quad DK = K^{-1}D, \quad D\mathcal{E}^{(n)} = \mathcal{E}^{(n)}D, \quad D\mathcal{F}^{(n)} = \mathcal{F}^{(n)}D. \]

Proof. We only prove the last isomorphism. Keep the notation (3.7). Since \(p : X_d^{r,r+n} \rightarrow X_d^r\) is an \(X_d^{r-r}\)-bundle and \(p' : X_d^{r,r+n} \rightarrow X_d^{r+n}\) is proper, we have

\[ Dp^*[n(d-n-r)] = p^*[n(d-n-r)]D, \quad Dp'_* = p'_*D. \]

The isomorphism \(D\mathcal{F}^{(n)} = \mathcal{F}^{(n)}D\) then follows. \(\square\)

This gives us immediately
Theorem 3.5.2. The Verdier duality functor $D$ induces an anti-$A$-linear isomorphism $\Psi : Q_d \to Q_d$, which satisfies

1. $\Psi(b_r) = b_r$, for $b_r = [\mathcal{IC}(X_r)]$, $X_r \in \mathcal{S}_d$;
2. $\Psi^2 = \text{Id}$;
3. $\Psi(xu) = \bar{x}\Psi(u)$, for $x \in U_A$, $u \in Q_d$.

Combining Proposition 3.4.4 and Proposition 3.5.1 we also obtain

Proposition 3.5.3. The functors $K$, $E^n(L)$, $F^n(L)$ have the functors $K_{-1}$, $K_nF^n(L)$, $K_{-n}E^n(L)$ as left adjoints and have the functors $K_{-1}$, $K_{-n}F^n(L)$, $K_nE^n(L)$ as right adjoints, respectively.

3.6. Abelian categorification. In this subsection, we categorify the $U$-modules $Q(q) \otimes_A Q_d$ via abelian categories.

First, we realize the additive category $Q_d$ as a full subcategory of an abelian category. We have a finite-dimensional graded $C$-algebra

$$A^\bullet = \text{Ext}^\bullet_D(L, L) = \bigoplus_r \text{Ext}^\bullet_D(L_r, L_r')$$

where

$$\mathcal{D} = D(\sqcup r \mathcal{X}_d) = \bigoplus_r \mathcal{D}(X_r')$$

and

$$L = \bigoplus_r \mathcal{IC}(\mathcal{X}_r), \quad L_r' = \bigoplus_{X_r \in \mathcal{S}_d} \mathcal{IC}(X_r).$$

The complex $L$ is by definition the direct sum of the simple perverse sheaves (up to isomorphism) from $Q_d$. The multiplication of $A^\bullet$ is given by

$$\text{Ext}_D^n(L, L) \otimes \text{Ext}_D^n(L, L) = \text{Hom}_D(L, L[n]) \otimes \text{Hom}_D(L[n], L[n + m])$$

$$\to \text{Hom}_D(L, L[n + m]) = \text{Ext}_D^{n+m}(L, L).$$

Then for every complex $C \in Q_d$,

$$\text{Ext}_D^\bullet(L, C) \quad (\text{resp. } \text{Ext}_D^\bullet(C, L))$$

defines a graded left (resp. right) $A^\bullet$-module.

Let $A^\bullet_{-}\text{mof}$ denote the category of finite-dimensional graded left $A^\bullet$-modules and let $A^\bullet_{-}\text{pmof}$ denote the full subcategory consisting of the projectives. By 2.2.(3)(4) we have

1. $A^\bullet$ is $\mathbb{Z}_{\geq 0}$-graded.
2. $A^0 = \bigoplus_r \bigoplus_{X_r \in \mathcal{S}_d} \text{Hom}_D(\mathcal{IC}(\mathcal{X}_r), \mathcal{IC}(\mathcal{X}_r))$, with each summand isomorphic to $\mathbb{C}$.

These further imply
(3) The units of the $\mathbb{C}$-summands of $A^0$ are the indecomposable idempotents of $A^\bullet$.

(4) The $\mathbb{C}$-summands of $A^0$ enumerate the simple left $A^\bullet$-modules.

(5) $\text{Ext}^\bullet_{\mathcal{D}}(L, IC(X_r))$, $X_r \in \sqcup_s \mathcal{F}_d^r$ enumerate the indecomposable projective left $A^\bullet$-modules.

(6) $\text{Hom}_{A^\bullet\text{-mof}}(\text{Ext}^\bullet_{\mathcal{D}}(L, C), \text{Ext}^\bullet_{\mathcal{D}}(L, C')) = \text{Hom}_{\mathcal{D}}(C, C')$ for $C, C' \in \mathcal{Q}_d$.

Therefore, we obtain

**Proposition 3.6.1.** The obvious functor $\mathcal{Q}_d \to A^\bullet\text{-pmof}$ is an equivalence of categories. Moreover, the equivalence identifies the Grothendieck group of $A^\bullet\text{-mof}$ with $\mathbb{Q}(q) \otimes_A \mathbb{Q}_d$.

In the proposition, the Grothendieck group of the abelian category $A^\bullet\text{-mof}$ means the $\mathbb{Q}(q)$-linear space defined by the generators each for an isomorphism class of objects from $A^\bullet\text{-mof}$ and the relations

(i) $[M^\bullet] = [M'^\bullet] + [M''^\bullet]$, for exact sequence $M'^\bullet \hookrightarrow M^\bullet \twoheadrightarrow M''^\bullet$;

(ii) $[M^{\bullet+1}] = q^{-1}[M^\bullet]$, for $M^\bullet \in A^\bullet\text{-mof}$.

Next, we translate the endofunctors $\mathcal{K}, \mathcal{K}^{-1}, \mathcal{E}, \mathcal{F}$ into exact endofunctors of $A^\bullet\text{-mof}$. Recall that every graded $A^\bullet$-bimodule defines an endofunctor of $A^\bullet\text{-mof}$ by tensoring on the left.

For an additive endofunctor $\mathcal{G} : \mathcal{Q}_d \to \mathcal{Q}_d$, compatible with the shift functor, there is a well-defined graded $A^\bullet$-bimodule

$$\mathcal{G}^\bullet = \text{Ext}^\bullet_{\mathcal{D}}(L, \mathcal{G}L) \quad (3.21)$$

of which the bimodule structure is given by

$$a \cdot x \cdot b = ax\mathcal{G}(b) \quad \text{for } x \in \mathcal{G}^\bullet, \ a, b \in A^\bullet.$$

The followings are easy to verify.

(7) $\mathcal{G}^\bullet \otimes A^\bullet_\ast \text{Ext}^\bullet_{\mathcal{D}}(L, C) = \text{Ext}^\bullet_{\mathcal{D}}(L, \mathcal{G}C)$ for $C \in \mathcal{Q}_d$. In particular, $\mathcal{G}^\bullet$ induces an endofunctor of $A^\bullet\text{-pmof}$.

(8) If $\mathcal{G}$ has a left adjoint then the $A^\bullet$-bimodule $\mathcal{G}^\bullet$ is (left and right) projective, hence is flat and defines an exact endofunctor of $A^\bullet\text{-mof}$.

As endofunctors of $\mathcal{Q}_d$, $\mathcal{K}, \mathcal{K}^{-1}, \mathcal{E}, \mathcal{F}$ are compatible with the shift functor and have left adjoints (Proposition 3.5.3), therefore they define projective graded $A^\bullet$-bimodules $\mathcal{K}^\bullet, \mathcal{K}^{-1}\ast, \mathcal{E}^\bullet, \mathcal{F}^\bullet$ and hence exact endofunctors of $A^\bullet\text{-mof}$. The results from Section 3.3 are then translated to

**Theorem 3.6.2.** We have isomorphisms of projective graded $A^\bullet$-bimodules

$$\mathcal{K}^\bullet \otimes \mathcal{K}^{-1}\ast \cong \mathcal{K}^{-1}\ast \otimes \mathcal{K}^\bullet \cong A^\bullet,$$

$$\mathcal{K}^\bullet \otimes \mathcal{E}^\bullet \cong A^{\ast-2} \otimes \mathcal{E}^\bullet \otimes \mathcal{K}^\bullet,$$

$$\mathcal{K}^\bullet \otimes \mathcal{F}^\bullet \cong A^{\ast+2} \otimes \mathcal{F}^\bullet \otimes \mathcal{K}^\bullet,$$
and
\[
\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet \oplus \bigoplus_{r>d/2} \bigoplus_{j=0}^{(2r-d)-1} \text{Ext}_D^\bullet((2r-d)-1-2j)(L^r, L^r)
\]
\[
\cong \mathcal{F}^\bullet \otimes \mathcal{E}^\bullet \oplus \bigoplus_{r<d/2} \bigoplus_{j=0}^{(d-2r)-1} \text{Ext}_D^\bullet((d-2r)-1-2j)(L^r, L^r).
\]

Therefore, the abelian category $A^\bullet$-mof together with the exact endofunctors $K^\bullet$, $\mathcal{K}^{-1\bullet}$, $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ categorifies the $U$-module $\mathbb{Q}(q) \otimes_A Q_d$.

**Example 3.6.3.** For $d = (d)$, we have $A^\bullet = \bigoplus_{0 \leq r \leq d} H^\bullet(X_d^r, \mathbb{C})$ and
\[
\mathcal{K}^\bullet = \bigoplus_r H^{\bullet-d+2r}(X_d^r, \mathbb{C}),
\]
\[
\mathcal{E}^\bullet = \bigoplus_r H^{\bullet+d-r-1}(X_d^r, \mathbb{C}),
\]
\[
\mathcal{F}^\bullet = \bigoplus_r H^\bullet+d(r, \mathbb{C}),
\]
in which $H^\bullet(X_d^{r+1}, \mathbb{C})$ is regarded as a graded $H^\bullet(X_d^r, \mathbb{C})$-bimodule for $\mathcal{E}^\bullet$ and a graded $H^\bullet(X_d^{r+1}, \mathbb{C})$-bimodule for $\mathcal{F}^\bullet$.

### 3.7. The functor Res.

Keep the notations of Section 3.1. We split the composition $d = (d_1, d_2, \ldots, d_l)$ into a couple of compositions
\[
d' = (d_1, d_2, \ldots, d_{l'}), \quad d'' = (d_{l'+1}, d_{l'+2}, \ldots, d_l)
\] (3.22)
of $d' = \sum_{i=1}^{l'} d_i$, $d'' = \sum_{i=l'+1}^l d_i$ respectively.

Set $W' = W_{d'}$ and $W'' = W/W_{d''}$, either inheriting a complete flag from $W$:
\[
0 = W_0 \subset W_1 \subset \cdots \subset W_{d'} = W',
\] (3.23)
\[
0 = W_{d'}/W_{d'} \subset W_{d'+1}/W_{d'} \subset \cdots \subset W_d/W_{d'} = W''.
\] (3.24)

We associate a collection of data $X_{d'}', P_{d'}$, $\mathcal{D}_{d'}'$ resp. $X_{d''}', P_{d''}, \mathcal{D}_{d''}$ to $W'$ resp. $W''$ as in Section 3.1. In this way, $P_{d'} \times P_{d''}$ is regarded as a quotient of $P_d$ and we have
\[
\mathcal{M}_{P_d}(X_{d'}' \times X_{d''}') = \mathcal{M}_{P_{d'} \times P_{d''}}(X_{d'}' \times X_{d''}').
\]

Let $\mathcal{Q}_{d',d''}$ be the full subcategory of $\mathcal{D}(X_{d'}' \times X_{d''}')$ whose objects are the $P_{d'} \times P_{d''}$-equivariant semisimple complexes. Up to isomorphism the objects from $\mathcal{Q}_{d',d''}$ are finite direct sums of $\mathcal{I}C(X_{d'}, X_{d''})[j]$ for various $P_{d'} \times P_{d''}$-orbits $X_{d'} \times X_{d''}$ and integers $j$. Then set
\[
\mathcal{Q}_{d',d''} = \bigoplus_{r', r''} \mathcal{Q}_{r', r''}^{d', d''}.
\] (3.25)

In the same way as we have done for $\mathcal{Q}_d$, we can define endofunctors $\mathcal{K}'$, $\mathcal{E}'(n)$, $\mathcal{F}'(n)$ and $\mathcal{K}''$, $\mathcal{E}''(n)$, $\mathcal{F}''(n)$.
of $Q_{d',d''}$ which categorify the generators
\[ K \otimes 1, E^{(n)} \otimes 1, F^{(n)} \otimes 1 \quad \text{and} \quad 1 \otimes K, 1 \otimes E^{(n)}, 1 \otimes F^{(n)} \]
of $U_A \otimes U_A$, so as to endow the Grothendieck group $Q_{d',d''}$ of $Q_{d',d''}$ with a $U_A$-module structure. We can also define an inner product in terms of Ext groups. The $U_A$-module $Q_{d',d''}$ has a basis
\[ b_{r',r''} = [IC(\overline{X}_{r'} \times \overline{X}_{r''})] = [IC(\overline{X}_{r'}) \otimes IC(\overline{X}_{r''})], \quad X_{r'} \in \cup_r \mathcal{D}_{d'}, \quad X_{r''} \in \cup_r \mathcal{D}_{d''}. \quad (3.26) \]

Notice the equivalence of categories
\[ Q_{d'} \times Q_{d''} \rightarrow Q_{d',d''} \quad ((C',C'') \mapsto C' \otimes C''). \quad (3.27) \]
It follows that $Q_{d'} \otimes Q_{d''}$ together with all algebraic structures defined on it is naturally identified with $Q_{d',d''}$. In particular, the basis element $b_{r'} \otimes b_{r''}$ is identified with $b_{r',r''}$.

Now we define the functor Res by using the diagram
\[ X_d \rightarrow X_d' \times X_d'' \quad \text{where} \quad Y^{r',r''} = \{ V \in X^r_{d'} + r'' \mid \dim(V \cap W') = r' \}, \]
and $\iota$ is the inclusion, $\pi(V) = (V \cap W', V/(V \cap W'))$. Define for each pair of integers $r'$, $r''$ a functor
\[ \text{Res}^{r',r''} = \pi \iota^*[\pi'(d'-r')r''] : D(X_d^{r'+r''}) \rightarrow D(X_d' \times X_d'') \quad (3.29) \]
and assemble them together
\[ \text{Res}^{r',r''} = \bigoplus_{r',r''} \text{Res}^{r',r''} : \bigoplus_{r'} D(X_d^r) \rightarrow \bigoplus_{r'} D(X_d' \times X_d''). \quad (3.30) \]

The following proposition states that Res induces a functor $Q_d \rightarrow Q_{d',d''}$, hence induces an $A$-linear map
\[ \Upsilon_{d',d''} : Q_d \rightarrow Q_{d'} \otimes Q_{d''}. \quad (3.31) \]

**Proposition 3.7.1.** We have $\text{Res}^{r',r''} C \subset Q^{r',r''}_{d',d''}$ for $C \in Q^{r'+r''}_d$.

The proof is immediate from the next two lemmas.

**Lemma 3.7.2.** The functor $\pi^*[d'-r')r''$ and the functor $\pi^*[d'-r')r'']$ define an equivalence of the categories $\mathcal{M}_{P_d}(Y^{r',r''})$, $\mathcal{M}_{P_{d'} \times P_{d''}}(X^{r'}_{d'} \times X^{r''}_{d''})$.

**Proof.** Since $\pi$ is a $\mathbb{C}^{d'-r')r''}$-bundle, $\pi^*[d'-r')r'']$ induces a fully faithful functor from $\mathcal{M}(X^{r'}_{d'} \times X^{r''}_{d''})$ to $\mathcal{M}(Y^{r',r''})$. Moreover, the kernel of the group homomorphism $P_d \rightarrow P_{d'} \times P_{d''}$ acts transitively on each fiber of $\pi$. Hence $\pi^*[d'-r')r'']$ defines an equivalence of the categories $\mathcal{M}_{P_{d'} \times P_{d''}}(X^{r'}_{d'} \times X^{r''}_{d''})$, $\mathcal{M}_{P_d}(Y^{r',r''})$.

On the other hand, $\pi_\pi^* = [-2(d'-r')r'']$. Thus the functor $\pi_\pi^*[d'-r')r'']$ gives an inverse for the equivalence. \hfill \Box
Lemma 3.7.3. \(i^* C\) is a \(P_d\)-equivariant semisimple complex for \(C \in Q_d^{r'+r''}\).

Proof. We may assume \(C = IC(\overline{X}_w)\) where \(X_w\) is a Schubert cell of \(X_d^{r'+r''}\). By Lemma 2.3.4 there exists a resolution of singularities \(f : Z \to \overline{X}_w\) such that
\[
Z' = f^{-1}(Y^{r',r''})
\]
is a smooth variety. Then \(C\) is a direct summand of \(f_! C_Z[\dim Z]\) and \(i^* C\) is therefore a direct summand of \(i^* f_! C_Z[\dim Z]\). By the decomposition theorem and proper base change

\[
i^* f_! C_Y[\dim Z] = (f|_{Z'})_! C_{Z'}[\dim Z]
\]
is a semisimple complex. It follows that \(i^* C\) is a semisimple complex, whose \(P_d\)-equivariance is obvious. \(\square\)

Example 3.7.4. For \(d = (1,1,1)\), the \(A\)-linear map \(\Upsilon_{1,2}\) at level \(r' + r'' = 1\) is as follows.

\[
b_{(1,0,0)} \mapsto b_{(1)} \otimes b_{(0,0)},
b_{(0,1,0)} \mapsto b_{(0)} \otimes b_{(1,0)} + q^{-1} b_{(1)} \otimes b_{(0,0)},
b_{(0,0,1)} \mapsto b_{(0)} \otimes b_{(0,1)} + q^{-2} b_{(1)} \otimes b_{(0,0)}.
\]

3.8. The isomorphism \(Q_d \cong Q_{d'} \otimes Q_{d''}\). Recall that we split the composition \(d = (d_1, d_2, \ldots, d_l)\) into

\[
d' = (d_1, d_2, \ldots, d_{l'}), \quad d'' = (d_{l'+1}, d_{l'+2}, \ldots, d_l).
\]

In the next proposition, we associate to each composition \(r = (r_1, r_2, \ldots, r_l)\) a pair of compositions

\[
r' = (r_1, r_2, \ldots, r_{l'}), \quad r'' = (r_{l'+1}, r_{l'+2}, \ldots, r_l).
\]

Proposition 3.8.1. For \(b_r = [IC(\overline{X}_r)], X_r \in \mathcal{F}_d\) we have

\[
\Upsilon_{d',d''}(b_r) = b_{r'} \otimes b_{r''} + \sum_{X_s \in \mathcal{F}_d' : X_s \subset \overline{X}_r} c_{r,s} \cdot b_{r'} \otimes b_{r''}
\]

where \(c_{r,s} \in q^{-1} \mathbb{Z}_{\geq 0}[q^{-1}]\). Therefore, \(\Upsilon_{d',d''} : Q_d \to Q_{d'} \otimes Q_{d''}\) is an \(A\)-linear isomorphism.

Proof. We set

\[
c_{r,s} = \sum_k n_{r,s}^k \cdot q^k
\]

where \(n_{r,s}^k\) are the multiplicities appearing in the decomposition

\[
\text{Res}_{d',d''} IC(\overline{X}_r) \cong \bigoplus_k \bigoplus_{X_s \in \mathcal{F}_d'} n_{r,s}^k \cdot IC(\overline{X}_{s'}) \boxtimes IC(\overline{X}_{s''})[-k].
\]
Then
\[ \gamma_{d',d''}(b_r) = \sum_{X_s \in \mathcal{F}_d} c_{r,s} \cdot b_s \otimes b_{s''}. \]

The simple perverse sheaf \( \mathcal{IC}(\Xi_r) \) is by definition the intermediate extension of the shifted constant sheaf \( \mathbb{C}_{X_r}[\dim X_r] \). It follows that

\begin{enumerate}[(i)]
  \item \( \mathcal{IC}(\Xi_r)|_{X_r} = \mathbb{C}_{X_r}[\dim X_r]; \)
  \item \( \mathcal{IC}(\Xi_r)|_{X_s} = 0 \) if \( X_s \not\subseteq \Xi_r \); and
  \item \( \varphi^h(\mathcal{IC}(\Xi_r)|_{X_s}) = 0 \) for \( k \geq 0 \) if \( X_s \not\subseteq \Xi_r \).
\end{enumerate}

Further, by proper base change,

\begin{enumerate}[(i)]
  \item \( \text{Res}_{d',d''}^* \mathcal{IC}(\Xi_r)|_{X_{d'} \times X_{d''}} = \mathbb{C}_{X_{d'} \times X_{d''}}[\dim X_{d'} \times X_{d''}]; \)
  \item \( \text{Res}_{d',d''}^* \mathcal{IC}(\Xi_r)|_{X_{d'} \times X_{d''}} = 0 \) if \( X_s \not\subseteq \Xi_r \); and
  \item \( \varphi^h(\text{Res}_{d',d''}^* \mathcal{IC}(\Xi_r)|_{X_{d'} \times X_{d''}}) = 0 \) for \( k \geq 0 \) if \( X_s \not\subseteq \Xi_r \).
\end{enumerate}

This gives \( c_{r,r} = 1; c_{r,s} = 0 \) if \( X_s \not\subseteq \Xi_r \); and \( c_{r,s} \in q^{-1}\mathbb{Z}_{\geq 0}[q^{-1}] \) if \( X_s \subseteq \Xi_r \).

The claim of isomorphism follows from that the \( \mathcal{A} \)-linear map \( \gamma_{d',d''} \) can be represented by a triangular \( \mathcal{A} \)-matrix with unit diagonal.

**Proposition 3.8.2.** The \( \mathcal{A} \)-linear map \( \gamma_{d',d''} : Q_{d'} \to Q_{d''} \otimes Q_{d'} \) preserves inner product.

**Proof.** For \( C', C'' \in \mathcal{Q}_d \), keeping the notation (3.28) we have

\[ \text{Ext}_{D(X_d)}^*(C, D C') \]
\[ \cong \bigoplus_{r+r''=r} \text{Ext}_{D(Y_{d'}, d'')}^*(t^* C, D t^* C') \]
\[ \cong \bigoplus_{r+r''=r} \text{Ext}_{D(Y_{d'}, d'')}^*(\pi^* \pi_! [2(d' - r')r''_!] t^* C, D t^* C') \]
\[ = \bigoplus_{r+r''=r} \text{Ext}_{D(X_{d'} \times X_{d''})}^*(\pi t^* \pi_! [(d' - r')r''_!] C, D \pi t^* \pi_! [(d' - r')r''_!] C') \]
\[ = \bigoplus_{r+r''=r} \text{Ext}_{D(X_{d'} \times X_{d''})}^*(\text{Res}_{d',d''}^* C, D \text{Res}_{d',d''}^* C''). \]

The first isomorphism is by applying Corollary 2.3.3 to the decomposition

\[ X'_d = \bigsqcup_{r+r''=r} Y'_{d'}, Y''_{d''}, \]

the second is by Lemma 3.7.2 and Lemma 3.7.3. This gives

\[ ([C], [C'']) = ([\text{Res}_{d',d''} C], [\text{Res}_{d',d''} C']). \]

Thus the proposition follows.

It remains to check the compatibility of \( \gamma_{d',d''} \) with the comultiplication (2.2).

**Proposition 3.8.3.** For \( C \in \mathcal{Q}_d \) we have isomorphisms

\[ \text{Res}_{d',d''}^* \mathcal{K} C \cong \mathcal{K}' \mathcal{K}'' \text{Res}_{d',d''} C, \]
\[ \text{Res}_{d',d''}^* \mathcal{E} C \cong (\mathcal{E}' \oplus \mathcal{K} \mathcal{E}'') \text{Res}_{d',d''} C, \]
\[ \text{Res}_{d',d''}^* \mathcal{F} C \cong (\mathcal{F} \mathcal{K}' - 1 \oplus \mathcal{F}'') \text{Res}_{d',d''} C. \]
Therefore, the $\mathcal{A}$-linear map $\Upsilon_{d',d''} : Q_d \to Q_{d'} \otimes Q_{d''}$ is a homomorphism of $U_{\mathcal{A}}$-modules.

**Proof.** We prove the third isomorphism, which by Corollary 3.4.3 is equivalent to that the equality

$$([\text{Res}_{d',d''} \mathcal{F}C], [C']) = ([(\mathcal{F}'K''-1 \oplus \mathcal{F}'') \text{Res}_{d',d''} C], [C'])$$

holds for all $C' \in Q_{d',d''}$.

Suppose $C \in Q_d$ and $C' \in Q_{d',d''}$. We assume $r' + r'' = r + 1$ as well; otherwise both sides of the above equality vanish. Our first commutative diagram is

\[
\begin{array}{ccccccc}
X_d & \xleftarrow{p} & X_d^{r,r+1} & \xrightarrow{p'} & X_d^{r+1} & \xleftarrow{\iota} & Y_{r',r''} & \xrightarrow{\pi} & X_{d'} \times X_{d''} \\
\downarrow{j} & & \downarrow{\rho} & & \downarrow{\iota} & & \downarrow{\pi} & & \\
Z & & & & & & & & 
\end{array}
\]

where

$$Z = \{(V_1, V_2) \in X_d^{r,r+1} \mid \dim(V_2 \cap W') = r'\},$$

$j$ is the inclusion, $\rho(V_1, V_2) = V_2$, and $p,p'$ are given as (3.28). The middle part of the diagram is a cartesian square, thus by proper base change we have

$$\text{Res}_{d',d''} \mathcal{F}C = \pi_1 j^*[d' - r')r''p_1^*p^*[d - r - 1]C = \pi_1 \rho_1 j^*p^*C[k]$$

where we set $k = (d' - r')r'' + (d - r - 1)$.

Note that $\rho : Z \to Y_{r',r''}$ is a $\mathbb{P}^r$-bundle, $\rho^*\pi^*C'$ is therefore a semisimple complex. Since $Z$ is a locally closed smooth subvariety of $X_d^{r,r+1}$, $\rho^*\pi^*C'$ is the restriction of a $P_d$-equivariant semisimple complex on $X_d^{r,r+1}$. Applying Corollary 2.3.3 to the decomposition $Z = Z_1 \sqcup Z_2$ where

$$Z_1 = \{(V_1, V_2) \in Z \mid \dim(V_1 \cap W') = r' - 1\},$$

$$Z_2 = \{(V_1, V_2) \in Z \mid \dim(V_1 \cap W') = r'\},$$

we deduce that

$$\text{Ext}_D^*(X_d^{r,r+1}, \text{Res}_{d',d''} \mathcal{F}C, DC')$$

$$= \text{Ext}_D^*(Z_1) (j_1^*\rho_1^*C[k], D\rho_1^*\pi^*C')$$

$$\cong \text{Ext}_D^*(Z_1) (j_1^*\rho_1^*C[k], D\rho_1^*\pi^*C') \oplus \text{Ext}_D^*(Z_2) (j_2^*\rho_2^*C[k], D\rho_2^*\pi^*C')$$

$$= \text{Ext}_D^*(X_d^{r,r+1}) (\pi_1 \rho_1 j_1^*\pi^*C[k] \oplus \pi_1 \rho_2 j_2^*\pi^*C[k], DC').$$

In the above equation, we set $j_i = j|_{Z_i}$ and $\rho_i = \rho|_{Z_i}$, $i = 1, 2$. 

\[\text{(3.34)}\]
Then we consider the following commutative diagrams of which the top left corners are cartesian squares.

\[
\begin{array}{ccc}
X_{d''}^{r' - 1} \times X_{d''}^{r''} & \xrightarrow{p_1} & X_{d''}^{r' - 1, r''} \times X_{d''}^{r''} \\
\pi_1 \downarrow & & \downarrow \pi \\
Y_{r' - 1, r''} & \cong & Y_{r', r''}
\end{array}
\]

\[
\begin{array}{ccc}
X_{d'}^{r'} \times X_{d''}^{r''} & \xrightarrow{p_2} & X_{d'}^{r'} \times X_{d''}^{r'' - 1, r''} \\
\pi_2 \downarrow & & \downarrow \pi \\
Y_{r', r'' - 1} & \cong & Y_{r', r''}
\end{array}
\]

where

\[
Z_1' = \{V_1 \in Y_{r' - 1, r''}, V_2 \in X_{d'}^{r'} | V_1 \cap W' \subset V_2\}, \\
Z_2' = \{V_1 \in Y_{r', r'' - 1}, V_2 \in X_{d''}^{r''} | V_1/(V_1 \cap W') \subset V_2\},
\]

\[\tilde{\rho}_1(V_1, V_2) = (V_1, V_2 \cap W'), \tilde{\rho}_2(V_1, V_2) = (V_1, V_2/(V_2 \cap W'))\] and \(p_i, p_i'\) are given as (3.7), \(\iota_i, \pi_i\) are given as (3.28), \(i = 1, 2\). We have

\[
\pi_1(\rho_1)(j_1)^*p^*[k] = (p_1')^*(\rho_1)^*(\pi_1)^*(\iota_1)^*[k] = F_{r' - 1}^{r''}K_{r'' - 1}^{r'}\text{Res}_{d', d''}^{r', r'' - 1},
\]

\[
\pi_1(\rho_2)(j_2)^*p^*[k] = (p_2')^*(\rho_2)^*(\pi_2)^*(\iota_2)^*[k - 2(d' - r')] = F_{r'' - 1}^{r'}\text{Res}_{d', d''}^{r', r'' - 1}. \tag{3.35}
\]

Here we used the isomorphisms

\[(\tilde{\rho}_1)(\tilde{\rho}_1)^* = \text{Id}, \quad (\tilde{\rho}_2)(\tilde{\rho}_2)^* = [-2(d' - r')]\]

which follow from that \(\tilde{\rho}_1\) is an isomorphism and \(\tilde{\rho}_2\) is a \(\mathbb{C}^{d' - r'}\)-bundle.

Assembling isomorphisms (3.34) and (3.35), we obtain (3.33), hence prove our proposition. \(\square\)

Above propositions are summarized to

**Theorem 3.8.4.** *The \(\mathcal{A}\)-linear map \(Y_{d', d''} : Q_d \to Q_{d'} \otimes Q_{d''}\) induced by the functor \(\text{Res}_{d', d''} : Q_d \to Q_{d', d''}\) is an inner product preserving isomorphism of \(U_{\mathcal{A}}\)-modules.*
3.9. The isomorphism $Q_d \cong \Lambda_d$. We have constructed in a purely geometric way various finite-dimensional $U_A$-modules and established isomorphisms among them. Now we relate them to the $U_A$-modules $\Lambda_d$ introduced in Section 2.1.

Recall that the $U_A$-module $Q_d$ has a basis

$$b_{(r)} = [\mathcal{IC}(X_r^d)], \quad 0 \leq r \leq d$$

and the $U_A$-module $\Lambda_d$ has a basis

$$v_r = \bar{F}(r), \quad 0 \leq r \leq d.$$ 

By Example 3.2.2 and Example 3.4.6, the $A$-linear map

$$\varphi_d : Q_d(d) \rightarrow \Lambda_d, \quad b_{(r)} \mapsto v_r$$

is an inner product preserving isomorphism of $U_A$-modules.

For general cases, we apply the Res functor repeatedly to form an inner product preserving isomorphism of $U_A$-modules

$$\Upsilon_{d_1,d_2,...,d_l} : Q_d(d_1) \otimes Q(d_2) \otimes \cdots \otimes Q(d_l).$$

Followed by $\varphi_{d_1} \otimes \varphi_{d_2} \otimes \cdots \otimes \varphi_{d_l}$, it gives rise to an inner product preserving isomorphism of $U_A$-modules

$$\varphi_d = (\varphi_{d_1} \otimes \varphi_{d_2} \otimes \cdots \otimes \varphi_{d_l}) \circ \Upsilon_{d_1,d_2,...,d_l} : Q_d \rightarrow \Lambda_d.$$  (3.38)

In the rest of this subsection, we give a straightforward description of this isomorphism.

For each $P_d$-orbit $X_r \in \mathcal{I}_d^r$ and for each complex $C \in \mathcal{Q}_d$, there are a set of integers $n^k_r(C)$ appearing as multiplicities in the isomorphism

$$pH^k(C|X_r) \cong n^k_r(C) \cdot \mathbb{C}_{X_r}[[\text{dim } X_r]],$$

they forming a polynomial

$$n_r(C) = \sum_k n^k_r(C) \cdot q^k \in \mathbb{Z}_{\geq 0}[q, q^{-1}].$$

In the above notations, the isomorphism $\varphi_d : Q_d \rightarrow \Lambda_d$ is such that

$$\varphi_d([C]) = \sum_{X_r \in \mathcal{I}_d^r} n_r(C) \cdot v_r$$

for $C \in \mathcal{Q}_d$, where $v_r$ are the elements of $\Lambda_d$ defined in (2.10).

Example 3.9.1. For $d = (2,2)$, the isomorphism $\varphi_d$ at level $r = 2$ is as follows.

$$b_{(2,0)} \mapsto v_2 \otimes v_0,$$

$$b_{(1,1)} \mapsto v_1 \otimes v_1 + (q^{-1} + q^{-3})v_2 \otimes v_0,$$

$$b_{(0,2)} \mapsto v_0 \otimes v_2 + q^{-1}v_1 \otimes v_1 + q^{-4}v_2 \otimes v_0.$$ 

The following proposition is a specialization of Proposition 3.8.1.
Proposition 3.9.2. For \( b_r = [IC(X_r)] \), \( X_r \in \mathcal{A}_d \) we have
\[
\varphi_d(b_r) = v_r + \sum_{X_s \in \mathcal{A}_d : X_s \supsetneq X_r} c_{r,s} \cdot v_s
\] (3.40)
where \( c_{r,s} \in q^{-1}Z_{\geq 0}[q^{-1}] \).

Proof. Set \( c_{r,s} = n_s(\mathcal{IC}(X_r)) \), then \( \varphi_d(b_r) = \sum_{X_s \in \mathcal{A}_d} c_{r,s} \cdot v_s \). The simple perverse sheaf \( \mathcal{IC}(X_r) \) is by definition the intermediate extension of the shifted constant sheaf \( \mathbb{C}_{X_r}[\dim X_r] \). It follows that
(i) \( \mathcal{IC}(X_r)|_{X_r} = \mathbb{C}_{X_r}[\dim X_r] \);
(ii) \( \mathcal{IC}(X_r)|_{X_s} = 0 \) if \( X_s \not\subset X_r \); and
(iii) \( p^Hk(\mathcal{IC}(X_r)|_{X_s}) = 0 \) for \( k \geq 0 \) if \( X_s \not\subset X_r \).
Therefore, \( c_{r,r} = 1 \); \( c_{r,s} = 0 \) if \( X_s \not\subset X_r \); and \( c_{r,s} \in q^{-1}Z_{\geq 0}[q^{-1}] \) if \( X_s \not\subset X_r \). \( \square \)

Remark 3.9.3. It is not difficult by interpreting the coefficients \( c_{r,s} \) as parabolic Kazhdan-Lusztig polynomials \([KL79],[Deo87]\) to identify the canonical basis (3.5) of \( Q_d \) with the one introduced by Lusztig \([Lu93]\). Cf. \([FKK98]\). Moreover, since the anti-\( \mathcal{A} \)-linear isomorphism \( \Psi \) from Theorem 3.5.2 is uniquely determined by property (1) of the theorem, it is therefore the same as the one from \([Lu93]\) defined by means of quasi-universal \( R \)-matrix.

4. Categorification of \( R \)-matrices

One remarkable achievement (and impetus) on the topic of categorification is the discovery of Khovanov homology of knots and links \([Kh00]\), which has become of particular interest after Rasmussen’s elementary proof \([Ras03]\) of Milnor’s conjecture. In fact, the only solutions to the conjecture known before are using gauge theory or Floer homology.

Khovanov homology is able to be realized in many different ways and has been generalized to the categorification of several other quantum invariants of knots and links. See for instance \([Str03],[CK07],[Kh03],[KR04]\). However, the quantum invariants under consideration are still very limited, and the machinery used is apparently hard to be applied to general cases.

To give a uniform treatment for the categorification of general quantum invariants, one possible approach is to follow Reshetikhin-Turaev’s principle \([RT90]\) for building tangle invariants from representations of quantum groups. This means to categorify, besides representations of quantum groups, \( R \)-matrices and “cup/cap” homomorphisms among them.

In this section, we deal with the issue of \( R \)-matrices on \( U_q(sl_2) \)-modules. This part of work is new in many aspects.

Formally speaking, a system of \( R \)-matrices on the \( U_{\mathcal{A}} \)-modules \( \Lambda_d \) is a collection of \( U_{\mathcal{A}} \)-module isomorphisms \( R(d, \sigma) : \Lambda_d \rightarrow \Lambda_{\sigma(d)} \), each for a composition \( d = \)
(d_1, d_2, \ldots, d_l) and a permutation σ ∈ S_l, such that

\[ R(σ_2(d), σ_1) ∘ R(d, σ_2) = R(d, σ_1σ_2), \quad (4.1) \]

whenever ℓ(σ_1σ_2) = ℓ(σ_1) + ℓ(σ_2).

The standard algebraic approach to the realization of R-matrices is by Drinfeld’s universal R-matrix (cf. [Kas95, XVII.4.2] and formula (4.31) below), which assigns to each pair of U_A-modules Λ_{d_1}, Λ_{d_2} an isomorphism

\[ R_{d_1,d_2} : \Lambda_{d_1} ⊗ \Lambda_{d_2} \to \Lambda_{d_2} ⊗ \Lambda_{d_1}, \quad (4.2) \]

then composes them in the obvious way to give the others.

The major difficulty underlying categorification of R-matrices is the failure of their positivity over canonical basis; that is, a canonical basis element may be sent to a linear combination in which both positive and negative coefficients occur (cf. Example 4.4.3). This forces us to settle the categorification problem by using complexes of functors rather than merely functors.

Section 4.2 constitutes the heart part of this section, in which we introduce the notion of pure resolution of mixed complexes and establish a uniqueness theorem for mixed perverse sheaves. Then, we categorify the braiding relation (4.1) in Section 4.3 and establish categorification theorem in the reminder subsections.

4.1. Some homological algebra. Below are some elementary facts that will be used in this section.

**Lemma 4.1.1.** Suppose we are given a morphism of complexes forming by objects and morphisms from a triangulated category.

\[ \cdots \to C^1 \to C^2 \xrightarrow{a} C^3 \to C^4 \to \cdots \]
\[ \cdots \to \tilde{C}^1 \to \tilde{C}^2 \xrightarrow{b} \tilde{C}^3 \to \tilde{C}^4 \to \cdots \]

If there is a triangle morphism

\[ B \to C^2 \xrightarrow{b} \tilde{C}^2 \xrightarrow{c} \tilde{C}^3 \xrightarrow{d} \to \tilde{C}^4 \]

then the above complex morphism is a homotopy equivalence.

**Proof.** Rewrite the triangle morphism as follows, in which e, f and b, c are the obvious inclusions and projections, respectively.

\[ B \xrightarrow{e} B \oplus \tilde{C}^2 \xrightarrow{b} \tilde{C}^2 \xrightarrow{c} \tilde{C}^3 \xrightarrow{d} \to \tilde{C}^4 \]
Then \( a \) must be in the form \( \begin{pmatrix} \text{Id} & 0 \\ \ast & d \end{pmatrix} \). Then a direct computation. \( \square \)

**Lemma 4.1.2.** Any sequence \( C^{\leq w-2} \to C^{\leq w-1} \to C^{\leq w} \) in a triangulated category extends to commutative diagrams

\[
\begin{array}{cccc}
C^w[-1] & \to & C^{\leq w-1} & \to C^{w-1} \\
\downarrow d' & & \downarrow e & \downarrow c' \\
C^{\leq w-2} & \to & C^{\leq w-1} & \to C^w \\
\downarrow b' & & \downarrow e & \downarrow d \\
C^w[-1] & \to & C^{\leq w-1} & \to C^{w-1} \\
\end{array}
\]

in which the vertical and the slash lines are exact triangles.

**Proof.** Follows directly from the defining axioms of triangulated category. \( \square \)

**Lemma 4.1.3** (Postnikov system). Suppose we are given a system of exact triangles from a triangulated category

\[
C^{\leq w-1} \to C^{\leq w} \to C^w. \quad (4.3)
\]

Then the following sequence

\[
\cdots \to C^{w+1}[-w-1] \to C^w[-w] \to C^{w-1}[-w+1] \to \cdots
\]

in which the morphisms are the compositions

\[
C^w[-w] \to C^{\leq w-1}[-w+1] \to C^{w-1}[-w+1],
\]

is a complex.

**Proof.** Observe that the morphisms

\[
C^{w+1}[-w-1] \to C^w[-w] \to C^{w-1}[-w+1]
\]

are realized by the sequence

\[
C^{w+1}[-w-1] \to C^{\leq w}[-w] \to C^w[-w] \to C^{\leq w-1}[-w+1] \to C^{w-1}[-w+1],
\]

which composes to zero because its middle part is an exact triangle. \( \square \)
Lemma 4.1.4. Suppose in addition to the assumption of the above lemma, there are triangle morphisms

\[
\begin{array}{ccc}
B^{\leq w-1} & \rightarrow & C^{\leq w-1} \\
\downarrow & & \downarrow \\
B^w & \rightarrow & C^w
\end{array}
\]

Then they extend to give a system of exact triangles

\[
B^{\leq w-1} \rightarrow B^w \rightarrow C^w
\]

which induce the same complex as \((4.3)\).

Proof. By the octahedron axiom of triangulated category, the given triangle morphisms extend to commutative diagrams with exact triangles on their rows and columns

\[
\begin{array}{ccc}
B^{\leq w-1} & \rightarrow & C^{\leq w-1} \\
\downarrow & & \downarrow \\
B^w & \rightarrow & C^w \\
\downarrow & & \downarrow \\
C^w & \rightarrow & C^w
\end{array}
\]

This gives the exact triangles \((4.4)\) and, further, commutative diagrams

\[
\begin{array}{ccc}
C^w & \rightarrow & B^{w-1}[1] \\
\downarrow & & \downarrow \\
C^w & \rightarrow & C^{w-1}[1]
\end{array}
\]

saying that \((4.4)\) induce the same complex as \((4.3)\).

\[\square\]

4.2. Pure resolution of mixed complexes. Let \(\mathbb{F}_q\) be a finite field with \(q\) elements and \(\mathbb{F}\) be its algebraic closure. Let \(X_0\) be a scheme of finite type over \(\mathbb{F}_q\) and let \(X\) be the scheme \(X_0 \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(\mathbb{F})\) over \(\mathbb{F}\). We denote by \(\mathcal{D}(X_0) = D^b_c(X_0, \mathbb{Q}_l)\) (resp. \(\mathcal{D}(X) = D^b_c(X, \mathbb{Q}_l)\)) the triangulated category of \(\mathbb{Q}_l\)-sheaves \([\text{BBD82} 2.2.18]\) on \(X_0\) (resp. \(X\)), where \(l\) is a prime number invertible in \(\mathbb{F}_q\). For a complex \(C_0 \in \mathcal{D}(X_0)\) we denote by \(C \in \mathcal{D}(X)\) its pullback to \(X\).

A complex from \(\mathcal{D}(X_0)\) is called mixed if all its cohomology sheaves are mixed \(\mathbb{Q}_l\)-sheaves. Mixed complexes from \(\mathcal{D}(X_0)\) form a full triangulated subcategory \(\mathcal{D}_m(X_0)\). It inherits the perverse \(t\)-structure from \(\mathcal{D}(X_0)\) thus gives rise to the category \(\mathcal{M}_m(X_0)\) of mixed perverse sheaves.

We denote by \(\mathcal{D}_{\leq w}(X_0)\) the full subcategory of \(\mathcal{D}_m(X_0)\) consisting of those mixed complexes whose \(i\)-th cohomology sheaf is of weight \(\leq w + i\) for all \(i\) and denote by \(\mathcal{D}_{\geq w}(X_0)\) the full subcategory consisting of those mixed complexes \(C\) such that \(DC \in \mathcal{D}_{\leq -w}(X_0)\). The complexes from \(\mathcal{D}_{\leq w}(X_0) \cap \mathcal{D}_{\geq w}(X_0)\) are called pure of
weight \( w \). Be careful of that the purity of a mixed \( \mathcal{O}_\ell \)-sheaf does not necessarily agree with the one as a mixed complex.

Listed below are some properties of mixed complexes (cf. [BED82, KW01]). The key step to them is the proof of (3) for \( f \), which is the main result of [De80]. Be aware of that the decomposition theorem is immediate from (1)-(3).

1. Simple mixed perverse sheaves are pure.
2. If \( C_0 \in \mathcal{D}_m(X_0) \) is pure, then \( C \cong \oplus_n^p H^n(C)[-n] \) and \( \mathcal{P}H^n(C) \) is a semisimple perverse sheaf for all \( n \).
3. For a morphism \( f : X_0 \to Y_0 \), the functors \( f_* \), \( f^* \) preserve \( \mathcal{D}_{\leq w} \) and the functors \( f^{-1} \) preserve \( \mathcal{D}_{\geq w} \).
4. For a smooth morphism \( f : X_0 \to Y_0 \) of relative dimension \( d \), we have \( f^*[d] = f^![-d](-d) \), where \((−d)\) indicates the Tate twist (increasing the weight by \( 2d \)).
5. The outer tensor product functor \( \boxtimes \) sends \( \mathcal{D}_{\leq w_1} \times \mathcal{D}_{\leq w_2} \) to \( \mathcal{D}_{\leq w_1+w_2} \) and sends \( \mathcal{D}_{\geq w_1} \times \mathcal{D}_{\geq w_2} \) to \( \mathcal{D}_{\geq w_1+w_2} \).
6. There are exact sequences for \( C_0, C'_0 \in \mathcal{D}_m(X_0) \)

\[
\text{Ext}^{n-1}_{\mathcal{D}(X)}(C, C')_F \leftrightarrow \text{Ext}^n_{\mathcal{D}(X_0)}(C_0, C'_0) \leftrightarrow \text{Ext}^n_{\mathcal{D}(X)}(C, C')^F
\]

where \( F \) is the geometric Frobenius.

7. Assume \( C_0 \in \mathcal{D}_{\leq w}(X_0) \), \( C'_0 \in \mathcal{D}_{\geq w}(X_0) \). We have \( \text{Ext}^n_{\mathcal{D}(X)}(C, C')_F = \text{Ext}^n_{\mathcal{D}(X)}(C, C')^F = 0 \) for \( n \geq 0 \). Therefore, \( \text{Ext}^n_{\mathcal{D}(X_0)}(C_0, C'_0) = 0 \) for \( n > 0 \). Further, \( \text{Ext}^n_{\mathcal{D}(X_0)}(C_0, C'_0) = 0 \) provided in addition that \( C_0, C'_0 \) are mixed perverse sheaves.

8. The pullback \( \text{Ext}^n_{\mathcal{D}(X_0)}(C_0, C'_0) \to \text{Ext}^n_{\mathcal{D}(X)}(C, C') \) is the zero map for \( C_0 \in \mathcal{D}_{\leq w}(X_0), C'_0 \in \mathcal{D}_{\geq w}(X_0) \) and \( n > 0 \).

9. A subquotient of a mixed perverse sheaf of weight \( \leq w \) (resp. \( \geq w \)) is of weight \( \leq w \) (resp. \( \geq w \)).

10. A mixed perverse sheaf \( C_0 \in \mathcal{D}_m(X_0) \) admits a unique weight filtration \( W^\bullet C_0 \) whose grade piece \( Gr^i_W C_0 = W^i C_0/W^{i-1} C_0 \) is pure of weight \( i \). A morphism \( C_0 \to C'_0 \) of mixed perverse sheaves maps \( W^i C_0 \) to \( W^i C'_0 \).

11. A mixed complex \( C_0 \) is of weight \( \leq w \) (resp. \( \geq w \)) if and only if \( \mathcal{P}H^i(C_0) \) is of weight \( \leq w+i \) (resp. \( \geq w+i \)) for all \( i \).

Now we introduce the notion of pure resolution of mixed complexes. Suppose we are given a system of exact triangles

\[
C^{\leq w-1}_0 \to C^w_0 \to C^w_0
\]

with \( C^{\leq w}_0 \in \mathcal{D}_{\leq w}(X_0) \) and \( C^w_0 \) being pure of weight \( w \). Assume further that the exact triangles (4.5) are identical to \( 0 \to 0 \to 0 \) for \( w \ll 0 \) and are identical to \( C_0 \to C_0 \to 0 \) for \( w \gg 0 \), where \( C_0 \) is a prescribed mixed complex. Following
Lemma 4.1.3 we have a complex forming by objects and morphisms from \(D_m(X_0)\)
\[
\cdots \to C_0^{w+1}[-w-1] \to C_0^w[-w] \to C_0^{w-1}[-w+1] \to \cdots
\]
(4.6)
in which the differentials are the compositions
\[
C_0^w[-w] \to C_0^{w-1}[-w+1] \to C_0^{w-1}[-w+1].
\]

**Definition 4.2.1.** In the above notations, we assign degree \(-w\) to \(C_0^w[-w]\) and call (4.6) a pure resolution of \(C_0\).

In particular, given a mixed perverse sheaf \(C_0 \in \mathcal{M}_m(X_0)\), the unique weight filtration \(W^\bullet C_0\) gives rise to a system of exact sequences in \(\mathcal{M}_m(X_0)\) (hence exact triangles in \(D_m(X_0)\))
\[
W^{w-1}C_0 \hookrightarrow W^wC_0 \twoheadrightarrow Gr^w_W C_0,
\]
then a pure resolution of \(C_0\)
\[
\cdots \to Gr^{w+1}_W C_0[-w-1] \to Gr^w_W C_0[-w] \to Gr^{w-1}_W C_0[-w+1] \to \cdots
\]
(4.8)

**Definition 4.2.2.** We call (4.8) the canonical pure resolution of the mixed perverse sheaf \(C_0\). Moreover, when \(C_0\) is clear from context, we slightly abuse language to call the pullback of (4.8) the canonical pure resolution of the perverse sheaf \(C\).

**Proposition 4.2.3.** We have the followings.

1. If \(f : X_0 \to Y_0\) is a proper morphism, then \(f_! C_0 \in \mathcal{D}_m(Y_0)\) transforms a pure resolution of \(C_0 \in \mathcal{D}_m(X_0)\) into a pure resolution of \(f_! C_0 \in \mathcal{D}_m(Y_0)\).
2. If \(f : X_0 \to Y_0\) is a smooth morphism of relative dimension \(d\), then \(f^*[d]\) transforms a pure resolution of \(C_0 \in \mathcal{D}_m(Y_0)\) into a pure resolution of \(f^*[d]C_0 \in \mathcal{D}_m(X_0)\), up to a grade shifting.
3. The Verdier duality functor \(D\) transforms a pure resolution of \(C_0 \in \mathcal{D}_m(X_0)\) into a pure resolution of \(DC_0 \in \mathcal{D}_m(X_0)\).
4. The outer tensor product functor \(\boxtimes\) transforms the canonical pure resolutions of \(C_0 \in \mathcal{M}_m(X_0), C'_0 \in \mathcal{M}_m(Y_0)\) into the canonical pure resolution of \(C_0 \boxtimes C'_0 \in \mathcal{M}_m(X_0 \times Y_0)\).

Moreover, for a mixed perverse sheaf \(C_0\), its canonical pure resolution is transformed into canonical ones in (2) and (3).

**Proof.** Claim (1)(2) follow directly from (1.2)(3)(4).

We show then the third claim. Suppose \(C_0^\bullet\) is a pure resolution of \(C_0\) derived from a system of exact triangles
\[
C_0^{w-1} \to C_0^w \to C_0^w.
\]
We form triangle morphisms

\[ B^w_0 \rightarrow C^w_0 \rightarrow C^w_0 \]

such that \( d_w \) is the identity for large \( w \). By Lemma 4.1.4 there are exact triangles \( B^{w-1}_0 \rightarrow B^w_0 \rightarrow C^w_0 \) inducing the given complex \( C_0^* \). Then the exact triangles

\[ (DB^{w-1}_0)[-1] \rightarrow (DB^w_0)[-1] \rightarrow DC^w_0 \]

(4.9) induce the Verdier dual of \( C_0^* \).

Using 4.2(9)-(11) we can show by induction that \( \text{Img} pH^i(d_w) = W^{w+i}H^i(C_0) \) and that \( \text{Ker} pH^i(d_w) \) is pure of weight \( w + i \). Therefore, \( pH^i(B^w_0) \) is of weight \( \geq w + i \). Hence \( B^{w-1}_0 \in D_{\geq w}(X_0); (DB^w_0)[-1] \in D_{\leq w-1}(X_0) \). That being said, the exact triangles (4.9) define a pure resolution of \( DC_0 \). This proves Claim (3).

Below we prove Claim (4) by using 4.2(5). First, we show

\[ Gr^*_W(C_0 \boxtimes C_0') = \oplus_j Gr^i_W C_0 \boxtimes Gr^*_W C_0'. \]

(4.10)

Assume \( C_0 \) is of weight \( \leq i \). Then by 4.2(5) we have exact sequences

\[ W^*(W^{i-1}C_0 \boxtimes C_0') \hookrightarrow W^*(C_0 \boxtimes C_0') \rightarrow W^*(Gr^i_W C_0 \boxtimes C_0') \]

(4.11)

and thus

\[ Gr^*_W(W^{i-1}C_0 \boxtimes C_0') \hookrightarrow Gr^*_W(C_0 \boxtimes C_0') \rightarrow Gr^*_W(Gr^i_W C_0 \boxtimes C_0'). \]

Clearly

\[ Gr^*_W(Gr^i_W C_0 \boxtimes C_0') = Gr^i_W C_0 \boxtimes Gr^{*-i} W C_0'. \]

By induction on weight we may assume further

\[ Gr^*_W(W^{i-1}C_0 \boxtimes C_0') = \oplus_{j<i} Gr^j_W C_0 \boxtimes Gr^{*-j} W C_0'. \]

Moreover, from the Künneth formula

\[ \text{Ext}^\bullet_D(X_0 \times Y_0)(A_0 \boxtimes B_0, A'_0 \boxtimes B'_0) = \text{Ext}^\bullet_D(X_0)(A_0, A'_0) \otimes \text{Ext}^\bullet_D(Y_0)(B_0, B'_0) \]

and 4.2(7) we deduce that

\[ \text{Ext}^1_D(X_0 \times Y_0)(Gr^*_W C_0 \boxtimes Gr^{*-i} W C_0', \oplus_{j<i} Gr^j_W C_0 \boxtimes Gr^{*-j} W C_0') = 0. \]

Hence (4.10) follows.

Next, we determine the differentials in the canonical pure resolution of \( C_0 \boxtimes C_0' \)

\[ Gr^*_W(C_0 \boxtimes C_0')[-w] \rightarrow Gr^{w-1}_W(C_0 \boxtimes C_0')[-w + 1]. \]

By the Künneth formula and 4.2(7) again, the morphisms by restriction

\[ Gr^j_W C_0 \boxtimes Gr^{w-j}_W C_0'[-w] \rightarrow Gr^k_W C_0 \boxtimes Gr^{w-1-k}_W C_0'[-w + 1] \]

(4.12)
must be zero unless \( j = k \) or \( k + 1 \). For \( j = k \), by using the exact sequences (4.11) and by induction on weight, one verifies that the induced morphisms (4.12) coincide with the differentials in the canonical pure resolutions of \( G^j_W C_0 \otimes C'_0 \). Similarly for \( j = k + 1 \). This concludes Claim (4).

The moreover part of the proposition is clear. □

The rest of this subsection is dedicated to the proof of the following theorem.

**Theorem 4.2.4.** Let \( C_0 \in \mathcal{M}_m(X_0) \) be a mixed perverse sheaf. Then the pullbacks (to \( X \)) of the pure resolutions of \( C_0 \) are all homotopy equivalent.

**Proof.** Suppose we are given a pure resolution of \( C_0 \) derived from a system of exact triangles

\[
C_0^{\leq w-1} \rightarrow C_0^{\leq w} \rightarrow C_0^w. \tag{4.13}
\]

We shall show that its pullback to \( X \) is homotopy equivalent to the pullback of the canonical one.

Let \( p_{\tau \leq n}, p_{\tau \geq n} \) denote the truncation functors of \( \mathcal{D}(X_0) \) for the perverse t-structure. If all \( C_0^w \) are mixed perverse sheaves, an easy induction shows that so are \( C_0^{\leq w} \). Property 4.2.(9),(10) then imply in this case that the exact triangles (4.13) must be the canonical ones \( W^{w-1} C_0 \rightarrow W^w C_0 \rightarrow G^w_W C_0 \); we are done. Otherwise, there exists \( C_0^w \) such that \( p_{\tau \leq -1} C_0^w \neq 0 \) or \( p_{\tau \geq 1} C_0^w \neq 0 \). Without loss of generality we consider the former case; the latter can be treated by passing to Verdier dual.

Let \( w \) be maximal such that \( p_{\tau \leq -1} C_0^w \neq 0 \). We form a triangle morphism

\[
\begin{array}{c}
C_0^{\leq w-1} \\
\downarrow e \\
\tilde{C}_0^{\leq w-1}
\end{array} \rightarrow \begin{array}{c}
C_0^{\leq w} \\
\downarrow p \\
\tilde{C}_0^{\leq w}
\end{array} \rightarrow \begin{array}{c}
C_0^w \\
\downarrow p_{\tau \geq 0} C_0^w
\end{array} \tag{4.14}
\]

where \( p \) is the morphism in the exact triangle

\[
p_{\tau \leq -1} C_0^w \rightarrow C_0^w \rightarrow p_{\tau \geq 0} C_0^w \rightarrow p_{\tau \leq -1} C_0^w.
\]

Applying Lemma 4.1.2 to the sequence

\[
C_0^{\leq w-2} \rightarrow C_0^{\leq w-1} \rightarrow \tilde{C}_0^{\leq w-1} \rightarrow C_0^{\leq w},
\]

then gives an exact triangle

\[
C_0^{\leq w-2} \rightarrow \tilde{C}_0^{\leq w-1} \rightarrow \tilde{C}_0^{w-1}, \tag{4.15}
\]

a triangle morphism

\[
\begin{array}{c}
p_{\tau \leq -1} C_0^w \\
\downarrow \\
p_{\tau \leq -1} C_0^w
\end{array} \rightarrow \begin{array}{c}
C_0^w \\
\downarrow p \\
C_0^w \rightarrow \tilde{C}_0^{w-1}[1]
\end{array} \rightarrow \begin{array}{c}
\tilde{C}_0^{w-1}[1] \\
\downarrow v
\end{array} \rightarrow \begin{array}{c}
\tilde{C}_0^{w-1}
\end{array} \tag{4.16}
\]
and a complex morphism

\[
\cdots \to C_0^{w+1}[-1] \to C_0^w \to C_0^{w-1}[1] \to C_0^{w-2}[2] \to \cdots \tag{4.17}
\]

\[
\cdots \to C_0^{w+1}[-1] \to p_{\tau \geq 0} C_0^w \to \tilde{C}_0^{w-1}[1] \to C_0^{w-2}[2] \to \cdots
\]

of which the bottom row is the one derived from the exact triangles (4.13) with

\[C_0^{w-1}, C_0^{\leq w-1}, C_0^w\]

replaced by \[\tilde{C}_0^{w-1}, \tilde{C}_0^{\leq w-1}, p_{\tau \geq 0} C_0^w\], respectively.

By the maximality of \(w\), an inductive argument shows \(p_{\tau \leq -1} C_0^w = 0\). Then from (4.14) and the bottom row of (4.16) we conclude that \(\tilde{C}_0^w\) is pure of weight \(w - 1\).

Hence \(\tilde{C}_0^w\) is of weight \(\leq w - 1\). Further, from (4.15) and the bottom row of (4.16) we conclude that \(C_0^{w-1}\) is pure of weight \(w - 1\).

It follows on the one hand that, up to a grade shifting, the bottom row of (4.17) is a pure resolution of \(C_0\); and on the other hand that the pullbacks of \(u, v\) to \(X\) are zero by (4.2)(8), thus by Lemma 4.1.1 the pullback of (4.17) to \(X\) is a homotopy equivalence. Summarizing, we obtain a new pure resolution of \(C_0\) whose pullback to \(X\) is homotopy equivalent to that of the original one.

Note that the above process remains all \(C_i\) untouched but truncates off nontrivial direct summands from \(C_0^{w-1}\) and \(C_0^w\). Therefore, after finitely many repetitions, the original pure resolution can be deformed to the canonical one. This completes the proof of our theorem. \(\square\)

Remark 4.2.5. The claim of the theorem may not be true if we do not pull back pure resolutions to \(X\). For example, let \(X_0 = Spec(\mathbb{F}_q)\) and, accordingly, \(X = Spec(\mathbb{F})\). We can form an exact sequence of pure perverse sheaves (of weight 0)

\[
\cdots \to 0 \to \mathbb{Q}_{l,X_0} \to A_0 \to \mathbb{Q}_{l,X_0} \to 0 \to \cdots
\]

with \(A_0\) indecomposable [BBD82, 5.3.9]. It is easy to realize the above sequence as a pure resolution of the zero mixed perverse sheaf, which is, however, may not be homotopic to zero. Indeed, the existence of such indecomposable \(A_0\) is the obstacle preventing the morphisms \(u, v\) in (4.16) from being zero. If we pull back the above sequence to \(X\), it yields now a complex homotopic to zero

\[
\cdots \to 0 \to \mathbb{Q}_{l,X} \to \mathbb{Q}_{l,X} \oplus \mathbb{Q}_{l,X} \to \mathbb{Q}_{l,X} \to 0 \to \cdots
\]
4.3. The complex $T^\bullet$. First, let us remark that, according to the standard reduction technique [BBD82, 6.1] from the base field $\mathbb{C}$ to finite fields, it makes sense to pull back a mixed complex to a complex algebraic variety $X$ thus form a complex of $\mathbb{C}$-sheaves, as if $X$ is obtained from a scheme over a finite field by base field extension.

Readers who are unsatisfactory with such reduction may simply bypass it by transferring from the very beginning of this paper to the setting of varieties over algebraic closures of finite fields and categories of $\mathbb{Q}_l$-sheaves.

Keep the notations of Section 2.3. Let $X = G/P$ be a partial flag variety. For each $w \in W$ set

$$
\Delta_w^+ = j_w!*\mathbb{C}_{X_w}[\dim X_w], \quad \Delta_w^- = D\Delta_w^+
$$

(4.18)

where $j_w : X_w \to X$ is the inclusion. Since $j_w$ is an affine morphism, $\Delta_w^\pm$ are perverse sheaves on $X$.

By regarding $\mathbb{C}_{X_w}$ as the pullback of a constant $\mathbb{Q}_l$-sheaf (pure of weight 0) for each $w \in W$, we are clear from which mixed perverse sheaves $\Delta_w^\pm$ are pulled back. Then we define $T^\bullet(P, \Delta_w^\pm)$ to be the canonical pure resolutions of $\Delta_w^\pm \in M_B(X)$.

The first properties of these complexes are as follows.

**Proposition 4.3.1.** Let $P \subset G$ be parabolic subgroups containing $B$.

1. $T^n(P, \Delta_w^\pm)[-n]$ are semisimple subquotients of $\Delta_w^\pm$. In particular, they are self dual and $B$-equivariant.

2. $DT^\bullet(P, \Delta_w^\pm) = T^\bullet(P, \Delta_w^\pm)$.

**Proof.** Claim (1) is immediate from the definition of canonical pure resolution and [12](2). Claim (2) follows from Proposition [12](3). \qed

**Example 4.3.2.** For the unit element $e \in W$, $T^\bullet(P, \Delta_e^\pm)$ are clearly the complex concentrated at degree 0

$$
\cdots \to 0 \to 0 \to \Delta_e^+ \to 0 \to 0 \to \cdots
$$

**Example 4.3.3.** For a simple reflection $s \in W$, note that $\overline{X_s} = X_s \sqcup X_e \cong \mathbb{P}^1$ and $\mathcal{IC}(\overline{X_s}) = \mathcal{IC}_{\overline{X_s}}[1]$. The complex $T^\bullet(P, \Delta_s^+)$ is the one concentrated at degree $-1$, 0

$$
\cdots \to 0 \to \mathcal{IC}(\overline{X_s})[-1] \xrightarrow{a} \Delta_e^+ \to 0 \to 0 \to \cdots
$$

where $a$ is the adjunction morphism $\mathbb{C}_{\overline{X_s}} \to j_{es}j_e^*\mathbb{C}_{\overline{X_s}}$. More precisely, $T^\bullet(P, \Delta_s^+)$ is the one derived from the exact sequences

$$
0 \hookrightarrow \Delta_e^+ \twoheadrightarrow \Delta_s^+,
\Delta_e^+ \hookrightarrow \Delta_s^+ \twoheadrightarrow \mathcal{IC}(\overline{X_s}),
$$

of which the latter is actually the adjunction triangle

$$
j_{es}j_e^!\mathcal{IC}(\overline{X_s}) \to \mathcal{IC}(\overline{X_s}) \to j_{es}j_e^*\mathcal{IC}(\overline{X_s}).$$
Example 4.3.4. In the notations of Section 3.1 we have the followings, where \( I\mathcal{C}(X_w) \) is abbreviated to \( I\mathcal{C}_w \).

\[
\begin{align*}
T^\bullet(P_1,2), \Delta^+_{s,s_1} & = \cdots \to 0 \to 0 \to I\mathcal{C}_{s_1}[-2] \to I\mathcal{C}_{s_1}[-1] \to \cdots \\
T^\bullet(P_1,3), \Delta^+_{s,s_2,s_3} & = \cdots \to 0 \to I\mathcal{C}_{s_1,s_2,s_3}[-3] \to I\mathcal{C}_{s_1,s_3}[-2] \to 0 \to \cdots \\
T^\bullet(P_2,2), \Delta^+_{s,s_1,s_3} & = \cdots \to I\mathcal{C}_{s_1,s_3,s_2}[-4] \to I\mathcal{C}_{s_1,s_3}[-3] \to I\mathcal{C}_w[-2] \to 0 \to \cdots 
\end{align*}
\]

For a collection of parabolic subgroups \( P, P_1, \ldots, P_k \subset G \) containing \( B \), there is a principal \( P \)-bundle

\[
\mu^P,P_1,\ldots,P_k : G \times G/P_1 \times \cdots \times G/P_k \to G/P \times G/P_1 \times \cdots \times G/P_k \\
(g, x_1, \ldots, x_k) \mapsto ([g], gx_1, \ldots, gx_k).
\]

Recall that given a principal \( P \)-bundle \( \mu : X \to Y \), the functor \( \mu^*([\dim P]) \) is perverse \( t \)-exact and, moreover, together with the functor \( \mu_p = pH^{-\dim P}\mu^*_p \) it defines an equivalence of the categories \( \mathcal{M}_P(X), \mathcal{M}(Y) \). By abusing notations, when \( T^\bullet \) is the complex derived from a system of exact sequences \( C^{\leq w-1} \hookrightarrow C^{\leq w} \to C^w \) in \( \mathcal{M}_P(X) \) (cf. Lemma 4.1.3), we denote by \( \mu^T \) the complex derived from the system of exact sequences \( \mu^*_p C^{\leq w-1} \hookrightarrow \mu^*_p C^{\leq w} \to \mu^*_p C^w \) in \( \mathcal{M}(Y) \).

Let \( P, Q \subset G \) be parabolic subgroups containing \( B \) and let \( G \) acts on \( G/P \times G/Q \) diagonally. Let \( \mathcal{W}^p \) be the set of shortest representatives of the double cosets \( \mathcal{W}^p \mathcal{W}/\mathcal{W}^q \). Then we have a decomposition by \( G \)-orbits

\[
G/P \times G/Q = \bigsqcup_{w \in \mathcal{W}^p \mathcal{W}} O_w 
\]

(4.19)

where \( O_w \) is the \( G \)-orbit of \( (P, wQ) \).

Notice the one-to-one correspondence between the \( G \)-orbits of \( G/P \times G/Q \) and the \( P \)-orbits of \( G/Q \)

\[
O_w \leftrightarrow P \hat{w} Q / Q. 
\]

(4.20)

Assume \( w \in \mathcal{W}^p \mathcal{W} \) is such that \( w_P w = w w_Q \) where \( w_P, w_Q \) are the longest elements in \( \mathcal{W}_P, \mathcal{W}_Q \), respectively. Then \( \Delta^+_w \in \mathcal{M}_P(G/Q) \). We define \( T^\bullet(P, Q, \Delta^+_w) \) to be the canonical pure resolutions of

\[
\mu^P,Q(\mathcal{C}_G[\dim G] \boxtimes \Delta^+_w) \in \mathcal{M}_G(G/P \times G/Q).
\]

The following proposition follows easily from (4.2) and Proposition 4.2.3(2).

**Proposition 4.3.5.** Let \( P, Q \subset G \) be parabolic subgroups containing \( B \).

1. \( T^n(P, Q, \Delta^+_w)[-n] \) are semisimple \( G \)-equivariant perverse sheaves.
2. \( \tau^*T^\bullet(P, Q, \Delta^+_w) = T^\bullet(Q, P, \Delta^+_w) \), where \( \tau : G/P \times G/Q \to G/Q \times G/P \) is the transposition.
3. \( T^\bullet(-\dim G/P)(P, Q, \Delta^+_w) = \mu^P,Q(\mathcal{C}_G[\dim G] \boxtimes T^\bullet(Q, \Delta^+_w))[-\dim G/P] \).
Thus, the canonical pure resolution of (4.25) is
\[
C \ast C' = \pi_{13!}(\pi_{12}^*C \otimes \pi_{23}^*C') \in \mathcal{D}(X \times Z)
\] (4.21)
where \(\pi_{ij}\) is the projection of \(X \times Y \times Z\) onto the \(i, j\)-th coordinates. In this way, each \(C \in \mathcal{D}(X \times Y)\) gives rise to a functor
\[
C \ast : \mathcal{D}(Y \times Z) \to \mathcal{D}(X \times Z).
\] (4.22)
It is left adjoint to the functor
\[
D \circ \tau^*C \ast \circ D : \mathcal{D}(X \times Z) \to \mathcal{D}(X \times Z)
\] (4.23)
where \(\tau : X \times Y \to Y \times X\) is the transposition.

**Lemma 4.3.6.** Let \(P, Q, R \subset G\) be parabolic subgroups containing \(B\). There are natural isomorphisms for \(C_1 \in \mathcal{M}_P(G/Q), C_2 \in \mathcal{M}_Q(G/R),\)
\[
\pi_{12}^!\mu_y^{P,Q}(\mathcal{C}_G[\dim G] \boxtimes C_1)[- \dim G/P] \otimes \pi_{23}^!\mu_y^{Q,R}(\mathcal{C}_G[\dim G] \boxtimes C_2)[- \dim G/Q]
\]
\[
\cong \mu_y^{P,Q,R}(\mathcal{C}_G[\dim G] \boxtimes \mu_y^{Q,R}(\mu^Q[y^*\dim Q]C_1 \boxtimes C_2))[- \dim G/P].
\]

*Proof.* A direct computation. \(\square\)

**Lemma 4.3.7.** Let \(P, Q, R \subset G\) be parabolic subgroups containing \(B\). The complex
\[
\pi_{12}^!T^\bullet(P, Q, \Delta_{u_1}^\epsilon) \otimes \pi_{23}^!T^\bullet-\dim G/Q(Q, R, \Delta_{u_2}^\epsilon)
\] (4.24)
is the canonical pure resolution of the perverse sheaf on \(G/P \times G/Q \times G/R\)
\[
\mu_y^{P,Q,R}(\mathcal{C}_G[\dim G] \boxtimes \mu_y^{Q,R}(\mu^Q[y^*\dim Q]\Delta_{w_1}^\epsilon \boxtimes \Delta_{w_2}^\epsilon)).
\] (4.25)

*Proof.* By Proposition 4.2.3(2)(4), the canonical pure resolution of
\[
\mu_y^{Q,R}(\mu^{Q[y^*\dim Q]}\Delta_{w_1}^\epsilon \boxtimes \Delta_{w_2}^\epsilon)
\]
is
\[
\mu_y^{Q,R}(\mu^{Q[y^*\dim Q]}T^\bullet(Q, \Delta_{w_1}^\epsilon) \boxtimes T^\bullet(R, \Delta_{w_2}^\epsilon))
\]
Thus, the canonical pure resolution of (4.25) is
\[
\mu_y^{P,Q,R}(\mathcal{C}_G[\dim G] \boxtimes \mu_y^{Q,R}(\mu^Q[y^*\dim Q]\Delta_{w_1}^\epsilon \boxtimes \Delta_{w_2}^\epsilon)))[- \dim G/P]
\]
which by Proposition 4.3.5(3) and Lemma 4.3.6 is equal to (4.24). \(\square\)

**Proposition 4.3.8.** Let \(P, Q, R \subset G\) be parabolic subgroups containing \(B\). Then
\[
T^\bullet(P, Q, \Delta_{u_1}^\pm) \ast T^\bullet-\dim G/Q(Q, R, \Delta_{u_2}^\pm) \simeq T^\bullet(P, R, \Delta_{w_1}^\pm)
\]
whenever \(\ell(w_1w_2) = \ell(w_1) + \ell(w_2).\)
Proof. Suppose $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. Thus we have
\[ \pi_{2!}\mu_b^{Q,R}(\mu^{Q*}[\dim Q]\Delta_{w_1}^\pm \boxtimes \Delta_{w_2}^\pm) \cong \Delta_{w_1w_2}^\pm, \]
where $\pi_2 : G/Q \times G/R \to G/R$ is the projection onto the second coordinate. It follows that
\[ \pi_{13!}\mu_b^{P,Q,R}\left( \mathcal{C}_{[\dim G]} \boxtimes \mu_b^{Q,R}(\mu^{Q*}[\dim Q]\Delta_{w_1}^\pm \boxtimes \Delta_{w_2}^\pm) \right) \cong \mu_b^{P,R}\left( \mathcal{C}_{[\dim G]} \boxtimes \Delta_{w_1w_2}^\pm \right). \]
Then apply Lemma 4.3.7 and Proposition 4.2.3(1), Theorem 4.2.4. \qed

The next lemma remains true if we change the base field $\mathbb{C}$ to finite fields. To make the point clear, we write down the Tate twist terms in its proof.

Lemma 4.3.9. For each simple reflection $s \in \mathcal{W}$, we have
\[ \pi_{2!}\mu_b^{P,B}(\mu^{B*}[\dim B]\Delta_s^+ \boxtimes \Delta_s^-) \cong \Delta_e^\pm \]
where $\pi_2 : G/B \times G/B \to G/B$ is the projection onto the second coordinate.

Proof. We put $\Delta = \mu_b^{P,B}(\mu^{B*}[\dim B]\Delta_s^+ \boxtimes \Delta_s^-)$. Then
\[ \Delta = \mu_b^{P,B}(\mu^{B*}[\dim B][j_s!\mathbb{C}_{X_s}[1] \boxtimes j_s*\mathbb{C}_{X_s}[1](1))] = j_{Z_1!}j_{Z_0*}\mathbb{C}_{Z_0}[2](1) \]
where $j_{Z_0} : Z_0 \to Z_1, j_{Z_1} : Z_1 \to G/B \times G/B$ are the inclusions of subvarieties
\[
Z_0 = \{(g_1),[g_1g_2]) | g_1 \in B\hat{s}B, g_2 \in B\hat{s}B\} \subset G/B \times G/B,
\]
\[
Z_1 = \{(g_1),[g_1g_2]) | g_1 \in B\hat{s}B, g_2 \in \overline{B\hat{s}B}\} \subset G/B \times G/B.
\]
Set $Y = Z_1 \setminus Z_0$ and let $i_Y : Y \to Z_1$ be the inclusion. Note that via the morphism $\pi_2$, $Z_1$ becomes a $\mathbb{C}$-bundle over $\overline{X}_s \cong \mathbb{P}^1$. The functor $\pi_{2!}j_{Z_1!}$ transforms the adjunction triangle
\[
i_{Y!}\mathbb{C}_Y \to \mathbb{C}_{Z_1}[2](1) \to j_{Z_0*}\mathbb{C}_{Z_0}[2](1)
\]
to an exact triangle
\[
j_{s!}\mathbb{C}_{X_s} \to \mathbb{C}_{\overline{X}_s} \to \pi_{2!}\Delta.
\]
Applying the functors $j^*_e, j^*_s$, one verifies that
\[
j^*_e\pi_{2!}\Delta \cong j^*_e\Delta_e^+, \quad j^*_s\pi_{2!}\Delta \cong 0.
\]
Hence the isomorphism $\pi_{2!}\Delta \cong \Delta_e^+$ follows.
Applying the Verdier duality functor $D$ yields the other isomorphism. \qed

Proposition 4.3.10. Let $P, Q \subset G$ be parabolic subgroups containing $B$. Then
\[ T^*(P, Q, \Delta_{w_1}^\pm) * T^{*-\dim G/Q}(Q, P, \Delta_{w}^\pm) \cong T^*(P, P, \Delta_e^\pm). \]
Proof. Choose a reduced word \( w = s_{i_1} s_{i_2} \cdots s_{i_t} \). Defines a principal \( B^{2t} \)-bundle
\[
\mu : G^{2t} \to (G/B)^{2t}
\]
\[(g_1, g_2, \ldots, g_{2t}) \mapsto ([g_1], [g_1 g_2], \ldots, [g_1 \cdots g_{2t}]).\]
Let \( \pi_2 : G/Q \times G/P \to G/P \) and \( \pi_2 : (G/B)^{2t} \to G/B \) be the projections onto the last coordinates and let \( \rho : G/B \to G/P \) be the obvious projection. By a direct computation,
\[
\pi_2 \mu_\rho^{Q,P} \left( \mu^{Q,*} [\dim Q] \Delta^\pm_{w-1} \boxtimes \Delta^\pm_w \right)
\]
\[
\cong \rho_! \pi_2 \mu_\rho \left( \mu^{B,*} \Delta^\pm_{s_{i_1}} \boxtimes \cdots \boxtimes \mu^{B,*} \Delta^\pm_{s_{i_t}} \boxtimes \mu^{B,*} \Delta^\pm_{s_{i_1}} \boxtimes \cdots \boxtimes \mu^{B,*} \Delta^\pm_{s_{i_t}} [2t \dim B] \right).
\]
Applying Lemma 4.3.9 repeatedly, we deduce that the right hand side is further isomorphic to \( \Delta^\pm \).

Then, the same argument as the proof of Proposition 4.3.8 concludes the proposition. \( \square \)

**Proposition 4.3.11.** Let \( P, Q, R_1, R_2 \subset G \) be parabolic subgroups containing \( B \). Assume \( R_1 \subset R_2 \) and let \( p : G/R_1 \to G/R_2 \) be the obvious projection. We have natural isomorphisms for \( C \in \mathcal{D}(G/P \times G/Q) \) and \( C_i \in \mathcal{D}(G/Q \times G/R_i) \)
\[
(\text{Id}_{G/P} \times p)_!(C \ast C_1) = C \ast (\text{Id}_{G/Q} \times p)_! C_1,
\]
\[
(\text{Id}_{G/P} \times p)^*(C \ast C_2) = C \ast (\text{Id}_{G/Q} \times p)^* C_2.
\]

**Proof.** An easy computation. \( \square \)

4.4. **Categorification theorem.** Now we transfer to the notations of Section 3.1. Let \( X = \bigsqcup_r X^r_d \) be the union of Grassmannian varieties.

First, we enhance the category \( \mathcal{Q}_d \) to \( \mathcal{Q}_d \), which is defined to be the full subcategory of \( \mathcal{D}(G/P_d \times X) \) consisting of the \( G \)-equivariant semisimple complexes. Notice the canonical correspondence between the \( G \)-orbits of \( G/P_d \times X \) and the \( P_d \)-orbits of \( X \) given by (4.20). The key observation is that the functor
\[
\mu_\rho^{P_d} (\mathcal{C}_G[\dim G] \boxtimes -) : \mathcal{M}_{P_d}(X) \to \mathcal{M}_G(G/P_d \times X)
\]
gives rise to a one-to-one correspondence between the (isomorphism classes of) perverse sheaves from \( \mathcal{Q}_d \) and \( \mathcal{Q}_d \). In particular, it identifies the Grothendieck group of \( \mathcal{Q}_d \) with \( \mathcal{Q}_d \).

All concepts and claims from Section 3 can be migrated word by word to a version for \( \mathcal{Q}_d \) in the obvious way. For example, \( K_{d,r}, \mathcal{E}^{(n)}_d, \mathcal{F}^{(n)}_d \) are endofunctors of \( \mathcal{D}(G/P_d \times X) \) defined by assembling
\[
K_{d,r} = [2r - d] : \mathcal{D}(G/P_d \times X^r_d) \to \mathcal{D}(G/P_d \times X^r_d),
\]
\[
\mathcal{E}^{(n)}_{d,r+n} = p^! p^*[nr] : \mathcal{D}(G/P_d \times X^{r+n}_d) \to \mathcal{D}(G/P_d \times X^{r+n}_d),
\]
\[
\mathcal{F}^{(n)}_{d,r} = p^! p^*[n(d - n - r)] : \mathcal{D}(G/P_d \times X^r_d) \to \mathcal{D}(G/P_d \times X^{r+n}_d).
\]
where \( p, p' \) are the projections

\[
G/P_d \times X_d^r \xrightarrow{p} G/P_d \times X_d^{r,n} \xrightarrow{p'} G/P_d \times X_d^{n}
\]  \hspace{1cm} (4.28)

They satisfy the functor isomorphisms stated in the propositions from Section 3.3, hence induce the same \( U_A \)-module structure on \( Q_d \) as the functors \( K, E(n), F(n) \).

Next, we identify the Weyl group \( W \) of \( G = GL(W) \) with the symmetric group \( S_d \) of the symbols \( \{1, 2, \ldots, d\} \). For each composition \( d = (d_1, d_2, \ldots, d_l) \) of \( d \) and for each permutation \( \sigma \in S_l \), we let \( \sigma \) act on the symbols \( \{1, 2, \ldots, d\} \) by permuting the blocks

\[
\{d_1 + \cdots + d_{i-1} + j \mid 1 \leq j \leq d_i\}_{1 \leq i \leq l}
\]

thus yield an element \( w(d, \sigma) \in W \), then define a couple of complexes formed by functors from \( D(G/P_d \times X) \) to \( D(G/P_{d(d)} \times X) \)

\[
\mathcal{R}^\bullet_{\sigma}(d, \sigma) = D \circ T^{-\dim G/P_d}(P_{\sigma(d)}, P_d, \Delta_{\sigma(d, \sigma)}) \circ D.
\]  \hspace{1cm} (4.29)

They are understood in the standard way as functors of bounded homotopic categories

\[
\mathcal{R}^\bullet_{\sigma}(d, \sigma) : K^b(D(G/P_d \times X)) \to K^b(D(G/P_{\sigma(d)} \times X)).
\]  \hspace{1cm} (4.30)

**Theorem 4.4.1.** The functor complexes \( \mathcal{R}^\bullet_{\sigma}(d, \sigma) \) satisfy the followings.

1. \( \mathcal{R}^\bullet_{\sigma}(\sigma_2(d), \sigma_1) \circ \mathcal{R}^\bullet_{\sigma}(d, \sigma_2) \simeq \mathcal{R}^\bullet_{\sigma}(d, \sigma_1 \sigma_2) \) if \( \ell(\sigma_1 \sigma_2) = \ell(\sigma_1) + \ell(\sigma_2) \).
2. \( \mathcal{R}^\bullet_{\sigma}(\sigma(d), \sigma^{-1}) \circ \mathcal{R}^\bullet_{\sigma}(d, \sigma) \simeq \text{Id} \) (concentrated at degree 0).
3. \( \mathcal{G}_{\sigma(d)} \circ \mathcal{R}^\bullet_{\sigma}(d, \sigma) \simeq \mathcal{R}^\bullet_{\sigma}(d, \sigma) \circ \mathcal{G}_d \) for \( \mathcal{G} \in \{K, E(n), F(n)\} \).
4. \( \mathcal{R}^\bullet_{\sigma}(d, \sigma) \) are right adjoint to \( D \mathcal{R}^\bullet_{\sigma}(\sigma(d), \sigma^{-1}) D \).
5. \( \mathcal{R}^\bullet_{\sigma}(d, \sigma) \) restrict to functors from \( \tilde{Q}_d \) to \( \tilde{Q}_{\sigma(d)} \).

**Proof.** (1)(2)(3) follow from Propositions 4.3.8, 4.3.10, 4.3.11, respectively.

(4) follows from Propositions 4.3.5(2).

(5) follows from Lemma 4.3.6 and the decomposition theorem. \( \square \)

**Corollary 4.4.2.** Let \( K^b(\tilde{Q}_d) \) be the bounded homotopic category of \( \tilde{Q}_d \). Via the valuation

\[
K^b(\tilde{Q}_d) \to Q_d, \quad C^\bullet \mapsto \sum_n (-1)^n [C^m],
\]

the system of functors

\[
\mathcal{R}^\bullet_{\sigma}(d, \sigma) : K^b(\tilde{Q}_d) \to K^b(\tilde{Q}_{\sigma(d)})
\]

induce a system of \( R \)-matrices on the \( U_A \)-modules \( Q_d \).

**Proof.** It suffices to show the valuation only depends on the homotopy equivalence class of \( C^\bullet \). But this is evident from Proposition 4.5.1 below. \( \square \)
Example 4.4.3. For the simplest nontrivial case \( d = (1, 1) \), \( S_2 = \{ \epsilon, \sigma \} \), it is not difficult to derive from Example 4.3.3 and Lemma 4.3.6 that \( R_+^\ast (d, \sigma) \) (left column) and \( R_-^\ast (d, \sigma) \) (right column) induce the following maps

\[
\begin{align*}
    b_{(0,0)} & \mapsto -q^2 b_{(0,0)}, & b_{(0,0)} & \mapsto -q^{-2} b_{(0,0)}, \\
    b_{(1,0)} & \mapsto b_{(1,0)} - q b_{(1,1)}, & b_{(1,0)} & \mapsto b_{(1,0)} - q^{-1} b_{(0,1)}, \\
    b_{(0,1)} & \mapsto -q^2 b_{(0,1)}, & b_{(0,1)} & \mapsto -q^{-2} b_{(0,1)}, \\
    b_{(1,1)} & \mapsto -q^2 b_{(1,1)}, & b_{(1,1)} & \mapsto -q^{-2} b_{(1,1)}.
\end{align*}
\]

In what follows we write \( 1^d = (1, 1, \ldots, 1) \) for the composition of \( d \) consisting of the \( 1 \)'s. The category \( Q_d \) is by definition a full subcategory of \( Q_{1^d} \), thus \( Q_d \) is canonically embedded in \( Q_{1^d} \). Observe that the same embedding is induced by the functor

\[
(\rho_d \times \text{Id}_X)^* \colon Q_d \to Q_{1^d}
\]

where \( \rho_d : G/B \to G/P_d \) is the obvious projection. The next proposition states that \( R_\pm^\ast (d, \sigma) \) and \( R_\pm^\ast (1^d, w(d, \sigma)) \) induce the same isomorphisms \( Q_d \to Q_{\sigma(d)} \).

**Proposition 4.4.4.** Let \( \rho_d : G/B \to G/P_d \) be the obvious projection. Then

\[
(\rho_{\sigma(d)} \times \text{Id}_X)^* \circ R_\pm^\ast (d, \sigma) \simeq R_\pm^\ast (1^d, w(d, \sigma)) \circ (\rho_d \times \text{Id}_X)^*.
\]

**Proof.** Put \( w = w(d, \sigma) \). We have \( \rho_d \Delta_w^\pm = \Delta_w^\pm \) (\( \Delta_w^\pm \) defined respectively on \( G/B \) and \( G/P_d \)). Hence

\[
(\text{Id}_{G/B} \times \rho_d)^! \mu_{B,B}^{|\dim G|} (C_G[\dim G] \boxtimes \Delta_w^\mp) = (\rho_{\sigma(d)} \times \text{Id}_{G/P_d})^! \mu_{B,B}^{|\dim P_d/B|} (C_G[\dim G] \boxtimes \Delta_w^\mp).
\]

Hence, by Proposition 4.2.3(1)(2) and Theorem 4.2.4

\[
(\text{Id}_{G/B} \times \rho_d)_! T_{\ast - \dim G/B} (B, B, \Delta_w^\mp) \simeq (\rho_{\sigma(d)} \times \text{Id}_{G/P_d})_! T_{\ast - \dim G/P_d} (P_{\sigma(d)}, P_d, \Delta_w^\mp).
\]

It follows that

\[
R_\pm^\ast (1^d, w(d, \sigma)) \circ (\rho_d \times \text{Id}_X)^* = D_{\pi_{13d}} \left( \pi_{12}^* T_{\ast - \dim G/B} (B, B, \Delta_w^\mp) \otimes \pi_{23}^* (\rho_d \times \text{Id}_X)^* [2 \dim P_d/B] D - \right) = D_{\pi_{13d}} \left( \pi_{12}^* (\text{Id}_{G/B} \times \rho_d)_! T_{\ast - \dim G/B} (B, B, \Delta_w^\mp) \otimes \pi_{23}^* [2 \dim P_d/B] D - \right) \simeq D_{\pi_{13d}} \left( \pi_{12}^* (\rho_{\sigma(d)} \times \text{Id}_{G/P_d})_! T_{\ast - \dim G/P_d} (P_{\sigma(d)}, P_d, \Delta_w^\mp) \otimes \pi_{23}^* [2 \dim P_d/B] D - \right) = \rho_{\sigma(d)} \times \text{Id}_X)^* D_{\pi_{13d}} \left( \pi_{12}^* T_{\ast - \dim G/P_d} (P_{\sigma(d)}, P_d, \Delta_w^\mp) \otimes \pi_{23}^* D - \right) = (\rho_{\sigma(d)} \times \text{Id}_X)^* \circ R_\pm^\ast (d, \sigma). \]

\[\square\]
The main results from the previous subsection are now stated as follows.

**Example 4.4.5.** Assume \( d = (d_1, d_2), S_2 = \{e, \sigma\} \) and \( d = d_1 + d_2 \). The Grassmannian \( X^0_d \) is a single point. It follows that, for each simple reflection \( s \in W \),

\[
\mathcal{R}_+^\bullet (1^d, s) \mathcal{IC}(G/B \times X^0_d) \simeq \mathcal{IC}(G/B \times X^0_d)[-2]
\]

where the right hand side is a complex concentrated at degree \(-1\). Clearly, \( \ell(w(d, \sigma)) = d_1 d_2 \). Therefore, by the above proposition, the highest weight vector \( b_{(0, 0)} \in Q_d \) is sent to \((-q^2)^{d_1 d_2} \cdot b_{(0, 0)}\) by the \( R \)-matrix induced from \( \mathcal{R}_+^\bullet (d, \sigma) \).

**Remark 4.4.6.** From Example 4.4.3, Proposition 4.4.4 and a compatibility result (whose statement and proof are left to the reader) between the functor complex \( \mathcal{R}_+^\bullet \) and the functor \( \text{Res} \) from Section 3.7, it can be shown that the system of \( R \)-matrices claimed in Corollary 4.4.2 are those composed of the isomorphisms

\[
R_{d_1, d_2} = (-q^2)^{d_1 d_2} \cdot \tau \cdot q^{1/2} H \otimes H \sum_{n=0}^{\infty} q^{n(n-1)/2} (q - q^{-1})^n [n]_q F^{(n)} \otimes E^{(n)}
\]

where \( \tau : \Lambda_{d_1} \otimes \Lambda_{d_2} \to \Lambda_{d_2} \otimes \Lambda_{d_1} \) is the transposition, \( H \) is formally defined by \( K = q^H \).

### 4.5. Abelian categorification

In this subsection we translate the categorification theorem into an abelian version.

Define algebra \( A_d^\bullet \) as in Section 3.6

\[
A_d^\bullet = \text{Ext}_D^\bullet (G/P_d \times X) (L_d, L_d)
\]

where

\[
L_d = \oplus_{S \in \mathcal{A}} \mathcal{IC}(S).
\]

Here \( \mathcal{A}_d \) denotes the set of the \( G \)-orbits of \( G/P_d \times X \).

Let \( A_d^\bullet \text{mof} \) denote the category of finite-dimensional graded left \( A_d^\bullet \)-modules and let \( A_d^\bullet \text{pmof} \) denote the full subcategory consisting of the projectives. We apply the same arguments as in Section 3.6

**Proposition 4.5.1.** The obvious functor \( \mathcal{Q}_d \to A_d^\bullet \text{pmof} \) is an equivalence of categories. Moreover, the equivalence identifies the Grothendieck group of \( A_d^\bullet \text{mof} \) with \( \mathbb{Q}(q) \otimes_A \mathbb{Q}_d \).

The main results from the previous subsection are now stated as follows.

**Theorem 4.5.2.** The followings define complexes of projective graded \( A_{\sigma(d)}^\bullet \)-\( A_d^\bullet \)-bimodules, hence complexes of exact functors from \( A_d^\bullet \text{mof} \) to \( A_{\sigma(d)}^\bullet \text{mof} \)

\[
\mathcal{R}_\pm^\bullet (d, \sigma) = \text{Ext}_D^\bullet (G/P_{\sigma(d)} \times X) (L_{\sigma(d)}, \mathcal{R}_\pm^\bullet (d, \sigma)L_d).
\]

They satisfy the homotopy equivalences and isomorphisms

\[
\mathcal{R}_\pm^\bullet (\sigma_2(d), \sigma_1) \otimes \mathcal{R}_\pm^\bullet (d, \sigma_2) \simeq \mathcal{R}_\pm^\bullet (d, \sigma_1 \sigma_2) \quad \text{if} \quad \ell(\sigma_1 \sigma_2) = \ell(\sigma_1) + \ell(\sigma_2),
\]

\[
\mathcal{R}_\pm^\bullet (d, \sigma) \otimes \mathcal{R}_\mp^\bullet (d, \sigma) \simeq A_d^\bullet \quad \text{(concentrated at degree} \ 0) \quad \text{and}
\]

\[
G_{\sigma(d)} \otimes \mathcal{R}_\pm^\bullet (d, \sigma) \simeq \mathcal{R}_\pm^\bullet (d, \sigma) \otimes G_d^\bullet \quad \text{for} \quad G \in \{ \mathcal{K}, \mathcal{K}^{-1}, \mathcal{E}, \mathcal{F} \}.
\]
Corollary 4.5.3. Let $\mathcal{K}^b(A_d^{\ast}\text{-mof})$ be the bounded homotopic category of $A_d^{\ast}\text{-mof}$ and identify the Grothendieck group of $A_d^{\ast}\text{-mof}$ with $\mathbb{Q}(q) \otimes_A Q_d$. Via the valuation

$$\mathcal{K}^b(A_d^{\ast}\text{-mof}) \to \mathbb{Q}(q) \otimes_A Q_d, \quad M^{\bullet \bullet} \mapsto \sum_{n} (-1)^n [H^n(M^{\bullet \bullet})],$$

the system of functors

$$\mathcal{R}_{+}^{\bullet \bullet}(d, \sigma) : \mathcal{K}^b(A_d^{\ast}\text{-mof}) \to \mathcal{K}^b(A_{\sigma(d)}^{\ast}\text{-mof})$$

induce a system of $R$-matrices on the $U$-modules $\mathbb{Q}(q) \otimes_A Q_d$.

Remark 4.5.4. The same is true if we replace $\mathcal{K}^b(A_d^{\ast}\text{-mof})$ by the bounded derived category $\mathcal{D}^b(A_d^{\ast}\text{-mof})$.

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