A SHARP FORM OF THE MARCINKIEWICZ INTERPOLATION THEOREM FOR ORLICZ SPACES

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Abstract. An extension of Marcinkiewicz Interpolation Theorem, allowing intermediate spaces of Orlicz type, is proved. This generalization yields a necessary and sufficient condition so that every quasilinear operator, which maps the set, $S(X, \mu)$, of all $\mu$-measurable simple functions on $\sigma$-finite measure space $(X, \mu)$ into $M(Y, \nu)$, the class of $\nu$-measurable functions on $\sigma$-finite measure space $(Y, \nu)$, and satisfies endpoint estimates of type: $1 < p < \infty$, $1 \leq r < \infty$,
\[
\lambda \nu \left( \{ y \in Y : |(Tf)(y)| > \lambda \} \right)^{\frac{1}{r}} \leq C_{p,r} \left( \int_{R_+} \mu \left( \{ x \in X : |(f)(x)| > t \} \right)^{\frac{1}{t}} t^{-1} dt \right)^{\frac{1}{p}},
\]
for all $f \in S(X, \mu)$ and $\lambda \in R_+$; is bounded from an Orlicz space into another.

1. Introduction

Let $(X, \mu)$ and $(Y, \nu)$ be two $\sigma$-finite measure spaces and denote by $T$ a quasilinear operator that maps the set, $S(X, \mu)$, of all $\mu$-measurable simple functions on $X$ into $M(Y, \nu)$, the class of $\nu$-measurable functions on $Y$.

An important special case of the classical Marcinkiewicz interpolation theorem asserts that, if $0 < p_0, p_1 < \infty$ with $p_0 < p_1$, then every quasilinear operator $T$ of weak-types $(p_0, p_0)$ and $(p_1, p_1)$, namely, satisfying the inequalities
\[
\lambda \nu \left( \{ y \in Y : |(Tf)(y)| > \lambda \} \right)^{\frac{1}{r}} \leq M_i \left( \int_X |f(x)|^{p_i} d\mu(x) \right)^{\frac{1}{p_i}}, \quad i = 0, 1,
\]
in which the positive constants $M_0$ and $M_1$ are independent of $f \in S(X, \mu)$ and $\lambda > 0$, is bounded on Lebesgue space $L_{p_0}(X, \mu)$, provided
\[
\frac{1}{p_0} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]
for some $\theta \in (0, 1)$ [Ma39], [Zy56].

In his 1956 paper Zygmund proved the following partial generalization of Marcinkiewicz interpolation theorem, the formulation of which, [Zy57], he also attributes to Marcinkiewicz. The principal result of this paper, Theorem A, is partly modelled on this generalization.

Theorem 1.1. Let $(X, \mu)$ and $(Y, \nu)$ be finite measure spaces and suppose $T$ is a quasilinear operator of weak-types $(p_0, p_0)$ and $(p_1, p_1)$, $1 \leq p_0 < p_1 < \infty$. Let $\Phi$ be a increasing continuous function on $\mathbb{R}_+ = (0, \infty)$ satisfying $\Phi(0^+) = 0$. Then, $Tf$ is defined for every $f$ with $\int_X \Phi(|f(x)|) d\mu(x) < \infty$ and
\[
\int_Y \Phi(|(Tf)(y)|) d\nu(y) \leq K \int_X \Phi(|f(x)|) d\mu(x) + K,
\]
$K > 0$ being independent of $f$, provided
\[
\Phi(2t) = O(\Phi(t)),
\]
\[
\int_1^t \frac{\Phi(s)}{s^{p_0+1}} ds = O \left( \frac{\Phi(t)}{t^{p_0}} \right).
\]

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1.6
\int_t^\infty \frac{\Phi(s)}{sp+1} ds = O \left( \frac{\Phi(t)}{tp} \right),

as \( t \to \infty \).

Strömberg [Str79] proved a similar result when both the measure spaces \((X, \mu)\) and \((Y, \nu)\) are \((\mathbb{R}^n, m)\), \(m\) being Lebesgue measure on \(\mathbb{R}^n\). His result can be adapted to incorporate two increasing continuous functions \(\Phi_1\) and \(\Phi_2\), and then it reads

**Theorem 1.2** (Strömberg, [Str79]). Let \((X, \mu)\) and \((Y, \nu)\) be totally \(\sigma\)-finite measure spaces with \(\mu(X) = \nu(Y) = \infty\). Suppose \(T\) is a quasilinear operator from \(S(X, \mu)\) into \(M(Y, \nu)\) that is of weak-types \((p_0, p_0)\) and \((p_1, p_1)\), \(1 \leq p_0 < p_1 < \infty\). Then,

\[
\int_Y \Phi_1(|(Tf)(y)|) d\nu(y) \leq \int_X \Phi_2(|f(x)|) d\mu(x),
\]

where \(K > 0\) is independent of \(f \in S(X, \mu)\), provided there exist \(A, B > 0\) such that

\[
t^{p_0} \int_0^t \frac{\Phi_1(s)}{sp_0+1} ds \leq \Phi_2(At) \\
t^q \int_t^\infty \frac{\Phi_1(s)}{s^{p_1}+1} ds \leq \Phi_2(Bt),
\]

for all \(t \in \mathbb{R}_+\).

The conditions in (1.6) will be referred to as *Zygmund-Strömberg conditions* in the sequel.

A (complete) generalization of Marcinkiewicz interpolation theorem in a different direction was given by Stein and Weiss [StW59]. They arrived at the same conclusion as Marcinkiewicz assuming that the weak-type inequalities hold only for characteristic functions, \(\chi_E\), of sets \(E \subset X\) with \(\mu(E) < \infty\). This leads to the so-called restricted weak-type conditions. Calderón in [Ca66] (see also Hunt [Hu64]) showed that if a nonnegative sublinear operator \(T\) is of restricted weak-type \((p, q)\) then it satisfies the stronger inequality

\[
\lambda \nu \left( \{ y \in Y : |(Tf)(y)| > \lambda \} \right)^\frac{1}{q} \leq M \int_{\mathbb{R}_+} \mu \left( \{ x \in X : |f(x)| > t \} \right)^\frac{1}{p} dt.
\]

In [Ci98] and [Ci99], Cianchi obtained an interpolation theorem, in the spirit of those of Zygmund and Strömberg, for quasilinear operators of restricted weak-types \((p_0, p_0)\) and \((p_1, p_1)\), \(1 \leq p_0 < p_1 < \infty\). It concerns two increasing functions on \(\mathbb{R}_+\), \(\Phi_1\) and \(\Phi_2\), that are Young functions. A Young function, \(\Phi\), is a function from \(\mathbb{R}_+\) into \(\mathbb{R}_+\) having the form

\[
\Phi(t) = \int_0^t \phi(s) ds,
\]

for all \(t \in \mathbb{R}_+\), in which \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) is an increasing, left-continuous, with \(\phi(0^+) = 0\) and which is neither identically zero nor identically infinite.

Given a Young function \(\Phi\) and a totally \(\sigma\)-finite measure space \((X, \mu)\) one defines the Orlicz class

\[
L_\Phi(X, \mu) = \left\{ f \in M(X, \mu) : \int_X \Phi(|f(x)|) \, d\mu(x) < \infty \text{ for some } k \in \mathbb{R}_+ \right\}.
\]

Under the gauge norm

\[
\|f\|_{L_\Phi(X, \mu)} = \inf \left\{ \lambda > 0 : \int_X \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \right\},
\]

\(L_\Phi(X, \mu)\) becomes a Banach function space as in [BS88, Theorem 8.9, p. 269].

We observe here that if the \(\Phi_1\) and \(\Phi_2\) in Theorem 1.2 are Young functions, then the assumptions of the theorem guarantee the norm inequality which is the subject of
Theorem 1.3 (Cianchi, [Ci98], [Ci99]). Let $(X, \mu)$ and $(Y, \nu)$ be nonatomic $\sigma$-finite measure spaces with (for simplicity) $\mu(X) = \nu(Y) = \infty$. Fix the indices $p_0$ and $p_1$, $1 \leq p_0 < p_1 < \infty$. Suppose $\Phi_1(t) = \int_0^t \phi_1(s)ds$ are Young functions with complementary functions $\Psi_1(t) = \int_0^t \phi_1^{-1}(s)ds$, $i = 1, 2$. Assume $T$ is a quasilinear operator from $S(X, \mu)$ into $M(Y, \nu)$ which is of restricted weak-types $(p_0, p_0)$ and $(p_1, p_1)$, that is, $(1.7)$ holds for pairs $(p_0, p_0)$ and $(p_1, p_1)$. Then, there exists a constant $C > 0$, independent of $f \in L_{\Phi_2}(X, \mu)$, with
\[
\|Tf\|_{L_{\Phi_1}(Y, \nu)} \leq C\|f\|_{L_{\Phi_2}(X, \mu)},
\]
if, for some $K > 0$,
\[
\left( \int_{Kt}^\infty \left( \frac{s}{\Phi_1(s)} \right)^{p_1-1} ds \right)^{\frac{1}{p_1}} \left( \int_0^t \frac{\Psi_2(s)}{s^{p_1+1}} ds \right)^{\frac{1}{p_1}} \leq K,
\]
\[(1.10)\]
and
\[
t \int_0^t \frac{\Phi_1(s)}{s^2} ds \leq \Phi_2(Kt),
\]
or
\[
\left( \int_{Kt}^\infty \left( \frac{s}{\Phi_2(s)} \right)^{p_0-1} ds \right)^{\frac{1}{p_0}} \left( \int_0^t \frac{\Phi_1(s)}{s^{p_0+1}} ds \right)^{\frac{1}{p_0}} \leq K,
\]
\[(1.11)\]
for all $t \in \mathbb{R}_+$, depending on whether $p_0 = 1$ or $1 < p_0 < \infty$.

In this paper, we have considered operators such as
\[
f(x) \to \int_t^\infty f^*(s) \frac{ds}{s},
\]
where $f^*$ is the decreasing rearrangement of $f$ (see (1.14)). These are not quasilinear, however, they are $r$-quasilinear, in the sense that
\[
[T(f_1 + f_2)]^r(t) \leq C [(Tf_1)^r(ct) + (Tf_2)^r(ct)],
\]
in which $C > 0$, $0 < c < 1$ are independent of $f_1, f_2 \in S(X, \mu)$ and $t \in \mathbb{R}_+$. We observe that a quasilinear operator is $r$-quasilinear.

We now prepare to state our principal result, Theorem A, below. In it we consider weak-type conditions on a so-called $r$-quasilinear operator (see (1.12)) that are refinements of the weak type $(p, p)$ and restricted weak-type $(p, p)$ conditions. Thus, for $\sigma$-finite measure spaces $(X, \mu)$ and $(Y, \nu)$ and an $r$-quasilinear operator $T$ from $S(X, \mu)$ into $M(Y, \nu)$ and $1 \leq p < \infty$, $1 \leq r < \infty$, we are interested in the following weak type conditions:

\[
\lambda \nu \{(y \in Y : |(Tf)(y)| > \lambda)\} \overset{\lambda}{\frac{1}{p}} \leq C_{p,r} \left( \int_{\mathbb{R}_+} \mu \{|(f)(x)| > t\} \frac{dt}{t^{r-1}} \right)^{\frac{1}{r}},
\]
\[(1.13)\]
where $C_{p,r} = C_{p,r}(T) > 0$ is independent of $f \in S(X, \mu)$ and $\lambda \in \mathbb{R}_+$. For $r = 1$ and $r = p$ this is (1.7) and (1.1), respectively. When $1 < r < p$, it will be seen that (1.13) is intermediate in strength between the two; when $r > p$, then (1.13) is stronger than (1.1).

Finally, when $1 \leq p_0 < p_1 < \infty$ and $1 \leq r_0, r_1 < \infty$, we denote by $W((p_0, r_0), (p_1, r_1); \mu, \nu)$ the class of all $r$-quasilinear operators $T$, mapping $S(X, \mu)$ into $M(Y, \nu)$, which satisfy weak type estimates (1.13) for the pairs $(p_0, r_0)$ and $(p_1, r_1)$. Also, denote by $W((p_0, r_0), (\infty, \infty); \mu, \nu)$ the class of all $r$-quasilinear operators $T$, mapping $S(X, \mu)$ into $M(Y, \nu)$, which satisfy the inequality (1.13) for the pair $(p_0, r_0)$ and are bounded from $L_\infty(X, \mu)$ to $L_\infty(Y, \nu)$.

In this paper, we assume that the measure spaces $(X, \mu)$ and $(Y, \nu)$ are such that $\mu(X) = \nu(Y) = \infty$. 
Theorem A. Let \((X, \mu)\) and \((Y, \nu)\) be \(\sigma\)-finite measure spaces with \(\mu(X) = \nu(Y) = \infty\), the latter being nonatomic and separable. Fix the indices \(p_0, p_1, r_0\) and \(r_1\), with \(1 < p_0 < p_1 < \infty\) and \(1 \leq r_0, r_1 < \infty\). Suppose \(\Phi_i(t) = \int_0^t \phi_i(s)ds, i = 0, 1\), are Young functions. Then, the following are equivalent:

1. To each \(T \in W((p_0, r_0), (p_1, r_1); \mu, \nu)\) there corresponds \(C > 0\), depending on \(C_{p_0, r_0}\) and \(C_{p_1, r_1}\), such that
   \[\|Tf\|_{L_{\Phi_1}(Y, \nu)} \leq C\|f\|_{L_{\Phi_2}(X, \mu)},\]
   for all \(f \in S(X, \mu)\);
2. To each \(T \in W((p_0, r_0), (p_1, r_1); \mu, \nu)\) there corresponds \(K > 0\), depending on \(C_{p_0, r_0}\) and \(C_{p_1, r_1}\) such that
   \[\int_Y \Phi_1(|(Tf)(y)|)d\nu(y) \leq \int_X \Phi_2(K|f(x)|)d\mu(x),\]
   for all \(f \in S(X, \mu)\);
3. There exist constants \(L, B, D > 0\) such that, for all \(t > 0\),
   \[
   \chi_{[0,\infty)}(r_0 - p_0) \frac{tp_0}{\Phi_2(t)} \int_0^t \frac{\Phi_1(s)}{s^{p_0+1}} ds \\
   + \chi_{(0,\infty)}(p_0 - r_0) \left( \int_t^\infty \frac{\phi_2(Ls)}{\Phi_2(Ls)^{p_0}} s^{p_0} ds \right)^{\frac{1}{p_0}} \left( \int_0^t \frac{\Phi_1(s)}{s^{p_0+1}} ds \right)^{\frac{1}{p_0}} \leq B
   
   \]
   and
   \[
   \chi_{[0,\infty)}(r_1 - p_1) \frac{tp_1}{\Phi_2(t)} \int_0^t \frac{\Phi_1(s)}{s^{p_1+1}} ds \\
   + \chi_{(0,\infty)}(p_1 - r_1) \left( \int_0^t \frac{\phi_2(Ls)}{\Phi_2(Ls)^{p_1}} s^{p_1} ds \right)^{\frac{1}{p_1}} \left( \int_0^t \frac{\Phi_1(s)}{s^{p_1+1}} ds \right)^{\frac{1}{p_1}} \leq D,
   
   \]
   where \(\rho_0 = p_0/r_0\) and \(\rho_1 = p_1/r_1\).

We remark here that for \(r_0 = \infty\), (1.13) should be replaced by
\[
\lambda \nu (\{y \in Y : (Tf)(y) > \lambda\})^{\frac{1}{t}} \leq C_{p,\infty} \sup_{t > 0} t \nu (\{x \in X : |f(x)| > t\})^{\frac{1}{t}}.
\]

An \(r\)-quasilinear operator \(T\) satisfying such an inequality is said to be of weak-type \((p, \infty)\).

The principal result in the paper [KPP14] of Kerman, Phipps and Pick yields an analogue of theorem A for the class \(W((p_0, \infty), (p_1, \infty); \mu, \nu)\), \(1 < p_0 < p_1 < \infty\), and interestingly Zygmund-Strömberg condition (1.6) characterizes this class as well.

Let \((X, \mu)\), \((Y, \nu)\), \(p_0\), \(r_0\), \(p_1\) and \(r_1\) be as in Theorem A. To study the behaviour of an \(r\)-quasilinear operator \(T\) in \(W((p_0, r_0), (p_1, r_1); \mu, \nu)\) we transfer considerations to the measure space \((R_+, m)\), \(m\) being Lebesgue measure. This is done using rearrangements.

So, if \(f \in M(X, \mu)\), say, its distribution function, \(\lambda_{f, \mu}\), with respect to \(\mu\), is given at \(s \in R_+\) by \(\lambda_{f, \mu}(s) = \mu(\{x \in X : |f(x)| > s\})\). This nonnegative function is nonincreasing on \(R_+\) and so has a unique right-continuous generalized inverse,
\[
f^{*\mu} = \lambda_{f, \mu}^{-1},
\]
called the decreasing rearrangement of \(f\). We shall also use notation \(f^*\) instead of \(f^{*\mu}\) whenever there is no possible confusion.

Following Calderón [Ca66] we show that given \(T\) in \(W((p_0, r_0), (p_1, r_1); \mu, \nu)\) and \(f \in S(X, \mu)\),
we are then able to show that (1.15) and other considerations we are able to show that, for a pair of Young functions, Φ₁ and Φ₂, one has

\[ \| H^{\rho_0,\tau_0} g \|_{L_{\Phi_1}(\mathbb{R}^+,\mu)} \leq C_1 \| g \|_{L_{\Phi_2}(\mathbb{R}^+,\mu)} \]

for all nonnegative, nonincreasing g on \( \mathbb{R}^+ \).

The inequalities in (1.17) suggest working with the class \( W((p_0, r_0), (\infty, \infty); \mu, \nu) \) and \( W((1, 1), (p_1, r_1); \mu, \nu) \) instead of \( W((p_0, r_0), (p_1, r_1); \mu, \nu) \). This, indeed, allows us to solve the problem at hand, Theorem A.

In Section 3 we take up the class \( W((p_0, r_0), (\infty, \infty); \mu, \nu) \) and the first inequality in (1.17).

This in turn leads us to considering the boundedness of the operator \( H^{\rho_0,\tau_0} \) between Orlicz spaces. As the operator \( H^{\rho_0,\tau_0} \) is dilation-commuting, the first inequality in (1.17) is equivalent to the modular inequality

\[ \int_{\mathbb{R}^+} \Phi_1((H^{\rho_0,\tau_0} f^*)(t)) dt \leq \int_{\mathbb{R}^+} \Phi_2(K f^*(s)) ds, \]

where \( f \in M(X, \mu) \).

Further work allows us to reduce this inequality to a weighted dual Hardy inequality for nonnegative, nonincreasing functions \( g \), namely, to an inequality of the form

\[ \int_{\mathbb{R}^+} \left( \int_{x}^{\infty} g(y) dy \right)^{p_0/\rho_0} w(x) dx \leq B \int_{\mathbb{R}^+} g(y)^{p_0/\rho_0} v(y) dy, \]

involving certain weights \( w \) and \( v \) on \( \mathbb{R}^+ \).

As seen from the statement of Theorem A we are then able to show that (1.18) (and hence the first inequality in (1.17)) holds if and only if

\[ \left( \int_{t}^{\infty} \frac{\Phi_2(Ls)}{\Phi_2(Ls)^{\rho_0}} s^{\rho_0/\rho_0} ds \right)^{1/\rho_0} \left( \int_{0}^{t} \frac{\Phi_1(s)}{s^{\rho_0+1}} ds \right)^{1/\rho_0} \leq A, \]

for all \( t \in \mathbb{R}^+ \), where \( \rho_0 = \frac{\rho_0}{\tau_0} \), \( 1 \leq r_0 < p_0 \).

By other means we prove (1.18) equivalent to

\[ t^{p_0} \int_{0}^{t} \frac{\Phi_1(s)}{s^{\rho_0+1}} ds \leq B \Phi_2(Kt), \]

for all \( t \in \mathbb{R}^+ \), when \( r_0 \geq p_0 \).
In Section 4, we consider the class $W((1,1),(p_1,r_1);\mu,\nu)$. The corresponding Calderón operator that arises in this case is $P + H_{p_1,r_1}$, where operator $P = H^{1,1}$. The necessary and sufficient conditions for the boundedness of $P$ between Orlicz spaces are already known, for example in [C39]. The treatment of the operator $H_{p_1,r_1}$ is similar to the one for $H_{p_0,r_0}$. The final result, then, combines the conditions for $P$ and $H_{p_1,r_1}$ to arrive at conditions for the class $W((1,1),(p_1,r_1);\mu,\nu)$.

In Section 5, we present the Marcinkiewicz interpolation theorem for the joint class $W((p_0,r_0),(p_1,r_1);\mu,\nu)$, $1 < p_0 < p_1 < \infty$, $1 \leq r_1, r_2 < \infty$, combining the results of Section 3 and Section 4.

In the concluding Section 6, we give an example to compare various conditions obtained in Theorem A.

2. Background

2.1. Suppose $(X,\mu)$ is a $\sigma$-finite measure space. Given $f \in M(X,\mu)$, we define the decreasing rearrangement, $f^{*\mu}$, of $f$ by

$$f^{*\mu}(t) = \inf\{s > 0 : \lambda_{f,\mu}(s) \leq t\},$$

where, $t \in \mathbb{R}_+$ and $\lambda_{f,\mu}$ is the distribution function of $f$ given by

$$\lambda_{f,\mu}(s) = \mu\{x \in X : |f(x)| > s\}, \text{ for } s \in \mathbb{R}_+.$$  

We remark here that the dependence of $f^*$ on $\mu$ will usually be clear from the context in which it appears. When we wish to emphasize the dependence we will use the notation $f^{*\mu}$ rather than $f^*$.

Two functions $f \in M(X,\mu)$ and $g \in M(Y,\nu)$ are said to be equimeasurable if they have the same distribution function, that is, if $\lambda_{f,\mu}(s) = \lambda_{g,\nu}(s)$ for all $s \in \mathbb{R}_+$.

The decreasing rearrangement $f^*$ satisfies the following inequality of Hardy and Littlewood: For $f,g \in M_+(X,\mu)$,

$$\int_X f(x)g(x)d\mu(x) \leq \int_{\mathbb{R}_+} f^*(t)g^*(t)dt.$$  

The operation of rearrangement is not sublinear though it satisfies

$$(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2),$$

where $f, g \in M(X,\mu)$ and $t_1, t_2 \in \mathbb{R}_+$.

However, the operator $f \mapsto f^{**}$ is sublinear, namely,

$$(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t),$$

for all $t \in \mathbb{R}_+$, where the maximal function $f^{**}$ of $f$ (or the Hardy averaging operator at $f$) is defined, at $t \in \mathbb{R}_+$, by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds.$$  

2.2. Let $(X,\mu)$ be a $\sigma$–finite measure space and suppose $0 < p < \infty$, $0 < r \leq \infty$. The Lorentz space $L_{p,r}(X,\mu)$ consists of all $f$ in $M(X,\mu)$ for which the quantity

$$\|f\|_{p,r} = \begin{cases} \left( \frac{r}{p} \frac{1}{\mathbb{R}_+} \left( \int_{\mathbb{R}_+} \left( t^p f^*(t) \right) \frac{dt}{t} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}, & 0 < r < \infty, \\
\sup_{0 < t < \infty} t^r f^*(t), & r = \infty, \end{cases}$$

is finite.
A useful alternative expression for $\|f\|_{p,r}, r < \infty$, is
\[\|f\|_{p,r} = \left( r \int_{\mathbb{R}^+} \lambda_{f,\mu}(s) s^{r-1} ds \right)^{\frac{1}{r}}.\]

This indeed was the form of $\|\cdot\|_{p,r}$ used in Introduction.

For any fixed $p$, $0 < p < \infty$, the Lorentz space $L_{p,r}$ gets bigger as the secondary exponent $r$ increases: If $0 < r_1 < r_2 \leq \infty$. Then,
\[\|f\|_{p,r_2} \leq \|f\|_{p,r_1},\]
for all $f \in M(X,\mu)$. In particular, one has the imbedding $L_{p,r_1}(X,\mu) \hookrightarrow L_{p,r_2}(X,\mu)$.

2.3. A Young function is convex and that for $t \in \mathbb{R}^+$,
\[\Phi(t) \leq t\phi(t) \leq \Phi(2t).\]

We associate to the Orlicz space $L_{\Phi}$ an another Orlicz space $L_{\Psi}$ that has the same relationship to $L_{\Phi}$ as the Lebesgue space $L_{p'}$ does to the Lebesgue spaces $L_p$, where $p' = \frac{p}{p-1}$.

Let $\Phi(t) = \int_0^t \phi(s) ds$ be a Young function. Let $\psi$ be left-continuous inverse of $\phi$, that is, for $t \in \mathbb{R}^+$
\[\psi(t) = \inf\{s: \phi(s) \geq t\}.\]

Then the function $\Psi$ defined as
\[\Psi(t) = \int_0^t \psi(s) ds,\]
for all $t \in \mathbb{R}^+$, is called the complementary Young function of $\Phi$ and satisfies the following basic inequality, known as Young’s inequality: For every $s, t \in \mathbb{R}^+$,
\[st \leq \Phi(s) + \Psi(t).\]

Let $\Phi$ and $\Psi$ be complementary Young functions. The Orlicz norm on $L_{\Phi}$ is defined as
\[\|f\|_{L_{\Phi}} := \sup \left\{ \int_X |f(x)g(x)| d\mu(x) : \int_X \Psi(|g(x)|) d\mu(x) \leq 1 \right\}.
\]

The Orlicz norms and gauge norm, (1.9), are equivalent:
\[\|f\|_{L_{\Phi}} \leq \|f\|_{L_{\Psi}} \leq 2\|f\|_{L_{\Phi}}.\]

2.4. Let $(X,\mu)$ and $(Y,\nu)$ be $\sigma$-finite measure spaces with $\mu(X) = \nu(Y) = \infty$. It will be useful in our work to be able to associate to an operator $S : M_+(\mathbb{R}^+,\mu) \to M_+(\mathbb{R}^+,\mu)$ an operator $\tilde{S} : M_+(X,\mu) \to M_+(Y,\nu)$, having the property
\[\tilde{(Sf)^*}\nu(t) = (Sf^*)^*\mu(t),\]
for all $f \in M_+(X,\mu)$ and $t \in \mathbb{R}^+$.

To do this we require a result on measure-preserving transformations from [Ha50, page 174], which we now describe.

Denote by $Y_{\text{Fin}}$, the class of $\nu$-measurable subsets $E$ of $Y$ with $\nu(E) < \infty$. The functional $d$ defined on $Y_{\text{Fin}} \times Y_{\text{Fin}}$ by
\[d(E,F) = \nu(E \Delta F),\]
for all $E,F \in Y_{\text{Fin}}$, is a metric on $Y_{\text{Fin}}$. If the metric space $(Y_{\text{Fin}},d)$ is separable, then the measure space $(Y,\nu)$ is said to be separable.

It is shown in [Ha50, page 174] that, when $(Y,\nu)$ is separable and nonatomic, there exists a 1-1 transformation $\tau$ from $Y_{\text{Fin}}$ onto $\mathbb{R}^+_\text{Fin}$, with
\[\tau(E_1 - E_2) = \tau(E_1) - \tau(E_2), \quad \tau\left(\bigcup_{n=1}^\infty E_n\right) = \bigcup_{n=1}^\infty \tau(E_n)\]
and \( m(\tau(E)) = \nu(E), \) for \( E_n, E \in Y_{\text{fin}}, \ n = 1, 2, \ldots \). As \((Y, \nu)\) is \( \sigma\)-finite, \( \tau \) can be extended to all \( \nu\)-measurable subsets of \( Y \).

We now state and prove our result concerning \( S \) and \( \tilde{S} \).

**Theorem 2.1.** Suppose \((X, \mu)\) and \((Y, \nu)\) are \( \sigma\)-finite measure spaces, with \( \mu(X) = \nu(Y) = \infty \). Assume, in addition, that \((Y, \nu)\) is nonatomic and separable and denote by \( \tau \) the measure-preserving transformation between the \( \nu\)-measurable subsets of \( Y \) and the Lebesgue measurable subsets of \( \mathbb{R}_+ \) described above.

Given the operator \( S : M_+(\mathbb{R}_+, m) \to M_+(\mathbb{R}_+, m) \), define the operator \( \tilde{S} \) on \( f \in M_+(X, \mu) \) to be the Radon-Nikodym derivative of the \( \nu\)-absolutely continuous measure \( \lambda \), given at the \( \nu\)-measurable set \( E \) by

\[
\lambda(E) = \int_{\tau(E)} Sf^* \mu.
\]

Then,

\[
(\tilde{S}f)^* \nu = (Sf^* \mu)^* m, \quad \text{m.a.e.}
\]

**Proof.** That \( \lambda \) is a measure on the \( \nu\)-measurable subsets of \( Y \) follows from the properties of \( \tau \), as does the absolute continuity of \( \lambda \) with respect to \( \nu \).

Now, (2.11) is equivalent to

\[
\int_E (\tilde{S}f)(y) d\nu(y) = \lambda(E) = \int_{\tau(E)} Sf^* \mu(t) dt
\]

for all \( \nu\)-measurable \( E \subset E_{\tilde{S}f}(u) \) and the Lebesgue differentiation theorem, we conclude that, modulo sets of measure zero,

\[
E_{Sf^* \mu}(u) = \tau \left( E_{\tilde{S}f}(u) \right),
\]

which ensures (2.12). Indeed, from

\[
\int_E (\tilde{S}f)(y) d\nu(y) = \lambda(E) = \int_{\tau(E)} Sf^* \mu(t) dt
\]

for all \( \nu\)-measurable \( E \subset E_{\tilde{S}f}(u) \) and the Lebesgue differentiation theorem, we conclude that, modulo sets of measure zero,

\[
\tau \left( E_{\tilde{S}f}(u) \right) \subset E_{Sf^* \mu}(u).
\]

A similar argument yields

\[
\tau^{-1} \left( E_{Sf^* \mu}(u) \right) \subset E_{\tilde{S}f}(u),
\]

that is,

\[
E_{Sf^* \mu}(u) \subset \tau \left( E_{\tilde{S}f}(u) \right),
\]

once again, modulo sets of measure zero. \( \square \)

**Lemma 2.2.** Let the measures \( \mu \) and \( \nu \) and the operators \( S \) and \( \tilde{S} \) be as in Theorem 2.1. Then, \( \tilde{S} \) is \( r\)-quasilinear on \( M_+(X, \mu) \), provided \( S \) is monotone and dilation-commuting as well as \( r\)-quasilinear on \( M_+(X, \mu) \).
Proof. Given $f$ and $g$ in $M_+(X, \mu)$ one has, by (2.11),
\begin{align*}
(\tilde{S}(f + g))^\nu(t) &= [S((f + g)^\nu)]^m(t) \\
&\leq [S(f^\nu(\frac{1}{t(\tilde{S})}) + g^\nu(\frac{1}{t(\tilde{S})}))]^m(t) \\
&\leq C \left[ [S(f^\nu(\frac{1}{t}))]^m(ct) + [S(g^\nu(\frac{1}{t}))]^m(ct) \right]^{m} \\
&= C \left[ [(Sf^\nu)(\frac{1}{t})]^m(ct) + [(Sg^\nu)(\frac{1}{t})]^m(ct) \right]^{m} \\
&= C \left[ (\tilde{S}f)^\nu(\frac{1}{t}) + (\tilde{S}g)^\nu(\frac{1}{t}) \right].
\end{align*}

\[ \square \]

Recall that given indices $p_0, p_1, r_0$ and $r_1$ satisfying $1 < p_0 < p_1 < \infty$, $1 \leq r_1, r_2 < \infty$, we say that an $r$-quasilinear operator $T : S(X, \mu) \to M(Y, \nu)$ is in the class $W((p_0, r_0), (p_1, r_1); \mu, \nu)$ if
\[
\|Tf\|_{L_{p_i}(Y, \nu)} \leq C_i \|f\|_{L_{p_i, r_i}(X, \mu)}
\]
where $C_i > 0$ is independent of $f \in S(x, \mu), i = 1, 2$. Such weak-type $(p, r)$ inequalities are equivalent to those in (1.13).

We will be concerned with the action of the operators $T \in W((p_0, r_0), (p_1, r_1); \mu, \nu)$ on Orlicz spaces
\[
L_{\Phi}(X, \mu) \subset L_{p_0, r_0}(X, \mu) + L_{p_1, r_1}(X, \mu).
\]

Our next result treats an important decomposition of functions in $L_{p_0, r_0}(X, \mu) + L_{p_1, r_1}(X, \mu)$.

Lemma 2.3. Fix indices $p_0, p_1, r_0$ and $r_1$ satisfying $1 < p_0 < p_1 < \infty$, $1 \leq r_0, r_1 < \infty$. Let $(X, \mu)$ be a $\sigma$-finite measure space with (for simplicity) $\mu(X) = \infty$ and suppose $f \in (L_{p_0, r_0} + L_{p_1, r_1})(X, \mu), t \in \mathbb{R}_+$. At $x \in X$, set
\[
f_1(x) = \min[|f(x)|, f^*(t)] \cdot \text{sgn}f(x) \quad \text{and} \quad f_0(x) = f(x) - f_1(x).
\]
Then, $f_0 \in L_{p_0, r_0}(X, \mu)$ and $f_1 \in L_{p_1, r_1}(X, \mu)$.

3. Interpolation results for the class $W((p, r), (\infty, \infty); \mu, \nu)$

In the present section we shall give a description of the interpolation pairs, $(L_{\Phi_2}(X, \mu), L_{\Phi_1}(Y, \nu))$, of Orlicz spaces for which every $T \in W((p, r), (\infty, \infty); \mu, \nu)$ maps $L_{\Phi_2}(X, \mu)$ boundedly into $L_{\Phi_1}(Y, \nu)$. This class naturally arises, as explained in the Introduction, as an intermediate step in determining the interpolation pairs $(L_{\Phi_2}(X, \mu), L_{\Phi_1}(Y, \nu))$ for $W((p_1, r_1), (q, r_2); \mu, \nu)$.

The conditions imposed on an operator $T \in W((p, r), (\infty, \infty); \mu, \nu)$ ensure that it is dominated by a Calderón operator, $H^{p,r}$, (in the sense of [BS88, page 141]), namely,
\[
(Tf)^*(t) \leq C H^{p,r}f^*(t),
\]
in which $C > 0$ is independent of $f \in (L_{p,r} + L_\infty)(X, \mu)$ and $t \in \mathbb{R}_+$. Moreover, $H^{p,r} \in W((p, r), (\infty, \infty); m, m)$ and it is essentially the smallest operator such that (3.1) holds.

The fundamental interpolation theorem of Calderón [Ca66], see also [BS88, Chapter 3, Theorem 5.7], describes the action of operators satisfying (3.1), when $r = 1$, on rearrangement-invariant spaces in terms of the boundedness of the Calderón operator on their representative spaces. In Theorem 3.2, we formulate a Calderon-type theorem for the operators of the type in (3.1). Thus, it is enough to characterise those $\Phi_1$ and $\Phi_2$ for which $H^{p,r}$ maps $L_{\Phi_2}(\mathbb{R}_+, m)$ boundedly into $L_{\Phi_1}(\mathbb{R}_+, m)$.

As we shall see in Lemma 3.3, $H^{p,r}$ is a dilation-commuting operator, and therefore it suffices to work with a modular inequality rather than a norm inequality, as explained in Theorem 3.4.

Using the estimates of the distribution function for $H^{p,r}f^*$ in Lemma 3.5 and Theorem 3.6 we are able to reformulate the modular inequality for $H^{p,r}$ involving $\Phi_1$ and $\Phi_2$ as a weighted
Hardy inequality (with weights involving $\phi_1$ and $\phi_2$) on nonnegative, nonincreasing functions. The duality principle of Sawyer [Sw90] then allows us to pass to an equivalent inequality for a Hardy-type operator on nonnegative measurable functions. To conclude, we then arrive at our desired necessary and sufficient conditions by invoking the results of Stepanov, [Stp90] for such inequalities.

Our conditions depend on $r$ for $1 \leq r < p$, see Theorem 3.9, and are, interestingly, independent of $r$ for $p \leq r < \infty$, see Theorem 3.10. For $1 \leq r < p$, conditions, that we get, can readily be seen as an extension of the earlier results of Cianchi [Ci99] for the case $r = 1$. For $p \leq r < \infty$, we get the well-known conditions as given by Zygmund [Zy57, Theorem 4.22, page 116] and Stromberg [Str79].

3.1. **A Calderón-type theorem.** The following result is modeled on Theorem 4.11 in [BS88, page 223] or Theorem 8 in [Ca66], and the proof carries over almost verbatim to this slightly more general case.

**Theorem 3.1.** Let $(X, \mu)$ and $(Y, \nu)$ be $\sigma$-finite measure spaces and suppose $1 \leq p, r < \infty$. If $T \in W((p, r), (\infty, \infty); \mu, \nu)$, then

$$
(Tf)^*(t) \leq KH^{p,r}f^*(t),
$$

where $K > 0$ is independent of $f \in (L_{p,r} + L_\infty)(X, \mu)$ and $t \in \mathbb{R}_+$. Further, the operator $H^{p,r}$ is in the class $W((p, r), (\infty, \infty); m, m)$.

**Proof.** Let $f \in (L_{p,r} + L_\infty)(X, \mu)$ and fix $t > 0$. Set

$$
f_1(x) = \min([f(x)], f^*(t)) \cdot \text{sgn } f(x)
$$

and

$$
f_0(x) = f(x) - f_1(x) = \left([f(x)] - f^*(t)\right)^+ \cdot \text{sgn } f(x),
$$

for $x \in X$. Then, $f = f_0 + f_1$ and for all $s > 0$

$$
f_0^s(s) = [f^*(s) - f^*(t)]^+,
$$

$$
f_1^s(s) = \min(f^*(s), f^*(t)).
$$

Clearly $f_1 \in L_\infty(X, \mu)$; moreover, $f_0 \in L_{p,r}(X, \mu)$ follows from Lemma 2.3.

We next establish inequality (3.2). Since $T$ is a $r$-quasilinear operator with, say, constant of $r$-quasilinearity $C > 0$ and $0 < c < 1$, we have for $t > 0$,

$$
(Tf)^*(t) = (T(f_0 + f_1))^*(t)
$$

$$
T[(Tf_0)^*(ct) + (Tf_1)^*(ct)].
$$

Since

$$
T : L_{p,r}(X, \mu) \to L_{p,\infty}(Y, \nu),
$$

it follows that

$$
(Tf_0)^*(ct) \leq M_{p,r}(ct)^{-\frac{1}{p}}\left(\frac{1}{p} \int_{\mathbb{R}_+} (s^p f_0^*(s))^r \frac{ds}{s}\right)^{\frac{1}{r}}
$$

$$
\leq M_{p,r}c^{-\frac{1}{p}} \left(\frac{t}{p}\right)^{\frac{1}{p}} \left(\int_{0}^{t} (s^p f^*(s))^r \frac{ds}{s}\right)^{\frac{1}{r}}
$$

$$
= M_{p,r}c^{-\frac{1}{p}} \left(\frac{t}{p}\right)^{\frac{1}{p}} H^{p,r} f^*(t).
$$
Again, \( f^* \) is a decreasing function, so
\[
H^{p,r} f^*(t) = t^{-\frac{1}{r}} \left( \int_0^t (s^\frac{1}{p} f^*(s)) r \frac{ds}{s} \right)^{\frac{1}{r}}
\]
(3.3)
\[
\geq f^*(t) t^{-\frac{1}{r}} \left( \int_0^t s^\frac{r}{p} - 1 \frac{ds}{s} \right)^{\frac{1}{r}} = \left( \frac{t}{t} \right)^{\frac{1}{r}} f^*(t).
\]
Using the fact that
\[ T : L_\infty(X,\mu) \to L_\infty(Y,\nu), \]
and (3.3), we get
\[
(T f_1)^*(ct) \leq M_\infty \| f_1 \|_{L_\infty(X,\mu)} = M_\infty f^*(t)
\]
\[
\leq M_\infty \left( \frac{t}{t} \right)^{\frac{1}{r}} H^{p,r} f^*(t).
\]
Combining these yields
\[
(T f)^*(t) \leq C [(T f_0)^*(ct) + (T f_1)^*(ct)] = K H^{p,r} f^*(t),
\]
where \( K = \left( M_{p,r} c^{-\frac{1}{p}} + M_\infty \right) \left( \frac{t}{t} \right)^{\frac{1}{r}} C. \)

We are now in a position to formulate a Calderón-type theorem for operators in \( W((p,r),(\infty,\infty);\mu,\nu) \).

**Theorem 3.2.** Fix \( p \) and \( r, 1 \leq p, r < \infty \), and suppose \((X,\mu)\) and \((Y,\nu)\) are \( \sigma \)-finite measure spaces with \( \mu(X) = \nu(Y) = \infty \), the latter being nonatomic and separable. Then, the following are equivalent:

1. Every operator \( T \) in the class \( W((p,r),(\infty,\infty);\mu,\nu) \) is bounded from \( L_{\Phi_2}(X,\mu) \) to \( L_{\Phi_1}(Y,\nu) \).
2. The operator \( H^{p,r} \) is bounded from \( L_{\Phi_2}(\mathbb{R}_+,m) \) to \( L_{\Phi_1}(\mathbb{R}_+,m) \).

**Proof.** We first show (2) implies (1). Let \( T \) be an operator in the class \( W((p,r),(\infty,\infty);\mu,\nu) \). Then, by Theorem 3.1,
\[
(T f)^*(t) \leq K H^{p,r} f^*(t),
\]
for all \( f \in (L_{p,r} + L_\infty)(X,\mu) \) and for all \( t \in \mathbb{R}_+ \). Now, from (2), it follows that \( L_{\Phi_2}(X,\mu) \subseteq (L_{p,r} + L_\infty)(X,\mu) \). Indeed, we have,
\[
C \| f^* \|_{L_{\Phi_2}(\mathbb{R}_+,m)} \geq H^{p,r} f^* \|_{L_{\Phi_1}(\mathbb{R}_+,m)} \geq \frac{1}{2} \sup \left\{ \int_{\mathbb{R}_+} H^{p,r} f^*(s) g^*(s) ds : \| g^* \|_{L_{\Phi_1}(\mathbb{R}_+,m)} \leq 1 \right\}.
\]
Taking \( g^* = \chi_{(0,1)} / \| \chi_{(0,1)} \|_{L_{\Phi_1}(\mathbb{R}_+,m)} \), we get
\[
H^{p,r} f^*(1) \leq \int_0^1 H^{p,r} f^*(s) ds \leq D \| f^* \|_{L_{\Phi_2}(\mathbb{R}_+,m)} < \infty,
\]
with \( D = 2 C \| \chi_{(0,1)} \|_{L_{\Phi_1}(\mathbb{R}_+,m)} \). From the estimate of the K-functional for the pair \( (L_{p,r}(X,\mu), L_\infty(X,\mu)) \) (see [Ho70, Theorem 4.2]), we have that
\[
H^{p,r} f^*(1) = \int_0^1 f^*(s) s^{r/p-1} ds \approx \| f \|_{(L_{p,r} + L_\infty)(X,\mu)} < \infty,
\]
for all \( f \in (L_{p,r} + L_\infty)(X,\mu) \).
Next, (2), together with (3.4), implies that, given \( f \) in \( L_{\Phi_2}(X, \mu) \) and hence in \( (L_{p,r} + L_\infty)(X, \mu) \), one has
\[
\|Tf\|_{L_{\Phi_2}(Y, \nu)} = \|(Tf)^*\|_{L_{\Phi_2}(\mathbb{R}^+, m)} \leq K \|H^{p,r}f^*\|_{L_{\Phi_2}(\mathbb{R}^+, m)} \leq KC\|f^*\|_{L_{\Phi_2}(\mathbb{R}^+, m)} = KC\|f\|_{L_{\Phi_2}(X, \mu)},
\]
so that the operator \( T \) is bounded from \( L_{\Phi_2}(X, \mu) \) to \( L_{\Phi_2}(Y, \nu) \).

Conversely, assume that (1) holds. In Theorem 2.1 take \( S = H^{p,r} \) and denote by \( \tilde{H}^{p,r} \) the operator \( \tilde{S} \) guaranteed to exist by that theorem. In particular, then,
\[
(\tilde{H}^{p,r}f)^* = H^{p,r}f^*, \quad m\text{-a.e.,}
\]
for all \( f \in M(X, \mu) \), since \((H^{p,r}f^*)(t) = \left( \int_0^1 f^*(ts) \frac{ds}{s} - \frac{1}{t} \right)^{\frac{1}{p}} \) is nonincreasing, so \((H^{p,r}f^*)^* = H^{p,r}f^*\). Moreover, since \( H^{p,r} \) is \( r \)-quasilinear, \( \tilde{H}^{p,r} \) will, according to Lemma 2.2, be \( r \)-quasilinear.

Next,
\[
\tilde{H}^{p,r} : L_{p,r}(X, \mu) \to L_{p,\infty}(Y, \nu) \quad \text{and} \quad \tilde{H}^{p,r} : L_\infty(X, \mu) \to L_\infty(Y, \nu)
\]
boundedly.

Indeed, from Theorem 3.1, \( H^{p,r} : L_{p,r}(\mathbb{R}^+, m) \to L_{p,\infty}(\mathbb{R}^+, m) \), so given \( f \in L_{p,r}(X, \mu) \), one has
\[
\|\tilde{H}^{p,r}f\|_{L_{p,\infty}(Y, \nu)} = \|(\tilde{H}^{p,r}f)^*\|_{L_{p,\infty}(\mathbb{R}^+, m)} = \|H^{p,r}f^*\|_{L_{p,\infty}(\mathbb{R}^+, m)} \leq C\|f^*\|_{L_{p,r}(\mathbb{R}^+, m)} = C\|f\|_{L_{p,r}(X, \mu)},
\]
that is, \( \tilde{H}^{p,r} : L_{p,r}(X, \mu) \to L_{p,\infty}(Y, \nu) \) boundedly. Similarly, \( \tilde{H}^{p,r} : L_\infty(X, \mu) \to L_\infty(Y, \nu) \) boundedly. We have now shown \( \tilde{H}^{p,r} \in W((p, r), (\infty, \infty) ; \mu, \nu) \), whence, by (1), \( \tilde{H}^{p,r} : L_{\Phi_2}(X, \mu) \to L_{\Phi_2}(Y, \nu) \) boundedly.

In Theorem 2.1, take \( X \) to be \( \mathbb{R}^+, \mu \) to be \( m \), \( Y \) to be \( X, \nu \) to be \( \mu \) and \( S \) to be the operator \( g \to g^{*m} \). Given \( f \in M_+(\mathbb{R}^+, m) \), set \( \tilde{f} = \tilde{S}f \in M_+(X, \mu) \), so that \( \tilde{f}^{*m} = f^{*m} \).

Thus,
\[
\|H^{p,r}f\|_{L_{\Phi_2}(\mathbb{R}^+, m)} \leq \|H^{p,r}f^{*m}\|_{L_{\Phi_2}(\mathbb{R}^+, m)} = \|H^{p,r}\tilde{f}^{*m}\|_{L_{\Phi_2}(\mathbb{R}^+, m)} = \|(H^{p,r}\tilde{f})^{*m}\|_{L_{\Phi_2}(\mathbb{R}^+, m)} = \|(\tilde{H}^{p,r}\tilde{f})^{*m}\|_{L_{\Phi_2}(\mathbb{R}^+, m)} = \|\tilde{H}^{p,r}\tilde{f}\|_{L_{\Phi_2}(Y, \nu)} \leq C\|\tilde{f}\|_{L_{\Phi_2}(X, \mu)} = C\|f^{*m}\|_{L_{\Phi_2}(\mathbb{R}^+, m)} = C\|f^{*m}\|_{L_{\Phi_2}(\mathbb{R}^+, m)},
\]
whence \( H^{p,r} : L_{\Phi_2}(X, \mu) \to L_{\Phi_2}(Y, \nu) \) boundedly.

This completes the proof. \( \square \)
3.2. The Calderón operator $H^{p,r}$ and an associated Hardy inequality. Our next result shows that it is enough to work with the modular inequality for $H^{p,r}$.

**Lemma 3.3.** Let $1 \leq p < \infty$ and $1 \leq r < \infty$. Then, $H^{p,r}$ is a dilation-commuting operator.

**Proof.** The proof is an easy exercise in change of variable, hence we omit it. \hfill \Box

**Theorem 3.4.** Let $\Phi_1$ and $\Phi_2$ be Young functions. For $1 \leq p, r < \infty$, we have that the norm inequality

$$
(3.5) \quad \| H^{p,r} f \|_{L_{\Phi_1}(\mathbb{R}^+, m)} \leq C \| f \|_{L_{\Phi_2}(\mathbb{R}^+, m)},
$$

holds for all $f$ in $M_+(\mathbb{R}^+, m)$ if and only if the modular inequality

$$
(3.6) \quad \int_{\mathbb{R}^+} \Phi_1(\|H^{p,r} f\|) \, dt \leq \int_{\mathbb{R}^+} \Phi_2(\|K f\|) \, ds,
$$

holds for all $f$ in $M_+(\mathbb{R}^+, m)$.

**Proof.** The proof follows from [KRS17, Theorem A], as the norm and the modular inequalities are equivalent for a dilation-commuting operator. \hfill \Box

We now seek an expression equivalent to the distribution function of $H^{p,r} g$, when $g$ is non-negative and nonincreasing on $\mathbb{R}_+$.

**Lemma 3.5.** Let $1 \leq p < \infty$, $1 \leq r < \infty$ and suppose $T \in W((p, r), (\infty, \infty); \mu, \nu)$. Then, for every $f$ in the domain of $T$,

$$
(3.7) \quad \nu_{T f}(t) \leq \frac{1}{c} \left(2^{1+1/r} r^{1/r} CM_{p, r}\right) \frac{1}{t^{p/r}} \left(\int_{t/4CM_{\infty}}^{\infty} \mu_f(s)^{r/p} s^{r-1} ds\right)^{p/r},
$$

where $C$ and $c$ are the constant of $r$-quasilinearity of $T$ and $M_{p, r}, M_\infty$ are the operator norms in $T : L_{p, r}(X, \mu) \rightarrow L_{p, \infty}(Y, \nu)$ and $T : L_\infty(X, \mu) \rightarrow L_\infty(Y, \nu)$, respectively.

**Proof.** Fix $t > 0$ and $f \in L_{p, r} + L_\infty(X, \mu)$. Let $k$ be any positive number. Write $f = f_t + f^t$, with

$$
(f^t(x) = \begin{cases} f(x), & |f(x)| \geq \frac{t}{2Ck}, \\ 0, & |f(x)| < \frac{t}{2Ck}, \end{cases}
$$

and $f_t(x) = f(x) - f^t(x)$. Observe that the distribution functions of $f_t$ and $f^t$ are as follows:

$$
\mu_{f_t}(s) = \begin{cases} \mu_f(s) - \mu_f\left(\frac{t}{2Ck}\right), & s < \frac{t}{2Ck}, \\ 0, & s \geq \frac{t}{2Ck}, \end{cases}
$$

and

$$
\mu_{f^t}(s) = \begin{cases} \mu_f\left(\frac{t}{2Ck}\right), & s < \frac{t}{2Ck}, \\ \mu_f(s), & s \geq \frac{t}{2Ck}. \end{cases}
$$

Then, from the $r$-quasilinearity of $T$

$$
\nu_{T f}(t) \leq \frac{1}{c} \left[ \nu_{T f^t} \left(\frac{t}{2Ck}\right) + \nu_{T f_t} \left(\frac{t}{2Ck}\right) \right].
$$

For any $t > 0$ and $x$ such that $|T f_t(x)| > \frac{t}{2Ck}$, we have $\frac{t}{2Ck} < |T f_t(x)| \leq M_\infty \|f_t\|_{L_\infty(X, \mu)} \leq M_\infty \frac{t}{2Ck}$. Therefore, $\nu_{T f_t} \left(\frac{t}{2Ck}\right) = 0$ when $k \geq M_\infty$. So, for such a $k$,

$$
\nu_{T f}(t) \leq \frac{1}{c} \nu_{T f^t} \left(\frac{t}{2Ck}\right).
$$
Since $T : L_{p,r}(X, \mu) \to L_{p,\infty}(Y, \nu)$, with operator norm, say, $M_{p,r}$, we have, for any $y > 0$,
\[
y \nu_{Tf}(y)^{\frac{1}{p}} \leq M_{p,r} \|f\|_{L_{p,r}}
\]
\[
= \tau^{1/r} M_{p,r} \left( \int_0^\infty \mu_f(s)^{r/p} s^{r-1} ds \right)^{1/r}
\]
\[
= \tau^{1/r} M_{p,r} \left( \mu_f \left( \frac{t}{2\overline{C}k} \right)^{r/p} \left( \frac{t}{2\overline{C}k} \right)^r + \int_{\frac{t}{2\overline{C}k}}^\infty \mu_f(s)^{r/p} s^{r-1} ds \right)^{1/r}
\]
\[
\leq \tau^{1/r} M_{p,r} \left( 2 \int_{\frac{t}{4\overline{C}k}}^\infty \mu_f(s)^{r/p} s^{r-1} ds \right)^{1/r}.
\]
Indeed, for any $x > 0$
\[
\int_{x/2}^\infty \mu_f(s)^{r/p} s^{r-1} ds = \int_{x/2}^x \mu_f(s)^{r/p} s^{r-1} ds + \int_x^\infty \mu_f(s)^{r/p} s^{r-1} ds
\]
\[
\geq \mu_f(x)^{r/p} \left( 1 - \frac{1}{2^r} \right) + \int_x^\infty \mu_f(s)^{r/p} s^{r-1} ds
\]
\[
\geq \frac{1}{2} \left( \mu_f(x)^{r/p} \frac{2^r}{r} + \int_x^\infty \mu_f(s)^{r/p} s^{r-1} ds \right),
\]
which yields the assertion on taking $x = \frac{t}{2\overline{C}k}$. Again, with $y = \frac{t}{2\overline{C}}$, we get
\[
\frac{t}{2\overline{C}} \nu_{Tf} \left( \frac{t}{2\overline{C}} \right)^{\frac{1}{p}} \leq \tau^{1/r} M_{p,r} \left( 2 \int_{\frac{t}{4\overline{C}k}}^\infty \mu_f(s)^{r/p} s^{r-1} ds \right)^{1/r},
\]
which implies
\[
\nu_{Tf}(t) \leq \frac{1}{c} \left( \frac{M_{p,r}(2\overline{C})^{1/r}}{\frac{t}{2\overline{C}}} \right)^p \left( \int_{\frac{t}{4\overline{C}k}}^\infty \mu_f(s)^{r/p} s^{r-1} ds \right)^{p/r}.
\]

\[\text{Theorem 3.6.}\] Fix $p$ and $r$, with $1 < p < \infty$ and $1 \leq r < \infty$. Then, for any nonnegative, nonincreasing $g \in (L_{p,r} + L_{\infty})(\mathbb{R}_+, m)$ and $t \in \mathbb{R}_+$, one has
\[
p^{p/r} \left( \int_t^\infty m_{g^*}(s)^{r/p} s^{r-1} ds \right)^{p/r} \leq t^p m_{H^p g^*}(t)
\]
\[
\leq 2^{p+1} p^{p/r} \left( \int_{\frac{t}{2^{p+1}}}^\infty m_{g^*}(s)^{r/p} s^{r-1} ds \right)^{p/r}
\]
(3.8)

Proof. The operator $H^p g$ is in $W((p, r), (\infty, \infty); m, m)$, with $r$-quasilinearity constants $C = 2^{1-\frac{1}{r}}, \quad c = \frac{1}{2}$ and $M_{p,r}, M_{\infty}$ are less than or equal to $(\frac{2}{r})^{1/r}$. Also, observe that
\[
H^p f \leq H^p f^*, \quad f \in M_+(\mathbb{R}_+, m).
\]
This suffices to establish the first of the inequalities in (3.8), in view of Lemma 3.5.

To prove the first inequality we begin by letting $\tau_0$ be the least $\tau$ for which $(H^p g)(\tau) = t$. Then,
\[
\tau_0 = m_{H^p g}(t)
\]
and
\[(H^{p,r}g)(\tau_0) = t \iff \left( \tau_0^{-r/p} \int_0^{\tau_0} g(s)^r s^{p-1} ds \right)^{1/r} = t.\]

Since $H^{p,r}g(t) \geq g(t)$, implying thereby $\tau_0 = m_{H^{p,r}g}(t) \geq m_g(t)$,
\[
m_{H^{p,r}g}(t) = \tau_0 = \frac{1}{tp} \left( \int_0^{\tau_0} g(s)^r s^{p-1} ds \right)^{\frac{1}{p}} \geq \frac{1}{tp} \left( \int_0^{m_g(t)} g(s)^r s^{p-1} ds \right)^{\frac{1}{p}} = \frac{1}{tp} \left( \int_{\mathbb{R}^+} (\chi(0,m_g(t)) g)(s)^r s^{p-1} ds \right)^{\frac{1}{p}} = \left( \frac{p}{r} \right)^{\frac{1}{p}} \frac{1}{tp} \left( \frac{r}{p} \int_{\mathbb{R}^+} (\chi(0,m_g(t)) g)^*(s)^r s^{p-1} ds \right)^{\frac{1}{p}} = \left( \frac{p}{r} \right)^{\frac{1}{p}} \frac{1}{tp} ||\chi(0,m_g(t)) g||_{L^{p,r}(\mathbb{R}^+)}^{p} = \frac{p}{r} \frac{1}{tp} \left( \int_{\mathbb{R}^+} m_g(s)^r s^{p-1} ds \right)^{\frac{1}{p}} = \frac{p}{r} \frac{1}{tp} \left( \int_t^\infty m_g(s)^r s^{p-1} ds + \int_t^\infty m_g(s)^{r/p} s^{r-1} ds \right)^{\frac{1}{p}}\]

where in the last but one equality integral is realized as a Lorentz space norm, so finally we get
\[
m_{H^{p,r}g}(t) \geq p^{p/r} \frac{1}{tp} \left( \int_t^\infty m_g(s)^{r/p} s^{r-1} ds \right)^{\frac{1}{p}}.\]

Using Theorem 3.6 we can reduce a modular inequality (and hence the equivalent norm inequality) involving $H^{p,r}$ to a weighted Hardy inequality.

**Theorem 3.7.** Fix $p$ and $r$, where $1 < p < \infty$ and $1 \leq r < \infty$. Suppose $\Phi_i(t) = \int_0^t \phi_i(s)ds$, $i = 1, 2$ are Young functions. Then, the following are equivalent:

1. There exists a constant $C > 0$ such that
\[(3.11) \int_{\mathbb{R}^+} \Phi_1((H^{p,r}f^*)(t)) dt \leq \int_{\mathbb{R}^+} \Phi_2(Cf^*(s)) ds,\]

holds for all $f$ in $M^+_1(\mathbb{R}^+, m)$;

2. There exist $C_1, C_2 > 0$, such that the weighted Hardy inequality
\[(3.12) \int_{\mathbb{R}^+} \left( \int_x^\infty g(s) ds \right)^{p/r} \frac{\phi_1(y^{\frac{1}{r}})}{x^{1+\frac{1}{r}}} dx \leq C_1 \int_{\mathbb{R}^+} g(y)^{p/r} \phi_2 \left( C_2 y^{\frac{1}{r}} \right) y^{\frac{r-1}{r}} dy,\]

holds for all nonnegative, nonincreasing function $g$ on $\mathbb{R}^+$.

Moreover, $C_2 = \frac{C_1}{4K_{M_{\infty}}^r}$ and $C_1 = \frac{2K_{M_{p,r}}^{r-1}}{M_{p,r}^{r-1}} \left( \frac{C_1}{4K_{M_{\infty}}^r} \right)^{\frac{r}{p}}$, where $k = 1$ and $K$ are the constants of $r$-quasilinearity for the operator $H^{p,r}M_{p,r}$, $M_{\infty}$ are operator norms of $H^{p,r} : L^{p,r}(\mathbb{R}^+, m) \to L^{p,r}(\mathbb{R}^+, m)$ and $H^{p,r} : L^\infty(\mathbb{R}^+, m) \to L^\infty(\mathbb{R}^+, m)$ respectively.
Proof. Let $\alpha = 2^{2+\frac{1}{p}}\frac{p^p}{r}$ and $\beta = 2^{3-\frac{1}{r}}\frac{p^p}{r}$. Then, in view of Theorem 3.6,

$$\int_{\mathbb{R}^+} \Phi_1((H^{p,r}f^*)(t))dt = \int_{\mathbb{R}^+} \phi_1(t)m_{H^{p,r}f^*}(t)dt$$

$$\leq \alpha \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^p} \left( \int_{t^{1/\beta}}^{\infty} m_{f^*}(s/r^p)s^{r-1}ds \right)^{p/r} dt$$

$$= \left( \frac{\alpha}{\beta} \right)^p \frac{1}{r^{1+p/r}} \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^p} \left( \int_{t^{1/\beta}}^{\infty} m_{f^*}(s/r^p)s^{r-1}ds \right)^{p/r} dt$$

$$= \left( \frac{\alpha}{\beta} \right)^p \frac{1}{r^{1+p/r}} \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^p} \left( \int_{t^{1/\beta}}^{\infty} m_{f^*}(y^{\frac{1}{r}})y^{\frac{1}{r}-1}dy \right)^{p/r} \frac{x^{1/\beta}}{x^{1/\beta}} dx$$

Now, given (3.12), the latter will be

$$\leq \left( \frac{\alpha}{\beta} \right)^p \frac{1}{r^{1+p/r}} C_1 \int_{\mathbb{R}^+} m_{f^*}(y^{\frac{1}{r}}) \phi_2 \left( C_2 y^{\frac{1}{r}} \right) y^{\frac{1}{r}-1}dy$$

$$\leq \left( \frac{\alpha}{\beta} \right)^p \frac{1}{r^{p/r}} C_1 \int_{\mathbb{R}^+} m_{f^*}(s) \phi_2 \left( C_2 s \right) ds$$

$$\leq \left( \frac{\alpha}{\beta} \right)^p \frac{1}{r^{p/r}} \frac{C_1}{C_2} \int_{\mathbb{R}^+} \Phi_2 \left( \beta C_2 f^*(s) \right) ds$$

$$\leq \int_{\mathbb{R}^+} \Phi_2 \left( C f^*(s) \right) ds,$$

$$C = \max \left[ \beta C_2, \left( \frac{\alpha}{\beta} \right)^p \frac{1}{r^{p/r}} \frac{C_1}{C_2} \beta C_2 \right].$$

Thus, (3.12) implies (3.11).

Suppose, next, that (3.11) holds. The nonnegative, nonincreasing $g$ in (3.12) is of the form $m_{\beta f^*}(y^{\frac{1}{r}})^{r/p}$ for some $f^*$, namely, for

$$f^*(t) = \frac{g(s^{p/r})^{-1}(t)}{\beta}, \quad t \in \mathbb{R}^+.$$

So, (3.12) is equivalent to the inequality

$$\int_{\mathbb{R}^+} \left( \int_{t^{1/\beta}}^{\infty} m_{f^*}(y^{\frac{1}{r}})^{r/p}dy \right)^{p/r} \frac{\phi_1(t)}{t^p} dx \leq C_1 \int_{\mathbb{R}^+} m_{f^*}(y^{\frac{1}{r}}) \phi_2 \left( C_2 y^{\frac{1}{r}} \right) y^{\frac{1}{r}-1}dy.$$

Taking $x = t^r$ in the first integral we get

$$\int_{\mathbb{R}^+} \left( \int_{t^{1/r}}^{\infty} m_{f^*}(y^{\frac{1}{r}})^{r/p}dy \right)^{p/r} \frac{\phi_1(t)}{t^p} dt \leq C_1 \int_{\mathbb{R}^+} m_{f^*}(y^{\frac{1}{r}}) \phi_2 \left( C_2 y^{\frac{1}{r}} \right) y^{\frac{1}{r}-1}dy.$$

Again, with $y = s^{r}$ in either side of this last inequality we arrive at

$$\int_{\mathbb{R}^+} \left( \int_{t}^{\infty} m_{f^*}(s)^{r/p} ds \right)^{p/r} \frac{\phi_1(t)}{t^p} dt \leq C_1 \frac{1}{r^{p/r}} \int_{\mathbb{R}^+} m_{f^*}(s) \phi_2 \left( C_2 s \right) ds.$$
In view of (3.8) and (3.11) we have that
\[
\int_{\mathbb{R}^+} \left( \int_t^\infty m_{\beta f^*}(s)^{r/p} s^{r-1} ds \right)^{p/r} \frac{\phi_1(t)}{t^p} dt \leq p^{-p/r} \int_{\mathbb{R}^+} m_{H^{p,r}(\beta f^*)}(t) \phi_1(t) dt
= p^{-p/r} \int_{\mathbb{R}^+} \Phi_1(H^{p,r}(\beta f^*)(t)) dt
\leq p^{-p/r} \int_{\mathbb{R}^+} \Phi_2(C\beta f^*)(t)) dt
= \frac{C}{p^{p/r}} \int_{\mathbb{R}^+} m_{\beta f^*}(s) \phi_2(Cs) ds.
\]

So, if we choose \( C_1 = \left( \frac{\Phi}{p} \right)^{-p/r} C \) and \( C_2 = C \), we have (3.12) is implied by (3.11). \( \square \)

In the next two sections, we will be taking up the two-weight Hardy inequality (3.12) on nonnegative, nonincreasing functions with weights being functions involving \( \Phi_1 \) and \( \Phi_2 \). As in the classical case, the inequality (3.12) needs to be studied in two cases depending on whether \( \frac{p}{r} > 1 \) or \( \frac{p}{r} \leq 1 \).

3.3. The case \( 1 \leq r < p \). The dual Hardy operator, \( g \mapsto (Qg)(y) := \int_y^\infty g^*(s) ds \), in (3.12) is an example of a kernel operator, namely, an operator \( T \) of the form
\[
Tf(x) = \int_{\mathbb{R}^+} K(x,y) f(y) dy,
\]
in which \( f \in M_{+}(\mathbb{R}_+, m) \), \( x \in \mathbb{R}_+ \) and the nonnegative kernel \( K(x,y) \in M(\mathbb{R}_+ \times \mathbb{R}_+, m \times m) \).

The following result of E. T. Sawyer [Sw90] reduces the study of a weighted norm inequality for such a \( T \) on nonnegative nonincreasing functions on \( \mathbb{R}_+ \), as in (3.12), to one on nonnegative functions in \( M(\mathbb{R}_+, m) \).

**Theorem 3.8** (E. T. Sawyer, [Sw90]). Fix \( p_1 \) and \( q_1 \), \( 1 < p_1, q_1 < \infty \), and suppose \( w(x) \) and \( v(x) \) are weights on \( \mathbb{R}_+ \). Then, the inequality
\[
\left( \int_{\mathbb{R}^+} Tf(x)^{q_1} w(x) dx \right)^{1/q_1} \leq C \left( \int_{\mathbb{R}^+} f(x)^{p_1} v(x) dx \right)^{1/p_1}
\]
holds with \( C > 0 \) independent of the nonnegative and nonincreasing function \( f \) on \( \mathbb{R}_+ \) if and only if
\[
\int_{\mathbb{R}_+} \left( \int_0^x T^* g \right)^{p_1} \left( \frac{v(x)}{\int_0^x v} \right)^{q_1} dx \right)^{1/p_1} + \left( \int_{\mathbb{R}_+} \frac{T^* g}{V(\infty)^{1/q_1}} dx \right) \leq C \left( \int_{\mathbb{R}_+} g(x)^{q_1} w(x)^{1-q_1} dx \right)^{1/q_1},
\]
where \( C > 0 \) does not depend on nonnegative \( g \) in \( M(\mathbb{R}_+) \). Here \( T^* \) is the adjoint of \( T \) given by
\[
T^* g(y) = \int_{\mathbb{R}_+} K(z,y) g(z) dz, \text{ for all } y \in \mathbb{R}_+,
\]
and
\[
V(x) = \int_0^x v, \text{ for all } x \in \mathbb{R}_+
\]
and
\[
V(\infty) = \lim_{x \to \infty} V(x).
\]
The inequality (3.12) can now be rephrased as (3.13) with \((Tg)(x) = \int_x^\infty g^*(s)ds\), the dual Hardy operator, \(p_1 = q_1 = p/r > 1\), \(w(y) = \frac{\phi_1(y^{\frac{1}{p}})}{y^{\frac{1}{p_1} - 1}}\) and \(v(x) = \frac{\phi_2(C_2 x^{\frac{1}{r}})}{x^{1 - \frac{1}{r}}}\).

With this,
\[
V(x) = \int_0^x \phi_2(C_2 y^{\frac{1}{r}}) y^{\frac{1}{r} - 1} dy = \frac{r}{C_2} \int_0^{C_2 x^{\frac{1}{r}}} \phi_2(y) dy = \frac{r}{C_2} \Phi_2(C_2 x^{\frac{1}{r}})
\]
and
\[
\int_0^x T^*g = \int_0^x \left[ \int_0^y g(t) dt \right] dy = \int_0^x (x - y)g(y) dy = (I_2g)(x),
\]
is the Riemann-Liouville fractional integral operator of order 2.

Since
\[
\lim_{x \to \infty} V(x) = \lim_{x \to \infty} \Phi_2(C_2 x^{\frac{1}{r}}) = \infty,
\]
(3.14) amounts to the inequality
\[
\left( \int_0^\infty I_2g(x)^{p_1} \left( \frac{v(x)}{V(x)^{p_1}} \right) dx \right)^{1/p_1} \leq C \left( \int_0^\infty g(x)^{q_1} w(x)^{1 - q_1} dx \right)^{1/q_1}
\]
for \(0 \leq g \in M(\mathbb{R}_+, m)\), with \(p_1, w, v\) and \(V\) as specified above.

Now, a special case of the main result in Stepanov [Stp90], asserts that (3.15) holds if and only if for all \(x \in \mathbb{R}_+\)
\[
\left( \int_t^\infty (y - t)^{p_1} \frac{v(y)}{V(y)^{p_1}} dy \right)^{1/p_1} \left( \int_0^t w(y) dy \right)^{1/p_1} < \infty,
\]
and
\[
\left( \int_t^\infty \frac{v(y)}{V(y)^{p_1}} dy \right)^{1/p_1} \left( \int_0^t (t - y)^{p_1} w(y) dy \right)^{1/p_1} < \infty,
\]
Making the change of variable \(x \to x^{\frac{1}{r}}\) and \(y \to y^{\frac{1}{r}}\), in the expressions for \(w, v\) and \(V\), we arrive at the conditions
\[
\left( \int_x^\infty (y^r - x^r)^{p_1} \frac{\phi_2(C_2 y)}{\Phi_2(C_2 y)^{p_1}} dy \right)^{1/p_1} \left( \int_0^x \frac{\phi_1(y)}{y^{p_1}} dy \right)^{1/p_1} \leq A,
\]
\[
\left( \int_x^\infty \frac{\phi_2(C_2 y)}{\Phi_2(C_2 y)^{p_1}} dy \right)^{1/p_1} \left( \int_0^x (x^r - y^r)^{p_1} \frac{\phi_1(y)}{y^{p_1}} dy \right)^{1/p_1} \leq A.
\]

The operator \(H^{p_1, 1}\), given at \(f \in M_+(\mathbb{R}_+, m)\) by
\[
H^{p_1, 1}f(t) = t^{\frac{1}{p} - \frac{1}{p_1}} \int_0^t f(s) s^{\frac{1}{p} - 1} ds,
\]
was found by A. Cianchi, [Ci99], to satisfy the norm inequality
\[
\|H^{p_1, 1}f\|_{L_{\Phi_1}(\mathbb{R}_+, m)} \leq C \|f\|_{L_{\Phi_1}(\mathbb{R}_+, m)}
\]
if and only if there exist constants \(D, B > 0\) such that for all \(x \in \mathbb{R}_+\),
\[
\left( \int_x^\infty \frac{\phi_2(D y)}{\Phi_2(D y)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p}} \left( \int_0^x \frac{\phi_1(y)}{y^{p_1}} dy \right)^{\frac{1}{p_1}} \leq B.
\]
This suggests the possibility that the two conditions for the norm boundedness of \(H^{p_1, r}\), \(1 \leq r < p\) in Theorem 4.3.2 can be replaced by a single condition. That this is the case is the content of
Theorem 3.9. Let \((X, \mu)\) and \((Y, \nu)\) be \(\sigma\)-finite measure spaces with \(\mu(X) = \nu(Y) = \infty\), the latter being nonatomic and separable. Fix the indices \(p\) and \(r\), \(1 < p < \infty\) and \(1 \leq r < p\). Suppose \(\Phi_i(t) = \int_0^t \phi_i(s)ds\), \(i = 1, 2\), are Young functions. Then, with \(p_1 = \frac{p}{r}\), the following are equivalent:

1. To each \(T \in W((p, r), (\infty, \infty); \mu, \nu)\) there corresponds \(C > 0\) such that

\[
\|Tf\|_{L_{\Phi_1}(Y, \nu)} \leq C\|f\|_{L_{\Phi_2}(X, \mu)},
\]

for all \(f \in L_{\Phi_2}(X, \mu)\);

2. There exist \(B, D > 0\) such that for all \(x \in \mathbb{R}_+\),

\[
\left( \int_x^\infty \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{r p_1} dy \right)^{\frac{1}{p'}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p}} \leq B,
\]

namely,

\[
\left( \int_x^\infty \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{r p_1} dy \right)^{1 - \frac{r}{p'}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p}} \leq B.
\]

Proof. We have shown in Theorem 3.2 and Theorem 3.7 that the inequality in (1) is equivalent to the weighted Hardy inequality (3.3) for nonnegative, nonincreasing functions and hence to the inequality (3.15) for nonnegative, measurable functions. We need thus only show that (3.18) holds if and only if (3.22) does. It is important to observe that it follows from our previous considerations, (3.22) holds for \(D' > D\) whenever it holds for \(D\).

Suppose, first, we have (3.18). Now, with \(D = 2^+ C\)

\[
\int_x^\infty \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{r p_1} dy = \left( \int_x^{2^+ x} + \int_{2^+ x}^\infty \right) \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{r p_1} dy = I(x) + II(x).
\]

Thus, the left side of (3.22) is dominated by

\[
I(x)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}} + II(x)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}.
\]

But, for \(y \geq 2^+ x\),

\[
II(x) \leq 2^{p_1} \int_{2^+ x}^\infty \left( 1 - \frac{x^r}{y^r} \right)^{\frac{p_1}{r'}} \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{r p_1} dy
\]

\[
\leq 2^{p_1} \int_x^\infty (y^r - x^r)^{p_1} \frac{\phi_2(Cy)}{\Phi_2(Cy)^{p_1}} dy.
\]

Therefore, the first condition in (3.18) ensures that

\[
II(x)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}} \leq 2A.
\]
Again, with $D = 2^{\frac{1}{r}}C$,

$$I(x)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}} = \left( \int_x^{2^{\frac{1}{r}}x} \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}$$

$$\leq 2x^r \left( \int_x^{2^{\frac{1}{r}}x} \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}$$

$$\leq 2^{1 - \frac{1}{rp_1}} \left( \int_x^{2^{\frac{1}{r}}x} \frac{\phi_2(Cy)}{\Phi_2(Cy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}.$$

Now,

$$x^p \int_0^x \frac{\phi_1(y)}{y^p} dy \leq \int_0^x (2x^r - y^r)p_1 \frac{\phi_1(y)}{y^p} dy \leq \int_0^{2^{\frac{1}{r}}x} ((2^{\frac{1}{r}}x)^r - y^r)p_1 \frac{\phi_1(y)}{y^p} dy.$$

Altogether, then,

$$I(x)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}} \leq 2^{1 - \frac{1}{rp_1}} \left( \int_x^{2^{\frac{1}{r}}x} \frac{\phi_2(Cy)}{\Phi_2(Cy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}$$

$$\leq 2^{1 - \frac{1}{rp_1}} A.$$

Next, assume (3.22) holds with constant $D > 0$. Fix $x \in \mathbb{R}_+$. The left side of the first condition in (3.18) is equal to

$$\left( \int_x^\infty \left( 1 - \frac{x^r}{y^r} \right)^{\frac{1}{p_1}} \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}} \leq B.$$

Again,

$$\left( \int_x^\infty \frac{\phi_2(Cy)}{\Phi_2(Cy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x (x^r - y^r)p_1 \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}$$

$$\leq \left( \int_x^\infty \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} x^r dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}}$$

$$\leq \left( \int_x^\infty \frac{\phi_2(Dy)}{\Phi_2(Dy)^{p_1}} y^{p_1} dy \right)^{\frac{1}{p_1}} \left( \int_0^x \frac{\phi_1(y)}{y^p} dy \right)^{\frac{1}{p_1}} \leq B.$$

\[ \square \]

3.4. **The case** $p \leq r < \infty$. We have shown in Section 3.1 and Section 3.2 that $(L_{\phi_2}(X, \mu), L_{\phi_1}(Y, \nu))$ is an interpolation pair for $W((p, r), (\infty, \infty); \mu, \nu)$ if and only if there holds the modular inequality

$$\int_{\mathbb{R}_+} \Phi_1(H^p f^*(t)) dt \leq \int_{\mathbb{R}_+} \Phi_2(K f^*(s)) ds,$$
for all \( f \in M_+ (\mathbb{R}_+, m) \). We will prove that this modular inequality holds if and only if \( \Phi_1 \) and \( \Phi_2 \) satisfy the Zygmund-Strömberg condition: There exist \( A, B > 0 \) such that for all \( t \in \mathbb{R}_+ \),

\[
t^p \int_0^t \frac{\Phi_1(s)}{s^{p+1}} \, ds \leq A \Phi_2(Bt).
\]

Our complete result is

**Theorem 3.10.** Fix the indices \( p \) and \( r \), where \( 1 < p < \infty \) and \( p \leq r < \infty \). Let \( (X, \mu) \) and \( (Y, \nu) \) be \( \sigma \)-finite measure spaces with \( \mu(X) = \nu(Y) = \infty \), the latter being nonatomic and separable. Suppose \( \Phi_1 \) and \( \Phi_2 \) are Young functions. Then, the following are equivalent:

1. To each \( T \in W((p, r), (\infty, \infty); \mu, \nu) \) there corresponds \( C > 0 \) such that
   \[
   \|Tf\|_{L_{\Phi_1}(Y, \nu)} \leq C \|f\|_{L_{\Phi_2}(X, \mu)},
   \]
   whenever \( f \in L_{\Phi_2}(X, \mu) \);
2. \[
   \int_{\mathbb{R}_+} \Phi_1(H^{p,r} f^*(t)) \, dt \leq \int_{\mathbb{R}_+} \Phi_2(K f^*(s)) \, ds,
   \]
   for all \( f \in M_+ (\mathbb{R}_+, m) \)
3. One has
   \[
   t^p \int_0^t \frac{\Phi_1(s)}{s^{p+1}} \, ds \leq A \Phi_2(Bt),
   \]
   in which \( A, B > 0 \) are independent of \( t \in \mathbb{R}_+ \).

**Proof.** Only the equivalence of (2) and (3) needs proving at this point.

The argument that (3.25) implies (3.24) is essentially that of Strömberg [Str79] for the case \( r = p \). Thus, by Theorem 3.6,

\[
\int_{\mathbb{R}_+} \Phi_1((H^{p,r} f^*)^*)(t)) \, dt \leq \int_{\mathbb{R}_+} \Phi_1((H^{p,p} f^*)^*)(t)) \, dt
\]

\[
= \int_{\mathbb{R}_+} \phi_1(t) m_{H^{p,p} f^*}(t) \, dt
\]

\[
\leq 2^{2p+1} p \int_{\mathbb{R}_+} \frac{\phi_1(t)}{t^p} \int_{t/8}^{\infty} m_{f^*}(s) s^{p-1} \, ds \, dt
\]

\[
= 2^{2p+1} p \int_{\mathbb{R}_+} m_{f^*}(s) s^{p-1} \int_0^{8s} \phi_1(t) \frac{dt}{t^p} \, ds
\]

\[
= \frac{p}{2^{p-1}} \int_{\mathbb{R}_+} m_{f^*}(s)(8s)^p \int_0^{8s} \frac{\phi_1(t)}{t^{p+1}} \, dt \, ds
\]

\[
\leq \frac{p}{2^{p-1}} A \int_{\mathbb{R}_+} m_{f^*}(s) \Phi_2(8Bs) \frac{ds}{s}
\]

\[
\leq \frac{p}{2^{p-1}} A \int_{\mathbb{R}_+} m_{f^*}(s) \phi_2(8Bs) d(8Bs)
\]

\[
= \frac{p}{2^{p-1}} A \int_{\mathbb{R}_+} m_{8Bf^*}(s) \phi_2(s) \, ds
\]

\[
= \frac{p}{2^{p-1}} A \int_{\mathbb{R}_+} \Phi_2(8B f^*(s)) \, ds
\]

\[
\leq \int_{\mathbb{R}_+} \Phi_2(K f^*(s)) \, ds.
\]
where \( K = \frac{p}{q} \cdot 8AB \) or \( 8B \), according as \( \frac{p}{q} \cdot A \geq 1 \) or not.

To obtain (3.25) from (3.24) we substitute \( f(s) = f^*(s) = t\chi_{(0,1)}(s) \) in (3.24) to get

\[
\int_{\mathbb{R}^+} \Phi_2(K f^*(u))du \geq \int_1^\infty \Phi_1(H^{p,r} f^*(u))du
\]

\[
= \int_1^\infty \Phi_1 \left( u^{-\frac{1}{p}} \left( \int_0^u f^*(s)^r s^{\frac{q}{p} - 1} ds \right)^{\frac{1}{r}} \right) du
\]

\[
= \int_1^\infty \Phi_1 \left( u^{-\frac{1}{p}} \left( \frac{1}{\gamma} t \right)^{\frac{1}{r}} \right) du
\]

\[
= \int_1^\infty \Phi_1 \left( u^{-\frac{1}{p}} \gamma t \right) du
\]

\[
= p(\gamma t)^p \int_0^{\gamma t} \Phi_1(s) \frac{1}{s^{\frac{1}{r}} + 1} ds,
\]

where \( \gamma = \left( \frac{1}{\gamma} \right)^{\frac{1}{r}} \). Since \( \int_{\mathbb{R}^+} \Phi_2(K f^*(u))du = \int_0^1 \Phi_2(Kt)du = \Phi_2(Kt) \), we find, on replacing \( t \) by \( \gamma \), that (3.25) is satisfied with \( A = \frac{1}{p} \) and \( B = \frac{\Delta}{\gamma} = K \left( \frac{\Delta}{\gamma} \right)^{\frac{1}{r}} \). \( \qed \)

4. Interpolation results for the class \( W((1,1),(q,r);\mu,\nu) \)

Recall that a \( r \)-quasilinear operator \( T \) is in \( W((1,1),(q,r);\mu,\nu) \) if

\[
T : L_1(X,\mu) \rightarrow L_{1,\infty}(Y,\nu) \quad \text{and} \quad T : L_{q,r}(X,\mu) \rightarrow L_{q,\infty}(Y,\nu).
\]

Our main results in this section are Theorem 4.9 and Theorem 4.10 which give necessary and sufficient conditions on Young functions \( \Phi_1 \) and \( \Phi_2 \) so that every \( T \in W((1,1),(q,r);\mu,\nu) \) maps \( L_{q,r}(X,\mu) \) boundedly into \( L_{q,\infty}(Y,\nu) \); the first theorem deals with \( 1 \leq r < q \), the second with \( q \leq r < \infty \).

We proceed as in Section 3. First, in Section 4.1, we reduce the problem to the boundedness of a Calderón operator, \( S_{q,r} \), which corresponds to the class \( W((1,1),(q,r);\mu,\nu) \). We then establish, in Section 4.2, the equivalence between a gauge norm inequality involving \( S_{q,r} \) and a certain weighted Hardy inequality. The desired characterizations are then obtained in much the same way as those in the previous section.

4.1. A Calderón-type theorem. We use the notation \( S_{q,r} \) for the Calderón operator \( P + H_{q,r} \), \( (1 < q < \infty, 1 \leq r < \infty) \) where, for \( g \in M_+(\mathbb{R}_+,m), t \in \mathbb{R}_+, \)

\[
(Pg)(t) = \frac{1}{t} \int_0^t g(s)ds \quad \text{and} \quad (H_{q,r}g)(t) = \left( t^{-\frac{q}{r}} \int_t^\infty g(s)^r s^{\frac{q}{r} - 1} ds \right)^{\frac{1}{r}}.
\]

The operator \( P \) is same as \( H^{1,1} \), the Hardy averaging operator, but we prefer to use the more familiar notation \( P \) for it.

We begin with the following analogue of Theorem 3.1.

Theorem 4.1. Let \( (X,\mu) \) and \( (Y,\nu) \) be \( \sigma \)-finite measure spaces and fix the indices \( q \) and \( r \), \( 1 < q < \infty \) and \( 1 \leq r < \infty \). Suppose \( T \) is an \( r \)-quasilinear operator in the class \( W((1,1),(q,r);\mu,\nu) \). Then,

\[
(4.1) \quad (Tf)^*(t) \leq K (S_{q,r}f^*)(ct),
\]

where \( K \) is independent of \( f \in (L_1 + L_{q,r})(X,\mu) \) and \( t \in \mathbb{R}_+ \).

Further, the operator \( S_{q,r} \) is in the class \( W((1,1),(q,r);m,m) \).
Remark 4.1. The argument in proving the above theorem is the same as that of Theorem 3.1, so we have skipped its proof. We point out that, here, $K = \max \left[ M_{q,r} \left( \frac{2}{q} \right)^{\frac{1}{p}} \frac{1}{(1-\varepsilon r)^{\frac{1}{q}}}, 4M_1 \right] C$, where $c$ and $C$ are the constant of $r$-quasilinearity of $T$ and $M_{q,r}$ are operator norms of the mappings $T : L^1(X,\mu) \to L^{1,\infty}(Y,\nu)$ and $T : L_{q,r}(X,\mu) \to L_{q,\infty}(Y,\nu)$ respectively.

Now we establish a Calderón-type interpolation theorem for operators in $W((1,1),(q,r);\mu,\nu)$.

Theorem 4.2. Fix the indices $q$ and $r$, where $1 < q < \infty$ and $1 \leq r < \infty$. Suppose $(X,\mu)$ and $(Y,\nu)$ are $\sigma$-finite measure spaces with $\mu(X) = \nu(Y) = \infty$, the latter being nonatomic and separable. Then, the following are equivalent:

1. Every operator $T$ in the class $W((1,1),(q,r);\mu,\nu)$ is bounded from $L_{\Phi_2}(X,\mu)$ to $L_{\Phi_1}(Y,\nu)$;
2. The operator $S_{q,r}$ is bounded from $L_{\Phi_2}(\mathbb{R}^+,m)$ to $L_{\Phi_1}(\mathbb{R}^+,m)$.

Proof. We only show (1) implies (2), since the converse can be settled as in Theorem 3.2.

In Theorem 2.1 take $S = S_{q,r}$ and denote by $\tilde{S}_{q,r}$ the operator $\tilde{S}$ guaranteed to exist by that theorem, so that

$$(\tilde{S}_{q,r} f)^{\ast s} = S_{q,r} f^{\ast s}, \quad m\text{-a.e.},$$

for all $f \in M(X,\mu)$, since $(S_{q,r} f^{\ast s})(t) = \int_0^1 f^{\ast s}(ts)ds + \left( \int_{1}^{\infty} f^{\ast s}(ts)^r s^{\frac{r-1}{p}}ds \right)^{\frac{1}{r}}$, is nonincreasing, so $(S_{q,r} f^{\ast s})^{\ast r} = S_{q,r} f^{\ast \mu}$. Moreover, Lemma 2.2 ensures that $\tilde{S}_{q,r}$ is $r$-quasilinear, since $S_{q,r}$ is.

Again arguing as in the proof of Theorem 3.2, one obtains

$$\tilde{S}_{q,r} : L^1(X,\mu) \to L^{1,\infty}(Y,\nu) \quad \text{and} \quad \tilde{S}_{q,r} : L_{q,r}(X,\mu) \to L_{q,\infty}(Y,\nu)$$

boundedly, hence

$$\tilde{S}_{q,r} : L_{\Phi_2}(X,\mu) \to L_{\Phi_1}(Y,\nu)$$

boundedly and so

$$S_{q,r} : L_{\Phi_2}(\mathbb{R}^+,m) \to L_{\Phi_1}(\mathbb{R}^+,m),$$

thereby completing the proof. \qed

4.2. The Calderón operator $S_{q,r}$ and an associated Hardy inequality. In this section, we give the connection between the norm inequality of the operator $S_{q,r}$ and a weighted Hardy inequality.

An easy exercise in changes of variable shows that $S_{q,r}$ is a dilation-commuting operator, so from [KRS17, Theorem A], we get the following result.

Theorem 4.3. Let $\Phi_1$ and $\Phi_2$ be Young functions, and fix indices $q$ and $r$, with $1 < q < \infty$, $1 \leq r < \infty$. Then, norm inequality

$$(4.2) \quad \| S_{q,r} f \|_{L_{\Phi_1}(\mathbb{R}^+,m)} \leq C \| f \|_{L_{\Phi_2}(\mathbb{R}^+,m)}$$

holds for all $f \in M_+(\mathbb{R}^+,m)$ if and only if the modular inequality

$$(4.3) \quad \int_{\mathbb{R}^+} \Phi_1(S_{q,r}(f^{\ast s}))(t)dt \leq \int_{\mathbb{R}^+} \Phi_2(Kf^{\ast s})(s)ds,$$

holds for all $f$ in $M_+(\mathbb{R}^+,m)$.

In the next theorem we estimate the distribution function of $Tf$, where $T \in W((1,1),(q,r);\mu,\nu)$ and $f \in (L^1 + L_{q,r})(X,\mu)$.

Lemma 4.4. Let $1 < q < \infty, 1 \leq r < \infty$ and suppose $T \in W((1,1),(q,r);\mu,\nu)$. Then, for every $f \in (L^1 + L_{q,r})(X,\mu)$, $t \in \mathbb{R}^+$ and for every $k > 0$,
(4.4) \( \nu_T(t) \leq c^{-1} \frac{4CM_1}{t} \left( \int_0^\infty \mu_f(s)ds \right) + c^{-1} \left( \frac{4C^{1/r}M_q r}{t} \right)^q \left( \int_0^{t/4Ck} \mu_f(s^{r/q})s^{r-1}ds \right)^{q/r} \),

where \( c \) and \( C \) are the constant of \( r \)-quasilinearity of \( T \) and \( M_1, M_q r \) are operator norms of the mappings \( T : L_1(X, \mu) \rightarrow L_{1,\infty}(Y, \nu) \) and \( T : L_{q,r}(X, \mu) \rightarrow L_{q,\infty}(Y, \nu) \) respectively.

In particular, for \( k = M_1 \),

(4.5) \( \nu_T(t) \leq c^{-1} \left( \frac{4CM_1}{t} \right)^{-1} \int_0^\infty \mu_f(s)ds \)

\[ + c^{-1} \left( \frac{r^{1/r}M_q}{M_1} \right)^q \left( \frac{t}{4CM_1} \right)^{-q} \left( \int_0^{t/4Ck} \mu_f(s^{r/q})s^{r-1}ds \right)^{q/r} \),

Proof. Fix \( t > 0 \) and \( f \in (L_1 + L_{q,r})(X, \mu) \). Let \( k \) be any positive number. Write \( f = f_t + f^t \), with

\[ f^t(x) = \begin{cases} f(x) & |f(x)| > \frac{t}{2Ck}, \\ 0 & |f(x)| \leq \frac{t}{2Ck}, \end{cases} \]

and \( f_t(x) = f(x) - f^t(x) \). Then \( f^t \in L_1(X, \mu) \) and \( f_t \in L_{q,r}(X, \mu) \).

Now, by the \( r \)-quasilinearity of \( T \),

\[ \nu_T(t) \leq c^{-1} \left[ \nu_T f^t \left( \frac{t}{2C} \right) + \nu_T f_t \left( \frac{t}{2C} \right) \right]. \]

Since, \( T : L_1(X, \mu) \rightarrow L_{1,\infty}(Y, \nu) \) with operator norm \( M_1 \), we have,

\[ \sup_{y > 0} \nu_T f^t(y) \leq M_1 \| f^t \|_{L_1} \]

\[ = M_1 \left( \int_0^{t/2Ck} \mu_f \left( \frac{t}{2Ck} \right) ds + \int_{t/2Ck}^\infty \mu_f(s)ds \right). \]

It follows that

\[ \nu_T f^t \left( \frac{t}{2C} \right) \leq \frac{M_1}{(2C)^r} \left( \frac{t}{2Ck} \mu_f \left( \frac{t}{2Ck} \right) + \int_{t/2Ck}^\infty \mu_f(s)ds \right) \]

\[ \leq \frac{M_1}{(2C)^r} \left( 2 \int_{t/4C}^\infty \mu_f(s)ds \right) \]

(4.6) \[ = 4CM_1 \left( \frac{t}{4Ck} \right)^{-1} \int_{t/4C}^\infty \mu_f(s)ds \],

where the last but one inequality follows from the fact that, for any \( x \in \mathbb{R}_+ \),

\[ \int_x^\infty \mu_f(s)ds = \int_x^{2x} \mu_f(s)ds + \int_{2x}^\infty \mu_f(s)ds \]

\[ \geq \mu_f(2x)x + \int_{2x}^\infty \mu_f(s)ds \]

\[ \geq \frac{1}{2} \left[ \mu_f(2x)2x + \int_{2x}^\infty \mu_f(s)ds \right], \]

which yields the assertion on taking \( x = t/4Ck \).
Again, since $T : L_{q,r}(X, \mu) \to L_{q,\infty}(Y, \nu)$, with operator norm $M_{q,r}$, we have,

$$\sup_{y > 0} \nu_{Tf}(y)^{\frac{1}{q}} \leq M_{q,r} \|f\|_{L_{q,r}(X, \mu)}$$

$$= M_{q,r} \left( r \int_{\mathbb{R}^+} \mu_{f}(s^{q/r} s^{r-1}) ds \right)^{1/r}$$

$$= r^{1/r} M_{q,r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s) - \mu_{f} \left( \frac{s}{2Ce^{r}} \right) s^{r-1} ds \right)^{1/r}$$

$$\leq r^{1/r} M_{q,r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s^{q/r} s^{r-1}) ds \right)^{1/r}$$

whence,

$$\nu_{Tf} \left( \frac{t}{2Ce^{r}} \right) \leq \left( \frac{2Ce^{r} M_{q,r}}{t} \right)^{q/r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s^{q/r} s^{r-1}) ds \right)^{1/r}$$

$$= \left( \frac{4Ce^{r} M_{q,r}}{t} \right)^{q/r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s^{r/q} s^{r-1}) ds \right)^{1/r}$$

(4.7)

$$\leq \left( \frac{4Ce^{r} M_{q,r}}{t} \right)^{q/r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s^{r/q} s^{r-1}) ds \right)^{1/r}$$

From the estimates (4.6) and (4.7), we have

(4.8) \quad $$\nu_{Tf}(t) \leq c^{-1} \frac{4CM_{1}}{t} \left( \int_{1}^{\infty} \mu_{f}(s) ds \right) + c^{-1} \left( \frac{4Ce^{r} M_{q,r}}{t} \right)^{q/r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s^{r/q} s^{r-1}) ds \right)^{1/r}.$$

Whence, setting $k = M_{1}$, we get the following pointwise estimate for $\nu_{Tf}$,

$$\nu_{Tf}(t) \leq c^{-1} \frac{4CM_{1}}{t^{1/4CM_{1}}} \int_{0}^{\infty} \mu_{f}(s) ds + c^{-1} \left( \frac{4Ce^{r} M_{q,r}}{t^{1/4CM_{1}}} \right)^{q/r} \left( \int_{0}^{2Ce^{r}} \mu_{f}(s^{r/q} s^{r-1}) ds \right)^{1/r}.$$

□

**Remark 4.2.** In Lemma 4.4 take $X = Y = \mathbb{R}^+$, $\mu = \nu = m$ and

$$(Tf)(t) = (S_{q,r} f^{*})(t), \quad f \in (L_{1} + L_{q,r})(\mathbb{R}^+, m).$$

For this operator we have $C = 2^{1 - \frac{1}{q}}, \quad c = \frac{1}{2}, \quad M_{1} \leq 1 + \left( \frac{q}{2} \right)^{\frac{1}{q}}$ and

$$M_{q,r} \leq \begin{cases} \left[ \left( \frac{q}{2} \right)^{\frac{1}{q}} \right]^{q/r} + 1, & \text{if } r > 1, \\ 2q, & \text{if } r = 1; \end{cases}$$

indeed,

$$t^{\frac{1}{q}}(Pf^{*})(t) = t^{\frac{1}{q}} t^{-1} \int_{0}^{t} f^{*}(s) ds$$

$$= t^{-\frac{1}{q}} \int_{0}^{t} \left[ \left( s^{q/r} \right)^{\frac{1}{q}} - \frac{1}{q} \right] s^{r-\frac{1}{q}} ds$$
\[
\alpha = \begin{cases} 
q^\frac{2}{r} \left( \left( \frac{q'}{r'} \right)^{\frac{1}{r}+1} \right)^q, & r > 1 \\
q^q \left( \frac{2}{r+1} \right)^q, & r = 1,
\end{cases}
\]

Then, such \(\alpha\) satisfy \(2\alpha \beta^q \geq c^{-1} (4C r^{-1/r} M_{q,r})^q\) and we arrive at the estimate

\[
m_{S_{q,r} f^*}(t) \leq 2 \left[ \frac{1}{t/\beta} \int_0^\infty m_{f^*}(s) ds + \alpha \left( \int_0^{t/\beta} m_{f^*}(s)^{r/q} s^{r-1} ds \right)^{q/r} \right].
\]

**Theorem 4.5.** Fix the indices \(q\) and \(r\), with \(1 < q < \infty\) and \(1 \leq r < \infty\). Then, for \(f \in (L_1 + L_{q,r})(\mathbb{R}_+, m)\) and \(t \in \mathbb{R}_+\), one has we have,

\[
\frac{1}{2^q \left( \frac{q}{r} \right)^q} \left[ t^{-1} \int_0^\infty m_{f^*}(\lambda) d\lambda + t^{-q} \left( \int_0^t m_{f^*}(\lambda)^{r/q} \lambda^{r-1} d\lambda \right)^{q/r} \right] \leq m_{S_{q,r} f^*}(t)
\]

\[
\leq E \left[ \frac{1}{t/\beta} \int_0^\infty m_{f^*}(s) ds + \frac{1}{(t/\beta)^q} \left( \int_0^{t/\beta} m_{f^*}(s)^{r/q} s^{r-1} ds \right)^{q/r} \right].
\]

in which \(E = 2 \max \left[ 1, q^\frac{2}{r} \left( \left( \frac{q'}{r'} \right)^{\frac{1}{r}+1} \right)^q \right] \) and \(\beta = 2^{\frac{1}{r'}} \left( 1 + \left( \frac{q'}{r'} \right)^{\frac{1}{r'}} \right)\).

**Proof.** Fix \(f \in (L_1 + L_{q,r})(\mathbb{R}_+, m)\) and \(t \in \mathbb{R}_+\). To the end of establishing the first estimate in (4.10), let \(\tau_0\) be the least \(\tau\) such that \(S_{q,r} f^*(\tau) = t\). Then,

\[
m_{S_{q,r} f^*}(t) = \tau_0 \quad \text{and} \quad (S_{q,r} f^*)(\tau_0) = t.
\]
Observe that
\[
\int_0^{\tau_0} f^*(s)ds = \tau_0 f^*(\tau_0) + \int_{f^*(\tau_0)}^{\infty} m_{f^*}(\lambda)d\lambda.
\]
Now,
\[
\int_{\tau}^{\infty} f^*(s)^r s^{\frac{r-1}{r}} ds = \frac{q}{r} \int_{\tau}^{\infty} f^*(u^\frac{q}{r})^r du = \frac{q}{r} \int_{0}^{\infty} g(u)du = \frac{q}{r} \int_{0}^{\infty} m_g(\lambda)d\lambda
\]
where
\[
g(u) = \begin{cases}
0, & u < \tau^\frac{q}{r}, \\
f^*(u^\frac{q}{r})^r, & u \geq \tau^\frac{q}{r}
\end{cases}
\]
and
\[
m_g(\lambda) = \begin{cases}
0, & \lambda \geq f^*(\tau)^r, \\
m_{f^*}(\lambda^\frac{q}{r}) - \tau^\frac{q}{r}, & \lambda < f^*(\tau)^r,
\end{cases}
\]
so,
\[
\int_{\tau}^{\infty} f^*(s)^r s^{\frac{r-1}{r}} ds = \frac{q}{r} \left[ \int_{0}^{f^*(\tau)^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda - \tau^\frac{q}{r} f^*(\tau)^r \right].
\]
Thus,
\[
t = (S_{q,r}f^*)(\tau_0) = f^*(\tau_0) + \tau_0^{-1} \int_{f^*(\tau_0)}^{\infty} m_{f^*}(\lambda)d\lambda
\]
\[\quad + \tau_0^{-\frac{1}{q}} \left( \frac{q}{r} \right)^\frac{1}{r} \left[ \int_{0}^{f^*(\tau_0)^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda - \tau_0^\frac{q}{r} f^*(\tau_0)^r \right]^{\frac{1}{r}} \]
\[\quad \geq \min \left( 1, \left( \frac{q}{r} \right)^\frac{1}{r} \right) \left[ \tau_0^{-1} \int_{f^*(\tau_0)}^{\infty} m_{f^*}(\lambda)d\lambda + \tau_0^{-\frac{1}{q}} \left( \int_{0}^{f^*(\tau_0)^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda \right)^\frac{1}{r} \right].
\]
Since
\[
f^*(\tau_0) \leq \tau_0^{-1} \int_{0}^{\tau_0} m_{f^*}(s)ds \leq (S_{q,r}f^*)(\tau_0) = t,
\]
we have
\[
\int_{0}^{f^*(\tau_0)^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda = \int_{t}^{t^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda - \int_{f^*(\tau_0)^r}^{t^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda
\]
\[\quad \geq \int_{0}^{t^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda - \tau_0^\frac{q}{r} (t^r - f^*(\tau_0)^r).\]
Altogether, then,
\[
\max \left( 1, \left( \frac{q}{r} \right)^\frac{1}{r} \right) t = \max \left( 1, \left( \frac{q}{r} \right)^\frac{1}{r} \right) (S_{q,r}f^*)(\tau_0)
\]
\[\quad \geq \tau_0^{-1} \int_{f^*(\tau_0)}^{\infty} m_{f^*}(\lambda)d\lambda + \tau_0^{-\frac{1}{q}} \left( \int_{0}^{t^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda - \tau_0^\frac{q}{r} (t^r - f^*(\tau_0)^r) \right)^\frac{1}{r}
\]
\[\quad \geq \tau_0^{-1} \int_{t}^{\infty} m_{f^*}(\lambda)d\lambda + \tau_0^{-\frac{1}{q}} \left( \int_{0}^{t^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda \right)^\frac{1}{r} - t
\]
or
\[
\max \left( 2, 1 + \left( \frac{q}{r} \right)^\frac{1}{r} \right) t \geq \tau_0^{-1} \int_{t}^{\infty} m_{f^*}(\lambda)d\lambda + \tau_0^{-\frac{1}{q}} \left( \int_{0}^{t^r} m_{f^*}(\lambda^\frac{q}{r})^\frac{1}{r} d\lambda \right)^\frac{1}{r}.
\]
From
\[ \max \left( 2, 1 + \left( \frac{r}{q} \right) \frac{1}{q} \right) t \geq \tau_0^{-1} \int_t^{\infty} m_{f^*}(\lambda) d\lambda \]
and
\[ \max \left( 2, 1 + \left( \frac{r}{q} \right) \frac{1}{q} \right) t \geq \tau_0^{-\frac{1}{q}} \left( \int_0^{t^r} m_{f^*}(\lambda^\frac{1}{r})^{\frac{r}{q}} d\lambda \right)^{\frac{1}{r}} \],
we deduce that, with \( \gamma = \max \left( 2, 1 + \left( \frac{r}{q} \right) \frac{1}{q} \right) \),
\[ \tau_0 \geq \frac{1}{\gamma} \int_t^{\infty} m_{f^*}(\lambda) d\lambda \geq \frac{1}{\gamma} t^{-1} \int_t^{\infty} m_{f^*}(\lambda) d\lambda \]
and
\[ \tau_0 \geq \frac{1}{\gamma} t^{-\frac{1}{q}} \left( \int_0^{t^r} m_{f^*}(\lambda^\frac{1}{r})^{\frac{r}{q}} d\lambda \right)^{\frac{2}{r}}, \]
so
\[ \tau_0 \geq \frac{1}{2\gamma^q} \left[ t^{-1} \int_t^{\infty} m_{f^*}(\lambda) d\lambda + t^{-\frac{1}{q}} \left( \int_0^{t^r} m_{f^*}(\lambda^\frac{1}{r})^{\frac{r}{q}} d\lambda \right)^{\frac{2}{r}} \right]. \]
Now,
\[ 2\gamma^q = 2 \max \left[ 2^q, \left( 1 + \left( \frac{r}{q} \right) \frac{1}{q} \right)^q \right] \]
\[ \leq 2 \max \left[ 2^q, 2^{q-1} \left( 1 + \left( \frac{r}{q} \right) \frac{2}{r} \right)^q \right] \]
\[ = \max \left[ 2^{q+1}, 2^q \left( 1 + \left( \frac{r}{q} \right) \frac{2}{r} \right)^q \right]. \]
Therefore,
\[ \tau_0 \geq \frac{1}{2^{q+1}} \min \left[ \frac{1}{2}, \left( 1 + \left( \frac{r}{q} \right) \frac{2}{r} \right)^{-1} \right] \left[ t^{-1} \int_t^{\infty} m_{f^*}(\lambda) d\lambda + t^{-\frac{1}{q}} \left( \int_0^{t^r} m_{f^*}(\lambda^\frac{1}{r})^{\frac{r}{q}} d\lambda \right)^{\frac{2}{r}} \right]. \]
This, together with the upper bound obtained for \( (S_{q,r} f^*)(t) \) in Remark 4.2, completes the proof. \( \square \)

**Theorem 4.6.** Let \((X, \mu)\) and \((Y, \nu)\) be \(\sigma\)-finite measure spaces, with \((Y, \nu)\) being nonatomic and separable. Fix indices \(q\) and \(r\), where \(1 < q < \infty\) and \(1 \leq r < \infty\). Suppose \(\Phi_1\) and \(\Phi_2\) are Young functions, with \(\phi_i(t) = \frac{d\Phi_i}{dt}(t), i = 1, 2\). Then, the following are equivalent:

1. To each \(T \in W((1,1),(q,r);\mu,\nu)\) there corresponds a \(C > 0\) such that

\[ \|Tf\|_{L_{\Phi_1}(Y,\nu)} \leq C\|f\|_{L_{\Phi_2}(X,\mu)}, \]

for all \(f \in L_{\Phi_2}(X,\mu);\)

2. There exist constants \(C_1, C_2 > 0\), such that both of the Hardy type inequalities

\[ \int_{\mathbb{R}^+} \int_t^{\infty} g(s)ds \frac{\phi_1(t)}{t} dt \leq C_1 \int_{\mathbb{R}^+} g(t)\phi_2 \left( C_2 t \right) dt \]

and

\[ \int_{\mathbb{R}^+} \left( \int_0^{t} g(s)ds \right)^{q/r} \frac{\phi_1(t)}{t^{1+\frac{1}{r}}} dt \leq C_1 \int_{\mathbb{R}^+} g(t)^{q/r} \phi_2 \left( C_2 t^{\frac{r}{q}} \right) t^{\frac{1}{r}-1} dt, \]

hold for all nonnegative, nonincreasing functions \(g\) on \(\mathbb{R}^+\).
Proof. In view of Theorem 4.2, the necessary and sufficient conditions on Young functions $\Phi_1, \Phi_2$ such that the norm inequality (4.11) holds are the same as those for which the following norm inequality for $S_{q,r}$ holds,

$$\|S_{q,r}f^*\|_{L_{\Phi_1}(\mathbb{R}^+)} \leq C\|f^*\|_{L_{\Phi_2}(\mathbb{R}^+)}.$$  

This norm inequality is, in turn, equivalent to the modular inequality,

$$\int_{\mathbb{R}^+} \Phi_1(S_{q,r}f^*(t))dt \leq \int_{\mathbb{R}^+} \Phi_2(Cf^*(s))ds. \tag{4.14}$$  

In view of Theorem 4.5,

$$\int_{\mathbb{R}^+} \Phi_1(S_{q,r}f^*(t))dt = \int_{\mathbb{R}^+} \phi_1(t)m_{S_{q,r}f^*}(t)dt \leq E\int_{\mathbb{R}^+} \phi_1(t) \left[ \left( \frac{t}{\beta} \right)^{-1} \int_{t/\beta}^{\infty} m_f(\lambda)d\lambda + \left( \frac{t}{\beta} \right)^{-q} \left( \int_0^{t/\beta} m_f(\lambda)^{r/q} \lambda^{r-1}d\lambda \right)^{q/r} \right] dt$$

$$= E\left[ \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t} \int_t^{\infty} m_f(\lambda)d\lambda dt + \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^q} \left( \int_0^{t} m_f(\lambda)^{r/q} \lambda^{r-1}d\lambda \right)^{q/r} dt \right]$$

$$= E\left[ \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t} \int_t^{\infty} m_{\beta_f}(\lambda)d\lambda dt + \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^q} \left( \int_0^{t} m_{\beta_f}(\lambda)^{r/q} \lambda^{r-1}d\lambda \right)^{q/r} dt \right].$$

Now, by (4.12),

$$\int_{\mathbb{R}^+} \frac{\phi_1(t)}{t} \int_t^{\infty} m_{\beta_f}(\lambda)d\lambda dt \leq C_1 \int_{\mathbb{R}^+} m_{\beta_f}(\lambda)\phi_2(C_2\lambda) d\lambda.$$  

Again,

$$\int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^q} \left( \int_0^{t} m_{\beta_f}(\lambda)^{r/q} \lambda^{r-1}d\lambda \right)^{q/r} dt = \frac{1}{r^{1+\frac{1}{r}}+1} \int_{\mathbb{R}^+} \frac{\phi_1(t)}{t^q} \left( \int_0^{t} m_{\beta_f}(\lambda)^{r/q} \lambda^{r-1}d\lambda \right)^{q/r} dt$$

which, by (4.13), is dominated by

$$\left( \frac{C_1}{r^{1+\frac{1}{r}}+1} \right) \int_{\mathbb{R}^+} \lambda^{r-1}d\lambda = \left( \frac{C_1}{r^{1+\frac{1}{r}}+1} \right) C_2.$$

So, (4.14) holds with $C = \max \left\{ 1, \frac{2EC_1}{C_2} \right\} C_2.$

An argument similar to the one above yields, on making the change of variable $t \to t^r$, then $s \to s^r,$

$$\int_{\mathbb{R}^+} \left( \int_0^t g(s)ds \right)^{q/r} \frac{\phi_1(t)}{t^{q/r}} dt = t^{1+\frac{2}{q/r}} \int_{\mathbb{R}^+} \left( \int_0^{t} g(s^r) s^{r-1} ds \right)^{q/r} \frac{\phi_1(t)}{t^{q/r}} dt$$

$$= t^{1+\frac{2}{q/r}} \int_{\mathbb{R}^+} \left( \int_0^{t} m_{f^*}(s)^{r/q} s^{r-1} ds \right)^{q/r} \frac{\phi_1(t)}{t^{q/r}} dt$$

$$\leq 2q^{1+\frac{2}{q/r}} \left( 1 + \left( \frac{t}{q} \right)^{\frac{q}{r}} \right) \int_{\mathbb{R}^+} m_{S_{q,r}f^*}(t)\phi_1(t) dt$$

$$= 2q^{1+\frac{2}{q/r}} \left( 1 + \left( \frac{t}{q} \right)^{\frac{q}{r}} \right) \int_{\mathbb{R}^+} \Phi_1(S_{q,r}f^*(t)) dt$$
Theorem 4.7. Fix the indices $q$ and $r$, $1 < q < \infty$, $1 \leq r < \infty$. Let $\Phi_1$ and $\Phi_2$ be Young functions such that

\begin{equation}
\| S_{q,r}f\|_{L_{\Phi_1}(\mathbb{R}_+,m)} \leq C \| f\|_{L_{\Phi_2}(\mathbb{R}_+,m)},
\end{equation}

in which $C > 0$ is independent of $f \in L_{\Phi_2}(\mathbb{R}_+,m)$. Then, there holds the Zygmund-Strömberg condition

\begin{equation}
t^{q} \int_{t}^{\infty} \frac{\Phi_1(s)}{s^{q+1}} ds \leq A\Phi_2(Bt),
\end{equation}

where the constants $A, B > 0$ does not depend on $t \in \mathbb{R}_+$.

Proof. According to Theorem 4.3, the norm inequality (4.15) holds if and only if one has the modular inequality

\begin{equation}
\int_{\mathbb{R}_+} \Phi_1(S_{q,r}f^*(t)) dt \leq \int_{\mathbb{R}_+} \Phi_2(Kf^*(s)) ds,
\end{equation}

for all $f \in M_+(\mathbb{R}_+,m)$.

Fix $t \in \mathbb{R}_+$. We will obtain (4.16) from (4.17) by substituting the function $f(s) = f^*(s) = t\chi_{(0,1)}(s)$ in the modular inequality. Indeed,

$$\int_{\mathbb{R}_+} \Phi_2(Kf^*(s)) ds = \int_{0}^{1} \Phi_2(Kt) ds = \Phi_2(Kt).$$

Again, for $y < 1$,\n
$$\begin{align*}
(S_{q,r}f^*)(y) &= t \int_{0}^{y} \chi_{(0,1)}(s) ds + t \left( y^{-r/q} \int_{y}^{1} s^{r/q-1} ds \right)^{1/r} \\
&= t + t \left( \frac{2}{r} y^{-r/q} (1 - y^{r/q}) \right)^{1/2} \\
&= t + t \left( \frac{2}{r} \right)^{1/2} \left( (1/y)^{r/q} - 1 \right)^{1/2} \\
&\geq ct \left[ 1 + \left( (1/y)^{r/q} - 1 \right)^{1/2} \right] \\
&= ct(1/y)^{1/q},
\end{align*}$$

in which we have taken $f^*(t) = \left( g(s^{r/q})^{2} \right)^{-1}(t)$ and has made use of (4.10) to get (4.13), with $C_1 = 2^{q}\frac{2}{r} \left( 1 + \frac{2}{q} \right) C$ and $C_2 = C$. \qed

4.3. The necessity of the Zygmund-Strömberg condition for the boundedness of $S_{q,r}$. 

Theorem 4.7. Fix the indices $q$ and $r$, $1 < q < \infty$, $1 \leq r < \infty$. Let $\Phi_1$ and $\Phi_2$ be Young functions such that

\begin{equation}
\| S_{q,r}f\|_{L_{\Phi_1}(\mathbb{R}_+,m)} \leq C \| f\|_{L_{\Phi_2}(\mathbb{R}_+,m)},
\end{equation}

in which $C > 0$ is independent of $f \in L_{\Phi_2}(\mathbb{R}_+,m)$. Then, there holds the Zygmund-Strömberg condition

\begin{equation}
t^{q} \int_{t}^{\infty} \frac{\Phi_1(s)}{s^{q+1}} ds \leq A\Phi_2(Bt),
\end{equation}

where the constants $A, B > 0$ does not depend on $t \in \mathbb{R}_+$. 

Proof. According to Theorem 4.3, the norm inequality (4.15) holds if and only if one has the modular inequality

\begin{equation}
\int_{\mathbb{R}_+} \Phi_1(S_{q,r}f^*(t)) dt \leq \int_{\mathbb{R}_+} \Phi_2(Kf^*(s)) ds,
\end{equation}

for all $f \in M_+(\mathbb{R}_+,m)$.

Fix $t \in \mathbb{R}_+$. We will obtain (4.16) from (4.17) by substituting the function $f(s) = f^*(s) = t\chi_{(0,1)}(s)$ in the modular inequality. Indeed,

$$\int_{\mathbb{R}_+} \Phi_2(Kf^*(s)) ds = \int_{0}^{1} \Phi_2(Kt) ds = \Phi_2(Kt).$$

Again, for $y < 1$,\n
$$\begin{align*}
(S_{q,r}f^*)(y) &= t \int_{0}^{y} \chi_{(0,1)}(s) ds + t \left( y^{-r/q} \int_{y}^{1} s^{r/q-1} ds \right)^{1/r} \\
&= t + t \left( \frac{2}{r} y^{-r/q} (1 - y^{r/q}) \right)^{1/2} \\
&= t + t \left( \frac{2}{r} \right)^{1/2} \left( (1/y)^{r/q} - 1 \right)^{1/2} \\
&\geq ct \left[ 1 + \left( (1/y)^{r/q} - 1 \right)^{1/2} \right] \\
&= ct(1/y)^{1/q},
\end{align*}$$

in which we have taken $f^*(t) = \left( g(s^{r/q})^{2} \right)^{-1}(t)$ and has made use of (4.10) to get (4.13), with $C_1 = 2^{q}\frac{2}{r} \left( 1 + \frac{2}{q} \right) C$ and $C_2 = C$. \qed
where $c = \min\{1, \left(\frac{2}{q}\right)^{\frac{1}{q}}\}$. So,

$$
\int_{\mathbb{R}^+} \Phi_1(S_{q,r}(f^*)(y))dy \geq \int_{0}^{1} \Phi_1(S_{q,r}(f^*)(y))dy \\
\geq \int_{0}^{1} \Phi_1(ckt^{1/q})dy \\
= q(ckt)^q \int_{ct}^{\infty} \frac{\Phi_1(z)}{z^{q+1}}dz,
$$

where we have made the change of variable $ckt^{1/q} = z$. Altogether, then, we have

$$
q(ckt)^q \int_{ct}^{\infty} \frac{\Phi_1(z)}{z^{q+1}}dz \leq \Phi_2(Kt).
$$

Replacing $ct$ by $t$ yields (4.16), with $A = \frac{q}{q}$ and $B = \frac{K}{c}$. \qed

4.4. The case $1 \leq r < q$. In this section, we prove our interpolation result for the class $W((1,1),(q,r);\mu,\nu)$ in the case of $1 \leq r < q$, by characterizing the weighted Hardy inequalities obtained in Theorem 4.6, using a result of Sawyer [Sw90, Theorem 2], which we now state.

**Theorem 4.8** (E. T. Sawyer, [Sw90]). Suppose that $w_1(x)$ and $v_1(x)$ are nonnegative measurable functions on $\mathbb{R}^+$. If $1 < p_1 < q_1 < \infty$, then

$$
\left(\int_0^t f(t)dt\right)^{\frac{q_1}{q_1}} \leq C \left(\int_0^\infty f(x)^{p_1}v_1(x)dx\right)^{\frac{1}{p_1}},
$$

holds for all nonnegative and nonincreasing functions $f$, if and only if both of the following conditions hold:

$$
\left(\int_0^t v_1(x)dx\right)^{\frac{1}{p_1}} \leq A \left(\int_0^t v_1(x)dx\right)^{\frac{1}{p_1}}, \quad \text{for all } t > 0;
$$

$$
\left(\int_t^\infty x^{-q_1}w_1(x)dx\right)^{\frac{1}{q_1}} \left(\int_0^t (x^{-1}V_1(x))^{-p_1}v_1(x)dx\right)^{\frac{1}{p_1}} \leq B, \quad \text{for all } t > 0,
$$

where $V_1(x) = \int_0^x v_1(y)dy$. Moreover, if $C$ is the best constant in (4.18), then $C \approx A + B$.

Next we prove our interpolation result.

**Theorem 4.9.** Let $(X,\mu)$ and $(Y,\nu)$ be $\sigma$-finite measure spaces with $\mu(X) = \nu(Y) = \infty$, the latter being nonatomic and separable. Fix indices $p$ and $r$, where $1 < q < \infty$ and $1 \leq r < q$. Suppose $\Phi_i(t) = \int_0^t \phi_i(s)ds$, $i = 1,2$, are Young functions satisfying Zygmund-Strömberg condition: There exist $A > 0$ such that for all $t \in \mathbb{R}_+$,

$$
t \int_0^t \frac{\Phi_1(s)}{s^2}ds \leq \Phi_2(At).
$$

Then, setting $q_1 = q/r$, the following are equivalent:

1. To each $T \in W((1,1),(q,r);\mu,\nu)$ there corresponds $C > 0$ such that

$$
\|Tf\|_{L_{\Phi_1}(Y,\nu)} \leq C\|f\|_{L_{\Phi_2}(X,\mu)},
$$

for all $f \in L_{\Phi_2}(X,\mu)$. 

There exist $C_2 > 0$ such that the following condition, below, holds

\begin{equation}
\left( \int_0^t \frac{\phi_2(C_2y)}{\Phi_2(C_2y)^{q_1}} y^{r_1} \, dy \right)^{\frac{1}{q_1}} \left( \int_t^\infty \frac{\phi_1(y)}{y^q} \, dy \right)^{\frac{1}{q}} \leq F < \infty,
\end{equation}

namely,

\begin{equation}
\left( \int_0^t \frac{\phi_2(C_2y)}{\Phi_2(C_2y)^{r_1}} y^{q_2} \, dy \right)^{\frac{1}{r_1}} \left( \int_t^\infty \frac{\phi_1(y)}{y^q} \, dy \right)^{\frac{1}{q}} \leq F < \infty.
\end{equation}

Moreover, if $C$ is the least constant for which \((4.22)\) holds, then the ratio $C/(A+B)$ is bounded between two positive constants depending only on $q,r$.

**Proof.** In view of Theorem 4.6, we need necessary and sufficient conditions on the appropriate weights in order that the inequalities

\begin{equation}
\int_{\mathbb{R}_+} \int_t^\infty g(s) ds \frac{\phi_1(t)}{t} dt \leq C_1 \int_{\mathbb{R}_+} g(t) \phi_2(C_2t) dt
\end{equation}

and

\begin{equation}
\int_{\mathbb{R}_+} \left( \int_0^t g(s) ds \right)^{\frac{q}{r}} \frac{\phi_1(t)}{t} dt \leq C_1 \int_{\mathbb{R}_+} g(t) \phi_2(C_2t) t^{\frac{1}{r}-1} dt,
\end{equation}

hold with $C_1, C_2 > 0$ independent of the nonnegative, nonincreasing functions $g$ on $\mathbb{R}_+$.

Interchanging the order of integration in the integral on the left side of \((4.24)\) leads to the inequality

\begin{equation}
\int_{\mathbb{R}_+} g(s) \int_0^s \frac{\phi_1(t)}{t} dt ds \leq C_1 \int_{\mathbb{R}_+} g(t) \phi_2(C_2t) dt.
\end{equation}

The most general nonnegative, nonincreasing $g$ for which this latter inequality holds essentially has the form

\begin{equation}
g(s) = \int_s^\infty h(y) dy, \quad \text{for some } h \in M_+ (\mathbb{R}_+, m),
\end{equation}

in which case \((4.24)\) changes to

\begin{equation}
\int_{\mathbb{R}_+} h(y) \int_0^y \left( \int_0^s \frac{\phi_1(t)}{t} dt \right) ds dy \leq C_1 \int_{\mathbb{R}_+} h(y) \int_0^y \phi_2(C_2t) dt dy.
\end{equation}

One readily shows this is satisfied if and only if one has \((4.21)\).

As for the inequality, \((4.25)\), Theorem 2 of [Sw90] shows it holds if and only if for $t \in \mathbb{R}_+$,

\begin{equation}
\int_0^t \phi_1(s^\frac{1}{r}) s^{\frac{1}{r}-1} ds \leq A \int_0^t \phi_2(C_2s^\frac{1}{r}) s^{\frac{1}{r}-1} ds
\end{equation}

and

\begin{equation}
\left( \int_t^\infty s^{-q_1} \phi_1(s^\frac{1}{r}) s^{\frac{1}{r}-1} ds \right)^{\frac{1}{q_1}} \left( \int_0^t \left( \int_0^s \phi_2(C_2y^\frac{1}{r}) y^{\frac{1}{r}-1} dy \right) \phi_2(C_2s^\frac{1}{r}) s^{\frac{1}{r}-1} ds \right)^{\frac{1}{q}} \leq B,
\end{equation}

hold, where $q_1 = \frac{q}{r}$.

After suitable change of variable, \((4.26)\) reads

\begin{equation}
\Phi_1(t^\frac{1}{r}) \leq \frac{A}{C_2} \Phi_2(C_2t^\frac{1}{r}),
\end{equation}

or, on replacing $t^\frac{1}{r}$ by $t$,

\begin{equation}
\Phi_1(t) \leq \frac{A}{C_2} \Phi_2(C_2t).
\end{equation}
But, this condition is implied by the Zygmund-Strömberg condition (4.21), which is one of our hypothesis.

The change of variable \( s \to s^r \) in the left hand integral in (4.27) yields
\[
\int_{t^\frac{1}{r}}^s s^{-q} \phi_1(s) \, ds
\]
Again, as observed above,
\[
\int_0^s \phi_2(C_2 y^\frac{1}{r}) y^{-1} \, dy = \frac{r}{C_2} \Phi_2(C_2 s^\frac{1}{r}),
\]
so that
\[
\int_0^t \left( s^{-1} \int_0^s \phi_2(C_2 y^\frac{1}{r}) y^{-1} \, dy \right)^{-q'_1} \phi_2(C_2 s^\frac{1}{r}) s^{-1} \, ds
\]
\[
= \int_0^t \left( \frac{r}{C_2} \Phi_2(C_2 s^\frac{1}{r}) \right)^{-q'_1} \phi_2(C_2 s^\frac{1}{r}) s^{-1} \, ds
\]
\[
= r \left( \frac{C_2}{r} \right)^{q'_1} \int_0^t \frac{\phi_2(C_2 s)}{\Phi_2(C_2 s)^{q'_1}} s^{r q'_1} \, ds
\]
Thus, (4.27) amounts to (4.23).

4.5. The case \( r \geq q \). Our result in this case is independent of \( r \). It is given in

**Theorem 4.10.** Let \((X, \mu)\) and \((Y, \nu)\) be \(\sigma\)-finite measure spaces with \(\mu(X) = \nu(Y) = \infty\), the latter being nonatomic and separable. Fix the indices \( q \) and \( r \), \( r \geq q > 1 \). Suppose \( \Phi_i(t) = \int_0^t \phi_i(s) \, ds \), \( i = 1, 2 \), are Young functions satisfying Zygmund-Strömberg condition (4.21). Then, the following are equivalent:

1. To each \( T \in W((1, 1), (q, r); \mu, \nu) \) there corresponds \( C > 0 \) such that

\[
\| T f \|_{L_{q,r}(Y, \nu)} \leq C \| f \|_{L_{q,r}(X, \mu)};
\]

   for all \( f \in L_{q,r}(X, \mu) \);

2. There exist \( A, B > 0 \) such that (4.16) holds.

The result for \( r \geq q \) can be reduced to the case of \( r = q \). This is a consequence of

**Proposition 4.11.** Given \( r \geq q \), there exists a constant \( K > 0 \), depending on \( r \), such that
\[
\| S_{q, r} f^* \| (t) \leq K \| S_{q, q} f^* \| \left( \frac{2^q}{2^q - 1} t \right),
\]
for all \( f \in M(\mathbb{R}_+, m) \) and all \( t \in \mathbb{R}_+ \).

**Proof.** It suffices to verify the inequality
\[
\| H_{q, r} f^* \| (t) \leq K \| H_{q, q} f^* \| (t),
\]
for all \( f \in M(\mathbb{R}_+, m) \) and all \( t \in \mathbb{R}_+ \). Recall,
\[
\| H_{q, r} f^* \| (t) = \left( \frac{2^q}{q} \right)^{\frac{r}{q}} \int_t^{\infty} f^*(s)^{\frac{r}{q}} s^{\frac{r}{q} - 1} \, ds \right)^{\frac{1}{r}}.
\]
Letting \( s = u^q \), we get
\[
\| H_{q, r} f^* \| (t) = \left( \frac{2^q}{q} \right)^{\frac{r}{q}} \int_t^{\infty} f^*(u^q)^{\frac{r}{q}} \, du \right)^{\frac{1}{r}}.
\]
or
\[
\left[(H_{q,r} f^*)(t^{\frac{q}{r}})\right]^r = \left(\frac{q}{r}\right) t^{-1} \int_1^\infty f^*(u^{\frac{q}{r}})^r du.
\]

Now,
\[
\left[(H_{q,r} f^*)(t^{\frac{q}{r}})\right]^r = \left(\frac{q}{r}\right) t^{-1} \int_{\mathbb{R}^+} f^* \left((t + u)^{\frac{q}{r}}\right)^r du
\leq \left(\frac{q}{r}\right) t^{-1} \int_{\mathbb{R}^+} f^* \left(2^{\frac{q}{r}-1}t^{\frac{q}{r}} + 2^{\frac{q}{r}-1}u^{\frac{q}{r}}\right)^r du \quad \text{(since } r \geq q\text{)}
\]
\[= \gamma^{-\frac{r}{q}} t^{-1} \int_{\mathbb{R}^+} f^* \left(x + \gamma t^{\frac{q}{r}}\right)^r x^{\frac{r}{q}-1} dx,
\]
where \(\gamma = 2^{\frac{q}{r}-1}\).

Thus,
\[\left[(H_{q,r} f^*)(t^{\frac{q}{r}})\right]^r \leq \gamma^{-\frac{1}{q}} \left(\frac{q}{r}\right)^{\frac{1}{q}} t^{-\frac{1}{q}} \|f^* (\cdot + \gamma t^{\frac{q}{r}})\|_{L_{q,r}(\mathbb{R}^+, m)} \]
\[\leq \gamma^{-\frac{1}{q}} \left(\frac{q}{r}\right)^{\frac{1}{q}} t^{-\frac{1}{q}} \|f^* (\cdot + \gamma t^{\frac{q}{r}})\|_{L_{q,q}(\mathbb{R}^+, m)}
\]
\[= \gamma^{-\frac{1}{q}} \left(\frac{q}{r}\right)^{\frac{1}{q}} t^{-\frac{1}{q}} \left(\int_{\gamma t^{\frac{q}{r}}}^{\infty} f^* (s)^q ds\right)^\frac{1}{q}.
\]

Replacing \(t\) by \(t^{r/q}\) yields
\[\left[(H_{q,r} f^*)(t)\right] \leq \left(\frac{q}{r}\right)^{\frac{1}{q}} (\gamma t)^{\frac{1}{q}} \left(\int_{\gamma t}^{\infty} f^* (s)^q ds\right)^\frac{1}{q} = \left(\frac{q}{r}\right)^{\frac{1}{q}} (H_{q,q} f^*)(\gamma t).
\]

**Proof of Theorem 4.10.** In view of Theorems 4.2 and 4.3, the assertion in 1 is equivalent to the requirement that
\[(4.29) \quad \int_{\mathbb{R}^+} \Phi_1(S_{q,r} f^*(t))dt \leq \int_{\mathbb{R}^+} \Phi_2(K f^*(s))ds,
\]
for all \(f \in M(\mathbb{R}^+, m)\).

Suppose first that \((4.29)\) holds. This, together with Proposition 4.11, shows 2 holds, given 1. Assume, next, we have 2. According to Theorem 4.5,
\[
\int_{\mathbb{R}^+} \Phi_1(S_{q,q} f^*(t))dt = \int_{\mathbb{R}^+} \phi_1(t) m_{S_{q,q} f^*}(t)dt
\;
\leq E \left[\int_{\mathbb{R}^+} \phi_1(t) \left(\frac{t}{\beta}\right)^{-1} \int_{\frac{t}{\beta}}^{\infty} m_{f^*}(\lambda)d\lambda + \int_{\mathbb{R}^+} \frac{\phi_1(t)}{(t/\beta)^{\eta}} \int_0^{\frac{t}{\beta}} m_{f^*}(\lambda)\lambda^{q-1}d\lambda dt\right]dt.
\]

To begin with the first term in the last expression, we have
\[
\int_{\mathbb{R}^+} \phi_1(t) \left(\frac{t}{\beta}\right)^{-1} \int_{\frac{t}{\beta}}^{\infty} m_{f^*}(\lambda)d\lambda dt = \beta \int_{\mathbb{R}^+} m_{f^*}(\lambda) \int_0^{\beta \frac{\lambda}{t}} \frac{\phi_1(t)}{t} dt d\lambda
\;
\leq \beta \int_{\mathbb{R}^+} m_{f^*}(\lambda) \int_0^{\beta \frac{\lambda}{t}} \frac{\phi_1(2t)}{t^2} dt d\lambda
\;
= \int_{\mathbb{R}^+} m_{f^*}(\lambda/2\beta) \int_0^{\beta \lambda} \frac{\phi_1(t)}{t^2} dt d\lambda
\]

Proof. Let $t \in \mathbb{R}_+$. Then, given $\Phi_2(t) = \frac{\Phi_2(B\lambda)}{\lambda}$ and $m_2f^*(\lambda) \leq A \int_{\mathbb{R}_+} m_2f^*(\lambda) \frac{\Phi_2(B\lambda)}{\lambda} d\lambda$

$$\leq A \int_{\mathbb{R}_+} \Phi_2(2\beta B f^*(t)) dt. \quad \square$$

5. Interpolation pairs of Orlicz spaces for the class $W((p_0, r_0), (p_1, r_1); \mu, \nu)$

5.1. A Calderón-type theorem.

**Theorem 5.1.** Let $(X, \mu)$ and $(Y, \nu)$ be $\sigma$-finite measure spaces. Fix the indices $p_0, p_1, r_0$ and $r_1$, $1 < p_0 < p_1 < \infty$ and $1 \leq r_1, r_2 < \infty$. Then, given $T \in W((p_0, r_0), (p_1, r_1); \mu, \nu)$, one has

$$(Tf)^*(t) \leq K[(H_{p_0, r_0} + H_{p_1, r_1})f^*(ct)],$$

in which $K = K(T) > 0$ and $c = c(T) > 0$ are independent of $f \in (L_{p_0, r_0} + L_{p_1, r_1})(X, \mu)$ and of $t \in \mathbb{R}_+$.

**Proof.** Let $f \in (L_{p_0, r_0} + L_{p_1, r_1})(X, \mu)$ and fix $t \in \mathbb{R}_+$. At $x \in X$, set

$$f_1(x) = \min[|f(x)|, f^*(t)] \text{sgn} f(x)$$

and

$$f_0(x) = f(x) - f_1(x) = [|f(x)| - f^*(t)]^+ \text{sgn} f(x).$$

Then, $f = f_0 + f_1$ and for all $s \in \mathbb{R}_+$

$$f_0^*(s) = [f^*(s) - f^*(t)]^+,$$

$$f_1^*(s) = \min(f^*(s), f^*(t)).$$

Moreover, as shown in Lemma 2.3, $f_0 \in L_{p_0, r_0}(X, \mu)$ and $f_1 \in L_{p_1, r_1}(X, \mu)$. So, if $T$ has $r$-quasilinearity constants $C$ and $c$ (see (1.12), p. 3),

$$(Tf)^*(t) \leq C[(Tf_0)^*(ct) + (Tf_1)^*(ct)] \leq C \left[ (ct)^{-\frac{1}{p_0}} M_{p_0, r_0} \|f_0\|_{L_{p_0, r_0}(X, \mu)} + (ct)^{-\frac{1}{p_1}} M_{p_1, r_1} \|f_1\|_{L_{p_1, r_1}(X, \mu)} \right],$$

in which $M_{p_i, r_i}$ is the norm of $T$ as a mapping from $L_{p_i, r_i}(X, \mu)$ to $L_{p_i, \infty}(Y, \nu)$, $i = 0, 1$. 


Now,
\[
\| f_0 \|_{L_{p_0,r_0}(X,\mu)} = \| f_0^* \|_{L_{p_0,r_0}(\mathbb{R}^+, m)}
= \left( \frac{r_0}{p_0} \right)^\frac{1}{r_0} \left( \int_0^t (f^*(s) - f^*(t))^{r_0} \frac{r_0}{p_0} s^{\frac{r_0}{p_0} - 1} ds \right)^{\frac{1}{r_0}}
\leq \left( \frac{r_0}{p_0} \right)^\frac{1}{r_0} \left( \int_0^t f^*(s)^{r_0} \frac{r_0}{p_0} s^{\frac{r_0}{p_0} - 1} ds \right)^{\frac{1}{r_0}}
\leq \left( \frac{r_0}{p_0} \right)^\frac{1}{r_0} c^{-\frac{1}{r_0}} \left( \int_0^t f^*(s)^{r_0} \frac{r_0}{p_0} s^{\frac{r_0}{p_0} - 1} ds \right)^{\frac{1}{r_0}}
\leq \left( \frac{r_0}{p_0} \right)^\frac{1}{r_0} \left( f^*(t)^{r_1} \frac{r_0}{p_1} s^{\frac{r_0}{p_1} - 1} ds + \int_t^\infty f^*(s)^{r_1} \frac{r_0}{p_1} s^{\frac{r_0}{p_1} - 1} ds \right)^{\frac{1}{r_1}}
\leq \left( \frac{r_0}{p_0} \right)^\frac{1}{r_0} \left( 1 - c \frac{r_1}{p_1} \right)^{-\frac{1}{r_1}} \left( \int_t^\infty f^*(s)^{r_1} \frac{r_0}{p_1} s^{\frac{r_0}{p_1} - 1} ds \right)^{\frac{1}{r_1}}.
\]
and
\[
\| f_1 \|_{L_{p_1,r_1}(X,\mu)} = \| f_1^* \|_{L_{p_1,r_1}(\mathbb{R}^+, m)}
= \left( \frac{p_1}{p_0} \right)^\frac{1}{r_1} \left( \int_0^t f^*(t)^{r_1} \frac{r_0}{p_0} s^{\frac{r_0}{p_0} - 1} ds + \int_t^\infty f^*(s)^{r_1} \frac{r_0}{p_0} s^{\frac{r_0}{p_0} - 1} ds \right)^{\frac{1}{r_1}}
\leq \left( \frac{p_1}{p_0} \right)^\frac{1}{r_1} \left( \frac{r_0}{p_0} f^*(t)^{r_1} \frac{r_0}{p_1} s^{\frac{r_0}{p_1} - 1} ds + \int_t^\infty f^*(s)^{r_1} \frac{r_0}{p_1} s^{\frac{r_0}{p_1} - 1} ds \right)^{\frac{1}{r_1}}
\leq \left( \frac{p_1}{p_0} \right)^\frac{1}{r_1} \left( \int_t^\infty f^*(s)^{r_1} \frac{r_0}{p_1} s^{\frac{r_0}{p_1} - 1} ds \right)^{\frac{1}{r_1}}.
\]
Altogether, then,
\[
(Tf)^*(t) \leq K \left[ \left( H_{p_0,r_0} + H_{p_1,r_1} \right) f^*(ct) \right],
\]
with
\[
K = \left( \frac{r_0}{p_0} \right)^\frac{1}{r_0} c^{-\frac{1}{r_0}} + \left( \frac{p_1}{p_0} \right)^\frac{1}{r_1} \left( 1 - c \frac{r_1}{p_1} \right)^{-\frac{1}{r_1}} \left( M_{p_0,r_0} + M_{p_1,r_1} \right) C.
\]

**Theorem 5.2.** Fix the indices \( p_0, p_1, r_0 \) and \( r_1, 1 < p_0 < p_1 < \infty, 1 \leq r_0, r_1 < \infty \). Let \( (X, \mu) \) and \( (Y, \nu) \) be \( \sigma \)-finite measure spaces, with \( \mu(X) = \nu(Y) = \infty \) and \( (Y, \nu) \) being nonatomic and separable. Assume \( \Phi_i(t) = \int_0^t \phi_i(s) ds, i = 1, 2, \) are Young functions. Then, the following are equivalent:

1. Every operator \( T \in W((p_0,r_0),(p_1,r_1); \mu, \nu) \) maps \( L_{\Phi_2}(X, \mu) \) boundedly into \( L_{\Phi_1}(Y, \nu) \);
2. The operator \( H_{p_0,r_0} + H_{p_1,r_1} \) maps the nonincreasing functions in \( L_{\Phi_2}(\mathbb{R}^+, m) \) boundedly into \( L_{\Phi_1}(\mathbb{R}^+, m) \).

**Proof.** Suppose (2) holds. Then, given \( T \in W((p_0,r_0),(p_1,r_1); \mu, \nu) \) and \( f \in L_{\Phi_2}(X, \mu) \), one has, by Theorem 5.1,
\[
\| Tf \|_{L_{\Phi_1}(Y,\nu)} = \|(Tf)^*(t)\|_{L_{\Phi_1}(\mathbb{R}^+, m)}
\leq K \left[ \left( H_{p_0,r_0} + H_{p_1,r_1} \right) f^*(ct) \right] \| f \|_{L_{\Phi_2}(\mathbb{R}^+, m)}
\leq K h(c) \left[ \left( H_{p_0,r_0} + H_{p_1,r_1} \right) f^*(t) \right] \| f \|_{L_{\Phi_2}(\mathbb{R}^+, m)}
\leq K h(c) \| f \|_{L_{\Phi_2}(\mathbb{R}^+, m)}
= K h(c) \| f \|_{L_{\Phi_2}(X, \mu)},
\]
namely, (1) holds.

The argument that (1) implies (2) is by now a familiar one. First, one readily proves that
\[
H_{p_0,r_0} + H_{p_1,r_1} \in W((p_0,r_0),(p_1,r_1); m, m).
\]
Thus, second, the operator \((H^{p_0,r_0} + H_{p_1,r_1})\), constructed in Theorem 2.1, is in \(W((p_0, r_0),(p_1, r_1);\mu, \nu)\) and so maps \(L_{\Phi_2}(X, \mu)\) boundedly into \(L_{\Phi_1}(Y, \nu)\). Third, taking in Theorem 2.1, \(X = \mathbb{R}_+, \mu = m, (Y, \nu)\) to be \((X, \mu)\) and, as \(S\), the operator \(f \rightarrow f^{*m}\), one gets, for \(f \in M_+(\mathbb{R}_+, m)\), a function \(\tilde{f} \in M_+(X, \mu)\) such that for all \(t \in \mathbb{R}_+\)

\[
\tilde{f}^{*m}(t) = f^{*m}(t).
\]

Therefore, since

\[
\left(\frac{1}{(H^{p_0,r_0} + H_{p_1,r_1})f}\right)^{*m}(t) = \left(\frac{1}{(H^{p_0,r_0} + H_{p_1,r_1})f^{*m}}\right)(t) = \left(\frac{1}{(H^{p_0,r_0} + H_{p_1,r_1})f^{*m}}\right)(t),
\]

\(f \in M_+(\mathbb{R}_+, m), t \in \mathbb{R}_+,\) we get, for \(g \in M_+(\mathbb{R}_+, m), g\) nonincreasing,

\[
\| (H^{p_0,r_0} + H_{p_1,r_1})g \|_{L_{\Phi_1}([\mathbb{R}_+, \infty)} = \| (H^{p_0,r_0} + H_{p_1,r_1})\tilde{g}^{*m} \|_{L_{\Phi_1}([\mathbb{R}_+, \infty)} \\
= \| (H^{p_0,r_0} + H_{p_1,r_1})\tilde{g}^{*m} \|_{L_{\Phi_1}([\mathbb{R}_+, \infty)} \\
= \| (H^{p_0,r_0} + H_{p_1,r_1})\tilde{g}^{*m} \|_{L_{\Phi_1}([\mathbb{R}_+, \infty)} \\
\leq C\| \tilde{g}^{*m} \|_{L_{\Phi_1}(\mathbb{R}_+, \infty)} \\
= C\| g^{*m} \|_{L_{\Phi_1}([\mathbb{R}_+, \infty)} \\
\leq C\| g \|_{L_{\Phi_2}([\mathbb{R}_+, \infty)} \\
given (1).
\]

5.2. **Proof of the main Theorem.** We are now able to verify the main result of this thesis, namely, to give the

**Proof of theorem A.** Theorem 5.2 ensures that (1) amounts to the assertion that

\[
\| (H^{p_0,r_0} + H_{p_1,r_1})f^* \|_{L_{\Phi_1}([\mathbb{R}_+, \infty)} \leq C\| f^* \|_{L_{\Phi_2}([\mathbb{R}_+, \infty)}
\]

with \(C > 0\) independent of \(f \in L_{\Phi_2}([\mathbb{R}_+, \infty)}\).

Since \(H^{p_0,r_0} + H_{p_1,r_1}\) commutes with dilations, Theorem A in [KRS17] guarantees this assertion equivalent to the inequality

\[
\int_{\mathbb{R}_+} \Phi_1 \left(\left(\frac{1}{H^{p_0,r_0} + H_{p_1,r_1}}f^*(t)\right) \right) dt \leq \int_{\mathbb{R}_+} \Phi_1 \left(\left(\frac{1}{Kf^*(s)}\right) \right) ds,
\]

in which \(K > 0\) is independent of \(f \in M_+(\mathbb{R}_+, m)\). Indeed, the methods of Theorem 5.2 shows (5.3) equivalent to (2).

Finally, it follows from Theorems 3.9, 3.10 and 4.9, 4.10 that (5.3) is equivalent to (3).

\[
6. \textbf{On the monotonicity in } r \text{ of the condition for } H^{p,r} : L_{\Phi_2} \rightarrow L_{\Phi_1}
\]

We will show in this section that the necessary and sufficient condition for

\[
H^{p,r} : L_{\Phi_2}(\mathbb{R}_+, m) \rightarrow L_{\Phi_1}(\mathbb{R}_+, m), \quad 1 \leq r < p,
\]

namely, with \(p_1 = \frac{p}{r}\)

\[
\left(\int_{t}^{\infty} \frac{\phi_2(Ds)}{\Phi_2(Ds)^{p_1}} \left(\frac{1}{s^{p'}} \int_{0}^{t} \frac{\phi_1(s)}{s^p} ds \right) \frac{1}{p_1} \right) \leq B, \quad \text{for all } t \in \mathbb{R}_+,
\]

decreases strictly in strength as \(r\) increases in \([1,p)\).
The same can be shown about the condition for

\[ H_{q,r} : L_{\Phi_2}(\mathbb{R}_+, m) \rightarrow L_{\Phi_1}(\mathbb{R}_+, m). \]

Now, the inequality

\[ (H^{p,r_2} f^*) (t) \leq C (H^{p,r_1} f^*) (t), \quad \text{for all } t \in \mathbb{R}_+, \ 1 \leq r_1 < r_2 < \infty, \]

which follows from (2.6), implies the condition (6.2) must decrease in strength. That the decrease is strict in \([1,p]\) is demonstrated by

**Example 6.1.** Let \(1 < p < \infty\) and \(1 \leq r_1 < r_2 < p\). Let us denote \(p_1 = p/r_1\) and \(p_2 = p/r_2\). Fix indices \(\alpha_1, \alpha_2\) and \(\beta\) with \(0 < 1 + \alpha_1 \leq p_1 - p_2, p_2 < 1 + \alpha_2 \leq p_1\) and \(\beta > p\). Consider the Young functions defined by

(6.3) \(\Phi_1(t) = \begin{cases} t^\beta, & t < e, \\ p^\beta (\log t)^{\alpha_1}, & t > e \end{cases}\)

and

(6.4) \(\Phi_2(t) = \begin{cases} t^\beta, & t < e, \\ p^\beta (\log t)^{\alpha_2}, & t > e. \end{cases}\)

Here, we will see that for the pair of the Young functions defined in (6.3) and (6.4), the condition (6.2) holds with \(r = r_2\) but not with \(r = r_1\).

Observe that the condition (6.2) can be rewritten as

(6.5) \[ \left( \int_{D_i} \left( \frac{y}{\Phi_2(y^{\frac{1}{r_i}})} \right)^{p_i'-1} dy \right)^{\frac{1}{p_i'}} \left( \int_0^x \frac{\Phi_1(y^{\frac{1}{r_i}})}{y^{p_i+1}} dy \right)^{\frac{1}{p_i}} \leq B_i', \]

for some \(B_i' > 0, i = 1, 2\) and all \(x \geq 0\). We can ignore \(D_i\) in (6.5) for our purpose in this proposition.

First we will see that for these Young functions, the condition (6.5) with \(i = 1\) (that is for \(r_1\)) does not hold. Let us consider the second integral in the right hand side of (6.5). For large \(x\) we have,

\[ I_{r_1}(x) := \int_0^x \frac{\Phi_1(y^{\frac{1}{r_1}})}{y^{p_1+1}} dy \]

\[ = \int_0^{e^{r_1}} y^{\beta/r_1 - p_1 - 1} dy + \int_{e^{r_1}}^x \frac{y^{p/r_1} (\log(y^{\frac{1}{r_1}}))^{\alpha_1}}{y^{p_1+1}} dy \]

\[ = \frac{e^{\beta - p}}{\beta / r_1 - p_1} + \frac{1}{r_1^{\alpha_1}} \int_{e^{r_1}}^x (\log y)^{\alpha_1} dy \]

\[ = \frac{e^{\beta - p}}{\beta / r_1 - p_1} + \frac{1}{r_1^{\alpha_1}} \int_{r_1}^{\log x} \frac{1}{y^{-\alpha_1}} dy \]

\[ \approx (\log x)^{1 + \alpha_1}. \]
For the first integral in the left hand side of (6.5), we have

\[ J_{r_1}(x) := \int_{x}^{\infty} \left( \frac{y}{\Phi_2(y^{r_1})} \right)^{p_1' - 1} dy \]

\[ = \int_{x}^{\infty} \left( \frac{y^{p/r_1}(\log(y^{1/r_1}))^{\alpha_2}}{(\log y)^{\alpha_2}(p_1' - 1)} \right)^{p_1' - 1} dy \]

\[ = r_1^{\alpha_2(p_1' - 1)} \int_{x}^{\infty} \frac{1}{(\log y)^{\alpha_2}(p_1' - 1)} dy \]

\[ \approx \int_{\log x}^{\infty} \frac{1}{y^{\alpha_2}(p_1' - 1)} dy \to \infty, \]

as \( \alpha_2(p_1' - 1) \leq 1 \). So from (6.6) and (6.7) we have that the condition (6.5) for \( r_1 \) does not hold.

It remains to show that the condition (6.5) holds for \( r_2 \). Which follows from the following expressions for the two integrals appearing in (6.5) and the assumption \( 1 - \left( \frac{\alpha_2 - \alpha_1}{p_2} \right) \leq 0 \).

\[ I_{r_2}(x) := \int_{0}^{x} \frac{\Phi_1(y^{r_2})}{y^{p_2 + 1}} dy \]

\[ = \begin{cases} 
  x^{\beta/r_2 - p_2}, & x < e^{r_2}, \\
  e^{\beta - p} + \frac{1}{r_2} \frac{(\log x)^{1+\alpha_2 - r_2} - 1}{1+\alpha_2}, & x > e^{r_2}
\end{cases} \]

and as in (6.7)

\[ J_{r_2}(x) := \int_{x}^{\infty} \left( \frac{y}{\Phi_2(y^{r_2})} \right)^{p_2' - 1} dy \]

\[ = \begin{cases} 
  \int_{e^{r_2}}^{x} \left( \frac{y^{p/r_2}}{\Phi_2(y^{r_2})} \right)^{p_2' - 1} dy + \int_{e^{r_2}}^{\infty} \left( \frac{y^{p/r_2}(\log(y^{1/r_2}))^{\alpha_2}}{(\log y)^{\alpha_2}(p_2' - 1)} \right)^{p_2' - 1} dy, & x < e^{r_2}, \\
  \int_{x}^{\infty} \frac{1}{(\log y)^{\alpha_2}(p_2' - 1)} dy, & x > e^{r_2}
\end{cases} \]

\[ = \begin{cases} 
  \frac{1}{p_2^{(\beta/p - 1)}} \left[ x^{-p_2'(\beta/p - 1)} - e^{r_2}p_2'(\beta/p - 1) \right] + A, & x < e^{r_2}, \\
  \frac{r_2^{\alpha_2(p_2' - 1)}}{1+\alpha_2 - p_2} \left( \frac{1}{\log x} \right)^{p_2'(1+\alpha_2 - p_2)} p_2', & x > e^{r_2},
\end{cases} \]

where \( A = r_2^{\alpha_2(p_2' - 1)} \int_{e^{r_2}}^{\infty} \frac{1}{(\log y)^{\alpha_2}(p_2' - 1)} dy = \frac{p_2 r_2}{p_2'(1+\alpha_2 - p_2)} \to \infty \) as \( \alpha_2(p_2' - 1) > 1 \).

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