SU(N) BPS Monopoles in $\mathcal{M}^2 \times S^2$

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April 3, 2015

Abstract

We extend the investigation of BPS saturated t'Hooft-Polyakov monopoles in $\mathcal{M}^2 \times S^2$ to the general case of $SU(N)$ gauge symmetry. This geometry causes the resulting $N-1$ coupled non-linear ordinary differential equations for the $N-1$ monopole profiles to become autonomous. One can also define a flat limit in which the curvature of the background metric is arbitrarily small but the simplifications brought in by the geometry remain. We prove analytically that non-trivial solutions in which the profiles are not proportional can be found. Moreover, we construct numerical solutions for $N=2, 3$ and $4$. The presence of the parameter $N$ allows one to take a smooth large $N$ limit which greatly simplifies the treatment of the infinite number of profile function equations. We show that, in this limit, the system of infinitely many coupled ordinary differential equations for the monopole profiles reduces to a single two-dimensional non-linear partial differential equation.

1 Introduction

The subject of 't Hooft-Polyakov monopoles [1] [2] has been extensively studied in the literature, (see for reviews [3] [4] [5]). The fundamental role these objects are believed to play in modern theoretical physics ranges between all branches of the subject (for a comprehensive review see, for example, [6]). These solitonic objects have become predominant characters of most modern-day non-Abelian gauge theories, including supersymmetry [7] and string theory (see for example [8]) despite having never been seen in experiments. One aspect of their description which is less studied is their intimate link with the topology of the underlying space. This paper is an extension of our previous work [9] which investigates solitonic t'Hooft-Polyakov monopoles in a cylindrical topology with spherical cross-sections (for related work see also [10] [11]). In our previous investigation we showed that for $SU(2)$ monopoles living in this topology all field profile equations become autonomous (that is, they don’t depend on the radial variable $r$ explicitly) which allows one to treat them by analytical tools normally not available. In this work we extend this set-up to the general case of $SU(N)$ gauge symmetry. The main mathematical tool used to achieve this is the formalism of harmonic maps which is reviewed in section 2. We will show that, as usually happens, by introducing an arbitrary dependence on the parameter $N$, one can find find a simplifying limit in which $N$ becomes large. The simplifications
brought about by this limit are especially treatable in the harmonic map formalism and constitute the main reason for choosing this over other methods.

The paper is organized as follows: section 2 is devoted to introducing the system with its Lagrangian and the aforementioned topology. This section also includes a review of the harmonic map formalism used to describe $SU(N)$ monopoles. In section 3 we discuss in detail the consequences of the large $N$ limit and in section 4 we find solutions numerically for the cases $N = 2, 3, 4$. Finally we provide some conclusions in section 5. In the appendix we provide a rigorous mathematical proof that the resulting equations for field profiles always allow non-trivial solutions (given a suitable bound on the shape of the cylinder).

2 The System

We begin this section by reviewing the general algorithm to obtain spherically symmetric $SU(N)$ monopole solutions using rational maps of the Riemann sphere into flag manifolds. We refer the reader to [12], on which this review is based and from which we borrow our notation, for the relevant mathematical details.

The action of the $SU(N)$ Yang-Mills-Higgs system in four dimensional space-time is

$$S_{\text{SYM}} = \int d^4x \sqrt{-g} \, \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \Phi D^\mu \Phi - \frac{\lambda^2}{8} (\Phi \Phi - v^2)^2 \right),$$

(1)

where the Planck constant, the speed of light and the gauge coupling constant have been set to 1. The remaining dimensionless coupling constant is $\lambda$. In this equation, $\Phi$ is an $su(N)$ valued scalar field, the covariant derivative is $D_i = \partial_i + [A_i,]$ and $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. The $SU(N)$ BPS monopoles are finite energy solutions of the BPS equation

$$D_i \Phi = -\frac{1}{2} \epsilon_{ijk} F^{jk},$$

(2)

obtained by an energy minimization argument from eq.(1) when $\lambda = 0$. In polar coordinates with $z = e^{i\varphi} \tan(\theta/2)$ the flat space-time metric reduces to

$$ds^2 = -dt^2 + dr^2 + \frac{4r^2}{(1 + |z|^2)^2} dz d\bar{z},$$

(3)

and one can use the ansatz

$$\Phi = -i A_r = -\frac{i}{2} H^{-1} \partial_r H, \quad A_z = H^{-1} \partial_z H, \quad A_{\bar{z}} = 0,$$

(4)

for $H \in \text{SL}(N, \mathbb{C})$, to reduce the matrix system of BPS equations to

$$\partial_r (H^{-1} \partial_r H) + \frac{(1 + |z|^2)^2}{r^2} \partial_{\bar{z}} (H^{-1} \partial_z H) = 0.$$

(5)
The ansatz used in eq.(4) (which is the most natural generalization of the ’t Hooft-Polyakov hedgehog ansatz for $SU(2)$) describes three-dimensional topological defects with non-Abelian magnetic charge given by

$$Q_M = \int_{S_R} d^2 S_i B_i^a \Phi^a,$$  

(6)

where $S_R$ denotes a sphere centered around the monopole and $B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a$ is the non-abelian magnetic field. More generally, the dimensionality of a topological defect (see, for a detailed explanation, [4] and [13]) is determined by the homotopy class of the corresponding ansatz. In general, for systems living in a spacetime $\mathcal{M}$ and with a gauge symmetry $G$ broken down to a subgroup $H$ one can label the topological charges of solitonic solutions by the degree of maps of $S^n \in \mathcal{M}$ into the coset space $G/H$,

$$\pi_n (G/H),$$  

(7)

where $n$ varies according to $\mathcal{M}$ and which pattern of symmetry breaking is considered\footnote{For instance, in flat space, vortex-like objects have a non-trivial first homotopy class $\pi_1(U(1))$, monopole-like objects (which are point-like in space) have a non-trivial second homotopy class $\pi_2(SU(2)/U(1))$ while instanton-like solutions (which are point-like in space-time) have a non-trivial third homotopy class $\pi_3(SU(N))$.}. For the case of $SU(N)$ monopoles considered here, we will be interested in the homotopy class

$$\pi_2 (SU(N)/J),$$  

(8)

where $J$ can vary between $U(N-1)$ and $U(1)^{N-1}$ corresponding to how the gauge symmetry is broken. The case $U(N-1)$ corresponds to minimal symmetry breaking whilst $U(1)^{N-1}$ is the case of maximum symmetry breaking and these depend on the vacuum expectation value of $\Phi$. These solitonic objects are stable unrelated solutions of different energy minimization requirements, however one of the interesting outcomes of the present analysis is that different topological objects can actually be difficult to tell apart (at least by looking at the equations of motion) when such topological objects are analyzed within space-like regions with non-trivial topology. In particular, we will show that the field equations for non-Abelian BPS monopoles (possessing non-trivial second homotopy class as per eq.(8)) within the bounded tube-shaped region defined in the next section (see [9]) are related to the field equations of domain-wall objects in flat topology and with a similar homotopy class eq.(7) (see [14]) through a simple field redefinition.

In order to find spherically symmetric monopole solutions to equation [5] one can follow a simple algorithm: first, one needs the spherically symmetric maps into $\mathbb{CP}^{N-1}$ (see [12]) which are given by

$$f = (f_0, ..., f_j, ..., f_{N-1})^t, \quad f_j = z^j \sqrt{\binom{N - 1}{j}},$$  

(9)

where the expression in the square root denotes the standard binomial coefficient. Then using the $\Delta$
operator defined as
\[ \Delta f = \partial_z f - \frac{f (f^\dagger \partial_z f)}{|f|^2}, \] (10)
and applying it iteratively \( \Delta^k f = \Delta (\Delta^{k-1} f) \), \( k = 0, ..., N - 1 \), one can construct a projector matrix
\[ P_k = P (\Delta^k f), \quad P f = \frac{f f^\dagger}{|f|^2}, \] (11)
satisfying \( P^\dagger = P = P^2 \) which is used to parametrize the general \( N \times N \ SL(N, \mathbb{C}) \) Hermitian matrix \( H \) appearing in eq.(5) by
\[ H = \exp \left( \sum_{i=0}^{N-2} g_i (P_i - 1/N) \right), \] (12)
where \( g_i \) are general profile functions which depend only on \( r \) and the \( 1/N \) factor denotes the identity matrix divided by \( N \). Substituting eq.(12) into eq.(5) gives a general matrix of equations which can be decoupled for each \( \ddot{g}_i \) (here \( \dot{\cdot} \) denotes differentiation w.r.t \( r \)). More generally, following [16] one can find a convenient form of the resulting equations in terms of \( N \) and \( l = 0, ..., N - 2 \), the index labelling the profile function \( g_l \),
\[ -\frac{2(l+1)}{N} \sum_{i=0}^{N-2} (i+1)\ddot{F}_i + 2 \sum_{k=0}^{l} \sum_{i=k}^{N-2} \ddot{F}_i - \frac{2}{r^2} (l+1)(N-l-1)(\exp(F_l)-1) = 0, \] (13)
where \( F_l = g_l - g_{l+1} \) and \( F_{N-2} = g_{N-2} \). Decoupling these equations gives \( N - 1 \) equations for the profile functions, we include below the examples for \( SU(N) \) with \( N = 2, 3 \), which are respectively
\[ \ddot{g}_0 + \frac{2}{r^2} (1 - e^{g_0}) = 0, \] (14)
and
\[ -\ddot{g}_0 + \frac{2}{r^2} (e^{g_0-g_1} - 1) + \frac{2}{r^2} (e^{g_1} - 1) = 0, \] (15)
\[ -\ddot{g}_1 + \frac{2}{r^2} (e^{g_0-g_1} - 1) + \frac{4}{r^2} (e^{g_1} - 1) = 0. \] (16)

As shown in [12], these equations have analytic solutions corresponding to magnetic monopoles. In the simplest case of \( N = 2 \), the solution is
\[ g_0 = 2 \log (2r/\sinh 2r), \] (17)
which describes a single BPS saturated monopole, with energy equal to its charge \( Q_M \). In the case where \( N > 2 \), the equations still allow for at least one analytic solution in which the profile functions are chosen proportional to each other. Indeed if, for example, we choose \( N = 3 \) then there exists the solution with \( g_0 = 2g_1 \). In general, one has a solution for all field profiles proportional when
\[ \frac{g_l}{N - (i+1)} = g \] (18)
with \( g \) given by eq.(17) and \( i = 0, \ldots, N - 2 \).

### 2.1 \( SU(N) \) monopoles on \( M^2 \times S^2 \)

In this section we discuss a simple geometrical modification of the above set up and its consequences. We wish to consider the above system in \( M^2 \times S^2 \) (or \( M^1 \times S^1 \times S^2 \)) with metric

\[
ds^2 = -dt^2 + dr^2 + R_0^2(d\theta^2 + \sin^2 \theta \ d\phi^2), \quad 0 \leq r \leq L,
\]

\[
0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,
\]

where \( L \) is a longitudinal length and \( R_0 \) is a constant with the dimension of length related to the size of the transverse sections of this topology. In complexified coordinates, those appropriate to this paper, the metric reads

\[
ds^2 = -dt^2 + dr^2 + \frac{4R_0^2}{(1 + |z|^2)^2}dzd\bar{z}.
\]

This metric describes a tubular geometry with spherical caps as cross sectional slices. The non-vanishing components of the Riemann tensor \( R_{\mu\nu\rho\sigma} \) of this space are proportional to \( 1/R_0^2 \):

\[
R_{\mu\nu\rho\sigma} \sim \frac{1}{R_0^2}.
\]

This simple modification of the geometry leads to a dramatic simplification in the resulting equations for the profile functions. As shown in [9], for the case of \( SU(2) \), the resulting energy minimization equations for the monopole profile functions are autonomous, that is they don’t involve any explicit powers of \( r \). Below we find that this also happens for the \( SU(N) \) case. As shown later in the paper, one can easily define a flat limit in which the curvature of the background metric is as small as one wants keeping, at the same time, all the simplifications brought in by the background metric in eq. (21) (two different ways to achieve the flat limit will be described in the following sections). Moreover, the present formalism introduces the possibility to define a large \( N \) limit. We will consider this in detail in a further section.

Equation (5) when analyzed in the background metric given by eq.(21) is modified to

\[
\partial_r \left( H^{-1}\partial_r H \right) + \frac{(1 + |z|^2)^2}{R_0^2} \partial_z \left( H^{-1}\partial_z H \right) = 0,
\]

and equation (13) becomes,

\[
-\frac{2(l + 1)}{N} \sum_{i=0}^{N-2} (i + 1) \ddot{F}_i + 2 \sum_{k=0}^{l} \sum_{i=k}^{N-2} \ddot{F}_i - \frac{2(l + 1)}{R_0^2} (N - l - 1)(\exp(F_l) - 1) = 0,
\]

where, as in the previous section, \( F_l = g_l - g_{l+1} \). The equations now become autonomous. In line with the discussion around eq.(18) we find that for \( N > 2 \) there exist analytical solutions in which all the profile functions are proportional. However, in the following we will show both numerically and
analytically that non-trivial solutions in which the profiles are not equal (and hence do not correspond to trivial embeddings of $SU(2)$ into $SU(N)$) also exist.

Let us take a moment here to connect the previous results (and the more standard monopole notation) to the current notation. For the simplest case of $SU(2)$ the equation reduces to

$$\ddot{g}_0 + \frac{2}{R_0^2} (1 - e^{2g_0}) = 0. \quad (25)$$

It was shown in [9] that for $N = 2$ using standard polar coordinates $(r, \theta, \phi)$ and an ansatz of the form

$$A_\mu = (k(r) - 1) U^{-1} \partial_\mu U, \quad \Psi = \psi(r) U, \quad (26)$$

$$U = \hat{n}^i t_i, \quad U^{-1} = U, \quad (27)$$

$$\hat{n}^1 = \sin \theta \cos \phi, \quad \hat{n}^2 = \sin \theta \sin \phi, \quad \hat{n}^3 = \cos \theta, \quad (28)$$

where the $t^i$ are the standard Pauli matrices, the BPS equations reduced to

$$\partial_r \psi + \frac{1 - k^2}{R_0^2} = 0, \quad (29)$$

$$\partial_r k - k \psi = 0, \quad (30)$$

for which a general solution was proposed of the form

$$k = \exp(u), \quad (31)$$

$$\psi = \partial_r u, \quad (32)$$

where the function $u(r)$ is the inverse of the following integral

$$\int_{u(0)}^{u(r)} \left[ 2 \left( I_0 + \frac{\exp(2z) - z}{R_0^2} \right) \right]^{-1/2} dz = \pm r, \quad (33)$$

with $I_0$ an integration constant. This is consistent with the above construction, as one indeed expects, upon the identification $u = g_0/2$ where eqs.(29) and (30) reduce to eq.(25).

The autonomous set of equations (24), of which eq.(25) is both the $N = 2$ and the degenerate equation for proportional fields representative, has some interesting properties which we discuss throughout the paper. One of which is that it coincides with the equation for a domain wall separating the Higgs and Coulomb phases in the Abelian - Higgs model\textsuperscript{2} In [14], the equation for the scalar field describing the domain wall was found to be

$$(\log \phi)'^2 = e^2 (\phi^2 - v^2), \quad (34)$$

\textsuperscript{2}We thank S. Bolognesi for pointing this out to us.
where $e$ is the $U(1)$ gauge coupling and $v^2$ is the constant appearing in the quartic potential (similar to that appearing in eq.\((1)\)). The above equation is a particular case of the well known Taubes equation \([15]\). Upon identifying $\phi = v \exp(g_0/2)$ and $v^2 e^2 = 1/R_0^2$ we see that eq.\((34)\) becomes eq.\((25)\). What is intriguing about this observation is that topological objects which are quite different (possessing different non-trivial homotopy classes) can be described by identical solutions if they are constrained to live in space-time regions with non-trivial topology. In fact, since there is always a solution of the $N$ equations in which the field profiles are proportional as in eq.\((18)\), this relation can be trivially extended to monopoles of topological charge $N$.

3 The large $N$ limit

The large $N$ limit introduced in Yang-Mills theory in \([19]\) (see also \([20]\) and \([21]\); for two detailed reviews see \([22]\)) is a very powerful tool to analyze non-perturbative features in gauge theories (such as confinement, bound states and so on). The non-trivial scaling behavior of physical quantities within the large $N$ expansion arises from the fact that Feynman diagrams have different weights depending on the topology of the surfaces they can be drawn on (once $N$ is considered as a large number). Thus, this non-trivial behavior is purely “quantum” in nature and one would expect that no non-trivial large $N$ behavior should be found when analyzing classical BPS equations (as we are doing in the present paper). In fact, the present formalism shows that a non-trivial scaling with $N$ emerges already at the level of the BPS field equations eq.\((24)\). Naively taking $N$ very large means increasing the number of non-linear coupled equations, or equivalently the number of monopole profiles, which seems like an un-neccesary complication.

Fortunately, there is another dramatic simplification in this limit which makes this problem treatable. The key issue is that in this limit, through an appropriate ansatz of the interpolating discrete function, one can replace the discrete label $l = 0, ..., N-2$ in eq.\((24)\) by a continuous label. The main reason is that in the limit in which $N \to \infty$, where the range of values of $l$ becomes infinite, each discrete jump in its value (which is just 1 of course) becomes infinitesimal compared to the range. In other words, were we to rescale each step $\Delta l = 1$ of the complete range by $N$, this would become infinitesimal in this limit.

Let us consider this simplification in detail. First we take eq.\((24)\) and replace the discrete labels $F_i$ by a discrete function $F(i)$ (we omit the dependence on the radial variable for simplicity). This is simply a renaming which is useful to understand the subsequent manipulations. Clearly, since $i$ is integer, this is a function on $\mathbb{Z}$. Then we multiply eq.\((24)\) by $1/N^2$ and define the quantity

$$x = l/N.$$  \hspace{1cm} (35)

Then, in the limit in which $N \to \infty$, the discrete label $l$ can be replaced by the continuous variable $x \in \mathbb{R}_{[0,1]}$. Consequently we can replace the discrete function $F(l)$ living in $\mathbb{Z}$ by the continuous
function \( F(x) \) living in \( \mathbb{R}_{[0,1]} \). The equation then becomes

\[
-2 \left( x + \frac{1}{N} \right) \sum_{i=0}^{N-2} \frac{1}{N} F(i) + 2 \sum_{k=0}^{l} \sum_{i=k}^{N-2} \frac{1}{N} \tilde{F}(i) - \frac{2}{R_0^2} \left( x + \frac{1}{N} \right) \left( 1 - (x + \frac{1}{N}) \right) \left( \exp(F(Nx) - 1) \right) = 0.
\]

(36)

Now let us manipulate the sums, define

\[
i/N = y, \quad z = k/N, \quad p = i/N,
\]

then simple manipulations lead to

\[
-2 \left( x + \frac{1}{N} \right) \sum_{y=0}^{1-1/N} (y + 1/N) \frac{1}{N} \tilde{F}(Ny) + 2 \sum_{z=0}^{x} \sum_{p=z}^{1-2/N} \frac{1}{N} \tilde{F}(Np)
\]

\[
- \frac{2}{R_0^2} \left( x + \frac{1}{N} \right) \left( 1 - (x + \frac{1}{N}) \right) \left( \exp(F(Nx) - 1) \right) = 0.
\]

(38)

Finally, taking the large \( N \to \infty \) limit, in which we can safely replace the sums by integrals

\[
\sum_{y=0}^{1-1/N} \frac{1}{N} [G(y)] \rightarrow \int_0^1 dy \ G(y),
\]

since, from eq.(35), \( 1/N \) plays the role of “\( dx \)” in the mathematical definition of Riemann and Lebesgue integrals, expression eq.(38) greatly simplifies and becomes

\[
-2x \int_0^1 y \left[ \frac{\partial^2}{\partial r^2} G(y,r) \right] dy + 2 \int_0^x \int_0^1 \left[ \frac{\partial^2}{\partial r^2} G(y,r) \right] dy dz - \frac{2x}{R_0^2} (1 - x) \left( \exp(G(x,r)) - 1 \right) = 0,
\]

(39)

where we defined \( G(x,r) = F(Nx,r) \): \( G(x,r) \) will be denoted as the complete profile function since it encodes information about all the elementary profiles \( F_i(r) \) at the same time.

From here throughout the rest of the paper (unless specified) we switch to dimensionless units

\[
\rho = vr, \quad \tilde{R}_0 = vR_0, \quad \tilde{E} = E/v,
\]

(40)

where \( v \) is the parameter appearing in the potential with dimensions of mass and \( E \) is the energy of the solution. Then eq.(39) becomes

\[
-2x \int_0^1 y \left[ \frac{\partial^2}{\partial \rho^2} G(y,\rho) \right] dy + 2 \int_0^x \int_0^1 \left[ \frac{\partial^2}{\partial \rho^2} G(y,\rho) \right] dy dz - \frac{2x}{R_0^2} (1 - x) \left( \exp(G(x,\rho)) - 1 \right) = 0.
\]

(41)

Taking two derivatives with respect to \( x \) gives single non-linear elliptic partial differential equation for
the complete profile function \( G(x, \rho) \) which reads

\[
\frac{\partial^2}{\partial \rho^2} G + \frac{\partial^2}{\partial x^2} \left[ \frac{x(1-x)}{\tilde{R}_0^2} \left( \exp(G(x, \rho)) - 1 \right) \right] = 0 \tag{42}
\]

Solving this equation with the boundary conditions equal to the profile functions in the radial direction is expected to yield a function which interpolates smoothly between the monopole profiles. This is a non-trivial numerical task which we defer to a later publication. The main important advantage of eq. (42) over the original system in eq.(24) is that instead of having, in the large \( N \) limit, \textit{infinitely many coupled equations for infinitely many unknowns} one has \textit{just one non-linear partial differential equation for the complete profile function} \( G(x, \rho) \). Eq. (42) is well suited for numerical analysis and one could also easily apply the techniques described in the previous section to prove the existence of solutions with suitable properties.

It is worth emphasizing here that the only assumption used to derive eq. (39) starting from the original system in eq.(24) is that there exists a smooth interpolating function \( F(x) \) in which the group theoretical label \( \frac{l}{N} \) becomes a continuous variable \( x \). A necessary condition is the existence of a large \( N \) limit of the field theory. Although the proof of this result is not available yet, there is a huge amount of evidence in the literature (see [22] and references therein) that it is actually possible to define a smooth large \( N \) limit in many different areas (from gauge theories to matrix model and so on). Whether the profile functions \( F_l(r) \) as functions of the discrete label \( l \) follow a pattern regular enough to be interpolated by a continuous (smooth) function depends on the \( g_i \) solutions at large \( N \). Therefore, as used in this paper, this assumption translates to an ansatz on the solutions we consider in the large \( N \) limit which, in detail, involves the function \( F_l(r) \) to be doubly differentiable. This ansatz leads directly to eq.(42). Remarkably the above argument can also be directly extended to the case in which one starts off with a flat topology and considers standard \( SU(N) \) BPS monopoles in flat space.

We will now discuss two different ways to consider the flat limit in the background geometry in eq.(21) in which \( \tilde{R}_0^2 \to \infty \).

### 3.1 The flat limit as a perturbative limit

Another important advantage of eq.(42) over the original system eq.(24) is that, in the large \( N \) limit, it discloses the role of the adimensional curvature parameter \( \tilde{R}_0^{-2} \) as coupling constant of the master equation given by eq.(42) in such a way that the flat limit corresponds to a “weak field” limit.

Indeed, from eq.(42) it is clear that in the large \( \tilde{R}_0^2 \) limit (in which the curvature vanishes) the equation for \( G(x, \rho) \) becomes just a linear partial differential equation while the limit in which \( \tilde{R}_0^2 \) is small (so that the curvature is large) the non-linear effects become strong as well. Thus, this formalism provides one with a clear perturbative scheme in which the equation for the complete profile function \( G(x, \rho) \) becomes linear in the flat limit in such a way that the curvature parameter \( 1/\tilde{R}_0^2 \) plays the role of a coupling constant.
In fact, if one would put the monopoles on a flat geometry from the very beginning the corresponding system would be eq.(13) and such a system does not admit any obvious perturbative scheme in the large $N$ limit although one would arrive at an equation similar to eq.(42). Indeed, following similar steps to the previous section, if one would start from eq.(13) assuming in the large $N$ limit a continuous dependence on the group label $l$, then one would arrive at the following large $N$ limit for the complete profile function $G_F(x, \rho)$:

$$\frac{\partial^2}{\partial \rho^2} G_F + \frac{\partial^2}{\partial x^2} \left[ x(1-x) \left( \frac{\exp(G_F(x, \rho)) - 1}{\rho^2} \right) \right] = 0,$$

(43)

where the label $F$ has been added to emphasize that $G_F(x, \rho)$ is the complete profile function of the system of monopoles described by eq.(13) which live, from the very beginning, on a flat metric. The difference between eq.(42) and eq.(43) is then apparent: in the former equation $\tilde{R}_0^{-2}$ plays the role of coupling constant allowing a perturbative analysis of the equation while in the latter equation one cannot do this since $\tilde{R}_0^{-2}$ has been replaced by $\rho^{-2}$ where $\rho$ is one of the independent variables of the equation.

### 3.2 The flat limit as a geometrical bound

An alternative way to define a flat limit for the background metric corresponds to rescaling the original longitudinal variable in eq.(24) such that

$$\tilde{\rho} = r/R_0.$$  

(44)

Since the length of the tube-shaped region in which the monopoles are living is $L$, the above rescaling is equivalent to considering a tube of adimensional length

$$\tilde{L} = \frac{L}{R_0}.$$  

(45)

Then, one can take the flat limit taking $R_0$ and $L$ simultaneously large in such a way that $\tilde{L}$ stays finite:

$$R_0 \to \infty, \quad L \to \infty \quad | \quad \tilde{L} = \text{const} \neq 0, \infty.$$  

(46)

In this limit, the master equation eq.(42) for the complete profile function simply becomes

$$\frac{\partial^2}{\partial \rho^2} G + \frac{\partial^2}{\partial x^2} \left[ x(1-x) \left( \frac{\exp(G(x, \rho)) - 1}{\rho^2} \right) \right] = 0.$$  

(47)

As shown in the appendix it is important to note however that, in order to be sure that when considering the flat limit in eqs.(44), (45) and (46) non-trivial solutions always exist, one should take the limit in such a way that the inequalities$^3$ in eqs.(85) and (86) are never violated. This fact can be interpreted as a sort of bound on the shape of the cylinder which cannot be too “slim” since $\tilde{L} \gg 1$

$^3$In the generic $SU(N)$ case, the application of the Schauder theorem would give very similar inequalities.
would violate eqs. (85) and (86).

4 Numerical Solutions

In this section we provide numerical solutions for the profile functions in the specific cases of $N = 2, 3$ and $4$. For every value of $\tilde{R}_0$ there exist a unique solution for the profile functions with an integer value of $Q_M$. The numerical strategy is therefore the following: we provide boundary conditions for the profile functions and vary $\tilde{R}_0$ until a solution with the desired topological charge is found. This guarantees that the solution obtained is a solution (for a given $\tilde{R}_0$) which represents a BPS monopole with topological charge $Q_M$. The finite length cylindrical topology is implemented numerically by imposing the boundary conditions at a finite cutoff. The numerical procedure is a second-order finite difference procedure with accuracy $O(10^{-4})$.

4.1 $N = 2$

Let us proceed to solve equation (25) numerically. We fix $vL = 50$. We fix the boundary conditions on the profile function to be 

$$g_0(0) = 1, \quad g_0(50) = 0,$$

and we find that at $\tilde{R}_0 = 1.45$ the energy reads

$$\tilde{E} = \frac{Q_M}{4\pi} = \left[ (e^{g_0} - 1) \frac{g_0'}{2} \right]_{\rho=50}^{\rho=0} = 1.000,$$

which is the expected one-monopole solution. This solution is shown in figure 1.

For this value of $\tilde{R}_0$ one can use the topology of the system to find the solution corresponding to placing the monopole at the other end of the cylinder. This solution is found by imposing the
conditions

\[ g_0(0) = 0, \quad g_0(50) = 1, \]  

and is shown in figure 2. For this solution, which is just an inversion of the previous solution about the center of the cylinder, the energy is the same, as expected. Note that this is not an anti-monopole solution as the topological charge is the same.

At this point a comment is in order regarding the SU(2) solutions of [9]. Below eq.(33) we pointed out that there is a direct map between the profile function \( g_0 \) and the solutions obtained without using the harmonic map formalism. Namely we showed that using \( u = g_0/2 \) one can map the two first order BPS equations into the profile equation for \( g_0 \). However, the solution for \( g_0 \) presented in figure 1, when translated to the standard Higgs and gauge fields \( \psi = g_0'/2 \) and \( k = \exp(g_0/2) \) does not reproduce the solution found in [9] even though both solutions have the same topological charge (the reader may argue that the values of \( \tilde{R}_0 \) used for both plots are not the same, but this comment applies also when these values are made equal). This is however expected, in order for both solutions to match, since \( k(0) = 0 \), one would have to provide a singular boundary condition for the profile function \( g_0(0) = -\infty \), which is numerically impossible. The non-uniqueness of the solution with given charge is made possible because in this topology one does not require that the profile function be regular at the radial origin, i.e. \( g_0' = 0 \) precisely because (unlike what happens in flat metric in spherical coordinates) there is no preferred origin.
4.2 $N = 3$

In this section we wish to solve

$$- g_0'' + \frac{2}{R_0^2} \left( e^{g_0} - g_1 - 1 \right) + \frac{2}{R_0^2} \left( e^{g_1} - 1 \right) = 0$$  \hfill (51)

$$- g_1'' - \frac{2}{R_0^2} \left( e^{g_0} - g_1 - 1 \right) + \frac{4}{R_0^2} \left( e^{g_1} - 1 \right) = 0.$$  \hfill (52)

where $'$ denote differentiation w.r.t $\rho$. In Figure 2 we show the solution corresponding to the case where the profile functions are proportional. This is easily obtained by demanding boundary conditions of the form

$$g_0(0) = 1, \quad g_1(0) = 0.5$$  \hfill (53)

$$g_0(50) = g_1(50) = 0.$$  \hfill (54)

The general energy equation reads

$$\tilde{E} = \frac{QM}{4\pi} = e^{-g_1} \left[ (e^{g_0} - e^{g_1}) g_0' - (e^{g_0} - e^{2g_1}) g_1' \right] \bigg|_{\rho=50}^{\rho=0}$$  \hfill (55)

For the solution in Figure 3, by construction we find numerically

$$\tilde{E} = 1.0006.$$  \hfill (56)

We can however also look for solutions which are not proportional. These are more interesting solutions from the numerical point of view as they are full solutions of the coupled equations rather than the reduction of all of these to one equation for a single profile function.

Using the topology of the system we may also find solutions which have non-vanishing boundary conditions at both ends of the cylinder and therefore not proportional. In this case we seek solutions with boundary conditions of the form

$$g_0(0) = 1, \quad g_1'(0) = 0,$$  \hfill (57)

$$g_0'(50) = 0, \quad g_1(50) = 1,$$  \hfill (58)

these are shown in Figure 4 for a very specific choice of $R_0$. The energy for this solution, calculated numerically using eq.(55) is

$$\tilde{E} = 2.001,$$  \hfill (59)

and hence it is tempting to identify this solution as a monopole-anti-monopole state, with each species of monopole living at each end of the cylinder. However once again the boundary conditions lead to monopole charges which are not consistent with this picture.
Figure 3: Proportional monopole profile functions for $SU(3)$. The plots correspond to the numerical value $\tilde{R}_0 = 1$. For these plots $g_0 = 2g_1$. $g_0$ is the solid line whilst $g_1$ is the dashed line.

Figure 4: A solution with non-vanishing boundary conditions at both ends of the cylinder at $\tilde{R}_0 = 4.28$. Dashed line is $g_1$, solid line is $g_0$. 
Figure 5: An SU(4) two monopole solution at $\tilde{R}_0 = 5.9$. Thin dashed line is $g_2$, medium dashed line is $g_1$ and solid line is $g_0$.

4.3 $N = 4$

In this case we wish to solve the system of equations

$$-g''_0 + \frac{3}{R_0^2} (e^{g_0-g_1} - 1) + \frac{3}{R_0^2} (e^{g_2} - 1) = 0,$$

$$-g''_1 - \frac{3}{R_0^2} (e^{g_0-g_1} - 1) + \frac{4}{R_0^2} (e^{g_1-g_2} - 1) + \frac{3}{R_0^2} (e^{g_2} - 1) = 0,$$

$$-g''_2 - \frac{4}{R_0^2} (e^{g_1-g_2} - 1) + \frac{6}{R_0^2} (e^{g_2} - 1) = 0,$$

and the energy evaluates to

$$\tilde{E} = \frac{QM}{4\pi} = e^{-g_1-g_2} \left[ 3e^{g_0+g_2} (g'_0 - g'_1) + 4e^{2g_1} (g'_1 - g'_2) + e^{g_1+g_2} (-3g'_0 - g'_1 + (1 + 3e^{g_2}) g'_2) \right] \bigg|_{\rho=50}^{\rho=0}. \tag{63}$$

We look for the analogous two-monopole solution of the previous section. Therefore we impose the boundary conditions

$$g_0(0) = 1, \quad g'_1(0) = 0, \quad g'_2(0) = 0, \tag{64}$$

$$g'_0(50) = 0, \quad g_1(50) = 1, \quad g'_2(50) = 0. \tag{65}$$

The corresponding solution shown in Figure 5 is found to have $\tilde{E} = 2.000$ at $\tilde{R}_0 = 5.9$. If one calculates the topological charge at each end of the cylinder one once again does not find equal contributions.

In each case, it appears that the profile functions concentrate around the end-points of the cylindrical topology and want to vanish in the intermediate region.
5 Conclusions

In the present paper, using the harmonic map formalism, we have extended the investigation of BPS saturated t’Hooft-Polyakov monopoles in $\mathcal{M}^2 \times S^2$ to the general case of $SU(N)$ gauge symmetry. We found that, as per the $N = 2$ case investigated previously in [9], all equations for the monopole profile functions become autonomous. For some specific cases we solved these equations numerically and in all cases we demonstrated analytically the existence of non-trivial solutions in which the field profiles are not proportional. Furthermore, we investigated the remarkable power of the large $N$ limit in this scenario, where we have shown that one can, under a suitable condition of smoothness, reduce the infinite set of profile equations to a single partial differential equation for an interpolating function. This equation encodes the subtle role played by the curvature parameter $\tilde{R}_0$ and elucidates the flat $\tilde{R}_0 \to \infty$ limit both from a perturbative and a geometrical aspect. We leave the numerical treatment of this equation to further work.

Acknowledgements

This work has been funded by the Fondecyt grants 1120352 and 3140122. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

A Existence of solutions

In this appendix we will provide a rigorous mathematical proof that solutions to the set of equations obtained from eq.(24) exist even in the case that the fields are not proportional. This procedure does not yield the solutions themselves, these are shown numerically in a previous section of the main text, but is nonetheless an important part of the analysis of the field profile equations and the space of their solutions. In particular, one of the inequalities derived in this section will be useful in the discussion of the flat limit.

The basic mathematical tool used here is the Schauder theorem (see for a detailed pedagogical review [17]). For simplicity, we will focus on the $SU(3)$ case but the same argument can be easily extended to the general case.

The statement of the Schauder theorem ([17] [18]) is the following: let $S$ be a complete metric (Banach) space so that a distance $d(X,Y)$ between any pair of elements of the space is defined by

$$d(X,Y) \in \mathbb{R} , \quad X,Y \in S ,$$

and such that, with respect to the chosen metric, from every Cauchy sequence one can extract a convergent subsequence (this is the inclusion of “complete” in the definition of the metric space). Let
Let $C$ be a bounded closed convex set in $S$ and let $T$ be a compact operator from the Banach space $S$ into itself such that $T$ maps $C$ into itself:

$$T[.] : C \to C.$$  \hspace{1cm} (67)

Then the map $T[.]$ has (at least) one fixed point in $C$. In other words, under the above hypothesis, there always exist a solution to the equation

$$T[X] = X.$$ \hspace{1cm} (68a)

Our task is to determine under what precise conditions this applies to the system of equations obtained by decoupling the field profiles in eq. (24) for the case of $N = 3$. To do so, let us rewrite the system in eqs. (15) and (16) (with the factor $r$ replaced by $R_0$ everywhere) as coupled integral equations:

$$g_0 (x) = a_0 + b_0 x + \int_0^x \int_0^s \left[ \frac{2}{R_0^2} \{ \exp [g_0 (\rho) - g_1 (\rho)] - 1 \} + \frac{2}{R_0^2} [\exp (g_1 (\rho)) - 1] \right] \, dpds , \quad (69)$$

$$g_1 (x) = a_1 + b_1 x + \int_0^x \int_0^s \left[ \frac{4}{R_0^2} [\exp (g_1 (\rho)) - 1] - \frac{2}{R_0^2} \{ \exp [g_0 (\rho) - g_1 (\rho)] - 1 \} \right] \, dpds , \quad (70)$$

where $a_i$ and $b_i$ represent the initial data for the two profiles $g_0$ and $g_1$ and their derivatives at $x = 0$. It is a trivial computation to show that the above system of integral equations is equivalent to the system in eq. (24) with $N = 3$. The system of eqs. (69) and (70) can be written as a fixed point condition for the following vectorial operator $\vec{T}$ acting on pairs of continuous functions $\vec{g}(x) = (g_0 (x), g_1 (x))$:

$$\vec{T} : C^0 [0, L] \times C^0 [0, L] \to C^0 [0, L] \times C^0 [0, L] ,$$

$$\vec{g}(x) = (g_0 (x), g_1 (x)) \in C^0 [0, L] \times C^0 [0, L] ,$$

$$\vec{T} [g_0, g_1] = \vec{T} [\vec{g}(x)] = (T_0 (x), T_1 (x)) , \quad (71)$$

where

$$T_0 (x) = a_0 + b_0 x + \int_0^x \int_0^s \left[ \frac{2}{R_0^2} \{ \exp [g_0 (\rho) - g_1 (\rho)] - 1 \} + \frac{2}{R_0^2} [\exp (g_1 (\rho)) - 1] \right] \, dpds , \quad (72)$$

$$T_1 (x) = a_1 + b_1 x + \int_0^x \int_0^s \left[ \frac{4}{R_0^2} [\exp (g_1 (\rho)) - 1] - \frac{2}{R_0^2} \{ \exp [g_0 (\rho) - g_1 (\rho)] - 1 \} \right] \, dpds . \quad (73)$$

The fixed-point condition is then simply

$$\vec{g}(x) = \vec{T} [\vec{g}(x)] , \quad (74)$$

\footnote{An operator $T$ from a Banach space into itself (see, for a detailed discussion, [17] [18]) is called \textit{compact} if and only if, for any bounded sequence $\{ x_n \}$, the sequence $\{ T(x_n) \}$ has a convergent subsequence.}
where the operator $\overrightarrow{T}$ has been defined in eqs. (71), (72) and (73). Whilst this proves that a fixed point operator condition can be found, in order to apply Schauder’s theorem we have to show that $\overrightarrow{T}$ is a compact operator from a bounded closed convex sub-set of a Banach space into itself. This is not a trivial task, let us begin by defining the following metric in the space $C^0 [0, L] \times C^0 [0, L]$ (which is the Cartesian product of the space of the continuous function on $[0, L]$ with itself):

$$d \left( \overrightarrow{g}, \overrightarrow{h} \right) = \sup_{x \in [0, L]} |g_0(x) - h_0(x)| + \sup_{x \in [0, L]} |g_1(x) - h_1(x)|, \quad (75)$$

$$\overrightarrow{g} = (g_0(x), g_1(x)), \quad \overrightarrow{h} = (h_0(x), h_1(x)).$$

With respect to this norm, the space $C^0 [0, L] \times C^0 [0, L]$ is a Banach space (which we call $S$).

Then we define a bounded closed convex sub-set $C$ of the Banach space defined above (using the metric in eq. (75)) such that $\overrightarrow{T}$ maps $C$ into itself,

$$C \equiv \{ \overrightarrow{g} (x) = (g_0(x), g_1(x)) \in S | \forall x \in [0, L] \ |g_0(x) - a_0| \leq B, \ |g_1(x) - a_1| \leq B \}, \quad B \in \mathbb{R}_+, \quad (76)$$

where $a_i$ are the initial data appearing in eqs. (69) and (70) so that $C$ is closed by definition. It is easy to see that $C$ is bounded since

$$B \geq |g_0(x) - a_0| \geq |g_0(x)| - |a_0| \Rightarrow |g_0(x)| \leq B + |a_0|, \quad (77)$$

$$B \geq |g_1(x) - a_1| \geq |g_1(x)| - |a_1| \Rightarrow |g_1(x)| \leq B + |a_1|. \quad (78)$$

In order to prove that $C$ is convex we have to check that if $\overrightarrow{g} (x)$ and $\overrightarrow{h} (x)$ both belong to $C$ then $\theta \overrightarrow{g} (x) + (1 - \theta) \overrightarrow{h} (x)$ also belongs to $C \forall \theta \in [0, 1]$. This is easily verified as

$$|\theta g_1 (x) + (1 - \theta) h_1 (x) - a_1| \leq |\theta (g_1(x) - a_1)| + |(1 - \theta) (h_1(x) - a_1)| \leq \theta B + (1 - \theta) B \leq B, \quad (79)$$

$$|\theta g_0 (x) + (1 - \theta) h_0 (x) - a_0| \leq |\theta (g_0(x) - a_0)| + |(1 - \theta) (h_0(x) - a_0)| \leq \theta B + (1 - \theta) B \leq B. \quad (80)$$

Now we can proceed to show that $T$ is compact.

First of all, we must show that if $\overrightarrow{g}_n (x) = (g_{0n}(x), g_{1n}(x))$ is a sequence in $C$ then the sequence $\overrightarrow{T} [\overrightarrow{g}_n (x)] = (T_0 (n; x), T_1 (n; x))$ is uniformly bounded in $C$ (namely, the absolute values of both components of $\overrightarrow{T} [\overrightarrow{g}_n (x)]$ are bounded by a constant which does not depend on $n$ hence ensuring that $\overrightarrow{T} [\overrightarrow{g}_n (x)]$ belong to $C \forall n$ as well). Therefore we consider

$$|T_0 (n; x)| = \left| a_0 + b_0 x + \int_0^x \int_0^s \frac{2}{R_0^2} \{\exp [g_{0n}(\rho) - g_{1n}(\rho)] - 1\} + \frac{2}{R_0^2} \{\exp (g_{1n}(\rho)) - 1\} \right| ds x \leq$$
In order to derive eqs. (81) and (82) we used that, because of eqs. (77) and (78) (which are equivalent to saying that the \( \tilde{g}_n^\pm (x) \) belong to \( C \) \( \forall n \)), one has

\[
\left| \frac{2}{R_0^2} \{ \exp [g_0^n (\rho) - g_1^n (\rho)] - 1 \} + \frac{2}{R_0^2} \{ \exp (g_1^n (\rho)) - 1 \} \right| dpds \Rightarrow 
\]

\[
|T_0 (n; x)| \leq |a_0| + |b_0 L| + \frac{2L^2}{R_0^2} \left[ \{ \exp (2B + |a_0| + |a_1|) - 1 \} + \exp (B + |a_1|) - 1 \right] \ , 
\]

and, similarly,

\[
|T_1 (n; x)| \leq |a_1| + |b_1 L| + \frac{2L^2}{R_0^2} \left[ 2 \{ \exp (B + |a_1|) - 1 \} + \exp (2B + |a_0| + |a_1|) - 1 \right] 
\]

Eqs. (81) and (82) show that, if \( \tilde{g}_n^\pm (x) = (g_0^n (x), g_1^n (x)) \) is a sequence in \( C \), the sequence \( \tilde{T} \left[ \tilde{g}_n^\pm (x) \right] \) is uniformly bounded. Moreover, one has to require that the sequence of images \( \tilde{T} \left[ \tilde{g}_n^\pm (x) \right] \) belongs to \( C \) as well. As always happens (see \[17\] and \[18\]) this will give some constraints on the range on the parameters \( B, L \) and \( R_0 \). In order for the sequence of images to belong to \( C \) the following inequalities must be satisfied (as can be easily seen by comparing eqs. (76), (77) and (78) with eqs. (81) and (82)):

\[
|a_0| + |b_0 L| + \frac{2L^2}{R_0^2} \left[ \{ \exp (2B + |a_0| + |a_1|) - 1 \} + \exp (B + |a_1|) - 1 \right] \leq B 
\]

\[
|b_1 L| + \frac{2L^2}{R_0^2} \left[ 2 \{ \exp (B + |a_1|) - 1 \} + \exp (2B + |a_0| + |a_1|) - 1 \right] \leq B 
\]

These imply that in order for this theorem to work, the length \( L \) of the cylindrically-shaped region in which these non-Abelian BPS monopoles are living cannot exceed the bounds defined in eqs. (85) and (86). One cannot obtain a very large value for the allowed \( L \) by simply increasing \( B \) since the left hand sides of eqs. (85) and (86) increase faster than the right hand sides, but the situation improves if \( \tilde{R}_0^2 \) is very large, namely in the flat limit (in which case the exponentials containing \( B \) are suppressed).

What is important however is that it is always possible the choose \( B, L \) and \( \tilde{R}_0 \) in such a way that eqs. (85) and (86) are fulfilled.

The next step to prove that \( T \) is compact is to show that if \( \tilde{g}_n^\pm (x) = (g_0^n (x), g_1^n (x)) \) is a sequence
in $C$ then the sequence $\tilde{T} [ \tilde{g}_n (x) ]$ is equicontinuous.

To show this, we must evaluate, for a generic $n$, the absolute values of following differences:

$$|T_0 (n; x) - T_0 (n; y)| = b_0 (x - y) + \int_x^y \int_0^s \left[ \frac{2}{R_0} \{ \exp [g_0^n (\rho) - g_1^n (\rho)] - 1 \} + \frac{2}{R_0} \{ \exp (g_0^n (\rho)) - 1 \} \right] d\rho ds,$$

$$|T_1 (n; x) - T_1 (n; y)| = b_1 (x - y) + \int_x^y \int_0^s \left[ \frac{4}{R_0} \{ \exp (g_1^n (\rho)) - 1 \} - \frac{2}{R_0} \{ \exp [g_0^n (\rho) - g_1^n (\rho)] - 1 \} \right] d\rho ds,$$

where $0 < x < y < L$. After some trivial manipulations (which use the fact that all the functions $\tilde{g}_n (x)$ belong to $C$ and consequently eqs. [77] and [78] are satisfied) one arrives at

$$|T_0 (n; x) - T_0 (n; y)| \leq |x - y| \left[ b_0 + \frac{2L}{R_0} \{ \exp (2B + |a_0| + |a_1|) - 1 \} + \exp (B + |a_1|) - 1 \right],$$

$$|T_1 (n; x) - T_1 (n; y)| \leq |x - y| \left[ b_1 + \frac{2L}{R_0} [2 \{ \exp (B + |a_1|) - 1 \} + \exp (2B + |a_0| + |a_1|) - 1] \right],$$

Thus, given any $\epsilon > 0$, we can choose

$$\delta < \frac{\epsilon}{2 \left\{ b_0 + |b_1| + \frac{2L}{R_0} [2 \{ \exp (B + |a_1|) - 1 \} + \exp (2B + |a_0| + |a_1|) - 1] \right\}},$$

in such a way that both the choice of $\delta$ in eq. 91 does not depend on $n$ and,

$$|x - y| < \delta \Rightarrow |T_i (n; x) - T_i (n; y)| \leq \frac{\epsilon}{2}, \forall n, \forall i = 0, 1.$$

In summary, eqs. 81, 82, 85 and 86 show that, if $\tilde{g}_n (x) = (g_0^n (x), g_1^n (x))$ is any sequence in $C$, then the sequence $\tilde{T} [ \tilde{g}_n (x) ]$ is uniformly bounded in $C$. Subsequently, eqs. 89, 90, 91 and 92 show that, if $\tilde{T} [ \tilde{g}_n (x) ]$ is equicontinuous. Consequently, using the Ascoli-Arzela theorem (see 17), from any sequence $\tilde{T} [ \tilde{g}_n (x) ]$ one can extract a convergent subsequence: this implies that the operator $\tilde{T}$ is a compact operator from a bounded closed convex set into itself.

Finally, the Schauder theorem ensures that eq. 74 (which is equivalent to our original system) has at least one solution. Moreover, it is always possible to choose appropriately the initial data $a_i$ and $b_i$ in such a way that the two profiles are not proportional. This concludes the proof on existence of solutions of the monopole profile equations. Moreover, one can also show by a similar procedure that the solutions are actually not just continuous but they also have continuous first and second

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5 A sequence of functions $\{f_n\}$ is said to be equicontinuous if, given $\epsilon > 0$, $\exists \delta > 0$ such that $|f_n (x) - f_n (y)| < \epsilon$ whenever $|x - y| < \delta$ and, moreover, $\delta$ does not depend on $n$ (otherwise the sequence would be continuous but not equicontinuous: see 17).
derivatives.\footnote{There are many standard ways to prove this result (see [17] and [18]). However, the easiest way to argue that this is indeed the case is by observing that one can take the double derivative of the fixed-point formula directly since the profiles are continuous and the right hand side of the fixed point condition is a double integral of a continuous bounded function.}

The present rigorous argument can be easily extended to the $SU(N)$ case with $N > 3$. Besides the intrinsic mathematical elegance of the fixed-point Schauder-type argument, the present procedure also discloses the presence of the bounds in eqs. (85) and (86) on the length $L$ of the tube-shaped region in which these non-Abelian BPS monopoles are living. At the present stage of the analysis, it is not possible yet to say whether such a bound is just a limitation of the method or it signals some deeper physical limitation on the volume of the regions in which one constrains these non-Abelian BPS monopoles to live. Understanding whether or not such BPS monopoles can fit into very large cylindrically-shaped regions is certainly a very interesting question on which we hope to come back in a future investigation.

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