On the critical difference of almost bipartite graphs

Vadim E. Levit\(^1\) · Eugen Mandrescu\(^2\)

Received: 5 June 2019 / Accepted: 4 August 2020 / Published online: 4 September 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract

A set \(S \subseteq V\) is independent in a graph \(G = (V, E)\) if no two vertices from \(S\) are adjacent. The independence number \(\alpha(G)\) is the cardinality of a maximum independent set, while \(\mu(G)\) is the size of a maximum matching in \(G\). If \(\alpha(G) + \mu(G)\) equals the order of \(G\), then \(G\) is called a König–Egerváry graph (Deming in Discrete Math 27:23–33, 1979; Sterboul in J Combin Theory Ser B 27:228–229, 1979). The number \(d(G) = \max\{|A| − |N(A)| : A \subseteq V\}\) is called the critical difference of \(G\) (Zhang in SIAM J Discrete Math 3:431–438, 1990) (where \(N(A) = \{v : v \in V, N(v) \cap A \neq \emptyset\}\)). It is known that \(\alpha(G) − \mu(G) \leq d(G)\) holds for every graph (Levit and Mandrescu in SIAM J Discrete Math 26:399–403, 2012; Lorentzen in Notes on covering of arcs by nodes in an undirected graph, Technical report ORC 66-16. University of California, Berkeley, CA, Operations Research Center, 1966; Schrijver in Combinatorial optimization. Springer, Berlin, 2003). In Levit and Mandrescu (Graphs Combin 28:243–250, 2012), it was shown that \(d(G) = \alpha(G) − \mu(G)\) is true for every König–Egerváry graph. A graph \(G\) is (i) unicyclic if it has a unique cycle and (ii) almost bipartite if it has only one odd cycle. It was conjectured in Levit and Mandrescu (in: Abstracts of the SIAM conference on discrete mathematics, Halifax, Canada, p 40, abstract MS21, 2012, 3rd international conference on discrete mathematics, June 10–14, Karnataka University. Dharwad, India, 2013) and validated in Bhattacharya et al. (Discrete Math 341:1561–1572, 2018) that \(d(G) = \alpha(G) − \mu(G)\) holds for every unicyclic non-König–Egerváry graph \(G\). In this paper, we prove that if \(G\) is an almost bipartite graph of order \(n(G)\), then \(\alpha(G) + \mu(G) \in \{n(G) − 1, n(G)\}\). Moreover, for each of these two values, we characterize the corresponding graphs. Further, using these findings, we show that the critical difference of an almost bipartite graph \(G\) satisfies

\[
d(G) = \alpha(G) − \mu(G) = |\text{core}(G)| − |\text{N(core}(G))|,
\]

where by \(\text{core}(G)\) we mean the intersection of all maximum independent sets.

\(^1\) Vadim E. Levit
levitv@ariel.ac.il

Extended author information available on the last page of the article
Keywords  Independent set · Core · Matching · Critical set · Critical difference · Bipartite graph · König–Egerváry graph

1 Introduction

Throughout this paper, $G = (V, E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V = V(G)$ of cardinality $n(G)$ and edge set $E = E(G)$ of size $m(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$, we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the subgraph of $G$ obtained by deleting the edges of $F$, and we use $G - e$, if $F = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then $(A, B)$ stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : v \in V \text{ and } vw \in E\}$, and $N(A) = \bigcup\{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subset V$. By $C_n$, $K_n$, we mean the chordless cycle on $n \geq 3$ vertices, and, respectively, the complete graph on $n \geq 1$ vertices. In order to avoid ambiguity, we use also $N_G(v)$ instead of $N(v)$, and $N_G(A)$ instead of $N(A)$.

Let us define the trace of a family $\mathcal{F}$ of sets on the set $X$ as $\mathcal{F}|_X = \{F \cap X : F \in \mathcal{F}\}$. A set $S$ of vertices is independent if no two vertices from $S$ are adjacent, and an independent set of maximum size will be referred to as a maximum independent set. The independence number of $G$, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of $G$.

Let $\Omega(G) = \{S : S \text{ is a maximum independent set of } G\}$, core$(G) = \bigcap\{S : S \in \Omega(G)\}$ [11], and corona$(G) = \bigcup\{S : S \in \Omega(G)\}$ [4]. An edge $e \in E(G)$ is $\alpha$-critical whenever $\alpha(G - e) > \alpha(G)$. Notice that $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ holds for each edge $e$.

The number $d_G(X) = |X| - |N(X)|$, $X \subseteq V(G)$ is the difference of the set $X$, $d(G) = \max\{d_G(X) : X \subseteq V\}$ is the critical difference of $G$, and a set $U \subseteq V(G)$ is critical if $d_G(U) = d(G)$ [23]. The number $id(G) = \max\{d_G(I) : I \in \text{Ind}(G)\}$ is the critical independence difference of $G$. If $A \subseteq V(G)$ is independent and $d_G(A) = id(G)$, then $A$ is called a critical independent set [23]. Clearly, $d(G) \geq id(G)$ is true for every graph $G$.

**Theorem 1.1** [23] The equality $d(G) = id(G)$ holds for every graph $G$.

A matching (i.e., a set of non-incident edges of $G$) of maximum cardinality $\mu(G)$ is a maximum matching, and a perfect matching is the one covering all vertices of $G$. An edge $e \in E(G)$ is $\mu$-critical provided $\mu(G - e) < \mu(G)$.

**Theorem 1.2** For any graph $G$, the following assertions are true:

(i) [13] no $\alpha$-critical edge has an endpoint in $N[\text{core}(G)]$;
(ii) [4] there is a matching from $S - \text{core}(G)$ into $\text{corona}(G) - S$, for each $S \in \Omega(G)$;
(iii) [11] if $G$ is a connected bipartite graph with $n(G) \geq 2$, then $\alpha(G) > \frac{n(G)}{2}$ if and only if $|\text{core}(G)| \geq 2$.

It is well known that $\left\lceil \frac{n(G)}{2} \right\rceil + 1 \leq \alpha(G) + \mu(G) \leq n(G)$ hold for every graph $G$. If $\alpha(G) + \mu(G) = n(G)$, then $G$ is called a König–Egerváry graph [6,21]. Various
properties of König–Egerváry graphs are presented in [2,3,12,13,15]. It is known that every bipartite graph is a König–Egerváry graph [8,9]. This class includes also non-bipartite graphs (see, for instance, the graph $G$ in Fig. 1).

Theorem 1.3 If $G$ is a König–Egerváry graph, then

(i) [12] every maximum matching matches $N(\text{core}(G))$ into $\text{core}(G)$;

(ii) [15] $d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G)$.

A cycle is a trail, where the only repeated vertices are the first and last ones. The graph $G$ is unicyclic if it has a unique cycle. We call a graph $G$ almost bipartite if it has a unique odd cycle, denoted $C = (V(C), E(C))$. Since $C$ is unique, it is chordless, and there is no other cycle of $G$ sharing edges with $C$. For every $y \in V(C)$, let us define $D_y = (V_y, E_y)$ as the connected bipartite subgraph of $G - E(C)$ containing $y$, and

$$N_1(C) = \{v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset\}.$$  

Clearly, every unicyclic graph with an odd cycle is almost bipartite.

The smallest number of edges that have to be deleted from a graph to obtain a bipartite graph is called the bipartite edge frustration of $G$ and denoted by $\varphi(G)$ [7,22]. If $G$ is an almost bipartite graph, then $\varphi(G) = 1$. The diamond graph tells us that the opposite is not true.

In this paper, we analyze the relationship between several parameters of an almost bipartite graph $G$, namely $\text{core}(G)$, $d(G)$, $\alpha(G)$ and $\mu(G)$.

2 Results

Lemma 2.1 If $G$ is an almost bipartite graph, then there is an edge $e \in E(C)$, such that $\mu(G - e) = \mu(G)$.

Proof For every pair of edges, consecutive on $C$, only one of them may belong to every maximum matching of $G$. In other words, at most one of the edges could be $\mu$-critical. □

Notice that $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ holds for each edge $e$. Every edge of the unique odd cycle could be $\alpha$-critical, e.g., the graph $G$ from Fig. 2.

Lemma 2.2 [17] For every bipartite graph $H$, a vertex $v \in \text{core}(H)$ if and only if there exists a maximum matching that does not saturate $v$.

Lemma 2.2 fails for non-bipartite König–Egerváry graphs, e.g., every maximum matching of the graph $G$ from Fig. 1 saturates $c \in \text{core}(G) = \{a, b, c\}$. 
Lemma 2.3 If $G$ is an almost bipartite graph, then $n(G) - 1 \leq \alpha(G) + \mu(G) \leq n(G)$.

Proof If $e = xy \in E(C)$, then $G - e$ is bipartite, and hence, $\alpha(G - e) + \mu(G - e) = n(G)$. Clearly, $\alpha(G - e) \leq \alpha(G) + 1$, while $\mu(G - e) \leq \mu(G)$. Consequently, we get that

$$n(G) = \alpha(G - e) + \mu(G - e) \leq \alpha(G) + \mu(G) + 1,$$

which leads to $n(G) - 1 \leq \alpha(G) + \mu(G)$. The inequality $\alpha(G) + \mu(G) \leq n(G)$ is true for every graph $G$. \hfill \square

Lemma 2.4 Let $G$ be an almost bipartite graph. Then, $n(G) - 1 = \alpha(G) + \mu(G)$ if and only if each edge of its unique odd cycle is $\alpha$-critical.

Proof Assume that $n(G) - 1 = \alpha(G) + \mu(G)$. For each $e \in E(C)$, $G - e$ is bipartite, and then, we have

$$\alpha(G - e) - \alpha(G) + \mu(G - e) - \mu(G) = 1,$$

which implies $\mu(G - e) = \mu(G)$ and $\alpha(G - e) = \alpha(G) + 1$, since

$$-1 \leq \mu(G - e) - \mu(G) \leq 0 \leq \alpha(G - e) - \alpha(G) \leq 1.$$

In other words, every $e \in E(C)$ is $\alpha$-critical.

Conversely, let us choose $e \in E(C)$ satisfying $\mu(G - e) = \mu(G)$. By Lemma 2.1 such an edge exists. Since $e$ is $\alpha$-critical, and $G - e$ is bipartite, we infer that

$$n(G) - 1 = \alpha(G - e) + \mu(G - e) - 1 = \alpha(G) + \mu(G),$$

and this completes the proof. \hfill \square

A coalescence of disjoint graphs $G$ and $H$ is the graph $G \cdot H$ obtained by identifying one vertex of $G$ and one vertex of $H$ [5].

Lemma 2.5 Let $y$ be the identified vertex of the coalescence $G \cdot H$ and $x \in N_H(y)$. If $y \notin \text{core}(G)$, then $yx$ is not $\alpha$-critical in $G \cdot H$.

Proof Suppose, to the contrary, that $\alpha(G \cdot H) + 1 = \alpha(G \cdot H - yx)$. In other words, there exist some independent sets $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ such that $\{y\} \cup S_G \cup$
\(S_H \cup \{x\}\) is a maximum independent set in \(G \cdot H - xy\). Hence, \(S_G \cup S_H \cup \{x\}\) is a maximum independent set in \(G \cdot H\), because

\[|S_G \cup S_H \cup \{x\}| = \alpha (G \cdot H - xy) - 1.\]

Since \(y \notin \text{core}(G)\), there exists some maximum independent set \(A\) in \(G\) such that \(y \notin A\). If \(|S_G| < |A| = \alpha (G)\), then \(S_G\) may be replaced by \(A\), in the union \(S_G \cup S_H \cup \{x\}\). In this case, \(A \cup S_H \cup \{x\}\) is independent in \(G \cdot H\), while \(|A \cup S_H \cup \{x\}| > \alpha (G \cdot H)\), which is impossible. Thus, \(|S_G| = \alpha (G)\). On the other hand, \(\{y\} \cup S_G\) is an independent set in \(G\), which is a contradiction. \(\square\)

**Lemma 2.6** Let \(G\) be an almost bipartite graph. There exists some \(y \in V(C)\), such that \(y \notin \text{core}(D_y)\) if and only if \(G\) is a König–Egerváry graph.

**Proof** Assume that \(y \in V(C)\) and \(y \notin \text{core}(D_y)\). First, \(G = D_y \cdot (G - (D_y - y))\). By Lemma 2.5, there is a non-\(\alpha\)-critical edge belonging to \(C\). In accordance with Lemma 2.3 and Lemma 2.4, we conclude that \(n(G) = \alpha(G) + \mu(G)\), i.e., \(G\) is a König–Egerváry graph.

Let \(G\) be a König–Egerváry graph. Suppose to the contrary that \(y \in \text{core}(D_y)\) for every \(y \in V(C)\).

Clearly,

\[\mu(C) + \sum_{y \in V(C)} \mu(D_y - y) \leq \mu(G) \leq \mu(C) + \sum_{y \in V(C)} \mu(D_y).\]

By Lemma 2.2, \(\mu(D_y) = \mu(D_y - y)\), because \(D_y\) is bipartite and \(y \in \text{core}(D_y)\) for every \(y \in V(C)\). Consequently, we obtain

\[\mu(G) = \mu(C) + \sum_{y \in V(C)} \mu(D_y) = \left\lfloor \frac{|V(C)|}{2} \right\rfloor + \sum_{y \in V(C)} \mu(D_y).\]

Clearly,

\[\alpha(C) + \sum_{y \in V(C)} \alpha(D_y - y) \leq \alpha(G) \leq \sum_{y \in V(C)} \alpha(D_y) - (|V(C)| - \alpha(C)).\]

The fact that \(y \in \text{core}(D_y)\) means \(\alpha(D_y) - 1 = \alpha(D_y - y)\). Consequently, we obtain

\[\alpha(G) = \sum_{y \in V(C)} \alpha(D_y) - (|V(C)| - \alpha(C)) = \sum_{y \in V(C)} \alpha(D_y) - \left\lfloor \frac{|V(C)|}{2} \right\rfloor - 1.\]

Since every \(D_x\) is bipartite, we conclude with
In accordance with Lemma 2.6, we know that
\[ \alpha(G) + \mu(G) = \sum_{y \in V(C)} \alpha(D_y) + \sum_{y \in V(C)} \mu(D_y) - 1 \]
\[ = \sum_{y \in V(C)} n(D_y) - 1 = n(G) - 1. \]

Thus, \( G \) is not König–Egerváry, which is a contradiction. \( \Box \)

**Theorem 2.7** If \( G \) is an almost bipartite non-König–Egerváry graph, then \( K \{ D_y \} \) for every \( y \in V(C) \).

**Proof** In accordance with Lemma 2.6, \( y \in \text{core}(D_y) \) for every \( y \in V(C) \).

First, one has to prove that every maximum independent set of \( D_y \) may be enlarged to some maximum independent set of \( G \).

Let \( A \in \Omega(D_y - y), y \in V(C), \) and \( z \in N(y) \cap V(C) \). According to Lemma 2.4, the edge \( yz \) is \( \alpha \)-critical. Hence, there exist \( S_y \in \Omega(G), S_{yz} \in \Omega(G - yz) \), such that \( y \in S_y \) and \( y, z \in S_{yz} \).

**Case 1** Assume that \( A \cap N(y) = \emptyset \).

If \( |S_y - V(D_y - y)| < \alpha(G - D_y - y) \) and \( S_0 \in \Omega(G - D_y - y) \), then \( S_0 \cup (S_y - V(D_y - y)) \) is independent in \( G \) that causes the contradiction
\[ \alpha(G) = |S_y| - |S_y - V(D_y - y)| + |S_y \cap V(D_y - y)| \]
\[ < |S_0| + |S_y \cap V(D_y - y)| = |S_0 \cup (S_y \cap V(D_y - y))| \]

Therefore, we have \( |S_y - V(D_y - y)| = \alpha(G - D_y - y) \).

The set \( A \cup (S_y - V(D_y - y)) \) is independent, since \( A \cap N(y) = \emptyset \). Moreover, \( A \cup (S_y - V(D_y - y)) \in \Omega(G) \), otherwise we get the following contradiction
\[ |S_y - V(D_y - y)| + |A| < \alpha(G) \]
\[ \leq \alpha(G - D_y - y) + \alpha(D_y - y) = |S_y - V(D_y - y)| + |A| \]

**Case 2** Assume now that \( A \cap N(y) \neq \emptyset \).

Then, \( |A| \geq |S_{yz} \cap V(D_y - y)| \). Hence,
\[ \alpha(G) = |S_{yz} \cap \{ y \}| \leq |(S_{yz} \cap \{ y \}) - (S_{yz} \cap V(D_y - y))| \cup A| \]
\[ = |(S_{yz} \cap \{ y \}) - V(D_y - y))| \cup A| \]

Since the set \((S_{yz} \cap \{ y \}) - V(D_y - y)) \cup A \) is independent and its size is \( \alpha(G) \) at least, it is also maximum independent, i.e., \((S_{yz} \cap \{ y \}) - V(D_y - y)) \cup A \in \Omega(G) \).

Second, it is left to prove that \( S \cap V(D_y - y) \in \Omega(D_y - y) \) for every \( S \in \Omega(G) \). Let \( S \in \Omega(G) \), and suppose, to the contrary, that \( A = S \cap V(D_y - y) \neq \Omega(D_y - y) \).

By Lemma 2.6, we know that \( y \in \text{core}(D_y) \). Hence, we can change \( A \) for some \( B \in \Omega(D_y - y) \) such that \( B \cap N(y) = \emptyset \). Therefore, the set \((S - A) \cup B \) is independent, and \(|(S - A) \cup B| = |S - A| + |B| > |S| = \alpha(G) \). This contradiction completes the proof. \( \Box \)
Let $G$ be an almost bipartite graph. Then, the following assertions

(i) $\text{core}(G) \cap N[V(C)] = \emptyset$;

(ii) $\text{core}(G) = \bigcup_{y \in V(C)} \text{core}(D_y - y)$.

**Proof** (i) Let $ab \in E(C)$. By Lemma 2.4, the edge $ab$ is $\alpha$-critical. Hence, there exist $S_a, S_b \in \Omega(G)$, such that $a \in S_a$ and $b \in S_b$. Since $a \notin S_b$, it follows that $a \notin \text{core}(G)$, and because $a \in S_a$, we infer that $N(a) \cap \text{core}(G) = \emptyset$. Consequently, we obtain that $\text{core}(G) \cap N[V(C)] = \emptyset$.

(ii) By Theorem 2.7, we infer that:

$$\text{core}(D_y - y) = \bigcap\{A : A \in \Omega(D_y - y)\} = \bigcap\{S \cap V(D_y - y) : S \in \Omega(G)\} = \left(\bigcap\{S : S \in \Omega(G)\}\right) \cap V(D_y - y) = \text{core}(G) \cap V(D_y - y).$$

Together with Part (i), it implies $\text{core}(G) = \bigcup_{y \in V(C)} \text{core}(D_y - y)$. $\square$

The assertion in Corollary 2.8(ii) may fail for connected unicyclic König–Egerváry graphs. For instance,

$$\text{core}(G_2) \neq \{u, w\} = \bigcup_{y \in V(C)} \text{core}(D_y - y),$$

while $\text{core}(G_1) = \bigcup_{y \in V(C)} \text{core}(D_y - y)$, where $G_1$ and $G_2$ are from Fig. 3.

**Proposition 2.9** Let $G$ be an almost bipartite graph. Then, the following assertions are equivalent:

(i) $y \in \text{core}(D_y)$, for every $y \in V(C)$;

(ii) there exists some $S \in \Omega(G)$, such that $S \cap N_1(C) = \emptyset$;

(iii) $\alpha(G) - 1 = \alpha(G) + \mu(G)$, i.e., $G$ is not a König–Egerváry graph.

**Proof** (i) $\Leftrightarrow$ (iii) has been proved in Lemma 2.6.

(i) $\Rightarrow$ (ii) Let $y \in V(C)$ and assume that there is $S_1 \in \Omega(G)$, such that $N(y) \cap N_1(C) \cap S_1 \neq \emptyset$. Since $y \in \text{core}(D_y)$, there exists some $S_y \in \Omega(D_y - y)$, such that $N(y) \cap S_y = \emptyset$. Hence, we infer that $|S_1 \cap V(D_y - y)| \leq \alpha(D_y - y) = |S_y|$, $(S_1 - (S_1 \cap V(D_y - y))) \cup S_y$ is independent in $G$, and then,

$$|(S_1 - (S_1 \cap V(D_y - y))) \cup S_y| = |S_1 - (S_1 \cap V(D_y - y))| + |S_y|.$$
\[ |S_1| - |S_1 \cap V(D_y - y)| + \alpha(D_y - y) \geq \alpha(G). \]

Therefore, \( S_2 = (S_1 - (S_1 \cap V(D_y - y))) \cup S_1 \in \Omega(G), \) and \( N(y) \cap N_1(C) \cap S_2 = \emptyset. \)

In this way, considering more vertices belonging to \( N_1(C) \cap S_1, \) one can build some \( S \in \Omega(G), \) such that \( S \cap N_1(C) = \emptyset. \)

(ii) \( \Rightarrow \) (iii) We have that \( |S \cap V(C)| = \left\lfloor \frac{|V(C)|}{2} \right\rfloor, \) because \( S \cap N_1(C) = \emptyset. \)

Let \( ab \in E(C). \) Since \( C \) is a chordless odd cycle, say \( C = C_{2k+1}, k \geq 1, \) the edge \( ab \) is \( \alpha- \) critical in \( C, \) i.e., there is \( S_{ab} \subset V(C - ab), \) such that \( a, b \in S_{ab} \) and \( |S_{ab}| = k + 1. \)

Then, \( W_a = (S - V(C)) \cup S_{ab} \) is an independent set in \( G - ab, \) with

\[ |W_a| = |S - V(C)| + |S_{ab}| = |S| - \left\lfloor \frac{|V(C)|}{2} \right\rfloor + \left\lfloor \frac{|V(C)|}{2} \right\rfloor + 1 = 1 + \alpha(G), \]

which implies that the edge \( ab \) is \( \alpha- \) critical in \( G. \) Since \( ab \) was an arbitrary edge on \( C, \) it follows that every edge of \( C \) is \( \alpha- \) critical in \( G. \) By Lemma 2.4, it follows that \( n(G) - 1 = \alpha(G) + \mu(G). \)

\[ \square \]

**Theorem 2.10** Let \( G \) be a connected almost bipartite graph. Then, the following assertions are true:

(i) \( \mu(G) \leq \alpha(G); \)

(ii) there exists a matching from \( N(\text{core}(G)) \) into \( \text{core}(G); \)

(iii) there is a maximum matching of \( G \) that matches \( N(\text{core}(G)) \) into \( \text{core}(G). \)

**Proof** If \( G \) is a König–Egerváry graph, then (i) follows from the definition and the fact that \( \mu(G) \leq \frac{n(G)}{2}, \) while (ii), (iii) are true, by Theorem 1.3(i).

For the rest of the proof, we suppose that \( G \) is not a König–Egerváry graph.

(i) By Lemma 2.3, we have \( n(G) - 1 = \alpha(G) + \mu(G). \) According to Lemma 2.4, \( \alpha(G - xy) = \alpha(G) + 1 \) holds for each edge \( xy \in E(C). \) Consequently, we get that \( x, y \in \text{core}(G - xy). \) Since \( G - xy \) is bipartite, Theorem 1.2(iii) ensures that

\[ \alpha(G) + 1 = \alpha(G - xy) > \frac{n(G - xy)}{2} = \frac{n(G)}{2} \geq \mu(G - xy) = n(G) - \alpha(G - xy) = n(G) - \alpha(G) - 1 = \mu(G), \]

which results in \( \alpha(G) \geq \mu(G). \)

(ii) If \( \text{core}(G) = \emptyset, \) then the conclusion is clear.

Assume that \( \text{core}(G) \neq \emptyset. \) By Theorem 1.3(i), in each \( D_y - y \) there is a matching \( M_y \) from \( N(\text{core}(D_y - y)) \) into \( \text{core}(D_y - y). \) By Theorem 1.2(i), it follows that \( V(C) \cap N[\text{core}(G)] = \emptyset. \) Taking into account Corollary 2.8(ii), we see that the union of all these matchings \( M_y \) gives a matching from \( N(\text{core}(G)) \) into \( \text{core}(G). \)

(iii) Let \( M \) be a maximum matching of \( G \) and \( M_1 \) be a matching from \( N(\text{core}(G)) \) into \( \text{core}(G) \) that exists by Part (ii). The matching \( M \) must saturate \( N(\text{core}(G)), \)

because otherwise it can be enlarged with edges from \( M_1. \) Hence, all the edges of \( M \)
Let graph, then $y$ ∈ $\Omega_1(G)$ and this completes the proof. □

The almost bipartite graph $G$ from Fig. 2 has $M_1 = \{uv, cx, dt, wy\}$ and $M_2 = \{uv, ac, dt, wy\}$ as maximum matchings, but only $M_2$ matches $N(\text{core}(G)) = \{c\}$ into $\text{core}(G) = \{a, b\}$. Notice that $G$ is not a König–Egerváry graph.

**Proposition 2.11** If there is a matching from $N(\text{core}(G))$ into $\text{core}(G)$, then

$$\alpha(G) - \mu(G) \leq |\text{core}(G)| - |N(\text{core}(G))| .$$

**Proof** Let $M_1$ be a matching from $N(\text{core}(G))$ into $\text{core}(G)$. According to Theorem 1.2(ii), there is a matching, say $M_2$, from $S - \text{core}(G)$ into corona$(G) - S$ for $S \in \Omega(G)$. Consequently, we get that

$$|M_1| + |M_2| = |N(\text{core}(G))| + |S - \text{core}(G)| = |N(\text{core}(G))| + \alpha(G) - |\text{core}(G)| \leq \mu(G),$$

and this completes the proof. □

**Lemma 2.12** Let $G$ be a connected non-König–Egerváry almost bipartite graph with the unique odd cycle $C$.

(i) If $A$ is a critical independent set, then $A \cap V(C) = \emptyset$.

(ii) $\text{core}(G)$ is critical.

**Proof** Let $I$ be an independent set of $G$ such that $I = B \cup Q$, $B \cap Q = \emptyset$, and $Q \subset V(C)$.

(i) Suppose that $Q$ is not empty. Since $G$ is a non-König–Egerváry almost bipartite graph, then $y \in \text{core}(D_y)$ for every $y \in V(C)$, in accordance with Lemma 2.6. By Corollary 2.8(ii), $\text{core}(G) = \bigcup_{y \in V(C)} \text{core}(D_y - y)$.

Let us show that $d_G(I) < d_G(U)$, where $U = \text{core}(G)$.

First, $|Q| - |N_G(I) \cap V(C)| \leq |Q| - |N_C(Q)| < 0$, since $I$ is independent in $G$ and $C$ is an odd cycle.

Second, for every $y \in V(C)$

$$d_G(I \cap V(D_y - y)) = |I \cap V(D_y - y)| - |N_G(I \cap V(D_y - y))| \leq |I \cap V(D_y - y)| - |N_{D_y - y}(I \cap V(D_y - y))| = d_{D_y - y}(I \cap V(D_y - y)) \leq |\text{core}(D_y - y)| - |N_{D_y - y}(\text{core}(D_y - y))| = |\text{core}(D_y - y)| - |N_G(\text{core}(D_y - y))| ,$$

since $N_{D_y - y}(I \cap V(D_y - y)) \subseteq N_G(I \cap V(D_y - y))$, $D_y - y$ is bipartite, and $y \notin N_G(\text{core}(D_y - y))$.

Consequently,
If $G$ is a connected almost bipartite graph, then

Theorem 2.13 If $G$ is a connected almost bipartite graph, then

$$\alpha(G) - \mu(G) \leq |\text{core}(G)| - |N(\text{core}(G))| = d(G) \leq \alpha(G) - \mu(G) + 1.$$  

Proof If $G$ is a König–Egerváry graph, the result is true by Theorem 1.3(ii).

Otherwise, let $e \in E(C)$. Then, $H = G - e$ is a bipartite graph, and by Lemma 2.4, we get that $\alpha(H) = \alpha(G) + 1$ and $\mu(H) = \mu(G)$. For every $A \subseteq V(G)$, it follows that $|N_H(A)| \leq |N_G(A)|$, which implies

$$|A| - |N_G(A)| \leq |A| - |N_H(A)|,$$

and, consequently, $d(G) \leq d(H)$. Hence, using Proposition 2.11 and Theorem 1.3(ii), we obtain

$$\alpha(G) - \mu(G) \leq |\text{core}(G)| - |N(\text{core}(G))| \leq d(G) \leq d(H)$$

$$= \alpha(H) - \mu(H) = \alpha(G) - \mu(G) + 1.$$
By Lemma 2.12(ii), \( d(G) = d(\text{core}(G)) \), which completes the proof. 

**Theorem 2.14** [1] If \( G \) is unicyclic and non-König–Egerváry, then \( d(G) = \alpha(G) - \mu(G) \).

**Lemma 2.15** [10] Every connected bipartite graph has a spanning tree with the same independence number.

**Theorem 2.16** If \( G \) is an almost bipartite non-König–Egerváry graph, then

\[
d(G) = \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|.
\]

**Proof** Case 1 \( G \) is connected.

By Lemma 2.15, every bipartite subgraph \( D_y - y \) of \( G \) has a spanning forest \( F_y \), having the same independence number and hence the same matching number, i.e.,

\[
\alpha(F_y) = \alpha(D_y - y) \quad \text{and} \quad \mu(F_y) = \mu(D_y - y).
\]

Consequently, \( \Omega(D_y - y) \subseteq \Omega(F_y) \), which gives \( \text{core}(F_y) \subseteq \text{core}(D_y - y) \). By Theorem 2.7, we have that \( \Omega(G)|_V(D_y - y) = \Omega(D_y - y) \).

Let \( H \) be the graph obtained from \( G \) by substituting every \( D_y - y \) with an appropriate \( F_y \). Thus, \( H \) is a connected unicyclic graph, having \( C \) as its unique cycle.

**Claim 1** \( d(G) \leq d(H) \). Every independent set \( S \) of \( G \) is independent in \( H \) as well, while \( N_H(S) \subseteq N_G(S) \). Hence,

\[
d_G(S) = |S| - |N_G(S)| \leq |S| - |N_H(S)| = d_H(S).
\]

Thus, \( d(G) \leq d(H) \).

**Claim 2** \( \alpha(G) = \alpha(H) \). Since \( G \) and \( H \) have the same vertex sets and \( E(H) \subseteq E(G) \), we get that \( \alpha(G) \leq \alpha(H) \).

By Proposition 2.9(ii), there exists some \( A \in \Omega(G) \), such that \( A \cap N_1(C) = \emptyset \). Hence, by Theorem 2.7,

\[
A \cap V(F_y) = A \cap V(D_y - y) \in \Omega(D_y - y) \subseteq \Omega(F_y) \quad \text{for every} \ y \in V(C),
\]

\[
|A \cap V(C)| = \left\lfloor \frac{|V(C)|}{2} \right\rfloor, \quad \text{and}
\]

\[
A = (A \cap V(C)) \cup \bigcup_{y \in V(C)} (A \cap V(D_y - y)).
\]

Clearly, \( A \) is an independent set in \( H \) as well.

Let \( S \in \Omega(H) \). Then, \( |S \cap V(C)| \leq \left\lfloor \frac{|V(C)|}{2} \right\rfloor = |A \cap V(C)| \), and also

\[
|S \cap V(D_y - y)| \leq |A \cap V(F_y)| \quad \text{for every} \ y \in V(C).
\]
Thus,
\[
\alpha(H) = |S| = |S \cap V(C)| \cup \bigcup_{y \in V(C)} |S \cap V(F_y)|
\]
\[
= |S \cap V(C)| + \sum_{y \in V(C)} |S \cap V(F_y)|
\]
\[
\leq |A \cap V(C)| + \sum_{y \in V(C)} |(A \cap V(F_y))| = |A| = \alpha(G).
\]

In conclusion, we get that \(\alpha(G) = \alpha(H)\).

**Claim 3** \(\mu(G) = \mu(H)\).

Along the lines of the proof of Claim 2, we know that there exists a set \(A \in \Omega(H)\), such that \(A \cap N_1(C) = \emptyset\). Therefore, Proposition 2.9 implies that \(H\) is a non-König–Egerváry graph. Hence,
\[
\alpha(G) + \mu(G) - 1 = n(G) = n(H) = \alpha(H) + \mu(H) - 1.
\]

By Claim 2, it means that \(\mu(G) = \mu(H)\).

**Claim 4** \(d(G) = \alpha(G) - \mu(G)\).

By Claim 2, Claim 3, Theorem 2.13, Claim 1, and Theorem 2.14, we finally obtain the following:
\[
\alpha(H) - \mu(H) = \alpha(G) - \mu(G) \leq |\text{core}(G)| - |N(\text{core}(G))|
\]
\[
= d(G) \leq d(H) = \alpha(H) - \mu(H),
\]

which completes the proof.

*Case 2* \(G\) is disconnected.

Clearly, \(G = G_1 \cup G_2\), where \(G_1\) is the connected component of \(G\) containing the unique odd cycle, and \(G_2\) is a nonempty bipartite graph. By Case 1,
\[
d(G_1) = \alpha(G_1) - \mu(G_1) = |\text{core}(G_1)| - |N(\text{core}(G_1))|,
\]

while Theorem 1.3(ii) implies
\[
d(G_2) = \alpha(G_2) - \mu(G_2) = |\text{core}(G_2)| - |N(\text{core}(G_2))|.
\]

Since
\[
d(G) = d(G_1) + d(G_2), \quad \alpha(G) = \alpha(G_1) + \alpha(G_2), \quad \mu(G) = \mu(G_1) + \mu(G_2),
\]
\[
\text{core}(G) = \text{core}(G_1) \cup \text{core}(G_2), \quad N(\text{core}(G)) = N(\text{core}(G_1)) \cup N(\text{core}(G_2)),
\]

\[\square\] Springer
Fig. 4  \( \text{core}(G_1) = \emptyset \), while \( \text{core}(G_2) = \{x, y, z\} \) and \( N(\text{core}(G_2)) = \{v\} \)

we conclude with

\[
d(G) = \alpha(G_1) - \mu(G_1) + \alpha(G_2) - \mu(G_2) = \alpha(G) - \mu(G) \\
= |\text{core}(G_1)| - |N(\text{core}(G_1))| + |\text{core}(G_2)| - |N(\text{core}(G_2))| \\
= |\text{core}(G)| - |N(\text{core}(G))|,
\]

as required. \( \square \)

3 Conclusions

It is known that for every graph, \( \max\{0, \alpha(G) - \mu(G)\} \leq d(G) \) \([16,19,20]\), while \( |\text{core}(G)| - |N(\text{core}(G))| \leq d(G) \) by definition of \( d(G) \).

By Theorems 1.3, 2.16, \( d(G) = \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))| \) for both König–Egerváry graphs and almost bipartite graphs. Otherwise, every relation between \( \alpha(G) - \mu(G) \) and \( |\text{core}(G)| - |N(\text{core}(G))| \) is possible. For instance, the non-König–Egerváry graphs from Fig. 4 satisfy

\[
\alpha(G_1) - \mu(G_1) = 0 = |\text{core}(G_1)| - |N(\text{core}(G_1))| = d(G_1), \\
\alpha(G_2) - \mu(G_2) = 1 < 2 = |\text{core}(G_2)| - |N(\text{core}(G_2))| = d(G_2).
\]

The opposite direction of the displayed inequality may be found in \( G_3 = K_{2n} - e, n \geq 3 \), where

\[
d(K_{2n} - e) = 0 > \alpha(G_3) - \mu(G_3) = 2 - n > 2 - (2n - 2) \\
= |\text{core}(G_3)| - |N(\text{core}(G_3))|.
\]

Another example reads as follows:

\[
\alpha(G) - \mu(G) = 2 < |\text{core}(G)| - |N(\text{core}(G))| = 3 < 4 = d(G),
\]

where \( G \) is from Fig. 5.

Problem 3.1 Characterize graphs enjoying \( d(G) = \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))| \).
Fig. 5  $\text{core}(G) = \{x, y, z, u, v, w\}$, $\alpha(G) = 8$ and $\mu(G) = 6$

Acknowledgements  We express our gratitude to anonymous reviewers for their comments that substantially improved the presentation of the paper. Our special thanks are to one of them that suggested to generalize our findings from graphs with a unique odd cycle and each of the other bipartite components connected to a vertex of the cycle via a cut-edge to the family of graphs with exactly one odd cycle.

References

1. Bhattacharya, A., Mondal, A., Murthy, T.S.: Problems on matchings and independent sets of a graph. Discrete Math. 341, 1561–1572 (2018)
2. Bourjolly, J.M., Hammer, P.L., Simeone, B.: Node weighted graphs having König–Egerváry property. In: Korte, B., Ritter, K. (eds.) Mathematical Programming at Oberwolfach II. Mathematical Programming Studies, vol. 22, pp. 44–63. Springer, Berlin, Heidelberg (1984). https://doi.org/10.1007/BFb0121007
3. Bourjolly, J.M., Pulleyblank, W.R.: König–Egerváry graphs, 2-bicritical graphs and fractional matchings. Discrete Appl. Math. 24, 63–82 (1989)
4. Boros, E., Columbic, M.C., Levit, V.E.: On the number of vertices belonging to all maximum stable sets of a graph. Discrete Appl. Math. 124, 17–25 (2002)
5. Brigham, R.C., Chinn, P.Z., Dutton, R.D.: Vertex domination-critical graphs. Networks 18, 173–179 (1988)
6. Deming, R.W.: Independence numbers of graphs—an extension of the König–Egerváry theorem. Discrete Math. 27, 23–33 (1979)
7. Došlić, T., Vukičević, D.: Computing the bipartite edge frustration of fullerene graphs. Discrete Appl. Math. 155, 1294–1301 (2007)
8. Egerváry, E.: On combinatorial properties of matrices. Mat. Lapok 38, 16–28 (1931)
9. König, D.: Graphen und Matrizen. Mat. Lapok 38, 116–119 (1931)
10. Levit, V.E., Mandrescu, E.: On the structure of $\alpha$-stable graphs. Discrete Math. 236, 227–243 (2001)
11. Levit, V.E., Mandrescu, E.: Combinatorial properties of the family of maximum stable sets of a graph. Discrete Appl. Math. 117, 149–161 (2002)
12. Levit, V.E., Mandrescu, E.: On $\alpha^+$-stable König–Egerváry graphs. Discrete Math. 263, 179–190 (2003)
13. Levit, V.E., Mandrescu, E.: On $\alpha$-critical edges in König–Egerváry graphs. Discrete Math. 306, 1684–1693 (2006)
14. Levit, V.E., Mandrescu, E.: Independent sets in almost König–Egerváry graphs. In: Abstracts of the SIAM Conference on Discrete Mathematics, Halifax, Canada, p 40, Abstract MS21 (2012)
15. Levit, V.E., Mandrescu, E.: Critical independent sets and König–Egerváry graphs. Graphs Combin. 28, 243–250 (2012)
16. Levit, V.E., Mandrescu, E.: Vertices belonging to all critical independent sets of a graph. SIAM J. Discrete Math. 26, 399–403 (2012)
17. Levit, V.E., Mandrescu, E.: On the core of a unicyclic graph. Ars Math. Contemp. 5, 321–327 (2012)
18. Levit, V.E., Mandrescu, E.: Critical independent sets in a graph. In: 3rd International Conference on Discrete Mathematics, June 10–14, Karnataka University. Dharwad, India (2013)
19. Lorentzen, L.C.: Notes on Covering of Arcs by Nodes in an Undirected Graph, Technical report ORC 66–16. University of California, Berkeley, CA, Operations Research Center (1966)
20. Schrijver, A.: Combinatorial Optimization. Springer, Berlin (2003)
21. Sterboul, F.: A characterization of the graphs in which the transversal number equals the matching number. J. Combin. Theory Ser. B 27, 228–229 (1979)
22. Yarahmadi, Z., Došlić, T., Ashrafi, A.R.: The bipartite edge frustration of composite graphs. Discrete Appl. Math. 158, 1551–1558 (2010)
23. Zhang, C.Q.: Finding critical independent sets and critical vertex subsets are polynomial problems. SIAM J. Discrete Math. 3, 431–438 (1990)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

Vadim E. Levit$^1$ · Eugen Mandrescu$^2$

Eugen Mandrescu
eugen_m@hit.ac.il

1 Department of Computer Science, Ariel University, Ariel, Israel
2 Department of Computer Science, Holon Institute of Technology, Holon, Israel