An optimal bound for the ratio between ordinary and uniform exponents of Diophantine approximation

Antoine Marnat* and Nikolay G. Moshchevitin†

Abstract

We provide a lower bound for the ratio between the ordinary and uniform exponents of both simultaneous Diophantine approximation to \( n \) real numbers and Diophantine approximation for one linear form in \( n \) variables. This question was first considered in the 50’s by V. Jarník who solved the problem for two real numbers and established certain bounds in higher dimension. Recently different authors reconsidered the question, solving the problem in dimension three with different methods. Considering a new concept of parametric geometry of numbers, W. M. Schmidt and L. Summerer conjectured that the optimal lower bound is reached at regular systems. It follows from a remarkable result of D. Roy that this lower bound is then optimal. In the present paper we give a proof of this conjecture by W. M. Schmidt and L. Summerer.

1 Introduction

In the 50’s, V. Jarník [8, 9, 10] considered exponents of Diophantine approximation, and in particular the ratio between ordinary and uniform exponent. An optimal lower bound expressed as a function of the uniform exponent was established for simultaneous approximation to two real numbers and for one linear form in two variables. The question was reconsidered recently by different authors [13, 17, 18, 27, 7, 5]. The optimality of V. Jarník’s inequalities for two numbers was shown by M. Laurent [13]. The inequality for simultaneous approximation to three real numbers was obtained by the second named author [17]. Introducing parametric geometry of numbers [27, 26], W. M. Schmidt and L. Summerer considered recently a new method to obtain the optimal lower bounds for the approximation to three numbers (both in the cases of simultaneous approximation and approximation for one linear form in three variables), and improve the general lower bound in any dimension. They conjectured in this context that the general lower bound in the problem of approximation to \( n \)

*supported by Austrian Science Fund (FWF), Project I 3466-N35 and EPSRC Programme Grant EP/J018260/1
†supported by Russian Science Foundation (RNF) Project 18-41-05001 in Pacific National University
real numbers arise from so-called regular systems. The goal of the present paper is to prove this conjecture. To do this we use Schmidt’s inequality on heights [24] applied to a well-chosen subsequence of best approximation vectors. Our main result is stated in Theorem 1 below. The optimality of our bound follows from a recent breakthrough paper by D. Roy [22].

Throughout this paper, the integer \( n \geq 1 \) denotes the dimension of the ambient space, and \( \theta = (\theta_1, \ldots, \theta_n) \) denotes an \( n \)-tuple of real numbers such that 1, \( \theta_1, \ldots, \theta_n \) are \( \mathbb{Q} \)-linearly independent.

Given \( n \geq 1 \) and \( \theta \in \mathbb{R}^n \), we consider the irrationality measure function

\[
\psi(t) = \min_{q \in \mathbb{Z}^n} \max_{1 \leq j \leq n} \| q \theta_j \|,
\]

which gives rise to the ordinary exponent of simultaneous Diophantine approximation

\[
\lambda(\theta) = \sup \{ \lambda : \liminf_{t \to +\infty} t^\lambda \psi(t) < +\infty \}
\]

and the uniform exponent of simultaneous Diophantine approximation

\[
\hat{\lambda}(\theta) = \sup \{ \lambda : \limsup_{t \to +\infty} t^\lambda \psi(t) < +\infty \}.
\]

The irrationality measure function

\[
\varphi(t) = \min_{q \in \mathbb{Z}^n, 0 < \max_{1 \leq j \leq n} |q_j| \leq t} \| q_1 \theta_1 + \cdots + q_n \theta_n \|
\]

gives rise to the ordinary exponent of Diophantine approximation by one linear form

\[
\omega(\theta) = \sup \{ \omega : \liminf_{t \to +\infty} t^\omega \varphi(t) < +\infty \}
\]

and the uniform exponent of Diophantine approximation by one linear form

\[
\hat{\omega}(\theta) = \sup \{ \omega : \limsup_{t \to +\infty} t^\omega \varphi(t) < +\infty \}.
\]

These exponents were first introduced and studied by A. Khintchine [11, 12] and V. Jarník [8]. Dirichlet’s Schubfachprinzip ensures that for any \( \theta \) with \( \mathbb{Q} \)-linearly independent coordinates with 1

\[
\omega(\theta) \geq \hat{\omega}(\theta) \geq n \quad \text{and} \quad \lambda(\theta) \geq \hat{\lambda}(\theta) \geq 1/n.
\]

Exponents of Diophantine approximation give more detailed information about approximation to \( \theta \) in the case when \( \theta \) admits approximations better than the approximations provided by Dirichlet’s Schubfachprinzip. The ordinary exponent deals with the question whether Dirichlet’s Schubfachprinzip can be improved for approximation vectors of arbitrarily large size \( t \), while uniform exponents deals with the question whether it can be improved for any
sufficiently large upper bound $t$ for the size of approximation vectors. The aim of this paper is to provide a lower bound for the ratios $\lambda(\theta)/\hat{\lambda}(\theta)$ and $\omega(\theta)/\hat{\omega}(\theta)$ as a function of $\hat{\lambda}(\theta)$ and $\hat{\omega}(\theta)$ respectively, in any dimension. In dimension $n = 1$ simultaneous approximation and approximation by one linear form coincide. Khintchine [12] observed that the uniform exponent for an irrational $\theta$ always takes the value 1 and it follows from Dirichlet’s Schubfachprinzip that the ordinary exponent satisfy $\omega(\theta) = \lambda(\theta) \geq 1 = \hat{\omega}(\theta) = \hat{\lambda}(\theta)$. In dimension $n = 2$, Jarník proved in [9, 10] the inequalities

$$\lambda(\theta) \geq \frac{\hat{\lambda}(\theta)}{1 - \hat{\lambda}(\theta)} \quad \text{and} \quad \frac{\omega(\theta)}{\hat{\omega}(\theta)} \geq \frac{\hat{\omega}(\theta) - 1}{\hat{\lambda}(\theta)}.$$  

(1)

These inequalities are optimal by a result of M. Laurent [13]. In [17], Moshchevitin proved the optimal bound for simultaneous approximation in dimension $n = 3$:

$$\frac{\lambda(\theta)}{\hat{\lambda}(\theta)} \geq \frac{\hat{\lambda}(\theta) + \sqrt{4\hat{\lambda}(\theta) - 3\hat{\lambda}(\theta)^2}}{2(1 - \hat{\lambda}(\theta))} = \frac{1}{2} \left( \frac{\hat{\lambda}(\theta)}{1 - \hat{\lambda}(\theta)} + \frac{\left( \frac{\hat{\lambda}(\theta)}{1 - \hat{\lambda}(\theta)} \right)^2 + 4\hat{\lambda}(\theta)}{1 - \hat{\lambda}(\theta)} \right).$$

(1)

The proof is based on consideration of a special pattern of best approximation vectors. This pattern was discovered in an earlier paper by D. Roy [23], where another problem was considered. We discuss this pattern in Section 3.1 when explaining our proof in low dimensions.

Schmidt and Summerer provided an alternative proof using parametric geometry of numbers in [28], and found the following bound for approximation by one linear form in 3 variables:

$$\frac{\omega(\theta)}{\hat{\omega}(\theta)} \geq \frac{\sqrt{4\hat{\omega}(\theta) - 3} - 1}{2}.$$ 

(2)

A simple proof of this bound was given in [18]. In [10], Jarník also provided a lower bound in arbitrary dimension $n \geq 2$.

$$\frac{\omega(\theta)}{\hat{\omega}(\theta)} \geq \hat{\omega}(\theta)^{1/(n-1)} - 3, \quad \text{provided that} \quad \hat{\omega}(\theta) > (5n^2)^{n-1},$$ 

(3)

$$\frac{\lambda(\theta)}{\hat{\lambda}(\theta)} \geq \frac{\hat{\lambda}(\theta)}{1 - \hat{\lambda}(\theta)}.$$ 

(4)

In fact, these bounds also apply in a more general setting of simultaneous Diophantine approximation by a set of linear forms.

Using their new tools of parametric geometry of numbers, Schmidt and Summerer [26] provided the first general improvement valid for the whole admissible interval of values of the
uniform exponents $\hat{\omega}$ and $\hat{\lambda}$.

\[
\frac{\omega(\theta)}{\hat{\omega}(\theta)} \geq \frac{(n-2)(\hat{\omega}(\theta) - 1)}{1 + (n-3)\hat{\omega}(\theta)}, \tag{5}
\]

\[
\frac{\lambda(\theta)}{\hat{\lambda}(\theta)} \geq \frac{\hat{\lambda}(\theta) + n - 3}{(n-2)(1 - \hat{\lambda}(\theta))}. \tag{6}
\]

Here relation (6) is sharper than relation (4). Relation (5) is valid for the whole interval of possible values of $\hat{\omega}(\theta)$, but Jarník's asymptotic relation (3) is better for large $\hat{\omega}(\theta)$. A simple proof of (6) was given in [7].

In [28] Schmidt and Summerer conjecture that, as in dimension $n = 3$, the general optimal lower bound is reached at regular systems. In this paper we show that this conjecture holds. Let us first introduce some notation.

For given $n \geq 1$ and parameters $\alpha^* \geq n$ and $1/n \leq \alpha < 1$, we consider the polynomials

\[
R_{n,\alpha}(x) = x^{n-1} - \frac{\alpha}{1 - \alpha} (x^{n-2} + \cdots + x + 1), \tag{7}
\]

\[
R_{n,\alpha^*}(x) = x^{n-1} + x^{n-2} + \cdots + x + 1 - \alpha^*. \tag{8}
\]

Note that

\[
R_{n,\alpha}(x) = \frac{\alpha}{\alpha - 1} x^{n-1} R_{n,\frac{1}{\alpha}}(\frac{1}{x}).
\]

Denote by $G(n, \alpha)$ the unique real positive root of $R_{n,\alpha}(x)$ and by $G^*(n, \alpha^*)$ the unique positive root of $R_{n,\alpha^*}(x)$.

Some further necessary properties of these polynomials are discussed in Subsection 2.4 below.

Now we are able to formulate the main result of our paper.

**Theorem 1.** For $\theta = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent, one has

\[
\frac{\lambda(\theta)}{\hat{\lambda}(\theta)} \geq G(n, \hat{\lambda}(\theta)) \quad \text{and} \quad \frac{\omega(\theta)}{\hat{\omega}(\theta)} \geq G^*(n, \hat{\omega}(\theta)). \tag{9}
\]

Furthermore, for any $\hat{\omega} \geq n$ and any $C \geq G^*(n, \hat{\omega})$, there exists infinitely many $\theta = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent and

$\hat{\omega}(\theta) = \hat{\omega}$ and $\omega(\theta) = C\hat{\omega}$

and for any $1/n \leq \hat{\lambda} \leq 1$ and any $C \geq G(n, \hat{\lambda})$, there exists infinitely many $\theta = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are $\mathbb{Q}$-linearly independent and

$\hat{\lambda}(\theta) = \hat{\lambda}$ and $\lambda(\theta) = C\hat{\lambda}$.
It follows from Roy’s Theorem 2 [22] applied to Schmidt-Summerer’s regular systems [28] [21] that the lower bound is reached and thus optimal. The second part of Theorem 1 refines this observation. Note that for any \( \theta = (\theta_1, \ldots, \theta_n) \) such that \( 1, \theta_1, \ldots, \theta_n \) are \( \mathbb{Q} \)-linearly independent, we have \( \hat{\omega}(\theta) \geq n \) and \( \hat{\lambda}(\theta) \in [1/n, 1] \), (see [6], [14]) hence the constraint on \( \hat{\lambda} \) and \( \hat{\omega} \) is not restrictive.

The main part of Theorem 1 is the lower bound. The proof uses determinants of best approximation vectors, following the idea of [17]. It deeply relies on an inequality of Schmidt [24] applied inductively to a well chosen subsequence of best approximation vectors. The second part of Theorem 1 is a consequence of the parametric geometry of numbers, and is proved independently in Section 6.

In the next section, we define the main tools needed for the proof: best approximation vectors and their properties. With examples of approximation to 3 and 4 numbers in Section 3, we then provide a proof of Theorem 1 in the important case of simultaneous approximation (Section 4). In Section 5, we explain how a hyperbolic rotation reduces the case of approximation by one linear form to the case of simultaneous approximation.

2 Main tools

2.1 Sequences of best approximations

We denote by \((z_l)_{l \in \mathbb{N}}\) the sequence of best approximations (or minimal points) to \( \theta \in \mathbb{R}^n \). This notion was introduced by Voronoi [29] as minimal points in lattices, it was first defined in our context by Rogers [20]. It has been used implicitly or explicitly in many proofs concerning exponents of Diophantine approximation. Many important properties of best approximation vectors are discussed in a survey by Chevallier [1].

Let \( k \geq 1 \) be an integer. Let \( L \) and \( N \) be two maps from \( \mathbb{Z}^k \) to \( \mathbb{R}_+ \), where \( N \) represents the height of an approximation vector in \( \mathbb{Z}^k \) and \( L \) represents the approximation error. We call a sequence of best approximation vectors \((z_l)_{l \geq 0} \in (\mathbb{Z}^k)^\mathbb{N}\) with respect to \( L \) and \( N \) a sequence such that

- \( N(z_l) \) is a strictly increasing sequence with \( N(z_1) \geq 1 \),
- \( L(z_l) \) is a strictly decreasing sequence with \( L(z_1) \leq 1 \),
- for any approximation vector \( z \in \mathbb{Z}^k \), if \( N(z) < N(z_{l+1}) \) then \( L(z) \geq L(z_l) \).

In general we do not have uniqueness of such a sequence, and existence follows if \( L \) reaches a minimum on sets of the form

\[
E_B = \{ X \in \mathbb{Z}^k | N(X) \leq B \},
\]
where $B$ is any real bound.

In the context of best approximation vectors for simultaneous Diophantine approximation for $\mathbb{Q}$-independent numbers $1, \theta_1, \ldots, \theta_n$ we set

$$L_{\lambda, \theta}(z) = \max_{1 \leq i \leq n} |q_{\theta_i} - a_i| \quad \text{and} \quad N_{\lambda, \theta}(z) = q$$

for

$$z = (q, a_1, a_2, \ldots, a_n) \in \mathbb{Z}^{n+1}, l \in \mathbb{N}$$

and define the unique sequence of best approximation vectors

$$z_l = (q_l, a_{1,l}, a_{2,l}, \ldots, a_{n,l}) \in \mathbb{Z}^{n+1}, l \in \mathbb{N}$$

with $q_1 = 1$. So $L_{\lambda, \theta}(z_l) = \xi_l = \max_{1 \leq i \leq n} |q_{\theta_i} - a_{i,l}|$, $N_{\lambda, \theta}(z_l) = q_l$ and

$$1 = q_1 < q_2 < \cdots < q_l < q_{l+1} < \cdots \quad \text{and} \quad 1 > \xi_1 > \xi_2 > \cdots > \xi_l > \xi_{l+1} > \cdots. \quad (10)$$

We may also assume that for every $l$ large enough one has

$$\xi_l \leq q_{l+1}^\alpha, \quad (11)$$

where $\alpha < \hat{\lambda}(\theta)$.

In the context of best approximation vector for approximation by one linear form, we can set

$$L_{\omega, \theta}(z) = |q_1 \theta_1 + \cdots + q_n \theta_n - a| \quad \text{and} \quad N_{\omega, \theta}(z) = \max_{1 \leq j \leq n} |q_j|$$

for

$$z = (q_1, q_2, \ldots, q_n, a) \in \mathbb{Z}^{n+1}.$$

In such a way we define the sequence

$$z_l = (q_{1,l}, q_{2,l}, \ldots, q_{n,l}, a_l) \in \mathbb{Z}^{n+1}, \ l = 1, 2, 3...$$

with

$$L_{\omega, \theta}(z_l) = L_l = q_{1,l} \theta_1 + \cdots + q_{n,l} \theta_n - a_l \quad \text{and} \quad N_{\omega, \theta}(z_l) = M_l = \max_{1 \leq j \leq n} |q_j|.$$ 

Here due to the symmetry, we may assume that $L_l > 0$. In the $\mathbb{Q}$-independent case this defines vectors $z_l$ uniquely. By definition of best approximations

$$1 \leq M_1 < M_2 < \cdots < M_l < M_{l+1} < \cdots \quad \text{and} \quad 1 > L_1 > L_2 > \cdots > L_l > L_{l+1} > \cdots.$$

We may also assume that $M_1$ is large enough so that for every $l \geq 1$

$$L_l \leq M_{l+1}^{-\alpha^*}. \quad (12)$$
where \( \alpha^* < \hat{\omega}(\theta) \).

In the context of simultaneous Diophantine approximation, provided that \( 1, \theta_1, \ldots, \theta_n \) are linearly independent over \( \mathbb{Q} \), it is known that a sequence of best approximation vectors ultimately spans the whole space \( \mathbb{R}^{n+1} \). However in the context of approximation by one linear form, the situation is different: it may happen that vectors of best approximation span a strictly lower dimensional subspace of \( \mathbb{R}^{n+1} \). See the surveys \([15, 16]\) by Moshchevitin and the paper \([1]\) by Chevallier for more details. Fortunately, if best approximation vectors do not span the whole space \( \mathbb{R}^{n+1} \) we get a sharper result, since \( G(n, \alpha) \) is a decreasing function of \( n \) (see Proposition \( 2 \)). Thus, we may assume without loss of generality that in both contexts best approximation vectors ultimately span the whole space \( \mathbb{R}^{n+1} \).

Using sequences of best approximation vectors, to prove that \( \lambda(\theta) \geq G \) it is enough to show that given arbitrary \( g < G \), there exists arbitrarily large indices \( k \) with \( q_{k+1} \gg q_g^g \). Similarly, to prove that \( \hat{\omega}(\theta) \geq G^* \) it is enough to show that given arbitrary \( g^* < G^* \) and \( \alpha^* < \hat{\omega}(\theta) \), there exists arbitrarily large indices \( k \) with \( M_{k+1} \gg M_k^{g^*} \) or \( L_k \ll M_k^{-\alpha^* g^*} \).

Here and below, the Vinogradov symbols \( \ll, \gg \) and \( \asymp \) refer to constants depending on \( \theta \) but not the index \( k \). This observation relies on the expression of exponents of Diophantine approximation in terms of best approximation vectors

\[
\omega(\theta) = \limsup_{k \to \infty} \left( -\frac{\log(L_k)}{\log(M_k)} \right), \quad \hat{\omega}(\theta) = \liminf_{k \to \infty} \left( -\frac{\log(L_k)}{\log(M_{k+1})} \right),
\]

\[
\lambda(\theta) = \limsup_{k \to \infty} \left( -\frac{\log(\xi_k)}{\log(q_k)} \right), \quad \hat{\lambda}(\theta) = \liminf_{k \to \infty} \left( \frac{\log(\xi_k)}{\log(q_{k+1})} \right).
\]

For the sake of convenience, if it does not make confusion, we may omit \( \theta \) in exponents \( \lambda(\theta), \hat{\lambda}(\theta), \omega(\theta), \hat{\omega}(\theta) \).

The proofs in the case of simultaneous approximation and approximation by one linear form rely on the same geometric analysis. The idea is to take \( \alpha < \hat{\lambda}(\theta) \) or \( \alpha^* < \hat{\omega}(\theta) \). For an arbitrarily large index \( k \), we construct a pattern of best approximation vectors in which at least one pair of successive best approximation vectors satisfies

\[
q_{k+1} \gg q_k^g \quad (13)
\]

where \( g = G(n, \alpha) < G(n, \hat{\lambda}) \), in the case of simultaneous approximation and

\[
M_{k+1} \gg M_k^{g^*} \quad \text{or} \quad L_k \ll M_k^{-\alpha^* g^*} \quad (14)
\]

where \( g^* = G^*(n, \alpha^*) < G(n, \hat{\omega}) \), in the case of approximation by one linear form.
Given a sublattice $\Lambda \subset \mathbb{Z}^{n+1}$, we denote by $\det(\Lambda)$ the fundamental volume of the lattice $\Lambda$ in the linear subspace $\langle \Lambda \rangle_\mathbb{R}$. We recall well known facts about best approximation vectors and fundamental determinants of the related lattices.

**Lemma 1.** Two consecutive best approximation vectors $z_i$ and $z_{i+1}$ are $\mathbb{Q}$-linearly independent and form a basis of the integer points of the rational subspace they span.

\[
\langle z_i, z_{i+1} \rangle_\mathbb{Z} = \langle z_i, z_{i+1} \rangle_\mathbb{R} \cap \mathbb{Z}^{n+1}.
\]

See for example [4, Lemma 2].

**Lemma 2.** For any $l \geq 1$, consider the lattice $\Lambda_l$ with basis $z_l, z_{l+1}$ and the lattice $\Gamma_l$ with basis $z_{l-1}, z_l, z_{l+1}$. In the context of simultaneous approximation we have the estimates of their fundamental volumes

\[
\det(\Lambda_l) \asymp \xi_l q_{l+1} \quad \text{and} \quad \det(\Gamma_l) \ll \xi_{l-1} \xi_l q_{l+1}.
\]  

(15)

In the context of approximation by one linear form, we do not have directly such estimates. In section 5 we explain how hyperbolic rotation provides a helpful analogue.

The proof of Lemma 2 is well known, see for example [1] or [16]. For the sake of completeness, and because we want to adapt the proof for the case of approximation by one linear form, we provide a detailed proof. The upper bounds rely on the following lemma (see [25, Lemma 1]), while the lower bound comes from Minkowski’s first convex body theorem.

**Lemma 3.** Assume $X_1, \ldots, X_m$ are linearly independent vectors of an Euclidean space $E^n$, and have coordinates $X_t = (x_{t,1}, \ldots, x_{t,n})$ for $1 \leq t \leq m \leq n$ in some Cartesian coordinate-system of $E^n$. Then $\det^2(X_1, \ldots, X_m)$ is the sum (with $\binom{m}{n}$ summands) of the squares of the absolute values of the determinants of the $(m \times m)$-submatrices of the matrix $(x_{t,j})_{1 \leq t \leq m, 1 \leq j \leq n}$.

**Proof of Lemma 2.** The proof relies on the geometric fact that the best approximation $z_l = (q_l, a_{1,l}, a_{2,l}, \ldots, a_{n,l}) \in \mathbb{Z}^{n+1}$ satisfy (10). We first prove the upper bounds.

Consider the 2-dimensional fundamental volume of the lattice spanned by $z_l, z_{l+1}$. The coordinates of these vectors form the matrix

\[
\begin{pmatrix}
q_l & a_{1,l} & \cdots & a_{n,l} \\
q_{l+1} & a_{1,l+1} & \cdots & a_{n,l+1}
\end{pmatrix}.
\]

However it is not convenient to use this matrix to apply Lemma 3. We consider a special choice of Cartesian coordinates. We take the system of orthogonal unit vector $(e_0, e_1, \ldots, e_n)$ in the following way: $e_0$ is parallel to $(1, \theta_1, \ldots, \theta_n)$ and $e_1, \ldots, e_n$ are arbitrary. Then, in our new coordinates

\[
z_l = (Z_l, \Xi_{1,l}, \ldots, \Xi_{n,l})
\]
where $Z_l \times q_l$ and $|\Xi_{i,l}| \ll \xi_l$.

Now we consider the $2 \times (n+1)$ matrix

$$
\begin{pmatrix}
Z_l & \Xi_{1,l} & \cdots & \Xi_{n,l} \\
Z_{l+1} & \Xi_{1,l+1} & \cdots & \Xi_{n,l+1}
\end{pmatrix},
$$

If $M_{i,j}$ is the $(2 \times 2)$ minor of index $i, j$, we have by Lemma 3

$$
\det(\Lambda_l)^2 = \sum_{0 \leq i < j \leq n} M_{i,j}^2 \ll \max_{0 \leq i < j \leq n} |Z_l+1|^2 \max_{1 \leq i \leq n} |\Xi_i,l|^2 \ll (\xi_l q_{l+1})^2.
$$

Consider the 3-dimensional fundamental volume $\det(\Gamma_l)$ of the lattice spanned by $z_{l-1}, z_l, z_{l+1}$. Denote by $M_{i,j,k}$ the $3 \times 3$ minors of the matrix

$$
\begin{pmatrix}
Z_{l-1} & \Xi_{1,l-1} & \cdots & \Xi_{n,l-1} \\
Z_l & \Xi_{1,l} & \cdots & \Xi_{n,l} \\
Z_{l+1} & \Xi_{1,l+1} & \cdots & \Xi_{n,l+1}
\end{pmatrix}
$$

By Lemma 3 we have

$$
\det(\Gamma_l)^2 \ll \sum_{0 \leq i < j < k \leq n} M_{i,j,k}^2 \ll \max_{0 \leq i < j < k \leq n} M_{i,j,k}^2 \ll |Z_{l+1}^2 \Xi_{l,l-1}|^2 \ll |q_{l+1} \xi_l \xi_l-1|^2.
$$

We now prove the lower bound for $\det(\Lambda_l)$. Consider the symmetric convex body

$$
\Pi = \{z \mid |z_0| < q_{l+1}, \max_{1 \leq j \leq n} |z_0 \theta_i - z_i| < \xi_l\}
$$

and the intersection $\mathcal{P} = \Pi \cap \langle z_l, z_{l+1} \rangle$. The intersection $\mathcal{P} \cap \langle z_l, z_{l+1} \rangle$ is reduced to zero by definition of the best approximation. Hence Minkowski’s first convex body theorem ensures that for the two-dimensional convex set $\mathcal{P}$ we have

$$
\text{area}(\mathcal{P}) \leq 4 \det(\Lambda_l).
$$

The intersection of $\mathcal{P}$ with the coordinate hyperplane $\{z_0 = 0\}$ is an interval with endpoints $A$ and $B$ of length $|AB| \geq 2\xi_l$. So $\mathcal{P}$ contains a polygon $\mathcal{P}' \subset \mathcal{P}$ with vertices $A, B, -z_{l+1}, z_{l+1}$. It is clear that the Euclidean distance between the point $z_{l+1}$ and the line $AB$ is greater than $q_{l+1}$. We deduce the lower bound for the area of $\mathcal{P}'$

$$
\text{area}(\mathcal{P}') \geq 4q_{l+1}\xi_l.
$$

This yields $q_{l+1}\xi_l \leq \det(\Lambda_l)$. 

**Notation** We denote by calligraphic letter $\mathcal{S}$ the sets of best approximation vectors $\{z_k, \ldots, z_m\}$. Given such a set $\mathcal{S}$, we denote by Greek letter $\Gamma = \langle z_k, \ldots, z_m \rangle_{\mathbb{Z}}$ the lattice spanned by its elements, and by bold Roman letter $\mathbf{S} = \langle z_k, \ldots, z_m \rangle_{\mathbb{R}}$ the rational subspace spanned over $\mathbb{R}$. Finally, we denote with Gothic letter $\mathfrak{S}$ the underlying lattice of integer points $\mathfrak{S} = \mathcal{S} \cap \mathbb{Z}^n$. Note that $\Gamma \subset \mathfrak{S}$. We should note that two- and three-dimensional objects play a special role in our proofs (see e.g. Lemma 2 and Lemma 4). Therefore, if our objects are 2-dimensional, we rather use the letters $L, \Lambda, L$ and $L$, following notation of previous papers [5, 15, 16, 17, 18] dealing with low-dimensional cases. For certain sets $\mathcal{S}$ of consecutive best approximation vectors we will use the word pattern. For example three successive independent best approximation vectors $z_{l-1}, z_l, z_{l+1}$ form a simplest pattern. More complicated patterns may consist of combinations of triples of successive best approximation vectors connected by certain rules. If a pattern $\mathcal{S}$ is the union of say four patterns $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and $\mathcal{S}_4$, we denoted it by $\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2 - \mathcal{S}_3 - \mathcal{S}_4$. If moreover the two patterns $\mathcal{S}_2$ and $\mathcal{S}_3$ generate the same rational subspace, we denoted by $\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2 \equiv \mathcal{S}_3 - \mathcal{S}_4$. Finally, if the rational subspaces generated by $\mathcal{S}_1$ and $\mathcal{S}_2$ have intersection $Q$ and $\mathcal{Q} = Q \cap \mathbb{Z}^n$ is its lattice of integer points, we denote it by either $\mathcal{S}_1 - Q \mathcal{S}_2$ or $\mathcal{S}_1 - \mathcal{Q} \mathcal{S}_2$.

### 2.2 Key lemma

The following lemma plays a key role in the proof of Theorem 1.

**Lemma 4** ($\Gamma_+ - \Gamma_-$). *In the context of simultaneous Diophantine approximation, consider $(z_l)_{l \in \mathbb{N}}$ the sequence of best approximations to the point $\theta \in \mathbb{R}^n$. Suppose that $k > \nu$ and triples

$$S_- := \{z_{\nu-1}, z_\nu, z_{\nu+1}\} \quad \text{and} \quad S_+ := \{z_{k-1}, z_k, z_{k+1}\}$$

consist of linearly independent consecutive best approximation vectors. Suppose that

$$\langle z_\nu, z_{\nu+1} \rangle_{\mathbb{Z}} = \langle z_{k-1}, z_k \rangle_{\mathbb{Z}} =: \Lambda. \quad (16)$$

and consider the three-dimensional lattices

$$\mathfrak{G}_- = \langle z_{\nu-1}, z_\nu, z_{\nu+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^n, \quad \text{and} \quad \mathfrak{G}_+ = \langle z_{k-1}, z_k, z_{k+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^n.$$  

Suppose that for positive $s$ and $t$ the following estimate holds

$$(\det \mathfrak{G}_-)^s (\det \mathfrak{G}_+)^t \gg \det \Lambda. \quad (17)$$
Suppose that the index of our vectors are large enough so that for \( \alpha < \hat{\lambda}(\theta) \).

\[ \xi_j \leq q_j^{-\alpha} \quad \text{for} \quad j = \nu - 1, \nu, k - 1, k. \tag{18} \]

Define

\[ g(s,t) = \frac{\alpha s}{(1 - \alpha)(s - w(s,t))} = \frac{\alpha(t + w(s,t) - 1) - w(s,t) + 1}{(1 - \alpha)t}. \tag{19} \]

where the second equality comes from \( w(s,t) \in (0,1) \) being the root of the equation

\[ w^2 - \left( s + 1 + \frac{\alpha}{1 - \alpha}t \right) w + s = 0. \tag{20} \]

Assume the positivity condition

\[ (1 - \alpha)s + t - 1 \geq 0. \tag{21} \]

Then

either \( q_{\nu+1} \gg q_{\nu}^{g(s,t)} \) or \( q_{k+1} \gg q_{k}^{g(s,t)}. \tag{22} \]

When the parameters are \( s = t = 1 \), this lemma directly provides the result for the approximation to 3 numbers (Proof from [17], see subsection 3.1 for details). Parameters \( s \) and \( t \) are needed in higher dimension. We exhibit a range of pairs of triples of consecutive best approximation vectors, denoted by an index, satisfying conditions of Lemma 4. Parameters \( s \) and \( t \) appear with values depending on dimension and the geometry of best approximation vectors that need to be optimize with respect to \( g(s,t) \). To prove Theorem 1, we show inductively that the optimized parameter \( g(s,t) \) is root of the polynomial \( R_{n,\alpha} \) defined by (7) for \( \alpha < \hat{\lambda} \) arbitrarily close to \( \hat{\lambda} \).

**Proof of Lemma 4.** From (20) it follows that \( s > w(s,t) \) and hence \( g > 0 \). Now we use Lemma 2. By (15) together with (16), \( \langle z_{\nu-1}, z_{\nu}, z_{\nu+1} \rangle \subset \Theta_- \) and \( \langle z_{k-1}, z_k, z_{k+1} \rangle \subset \Theta_+ \), the estimate (17) can be rewritten as:

\[ (\xi_{\nu-1}\xi_{\nu}q_{\nu+1})^s(\xi_{k-1}\xi_kq_{k+1})^t \gg (\xi_{\nu}q_{\nu+1})^{w(s,t)}(\xi_{k-1}q_{k})^{1-w(s,t)}. \]

This means that either

\[ (\xi_{\nu-1}\xi_{\nu}q_{\nu+1})^s \gg (\xi_{\nu}q_{\nu+1})^{w(s,t)} \quad \text{[case (a)]} \]

or

\[ (\xi_{k-1}\xi_kq_{k+1})^t \gg (\xi_{k-1}q_{k})^{1-w(s,t)} \quad \text{[case (b)]}. \]

Now we take into account (18). In case (a), we use \( s > w(s,t) \) to deduce that

\[ 1 \ll \xi_{\nu-1}^s(\xi_{\nu}q_{\nu+1})^{s-w(s,t)} \ll q_{\nu}^{-\alpha s}(1-\alpha)(s-w(s,t)) \]

\[ q_{\nu+1} \ll q_{\nu}^{g(s,t)} \]

\[ q_{k+1} \ll q_{k}^{g(s,t)} \]

\[ q_{\nu+1} \gg q_{\nu}^{g(s,t)} \]

\[ q_{k+1} \gg q_{k}^{g(s,t)} \]

\[ \text{for} \quad \alpha < \hat{\lambda} \text{ arbitrarily close to } \hat{\lambda}. \]
and so \( q_{\nu+1} \ll q_{\nu}^{(s,t)} \). In the case (b) we use condition (21) to deduce

\[
w(s, t) \geq 1 - t.
\]  

(23)

Indeed, consider the function

\[
U_{s,t}(w) = w^2 - \left( s + 1 + \frac{\alpha}{1-\alpha} t \right) w + s,
\]

which is a polynomial in \( w \) of degree two. We see that

\[
U_{s,t}(0) = s > 0, \quad U_{s,t}(1) = -\frac{\alpha}{1-\alpha} t < 0.
\]

Moreover by (21) we have

\[
U_{s,t}(1 - t) = \frac{t}{1-\alpha}((1-\alpha)s + t - 1) \geq 0
\]

and \( w(s, t) \in (0, 1) \) is a root of equation \( U_{s,t}(w(s, t)) = 0 \). So we get (23). Now by means of (23) we get

\[
1 \ll \xi_k^{t+w(s,t)-1} \xi_k^t q_k^{w(s,t)-1} \ll q_k^{w(s,t)-1-\alpha(t+w(s,t)-1)} q_{k+1}^{t(1-\alpha)}
\]

and \( q_{k+1} \ll q_k^{g(s,t)} \).

\[\square\]

2.3 About the values of \( g(s, t) \)

This subsection is rather technical and deals with some properties of \( g(s, t) \). First of all we should note that the value of \( g = g(s, t) \) defined in Lemma 4 satisfies the relation

\[
g^2 - \left( \frac{\alpha}{1-\alpha} + \frac{1-s}{t} \right) g - \frac{s}{t} \frac{\alpha}{1-\alpha} = 0.
\]  

(24)

Indeed, equation (24) immediately follows from (19) and (20).

Then we should point out that if either \( s \) or \( t \) is 1, we can use (24) to express back the value of the other parameter \( s \) or \( t \) in terms of the value \( g = g(s, t) \) defined in (19). Namely, we have the following equalities which are equivalent to (19) in the special cases \( s = 1 \) or \( t = 1 \):

\[
s = \frac{g^2 - \frac{\alpha}{1-\alpha} g - g}{\frac{\alpha}{1-\alpha} - g}, \quad \text{for} \quad g = g(s, 1),
\]  

(25)

\[
t = \frac{\frac{\alpha}{1-\alpha}}{g(g - \frac{\alpha}{1-\alpha})}, \quad \text{for} \quad g = g(1, t),
\]  

(26)

\[
s = \frac{g^2 - \frac{\alpha}{1-\alpha} g - \frac{\alpha}{1-\alpha}}{g - \frac{\alpha}{1-\alpha}} = \frac{R_{3,\alpha}(g)}{R_{2,\alpha}(g)}, \quad \text{for} \quad g = g(1-s, 1),
\]  

(27)

\[
t = \frac{g^2 - \frac{\alpha}{1-\alpha} g - \frac{\alpha}{1-\alpha}}{g(g - \frac{\alpha}{1-\alpha})} = \frac{R_{3,\alpha}(g)}{gR_{2,\alpha}(g)}, \quad \text{for} \quad g = g(1, 1-t).
\]  

(28)
2.4 About polynomials $R_{n,\alpha}(x)$ and $R_{n,\alpha}^*(x)$

To continue with our exposition we need to establish some further properties of polynomials $R_{n,\alpha}(x)$, $R_{n,\alpha}^*(x)$ defined in (7) and (8).

**Proposition 1.** The polynomials $R_{n,\alpha}(x)$ and $R_{n,\alpha}^*(x)$ can be defined inductively for all $n \geq 2$ in the following way:

\[
\begin{align*}
R_2,\alpha(x) &= x - \frac{\alpha}{1-\alpha} \\
R_{n+1,\alpha}(x) &= xR_{n,\alpha}(x) - \frac{\alpha}{1-\alpha} \\
R_2,\alpha^*(x) &= x + (1 - \alpha^*) \\
R_{n+1,\alpha}^*(x) &= R_{n,\alpha}^*(x) + x^n.
\end{align*}
\]

(29)

The result of the proposition above follows from easy calculations.

Recall that by $G(n, \alpha)$ we have denoted the unique real positive root of $R_{n,\alpha}(x)$ and by $G^*(n, \alpha^*)$ the unique positive root of $R_{n,\alpha}^*(x)$.

**Proposition 2.** The values $G(n, \alpha)$ and $G^*(n, \alpha^*)$ are decreasing functions in $n$.

**Proof.** Since $G^*(n, \alpha^*)$ is the unique positive root of $R_{n,\alpha}^*$, and $R_{n,\alpha}^*(x) \to x \to \infty \infty$, it follows from $R_{n+1,\alpha}^*(G^*(n, \alpha^*)) = G^*(n, \alpha^*)^n > 0$. The proof is analogous for $G(n, \alpha)$.

**Proposition 3.** Suppose that $g = G(n, \alpha)$ is the positive root of the polynomial $R_{n,\alpha}(x)$. Then

\[
\frac{\alpha}{1-\alpha} \leq g \leq \frac{1}{1-\alpha}
\]

(30)

and

\[
g((1-\alpha)g - \alpha) \leq 1
\]

(31)

**Proof.** Inequalities (30) follow from

\[
R_{n,\alpha} \left( \frac{\alpha}{1-\alpha} \right) < 0 < R_{n,\alpha} \left( \frac{1}{1-\alpha} \right).
\]

Calculations show that

\[
(1-\alpha) \cdot 1 + \left(1 - \frac{R_{3,\alpha}(g)}{gR_{2,\alpha}(g)}\right) - 1 \geq 0
\]

(32)

and this is equivalent to (31). To see this one should take into account the right inequality from (30).

**Proposition 4.** Suppose that $g^* = G(n, \alpha^*)$ is the positive root of the polynomial $R_{n,\alpha}^*(x)$. Then

\[
\frac{\alpha^* - 1}{\alpha^*} < 1 \leq g^* \leq \alpha^* - 1.
\]

(33)
2.5 Schmidt’s inequality on heights

The proof of Theorem 1 essentially relies on Lemma 4 as well as on Schmidt’s inequality on height (see [24], in fact this inequality was already used in the last section in [17]). It provides the setting to apply Lemma 4 for different parameters \((s, t)\) to be determined later.

Proposition 5 (Schmidt’s inequality). Let \(A, B\) be two rational subspaces in \(\mathbb{R}^n\), we have

\[
H(A + B) \cdot H(A \cap B) \ll H(A) \cdot H(B).
\]

(34)

where the height \(H(A)\) is the fundamental volume of the lattice of integer points \(\det(\mathfrak{A}) = \det(A \cap \mathbb{Z}^n)\).

3 Examples: simultaneous approximation to three and four numbers.

In this section, we describe in details the proofs in the cases of simultaneous approximation to three and four numbers.

An example for approximation by one linear form will be presented in Section 5.3.2.

3.1 Simultaneous approximation to three numbers

Consider \(\theta \in \mathbb{R}^3\) with \(\mathbb{Q}\)-linearly independent coordinates with 1. Consider a sequence \((z_l)_l \in \mathbb{N}\) of best approximations vectors to \(\theta\). Recall that as we consider simultaneous approximation, the sequence \((z_l)_l \in \mathbb{N}\) spans the whole space \(\mathbb{R}^4\).

Lemma 5. For arbitrarily large indices \(k_0\), there exists indices \(k > \nu > k_0\) and triples of consecutive best approximation vectors

\[
S_- := \langle z_{\nu-1}, z_{\nu}, z_{\nu+1} \rangle \quad \text{and} \quad S_+ := \langle z_{k-1}, z_k, z_{k+1} \rangle
\]

consisting of linearly independent vectors such that

\[
S_- \cap S_+ =: \Lambda = \langle z_{\nu}, z_{\nu+1} \rangle_{\mathbb{Z}} = \langle z_{k-1}, z_k \rangle_{\mathbb{Z}} \quad \text{and} \quad (S_- \cup S_+)_{\mathbb{R}} = \mathbb{R}^4,
\]

(35)

where

\[
S_- := \langle z_{\nu-1}, z_{\nu}, z_{\nu+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^n \quad \text{and} \quad S_+ := \langle z_{k-1}, z_k, z_{k+1} \rangle_{\mathbb{R}} \cap \mathbb{Z}^n.
\]

This was proved in [17].

Denote by \(S_4\) the pattern of best approximation vectors described in Lemma 5 (see Figure 1 ). Lemma 5 ensures that the pattern \(S_4\) suites the first conditions to apply Lemma 4 for arbitrarily large indices.
Here we chose $k_0$ sufficiently large for (18) to hold. Schmidt’s inequality (34) provides (17) with parameters $s = t = 1$. Inequality (21) is obvious.

Lemma 4 provides that for any $\alpha < \hat{\lambda}(\theta)$,

$$q_{l+1} \gg q_l^{g_\alpha}$$

for $l = \nu$ or $k$, where $g_\alpha$ is solution of the equation (24) with $s = t = 1$. Namely

$$g_\alpha^2 - \frac{\alpha}{1 - \alpha} g_\alpha - \frac{\alpha}{1 - \alpha} = R_{3,\alpha}(g_\alpha) = 0,$$

which provides

$$g_\alpha = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2(1 - \alpha)}.$$

Hence for every $\alpha < \lambda(\theta)$, we have

$$\frac{\lambda(\theta)}{\hat{\lambda}(\theta)} \geq g_\alpha = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2(1 - \alpha)}.$$

We deduce the lower bound (1).

We now explain how to obtain the pattern of best approximation vectors in Lemma 5. It is the basic step for a more general construction in higher dimension.

**Proof of Lemma 5.** Figure 1 may be usefull to understand the construction. Consider $(z_l)_{l \in \mathbb{N}}$ a sequence of best approximation vectors to $\theta \in \mathbb{R}^3$, and an arbitrarily large index $k_0$. Since $(z_l)_{l \geq k_0}$ spans a 4-dimensional subspace, we can define $k$ to be the smallest index such that

$$\dim \langle z_{k_0}, z_{k_0+1}, \ldots, z_k, z_{k+1} \rangle_{\mathbb{R}} = 4.$$ 

Note that by minimality, $z_{k+1}$ is not in the 3-dimensional subspace spanned by $(z_l)_{k_0 \leq l \leq k}$.

In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $z_{k-1}, z_k, z_{k+1}$ are linearly independent. Set $\nu > k_0$ to be the largest index such that

$$\dim \langle z_{\nu-1}, z_{\nu}, \ldots, z_{k}, z_{k+1} \rangle_{\mathbb{R}} = 4.$$ 

Note that by maximality, $z_{\nu-1}$ is not in the 3-dimensional subspace spanned by $(z_l)_{\nu \leq l \leq k+1}$.

In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors $z_{\nu-1}, z_{\nu}, z_{\nu+1}$ are linearly independent. Moreover, combining both observations we deduce that the lattice

$$\Lambda := \langle z_{\nu}, z_{\nu+1}, \ldots, z_{k-1}, z_k \rangle_{\mathbb{R}} \cap \mathbb{Z}^4 = \langle z_{\nu}, z_{\nu+1} \rangle_{\mathbb{Z}} = \langle z_{k-1}, z_k \rangle_{\mathbb{Z}}$$

is 2-dimensional, and is spanned by two consecutive best approximation vectors (see Lemma 1). Hence, the considered indices $\nu$ and $k$ provide 6 best approximation vectors satisfying Lemma 5.
3.2 Simultaneous approximation to four numbers

In the case of simultaneous approximation to four numbers, we select a pattern $S_5$ of best approximation vectors that combines two patterns $S_4$ coming from Lemma 5. This is the first step of the induction for arbitrary dimension, where we combine two patterns of lower dimension. Thus, it is an enlightening example. Note that in this simple case, a proper choice of parameters was made in [5, equalities after formula (13) from the case $i(\Theta) = 1]$. Consider $\theta \in \mathbb{R}^4$ with $\mathbb{Q}$-linearly independent coordinates with 1. Consider $(z_l)_{l \in \mathbb{N}}$ a sequence of best approximation vectors to $\theta$.

Lemma 6. Let $k_0$ be an arbitrarily large index. There exists indices $k_0 < r_0 < r_1 \leq r_2 < r_3$ such that the following holds.

1. The triples of consecutive best approximation vectors $$S_{r_i} := \{z_{r_i-1}, z_{r_i}, z_{r_i+1}\}, \quad 0 \leq i \leq 3$$ consist of linearly independent vectors spanning a 3-dimensional subspace $S_{3,i} := \langle S_{r_i} \rangle_{\mathbb{R}}$.

2. The two triples of consecutive best approximation vectors $S_{r_1}$ and $S_{r_2}$ generate the same rational subspace $$Q := S_{3,1} = S_{3,2}.$$

3. The pairs of consecutive best approximation vectors $z_{r_0}, z_{r_0+1}$ and $z_{r_1-1}, z_{r_1}$ span the same 2-dimensional lattice $$\Lambda_0 := \langle z_{r_0}, z_{r_0+1} \rangle_{\mathbb{Z}} = \langle z_{r_1-1}, z_{r_1} \rangle_{\mathbb{Z}} = S_{3,0} \cap S_{3,1} \cap \mathbb{Z}^5.$$

Figure 1: All best approximation vectors with index between $\nu$ and $k$ lie in the 2-dimensional lattice $\Lambda$. The four bold vectors are linearly independent and span the whole space.
4. The pairs of consecutive best approximation vectors $z_{r_2}, z_{r_2+1}$ and $z_{r_3-1}, z_{r_3}$ span the same 2-dimensional lattice

$$\Lambda_1 := \langle z_{r_2}, z_{r_2+1} \rangle_Z = \langle z_{r_3-1}, z_{r_3} \rangle_Z = S_{3,2} \cap S_{3,3} \cap \mathbb{Z}^5.$$ 

5. Both quadruples of best approximation $\{z_{r_0-1}, z_{r_0}, z_{r_0+1}, z_{r_1+1}\}$ and $\{z_{r_2-1}, z_{r_3-1}, z_{r_3}, z_{r_3+1}\}$ consist of linearly independent vectors.

6. The whole space $\mathbb{R}^5$ is spanned by $z_{r_0-1}, z_{r_0}, z_{r_0+1}, z_{r_1+1}, z_{r_3+1}$ that is

$$\langle z_{r_0-1}, z_{r_0}, z_{r_0+1}, z_{r_1+1}, z_{r_3+1} \rangle_{\mathbb{R}} = \langle S_{3,0} \cup Q \cup S_{3,2} \rangle_{\mathbb{R}} = \mathbb{R}^5.$$ 

We discuss the meaning of the lemma, and apply it to the proof of the main result for simultaneous approximation to four numbers. The proof is postponed at the end of the section.

The 5-dimensional pattern described in Lemma 6 is denoted by

$$S_5 : S_{3,0} - S_{3,1} = S_{3,2} - S_{3,3}$$

till the end of the section. Note that it consists of two 4-dimensional patterns

$$S_{4,0} : S_{3,0} - S_{3,1}$$

given by indices $\nu = r_0$ and $k = r_1$ in Lemma 5 and

$$S_{4,1} : S_{3,2} - S_{3,3}$$

given by indices $\nu = r_2$ and $k = r_3$ in Lemma 5. These two 4-dimensional patterns $S_{4,0}$ and $S_{4,1}$ intersect on the 3-dimensional subspace $Q$. Thus,

$$S_5 : S_{4,0} - S_{4,1}.$$ 

For the pattern $S_5$, Schmidt’s inequality (34) provides

$$\det S_{3,0} \det \Omega \det S_{3,3} \gg \det \Lambda_0 \det \Lambda_1$$

where $\mathfrak{S}_{i,j} = S_{i,j} \cap \mathbb{Z}^5$ and $\Omega = S_{3,1} = S_{3,2}$. It can be rewritten as

$$\frac{\det \mathfrak{S}_{3,0} (\det \mathfrak{S}_{3,1})^x}{\det \Lambda_{0}} \cdot \frac{(\det \mathfrak{S}_{3,2})^{1-x} \det \mathfrak{S}_{3,3}}{\det \Lambda_{2}} \gg 1$$

with arbitrary $x \in (0, 1)$. This means that

either

$$\frac{\det \mathfrak{S}_{3,0} (\det \mathfrak{S}_{3,1})^x}{\det \Lambda_{0}} \gg 1$$

or

$$\frac{(\det \mathfrak{S}_{3,2})^{1-x} \det \mathfrak{S}_{3,3}}{\det \Lambda_{2}} \gg 1.$$
Hence conditions (16), (17) and (18) are satisfied either for \((\mathfrak{S}_{3,0}, \mathfrak{S}_{3,1})\) and \((s, t) = (1, x)\) or for \((\mathfrak{S}_{3,2}, \mathfrak{S}_{3,3})\) and \((s, t) = (1 - x, 1)\). For \(g\) satisfying the equation \(R_{4, \alpha}(g) = gR_{3, \alpha}(g) - \frac{\alpha}{1-\alpha} = 0\), we set
\[
x = \frac{\alpha}{g(g - \frac{\alpha}{1-\alpha})} = \frac{R_{3, \alpha}(g)}{g - \frac{\alpha}{1-\alpha}}.
\]

From (26), (27), we deduce that
\[
g = G(4, \alpha) = g(1, x) = g(1 - x, 1).
\]

We should mention that as now \(g = G(4, \alpha)\) is the root of equation \(R_{4, \alpha}(x) = 0\), we have (30). Hence for parameters \((s, t) = (1, x)\) and \((s, t) = (1 - x, 1)\), the positivity condition (21) follows from (30). The first part of Theorem 1 for simultaneous approximation to four numbers follows from Lemma 4.

Here, there is one parameter \(x\) to optimize. In higher dimension, we have many more, and need to compute the optimal values of that parameters inductively.

**Proof of Lemma 6.** Figure 3 may be useful to understand the construction. Consider a sequence \((z_l)_{l \in \mathbb{N}}\) of best approximation vectors to \(\theta \in \mathbb{R}^4\), and an arbitrarily large index \(k_0\). Set \(r_3\) to be the smallest index such that
\[
\dim(\langle z_{k_0}, z_{k_0+1}, \ldots, z_{r_3}, z_{r_3+1} \rangle_{\mathbb{R}}) = 5.
\]
Note that by minimality, \(z_{r_3+1}\) is not in the 4-dimensional subspace spanned by \((z_l)_{k_0 \leq l \leq r_3}\). In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors \(z_{r_3-1}, z_{r_3}, z_{r_3+1}\) are linearly independent and span a 3-dimensional lattice denoted by \(\Gamma_3\). Set \(r_0 > k_0\) to be the largest index such that
\[
\dim(\langle z_{r_0-1}, z_{r_0}, \ldots, z_{r_3}, z_{r_3+1} \rangle_{\mathbb{R}}) = 5.
\]
Note that by maximality, \( z_{r_0-1} \) is not in the 4-dimensional subspace spanned by \( \langle z_l \rangle_{r_0 \leq l \leq r_1+1} \). In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors \( z_{r_0-1}, z_{r_0}, z_{r_0+1} \) are linearly independent and span a 3-dimensional lattice denoted by \( \Gamma_0 \). Moreover, combining both observations we deduce that

\[
Q := \langle z_{r_0}, z_{r_0+1}, \ldots, z_{r_3-1}, z_{r_3} \rangle_{\mathbb{R}}
\]

is a 3-dimensional rational subspace.

Now appears the **induction step**: we apply the same procedure in lower dimension to the two 4-dimensional subspaces

\[
S_{4,0} := \langle z_{r_0-1}, z_{r_0}, \ldots, z_{r_3-1}, z_{r_3} \rangle_{\mathbb{R}} \quad \text{and} \quad S_{4,1} := \langle z_{r_0}, z_{r_0+1}, \ldots, z_{r_3}, z_{r_3+1} \rangle_{\mathbb{R}}.
\]

Note that it gives a proof of Lemma 5.

Set \( r_1 \) to be the smallest index such that

\[
\langle z_{r_0-1}, z_{r_0}, \ldots, z_{r_1}, z_{r_1+1} \rangle_{\mathbb{R}} = S_{4,0}.
\]

Note that by minimality, \( z_{r_1+1} \) is not in the 3-dimensional subspace \( S_{3,0} \) spanned by \( \langle z_l \rangle_{r_0 \leq l \leq r_1} \). In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors \( z_{r_1-1}, z_{r_1}, z_{r_1+1} \) are linearly independent and span a 3-dimensional lattice \( \Gamma_1 \) included in \( Q = S_{3,1} \). By construction, \( r_0 \) is already the largest index such that

\[
\langle z_{r_0-1}, z_{r_0}, \ldots, z_{r_1-1}, z_{r_1} \rangle_{\mathbb{R}} = S_{4,0}.
\]

Hence, \( \langle z_{r_0}, z_{r_0+1}, \ldots, z_{r_1-1}, z_{r_1} \rangle_{\mathbb{Z}} =: \Lambda_0 \) is a 2-dimensional lattice spanned by either \( \langle z_{r_0}, z_{r_0+1} \rangle_{\mathbb{Z}} \) or \( \langle z_{r_1-1}, z_{r_1} \rangle_{\mathbb{Z}} \), and is the intersection \( S_{3,0} \cap S_{3,1} \cap \mathbb{Z}^5 \) (see Lemma 1).

Set \( r_2 \) to be the largest index such that

\[
\langle z_{r_2-1}, z_{r_2}, \ldots, z_{r_3}, z_{r_3+1} \rangle_{\mathbb{R}} = S_{4,1}.
\]

Note that \( z_{r_2-1} \) is not in the 3-dimensional subspace \( S_{3,3} \) spanned by \( \langle z_l \rangle_{r_2 \leq l \leq r_3+1} \). In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors \( z_{r_2-1}, z_{r_2}, z_{r_2+1} \) are linearly independent and span a 3-dimensional lattice \( \Gamma_2 \) included in \( Q = S_{3,1} \). By construction, \( r_3 \) is already the smallest index such that

\[
\langle z_{r_2-1}, z_{r_2}, \ldots, z_{r_3}, z_{r_3+1} \rangle_{\mathbb{R}} = S_{4,1}.
\]

Hence, \( \langle z_{r_2}, z_{r_2+1}, \ldots, z_{r_3-1}, z_{r_3} \rangle_{\mathbb{Z}} =: \Lambda_1 \) is a 2-dimensional lattice spanned by \( \langle z_{r_2}, z_{r_2+1} \rangle_{\mathbb{Z}} \) or \( \langle z_{r_3-1}, z_{r_3} \rangle_{\mathbb{Z}} \), and is the intersection \( Q \cap S_{3,3} \cap \mathbb{Z}^5 \) (see Lemma 1).

Note that we may have \( r_1 = r_2 \). Lattices \( \Gamma_1 \) and \( \Gamma_2 \) may not coincide, but they are both sub-lattices of \( \Omega = Q \cap \mathbb{Z}^5 \).
In Figure 3, the dashed lines should be interpreted as follows. The best approximation vectors \((z_l)_{r_0 \leq l \leq r_1}\) generate the 2-dimensional lattice \(\Lambda_0\). The best approximation vectors \((z_l)_{r_2 \leq l \leq r_3}\) generate the 2-dimensional lattice \(\Lambda_1\). The best approximation vectors \((z_l)_{r_1 - 1 \leq l \leq r_2 + 1}\) generate the 3-dimensional rational subspace \(Q = S_{3,1} = S_{3,2}\). The five bold vectors span the whole space \(\mathbb{R}^5\).

4 Arbitrary dimension

4.1 Two lemmas

Consider \(\theta \in \mathbb{R}^n\) with \(\mathbb{Q}\)-linearly independent coordinates with 1. Consider \((z_l)_{l \in \mathbb{N}}\) a sequence of best approximation vectors to \(\theta\).

**Lemma 7.** Let \(k_0\) be an arbitrarily large index. There exists \(2^{n-2}\) indices \(k_0 < r_0 < r_1, \ldots, r_{2n-2-2} < r_{2n-2-1}\) such that the following holds.

1. The triples of consecutive best approximation vectors

\[ S_{3,l} = \{z_{r_{l-1}}, z_{r_l}, z_{r_{l+1}}\}, \quad 0 \leq l \leq 2^{n-2} - 1 \]

consist of linearly independent vectors spanning a 3-dimensional rational subspace \(S_{3,l}\).

2. For \(4 \leq k \leq n + 1\) and \(0 \leq l \leq 2^{n-k+1} - 1\), denote by \(S_{k,l}\) the set of best approximation vectors

\[ S_{k,l} = \bigcup_{\nu=0}^{2^{k-3}-1} S_{3,2^{k-3}l+\nu}. \]

\(S_{k,l}\) spans the \(k\)-dimensional rational subspace \(S_{k,l}\).
3. The rational subspaces $S_{k,l}$ satisfy the relations
\[
S_{k,2l} \cup S_{k,2l+1} = S_{k+1,l}, \quad S_{k,2l} \cap S_{k,2l+1} = S_{k-1,4l+2} =: Q_{k-1,l}.
\]

In particular, $Q_{2,l}$ is spanned by both $z_{r2l}, z_{r2l+1}$ and $z_{r2l+1-1}, z_{r2l+1}$.

4. The full space $\mathbb{R}^{n+1}$ is spanned by
\[
(z_{r0-1}, z_{r0}, z_{r0+1}, z_{r1+1}, z_{r2+1}, \ldots, z_{r2^{n-3},l+1})\mathbb{R} = (\cup_{l=0}^{2^{n-k+1}-1} S_{k,l}) \mathbb{R}, \quad 3 \leq k \leq n + 1.
\]

In particular, $S_{n+1,0} = \mathbb{R}^{n+1}$.

Here, the first index always denotes the dimension of the considered object. For a given dimension $k$, there are $2^{n-k+1}$ subspaces $S_{k,l}$ and $2^{n-k-1}$ subspaces $Q_{k,l}$ of dimension $k$.

Another important pattern of best approximation vectors which may be useful for the considered problem was already discovered for any dimension in 2013 by V. Nguyen in [19, §2.3] while studying simultaneous approximation to the basis of an algebraic number field and an extra real number.

Lemma 7 coincide with Lemma 5 for the approximation to three numbers and with Lemma 6 for the approximation to four numbers. In the later case, we have $\Lambda_j = \Omega_{2,j}$ for $0 \leq j \leq 1$.

We can partially describe the situation with the binary tree from Figure 4, where each child is included in its parent. In particular, the parent of a given rational subspace $S_{k,l}$ is $S_{k+1,\sigma(l)}$ where $\sigma$ is the usual shift on the binary expansion.

We may write the recursive step of the construction of patterns as follows:
\[
S_{n+1,0} : S_{n,0} - Q_{n-1,0}
\]
where $Q_{n-1,0}$ is a $n-1$ dimensional subspace. For the rational subspaces $S_{n,0}, S_{n,1}$ and $Q_{n-1,0}$ and their underlying lattices $\mathcal{S}_{n,0}, \mathcal{S}_{n,1}$ and $\mathcal{Q}_{n-1,0}$, Schmidt’s inequality (34) provides
\[
\frac{\det \mathcal{S}_{n,0} \cdot \det \mathcal{S}_{n,1}}{\det \mathcal{Q}_{n-1,0}} \gg 1.
\]

This relation enables us to shift the optimization equation in the next dimension as obtained in the next lemma.

**Lemma 8.** Let $n \geq 4$. Consider the pattern of best approximation vectors $S_{n+1,0}$ and its sub-patterns given by Lemma 7. Here as before, $\mathcal{S}_{k,l} = S_{k,l} \cap \mathbb{Z}^{n+1}$ and $\mathcal{Q}_{k,l} = Q_{k,l} \cap \mathbb{Z}^{n+1}$ are the integer points lattices of the rational subspace $S_{k,l}$ and $Q_{k,l}$. Then
\[
\prod_{l=0}^{2^{n-4}-1} \frac{\det (\mathcal{S}_{3,l}) \cdot \det (\mathcal{Q}_{3,l})^{1-y_{n-4}}}{\det (\mathcal{Q}_{2,l})} \left( \frac{\det (\mathcal{Q}_{3,l})^{1-z_{n-4}} \det (\mathcal{S}_{3,l+3})}{\det (\mathcal{Q}_{2,l+1})} \right)^{w_{n-4,l}} \gg 1,
\]
where the parameters $w_{k,l}, w'_{k,l}, y_k$ and $z_k$ are defined inductively as follows.

Parameters $y_0, z_0 \in (0, 1)$ are arbitrary such that

$$0 = y_0 + z_0 - 1$$  \hfill (42)

and then

$$(y_{k+1}, z_{k+1}) = F(y_k, z_k) = \left(\frac{y_k}{y_k + z_k - y_k z_k}, \frac{z_k}{y_k + z_k - y_k z_k}\right),$$  \hfill (43)

$$1 = w_{0,0} = w'_{0,0},$$  \hfill (44)

$$w_{k+1,2l} = w_{k,l}, \quad w_{k+1,2l+1} = (1 - z_k) w'_{k,l}, \quad w'_{k+1,2l} = (1 - y_k) w_{k,l} \quad \text{and} \quad w'_{k+1,2l+1} = w'_{k,l}.$$  \hfill (45)

Furthermore, the parameters satisfy the relations

$$\sum_{l=0}^{2^n-4-1} (2 - y_{n-4}) w_{n-4,l} + (2 - z_{n-4}) w'_{n-4,l} = n - 1,$$  \hfill (46)

$$\sum_{l=0}^{2^n-4-1} w_{n-4,l} + w'_{n-4,l} = n - 2.$$  \hfill (47)
We prove Lemma 7 in subsection 4.3 and then Lemma 8 in subsection 4.4. We first finish the proof of Theorem 1 in the case of simultaneous approximation.

4.2 Proof of Theorem 1

Consider $\theta \in \mathbb{R}^n$ with $\mathbb{Q}$-linearly independent coordinates with 1, and take $\alpha < \hat{\lambda}(\theta)$. Denote by $g$ the unique positive root of $R_{n,\alpha}$ defined by (7).

Choose $y_0$ and $z_0$ in the following way:

$$y_0 = \frac{R_{n-1,\alpha}(g)}{gR_{n-2,\alpha}(g)}, \quad z_0 = \frac{R_{n-1,\alpha}(g)}{R_{n-2,\alpha}(g)}.$$  \hspace{1cm} (48)

Using (29), one can check the condition

$$y_0 + z_0 - 1 = \frac{R_{n-1,\alpha}(g)}{gR_{n-2,\alpha}(g)} - \frac{gR_{n-1,\alpha}(g) - gR_{n-2,\alpha}(g)}{gR_{n-2,\alpha}(g)} = \frac{R_{n,\alpha}(g)}{gR_{n-2,\alpha}(g)} = 0.$$  

By the induction formula (43), we deduce that for every $4 \leq k \leq n$

$$y_{n-k} = \frac{R_{k-1,\alpha}(g)}{gR_{k-2,\alpha}(g)} \quad \text{and} \quad z_{n-k} = \frac{R_{k-1,\alpha}(g)}{R_{k-2,\alpha}(g)}.$$  \hspace{1cm} (49)

Indeed, the formula (49) is satisfied for $n-k=0$. Suppose that it is valid for a certain value of $k$. Then $\frac{z_{n-k}}{y_{n-k}} = g$ and recursive formula (43) gives us $\frac{z_{n-k-1}}{y_{n-k-1}} = g$. It is enough for verifying (43) with $k$ replaced by $k+1$ by means of the first group of equalities from (29).

In particular, we have

$$y_{n-4} = \frac{R_{3,\alpha}(g)}{gR_{2,\alpha}(g)} \quad \text{and} \quad z_{n-4} = \frac{R_{3,\alpha}(g)}{R_{2,\alpha}(g)}.$$  \hspace{1cm} (50)

We consider $g(s, t)$ defined in (19) (Lemma 4) for the parameters

$$s = 1, \; t = 1 - y_{n-4}$$  \hspace{1cm} (50)

and

$$s = 1 - z_{n-4}, \; t = 1.$$  \hspace{1cm} (51)

From (27) and (28), it follows that

$$g = G(n, \alpha) = g(1, 1 - z_{n-4}) = g(1 - y_{n-4}, 1).$$  \hspace{1cm} (52)

Recall that now $g = G(n, \alpha)$ is the root of the polynomial $R_{n,\alpha}(x)$. So the positivity condition (21) for parameters (50) follows from (31). At the same time for parameters (51)
the positivity condition (21) is clearly true.

According to Lemma 8, we have (41) and therefore there exists an index \( 0 \leq l \leq 2^n - 1 \) such that either

\[
\frac{\det(\mathcal{G}_{3,4}) \det(\Omega_{3,l})^{1-y_{n-4}}}{\det(\Omega_{2,2l})} \gg 1 \quad \text{or} \quad \frac{\det(\mathcal{Q}_{3,l})^{1-z_{n-4}} \det(\mathcal{G}_{3,4,l+3})}{\det(\Omega_{2,2l+1})} \gg 1.
\]

To summarize, all the conditions are met to apply Lemma 4 for either

\[
\mathcal{G}_- = \mathcal{G}_{3,4}, \quad \mathcal{G}_+ = \Omega_{3,l}, \quad s = 1, \quad t = 1 - y_{n-4}
\]

or

\[
\mathcal{G}_- = \Omega_{3,l}, \quad \mathcal{G}_+ = \mathcal{G}_{3,4,l+3}, \quad s = 1 - z_{n-4}, \quad t = 1.
\]

Hence, there exists \( \nu \) with \( q_{\nu+1} \gg q_0^l \) and (13) is met, proving Theorem 1.

\[ \square \]

### 4.3 Proof of Lemma 7

Figure 4 may be useful to understand the construction.

Let \( k_0 \gg 1 \). We prove the lemma by induction in the dimension \( n \). Suppose that we can construct a pattern \( S_{m,0} \) of \( 2^{m-3} \) triples of consecutive best approximation vectors given by indices \( k_0 < r_0 < r_1, \ldots, r_{2m-3-2} < r_{2m-3-1} \) spanning a \( m \)-dimensional rational space. Such a construction for \( m = 4, 5 \) holds via Lemmas 5 and 6. This provides the base of induction.

Consider \( (z_l)_{l \in \mathbb{N}} \) a sequence of best approximation vectors spanning a \((m+1)\)-dimensional rational space \( S_{m+1} \). Set \( r_{2m-2-1} \) to be the smallest index such that

\[
\langle z_{k_0}, z_{k_0+1}, \ldots, z_{r_{2m-2-1}}, z_{r_{2m-2-1}+1} \rangle_{\mathbb{R}} = S_{m+1}.
\]

Note that \( z_{r_{2m-2-1}+1} \) is not in the \( m \)-dimensional subspace spanned by \( (z_l)_{k_0 \leq l \leq r_{2m-2}} \). In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors \( z_{r_{2m-2-1}+1}, z_{r_{2m-2-1}}, z_{r_{2m-2-1}+1} \) are linearly independent and span a 3-dimensional subspace denoted by \( S_{3,2m-2-1} \). Set \( r_0 > k_0 \) to be the largest index such that

\[
\langle z_{r_0-1}, z_{r_0}, \ldots, z_{r_{2m-2-1}}, z_{r_{2m-2-1}+1} \rangle_{\mathbb{R}} = S_{m+1}.
\]

Note that \( z_{r_0-1} \) is not in the \( m \)-dimensional subspace spanned by \( (z_l)_{r_0 \leq l \leq r_{2m-2-1}+1} \). In particular, since two consecutive best approximation vectors are linearly independent the three consecutive best approximation vectors \( z_{r_0-1}, z_{r_0}, z_{r_0+1} \) are linearly independent and span a 3-dimensional subspace denoted by \( S_{3,0} \). Moreover, combining both observations we get that

\[
Q_{m-1,0} := \langle z_{r_0}, z_{r_0+1}, \ldots, z_{r_{2m-2-1}-1}, z_{r_{2m-2-1}} \rangle_{\mathbb{R}}
\]

24
is a $m - 1$-dimensional subspace.

We use the induction hypothesis for the two $m$-dimensional subspaces

$$S_m' := \langle z_{r_0-1}, z_{r_0}, \ldots, z_{r_{2m-2}-1}, z_{r_{2m-2}+1} \rangle_{\mathbb{R}} \text{ and } S_m'' := \langle z_{r_0}, z_{r_0+1}, \ldots, z_{r_{2m-2}-1}, z_{r_{2m-2}+1} \rangle_{\mathbb{R}}$$

for $k_0' = r_0 - 1$ and $k_0'' = r_0$ respectively. This provides two patterns $S_m'$ and $S_m''$ of triples of best approximation vectors defined by indices $r_0 \leq r_0' < r_1', \ldots, r_{2m-3} < r_2''$ and $r_0 + 1 \leq r_0'' < r_1''$, $\ldots, r_{2m-3}'' < r_2''$ satisfying the conditions of Lemma 7. A key observation is that by definition of $r_0$, we necessarily have $r_0' = r_0$. Similarly, by definition of $r_{2m-2}$, we necessarily have $r_{2m-2} = r_{2m-3}''$. It follows that both sub-patterns $S_{m-1,1}'$ and $S_{m-1,0}''$ span the rational subspace $Q_{m-1,0}$. Hence, the pattern $S$ defined by the triples given by indices

$$r_i = r_i' \text{ and } r_{i+2m-3} = r_i'' \text{ for } 0 \leq i \leq 2m-3 - 1$$

combining the two sub-patterns $S_m'$ and $S_m''$ satisfies the required properties at the rank $m + 1$.

$$S : S_{m-1,0}' S_{m-1,0}''$$

Since $(z_l)_{l \in \mathbb{N}}$ a sequence of best simultaneous approximation vectors to $\theta \in \mathbb{R}^n$ spans the whole space $\mathbb{R}^{n+1}$, Lemma 7 follows.

Remark. Note that the proof provides a $m$-dimensional pattern for $\theta \in \mathbb{R}^n$ where $m$ is the dimension of the space spanned by its best approximation vectors. Furthermore, note that this construction holds for both simultaneous approximation and approximation by one linear form.

4.4 Proof of Lemma 8

By induction on $k$ we prove a more general formula

$$\prod_{l=0}^{2k-1-1} \left( \frac{\det(\mathcal{G}_{n-k,4l}) \det(\mathcal{G}_{n-k,4l+1})^{1-y_{k-1}}}{\det(\mathcal{G}_{n-k,1,8l+1})} w_{k-1,l} \right) \times \prod_{l=0}^{2k-1-1} \left( \frac{\det(\mathcal{G}_{n-k,4l+2})^{1-y_{k-1}} \det(\mathcal{G}_{n-k,4l+3})}{\det(\mathcal{G}_{n-k,1,8l+3})} w_{k-1,l} \right) \geq 1 \quad (53)$$

If we write it in terms of $\Omega_{i,j} = \mathcal{G}_{i,4j+1} = \mathcal{G}_{i,4j+2}$, we have

$$\prod_{l=0}^{2k-1-1} \left( \frac{\det(\mathcal{G}_{n-k,4l}) \det(\Omega_{n-k,l})^{1-y_{k-1}}}{\det(\Omega_{n-k,1,2l})} w_{k-1,l} \right) \times \prod_{l=0}^{2k-1-1} \left( \frac{\det(\Omega_{n-k,l})^{1-y_{k-1}} \det(\mathcal{G}_{n-k,4l+3})}{\det(\Omega_{n-k,1,2l+1})} w_{k-1,l} \right) \geq 1 \quad (54)$$
Lemma 8 is the latter formula for \( k = n - 3 \).

We call factors of the first product, of the form
\[
\frac{\det (S_{n-k,4l}) \det (Q_{n-k,l})^{1-y_{k-1}}}{\det (Q_{n-k-1,2l})^{w_{k-1,l}}} \]
factors of Type I, and factors of the second product of the form
\[
\frac{\det (Q_{n-k,l})^{1-z_{k-1}} \det (S_{n-k,4l+3})^{w'_{k-1,l}}}{\det (Q_{n-k-1,2l+1})} \]
factors of Type II.

**Base of induction** follows the steps of approximation to four numbers. Namely, Schmidt’s inequality (34) provides
\[
\begin{align*}
\det (S_{n,0}) \det (S_{n,1}) & \gg \det (Q_{n-1,0}) \det (S_{n+1,0}), \\
\det (S_{n-1,0}) \det (S_{n-1,1}) & \gg \det (Q_{n-2,0}) \det (S_{n,0}), \\
\det (S_{n-2,1}) \det (S_{n-1,3}) & \gg \det (Q_{n-2,1}) \det (S_{n,1}).
\end{align*}
\]

(55)

Since \( S_{n+1,0} \) spans the whole space \( \mathbb{R}^{n+1} \), we have \( \det S_{n+1,0} = 1 \) and using the fact that \( \det Q_{n-1,0} = \det S_{n-1,1} = \det S_{n-2,1} \) (by (39)), we get the formula
\[
\frac{\det (S_{n-1,0}) \det (Q_{n-1,0}) \det (S_{n-1,3})}{\det (Q_{n-2,0}) \det (Q_{n-2,1})} \gg 1.
\]

Setting \( w_{0,0} = w'_{0,0} = 1 \) and \( y_0 \) and \( z_0 \) such that \( y_0 + z_0 - 1 = 0 \), we can rewrite
\[
\left( \frac{\det (S_{n-1,0}) \det (Q_{n-1,0})^{1-y_0}}{\det (Q_{n-2,0})} \right)^{w_{0,0}} \left( \frac{\det (Q_{n-1,0})^{1-z_0} \det (S_{n-1,3})}{\det (Q_{n-2,1})} \right)^{w'_{0,0}} \gg 1.
\]

This establishes the expected formula for \( k = 1 \). In the inductive step, Schmidt’s inequality (34) splits each term of the product in two terms involving rational subspaces of lower dimension, and shift the values of the parameters \( y_k \) and \( z_k \).

Indeed, for \( 3 \leq i \leq n + 1 \) and \( 0 \leq j \leq 2^{n+1-i} - 1 \), Schmidt’s inequality provides
\[
\frac{\det (S_{i-1,2j}) \det (S_{i-1,2j+1})}{\det (Q_{i-2,j})} \gg \det (S_{i,j}).
\]

(56)
Inductive step. Assume that (53) holds for some $1 \leq k < n - 3$. In the product (53), there are two types of factors: factors of Type I and of Type II. Each of these factors splits into two factors, one of Type I and one of Type II. We first deal with factors of Type I. For every $0 \leq l \leq 2^{k-1} - 1$, we can apply Schmidt’s inequality (56) with parameters $i = n - k$ and $j = 4l$ and $j = 4l + 1$ respectively to split

\[
\begin{pmatrix}
\frac{\det(\mathcal{S}_{n-k,4l}) \det(\mathcal{S}_{n-k,4l+1})^{1-y_k}}{\det(\Omega_{n-k-1,2l})}^{w_{k,l}} \\
\frac{\det(\mathcal{S}_{n-k-1,8l}) \det(\mathcal{S}_{n-k-1,8l+1})}{\det(\Omega_{n-k-2,4l})}
\end{pmatrix}
\begin{pmatrix}
\frac{\det(\mathcal{S}_{n-k-1,8l+2}) \det(\mathcal{S}_{n-k-1,8l+3})}{\det(\Omega_{n-k-2,4l+1})}^{1-y_k} \\
\frac{\det(\mathcal{S}_{n-k-1,8l+2})^{1-u} \det(\mathcal{S}_{n-k-1,8l+3})}{\det(\Omega_{n-k-2,4l+1})}^{(1-y_k)w_{k,l}}
\end{pmatrix}.
\]

Considering that $Q_{n-k-1,2l} = S_{n-k-1,8l+1} = S_{n-k-1,8l+2}$, for any $u \in (0,1)$ we can write

\[
\begin{pmatrix}
\frac{\det(\mathcal{S}_{n-k-1,8l}) \det(\mathcal{S}_{n-k-1,8l+1})^{u(1-y_k)}}{\det(\Omega_{n-k-2,4l})}^{w_{k,l}} \\
\frac{\det(\mathcal{S}_{n-k-1,8l+2})^{1-u} \det(\mathcal{S}_{n-k-1,8l+3})}{\det(\Omega_{n-k-2,4l+1})}^{(1-y_k)w_{k,l}}
\end{pmatrix} \gg (57).
\]

Similarly, for factors of Type II, for any $v \in (0,1)$, using (56) with $i = n - k$ and $j = 4l + 2$ and $j = 4l + 3$ respectively, and the fact that $Q_{n-k-1,2l+1} = S_{n-k-1,8l+5} = S_{n-k-1,8l+6}$ we get
\[
\left( \frac{\det (S_{n-k,4l+2})^{1-z_k-1} \det (S_{n-k,4l+3})}{\det (\Omega_{n-k,1,2l+1})} \right)^{w'_{k,l}} \ll \left( \frac{\det (S_{n-k,1,8l+4}) \det (S_{n-k-1,8l+5})^{1-v}}{\det (\Omega_{n-k-2,4l+2})} \right)^{(1-z_k)w'_{k,l}}

\times \left( \frac{\det (S_{n-k-1,8l+6})^{v(1-z_k)} \det (S_{n-k-1,8l+7})}{\det (\Omega_{n-k-2,4l+3})} \right)^{w'_{k,l}}.
\]

Combining the splitting of Type I (58) and Type II (59) factors in the induction hypothesis (53), it appears that we should define the parameters \((y_{k+1}, z_{k+1})\) to be solutions of the system in variables \((v, u)\)

\[u(1 - y_k) = 1 - v \quad \text{and} \quad 1 - u = v(1 - z_k).\]

That is

\[y_k = \frac{y_{k+1} + z_{k+1} - 1}{z_{k+1}} \quad \text{and} \quad z_k = \frac{y_{k+1} + z_{k+1} - 1}{y_{k+1}}\]

or equivalently

\[y_{k+1} = \frac{y_k}{y_k + z_k - y_k z_k} \quad \text{and} \quad z_{k+1} = \frac{z_k}{y_k + z_k - y_k z_k}.\]

The last equality coincide with the definition \(F(y, z)\) in (43) and (59) gives formulae (45) for \(w_{k+1,l}\) and \(w'_{k+1,l}\).

This and the parameters (45) establish formula (53) for \(k + 1\).

We now prove the relation (46) and (47) by descending induction, showing that for any \(4 \leq k \leq n\)

\[\sum_{l=0}^{2^n-k-1} \left( (2 - y_{n-k}) w_{n-k,l} + (2 - z_{n-k}) w'_{n-k,l} \right) = n - k + 3, \quad (60)\]

\[\sum_{l=0}^{2^n-k-1} \left( w_{n-k,l} + w'_{n-k,l} \right) = n - k + 2. \quad (61)\]

First, note that

\[w_{0,0} + w'_{0,0} = 2, \quad \text{and} \quad w_{0,0}(2 - y_0) + w'_{0,0}(2 - z_0) = 3,\]

hence we have the base of induction at \(k = n\).

Assume that for some \(4 \leq k \leq n\) (60) and (61) holds. The two sums represent the degrees of determinants that appears respectively at the numerator and at the denominator in (41). The key is to observe the splitting in (57): the new sum for the denominator is the sum

\[\sum_{l=0}^{2^n-k-1} \left( (2 - y_{n-k}) w_{n-k,l} + (2 - z_{n-k}) w'_{n-k,l} \right).\]
from the previous numerator, while at the numerator, the previous denominator is doubled
but we have a cancellation with one denominator. Namely, using the recurrence formula (43)
and (45) for the parameters, we get

\[
\sum_{l=0}^{2^{n-k+1}-1} \left( w_{n-k+1,l} + w'_{n-k+1,l} \right) = \sum_{l=0}^{2^{n-k}-1} \left( w_{n-k+1,2l} + w_{n-k+1,2l+1} + w'_{n-k+1,2l} + w'_{n-k+1,2l+1} \right)
\]

\[
= \sum_{l=0}^{2^{n-k+1}-1} \left( w_{n-k,l} + w'_{n-k,l}(1 - z_{n-k}) + w'_{n-k,l} + w_{n-k,l}(1 - y_{n-k}) \right)
\]

\[
= \sum_{l=0}^{2^{n-k}-1} \left( (2 - y_{n-k})w_{n-k,l} + (2 - z_{n-k})w'_{n-k,l} \right) = n - k + 3,
\]

\[
\sum_{l=0}^{2^{n-k+1}-1} \left( w_{n-k+1,l}(2 - y_{n-k+1}) + w'_{n-k+1,2l+1}(2 - z_{n-k+1}) \right) = \sum_{l=0}^{2^{n-k}-1} \left( w_{n-k+1,2l} + w_{n-k+1,2l+1}(2 - y_{n-k+1}) \right) + \sum_{l=0}^{2^{n-k}-1} \left( w'_{n-k+1,2l} + w'_{n-k+1,2l+1}(2 - z_{n-k+1}) \right)
\]

\[
= \sum_{l=0}^{2^{n-k}-1} \left( w_{n-k,l}(3 - 2y_{n-k}) + w'_{n-k,l}(3 - 2z_{n-k}) \right) = 2(n - k + 3) - (n - k + 2) = n - k + 4.
\]

Hence the result by descending induction. \qed

5 Approximation by one linear form

In this section, we explain how the very same geometry of a sequence of best approximation
vectors provides Theorem 1 for approximation by one linear form. We need to consider a
hyperbolic rotation to get a suitable analogue of the estimates in Lemma 2. For this, we use
Schmidt’s inequalities on heights in a slightly larger context than rational subspaces.

5.1 About Schmidt’s inequalities on heights

As stated in Proposition 5, Schmidt’s inequality deals with the intersections of rational sub-
spaces with the lattice \( \mathbb{Z}^d \) of integer points. Here we need to deal with a more general situation.
Let \( \Lambda \subset \mathbb{R}^d \) be a complete lattice, that plays the role of integer points. Let \( M \subset \mathbb{R}^d \) be a
subspace, it is called \( \Lambda \)-rational if the lattice \( \mathfrak{M} = M \cap \Lambda \) is complete, i.e. if \( \langle \mathfrak{M} \rangle_\mathbb{R} = M \).

Lemma 9. The intersection of two \( \Lambda \)-rational subspaces is \( \Lambda \)-rational.
The proof is the same as for rational subspaces, and use the description of subspaces by their orthogonal vectors.

**Definition.** Given a fixed complete lattice \( \Lambda \), we define the height \( H_\Lambda \) of a \( \Lambda \)-rational subspace \( M \) to be the fundamental volume

\[
H_\Lambda(M) = \det(\mathfrak{M}) = \det(M \cap \Lambda)
\]

of the \( \Lambda \)-points of \( M \).

**Proposition 6** (Schmidt’s inequality). Let \( \Lambda \) be a complete lattice. Let \( M_1, M_2 \) be two \( \Lambda \)-rational subspaces in \( \mathbb{R}^d \), we have

\[
H_\Lambda(M_1 + M_2) \cdot H_\Lambda(M_1 \cap M_2) \ll H_\Lambda(M_1) \cdot H_\Lambda(M_2).
\]

(62)

**Proof.** Let \( N = M_1 \cap M_2 \) and denote

\[
dim M_j = m_j, \ j = 1, 2, \quad dim N = n, \quad dim (M_1 + M_2) = f.
\]

Then

\[
f = m_1 + m_2 - n.
\]

Consider the orthogonal complement \( K \) to \( N \), \( \dim K \cap M_j = m_j - n \). Let \( N \) be a basis in \( N \). For \( j = 1, 2 \), we take a collection of vectors \( \mu_j \subset M_j \) in such a way that the collection \( M_j = N \cup \mu_j \) forms a basis of \( M_j \). This means that we complete \( N \) by \( \mu_j \) to a basis of \( M_j \). Let \( \mu_j^* \) be a collection of independent vectors in \( K \cap M_j \) which can be obtained from \( \mu_j \) by orthogonal projection on \( K \) parallel to \( N \). Let us consider the parallelepiped \( \Pi \subset M_1 + M_2 \) generated by all the vectors from the collection \( N \cup \mu_1 \cup \mu_2 \), and the parallelepiped \( \Pi^* \subset M_1 + M_2 \) generated by all the vectors from the collection \( N \cup \mu_1^* \cup \mu_2^* \). We consider also the parallelepipeds

\[
\Pi_N, \quad \Pi_1^*, \quad \Pi_2^*
\]

generated by the collections of independent vectors

\[
N, \quad \mu_1^*, \quad \mu_2^*
\]

correspondingly. Also we need to consider parallelepipeds

\[
\Pi_{M_j}, \ j = 1, 2
\]

corresponding to the collections

\[
M_j = N \cup \mu_j, \ j = 1, 2.
\]

It is clear that

\[
\text{vol}_{m_j} \Pi_{M_j} = \text{vol}_{m_j-n} \Pi_j^* \cdot \text{vol}_n \Pi_N, \ j = 1, 2
\]

30
and

$$\text{vol}_f \Pi = \text{vol}_f \Pi^* \leq \text{vol}_{m_1-n} \Pi_1^* \cdot \text{vol}_{m_2-n} \Pi_2^* \cdot \text{vol}_n \Pi_N = \frac{\text{vol}_{m_1} \Pi_{M_1} \cdot \text{vol}_{m_2} \Pi_{M_2}}{\text{vol}_n \Pi_N},$$

where $\text{vol}_k(\cdot)$ stands for $k$-dimensional volume. So

$$\text{vol}_f \Pi \cdot \text{vol}_n \Pi_N \leq \text{vol}_{m_1} \Pi_{M_1} \cdot \text{vol}_{m_2} \Pi_{M_2},$$

To obtain (62) we need to apply the last inequality in the case when $\mathcal{N}$ is a basis of the lattice $\Lambda \cap (\mathbf{M}_1 \cap \mathbf{M}_2)$ and $\mu_1, \mu_2$ complete $\mathcal{N}$ to the basises of lattices $\Lambda \cap \mathbf{M}_1$ and $\Lambda \cap \mathbf{M}_2$ correspondingly.

5.2 Hyperbolic rotation

Given a sequence $(z_l)_{l \in \mathbb{N}} = (q_{1,l}, \ldots, q_{n,l}, a_l)$ of best approximations to a point $\theta \in \mathbb{R}^n$ for the approximation by one linear form, we can extract a subsequence satisfying Lemma 7. For approximation by one linear form, it may happen that the sequence of best approximation vectors spans a subspace of dimension $m < n + 1$ in $\mathbb{R}^{n+1}$ (see [1]). In this case, Proposition 2 provides that Theorem 1 holds with the stronger lower bound $G^*(m, \hat{\omega}(\theta))$ instead of $G^*(n, \hat{\omega}(\theta))$. See the remark after the proof of Lemma 7. In the sequel, we suppose that the best approximation vectors span the full space. In particular the coordinates $\theta_1, \ldots, \theta_n$ are linearly independent with 1.

Consider the matrix

$$L = \begin{pmatrix}
1 & \cdots & 0 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
\theta_1 & \cdots & \theta_n & 1
\end{pmatrix}.$$ 

We can consider the sequence of best approximation as points of the lattice $\mathcal{L} = L\mathbb{Z}^{n+1}$ with

$$(\tilde{z}_l)_{l \in \mathbb{N}} = L.(z_l)_{l \in \mathbb{N}} \in \mathcal{L}.$$ 

Here, we simply replace the last coordinate $a_l$ by the error of approximation $L_l$.

Consider a large parameter $T$, and the hyperbolic rotation

$$\mathcal{G}_T = \begin{pmatrix}
T^{-1} & \cdots & 0 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
0 & \cdots & T^{-1} & 0 \\
0 & \cdots & 0 & T^n
\end{pmatrix}.$$ 

The lattice $\mathcal{L}' = \mathcal{G}_T \mathcal{L}$ is complete since the determinants of $L$ and $\mathcal{G}_T$ are 1.
Consider the sequence \((z'_l)_{l \in \mathbb{N}} = G_T L(z_l)_{l \in \mathbb{N}} \in \mathcal{L}'\) where
\[
z'_l = \langle z'_1, \ldots, z'_{n,l}, z'_{n+1,l} \rangle = \langle T^{-1} q_{1,l}, \ldots, T^{-1} q_{n,l}, T^n L_l \rangle.
\]

For best approximation by one linear form we defined \(M_l = \max_{1 \leq i \leq n} |z_i,l|\), and after hyperbolic rotation we have
\[
\max_{1 \leq i \leq n} |z_i,l| \leq M_l T^{-1}.
\]

Since we assume that the best approximation vectors \((z_l)_{l \in \mathbb{N}}\) span the full space \(\mathbb{R}^{n+1}\), we can apply Lemma 7 to \((z_l)_{l \in \mathbb{N}}\) and obtain a set of indices \((r_k)_{0 \leq 2^{n-2}-1}\). Denote
\[
S'_3,l = \{ z'_1, z'_2, z'_{r_1+1} \} = G_T L S_{3,l}, \quad 0 \leq l \leq 2^{n-2} - 1
\]
and for \(4 \leq k \leq n+1\) and \(0 \leq l \leq 2^{n-k+1} - 1\), denote by \(S'_k,l\) the set of best approximation vectors
\[
S'_k,l = \bigcup_{r=0}^{2^{k-3}-1} S'_{3,2^{k-3}l+r} = G_T L S_{k,l}.
\]

Since \(G_T\) and \(L\) have determinant 1, these sets satisfy the properties of linear independence and inclusion listed in Lemma 7.

Further in the proof of Theorem 1, we need an estimate on the fundamental volumes of the lattices \(\Lambda'_k = \langle z'_k, z'_{k+1} \rangle\) and \(\Gamma'_k = \langle z'_{k-1}, z'_k, z'_{k+1} \rangle\) spanned by consecutive independent vectors \(z'_l\). For large \(T\), we can follow a similar proof as in Lemma 2.

**Lemma 10.** Fix an index \(k\). Let \(T\) be large enough so that
\[
T > M_{k+1} \quad \text{and} \quad T > L_{k-1}^{-1/n}. \quad (63)
\]

Given two consecutive and linearly independent best approximation vectors \(z_k, z_{k+1}\), the fundamental volume \(\det \Lambda'_k\) satisfies
\[
\det \Lambda'_k \asymp L_k T^n M_{k+1} T^{-1} = L_k M_{k+1} T^{n-1}. \quad (64)
\]

Given three consecutive and linearly independent best approximation vectors \(z_{k-1}, z_k, z_{k+1}\), the fundamental volume \(\det \Gamma'_k\) satisfies
\[
\det \Gamma'_k \ll L_{k-1} T^n M_{k+1} T^{-1} = L_{k-1} M_k M_{k+1} T^{n-2}. \quad (65)
\]

**Proof.** For \(T\) satisfying (63), we see that \(z'_k = t \langle T^{-1} q_{1,k}, \ldots, T^{-1} q_{n,k}, T^n L_k \rangle\) satisfies
\[
|T^n L_k| > 1 \quad \text{and} \quad |T^{-1} q_{i,k}| < 1 \quad \text{for} \quad 1 \leq i \leq n. \quad (66)
\]

Consider the \(2 \times (n + 1)\) matrix
\[
\begin{pmatrix}
v'_{1,k} & \cdots & v'_{n,k} & v'_{n+1,k} \\
v'_{1,k+1} & \cdots & v'_{n,k+1} & v'_{n+1,k+1}
\end{pmatrix} = t \begin{pmatrix}
T^{-1} q_{1,k} & \cdots & T^{-1} q_{n,k} & T^n L_k \\
T^{-1} q_{1,k+1} & \cdots & T^{-1} q_{n,k+1} & T^n L_{k+1}
\end{pmatrix} \quad (67)
\]
and the $3 \times (n + 1)$ matrix
\[
\begin{pmatrix}
    z_{1,k-1}' & \cdots & z_{n,k-1}' & z_{n+1,k-1}' \\
    z_{1,k}' & \cdots & z_{n,k}' & z_{n+1,k}' \\
    z_{1,k+1}' & \cdots & z_{n,k+1}' & z_{n+1,k+1}' \\
\end{pmatrix}
= \begin{pmatrix}
    T^{-1}q_{1,k-1} & \cdots & T^{-1}q_{n,k-1} & T^nL_{k-1} \\
    T^{-1}q_{1,k} & \cdots & T^{-1}q_{n,k} & T^nL_k \\
    T^{-1}q_{1,k+1} & \cdots & T^{-1}q_{n,k+1} & T^nL_{k+1} \\
\end{pmatrix}.
\] (68)

The rest of the proof is completely analogous to the proof of Lemma 2. To obtain the upper bounds in (64) and (65) we need to get upper bounds for $2 \times 2$ minors of the matrix (67) and for $3 \times 3$ minors of the matrix (68) by taking into account inequalities (66). This bounds will be of the form

\[
2 \times 2 \text{ minors of } (67) \ll L_k T^n M_{k+1} T^{-1} = L_k M_{k+1} T^{n-1}
\]
and

\[
3 \times 3 \text{ minors of } (68) \ll L_{k-1} T^n M_k T^{-1} M_{k+1} T^{-1} = L_{k-1} M_k M_{k+1} T^{n-2}.
\]

Then application of Lemma 3 gives upper bounds in (64) and (65).

The lower bound for $\det A_k'$ from (64) follows from Minkowski’s first convex body theorem as well, analogously to the argument of the final part of the proof of Lemma 2. One should consider the symmetric convex body

\[
\Pi = \left\{ z : \max_{1 \leq j \leq n} |z_j| < M_{k+1}, |z_{n+1}| < L_k \right\}
\]
and its image

\[
\mathcal{G}_T \Pi = \left\{ z : \max_{1 \leq j \leq n} |z_j| < \frac{M_{k+1}}{T}, |z_{n+1}| < T^n L_k \right\}.
\]

It is clear that

\[
\mathcal{G}_T \Pi \cap L' = \Pi \cap L = \{0\}.
\]

Consider the section

\[
P = \mathcal{G}_T \Pi \cap \langle z_k', z_{k+1}' \rangle_{\mathbb{R}},
\]
by means of Minkowski’s theorem, we obtain the upper estimate for its area

\[
\text{area } P \leq 4A_k'.
\]

The lower bound

\[
\text{area } P \gg L_k M_{k+1} T^{n-1}
\]
comes from (63).

\[ \square \]

Here, we need a large parameter $T$ to obtain a good upper bound for the minors. If $T = 1$, such upper bounds are false.

**Remark.** In the case of a lattice generated by both $\Lambda := \langle z_{\nu}', z_{\nu+1}' \rangle_{\mathbb{Z}} = \langle z_{k-1}', z_k' \rangle_{\mathbb{Z}}$ we have

\[
\det \Lambda \asymp L_\nu M_{\nu+1} T^{n-1} \asymp L_{k-1} M_k T^{n-1}.
\] (69)
5.3 Proof of Theorem 1 for approximation by one linear form

The proof in the case of approximation by one linear form follow the same steps as in the case of simultaneous approximation. Hence, we give a sketch of the proof in general, but to make the ideas of the proof clearer, in Section 5.3.2 we give a very detailed proof in the simplest case of approximation to 4 numbers. Idea of the argument comes from [18]. Note that by reversing time, we get two inequalities in term of coefficients, and two in term of linear forms.

5.3.1 Proof in any dimension

Consider \( \theta \in \mathbb{R}^n \) with \( \mathbb{Q} \)-linearly independent coordinates with 1, and take \( \alpha^* < \hat{\omega}(\theta) \). Let \( g^* = G^*(n, \alpha^*) \) be the unique positive root of the polynomial \( R_{n,\alpha}^* \) defined in (8), recall (33).

We define for \( 4 \leq k \leq n \) the parameters

\[
z_{n-k} = \frac{R_{k-1,\alpha}^*(g^*)}{R_{k-2,\alpha}^*(g^*)} \quad \text{and} \quad y_{n-k} = \frac{R_{k-1,\alpha}^*(g^*)}{g^* R_{k-2,\alpha}^*(g^*)}
\]

which satisfy the assumptions (42) and (43) of Lemma 8 because of the induction formula (29) and \( R_{n,\alpha}^*(g^*) = 0 \).

Considering a sequence \((z_l)_l \in \mathbb{N}\) of best approximations to a point \( \theta \in \mathbb{R}^n \), we obtain via Lemma 7 a set of indices satisfying good properties. Suppose that \( k_0 \) is large enough so that for \( \alpha^* < \hat{\omega}(\theta) \),

\[
L_j \leq M_{j+1}^{-\alpha^*}, \quad \text{for} \quad j \geq k_0.
\]

For any fixed \( T \gg 1 \), the hyperbolic rotation \((z_l')_l \in \mathbb{N} = \mathcal{G}_T \mathcal{L} \cdot (z_l)_l \in \mathbb{N}\) preserves the property of linear independence, and hence the structure of the pattern of best approximation vectors constructed in Lemma 7. We consider the rotated sets \( S_{k,l}' = \mathcal{G}_T \mathcal{L} \cdot S_{k,l} \), \( Q_{k,l}' = \mathcal{G}_T \mathcal{L} \cdot Q_{k,l} \) from the sets \( S_{k,l} \) and \( Q_{k,l} \) defined in Lemma 7. We denote respectively by \( S_{k,l}' \) and \( Q_{k,l}' \) the lattices of their \( \mathcal{G}_T \mathcal{L} \)-points. Section 5.1 explains that we can modify the proof of Lemma 8 so that

\[
\prod_{l=0}^{2^{n-4}-1} \left( \frac{\det(S_{3,4,l}') \det(Q_{3,l}')^{1-y_{n-4,l}}}{\det(\Omega_{2,2l}')} w_{n-4,l} \right) \left( \frac{\det(Q_{3,l}')^{1-z_{n-4,l}} \det(S_{3,4,l}'+3)}{\det(\Omega_{2,2l+1}')} w_{n-4,l}' \right) \geq 1
\]

where the parameters \( y_{n-4,l}, z_{n-4,l} \) are defined in (70) and \( w_{n-4,l}, w_{n-4,l}' \) are defined by (45) and satisfy (46) and (47).

As for the proof of the analogue of Lemma 4, we express the denominators in two different ways. Indeed,

\[
\begin{align*}
\Omega_{2,2l}' &= \langle z_{l+4}'; z_{l+4l+1}'; z \rangle = \langle z_{l+4l+1}'; z_{l+4l+3}; z \rangle \\
\Omega_{2,2l+1}' &= \langle z_{l+4l+2}; z_{l+4l+4}; z \rangle = \langle z_{l+4l+2}; z_{l+4l+3}; z \rangle
\end{align*}
\]
and we write both det \((\Omega_{2,2l})\) and det \((\Omega_{2,2l+1})\) with their two expressions coming from (69). Given \(s, t \in [0, 1]\), analogously to Lemma 4, we define

\[
g^*(s, t) = \frac{(1 - \alpha^*)s}{(1 - \alpha^*)w^*(s, t) - s} = \frac{(1 - \alpha^*)(1 - w^*(s, t) - t)}{t},
\]

where the second equality comes from \(w^*(s, t) \in [0, 1]\) being the root of the equation

\[
w^2 - \left(1 - t - \frac{s}{\alpha^* - 1}\right)w - \frac{s}{\alpha^* - 1} = 0.
\]

We obtain the analogue of (24) for \(g^*(s, t)\)

\[
1/g^*(s, t)^2 = \left(\frac{1}{\alpha^* - 1} + \frac{1 - t}{s}\right) \cdot \frac{1}{g^*(s, t)} - \frac{t}{s(\alpha^* - 1)} = 0.
\]

from which we deduce the analogue of (27) and (28), that is

\[
s = \frac{R_{3,\alpha^*}(g^*)}{R_{2,\alpha^*}(g^*)}, \quad \text{for} \quad g^* = g^*(1 - s, 1),
\]

\[
t = \frac{R_{3,\alpha^*}(g^*)}{g^*R_{2,\alpha^*}(g^*)}, \quad \text{for} \quad g^* = g^*(1, 1 - t).
\]

Consider

\[
w_1 = w^*(1, 1 - y_{n-4}) \quad \text{and} \quad w_2 = w^*(1 - z_{n-4}, 1)
\]

and the associated values \(g_1 = g^*(1, 1 - y_{n-4})\) and \(g_2 = g^*(1 - z_{n-4}, 1)\).

From (78) and (79), following similar argument as in Section 4.2, we get the analogue of (52):

\[
g^* = g_1 = \frac{\alpha^* - 1}{(\alpha^* - 1)w_1 + 1} = \frac{(\alpha^* - 1)(w_1 - y_{n-4})}{1 - y_{n-4}},
\]

\[
g^* = g_2 = \frac{\alpha^* - 1(1 - z_{n-4})}{(\alpha^* - 1)w_2 + 1 - z_{n-4}} = (\alpha^* - 1)w_2.
\]

Applying the estimates of Lemma 10, and weighting the two ways to write the denominators coming from (69) with parameters \(w_1\) and \(w_2\), we get

\[
\prod_{l=0}^{2^{n-4} - 1} \left(\frac{(L_{r_{d+1}-1}M_{r_{d+1}}M_{r_{d+1}+1}T_{n-2}^l) (L_{r_{d+1}-1}M_{r_{d+1}}M_{r_{d+1}+1}T_{n-2}^l)^l_{y_{n-4}}}{(L_{r_{d+1}}M_{r_{d+1}+1}T_{n-1})^{w_1} (L_{r_{d+1}-1}M_{r_{d+1}+1}T_{n-1})^{1-w_1}}\right)^{w_{n-4,l}}.
\]

\[
\prod_{l=0}^{2^{n-4} - 1} \left(\frac{(L_{r_{d+2}-1}M_{r_{d+2}}M_{r_{d+2}+1}T_{n-2}^l)^l_{z_{n-4}}}{(L_{r_{d+2}}M_{r_{d+2}+1}T_{n-1})^{w_2} (L_{r_{d+3}}M_{r_{d+3}}T_{n-1})^{1-w_2}}\right)^{w'_{n-4,l}} \gg 1.
\]
Furthermore, by (46) and (47), $T$ has the same power $(n - 1)(n - 2)$ at numerator and denominator and can be cancelled out.

$$
\prod_{l=0}^{2^n-1} \left( \frac{(L_{r_d+1} M_{r_d+1}) (L_{r_d+3-1} M_{r_d+3}) (1-y_{n-4})}{(L_{r_d+1} M_{r_d+1})^{y_1} (L_{r_d+3-1} M_{r_d+3})^{1-y_1}} \right) w_{n-4,l}^{w_{n-4,l}}.
\prod_{l=0}^{2^n-1} \left( \frac{(L_{r_d+2-1} M_{r_d+2} M_{r_d+3}) (1-y_{n-4})}{(L_{r_d+2} M_{r_d+2} M_{r_d+3})^{y_2} (L_{r_d+3} M_{r_d+3})^{1-y_2}} \right) w_{n-4,l}^{w_{n-4,l}} \gg 1
$$

Hence, at least one of the following four inequalities holds:

1. inequality (82) leads to $L_{r_d} \ll M_{r_d}^{-\alpha^* g^*}$;
2. inequality (83) leads to $M_{r_d+1} \gg M_{r_d}^{g^*}$;
3. inequality (84) leads to $L_{r_d+2} \ll M_{r_d+2}^{-\alpha^* g^*}$;
4. inequality (85) leads to $M_{r_d+3} \gg M_{r_d+3}^{g^*}$.

We explain how to get the first two inequalities of this group from the first two inequality of the previous group. The others are obtained in a similar way, however one should note that the inequality (33) is crucial for checking the positivity of exponents in case 3).

1) Indeed, suppose (82). Then as $L_{r_d} < M_{r_d}^{-\alpha^*}$ or $M_{r_d+1} < L_{r_d}^{-1/\alpha^*}$, we deduce the upper bound for the linear form $L_{r_d}$ by means of the estimate

$$
L_{r_d}^{y_1} \ll L_{r_d-1} M_{r_d}^{-1} M_{r_d+1}^{1-y_1} \ll M_{r_d}^{-\alpha^*} L_{r_d}^{\frac{1-y_1}{\alpha^*}}, \text{ or } L_{r_d} \ll M_{r_d}^{y_1 + (1-y_1)/\alpha^*}.
$$

2) Suppose (83). Then we use $L_{r_d+1} M_{r_d+1} < M_{r_d+1}^{-\alpha^*}$. Now

$$
1 \ll M_{r_d+1}^{y_{n-4}} (L_{r_d+2} M_{r_d+1})^{w_1-y_{n-4}} \ll M_{r_d+1}^{-y_{n-4}} M_{r_d+1}^{-1/\alpha^*} (w_1-y_{n-4})
$$

(here we use the inequality $w_1 - y_{n-4} > 0$ which follows from $g^* > 0$ and the second inequality form (80)). We use the second inequality form (80) to conclude that

$$
M_{r_d+1} \gg M_{r_d+1}^{(\alpha^*-1) w_1-y_{n-4}} L_{r_d+1}^{1-y_{n-4}} = M_{r_d+1}^{g^*}.
$$

We have checked (14) and the result follows.

\[\square\]
5.3.2 Example of approximation to 4 numbers

Consider a sequence of best approximation vectors to $\theta \in \mathbb{R}^4$ by one linear form. We may assume that it spans $\mathbb{R}^5$. Take $\alpha^* < \hat{\omega}(\theta)$.

We consider the unique positive real number $g^*$ such that $\alpha^* - 1 - g^* - (g^*)^2 - (g^*)^3 = 0$. Set

$$x := \frac{\alpha^* - 1 - g^* - (g^*)^2}{\alpha^* - 1 - g^*} = \frac{\alpha^* - 1 - \alpha^* g^*}{g^* (g^* - \alpha^* + 1)} = \frac{R_{3,\alpha^*}^* (g^*)}{R_{2,\alpha^*}^* (g^*)} = 1 - \frac{R_{3,\alpha^*}^* (g^*)}{g^* R_{2,\alpha^*}^* (g^*)}.$$

Set the parameters (using (76))

$$w_1 = w^*(1, x) = \frac{\alpha^* - 1 - g^*}{g^* (\alpha^* - 1)} \quad \text{and} \quad w_2 = w^*(1 - x, 1) = \frac{g^*}{\alpha^* - 1}.$$

One can check that

$$g^* = \frac{\alpha^* - 1}{(\alpha^* - 1) w_1 + 1} = (\alpha^* - 1) (1 + (w_1 - 1)/x) = \frac{(\alpha^* - 1) (1 - x)}{(\alpha^* - 1) w_2 + 1 - x} = (\alpha^* - 1) w_2. \quad (86)$$

As $0 < g^* = (\alpha^* - 1) \frac{x + w_1 - 1}{x}$, we deduce that

$$w_1 + x - 1 > 0. \quad (87)$$

As $\frac{\alpha^* - 1}{\alpha^*} < 1 \leq g^* \leq \alpha^* - 1$ we have

$$1 - x - w_2 = 1 - \frac{R_{3,\alpha^*}^* (g^*)}{R_{2,\alpha^*}^* (g^*)} - w_2 = \frac{\alpha^* g^* \left( g^* - \frac{\alpha^* - 1}{\alpha^*} \right)}{(\alpha^* - 1)(\alpha^* - 1 - g^*)} > 0 \quad (88)$$

Now we are able to start the proof. For an index $k_0 \gg 1$ we apply Lemma 6. It provides a pattern of best approximation vectors

$$z_{\tau_0 - 1}, z_{\tau_0}, z_{\tau_0 + 1}; \quad z_{\tau_1 - 1}, z_{\tau_1}, z_{\tau_1 + 1}; \quad z_{\tau_2 - 1}, z_{\tau_2}, z_{\tau_2 + 1}; \quad z_{\tau_3 - 1}, z_{\tau_3}, z_{\tau_3 + 1};$$

of linearly independent triples satisfying properties of Lemma 6. Consider $T$ such that $T > M_{r_3 + 1}$ and $T > L_{r_3 - 1}^{-1/n}$. For $j \geq r_0 - 1$, we apply the hyperbolic rotation to the integer vectors $z_j$ to get

$$z'_j = G_T L \cdot z_j.$$

For $0 \leq i \leq 3$ we consider the subspace

$$S_{3,i} = \langle z'_{r_1 - 1}, z'_{r_1}, z'_{r_1 + 1} \rangle \mathbb{R}$$
and its lattice of $\mathcal{G}_T\mathcal{L}$ points
\[ \mathcal{G}_{3,i} = S_{3,i} \cap \mathcal{G}_T\mathcal{L}. \]

We recall that
\[ S_{3,1} = S_{3,2} = Q. \]

Consider the 2-dimensional lattices
\[ \Lambda_0 := \langle z'_{r_0}, z'_{r_0+1} \rangle = \langle z'_{r_1-1}, z'_{r_1} \rangle = S_{3,0} \cap S_{3,1} \cap \mathcal{G}_T\mathcal{L} \]
and
\[ \Lambda_1 := \langle z'_{r_2}, z'_{r_2+1} \rangle = \langle z'_{r_3-1}, z'_{r_3} \rangle = S_{3,2} \cap S_{3,3} \cap \mathcal{G}_T\mathcal{L}. \]

We apply Schmidt’s inequality (Proposition 6) with underlying lattice $\mathcal{G}_T\mathcal{L}$ to obtain the analogue of (36)
\[ \frac{\det \mathcal{G}_{3,0}(\det \mathcal{G}_{3,1})^x}{\det \Lambda_0} \cdot \frac{(\det \mathcal{G}_{3,2})^{1-x} \det \mathcal{G}_{3,3}}{\det \Lambda_1} \gg 1. \]

By Lemma 10, we get
\[ \frac{L_{r_0-1}M_{r_0}M_{r_0+1}T^2(L_{r_1-1}M_{r_1}M_{r_1+1})^x}{L_{r_0}M_{r_0+1}T^3} \cdot \frac{(L_{r_2-1}M_{r_2}M_{r_2+1}T^2)^{1-x}L_{r_3-1}M_{r_3}M_{r_3+1}T^2}{L_{r_3-1}M_{r_3}T^3} \gg 1. \]

Here, $T$ disappears as it has power 6 at numerator and denominator:
\[ 3 + 3 = 6 = 2 + 2x + 2(1 - x) + 2. \]

We deduce
\[ \frac{L_{r_0-1}M_{r_0}M_{r_0+1}(L_{r_1-1}M_{r_1}M_{r_1+1})^x}{L_{r_0}M_{r_0+1}} \cdot \frac{(L_{r_2-1}M_{r_2}M_{r_2+1})^{1-x}L_{r_3-1}M_{r_3}M_{r_3+1}}{L_{r_3-1}M_{r_3}} \gg 1. \]

$\Lambda_0 = \langle z'_{r_0}, z'_{r_0+1} \rangle = \langle z'_{r_1-1}, z'_{r_1} \rangle$ therefore $L_{r_0}M_{r_0+1} \geq L_{r_1-1}M_{r_1}$ according to (69) and by analogous arguments applied to $\Lambda_1$ we get the second equation $L_{r_2}M_{r_2+1} \geq L_{r_3-1}M_{r_3}$. Hence we can replace
\[ L_{r_0}M_{r_0+1} \text{ by } (L_{r_0}M_{r_0+1})^{w_1}(L_{r_1-1}M_{r_1})^{1-w_1} \]
and
\[ L_{r_3-1}M_{r_3} \text{ by } (L_{r_2}M_{r_2+1})^{w_2}(L_{r_3-1}M_{r_3})^{1-w_2}. \]

We deduce that at least one of the four following inequalities holds
\[ \begin{align*}
L_{r_0-1}M_{r_0}M_{r_0+1} & \gg (L_{r_0}M_{r_0+1})^{w_1}, \\
(L_{r_1-1}M_{r_1}M_{r_1+1})^x & \gg (L_{r_1-1}M_{r_1})^{1-w_1}, \\
(L_{r_2-1}M_{r_2}M_{r_2+1})^{1-x} & \gg (L_{r_2}M_{r_2+1})^{w_2}, \\
L_{r_3-1}M_{r_3}M_{r_3+1} & \gg (L_{r_3-1}M_{r_3})^{1-w_2}.
\end{align*} \]
1) From (89) and (86), as $L_{r_0} < M^{-\alpha}_{r_0+1}$ or $M_{r_0+1} < L_{r_0}^{-1/\alpha}$, we deduce the upper bound for the linear form

$$L_{r_0}^{w_1} \ll L_{r_0}^{-1} M_{r_0} M_{r_0+1}^{1-w_1} \ll M_{r_0}^{1-\alpha} L_{r_0}^{1-w_1},$$

or $L_{r_0} \ll M_{r_0}^{-1+1/(1-w_1)/\alpha} = M_{r_0}^{-\alpha} g^\ast$.

2) From (90) and (86), as $L_{r_1-1} M_{r_1} < M_{r_1}^{1-\alpha}$, we deduce the lower bound for the coefficient

$$1 \ll M_{r_1-1}^{x} (L_{r_1}^{1-1})^{x+w_1-1} \ll M_{r_1+1}^{x} M_{r_1}^{(1-\alpha^\ast)(x+w_1-1)},$$

or $M_{r_1+1} \gg M_{r_1}^{(\alpha^\ast-1) \frac{x+w_1-1}{x}} = M_{r_1} g^\ast$.

The second inequality is satisfied because of (87).

3) From (91) and (86), as $L_{r_2-1} < M_{r_2}^{\alpha}$ or $M_{r_2+1} < L_{r_2}^{-1/\alpha}$, we deduce the upper bound for the linear form

$$L_{r_2}^{w_2} \ll (L_{r_2}^{-1} M_{r_2})^{1-x} M_{r_2+1}^{1-x-w_2} \ll M_{r_2}^{1-x} M_{r_2}^{(1-x)(1-\alpha^\ast)} \frac{x+w_2-1}{x},$$

or $L_{r_2} \ll M_{r_2}^{-2+1/(1-x-w_2)/\alpha} = M_{r_2}^{-\alpha^\ast} g^\ast$.

Here we use (88).

4) From (92) and (86), as $L_{r_3-1} M_{r_3} < M_{r_3}^{\alpha^\ast-1}$, we deduce the lower bound for the coefficient

$$1 \ll (L_{r_3-1} M_{r_3})^{w_2} M_{r_3+1} \ll M_{r_3}^{(\alpha^\ast-1) w_2} M_{r_3+1},$$

or $M_{r_3+1} \gg M_{r_3}^{(\alpha^\ast-1) w_2} = M_{r_3}^g$.

Hence, we proved that one of the following four inequalities holds:

$$M_{r_1+1} \gg M_{r_1}^g, \quad M_{r_3+1} \gg M_{r_3}^g, \quad L_{r_0} \ll M_{r_0}^{-\alpha^\ast} g^\ast, \quad L_{r_2} \ll M_{r_2}^{-\alpha^\ast} g^\ast.$$

So we have checked (14) and the result follows.

\[\square\]

**6 Construction of points with given ratio**

In this last section, we prove the second part of Theorem 1. To construct points with given ratio, we place ourselves in the context of parametric geometry of numbers introduced by Schmidt and Summerer in [27, 26]. For the convenience of the reader and the sake of self-containment, we briefly present the parametric geometry of numbers in section 6.1. An important theorem by Roy [22] enables to construct points with computable exponents of Diophantine approximation out of Roy-systems, a combinatorial family of piecewise linear applications. For our purpose, we construct explicitly in Section 6.2 a family of Roy-systems with three parameters. The construction shows how the values $G_n(\alpha)$ and $G^\ast_n(\alpha^\ast)$ appear naturally in the context of parametric geometry of numbers, and why they are reached at regular systems.
6.1 Parametric geometry of numbers

The Parametric Geometry of Numbers answers a question of W. M. Schmidt [24]. Given a convex body and a lattice, we deform either of them with a one parameter diagonal map. We study the behavior of the successive minima in terms of this parameter. It was developed by W. M. Schmidt and L. Summerer [26, 27], and further by D. Roy [22].

We use the notation introduced by D. Roy in [22] which is essentially dual to the one of W. M. Schmidt and L. Summerer [26]. It follows the presentation in [14, §2]. We refer the reader to these papers for further details.

Here $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ is the usual scalar product of vectors $x$ and $y$, and $\|x\|_2 = \sqrt{x \cdot x}$ is the usual Euclidean norm.

Let $u = (u_0, \ldots, u_n)$ be a vector in $\mathbb{R}^{n+1}$, with Euclidean norm $\|u\|_2 = 1$. For a real parameter $Q \geq 1$ we consider the convex body

$$C_u(Q) = \left\{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq 1, |x \cdot u| \leq Q^{-1} \right\}.$$

For $1 \leq d \leq n+1$ we denote by $\lambda_d(C_u(Q))$ the $d$-th minimum of $C_u(Q)$ relatively to the lattice $\mathbb{Z}^{n+1}$. For $q \geq 0$ and $1 \leq d \leq n+1$ we set

$$L_{u,d}(q) = \log \lambda_d(C_u(e^q)).$$

Finally, we define the successive minima function $L_u$ associated with $u$:

$$L_u : [0, \infty) \to \mathbb{R}^{n+1} \quad q \mapsto (L_{u,1}(q), \ldots, L_{u,n+1}(q)).$$

The lattice $\mathbb{Z}^{n+1}$ is invariant under permutation of coordinates. Hence, $L_u$ remains the same if we permute the coordinates in $u$. Since $\|u\|_2 = 1$ we can thus assume that $u_0 \neq 0$.

The following proposition links the exponents of Diophantine approximation associated with $\theta = \begin{pmatrix} u_1 & \ldots & u_n \\ u_0 & \ldots & u_0 \end{pmatrix}$ to the behavior of the map $L_u$, assuming $u_0 \neq 0$.

Proposition 7. Let $u = (u_0, \ldots, u_n) \in \mathbb{R}^{n+1}$, with Euclidean norm $\|u\|_2 = 1$ and $u_0 \neq 0$. Set $\theta = \begin{pmatrix} u_1 & \ldots & u_n \\ u_0 & \ldots & u_0 \end{pmatrix}$. We have the following relations:

\[\text{In [2, 3], Das, Fishman, Simmons and Urbański introduce a variational principle in parametric geometry of numbers that extends Theorem 2. They both extend to the case of approximation to a matrix $\theta$, and provide a quantitative result. Applying the variational principle to our construction, we obtain a lower bound for the Hausdorff dimension of points with given pair of exponents $(\omega, \hat{\omega})$ or $(\lambda, \hat{\lambda})$ satisfying (9). However, for $c > 1$ it is probably not optimal.}\]
Thus, if we know an explicit map \( \mathbf{P} = (P_1, \ldots, P_{n+1}) : [0, \infty) \to \mathbb{R}^{n+1} \), such that \( L_\mathbf{u} - \mathbf{P} \) is bounded, then we can compute the 4 exponents \( \hat{\lambda}, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3 \) upon replacing \( L_{\mathbf{u},i} \) by \( P_i \) in the above formulas for \( i = 1 \) or \( n + 1 \). For this purpose, Roy introduced [22] what we will call Roy-systems.

**Definition.** Let \( I \) be a subinterval of \([0, \infty)\) with non-empty interior. A generalized \((n+1)\)-system on \( I \) is a continuous piecewise linear map \( \mathbf{P} = (P_1, \ldots, P_{n+1}) : I \to \mathbb{R}^{n+1} \) with the following three properties.

\( \text{(S1)} \) For each \( q \in I \), we have \( 0 \leq P_1(q) \leq \cdots \leq P_{n+1}(q) \) and \( P_1(q) + \cdots + P_{n+1}(q) = q \).

\( \text{(S2)} \) If \( H \) is a non-empty open subinterval of \( I \) on which \( \mathbf{P} \) is differentiable, then there are integers \( \underline{r}, \bar{r} \) with \( 1 \leq \underline{r} \leq \bar{r} \leq n + 1 \) such that \( P_{\underline{r}}, P_{\bar{r}+1}, \ldots, P_{\bar{r}} \) coincide on the whole interval \( H \) and have slope \( 1/(\bar{r} - \underline{r} + 1) \) while any other component \( P_k \) of \( \mathbf{P} \) is constant on \( H \).

\( \text{(S3)} \) If \( q \) is an interior point of \( I \) at which \( \mathbf{P} \) is not differentiable, if \( \underline{r}, \bar{r}, \underline{s}, \bar{s} \) are the integers for which

\[
P_k'(q^-) = \frac{1}{\bar{r} - \underline{r} + 1} \quad (\underline{r} \leq k \leq \bar{r}) \quad \text{and} \quad P_k'(q^+) = \frac{1}{\bar{s} - \underline{s} + 1} \quad (\underline{s} \leq k \leq \bar{s}),
\]

and if \( \underline{r} < \bar{r} \), then we have \( P_{\underline{r}}(q) = P_{\underbar{r}+1}(q) = \cdots = P_{\bar{s}}(q) \).

Here \( P_k'(q^-) \) (resp. \( P_k'(q^+) \)) denotes the left (resp. right) derivative of \( P_k \) at \( q \).

**Theorem 2** (Roy, 2015). For each non-zero point \( \mathbf{u} \in \mathbb{R}^{n+1} \), there exists \( q_0 \geq 0 \) and a generalized \((n+1)\)-system \( \mathbf{P} \) on \([q_0, \infty)\) such that \( L_\mathbf{u} - \mathbf{P} \) is bounded on \([q_0, \infty)\). Conversely, for each generalized \((n+1)\)-system \( \mathbf{P} \) on an interval \([q_0, \infty)\) with \( q_0 \geq 0 \), there exists a non-zero point \( \mathbf{u} \in \mathbb{R}^{n+1} \) such that \( L_\mathbf{u} - \mathbf{P} \) is bounded on \([q_0, \infty)\).

In view of the remark following Proposition 7, this result reduces the construction of points with prescribed exponents of Diophantine approximation to a combinatorial study of Roy-systems.
6.2 Construction of a family of Roy-systems with three parameters

In this section, we construct explicitly a family of Roy-systems with parameters. According to Proposition 7 and Theorem 2, these Roy-systems provide the existence of points with requested exponents, proving the second part of Theorem 1.

**Approximation by one linear form.** Fix the dimension $n \geq 2$, and consider the case of approximation by one linear form. Fix the three parameters $\hat{\omega} \geq n$, $\rho = G^*(n, \hat{\omega})$ and $c \geq 1$. Consider the Roy-system $P$ on the interval $[1, c\rho]$ depending on these parameters whose combined graph is given below by Figure 6, where

$$P_1(1) = \frac{1}{1 + \hat{\omega}}, \quad P_k(1) = \rho^{k-2}P_1(1) \quad \text{for} \quad 2 \leq k \leq n+1 \quad \text{and} \quad P_k(c\rho) = c\rho P_k(1) \quad \text{for} \quad 1 \leq k \leq n+1.$$ 

![Figure 6: Pattern of the combined graph of $P$ on the fundamental interval $[1, c\rho]$](image)

The fact that all coordinates sum up to 1 for $q = 1$ follows from $\rho$ being a root of the polynomial $R^*_n, \hat{\omega}$ defined in (8). On each interval between two consecutive division points, there is only one line segment with slope 1. On $[1, q_0]$, there is one line segment of slope 1.
starting from the value \( \frac{1}{1 + \hat{\omega}} \) and reaching the value \( \frac{c \rho^n}{1 + \hat{\omega}} \). Then, each component \( P_k \) increases from \( \rho_k^{1 - \frac{1}{1 + \hat{\omega}}} \) to \( \frac{c \rho^{k-1}}{1 + \hat{\omega}} \) with slope 1 where \( k \) decreases from \( k = n \) down to \( k = 2 \).

We extend \( P \) to the interval \([1, \infty)\) by self-similarity. This means, \( P(q) = (c \rho)^m P((c \rho)^{-m} q) \) for all integers \( m \). In view of the value of \( P \) and its derivative at 1 and \( c \rho \), one sees that the extension provides a Roy-system on \([1, \infty)\).

Note that for \( c = 1 \), the parameters \( q_0 \) and \( q_1 \) coincide and we constructed a regular system.

Roy’s Theorem \([22]\) provides the existence of a point \( \theta \) in \( \mathbb{R}^n \) such that

\[
\frac{1}{1 + \hat{\omega}(\theta)} = \limsup_{q \to +\infty} \frac{P_1(q)}{q} \quad \text{and} \quad \frac{1}{1 + \omega(\theta)} = \liminf_{q \to +\infty} \frac{P_1(q)}{q}.
\]

Here, self-similarity ensures that the \( \limsup \) (resp. \( \liminf \)) is in fact the maximum (resp. the minimum) on the interval \([1, c \rho]\). Thus,

\[
\frac{1}{1 + \hat{\omega}} = \max_{[1, c \rho]} \frac{P_1(q)}{q} = \frac{P_1(1)}{1 + \hat{\omega}} \quad \text{and} \quad \frac{1}{1 + \omega} = \min_{[1, c \rho]} \frac{P_1(q)}{q} = \frac{P_1(q_1)}{q_1} = \frac{1}{c \rho \hat{\omega} + 1}
\]

where

\[
q_1 = \frac{c(\rho^n + \cdots + \rho^2 + \rho) + 1}{1 + \hat{\omega}} = \frac{c(\rho \hat{\omega}) + 1}{1 + \hat{\omega}}.
\]

The simplification comes from \( \rho \) being a root of the polynomial \( R_{n, \hat{\omega}} \) defined in (8). Hence, \( \hat{\omega}(\theta) = \hat{\omega} \) and \( \omega(\theta) = c \rho \hat{\omega} \), and we constructed the required points since \( c \geq 1 \) and \( \rho = G^*(n, \hat{\omega}) \).

**Simultaneous approximation** Consider the case of simultaneous approximation. Fix the three parameters \( 1 \geq \hat{\lambda} \geq 1/n, \rho = G(n, \hat{\lambda}) \) and \( c \geq 1 \). Consider the Roy-system \( P \) on the interval \([1, c \rho]\) depending on these parameters whose combined graph is given below by Figure 7, where

\[
P_{n+1}(1) = \hat{\lambda}, \quad P_k(1) = \rho^{n-k} P_1(1) \quad \text{for} \ 1 \leq k \leq n \quad \text{and} \quad P_k(c \rho) = c \rho P_k(1) \quad \text{for} \ 2 \leq k \leq n + 1.
\]

The fact that all coordinates sum up to 1 for \( q = 1 \) follows from \( \rho \) being the root of the polynomial \( R_{n, \hat{\lambda}} \) defined in (7). Up to change of origin and rescaling, this is the same pattern as shown by Figure 6. We extend \( P \) to the interval \([1, \infty)\) by self-similarity. This means, \( P(q) = (c \rho)^m P((c \rho)^{-m} q) \) for all integers \( m \). In view of the value of \( P \) and its derivative at 1 and \( c \rho \), one sees that the extension provides a Roy-system on \([1, \infty)\).
Figure 7: Pattern of the combined graph of $P$ on the fundamental interval $[1, c\rho]$, where $\beta = \frac{\hat{\lambda}}{1+\lambda}$.

For $c = 1$, the parameters $q_0$ and $q_1$ coincide and we constructed a regular system.

Roy’s Theorem [22] provides the existence of a point $\theta$ in $\mathbb{R}^n$ such that

$$\frac{\hat{\lambda}(\theta)}{1 + \hat{\lambda}(\theta)} = \liminf_{q \to +\infty} \frac{P_{n+1}(q)}{q} \quad \text{and} \quad \frac{\lambda(\theta)}{1 + \lambda(\theta)} = \limsup_{q \to +\infty} \frac{P_{n+1}(q)}{q}$$

Again, self-similarity ensures that the lim sup (resp. lim inf) is in fact the maximum (resp. the minimum) on the interval $[1, c\rho]$. Thus,

$$\frac{\hat{\lambda}(\theta)}{1 + \hat{\lambda}(\theta)} = \min_{[1, c\rho]} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(1)}{1} = \frac{\hat{\lambda}}{1 + \hat{\lambda}},$$

$$\frac{\lambda(\theta)}{1 + \lambda(\theta)} = \max_{[1, c\rho]} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(q_0)}{q_0} = \frac{cp\hat{\lambda}}{1 + cp\lambda}$$

where

$$q_0 = \frac{\hat{\lambda}(c\rho + (1 + \rho^{-1} + \cdots + \rho^{-(n-1)}))}{1 + \hat{\lambda}} = \frac{1 + cp\hat{\lambda}}{1 + \hat{\lambda}}.$$
The simplification comes from $\rho$ being the root of the polynomial $R_{n,\hat{\lambda}}$ defined in (7). Hence, $\hat{\lambda}(\theta) = \hat{\lambda}$ and $\lambda(\theta) = c\rho\hat{\lambda}$, and we constructed the required points since $c \geq 1$ and $\rho = G(n, \hat{\lambda})$.

For both simultaneous approximation and approximation by a linear form, the constructed 3-parameters families of self-similar Roy-systems provide infinitely many distinct points $\theta \in \mathbb{R}^n$ via Roy’s theorem with $\mathbb{Q}$-linearly independent coordinates with 1, as explained in [14, end of §3]. The $\mathbb{Q}$-linear independence comes from $P_1(q) \to \infty$ when $q \to \infty$. The construction of infinitely many points follows from a change of origin with the same pattern and self-similarity. The degenerate cases when some of the exponents are infinite is managed by (non self-similar) Roy-systems consisting in patterns described by Figure 6 or 7, where the parameter $c$ and/or $\hat{\lambda}$ or $\hat{\omega}$ increases to infinity at each repetition. An explicit example of this trick is to be found in [14, end of §3].

Acknowledgement We are very grateful for the hospitality of Mathematisches Forschungsinstitut Oberwolfach. An important part of this work has been done during Research in Pairs stay 1823r. We are also very grateful for the hospitality of the Centro Internazionale per la Ricerca Matematica (C.I.R.M.) of Trento, as the last part of the work was done during Research in Pairs stay there during May 19-June 1, 2019.

References

[1] N. Chevallier : Best simultaneous Diophantine approximations and multidimensional continued fraction expansions, Mosc. J. Comb. Number Theory, 3:1 pp. 3–56 (2013).

[2] T. Das, L. Fishman, D. Simmons and M. Urbański : A variational principle in the parametric geometry of numbers, with applications to metric Diophantine approximation, Comptes Rendus Mathématique, 355 (8), pp. 835–846 (2017).

[3] T. Das, L. Fishman, D. Simmons and M. Urbański : A variational principle in the parametric geometry of numbers, ArXiV preprint 1901.06602.

[4] H. Davenport and W. M. Schmidt : Approximation to real numbers by quadratic irrationals, Acta Arithmetica 13, pp. 169 – 176 (1967).

[5] D. Gayfulin and N. G. Moshchevitin : On Diophantine exponents in dimension 4, Preprint arXiv 1309.7826.

[6] O. N. German : On Diophantine exponents and Khintchine’s tranference principle, Mosc. J. Comb. Number Theory, 2(2): pp. 22-51 (2012).

[7] O. N. German and N.G. Moshchevitin : A simple proof of Schmidt-Summerer’s inequality, Monatshfte für Mathematik 170: 3–4, pp. 361 – 370 (2013).

[8] V. Jarník : Zum Khintchineschen Übertragungssatz, Trav. Inst. Math. Tbilissi 3, p. 193-212 (1938).
[9] V. Jarník: *Une remarque sur les approximations diophantiennes linéaires*, Acta Scientarium Mathem. Szeged 12 (1949) pp. 82–86.

[10] V. Jarník: *Contribution à la théorie des approximations diophantiennes linéaires et homogènes*, Czechoslovak. Math. J. 4, pp. 330 – 353 (1954).

[11] A. Ya. Khintchine: *Zur metrischen Theorie der Diophantischen Approximationen*, Math. Z. 24, pp. 706 – 714 (1926).

[12] A. Ya. Khintchine: *Über eine Klasse linearer Diophantischer Approximationen*, Rend. Circ. Math. Palermo 50, pp. 170 –195 (1926).

[13] M. Laurent: *Exponents of Diophantine approximations in dimension two*, Canad. J. Math. 61, 1,165 – 189 (2009).

[14] A. Marnat: *About Jarník’s type relation in higher dimension*, Annales de l’Institut Fourier, 68 no. 1 (2018), p. 131–150.

[15] N. G. Moshchevitin: *Best Diophantine approximation: the phenomenon of degenerate dimension*, London Math. Soc. Lecture Note Ser., 338, pp. 158 – 182. Cambridge Univ. Press (2007).

[16] N. G. Moshchevitin: *Khinchine’s singular Diophantine systems and their applications* Russian Math. Surveys, 65:3n pp. 433 – 511 (2010).

[17] N. G. Moshchevitin: *Exponents for three-dimensional simultaneous Diophantine approximations*, Czechoslovak Math. J. 62(137), no. 1, pp. 127–137 (2012).

[18] N. G. Moshchevitin: *Über eine Ungleichung von Schmidt und Summerer für diophantische Exponenten von Linearformen in drei Variable*, Preprint arXiv 1312.1841.

[19] N. A. V. Nguyen: *On some problems in Transcendental Number Theory and Diophantine Approximation*, PhD Thesis Ottawa, https://ruor.uottawa.ca/handle/10393/30350 (2014).

[20] C. A. Rogers: *The signature of the errors of some Diophantine approximations*, Proc. London Math. Soc. 52, pp. 186 – 190 (1951).

[21] D. Roy: *Construction of points realizing the regular systems of Wolfgang Schmidt and Leonard Summerer*, J. Théor. Nombres Bordeaux, 27 (2): pp. 591–603 (2015).

[22] D. Roy: *On Schmidt and Summerer parametric geometry of numbers*, Ann. of Math., 182: pp. 739–786 (2015).

[23] D. Roy: *Simultaneous approximation to a real number, its square and its cube*, Acta Arithmetica, 133 (2008), pp. 185–197.

[24] W. M. Schmidt: *Diophantine approximation and Diophantine equations*, Lecture Notes in Mathematics (1467), Springer. (1991).

[25] W. M. Schmidt: *On heights of algebraic subspaces and diophantine approximations*, Ann. of Math., 85(2): pp. 430–472 (1967).
[26] W. M. Schmidt and L. Summerer: *Diophantine approximation and parametric geometry of numbers*, Monatsh. Math 169:1, pp. 51 – 104 (2013).

[27] W. M. Schmidt and L. Summerer: *Parametric geometry of numbers and applications*, Acta Arithmetica, 140(1): pp. 67–91 (2009).

[28] W. M. Schmidt and L. Summerer: *Simultaneous approximation to three numbers*, Mosc. J. Comb. Number Theory 3, no. 1, pp. 84–107 (2013).

[29] G. F. Voronoi: *On one generalization of continued fractions’ algorithm*, Warsaw, 1896 (in russian).