Classical Electrodynamics from the Motion of a Relativistic Fluid

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Abstract. We show that there exists a choice of gauge in which the electromagnetic 4-potential may be written as the difference of two 4-velocity vector fields describing the motion of a two-component space-filling relativistic fluid. Maxwell’s equations are satisfied immediately, while the Lorentz force equation follows from the interactions of sources and sinks. The usual electromagnetic quantities then admit new interpretations as functions of the local 4-velocities. Electromagnetic waves are found to be described by oscillations of the underlying medium which can therefore be identified with the ‘luminiferous aether’. The formulation of electrodynamics in terms of 4-velocities is more general than that of the standard 4-potential in that it also allows for a classical description of a large class of vacuum energy configurations. Treated as a self-gravitating fluid, the model can be explicitly identified with Nelson’s stochastic formulation of quantum mechanics, making it a promising candidate as the classical field theory unifying gravitation, electromagnetism and quantum theory which Einstein had sought.

INTRODUCTION

Motivated by the fact that macroscopic waves tend to propagate in a medium of some kind, many physicists have in the past attempted to find a description of electrodynamics in which electromagnetic waves may also be described by the motion of an underlying medium, commonly referred to as the ‘aether’. Despite significant efforts, no such description was found, and the negative results of the Michelson-Morley experiment put the final nail in the coffin of the aether concept as it was then understood. The failure of this program eventually led to the introduction of the concept of a ‘field’ requiring no underlying medium, and on quantisation of these fields, to ‘quantum field theory’. We demonstrate here that Maxwell’s equations can in fact be derived from the motion of an underlying medium if we assume from the outset that the underlying spacetime is Lorentzian rather than Galilean. While our formulation differs from earlier conceptions of the aether, it does show that an alternative description of electrodynamics did (and does) exist which did not require the introduction of the field concept. The medium can be interpreted as a two-component relativistic fluid with sources and sinks playing the role of charges. This fluid dynamical model allows an explicit connection to be made with Nelson’s stochastic formulation of quantum mechanics, suggesting that the foundations of quantum electrodynamics may lie purely in classical gravity as Einstein had long believed.

We assume a metric with signature \((+,-,-,-)\), and follow the conventions of Jackson throughout.

THE RELATIVISTIC CONTINUUM

We first establish the reference system and coordinates that will be used to describe the motion of the continuum. We then define the ‘continuum gauge’, showing how Maxwell’s equations appear. We also show that the Lorentz force equation translates into a third order partial differential equation constraining the continuum’s motion.

Coordinates and Reference Frames

Consider an arbitrary relativistic inertial frame with 4-coordinates \(x^\mu = (ct,x,y,z)\), so that the spacetime partial derivatives are given by \(\partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla\right)\) and \(\partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla\right)\) respectively. Suppose that there exists a continuous
medium in relative motion to this frame which spans the entire spacetime. To describe positive and negative charge configurations we will see that this continuum must consist of two superimposed but mutually independent components which we shall refer to as the ‘positive continuum’ and the ‘negative continuum’ respectively.

Let τ be the proper time in the inertial frame, and consider the instantaneous motion at proper time τ of either component of the continuum at any 3-position \( \mathbf{r} = (x, y, z) \). Then the 3-velocity of that component at \( \mathbf{r} \) as measured by the inertial frame is,

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt},
\]

where \( t \) is the time as measured by a clock moving with the continuum component. We can therefore define the interval,

\[
ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.
\]

Similarly, we can define a 4-velocity vector field describing the motion of the continuum component as,

\[
\mathbf{u}^\mu = \frac{dx^\mu}{d\tau} = \left( c \frac{dt}{d\tau}, \frac{d\mathbf{r}}{d\tau} \right) = \left( c \gamma, \gamma \mathbf{v} \right),
\]

where \( \gamma = \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \) is the Lorentz factor at each point. This 4-velocity clearly satisfies,

\[
\mathbf{u}^\mu \mathbf{u}_\mu = c^2, \quad \mathbf{u}^\mu \mathbf{u}_\nu = 0,
\]

where partial derivatives \( \partial^\nu \mathbf{u}_\mu \) are written as \( \mathbf{u}^\mu,\nu \) for convenience.

The above definitions and identities hold for both the positive and negative continua independently, and henceforth we distinguish the two sets of variables by a ‘+’ or ‘−’ subscript respectively as necessary.

### The Continuum Gauge

The key to what follows is to split the electromagnetic potential 4-vector \( A^\mu \) into the sum of two components \( A^\mu_+ \) and \( A^\mu_- \), which we identify (up to a dimensional constant \( k \)) with the two continuum 4-velocities \( u^\mu_+ \) and \( u^\mu_- \),

\[
A^\mu = A^\mu_+ + A^\mu_-,
\]

where \( A^\mu_+ = ku^\mu_+ = (\phi_+, \mathbf{A}_+) \), \( A^\mu_- = -ku^\mu_- = (\phi_-, \mathbf{A}_-) \).

The condition \( 4 \) implies the following covariant constraint for both \( A^\mu_+ \) and \( A^\mu_- \),

\[
A^\mu_+ A^\mu_- = A^\mu_- A^\mu_+ = k^2 c^2.
\]

We will refer to conditions \( 5 \) and \( 6 \) as the ‘continuum gauge’. This is a non-standard choice of gauge, and we will demonstrate its consistency in the next section. The antisymmetric field-strength tensor can now be defined as,

\[
F^{\mu\nu} = A^{\nu,\mu} - A^{\mu,\nu} \sim (\mathbf{E}, \mathbf{B}).
\]

Other standard properties now follow in the usual way. From the definition \( 7 \), \( F^{\mu\nu} \) satisfies the Jacobi identity,

\[
F^{\mu\nu,\lambda} + F^{\nu\lambda,\mu} + F^{\lambda\mu,\nu} = 0,
\]

and this is just the covariant form of the homogeneous Maxwell’s equations. One can define the 4-current as the 4-divergence of the field-strength tensor,

\[
\mathbf{J}^\mu = \frac{c}{4\pi} F^{\mu\nu} = (c \rho, \mathbf{j}),
\]

and this is the covariant form of the inhomogeneous Maxwell’s equations. Charge conservation is guaranteed by the antisymmetry of \( F^{\mu\nu} \). The covariant Lorentz force equation takes the following form,

\[
\frac{dV^\mu}{d\tau} = \frac{Q}{Mc} F^{\mu\nu} V_\nu,
\]
where \( Q, M \) and \( V^\mu = (c\gamma, \gamma V) \) are the charge, mass and 4-velocity vector of the observed particles. This cannot be derived directly from the definition of the 4-potential, and must be considered for now as an auxiliary constraint. The charge 4-velocity \( V^\mu \) and scalar charge \( Q \) are related to the 4-current density \( J^\mu \) through the following equation,

\[
J^\mu = QV^\mu, \quad \text{where} \quad V^\mu V_\mu = c^2, \quad V^0 \geq c.
\]

The constraint on \( V^\mu \) allows us to separate the 4-current uniquely into the charge and its 4-velocity. Indeed we have,

\[
Q = \text{sgn}(J^0) \cdot \left( \frac{1}{c^2} J^\mu J_\mu \right)^{1/2},
\]

where the sign of the 0-component of the 4-current appears to ensure that the 0-component \( V^0 \) of the charge 4-velocity is positive. Since the sign of \( J^0 \) cannot be flipped by a Lorentz transformation, each 4-velocity vector field can only account for either positive or negative charge configurations, and hence the need for two separate continuum components, each with opposite contributions to the 4-potential. The gauge based upon a single 4-vector field was precisely that introduced by Dirac in his classical model of the electron, and it is noteworthy that he was also led to speculate that this 4-velocity field described the motion of a real, physical, ‘aether’. The form of the charge 4-velocity in terms of the continuum 4-velocity now follows directly from (11).

Besides the mass \( M \) which is determined by initial conditions, each of the terms in (10) may be written in terms of the 4-velocities \( u^\alpha \) and \( u^\beta \). From the definitions of \( F^{\mu\nu}, J^\mu, Q \) and \( V^\mu \), we find that the Lorentz force equation (10) translates into a complicated third order partial differential equation constraining the 4-velocities. The conservation of mass follows from the continuity equation for mass density,

\[
(MV^\mu)_{,\mu} = 0,
\]

which is ensured if the flow of mass density follows the flow of charge density. We will see later that the Lorentz force equation follows from the fluid dynamical interactions between sources and sinks, and this will complete our picture of classical electrodynamics in this gauge.

**THE CONSISTENCY OF THE CONTINUUM GAUGE**

We have identified the components \( A^\mu_+ \) and \( A^\mu_- \) of the 4-potential with the 4-velocities \( u^\mu_+ \) and \( u^\mu_- \) of the continuum satisfying the conditions (5) and (6), and have referred to this gauge choice as the ‘continuum gauge’. It is not obvious that this gauge choice can be applied consistently to all electromagnetic field configurations, so we demonstrate its consistency here, and give explicit solutions given for the point charge and the plane electromagnetic wave.

**Proof of Consistency**

In order to prove consistency, it is necessary to find a decomposition of the 4-potential as the difference of two 4-velocity fields satisfying equations (5) and (6) simultaneously. Using the notation of (3), we therefore need to find, given any 4-potential \( A^\mu = (\phi, A) \) defined up to a gauge transformation \( A^\mu \rightarrow A^\mu + \partial^\mu \phi \), two 3-velocity fields \( v_+ \) and \( v_- \) satisfying the following conditions,

\[
\frac{\phi}{kc} = \gamma_+ - \gamma_-, \quad \frac{A}{k} = \gamma_+ v_+ - \gamma_- v_-.
\]

The second of these equations is a simple geometrical vector identity, and it is clear that any solution set for \( (\gamma_+ v_+ , \gamma_- v_-) \) will form a surface of revolution about the axis defined by \( A \). To find the solution surface explicitly for a given \( (\phi, A) \), it is convenient to take the origin to lie at \( A/2k \), and to use polar coordinates \( (r, \theta) \) in any plane containing \( A \), where \( r \in [0, \infty] \) is the radial distance from the origin and \( \theta \in [0, \pi] \) is the angle made with respect to the direction of \( A \). Note the following simple chain of identities,

\[
\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \Rightarrow v = c \sqrt{1 - \frac{1}{\gamma^2}} \Rightarrow \gamma v = c \sqrt{\gamma^2 - 1} \Rightarrow \gamma = \sqrt{1 + \left( \frac{\gamma v}{c} \right)^2},
\]

(15)
so that from (14) we have,
\[ \phi = \frac{\theta}{kc} = \sqrt{1 + \left(\frac{\gamma v_+}{c}\right)^2} - \sqrt{1 + \left(\frac{\gamma v_-}{c}\right)^2}. \]  
(16)

Applying standard trigonometric identities to our geometrical picture, we obtain,
\[ (\gamma v_+)^2 = r^2 + A^2/4k^2 + \frac{Ar}{k} \cos \theta; \quad (\gamma v_-)^2 = r^2 + A^2/4k^2 - \frac{Ar}{k} \cos \theta, \]
(17)

so that the set of solutions on the plane in question is determined by the condition,
\[ \phi = \sqrt{A^2/4 + Ak r \cos \theta + k^2(r^2 + c^2)} - \sqrt{A^2/4 - Ak r \cos \theta + k^2(r^2 + c^2)}. \]
(18)

Note that given any solution for \((\phi, A)\), a solution for \((-\phi, A)\) is obtained by letting \(\theta \to \pi - \theta\). Note also (i) that \(\phi = 0\) whenever \(\theta = \pi/2\) including when \(r = 0\), (ii) that for a given value of \(r\) the magnitude of \(\phi\) is maximum when \(\theta = 0\), (iii) that for \(\theta = 0\), \(\phi\) is a monotonically increasing function of \(r\), and (iv) that \(\phi \to A \cos \theta\) as \(r \to \infty\).

In conclusion, for a given value of \(A = ||A||\), equations (14) will have solutions whenever \(|A| \leq A\). In the special case \(\phi = 0\) the solution surface for \(v_+\) is just the plane perpendicular to \(A\) passing through the point \(A/2k\), throughout which \(\gamma v_+ = |v_-|\), and \(\gamma v_+ \geq A/2k\). For other values of \(|A| \leq A\) the solutions form a paraboloid-like surface of revolution about the \(A\) axis. The sign of \(\phi\) determines which side of the \(\theta = \pm \pi/2\) plane the solution surface lies.

It is always possible to choose the function \(\psi\) defining the choice of gauge in such a way that \(\phi = 0\) everywhere\(^7\). Since solutions to (14) always exist in this case, this proves that the continuum gauge is indeed a consistent one.

It is important to note that there is actually a significant additional degree of freedom inherent in the way the decomposition of \(A^\mu\) is made into 4-velocity fields, which goes beyond the standard gauge freedom. First of all, for each electromagnetic configuration there will be a continuum of gauge choices for which a continuum gauge solution set exists. Secondly, for any particular choice of gauge for which a solution does exist, there will in general be an entire two-parameter surface of possible solutions for \(v_+\) and \(v_-\) at each point in space. We will show later that these velocity vector fields correspond to the motion of massive discrete particles, so that this freedom may have a real physical significance as a possible classical source of dark matter.

### The Point Charge

Let us now find the vacuum configuration which describes a positive charge \(q\) positioned at the origin. The corresponding electromagnetic fields are given by,
\[ E = \frac{q\hat{r}}{r^2} = -\nabla \left(\frac{q}{r}\right), \quad B = 0. \]  
(19)

We seek a 4-potential of the following form which only has contributions from the motion of the positive continuum,
\[ A_+^\mu = (\phi_+, A_+) = (kc\gamma, k\gamma v), \quad A_-^\mu = (\phi_-, A_-) = (-kc, 0), \]  
(20)

where the velocity vector field \(v\) is to be found. The corresponding electromagnetic fields \(E\) and \(B\) are given by,
\[ E = -\nabla \phi_+ - \frac{1}{c} \frac{\partial A_+}{\partial t} = -\nabla (kc\gamma) - \frac{1}{c} \frac{\partial (k\gamma v)}{\partial t}, \quad B = \nabla \times A_+ = \nabla \times (k\gamma v). \]  
(21)

For any electrostatic configuration with stationary charges we have \(B = \nabla \times (k\gamma v) = 0\), so there must exist a scalar field \(\gamma\) such that \(k\gamma v = \nabla \psi\). After some algebraic manipulation this can be seen to imply that,
\[ \gamma = \frac{\nabla \psi}{\sqrt{k^2c^2 + (\nabla \psi)^2}} \leq 1, \]  
(22)

so that in terms of \(\psi\), the \(E\) field is given by,
\[ E = -\nabla \left( (k^2c^2 + (\nabla \psi)^2)^{1/2} \right) - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \psi). \]  
(23)
Because of the rotational and time invariance of the problem, we need only look for solutions of the form $\psi = \psi(r)$, so that $\nabla \psi = \partial \psi / \partial r$ and the second term of (23) vanishes. Comparing with (19), it is clear that $\psi$ must satisfy,

$$\left( k^2 c^2 + \left( \frac{\partial \psi}{\partial r} \right)^2 \right)^{1/2} = \frac{q}{r} + \alpha, \tag{24}$$

where $\alpha$ is a constant of integration. Since the charge is positive and the velocity of the continuum should vanish at infinity, we require $\alpha = kc$ for a real solution to exist. From (24), the resulting differential equation for $\psi$ is as follows,

$$\frac{\partial \psi}{\partial r} = \pm \left( \left( \frac{q}{r} + kc \right)^2 - k^2 c^2 \right)^{1/2}, \tag{25}$$

where either the positive or negative square root may be chosen, as the 4-potential depends only on the magnitude of the velocity and not its direction. There is therefore insufficient information to specify whether the positive charge acts as a source or a sink (or both). The solution for the velocity field and the corresponding Lorentz factor is therefore,

$$\frac{\psi}{c} = \pm \left( 1 - \left( 1 + \frac{q}{krc} \right)^{-2} \right)^{1/2}, \quad \gamma = 1 + \frac{q}{krc}. \tag{26}$$

Note that $q/krc$ becomes singular at the origin, implying that the continuum velocity in (26) becomes equal to $c$ there.

The above confirms that the electromagnetic fields outside a positive point charge can indeed be described by the potential $\psi$ with $\nabla \psi = \partial \psi / \partial r$. While in principle one can claim that all electromagnetic configurations ultimately originate from the presence of charges, there do exist nontrivial configurations in which no charges are present, the most obvious and important example being that of the electromagnetic wave. It is therefore important, both for this reason and from a historical perspective, to show explicitly how plane waves arise in the present context from the motion of the relativistic continuum. We turn to this problem now.

Let us consider a plane electromagnetic wave with wave-vector $\kappa$ travelling in the $x$-direction with the $E$-field plane-polarised in the $y$-direction. The 4-potential describing this plane wave is,

$$A^\mu = (0, A) = (0, 0, A_y \cos(\omega t - \kappa x), 0), \tag{27}$$

where $\omega = c \kappa$, with corresponding $E$ and $B$ fields,

$$E = (0, E_y, 0) = (0, \kappa A_y \sin(\omega t - \kappa x), 0), \quad B = (0, 0, B_z) = (0, 0, \kappa A_y \sin(\omega t - \kappa x)). \tag{28}$$

We therefore seek solutions of the form,

$$A^\mu_+ = (kc \gamma_+, k \gamma_+ v_+), \quad A^\mu_- = (-kc \gamma_-, k \gamma_- v_-). \tag{29}$$

Applying (25) and equating with (27) we obtain the two conditions,

$$\gamma_+ = \gamma_- , \quad k(\gamma_+ v_+ - \gamma_- v_-) = (0, A_y \cos(\omega t - \kappa x), 0). \tag{30}$$

Ignoring equal velocity motions of the positive and negative continua which have already been shown to have no electromagnetic consequences, these conditions allow us to restrict our attention to solutions of the form,

$$v_+ = -v_- = (0, v, 0), \quad \text{where} \quad \frac{v}{c} = \frac{A}{\sqrt{A^2 + 4k^2 c^2}}, \tag{31}$$

and we have defined $A = A_y \cos(\omega t - \kappa x)$ for convenience. The velocities of the positive continuum and the negative continuum here are equal in magnitude and opposite in direction, so that there is no net charge, with the motion of both being parallel to the electric field but $\pm \pi/2$ radians out of phase respectively. It also follows from (31) that the velocity...
of the continuum can never exceed the speed of light, irrespective of the intensity of the plane wave. Substituting \(31\) into \(29\) the motion of the continuum is given by,

\[
\begin{align*}
  u^\mu_+ &= (\sqrt{c^2/k^2 + A^2/4k^2}, 0, A/2k, 0), \\
  u^\mu_- &= (\sqrt{c^2/k^2 + A^2/4k^2}, 0, -A/2k, 0).
\end{align*}
\]  

(32)

These equations clearly show that the propagation of a plane electromagnetic wave is described by the oscillation of the medium in the direction of the electric field - the positive continuum oscillates \(\pi/2\) out of phase with \(E\) while the negative continuum oscillates with the same magnitude and precisely the opposite phase. Thus the propagation of electromagnetic waves is seen to be a direct manifestation of the oscillations of the underlying relativistic continuum.

**Gauge Redundancies and the Principle of Superposition**

While the usual principle of superposition obviously still holds for the 4-potential, we can now supplement this with the following continuum-gauge-inspired superposition principle.

Consider two 4-potential fields \(A^\mu = (c\gamma_+ - c\gamma_-, \gamma_+ v_+ - \gamma_- v_-)\) and \(A'^\mu = (c\gamma'_+ - c\gamma'_-, \gamma'_+ v'_+ - \gamma'_- v'_-)\) in the continuum gauge which describe two different 4-velocity field configurations. Then the superposition of the two field configurations is described by the 4-potential \(A''^\mu = (c\gamma''_+ - c\gamma''_-, \gamma''_+ v''_+ - \gamma''_- v''_-)\) where the velocity vector field \(v''_+\) (respectively \(v''_-\)) is given by the pointwise relativistic sum of \(v_+\) and \(v'_+\) (respectively \(v_-\) and \(v'_-\)),

\[
v''_\pm = \frac{v_\pm + v'_\pm}{1 + v_\pm \cdot v'_\pm/c^2}.
\]  

(33)

As mentioned earlier, the description of an electromagnetic configuration in terms of 4-velocities \(u^\mu_+\) and \(u^\mu_-\) is far from unique, as for each of the infinite number of 4-potentials \(A^\mu = (\phi, A)\) with \(|\phi| \leq |A|\) describing that particular configuration, there exists an entire two-parameter set of solutions at each point.

Recall the particular gauge choice in which \(\phi = 0\) everywhere. We saw that the simplest ‘lowest energy’ solution is given in this case by \(v_+ = -v_- = A/2k\). However, we also saw that it is possible to add, relativistically in the sense of \(33\), the same, arbitrary, possibly time-dependent, 3-velocity vector field to both \(v_+\) and \(v_-\) without changing the 4-potential. If these velocity fields have a real physical meaning then this additional freedom will correspond to a large class of vacuum configurations which can perhaps be interpreted in terms of the motion of an arbitrarily distributed ‘Dirac sea’ of particles and antiparticles. This provides a means of adding energy density to the vacuum without any observable electromagnetic effects.

**THE CONTINUUM AS A RELATIVISTIC FLUID**

In this section we show that the spacetime continuum must be a relativistic fluid of massive discrete particles, and that interactions between sources and sinks give rise to the Lorentz force equation. The fact that both Maxwell’s equations and the Lorentz force are consequences of the relativistic fluid model is a strong indication that there is more to this description than mere formalism, and that classical electrodynamics may in reality have a fluid dynamical basis.

**The Massive Continuum**

We saw in \(26\) that the velocity of the continuum decreases with radius outside of the point charge acting as its source. Had the continuum been massless, its velocity would have been constant and equal to \(c\) everywhere. We therefore conclude that the continuum has mass and that there is an attractive central force acting on the continuum outside of the charge.

It is possible to derive an expression for this attractive central force. In particular, if we assume the charged particle is centred at the origin, then the force \(F\) acting on an infinitesimal element of the continuum at radius \(r\) must satisfy\[7\],

\[
F = \frac{dp^i}{dt} = m\gamma^i \frac{dv^i}{dt}.
\]  

(34)
where \( m = \rho_m \delta V \) is the mass of the test element assuming that it has mass density \( \rho_m \) and occupies volume \( \delta V \). To find the value of \( dv/dt \), solve (26) for \( v \) and differentiate the resulting equation with respect to \( t \) to find an expression for \( dv/dt \) in terms of \( v \). Rearranging terms and simplifying, the field at radius \( r \) is found to have the form,

\[
\gamma^2 \frac{dv}{dt} = -\frac{qc}{kr^2}.
\]  

(35)

Thus there appears to be a Coulombic attraction between the charge and the continuum around it, with the continuum having a charge-to-mass ratio of \(-c/k\). This is quite mysterious as in our model charge is defined in terms of the motion of the continuum, so clearly the continuum itself cannot be charged. The mystery will be resolved in due course.

Assuming continuum conservation, the continuum density \( \rho_n \) will satisfy the following continuity equation,

\[
\partial_\mu (\rho_n \gamma^\mu) = \frac{\partial (\rho_n \gamma)}{\partial t} - \nabla \cdot (\rho_n \gamma \nu) = 0 \quad \Rightarrow \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_n \gamma) = 0.
\]  

(36)

where we ignore the time-derivative term as the system is in a steady state condition, and use the rotational symmetry to rewrite the divergence term in its spherical polar form. The solution is,

\[
\rho_n = \frac{S}{4\pi r^2 \gamma},
\]  

(37)

where \( S \) is a radius-independent proportionality factor. Now, the flux of continuum passing through a spherical shell at radius \( r \) is just \( \Phi = 4\pi r^2 \rho_n \gamma \) (where the factor of \( \gamma \) takes into account to the relativistic contraction in the radial direction). But this is precisely the constant \( S \) in (35) which can therefore be identified as the strength of the charged particle sink/source.

**The Relativistic Fluid**

We discovered in the previous subsection that there is an inverse-square law attraction of elements of the continuum towards the point charge. Given that the continuum density is greater closer to the charged particle, let us investigate the possibility that the continuum may be a continuous, compressible, medium whose attractive self-interactions result in the observed attraction. If the attractive force between two volume elements of the continuum is given by,

\[
dF(r_1, r_2) \sim \rho_n(r_1) \rho_n(r_2) f(|r_1 - r_2|).
\]  

(38)

where \( f(r) \) is some polynomial in \( r \), then a little calculation shows that an inverse square attraction is possible only if \( f(r) \sim r^{-4} \). However, the magnitude of the resultant force on any element turns out to be infinitely large in this case.

There are three sources of these (logarithmic) divergences - (i) the contribution from the core of the point charge where the continuum density becomes infinite, (ii) the contribution from the continuum at infinity, and (iii) the contribution from continuum elements in the immediate neighbourhood of that element. The first of these can be avoided if the charges are not pointlike, the second can be avoided if the universe is either bounded or homogeneous, and the third can be avoided by discarding the idea that the continuum is some kind of continuous elastic medium, but rather consists of a fluid of interacting discrete particles.

We are therefore led to conclude that our relativistic continuum is a space-filling relativistic fluid and the electromagnetic 4-potential must be defined in terms of the ensemble motion of the fluid as opposed to the motion of the individual discrete particles. If the instantaneous fluid velocity at \( x^\mu \) is \( \zeta^\mu(x) \), then the 4-velocity appearing in (5) is,

\[
u^\mu(x) = < \zeta^\mu > ,
\]  

(39)

where \( < \zeta^\mu > \) indicates the time-averaged motion of the particles in the neighbourhood of \( x^\mu \). All other electrodynamic quantities must be defined as time-averages in the same way.

Although the configuration representing a charged particle is in steady state, the fluid itself remains in constant motion. Recall that the motion of an individual particle in the co-moving frame is described by the total derivative \( \frac{d\zeta^\mu}{d\tau} = (\zeta^\nu \partial_\nu) \zeta^\mu = (\zeta^\nu \partial_\nu) \zeta^\mu - \frac{1}{2} \partial^\mu (\zeta^\nu \zeta_\nu) = (\partial^\mu (\zeta^\nu \zeta_\nu) - \partial^\nu \zeta^\mu) \zeta_\nu , \)

(40)

where we have added a vanishing term using the fact that \( \zeta^\mu \partial_\mu \zeta^\mu = c^2 \). If we now consider the time-averaged version of (40) and recall the definitions (39), (5) and (7), we find that,
which is in the form of the Lorentz force equation. In particular we find that, on average, each particle moves \( \text{as if} \) it were charged with \( q/m = -c/k \). This is precisely the charge-to-mass ratio observed in the Coulomb-like attraction of the self-gravitating fluid, hence in (35), and so the earlier mystery has been resolved. Because (40) is a basic identity valid for any motion of the relativistic fluid, this conclusion holds irrespective of the precise nature of the interactions between the fluid particles.

As further compelling evidence that classical electrodynamics has relativistic fluid dynamics as its basis, we will now show that the Lorentz force equation emerges automatically from the interaction between sources and sinks when they are \text{not} assumed to be fixed in position.

The integral momentum equation for a fluid tells us that the force on a target charged particle with charge \( Q \) due to a source particle of charge \( q \) at distance \( r \) is given by the rate of change of momentum transfer to the target by the particles entering or leaving the source. If we suppose that the target particle has an effective radius \( R \) then, assuming spherical symmetry, it will have an effective volume of \( \frac{4}{3}\pi R^3 \). In accordance with (37), the density of fluid particles encountering the target at distance \( r \) from the source is \( \rho_n(r) \). If we further assume that each fluid particle is identical with mass \( m \), then the 3-momentum carried by each is given by \( m\gamma v \). Finally, the collision rate will be determined by the strength \( S' \) of the target. Thus the force on the target will be given by the product of these contributions,

\[
F = \frac{4}{3}\pi R^3 \rho_n m\gamma v S' = \frac{mSS' R^3}{3r^2},
\]

where he have used (37). This takes precisely the form of Coulomb’s law if we make the following identification,

\[
Q = S\sqrt{\frac{mR^3}{3}},
\]

where the charge \( Q \) is expressed in terms of the strength of the source \( S \), the mass \( m \) of the fluid particles and the effective charge radius \( R \). Clearly for (42) to hold, positive charges must effectively act as sinks, and negative charges as sources, or vice versa\(^1\).

The validity of Coulomb’s law in turn implies the validity of the Lorentz force equation\(^9\), as we have assumed from the outset that relativity holds. This completes our description of classical electrodynamics.

### Stochastic Quantum Mechanics from the Self-Gravitating Fluid

We have shown that classical electrodynamics can be explained at the macroscopic level in terms of the ensemble motion of a relativistic fluid of massive (i.e. gravitationally interacting), discrete particles. At a microscopic level this allows us to make an explicit identification of our model with the stochastic formulation of quantum mechanics.

In his monograph\(^10\), Nelson gave a detailed derivation of quantum mechanics on the basis of the conservative diffusion of a classical fluid, wherein the Schrödinger wavefunction is identified with the density of the fluid thus, \( \psi = \sqrt{\rho}e^{iS/h} \),

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where \( S \) is the stochastic analogue of Hamilton’s principle function. However there remained a number of important unresolved problems. The first of these was “...to find a classical Lagrangian, of system + background field oscillators + interaction, that [...] produces a conservative diffusion system.” Traditionally the system is assumed to be coupled in some way to an electromagnetic background\(^11\). We have succeeded here in showing that the classical fluid \text{is} the electromagnetic background which Nelson sought.

Neither was he able to come to any definite conclusion about the nature of the interparticle interactions responsible for the conservative diffusion - except that it could \text{not} be gravitational on dimensional grounds and that it was possibly of electromagnetic origin. This second issue has also been resolved here as gravitational interactions are themselves responsible for electromagnetism. The possibility of a gravitational explanation for the conservative diffusion, and consequently for the observed magnitude of Planck’s constant, has also been proposed by Calogero\(^12\).

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\(^1\) The fact that equal velocity contributions of fluid particles from the positive and negative continua have no electromagnetic effects means that the net momentum transfer must be zero, which in turn implies that particles in the negative continuum must have equal and opposite mass to those in the positive continuum. The constant \( k \) in (5) must then be proportional to the mass of the fluid particles, so that the 4-potential \( A^\mu \) is nothing but the net 4-momentum of the two-component fluid. This is a radically different interpretation of the 4-potential from the one we are accustomed to.
SUMMARY AND CONCLUSION

We have demonstrated the simple yet profound result that all of the equations of classical electrodynamics follow from the motion of a two-component relativistic continuum satisfying the standard equations of relativistic fluid dynamics. Charged particles appear as sources and sinks of the continuum in this framework, while electromagnetic waves are associated with oscillations of the continuum. There is a freedom inherent in the 4-velocity description of electrodynamics which can potentially account for 'dark matter'. The identification of the vacuum as a self-gravitating fluid of discrete particles makes possible an explicit connection to Nelson’s stochastic formulation of quantum mechanics, raising the tantalising prospect of having a natural, unified, description of gravitation and quantum electrodynamics purely in terms of classical general relativity.

Maxwell and others had struggled to find a mathematical description of the underlying medium, the ‘aether’, in which electromagnetic waves were presumed to propagate. Although the continuum we have described is not precisely equivalent to the notion which the earlier proponents had had in mind, our analysis does show that a description of electrodynamics in terms of an underlying continuum is possible. This is particularly important as the failure to find such a formulation historically contributed to the origin of the concept of ‘fields’ postulated not to require such a medium. The field concept may not have been necessary after all.

A number of issues still remain open. We are still left to ponder the existence, interpretation and physical properties of the fluid particles and their sources and sinks, to explain the origin of quantised mass and charge, and the presence of the two continuum components. Considerable evidence has already been gathered that each of these issues can be resolved completely within the framework of general relativity, and we feel that a unified classical description of quantum theory and gravity is now close at hand. It would certainly be fitting if Einstein’s dream were finally to be realised on the 100th anniversary of the birth of his theory of relativity.

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