Explicit Föllmer–Schweizer Decomposition and Discrete-Time Hedging in Exponential Lévy Models

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Abstract. In a financial market driven by an exponential Lévy process, an explicit representation is shown for the Föllmer–Schweizer decomposition of European type options, implying a closed-form expression of the corresponding local risk-minimizing strategies. Using a jump-adjusted approximation scheme, the error caused by discretising the local risk-minimizing strategies is investigated in dependence of properties of the Lévy measure, the regularity of the pay-off function and the chosen random discretisation times. The rate of this error as the number of expected discretisation times increases is measured in weighted BMO spaces, implying also $L_p$-estimates. Moreover, the effect of a change of measure satisfying a reverse Hölder inequality is addressed.

1. Introduction

This article is concerned with hedging problems in financial markets driven by exponential Lévy processes. We investigate two problems corresponding to two typical types of risks for hedging an option. The first one comes from the incompleteness of the market. We consider the semimartingale setting and aim to determine an explicit form for the Föllmer–Schweizer decomposition of European type options which provides directly a closed form for the local risk-minimizing strategies (a similar closed form expression in the martingale setting has been established in [8, 19, 36, 37]). The second type of risk is due to the impossibility of continuously rebalancing a hedging portfolio which leads to the discrete-time hedging. The discretisation error we measure in weighted bounded mean oscillation spaces from which one can achieve good distributional tail estimates such as a $p$th-order polynomial decay, $p \in (2, \infty)$.

Let us introduce some notations to state the main results. Let $T \in (0, \infty)$ be a fixed time horizon and $X = (X_t)_{t \in [0,T]}$ a Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the augmented natural filtration of $X$ which satisfies the usual conditions (right continuity and completeness). Assume that $\mathbb{F} = \mathcal{F}_T$. Let $\sigma \geq 0$ be the coefficient of the standard Brownian component and $\nu$ the Lévy measure of $X$ (see (2.1)). We assume that the underlying discounted price process is modelled by the exponential $S = e^X$.

1.1. Explicit Föllmer–Schweizer (FS) decomposition. Because models with jumps correspond to incomplete markets, in general there is no hedging strategy which is self-financing and replicates an option at maturity. Hence, one has to look for certain strategies that minimize some types of risk. In the current work, we choose the quadratic hedging approach which is a popular method to deal with the problem in models.
with jumps. We refer the reader to the survey article [34] for this approach. Two typical types of quadratic hedging strategies are the local risk-minimizing (LRM) strategies and the mean-variance hedging (MVH) strategies. Roughly speaking, the LRM strategy is mean-self-financing, replicates an option at maturity and minimizes the riskiness of the cost process locally in time, while the MVH strategy is self-financing and minimizes the global hedging error in the mean square sense. Both types of those strategies are intimately related to the so-called FS decomposition. Namely, in our (exponential Lévy) setting, the FS decomposition gives directly the LRM strategy, and the MVH strategy can be determined based on this decomposition. This article discusses the FS decomposition and focuses on the LRM strategies only.

Assume that $S$ is square integrable so that it is a semimartingale satisfying the structure condition, and that the mean-variance trade-off process of $S$ is deterministic and bounded (see Remark 4.3). Then, the FS decomposition of an $H \in L^2(\mathbb{P})$ is of the form

$$H = H_0 + \int_0^T \vartheta^H_t dS_t + L^H_T,$$

(1.1) where $H_0 \in \mathbb{R}$, $\vartheta^H$ is an admissible integrand (specified in (4.2)), and $L^H$ is an $L^2(\mathbb{P})$-martingale starting at zero which is orthogonal to the martingale part of $S$. The integrand $\vartheta^H$ is called the LRM strategy of $H$, and it is unique up to a $\mathbb{P} \otimes \lambda$-null set. A key tool to study the FS decomposition is the minimal (signed) local martingale measure for $S$ (see [33]), and we denote this signed measure by $\mathbb{P}^*$ from now on. Recently, [6, Theorem 4.3] indicated that under a regularity condition for $\mathbb{P}^*$, we can determine the LRM strategy $\vartheta^H$ based on the martingale representation of $H$ with respect to $\mathbb{P}^*$.

There are many works interested in finding an explicit representation for the FS decomposition and the LRM strategy in the semimartingale framework (see, e.g., [2, 16, 17, 20, 36]). In the exponential Lévy setting and in the case of a European type option $H = g(S_T)$, Hubalek et al. [17] assumed that the function $g$ can be represented as an integral transform of finite complex measures from which one can determine a closed form for the LRM strategy. The key idea of this approach is the separation of the function $g$ and the underlying price process $S$ by using a kind of inverse Fourier transform. An advantage of this method is that one gains much flexibility for choosing the underlying Lévy process where there is no extra regularity required for the driving process $S$ except some mild integrability.

As our first main result, Theorem 1.1 below provides a closed form for the LRM strategy $\vartheta^H$ of an $H = g(S_T)$. To obtain this result, except of some mild integrability conditions, we neither assume any regularity for the payoff function $g$ nor require any extra condition for the small jump behavior of $X$. However, the price one has to pay is the condition that $\mathbb{P}^*$ exists as a true probability measure (see Assumption 4.5) which leads to a constraint for the characteristics of $X$. This result might be regarded as a counterpart of [17, Proposition 3.1] in which only the square integrability is required for $S$ while the function $g$ are supposed to be the integral transform of finite complex measures. The notation $\mathbb{E}^*$ below means the expectation with respect to $\mathbb{P}^*$.

**Theorem 1.1.** Assume that $X$ is not a.s. deterministic and $S = e^X$ is square $\mathbb{P}$-integrable. Under Assumption 4.5, if $g: (0, \infty) \to \mathbb{R}$ is a Borel function with $\mathbb{E}^*[|g(yS_t)|] < \infty$ for all $(t, y) \in [0, T] \times (0, \infty)$ and $g(S_T) \in L^2(\mathbb{P}) \cap L^2(\mathbb{P}^*)$, then the following assertions hold:
(1) The LRM strategy $\vartheta^H$ corresponding to $H = g(S_T)$ is of the form

$$\vartheta^H_t = \frac{1}{\|(\sigma, \nu)\|} \left( a^2 \partial_y G^*(t, S_{t-}) + \int_{\mathbb{R}} \frac{G^*(t, e^x S_{t-}) - G^*(t, S_{t-})}{S_{t-}} (e^x - 1) \nu(dx) \right)$$

for $\mathbb{P} \otimes \lambda$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $\|(\sigma, \nu)\| := a^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \in (0, \infty)$, $G^*(t, y) := \mathbb{E}^* g(y S_{T-t})$, and we set $\partial_y G^* := 0$ when $\sigma = 0$ by convention.

(2) There exists a process $\tilde{\vartheta}^g$ which is adapted and càdlàg on $[0, T)$, satisfies $\tilde{\vartheta}^g_S = \vartheta^H$ for $\mathbb{P} \otimes \lambda$-a.e. $(\omega, t) \in \Omega \times [0, T)$, and $\tilde{\vartheta}^g$ is a $\mathbb{P}^*$-martingale.

According to Theorem 1.1(2), $\tilde{\vartheta}^g$ is also a LRM strategy of $H = g(S_T)$, and one can determine it at every time $t \in [0, T)$ as showed in Remark 4.6 below. Furthermore, the càdlàg property of $\tilde{\vartheta}^g$ is useful to design some Riemann-type approximations for $\int_0^T \tilde{\vartheta}^g_t dS_t$. For example, an approximation scheme based on tracking jumps of $\tilde{\vartheta}^g$ has been constructed in [30]. We also employ the càdlàg version of the LRM strategy for the discrete-time hedging problem in Section 5. Such a path regularity for the integrand in the martingale setting was also studied in [24].

Some formulas resembling (1.2) have been established in [19, Formula (2.12)], [8, Formula (4.1)], [36, Formula (45)], or in [37, Formula (4.2)]. But in fact they are different. The formulas in [19, 8, 36, 37] were obtained by projecting $H$ orthogonally down to the space of stochastic integrals driven by a (local) martingale, while the formula (1.2) is derived from the FS decomposition which is a different orthogonal decomposition in the semimartingale framework.

The proof of Theorem 1.1 is provided in Section 4, and the main tool we use is Proposition 1.2 where the square integrability of $e^X$ is not necessarily assumed. We denote by $W$ the standard Brownian motion and by $\tilde{N}$ the compensated Poisson random measure appearing in the Lévy–Itô decomposition of $X$ (see, e.g., [1, Theorem 2.4.16]).

Proposition 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\mathbb{E}|f(x + X_t)| < \infty$ for all $(t, x) \in [0, T] \times \mathbb{R}$. If $f(X_T) \in L_2(\mathbb{P})$, then

$$\mathbb{E} \int_0^T |\sigma \partial_x F(t, X_{t-})|^2 dt + \mathbb{E} \int_0^T \int_{\mathbb{R}} |F(t, x + x - F(t, x_{t-})|^2 \nu(dx)dt < \infty$$

and, a.s.,

$$f(X_T) = \mathbb{E} f(X_T) + \int_0^T \sigma \partial_x F(t, X_{t-}) dW_t$$

$$+ \int_0^T \int_{\mathbb{R}\{0\}} (F(t, x_{t-} + x) - F(t, x_{t-})) \tilde{N}(dt, dx),$$

where $F(t, x) := \mathbb{E} f(x + X_{T-t})$ for $(t, x) \in [0, T] \times \mathbb{R}$, and we set $\partial_x F := 0$ if $\sigma = 0$.

Proposition 1.2 provides a martingale representation for functionals of $X_T$ in which the integrands with respect to the Brownian part and the jump part are determined explicitly. Its proof is given in Section 3 by using Malliavin calculus. We also remark here that (1.3) is a Clark–Ocone type formula but $f(X_T)$ is not necessarily differentiable in the Malliavin sense.

Proposition 1.2 extends [8, Proposition 7] in which the function $f$ has a polynomial growth and $X$ satisfies a certain condition. A similar representation to (1.3) in a general framework (with different assumptions from ours) can be found in the proof of [19, Theorem 2.4]. On the other hand, when $f(X_T)$ is Malliavin differentiable then one can use the Clark–Ocone formula (see, e.g., [2, 3, 23]) to obtain its explicit martingale
representation. However, the Malliavin differentiability of $f(X_T)$ fails to hold in many contexts. For example, if $f(x) = 1_{[K,\infty)}(x)$ for some $K \in \mathbb{R}$, $X$ is of infinite variation and $X_T$ has a density satisfying a mild condition, then $f(X_T)$ is not Malliavin differentiable (see [22, Theorem 6(b)]).

1.2. Discrete-time hedging in weighted bounded mean oscillation (BMO) spaces. We investigate the discrete-time approximation problem for stochastic integrals driving by the exponential Lévy process $S$. Let $E = (E_t)_{t \in [0,T]}$ be the error given by

$$E_t := \int_0^t \vartheta_u dS_u - A_t, \quad t \in [0,T],$$

where $\vartheta$ is an admissible integrand and $A = (A_t)_{t \in [0,T]}$ is an approximation scheme for the stochastic integral. In mathematical finance, the stochastic integral can be interpreted as the theoretical hedging portfolio which is continuously readjusted. However, in practice one can only rebalance the portfolio finitely many times, and this leads to a discretisation of the stochastic integral, represented by $A$.

In case that $A = A_{Rm}$ is the Riemann approximation process, the caused error $E = E_{Rm}$ and its convergence rate have been investigated in the $L_2$-sense in several works. When $S$ is assumed to be a martingale, the error was examined in [5, 11]. The error was also considered in a more general setting in [30] where the driving process is a local martingale with jumps. In general, the $L_2$-approach for the error yields a second-order polynomial decay for its distributional tail by Markov’s inequality.

In the second part of this article, we aim to improve the distributional tail estimate for the approximation error by means of the weighted bounded mean oscillation (weighted BMO) approach. Moreover, the driving process $S$ is not necessarily a (local) martingale but a semimartingale. To do this, we use the approximation scheme introduced in [37], the so-called jump adjusted method which was constructed by tracking jumps of the driving process $S$. Moreover, we show how the theory of weighted BMO spaces can be used to obtain $L_p$-estimates, $p \in (2, \infty)$, for the corresponding error. This approach also allows a change of the underlying measure which leaves the error estimates unchanged provided the change of measure satisfies a reverse Hölder inequality (see Proposition 5.3). The latter is useful to switch the problem between the martingale setting and the semimartingale setting.

The main results of the second part are Theorems 5.7 and 5.12 below. In Theorem 5.7, we provide several estimates for the error measured in weighted BMO-norms and describe a situation so that the $L_p$-estimate can be achieved for $p \in (2, \infty)$. Theorem 5.12 serves as an application of Theorem 5.7 where we consider the approximation problem for the stochastic integral term in (1.1) and the chosen integrand is the LRM strategy of a European type option. The results show how the interplay between the regularity of payoff functions and the small jumps intensity of the underlying Lévy process affects the convergence rate.

1.3. Structure of the article. We introduce the notation and recall Malliavin–Sobolev spaces and exponential Lévy processes in Section 2. The proof of Proposition 1.2 is contained in Section 3. Section 4 is devoted to prove Theorem 1.1. Section 5 presents the discrete-time hedging problem with the weighted BMO-approach for exponential Lévy models. Some technical results used in this article are given in Appendix A.
2. Preliminaries

2.1. General notations. Denote \( \mathbb{R}_+ := (0, \infty) \) and \( \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \). For \( a, b \in \mathbb{R} \), we set \( a \lor b := \max\{a, b\} \) and \( a \land b := \min\{a, b\} \). For \( A, B \geq 0 \) and \( c \geq 1 \), by \( A \sim_c B \) we mean \( \frac{1}{c} A \leq B \leq cA \). Subindexing a symbol by a label indicates the place where that symbol appears (e.g., \( \epsilon_5 \)).

Let \( B(\mathbb{R}) \) be the Borel \( \sigma \)-algebra on \( \mathbb{R} \). The Lebesgue measure on \( B(\mathbb{R}) \) is denoted by \( \lambda \), and we also write \( dx \) instead of \( \lambda(dx) \) for simplicity. For \( p \in [1, \infty] \) and \( \mathbb{A} \in B(\mathbb{R}) \), the space \( L^p(\mathbb{A}) \) consists of all \( p \)-order integrable Borel functions on \( \mathbb{A} \) with respect to \( \lambda \), where the essential supremum is taken when \( p = \infty \). For a measure \( \mu \) defined on \( B(\mathbb{R}) \), its support is defined by

\[
\text{supp} \mu := \{ x \in \mathbb{R} : \mu((x-\epsilon, x+\epsilon)) > 0, \forall \epsilon > 0 \}.
\]

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space and \( \xi : \Omega \to \mathbb{R} \) a random variable. Denote by \( \mathbb{P}_{\xi} \) the push-forward measure of \( \mathbb{P} \) with respect to \( \xi \). If \( \xi \) is integrable (non-negative), then the (generalized) conditional expectation of \( \xi \) given a sub-\( \sigma \)-algebra \( \mathbb{G} \subseteq \mathcal{F} \) is denoted by \( \mathbb{E}_{\mathbb{P}}[\xi|\mathbb{G}] \). We set \( L^p_{\mathbb{P}}(\mathbb{A}) := L^p(\Omega, \mathcal{F}, \mathbb{P}) \).

For a non-empty and open interval \( U \subseteq \mathbb{R} \), let \( C^\infty(U) \) denote the family of all functions \( f \) which have derivatives of all orders on \( U \).

2.2. Notation for stochastic processes. Let \( T > 0 \) be a fixed finite time horizon, and let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space equipped with a right continuous filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]} \). Assume that \( \mathcal{F}_0 \) is generated by \( \mathbb{P} \)-null sets only. The conditions imposed on \( \mathcal{F} \) allow us to assume that every martingale adapted to this filtration is càdlàg (right continuous with left limits). We use the following notations and conventions where \( \mathbb{I} = [0, T] \) or \( \mathbb{I} = [0, T) \).

- For processes \( X = (X_t)_{t \in \mathbb{I}} \) and \( Y = (Y_t)_{t \in \mathbb{I}} \), we write \( X = Y \) to indicate that \( X_t = Y_t \) for all \( t \in \mathbb{I} \) a.s., and similarly when the relation \( \approx \) is replaced by some other standard relations such as \( \ll \), \( \gg \), etc.
- For a càdlàg process \( X = (X_t)_{t \in \mathbb{I}} \), the process \( X_- = (X_{t-})_{t \in \mathbb{I}} \) is defined by setting \( X_{t-} := X_0 \) and \( X_{t-} := \lim_{0<s \downarrow t} X_s \) for \( t \in \mathbb{I} \setminus \{0\} \). We set \( \Delta X := X - X_- \).
- \( \mathcal{CL}(\mathbb{I}) \) denotes the family of all càdlàg and \( \mathcal{F} \)-adapted processes.
- \( \mathcal{CL}_0(\mathbb{I}) \) (resp. \( \mathcal{CL}^+(\mathbb{I}) \)) consists of all \( X \in \mathcal{CL}(\mathbb{I}) \) with \( X_0 = 0 \) a.s. (resp. \( X \geq 0 \)).
- For \( p \in [1, \infty] \) and \( X \in \mathcal{CL}([0, T]) \), we set \( \|X\|_{L^p(\mathbb{P})} := \| \sup_{t \in [0,T]} |X_t| \|_{L^p(\mathbb{P})} \).

- \( \mathcal{P} \) is the predictable \( \sigma \)-algebra\(^1\) on \( \Omega \times [0, T] \) and \( \tilde{\mathcal{P}} := \mathcal{P} \otimes B(\mathbb{R}) \).

We recall some notions regarding semimartingales on the finite time interval \([0, T]\).

- A process \( M \in \mathcal{CL}([0, T]) \) is called a local (resp. locally square integrable) martingale if there is a sequence of non-decreasing stopping times \((\rho_n)_{n \geq 1}\) taking values in \([0, T]\) such that \( \mathbb{P}(\rho_n < T) \to 0 \) as \( n \to \infty \) and the stopped process \( M^{\rho_n} = (M_{t \wedge \rho_n})_{t \in [0,T]} \) is a martingale (resp. square integrable martingale) for all \( n \geq 1 \). Let \( \mathcal{M}^0_{\mathbb{P}}(\mathbb{R}) \) be the space of all square integrable \( \mathbb{P} \)-martingales \( M = (M_t)_{t \in [0,T]} \) with \( M_0 = 0 \) a.s.
- A process \( S \in \mathcal{CL}([0, T]) \) is called a semimartingale if \( S \) can be written as a sum of a local martingale and a process of finite variation a.s. The quadratic covariation of two semimartingales \( S \) and \( R \) is denoted by \([S, R]\). The predictable \( \mathbb{Q} \)-compensator of \([S, R]\), if it exists, is denoted by \([S, R] \circ \mathbb{Q}\), where \( \mathbb{Q} \) is a probability measure. We will omit the reference measure if there is no risk of confusion.

\(^1\)\( \mathcal{P} \) is the \( \sigma \)-algebra generated by \( \{ A \times \{0\} : A \in \mathcal{F}_0 \} \cup \{ A \times (s, t) : 0 \leq s < t \leq T, A \in \mathcal{F}_s \} \).
Let $M, N$ be locally square integrable martingales under a probability measure $Q$.
Then, $M$ and $N$ are said to be $Q$-orthogonal if $[M, N]$ is a local martingale under $Q$, or equivalently, $\langle M, N \rangle^Q = 0$.

2.3. Lévy process and Itô’s chaos expansion. Let $X = (X_t)_{t \in [0, T]}$ be a real-valued Lévy process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F})$, i.e., $X_0 = 0$, $X$ has independent and stationary increments and $X$ has càdlàg paths. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ denote the augmented natural filtration generated by $X$. From now on, we assume that $\mathbb{F} = \mathcal{F}_T$. According to the Lévy–Khintchine formula (see, e.g., [31, Theorem 8.1]), the characteristic exponent $\psi$ of $X$, which is defined by

$$Ee^{iuX_t} = e^{-\psi(u)}, \quad u \in \mathbb{R}, t \in [0, T],$$

is of the form

$$\psi(u) = -i\gamma u + \frac{\sigma^2 u^2}{2} - \int_{\mathbb{R}} \left(e^{ixu} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}}\right) \nu(dx), \quad u \in \mathbb{R}. \quad (2.1)$$

Here, $\gamma \in \mathbb{R}$, while $\sigma \geq 0$ is the coefficient of the Brownian component, and $\nu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ is a Lévy measure (i.e. $\nu(\{0\}) := 0$ and $\int_{\mathbb{R}} (x^2 + 1)\nu(dx) < \infty$). The triplet $(\gamma, \sigma, \nu)$ is also called the characteristics of $X$. To indicate explicitly the characteristics of $X$ under $\mathbb{P}$, we write

$$(X|\mathbb{P}) \sim (\gamma, \sigma, \nu) \quad \text{or} \quad (X|\mathbb{P}) \sim \psi.$$  

We present briefly the Malliavin calculus for Lévy processes by means of Itô’s chaos expansion which is the main tool to prove Proposition 1.2. For further details, we refer to [35, 27, 28, 1] and the references therein. Define the $\sigma$-finite measures $\mu$ on $\mathcal{B}(\mathbb{R})$ and $m$ on $\mathcal{B}([0, T] \times \mathbb{R})$ by setting

$$\mu(dx) := \sigma^2 \delta_0(dx) + x^2\nu(dx) \quad \text{and} \quad m := \lambda \otimes \mu,$$

where $\delta_0$ is the Dirac measure at zero. For $B \in \mathcal{B}([0, T] \times \mathbb{R})$ with $m(B) < \infty$, the random measure $M$ is defined by

$$M(B) := \sigma \int_{\{t \in [0, T] : (t, 0) \in B\}} dW_t + L_2(\mathbb{P})- \lim_{n \to \infty} \int_{B \cap ([0, T] \times \{|x| < \frac{1}{n}\})} x\tilde{N}(dt, dx),$$

where $W$ is the standard Brownian motion and $\tilde{N}$ is the compensated Poisson random measure appearing in the Lévy–Itô decomposition of $X$ (see, e.g., [1, Theorem 2.4.16]).

Set $L_2(\mu^0) = L_2(m^0) := \mathbb{R}$. For $n \geq 1$, we denote

$$L_2(\mu^{\otimes n}) := L_2(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu^{\otimes n}),$$

$$L_2(m^{\otimes n}) := L_2(([0, T] \times \mathbb{R})^n, \mathcal{B}(([0, T] \times \mathbb{R})^n), m^{\otimes n}).$$

The multiple integral $I_n : L_2(m^{\otimes n}) \to L_2(\mathbb{P})$ is defined in the sense of Itô [18] by using an approximation argument, where it is given for simple functions as follows: For

$$\xi^m_n := \sum_{k=1}^m a_k \mathbb{1}_{B^k_1 \times \cdots \times B^k_n},$$

where $a_k \in \mathbb{R}$, $B^k_i \in \mathcal{B}([0, T] \times \mathbb{R})$ with $m(B^k_i) < \infty$ and $B^k_i \cap B^k_j = \emptyset$ for $k = 1, \ldots, m$, $i, j = 1, \ldots, n$, $i \neq j$ and $m \geq 1$, we define

$$I_n(\xi^m_n) := \sum_{k=1}^m a_k M(B^k_1) \cdots M(B^k_n).$$
Lemma 3.3; secondly, the Malliavin derivative of a
Exponential Lévy processes.

Then, [18, Theorem 2] asserts the following Itô chaos expansion
$$L_2(P) = \bigoplus_{n=0}^{\infty} \{I_n(\xi_n) : \xi_n \in L_2(m^{\otimes n})\},$$
where $I_0(\xi_0) := \xi_0 \in \mathbb{R}$. For $n \geq 1$, the symmetrization $\tilde{\xi}_n$ of a $\xi_n \in L_2(m^{\otimes n})$
is
$$\tilde{\xi}_n((t_1, x_1), \ldots, (t_n, x_n)) := \frac{1}{n!} \sum_{\pi} \xi_n((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)})),$$
where the sum is taken over all permutations $\pi$ of $\{1, \ldots, n\}$, so that $I_n(\xi_n) = I_n(\tilde{\xi}_n)$ a.s.
The Itô chaos decomposition verifies that $\xi \in L_2(P)$ if and only if there are $\xi_n \in L_2(m^{\otimes n})$
such that $\xi = \sum_{n=0}^{\infty} I_n(\xi_n)$ a.s., and this expansion is unique if every $\xi_n$ is symmetric,
i.e. $\xi_n = \tilde{\xi}_n$. Furthermore, $\|\xi\|_{L_2(P)}^2 = \sum_{n=0}^{\infty} n! \|\tilde{\xi}_n\|_{L_2(m^{\otimes n})}^2$.

**Definition 2.1.** Let $\mathbb{D}_{1,2}$ be the Malliavin–Sobolev space of all $\xi = \sum_{n=0}^{\infty} I_n(\xi_n) \in L_2(P)$ such that
$$\|\xi\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \|\tilde{\xi}_n\|_{L_2(m^{\otimes n})}^2 < \infty.$$ The *Malliavin derivative operator* $D : \mathbb{D}_{1,2} \to L_2(P \otimes m)$, where $L_2(P \otimes m) := L_2(\Omega \times [0, T] \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R}), \mathbb{P} \otimes m)$, is defined for $\xi = \sum_{n=0}^{\infty} I_n(\xi_n) \in \mathbb{D}_{1,2}$ by
$$D_{t,x}\xi := \sum_{n=1}^{\infty} n I_{n-1}(\tilde{\xi}_n((t, x), \cdot)),$$
$$(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}.$$

**2.4. Exponential Lévy processes.** Let $X$ be a Lévy process with $(X|\mathbb{P}) \sim (\gamma, \sigma, \nu)$. The stochastic exponential of $X$, denoted by $\mathcal{E}(X)$, is the càdlàg process that satisfies the stochastic differential equation (SDE)
$$d\mathcal{E}(X) = \mathcal{E}(X) \cdot dX, \quad \mathcal{E}(X)_0 = 1.$$ We apply [1, Theorem 5.1.6] with the truncation function $x \mathbb{1}_{\{|x| \leq 1\}}$ instead of $x \mathbb{1}_{\{|x| < 1\}}$
to obtain that if $\mathcal{E}(X) > 0$, then there exists a Lévy process $Y$ with $(Y|\mathbb{P}) \sim (\gamma Y, \sigma_Y, \nu_Y)$
such that $\mathcal{E}(X) = e^Y$,
$$\nu_Y(B) = \int_{\mathbb{R}} \mathbb{1}_{\{|\ln(1+x)| \leq 1\}} \nu(dx), \quad B \in \mathcal{B}(\mathbb{R}),$$
$$\gamma_Y = \gamma - \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(\mathbb{1}_{\{|\ln(1+x)| \leq 1\}} \ln(1+x) - x \mathbb{1}_{\{|x| \leq 1\}}\right) \nu(dx).$$
Conversely, there is a Lévy process $Z$ with $(Z|\mathbb{P}) \sim (\gamma Z, \sigma_Z, \nu_Z)$ such that $e^X = \mathcal{E}(Z)$. Moreover, one has $\sigma_Z = \sigma$ and
$$\nu_Z(B) = \int_{\mathbb{R}} \mathbb{1}_{\{|e^x-1| \leq 1\}} \nu(dx), \quad B \in \mathcal{B}(\mathbb{R}),$$
$$\gamma_Z = \gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left((e^x-1) \mathbb{1}_{\{|e^x-1| \leq 1\}} - x \mathbb{1}_{\{|x| \leq 1\}}\right) \nu(dx).$$

**3. Martingale representation with explicit integrands**

This section is devoted to prove Proposition 1.2 by using Malliavin calculus. There are two key observations: first, the kernels in the chaos expansion of $f(X_T) \in L_2(P)$
do not depend on the time variables which implies the Malliavin differentiability of $\mathbb{E}_\mathbb{F}_t[f(X_T)]$ for any $t \in [0, T)$ (see Lemma 3.3); secondly, the Malliavin derivative of a
Lemma 3.2. Assume \( \sigma > 0 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a Borel function with \( \mathbb{E}|f(X_T)|^q < \infty \) for some \( q > 1 \). Then, \( \mathbb{E}|f(x + X_{T-t})| < \infty \) for all \( (t,x) \in [0,T] \times \mathbb{R} \), and the function \( x \mapsto F(t,x) := \mathbb{E}f(x + X_{T-t}) \) belongs to \( C^\infty(\mathbb{R}) \) for any \( t \in [0,T) \). Furthermore,

\[
\mathbb{E}_F[\partial_x F(t,X_t)] = \partial_x F(s,X_s) \quad \text{a.s.}
\]

for any \( 0 \leq s < t < T \).

Lemma 3.2 below was obtained in [21, Corollary 3.1 in the second article of this thesis] and it provides an equivalent condition such that a functional of \( X_t \) belongs to \( \mathcal{D}_{1,2} \). We refer to [25, Proposition V.2.3.1] when \( X \) is a Brownian motion and refer to [12, Lemma 3.2] when \( X \) has no Brownian component.

Lemma 3.2 ([21]). Let \( t \in [0,T) \) and a Borel function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(X_t) \in L_2(\mathbb{P}) \). Then, \( f(X_t) \in \mathcal{D}_{1,2} \) if and only if the following two assertions hold:

(a) when \( \sigma > 0 \), \( f \) has a weak derivative\(^2\) \( f'_w \) on \( \mathbb{R} \) with \( f'_w(X_t) \in L_2(\mathbb{P}) \).

(b) the map \( (s,x) \mapsto \frac{(X_i+x) - f(X_i)}{x} \mathbb{1}_{[0,t]}(s,x) \) belongs to \( L_2(\mathbb{P} \times \mathcal{M}) \).

Furthermore, if \( f(X_t) \in \mathcal{D}_{1,2} \), then for \( \mathbb{P} \otimes \mathcal{M} \)-a.e. \( (\omega,s,x) \in \Omega \times [0,T] \times \mathbb{R} \) one has

\[
D_{s,x} f(X_t) = f'(w)(X_t) \mathbb{1}_{[0,t]} \mathbb{1}_{\{s \}}(s,x) + \frac{f(X_t + x) - f(X_t)}{x} \mathbb{1}_{[0,t]}(s,x),
\]

where we set, by convention, \( f'_w := 0 \) when \( \sigma = 0 \).

Lemma 3.3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a Borel function with \( f(X_T) \in L_2(\mathbb{P}) \).

(1) There are symmetric \( \tilde{f}_n \in L_2(\mu^{\leq n}) \) such that \( f(X_T) = \sum_{n=0}^{\infty} \mathbb{I}_n(\tilde{f}_n \mathbb{1}_{[0,T]}) \) a.s.

(2) For \( t \in [0,T) \), one has \( \mathbb{E}_{F_t}[f(X_T)] = \sum_{n=0}^{\infty} \mathbb{I}_n(\tilde{f}_n \mathbb{1}_{[0,T]}) \) a.s. and \( \mathbb{E}_{F_t}[f(X_T)] \in \mathcal{D}_{1,2} \).

(3) For \( t \in (0,T) \), it holds

\[
\mathbb{E} \left( |\sigma \partial_x F(t,X_t)|^2 + \int_\mathbb{R} |F(t,X_t + x) - F(t,X_t)|^2 \nu(dx) \right) < \infty, \tag{3.1}
\]

where \( F(t,x) := \mathbb{E}f(x + X_{T-t}) \) if \( \sigma > 0 \), and in the case \( \sigma = 0 \) we let \( F(t,\cdot) \) be a Borel function such that \( F(t,X_t) = \mathbb{E}_{F_t}[f(X_T)] \) a.s. and set \( \partial_x F := 0 \).

Proof. Items (1) and (2) are due to [15, Lemma D.1]. For item (3), it is clear for the case \( \sigma = 0 \) that (3.1) is implied by Lemma 3.2. Let us turn to the case \( \sigma > 0 \). According to Lemma 3.1, one has \( F(t,\cdot) \in C^\infty(\mathbb{R}) \), and hence \( (F(t,\cdot))'_w = \partial_x F(t,\cdot) \) a.e. with respect to the Lebesgue measure \( \mathcal{L} \). Since the law of \( X_t \) is absolutely continuous with respect to \( \mathcal{L} \), it holds that \( (F(t,\cdot))'_w(X_t) = \partial_x F(t,X_t) \) a.s. Then, (3.1) follows from Lemma 3.2. 

We are now in a position to prove Proposition 1.2.

\(^2\)A locally integrable function \( h \) is called a weak derivative of a locally integrable function \( f \) on \( \mathbb{R} \) if \( \int_\mathbb{R} f(x)\phi'(x)dx = -\int_\mathbb{R} h(x)\phi(x)dx \) for all smooth functions \( \phi \) with compact support in \( \mathbb{R} \). When such an \( h \) exists (unique up to a \( \mathcal{L} \)-null set), then we denote \( f'_w := h \).
Proof of Proposition 1.2. For \((t, x) \in [0, T] \times \mathbb{R}\), denote
\[
\Delta F(t, x) := \partial_x F(t, X_t^-) \mathbbm{1}_{\{x = 0\}} + \frac{F(t, X_t^- + x) - F(t, X_t^-)}{x} \mathbbm{1}_{\{x \neq 0\}},
\]
where we recall that \(\partial_x F := 0\) if \(\sigma = 0\) by convention. The assumption \(\mathbb{E}[|f(x + X_t)|] < \infty\) for all \((t, x) \in [0, T] \times \mathbb{R}\) implies that \((F(t, X_t + x) - F(t, X_t))_{t \in [0, T]}\) is a martingale for each \(x \in \mathbb{R}\). Moreover, in the case \(\sigma > 0\), the assumption \(f(X_T) \in L^2(\mathbb{P})\) and Lemma 3.1 imply that \(F(t, \cdot) \in C^\infty(\mathbb{R})\) for all \(t \in [0, T]\) and \((\partial_x F(t, X_t))_{t \in [0, T]}\) is a martingale.

Step 1. We show that for any \(t \in (0, T)\),
\[
C(t) := \mathbb{E} \int_0^t \int_\mathbb{R} |\Delta F(s, x)|^2 \, \nu(dx) \, ds < \infty.
\]
Observe that \((t, x) \mapsto F(t, x)\) is Borel measurable by Fubini’s theorem. In addition, since \(X^-\) is predictable, we infer that \((\omega, t, x) \mapsto F(t, X_t^- + x)\) is \(\tilde{\mathcal{F}}\)-measurable. Therefore, \(\Delta F\) given in (3.2) is \(\tilde{\mathcal{F}}\)-measurable.

Remark that \(X_s = X^-\) a.s. for each \(s \in [0, T]\). Using Fubini’s theorem and the martingale property, together with (3.1), we obtain for any \(t \in (0, T)\) that
\[
C(t) = \mathbb{E} \int_0^t \int_\mathbb{R} \left|\Delta F(s, x)\right|^2 \, \nu(dx) \, ds \\
\leq t \left(\mathbb{E}[\sigma^2 \partial_x F(t, X_t)]^2 + \mathbb{E} \int_\mathbb{R} \left|F(t, X_t + x) - F(t, X_t)\right|^2 \, \nu(dx)\right) \\
< \infty.
\]
Hence, the stochastic integral \(\int_0^t \int_\mathbb{R} \Delta F(s, x) M(ds, dx)\) exists as an element in \(L^2(\mathbb{P})\).

Step 2. Fix \(t \in (0, T)\). We prove that, a.s.,
\[
F(t, X_t) = \mathbb{E} F(X_T) + \int_0^t \int_\mathbb{R} \Delta F(s, x) M(ds, dx).
\]
The representation (3.3) can be regarded as a consequence of the Clark–Ocone formula. However, this formula seems to be considered either when the Lévy process \(X\) is square integrable or when \(X\) has no Brownian component (i.e. \(\sigma = 0\)) (see, e.g., [3, 23, 27, 28, 35]). So, for the reader’s convenience, we present here a complete proof for (3.3) where neither square integrability nor \(\sigma = 0\) is assumed. Due to the denseness of the simple multiple stochastic integrals in \(L^2(\mathbb{P})\) (see [10, Lemma 2.1]), in order to obtain (3.3) it is sufficient to check that
\[
\mathbb{E} \left[I_m(k_m) F(t, X_t)\right] = \mathbb{E} \left[I_m(k_m) \int_0^t \int_\mathbb{R} \Delta F(s, x) M(ds, dx)\right]
\]
for all \(m \geq 1\) and all functions \(k_m\) of the form
\[
k_m = \mathbbm{1}_{B_1 \times \ldots \times B_m},
\]
where \(B_i = (s_i, t_i] \times (a_i, b_i]\) in which \((a_i, b_i]\) are finite intervals and the time intervals \((s_i, t_i] \subset [0, t]\) satisfy \(t_{i-1} \leq t_i, i = 2, \ldots, m\).

Since \(F(t, X_t) \in \mathbb{D}_{1,2}\) by Lemma 3.3(2), applying Lemma 3.2 we have for \(\mathbb{P} \otimes \mathfrak{m}\text{-a.e. } (\omega, s, x) \in \Omega \times [0, T] \times \mathbb{R}\),
\[
D_s F(t, X_t) = \partial_x F(t, X_t) \mathbbm{1}_{[0, t] \times \{0\}}(s, x) + \frac{F(t, X_t + x) - F(t, X_t)}{x} \mathbbm{1}_{[0, t] \times \mathbb{R}_{\neq 0}}(s, x).
\]
Lemma 3.3

is computed as

\[ \Delta F(s, x), \]

where the second equality comes from the fact that \( X_s = X_{s-} \text{ a.s.} \)

We let \( f(X_T) = \sum_{n=0}^{\infty} I_n(f_n \mathbb{1}_{[0,T]} \mathbb{1}_{0}) \) and \( F(t, X_t) = \sum_{n=0}^{\infty} I_n(f_n \mathbb{1}_{[0,T]} \mathbb{1}_{0}) \) as in Lemma 3.3(1) and (2) respectively, where \( f_n \in L_2(\mu^{\otimes n}) \) are symmetric. Let \( k_n \) be of the form as in (3.5). Since functions \( f_n \) are symmetric, the left-hand side of (3.4) is computed as follows

\[
\text{LHS}_{(3.4)} = m! \int_{B_1 \times \cdots \times B_m} \tilde{f}_m(x_1, \ldots, x_m) \mathbb{m}(ds_1, dx_1) \cdots \mathbb{m}(ds_m, dx_m). \tag{3.7}
\]

For the right-hand side of (3.4), writing \( I_m(k_m) = \int_{B_m} I_{m-1}(k_{m-1}) \mathbb{M}(ds, dx) \), where \( k_{m-1} := \mathbb{1}_{B_1 \times \cdots \times B_{m-1}} \), and using Fubini’s theorem we obtain

\[
\text{RHS}_{(3.4)} = \mathbb{E} \int_{B_m} I_{m-1}(k_{m-1}) \Delta F(s, x) \mathbb{m}(ds, dx)
= \int_{B_m} \mathbb{E} \left[ I_{m-1}(k_{m-1}) \mathbb{F}_s \left[ \partial_x F(t, X_t) \mathbb{1}_{\{x=0\}} + F(t, X_t + x) - F(t, X_t) \mathbb{1}_{\{x \neq 0\}} \right] \right] \mathbb{m}(ds, dx)
= \mathbb{E} \int_{B_m} I_{m-1}(k_{m-1}) D_{s,x} F(t, X_t) \mathbb{m}(ds, dx)
= \mathbb{E} \int_{B_m} I_{m-1}(k_{m-1}) \left( \sum_{i=1}^{\infty} \sum_{j=1}^{k_{m-1}} i I_{k_{m-1}} \left( \tilde{f}_j(s, x) \mathbb{1}_{[0,\Delta]}(s) \right) \right) \mathbb{m}(ds, dx)
= m! \int_{B_m} \int_{B_1 \times \cdots \times B_{m-1}} \tilde{f}_m(x_1, \ldots, x_m, x) \mathbb{m}(ds_1, dx_1) \cdots \mathbb{m}(ds_{m-1}, dx_{m-1}) \mathbb{m}(ds, dx). \tag{3.9}
\]

Here, one uses (3.6) and the fact that \( I_{m-1}(k_{m-1}) \) is \( \mathbb{F}_s \)-measurable for all \( s \in (s_m, t_m] \) to obtain (3.8). Combining (3.7) with (3.9) yields (3.4).

Step 3. For any \( t \in (0, T) \), Jensen’s inequality implies that \( \mathbb{E}[f(X_T)] \geq \mathbb{E}[F(t, X_t)] \). Then, we apply Step 2 and Itô’s isometry to obtain

\[
\mathbb{E}[f(X_T)]^2 \geq \mathbb{E}[F(X_T)]^2 + \mathbb{E} \int_0^t \int_{\mathbb{R}} |\Delta F(s, x)|^2 \mathbb{m}(ds, dx).
\]

Letting \( t \uparrow T \), we infer that the stochastic integral \( \int_0^T \int_{\mathbb{R}} \Delta F(s, x) \mathbb{M}(ds, dx) \) exists as an element in \( L_2(\mathbb{P}) \) and equals to \( L_2(\mathbb{P})\lim_{t \uparrow T} \int_0^t \int_{\mathbb{R}} \Delta F(s, x) \mathbb{M}(ds, dx) \). On the other hand, due to the martingale convergence theorem, \( F(t, X_t) = \mathbb{E}_{\mathbb{F}_t}[f(X_T)] \rightarrow \mathbb{E}_{\mathbb{F}_{T^-}}[f(X_T)] \text{ a.s. and in } L_2(\mathbb{P}) \text{ as } t \uparrow T, \) where \( \mathbb{F}_{T^-} := \sigma(\cup_{t<T} \mathbb{F}_t) \). Since \( (\mathbb{F}_t)_{t \in [0,T]} \) is the augmented natural filtration of the Lévy process \( X \), it holds that \( \mathbb{F}_{T^-} = \mathbb{F}_T \), and hence the desired conclusion follows.

4. Closed form for the local risk-minimizing strategy

This section gives the proof of Theorem 1.1. First, let us fix the setting of this section.

Setting 4.1. Let \( S = e^X \) be the exponential of a Lévy process \( X \) with \( (X|\mathbb{P}) \sim (\gamma, \sigma, \nu) \). Assume that \( \sigma^2 + \nu(\mathbb{R}) > 0 \) and \( \int_{|x|>1} e^{2x} \nu(dx) < \infty \).
The condition $\sigma^2 + \nu(\mathbb{R}) > 0$ is simply to exclude the trivial case that $X$ is a.s. deterministic. The condition $\int_{|x|>1} e^{2x} \nu(dx) < \infty$ is equivalent to the square integrability of $S$ (see [31, Theorem 25.3]).

By Itô’s formula, one has

$$S = 1 + \left( \int_0^t \sigma S_t \, dW_t + \int_0^t \int_{\mathbb{R}} S_t - (e^x - 1) \tilde{N}(dt, dx) \right) + \int_0^t \gamma S_t \, dt$$

$$=: 1 + S^m + S^{fv},$$

where $S^m$ and $S^{fv}$ respectively denote the martingale part and the predictable finite variation part in the representation of $S$, and where

$$\gamma_S := \gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x 1_{\{|x| \leq 1\}}) \nu(dx). \quad (4.1)$$

Recall from Theorem 1.1 the notation

$$\| (\sigma, \nu) \| = \sigma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \in (0, \infty).$$

4.1. Föllmer–Schweizer (FS) decomposition. We briefly present the FS decomposition of a random variable and the notion of the minimal local martingale measure which is the key tool to determine the FS decomposition. We refer the reader to [34] for a survey about these objects.

In this article, we follow [17, p.863] and use the family of admissible strategies as

$$\Sigma_{\text{adm}}^S(\mathbb{P}) := \{ \vartheta \text{ predictable : } \mathbb{E} \int_0^T \vartheta_t^2 S_t^2 \, dt < \infty \}.$$ \hfill (4.2)

It turns out that if $\vartheta \in \Sigma_{\text{adm}}^S(\mathbb{P})$, then

$$\mathbb{E} \int_0^T \vartheta_t^2 d[S,S]_t = \mathbb{E} \int_0^T \vartheta_t^2 d[S^m,S^m]_t = \mathbb{E} \int_0^T \vartheta_t^2 d\langle S^m, S^m \rangle_t$$

$$= \| (\sigma, \nu) \| \mathbb{E} \int_0^T \vartheta_t^2 S_t^2 \, dt < \infty. \quad (4.3)$$

The following definition is due to [34].

**Definition 4.2.** (1) An $H \in L_2(\mathbb{P})$ admits a FS decomposition if $H$ can be written as

$$H = H_0 + \int_0^T \vartheta_t^H \, dS_t + L^H_t,$$

where $H_0 \in \mathbb{R}$, $\vartheta^H \in \Sigma_{\text{adm}}^S(\mathbb{P})$ and $L^H \in \mathcal{M}_2^0(\mathbb{P})$ is $\mathbb{P}$-orthogonal to $S^m$.

(2) The integrand $\vartheta^H$ is called the local risk-minimizing strategy of $H$.

**Remark 4.3.** In our context, $S$ satisfies the structure condition and the mean-variance trade-off process $\hat{K}$ of $S$ in the sense of [34, p.553] is

$$\hat{K}_t = \frac{\gamma_S^2}{\| (\sigma, \nu) \|} t,$$

which is uniformly bounded in $(\omega, t) \in \Omega \times [0, T]$. Hence, it is known that any $H \in L_2(\mathbb{P})$ admits a unique FS decomposition (see [26, Theorem 3.4]).

We continue with the notion of the minimal martingale measure.
Definition 4.4 ([33], Section 2). Let $\mathcal{E}(U) \in \text{CL}([0,T])$ be the stochastic exponential of $U$, i.e. $d\mathcal{E}(U) = \mathcal{E}(U)\ dU$ with $\mathcal{E}(U)_0 = 1$, where

$$U = -\frac{\gamma_S}{\|\sigma,\nu\|} \left( \sigma W + \int_0^t \int\limits_{\mathbb{R}_0} (e^x - 1) \tilde{N}(ds, dx) \right). \quad (4.4)$$

If $\mathcal{E}(U) > 0$, then the probability measure $\mathbb{P}^*$ defined by

$$d\mathbb{P}^* := \mathcal{E}(U)_T \ d\mathbb{P}$$

is called the minimal martingale measure for $S$.

Since $U$ given in (4.4) is a Lévy process and belongs to $\mathcal{M}^0_{2}(\mathbb{P})$, it is known that $\mathcal{E}(U)$ is also an $\mathcal{L}^2(\mathbb{P})$-martingale (see, e.g., [29], Ch.V, Theorem 67] or [11, Lemma 1]).

We now give a condition imposed on the characteristics of $X$ such that $\mathbb{P}^*$ exists. Let $(U|\mathbb{P}) \sim (\gamma_U, \sigma_U, \nu_U)$ and denote $\alpha_U(x) := -\frac{\gamma_S(e^x - 1)}{\|\sigma,\nu\|}$, $x \in \mathbb{R}$.

Then, it follows from (4.4) that

$$\gamma_U = -\int_{\{\alpha_U(x) > 1\}} \alpha_U(x) \nu(dx), \quad \sigma_U = \frac{\|\gamma_S\sigma\|}{\|\sigma,\nu\|}, \quad \nu_U = \nu \circ \alpha_U^{-1}. \quad (4.5)$$

Since

$$\mathcal{E}(U) > 0 \iff \Delta U > -1 \iff \nu_U((-\infty, -1]) = 0 \iff \gamma_S(e^x - 1) < \|\sigma,\nu\|, \forall x \in \text{supp}\nu,$$

the following assumption ensures the existence of $\mathbb{P}^*$:

Assumption 4.5. $\gamma_S(e^x - 1) < \|\sigma,\nu\|$ for all $x \in \text{supp}\nu$.

Remark that a sufficient condition for Assumption 4.5 is

$$0 \geq \gamma_S \geq -\|\sigma,\nu\|.$$

Assume that Assumption 4.5 holds true. Then, by an application of Girsanov’s theorem (see, e.g., [9, Propositions 2 and 3]), $X$ is also a Lévy process under $\mathbb{P}^*$ with $(X|\mathbb{P}^*) \sim (\gamma^*, \sigma^*, \nu^*)$, where

$$\gamma^* = \gamma - \frac{\gamma_S \|\sigma,\nu\|}{\|\sigma,\nu\|} \left( \sigma + \int_{|x| \leq 1} x(e^x - 1) \nu(dx) \right),$$

$$\sigma^* = \sigma \quad \text{and} \quad \nu^*(dx) = \left(1 - \frac{\gamma_S(e^x - 1)}{\|\sigma,\nu\|}\right) \nu(dx). \quad (4.6)$$

Moreover, if $W^*$ and $\tilde{N}^*$ are the standard Brownian motion and the compensated Poisson random measure of $X$ under $\mathbb{P}^*$, then

$$W^*_t = W_t + \frac{\gamma_S \sigma \|\sigma,\nu\|}{\|\sigma,\nu\|} t, \quad (4.7)$$

$$\tilde{N}^*(dt, dx) = \tilde{N}(dt, dx) + \frac{\gamma_S \|\sigma,\nu\|}{\|\sigma,\nu\|}(e^x - 1) \nu(dx) dt. \quad (4.8)$$

In the sequel, let $\mathbb{E}^*$ (resp. $\mathbb{E}^*_G$) denote the expectation (resp. conditional expectation given a $\sigma$-algebra $G \subseteq \mathcal{F}$) with respect to $\mathbb{P}^*$.
4.2. Proof of Theorem 1.1. Let \( f(x) := g(e^x) \) and \( F^*(t, x) := E^*f(x + X_{T-t}) \) so that \( G^*(t, e^x) = F^*(t, x) \) for \( (t, x) \in [0, T] \times \mathbb{R} \). We define
\[
\Delta_j G^*(t, x) := G^*(t, e^x S_{t-}) - G^*(t, S_{t-}), \quad (t, x) \in [0, T] \times \mathbb{R}.
\]

(1) We present here a direct proof for this assertion, an alternative argument for more general settings can be found in [6, Proof of Theorem 4.3]. By assumption, \( f(X_T) = g(S_T) \in L_2(\mathbb{P}^*) \) and \( E^*[f(x + X_t)] = E^*[g(e^x S_t)] < \infty \) for any \( (t, x) \in [0, T] \times \mathbb{R} \), we apply Proposition 1.2 to obtain
\[
K^* = E^*g(S_T) + \int_0^T \sigma S_{t-} \partial_y G^*(t, S_{t-})dW_t^* + \int_0^T \int_{\mathbb{R}_0} \Delta_j G^*(t, x) \tilde{N}^*(dt, dx), \quad (4.9)
\]
where \( K^* = (K_t^*)_{t \in [0, T]} \) is the càdlàg version of the \( L_2(\mathbb{P}^*) \)-martingale \( (E^*[g(S_T)])_{t \in [0, T]} \), and where \( W^* \) and \( \tilde{N}^* \) are introduced in (4.7) and (4.8). Then, it holds that \( E(U)K^* \) is a martingale under \( \mathbb{P} \). Since the \( \mathbb{P} \)-martingale \( U \) given in (4.4) satisfies that
\[
\|\{U, U\}_T\|_{L_\infty(\mathbb{P})} = \frac{\gamma^2}{\|\{\sigma, \nu\}\|^2} \left( \sigma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \right) < \infty,
\]
it implies that \( E(U) \) is regular and satisfies \( R_2 \) in the sense of [7, Proposition 3.7]. Since \( K_T^* = g(S_T) \in L_2(\mathbb{P}) \) by assumption, we apply [7, Theorem 4.9((i)\( \iff \)(ii))] to obtain
\[
E[K^*, K^*_T] < \infty.
\]
Combining this with (4.9) yields
\[
E \int_0^T \sigma^2 |S_{t-} \partial_y G^*(t, S_{t-})|^2 dt + E \int_0^T \int_{\mathbb{R}_0} |\Delta_j G^*(t, x)|^2 N(dt, dx) = E[K^*, K^*_T] < \infty.
\]
Since \( dt \nu(dx) \) is the predictable \( \mathbb{P} \)-compensator of \( N(dt, dx) \), it implies that
\[
E \int_0^T \sigma^2 |S_{t-} \partial_y G^*(t, S_{t-})|^2 dt + E \int_0^T \int_{\mathbb{R}} |\Delta_j G^*(t, x)|^2 \nu(dx)dt < \infty. \quad (4.10)
\]
Using Cauchy–Schwarz’s inequality yields
\[
E \int_0^T \sigma^2 S_{t-}^2 |\partial_y G^*(t, S_{t-})|dt + E \int_0^T \int_{\mathbb{R}} |\Delta_j G^*(t, x) S_{t-}(e^x - 1)| \nu(dx)dt \\
\leq \sqrt{E \int_0^T S_{t-}^2 dt} \sqrt{E \int_0^T |\partial^2 G^*(t, S_{t-})|^2 dt} \\
+ \sqrt{\int_{\mathbb{R}} (e^x - 1)^2 \nu(dx)} \sqrt{E \int_0^T S_{t-}^2 dt} \sqrt{E \int_0^T |\Delta_j G^*(t, x)|^2 \nu(dx)dt} < \infty. \quad (4.11)
\]
On the other hand, the FS decomposition of \( H = g(S_T) \) is
\[
g(S_T) = H_0 + \int_0^T \partial^H_t dS_t + L^H_T \quad (4.12)
\]
where \( H_0 \in \mathbb{R} \), \( \partial^H \in \Sigma^{adm}_{S_T}(\mathbb{P}) \) and \( L^H \in \mathcal{M}^0_2(\mathbb{P}) \) is \( \mathbb{P} \)-orthogonal to the martingale component \( S^m \) of \( S \). According to [34, Eq. (3.10)], it holds that \( L^H \) is a local \( \mathbb{P}^* \)-martingale. We remark that \( \int_0^T \partial^H_t dS_t \) is also a local \( \mathbb{P}^* \)-martingale. Using Cauchy–Schwarz’s inequality and (4.3), we obtain
\[
E^*[L^H, L^H]_{T} \leq \|E(U)\|_{L_2(\mathbb{P})} \sqrt{E[L^H, L^H]_T} < \infty,
\]
Lemma 3.1 holds to obtain with (4.10) with \(t \in [0,T] \) for each \(t\)

By assumption, it is clear that \((g(S_t))\) which yields

Recall that the martingale part of \(S\) is \(S^m = \int_0^t \sigma S_t \cdot dW_t + \int_0^t \int_{\mathbb{R}} \eta_t - \frac{1}{2} \tilde{N}(dt, dx)\).
Since \(L^H, S^m\) is \(0\) by the definition of the FS decomposition, we take the predictable quadratic covariation on both sides of (4.13) with \(S^m\) under \(\mathbb{P}\) and notice that the integrability condition (4.11) holds to obtain

which yields (1.2) as desired.

(2) It follows from Cauchy–Schwarz’s inequality and (4.10) that

By assumption, it is clear that \((G^*(t, e^x S_t))\) is a \(\mathbb{P}^n\)-martingale for each \(t \in [0,T]\). In the case \(\sigma > 0\), due to \(g(S_T) \in L_2(\mathbb{P})\) and Lemma 3.1, \((S_t \partial_y G^*(t, S_t))\) is a \(\mathbb{P}^n\)-martingale. Hence, the function

is non-decreasing by the martingale property. In addition, noticing that \(S_t = S_t\) a.s. for each \(t \in [0,T]\), we infer from (4.14) and Fubini’s theorem that

for all \(t \in [0,T]\). Therefore,

\[
\left( \frac{1}{\| (\sigma, \nu) \|} \left( \int_0^T \| \sigma S_t \partial_y G^*(t, S_t) + \int_0^T \| G^*(t, e^x S_t) - G^*(t, S_t) \| dx \right) \right)_{t \in [0,T]}
\]
is a \(\mathbb{P}^n\)-martingale for which one can find a càdlàg modification, denoted by \(\varphi^g\). Then, the process \(\varphi^g\) defined by

satisfies the desired requirements. \(\square\)
Remark 4.6. Let \( \tilde{\vartheta} \in \text{CL}([0,T]) \) be such that \( \tilde{\vartheta} = \vartheta^g \) for \( \mathbb{P} \otimes \lambda \)-a.e. \((\omega, t) \in \Omega \times [0,T)\), where \( \vartheta^g \) given in (4.15). Then, \( \mathbb{P}(\tilde{\vartheta}_t = \vartheta^g_t, \forall t \in [0,T)) = 1 \) due to the càdlàg property. Hence, \( \tilde{\vartheta}_- \) is also a LRM of \( H = g(S_T) \), and it holds that, for any \( t \in [0,T) \),

\[
\tilde{\vartheta}_t = \frac{1}{\| (\sigma, \nu) \|} \left( \sigma^2 S_t \partial_p G^*(t, S_t) + \int_{\mathbb{R}} (G^*(t, e^x S_t) - G^*(t, S_t))(e^x - 1)\nu(dx) \right) \quad \text{a.s.}
\]

5. Discrete-time hedging in weighted bounded mean oscillation spaces

This section is a continuation of the work in [37] for the exponential Lévy models. First, we use the approximation scheme for stochastic integrals introduced in [37] and investigate the resulting error in weighted BMO spaces. Consequently, the \( L_p \)-estimates \((p \in (2, \infty))\) for the error are provided. Secondly, to illustrate the obtained results, we consider the stochastic integral term in the FS decomposition of a European type option. This integral can be interpreted as the hedgeable part of the option. Notice that we do not assume the (local) martingale property under the reference measure for the underlying price process.

5.1. Weighted bounded mean oscillation (BMO) spaces. Let \( S([0,T]) \) denote the family of all stopping times \( \rho: \Omega \rightarrow [0,T] \), and set \( \inf \emptyset := \infty \).

Definition 5.1 ([14, 15]). Let \( p \in (0, \infty) \).

1. For \( \Phi \in \text{CL}^+([0,T]) \), we denote by \( \text{BMO}^p_\rho(\mathbb{P}) \) the space of all \( Y \in \text{CL}_0([0,T]) \) with \( \| Y \|_{\text{BMO}^p_\rho(\mathbb{P})} < \infty \), where

\[
\| Y \|_{\text{BMO}^p_\rho(\mathbb{P})} := \inf \left\{ c > 0 : \mathbb{E}_{\mathbb{F}_\rho} \left[ | Y_T - Y_{\rho^-} |^p \right] \leq c^p \Phi^p_\rho \text{ a.s., } \forall \rho \in S([0,T]) \right\}.
\]

2. (Weight regularity) Let \( SM_p(\mathbb{P}) \) be the space of all \( \Phi \in \text{CL}^+([0,T]) \) with \( \| \Phi \|_{SM_p(\mathbb{P})} < \infty \), where

\[
\| \Phi \|_{SM_p(\mathbb{P})} := \inf \left\{ c > 0 : \mathbb{E}_{\mathbb{F}_a} \left[ \sup_{t \in [0,T]} \Phi^p_\rho \right] \leq c^p \Phi^p_\rho \text{ a.s., } \forall a \in [0,T] \right\}.
\]

The theory of non-weighted BMO-martingales (i.e. when \( \Phi \equiv 1 \) and \( Y \) is a martingale) can be found in [29, Ch.IV]. One remarks that the weighted BMO spaces above were introduced in [14] for general càdlàg processes which are not necessarily martingales.

Definition 5.2 ([14]). For \( s \in (1, \infty) \), we denote by \( \mathcal{R}\mathcal{H}_s(\mathbb{P}) \) the family of all probability measures \( \mathbb{Q} \) equivalent to \( \mathbb{P} \) such that \( d\mathbb{Q}/d\mathbb{P} =: U \in L_s(\mathbb{P}) \) and there exists a constant \( c_{(5.1)} > 0 \) such that \( U \) satisfies the following reverse Hölder inequality

\[
\mathbb{E}_{\mathbb{F}_\rho} [U^s] \leq c_{(5.1)}^s \mathbb{E}_{\mathbb{F}_\rho} [U]^s \quad \text{a.s., } \forall \rho \in S([0,T]).
\]

We refer the reader to [14, 15] for further properties of those quantities. Proposition 5.3 below recalls some features of weighted BMO which are crucial for our applications, and their proofs can be found in [15, Proposition A.6] and [37, Proposition 2.5].

Proposition 5.3 ([15, 37]). Let \( p \in (0, \infty) \).

1. There is a constant \( c_1 = c(p) > 0 \) such that \( \| \cdot \|_{L_p(\mathbb{P})} \leq c_1 \| \Phi \|_{L_p(\mathbb{P})} \cdot \| \text{BMO}^p_\rho(\mathbb{P}) \|. \)

2. If \( \Phi \in SM_p(\mathbb{P}) \), then for any \( r \in (0, p] \) there is a constant \( c_2 = c_2(r, p, \| \Phi \|_{SM_p(\mathbb{P})}) > 0 \) such that \( \| \cdot \|_{\text{BMO}^p_\rho(\mathbb{P})} \sim c_2 \| \cdot \|_{\text{BMO}^r_\rho(\mathbb{P})} \cdot \)

3. If \( \mathbb{Q} \in \mathcal{R}\mathcal{H}_s(\mathbb{P}) \) for some \( s \in (1, \infty) \) and \( \Phi \in SM_p(\mathbb{Q}) \), then there exists a constant \( c_3 = c(s, p) > 0 \) such that \( \| \cdot \|_{\text{BMO}^p_\rho(\mathbb{Q})} \leq c_3 \| \cdot \|_{\text{BMO}^p_\rho(\mathbb{P})} \cdot \)
Remark 5.4. The benefit of Proposition 5.3(2) is as follows: If $p \in [2, \infty)$ (this is usually the case in applications), then one can choose $r = 2$ so that $\| \cdot \|_{\text{BMO}_{p}^{2}(\mathbb{P})} \sim c_{2}$ and then we can still exploit some similar techniques as in the $L_{2}(\mathbb{P})$-theory to deal with $\| \cdot \|_{\text{BMO}_{p}^{2}(\mathbb{P})}$. Combining this observation with Proposition 5.3(1) yields the following estimate provided that $\Phi \in \mathcal{SM}_{p}(\mathbb{P})$, $p \in [2, \infty)$,

$$\| \cdot \|_{L_{p}(\mathbb{P})} \leq c_{1}c_{2}\| \Phi \|_{L_{p}(\mathbb{P})} \cdot \| \cdot \|_{\text{BMO}_{p}^{2}(\mathbb{P})}. \quad (5.2)$$

Proposition 5.3(3) gives a change of the underlying measure which might be of interest for further applications in mathematical finance.

5.2. Jump adjusted approximation. Let us recall from [37] the approximation scheme with the jump adjusted method. Roughly speaking, this method is constructed by adding suitable correction terms to the classical Riemann sum of the stochastic integral as soon as relatively large jumps of the driving process occur.

Time-nets. Let $\mathcal{T}_{\text{det}}$ denote the family of all deterministic time-nets $\tau = (t_{i})_{i=0}^{n}$: $0 = t_{0} < t_{1} < \cdots < t_{n} = T$, $n \geq 1$. The mesh size of $\tau = (t_{i})_{i=0}^{n} \in \mathcal{T}_{\text{det}}$ associated with a parameter $\theta \in (0, 1]$ is defined by

$$\| \tau \|_{\theta} := \max_{i=1, \ldots, n} \frac{t_{i} - t_{i-1}}{(T - t_{i-1})^{1-\theta}}.$$

Let $\tau_{n} \in \mathcal{T}_{\text{det}}$ with $\# \tau_{n} = n + 1$. By a short calculation we can find that $\| \tau_{n} \|_{\theta} \geq \frac{T^{\theta}}{n}$. Minimizing $\| \tau_{n} \|_{\theta}$ over $\tau_{n} \in \mathcal{T}_{\text{det}}$ with $\# \tau = n + 1$ leads us to the following adapted time-net $\tau_{n}^{\theta} = (t_{i,n}^{\theta})_{i=0}^{n}$ is defined by

$$t_{i,n}^{\theta} := T \left(1 - \frac{\theta}{\sqrt{1 - i/n}}\right), \quad i = 1, \ldots, n. \quad (5.3)$$

Then, a calculation gives

$$\frac{T^{\theta}}{n} \leq \| \tau_{n}^{\theta} \|_{\theta} \leq \frac{T^{\theta}}{\theta n}, \quad n \geq 1.$$

Jump adjusted approximation scheme. Let $S = e^{X}$ be the exponential Lévy process and assume Setting 4.1. Let $\tilde{\vartheta} \in \text{CL}([0, T])$ be such that $E \int_{0}^{T} \tilde{\vartheta}_{t}^{2} S_{t}^{2} dt < \infty$ (the tilde sign here indicates the cadlag property of the process $(\tilde{\vartheta}_{t})_{t \in [0, T]}$). For $\tau = (t_{i})_{i=0}^{n} \in \mathcal{T}_{\text{det}}$, the Riemann approximation $A_{\tau}^{\text{Rm}}(\tilde{\vartheta}, \tau)$ of $\int_{0}^{T} \tilde{\vartheta}_{t} \cdot dS_{t}$ is defined by

$$A_{\tau}^{\text{Rm}}(\tilde{\vartheta}, \tau) := \sum_{i=1}^{n} \tilde{\vartheta}_{t_{i-1}}(S_{t_{i} \wedge T} - S_{t_{i-1} \wedge T}), \quad t \in [0, T].$$

Before proceeding to the jump adjusted approximation, we need the following stopping times which capture the relative large jumps of $S$: For $\epsilon > 0$ and $\kappa \geq 0$, we define the family of stopping times $\rho(\epsilon, \kappa) = (\rho_{i}(\epsilon, \kappa))_{i \geq 0}$ by setting $\rho_{0}(\epsilon, \kappa) := 0$ and

$$\rho_{i}(\epsilon, \kappa) := \inf\{T \geq t > \rho_{i-1}(\epsilon, \kappa) : |\Delta S_{t}| \geq \epsilon(T - t)^{\kappa} S_{t-} \wedge T, \quad i \geq 1 \}.$$

By specializing $\kappa = 0$, the parameter $\epsilon$ can be regarded as the jump size threshold. When $\kappa > 0$, this threshold shrinks as $t \uparrow T$, and thus the parameter $\kappa$ indicates the jump size decay rate. The reason for using the decay function $\epsilon(T - t)^{\kappa}$ is to compensate the growth of integrands.

Definition 5.5. Let $\epsilon > 0$, $\kappa \in [0, \frac{1}{2})$ and $\tau = (t_{i})_{i=0}^{n} \in \mathcal{T}_{\text{det}}$. 


(1) Let $\tau \mid \rho(\varepsilon, \kappa)$ denote the combined time-net constructed by combining $\tau$ with $\rho(\varepsilon, \kappa)$ and re-ordering their time-knots.

(2) For $t \in [0, T]$, we define

$$
\tilde{\vartheta}(\tau)_t := \sum_{i=1}^n \tilde{\vartheta}_{i-1} \mathbb{1}_{(t_{i-1}, t_i]}(t),
$$

$$
A_t^{\text{Corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) := A_t^{\text{RM}}(\tilde{\vartheta}, \tau) + \sum_{\rho(\varepsilon, \kappa) \in [0, t] \cap (0, T)} \left( \tilde{\vartheta}_{\rho(\varepsilon, \kappa) -} - \tilde{\vartheta}(\tau)_{\rho(\varepsilon, \kappa)} \right) \Delta S_{\rho(\varepsilon, \kappa)},
$$

$$
E_t^{\text{Corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) := \int_0^t \tilde{\vartheta}_u dS_u - A_t^{\text{Corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa).
$$

Denote

$$
\mathcal{N}(\varepsilon, \kappa) := \inf \{ i > 1 : \rho_i(\varepsilon, \kappa) = T \}.
$$

We apply [37, Proposition 5.3] (with $\alpha = 2$) to conclude that $\mathcal{N}(\varepsilon, \kappa) < \infty$ a.s. for any $\varepsilon > 0$ and $\kappa \in \{ \frac{1}{2}, 1 \}$. Hence, the sum in the definition of $A_t^{\text{Corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa)$ is a finite sum a.s. By adjusting this sum on a set of probability zero, we may assume that $A_t^{\text{Corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa)$, and hence, $E_t^{\text{Corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa)$, belong to $\text{CL}_0([0, T])$.

5.3. Discrete-time approximation in weighted BMO: A general result. Let us introduce the main assumption to obtain the approximation results.

**Assumption 5.6.** Let $S = e^X$ with $(X||\mathbb{P}) \sim (\gamma, \sigma, \nu)$. Let $\tilde{\vartheta} \in \text{CL}([0, T])$ and $\theta \in (0, 1]$. Assume that

(i) $\int_{|x| > 1} e^{2x} \nu(dx) < \infty$.

(ii) $\Delta \tilde{\vartheta}_t = 0$ a.s. for each $t \in [0, T)$.

(iii) There exists a random measure $\Upsilon : \Omega \times \mathcal{B}((0, T)) \to [0, \infty]$ such that $\Upsilon(\omega, (0, t]) < \infty$ for all $(\omega, t) \in \Omega \times (0, T)$, and such that for any $0 \leq a < b < T$,

$$
\mathbb{E}_{\mathcal{F}_a} \left[ \int_{(a,b]} |\tilde{\vartheta}_t - \tilde{\vartheta}_a|^2 S_t^2 dt \right] \leq c_{(5.4)}(5.4) \mathbb{E}_{\mathcal{F}_a} \left[ \int_{(a,b]} (b-t) \Upsilon(\cdot, dt) \right] \text{ a.s.} \tag{5.4}
$$

(iv) There is an a.s. non-decreasing process $\Theta \in \text{CL}^+([0, T])$ such that

(1) (Growth condition) One has

$$
|\tilde{\vartheta}_a| \leq c_{(5.5)}(5.5) (T-a) \Theta_a \quad \text{a.s., } \forall a \in [0, T). \tag{5.5}
$$

(2) (Curvature condition) One has

$$
\mathbb{E}_{\mathcal{F}_a} \left[ \int_{(a,T]} (T-t)^{1-\theta} \Upsilon(\cdot, dt) \right] \leq c_{(5.6)}(5.6) \Phi_a^2 \quad \text{a.s., } \forall a \in [0, T), \tag{5.6}
$$

where

$$
\Phi := \Theta S. \tag{5.7}
$$

Here, $c_{(5.4)}$, $c_{(5.5)}$, $c_{(5.6)}$ are positive constants independent of $a, b$.

Condition (i) is equivalent to the square integrability of $S$. Condition (ii) means that the integrand $\tilde{\vartheta}$ has no fixed-time discontinuity, and this property is satisfied in various contexts. Conditions (iii)–(iv) are adapted from [37, Assumption 3.3], and the random measure $\Upsilon$ above describes some kind of curvature of the stochastic integral. Several specifications of $\Upsilon$ are provided in [15] (for the Brownian setting and the Lévy setting) and in [37] (for the exponential Lévy setting).
Theorem 5.7. Let Assumption 5.6 hold for some $\tilde{\theta} \in \text{CL}([0,T])$ and for some $	heta \in (0,1]$. For $\Phi$ given in (5.7), we define $\overline{\Phi}_t := \Phi_t + \sup_{u \in [0,t]} |\Delta \Phi_u|$, $t \in [0,T]$. Assume that $\Phi \in \mathcal{SM}_2(\mathbb{P})$. Then, the following assertions hold:

1. If $\int_{|x| \leq 1} |x|^r \nu(dx) < \infty$ for some $r \in [1,2]$, then there is a constant $c_{(5.8)} > 0$ such that for all $\tau \in T_{\text{det}}$, $\varepsilon > 0$,

$$\left\| E_{\text{Corr}} \left( \tilde{\theta}, \tau \varepsilon, \frac{1-\theta}{2} \right) \right\|_{\text{BMO}_2^\Phi(\mathbb{P})} \leq c_{(5.8)} \max \left\{ \varepsilon^{1-r} \|\tau\|_\theta, \sqrt{\|\tau\|_\theta} \varepsilon \right\}. \quad (5.8)$$

Consequently, choosing the adapted time-net $\tau^0_n$ and $\varepsilon = n^{-\frac{1}{2}}$ in (5.8) we obtain

$$\sup_{n \geq 1} \varepsilon \left\| E_{\text{Corr}} \left( \tilde{\theta}, \tau^0_n n^{-\frac{1}{2}}, \frac{1-\theta}{2} \right) \right\|_{\text{BMO}_2^\Phi(\mathbb{P})} < \infty. \quad (5.9)$$

2. If $\sup_{\tau > 0} \left| \int_{|x| > r}(e^{\tau x} - 1)\nu(dx) \right| < \infty$, then there is a constant $c_{(5.10)} > 0$ such that for all $\tau \in T_{\text{det}}$, $\varepsilon > 0$,

$$\left\| E_{\text{Corr}} \left( \tilde{\theta}, \tau \varepsilon, \frac{1-\theta}{2} \right) \right\|_{\text{BMO}_2^\Phi(\mathbb{P})} \leq c_{(5.10)} \max \left\{ \sqrt{\|\tau\|_0}, \varepsilon \right\}. \quad (5.10)$$

Consequently, choosing the adapted time-net $\tau^0_n$ and $\varepsilon = n^{-\frac{1}{2}}$ in (5.10) we obtain

$$\sup_{n \geq 1} \varepsilon \left\| E_{\text{Corr}} \left( \tilde{\theta}, \tau^0_n n^{-\frac{1}{2}}, \frac{1-\theta}{2} \right) \right\|_{\text{BMO}_2^\Phi(\mathbb{P})} < \infty. \quad (5.11)$$

3. If in addition $\Phi \in \mathcal{SM}_p(\mathbb{P})$ for some $p \in (2,\infty)$, then the conclusions of items (1), (2) hold for the $L_p(\mathbb{P})$-norm in place of the $\text{BMO}_2^\Phi(\mathbb{P})$-norm.

4. If in addition $\mathcal{Q} \in \mathcal{RH}_s(\mathbb{P})$ for some $s \in (1,\infty)$ and $\Phi \in \mathcal{SM}_2(\mathcal{Q})$, then the conclusions of items (1), (2) hold for the $\text{BMO}_2^\Phi(\mathcal{Q})$-norm in place of the $\text{BMO}_2^\Phi(\mathbb{P})$-norm.

Proof. By Subsection 2.4, one has $dS_t = S_t \, dB_t$, where $Z$ is a square integrable Lévy process with the Lévy measure $\nu_Z = \nu \circ h^{-1}$, where $h(x) := e^{\tau x} - 1$. Moreover, it is clear that $\int_{|x| \leq 1} |x|^r \nu(dx) < \infty \iff \int_{|x| \leq 1} |x|^r \nu_Z(dx) = \infty$. Then, we apply [37, Theorem 3.10] to obtain items (1) and (2). Items (3), (4) are due to Proposition 5.3 and Lemma A.2. \hfill \Box

Remark 5.8. The parameter $n$ in front of the $\text{BMO}_2^\Phi(\mathbb{P})$-norm in (5.9) and (5.11) can be regarded as the $L_2(\mathbb{P})$-norm of the cardinality of the combined time-net $\tau^0_n \cup \rho(n^{-\frac{1}{2}}, \frac{1-\theta}{2})$ and $\tau^0_n \cup \rho(n^{-\frac{1}{2}}, \frac{1-\theta}{2})$ respectively. This assertion is derived from [37, Proposition 3.13] (with $\mathcal{Q} = \mathbb{P}$, and $q = 2, r = \infty$).

5.4. Hölder spaces and $\alpha$-stable-like processes. We first define some classes of Hölder continuous functions and bounded Borel functions, where the payoff functions are contained in.

Definition 5.9. Let $U \subseteq \mathbb{R}$ be a non-empty open interval.

1. For $\eta \in [0,1]$, we let $C^{0,\eta}(U)$ denote the space of all Borel functions $f : U \to \mathbb{R}$ with $\|f\|_{C^{0,\eta}(U)} < \infty$, where

$$\|f\|_{C^{0,\eta}(U)} := \inf \{ c \geq 0 : |f(x) - f(y)| \leq c|x-y|^\eta, \forall x, y \in U, x \neq y \}.$$

2. For $q \in [1,\infty]$, we define $W^{1,q}(U) := \left\{ f : U \to \mathbb{R} : \exists k \in L_q(U), f(y) - f(x) = \int_x^y k(u)du, \forall x, y \in U, x < y \right\}$, and let $\|f\|_{W^{1,q}(U)} := \|k\|_{L_q(U)}$. 

It is obvious that \( C^{0,\eta}(U) \) is the space of all \( \eta \)-Hölder continuous functions on \( U \) for \( \eta \in (0,1) \), and \( C^{0,0}(U) \) consists of all bounded and Borel functions on \( U \). For \( \eta \in [0,1] \), Hölder’s inequality implies that
\[
\dot{W}^{1,\frac{1}{1-\eta}}(U) \subseteq C^{0,\eta}(U) \quad \text{with} \quad \|f\|_{C^{0,\eta}(U)} \leq \|f\|_{\dot{W}^{1,\frac{1}{1-\eta}}(U)}, \quad \forall f \in \dot{W}^{1,\frac{1}{1-\eta}}(U).
\]
In particular, \( \dot{W}^{1,\infty}(U) = C^{0,1}(U) \), which is the space of Lipschitz functions on \( U \).

We next introduce some classes of \( \alpha \)-stable-like Lévy measures.

**Definition 5.10.** Let \( \nu \) be a Lévy measure and \( \alpha \in (0,2) \).

1. We let \( \nu \in S_1(\alpha) \) if one can decompose \( \nu = \nu_1 + \nu_2 \), where \( \nu_1, \nu_2 \) are Lévy measures and satisfy that
\[
\limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \nu_2(\mathrm{d}x) < \infty, \quad (5.12)
\]
\[
\nu_1(\mathrm{d}x) = \frac{k(x)}{|x|^{\alpha+1}} \mathbb{1}_{\{x \neq 0\}} \mathrm{d}x, \quad (5.13)
\]
where \( 0 < \liminf_{x \to 0} k(x) \leq \limsup_{x \to 0} k(x) < \infty \), and the function \( x \mapsto \frac{k(x)}{|x|^\alpha} \) is non-decreasing on \((-\infty,0)\) and non-increasing on \((0,\infty)\).

2. We let \( \nu \in S_2(\alpha) \) if
\[
0 < \liminf_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \nu(\mathrm{d}x) \leq \limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \nu(\mathrm{d}x) < \infty. \quad (5.14)
\]

In fact, \( S_1(\alpha) \subseteq S_2(\alpha) \) for \( \alpha \in (0,2) \), and moreover, the inclusion is strict. This assertion and some further properties of \( S_1(\alpha) \), \( S_2(\alpha) \) are given in Lemma A.1.

**Example 5.11.** Let us provide some examples for those classes of Hölder functions and of \( \alpha \)-stable-like processes used in financial modelling.

1. The European call and put are Lipschitz, hence they belong to \( \dot{W}^{1,\infty}(\mathbb{R}_+) \).

   The power call \( g(y) := ((y-K) \vee 0)^q \) with \( K > 0 \) and \( q \in (0,1) \) belongs to \( C^{0,\eta}(\mathbb{R}_+) \), but \( g \notin \dot{W}^{1,\eta}(\mathbb{R}_+) \) for any \( \eta \in (1,\infty) \). However, we can decompose \( g = g_1 + g_2 \), where \( g_1 := ((y-K) \vee 0)^q \wedge 1 \) and \( g_2 := g - g_1 \), so that \( g_1 \in \cap_{1 < \eta < 1} \dot{W}^{1,\eta}(\mathbb{R}_+) \) and \( g_2 \) is Lipschitz. By the linearity, the LRM strategy of \( g \) is the sum of the LRM strategies corresponding to \( g_1 \) and \( g_2 \).

   The binary option \( g(y) := \mathbb{1}_{\{K,\infty]\}}(y) \) belongs to \( C^{0,0}(\mathbb{R}_+) \) obviously.

2. The CGMY process with parameters \( C, G, M > 0 \) and \( Y \in (0,2) \) (see [32, Section 5.3.9]) has the Lévy measure
\[
\nu_{\text{CGMY}}(\mathrm{d}x) = C e^{Gx} \mathbb{1}_{\{x < 0\}} + e^{-Mx} \mathbb{1}_{\{x > 0\}} \frac{1}{|x|} \mathbb{1}_{\{x \neq 0\}} \mathrm{d}x
\]
which belongs to \( S_1(Y) \) due to Lemma A.1(3).

   The Normal Inverse Gaussian (NIG) process (see [32, Section 5.3.8]) has the Lévy density \( p_{\text{NIG}}(x) := \nu_{\text{NIG}}(\mathrm{d}x)/\mathrm{d}x \) that satisfies
\[
0 < \liminf_{|x| \to 0} x^2 p_{\text{NIG}}(x) \leq \limsup_{|x| \to 0} x^2 p_{\text{NIG}}(x) < \infty.
\]
Hence, Lemma A.1(3) verifies that \( \nu_{\text{NIG}} \in S_1(1) \).
5.5. Discretisation of LRM strategies. Let $X$ be a Lévy process with $(X|\mathbb{P}) \sim (\gamma, \sigma, \nu)$ and $S = e^X$. In this subsection, we apply results of Subsection 5.3, and the stochastic integral being approximated is the integral term in the FS decomposition of $g(S_T)$. Moreover, we choose the càdlàg version $\tilde{\vartheta}^g$ of the LRM strategy as mentioned in Theorem 1.1(2) so that the integral we are going to approximate is of the form

$$\int_0^T \tilde{\vartheta}^g_t \, dS_t.$$ 

Under the assumptions of Theorem 1.1, it follows from Remark 4.6 that, for $t \in [0, T)$,

$$\tilde{\vartheta}^g_t = \frac{1}{\|\sigma, \nu\|} \left( \sigma^2 \partial_y G^*(t, S_t) + \int_{\mathbb{R}} G^*(t, e^x S_t) - G^*(t, S_t) \frac{e^x - 1}{S_t} \nu(dx) \right) \quad \text{a.s.} \ (5.15)$$

For $\eta \in [0, 1]$ and $t \in [0, T]$, we define

$$\Theta(\eta)_t := \sup_{u \in [0, t]} (S_u^{\eta-1}), \quad \Phi(\eta) := \Theta(\eta)_t S_t,$$

$$\Phi(\eta)_t := \Phi(\eta)_t + \sup_{u \in [0, t]} |\Delta \Phi(\eta)_u|.$$

The results about approximation are given in items (4)-(6) of Theorem 5.12 below. In fact, the LRM strategy $\tilde{\vartheta}^g$ is quite difficult to investigate directly under the original measure $\mathbb{P}$ but it fits well the main assumption Assumption 4.5 under the minimal martingale measure $\mathbb{P}^*$. Therefore, our idea is to switch between the original measure $\mathbb{P}$ and the minimal martingale measure $\mathbb{P}^*$ and use the fact that weighted BMO-norms allow a change of measure as given in Proposition 5.3(3). Moreover, regarding the drift coefficient $\gamma_S$ given in (4.1), we now focus on the case $\gamma_S \neq 0$ since the case $\gamma_S = 0$, which corresponds to the martingale setting, was investigated in [37, Section 4].

**Theorem 5.12.** Assume Setting 4.1, Assumption 4.5, $\gamma_S \neq 0$ and $\int_{|x| \geq 1} e^{3\gamma_x} \nu(dx) < \infty$. Let $g \in C^{0, \eta}(\mathbb{R}_+)$ with $\eta \in [0, 1]$. Then, the following assertions hold:

1. Both $\Phi(\eta)$ and $\Phi(\eta)$ belong to $\bar{\mathcal{S}} \mathcal{M}_3(\mathbb{P}) \cap \mathcal{S} \mathcal{M}_2(\mathbb{P}_*)$.
2. $\mathbb{P}^* \in \mathcal{R} \mathcal{H}_3(\mathbb{P})$ and $\|\cdot\|_{\text{BMO}\, \Phi(n)(\mathbb{P})} \leq c \|\cdot\|_{\text{BMO}\, \Phi(n)(\mathbb{P})}$ for some constant $c > 0$.
3. Let $M := \vartheta^g S$. Then, Assumption 5.6 is satisfied under $\mathbb{P}^*$ for the selection

$$\hat{\vartheta} = \vartheta^g, \quad \Upsilon(\cdot, dt) = d(M, M^*_{\hat{\vartheta}} + M^*_{\vartheta}) dt, \quad \Theta = \Theta(\eta), \quad \Phi = \Phi(\eta),$$

and for the parameter $\theta$ provided in Table 1.

4. With the adapted time-nets $\tau^0_\theta$ given in (5.3), one has

$$\sup_{n \geq 1} \frac{1}{n^{\frac{1}{2}}} \left\| E^{\text{corr}} \left( \vartheta^g, \tau^0_\theta \left| n^{-\frac{1}{2}}, \frac{1-\theta}{2} \right. \right) \right\|_{\text{BMO}\, \Phi(n)(\mathbb{P})} < \infty, \quad (5.16)$$

where the parameters $r$ and $\theta$ are provided in Table 1.

5. Let $s \in (1, \infty)$, and assume in addition when $\left\| \frac{\sigma, \nu}{\gamma^s} \right\| \in [-1, \infty)$ that $\int_{|x| \geq 1} e^{(1-s)x} \nu(dx) < \infty$. Then, $\mathbb{P} \in \mathcal{R} \mathcal{H}_s(\mathbb{P}^*)$ and

$$\|\cdot\|_{\text{BMO}\, \Phi(n)(\mathbb{P}^*)} \sim c \|\cdot\|_{\text{BMO}\, \Phi(n)(\mathbb{P})}$$

for some constant $c \geq 1$, and hence

$$\sup_{n \geq 1} \frac{1}{n^{\frac{1}{2}}} \left\| E^{\text{corr}} \left( \vartheta^g, \tau^0_\theta \left| n^{-\frac{1}{2}}, \frac{1-\theta}{2} \right. \right) \right\|_{\text{BMO}\, \Phi(n)(\mathbb{P})} < \infty, \quad (5.17)$$

where the parameters $r$ and $\theta$ are provided in Table 1. Moreover, (5.17) holds true for the $L_\beta(\mathbb{P})$-norm in place of the $\text{BMO}\, \Phi(n)(\mathbb{P})$-norm.
(6) If in addition \( \int_{|x|>1} e^{px} \nu(dx) < \infty \) for some \( p \in (3, \infty) \), then (5.16) (resp. (5.17)) is satisfied for the \( L_{p-1}(\mathbb{P}^*) \)-norm (resp. \( L_p(\mathbb{P}) \)-norm) in place of the \( \text{BMO}_2^{\Phi(\eta)}(\mathbb{P}^*) \)-norm (resp. \( \text{BMO}_2^{\Phi(\eta)}(\mathbb{P}) \)-norm).

| \( \sigma \) and \( \eta \) | Small jump condition | Regularity of \( g \) | Conclusions for \( r \) and \( \theta \) |
|---|---|---|---|
| C1 | \( \sigma > 0 \) \( \eta \in [0, 1] \) \( \int_{|x|\leq 1} |x|^{\alpha} \nu(dx) < \infty \) for some \( \alpha \in [1, 2] \) | \( g \in C^{0, \eta}(\mathbb{R}_+) \) | \( \forall r \in [\alpha, 2] \) \( \forall \theta \in (0, \eta) \) if \( \eta \in (0, 1) \) \( \theta = 1 \) if \( \eta = 1 \) |
| C2 | \( \sigma = 0 \) \( \eta \in [0, 1] \) \( \int_{|x|\leq 1} |x|^{\alpha} \nu(dx) < \infty \) for some \( \alpha \in [1, \eta + 1] \) | \( g \in C^{0, \eta}(\mathbb{R}_+) \) | \( \forall r \in [\alpha, 2] \) \( \theta = 1 \) |
| C3 | \( \sigma = 0 \) \( \eta \in [0, 1] \) \( \nu \in S_2(\alpha) \) for some \( \alpha \in [1 + \eta, 2] \) | \( g \in C^{0, \eta}(\mathbb{R}_+) \) | \( \forall r \in (\alpha, 2] \) \( \forall \theta \in \left(0, \frac{2(1+n)}{n} - 1\right) \) |
| C4 | \( \sigma = 0 \) \( \eta \in [0, 1] \) \( \nu \in S_2(\alpha) \) for some \( \alpha \in [1 + \eta, 2] \) | \( g \in \tilde{W}^{1, 1-\tau}(\mathbb{R}_+) \) | \( \forall r \in (\alpha, 2] \) \( \forall \theta \in \left(0, \frac{2(1+n)}{n} - 1\right) \) |

**Remark 5.13.** (1) Let us comment on the parameters \( r \) and \( \theta \) in Table 1. First, since we use the adapted time-net \( \tau_n^\theta \) which leads to better estimates (see (5.9)), it follows that the parameter \( r \) only depends on the behavior of \( \nu \) around zero. Moreover, the smaller \( r \) is, the better approximation accuracy one achieves. The parameter \( \theta \) is the outcome of the interplay between the behavior of \( \nu \) around zero and the Hölder regularity of the payoff function.

(2) Since \( X \) is a Lévy process under both measures \( \mathbb{P} \) and \( \mathbb{P}^* \), we apply [37, Proposition 5.3] (with \( \alpha = 2 \) and \( \kappa = \frac{1-\theta}{2}, \varepsilon = n^{-\frac{1}{2}} \)) to conclude that the parameter \( n \) in front of the \( \text{BMO}_2^{\Phi(\eta)}(\mathbb{P}^*) \)-norm in (5.16) can be regarded as the \( L_2(\mathbb{P}) \)-norm and the \( L_2(\mathbb{P}^*) \)-norm of the cardinality of the combine time-net \( \tau_n^\theta \cap \rho(n^{-\frac{1}{2}}, \frac{1-\theta}{2}) \). The parameter \( n \) in front of the \( \text{BMO}_2^{\Phi(\eta)}(\mathbb{P}) \)-norm in (5.17) can be interpreted in a similar manner.

For the proof of Theorem 5.12, we need the following lemmas where we recall \( \nu^*(dx) = \left(1 - \frac{\nu^*}{\|\nu^*\|_{\tilde{L}^p(\mathbb{R})}}(e^x - 1)\right)\nu(dx) \) from (4.6) and the classes \( S_1(\alpha), S_2(\alpha) \) from Definition 5.10.

**Lemma 5.14.** Under Assumption 4.5, the following assertions hold:

1. For \( \beta \in [0, 2] \), one has \( \int_{|x|\leq 1} |x|^{\beta} \nu(dx) < \infty \iff \int_{|x|\leq 1} |x|^{\beta} \nu^*(dx) < \infty \).
2. Assume \( \gamma_S \neq 0 \). Then, for \( r \in [1, \infty) \) one has

\[
\mathbb{E}e^{rX_t} < \infty, \forall t > 0 \iff \int_{|x|>1} e^{rx} \nu(dx) < \infty \iff \int_{|x|>1} e^{(r-1)x} \nu^*(dx) < \infty \iff \mathbb{E}e^{(r-1)X_t} < \infty, \forall t > 0.
\]

**Proof.** Item (1) is clear from the relation between \( \nu \) and \( \nu^* \). A short computation and [31, Theorem 25.3] imply item (2). \( \square \)

**Lemma 5.15.** Assume Assumption 4.5 and \( \int_{|x|>1} e^{r\nu}(dx) < \infty \). If \( \nu \in S_i(\alpha) \) for some \( \alpha \in (0, 2) \), then \( \nu^* \in S_i(\alpha) \) for \( i = 1, 2 \).
Lemma 5.14, we obtain that $\Phi(\eta) \in SM_3(\mathbb{P}) \cap SM_2(\mathbb{P}^*)$. Thanks to Lemma A.2, one has $\Phi(\eta) \in SM_3(\mathbb{P}) \cap SM_2(\mathbb{P}^*)$. 

Proof. We first prove the assertion for $i = 1$. Assume that $S_1(\alpha) \ni \nu = \nu_1 + \nu_2$, where $\nu_1, \nu_2$ are Lévy measures satisfying (5.13) and (5.12) respectively. Observe that $\text{supp} \nu_i \subseteq \text{supp} \nu$ for $i = 1, 2$. We define 

$$\nu^*_1(dx) := \begin{cases} \left(1 - \frac{\gamma_2}{\|\sigma, \nu\|}(e^x - 1)\right) \, \mathbb{1}_{\{x < 0\}} + \mathbb{1}_{\{x > 0\}} \nu_1(dx) & \text{if } \frac{\gamma_2}{\|\sigma, \nu\|} \leq 0 \\ \left(1 - \frac{\gamma_2}{\|\sigma, \nu\|}(e^x - 1)\right) \, \mathbb{1}_{\{x > 0\}} + \mathbb{1}_{\{x < 0\}} \nu_1(dx) & \text{if } \frac{\gamma_2}{\|\sigma, \nu\|} > 0, \end{cases}$$

and set 

$$\nu^*_2(dx) := \nu^*(dx) - \nu^*_1(dx)$$

It is clear that $\nu^*_1$ and $\nu^*_2$ are Lévy measures. Moreover, a short calculation shows that $\nu^*_1$ and $\nu^*_2$ satisfy (5.13) and (5.12) respectively, which verifies $\nu^* \in S_1(\alpha)$.

We now prove the statement for $i = 2$. Assume that $\nu \in S_2(\alpha)$. Let $\varepsilon \in (0, 1)$ and $\delta > 0$ be such that $\frac{\gamma_2(e^x - 1)}{\|\sigma, \nu\|} < \varepsilon$ for all $|x| < \delta$. Then, 

$$\int_\mathbb{R} (1 - \cos(ux))\nu^*(dx) \geq \int_{|x| < \delta} (1 - \cos(ux)) \left(1 - \frac{\gamma_2(e^x - 1)}{\|\sigma, \nu\|}\right) \nu(dx) \geq (1 - \varepsilon) \int_{|x| < \delta} (1 - \cos(ux))\nu(dx)$$

and hence,

$$\limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_\mathbb{R} (1 - \cos(ux))\nu^*(dx) \leq (1 + \varepsilon) \limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_\mathbb{R} (1 - \cos(ux))\nu(dx) < \infty.$$
(2) We recall $\mathcal{E}(U)$ from Definition 4.4 and notice that $\mathcal{E}(U) > 0$ due to Assumption 4.5. According to Subsection 2.4, there is a Lévy process $V$ with $(V|\mathbb{P}) \sim (\gamma_V, \sigma_V, \nu_V)$ such that $\mathcal{E}(U) = e^V$. Due to (4.5), by letting $h(x) := \ln(1 + x)$ for $x > -1$ one has

$$\nu_V = \nu_U \circ h^{-1} = (\nu \circ \alpha_U^{-1}) \circ h^{-1} = \nu \circ (h \circ \alpha_U)^{-1}. \quad (5.18)$$

Since $h(\alpha_U(x)) = \ln\left(1 - \frac{\gamma(x^\sigma - 1)}{\|\sigma, \nu\|}\right)$ for $x \in \text{supp} \nu$, there exists an $\varepsilon(5.19) > 0$ such that

$$\{x \in \text{supp} \nu : |h(\alpha_U(x))| > 1\} \subseteq \mathbb{R}\setminus(-\varepsilon(5.19), \varepsilon(5.19)). \quad (5.19)$$

Then, the assumption $\int_{|x| > 1} e^{3x} \nu(dx) < \infty$ implies that

$$\int_{|x| > 1} e^{3x} \nu(dx) = \int_{|h(\alpha_U(x))| > 1} e^{3(h(\alpha_U(x)))} \nu(dx) \leq \int_{|x| > \varepsilon(5.19)} \left(1 - \frac{\gamma(x^\sigma - 1)}{\|\sigma, \nu\|}\right)^3 \nu(dx) < \infty,$$

Let $(V|\mathbb{P}) \sim \psi_V$. Since $(e^{3V_t + \theta\psi_V(-3)}\mathbb{I}_{t \in [0,T]})$ is a càdlàg martingale, it follows from the optional stopping theorem that for any stopping time $\rho: \Omega \to [0, T]$, a.s.,

$$\mathbb{E}_\mathbb{P}[e^{3V_T}] = e^{-T\psi_V(-3)}\mathbb{E}_\mathbb{P}\left[e^{3V_{\rho T} + \theta\psi_V(-3)}\right] = e^{-T\psi_V(-3)}e^{3\psi_V + \theta\psi_V(-3)}$$

$$\leq e^{\theta\psi_V(-3)}e^{3\psi_V} = e^{T\psi_V(-3)}\mathbb{E}_\mathbb{P}[e^{\psi_V}]^3,$$

where we use the martingale property of $e^V$ for the last equality. According to Definition 5.2 and Proposition 5.3(3), $d\mathbb{P}^* = e^{V_T}d\mathbb{P} \in \mathcal{RH}_3(\mathbb{P})$.

(3) In the notations of Assumption 5.6, let

$$\dot{\vartheta} = \dot{\vartheta}^g, \quad \Upsilon(\cdot, dt) = d\langle M, M\rangle_t^{p^*} + M_t^2 dt, \quad \Theta = \Theta(\eta), \quad \Phi = \Phi(\eta).$$

We now verify the requirements of Assumption 5.6 under the measure $\mathbb{P}^*$.

Item (i) is clear. For item (ii), Theorem 1.1(2) verifies that $M = \dot{\vartheta}^gS$ is a $\mathbb{P}^*$-martingale adapted to the augmented natural filtration of $X$, which is a quasi-left continuous filtration (see [29, p.150]). This implies that $\dot{\vartheta}^g_{t-}S_{t-} = \dot{\vartheta}^g_t S_t$ a.s. for each $t \in [0, T)$ (see [29, p.191]), and hence $\dot{\vartheta}^g_t = \vartheta^g_t$ a.s. due to $S_t = S_t$ a.s.

For item (iii), we can prove (5.4) as in [37, Example 3.2] (with $\sigma(x) = x$), where the square $\mathbb{P}^*$-integrability of $M$ can be inferred from (5.20).

For item (iv), it follows from the proof of [37, Theorem 4.6(3)] that for any $a \in [0, T)$, a.s.,

$$\mathbb{E}_\mathbb{P}^*\left[\int_{(a, T]} (T - t)^{1-a}\Upsilon(\cdot, dt)\right]$$

$$\leq \left\{\begin{array}{ll}
\mathbb{E}_\mathbb{P}^*\left[\lim_{T \to T}\int_{(a, T]} M_t^2 dt\right] & \text{if } \theta = 1 \\
\mathbb{E}_\mathbb{P}^*\left[\int_{(a, T]} ((1 - \theta)(T - t)^{-\theta} + (T - t)^{1-\theta}) M_t^2 dt\right] & \text{if } \theta \in (0, 1).
\end{array}\right.$$}

Hence, in order to verify (iv), thanks to $\Phi(\eta) \in \mathcal{SM}_2(\mathbb{P}^*)$, it suffices to show that there is a constant $c(5.20) > 0$ which might depend on $\dot{\vartheta}$ but is independent of $t$ such that

$$|\dot{\vartheta}^g_t| \leq c(5.20)(T - t)^{\frac{\theta - 1}{\theta}} \Theta(\eta)_t \quad \text{a.s., } \forall t \in [0, T), \quad (5.20)$$

where $\dot{\vartheta}$ is given according to the cases C1 and C2 of Table 1 as follows

$$\dot{\vartheta} = \begin{cases}
\eta & \text{in the case C1} \\
1 & \text{in the case C2}.
\end{cases}$$

Regarding C3 and C4, it is sufficient to prove that (5.20) holds for any $\dot{\vartheta} \in \left(0, \frac{2(1+\eta)}{\alpha} - 1\right)$. 


Indeed, we first let \( Q := \mathbb{P}^* \) and \( \ell := \nu \) in (A.1) and then derive from (5.15) that
\[
\bar{\nu}_t^\theta = \frac{I_{v}^{\nu^*}(T - t, S_t)}{\| (\sigma, \nu) \|} \quad \text{a.s., } \forall t \in [0, T).
\]

**Case C1:** Since \( \sigma^* = \sigma > 0 \) and \( \int_{|x|>1} e^{2x} \nu^*(dx) < \infty \), Proposition A.4(1) implies (5.20) with \( \bar{\nu}_t^\theta = \eta_t \).

**Case C2:** Since \( \int_{|x|\leq 1} |x|^{\gamma+1} \nu(dx) < \infty \), combining Lemma 5.14(1) with Proposition A.4(2) we obtain (5.20) with \( \bar{\nu}_t^\theta = \bar{\nu}_t \).

**Case C3:** Due to Lemma 5.15, we have \( \nu^* \in S_1(\alpha) \). Let \( \varepsilon \in (0, 2 - \alpha) \) be arbitrary. Then it follows from Lemma A.1(2) that \( \int_{|x|\leq 1} |x|^{\alpha+\varepsilon} \nu(dx) < \infty \). We apply Proposition A.4(3) and Remark A.5 with \( \beta = \alpha + \varepsilon \) to obtain that, for any \( t \in [0, T) \), a.s.,
\[
|\bar{\nu}_t^\theta| = \frac{|I_{v}^{\nu^*}(T - t, S_t)|}{\| (\sigma, \nu) \|} \leq c_\varepsilon (T - t)^{-\frac{\gamma+1}{\alpha} - 1 - \frac{\alpha}{\nu} S_t^{-1}} \leq c_\varepsilon (T - t)^{-\frac{1}{2} \left( \frac{2(\gamma+1)}{\alpha} - 1 - \frac{2\beta}{\alpha} \right)} \Theta(\eta)_t,
\]
where \( c_\varepsilon > 0 \) is some constant which might depend on \( \varepsilon \). Since \( \varepsilon > 0 \) can be arbitrarily small, the assertion (5.20) holds for any \( \bar{\nu}_t^\theta \in (0, \frac{2(1+\eta)}{\alpha} - 1) \).

**Case C4:** Again, one has \( \nu^* \in S_2(\alpha) \) due to Lemma 5.15, and Lemma A.1(2) verifies \( \int_{|x|\leq 1} |x|^{\alpha+\varepsilon} \nu(dx) < \infty \) for all \( \varepsilon \in (0, 2 - \alpha) \). By Proposition A.4(4) and Remark A.5 with \( \beta = \alpha + \varepsilon \) and by the same reason as in the case C3 above, we get (5.20).

(4) By the relation between the behavior of \( \nu \) and of \( \nu^* \) around zero given in Lemma 5.14(1), we use item (3) and apply (5.9) to obtain (5.16).

(5) **Step 1.** For \( \nu^*_\nu \) given in (5.18), we first show that \( \int_{|x|>1} e^{(1-s)x} \nu^*_\nu(dx) < \infty \).

Indeed,
\[
\int_{|x|>1} e^{(1-s)x} \nu^*_\nu(dx) = \int_{|x|>1} \left( 1 - \frac{\gamma S(x)}{\| (\sigma, \nu) \|} \right)^{1-s} \nu(dx)
\]
\[
\leq \int_{|x|\geq \varepsilon(5.19)} \left( 1 - \frac{\gamma S(x)}{\| (\sigma, \nu) \|} \right)^{1-s} \nu(dx) =: I(5.21).
\]

We consider three cases regarding \( \frac{\| (\sigma, \nu) \|}{\gamma S} \) as follows:

**Case 1:** \( \frac{\| (\sigma, \nu) \|}{\gamma S} > -1 \). We denote \( x_0 := \ln (1 + \frac{\| (\sigma, \nu) \|}{\gamma S}) \). Then, Assumption 4.5 verifies that \( x_0 \notin \text{supp} \nu \), which means \( \nu((x_0 - \varepsilon_0, x_0 + \varepsilon_0)) = 0 \) for some \( \varepsilon_0 > 0 \). Moreover, using the mean value theorem we infer that \( 1 - \frac{\gamma S(x)}{\| (\sigma, \nu) \|} \geq |x - x_0|^{1-s} \frac{|\gamma S|}{\| (\sigma, \nu) \|} \| \xi^x \| \| \xi \| \) for all \( x \in \text{supp} \nu \). Hence,
\[
I(5.21) = \int_{|x|\geq \varepsilon(5.19), |x-x_0|\geq \varepsilon_0} \left( 1 - \frac{\gamma S(x)}{\| (\sigma, \nu) \|} \right)^{1-s} \nu(dx),
\]
\[
\leq \varepsilon_0^{1-s} \frac{|\gamma S|}{\| (\sigma, \nu) \|} \int_{|x|\geq \varepsilon(5.19), |x-x_0|\geq \varepsilon_0} e^{(1-s)(x\wedge x_0)} \nu(dx)
\]
\[
\leq \varepsilon_0^{1-s} \frac{|\gamma S|}{\| (\sigma, \nu) \|} \int_{|x|\geq \varepsilon(5.19)} e^{(1-s)(x\wedge x_0)} \nu(dx) < \infty,
\]
where the finiteness is due to the assumption \( \int_{|x|>1} e^{(1-s)x} \nu(dx) < \infty \).

**Case 2:** \( \frac{\| (\sigma, \nu) \|}{\gamma S} = -1 \). We have \( I(5.21) = \int_{|x|\geq \varepsilon(5.19)} e^{(1-s)x} \nu(dx) < \infty \).
Case 3: $\|\sigma\psi\|_{\gamma_7} < -1$. In this case, one has $\gamma_S < 0$, which implies that $\inf_{x \in \mathbb{R}} \left(1 - \frac{\gamma_S(e^{(1-s)} - 1)}{\|\sigma\psi\|_{\gamma_7}}\right) > 1 + \frac{\gamma_S}{\|\sigma\psi\|_{\gamma_7}} > 0$. Hence,

$$I_{(5.21)} \leq \left(1 + \frac{\gamma_S}{\|\sigma\psi\|_{\gamma_7}}\right)^{1-s} \int_{|x| \geq \varepsilon (5.19)} \nu(dx) < \infty.$$  

We conclude from three cases above that $\int_{|x| > 1} e^{(1-s)x} \nu(dx) < \infty$, or equivalently

$$e^{-t\nu(V((s-1)i))} = \mathbb{E}e^{(1-s)V_i} < \infty, \quad t > 0.$$

**Step 2.** We show $\mathbb{P} \in \mathcal{R}\mathcal{H}_s(\mathbb{P}^*)$. By writing $d\mathbb{P} = e^{-V_t}d\mathbb{P}^*$ and since $e^V = \mathcal{E}(U)$ is a $\mathbb{P}$-martingale, it implies that $e^V$ is a $\mathbb{P}^*$-martingale. We have for any $t \in [0, T]$ that, a.s.,

$$\mathbb{E}_{\mathbb{F}_t}[e^{s(-V_T)}] = e^{-V_t}\mathbb{E}_{\mathbb{F}_t}[e^{-sV_t}e^{V_T}] = e^{-V_t}\mathbb{E}_{\mathbb{F}_t}[e^{(1-s)V_T}] \leq e^{T|\nu(V((s-1)i))|}e^{-sV_t}.''

By a similar argument as in the proof of [15, Proposition A.1], we infer that

$$\mathbb{E}_{\mathbb{F}_t}[e^{s(-V_T)}] \leq e^{T|\nu(V((s-1)i))|}e^{-sV_t} = e^{T|\nu(V((s-1)i))|+s}\mathbb{E}_{\mathbb{F}_t}[e^{-V_T}].$$

for any stopping times $\rho: \Omega \rightarrow [0, T]$, which implies $\mathbb{P} \in \mathcal{R}\mathcal{H}_s(\mathbb{P}^*)$.

**Step 3.** Thanks to Step 2 and items (1), (2), we apply Proposition 5.3(3) to obtain

$$\|\cdot\|_{\text{BMO}^2(\mathbb{P}^*)} \sim_{c} \|\cdot\|_{\text{BMO}^2(\mathbb{P})}.$$  

Then, assertion (5.17) is clear due to (5.16). The “Moreover” part holds because of $\Phi(\eta) \in \mathcal{S}\mathcal{M}_2(\mathbb{P})$ and (5.2).

(6) A similar argument as in the proof of item (1) shows that both $\Phi(\eta)$ and $\overline{\Phi}(\eta)$ belong to $\mathcal{S}\mathcal{M}_p(\mathbb{P}) \cap \mathcal{S}\mathcal{M}_{p-1}(\mathbb{P}^*)$. We now apply (5.2) to derive the assertion. \(\square\)

**Appendix A. Some technical results**

A.1. **Some properties of classes $\mathcal{S}_1(\alpha)$ and $\mathcal{S}_2(\alpha)$**. We recall $\mathcal{S}_1(\alpha)$ and $\mathcal{S}_2(\alpha)$ from Definition 5.10.

**Lemma A.1** (See also [37], Remark 4.5). For $\alpha \in (0, 2)$, the following assertions hold:

1. $\mathcal{S}_1(\alpha) \subsetneq \mathcal{S}_2(\alpha)$.
2. If $\nu \in \mathcal{S}_2(\alpha)$, then $\alpha = \inf \{r \in [0, 2]: \int_{|x| < 1} |x|^r \nu(dx) < \infty\}$.
3. If a Lévy measure $\nu$ has a density $p(x) := \frac{\nu(dx)}{|x|^{\alpha}}$ which satisfies
   $$0 < \lim_{|x| \to 0} \inf |x|^{1+\alpha} p(x) \leq \lim_{|x| \to 0} \sup |x|^{1+\alpha} p(x) < \infty,$$
   then $\nu \in \mathcal{S}_1(\alpha)$.

**Proof.** (1) Let $\mathcal{S}_1(\alpha) \ni \nu = \nu_1 + \nu_2$. A short calculation shows that (5.14) holds for $\nu_1$ in place of $\nu$. Combining this with (5.12) yields that (5.14) holds for $\nu$, and hence $\nu \in \mathcal{S}_2(\alpha)$. Since $\nu(dx) := x^{-1-\alpha} \mathbb{I}_{(0, 1)}(x)dx \in \mathcal{S}_2(\alpha) \setminus \mathcal{S}_1(\alpha)$, the inclusion $\mathcal{S}_1(\alpha) \subseteq \mathcal{S}_2(\alpha)$ is strict.

(2) follows from [4, Theorem 3.2].

(3) By assumption, there exist constants $0 < c \leq C < \infty$ and $\varepsilon > 0$ such that

$$c|x|^{-1-\alpha} \leq p(x) \leq C|x|^{-1-\alpha}, \quad \forall |x| \leq \varepsilon.$$
We let
\[ \nu_1(dx) := c 1_{\{0 < |x| \leq \varepsilon\}} |x|^{-1-\alpha} dx \quad \text{and} \quad \nu_2(dx) := \nu(dx) - \nu_1(dx). \]
Then, \( \nu_1 \) satisfies (5.13). For \( \nu_2 \), we have
\[
\int_{\mathbb{R}} (1 - \cos(ux)) \nu_2(dx) \leq (C - c) \int_{|x| \leq \varepsilon} \frac{1 - \cos(ux)}{|x|^{1+\alpha}} dx + 2 \int_{|x| > \varepsilon} \nu(dx),
\]
which implies that (5.12) holds for \( \nu_2 \). Hence, \( \nu \in S_1(\alpha) \). \( \square \)

A.2. Regularity of weight processes. Let \( T \in (0, \infty) \). We assume that \( \mathcal{Q} \) is a probability measure and \( X = (X_t)_{t \in [0,T]} \) is a Lévy process with \( (X|Q) \sim (\gamma, \sigma^Q, \nu^Q) \).

The regularity of the weight \( \Phi \) used in Theorem 5.7 is verified by Lemma A.2 below. For \( \Phi \in \text{CL}^+([0,T]) \), we let \( \Phi \in \text{CL}^+([0,T]) \) by setting
\[
\Phi_t := \Phi_t + \sup_{u \in [0,t]} |\Delta \Phi_u|, \quad t \in [0,T].
\]
It is clear that \( \Phi \vee \Phi_- \leq \Phi \), and \( \Phi \equiv \Phi \) if and only if \( \Phi \) is continuous.

**Lemma A.2** ([37], Proposition 7.1). If \( \Phi \in \text{SM}_q(Q) \) for some \( q \in (0, \infty) \), then \( \Phi \in \text{SM}_q(Q) \).

We next recall the process \( \Phi(\eta) \in \text{CL}^+([0,T]) \) used in Theorem 5.12, that is
\[
\Phi(\eta)_t := e^{X_t} \sup_{u \in [0,t]} e^{(\eta-1)X_u}, \quad t \in [0,T], \eta \in [0,1].
\]

**Lemma A.3** ([37], Proposition 7.2). If \( \int_{|x| > 1} e^{\sigma^2 x^2} \nu^Q(dx) < \infty \) for some \( q \in (1, \infty) \), then \( \Phi(\eta) \in \text{SM}_q(Q) \) for all \( \eta \in [0,1] \).

A.3. Gradient type estimates for a Lévy semigroup on Hölder spaces. Assume that \( X = (X_t)_{t \geq 0} \) is a Lévy process with respect to a probability measure \( Q \) with \( (X|Q) \sim (\gamma^Q, \sigma^Q, \nu^Q) \). We let
\[
D^\exp(X|Q) := \left\{ g: \mathbb{R}^+ \to \mathbb{R} \text{ Borel} : \mathbb{E}^Q|g(ge^{X_t})| < \infty, \forall y > 0, t \geq 0 \right\},
\]
where \( \mathbb{E}^Q \) is the expectation computed under \( Q \). For \( t \geq 0 \), we define \( Q_t: D^\exp(X|Q) \to D^\exp(X|Q) \) by setting
\[
Q_t g(y) := \mathbb{E}^Q g(ge^{X_t}).
\]
It is clear that \( Q_{t+s} = Q_t \circ Q_s \) for all \( s,t \geq 0 \) which means that \( (Q_t)_{t \geq 0} \) is a semigroup.

For a Lévy measure \( \ell \) on \( B(\mathbb{R}) \) and a Borel function \( g \), we write symbolically
\[
\Gamma^Q_\ell(t,y) := |\sigma^Q|^2 \partial_y Q_t g(y) + \int_{\mathbb{R}} \frac{Q_{t\ell} (g(e^x y) - Q_t g(y)) (e^x - 1) \ell(dx)}{y}, \quad (t,y) \in \mathbb{R}^+_t, \quad \partial_y Q_t g := 0 \text{ if } \sigma^Q = 0.
\]
for \( (t,y) \in \mathbb{R}^+_t \), where \( \partial_y Q_t g \) : = 0 if \( \sigma^Q = 0 \). We recall \( C^{0,\eta}(\mathbb{R}^+) \) and \( W^{1,1}(\mathbb{R}^+) \) from Definition 5.9 and \( S_1(\alpha), S_2(\alpha) \) from Definition 5.10.

**Proposition A.4** ([37], Proposition 8.6). Let \( \ell \) be a Lévy measure and \( g \in C^{0,\eta}(\mathbb{R}^+) \) with \( \eta \in [0,1] \). Assume that \( \int_{|x| > 1} e^{(\eta+1)\sigma^2 x^2} \ell(dx) < \infty \). Then, for any \( T \in (0, \infty) \) there exists a constant \( c_{(A.2)} > 0 \) such that
\[
|\Gamma^Q_\ell(t,y)| \leq c_{(A.2)} R_t y^{\eta-1}, \quad \forall (t,y) \in (0,T] \times \mathbb{R}^+,
\]
where the cases for \( R_t \) are provided in the following cases:

1. If \( \sigma^Q > 0 \) and \( \int_{|x| > 1} e^{2\sigma^2 x^2} \ell(dx) < \infty \), then \( R_t = t^{\eta-1}. \)
2. If \( \sigma^Q = 0 \), \( \int_{|x| > 1} e^{2\sigma^2 x^2} \ell(dx) < \infty \) and \( \int_{|x| \leq 1} |x|^\eta \ell(dx) < \infty \), then \( R_t = 1. \)
(3) If $\sigma^Q = 0$ and if the following two conditions hold:
   (a) $\nu^Q \in S_1(\alpha)$ for some $\alpha \in (0, 2)$ and $\int_{|x|>1} e^{r\nu^Q}(dx) < \infty$,
   (b) there is a $\beta \in (1 + \eta, 2]$ such that
   \[
   0 < \sup_{r \in (0, 1]} r^\beta \int_{|x|\leq 1} \left( \frac{|x|^2}{r} \land \frac{|x|^\eta+1}{r} \right) \ell(dx) < \infty,
   \]
   then one has $R_\beta = t^{\frac{1+\eta-\alpha}{\alpha}}$.

(4) If $\sigma^Q = 0$, $g \in \hat{W}^{\frac{1}{1-\alpha}}(\mathbb{R}_+)$, and if the following two conditions hold:
   (a) $\nu^Q \in S_2(\alpha)$ for some $\alpha \in (0, 2)$ and $\int_{|x|>1} e^{r\nu^Q}(dx) < \infty$,
   (b) there is a $\beta \in (1 + \eta, 2]$ such that (A.3) is satisfied,
   then one has $R_\beta = t^{\frac{\eta+1-\alpha}{\alpha}}$.

Here, the constant $c_{(A.2)}$ might depend on $\beta$ in items (3) and (4).

Remark A.5. Since $\frac{|x|^2}{r} \land \frac{|x|^\eta+1}{r} \leq |x|^\beta$ for $\beta \in (1 + \eta, 2]$, a sufficient condition for (A.3) is that $0 < \int_{|x|\leq 1} |x|^\beta \ell(dx) < \infty$.

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References

1. D. Applebaum, *Lévy processes and stochastic calculus (2nd ed.)*, Cambridge University Press, Cambridge, 2009.
2. T. Arai, R. Suzuki, Local risk-minimization for Lévy markets, *Int. J. Financ. Eng.* 2(2), 2015, 1550015.
3. F. Benth, G. Di Nunno, A. Lokka, B. Øksendal, F. Proske, Explicit representation of the minimal variance portfolio in markets driven by Lévy processes, *Math. Finance* 13(1), 2003, 55–72.
4. R. Blumenthal, R. Getoor, Sample functions of stochastic processes with stationary independent increments, *J. Math. Mech.* 10, 1961, 493–516.
5. M. Brodén, P. Tankov, Tracking errors from discrete hedging in exponential Lévy model, *Int. J. Theor. Appl. Finance* 14(6), 2011, 803–837.
6. T. Choulli, N. Vandaele, M. Vanmaele, The Föllmer–Schweizer decomposition. Comparison and description, *Stochastic Process. Appl.* 120, 2010, 853–872.
7. T. Choulli, L. Krawczyk, C. Stricker, $\mathcal{E}$-martingales and their applications in mathematical finance, *Ann. Probab.* 26(2), 1998, 853–876.
8. R. Cont, P. Tankov, E. Voltchkova, *Hedging with options in models with jumps*, In: Stochastic Analysis and Applications – The Abel Symposium 2005, Springer, Berlin, 2007.
9. F. Esche, M. Schweizer, Minimal entropy preserves the Lévy property: how and why, *Stochastic Process. Appl.* 115, 2005, 299–327.
10. C. Geiss, E. Laukka, Denseness of certain smooth Lévy functionals in $\mathbb{D}_{1,2}$, *Probab. Math. Statist.* 31, 2011, 1–15.
11. C. Geiss, S. Geiss, E. Laukka, A note on Malliavin fractional smoothness for Lévy processes and approximation, *Potential Anal.* 39, 2013, 203–230.
12. C. Geiss, A. Steinicke, Malliavin derivative of random functions and applications to Lévy driven BSDEs, *Electron. J. Probab.* 21(10), 2016, 28 pp.
13. S. Geiss, Quantitative approximation of certain stochastic integrals, *Stochastics Stochastics Rep.* 73(3–4), 2002, 241–270.
14. S. Geiss, Weighted BMO and discrete time hedging within the Black-Scholes model, *Probab. Theory Related Fields* 132, 2005, 13–38.
15. S. Geiss, N. T. Thuan, On Riemann–Liouville operators, BMO, gradient estimates in the Lévy–Itô space, and approximation, 2020, arXiv:2009.00899.
16. S. Goutte, N. Oudjane, F. Russo, Variance optimal hedging for continuous time additive processes and applications, *Stochastics* 86(1), 2014, 147–185.
17. F. Hubalek, J. Kallsen, L. Krawczyk, Variance-optimal hedging for processes with stationary in-dependent increments, *Ann. Appl. Probab.* 16(2), 2006, 853–885.
and Lévy-type processes, *J. Theor. Probab.* 25, 2012, 144–170.

18. K. Itô, Spectral type of the shift transformation of differential processes with stationary increments, *Trans. Amer. Math. Soc.* 81, 1956, 253–263.

19. J. Jacob, S. Méléard, P. Protter, Explicit form and robustness of martingale representations, *Ann. Probab.* 28(4), 2000, 1747–1780.

20. J. Kallsen, A. Pauwels, Variance-optimal hedging in general affine stochastic volatility models, *Adv. Appl. Prob.* 42, 2010, 83–105.

21. E. Laukkanen, On Malliavin calculus and approximation of stochastic integrals for Lévy process, PhD thesis, University of Jyväskylä, 2013.

22. E. Laukkanen, Malliavin smoothness on the Lévy space with Hölder continuous or $BV$ functionals, *Stochastic Process. Appl.* 130(8), 2020, 4766–4792.

23. A. Lokka, Martingale representation of functionals of Lévy processes, *Stoch. Anal. Appl.* 22(4), 2004, 867–892.

24. J. Ma, P. Protter, J. Zhang, Explicit form and path regularity of martingale representations, In: *Lévy processes: Theory and applications*, Springer, New York, 2001.

25. P. Malliavin, H. Airault, L. Kay, G. Letac, *Integration and probability*, Springer, New York, 1995.

26. P. Monat, C. Stricker, Föllmer–Schweizer decomposition and mean-variance hedging for general claims, *Ann. Probab.* 23(2), 1995, 605–628.

27. D. Nualart, E. Nualart, *Introduction to Malliavin calculus*, Cambridge University Press, Cambridge, 2018.

28. G. Di Nunno, B. Øksendal, F. Proske, *Malliavin calculus for Lévy processes with applications to finance*, Springer-Verlag Berlin Heidelberg, 2009.

29. P. Protter, *Stochastic integration and differential equations (2nd ed., ver. 2.1)*, Springer-Verlag, Berlin, 2005.

30. M. Rosenbaum, P. Tankov, Asymptotically optimal discretization of hedging strategies with jumps, *Ann. Appl. Probab.* 24(3), 2014, 1002–1048.

31. K. Sato, *Lévy processes and infinitely divisible distributions (2nd ed.)*, Cambridge University Press, Cambridge, 2013.

32. W. Schoutens, *Lévy processes in finance: Pricing financial derivatives*, Wiley, 2003.

33. M. Schweizer, On the minimal martingale measure and the Föllmer–Schweizer decomposition, *Stoch. Anal. Appl.*, 13(5), 1995, 573–599.

34. M. Schweizer, A guided tour through quadratic hedging approaches, In: *Option pricing, interest rates and risk management*, Cambridge Univ. Press, Cambridge, 2001.

35. J. Solé, F. Utzet, J. Vives, *Chaos expansion and Malliavin calculus for Lévy processes*, In: Stochastic Analysis and Applications – The Abel Symposium 2005, Springer, Berlin, 2007.

36. P. Tankov, *Pricing and hedging in exponential Lévy models: review of recent results*, Paris-Princeton Lecture Notes in Mathematical Finance, Springer, 2010.

37. N. T. Thuan, Approximation of stochastic integrals with jumps in weighted bounded mean oscillation spaces, 2020. arXiv:2009.02116.

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