Booms and Crashes in Self–Similar Markets

S. Gluzman\textsuperscript{1} and V. I. Yukalov\textsuperscript{2}\textsuperscript{*}

\textsuperscript{1}International Center of Condensed Matter Physics
University of Brasilia, CP 04513, Brasilia, DF 70919-970, Brazil
and

\textsuperscript{2}Centre for Interdisciplinary Studies in Chemical Physics
University of Western Ontario, London, Ontario N6A 3K7, Canada

Sharp changes in time series representing market dynamics are studied by means of the self–similar analysis suggested earlier by the authors. These sharp changes are market booms and crashes. Such crises phenomena in markets are analogous to critical phenomena in physics. A simple classification of the market crisis phenomena is given.

\textsuperscript{*}The author to whom correspondence is to be addressed
I. INTRODUCTION

Booms and crashes of market structures are examples of dynamical transitions that remind phase transitions and critical phenomena in physical systems. This analogy is, of course, not direct, since markets are nonequilibrium systems. Nevertheless, it is possible to develop a renormalization group approach for analysing the dynamics of markets near the points of their critical changes, such as booms and crashes, somewhat analogous to the renormalization group technique used for describing critical phenomena. Since nonequilibrium markets are quite different from equilibrium systems of statistical mechanics, the renormalization group needed for the analysis of market behaviour is to be also very different from the statistical renormalization group. We show that such a natural tool for describing market critical phenomena is the algebraically invariant self–similar renormalization group. In this approach, a market critical restructuring is treated analogously to a critical phenomenon in physics, which permits us to present the market time evolution in the critical region by means of a group property called, in physical language, the property of self–similarity. Using this self–similar renormalization group allows us to renormalize the time series generated by the market and to describe the occurrence of booms and crashes in good quantitative agreement with the data available.

A powerful tool for describing critical phenomena in equilibrium statistical systems is the so–called statistical renormalization group, formulated by Wilson being based on the Kadanoff idea of scaling transformations. A detailed account of this technique can be found in many books, for instance, in Ma [1]. Markets can be considered as very complex nonequilibrium statistical systems, and sharp changes of market structure, such as crashes, can be thought of as a kind of nonequilibrium phenomena, comparable to earthquakes [2,3]. There have been suggested several other analogies between physical systems and markets [2-5]. However, to our knowledge, there has been no attempt to develop a renormalization group approach for treating the behaviour of markets near their critical points of sharp changes, that is near their booms and crashes. We have suggested such an approach in our previous papers [6,7].

As far as markets, despite some analogies with physical systems, are much more complex and, in addition, nonequilibrium, the standard statistical renormalization group cannot be applied to such transitions as booms and crashes. Fortunately, there exists another approach based on self–similar renormalization group, which provides an evolution equation in the space of approximations [8-12]. This approach, completed by the condition of algebraic invariance, makes the basis of the algebraic self–similar renormalization [13-15]. Dynamics in the space of approximations, for the case of a market, is nothing but the time evolution of this market. This is why the self–similar renormalization group seems to be an absolutely natural concept for treating market evolution.

Let us stress the main ideas explaining why our approach suites well for describing market dynamics: (i) Sharp structural changes in a market are equivalent to critical phenomena in a physical system. (ii) The evolution of a market in the critical region can be formulated by means of a group property with respect to time, similarly to the evolution of a Hamiltonian with respect to its parameters in statistical renormalization group. (iii) In the same way as in the latter group, where in order to get a good quantitative description, one needs to make just a few renormalization steps, in the self–similar renormalization group, we also need to take into account only a few temporal points.

II. SCHEME OF ANALYSIS

Here we present a short survey of the self–similar analysis to be used in what follows. More details can be found in our previous papers [6,7]. The main characteristic of any market activity is the price for a security or commodity, or some index related to this price. Let \( f(t) \) be such a characteristic of a market at the moment of time \( t \). A sharp change of the value of price, that is of \( f(t) \), occurring during the period of time comparable with the resolution of the time series corresponding to \( f(t) \), is commonly called a crisis. Booms are sharp upward moves and crashes are sharp downward moves of the price.

Suppose that the values of \( f(t) \) are known for \( n + 1 \) time points \( t = k = 0, 1, 2, \ldots, n \), so that

\[
 f(k) = a_k \quad (k = 0, 1, \ldots, n). \tag{1}
\]

The aim of our analysis is to predict, being based on the data (1), the behaviour of \( f(t) \) for \( t \geq n + 1 \).

At the first step, for the function \( f(t) \) in the time interval \( 0 \leq t \leq n \), we construct a polynomial representation

\[
 p_n(t) = \sum_{k=0}^{n} A_k t^k \quad (0 \leq t \leq n), \tag{2}
\]

in which the coefficients \( A_k \) are defined by the equations

\[
 p_k(k) = a_k \quad (k = 0, 1, \ldots, n). \tag{3}
\]
The coefficients of polynomial (2) characterize different tendencies, or trends, existing in the market. The plus or minus sign of $A_k$ describes the tendency to growth or to decrease, respectively. Such different trends may be called heterotrends. The latter are somewhat analogous to heterophase fluctuations in statistical systems [16], where they play a very important role, especially in the vicinity of phase transitions. Market models must also include some kind of heterogeneity in their structure [17,18].

The competition of different tendencies in a market determines the formation of following prices. If there are no constraints imposed from outside, the market should develop according to its own laws. We assume that this natural dynamics of a market can be formulated as the property of self-similarity for a given time series.

The algebraic self-similar renormalization group selects the most stable nonlinear mixture of tendencies prevailing over the less stable. Thus, evolving and competing via the self-similar renormalization dynamics, different trends form a self-similar heterotrend market, or simply self-similar market. The problem of forecasting the price for a real market is mapped to the problem of evolution of a self-similar market. Let us emphasize again that the possibility of so simplifying market dynamics is justified in the critical region where collective coherent effects become prevailing. The development of such a coherent behaviour with strong correlations between market agents is a necessary condition for the formation of a law of collective motion, which, in turn, can be expressed through the self-similar renormalization group.

For the sequence \{p_k(t)\} of polynomials $p_k(t) = \sum_{m=0}^{k} A_m t^m$, we may define [15] the self-similar exponential approximants

$$f_k(t, \tau) = A_0 \exp \left( \frac{A_1}{A_0} t \exp \left( \frac{A_2}{A_1} t \ldots \exp \left( \frac{A_k}{A_{k-1}} \tau t \right) \ldots \right) \right),$$

where it is assumed that $A_m \neq 0$ ($m = 0, 1, \ldots, k$). This kind of nested exponentials, because of their self-similar structure, is, as we think, the most suitable functional form for describing market dynamics. The effective time $\tau$ is to be found from a fixed-point condition, e.g., taken as the minimal-difference condition

$$|f_n(t, \tau_n) - f_{n-1}(t, \tau_n)| = \min \tau \left| f_n(t, \tau) - f_{n-1}(t, \tau) \right|,$$

giving $\tau_n = \tau_n(t)$. In many cases, condition (5) can be reduced to

$$f_n(t, \tau) = f_{n-1}(t, \tau), \quad \tau = \tau_n(t),$$

which, according to (4), is equivalent to the equation

$$\tau = \exp \left( \frac{A_n}{A_{n-1}} \tau t \right), \quad \tau = \tau_n(t).$$

The latter, as is obvious, possesses a solution if $A_n / A_{n-1} < 0$, but when $A_n / A_{n-1} > 0$, solutions may be absent. With the found effective time $\tau_n(t)$, we obtain the self-similar forecast

$$f_n^*(t) \equiv f_n(t, \tau_n(t)) \quad (t \geq n + 1).$$

It may happen that, when $A_n / A_{n-1} > 0$, equations (6) and (7) have no solutions. Then we need to return to condition (5) which, together with (4), yields $\tau_n = 0$. In this case, the forecast becomes

$$f_n^*(t) = f_{n-1}(t, 1) \quad (\tau = 0).$$

The stability analysis of the procedure is checked by analysing the multipliers

$$M_k(t, \tau) \equiv \frac{\delta f_k(t, \tau)}{\delta f_1(t, 1)}, \quad M_k(t) \equiv M_k(t, 1),$$

where $k = 1, 2, \ldots, n$. Note that $M_1(t, 1) = 1$ for any $t$. When Eq. (6) has a solution $\tau_n(t)$, then the multiplier at the fixed point (8) is defined as

$$M_n^*(t) \equiv \frac{1}{2} [M_n(t, \tau_n(t)) + M_{n-1}(t, \tau_n(t))].$$

When (6) has no solution, then the fixed point is given by (9), and the related multiplier is
\[ M^*_n(t) = M_{n-1}(t, 1) \quad (\tau = 0). \] (12)

It is also admissible to define a fixed point as an average
\[ \bar{f}_n(t) \equiv \frac{1}{2} [f_n(t, 1) + f_{n-1}(t, 1)]. \] (13)

In this case, the corresponding multiplier can be written as
\[ \bar{M}_n(t) \equiv \frac{1}{2} [M_n(t, 1) + M_{n-1}(t, 1)]. \] (14)

The forecasting procedure is stable if the multiplier at the fixed point satisfies the inequality \(|M^*_n(t)| \leq 1\) or, respectively, \(|\bar{M}_n(t)| \leq 1\). The case of an equality is called neutrally stable. As an optimal forecast, that corresponding to the minimal multiplier is to be taken.

Note that the difference \( \Delta_n(t) \equiv f_n(t, 1) - f_{n-1}(t, 1) \) describes the degree of a market volatility. For a very volatile market, \( \Delta_n(t) \) can be comparable with \( f_n(t, 1) \). Contrary to this, for a steady-state market, \( \Delta_n(t) \) is much less than \( f_n(t, 1) \).

In the following section, we pass to the consideration of particular examples of self-similar markets. All data, unless stated otherwise, are taken from the books of International Financial Statistics issued by the International Monetary Fund and from UNCTAD Commodity Yearbooks issued by the United Nations.

We start the self-similar analysis from the simplest case, when the values of \( f(t) \) are given for only three time points. So, we shall deal with the self-similar exponential approximants
\[ f_1(t, \tau) = a_0 \exp \left( \frac{A_1}{a_0} \tau t \right), \quad f_2(t, \tau) = a_0 \exp \left( \frac{A_1}{a_0} t \exp \left( \frac{A_2}{A_1} \tau t \right) \right) \]
and the multipliers
\[ M_1(t, \tau) = \tau \exp \left\{ -\frac{A_1}{a_0} (1 - \tau) t \right\}, \]
\[ M_2(t, \tau) = \left( 1 + \frac{A_2}{A_1} \tau t \right) \exp \left\{ -\frac{A_1}{a_0} \left[ 1 - \exp \left( \frac{A_2}{A_1} \tau t \right) \right] t + \frac{A_2}{A_1} \tau t \right\}. \]

It turns out that specific features of a market are directly related to the quantities
\[ X \equiv a_1 - a_0, \quad Y \equiv a_2 - a_1. \] (15)

The coefficients of the polynomial representation (2) can be expressed through these quantities as
\[ A_0 = a_0, \quad A_1 = \frac{1}{2} (3X - Y), \quad A_2 = \frac{1}{2} (Y - X). \]

Depending on the values of \( X \) and \( Y \), different possibilities can arise.

**III. UPWARD BOUND MARKETS**

This case implies that \( a_0 \leq a_1 < a_2 \), hence \( X \geq 0, Y > 0 \). The following subcases exist.
A. Balanced Bull Market (0 < Y < X)

Then $A_1 > 0$, $A_2 < 0$, and $|A_1| > |A_2|$. The bullish tendency presented by $A_1$ dominates over the bearish tendency given by $A_2$. For the balanced bull market, we have $|M_2(3)| < 1$, and prices rise. When a price goes up sharply, we get a boom. Some examples of such booms are considered below.

(i) The average index of the Swedish share prices (1980=100) from 1980 to 1982:

$$a_0 = 100 \ (1980), \quad a_1 = 149 \ (1981), \quad a_2 = 185 \ (1982).$$

Let us find the index for 1983. The coefficients of polynomial (2), defined by (3), are $A_1 = 55.5$ and $A_2 = -6.5$. For the exponential approximants (4), we have $f_1(3,1) = 528.567$ and $f_2(3,1) = 322.754$. Recall that $M_1(t) \equiv 1$ for any $t$, and using (10), we get $M_2(3) = 0.279$. For the average (13), we find $\tilde{f}_2(3) = 425.661$, with the multiplier (14) being $\tilde{M}_2(3) = 0.639$. From the minimal–difference condition (7), it follows that $\tau = 0.76445$. This gives for approximant (8) the value $f_2^*(3) = 357.087$, and for multiplier (11), we have $|M_2^*(3)| = 0.447$. Since $M_2^*(3)$ has the absolute value which is less than that of $\tilde{M}_2(3)$, we have to accept $f_2^*(3)$ as the optimal forecast. The actual index boomed in 1983 to 359. Thus, the error of our forecast is $-0.5\%$.

(ii) The average index of the Philippines share prices (1985 = 100) from the fourth quarter of 1986 till the second quarter of 1987:

$$a_0 = 224.9 \ (IV, \ 1986), \quad a_1 = 273 \ (I, \ 1987), \quad a_2 = 315.6 \ (II, \ 1987).$$

What is the index in the third quarter of 1987? Following the standard prescription, we find the polynomial coefficients $A_1 = 50.85$ and $A_2 = -2.75$. Then we get $f_1(3,1) = 443.172$ and $f_2(3,1) = 400.363$, with the multiplier $M_2(3) = 0.643$. From here, $\tilde{f}_2(3) = 421.767$ with $\tilde{M}_2(3) = 0.842$. The minimal–difference condition gives $\tau = 0.8686$, so that $f_2^*(3) = 405.371$ with $|M_2^*(3)| = 0.739$. The actual value of the index in the third quarter of 1987 was 424.53. Both our estimates are pretty close to this value, with an error less than 4.5\%.

B. Super–Bull Market (0 \leq X < 3X)

Only bullish tendencies are present, since $A_1 > 0$ and $A_2 > 0$. For this case, Eq. (7) may have no solution, then the minimal–difference condition (5) yields $\tau = 0$. Therefore, instead of (8), we must consider (9). Exponential approximants increase with time, and $|M_2(3)| > 1$. An interesting situation may occur – Despite an exponential growth of the approximants, the value of the sought function at $t = n + 1$ can be less than $a_n$. And even a crash may happen. We illustrate this by the examples below.

(i) The average index of the Canadian industrial share prices (1990 = 100) from 1967 to 1969:

$$a_0 = 25.9 \ (1967), \quad a_1 = 27.2 \ (1968), \quad a_2 = 30.3 \ (1969).$$

Let us look for the index in 1970. Again following the prescribed procedure, we get the polynomial coefficients $A_1 = 0.4$ and $A_2 = 0.9$. The exponential approximants are $f_1(3,1) = 27.128$ and $f_2(3,1) \sim 10^{18}$. The latter is not probable because of $M_2(3)$ being of similar order. The same holds true for $\tilde{f}_2(3)$ for which $|\tilde{M}_2(3)| \gg 1$. Consequently, the optimal forecast is $f_2^*(3) = f_1(3,1) = 27.128$, with $M_2^*(3) = 1$. The actual index in 1970 was down to 26.6. Our prediction gives an error of $2\%$.

(ii) The Malaysian gross domestic product (1985 = 100) from 1987 to 1989:

$$a_0 = 82.585 \ (1987), \quad a_1 = 89.967 \ (1988), \quad a_2 = 97.804 \ (1989).$$

What would be the value of GDP in 1990? The polynomial coefficients are $A_1 = 7.154$ and $A_2 = 0.227$. Then $f_1(3,1) = 107.096$ and $f_2(3,1) = 109.918$, with $M_2(3) = 1.237$. Thence, $\tilde{f}_2(3) = 108.507$ with $M_2^*(3) = 1.119$, while $\tau = 0$ and $f_2^*(3) = f_1(3,1) = 107.096$, with $M_2^*(3) = 1$. Consequently, the optimal forecast is $f_2^*(3)$, which deviates from the actual value 107.406 in 1990 by only 0.29\%.
(iii) The market tea prices (in US dollars per metric ton) at the Average Auction, London, in 1982–1985:

\[ a_0 = 1931.7 \text{ (1982)}, \quad a_1 = 2324.6 \text{ (1983)}, \quad a_2 = 3456.8 \text{ (1984)}. \]

Let us find the price in 1985. The polynomial coefficients are \( A_1 = 23.25 \) and \( A_2 = 369.65 \). Since for the approximant \( f_2 (3) \), we have \(|M_2 (3)| \gg 1\), the optimal forecast is \( f^*_2 (3) = f_1 (3, 1) = 2003\), with \( M^*_2 (3) = 1\). The actual price in 1985 was down to 1983.6. The error of our forecast is 0.98%.

C. Bull–Turned–to–Bear Market \((0 \leq X < Y)\)

This implies \( A_1 < 0\), \( A_2 > 0\), and \(|A_1| < |A_2|\). The bullish tendency presented by \( A_2 > 0\) is not strong enough to prevent the price from falling down, because of the influence of the bearish tendency given by \( A_1 < 0\). The growth of prices is too fast, forming a bubble that can burst. Examples of such bubble crashes are given below.

(i) The average index of the Swiss industrial share prices \((1980 = 100)\) from the first quarter of 1987 to the third quarter of 1987:

\[ a_0 = 206.9 \text{ (I, 1987)}, \quad a_1 = 211.5 \text{ (II, 1987)}, \quad a_2 = 245.5 \text{ (III, 1987)}. \]

What is the index in the fourth quarter of 1987? For the polynomial coefficients we have \( A_1 = -10.1 \) and \( A_2 = 14.7 \). The approximants are \( f_1 (3, 1) = 178.714 \) and \( f_2 (3, 1) = 206.516\), with \( M_2 (3) = -0.049 \). The average is \( \tilde{f}_2 (3) = 192.615\), with \( \tilde{M}_2 (3) = 0.476 \). From the minimal difference condition \((7)\), we get \( \tau = 0.28638\). Then, \( f^*_2 (3) = 198.402\), with \(|M^*_2 (3)| = 0.119\). The optimal forecast is \( f^*_2 (3)\). The actual index in the fourth quarter of 1987 was 202.5. The error of our forecast is \(-2.024\%\).

(ii) The tin prices (in US cents per pound) in Bolivia from 1987 to 1989:

\[ a_0 = 308.97 \text{ (1987)}, \quad a_1 = 320.22 \text{ (1988)}, \quad a_2 = 400.71 \text{ (1989)}. \]

Let us find the price in 1990. The polynomial coefficients are \( A_1 = -23.37 \) and \( A_2 = 34.62 \). For the approximants we have \( f_1 (3, 1) = 246.246 \) and \( f_2 (3, 1) = 308.148\), with \( M_2 (3) = -0.051\). Hence, \( \tilde{f}_2 (3) = 277.197\), with \( \tilde{M}_2 (3) = 0.475\). The minimal difference condition gives an effective time \( \tau = 0.2836 \). Then, \( f^*_2 (3) = 289.714\), with \(|M^*_2 (3)| = 0.124\). The optimal forecast is \( f^*_2 (3)\), deviating from the actual price 286.88 in 1990 by 0.99%.

(iii) The volume of the Argentine export (in billions of US dollars) in 1979–1981:

\[ a_0 = 7.81 \text{ (1979)}, \quad a_1 = 8.021 \text{ (1980)}, \quad a_2 = 9.143 \text{ (1981)}. \]

Let us look for the volume in 1982. The coefficients are \( A_1 = -0.244 \) and \( A_2 = 0.455 \). Then, \( f_1 (3, 1) = 7.110 \) and \( f_2 (3, 1) = 7.807\), with \( M_2 (3) = -0.019\). From here, \( \tilde{f}_2 (3) = 7.459\), with \( \tilde{M}_2 (3) = 0.491\). The effective time is \( \tau = 0.2489\), which yields \( f^*_2 (3) = 7.63\), with \(|M^*_2 (3)| = 0.082\). The optimal forecast is \( f^*_2 (3)\). The actual volume in 1982 was 7.625. The error of our forecast is 0.066%.

IV. DOWNWARD BOUND MARKETS

In this case \( a_0 \geq a_1 > a_2\), because of which \( X \leq 0 \) and \( Y < 0 \). Three possibilities can arise.

A. Balanced Bear Market \((X < Y < 0)\)

This means that \( A_1 < 0\), \( A_2 > 0\), and \(|A_1| > |A_2|\). The bearish tendency in the negative \( A_1\) dominates over the bullish tendency in the positive \( A_2\), which leads to the decrease of price. Examples are given below.

(i) The exchange rate of Japanese Yen to SDR (Special Drawing Rights) in 1984–1986:

\[ a_0 = 246.13 \text{ (1984)}, \quad a_1 = 220.23 \text{ (1985)}, \quad a_2 = 194.61 \text{ (1986)}. \]
Let us find the rate in 1987. The coefficients are $A_1 = -26.04$ and $A_2 = 0.14$. Repeating the same steps as above, we have $f_1(3, 1) = 179.194$, $f_2(3, 1) = 180.106$, with $M_2(3) = 0.973$, which yields $\bar{f}_2(3) = 179.65$ with $\bar{M}_2(3) = 0.986$. The effective time is $\tau = 0.98425$. Then $f_2^*(3) = 180.092$ with $|M_2^*(3)| = 0.981$. Both forecasts are very close to each other as well as to the actual rate 175.2 in 1987, the error being less than 3%.

(ii) The unit value of the Argentine wheat exports (1990 = 100) in 1982–1984:
\[
\begin{align*}
a_0 &= 115.6 \text{ (1982)}, & a_1 &= 99.4 \text{ (1983)}, & a_2 &= 91.6 \text{ (1984)}.
\end{align*}
\]
What would be the value in 1985? In the standard way, we get $A_1 = -20.4$, $A_2 = 4.2$, $f_1(3, 1) = 68.083$, $f_2(3, 1) = 86.892$, $M_2(3) = 0.263$, so that $f_2^*(3) = 77.488$, $\bar{M}_2(3) = 0.631$. Then $\tau = 0.663696$, and $f_2^*(3) = 81.351$, $|M_2^*(3)| = 0.631$. Again, both forecasts are equivalent from the viewpoint of stability, close to each other and to the actual value 79.8 in 1985, the error being less than 3%.

(iii) The average index of the Korean share prices (1985 = 100) in 1989–1991:
\[
\begin{align*}
a_0 &= 661.2 \text{ (1989)}, & a_1 &= 537.7 \text{ (1990)}, & a_2 &= 473 \text{ (1991)}.
\end{align*}
\]
Let us find the index for 1992 and compare it with that 422.63 actually happened. In the usual way, we get $A_1 = -152.9$, $A_2 = 29.4$, $f_1(3, 1) = 330.4$, $f_2(3, 1) = 447.83$, $M_2(3) = 0.322$, and $\bar{f}_2(3) = 389.116$, with $\bar{M}_2(3) = 0.661$. The effective time is $\tau = 0.6768$, which results in $f_2^*(3) = 413.453$, with $|M_2^*(3)| = 0.681$. The error of $f_2^*(3)$ is $-7.9\%$ and that of $f_2^*(3)$ is $-2.2\%$.

B. Super–Bear Market ($3X < Y < X \leq 0$)

This implies that both $A_1$ and $A_2$ are negative. Only bearish tendencies are presented. As an example, consider the volume of GDP (in millions of SDR) of the Yemen Republic in 1979–1981:
\[
\begin{align*}
a_0 &= 1084 \text{ (1979)}, & a_1 &= 1006 \text{ (1980)}, & a_2 &= 826 \text{ (1981)}.
\end{align*}
\]
Let us calculate the volume in 1982, comparing it with the actual one equal to 502. We have $A_1 = -27$, $A_2 = -51$, $f_1(3, 1) = 1006$, $M_2(3) = 4.5 \times 10^{-7}$, $M_2(3) = 8.6 \times 10^{-7}$, and $\bar{f}_2(3) = 502.976$, with $\bar{M}_2(3) = 0.5$. The minimal–difference condition (5) gives $\tau = 0$. So, $f_2^*(3) = f_1(3, 1) = 1006$, with $M_2^*(3) = M_1(3) = 1$. The optimal forecast is $f_2(3)$, whose error is 0.19%.

C. Bear–Turned–to–Bull Market ($Y < 3X \leq 0$)

Now $A_1 > 0$, $A_2 < 0$, and $|A_1| < |A_2|$. The bullish trend in positive $A_1$ becomes dominating, although the bearish trend in negative $A_2$ still remains. The motion looks like a downward bubble that bursts. Consider some examples.

(i) The average index of the US industrial share prices (1985 = 100) in 1968–1970:
\[
\begin{align*}
a_0 &= 51.7 \text{ (1968)}, & a_1 &= 51.2 \text{ (1969)}, & a_2 &= 43.9 \text{ (1970)}.
\end{align*}
\]
We shall find the index for 1971 and compare it with the actual value 52.1. We get $A_1 = 2.9$, $A_2 = -3.4$, $f_1(3, 1) = 61.175$, $f_2(3, 1) = 51.96$, $M_2(3) = -0.063$, and $\bar{f}_2(3) = 56.568$, with $\bar{M}_2(3) = 0.469$. The minimal–difference condition gives $\tau = 0.3221$, because of which $f_2^*(3) = 54.58$ and $|M_2^*(3)| = 0.124$. The optimal forecast is $f_2^*(3)$, with an error 4.8%.

(ii) The volume of the US imports (1985 = 100) in 1973–1975:
\[
\begin{align*}
a_0 &= 57.5 \text{ (1973)}, & a_1 &= 56.7 \text{ (1974)}, & a_2 &= 49.9 \text{ (1975)}.
\end{align*}
\]
Let us find the volume in 1976, comparing it with the actual value 60.8. We may mention that the decline in imports in these years could be related to the oil embargo and the energy crisis of 1973. Following the standard procedure, we
have \( A_1 = 2.2, \ A_2 = -3, \ f_1(3,1) = 64.494, \ f_2(3,1) = 57.61, \ M_2(3) = -0.046, \ \tilde{f}_2(3) = 61.052, \) and \( \bar{M}_2(3) = 0.477. \) The effective time is \( \tau = 0.2969. \) Then, \( f_2^*(3) = 59.493, \) with \( |M_2^*(3)| = 0.108. \) The optimal forecast is \( f_2^*(3). \) with an error \(-2\%\), though \( f_2(3) \) is also good.

(iii) The ratio of the total Non–Gold Reserves to Imports for Industrial Countries in 1967–1969:

\[
a_0 = 8.1 \ (1967), \quad a_1 = 8.1 \ (1968), \quad a_2 = 6.8 \ (1969).
\]

We make a prediction for 1970 and compare it with the actual value 8.9. As usual, we find \( A_1 = 0.65, \ A_2 = -0.65, \ f_1(3,1) = 10.305, \ f_2(3,1) = 8.198, \ M_2(3) = -0.079, \ \tilde{f}_2(3) = 9.252, \) and \( \bar{M}_2(3) = 0.461. \) Also, \( \tau = 0.35, \ f_2^*(3) = 8.812, \) with \( |M_2^*(3)| = 0.142. \) The forecast \( f_2^*(3) \) is optimal, with an error \(-0.989\%\).

V. YO–YO MARKETS

In such markets, time data are not consecutively ordered but behave oscillatory. With only three time points we consider here, there can be two possibilities.

A. Yo–Yo Decline Market \((Y \leq 0 < X)\)

In this case \( a_0 < a_1, \ a_1 \geq a_2, \) hence \( A_1 > 0, \ A_2 < 0, \) and \( |A_1| > |A_2|. \) The bullish trend in positive \( A_1 \) overweights the bearish trend in negative \( A_2. \) The price, after \( a_2, \) can go only up. This behaviour may be called \( \text{yo–yo boom}. \) Consider an example.

The average index of the UK share prices \((1958 = 100)\) in 1960–1962:

\[
a_0 = 166 \ (1960), \quad a_1 = 171 \ (1961), \quad a_2 = 158 \ (1962).
\]

Let us estimate the index for 1963 and compare it with the actual value 181. Calculations give us \( A_1 = 14, \ A_2 = -9, \ f_1(3,1) = 213.791, \ f_2(3,1) = 172.219, \ M_2(3) = -0.109, \) and \( \tilde{f}_2(3) = 193, \) with \( \bar{M}_2(3) = 0.446. \) The effective time is \( \tau = 0.433459, \) hence \( f_2^*(3) = 185.241, \) with \( |M_2^*(3)| = 0.215. \) The optimal forecast \( f_2^*(3) \) has an error 2.3%.

B. Yo–Yo Rise Market \((X < 0 \leq Y)\)

Here we have \( a_0 > a_1, \ a_1 \leq a_2, \) so that \( A_1 < 0, \ A_2 > 0, \) and \( |A_1| > |A_2|. \) The bearish trend prevails over bullish, as a result, the price falls below \( a_2. \) This type of behaviour may be termed \( \text{yo–yo crash}. \) An example is given below.

The Chilean total reserves (in millions of SDR) in 1972–1974:

\[
a_0 = 204 \ (1972), \quad a_1 = 137 \ (1973), \quad a_2 = 149 \ (1974).
\]

We make a forecast for 1975, comparing it with the actual value of 84. Recall that an upheaval happened in 1974 in Chile. We find \( A_1 = -106.5, \ A_2 = 39.5, \ f_1(3,1) = 42.604, \ f_2(3,1) = 121.918, \ M_2(3) = -0.106, \ \tilde{f}_2(3) = 82.261, \) and \( \bar{M}_2(3) = 0.447. \) Then \( \tau = 0.54519, \) which leads to \( f_2^*(3) = 86.856, \) with \( |M_2^*(3)| = 0.774. \) Here the optimal forecast is \( f_2^*(3) \) having an error \(-2\%\).

VI. BUBBLE BURST

The relatively simple three–point classification of market behaviour we have presented does not exhaust all possible types of market dynamics which might occur when more points of a time series are taken, but rather demonstrates how the self–similar analysis can be used for describing market crises. It is needless to recall that the possibility of predicting market crises is very much important. We would like now to pay a special attention to market bubble
bursts showing that, in the framework of the self–similar analysis, all of them occur in the same way. Such an analogy between bubble bursts of quite different markets makes it possible to predict these burst phenomena.

(i) The average index of the Belgian industrial share prices \((1990 = 100)\) in 1971–1973:

\[
a_0 = 34 (1971), \quad a_1 = 38 (1972), \quad a_2 = 46 (1973).
\]

Let us look for the index in 1974 and compare it with the actual value 37. Repeating the standard steps, we get \(f_1(3,1) = 40.562\), while \(f_2(3,1) \sim 10^3\), with \(M_2(3) \sim 10^3\). Therefore, the average estimate \(\bar{f}_2(3)\) has negligible probability to happen. The minimal–difference condition (5) gives \(\tau = 0\). So, \(f^*_2(3) = f_1(3,1) = 40.562\), with \(M^*_2(3) = 1\). The error of \(f^*_2(3)\) is 9.6%.

(ii) The average index of the Japanese share prices \((1985 = 100)\) in 1987–1989:

\[
a_0 = 196.4 (1987), \quad a_1 = 213.9 (1988), \quad a_2 = 257.8 (1989).
\]

During these years, Japan had, what it would have been termed later, the bubble economy [19]. In 1990, the index dropped to 218.8. Applying the self–similar analysis to the data above, we find \(f_1(3,1) = 209.733\), while \(f_2(3,1)\) as well as \(M_2(3)\) are again unreasonably large. For the effective time, we again get \(\tau = 0\). In this way, the sole stable estimate is \(f^*_2(3) = f_1(3,1) = 209.733\), whose error is \(-4.1\%\).

We should keep in mind that the error of three–point forecasts may be of about 10%. For a more accurate analysis, we need to consider a larger number of historical points. But in the present paper, we wish to limit ourselves by the simplest, although the crudest, type of the three–point analysis.

VII. CONCLUSION

We have shown how the self–similar analysis can be used for describing and forecasting various crises. The latter, according to their strength, can be roughly divided into two types: (1) First degree crash (boom), when the price falls (grows) by an amount comparable to the gain (loss) achieved during the period of observation. The majority of bubble crashes and bubble booms are of this type. Mainly those who buy stocks on margin, e.g. bears selling short, are hurt. (2) Second degree crash (boom), when the price loss (gain) is of the order of the price value at the last point of observation. Yo–yo crashes and booms are often of this category. Then, not only margin–players can be ruined, but those who own stocks outright can be hurt too.

We illustrated our approach by several examples. The number of the latter could be increased to any desirable quantity, since we have considered hundreds of such cases. In the majority of these cases, the simple three–point analysis gives reasonable predictions. When the accuracy of such a simplest analysis is not good enough, one has to invoke additional information taking into account more data on the market behaviour. However in this paper, we would like to limit ourselves by the most simple case involving just three points of data. How to deal with time series containing more points was briefly explained in Refs. [6,7] and will be discussed in detail in our following publications.

The principal difference of our approach to describing market crises, as compared to that based on modelling the market behaviour by a complicated system of nonlinear differential or difference equations [20], is in the following. Markets are so complex systems that they can be treated by dynamical models, with more or less success, only in a stationary state. But crises are principally nonstationary phenomena. The complexity of real markets makes it impossible, to our mind, to model them by any complicated system of nonlinear equations. Moreover, there is quite known property of nonlinear equations, when adding or omitting a negligibly small term drastically changes the behaviour of solutions. We do not try to invent, which, as we think, is impossible, a dynamical model for each market. Instead, we assume that each market develops self–similarly, according to its own laws. The result of this development is exhibited in market characteristics, which contain the hidden information on the laws governing the market. The self–similar analysis permits one to extract this hidden information. Thus, each market itself prescribes its future development – and this is the essence of the notion of a self–similar market.

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