EXPLICIT BREATHER SOLUTION OF THE NONLINEAR SCHRÖDINGER EQUATION

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We present a one-line closed-form expression for the three-parameter breather of the nonlinear Schrödinger equation. This provides an analytic proof of the time period doubling observed in experiments. The experimental check that some pulses generated in optical fibers are indeed such generalized breathers will be drastically simplified.

Keywords: modulational instability, nonlinear Schrödinger equation, nonlinear optics, breather, exact solutions

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1. Introduction

The complex amplitude of many nonlinear media displays two generic features. The first one is to obey an evolution equation (first order in the time variable \( t \)) with a (linear) dispersion term (second order in the space variable \( x \)) and the simplest nonlinearity preserving the phase invariance of \( A \),

\[
iA_t + pA_{xx} + q|A|^2A = 0, \quad pq \neq 0, \quad p, q \text{ real}\.
\] (1)

The second feature, observed in the “focusing” regime (\( pq > 0 \)) of this nonlinear Schrödinger (NLS) equation, is the “modulational instability” (MI) [1], also known as the Benjamin–Feir instability: an initial plane wave grows exponentially, then saturates and decreases to its original state, with only a shift of its phase. This MI has an enormous number of applications, which we now recall.

In the ocean, deep water waves are suitably described by the focusing NLS equation [2], where one observes “bright” solitons. Sailors have also reported the sudden occurrence of huge waves (“freak” or “rogue” waves) that disappear as quickly as they appear, and these solutions of very high amplitude and energy can also be described by the NLS equation [3]. However, experimental setups able to reproduce this rare observation are quite difficult to arrange. As to the “defocusing” regime (\( pq < 0 \)), it is more adapted to shallow water waves, where only “dark” solitons occur.

The situation is quite different in a Bose–Einstein condensate (BEC), where the wave function of the condensate obeys the Gross–Pitaevskii equation, a three-dimensional analogue of the NLS equation. It has

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been proved analytically [4] that MI is the mechanism that generates wave functions of soliton type in a BEC, a prediction confirmed by the experimental observation [5] of MI in a cigar-shaped BEC.

But nowadays the main playground of MI no more water waves nor even the BEC but nonlinear optics, for two reasons. The first one is the huge recent progress in manufacturing optical fibers with prescribed physical properties (refractive index, etc), making experiments easier, cheaper, and more easily reproducible. The second reason is more fundamental: as opposed to a three-dimensional BEC, a fiber is quasi-one-dimensional and thus well described by the NLS equation, $t$ being the propagation distance and $x$ the transverse coordinate. For instance, one has succeeded [6], [7] in generating rogue waves in optical fibers, an achievement with potentially important industrial applications. Nonlinear optics has become an excellent field to perform an experimental check of the beautiful analytic description of MI, which we first recall.

Indeed, the later stages of MI can be computed exactly, resulting in a two-parameter \(^1\) bright soliton localized in space and periodic in time, whose asymptotic behavior as $|x| \to +\infty$ is the plane wave $e^{-i\omega_0 t}$ (see (24) below). This achievement of Kuznetsov [8] was obtained in plasma physics, where the Langmuir waves are appropriately described by the focusing NLS equation. Changing the sign of one parameter converts this soliton into another quite important physical solution, localized in time and periodic in space, known as the Akhmediev breather [9] (see (25) below). Finally, using a quite simple ansatz, Akhmediev, Eleonskii, and Kulagin [10] extrapolated the Kuznetsov soliton to a three-parameter breather solution in which $Ae^{i\omega_0 t}$ is elliptic in $x$ and quasielliptic in $t$.\(^2\)

In a recent experiment [12] with an optical fiber, this breather was observed by matching the three arbitrary parameters with experimental data, showing a “good” agreement, however during only two quasiperiods of $t$. The difficulty arose not from the sophisticated experimental setup but from “the complexity of this class of solutions” [13]. Indeed, its current analytic representation [10] (Eqs. (3), (22), (24), and (25) there) does not clearly separate the elliptic dependence on $x$ and the quasielliptic dependence on $t$, despite several later attempts [14]–[17], forcing the authors to expand the amplitude in a Fourier series in $x$ and to retain only the first two coefficients.

In this article, we provide a one-line closed-form expression for this three-parameter solution (see Eq. (16)) and give a full classification of the solutions of the ansatz of Ref. [10], thus uncovering a new solution, Eq. (33), elliptic in $x$ and trigonometric in $t$, together with its degeneracy. The present three-parameter closed form makes it possible to check the agreement on a much larger number of quasiperiods of $t$, and therefore to more accurately determine the nonlinear range of validity of MI as sketched in [13].

Another puzzling phenomenon observed in Ref. [12], the time period twice the one expected, is naturally explained by our three-parameter solution.

2. The generic solution

It was assumed in [9] that the constraint between $A$ and $\mathbf{A}$ is defined by three real functions $\varphi(t)$, $\delta(t)$, and $Q(x,t)$ as

\[
\begin{align*}
\sin \varphi(t) \cdot \Re A - \cos \varphi(t) \cdot \Im A + \delta(t) &= 0, \\
\cos \varphi(t) \cdot \Re A + \sin \varphi(t) \cdot \Im A - Q(x,t) &= 0. 
\end{align*}
\]

Since $A = (Q/\delta + i)\delta e^{i\varphi}$ is single valued [18], [19], both terms $Q/\delta$ and $\delta e^{i\varphi}$ are single valued, while $Q$, $\delta$, and $e^{i\varphi}$ can be multivalued. Because of the absence of methods to handle multivaluedness, the strategy is therefore to only consider $\delta e^{i\varphi}$, its complex conjugate, and $Q/\delta$.

\(^1\)The scaling invariance $(x,t,A) \rightarrow (kx,k^2 t,kA)$ of the NLS equation reduces this number by one.

\(^2\)We never use the ambiguous term “periodic” for elliptic solutions, but always either “doubly periodic,” alias “elliptic” (example: Jacobi dn, Weierstrass \(\wp\)), or “quasi-doubly periodic,” alias “quasielliptic,” alias “elliptic of the second kind” in Hermite’s terminology [11, Vol. 1, p. 227; Vol. 2, p. 506] (example: the solution $H(t,a)$ of Lamé equation (10)).
The integrability condition for Eqs. (4a) and (5) gives rise to the ODEs in [10], and the second equation admits the first integral characterized by the three nonzero roots function [10] (Eq. (13) in [10]), which in the notation of Weierstrass Lamé equation of index

This system admits three real first integrals

where

Remark 1. The real and imaginary parts of \( A \) are

\[
A = \frac{Q - \delta \psi + i(Q \psi + \delta)}{\sqrt{1 + \psi^2}}, \quad \psi = \tan \varphi. \]  (3)

We first recall the result in [10] and then proceed to the explicit dependence on \( x \) and \( t \). By eliminating \( A \) from Eq. (1), the system to be solved is made of two coupled real PDEs for \( Q(x,t) \) [10] (see Eqs. (4) and (5) in [10]),

\[
\begin{align*}
Q_t + \delta Q^2 - \varphi' \delta + q \delta^3 &= 0, \quad (4a) \\
pQ_{xx} + qQ^3 + (q^2 - \varphi')Q - \delta' &= 0. \quad (4b)
\end{align*}
\]

and the second equation admits the first integral \( h(t), \)

\[
h = p(Q_x)^2 + q \frac{Q^4}{2} + (q \delta^2 - \varphi')Q^2 - 2 \delta' Q \quad (Q_x \neq 0). \]  (5)

The integrability condition for Eqs. (4a) and (5) gives rise to the ODEs

\[
\begin{align*}
\varphi'' + 4q \delta \delta' &= 0, \quad h' + 2 \delta \delta' - 2q \delta^3 \delta' &= 0, \\
\delta'' + 4(\varphi')^2 - 2q \delta^3 \varphi' + 2q \delta h + q^2 \delta^5 &= 0.
\end{align*}
\]  (6)

This system admits three real first integrals \( \omega_0, k_1, \) and \( k_2, \)

\[
\begin{align*}
g \delta^2 &= 2z, \quad \varphi' = -4z - \omega_0, \quad qh &= 2(3z^2 + \omega_0 z + k_2), \\
(z')^2 &= -4(4z + \omega_0)^2 z^2 - 16k_1 z^2 + 4k_1 z \quad (z' \neq 0)
\end{align*}
\]  (7)

characterized by the three nonzero roots \( z_1, z_2, \) and \( z_3 \) of \( z': \)

\[
\begin{align*}
\omega_0 &= -2(z_1 + z_2 + z_3), \quad k_1 = 16z_1 z_2 z_3, \quad k_2 = (z_1 + z_2 + z_3)^2 - 2(z_1^2 + z_2^2 + z_3^2).
\end{align*}
\]  (8)

In the generic case \( Q_x z k_1 \neq 0 \) (nongeneric cases are detailed in Sec. 3), the product \( \delta^2 \) is an elliptic function [10] (Eq. (13) in [10]), which in the notation of Weierstrass\(^3\) takes the simple form \((i a \text{ is real})\)

\[
\begin{align*}
z &= \frac{k_1}{\wp(t) - \wp(a)}, \quad \wp(a) = -\frac{\omega_0^2 + 4k_2}{3}, \quad \wp'(a) = -8ik_1, \\
g_2 &= \frac{4}{3}(|\omega_0^2 + 4k_2|^2 + 24k_1 \omega_0], \\
g_3 &= \frac{8}{27}(|\omega_0^2 + 4k_2|^3 + 36k_1 (\omega_0(\omega_0^2 + 4k_2) + 6k_1)], \\
\Delta^{(i)} &= g_3^3 - 27g_2^2 = -2^{12}k_1^2[16k_1^3 + 8\omega_0^2 k_2^2 + \omega_0^2 k_2 + 36\omega_0 k_2 k_1 + \omega_0^3 k_1 + 27k_1^2],
\end{align*}
\]  (9)

where \( k_1 \neq 0. \)

We next determine \( \delta e^{i\varphi} \) and \( \delta e^{-i\varphi} \) simultaneously, not by the multivalued quadrature \( \int \varphi' \, dt \) as is usually done but as the two complex-conjugate solutions of a real second-order ODE. The phase invariance of the NLS equation only allowing the contribution of \( \varphi' \), not of \( \varphi \), by eliminating \( z \) we easily obtain the Lamé equation of index \( n = 1, \)

\[
\left( \frac{d^2}{dt^2} - (2\wp(t) + \wp(a)) \right) (\delta^{-1} e^{i(\varphi + \omega_0 t)}) = 0. \]  (10)

\(^3\text{To convert to the notation of Jacobi, see [21], §18.9.11 and 18.10.8.}\)
Its two independent solutions are, generically,
\[ \delta^{-1} e^{\mp i\varphi} = \sqrt{-\frac{q}{k_1}} e^{\pm i\omega_0 t} H(t, \pm a), \]  
(11)

where (see [11], Vol. 2, p. 506)
\[ H(t, a) = e^{-\zeta(a)t} \frac{\sigma(t + a)}{\sigma(a)\sigma(t)}. \]  
(12)

At this point, the authors of [10] chose to integrate the \( x \)-elliptic ODE (5) with \( t \)-dependent coefficients. It is more efficient to integrate the \( t \)-Riccati ODE (4a) with \( x \)-independent coefficients; this allows us to uncover a new solution in what follows, Eq. (33). Indeed, an affine transformation on \( Q(x, t) \) maps Eq. (4a) to a canonical Riccati equation,
\[ \frac{Q(x, t)}{\delta(t)} = \frac{y(x, t)}{2z} + \frac{z'}{8z^2}, \quad \partial_t y + y^2 - \frac{3}{4} \varphi(t) = 0 \quad (z \neq 0) \]  
(13)
equivalent to a particular Lamé equation of index \( n = 1/2 \), whose solution is [20, §20, p. 104], [11, Vol. 2, p. 482],
\[ y = \partial_t \log \frac{4\sqrt{k_1} F(x) + \varphi(t/2) - \varphi(a)}{\sqrt{\varphi'(t/2)}}. \]

The real-valued function \( \sqrt{k_1} F(x) \) is defined by
\[ p(F')^2 + P(F) = 0, \quad P(F) \equiv F^4 + \omega_0 F^2 - 2\sqrt{k_1} F - k_2, \]  
(14)

and evaluates as (all signs of the square roots are allowed)
\[ F = \sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} - \frac{2}{p} \frac{(\sqrt{z_2} + \sqrt{z_3})(\sqrt{z_3} + \sqrt{z_1})(\sqrt{z_1} + \sqrt{z_2})}{\varphi(x, G_2, G_3) - \varphi(b, G_2, G_3)}, \]
\[ \varphi(b) = -\frac{z_1 + z_2 + z_3 + 3(\sqrt{z_2}\sqrt{z_3} + \sqrt{z_3}\sqrt{z_1} + \sqrt{z_1}\sqrt{z_2})}{3p}, \]
\[ \sqrt{k_1} = 4\sqrt{z_1}\sqrt{z_2}\sqrt{z_3}, \]
\[ G_2 = \frac{\omega_0^2 - 12k_2}{12p^2}, \quad G_3 = \frac{\omega_0^3 + 36 \omega_0 k_2 + 54 k_1}{6p^3}, \]
\[ \Delta^{(x)} \equiv G_2^3 - 27 G_3^2 = 2^{-16} p^{-6} k_1^{-2} \Delta(t). \]

To summarize, the complex amplitude is
\[ A = \left[ \frac{16\sqrt{k_1}}{\varphi'(t/2, g_2, g_3)} \left( \frac{P(V)}{F(x) - V(t)} + \frac{1}{4} \frac{dP(V)}{dV} \right) + i \sqrt{-\frac{k_1}{q} e^{-i\omega_0 t} H(t, a)} \right] \]
\[ V(t) = \frac{\varphi(a, g_2, g_3) - \varphi(t/2, g_2, g_3)}{4\sqrt{k_1}}, \]  
(16)

with \( P(V(t)) \) and \( F(x) \) defined in (14) and (15), \( a \) in (9), and \( H(t, a) \) in (12), and its complex conjugate results from the change \((i, a) \rightarrow (-i, -a)\).

The amplitude in (16) depends on three arbitrary real constants \( \omega_0, k_1, \) and \( k_2 \) and is elliptic in \( x \). The ratio two between the \( t \)'s in \( \varphi(t/2, g_2, g_3) \) and in \( H(t, a) \) makes the quasi-\( t \)-periods of \( A(x, t) e^{i\omega_0 t} \) twice the periods of \( \varphi(\cdot, g_2, g_3) \), thus proving the period doubling observation [12].
Remark 2.

- The generality of (16) is worth being emphasized. This unique formula (the advantage of the Weierstrass notation) covers both signs of the discriminant: $\Delta^{(i)} < 0$ ("B-type" solutions, one nonzero real $z_j$) and $\Delta^{(i)} > 0$ ("A-type" solutions, three nonzero real $z_j$), it involves no multivalued expressions, and even applies to both NLS regimes (focusing and defocusing).
- The argument doubling formula $\sigma(2y) = -\psi(y)\sigma^4(y)$ (see Eq. 18.4.8 in [21]) allows expressing (16) with the unique argument $t/2$.
- To be physically admissible, amplitude (16) must obey two constraints. The first one, $\delta^2(t) > 0$, implies that $pz_3 > 0$ [10, p. 811] and that $pz_1$ and $pz_2$ are positive or complex conjugate, with bounds $0 < pz(t) < (\text{the smallest positive } pz_1)$. The second constraint, that $Q/\delta$ is real, which was painful to implement [10], is equivalent to $\sqrt{k1}F(x)$ being real, i.e., the transposition to the four zeros $\sqrt{\omega_1} + \sqrt{\omega_2} + \sqrt{\omega_3}$ of $F'(x)$ of the constraints on the zeroes $0, z_1, z_2, z_3$ of $z'(t)$. The bounded solutions of this focusing regime result from (16) by shifting the origins of $x$ and $t$ by either zero or a nonreal half-period, depending on the common sign of the two discriminants $\Delta^{(x)}$ and $\Delta^{(t)}$ (see formulas 16.8 and 18.4.1 in [21]).

3. Nongeneric solutions

They are defined by either $Q_x = 0$ (nonexistence of $h(t)$) or $z'(t) = 0$ (nonexistence of $k_1$) or $z(t) = 0$ (undefined link (13) between $Q$ and $y$) or $k_1 = 0$ (the independence of (13) from $z$) or $\psi'(a) = 0$ (a linear dependence of the two solutions (11) of (10)) or $\Delta^{(t)} = 0$ (the degeneracy of elliptic functions to either trigonometric or rational functions). Because Eq. (13) was not considered in [10], the nongeneric case $k_1 = 0$ yields the new solution given in Eq. (33) below.

3.1. Degeneracies of the generic solution. They are characterized by $Q_x \delta^2 k_1 \neq 0$ and $\psi'(a)\Delta^{(t)} = 0$.

When $\psi'(a) = 0$, then $a$ is a purely imaginary half-period $\omega'$, the multipliers of $H(t, \omega')$ are $(-1, 1)$, but $k_1 = 0$, which is forbidden. Fortunately, the form invariance of the ODE for $\varphi$ under halving one period changes $\psi'(a)$ to $\varphi'(2a) \equiv 8i(k_1 + \omega_0k_2)$, now allowed to vanish. This “Landen transformation” [22, p. 39], [21, 16.14.2]

$$\varphi(t, g_2, g_3) \equiv \varphi(t|\omega, \omega') \to P(t, \gamma_2, \gamma_3) \equiv P(t|\omega, 2\omega'),$$

$$\varphi(t) = P(t) + \frac{(e_2 - e_1)(e_3 - e_1)}{P(t) - e_1}, \quad e_1 = \frac{8k_2 - \omega_0^2}{3},$$

$$(\varphi')^2 = 4(\varphi - e_1)(\varphi - e_2)(\varphi - e_3), \quad (P')^2 = 4(P + 2e_1)(P - e_2)(P - e_3),$$

$$g_2 = -4\gamma_2 + 60e_1^2, \quad g_3 = 8\gamma_3 + 56e_1^3,$$

makes both multipliers unity (i.e., $H$ elliptic), yielding Jacobi functions as solutions of Lamé ODE (10):

$$\sqrt{P(t) - e_2}, \quad \sqrt{P(t) - e_3}.$$  

This leads to the two elliptic breathers in an algorithmic way, instead of the kind of magic derivation in Ref. [10], and the notation of Halphen

$$h_\alpha(x) = \sqrt{\varphi(x, G_2, G_3) - E_\alpha}, \quad h_\alpha(t) = \sqrt{P(t, \gamma_2, \gamma_3) - \varepsilon_\alpha},$$

$$(\varphi'(x))^2 = 4\varphi^3 - G_2\varphi - G_3 = 4(\varphi - E_\alpha)(\varphi - E_b)(\varphi - E_c)$$

\textsuperscript{4}Under the addition of any of the two periods, a quasielliptic function is multiplied by a constant factor, the multiplier.
allows unifying them into a very symmetric expression. Characterized by the relation \( p_{21} + p_{22} = p_{32} > 0 \) between the three roots of \((z')^2\) in the third relation in (7), their two single-valued parts

\[
\delta e^{i\varphi} = \left( -\frac{\omega_0}{q} \right)^{1/2} \frac{h_1(t) e^{-i\omega_0 t}}{\omega_0 h_1(t) h_3(t) + i\omega_0 h_2(t)},
\]

\[
Q = -\frac{\omega_0^2 \mu h_2(t) h_2(x) + h_1^2(t) h_3(t) h_3(x)}{\omega_0 h_1(t) [\mu h_3(t) h_2(x) + h_2(t) h_3(x)]},
\]

\[
k_2 = \frac{(\mu - 1)^2 \omega_0^2}{2}, \quad k_1 = -\omega_0 k_2
\]

yield the amplitude (Eq. (18) in [9] and Eqs. (45) and (59) in [10])

\[
A = \left( -\frac{\omega_0}{q} \right)^{1/2} \frac{h_a(t) h_c(x) + i\mu h_0 h_b(x) e^{-i\omega_0 t}}{h_b(t) h_c(x) + \mu h_0 h_b(x)},
\]

\[
\frac{\varepsilon_0 - \varepsilon_{a\beta}}{(\mu^2 - 1)^2} = \frac{\varepsilon_0 - \varepsilon_{\beta\gamma}}{\mu^2(2 - \mu^2)} = \omega_0^2,
\]

\[
E_b - E_c = \frac{E_c - E_b}{-\mu^2} = \frac{E_a - E_b}{2 - \mu^2} = \frac{\omega_0^2}{2p},
\]

with \((\omega_0, \mu)\) and \((a, b, c)\) being two independent permutations of \((1, 2, 3)\). Its two arbitrary constants are \((\omega_0, K)\). The conversion to the Jacobi notation [23, Appendix B] yields the two types, A \((\Delta^{(1)} > 0)\) and B \((\Delta^{(1)} < 0)\).

Next, \(\Delta^{(1)} = 0\) can be represented in terms of \(\Omega\)

\[
k_1 = -\frac{\Omega(\Omega - \omega_0)^2}{2}, \quad k_2 = -\frac{\Omega(3\Omega - \omega_0)}{4},
\]

\[
(z')^2 = -64 \left( z - \frac{\Omega - \omega_0}{4} \right)^2 \left( z + \frac{\Omega}{2} \right) z.
\]

The first degeneracy \((k \neq 0)\)

\[
Q = \frac{k}{2(\Omega - \omega_0)} \left( \cot \frac{kt}{2} + \frac{6\Omega - 3\omega_0 + 3\Omega \cos(kt)}{3\Omega(\alpha \cosh(Kx) + \cos(kt/2) \sin(kt/2))} \right),
\]

\[
\delta e^{i\varphi} = \left( \frac{\Omega - \omega_0}{2q} \right)^{1/2} e^{-i\Omega t} \frac{e^{-i\Omega t}}{\sin(k(t - t_3)/2)},
\]

\[
k^2 = 4(\Omega - \omega_0)(3\Omega - \omega_0), \quad pK^2 = 3\Omega - \omega_0,
\]

\[
\cos(kt_3) = -\frac{2\Omega - \omega_0}{\Omega}, \quad \sin(kt_3) = \frac{ik}{2\Omega},
\]

is the first iteration of the plane wave (27) by a Bäcklund transformation; it depends on two arbitrary real constants \(\omega_0\) and \(\Omega\) restricted to \(0 < \Omega/\omega_0 < 1\) by the reality of \(y(x, t)\). Depending on the signs of \((K^2, k^2)\), this mathematical solution defines four physical solutions: two unbounded in the defocusing regime and two in the focusing regime: the Kuznetsov bright soliton solution [8], [24, Eq. (6.10)], [25, Eq. (41a)] localized in space and periodic in time,

\[
A = \sqrt{\frac{\Omega}{q}} e^{-i\Omega t} \left[ 1 - \frac{2(1 - \alpha^2)\Omega \cos(kt/2) + i(1/2) \sin(kt/2)}{\Omega(\alpha \cosh(Kx) + \cos(kt/2))} \right],
\]

\[
K^2 = 2\Omega(1 - \alpha^2), \quad k^2 = -16\Omega^2 \alpha^2(1 - \alpha^2), \quad \alpha^2 > 1,
\]
and the breather solution of Akhmediev [9, Eq. (11)] localized in time and periodic in space,

\[ A = \sqrt{-\frac{\Omega}{q}} e^{-it} \left[ 1 + \frac{2(\alpha^2 - 1)\Omega \cosh(kt/2) + i(\kappa/2) \sinh(kt/2)}{\Omega[\alpha \cos(K'x) + \cosh(kt/2)]} \right] \]  

\[(K')^2 = -2\Omega(1 - \alpha^2), \quad \kappa^2 = 16\Omega^2\alpha^2(1 - \alpha^2), \quad 0 < \alpha^2 < 1. \]

A rigorous proof of their instability under small perturbations can be found in [26].

The second degeneracy \((k = 0, \omega_0 = 3\Omega \neq 0)\) yields the Peregrine soliton [27], whose complex amplitude is rational in \(x\) and \(t\),

\[ A = \left(-\frac{\Omega}{q}\right)^{1/2} \left[ 1 + \frac{1 - 2i\Omega t}{2i\Omega x^2 - p(1 + (2\Omega t)^2)} \right] e^{-i\Omega t}. \]

and whose large maximum amplitude \(3\) above its background makes it a simple prototype of a rogue wave.

### 3.2. Nongeneric solutions, \(Q_x z' = 0\).

If \(Q_x = 0\), the solution is a particular plane wave,

\[ A = \sqrt{\frac{\omega_0}{q}} e^{-i\omega_0 t} \]  

which is also the limit \(\Omega \to \omega_0\) of both (24) and (25).

If \(Q_x \neq 0\) and \(z = z_0 \neq 0\), we obtain a two-parameter particular “dark” one-soliton solution (see Eq. (28) in [28])

\[ A = \left(-\frac{2p}{q}\right)^{1/2} \left( \frac{\lambda \tanh(\lambda(x - ct)) + i\frac{c}{2p}}{1 - 2i\Omega t} \right) e^{-i\Omega_0 t}, \]  

\[ \lambda^2 = \frac{\Omega_0}{2p} - \frac{c^2}{4p^2}, \quad \Omega_0 = \omega_0 + 2z_0. \]

and its one-parameter rational degeneracy \(\lambda = 0\)

\[ A = \left(-\frac{2p}{q}\right)^{1/2} \left( \frac{1}{x - ct} + i\frac{c}{2p} \right) e^{-i\Omega_0 t}, \]  

\[ c^2 = 2p\Omega_0. \]

The conditions \(Q_x \neq 0\) and \(z = 0\) define the envelope solution

\[ A = \sqrt{\frac{2p}{q}} \text{dn}(\lambda x, m_2) e^{-i\omega_0 t}, \quad \omega_0 = p\lambda^2(m_2 - 2), \quad k_2 = p^2\lambda^4(m_2 - 1), \]  

and its degeneracy, the “bright” one-soliton solution [29] for \(k_2 = 0\):

\[ A = \left(\frac{2p}{q}\right)^{1/2} \frac{\lambda}{\cosh(\lambda x)} e^{-i\omega_0 t}, \quad \lambda^2 = -\frac{\omega_0}{p}, \]  

\[ k_2 = -\frac{\omega_0^2}{4}. \]

The other trigonometric degeneracy \(k_2 = -\omega_0^2/4\) is identical to the limit \(z_0 = 0\) of (28), and their common rational degeneracy is also the limit \(\Omega_0 \to 0\) of (29).

### 3.3. Nongeneric solutions, \(Q_x z' \neq 0\) and \(k_1 = 0\).

We must distinguish between the cases where \(k_2(\omega_0^2 + 4k_2)\) is zero or nonzero.

For \(k_2(\omega_0^2 + 4k_2) \neq 0\), we obtain

\[ z^{-1} = 2a(\cos(kt) - \cos(kt_1)), \quad \sin(kt_1) = -\frac{4i}{ak}, \quad \cos(kt_1) = \frac{8\omega_0}{ak^2}, \]  

\[ y = \partial_\bar{t} \ln \left[ F(x) \sin \left( \frac{kt}{4} + \cos \left( \frac{kt}{4} \right) \right) \right], \]  

\[ 8p(F')^2 = 2\omega_0(F^2 + 1)^2 - a\frac{k^2}{4}(F^4 - 6F^2 + 1), \]  

\[ a^2 = -\frac{16k_2}{(\omega_0^2 + 4k_2)^2}, \quad k^2 = 4(\omega_0^2 + 4k_2), \]

and the reality of \(z(t)\) restricts \(k_2\) to be negative.
To our knowledge, this is a new solution, depending on two constants $\omega_0$ and $k_2$. The reason why it was not found earlier is the choice made by all authors to integrate the $x$-elliptic ODE (5) instead of the $t$-Riccati ODE (4a), preventing $k_1 = 0$ from being singled out. The physically admissible solutions, elliptic in $x$, exist in the focusing and defocusing regimes, but are not bounded. When $-\omega_0^2/4 < k_2 < 0$, the amplitude $A e^{i\omega_0 t}$ is periodic in time,

$$A = \left( -\frac{a}{q k_2} \right) \frac{1}{2} \frac{k^2}{16 \sinh(kt_1/2)} e^{-i\omega_0 t} \frac{ak (\cos(kt_1) - 1)[1 + F(x) c] + i[F(x) - c]}{4 F(x) + c},$$

and when $k_2 < -\omega_0^2/4$, only periodic in $x$:

$$A = \left( \frac{a}{q k_2} \right) \frac{1}{2} \frac{k^2}{16 \sinh(kt_1/2)} e^{-i\omega_0 t} \frac{ak (\cosh(kt_1) - 1)[1 + G(x) c] + i[G(x) + c]}{-G(x) + c},$$

where $G(x) = iF(x)$ real, $c = \coth \frac{kt}{4}$, and $k^2 = -k^2 > 0$. The degeneracy $k_1 = 0$, $k_2 = -\omega_0^2/4 \neq 0$ of (33),

$$A = \sqrt{-\frac{2p}{q} \frac{K}{2}} \left[ 1 - \frac{2(2\omega_0 t - i)}{\sinh(\sqrt{2\omega_0/p}x) + 2\omega_0 t} \right] e^{-i\omega_0 t},$$

is the limit $\Omega \to \omega_0$ of the degeneracy (24) of (16), obtained by

$$\Omega = \omega_0(1 - 2\varepsilon^2), \quad \alpha = \varepsilon, \quad k = -4i\omega_0\varepsilon \quad \text{for } \varepsilon \to 0$$

and $\cosh(Kx + i\pi/2) = i \sinh(\lambda x)$ and expanding sin and cos near $kt = 0$. Although we could not find (35) explicitly written in the literature, it is certainly not new (see, e.g., [30]).

Last, the degeneracy $k_2 = 0$ has a nonreal value of $z(t)$.

Table 1 displays all solutions generated by (2).

4. On higher-degree constraints

Since those singularities of $A$ and $\bar{A}$ that depend on the initial conditions are simple poles [18], [19], the next constraint after (2) should be

$$(g_{2,1}R^2 + 2g_{2,2}RI + g_{2,3}I^2 + g_{2,4}R_x + g_{2,5}I_x) + (g_{1,1}R + g_{1,2}I) + g_0 = 0, \quad (37)$$

where $R = \text{Re} A$ and $I = \text{Im} A$, in which the real coefficients $g_{ij}$ depend on $t$ (and maybe on $x$). Indeed, the relevant degree is the singularity degree (two in (37)), not the polynomial degree, which is why the restrictive assumption $g_{2,4} = g_{2,5} = 0$ in [10, Eq. (61)] does not lead to finding anything new. The larger freedom of (37) should generate more solutions, and this will be the subject of future work.

5. Conclusion and discussion

In this paper, making the three-parameter extrapolation (2) of the NLS breather explicit, we explained the $t$-period doubling experimentally observed in [12]. This should provide a much better precision in all the experiments on the phenomenon of modulational instability.

The Lamé equation is fundamental in the solution of constraint (2): (i) it leads to compact expression (16), (ii) it provides a natural derivation of the breather in Eq. (21), initially obtained by expert manipulations [10], [14].
Table 1. All solutions of constraint (2). Each solution is separated by a single line from its degeneracies. Columns display: the $x$- and $t$-dependences of $A e^{i\omega_0 t}$ (quasielliptic (Q), elliptic (E), trigonometric (T), rational (R), none (—)), the arbitrary constants, the complex amplitude, and the original reference.

| $Q_x$ | $z'(t)$ | $z$ | $k_1$ | $x$ | $t$ | Arbitrary constants | Amplitude | Reference |
|-------|---------|-----|-------|-----|-----|---------------------|-----------|-----------|
| $A$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | E | Q | $\omega_0$, $k_1$, $k_2$ | (16) | [10, Eqs. (3), (22), (24), (25)] |
| $B$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | E | E | $\omega_0$, $k_1$ | (21) | [9, Eq. (18)] |
| $C$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | T | T | $\Omega$, $\alpha$ | (24) | [8] |
| $D$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | T | T | $\Omega$, $\alpha$ | (25) | [9, Eq. (11)] |
| $E$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | R | R | $\omega_0$ | (26) | [27, Eq. (6.7)] |
| 1     | 0       | 0    | 0     | 0    | —  | —  | —       | $\omega_0$ | (27) | [10, Eqs. (37), (51)] |
| 2     | $\neq 0$ | 0    | $\neq 0$ | 0    | T | T | $\Omega_0$, $c$ | (28) | [28, Eq. (3)] |
| 3     | $\neq 0$ | 0    | $\neq 0$ | 0    | R | R | $\omega_0$ | (29) | [28] |
| 4     | $\neq 0$ | 0    | 0     | 0    | E | —  | $\omega_0$, $k_2$ | (30) | [10, Eqs. (54), (60)] |
| 5     | $\neq 0$ | 0    | 0     | 0    | T | —  | $\omega_0$ | (31) | [29], [10, Eq. (46)] |
| 6     | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | E | T | $\omega_0$, $k_2$ | (33) | New |
| 7     | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | — | T | $\omega_0$ | (35) |

Because Kuznetsov solution (24), identical in the complex plane to Akhmediev breather (25), is generated by plane wave (27) via a Bäcklund transformation, it is natural to ask which seed generates the three-parameter solution (16), an extrapolation of (24). We conjecture that this could be the general traveling wave

$$A = \left( -\frac{2p}{q} \right)^{1/2} \sigma(\xi + d) \sigma(\xi) e^{-i\omega t - \zeta(d)\xi + i(c/2p)\xi}, \quad \xi = x - ct, \quad (38)$$

with $id$ real (again Lamé!) for two reasons: (i) the Bäcklund transformation involves the integration of a linear differential system (the Lax pair) depending on the seed, and therefore this seed must be elliptic in $x$ and $t$; (ii) the elliptic discriminants $\Delta^{(x)}$ and $\Delta^{(t)}$ of (16) have a never vanishing ratio, just like the elliptic discriminants of (38) have the ratio given by a power of $c$.

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REFERENCES

1. V. I. Bespalov and V. I. Talanov, “Filamentary structure of light beams in nonlinear liquids,” JETP Lett., 3, 307–310 (1966).
2. V. E. Zakharov, “Stability of periodic waves of finite amplitude on the surface of a deep fluid,” J. Appl. Mech. Tech. Phys., 9, 190–194 (1968).
3. N. Akhmediev, A. Ankiewicz, and M. Taki, “Waves that appear from nowhere and disappear without a trace,” Phys. Lett. A, 373, 675–678 (2009).
4. V. V. Konotop and M. Salerno, “Modulational instability in Bose–Einstein condensates in optical lattices,” *Phys. Rev. A*, 65, 021602, 4 pp. (2002), arXiv:cond-mat/0106228.

5. P. J. Everitt, M. A. Sooriyabandara, M. Guasoni et al., “Observation of a modulational instability in Bose–Einstein condensates,” *Phys. Rev. A*, 96, 041601, 5 pp. (2017), arXiv:1703.07502.

6. D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, “Optical rogue waves,” *Nature*, 450, 1054–1057 (2007).

7. D.-I. Yeom and B. J. Eggleton, “Rogue waves surface in light,” *Nature*, 450, 953–954 (2007).

8. E. A. Kuznetsov, “On solitons in parametrically unstable plasma,” *Sov. Phys. Dokl.*, 22, 507–508 (1977).

9. N. N. Akhmediev and V. I. Korneev, “Modulation instability and periodic solutions of the nonlinear Schrödinger equation,” *Theoret. and Math. Phys.*, 69, 1089–1093 (1986).

10. N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, “Exact first-order solutions of the nonlinear Schrödinger equation,” *Theoret. and Math. Phys.*, 72, 809–818 (1987).

11. G. H. Halphen, *Traité des fonctions elliptiques et de leurs applications*, Vol. 1: Théorie des fonctions elliptiques et de leurs développements en série, Gauthier-Villars, Paris (1886); (1888); (1891).

12. G. Vanderhaegen, P. Szriftgiser, C. Naveau et al., “Observation of doubly periodic solutions of the nonlinear Schrödinger equation in optical fibers,” *Optics Lett.*, 45, 3757–3760 (2020).

13. M. Conforti, A. Mussot, A. Kudlinski, S. Trillo, and N. Akhmediev, “Doubly periodic solutions of the focusing nonlinear Schrödinger equation: recurrence, period doubling, and amplification outside the conventional modulation-instability band,” *Phys. Rev. A*, 101, 023843, 11 pp. (2020).

14. N. Akhmediev and A. Ankiewicz, “First-order exact solutions of the nonlinear Schrödinger equation in the normal-dispersion regime,” *Phys. Rev. A*, 47, 3213–3221 (1993).

15. D. Mihalache and N. C. Panoiu, “Exact solutions of nonlinear Schrödinger equation for positive group velocity dispersion,” *J. Math. Phys.*, 33, 2323–2328 (1992).

16. D. Mihalache and N. C. Panoiu, “Exact solutions of the nonlinear Schrödinger equation for the normal-dispersion regime in optical fibers,” *Phys. Rev. A*, 45, 6730–6734 (1992).

17. K. W. Chow, “A class of doubly periodic waves for nonlinear evolution equations,” *Wave Motion*, 35, 71–90 (2002).

18. D. V. Chudnovsky, G. V. Chudnovsky, and M. Tabor, “Painlevé property and multicomponent isospectral deformation equations,” *Phys. Lett. A*, 97, 268–274 (1983).

19. R. Conte and M. Musette, *The Painlevé Handbook*, Mathematical Physics Studies, Springer, Cham (2020).

20. G.-H. Halphen, “Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables,” *Mém. Acad. Sci. Inst. France*, 28, 1–301 (1884).

21. M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Appl. Math. Ser., Vol. 55, Dover, New York (1972).

22. L. Kiepert, “Auflösung der Transformationsgleichungen und Division der elliptischen Functionen,” *J. Reine Angew. Math.*, 1873, 34–44 (1873).

23. K. W. Chow, R. Conte, and N. Xu, “Analytic doubly periodic wave patterns for the integrable discrete nonlinear Schrödinger (Ablowitz–Ladik) model,” *Phys. Lett. A*, 349, 422–429 (2006), arXiv:nlin/0509005.

24. T. Kawata and H. Inoue, “Inverse scattering method for the nonlinear evolution equations under nonvanishing conditions,” *J. Phys. Soc. Japan*, 44, 1722–1729 (1978).

25. Y.-C. Ma, “The perturbed plane-wave solutions of the cubic Schrödinger equation,” *Stud. Appl. Math.*, 60, 43–58 (1979).

26. M. A. Alejo, L. Fanelli, and C. Muñoz, “The Akhmediev breather is unstable,” *São Paulo J. Math. Sci.*, 13, 391–401 (2019).

27. D. H. Peregrine, “Water waves, nonlinear Schrödinger equations and their solutions,” *J. Austral. Math. Soc. Ser. B*, 25, 16–43 (1983).

28. V. E. Zakharov and A. B. Shabat, “Interaction between solitons in a stable medium,” *Soviet Phys. JETP*, 37, 823–828 (1973).

29. V. E. Zakharov and A. B. Shabat, “Exact theory of two-dimensional self-focusing and onedimensional self-modulation of waves in nonlinear media,” *JETP*, 34, 62–69 (1972).

30. K. W. Chow, “Solitary waves on a continuous background,” *J. Phys. Soc. Japan*, 64, 1524–1528 (1995).