Modeling of thin-walled structures interacting with acoustic media as constrained two-dimensional continua

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Abstract. The transient interaction of acoustic media and elastic shells is considered on the basis of the transition function approach. The three-dimensional hyperbolic initial boundary-value problem is reduced to a two-dimensional problem of shell theory with integral operators approximating the acoustic medium effect on the shell dynamics. The kernels of these integral operators are determined by the elementary solution of the problem of acoustic waves diffraction at a rigid obstacle with the same boundary shape as the wetted shell surface. The closed-form elementary solution for arbitrary convex obstacles can be obtained at the initial interaction stages on the background of the so-called “thin layer hypothesis”. Thus, the shell–wave interaction model defined by integro-differential dynamic equations with analytically determined kernels of integral operators becomes hence two-dimensional but nonlocal in time. On the other hand, the initial interaction stage results in localized dynamic loadings and consequently in complex strain and stress states that require higher-order shell theories. Here the modified theory of I.N.Vekua–A.A.Amosov-type is formulated in terms of analytical continuum dynamics. The shell model is constructed on a two-dimensional manifold within a set of field variables, Lagrangian density, and constraint equations following from the boundary conditions “shifted” from the shell faces to its base surface. Such an approach allows one to construct consistent low-order shell models within a unified formal hierarchy. The equations of the $N$th-order shell theory are singularly perturbed and contain second-order partial derivatives with respect to time and surface coordinates whereas the numerical integration of systems of first-order equations is more efficient. Such systems can be obtained as Hamilton–de Donder–Weyl-type equations for the Lagrangian dynamical system. The Hamiltonian formulation of the elementary $N$th-order shell theory is here briefly described.

1. Introduction

In general, a transient problem of interaction of a shell with a fluid can be solved using the finite element discretization of both the liquid and the deformable solid together with appropriate numerical algorithms, the so-called “fluid–structure interaction” [1–6]. Such an approach results in models with many thousands of degrees of freedom and leads to several computational difficulties [3, 4] especially for thin-walled and light-weight structures [7] that force to develop new computational approaches (see, e.g., [8, 9]). In particular, special finite volume schemes are required to suppress high-frequency oscillations near multiple wave fronts in liquids [10, 11].

Another way to study the transient dynamics of thin-walled structures interacting with liquids is based on the use of the elementary solution apparatus and leads to time-boundary
integral equations. The kernels of integral operators are determined by Green’s functions of the corresponding unsteady problems of fluid dynamics with appropriate boundary conditions on the fluid-solid interfaces [12, 13]. Thus, the problem can be divided into two steps: at the first step, the initial boundary-value problem of wave diffraction at curvilinear obstacles must be solved, whereas the second step consists in solving the time-boundary integral equations. The diffraction problem can be linearized using the acoustic media approximation for liquids, so the solution can be finally determined as the superposition of incident waves, reflected waves, and waves radiated by the deformed solid [14, 15]. The closed-form definition of kernels of integral operators makes the time-boundary integral equations much more efficient. The appropriate analytical solutions of unsteady diffraction problems can be obtained on the basis of the integral transforms or integral equation [16–18] which are usually restricted by the obstacles on the surfaces of canonical geometry. In general, the solutions for arbitrary-shaped convex obstacles can be found using various simplifying assumptions estimated in [19–21]. The so-called “modified thin-layer hypothesis” [12, 13, 22–25] improving the known “plane reflection hypothesis” [26, 27] is used below; it consists in taking the medium curvature of the wetted surface of the obstacle into account and remains consistent with the wave-structure interaction for a short time after the contact initiation [27].

If the kernel of the integral operator is found, then the unsteady problem of shell dynamics is formulated as a system of integro-differential equations; in other words, the shell model is hence nonlocal [12, 13, 23–25]. The closed-form analytical solution for such problems can be constructed in the case of a shell with canonically shaped base surface, otherwise, numerical approaches [12, 13, 25, 28] or the eigenfunction expansion [13] can be useful. The numerical integration of singularly perturbed dynamic equations of various shell theories may be difficult due to high-frequency oscillatory solutions [10]; this drawback can be eliminated by using specific numerical methods [29, 30]. The eigenfunction expansion offers the possibility of reducing the problem by choosing an appropriate eigenfunction subset making a decisive effect on the dynamics of linear or nonlinear systems [31], but the shell dynamics at the initial diffraction stages is too complex to be adequately defined within a few eigenfunctions due to the local loading behavior. At the same time, the coupling of transverse shear and normal strains must be taken into account to compute higher-order eigenfunctions, moreover, the shell kinematics under high-frequency oscillations cannot be adequately defined within the assumptions of the classical shell model [32–37]. Thus, the higher-order shell theories are strongly required.

A wide range of nonclassical shell theories exists nowadays such as the well-known assumption-based approach [32, 38, 39], the asymptotic integration [40–42], the power or Fourier series expansion [43–36], etc. The asymptotic method allows a qualitative analysis of the shell dynamics [41, 47, 48]. On the other hand, the formal expansion approach offers the possibility of constructing a hierarchy of various-order shell theories using a unified formalism [43, 44, 49–51] and can be efficiently combined with numerical methods such as finite element modeling. This unified formalism consists in the Lagrangian continuum dynamics [52, 53]: a shell model is defined on the two-dimensional manifold within a set of field variables and Lagrangian density. The field variables are introduced as coefficients of a biorthogonal expansion for the three-dimensional displacement vector field [52], therefore the Legendre polynomials as compact functions can also be used as a base system. Such a theory shows its efficiency in steady problems of “thin-walled structure” dynamics [54–56] as in the transient dynamics [49]. An improved formalism of the so-called “extended shell theory” [57, 58] accounts for the boundary conditions on the shell faces as constraint equations for a variational problem of Lagrangian dynamics; moreover, it allows one to obtain the asymptotically consistent Kirchhoff model within a unified formal hierarchy without supplementary assumptions [59]. The extended Nth-order shell theory is used below to obtain an approximate model of transient fluid–shell interaction.
2. Approximate solution for the diffraction of a pressure wave at a rigid shell

2.1. Basics of the thin layer interaction model

Let us consider the diffraction of acoustic waves at a rigid shell which is a three-dimensional body \( V \subset \mathbb{R}^3 \), \( \partial V = S_+ \oplus S_B \), with faces \( S_+ \neq \emptyset \) and lateral surface \( S_B \):

\[
\forall M_x \in \bar{V} \quad \mathbf{R}(M_x) = \mathbf{R}(\xi^\alpha, \xi^3) = \xi^\alpha \mathbf{r}_\alpha + \xi^3 \mathbf{n}, \quad \mathbf{r}_\alpha(\xi^\alpha) = \partial_\alpha \mathbf{R}(\xi^\alpha, 0),
\]

where \( \xi^\alpha \in D_\xi \subseteq \mathbb{R}^2 \), \( \alpha = 1, 2 \), are curvilinear coordinates on the wetted surface \( S_+ \), considered as the base, \( \partial S = S' \cup S_B \); \( \xi^3 \in [-2h(\xi^\alpha), 0] \) is the normal coordinate, \( \forall M \in S_+ \xi^3 = 0 \); \( \mathbf{r}(\xi^\alpha) \) are base vectors in the tangent fibration \( T_M S \), and \( \mathbf{n} \) is a normal unit vector; \( \partial_i \equiv \partial / \partial \xi^i \) denote the partial differentiation with respect to the curvilinear coordinates \( \xi^i \), \( i = 1, 2, 3 \).

The equations for acoustic media dynamics can be written as [12, 13]

\[
\rho_0 \ddot{\mathbf{v}} = -\nabla p, \quad \dot{p} = -\rho_0 c^2 \nabla \cdot \mathbf{v},
\]

where \( p \) is the pressure of the acoustic media, \( \nabla \) is the Nabla operator, and \( \mathbf{v} = v^i \mathbf{R}_i \) is the velocity vector referred to the basis of the frame \( \xi^1, \xi^2, \xi^3 \) attached to the wetted surface \( S_+ \):

\[
\mathbf{R}_\alpha = A^\beta_\alpha \mathbf{r}_\beta, \quad \mathbf{R}_3 = \mathbf{n}, \quad A^\beta_\alpha = \delta^\beta_\alpha - \xi^3 b^\beta_\alpha, \quad |\mathbf{n}| = 1,
\]

where \( \mathbf{n} \) is the normal vector at a point \( M_x \in S_+ \), \( A^\beta_\alpha \) are components of the parallel shift tensor [43], \( b^\alpha_\beta \) is the curvature tensor of the surface \( S_+ \), and \( \delta^\beta_\alpha \) denotes the Kronecker delta.

Let us express equations (1) in terms of the physical components of the velocity vector \( \mathbf{v}_0 \):

\[
\rho_0 \ddot{v}_k = -\partial_k p \sqrt{g_{kk}}, \quad \dot{p} = -\rho_0 c^2 g^{-1/2} \partial_i \left( v_{j0} g^{ij} \sqrt{g_{jj}} \right), \quad v_{k0} = v_k \sqrt{g_{kk}}.
\]

Here \( c \) is the speed of sound, \( \rho_0 \) is the liquid mass density, \( g_{ij} = \mathbf{R}_i \cdot \mathbf{R}_j \) is the covariant metric, \( g^{ij} \) is the contravariant metric, and \( g = \det(g_{ij}) \). Let us note that \( g_{33} = g^{33} = 1 \) and \( g_{03} = 0 \). Here the summation rule is used for both Latin and Greek indices except for the indices \( k \) and \( \kappa \).

Hence let us consider the acoustic media dynamics in the layer of thickness \( \delta \) near the wetted surface \( S_+ \), and let \( L \) be a characteristic dimension of the shell. The dimensionless coordinates \( \xi^\alpha = \xi^\alpha / L^{\alpha} \), \( \alpha = 1, 2, \zeta = \xi^3 / \delta \), the time \( \tau = ct / \delta \), the pressure \( \bar{p} = p / (\rho_0 c^2) \), and the velocity \( \bar{v}_{k0} = v_{k0} / c \) can be introduced. Thus, we have the following formulation of equations (2):

\[
\bar{v}_k = -\gamma \partial_k \bar{p} \sqrt{\bar{g}}, \quad \bar{v}_0 = -\partial_3 \bar{p}, \quad \bar{p} = -g^{-1/2} \partial_\kappa \left( v_{00} g^{\alpha\kappa} \sqrt{g^{\alpha\alpha}} \right) + \partial_\zeta (v_{30} \sqrt{g}),
\]

where \( \gamma = \delta / L \), \( \kappa = 1, 2 \), and the partial differentiation is considered with respect to the dimensionless coordinates \( \xi^\alpha \) and time \( \tau \). Here and below, the tilde symbol is omitted for convenience.

Let the acoustic layer be sufficiently thin, so that \( \gamma \to 0 \), then the corresponding terms in equation (3) vanish. This hypothesis becomes the first postulate of the thin layer theory. Of course, the introduced assumption fails if the time \( \tau \) increases; therefore, the developed model can be used to study the initial stage of dynamic interaction between acoustic waves and shells. As a result, we can simplify equations (3) by omitting the terms with \( \gamma \) [12]:

\[
\bar{v}_0 = 0, \quad \bar{v}_3 = -\partial_\zeta \bar{p}, \quad \bar{p} = -g^{-1/2} \partial_\zeta (v_{30} \sqrt{g}).
\]

Considering equations (4) and letting \( \gamma \to 0 \), we obtain the following approximate equation for the dimensionless pressure \( \bar{p} \) (for more details, see [12] and [13]):

\[
\bar{p} = \partial_\zeta \bar{p} + \frac{\partial_\zeta g}{4g} \partial_\zeta \bar{p}, \quad g = \det(g_{ij}).
\]
Let us consider the known expressions for the metric determinant $g$ (see, e.g., [13, 52]):

$$\frac{\partial \varsigma}{4g} = -\frac{H - \zeta K}{1 - 2\zeta H + \zeta^2 K} \approx -H, \quad 2H = b^a_a, \quad K = \det(b^a_\beta),$$

(6)

where $H$ is the medium curvature of the wetted surface $S_+$. This quantity is invariant under the frame $O\xi^1\xi^2$, we can write the final equation of the thin layer theory that depends only on the intrinsic geometry of the wetted surface $S_+ [12, 13]$:

$$\ddot{\varsigma} = \partial_x G - 2H\partial_x G, (7)$$

2.2. Approximate transition functions based on the thin layer assumption

Let us pose the initial boundary-value problem for the transient function on the basis of (7):

$$\dot{G}^\varsigma = \partial_x G^\varsigma - 2H\partial_x G^\varsigma, \quad G^\varsigma|_{\tau=0} = \dot{G}^\varsigma|_{\tau=0} = 0, \quad (\delta_{1\varsigma} v_n + \delta_{2\varsigma} \ddot{p})|_{\zeta=0} = \delta(\tau)\delta(\xi^1 - \xi^1_0)\delta(\xi^2 - \xi^2_0),$$

(8) \quad (9) \quad (10)

$$G^\varsigma(\tau, r) = O(1), \quad r \to \infty, \quad r^2 = g_{ij}\xi^i\xi^j.$$  

(11)

Here $G^\varsigma(r, \tau)$ corresponds to the acoustic medium pressure, $v_n = \mathbf{v} \cdot \mathbf{n}$ is the velocity normal to the wetted surface $S_+$, and $\delta(\tau), \delta(\xi^1)$ are Dirac functions. The upper index $\varsigma = 1$ corresponds to the boundary-value problem of the first kind and $\varsigma = 2$ to the boundary-value problem of the second kind given by the general boundary condition (10) (for more details, see [12, 13]).

Let us apply the Laplace transform to problem (8)–(11) in the time domain:

$$\frac{d^2 G^\varsigma}{d\varsigma^2} - 2H(\xi^\alpha) \frac{dG^\varsigma}{d\varsigma} - s^2 G^\varsigma = 0,$$  

(12)

$$\langle \delta_{1\varsigma} v_{30} + \delta_{2\varsigma} \ddot{p} \rangle_{\zeta=0} = \delta(\xi^1 - \xi^1_0)\delta(\xi^2 - \xi^2_0),$$

(13)

$$\tilde{G}(r, s; \xi^\alpha) = O(1), \quad r \to \infty.$$  

(14)

Taking into account boundary conditions (13) and (14), we have the following general solution of linear equation (12) for the Laplace image $\tilde{G}^\varsigma$ of the transition function $G^\varsigma$:

$$\tilde{G}^\varsigma(\varsigma) = \frac{se^{-\lambda \varsigma}}{\delta_{1\varsigma} \lambda + \delta_{2\varsigma} s} \delta(\xi^1 - \xi^1_0)\delta(\xi^2 - \xi^2_0),$$

(15)

$$\lambda(\xi^\alpha, s) = -H + \sqrt{H^2 + s^2} \quad (\alpha = 1, 2),$$

(16)

where the root corresponds to $\Re\sqrt{\xi} > 0$. Let us define the surface transient function image $\tilde{G}^\varsigma_0$:

$$\tilde{G}^\varsigma_0 = \tilde{G}^\varsigma|_{\zeta=0} = \frac{1}{\delta_{1\varsigma} \lambda + \delta_{2\varsigma} s} \delta(\xi^1 - \xi^1_0)\delta(\xi^2 - \xi^2_0).$$

(17)

With regard to (16), for the initial boundary-value problem of the first kind, we have

$$\tilde{G}^\varsigma_1 = \frac{s}{-H + \sqrt{H^2 + s^2}} \delta(\xi^1 - \xi^1_0)\delta(\xi^2 - \xi^2_0),$$

(18)

whereas in case of the initial boundary-value problem of the second kind, we have

$$\tilde{G}^\varsigma_0 = \delta(\xi^1 - \xi^1_0)\delta(\xi^2 - \xi^2_0).$$

(19)
The transient function $G_0^1$ can be obtained by the inverse transform of (18). Bearing in mind
\[
\frac{H}{s^2} \triangleq H \tau, \quad \frac{1}{\sqrt{H^2 + s^2}} \triangleq J_0(H \tau),
\]
and the Laplace image multiplication theorem which implies
\[
\frac{H^2}{s^2 \sqrt{H^2 + s^2}} \triangleq H^2 \int_0^\tau (\tau - t)J_0(H t) \, dt = \tau^2 \left\{ J_0(z) - \frac{\pi}{2} [ J_0(z) H_1(z) - J_1(z) H_0(z) ] \right\} + \frac{\tau}{H} J_1(z),
\]
where $z = -H \tau$, $J_0(z)$, $J_1(z)$ are Bessel functions of the first kind, and $H_0(z)$, $H_1(z)$ are Struve functions of the zeroth and first orders (for more details, see [13]), we finally obtain
\[
G_0^1(\xi^\alpha, \tau; \xi_0^\alpha) = F^1(\xi^\alpha, \tau) \delta(\xi^1 - \xi_0^1) \delta(\xi^2 - \xi_0^2), \quad F^1(\xi^\alpha, \tau) = \Theta(\tau) \{ Q(z) - z[1 + J_1(z)] \}, \quad (20)
\]
\[
Q(z) = J_0(z) + z^2 \left\{ J_0(z) - \frac{\pi}{2} [ J_0(z) H_1(z) - J_1(z) H_0(z) ] \right\},
\]
where $\Theta(\tau)$ is the Heaviside function. For the time derivative $\dot{F}^1(\xi^\alpha, \tau)$, we have
\[
\dot{F}^1(\xi^\alpha, \tau) = \delta(\tau) - H[Q'(z) - zJ_0(z) - 1] \Theta(\tau), \quad (21)
\]
\[
Q'(z) = -J_1(z) + 2z \left\{ J_0(z) - \frac{\pi}{4} [ J_0(z) H_1(z) - J_1(z) H_0(z) ] \right\}.
\]

Thus, we can construct the following integral formula for the pressure $p$ on the wetted surface $S_+$:
\[
p(\xi^\alpha, \tau) = \int_0^\tau dt \int_{S_+} G_0^1(\xi^\alpha, \tau - t; \xi_0^\alpha) v_n(\xi_0^\alpha, t) \, dS_+ = v_n(\xi^\alpha, \tau) - H(\xi^\alpha) \int_0^\tau \{ Q'[\delta - H(\xi^\alpha)(\tau - t)] + H(\xi^\alpha)(\tau - t) J_0[\delta - H(\xi^\alpha)(\tau - t)] - 1 \} v_n(\xi^\alpha, t) \, dt. \quad (22)
\]

Here the first term corresponds to the plane reflection assumption, whereas the second term accounts for the curvature of the wetted surface of the obstacle [12, 13]. As a result, we must know the normal velocity $v_n$ on the wetted surface to approximate the pressure. In the case of elastic shell, taking the boundary condition on $S_+$ into account, we obtain
\[
\dot{u}_3 \big|_{S_+} = (v_s \cdot n + v_n) \big|_{S_+}, \quad (23)
\]
where $u_3$ is the displacement of the wetted shell surface and $v_s$ is the incident wave velocity. As a result, the shell is loaded by the summary pressure defined by the formula [13]
\[
p(\xi^\alpha, t) = p_s(\xi^\alpha, t) \big|_{S_+} + G^1(\xi^\alpha, t) \ast \ast \ast [ \dot{u}_3 - v_s \cdot n ] \big|_{S_+}, \quad (24)
\]
where $p_s$ is the pressure in the incident acoustic wave computed on the outer surface of the shell and the symbol “∗” denotes the integral convolution corresponding to (22).
3. Two-dimensional approximate model of the wave–shell interaction

3.1. Constrained Lagrangian model of the wave–shell interaction

Let the shell be elastic. Thus, let \( \mathbf{s} \) be a stress tensor, \( \mathbf{C} \) be an elasticity tensor, and \( \mathbf{u} = u\alpha \mathbf{r}^\alpha + u_3 \mathbf{n} \) be a displacement vector field. The higher-order shell model can be defined on the two-dimensional manifold corresponding to the base surface by the set of field variables \( u_{(k)}^{(i)}(\xi, \alpha, t) \), \( i = 1, 2, 3 \), \( k = 0, 1, \ldots, N \), \( \alpha = 1, 2 \) and the Lagrangian density \( \mathcal{L}_S(u_{(k)}^{(i)}, \dot{u}_{(k)}^{(i)}, \nabla_{\beta} u_{(k)}^{(i)}) \) in terms of the dimensional reduction concept \([52, 53]\). The field variables are introduced by means of the biorthogonal expansion of the covariant displacement components \( u_{(k)}(\xi, \alpha, z, t) \) with respect to the base system \( p_{(k)}(z) \); here \( z \) is the dimensionless thickness variable \([52]\). As a result, we have the following Lagrangian density formulation (for more details, see \([52]\) and \([53]\)):

\[
\mathcal{L}_M = \frac{1}{2} \rho (m) \left( \dot{u}_{(m)}^{(1)} + s_{33}^{(m)} \dot{u}_{(3)}^{(m)} \right) - \frac{1}{2} \left[ \gamma_{\beta}^{\gamma} \nabla_{\gamma} u_{(k)}^{(k)} - b_{\gamma\beta} u_{3}^{(k)} + s_{33}^{(m)} D_{(k)}^{(m)} u_{3}^{(k)} \right] + \mathcal{L}_S^{\alpha}\left( u_{(k)}^{(i)}, \dot{u}_{(k)}^{(i)}, \nabla_{\beta} u_{(k)}^{(i)} \right) + P_{(k)}^{\alpha} u_{(k)}^{(i)} + P_{(k)}^{3} u_{3}^{(k)},
\]

(25)

where \( H_{(k)}^{(m)} \) and \( D_{(k)}^{(m)} \) are linear operators and \( s_{ij}^{(k)} \) are generalized forces \([52]\): \n
\[
s_{ij}^{(k)} = \bar{C}_{(km)}^{ij\delta} \nabla_{\delta} u_{(m)}^{(\gamma)} + \bar{C}_{(km)}^{ij3} \nabla_{\delta} u_{3}^{(m)} + \bar{C}_{(km)}^{ij\gamma} u_{3}^{(m)} + \bar{C}_{(km)}^{ij3} u_{3}^{(m)}, \quad i, j = 1, 2, 3, \quad \gamma, \delta = 1, 2. \quad (26)
\]

The use of Legendre polynomials \( p_{(k)}(z) \) allows one to construct traditional shell theories \([43, 44, 49, 54, 55]\), whereas the compact functions result in “finite layer” shell models. Let us note that the boundary conditions on the faces \( S_{\pm} \) of a shell including the no-flow condition (23) can be satisfied only approximately as a result of convergence of the two-dimensional solution sequence as \( N \to \infty \) \([51]\), moreover, the first-order shell theory given in \([25]\) requires the plane stress assumption to be consistent \([43, 45]\). To eliminate this drawback of the dimensional reduction, the so-called extended shell theory of the \( N \)th order \([57–59]\) can be constructed.

Let us consider the boundary conditions on the faces \( S_{\pm} \) given as

\[
\mathbf{s}_{\pm} \cdot \mathbf{n}_{\pm} \equiv [\mathbf{C} : (\nabla \times \mathbf{u})]_{\pm} \cdot \mathbf{n}_{\pm} = \mathbf{q}_{\pm},
\]

(27)

where \( \mathbf{q}_{\pm} \) is the resultant force vector and \( \mathbf{n}_{\pm} \) are normal unit vectors on \( S_{\pm} \). The boundary conditions (27) can be “shifted” from \( S_{\pm} \) to the base surface by representing the vector \( \mathbf{u} \) in the basis \( \mathbf{r}^\alpha, \mathbf{n} \) \([57]\). The wetted surface \( S_{+} \) is considered below as the base surface according to the above-formulated statement of the diffraction problem, therefore, the no-flow condition does not require further transformations, and moreover \( q_{-} = 0 \) and \( q_{+} = 0 \) because the acoustic media are nonviscous. Substituting the biorthogonal expansion of the displacement vector \( u(x, \mathbf{r}^\alpha + u_3 \mathbf{n}) \) into (27) and taking (24) into account, we obtain the equations

\[
C_{+}^{ij\beta} \left( \nabla_{\delta} u_{\gamma}^{(m)} + H_{(m)}^{(k)} u_{\gamma}^{(m)} - b_{\gamma\beta} u_{3}^{(k)} \right) + \bar{C}_{+}^{ij3} \left( \nabla_{\delta} u_{3}^{(m)} + H_{(m)}^{(k)} u_{3}^{(m)} + b_{3} u_{\gamma}^{(k)} \right)
\]

+ \bar{C}_{+}^{ij3} h^{-1} D_{(m)}^{(k)} u_{3}^{(m)} - \delta_{\gamma}^{\alpha} p_{(k)}(z) + G^{1} \ast \ast \ast \left( u_{3}^{(k)} - \mathbf{v} \right) \cdot \mathbf{n} = 0, \quad (28)

\[
C_{-}^{ij\beta} \left( \nabla_{\delta} u_{\gamma}^{(m)} + H_{(m)}^{(k)} u_{\gamma}^{(m)} - b_{\gamma\beta} u_{3}^{(k)} \right) + \bar{C}_{-}^{ij3} \left( \nabla_{\delta} u_{3}^{(m)} + H_{(m)}^{(k)} u_{3}^{(m)} + b_{3} u_{\gamma}^{(k)} \right)
\]

+ \bar{C}_{-}^{ij3} h^{-1} D_{(m)}^{(k)} u_{3}^{(m)} = 0, \quad i, j = 1, 2, 3. \quad (29)

Here the “surface” values of the generalized elastic constants \( \bar{C}_{ijpq} \) are introduced \([57, 58]\) as

\[
\bar{C}_{ijpq}^{=\pm} = C_{ijpq}^{=\pm} p_{(k)}(z)_{S_{\pm}}, \quad \bar{C}_{ijpq}^{=\pm} = C_{ijpq}^{=\pm} p_{(k)}(z)_{S_{\pm}}.
\]
Equations (28), (29) become the constraint equations for the variational problem given by the Lagrangian (25); thus, the shell model becomes nonlocal due to the integral operators (22). The generalized Lagrange equations following from (25) and (28), (29) and their natural boundary conditions can be represented in the formulation [57, 58], where the pressure (24) is taken into account:

\[ p^{(m)}_{\lambda} = \bar{\nabla}^2 \beta s(k) \alpha^{\beta} - H^{(m)}_{\beta(k)} s^{\alpha \beta} - h^{-1} D^{(m)}_{\beta(k)} s^{\alpha \beta}, \]

\[ p^{(m)}_{\lambda} = \bar{\nabla}^2 \beta s(k) \alpha^{\beta} - H^{(m)}_{\beta(k)} s^{\alpha \beta} - h^{-1} D^{(m)}_{\beta(k)} s^{\alpha \beta}, \]

\[ \frac{\partial \dot{u}^{(k)}_i}{\partial \dot{u}^{(k)}_i} - G^1 \bigg|_{S_0} = 0, \quad i = 1, 2, 3, \quad \gamma, \beta = 1, 2. \]

The shell model consists in (30)–(32), the initial conditions (for more details, see [52, 57]):

\[ u^{(k)}_i \bigg|_{t=0} = U^{(k)}_i, \quad \dot{u}^{(k)}_i \bigg|_{t=0} = V^{(k)}_i, \quad i = 1, 2, 3, \quad k = 0, \ldots, N, \]

3.2. Hamilton–de Donder–Weyl shell model

The obtained system of equations (30), (31) contains the partial derivatives of the second order, whereas the efficient numerical integration requires systems of first-order partial differential equations [30]. The first-order equations can be obtained using the generalized Hamiltonian formalism [60].

Let us apply the Lagrange transform with respect to time and space derivatives to (25):

\[ H^{DW} = p^{(k)}_a u^{(k)}_a + p^{(k)}_3 u^{(k)}_3 - \gamma^{\alpha \beta} \bar{\nabla}_\alpha u^{(k)}_\beta - s^{\gamma \beta} \bar{\nabla}_\gamma u^{(k)}_3 - \mathcal{L}, \]

where the quasicanonical polymomenta are defined as follows:

\[ p^{(k)}_a = \frac{\partial \mathcal{L}}{\partial \dot{u}^{(k)}_a}, \quad p^{(k)}_3 = \frac{\partial \mathcal{L}}{\partial \dot{u}^{(k)}_3}, \quad s^{\gamma \beta} = - \frac{\partial \mathcal{L}}{\partial \bar{\nabla}_\gamma u^{(k)}_\beta}, \quad s_s^{\gamma \beta} = - \frac{\partial \mathcal{L}}{\partial \bar{\nabla}_\gamma u^{(k)}_3}. \]

Taking into account (25), (35), and (36), we obtain the system of equations

\[ u^{(k)}_a = \frac{\partial H^{DW}_{a}}{\partial p^{(k)}_a}, \quad u^{(k)}_3 = \frac{\partial H^{DW}_{a}}{\partial p^{(k)}_3}, \]

\[ \frac{\partial H^{DW}_{a}}{\partial s^{(k)}_a} - \bar{\nabla}_\beta s^{(k)}_a = - \frac{\partial H^{DW}_{a}}{\partial u^{(k)}_3} - \bar{\nabla}_\beta s^{(k)}_3 = - \frac{\partial H^{DW}_{a}}{\partial u^{(k)}_3}. \]

Here equations (37) define the generalized velocities \( u^{(k)}_a \), (38) are dynamics equations, and (39) are constitutive equations of the elementary \( N \)-th-order shell theory. Expressing the covariant derivatives in terms of partial derivatives and Christoffel symbols and then transposing them to the right-hand sides, we obtain the quasicanonical formulation of system (37)–(39) corresponding to the Hamilton–de Donder–Weyl formalism of analytical continuum dynamics [60, 61].
Conclusions
Thus, the approximate theory of transient interaction between acoustic waves and elastic anisotropic shells is formulated for higher-order shell theories. The proposed theory allows one to model the unsteady shell dynamics at the initial stage of the shell–wave interaction, considers the coupling of the transverse shear and normal deformation and takes the higher degrees of freedom into account in addition to the translation and rotation at a point of the base surface. Two formulations are shown, one is based on the Lagrangian formalism of analytical continuum mechanics and results in second-order partial differential equations, and the other is based on the Hamilton–de Donder–Weyl formalism and leads to dynamic equations with first-order partial derivatives.

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