STRONG CONVERGENCE OF WAVE OPERATORS FOR A FAMILY OF DIRAC OPERATORS

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Abstract. We consider a family of Dirac operators with potentials varying with respect to a parameter $h$. The set of potentials has different power-like decay independent of $h$. The proofs of existence and completeness of the wave operators are similar to that given in [4]. We are mainly interested in the asymptotic behavior of the wave operators as $h \rightarrow \infty$.

1. Introduction

In quantum mechanics, it is important to compare a given interacted operator with a simpler (free) operator for which many spectral features are known. Scattering theory is part of perturbation theory that concerns a comparative study of the absolutely continuous spectrum of operators. That is, for two self-adjoint operators $T$ and $T_0$ that are close to each other in an appropriate sense, scattering theory is mainly the study of existence and completeness of $s \lim_{t \to \pm \infty} e^{i T t} J e^{-i T_0 t} P_{0}^{(ac)}$ where $J$ is some bounded operator (identification), $P_{0}^{(ac)}$ is the orthogonal projection onto the absolutely continuous spectrum of $T_0$, and $s$ refers to the strong convergence sense. Another important issue is the case studied in the present work, where the operator $T$ is $h$-dependent, with $h$ a parameter allowed to grow to infinity. For such operators, in addition to study the existence of the time limit, a parallel question also emerges whether or not the limit $s \lim_{h \to \infty, t \to \pm \infty} e^{i T_h t} J_h e^{-i T_0 t} P_{0}^{(ac)}$ exists, where now $J_h$ is an $h$-dependent bounded identification.

Scattering theory for the Dirac operator with potentials decaying faster than the Coulomb potential (short-range potentials) goes straightforward. In this case, the proofs of existence and completeness of the wave operator (WO) $W(H, H_0) = s \lim_{t \to \pm \infty} e^{i H t} e^{-i H_0 t}$, where $H_0$ and $H$ are respectively the free and the interacted Dirac operators, are similar to that of the Schrödinger operator. In the Coulomb

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interaction case, the modified WO $W_{\pm} = W_{\pm}(H, H_0, \mathcal{J}) = s\lim_{t \to \pm \infty} e^{iHt}\mathcal{J} e^{-iH_0t}$ has been constructed in [2, 3], where $\mathcal{J}$ is a bounded identification. For potentials decaying as the Coulomb potential or slower (long-range potentials), the existence and completeness of the modified WO $W_{\pm}$ have been studied in [4, 8, 9, 12, 18]. The study of the asymptotic behavior of the WO $W_{\pm}$ with respect to the speed of light ($c$), as $c \to \infty$, has been studied for short-range potential in [17] and for long-range potential in [18].

Consider the family of Dirac operators $H_h = H_0 + V_h$ defined on the same Hilbert space and with the same domain as of $H_0$, where $H_0$ is the free Dirac operator defined on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$, and $V_h$ is a bounded interaction to $H_0$. Under suitable power-like decay assumption on the potential $V_h$ the WO $W_{\pm,h} = W_{\pm}(H_h, H_0; \mathcal{J}_{\pm,h}) = s\lim_{t \to \pm \infty} e^{iH_ht}\mathcal{J}_{\pm,h} e^{-iH_0t}$ exists and is complete [4], where $\mathcal{J}_{\pm,h}$ is a bounded identification. In other words, if for all $h > 0$, $|V_h| \leq \langle x \rangle^{-\rho}$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$, then the WO $W_{\pm,h}$ exists and is complete where $\mathcal{J}_{\pm,h}$ being just the identity operator for $\rho > 1$ (short-range). For $0 < \rho \leq 1$ (long-range), the identification $\mathcal{J}_{\pm,h}$ is a pseudo-differential operator (PSDO) defined as

$$
(\mathcal{J}_{\pm,h}\psi)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix\cdot\zeta + i\Phi_{\pm,h}(x, \zeta)} \mathcal{P}_{\pm,h}(x, \zeta) \mathcal{G}_{\pm}(x, \zeta) \psi(|\zeta|^2) \hat{\psi}(\zeta) \, d\zeta,
$$

where $\Phi_{\pm,h}$ is a phase function, $\mathcal{P}_{\pm,h}$ is an amplitude function, $\mathcal{G}_{\pm}$ is a cut-off function, and $\psi$ is a smooth function introduced to localize $\mathcal{J}_{\pm,h}$ in compact intervals of the continuous spectrum.

The goal of the present work is to study the asymptotic behavior of the WO $W_{\pm,h}$ and its adjoint $W_{\pm,h}^* = W_{\pm}(H_0, H_h; \mathcal{J}_{\pm,h})$ as $h \to \infty$. By the existence of $W_{\pm,h}$, the convergence of $H_h$ in the strong resolvent sense (SRS), and the strong convergence of the identification $\mathcal{J}_{\pm,h}$ we prove that the two strong limits $s\lim_{h \to \infty}$ and $s\lim_{t \to \pm \infty}$ are interchangeable. Hence, if the Dirac operator $H_h$ converges in the SRS to $H_\infty$ and $\mathcal{J}_{\pm,h}$ converges strongly to $\mathcal{J}_{\pm,\infty}$, then we have

$$
s\lim_{h \to \infty} W_{\pm,h} = W_{\pm}(H_\infty, H_0; \mathcal{J}_{\pm,\infty}).
$$

By the strong convergence of the identification $\mathcal{J}_{\pm,h}$ to $\mathcal{J}_{\pm,\infty}$, then also $\mathcal{J}_{\pm,h}^*$ converges strongly to $\mathcal{J}_{\pm,\infty}^*$ (this is not true in general, but it is valid for the type of identifications we consider in this work), hence we also have

$$
s\lim_{h \to \infty} W_{\pm,h}^* = W_{\pm}(H_0, H_\infty; \mathcal{J}_{\pm,\infty}^*).
$$

Further, we consider different cases of potential decay, for these cases, the identification $\mathcal{J}_{\pm,h}$ is simplified so that its strong convergence is easy to work out. In the case $\rho = 1$, we prove that the phase and amplitude functions, $\Phi_{\pm,h}$ and $\mathcal{P}_{\pm,h}$, can be chosen independently of $h$, thus the convergence of the WOs $W_{\pm,h}$ and $W_{\pm,h}^*$, as $h \to \infty$, is reduced to the convergence of the Dirac operator $H_h$ in the SRS. In the
case $\rho \in (1/2, 1)$, the amplitude function can be chosen independently of $h$, but not the phase function. In this case, even that the phase function still depends on $h$, but it can be simplified so that the convergence of the identification $J_{\pm,h}$ is simpler to consider.

The paper is arranged as follows; in Section 2 we provide the reader with brief preliminaries about the Dirac operator, scattering theory in general setting, basic calculus of PSDO, and the existence and completeness of the WO $W_{\pm,h}$. In Section 3, we state and prove the main results of the asymptotic limit of the WO $W_{\pm,h}$ as $h \to \infty$, also different cases of potential decay are discussed.

2. Preliminaries

By $R(A)$, $D(A)$, and $N(A)$ we refer respectively to the range, domain, and null space of a given operator $A$, also we denote by $X$ and $Y$ the Hilbert spaces $H^1(\mathbb{R}^3, \mathbb{C}^4)$ and $L^2(\mathbb{R}^3, \mathbb{C}^4)$ respectively.

2.1. The Dirac operator with an $h$-dependent potential. The free Dirac evolution equation is given by

\begin{equation}
    i\hbar \frac{\partial}{\partial t} u(x,t) = H_0 u(x,t), \quad u(x,0) = u^0(x),
\end{equation}

where $H_0 : X \to Y$ is the free Dirac operator defined as

\begin{equation}
    H_0 = hcD_\alpha + mc^2 \beta.
\end{equation}

Here, $\hbar$ is the Planck constant divided by $2\pi$, $D_\alpha = \alpha \cdot D$ where $D = (D_1, D_2, D_3)$ and $D_j = -i \frac{\partial}{\partial x_j}$ for $j = 1, 2, 3$, the constant $c$ is the speed of light, and $m$ is the particle rest mass. The notations $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ are the $4 \times 4$ Dirac matrices given by

\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]

Here $I$ and $0$ are the $2 \times 2$ unity and zero matrices respectively, and $\sigma_j$’s are the $2 \times 2$ Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Note that separation of the variables $x$ and $t$ in (1) yields the free Dirac eigenvalue problem

\begin{equation}
    H_0 u(x) = \lambda u(x),
\end{equation}

where $u(x)$ is the spatial part of the wave function $u(x,t)$, and $\lambda$ is the total energy of the particle. The free operator $H_0$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$.
and self-adjoint on $X$, its spectrum, $\sigma(H_0)$, is purely absolutely continuous and is given by

$$\sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, +\infty).$$

Let $\mathcal{F}$ be the Fourier transform operator

$$\mathcal{F}u(\zeta) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix \cdot \zeta} u(x) dx =: \hat{u}(\zeta),$$

then $\mathcal{F}H_0\mathcal{F}^*$ is the multiplication operator given by the matrix

$$h_0(\zeta) = \zeta \alpha + mc^2 \beta,$$

known as the symbol of $H_0$, where $\zeta = \alpha \cdot \zeta = \sum_{k=1}^{3} \alpha_k \zeta_k$. The symbol $h_0(\zeta)$ can be written as

$$h_0(\zeta) = \eta(\zeta) p_{+,0}(\zeta) - \eta(\zeta) p_{-,0}(\zeta),$$

where $p_{\pm,0}(\zeta)$ are the orthogonal projections onto the eigenspaces of $h_0(\zeta)$ and are given by

$$p_{\pm,0}(\zeta) = \frac{1}{2}(I \pm \eta^{-1}(\zeta)(\zeta_\alpha + mc^2 \beta)), $$

and $\pm \eta(\zeta) = \pm \sqrt{\lvert \zeta \rvert^2 + m^2 c^4}$ are the corresponding eigenvalues.

We consider an $h$-dependent potential, $V_h(x)$, added to the free Dirac operator and define

$$H_h = H_0 + V_h.$$ 

The potential $V_h$ is assumed to be real and say bounded, thus, for all $h > 0$, $H_h$ and $H_0$ have the same domain $X$ and $H_h$ is self-adjoint on $X$. For simplicity we assume $\hbar = c = 1$. The corresponding evolution equation reads

$$\begin{cases}
i \partial_t u_h(x,t) = H_h u_h(x,t), \\
u_h(x,0) = u^0_h(x).
\end{cases}$$

By the Stone theorem, there exists a unique solution to (9) given by

$$u_h(x,t) = U_h(t) u^0_h(x), \quad u^0_h \in X,$$

where the strongly continuous unitary operator $U_h(t) = e^{-iH_h t}$ is generated by the operator $-iH_h$, see e.g. [6, 11].

The potential $V_h$ is assumed to fulfill the following condition for all multi-index $\alpha$

$$|\partial^\alpha V_h(x)| \leq C \langle x \rangle^{-\rho - |\alpha|}, \quad \text{for all } h > 0, \text{ and } \rho \in (0,1],$$

the constant $C$ is independent of $x$ and $h$, and recall that $\langle x \rangle = (1 + |x|^2)^{1/2}$. This condition simply means that $V_h$ is of long-range type for all $h > 0$. 
2.2. Basic setting of scattering theory. Given self-adjoint operators $T_0$ and $T$ in Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$ respectively. Let $P_{0}^{(ac)}$ and $P^{(ac)}$ be the orthogonal projections onto the absolutely continuous subspaces, $\mathcal{H}_0^{(ac)}$ and $\mathcal{H}^{(ac)}$, of $T_0$ and $T$ respectively.

Definition 1. The WO for $T$ and $T_0$, with a bounded identification $\mathcal{J} : \mathcal{H}_0 \to \mathcal{H}$ is denoted by $W_{\pm}(T, T_0; \mathcal{J})$ and defined as

\[
W_{\pm}(T, T_0; \mathcal{J}) = s\lim_{t \to \pm\infty} U(-t)\mathcal{J}U_0(t)P_{0}^{(ac)},
\]

provided that the corresponding strong limits exist, where again the letter $s$ refers to the strong convergence sense, $U(t) = e^{-itT}$ and $U_0(t) = e^{-itT_0}$. If $\mathcal{H} = \mathcal{H}_0$ and $\mathcal{J}$ is the identity operator, then the WO is denoted by $W_{\pm}(T, T_0)$. Also if $T_0$ has only absolutely continuous spectrum, then $P_{0}^{(ac)}$ is superfluous.

If the WO exists, then it is bounded. Since the operator $U(-t)U_0(t)$ is unitary, the WO $W_{\pm}(T, T_0)$ is isometric. In the case that $\mathcal{J}$ is not the identity operator, the WO $W_{\pm}(T, T_0; \mathcal{J})$ is isometric if $\lim_{t \to \pm\infty} \|\mathcal{J}U_0(t)u_0\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}_0}$ for any $u_0 \in \mathcal{H}_0^{(ac)}$.

The WO admits the chain rule, i.e., if $W_{\pm}(T, T_1; \mathcal{J}_1)$ and $W_{\pm}(T_1, T_0; \mathcal{J}_0)$ exist, then the WO $W_{\pm}(T, T_0; \mathcal{J}_{10}) = W_{\pm}(T, T_1; \mathcal{J}_1)W_{\pm}(T_1, T_0; \mathcal{J}_0)$ also exists, where $\mathcal{J}_{10} = \mathcal{J}_1\mathcal{J}_0$.

The WO possesses the intertwining property, that is

\[
\phi(T)W_{\pm}(T, T_0; \mathcal{J}) = W_{\pm}(T, T_0; \mathcal{J})\phi(T_0),
\]

for any bounded Borel function $\phi$. Also for any Borel set $\triangle \subset \mathbb{R}$

\[
E(\triangle)W_{\pm}(T, T_0; \mathcal{J}) = W_{\pm}(T, T_0; \mathcal{J})E_0(\triangle),
\]

where $E$ and $E_0$ are the spectral families of $T$ and $T_0$ respectively. The following remark is about the equivalence between WOs with different identifications.

Remark 1. Assume that the WO $W_{\pm}(T, T_0; \mathcal{J}_1)$ exists with an identification $\mathcal{J}_1$, and let $\mathcal{J}_2$ be another identification such that $\mathcal{J}_1 - \mathcal{J}_2$ is compact, then the WO $W_{\pm}(T, T_0; \mathcal{J}_2)$ exists and $W_{\pm}(T, T_0; \mathcal{J}_1) = W_{\pm}(T, T_0; \mathcal{J}_2)$. Moreover, the condition that $\mathcal{J}_1 - \mathcal{J}_2$ is compact can be replaced by $s\lim_{t \to \pm\infty} (\mathcal{J}_1 - \mathcal{J}_2)U_0(t)P_{0}^{(ac)} = 0$.

Another task of scattering theory is to study the completeness of WOs.

Definition 2. The WO $W_{\pm}$ is said to be complete if $\mathcal{R}(W_{\pm}) = \mathcal{H}^{(ac)}$.

If the WO $W_{\pm}(T, T_0; \mathcal{J})$ is complete, then the absolutely continuous part of $T_0$ is unitary equivalent to that of $T$. We refer to [14] for the completeness criteria. For comprehensive materials on scattering theory we refer to [5, 6, 7, 10, 12, 14, 16].
Definition 3. Given a self-adjoint operator $T$ in a Hilbert space $\mathcal{H}$. A $T$-bounded operator, $A : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is an auxiliary Hilbert space, is called $T$-smooth if one of the following properties is fulfilled

\[ \sup_{\|v\|_{\mathcal{H}}=1, v \in D(T)} \left| \int_{-\infty}^{\infty} \| A e^{-i T t} v \|_H^2 \, dt \right| < \infty, \]

\[ \sup_{\epsilon > 0, \mu \in \mathbb{R}} \| A R_T(\mu \pm i \epsilon) \|_H^2 < \infty, \]

where $R_T(\mu \pm i \epsilon)$ is the resolvent operator of $T$.

2.3. Preliminaries regarding pseudo-differential operators. In this subsection we will introduce basic calculus about pseudo-differential operators (PSDOs) with symbols belonging to the class $S_{\rho, \delta}^r(\mathbb{R}^3, \mathbb{R}^3)$.

Definition 4. The class $S_{\rho, \delta}^r(\mathbb{R}^3, \mathbb{R}^3)$ is the vector space of all smooth functions $P(x, \zeta) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$ such that for all multi-indices $\alpha$ and $\gamma$

\[ |\partial_\alpha^\alpha \partial_\zeta^\gamma P(x, \zeta)| \leq c_{\alpha, \gamma} (|x|^{r-\rho|\alpha|+\delta|\gamma|}, \]

where $r \in \mathbb{R}$, $\rho > 0$, and $\delta < 1$. The function $P$ is called the symbol of the PSDO and $r$ is called the order of $P$.

Let $P(x, \zeta) \in S_{\rho, \delta}^r(\mathbb{R}^3, \mathbb{R}^3)$, the associated PSDO, $\mathcal{P}$, to $P$ in $Y$ is defined as the following inverse Fourier integral

\[ (\mathcal{P} f)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix \cdot \zeta} P(x, \zeta) \hat{f}(\zeta) \, d\zeta, \]

where $f \in Y$ and $\hat{f}(\zeta) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix \cdot \zeta} f(x) \, dx$ is the Fourier transform of $f$.

Definition 5. The class $C^r(\Phi) \subset S_{1-s,s}^1(\mathbb{R}^3, \mathbb{R}^3)$, $s \in (0, 1)$, consists of all oscillating symbols $P$ that have the representation $P(x, \zeta) = e^{i \Phi(x, \zeta)} b(x, \zeta)$, where $\Phi \in S_{1,0}^1$ and $b \in S_{r,0}^1$.

The following two propositions are important, their proofs can be found in [15].

Proposition 1. Let $P \in S_{0,0}^0$ be compactly supported in $x$, then the associated PSDO $\mathcal{P}$ is bounded in $Y$ if $r = 0$ and compact if $r < 0$.

Proposition 2. Let $P \in C^r(\Phi)$ be compactly supported in $x$, then the associated PSDO $\mathcal{P}$ is bounded in $Y$ if $r = 0$ and compact if $r < 0$.

2.4. The wave operator. In this subsection we adapt the proofs derived in [4] for the class of potentials that we intend to study. Since the potentials we assume are of long-range type for all $h > 0$, there is no need to discuss the construction of the identification $J_h$ upon the eikonal equation, and thus most of the materials set in this subsection are similar to that given in [4].
Let $\Phi_\pm(x, \zeta)$ be a cut-off function defined as

\begin{equation}
\Phi_\pm(x, \zeta) = \theta(x)\omega_\pm((\hat{x}, \hat{\zeta})), \quad \text{for all } y \in \mathbb{R}^3 \setminus \{0\}, \; \hat{y} = y/|y|.
\end{equation}

The function $\theta$ is a $C^\infty(\mathbb{R}^3)$-function such that $\theta(x) = 0$ near $x = 0$ and $\theta(x) = 1$ for large $x$, thus $\theta(x)$ is introduced to avoid the singularity of $\hat{x}$ at $x = 0$. The function $\omega_\pm \in C^\infty(-1,1)$ is such that $\omega_\pm(\tau) = 1$ near $\pm 1$ and $\omega_\pm(\tau) = 0$ near $\mp 1$. Thus the cut-off function $\Phi_\pm$ is supported in the cone

\begin{equation}
\Xi_\pm(\nu) = \{(x, \zeta) \in \mathbb{R}^6 : \pm \langle x, \zeta \rangle \geq \nu |x||\zeta|\}, \quad \nu \in (-1,1).
\end{equation}

The purpose of defining the cut-off function is that the eikonal equation of the phase function of the approximated eigenfunction of the Dirac equation is not solvable, and thus it is obliged to remove a neighborhood of $-\zeta$ or $\zeta$ in order to find a global solution, see [4] for more clarification. Let now $\Phi_{\pm,h}(x, \zeta)$ be defined as

\begin{equation}
\Phi_{\pm,h}(x, \zeta) = \sum_{n=1}^{N} \Phi_{\pm,h}^{(n)}(x, \zeta), \quad x \in \Xi_\pm(\nu),
\end{equation}

where $N$ satisfies $(N+1)\rho > 1$, and for $n \geq 0$

\begin{equation}
\Phi_{\pm,h}^{(n+1)}(x, \zeta) = Q_{\pm}(\zeta)F_{\pm,h}^{(n)},
\end{equation}

where

\begin{equation}
(Q_{\pm}(\zeta)F)(x) = \pm \int_{0}^{\infty} (F(x \pm t\zeta, \zeta) - F(\pm t\zeta, \zeta)) \, dt.
\end{equation}

Let $F_{\pm,h}^{(n)}$’s be defined as

\begin{equation}
F_{\pm,h}^{(0)}(x, \zeta) = \eta(\zeta)V_{h}(x) - \frac{1}{2}V_{h}^2(x), \quad F_{\pm,h}^{(1)}(x, \zeta) = \frac{1}{2} \nabla \Phi_{\pm,h}^{(1)}(x, \zeta),
\end{equation}

and for $n \geq 2$

\begin{equation}
F_{\pm,h}^{(n)}(x, \zeta) = \sum_{k=1}^{n-1} \langle \nabla \Phi_{\pm,h}^{(k)}(x, \zeta), \nabla \Phi_{\pm,h}^{(n)}(x, \zeta) \rangle + \frac{1}{2} |\nabla \Phi_{\pm,h}^{(n)}(x, \zeta)|^2.
\end{equation}

Define the amplitude function $P_{\pm,h}(x, \zeta)$ as

\begin{equation}
P_{\pm,h}(x, \zeta) = (I - S_{\pm,h}(x, \zeta))^{-1}p_0(\zeta), \quad x \in \Xi_\pm(\nu),
\end{equation}

where $p_0(\zeta) = p_{+,0}(\zeta)$ and

\begin{equation}
S_{\pm,h}(x, \zeta) = (2\eta(\zeta))^{-1} \left( V_{h}(x) + \sum_{k=1}^{3} \partial_h \Phi_{\pm,h}(x, \zeta) \alpha_k \right), \quad x \in \Xi_\pm(\nu).
\end{equation}

Note that, all estimates are uniform in $\zeta$ through out the paper for $0 < c_1 \leq \zeta \leq c_2 < \infty$. By the definitions of the phase function $\Phi_{\pm,h}(x, \zeta)$ and the amplitude function $P_{\pm,h}(x, \zeta)$ we have for all multi-indices $\alpha$ and $\gamma$

\begin{equation}
|\partial_x^\alpha \partial_{\zeta}^\gamma \Phi_{\pm,h}(x, \zeta)| \leq c_{\alpha,\gamma} |x|^{1-|\alpha|}, \quad x \in \Xi_\pm(\nu),
\end{equation}
and
\[
|\partial^2_x \partial^2_\zeta \mathcal{P}_{\pm,h}(x, \zeta)| \leq c_{\alpha, \gamma} \langle x \rangle^{-\rho - |\alpha|}, \quad x \in \Xi_{\pm}(\nu).
\]

Hence, the approximation
\[
u \pm \mathcal{P}_{\pm,h}(x, \zeta) = e^{ix \pm \gamma(x, \zeta)} \mathcal{P}_{\pm,h}(x, \zeta) := e^{ix \pm \Phi_{\pm,h}(x, \zeta)} \mathcal{P}_{\pm,h}(x, \zeta)
\]
of the corresponding eigenvalue problem of (8) satisfies, as $|x| \to \infty$,
\[
(H_h - \eta(\zeta)) \nu \pm \mathcal{P}_{\pm,h}(x, \zeta) = \Theta(|x|^{-1 - \varepsilon}), \quad \varepsilon = (N + 1)\rho - 1 > 0.
\]
Consider the identification $\mathfrak{g}_{\pm,h}$ given by the following PSDO
\[
(\mathfrak{g}_{\pm,h} \hat{g})(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix \cdot \zeta + i\Phi_{\pm,h}(x, \zeta)} \mathcal{P}_{\pm,h}(x, \zeta) \mathcal{G}_{\pm}(x, \zeta) \psi(|\zeta|^2) \hat{g}(\zeta) \, d\zeta,
\]
where $\psi \in C^0_0(\mathbb{R}_+)$ is introduced to localize the identification in a compact interval of $(m, \infty)$. Consider the WOs $W_{\pm,h}$ and $W^*_{\pm,h}$ defined as
\[
W_{\pm,h} = W_{\pm}(H_h, H_0; \mathfrak{g}_{\pm,h}) = s-\lim_{t \to \pm \infty} U_h(-t) \mathfrak{g}_{\pm,h} U_0(t)
\]
and
\[
W^*_{\pm,h} = W^*_{\pm}(H_h, H_0; \mathfrak{g}_{\pm,h}) = s-\lim_{t \to \pm \infty} U_0(-t) \mathfrak{g}_{\pm,h} U_h(t) P^{(ac)}_h,
\]
where $P^{(ac)}_h$ is the orthogonal projection onto the absolutely continuous subspace of $H_h$, $U_0(t) = e^{-iH_0 t}$, and $U_h(t) = e^{-iH_h t}$. Note that the free Dirac operator has only absolutely continuous spectrum, so there is no need to write the corresponding orthogonal projection onto the absolutely continuous subspace in the definition of the WO $W_{\pm,h}$. For the existence and completeness of the WO $W_{\pm,h}$ we have the following theorem.

**Theorem 1.** Let $V_h$ satisfy (14), and let $\mathfrak{g}_{\pm,h}$ be as defined in (30) where the functions $\Phi_{\pm,h}(x, \zeta)$ and $\mathcal{P}_{\pm,h}(x, \zeta)$ are given by (19) and (21) respectively. Then the WOs $W_{\pm}(H_h, H_0; \mathfrak{g}_{\pm,h})$ and $W_{\pm}(H_0, H_h; \mathfrak{g}_{\pm,h}^*)$ exist for all $h > 0$ and $W^*_{\pm}(H_h, H_0; \mathfrak{g}_{\pm,h}) = W_{\pm}(H_0, H_h; \mathfrak{g}_{\pm,h}^*)$. Moreover if $\triangle \subset (m, \infty)$ is a compact interval and $\psi(\mu^2 - m^2) = 1$ for all $\mu \in \triangle$, then the WO $W_{\pm,h}(H_h, H_0; \mathfrak{g}_{\pm,h})$, for all $h > 0$, is isometric on the subspace $E_0(\triangle) \mathcal{H}$ and is complete.

**Proof.** See [4].

**Remark 2.** It is worth to mention that the WOs defined above are for the positive part of the absolutely continuous spectrum, $(m, \infty)$. For the negative part of the absolutely continuous spectrum, $(-\infty, -m)$, the WOs operator can be defined in the same way as above but $\eta(\zeta)$ in the definition of $\Phi_{\pm,h}(x, \zeta)$ is replaced by $-\eta(\zeta)$, and $\eta(\zeta)$ and $p_0(\zeta') = p_{+0}(\zeta)$ in the definition of $\mathcal{P}_{\pm,h}(x, \zeta)$ are replaced respectively..
by $-\eta(\zeta)$ and $p_0(\zeta) = p_{-0}(\zeta)$. Consequently, and therefore, the corresponding
identification $\mathcal{J}_{\pm,h}$ is given by the following PSDO
\begin{equation}
(\mathcal{J}_{\pm,h},g)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i x \cdot \zeta + i \Phi_{\pm,h}(x,\zeta)} \mathcal{P}_{\pm,h}(x,\zeta) \mathcal{E}_{\pm}(x,\zeta) \psi(|\zeta|^2) \hat{g}(\zeta) \, d\zeta.
\end{equation}

In this work, we consider the WOs corresponding to the positive part of the absolutely continuous spectrum. However, the asymptotic study below can be carried out to the WOs on the negative part of the absolutely continuous spectrum in a similar way.

3. STRONG CONVERGENCE OF THE WOS

We study the existence of the WOs $W^\dagger_{\pm} := s\lim_{h \to \infty} W_{\pm,h}$ and $W^{*\dagger}_{\pm} := s\lim_{h \to \infty} W^*_{\pm,h}$ given the existence of the WOs $W_{\pm,h}$ and $W^*_{\pm,h}$. We refer to [1] for comprehensive materials on the asymptotic study of WOs.

Let the perturbed Dirac operator $H_h$ be convergent in the SRS, and let the identification $\mathcal{J}_{\pm,h}$ given by (33) be strongly convergent, then the WOs $W^\dagger_{\pm}$ and $W^{*\dagger}_{\pm}$ exist, i.e., the strong limits as $h \to \infty$ exist for both WOs $W_{\pm,h}$ and $W^*_{\pm,h}$. The question now is about characterizing $W^\dagger_{\pm}$ and $W^{*\dagger}_{\pm}$, in other words, characterizing the strong limits of the WOs $W_{\pm,h}$ and $W^*_{\pm,h}$ as $h \to \infty$.

**Theorem 2.** Let the WOs $W_{\pm,h}$ and $W^*_{\pm,h}$ be defined by (31) and (32) respectively, where the identification $\mathcal{J}_{\pm,h}$, the amplitude $\mathcal{P}_{\pm,h}$, and the phase $\Phi_{\pm,h}$ are given respectively by (31), (21), and (19). Suppose that, as $h \to \infty$, the Dirac operator $H_h$ converges to $H_{\infty}$ in the SRS, and $\mathcal{J}_{\pm,h}$ converges strongly to $\mathcal{J}_{\pm,\infty}$. Then the WOs $W^\dagger_{\pm}$ and $W^{*\dagger}_{\pm}$ exist,

\begin{equation}
W^\dagger_{\pm} = s\lim_{h \to \infty} W_{\pm}(H_h, H_0; \mathcal{J}_{\pm,h}) = W_{\pm}(H_{\infty}, H_0; \mathcal{J}_{\pm,\infty}),
\end{equation}

and

\begin{equation}
W^{*\dagger}_{\pm} = s\lim_{h \to \infty} W_{\pm}(H_0, H_h; \mathcal{J}^*_{\pm,h}) = W_{\pm}(H_0, H_{\infty}; \mathcal{J}^*_{\pm,\infty}).
\end{equation}

The proof follows Theorem 2.1 in [1], and is divided into several steps given by the following lemmas, corollaries, and discussion. Firstly, by (4),

\begin{equation}
H_h \mathcal{J}_{\pm,h} - \mathcal{J}_{\pm,h} H_0 = \sum_{j=1}^{3} T_j^* B_{1,h} T_j + \langle x \rangle^{-(1+\rho)/2} B_{2,h} \langle x \rangle^{-(1+\rho)/2},
\end{equation}

where $T_j = \langle x \rangle^{-1/2} \mathcal{N}_j$, $\mathcal{N}_j v(x) = \partial_j v(x) - |x|^2 x_j \sum_{k=1}^{3} x_k \partial_k v(x)$, $B_{1,h}$ and $B_{2,h}$ are bounded operators. Note that for all $h > 0$, $\langle x \rangle^{-(1+\rho)/2}$ for all $\rho > 0$ and $T_j$ for $j=1,2,3$, are $H_0$-smooth and $H_h$-smooth on any compact set $\Delta \subset (-\infty, -m) \cup (m, \infty)$ such that $\Delta \cap \sigma_p(H_h) = \emptyset$. The $H_0$-smoothness and $H_h$-smoothness of $\langle x \rangle^{-(1+\rho)/2}$ and
Since the operators in (41) are well-defined and bounded, thus for all $T$ of (38) and (43), $L$ belongs to $H$ as (35)

$$H_h \mathcal{D}_{\pm,h} - \mathcal{D}_{\pm,h} H_0 = \sum_{i=1}^{2} K_{2,i,h}^* K_{1,i,h}.$$

The operators $K_{2,1,h}$ and $K_{2,2,h}$ are $H_h$-smooth for all $h > 0$, and $K_{1,1,h}$ and $K_{1,2,h}$ are $H_0$-smooth for all $h > 0$. Without loss of generality, we assume that

$$H_h \mathcal{D}_{\pm,h} - \mathcal{D}_{\pm,h} H_0 = G_h^* G_{0,h}$$

such that $G_h$ and $G_{0,h}$ are $H_h$-smooth and $H_0$-smooth respectively for all $h > 0$.

**Lemma 1.** For all $h > 0$ and for all $u_0 \in X$ the function

$$\mathcal{K}_{u_0,h}^{(1)}(t) = \| (H_h - z) \phi (H_h) \mathcal{D}_{\pm,h} \phi (H_0) - \phi (H_h) \mathcal{D}_{\pm,h} H_0 \phi (H_0) \|_Y$$

belongs to $L^1 ((-\infty, \infty); dt)$ for some continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that $x \phi (x)$ is bounded on $\mathbb{R}$.

**Proof.** Let $\phi (x) = (x-z)^{-1}$, $z \in \text{Res}(H_h) \cap \text{Res}(H_0)$ where Res denotes the resolvent set. Therefore, and since

$$\mathcal{K}_{u_0,h}^{(1)}(t) = \| (H_h - z) \phi (H_h) \mathcal{D}_{\pm,h} \phi (H_0) - \phi (H_h) \mathcal{D}_{\pm,h} H_0 \phi (H_0) \|_Y,$$

to prove the assertion of the lemma, it is enough to prove that

$$\mathcal{K}_{u_0,h}^{(1)}(t) = \| (\phi (H_h) \mathcal{D}_{\pm,h} - \mathcal{D}_{\pm,h} \phi (H_0)) U_0 (t) u_0 \|_Y$$

belongs to $L^1 ((-\infty, \infty); dt)$. To this end, for all $u_0, u \in X$ we have

$$\langle (\mathcal{D}_{\pm,h} H_0 u_0, H_h u) - \langle \mathcal{D}_{\pm,h} H_0 u_0, u \rangle = \langle G_{0,h} u_0, G_h u \rangle.$$

By (40) we have for any $v_0, v \in Y$

$$\langle G_{0,h} R_0 (z) v_0, G_h R_h (\overline{z}) v \rangle = \langle \mathcal{D}_{\pm,h} R_0 (z) v_0, H_h R_h (\overline{z}) v \rangle - \langle \mathcal{D}_{\pm,h} H_0 R_0 (z) v_0, R_h (\overline{z}) v \rangle$$

$$= \langle \mathcal{D}_{\pm,h} R_0 (z) v_0, v \rangle + \langle \mathcal{D}_{\pm,h} H_0 R_0 (z) v_0, \overline{R_h (\overline{z})} v \rangle - \langle \mathcal{D}_{\pm,h} R_0 (z) v_0, \overline{R_h (\overline{z})} v \rangle.$$

Since $G_h$ and $G_{0,h}$ are $H_h$-bounded and $H_0$-bounded respectively for all $h > 0$, all operators in (41) are well-defined and bounded, thus for all $v_0, v \in Y$,

$$\langle (\mathcal{D}_{\pm,h} R_0 (z) - R_h (z) \mathcal{D}_{\pm,h}) v_0, v \rangle = \langle (G_h R_h (\overline{z}))^* G_{0,h} R_0 (z) v_0, v \rangle.$$

Hence

$$\mathcal{D}_{\pm,h} R_0 (z) - R_h (z) \mathcal{D}_{\pm,h} = (G_h R_h (\overline{z}))^* G_{0,h} R_0 (z).$$
Therefore, we have for all $w_0 \in X$ and for all $h > 0$,
\[
\int_0^{\pm \infty} \| R_h(z) \mathcal{J}_{\pm, h} - \mathcal{J}_{\pm, h} R_0(z) U_0(t) w_0 \|_Y dt = \int_0^{\pm \infty} \| (G_h R_h(z))^* G_{0,h} R_0(z) U_0(t) w_0 \|_Y dt 
\leq C \int_0^{\pm \infty} \| G_{0,h} U_0(t) w_0 \|_Y dt 
< \infty
\]
where $u_0 = R_0(z) w_0$ and $C \in \mathbb{R}$. Here we have used the $H_h$-boundedness of $G_h$ and in the last inequality the $H_0$-smoothness of $G_{0,h}$ for all $h > 0$.

The following corollary is a direct consequence of Lemma 1.

**Corollary 1.** Given the hypotheses of Lemma 1, then for any $\varepsilon > 0$ there exist $D_1, D_2 \in \mathbb{R}$ such that $\int_{D_1}^{D_2} \mathcal{K}_{\pm, h}^{(1)}(t) dt < \varepsilon$ and $\int_{-\infty}^{D_1} \mathcal{K}_{\pm, h}^{(1)}(t) dt < \varepsilon$.

**Lemma 2.** For all $h > 0$ and for all $u \in X$ the function
\[
\mathcal{K}_{a, h}^{(2)}(t) = \| (H_0 \phi(H_0) \mathcal{J}_{\pm, h}^* \phi(H_h) - \phi(H_0) \mathcal{J}_{\pm, h} H_h \phi(H_h)) U_0(t) u \|_Y
\]
belongs to $L^1([\infty, \infty); dt)$ for some continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that $x \phi(x)$ is bounded on $\mathbb{R}$.

**Proof.** The proof is similar to that of Lemma 1.

The following corollary is a direct consequence of Lemma 2.

**Corollary 2.** Given the hypotheses of Lemma 2, then for any $\varepsilon > 0$ there exist $D_3, D_4 \in \mathbb{R}$ such that $\int_{D_3}^{D_4} \mathcal{K}_{a, h}^{(2)}(t) dt < \varepsilon$ and $\int_{-\infty}^{D_3} \mathcal{K}_{a, h}^{(2)}(t) dt < \varepsilon$.

Since $H_h$ is convergent to $H_\infty$ in the SRS, then for any continuous bounded function $F$ on $\mathbb{R}$, $F(H_h)$ is strongly convergent to $F(H_\infty)$, thus $e^{-iH_h t}$ is strongly convergent to $e^{-iH_\infty t}$ for all $t \in \mathbb{R}$. Moreover, since $\mathcal{J}_{\pm, h}$ is strongly convergent to $\mathcal{J}_{\pm, \infty}$, then $W_{\pm}^f$ and $W_{\pm}^{f, *}$ exist, and the characterization of the asymptotic limits of $W_{\pm, h}$ and $W^{*, *}_{\pm, h}$ is reduced to the problem of interchanging $s-lim$ and $s-lim$. In this context, the following theorem is important, it is an adaptation of a result achieved in [1].

**Theorem 3.** Given self-adjoint operators $T_h$ and $T_0$ in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_0$ respectively, let the WOs $W_{\pm}(T_h, T_0; J_h)$ and $W_{\pm}(T_0, T_h; J_0^*)$ exist, where $J_h$ is some bounded identification. Assume that, as $h \to \infty$, the operator $T_h$ is convergent in the SRS to $T_\infty$ and that $J_h$ and $J_0^*$ converge strongly to $J_\infty$ and $J_\infty^*$ respectively. If, for $T_h$, $T_0$, $\mathcal{H}$, $\mathcal{H}_0$ and $J_h$, the functions $\mathcal{K}_{\pm, h}^{(1)}(t)$ and $\mathcal{K}_{a, h}^{(2)}(t)$, given respectively by (37) and (44), satisfy the conclusions of Corollaries 1 and 2 respectively, then
\[
s-lim_{h \to \infty} W_{\pm}(T_h, T_0; J_h) = W_{\pm}(T_\infty, T_0; J_\infty)
\]
Theorem 3. Fortunately, the identification operator $J$ with the adjoint operator $J^*$ (46) the strong convergence of $J$ convergence of its adjoint to the adjoint of its strong limit, therefore we have assumed and 

$$s-lim_{h \to \infty} W_\pm(T_0, T_h; J^*_h) = W_\pm(T_0, T_\infty; J^*_\infty).$$

Proof. See Theorem 2.1 and Remark 2.3 in [1].

Since, in general, the strong convergence of an operator does not imply the strong convergence of its adjoint to the adjoint of its strong limit, therefore we have assumed the strong convergence of $J^*_h$ to $J^*_\infty$ parallel to the strong convergence of $J_h$ to $J_\infty$ in Theorem 3. Fortunately, the identification operator $\tilde{J}_{\pm,h}$ defined by (30) is a PSDO with the adjoint operator $\tilde{J}_{\pm,h}^*$ given by

$$\tilde{J}_{\pm,h}^*(g)(\zeta) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot\zeta} \Phi_{\pm,h}(x, \zeta) p_{\pm,h}(x, \zeta) \epsilon_{\pm}(x, \zeta) \psi(|\zeta|^2) g(x) \, dx.$$ 

This implies that if $\tilde{J}_{\pm,h}$ is strongly convergent to $\tilde{J}_{\pm,\infty}$ as $h \to \infty$, then $\tilde{J}_{\pm,h}^*$ is also strongly convergent to $\tilde{J}_{\pm,\infty}^*$.

The proof of Theorem 2 follows from Lemmas 1-2, Corollaries 1-2, and Theorem 3. Thus, we may conclude that the strong convergence as $h \to \infty$ of $W_{\pm,h} = W_{\pm}(H_h, H_0; J_{\pm,h})$ and its adjoint $W_{\pm,h}^* = W_{\pm}(H_0, H_h; J_{\pm,h}^*)$ to the WOs $W_{\pm,\infty} = W_{\pm}(H_\infty, H_0; J_{\pm,\infty})$ and $W_{\pm,\infty}^* = W_{\pm}(H_0, H_\infty; J_{\pm,\infty}^*)$ respectively is guaranteed if $H_h$ is convergent to $H_\infty$ in the SRS and $\tilde{J}_{\pm,h}$ is strongly convergent to $\tilde{J}_{\pm,\infty}$.

In the coming discussion we assume convergence in the SRS of $H_h$ to $H_\infty$, and study cases of the identification $\tilde{J}_{\pm,h}$. In the second case we assume particular condition so that $\Phi_{\pm,h}$ is replaced with some $h$-free functions in the definition of $\tilde{J}_{\pm,h}$ to obtain new equivalent identification $\tilde{J}_{\pm,h}$. Even that this condition is stringent, but this replacement is advantageous if we can also replace the amplitude functions $p_{\pm,h}$ of the identification $\tilde{J}_{\pm,h}$ with $h$-free function $\tilde{p}_{\pm}$ so that the difference $(\tilde{J}_{\pm,h}(p_{\pm,h}) - \tilde{J}_{\pm}(\tilde{p}_{\pm}))$ is compact for all $h > 0$, and then applying Remark 1. In this case the study of the asymptotic behavior of the WO $W_{\pm}(H_h, H_0; J_{\pm,h}(p_{\pm,h}))$ is reduced to study the asymptotic behavior of the WO $W_{\pm}(H_h, H_0; J_{\pm}(\tilde{p}_{\pm}))$. Thus no convergence conditions are needed on the identification operator in finding the asymptotic limits of the WOs.

3.0.1. The case $\rho > 1$. Note that here we consider short-range potentials, while our main assumption, (11), assumes potentials of long-range type. However, for $\rho > 1$, we can set $J_{\pm,h} = I$, this is because for short-range potentials, the WOs $W_{\pm}(H_h, H_0)$ and $W_\pm(H_0, H_h)$ exist and are complete. The proofs of existence and completeness of the WOs for the Dirac operator with short-range potential are similar to that for the Schrödinger operator. Hence, the strong convergence of the WOs $W_{\pm}(H_h, H_0)$ and $W_{\pm}(H_0, H_h)$, as $h \to \infty$, is reduced to the convergence of the Dirac operator.
$H_h$ in the SRS. Therefore, by assuming the convergence in the SRS of $H_h$ to $H_\infty$, we have

\begin{equation}
W_\pm = W_\pm(H_\infty, H_0)
\end{equation}

and

\begin{equation}
W_\pm^* = W_\pm(H_0, H_\infty).
\end{equation}

3.0.2. The case $\rho = 1$. In this case, one can replace the $h$-dependent phase function $\Phi(x, \xi)$ by a $h$-free function by virtue of Remark 1 and Proposition 1 as follows.

**Theorem 4.** Let $\Phi(x, \xi)$ be an $h$-free function satisfying estimate (26), and let the identification $\mathcal{J}_{\pm,h}$ be given by (49) but where $\Phi_{\pm,h}(x, \xi)$ is replaced by $\Phi_{\pm}(x, \xi)$. Then for $\rho > 0$ such that $\rho/(1 - \rho) > |\gamma|/(1 + |\alpha|)$, the operator $\mathcal{J}_{\pm,h}^{(1)} - \mathcal{J}_{\pm,h}$ is compact.

**Proof.** By the estimate (27) the function $\mathcal{P}_{\pm,h}$ belongs to the class $S_{-d}(\mathbb{R}^3, \mathbb{R}^3)$ which is a subset of $S_{-d,\delta}(\mathbb{R}^3, \mathbb{R}^3)$ for $\delta = 1 - \rho$. Since both $\Phi_{\pm,h}(x, \xi)$ and $\Phi_{\pm}(x, \xi)$ satisfy the estimate (26), for $x \in \Xi(\nu)$,

\begin{equation}
|\partial_{x}^{\alpha,\gamma} e^{i\Phi_{\pm,h}(x, \xi)}| \leq c_{\alpha,\gamma} (1 + |\alpha|)^{\delta}|\gamma|
\end{equation}

and

\begin{equation}
|\partial_{x}^{\alpha,\gamma} e^{i\Phi_{\pm}(x, \xi)}| \leq c_{\alpha,\gamma} (1 + |\alpha|)^{\delta}|\gamma|.
\end{equation}

Thus $e^{i\Phi_{\pm,h}\mathcal{P}_{\pm,h}}$ and $e^{i\Phi_{\pm}\mathcal{P}_{\pm,h}}$ are elements of $S_{-d,\delta}(\mathbb{R}^3, \mathbb{R}^3)$, consequently the difference is also an element of $S_{-d,\delta}(\mathbb{R}^3, \mathbb{R}^3)$. By Proposition 1, the operator $\mathcal{J}_{\pm,h}^{(1)} - \mathcal{J}_{\pm,h}$ is compact if its symbol belongs to $S_{0,0}(\mathbb{R}^3, \mathbb{R}^3)$ for $r < 0$. This is achieved if

$-\rho - |\alpha| + \delta|\gamma| < 0$

where again $\delta = 1 - \rho$, i.e., $\mathcal{J}_{\pm,h}^{(1)} - \mathcal{J}_{\pm,h}$ is compact if $\rho/(1 - \rho) > |\gamma|/(1 + |\alpha|)$.

According to Theorem 4, it is noted that if $\rho = 1$ (also if $\rho \to 1^-$), then $\rho/(1 - \rho) > |\gamma|/(1 + |\alpha|)$ is satisfied for all multi-indices $\alpha$ and $\gamma$.

Regarding the amplitude function, we shall need the following proposition which is due to Gâtel and Yafaev [4]. This proposition is important in the convergence results in the sense that it replaces the $h$-dependent amplitude function, $\mathcal{P}_{\pm,h}(x, \xi)$, with another $h$-free function.

**Proposition 3.** For the identification

\begin{equation}
(\mathcal{J}_{\pm,h}g)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix\cdot\zeta + i\Phi_{\pm,h}(x, \xi)} p_0(\zeta) \mathcal{G}_{\pm}(x, \xi) \psi(|\zeta|^2) \hat{g}(\zeta) d\zeta
\end{equation}

the difference $\mathcal{J}_{\pm,h} - \mathcal{J}_{\pm,h}$ is a compact PSDO.
and then use the inequalities easier to show first satisfies estimate (26). The proof is just a simple elementar y calculus, where it is clear that for all $h > 0$, $\left| \partial_\zeta^\rho \partial_\zeta^\gamma (p_0(\zeta) - p_{\pm,h}(x, \zeta)) \right| \leq C_{\alpha, \gamma}(x)^{1-\rho-|\alpha|}$, thus the symbol of $J_{\pm,h} - J_{\pm,h}$ belongs to $C^{-\rho}(\Phi_{\pm,h})$ for all $h > 0$. By Proposition 2, $J_{\pm,h} - J_{\pm,h}$ is compact.

According to Remark 1, the WOs $W_{\pm}(H_h, H_0; J_{\pm,h})$ and $W_{\pm}(H_0, H_h; J^*_{\pm,h})$ exist, $W_{\pm}(H_h, H_0; J_{\pm,h}) = W_{\pm}(H_h, H_0; J^*_{\pm,h})$, and $W_{\pm}(H_0, H_h; J^*_{\pm,h}) = W_{\pm}(H_0, H_h; J^*_{\pm,h})$. To this end, if $\rho = 1$ (or $\rho \to 1^-$), we have after applying Remark 1 and Proposition 3

\begin{equation}
W_{\pm}^\dagger = s - \lim_{h \to \infty} W_{\pm}(H_h, H_0; J_{\pm,h}) = W_{\pm}(H_\infty, H_0; J_{\pm,h}^{(2)})
\end{equation}

and

\begin{equation}
W_{\pm}^{\dagger,*} = s - \lim_{h \to \infty} W_{\pm}(H_0, H_h; J^*_{\pm,h}) = W_{\pm}(H_0, H_\infty; J_{\pm,h}^{(2),*}),
\end{equation}

where $J_{\pm}^{(2)}$ (with adjoint operator denoted by $J_{\pm}^{(2),*}$) is given by

\begin{equation}
(J_{\pm}^{(2)} g)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix \cdot \zeta + i\Phi_{\pm}(x, \zeta)} p_0(\zeta) \mathcal{C}_{\pm}(x, \zeta) \psi(|\zeta|^2) \hat{g}(\zeta) \, d\zeta.
\end{equation}

**Example 1.** The function

\begin{equation}
\Phi_{\pm}(x, \zeta) = \pm \eta(\zeta) \int_0^\infty \left( \langle x \pm t\zeta \rangle - \langle \pm t\zeta \rangle \right) \, dt
\end{equation}

does not satisfy estimate (26). The proof is just a simple elementary calculus, where it is easier to show first

\begin{equation}
\left| \partial_\zeta^\rho \partial_\zeta^\gamma \Phi_{\pm}(x, \zeta) \right| \leq C_{\alpha, \gamma}(1 + |x|)^{1-\rho-|\alpha|}, \quad x \in \Xi_{\pm}(\nu),
\end{equation}

and then use the inequalities

\begin{equation}
\langle x \rangle \leq (1 + |x|) \leq \sqrt{2}(x).
\end{equation}

3.0.3. The case $\rho \in (1/2, 1)$. According to the definition of the phase function $\Phi_{\pm,h}(x, \zeta)$ given by (19), if $\rho \in (1/2, 1)$ then $N = 1$ satisfies the condition $(N + 1)\rho > 1$. In this case $\Phi_{\pm,h}(x, \zeta)$ can be chosen as, after neglecting the quadratic terms,

\begin{equation}
\Phi_{\pm,h}(x, \zeta) = \pm \eta(\zeta) \int_0^\infty (V_h(x \pm t\zeta) - V_h(\pm t\zeta)) \, dt.
\end{equation}

By Proposition 3, to study the strong convergence of $W_{\pm,h}$ with the identification $J_{\pm,h}$ (respectively $W^*_{\pm,h}$ with $J^*_{\pm,h}$) is equivalent to study its strong convergence with $J_{\pm,h}^{(3)}$ (respectively with $J_{\pm,h}^{(3),*}$, where $J_{\pm,h}^{(3),*}$ is the adjoint operator of $J_{\pm,h}^{(3)}$),

\begin{equation}
(J_{\pm,h}^{(3)} g)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix \cdot \zeta + i\Phi_{\pm,h}(x, \zeta)} p_0(\zeta) \mathcal{C}_{\pm}(x, \zeta) \psi(|\zeta|^2) \hat{g}(\zeta) \, d\zeta,
\end{equation}
where \( \Phi_{\pm,h} \) is given by (60). By dominated convergence theorem, and since the integrand in (61) is bounded, then if \( \Phi_{\pm,h}(x, \zeta) \), given by (60), converges to \( \Phi_{\pm,\infty}(x, \zeta) \) in the SRS, then the identification \( J^{(3)}_{\pm,h} \) is strongly convergent to \( J^{(3)}_{\pm,\infty} \), where

\[
(J^{(3)}_{\pm,\infty})_g(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix \cdot \zeta + i\Phi_{\pm,\infty}(x, \zeta)p_0(\zeta)} \mathcal{E}_{\pm}(x, \zeta)\psi(\zeta^2)\hat{g}(\zeta) \, d\zeta.
\]

Therefore, for those types of potentials \( V_h \) that satisfy (11) with \( \rho \in (1/2, 1) \) and such that \( H_h = H_0 + V_h \) and \( \Phi_{\pm,h}(x, \zeta) \) are convergent in the SRS respectively to \( H_\infty = H_0 + V_\infty \) and

\[
\Phi_{\pm,\infty}(x, \zeta) = \pm \eta(\zeta) \int_0^\infty (V_\infty(x \pm t\zeta) - V_\infty(\pm t\zeta)) \, dt,
\]

we have

\[
W^{\dagger}_{\pm} = s - \lim_{h \to \infty} W_{\pm}(H_h, H_0; J_{\pm,h}) = W_{\pm}(H_\infty, H_0; J^{(3)}_{\pm,\infty})
\]

and

\[
W^{\dagger,\ast}_{\pm} = s - \lim_{h \to \infty} W_{\pm}(H_0, H_h; J_{\pm,h}^\ast) = W_{\pm}(H_0, H_\infty; J^{(3),\ast}_{\pm,\infty}).
\]

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