Harnack Inequalities on Manifolds with Boundary and Applications

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November 1, 2009

Abstract

On a large class of Riemannian manifolds with boundary, some dimension-free Harnack inequalities for the Neumann semigroup is proved to be equivalent to the convexity of the boundary and a curvature condition. In particular, for \( p_t(x, y) \) the Neumann heat kernel w.r.t. a volume type measure \( \mu \) and for \( K \) a constant, the curvature condition \( \text{Ric} - \nabla Z \geq K \) together with the convexity of the boundary is equivalent to the heat kernel entropy inequality

\[
\int_M p_t(x, z) \log \frac{p_t(x, z)}{p_t(y, z)} \mu(dz) \leq \frac{K \rho(x, y)^2}{2(e^{2Kt} - 1)}, \quad t > 0, x, y \in M,
\]

where \( \rho \) is the Riemannian distance. The main result is partly extended to manifolds with non-convex boundary and applied to derive the HWI inequality.

AMS subject Classification: 60J60, 58G32.
Keywords: Curvature, Harnack inequality, heat kernel, second fundamental form.

1 Introduction

Let \( M \) be a connected complete Riemannian manifold possibly with a boundary \( \partial M \). Let \( L = \Delta + Z \) for a \( C^2 \) vector field \( Z \) on \( M \). Let \( P_t \) be the (Neumann if \( \partial M \neq \emptyset \)) diffusion

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*Supported in part by WIMCS, NNSFC(10721091) and the 973-Project.
semigroup generated by $L$. Then for any measure $\mu$ equivalent to the Riemannian volume, $P_t$ has a heat kernel $\{p_t(x,y) : x, y \in M\}$ with respect to $\mu$, i.e.

$$P_t f(x) = \int_M p_t(x,y) f(y) \mu(dy)$$

holds for any bounded measurable function $f$. When $\partial M = \emptyset$, there exist many equivalent statements on the semigroup $P_t$ for the following curvature condition (known as the $\Gamma_2$ condition of Bakry and Emery [2]):

$$(1.1) \quad \text{Ric}(X,X) - \langle \nabla_X Z, X \rangle \geq -K|X|^2, \quad X \in TM,$$

where $K \in \mathbb{R}$ is a constant. See e.g. [1, 3] for equivalent gradient and Poincaré/log-Sobolev inequalities, [13] for equivalent cost (or Wasserstein distance) inequalities, and [14] for equivalent dimension-free Harnack inequalities. These equivalences also hold if $M$ has a convex boundary (cf. [14]). The main purpose of this paper is to provide equivalent heat kernel inequalities for (1.1) and the convexity of $\partial M$. To this end we first recall two known Harnack type inequalities for $P_t$.

According to [13, Lemma 2.2], if $\partial M$ is either empty or convex, then (1.1) implies the Harnack inequality

$$(1.2) \quad \frac{(P_t f(x))^\alpha}{P_t f^{\alpha} f(y)} \leq \exp \left[ \frac{K \alpha \rho(x,y)^2}{2(\alpha - 1)(1 - e^{-2Kt})} \right], \quad f \in \mathcal{M}_b^+(M), \ t > 0, x, y \in M$$

for all $\alpha > 1$, where $\mathcal{M}_b^+(M)$ is the set of all positive measurable functions on $M$, and $\rho$ is the Riemannian distance on $M$. It is also proved in [14] that, if (1.2) holds for all $\alpha > 1$ then (1.1) holds. In this paper we shall prove that (1.2) is equivalent to (1.1) for each fixed $\alpha > 1$.

Next, when $\partial M$ is either empty or convex, we prove that (1.1) is also equivalent to the following log-Harnack inequality, a limit version of (1.2) as $\alpha \to \infty$ (see Section 2):

$$(1.3) \quad P_t(\log f)(x) \leq \log P_t f(y) + \frac{K \rho(x,y)^2}{2(1 - e^{-2Kt})}, \quad f \geq 1, t > 0, x, y \in M.$$

Note that this type inequality was used in [4] for the study of HWI inequalities on manifolds without boundary. In conclusion we have the following result.

**Theorem 1.1.** Assume that $\partial M$ is either empty or convex. Let $K \in \mathbb{R}$. Then the following statements are equivalent to each other:

(1) $\text{Ric}(X,X) - \langle \nabla_X Z, X \rangle \geq -K|X|^2, \quad X \in TM.$

(2) The Harnack inequality (1.2) holds for all $\alpha > 1$. 

2
The Harnack inequality (1.2) holds for some \( \alpha > 1 \).

The log-Harnack inequality (1.3) holds.

For any \( \alpha > 1 \),

\[
(1.4) \quad \int_M p_t(x, z) \left( \frac{p_t(x, z)}{p_t(y, z)} \right)^{-\frac{1}{\alpha}} \mu(dz) \leq \exp \left[ \frac{K\alpha \rho(x, y)^2}{2(\alpha - 1)^2(1 - e^{-2Kt})} \right],
\]

\( t > 0, x, y \in M \).

There exists \( \alpha > 1 \) such that (1.4) holds.

The following entropy inequality holds:

\[
(1.5) \quad \int_M p_t(x, z) \log \frac{p_t(x, z)}{p_t(y, z)} \mu(dz) \leq \frac{K\rho(x, y)^2}{2(1 - e^{-2Kt})}, \quad t > 0, x, y \in M.
\]

To see that the assumption on the boundary is essential, we intend to prove that when \( \partial M \) is non-empty, each of (1.2), (1.3), (1.4) and (1.5) implies the convexity of \( \partial M \). Due to technical reasons for estimates on local times, we assume that \( L\rho_\partial \) is bounded for small \( \rho_\partial \), where \( \rho_\partial \) is the Riemannian distance to \( \partial M \). This assumption is trivial when the manifold is compact. Moreover, by Kasue’s comparison theorems [7], this assumption follows if there exists \( r_0 > 0 \) such that \( \langle Z, \nabla \rho_\partial \rangle \) is bounded on the set \( \{ \rho_\partial \leq r_0 \} \), \( \partial M \) has a bounded second fundamental form and a strictly positive injectivity radius, the sectional curvature of \( M \) is bounded above, and the Ricci curvature of \( M \) is bounded below (see e.g. [15, 16] for details).

**Theorem 1.2.** Let \( M \) have a boundary \( \partial M \) such that for some constant \( r_0 > 0 \) the function \( \rho_\partial \) is smooth with bounded \( L\rho_\partial \) on the set \( \{ \rho_\partial \leq r_0 \} \). Then (1.3) implies that \( \partial M \) is convex. Consequently, each of statements (2)-(7) in Theorem 1.1 is equivalent to

(8) \( \partial M \) is convex and (1.1) holds.

Obviously, Theorem 1.2 implies the assertions claimed in Abstract. We remark that a formula for the second fundamental form was presented in a recent work [17] for compact manifolds with boundary by using the gradient estimate due to Hsu [5]. As a consequence, the manifold is convex if and only if the gradient estimate

\[
|\nabla P_tf|^p \leq e^{Kt}Pt|\nabla f|^p, \quad t \geq 0, f \in C^1_b(M)
\]

holds for some \( p \geq 1 \) and \( K \in \mathbb{R} \). When \( \partial M \) is empty it is well known that such a gradient estimate is equivalent to the curvature condition (1.1) (see e.g. [11]), but the equivalence with the convexity of boundary was first observed in [17]. Theorem 1.2 in this paper provides more equivalent semigroup (heat kernel) properties for (1.1) and the convexity of \( \partial M \) without using gradient.
In Section 2 we shall provide in the next section some general properties for Harnack type inequalities, which are interesting by themselves. Using these properties we are able to present complete proofs for the above two theorems in Sections 3 and 4 respectively. The log-Harnack inequality is established in Section 5 for a class of non-convex manifolds. As an application, the HWI inequality is presented in Section 6. Finally, two technical points, i.e. the exponential estimates of the local time and a simple proof of Hsu’s gradient estimate on non-compact manifolds, are addressed in the Appendix.

2 Some properties of Harnack Inequalities

Let \((E, \rho)\) be a metric space, and \(P(x, dy)\) a transition probability on \(E\), which provides a contractive linear operator \(P\) on \(B_b(E)\), the set of all bounded measurable functions on \(E\):

\[
P f(x) = \int_E f(y) P(x, dy), \quad f \in B_b(E), x \in E.
\]

Let \(B^+_b(E)\) be the set of nonnegative elements in \(B_b(E)\). We shall study the following Harnack inequality with a power \(\alpha > 1\):

\[
(P f(x))^\alpha \leq (P f^\alpha(y)) \exp \left[ \frac{\alpha c \rho(x, y)^2}{\alpha - 1} \right], \quad f \in B^+_b(E), x, y \in E,
\]

where \(c > 0\) is a constant. To state our first result in this section, we shall assume that \(E\) is a length space, i.e. for any \(x \neq y\) and any \(s \in (0, 1)\), there exists a sequence \(\{z_n\} \subset E\) such that \(\rho(x, z_n) \to s \rho(x, y)\) and \(\rho(z_n, y) \to (1 - s) \rho(x, y)\) as \(n \to \infty\).

**Proposition 2.1.** Assume that \((E, \rho)\) is a length space and let \(\alpha_1, \alpha_2 > 1\) be two constants. If (2.1) holds for \(\alpha = \alpha_1, \alpha_2\), it holds also for \(\alpha = \alpha_1 \alpha_2\).

**Proof.** Let

\[
s = \frac{\alpha_1 - 1}{\alpha_1 \alpha_2 - 1}, \quad 1 - s = \frac{\alpha_1 (\alpha_2 - 1)}{\alpha_1 \alpha_2 - 1},
\]

and let \(\{z_n\} \subset E\) such that \(\rho(x, z_n) \to s \rho(x, y)\) and \(\rho(z_n, y) \to (1 - s) \rho(x, y)\) as \(n \to \infty\). Since (2.1) holds for \(\alpha = \alpha_1\) and \(\alpha = \alpha_2\), for any \(f \in B^+_b(E)\) we have

\[
(P f(x))^\alpha \leq (P f^\alpha(z_n))^\alpha \exp \left[ \frac{\alpha_1 \alpha_2 c \rho(x, z_n)^2}{\alpha_1 - 1} \right]
\]

\[
\leq (P f^\alpha(z_n)) \exp \left[ \frac{\alpha_1 \alpha_2 c \rho(x, z_n)^2}{\alpha_1 - 1} + \frac{\alpha_2 c \rho(z_n, y)^2}{\alpha_2 - 1} \right].
\]

Letting \(n \to \infty\) we arrive at
\[(Pf(x))^\alpha_1 \alpha_2 \leq (Pf^{\alpha_1 \alpha_2}(y)) \exp \left[ \frac{\alpha_1 \alpha_2 \epsilon^2}{\alpha_1 - 1} \rho(x, y)^2 + \frac{\alpha_2 (1 - s)^2}{\alpha_2 - 1} \rho(x, y)^2 \right] \]

\[= (Pf^{\alpha_1 \alpha_2}(y)) \exp \left[ \frac{\alpha_1 \alpha_2 \epsilon}{\alpha_1 \alpha_2 - 1} \rho(x, y)^2 \right].\]

\[\square\]

**Proposition 2.2.** If (2.1) holds for some \(\alpha > 1\), then

\[P(\log f)(x) \leq \log Pf(y) + c\rho(x, y)^2, \quad x, y \in E, \ f \geq 1, \ f \in B_b(E).\]

**Proof.** By Proposition 2.1, (1.5) holds for \(\alpha^n (n \in \mathbb{N})\) in place of \(\alpha\). So,

\[Pf^{\alpha - n}(x) \leq (Pf(y))^{\alpha - n} \exp \left[ \frac{c\rho(x, y)^2}{\alpha^n - 1} \right].\]

Therefore, by the dominated convergence theorem

\[P(\log f)(x) = \lim_{n \to \infty} P \left( \frac{f^{\alpha - n} - 1}{\alpha^{-n}} \right)(x) \]

\[\leq \lim_{n \to \infty} \left\{ (Pf(y))^{\alpha - n} - 1 + (Pf(y))^{\alpha - n} \exp \left[ \frac{c\rho(x, y)^2}{\alpha^n - 1} \right] - 1 \right\} \]

\[= \log Pf(y) + c\rho(x, y)^2.\]

\[\square\]

**Proposition 2.3.** Let \(\Phi\) be a positive function on \(E \times E\) such that \(\Phi(x, y) \to 0\) as \(y \to x\) holds for any \(x \in E\). Then the log-Harnack inequality

\[(2.2) \quad P(\log f)(x) \leq \log Pf(y) + \Phi(x, y), \quad x, y \in E, \ f \geq 1, \ f \in B_b(E)\]

implies the strong Feller property of \(P\), i.e. \(P B_b(E) \subset C_b(E)\).

**Proof.** It suffices to prove that \(Pf \in C_b(E)\) for \(f \in B^+_b(E)\). Applying (2.2) for \(1 + \epsilon f\) in place of \(f\), we obtain

\[Pf(y) - \epsilon \|f\|_\infty^2 \leq P \frac{\log(1 + \epsilon f)}{\epsilon}(y) \leq \frac{1}{\epsilon} \log(1 + \epsilon Pf(x)) + \frac{\Phi(x, y)}{\epsilon}, \quad \epsilon > 0, \ x, y \in E.\]

Letting first \(y \to x\) then \(\epsilon \to 0\), we arrive at

\[\limsup_{y \to x} Pf(y) \leq Pf(x).\]

5
On the other hand, we have
\[
P \log \left(1 + \varepsilon f \right) \leq \frac{\Phi(x, y)}{\varepsilon} \leq \frac{1}{\varepsilon} \log(1 + \varepsilon P f(y)) \leq P f(y).
\]
Letting first \( y \to x \) then \( \varepsilon \to 0 \), we arrive at
\[
P f(x) \leq \liminf_{y \to x} P f(y).
\]

Obviously, each of (2.1) and (2.2) implies that \( P(x, \cdot) \) and \( P(y, \cdot) \) are equivalent to each other. Indeed, if \( P(y, A) = 0 \) then applying (2.1) to \( f = 1_A \) or applying (2.2) to \( f = 1 + n1_A \) and letting \( n \to \infty \), we conclude that \( P(x, A) = 0 \). By the same reason, \( P(x, \cdot) \) and \( P(y, \cdot) \) are equivalent for any \( x, y \in E \) if

\[
(2.3) \quad (P f(x))^\alpha \leq (P f^\alpha(y)) \Psi(x, y), \quad x, y \in E, f \in B^+(E)
\]
holds for some positive function \( \Psi \) on \( E \times E \). In these cases let
\[
p_{x,y}(z) = \frac{P(x, dz)}{P(y, dz)}
\]
be the Radon-Nikodym derivative of \( P(x, \cdot) \) with respect to \( P(y, \cdot) \).

**Proposition 2.4.** Let \( \Phi, \Psi \) be positive functions on \( E \times E \).

(1) (2.3) holds if and only if \( P(x, \cdot) \) and \( P(y, \cdot) \) are equivalent and \( p_{x,y} \) satisfies

\[
(2.4) \quad P \left\{ p_{x,y}^{1/(\alpha-1)} \right\}(x) \leq \Psi(x, y)^{1/(\alpha-1)}, \quad x, y \in E.
\]

(2) (2.2) holds if and only if \( P(x, \cdot) \) and \( P(y, \cdot) \) are equivalent and \( p_{x,y} \) satisfies

\[
(2.5) \quad P \{ \log p_{x,y} \}(x) \leq \Phi(x, y), \quad x, y \in E.
\]

**Proof.** (1) Applying (2.3) to \( f_n(z) := \{ n \wedge p_{x,y}(z) \}^{1/(\alpha-1)}, \ n \geq 1 \), we obtain
\[
(P f_n(x))^\alpha \leq \Psi(x, y) P f_n(x) = \Psi(x, y) \int_E \{ n \wedge p_{x,y}(z) \}^{\alpha/(\alpha-1)} P(y, dz)
\leq \Psi(x, y) \int_E \{ n \wedge p_{x,y}(z) \}^{1/(\alpha-1)} P(x, dz) = \Psi(x, y) P f_n(x).
\]
Thus,
\[ P\{p_x^1/(\alpha -1)\}(x) = \lim_{n \to \infty} P f_n(x) \leq \Psi(x, y)^{1/(\alpha -1)}. \]

So, (2.3) implies (2.4).

On the other hand, if (2.4) holds then for any \( f \in \mathcal{B}_b^+(E) \), by the Hölder inequality

\[ Pf(x) = \int_E \{p_{x,y}(z)f(z)\} P(y, dz) \leq (Pf^\alpha(y))^{1/\alpha} \left( \int_E p_{x,y}(z)^{\alpha/(\alpha -1)} P(y, dz) \right)^{(\alpha -1)/\alpha} \]

\[ = (Pf^\alpha(y))^{1/\alpha} ( Pf_{x,y}^1/(\alpha -1)(x) )^{(\alpha -1)/\alpha} \leq (Pf^\alpha(y))^{1/\alpha} \Psi(x, y)^{1/\alpha}. \]

Therefore, (2.3) holds.

(2) We shall use the following Young inequality: for any probability measure \( \nu \) on \( M \), if \( g_1, g_2 \geq 0 \) with \( \nu(g_1) = 1 \), then

\[ \nu(g_1g_2) \leq \nu(g_1 \log g_1) + \log \nu(e^{g_2}). \]

For \( f \geq 1 \), applying the above inequality for \( g_1 = p_{x,y}, g_2 = \log f \) and \( \nu = P(y, \cdot) \), we obtain

\[ P(\log f)(x) = \int_E \{p_{x,y}(z) \log f(z)\} P(y, dz) \]

\[ \leq P(\log p_{x,y})(x) + \log Pf(y). \]

So, (2.5) implies (2.2). On the other hand, applying (2.2) to \( f_n = 1 + np_{x,y} \), we arrive at

\[ P\{\log p_{x,y}\}(x) \leq P(\log f_n)(x) - \log n \]

\[ \leq \log Pf_n(y) - \log n + \Phi(x, y) = \log \frac{n+1}{n} + \Phi(x, y). \]

Therefore, by letting \( n \to \infty \) we obtain (2.5). \hfill \Box

3 Proof of Theorem 1.1

By [13, Lemma 2.2], if \( \partial M \) is either convex or empty then (1.1) implies (1.2). Combining this with Propositions 2.2 and 2.4 for \( P = P_t \) so that \( p_{x,y}(z) = \frac{p_t(x,z)}{p_t(y,z)} \), it remains to prove that (1.3) implies (1.1).

Let \( x \in M \) (when \( M \) has a convex boundary, we take \( x \) in the interior) and \( X \in T_x M \) be fixed. For any \( n \geq 1 \) we may take \( f \in C_b^\infty(M) \) such that \( f \geq 1 \), \( f \) is constant outside a compact set, and

\[ \nabla f(x) = X, \quad \text{Hess} f(x) = 0, \quad f(x) \geq n. \]
If $M$ has a convex boundary $\partial M$, we may assume further that $f$ is constant in a neighborhood of $\partial M$ so that the Neumann boundary condition is satisfied. Such a function can be constructed by using the exponential map as follows. Let $r_0 > 0$ be smaller than the injectivity radius at point $x$ such that the exponential map

$$\exp_x : \{Y \in T_x M : |Y| < r_0\} \to B(x, r_0) := \{z \in M : \rho(x, z) < r_0\} \subset M \setminus \partial M$$

is diffeomorphic. Then the function

$$g(z) := \langle X, \exp_x^{-1}(z) \rangle, \quad z \in B(x, r_0)$$

is smooth and satisfies $\nabla g(x) = X$, $\text{Hess}_g(x) = 0$. Let $F \in C^\infty_0(M)$ such that $F|_{B(x, r_0/4)} = 1$ and $F|_{B(x, r_0/2)} = 0$. Then $f := gF + R$ meets our requirements for a large enough constant $R > 0$.

Taking $\gamma_t = \exp_x[-2t \nabla \log f(x)]$, we have $\rho(x, \gamma_t) = 2t |\nabla \log f(x)|$ for $t \in [0, t_0]$, where $t_0 > 0$ is such that $2t_0 |X| < r_0 f(x)$. By (1.3) with $y = \gamma_t$, we obtain

$$P_t(\log f)(x) \leq \log P_t f(\gamma_t) + \frac{2Kt^2 |\nabla \log f|^2(x)}{1 - e^{-2Kt}}, \quad t \in (0,t_0].$$

Since $Lf \in C^2_0(M)$ and $L \log f = 0$ around $\partial M$, and noting that $\text{Hess}_f(x) = 0$ implies $\nabla |\nabla f|^2(x) = 0$, at point $x$ we have

$$\begin{align*}
\frac{d}{dt} P_t \log f|_{t=0} &= L \log f = \frac{Lf}{f} - |\nabla \log f|^2, \\
\frac{d^2}{dt^2} P_t \log f|_{t=0} &= L^2 \log f = \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} + \frac{2|\nabla f|^2 Lf}{f^3} + 2\langle \nabla Lf, \nabla f^{-1} \rangle - \frac{L|\nabla f|^2}{f^2} \\
&\quad + \frac{2|\nabla f|^2 Lf}{f^3} - \frac{6|\nabla f|^4}{f^4} - 2\langle \nabla |\nabla f|^2, \nabla f^{-2} \rangle \\
&= \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} + 2\frac{f^2}{f^2} \langle \nabla Lf, \nabla f \rangle - \frac{L|\nabla f|^2}{f^2} + \frac{4|\nabla f|^2 Lf}{f^3} - \frac{6|\nabla f|^4}{f^4} =: A.
\end{align*}$$

Thus, by Taylor’s expansions,

$$P_t(\log f)(x) = \log f(x) + t(f^{-1} Lf - |\nabla \log f|^2)(x) + \frac{t^2}{2} A + o(t^2)$$

holds for small $t > 0$. On the other hand, let $N_t = [x \mapsto \gamma_t \nabla \log f(x)]$, where $[x \mapsto \gamma_t]$ is the parallel displacement along the geodesic $t \mapsto \gamma_t$. We have $\gamma_t = -2N_t$ and $\nabla_{\gamma_t} N_t = 0$. So,
\[
\frac{d}{dt} \log P_t f(\gamma_t)|_{t=0} = \left( \frac{LP_t f(\gamma_t)}{P_t f(\gamma_t)} - \frac{2 \langle \nabla P_t f, N_t \rangle}{P_t f(\gamma_t)} \right)|_{t=0} = \frac{L f}{f} - 2 \| \nabla \log f \|^2,
\]
\[
\frac{d^2}{dt^2} \log P_t f(\gamma_t)|_{t=0} = \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - 2 \langle \nabla (f^{-1} Lf), \nabla \log f \rangle - \frac{2 f}{f} \langle \nabla L f, \nabla \log f \rangle + \frac{2}{f^2} \langle \nabla f, \nabla \log f \rangle L f + 4 \text{Hess}_{\log f}(\nabla \log f, \nabla \log f)
\]
\[
= \frac{L^2 f}{f} - \frac{(Lf)^2}{f^2} - 4 \frac{\langle \nabla L f, \nabla f \rangle}{f^2} + 4 \frac{\| \nabla f \|^2 L f}{f^3} - 4 \frac{\| \nabla f \|^4}{f^4} =: B,
\]
where, as in above, the functions take value at point \( x \) and we have used \( \text{Hess}_f(x) = 0 \) in the last step. Thus, we have
\[
\log P_t f(\gamma_t) = \log f(x) + t (f^{-1} Lf - 2 \| \nabla \log f \|^2)(x) + \frac{t^2}{2} B + o(t^2).
\]
Combining this with (3.2) and (3.3), we arrive at
\[
\frac{1}{t} \left( 1 - \frac{2Kt}{1 - e^{-2Kt}} \right) \| \nabla \log f \|^2(x) \leq \frac{1}{2} \left( \frac{L \| \nabla f \|^2 - 2 \langle \nabla L f, \nabla f \rangle}{f^2} + \frac{2 \| \nabla f \|^4}{f^4} \right)(x) + o(1).
\]
Letting \( t \to 0 \) we obtain
\[
\Gamma_2(f, f)(x) := \frac{1}{2} L \| \nabla f \|^2(x) - \langle \nabla L f, \nabla f \rangle(x) \geq -K \| \nabla f \|^2(x) - \frac{\| \nabla f \|^4}{f^2}(x).
\]
Since by the Bochner-Weitzenböck formula and (3.1) we have \( \nabla f(x) = X, f(x) \geq n \) and
\[
\Gamma_2(f, f)(x) = \text{Ric}(X, X) - \langle \nabla_X Z, X \rangle,
\]
it follows that
\[
\text{Ric}(X, X) - \langle \nabla_X Z, X \rangle \geq -K \| X \|^2 - \frac{\| X \|^4}{n}, \quad n \geq 1.
\]
This implies (1.1) by letting \( n \to \infty \).

4 Proof of Theorem 1.2

Since in the proofs of [17, Theorem 2.1 and Lemma 2.2] only the boundedness of \( L_{\rho_0} \) on \( \{ \rho_0 \leq r_0 \} \) rather than the compactness of \( M \) is used, these two results hold true in the setting of Theorem 1.2. More precisely, we have the following result.
Proposition 4.1. If there exists \( r_0 > 0 \) such that \( \rho_\theta \) is smooth with bounded \( L\rho_\theta \) on \( \{ \rho_\theta \leq r_0 \} \), then there exists a constant \( c > 0 \) such that \( E_t^2 \leq ct \) holds for all \( x_0 \in \partial M \) and \( t \in [0, 1] \), and

\[
\limsup_{t \to 0} \frac{1}{t} \left| E_t - \frac{2}{\sqrt{\pi}} \sqrt{t} \right| < \infty
\]

holds uniformly in \( x_0 \in \partial M \).

Let \( N \) be the unit inward normal vector field of \( \partial M \). Then

\[ \Pi(X, X) := -\langle \nabla_X N, X \rangle \geq 0, \quad X \in T\partial M \]

is the second fundamental form of \( \partial M \). By definition \( \partial M \) is called convex if \( \Pi \geq 0 \).

For any \( x \in \partial M \) and \( X \in T_{x\partial M} \), let \( f \in C^\infty(M) \) be such that \( f \geq 1 \), \( Nf|_{\partial M} = 0 \) and \( \nabla f(x) = X \). We may further assume that \( f \) is constant outside a compact set. To construct such a function, let \( \tilde{f} \in C^\infty(\partial M) \) such that \( \nabla_{\partial M} \tilde{f}(x) = X \), where \( \nabla_{\partial M} \) is the gradient on \( \partial M \) with respect to the induced metric. Let \( \tilde{f} \) be supported on \( \partial M \cap B(x, m) \) for some \( m > 0 \), where \( B(x, m) \) is the open geodesic ball around \( x \) with radius \( m \). Then there exists \( r_1 \in (0, 1) \) such that the exponential map

\[ U := (B(x, m + 3) \cap \partial M) \times [0, r_1) \ni (\theta, r) \mapsto \exp_{\theta}[rN] \]

is smooth and one-to-one, which is known as the local polar coordinates around \( B(x, m + 2) \cap \partial M \). Let \( h \in C^\infty([0, \infty)) \) such that \( h|_{0,(r_1\wedge r_0)/4} = 1 \) and \( h|_{((r_0\wedge r_1)/2, \infty)} = 0 \). Since \( \tilde{f} \) is supported on \( B(x, m) \) the function

\[ M \ni x \mapsto f(x) := R + \begin{cases} \tilde{f}(\theta)h(r), & \text{if there exists } (\theta, r) \in U \text{ such that } x = \exp_{\theta}[rN], \\ 0, & \text{otherwise} \end{cases} \]

for large enough constant \( R > 0 \) meets our requirements.

Let \( \exp^\partial_x : T_x \partial M \to \partial M \) be the exponential map on the Riemannian manifold \( \partial M \) with the induced metric, and let

\[ \gamma_t = \exp^\partial_x \left[ -2t \nabla \log f(x) \right], \quad t \geq 0. \]

Applying (4.3) to \( y = \gamma_t \) we obtain

\[ P_t \log f(x) \leq \log P_tf(\gamma_t) + \frac{2Kt^2|\nabla \log f|^2(x)}{1 - e^{-2Kt}}, \quad t \geq 0. \]

Since \( f \) and \( Lf \) satisfy the Neumann boundary condition, we have
\[ P_t \log f(x) = \log f(x) + \int_0^t P_s L \log f(x) ds \]
\[ = \log f(x) + \int_0^t P_s \frac{Lf}{f}(x) ds - \int_0^t P_s |\nabla \log f|^2(x) ds. \]  

(4.2)

Let \( X_s \) be the reflecting \( L \)-diffusion process with \( x_0 = x \), and let \( l_s \) be its local time on \( \partial M \). By the Itô formula for \( |\nabla \log f|^2(x) \) we obtain

\[ P_s |\nabla \log f|^2(x) = |\nabla \log f|^2(x) + \int_0^s P_r L |\nabla \log f|^2(x) dr + \mathbb{E} \int_0^s \langle N, \nabla |\nabla \log f|^2 \rangle(X_r)dl_r. \]

Since \( f \) satisfies the Neumann boundary condition so that

\[ \langle N, \nabla |\nabla \log f|^2 \rangle = 2 f^{-2} \text{Hess}_f(N, \nabla f), \]

and since \( \langle \nabla f, \nabla \langle N, \nabla f \rangle \rangle = 0 \) implies

\[ \text{Hess}_f(N, \nabla f) = -\langle \nabla_{\nabla f}N, \nabla f \rangle = \|\nabla f, \nabla f\|, \]

it follows that

\[ P_s |\nabla \log f|^2(x) = |\nabla \log f|^2(x) + O(s) + 2 f^{-2}(x) \|\nabla f, \nabla f\|(x) \mathbb{E} l_s + o(\mathbb{E} l_s). \]

Since due to Proposition 4.1 we have \( \lim_{t \to 0} t^{-1/2} \mathbb{E} l_t = \frac{2}{\sqrt{\pi}} \), this and (4.2) yield (recall that \( \nabla f(x) = X \))

\[ P_t \log f(x) = \log f(x) + \int_0^t P_s \frac{Lf}{f}(x) ds - |\nabla \log f|^2(x) - \frac{8t^{3/2}}{3\sqrt{\pi} f^2(x)} \|X, X\| + o(t^{3/2}). \]

(4.3)

On the other hand, we have

\[ P_t f(\gamma_t) = f(\gamma_t) + \int_0^t P_s L f(\gamma_t) ds \]
\[ = f(x) + t \langle \dot{\gamma}_s, \nabla f(\gamma_s) \rangle |_{s=0} + O(t^2) + \int_0^t P_s L f(x) ds \]
\[ = f(x) - \frac{2t}{f(x)} |\nabla f|^2(x) + \int_0^t P_s L f(x) ds + O(t^2). \]

Thus,
\[ \log P_t f(\gamma_t) = \log f(x) + \frac{1}{f(x)} \int_0^t P_s L f(x) \, ds - 2t|\nabla \log f|^2(x) + O(t^2). \]

Combining this with (4.1) and (4.3) we arrive at

\begin{equation}
\frac{1}{t \sqrt{t}} \int_0^t \left( P_s \frac{L f}{f} - \frac{P_s L p f}{f} \right)(x) \, ds + \frac{1}{\sqrt{t}} \left( 1 - \frac{2Kt}{1 - e^{-2Kt}} \right) |\nabla \log f|^2(x) \leq \frac{8}{3 \sqrt{\pi} f^2(x)} \mathbb{II}(X, X) + o(1).
\end{equation}

Obviously,

\[ \lim_{t \to 0} \frac{1}{\sqrt{t}} \left( 1 - \frac{2Kt}{1 - e^{-2Kt}} \right) = 0. \]

So, to derive \( \mathbb{II}(X, X) \geq 0 \) from (4.4) it remains to verify

\begin{equation}
\lim_{t \to 0} \frac{1}{t \sqrt{t}} \int_0^t \left( P_s \frac{L f}{f} - \frac{P_s L p f}{f} \right)(x) \, ds = 0.
\end{equation}

Noting that \( Z \) is \( C^2 \)-smooth and \( f \in C^\infty(M) \) is constant outside a compact set, we have \( L f \in C^2_0(M) \). Moreover, \( f \geq 1 \) and \( f \) satisfies the Neumann boundary condition. So, by the Itô formula we have

\begin{equation}
\left( P_s \frac{L f}{f} - \frac{P_s L p f}{f} \right)(x) = \int_0^s \left( L f \frac{L f}{f} - \frac{L f}{f} f \right)(x) \, dr + \mathbb{E} \int_0^s \left( \frac{1}{f(x_r)} - \frac{1}{f(x)} \right) \langle N, \nabla L f \rangle(x_r) \, dl_r.
\end{equation}

Since \( \frac{1}{f} \) is bounded and \( X_r \to x \) as \( r \to 0 \), it follows from Proposition 4.1 that

\[ \lim_{s \to 0} \frac{1}{\sqrt{s}} \mathbb{E} \int_0^s \left( \frac{1}{f(x_r)} - \frac{1}{f(x)} \right) \langle N, \nabla L f \rangle(x_r) \, dl_r \leq \lim_{s \to 0} \frac{\|\nabla L f\|_\infty}{\sqrt{s}} \mathbb{E} \left( l_s \sup_{r \in [0, s]} |f(X_r)^{-1} - f(x)^{-1}| \right) \leq \|\nabla L f\|_\infty \lim_{s \to 0} \left( \frac{\mathbb{E} l_s^2}{s} \right)^{1/2} \left( \mathbb{E} \sup_{r \in [0, s]} |f(X_r)^{-1} - f(x)^{-1}|^2 \right)^{1/2} = 0. \]

Therefore, (4.5) follows from (4.6) immediately.
5 An extension to non-convex manifolds

In this section we aim to established the log-Harnack inequality on a class of non-convex manifolds. To this end, we need the following assumption.

(A) The boundary $\partial M$ has a bounded second fundamental form and a strictly positive injectivity radius, the sectional curvature of $M$ is bounded above, and there exists $r > 0$ such that $Z$ is bounded on the $r$-neighborhood of $\partial M$.

Under this assumption, we have $\sup_x M e^{\lambda t} < \infty$ for all $\lambda > 0$ (see Proposition 7.1 in Appendix). Let

$$U_{x,y}(s) = \sup_{z: \rho(z,x) \vee \rho(z,y) \leq \rho(x,y)} \mathbb{E}^z e^{2\sigma s}, \quad x,y \in M, s \geq 0.$$

As a complement to known equivalent statements for lower bounds on curvature and second fundamental form derived recently in [18], the following result provides two more equivalent statements.

**Theorem 5.1.** Assume (A). Let $K, \sigma \in \mathbb{R}$ be two constants. Then the following statements are equivalent each other:

1. $\text{Ric} - \nabla Z \geq -K, I \geq -\sigma$.
2. $P_t(\log f)(x) \leq \log P_t f(y) + \frac{\rho(x,y)^2}{4 \int_0^t e^{-2Ks} \{U_{x,y}(s)\}^{-1} \, ds}$ holds for all $f \in \mathcal{B}_b^+(M)$ with $f \geq 1, t \geq 0$, and $x,y \in M$.
3. $\int_M P_t(x,z) \log \frac{\rho(x,z)}{\rho(y,z)} \mu(\mathrm{d}z) \leq \frac{\rho(x,y)^2}{4 \int_0^t e^{-2Ks} \{U_{x,y}(s)\}^{-1} \, ds}$ holds for all $t > 0, x,y \in M$.

**Proof.** Since Proposition 2.4 ensures that (2) and (3) are equivalent, it suffices to prove the equivalence of (1) and (2).

(a) (1) implies (2). According to (1), the following Hsu’s gradient estimate holds (see Proposition 7.2 in Appendix):

$$|\nabla P_t f|^2 \leq \left( \mathbb{E} \left\{ |\nabla f|(X_t) e^{Kt+2\sigma t} \right\} \right)^2 \leq (P_t |\nabla f|^2) \mathbb{E} e^{2Kt+2\sigma t}.$$  

Let $\gamma : [0,1] \to M$ be the minimal curve with constant such that $\gamma(0) = y$ and $\gamma(1) = x$. We have $|\gamma| = \rho(x,y)$. Let $h \in C^1([0,t])$ be such that $h(0) = 0$ and $h(t) = 1$. By (5.1) and the definition of $U_{x,y}$ we have

$$\frac{d}{ds} P_s \log P_{t-s} f(\gamma \circ h(s)) = -P_s |\nabla \log P_{t-s} f(\gamma \circ h(s))|^2 (\gamma \circ h(s)) + \dot{h}(s) (\gamma \circ h(s), \nabla P_s \log P_{t-s} f(\gamma \circ h(s)))$$

$$\leq -P_s |\nabla \log P_{t-s} f(\gamma \circ h(s))|^2 |\dot{h}(s)| \rho(x,y) e^{-Ks} \{U_{x,y}(s) P_s |\nabla \log P_{t-s} f|^2 (\gamma \circ h(s))\}^{1/2}$$

$$\leq \frac{1}{4} |\dot{h}(s)|^2 \rho(x,y)^2 U_{x,y}(s) e^{2Ks}, \quad s \in [0,t].$$
This implies

\[ P_t \log f(x) \leq \log P_t f(y) + \frac{\rho(x,y)^2}{4} \int_0^t |\dot{h}(s)|^2 U_{x,y}(s) e^{2Ks} \, ds. \]

Therefore, we prove (2) by taking

\[ h(s) = \frac{\int_0^s e^{-2Kr[G(x,y)]^{-1}} \, dr}{\int_0^t e^{-2Kr[G(x,y)]^{-1}} \, dr}, \quad s \in [0,t]. \]

(b) (2) implies (1). Let \( x \in M \setminus \partial M \). There exists \( \delta > 0 \) such that the closed geodesic ball \( \bar{B}(x,2\delta) \) at \( x \) with radius \( 2\delta \) is contained in \( M \setminus \partial M \), i.e. \( \bar{B}(x,2\delta) \cap \partial M = \emptyset \). Let \( \tau \) be the hitting time of \( X_t \) to the boundary, we have (cf. \cite[Proposition A.2]{17})

\[ \mathbb{P}^z(\tau \leq t) \leq Ce^{-\delta^2/(16t)}, \quad z \in B(x,\delta) \]

for some constant \( C > 0 \) and all \( t > 0 \). Moreover, by \cite[Proof of Lemma 2.1]{15}, we have

\[ C' := \sup_{z \in \partial M} \mathbb{E}^z e^{2\sigma_t} < \infty. \]

Since \( l_t = 0 \) for \( t \leq \tau \) and \( l_t \) is increasing in \( t \), it follows that

\[ \mathbb{E} e^{2\sigma_t} \leq \mathbb{P}(\tau > t) + \mathbb{E}1_{\{\tau \leq t\}} \mathbb{E}^{X_t} e^{2\sigma_t} \]

\[ \leq 1 + C'Ce^{-\delta^2/(16t)}, \quad t \in [0,1]. \]

Thus, for any \( y \in B(x,\delta) \),

\[ \int_0^t e^{-2Ks} \frac{e^{-2Ks}}{U_{x,y}(s)} \, ds = \int_0^t e^{-2Ks} \, ds + o(t^3) = \frac{e^{2Kt} - 1}{2K} + o(t^3), \]

where \( o(t^3) \) is uniform in \( y \in B(x,\delta) \). Combining this with the proof of Theorem 1.1 we derive \( \text{Ric} - \nabla Z \geq -K \) from (2).

Now, let \( x \in \partial M \). By Proposition 4.1 and (5.2) we have

\[ \sup_{z \in M} \mathbb{E}^z e^{2\sigma_t} \leq 1 + \frac{4\sigma}{\sqrt{\pi}} \sqrt{t} + o(t). \]

Then

\[ \int_0^t e^{2Ks} \{U_{x,y}(s)\}^{-1} \, ds \geq t + \frac{4\sigma}{\sqrt{\pi}} \int_0^t \sqrt{s} \, ds + o(t^{3/2}) = t + \frac{8\sigma}{3\sqrt{\pi}} t^{3/2} + o(t^{3/2}). \]

So, (2) implies
\[ P_t \log f(x) \leq \log P_t f(y) + \frac{\rho(x,y)^2}{1 + \frac{\kappa^2}{3\sqrt{\pi}} t^{3/2} + o(t^{3/2})}. \]

Thus, instead of (4.4) the proof of Theorem 1.2 yields

\[
\frac{1}{t\sqrt{t}} \int_0^t \left( P_s \frac{L f}{f} - P_s \frac{L f}{f} \right)(x) ds + \frac{1}{t\sqrt{t}} \left( t - t - \frac{8\sigma}{3\sqrt{\pi}} t^{3/2} + o(t^{3/2}) \right) |\nabla \log f|^2(x)
\]

\[
\leq \frac{8}{3\sqrt{\pi} f^2(x)} \mathbb{I}(X, X) + o(1).
\]

By this and (4.5) and letting \( t \to 0 \) we deduce that \( \mathbb{I}(X, X) \geq -\sigma |X|^2. \)

6 HWI inequality

To study the HWI inequality, we consider the symmetric case that \( Z = \nabla V \) for some \( V \in C^2(M) \) such that \( \mu(dx) = e^{V(x)} dx \) is a probability measure on \( M \), where \( dx \) is the Riemannian volume measure on \( M \). Let \( P_t \) be the semigroup of the reflecting diffusion process generated by \( L \) on \( M \), which is then symmetric in \( L^2(\mu) \). When \( \partial M \) is convex \((1.1)\) implies the following gradient estimate (cf. \[10, 13\])

\[(6.1) \quad |\nabla P_t f| \leq e^{K t} P_t |\nabla f|, \quad f \in C^1_b(M).\]

Combining this estimate and an argument of [4] (see also [3]), we can easily obtain the following HWI inequality:

\[(6.2) \quad \mu(f^2 \log f^2) \leq 2 \sqrt{\mu(|\nabla f|^2)} W_2(f^2 \mu, \mu) + \frac{K}{2} W_2(f^2 \mu, \mu)^2, \quad \mu(f^2) = 1,\]

where \( W_2 \) is the \( L^2 \)-Wasserstein distance induced by the Riemannian distance function \( \rho \) on \( M \). More precisely, for a probability measure \( \nu \) on \( M \) (note that we are using \( \rho^2 \) to replace \( \frac{1}{2} \rho^2 \) in [3])

\[ W_2(\nu, \mu)^2 := \inf_{\pi \in \mathcal{C}(\nu, \mu)} \int_{M \times M} \rho(x, y)^2 \pi(dx, dy), \]

where \( \mathcal{C}(\nu, \mu) \) is the class of all couplings of \( \nu \) and \( \mu \).

**Theorem 6.1.** Let \( Z = \nabla V \) for some \( V \in C^2(M) \) such that \( \mu \) is a probability measure. Assume (A) and (1.1). Let \( \mathbb{I} \geq -\sigma \) for some \( \sigma \in \mathbb{R} \). Then

\[ \eta_\lambda(s) := \sup_{x \in M} \mathbb{E}^x e^{\lambda s} < \infty, \quad s, \lambda \geq 0 \]
holds, and for any $t > 0$,

$$(6.3) \quad \mu(f^2 \log f^2) \leq 4 \left( \int_0^t e^{2Ks} \eta_2(s) ds \right) \mu(|\nabla f|^2) + \frac{W_2(f^2 \mu, \mu)^2}{4 \int_0^t e^{-2Ks} \eta_2(s) \cdot 1} ds, \quad \mu(f^2) = 1.$$ 

**Proof.** By Proposition 7.1 in Appendix, it remains to verify (6.3). Let $f \in C^1_b(M)$ and $t > 0$. We have

$$(6.4) \quad \frac{d}{ds} P_s \{(P_{t-s} f^2) \log P_{t-s} f^2 \} = P_s \left[ \frac{\nabla P_{t-s} f^2}{P_{t-s} f^2} \right], \quad s \in [0, t].$$

By Proposition 7.2 below and the Schwartz inequality we have

$$|\nabla P_{t-s} f^2|^2 \leq e^{2K(t-s)} E_y \left[ |\nabla f|^2 (X_{t-s}) e^{2\sigma l_{t-s}} \right] = 4e^{2K(t-s)} g_s(y), \quad s \in [0, t], y \in M.$$ 

Combining this with (6.4) we obtain

$$P_t (f^2 \log f^2) \leq (P_t f^2) \log P_t f^2 + 4 \int_0^t e^{2K(t-s)} P_s g_s ds.$$ 

Since $\mu$ is an invariant measure of $P_t$, taking integral for both sides with respect to $\mu$ we arrive at

$$(6.5) \quad \mu(f^2 \log f^2) \leq \mu((P_t f^2) \log P_t f^2) + 4 \int_0^t e^{2K(t-s)} \mu(g_s) ds.$$ 

Let $P_t^\sigma$ be defined by

$$P_t^\sigma h(x) = E_x[h(X_t)e^{2\sigma l_t}], \quad h \in C_b(M).$$

Then it is easy to see that $u(t, x) := P_t^\sigma h(x)$ solve the heat equation with Robin boundary condition

$$\partial_t u = Lu, \quad u(0, \cdot) = h, (Nu + 2\sigma u)|_{\partial M} = 0.$$ 

In particular, since $L$ is symmetric in $L^2(\mu)$ under the Robin boundary condition, so is $P_t^\sigma$. Therefore,

$$\mu(g_s) = mu(P_t^\sigma |\nabla f|^2) = \mu(|\nabla f|^2 P_t^\sigma 1) \leq \mu(|\nabla f|^2) \eta_{2\sigma}(t - s).$$

Combining this with (6.5) we obtain
The proof is completed by taking

\[ h_s = \frac{\int_s^t e^{2Ku} \eta_{2\sigma}(u)^{-1} du}{\int_0^t e^{-2Ku} \eta_{2\sigma}(u)^{-1} du}, \quad s \in [0, t]. \]
7 Appendix

We aim to confirm the exponential integrability of the local time and Hsu’s gradient estimate used in Section 5 and Section 6 for the non-convex case, which are known in [15] and [5] respectively for the compact case. Here we shall reprove them for the non-compact case under assumption (A).

To estimate \( E e^{\lambda l_t} \) for \( \lambda > 0 \), we introduce some concrete conditions in terms of assumption (A). Let \( M \) be the sectional curvature of \( M \) and \( \partial M > 0 \) be the injectivity radius of \( \partial M \). Let

\[
\delta_r(Z) := \sup_{\partial_r M} \langle Z, \nabla \rho_{\partial M} \rangle, \quad r > 0.
\]

**Proposition 7.1.** Let \( r_0, \sigma, k, > 0 \) be such that \( \delta_{r_0}(Z) < \infty, -\sigma \leq I \leq \gamma \) and \( \text{Sect}_M \leq k \). Then

\[
\sup_{x \in M} E x e^{\lambda l_t} \leq \exp \left[ \frac{\lambda dr}{2} + \left( \frac{\lambda d}{r} + \lambda \delta_r(Z) + 2\lambda^2 \right)t \right], \quad t \geq 0, \lambda \geq 0
\]

holds for any

\[
0 < r \leq \min \left\{ \text{i}_{\partial M}, \ r_0, \ \frac{1}{\sqrt{k}} \arcsin \left( \frac{\sqrt{k}}{\sqrt{k} + \gamma^2} \right) \right\}.
\]

**Proof.** Let

\[
h(s) = \cos \left( \sqrt{k} s \right) - \frac{\gamma}{\sqrt{k}} \sin \left( \sqrt{k} s \right), \quad s \geq 0.
\]

Then \( h \) is the unique solution to the equation

\[
h'' + kh = 0, \quad h(0) = 1, h'(0) = -\gamma.
\]

By the Laplacian comparison theorem for \( \rho_{\partial M} \) (cf. [?, Theorem 0.3] or [16]),

\[
\Delta \rho_{\partial M} \geq \frac{(d-1)h'}{h}(\rho_{\partial M}), \quad \rho_{\partial M} < i_{\partial M} \wedge h^{(-1)}(0).
\]

Thus,

\[
(7.1) \quad L \rho_{\partial M} \geq \frac{(d-1)h'}{h}(\rho_{\partial M}) - \delta_r(Z), \quad \rho_{\partial M} \leq r.
\]

Now, let

\[
\alpha = (1 - h(r))^{1-d} \int_0^r (h(s) - h(r))^{d-1}ds,
\]

\[
\psi(s) = \frac{1}{\alpha} \int_0^s (h(t) - h(r))^{1-d}dt \int_{t \wedge r}^r (h(u) - h(r))^{d-1}du, \quad s \geq 0.
\]
We have $\psi(0) = 0, 0 \leq \psi' \leq \psi'(0) = 1$. Moreover, as observed in [15, Proof of Theorem 1.1],

\begin{equation}
\alpha \geq \frac{r}{d}, \quad \psi(\infty) = \psi(r) \leq \frac{r^2}{2\alpha} \leq \frac{dr}{2}.
\end{equation}

Combining this with (7.1) we obtain (note that $\psi'(s) = 0$ for $s \geq r$)

\begin{equation}
L\psi \circ \rho_{\partial M} \partial M = \psi' \circ \rho_{\partial M} L \rho_{\partial M} + \psi'' \circ \rho_{\partial M} \geq -\frac{1}{\alpha} - \delta_{\partial}(Z) \geq -\frac{d}{r} - \delta_{\partial}(Z).
\end{equation}

On the other hand, since $\psi'(0) = 1$, by the Itô formula we have

\begin{equation}
d\psi \circ \rho_{\partial M}(X_t) = \sqrt{2} \psi' \circ \rho_{\partial M}(X_t)db_t + L\psi \circ \rho_{\partial M}(X_t)dt + dl_t,
\end{equation}

where $b_t$ is the one-dimensional Brownian motion. Then it follows from (7.2) and (7.3) that (note that $|\psi'| \leq 1$)

\[ \mathbb{E}e^{\lambda t} = \mathbb{E}\exp \left[ \lambda \psi \circ \rho_{\partial M}(X_t) + \left( \frac{d\lambda}{r} + \lambda \delta_r(Z) \right) t - \sqrt{2}\lambda \int_0^t \psi' \circ \rho_{\partial M}(X_s)db_s \right] \]
\[ \leq \exp \left[ \frac{1}{2} \lambda dt + \left( \frac{d\lambda}{r} + \lambda \delta_r(Z) \right) t \right] \left( \mathbb{E}\exp \left[ 4\lambda^2 \int_0^t (\psi' \circ \rho_{\partial M}(X_s))^2 ds \right] \right)^{1/2} \]
\[ \leq \exp \left[ \frac{1}{2} \lambda dt + \left( \frac{d\lambda}{r} + \lambda \delta_r(Z) + 2\lambda^2 \right) t \right]. \]

\[ \square \]

**Proposition 7.2.** Assume that (A). Let $\kappa_1, \kappa_2 \in C_b(M)$ be such that

\begin{equation}
\text{Ric} - \nabla Z \geq -\kappa_1, \quad \mathbb{I} \geq -\kappa_2
\end{equation}

hold on $M$ and $\partial M$ respectively. Then

\begin{equation}
|\nabla P_tf(x)| \leq \mathbb{E}^x \left\{ |\nabla f|(X_t) \exp \left[ \int_0^t \kappa_1(X_s)ds + \int_0^t \kappa_2(X_s)dl_s \right] \right\}
\end{equation}

holds for all $f \in C^1_b(M), t > 0, x \in M$.

We first provide a simple proof of (7.6) under a further condition that $|\nabla P_t f|$ is bounded on $[0, T] \times M$ for any $T > 0$, then drop this assumption by an approximation argument. Since this condition is trivial for compact $M$, our proof below is much shorter than that in [5].
Lemma 7.3. Assume that $f \in C^1_b(M)$ such that $|\nabla P.f|$ is bounded on $[0, T] \times M$ for any $T > 0$. Then (7.6) holds.

Proof. For any $\varepsilon > 0$, let

$$\zeta_s = \sqrt{\varepsilon + |\nabla P_{t-s}f|^2(X_s)}, \quad s \leq t.$$ 

By the Itô formula we have

$$d\zeta_s = d\zeta_s + \frac{L[|\nabla P_{t-s}f|^2 - 2\langle \nabla L P_{t-s}f, \nabla P_{t-s}f \rangle]}{2(\varepsilon + |\nabla P_{t-s}f|^2)^2}(X_s)ds \quad - \frac{|\nabla|\nabla P_{t-s}f|^2|^2}{4(\varepsilon + |\nabla P_{t-s}f|^2)^{3/2}}(X_s)ds + \frac{N|\nabla P_{t-s}f|^2}{2(\varepsilon + |\nabla P_{t-s}f|^2)}(X_s)dl_s, \quad s \leq t,$$

where $M_s$ is a local martingale. Combining this with (7.5) and (see [8, (1.14)])

$$(7.7) \quad L|\nabla u|^2 - 2\langle \nabla Lu, \nabla u \rangle \geq -2\kappa_1|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2},$$

we obtain

$$d\zeta_s \geq d\zeta_s - \frac{\kappa_1|\nabla P_{t-s}f|^2}{\varepsilon + |\nabla P_{t-s}f|^2}(X_s)\zeta_s ds - \frac{\kappa_2|\nabla P_{t-s}f|^2}{\varepsilon + |\nabla P_{t-s}f|^2}(X_s)\zeta_s dl_s, \quad s \leq t.$$

Since $\zeta_s$ is bounded on $[0, t]$, $\kappa_1$ and $\kappa_2$ are bounded, and by Proposition 7.1 below $\mathbb{E}e^{\lambda t} < \infty$ for all $\lambda > 0$, this implies that

$$[0, t] \ni s \mapsto \zeta_s \exp \left[ \int_0^s \frac{\kappa_1|\nabla P_{t-r}f|^2}{\varepsilon + |\nabla P_{t-r}f|^2}(X_r)dr + \int_0^s \frac{\kappa_2|\nabla P_{t-r}f|^2}{\varepsilon + |\nabla P_{t-r}f|^2}(X_r)dl_r \right]$$

is a submartingale for any $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we conclude that

$$[0, t] \ni s \mapsto |\nabla P_{t-s}f|(X_s) \exp \left[ \int_0^s \kappa_1(X_r)dr + \int_0^s \kappa_2(X_r)dl_r \right]$$

is a submartingale as well. This completes the proof. \qed

By Lemma 7.3 to prove Proposition 7.2 it suffices to confirm the boundedness of $|\nabla P.f|$ on $[0, T] \times M$ for $f \in C^1_b(M)$. Below we first consider $f \in C^\infty_0(M)$ satisfying the Neumann boundary condition.

Lemma 7.4. Assume (A). If (L.1) holds then for any $T > 0$ and $f \in C^\infty_0(M)$ such that $Nf|_{\partial M} = 0$, $|\nabla P.f|$ is bounded on $[0, T] \times M$. 

20
Proof. We shall take a conformal change of metric as in [16] to make the boundary convex, so that the known estimates for the convex case can be applied. As explained on page 1436 in [16], under assumption (A) there exists \( \phi \in C^\infty(M) \) and a constant \( R > 1 \) such that \( 1 \leq \phi \leq R, |\nabla \phi| \leq R \), \( N \log \phi_{|\partial M} \geq \sigma \), and \( \nabla \phi = 0 \) outside \( \partial, M \). Since \( \sigma \geq -\sigma \), by [16, Lemma 2.1] \( \partial M \) is convex under the new metric

\[
\langle \cdot, \cdot \rangle = \phi^{-2} \langle \cdot, \cdot \rangle.
\]

Let \( \Delta', \nabla', \text{Ric}' \) be corresponding to the new metric. By [16, Lemma 2.2]

\[
L' := \phi^2 L = \Delta' + (d - 2)\phi \nabla \phi + \phi^2 Z =: \Delta' + Z'.
\]

Following e.g. [16] we shall now calculate the curvature tensor \( \text{Ric}' - \nabla' Z' \) under the new metric. By [16, (9)], for any unit vector \( U \in TM \), \( U' := \phi U \) is unit under the new metric, and the corresponding Ricci curvature satisfies

\[
\text{Ric'}(U', U') \geq \phi^2 \text{Ric}(U, U) + \phi \Delta \phi - (d - 3)|\nabla \phi|^2 - 2(U\phi)^2 + (d - 2)\phi \text{Hess}_\phi(U, U).
\]

Noting that

\[
\nabla_X Y = \nabla_X Y - \langle X, \nabla \log \phi \rangle Y - \langle Y, \nabla \log \phi \rangle X + \langle X, Y \rangle \nabla \log \phi, \quad X, Y \in TM,
\]

we have

\[
\langle \nabla_u Z', U' \rangle = \langle \nabla_u Z', U \rangle - \langle Z', \nabla \log \phi \rangle = \phi^2 \langle \nabla_u Z, U \rangle + (U\phi^2)(Z, U) + (d - 2)(U\phi)^2 + (d - 2)\phi \text{Hess}_\phi(U, U) - \langle Z', \nabla \log \phi \rangle.
\]

Combining this with (7.8), (1.1), \( \|Z\|_r < \infty \) and the properties of \( \phi \) mentioned above, we find a constant \( K' \geq 0 \) such that

\[
\text{Ric'}(U', U'') - \langle \nabla_u Z', U'' \rangle' \geq -K', \quad \langle U', U'' \rangle' = 1.
\]

For any \( x, y \in M \), let \( (X'_t, Y'_t) \) be the coupling by parallel displacement of the reflecting diffusion processes generated by \( L' \) with \( (X'_0, Y'_0) = (x, y) \). Let \( \rho' \) be the Riemannian distance induced by \( \langle \cdot, \cdot \rangle' \). Since \( (M, \langle \cdot, \cdot \rangle') \) is convex, we have (see [13, (3.2)])

\[
\rho'(X'_t, Y'_t) \leq e^{K't} \rho'(x, y), \quad t \geq 0.
\]

Since \( 1 \leq \phi \leq R \), we have \( R^{-1} \rho \leq \rho' \leq \rho \) so that

\[
\rho(X'_t, Y'_t) \leq Re^{K't} \rho(x, y), \quad t \geq 0.
\]
To derive the gradient estimate of $P_t$, we shall make time changes

$$
\xi_x(t) = \int_0^t \phi^2(X_s')ds, \quad \xi_y(t) = \int_0^t \phi^2(Y_s')ds.
$$

Since $L' = \phi^2 L$, we see that $X_t := X_{\xi^{-1}_x(t)}$ and $Y_t := Y_{\xi^{-1}_y(t)}$ are generated by $L$ with reflecting boundary. Again by $1 \leq \phi \leq R$ we have

$$
R^{-2}t \leq \xi_x^{-1}(t), \xi_y^{-1}(t) \leq t, \quad t \geq 0.
$$

Combining this with $|\nabla \phi| \leq R, 1 \leq \phi \leq R$ and (7.9) we arrive at

$$
|\xi_x^{-1}(t) - \xi_y^{-1}(t)| \leq \int_{\xi_x^{-1}(t) \land \xi_y^{-1}(t)} \phi^2(Y_s')ds = |\xi_y \circ \xi_x^{-1}(t) - \xi_y \circ \xi_y^{-1}(t)|
$$

(7.10)

$$
= |\xi_x \circ \xi_x^{-1}(t) - \xi_y \circ \xi_x^{-1}(t)| \leq \int_0^{\xi_x^{-1}(t)} |\phi^2(X_s') - \phi^2(Y_s')|ds
$$

$$
\leq 2R^2 \rho(x, y) \int_0^t e^{K's}ds \leq 2te^{K't}R^2 \rho(x, y).
$$

Therefore,

$$
\begin{align*}
|P_t f(x) - P_t f(y)| &= |\mathbb{E}\{f(X_{\xi_x^{-1}(t)}) - f(Y_{\xi_y^{-1}(t)})\}| \\
&\leq \mathbb{E}|f(X_{\xi_x^{-1}(t)}) - f(Y_{\xi_y^{-1}(t)})| + \mathbb{E}\{f(X_{\xi_x^{-1}(t)}) - f(X_{\xi_y^{-1}(t)})\} =: I_1 + I_2.
\end{align*}
$$

(7.11)

By (7.9) and $\xi_y^{-1}(t) \leq t$ we obtain

$$
I_1 \leq \|\nabla f\|_{\infty} e^{K't} R \rho(x, y).
$$

(7.12)

Moreover, since $f \in C^\infty(M)$ with $Nf|_{\partial M} = 0$, it follows from the Itô formula and (7.10) that

$$
I_2 \leq |\mathbb{E}\int_{\xi_x^{-1}(t) \land \xi_y^{-1}(t)} L' f(X_s')ds| \leq \|L' f\|_{\infty} \mathbb{E}|\xi_x^{-1}(t) - \xi_y^{-1}(t)| \leq c_1 te^{K't} \rho(x, y)
$$

holds for some constant $c_1 > 0$. Combining this with (7.11) and (7.12) we conclude that

$$
\|\nabla P_t f\|_{\infty} \leq c_2 (1 + t)e^{K't}, \quad t \geq 0
$$

for some constant $c_2 > 0$. 

\[\square\]
Proof of Proposition 7.2. Let $f \in C_b^1(M)$. By Lemma 7.3 we only have to prove the boundedness of $|\nabla P.f|$ on $[0,T] \times M$.

(a) Let $f \in C_0^\infty(M)$. In this case there exist a sequence of functions $\{f_n\}_{n \geq 1} \subset C_0^\infty(M)$ such that $N f_n|_{\partial M} = 0$, $f_n \to f$ uniformly as $n \to \infty$, and $\|\nabla f_n\|_\infty \leq 1 + \|\nabla f\|_\infty$ holds for any $n \geq 1$, see e.g. [12]. By Lemmas 7.3 and 7.4, (7.6) holds for $f_n$ in place of $f$ so that Proposition 7.1 implies

$$\frac{|P_t f_n(x) - P_t f_n(y)|}{\rho(x,y)} \leq C, \quad t \leq T, n \geq 1, x \neq y$$

for some constant $C > 0$. Letting first $n \to 0$ then $y \to x$, we conclude that $|\nabla P.f|$ is bounded on $[0,T] \times M$.

(b) Let $f \in C_b^\infty(M)$. Let $\{g_n\}_{n \geq 1} \subset C_0^\infty(M)$ be such that $0 \leq g_n \leq 1$, $|\nabla g_n| \leq 2$ and $g_n \uparrow 1$ as $n \uparrow \infty$. By (a) and Lemma 7.3 we may apply (7.6) to $g_n f$ in place of $f$ such that Proposition 7.1 implies

$$\frac{|P_t(g_n f)(x) - P_t(g_n f)(y)|}{\rho(x,y)} \leq C, \quad t \leq T, n \geq 1, x \neq y$$

holds for some constant $C > 0$. By the same reason as in (a) we conclude that $|\nabla P.f|$ is bounded on $[0,T] \times M$.

(c) Finally, for $f \in C_b^1(M)$ there exist $\{f_n\}_{n \geq 1} \subset C_0^\infty(M)$ such that $f_n \to f$ uniformly as $n \to \infty$ and $\|\nabla f_n\|_\infty \leq \|\nabla f\|_\infty + 1$ for any $n \geq 1$. Therefore, the proof is complete by the same reason as in (a) and (b). \qed

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