KOPPELMAN FORMULAS AND THE $\bar{\partial}$-EQUATION
ON AN ANALYTIC SPACE

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Abstract. Let $X$ be an analytic space of pure dimension. We introduce a formalism to generate intrinsic weighted Koppelman formulas on $X$ that provide solutions to the $\bar{\partial}$-equation. We prove that if $\phi$ is a smooth $(0,q+1)$-form on a Stein space $X$ with $\bar{\partial}\phi = 0$, then there is a smooth $(0,q)$-form $\psi$ on $X_{\text{reg}}$ with at most polynomial growth at $X_{\text{sing}}$ such that $\bar{\partial}\psi = \phi$. The integral formulas also give other new existence results for the $\bar{\partial}$-equation and Hartogs theorems, as well as new proofs of various known results.

1. Introduction

Let $X$ be an analytic space of pure dimension $d$ and let $O_X$ be the structure sheaf of (strongly) holomorphic functions. Locally $X$ is a subvariety of a domain $\Omega$ in $\mathbb{C}^n$ and then $O_X = O/J$, where $J$ is the sheaf in $\Omega$ of holomorphic functions that vanish on $X$. In the same way we say that $\phi$ is a smooth $(0,q)$-form on $X$, $\phi \in E_{0,q}(X)$, if given a local embedding, there is a smooth form in a neighborhood in the ambient space such that $\phi$ is its pull-back to $X_{\text{reg}}$. It is well-known that this defines an intrinsic sheaf $E_{0,q}$ on $X$. It was proved in [13] that if $X$ is embedded as a reduced complete intersection (see Example [11]) in a pseudoconvex domain and $\phi$ is a $\bar{\partial}$-closed smooth form on $X$, then there is a solution $\psi$ to $\bar{\partial}\psi = \phi$ on $X_{\text{reg}}$. It has been an open question since then whether this holds more generally. In this paper we prove that this is indeed true for any Stein space $X$.

We introduce Koppelman formulas with weight factors on $X$ by means of which we can obtain intrinsic solutions operators for the $\bar{\partial}$-equation. We begin with a semi-global existence result.

Theorem 1.1. Let $Z$ be an analytic subvariety of pure dimension of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and assume that $\omega \subset\subset \Omega$. There are linear operators $\mathcal{K}: E_{0,q+1}(Z) \to E_{0,q}(Z_{\text{reg}} \cap \omega)$ and $\mathcal{P}: E_{0,0}(Z) \to O(\omega)$
such that
\begin{equation}
\phi(z) = \overline{\partial}K\phi(z) + \mathcal{K}(\overline{\partial}\phi)(z), \quad z \in Z_{\text{reg}} \cap \omega, \phi \in \mathcal{E}_{0,q}(Z), \ q > 0,
\end{equation}
and
\begin{equation}
\phi(z) = \mathcal{K}(\overline{\partial}\phi)(z) + \mathcal{P}\phi(z), \quad z \in Z_{\text{reg}} \cap \omega, \phi \in \mathcal{E}_{0,0}(Z).
\end{equation}

Moreover, there is a number \( M \) such that
\begin{equation}
\mathcal{K}\phi(z) = \mathcal{O}(\delta(z)^{-M}),
\end{equation}
where \( \delta(z) \) is the distance to \( Z_{\text{sing}} \).

The operators are given as
\begin{equation}
\mathcal{K}\phi(z) = \int_{\zeta} K(\zeta, z) \wedge \phi(\zeta), \quad \mathcal{P}\phi(z) = \int_{\zeta} P(\zeta, z) \wedge \phi(\zeta),
\end{equation}
where \( K \) and \( P \) are intrinsic integral kernels on \( Z \times (Z_{\text{reg}} \cap \omega) \) and \( Z \times \omega \), respectively. They are locally integrable with respect to \( \zeta \) on \( Z_{\text{reg}} \) and the integrals in \( 1.4 \) are principal values at \( Z_{\text{sing}} \). If \( \phi \) vanishes in a neighborhood of a point \( x \), then \( \mathcal{K}\phi \) is smooth at \( x \).

There is an integer \( N \) only depending on \( Z \) such that \( \mathcal{K} : C_{0,q+1}^k(Z) \to C_{0,q}^k(Z_{\text{reg}} \cap \omega) \) for each \( k \geq N \) and \( \mathcal{P} : C_{0,0}^k(Z) \to \mathcal{O}(\omega) \). Here \( \phi \in C_{0,q}^k(Z) \) means that \( \phi \) is the pullback to \( Z_{\text{reg}} \) of a \((0,q)\)-form of class \( C^k \) in a neighborhood of \( Z \) in the ambient space. As a corollary we have

**Corollary 1.2.** (i) If \( \phi \in C_{0,q}^k(Z) \), \( k \geq N + 1 \), and \( \overline{\partial}\phi = 0 \), then there is \( \psi \in C_{0,q}^k(Z_{\text{reg}} \cap \omega) \) with \( \psi(z) = \mathcal{O}(\delta(z)^{-M}) \) and \( \overline{\partial}\psi = \phi \).

(ii) If \( \phi \in C_{0,0}^{N+1}(Z) \) and \( \overline{\partial}\phi = 0 \) then \( \phi \) is strongly holomorphic.

Part (ii) is well-known, \[15\] and \[22\], but \( \mathcal{P}\phi \) provides an explicit holomorphic extension of \( \phi \) to \( \omega \). The existence result in \[13\] for a reduced complete intersection is also obtained by an integral formula, which however does not give an intrinsic solution operator on \( Z \).

We cannot expect our solution \( \mathcal{K}\phi \) to be smooth across \( Z_{\text{sing}} \). For instance, let \( Z \) be the germ of a curve at \( 0 \in \mathbb{C}^2 \) defined by \( t \mapsto (t^3, t^7 + t^8) \). If \( \phi = \overline{\omega}dz = 3(\overline{t}^3 + \overline{t}^{10})d\overline{t} \) then there is no solution \( \psi = f(t^3, t^7 + t^8) \) with \( f \) smooth. See \[20\] for other examples. However, it turns out that the difference of two of our solutions is anyway \( \partial \)-exact on \( Z_{\text{reg}} \) if \( q > 1 \) and strongly holomorphic if \( q = 1 \). By an elaboration of these facts we can prove:

**Theorem 1.3.** Assume that \( X \) is an analytic space of pure dimension. Any smooth \( \partial \)-closed \((0,q)\)-form \( \phi \) on \( X \), \( q \geq 1 \), defines a canonical class in \( H^q(X, \mathcal{O}_X) \), and if this class vanishes then there is a global smooth form \( \psi \) on \( X_{\text{reg}} \) such that \( \overline{\partial}\psi = \phi \). In particular, there is always such a solution if \( X \) is a Stein space.

We can use our integral formulas to solve the \( \partial \)-equation with compact support. As usual this leads to Hartogs results for holomorphic functions.
Theorem 1.4. Assume that $X$ is a Stein space of pure dimension $d$ with globally irreducible components $X^\ell$ and let $K$ be compact subset such that $X^\ell_{\text{reg}} \setminus K$ is connected for each $\ell$. Let $\nu$ be the (minimal) depth of the rings $\mathcal{O}_{X,x}$, $x \in X_{\text{sing}}$.

(i) If $\nu \geq 2$, then for each holomorphic function $\phi \in \mathcal{O}(X \setminus K)$ there is $\Phi \in \mathcal{O}(X)$ such that $\Phi = \phi$ in $X \setminus K$.

(ii) Assume that $\nu = 1$ and let $\chi$ be a cutoff function that is identically 1 in a neighborhood of $K$. There is a smooth $(d, d-1)$-form $\alpha$ on $X_{\text{reg}}$ such that the function $\phi \in \mathcal{O}(X \setminus K)$ has a holomorphic extension $\Phi$ across $K$ if and only if

\[(1.5) \quad \int_Z \bar{\partial} \chi \wedge \alpha \phi h = 0, \quad h \in \mathcal{O}(X),\]

where the integrals exist as principal values at $X_{\text{sing}}$.

If $X$ is normal and $X \setminus K$ is connected, then the conditions in (i) are fulfilled, and so we get a Hartogs theorem that was proved by other methods by Merker and Porten in [16]. Recently, Ruppenthal, [19], also gave a proof by $\bar{\partial}$-methods in case $X_{\text{sing}}$ discrete. If $X$ is not normal it is necessary to assume that $X^\ell_{\text{reg}} \setminus K$ is connected; see Example 3 in Section 8 below.

In the same way we can obtain the existence of $\bar{\partial}$-closed extensions across $X_{\text{sing}}$ of $\bar{\partial}$-closed forms in $X_{\text{reg}}$. This leads to existence results for the $\bar{\partial}$-equation in $X_{\text{reg}}$ via Theorem 1.1. In this way we obtain the following vanishing theorem that was proved already in [21] by analyzing the Cech cohomology of the sheaf $\mathcal{O}/\mathcal{J}$ in a local embedding of $X$.

Theorem 1.5. Assume that $X$ is a Stein space of pure dimension $d$. Let $\nu$ be the (minimal) depth of the rings $\mathcal{O}_{X,x}$, $x \in X_{\text{sing}}$. Assume that $\phi$ is a smooth $\bar{\partial}$-closed $(0, q)$-form in $X_{\text{reg}}$. If $0 < q < \nu - 1 - \dim X_{\text{sing}}$, then there is a smooth solution to $\bar{\partial} \psi = \phi$ in $X_{\text{reg}}$. If $q = 0 < \nu - 1 - \dim X_{\text{sing}}$, then $\phi$ extends to a strongly holomorphic function.

If $q = \nu - 1 - \dim X_{\text{sing}}$, then the same conclusion is true if and only if a certain moment condition, similar to (1.5), is fulfilled locally at $Z_{\text{sing}}$. The sufficient condition in case $q = 0$ is not necessary. The precise condition is Serre’s criterion; see Section 9, where we also present a conjecture about an analogous sharp(er) criterion for solvability of $\bar{\partial}$ for $q > 0$.

We have the following new vanishing result:

Theorem 1.6. Assume that $X$ is a Stein space of pure dimension $d$. If $\dim X_{\text{sing}} = 0$, then for each smooth $(0, d)$-form on $X_{\text{reg}}$ there is a smooth solution to $\bar{\partial} \psi = \phi$ on $X_{\text{reg}}$. 
If $\nu = \dim X$ (i.e., $X$ is Cohen-Macaulay) and $X_{\text{sing}}$ is discrete, then there is thus a local obstruction only when $q = \dim X - 1$ (as at a regular point).

Our solution operator $K$ behaves like a classical solution operator on $X_{\text{reg}}$ and by appropriate weights we get

**Theorem 1.7.** Assume that $Z$ is subvariety of pure dimension of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and let $\omega \subset \subset \Omega$. Given $M \geq 0$ there is an $N \geq 0$ and a linear operator $K$ such that if $\phi$ is a $\bar{\partial}$-closed $(0,q)$-form on $Z_{\text{reg}}$ with $\delta^{-N}\phi \in L^p(Z_{\text{reg}})$, $1 \leq p \leq \infty$, then $\bar{\partial}K\phi = \phi$ and $\delta^{-M}K\phi \in L^p(Z_{\text{reg}})$.

The existence of such solutions was proved in [9] (even for $(r,q)$-forms) by resolution of singularities and cohomological methods (for $p = 2$, but the same method surely gives the more general results). By a standard technique this theorem implies global results for a Stein space $X$.

In case $Z_{\text{sing}}$ is a single point more precise result are obtained in [18] and [8]. In particular, if $\phi$ has bidegree $(0,q)$, $q < \dim Z$, then the image of $L^2(Z_{\text{reg}})$ under $\bar{\partial}$ has finite codimension in $L^2(Z_{\text{reg}})$. See also [17], and the references given there, for related results. In [7], Fornæss and Gavosto show that, for complex curves, a Hölder continuous solution exists if the right hand side is bounded. Recently, certain hypersurfaces have also been considered, e.g., in [20].

In [24] Tsikh obtained a residue criterion for a weakly holomorphic function (or even a meromorphic function) to be strongly holomorphic in case $Z$ is a (reduced) complete intersection. This result was recently extended to a general variety in [3]. By formula (1.2) we get a new proof of this result and an explicit representation of the holomorphic extension.

The main ingredients in the construction of the integral operators $K$ and $P$ in Theorem 1.1 are a certain residue current $R$, introduced in [4] and [5], that is associated to the variety $Z$, and the integral representation formulas from [2]. We discuss the current $R$ in Section 2, and in Section 3 we obtain the Koppelman formula as the restriction to $Z$ of a certain global formula in the ambient set $\Omega$. In Section 6 we compute our Koppelman formulas more explicitly in case $Z$ is a reduced complete intersection. The resulting formula for $P$ coincides with the representation formula by Stout [23] and Hatziafratis [11] when $Z_{\text{sing}}$ is discrete.

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2. A residue current associated to $Z$

Let $Z$ be a subvariety of pure codimension $p = n - d$ of a pseudoconvex set $\Omega \subset \mathbb{C}^n$. The Lelong current $[Z]$ is a classical analytic object
that represents \( Z \). It is a \( d \)-closed \((p,p)\)-current such that

\[
[Z] \xi = \int_Z \xi
\]

for test forms \( \xi \). If \( \text{codim} \, Z = 1 \), \( Z = \{ f = 0 \} \) and \( df \neq 0 \) on \( Z_{\text{reg}} \), then a simple form of the Poincare-Lelong formula states that

\[
(2.1) \quad \overline{\partial} \frac{1}{f} \wedge \frac{df}{2\pi i} = [Z].
\]

To construct integral formulas we will use an analogue of the current \( \overline{\partial}(1/f) \), introduced in [4], for a general variety \( Z \). It turns out that this current, contrary to \( [Z] \), also reflects certain subtleties of the variety at \( Z_{\text{sing}} \) that are encoded by the algebraic description of \( Z \). Let \( J \) be the ideal sheaf over \( \Omega \) generated by the variety \( Z \). In a slightly smaller set, still denoted \( \Omega \), one can find a free resolution

\[
(2.2) \quad 0 \to \mathcal{O}(E_N) \overset{f_N}{\to} \cdots \overset{f_3}{\to} \mathcal{O}(E_2) \overset{f_2}{\to} \mathcal{O}(E_1) \overset{f_1}{\to} \mathcal{O}(E_0) \to 0
\]

of the sheaf \( \mathcal{O}/J \). Here \( E_k \) are trivial vector bundles over \( \Omega \) and \( E_0 = \mathbb{C} \) is the trivial line bundle. This resolution induces a complex of trivial vector bundles

\[
(2.3) \quad 0 \to E_N \overset{f_N}{\to} \cdots \overset{f_3}{\to} E_2 \overset{f_2}{\to} E_1 \overset{f_1}{\to} E_0 \to 0
\]

that is pointwise exact outside \( Z \). Let \( Z_k \) be the set where \( f_k \) does not have optimal rank. Then

\[
\cdots Z_{k+1} \subset Z_k \subset \cdots \subset Z_p = Z,
\]

and these sets are independent of the choice of resolutions, thus invariants of the sheaf \( \mathcal{F} = \mathcal{O}/J \). The Buchsbaum-Eisenbud theorem claims that \( \text{codim} \, Z_k \geq k \) for all \( k \), and since furthermore \( \mathcal{F} \) has pure codimension \( p \) in our case, \( Z_k \subset Z_{\text{sing}} \) for \( k > p \), and (see Corollary 20.14 in [6])

\[
(2.4) \quad \text{codim} \, Z_k \geq k + 1, \quad k \geq p + 1.
\]

There is a resolution (2.2) if and only if \( Z_k = \emptyset \) for \( k > N \), and this number is equal to \( n - \nu \), where \( \nu \) is the minimal depth of \( \mathcal{O}/J \). In particular, the variety is Cohen-Macaulay, or equivalently, the sheaf \( \mathcal{F} = \mathcal{O}/J \) is Cohen-Macaulay if and only if \( Z_k = \emptyset \) for \( k \geq p + 1 \). In this case we can thus choose the resolution so that \( N = p \).

**Remark 1.** Let us define \( Z^0 = Z_{\text{sing}} \) and \( Z^r = Z_{p+r} \) for \( r > 0 \). One can prove that these sets are independent of the embedding and thus intrinsic objects of the analytic space \( Z \) that describe the complexity of the singularities. In fact, by the uniqueness of minimal embeddings, it is enough to verify that these sets are unaffected if we add nonsense variables and consider \( Z \) as embedded into \( \Omega \times \mathbb{C}^m \). This follows, e.g., from the proof of Theorem 1.6 in [3]. \[\Box\]
Proposition 2.1. If \( V \) variety \( V \) has support on \( V \) such that \( \lambda > \lambda \) at \( \lambda \), there is a natural restriction \( V \) and it was pointed out that the currents \( U \) pseudomoromorphic. For each pseudomoromorphic current \( \overline{\mu} \) of codimension \( k > p \) as measures in \( Z \) is Cohen-Macaulay and \( \lambda > \lambda \) is again a reduced complete intersection, i.e., the classical Coleff-Herrera product (times \( e \)) is the minimal solution to \( \delta_\lambda s_a = 1 \) outside \( Z \) (with respect to the trivial metric on \( A \)). If we consider all forms as sections of the bundle \( A(T^*(\Omega) \oplus A) \), see [4], then \( u_k = s_a \delta s_a \) is \( k \)-1. If \( F \) is any holomorphic tuple such that \( [F] \sim [a] \), then, see, e.g., [4],

\[
R = R_p = \overline{\delta}[F]^{2\lambda} \wedge u_p |_{\lambda = 0} = \overline{\delta} \frac{1}{a_p} \ldots \overline{\delta} \frac{1}{a_1} \wedge da_1 \wedge \ldots \wedge da_p / (2\pi)^p = [Z].
\]

For further reference we also observe that

\[
\overline{\delta}[F]^{2\lambda} \wedge up \rightarrow \overline{\delta} \frac{1}{a_p} \ldots \overline{\delta} \frac{1}{a_1}
\]

as measures in \( Z_{\text{reg}} \) when \( \lambda \) is the classical Coleff-Herrera product (times \( e_1 \ldots \wedge e_p \)). It is well-known that

\[
\overline{\delta} \frac{1}{a_p} \ldots \overline{\delta} \frac{1}{a_1} \wedge \overline{\delta} \frac{1}{a_1} \wedge da_1 \wedge \ldots \wedge da_p / (2\pi)^p = [Z].
\]

Proposition 2.1. If \( \mu \in \mathcal{PM} \) with bidegree \((r,p)\) has support on a variety \( V \) of codimension \( k > p \) then \( \mu = 0 \).
It is proved in [5] that the restriction $R1_V$ of $R$ to any subvariety $V$ of $Z$ (of higher codimension) must vanish; we say that $R$ has the standard extension property, SEP, with respect to $Z$. For the component $R_p$ of $R$ the SEP follows immediately from Proposition 2.1, but the general statement is deeper; it depends on the assumption that $Z$ has pure codimension. In particular, if $h$ is a holomorphic function that does not vanish identically on any component of $Z$ (the interesting case is when $\{h = 0\}$ contains $Z_{\text{sing}}$), and $\chi$ is a smooth approximand of the characteristic function for $[1, \infty)$, then

\begin{equation}
\lim_{\delta \to 0} \chi(||h||/\delta)R = R.
\end{equation}

**Proposition 2.2.** For the residue current $R$ associated to (2.2) the following hold:

(i) There are smooth currents $\gamma_k$ on $Z_{\text{reg}}$ such that

\begin{equation}
R_k = \gamma_k \wedge [Z]
\end{equation}

there. Moreover, there is a number $M > 0$ such that

\begin{equation}
|\gamma_k| \leq C\delta^{-M},
\end{equation}

where $\delta$ is the distance to $Z_{\text{sing}}$.

(ii) If $\Phi$ is a smooth $(0, q)$-form whose pull-back to $Z_{\text{reg}}$ vanishes, then $R \wedge \Phi = 0$.

To be precise, $\gamma_k$ is a section of the bundle $\Lambda^{0-k-p}T^*(X) \otimes E_k \otimes \Lambda^pT_{1,0}(X)$. Part (ii) means that for each $\phi \in E_{0,q}(Z)$ we have an intrinsically defined current $R \wedge \phi$.

**Proof.** In a neighborhood of a given point $x \in Z_{\text{reg}}$ we can choose coordinates $(w', w'')$ such that $Z = \{w''_1 = \ldots = w''_p = 0\}$. Then $\mathcal{J}$ is generated by $w''$, the associated Koszul complex provides a (minimal) resolution of $\mathcal{O}/\mathcal{J}$ there, and the corresponding residue current $R = R_p$ is just the Coleff-Herrera product formed from the tuple $a = w''$, see Example 1 above. An arbitrary resolution at $x$ will contain the Koszul complex as a direct summand, and it follows, see Theorem 4.4 in [4] or Section 5 below, that therefore

\[ R_p = \alpha \bar{\partial} \frac{1}{w''_1} \wedge \ldots \wedge \bar{\partial} \frac{1}{w''_p}, \]

where $\alpha$ is a smooth $E_p$-valued form. It follows that we can take $\gamma_p$ as

\[ \tau = \alpha \otimes \frac{\partial}{\partial w''_1} \wedge \ldots \wedge \frac{\partial}{\partial w''_p} / (2\pi i)^p. \]

To obtain a global form, for $x \in Z_{\text{reg}}$, let $L_x$ be the orthogonal complement in $(T(X)_{1,0})_x$ of $(T(Z)_{1,0})_x$ (with respect to the usual metric in the ambient space). We can then modify $\tau$ so that it takes values in $\Lambda^pL$ without affecting (2.10), and $\gamma_p$ so defined is pointwise unique and hence a global smooth form on $Z_{\text{reg}}$. For further reference we also
notice that the norm of $\gamma_p$ will not exceed the norm of the locally defined form $\tau$. The proof of the asymptotic estimate (2.11) for $k = p$ is postponed to Section 5.

Outside $Z_{k+1}$ there is a smooth $(0,1)$-form $\alpha_{k+1}$ (with values in $\text{Hom}(E_k, E_{k+1})$) such that $R_{k+1} = \alpha_{k+1} R_k$. Moreover, the denominator of $\alpha_{k+1}$ is the modulus square of a tuple of subdeterminants of the matrix $f_k$, see [4], and hence $\alpha_k$ has polynomial growth when $\zeta \to Z_{k+1}$, see [4] Theorem 4.4. It follows that we can take

$$\gamma_k = \pm \alpha_k \cdots \alpha_{p+1} \gamma_p$$

for $k \geq p + 1$, and (2.11) for $k > p$ follows from the case $k = p$.

To see (ii), assume that $\Phi$ vanishes on $Z_{\text{reg}}$. Since $\Phi$ is $(0,q)$ we have that $R_k \wedge \Phi = \gamma_k \wedge [Z] \wedge \Phi = \gamma_k \wedge ([Z] \wedge \Phi) = 0$ on $Z_{\text{reg}}$. Now (ii) follows from (2.9). \[\square\]

3. CONSTRUCTION OF KOPPELMAN FORMULAS ON $Z$

We now recall the construction of integral formulas in [2] on an open set $\Omega$ in $\mathbb{C}^n$. Let $(\eta_1, \ldots, \eta_n)$ be a holomorphic tuple in $\Omega_\zeta \times \Omega_z$ that span the ideal associated to the diagonal $\Delta \subset \Omega_\zeta \times \Omega_z$. For instance, one can take $\eta = \zeta - z$. Following the last section in [2] we consider forms in $\Omega_\zeta \times \Omega_z$ with values in the exterior algebra $\Lambda_\eta$ spanned by $T^\ast_0(\Omega \times \Omega)$ and the $(1,0)$-forms $d\eta_1, \ldots, d\eta_n$. On such forms interior multiplication $\delta_\eta$ with

$$\delta_\eta = 2\pi i \sum_{1}^{n} \eta_j \frac{\partial}{\partial \eta_j}$$

has a meaning. We introduce $\nabla_\eta = \delta_\eta - \bar{\partial}$. Let $g = g_0 + \cdots + g_n$ be a smooth form (in $\Lambda_\eta$) defined for $z$ in $\omega \subset \subset \Omega$ and $\zeta \in \Omega$, such that $g_0 = 1$ on the diagonal $\Delta$ in $\omega \times \Omega$ (lower indices denote degree in $d\eta$) and $\nabla_\eta g = 0$. Such a form will be called a weight with respect to $\omega$. Notice that if $g$ and $g'$ are weights, then $g \wedge g'$ is again a weight. We will use one weight that has compact support in $\Omega$, and one weight which gives a division-interpolation type formula with respect to the ideal sheaf $\mathcal{J}$ associated to the variety $Z \subset \Omega$.

Example 2. If $\Omega$ is pseudoconvex and $K$ is a holomorphically convex compact subset, then one can find a weight with respect to some neighborhood $\omega$ of $K$, depending holomorphically on $z$, that has compact support (with respect to $\zeta$) in $\Omega$, see, e.g., Example 2 in [2]. Here is an explicit choice when $K$ is the closed ball $\overline{B}$ and $\eta = \zeta - z$. If $\sigma = \zeta \cdot d\eta/(2\pi i(\zeta|\zeta|^2 - \zeta \cdot z))$, then $\delta_\eta \sigma = 1$ for $\zeta \neq z$ and

$$\sigma \wedge (\bar{\partial} \sigma)^{k-1} = \frac{1}{(2\pi i)^k} \frac{\zeta \cdot d\eta \wedge (\bar{\zeta} \cdot d\eta)^{k-1}}{(|\zeta|^2 - \zeta \cdot z)^k}.$$
If \( \chi \) is a cutoff function that is 1 in a slightly larger ball, then we can take
\[
g = \chi - \bar{\partial} \chi \wedge \sigma = \chi - \bar{\partial} \chi \wedge [\sigma + \sigma \wedge \bar{\partial} \sigma + \sigma \wedge (\bar{\partial} \sigma)^2 + \cdots + \sigma \wedge (\bar{\partial} \sigma)^{n-1}].
\]
One can find a \( g \) of the same form in the general case. \( \square \)

Assume now that \( \Omega \) is pseudoconvex. Let us fix global frames for the bundles \( E_k \) in (2.3) over \( \Omega \). Then \( E_k \simeq \mathbb{C}^{\text{rank} E_k} \), and the morphisms \( f_k \) are just matrices of holomorphic functions. One can find (see [2] for explicit choices) \((k - \ell, 0)\)-form-valued Hefer morphisms, i.e., matrices, \( H_{k}^{\ell} : E_k \to E_\ell \) depending holomorphically on \( z \) and \( \zeta \), such that \( H_{k}^{\ell} = 0 \) for \( k < \ell \), \( H_{\ell}^{\ell} = \mathbb{I}_{E_\ell} \), and in general,
\[
(3.1) \quad \delta_\eta H_{k}^{\ell} = H_{k-1}^{\ell} f_k - f_{\ell+1}(z) H_{k+1}^{\ell+1};
\]
here \( f \) stands for \( f(\zeta) \). Let
\[
\begin{align*}
HU &= \sum_k H_{1}^{0} U_{k}, & HR &= \sum_k H_{0}^{0} R_{k}.
\end{align*}
\]
Thus \( HU \) takes a section \( \Phi \) of \( E_0 \), i.e., a function, depending on \( \zeta \) into a (current-valued) section \( HU \Phi \) of \( E_{1} \) depending on both \( \zeta \) and \( z \), and similarly, \( HR \) takes a section of \( E_0 \) into a section of \( E_{0} \).

Let \( s \) be a smooth \((1, 0)\)-form in \( \Lambda_\eta \) such that \( |s| \leq C|\eta| \) and \( |\delta_\eta s| \geq C|\eta|^2 \); such an \( s \) is called admissible. Then \( B = s/\nabla_\eta s \) is a locally integrable form and
\[
(3.2) \quad \nabla_\eta B = 1 - |\Delta|,
\]
where \( |\Delta| \) is the \((n, n)\)-current of integration over the diagonal in \( \Omega \times \Omega \). If \( \eta = \zeta - z \), \( s = \partial|\eta|^2 \) will do, and we then refer to the resulting form \( B \) as the Bochner-Martinelli form.

Let \( g \) be any smooth weight (with respect to \( \omega \subset\subset \Omega \), but not necessarily holomorphic in \( z \)), and with compact support in \( \Omega \). For a smooth \((0, q)\)-form \( \phi \) on \( Z \) we want to define
\[
(3.3) \quad \mathcal{K}\phi(z) = \int_\zeta (HR \wedge g \wedge B)_n \wedge \phi, \quad z \in Z_{\text{reg}} \cap \omega,
\]
and
\[
(3.4) \quad \mathcal{P}\phi(z) = \int_\zeta (HR \wedge g)_n \wedge \phi, \quad z \in \omega.
\]
Here the lower index denotes degree in \( d\eta \). To this end, let \( \Phi \) be any smooth form in \( \Omega \) whose pull-back to \( Z_{\text{reg}} \) is equal to \( \phi \). If \( \Phi \) is vanishing in a neighborhood of some given point \( x \) on \( Z_{\text{reg}} \), then \( B \wedge \Phi \) is smooth in \( \zeta \) for \( z \) close to \( x \), and the integral is to be interpreted as the current \( R \) acting on a smooth form. It is clear that this integral depends smoothly on \( z \in Z_{\text{reg}} \cap \omega \) and in view of Proposition 2.2 it only depends on \( \phi \).
Let us then assume that $\Phi$ has support in a neighborhood of $x$ in which $R = \gamma \omega |Z|$. Notice that

$$(HR \wedge g \wedge B)_n = H_0^0 R_0 \wedge (g \wedge B)_n - p + H_0^0 R_{p+1} \wedge (g \wedge B)_{n-p} + \cdots ,$$
cf., (2.5), and that

$$(3.5) \quad (g \wedge B)_{n-k} = O(1/|\eta|^{2n-2k-1})$$

so it is integrable on $Z_{\text{reg}}$ for $k \geq p$. Thus

$$(3.6) \quad \int_{\zeta} H_0^0 R_k \wedge (g \wedge B)_{n-k} \wedge \Phi = \pm \int_{\zeta \in Z} \gamma_k \cdot (H_0^0 \wedge (g \wedge B)_{n-k}) \wedge \Phi$$
is defined pointwise and depends continuously on $z \in \omega$, and it is in fact smooth on $Z_{\text{reg}} \cap \omega$ according to Lemma 3.2 below. It is also clear from (3.6) that the integral only depends on the pullback of $\Phi$ to $Z_{\text{reg}}$.

In the same way one gives a meaning to (3.4).

Since $B$ has bidegree $(\ast, \ast - 1)$, $K \phi$ is a $(0, q-1)$-form and $P \phi$ is $(0, q)$-form. It follows from (2.9) that (1.4) holds as principal values at $Z_{\text{sing}}$ with

$$(3.7) \quad K(\zeta, z) = \pm \gamma \cdot (H \wedge g \wedge B)_n, \quad P(\zeta, z) = \pm \gamma \cdot (H \wedge g)_n .$$

**Proposition 3.1.** Let $g$ be any smooth weight in $\Omega$ with respect to $\omega \subset \subset \Omega$ and with compact support in $\Omega$. For any smooth $(0, q)$-form on $Z$, $K \phi$ is a smooth $(0, q-1)$-form in $Z_{\text{reg}} \cap \omega$, $P \phi$ is a smooth $(0, q)$-form in $\omega$, and we have the Koppelman formula

$$(3.8) \quad \phi(z) = \frac{\bar{\partial}}{\partial} \int (HR \wedge g \wedge B)_n \wedge \phi + \int (HR \wedge g \wedge B)_n \wedge \bar{\partial} \phi + \int (HR \wedge g)_n \wedge \bar{\partial} \phi ,$$

for $z \in Z_{\text{reg}} \cap \omega$.

**Proof.** On a formal level the Koppelman formula follows from Section 7.4 in [2] by just restricting to $z \in Z_{\text{reg}} \cap \omega$, but for a strict argument one must be careful with the limit processes. Let $U^\lambda = |F|^{2\lambda} u$

and

$$R^\lambda = \sum_{k=0}^{N} R_k^\lambda = 1 - |F|^{2\lambda} + \bar{\partial}|F|^{2\lambda} \wedge u ,$$

so that $\nabla_f U^\lambda = 1 - R^\lambda$. We can have

$$g^\lambda = f(z) HU^\lambda + HR^\lambda$$
as smooth as we want by just taking Re $\lambda$ large enough. If Re $\lambda >> 0$, then, cf., [2] p.325, $g^\lambda$ is a weight, and thus, cf., (3.2),

$$\nabla_\eta (g^\lambda \wedge g \wedge B) = g^\lambda \wedge g - [\Delta]$$

from which we get

$$(3.9) \quad \bar{\partial}(g^\lambda \wedge g \wedge B)_n = [\Delta] - (g^\lambda \wedge g)_n .$$
As in [2] we get the Koppelman formula

\[ (3.10) \quad \Phi(z) = \int_{\zeta} (g^\lambda \wedge g \wedge B)_n \wedge \bar{\partial} \Phi + \bar{\partial}_z \int_{\zeta} (g^\lambda \wedge g \wedge B)_n \wedge \Phi + \int_{\zeta} (g^\lambda \wedge g)_n \wedge \Phi \]

for \( z \in \omega \), and since \( g^\lambda = HR^\lambda \) when \( z \in Z_{\text{reg}} \) we get

\[ \Phi(z) = \int_{\zeta} (HR^\lambda \wedge g \wedge B)_n \wedge \bar{\partial} \Phi + \int_{\zeta} (HR^\lambda \wedge g)_n \wedge \Phi, \quad z \in Z_{\text{reg}} \cap \omega. \]

It is now enough to check that

\[ (3.11) \int_{\zeta} (HR^\lambda \wedge g \wedge B)_n \wedge \Phi, \int_{\zeta} (HR^\lambda \wedge g)_n \wedge \Phi \]

have analytic continuations to \( \Re \lambda > 0 \) and tend weakly to \( K\Phi \) and \( P\Phi \), respectively, when \( \lambda \searrow 0 \). To this end, fix a point \( x \) on \( Z_{\text{reg}} \cap \omega \). If \( \Phi \) vanishes identically in a neighborhood of \( x \), then the first integral in (3.11) is just the current \( R^\lambda \) acting on a smooth form, and hence the continuation exists to \( \Re \lambda > -\epsilon \) and has the desired value at \( \lambda = 0 \). Therefore, we can assume that \( \Phi \) has compact support in a neighborhood of \( x \) where \( R = \gamma_\omega [Z] \). Let \( \psi(z) \) be a test form of bidegree \((n - p, n - p - q + 1)\) with support in \( Z_{\text{reg}} \cap \omega \). We have to prove that

\[ \int_{z \in Z} \psi(z) \wedge \sum_{k=0}^{N} \int_{\zeta} H^0_k R^\lambda_k \wedge (g \wedge B)_{n-k} \wedge \Phi \]

is analytic for \( \Re \lambda > 0 \) and tends to

\[ \int_{z \in Z} \psi(z) \wedge K\Phi(z) \]

when \( \lambda \searrow 0 \). For \( k \geq p \) we have, as before, cf., (3.5) that

\[ \int_{z \in Z} \psi(z) \wedge \int_{\zeta} H^0_k R^\lambda_k \wedge (g \wedge B)_{n-k} \wedge \Phi = \int_{\zeta} R^\lambda_k \wedge \Phi \wedge T\psi, \]

where \( T\psi(\zeta) \) is continuous. If \( a_j = w^j \) defines \( Z \) locally as in the proof of Proposition 2.2, then \( |F| \sim |a| \), and (see [4])

\[ u_k = \alpha_k (u_p \oplus \alpha) \]

where \( \alpha, \alpha_k \) are smooth and \( u_p \) is the form from Example 1. For \( \Re \lambda > 0 \), the form \( R^\lambda_k \) is locally integrable, and in view of (2.8) we have that \( R^\lambda_k \to R_k \) as measures when \( \lambda \searrow 0 \). On the other hand, if \( 1 \leq k < p \), then

\[ T\psi(\zeta) = \int_{z \in Z} H^0_k \wedge (g \wedge B)_{n-k} \wedge \Psi(z) = \mathcal{O}(|a(\zeta)|^{-(2p-2k-1)}). \]
Moreover, \( u_k = \alpha_k(u_k \oplus \alpha) = \mathcal{O}(1/|a|^{2k-1}) \). Thus

\[
\int_{z \in Z} \Psi(z) \int_{\zeta} H_0^0 R_1^k(g \wedge B)_{n-k} \wedge \Phi = \int_{\zeta} \mathcal{O}(\lambda |a|^{2\lambda-2p+1})
\]

which tends to 0 when \( \lambda \to 0 \). Finally, the case \( k = 0 \) is handled by dominated convergrence. The second integral in (3.11) is treated in a similar way.

**Lemma 3.2.** Let \( \Phi \) be a non-negative function in \( \mathbb{R}^N_x \times \mathbb{R}^N_y \) such that \( \Phi^2 \) is smooth and \( \sim |x - y|^2 \). For each integer \( \ell \geq 0 \), let \( \alpha_\ell \) denote a smooth function that is \( \mathcal{O}(|x - y|^\ell) \), and let \( \mathcal{E}_\nu \) denote a finite sum \( \sum_{\ell \geq 0} \alpha_\ell / \Phi^{\nu + \ell} \). If \( \nu < N \) and \( \xi \in C^k_c(\mathbb{R}^N) \), then

\[
T\xi(x) = \int_y \mathcal{E}_\nu(x, y) \xi(y) dy
\]

is in \( C^k(\mathbb{R}^N) \).

This lemma should be well-known, but for the reader’s convenience we sketch a proof. Let \( L_j = (\partial/\partial x_j + \partial/\partial y_j) \). It is readily checked that \( L_k \alpha_\ell = \alpha_\ell \) from which we conclude that \( L_k \mathcal{E}_\nu = \mathcal{E}_\nu \). The lemma then follows.

### 4. Proofs of Theorems 1.1 and 1.3

**Proof of Theorem 1.1.** If we choose \( g \) as the weight from Example 2 then \( P\phi \) will vanish for degree reasons unless \( \phi \) has bidegree \( (0, 0) \), i.e., is a function, and in that case clearly \( P\phi \) will be holomorphic for all \( z \) in a \( \omega \). Now Theorem 1.1 follows from the Koppelman formula (3.8) except for the asymptotic estimate (1.3).

After a slight regularization we may assume that \( \delta(z) \) is smooth on \( Z_{reg} \) or alternatively we can replace \( \delta \) by \( |h| \) where \( h \) is a tuple of functions in \( \Omega \) such that \( Z_{sing} = \{ h = 0 \} \), by virtue of Lojasiewicz’ inequality, [14] and [15]. Let \( \mu = HR \). We have to estimate

\[
\int_{\zeta} \mu(\zeta) \frac{\mathcal{O}(|\eta|)}{|\eta|^{2n-2p}}
\]

when \( z \to Z_{sing} \). To this end we take a smooth approximand of \( \chi_{[1/\sqrt{2}, \infty)}(t) \) and write (1.1) as

\[
\int_{\zeta} \chi(\delta(\zeta)/\delta(z)) \mu(\zeta) \frac{\mathcal{O}(|\eta|)}{|\eta|^{2n-2p}} + \int_{\zeta} (1 - \chi(\delta(\zeta)/\delta(z))) \mu(\zeta) \frac{\mathcal{O}(|\eta|)}{|\eta|^{2n-2p}}.
\]

In the first integral \( \delta(\zeta) \sim \delta(z) \) and since (2.10) holds here and the integrand is integrable we can use (2.11) and get the estimate \( \lesssim \delta(z)^{-M} \) for some \( M \). In the second integral we use instead that \( \mu \) has some fixed finite order so that the action can be estimates by a finite number of derivatives of \( (1 - \chi)\mathcal{O}(|\eta|)/|\eta|^{2n-2p} \), which again is like \( \delta(z)^{-M} \) for
some $M$, since here $C|\eta| \geq |\delta(\zeta) - \delta(z)| \geq \delta(z)/2$. Thus we have $|K\phi(z)| \lesssim \delta(z)^{-M}$.

**Proof of Corollary 1.2.** Suppose that $\nu$ is the order of the current $R$. Since $K\Phi$ essentially is the current $R$ acting on $\Phi$ times a smooth form, it is clear that the Koppelman formula remains true even if $\Phi$ is just of class $C^{\nu+1}$ in a neighborhood of $Z$. However, it seems to be more delicate matter to check that $K\Phi$ only depends on the pullback of $\Phi$ to $Z$. In order to copy the argument in the proof of Proposition 2.2 one may need (possibly just for technical reasons) some more regularity. After appropriate resolutions of singularities, the current $R$ is (locally) the push-forward of a finite sum of simple current of the form

$$\bar{\partial}_{t_1}^{a_1} \wedge \ldots \wedge \bar{\partial}_{t_r}^{1} \wedge \alpha_{\bar{a}_r+1} \wedge \ldots \wedge \bar{a}_m,$$

where $\alpha$ is smooth. If we choose $N$ as the sum of the powers of the denominators then the argument will work. This follows from an inspection of the arguments in [5] but we omit the details. In general, however, the number $N$ is much higher than the order of $R$. □

We now turn our attention to the proof of Theorem 1.3. We first assume that $X = Z$ is a subvariety of some domain $\Omega$ in $\mathbb{C}^n$. A basic problem with the globalization is that we cannot assume that there is one single resolution $[2,2]$ of $\mathcal{O}/\mathcal{J}$ in the whole domain $\Omega$. We therefore must patch together local solutions. To this end we will use Cech cohomology. Recall that if $\Omega_j$ is an open cover of $\Omega$, then a $k$-cochain $\xi$ is a formal sum

$$\xi = \sum_{|I|=k+1} \xi_I \wedge \epsilon_I$$

where $I$ are multi-indices and $\epsilon_I$ is a nonsens basis, cf., e.g., [1] Section 8. Moreover, in this language the coboundary operator $\partial$ is defined as $\rho \xi = \epsilon \wedge \xi$, where $\epsilon = \sum_j \epsilon_j$.

If $g$ is a weight as in Example 2 and $g' = (1 - \chi)s/\nabla_{\eta}s$, then

$$\nabla_{\eta}g' = 1 - g.$$  

(4.2)

Notice that the relations (3.1) for the Hefer morphism(s) can be written simply as

$$\delta_{\eta}H = Hf - f(z)H = Hf$$

if $z \in Z$.

**Proof of Theorem 1.3 in case $Z \subset \Omega \subset \mathbb{C}^n$.** Let $\Omega_j$ be a locally finite open cover of $\Omega$ with convex polydomains (Cartesian products of convex domains in each variable), and for each $j$ let $g_j$ be a weight with support in a slightly larger convex polydomain $\tilde{\Omega}_j \supset \Omega_j$ and holomorphic in $z$ in a neighborhood of $\mathbb{N}_j$. Moreover, for each $j$ suppose that we have
a given resolution \((\text{2.2})\) in \(\tilde{\Omega}_j\), choice of Hermitian metric, a choice of Hefer morphism, and let \((HR)_j\) be the resulting current. If \(\phi\) is a \(\bar{\partial}\)-closed \((0,q)\)-form in \(\Omega\), then

\[
(4.3) \quad u_j(z) = \int (hr^j \wedge g_j \wedge B)_n \wedge \phi
\]

is a solution in \(\Omega_j\) to \(\bar{\partial}u_j = \phi\). We will prove that \(u_j - u_k\) is (strongly) holomorphic on \(\Omega_{jk} \cap Z\) if \(q = 1\) and \(u_j - u_k = \bar{\partial}u_{jk}\) on \(\Omega_{jk} \cap Z_{\text{reg}}\) if \(q > 1\), and more generally:

**Claim I** Let \(u^0\) be the 0-cochain \(u^0 = \sum u_j \wedge \epsilon_j\). For each \(k \leq q - 1\) there is a \(k\)-cochain of \((0,q-k-1)\)-forms on \(Z_{\text{reg}}\) such that \(pu^k = \bar{\partial}u^{k+1}\) if \(k < q - 1\) and \(pu^{q-1}\) is a (strongly) holomorphic \(q\)-cocycle.

The holomorphic \(q\)-cocycle \(pu^{q-1}\) defines a class in \(H^q(\Omega, \mathcal{O}/\mathcal{J})\) and if \(\Omega\) is pseudoconvex this class must vanish, i.e., there is a holomorphic \(q-1\)-cochain \(h\) such that \(\rho h = pu^{q-1}\). By standard arguments this yields a global solution to \(\bar{\partial}\psi = \phi\). For instance, if \(q = 1\) this means that we have holomorphic functions \(h_j\) in \(\Omega_j\) such that \(u_j - u_k = h_j - h_k\) in \(\Omega_{jk} \cap Z\). It follows that \(u_j - h_j\) is a global solution in \(Z_{\text{reg}}\).

We thus have to prove Claim I. To begin with we assume that we have a fixed resolution with a fixed metric and Hefer morphism; thus a fixed choice of current \(HR\). Notice that if

\[
g_{jk} = g_j \wedge g_k' - g_k \wedge g_j';
\]

cf., \((4.2)\), then

\[
\nabla_{\eta} g_{jk} = g_j - g_k
\]

in \(\tilde{\Omega}_{jk}\). With \(g^{\lambda}\) as in Section 3 and in view of \((3.2)\), we have

\[
\nabla_{\eta}(g^{\lambda} \wedge g_{jk} \wedge B) = g^{\lambda} \wedge g_j \wedge B - g^{\lambda} \wedge g_k \wedge B - g^{\lambda} \wedge g_{jk} + g^{\lambda} \wedge g_{jk} \wedge [\Delta].
\]

However, the last term must vanish since \([\Delta]\) has full degree in \(d\eta\) and \(g_{jk}\) has at least degree 1. Therefore

\[
-\bar{\partial}(g^{\lambda} \wedge g_{jk} \wedge B)_n = (g^{\lambda} \wedge g_j \wedge B)_n - (g^{\lambda} \wedge g_k \wedge B)_n - (g^{\lambda} \wedge g_{jk})_n
\]

and as before we can take \(\lambda = 0\) and get, assuming that \(\bar{\partial}\phi = 0\),

\[
(4.4) \quad u_j - u_k = \int (hr^j \wedge g_{jk} \wedge B)_n \wedge \phi + \bar{\partial}_z \int (hr^j \wedge g_{jk} \wedge B)_n \wedge \phi.
\]

Since \(g_{jk}\) is holomorphic in \(z\) in \(\Omega_{jk}\) it follows that \(u_j - u_k\) is (strongly) holomorphic in \(\Omega_{jk} \cap Z\) if \(q = 1\) and \(\bar{\partial}\)-exact on \(\Omega_{jk} \cap Z_{\text{reg}}\) if \(q > 1\).

**Claim II** Assume that we have a fixed resolution but different choices of Hefer forms and metrics and thus different \(a_j = (HR)_j\) in \(\tilde{\Omega}_j\). Let \(\epsilon'_j\) be a nonsense basis. If \(A^0 = \sum a_j \wedge \epsilon'_j\), then for each \(k > 0\) there is a \(k\)-cochain

\[
A^k = \sum_{|I|=k+1} A_I \wedge \epsilon'_I,
\]
where \( A_I \) are currents on \( \tilde{\Omega}_I \) with support on \( \tilde{\Omega}_I \cap Z \) and holomorphic in \( z \) in \( \Omega_I \), such that

\[
\rho' A^k = \epsilon' \wedge A^k = \nabla_\eta A^{k+1}.
\]

In particular we have currents \( a_{jk} \) with support on \( Z \) and such that

\[
\nabla_\eta a_{jk} = a_j - a_k \text{ in } \tilde{\Omega}_{jk}.
\]

If

\[
w_{jk} = a_{jk} \wedge g_j \wedge g_k + a_j \wedge g_j \wedge g_k' - a_k \wedge g_k \wedge g_j';
\]

then

\[
\nabla_\eta w_{jk} = a_j \wedge g_j - a_k \wedge g_k.
\]

Notice that \( w_{jk} \) is a globally defined current. By a similar argument as above (and via a suitable limit process) one gets that

\[
u_j - u_k = \int (w_{jk})_n \wedge \phi + \bar{\partial}_z \int (w_{jk} \wedge B)_n \wedge \phi
\]

in \( \Omega_{jk} \) as before. In general we put

\[
\epsilon' = g = \sum g_j \wedge \epsilon_j.
\]

If, cf., (4.2),

\[
g' = \sum g'_j \wedge \epsilon_j
\]

then

\[
\nabla_\eta g' = \epsilon - g = \epsilon - \epsilon'.
\]

If \( a_I \) is a form on \( \tilde{\Omega}_I \), then \( a_I \wedge \epsilon'_j \) is a well-defined global form. Therefore \( A \) and hence

\[
W = A \wedge e^{g'},
\]

(i.e., \( W^k = \sum_j A^{k-j} (g')^j/j! \)) has globally defined coefficients and

\[
\rho W = \nabla_\eta W.
\]

In fact, since \( A \) and \( g' \) have even degree,

\[
\nabla_\eta(A \wedge e^{g'}) = \epsilon' \wedge A \wedge e^{g'} + A \wedge e^{g'} \wedge (\epsilon - \epsilon') = \epsilon \wedge A \wedge e^{g'}.
\]

By the yoga above then the \( k \)-cochain

\[
u^k = \int (W^k \wedge B)_n \wedge \phi
\]

satisfies

\[
\rho u^k = \bar{\partial}_z \int (W^{k+1} \wedge B)_n \wedge \phi + \int (W^{k+1})_n \wedge \phi.
\]

Thus \( \rho u^k = \bar{\partial} u^{k+1} \) for \( k < q - 1 \) whereas \( \rho \wedge u^{q-1} \) is a holomorphic \( q \)-cocycle as desired.

It remains to consider the case when we have different resolutions in \( \Omega_j \). For each pair \( j, k \) choose a weight \( g_{jk} \) with support in \( \tilde{\Omega}_{jk} \) that is holomorphic in \( z \) in \( \Omega_{s_{jk}} = \Omega_{jk} \). By Theorem 3 Ch. 6 Section F in [10] we can choose a resolution in \( \tilde{\Omega}_{s_{jk}} = \tilde{\Omega}_{jk} \) in which both of the
resolutions in $\tilde{\Omega}_j$ and $\tilde{\Omega}_k$ restricted to $\Omega_{s_{jk}}$ are direct summands. Let us fix metric and Hefer form and thus a current $a_{s_{jk}} = (HR)_{s_{jk}}$ in $\Omega_{s_{jk}}$ and thus a solution $u_{s_{jk}}$ corresponding to $(HR)_{s_{jk}} \wedge g_{s_{jk}}$. If we extend the metric and Hefer form from $\tilde{\Omega}_j$ in a way that respects the direct sum, then $(HR)_{j}$ with these extended choices will be unaffected, cf., Section 4 in [4]. On $\tilde{\Omega}_{j s_{jk}}$ we therefore practically speaking have just one single resolution and as before thus $u_j - u_s$ is holomorphic (if $q = 1$) and $\bar{\partial}u_{j s_{jk}}$ if $q > 1$. It follows that $u_j - u_k = u_j - u_s + u_s - u_k$ is holomorphic on $\Omega_{j k}$ if $q = 1$ and equal to $\bar{\partial}$ of

$$u_{j k} = u_{j s_{jk}} + u_{s_{j k}}$$

if $q > 1$. We now claim that each 1-cocycle

(4.6)  
\[ u_{j k} + u_{k l} + u_{l j} \]

is holomorphic on $\Omega_{j k l}$ if $q = 2$ and $\bar{\partial}$-exact on $\Omega_{j k l} \cap Z_{\text{reg}}$ if $q > 2$. On $\tilde{\Omega}_{s_{j k l}} = \tilde{\Omega}_{j k l}$ we can choose a resolution in which each of the resolutions associated with the indices $s_{jk}$, $s_{kl}$ and $s_{kj}$ are direct summands. It follows that $u_{j s_{jk}} + u_{s_{jk} s_{kl}} + u_{s_{kl} j}$ is holomorphic if $q = 2$ and $\bar{\partial}u_{j s_{jk} s_{kl}}$ if $q > 2$. Summing up, the statement about (4.6) follows. If we continue in this way Claim I follows.

It remains to prove Claim II. It is not too hard to check by an appropriate induction procedure, cf., the very construction of Hefer morphisms in [2], that if we have two choices of (systems of) Hefer forms $H_j$ and $H_k$ for the same resolution $f$, then there is a form $H_{jk}$ such that

(4.7)  
\[ \delta \eta H_{j k} = H_j - H_k + f(z)H_{j k} - H_{j k} f. \]

More generally, if

$$H^0 = \sum H_j \wedge \epsilon_j$$

then for each $k$ there is a (holomorphic) $k$-cochain $H^k$ such that (assuming $f(z) = 0$ for simplicity)

(4.8)  
\[ \delta \eta H^k = \epsilon \wedge H^{k-1} - H^k f \]

(the difference in sign between (4.7) and (4.8) is because in the latter one $f$ is to the right of the basis elements).

Elaborating the construction in Section 4 in [4], cf., Section 8 in [1], one finds, given $R^0 = \sum R_j \wedge \epsilon_j$, $k$-cochains of currents $R^k$ such that

(4.9)  
\[ \nabla f R^{k+1} = \epsilon \wedge R^k. \]

We now define a product of forms in the following way. If the multiindices $I, J$ have no index in common, then $(\epsilon_I, \epsilon_J) = 0$, whereas

$$\left( \epsilon_I \wedge \epsilon_I^*, \epsilon_I^* \wedge \epsilon_J \right) = \frac{|I||J|!}{(|I| + |J| + 1)!} \epsilon_I \wedge \epsilon_J.$$
We then extend it to any forms bilinearly in the natural way. It is easy to check that
\[(H^k f, R^\ell) = -(H^k, f R^\ell)\].
Using (4.8) and (4.9) (and keeping in mind that \(H^k\) and \(R^\ell\) have odd order) one can verify that
\[\nabla_\eta(H^k, R^\ell) = (\epsilon \wedge H^{k-1}, R^\ell) + (H^k, \epsilon \wedge R^\ell)\].
By a similar argument one can finally check that
\[A^k = \sum_{j=0}^{k} (H^j, R^{k-j})\]
will satisfy (4.5). Thus Claim II and hence Theorem 1.3 is proved in case \(Z = X\) is a subvariety of \(\Omega \subset \mathbb{C}^n\).

The extension to a general analytic space \(X\) is done in pretty much the same way and we just sketch the basic idea. First assume that we have a fixed \(\eta\) as before but two different choices \(s\) and \(\tilde{s}\) of admissible form, and let \(B\) and \(\tilde{B}\) be the corresponding locally integrable forms. Then, see [3],

\[(4.10) \quad \nabla_\eta(B \wedge \tilde{B}) = \tilde{B} - B\]
in the current sense, and by a minor modification of Lemma 3.2 one can check that
\[\int (H R \wedge g \wedge B \wedge \tilde{B})_n \wedge \phi\]
is smooth on \(X_{reg} \cap \omega\); for degree reasons it vanishes if \(q = 1\). It follows from (4.10) that \(\nabla_\eta(H R \lambda \wedge g \wedge B \wedge \tilde{B}) = H R \lambda \wedge g \wedge \tilde{B} - H R \lambda \wedge g \wedge B\) from which we can conclude that

\[(4.11) \quad \bar{\partial}_z \int (H R \wedge g \wedge B \wedge \tilde{B})_n \wedge \phi = \]
\[\int (H R \wedge g \wedge B)_n \wedge \phi - \int (H R \wedge g \wedge \tilde{B})_n \wedge \phi, \quad z \in \omega \cap Z_{reg}.\]

Now let us assume that we have two local solutions, in say \(\omega\) and \(\omega'\), obtained from two different embeddings of slightly larger sets \(\bar{\omega}\) and \(\bar{\omega}'\) in subsets of \(\mathbb{C}^n\) and \(\mathbb{C}^n',\) respectively. We want to compare these solutions on \(\omega \cap \omega'\). Localizing further, as before, we may assume that the weights both have support in \(\bar{\omega} \cap \bar{\omega}'\). After adding nonsense variables we may assume that both embeddings are into the same \(\mathbb{C}^n\), and after further localization there is a local biholomorphism in \(\mathbb{C}^n\) that maps one embedding onto the other one, see [10]. (Notice that a solution obtained via an embedding in \(\mathbb{C}^{n1}\) also can be obtained via an embedding into a larger \(\mathbb{C}^n\), by just adding dummy variables in the first formula.) In other words, we may assume that we have the same embedding in some open set \(\Omega \subset \mathbb{C}^n\) but two solutions obtained from different \(\eta\) and \(\eta'\). (Arguing as before, however, we may assume that
we have the same resolution and the same residue current \( R \). Locally there is an invertible matrix \( h_{jk} \) such that

\[
(4.12) \quad \eta'_j = \sum h_{jk} \eta_k.
\]

We define a vector bundle mapping \( \alpha^* : \Lambda_{\eta'} \to \Lambda_{\eta} \) as the identity on \( T^*_{0,+}(\Omega \times \Omega) \) and so that

\[
\alpha^* \eta'_j = \sum h_{jk} \eta_k.
\]

It is readily checked that

\[
\nabla_{\eta'} \alpha^* = \alpha^* \nabla_{\eta'}.
\]

Therefore, \( \alpha^* g' \) is an \( \eta \)-weight if \( g' \) is an \( \eta' \)-weight. Moreover, if \( H \) is an \( \eta' \)-Hefer morphism, then \( \alpha^* H \) is an \( \eta \)-Hefer morphism, cf., (3.1). If \( B' \) is obtained from an \( \eta' \)-admissible form \( s' \), then \( \alpha^* s' \) is an \( \eta \)-admissible form and \( \alpha^* B' \) is the corresponding locally integrable form. We claim that the \( \eta' \)-solution

\[
(4.13) \quad v' = \int (H'R \wedge g' \wedge B')_n \wedge \phi
\]

is comparable to the \( \eta \)-solution

\[
(4.14) \quad v = \int \alpha^* (H'R) \wedge \alpha^* g' \wedge \alpha^* B' \wedge \phi.
\]

Notice that we are only interested in the \( d\zeta \)-component of the kernels. We have that \( d\eta = d\eta_1 \wedge \ldots \wedge d\eta_n \) etc

\[
(H'R \wedge g' \wedge B')_n = A \wedge d\eta' \sim A \wedge \det(\partial \eta' / \partial \zeta) \wedge d\zeta
\]

and

\[
\alpha^* (H'R \wedge g' \wedge B')_n = A \wedge \det h \wedge \eta \sim A \wedge \det h \wedge \det(\eta / \partial \zeta) \wedge d\zeta.
\]

Thus

\[
\alpha^* (H'R \wedge g' \wedge B')_n \sim \gamma(\zeta, z)(H'R \wedge g' \wedge B')_n
\]

with

\[
\gamma = \det h \det \frac{\partial \eta}{\partial \zeta} \left( \det \frac{\partial \eta'}{\partial \zeta} \right)^{-1}.
\]

From (4.12) we have that \( \partial \eta'_j / \partial \zeta_k = \sum h_{jk} \partial \eta_k / \partial \zeta_k + O(|\eta|) \) which implies that \( \gamma \) is 1 on the diagonal. Thus \( \gamma \) is a smooth (holomorphic) weight and therefore (4.13) and (4.14) are comparable, and thus the claim is proved. This proves Theorem 1.3 in the case \( q = 1 \), and elaborating the idea as in the previous proof we obtain the general case.

Remark 2. In case \( X \) is a Stein space and \( X_{\text{sing}} \) is discrete there is a much simpler proof of Theorem 1.3. To begin with we can solve \( \overline{\partial} v = \phi \) locally, and modifying by such local solutions we may assume that \( \phi \) is vanishing identically in a neighborhood of \( X_{\text{sing}} \). There exists a sequence of holomorphically convex open subsets \( X_j \) such that \( X_j \) is
relatively compact in $X_{j+1}$ and $X_j$ can be embedded as a subvariety of some pseudoconvex set $\Omega_j$ in $\mathbb{C}^{n_j}$. Let $K_\ell$ be the closure of $X_\ell$. By Theorem 1.1 we can solve $\bar{\partial}u_\ell = \phi$ in a neighborhood of $K_\ell$ and $u_\ell$ will be smooth. If $q > 1$ we can thus solve $\bar{\partial}w_\ell = u_{\ell+1} - u_\ell$ in a neighborhood of $K_\ell$, and since $Z_{sing}$ is discrete we can assume that $\bar{\partial}w_\ell$ is smooth in $X$. Then $v_\ell = u_\ell - \sum_{\ell=1}^{\ell-1} \bar{\partial}w_k$ defines a global solution. If $q = 1$, then one obtains a global solution in a similar way by a Mittag-Leffler type argument.

5. The asymptotic estimate

To catch the asymptotic behaviour we have to globalize the proof of the first part of Proposition 2.2.

Since the functions $f_j^i$ generate the ideal $J$, given any fixed point $x$ on $Z_{reg}$ we can extract $h_1, \ldots, h_p$ from $f_j^i$ such that $dh_1 \wedge \ldots \wedge dh_p \neq 0$ at $x$. After a reordering of the variables we may assume that $\zeta = (\zeta', \zeta'') = (\zeta'_1, \zeta''_1, \ldots, \zeta''_p)$ such that $H = \det(\partial h_1/\partial \zeta'') \neq 0$ at $x$. Outside the hypersurface $\{H = 0\}$ we can (locally) make the change of coordinates $(\omega', \omega'') = (\zeta', h(\zeta', \zeta''))$ since

$$d(\omega', \omega'') = d(\zeta', \zeta'').$$

Moreover,

$$\frac{\partial}{\partial \omega''_j} = \frac{1}{H} \sum_k A_{jk} \frac{\partial}{\partial \zeta''_k},$$

where $A_{jk}$ are global holomorphic functions. Therefore, anywhere outside $\{H = 0\}$ we have that

$$\bar{\partial} \frac{1}{h_p} \wedge \ldots \wedge \bar{\partial} \frac{1}{h_1} = \frac{\det A_{jk}}{H^p} \wedge \frac{\partial}{\partial \zeta''_1} \wedge \ldots \wedge \frac{\partial}{\partial \zeta''_p} \wedge [Z].$$

Proposition 5.1. Given a point $x \in Z_{reg}$, there is a hypersurface $Y = \{H = 0\}$ avoiding $x$ such that

$$R_p = \tau \bar{\partial} \frac{1}{h_p} \wedge \ldots \wedge \bar{\partial} \frac{1}{h_1},$$

where $\tau$ is smooth outside $Y$ and $\tau = O(H^{-M})$ for some $M > 0$.

It follows from (5.1) and (5.2), cf., the proof of Proposition 2.1 that

$$|\gamma_p| \leq C|H|^{-M}.$$

With a finite number of such choices $H_j$ we have that

$$Z_{sing} = \bigcap_j \{H_j = 0\}$$

and thus

$$|\gamma_p(z)| \lesssim \min_j |H_j(z)|^{-M_j} \leq C|H(z)|^{-M},$$
where $H = (H_1, \ldots, H_\nu)$. However $|H| \geq \delta^N$ for some $N$ and hence \cite{2,11} follows for $k = p$.

It remains to prove Proposition \ref{prop-5.1}. We begin with the following simple lemma.

**Lemma 5.2.** Assume that $F_1, \ldots, F_m, \Phi$ are holomorphic $r$-columns at $x \in \Omega$ and that the germ $\Phi_x$ is in the submodule of $O_x^{\text{ev}}$ generated by $(F_j)_x$. If $F_j, \Phi$ have meromorphic extensions to $\Omega$, then there are holomorphic $A_j$ with meromorphic extension to (a possibly somewhat smaller neighborhood) $\Omega$ such that $\Phi = A_1 F_1 + \cdots + A_m F_m$.

**Proof.** The analytic sheaf $\mathcal{F} = (F_1, \ldots, F_m, \Phi)/(F_1, \ldots, F_m)$ is coherent in $\Omega$ and vanishing at $x$ so it must have support on a variety $Y$ outside $x$. If $h$ is holomorphic and vanishing on $Y$, then $h^M \mathcal{F} = 0$ in a Stein neighborhood $\Omega'$ of the closed ball if $M$ is large enough. Therefore there are holomorphic functions $a_j$ in $\Omega'$ such that $h^M \Phi = \sum a_j F_j$. \hfill \Box

Suppose that the holomorphic $r$-columns $F = (F_1, \ldots, F_m)$ and $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_m)$ are minimal generators of the same sheaf at $x$. It is well-known that then $\tilde{m} = m$ and there is a holomorphic invertible $m \times m$-matrix $a$ at $x$ such that $\tilde{F} = Fa$.

**Claim I** If $F, \tilde{F}$ have meromorphic extensions to $\Omega$, then we may assume that $a$ has as well.

**Proof.** By Lemma \ref{lem-5.2} we have global meromorphic matrices $a$ and $b$, holomorphic at $x$, such that $\tilde{F} = Fa$ and $F = Fb$. Thus $F = Fab$, and since $F$ is minimal, it follows that $ab = I + \alpha$ where the entries in $\alpha$ belong to the maximal ideal at $x$, i.e., $\alpha(x) = 0$. Therefore the matrix $I + \alpha$ is invertible at $x$, and so $b(I + \alpha)^{-1}$ is a meromorphic inverse to $a$ that is holomorphic and an isomorphism at $x$. \hfill \Box

Assume that $\mathcal{F}$ is a coherent sheaf in $\Omega$ of codimension $p$ at $x$ and let $O(E_k), f_k$ and $O(\tilde{E}_k), \tilde{f}_k$, $k = 0, \ldots, p$, be two minimal free resolutions of $\mathcal{F}$ at $x \in \Omega$. Moreover, assume that all $f_k, \tilde{f}_k$ have meromorphic extensions to $\Omega$. By iterated use of Claim I we get:

**Claim II** There are isomorphisms $g_k: O(E_k) \rightarrow O(\tilde{E}_k)$ holomorphic at $x$ and with meromorphic extensions to $\Omega$ such that $g_{k-1} f_k = \tilde{f}_k g_k$.

Assume for simplicity that $E_0 = \tilde{E}_0$. Outside some hypersurface $Y$ all the mappings $f_k, \tilde{f}_k, g_k$ are holomorphic, and there we have well-defined currents $R_p$ and $\tilde{R}_p$, and $R_p = g_p R_p$ there, cf., Section 4 in \cite{4}. Since the codimension is $p$ and the complexes end up at $p$ the residue currents $R_p$ and $\tilde{R}_p$ are independent of the choice of Hermitian metrics.

Now let $O(E_k), f_k$ be an arbitrary free resolution of $\mathcal{F}$ in $\Omega$. It is well-known that, given $x \in \Omega$, there is locally a holomorphic decomposition $E_k = E'_k \oplus E''_k$, $f_k = f'_k \oplus f''_k$ such that $O(E'_k), f'_k$ is a minimal free resolution of $\mathcal{F}$ at $x$ and $O(E''_k), f''_k$ is a free resolution of $0$. In other words, if we fix global holomorphic frames $e_k$ for $E_k$ to begin with,
then there are holomorphic $G_k$ with values in $GL(\text{rank } E_k, \mathbb{C})$ such that the first rank $E'_k$ elements in $e_k G_k$ generate $E'_k$ whereas the last ones generate $E''_k$. We claim, as the reader may expect at this stage, that

**Claim III** The $G_k$ can be assumed to have meromorphic extensions to $\Omega$.

**Proof of Claim III.** We proceed by induction. Suppose that we have found the desired decomposition up to $E_k$ and consider the mapping $f_{k+1}$ expressed in the new frame of $E_k$ and the original frame for $E_{k+1}$. Thus (the matrix for) $f_{k+1}$ is holomorphic at $x$ and globally meromorphic. Choose a minimal number of columns of $f_{k+1}$ such that the restrictions to $E'_k$ generate the stalk of Ker $f_k$ at $x$. After a trivial reordering of the columns we may assume that

$$f_{k+1} = \begin{pmatrix} f'_{k+1} & \Phi' \\ \Psi & \Phi'' \end{pmatrix}$$

By Lemma 5.2 there is a meromorphic matrix $a$, holomorphic and invertible at $x$, such that $\Phi' = f'_{k+1}a$. Therefore we can make the meromorphic change of frame

$$\begin{pmatrix} f'_{k+1} & \Phi' \\ \Psi & \Phi'' \end{pmatrix} \begin{pmatrix} I & -a \\ 0 & I \end{pmatrix} = \begin{pmatrix} f'_{k+1} & 0 \\ \Psi & f''_{k+1} \end{pmatrix}.$$ We now claim that

(5.3) $\text{Im } f''_{k+1} = \text{Ker } f''_k$

at $x$. By the lemma again we can then find a meromorphic matrix $a$, holomorphic and invertible at $x$ such that $\Psi = f''_{k+1}a$, and then after a similar meromorphic change of frame as before we get that the mapping $f_{k+1}$ has the matrix

$$\begin{pmatrix} f'_{k+1} & 0 \\ 0 & f''_{k+1} \end{pmatrix}$$

in the new frames. Thus it remains to check (5.3) which is indeed a statement over the local ring $\mathcal{O}_x$ and therefore “wellknown”. In any case, for each $z \in \text{Ker } f''_k$ we can solve

$$\begin{pmatrix} f'_{k+1} & 0 \\ \Psi & f''_k \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ z \end{pmatrix}. $$

Since $f'_{k+1}$ is minimal this implies that $\xi$ is in the maximal ideal at $x$ and hence $\Psi \xi$ is in the maximal ideal. Thus we can solve $f''_{k+1} = z - \alpha$ with $\alpha$ in the maximal ideal for each $z \in \text{Ker } f''_k$. However, since $f''_k$ is a resolution of 0 it follows that each $\text{Ker } f''_k$ is a free module. Expressed in a basis for $\text{Ker } f''_k$ we can solve then $f''_{k+1} \eta = I - \alpha$ and since $\alpha$ is in the maximal ideal it follows that $I - \alpha$ is invertible; hence (5.3) follows. □
We can now conclude the proof of Proposition 5.1. Let us equip the bundles $E_k = E'_k \oplus E''_k$ with some metrics that respect the decomposition, for instance the trivial metric with respect to the “new” frame. Both $\mathcal{O}(E'_k), f'_k$ and the Koszul complex generated by $h$ are minimal resolutions of $F = \mathcal{O}/\mathcal{J}$ at $x$, and since both of them have meromorphic extensions to $\Omega$ by Claim II there is a meromorphic $g_p$, holomorphic at $x$, such that

$$R'_p = g_p \bar{\partial} (1/h_p) \wedge \ldots \bar{\partial} (1/h_1).$$

Here $R'_p$ is the current obtained from the resolution $f'_k$. If $\tilde{R}_p$ is the current with respect to the new metric, then

$$\tilde{R}_p = \left( \begin{array}{c} g_p \\ 0 \end{array} \right) \bar{\partial} \frac{1}{h_p} \wedge \ldots \bar{\partial} \frac{1}{h_1}$$

with respect to the new frame, and hence we obtain the matrix for $\tilde{R}_p$ with respect to the original frame after multiplying with the matrix $G_p$. Notice that outside $Z_{p+1}$, the image of $f_{p+1}$ is a smooth (holomorphic) subbundle $H$ of $E_p$, and let $\pi$ be the orthogonal projection onto the orthogonal complement (with respect to the original metric) of $H$. Then, cf., [4], $R_p = \pi \tilde{R}_p$. Thus $\tau$ in (5.2) is $\pi G_p (g_p)^T$, and since $\pi$ does not increase norms, the estimate in Proposition 5.1 follows.

6. Examples

We explain what the currents $U$ and $R$ and our Koppelman formulas mean in the case of a reduced complete intersection. We also illustrate the techniques of Section 8 where $\bar{\partial}$-closed extensions and solutions with compact support are considered.

Let $f_1, \ldots, f_p$ be holomorphic functions, defined in a suitable neighborhood of $\mathbb{B} \subset \mathbb{C}^n$, and assume that $Z = \{f_1 = \cdots = f_p = 0\}$ has dimension $d = n - p$ and $df_1 \wedge \ldots \wedge df_p \neq 0$ on $Z_{\text{reg}}$, cf., Example [1]. Then $R = R_p$ is given by (2.6) with $a$ replaced by $f$.

Let $h_j$ be Hefer $(0,1)$-forms so that $\delta_{\eta} h_j = f_j (\xi) - f_j (z)$ and let $\tilde{h} = \sum h_j \wedge e^*_j$; recall that $\tilde{h}$ is a section of $\Lambda (A^* \oplus T^*(\Omega))$. The Hefer morphisms $H^\ell_k$ can be described as interior multiplication with $\tilde{h}^{k-\ell}/(k-\ell)!$ and a straightforward computation shows that

$$HR = H^0_p R_p = \bar{\partial} \frac{1}{f_p} \wedge \ldots \bar{\partial} \frac{1}{f_1} \wedge h_1 \wedge \ldots \wedge h_p = \gamma_{\wedge} [Z] \wedge h,$$

where $\gamma$ is a smooth $(p,0)$-vector field on $Z_{\text{reg}}$ such that $\gamma_{\wedge} df_p \wedge \ldots \wedge df_1 = (2\pi i)^p$ and $h = h_1 \wedge \ldots \wedge h_p$. According to the proof of the Koppelman formula(s) above, our solution operator to $\bar{\partial}$ on $Z_{\text{reg}}$ is

$$\mathcal{K} \phi (z) = \int_Z \gamma_{\wedge} [h \wedge (g \wedge B) d] \wedge \phi.$$
and the projection operator is

\begin{equation}
\mathcal{P}\phi = \int_Z \gamma_\omega [h \wedge g_d] \wedge \phi = \int_Z \gamma_\omega \left[ h \wedge \partial \bar{\zeta} \wedge (d\bar{\zeta} \cdot d\zeta)^{d-1} \right] \wedge \partial \chi \wedge \phi.
\end{equation}

Here \( g = \chi(\zeta) - \partial \bar{\chi}(\zeta) \wedge (\sigma / \nabla_\eta \sigma) \) from Example 2 and \( B \) is the Bochner-Martinelli form associated with \( \eta = \zeta - z \).

In particular, the right hand side of \((6.2)\) is a quite simple representation formula for a strongly holomorphic function \( \phi \) on \( Z \).

If \( Z_{\text{sing}} \) is discrete, avoids the boundary, \( \partial \mathbb{B} \), of the ball, and \( Z \) intersects \( \partial \mathbb{B} \) transversally, then we get back the representation formula for strongly holomorphic functions of Stout \cite{23} and Hatziafratis \cite{11} since then we may let \( \chi \) in \((6.2)\) be the characteristic function for \( \mathbb{B} \) and the integral becomes an integral over \( Z \cap \partial \mathbb{B} \).

Suppose in addition that \( d = 1 \). Let \( \xi = \sum \xi_j d\eta_j \) be a form satisfying \( \delta_\eta \xi = 1 \) outside \( \Delta \), e.g., \( \xi = B_1 \) or \( \xi = \sigma \). For some function \( C(z, \zeta) \) (a priori depending on \( \xi \)) we have \( h \wedge \xi = C d\eta_1 \wedge \cdots \wedge d\eta_n \). Applying \( \delta_\eta \) to this equality we get \((-1)^{n-1} h = C \delta_\eta (d\eta_1 \wedge \cdots \wedge d\eta_n) \) for \((z, \zeta) \in Z \times Z \) since \( \delta_\eta h = 0 \) for such \((z, \zeta) \). From this we read off that \( C \mid_{Z \times Z} \) is meromorphic, independent of \( \xi \), and with (at most) a first order singularity along the diagonal. We conclude that \( h \wedge (g \wedge B)_1 = \chi Cd\eta_1 \wedge \cdots \wedge d\eta_n \) and \( h \wedge g_1 = \bar{\partial} \chi \wedge C d\eta_1 \wedge \cdots \wedge d\eta_n \) on \( Z \times Z \) and our solution kernels \( K \) and \( P \) become

\[ K(z, \zeta) = \chi(\zeta) C(z, \zeta) \cdot (\gamma_\omega d\zeta), \quad P(z, \zeta) = \pm \bar{\partial} \chi(\zeta) \wedge C(z, \zeta) \cdot (\gamma_\omega d\zeta). \]

Notice that \( \gamma_\omega d\zeta \) is a holomorphic 1-form on \( Z_{\text{reg}} \) since \( \gamma_\omega [Z] = \bar{\partial} (1/f) \) there. If \( Z \) is the cusp \( Z = \{ f(z) = z_1^r - z_2^s = 0 \} \subset \mathbb{C}^2 \), where \( r \) and \( s \) are relatively prime integers \( 2 \leq r < s \), one readily checks that

\[ h = \frac{1}{2\pi i} \left( \frac{z_1^r - z_1^s}{\zeta_1 - z_1} d\eta_1 - \frac{z_2^s - z_2^s}{\zeta_2 - z_2} d\eta_2 \right), \quad \frac{\gamma}{2\pi i} = \frac{r z_1^{r-1} \partial / \partial \zeta_1 - s z_2^{s-1} \partial / \partial \zeta_2}{r^2 |\zeta_1|^{2(r-1)} + s^2 |\zeta_2|^{2(s-1)}}. \]

Using the parametrization \( \tau \mapsto (\tau^s, \tau^r) = (\zeta_1, \zeta_2) \) of \( Z \), a straight forward computation shows that \( \gamma_\omega d\zeta_1 \wedge d\zeta_2 = 2\pi i d\tau / \tau^{(r-1)(s-1)} \), yielding the following Cauchy formula

\[ \phi(t) = \int_{|\tau| = \rho} \frac{\phi(\tau) C(\tau, t) d\tau}{\tau^{(r-1)(s-1)}} - \lim_{\epsilon \to 0} \int_{\epsilon < |\tau| < \rho} \frac{\bar{\partial} \phi(\tau) \wedge C(\tau, t) d\tau}{\tau^{(r-1)(s-1)}}, \]

on \( Z \), where

\[ C(\tau, t) = \frac{1}{2\pi i} \frac{\tau^{r^s} - t^{r^s}}{(\tau^r - t^r)(\tau^s - t^s)}. \]

Assume now, cf., Section \( \S \) that \( Z_{\text{sing}} \subset K \subset \subset \mathbb{B} \) and let \( \varphi \) be a smooth \( \bar{\partial} \)-closed \((0, q - 1)\)-form on \( Z \setminus K \). Let \( \chi \) and \( \bar{\chi} \) be cutoff functions in \( \mathbb{B} \) such that \( \chi \) is 1 in a neighborhood of \( K \) and \( \bar{\chi} \) is 1 in a
neighborhood of \( \text{supp}(\chi) \). Put
\[
\tilde{g} = \tilde{\chi}(z) - \partial \tilde{\chi}(z) \wedge \sum_1^n \frac{\bar{z} \cdot d\eta \wedge (d\bar{z} \cdot d\eta)^{k-1}}{(2\pi i(|z|^2 - \bar{z} \cdot \zeta))^k},
\]
i.e., \( \tilde{g} \) is the weight from Example 2 with \( z \) and \( \zeta \) interchanged. Our formulas show that (6.1), with \( g \) replaced by \( \tilde{g} \) and \( \phi \) replaced by \( \partial \chi \wedge \phi \), is a solution with compact support in \( B \) (and in fact also smooth across \( Z_{\text{sing}} \)) to the equation \( \partial u = \partial \chi \wedge \phi \) on \( Z_{\text{reg}} \) provided that the corresponding projection term, cf., (6.2), (6.3)
\[
- \partial \tilde{\chi}(z) \wedge \int_{Z} \gamma \cdot \left[ h \wedge \frac{\bar{z} \cdot d\zeta \wedge (d\bar{z} \cdot d\zeta)^{d-1}}{(2\pi i(|z|^2 - \bar{z} \cdot \zeta))^d} \right] \wedge \partial \chi \wedge \phi.
\]
vanishes. Then \((1 - \chi)\phi + u\) is smooth and \( \partial \)-closed on \( Z \), and coincides with \( \phi \) outside a neighborhood in \( Z \) of \( K \). As long as \( q < d \), (6.3) is trivally zero; if \( q = d \), then it is clearly sufficient that
\[
\int Z \partial \chi \wedge \phi \xi \wedge (\gamma \cdot d\zeta) = 0, \quad \xi \in \mathcal{O}(Z).
\]
On the other hand, if \( \phi \) has a smooth \( \partial \)-closed extension, then (6.4) holds. In particular we see that if \( \phi \) is holomorphic on the regular part of the cusp \( Z \), then \( \phi \) is strongly holomorphic if and only if
\[
\int_{|\tau| = \epsilon} \phi \xi d\tau / \tau^{(r-1)(s-1)} = 0, \quad \xi \in \mathcal{O}(Z).
\]

7. Solutions formulas with weights

For the proof of Theorem 1.7 we use extra weight factors. Let \( A \) be any subvariety of \( Z \) that contains \( Z_{\text{sing}} \), in particular \( A \) may be \( Z_{\text{sing}} \) itself. Let \( a \) be a holomorphic tuple in \( \Omega \) such that \( \{ a = 0 \} \cap Z = Z_{\text{sing}} \), and let \( H^a \) be a holomorphic \((1,0)\)-form in \( \Omega \) such that \( \delta_q H^1 = a(\zeta) - a(z) \). If \( \psi \) is a \((0,q)\)-form that vanishes in a neighborhood of \( Z_{\text{sing}} \) we can incorporate the weight
\[
g^\mu_a = \left( \frac{a(z) \cdot a}{|a|^2} + H^a \cdot \bar{a} \right)^\mu
\]
in (3.8), i.e., we use the weight \( g^\mu_a \wedge g \) instead of just \( g \), the usual weight with compact support that is holomorphic in \( z \). Since the operators in Lemma 3.2 are bounded on \( L^p_{\text{loc}} \), we have that
\[
\psi = \partial \int_{Z_{\text{reg}}} \gamma \cdot (H \wedge g^\mu_a \wedge g \wedge B)_n \wedge \psi + \int_{Z_{\text{reg}}} \gamma \cdot (H \wedge g^\mu_a \wedge g \wedge B)_n \wedge \partial \psi,
\]
for \((0,q)\)-forms \( \psi \), \( q \geq 1 \), in \( L^p(Z_{\text{reg}}) \) that vanish in a neighborhood of \( Z_{\text{sing}} \). If \( \phi \) is as in Theorem 1.7 thus (7.2) holds for \( \psi = \chi(|a|^2 / \epsilon)\phi \) for each \( \epsilon > 0 \). For each natural number \( \mu \) we get a solution when \( \epsilon \to 0 \) in view of the asymptotic estimate of \( |\gamma| \) if just \( N \) is large enough. If \( \mu \) is
large, then the solution will vanish to high order at $Z_{\text{sing}}$ and therefore Theorem 1.7 follows.

8. Solutions with compact support

Theorems 1.4, 1.5, and 1.6 are Hartogs type theorems, because solvability of $\bar{\partial}\psi = \phi$ in $X_{\text{reg}}$ roughly speaking means that $\psi$ has a $\bar{\partial}$-closed smooth extension across $X_{\text{sing}}$. As usual therefore the proofs rely on the possibility to solve the $\bar{\partial}$-equation with compact support.

To begin with we assume that $Z$ is defined in a neighborhood of the closed unit ball $\overline{B}$. Since the depth of $\mathcal{O}/\mathcal{J}$ is at least $\nu$ we can choose, see, e.g., [6], a resolution (2.2) with $N = n - \nu$, and the associated residue current then is $R = R_p + \cdots + R_{n-\nu}$. Notice that $\bar{\partial}R_{n-\nu} = 0$.

Proposition 8.1. Let $Z$ be a subvariety of a neighborhood of $\overline{B}$ with the single singular point $0$. Assume that $\phi$ is a smooth $\bar{\partial}$-closed $(0,q)$-form in $Z \cap \overline{B \setminus \epsilon}$.

(i) If $q \leq \nu - 2$ there is a smooth $\bar{\partial}$-closed form $\Phi$ in $Z \cap \overline{B}$ that coincides with $\phi$ outside a neighborhood in $Z$ of $Z \cap \overline{B \setminus \epsilon}$.

(ii) If $q = \nu - 1$ the same is true if and only if

\begin{equation}
\int R_{n-\nu} \wedge \bar{\partial} \chi \wedge h \phi \wedge d\zeta = 0, \quad h \in \mathcal{O}(B),
\end{equation}

if $\chi$ is a cutoff function in $B$ that is 1 in a neighborhood of $\overline{B \setminus \epsilon}$.

Notice that (8.1) holds for all such $\chi$ if it holds for one single $\chi$.

Proof. First notice that if $q = \nu - 1$ and the extension $\Phi$ of $\phi$ exists, then choosing $\chi$ such that $\Phi = \phi$ on the support of $\bar{\partial} \chi$ we have that

$$R_{n-\nu} \wedge \bar{\partial} \chi \wedge h \phi \wedge d\zeta = \int Z \bar{\partial} \chi \wedge h \phi \wedge (\gamma_{n-\nu} \wedge d\zeta) = 0,$$

and since $R_{n-\nu} \wedge \bar{\partial} \chi \wedge h \phi \wedge d\zeta$ has compact support (8.1) must hold.

If $\chi$ is as in the theorem, then $(1 - \chi)\phi$ is a smooth extension of $\phi$ across $\overline{B \setminus \epsilon}$, and to find the $\bar{\partial}$-closed extension we have to solve $\bar{\partial}u = f$ with compact support, where $f = \bar{\partial} \chi \wedge \phi$. To this end, let $\tilde{\chi}$ be a cutoff function that is 1 in a neighborhood of a closed ball that contains the support of $f$ and let $g$ be the weight from Example 2 with this choice of $\tilde{\chi}$ but with $z$ and $\zeta$ interchanged. It does not have compact support with respect to $\zeta$, but since $f$ has compact support itself we still have the Koppelman formula (3.8). Clearly

$$u(z) = \int (HR \wedge g \wedge B)_n \wedge f$$

has support in a neighborhood of the support of $f$, and it follows from Koppelman’s formula that it is indeed a solution if the associated integral $\mathcal{P} f$ vanishes. However, since now $s$ is holomorphic in $\zeta$, for degree
reasons we have that
\[ Pf(z) = \int (HR^q \wedge)_n f = \pm \bar{\partial} \chi(z) \wedge \int HR_{n-1}^q \wedge s \wedge (\bar{s})^q \wedge \bar{\partial} \chi \wedge \phi \]
\[ = \pm \bar{\partial} \chi(z) \wedge \int HR_{n-1}^q \wedge \frac{\bar{z} \cdot d\zeta \wedge (d\bar{z} \cdot d\zeta)^q}{2\pi i (|z|^2 - \bar{z} \cdot \zeta)} \wedge \bar{\partial} \chi \wedge \phi. \]
If \( q < \nu - 1 \), then this integral vanishes since then \( R_{n-1}^q = 0 \). If \( q = \nu - 1 \), then \( Pf \) vanishes if \( (8.1) \) holds, keeping in mind that \( H \) is holomorphic in the ball. Since \( f = 0 \) in a neighborhood of \( 0 \) in \( Z \) we have that \( u \) is smooth, and \( \Phi = (1 - \chi)\phi + u \) is the desired \( \bar{\partial} \)-closed extension.

In particular we have proved a simple case of Theorem 1.4 and we obtain the general case along the same lines.

**Proof of Theorem 1.4.** Since \( X \) can be exhausted by holomorphically convex subsets each of which can be embedded in some affine space, we can assume from the beginning that \( X \subset \Omega \subset \mathbb{C}^n \), where \( \Omega \) is pseudo-convex. Let \( \omega \subset \subset \Omega \) be a holomorphically convex open set in \( \Omega \) that contains \( K \). Let \( \chi \) be a cutoff function in \( \omega \) that is 1 in a neighborhood of \( K \). Choose a cutoff function \( \tilde{\chi} \) that is 1 in a neighborhood of the holomorphically convex hull of the support of \( f \) and let \( g \) be the weight from Example 2. With this choice of \( \tilde{\chi} \) but with \( z \) and \( \zeta \) interchanged. As in the previous proof we get a solution with support in \( \omega \), provided that the corresponding projection term \( Pf \) vanishes. If \( \nu > 1 \), then \( Pf \) vanishes automatically and if \( \nu = 1 \), then \( Pf = 0 \) if

\[ (8.2) \int R_{n-1}^q \wedge d\zeta \wedge \bar{\partial} \chi \wedge \phi h = \pm \int_X \bar{\partial} \chi \wedge \phi h \wedge (\gamma_{n-1} \wedge d\zeta) = 0 \]

for all \( h \in \mathcal{O}(\omega \cap X) \), and by approximation it is enough to assume that \( (8.2) \) holds for \( h \in \mathcal{O}(X) \), i.e., that \( (1.5) \) holds.

Since \( X_{\text{sing}} \) is not contained in \( K \), our solution \( u \) is, outside of \( K \), only defined on \( X_{\text{reg}} \). Therefore \( \Phi = (1 - \chi)\phi + u \) is holomorphic in \( X_{\text{reg}} \), in a neighborhood of \( K \), and outside \( \omega \). Since \( X_{\text{reg}} \cap K \) is connected, \( \Phi = \phi \) there. (Even without the connectedness assumptions it follows that \( \Phi \) is in \( \mathcal{O}(X) \), since it has at most polynomial growth at \( Z_{\text{sing}} \), hence is meromorphic and its pole set is contained in \( \omega \cap X \).) The necessity of the moment condition follows as in the previous proof.

**Example 3.** Let \( X \subset \mathbb{C}^2 \) be an irreducible curve with one transverse self intersection at 0 in \( \mathbb{C}^2 \). Close to 0, \( X \) has two irreducible components, \( A_1, A_2 \), each isomorphic to a disc in \( \mathbb{C} \). Let \( K \subset A_1 \) be a closed annulus surrounding the intersection point \( A_1 \cap A_2 \). Then \( X \cap K \) is connected but \( X_{\text{reg}} \cap K \) is not. Denote the “bounded component” of \( A_1 \cap K \) by \( U_1 \) and put \( U_2 = X \setminus (K \cup U_1) \). Let \( \tilde{\phi} \in \mathcal{O}(X) \) satisfy \( \tilde{\phi}(0) = 0 \) and define \( \phi \) to be 0 on \( U_1 \) and equal to \( \tilde{\phi} \) on \( U_2 \). Then \( \phi \in \mathcal{O}(X \setminus K) \) and a straight forward verification shows that \( \phi \) satisfies the compatibility
condition (1.5); cf. also (6.4). But clearly, $\phi$ cannot be extended to a strongly holomorphic function on $X$. 

We now consider the case when $X_{\text{sing}}$ has positive dimension more closely. Locally we have an analogue of Proposition 8.1. For convenience we first consider the technical part concerning solutions with compact support.

**Proposition 8.2.** Let $Z$ be an analytic set defined in a neighborhood of $\overline{\mathbb{B}} \subset \mathbb{C}^n$, let $x \in Z_{\text{sing}}$, and let $a$ be a holomorphic tuple such that $Z_{\text{sing}} = \{ a = 0 \}$ in a neighborhood of $x$ and let $d' = \dim Z_{\text{sing}}$. Assume that $f$ is a smooth $\bar{\partial}$-closed $(0,q)$-form in a neighborhood of $x$ with $f = 0$ close to $Z_{\text{sing}}$ and with $f$ supported in $\{|a| < t\}$ for some small $t$.

(i) If $1 \leq q \leq \nu - d' - 1$, then in a neighborhood $U$ of $x$ one can find a smooth $(0,q-1)$-form, $u$, with support in $\{|a| < t\}$ and $\bar{\partial} u = f$ in $U \cap Z_{\text{reg}}$.

(ii) If $q = \nu - d'$, then one can find such a solution if and only if

\begin{equation}
\int \mathbb{R}^{n-\nu} \wedge h \wedge f = \pm \int_Z f \wedge h \wedge (\gamma_{n-\nu} \wedge d\zeta) = 0
\end{equation}

for all smooth $\bar{\partial}$-closed $(0,d')$-forms, $h$, such that $\text{supp}(h) \cap \{|a| \leq t\}$ is compact.

**Proof.** Let $\chi_a$ be a cutoff function in $\mathbb{B}$, which in a neighborhood of $x$ satisfies that $\chi_a = 1$ in a neighborhood of the support of $f$ and $\chi_a = 0$ in a neighborhood of $\{|a| \geq t\}$. Let also $H^a$ be a holomorphic $(1,0)$-form, as in the previous section, and define

\[ g^a = \chi_a(z) - \bar{\partial} \chi_a(z) \wedge \frac{\sigma_a}{\nabla_\eta \sigma_a}, \quad \sigma_a = \frac{a(z) \cdot H^a}{|a(z)|^2 - a(\zeta) \cdot a(z)}. \]

Then $g^a$ is a smooth weight for $\zeta$ in the support of $f$. Close to $x$ we can choose coordinates $(z',z'') = (z'_1, \ldots, z'_{d'}, z''_1, \ldots, z''_{p+r})$ centered at $x$ so that $Z_{\text{sing}} \subset \{|z''| \leq |z'|\}$. Since $f$ is supported close to $Z_{\text{sing}}$ we can choose a function $\chi = \chi(\zeta')$, which is 1 close to $x$ and $f \chi$ has compact support. Let now $g = \chi - \bar{\partial} \chi \wedge \sigma / \nabla_\eta \sigma$ be the weight from Example 2 but built from $z'$ and $\zeta'$. Our Koppelman formula now gives that

\[ u = Kf = \int (\mathbb{R} \wedge g^a \wedge g \wedge B)_n \wedge f \]

has the desired properties provided that the obstruction term

\[ \mathcal{P} f = \int (\mathbb{R} \wedge g^a \wedge g)_n \wedge f \]

vanishes. Since $g$ is built from $\zeta'$, $g$ has at most degree $d'$ in $d\zeta$. Moreover, $\mathbb{R}$ has at most degree $n - \nu$ in $d\zeta$ and $g^a$ has no degree in $d\zeta$. 

Thus, if \( q < \nu - d' \), then \((HR \wedge g^a \wedge g)_n \wedge f\) cannot have degree \( n \) in \( d\zeta \) and so \( P f = 0 \) in that case. This proves (i).

To show (ii), note that if \( q = \nu - d' \), then

\[
P f = \chi_a(z) \int HR_{n-\nu} \wedge g d^* \wedge f.
\]

Now, \( H \) depends holomorphically on \( \zeta \) and \( g d^* \) is \( \bar{\partial} \)-closed since it is the top degree term of a weight. Also, \( g \) has compact support in the \( \zeta' \)-direction, so \( \text{supp}(g) \cap \{|a| \leq t\} \) is compact and thus \( P f = 0 \) if (8.3) is fulfilled. On the other hand, it is clear that the existence of a solution with support in \( \{|a| < t\} \) implies (8.3).

**Proof of Theorem 1.5.** We first assume that \( \Omega = B \) and \( Z \subset \Omega \) has the single singular point 0. If \( q = 0 < \nu - 1 \) (or \( q = 0 = \nu - 1 \) and (8.1) holds), then it is clear from Proposition 8.1 that \( \phi \) is strongly holomorphic.

Fix \( r < 1 \) and let \( K_\ell = Z \cap (B_r \setminus B_{1/\ell}) \). If now \( q < \nu - 1 \) it follows from Proposition 8.1 that there is a \( \bar{\partial} \)-closed form \( \Phi_\ell \) in a neighborhood in \( Z \) of \( B_r \cap Z \) that coincides with \( \phi \) in a neighborhood of \( K_\ell \), and by Theorem 1.1 we therefore have a smooth solution \( u'_\ell \) to \( \bar{\partial}u'_\ell = \phi \) in a neighborhood of \( K_\ell \). Now \( u'_{\ell+1} - u'_\ell \) is a \( \bar{\partial} \)-closed \((0, q - 1)\)-form in a neighborhood of \( K_\ell \) and thus there is a global smooth \( \bar{\partial} \)-closed form \( w_\ell \) that coincides with \( u'_{\ell+1} - u'_\ell \) in a neighborhood of \( K_\ell \). If we let \( u_k = u'_k - (w_1 + \cdots + w_{k-1}) \) then \( u = \lim u_k \) exists and solves \( \bar{\partial}u = \phi \) in \( Z \cap B_r \setminus \{0\} \).

Notice that if the desired solution exists, then (8.1) must be fulfilled.

Assume now that \( X \) is an analytic space with arbitrary singular set. Arguing as in the proof of the case \( \dim X_{\text{sing}} = 0 \) above, we can conclude from Proposition 8.2: *Given a point \( x \) there is a neighborhood \( U \) such that if \( \phi \) is a \( \bar{\partial} \)-closed smooth \((0, q)\)-form in \( U \cap X_{\text{reg}} \), \( 0 \leq q < \nu - d' - 1 \), then \( \phi \) is strongly holomorphic if \( q = 0 \) and exact in \( X_{\text{reg}} \cap U' \), for a possibly slightly smaller neighborhood \( U' \) of \( x \), if \( q \geq 1 \).*

We define the analytic sheaves \( \mathcal{F}_k \) on \( X \) by \( \mathcal{F}_k(U) = \mathcal{E}_{0,k}(U \cap X_{\text{reg}}) \) for open sets \( U \subset X \). Then \( \mathcal{F}_k \) are fine sheaves and

\[
0 \to \mathcal{O}_X \to \mathcal{F}_0 \xrightarrow{\partial} \mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_2 \xrightarrow{\partial} \cdots
\]

is exact for \( k < \nu - d' - 1 \). It follows that

\[
H^k(X, \mathcal{O}_X) = \frac{\text{Ker } \partial \mathcal{F}_k(X)}{\partial \mathcal{F}_{k-1}(X)}
\]

for \( k < \nu - d' - 1 \). Hence Theorem 1.5 follows. □

**Proof of Theorem 1.6.** We first assume that \( X \subset \Omega \subset \mathbb{C}^n \) has an isolated singularity at 0. After a linear change of coordinates in \( \mathbb{C}^n \), and shrinking \( \Omega \), we may assume that the \( d \)-tuple \( a(z) = (z_1, \ldots, z_d) \) vanishes only at 0 on \( X \). Let \( U_\ell = \{|a| < 2^{-\ell}\} \cap \Omega \). We claim that if \( f \)
is a smooth \((0,d)\)-form in \(U_\ell \setminus \{0\}\), with support in \(U_\ell\), then there is a smooth form \(v_\ell\) such that \(f - \bar{\partial} v_\ell\) has support in \(U_{\ell+1}\) and \(v_\ell\) together with its derivatives up to order \(\ell\) are bounded by \(2^{-\ell}\) outside \(U_\ell\).

From the beginning we assume that \(\phi\) has support in \(U_1\). Taking the claim for granted we choose inductively \(f\) as \(\phi - \bar{\partial} v_1 - \ldots - \bar{\partial} v_{\ell-1}\), and we then obtain a solution \(v = v_1 + v_2 + \ldots\) in \(U \setminus \{0\}\) to \(\bar{\partial} v = \phi\).

To see the claim we use the weight (7.1) but with \(z\) and \(\zeta\) interchanged, i.e.,

\[
g^\mu = (\sigma(z) \cdot a(\zeta) + \bar{\partial} \sigma(z) \cdot H)^\mu,
\]

where \(\sigma = \bar{a}/|a|^2\). After a small modification we may assume that \(f\) vanishes identically in a neighborhood of 0. Then since \(f\) has support in \(U_\ell\),

\[
K f(z) = \int (HR \wedge g^\mu \wedge B)_n \wedge f
\]

together with a finite number of derivatives will be small outside \(U_\ell\) if \(\mu\) is chosen large enough. As before it is smooth since \(f = 0\) close to \(Z_{\text{sing}}\). Moreover it is a solution, because

\[
P f(z) = \int (HR \wedge g^\mu)_n \wedge f
\]

will vanish for degree reasons since \(\bar{\partial} \sigma_1 \wedge \ldots \wedge \bar{\partial} \sigma_d = 0\).

Finally assume that \(X\) is a general Stein space. Since we can solve \(\bar{\partial} u = \phi\) in a neighborhood of each singular point, we can find a global \(u\) such that \(f = \phi - \bar{\partial} u\) is smooth and vanishes in a neighborhood of \(X_{\text{sing}}\). By Theorem 1.3 we can solve \(\bar{\partial} v = f\) on \(X_{\text{reg}}\) and thus \(\bar{\partial} (v + u) = \phi\) in \(X_{\text{reg}}\).

9. Meromorphic and strongly holomorphic functions

A meromorphic function \(\phi\) on \(Z \subset \Omega\) can be represented by a meromorphic \(\Phi\) in the ambient space that is generically holomorphic on \(Z_{\text{reg}}\). Let \(R\) be the residue current associated with \(Z\). We show in [3] that \(R\phi\) is well-defined for any meromorphic \(\phi\). In fact, it can be defined as the analytic continuation to \(\lambda = 0\) of the current \(|h|^{2\lambda} \Phi R\), if \(\Phi\) is a representative of \(\phi\) in the ambient space and \(h\) is a holomorphic function in \(\Omega\) such that \(h \Phi\) is holomorphic and generically non-vanishing on \(Z\).

One also has a well-defined current

\[
R \wedge \bar{\partial} \phi = -\nabla_f (R \phi) = \bar{\partial} |h|^{2\lambda} \wedge R \phi |_{\lambda = 0}
\]

with support on the pole set \(P_\phi\) of \(\phi\).

In [3] we proved the following result that generalizes a previous result by Tsikh in the case of a complete intersection, see [24] and [12].

**Theorem 9.1.** If \(\phi\) is meromorphic on \(Z\), then \(\phi\) is strongly holomorphic if and only if \(R \wedge \bar{\partial} \phi = 0\).
By our Koppelman formula we can give a proof that provides an explicit analytic extension of \( \phi \) to \( \Omega \).

**Proof.** Assume that \( \phi \) is meromorphic on \( Z \) and let \( \Phi \) be a representative. For \( \text{Re} \lambda >> 0 \) we have from Theorem 1.1,

\[
|h(z)|^{2\lambda} \Phi(z) = \int |h|^{2\lambda} HR\Phi \wedge g + \int \bar{\partial}|h|^{2\lambda} HR\Phi \wedge g \wedge B.
\]

For \( z \in Z_{\text{reg}} \setminus \{ h = 0 \} \) we can take \( \lambda = 0 \) and we get (after choosing various \( h \)) the formula

\[
\phi(z) = \int HR\phi \wedge g + \int H(R \wedge \bar{\partial}\phi) \wedge g \wedge B, \quad z \in Z_{\text{reg}} \setminus P_\phi.
\]

If \( R \wedge \bar{\partial}\phi = 0 \) it follows that \( \phi(z) \) generically is equal to the first term on the right hand side which is a strongly holomorphic function.

We conclude by formulating a conjecture. If \( \phi \) is weakly holomorphic then \( P_\phi \subset Z_{\text{sing}} \) so \( R \wedge \bar{\partial}\phi \) has support on \( Z_{\text{sing}} \). Since \( R \wedge \bar{\partial}\phi \) is a \( \mathcal{P}\mathcal{M} \)-current it follows for degree reasons that it must vanish if

\[
\text{codim} Z_{\text{sing}} \geq 2 + p, \quad \text{codim} Z_k \geq 2 + k, \quad k > p,
\]

see [3]. This means that all weakly holomorphic functions are indeed strongly holomorphic if (9.1) is fulfilled. One can check that (9.1) is equivalent to the conditions \( R1 \) and \( S2 \) in Serre’s criterion, see, e.g., [6]. Therefore (9.1) is indeed equivalent to that all (germs of) weakly holomorphic functions are holomorphic, i.e., \( Z \) is a normal variety.

We suppose that \( \phi \) is a smooth \( \bar{\partial} \)-closed \((0,q)\)-form in \( Z_{\text{reg}} \) and assume that \( \phi \) admits some reasonable extension across \( Z_{\text{sing}} \) so that \( R \wedge \bar{\partial}\phi \) is a hypermeromorphic current. Arguing as in [3] it follows that \( R \wedge \bar{\partial}\phi \) must vanish if

\[
\text{codim} Z_{\text{sing}} \geq 2 + q + p, \quad \text{codim} Z_k \geq 2 + q + k, \quad k > p,
\]

which is (equivalent to) the conditions \( R_{q-1} \) and \( S_{q} \). The Koppelman formula will then produce a smooth solution to \( \bar{\partial}\psi = \phi \) on \( Z_{\text{reg}} \). One could therefore conjecture that the Dolbeault cohomology groups \( H^{0,\ell}(Z_{\text{reg}}) \) vanish for \( \ell \leq q \) if (and only if?) (9.2) holds.

If we consider \( Z \) as an intrinsic analytic space, then in the notation in Remark 1 the condition (9.2) means that codim \( Z^r \geq 2 + q + r \) for \( r \geq 0 \).

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