A Random Matrix Model for $\kappa$-$\mu$ Shadowed Fading

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Abstract—This paper presents a generalization of the $\kappa$-$\mu$ shadowed model when multiple antennas are present at both the transmitter and receiver sides, i.e., for a multiple-input multiple-output (MIMO) scenario. Using multivariate statistical theory, the MIMO $\kappa$-$\mu$ shadowed model is defined. Its probability density function (pdf) can be expressed in terms of the well-known gamma-Wishart distribution and the moment generating function is carried out from it. Closed-form expressions for the cumulative distribution function (cdf) and the pdf of the maximum eigenvalue are derived. Like the single-input single-output (SISO) model present in the literature, the MIMO $\kappa$-$\mu$ shadowed model allows the unification of some MIMO stochastic channels. In fact, the MIMO Rayleigh, MIMO Nakagami-$m$, MIMO Rician, MIMO $\kappa$-$\mu$ and MIMO Rician-Shadowed models can be derived from it, and so their SISO counterparts, i.e., the Rayleigh, Nakagami-$m$, Rician, $\kappa$-$\mu$ and Rician-Shadowed, respectively.

I. INTRODUCTION

The scientific community has been recently interested in the definition of new general fading models, aiming to provide a better fit to real measurements observed in different scenarios [1]–[3]. In such context, the $\kappa$-$\mu$ fading model [1] is one of those new models which has been paid more attention [4]–[7].

The $\kappa$-$\mu$ is a general fading model for a Line-Of-Sight (LOS) scenario, which includes some classic fading distributions. The Rician, Nakagami-$m$, one-sided Gaussian and Rayleigh can be derived from the $\kappa$-$\mu$ by setting its shape parameters $\kappa$ and $\mu$ to specific values. Moreover, using this fading model leads to better performance in numerous scenarios thanks to its flexible parameters which may be fixed to any real positive value [1].

In turn, the $\kappa$-$\mu$ model has been generalized under the name of $\kappa$-$\mu$ shadowed [3], thus jointly including the effects of small and large-scale fading. Recently, this new model has shown excellent performance when compared to measured fading channels in underwater acoustic communications [3], [8] and body communications channels [9]. The main novelty of this model is that it takes into account a possible shadowing in the LOS path. Thus the $\kappa$-$\mu$ shadowed fading is also suitable for land-mobile satellite channels because the Rician-Shadowed case, which postulates the Rician distribution for the multipath fading and the Nakagami-$m$ distribution for the shadowing of the LOS path, is included in the model [3].

In the literature, the statistical characterization of the aforementioned channel models is usually tackled on a single-link fashion, i.e., for a single-input single-output (SISO) communication system. However, modern communication systems like Wi-Fi standards or 4G always use several antennas at both the transmitter and receiver sides, i.e. multiple-input multiple-output (MIMO) systems. In this scenario, the channel is no longer a scalar random variable; instead, it is characterized by a random matrix.

While the statistical characterization of MIMO fading channels is of extreme interest, it is very challenging to generalize existing fading models to a MIMO scenario. For this reason, random matrix models for fading channels other than Rayleigh are scarce in the literature [10], [11].

In this paper, we introduce a random matrix model for the $\kappa$-$\mu$ shadowed fading model, suitable for MIMO communication systems. Not only the probability density function (pdf) and the moment generating function (mgf) of such model will be derived in closed form, but also the cumulative density function (cdf) and pdf of the maximum eigenvalue distribution. We show that the model here presented unifies the MIMO Rayleigh, MIMO Nakagami-$m$, MIMO Rician, MIMO $\kappa$-$\mu$ and MIMO Rician shadowed models when its parameters are set to specific values.

This paper is structured as follows. In Section II, we introduce some preliminary results needed in our following derivations. In Section III, the MIMO $\kappa$-$\mu$ model is introduced. In Section IV, the MIMO $\kappa$-$\mu$ shadowed model is defined and then closed-form expressions for the pdf, mgf, and the maximum eigenvalue distribution of the random matrix model are derived. In Section V, we present some numerical results. Finally, conclusions are drawn.

Throughout this paper, matrices are denoted in bold upper-case. The matrix $I_p$ symbolizes the $p \times p$ identity matrix, while $0_p$ is the $p \times p$ null matrix. When the operator $|\cdot|$ is used around a matrix, it indicates the determinant of that square matrix; otherwise, it is the complex modulus. The matrix $\bar{A}$ is the expectation matrix of $A$. The conditional matrix $A|B$ means the matrix $A$ given matrix $B$. The operator $tr(\cdot)$ represents the matrix trace while $etr(\cdot)$ is the exponential of the matrix trace. The super-index $H$ means the conjugate transpose and the symbol $\sim$ expresses statistically distributed as. If $H$ is a $p \times n$ matrix, we refer to $H^H H$ as its Gram matrix. Finally, $V(\cdot)$ is the Vandermonde determinant [12] p. 29] and $A > 0$ indicates positive definiteness.

II. PRELIMINARIES

Definition 1: Noncentral Complex Wishart Matrix.

Let $H$ be a $p \times n$ ($p \geq n$) complex Gaussian matrix distributed as $CN(H, I_p \otimes \Sigma)$, where $H \in \mathbb{C}^{p \times n}$ is the expectation matrix and $I_p \otimes \Sigma$ is the covariance matrix, with $\Sigma \in \mathbb{C}^{n \times n} > 0$. The
Gram matrix \( \mathbf{W} = \mathbf{H}_1^H \mathbf{H}_1 \) has a noncentral complex Wishart distribution with \( p \) degrees of freedom, covariance matrix \( \mathbf{\Sigma} \) and matrix of noncentrality parameters \( \mathbf{\Theta} = \mathbf{\Sigma}^{-1} \mathbf{H}_1^H \mathbf{H}_1 \), i.e., \( \mathbf{W} \sim \mathcal{W}_n(p, \mathbf{\Sigma}, \mathbf{\Theta}) \), if its pdf is given by [13, eq. (99)]

\[
    f_{\mathbf{W}}(\mathbf{W}) = \frac{\text{etr}(\mathbf{W}^-1)}{\Gamma_n(p) |\mathbf{\Sigma}|^p} \times \text{etr}(\mathbf{\Theta}) \cdot F_1(p; \mathbf{\Theta} \mathbf{\Sigma}^{-1} \mathbf{W})
\]

where \( \bar{n}_n(p) \) is the complex multivariate gamma function [13, eq. (83)], and \( q \cdot F_1(\cdot; \cdot) \) is the complex confluent hypergeometric function of matrix argument [13]. Notice that the first line expression of the eq. (1) corresponds to the pdf of a central complex Wishart matrix [13, eq. (94)].

Definition 2: Complex Gamma-variate Matrix.

The \( n \times n \) Hermitian positive-definite matrix \( \mathbf{B} \) is a complex gamma-variate matrix, with scalar parameter \( \beta (\beta \geq n) \) and matrix parameter \( \mathbf{\Omega} \), if \( \mathbf{B} \) follows the complex gamma distribution \( \Gamma_n(\beta, \mathbf{\Omega}) \) [14, p. 254, p. 356], i.e.,

\[
    f_{\mathbf{B}}(\mathbf{B}) = \frac{|\mathbf{B}|^{\beta-n}|\mathbf{\Omega}|^{\beta}}{\Gamma_n(\beta)} \cdot \text{etr}(\mathbf{\Omega} \mathbf{B}).
\]

Notice that the complex gamma distribution \( \Gamma_n(\beta, \mathbf{\Omega}) \) can be seen as the continuous extension of the central Wishart distribution when its scalar parameter takes real positive values. In fact, eq. (2) equals the first line of eq. (1) when \( \beta = p \) and \( \mathbf{\Omega} = \mathbf{\Sigma}^{-1} \).

Definition 3: Complex Gamma-Wishart Matrix.

Assume to have a \( q \times n \) \( (q \geq n) \) matrix, \( \mathbf{H} \), defined as

\[
    \mathbf{H} = \mathbf{\bar{H}} + \mathbf{\bar{\bar{H}}}
\]

where \( \mathbf{\bar{H}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_q \otimes \mathbf{\Sigma}) \) and \( \mathbf{\bar{H}}^H \mathbf{\bar{\bar{H}}} \sim \Gamma_n(\alpha, \mathbf{\Omega}) \) are statistically independent. Then the Gram matrix \( \mathbf{A} = \mathbf{H}^H \mathbf{H} \) follows the gamma-Wishart distribution \( \mathcal{GW}_n(\alpha, q, \mathbf{\Theta}, \mathbf{\Sigma}) \) given by [10].

\[
    f_{\mathbf{A}}(\mathbf{A}) = \frac{\text{etr}(\mathbf{A}^{-1}) |\mathbf{A}|^{\alpha-n} |\mathbf{\Sigma}^{-1}|^{\alpha}}{\Gamma_n(q) |\mathbf{\Theta}| |\mathbf{\Sigma}^{-1} + \mathbf{\Omega}|^{\alpha}} \times \text{etr}(\mathbf{\Theta}) \cdot 1 \cdot F_1(q; \alpha; \mathbf{\Sigma}^{-1}(\mathbf{\Sigma}^{-1} + \mathbf{\Omega})^{-1} \mathbf{\Sigma}^{-1} A).
\]

where \( q \cdot F_1(\cdot; \cdot) \) is the complex confluent hypergeometric function of matrix argument [13].

Theorem 1: Sum of noncentral Wishart matrices.

If the \( n \times n \) Hermitian positive-definite matrices \( \mathbf{W}_1, \ldots, \mathbf{W}_r \) are all independent and \( \mathbf{W}_i \sim \mathcal{W}_n(p_i, \mathbf{\Sigma}, \mathbf{\Theta}_i) \), \( i = 1, \ldots, r \), then \( \sum_{i=1}^r \mathbf{W}_i \sim \mathcal{W}_n(p, \mathbf{\Sigma}, \mathbf{\Theta}) \), with \( p = \sum_{i=1}^r p_i \) and \( \mathbf{\Theta} = \sum_{i=1}^r \mathbf{\Theta}_i \).

Proof: The proof is an immediate generalization of the proof in [15] Theor. 10.3.4] for real matrices.

III. MIMO \( \kappa-\mu \) MODEL

The \( \kappa-\mu \) fading model is based on the physical scenario described in [11]. The signal is divided into different clusters of waves. In each cluster, there is a deterministic LOS component with arbitrary power which propagates in an homogeneous environment, leading to complex Gaussian processes with some non-zero mean.

In fact, the SISO \( \kappa-\mu \) in [1] can be seen as a generalization of the well-known Rician model. A \( \kappa-\mu \) random variable can be obtained by a sum of \( \mu \) Rician random variables.

Bringing this environment to a MIMO scenario, the channel can be divided into \( \mu \) different clusters. In each cluster \( i \), a different non-zero mean complex Gaussian matrix of \( p \times n \) elements, \( \mathbf{H}_i \), is defined. Thus, the MIMO \( \kappa-\mu \) physical model can be expressed as

\[
    \mathbf{Z} = \sum_{i=1}^\mu \mathbf{H}_i^H \mathbf{H}_i
\]

where \( \mathbf{H}_i \sim \mathcal{CN}(\mathbf{H}_i, \mathbf{I}_p \otimes \mathbf{\Sigma}) \).

In the rest of the paper, we consider that the receiver is equipped with \( n \) antennas, i.e., we consider that the minimum number of antennas of such system stays in the receiver side. Notice that this does not imply any loss of generality, since every Gram matrix will have \( n \times n \) elements, with \( n = \min(p, n) \), but we take this assumption for the sake of notational simplicity.

However, it is convenient to define the MIMO \( \kappa-\mu \) model in terms of the parameter \( \kappa \), which has not been introduced yet. In a SISO scenario, this parameter can be interpreted as the ratio between the total power of the dominant components and the total power of the scattered waves [1]. In a MIMO scenario, the parameter \( \kappa \) has not a straightforward physical interpretation and becomes a matrix denoted \( \mathbf{K} \) which can be expressed as

\[
    \mathbf{K} = \mu^{-1} \mathbf{\Sigma}^{-1} \mathbf{D}
\]

where the product \( \mathbf{\Sigma}^{-1} \mathbf{D} = \sum_{i=1}^\mu \mathbf{H}_i^H \mathbf{H}_i \) is the sum of the \( \mu \) matrices of noncentrality parameters and \( \mathbf{\Sigma} \) is the covariance matrix of all the scattered waves. In fact, through the matrices \( \mathbf{K} \) and \( \mathbf{\Sigma} \), the MIMO \( \kappa-\mu \) model is considering the spatial correlation at the receiver side of such multi-antenna system. The deduction of the pdf follows.

Lemma 1: Let \( \mathbf{L} \) be a \( n \times n \) Gram matrix of a MIMO \( \kappa-\mu \) channel, with a covariance matrix of the scattered waves \( \mathbf{\Sigma} \) and a matrix parameter \( \mathbf{K} \). Then \( \mathbf{Z} \sim \mathcal{W}_n(r, \mathbf{\Sigma}, \mathbf{K}) \), where \( r = \mu \cdot p \).

Proof: Applying Theorem 1, the pdf of \( \mathbf{Z} \) is directly derived. Notice that this result can be extended for \( r \) taking real positive values [15], even though the physical model interpretation is lost. Also notice that, in case of considering zero mean Gaussian matrices in eq. (5), we obtain the channel model that we will call MIMO Nakagami-\( m \) for \( \mu = m \).

Moreover, if we fix \( n = p = 1 \), then the SISO \( \kappa-\mu \) model presented in [1] is obtained thanks to the relationship between the Bessel hypergeometric function and the modified Bessel function [16, eq. (9.1.69)]. In this case, \( \mathbf{K} \) is reduced to the scalar parameter \( \kappa \) and \( \mathbf{\Sigma} \) is reduced to the parameter \( 2\sigma^2 \) defined in [1, 3].

In spite of its relatively simple derivation, the MIMO \( \kappa-\mu \) model is here defined for the first time in the literature, to the best of our knowledge. Since the Gram matrix of the MIMO \( \kappa-\mu \) fading model is shown to follow a noncentral Wishart distribution, the joint eigenvalue distribution is well-known.
matrices, thus it is distributed as the noncentral Wishart by
of physical meaning.

channel clusters, iii) corresponds to
mgf of the MIMO SISO model \[3\] but it is introduced in this new matrix model
at the receiver side.

\[ W \]

represents the scattered components of the
[17]. Once the MIMO \( \kappa - \mu \) is stated, the MIMO \( \kappa - \mu \) shadowed
is now introduced.

IV. MIMO \( \kappa - \mu \) SHADOWED MODEL

In this section, the \( \kappa - \mu \) shadowed model is defined for a
MIMO scenario. First, the pdf is presented and then used
to derive the mgf. Then, the cdf and pdf of the maximum
eigenvalue distribution are carried out for two interesting cases
from a physical model viewpoint.

A. Model definition

The \( \kappa - \mu \) shadowed fading model arises when each dominant
component of all the clusters is considered to suffer from
shadowing. From the SISO physical model \[3\], all the LOS
components are subject to shadowing, that is modeled using
the Nakagami-\( m \) distribution, and which can be related to the
univariate gamma-distribution by a square root. The MIMO
\( \kappa - \mu \) shadowed model can be similarly defined by separating
the shadowed dominant component from the scattered waves, in form of

\[ Y = \sum_{i=1}^{\mu} (\hat{H}_i + s_i \Xi) H_i + s_i \Xi \] (7)

where \( \hat{H}_i \sim CN(0, I_p \otimes \Sigma) \), and \( W_S = \Xi H \Xi \sum_{i=1}^{\mu} |s_i|^2 \sim \Gamma_n(m, M) \). In fact, this MIMO physical model can be identified
with the one introduced in \[3\] for a SISO case. \( \hat{H}_i \)
represents the scattered components of the \( i \)th cluster, which
corresponds to \( X_i + jY_i \); \( \Xi \) is interpreted as the shadowed
component of all the cluster LOS, which is reduced to \( \xi \); \( s_i \)
corresponds to \( p_i + jq_i \) in \[3\]; finally, \( M \) is not present in the
SISO model \[3\] but it is introduced in this new matrix model
to take into account the spatial correlation of the shadowing
at the receiver side.

The parameters of the physical \( \kappa - \mu \) shadowed model are:

- i) \( K \), the kappa matrix parameter,
- ii) \( \mu \), the number of
cchannel clusters,
- iii) \( m \), the gamma scalar parameter,
- iv) \( \Sigma \)
the covariance matrix of the scattered waves, and

- v) \( p \), the
scattering degree of freedom. The pdf of the Gram channel
matrix, \( \bar{Y} \), can be deduced as follows.

**Lemma 2:** Let \( Y \) be the Gram matrix of the \( \kappa - \mu \) shadowed
gchannel given by (7), where \( \hat{H}_i \sim CN(0, I_p \otimes \Sigma) \)
and \( W_S = \Xi H \Xi \sum_{i=1}^{\mu} |s_i|^2 \sim \Gamma_n(m, M) \). Then,
the Gram channel matrix \( Y \sim \Gamma_W(m, r, \Sigma, \frac{m}{\mu} K^{-1} \Sigma^{-1}) \),
where \( r = \mu \cdot p \) and \( K \) is defined in eq. (6) with \( D = m \cdot M^{-1} \).

**Proof:** Since \( Y | W_S \) is the sum of noncentral Wishart
matrices, thus it is distributed as the noncentral Wishart by
virtue of Theorem 1. Then, the proof based on conditional
forms presented in [10] can be followed step by step, which
leads to the gamma-Wishart distribution. Notice that this result
can be also extended to \( r \) taking real values, despite the lack
of physical meaning.

Furthermore, if we fix \( n = p = 1 \), then \( r = \mu \) and the SISO
\( \kappa - \mu \) shadowed model presented in \[3\] is obtained. Next, the
mgf of the MIMO \( \kappa - \mu \) shadowed model is derived from its pdf.

**Lemma 3:** Let \( Y \sim \Gamma W_n(m, r, \Sigma, \frac{m}{\mu} K^{-1} \Sigma^{-1}) \); then, its
mgf is given by

\[ \mathcal{M}_Y(S) = \mathbb{E}[\exp(YS)] = \frac{1}{\Gamma_n(r) |\Sigma|^r |I_n + \frac{m}{\mu} K|^m} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{m_\kappa}{|\kappa|^\kappa} \times \right| Y^n = Y > 0 \right| \cdot f_Y(Y)(dY) \] (9)

where \( f_Y(Y) \) is the pdf of the matrix \( Y \), which depends on the
hypergeometric \( _1F_1(\cdot;\cdot) \). Eq. (9) is carried out by expressing
this hypergeometric by infinite series \[13\] eq. (87), so that

\[ \mathcal{M}_Y(S) = \frac{1}{\Gamma_n(r) |\Sigma|^r |I_n + \frac{m}{\mu} K|^m} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{m_\kappa}{|\kappa|^\kappa} \times \right| Y^n = Y > 0 \right| \cdot f_Y(Y)(dY) \] (10)

where \( \Sigma^{-1} = \Sigma^{-1}(I_n + \frac{m}{\mu} K)^{-1} \), \( T = -S + \Sigma^{-1} \)
and \( C_\kappa(A) \) is the complex zonal polynomial of \( A \) \[13\] eq.
(85)]. With the help of \[14\] eq. (6.1.20), the integration of
the zonal polynomial is evaluated. The complex multivariate
gamma function, \( \Gamma_n(r) \), and the complex Pochhammer symbol
\[13\] eq. (84)], \( |r|^\kappa \), are then simplified. Thus, the Binomial
hypergeometric function \( _1F_1(\cdot;\cdot) \) is obtained and expressed in
turn by a determinant \[13\] eq. (90)).

Again, the mgf of the SISO \( \kappa - \mu \) shadowed presented in \[3\]
is obtained when \( n = p = 1 \).

The fundamental statistical results presented in Lemmas 2
and 3 bring a new model unification for some MIMO common
channels. Actually, the MIMO Rayleigh, MIMO Nakagami-
m, MIMO Rician, MIMO \( \kappa - \mu \) and MIMO Rician-Shadowed
models can be deduced from the MIMO \( \kappa - \mu \) shadowed
fading model when its parameters are set to specific values
and/or taken to limit. Table I summarizes these MIMO fading
derivations, where the \( \kappa - \mu \) shadowed model parameters are
underlined for the sake of clarity.

Due to space constraints, we only outline the proofs which
are required to obtain the results in Table I. On the one hand,
the derivations for the MIMO Rayleigh and MIMO Nakagami-
m fading models are carried out thanks to the next properties
of the hypergeometric functions. When the case \( K \rightarrow 0_n \) is
considered, we apply

\[ \lim_{\epsilon \rightarrow 0} \epsilon \tilde{F}_r(a_1 \ldots a_p; b_1 \ldots b_q; \epsilon X) = 1. \] (11)

When \( m = r \), we use

\[ _1F_1(a; a; X) = \exp(X) \tilde{F}_r(a - a; a; -X) = \exp(X). \] (12)

In fact, the eq. (11) can be handled by simply exploiting
the series expression of the hypergeometric function of matrix
argument \[^{13}\text{eq. (87)}\], where the first term has the unit value and the rest of the terms depend on the eigenvalues of the matrix argument, which become zero when \( K \to 0_n \). The eq. (12), which is usually referred to as the Kummer relation derived by using the integral representation of the hypergeometric function \[^{14}\text{eq. (6.2.4)}\]. Actually, \( \text{eq. (13)} \) can be proved by expressing the hypergeometric function in series form \[^{13}\text{eq. (87)}\]. The constant of the zonal polynomial argument can be then extracted from it, so that the complex Pochhammer symbol vanishes when taking the limit. The relation \( \text{eq. (14)} \) can be derived by expressing the determinant as the eigenvalue product \( \prod_{i=1}^{m} (1 + \frac{1}{m} \lambda_i) \) and then the limit in \( \text{eq. (14)} \) is straightforward, by observing each product component tends to the exponential function.

Finally, the MIMO Rician-Shadowed is but a particular case of the MIMO \( \kappa \)-\( \mu \) shadowed when \( \mu = 1 \).

### B. Maximum eigenvalue distribution

The study of the maximum eigenvalue distribution focuses on two main cases: i) \( M \) is a diagonal matrix with distinct elements, denominated as nonhomogeneous shadowing case, and ii) \( M \) is a diagonal matrix with equal elements, called homogeneous shadowing case. In the first case, each receiver antenna collects distinct shadowed LOS power, while in the second case, each receiver antenna collects the same shadowed LOS power.

Furthermore, when resolving the two cases aforementioned, we give solutions to more general problems. Actually, the solutions which will be stated depend on whether the matrix \( M \) has distinct or equal eigenvalues, and not necessarily the shadowing has to be spatially uncorrelated, i.e, \( M \) has not to be a diagonal matrix.

1) **Nonhomogeneous shadowing case**: When a nonhomogeneous shadowing is present, the matrix \( M \) has distinct eigenvalues. This case can be derived by following the mathematical analysis presented in \[^{10}\text{eq. (11), (13)}\], where the cdfs of the extreme eigenvalues are carried out for the MIMO Rician shadowed case \( (\mu = 1) \) when \( \Sigma = I_n \). For the case where \( \mu \neq 1 \), the expressions are still the same. In fact, for any positive integer value of the parameter \( r = \mu \cdot p \), closed-form expressions for the cdfs of the extreme eigenvalues when \( m < r \) are given in \[^{10}\text{eq. (11), (13)}\]. For \( m > r \), the cdfs of the extreme eigenvalues are expressed as infinite series \[^{10}\text{eq. (12)}\]. However, with this set of expressions, it is not possible to compute the cdfs of the extreme eigenvalues of \( M \) when any pair of eigenvalues are equal. Thus, we consider the next case.

2) **Homogeneous shadowing case**: When an homogeneous shadowing is present, the matrix \( M \) has equal eigenvalues.

**Corollary 1**: The joint distribution of the ordered eigenvalues \( \phi_1 < \phi_2 < \ldots < \phi_n \) of \( Y \sim I_{\Sigma}(m, r, \Sigma, \Sigma^{-1} - \Sigma^{-1}) \), when \( \Sigma = \sigma^2 I_n \) and \( K = \kappa I_n \) is given by

\[
\phi(\Phi) = \frac{n^{(n-1)}}{\sigma^{2n}} \prod_{i<j} (\phi_i - \phi_j)^2 \frac{1}{\Gamma(n)} \Gamma(n)(1 + \frac{nr}{m})^{nm} \times \Phi \mid - \Phi \mid - \sigma^2 \Phi \mid 1 + \frac{nr}{m} \mid
\]

where the confluent hypergeometric function is of one matrix argument, \( \Phi = \text{diag}(\phi_i) \).

**Proof**: Applying \[^{13}\text{eq. (88)}\] for deriving the ordered eigenvalue distribution, the integration over the space of unitary matrices leads to the hypergeometric function of one matrix argument. Next, the cdf of the maximum eigenvalue is derived.

**Lemma 4**: Let \( r = r + n \), the cdf of the maximum eigenvalue of \( Y \sim I_{\Sigma}(m, r, \Sigma, \Sigma^{-1} - \Sigma^{-1}) \), when \( \Sigma = \sigma^2 I_n \) and \( K = \kappa I_n \), can be expressed as

\[
F_{\phi_n}(\phi_n) = C|\text{Y}(\phi_n)|.
\]

where the constant \( C \) can be expressed as

\[
C = \frac{n^{(n-1)}}{\sigma^{2n}} \Gamma(n) \Gamma(n,r)(1 + \frac{nr}{m})^{nm}.
\]

When \( m < r \), the entries of the \( n \times n \) matrix \( \text{Y}(x) \) are given by the eq. \[^{15}\text{eq. (15.1.1)}\], \( \Phi_1(\cdot, \cdot, \cdot, \cdot) \) is the confluent hypergeometric function of two scalar variables \[^{13}\text{eq. (9.261.1)}\], and \( \Gamma(a) \) is the univariate gamma function. When \( m > r \), no closed-form expression

| TABLE I | THE MIMO CHANNELS DERIVED FROM MIMO \( \kappa \)-\( \mu \) SHADOWED MODEL |
|----------------|-------------------------------------------------|
| MIMO Channels (Distributions) | MIMO \( \kappa \)-\( \mu \) Shadowed Parameters |
| MIMO Rayleigh (Central Wishart) | \( \mu = 1 \), \( K \to 0_n \), \( m \to \infty \) |
| MIMO Nakagami-\( m \), with \( m \) parameter (Gamma) | \( \mu = m \), \( K \to 0_n \), \( m \to \infty \) |
| MIMO Rician, with mean \( \bar{H} \) (Noncentral Wishart) | \( \mu = \mu \), \( K = \bar{H} \), \( m \to \infty \) |
| MIMO \( \kappa \)-\( \mu \) (Noncentral Wishart) | \( \mu = \mu \), \( K = K \), \( m = m \) |
| MIMO Rician-Shadowed (Gamma-Wishart) | \( \mu = 1 \), \( K \to 0_n \), \( m \to \infty \) |

where \( \kappa \) stands for the number of antennas and \( \mu \) for the number of receiving antennas.
\[
\{ \Psi_{i,j}(x) \} = \sigma^{2r-2j+2} \left( 1 + \frac{m}{\kappa \mu} \right)^{i-n} \Gamma(\tau - i - j + 1) \left[ 2 \mathcal{F}_1 \left( \tau - i - j + 1, m - i + 1; r - i + 1; \frac{1}{1 + \frac{m}{\kappa \mu}} \right) \right] \\
- e^{-\sigma^2 x} \sum_{k=0}^{\tau - i - j} \frac{(\sigma^2 x)^k}{k!} \Phi_1 \left( m - i + 1, \tau - i - j - k + 1, r - i + 1, \frac{1}{1 + \frac{m}{\kappa \mu}}, \frac{\sigma^2 x}{1 + \frac{m}{\kappa \mu}} \right) 
\]

(18)

is obtained, so we give the entries of \( \Psi(x) \) in the following integral form

\[
\{ \Psi_{i,j}(x) \} = \left[ \sigma^2 \left( 1 + \frac{m}{\kappa \mu} \right) \right]^{i-n} \\
\times \int_0^x y^{r-i-j} e^{-\sigma^2 y} \mathcal{F}_1 \left( m - i + 1; r - i + 1; \frac{\sigma^{-2 y}}{1 + \frac{m}{\kappa \mu}} \right) dy.
\]

(19)

Finally, when \( m = r \), \( \Psi \) follows a central Wishart distribution, so that its extreme eigenvalue distributions are given in [19].

**Proof:** The cdf of the maximum eigenvalue is derived by integrating the joint eigenvalue distribution in eq. (15) multiple times such as

\[
F_{\phi_n}(\phi_n) = \Pr(\phi_n \leq x) \\
= \int_{0 < \phi_1 < \ldots < \phi_n \leq x} f_{\phi_1, \ldots, \phi_n}(\phi_1, \ldots, \phi_n) d\phi_1 \ldots d\phi_n.
\]

(20)

Thanks to [20 eq. (2.9)], the hypergeometric function of one matrix argument in eq. (15) can be expressed by two determinants of the form

\[
\mathcal{F}_1(m; r; \frac{\sigma^2 - \Phi}{1 + \frac{m}{\kappa \mu}})^{n-i} = \frac{|\Psi(\Phi)|}{\psi \left( \frac{\sigma^2 - \Phi}{1 + \frac{m}{\kappa \mu}} \right)}
\]

(21)

where the entries of the \( n \times n \) matrix \( \Psi(\Phi) \) are given by

\[
\{ \Psi_{i,j}(\Phi) \} = \left( \frac{\sigma^2 - \Phi}{1 + \frac{m}{\kappa \mu}} \right)^{n-i} \\
\times \mathcal{F}_1 \left( m - i + 1; r - i + 1; \frac{\sigma^2 - \Phi}{1 + \frac{m}{\kappa \mu}} \right).
\]

(22)

Then, the Vandermonde determinant in eq. (21) is simplified by the square Vandermonde determinant in eq. (15), leading to a product of two determinants. Since the multiple integrals of a product of two determinants can be expressed as a determinant of a single integral [21], we finally obtain the integral form of eq. (19), which can be expressed as a finite sum of confluent hypergeometric functions of two scalar variables when \( m < r \).

Notice that this result is carried out by a new mathematical analysis. In fact, it avoids at first step the indetermination 0/0 produced when the eigenvalues of \( M \) are equal by applying the formula (21), instead of using the well-known formula of Gross and Richards for complex hypergeometric functions of two matrix arguments [22 eq. (4.8)].

**Lemma 5:** Let \( \tau = r + n \), the pdf of the maximum eigenvalue \( Y \sim \Gamma W_n(m, r, \Sigma, \frac{m}{\mu} K^{-1} \Sigma^{-1}) \), when \( \Sigma = \sigma^2 I_n \), and \( K = \kappa I_n \), can be expressed as

\[
f_{\phi_n}(\phi_n) = C e^{-\sigma^2 \phi_n} \left| \Psi_{i,j}(\phi_n) \right| \times \text{tr} \left\{ \Psi^{-1}(\phi_n) J(\phi_n) \right\} U(\phi_n).
\]

(23)

where the \( U(\cdot) \) is the unit step function and the entries of the \( n \times n \) matrix \( J(x) \) are given by

\[
\{ J_{i,j}(x) \} = \left[ \frac{\sigma^2 - \Phi}{1 + \frac{m}{\kappa \mu}} \right]^{i-n} x^{r-i-j} \\
\times \mathcal{F}_1 \left( m - i + 1; r - i + 1; \frac{\sigma^2 - \Phi}{1 + \frac{m}{\kappa \mu}} \right).
\]

(24)

**Proof:** The proof is straightforward by using the derivative formula of a determinant given by [23 eq. (9)]

\[
\frac{d |A(x)|}{dx} = |A(x)| \text{tr} \left( A^{-1}(x) \frac{d|A(x)|}{dx} \right).
\]

(25)

Notice that this can be also applied in the nonhomogeneous shadowing case to obtain the pdf of the maximum eigenvalue.

**V. Numerical Results**

In order to validate our analytical results, we compare them with Monte-Carlo simulations. Fig. 1 shows different simulated and theoretical curves of the cdf of the maximum eigenvalue when the shadowing is considered to be homogeneous. We appreciate a perfect match between simulated and theoretical values. In turn, Fig. 2 allows to validate our theoretical expression for the pdf of the maximum eigenvalue.

Once the model is checked, it is interesting to see how the MIMO \( \kappa-\mu \) shadowed model unifies the common MIMO channels of Table I. For instance, in case that \( m \rightarrow \infty \), the MIMO \( \kappa-\mu \) fading model converges to the MIMO Rician fading model when \( \mu = 1 \). In fact, Fig. 3 shows the evolution of the pdf of the maximum eigenvalue as the parameter \( m \) grows. We can observe that the pdf of maximum eigenvalue of the MIMO \( \kappa-\mu \) shadowed fading tends to the one of the MIMO Rician model when \( m \rightarrow \infty \), i.e. at the limit, the pdf of the maximum eigenvalue follows the distribution of a noncentral Wishart maximum eigenvalue, which can be found in [17].

**VI. Conclusion**

We have presented a random matrix model for the \( \kappa-\mu \) shadowed model, that finds application in MIMO communication systems affected by small and large scale fading, when operating in wireless environments. Closed-form expressions for the pdf and mgf of the Gram channel matrix have been derived. Concerning the maximum eigenvalue distribution, closed-form expressions for the cdf and pdf have been obtained.
in terms of the confluent hypergeometric function of two scalar variables. Since this model unifies some other common MIMO fading channels, it gives more flexibility to model any MIMO channel affected by different propagation conditions. Actually, by taking some limits and/or fixing some parameters to some specific values, the MIMO Rayleigh, MIMO Nakagami-$m$, MIMO Rician, MIMO $\kappa$-$\mu$ and MIMO Rician-Shadowed are derived from the MIMO $\kappa$-$\mu$ shadowed fading model.

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