Complete minors and average degree: A short proof

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Abstract
We provide a short and self-contained proof of the classical result of Kostochka and of Thomason, ensuring that every graph of average degree $d$ has a complete minor of order $\Omega(d/\sqrt{\log d})$.

**Keywords**
clique minors, probabilistic methods

Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$. How large a complete minor are we guaranteed to find in $G$? This classical question, closely related to the famed Hadwiger’s conjecture, has been thoroughly studied over the last half a century. It is quite easy to see the answer is at least logarithmic in $d$. Mader [3] proved it is of order at least $d/\log d$. The right order of magnitude was established independently by Kostochka [1, 2] and by Thomason [4] to be $d/\sqrt{\log d}$, its tightness follows by considering random graphs. Finally, Thomason found in [5] the asymptotic value of this extremal function.

Here we provide a short and self-contained proof of the celebrated Kostochka–Thomason bound.

**Theorem 1.** Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$, where $d$ is a sufficiently large integer. Then $G$ contains a minor of the complete graph on at least $d/10\sqrt{\ln d}$ vertices.

The constant $1/10$ in the above statement is inferior to the best constant $3.13...$ found by Thomason [5] (yet is better than the constants in [1, 2]); we did not make any serious attempt to
optimize it in our arguments. The main point here is to give a short proof of the tight $\Omega(d/\sqrt{\log d})$ bound for this classical extremal problem.

Throughout the proof we assume, whenever this is needed, that the parameters $n$ and $d$ are sufficiently large. To simplify the presentation we omit all floor and ceiling signs in several places. For a graph $G = (V, E)$, its minimum degree is denoted by $\delta(G)$, and for $v \in V$ we use $N_G(v)$ for the external neighborhood of $v$ in $G$.

We need the following lemma proven by simple probabilistic arguments.

**Lemma 2.** Let $H = (V, E)$ be a graph on at most $n$ vertices with $\delta(H) \geq n/6$. Let $t \leq n/\sqrt{\ln n}$, and let $A_1, ..., A_t \subseteq V$ with $|A_j| \leq ne^{-\sqrt{\ln n}/3}$ for all $1 \leq j \leq t$. Then there is a set $B \subseteq V$ of size $|B| \leq 3.1 \sqrt{\ln n}$ such that $B$ dominates all but at most $n e^{-\sqrt{\ln n}/3}$ vertices of $V$, $B \setminus A_j \neq \emptyset$ for all $j = 1, ..., t$, and the induced subgraph $G[B]$ has at most six connected components.

**Proof.** Set $s = 3.1 \sqrt{\ln n}$ and choose $s$ vertices of $V$ independently at random with repetitions. Let $B$ be the set of chosen vertices. Observe that for every vertex $v \in V$,

$$\Pr[N(v) \cap B = \emptyset] \leq \left(1 - \frac{d(v)}{n}\right)^s \leq e^{-sd(v)/n} \leq e^{-s/6}.$$ 

Hence the expected number of vertices not dominated by $B$ is at most $n e^{-s/6} < ne^{-3.1 \sqrt{\ln n}/6} < ne^{-\sqrt{\ln n}/2}$, and by Markov’s inequality, it is at most $n e^{-\sqrt{\ln n}/3}$ with probability exceeding $1/2$ (with room to spare). Also, since $|V| > \delta(H) \geq n/6$, for every subset $A_j$, 

$$\Pr[B \subseteq A_j] = \left(\frac{|A_j|}{|V|}\right)^s \leq \left(\frac{6|A_j|}{n}\right)^s \leq 6^s e^{-s\sqrt{\ln n}/3} \leq 6^s e^{-3.1 \ln n/3} < \frac{1}{n}.$$ 

Therefore the probability that $B \setminus A_j \neq \emptyset$ for all $j$ is at least $1 - t/n \geq 1 - 1/\sqrt{\ln n}$.

We now argue about the number of connected components in $G[B]$. Writing $B = (v_1, ..., v_t)$, for $1 \leq i \leq s$ let $x_i$ be the random variable counting the number of indices $1 \leq j \neq i \leq s$ for which $v_j$ is a neighbor of $v_i$. Conditioning on $v_i$, we see that $x_i$ is distributed as a binomial random variable with parameters $s - 1$ and $d(v_i)/|V| > 1/6$. Hence invoking the standard Chernoff-type bound on the lower tail of the binomial distribution, we derive that $\Pr[x_i < s/7] \leq e^{-\Theta(s)}$. Applying the union bound over all $1 \leq i \leq s$, we conclude that with probability $1 - o(1)$, we have $x_i \geq s/7$ for all $i$. Finally, observe that since $s \ll \sqrt{|V|}$, with probability $1 - o(1)$ there are no repetitions in $B$, and hence $d(v_i, B) = x_i \geq s/7$ for all $1 \leq i \leq s$. But then all connected components of $G[B]$ are of size exceeding $s/7$, and therefore $G[B]$ has at most six connected components.

Combining the above three estimates, the desired result follows.

**Proof of Theorem 1.** Let $G' = (V', E')$ be a minor of $G$ such that $|E'| \geq d|V'|$ and $|V'| + |E'|$ is minimal. If an edge $e$ of $G'$ is contained in $t$ triangles then contracting $e$ gives a minor of $G$ with one vertex and $t + 1$ edges less. By the minimality of $G'$ we have $t + 1 > d$, implying $t \geq d$. Hence for every edge $e = (u, v) \in E(G')$, the vertex $u$ is connected by an edge of $G'$ to at least $d$ neighbors of $v$. The minimality of $G'$ also implies $|E'| = d|V'|$, hence $G'$ has a vertex $v$ of degree at most $2d$. Let $H$ be the subgraph of $G'$
induced by $N_G(v)$. Then $H$ has at most $2d$ vertices and minimum degree at least $d$. Obviously a minor of $H$ is a minor of $G$ as well.

We now argue that $H$ contains a $d/3$-connected subgraph $H_1$ with $\delta(H_1) \geq 2d/3$. If $H$ itself is $d/3$-connected this holds for $H_1 = H$. Otherwise, there is a partition $V(H) = A \cup B \cup S$, where $A, B \neq \emptyset$, $|S| < d/3$, and $H$ has no edges between $A$ and $B$. Assume without loss of generality $|A| \leq |B|$. Then $|A| \leq d$, and since $\delta(H) \geq d$, every vertex $v \in A$ has at least $2d/3$ neighbors in $A$, implying that every pair of vertices of $A$ has at least $d/3$ common neighbors in $A$. Hence the induced subgraph $H_1 := H[A]$ is $d/3$-connected, has at most $2d$ vertices and satisfies $\delta(H_1) \geq 2d/3$.

Set $i = 1$ and repeat the following iteration $d/10\sqrt{\ln d}$ times. Let $H_i = (V_i, E_i) \subseteq H_1$ be the current graph, and suppose $A_1, \ldots, A_{i-1}$ are subsets of $V_i$ of cardinalities $|A_j| \leq 2de^{-\sqrt{\ln(2d)}/3}$ (representing the nonneighbors of the previously found branch sets $B_j$ in $V_j$). We assume (and justify it later) that $H_i$ is connected and has $\delta(H_i) > d/3$. Then the diameter of $H_i$ is at most 14, as on any shortest path $P = (v_0, v_1, \ldots)$ in $H_i$ the vertices at positions divisible by three have pairwise disjoint neighborhoods. Since $|V(H_i)|/\delta(H_i) < 6$, the number of such neighborhoods is at most 5, and therefore any shortest path has at most 15 vertices. Applying Lemma 2 with $H := H_i$, $n := 2d$, $t := i - 1$, and $A_1, \ldots, A_{i-1}$ (for the initial step $i = 1$ there are no $A_j$'s to plug into Lemma 2—which of course does not hinder its application) we get a subset $B_i$ of cardinality $|B_i| \leq 3.1\sqrt{\ln(2d)}$ as promised by the lemma. We now turn $B_i$ into a connected set by adding few vertices of $H_i$ if necessary. Recall that $H_i[B_i]$ has at most six connected components. Connecting one of them by shortest paths in $H_i$ to all others and recalling that $H_i$ has diameter at most 14, we conclude that by appending to $B_i$ all the vertices of these paths we make it connected by adding to it at most $13 \times 5 = 65$ vertices. Altogether we obtain a connected subset $B_i$ of cardinality $|B_i| \leq (3.1 + o(1))\sqrt{\ln(2d)}$, dominating all but at most $2de^{-\sqrt{\ln(2d)}/3}$ vertices of $V_i$ and having a vertex outside every $A_j$ (these properties are preserved under vertex addition when making $B_i$ into a connected subset)—meaning connected to every previous $B_j$. We now update $V_{i+1} := V_i - B_i$, $A_i := V_{i+1} - N_{H_i}(B_i)$, and $A_j := A_j \cap V_{i+1}$, $j = 1, \ldots, i - 1$, and finally increment $i := i + 1$, set $H_i := H[V_i]$, and proceed to the next iteration. The total number of vertices deleted in all iterations satisfies:

$$|\cup_i B_i| \leq \frac{d}{10\sqrt{\ln d}} \cdot (3.1 + o(1))\sqrt{\ln(2d)} < \frac{d}{3},$$

and since we started with the $d/3$-connected graph $H_1$ with $\delta(H_1) \geq 2d/3$, we indeed have that at each iteration the graph $H_i$ is connected and has $\delta(H_i) > d/3$.

After having completed all $d/10\sqrt{\ln d}$ iterations, we get a family of $d/10\sqrt{\ln d}$ branch sets $B_i$, all connected, and with an edge of $H_i$ between every pair of branch sets. Hence they form a complete minor of order $d/10\sqrt{\ln d}$ as promised.

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