Structure of the Effective Potential in Nonrelativistic Chern-Simons Field Theory

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Submitted to: Physical Review D

* This work is supported in part by funds provided by the Natural Sciences and Engineering Research Council of Canada and the Fonds pour la Formation de Chercheurs et l’Aide à la Recherche.
We present the scalar field effective potential for nonrelativistic self-interacting scalar and fermion fields coupled to an Abelian Chern-Simons gauge field. Fermions are non-minimally coupled to the gauge field via a Pauli interaction. Gauss’s law linearly relates the magnetic field to the matter field densities; hence, we also include radiative effects from the background gauge field. However, the scalar field effective potential is transparent to the presence of the background gauge field to leading order in the perturbative expansion. We compute the scalar field effective potential in two gauge families. We perform the calculation in a gauge reminiscent of the $R_\xi$-gauge in the limit $\xi \to 0$ and in the Coulomb family gauges. The scalar field effective potential is the same in both gauge-fixings and is independent of the gauge-fixing parameter in the Coulomb family gauge. The conformal symmetry is spontaneously broken except for two values of the coupling constant, one of which is the self-dual value. To leading order in the perturbative expansion, the structure of the classical potential is deeply distorted by radiative corrections and shows a stable minimum around the origin, which could be of interest when searching for vortex solutions. We regularize the theory with operator regularization and a cutoff to demonstrate that the results are independent of the regularization scheme.
I. Introduction

Chern-Simons theories have been studied in many contexts in the last decade from the study of general relativity to condensed matter systems. An important line of developments occurred when it was shown that classical relativistic charged scalars minimally coupled to an Abelian Chern-Simons gauge field in (2+1) spacetime dimensions have vortex (soliton) solutions for self-dual equations when the coupling constant takes special values in a $\phi^6$-theory [1,2]. The presence of vortex solutions permits the emergence of new mechanisms for anyons superconductivity [3]. Evidence has been found showing that the existence of such systems possessing vortex solutions is due to the presence of an $N = 2$ supersymmetry obtained by adding fermion fields in an appropriate way [4,5].

It is more reasonable to think that the physics of superconductors should be at lower energies and described by a nonrelativistic system. It turns out that the same statements as above can be made for the corresponding nonrelativistic field theory. Specifically, by taking the limit $c \to \infty$ ($c$ being the speed of light), one obtains a field theory of interacting nonrelativistic scalar fields minimally coupled to an Abelian Chern-Simons gauge field [6,7]. This theory also contains self-dual vortex (soliton) solutions when the coupling constant takes a special value [7]. Perhaps more surprisingly in the nonrelativistic case, the self-duality originates also from $N = 2$ supersymmetry [8].

The systematic functional method of Jackiw is an efficient, concise and useful way to evaluate the effective potential without having to use a classical background field [9]. The functional evaluation of the effective potential provides us with an exact expression for the one-loop order correction by summing all 1PI $n$-point graphs evaluated at zero-momentum.

The goal of this paper is to compute the scalar field effective potential of the nonrelativistic Chern-Simons matter system and to look for deformation of the effective potential.
We start with the action

\[ S = \int dtd^2x \left\{ \frac{\kappa}{2c} (\partial_t A) \times A - \kappa A^0 \nabla \times A + i\phi^*(\partial_t + icA^0)\phi + i\psi^*(\partial_t + icA^0)\psi \right. \\
- \frac{1}{2} |D\phi|^2 - \frac{1}{2} |D\psi|^2 + \frac{e}{2c} B|\psi|^2 - \frac{\lambda_1}{4} (|\phi|^2)^2 - \lambda_2 |\phi|^2 |\psi|^2 \left. \right\} \tag{1.1} \]

where \( D = \nabla - ieA \) is the covariant derivative and \( B = \nabla \times A \) is the magnetic field. The action (1.1) represents a system of self-interacting scalar field and fermionic field coupled to an Abelian Chern-Simons gauge field. Note that the latter is non-minimally coupled to the gauge field through the Pauli term. The action (1.1) is \( N = 2 \) supersymmetric when \( \lambda_1 = \frac{-2e^2}{c\kappa} \) and \( \lambda_2 = \frac{3}{4} \lambda_1 [8] \). We use a vector notation: For instance, in the plane the cross product is \( V \times W = \epsilon^{ij} V^i W^j \), the curl of a vector is \( \nabla \times V = \epsilon^{ij} \partial_i V^j \), the curl of a scalar is \( (\nabla \times S)^i = \epsilon^{ij} \partial_j S \) and we shall introduce the notation \( (A \times \hat{z})^i = \epsilon^{ij} A^j \). The notation \( x = (t, x) \) will also be used unless stated otherwise.

To analyze the structure of the scalar field effective potential of the action (1.1), we proceed with two regularization methods in conjunction with the functional evaluation of Jackiw. We use 1) operator regularization (OR) [10,11] and 2) the cutoff regularization method. The first method was proposed a few years ago and regularizes the operators present in the theory instead of modifying the original Lagrangian by either adding counterterms or massive fields. It is a powerful method for regularization since it preserves classical symmetries modulo anomalies and at every stage of the calculation all ultraviolet divergences are absent even when the regulator is removed. OR has been applied successfully in many examples of relativistic field theories such as gauge theories, supersymmetry, chiral anomalies, curved-spacetime QFTs, quantum gravity, and for higher loop contributions in many scalar field theories [see 10 and 11, and refs. therein]. However, OR has never been applied to nonrelativistic theories such as the present case. Let us summarize the method. The effective potential is given to \( \mathcal{O}(\hbar) \) by the trace of the logarithm
of operators occurring in the theory. The logarithm is then expressed in the form

\[ \ln H = -\lim_{s \to 0} \frac{d}{ds} H^{-s} \]  

(1.2)

where the operator \( H \) is dimensionless in OR once a mass parameter \( \mu \) is introduced which plays the role of the renormalization scale. Next, we use the \( \Gamma \)-function representation of an operator,

\[ H^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \ t^{s-1} e^{-Ht} \]  

(1.3)

and the trace of the exponential in Eq. (1.3) is given by the Schwinger expansion

\[
\text{Tr} e^{- (H_0 + H_I) t} = \text{Tr} \left\{ e^{-H_0 t} + (-t) e^{-H_0 t} H_I + \frac{(-t)^2}{2} \int_0^1 du e^{-(1-u)H_0 t} H_I e^{-uH_0 t} H_I + \cdots \right\}
\]

(1.4)

upon defining \( H = H_0 + H_I \). The \( n \)-point function is easily extracted from the Schwinger expansion as each \( H_I \) corresponds to a 1-point insertion.

The second method of regularization consists in cutting the region of integration for the momentum vector. In \( D \)-dimensional relativistic theories, the cutoff is put on the \( D \)-vector \( p^2 = E^2 - p^2 c^2 < \Lambda^2 \), which preserves Lorentz-transformations. In the nonrelativistic theory, the cutoff is put only on the \((D-1)\)-vector, leaving the energy unconstrained. The cutoff method regulates ultraviolet divergences in such a way that the radiative corrections are invariant under rotations when used in nonrelativistic theories. However, it is not clear that this method preserves Galilean boosts when the regularization is performed (or at least no proof exists that the method is Galilean invariant). Recently, the cutoff method has been used by Lozano [12] and Bergman and Lozano [13] in a recent investigation of a similar problem. In Lozano’s analysis, the ground state energy is evaluated in the same model (except for the inclusion of fermi fields)
only at the value of classical solution in the thermodynamic limit with the introduction of a
chemical potential, and of an external charge at spatial infinity to maintain charge neutrality.
In the second approach, Bergmann and Lozano used the Feynman diagrammatic with the cutoff
method to evaluate the scattering of two scalars in the same theory.

We would like to show that in the context of the effective potential method without an
external charge but including a background gauge field consistent with Gauss’s law, the answer
is the same irrespective of the method one uses. In this way, we infer that the cutoff method
preserves Galilean invariance since OR is a regularization scheme that preserves all classical
symmetries up to anomalies. We conclude further that the external charge is unnecessary since
we obtain an answer consistent with Gauss’s law. We generalize their results by providing the
one-loop evaluation of the scalar field effective potential in the $R_\xi$-gauge (in the $\xi \to 0$ limit)
including radiative effects with external gauge field and in the Coulomb-gauge family. The
remainder of the paper is divided in 3 sections. In the next section, we set up the problem
illustrating clearly what is the procedure suggested by the classical equation of motion when
the scalar field is constant. In the third section, we calculate the scalar field effective potential
in the $R_\xi$-gauge using both regulators. In the fourth section, we discuss the scale anomaly and
the structure of the effective potential. In the appendix, we derive the same effective potential
in the Coulomb-gauge family and show that the result is gauge-parameter independent in both
methods.

II. Constant Background Matter Field and Equation of Motions

The coupling constants of the theory of Eq. (1.1) are $\lambda_1$, the strength of the scalar field
self-interaction, $\lambda_2$, the fermion-scalar interaction and $\kappa$, the Chern-Simons coupling constant,
which we take positive for convenience. It is easy to see that these are the only three coupling
constants: making the following substitution

\[ A' = \frac{e}{c} A; \quad A'^0 = eA^0; \quad \kappa' = \frac{\kappa c}{e^2} \]  \hspace{1cm} (2.1)
in the classical action (1.1) and dropping the primes gives

\[
S = \int dt d^2 x \left\{ \frac{k}{2} (\partial_t A) \times A - \kappa A^0 \nabla \times A - A^0 (|\phi|^2 + |\psi|^2) 
\right. \\
- \frac{i}{2} A \cdot \left( \phi^* \nabla \phi - (\nabla \phi^*) \phi + \psi^* \nabla \psi - (\nabla \psi^*) \psi \right) - \frac{1}{2} A \cdot A (|\phi|^2 + |\psi|^2) + \frac{1}{2} \nabla \times A |\psi|^2 \\
+ \phi^* (i \partial_t + \frac{1}{2} \nabla^2) \phi + \psi^* (i \partial_t + \frac{1}{2} \nabla^2) \psi - \frac{\lambda_1}{4} (|\phi|^2)^2 - \lambda_2 |\phi|^2 |\psi|^2 \right\} 
\]  

(2.2)

where an integration by parts has been performed.

The effective potential method starts with the definition of a new shifted action:

\[
S_{\text{new}} = S \left\{ \phi(x) = \varphi + \pi(x); \psi(x); A^\mu(x) = a^\mu(x) + Q^\mu(x) \right\} \\
- S \left\{ \varphi; a^\mu(x) \right\} - \text{terms linear in quantum fields}, 
\]

(2.3)

where we shift the scalar field by a constant field and we shift the gauge field by a solution to the classical equations of motion for the electromagnetic fields. The fermionic field is not shifted because we consider only quantum corrections to the scalar field effective potential.

The important question is to know how to choose the background gauge field \( a^\mu(x) \). For this purpose, we need to analyse the classical equations of motion with constant scalar field configuration. The classical equations of motion for arbitrary scalar field and arbitrary gauge field (with vanishing classical fermion field) are

\[
B = \nabla \times a = -\frac{1}{\kappa} |\phi|^2 
\]

(2.4a)

\[
E \equiv -\nabla a^0 - \partial_t a = \frac{1}{\kappa} J \times \hat{z} 
\]

(2.4b)

\[
\left[ i(\partial_t + ia^0) + \frac{1}{2} D^2 (a^\mu) - \frac{\lambda_1}{2} |\phi|^2 \right] \phi = 0 
\]

(2.4c)

where the current is given by \( J = \frac{1}{\kappa} [\phi^* D(a^\mu) \phi - (D(a^\mu) \phi)^* \phi] \) and \( D(a^\mu) \) is the covariant derivative with respect to the background gauge field. The equation for the magnetic field
(2.4a) is recognized as Gauss's law. In order to apply the functional method of Jackiw, the scalar field needs to be shifted by a constant and Eqs. (2.4) are read with \( \phi = \varphi = \text{constant} \). To maintain consistency with Gauss’s law, we need to choose a background gauge field \( a^\mu(x) \) such that the magnetic field is constant throughout the plane. We set

\[
\mathbf{a}(\mathbf{x}) = -\frac{B}{2} \mathbf{x} \times \hat{z} = \frac{\varphi^* \varphi}{2\kappa} \mathbf{x} \times \hat{z}
\]

where \( B \) is the constant magnetic field. Such a choice is also consistent with the electric field equation of motion if

\[
a^0(\mathbf{x}) = -\frac{(\varphi^* \varphi)^2}{4\kappa^2} \mathbf{x}^2.
\]

The above choice for \( a^\mu(\mathbf{x}) \) provides a solution for the equation of motion for the electromagnetic field when the system has constant scalar field configuration. However, the equation of motion for the scalar field Eq. (2.4c) is not satisfied with constant scalar field and with the above gauge field choice unless \( \varphi = 0 \). This is not too surprising since the extremum of our classical potential is located at the origin.

The advantage of the present setup for calculating radiative corrections to the scalar field effective potential is that we do not need to introduce an external charge at spatial infinity, nor do we need to introduce a chemical potential thus of an unnatural classical symmetry-breaking solution away from the origin [12]. However, we need to consider the background gauge field \( a^\mu(\mathbf{x}) \) because it could lead \( \varphi \)-dependence for the scalar field effective potential. We now turn to the calculation of the effective potential in the \( R_\xi \)-gauge using the functional method in conjunction with OR.

**III. Effective Potential in the \( R_\xi \)-gauge.**

In order to quantize the theory, we need to gauge-fix the action (2.3) on the quantum gauge field. A simplification in the calculation occurs when the gauge-fixing term

\[
\mathcal{L}_{G.F.} = \frac{1}{2\xi} \left[ \nabla \cdot \mathbf{Q} + i \xi \varphi \pi^* \right] \left[ \nabla \cdot \mathbf{Q} - i \xi \varphi^* \pi \right]
\]

reminiscent of the \( R_\xi \)-gauge is chosen (the Coulomb family gauges \( \mathcal{L}_{G.F.} = \frac{1}{2\xi} (\nabla \cdot \mathbf{Q})^2 \) will be treated in the appendix). The choice of the \( R_\xi \)-gauge enables us to eliminate some of the
cross-terms mixing gauge and scalar fields. The quadratic action in the quantum fields then
becomes

$$S = \int dt \, d^2 x \left\{ \frac{\kappa}{2} (\partial_t Q) \times Q - \kappa Q^0 \nabla \times Q + \frac{1}{4\xi} (\nabla \cdot Q)^2 - \frac{B}{2} Q \cdot Q \\
+ i\pi^*(\partial_t + i\alpha^0)\pi - \frac{1}{2} |D\pi|^2 - \frac{\lambda_1}{4} (\varphi^2(\pi^*)^2 + 4\rho|\pi|^2 + (\varphi^*)^2(\pi)^2) + \frac{\xi}{2} \rho|\pi|^2 \\
+ i\psi^*(\partial_t + i\alpha^0)\psi - \frac{1}{2} |D\psi|^2 - \lambda_2 \rho|\psi|^2 + \frac{B}{2} |\psi|^2 \\
+ c^*(-\nabla^2 + \xi\rho)c + J^0 Q^0 + J \cdot Q \right\}$$  (3.1)

where the current has the form $J^0 = -[\varphi^*\pi + \pi^*\varphi]$, $J = aJ^0$ and the notation $\rho = \varphi^*\varphi$ is used. The $c$-field term in Eq.(3.1) is the ghost compensating term arising from the choice of gauge-fixing condition. Thus the quadratic part in the spacetime varying field of the shifted action (3.1) in the $R_\xi$-gauge has the form

$$\int dt d^2 x \, \mathcal{L}'(\varphi, a^\mu(x), \pi^a(x), \psi^a(x), c(x), Q^\mu(x)) = \int dt dt' d^2 x d^2 x' \left\{ \frac{1}{2} \pi^{*a}(x)D^{-1}_{ab}(x-x')\pi^b(x') \\
+ \frac{1}{2} \psi^{*a}(x)S^{-1}_{ab}(x-x')\psi^b(x') - \frac{1}{2} Q^\mu(x)\Delta^{-1}_{\mu\nu}(x-x')Q^\nu(x') \\
+ c^*(x)P^{-1}(x-x')c(x') + \frac{1}{2} J^\mu(x)\Delta^{\mu\nu}(x-x')J'^\nu(x') \right\}$$  (3.2)

where the notation for the scalar and fermion fields is $\pi^a = (\pi, \pi^*)$; $\psi^a = (\psi, \psi^*)$; $a = 1, 2$; $Q^\mu = (Q^0, Q^1, Q^2)$. The last term involving currents is obtained by a conventional shift of variable involving the quantum gauge field (in contrast to the procedure used in the appendix). The matrices $D^{-1}$, $S^{-1}$, $\Delta^{-1}$ and $P^{-1}$ can be read from the action (3.1) and will be used below.

We now discuss the structure of the perturbative expansion. The effective potential is related to the effective action by $V_{\text{eff}} \int d^3 x = -\Gamma_{\text{eff}}$ when defined on constant background fields. In the present case, the background gauge field $a^\mu(x)$ is space-dependent; hence, we cannot use directly the functional method of Jackiw. We adopt the following strategy. We will compute the effective action by factoring out a matrix that is background gauge field independent and
perturbatively expand the gauge field dependent part in powers of small coupling constants
\( \lambda_1 \ll 1, \kappa^{-1} \ll 1 \) (recall that \( a^0 \sim \frac{\rho^2}{\kappa^2} \) and \( a \sim \frac{\rho}{\kappa} \)). The computation is up to \( \mathcal{O}(\rho^3) \) because
each term of \( \mathcal{O}(\rho^3) \) is either of \( \mathcal{O}(\lambda_1^2), \mathcal{O}(\frac{\lambda_1^2}{\kappa}) \) or \( \mathcal{O}(\frac{1}{\kappa^3}) \). Therefore, for the rest of the paper, we will use the terminology \( \mathcal{O}(\rho^3) \) to mean that the expansion is in small coupling constants to \( \mathcal{O}(\lambda_1^2), \mathcal{O}(\frac{\lambda_1^2}{\kappa}) \) or \( \mathcal{O}(\frac{1}{\kappa^3}) \). We do not introduce the parameter \( \lambda_2 \) in these expressions for simplicity since \( \lambda_2 \) does not enter the scalar field effective potential, as we will see below. Thus, as again we will see, we need only to consider 1-point or 2-point functions in background
gauge fields that contribute to the effective action. The gauge-independent part will be treated
following Jackiw’s method [9].

From the fact that the Lagrangian is quadratic in quantum fields, we can easily perform
the Gaussian integrals. The effective action to \( \mathcal{O}(\hbar) \) is given as usual by
\[
\Gamma_{\text{eff}} = S(\varphi, a^\mu(x)) + \frac{i}{2} \ln \text{Det}\{D_{ab}^{-1} + M_{ab}\} - i \ln \text{Det}S^{-1}_{ab} + \frac{i}{2} \ln \text{Det}\Delta_{\mu\nu}^{-1} - i \ln \text{Det}\mathcal{P}^{-1} \quad (3.3)
\]
where the determinant \( \text{Det} \) is functional. The matrices are defined at the operator level where
\( \omega \) and \( p \) are operators acting on functions on the right as follows:
\[
D^{-1}(\varphi, a^\mu(x); \omega, p) = \left( \begin{array}{cc}
\omega + \frac{1}{2}D \cdot D + (\frac{\xi}{2} - \lambda_1)\rho - a^0 & -\frac{\lambda_2}{2}\varphi\varphi^*

-\frac{\lambda_2}{2}\varphi^*\varphi & -\omega + \frac{1}{2}D^* \cdot D^* + (\frac{\xi}{2} - \lambda_1)\rho - a^0
\end{array} \right) \quad (3.4)
\]
where \( D = i(p - a(x)) \). The matrix \( M \) is obtained from the current-current term of Eq.(3.2)
\[
M(\varphi, a^\mu(x); \omega, p) = \frac{\left[ -\rho + i\kappa(a \times p - p \times a) + \mathcal{O}(\xi) \right]}{\kappa(p^2 - \xi\rho)} \left( \begin{array}{cc}
\rho & \varphi\varphi^*

\varphi^*\varphi & \rho
\end{array} \right) \quad , \quad (3.5)
\]
and the matrix for fermions is
\[
S^{-1}(\varphi, a^\mu(x); \omega, p) = \left( \begin{array}{cc}
\omega + \frac{1}{2}D \cdot D - \lambda_2\rho + \frac{1}{2}B - a^0 & 0

0 & \omega - \frac{1}{2}D^* \cdot D^* + \lambda_2\rho - \frac{1}{2}B + a^0
\end{array} \right) \quad . \quad (3.6)
\]
The gauge field matrix is

$$\Delta^{-1}(\varphi; \omega, p) = \begin{pmatrix}
0 & -i\kappa p^2 & i\kappa p^1 \\
i\kappa p^2 & \rho - \frac{1}{\xi} p^1 p^1 & -i\kappa \omega - \frac{1}{\xi} p^1 p^2 \\
-i\kappa p^1 & i\kappa \omega - \frac{1}{\xi} p^1 p^2 & \rho - \frac{1}{\xi} p^2 p^2
\end{pmatrix}, \quad (3.7)$$

and the ghost matrix is

$$\mathcal{P}^{-1}(\varphi; \omega, p) = (p^2 + \xi \rho). \quad (3.8)$$

Now, let us use the limit $\xi \to 0$. In this limit, contributions to the effective action from integrating the quantum gauge fields and ghosts disappear since the $\ln \det \Delta^{-1}$ and the $\ln \det \mathcal{P}^{-1}$ become $\rho$-independent. The remaining contributions are from the matter fields. We transform the $\ln \text{Det}$ as a functional trace and logarithm, $\ln \text{Det} = \text{Tr} \ln$. Then, we separate a background gauge field dependent matrix as $\mathcal{D}^{-1} + \mathcal{M} = \Theta^{-1}(1 + \Theta X)$ and similarly for the fermions $\mathcal{S}^{-1} = \Omega^{-1}(1 + \Omega Y)$ where the matrices $\Theta^{-1}$ and $\Omega^{-1}$ are background gauge field independent. Then, for example, for the scalar field

$$\text{Tr} \ln (\mathcal{D}^{-1} + \mathcal{M}) = \text{Tr} \ln \Theta^{-1} + \text{Tr} \ln (1 + \Theta X)$$

$$= \text{tr} \ln \det \Theta^{-1} + \text{Tr} \ln (1 + \Theta X) \quad (3.9)$$

where for the first term of the final line, the trace is on momentum/energy space and the determinant on internal space indices. The last term of Eq.(3.9) is the background gauge field dependent part and we find its contribution to the effective action by a perturbative expansion. A similar expression for the fermions also arises. It is easy to find the matrices $\Theta^{-1}$, $\Theta$, $\Omega^{-1}$, $\Omega$, $X$, and $Y$ to the order needed. They are

$$\Theta^{-1} = \begin{pmatrix}
\omega - \frac{1}{2} p^2 & -\lambda_1 \rho - \frac{\rho^2}{\kappa^2 p^2} & \frac{\rho - \rho^2}{\kappa^2 p^2} \phi \phi^*\\
\frac{\rho - \rho^2}{\kappa^2 p^2} & -\omega - \frac{\lambda_1 \rho}{\kappa^2 p^2} & \frac{1}{2} p^2 - \lambda_1 \rho - \frac{\rho^2}{\kappa^2 p^2}
\end{pmatrix}, \quad (3.10a)$$
\[
\Omega^{-1} = \begin{pmatrix}
\omega - \frac{1}{2} p^2 - \lambda_2 \rho + \frac{1}{2} B & 0 \\
0 & \omega + \frac{1}{2} p^2 + \lambda_2 \rho - \frac{1}{2} B
\end{pmatrix},
\]

(3.10b)

and for their inverse,

\[
\Theta = \frac{1}{[-\omega^2 + (\frac{1}{2} p^2 + \lambda_1 \rho)^2]} \left[ \begin{pmatrix}
-\omega - \frac{1}{2} p^2 - \lambda_1 \rho & \frac{\lambda_1}{2} \varphi \varphi^* \\
\frac{\lambda_1}{2} \varphi^* \varphi & \omega - \frac{1}{2} p^2 - \lambda_1 \rho
\end{pmatrix} + \Theta_1(\mathcal{O}(\rho^2)) \right],
\]

(3.10c)

where \( \Theta_1 \) is of \( \mathcal{O}(\rho^2) \) and

\[
\Omega = \frac{1}{[\omega^2 - (\frac{1}{2} p^2 + \lambda_2 \rho - \frac{1}{2} B)^2]} \begin{pmatrix}
\omega + \frac{1}{2} p^2 + \lambda_2 \rho - \frac{1}{2} B & 0 \\
0 & \omega - \frac{1}{2} p^2 - \lambda_2 \rho + \frac{1}{2} B
\end{pmatrix}.
\]

(3.10d)

The matrices \( X \) and \( Y \) are the gauge field dependent matrices of a similar form

\[
X = \begin{pmatrix}
\frac{1}{2} (a \cdot p + a \cdot p) & 0 \\
0 & -\frac{1}{2} (a \cdot p + a \cdot p)
\end{pmatrix} + X_1(\mathcal{O}(\rho^2))
\]

(3.11a)

and

\[
Y = \begin{pmatrix}
\frac{1}{2} (a \cdot p + a \cdot p) & 0 \\
0 & \frac{1}{2} (a \cdot p + a \cdot p)
\end{pmatrix} + Y_1(\mathcal{O}(\rho^2))
\]

(3.11b)

We have included the background constant magnetic field in the fermion \( \Omega^{-1} \)–matrix for simplicity and the two matrices \( X_1 \) and \( Y_1 \) are background gauge field dependent and of \( \mathcal{O}(\rho^2) \). Let us first consider the background gauge field contribution coming from the last term of Eq. (3.9)

\[
\text{TrLn}(1 + \Theta X) = \text{Tr}[\Theta X - \frac{1}{2} (\Theta X)(\Theta X) + ...]
\]

(3.12)

where the matrix \( X \) is at least of \( \mathcal{O}(\rho) \). The first term in (3.12) represents 1-point function contributions (alternatively called “tadpole” graphs) with one external gauge field dependent
line. No external momentum enters in those graphs. For example, one of the tadpole graphs is
given by
\[
\int d^2p \, d\omega \, \langle \omega | p | 1 \rangle \left( -\omega - \frac{1}{2} p^2 - \lambda_1 \rho \right) (a \cdot p) | \omega p \rangle
\]
which vanishes either because of symmetric integration over the momentum space or the energy
space, or is neglected because it simply gives \( \mathcal{O}(\rho^3) \) contributions. This is in fact true for all
tadpole graphs encountered in this theory.

The \( \mathcal{O}(\rho^2) \) contribution coming from the second term in Eq.(3.12) is the 2-point function
and is given by
\[
- \frac{1}{8} \int d^2 p \, d^2 q \, d\omega \left\{ \frac{1}{(-\omega^2 + \frac{1}{2} p^2 + \lambda_1 \rho)^2} (-\omega - \frac{1}{2} p^2 - \lambda_1 \rho) (a \cdot p + p \cdot a) \right.
\]
\[
\times \int d^2 q \, d\omega \, \langle q \omega | q | 1 \rangle \left( -\omega + \frac{1}{2} q^2 + \lambda_1 \rho \right) \left( -\omega + \frac{1}{2} q^2 + \lambda_1 \rho \right)
\]
\[
\left. \times \frac{1}{(-\omega^2 + \frac{1}{2} q^2 + \lambda_1 \rho)^2} (-\omega^2 + \frac{1}{2} q^2 + \lambda_1 \rho) (a \cdot p + p \cdot a) \right| \omega p \rangle
\] (3.14)

Upon acting on states, integrating \( \delta(\omega_q - \omega_p) \) which arises because \( a \) is only \( x \)-dependent, we
get
\[
- \frac{1}{8} \int d^2 p \, d^2 q \, d\omega \left\{ \frac{1}{(-\omega + \frac{1}{2} p^2 + \lambda_1 \rho)} \left( -\omega + \frac{1}{2} q^2 + \lambda_1 \rho \right) \right.
\]
\[
\times \frac{1}{(-\omega + \frac{1}{2} q^2 + \lambda_1 \rho)} \left( -\omega + \frac{1}{2} q^2 + \lambda_1 \rho \right)
\]
\[
\left. \times \frac{1}{(-\omega + \frac{1}{2} p^2 + \lambda_1 \rho)} \left( -\omega + \frac{1}{2} p^2 + \lambda_1 \rho \right) \right\} [p + q] \cdot a(p - q) [p + q] \cdot a(q - p)
\] (3.15)

Now, Eq. (3.15) is easily evaluated if one notices that upon using the \( i\epsilon \)-prescription \( \pm \omega + \frac{1}{2} p^2 + i\epsilon \)
a contour integration on \( \omega \) necessarily gives zero for each integral since all the poles (for each
integrals) in \( \omega \) are located on the same side of the real axis. A similar analysis for the fermions gives the same result for the background gauge field contribution. We therefore conclude that to \( \mathcal{O}(\rho^3) \) the background gauge field does not contribute to the effective action/potential in the \( R_\xi \)-gauge with \( \xi \to 0 \); hence, all contributions come from the first term in Eq. (3.9), i.e., the background gauge field independent term.

We evaluate the first term of Eq. (3.9) and the similar expression coming from the fermions

\[
V_{\text{eff}} = -\frac{\Gamma_{\text{eff}}}{\int d^3x} = V_0(\rho) - \frac{i}{2} \int \frac{d^2p}{(2\pi)^2} \frac{d\omega}{2\pi} \ln \frac{1}{\mu^4} \left\{ -\omega^2 + \left( \frac{p^2}{2} + \lambda_1 \rho \right)^2 + \left( -\frac{\lambda_1^2}{4} + \frac{1}{\kappa^2} \right) \rho^2 + \mathcal{O}(\rho^3) \right\}
\]

\[
+ \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \frac{d\omega}{2\pi} \ln \frac{1}{\mu^4} \left\{ -\omega^2 + \left( \frac{p^2}{2} + \lambda_2 \rho - \frac{1}{2} \Lambda \right)^2 \right\},
\]

(3.16)

where we have performed the determinant of the matrix \( \Theta^{-1} \) and \( \Omega^{-1} \) and \( V_0(\rho) = \frac{1}{2} \delta m \rho + \frac{1}{4} (\lambda_1 + \delta \lambda_1) \rho^2 \) with \( \delta m \) and \( \delta \lambda_1 \) the counterterms, anticipating use of the cutoff regularization method. Up to this point in the evaluation of the effective potential, we did not use any regularization prescription. Divergences, however, now occur in the momentum integration. We perform the computation first in the cutoff and then in the OR method. In the cutoff method, both integrals of Eq. (3.16) are simplified using

\[
-\frac{i}{2} \int \frac{d\omega}{2\pi} \ln\left(-\omega^2 + E^2 - i\epsilon\right) = \frac{1}{2} E
\]

The scalar field effective potential then becomes

\[
V_{\text{eff}} = V_0(\rho) + \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \left[ \left( \frac{p^2}{2} + \lambda_1 \rho \right)^2 + \left( -\frac{\lambda_1^2}{4} + \frac{1}{\kappa^2} \right) \rho^2 \right]^{1/2}
\]

\[
- \int \frac{d^2p}{(2\pi)^2} \left( \frac{p^2}{2} + \lambda_2 \rho - \frac{1}{2} \Lambda \right)^2 + \mathcal{O}(\rho^3)
\]

(3.17)

The second integral from fermions leads to a field-independent infinite term and a contribution

\[
a_1(\lambda_2 \rho - \frac{1}{2} \Lambda) \Lambda^2,
\]

which will be removed upon renormalization. The first integral can be expanded in powers of coupling constants and the scalar field effective action before renormal-
\( V_{\text{eff}} = V_0(\rho) + (a_1(\lambda_2 \rho - \frac{1}{2} B) + a_2 \lambda_1 \rho) \lambda^2 + \frac{1}{32\pi} \left(-\lambda_1^2 + \frac{4}{\kappa^2}\right) \rho^2 \ln \frac{\Lambda}{2\lambda_1 \rho} + O(\rho^3) \) . \hspace{1cm} (3.18)

The renormalization is performed with normalization conditions

\[
\frac{d^2}{d\rho^2} V_{\text{eff}}\big|_{\rho = \mu} = \frac{1}{2} \lambda_1(\mu) \tag{3.19}
\]

and vanishing mass. We obtain to one-loop order, in the cutoff method, the renormalized scalar field effective potential in the \( R_\xi \)-gauge in the \( \xi \to 0 \) limit

\[
V_{\text{eff}}^R(\rho) = \frac{\lambda_1(\mu)}{4} \rho^2 + \frac{1}{32\pi} \left(\lambda_1^2(\mu) - \frac{4}{\kappa^2}\right) \rho^2 \left(\ln \frac{\rho}{\mu^2} - \frac{3}{2}\right) \tag{3.20}
\]

with \( \lambda_1(\mu) \approx \lambda_1 + O(\lambda_1^2) \).

We next use OR with the following identification. For the first logarithm of Eq. (3.16), we use \( H_0 = \left[-\omega^2 + (\frac{p^2}{2} + \lambda_1 \rho)^2\right]/\mu^4 \) and \( H_I = \left(-\lambda_1^2 + \frac{1}{\kappa^2}\right) \rho^2/\mu^4 \) and for the second logarithm, we use \( H_0 = \left[-\omega^2 + (\frac{p^2}{2} + \lambda_2 \rho - \frac{1}{2} B)^2\right]/\mu^4 \); however, for the fermions, there is no \( H_I \). Now, we use Eqs. (1.2-4) and expand in powers of \( \rho \), i.e., compute only up to the 1-point function in the Schwinger expansion we get

\[
V_{\text{eff}}(\rho) = \frac{\lambda_1}{4} \rho^2 + a_3 \rho^2 - \frac{i}{2} \lim_{s \to 0} \int \frac{d^2p}{(2\pi)^2} \frac{d\omega}{2\pi} \frac{d}{ds} \frac{1}{\Gamma(s)} \times \left\{ \mu^{4s} \rho^2 \frac{(-\frac{1}{2} \lambda_1^2 + \frac{1}{\kappa^2})}{[-\omega^2 + (\frac{p^2}{2} + \lambda_1 \rho)^2]^{1+s}} \Gamma(1 + s) + O(\rho^3) \right\} \tag{3.21}
\]

where an unimportant constant \( a_3 \) [see below] arises from the first term independent of \( H_I \) in the Schwinger expansion. The integration over \( \omega \) is easy to perform in the complex plane using
the residue theorem. The integral we need is

$$I \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{[-\omega^2 + (P^2 + a)^2]^{1+s}} = \frac{i(2s)!}{s!s!s!} (P^2 + 2a)^{-(1+2s)}.$$  

(3.22)

Substituting this integral in Eq. (3.21) and integrating over momentum space gives

$$V_{\text{eff}}(\rho) = \frac{\lambda_1}{4} \rho^2 + a_3 \rho^2 - \frac{1}{64\pi} \lim_{s \to 0} \frac{d}{ds} (s) \rho^2 \left\{ (-\lambda_1^2 + \frac{4}{\kappa^2}) \right\} \frac{1}{s} \left( \frac{\mu^2}{2\lambda\rho} \right)^{2s}.$$  

(3.23)

Taking the derivative with respect to $s$ and the limit $s \to 0$, we obtain

$$V_{\text{eff}}(\rho) = \frac{\lambda_1}{4} \rho^2 + \frac{1}{32\pi} \left( \lambda_1^2 - \frac{4}{\kappa^2} \right) \rho^2 \ln \frac{2\lambda\rho}{\mu^2} + a_3 \rho^2.$$  

(3.24)

Note that the fermions did not contribute to the $\mu'$-dependent expression of the scalar field effective potential and their addition into the problem has no effect on the scalar field effective potential.

Ultraviolet divergences are absent in this method and renormalization is unnecessary. The $a_3\rho^2$-term and $\mu'$ dependences are eliminated to the expense of a new scale $\mu$ by normalizing the scalar field effective potential as in Eq. (3.19). The renormalized scalar field effective potential computed using OR in the $R_\xi$-gauge in the $\xi \to 0$ limit is then given by Eq. (3.20), which agrees with the result given by the cutoff method in the same gauge and with the scalar field effective potential computed with the use of both methods in the Coulomb family gauges [see the appendix].

IV. Scale Anomaly and Structure of the Effective Potential.

The advantage of Jackiw’s method is the freedom in choosing the shifted field to be independent of the classical equation of motion, at least in this model, for the scalar field.
permits us to find the scalar field effective potential as a functional of the shifted field and look for a deformation of the scalar field effective potential. We have computed the scalar field effective potential in the $R_\xi$-gauge in the $\xi \to 0$ limit and in the appendix we perform the calculation in the Coulomb family gauges for comparison and completeness, and show that in this gauge fixing choice, the scalar field effective potential is invariant with respect to changes in the parameter $\xi$. We regulate the theory in both gauge fixings using two regularization methods: a non-relativistic cutoff and operator regularization. We find that the scalar field effective potential calculated in either gauge fixing or either regularization methods is the same [see Eqs. (3.20) and (A.11)]. Furthermore, the scalar field effective potential is transparent to the presence of a constant magnetic field, i.e., the scalar field effective potential does not depend, to the order considered, on a background gauge field, which satisfies Gauss’s law. As a spin-off of this result, we demonstrate that the results are independent of the regularization scheme and hence prove indirectly that the cutoff method regulates in a Galilean invariant way since OR preserves all symmetries up to anomalies.

The first advantage of OR is that the method preserves all symmetries up to anomalies. Another advantage of OR is the absence of a mass $\mu'$-dependent term. In the cutoff method, we show the appearance of a cutoff dependent mass term [see Eq. (3.18) or Eq. (A.10)] necessitating the introduction of a counterterm, which is not present at the classical level. Hence contrary to OR, the cutoff method requires imposing a vanishing mass normalization condition.

Let us now analyse the scale anomaly. Conformal symmetry is related to the $\beta$-function. A non-vanishing $\beta$-function indicates conformal symmetry breaking [14]. Using the renormalization group equation

$$0 = \mu \frac{d}{d\mu} V^R_{\text{eff}}(\rho) = \left[ \mu \frac{\partial}{\partial \mu} + \beta(\lambda_1(\mu)) \frac{\partial}{\partial \lambda_1(\mu)} \right] V^R_{\text{eff}}(\rho) \tag{4.1}$$
the $\beta$-function reads

$$\beta(\lambda_1(\mu)) = \frac{1}{4\pi} \left( \lambda_1^2(\mu) - \frac{4}{\kappa^2} \right).$$

(4.2)

For unrelated coupling constants the theory loses conformal symmetry. At the self-dual point $\lambda_1(\mu) = -\frac{2}{\kappa}$ and at $\lambda_1(\mu) = \frac{2}{\kappa}$ the $\beta$-function vanishes; hence, the theory is conformally symmetric, recovering the result of Lozano [12] and Bergman and Lozano [13].

We are now in a position to discuss the structure of the scalar field effective potential of Eq. (3.20) following Coleman and Weinberg [15] since we have computed radiative effects consistently with Gauss’s law by including a constant magnetic field on the plane. In the present Chern-Simons theory, the coupling constant $\lambda_1(\mu)$ can be either positive or negative in the range $\lambda_1(\mu) > -\frac{2}{\kappa}$. The theory with negative coupling constant is well-defined since the Chern-Simons gauge field renders the Hamiltonian positive definite. Consider first the simple case of the scalar field theory of Eq. (3.20) when $\frac{1}{\kappa} \to 0$. The theory with negative coupling constant is undefined since the gauge field is absent. For the theory with positive coupling constant, it seems that for small values of $\rho$ the scalar field effective potential could generate a minimum away from the origin. However, the location of this minimum is outside the regime of validity since it is given by $\rho_{\text{min}} = \mu \exp\left(-\frac{8\pi}{\lambda_1(\mu)}\right)$. For small values of $\lambda_1(\mu)$, $\rho_{\text{min}}$ is driven towards the origin and the scalar field effective potential ceases to be trustworthy.

In the case when the Chern-Simons term is present a similar scenario arises when $\lambda_1(\mu) > 0$ and $\beta(\lambda_1(\mu)) > 0$. Then $\rho_{\text{min}} = \mu^2 \exp\left(-2 \frac{\lambda_1(\mu)}{\beta(\lambda_1(\mu))}\right)$ and this minimum is in the forbidden region.

An interesting scenario can arise when $\beta(\lambda_1(\mu))$ is negative. The scalar field effective potential is the negative image of the usual Mexican hat potential. There are three possibilities. When $\lambda_1(\mu)$ is positive with $\lambda_1^2(\mu) \approx \frac{4}{\kappa^2}$, the location of the maximum is given by
\( \rho_{\text{max}} = \mu^2 \exp\left(2 \frac{\lambda_1(\mu)}{|\beta(\lambda_1(\mu))|}\right) \), which may be driven too far away from the origin. The two other cases arise when the Chern-Simons coupling diminishes such that \( |\lambda_1(\mu)| \approx \frac{1}{2\pi \kappa^2} \). In this way, the radiative corrections are dominated by the Chern-Simons contributions and are of the same order as the classical term in the scalar field effective potential. For \( \lambda_1(\mu) \) positive the maximum is located at \( \rho_{\text{max}} \approx 7\mu^2 \) and for \( \lambda_1(\mu) \) negative the maximum is located at \( \rho_{\text{max}} \approx \mu^2 \). In the case where the classical potential is unbounded from below, the region and an important distortion of the scalar field effective potential around \( \rho = 0 \). This feature could be important when searching for topological vortex solutions.

In all three of these cases, when \( \rho \) goes beyond the maximum, the scalar field effective potential gets arbitrarily negative: it is unbounded from below. One could expect higher-loop contributions to cure this behavior. In any case, higher loop contributions would not perturb too much the scalar field effective potential near the location of the maximum.
Appendix. Effective Potential in the Coulomb Family Gauges

For sake of completeness, we compute the scalar field effective potential in the Coulomb family gauges. The authors of ref. [13] prove that in the $\xi \to 0$ limit of the Coulomb family gauges, the background gauge field does not contribute to the scattering of scalar-scalar fields, therefore in the functional evaluation of the effective action, they should not contribute either to the order considered in this paper. In any case, for arbitrary $\xi$, a similar proof as the one presented in section III can be used to show that the background gauge field does not contribute to scalar field effective potential to the order considered in the paper. Hence, we set $a^\mu(x) = 0$ in this appendix. Also we ignore the fermion field, which was seen to have no effect above. We find then that the scalar field effective potential is $\xi$-independent.

We gauge fix this time the action (2.3) with the Galilean-invariant Coulomb family gauges $L_{G.F.} = \frac{1}{2}(\nabla \cdot Q)^2$. The quadratic action in the quantum fields becomes

$$\int dt d^2x \left\{ \frac{K}{2} (\partial_t Q) \times Q - \kappa Q^0 \nabla \times Q + \frac{1}{2\xi} (\nabla \cdot Q)^2 - \frac{1}{2} \rho Q \cdot Q \\
+ i\pi^* \partial_t \pi - \frac{1}{2} \nabla \pi^* \nabla \pi - \frac{\lambda_1}{4} (\psi^2 (\pi^*)^2 + 4 \psi^* \varphi \pi^* \pi + (\varphi^*)^2 (\pi^2)) \\
- \varphi (Q^0 + \frac{i}{2} \nabla \cdot Q) \pi^* - \varphi^* (Q^0 - \frac{i}{2} \nabla \cdot Q) \pi \right\}$$

(A.0)

hence the quadratic part in the spacetime varying fields of the shifted action (A.0) in the Coulomb family gauges has the form

$$\int dt d^2x L' (\varphi, \pi^a(x), Q^\mu(x)) = \int dt dt' d^2x d^2x' \left\{ \frac{1}{2} \pi^* a(x) D^{-1}_{ab}(x-x') \pi^b(x') \\
- \frac{1}{2} Q^\mu(x) \Delta^{-1}_{\mu\nu} (x-x') Q^\nu(x') \right\} + \int dt d^2x \left\{ J(x) \pi^* (x) + J^* (x) \pi (x) \right\}$$

(A.1)

where the notation is the same as in section III, i.e., for the scalar field $\pi^a = (\pi, \pi^*)$; $a = 1, 2$, $Q^\mu = (Q_0^\mu, Q_1^\mu, Q_2^\mu)$ and $J(x) = -\varphi (Q^0(x) + \frac{i}{2} \nabla \cdot Q(x))$ with $J^* (x)$ its complex conjugate. However, we set the coupling constant $\lambda_1 = \lambda$. 

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The boson-gauge field transition induced by the last term is eliminated by a conventional shift of variables, so that the quadratic part of the resulting Lagrangian contains no such cross-terms. The functional integral is then elementary. The scalar field effective potential to $O(\hbar)$ is then the sum of all $n$-point one-loop graphs, and is given as

\[
V_{\text{eff}}(\rho) = \frac{-\Gamma}{\int d^3x} = V_0(\rho) \tag{A.2}
\]

\[
-\frac{i}{2} \int \frac{d^2p}{(2\pi)^2} \frac{d\omega}{2\pi} \left\{ \ln \det D^{-1}(\varphi; \omega, \mathbf{p}^2) + \ln \det \left( \Delta^{-1}(\varphi; \omega, \mathbf{p}^i) + N(\varphi; \omega, \mathbf{p}^i) \right) \right\}
\]

where again $V_0(\rho) = \frac{1}{2}\delta m \rho + \frac{1}{4}(\lambda + \delta \lambda)\rho^2$. $\delta m$ and $\delta \lambda$ are the counterterms, anticipating the cutoff regularization method necessary to render $V_{\text{eff}}(\rho)$ finite. The propagators in Fourier space are

\[
D^{-1}(\varphi; \omega, \mathbf{p}^i) = \begin{pmatrix} \omega - \frac{1}{2}\mathbf{p}^2 - \lambda \rho & -\frac{1}{2}\partial_i \varphi \\ -\frac{1}{2}\partial^* \varphi^* & -\omega - \frac{1}{2}\mathbf{p}^2 - \lambda \rho \end{pmatrix}, \tag{A.3}
\]

\[
\Delta^{-1}(\varphi; \omega, \mathbf{p}^i) = \begin{pmatrix} 0 & -ic\kappa \mathbf{p}^2 & ic\kappa \mathbf{p}^1 \\ ic\kappa \mathbf{p}^2 & \rho - \frac{1}{2}\mathbf{p}^1 \mathbf{p}^1 & -ic\kappa \omega - \frac{1}{2}\mathbf{p}^1 \mathbf{p}^2 \\ -ic\kappa \mathbf{p}^1 & ic\kappa \omega - \frac{1}{2}\mathbf{p}^1 \mathbf{p}^2 & \rho - \frac{1}{2}\mathbf{p}^2 \mathbf{p}^2 \end{pmatrix}, \tag{A.4}
\]

and

\[
N(\varphi; \omega, \mathbf{p}^i) = \frac{1}{\det D^{-1}} \begin{pmatrix} -c^2 \rho (\mathbf{p}^2 + \lambda \rho) & c\rho \mathbf{p}^1 & c\rho \mathbf{p}^2 \\ c\rho \mathbf{p}^1 & -\frac{c^2}{4}(\mathbf{p}^2 + 3\lambda \rho) \mathbf{p}^1 \mathbf{p}^1 & -\frac{c\rho}{4}(\mathbf{p}^2 + 3\lambda \rho) \mathbf{p}^1 \mathbf{p}^2 \\ c\rho \mathbf{p}^2 & -\frac{c^2}{4}(\mathbf{p}^2 + 3\lambda \rho) \mathbf{p}^1 \mathbf{p}^2 & -\frac{c\rho}{4}(\mathbf{p}^2 + 3\lambda \rho) \mathbf{p}^2 \mathbf{p}^2 \end{pmatrix}, \tag{A.5}
\]

where $\det D^{-1}(\rho; \omega, \mathbf{p}^2) = [-\omega^2 + \frac{1}{4}(\mathbf{p}^2 + \lambda \rho)^2 + \frac{\lambda\rho}{2}(\mathbf{p}^2 + \lambda \rho)]$ and again $\varphi$ does not necessarily satisfy the classical equation of motion. A simple calculation of the determinants appearing in Eq. (A.2) shows that the first logarithm eliminates a term of the second logarithm; after the cancellation, the remaining part of the second logarithm reads.
We have dropped integrals that have no field dependence. Note that Eq. (A.6) is invariant under rotations and is a function of $\rho$ only. Eq. (A.6) contains all the necessary information to get the full effective potential.

If one is interested in using the cutoff regularization method, it is necessary to do the $\omega$ integration. We perform the following change of variable $\tilde{\omega} = \omega \sqrt{\rho^4 - \xi \lambda \rho^2}$ and then make use of the relation

$$-\frac{i}{2} \int \frac{d\omega}{2\pi} \ln \{-\omega^2 + (\frac{\rho^2}{2})^2 + \frac{1}{(\rho^4 - \xi \lambda \rho^2)} [\lambda \rho^6$$

$$+ (\frac{3}{4} \lambda^2 + \frac{1}{\kappa^2}) \rho^2 \rho^4 + \frac{\lambda}{\kappa^2} \rho^6 (1 + \frac{3}{4} \lambda^2)]\}$$

(A.6)

The effective potential is infrared convergent since as $p^2 \rightarrow 0$ the biggest contribution comes from the last term $V_{\text{eff}} - V_0 \approx \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{p^4 - \xi \lambda \rho^2}} \left(-\frac{\xi \lambda}{\kappa^2} \rho^4 (1 + \frac{3}{4} \lambda^2)\right)^{1/2}$ which converges in the infrared sector for $\xi \neq 0$. In the Landau gauge $\xi = 0$, Eq. (A.7) takes the simple form

$$V_{\text{eff}}(\rho) = V_0(\rho) + \frac{1}{4} \int \frac{d^2p}{(2\pi)^2} \frac{1}{\sqrt{p^2 + \lambda \rho \sqrt{p^2 + 3 \lambda \rho}}} \sqrt{p^2 (p^2 + 3 \lambda \rho) + 4 \frac{\rho^2}{\kappa^2}}$$

(A.8)

which is again obviously infrared convergent since the element $d^2p$ contain $\sqrt{p^2}$ and therefore cancels the denominator.

In the regime of small coupling constants $\lambda \ll 1$ and $\kappa^{-1} \ll 1$, we can expand expression (A.7) up to $O(\rho^3)$ with $\xi \neq 0$ to obtain
\[
V_{\text{eff}}(\rho) = V_0(\rho)
\]
\[
+ \frac{1}{4} \int \frac{d^2 p}{(2\pi)^2} \sqrt{p^2 + \lambda \rho} \sqrt{p^2 + 3\lambda \rho} \left\{ 1 + \frac{4\rho^2}{\kappa^2} \frac{p^2}{(p^2 + 3\lambda \rho)(p^4 - \xi \lambda \rho^2)} \right\}^{1/2}
\]

The evaluation of Eq. (A.9) is then straightforward and gives to \(O(\rho^3)\)
\[
V_{\text{eff}}(\rho) = V_0(\rho) + \frac{1}{32\pi} \left\{ 4\Lambda^2 \lambda \rho + a_4 \lambda^2 \rho^2 + (-\lambda^2 + \frac{4}{\kappa^2}) \rho^2 \ln \frac{\lambda^2}{\lambda \rho} - 2\frac{\rho^2}{\kappa^2} \ln \left( \frac{\rho}{\lambda} \right) \right\} \quad (A.10)
\]

where \(a_4\) is a constant that could be evaluated if Eq. (A.7) is solved exactly. We do not compute it here since it does not contribute to the renormalized effective potential [see below]. Once the renormalization is performed with the normalization conditions of Eq. (3.19) and vanishing mass, we find to \(O(\rho^3)\), in the cutoff method, in the Coulomb family gauges
\[
V_{\text{eff}}^R(\rho) = \frac{1}{4} \lambda_1(\mu) \rho^2 + \frac{1}{32\pi} \left( \lambda_1(\mu)^2 - \frac{4}{\kappa^2} \right) \rho^2 \left( \ln \frac{\rho}{\mu^2} - \frac{3}{2} \right) \quad . \quad (A.11)
\]

Notice that the renormalized scalar field effective potential is \(\xi\)-independent and agrees with Eq. (3.20).

We now turn to evaluate the scalar field effective potential using OR. We start with the exact expression for \(V_{\text{eff}}(\rho)\) in Eq. (A.6) with the counterterms of \(V_0(\rho)\) set to zero since there are no ultraviolet-divergences in this method and hence no need for them. We combine the two logarithms into one expression and expand it to \(O(\rho^3)\)
\[
V_{\text{eff}}(\rho) = \frac{\lambda}{4} \rho^2 - \frac{i}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{d\omega}{2\pi} \ln \frac{1}{\mu^4} \left\{ -\omega^2 + \left( \frac{p^2}{2} + \lambda \rho \right)^2 + \left( -\frac{\lambda^2}{4} + \frac{1}{\kappa^2} - \frac{1}{2} \xi \lambda + \xi \lambda \frac{\omega^2}{p^4} \right) \rho^2 \right. \\
\left. + O(\rho^3) \right\} \quad (A.12)
\]

Again, the \(n\)-point function is easily extracted from the Schwinger expansion as each \(H_I\) corresponds to a 1-point insertion. In the case at hand, \(H_0 = \frac{-\omega^2 + \left( \frac{p^2}{2} + \lambda \rho \right)^2}{\mu^4}\) and \(H_I\) is the rest
of the expression in Eq. (A.12). Now, we use Eqs. (1.2-4) and get

\[
V_{\text{eff}}(\rho) = \frac{\lambda}{4} \rho^2 - \frac{i}{2} \lim_{s \to 0} \left( \frac{d^2 p}{d\omega^2} \right) \left( \frac{d}{ds} \right) \Gamma(s) \times \]

\[
\left\{ \mu' s \rho^2 \left( \frac{-\lambda^2 + 4\kappa^2}{\mu' \rho^2} \right) \Gamma(1+s) + \mathcal{O}(\rho^3) \right\}
\]

The integration over \( \omega \) is again easy to perform in the complex plane using the residue theorem.

The two integrals we need are the one in Eq. (3.22) and

\[
J \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{[-\omega^2 + (\frac{p^2}{2} + \lambda \rho)^2]^{1+s}} = \frac{i}{s!s!} \frac{1}{4(1-2s)} \left( p^2 + 2a \right)^{(1-2s)}
\]

\[
+ (-)^{1+s} \frac{R^{1-2s}}{(1-2s)} (1 + e^{i2\pi s}) . \quad (A.14)
\]

The quantity \( R \) in \( J \) is the radius of the curve enclosing the singularity, which goes to infinity.

We leave it there; however, it will disappear from the final result.

Substituting these integrals in Eq. (A.13), cancelling powers in \( p^2 \), and integrating over the momentum space, we obtain

\[
V_{\text{eff}}(\rho) = \frac{\lambda}{4} \rho^2 + a_5 \rho^2 - \frac{1}{64\pi} \lim_{s \to 0} \left( \frac{d}{ds} \right) \frac{2s!}{s!s!} \rho^2 \left\{ (-\lambda^2 + \frac{4}{\kappa^2}) + 2\xi \lambda s \right\} \left( \frac{\mu'^2}{2\lambda \rho} \right)^{(2s)} . \quad (A.15)
\]

The \( R \)-dependent expression coming from \( J \) is convergent in the momentum plane, and hence does not appear here because it is \( \mu' \)-independent. Again the \( a_5 \rho^2 \) term arises from the first term independent of \( H_I \) in the Schwinger expansion. Taking the derivative with respect to \( s \) and the limit \( s \to 0 \), we obtain

\[
V_{\text{eff}}(\rho) = \frac{\lambda}{4} \rho^2 + \frac{1}{32\pi} \left( \lambda^2 - \frac{4}{\kappa^2} \right) \rho^2 \ln \frac{2\lambda \rho}{\mu'^2} + \left( \frac{\xi \lambda}{32\pi} + a_5 \right) \rho^2 . \quad (A.16)
\]

Note that ultraviolet divergences are absent in this method and that renormalization is unnecessary. The \( \xi \) and \( \mu' \) dependences are eliminated by normalizing the scalar field effective
potential as in Eq. (3.19). The result agrees with the result given by the cutoff method in Eq. (A.11) and with the result given in Eq. (3.20) when the calculation is performed in the $R_\zeta$-gauge.
Acknowledgements

We thank G. Lozano, R.B. Mackenzie, and M.B. Paranjape for useful comments.

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