A Systematic Approach to Programming

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Abstract. We show how to systematically implement a mental representation of an algorithm. The first step is to write down how the algorithm proceeds on a set of concrete inputs and then generalize the literals in these examples. Then, the control flow graph is synthesized from the generalized examples. The final step is to determine the edge predicates of the control flow graph, which determine the execution path through the control flow graph. This method does not rely on a particular programming language. Compared to classical programming methods this approach provides a higher level of confidence that the program works as intended without requiring the full rigor of a formal proof. Additionally, semantical programming errors can be immediately spotted.

Keywords: control flow graph, execution trace, programming by example, bottom-up, mental representation, error sources

1 Introduction

Let us begin by distinguishing between algorithm design and programming. When confronted with an algorithmic problem, the first step is to find an algorithm which solves it (algorithm design) and the second step is to implement that algorithm on a computer (programming). More specifically, we consider programming to be the thought process in which one converts a mental representation of an algorithm to a computer program (code). To further illustrate this distinction consider the following metaphor. Suppose you want a donkey to carry a load to a plateau on a higher level. Assume that everything the donkey can do, you can do as well and vice versa (Church-Turing thesis). The first step is to figure out a method how you would accomplish this task yourself. The second step is to teach the donkey this method. A particular difficulty of the second step is that, unlike another human, the donkey only understands a limited set of instructions and the challenge is to express your method in terms of this set of instructions. To exemplify this (relative) difficulty consider the following algorithmic problem.

Confidential String Matching. You are given two arrays of strings $A$ and $B$ as input where $A$ has $n$ elements and $B$ has $m$ elements. Decide whether $A[1] \ldots A[n] = B[1] \ldots B[m]$. For example, if $A = [ab, ab, a]$ and $B = [aba, ba]$ the output should be ‘true’ since the concatenation of the strings in $A$ and $B$ both equal $ababa$. The trivial solution would be to concatenate the strings in $A$ and $B$ and check for equality. However, there is a twist. The strings in the input arrays $A$ and $B$ are confidential. In order to protect their confidentiality you are only granted indirect access. Namely, your program can only access the following information:

- the number of elements in $A$ and $B$
- the length of the strings in $A$ and $B$

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whether $\text{substr}(A[i], x, z) = \text{substr}(B[j], y, z)$ (you supply the integers $i, j, x, y, z$ and get the boolean result)

where $\text{substr}(s, o, l)$ denotes the substring of $s$ which starts at offset $o$ and has length $l$. Since the third operation is quite costly you shall not use more than $n + m$ of those queries for any input.

It is trivial to come up with an algorithm which solves this problem. Now, consider the following two tasks. The first one is to explain your algorithm to someone else such that this person is able to execute the algorithm. The second task is to implement your algorithm on a computer. You will probably find the first task to be intellectually less taxing and time-consuming.

Programming itself can be seen as algorithmic problem. Given a mental representation of an algorithm and a target programming language (or a target model of computation), output a representation of that algorithm in the target language. What methods are known for this problem?

One such method is program development by step-wise refinement [Wir71]. It is a top-down approach where the initial step is to express the program in terms of high-level predicates and statements which are on the same level of abstraction as the mental representation of the algorithm to be implemented. These high-level entities are composed using the control structures of the chosen programming language. The next step is to break down each high-level predicate and statement further into ‘less’ high-level ones. This recursive procedure terminates after every high-level entity has been expressed in terms of primitive instructions provided by the programming language. For example, the algorithms in Knuth’s “The Art of Computer Programming” are presented in this fashion. In essence, this method is also commonly used to express algorithms in theoretical computer science. More specifically, to describe an algorithm for an algorithmic problem $X$ it is shown that $X$ reduces to ‘easier’ algorithmic problems $X_1, \ldots, X_k$ via a reduction algorithm $R$. Informally, this means $R$ can call subroutines which solve $X_1, \ldots, X_k$. Here, the high-level entities are the problems $X_1, \ldots, X_k$ instead of predicates and statements, and the composition of these entities is described by the reduction algorithm $R$.

Another method is to translate the mental representation into code ad hoc and then verify its correctness using some test cases. If a test case fails the code is modified until all test cases are satisfied. We call this ad hoc or freestyle programming. Step-wise refinement combined with this method is probably the de facto standard used for programming. If a certain high-level instruction is estimated to be sufficiently simple then it is implemented ad hoc rather than continuing with the refinement process. Under certain circumstances this can be very efficient and sufficiently reliable. For instance, if one regularly programs the same kind of applications using the same programming language, one will eventually memorize all the relevant code patterns which can be quickly recalled. However, if the programming language is drastically changed or the algorithmic problem to be solved is unfamiliar then this method performs poorly. In such cases ad hoc programming reduces to educated guessing.

A third method is invariant-based programming [Gri87, Bac09]. Roughly speaking, it is a formal method where proving correctness is part of the programming process. The code is derived from invariants which are determined beforehand. An invariant is a statement which remains true throughout the execution of a program.

We present another method which proceeds as follows. First, the algorithm is executed ‘by hand’ for certain inputs. This yields a set of execution traces from which the program is built. More specifically, the executions traces are generalized and then used to build the control flow graph of the program. The program is built in such a way that it must be consistent with the execution traces. The premise of this method is that it is easy to produce execution traces and then generalize them. The difference between our method and programming by example (PbE) is that the latter focuses on supporting the generalization process using AI whereas our method is a general programming method that can be applied using only pen and paper.
2 Method Demonstration

We show how to systematically convert a mental representation of an algorithm to code. We demonstrate this method for the confidential string matching problem. The algorithm that we want to implement is shown in Figure 1.

Step 1: Determine the set of variables used by the algorithm. For this algorithm we need to remember the current strings under consideration in array $A$ and $B$ ($ca, cb$), how much of them has been read (offsets $oa, ob$), the length for the next substring call ($l$) and a boolean variable for the result ($r$). For example, in the second step of Figure 1 the algorithm is at the first string of $A$ ($ca = 1$) with an offset of 2 ($oa = 2$), at the second string of $B$ ($cb = 2$) with no offset ($ob = 0$) and the length for the next substring call is $l = 2$. The input arrays $A$ and $B$ are also variables that the algorithm needs to access. However, since the algorithm does not modify them we will not explicitly mention them in the following.

Step 2: Produce execution traces for selected inputs. When presented with two arrays of strings $A, B$ it is easy to produce something analogous to Figure 1. This figure can be seen as graphical representation of an execution trace, i.e. how the algorithm proceeds step by step on a certain input. More specifically, we consider an execution trace to be a description of the contents of the variables after each execution step; such a trace can be conveniently represented by a table which has a column for each variable and the $i$-th row describes the contents of the variables after having executed $i$ steps of the algorithm. The execution traces will serve as basis for building the desired program.

Let us consider the execution trace for the input in Figure 1. The first step of the algorithm is to initialize the variables $ca, cb, oa$ and $ob$. After the first step it holds that $(ca, cb, oa, ob, l, r) = (1, 1, 0, 0, ·, ·)$ since the algorithm starts with the first string in $A$ and $B$ and no offset ('·' indicates the corresponding variable has not been assigned a value by the algorithm in the current step; $l$ and $r$ are undefined so far). The next step is to determine the length for the next substring call, i.e. the number of characters until the end of one of the two current strings is reached. In
this case this is two and thus after the second step it holds that \((ca, cb, oa, ob, l, r) = (\cdot, \cdot, \cdot, 2, \cdot)\).

Since the two substrings match, i.e. substr\((A[1], 0, 2) = \text{substr}(B[1], 0, 2)\), the algorithm increases the offset of the current string in A and moves on to the next string in B and resets the offset. This means after the third step it holds that \((ca, cb, oa, ob, l, r) = (\cdot, 2, 2, 0, \cdot)\). In the fourth step the algorithm determines the length for the next substring call which is two again and therefore after this step it holds that \((ca, cb, oa, ob, l, r) = (\cdot, \cdot, 2, 2, \cdot)\). After the fifth step it holds that \((ca, cb, oa, ob, l, r) = (2, \cdot, 0, 2, \cdot)\) since the algorithm has moved on to the next string in A and adjusted the offset for the current string in B. The full execution trace is shown in Table 1 (‘’ is represented by an empty cell).

The set of inputs for which execution traces are produced should be chosen in such a way that most of the behavior of the algorithm is exhibited (high code coverage). In this case one should choose positive and negative inputs. An input is negative because either the length of the concatenated strings differ or they have identical length but are different. We produce additional execution traces for: \([a, a, a]\), \([ba, a, [b, ab]]\) and \([a], [aa]\). Their execution traces can be found in Table 3, 5, 7 respectively. Note, for the input \([a], [aa]\) the algorithm recognizes after the initialization step that the length of the concatenation of the strings in A differs from the one in B and therefore sets the result flag to false ‘⊥’ in the second step. This length check can also be done before the initialization step, however, we do it afterwards for expository purposes.

**Step 3: Generalize the literals in the execution traces.** In the following we refer to the execution trace in Table 1. Let us write \(x(i)\) to denote the value of variable \(x\) after the \(i\)-th execution step of the algorithm and let \((x, i)\) denote the corresponding cell in the table. The goal of this step is to express the values in the \(i\)-th row of the table in terms of the values in row \(i - 1\). The literals in the execution trace are instantiations of abstract quantities. For example, for \(i = 2\) the value \(l(i) = 2\) represents the abstract quantity \(|B[1]|\). This can be further generalized to \(|B[cb]|\) since \(cb(i - 1) = 1\). We add this information to the table by writing ‘2 := \(|B[cb]|\)’ in cell \((l, 2)\). We call expressions after ‘:=’ atomic operations.

To see how the values in the third row emerge consider the first step in Figure 1. After two characters have been read the algorithm is still at the first string in A and has moved on to the next string in B, therefore we write ‘2 := cb + 1’ in cell \((cb, 3)\). Since two characters have been read the offset \(oa\) has to be increased by this value, we write ‘2 := oa + 1’ in cell \((oa, 3)\). The offset for the current string in B needs to be reset since nothing of it has been read yet, we write ‘0 := 0’ in cell \((ob, 3)\). The value \(l(4)\) represents the length of the remainder of the current string in A, i.e. \(A[1]\) starting from offset 2; write ‘2 := |A[ca]| − oa’ in cell \((l, 4)\). In row 5 the situation is symmetric to row 3; instead of moving to the next string in B one moves to the next string in A. In analogy to \((l, 4)\) write ‘2 := |B[cb]| − ob’ in cell \((l, 6)\). At this point one might notice that the contents of cell \(l(2)\) can be further generalized from ‘2 := |B[cb]|’ to ‘2 := |B[cb]| − ob’, which is the same expression as in cell \((l, 6)\). After completing this step we arrive at Table 2. This step is conducted for every execution trace.

**Step 4: Gather and identify state operations.** Remove the literals and ‘:=’ from the generalized execution trace tables. We call a row of atomic operations a state operation. Observe that certain rows contain the same state operation. For example, this is the case for the rows 2, 6 and 3, 7 in Table 2. Collect the different state operations from all execution traces in a table and name them. We call the state operation \((\cdot, \cdot, \cdot, |A[ca]| − oa, \cdot)\) ALEN because it determines the length of the remaining current string in A. Consequently, we name \((\cdot, \cdot, \cdot, |B[cb]| − ob, \cdot)\) BLEN. The state operation \((ca + 1, \cdot, 0, ob + l, \cdot)\) is called ANBS (A Next, B Stay). Analogously, the state operations in row 3 and 11 are named ASBN and ANBN, respectively. See Table 9 for the list of state operations.
During this step one might notice that certain state operations are missing because they do not appear in the considered execution traces. In that case one should find inputs which lead to execution traces with the missing state operations and go back to step 2. For example, in this case one might wonder whether a state operation ASBS is missing. After trying to come up with an input whose execution trace contains this state operation one should eventually realize that it can never occur. We recommend to prove the existence of every state operation by an actual input rather than justifying it by analogy since the latter might lead to phantom state operations, i.e. those which are never executed by the program.

**Step 5: Synthesize the control flow graph.** Instead of writing lines of code we build the desired program by iteratively constructing its control flow graph $C$. At each node of $C$ a state operation is executed and an edge from $u$ to $v$ indicates that the program might proceed with $v$ after $u$. An execution trace corresponds to a path through $C$. The information that we have collected so far can be used to partially synthesize $C$. More specifically, let the vertex set of $C$ be the set of state operations and add an edge for every two consecutive state operations in the execution traces. For instance, in Table 2 the state operation of row 1 is INIT and the state operation of row 2 is BLEN, thus we add an edge from INIT to BLEN in $C$. This leads to the graph shown in Figure 2 (the state operation of row 1 is INIT and the state operation of row 2 is BLEN). Additionally, we have marked the states YES and NO to indicate that they are terminating states, i.e. whenever the program reaches one of these states it has correctly terminated (‘correctly’ here does not mean that the result is correct). The edge (INIT,ALEN) is from Table 4 and the edge (INIT,NO) is from Table 8.

**Step 6: Complete the control flow graph.** The objective of this step is to identify missing edges in the control flow graph and find inputs whose execution traces witness these missing edges. Let us start by considering the node INIT. Should there be an edge from INIT to YES? If we allow the input which consists of two empty arrays then there should be an edge from INIT to YES. Otherwise, the program has to do at least one substring check. In order to do this check the length $l$ has to be determined first which occurs at ALEN or BLEN. Therefore there is no edge from INIT to YES if we disallow the empty input (which we shall do). For the same reason there is no edge from INIT to ANBS, ANBN or ASBN because these operations are performed after a substring check which requires determining $l$ first. Since YES and NO are terminating states they have no outgoing edges.

Next, let us consider ALEN and BLEN. There is an edge missing from ALEN to NO, which is
witnessed by the input ([a], [b]) (we make the arbitrary choice that ALEN is executed whenever the remaining current strings in A and B are of equal length). There is no edge from ALEN to ASBN because after ALEN the rest of the current string in A is read which implies that one has to move to the next string in A. There is no edge from BLEN to YES because the substring call after BLEN cannot be the last one. This holds because if BLEN is visited than the length of the remaining current string in B is strictly shorter than the length of the one in A. This means there is still a part of the current string in A to be checked. There is no edge from BLEN to ANBS for the same reason that there is no edge from ALEN to ASBN. Additionally, there is also no edge from BLEN to ANBN due to the arbitrary choice that we made above.

Lastly, we consider ANBS, ANBN and ASBN. The program should only terminate right after ALEN or BLEN and that INIT is never visited a second time. Thus there can only be edges from these three nodes to ALEN or BLEN. All these six edges should be present which means the edges (ANBN,BLEN) and (ASBN,BLEN) are missing. These two edges are witnessed by the input ([a, a, a], [a, a, a, a]).

**Step 7: Determine the edge predicates in the control flow graph.** In this step the edges of the control flow graph are labeled with boolean expressions which specify what path the program takes through the control flow graph during execution for a given input. For example, as previously stated the program goes from INIT to NO if the length of the concatenation of the strings in A and the one in B differs. Let EQLEN denote the predicate which is true iff the concatenated strings are of equal length, i.e. $\sum_{i=1}^{|A|} |A[i]| = \sum_{i=1}^{|B|} |B[i]|$. This means we label the edge (INIT,NO) with ‘$\neg$EQLEN’. After labeling an edge with a predicate we have to verify that it is consistent with the execution traces. More specifically, it has to hold that one goes from INIT to NO in an execution trace iff the predicate ‘$\neg$EQLEN’ is satisfied, which is the case.

The program goes from INIT to ALEN if the length of the remaining current string in A is not larger than the one in B, i.e. $|A[ca]| - oa \leq |B[cb]| - ob$. Let ALEQ denote this predicate. We label (INIT,ALEN) with ‘ALEQ’ and (INIT,BLEN) with ‘$\neg$ALEQ’. Again, we have to verify that the newly added edge predicates are consistent with the execution traces. In Table 2 one goes from INIT (row 1) to BLEN (row 2). It holds that ALEQ is false ($|A[ca(1)]| - oa(1) = 4 \not\leq 2 = |B[cb(1)]| - ob(1)$). In Table 3 one goes from INIT to ALEN. In that case ALEQ is true because $|A[ca(1)]| - oa(1) = 1 \leq 3 = |B[cb(1)]| - ob(1)$. Similarly, in Table 4 one goes from INIT to BLEN and ALEQ is false. Therefore the newly added predicates are consistent with the first three traces. In Table 5 one goes from INIT to NO and ALEQ holds. This is incorrect because it implies that one should go from INIT to ALEN. More generally, there is an ambiguity in the control flow graph, i.e. at a certain node more than one of the edge predicates of its outgoing edges are true for certain inputs. To resolve this we relabel the edge (INIT,BLEN) with ‘$\neg$ALEQ ∧ EQLEN’ (one should only visit ALEN if the concatenated strings are of equal length). Similarly, we relabel the edge (INIT,BLEN) with ‘$\neg$ALEQ ∧ EQLEN’. Now the edge predicates of (INIT,ALEN) and (INIT,BLEN) are consistent with all four traces.

For $X \in \{ANBS, ANBN, ASBN\}$ label (X, ALEN) with ‘ALEQ’ and (X, BLEN) with ‘$\neg$ALEQ’. It holds that INIT, ANBS, ANBN and ASBN go to ALEN if ALEQ holds and to BLEN otherwise. Additionally, in the case of INIT the predicate EQLEN has to hold. We restructure the control flow graph by adding a new node called NOOP (no operation) which has incoming edges from INIT, ANBS, ANBN and ASBN and outgoing edges to ALEN and BLEN. The program does not modify any variable at this node and thus we assume that the state operation at NOOP does not add a row to the execution trace. We label (INIT,NOOP) with ‘EQLEN’, (X,NOOP) with ‘⊤’ (constant true) for $X \in \{ANBS, ANBN, ASBN\}$, (NOOP,ALEN) with ‘ALEQ’ and (NOOP,BLEN) with ‘$\neg$ALEQ’. Again, it can be checked that consistency is maintained after these modifications.

It remains to specify the edge predicates for the outgoing edges of ALEN and BLEN. Let us start
with BLEN which either goes to NO or ASBN. The only reason the program should go to NO if the concatenated strings have equal length is if one of the substring checks fails. Let SS be true if the substring check succeeds, i.e. $\text{substr}(A[ca], oa, l) = \text{substr}(B[cb], ob, l)$. Label (BLEN,NO) with ‘¬SS’ and (BLEN,ASBN) with ‘SS’. Let us consider ALEN. For the same reason as before we can label (ALEN,NO) with ‘¬SS’. The program should go to YES if every substring check succeeded. In particular, the program should go from ALEN to YES if the next substring check succeeds and it is the final one. Let FC (final call) be the predicate which denotes that the next substring check is the last one. Label (ALEN,YES) with ‘SS ∧ FC’. Whether one goes from ALEN to ANBS or ANBN depends on whether the end of the current string in $B$ has been reached. Label (ALEN,ANBS) with ‘¬EOB ∧ SS ∧ ¬FC’ and (ALEN,ANBN) with ‘EOB ∧ SS ∧ ¬FC’ where EOB is the predicate which expresses that the end of the current string in $B$ has been reached. Notice, that the part ‘∧¬FC’ for the edge predicate of (ALEN,ANBS) can be omitted since ¬EOB implies that the next substring call cannot be the final one (¬FC). The definition of all predicates is given in Table 10.

The final control flow graph with edge predicates is shown in Figure 3. An alternative control flow graph which describes the same algorithm is shown in Figure 4. In this case the state operations ALEN and BLEN have been further generalized to $l := \min\{|A[ca]| - oa, |B[cb]| - ob\}$ and merged into a single one called LEN.

Completeness and correctness. If the consistency checks have been faithfully conducted at each step the resulting program is consistent with all execution traces. However, this does not necessarily imply that the resulting program works correctly for all inputs. One problem could be that certain aspects of the algorithm are not exhibited by the execution traces. For instance, if we would not have considered the input [a, aa] the resulting program might have failed to work if the concatenated strings of the input arrays have different lengths. Another issue could be that certain literals are wrongly generalized. Consider the following situation: we want to implement an algorithm which has three variables $a, b, c$ and for some atomic operation the variable $c$ is assigned a value which is functionally dependent on $a$ and $b$, i.e. $c := f(a, b)$. Suppose the correct generalization
is \( f(a, b) = a \cdot b \). From the execution traces it is known that \( f(0, 0) = 0 \) and \( f(2, 2) = 4 \). In that case one might wrongly choose \( f(a, b) = a + b \) which would also be consistent with the traces. By choosing a sufficiently ‘diverse’ set of example inputs the likelihood of these two types of problems can be minimized. We discuss potential error sources at the end of the next section.

**Translation into code.** There are different ways to code the program that we just built using a conventional programming language. The straightforward approach is to write a block of code for each state operation. Each such block is followed by a set of conditional jumps which represents the outgoing edges of the preceding state operation in the control flow graph. Another way is to implement the program as a class where each state operation is a class function. The corresponding class function of a state operation executes the state operation and then checks the edge predicates and calls the next class function. The variables are part of the class.

### 3 General Method

Building a computer program to solve a given problem can be seen as a search process. Let \( \mathbb{P} \) denote the space of computer programs w.r.t. a fixed programming language. For a given problem \( X \) the goal is to find a program \( P \in \mathbb{P} \) which solves \( X \). Let \( \mathbb{P}(X) \) denote the set of such programs. Usually, there are additional constraints \( C \) such as efficiency or security. Let \( \mathbb{P}(X, C) \) denote the set of programs which solve \( X \) and satisfy the constraints \( C \). The first step is to find an algorithm \( A \) which solves \( X \) under the given constraints \( C \). Let \( \mathbb{P}(A) \) denote the set of programs which implement \( A \). Saying that \( A \) works correctly is equivalent to saying that \( \mathbb{P}(A) \) is a subset of \( \mathbb{P}(X, C) \). In general, there can be more than one program which implements \( A \). For example, consider the two programs in the previous section (Figure 3 and 4) which both implement the algorithm shown in Figure 1.

The core idea behind our method is as follows. To facilitate finding a program \( P \) which implements \( A \), we systematically reduce the search space from \( \mathbb{P} \) to \( \mathbb{P}(\mathcal{E}) \) where \( \mathcal{E} \) is a set of execution traces consistent with \( A \) and \( \mathbb{P}(\mathcal{E}) \) denotes the set of programs consistent with \( \mathcal{E} \). The fact that \( A \) is consistent with \( \mathcal{E} \) means that \( \mathbb{P}(A) \subseteq \mathbb{P}(\mathcal{E}) \). The set \( \mathcal{E} \) is iteratively constructed during the development of \( P \). The resulting program will be consistent with \( \mathcal{E} \), thus reducing the space of erroneous implementations of \( A \) to \( \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(A) \). Moreover, the program will be consistent with \( \mathcal{E} \) at each step of the development process, i.e. every atomic modification of the program is verified.
This determines the set of atomic operations and predicates. The choice of a programming language determines how the atomic operations and predicates are represented as strings (code). Since any Turing-complete programming language represents exactly one element or \([0,1]^*\) and for \(n \geq 1\) let \([n] = \{1, \ldots, n\}\).

**Definition 3.1.** Let \(k, n \in \mathbb{N}\). A \(k\)-execution trace is a sequence \((\vec{x}_0, \ldots, \vec{x}_n)\) with \(\vec{x}_i \in \mathbb{B}^k\). The \(k\)-ary vector \(\vec{x}_i\) represents the contents of \(k\) variables after the \(i\)-th execution step for \(i \in [n]\). The vector \(\vec{x}_0\) represents the input.

**Definition 3.2.** Let \(k \in \mathbb{N}\). A \(k\)-program \(P\) is a tuple \((A, S, P, C, s, F, \alpha, \beta)\) where

- \(A\) is a set of partial, computable functions \(\mathbb{B}^k \to \mathbb{B}\); an element of \(A\) is called atomic operation,
- \(S \subseteq A^k\); an element of \(S\) is called state operation and we interpret it as a function \(\mathbb{B}^k \to \mathbb{B}^k\),
- \(P\) is a set of partial, computable functions \(\mathbb{B}^k \to \{0, 1\}\); an element of \(P\) is called predicate,
- \(C = (V, E)\) is a directed graph called control flow graph,
- \(s\) is a node of \(C\) called start node,
- \(F\) is a set of nodes of \(C\); a node in \(F\) is called terminating node,
- \(\alpha : V \to S\) and \(\beta : E \to P\) are total functions.

We give an informal description of the semantics of \(P\). On input \(\vec{x}_0 \in \mathbb{B}^k\) the program \(P\) starts at node \(s\) in \(C\). Assume that \(P\) is at node \(v\) after the \(i\)-th execution step. It computes \(\vec{x}_{i+1} = [\alpha(v)](\vec{x}_i)\). If \(\vec{x}_i\) is not in the domain of \([\alpha(v)]\) the program crashes (functions from \(S\) need not be total). For instance, after the \(0\)-th execution step \(P\) is at \(s\) and thus computes \(\vec{x}_1 = [\alpha(s)](\vec{x}_0)\). If \(v\) is a terminating node the program terminates. Otherwise, it needs to determine the next node. Let \(W\) be the set of outgoing neighbors \(w\) of \(v\) such that the edge predicate \((v, w)\) is satisfied, i.e. \(\{w \in V \mid (v, w) \in E \land [\beta(v, w)](\vec{x}_{i+1}) = 1\}\). If \(W\) does not contain exactly one element or \([\beta(v, w)](\vec{x}_{i+1})\) is undefined for some \(w\) with \((v, w) \in E\) the program crashes. Otherwise \(P\) continues with the single node in \(W\). We write \(P(\vec{x}_0)\) to denote the execution trace \((\vec{x}_0, \ldots, \vec{x}_n)\) produced by \(P\) on input \(\vec{x}_0\), assuming that \(P\) eventually terminates.

Observe that our definition of program makes no reference to a particular programming language. The choice of a programming language determines how the atomic operations and predicates are represented as strings (code). Since any Turing-complete programming language represents exactly the set of computable functions this is solely a matter of encoding.

**Step 1 and 2:** Determine the set of variables and produce execution traces. For simplicity, we assume the variables to be binary strings and there are \(k\) variables \(x_1, \ldots, x_k\). To determine the set of variables used by the algorithm it can be helpful to produce an execution trace as one might discover additional variables during this process. Let \(E\) denote the set of executions traces.

**Step 3 and 4:** Generalize literals and identify state operations. The first part of this step is to find consistent atomic operations for each literal in all execution traces. More precisely, for every execution trace \(E = (\vec{x}_0, \ldots, \vec{x}_n)\) in \(E\) and all \(i \in [n], j \in [k]\) one has to find an atomic operation \(f_{E,i,j} : \mathbb{B}^k \to \mathbb{B}\) such that \(\vec{x}_{i,j} = f_{E,i,j}(\vec{x}_{i-1})\) where \(\vec{x}_{i,j}\) denotes the \(j\)-th variable of \(\vec{x}_i\). This determines the set of atomic operation \(A\) of \(P\): \(\{f_{E,i,j} \mid E \in E, i \in |E| - 1, j \in [k]\}\). The set of state operations \(S\) is \(\{(f_{E,i,1}, \ldots, f_{E,i,k}) \mid E \in E, i \in |E| - 1\}\).
Step 5 and 6: Synthesize and complete the control flow graph. Determine the vertex set of the control flow graph. Unlike implicitly assumed in the previous section, it is not necessarily the case that the set of nodes of the control flow graph coincides with the set of state operations. Intuitively, a node corresponds to a line of code in a conventional program. It is conceivable that the same piece of code such as incrementing a counter occurs at two different lines in a program and how the program proceeds afterwards depends on the line it came from. In such a case there should also be two different nodes with the same state operation in the control flow graph. In general, one has to determine a mapping $h$ from each row of every execution trace (except the first row $\vec{x}_0$ which represents the input) to $\mathbb{N}$ with the constraint that two rows that have different state operations cannot be mapped to the same number. Intuitively, $h(E, i)$ describes the line of code at which the program is at the $i$-th execution step of $E$ with $E \in \mathcal{E}$ and $i \in [|E| - 1]$. The vertex set of $C$ is the image of $h$. Add an edge $(u, v)$ to $C$ whenever there exists an $E \in \mathcal{E}$ and $i \in [1, \ldots, |E| - 2]$ such that $h(E, i) = u$ and $h(E, i + 1) = v$.

Another constraint that $h$ needs to satisfy is that $h(E, 1) = h(E', 1)$ for all $E, E' \in \mathcal{E}$. Intuitively, this means that the execution of the program always starts at the same line. The start node $s$ of $C$ is $h(E, 1)$ for an arbitrary $E \in \mathcal{E}$. A node $u$ of $C$ is a terminating node if there exists an $E \in \mathcal{E}$ such that $u = h(E, |E| - 1)$. Additionally, it should hold that terminating nodes have out-degree 0.

To complete the control flow graph one has to determine for every $(u, v) \notin E(C)$ whether there is a possible input whose execution trace leads to an edge from $u$ to $v$. If such an input exists then its execution trace should be added to $\mathcal{E}$ (go back to step 2). Additionally, it has to be determined whether there are nodes missing in $C$ because they have not been visited in any of the existing execution traces. Again, in such a case one should determine an input which leads to the missing node and go back to step 2.

Step 7: Determine the edge predicates. In this step one has to determine the set of predicates $\mathcal{P}$ and the mapping $\beta$ from $E(C)$ to $\mathcal{P}$. Let $g: \mathbb{B}^k \to \{0, 1\}$ be in $\mathcal{P}$ and assume that $\beta(u, v) = g$ for some edge $(u, v) \in E(C)$. We say the edge predicate $\beta(u, v)$ of $(u, v)$ is consistent with an execution trace $E = (\vec{x}_0, \ldots, \vec{x}_n)$ if for all $i \in [n - 1]$ with $h(E, i) = u$ it holds that $h(E, i + 1) = v \Leftrightarrow g(\vec{x}_i) = 1$. Each edge predicate must be consistent with all execution traces.

Error sources. Programming errors can be categorized as either syntactical, semantical or logical. We consider a semantical error to be an error where the programmer wrongly interprets how a statement is executed. For example, referencing the last element of an array $A$ with $n$ elements with $A[n]$ would be such an error if the used programming language starts indexing arrays at 0. A logical error is one that is neither syntactical nor semantical. We distinguish between two types of logical errors. The first one (implementation error) is a wrong implementation of the mental representation of the algorithm, i.e. $P \in \mathcal{P}(E) \setminus \mathcal{P}(A)$. It is similar to a semantical error in the sense that the programmer wrongly describes an operation carried out during execution of the mental representation. The second one (incorrect algorithm) is that the algorithm to be implemented is incorrect, i.e. $\mathcal{P}(A) \cap \mathcal{P}(X, C) = \emptyset$.

Semantical errors can be quickly spotted using this method because they become visible as an inconsistency between a manually produced execution trace and the program. An incorrect algorithm can be identified by the fact that a manually produced execution trace leads to a wrong result. An implementation error also leads to an inconsistency between a manually produced execution trace and the program. An implementation error can be caused by wrong generalization or a missing dependency. An example of a wrong generalization is given in the previous section in the paragraph ‘Completeness and correctness’. A missing dependency is caused by implicitly assuming a certain value to be constant. For example, assume you have three variables $a, b, c$ and
c is determined by a and b at some point. From the execution traces you know that \((a, b, c) \in \{(0, 0, 0), (1, 0, 1), (2, 0, 2)\}\). From these traces it is not clear whether c is functionally dependent on b and thus you might wrongly think that \(c := a\) is the correct state operation.

Another way to classify errors is by where they occur in \(P\). The following can be wrong with \(P\): missing node, missing edge, incorrect state operation or incorrect edge predicate. For instance, a node can be missing from \(P\) because it is never visited in the given execution traces or two lines with the same state operation have been mistakenly assigned the same node. An edge can only be missing because it is not taken in the given execution traces. The causes of an incorrect edge predicate or state operation might be wrong generalization or a missing dependency.

4 Applications

Software development. Debugging can be a time-consuming process. While syntactical errors have become a minor nuisance due to modern IDEs which can check the syntax as you type, semantical and logical errors are still haunting developers. A major issue with the latter two types of errors is to identify where they occur. An IDE which supports our method would be able to detect semantical errors on the fly by evaluating the state operations and checking whether they are consistent with the manually produced execution traces. The additional cost for this service is to create the execution traces beforehand. Moreover, spotting logical errors becomes easier. If the program behaves wrongly on a certain input, one can manually produce an execution trace for this input. If the manually produced execution trace coincides with the program’s execution trace this means the underlying algorithm was incorrect. Otherwise it must have been an implementation error. A downside of our method is that it might be infeasible to produce execution traces in certain cases because they are too large. A trade-off could be to use partial traces.

In [Gri87, p. 164] Gries postulates that “a program and its proof should be developed hand-in-hand, with the proof usually leading the way”. As explained in the previous section, we distinguish between two kinds of proof (or correctness). The first one shows that the algorithm correctly solves the problem. The second one shows that the program correctly implements the algorithm. If the program’s proof is considered to be the latter kind then the execution traces can be seen as some form of (non-rigorous) proof. In that sense our method respects Gries’ postulate. If one strictly follows the principle that every state operation and edge predicate must be supported by at least one execution trace then the execution traces can be seen as a set of test cases which yield 100% code coverage. Additionally, this facilitates comprehending someone else’s code since for each abstract expression there is a concrete example to back it up.

In terms of confidence of correctness our method lies somewhere between ad hoc and invariant based programming. The production of execution traces requires additional time compared to the ad hoc approach but also provides a higher level of confidence. Compared to invariant-based programming our method is not as rigorous, i.e. it does not yield a formal proof of correctness, but also requires a much lower level of mathematical maturity. In a sense, our method can be seen as epitome of test-driven development, which complements existing programming methods.

Education. We believe that this method can reduce the steepness of the learning curve for programming and thus make programming more accessible to the general population. To accomplish this, we deem it important to separate algorithm design from programming. After having verified that the learner has a mental representation of a correct algorithm, the next step is to guide him or her through the implementation process. In the ad hoc variant the learner is left alone with finding a way to accomplish this. This would be similar to teaching addition to children by explaining them the semantics and providing them with sets of triples \((x, y, z)\) such that \(x + y = z\) and expecting them to infer an algorithm for addition from this information. In the case of stepwise refinement
the difficulty for the learner is to find the right level of abstraction which might additionally be dependent on the control structures provided by the particular programming language. With our method the learner can focus separately on the various tasks that occur when translating the mental representation to code. For instance, the task of finding state operations and edge predicates is done separately whereas in the case of classical programming these two tasks are interleaved. Moreover, our method does not require the introduction of idiomatic control structures such as ‘if’, ‘for’ and ‘while’ since they are implicit in the control flow graph. This means the learner just needs to be aware of the possible state operations and how to express edge predicates (either as boolean formulas or binary decision diagrams). In the beginning, a minimalistic yet universal model of computation such as stack machines can be chosen to put the emphasis on the thinking process rather than remembering an arbitrary set of non-essential details.

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Appendix
Table 1: Execution trace for input from Fig. 1

|   | ca | cb | oa | ob | l | r |
|---|----|----|----|----|---|---|
| 1 | 1  | 1  | 0  | 0  |   |   |
| 2 | 2  |    |    |    | 2 |   |
| 3 | 2  | 2  | 0  |    |   |   |
| 4 | 2  |    |    |    | 2 |   |
| 5 | 2  | 0  | 2  |    |   |   |
| 6 | 3  | 1  | 0  |    | 1 |   |
| 7 | 3  | 0  | 1  |    |   |   |
| 8 | 4  | 4  | 0  | 0  |   |   |
| 9 |    |    |    |    | 2 |   |
| 10|    |    |    |    |   | T |
| 11|    |    |    |    |   |   |

Table 2: Generalized execution trace for input from Fig. 1

|   | ca  | cb  | oa  | ob  | l  | r  |
|---|-----|-----|-----|-----|----|----|
| 1 | 1 := 1 | 1 := 1 | 0 := 0 | 0 := 0 |     |   |
| 2 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 3 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 4 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 5 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 6 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 7 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 8 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 9 | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 10| 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 11| 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 12| 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |
| 13| 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob | 2 := \(B[cb]\) − ob |

Table 3: Execution trace for \(A = [a, a, a]\), \(B = [aaa]\)

|   | ca | cb | oa | ob | l | r |
|---|----|----|----|----|---|---|
| 1 | 1  | 1  | 0  | 0  |   |   |
| 2 | 2  |    |    |    | 1 |   |
| 3 | 2  | 0  | 1  |    |   |   |
| 4 | 3  |    |    |    |   |   |
| 5 | 3  | 0  | 2  |    |   |   |
| 6 | 3  | 0  | 2  |    |   |   |
| 7 | 3  | 0  | 2  |    |   |   | T |
Table 4: Generalized execution trace for $A = [a, a, a], B = [aaa]$

|   | $ca$ | $cb$ | $oa$ | $ob$ | $l$ | $r$ |
|---|------|------|------|------|-----|-----|
| 1 | 1 := 1 | 1 := 1 | 0 := 0 | 0 := 0 | 1 := $|A[ca]| - oa$ |
| 2 | 2 := $ca + 1$ | 0 := 0 | 1 := $ob + l$ | 1 := $|A[ca]| - oa$ |
| 3 | 3 := $ca + 1$ | 0 := 0 | 2 := $ob + l$ | 1 := $|A[ca]| - oa$ |

Table 5: Execution trace for $A = [ba, a], B = [b, ab]$

|   | $ca$ | $cb$ | $oa$ | $ob$ | $l$ | $r$ |
|---|------|------|------|------|-----|-----|
| 1 | 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | | 1 |
| 3 | | | | | 1 |
| 4 | 2 | 0 | 1 | | 1 |
| 5 | | | | | 1 |
| 6 | | | | | 1 |
| 7 | | | | | 1 |

Table 6: Generalized execution trace for $A = [ba, a], B = [b, ab]$

|   | $ca$ | $cb$ | $oa$ | $ob$ | $l$ | $r$ |
|---|------|------|------|------|-----|-----|
| 1 | 1 := 1 | 1 := 1 | 0 := 0 | 0 := 0 | 1 := $|B[cb]| - ob$ |
| 2 | | | | | 1 := $|A[ca]| - oa$ |
| 3 | 2 := $cb + 1$ | 1 := $oa + l$ | 0 := 0 | 1 := $|B[cb]| - ob$ |
| 4 | | | | | 1 := $|B[cb]| - ob$ |
| 5 | 2 := $ca + 1$ | 0 := 0 | 1 := $ob + l$ | | 1 := $|B[cb]| - ob$ |
| 6 | | | | | 1 := $|B[cb]| - ob$ |
| 7 | | | | | 1 := $|B[cb]| - ob$ |

Table 7: Execution trace for $A = [a], B = [aa]$

|   | $ca$ | $cb$ | $oa$ | $ob$ | $l$ | $r$ |
|---|------|------|------|------|-----|-----|
| 1 | 1 | 1 | 0 | 0 | 1 := $⊥$ |
| 2 | 1 := $⊥$ | | | | 1 := $⊥$ |
Table 8: Generalized execution trace for $A = [a], B = [aa]$

|   | ca | cb | oa | ob | l  | r  |
|---|----|----|----|----|----|----|
| 1 | 1 := 1 | 0 := 0 | 1 := 1 | 0 := 0 | ⊥ := ⊥ | 0 := 0 |

Table 9: State operations

| id  | ca | cb | oa | ob | l  | r  |
|-----|----|----|----|----|----|----|
| INIT | 1  | 1  | 0  | 0  | 0  | 0  |
| ALEN | $|A[ca]| - oa$ | 0  | 0  | 0  | 0  | 0  |
| BLEN | $|B[cb]| - ob$ | 0  | 0  | 0  | 0  | 0  |
| ASBN | $cb + 1$ | $oa + l$ | 0  | 0  | 0  | 0  |
| ANBS | $ca + 1$ | $ob + l$ | 0  | 0  | 0  | 0  |
| ANBN | $ca + 1$ | $cb + 1$ | 0  | 0  | 0  | 0  |
| YES  | $\top$ | $\top$ | 0  | 0  | 0  | 0  |
| NO   | $\bot$ | $\bot$ | 0  | 0  | 0  | 0  |

Table 10: Predicates

| id  | boolean expression |
|-----|---------------------|
| EQLLEN | $\sum_{i=1}^{A} |A[i]| = \sum_{i=1}^{B} |B[i]|$ |
| SS   | $\text{substr}(A[ca], oa, l) = \text{substr}(B[cb], ob, l)$ |
| ALEQ | $|A[ca]| - oa \leq |B[cb]| - ob$ |
| EOA  | $|A[ca]| - oa = l$ |
| EOB  | $|B[cb]| - ob = l$ |
| LASTA| $ca = |A|$ |
| LASTB| $cb = |B|$ |
| FC   | $\text{EOA} \land \text{EOB} \land \text{LASTA} \land \text{LASTB}$ |