THE PROPAGATION OF RELATIVISTIC JETS IN EXTERNAL MEDIA

Omer Bromberg\textsuperscript{1}, Ehud Nakar\textsuperscript{2}, Tsvi Piran\textsuperscript{1}, and Re’em Sari\textsuperscript{1}

\textsuperscript{1} Racah Institute of Physics, The Hebrew University, 91904 Jerusalem, Israel
\textsuperscript{2} The Raymond and Beverly Sackler School of Physics and Astronomy, Tel Aviv University, 69978 Tel Aviv, Israel

Received 2011 April 11; accepted 2011 July 20; published 2011 October 4

ABSTRACT

Relativistic jets are ubiquitous in astrophysical systems that contain compact objects. They transport large amounts of energy to large distances from the source and their interaction with the ambient medium has a crucial effect on the evolution of the system. The propagation of the jet is characterized by the formation of a shocked “head” at the front of the jet which dissipates the jet’s energy and a cocoon that surrounds the jet and potentially collimates it. We present here a self-consistent, analytic model that follows the evolution of the jet and its cocoon, and describes their interaction. We show that the critical parameter that determines the properties of the jet–cocoon system is the dimensionless ratio between the jet’s energy density and the rest-mass energy density of the ambient medium. This parameter, together with the jet’s injection angle, also determines whether the jet is collimated by the cocoon or not. The model is applicable to relativistic, unmagnetized jets on all scales and may be used to determine the conditions in active galactic nucleus (AGN) jets as well as in gamma-ray bursts (GRBs) or microquasars. It shows that AGN and microquasar jets are hydrodynamically collimated due to the interaction with the ambient medium, while GRB jets can be collimated only inside a star and become uncollimated once they break out.

Key words: galaxies: jets – gamma-ray burst: general – hydrodynamics – ISM: jets and outflows – relativistic processes

Online-only material: color figures

1. INTRODUCTION

Relativistic jets are observed in many astrophysical systems which host compact objects, such as radio galaxies, gamma-ray bursts (GRBs), and microquasars (MCs). These jets appear in a variety of lengths, durations, and energy scales, and propagate in different types of media. However, despite the differences in their characteristics, which are many orders of magnitude apart, the interactions of the jets with their surroundings lead to similar results, such as energy injection into the ambient medium and its feedback on the jet. Understanding these common phenomena can provide valuable insights on puzzles such as what type of medium can collimate the jet and how much energy is injected into the surrounding matter. These in turn can be used to study the heating of the interstellar medium (ISM) and intergalactic medium (IGM) by active galactic nuclei (AGNs) and the fate of jets and stellar envelopes of collapsing massive stars during GRBs.

The interaction of a jet with the external medium was studied extensively in many different scales, using analytic and numerical methods. Initially, it was studied in the context of radio-loud AGNs (Blandford & Rees 1977; Scheuer 1974). They showed that the propagation of the jet generates a double bow-shock structure at the head of the jet. Energy and matter that enter this structure are pushed aside due to a high pressure gradient and create a hot cocoon around the jet. The cocoon, in turn, applies pressure on the jet and compresses it. Begelman & Cioffi (1989) calculated the head’s velocity and the cocoon expansion rate, assuming that the cocoon and the head are supported by the ram pressure of the ambient medium. Their calculations assume Newtonian head velocities and rely on the knowledge of the cross-section of the head, which they estimated phenomenologically using the size of the radio lobes observed around jets of radio-loud AGNs. Mészáros & Waxman (2001) calculated the Lorentz factor of a relativistic head assuming a conical jet (i.e., no collimation). Matzner (2003) extended the model of Begelman & Cioffi (1989) to include both Newtonian and relativistic head velocities, assuming a conical jet as in Mészáros & Waxman (2001). Later, Lazzati & Begelman (2005) used this model to calculate the opening angle and the properties of the jet at breakout from a surface of a collapsing star. They accounted for the reduction in the opening angle of the jet due to collimation, but ignored the dissipation of energy in shocks that form within the jet as a result of such a collimation. This led to an incorrect dependence of the opening angle on the cocoon’s pressure. An attempt to include the effects of these shocks was made by Morsony et al. (2007).

Apart from the analytical studies, the numerical approach was also given a lot of attention over the years. Simulations were conducted in the context of extragalactic jets (e.g., Marti et al. 1995, 1997; Aloy et al. 1999; Hughes et al. 2002) as well as GRB jets that propagate inside a star (e.g., Zhang et al. 2003; Morsony et al. 2007; Mizuta & Aloy 2009). All these works showed the basic features discussed above, i.e., the formation of a jet head and a hot cocoon. A collimation shock at the base of the jet is also evident in the simulations whenever the jet is collimated by the cocoon. However, despite the extensive numerical and analytical efforts, to this date there is no simple analytic description of the time evolution of the jet–cocoon system. As a result, there is no quantitative understanding of the collimation process and its effect on the jet angle.

The goal of this work is to provide a simple, self-consistent, analytic description for the time evolution of an unmagnetized relativistic jet and the cocoon that forms around it when it propagates in a density profile that is suitable for most astrophysical systems. The key point in our analysis is the treatment of the collimation shock that forms at the base of the jet. This shock is an inevitable consequence of collimation in a supersonic jet, as it dissipates part of the jet’s energy, generating the pressure needed to counterbalance the cocoon’s pressure. It
is crucial for the proper modeling of the system, since it sets the width of the jet and controls its propagation velocity. We base our solution on the earlier analysis of Begelman & Cioffi (1989) and Matzner (2003), and incorporate the geometry of the jet and ambient medium. We assume that the entire system (jet, cocoon, and ambient medium) is axisymmetric and use cylindrical coordinates \((z, r)\). We further approximate the ambient medium density, \(\rho_a\), to depend only on the height, i.e., \(\rho_a(z)\). This is a good approximation even in cases where \(\rho_a\) is spherically symmetric (e.g., a stellar envelope), since the opening angles of the jet and the cocoon are small whenever the ambient medium plays an important role and collimates the jet.

The jet propagates by pushing the matter in front of it, leading to the formation of a forward shock and a reverse shock at the jet’s front that are separated by a contact discontinuity. We refer to this structure as the jet’s head. Matter that enters the head through the shocks is heated and flows sideways since its pressure is higher than that of the surrounding matter (see Figure 1). This leads to the formation of a pressured cocoon around the jet. A contact discontinuity divides the cocoon into an inner, light part containing the jet material which crossed the reverse shock and an outer part of the heavier shocked medium. The pressure on both sides of the discontinuity is equal, but since the plasma is much more tenuous in the inner cocoon, the sound speed there is much larger than in the outer cocoon. This allows a causal connection along the cocoon, leading to pressure equilibration in the vertical direction and a more uniform distribution of the energy (see further discussion in Section 3). Since our analysis needs only the cocoon’s pressure, in the following we disregard the cocoon’s inner structure. If the cocoon’s pressure is sufficiently high, it collimates the jet and reduces its opening angle. This changes the jet’s propagation velocity and the energy flow into the cocoon. We classify two regimes of the system: collimated and uncollimated, according to how strongly the jet is affected by the cocoon’s pressure.

Figure 1 depicts a schematic description of the system in these two regimes.

In the collimated regime (Figure 1, right panel), the cocoon’s pressure collimates the jet and reduces substantially its opening angle. Since the jet is supersonic, the collimation leads to the formation of an oblique shock at the base of the jet. This shock deflects the jet flow lines and generates a pressure that counterbalances the cocoon’s pressure. To maintain the required pressure the shock curves toward the jet’s axis, until it converges

---

### Table 1

The System’s Characteristics in the Collimated and Uncollimated Regimes

| Parameter | Collimated Jet | Uncollimated Jet |
|-----------|----------------|------------------|
| \(\bar{L}\) | \(L < \bar{L} < \theta_0^{-4/3}\) | \(\theta_0^{-4/3} < \bar{L} < 4\theta_0^{-4}\) | \(4\theta_0^{-4} < \bar{L} < 4\theta_0^{-4}\) | \(\Gamma_j^+ \ll \bar{L}\) |
| \(\beta_\theta\) | \(\bar{L}^{1/2}\) | \(\bar{L}^{1/2}\) | \(\bar{L}^{1/2}\) | \(\bar{L}^{1/2}\) |
| \(\Gamma_j^+\) | 1 | \(\frac{1}{2}\bar{L}^{1/2}\) | \(\frac{1}{2}\bar{L}^{1/2}\) | \(\frac{1}{2}\bar{L}^{1/2}\) |
| \(\theta_j\) | \(L^{1/4}\theta_0^2\) | \(L^{1/4}\theta_0^2\) | \(L^{1/4}\theta_0^2\) | \(\theta_0\) |
| \(P_c\) | \(\bar{L}\theta_0^2\rho_a c^2\) | \(\bar{L}^{1/4}\theta_0\rho_a c^2\) | \(\Gamma_j\theta_0\rho_a c^2\) | \(\Gamma_j\theta_0\rho_a c^2\) |
| \(\bar{\beta_\theta}\) | \(\bar{L}^{1/2}\theta_0\) | \(\bar{L}^{1/2}\theta_0\) | \(\bar{L}^{1/2}\theta_0\) | 1 |
| \(\eta\) | 1 | 1 | 1 | \(\theta_0^{-4}\) |
| \(\Gamma_j\) | \(\Gamma_j = \theta_0^{-1}\) | \(\Gamma_j\) | \(\Gamma_j\) | \(\Gamma_j\) |

**Notes.** All quantities are missing order of unity constants of integration over the density profile. In the case of a power-law density profile these constants can be calculated analytically and they are given in Appendix B.

a \(\Gamma_j\) is the jet Lorentz factor just below the head.

b When \(\bar{L} > 4\theta_0^{-4}\) this quantity represents the total energy in the cocoon divided by the volume \(V_c = (ct)^3\) and not the real pressure, which is not calculated in this regime.
Five main elements (see Figure 1) of the model are evident: the jet (divided into a shocked and unshocked part), the jet’s head, the cocoon, and the ambient medium. Also shown are the collimation shock and the contact discontinuity. The collimation shock splits the jet to an unshocked region and a shocked region. The contact discontinuity separates the jet material that enters the head from the ambient medium. This discontinuity extends to the cocoon and divides it to an inner and an outer part. The cocoon expands into the ambient medium behind a shock that extends the forward shock at the head. All shocks are marked with dashed lines and the contact discontinuities with solid lines. Right: a closeup of the jet’s head and the contact discontinuity. The matter flows into the head through a forward and a reverse shock, and from there to the cocoon as illustrated by the four arrows.

Figure 1. Schematic description of the jet geometry in the two collimation regimes. Left: a collimated jet; center: an uncollimated jet. In both panels, the basic ingredients of the model are evident: the jet (divided into a shocked and unshocked part), the jet’s head, the cocoon, and the ambient medium. Also shown are the collimation shock and the contact discontinuity.

3. THE JET–COCOON MODEL

Our model contains the five elements discussed above, where the ambient medium serves as a fixed background. Given a jet with a luminosity $L_j$, an injection angle, $\theta_0$, and a medium density profile, $\rho_a(z)$, we calculate the time-dependent quantities: the head velocity, $\beta_h$ (predominantly in the $z$-direction), the cocoon’s pressure, $P_c$, the cocoon expansion velocity, $\beta_c$ (predominantly in the $r$-direction), and the jet’s cross-section $\Sigma_j$. We use the subindices $j$, $h$, $c$, $a$ to designate quantities related to the jet, the jet’s head, the cocoon, and the ambient medium, respectively. The distinction between the shocked and unshocked jet is relevant only in the collimated regime, and we use it when we discuss this regime. The subindex $j$ is used to describe general properties of the jet (like the jet dimensions) and when it is unimportant which of the two region (shocked or unshocked jet) is considered, e.g., the jet luminosity is equal in the two regions and is denoted as $L_j$. 

at some altitude, above which the collimation is complete (see Section 3.1). The geometry of the collimation shock sets the jet’s cross-section at the head to be much smaller than the cross-section of an uncollimated jet. Consequently, the jet applies a larger ram pressure on the head and pushes it to higher velocities. The faster motion reduces the rate of energy flow into the cocoon. At the same time the cocoon’s height increases at a faster rate, resulting in a larger volume and a decrease in the cocoon’s pressure. There is a limit to the head velocity above which the cocoon’s pressure is too low to effectively collimate the jet. We show that this occurs when $L_j / (z^2 \rho_a c^3) \simeq \theta_0^{5/3}$, corresponding to a head’s Lorentz factor $\Gamma_h \simeq \theta_0^{-1/3}$. Therefore for typical initial opening angles $\theta_0 > 1^\circ$, the head velocities in the collimated regime can be at most mildly relativistic.

The uncollimated regime is characterized by larger values of $\Gamma_h$ and a cocoon pressure which is insufficient to collimate the jet in an appreciable amount. The jet remains conical to a good approximation and the collimation shock remains at the edges of the jet and does not converge onto the jet’s axis. This results in a coaxial jet structure composed of an inner fast spine surrounded by a denser layer of the shocked jet material, having a lower Lorentz factor (Figure 1, left panel). When $\Gamma_h > \theta_0^{-1}$, the head moves so fast that different parts of the jet’s head become causally disconnected and energy can flow into the cocoon only from a small region of the head. This further reduces the cocoon’s pressure. At even larger Lorentz factors the reverse shock at the head becomes weak, and it no longer affects the jet, which can be considered as propagating in a vacuum. The forward shock continues, however, to gather matter from the ambient medium in front of the jet and to accelerates it to a Lorentz factor similar to that of the jet. A small fraction of this shocked matter continues to stream into the cocoon and feeds with relativistic particles.

Following the above description, we divide the system into five main elements (see Figure 1): unshocked jet, shocked jet (separated by the collimation shock), jet’s head, cocoon (containing an inner light part and an outer heavy part), and ambient medium. The system dynamics are determined by the following relations between these region. (1) The jet’s head velocity is set by balancing the ram pressure applied on the forward shock (by the ambient medium) with the ram pressure applied on the reverse shocks (by the shocked or unshocked jet, depending on the collimation regime). The head velocity determines the cocoon height and the energy injection rate into it. The jet head has the highest pressure in the system. (2) The pressure in the cocoon is set by its size and by the energy injected into it from the head. (3) The velocity of the shock, which inflates the cocoon into the ambient medium, is set by the balance between the ram pressure of the ambient medium on that shock and the cocoon pressure. (4) The pressure in the shocked jet is equal to the cocoon pressure. The collimation shock structure is set to build up this pressure in the jet. This structure, in turn, determines the jet head cross-section and thus the ram pressure applied on the reverse shock. (5) The unshocked jet properties are determined by the inner engine.
We begin by discussing the main assumptions of the model. We take $L_j$, $\theta_0$, and $\rho(\zeta)$ as given and derive equations for $\beta_h$, $P_c$, $\beta_c$, and $\Sigma_j$. The latter differs between the two collimation regimes, and therefore we present the relevant solution separately in each regime.

The head velocity is set by balancing the ram pressure applied on the forward and reverse shocks (Begelman & Cioffi 1989; Matzner 2003):

$$\rho_j h_j \Gamma_j^2 \beta_j^2 (\beta_j - \beta_0)^2 + P_j = \rho_a h_a \Gamma_a^2 \beta_a^2 + P_a,$$  \tag{1}

where $\rho$, $P$, and $h = 1 + 4P/\rho c^2$ are the mass density, pressure, and dimensionless specific enthalpy of each fluid, measured in the fluid’s rest frame, and $\Gamma$ is the proper fluid velocity in the frame of the ambient medium. When the temperature of the ambient medium is non-relativistic and as long as the reverse shock is strong, $P_a$ and $P_j$ can be ignored. Under these conditions, the head’s velocity is (Matzner 2003)

$$\beta_h = \frac{\beta_j}{1 + L^{-1/2}},$$  \tag{2}

where the dimensionless parameter

$$\bar{L} \equiv \frac{\rho_j h_j \Gamma_j^2}{\rho_a} \sim \frac{L_j}{\Sigma_j \rho_a c^2}$$  \tag{3}

represents the ratio between the energy density of the jet ($L_j/\Sigma_j c$) and the rest-mass energy density of the surrounding medium at the location of the head. Note that by defining $\bar{L}$ as such, we implicitly assume that the energy lost in the jet due to work against the pressure of the cocoon is negligible and that radiation losses are negligible as well. We discuss this assumption in Appendix A and show that it is valid as long as the jet injection angle, $\theta_0$, is not too large. As we show in the following sections, $\bar{L}$ is the critical parameter that determines the evolution of the jet and the cocoon, while the combination $L \theta_0^{1/3}$ distinguishes between collimated and uncollimated jets. We stress that $\bar{L}$ may vary with the propagation of the jet, even if the jet luminosity is constant, depending on the density profile of the ambient medium and on the behavior of the jet’s cross-section. Thus, a jet can switch between the different collimation regimes from collimated to uncollimated and vice versa. Specifically, a collimated jet that encounters a sharp density decline may become uncollimated.

In the limits $\bar{L} \ll 1$ or $\bar{L} \gg 1$, Equation (2) can be linearized (Matzner 2003): when $\bar{L} \ll 1$

$$\beta_h \simeq \bar{L}^{1/2},$$  \tag{4a}

corresponding to a non-relativistic head velocity. If $1 \ll \bar{L} \ll 4 \Gamma_j^4$

$$\Gamma_h \simeq \sqrt{\bar{L}} \bar{L}^{1/4},$$  \tag{4b}

which corresponds to a relativistic proper velocity of the head, $\Gamma_h \beta_h > 1$ (e.g., Mészáros & Waxman 2001). When $\bar{L} \gg 4 \Gamma_j^4$, it follows from Equation (2) that $\Gamma_h \simeq \Gamma_j$, implying a non-relativistic reverse shock and a negligible energy flow of the shocked jet material into the cocoon. This is equivalent to the requirement that the volumetric-enthalpy ratio of the jet

and the ambient medium, $\rho_j h_j/\rho_a h_a > \Gamma_j^2$, and is similar to the criterion given by Sari & Piran (1995) for the case of a cold jet.

The pressure in the cocoon is sustained by a continuous flow of energy from the head. At any given moment the total energy in the cocoon is $E_c = \eta L_j (t - z_h/c)$, where the second term in the brackets represents the fraction of the energy which is carried by the relativistic jet and has not reached the cocoon yet (Lazzati & Begelman 2005). The parameter, $\eta$, varies between 0 and 1. It stands for the fraction of the energy that flows into the head and enters later into the cocoon. We assume that all the energy in the region of the head that is in causal connection with the cocoon flows into the cocoon, so $\eta$ is the fraction of that region out of the total head volume. When $\Gamma_h \gg 1$, most of the energy that flows out is initially tapped to the bulk motion of the fluid and it cannot contribute to the pressure. However, later, as the matter enters the cocoon, it spreads sideways and decelerates exponentially to $\Gamma \simeq 1$, in a similar manner to a rapidly spreading jet in a GRB afterglow (e.g., Granot et al. 2001; Piran 2000). Therefore, practically all the energy that flows into the cocoon is available to generate pressure.

The energy in the cocoon is shared between two parts which are separated by a contact discontinuity: an outer part made of the shocked ambient medium with a typical density $\sim \rho_a$ and an inner part that consists of the lighter jet material that crossed the reverse shock and enters from the head (see Figure 1). The outer cocoon is supported from the sides by the ram pressure of the surrounding medium, which balances its pressure. This results in a lateral expansion velocity of $\beta_c = \sqrt{P_c/\rho_a c^2}$ which is just below the sound speed in the outer cocoon as long as $\beta_c$ is sub-relativistic. In the inner cocoon, the sound speed is relativistic and typically much faster than $\beta_c$. Thus, matter on both sides of the contact discontinuity is in causal contact in the lateral direction and any pressure difference in this direction is smoothed out. In the longitude direction, the inner cocoon connects the upper parts of the cocoon with lower parts which otherwise would have been out of causal contact. This suppresses the pressure drop in the vertical direction, and in cases where the head is sub-relativistic leads to a uniform distribution of energy in the cocoon. The size of the region connected by the inner cocoon is determined by the survival length of the contact discontinuity, which is set by the growth rate of Kelvin–Helmholtz instabilities. The small shear-velocity differences on both sides of the contact render the growth rate small and stabilize the contact discontinuity. Indeed numerical simulations show that the contact discontinuity remains intact for a considerable fraction of the cocoon’s length and that the pressure drop is suppressed (see Section 4). We stress that the uniform pressure approximation should also hold rather well in cases where $P_c$ develops some vertical gradient since the conditions in the jet’s head depend only weakly on the pressure gradient.

To calculate the cocoon’s pressure we make the following approximations. First, we take the cocoon’s shape to be a cylinder with a height $z_h = \int \beta_h dt$ and a radius $r_c = \int \beta_c dt$. Second, based on the discussion above, we approximate the energy density to be uniformly distributed within the cocoon. Third, over a wide range of parameters and especially in the systems relevant for us, the pressure in the cocoon is radiation dominated, thus we use in our analysis an adiabatic index of $4/3$. Under these approximations, the cocoon’s pressure satisfies

$$P_c = \frac{E}{3V_c} = \frac{\eta}{3\pi c^3} \int \beta_h dt \int (\beta_c dt)^2.$$  \tag{5}
To calculate the transverse velocity of the cocoon, we take the average density of the medium $\rho_a = 1/V_c(z) \int \rho_a(z) dV$, where $V_c(z)$ is the volume of the cocoon and obtain the lateral expansion velocity (e.g., Begelman & Cioffi 1989):

$$
\beta_c = \sqrt{\frac{P_c}{\rho_a(z)c^2z^2}}.
$$

Approximating $\int \beta_0 dt \sim \beta_0 t$ and $\int \beta_0 dt \sim \beta_0 t$ and substituting into Equation (5) we obtain the cocoon’s pressure:

$$
P_c = \Xi_a \left( \frac{L_j \beta_0}{3\pi c^2} \right)^{1/2} \hat{L}^{-1/4} t^{-1},
$$

where $\Xi_a$ is a constant of order unity that depends on the density profile of the ambient medium and on the collimation regime. The value of $\Xi_a$ in each regime is given in the Appendix B, where we list the full time-dependent solutions of Equations (2), (5), and (6) in the limits: $\hat{L} \ll 1$ and $\hat{L} \gg 1$.

### 3.1. The Collimated Regime ($\hat{L} < \theta_0^{-1/3}$)

In the collimated regime, the pressure in the cocoon is sufficiently strong to compress the jet and form a strong oblique shock at the base of the jet. The jet’s head is then at full causal contact and therefore $\eta = 1$. The collimation shock divides the jet into a pre-shocked and a shocked region. We designate them by subindices $j0$ and $j1$, respectively (See Figure 2). The flow lines in area $j0$ are radial and they encounter the shock at an angle $\Psi(z)$, defined as the angle between the original direction of the flow line and the shock surface. As the flow lines pass through the shock, they are deflected toward the jet’s axis. Some of the flow energy is dissipated generating the pressure that supports the jet against the compression of the cocoon. The shock’s geometry is set by the balance between the cocoon’s pressure and the upstream momentum flux, normal to the shock, that enters at the upstream (Komissarov & Falle 1997; Bromberg & Levinson 2007):

$$
\rho_j c^2 h_j \Gamma_j^2 \beta_j^2 \sin^2 \Psi + P_j0 = P_c.
$$

Since the jet is relativistic and its ram pressure is much larger than its internal pressure, $P_j0$ can be neglected on the left-hand side of Equation (8). The radial geometry of the unshocked flow lines implies that $\rho_j c^2 h_j \Gamma_j^2 \beta_j^2 \sin^2 \Psi \approx \pi/P_c$. Therefore to keep a uniform pressure at the downstream $\Psi$ must increase with $z$. Consequently, the shock converges toward the axis until it converges on it at some point. In the small angle approximation, to a first order, $\sin \Psi = (z - d^2 c)^{1/2}$, where $r_s$ is the cylindrical radius of the collimation shock (see Figure 2). In this limit Equation (8) becomes a first-order ordinary differential equation whose solution is (Komissarov & Falle 1997; Bromberg & Levinson 2009)

$$
r_s = \theta_0 z (1 + A z^* - A \theta_0 z^2),
$$

where $A \equiv \sqrt{\frac{\pi \rho_a c^2}{\rho_a \Gamma}}$ and $z_*$ is the height where the jet is first affected by the cocoon and the collimation shock forms. As long as the pressure in the jet is larger than the pressure in the cocoon, the jet is unaware of the cocoon. As the jet accelerates pressure is gradually converted to kinetic energy until eventually it reaches a point where $P_j0(z) = P_c$. At this point, the jet’s compression by the cocoon becomes significant. Therefore, if the jet–medium interaction begins at the injection point $z_*$ can be tracked to the point where $P_j0 = P_c$. Alternatively, if the jet is injected into a cavity of radius $R_\ast$, $z_\ast = \max(R_\ast, z(P_j0 = P_c))$. Note that once the shock forms, its geometry depends only on the jet luminosity and it is the same for an accelerating or a non-accelerating jet. The shock converges at $r_s = 0$, namely, at

$$
\hat{z} = A^{-1} + z_\ast.
$$

As long as $z_\ast < A^{-1}$ it can be neglected from Equations (9) and (10), which implies that $R_\ast$ must be smaller than $A^{-1}$ as well. When $z_\ast > R_\ast$, we can use the jet luminosity at $z_\ast$ which holds $L_j \geq 4\pi P_j0^2 j0^2 \beta_j^2 z^2 \theta_0^2 c^4$ together with the condition that $P_j0 = P_c$ and extract $A$. Substituting $A$, we then get that $z_\ast < A^{-1}$ if

$$
\Gamma_j0(z_\ast) \beta_j0(z_\ast) \geq \theta_0^{-1}.
$$

We define a “sufficiently fast” jet as one that satisfies this condition. In this situation, we can ignore $z_\ast$ as long as $R_\ast < A^{-1}$. Since we work in the limit where $\theta_0$ is small, Equation (11) also implies that the jet is relativistic and therefore in the following we approximate $\beta_j0 = 1$ and $\beta_j1 = 1$. Note that as long as $\Gamma_j0(z) \leq \theta_0^{-1}$, the jet expands sideways under its own pressure and its opening angle increases as it propagates. In such a case, it is meaningless to discuss a constant initial opening angle, and our model does not hold. The jet stops expanding when the angle of the flow lines is of the order of $\Gamma_0^{-1}$, one over the injected Lorentz factor. Beyond this point it corresponds with our condition for neglecting $z_\ast$. In this case we can approximate $\theta_0$ to be $\sim \Gamma_0^{-1}$ and neglect $z_\ast$.

To estimate the jet’s cross-section, we approximate the jet to be conical up to the point where collimation has a sizable effect on the jet’s geometry, which is where the collimation shock is roughly parallel to the z-axis. From Equation (9) this occurs at $\hat{z}/2$. Above this point the jet is collimated and since the cocoon pressure is roughly uniform, the jet cross-section does not change much, and it can be taken to be constant. In this approximation, the cylindrical radius of the jet is $r_j(z > \hat{z}) \simeq \theta_0 \hat{z}/2$ and its cross-section is

$$
\Sigma_j(z > \hat{z}) \simeq \frac{1}{4} \pi \hat{z}^2 \theta_0^2 \simeq \frac{L_j \beta_0^2}{4c P_c}.
$$

---

**Figure 2.** Schematic description of the collimation shock in the collimated regime. The collimation shock is marked in a dashed black line, and it separates the jet into an unshocked region ($j0$) and a shocked region ($j1$). The jet flow lines, marked with light blue arrows, are initially radial. They intersect the shock at point $(r_s, z)$ and form an angle $\Psi$ with the shock’s surface. Downstream of the shock, the flow lines are deflected toward the jet’s axis. The shock crossing the average density of the medium

---

**Note:** The image contains additional figures and diagrams that are not transcribed here due to the nature of the text. The equations and text explain the dynamics of a collimated jet, including its interaction with ambient medium and the formation of a cocoon and shock. The calculations and approximations are used to describe the jet’s behavior and the conditions under which it becomes collimated. The text also refers to specific equations and constants, which are used to model the jet’s expansion and interaction with its environment.
The luminosity above \( \hat{z} \) is

\[
L_j \simeq 4P_c \Gamma_j^2 \Sigma_j c. \tag{13}
\]

Substituting \( L_j \) in Equation (12), we obtain the Lorentz factor of the shocked jet material above \( \hat{z} \):

\[
\Gamma_j(z > \hat{z}) = \frac{1}{\theta_0}. \tag{14}
\]

This implies that, as long as the jet is sufficiently fast at the injection point, the Lorentz factor of the collimated jet depends only on the injection angle, and it is independent of the initial Lorentz factor.

The remaining system parameters are calculated by substituting Equation (12) for the jet’s cross-section into Equations (3)–(7). Equations (3) and (7) yield

\[
\tilde{L} = \frac{4}{\theta_0^2} P_c \rho_a c^2 \simeq \frac{4}{c^2} \tilde{\Gamma}^{4/5} \tilde{L}_j^{2/5} \rho_a^{-2/5} \theta_0^{-8/5}, \tag{15}
\]

and

\[
P_c \simeq t^{-4/5} \tilde{L}_j^{2/5} \rho_a^{-3/5} \theta_0^{-2/5}. \tag{16}
\]

The head’s position is calculated using \( z_h \sim b_\theta t \), and the linearized expression for \( \beta_\theta \) is given in Equations (4a) and (4b):

\[
z_h \sim \begin{cases} \tilde{L}^{1/5} c t, & \tilde{L} \ll 1 \\ \tilde{L} < \theta_0^{-4/3}, \end{cases} \tag{17}
\]

Taking the cocoon’s cylindrical radius \( r_c \sim t \sqrt{P_c/\rho_a} \) and the jet’s cylindrical radius \( r_j \sim \theta_0 \tilde{L}_j / c P_c \), we calculate the opening angle of the cocoon and the jet, respectively:

\[
\theta_c \equiv \frac{r_c}{z_h} \sim \begin{cases} \theta_0, & \tilde{L} \ll 1 \\ \tilde{L}^{1/2} \theta_0, & 1 \ll \tilde{L} < \theta_0^{-4/3}, \end{cases} \tag{18}
\]

and

\[
\theta_j \equiv \frac{r_j}{z_h} \sim \begin{cases} \tilde{L}^{1/4} \theta_0^2, & \tilde{L} \ll 1 \\ \tilde{L}^{3/4} \theta_0^2, & 1 \ll \tilde{L} < \theta_0^{-4/3}. \end{cases} \tag{19}
\]

Note that if the head is non-relativistic, the cocoon’s aspect ratio is constant and it equals the jet’s injection angle, up to a constant of order unity (see Appendix B).

We can now infer the parameter regime for which the jet is collimated by the cocoon. The jet is considered collimated if the collimation shock converges below the jet’s head. Therefore, the transition to the uncollimated regime occurs when \( \hat{z} = z_h \).

We find that this equality can take place only when \( \tilde{L} > 1 \), thus we substitute \( t = z_h/c \) into Equation (16) and get that for \( z_h \geq \hat{z}, P_c \geq \rho_a c^2 \theta_0^{2/3} \). Substituting \( P_c \) in Equation (15), we find, up to a constant of order unity, that the condition for collimation is

\[
\tilde{L} \lesssim \theta_0^{-4/3}. \tag{20}
\]

The numerical factor missing in this equation depends on the density profile (see Appendix B). This condition corresponds to a limit on the head’s Lorentz factor

\[
\Gamma_h \lesssim \theta_0^{-1/3}. \tag{21}
\]

If the head accelerates beyond \( \theta_0^{-1/3} \) the collimation shock fails to converge below the head and the jet can be approximated as having a conical shape.

The jet can shift from a collimated to an uncollimated state and vice versa, depending on the density profile of the ambient medium. According to Equations (15) and (17), \( \tilde{L} \propto P_c / \rho_a \propto (z^{-4} \rho_a^{-2})^{\delta} \), where \( \delta = 1/3 \) if \( \tilde{L} \ll 1 \) and \( \delta = 1/5 \) if \( 1 \ll \tilde{L} < \theta_0^{-4/3} \). If the density profile is steeper than \( z^{-2} \), then \( \tilde{L} \) increases with \( z \) and the jet’s head accelerates. In such a case \( \theta_j = \theta_0 \rho_a \sqrt{\tilde{L}} \) increases with time. When \( \tilde{L} \simeq \theta_0^{-4/3} \), the collimation shock converges at the head, and \( \theta_j = \theta_0/2 \). At higher values of \( \tilde{L} \), the shock opens up and the jet becomes conical. In the limit of \( \rho_a \propto z^{-2} \), \( \tilde{L} \) is constant and the jet’s head velocity is constant. Equation (19) shows that in this limit \( \theta_j \) is constant as well, implying that \( r_j \propto z_h \). This makes an interesting case where the jet expands sideways at the same rate it propagates upward. Although the opening angle of the jet, \( \theta_j \ll \theta_0 \), it remains constant like in the case of a conical jet.

3.2. The Uncollimated Regime

When \( \tilde{L} \gg \theta_0^{-4/3} (\Gamma_h \gg \theta_0^{-1/3}) \) the pressure in the cocoon is too weak to significantly affect the geometry of the jet. In this case the jet remains conical to a good approximation and

\[
\Sigma_j(z_h) = \pi z_h^2 \theta_0^2. \tag{22}
\]

The collimation shock remains at the edge of the jet, resulting in a coaxial jet structure of a cold, fast, inner spine surrounded by a hotter and denser layer of the shocked jet material moving with a lower Lorentz factor (see Figure 1). The layer of shocked material becomes thinner and thinner at higher values of \( \tilde{L} \). In this regime, most of the jet’s plasma streams freely all the way to the head (\( j_0 \) extends to the head) and dissipates all its energy in the reverse shock. The jet is therefore cold below the head with some brightening at its limb, as opposed to a collimated jet which is hot below the head, since its plasma is first shocked much closer to the base by the collimation shock. Substituting \( \Sigma_j \) in Equation (3) and taking \( z_h = ct \) gives

\[
\tilde{L} = \frac{L_j}{\pi \theta_0^2 \rho_a c^2 c^5}. \tag{23}
\]

This implies that also in the uncollimated regime \( \rho_a \sim z^{-2} \) remains the limiting profile which distinguishes between an accelerating and a decelerating head.

The cocoon has some distinctive properties which can be classified into three subcases, according to the values of \( \tilde{L} \) and \( \eta \), the fraction of energy that flows from the head into the cocoon (see also Table 1). In each subcase the conditions in the head are different, and this affects the amount of energy that flows out of the head.

1. \( \theta_0^{-4/3} \ll \tilde{L} \ll \theta_0^{-4} (\theta_0^{-1/3} \ll \Gamma_h \ll \theta_0^{-1}) \): the head is sufficiently slow. Different regions of the head are in a causal contact with each other and all the energy in the head can flow into the cocoon, thus \( \eta \simeq 1 \). The cocoon’s pressure is calculated by substituting \( \tilde{L} \) from Equation (23) in Equation (7) and using \( z_h = ct \):

\[
P_c \simeq t^{-1/2} L_j^{1/4} \rho_a^{-3/4} \theta_0^{1/2} c^{3/4} \simeq \tilde{L}^{1/4} \theta_0 \rho_a c^2. \tag{24}
\]

Under this pressure, the temperature at the outer cocoon remains sub-relativistic and the different parts of the cocoon maintain causal connection in the lateral direction. As \( \tilde{L} \) increases \( P_c / \rho_a c^2 \) grows and it approaches unity as
\( \tilde{L} \rightarrow \theta_j^{-4} \). In this limit, the pressure becomes mildly relativistic and it pushes the edge of the cocoon to a velocity \( \beta_c \rightarrow 1 \), which is above the local sound speed, \( c/\sqrt{3} \). This results in a loss of causality in the transverse direction which implies that the approximation of a uniform pressure no longer holds. In addition the cocoon’s aspect ratio approaches unity and it can no longer be considered as cylindrical. Therefore above the limit of \( \tilde{L} \approx \theta_0^{-4} \) our model can only provide the total energy in the cocoon. This has no effect on the conditions in the jet and the jet’s head, which are well described by our model, since in the uncollimated regime the jet is not sensitive to the pressure in the cocoon.

2. \( \theta_0^{-4} \ll \tilde{L} \ll \Gamma_j \theta_0^{-1} \Gamma_h^{-1} \): at this limit the reverse shock is still strong and pressure in the head is large, but the jet’s head loses causal connectivity in the transverse direction. Energy can flow to the cocoon only from a thin annulus on the edge of the head with an opening angle \( \Gamma_h^{-1} \), which corresponds to a fraction \( \eta = 2(\Gamma_h \theta_0)^{-1} \) of the total energy that enters the head. Most of the energy that flows into the head remains trapped and accumulates at a rate of \( \tilde{L} \Gamma_h^{-2} \). The total energy that flows into the cocoon can thus be estimated as

\[
E_c = \eta L_j \Gamma_h^{-2} t \approx \tilde{L}^{1/4}\theta_0 \rho_a c^5 t^3, \tag{25}
\]

where we substitute \( L_j \) from Equation (23) and used the relation \( \tilde{L} \approx 4 \Gamma_j^4 \) from Equation (4b). Since \( \beta_c = 1 \), the cocoon has a spherical rather than cylindrical shape and it occupies a volume \( V_c \approx (ct)^3 \).

3. \( \Gamma_j \theta_0 \ll \tilde{L} (\Gamma_h \Sigma_j \approx \Gamma_j \theta_0) \): in this regime the reverse shock becomes Newtonian while the forward shock remains relativistic and moves with a Lorentz factor \( \sim \Gamma_j \theta_0 \). This leads to different sound speeds in the head, above and below the contact discontinuity. Below, within the shocked jet material, the sound speed \( \ll c \). Above, within the shocked medium, the sound speed is \( c/\sqrt{3} \). As a result the energy that flows into the cocoon comes mostly from the ambient medium part of the head and it flows into the outer cocoon, while the inner cocoon which is fed by the shocked jet material in the head becomes insignificant. To measure the energy that enters the cocoon, we first estimate the energy per unit time that flows into the head through the forward shock: \( E_{ha} \approx \rho_a \Gamma_j \theta_0^3 \Gamma_j^{-2} c^5 \). The total energy in the cocoon is therefore

\[
E_c = \eta E_{ha} t \approx \Gamma_j \theta_0 \rho_a t^3 c^5, \tag{26}
\]

where \( \eta = 2(\Gamma_j \theta_0)^{-1} \) in this case.

Table 1 summarizes the different characteristics of the jet–cocoon in the four different collimation regimes.

### 3.3. Comparison with Previous Analytical Works

The propagation and interaction of a jet with an ambient medium was studied in the past in various parameter regimes. Here, we briefly comment on some relevant works and discuss their compatibility to our results.

The main new feature introduced in this work is a proper closure of the set of equations using the conditions at the collimation shock. This allows us to determine if the jet is collimated or not, given its luminosity, initial opening angle, and the ambient density. When the jet is collimated it also allows us to determine its cross-section, \( \Sigma_j \). Earlier works differ from ours in the way they deal with this issue. Most ignore this collimation shock and assume a specific value or functional form for one of the variables (e.g., the jet cross-section), obtaining a closure in this way. Such solutions typically agree with ours if we substitute the value we obtain from the full system of equations instead of the free variable in these solutions.

Begelman & Cioffi (1989) analyzed the propagation of a non-relativistic galactic jet in the IGM, assuming that the jet is collimated and that it has a constant velocity \( \nu_j \ll c \). When using our expression for \( \Sigma_j \) (Equation (12)) in their solution (and taking the limit where \( \nu_j \rightarrow c \)), their results agree with ours for \( \tilde{L} \ll 1 \). Matzner (2003) extended this model to relativistic jets. Like Begelman & Cioffi (1989), he did not model the collimation of the jet and used \( \theta_j \) (which is related to \( \Sigma_j \)) as a given parameter. This solution is consistent with ours (for \( \tilde{L} \ll 1 \)) upon substitution of \( \theta_j \) from our solution. However, Matzner (2003) assumed that the jet is in fact conical, with \( \theta_j = \theta_0 \), which is inconsistent in this limit of \( \tilde{L} \ll 1 \).

Lazzati & Begelman (2005) considered the collimation of the jet by the cocoon’s pressure. However they ignored the dissipation in the collimation shock. Instead they assumed that the jet material expands adiabatically, leading to a relation \( \Gamma_j \propto \Sigma_j^{1/2} \). As we show in Section 3.1 (Equations (12)–(14)), the collimation shock renders this relation invalid. Consequently, the solution they obtained differs from ours in all regimes. Their solution has a smaller opening angle and a larger value of \( \Gamma_j \) at breakout. Later on, Morsony et al. (2007) attempted to calculate the geometry of the cocoon shock, but they used an incorrect expression for the momentum flux that crosses this shock, which led to a shock that never converged to the axis (\( \tilde{z} \rightarrow \infty \)).

Finally, Mészáros & Waxman (2001) analyzed the propagation of an uncollimated GRB jet in the outer envelope of a red supergiant. They considered only the properties of the jet’s head, ignoring the surrounding cocoon. Their solution for \( \Gamma_j \) is valid for \( \theta_0^{-4/3} \ll \tilde{L} \ll \Gamma_j \).

### 4. Comparison with Numerical Simulations

Jet simulations have been carried out extensively by various authors. We turn now to compare our results with two recent numerical simulations. First, we consider the simulation by Mizuta & Aloy (2009) who modeled the propagation of a relativistic jet in the envelope of a massive star. To model the star they used a numerically calculated density profile from Woosley & Heger (2006) (model number HE166N in Mizuta & Aloy 2009). This star has a radius of \( \sim 6 \times 10^{10} \) cm, and it has a density profile which can be divided into three parts. The inner part (up to \( \sim 1.2 \times 10^{10} \) cm) has an average power-law profile with an index \( \alpha = -2.5 \). Above it (up to \( 4 \times 10^{10} \) cm) the profile is steeper with an averaged index \( \alpha \approx -4.5 \), and it drops sharply from there to the edge of the star. The jet in the simulation is hot and it is injected with a Lorentz factor, \( \Gamma_0 = 5 \) into a cone of an opening angle \( \theta_0 = 5^\circ \) and an initial altitude of \( z_0 = 10^3 \) cm. Since \( \theta_0 < \Gamma_0^{-1} \) the jet goes through an initial transitory phase where its opening angle increases until it stabilizes when \( \theta_j(z) \approx \Gamma_j^{-1}(z) \approx 10^{-4} \). We therefore take in our calculations \( \theta_0 = 10^4 \). Figure 3 shows two snapshots of the jet and the cocoon at times 0.8 s and 1.2 s after the injection.\(^5\)

\(^5\) The snapshots are by courtesy of A. Mizuta and are taken from a simulation published in Mizuta & Aloy (2009).
injection point and the opening angle at the jet is far enough from the jet, the cocoon, and the collimation shock is drawn in black lines on top of a simulated jet by Mizuta & Aloy (2009). The left (right) panel shows a snapshot of the jet and the cocoon after 0.8(1.2) s. Color codes are of equal pressure divided by $c^2$ on the left and equal density on the right. The black arrow shows the average value of the cocoon’s pressure from our calculation. Figures are courtesy of A. Mizuta.

(A color version of this figure is available in the online journal.)

The times were chosen so that the jet is far enough from the contact discontinuity becomes unstable and below this point the pressure remains uniform to a good approximation through the region with the steeper density profile, just before the sharp drop. The steeper gradient in the stellar density profile leads to an acceleration of the jet’s head, which changes the rate of energy flow into the cocoon. Nevertheless it can be seen that the pressure remains uniform to a good approximation throughout most of the cocoon’s height, due to the survival of the contact discontinuity that keeps the inner cocoon intact. The pressure calculated by our model fits well with the simulated pressure in the cocoon fits well with the pressure profile in the simulation. Our results agree with the numerical results to a good accuracy.

Figure 4 presents two snapshots of the jet and the cocoon in the region with the steeper density profile, just before the sharp drop. The steeper gradient in the stellar density profile leads to an acceleration of the jet’s head, which changes the rate of energy flow into the cocoon. Nevertheless it can be seen that the pressure remains uniform to a good approximation throughout most of the cocoon’s height, due to the survival of the contact discontinuity that keeps the inner cocoon intact. The pressure calculated by our model fits well with the simulated pressure in the region with the steeper density profile, just before the sharp drop.

The black arrow shows the average value of the cocoon’s pressure from our calculation. Figures are courtesy of A. Mizuta.

(A color version of this figure is available in the online journal.)

The Astrophysical Journal, 740:100 (12pp), 2011 October 20

Table 2

| Model | $L_j/10^{51}$ (erg) | $\theta_0$ (°) | $\rho^*$ (s) | $t^*$ (s) |
|-------|---------------------|----------------|-------------|---------|
| JA    | 1.0                 | 20°            | 6.9         | 6.3     |
| JB    | 1.0                 | 5°             | 3.5         | 3.8     |
| JC    | 0.3                 | 10°            | 5.7         | 5.2     |

Notes.

1 Data from Zhang et al. (2003).
2 The breakout time according to the simulation.
3 The breakout time according to the analytic calculation.

and separates the shallow profile of $\alpha = -2.5$ from the steeper one with $\alpha = -4.5$. At this step, the growth rate of instabilities on the contact discontinuity increases.

We also compared our model with the simulations of Zhang et al. (2003). They examined the consequences of changing the injection angle and the luminosity of the jet on the propagation in a stellar mantle. Table 2 shows the initial parameters and the breakout times of the jet from the stellar surface in the three cases examined. In the right column we added the breakout times calculated by our analytical model, using the same stellar density profile and initial jet parameters. It can be seen that our results agree to within 10% with the simulated ones.

When comparing our results with numerical simulations we should recall that these simulations usually use a relatively large value of $z_*$, the radius in which the jet is injected to the stellar envelope. Such a jet is initially wider than a similar jet that is injected at a smaller radius and at this stage it propagates slower. Eventually the solution converges but the resulting breakout time is longer by $\approx z_*/c_0 \sqrt{\frac{\pi \rho_c c^2 \rho_* c^2}{L_j}}$, which becomes significant for $z_* \sim 10^{-3} R$, where $R$ is the stellar radius. The values of $z_*$ in jet simulations are usually larger than that and therefore we integrate numerically Equations (2), (5), (6), (10), and (12) to calculate the breakout times of the jet presented in Table 2.
Though our analytical model greatly simplifies the conditions in the jet and the cocoon, it shows a remarkable agreement with the results of the simulations presented above, which is slightly better than what we would expect in the general case. Generally, we expect an order of unity agreement for all density profiles that are not too steep, so that the forward and reverse shocks of the head remain in causal contact and that the approximation of uniform cocoon pressure is reasonable. As we see in the comparison to Mizuta & Aloy (2009) even a power-law density gradient as steep as \( \rho \propto z^{-4.5} \) satisfies these conditions. These conditions break down however in a very steep density profile, such as the one at the edge of a stellar envelope, where the forward shock accelerates and loses causal contact with the jet.

5. JET COLLIMATION IN ASTROPHYSICAL ENVIRONMENTS

According to our model when \( \tilde{L} > \theta_0^{-4/3} \) the jet is uncollimated and it has a conical shape with \( \theta_j = \theta_0 \). It implies that in this regime \( \tilde{L} = L_j / \pi \tilde{z}^2 \theta_0^2 \rho_0 c^3 \). When \( \tilde{L} < \theta_0^{-4/3} \), on the other hand, the jet is collimated and \( \theta_j < \theta_0 \). We can therefore formulate the condition for collimation as

\[
L_j / \pi \tilde{z}^2 \rho_0 c^3 < \theta_0^{2/3},
\]

and evaluate the conditions in the jet that lead to its collimation in different astrophysical media.

The conventional view about long GRBs is that they originate from massive stars that collapse (e.g., Woosley 1993). The jet propagation in this model is characterized by two stages: pre-breakout and post-breakout. Possible signatures of the breakout of the jet and the cocoon were discussed by several authors (e.g., Ramirez-Ruiz et al. 2002; Lazzati et al. 2009). Prior to the breakout the jet propagates inside the star which is considered to be massive (\( M \sim 10 M_\odot \)) and rather compact (\( R \sim R_\odot \)). Wolf-Rayet star (e.g., Woosley & Heger 2006; Crowther 2007).

The density profile in the stellar envelope can be approximated as \( \rho_0 = \tilde{\rho}(z/R)^{-\alpha} \), with \( 2 \lesssim \alpha \lesssim 3 \) (e.g., Matzner & McKee 1999), and \( \tilde{\rho} = \left( \left(3 - \alpha \right) / 4 \pi \right) \cdot \left( M R_\odot \right)^{-3} \) is the average density. Substituting this in Equation (27) we get that the jet is collimated if

\[
L_j \lesssim 10^{54} \left( \frac{\tilde{z}_{\odot}}{R} \right)^{2-\alpha} \left( \frac{R}{M \odot} \right)^{-1} \left( \frac{\theta_0}{10^\circ} \right)^{2/3} \left( \frac{M}{10 M_\odot} \right) \text{erg s}^{-1}.
\]

The luminosity we measure in GRBs reflects the luminosity of the jet at a time when it breaks out. We take here the simple approach that this luminosity is constant over time. Under this assumption, we can use the observed jet luminosities to estimate the collimation of the jet inside the star. Correcting to a typical beaming angle of \( 5^\circ - 10^\circ \), the observed values give \( L_j \lesssim 10^{52} \text{erg s}^{-1} \). In addition, as we show below, the measured values of the beaming angle of the jet are similar to the size of the injection angle at the base of the jet, \( \theta_0 \). Substituting this in Equation (28), we get that GRB jets are collimated before they break out. The propagation of the jet inside a stellar envelope and the implications on GRB observations are discussed to a greater extent in O. Bromberg et al. (2011, in preparation).

Once the jet breaks out its opening angle, \( \theta_j \), can be measured, for example by identifying a “jet break” in the afterglow light curve. Therefore, it is better to express the condition for collimation (Equation (27)) in terms of \( \theta_j \) rather than \( \theta_0 \).

Generally \( \theta_j \leq \theta_0 \) and \( \theta_j = \theta_0 \) if the jet is uncollimated. Therefore if \( \tilde{L} > \theta_0^{-4/3} \) it follows that \( \tilde{L} > \theta_0^{-4/3} \). On the other hand if \( \tilde{L} < \theta_0^{-4/3} \) then necessarily \( \tilde{L} < \theta_0^{-4/3} \) (otherwise the jet is uncollimated and \( \theta_j = \theta_0 \)). Thus Equation (27) can be used with \( \theta_j \) replacing \( \theta_0 \).

The medium outside the star can either be a dense wind ejected from the surface of the star, having a typical density profile \( \rho_a = a^* / z^2 \), where \( a^* \approx 5 \times 10^{47} \text{ g cm}^{-3} \), or it can be the constant density ISM. If the jet breaks out into a stellar wind environment it is collimated if

\[
L_j \leq 10^{43} \left( \frac{\theta_j}{10^\circ} \right)^{2/3} \left( \frac{a^*}{5 \times 10^{47} \text{ g cm}^{-3}} \right) \text{erg s}^{-1}.
\]

(29)

If alternatively it propagates in the ISM, it is collimated if

\[
L_j \leq 3 \times 10^{33} \left( \frac{z}{10^{13} \text{ cm}} \right)^2 \left( \frac{\theta_j}{10^\circ} \right)^{2/3} \left( \frac{\rho}{10^{-24} \text{ g cm}^{-3}} \right) \text{erg s}^{-1}.
\]

(30)

Since the observed luminosity is orders of magnitude higher than both of these limits, the jet becomes uncollimated when it propagates in these media. Therefore, as the jet, which was collimated inside the star, breaks out, it rapidly expands sideways and accelerates. Without any external interference the opening angle stabilizes when \( \theta_j \approx \Gamma_j^{-1} \), where \( \Gamma_j \) is the Lorentz factor of the jet at breakout, and since \( \Gamma_j \approx \theta_0^{-1} \) (Equation (14)) we get that the opening angle after breakout should be \( \approx \theta_0 \). However the breakout of the cocoon, which occurs simultaneously with the jet, limits the sideways expansion. Thus, the final opening angle of the jet at early times may be smaller than \( \theta_0 \). Note that the smaller opening angle only tightens the constraints for collimation given in Equations (29) and (30). At late times, when the cocoon clears out, the jet is no longer confined and maintains its initial injection angle. Therefore measuring \( \theta_j \) at late times gives information about the conditions at the injection site of the jet, even before the breakout when the jet was still buried in the star and was in fact collimated.

MCs and X-ray binaries jets have luminosities of \( \sim 10^{39} \text{ erg s}^{-1} \), and their opening angles \( \lesssim 10^\circ \). Using Equation (30) we get that these jets may begin uncollimated, but beyond \( \sim 2 \times 10^{-3} \) pc they become collimated due to the interaction with the ISM. The collimation of MC and X-ray binaries jets was also proposed by Miller-Jones et al. (2006), who showed that if the values of \( \Gamma_j \) are low (\( \sim 10 \)) it implies that the jets are collimated. Our calculation therefore supports this claim regardless of the Lorentz factor, which cannot be measured reliably at the present time.

Powerful quasar jets extend to distances of hundreds of kiloparsec from their sources, and they propagate in a much thinner medium. Their typical power is \( \lesssim 10^{47} \text{ erg s}^{-1} \), and their opening angle \( \lesssim 10^\circ \). The density profile in the galactic halo at such distances is usually considered to be isothermal with \( \alpha < 2 \), and a mass density of the order of \( \sim 10^{-27} \) to \( \sim 10^{-28} \) g cm\(^{-3} \) at a distance of \( \sim 10 \) kpc (e.g., Bulbul et al. 2010; Capelo et al. 2010). Substituting these values in Equation (30), we get that beyond this distance quasars jets are hydrodynamically collimated by their cocoons.

6. SUMMARY

In this work, we present an analytical study of the propagation of a relativistic hydrodynamic jet in an external medium with a general density profile. The interaction of the jet with the
medium results in the formation of a shocked “head” at the front of the jet and an overpressured cocoon with a rather uniform distribution of energy surrounding the jet. When the pressure of the cocoon, \( P_a \), is larger than the internal pressure of the jet, it compresses the jet and leads to the formation of an oblique shock at the base of the jet. Under some conditions, determined in this work, the shock converges to the jet’s axis, and the jet becomes collimated. This changes the jet’s properties and affects the conditions in the cocoon. Our model follows the evolution of the jet, the jet’s head, the cocoon, and the collimation shock, given three initial parameters: the jet’s luminosity, \( L_j \), the jet’s opening angle at the injection point, \( \theta_0 \), and the density of the ambient medium, \( \rho_a(z) \). We determine the condition for collimation and provide a self-consistent, time-dependent, solution to the system’s parameters that depends on these three initial parameters alone. The main results of our analysis can be summarized as follows.

1. The jet’s evolution can be classified into two regimes: a collimated and an uncollimated, according to the strength of interaction of the jet with the ambient medium. The two regimes are distinguished by the parameter \( \hat{L} \) and by \( \theta_0 \) (or \( \theta_h \)), which represents the ratio of the jet’s energy density to the rest-mass energy density of the ambient medium at the location of the head.

2. When \( \hat{L} < \theta_0^{-4/3} \), the interaction is strong and the jet is collimated. In this regime the collimation shock converges on the jet’s axis at some point, \( \hat{z} \), below the head. Above \( \hat{z} \) the jet is cylindrical to a good approximation and its width is estimated as \( \hat{z} \theta_0^2 / 2 \). This width implies that the Lorentz factor of the collimated ejecta satisfies \( \Gamma_{j1} = \theta_0^{-1} \).

3. If \( \hat{L} \ll 1 \), the head of the collimated jet is non-relativistic (\( \beta_h \ll 1 \)). In this limit the cocoon’s expansion velocity is proportional to the head’s velocity, and it has a constant opening angle \( \theta_t \simeq \theta_0 \).

4. When \( \hat{L} \gg \theta_0^{-4/3} \), the collimation shock fails to converge and it remains at the edge of the jet. The jet in this regime in uncollimated and it remains conical to a good approximation. It has a coaxial structure of a fast unshocked spine surrounded by a denser layer of shocked material with lower Lorentz factor.

5. When \( \hat{L} \gg \theta_0^{-4} \) and \( \Gamma_h > \theta_0^{-1} \), and the head is not causal in the transverse direction. Here only a fraction \( \sim (\Gamma_h \theta_0)^{-1} \) of the jet energy flows into the cocoon. This energy is enough to produce a mildly relativistic pressure in the cocoon, and accelerate cocoon’s edge to a velocity \( \beta_c \sim 1 \).

6. When \( \hat{L} > 4 \Gamma_h^{-4} \), the reverse shock becomes Newtonian and the head no longer affects the jet’s propagation. Matter continues to stream into the cocoon only from the forward shock, and it generates a relativistic pressure in the cocoon.

7. The value of \( \hat{L} \) can change with time, even for a constant \( L_j \) and \( \theta_0 \), according to the slope of the density profile. In a density profile \( \rho_a \sim z^{-\alpha} \) with \( \alpha > 2 \), \( \hat{L} \) increases with time. This corresponds to an acceleration of the jet’s head. In such cases, a collimated jet opens up and becomes uncollimated once \( \hat{L} \) becomes larger than \( \theta_0^{-4/3} \). The opposite evolution takes place when \( \alpha < 2 \). In the case of \( \alpha = 2 \), \( \hat{L} \) is constant, the head maintains its velocity, and the opening angle of a collimated jet remains constant.

8. The choice of the initial injection radius of the jet, \( z_a \), can affect the jet’s breakout time, where larger values of \( z_a \) increase this time. This effect becomes important when the ratio of \( z_a \) to the stellar radius \( \gtrsim 10^{-3} \). Numerical simulations typically use values of \( z_a \) which are above this limit and therefore their resultant breakout times are affected by this choice.

Our model provides a general framework to study the properties of relativistic jets in different media. It can be used to examine various phenomena, such as the minimal energy required by the jet to break out of a boundary surface at a finite distance, like the edge of a star. It can also be used to test the energy feedback into the stellar envelope in the case of a GRB jet that penetrates a star or the IGM in the case of AGN jets. This may help better understand issues such as the problem of IGM heating. Our model confirms that GRB jets, which form inside a star, are collimated before they break out and become uncollimated afterward. MCs and X-ray binary jets, however, may start uncollimated, but the interaction with the ISM leads to their collimation beyond \( \sim 2 \times 10^{-3} \) pc. We also show that quasar jets are collimated as well at large distances from their sources (\( \gtrsim 10 \) kpc). We stress that in our calculations we assume that the magnetic fields in the jet, in the region where the jet undergoes the collimation and above, are dynamically unimportant and therefore can be ignored. High magnetization in this region will alter our results, for example by preventing the formation of the collimation shock, but will keep the other basic properties of the model, i.e., the formation of the cocoon and the collimation of the jet.

We thank A. Mizuta for providing us results from his numerical simulations. We also thank Y. Lubarsky for useful discussions and the anonymous referee for helpful comments. O.B. and T.P. were supported by the Israel Center for Excellence for High Energy Astrophysics and by an ERC advanced research grant. E.N. was supported in part by the Israel Science Foundation (grant no. 174/08) and by an EU International Reintegration Grant. R.S. is partially supported by IRG and ERC grant, and a Packard fellowship.

**APPENDIX A**

**TESTING THE ASSUMPTION OF ENERGY CONSERVATION IN THE JET**

In calculating the parameter \( \hat{L} \) and the velocity of the head, we assume that the jet material does not lose energy to work as it flows from the injection point to the head. Such an assumption is natural if the jet is conical, since the jet has a constant opening angle and its envelope at a given point does not expand with time. However when the jet is collimated its width changes with time, and the jet loses or gains energy due to mechanical work preformed against the pressure of the cocoon. When \( \hat{L} \ll 1 \) the head moves at a sub-relativistic velocity. In this limit, to a first order in \( \beta_h \), the work done due to sideways expansion of the jet is \( dW = P_c dV = P_c z_a d\Sigma_j \). This work can be neglected as long as it is much smaller than the energy added to the system: \( L_{jdt} \). We therefore define \( \epsilon_j = (P_c z_a / L_j) \cdot (d\Sigma_j / dt) \) as the relative amount of energy lost to expansion of the jet. Our approximation holds as long as \( \epsilon_j \ll 1 \). Substituting \( \Sigma_j \) from Equation (12) and \( P_c \) from Equation (16) gives

\[
\epsilon_j \simeq \theta_0^{-1} \beta_h \simeq 0.03 \left( \frac{\theta_0}{10^2} \right)^2 \beta_h.
\]  

(A1)

Implying that the assumption of negligible energy losses in the jet is always valid in the limit of small injection angle.
When $1 \ll \tilde{L} \ll \theta_0^{-4/3}$, the head is relativistic and the energy flux that enters the head through the reverse shock is reduced by a factor of $\sim \Gamma_h^{-2}$. Since we are interested in the energy that enters the head we compare the work lost to expansion with $L_j \Gamma_h^{-2} dt$. This gives $\epsilon_j = (\Gamma_h^2 P_c z_h / L_j) (d\Sigma_j / dt)$, and applying the same consideration as before we get that

$$\epsilon_j \simeq \Gamma_h^2 \theta_0^2. \quad (A2)$$

In this regime $\Gamma_h \ll \theta_0^{-1/3}$ (see Table 1), therefore the relative energy lost to work in a relativistic collimated jet maintains $\epsilon_j < \theta_0^{-1/3}$. But in this regime of a collimated jet with a relativistic head, the injection angle is limited by $\theta_0^{-1/3} \ll 1$. This guarantees that $\epsilon_j \ll 1$ and implies that our approximation is always valid in this type of jets as well.

### APPENDIX B

### ANALYTIC SOLUTIONS TO THE RELATIVISTIC AND NON-RELATIVISTIC LIMITS

The system’s behavior is determined by the Equations (2), (5), (6), and (12). Generally these equations should be solved numerically, due to the nonlinearity of $\beta_h$. But in the limits of $\tilde{L} \ll 1$ and $\tilde{L} \gg 1$, Equation (2) can be linearized and the integration over time can be solved analytically. We define the following integration parameters: $\bar{\rho}(z_h) = \int \rho_d V / V \equiv \rho_0 \rho(z_h)$, $z_h = \int \rho_h dt \equiv \xi \beta_h t$, $r_c = \int \beta_c dt \equiv \xi \beta_c t$, and $\int \Gamma_h^{-2} dt \equiv \xi \Gamma_h^{-2} t$. Using these parameters, Equation (7) can be written as

$$P_c = \left( \frac{L_j \rho_h}{3 \pi} \right)^{1/2} \tilde{L}^{1/4} t^{-1} \left( \frac{\xi c \bar{\rho}}{\xi \bar{\rho}} \right)^{1/2}. \quad (B1)$$

Each of these parameters takes a different value when $\tilde{L} \ll 1$ and when $\tilde{L} \gg 1$. If the density profile is a power law of the sort $\rho \sim z^{-\alpha}$, the parameters become constants and their value depends on $\alpha$. The resulting solutions including the integration parameters in each regime are presented below, for convenience in presenting $\rho$ we took density profiles with $\alpha < 3$.

#### B.1. A Collimated Jet with a Non-relativistic Head ($\tilde{L} \ll 1$)

In this regime $\xi = 1$ and the rest of the integration parameters take the following values: $\xi = \epsilon = \frac{5-\alpha}{3-\alpha}$, $\rho \equiv \frac{3}{(3-\alpha)}$. The solutions to the system’s parameters are

$$z_h = \left( \frac{L_j \rho_h}{\rho_0^3} \right)^{1/5} \left[ \frac{2^4}{3 \pi} \rho \xi^2 \right]^{1/5} \quad (B2)$$

$$\beta_h = \left( \frac{L_j \rho_h}{\rho_0^3} \right)^{1/5} \left[ \frac{2^4}{3 \pi} \rho \xi^{-3} \right]^{1/5} \frac{1}{c} \quad (B3)$$

$$P_c = \left( \frac{L_j \rho_h}{\rho_0^3} \right)^{1/5} \left[ \frac{1}{6 \pi} \rho \xi^2 \right]^{2/5} \quad (B4)$$

$$r_c = z \theta_0 \left( \frac{1}{2 \sqrt{\theta}} \right) \quad (B5)$$

$$\beta_c = \beta_h \theta_0 \left( \frac{1}{2 \sqrt{\theta}} \right) \quad (B6)$$

#### B.2. A Collimated Jet with a Relativistic Head ($\tilde{L} \gg 1$)

In this regime $\xi = 1$, $\epsilon = \frac{5-\alpha}{3-\alpha}$, $\rho \equiv \frac{3}{(3-\alpha)}$. The solutions to the system’s parameters are

$$\theta_c = \frac{\theta_0}{2 \sqrt{\theta}} \quad (B7)$$

$$r_j = \left( \frac{t^4 L_j \theta_0^8}{\rho_0^3} \right)^{1/10} \left[ \frac{2^4 \pi \rho \xi^2 \xi^{-3}}{3} \right]^{1/5} \left( \frac{1}{\sqrt{c}} \right) \quad (B8)$$

$$\theta_j = \left( \frac{L_j \theta_0}{t^4} \right)^{1/10} \left[ \frac{2^4 \pi \rho \xi^2 \xi^{-3}}{3} \right]^{1/5} \left( \frac{1}{\sqrt{c}} \right) \quad (B9)$$

$$\Gamma_1 = \theta_0^{-1} \quad (B10)$$

$$\tilde{z} = \frac{2 r_j}{\theta_0} = \sqrt{\frac{L_j}{\pi c P_c}} \quad (B11)$$

$$\frac{L_j \rho_h}{\rho_0^3} \left( \frac{1}{3 \pi} \right)^{1/5} \left[ \frac{2^4}{3 \pi} \rho \xi^2 \xi^{-3} \right]^{1/5} \frac{1}{c} \quad (B12)$$

$$\beta_h = \left( \frac{L_j \rho_h}{\rho_0^3} \right)^{1/5} \left[ \frac{2^4}{3 \pi} \rho \xi^{-3} \xi^{-3} \right]^{1/5} \frac{1}{c} \quad (B13)$$

$$P_c = \left( \frac{L_j \rho_h}{\rho_0^3} \right)^{1/5} \left[ \frac{1}{6 \pi} \rho \xi^2 \xi^{-2} \right]^{2/5} \quad (B14)$$

$$r_c = \left( \frac{t^3 L_j \theta_0}{L_a} \right)^{1/5} \left[ \frac{1}{6 \pi} \rho \xi^2 \xi^{-2} \right]^{2/5} \quad (B15)$$

$$\beta_c = \left( \frac{L_j \rho_h}{\rho_0^3} \right)^{1/5} \left[ \frac{2^4}{3 \pi} \rho \xi^{-3} \xi^{-3} \right]^{1/5} \frac{1}{c} \quad (B16)$$

$$\theta_c = \beta \quad (B17)$$

$$r_j = \left( \frac{t^4 L_j \theta_0^8}{\rho_0^3} \right)^{1/10} \left[ \frac{3 \rho \xi^2}{16 \pi^2 \xi^2} \right]^{1/5} \left( \frac{1}{\sqrt{c}} \right) \quad (B18)$$

$$\theta_j = \left( \frac{L_j \theta_0}{t^4} \right)^{1/10} \left[ \frac{3 \rho \xi^2}{16 \pi^2 \xi^2} \right]^{1/5} \left( \frac{1}{\sqrt{c}} \right) \quad (B19)$$

$$\Gamma_1 = \theta_0^{-1} \quad (B20)$$

$$\tilde{z} = \frac{2 r_j}{\theta_0} = \sqrt{\frac{L_j}{\pi c P_c}} \quad (B21)$$

The jet is collimated as long as $\tilde{z} \leq z_h$. Substituting Equations (B12) and (B15) into Equation (B22), and using the relation $L = \frac{4 \pi P_c}{\rho_0 c}$ (Equation (15)), we get that the jet is collimated as long as

$$\tilde{L} \leq \theta_0^{-4/3} \left[ \frac{16 \pi^2 \xi^2}{3 \rho \xi^2} \right]^{2/3} \quad (B23)$$
B.3. An Uncollimated Jet with a Causally Connected Relativistic Head ($\theta_0^{-4/3} \ll \bar{L} \ll \theta_0^{-4}$)

The integration parameters in this regime are the same as in the relativistic, collimated regime, i.e., $\zeta = 1$, $\xi = 5/(7 - \alpha)$, $\epsilon = 5/(3 + \alpha)$, and $\varphi = 3/(3 - \alpha)$, but since the jet is no longer conical the solutions are

$$z_h \simeq c t$$  \hspace{1cm} (B24)$$
$$\beta_h \simeq 1$$  \hspace{1cm} (B25)$$
$$\Gamma_h = \left( \frac{L_j \rho_0^2}{t^2 \rho} \right)^{1/4} \left( \frac{1}{4 \pi c^5} \right)^{1/4}$$  \hspace{1cm} (B26)$$
$$P_c = \left( \frac{L_j \rho_0^3}{t^2} \right)^{1/4} \left( \frac{\xi \varphi}{3 \pi \xi \varphi^2} \right)^{1/2} c^{3/4}$$  \hspace{1cm} (B27)$$
$$\beta_c = \left( \frac{L_j \rho_0}{t^2 \rho_a} \right)^{1/8} \left( \frac{\xi \varphi}{3 \pi \xi \varphi^2} \right)^{1/2} c^{-5/8}$$  \hspace{1cm} (B28)$$
$$r_c = \left( \frac{t^6 L_j \rho_0^3}{\rho_a} \right)^{1/8} \left( \frac{\xi \varphi^2}{3 \pi \xi \varphi^2} \right)^{1/2} c^{3/8}$$  \hspace{1cm} (B29)$$
$$\theta_c = \beta_c$$  \hspace{1cm} (B30)$$
$$\theta_j = \theta_0.$$  \hspace{1cm} (B31)

B.4. An Uncollimated Jet with an Uncausal Relativistic Head ($4 \theta_0^{-4} \ll \bar{L}$)

In this regime, different parts in the head are not in causal connection and therefore only a fraction $2/(\Gamma_0 \theta_0)$ of the energy in the head goes into the cocoon. The cocoon’s pressure is relativistic and it pushes the edge of the cocoon to a velocity $\beta_c \to 1$ which implies that our approximation of the cylindrical cocoon breaks down. Moreover, since the $\beta_c > c/\sqrt{3}$, the cocoon is no longer causally connected in the lateral direction and the pressure is no longer uniform. In this case we are able to provide the average energy density in the cocoon, by taking the total energy that enters the cocoon and dividing it with $V_c$, the cocoon’s volume approximated as a sphere in this regime. Under these approximations, the model parameters are

$$z_h \simeq c t$$  \hspace{1cm} (B32)$$
$$\beta_h \simeq 1$$  \hspace{1cm} (B33)$$
$$\Gamma_h = \left( \frac{L_j \rho_0^2}{t^2 \rho} \right)^{1/4} \left( \frac{1}{4 \pi c^5} \right)^{1/4}$$  \hspace{1cm} (B34)$$

When $\bar{L} \gg 4 \Gamma_0 \theta_0$, the Lorentz factor of the head is $\sim \Gamma_0 \theta_0$, and the reverse shock becomes Newtonian. In this limit, most of the energy that flows into the cocoon comes from the material behind the forward shock which remains relativistic. Energy enters the head through the forward shock at a rate $E_{\text{in}} = \pi t^3 \theta_0^4 \Gamma_0^2 \rho_0^2 c^5$ and a fraction $2/(\Gamma_0 \theta_0)$ of that flows into the cocoon. Approximating the cocoon’s volume as a sphere, we get

$$\Gamma_h = \Gamma_0$$  \hspace{1cm} (B39)$$
$$E_c = \frac{3}{2} \frac{\Gamma_0 \theta_0 \rho_a c^2}{\bar{L}}.$$  \hspace{1cm} (B40)$$

where all the rest of the parameters remain as before. Note that in this regime, the average energy density in the cocoon remains constant.

REFERENCES

Aloy, M. A., Ibáñez, J. M., Martí, J. M., Gómez, J. L., & Müller, E. 1999, ApJ, 523, L125
Begelman, M. C., & Cioffi, D. F. 1989, ApJ, 345, L21
Blandford, R. D., & Rees, M. J. 1974, MNRAS, 169, 395
Bromberg, O., & Levinson, A. 2007, ApJ, 671, 678 (BL07)
Bromberg, O., & Levinson, A. 2009, ApJ, 699, 1274 (BL09)
Bulbul, G. E., Hasler, N., Bonamente, M., & Joy, M. 2010, ApJ, 720, 1038
Capelo, P. R., Natarajan, P., & Coppi, P. S. 2010, MNRAS, 407, 1148
Crowther, P. A. 2007, ARA&A, 45, 177
Granot, J., Miller, M., Piran, T., Suen, W.-M., & Hughes, P. A. 2001, in ESO Astrophysics Symp., Gamma-Ray Bursts in the Afterglow Era Conf., ed. E. Costa, F. Frontera, & J. Hjorth (Berlin: Springer), 312
Hughes, P. A., Miller, M. A., & Duncan, G. C. 2002, ApJ, 572, 713
Komissarov, S. S., & Falle, S. A. E. G. 1997, MNRAS, 288, 833
Lazzati, D., & Begelman, M. 2005, ApJ, 629, 903
Lazzati, D., Morsony, B. J., & Begelman, M. C. 2009, ApJ, 700, L47
Marti, J. M. A., Muller, E., Font, J. A., & Ibanez, J. M. 1995, ApJ, 448, L105
Marti, J. M. A., Muller, E., Font, J. A., Ibanez, J. M. A., & Marquina, A. 1997, ApJ, 479, 151
Matzner, C. D. 2003, MNRAS, 345, 575
Matzner, C. D., & McKee, C. F. 1999, ApJ, 510, 379
Mészáros, P., & Waxman, E. 2001, Phys. Rev. Lett., 87, 171102
Miller-Jones, J. C. A., Fender, R. P., & Nakar, E. 2006, MNRAS, 367, 1432
Mizuta, A., & Aloy, M. A. 2009, ApJ, 699, 1261
Morsony, B. J., Lazzati, D., & Begelman, M. C. 2007, ApJ, 665, 569
Piran, T. 2000, Phys. Rep., 333, 529
Ramirez-Ruiz, E., Celotti, A., & Rees, M. J. 2002, MNRAS, 337, 1349
Sari, R., & Piran, T. 1999, ApJ, 455, L143
Scheuer, P. A. G. 1974, MNRAS, 166, 513
Woosley, S. E. 1993, ApJ, 405, 273
Woosley, S. E., & Heger, A. 2006, ApJ, 637, 914
Zhang, W., Woosley, S. E., & MacFadyen, A. I. 2003, ApJ, 586, 356