SOME PROPERTIES OF A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

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Abstract. We prove Schwarz-Pick type estimates and coefficient estimates for a class of elliptic partial differential operators introduced by Olofsson. Then we apply these results to obtain a Landau type theorem.

1. Introduction and main results

Let $\mathbb{C}$ be the complex plane. For $a \in \mathbb{C}$, let $r > 0$ and $D(a, r) = \{z : |z - a| < r\}$. In particular, we use $D_r$ to denote the disk $D(0, r)$ and $D$, the open unit disk $D_1$.

For a real $2 \times 2$ matrix, we will consider the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$ with $x$ and $y$ real, we denote the complex differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

If we denote the formal derivative of $f = u + iv$ by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then $\|D_f\| = |f_z| + |f_{\overline{z}}|$ and $l(D_f) = |f_z| - |f_{\overline{z}}|$, where $u, v$ are real functions, $f_z = \partial f/\partial z$ and $f_{\overline{z}} = \partial f/\partial \overline{z}$. Throughout this paper, we denote by $C^n(D)$ the set of all $n$-times continuously differentiable complex-valued functions in $D$, where $n \in \{1, 2, \cdots\}$.

For $\alpha \in \mathbb{R}$ and $z \in D$, let

$$T_{\alpha} = -\frac{\alpha^2}{4} (1 - |z|^2)^{-\alpha - 1} + \frac{\alpha}{2} (1 - |z|^2)^{-\alpha - 1} \left( z \frac{\partial}{\partial z} + \overline{z} \frac{\partial}{\partial \overline{z}} \right) + \frac{1}{4} (1 - |z|^2)^{-\alpha} \Delta$$

be the second order elliptic partial differential operator, where $\Delta$ is the usual complex Laplacian operator

$$\Delta := 4 \frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

We consider the Dirichlet boundary value problem of distributional sense as follows:

2000 Mathematics Subject Classification. Primary: 31A05; Secondary: 35J25.

Key words and phrases. Differential operator, coefficient estimate, Schwarz-Pick estimate.
Here, the boundary data $f^* \in \mathcal{D}'(\partial \mathbb{D})$ is a distribution on the boundary $\partial \mathbb{D}$ of $\mathbb{D}$, and the boundary condition in (1.1) is interpreted in the distributional sense that $f_r \to f^*$ in $\mathcal{D}'(\partial \mathbb{D})$ as $r \to 1^-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta}), \quad e^{i\theta} \in \partial \mathbb{D},$$

for $r \in [0, 1)$ (see [21]).

In [21], Olofsson proved that, for parameter values $\alpha > -1$, a function $f \in C^2(\mathbb{D})$ satisfies (1.1) if and only if it has the form of a Poisson type integral

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-i\tau}) f^*(e^{i\tau}) d\tau, \quad z \in \mathbb{D},$$

where

$$K_\alpha(z) = c_\alpha \frac{(1 - |z|^2)^{\alpha+1}}{|1 - z|^{\alpha+2}},$$

$c_\alpha = (\Gamma(\alpha/2 + 1))^2 / \Gamma(1 + \alpha)$ and $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ for $s > 0$ is the standard Gamma function. If we take $\alpha = 2(n-1)$, then $f$ is polyharmonic (or $n$-harmonic), where $n \in \{1, 2, \ldots\}$ (cf. [1, 2, 5, 14, 18, 19, 20, 22]). In particular, if $\alpha = 0$, then $f$ is harmonic.

For $a, b, c \in \mathbb{R}$ with $c \not= 0, -1, -2, \ldots$, the hypergeometric function is defined by the power series

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

where $(a)_n = a(a+1) \cdots (a+n-1)$ for $n = 1, 2, \ldots$ are the Pochhammer symbols. Obviously, for $n = 0, 1, 2, \ldots$, $(a)_n = \Gamma(a+n)/\Gamma(a)$. In particular, for $a, b, c > 0$ and $a + b < c$, we have (cf. [3, 4])

$$F(a, b; c; 1) = \lim_{x \to 1} F(a, b; c; x) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} < \infty.$$

On the basis of Olofsson’s research, we continue to investigate some properties of solutions to (1.1). The following is a Schwarz-Pick type estimate on the solutions to (1.1).

**Theorem 1.** For $\alpha > -1$, let $f \in C^2(\mathbb{D})$ satisfy (1.1) and $\sup_{z \in \mathbb{D}} |f(z)| \leq M$, where $M$ is a positive constant. Then, for $z \in \mathbb{D},$

$$|f(z) - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|} f(0)| \leq M \left[ \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-i\tau}) dt - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|} K_\alpha(0) \right]$$

and

$$\|Df(z)\| \leq \frac{M M_\alpha(|z|) [2 + \alpha + (4 + 3\alpha)|z|]}{1 - |z|^2} \leq \frac{M [2 + \alpha + (4 + 3\alpha)|z|]}{1 - |z|^2},$$

where $M_\alpha(|z|) = \sup_{|z| < 1} |f(z)|$. 


where
\[ M_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\theta}) d\theta = \frac{\Gamma(1 + \frac{\alpha}{2})^2}{\Gamma(1 + \alpha)} F \left( -\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2 \right), \quad r \in [0, 1). \]

Let \( f \) be a harmonic mapping of \( \mathbb{D} \) onto \( \mathbb{D} \) with \( f(0) = 0 \). In [15], Heinz showed that, for \( z \in \mathbb{D} \),
\[ \|D_f(z)\| \geq \frac{2}{\pi}. \]

By using Theorem 1, we get a Heinz type inequality on \( \partial \mathbb{D} \) as follows.

**Theorem 2.** For \( \alpha \geq 0 \), let \( f \in C^2(\mathbb{D}) \) satisfying (1.1). Suppose that \( f(0) = 0 \), \( f(\overline{\mathbb{D}}) = \overline{\mathbb{D}} \), \( f(\partial \mathbb{D}) = \partial \mathbb{D} \).

(a) If \( \alpha = 0 \), then, for \( \theta \in [0, 2\pi] \),
\[ \|D_f(e^{i\theta})\| \geq \frac{2}{\pi}; \]

(b) If \( \alpha > 0 \), then, for \( \theta \in [0, 2\pi] \),
\[ \|D_f(e^{i\theta})\| \geq \lim_{r \to 1^{-}} \frac{d}{dr} M_\alpha(r) = \frac{\alpha}{2}, \]
where \( M_\alpha(r) \) is given by (1.7).

The following result is the homogeneous expansion of solutions to (1.1).

**Theorem A.** ([21, Theorem 2.2]) Let \( \alpha \in \mathbb{R} \) and \( f \in C^2(\mathbb{D}) \). Then \( f \) satisfies (1.1) if and only if it has a series expansion of the form
\[ f(z) = \sum_{k=0}^{\infty} c_k F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; |z|^2 \right) z^k \]
\[ + \sum_{k=1}^{\infty} c_{-k} F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; |z|^2 \right) \overline{z}^k, \quad z \in \mathbb{D}, \]
for some sequence \( \{c_k\}_{k=-\infty}^{\infty} \) of complex numbers satisfying
\[ \lim_{|k| \to \infty} \sup |c_k|^{|k|} \leq 1. \]

In particular, the expansion (1.8), subject to (1.9), converges in \( C^{\infty}(\mathbb{D}) \), and every solution \( f \) of (1.1) is \( C^{\infty} \)-smooth in \( \mathbb{D} \).

For \( \alpha = 0 \), there are numerous discussions on coefficient estimates of harmonic mappings in the literature, see for example [7, 8, 9, 11, 13, 17, 24]. We investigate the problem of coefficient estimates on the solutions to (1.1) as follows.

**Theorem 3.** For \( \alpha > -1 \), let \( f \in C^2(\mathbb{D}) \) be a solution to (1.1) with the series expansion of the form (1.8) and \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M \) is a positive constant. Then, for \( k \in \{1, 2, \ldots\} \),
\[ |c_k F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1 \right)| + |c_{-k} F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1 \right)| \leq \frac{4M}{\pi} \]
functions are \( \alpha \).

In particular, if \( f \) is measurable, then the estimate of (1.10) is sharp and all the extreme functions are

\[
f_k(z) = \frac{2\varepsilon M}{\pi} \Im \left( \log \frac{1 + \partial z^k}{1 - \partial z^k} \right),
\]

where \( |\varepsilon| = |\vartheta| = 1 \).

The following result easily follows from Theorem 3 and [21, Proposition 1.4].

**Corollary 1.1.** For \( \alpha > -1 \), let \( f \in C^2(\mathbb{D}) \) be a solution to (1.1) with the series expansion of the form (1.8) and \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M \) is a positive constant. Then, for \( k \in \{1, 2, \ldots \} \),

\[
|c_k| + |c_{-k}| \leq \frac{4M\Gamma(1 + \frac{\alpha}{2}) \Gamma(k + 1 + \frac{\alpha}{2})}{k!\Gamma(\alpha + 1)\pi}.
\]

For \( p \in (0, \infty) \), the Hardy space \( \mathcal{H}^p \) consists of those functions \( f : \mathbb{D} \to \mathbb{C} \) such that \( f \) is measurable, \( M_p(r, f) \) exists for all \( r \in (0, 1) \) and \( \|f\|_p \leq M \), where\n
\[
\|f\|_p = \begin{cases} \sup_{0<r<1} M_p(r, f), & \text{if } p \in (0, \infty), \\ \sup_{z \in \mathbb{D}} |f(z)|, & \text{if } p = \infty, \end{cases}
\]

and

\[
M_p^* = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.
\]

The classical theorem of Landau shows that there is a \( \rho = \frac{1}{M + \sqrt{M^2 - 1}} \) such that every function \( f \), analytic in \( \mathbb{D} \) with \( f(0) = f'(0) = 1 = 0 \) and \( |f(z)| < M \) in \( \mathbb{D} \), is univalent in the disk \( \mathbb{D}_\rho \) and in addition, the range \( f(\mathbb{D}_\rho) \) contains a disk of radius \( M\rho^2 \) (see [16]), where \( M \geq 1 \) is a constant. Recently, many authors considered Landau type theorem for planar harmonic mappings (see [6, 7, 8, 9]), biharmonic mappings (see [11]) and polyharmonic mappings (see [1]). Applying Theorems 1 and 3, we get the following Landau type theorem.

**Theorem 4.** For \( \alpha \in (-1, 0] \), let \( f \in C^2(\mathbb{D}) \) be a solution to (1.1) satisfying \( f(0) = |J_f(0)| - \lambda = 0 \) and \( f \in \mathcal{H}^p \), where \( \lambda \) is a positive constant and \( J_f \) is the Jacobian of \( f \). Then \( f \) is univalent in \( \mathbb{D}_{\gamma_0\rho_0} \), where \( \rho_0 \) satisfies the following equation

\[
\frac{\lambda}{M^*(2 + \alpha)} - \frac{4M^*\rho_0}{\pi} \left[ \frac{2 - \rho_0}{(1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] = 0,
\]

where \( \mu(\gamma) = (1+\gamma)^{\frac{\alpha-1}{\alpha}} / \left[ \gamma(1 - \gamma)^{\frac{1}{\alpha}} \right] \), \( \mu(\gamma_0) = \min_{0<\gamma<1} \mu(\gamma) \) and \( M^* = c_0^\gamma \|f\|_p \mu(\gamma_0) \).

Moreover, \( f(\mathbb{D}_{\gamma_0\rho_0}) \) contains a univalent disk \( \mathbb{D}_{\gamma_0\rho_0} \) with

\[
R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\lambda}{M^*(2 + \alpha)} - \frac{M^*\rho_0(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right].
\]
We remark that Theorem 4 is a generalization of [6, Theorem 2] and [10, Theorem 5].

The proofs of Theorems 1, 2 and 3 will be presented in Section 2, and the proof of Theorem 4 will be given in Section 3.

2. SCHWARTZ–PICK TYPE ESTIMATES AND COEFFICIENT ESTIMATES

Proof of Theorem 1. We first prove (1.5). By the assumption, we see that \( f_r \to f^* \) in \( \mathcal{D}'(\partial \mathbb{D}) \) as \( r \to 1^- \), where \( f_r \) is given by (1.2) for \( r \in [0, 1) \). By (1.3), for \( z = re^{i\theta} \in \mathbb{D} \), we have

\[
|f(z) - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|}f(0)|
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it})f^*(e^{it})dt - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|} \int_0^{2\pi} K_\alpha(0)f^*(e^{it})dt
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \left( K_\alpha(ze^{-it}) - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|}K_\alpha(0) \right) f^*(e^{it})dt
\]

Next we prove (1.6). By the proof of [21, Theorem 3.1], we observe that

\[
M_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\theta})de = \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2\right)
\]

and \( M_\alpha(r) \) is increasing on \( r \in [0, 1) \) with \( \lim_{r \to 1^-} M_\alpha(r) = 1 \).

By elementary calculations, for \( z \in \mathbb{D} \), we have

\[
\frac{\partial}{\partial z} K_\alpha(ze^{-it}) = c_\alpha \frac{(1 - |z|^2)^\alpha \left[(1 + \frac{\alpha}{2})e^{-it}(1 - \overline{z}e^{it})(1 - |z|^2) - (\alpha + 1)\overline{z}|1 - ze^{-it}|^2\right]}{|1 - ze^{-it}|^{4+\alpha}}
\]

and

\[
\frac{\partial}{\partial \overline{z}} K_\alpha(ze^{-it}) = c_\alpha \frac{(1 - |z|^2)^\alpha \left[(1 + \frac{\alpha}{2})e^{it}(1 - ze^{-it})(1 - |z|^2) - (\alpha + 1)z|1 - ze^{-it}|^2\right]}{|1 - ze^{-it}|^{4+\alpha}}
\]

which, together with (1.3) and (2.1), imply that

\[
\|D_f(z)\| \leq \frac{M c_\alpha}{\pi} \int_0^{2\pi} \frac{(1 - |z|^2)^\alpha}{|1 - ze^{-it}|^{4+\alpha}} \left[(1 + \alpha)|z||1 - ze^{-it}|^2 + (1 + \frac{\alpha}{2})|1 - ze^{-it}|(1 - |z|^2)\right] dt
\]

\[
\leq \frac{M c_\alpha}{\pi} \int_0^{2\pi} \frac{(1 - |z|^2)^\alpha}{|1 - ze^{-it}|^{4+\alpha}} \left[(1 + \alpha)|z||1 - ze^{-it}|^2 + (1 + \frac{\alpha}{2})|1 - ze^{-it}|^2\right] dt
\]
The proof of this theorem is complete. □

Proof of Theorem 2. Since (a) easily follows from the inequality (15) in [15], we only need to prove (b). Let \( \alpha > 0 \). By Theorem 1 (1.5), we have

\[
\|Df(e^{i\theta})\| \geq \left( \left| \frac{\partial f(re^{i\theta})}{\partial r} \right| \right)_{r=1}
\]

\[
= \lim_{r \to 1^-} \frac{|f(e^{i\theta}) - f(re^{i\theta})|}{1 - r}
\]

\[
\geq \lim_{r \to 1^-} \frac{1 - \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i(\theta-t)})dt + \frac{(1-r)^{\alpha+1}}{1+|z|}K_\alpha(0)}{1 - r}
\]

\[
= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\eta})d\eta + \frac{(1-r)^{\alpha+1}}{1+|z|}K_\alpha(0)
\]

which, together with L'Hopital's rule and (2.2), yield that

\[
\|Df(e^{i\theta})\| \geq \left( \left| \frac{\partial f(re^{i\theta})}{\partial r} \right| \right)_{r=1}
\]

\[
= \lim_{r \to 1^-} \frac{|f(e^{i\theta}) - f(re^{i\theta})|}{1 - r}
\]

\[
\geq \lim_{r \to 1^-} \frac{1 - \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i(\theta-t)})dt + \frac{(1-r)^{\alpha+1}}{1+|z|}K_\alpha(0)}{1 - r}
\]

\[
= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\eta})d\eta + \frac{(1-r)^{\alpha+1}}{1+|z|}K_\alpha(0)
\]
Differential operators

\[ = \frac{1}{2\pi} \lim_{r \to 1^-} \frac{d}{dr} \int_0^{2\pi} K_\alpha(re^{i\eta}) d\eta \]

\[ = \lim_{r \to 1^-} \frac{d}{dr} M_\alpha(r). \]

where \( M_\alpha(r) \) is given by (1.7). It follows from the proof of [21, Theorem 3.1] that

\[ M_\alpha(r) = \frac{\lbrack \Gamma(1 + \frac{\alpha}{2}) \rbrack^2}{\Gamma(1 + \alpha)} F \left( -\frac{\alpha}{2}, \frac{\alpha}{2}; 1; r^2 \right) = \frac{\lbrack \Gamma(1 + \frac{\alpha}{2}) \rbrack^2}{\Gamma(1 + \alpha)} \sum_{n=0}^{\infty} \lbrack (\frac{-\alpha}{2}) \rbrack^n (n!)^2, \]

which yields that

\[ \frac{d}{dr} M_\alpha(r) = \frac{\alpha^2 \lbrack \Gamma(1 + \frac{\alpha}{2}) \rbrack^2}{2 \Gamma(1 + \alpha)} r F \left( 1 - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 2; r^2 \right), \]

where \( r \in (0, 1) \). By (1.4), for \( \alpha > 0 \), we see that

\[ \lim_{r \to 1^-} \frac{d}{dr} M_\alpha(r) = \frac{\alpha \lbrack \Gamma(1 + \frac{\alpha}{2}) \rbrack^2}{2 \Gamma(1 + \alpha)} r F \left( 1 - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 2; 1 \right) \]

\[ = \frac{\alpha^2 \Gamma(2) \Gamma(\alpha)}{2 \Gamma(1 + \alpha) \lbrack \Gamma(1 + \frac{\alpha}{2}) \rbrack^2} \]

\[ = \frac{\alpha^2 \Gamma(\alpha)}{2 \Gamma(1 + \alpha)} = \frac{\alpha}{2}. \]

Therefore, for \( \theta \in [0, 2\pi] \),

\[ \| D_\theta (e^{i\theta}) \| \geq \lim_{r \to 1^-} \frac{d}{dr} M_\alpha(r) = \frac{\alpha}{2}, \]

where \( \alpha > 0 \). The proof of this theorem is complete. \( \square \)

**Proof of Theorem 3.** For \( r \in [0, 1) \), let

\[ A_k(r, \alpha) = c_k F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; r^2 \right) \]

and

\[ B_k(r, \alpha) = c_k F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; r^2 \right), \]

where \( r = |z| \). Then

\[ A_k(r, \alpha) r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{-ik\theta} d\theta \]

and

\[ B_k(r, \alpha) r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{ik\theta} d\theta, \]

which imply that

\[ |A_k(r, \alpha)| r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{-ik\theta} e^{-i\arg A_k(r, \alpha)} d\theta \]
and

\begin{equation}
|B_k(r, \alpha)|r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z)e^{ik\theta}e^{-i\arg B_k(r, \alpha)}d\theta,
\end{equation}

where \(A_k(r, \alpha) = |A_k(r, \alpha)|e^{i\arg A_k(r, \alpha)}\), \(B_k(r, \alpha) = |B_k(r, \alpha)|e^{i\arg B_k(r, \alpha)}\) and \(z = re^{i\theta}\).

By (2.3), (2.4) and [12, Lemma 1], we have

\begin{equation}
\left|\left(|A_k(r, \alpha)| + |B_k(r, \alpha)|\right)r^k\right|
\end{equation}

\begin{align*}
&= \left|\frac{1}{2\pi} \int_0^{2\pi} f(z)\left[e^{-i\left(k\theta + \arg A_k(r, \alpha)\right)} + e^{i\left(k\theta - \arg B_k(r, \alpha)\right)}\right]d\theta\right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z)|\left|e^{-i\left(k\theta + \arg A_k(r, \alpha)\right)} + e^{i\left(k\theta - \arg B_k(r, \alpha)\right)}\right|d\theta \\
&\leq \frac{M}{2\pi} \int_0^{2\pi} \left|1 + e^{i\left(2k\theta + \arg A_k(r, \alpha) - \arg B_k(r, \alpha)\right)}\right|d\theta \\
&= \frac{M}{\pi} \int_0^{2\pi} \left|\cos\left(k\theta + \frac{\arg A_k(r, \alpha) - \arg B_k(r, \alpha)}{2}\right)\right|d\theta \\
&= \frac{4M}{\pi}.
\end{align*}

By letting \(r \to 1^-\) on (2.5), we obtain

\[|A_k(1, \alpha)| + |B_k(1, \alpha)| \leq \frac{4M}{\pi}.
\]

On the other hand, for \(k = 0\), we have

\begin{equation}
\frac{1}{2\pi} \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2d\theta = \left|c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right)\right|^2 \\
+ \sum_{k=1}^{\infty} \left|c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1\right)\right|^2 \\
+ \left|c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1\right)\right|^2
\end{equation}

\[\leq M^2,
\]

where \(r \in [0, 1)\). It follows from (2.6) that

\[\left|c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right)\right| \leq M.
\]

If \(\alpha = 0\), then the sharpness part follows from [11, Lemma 1]. The proof of this theorem is complete. □
3. The Landau type theorem

Lemma 1. For $x \in [0, 1)$, let

$$\varphi(x) = \frac{\delta}{M(2 + \alpha)} - \frac{4Mx}{\pi} \left[ \frac{2 - x}{(1 - x)^2} + \frac{2x}{(1 - x)(1 - x^2)^2} \right],$$

where $\alpha > -2$, $\delta > 0$ and $M > 0$ are constant. Then $\varphi$ is strictly decreasing and there is an unique $x_0 \in (0, 1)$ such that $\varphi(x_0) = 0$.

Proof. For $x \in [0, 1)$, let

$$f_1(x) = \frac{4Mx(2 - x)}{\pi (1 - x)^2} \quad \text{and} \quad f_2(x) = \frac{4M2x^2}{\pi (1 - x)(1 - x^2)^2}.$$

Since, for $x \in [0, 1)$,

$$f_1'(x) = \frac{8M}{\pi} \frac{1}{(1 - x)^3} > 0,$$

we see that $f_1$ is continuous and strictly increasing in $[0, 1)$. We observe that $f_2$ is also continuous and strictly increasing in $[0, 1)$. Then

$$\varphi(x) = \frac{\delta}{M(2 + \alpha)} - f_1(x) - f_2(x)$$

is continuous and strictly decreasing in $[0, 1)$, which, together with

$$\lim_{x \to 0^+} \varphi(x) = \frac{\delta}{M(2 + \alpha)} > 0 \quad \text{and} \quad \lim_{x \to 1^-} \varphi(x) = -\infty,$$

imply that there is an unique $x_0 \in (0, 1)$ such that $\varphi(x_0) = 0$. \hfill \Box

Lemma 2. For $\alpha \in (-1, 0]$, let $f \in C^2(\mathbb{D})$ be a solution to (1.1) satisfying $f(0) = |J_f(0)| - \beta = 0$ and sup$_{z \in \mathbb{D}} |f(z)| \leq M$, where $M, \beta$ are positive constants and $J_f$ is the Jacobian of $f$. Then $f$ is univalent in $\mathbb{D}_{\rho_0}$, where $\rho_0$ satisfies the following equation

$$\frac{\beta}{M(2 + \alpha)} - \frac{4M\rho_0}{\pi} \left[ \frac{2 - \rho_0}{(1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] = 0.$$

Moreover, $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\beta}{M(2 + \alpha)} - \frac{M\rho_0(2 - \rho_0)}{\pi(1 - \rho_0^2)^2} \right].$$

Proof. By Theorem A, we can assume that

$$f(z) = \sum_{k=0}^{\infty} c_k F \left( \frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; |z|^2 \right) z^k + \sum_{k=1}^{\infty} c_{-k} F \left( \frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; |z|^2 \right) \bar{z}^k, \; z \in \mathbb{D},$$
for some sequence \( \{c_k\}_{k=-\infty}^{\infty} \) of complex numbers satisfying

\[
\lim_{|k| \to \infty} \sup |c_k|^{1/|k|} \leq 1.
\]

For \( \alpha \in (-1, 0] \), by [21, Proposition 1.4], we observe that

\[
F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; r^2) = \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k - \frac{\alpha}{2})_n r^{2n}}{(k+1)_n n!} \geq 0
\]

is bounded and increasing on \( r \in [0, 1) \), which imply that

\[
(|c_k| + |c_{-k}|) F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; r^2\right) \leq \frac{4M}{\pi}, \tag{3.1}
\]

where \( r = |z| \) and \( k \in \{1, 2, \ldots\} \).

By (3.1) and Theorem 3, we see that, for each \( k \in \{1, 2, \ldots\} \),

\[
(|c_k| + |c_{-k}|) \frac{(-\frac{\alpha}{2})_n(k - \frac{\alpha}{2})_n 1}{(k+1)_n n!} \leq \frac{4M}{\pi}, \tag{3.2}
\]

where \( n \in \{1, 2, \ldots\} \).

Since \( c_0 = f(0) = 0 \), we see that

\[
f_z(z) - f_z(0) = \sum_{k=2}^{\infty} kc_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; w\right) z^{k-1}
\]

\[
+ \sum_{k=1}^{\infty} c_k \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; w\right) z^k \frac{z}{z}
\]

\[
+ \sum_{k=1}^{\infty} c_{-k} \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; w\right) z^{k+1}
\]

and

\[
f_{\bar{z}}(z) - f_{\bar{z}}(0) = \sum_{k=2}^{\infty} kc_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; w\right) \bar{z}^{k-1}
\]

\[
+ \sum_{k=1}^{\infty} c_k \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; w\right) z^k \bar{z}
\]

\[
+ \sum_{k=1}^{\infty} c_{-k} \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; w\right) \bar{z}^{k+1}
\]

where \( w = |z|^2 \).

Applying (3.2), (3.3) and (3.4), we obtain
which gives that

\[ |f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)| \leq \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|)F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}, k + 1; w\right)|z|^{k-1} + 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|) \frac{d}{dw}F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}, k + 1; w\right)|z|^{k+1} \]

\[ \leq \frac{4M}{\pi} \sum_{k=2}^{\infty} k|z|^{k-1} + 2\sum_{k=1}^{\infty} \left[ \frac{4M}{\pi} \sum_{n=1}^{\infty} n|z|^{2(n-1)} \right] |z|^{k+1} \]

\[ = \frac{4M}{\pi} \frac{|z|(2 - |z|)}{(1 - |z|)^2} + \frac{8M}{\pi} \sum_{k=1}^{\infty} \frac{|z|^{k+1}}{(1 - |z|^2)^2} \]

\[ = \frac{4M}{\pi} \frac{|z|(2 - |z|)}{(1 - |z|)^2} + \frac{8M}{\pi} \frac{|z|^2}{(1 - |z|)(1 - |z|^2)^2}. \]

Applying Theorem 1 (1.6), we get

\[ \beta = |J_f(0)| = \|\det D_f(0)\| = \|D_f(0)\||l(D_f(0))| \leq M(2 + \alpha)l(D_f(0)), \]

which gives that

\[ l(D_f(0)) \geq \frac{\beta}{M(2 + \alpha)}. \]

In order to prove the univalence of \( f \) in \( \mathbb{D}_{\rho_0} \), we choose two distinct points \( z_1, z_2 \in \mathbb{D}_{\rho_0} \) and let \( [z_1, z_2] \) denote the segment from \( z_1 \) to \( z_2 \) with the endpoints \( z_1 \) and \( z_2 \), where \( \rho_0 \) satisfies the following equation

\[ \frac{\beta}{M(2 + \alpha)} - \frac{4M\rho_0}{\pi} \left[ \frac{2 - \rho_0}{(1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] = 0. \]

By (3.5), (3.6) and Lemma 1, we have

\[ |f(z_2) - f(z_1)| = \left| \int_{[z_1,z_2]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z} \right| \]

\[ = \left| \int_{[z_1,z_2]} f_z(0)dz + f_{\overline{z}}(0)d\overline{z} \right| \]

\[ - \left| \int_{[z_1,z_2]} (f_z(z) - f_z(0))dz + (f_{\overline{z}}(z) - f_{\overline{z}}(0))d\overline{z} \right| \]

\[ \geq l(D_f)(0)|z_2 - z_1| \]

\[ - \int_{[z_1,z_2]} (|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)|)|dz| \]
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\[ > |z_2 - z_1| \left\{ \frac{\beta}{M(2 + \alpha)} - 4M\rho_0 \frac{2 - \rho_0}{\pi (1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right\} \]

\[ = 0. \]

Thus, \( f(z_2) \neq f(z_1) \). The univalence of \( f \) follows from the arbitrariness of \( z_1 \) and \( z_2 \). This implies that \( f \) is univalent in \( \mathbb{D}_{\rho_0} \).

Now, for any \( \zeta' = \rho_0 e^{i\theta} \in \partial \mathbb{D}_{\rho_0} \), we obtain that

\[ |f(\zeta') - f(0)| = \left| \int_{[0,\zeta']} f_z(z)dz + f_{\bar{z}}(z)d\bar{z} \right| \]

\[ = \left| \int_{[0,\zeta']} f_z(0)dz + f_{\bar{z}}(0)d\bar{z} \right| 
\]

\[ - \left| \int_{[0,\zeta']} (f_z(z) - f_z(0))dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0))d\bar{z} \right| \]

\[ \geq l(D_f)(0)\rho_0 - \int_{[0,\zeta']} (|f_z(z) - f_z(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)|)|dz| \]

\[ \geq l(D_f)(0)\rho_0 - \frac{4M\rho_0^2}{\pi} \int_0^1 \frac{t(2 - \rho_0 t)}{(1 - \rho_0 t)^2} + \frac{2\rho_0 t^2}{(1 - \rho_0 t)(1 - \rho_0^2 t^2)} \] \( dt \)

\[ \geq \frac{\beta\rho_0}{M(2 + \alpha)} - \frac{4M\rho_0^2}{\pi} \left[ \frac{(2 - \rho_0)}{(1 - \rho_0)^2} \int_0^1 tdt + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] \]

\[ = \rho_0 \left\{ \frac{\beta}{M(2 + \alpha)} - \frac{4M\rho_0}{\pi} \left[ \frac{2 - \rho_0}{2(1 - \rho_0)^2} + \frac{2\rho_0}{3(1 - \rho_0)(1 - \rho_0^2)^2} \right] \right\} \]

\[ = \frac{2\rho_0}{3} \left[ \frac{\beta}{M(2 + \alpha)} - \frac{M\rho(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right]. \]

Hence \( f(\mathbb{D}_{\rho_0}) \) contains a univalent disk \( \mathbb{D}_{R_0} \) with

\[ R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\beta}{M(2 + \alpha)} - \frac{M\rho(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right]. \]

The proof of this lemma is complete. \( \square \)

Let us recall the following result which is referred to as \textit{Jensen’s inequality} (cf. [23]).
Lemma B. Let \((\Omega, A, \mu)\) be a measure space such that \(\mu(\Omega) = 1\). If \(g\) is a real-valued function that is \(\mu\)-integrable, and if \(\chi\) is a convex function on the real line, then

\[
\chi \left( \int_{\Omega} g \, d\mu \right) \leq \int_{\Omega} \chi \circ g \, d\mu.
\]

Proof of Theorem 4. For \(z \in \mathbb{D}_r\), we have

\[
f(z) = \frac{c_\alpha}{2\pi r^\alpha} \int_0^{2\pi} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) \, dt,
\]

where \(r \in (0, 1)\). Let

\[
\phi_z(r) = \frac{c_\alpha}{2\pi r^\alpha} \int_0^{2\pi} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} \, dt,
\]

where \(z \in \mathbb{D}_r\). Applying [21, Theorem 3.1], we see that, for \(z \in \mathbb{D}\),

\[
(3.7) \quad \phi_z(1) = \lim_{|z| \to 1-} \phi_z(1) = 1.
\]

By using Jensen’s inequality (see Lemma B), for \(p \geq 1\), we get

\[
\left| \frac{f(z)}{\phi_z(r)} \right|^p = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{c_\alpha}{r^\alpha \phi_z(r)} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) \, dt \right|^p
\]

\[
\quad \leq \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{c_\alpha}{r^\alpha \phi_z(r)} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} |f(re^{it})|^p \, dt \right|
\]

\[
\quad \leq \frac{c_\alpha}{r^\alpha \phi_z(r)} \frac{(r^2 - |z|^2)^{\alpha+1}}{(r - |z|)^{2+\alpha}} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)
\]

\[
\quad \leq \frac{c_\alpha \|f\|_p^p (r + |z|)^{\alpha+1}}{r^\alpha \phi_z(r) (r - |z|)},
\]

which implies that

\[
|f(z)| \leq \left[ \frac{c_\alpha \|f\|_p^p (\phi_z(r))^{p-1}}{r^\alpha} \right]^{\frac{1}{p}} \left( \frac{r + |z|}{r - |z|} \right)^{\frac{\alpha+1}{p}},
\]

where \(z \in \mathbb{D}_r\). By letting \(r \to 1-\) and (3.7), for \(z \in \mathbb{D}\), we have

\[
(3.8) \quad |f(z)| \leq \left[ \frac{c_\alpha \|f\|_p^p (\phi_z(1))^{p-1}}{r^\alpha} \right]^{\frac{1}{p}} \left( \frac{1 + |z|}{1 - |z|} \right)^{\frac{\alpha+1}{p}} \leq \frac{c_\alpha \|f\|_p^p (1 + |z|)^{\frac{\alpha+1}{p}}}{(1 - |z|)^{\frac{1}{p}}}
\]

For \(\zeta \in \mathbb{D}\), let \(Q(\zeta) = f(\gamma \zeta)/\gamma\), where \(\gamma \in (0, 1)\). It is not difficult to know that \(Q(0) = |J_Q(0)| - \lambda = 0\). By (3.8), for \(\zeta \in \mathbb{D}\), we obtain

\[
|Q(\zeta)| = \frac{|f(\gamma \zeta)|}{\gamma} \leq c_\alpha \|f\|_p^p \frac{(1 + \gamma)^{\frac{\alpha+1}{p}}}{\gamma (1 - \gamma)^{\frac{1}{p}}},
\]
which gives that
\[ |Q(\zeta)| \leq c^\frac{1}{\alpha} \|f\|_p \min_{0<\gamma<1} \mu(\gamma), \]
where
\[ \mu(\gamma) = \frac{(1+\gamma)^{\alpha/2}}{\gamma(1-\gamma)^{\frac{1}{2}}} \cdot \]

Let \( \gamma_0 \in (0,1) \) satisfy
\[ \mu(\gamma_0) = \min_{0<\gamma<1} \mu(\gamma). \]

By using Lemma 2, we observe that \( Q \) is univalent in \( \mathbb{D}_{\rho_0} \), where \( \rho_0 \) satisfies the following equation
\[ \frac{\lambda}{M^*(2+\alpha)} - \frac{4M^*\rho_0}{\pi} \left[ \frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2} \right] = 0, \]
where \( M^* = c^\frac{1}{\alpha} \|f\|_p \mu(\gamma_0) \). Moreover, \( Q(\mathbb{D}_{\rho_0}) \) contains a univalent disk \( \mathbb{D}_{R_0} \) with
\[ R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\lambda}{M^*(2+\alpha)} - \frac{M^*\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2} \right]. \]

Hence \( f \) is univalent in \( \mathbb{D}_{\gamma_0\rho_0} \) and \( f(\mathbb{D}_{\gamma_0\rho_0}) \) contains a univalent disk \( \mathbb{D}_{\gamma_0R_0} \). The proof of this theorem is complete. \( \square \)

Acknowledgements: This research was partly supported by the National Natural Science Foundation of China (No. 11401184 and No. 11326081), the Hunan Province Natural Science Foundation of China (No. 2015JJ3025), the Excellent Doctoral Dissertation of Special Foundation of Hunan Province (No: 100017), the Construct Program of the Key Discipline in Hunan Province, The Väisälä Foundation of The Finnish Academy of Sciences and Letters.

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