Central Limit Theorem for Linear Statistics of Eigenvalues of Band Random Matrices

Lingyun Li * Alexander Soshnikov †

May 11, 2014

Abstract

We prove the Central Limit Theorem for linear statistics of the eigenvalues of band random matrices provided \( \sqrt{n} \ll b_n \ll n \) and test functions are sufficiently smooth.

1 Introduction

The goal of this paper is to prove the Central Limit Theorem for linear statistics of the eigenvalues of real symmetric band random matrices with independent entries.

First, we define a real symmetric band random matrix. Let \( \{b_n\} \) be a sequence of integers satisfying \( 0 \leq b_n \leq n/2 \) such that \( b_n \to \infty \) as \( n \to \infty \). Define

\[
   d_n(j, k) := \min\{|k - j|, n - |k - j|\},
\]

\[
   I_n := \{(j, k) : d_n(j, k) \leq b_n, \ j, k = 1, \ldots, n\}, \quad \text{and} \quad I_n^+ := \{(j, k) : (j, k) \in I_n, \ j \leq k\}.
\]

In particular, \( d_n \) has the following natural interpretation: if the first \( n \) positive integers are evenly spread out on a circle of radius \( n^{3/2}/\pi \), then \( d_n(j, k) \) is the distance between the integers \( j \) and \( k \).

The quantity \( b_n \) will be the radius of a band of our random matrix. In other words, all entries of the matrix with \( j, k \notin I_n \) are going to be zero. We define a real symmetric band random matrix

\[
   M = (M_{jk}), \ 1 \leq j, k \leq n,
\]

(1.3)

in such a way that for \( j \leq k \) one has

\[
   M_{jk} = M_{kj} = b_n^{-1/2}W_{jk} \quad \text{if} \quad d_n(j, k) \leq b_n,
\]

(1.4)

and \( M_{jk} = 0 \) otherwise, where \( \{W_{jk}\}_{(j, k) \in I_n^+} \) is a sequence of independent real valued random variables satisfying

\[
   \mathbb{E}\{W_{jk}\} = 0, \quad \mathbb{E}\{W_{jk}^2\} = (1 + \delta_{jk})\sigma^2.
\]

(1.5)

In general, the distribution of the entries \( W_{jk} \) might depend on the size \( n \) of the matrix but we will not indicate this dependence in our notations, unless it is necessary. An important special case corresponds to \( b_n = \lfloor (n - 1)/2 \rfloor \). Then \( M \) is standard Wigner random matrix (see e.g. [44], [1], [10], [1]).

*Department of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA 95616-8633, llyli@math.ucdavis.edu
†Department of Mathematics, University of California, Davis, One Shields Avenue, Davis, CA 95616-8633, soshniko@math.ucdavis.edu; research has been supported in part by the NSF grant DMS-107558
For a real symmetric (Hermitian) matrix $M$ of order $n$, its empirical distribution of the eigenvalues is defined as

$$\mu_M = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

where $\lambda_1 \leq \ldots \leq \lambda_n$ are the (ordered) eigenvalues of $M$. The Wigner semicircle law states that for any bounded continuous test function $\varphi : \mathbb{R} \to \mathbb{R}$, the linear statistic

$$\frac{1}{n} \sum_{i=1}^{n} \varphi(\lambda_i) = \frac{1}{n} \text{Tr}(\varphi(M)) =: \text{tr}_n(\varphi(M))$$

converges to $\int \varphi(x) d\mu_{sc}(dx)$ in probability, where $\mu_{sc}$ is determined by its density

$$\frac{d\mu_{sc}}{dx}(x) = \frac{1}{4\pi\sigma^2} \sqrt{8\sigma^2 - x^2} \mathbf{1}_{[-2\sqrt{2\sigma}, 2\sqrt{2\sigma}]}(x).$$

We refer the reader to [44], [4], [10], [1] for the proof in the full matrix case and to [11], [29] for the proof in the band matrix case.

Band random matrices have important applications in physics (see e.g. [32], [15], [16], [22], [28], [41]), in particular as a model of quantum chaos. It is conjectured that the eigenvectors are localized and local eigenvalue statistics are Poisson for $b_n \ll \sqrt{n}$. On the other hand, it is expected that the eigenvectors are delocalized and local eigenvalue statistics follow GUE (GOE) law for $b_n \gg \sqrt{n}$ (see e.g. [22]). Throughout the paper, the relation $a_n \ll b_n$ for two $n$-dependent quantities $a_n$ and $b_n$ means that $a_n/b_n \to 0$ as $n \to \infty$. For recent mathematical progress on local spectral properties of band random matrices, we refer the reader to [19], [20], [21], [35], [39], [40].

The linear eigenvalues statistics corresponding to a test function $\varphi$ is defined as

$$\mathcal{N}_n[\varphi] = \sum_{i=1}^{n} \varphi(\lambda_i).$$

In the Wigner (full matrix) case, the variance of $\mathcal{N}_n[\varphi]$ stays bounded as $n \to \infty$ for sufficiently smooth $\varphi$. Moreover, the fluctuation of the linear statistic is Gaussian in the limit (see e.g. [36], [2], [7], [26], [33], and references therein). Similar results have been established for other ensembles of random matrices ([23], [38], [34], [6]). In addition, we note recent results on partial linear eigenvalue statistics ([8], [30]) and the properties of the eigenvectors of Wigner matrices ([5]).

In this paper, we prove that the normalized linear statistic

$$\mathcal{M}_n[\varphi] := (b_n/n)^{1/2} \mathcal{N}_n[\varphi]$$

has an asymptotic normal distribution, as $n \to \infty$ provided $b_n \gg \sqrt{n}$, and $\varphi$, $W_{jk}$ satisfy some conditions.

## 2 Statement of Main Results

For the first theorem, we assume that the matrix entries satisfy the Poincaré inequality. We refer the reader to Section A of the Appendix for the definition and basic facts about the Poincaré inequality.

Theorem 2.1. Let $M = W/\sqrt{b_n}$ be a real symmetric random band matrix $[1.3, 1.5]$, where $\{b_n\}$ is a sequence of integers satisfying $\sqrt{n} \ll b_n \ll n$. Assume the following:

1. Diagonal and non-zero off-diagonal entries of $W$ are two sets of i.i.d random variables;

2. The marginal probability distribution of $W_{jk}$ satisfies the Poincaré Inequality with some uniform constant $m > 0$ which does not depend on $n, j, k$;

3. The fourth moment of the non-zero off-diagonal entries does not depend on $n$:

$$\mu_4 = \mathbb{E}\{W_{12}^4\}.$$
Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a test function with continuous bounded derivative. Then the corresponding centered normalized linear statistic of the eigenvalues

\[
\mathcal{M}_n^\varphi \coloneqq (b_n/n)^{1/2}(\mathcal{M}_n^\varphi - \mathbb{E}(\mathcal{M}_n^\varphi)) \tag{2.2}
\]
converges in distribution to the Gaussian random variable with zero mean and the variance

\[
\text{Var}_{\text{band}}[\varphi] = \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{(\varphi(x) - \varphi(\lambda))\varphi'(y)\sqrt{8\sigma^2 - x^2}\sqrt{8\sigma^2 - y^2}}{4\pi^4(x - \lambda)\sqrt{8\sigma^2 - \lambda^2}} F_\sigma(x, y)1_{\{x \neq y\}} \, dx dy d\lambda + \frac{\kappa_4}{16\pi^2\sigma^5} \left( \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{\varphi(\lambda)(4\sigma^2 - \lambda^2)}{\sqrt{8\sigma^2 - \lambda^2}} \, d\lambda \right)^2, \tag{2.3}
\]

where for \( x \neq y \)

\[
F_\sigma(x, y) := \int_{-\infty}^{\infty} \frac{(s^3 \sin s - s \sin^3 s) \, ds}{2\sigma^2 (s^2 - \sin^2 s)^2 - (s^4 \sin s + s \sin^3 s) \, xy + s^2 \sin^2 s (x^2 + y^2)}, \tag{2.4}
\]

and \( \kappa_4 \) is the fourth cumulant of off-diagonal entries, i.e.

\[
\kappa_4 = \mu_4 - 3\sigma^4. \tag{2.5}
\]

Next, we extend this result to the non-i.i.d. case when the fifth moment of the matrix entries is uniformly bounded. Here we do not assume that marginal distributions of the non-zero entries satisfy the Poincaré inequality. For technical reasons, we assume that the fourth cumulant of the matrix entries is zero. Also we require that \( \sqrt{n} \ln n \ll b_n \) (thus, we have additional \( \ln n \) factor at the l.h.s. as compared to the corresponding assumption in Theorem 2.1).

**Theorem 2.2.** Let \( M = W/\sqrt{b_n} \) be a real symmetric band matrix \([12] \ldots [13]\), where \( \{b_n\} \) is a sequence of positive integers satisfying \( \sqrt{n} \ln n \ll b_n \ll n \). Assume the following:

1. 

\[
\sigma_5 := \sup_{n \in \mathbb{N}} \max_{(j, k) \in I_n} \mathbb{E}(|W_{jk}^{(n)}|^5) < \infty. \tag{2.6}
\]

2. The third cumulant of the non-zero off-diagonal entries does not depend on \( j, k \):

\[
\kappa_3 = \kappa_{3, jk}, \quad (j, k) \in I_n, \ j \neq k.
\]

3. The fourth cumulant is zero: \( \kappa_4 = \mathbb{E}(|W_{jk}^{(n)}|^4) - 3\sigma^4 = 0. \)

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a test function with the Fourier transform

\[
\hat{\varphi}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\lambda} \varphi(\lambda) \, d\lambda \tag{2.7}
\]

satisfying

\[
\int_{-\infty}^{\infty} (1 + |t|^4) |\hat{\varphi}(t)| \, dt < \infty. \tag{2.8}
\]

Then the corresponding centered normalized linear eigenvalues statistic \( \mathcal{M}_n^\varphi \) converges in distribution to the Gaussian random variable with zero mean and variance \( \text{Var}_G \)

\[
\text{Var}_G[\varphi] = \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \int_{-2\sqrt{2}\sigma}^{2\sqrt{2}\sigma} \frac{(\varphi(x) - \varphi(\lambda))\varphi'(y)\sqrt{8\sigma^2 - x^2}\sqrt{8\sigma^2 - y^2}}{4\pi^4(x - \lambda)\sqrt{8\sigma^2 - \lambda^2}} F_\sigma(x, y)1_{\{x \neq y\}} \, dx dy d\lambda. \tag{2.9}
\]
Remark 2.3. It should be noted that in [24] the authors claimed to compute the asymptotic formula for the variance of the trace of the resolvent of a band random matrix (see the formulas (3.5)-(3.7) therein). In particular, they claim that the (normalized) variance has the same limiting expression as in the GOE case. We disagree with this statement. In fact, it is not hard to see that the limit of $\frac{1}{n} \text{Var} \text{Tr} M^3$ is different in the band and the full Wigner matrix cases.

In our computations, the difference is highlighted by the fact that the limiting behavior of the expression 

$$\frac{1}{n} \text{Var} \text{Tr} M^3$$

is different in the band and the full Wigner matrix cases. In the full Wigner case, the formula for the limit of $A_n(t)$ immediately follows from the Wigner semicircle law. In the band case, the fact that the summation in (3.29) is restricted to $(j,k)$: $d_n(j,k) \leq b_n$ leads to a different limit formula (see Subsection 3.5, in particular Proposition 3.9 and Lemma 3.12).

Remark 2.4. Similar results with little modification hold for Hermitian band random matrices. In particular, the variance (2.9) in Theorem 2.2 gets an additional factor $1/2$, provided (1.5) is replaced by $E\{W_{jk}\} = 0$, $E\{|W_{jk}|^2\} = (1 + \delta_{jk}) \sigma^2$, $E\{W_{jk}^2\} = 0$. (2.10)

In addition, it should be noted that the results of Theorems 2.1 and 2.2 hold if one replaces the condition $M_{jk} = 0$ for $d_n(j,k) > b_n$ by $M_{jk} = 0$ for $|j-k| > b_n$.

The proofs are very similar and left to the reader.

The rest of the paper is organized as follows. We prove Theorem 2.1 in Section 3 and Theorem 2.2 in Section 4. In the Appendix, we list basic facts about the Poincaré inequality and decoupling formula.

3 Proof of Theorem 2.1

3.1 Stein’s Method

We follow the approach used by A. Lytova and L. Pastur in [26] in the full matrix (Wigner) case. Essentially, it is a modification of the Stein’s method ([42], [9]). While several steps of our proof are similar to the ones in [26], the fact that we are dealing with band matrices raises new significant difficulties (see e.g. Lemmas 3.11 and 3.12 in Subsection 3.5).

First, we prove the result of Theorem 2.1 under an additional technical condition on the smoothness of a test function. Namely, we assume that the Fourier transform of $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty} (1 + |t|^{4+\varepsilon})|\hat{\varphi}(t)| dt < \infty,$$

where $\varepsilon$ is an arbitrary small positive number. Once the result is established for such test functions, it can be easily extended to the case of functions with bounded continuous derivative using (3.10).

Let $Z_n(x), Z(x)$ be the characteristic functions of the normalized linear statistic (1.9) and the Gaussian distribution with zero mean and $\text{Var}_{\text{band}}[\varphi]$ variance, respectively, i.e.

$$Z_n(x) = \mathbb{E}\{e^{ix \cdot \mathcal{M}^n_{\varphi}}\},$$

(3.2)

and

$$Z(x) = \exp\{-x^2 \text{Var}_{\text{band}}[\varphi]/2\}.$$  (3.3)

It is sufficient to show that for any $x \in \mathbb{R}$

$$\lim_{n \to \infty} Z_n(x) = Z(x).$$  (3.4)

We note that $Z(x)$ is the unique solution of the integral equation

$$Z(x) = 1 - \text{Var}_{\text{band}}[\varphi] \int_0^x yZ(y) dy$$  (3.5)
in the class of bounded continuous functions. It follows from (3.2) that the derivative of \( Z_n(x) \) can be written as

\[
Z'_n(x) = iE\{M_n^\circ \varphi e^{ix.M_n^\circ \varphi}\}. \tag{3.6}
\]

To bound the derivative of \( Z_n \), we use the Poincaré inequality. Since the Poincaré Inequality tensorises (see e.g. [1]), the joint distribution of \( \{W_{jk}\}_{(j,k)\in I^+} \) on \( \mathbb{R}^{(b_n+1)} \) satisfies the Poincaré Inequality with the same constant \( m > 0 \), i.e. for all continuously differentiable function \( \Phi \), we have

\[
\text{Var}\{\Phi(\{W_{jk}\}_{(j,k)\in I^+})\} \leq \frac{1}{m} \sum_{(j,k)\in I^+} E\{\left|\frac{\partial \Phi}{\partial W_{jk}}(\{W_{jk}\})\right|^2\}. \tag{3.7}
\]

Let

\[
\beta_{jk} = (1 + \delta_{jk})^{-1} = \begin{cases} 1 & j \neq k, \\ 1/2 & j = k. \end{cases} \tag{3.8}
\]

Since

\[
\frac{\partial M_n^\circ \varphi}{\partial W_{jk}} = \frac{2\beta_{jk}}{\sqrt{n}} \varphi'_{jk}(M), \tag{3.9}
\]

we have

\[
\text{Var}\{M_n^\circ \varphi\} \leq \frac{2m}{mn} E\{\text{Tr}(\varphi'(M)\varphi'(M)^\ast)\} \\
\leq \frac{2}{m} (\sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)|)^2. \tag{3.10}
\]

Applying the Cauchy-Schwarz inequality, we obtain

\[
|Z'_n(x)| \leq \sqrt{\frac{2}{m}} (\sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)|). \tag{3.11}
\]

In addition, (3.10) implies

\[
|Z''_n(x)| \leq \frac{2}{m} (\sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)|)^2. \tag{3.12}
\]

Taking into account \( Z_n(0) = 1 \), we have

\[
Z_n(x) = 1 + \int_0^x Z_n'(y)dy. \tag{3.13}
\]

We note that the sequence \( \{(Z_n(x), Z'_n(x))\} \) is pre-compact in \( C([-T, T], \mathbb{R}^2) \) for any \( T > 0 \). Therefore, it is enough to show that for any converging subsequence one has

\[
\lim_{n_j \to \infty} Z'_{n_j}(x) = -x\text{Var}_{\text{band}}[\varphi] \lim_{n_j \to \infty} Z_{n_j}(x). \tag{3.14}
\]

For the convenience of the reader, we use the same notations as in [26]:

\[
D_{jk} := \partial/\partial M_{jk}; \tag{3.15}
\]

\[
U(t) := e^{itM}, U_{jk}(t) := (U(t))_{jk}; \tag{3.16}
\]

\[
u_n(t) := \text{Tr}U(t), \quad u_n^\circ(t) := u_n(t) - E\{u_n(t)\}. \tag{3.17}
\]

Since \( U(t) \) is a unitary matrix, we have

\[
\|U\| = 1; \quad |U_{jk}| \leq 1; \quad \sum_{k=1}^n |U_{jk}|^2 = 1. \tag{3.18}
\]
Moreover,
\[ D_{jk}U_{ab}(t) = i\beta_{jk}(U_{aj} * U_{bk} + U_{ak} * U_{bj})(t), \]
where
\[ f * g(t) := \int_0^t f(s)g(t - s)ds. \]

Applying the Fourier inversion formula
\[ \hat{\varphi}(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda} \hat{\varphi}(t)dt, \]
we can write
\[ \mathcal{M}_n^\varphi = (b_n/n)^{1/2} \int_{-\infty}^{\infty} \hat{\varphi}(t)u_n^2(t)dt. \]

Therefore,
\[ Z'_n(x) = i \int_{-\infty}^{\infty} \hat{\varphi}(t)Y_n(x, t)dt, \]
where
\[ Y_n(x, t) := \mathbb{E}\{(b_n/n)^{1/2}u_n^2(t)e_n(x)\}, \]
and
\[ e_n(x) = e^{ix\mathcal{M}_n^\varphi}. \]

Taking into account (3.14) and (3.23), we conclude that the result of the theorem follows if we can establish the following two facts. First, we have to show that the sequence \( Y_n \) is bounded and equicontinuous on any bounded subset of \( \{t \geq 0, x \in \mathbb{R}\} \). Second, we have to show that any uniformly converging subsequence of \( Y_n \) has the same limit
\[ Y(x, t) = Y(-x, -t) \]
such that
\[ i \int_{-\infty}^{\infty} \hat{\varphi}(t)Y(x, t)dt = -x\text{Var}_{\text{band}}[\varphi]Z(x). \]

The main technical part of the proof of Theorem 2.1 is the following proposition.

**Proposition 3.1.** \( Y_n(x, t) \) satisfies the equation
\begin{align*}
Y_n(x, t) + \frac{2(2b_n + 1)}{b_n^2} \int_0^t \int_0^{t_1} \bar{v}_n(t_1 - t_2)Y_n(x, t_2)dt_2dt_1 \\
= xZ_n(x)A_n(t) + 2i\kappa_4xZ_n(x) \int_0^t \bar{v}_n + \bar{v}_n(t_1)dt_1 \int_{-\infty}^{t} t_2 \bar{v}_n + \bar{v}_n(t_2)\hat{\varphi}(t_2)dt_2 + r_n(x, t), \tag{3.28}
\end{align*}
where
\begin{align*}
A_n(t) &:= -\frac{2\sigma^2}{n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jk}(t_1)\varphi'_{jk}(M)\}dt_1, \tag{3.29} \\
\bar{v}_n(t) &:= n^{-1}\mathbb{E} \text{Re} e^{itM}, \tag{3.30}
\end{align*}
\( U_{jk}(t) \) is defined in (3.16), and \( r_n(x, t) \) converges to zero uniformly on any bounded subset of \( \{t \geq 0, x \in \mathbb{R}\} \).

The proof of Proposition 3.1 will be given in the remaining part of this subsection and in the next three subsections.
Proof. First, we show that \( Y_n(x, t) \) is bounded and uniformly equicontinuous on bounded subsets of \( \mathbb{R}^2 \). Indeed, applying inequality (3.10) to \( \varphi(\lambda) = e^{it\lambda} \) and \( \varphi(\lambda) = i\lambda e^{it\lambda} \), we get

\[
Var\{(b_n/n)^{1/2}u_n(t)\} \leq \frac{2t^2}{m} \tag{3.31}
\]

and

\[
Var\{(b_n/n)^{1/2}u'_n(t)\} \leq \frac{2}{m}(1 + 3\sigma^2t^2). \tag{3.32}
\]

This implies

\[
|Y_n(x, t)| \leq Var^{1/2}\{(b_n/n)^{1/2}u_n(t)\} \leq \sqrt{\frac{2}{m}|t|}, \tag{3.33}
\]

\[
\left| \frac{\partial}{\partial t}Y_n(x, t) \right| \leq Var^{1/2}\{(b_n/n)^{1/2}u'_n(t)\} \leq \sqrt{\frac{2}{m}(1 + 3\sigma^2t^2)}, \tag{3.34}
\]

and

\[
\left| \frac{\partial}{\partial x}Y_n(x, t) \right| \leq \frac{2}{m}|t| \sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)|. \tag{3.35}
\]

Therefore, we have shown that \( \{Y_n\} \) is bounded and equicontinuous on any bounded subset of \( \mathbb{R}^2 \). Applying the identity \( e^{itM} = 1 + i \int_0^t Me^{isM}ds \), we have

\[
u_n(t) = n + i \int_0^t \sum_{(j,k) \in I_n} M_{jk}U_{jk}(t_1)dt_1, \tag{3.36}
\]

and

\[
Y_n(x, t) = \frac{i}{\sqrt{n}} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{W_{jk}U_{jk}(t_1)e^n(x)\}dt_1, \tag{3.37}
\]

where \( e^n = e_n - \mathbb{E}\{e_n\} \). To analyze \( (3.37) \), we use the decoupling formula \( (B.1) \) with \( p = 3 \) to obtain

\[
Y_n(x, t) = \frac{i}{\sqrt{n}} \int_0^t \sum_{(j,k) \in I_n} \left\{ \sum_{l=0}^3 \frac{\kappa_{l+1,jk}}{b_n^{l/2}l!} \mathbb{E}\{D_{jk}(U_{jk}(t_1)e^n(x))\} + \varepsilon_{3,jk} \right\} \right. dt_1, \tag{3.38}
\]

where \( \kappa_{l,jk} \) is the \( l \)th cumulant of \( W_{jk} \), i.e.

\[
\kappa_{1,jk} = 0, \quad \kappa_{2,jk} = (1 + \delta_{jk})\sigma^2;
\]

in addition, for \( j \neq k \) one has

\[
\kappa_{3,jk} = \mathbb{E}\{(W_{12}^{(n)})^3\} =: \kappa_3, \quad \kappa_{4,jk} = \kappa_4;
\]

and for \( j = k \)

\[
\kappa_{3,jj} = \mathbb{E}\{(W_{11}^{(n)})^3\} =: \kappa'_3, \quad \kappa_{4,jj} = \mathbb{E}\{(W_{11}^{(n)})^4\} - 12\sigma^2 =: \kappa'_4.
\]

Moreover, we note that the remainder term \( \varepsilon_{3,jk} \) in \( (3.38) \) is bounded as

\[
|\varepsilon_{3,jk}| \leq C_3 \mathbb{E}\{|W_{jk}|^5\} \sup_{W_{jk} \in \mathbb{R}} \left| \frac{D_{jk}^4U_{jk}(t)e^n(x)}{b_n^2} \right|. \tag{3.39}
\]

Since the marginal distribution of the matrix entries satisfies the PI with constant \( m \) independent of \( n \), the third and fourth cumulants are uniformly bounded in \( n \), i.e. there exist \( \sigma_3 \) and \( \sigma_4 \) independent of \( n \) such that

\[
|\kappa_3|, |\kappa'_3| \leq \sigma_3, |\kappa'_4| \leq \sigma_4, \tag{3.40}
\]

and \( \sigma_5 := \max_{j,k,n} \mathbb{E}\{|W_{jk}|^5\} < \infty \).

We need the following technical lemma.
Lemma 3.2.

$$|D^l_{jk}(U_{jk}(t)e^o_n(x))| \leq C_l(\sqrt{b_n/n}x, t), \quad l = 1, 2, 3, 4,$$

where $C_l(x, t)$ is some polynomial in $|x|, |t|$ of degree $l$ with positive coefficients independent of $n$.

Proof. (3.19) implies

$$|D^l_{jk}U_{jk}(t)| \leq c_l|t|^l.$$

In addition,

$$D_{jk}e_n(x) = -2(b_n/n)^{1/2}\beta_{jk}xe_n(x)\int_{-\infty}^{\infty}sU_{jk}(s)\tilde{\varphi}(s)ds = 2i(b_n/n)^{1/2}\beta_{jk}xe_n(x)\varphi'_{jk}(M).$$

It follows from (3.1) that $\varphi$ has fourth bounded derivative. Thus, for $l = 1, 2, 3, 4$

$$|D^l_{jk}e_n(x)| \leq c'_l(1 + |(b_n/n)^{1/2}x|).$$

Combining (3.42) and (3.44) we obtain Lemma 3.2.

Lemma 3.2 and (3.39) imply

$$|\varepsilon_{3jk}| \leq C_5\sigma_2C_4((b_n/n)^{1/2}x, t)/b_n^2.$$  (3.45)

We can rewrite (3.38) as

$$Y_n(x, t) = T_1 + T_2 + T_3 + \varepsilon_3,$$  (3.46)

where

$$T_l := \frac{i}{b_n}\frac{l}{\sqrt{nb_n}}\int_0^t \sum_{(j,k) \in I_n} \kappa_{l+1,jk} E\{D^l_{jk}(U_{jk}(t_1)e^o_n(x))\} dt_1, \quad l = 1, 2, 3,$$  (3.47)

and

$$\varepsilon_3 = im^{-1/2}\int_0^t \sum_{(j,k) \in I_n} \varepsilon_{3,jk} dt_1.$$  (3.48)

By (3.45), we have

$$|\varepsilon_3| \leq \frac{\sqrt{n}}{b_n}C_5((b_n/n)^{1/2}x, t).$$  (3.49)

Since $n/b_n^2 \to 0$, we obtain that $\varepsilon_3 \to 0$ on any bounded subset of $\mathbb{R}^2$ as $n \to \infty$. In the next three subsections, we consider separately each of the terms $T_l, \quad l = 1, 2, 3$ in (3.40) and finish the proof of Proposition 3.1.

3.2 Estimate of $T_1$

The main result of this subsection is contained in the following proposition.

Proposition 3.3. Let $T_1$ be defined as in (3.40) with $l = 1$. Then

$$T_1 = -2(b_n + 1)^2\sigma^2b_n \int_0^t \int_0^{t_1} \bar{v}_n(t_1 - t_2)Y_n(x, t_2)dt_2dt_1 + xZ_n(x)A_n(t) + \varepsilon_n(x, t),$$  (3.50)

where $\bar{v}_n(t)$ is defined in (3.30), $A_n(t)$ is defined in (3.24), and $\varepsilon_n(x, t) \to 0$ as $n \to \infty$ uniformly on any bounded subset of $\{(x, t), t \geq 0\}$.
Proof. First, by (3.19) we write

$$T_1 = T_{11} + T_{12} + T_{13},$$

where

$$T_{11} = -\frac{a^2}{\sqrt{n}b_n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jk} \ast U_{jk}(t_1)e_n^2(x)\} dt_1,$$

$$T_{12} = -\frac{a^2}{\sqrt{n}b_n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jj} \ast U_{kk}(t_1)e_n^2(x)\} dt_1,$$

$$T_{13} = -\frac{2a^2}{n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jk}(t_1)x e_n(x)\varphi_{jk}'(M)\} dt_1.$$

It follows from the Cauchy-Schwarz inequality and $|e_n(x)| \leq 1$ that

$$|T_{11}| \leq \frac{a^2}{\sqrt{n}b_n} \int_0^t \sum_{k=1}^n \text{Var}^{1/2} \left\{ \sum_{j:(j,k) \in I_n} (U_{jk} \ast U_{jk})(t_1) \right\} dt_1.$$

Let us fix $k$ and define

$$U^{(k)}(t) := (U_{j}^{(k)}(t))_{j,l=1,...,n},$$

where

$$U_{j}^{(k)}(t) = \begin{cases} U_{j}^{(k)}(t) & \text{if } (j,k) \in I_n, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$||U^{(k)}(t)|| \leq 1.$$ (3.58)

By the Poincaré Inequality (3.7), (3.8), (3.19), and the Cauchy-Schwarz Inequality, we have

$$\text{Var}\left\{ \sum_{j:(j,k) \in I_n} U_{jk} \ast U_{jk}(t_1) \right\} \leq \frac{4}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\left\{ \left| \sum_{j:(j,k) \in I_n} U_{jp} \ast U_{jk}(t_1) + U_{js} \ast U_{kp} \ast U_{jk}(t_1) \right|^2 \right\}$$

$$\leq \frac{8a^2}{mb_n} \mathbb{E}\left\{ \int_0^{t_1} \int_0^{t_2} \sum_{p=1}^n |(U(t_1 - t_2)U^{(k)}(t_3))_{pk}|^2 \sum_{s=1}^n |U_{ks}(t_2 - t_3)|^2 dt_3 dt_2 \right\}.$$

It follows from (3.59), and

$$\sum_{p=1}^n \left| (U(t_1 - t_2)U^{(k)}(t_3))_{pk} \right|^2 = \|U(t_1 - t_2)U^{(k)}(t_3)e_k\|^2 = \|U^{(k)}(t_3)e_k\|^2 \leq \|e_k\|^2 = 1$$

that we have

$$\text{Var}\left\{ \sum_{j:(j,k) \in I_n} U_{jk} \ast U_{jk}(t_1) \right\} \leq \frac{4a^4}{mb_n}.$$ (3.61)

Hence,

$$|T_{11}| \leq \frac{2\sqrt{n}a^2t^3}{3\sqrt{mb_n}}.$$ (3.62)

Recall that $n/b_n^2 \to 0$, so $T_{11} \to 0$ as $n \to \infty$ if $t$ is bounded.

Now, we turn our attention to (3.55). We write $T_{12}$ as follows

$$T_{12} = -\frac{2a^2}{\sqrt{n}b_n} \int_0^t \int_0^{t_1} \sum_{k=1}^n \left\{ \mathbb{E}\{U_{kk}(t_1 - t_2)\} \sum_{j:(j,k) \in I_n} \mathbb{E}\{U_{jj}(t_2)e_n^2(x)\} \right\} dt_2 dt_1 + T_{12}',$$

(3.63)
where

$$T_{12} = -\frac{\sigma^2}{\sqrt{nb_n}} \int_0^t \int_0^{t_1} \sum_{(j,k) \in I_n} \mathbb{E}\{U_{kk}(t_1 - t_2)U_{jj}^0(t_2)e_n^0(x)\} dt_2 dt_1. \quad (3.64)$$

Since $\mathbb{E}\{U_{kk}(t)\}$ and $\mathbb{E}\{U_{kk}(t)e_n^0(x)\}$ are $k$-independent, $\mathbb{E}\{U_{kk}(t)\} = \bar{v}_n(t)$, and

$$\sum_{j:(j,k) \in I_n} \mathbb{E}\{U_{jj}(t_2)e_n^0(x)\} = \frac{2b_n + 1}{n} \mathbb{E}\{u(t_2)e_n^0(x)\} = \frac{2b_n + 1}{\sqrt{nb_n}} Y_n(x, t). \quad (3.65)$$

Thus, the first term in (3.63) can be written as

$$- \frac{2(2b_n + 1)\sigma^2}{b_n} \int_0^t \int_0^{t_1} \bar{v}_n(t_1 - t_2)Y_n(x, t_2) dt_2 dt_1. \quad (3.66)$$

We are left to bound $T_{12}'$. The Cauchy-Schwarz inequality and $|e_n^0(x)| \leq 2$ imply

$$|T_{12}'| \leq \frac{2\sigma^2}{\sqrt{nb_n}} \int_0^t \int_0^{t_1} \sum_{k=1}^n \text{Var}^{1/2}\{U_{kk}(t_2)\} \text{Var}^{1/2}\{ \sum_{j:(j,k) \in I_n} U_{jj}(t_1 - t_2) \} dt_2 dt_1. \quad (3.67)$$

Applying (3.7), (3.19), (3.8) and the Cauchy-Schwarz inequality, we get

$$\text{Var}\{U_{jk}(t)\} \leq \frac{4}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\{|U_{jp} \ast U_{ks}(t)|^2\} \leq \frac{4t}{mb_n} \mathbb{E}\{\int_0^t \sum_{(p,s) \in I_n^+} |U_{jp}(t_1)U_{ks}(t-t_1)|^2 dt_1\} \leq \frac{4t}{mb_n} \mathbb{E}\{\int_0^t |U_{jp}(t_1)|^2 \sum_{s=1}^n |U_{ks}(t-t_1)|^2 dt_1\} = \frac{4t^2}{mb_n}, \quad (3.68)$$

and for fixed $k$,

$$\text{Var}\left\{ \sum_{j:(j,k) \in I_n} U_{jj}(t) \right\} \leq \frac{4}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\{| \sum_{j:(j,k) \in I_n} U_{jp} \ast U_{js}(t)|^2\}$$

$$= \frac{4}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\{ | \int_0^t \sum_{j:(j,k) \in I_n} U_{jp}(t_1)U_{js}(t-t_1) dt_1|^2 \}$$

$$= \frac{4}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\{ | \int_0^t (U(t_1)U^{(k)}(t-t_1))_{ps} dt_1|^2 \}$$

$$\leq \frac{4t}{mb_n} \mathbb{E}\{ \int_0^t \sum_{p,s=1}^n |(U(t_1)U^{(k)}(t-t_1))_{ps}|^2 dt_1 \}$$

$$= \frac{4t}{mb_n} \mathbb{E}\{ \int_0^t \text{Tr}U(t_1)U^{(k)}(t-t_1)U(t_1)^*U^{(k)}(t-t_1)^* dt_1 \}$$

$$= \frac{4t}{mb_n} \mathbb{E}\{ \int_0^t \text{Tr}U^{(k)}(t-t_1)U^{(k)}(t-t_1)^* dt_1 \}. \quad (3.69)$$

Since

$$\text{Tr}(U^{(k)}(t)U^{(k)}(t)^*) \leq (2b_n + 1)||U^{(k)}(t)||^2 \leq (2b_n + 1)||U^{(k)}(t)||^2 \leq 3b_n, \quad (3.70)$$

we conclude that

$$\text{Var}\left\{ \sum_{j:(j,k) \in I_n} U_{jj}(t) \right\} \leq \frac{12t^2}{m}. \quad (3.71)$$
Therefore,

\[ |T'_{12}| \leq \frac{2\sigma^2}{\sqrt{nb_n}} \int_0^t \int_0^{t_1} \int_0^n \sqrt{\frac{4t_2^2}{mb_n}} \sqrt{\frac{12(t_1 - t_2)^2}{m}} dt_2 dt_1 = \frac{Const\sigma^2 t^4}{mb_n}. \]  

(3.72)

Now, we turn our attention to \( T_{13} \). We can rewrite \( T_{13} \) in the following form

\[ T_{13} = x Z_n(x) A_n(t) + T'_{13}, \]  

(3.73)

where \( Z_n(x) \) is given by \( (3.2) \), \( A_n(t) \) is defined in \( (3.29) \), and

\[ T'_{13} = -\frac{2i\sigma^2 x}{n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jk}(t_1)e^\mu_n(x)\} \int_{-\infty}^\infty t_2 U_{jk}(t_2) \hat{\varphi}(t_2) dt_2 dt_1. \]  

(3.74)

Then

\[ |T'_{13}| \leq \frac{2\sigma^2 x}{n} \int_0^t \int_{-\infty}^\infty \sum_{k=1}^n \text{Var}^{1/2}\left\{ \sum_{j:(j,k) \in I_n} U_{jk}(t_1) U_{jk}(t_2) \right\} |t_2||\hat{\varphi}(t_2)| dt_2 dt_1. \]  

(3.75)

Let us fix \( k \). The Poincaré Inequality \( (3.7) \), together with \( (3.19) \) and \( (3.8) \) imply

\[
\begin{align*}
\text{Var}\left\{ \sum_{j:(j,k) \in I_n} U_{jk}(t_1) U_{jk}(t_2) \right\} & \leq \frac{1}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\left\{ \sum_{j:(j,k) \in I_n} [U_{jp} * U_{ks}(t_1) + U_{js} * U_{kp}(t_1)]U_{jk}(t_2) + U_{jk}(t_1)[U_{jp} * U_{ks}(t_2) + U_{js} * U_{kp}(t_2)] \right\}^2 \\
& \leq \frac{4}{mb_n} \sum_{(p,s) \in I_n^+} \mathbb{E}\left\{ \left| \sum_{j:(j,k) \in I_n} U_{jp} * U_{ks}(t_1)U_{jk}(t_2) \right|^2 + \left| \sum_{j:(j,k) \in I_n} U_{js} * U_{kp}(t_1)U_{jk}(t_2) \right|^2 \\
& \quad + \left| \sum_{j:(j,k) \in I_n} U_{jk}(t_1)U_{jp} * U_{ks}(t_2) \right|^2 + \left| \sum_{j:(j,k) \in I_n} U_{jk}(t_1)U_{js} * U_{kp}(t_2) \right|^2 \right\} \\
& \leq \frac{8}{mb_n} \sum_{p,s=1}^n \mathbb{E}\left\{ \left| \sum_{j:(j,k) \in I_n} U_{jp} * U_{ks}(t_1)U_{jk}(t_2) \right|^2 + \left| \sum_{j:(j,k) \in I_n} U_{jk}(t_1)U_{jp} * U_{ks}(t_2) \right|^2 \right\}. \end{align*}
\]  

(3.76)

Note that the Cauchy-Schwarz inequality gives us

\[
\left| \sum_{j:(j,k) \in I_n} U_{jp} * U_{ks}(t_1)U_{jk}(t_2) \right|^2 = \left| \int_0^{t_1} \sum_{j:(j,k) \in I_n} U_{jp}(t_3)U_{ks}(t_1 - t_3)U_{jk}(t_2) dt_3 \right|^2
\]

\[
= \left| \int_0^{t_1} (U(t_3)U^{(k)}(t_2))_{pk} U_{ks}(t_1 - t_3) dt_3 \right|^2 \leq t_1 \int_0^{t_1} |(U(t_3)U^{(k)}(t_2))_{pk}|^2 U_{ks}(t_1 - t_3)^2 dt_3 \leq \frac{t_1^2}{2}. \]  

(3.77)

Using \( (8.15) \) and \( (15.41) \), we obtain

\[
\sum_{p,s=1}^n \left| \sum_{j:(j,k) \in I_n} U_{jp} * U_{ks}(t_1)U_{jk}(t_2) \right|^2 \leq t_1 \int_0^{t_1} \sum_{p=1}^n |(U(t_3)U^{(k)}(t_2))_{pk}|^2 \sum_{s=1}^n |U_{ks}(t_1 - t_3)|^2 dt_3 \leq \frac{t_1^2}{2}. \]  

(3.78)

Hence,

\[ \text{Var}\left\{ \sum_{j:(j,k) \in I_n} U_{jk}(t_1)U_{jk}(t_2) \right\} \leq \frac{4(t_1^2 + t_2^2)}{mb_n}. \]  

(3.79)
Therefore,

\[
|T'_{13}| \leq 2 \sigma^2 |x| \int_0^t \int_{|t_2| \leq t_1} \left| \frac{4(t_1^2 + t_2^2)}{mb_n} \varphi(t_2) \right| dt_2 dt_1
\]

\[
= 2 \sigma^2 |x| \int_0^t \left[ \int_{|t_2| \leq t_1} |t_2| \left| \frac{4(t_1^2 + t_2^2)}{mb_n} \varphi(t_2) \right| dt_2 + \int_{|t_2| > t_1} |t_2| \left| \frac{4(t_1^2 + t_2^2)}{mb_n} \varphi(t_2) \right| dt_2 \right] dt_1
\]

\[
\leq \frac{4 \sqrt{2} \sigma^2 |x|}{\sqrt{mb_n}} \int_0^t \left[ t_1^2 \int_{|t_2| \leq t_1} |\varphi(t_2)| dt_2 + \int_{|t_2| > t_1} |t_2|^2 |\varphi(t_2)| dt_2 \right] dt_1
\]

\[
\leq \frac{\sigma^2 |x|}{\sqrt{mb_n}} C_3(t). \tag{3.80}
\]

Combining the bounds obtained in this subsection, we get

\[
T_1 = - \frac{2(2b_n + 1) \sigma^2}{b_n} \int_0^t \int_0^{t_1} v_n(t_1 - t_2) Y_n(x, t_2) dt_2 dt_1 + x Z_n(x) A_n(t) + \varepsilon_n(x, t). \tag{3.81}
\]

Since \( b_n/n \to 0, n/b_n^2 \to 0 \), using (3.80), (3.82), and (3.83), we have that \( \varepsilon_n(x, t) = T_{11} + T_{12} + T_{13}' \to 0 \) on any bounded subset of \( \{(x, t), t \geq 0 \} \). Proposition 3.3 is proven.

### 3.3 Estimate of \( T_2 \)

The main result of this subsection is the following proposition.

**Proposition 3.4.** Let \( T_2 \) be defined as in (3.47) with \( l = 2 \). Then \( T_2 \) converges to zero as \( n \to \infty \) uniformly on any bounded subset of \( \{t \geq 0, x \in \mathbb{R} \} \).

**Proof.**

\[
T_2 = \frac{i \kappa_3}{2 \sqrt{nb_n}} \int_0^t \sum_{(j,k) \in I_n} E\{D^2_{jk}(U_{jk}(t_1) e_n^c(x))\} dt_1 + \frac{i(n' - \kappa_3)}{2 \sqrt{nb_n}} \int_0^t \sum_{j=1}^n E\{D^2_{jj}(U_{jj}(t_1) e_n^c(x))\} dt_1. \tag{3.82}
\]

By Lemma 3.2, the second term in \( T_2 \) is bounded by \( \frac{n}{b_n} C_3((b_n/n)^{1/2}, x, t) \). As for the first term in \( T_2 \), it can be written as the sum of \( T_{21} \) and \( T_{22} \), where

\[
T_{21} = \frac{-i \kappa_3}{\sqrt{nb_n}} \int_0^t \sum_{(j,k) \in I_n} E\{\beta^2_{jk}(U_{jk} * U_{jk} * U_{jk})(t_1) e_n^c(x)\} dt_1
\]

\[
+ \frac{2 \kappa_3}{n \sqrt{b_n}} \int_0^t \sum_{(j,k) \in I_n} E\{\beta^2_{jk}(U_{jk} * U_{jk})(t_1) x e_n(x) \int_{-\infty}^t t_2 U_{jk}(t_2) \varphi(t_2) dt_2 \} dt_1
\]

\[
+ \frac{2 i \kappa_3}{n \sqrt{b_n}} \int_0^t \sum_{(j,k) \in I_n} E\{\beta^2_{jk} x U_{jk}(t_1) e_n(x) \int_{-\infty}^t t_2 U_{jk}(t_2) \varphi(t_2) dt_2 \} dt_1
\]

\[
+ \frac{\kappa_3}{n \sqrt{b_n}} \int_0^t \sum_{(j,k) \in I_n} E\{\beta^2_{jk} x U_{jk}(t_1) e_n(x) \int_{-\infty}^t t_2 (U_{jk} * U_{jk})(t_2) \varphi(t_2) dt_2 \} dt_1. \tag{3.83}
\]

12
To estimate \( T_{22} \)

\[
T_{22} = -\frac{i\kappa_3}{\sqrt{n}b_n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{\beta_{jk}^2 (3U_{jj} * U_{jk} * U_{kk})(t_1)e_n(x)\} dt_1 \\
+ \frac{2\kappa_3}{n\sqrt{b_n}} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{\beta_{jk} (U_{jj} * U_{kk})(t_1)x e_n(x) \int_{-\infty}^{\infty} t_2 U_{jk}(t_2)\phi(t_2) dt_2\} dt_1 \\
+ \frac{\kappa_3}{n\sqrt{b_n}} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{\beta_{jk}^2 x U_{jk}(t_1)e_n(x) \int_{-\infty}^{\infty} t_2 (U_{jj} * U_{kk})(t_2)\phi(t_2) dt_2\} dt_1.
\]

(3.84)

Since for any \( s_1, s_2, s_3 \in \mathbb{R} \), one has

\[
| \sum_{(j,k) \in I_n} U_{jk}(s_1)U_{jk}(s_2)U_{jk}(s_3) | \leq \sum_{j,k=1}^{n} |U_{jk}(s_1)U_{jk}(s_2)| \leq \left( \sum_{j,k=1}^{n} |U_{jk}(s_1)|^2 \right)^{1/2} \left( \sum_{j,k=1}^{n} |U_{jk}(s_2)|^2 \right)^{1/2} = n.
\]

(3.85)

We have

\[
|T_{21}| \leq \sqrt{n}/b_n C_2(\sqrt{b_n/n}x)t.
\]

(3.86)

To estimate \( T_{22} \), we note that

\[
\mathbb{E} \left\{ \sum_{(j,k) \in I_n} \beta_{jk}^2 U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3)e_n(x) \right\} \leq \text{Var}^{1/2} \left\{ \sum_{(j,k) \in I_n} \beta_{jk}^2 U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3) \right\}.
\]

(3.87)

Using the Poincaré inequality, we obtain an upper bound

\[
\text{Var} \left\{ \sum_{(j,k) \in I_n} \beta_{jk}^2 U_{jj}(s_1)U_{jk}(s_2)U_{kk}(s_3) \right\} \\
\leq \frac{12}{nb_n} \sum_{(p,q) \in I_n^+} \mathbb{E} \left\{ | \sum_{(j,k) \in I_n} U_{jp} * U_{jq}(s_1)U_{jk}(s_2)U_{kk}(s_3) |^2 + | \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jp} * U_{kq}(s_2)U_{kk}(s_3) |^2 \\
+ | \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jq} * U_{kp}(s_2)U_{kk}(s_3) |^2 + | \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jk}(s_2)U_{kp} * U_{kq}(s_3) |^2 \right\}.
\]

(3.88)

Let \( \alpha_j = \sum_{(j,k) \in I_n} U_{jk}(s_2)U_{kk}(s_3) \) and \( D = \text{diag} \{ \alpha_1, ... , \alpha_n \} \). Then \( |\alpha_j| \leq \sqrt{2b_n + 1} \), and

\[
\sum_{(p,q) \in I_n^+} | \sum_{(j,k) \in I_n} U_{jp} * U_{jq}(s_1)U_{jk}(s_2)U_{kk}(s_3) |^2 \leq \sum_{(p,q) \in I_n^+} |s_1| \int_0^{s_1} \sum_{(j,k) \in I_n} U_{jp}(t)U_{jq}(s_1-t)U_{jk}(s_2)U_{kk}(s_3) |^2 dt \\
= |s_1| \int_0^{s_1} \sum_{j=1}^{n} \sum_{(p,q) \in I_n^+} U_{jp}(t)U_{jq}(s_1-t)\alpha_j |^2 dt = |s_1| \int_0^{s_1} \sum_{(p,q) \in I_n^+} |(U(t)DU(s_1-t))_{pq} |^2 dt \\
\leq |s_1| \int_0^{s_1} n||U(t)DU(s_1-t)||^2 dt \leq s_1^2 n(2b_n + 1).
\]

(3.89)

Now let \( D_1 = \text{diag} \{ U_{11}(s_1), ... , U_{nn}(s_1) \} \), \( D_2 = \text{diag} \{ U_{11}(s_3), ... , U_{nn}(s_3) \} \), and \( B = (B_{jk})_{j,k=1}^n \) be a 0-1 band matrix, such that \( B_{jk} = 1_{(j,k) \in I_n} \). Then

\[
\sum_{(p,q) \in I_n^+} | \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jp} * U_{kq}(s_2)U_{kk}(s_3) |^2 \leq \sum_{(p,q) \in I_n^+} |s_2| \int_0^{s_2} \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jp}(t)U_{kq}(s_2-t)U_{kk}(s_3) |^2 dt \\
= |s_2| \int_0^{s_2} \sum_{(p,q) \in I_n^+} |(U(t)D_1BD_2U(s_2-t))_{pq} |^2 dt \leq s_2^2 n |B|^2 = s_2^2 n(2b_n + 1)^2.
\]

(3.90)
Hence,

\[
\text{Var}\left\{ \sum_{(j,k) \in I_n} \beta_{jk}^2 U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3) \right\} \leq \frac{12n(2b_n + 1)}{mb_n} \left( s_1^2 + s_2^2 + 2s_2^2(2b_n + 1) \right) \ \ (3.91)
\]

\[
\leq nb_n C_2(s_1, s_2, s_3), \ \ (3.92)
\]

\[
|\mathbb{E}\left\{ \sum_{(j,k) \in I_n} \beta_{jk}^2 U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3) e_n(x) \right\}| \leq \sqrt{nb_n C_2^{1/2}(s_1, s_2, s_3)}. \ \ (3.93)
\]

For the last two terms in \( T_{22} \), taking into account that

\[
\mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3) e_n(x)\} = \mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3) e_n(x)\} + \mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3)\} \mathbb{E}\{e_n(x)\}, \ \ (3.94)
\]

we have

\[
|\sum_{(j,k) \in I_n} \beta_{jk}^2 \mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3) e_n(x)\}| \leq \sqrt{nb_n C_2^{1/2}(s_1, s_2, s_3)} + n/4 + \sum_{(j,k) \in I_n, j \neq k} |\mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3)\}|. \ \ (3.95)
\]

For \( j \neq k \), we can write

\[
\mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3)\} = \mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3)\} + \mathbb{E}\{U_{jj}(s_1)\} \mathbb{E}\{U_{jk}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\}
\]

\[
+ \mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\} + \mathbb{E}\{U_{jj}(s_1)\} \mathbb{E}\{U_{jk}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\}. \ \ (3.96)
\]

So the left hand side of \( 3.96 \) is bounded by

\[
(\text{Var}\{U_{jj}(s_1)\} \text{Var}\{U_{kk}(s_3)\})^{1/2} + (\text{Var}\{U_{jj}(s_1)\} \text{Var}\{U_{jk}(s_2)\})^{1/2} + (\text{Var}\{U_{jk}(s_2)\})^{1/2} + |\mathbb{E}\{U_{jk}(s_2)\}|. \ \ (3.97)
\]

By \( 3.68 \) it is bounded from above by

\[
\frac{4(|s_1 s_2| + |s_1 s_3| + |s_2 s_3|)}{mb_n} + |\mathbb{E}\{U_{jk}(s_2)\}|. \ \ (3.98)
\]

To bound the second term in the last expression, we use the following auxiliary proposition.

**Proposition 3.5.** Let \( M = W/\sqrt{b_n} \) be a real symmetric band random matrix defined as Theorem 2.1 and \( U(t) = e^{itM} \). Then

\[
\sup_{j \neq k} |\mathbb{E}\{U_{jk}(t)\}| = O\left( \frac{1 + t^6}{b_n} \right). \ \ (3.99)
\]

The proof of Proposition 3.3 is given in Appendix C. Assuming \( 3.99 \), we conclude that

\[
|\mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3)\}| \leq \frac{4(|s_1 s_2| + |s_1 s_3| + |s_2 s_3|)}{mb_n} + O\left( \frac{1 + s_2^6}{b_n} \right). \ \ (3.100)
\]

Therefore,

\[
|\sum_{(j,k) \in I_n} \beta_{jk}^2 \mathbb{E}\{U_{jj}(s_1) U_{jk}(s_2) U_{kk}(s_3) e_n(x)\}| \leq \sqrt{nb_n C_2^{1/2}(s_1, s_2, s_3)} + n/4 + \frac{8n(|s_1 s_2| + |s_1 s_3| + |s_2 s_3|)}{m} + O\left( \frac{s_2^6 n}{mb_n} \right), \ \ (3.101)
\]

and

\[
|T_{22}| \leq \frac{n}{b_n} C_4(t) + \frac{|x|}{b_n^2} C_4(t). \ \ (3.102)
\]

This and \( 3.86 \) imply that \( T_2 \) converges to zero uniformly on any bounded subset of \( \{ t \geq 0, x \in \mathbb{R} \} \). Proposition 3.4 is proven. \( \square \)
3.4 Estimate of $T_3$.

Finally, let us consider $T_3$. The main result of this subsection is the following bound.

**Proposition 3.6.** Let $T_3$ be defined as in (3.10) (with $l = 3$). Then

$$T_3(x, t) = 2i\kappa_4 x Z_n(x) \int_0^t \bar{v}_n * \bar{v}_n(t_1) dt_1 \int_{-\infty}^t t_2 \bar{v}_n * \bar{v}_n(t_2) \hat{\varphi}(t_2) dt_2 + \epsilon_n(x, t),$$

(3.103)

where $\epsilon_n(x, t) \to 0$ as $n \to \infty$ uniformly on any bounded subset of $\{(x, t), t \geq 0\}$.

**Proof.** One has

$$T_3(x, t) = \frac{ik_4}{6nb_n^3} \int_0^t \sum_{(j, k) \in I_n} \mathbb{E}\{\beta_{jk}^3(U_{jj} * U_{jk} * U_{kk} * U_{jj})(t_1)e_n^2(x)\} dt_1$$

(3.104)

where the term $T'_3$ comes from the summation over $j = k$ and corresponds to the fact that the marginal distribution of the diagonal entries is different from the marginal distribution of the off-diagonal entries. It follows form Lemma 3.2 that $T'_3$ can be bounded as

$$|T'_3(x, t)| \leq \sqrt{\frac{n}{b_n}} \sqrt{C_4((b_n/n)^{1/2}x, t)}.$$  

(3.105)

By (3.19), the first term in (3.104) can be written as the sum of

$$T_{31}(x, t) = \frac{k_4}{\sqrt{nb_n^3}} \int_0^t \sum_{(j, k) \in I_n} \mathbb{E}\{\beta_{jk}^3(U_{jj} * U_{jk} * U_{kk} * U_{jj})(t_1)e_n^2(x)\} dt_1$$

(3.106)

and

$$T_{32}(x, t) = \frac{\kappa_4}{nb_n} \int_0^t \sum_{(j, k) \in I_n} \mathbb{E}\{\beta_{jk}^3(6U_{jj} * U_{jk} * U_{kk} + U_{jk} * U_{jk} * U_{jk})(t_1)e_n^2(x)\} dt_1$$

(3.107)

and

$$T_{33}(x, t) = \frac{\kappa_4}{3nb_n} \int_0^t \sum_{(j, k) \in I_n} \mathbb{E}\{\beta_{jk}^3(U_{jk} * U_{jk} * U_{kk})(t_1)e_n^2(x)\} dt_1$$

(3.108)
Thus, $T_3 = T_{31} + T_{32} + T'_3$. Since we have already bounded $T'_3$ in (3.105), we are left with estimating the first two terms in the sum. There are two types of sums over $(j, k) \in I_n$ in (3.108), namely the first one corresponding to $U_{j}U_{jk}U_{k}U_{kk}$ and the second one to $U_{j}U_{jk}U_{j}U_{jk}$. Define

$$J_1(s_1, s_2, s_3, s_4) = \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jk}(s_2)U_{jk}(s_3)U_{kk}(s_4),$$

$$J_2(s_1, s_2, s_3, s_4) = \sum_{(j,k) \in I_n} U_{jk}(s_1)U_{jk}(s_2)U_{jk}(s_3)U_{jk}(s_4).$$

We note that

$$|J_1| \leq \sum_{(j,k) \in I_n} |U_{jk}(s_2)U_{jk}(s_3)| \leq \left\{ \sum_{j,k=1}^n |U_{jk}(s_2)|^2 \sum_{j,k=1}^n |U_{jk}(s_3)|^2 \right\}^{1/2} = n. \quad (3.111)$$

Similarly,

$$|J_2| \leq n. \quad (3.112)$$

It follows from the last two inequalities and (3.1) that

$$|T_{32}(x,t)| \leq \sqrt{n \frac{b_n}{n}} C_4((b_n/n)^{1/2} x, t). \quad (3.113)$$

Now, we estimate $T_{31}$. Recall that $T_{31}$ is defined in (3.106,3.107). Let us denote

$$v_n(s_1, s_2, s_3, s_4) = (nb_n)^{-1} \sum_{(j,k) \in I_n} U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4),$$

and

$$\bar{v}_n(s_1, s_2, s_3, s_4) := \mathbb{E}\{v_n(s_1, s_2, s_3, s_4)\}. \quad (3.115)$$

The rest of the proof of Proposition 3.6 follows from the next two lemmas.

**Lemma 3.7.**

$$\bar{v}_n(s_1, s_2, s_3, s_4) = 2\bar{v}_n(s_1)\bar{v}_n(s_2)\bar{v}_n(s_3)\bar{v}_n(s_4) + h(s_1, s_2, s_3, s_4), \quad (3.116)$$

where $\bar{v}_n(\cdot)$ is given by (3.30) and

$$|h(s_1, s_2, s_3, s_4)| \leq \frac{6(|s_1| + |s_2| + |s_3|)}{\sqrt{mb_n}} + \frac{1}{b_n}. \quad (3.117)$$

**Proof.**

$$\mathbb{E}\{U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\} = \mathbb{E}\{U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\} + \mathbb{E}\{U_{jj}(s_1)\} \mathbb{E}\{U_{jj}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\} \mathbb{E}\{U_{kk}(s_4)\} + \mathbb{E}\{U_{jj}(s_1)\} \mathbb{E}\{U_{jj}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\} \mathbb{E}\{U_{kk}(s_4)\}. \quad (3.118)$$

It follows from (3.08) that

$$|\mathbb{E}\{U_{jj}(s_1)U_{jj}(s_2)U_{kk}(s_3)U_{kk}(s_4)\} - \mathbb{E}\{U_{jj}(s_1)\} \mathbb{E}\{U_{jj}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\} \mathbb{E}\{U_{kk}(s_4)\}| \leq Var^{1/2}\{U_{jj}(s_1)\} + Var^{1/2}\{U_{jj}(s_2)\} + Var^{1/2}\{U_{kk}(s_3)\} \leq \frac{2(|s_1| + |s_2| + |s_3|)}{\sqrt{mb_n}}. \quad (3.119)$$

Therefore, we obtain

$$|\bar{v}_n(s_1, s_2, s_3, s_4) - (nb_n)^{-1} \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jj}(s_1)\} \mathbb{E}\{U_{jj}(s_2)\} \mathbb{E}\{U_{kk}(s_3)\} \mathbb{E}\{U_{kk}(s_4)\}| \leq \frac{6(|s_1| + |s_2| + |s_3|)}{\sqrt{mb_n}}. \quad (3.120)$$
In addition,
\[(nb_n)^{-1} \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jj}(a)\} \mathbb{E}\{U_{jj}(b)\} \mathbb{E}\{U_{kk}(c)\} \mathbb{E}\{U_{kk}(d)\} = (2 + 1/b_n) \bar{v}_n(a) \bar{v}_n(b) \bar{v}_n(c) \bar{v}_n(d)\] (3.121)

Now the lemma follows from (3.120-3.121) and \( \bar{v}_n(t) \leq 1 \). \( \square \)

The next lemma deals with \( T_{31} \) defined in (3.105-3.107).

**Lemma 3.8.**

\[ T_{31} = 2i\kappa_4xZ_n(x) \int_0^t \int_{-\infty}^\infty t_2 \bar{v}_n * \bar{v}_n(t_1) \bar{v}_n(t_2) \hat{\varphi}(t_2) dt_2 dt_1 + \delta_n(x,t), \] (3.122)

where
\[ \delta_n(x,t) \to 0 \] (3.123)

uniformly on any bounded subset of \( \{(x,t), t \geq 0\} \).

**Proof.** \( T_{31} \) can be written as
\[
T_{31}(x,t) = \frac{\kappa_4}{\sqrt{nb_n}} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{(U_{jj} * U_{jj} * U_{kk} * U_{kk})(t_1)e_n^o(x)\} dt_1 \\
+ \frac{i\kappa_4}{nb_n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{(U_{jj} * U_{kk})(t_1)x e_n^o(x)\} \int_{-\infty}^\infty t_2 (U_{jj} * U_{kk})(t_2) \hat{\varphi}(t_2) dt_2 dt_1 \\
+ \frac{i\kappa_4}{nb_n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{(U_{jj} * U_{kk})(t_1)x \mathbb{E}\{e_n(x)\}\} \int_{-\infty}^\infty t_2 (U_{jj} * U_{kk})(t_2) \hat{\varphi}(t_2) dt_2 dt_1 \\
+ T_{31}'(x,t) \\
=: T_{311} + T_{312} + T_{313} + T_{314}'(x,t). \] (3.124)

where \( T_{31}' \) comes from the diagonal terms, so \( |T_{31}'| \leq \sqrt{\frac{n}{\kappa_4}} C_4(\sqrt{b_n}/nx,t) \). Then

\[ T_{311} = \frac{\kappa_4\sqrt{n}}{\sqrt{b_n}} \int_0^t \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \mathbb{E}\{v_n(t_1 - t_2, t_3 - t_4) e_n(x)\} dt_4 dt_3 dt_2 dt_1, \] (3.125)

\[ T_{312} = i\kappa_4 x \int_0^t \int_{-\infty}^\infty \int_0^{t_1} \int_0^{t_2} t_2 \mathbb{E}\{v_n(t_3, t_4, t_1 - t_3, t_2 - t_4) e_n(x)\} \hat{\varphi}(t_2) dt_4 dt_3 dt_2 dt_1, \] (3.126)

\[ T_{313} = i\kappa_4 x Z_n(x) \int_0^t \int_{-\infty}^\infty \int_0^{t_1} \int_0^{t_2} t_2 \bar{v}_n(t_3, t_4, t_1 - t_3, t_2 - t_4) \hat{\varphi}(t_2) dt_4 dt_3 dt_2 dt_1. \] (3.127)

Moreover, by Lemma 3.8

\[ T_{313} = 2i\kappa_4 x Z_n(x) \int_0^t \int_{-\infty}^\infty \int_0^{t_1} \int_0^{t_2} t_2 \bar{v}_n(t_3) \bar{v}_n(t_4) \bar{v}_n(t_1 - t_3) \bar{v}_n(t_2 - t_4) \hat{\varphi}(t_2) dt_4 dt_3 dt_2 dt_1 + \tau_n(x,t) \]

\[ = 2i\kappa_4 x Z_n(x) \int_0^t \int_{-\infty}^\infty \int_0^{t_2} \bar{v}_n(t_1) \bar{v}_n(t_2) \hat{\varphi}(t_2) dt_2 dt_1 + \tau_n(x,t), \] (3.128)

where
\[ \tau_n(x,t) = i\kappa_4 x Z_n(x) \int_0^t \int_{-\infty}^\infty \int_0^{t_1} \int_0^{t_2} t_2 h(t_3, t_4, t_1 - t_3, t_2 - t_4) \hat{\varphi}(t_2) dt_4 dt_3 dt_2 dt_1. \]
Thus,
\[
|\tau_n(x,t)| \leq |\kappa_4 x Z_n(x)| \int_0^t \int_{-\infty}^\infty \left| \int_{t_1}^{t_2} t_2 h(t_3, t_4, t_1 - t_3, t_2 - t_4) \bar{\phi}(t_2) dt_4 \right| dt_3 dt_2 dt_1 
\leq \frac{|x|}{\sqrt{b_n}} C(t^4 + 1).
\] (3.129)

Since \(|e_n(x)| \leq 2,
\[
|E\{v_n(s_1, s_2, s_3, s_4)e_n(x)\}| \leq 2E\{v_n(s_1, s_2, s_3, s_4)\} \leq \frac{2}{mb_n} \sum_{(j,k) \in I_n} E\{|(U_{jj}(s_1)U_{jk}(s_3)U_{kk}(s_4))\}^2\}
\leq \frac{4}{mb_n} \sum_{(j,k) \in I_n} E\{|(U_{jj}(s_1)U_{jj}(s_2))\}^2\} + E\{|(U_{kk}(s_3)U_{kk}(s_4))\}^2\}.
\] (3.130)

Also,
\[
\text{Var}\{U_{jj}(s_1)U_{jj}(s_2)\} \leq \frac{4}{mb_n} \sum_{(p,q) \in I_n^x} E\{|U_{jp} * U_{jq}(s_1)U_{jj}(s_2) + U_{jj}(s_1)U_{jp} * U_{jq}(s_2)\}^2\}
\leq \frac{8}{mb_n} \sum_{(p,q) \in I_n} E\{|U_{jp} * U_{jq}(s_1)U_{jj}(s_2)\}^2\} + E\{|U_{jj}(s_1)U_{jp} * U_{jq}(s_2)\}^2\}\text{(3.131)}
\]

and
\[
\sum_{(p,q) \in I_n} E\{|U_{jp} * U_{jq}(s_1)U_{jj}(s_2)\}^2\} = E\{\sum_{(p,q) \in I_n} \int_0^\infty U_{jp}(s)U_{jq}(s_1 - s)U_{jj}(s_2) ds^2\}
\leq |s_1| E\{\sum_{(p,q) \in I_n} \int_0^{s_1} |U_{jp}(s)U_{jq}(s_1 - s)|^2 ds\} \leq s_1^2.\] (3.132)

Thus,
\[
\text{Var}\{U_{jj}(s_1)U_{jj}(s_2)\} \leq \frac{8(s_1^2 + s_2^2)}{mb_n},\] (3.133)

and we obtain
\[
|E\{v_n(s_1, s_2, s_3, s_4)e_n(x)\}| \leq 12 \sqrt{\frac{8(s_1^2 + s_2^2)}{mb_n}} + 12 \sqrt{\frac{8(s_3^2 + s_4^2)}{mb_n}} = \frac{C}{\sqrt{mb_n}} (\sqrt{s_1^2 + s_2^2} + \sqrt{s_3^2 + s_4^2}).\] (3.134)

So
\[
|T_{311}| \leq C|\kappa_4| \sqrt{n} \frac{1}{\sqrt{mb_n}} \int_0^t \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \sqrt{(t_1 - t_2)^2 + (t_2 - t_3)^2} + \sqrt{(t_3 - t_4)^2} + t_2^2 dt_4 dt_3 dt_2 dt_1 \leq \frac{C \sqrt{n}}{b_n} |t|^5,
\] (3.135)

and
\[
|T_{312}| \leq C|\kappa_4| \sqrt{n} \frac{1}{\sqrt{mb_n}} \int_0^t \int_{-\infty}^t \int_0^{t_1} \int_0^{t_2} |t_2| (\sqrt{t_3^2 + t_4^2} + \sqrt{(t_1 - t_3)^2 + (t_2 - t_4)^2}) \bar{\phi}(t_2) dt_4 dt_3 dt_2 dt_1 \leq \frac{|x|}{\sqrt{b_n}} C(|t|^3 + 1).
\] (3.136)

Since \(n/b_n^2 \to 0\), we observe that \(\delta = T_{311} + T_{312} + T_n + T_{31}'\) goes to zero uniformly on any bounded subset of \(\{t \geq 0, x \in \mathbb{R}\}\). This finishes the proof of Lemma 3.8.
Now, we are ready to finish the proof of Proposition 3.9. Indeed,
\[
T_3 = T_{31} + T_{32} + T_3'
= 2i\kappa_4 x Z_n(x) \int_0^t \int_{-\infty}^t t_2 \bar{v}_n * \bar{v}_n(t_1) \bar{v}_n * \bar{v}_n(t_2) \hat{f}_2 dt_2 dt_1 + \delta_n(x,t) + T_{32} + T_3'.
\]
(3.137)

The statement of Proposition 3.6 now follows from (3.105), (3.113), and (3.123). \(\square\)

To finish the proof of Proposition 3.1, we observe that the equation (3.28) follows from (3.46), (3.49), and Propositions 3.3-3.6. Proposition 3.1 is proven. \(\square\)

### 3.5 The limit of \(A_n\)

In this subsection, we study the limit of \(A_n(t)\) as \(n \to \infty\). This, in turn, will allow us to study the limiting behavior of \(Y_n(x,t)\). The main result of subsection is the following proposition.

**Proposition 3.9.** Let \(A_n(t)\) be as defined in (3.29). Then the limit of \(A_n(t)\) as \(n \to \infty\) exists and
\[
A(t) := \lim_{n \to \infty} A_n(t)
= -2\sigma^2 \int_0^t \frac{1}{8\pi^2 \sigma^2} \int_{-2\sqrt{t} \sigma}^{2\sqrt{t} \sigma} \int_{-2\sqrt{x} \sigma}^{2\sqrt{x} \sigma} e^{it_1 x} \varphi'(y) \sqrt{8\sigma^2 - x^2} \sqrt{8\sigma^2 - y^2} F_{\sigma}(x,y) 1_{(x \neq y)} dx dy dt_1,
\]
where for \(x \neq y\)
\[
F_{\sigma}(x,y) = \frac{\pi}{2\sigma^2} \sum_{k=0}^{\infty} U_k(x) U_k(y) \gamma_k
= \int_{-\infty}^{\infty} -\frac{\sin^3 s}{s^2} - \frac{\sin s}{s^2} ds.
\]

**Proof.** We recall that \(A_n(t)\) is defined in (3.29) as \(A_n(t) = -\frac{2\sigma^2}{n} \int_0^t \sum_{(j,k) \in I_n} \mathbb{E}\{U_{jk}(t_1) \varphi'(M_{jk})\} dt_1\). In the full Wigner matrix case, one has \(A_n = -2\sigma^2 \int_0^t \mathbb{E}\{e^{it_1 M} \varphi(M)\} dt_1\), and the limiting behavior of \(A_n\) immediately follows from the Wigner semi-circle law. In the band matrix case, there are additional difficulties due to the fact that the summation in the formula for \(A_n\) is restricted to the band entries, i.e. to \((j,k) \in I_n\).

We start with the definition of a bilinear form on \(C_b(\mathbb{R})\), the space of bounded continuous functions on \(\mathbb{R}\).

**Definition 3.10.** Let \(f, g \in C_b(\mathbb{R})\). Define
\[
< f, g > = n^{-1} \mathbb{E}\{ \sum_{(j,k) \in I_n} f(M_{jk}) g(M_{jk})\}.
\]

It follows from the above definition that
\[
A_n(t) = -2\sigma^2 \int_0^t < e^{it_1 x}, \varphi'(x) > dt_1.
\]
(3.142)

The bilinear form (3.141) is an inner product (perhaps, degenerate).

1. \(< f, f > \geq 0\),
2. \(< f, g > = < g, f >\),
3. \(< f, g_1+g_2 > = < f, g_1 > + < f, g_2 >\), \(< k f, g > = k < f, g >\), \(k \in \mathbb{R}\),
4. (Cauchy-Schwarz Inequality) \(|< f, g >| \leq < f, f >^{1/2} < g, g >^{1/2}\).

The proof of Proposition 3.9 relies on two auxiliary lemmas.
Lemma 3.11. For all $f, g \in C_b(\mathbb{R})$ the limit
\[
<f, g> := \lim_{n \to \infty} <f, g>_n
\] (3.143)
exists.

Proof. We start with monomials. While monomials do not belong to $C_b(\mathbb{R})$, the expression still makes sense since all moments of the matrix entries of $M$ are finite. For $l, m \in \mathbb{N}$, consider $f(x) = x^l$, $g(x) = x^m$. Then
\[
<x^l, x^m>_n = \frac{1}{n b_n^{(l+m)/2}} \sum_{(i_0, i_1), \ldots, (i_{l+m-1}, i_l) \in I_n} \mathbb{E} \{ W_{i_0, i_1, \ldots, W_{i_l, i_{l+m-1}, i_l}} \}. \tag{3.144}
\]

Let us fix $i_0 \in \{1, \ldots, n\}$. For $k = 1, \ldots, l + m$, define
\[
x_k = \begin{cases} i_k - i_{k-1} & \text{if } |i_k - i_{k-1}| \leq b_n, \\ i_k - i_{k-1} - n & \text{if } |i_k - i_{k-1}| > b_n, \\ n + (i_k - i_{k-1}) & \text{if } |i_k - i_{k-1}| < -b_n, \end{cases}
\] (3.145)
where $i_{l+m} = i_0$. Since $l, m$ are fixed and $n/b_n \to \infty$, for sufficiently large $n$ the restriction $(i_0, i_1), \ldots, (i_{l+m-1}, i_0), (i_1, i_0) \in I_n$ is equivalent to $|x_1|, \ldots, |x_{l+m}| \leq b_n$, $x_1 + \ldots + x_{l+m} = 0$, and $|x_1 + \ldots + x_l| \leq b_n$. Therefore, for sufficiently large $n$,
\[
x_l, x^m>_n = \frac{1}{n b_n^{(l+m)/2}} \sum_{i_0=1}^n \sum_{i_1, \ldots, i_{l+m-1}} \mathbb{E} \{ W_{i_0, i_1, \ldots, W_{i_l, i_{l+m-1}, i_l}} \}. \tag{3.146}
\]

Each $(i_0, i_1, \ldots, i_{l+m-1}, i_0)$ is a closed path such that the distance between the endpoints of each edge is bounded by $b_n$ and, in addition, the distance between $i_0$ and $i_l$ is also bounded by $b_n$. If $l + m$ is odd, one can show that $<x^l, x^m>_n \to 0$ using power counting and independence of matrix entries. The proof is very similar to the combinatorial argument in the proof of the Wigner semicircle law and is left to the reader.

Now consider the case when $l + m$ is even. Without loss of generality we can assume that $l \leq m$. As in the proof of the semicircle law, only the paths where every edge appears exactly twice contribute to the limit. For each such path,
\[
\mathbb{E} \{ W_{i_0, i_1, \ldots, W_{i_l, i_{l+m-1}, i_0}} \} = \sigma^{l+m}.
\]
Moreover, each such $(i_0, i_1, \ldots, i_{l+m-1}, i_0)$ corresponds to a Dyck path of length $l + m$ (see e.g. [1]). Recall that a Dyck path $(s(0), \ldots, s(l + m))$ of length $m + l$ satisfies
\[
s(0) = s(l + m) = 0, \ s(1), \ldots, s(l + m - 1) \geq 0, \ \text{and } |s(t + 1) - s(t)| = 1, \ i_0, \ldots, l + m - 1.
\]
Specifically, $s(t + 1) - s(t) = 1$ if the non-oriented edge $(i_t, i_{t+1})$ appears in $(i_0, i_1, \ldots, i_{l+m-1}, i_0)$ for the first time and $s(t + 1) - s(t) = -1$ if the edge $(i_t, i_{t+1})$ appears in $(i_0, i_1, \ldots, i_{l+m-1}, i_0)$ for the second time.

If one removes in (3.144) the condition that $(i_t, i_0) \in I_n$ then the l.h.s. in (3.144) becomes $\frac{1}{n} Tr M^{l+m}$ and each Dyck path gives equal contribution in the limit $n \to \infty$. However, we have to take into account the condition $(i_t, i_0) \in I_n$. As a result, the combinatorial analysis becomes more involved. Suppose $s(l) = k$, $0 \leq k \leq l$. Then during the first $l$ steps of the path $(i_0, i_1, \ldots, i_{l+m-1}, i_0)$, $(l - k)/2$ edges appear twice and $k$ edges appear only once. For each of the edges appearing twice, the corresponding two numbers $x_i$ have the same absolute value but differ in sign. The remaining $k$ numbers $x_i$ will be renumerated (in the order of their appearance) by $y_1, y_2, \ldots, y_k$. One obtains
\[
<x^l, x^m>_n = \frac{\sigma^{l+m}}{b_n^{l+m}} \sum_{k=0}^{l} \# \text{Dyck paths of length } l + m \text{ with } s(l) = k \times \# \text{of integers } |y_1| \leq b_n, \ldots, |y_k| \leq b_n, \ldots, |y_{l+m}| \leq b_n, |y_1 + \ldots + y_k| \leq b_n + O(b_n^{-1}). \tag{3.147}
\]
Therefore, \(< x^l, x^m > = \lim_{n \to \infty} < x^l, x^m >_n\) exists, and

\[
< x^l, x^m > = (\sqrt{2\sigma})^{l+m} \sum_{k=0}^{l} \nabla \left(\text{number of Dyck paths of length } l + m \text{ with } s(l) = k\right) \times \text{Vol} \left\{ t_1 \leq 1/2, t_2 \leq 1/2, \ldots, t_{l+m}/2 \leq 1/2, |t_1 + t_2 + \ldots + t_k| \leq 1/2 \right\}. 
\] (3.148)

The number of Dyck paths with \(s(l) = k\) is

\[
\left(\frac{l}{l+k} - \frac{l}{l+k+2}\right) \frac{(m)}{m+k+2} = \frac{(k+1)^2}{(l+1)(m+1)} \left(\frac{l+1}{l+k+2}\right) \left(\frac{m+1}{m+k+2}\right). 
\] (3.149)

Let \(T_1, \ldots, T_{l+m}/2\) be i.i.d random variables uniformly distributed on \([-1/2, 1/2]\). Then

\[
\text{Vol} \left\{ t_1 \leq 1/2, \ldots, t_k \leq 1/2, |t_1 + t_2 + \ldots + t_k| \leq 1/2 \right\} = \mathbb{P}(T_1 + T_2 + \ldots + T_k \leq 1/2). 
\] (3.150)

Let \(S_k = T_1 + \ldots + T_k\). Then the characteristic function of \(S_k\) is

\[
\mathbb{E}\{e^{ixS_k}\} = (\mathbb{E}\{e^{ixY_i}\})^k = \left(\frac{\sin x/2}{x/2}\right)^k. 
\] (3.151)

Hence, the density function of \(S_k\) is given by

\[
f_k(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixs} \left(\frac{\sin x/2}{x/2}\right)^k dx. 
\] (3.152)

Define \(\gamma_k := \mathbb{P}(|S_k| \leq 1/2)\). Then

\[
\gamma_k = \int_{-1/2}^{1/2} f_k(s) ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin x/2}{x/2}\right)^{k+1} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^{k+1} dx = f_{k+1}(0). 
\] (3.153)

The exact formula for \(\gamma_k\) is well known (see e.g. [17]):

\[
\gamma_k = \left\{ \begin{array}{ll}
[(2t)!]^{-1} \sum_{s=0}^{t} (-1)^s (2t+1) (t-s+1/2)^{2t}, & \text{if } k = 2t, \\
[(2t+1)!]^{-1} \sum_{s=0}^{t} (-1)^s (2t+2) (t-s+1)^{2t+1}, & \text{if } k = 2t+1.
\end{array} \right. 
\] (3.154)

Therefore, we conclude that

\[
< x^l, x^m > = (\sqrt{2\sigma})^{l+m} C_{l,m}, 
\] (3.155)

where \(C_{l,m}\) is defined in the following way. For \(l+m\) is odd, \(C_{l,m} = 0\). For \(l+m\) is even, \(l \leq m\),

\[
C_{l,m} = \left\{ \begin{array}{ll}
[(l+1)(m+1)]^{-1} \sum_{k=0}^{l/2} (2k+1)^2 (l+1)(m+1) \gamma_{2k}, & \text{if } l \text{ is even,} \\
[(l+1)(m+1)]^{-1} \sum_{k=0}^{(l-1)/2} (2k+2)^2 (l+1)(m+1) \gamma_{2k+1}, & \text{if } l \text{ is odd.}
\end{array} \right. 
\] (3.156)

For \(l+m\) is even, \(l > m\), \(C_{l,m} = C_{m,l}\). It follows from the definition that \(0 \leq C_{l,m} \leq C_{l+m,}\), where \(C_s = \frac{2^{2s}}{s+1}\) is the Catalan number.

If \(f, g\) are polynomials, \(f(x) = \sum_{i=0}^{p} a_i x^i, g(x) = \sum_{j=0}^{q} b_j x^j\), then by linearity

\[
< f, g > = \sum_{i=0}^{p} \sum_{j=0}^{q} a_i b_j (\sqrt{2\sigma})^{i+j} C_{i,j}. 
\] (3.157)

Thus, the result of Lemma 3.11 holds when \(f\) and \(g\) are arbitrary polynomials.
For general bounded continuous functions $f, g$, we will show that \( \{< f, g >\} \) is a Cauchy sequence. To this end, we choose a sufficiently large $B$ independent of $n$ (it will be enough to take $B = 4\sigma^2 + 1$). Fix $\delta > 0$. By the Stone-Weierstrass theorem, there exist polynomials $f_\delta, g_\delta$ such that
\[
\sup_{x:|x| \leq B+1} |f(x) - f_\delta(x)| \leq \delta, \quad \sup_{x:|x| \leq B+1} |g(x) - g_\delta(x)| \leq \delta. \tag{3.158}
\]
Let $h$ be an infinitely differentiable function such that $|h| \leq 1$, $h(x) = 1$ for $|x| \leq B$, $h(x) = 0$ for $|x| \geq B + 1$. We write
\[
< f, g > = < f - f_\delta h, g - g_\delta h > + < f - f_\delta h, g_\delta h > + < f_\delta h, g - g_\delta h > + < f_\delta h, g_\delta h >. \tag{3.159}
\]
Below we show that the first three terms on the r.h.s. of (3.159) are small provided $\delta$ is small. It follows from (3.111) that
\[
< (f - f_\delta)h, (f - f_\delta)h > \leq \delta^2, \tag{3.160}
\]
\[
< (g - g_\delta)h, (g - g_\delta)h > \leq \delta^2. \tag{3.161}
\]
Since $f, g$ are bounded on $\mathbb{R}$ and $f_\delta, g_\delta$ are polynomials, there exists sufficiently large $N \in \mathbb{Z}_+$, such that $(f - f_\delta)^2(1 - h) \leq x^{2N}(1 - h)$, and $(g - g_\delta)^2(1 - h) \leq x^{2N}(1 - h)$. Then
\[
< (f - f_\delta)(1 - h), (f - f_\delta)(1 - h) > \leq x^{2N}(1 - h), x^{2N}(1 - h) > \leq \frac{1}{nB^{2N}} \text{Tr} M^6 N \leq \delta^2 \tag{3.162}
\]
for sufficiently large $n$, where the last inequality follows from the semicircle law provided $N$ is chosen so that $\frac{\sqrt{2\sigma}^N}{B^{2N}} < \delta^2$. In a similar fashion,
\[
< (g - g_\delta)(1 - h), (g - g_\delta)(1 - h) > \leq \delta^2, \tag{3.163}
\]
\[
< f(1 - h), f(1 - h) > \leq \delta^2, \quad < g(1 - h), g(1 - h) > \leq \delta^2. \tag{3.164}
\]
for sufficiently large $n$. The bounds (3.160-3.164) imply
\[
< f - f_\delta h, f - f_\delta h > \leq \text{const}\delta^2, \tag{3.165}
\]
\[
< g - g_\delta h, g - g_\delta h > \leq \text{const}\delta^2. \tag{3.166}
\]
Now, applying (3.165-3.166) and the Cauchy-Schwarz inequality, we obtain
\[
|< f, g > - < f_\delta h, g_\delta h >| \leq \text{Const}\delta, \tag{3.167}
\]
\[
|< f_\delta h, g_\delta h > - < f_\delta h, g_\delta h >| \leq \text{Const}\delta,
\]
and, as a result,
\[
|< f, g > - < f_\delta h, g_\delta h >| \leq 2\text{Const}\delta. \tag{3.170}
\]
Therefore $< f, g >$ is a Cauchy sequence and $< f, g >$ exists.

In the next lemma, we diagonalize the bilinear form $< f, g >$.

**Lemma 3.12.** Let $\{U_n(x)\}$ be the (rescaled) Chebyshev polynomials of the second kind on $[-2\sqrt{2}\sigma, 2\sqrt{2}\sigma]$,
\[
U_n^\sigma(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \left(\frac{x}{2\sqrt{2}\sigma}\right)^{n-2k}. \tag{3.168}
\]
Then $\{U_n^\sigma(x)\}_{n \geq 0}$ are orthogonal with respect to the bilinear form (3.155), i.e.
\[
< U_n, U_m > = \delta_{nm}\gamma_n, \tag{3.169}
\]
where $\gamma_n$ is given by (3.154).
Remark 3.13. Note that $< f_1, g_1 > = < f_2, g_2 >$ if $f_1 = f_2$ and $g_1 = g_2$ in a neighborhood of $[-2\sqrt{\sigma}, 2\sqrt{\sigma}]$. Thus, one can reformulate Lemma 3.12 in such a way that $\{hU_n(\sigma)\}_{n \geq 0}$ are orthogonal with respect to the bilinear form $\langle \cdot, \cdot \rangle_{T_2(n)}$.

Remark 3.14.

Recall that the rescaled Chebyshev polynomials are orthonormal with respect to the Wigner semicircle law, i.e.

$$\int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} U_n^\sigma(x)U_m^\sigma(x) \frac{1}{4\pi \sigma^2} \sqrt{8\sigma^2 - x^2} = \delta_{nm}. \quad (3.170)$$

Also,

$$U_n^\sigma(x) = \frac{\sin((n+1)\theta)}{\sin \theta}, \quad x = 2\sqrt{\sigma} \cos \theta. \quad (3.171)$$

When it does not lead to ambiguity, we will omit the super-index in the notation for the rescaled Chebyshev polynomials (alternatively, the reader can assume that $2\sqrt{\sigma} = 1$).

Proof. Since $< x^l, x^m > = 0$ if $l + m$ is odd, it follows by linearity that

$$< U_n, U_m > = 0, \quad \text{if } n + m \text{ is odd}. \quad (3.172)$$

We are left to compute $< U_{2n}, U_{2m} >$ and $< U_{2n+1}, U_{2m+1} >$. We first compute $< x^{2l}, U_{2n} >$ and $< x^{2l+1}, U_{2n+1} >$ for $l = 0, 1, \ldots, n$. One has

$$< x^{2l}, U_{2n} > = (\sqrt{\sigma})^{2l} \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} C_{2l, 2n-2k}$$

$$= \frac{(\sqrt{\sigma})^{2l}}{2l+1} \sum_{k=0}^{n} \sum_{t=0}^{l} \binom{2n-k}{k} \binom{2l+1}{l-t} \gamma_{2t} \left[ \sum_{k=n-t}^{n} \frac{(-1)^k (2n-k)!}{k!(n-k-t)!(n-k-t+1)!} \right] \gamma_{2t}, \quad (3.173)$$

and

$$< x^{2l+1}, U_{2n+1} > = (\sqrt{\sigma})^{2l+1} \sum_{k=0}^{n} (-1)^k \binom{2n+1-k}{k} C_{2l+1, 2n+1-2k}$$

$$= \frac{(\sqrt{\sigma})^{2l+1}}{2l+2} \sum_{k=0}^{n} \sum_{t=0}^{l} \binom{2n+1-k}{k} \binom{2l+2}{l-t} \gamma_{2t+1} \left[ \sum_{k=n-t}^{n} \frac{(-1)^k (2n+1-k)!}{k!(n-k-t)!(n-k-t+1)!} \right] \gamma_{2t+1}. \quad (3.174)$$

Denote

$$G_l(n, t) = \sum_{k=0}^{n-t} \frac{(-1)^k (2n-k)!}{k!(n-k-t)!(n-k-t+1)!}. \quad (3.175)$$
\[ G_2(n, t) = \sum_{k=0}^{n-t} \frac{(-1)^k (2n+1-k)!}{k!(n-k-t)!(n-k+t+2)!}. \]  

(3.176)

Then
\[ <x^{2l}, U_{2n}> = \frac{(\sqrt{2\sigma})^{2l}}{2l+1} \sum_{l=0}^{l} (2l+1)^2 \binom{2l+1}{l-t} G_1(n, t) \gamma_{2l}, \]  

(3.177)

\[ <x^{2l+1}, U_{2n+1}> = \frac{(\sqrt{2\sigma})^{2l+1}}{2l+2} \sum_{l=0}^{l} (2l+2)^2 \binom{2l+2}{l-t} G_2(n, t) \gamma_{2l+1}. \]  

(3.178)

It follows from (3.175-3.176) that
\[ G_1(n, t) = \frac{(2n)!}{(n-t)!(n+t+1)!} {}_2F_1 \left( -n-t, -(n+t+1) ; -2n \right), \]  

(3.179)

\[ G_2(n, t) = \frac{(2n+1)!}{(n-t)!(n+t+2)!} {}_2F_1 \left( -n-t, -(n+t+2) ; -2n-1 \right), \]  

(3.180)

where \( {}_2F_1 \) is a hypergeometric function. By the Chu-Vandermonde identity (see e.g. [3]), we have
\[ {}_2F_1 \left( -n-t, -(n+t+1) ; -2n \right) = \frac{(-n+t+1)_{n-t}}{(-2n)_{n-t}}, \]  

(3.181)

\[ {}_2F_1 \left( -n-t, -(n+t+2) ; -2n-1 \right) = \frac{(-n+t+1)_{n-t}}{(-2n-1)_{n-t}}, \]  

(3.182)

where \((a)_n = a(a+1) \cdots (a+n-1)\). Since
\[ (-n+t+1)_{n-t} = \begin{cases} 0 & \text{if } t = 0, 1, \ldots, n-1 \\ 1 & \text{if } t = n, \end{cases} \]  

(3.183)

we obtain
\[ G_1(n, t) = 0, \ G_2(n, t) = 0, \text{ for } t = 0, 1, \ldots, n-1, \]  

(3.184)

and
\[ G_1(n, n) = \frac{1}{2n+1}, \ G_2(n, n) = \frac{1}{2n+2}. \]  

(3.185)

Therefore, for \( l = 0, 1, \ldots, n-1 \), each term at the r.h.s. of (3.177) is zero, and
\[ <x^{2n}, U_{2n}> = \frac{(\sqrt{2\sigma})^{2n}}{2n+1} \binom{2n+1}{0} G_1(n, n) \gamma_{2n} = (\sqrt{2\sigma})^{2n} \gamma_{2n}, \]  

(3.186)

\[ <x^{2n+1}, U_{2n+1}> = \frac{(\sqrt{2\sigma})^{2n+1}}{2n+2} \binom{2n+2}{0} G_2(n, n) \gamma_{2n+1} = (\sqrt{2\sigma})^{2n+1} \gamma_{2n+1}. \]  

(3.187)

Hence, for \( m < n \),
\[ <U_{2m}, U_{2n}> = 0, <U_{2m+1}, U_{2n+1}> = 0, \]  

(3.188)

and
\[ <U_{2n}, U_{2n}> = \left( \frac{x}{\sqrt{2\sigma}} \right)^{2n}, U_{2n} > = \gamma_{2n}, \]  

(3.189)

\[ <U_{2n+1}, U_{2n+1}> = \left( \frac{x}{\sqrt{2\sigma}} \right)^{2n+1}, U_{2n+1} > = \gamma_{2n+1}. \]  

(3.190)

Combining (3.188), (3.172), (3.189) and (3.190), we complete the proof of Lemma 3.12. \( \square \)
Now, we are ready to finish the proof of Proposition 3.9. Let \( f, g \in C_b(\mathbb{R}) \), and

\[
f_k = \frac{1}{4\pi^2} \int_{-2\sqrt{2} \sigma}^{2\sqrt{2} \sigma} f(x) U_k(x) \sqrt{8\sigma^2 - x^2} dx, \quad g_k = \frac{1}{4\pi^2} \int_{-2\sqrt{2} \sigma}^{2\sqrt{2} \sigma} g(x) U_k(x) \sqrt{8\sigma^2 - x^2} dx.
\] (3.191)

Then

\[
< f, g > = \sum_{k=0}^{\infty} f_k g_k \gamma_k
\] (3.192)

\[
= \frac{1}{8\pi^3 \sigma^2} \int_{-2\sqrt{2} \sigma}^{2\sqrt{2} \sigma} \int_{-2\sqrt{2} \sigma}^{2\sqrt{2} \sigma} f(x) g(y) \sqrt{8\sigma^2 - x^2} \sqrt{8\sigma^2 - y^2} F_\sigma(x, y) 1_{(x \neq y)} dx dy dt,
\] (3.193)

where, for \( x \neq y \),

\[
F_\sigma(x, y) = \frac{\pi}{2\sigma^2} \sum_{k=0}^{\infty} U_k(x) U_k(y) \gamma_k
\]

\[
= \int_{-\infty}^{\infty} 2\sigma^2 \left(1 - \frac{\sin^2 s}{s^2}\right)^2 - \frac{\sin s}{s} \frac{\sin^3 s}{s^3} xy + \frac{\sin^2 s}{s^2} (x^2 + y^2) ds.
\] (3.194)

Formula (3.192) follows for polynomials from (3.169) and (3.170), and then by continuity, by repeating the arguments at the end of the proof of Lemma 3.11, for general continuous bounded functions. Formula (3.194) is a straightforward consequence of the Fourier analysis. It follows from (3.171) that the r.h.s. of (3.192) can be rewritten as

\[
< f, g > = -2 \sum_{l \neq 0} \tilde{\alpha}_l \tilde{\beta}_l \gamma_{|l|-1},
\] (3.195)

where

\[
\alpha(\theta) = f(2\sqrt{2} \sigma \cos \theta), \quad \beta(\theta) = g(2\sqrt{2} \sigma \cos \theta),
\] (3.196)

\[
\tilde{\alpha}_l = \frac{1}{2\pi} \int_0^{2\pi} \alpha(\theta) e^{-i l \theta} d\theta, \quad \tilde{\beta}_l = \frac{1}{2\pi} \int_0^{2\pi} \beta(\theta) e^{-i l \theta} d\theta.
\] (3.197)

In particular, the trigonometric series \( \sum_{l \neq 0} \gamma_{|l|-1} e^{i l \theta} \) represents an \( L^1 \) function \( h \) which has \( O(\theta^{-1/2}) \) singularity near the origin. The convergence is pointwise for all \( \theta \neq 0 \),

\[
h(\theta) = \sum_{l \neq 0} \gamma_{|l|-1} e^{i l \theta}, \quad \theta \neq 0,
\]

\[
\tilde{h}_l = \gamma_{|l|-1}, \quad \text{if } l \neq 0, \quad \tilde{h}_0 = 0.
\]

The convolution of \( \beta \) and \( h \) is then a continuous function on the unit circle, and one can rewrite (3.195) in the integral form by applying the Parseval’s theorem.

Finally, it follows from (3.20) and (3.183) that the limit of \( A_n(x) \) exists and equals

\[
A(t) = -2\sigma^2 \int_0^t < e^{i t_1 x}, \varphi > dt_1
\] (3.198)

with

\[
< e^{i t_1 x}, \varphi > = \frac{1}{8\pi^3 \sigma^2} \int_{-2\sqrt{2} \sigma}^{2\sqrt{2} \sigma} \int_{-2\sqrt{2} \sigma}^{2\sqrt{2} \sigma} e^{i t_1 x} \varphi(y) \sqrt{8\sigma^2 - x^2} \sqrt{8\sigma^2 - y^2} F_\sigma(x, y) 1_{(x \neq y)} dx dy.
\] (3.199)
3.6 Variance

The rest of the proof of Theorem 2.1 follows the steps in [26]. Using pre-compactness of \( \{Y_n, Z_n\}_{n \geq 1} \), we consider a converging subsequence. Our goal is to show that the limit is unique. Let

\[
Y_{n_j}(x, t) \to Y(x, t), \quad Z_{n_j}(x) \to Z(x).
\]

(3.200)

By Wigner semicircle law,

\[
\tilde{v}_n(t) \to \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \frac{e^{it\mu}}{4\pi\sigma^2} \sqrt{8\sigma^2 - y^2} dy := v(t).
\]

(3.201)

So the limit of \( Y_n(x, t) \) satisfies the following equation:

\[
Y(x, t) + 4\sigma^2 \int_{0}^{t} \int_{0}^{t_1} v(t_1 - t_2) Y(x, t_2) dt_2 dt_1 = xZ(x)A(t) + 2i\kappa_4 xZ(x) \int_{0}^{t} v(t_1) dt_1 \int_{-\infty}^{\infty} t_2 v * v(t_2) \hat{\varphi}(t_2) dt_2,
\]

(3.202)

where \( v * v \) is defined in (3.21),

\[
v * v(t) = -\frac{i}{8\pi\sigma^4} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} e^{it\mu} \sqrt{8\sigma^2 - \mu^2} d\mu.
\]

(3.203)

Let

\[
B = \int_{-\infty}^{\infty} t_2 v * v(t_2) \hat{\varphi}(t_2) dt_2 = \frac{1}{4\pi\sigma^4} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \varphi(\mu) \frac{4\sigma^2 - \mu^2}{\sqrt{8\sigma^2 - \mu^2}} d\mu.
\]

(3.204)

As in [26] (see formulas (2.82)-(2.86) and Proposition 2.1 there), we can solve (3.202) to obtain

\[
Y(x, t) = -\frac{2\sigma^2 xZ(x)}{\pi} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \int_{-\infty}^{t} e^{i\lambda(t-t_1)} \frac{\varphi(t) e^{i\lambda(t-t_1)}}{\sqrt{8\sigma^2 - \lambda^2}} dt_1 d\lambda + \frac{i\kappa_4 xZ(x) B}{4\pi\sigma^4} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} e^{it\lambda} \frac{(4\sigma^2 - \lambda^2)}{\sqrt{8\sigma^2 - \lambda^2}} d\lambda.
\]

(3.205)

It then follows from (3.20) that

\[
Z'(x) = -\frac{2i\sigma^2 xZ(x)}{\pi} \int_{-\infty}^{\infty} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \int_{-\infty}^{t} \hat{\varphi}(t) e^{i\lambda(t-t_1)} \frac{\varphi(t) e^{i\lambda(t-t_1)}}{\sqrt{8\sigma^2 - \lambda^2}} < e^{it_1 x}, \varphi' > dt_1 d\lambda dt
\]

\[
-\kappa_4 xZ(x) \left( \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \varphi(\lambda) \frac{(4\sigma^2 - \lambda^2)}{\sqrt{8\sigma^2 - \lambda^2}} d\lambda \right)^2.
\]

(3.206)

One can rewrite the last formula in the form (3.5) with

\[
Var_{\text{band}}[\varphi] = \frac{2\sigma^2 xZ(x)}{\pi} \int_{-\infty}^{\infty} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \int_{-\infty}^{t} \hat{\varphi}(t) e^{i\lambda(t-t_1)} \frac{\varphi(t) e^{i\lambda(t-t_1)}}{\sqrt{8\sigma^2 - \lambda^2}} < e^{it_1 x}, \varphi' > dt_1 d\lambda dt
\]

\[
+ \frac{\kappa_4}{16\pi^2\sigma^8} \left( \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \varphi(\lambda) \frac{(4\sigma^2 - \lambda^2)}{\sqrt{8\sigma^2 - \lambda^2}} d\lambda \right)^2.
\]

(3.207)

Plugging in (3.199), finally we have

\[
Var_{\text{band}}[\varphi] = \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \frac{\varphi(x) - \varphi(y)}{4\pi^4(x - \lambda)(\sqrt{8\sigma^2 - \lambda^2})} \frac{\varphi(y) \sqrt{8\sigma^2 - x^2} \sqrt{8\sigma^2 - y^2}}{\varphi(\lambda) \sqrt{8\sigma^2 - \lambda^2}} F(x, y) 1_{\{x \neq y\}} dx dy d\lambda
\]

\[
+ \frac{\kappa_4}{16\pi^2\sigma^8} \left( \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \varphi(\lambda) \frac{(4\sigma^2 - \lambda^2)}{\sqrt{8\sigma^2 - \lambda^2}} d\lambda \right)^2.
\]

(3.208)
This finishes the proof of Theorem 2.1 for test functions satisfying (3.1).

Now, let \( \varphi \) be an arbitrary function with bounded continuous derivative. It follows from Lemma C.2 that we can assume that \( \varphi \) has compact support inside the interval \([-10\sigma, 10\sigma]\). One then approximates \( \varphi \) in the \( C_1([-10\sigma, 10\sigma]) \) norm by functions satisfying (3.1) and uses the bound (3.10) to control the variance of the error term. Theorem 2.1 is proven.

4 Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. Thus, our goal is to extend the result of Theorem 2.1 to the case of non-i.i.d. entries with uniformly bounded fifth moment. For technical reasons, we require that the fourth cumulant is zero and \( \sqrt{n} \ln n \ll b_n \).

First, we establish two auxiliary lemmas.

4.1

The first lemma is a simple statement about the norm of a sub-matrix of a unitary matrix.

**Lemma 4.1.** Let \( U \) be an \( n \times n \) unitary matrix and \( V \) be any \( k \times k \) block of \( U \). Then

\[
\|V\| \leq 1 \tag{4.1}
\]

**Proof.** Suppose the indices of \( V \) in \( U \) are \((s, s+1, \ldots, s+k-1) \times (t, t+1, \ldots, t+k-1)\). Then

\[
V = P_1 UP_2 \tag{4.2}
\]

where \( P_1 \) is the orthogonal projection onto the subspace spanned by \( e_s, \ldots, e_{s+k-1} \), and \( P_2 \) is the orthogonal projection onto the subspace spanned by \( e_t, \ldots, e_{t+k-1} \). Then

\[
\|V\| \leq \|P_1\|\|U\|\|P_2\| = 1. \tag{4.3}
\]

The second lemma gives an upper bound on the norm of a band matrix built from a unitary matrix.

**Lemma 4.2.** Let \( U \) be an \( n \times n \) unitary matrix. Let \( b \) be a positive integer smaller than \( n/2 \). Denote

\[
I := \{(j,k)| \ j, k = 1, \ldots, n, \ |j-k| \leq b \ or \ n - |j-k| \leq b \}. \tag{4.4}
\]

Then there exist positive constants \( C_1 \) and \( C_2 \), independent from \( n \) and \( b \), such that

\[
\|U^{\text{band}}\| \leq C_1 \ln b + C_2. \tag{4.5}
\]

**Proof.** Define

\[
A = \{U_{jk}, \ |j-k| \leq b : 0 \ otherwise\}_{j,k=1}^n, \tag{4.6}
\]

\[
B = \{U_{jk}, \ n - |j-k| \leq b : 0 \ otherwise\}_{j,k=1}^n. \tag{4.7}
\]

Then

\[
U^{\text{band}} = A + B, \tag{4.8}
\]

and

\[
\|U^{\text{band}}\| \leq \|A\| + \|B\|. \tag{4.9}
\]

Matrix \( B \) can be written as

\[
B = \begin{bmatrix}
0 & 0 & B_1 \\
0 & 0 & 0 \\
B_2 & 0 & 0
\end{bmatrix} \tag{4.10}
\]
where $B_1, B_2$ are $(b+1) \times (b+1)$ matrices and $B_1$ ($B_2$) is a strictly upper (lower) triangular matrix obtained from the corresponding $(b+1) \times (b+1)$ block of $U$ by making all entries below (above) the main diagonal zero. It is known (see e.g. [27]) that if $B_{\text{upper}}$ is an upper triangular matrix constructed in such a way from an $N \times N$ matrix $B$ then $\|B_{\text{upper}}\| \leq O(\log N)\|B\|$. Applying Lemma [11] we obtain

$$\|B_1\|, \|B_2\| \leq \text{Const} \ln(b+1),$$
$$\|B\| \leq \|B_1\| + \|B_2\| \leq 2\text{Const} \ln(b+1).$$

(4.11)

Now we turn our attention to the norm of $A$. Write $n = m \times (b+1) - r$, $0 \leq r \leq b$. Define

$$U' = \left[ \begin{array}{cc} U & 0_{n \times r} \\ 0_{r \times n} & 0_{r \times r} \end{array} \right], A' = \left[ \begin{array}{cc} A & 0_{n \times r} \\ 0_{r \times n} & 0_{r \times r} \end{array} \right].$$

(4.12)

Then $\|U'\| = \|U\| = 1, \|A'\| = \|A\|$ and $A'$ can be written as a block matrix

$$A' = \left[ \begin{array}{cccc} A_{11} & A_{12} & 0 & \ldots \\ A_{21} & A_{22} & A_{23} & \ldots \\ 0 & A_{32} & A_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right].$$

(4.13)

where $A'_{ij}$'s are $(b+1) \times (b+1)$ blocks of $U'$. Moreover, $A'_{ij}$'s ($j \neq k$) are strictly lower or upper triangular submatrices of the corresponding blocks in $U'$. Again, applying the Mathias bound in [27], we have

$$\|A_{jj}\| \leq 1, \quad \|A_{jk}\| \leq \text{Const} \ln(b+1).$$

(4.14)

Let

$$D = \text{Diag}\{A_{11}, \ldots, A_{mm}\}.$$ 

(4.15)

Then

$$\|D\| \leq \max_{1 \leq i \leq m} \|A_{ii}\| \leq 1.$$ 

(4.16)

Let

$$A_i = \left[ \begin{array}{cc} 0 & A_{i,i+1} \\ A_{i+1,i} & 0 \end{array} \right], i = 1, \ldots, m-1.$$ 

(4.17)

Then $A_i$'s are $2(b+1) \times 2(b+1)$ matrices, and

$$\|A_i\| \leq \|A_{i,i+1}\| + \|A_{i+1,i}\| = 2\text{Const} \ln(b+1).$$

(4.18)

If $m$ is even, let

$$E = \text{Diag}\{A_1, A_3, \ldots, A_{m-1}\},$$
$$F = \text{Diag}\{0_{1 \times 1}, A_2, A_4, \ldots, A_{m-2}, 0_{1 \times 1}\}.$$ 

If $m$ is odd, let

$$E = \text{Diag}\{A_1, A_3, \ldots, A_{m-2}, 0_{1 \times 1}\},$$
$$F = \text{Diag}\{0_{1 \times 1}, A_2, A_4, \ldots, A_{m-1}\}.$$ 

Then

$$A' = D + E + F,$$

and

$$\|E\|, \|F\| \leq \max_{1 \leq i \leq m} \{\|A_i\|\} = 2\text{Const} \ln(b+1).$$

(4.19)
Therefore, we have
\[ \|A'\| \leq \|D\| + \|E\| + \|F\| = 1 + 4\text{Const} \ln(b + 1) \leq C \ln(b + 1). \]  
\[ (4.20) \]

Therefore,
\[ \|A\| \leq C \ln(b + 1). \]  
\[ (4.21) \]

Finally, (4.9), (4.11), and (4.21) imply (4.5).
\[ \square \]

4.2

Now, we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** Let \( \hat{M} = b_n^{-1/2} \hat{W} \) be a band random real symmetric matrix with independent Gaussian random variables, and \( M \) be an arbitrary band real symmetric random matrix satisfying the conditions in Theorem 2.2. We denote, respectively, by \( \mathcal{M}_n^0[\varphi] \) and \( \mathcal{M}_n^0[\varphi] \) the centered normalized linear eigenvalue statistics of \( \hat{M} \) and \( M \) defined as in (2.2). Since Gaussian distribution satisfies the Poincaré inequality, Theorem 2.1 establishes the Central Limit Theorem for \( \mathcal{M}_n^0[\varphi] \). Thus, it suffices to show that, for every \( x \in \mathbb{R} \),
\[ R_n(x) := \mathbb{E}\{ e^{ix\mathcal{M}_n^0[\varphi]} \} - \mathbb{E}\{ e^{ix\mathcal{M}_n[\varphi]} \} \to 0, \ n \to \infty. \]
\[ (4.22) \]

Let us denote 
\[ e_n(s, x) = \exp\{ (b_n/n)^{1/2} x \text{Tr} \varphi(M(s)^0) \}, \]  
\[ (4.23) \]

where \( M(s) \) is the interpolating matrix \( M(s) = s^{1/2} + (1 - s)^{1/2} \hat{M}, \ 0 \leq s \leq 1 \). We have
\[ R_n(x) = \int_0^1 \frac{\partial}{\partial s} \mathbb{E}\{ e_n(s, x) \} ds. \]
\[ (4.24) \]

Taking into account that
\[ \frac{\partial}{\partial s} e_n(s, x) = \sum_{(j,k) \in I^+} \frac{\partial e_n(s, x)}{\partial M_{jk}(s)} \frac{\partial M_{jk}(s)}{\partial s} \]
\[ = (b_n/n)^{1/2} ix \sum_{(j,k) \in I^+} \frac{\partial \text{Tr} \varphi(M(s)^0)}{\partial M_{jk}(s)} \frac{\partial M_{jk}(s)}{\partial s} \]
\[ = (b_n/n)^{1/2} ix e_n(s, x) \sum_{(j,k) \in I^+} 2\beta_{jk}(\varphi'_{jk}(M(s)))^0 \frac{1}{2} (s^{-1/2} M_{jk} - (1 - s)^{-1/2} \hat{M}_{jk}) \]
\[ = \sqrt{\frac{b_n}{2\sqrt{n}}} ix e_n(s, x) \text{Tr}(\varphi'(M(s))^0 (s^{-1/2} M - (1 - s)^{-1/2} \hat{M})), \]  
\[ (4.25) \]

we can write
\[ R_n(x) = \frac{ix}{2\sqrt{n}} \int_0^1 \mathbb{E}\{ e_n^0(s, x) \text{Tr} \varphi'(M(s)) (s^{-1/2} W - (1 - s)^{-1/2} \hat{W}) \} ds. \]
\[ (4.26) \]

Since 
\[ \varphi'(M) = i \int_{-\infty}^{+\infty} \hat{\varphi}(t) U(t) dt, \]
\[ (4.27) \]

we can rewrite (4.26) as
\[ R_n(x) = -\frac{x}{2\sqrt{n}} \int_0^1 \hat{\varphi}(t) \mathbb{E}\{ e_n^0(s, x) \text{Tr} U(s, t) (s^{-1/2} W - (1 - s)^{-1/2} \hat{W}) \} dt ds \]
\[ (4.28) \]

\[ = -\frac{x}{2} \int_0^1 \hat{\varphi}(t) [K_n - L_n] dt ds, \]  
\[ (4.29) \]
where

\[ K_n = \frac{1}{\sqrt{n}s} \sum_{(j,k) \in I_n} \mathbb{E}\{W_{jk}\Phi_n\}, \]

\[ L_n = \frac{1}{\sqrt{n}(1-s)} \sum_{(j,k) \in I_n} \mathbb{E}\{W_{jk}\Phi_n\}, \quad \text{and} \]

\[ \Phi_n = U_{jk}(s,t)e_n^o(s,x), \quad U(s,t) = e^{itM(s)}. \]

Applying the decoupling formula with \( p = 3 \) to every term in \( K_n \) and \( L_n \), we obtain

\[ K_n - L_n = I_2 + I_3 + \varepsilon_3, \quad (4.30) \]

where

\[ I_l = \frac{s(t-1)/2}{l!n^{1/2}b_n^{1/2}} \sum_{(j,k) \in I_n} \kappa_{l+1,jk} \mathbb{E}\{D^l_{jk}(s)\Phi_n\}, D_{jk}(s)\theta/\partial M_{jk}(s), \quad l = 2, 3, \quad (4.31) \]

and

\[ |\varepsilon_3| \leq \frac{C_3\sigma_5}{\sqrt{n}b_n^2} \sum_{(j,k) \in I_n} \sup_{M \in \mathbb{R}} |D^4_{jk}(s)\Phi_n|_{M(s)=M}|. \quad (4.32) \]

Let us consider \( I_2 \) first.

\[ I_2 = \frac{\sqrt{s}k_3/2}{n^{1/2}\sigma_2} \sum_{(j,k) \in I_n} \beta^2_{jk} \mathbb{E}\{e_n(s,x)U_{jk}(s,t) \left[ \int \theta \hat{\varphi}(\theta) U_{jk}(s,\theta) d\theta \right]\}^2 \]

\[ -\frac{\sqrt{s}k_3}{n\sqrt{b_n}} \sum_{(j,k) \in I_n} \beta^2_{jk} \mathbb{E}\{e_n(s,x)[U_{j}+U_{kk}+U_{j}+U_{j}]U_{jk}(s,\theta) d\theta\} \]

\[ -\frac{\sqrt{s}k_3}{n\sqrt{b_n}} \sum_{(j,k) \in I_n} \beta^2_{jk} \mathbb{E}\{e_n^o(s,x)[U_{j}+U_{kk}+U_{j}+U_{j}]U_{jk}(s,t) \left[ \int \theta \hat{\varphi}(\theta) U_{jk}(s,\theta) d\theta \right]\} \]

\[ -\frac{\sqrt{s}k_3}{n\sqrt{b_n}} \sum_{(j,k) \in I_n} \beta^2_{jk} \mathbb{E}\{e_n^o(s,x)[U_{j}+U_{kk}+U_{j}+U_{j}]U_{jk}(s,t) \left[ \int \theta \hat{\varphi}(\theta) U_{jk}(s,\theta) d\theta \right]\} + I'_2, \quad (4.33) \]

where

\[ I'_2 = \frac{\sqrt{s}}{2\sqrt{b_n}} \sum_{j=1}^{n} (k_{3,jj} - k_3) \mathbb{E}\{D^2_{jk}(s)\Phi_n\}. \]

Recall \( k_{3,jj} \) is the third cumulant of the \( j \)th diagonal entry and \( k_3 \) is the third cumulant of the off-diagonal entries. Note that

\[ |D^l_{jk}(s)\Phi_n| \leq C_l(\sqrt{b_n/nx, t}), 0 \leq l \leq 4, \quad (4.34) \]

So

\[ |I'_2| \leq \frac{\sqrt{n}}{b_n} C_2(\sqrt{b_n/nx, t}). \]

Consider two types of the sums above:

\[ I_{21} = \sum_{(j,k) \in I_n} U_{j}(s,t_1)U_{jk}(s,t_2)U_{kk}(s,t_3), \quad (4.35) \]

\[ I_{22} = \sum_{(j,k) \in I_n} U_{jk}(s,t_1)U_{jk}(s,t_2)U_{jk}(s,t_3). \quad (4.36) \]
It follows from the Cauchy-Schwarz inequality that

\[ |I_{22}| \leq \left( \sum_{j,k=1}^{n} |U_{jk}(s,t_1)|^2 \right)^{1/2} \left( \sum_{j,k=1}^{n} |U_{jk}(s,t_2)|^2 \right)^{1/2} = n. \]  

(4.37)

In addition,

\[ I_{21} = n(U^{(B)}(s,t_2)V(t_1), V(t_3)), V(t) = n^{-1/2}(U_{11}(t), ..., U_{nn}(t))^t. \]  

(4.38)

Since \( \|U^{(B)}\| \leq C \ln n, \|V(t)\| \leq 1 \), we have \( |I_{21}| \leq Cn \ln b_n \). Therefore,

\[ |I_2| \leq C_1 \frac{x^2}{\sqrt{n}} + C_2 \frac{|x| \ln b_n}{\sqrt{n}} + C_3 \frac{\sqrt{n} \ln b_n}{b_n} + \sqrt{n} b_n C_2 (\sqrt{b_n/nt}), \]

Since \( \frac{\sqrt{n} \ln n}{b_n} \to 0 \), then \( I_2 \to 0 \) on any bounded subset of \( \{(x,t)| t \geq 0\} \).

Recall that \( \kappa_{4,jk} = 0, j \neq k \). Thus,

\[ I_3 = \frac{s}{3!n^{1/2}b_n^{3/2}} \sum_{j=1}^{n} \kappa_{4,j} \mathbb{E}\{D_{jj}^3(s)\Phi_n\}, \]  

(4.39)

and

\[ |I_3| \leq \frac{\sqrt{n}}{b_n^{3/2}} C_4 (\sqrt{b_n/nt}), \]  

(4.40)

Taking into account that

\[ |\varepsilon_3| \leq \frac{\sqrt{n}}{b_n} C_4 (\sqrt{b_n/nt}), \]  

(4.41)

we conclude that \( I_2, I_3, \varepsilon \to 0 \) on any bounded subset of \( \{(x,t) : t \geq 0\} \). It then follows from (4.30) and (4.29), that \( R_n, \) defined in (4.22), converges to 0 as \( n \to \infty \). Theorem 2.2 is proven.

\[ \Delta \]

5 Appendix

A Poincaré Inequality

**Definition A.1.** A probability measure \( P \) on \( \mathbb{R}^M \) satisfies the Poincaré Inequality (PI) with constant \( m > 0 \) if, for all continuously differentiable functions \( f \),

\[ \text{Var}_P(f) := E_P(|f(x) - E_P(f(x))|^2) \leq \frac{1}{m} E_P(|\nabla f|^2). \]  

(A.1)

We note that the Poincaré inequality tensorises and the probability measures satisfying the Poincaré inequality have sub-exponential tails (see e.g. [1]). In particular, if \( P \) satisfies the PI on \( \mathbb{R}^M \) with constant \( m \), then for any Lipschitz continuous function \( G \), and \( |t| \leq \sqrt{m}/\sqrt{2}G|z| \), we have

\[ E_P(e^{t(G - E_P(G))}) \leq K, \]  

(A.2)

with \( K = -\sum_{i \geq 0} 2^i \log(1 - 2^{-1}4^{-i}) \). Consequently, for all \( \delta > 0 \),

\[ P(|G - E_P(G)| \geq \delta) \leq 2Ke^{-\sqrt{n}b_n \delta}. \]  

(A.3)
B Decoupling formula

Definition B.1. Let \( \xi \) be a random variable such that \( \mathbb{E}\{|\xi|^{p+2}\} < \infty \) for a certain nonnegative integer \( p \). Then for any function \( f : \mathbb{R} \to \mathbb{C} \) of the class \( C^{p+1} \) with bounded derivatives \( f^{(l)}, l = 1, \ldots, p+1 \), we have

\[
\mathbb{E}\{\xi f(\xi)\} = \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbb{E}\{f^{(l)}(\xi)\} + \varepsilon_p. \tag{B.1}
\]

where \( \kappa_l \) denotes the \( l \)th cumulant of \( \xi \) and the remainder term \( \varepsilon_p \) admits the bound

\[
|\varepsilon_p| \leq C_p \mathbb{E}\{|\xi|^{p+2}\} \sup_{t \in \mathbb{R}} |f^{(p+1)}(t)| \leq \frac{1 + (3 + 2p)^{p+2}}{(p+1)!}. \tag{B.2}
\]

If \( \xi \) is a Gaussian random variable with zero mean,

\[
\mathbb{E}\{\xi f(\xi)\} = \mathbb{E}\{\xi^2\} \mathbb{E}\{f'(\xi)\}. \tag{B.3}
\]

C Proof of Proposition 3.5

The goal of this section is to derive a bound

\[
\sup_{j \neq k} |\mathbb{E}\{U_{jk}(t)\}| = O\left(1 + \frac{t^6}{b_n}\right). \tag{C.1}
\]

To achieve this, we first bound the mathematical expectation of the off-diagonal entries of the resolvent matrix. Then, we use the Helffer-Sjöstrand functional calculus to extend the bound to the off-diagonal entries of the unitary matrix \( U(t) \).

Consider \( R(z) = (z - M)^{-1}, \text{Im}(z) \neq 0 \). The main part of the proof of proposition is the following lemma.

Lemma C.1. Let \( |\text{Im}z| \leq 2 \). Then

\[
|\mathbb{E}\{R_{ps}\}| \leq \frac{C}{|3\text{Im}z|^2 b_n}, \tag{C.2}
\]

where \( C > 0 \) is a constant independent from \( p \neq s \) and \( n \).

Proof. We start with the resolvent identity

\[
z R(z) = I + M R(z). \tag{C.3}
\]

Therefore, the off-diagonal entries of \( R(z) \) satisfy the following equation

\[
z \mathbb{E}\{R_{ps}\} = \sum_{j: (j, p) \in I_n} \mathbb{E}\{M_{pj} R_{js}\}, p \neq s. \tag{C.4}
\]

Applying the decoupling formula, we obtain

\[
\mathbb{E}\{M_{pj} R_{js}\} = \left\{ \begin{array}{ll}
\frac{\sigma_n^2}{b_n} \mathbb{E}\{R_{jp} R_{js} + R_{jj} R_{ps}\} + \frac{\mu_n}{b_n} \mathbb{E}\{2 R_{jp}^2 R_{js} + 2 R_{jj} R_{pp} R_{js} + 4 R_{jp} R_{jj} R_{ps}\} + \varepsilon_{2,j} & j \neq p \\
\frac{2\sigma_n^2}{b_n} \mathbb{E}\{R_{pp} R_{ps}\} + \frac{2\mu_n}{b_n} \mathbb{E}\{R_{pp}^2\} + \varepsilon_{2,p} & j = p,
\end{array} \right. \tag{C.5}
\]

where

\[
|\varepsilon_{2,j}| \leq C_2 \max\{\kappa_4, \kappa'_4\} \sup_{M_{pj} \in \mathbb{R}} \left| \frac{\partial^3 R_{js}}{\partial M_{pj}^3} \right| = O\left(\frac{1}{b_n^4 |\text{Im}z|^4}\right). \tag{C.6}
\]
We note that
\[ |\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jp}R_{jp}\}| \leq \mathbb{E}\left[ \sqrt{\sum_{j:(j,p)\leq b_n} |R_{jp}|^2} \sqrt{\sum_{j:(j,p)\leq b_n} \mathbb{E}\{|R_{jp}|^2\}} \right] \leq \frac{1}{|Imz|^2}, \] (C.7)
\[ |\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jp}^2R_{jp}\}| \leq \frac{1}{|Imz|^2} \sum_{j:(j,p)\leq b_n} \mathbb{E}\{|R_{jp}|^2\} \leq \frac{\sqrt{2b_n + 1}}{|Imz|^2} \] (C.8)
and similarly,
\[ |\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}R_{pp}R_{js}\}| \leq \frac{\sqrt{2b_n + 1}}{|Imz|^2}, \quad |\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}^2R_{pp}\}| \leq \frac{\sqrt{2b_n + 1}}{|Imz|^2}. \] (C.9)

Thus, for \( p \neq s \),
\[ z \mathbb{E}\{R_{ps}\} = \sum_{j:(j,p)\in I_n} \frac{\sigma^2}{b_n} \mathbb{E}\{R_{jj}R_{ps}\} + O\left(\frac{1}{b_n|Imz|^2}\right) + O\left(\frac{1}{b_n|Imz|^4}\right) + O\left(\frac{1}{b_n^2|Imz|^4}\right) \]
\[ = \sum_{j:(j,p)\in I_n} \frac{\sigma^2}{b_n} \mathbb{E}\{R_{jj}R_{ps}\} + O\left(\frac{1}{b_n|Imz|^2}\right). \] (C.10)

Since the diagonal entries \( R_{jj}'s \) have the same distribution, we can write \( g_n(z) := \frac{1}{n}\mathbb{E}\{TrR\} = \mathbb{E}\{R_{jj}\} \).

From the Wigner semicircle law for band random matrices,
\[ g_n(z) \to \int_{-2\sqrt{\sigma}}^{2\sqrt{\sigma}} \frac{\sqrt{8\sigma^2 - x^2}}{4\pi \sigma^2(z - x)} dx. \] (C.11)

We have
\[ \sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}R_{ps}\} = (2b_n + 1)g_n(z)\mathbb{E}\{R_{ps}\} + \sum_{|j-p|\leq b_n} \mathbb{E}\{R_{jj}^2R_{ps}\}, \] (C.12)
and
\[ |\sum_{j:(j,p)\in I_n} \mathbb{E}\{R_{jj}^2R_{ps}\}| \leq (2b_n + 1)\text{Var}^{1/2}\{R_{11}\}\text{Var}^{1/2}\{R_{ps}\}. \] (C.13)

The Poincaré inequality implies that
\[ \text{Var}\{R_{ps}\} \leq \frac{1}{mb_n} \sum_{j:(j,p)\in I_n} \mathbb{E}\{\beta_{jk}^2 R_{jj}R_{ks} + R_{pj}R_{js}\}^2 \leq \frac{2}{mb_n|Imz|^4}. \] (C.14)

Hence,
\[ z \mathbb{E}\{R_{ps}\} = \frac{\sigma^2(2b_n + 1)}{b_n} g_n(z)\mathbb{E}\{R_{ps}\} + O\left(\frac{1}{3mz^4b_n}\right), \] (C.15)
which implies
\[ |z - \frac{2b_n + 1}{b_n} \sigma^2 g_n(z)|\mathbb{E}\{R_{ps}\} = O\left(\frac{1}{3mz^4b_n}\right). \] (C.16)

In a similar fashion,
\[ |z - \frac{2b_n + 1}{b_n} \sigma^2 g_n(z)|g_n(z) = 1 + O\left(\frac{1}{3mz^4b_n}\right). \] (C.17)

If the term \( O\left(\frac{1}{3mz^4b_n}\right) \) at the r.h.s. of (C.17) is bounded in absolute value from above by 1/2, then there exists a constant \( C_1 \) such that
\[ \frac{C_1}{b_n|Imz|^4} \leq 1/2. \] (C.18)
Then
\[ |z - \frac{2b_n + 1}{b_n} \sigma^2 g_n(z) g_n(z)| \geq 1/2, \]  
\[ |z - \frac{2b_n + 1}{b_n} \sigma^2 g_n(z)| \geq \frac{1}{2|g_n(z)|} \geq \frac{|Im z|}{2}, \]  
(C.19)
(C.20)
and (C.20) and (C.16) imply
\[ |E\{R_{ps}\}| = O\left(\frac{1}{3m |z|^5 b_n}\right). \]  
(C.21)

Now assume that
\[ \frac{C_1}{b_n |Im z|^4} > 1/2. \]  
(C.22)
Then
\[ |E\{R_{ps}\}| \leq \frac{1}{|Im z|} < \frac{2C_1}{b_n |Im z|^5}. \]  
(C.23)
Lemma (C.1) is proven.

Now, we extend the bound in the last lemma to the off-diagonal entries of \( f(M) \), where \( f \) is sufficiently smooth function with compact support. To this end, we use the Helffer-Sjöstrand functional calculus (see e.g. [18], [31]). We write
\[ E\{f(M)_{jk}\} = -\pi \int_{\mathbb{R} \times [-1,1]} \frac{\partial \tilde{f}}{\partial \bar{z}} R_{jk} dx dy = -\frac{1}{\pi} \int_{\mathbb{R} \times [-1,1]} \frac{\partial \tilde{f}}{\partial \bar{z}} \sigma(y) dx dy = O\left(\frac{1}{3m |z|^5 b_n}\right) dx dy, \]  
(C.24)
where
i) \( z = x + iy \) with \( x, y \in \mathbb{R} \);
ii) \( \tilde{f}(z) \) is the extension of the function \( f \) defined as
\[ \tilde{f}(z) := \left( \sum_{n=0}^{l} \frac{f^{(n)}(x)(iy)^n}{n!} \right) \sigma(y); \]  
(C.25)
here \( \sigma \in C^\infty(\mathbb{R}) \) is a nonnegative function equal to 1 for \( |y| \leq 1/2 \) and equal to zero for \( |y| \geq 1 \).

Since
\[ \frac{\partial \tilde{f}}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial \tilde{f}}{\partial x} + i \frac{\partial \tilde{f}}{\partial y} \right), \]  
(C.26)
one has (with \( l = 5 \))
\[ \frac{\partial \tilde{f}}{\partial \bar{z}} = \frac{1}{2} \left( \sum_{n=0}^{5} \frac{f^{(n)}(x)(iy)^n}{n!} \right) i \frac{d\sigma}{dy} + \frac{1}{2} f^{(6)}(x)(iy)^5 \frac{\sigma(y)}{5!}. \]  
(C.27)
In particular,
\[ \left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| \leq const\|f\|_{C^5(\mathbb{R})}|y|^5, \]  
(C.28)
for six times continuously differentiable function \( f \) with compact support, where
\[ \|f\|_{C^5(\mathbb{R})} = \max_{0 \leq k \leq 6} \max_{x \in \mathbb{R}} |f^{(k)}(x)|. \]  
(C.29)
Combining (C.23), (C.24), and (C.29), we arrive at
\[ E\{f(M)_{jk}\} = O\left(\frac{\|f\|_{C^5(\mathbb{R})} b_n}{b_n}\right). \]  
(C.30)
This bound is not sufficient for our purposes since \( g(x) = e^{ix} \) is not compactly supported. Let \( f(x) \in C^\infty(\mathbb{R}) \) be a function satisfying \( f(x) \equiv g(x) \) if \( x \in [-10\sigma, 10\sigma] \), \( f(x) = 0 \) if \( |x| > 20\sigma \). If \( \text{Spec}(M) \subset [-10\sigma, 10\sigma] \), we clearly have \( f(M) = g(M) \). Hence,

\[
|\mathbb{E}\{g(M)_{jk}\}| \leq |\mathbb{E}\{f(M)_{jk}\}| + \sup_{x \in \mathbb{R}}|g(x)| \mathbb{P}(\|M\| \geq 10\sigma). \quad (C.31)
\]

In the next lemma, we show that \( \mathbb{P}(\|M\| \geq 10\sigma) \) is negligibly small.

**Lemma C.2.** There exists a positive constant \( C \) such that

\[
\mathbb{P}(\|M\| \geq 10\sigma) \leq Ce^{-C\sqrt{n}b}. \quad (C.32)
\]

Clearly, (C.31) and (C.32) finish the proof of Proposition 3.5. Thus, we are left with proving (C.32).

**Proof.** We note that \( \|M\| \) is a Lipschitz function of the matrix entries and the distribution of the entries of \( M \) satisfies the Poincaré inequality. Therefore, we have

\[
\mathbb{P}(\|M\| - \mathbb{E}\{\|M\|\} \geq \delta) \leq c_1\mathbb{E}\{\|M\|\} \leq c_2 \mathbb{E} \mathbb{E}\{\|M\|\} \leq c_2 \mathbb{E}\{\|M\|\}
\]

with some positive constants \( c_1 \) and \( c_2 \). Below we show that \( \mathbb{E}\{\|M\|\} \leq 5\sigma \) for all sufficiently large \( n \).

Let \( \tilde{M} \) be an independent copy of \( M \). Using a symmetrization argument (see e.g. [43]), we have

\[
\mathbb{E}\{\|M - \tilde{M}\|\} \geq \mathbb{E}\{\|M\|\} \quad (C.34)
\]

Denote \( B = M - \tilde{M} \). Applying the method of moments ([36], [37]), one can show that

\[
\mathbb{E}\{\text{Tr}B^{2s}\} = \frac{(16\sigma^2)^s n}{\sqrt{\pi s^3}} (1 + o(1)), \quad (C.35)
\]

as \( n \to \infty \) provided \( s \to \infty \) so that \( s = o\left(b_n^{1/3}\right) \). The computations are standard and left to the reader. Then

\[
\mathbb{E}\{\|B\|^{2s}\} \leq \frac{(16\sigma^2)^s n}{\sqrt{\pi s^3}} (1 + o(1)), \quad (C.36)
\]

which implies

\[
\mathbb{E}\{\|B\|\} \leq 4\sigma \left[ \frac{n}{\sqrt{\pi s^3}} (1 + o(1)) \right]^{1/2s}. \quad (C.37)
\]

Therefore, for sufficiently large \( n \),

\[
\mathbb{E}\{\|M\|\} \leq \mathbb{E}\{\|B\|\} \leq 5\sigma. \quad (C.38)
\]

The last inequality and (C.33) finish the proof of Lemma C.2 and Proposition 3.5.

**References**

[1] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices*, Cambridge Studies in Advanced Mathematics 118, Cambridge University Press, New York, 2010.

[2] G. W. Anderson and O. Zeitouni. A CLT for a Band Matrix Model. *Probab. Theory Relat. Fields*, 134:283–338, 2006.

[3] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*. Cambridge University Press, 2000.

[4] Z. D. Bai. Methodologies in Spectral Analysis of Large-Dimensional Random Matrices, a Review. *Statist. Sinica*, 9, 611–677, 1999.
[5] Z. D. Bai and G. M. Pan. Limiting Behavior of Eigenvectors of Large Wigner Matrices. *Journal of Statistical Physics*, 146(3):519-549, 2012.

[6] Z. D. Bai and J. Silverstein. CLT for Linear Spectral Statistics of Large-Dimensional Sample Covariance Matrix. *Ann. Probab.*, 32, 553-605, 2004.

[7] Z. D. Bai, X. Wang, and W. Zhou. CLT for Linear Spectral Statistics of Wigner Matrices. *Electronic Journal of Probability*, 14(83):2391–2417, 2009.

[8] Z. G. Bao, G. M. Pan, and W. Zhou. Central Limit Theorem for Partial Linear Eigenvalue Statistics of Wigner Matrices. *Journal of Statistical Physics*, 150(1):88-120,2013.

[9] A. D. Barbour and L. H. Y. Chen. *An Introduction to Stein’s method*, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, 4, Singapore University Press, 2005.

[10] G. Ben Arous and A. Guionnet. Wigner Matrices, in *Oxford Handbook on Random Matrix Theory*, edited by Akemann G., Baik J. and Di Francesco P., Oxford University Press, New York, 2011.

[11] L. V. Bogachev, S. A. Molchanov, and L. A. Pastur. On the Level Density of Random Band Matrices. *Mat. Zametki*, 50(6):31–42, 1991.

[12] A. Borodin. CLT for Spectral of Submatrices of Wigner Random Matrices. arXiv:1010.0898, 2010.

[13] A. Borodin. CLT for Spectral of Submatrices of Wigner Random Matrices II. Stochastic Evolution. arXiv:1011.3544, 2010.

[14] M. Capitaine, C. Donati-Martin, and D. Féral. The Largest Eigenvalues of Finite Rank Deformation of Large Wigner Matrices: Convergence and Nonuniversality of the Fluctuations. *The Annals of Probability*, 37(1):1–47, 2009.

[15] G. Casati, F. Izrailev, and L. Molinari, Scaling Properties of Band Random Matrices. *Phys. Rev. Lett.* 64: 1851–1854, 1990.

[16] G. Casati, F. Izrailev, and L. Molinari, Scaling Properties of the Eigenvalue Spacing Distribution for Band Random Matrices *J. Phys. A: Math. Gen.* 24(20): 4755–4762, 1991.

[17] H. Cramer. *Mathematical Methods of Statistics*. Princeton University Press, 1946.

[18] E.B. Davies, The Functional Calculus. *J. London Math. Soc.*, 52: 166-176, 1995

[19] L. Erdos, A. Knowles, H.-T. Yau, and Jun Yin. Delocalization and Diffusion Profile for Random Band Matrices. arXiv:1205.5669, 2012.

[20] L. Erdos, A. Knowles, and H.-T. Yau. Averaging Fluctuations in Resolvents of Random Band Matrices. arXiv:1205.5664, 2012.

[21] L. Erdos, A. Knowles, H.-T. Yau, and J. Yin. The Local Semicircle Law for a General Class of Random Matrices, arXiv:1212.0164, 2012.

[22] Y. V. Fyodorov and A. D. Mirlin. Scaling Properties of Localization in Random Band Matrices: A σ-Model Approach. *Physical Review Letters*, 67(18):2405–2409, 1991.

[23] K. Johansson. On Fluctuations of Eigenvalues of Random Hermitian Matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998.

[24] A. Khorunzhy and W. Kirsch. On Asymptotic Expansions and Scales of Spectral Universality in Band Random Matrices. *Commun. Math. Phys.*, 231(2):223–255, 2002.
[25] A. Lytova. On Non-Gaussian Limiting Law for the Certain Statistics of the Wigner Matrices. arXiv:1201.3027.

[26] A. Lytova and L. Pastur. Central Limit Theorem for Linear Eigenvalue Statistics of Random Matrices With Independent Entries. *The Annals of Probability, 37*(5):1778–1840, 2009.

[27] R. Mathias. The Hadamard Operator Norm of A Circulant and Applications. *SIAM J. Matrix Anal Appl*, 14(4):1152–1157, 1993.

[28] A. D. Mirlin, Y. V. Fyodorov, F.-M. Dittes, J. Quezada, and T. H. Seligman. Transition from localized to extended eigenstates in the ensemble of power-law random banded matrices. *Physical Review E*, 54(4):3221–3230, 1996.

[29] S. A. Molchanov, L. A. Pastur, and A. M. Khorunzhii. Limiting Eigenvalue Distribution for Band Random Matrices. *Teor. Mat. Fizika*, 90(2):163–178, 1992.

[30] S O’Rourke and A. Soshnikov. Partial Linear Eigenvalue Statistics for Wigner and Sample Covariance Random Matrices. arXiv:1301.0368, to appear in *J. Theor. Probab.*, 2013.

[31] A. Pizzo, D. Renfrew, and A. Soshnikov. Fluctuation of Matrix Entries of Regular Functions of Wigner Matrices. *Journal of Statistical Physics*, 146(3):550–591, 2012.

[32] T. Seligman, J. Verbaarschot, and M. Zirnbauer Spectral Fluctuation Properties of Hamiltonian Systems, *J. Phys. A: Math. Gen.*, 18: 2751, 1985.

[33] M. Shcherbina. Central Limit Theorem for Linear Eigenvalue Statistics of the Wigner and Sample Covariance Random Matrices. *Journal of Mathematical Physics, Analysis, Geometry*, 7(2):176–192, 2011.

[34] M. Shcherbina and B. Tirozzi. Central Limit Theorem for Fluctuations of Linear Eigenvalue Statistics of Large Random Graphs. *Journal of Mathematical Physics*, 51, 2010.

[35] J. Schenker, Eigenvector Localization for Random Band Matrices with Power Law Band Width. *Commun. Math. Phys.*, 290: 1065–1097, 2009.

[36] Ya. Sinai and A. Soshnikov. Central Limit Theorem for Traces of Large Random Symmetric Matrices with Independent Matrix Elements. *Bol. Soc. Bras. Mat.*, 29:1–24, 1998.

[37] Ya. G. Sinai and A. Soshnikov. A Refinement of Wigner’s Semicircle Law in a Neighborhood of the Spectrum Edge for Random Symmetric Matrices. *Functional Analysis and Application*, 32(2):114–131, 1998.

[38] A. B. Soshnikov. The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities. *Annals of Probability*, 28(3):1353–1370, 2000.

[39] S. Sodin. The spectral edge of some random band matrices. *Ann. Math.*, 172(3):2223–2251, 2010.

[40] S. Sodin. An Estimate for the Average Spectral Measure of Random Band Matrices. *J. Stat. Phys.*, 144(1): 46–59, 2011.

[41] T. Spencer. Random Banded and Sparse Matrices (Chapter 23). Oxford Handbook of Random Matrix Theory, eds. G.Akemann, J. Baik, and P. Di Francesco, 2011.

[42] C. Stein. *Approximate computation of expectations*. Institute of Mathematical Statistics Lecture Notes, Monograph Series, 7. Hayward, 1986.

[43] T. Tao. *Topics in Random Matrix Theory*. American Mathematical Society, 2012.

[44] E. P. Wigner. On the distribution of the roots of certain symmetric matrices. *Ann. Math.*, 67, 325–327, 1958.