Stability of the two-dimensional Fermi polaron

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Abstract

A system composed of an ideal gas of \( N \) fermions interacting with an impurity particle in two space dimensions is considered. The interaction between impurity and fermions is given in terms of two-body point interactions whose strength is determined by the two-body binding energy, which is a free parameter of the model. If the mass of the impurity is 1.225 times larger than the mass of a fermion, it is shown that the energy is bounded below uniformly in the number \( N \) of fermions. This result improves previous, \( N \)-dependent lower bounds and it complements a recent, similar bound for the Fermi polaron in three space dimensions.

1 Introduction

The system considered in this paper is composed of an ideal gas of \( N \) fermions and one additional particle, called impurity, in two space dimensions. The impurity interacts with the fermions by two-body point interactions. Informally, the Hamiltonian of the system may thus be written as

\[
- \frac{1}{M} \Delta_y - \sum_{i=1}^{N} \Delta x_i - g \sum_{i=1}^{N} \delta(x_i - y),
\]

(1)

where \( M > 0 \) is the mass of the impurity and \( g \) plays the role of a coupling constant. The problem of defining a self-adjoint Hamiltonian describing (1) is discussed and solved in [4, 5]. We are interested in its ground state energy, and we show, in this paper, that it is bounded below uniformly in \( N \), provided that \( M > 1.225 \). In the physics literature the system described above is called Fermi polaron [8]. It is a model for an ultra-cold gas of fermionic atoms interacting with an additional, impurity atom. One is interested in the form of the ground state as a function of the coupling strength and in two space dimensions one expects a sharp transition, related to the BEC-BCS crossover [6, 12].

Our approach for defining a self-adjoint Hamiltonian describing (1) follows [4, 5] and it is described in [7]. Here we only summarise the ingredients and facts needed in this paper.

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Some more details are given in the appendix. We work in the language of second quantisation. Let $a_k$ and $a_k^*$ denote the usual annihilation and creation operators in antisymmetric Fock space over $L^2(\mathbb{R}^2)$. Let $H_f = \int k^2 a_k^* a_k \, dk$ and $P_f = \int k a_k^* a_k \, dk$. The main operator to be analysed in this paper is not the Hamiltonian of the system, which is not available explicitly, but a self-adjoint operator $\phi(E)$ in $\mathcal{H}_{N-1} = \bigwedge^{N-1} L^2(\mathbb{R}^2)$, depending on a parameter $E < 0$ and defined by

$$\phi(E) = \alpha + \phi^0(E) + \phi^I(E),$$  

where $\alpha \in \mathbb{R}$ and

$$\phi^0(E) := \frac{\pi}{1 + \frac{1}{M}} \log \left( \frac{1}{M + 1} P_f^2 + H_f - E \right)$$  

$$\phi^I(E) := \int dp dq \ a_p^* \left( \frac{1}{M} (P_f + p + q)^2 + H_f + p^2 + q^2 - E \right) a_q.$$  

Here $\alpha$ is a free parameter of the model that parametrises the coupling strength between fermions and impurity. To give it a physical interpretation we mention that $\alpha = -(\pi/(1 + M^{-1})) \cdot \log |E_B|$, where $E_B < 0$ is the ground state energy of the two-body system consisting of only one fermion and the impurity. A negative energy state is always present in the two-body system in two dimensions \[1\]. The point about $\phi(E)$ is that, for $E < 0$,

$$\phi(E) \geq 0 \quad \Rightarrow \quad H_N \geq E$$  

where $H_N$ denotes the self-adjoint realisation of \[1\] after separating off the center-of-mass motion (see Appendix \[A\]). The main result of this paper, Theorem \[1\] below, provides us with a number $E$, depending on $M$ only, such that $\phi(E) \geq 0$ for $M > 1.225$. This implies, by \[5\], that $H_N \geq E$ uniformly in $N$. It is an open problem, whether or not an $N$-independent lower bound on $H_N$ exists for arbitrary positive values of $M$. For a possible approach to this problem see \[7\].

For arbitrary $M > 0$ the Hamiltonian $H_N$ is bounded below, but the lower bound may depend on $N$: using that $\phi^I(E)$ is bounded with $\|\phi^I(E)\| \leq \text{const} \cdot (N - 1)$ and $\phi^0(E) \geq (\pi/(1 + M^{-1})) \cdot \log(-E)$, we conclude, by \[5\], that

$$H_N \geq E_B \cdot \exp(C \cdot (1 + M^{-1}) \cdot (N - 1)).$$  

This result can already be inferred from \[4\].

In three space dimensions, \[5\] still holds with $\phi(E)$ defined by \[2\], \[1\], see \[4\][11], and

$$\phi^0(E) = \frac{2\pi^2}{(1 + \frac{1}{M})^{3/2}} \cdot \sqrt{\frac{1}{M + 1} P_f^2 + H_f - E}.$$  

In contrast to the two-dimensional case, however, the operator $\phi^I(E)$ is not bounded anymore. Instead, from \[2\] it follows that

$$\phi^I(E) \geq -C(M,N) \cdot \phi^0(E),$$  

which implies $\phi(E) \geq \alpha + (1 - C(M,N)) \phi^0(E)$. Provided that $C(M,N) < 1$, we may choose $|E|$ large enough, using that $\phi^0(E) \geq (2\pi^2/(1 + M^{-1})^{3/2})\sqrt{-E}$, such that $\phi(E) \geq 0$.
and hence $H_N \geq E$. This improves an earlier result [9,10]. The condition $C(M,N) < 1$ is satisfied if $M$ is larger than the critical mass $M^*(N)$ defined by $C(M^*(N),N) = 1$. In [2], (7) is shown for a function $C(M,N)$ for which $M^*(N) \propto N$ as $N \to \infty$, and, moreover, it is shown that $\phi(E)$ is unbounded from below if $N \geq 2$ and $M < M^*(2) \approx 0.0735$. Recently, (7) was shown to hold with a constant $C(M)$ that is independent of $N$ [11]. This constant satisfies $C(M) < 1$ if $M > M^* \approx 0.36$. It follows that $H_N$ is bounded below uniformly in $N$ provided that $M > 0.36$. In fact, $H_N \geq 0$ if $M < M^*(2) \approx 0.0735$. Recently, (7) was shown to hold with a constant $C(M)$ that is independent of $N$ [11]. This constant satisfies $C(M) < 1$ if $M > M^* \approx 0.36$. It follows that $H_N$ is bounded below uniformly in $N$ provided that $M > 0.36$. In fact, $H_N \geq 0$ if $\alpha \geq 0$ and $H_N \geq -\tilde{C}(M) \cdot \alpha^2$ for a constant $\tilde{C}(M) > 0$ if $\alpha < 0$. While the present paper, on a technical level (i.e. in the proof of Lemma 5 below), has strongly benefited from [11], it solves an additional infrared problem, which arises due to the lower dimension. The main result can also be found in [7].

2 An $N$-independent lower bound for the Fermi polaron in $\mathbb{R}^2$

Theorem 1. Let $E_B < 0$. Set

$$\alpha(M) := \frac{1}{2(M+1)} + \frac{1}{2} \int_0^1 \frac{1}{\beta(u)(M+1-u)} du,$$

where

$$\beta(u) := \min \left\{ 1, \frac{(M+1-u)(M+2)}{M^2+3M+1-u} \right\},$$

and suppose that $\alpha(M) < M/(M+1)$, which is satisfied if $M > 1.225$. Then, for every $\lambda > 0$, the unique solution $\mu < 0$ of the equation

$$\left( \frac{M}{M+1} - \alpha(M) \right) \log \left( \frac{\mu}{E_B} \right) - \sqrt{\frac{\lambda}{-\mu}} - \sqrt{\frac{\lambda}{\lambda-\mu}} - \alpha(M) \log \left( E_B \left( \frac{1}{\mu} - \frac{1}{\lambda} \right) \right) - \alpha(M) = 0$$

(10)

satisfies $\phi(\mu) \geq 0$ and hence $H_N \geq \mu$ for all $N \in \mathbb{N}$.

Remarks.

(i) The left hand side of (10) can be written as

$$\frac{M}{M+1} \log \left( \frac{\mu}{E_B} \right) - \sqrt{\frac{\lambda}{-\mu}} - \sqrt{\frac{\lambda}{\lambda-\mu}} - \alpha(M) \log \left( 1 - \frac{\mu}{\lambda} \right) - \alpha(M),$$

which is obviously negative for $E_B \leq \mu < 0$. Thus, all solutions of (10) satisfy $\mu < E_B$.

(ii) For fixed $E_B < 0$ and $\lambda > 0$ the left hand side of (10) is a strictly monotonically decreasing function of $\mu$ on the interval $(-\infty,E_B]$. It tends to $+\infty$ as $\mu \to -\infty$ and it attains a negative value for $\mu = E_B$. Thus, there is a unique solution $\mu < E_B$ of (10) for fixed $\lambda$ and $E_B$.

(iii) The choice of the parameter $\lambda > 0$ is an opportunity for optimization of the lower bound $\mu$. 
(iv) The fermionic nature of the $N$ identical particles enters the model through the anti-symmetric product in the definition of $\mathcal{F}_N$ and $\mathcal{F}_{N-1}$ and the sign of $\phi^j(E)$ in (9). In fact in the case of $N$ identical bosons, in which an $N$-independent lower bound for $H_N$ cannot be established, $\phi^j(E)$ has to be replaced by $-\phi^j(E)$. Therefore, it is important to consider only the negative part of $\phi^j(E)$ when deriving a lower bound for $\phi(E)$ in the proof of Theorem 1.

Choosing $\lambda = -E_B$ in Theorem 1 (10) turns into an equation for $M$ and $\mu/E_B$ only and we obtain the following statement.

**Corollary 2.** Let $E_B < 0$ and assume that $\alpha(M) < M/(M + 1)$. Then,

$$H_N \geq \gamma_M \cdot E_B,$$

where $\gamma_M > 1$ depends on $M$ only and is defined as the unique positive solution of

$$\left(\frac{M}{M + 1} - \alpha(M)\right) \log(\gamma_M) - \frac{1}{\sqrt{\gamma_M}} - \frac{1}{\sqrt{1 + \gamma_M}} - \alpha(M) \log \left(1 + \frac{1}{\gamma_M}\right) = \alpha(M).$$

### 3 Proofs

This section is devoted to the proof of Theorem 1. We show that $\pi$ times the left hand side of (10) is a lower bound for $\phi(\mu)$ if $\mu < 0$. In view of (13), we then obtain $H_N \geq \mu$ for every solution $\mu$ of (10). For $r > 0$ let $\chi_r := \chi_{B(0, \sqrt{r})}$ be the characteristic function of the ball $B(0, \sqrt{r}) \subset \mathbb{R}^2$. The parameter $\lambda > 0$ is fixed in the following and plays the role of an infrared cutoff. For $p^2 \leq \lambda$ and $q^2 \leq \lambda$ we rewrite the part (4) of the operator $\phi(\mu)$, making use of the pull-through formulas

$$a_p f(P_f) = f(P_f + p)a_p \quad \text{and} \quad a_p g(H_f) = g(H_f + p^2)a_p,$$

and the canonical anti-commutation relations, in such a way that

$$\phi(\mu) = \frac{\pi}{1 + \frac{1}{\mathcal{M}}} \log \left(\frac{\mathcal{M}}{\mathcal{M} + 1} + \frac{P_f^2 + H_f - \mu}{-E_B}\right) + \int \frac{dp}{\mathcal{M}} \frac{1}{(P_f + p)^2 + H_f + p^2 - \mu}

- a(\chi_n) \frac{1}{\mathcal{M}} \frac{P_f^2 + H_f - \mu}{a^*(\chi_n)} - a(\chi_n - \chi_n) \frac{1}{\mathcal{M} P_f^2 + H_f - \mu}

= \frac{\int \frac{dp dq}{\mathcal{M}} a^*_p \left(\frac{1}{(P_f + p + q)^2 + H_f + p^2 + q^2 - \mu}\right) a_q + o(1)}{\lambda < p^2, q^2 \leq n}$$

as $n \to \infty$. The remainder term converges to zero strongly. Here and in the following $\lambda < p^2, q^2 \leq n$ means that $\lambda < p^2 \leq n$ and $\lambda < q^2 \leq n$. The first two terms of (12) are positive for $\mu < E_B$. The last three terms of (12) are estimated, uniformly in $n$ and $N$, in Lemma 3 and Lemma 5 below.

**Lemma 3.** For $n > \lambda \geq 0$ and $\mu < 0$,

$$\left\|\left(\frac{1}{\mathcal{M}} P_f^2 + H_f - \mu\right)^{-1} a^* (\chi_n - \chi_\lambda)\right\| \leq \sqrt{\frac{\pi}{\lambda - \mu}}.$$
Proof. The lemma follows from

\[
\left\|(\frac{1}{M} P_f^2 + H_f - \mu)^{-1} a^* (\chi_n - \chi)\right\| \leq \left\|(H_f - \mu)^{-1} a^* (\chi_n - \chi)\right\|
\]

\[
= \left\| a(\chi_n - \chi)(H_f - \mu)^{-2} a^* (\chi_n - \chi)\right\|^{1/2},
\]

and

\[
a(\chi_n - \chi) \frac{1}{(H_f - \mu)^2} a^* (\chi_n - \chi)
\]

\[
= \int dp\, dq\, a_p \frac{1}{(H_f - \mu)^2} a^*_q = \int dp\, \frac{1}{(H_f + p^2 - \mu)^2} - \int dp\, dq\, a^*_q \frac{1}{(H_f + p^2 + q^2 - \mu)^2} a_p
\]

\[
\leq \int dp\, \frac{1}{(p^2 - \mu)^2} = \frac{\pi}{\lambda - \mu},
\]

which is true because of the positivity of

\[
\int dp\, dq\, a^*_q \frac{1}{(H_f + p^2 + q^2 - \mu)^2} a_p
\]

\[
= \int ds\, \int dt\, \int dp\, dq\, a^*_q e^{-(s+t)q^2} e^{-(s+t)(H_f - \mu)} e^{-(s+t)p^2} a_p.
\]

For the proof of Lemma 4 below, we need the following lemma, which is a version of the Schur test.

Lemma 4. Let \( \Omega \subseteq \mathbb{R}^d \) be a measurable set and let \( G : \Omega \times \Omega \to \mathcal{L}(\mathcal{F}(L^2(\mathbb{R}^d))) \) be a measurable map. Thus for every \((p, q) \in \Omega \times \Omega\), \( G(p, q) \) is a bounded operator on the (antisymmetric) Fock space over \( L^2(\mathbb{R}^d) \). Assume that \( G(p, q)^* = G(p, q) = G(q, p) \) for all \( p, q \in \Omega \). Moreover, let \( h : \Omega \to \mathbb{R}_+ \) be a positive measurable function. Then,

\[
\int_{\Omega \times \Omega} dp\, dq\, a^*_p G(p, q) a_q \leq \int_{\Omega} dp\, h(p) a^*_p \left( \int_{\Omega} dq\, |G(p, q)| \frac{|G(p, q)|}{h(q)} \right) a_p.
\]

Proof. Let \( \psi \in \mathcal{F}(L^2(\mathbb{R}^d)) \). Writing \( G(p, q) = \text{sgn}(G(p, q)) \cdot |G(p, q)| \) with the help of the functional calculus, we obtain

\[
\int_{\Omega \times \Omega} dp\, dq\, \langle a_p \psi, G(p, q) a_q \psi \rangle \leq \int_{\Omega \times \Omega} dp\, dq\, \left\| |G(p, q)|^{1/2} a_p \psi \right\| \cdot \left\| |G(p, q)|^{1/2} a_q \psi \right\|
\]

\[
\leq \left( \int_{\Omega \times \Omega} dp\, dq\, \frac{h(p)}{h(q)} \left| |G(p, q)|^{1/2} a_p \psi \right|^2 \right)^{1/2} \left( \int_{\Omega \times \Omega} dp\, dq\, \frac{h(q)}{h(p)} \left| |G(p, q)|^{1/2} a_q \psi \right|^2 \right)^{1/2}
\]

\[
= \int_{\Omega \times \Omega} dp\, dq\, h(p) \langle \psi, a^*_p \frac{|G(p, q)|}{h(q)} a_q \psi \rangle.
\]
Lemma 5. Let $\mu < 0$. Then the operator

$$P := \int \frac{dp \, dq \, a_p^*}{\lambda < p^2, q^2 \leq n} \frac{1}{M^2 (P_f + p + q)^2 + H_f + p^2 + q^2 - \mu} a_q$$

admits the estimate

$$P \geq -\pi \alpha(M) \left(1 + \log \left(1 + \frac{H_f - \mu}{\lambda}\right)\right).$$

Remark. Our proof of Lemma 5 follows the arguments in [11], but in contrast to the three-dimensional case, the infrared contributions with $p^2 \leq \lambda$ or $q^2 \leq \lambda$ require a separate treatment.

Proof. Setting $\widehat{p} := p + \frac{1}{M+2} P_f$ and $\widehat{q} := q + \frac{1}{M+2} P_f$ we can rewrite the denominator in the expression defining $P$ as

$$(1 + \frac{1}{M})(\widehat{p}^2 + \widehat{q}^2) + \frac{2}{M} \widehat{p} \cdot \widehat{q} + \frac{1}{M+2} P_f^2 + H_f - \mu.$$ 

For $\psi \in \bigwedge^{N-1} L^2(\mathbb{R}^2)$, we define $\tilde{\psi} \in L^2(\mathbb{R}^2; \bigwedge^{N-2} L^2(\mathbb{R}^2))$ by $\tilde{\psi}(p) := a_p \psi$. Moreover, we define a unitary operator $T \in \mathcal{L}(L^2(\mathbb{R}^2; \bigwedge^{N-2} L^2(\mathbb{R}^2)))$ by

$$(T \varphi)(p; k_1, \ldots, k_{N-2}) := \varphi(p) + \frac{1}{M+2} \sum_{i=1}^{N-2} k_i; k_1, \ldots, k_{N-2},$$

where $(T \varphi)(p; k_1, \ldots, k_{N-2})$ and $\varphi(p; k_1, \ldots, k_{N-2})$ denote values of the functions $(T \varphi)(p)$ and $\varphi(p) \in \bigwedge^{N-2} L^2(\mathbb{R}^2)$, respectively. We obtain

$$\langle \psi, P \psi \rangle = \int \frac{dp \, dq \, \langle \tilde{\psi}(p), (1 + \frac{1}{M})(\widehat{p}^2 + \widehat{q}^2) + \frac{2}{M} \widehat{p} \cdot \widehat{q} + \frac{1}{M+2} P_f^2 + H_f - \mu \rangle \tilde{\psi}(q)}{\lambda < p^2, q^2 \leq n}$$

$$= \langle (\chi_n - \chi_\lambda) \tilde{\psi}, T \sigma T^* (\chi_n - \chi_\lambda) \tilde{\psi} \rangle,$$

where $\sigma$ is the operator on $L^2(\mathbb{R}^2; \bigwedge^{N-2} L^2(\mathbb{R}^2))$ with operator-valued integral kernel

$$\sigma(p, q) = \frac{1}{(1 + \frac{1}{M})(p^2 + q^2) + \frac{2}{M} p \cdot q + \frac{1}{M+2} P_f^2 + H_f - \mu}.$$ 

Following [11] (3.9), we compute the negative part of $\sigma$ explicitly. Its kernel is given by $\sigma^-(p, q) = \frac{1}{2}(\sigma(-p, q) - \sigma(p, q))$. We write $\sigma^-(p, q)$ as

$$\sigma^-(p, q) = \frac{1}{2} \left[ \frac{1}{(1 + \frac{1}{M})(p^2 + q^2) - \frac{2}{M} p \cdot q + \frac{1}{M+2} P_f^2 + H_f - \mu} \right]_{n=-1}^{u=1}$$

$$= \frac{1}{2} \int_{-1}^{1} \frac{du}{du} \left[ \frac{1}{(1 + \frac{1}{M})(p^2 + q^2) - \frac{2}{M} p \cdot q + \frac{1}{M+2} P_f^2 + H_f - \mu} \right]$$

$$= M p \cdot q \int_{-1}^{1} \frac{1}{[(M+1)(p^2 + q^2) - 2up \cdot q + B]^2},$$
where \( B := \frac{M}{M+2} P^2_f + MH_f - M\mu \). Then,

\[
P \geq - \int_{\lambda < p^2, q^2 \leq n} dp \, dq \, a^*_{p} \sigma^-(\hat{p}, \hat{q}) \, a_q \tag{13}
\]

\[
= -M \int_{\lambda < p^2, q^2 \leq n} dp \, dq \, a^*_{p} \left( \int_{-1}^{1} du \, \frac{\hat{p} \cdot \hat{q}}{((M+1)(\hat{p}^2 + \hat{q}^2) - 2u\hat{p} \cdot \hat{q} + B)^2} \right) a_q,
\]

and with Lemma [4] and \( h(p) = p^2 \) we obtain

\[
P \geq -M \int_{\lambda < p^2 \leq n} dp \, p^2 \, a^*_{p} f(p, P_f, H_f) \, a_p,
\]

where

\[
f(p, P_f, H_f) := \int_{\lambda < q^2 \leq n} dq \int_{-1}^{1} du \, \frac{\hat{p} \cdot \hat{q}}{q^2((M+1)(\hat{p}^2 + \hat{q}^2) - 2u\hat{p} \cdot \hat{q} + B)^2}.
\]

Our goal is now to find a function \( g \) with \( f(p, Q, E) \leq g(E + p^2) \). It then follows that

\[
P \geq -M \int_{\lambda < p^2 \leq n} dp \, p^2 \, a^*_{p} g(H_f + p^2) \, a_p
\]

\[
\geq -M \int_{\lambda < p^2 \leq n} dp \, p^2 \, a^*_{p} a_p \, g(H_f) \geq -MH_f g(H_f). \tag{14}
\]

To find such a function \( g \) we first note that \( 2u\hat{p} \cdot \hat{q} \leq 0 \) on half of the \( u \)-interval \([-1, 1]\) and hence the quotient in the definition of \( f \) goes up and becomes independent of \( u \) if we drop this term. Second, we use \( \hat{p}^2 + \hat{q}^2 \geq 2|\hat{p} \cdot \hat{q}| \) and \( B \geq 0 \) in the denominators. Explicitly,

\[
\int_{-1}^{1} du \, \frac{\hat{p} \cdot \hat{q}}{q^2((M+1)(\hat{p}^2 + \hat{q}^2) - 2u\hat{p} \cdot \hat{q} + B)^2}
\]

\[
\leq \frac{\hat{p} \cdot \hat{q}}{q^2((M+1)(\hat{p}^2 + \hat{q}^2) + B)^2} + \int_{0}^{1} du \, \frac{\hat{p} \cdot \hat{q}}{q^2((M+1)(\hat{p}^2 + \hat{q}^2) - 2u|\hat{p} \cdot \hat{q}| + B)^2}
\]

\[
\leq \frac{1}{2q^2(M+1)((M+1)(\hat{p}^2 + \hat{q}^2) + B)} + \int_{0}^{1} du \, \frac{1}{2q^2(M+1-u)((M+1-u)(\hat{p}^2 + \hat{q}^2) + B)}.
\tag{15}
\]

One can easily verify that

\[
(M + 1)\hat{p}^2 + \frac{M}{M+2} P^2_f \geq \frac{M(M+1)(M+2)}{M^2 + 3M + 1} p^2 \geq Mp^2 \tag{16}
\]

and, more generally,

\[
(M + 1 - u)\hat{p}^2 + \frac{M}{M+2} P^2_f \geq \frac{M(M + 1 - u)(M+2)}{M^2 + 3M + 1 - u} p^2 \geq M\beta(\mu)p^2, \tag{17}
\]
where $\beta(u)$ was defined in (9). From (15), (16) and (17) we obtain the estimate

$$f(p, P_f, H_f) \leq \int dq \frac{1 - \chi\lambda(q)}{q^2} \left( \tilde{f}(\hat{q}, 0) + \int_0^1 du \tilde{f}(\hat{q}, u) \right)$$

(18)

with

$$\tilde{f}(q, u) = \frac{1}{2(M + 1 - u)^2} \cdot \frac{1}{q^2 + A(u)} \quad \text{and} \quad A(u) = \frac{M[H_f + \beta(u)p^2 - \mu]}{M + 1 - u}.$$

In order to estimate (18), we replace $(1 - \chi\lambda(q))/q^2$ by the symmetric decreasing function $j\lambda(q) := (1 - \chi\lambda(q))/q^2 + \chi\lambda(q)/\lambda$. We then employ a rearrangement inequality that allows us to replace $\hat{q} = q + \frac{1}{M+2}P_f$ by $q$ in the argument of $\tilde{f}$. For an arbitrary $u \in [0, 1]$ this reads

$$\int dq \frac{1 - \chi\lambda(q)}{q^2} \tilde{f}(\hat{q}, u) \leq \int dq j\lambda(q) \tilde{f}(q, u)$$

$$= \frac{\pi}{2(M + 1 - u)^2 A(u)} \left( \frac{A(u)}{\lambda} \log \left( 1 + \frac{\lambda}{A(u)} \right) + \log \left( 1 + \frac{A(u)}{\lambda} \right) \right)$$

$$\leq \frac{\pi}{2M(M + 1 - u)\beta(u)} \frac{1}{H_f + p^2} \left( 1 + \log \left( 1 + \frac{H_f + p^2 - \mu}{\lambda} \right) \right),$$

(19)

where we used $\log(1 + x) \leq x$ if $x \geq 0$ for the first logarithm in (19), $A(u) \leq H_f + p^2 - \mu$ in the argument of the second logarithm and $(M + 1 - u)A(u) \geq M\beta(u)(H_f + p^2)$ in the overall prefactor $1/A(u)$. Combining (18) and (19), we arrive at

$$f(p, P_f, H_f) \leq \frac{\pi \alpha(M)}{M} \frac{1}{H_f + p^2} \left( 1 + \log \left( \frac{H_f + p^2 - \mu}{\lambda} \right) \right),$$

which is of the form $g(H_f + p^2)$ as desired. In view of (14) the lemma is proven.

**Proof of Theorem**

We combine (12), Lemma 3, Lemma 5 and $\|a(\chi\lambda)\| = \|a^*(\chi\lambda)\| = \sqrt{\pi\lambda}$. In the limit $n \to \infty$ we find

$$\phi(\mu) \geq \frac{\pi}{1 + \frac{\pi}{\lambda}} \log \left( \frac{1 + P_f^2 + H_f - \mu}{-E_B} \right) - \pi \sqrt{\frac{\lambda}{-\mu}} - \pi \sqrt{\frac{\lambda}{\lambda - \mu}}$$

$$- \pi \alpha(M) \left( 1 + \log \left( 1 + \frac{H_f - \mu}{\lambda} \right) \right)$$

$$\geq \pi \left( \frac{M}{M + 1} - \alpha(M) \right) \log \left( \frac{\mu}{E_B} \right) - \pi \sqrt{\frac{\lambda}{-\mu}} - \pi \sqrt{\frac{\lambda}{\lambda - \mu}}$$

$$- \pi \alpha(M) \log \left( -E_B \left( \frac{1}{\lambda} + \frac{1}{-\mu} \right) \right) - \pi \alpha(M).$$

By (5), this completes the proof of Theorem.
It would be very interesting to know whether the conclusion of Theorem 1 still holds for \( M < 1.225 \). One could address this question with the help of some numerics as follows. Using the pull-through formula and Lemma 4 with \( h(p) = p^2 \) one obtains from (13)

\[
\sqrt{\frac{H_f - \mu}{\log(1 + \frac{H_f - \mu}{\lambda})}} P \sqrt{\frac{H_f - \mu}{\log(1 + \frac{H_f - \mu}{\lambda})}} \geq -\int_{\lambda < p^2 \leq n} dp \int_{\lambda < q^2 \leq n} dq \frac{1}{q^2} \sqrt{\frac{H_f + p^2 - \mu}{\log(1 + \frac{H_f + p^2 - \mu}{\lambda})}} \sigma^-(\tilde{p}, \tilde{q}) \sqrt{\frac{H_f + q^2 - \mu}{\log(1 + \frac{H_f + q^2 - \mu}{\lambda})}} a_p
\]

with

\[
C := \sup_{p,Q \in \mathbb{R}^2, \tau > 0} \sqrt{\frac{\tau + p^2 - \mu}{\log(1 + \frac{\tau + p^2 - \mu}{\lambda})}} \times \int_{\lambda < q^2} dq \frac{1}{q^2} \sqrt{\frac{\tau + q^2 - \mu}{\log(1 + \frac{\tau + q^2 - \mu}{\lambda})}} \left( (1 + \frac{1}{\lambda})(\tilde{p}^2 + \tilde{q}^2) + \frac{2}{\lambda^2}(\tilde{p} \cdot \tilde{q}) \right)
\]

where \( \tilde{p} := p + \frac{1}{M+2} Q \) and \( \tilde{q} := q + \frac{1}{M+2} Q \). This yields \( P \geq -C \cdot \log(1 + \frac{H_f - \mu}{\lambda}) \). One could now attempt to evaluate the constant \( C \) numerically and compare it with the prefactor \( \pi/(1 + \frac{1}{\lambda^2}) \) in the definition of \( \phi^0(E) \) given in (3). A corresponding numerical analysis was done successfully in the three-dimensional case (11).

## A Appendix

In this appendix we briefly explain the connection between \( \phi(E) \) defined in the introduction, the Hamiltonian \( H_N \) that occurs in (3), and (11), see also Section 5.1 of [7].

Let \( H_0 := M^{-1} P_f^2 + H_f \), and for \( E < 0 \) let \( R_E:= V(H_0 - E)^{-1} \in \mathcal{L}(\mathcal{H}_N, \mathcal{H}_{N-1}) \), where \( V : D(H_0) \cap \mathcal{H}_N \to \mathcal{H}_{N-1} \) is defined by

\[
V\psi := \lim_{n \to \infty} \int_{k^2 \leq n} dk a_k \psi.
\]

The existence of this limit is easily established with the help of the pull-through formula \( a_k(H_0 - E)^{-1} = (H_0 + k^2 - E)^{-1} a_k \) [5,7]. The domain \( D(H_N) \) of \( H_N \) can be characterised as follows: a vector \( \psi \in \mathcal{H}_N \) belongs to \( D(H_N) \) if and only if there is a vector \( w_\psi \in D(\phi) \subseteq \mathcal{H}_{N-1} \) such that for some (and hence all) \( E < 0 \)

\[
\psi - R_E^*w_\psi \in D(H_0), \tag{20}
\]

and

\[
V(\psi - R_E^*w_\psi) = \phi(E)w_\psi. \tag{21}
\]

For \( \psi \in D(H_N) \) the action of \( H_N \) is given by

\[
(H_N - E)\psi = (H_0 - E)(\psi - R_E^*w_\psi). \tag{22}
\]
By \((21), (22)\), and the definition of \(R_E\),

\[
\langle \psi, (H_N - E)\psi \rangle = \langle \psi - R_E^* w_\psi, (H_0 - E)(\psi - R_E w_\psi) \rangle + \langle w_\psi, \phi(E)w_\psi \rangle
\]

which proves Condition \((5)\).

The Hamiltonian \(H_N\) as described above is a self-adjoint operator \([2, 7]\) and it represents the formal expression \((1)\) in the center-of-mass frame, or, which is the same, in the sector of total momentum zero. To explain this let us rewrite \((1)\) in terms of center-of-mass and relative coordinates, \(R = (My + \sum_{i=1}^{N} x_i)/(M + N)\) and \(r_i = x_i - y\), respectively. One obtains the sum of the kinetic energy of the center-of-mass motion, \(-(M + N)^{-1} \Delta_R\), and

\[
\frac{1}{M} \left( \sum_{i=1}^{N} i \nabla r_i \right)^2 - \sum_{i=1}^{N} \Delta r_i - g \sum_{i=1}^{N} \delta(r_i). \tag{23}
\]

Here we recognise in the first two terms the free Hamiltonian \(H_0\). We expect that \(H_N\) agrees with \(H_0\) away from the support of the \(\delta\)-potentials. Indeed, for \(\psi \in D(H_0) \cap \text{Ker}(V)\) we may choose \(w_\psi = 0\). It follows that \(\psi \in D(H_N)\) and that \(H_N \psi = H_0 \psi\). Thus \(H_N\) is an extension of \(H_0\) restricted to \(D(H_0) \cap \text{Ker}(V)\). Now, for a smooth function \(\psi \in \mathcal{H}\), the condition \(\psi \in \text{Ker}(V)\) is equivalent to \(\psi(x_1, \ldots, x_N) = 0\) whenever \(x_k = 0\) for some \(k\). In the literature, an extension of \(H_0\), characterized by a condition of the form \((21)\), is known as Skornyakov-Ter-Martirosyan (STM) extension. In the analogous situation in three dimensions with suitable values of the system parameters, a variety of different STM extensions is known to exist \([3]\).

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