LATTICE STRUCTURES FOR QUANTUM CHANNELS

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Abstract. We suggest that a certain one-to-one parametrization of completely positive maps on the matrix algebra $M_n$ might be useful in the study of quantum channels. This is illustrated in the case of binary quantum channels. While the algorithm is quite intricate, it admits a simple, lattice structure representation.

1. Introduction

Some recent papers deal with the analysis of the completely positive, trace-preserving linear maps on the matrix algebra $M_n$. The analysis is quite complete in the case $n = 2$, as it can be seen in the paper. The purpose of this paper is to introduce an algorithm that tests the complete positivity of a linear map on $M_n$, for any $n \geq 2$. This appears as a sort of Schur-Cohn test and it allows the introduction of certain lattice structures associated to completely positive linear maps. The algorithm is applied to $M_2$ and the result is compared with the analysis in. Since our algorithm produces a "free" parametrization of the completely positive maps on $M_n$, it is nonlinear in nature and other applications in order to check its usefulness remain to be investigated.

2. Completely positive maps on $M_n$

Let $M_n$ denote the algebra of complex $n \times n$ matrices. A linear map $\Phi : A \to \mathcal{L}(\mathcal{H})$ from a $C^*$-algebra into the set $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on the Hilbert space $\mathcal{H}$ is called completely positive if for every positive integer $n$, the map $\Phi \otimes I_{M_n} : A \otimes M_n \to \mathcal{L}(\mathcal{H}) \otimes M_n$ is positivity preserving. By Stinespring Theorem, any such map is the compression of a $*$-homomorphism. For linear completely positive maps on $M_n$, this implies a somewhat more explicit representation of the form:

$$\Phi(X) = \sum_j A_j^* X A_j,$$

where $\{A_j\}$ is a finite set of elements in $M_n$. The representation (2.1) is not unique and another characterization of linear completely positive maps on $M_n$ is also useful. Thus, by a result of Choi, the linear map $\Phi : M_n \to M_n$ is completely
adjoint $\hat{\Phi}$ of a linear map $\Phi$ on $M_n$ that preserve the trace. Such maps are usually called quantum channels $\mathcal{M}_n$ given by the Hilbert-Schmidt inner product (linear in the first variable), $\langle A, B \rangle = TrAB^*$, $A, B \in \mathcal{M}_n$, where $B^*$ denotes the usual adjoint of the operator $B$. It follows that $\Phi$ is trace-preserving if and only if $\hat{\Phi}$ is unital ($\hat{\Phi}(I) = I$).

We will use some standard notation associated to contractions on Hilbert spaces. Thus, let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all bounded linear maps operators from the Hilbert space $\mathcal{H}_1$ into the Hilbert space $\mathcal{H}_2$. The operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is called a contraction if $\|T\| \leq 1$. The defect operator of $T$ is $D_T = (I - T^*T)^{1/2}$ and $\mathcal{D}_T$ denotes the closure of the range of $D_T$. To any contraction $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ one associates the unitary operator $U(T) : \mathcal{H}_1 \oplus \mathcal{D}_T^* \to \mathcal{H}_2 \oplus \mathcal{D}_T$ by the formula:

$$U(T) = \begin{bmatrix} T & D_T^* \\ D_T & -T^* \end{bmatrix}.$$  

3. **Lattice structures**

Let $\Phi$ be a linear completely positive map on $\mathcal{M}_n$. The matrix $S = S_\Phi$ given by (2.2) is positive, and by Theorem 1.5.3 in [1], there exists a uniquely determined family $\Gamma = \{\Gamma_{kj} \mid 1 \leq k \leq j \leq n^2\}$ of complex numbers with the following properties. Thus, $S_{kk} = \Gamma_{kk}$, $1 \leq k \leq n^2$, and for $1 \leq k < j \leq n^2$, $\Gamma_{kj} \in \mathcal{L}(\mathcal{D}_{\Gamma_{k+1,j}}, \mathcal{D}_{\Gamma_{k,j-1}})$ are contractions such that

$$S_{kj} = \Gamma_{kk}^{1/2} (R_{k,j-1}U_{k+1,j-1}C_{k+1,j} + D_{\Gamma_{k+1,j}}^* \ldots D_{\Gamma_{k,j-1}}^* \Gamma_{kj} D_{\Gamma_{k+1,j}} \ldots D_{\Gamma_{j-1,j}}) \Gamma_{jj}^{1/2}.$$  

We use the convention that $\mathcal{D}_{\Gamma_{kk}}$ is just (the closure of) the range of $\Gamma_{kk}$. This algorithm is valid in higher dimensions as well, that is the entries of $S$ could be bounded operators and then the parameters $\Gamma_{kj}$ would be also operators. The notation used in (3.1) is
quite involved but easy to explain. Also, this formula shows that each $S_{kj}$ belongs to a certain disk.

For a fixed $k$, the operator $R_{kj}$ which appears in (3.1) is the row contraction

$$R_{kj} = \begin{bmatrix} \Gamma_{k,k+1}, & D_{\Gamma_{k,k+1}}\Gamma_{k,k+2}, & \cdots, & D_{\Gamma_{k,k+1}}\cdots D_{\Gamma_{k,j-1}}\Gamma_{kj} \end{bmatrix}.$$  

Analogously, for a fixed $j$, the operator $C_{kj}$ is the column contraction

$$C_{kj} = \begin{bmatrix} \Gamma_{j-1,j}, & \Gamma_{j-2,j}D_{\Gamma_{j-1,j}}, & \cdots, & \Gamma_{kj}D_{\Gamma_{k-1,j}}\cdots D_{\Gamma_{j-1,j}} \end{bmatrix}^t,$$

where "$t$" stands for matrix transpose. The operators $U_{ij}$ are defined by the recursion:

$$U_{kk} = 1$$

and for $j > k$,

$$U_{kj} = U_j(\Gamma_{j,j+1})U_j(\Gamma_{j,j+2})\cdots U_j(\Gamma_{kj})(U_{k+1,j} \oplus I_{\Gamma_{k,j}}),$$

where the subscript $j$ at $U_j(\Gamma_{j,j+1})$ means that for $1 \leq l \leq j - k$ the unitary operator $U_j(\Gamma_{k,k+l})$ is defined from

$$(\oplus_{m=1}^{l-1} D_{\Gamma_{k+1,k+m}}) \oplus (D_{\Gamma_{k+1,k+l}} \oplus D_{\Gamma_{k,k+l}}) \oplus (\oplus_{m=l+1}^{j-1} D_{\Gamma_{k,k+m}})$$

into

$$(\oplus_{m=1}^{l-1} D_{\Gamma_{k+1,k+m}}) \oplus (D_{\Gamma_{k,k+l-1}} \oplus D_{\Gamma_{k,k+l}}) \oplus (\oplus_{m=l+1}^{j-1} D_{\Gamma_{k,k+m}})$$

by the formula

$$U_j(\Gamma_{k,k+l}) = I \oplus U(\Gamma_{k,k+l}) \oplus I.$$  

We note that the above formula for $U_{kj}$ comes from the familiar Euler factorization of $SO(N)$, [7].

We obtain the following result.

**Theorem 3.1.** There exists a one-to-one correspondence between the set of linear completely positive maps on $M_n$ and the families $\Gamma = \{\Gamma_{kj} \mid 1 \leq k \leq n^2\}$ of complex numbers such that $\Gamma_{kk} \geq 0$ for $1 \leq k \leq n^2$, and for $1 \leq k < j \leq n^2$, $\Gamma_{kj} \in L(D_{\Gamma_{k+1,j}}, D_{\Gamma_{k,j-1}})$ are contractions. The correspondence is given by (2.3) and (3.1).

This result can be rephrased as a Schur-Cohn type test for complete positivity.

**Algorithm 3.2.** Consider a linear map $\Phi$ on $M_n$. The complete positivity of $\Phi$ can be verified as follows:

1. Consider the matrix $S = S_\Phi$ given by formula (2.2).

2. Check $\Gamma_{kk} \geq 0$ for each $k$. If for some $k$, $\Gamma_{kk} < 0$, then $\Phi$ is not completely positive. If for some $k$, $\Gamma_{kk} = 0$, then the whole $k$th row (and column) of $S$ must be zero.

3. Calculate the numbers $\Gamma_{kj}$ according to formula (3.1). At each step check the condition $|\Gamma_{kj}| \leq 1$ and keep track of the compatibility condition $\Gamma_{kj} \in L(D_{\Gamma_{k+1,j}}, D_{\Gamma_{k,j-1}})$. If this can be done for all indices $kj$, then $\Phi$ is completely positive. Otherwise, $\Phi$ is not completely positive.
We illustrate the applicability of this algorithm for the case of completely positive maps on $M_2$.

**Example 3.3.** A detailed analysis of quantum binary channels is given in [9]. We show here how Theorem 3.1 relates to that analysis. It is showed in [5] that any quantum binary channel $\Phi$ has a representation

$$\Phi(A) = U[\Phi_{t,A}(VAV^*)]U^*,$$

where $U, V \in U(2)$ and $\Phi_{t,A}$ has the matrix representation

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{bmatrix},$$

with respect to the Pauli basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$ of $M_2$. We can obtain (formula (26) in [9]) that

$$S_{\Phi_{t,A}} = \frac{1}{2} \begin{bmatrix} 1 + t_3 + \lambda_3 & t_1 - it_2 & 0 & \lambda_1 + \lambda_2 \\ t_1 + it_2 & 1 - t_3 - \lambda_3 & \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_1 - \lambda_2 & 1 + t_3 - \lambda_3 & t_1 - it_2 \\ \lambda_1 + \lambda_2 & 0 & t_1 + it_2 & 1 - t_3 + \lambda_3 \end{bmatrix}.$$

Similarly, by formula (27) in [9],

$$S_{\Phi_{t,A}} = \frac{1}{2} \begin{bmatrix} 1 + t_3 + \lambda_3 & 0 & t_1 + it_2 & \lambda_1 + \lambda_2 \\ 0 & 1 + t_3 - \lambda_3 & \lambda_1 - \lambda_2 & t_1 + it_2 \\ t_1 - it_2 & \lambda_1 - \lambda_2 & 1 - t_3 - \lambda_3 & 0 \\ \lambda_1 + \lambda_2 & t_1 + it_2 & 0 & 1 - t_3 + \lambda_3 \end{bmatrix}.$$  

It is slightly more convenient to deal with $S = [S_{k,j}]_{k,j=1}^4 = 2S_{\Phi_{t,A}}$. Formula (3.1) gives:

$$S_{11} = \Gamma_{11} = 1 + t_3 + \lambda_3; \quad S_{22} = \Gamma_{22} = 1 + t_3 - \lambda_3;$$

$$S_{33} = \Gamma_{33} = 1 - t_3 - \lambda_3; \quad S_{44} = \Gamma_{44} = 1 - t_3 + \lambda_3;$$

$$\Gamma_{12} = 0, \quad \Gamma_{34} = 0;$$

$$S_{23} = \Gamma_{22}^{1/2} \Gamma_{23} \Gamma_{33}^{1/2},$$

so that

$$\Gamma_{23} = \frac{\lambda_1 - \lambda_2}{(1 + t_3 - \lambda_3)^{1/2}(1 - t_3 - \lambda_3)^{1/2}};$$

$$S_{13} = \Gamma_{11}^{1/2} \Gamma_{13} D_{\Gamma_{23}} \Gamma_{33}^{1/2},$$

so that

$$\Gamma_{13} = \frac{(t_1 + it_2)(1 + t_3 - \lambda_3)^{1/2}}{((1 + t_3 - \lambda_3)(1 - t_3 - \lambda_3) - (\lambda_1 - \lambda_2)^2)^{1/2}(1 + t_3 + \lambda_3)^{1/2}};$$

$$S_{24} = \Gamma_{22}^{1/2} D_{\Gamma_{24}} \Gamma_{44}^{1/2},$$
so that
\[ \Gamma_{24} = \frac{(t_1 + it_2)(1 - t_3 - \lambda_3)^{1/2}}{((1 + t_3 - \lambda_3)(1 - t_3 - \lambda_3) - (\lambda_1 - \lambda_2)^2)^{1/2}(1 - t_3 + \lambda_3)^{1/2}}. \]

Finally,
\[ S_{14} = \Gamma_{11}^{1/2}(-\Gamma_{13}\Gamma_{23}^{*}\Gamma_{24} + D_{\Gamma_{13}}\Gamma_{14}D_{\Gamma_{24}})\Gamma_{44}^{1/2}. \]

By now, the formula for \( \Gamma_{14} \) becomes quite intricate, but there is no problem to write it explicitly. We deduce that \( \Phi_{t,\Lambda} \) is completely positive if and only if the following eighth inequalities hold:

\[ \Gamma_{kk} \geq 0, \quad k = 1, \ldots, 4, \]
\[ |\Gamma_{23}| \leq 1, \quad |\Gamma_{13}| \leq 1, \quad |\Gamma_{24}| \leq 1, \quad |\Gamma_{14}| \leq 1. \]

Also, we know what happens in the degenerate cases. Thus, the implication of \( \Gamma_{kk} = 0 \) for some \( k \) on the structure of \( \Phi_{t,\Lambda} \) is clear. Also, if \( |\Gamma_{23}| = 1 \), then necessarily \( t_1 = t_2 = 0 \) and \( \lambda_1 + \lambda_2 = (1 + t_3 + \lambda_3)^{1/2}\Gamma_{14}(1 - t_3 + \lambda_3)^{1/2} \) for some contraction \( \Gamma_{14} \). If either \( |\Gamma_{13}| = 1 \) or \( |\Gamma_{24}| = 1 \), then necessarily \( \Gamma_{14} = 0 \) and \( S_{14} = \Gamma_{11}^{1/2}(-\Gamma_{13}\Gamma_{23}^{*}\Gamma_{24})\Gamma_{44}^{1/2} \).

We notice that this result is of about the same nature as that in [9]. This is because the first step of (3.1) is precisely Lemma 6 in [9] which is used for the analysis in [9]. If we used the block version of (3.1) then we would deduce precisely Theorem 1 of [9]. What we basically have done here is that we used (3.1) in order to deduce in a systematic way the condition that \( R_{\Phi_{t,\Lambda}} \) in Theorem 1 of [9] is a contraction. One advantage of doing this is that it works in higher dimensions.

We also have to note that the correspondence between \( S_{\Phi} \) and the parameters \( \Gamma \) is nonlinear. Only for the first step the correspondence is affine and therefore can be used in the analysis of extreme points in the case \( n = 2 \), as it was done in [9]. This seems to be unclear for \( n \geq 2 \).

We conclude with the presentation of so-called lattice structures that can be associated to completely positive maps on \( \mathcal{M}_n \). This comes from the remark that \( S_{\Phi} \) has displacement structure as described in [10] and the general lattice structures associated to matrices with displacement structure in [10] can be used in our particular case. We can omit the details. In Figure 1 we show the lattice structure of completely positive maps on \( \mathcal{M}_2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{lattice_structure.png}
\caption{Lattice structure for completely positive maps on \( \mathcal{M}_2 \)}
\end{figure}

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