On the well-posedness for the viscous shallow water equations

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Abstract

In this paper, we prove the existence and uniqueness of the solutions for the 2D viscous shallow water equations with low regularity assumptions on the initial data as well as the initial height bounded away from zero.

1 Introduction

In this paper, we study the 2D viscous shallow water equations with a more general diffusion

\[
\begin{align*}
  h_t + \text{div}(hu) &= 0, \\
  h(u_t + u \cdot \nabla u) - \nu \nabla \cdot (hD(u)) - \nu \nabla (h\text{div}(u)) + h \nabla h &= 0, \\
  u(0, \cdot) &= u_0, \quad h(0, \cdot) = h_0,
\end{align*}
\]

(1.1)

where \( h(t, x) \) is the height of fluid surface, \( u(t, x) = (u_1(t, x), u_2(t, x)) \) is the horizontal velocity vector field, \( D(u) = \frac{1}{2}(\nabla u + \nabla u^t) \) is the deformation tensor, and \( \nu > 0 \) is the viscous coefficient. If the diffusion terms in (1.1) are replaced by \( -\nu \nabla \cdot (h \nabla u) \), then (1.1) turns into the usual viscous shallow water equations.

Recently, the viscous shallow water equations have been widely studied by Mathematicians, see the review paper [4]. Bui [5] proved the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for the shallow water equations with initial data \( h_0, u_0 \) in H"older spaces as well as \( h_0 \) bounded away from vacuum. Kloeden [17] and Sundbye [20] independently proved global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem in Sobolev spaces. Later, Sundbye [21] also proved global existence and uniqueness of classical solutions to the Cauchy problem. However, for all above results (except [5]), the authors only consider the case when the initial data \( h_0 \) is a small perturbation of some positive constant \( \bar{h}_0 \) and \( u_0 \) is small in some sense. Very recently, Wang and Xu [23] proved the local well-posedness of the Cauchy problem in Sobolev spaces for the large data \( u_0 \) and \( h_0 \) closing to \( \bar{h}_0 \). More precisely, they obtained the following result.

**Theorem 1.1** [23] Let \( \bar{h}_0 \) be a strictly positive constant and \( s > 2 \). Assume that

\[
\begin{align*}
  (i) \quad (u_0, h_0 - \bar{h}_0) &\in H^s(\mathbb{R}^2) \otimes H^s(\mathbb{R}^2); \\
  (ii) \quad \|h_0 - \bar{h}_0\|_{H^s} &\ll \bar{h}_0.
\end{align*}
\]

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Then there exist a positive time $T$ and a unique solution $(u, h)$ of (1.1) such that
\[ u, \quad h - \bar{h}_0 \in L^\infty([0, T], H^s), \quad \nabla u \in L^2([0, T]; H^s). \] (1.2)
Moreover, there exists a strictly positive constant $c$ such that if
\[ \|u_0\|_{H^s} + \|h_0 - \bar{h}_0\|_{H^s} \leq c, \] (1.3)
then we can choose $T = +\infty$.

One purpose of this paper is to study the well-posedness of (1.1) for the initial data with the minimal regularity. For the incompressible Navier-Stokes equations, such research has been initiated by Fujita and Kato\[16\], see also \[6, 7, 18\] for other relevant results. They proved local well-posedness for the incompressible Navier-Stokes equations in the scaling invariant space. The scaling invariance means that if $(u, p)$ is a solution of the incompressible Navier-Stokes equations with initial data $u_0(x)$, then
\[ u_\lambda(t, x) \triangleq \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) \triangleq \lambda^2 p(\lambda^2 t, \lambda x) \] (1.4)
is also a solution of the incompressible Navier-Stokes equations with $u_{0, \lambda} \triangleq \lambda u_0(\lambda x)$. Obviously, $\dot{H}^{d-1}(\mathbb{R}^d)$ is a scaling invariant space under the scaling of (1.4), i.e.
\[ \|u_\lambda\|_{\dot{H}^{d-1}} = \|u\|_{\dot{H}^{d-1}}. \]
The equations (1.1) have no scaling invariance like the incompressible Navier-Stokes equations. However, due to the similarity of the structure between (1.1) and the incompressible Navier-Stokes equations, we still solve (1.1) for initial data whose regularity fits with the scaling of (1.4). It should be pointed out that R. Danchin was the first to consider the similar problem for the compressible Navier-Stokes equations, and some ideas of this paper is motivated by \[11\].

The second purpose of this paper is to prove the local well-posedness of (1.1) under more natural assumption that the initial height is bounded away from zero. For the initial data with slightly higher regularity, this can be easily obtained by modifying the argument of Danchin\[13\]. However, for the initial data with low regularity, his method is not applicable any more, since the proof of \[13\] relies on the fact that some profits can be gained from the inclusion map $B^s \hookrightarrow L^\infty$ in the case of $s > \frac{d}{2}$. For this reason, we have to introduce some kind of weighted Besov space $E^s_T(\mathbb{R}^d)$ (see Section 3) which is crucial to get rid of the condition that the initial height $h_0$ is close to $\bar{h}_0$. One important observation is that the $E^s_T$ norm of the solution is small for small time $T$.

Before stating our main result, let us first introduce some notations and definitions. Choose a radial function $\varphi \in S(\mathbb{R}^d)$ such that
\[ \text{supp} \varphi \subset \{ \xi \in \mathbb{R}^d; \frac{5}{6} \leq |\xi| \leq \frac{12}{5} \}, \quad \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \]
Here $\varphi_k(\xi) = \varphi(2^{-k} \xi)$, $k \in \mathbb{Z}$.

**Definition 1.1** Let $k \in \mathbb{Z}$, the Littlewood-Paley projection operators $\Delta_k$ and $S_k$ are defined as follows
\[ \Delta_k f = \varphi(2^{-k} D) f, \quad S_k f = \sum_{j \leq k-1} \Delta_j f, \quad \text{for} \quad f \in S'(\mathbb{R}^d). \]
We denote the space \( Z'(\mathbb{R}^d) \) by the dual space of \( Z(\mathbb{R}^d) = \{ f \in S(\mathbb{R}^d); D^\alpha f(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index} \} \), it also can be identified by the quotient space of \( S'(\mathbb{R}^d)/\mathcal{P} \) with the polynomials space \( \mathcal{P} \). The formal equality

\[
    f = \sum_{k \in \mathbb{Z}} \Delta_k f
\]

holds true for \( f \in Z'(\mathbb{R}^d) \) and is called the homogeneous Littlewood-Paley decomposition. It has nice properties of quasi-orthogonality: with our choice of \( \varphi \),

\[
    \Delta_j \Delta_k f = 0 \quad \text{if} \quad |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} \Delta_k f) = 0 \quad \text{if} \quad |j - k| \geq 4. \quad (1.5)
\]

**Definition 1.2** Let \( s \in \mathbb{R} \), \( 1 \leq p, r \leq +\infty \). The homogeneous Besov space \( \dot{B}_{p,r}^s \) is defined by

\[
    \dot{B}_{p,r}^s = \{ f \in Z'(\mathbb{R}^d) : \| f \|_{\dot{B}_{p,r}^s} < +\infty \},
\]

where

\[
    \| f \|_{\dot{B}_{p,r}^s} = \begin{cases} 
    \left( \sum_{k \in \mathbb{Z}} 2^{ksr} \| \Delta_k f \|_p^r \right)^{\frac{1}{r}}, & \text{for } r < +\infty, \\
    \sup_{k \in \mathbb{Z}} 2^{ks} \| \Delta_k f \|_p, & \text{for } r = +\infty.
\end{cases}
\]

If \( p = r = 2 \), \( \dot{B}_{2,2}^s = \dot{H}^s \), and if \( d = 2 \), we have \( \dot{B}_{1,1}^1 \hookrightarrow L^\infty \) and

\[
    \| f \|_\infty \leq C \| f \|_{\dot{B}_{1,1}^1}.
\]

We refer to [8, 22] for more details.

In addition to the general time-space space such as \( L^\rho(0, T; \dot{B}_{p,r}^s) \), we introduce a useful mixed time-space homogeneous Besov space \( \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \) which is initiated in [10] and is used in the proof of the uniqueness.

**Definition 1.3** Let \( s \in \mathbb{R} \), \( 1 \leq p, r, \rho \leq +\infty \), \( 0 < T \leq +\infty \). The mixed time-space homogeneous Besov space \( \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \) is defined by

\[
    \tilde{L}_T^\rho(\dot{B}_{p,r}^s) = \{ f \in Z'(\mathbb{R}^{d+1}) : \| f \|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} < +\infty \},
\]

where

\[
    \| f \|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} = \left\| 2^{ks} \left( \int_0^T \| \Delta_k f(t) \|_p^\rho dt \right)^{\frac{1}{\rho}} \right\|_{\ell^r}.
\]

Using the Minkowski inequality, it is easy to verify that

\[
    L_T^\rho(\dot{B}_{p,r}^s) \subseteq \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \quad \text{if} \quad \rho \leq r \quad \text{and} \quad \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \subseteq L_T^\rho(\dot{B}_{p,r}^s) \quad \text{if} \quad \rho \geq r.
\]

Next, we introduce a hybrid-index Besov space which plays an important role in the study of compressible fluids and is initiated in [11, 12].
Definition 1.4 Let $s$, $\sigma \in \mathbb{R}$, and set
\[
\|f\|_{\tilde{B}_2^{s,\sigma}} \triangleq \sum_{k \leq 0} 2^{ks} \|\Delta_k f\|_2 + \sum_{k > 0} 2^{k\sigma} \|\Delta_k f\|_2.
\]
Let $m = -\left[\frac{d}{2} + 1 - s\right]$, we define
\[
\tilde{B}_2^{s,\sigma}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\tilde{B}_2^{s,\sigma}} < +\infty \right\} \quad \text{if} \quad m < 0,
\]
\[
\tilde{B}_2^{s,\sigma}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_m : \|f\|_{\tilde{B}_2^{s,\sigma}} < +\infty \right\} \quad \text{if} \quad m \geq 0,
\]
where $\mathcal{P}_m$ denotes the set of polynomials of degree $\leq m$.

Throughout this paper, we will denote $\dot{B}_2^{s,1}$ by $B_2^s$, and $\tilde{B}_2^{s,\sigma}$ by $\tilde{B}_2^{s,\sigma}$. The following facts can be easily verified by using the definition of $\tilde{B}_2^{s,\sigma}$:

(i) $\tilde{B}_2^{s,s} = B_2^{s,1}$;
(ii) If $s \leq \sigma$, then $\tilde{B}_2^{s,\sigma} = \dot{B}_2^{s,1} \cap \dot{B}_2^{\sigma,1}$. Otherwise, $\tilde{B}_2^{s,\sigma} = \dot{B}_2^{s,1} + \dot{B}_2^{\sigma,1}$.

Now we state our main result as follows.

Theorem 1.2 Let $\bar{h}_0$ be a positive constant. Assume that

(i) $(u_0, h_0 - \bar{h}_0) \in B^0(\mathbb{R}^2) \otimes \tilde{B}^{0,1}(\mathbb{R}^2)$;
(ii) $h_0 \geq \bar{h}_0$.

Then there exist a positive time $T$ and a unique solution $(u, h)$ of (1.1) such that

\[
\begin{align*}
& u \in C([0,T]; B^0) \cap L^1(0,T; B^2), \quad h - \bar{h}_0 \in C([0,T]; \tilde{B}^{0,1}) \cap L^1(0,T; \tilde{B}^{2,1}), \quad h \geq \frac{1}{2} \bar{h}_0. \quad (1.6)
\end{align*}
\]

Moreover, there exists a strictly positive constant $c$ such that if

\[
\|u_0\|_{B^0} + \|h_0 - \bar{h}_0\|_{\tilde{B}^{0,1}} \leq c,
\]

then we can choose $T = +\infty$.

The structure of this paper is as follows.

In Section 2, we recall some useful multilinear estimates in the Besov spaces. In Section 3, we prove the existence of solution. In Section 4, we prove the uniqueness of the solution. Finally, in the Appendix, we prove some multilinear estimates in the weighted Besov spaces.

Throughout the paper, $C$ denotes various “harmless” large finite constants, and $c$ denotes various “harmless” small constants. We shall sometimes use $X \lesssim Y$ to denote the estimate $X \leq CY$ for some constant $C$. We denote $\| \cdot \|_p$ by the $L^p$ norm of a function.

2 Multilinear estimates in the Besov spaces

Let us first recall the Bony’s paraproduct decomposition.
Definition 2.1  We shall use the following Bony’s paraproduct decomposition (see [1, 3])
\[ fg = T_fg + T_g f + R(f, g), \] (2.1)
with
\[ T_fg = \sum_{k \in \mathbb{Z}} S_{k-1} f \Delta_k g \quad \text{and} \quad R(f, g) = \sum_{k \in \mathbb{Z}} \sum_{|k' - k| \leq 1} \Delta_k f \Delta_{k'} g. \] (2.2)

Next, let us recall some useful lemmas and multilinear estimates in the Besov spaces.

Lemma 2.1  (Bernstein’s inequality) Let \( 1 \leq p \leq q \leq +\infty \). Assume that \( f \in \mathcal{S}'(\mathbb{R}^d) \), then for any \( \gamma \in \mathbb{Z}^d \), there exist constants \( C_1, C_2 \) independent of \( f, j \) such that
\[
\supp \hat{f} \subseteq \{|\xi| \leq A_0 2^j\} \Rightarrow \|\partial^\gamma f\|_q \leq C_1 2^{j|\gamma| + jd(\frac{d}{2} - \frac{1}{2})}\|f\|_p,
\]
\[
\supp \hat{f} \subseteq \{|A_1 2^j \leq |\xi| \leq A_2 2^j\} \Rightarrow \|f\|_p \leq C_2 2^{-j|\gamma|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_p.
\]
The proof can be found in [8].

Proposition 2.2  If \( s > 0, f, g \in B^s \cap L^\infty \). Then \( fg \in B^s \cap L^\infty \) and
\[
\|fg\|_{B^s} \leq C(\|f\|_\infty\|g\|_{B^s} + \|g\|_\infty\|f\|_{B^s}). \] (2.3)
If \( s_1, s_2 \leq \frac{d}{2} \) such that \( s_1 + s_2 > 0 \), \( f \in B^{s_1} \), and \( g \in B^{s_2} \). Then \( fg \in B^{s_1 + s_2 - \frac{d}{2}} \) and
\[
\|fg\|_{B^{s_1 + s_2 - \frac{d}{2}}} \leq C\|f\|_{B^{s_1}}\|g\|_{B^{s_2}}. \] (2.4)
If \( |s| < \frac{d}{2} \), \( 1 \leq r \leq +\infty \), \( f \in \dot{B}^s_{2,r} \), and \( g \in \dot{B}^s_{2,r} \). Then \( fg \in \dot{B}^s_{2,r} \) and
\[
\|fg\|_{\dot{B}^s_{2,r}} \leq C\|f\|_{\dot{B}^s_{2,r}}\|g\|_{\dot{B}^s_{2,r}}. \] (2.5)
If \( s \in (-\frac{d}{2}, \frac{d}{2}] \), \( f \in B^s \), and \( g \in \dot{B}^{-s}_{2,\infty} \). Then \( fg \in \dot{B}^{-s}_{2,\infty} \) and
\[
\|fg\|_{\dot{B}^{-s}_{2,\infty}} \leq C\|f\|_{B^s}\|g\|_{\dot{B}^{-s}_{2,\infty}}. \] (2.6)
If \( 1 \leq \rho_1, \rho_2, \rho \leq \infty \), \( s \in (-\frac{d}{2}, \frac{d}{2}] \), \( f \in \mathcal{L}^{\rho_1}_{\rho} (B^s) \), and \( g \in \mathcal{L}^{\rho_2}_{\rho} (\dot{B}^{-s}_{2,\infty}) \). Then there holds
\[
\|fg\|_{\mathcal{L}^{\rho_1}_{\rho}(\dot{B}^{-s}_{2,\infty})} \leq C\|f\|_{\mathcal{L}^{\rho_1}_{\rho}(B^s)}\|g\|_{\mathcal{L}^{\rho_2}_{\rho}(\dot{B}^{-s}_{2,\infty})}, \] (2.7)
where \( \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho} \).

Proof.  For the sake of simplicity, we only present the proof of (2.4) below, the others can be deduced in the same way (see also [14, 19]). By the Bony’s paraproduct decomposition and the property of quasi-orthogonality (1.5), for fixed \( j \in \mathbb{Z} \), we write
\[
\Delta_j(fg) = \sum_{|k-j| \leq 3} \Delta_j(S_{k-1} f \Delta_k g) + \sum_{|k-j| \leq 3} \Delta_j(S_{k-1} g \Delta_k f) + \sum_{k \geq j-2} \sum_{|k-k'| \leq 1} \Delta_j(\Delta_k f \Delta_{k'} g)
\]
\[ \equiv I + II + III. \]
Thanks to the definition of Besov space $B^s$, we have

$$\|fg\|_{B^{s_1+s_2-\frac{d}{2}}} \leq \left( \sum_{j \in \mathbb{Z}} 2^{(s_1+s_2-\frac{d}{2})j} \|I\|_2 \right) + \cdots + \left( \sum_{j \in \mathbb{Z}} 2^{(s_1+s_2-\frac{d}{2})j} \|III\|_2 \right) \triangleq I' + II' + III'. \quad (2.8)$$

It suffices to estimate the above three terms separately. Using the Young’s inequality and lemma [2.1] we have

$$\|\Delta_j(S_{k-1}f \Delta_k g)\|_2 \lesssim \|S_{k-1}f\|_\infty \|\Delta_k g\|_2 \lesssim \sum_{k' \leq k-2} 2^{k's_1} \|\Delta_{k'} f\|_2 2^{k'(\frac{d}{2}-s_1)} \|\Delta_k g\|_2$$

$$\lesssim \|f\|_{B^{s_1}} \|\Delta_k g\|_2 2^{k'(\frac{d}{2}-s_1)},$$

where we have used the fact $s_1 \leq \frac{d}{2}$ in the last inequality. Hence, we get

$$I' \lesssim \|f\|_{B^{s_1}} \sum_{j \in \mathbb{Z}} 2^{(s_1+s_2-\frac{d}{2})j} \sum_{|k-j| \leq 3} 2^{k'(\frac{d}{2}-s_1)} \|\Delta_k g\|_2$$

$$\lesssim \|f\|_{B^{s_1}} \sum_{|\ell| \leq 3} 2^{-(s_1+s_2-\frac{d}{2})\ell} \sum_{j \in \mathbb{Z}} 2^{s_2(j+\ell)} \|\Delta_{j+\ell} g\|_2 \lesssim \|f\|_{B^{s_1}} \|g\|_{B^{s_2}}. \quad (2.9)$$

Similarly, using the fact $s_2 \leq \frac{d}{2}$, we can obtain

$$II' \lesssim \|f\|_{B^{s_1}} \|g\|_{B^{s_2}}. \quad (2.10)$$

Now we turn to estimate $III'$. From Lemma [2.4] and Hölder inequality, it follows that

$$\|\Delta_j(\Delta_k f \Delta_k g)\|_2 \lesssim 2^{j\frac{d}{2}} \|\Delta_k f \Delta_k g\|_1 \lesssim 2^{j\frac{d}{2}} \|\Delta_k f\|_2 \|\Delta_k g\|_2.$$

So, we get by Minkowski inequality that for $s_1 + s_2 > 0$

$$III' \lesssim \sum_{j \in \mathbb{Z}} 2^{(s_1+s_2-\frac{d}{2})j} 2^j \left( \sum_{k \geq j-1} \sum_{|k'-k| \leq 1} \|\Delta_k f\|_2 \|\Delta_{k'} g\|_2 \right)$$

$$\lesssim \sum_{\ell \geq -2} 2^{-(s_1+s_2)\ell} \sum_{j \in \mathbb{Z}} 2^{s_1(j+\ell)} \|\Delta_{j+\ell} g\|_2 \|g\|_{B^{s_2}} \lesssim \|f\|_{B^{s_1}} \|g\|_{B^{s_2}}. \quad (2.11)$$

Summing up (2.8)-(2.11), we get the desired inequality (2.4).

**Proposition 2.3** (1) Let $s > 0$. Assume that $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^d)$ such that $F(0) = 0$. Then there exists a constant $C(s,d,F)$ such that if $u \in B^s \cap L^\infty$, there holds

$$\|F(u)\|_{B^s} \leq C(1 + \|u\|_\infty)^{[s]+1} \|u\|_{B^s}; \quad (2.12)$$

and if $u \in \dot{B}^s_{2,\infty} \cap L^\infty$, there holds

$$\|F(u)\|_{\dot{B}^s_{2,\infty}} \leq C(1 + \|u\|_\infty)^{[s]+1} \|u\|_{\dot{B}^s_{2,\infty}}. \quad (2.13)$$
(2) Assume that \( G \in W^{[\frac{d}{2}]+3,\infty}_\text{loc}(\mathbb{R}^d) \) such that \( G'(0) = 0 \). Then there exists a functions \( C(s,d,G) \) such that if \(-\frac{d}{2} < s \leq \frac{d}{2}\), \( u, v \in B^s_2 \cap L^\infty \) and \( u - v \in \dot{B}^s_2 \), there holds

\[
\|G(u) - G(v)\|_{B^s_2} \leq C(\|u\|_\infty, \|v\|_\infty)(\|u\|_{B^s_2} + \|v\|_{B^s_2})\|u - v\|_{B^s_2};
\]  

(2.14)

and if \(|s| < \frac{d}{2}\), \( u, v \in B^s_2 \cap L^\infty \) and \( u - v \in \dot{B}^s_2 \), there holds

\[
\|G(u) - G(v)\|_{\dot{B}^s_2} \leq C(\|u\|_\infty, \|v\|_\infty)(\|u\|_{\dot{B}^s_2} + \|v\|_{\dot{B}^s_2})\|u - v\|_{\dot{B}^s_2}. 
\]  

(2.15)

Proof. We can refer to [2] for the proof of (1). For (2), we refer to [11, 15]. For example, we write

\[ G(u) - G(v) = (u - v) \int_0^1 G'(v + \tau(u - v))d\tau, \]

then it follows from (2.5) that for \(|s| < \frac{d}{2}\)

\[
\|G(u) - G(v)\|_{\dot{B}^s_2} \leq C\|u - v\|_{\dot{B}^s_2} \|G'(v + \tau(u - v))\|_{\dot{B}^s_2},
\]

which together with (2.12) implies (2.15). \( \blacksquare \)

Proposition 2.4  Let \( A \) be a homogeneous smooth function of degree \( m \). Assume that \(-\frac{d}{2} < s_1, t_1, s_2, t_2 \leq 1 + \frac{d}{2}\), then there hold: if \( k \geq 1 \),

\[
|\langle A(D)\Delta_k (v \cdot \nabla f), A(D)\Delta_k f \rangle| \lesssim \alpha_k 2^{-k(s_2 - m)}\|v\|_{\dot{B}^{s_1 + t_2}_2} \|\Delta_k f\|_2,
\]

(2.16)

and if \( k \leq 0 \),

\[
|\langle A(D)\Delta_k (v \cdot \nabla f), A(D)\Delta_k f \rangle| \lesssim \alpha_k 2^{-k(s_1 - m)}\|v\|_{\dot{B}^{s_1 + t_2}_2} \|\Delta_k f\|_2.
\]

(2.17)

and if \( k \geq 1 \),

\[
|\langle A(D)\Delta_k (v \cdot \nabla f), \Delta_k g \rangle + \langle \Delta_k (v \cdot \nabla g), A(D)\Delta_k f \rangle| \lesssim \alpha_k \|v\|_{\dot{B}^{s_1 + t_2}_2} \|A(D)\Delta_k f\|_2 + 2^{-k(s_2 - m)}\|f\|_{\dot{B}^{s_1 + t_2}_2} \|\Delta_k g\|_2,
\]

(2.18)

and if \( k \leq 0 \),

\[
|\langle A(D)\Delta_k (v \cdot \nabla f), \Delta_k g \rangle + \langle \Delta_k (v \cdot \nabla g), A(D)\Delta_k f \rangle| \lesssim \alpha_k \|v\|_{\dot{B}^{s_1 + t_2}_2} \|A(D)\Delta_k f\|_2 + 2^{-k(s_1 - m)}\|f\|_{\dot{B}^{s_1 + t_2}_2} \|\Delta_k g\|_2,
\]

(2.19)

where \( \sum_{k \in \mathbb{Z}} \alpha_k \leq 1 \).

For the proof we refer to [12].
3 Existence

In this section, we prove the existence of the solution for the 2D viscous shallow water equations. Without loss of generality, we assume that $h_0 = 1$ and $\nu = 1$. Replacing $h$ by $h + 1$ in (1.1), we rewrite (1.1) as

$$\begin{align*}
\begin{cases}
h_t + \text{div} u + \text{div}(hu) = 0, \\
u_t - \nabla \cdot D(u) + \nabla \text{div} u + u \cdot \nabla u + \nabla h = 0,
\end{cases}
\end{align*}$$

(3.1)

3.1 The linearized system

In this subsection, we consider the linearized system of (3.1):

$$\begin{align*}
\begin{cases}
h_t + v \cdot \nabla h + \text{div} u = \mathcal{H}, \\
u_t - \nabla \cdot D(u) + \nabla \text{div} u + v \cdot \nabla u + \nabla h = \mathcal{G},
\end{cases}
\end{align*}$$

(3.2)

Let us first introduce some definitions. Set

$$e^r_k(t) \triangleq (1 - e^{-cr^2k^2t})^\frac{1}{r}, \quad \omega_k(t) = \sum_{k \geq k} 2^{-k-1} (e^1_k(t) + e^2_k(t)),$$

where $c$ is a positive constant which will be determined later. We remark that

$$\omega_k(t) \leq C, \quad \text{for any } k \in \mathbb{Z},$$

which will be constantly used in the following.

Definition 3.1 Let $s \in \mathbb{R}$ and $T > 0$. The function space $E_T^s$ is defined by

$$E_T^s = \{ f \in \mathcal{Z}'((0, T) \times \mathbb{R}^d) : \| f \|_{E_T^s} < +\infty \},$$

where

$$\| f \|_{E_T^s} \triangleq \sum_{k \in \mathbb{Z}} 2^{ks} \omega_k(T) \| \Delta_k f \|_{L_T^\infty(L^2)}.$$

Definition 3.2 Let $s_1, s_2 \in \mathbb{R}$ and $T > 0$. The function space $\tilde{E}_T^{s_1, s_2}$ is defined by

$$\tilde{E}_T^{s_1, s_2} = \{ f \in \mathcal{Z}'((0, T) \times \mathbb{R}^d) : \| f \|_{\tilde{E}_T^{s_1, s_2}} < +\infty \},$$

where

$$\| f \|_{\tilde{E}_T^{s_1, s_2}} \triangleq \sum_{k \leq 0} 2^{ks_1} \omega_k(T) \| \Delta_k f \|_{L_T^\infty(L^2)} + \sum_{k \geq 1} 2^{ks_2} \omega_k(T) \| \Delta_k f \|_{L_T^\infty(L^2)}.$$

Remark 3.1 If $s_1 \leq s_2$, then $\tilde{E}_T^{s_1, s_2} = E_T^{s_1} \cap E_T^{s_2}$. Otherwise, $\tilde{E}_T^{s_1, s_2} = E_T^{s_1} + E_T^{s_2}$. 

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Let \((u, h)\) be a smooth solution of \((3.2)\). We want to establish the following \textit{a-priori} estimates for \((h, u)\):

\[
\|u\|_{L^2_t(B^2)} + \|u\|_{L^2_t(B^1)} + \|h\|_{E_T^{0,1}} \\
\leq C \sum_{k \in \mathbb{Z}} \omega_k(T) E_k(0) + C \sum_{k \in \mathbb{Z}} \omega_k(T) \|\Delta_k G(t)\|_{L^2_t(L^2)} \\
+ C \sum_{k \geq 1} \omega_k(T) \|\nabla \Delta_k H(t)\|_{L^2_t(L^2)} + C \sum_{k < 1} \omega_k(T) \|\Delta_k H(t)\|_{L^2_t(L^2)} \\
+ C \|u\|_{L^2_t(B^1)} \|v\|_{L^2_t(B^1)} + C \|h\|_{E_T^{0,1}} \|v\|_{L^2_t(B^2)},
\]

and

\[
\|u\|_{L^\infty_t(B^0)} + \|h\|_{L^\infty_t(\tilde{B}^{0,1})} + \|h\|_{L^1_t(\tilde{B}^{2,1})} \\
\leq E_0 + C \left( \|H\|_{L^1_t(\tilde{B}^{0,1})} + \|G\|_{L^1_t(\tilde{B}^{0,1})} + \int_0^T V'(t)(\|u(t)\|_{B^0} + \|h(t)\|_{\tilde{B}^{0,1}}) dt \right),
\]

where \(V(t) = \|v(t')\|_{L^1_t(B^2)}\) and

\[
E_0 = \sum_{k \in \mathbb{Z}} E_k(0), \quad E_k(t) = \begin{cases} E_{kk}(t) & k \geq 1, \\ E_{0k}(t) & k < 1, \end{cases}
\]

with

\[
E_{kk}^2(t) = \frac{1}{2} \|u_k(t)\|^2_2 + \|\nabla h_k(t)\|^2_2 + (u_k(t), \nabla h_k(t)), \quad \text{and}
\]

\[
E_{0k}^2(t) = \frac{1}{2} \|u_k(t)\|^2_2 + \frac{1}{2} \|h_k(t)\|^2_2 + \frac{1}{8} (u_k(t), \nabla h_k(t)).
\]

Let us begin with the proof of \((3.3)\) and \((3.4)\). Set

\[
\begin{align*}
 u_k &= \Delta_k u, \quad h_k = \Delta_k h, \quad H^k = \Delta_k H, \quad G^k = \Delta_k G.
\end{align*}
\]

Then we get by applying the operator \(\Delta_k\) to \((3.2)\) that

\[
\begin{cases}
\partial_t h_k + \Delta_k (v \cdot \nabla h) + \Div u_k = H_k, \\
\partial_t u_k - (\nabla \cdot D(u_k)) + \Div u_k + \Delta_k (v \cdot \nabla u) + \nabla h_k = G_k, \\
u_k(0, \cdot) = \Delta_k u_0, \quad h_k(0, \cdot) = \Delta_k h_0.
\end{cases}
\]

(3.5)

Multiplying the second equation of \((3.5)\) by \(u_k\), and integrating the resulting equation over \(\mathbb{R}^2\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u_k\|^2 + \frac{1}{2} \|\nabla u_k\|^2_2 + \frac{3}{2} \|\Div u_k\|^2_2 + (\nabla h_k, u_k) = (G_k, u_k) - (\Delta_k (v \cdot \nabla u), u_k).
\]

(3.6)

In the following, we will deal with the high frequency and the low frequency of \(h\) in a different manner.

\textbf{High frequencies:} \(k \geq 1\).
Firstly, applying $\nabla$ to the first equation of (3.5), and multiplying it by $\nabla h_k$, then integrating the resulting equation over $\mathbb{R}^2$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla h_k\|_2^2 + (\nabla \text{div} u_k, \nabla h_k) = (\nabla H_k, \nabla h_k) - (\nabla \Delta_k (v \cdot \nabla h), \nabla h_k). \tag{3.7}
\]

Secondly, applying the operator $\nabla$ to the first equation of (3.5) and taking the $L^2$ product of the resulting equation with $u_k$; then taking the $L^2$ product of second equation of (3.5) with $\nabla h_k$, we get by summing them up that
\[
\frac{d}{dt} (u_k, \nabla h_k) - \| \text{div} u_k \|_2^2 - 2(\nabla \text{div} u_k, \nabla h_k) + \| \nabla h_k \|_2^2 \\
= (\nabla H_k, u_k) + (\mathcal{G}_k, \nabla h_k) - (\nabla \Delta_k (v \cdot \nabla h), u_k) - (\Delta_k (v \cdot \nabla u), \nabla h_k), \tag{3.8}
\]
where we used the fact that
\[
(\nabla \cdot D(u_k) + \nabla \text{div} u_k, \nabla h_k) = 2(\nabla \text{div} u_k, \nabla h_k).
\]

Then we get by summing up (3.6), (3.7)×2, and (3.8) that
\[
\frac{d}{dt} \frac{1}{2} \|u_k\|_2^2 + \|\nabla h_k\|_2^2 + (u_k, \nabla h_k) \\
+ \left[\|\nabla h_k\|_2^2 + \frac{1}{2} \|\nabla u_k\|_2^2 + \frac{1}{2} \|\text{div} u_k\|_2^2 + (\nabla h_k, u_k)\right] \\
= \left[(\nabla H_k, u_k) + 2(\nabla H_k, \nabla h_k) + (\mathcal{G}_k, u_k) + (\mathcal{G}_k, \nabla h_k)\right] \\
- (\Delta_k (v \cdot \nabla u), u_k) - 2(\nabla \Delta_k (v \cdot \nabla h), \nabla h_k) \\
- \left[(\nabla \Delta_k (v \cdot \nabla h), u_k) + (\Delta_k (v \cdot \nabla u), \nabla h_k)\right] \\
\triangleq I + II + III + IV. \tag{3.9}
\]

Note that
\[
(u_k, \nabla h_k) \leq \frac{1}{3} \|u_k\|_2^2 + \frac{3}{4} \|\nabla h_k\|_2^2,
\]
hence, we get by the definition of $E_{hk}$ that
\[
\frac{1}{6} (\|u_k\|_2^2 + \|\nabla h_k\|_2^2) \leq E_{hk}^2 \leq 2(\|u_k\|_2^2 + \|\nabla h_k\|_2^2). \tag{3.10}
\]

Similarly, using the fact that $\frac{5}{6}2^k \geq \frac{5}{3}$ and (3.10), we have
\[
\|\nabla h_k\|_2^2 + \frac{1}{2} \|\nabla u_k\|_2^2 + \frac{1}{2} \|\text{div} u_k\|_2^2 + (\nabla h_k, u_k) \geq \frac{1}{8} E_{hk}^2. \tag{3.11}
\]

By summing up (3.9)-(3.11), we obtain
\[
\frac{d}{dt} E_{hk}^2 + c E_{hk}^2 \leq C[I + II + III + IV]. \tag{3.12}
\]

In order to obtain (3.3), we use Lemma 5.1 to deal with the right hand terms of (3.12). Firstly, we get by using the Cauchy-Schwartz inequality and (3.10) that
\[
|I| \leq C(\|\nabla H_k(t)\|_2 + \|\mathcal{G}_k(t)\|_2) E_{hk}. \tag{3.13}
\]

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From Lemma 5.1 and (3.10), it follows that
\[
|II + III + IV| \leq C(\| \mathcal{F}_k^1(t) \|_2 + \| \bar{\mathcal{F}}_k^0(t) \|_2) E_{hk}. \tag{3.14}
\]

By summing up (3.12), and (3.13)-(3.14), we obtain
\[
\frac{d}{dt} E_{hk} + c E_{hk} \leq C \left( \| \nabla h_k(t) \|_2 + \| \mathcal{G}_k(t) \|_2 + \| \mathcal{F}_k^1(t) \|_2 + \| \bar{\mathcal{F}}_k^0(t) \|_2 \right), \tag{3.15}
\]
which implies that
\[
\| E_{hk}(t) \|_{L_t^\infty} \leq E_{hk}(0) + C \left( \| \nabla h_k(t) \|_{L_t^1(L^2)} + \| \mathcal{G}_k(t) \|_{L_t^1(L^2)} \right.
+ \| \mathcal{F}_k^1(t) \|_{L_t^1(L^2)} + \| \bar{\mathcal{F}}_k^0(t) \|_{L_t^1(L^2)} \biggr). \tag{3.16}
\]

Furthermore, by (5.4) and (5.5), there holds
\[
\sum_{k \in \mathbb{Z}} \omega_k(T) \left( \| \mathcal{F}_k^1(t) \|_{L_t^1(L^2)} + \| \bar{\mathcal{F}}_k^0(t) \|_{L_t^1(L^2)} \right) \leq C \left( \| u \|_{L_t^\infty(B^1)} \| v \|_{L_t^\infty(B^1)} + \| h \|_{E_1} \| v \|_{L_t^1(B^2)} \right).
\]

Multiplying \( \omega_k(T) \) on both sides of (3.16), then summing up the resulting equation over \( k \geq 1 \), we obtain
\[
\sum_{k \geq 1} \omega_k(T) \| E_{hk}(t) \|_{L_t^\infty} \leq \sum_{k \geq 1} \omega_k(T) E_{hk}(0)
+ C \sum_{k \geq 1} \omega_k(T) \left( \| \nabla h_k(t) \|_{L_t^1(L^2)} + \| \mathcal{G}_k(t) \|_{L_t^1(L^2)} \right)
+ C \left( \| u \|_{L_t^\infty(B^1)} \| v \|_{L_t^\infty(B^1)} + \| h \|_{E_1} \| v \|_{L_t^1(B^2)} \right). \tag{3.17}
\]

Next, we use the decay effect of the parabolic operators to estimate \( \| u \|_{L_t^\infty(B^1) \cap L_t^1(B^2)} \). It follows from (3.6) and Lemma 5.1 that
\[
\frac{d}{dt} \| u_k \|_2 + c 2^k \| u_k \|_2 \leq C(\| \nabla h_k(t) \|_2 + \| \mathcal{G}_k(t) \|_{L^2} + \| \bar{\mathcal{F}}_k^0(t) \|_{L^2}),
\]
which implies that
\[
\| u_k \|_2 \leq e^{-ct2^k} \| u_k(0) \|_2 + C e^{-ct2^k} *_{t} \left( \| \nabla h_k(t) \|_2 + \| \mathcal{G}_k(t) \|_2 + \| \bar{\mathcal{F}}_k^0(t) \|_2 \right),
\]
where the sign * denotes the convolution of functions defined in \( \mathbb{R}^+ \), more precisely,
\[
\text{e}^{-ct2^k} *_{t} f \triangleq \int_0^t \text{e}^{-c(t-\tau)2^k} f(\tau) d\tau.
\]

Taking the \( L^r \) norm for \( r = 1, 2 \) with respect to \( t \), we get by using the Young’s inequality that
\[
\| u_k \|_{L_t^r(B^2)} \leq C 2^{-2k/r} e_k^r(T) \left( \| u_k(0) \|_2 + \| \nabla h_k \|_{L_t^1(B^2)} + \| \mathcal{G}_k \|_{L_t^1(B^2)} + \| \bar{\mathcal{F}}_k^0 \|_{L_t^1(B^2)} \right),
\]

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which together with (5.5) implies that
\[
\sum_{k \geq 1} \left( 2^{2k} \|u_k\|_{L^2} + 2^k \|u_k\|_{L^2} \right) \leq C \sum_{k \geq 1} \omega_k(T) \|u_k(0)\|_2
\]
\[+ C \sum_{k \geq 1} \omega_k(T) \left( \|\nabla h_k\|_{L^1} + \|G_k\|_{L^1} \right) + C \|u\|_{L^2(B)} \|v\|_{L^2(B)}, \tag{3.18}\]
where we used the fact that
\[e_1^k + e_2^k \leq \omega_k(T).\]

On the other hand, it follows from (3.15) that
\[\|E_{hk}\|_2 \leq e^{-ct} E_{hk}(0) + C e^{-ct} \|\nabla H_k(t)\|_2 + \|G_k(t)\|_2 + \|F_k(t)\|_2 + \|\bar{F}_k(t)\|_2.\]

Taking the \(L^1\) norm with respect to \(t\), we get by using the Young’s inequality that
\[\|E_{hk}\|_{L^1_t} \leq C(1 - e^{-ctT}) E_{hk}(0) + C \left( \|\nabla H_k(t)\|_{L^1_t} + \|G_k(t)\|_{L^1_t} \right)
\[+ \|F_k(t)\|_{L^1_t} + \|\bar{F}_k(t)\|_{L^1_t}. \tag{3.19}\]

Note that for \(k \geq 1\)
\[1 - e^{-ct} \leq 1 - e^{-ct2^k} \leq \omega_k(t),\]

which together with (3.19) and Lemma 5.1 gives
\[\sum_{k \geq 1} \|E_{hk}\|_{L^1_t} \leq C \sum_{k \geq 1} \omega_k(T) E_{hk}(0) + C \sum_{k \geq 1} \omega_k(T) \left( \|\nabla H_k(t)\|_{L^1_t} + \|G_k(t)\|_{L^1_t} \right)
\[+ C \left( \|u\|_{L^2(B)} \|v\|_{L^2(B)} + \|h\|_{L^2(B)} \|v\|_{L^2(B)} \right). \tag{3.20}\]

Plugging (3.20) into (3.18), we obtain
\[\sum_{k \geq 1} \left( 2^{2k} \|u_k\|_{L^2} + 2^k \|u_k\|_{L^2} \right)
\[\leq C \sum_{k \geq 1} \omega_k(T) E_{hk}(0) + C \sum_{k \geq 1} \omega_k(T) \left( \|\nabla H_k(t)\|_{L^1_t} + \|G_k(t)\|_{L^1_t} \right)
\[+ C \left( \|u\|_{L^2(B)} \|v\|_{L^2(B)} + \|h\|_{L^2(B)} \|v\|_{L^2(B)} \right). \tag{3.21}\]

On the other hand, in order to obtain (3.3), we use Proposition 2.4 to deal with the right hand terms of (3.12). Applying (2.16) with \(s_1 = s_2 = 0\) to II, (2.16) with \(s_1 = 0, s_2 = 1\) to III, (2.18) with \(t_1 = t_2 = 0, s_1 = 0, s_2 = 1\) to IV, we obtain
\[|II + III + IV| \leq C E_{hk} \alpha_k V'(t)(\|u\|_{B_0} + \|h\|_{B_0}), \tag{3.22}\]

with \(\sum_{k \in \mathbb{Z}} \alpha_k \leq 1\) and \(V(t) = \|v(t')\|_{L^1_t(B^2)}\). From (3.13) and (3.22), it follows that
\[\frac{d}{dt} E_{hk} + c E_{hk} \leq C \left( \|\nabla H_k(t)\|_2 + \|G_k(t)\|_2 + \alpha_k V'(t)(\|u\|_{B_0} + \|h\|_{B_0}) \right), \tag{12}\]
from which, a similar proof of \([3.21]\) ensures that
\[
\sum_{k \geq 1} \left( \|E_{hk}\|_{L^1_T} + \|E_{hk}\|_{L^\infty_T} \right) \leq C \sum_{k \geq 1} E_{hk}(0) \\
+ C \left( \|H\|_{L^1_T(B^0)} + \|G\|_{L^1_T(B^0)} + \int_0^T V'(t)(\|u(t)\|_{B^0} + \|h(t)\|_{B^0,1}) dt \right). \tag{3.23}
\]

**Low frequencies:** \(k < 1\).

Multiplying the first equation of \([3.5]\) by \(h_k\), we get by integrating the resulting equation over \(\mathbb{R}^2\) that
\[
\frac{1}{2} \frac{d}{dt} \|h_k\|^2 + (\text{div} u_k, h_k) = (\mathcal{H}_k, h_k) - (\Delta_k (v \cdot \nabla h), h_k). \tag{3.24}
\]

Summing up \([3.6], \tag{3.8} \times \frac{1}{8}, \text{ and } \tag{3.24}\), we obtain
\[
\frac{d}{dt} \left[ \frac{1}{2} \|u_k\|^2 + \frac{1}{2} \|h_k\|^2 + \frac{1}{8} (u_k, \nabla h_k) \right] \\
+ \left[ \frac{1}{8} \|\nabla h_k\|^2 + \frac{1}{2} \|\nabla u_k\|^2 + \frac{11}{8} \|\text{div} u_k\|^2 - \frac{1}{4} (\nabla \text{div} u_k, \nabla h_k) \right] \\
= \left[ \frac{1}{8} (\nabla \mathcal{H}_k, u_k) + (\mathcal{H}_k, h_k) + (\mathcal{G}_k, u_k) + \frac{1}{8} (\mathcal{G}_k, \nabla h_k) \right] \\
- (\Delta_k (v \cdot \nabla u), u_k) - (\Delta_k (v \cdot \nabla h), h_k) \\
- \frac{1}{8} \left[ (\nabla \Delta_k (v \cdot \nabla h), u_k) + (\Delta_k (v \cdot \nabla u), \nabla h_k) \right] \\
\triangleq I + II + III + IV. \tag{3.25}
\]

Note that \(2^k \leq 1\), we get by the Cauchy-Schwartz inequality that
\[
\frac{1}{8} (u_k, \nabla h_k) \leq \frac{3}{10} \|u_k\|_2 \|h_k\|_2 \leq \frac{1}{4} \|u_k\|^2 + \frac{1}{4} \|h_k\|^2,
\]
hence, we get by the definition of \(E_{ik}\) that
\[
\frac{1}{4} (\|u_k\|^2 + \|h_k\|^2) \leq E_{ik}^2 \leq 2(\|u_k\|^2 + \|h_k\|^2). \tag{3.26}
\]

Similarly, we can prove
\[
\frac{1}{4} (\nabla \text{div} u_k, \nabla h_k) \leq \frac{3}{5} \|\nabla u_k\|_2 \|\nabla h_k\|_2 \leq \frac{9}{10} \|\nabla u_k\|^2 + \frac{1}{10} \|\nabla h_k\|^2,
\]
which together with \([3.26]\) implies that
\[
\frac{1}{8} \|\nabla h_k\|^2 + \frac{1}{2} \|\nabla u_k\|^2 + \frac{11}{8} \|\text{div} u_k\|^2 - \frac{1}{4} (\nabla \text{div} u_k, \nabla h_k) \geq \frac{1}{160} 2^{2k}(\|u_k\|^2 + \|h_k\|^2) \geq \frac{1}{320} 2^{2k} E_{ik}^2. \tag{3.27}
\]

By summing up \([3.25], \tag{3.27}\), we obtain
\[
\frac{d}{dt} E_{ik}^2 + c2^{2k} E_{ik}^2 \leq C|I + II + III + IV|. \tag{3.28}
\]

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In order to obtain (3.33), we use Lemma 5.1 to estimate the right hand terms of (3.28). Using the fact that $2^k \leq 1$, we get by the Cauchy-Schwartz inequality and (3.26) that

$$|I| \leq C(||H_k(t)||_2 + ||G_k(t)||_2)E_{lk}.$$  \hspace{1cm} (3.29)

Using Lemma 5.1 and (3.26), we have

$$|II + III + IV| \leq C(||F_k^1(t)||_2 + ||F_k^0(t)||_2 + ||\tilde{F}_k^0(t)||_2)E_{lk}.$$  \hspace{1cm} (3.30)

By summing up (3.28)−(3.30), we obtain

$$\frac{d}{dt} E_{lk} + c 2^{2k} E_{lk} \leq C\left(||H_k(t)||_2 + ||G_k(t)||_2 + ||F_k^1(t)||_2 + ||F_k^0(t)||_2 + ||\tilde{F}_k^0(t)||_2\right),$$

which implies that

$$E_{lk} \leq e^{-c 2^{2k} t} E_{lk}(0) + C e^{-c 2^{2k} t} \left(||H_k(t)||_2 + ||G_k(t)||_2 + ||F_k^1(t)||_2 + ||F_k^0(t)||_2 + ||\tilde{F}_k^0(t)||_2\right).$$

Taking the $L^r$ norm with respect to $t$, we get by using the Young’s inequality that

$$||E_{lk}||_{L^r_t} \leq C 2^{-2k/r} e^{c_k^*(T)} \left(||H_k(t)||_{L^r_t(L^2)} + ||G_k(t)||_{L^r_t(L^2)} + ||F_k^1(t)||_{L^r_t(L^2)} + ||F_k^0(t)||_{L^r_t(L^2)} + ||\tilde{F}_k^0(t)||_{L^r_t(L^2)}\right),$$

from which and Lemma 5.1, it follows that

$$\sum_{k < 1} \omega_k(T)||E_{lk}||_{L^r_t} \leq C \sum_{k < 1} \omega_k(T)E_{lk}(0) + \sum_{k < 1} \omega_k(T)(||H_k(t)||_{L^r_t(L^2)} + ||G_k(t)||_{L^r_t(L^2)})$$

$$+ C\left(||u||_{L^r_t(B^1)}||v||_{L^r_t(B^1)} + ||h||_{E_{r_t}^0}||v||_{L^r_t(B^1)}\right),$$  \hspace{1cm} (3.31)

and

$$\sum_{k < 1} (2^k ||E_{lk}||_{L^r_t} + 2^k ||E_{lk}||_{L^r_t})$$

$$\leq C \sum_{k < 1} \omega_k(T)E_{lk}(0) + \sum_{k < 1} \omega_k(T)(||H_k(t)||_{L^r_t(L^2)} + ||G_k(t)||_{L^r_t(L^2)})$$

$$+ C\left(||u||_{L^r_t(B^1)}||v||_{L^r_t(B^1)} + ||h||_{E_{r_t}^0}||v||_{L^r_t(B^1)}\right).$$  \hspace{1cm} (3.32)

On the other hand, in order to obtain (3.33), we use Proposition 5.4 to deal with the right hand terms of (3.28). Applying (2.17) with $s_1 = s_2 = 0$ to $II$, (2.17) with $s_1 = 0, s_2 = 1$ to $III$, (2.19) with $t_1 = t_2 = 0, s_1 = 0, s_2 = 1$ to $IV$, we obtain

$$|II + III + IV| \leq CE_{lk} \alpha_k V'(t)(||u||_{B^0} + ||h||_{B_{0,1}}),$$  \hspace{1cm} (3.33)

with $\sum_{k \in \mathbb{Z}} \alpha_k \leq 1$ and $V(t) = ||v(t)||_{L^1_t(B^2)}$. From (3.32) and (3.36), it follows that

$$\frac{d}{dt} E_{lk} + c 2^{2k} E_{lk} \leq C\left(||H_k(t)||_2 + ||G_k(t)||_2 + \alpha_k V'(t)(||u||_{B^0} + ||h||_{B_{0,1}})\right),$$

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from which and a similar proof of (3.21) ensure that
\[
\sum_{k<1} \left( 2^{2k} \| E_k \|_{L^2_T} + \| E_{h_k} \|_{L^\infty_T} \right) \leq \sum_{k<1} E_{h_k}(0)
+ C \left( \| \mathcal{H} \|_{L^1_0(\tilde{B}^{0,1})} + \| \mathcal{G} \|_{L^1_0(B^0)} + \int_0^T V'(t)(\| u(t) \|_{B^0} + \| h(t) \|_{\tilde{B}^{0,1}}) \right). \tag{3.34}
\]

The completion of the a-priori estimates

Firstly, adding up (3.17), (3.21), (3.31), and (3.32) yields that
\[
\| u \|_{L^2_T(B^2)} + \| u \|_{L^2_T(B^1)} + \| h \|_{E_T^{0,1}} \\
\leq C \sum_{k \in \mathbb{Z}} \omega_k(T) E_k(0) + C \sum_{k \in \mathbb{Z}} \omega_k(T) \| \mathcal{G}_k(t) \|_{L^1_0(L^2)} + \| \mathcal{H}_k(t) \|_{L^1_0(L^2)} + \| u \|_{L^2_T(B^1)} \| v \|_{L^2_T(B^1)} + C \| h \|_{E_T^{0,1}} \| v \|_{L^1_T(B^2)},
\tag{3.35}
\]
where we used the fact that
\[
\| h \|_{E^1_T} \leq C \| h \|_{E^{0,1}_T}.
\]

On the other hand, adding up (3.23) and (3.34) gives rise to
\[
\| u \|_{L^\infty_T(B^0)} + \| h \|_{L^\infty_T(\tilde{B}^{0,1})} + \| h \|_{L^1_T(\tilde{B}^{2,1})} \\
\leq E_0 + C \left( \| \mathcal{H} \|_{L^1_0(\tilde{B}^{0,1})} + \| \mathcal{G} \|_{L^1_0(B^0)} + \int_0^T V'(t)(\| u \|_{B^0} + \| h \|_{\tilde{B}^{0,1}}) dt \right), \tag{3.36}
\]
which together with the Gronwall inequality implies that
\[
\| u \|_{L^\infty_T(B^0)} + \| h \|_{L^\infty_T(\tilde{B}^{0,1})} + \| h \|_{L^1_T(\tilde{B}^{2,1})} \\
\leq C e^{C \| v \|_{L^1_T(B^2)}} \left( E_0 + \| \mathcal{H} \|_{L^1_0(\tilde{B}^{0,1})} + \| \mathcal{G} \|_{L^1_0(B^0)} \right). \tag{3.37}
\]

Finally, let us remark that
\[
E_0 \approx (\| h_0 \|_{\tilde{B}^{0,1}} + \| u_0 \|_{B^0}).
\]

### 3.2 The uniform estimate of the approximate sequence of solutions

In this subsection, we will construct the approximate solutions of (3.1) and present the uniform estimate of the approximate solutions. Let us first define the approximate sequence \((h^n, u^n)_{n \in \mathbb{N}}\) of (3.1) by the following system:
\[
\begin{dcases}
\partial_t h^{n+1} + u^n \cdot \nabla h^{n+1} + \text{div} u^{n+1} = \mathcal{H}^n, \\
\partial_t u^{n+1} - (\nabla \cdot D(u^{n+1}) + \nabla \text{div} u^{n+1}) + u^n \cdot \nabla u^{n+1} + \nabla h^{n+1} = \mathcal{G}^n, \\
(h^{n+1}, u^{n+1})|_{t=0} = \sum_{|k| \leq n+1} \Delta_k(h_0, u_0),
\end{dcases}
\tag{3.38}
\]
Assume that (3.39)-(3.41) hold for $(h^n, u^n)_{n \in N_0}$ and that the following bounds hold for all $n$ and $\eta$.

Set $(h_0, u_0) = (0, 0)$ and solve the linear system, we can define $(h^n, u^n)_{n \in N_0}$ by the induction. Next, we are going to prove by the induction that there exist positive constants $\eta$, $K$, and $T$ such that the following bounds hold for all $n \in N_0$:

\[ 1 + h^n \geq \frac{1}{2}, \quad \text{(3.39)} \]
\[ \|u^n\|_{L^1(B^2)} + \|h^n\|_{\tilde{E}^0} \leq \eta, \quad \text{(3.40)} \]
\[ \|u^n\|_{L^\infty(B^2)} + \|h^n\|_{L^1(\tilde{B}^0, \tilde{B}^2)} \leq KE_0. \quad \text{(3.41)} \]

Assume that (3.39)-(3.41) hold for $(h^n, u^n)$, we need to prove that (3.39)-(3.41) also hold for $(h^{n+1}, u^{n+1})$. Applying the a-priori estimates (3.35) and (3.37) to $(h^{n+1}, u^{n+1})$, we obtain

\[
\|u^{n+1}\|_{L^1(B^2)} + \|u^{n+1}\|_{L^2(B^1)} + \|h^{n+1}\|_{\tilde{E}^0} \leq C \sum_{k \in \mathbb{Z}} \omega_k(T) \|G^n_k(t)\|_{L^1(B^2)} + C \sum_{k \geq 1} \omega_k(T) \|\nabla H^n_k(t)\|_{L^1(B^2)} \\
+ C \sum_{k < 1} \omega_k(T) \|\tilde{H}^n_k(t)\|_{L^1(B^2)} + C \|u^{n+1}\|_{L^2(B^1)} \|u^n\|_{L^2(B^1)} + C \|h^{n+1}\|_{\tilde{E}^0} \|u^n\|_{L^1(B^2)},
\]

(3.42)

and

\[
\|u^{n+1}\|_{L^\infty(B^2)} + \|h^{n+1}\|_{L^1(\tilde{B}^0, \tilde{B}^2)} + \|h^{n+1}\|_{L^1(\tilde{B}^0, \tilde{B}^2)} \leq C e^{CE_0} \left( E_0 + \|H^n\|_{L^1(\tilde{B}^0, \tilde{B}^2)} + \|G^n\|_{L^1(B^2)} \right),
\]

(3.43)

with

\[ Q_0(T) \triangleq \sum_{k \in \mathbb{Z}} \omega_k(T) E_k(0). \]

Thanks to (2.4), we have

\[ \|H^n\|_{B^0} \leq C \|H^n\|_{B^0} \|u^n\|_{B^2}, \quad \text{and} \quad \|H^n\|_{B^1} \leq C \|h^n\|_{B^1} \|u^n\|_{B^2}, \]

which together with the fact that $\tilde{B}^0 = B^0 \cap B^1$ yields

\[ \|H^n\|_{L^1(\tilde{B}^0, \tilde{B}^2)} \leq C \|h^n\|_{L^1(\tilde{B}^0, \tilde{B}^2)} \|u^n\|_{L^1(B^2)} \leq CKE_0 \eta. \quad \text{(3.44)} \]

We rewrite $G^n$ as

\[ \frac{\nabla h^n}{1 + h^n} \tilde{\nabla} u^n = (1 + h^n) \nabla \left( \frac{h^n}{1 + h^n} \right) \tilde{\nabla} u^n. \]
Using (2.4) and (2.12), we get
\[
\|G^n\|_{L^1_t(B^0)} \leq C \left\| \nabla \left( \frac{h^n}{1 + h^n} \right) \right\|_{L^\infty_t(B^0)} \left\| (1 + h^n) \tilde{\nabla} u^n \right\|_{L^1_t(B^1)}
\leq C \left( 1 + \|h^n\|_{L^\infty_t(L^\infty)}^2 \|h^n\|_{L^\infty_t(B^1)} (1 + h^n) \|L^1_t(B^1) \right) \leq C \left( 1 + \|h^n\|_{L^\infty_t(L^\infty)}^3 \|h^n\|_{L^\infty_t(B^1)} \|u^n\|_{L^1_t(B^2)} \leq CKE0(1 + KE0)^3\eta. \right.
\]
(3.45)

Plugging (3.44) and (3.45) into (3.33) yields that
\[
\|u^{n+1}\|_{L^\infty_t(B^0)} + \|h^{n+1}\|_{L^\infty_t(B^0)} + \|h^{n+1}\|_{L^1_t(B^2)} \leq C e^{C\eta} \left( E_0 + KE0(1 + KE0)^3\eta \right). \tag{3.46}
\]

We take \( T, \eta > 0 \) small enough and \( K = 4C \) such that
\[
e^{C\eta} \leq 2, \quad K(1 + KE0)^3\eta \leq 1, \quad (3.47)
\]
from which and (3.46), it follows that
\[
\|u^{n+1}\|_{L^\infty_t(B^0)} + \|h^{n+1}\|_{L^\infty_t(B^0)} + \|h^{n+1}\|_{L^1_t(B^2)} \leq KE0.
\]
This proves (3.41) for \((u^{n+1}, h^{n+1})\).

Next, we prove (3.40) for \((u^{n+1}, h^{n+1})\). Applying Lemma 5.2 with \( s_1 = 0 \) and \( s_2 = 1 \), (2.4) with \( s_1 = s_2 = 1 \), and Lemma 5.4 with \( s = 1 \), we obtain
\[
\sum_{k \in \mathbb{Z}} \omega_k(T) \|G^n_k(t)\|_{L^1_t(L^2)} \leq C \left\| \nabla \left( \frac{h^n}{1 + h^n} \right) \right\|_{E^0_t} \left\| (1 + h^n) \tilde{\nabla} u^n \right\|_{L^1_t(B^1)}
\leq C \left( 1 + \|h^n\|_{L^\infty_t(L^\infty)}^3 \|h^n\|_{E^1_t}(1 + h^n) \|L^1_t(B^1) \right) \leq C \left( 1 + \|h^n\|_{L^\infty_t(L^\infty)}^4 \|h^n\|_{E^1_t} \|u^n\|_{L^1_t(B^2)} \right) \leq C(1 + KE0)^4\eta^2. \tag{3.47}
\]

On the other hand, we apply Lemma 5.2 with \( s_1 = 0 \), \( s_2 = 1 \) to get
\[
\sum_{k \geq 1} \omega_k(T) \|\nabla H^n_k(t)\|_{L^1_t(L^2)} + \sum_{k < 1} \omega_k(T) \|H^n_k(t)\|_{L^1_t(L^2)}
\leq C \sum_{k \in \mathbb{Z}} \omega_k(T) \left( \|\nabla h^n_k\|_{L^\infty_t(L^\infty)} \|h^n_k\|_{L^\infty_t(L^\infty)} \|u^n\|_{L^1_t(B^1)} \right) + C \sum_{k \in \mathbb{Z}} \omega_k(T) 2^{2k} \|u^n_k\|_{L^1_t(L^2)} \|h^n\|_{L^\infty_t(B^0)} \leq I + II.
\]

Obviously, we have
\[
I \leq C \left\| h^n \right\|_{E^0_t} \|u^n\|_{L^1_t(B^2)} \leq C\eta^2. \tag{3.48}
\]
In order to estimate \( II \), we first fix \( k_0 \geq 1 \) such that
\[
\sum_{k \geq k_0} \|u_k(0)\|_2 \leq \frac{\eta}{16CKE0}. \tag{3.49}
\]
Then we write
\[ II = \sum_{k \geq k_0} \omega_k(T)2^{2k} \| u^n_k \|_{L^1_t(L^2_x)} \| \nabla u^n \|_{L^\infty_t(B^{\frac{3}{2}})} + \sum_{k \leq k_0} \omega_k(T)2^{2k} \| u^n_k \|_{L^1_t(L^2_x)} \| \nabla u^n \|_{L^\infty_t(B^{\frac{3}{2}})} \]
\[ \triangleq I_{I1} + I_{I2}. \]

Using (3.18), (3.37), and (3.49), we obtain
\[ II_1 \leq CKE_0 \left[ \sum_{k \geq k_0} \omega_k(T) \| u^n_k(0) \|_2 + \sum_{k \geq k_0} \omega_k(T) \left( \| \nabla h^n_k \|_{L^1_t(L^2_x)} + \| G^{n-1}_k \|_{L^1_t(L^2_x)} \right) \right. \]
\[ + \left. \| u^n \|_{L^2_t(B^1)} \| u^{n-1} \|_{L^2_t(B^1)} \right] \leq CKE_0 \left[ \frac{\eta}{16 CKE_0} + \sum_{k \geq k_0} \omega_k(T) \| \nabla h^n_k \|_{L^1_t(L^2_x)} + (1 + K E_0)^4 \eta^2 \right]. \quad (3.50) \]

On the other hand, thanks to (3.19) and Lemma 5.1, we have
\[ \sum_{k \geq k_0} \omega_k(T) \| \nabla h^n_k \|_{L^1_t(L^2_x)} \leq C(1 - e^{-cT}) \sum_{k \geq k_0} \omega_k(T) E_{h,k}(0) \]
\[ + C(1 - e^{-cT}) \sum_{k \geq k_0} \omega_k(T) \left( \| \nabla h^{n-1}_k(t) \|_{L^1_t(L^2_x)} + \| G^{n-1}_k(t) \|_{L^1_t(L^2_x)} \right) \]
\[ + C \| u^n \|_{L^2_t(B^1)} \| u^{n-1} \|_{L^2_t(B^1)} + C \| h^n \|_{E^1_t} \| u^{n-1} \|_{L^2_t(B^1)} \]
\[ \leq C(1 - e^{-cT}) E_0 + C(1 - e^{-cT}) K E_0 \eta + C(1 + K E_0)^4 \eta^2, \]
where we used (3.44) and (3.47) in the second inequality. Plugging the above inequality into (3.50) yields that
\[ II_1 \leq CKE_0 \left[ \frac{\eta}{16 CKE_0} + (1 - e^{-cT})(E_0 + K E_0 \eta) + (1 + K E_0)^4 \eta^2 \right]. \quad (3.51) \]

Note for \( k \leq k_0 \), we can choose \( T > 0 \) small enough so that
\[ \omega_k(T) \leq \frac{1}{16 CKE_0 \eta}, \quad (R_2) \]
so we get
\[ |II_2| \leq \frac{\eta}{16}. \quad (3.52) \]

Plugging (3.47), (3.48), (3.51), (3.52) into (3.42), we get
\[ \| u^{n+1} \|_{L^1_t(B^2)} + \| u^{n+1} \|_{L^2_t(B^1)} + \| h^{n+1} \|_{E^0_t} \]
\[ \leq C \mathcal{Q}_0(T) + \frac{\eta}{8} + C(1 + K E_0)^5 \eta^2 + C K E_0 (1 - e^{-cT})(E_0 + K E_0 \eta) \]
\[ + C \eta \| u^{n+1} \|_{L^2_t(B^1)} + \| h^{n+1} \|_{E^0_t}. \quad (3.53) \]

Note that \( \mathcal{Q}_0(0) = 0 \), we can take \( T, \eta \) small enough such that
\[ C \eta \leq \frac{1}{2}, \quad C \mathcal{Q}_0(T) \leq \frac{\eta}{8}, \quad C(1 + K E_0)^5 \eta < \frac{1}{8}, \quad \text{and} \]
\[ C K E_0 (1 - e^{-cT})(E_0 + K E_0 \eta) \leq \frac{\eta}{8}, \quad (R_3) \]
which together with (3.53) gives
\[ \|u^{n+1}\|_{L^1_t(B^2)} + \|u^{n+1}\|_{L^2_t(B^1)} + \|h^{n+1}\|_{\tilde{E}^{0,1}_T} \leq \eta. \]

Finally, let us prove (3.39) for \( h^{n+1} \). We rewrite the first equation of (3.38) as
\[ \partial_t (1 + h^{n+1}) + u^n \cdot \nabla (1 + h^{n+1}) + \text{div} u^{n+1} - \mathcal{H} = 0. \]

Then \( 1 + h^{n+1} \) can be represented as
\[ (1 + h^{n+1})(t, x) = (1 + h^{n+1}_0)((\psi^n_t)^{-1}(x)) + \int_0^t \text{div} u^{n+1}(\tau, \psi^n_\tau((\psi^n_t)^{-1}(x))) d\tau \]
\[ + \int_0^t \mathcal{H}(\tau, \psi^n_\tau((\psi^n_t)^{-1}(x))) d\tau, \]
(3.54)
where the flow map \( \psi^n_t \) is defined by
\[ \begin{cases} 
\partial_t \psi^n_t(x) = u^n(t, \psi^n_0(x)) \\
\psi^n_t|_{t=0} = x.
\end{cases} \]

Thanks to the inclusion map \( \tilde{B}^1 \hookrightarrow L^\infty \) and (2.4), we get
\[ \int_0^t \|\text{div} u^{n+1}(\tau, \psi^n_\tau((\psi^n_t)^{-1}(x)))\|_\infty d\tau \leq \|u^{n+1}\|_{L^1_t(B^2)} \leq \eta, \]
\[ \int_0^t \|\mathcal{H}(\tau, \psi^n_\tau((\psi^n_t)^{-1}(x)))\|_\infty d\tau \leq \|h^n\|_{L^1_t(B^1)} \]
\[ \leq C\|h^n\|_{L^\infty_t(\tilde{B}^{0,1})} \|u^n\|_{L^1_t(B^2)} \leq CKE_0\eta, \]

from which and (3.54), it follows that
\[ 1 + h^{n+1} \geq \frac{3}{4} - (1 + CKE_0)\eta. \]
(3.55)

We take \( \eta \) small enough such that
\[ (1 + CKE_0)\eta \leq \frac{1}{4}, \]
(\( \mathcal{R}_4 \))
which together with (3.55) ensures that
\[ 1 + h^{n+1} \geq \frac{1}{2}. \]

So far, we have show that \( T, \eta \) can be chosen small enough such that the assumption (\( \mathcal{R}_1 \)) – (\( \mathcal{R}_4 \)) hold under which the approximate solutions \((u^n, h^n)_{n \in \mathbb{N}_0}\) is uniformly bounded in
\[ \mathcal{E}_T \triangleq \left( L^\infty_t(\tilde{B}^0) \cap L^1_t(B^2) \right) \times \left( L^\infty_t(\tilde{B}^{0,1}) \cap L^1_t(\tilde{B}^{2,1}) \right). \]

It should be pointed out that if \( \|u_0\|_{B^0} + \|h_0\|_{\tilde{B}^{0,1}} \) is small enough, we can take \( T = +\infty \) such that the assumption (\( \mathcal{R}_1 \)) – (\( \mathcal{R}_4 \)) hold.
### 3.3 The existence of the solution

Now let us turn to prove the existence of the solution, and the standard compact arguments will be used. In the section 3.2, we have showed that the approximate solutions \((h^n, u^n)_{n \in \mathbb{N}}\) satisfy (3.39)-(3.41), and without loss of generality, we can assume the following:

\[
1 + h^n \geq \frac{1}{2}, \quad (3.56)
\]

\[
\|u^n\|_{L_T^\infty(B^0) \cap L_T^2(B^2)} + \|h^n\|_{L_T^\infty(\overline{B}^0,1) \cap L_T^2(\overline{B}^2,1)} \leq KE_0. \tag*{(3.57)}
\]

Using the interpolation and the fact that \(B^0 \cap B^1 = \overline{B}^{0,1}\), we have

\[
\|h^n\|_{L_T^2(B^1)} \lesssim \|h^n\|_{L_G^\infty(\overline{B}^{0,1})} \|h^n\|_{L_T^2(B^2,1)}, \quad \|u^n\|_{L_T^2(B^1)} \lesssim \|u^n\|_{L_T^2(\overline{B}^0,1)} \|u^n\|_{L_T^2(B^2)},
\]

\[
\|h^n\|_{L_T^2(B^{1,2})} \lesssim \|h^n\|_{L_G^\infty(\overline{B}^{0,1})} \|h^n\|_{L_T^2(B^2)}, \quad \|u^n\|_{L_T^2(B^{1,2})} \lesssim \|u^n\|_{L_T^2(\overline{B}^0,1)} \|u^n\|_{L_T^2(B^2)},
\]

from which and (3.56), it follows that

\[
\|h^n\|_{L_T^2(B^1)} + \|u^n\|_{L_T^2(B^1)} + \|h^n\|_{L_T^2(B^{1,2})} + \|u^n\|_{L_T^2(B^{1,2})} \lesssim KE_0. \tag*{(3.58)}
\]

Now, we show that \((h^n, u^n)\) is uniformly bounded in \(C_T^{\frac{1}{2}}(B^0) \times C_T^{\frac{1}{2}}(B^{-\frac{1}{2}})\). Using (2.4), (3.57) and (3.58), it is easy to verify that

\[
\|u^n \cdot \nabla h^{n+1}\|_{L_T^2(B^{-\frac{1}{2}})} \lesssim \|u^n\|_{L_T^2(B^0)} \|h^{n+1}\|_{L_T^\infty(\overline{B}^{0,1})} \lesssim (KE_0)^2,
\]

\[
\|h^n \nabla u^n\|_{L_T^2(B^0)} \lesssim \|u^n\|_{L_T^2(B^1)} \|h^n\|_{L_T^\infty(\overline{B}^{0,1})} \lesssim (KE_0)^2,
\]

from which and the first equation of (3.38), it follows that \(\partial_t h^n\) is uniformly bounded in \(L_T^2(B^0)\) which implies \(h^n\) is uniformly bounded in \(C_T^{\frac{1}{2}}(B^0)\). On the other hand, thanks to (2.4), (3.56) and (2.12), we have

\[
\|u^n \cdot \nabla u^{n+1}\|_{L_T^2(B^{-\frac{1}{2}})} \lesssim \|u^n\|_{L_T^2(B^0)} \|u^{n+1}\|_{L_T^2(B^{1,2})} \lesssim (KE_0)^2,
\]

\[
\left\|\nabla h^n \right\|_{L_T^2(B^{-\frac{1}{2}})} \lesssim C(1 + \|h^n\|_{L_T^\infty(B^{0,1})}^{3}) \|u^n\|_{L_T^2(B^{1,2})} \lesssim C(1 + KE_0)^3 KE_0,
\]

from which and the second equation of (3.38), it follows that \(\partial_t u^n\) is uniformly bounded in \(L_T^2(B^{-\frac{1}{2}})\) which implies \(u^n\) is uniformly bounded in \(C_T^{\frac{1}{2}}(B^{-\frac{1}{2}})\).

Next, we claim that the inclusions \(B^0 \cap B^1 \hookrightarrow L^2\) and \(B^{-\frac{1}{2}} \cap B^0 \hookrightarrow \dot{H}^{-\frac{1}{2}}\) are locally compact. Indeed, these can be proved by noting that for \(s' < s\), \(\dot{H}^{s'} \cap \dot{H}^s \hookrightarrow \dot{H}^{s'}\) is locally compact and for \(s' \in \mathbb{R}, B^s \hookrightarrow \dot{H}^s\). Then, by the Arzela-Ascoli theorem and Cantor’s diagonal process, there exist a subsequence \((u^{n_k}, h^{n_k})\) and a function \((u, h)\) such that

\[
(u^{n_k}, h^{n_k}) \rightarrow (u, h) \quad \text{in} \quad C_{loc}(\dot{H}_{loc}^{-\frac{1}{2}}) \times C_{loc}(L_T^2_{loc}), \tag*{(3.59)}
\]

as \(n_k \rightarrow \infty\). On the other hand, \((u^{n_k}, h^{n_k})\) is uniformly bounded in \(E_T\), then there exists a subsequence (which still denoted by \((u^{n_k}, h^{n_k})\)) such that

\[
(u^{n_k}, h^{n_k}) \rightarrow (u, h) \quad \text{in} \quad E_T,
\]
where “$\to$” denotes weak* convergence.

Finally, let us prove that $(u, h)$ solves (1.1) in the sense of distribution. We only need to prove the nonlinear terms such as $u^n \cdot \nabla h^n$, $\frac{\nabla h^n}{1+h^n} \nabla u^n$, etc tend to the corresponding nonlinear terms in the sense of distribution. This can be done by using the uniform estimates of $(u^n, h^n)$, $(u, h)$ in $E_T$ and the convergence result (3.59). Here, we only show the case of the term $Y(h^n)\nabla u^n$ (where $Y(z) \triangleq \nabla z/(1 + z)$), the other terms can be treated in the same way. For any test function $\theta \in C_0^\infty([0, T^*] \times \mathbb{R}^2)$, we write

$$
\langle Y(h^n)\nabla u^n - Y(h)\nabla u, \theta \rangle
\quad = \quad \left( (1 + h^n)\nabla \left( \frac{h^n}{1 + h^n} - \frac{h}{1 + h} \right) \nabla u^n, \theta \right)
\quad + \quad \left( (h^n - h)\nabla \left( \frac{h}{1 + h} \right) \nabla u^n, \theta \right) + \left( (1 + h)\nabla \left( \frac{h}{1 + h} \right) \nabla (u^n - u), \theta \right)
\quad \triangleq I_1 + I_2 + I_3.
$$

Thanks to (2.4) and (3.56), we have

$$I_1 \leq \left\| \frac{\psi(h^n - h)}{(1 + h^n)(1 + h)} \| \nabla((1 + h^n)\nabla u^n) \|_2 \right\| \| \theta(h^n - h) \|_2 \| (1 + h^n)\nabla u^n \|_{B^1}
\lesssim \| \theta(h^n - h) \|_2 (1 + \| h^n \|_{\dot{B}^{0, 1}_0}) \| u^n \|_{B^2},
$$

where $\psi \in C_0^\infty([0, T^*] \times \mathbb{R}^2)$, and $\psi = 1$ on supp $\theta$. For $I_2$, we have

$$I_2 \leq \| \theta(h^n - h) \|_2 \| \nabla \left( \frac{h}{1 + h} \right) \nabla u^n \|_2 \lesssim \| \theta(h^n - h) \|_2 \| \nabla h \|_2 \| \nabla u^n \|_{L^\infty}
\lesssim \| \theta(h^n - h) \|_2 \| h \|_{\dot{B}^{0, 1}_0} \| u^n \|_{B^2}.
$$

Using (3.56) and the interpolation, we get

$$I_3 \leq \left\| (1 + h)\nabla \left( \frac{h}{1 + h} \right) \right\|_2 \| \nabla (u^n - u) \|_2 \lesssim (1 + \| h \|_{H^\infty}) \| \nabla h \|_2 \| (u^n - u) \|_{H^1}
\lesssim (1 + \| h \|_{\dot{B}^{0, 1}_0}) \| h \|_{B^1} \| u^n - u \|_{\dot{B}^{3/2}_1} \| (u^n - u) \|_{H^{-1/2}}^2.
$$

Thus, by (3.59), we get as $n \to 0$

$$\langle Y(h^n)\nabla u^n - Y(h)\nabla u, \theta \rangle \to 0.
$$

Following the argument in [11], we can also prove that $(u, h)$ is continuous in time with values in $B^0 \times \dot{B}^{0, 1}$. 

### 4 Uniqueness

In this section, we will prove the uniqueness of the solution. Firstly, let us recall some known results.
Lemma 4.1 (Osgood’s lemma) Let \( \rho \) be a measurable positive function and \( \gamma \) a positive locally integrable function, each defined on the domain \([t_0, t_1]\). Let \( \mu : [0, \infty) \rightarrow [0, \infty) \) be a continuous nondecreasing function, with \( \mu(0) = 0 \). Let \( a \geq 0 \), and assume that for all \( t \) in \([t_0, t_1]\),

\[
\rho(t) \leq a + \int_{t_0}^{t} \gamma(\tau) \mu(\rho(\tau)) d\tau.
\]

If \( a > 0 \), then

\[
-M(\rho(t)) + M(a) \leq \int_{t_0}^{t} \gamma(\tau) d\tau, \quad \text{where} \quad M(x) = \int_{x}^{1} \frac{d\tau}{\mu(\tau)}.
\]

If \( a = 0 \) and \( M = \infty \), then \( \rho \equiv 0 \).

This Lemma can be understood as a generalization of classical Gronwall Lemma and can be found in [8].

Proposition 4.2 Let \( s \in (\frac{-d}{p}, 1 + \frac{d}{p}) \), and \( 1 \leq p, r \leq +\infty \). Let \( v \) be a vector field such that \( \nabla v \in L^1_{\mathcal{F}}(\tilde{B}^s_{p,r} \cap L^\infty) \). Assume that \( f_0 \in \tilde{B}^s_{p,r} \), \( g \in L^1_{\mathcal{F}}(\tilde{B}^s_{p,r}) \) and \( f \in L^\infty_{\mathcal{F}}(\tilde{B}^s_{p,r}) \cap C([0, T]; S') \) is the solution of

\[
\begin{cases}
\partial_t f + v \cdot \nabla f = g, \\
f(0, x) = f_0.
\end{cases}
\]

Then there exists a constant \( C(s, p, d) \) such that for \( t \in [0, T] \)

\[
\|f\|_{L^\infty_{\mathcal{F}}(\tilde{B}^s_{p,r})} \leq C e^{CV(t)} \left( \|f_0\|_{\tilde{B}^s_{p,r}} + \int_{0}^{t} e^{-CV(\tau)} \|g(\tau)\|_{\tilde{B}^s_{p,r}} d\tau \right),
\]

where \( V(t) \triangleq \int_{0}^{t} \|\nabla v(\tau)\|_{\tilde{B}^s_{p,r} \cap L^\infty} d\tau \). If \( r < +\infty \), then \( f \) belongs to \( C([0, T]; \tilde{B}^s_{p,r}) \).

The proof can be found in [15].

Proposition 4.3 Let \( T > 0 \), \( s \in \mathbb{R} \), and \( 1 \leq q, r \leq +\infty \). Assume that \( u_0 \in \tilde{B}^s_{2,q} \), \( g \in \mathcal{L}^1_{T}(\tilde{B}^s_{2,q}) \) and \( u \) is the solution of

\[
\begin{cases}
\partial_t u - \nu \Delta u = g, \\
u(0, x) = u_0,
\end{cases}
\]

where \( \Delta u = \nabla \cdot D(u) + \nabla \text{div} u \). Then there exists a constant \( C(s, d, \nu) \) such that

\[
(r\nu)^{\frac{d}{2}} \|u\|_{\mathcal{L}^1_{T}(\tilde{B}^{s+\frac{\nu}{2}}_{2,q})} \leq \left( \sum_{k \in \mathbb{Z}} (1 - e^{-r\nu 2^k T})^\frac{q}{2} 2^{qk} \|\Delta ku_0\|_{L^2}^\frac{q}{2} \right)^\frac{1}{q} + C \left( \sum_{k \in \mathbb{Z}} (1 - e^{-r\nu 2^k T})^\frac{q}{2} 2^{qk} \|\Delta k g\|_{\mathcal{L}^1_{T}(L^2)}^\frac{q}{2} \right)^\frac{1}{q}.
\]

If \( q < +\infty \), then \( u \) belongs to \( C([0, T]; \tilde{B}^s_{2,q}) \).
Recall that \( u \) where we have used \( \nu > 0 \). For any Proposition 4.4 For any \( 1 \leq p, \rho \leq +\infty, s \in \mathbb{R} \) and \( 0 < \epsilon \leq 1 \), we have
\[
\|f\|_{\dot{L}_t^p(\dot{B}^s_{p,1})} \leq C \frac{\|f\|_{\tilde{L}_t^p(\dot{B}^s_{p,1})}}{\epsilon} \log \left( e + \frac{\|f\|_{\tilde{L}_t^p(\dot{B}^{s+1}_{p,1})}}{\|f\|_{\tilde{L}_t^p(\dot{B}^{s+1}_{p,1})}} \right). \tag{4.3}
\]

Now, let us prove the uniqueness of the solution of (3.1). Let \((u_1, h_1)\), \((u_2, h_2)\) \(\in \left(L^\infty_T(B^0) \cap \dot{L}_t^2(B^2)\right) \times \dot{L}_t^\infty(\dot{B}^{0,1})\) be two solutions of (3.1) with the same initial data. The difference \( \vartheta \triangleq h_2 - h_1, w \triangleq u_2 - u_1 \) satisfies the following system:
\[
\begin{aligned}
\vartheta_t + u_2 \cdot \nabla \vartheta &= -\text{div} w - w \nabla h_1 - \vartheta \text{div} u_2 - h_1 \text{div} w, \\
\vartheta w - \nu \tilde{\Delta} w &= -\nabla \vartheta - u_2 \cdot \nabla w - w \cdot \nabla u_1 + \nu(1 + h_1) \nabla \left( \frac{h_1}{1 + h_1} \right) \nabla w \\
&\quad + \nu(1 + h_1) \nabla \left( \frac{h_2}{1 + h_2} - \frac{h_1}{1 + h_1} \right) \nabla u_2 + \nu \vartheta \nabla \left( \frac{h_2}{1 + h_2} \right) \nabla u_2, \\
\vartheta(0, x) &= 0, \quad w(0, x) = 0.
\end{aligned} \tag{4.4}
\]
Without loss of generality, we assume that there holds for sufficiently small \( T \)
\[
1 + h_1 \geq \frac{1}{2}, \tag{4.5}
\]
\[
\|h_1\|_{\dot{B}^{0,1}} \leq \epsilon, \tag{4.6}
\]
where \( \epsilon > 0 \) is small enough. Applying the Proposition 4.2 to the first equation of (4.4) yields
\[
\|\vartheta(t)\|_{\dot{B}^{0,1}_2} \lesssim \int_0^t e^{C(V_2(t) - V_2(\tau))} \|w \cdot \nabla h_1 + \vartheta \text{div} u_2 + h_1 \text{div} w + \text{div} w\|_{\dot{B}^{0,1}_2} \, d\tau, \tag{4.7}
\]
with \( V_2(t) \triangleq \int_0^t \|\nabla u_2\|_{\dot{B}^{1,1}_2} \cap \dot{L}_t^\infty \, d\tau \). It follows from (2.5) with \( s = 0 \) that
\[
\|w \cdot \nabla h_1\|_{\dot{B}^{0,1}_2} \lesssim \|\nabla h_1\|_{\dot{B}^{0,0}_2} \|w\|_{\dot{B}^{1,1}_2} \lesssim \|w\|_{\dot{B}^{1,1}_2} \|h_1\|_{\dot{B}^{1,1}_2},
\]
\[
\|\vartheta \text{div} u_2\|_{\dot{B}^{0,1}_2} \lesssim \|\vartheta\|_{\dot{B}^{0,1}_2} \|u_2\|_{\dot{B}^{2,1}_2},
\]
\[
\|h_1 \text{div} w\|_{\dot{B}^{0,1}_2} \lesssim \|\text{div} w\|_{\dot{B}^{0,0}_2} \|h_1\|_{\dot{B}^{1,1}_2} \lesssim \|w\|_{\dot{B}^{1,1}_2} \|h_1\|_{\dot{B}^{1,1}_2},
\]
where we have used \( \dot{B}^{1,1}_2 \hookrightarrow \dot{B}^{0,1}_2 \). Plugging the above estimates into (4.7), we get
\[
\|\vartheta(t)\|_{\dot{B}^{0,1}_2} \lesssim \int_0^t e^{C(V_2(t) - V_2(\tau))} \left[ \|w\|_{\dot{B}^{1,1}_2} (1 + \|h_1\|_{\dot{B}^{1,1}_2}) + \|\vartheta\|_{\dot{B}^{0,1}_2} \|u_2\|_{\dot{B}^{2,1}_2} \right] \, d\tau. \tag{4.8}
\]
Recall that \( u^i \in L^1_T(B^2) \), we can take a \( T \in (0, \infty) \) small enough so that
\[
C \|u_2\|_{L^1_T(B^2)} \leq \frac{1}{4},
\]
\[23\]
which together with (4.3) implies that for $t \leq T$

$$
\|\partial\|_{L_t^\infty(B^0_{2,\infty})} \lesssim \|w\|_{L_t^1(B^1)}(1 + \|h_1\|_{L_t^\infty(B^1)}).
$$

(4.9)

Applying (4.3) to the term $\|w\|_{L_t^1(B^1)}$ yields

$$
\|\partial\|_{L_t^\infty(B^0_{2,\infty})} \lesssim \|w\|_{L_t^1(B^1)} \log \left( e + \frac{\|w\|_{L_t^1(B^0_{2,\infty})} + \|w\|_{L_t^1(B^2_{2,\infty})}}{\|w\|_{L_t^1(B^1)}} \right)(1 + \|h_1\|_{L_t^\infty(B^1)}).
$$

(4.10)

Thanks to $B^s \hookrightarrow B^s_2 \hookrightarrow B^1$ and $B^{0.1}_1 \hookrightarrow B^1$, we have

$$
\|\partial\|_{L_t^\infty(B^0_{2,\infty})} \lesssim \|w\|_{L_t^1(B^1)} \log \left( e + \frac{W(t)}{\|w\|_{L_t^1(B^1)}} \right)
$$

(4.11)

with

$$
W(t) \triangleq \|u^i\|_{L_t^1(B^0)} + \|u^i\|_{L_t^2(B^2)}.
$$

and for finite $t$, $W(t) < +\infty$.

Next, we deal with the second equation of (4.4). We get by applying (2.7) with $s = 1$, $s = 0$ respectively that

$$
\|u_2 \cdot \nabla w\|_{L_t^1(B^1_0)} \lesssim \|u_2\|_{L_t^2(B^1)} \|w\|_{L_t^2(B^2)},
$$

(4.12)

$$
\|w \cdot \nabla u_1\|_{L_t^1(B^1_0)} \lesssim \|u_1\|_{L_t^2(B^1)} \|w\|_{L_t^2(B^2)}.
$$

(4.13)

We can deduce $h_1 \in C(0, T; \mathbb{R}^2)$ ($i = 1, 2$) from the fact $B^1 \hookrightarrow C$. Moreover, due to (4.5), we can assume $h_1(t, x) + 1 \geq \delta$ for all $t \leq T$, $x \in \mathbb{R}^2$. Since $h_1, h_2$ have the same initial data, from the continuity of $h_2$, there exists a $\bar{T} \leq T$ such that

$$
h_2(x, t) + 1 \geq \delta, \quad \text{for all} \quad t \in [0, \bar{T}], \quad x \in \mathbb{R}^2.
$$

It follows from (2.6) with $s = 1$, (2.13) and $B_1^1 \hookrightarrow B_{2,\infty}^1$ that

$$
\left\| (1 + h_1)\nabla \left( \frac{h_2}{1 + h_2} - \frac{h_1}{1 + h_1} \right) \nabla u_2 \right\|_{B_{2,\infty}^1} \lesssim \left\| (1 + h_1)\nabla \left( \frac{h_2}{1 + h_2} - \frac{h_1}{1 + h_1} \right) \right\|_{B_{2,\infty}^1} \left\| \nabla u_2 \right\|_{B^1}
$$

$$
\lesssim (1 + \|h_1\|_{B^1}) \left( \frac{h_2}{1 + h_2} - \frac{h_1}{1 + h_1} \right) \left\| u_2 \right\|_{B^2}
$$

$$
\lesssim (1 + \|h_1\|_{B^1}) (\|h_1\|_{B^1} + \|h_2\|_{B^1}) \|\partial\|_{B_{2,\infty}^1} \|u_2\|_{B^2}
$$

which together with $L_t^1(B_{2,\infty}^1) \subset L_t^1(B_{2,\infty}^1)$ yields

$$
\left\| (1 + h_1)\nabla \left( \frac{h_2}{1 + h_2} - \frac{h_1}{1 + h_1} \right) \nabla u_2 \right\|_{B_{2,\infty}^1} \lesssim \int_0^t (1 + \|h_1\|_{B^1}) (\|h_1\|_{B^1} + \|h_2\|_{B^1}) \|\partial\|_{B_{2,\infty}^1} \|u_2\|_{B^2} dt.
$$

(4.14)
Thanks to (2.6), (2.12), and $L^1_t(\vec{B}^{-1}_{2,\infty}) \subset \widetilde{L}^1_t(\vec{B}^{-1}_{2,\infty})$, we get
\[
\| \vartheta \nabla \left( \frac{h_2}{1 + h_2} \right) \vec{\nabla} u_2 \|_{\widetilde{L}^1_t(\vec{B}^{-1}_{2,\infty})} \lesssim \int_0^t \| \vartheta \|_{\vec{B}^0_{2,\infty}} \| h_2 \|_{B^1} \| u_2 \|_{B^2} d\tau. \tag{4.15}
\]

Thanks to Lemma 5.3 with $s_1 = s_2 = 0$, Lemma 5.4 with $s = 1$, and (2.3) with $s = 0$, we have
\[
\sup_{k \in \mathbb{Z}} \omega_k(t) 2^{-k} \left\| \Delta \left( (1 + h_1) \nabla \left( \frac{1}{1 + h_1} \right) \vec{\nabla} w \right) \right\|_{L^1_t(L^2)} \\
\lesssim \left\| \nabla \left( \frac{h_1}{1 + h_1} \right) \right\|_{E^1_t} \left\| (1 + h_1) \vec{\nabla} w \right\|_{\vec{L}^1_t(\vec{B}^0_{2,\infty})} \\
\lesssim \left\| \frac{h_1}{1 + h_1} \right\|_{E^1_t} \left\| (1 + \| h_1 \|_{L^\infty(B^1)}) \nabla w \right\|_{\vec{L}^1_t(\vec{B}^0_{2,\infty})} \\
\lesssim \| h_1 \|_{E^0_t} (1 + \| h_1 \|_{L^\infty(L^{0,1})})^4 \| w \|_{\vec{L}^1_t(\vec{B}^1_{2,\infty})}. \tag{4.16}
\]

In terms of Proposition 4.3, (4.12)-(4.16) and $\vec{B}^{0,1} \hookrightarrow B^1$, we finally obtain
\[
\| w \|_{\vec{L}^1_t(\vec{B}^1_{2,\infty})} + \| w \|_{\vec{L}^2_t(\vec{B}^0_{2,\infty})} \\
\lesssim \| u_2 \|_{\vec{L}^2_t(B^1)} \| w \|_{\vec{L}^2_t(\vec{B}^0_{2,\infty})} + \| u_1 \|_{\vec{L}^2_t(B^1)} \| w \|_{\vec{L}^2_t(\vec{B}^0_{2,\infty})} \\
+ \| h_1 \|_{E^0_t} (1 + \| h_1 \|_{L^\infty(L^{0,1})})^4 \| w \|_{\vec{L}^1_t(\vec{B}^1_{2,\infty})} \\
+ \int_0^t (1 + \| h_1 \|_{\vec{B}^{0,1}}) (1 + \| h_1 \|_{\vec{B}^{0,1}} + \| h_2 \|_{\vec{B}^{0,1}}) (1 + \| u_2 \|_{B^2}) \| \vartheta \|_{\vec{B}^0_{2,\infty}} d\tau. \tag{4.17}
\]

Let us define
\[
Z(t) \triangleq \| w \|_{\vec{L}^1_t(\vec{B}^1_{2,\infty})} + \| w \|_{\vec{L}^2_t(\vec{B}^0_{2,\infty})}.
\]

Due to (4.6), if $T$ is chosen small enough, then the first four terms of the right side of (4.17) can be absorbed by the left side $Z(t)$. Noting that $r \log(e + \frac{W(T)}{r})$ is increasing, from (4.14) and (4.17), it follows that
\[
Z(t) \lesssim \int_0^t (1 + W'(\tau)) Z(\tau) \log \left( e + \frac{W(\tau)}{Z(\tau)} \right) d\tau \\
\lesssim \int_0^t (1 + W'(\tau)) Z(\tau) \log \left( e + \frac{W(T)}{Z(\tau)} \right) d\tau. \tag{4.18}
\]

It is easy to verify that
\[
1 + W'(\tau) \in L^1_{loc}(\mathbb{R}^+) \quad \text{and} \quad \int_0^1 \frac{dr}{r \log(e + \frac{W(T)}{r})} = +\infty.
\]

Hence by Osgood Lemma, we have $Z \equiv 0$ on $[0, \bar{T}]$, i.e. $w \equiv 0$, then from (4.9), $\vartheta = h_2 - h_1 \equiv 0$. Then a standard continuous argument gives the uniqueness.
5 Appendix

In this appendix, we prove some multilinear estimates in the weighted Besov space.

Lemma 5.1 Let $A$ be a homogeneous smooth function of degree $m$. Assume that $-\frac{d}{2} < \rho \leq \frac{d}{2}$. Then there hold

$$
\left| (A(D)\Delta_k (v \cdot \nabla h), A(D)\Delta_k h) \right| \leq C \|F_k^m (t)\|_2 \|A(D)\Delta_k h\|_2, \quad (5.1)
$$

and

$$
\left| (A(D)\Delta_k (v \cdot \nabla u), A(D)\Delta_k u) \right| \leq C \|\tilde{F}_k^m (t)\|_2 \|A(D)\Delta_k u\|_2, \quad (5.2)
$$

and

$$
\left| (A(D)\Delta_k (v \cdot \nabla h), A(D)\Delta_k h) \right| \leq C \left( \|F_k^m (t)\|_2 + \|\tilde{F}_k^m (t)\|_2 \right) \left( \|\Delta_k u\|_2 + \|A(D)\Delta_k h\|_2 \right), \quad (5.3)
$$

where $F_k^m (t)$ and $\tilde{F}_k^m (t)$ satisfy

$$
\sum_{k \in \mathbb{Z}} \omega_k (T) 2^{k(\rho-m)} \|F_k^m (t)\|_{L^1_\gamma (L^2)} \leq C \|h\|_{E^m_\rho} \|v\|_{L^1_\gamma (B^{\rho+1}_\gamma)}, \quad (5.4)
$$

and

$$
\sum_{k \in \mathbb{Z}} 2^{k(\rho-m)} \|\tilde{F}_k^m (t)\|_{L^1_\gamma (L^2)} \leq C \|u\|_{L^1_\gamma (B^{\rho+1}_\gamma)} \|v\|_{L^1_\gamma (B^{\rho+1}_\gamma)} \quad (5.5)
$$

Proof. Let us first prove (5.1). Using the Bony’s paraproduct decomposition, we write

$$
(A(D)\Delta_k (v \cdot \nabla h), A(D)\Delta_k h) = (A(D)\Delta_k (T_{\partial_j h} v^j), A(D)\Delta_k h) + J_k, \quad (5.6)
$$

where

$$
T'_f g = Tfg + R(f, g), \quad \text{and}
$$

$$
J_k = \sum_{|k' - k| \leq 3} ([A(D)\Delta_k, S_{k' - 1} v^j] \Delta_k \partial_j h, A(D)\Delta_k h)
$$

$$
+ \sum_{|k' - k| \leq 3} ((S_{k' - 1} - S_{k - 1}) v^j A(D)\Delta_k \Delta_k \partial_j h, A(D)\Delta_k h)
$$

$$
+ (S_{k - 1} v^j A(D)\Delta_k \partial_j h, A(D)\Delta_k h)
$$

We get by integration by parts that

$$
(S_{k - 1} v^j A(D)\Delta_k \partial_j h, A(D)\Delta_k h) = \frac{1}{2} (S_{k - 1} \text{div} v A(D)\Delta_k h, A(D)\Delta_k h).
$$

Let us set

$$
F_{k,0}^m (t) = A(D)\Delta_k (T_{\partial_j h} v^j),
$$

$$
F_{k,1}^m (t) = \sum_{|k' - k| \leq 3} [A(D)\Delta_k, S_{k' - 1} v^j] \Delta_k \partial_j h,
$$

$$
F_{k,2}^m (t) = \sum_{|k' - k| \leq 3} (S_{k' - 1} - S_{k - 1}) v^j A(D)\Delta_k \Delta_k \partial_j h,
$$

$$
F_{k,4}^m (t) = \frac{1}{2} S_{k - 1} \text{div} v A(D)\Delta_k h.
$$

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By the Cauchy-Schwartz inequality, we get
\[
(A(D)\Delta_k (v - \nabla h), A(D)\Delta_k h) \leq \|F^m_k(t)\|_2 \|A(D)\Delta_k h\|_2,
\]
with \(F^m_k(t) = \sum_{i=0}^{3} F^m_{k,i}(t)\). So, it remains to prove that \(F^m_k(t)\) satisfies (5.4). For the simplicity, we set
\[
\tilde{\Delta}_k = \sum_{|k'-k| \leq 1} \Delta_{k'}, \quad \Delta_k = \sum_{|k'-k| \leq 3} \Delta_{k'}.
\]
Thanks to the definition of \(F^m_{k,0}(t)\) and Lemma 2.1, we have
\[
\|F^m_{k,0}(t)\|_{L^1(L^2)} \leq \sum_{|k'-k| \leq 3} 2^{km} \|S_{k'-1} \partial_j h\|_{L^\infty(L^\infty)} \|\Delta_{k'} v^j\|_{L^1(L^2)}
\]
\[
+ \sum_{k' \geq k-2} 2^{(m+\frac{d}{2}) \rho} \|\Delta_k(\Delta_{k'} \partial_j \tilde{\Delta}_k v^j)\|_{L^1(L^1)}
\]
\[
\triangleq I + II.
\]
Thanks to Lemma 2.1, we have
\[
2^{k(\rho - m)} \leq 2^{k \rho} \sum_{k' \leq k+1} 2^{k'(1+\frac{d}{2})} \|\Delta_{k'} h\|_{L^\infty(L^2)} \|\tilde{\Delta}_k v\|_{L^1(L^2)}
\]
\[
\leq \sum_{k' \leq k+1} 2^{(k'-k)(1+\frac{d}{2})} 2^{k' \rho} \|\Delta_{k'} h\|_{L^\infty(L^2)} 2^{k(1+\frac{d}{2})} \|\tilde{\Delta}_k v\|_{L^1(L^2)}.
\]
from which and the definition of \(\omega_k(T)\), it follows that
\[
\sum_{k} \omega_k(T) 2^{k(\rho - m)} I
\]
\[
\leq \sum_{k' \in \mathbb{Z}} 2^{k \rho} \|\Delta_{k'} h\|_{L^\infty(L^2)} \sum_{k \geq k'-1} \omega_k(T) 2^{(k'-k)(1+\frac{d}{2})} 2^{k(1+\frac{d}{2})} \|\tilde{\Delta}_k v\|_{L^1(L^2)}
\]
\[
\leq \sum_{k' \in \mathbb{Z}} \omega_{k'}(T) 2^{k' \rho} \|\Delta_{k'} h\|_{L^\infty(L^2)} \sum_{k \geq k'-1} 2^{(k'-k)(\frac{d}{2})} 2^{k(1+\frac{d}{2})} \|\tilde{\Delta}_k v\|_{L^1(L^2)}
\]
\[
\leq \|h\|_{\mathcal{E}_{\rho}^{\infty}} \|v\|_{L^1(\mathbb{R}^{\frac{d}{2}+1})}, \quad (5.7)
\]
where we used the assumption \(\rho \leq \frac{d}{2}\) in the last inequality. Set \(e_k(T) = e_k^1(T) + e_k^2(T)\). Using Lemma 2.1, we also have
\[
\omega_k(T) 2^{k(\rho - m)} II \leq \omega_k(T) 2^{k(\rho + \frac{d}{2})} \sum_{k' \geq k-2} 2^{k'} \|\Delta_{k'} h\|_{L^\infty(L^2)} \|\tilde{\Delta}_k v\|_{L^1(L^2)}
\]
\[
\leq 2^{k(\rho + \frac{d}{2})} \sum_{k' \geq k-2} 2^{k'} \|\Delta_{k'} h\|_{L^\infty(L^2)} \|\tilde{\Delta}_k v\|_{L^1(L^2)} \sum_{k \geq k'} 2^{-(\tilde{k}-k)} e_{\tilde{k}}(T)
\]
\[
+ 2^{k(\rho + \frac{d}{2})} \sum_{k' \geq k-2} 2^{k'} \|\Delta_{k'} h\|_{L^\infty(L^2)} \|\tilde{\Delta}_k v\|_{L^1(L^2)} \sum_{\tilde{k} \geq k, k' \geq \tilde{k}'} 2^{-(\tilde{k}-k)} e_{\tilde{k}}(T)
\]
\[
\triangleq II_1 + II_2.
\]
Note that for $k' \leq k$

$$e_k(T) \leq e_{k'}(T) \leq \omega_{k'}(T),$$

from which and $\rho > -\frac{d}{2}$, we deduce that

$$\sum_{k' \in \mathbb{Z}} II_1 \lesssim \sum_{k' \in \mathbb{Z}} \omega_{k'}(T)2^k\rho \|\Delta_k h\|_{L^\infty_T(L^2)}2^{k'(\frac{d}{2}+1)}\|\Delta_k' v\|_{L^1_T(L^2)} \sum_{k \leq k'+2} 2^{(k-k')\rho+\frac{d}{2}}$$

$$\lesssim \|h\|_{E^1_T}\|v\|_{L^1_T(B^\frac{d}{2}+1)},$$

Similarly, we can obtain

$$\sum_{k' \in \mathbb{Z}} II_2 \lesssim \sum_{k' \in \mathbb{Z}} 2^k\rho \|\Delta_k h\|_{L^\infty_T(L^2)} \sum_{k \leq k'+2} 2^{(k-k')\rho+\frac{d}{2}} \sum_{k \geq k'} 2^{-(\rho-k)}e_k(T)\|v\|_{L^1_T(B^\frac{d}{2}+1)}$$

$$\lesssim \sum_{k' \in \mathbb{Z}} \omega_{k'}(T)2^k\rho \|\Delta_k h\|_{L^\infty_T(L^2)} \sum_{k \leq k'+2} 2^{(k-k')(\frac{d}{2}+\rho+1)}\|v\|_{L^1_T(B^\frac{d}{2}+1)}$$

$$\lesssim \|h\|_{E^1_T}\|v\|_{L^1_T(B^\frac{d}{2}+1)},$$

By summing up (5.8)-(5.9), we obtain

$$\sum_{k \in \mathbb{Z}} \omega_k(T)2^{(\rho-m)}\|F^m_{k,1}(t)\|_{L^1_T(L^2)} \lesssim \|h\|_{E^1_T}\|v\|_{L^1_T(B^\frac{d}{2}+1)}.$$  

Note that $A(D)\Delta_k = 2^{km}\tilde{\varphi}(2^{-k}D)$ with $\tilde{\varphi}(\xi) = A(\xi)\varphi(\xi)$. Set $\tilde{\theta} = F^{-1}\tilde{\varphi}$, we get by using the Taylor’s formula that

$$F^m_{k,1}(t) = \sum_{|k' - k| \leq 3} 2^{(m-1)} \int_{\mathbb{R}^d} \int_0^1 \tilde{\theta}(y \cdot S_{k'} - v) \Delta_k \partial_y h(x - 2^{-k}y)d\tau dy,$$

from which and Lemma 22.1 it follows that

$$\|F^m_{k,1}(t)\|_{L^1_T(L^2)} \lesssim 2^{(m-1)} \sum_{|k' - k| \leq 3} \|S_{k'} - v\|_{L^\infty_T(L^2)} \|\Delta_k\partial_y h\|_{L^1_T(L^2)}$$

$$\lesssim 2^{km} \sum_{|k' - k| \leq 3} \|\Delta_k h\|_{L^\infty_T(L^2)}\|v\|_{L^1_T(B^\frac{d}{2}+1)},$$

thus, we get

$$\sum_{k \in \mathbb{Z}} \omega_k(T)2^{(\rho-m)}\|F^m_{k,1}(t)\|_{L^1_T(L^2)} \lesssim \|h\|_{E^1_T}\|v\|_{L^1_T(B^\frac{d}{2}+1)}.$$  

Thanks to the fact $|k' - k| \leq 3$ and Lemma 22.1 we have

$$\|(S_{k'} - S_{k-1})v\| A(D)\Delta_k\Delta_k\partial_y h\|_{L^1_T(L^2)} \lesssim 2^{km}\|\Delta_k h\|_{L^\infty_T(L^2)}\|v\|_{L^1_T(B^\frac{d}{2}+1)},$$

from which, it follows that

$$\sum_{k \in \mathbb{Z}} \omega_k(T)2^{(\rho-m)}\left(\|F^m_{k,2}(t)\|_{L^1_T(L^2)} + \|F^m_{k,3}(t)\|_{L^1_T(L^2)}\right) \lesssim \|h\|_{E^1_T}\|v\|_{L^1_T(B^\frac{d}{2}+1)},$$
which together with \((5.10)\) and \((5.11)\) yields \((5.4)\).

Using the decomposition \((5.6)\) with \(h\) instead of \(u\) and Lemma 2.1, \((5.2)\) can be easily proved. We omit it here. In order to prove \((5.3)\), we use the decomposition

\[
(A(D)\Delta_k(v \cdot \nabla h), \Delta_k u) + (\Delta_k(v \cdot \nabla u), A(D)\Delta_k h) = I_k + J_k,
\]

with

\[
I_k = (A(D)\Delta_k(T_{\partial \nu}h v^j), \Delta_k u) + (\Delta_k(T_{\partial \nu}v^j), A(D)\Delta_k h)
\]

\[
\triangleq (F_{k,0}^m(t), \Delta_k u) + (\tilde{F}_{k,0}^0(t), A(D)\Delta_k h)
\]

\[
J_k = \sum_{|k'\cdot k| \leq 3} \left( [A(D)\Delta_k, S_{k' - 1}v^j] \Delta_k \partial_j h, \Delta_k u \right) + \left( (S_{k' - 1} - S_{k - 1})v^j A(D)\Delta_k \Delta_k \partial_j h, \Delta_k u \right) + \left( (S_{k' - 1} - S_{k - 1})v^j \Delta_k \Delta_k \partial_j u, A(D)\Delta_k h \right)
\]

\[+ \sum_{|k'\cdot k| \leq 3} \left( [\Delta_k, S_{k' - 1}v^j] \Delta_k \partial_j u, A(D)\Delta_k h \right) + \left( (S_{k' - 1} - S_{k - 1})v^j \Delta_k \Delta_k \partial_j u, A(D)\Delta_k h \right) - \left( S_{k - 1} \text{div} A(D)\Delta_k h, \Delta_k u \right)
\]

\[
\triangleq (F_{k,1}^m(t), \Delta_k u) + (F_{k,2}^m(t), \Delta_k u) + (\tilde{F}_{k,1}^0(t), A(D)\Delta_k h) + (\tilde{F}_{k,1}^2(t), A(D)\Delta_k h) + (F_{k,3}^m(t), \Delta_k u),
\]

from which, a similar proof of \((5.4)\) gives \((5.3)\). This completes the proof of Lemma 5.1 ■

Lemma 5.2 Let \(s_1 \leq \frac{d}{2} - 1\), \(s_2 \leq \frac{d}{2}\), and \(s_1 + s_2 > 0\). Then there holds

\[
\sum_{k \in \mathbb{Z}} \omega_k(T)2^{k(s_1 + s_2 - \frac{d}{2})}\|\Delta_k(fg)\|_{L^r_k(L^2)} \leq C \sum_{k \in \mathbb{Z}} \omega_k(T)2^{ks_1}\|\Delta_k f\|_{L^r_k(L^2)}\|g\|_{L^r_k(B^{s_2})},
\]

\[(5.13)\]

where \(1 \leq r_1, r_2 < \infty\) and \(\frac{1}{r_1} + \frac{1}{r_2} = 1\).

Proof. Using the Bony’s paraproduct decomposition, we write

\[
\Delta_k(fg) = \sum_{|k'\cdot k| \leq 3} \Delta_k(S_{k' - 1}f \Delta_k g) + \sum_{|k'\cdot k| \leq 3} \Delta_k(S_{k' - 1}g \Delta_k f)
\]

\[+ \sum_{k' \geq k - 2} \Delta_k(\Delta_k f \Delta_k g) \triangleq I + II + III.
\]

A similar proof of \((5.7)\) ensures that for \(s_1 \leq \frac{d}{2} - 1\)

\[
\sum_{k \in \mathbb{Z}} \omega_k(T)2^{k(s_1 + s_2 - \frac{d}{2})}\|I\|_{L^r_k(L^2)} \lesssim \sum_{k \in \mathbb{Z}} \omega_k(T)2^{ks_1}\|\Delta_k f\|_{L^r_k(L^2)}\|g\|_{L^r_k(B^{s_2})},
\]

while II can be directly deduced for \(s_2 \leq \frac{d}{2}\). On the other hand, a similar proof of \((5.8)\) and \((5.9)\) gives for \(s_1 + s_2 > 0\)

\[
\sum_{k \in \mathbb{Z}} \omega_k(T)2^{k(s_1 + s_2 - \frac{d}{2})}\|III\|_{L^r_k(L^2)} \lesssim \sum_{k \in \mathbb{Z}} \omega_k(T)2^{ks_1}\|\Delta_k f\|_{L^r_k(L^2)}\|g\|_{L^r_k(B^{s_2})}.
\]

This completes the proof of Lemma 5.2 ■

Similarly, we can also prove the following lemma.
Lemma 5.3 Let \( s_1 \leq \frac{d}{2} - 1, \ s_2 < \frac{d}{2} \) and \( s_1 + s_2 \geq 0 \). Then there holds
\[
\sup_{k \in \mathbb{Z}} \omega_k(T)2^k(s_1+s_2-\frac{d}{2})\|\Delta_k(fg)\|_{L_T^r(L^2)} \leq C\|f\|_{E_T^{s_1}}\|g\|_{\tilde{L}_T^{s_2}(E_{T}^{s_2})}.
\] (5.14)

Lemma 5.4 Let \( s > 0 \). Assume that \( F \in W'^{[s] + 3, \infty}_{\text{loc}}(\mathbb{R}^d) \) with \( F(0) = 0 \). Then there holds
\[
\|F(f)\|_{E_T^{s}} \leq C(1 + \|f\|_{L_{T}^{\infty}(L^{\infty})})|s|^{2}\|f\|_{E_T^{s}}.
\] (5.15)

Proof. We decompose \( F(f) \) as
\[
F(f) = \sum_{k' \in \mathbb{Z}} F(S_{k'+1}f) - F(S_kf) = \sum_{k' \in \mathbb{Z}} \Delta_k f \int_0^1 F'(S_k f + \tau \Delta_k f) d\tau \triangleq \sum_{k' \in \mathbb{Z}} \Delta_k f m_{k'},
\]
where \( m_{k'} = \int_0^1 F'(S_k f + \tau \Delta_k f) d\tau \). Furthermore, we write
\[
\Delta_k F(f) = \sum_{k' < k} \Delta_k(\Delta_k' f m_{k'}) + \sum_{k' \geq k} \Delta_k(\Delta_k' f m_{k'}) \triangleq I + II.
\]

By Lemma 2.1, we have
\[
\|I\|_{L_T^{r}(L^2)} \leq \sum_{k' < k} \|\Delta_k(\Delta_k' f m_{k'})\|_{L_T^{r}(L^2)} \leq \sum_{k' < k} 2^{-k|\alpha|} \sup_{|\gamma| = |\alpha|} \|D^\gamma \Delta_k(\Delta_k' f m_{k'})\|_{L_T^{r}(L^2)},
\] (5.16)
with \( \alpha \) to be determined later. Note that for \(|\gamma| > 0\), we have
\[
\|D^\gamma m_{k'}\|_{\infty} \lesssim 2^{k|\gamma|}(1 + \|f\|_{\infty})|\gamma|\|F'\|_{W'[\gamma], \infty},
\]
from which and (5.16), it follows that
\[
2^{k_1}\|I\|_{L_T^{r}(L^2)} \lesssim 2^{k_1(s-|\alpha|)} \sum_{k' < k} 2^{k'|\alpha|}\|\Delta_k' f\|_{L_T^{r}(L^2)}(1 + \|f\|_{L_T^{r}(L^\infty)})^{s-|\alpha|}\|F'\|_{W'[\alpha], \infty},
\]
thus, if we take \(|\alpha| = [s] + 2\), we get
\[
\sum_{k \in \mathbb{Z}} \omega_k(T)2^{ks}\|I\|_{L_T^{r}(L^2)} \lesssim \sum_{k' \in \mathbb{Z}} 2^{k's}\omega_k(T)\|\Delta_k f\|_{L_T^{r}(L^2)} \sum_{k > k'} 2^{k-k'(s-|\alpha|)+1}(1 + \|f\|_{L_T^{r}(L^\infty)})^{s-|\alpha|}\|F'\|_{W'[\alpha], \infty} \lesssim (1 + \|f\|_{L_T^{r}(L^\infty)})^{s+2}\|F'\|_{W'[s+2, \infty]}\|f\|_{E_T^{s}}.
\] (5.17)

Next, let us turn to the proof of \( II \). We get by using Lemma 2.1 that
\[
\|II\|_{L_T^{r}(L^2)} \lesssim \sum_{k \geq k'} \|\Delta_k f\|_{L_T^{r}(L^2)}.
\]

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Then we write
\[
\sum_{k \in \mathbb{Z}} \omega_k(T) 2^{ks} \| II \|_{L^\infty_T(L^2)} \lesssim \sum_{k \in \mathbb{Z}} \sum_{k' \geq k} 2^{ks} \| \Delta k' f \|_{L^\infty_T(L^2)} \sum_{\tilde{k} \geq k', k \geq k} 2^{-(\tilde{k}-k)} e_{\tilde{k}}(T)
+ \sum_{k \in \mathbb{Z}} \sum_{k' \geq k} 2^{ks} \| \Delta k' f \|_{L^\infty_T(L^2)} \sum_{\tilde{k} \geq k', k \geq k} 2^{-(\tilde{k}-k)} e_{\tilde{k}}(T),
\]
from which, a similar proof of (5.8) and (5.9) ensures that
\[
\sum_{k \in \mathbb{Z}} \omega_k(T) 2^{ks} \| II \|_{L^\infty_T(L^2)} \lesssim \| f \|_{E^s}.
\tag{5.18}
\]
By summing up (5.17) and (5.18), we deduce the inequality (5.15). This completes the proof of Lemma 5.4.

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