SPECTRAHEDRAL SHADOWS

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Abstract. We show that there are many (compact) convex semi-algebraic
sets in euclidean space that are not spectrahedral shadows. This gives a neg-
ative answer to a question by Nemirovski, resp. it shows that the Helton-Nie
conjecture is false.

Introduction

Semidefinite programming is a far-reaching generalization of linear programming.
While a linear program optimizes a linear function over a polyhedron, a semi-
definite program optimizes it over a convex region described by symmetric linear
matrix inequalities. Under mild conditions, semidefinite programs can be solved in
polynomial time up to any prescribed accuracy. They have numerous applications in
applied mathematics, engineering, control theory and so forth (see [1, Chapter 1]).

The feasible regions of semidefinite programs are called spectrahedral shadows,
or also semidefinitely (or SDP) representable sets. These are the sets
\[ K \subseteq \mathbb{R}^n \]
that can be written
\[ K = \left\{ \xi \in \mathbb{R}^n : \exists \eta \in \mathbb{R}^m \; A + \sum_{i=1}^{n} \xi_i B_i + \sum_{j=1}^{m} \eta_j C_j \succeq 0 \right\} \] (1)
where \( m \geq 0, A, B_i, C_j \) are real symmetric matrices of the same size and \( M \succeq 0 \)
means that \( M \) is positive semidefinite. Any representation as in (1) is called a
semidefinite representation of \( K \).

There has been considerable interest in characterizing spectrahedral shadows
by their geometric properties. Essentially, this is the question of what problems
in optimization can be modeled as semidefinite programs. Nemirovski [19] in his
2006 plenary address at ICM Madrid remarked: "A seemingly interesting question
is to characterize SDP-representable sets. Clearly, such a set is convex and semi-
algebraic. Is the inverse also true? ( . . . ) This question seems to be completely
open." Helton and Nie ([11, p. 790]) conjectured that the answer is in fact yes, i.e.,
that every convex semi-algebraic set in \( \mathbb{R}^n \) is a spectrahedral shadow.

Although the general question has so far been elusive, many results have been ob-
tained in support of the Helton-Nie conjecture. The class of spectrahedral shadows
is known to be closed under taking linear images or preimages, finite intersections,
or convex hulls of finite unions ([11, 24]). It is also closed under convex duality
respectively polarity, and under taking topological closures. Helton and Nie ([11],
[12]) gave a series of sufficient conditions for semidefinite representability of a convex

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semi-algebraic set $K$, in terms of curvature conditions for the boundary. Roughly, their results are saying that when $K$ is compact and its boundary is sufficiently nonsingular and has strictly positive curvature, then $K$ is a spectrahedral shadow. Netzer [21] proved that the interior of a spectrahedral shadow is again a spectrahedral shadow, and more generally, that removing suitably parametrized families of faces from a spectrahedral shadow results again in a spectrahedral shadow. By applying the criteria of Helton-Nie, Netzer and Sanyal [23] showed that smooth hyperbolicity cones are spectrahedral shadows. Scheiderer [36] showed that closed convex hulls of one-dimensional semi-algebraic sets are always spectrahedral shadows, and that the Helton-Nie conjecture is true for subsets of the plane.

In addition there are plenty of further results on semidefinite representations for particular kinds of sets. See, for example, [6, 7, 9, 14, 22, 25, 26, 29, 30, 35], and see [20, 3, 19, 4, ch. 6] or [1, ch. 2, 4 and 5] for surveys on semidefinite representation.

An important general technique for constructing semidefinite representations was introduced by Lasserre [17], and independently by Parrilo [27]. It is based on a dual relaxation principle and is generally known as the moment relaxation method. Starting with a (basic closed) semi-algebraic set $S \subseteq \mathbb{R}^n$, it produces outer approximations of the convex hull of $S$ that have explicit semidefinite representations. When $S$ is compact, these approximations can be made arbitrarily close. Under favorable conditions, moment relaxation is known to become exact, meaning that a suitable such approximation coincides with the convex hull of $S$, up to taking closures.

In this paper we exhibit, for the first time, non-trivial conditions that are necessary for semidefinite representability. They are based on semidefinite duality, and they imply that there are no more closed spectrahedral shadows than those obtainable from exact moment relaxation in a generalized sense. We then use arguments from algebraic geometry, in particular properties of smooth morphisms of varieties, to show that these conditions are indeed non-trivial, and to produce concrete examples of convex sets that fail to be spectrahedral shadows. Among them are natural prominent sets like the cone of non-negative forms of fixed degree in $\mathbb{R}[x_1, \ldots, x_n]$, in every case where this cone is different from the sums of squares cone (Corollary 4.25). In fact, for every semi-algebraic set $S \subseteq \mathbb{R}^n$ of dimension at least two we prove that there exist polynomial maps $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ for which the closed convex hull of $\varphi(S)$ in $\mathbb{R}^m$ has no semidefinite representation. This is in marked contrast to the case where $S$ has dimension one, when it is known that the closed convex hull of $S$ is always a spectrahedral shadow [36].

For optimization, our results imply that there exist natural semi-algebraic optimization problems that cannot be modeled exactly as semidefinite programs. For example, the problem of minimizing a general polynomial of degree $d \geq 4$ in $n \geq 3$ variables (or of degree $d \geq 6$ in $n = 2$ variables) over the unit ball in $\mathbb{R}^n$ is of this sort.

The paper is organized as follows. In Section 2 we recall and generalize the moment relaxation construction, arriving at general sufficient conditions for semidefinite representability. In Section 3 we show that the conditions obtained in Section 2 are also necessary. The main result is Theorem 3.4. In Section 4 we
present concrete constructions of closed convex sets that violate the necessary conditions from Section 3, and we give a few explicit examples. Finally, Section 5 contains a number of open questions.

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1. Preliminaries and notation

1.1. A symmetric matrix \( A \in \text{Sym}_d(\mathbb{R}) \) is said to be positive semidefinite (psd), denoted \( A \succeq 0 \), if all its eigenvalues are non-negative. If in addition all eigenvalues are nonzero then \( A \) is positive definite, written \( A \succ 0 \). The canonical inner product on \( \text{Sym}_d(\mathbb{R}) \) is denoted \( \langle A, B \rangle = \text{tr}(AB) \), for \( A, B \in \text{Sym}_d(\mathbb{R}) \). The set \( \text{Sym}_d^+(\mathbb{R}) = \{ A \in \text{Sym}_d(\mathbb{R}) : A \succeq 0 \} \) is a closed convex cone in \( \text{Sym}_d(\mathbb{R}) \), self-dual with respect to the inner product. The same terminology applies when the field \( \mathbb{R} \) of real numbers is replaced by a real closed field \( R \).

1.2. A set \( K \subseteq \mathbb{R}^n \) is called a spectrahedron if there exist \( d \geq 1 \) and \( M_1, \ldots, M_n \in \text{Sym}_d(\mathbb{R}) \) such that \( K = \{ \xi \in \mathbb{R}^n : M_0 + \sum_{i=1}^n \xi_i M_i \succeq 0 \} \). The set \( K \) is said to be a spectrahedral shadow (or to have a semidefinite representation) if there exists a spectrahedron \( S \subseteq \mathbb{R}^m \) for some \( m \) and a linear map \( f : \mathbb{R}^m \to \mathbb{R}^n \) such that \( K = f(S) \).

Spectrahedra have also been called LMI-sets or LMI-representable sets. In the plane, spectrahedra are characterized by the former Lax conjecture, which has been proved by Helton-Vinnikov \[13\] in 2007. In higher dimension there exist only conjectural characterizations of spectrahedra (so-called generalized Lax conjecture).

Spectrahedral shadows have as well occurred under various different names, such as projected spectrahedra, SDP representable sets or lifted-LMI representable sets. These sets are convex and semi-algebraic, but so far no other restrictions were known.

1.3. For \( V \) a vector space over a field \( k \) we denote the dual space of \( V \) by \( V^\vee = \text{Hom}_k(V, k) \). Let \( V \) be a finite-dimensional \( k \)-vector space. By a (convex) cone \( C \) in \( V \) we mean a non-empty set \( C \subseteq V \) with \( C + C \subseteq C \) and \( aC \subseteq C \) for all real numbers \( a \geq 0 \). Given any set \( M \subseteq V \) let \( \text{conv}(M) \) denote the convex hull of \( M \), and let \( \text{cone}(M) \) be the convex cone generated by \( M \) (consisting of all finite linear combinations of elements of \( M \) with non-negative coefficients). Moreover, \( M^* \subseteq V^\vee = \text{Hom}_k(V, \mathbb{R}) \) denotes the (closed convex) cone dual to \( M \), i.e. \( M^* = \{ \lambda \in V^\vee : \forall x \in M \lambda(x) \geq 0 \} \). When \( M \) is a semi-algebraic set then so are \( \text{conv}(M) \) and \( \text{cone}(M) \), by Carathéodory’s lemma, and also \( M^* \).

1.4. Given a field \( k \), a \( k \)-algebra is a commutative ring \( A \) with a fixed ring homomorphism \( k \to A \). The \( k \)-algebra \( A \) is said to be finitely generated if it is finitely generated as a ring over \( k \). The ideal in a ring \( A \) generated by a family of elements \( a_i \in A \ (i \in I) \) is denoted \( (a_i : i \in I) \).

We use standard terminology for algebraic varieties, that we briefly recall. Generally, we use the language of schemes. See \[1.5\] below for informal rephrasings of the most important concepts in non-technical language. Note however that it would be most cumbersome and awkward to formulate parts of Section \[4\] in such language, which is why we will not make such an attempt.
Our fields $k$ will be real closed fields, and in particular they have characteristic zero. All our $k$-varieties will be affine, and we assume them to be reduced but not necessarily irreducible. Thus, an affine $k$-variety $V$ is the spectrum of a reduced finitely generated $k$-algebra $A$, and $A = k[V] = \Gamma(V, \mathcal{O}_V)$ is the affine coordinate ring of $V$. As usual, $V(k) = \text{Hom}_k(A, k)$ is the set of $k$-rational points of $V$. Given $\xi \in V(k)$, the local ring of $V$ at $\xi$ is denoted $\mathcal{O}_{V, \xi}$. By $\mathcal{m}_{V, \xi}$ we invariantly denote both the maximal ideal of $\mathcal{O}_{V, \xi}$ and its preimage in $k[V]$. Given a morphism $\phi: V \to W$ of affine $k$-varieties, the associated homomorphism $k[W] \to k[V]$ of $k$-algebras is denoted $\phi^*$. Given a field extension $K/k$ we write $V_K := V \times_k \text{Spec}(K)$ for the base field extension of $V$, and similarly $\phi_K: V_K \to W_K$ for the base field extension of $\phi$. Upon identifying $K[V] = k[V] \otimes_k K$ we have $\phi_K^\ast = \phi^* \otimes 1$ for the induced map $K[Y] \to K[X]$.

1.5. To make the paper more accessible to readers who are not familiar with the basic notions of algebraic geometry, here are some explanations. Let $W \subseteq \mathbb{R}$ be the base field extension of $k$. To make the paper more accessible to readers who are not familiar with the

Given $\xi \in V(k)$, let $\mathcal{m}_{V, \xi} = \{f \in k[V]: f(\xi) = 0\}$, a maximal ideal of $k[V]$. The local ring of $V$ at $\xi$ is by definition the localization $\mathcal{O}_{V, \xi} = k[V]_{\mathcal{m}_{V, \xi}}$ of $k[V]$ at its maximal ideal $\mathcal{m}_{V, \xi}$. Given a field extension $K/k$, the base field extension $V_K$ of $V$ is the affine $k$-variety in $\mathbb{K}^n$ defined by the same $k$-polynomials as $V$, and it has coordinate ring $K[V] = k[V] \otimes_k K$.

For example, affine $n$-space is the affine $k$-variety $\mathbb{A}^n = \mathbb{K}^n$ and has $k[\mathbb{A}^n] = k[x_1, \ldots, x_n] = k[x]$ and $\mathbb{A}^n(k) = k^n$. Given $\xi \in k^n$, the local ring $\mathcal{O}_{\mathbb{A}^n, \xi}$ is the ring of fractions $\frac{f}{g}$ with $f, g \in k[x]$ and $g(\xi) \neq 0$.

1.6. When $V$ is an affine variety over $k = \mathbb{R}$, we will always equip the set $V(\mathbb{R})$ of $\mathbb{R}$-rational points with the euclidean topology (induced by $V(\mathbb{R}) \subseteq \mathbb{R}^n$ if $V \subseteq \mathbb{A}^n$ is a closed subvariety). This topology is independent of the choice of a closed embedding $V \subseteq \mathbb{A}^n$. A subset $S \subseteq V(\mathbb{R})$ is called semi-algebraic if it can be written as a finite boolean combination of sets of the form $\{\xi \in V(\mathbb{R}): f(\xi) > 0\}$ with $f \in \mathbb{R}[V]$.

1.7. Let $V$ be an affine $k$-variety, and let $L \subseteq k[V]$ be a $k$-linear subspace of finite dimension. Let $S^\ast L = \bigoplus_{d \geq 0} S^d L$ be the symmetric $k$-algebra over $L$, and let $S^\ast L \to k[V]$ be the natural $k$-homomorphism induced by the inclusion $L \subseteq k[V]$. The associated morphism of affine $k$-varieties will be denoted $\varphi_L: V \to \mathbb{A}_L$, where $\mathbb{A}_L := \text{Spec}(S^\ast L)$ is the affine space with coordinate ring $S^\ast L$. In plainer terms, upon fixing a linear basis $g_1, \ldots, g_m$ of $L$, we may identify $\varphi_L$ with the map $V \to \mathbb{A}^m$ given by $\xi \mapsto (g_1(\xi), \ldots, g_m(\xi))$. Note that $\mathbb{A}_L(k) = L^\vee = \text{Hom}_k(L, k)$, the linear space dual to $L$, so that on $k$-rational points the map $\varphi_L: V(k) \to L^\vee$ sends $\xi \in V(k)$ to the evaluation map $L \to k$ at $\xi$. 


2. Sufficient Conditions for Semidefinite Representability

Let $V$ be an affine $\mathbb{R}$-variety. Given a semi-algebraic subset $S \subseteq V(\mathbb{R})$ we write $\mathcal{P}(S) = \{ f \in \mathbb{R}[V] : f \geq 0 \text{ on } S \}$.

2.1. We start by informally recalling the moment relaxation construction, due to Lasserre [17] and independently Parrilo [27]. Let $L \subseteq \mathbb{R}[V]$ be a linear subspace with $\dim(L) = m < \infty$, and let $\varphi_L : V \to \mathbb{A}_L \cong \mathbb{A}^m$ be the associated morphism, see [17]. Assume that $S \subseteq V(\mathbb{R})$ is a basic closed semi-algebraic set, say $S = \{ \xi \in V(\mathbb{R}) : h_i(\xi) \geq 0 \ (i = 1, \ldots, r) \}$ where $h_1, \ldots, h_r \in \mathbb{R}[V]$. We are trying to find a semidefinite representation of the convex hull $K$ of $\varphi_L(S)$ in $\mathbb{A}_L(\mathbb{R}) = L^\vee \cong \mathbb{R}^m$, or at least an approximate such representation.

Without any serious restriction we can assume $1 \notin L$. Fix a sequence $W_0, \ldots, W_r$ of finite-dimensional linear subspaces of $\mathbb{R}[V]$. Any $f \in L_1 := \mathbb{R}1 + L \subseteq \mathbb{R}[V]$ that has a representation $f = s_0 + \sum_{i=1}^r s_i h_i$ with $s_i$ a sum of squares of elements of $W_i$ ($i = 0, \ldots, r$) is obviously non-negative on $S$. So the set of all such $f$ is a convex cone $C = C(W_0, \ldots, W_r)$, contained in $L_1 \cap \mathcal{P}(S)$. By construction, the dual cone $C^* \subseteq L_1^\vee$ has an explicit semidefinite representation. For $\lambda \in L^\vee$ let $\lambda' \in L_1^\vee$ be defined by $\lambda'|_L = \lambda$ and by $\lambda'(1) = 1$. The set $K' = K'(W_0, \ldots, W_r)$ of all $\lambda \in \mathbb{A}_L(\mathbb{R}) = L^\vee$ for which $\lambda' \in C^*$ is a closed spectrahedral shadow that contains $K$. Enlarging the spaces $W_i$, or adding more inequalities $h_i$ to the description of $S$, results in $K'$ getting smaller, and therefore becoming a closer approximation to $K$. Of particular interest is the case where $C = L_1 \cap \mathcal{P}(S)$. This condition is usually rephrased by saying that the linear polynomials non-negative on $\varphi_L(S)$ (i.e. the elements of $L_1 \cap \mathcal{P}(S)$) have weighted sum of squares representations of uniformly bounded degrees (the “degree bounds” being given by the subspaces $W_i$). The moment relaxation is exact in this case, which means that $K' = \overline{K}$, the closure of $K$. Therefore, under the assumption $C = L_1 \cap \mathcal{P}(S)$, the closure $\overline{K}$ is a spectrahedral shadow.

We now generalize this procedure, to arrive at a general sufficient condition for semidefinite representability. First two auxiliary lemmas.

Lemma 2.2. Let $V$ be an affine $\mathbb{R}$-variety, and let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. Let $L \subseteq \mathbb{R}[V]$ be a finite-dimensional linear subspace with $1 \notin L$, and write $L_1 := \mathbb{R}1 + L$.

(a) $\varphi_L(S)$ is a semi-algebraic subset of $L^\vee$.

(b) The closed convex hull $\text{conv}(\varphi_L(S))$ of $\varphi_L(S)$ in $L^\vee$ consists of all $\lambda \in L^\vee$ that satisfy $\lambda'(g) \geq 0$ for every $g \in L_1 \cap \mathcal{P}(S)$.

(c) The closed conic hull $\text{cone}(\varphi_L(S))$ of $\varphi_L(S)$ in $L^\vee$ consists of all $\lambda \in L^\vee$ that satisfy $\lambda'(g) \geq 0$ for every $g \in L_1 \cap \mathcal{P}(S)$.

Proof. In (b), $\lambda' \in L_1^\vee$ denotes the extension of $\lambda \in L^\vee$ defined by $\lambda'(1) = 1$, see 2.1 (a) follows from the Tarski-Seidenberg theorem, and (b), (c) are consequences of convex duality. \qed

Lemma 2.3. Let $A$ be an $\mathbb{R}$-algebra, let $U \subseteq A$ be a linear subspace with $\dim(U) < \infty$, and let $UU$ be the linear subspace of $A$ spanned by all products $uu'$ ($u, u' \in U$). Then the cone $\Sigma U^2$, consisting of all finite sums of squares of elements of $U$, is a spectrahedral shadow in $UU$. 

Proof. Choose a linear basis \( u_1, \ldots, u_n \) of \( U \). The linear map \( f : \text{Sym}_n(\mathbb{R}) \rightarrow U \), 
\( (a_{ij}) \mapsto \sum_{i,j} a_{ij} u_i u_j \) satisfies \( \Sigma u^2 = f(\text{Sym}_n(\mathbb{R})) \), which shows the claim. \( \square \)

We keep fixing an affine \( \mathbb{R} \)-variety \( V \), a semi-algebraic set \( S \subseteq V(\mathbb{R}) \) and a finite-dimensional linear subspace \( L \subseteq \mathbb{R}[V] \). As before write \( L_1 = L + \mathbb{R}1 \).

**Proposition 2.4.** Let \( \phi_i : X_i \rightarrow V \) \( (i = 1, \ldots, m) \) be finitely many morphisms of affine \( \mathbb{R} \)-varieties. For every \( i = 1, \ldots, m \) let \( U_i \subseteq \mathbb{R}[X_i] \) be a finite-dimensional linear subspace, and assume that the following two conditions hold:

1. \( S \subseteq \phi_i(X_i(\mathbb{R})) \) for \( i = 1, \ldots, m \);
2. For every \( f \in L_1 \cap \mathcal{P}(S) \) there exists \( i \in \{1, \ldots, m\} \) such that \( \phi_i^*(f) \in \mathbb{R}[X_i] \) is a sum of squares of elements of \( U_i \) (in \( \mathbb{R}[X_i] \)).

Then \( \text{conv}(\varphi_L(S)) \), the closed convex hull of \( \varphi_L(S) \) in \( \mathbb{A}_L(\mathbb{R}) = L^\vee \), is a spectrahedral shadow.

**Proof.** Write \( C := L_1 \cap \mathcal{P}(S) \), which is a closed convex cone in \( L_1 \). For a given index \( i \in \{1, \ldots, m\} \) let \( C_i \subseteq L_1 \) be the cone of all \( f \in L_1 \) for which \( \phi_i^*(f) \) is a sum of squares of elements of \( U_i \) in \( \mathbb{R}[X_i] \). By Lemma 2.2 and since linear preimages of spectrahedral shadows are again spectrahedral shadows, \( C_i \) is a spectrahedral shadow in \( L_1 \). By condition (1), elements of \( C_i \) are non-negative on \( S \), which means \( C_i \subseteq C \). Therefore \( C = \bigcup_{i=1}^m C_i \) by (2), and hence we have \( C^* = \bigcap_{i=1}^m C_i^* \) for the dual cones. For every index \( i \) the cone \( C_i^* \), being the dual cone to a spectrahedral shadow cone, is itself a spectrahedral shadow. So it follows that \( C^* \) is a spectrahedral shadow in \( L_1 \).

For the convex hull \( K := \text{conv}(\varphi_L(S)) \subseteq L^\vee \) we have \( \overline{K} = \{ \lambda \in L^\vee : \lambda' \in C^* \} \), see Lemma 2.2(b). So \( \overline{K} \) is the preimage of the spectrahedral shadow \( C^* \) under the affine-linear map \( L^\vee \rightarrow L_1^\vee, \lambda \mapsto \lambda' \), and hence is a spectrahedral cone, as asserted. \( \square \)

**Corollary 2.5.** If, in Proposition 2.4, condition (2) is only required to hold for every \( f \in L_1 \cap \mathcal{P}(S) \), then \( \text{cone}(\varphi_L(S)) \), the closed convex cone generated by \( \varphi_L(S) \), is a spectrahedral shadow.

**Proof.** The proof is completely analogous to the proof of 2.4, defining the respective cones \( C \) and \( C_i \) to be subcones of \( L \) instead of \( L_1 \), and applying Lemma 2.2(c). \( \square \)

The following examples and remarks illustrate Proposition 2.4.

**Remarks 2.6.**

1. Proposition 2.4 can be seen as a generalization of the moment relaxation construction. To explain this, assume that we are in the situation of 2.4 in particular \( S = \{ \xi \in V(\mathbb{R}) : h_i(\xi) \geq 0 \ (i = 1, \ldots, r) \} \) with \( h_i \in \mathbb{R}[V] \). Let \( X \) be the affine \( \mathbb{R} \)-variety obtained by formally adjoining square roots of \( h_1, \ldots, h_r \) to \( \mathbb{R}[V] \), i.e. \( \mathbb{R}[X] = \mathbb{R}[V][t_1, \ldots, t_r]/(t_i^2 - h_i, i = 1, \ldots, r) \), and let \( \phi : X \rightarrow V \) be the natural map. Then clearly \( \phi(X(\mathbb{R})) = S \). If subspaces \( W_i \subseteq \mathbb{R}[V] \) as in 2.4 have been found such that the sufficient exactness condition from 2.4 is satisfied, i.e. if \( L_1 \cap \mathcal{P}(S) = C(W_0, \ldots, W_r) \) (in the notation of 2.1), this implies that for every \( f \in L_1 \cap \mathcal{P}(S) \) the pull-back \( \phi^*(f) \in \mathbb{R}[X] \) is a sum of squares in \( \mathbb{R}[X] \) of elements from the subspace \( U := \phi^*(W_0) + \sum_{i=1}^r \phi^*(W_i)\sqrt{h_i} \) of \( \mathbb{R}[X] \). So under this assumption, the conditions of Proposition 2.4 are fulfilled with \( m = 1 \) and these particular choices of \( \phi \) and \( U \).
2. Conversely, the more general construction of a semidefinite representation in Proposition 2.4 is achieved essentially by reduction to a construction of moment relaxation type, as in Proposition 2.4. We leave it to the reader to make this statement precise.

3. The proof of Proposition 2.4 is constructive in the following sense. If the morphisms \( \phi_i : X_i \to V \) as well as the linear subspaces \( U_i \subseteq \mathbb{R}[x] \) are given explicitly, we can deduce from this data an explicit semidefinite representation of \( \text{conv}(\mathcal{P}(S)) \).

Example 2.7. Let \( C \) be a nonsingular affine curve over \( \mathbb{R} \) for which \( C(\mathbb{R}) \) is compact. Let \( L \subseteq \mathbb{R}[C] \) be a finite-dimensional linear subspace, and consider the associated map \( \varphi_L : C(\mathbb{R}) \to \mathbb{R}[L] = L_\varphi \). By [36, Corollary 4.4] there exists a finite-dimensional linear subspace \( U \subseteq \mathbb{R}[C] \) such that every \( f \in L + \mathbb{R}1 \) that is non-negative on \( C(\mathbb{R}) \) is a sum of squares of elements from \( U \). Using this fact, Proposition 2.4 applies with \( \psi \) that the convex hull of \( \varphi_L(C(\mathbb{R})) \) in \( L_\varphi \) is a spectrahedral shadow. (This consequence was already drawn in [36].)

Remark 2.8. Later (Remark 3.7 below) we’ll see that it is not enough in Proposition 2.4 to replace condition (2) by the weaker condition that every \( f \in L_1 \cap \mathcal{P}(S) \) becomes a sum of squares in one of the \( \mathbb{R}[x_i] \). Rather, it is essential that such sum of squares representations exist in a uniform way.

3. Necessary conditions for semidefinite representability

In the previous section we stated sufficient conditions for semidefinite representability. We now show that these conditions are also necessary. In the sequel let \( x = (x_1, \ldots, x_n) \) be a tuple of variables. We start by recalling one form of duality in semidefinite programming (see [28]):

Proposition 3.1. Let \( M_1, \ldots, M_n \in \text{Sym}_d(\mathbb{R}) \), write \( M(\xi) = \sum_{i=1}^n \xi_i M_i \) for \( \xi \in \mathbb{R}^n \), and let \( C = \{ \xi \in \mathbb{R}^n : M(\xi) \geq 0 \} \) be the associated spectrahedral cone. Assume that \( M(\xi^0) > 0 \) for some \( \xi^0 \in \mathbb{R}^n \). Then the dual cone of \( C \) has the following semidefinite representation:

\[
C^* = \left\{ (B, M_1), \ldots, (B, M_n) \in \text{Sym}_d(\mathbb{R}), B \geq 0 \right\} \subseteq \mathbb{R}^n.
\]

Proposition 3.2. Assume that \( S \subseteq \mathbb{R}^n \) is a semi-algebraic set for which the closed conical hull \( \text{cone}(S) \subseteq \mathbb{R}^n \) of \( S \) is a spectrahedral shadow. Then there exists a morphism \( \phi : X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties, together with a finite-dimensional \( \mathbb{R} \)-linear subspace \( U \) of \( \mathbb{R}[x] \), such that \( S \subseteq \phi(X(\mathbb{R})) \) and the following holds: For every homogeneous linear polynomial \( f \in \mathbb{R}[x] \) with \( f \geq 0 \) on \( S \), the pull-back \( \phi^*(f) \in \mathbb{R}[x] \) is a sum of squares of elements from \( U \).

Our original proof for Proposition 3.2 (see version 1 of arXiv:1612.07048) was non-constructive and used a compactness argument for the real spectrum. The following explicit construction is much more elegant and transparent. It was suggested by Christoph Hanselka, who kindly agreed that his argument may be included here. Independently, the original approach may still have its merits, as we plan to demonstrate in follow-up work.

Proof. Let \( C = \text{cone}(S) \), the convex cone generated by \( S \) in \( \mathbb{R}^n \). We may assume that \( \mathbb{R}^n \) is affinely spanned by \( S \). By assumption, \( \overline{C} \) is the linear image of a
spectrahedron $T \subseteq \mathbb{R}^N$ under a linear map $\pi: \mathbb{R}^N \to \mathbb{R}^n$, for some $N$. We may assume that $T$ is a cone, and we may replace $\mathbb{R}^N$ by the linear hull of $T$. Then $T$ can be represented by a homogeneous linear matrix inequality that is strictly feasible. So we can assume that there are integers $d \geq 1$ and $m \geq 0$, together with linear matrix pencils $M(x) = \sum_{i=1}^n x_i M_i$, $N(y) = \sum_{j=1}^m y_j N_j$ in $\text{Sym}_d(\mathbb{R})$, such that

$$T = \{ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m : M(\xi) + N(\eta) \succeq 0 \},$$

such that $\overline{C} = \pi(T)$ where $\pi(\xi, \eta) = \xi$, and such that there exists $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$ with $M(\xi) + N(\eta) \succ 0$.

Consider the closed subvariety $X$ of $\mathbb{A}^n \times \mathbb{A}^m \times \text{Sym}_d$ defined over $\mathbb{R}$ whose $\mathbb{C}$-points are the triples $(\xi, \eta, A)$, where $A$ is a symmetric $d \times d$-matrix satisfying

$$A^2 = \sum_{i=1}^n \xi_i M_i + \sum_{j=1}^m \eta_j N_j.$$

We shall denote the coordinate functions on $X$ by

$$(x_1, \ldots, x_n; y_1, \ldots, y_m; (z_{\mu \nu})_{1 \leq \mu, \nu \leq d}) = (x, y, Z)$$

with $z_{\mu \nu} = z_{\nu \mu}$ for $1 \leq \mu, \nu \leq d$. Let $\phi: X \to \mathbb{A}^n$ be the projection $\phi(\xi, \eta, A) = \xi$. Then $\phi(X(\mathbb{R})) = \pi(T) = \overline{C}$, since a real symmetric matrix is psd if and only if it is the square of some real symmetric matrix. Let $U \subseteq \mathbb{R}[X]$ be the linear subspace spanned by the coefficient functions $z_{\mu \nu} = z_{\nu \mu}$ $(1 \leq \mu, \nu \leq d)$ of $Z$. We claim that the assertion of 3.2 holds with these choices of $\phi$ and $U$.

To see this, let $f = \sum_{i=1}^n a_i x_i$ be a linear homogeneous polynomial in $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ with $f \geq 0$ on $S$ (and hence $f \geq 0$ on $\overline{C}$). So the tuple $(a, 0) = (a_1, \ldots, a_n; 0, \ldots, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ lies in the dual cone $T^*$ of $T$. Since the linear matrix inequality is strictly feasible, there exists $B \in \text{Sym}_d(\mathbb{R})$ with $B \succeq 0$ such that $a_i = \langle B, M_i \rangle$ $(1 \leq i \leq n)$ and $0 = \langle B, N_j \rangle$ $(1 \leq j \leq m)$, by Proposition 3.1. Let $V \in \text{Sym}_d(\mathbb{R})$ with $B = V^2$. Then, as an element of $\mathbb{R}[X]$, $\phi^*(f)$ is equal to

$$\sum_{i=1}^n \langle B, M_i \rangle x_i + \sum_{j=1}^m \langle B, N_j \rangle y_j = \langle B, M(x) + N(y) \rangle = \langle V^2, Z^2 \rangle = \langle ZV, ZV \rangle.$$ 

This means that

$$\phi^*(f) = \sum_{\mu, \nu = 1}^d ((ZV)_{\mu \nu})^2$$

is a sum of squares in $\mathbb{R}[X]$ from the linear subspace $U \subseteq \mathbb{R}[X]$.

Combining Propositions 2.5 and 3.2, we therefore get:

**Theorem 3.3.** Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set, and let $C = \text{cone}(S)$ be the convex cone in $\mathbb{R}^n$ generated by $S$. The closure $\overline{C}$ is a spectrahedral shadow if and only if there exists a morphism $\phi: X \to \mathbb{A}^n$ of affine $\mathbb{R}$-varieties, together with an $\mathbb{R}$-linear subspace $U \subseteq \mathbb{R}[X]$ of finite dimension, such that

1. $S \subseteq \phi(X(\mathbb{R}))$.
2. For every homogeneous linear polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ with $f \geq 0$ on $S$, the pull-back $\phi^*(f) \in \mathbb{R}[X]$ is a sum of squares of elements from $U$.

**Proof.** The second condition is necessary for $\overline{C}$ to be a spectrahedral shadow by Proposition 3.2, and it is sufficient by 2.5. □
Instead of working with convex cones we may also dehomogenize and derive a non-homogeneous version from Theorem 3.3. Alternatively, we could as well have worked in an inhomogeneous setting from the beginning:

**Theorem 3.4.** Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set, and let $K = \text{conv}(S)$ be its convex hull in $\mathbb{R}^n$. The closure $\overline{K}$ is a spectrahedral shadow if and only if there exists a morphism $\phi : X \to \mathbb{A}^n$ of affine $\mathbb{R}$-varieties and an $\mathbb{R}$-linear subspace $U \subseteq \mathbb{R}[X]$ of finite dimension such that

1. $S \subseteq \phi(X(\mathbb{R}))$,
2. for every (inhomogeneous) linear polynomial $f \in \mathbb{R}[x]$ with $f \geq 0$ on $S$, the element $\phi^*(f)$ of $\mathbb{R}[X]$ is a sum of squares of elements from $U$.

**Proof.** If there exist $\phi$ and $U$ satisfying (1) and (2), $\overline{K}$ has a semidefinite representation by Proposition 2.4. Conversely, assume that $\overline{K}$ has a semidefinite representation, and let $K_1 = \{1\} \times K \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$. Since $\overline{K}_1$ is a spectrahedral shadow, it is easy to see that $\text{cone}(\overline{K}_1)$ is a spectrahedral shadow in $\mathbb{R}^{n+1}$. Hence the closure of $\text{cone}(\overline{K}_1)$ is a spectrahedral shadow as well. Clearly, this last cone coincides with $\overline{C}_1$, where $C_1$ is the convex cone in $\mathbb{R}^{n+1}$ generated by $S_1 = \{1\} \times S$.

Now we can apply the “only if” part of Theorem 3.3 to $S_1$ and $C_1$ and deduce the converse in Theorem 3.4. \qed

**Remark 3.5.** In Theorem 3.4 we may sharpen conditions (1) and (2) further. Assume we are given a morphism $\phi : X \to \mathbb{A}^n$ and a linear subspace $U \subseteq \mathbb{R}[X]$ as in Theorem 3.3. From (1) we deduce that there exists a semi-algebraic set $M \subseteq X(\mathbb{R})$ with $\phi(M) = S$ and with $\dim(M) = \dim(S)$ (use a semi-algebraic section $S \to X(\mathbb{R})$ of $\phi$ over $S$). Let $X'$ be the Zariski closure of $M$ in $X$, and let $U' \subseteq \mathbb{R}[X']$ be the image of $U$ under $\mathbb{R}[X] \to \mathbb{R}[X']$. Then (1) and (2) hold as well for the restriction $\phi' : X' \to \mathbb{A}^n$ of $\phi$ and for $U'$. Therefore, we can achieve in addition that $\dim(X) = \dim(S)$. On the other hand, condition (1) can be replaced by either $\overline{K} = \phi(X(\mathbb{R}))$ (to make it seemingly stronger), or by $K \subseteq \text{conv}(\phi(X(\mathbb{R})))$ (to make it seemingly weaker).

Note that the inclusion $\phi(X(\mathbb{R})) \subseteq \overline{K}$ holds for any $\phi$ satisfying (2). Indeed, given any $\xi \in \mathbb{R}^n$, $\xi \notin \overline{K}$, there exists $f \in \mathbb{R}[x]$ linear with $f|_S \geq 0$ and $f(\xi) < 0$, so (2) implies $\xi \notin \phi(X(\mathbb{R}))$.

**3.6.** In the next section we need to work not only over the field $\mathbb{R}$ of real numbers, but also over real closed extension fields $R \supseteq \mathbb{R}$. Given an affine $\mathbb{R}$-variety $V$ and a semi-algebraic set $M \subseteq V(\mathbb{R})$, the base field extension of $M$ to $R$ is denoted $M_R$ (see Section 5.1). If $M$ is described by a finite boolean combination of inequalities $f_i > 0$ (with $f_i \in \mathbb{R}[V]$), the set $M_R \subseteq V(R)$ is described by the same system of inequalities.

**Remark 3.7.** Let $\phi : X \to V$ be a morphism of affine $\mathbb{R}$-varieties, let $L \subseteq \mathbb{R}[V]$ and $U \subseteq \mathbb{R}[X]$ be finite-dimensional linear subspaces, and let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. Assume that the following condition holds:

(*) For every $f \in L$ with $f \geq 0$ on $S$, the pull-back $\phi^*(f) \in \mathbb{R}[X]$ is a sum of squares of elements of $U$.

Then the extension of (*) to any real closed field extension $R$ of $\mathbb{R}$ holds as well. More precisely, any $f \in L_R = L \otimes R \subseteq R[V]$ with $f \geq 0$ on $S_R \subseteq V(R)$ becomes a sum of squares of elements of $U_R = U \otimes R$ in $R[X]$, by the Tarski principle.

In particular, any $f \in L \otimes R$ with $f \geq 0$ on $S_R$ becomes a sum of squares in $R[X]$. We remark that this last conclusion would fail in general if in (*) we had
only required that \( \phi^* (f) \) is a sum of squares in \( \mathbb{R} [X] \). For instance, taking \( \phi \) to be the identity of \( X = V = \mathbb{A}^2 \) and \( S \) the unit disk would give counter-examples: Every \( f \in \mathbb{R} [x_1, x_2] \) with \( f \geq 0 \) on \( S \) can be written \( f = p + (1 - x_1^2 - x_2^2) q \) with sums of squares \( p, q \in \mathbb{R} [x_1, x_2] \), but the analogous statement fails over any proper real closed extension \( R \) of \( \mathbb{R} \) (see [33] and [34]). Rather, one needs that uniform sums of squares expressions exist as in (\( * \)), to guarantee that the condition is stable under real closed field extension.

The following version is essentially identical with the “only if” part of Theorem 3.4, but will be more convenient in the next section. Let \( V \) be an affine \( \mathbb{R} \)-variety, and let \( L \subseteq \mathbb{R} [V] \) be a finite-dimensional linear subspace. Let \( \varphi_L: V \rightarrow \mathbb{A}^n \) be the associated morphism, see 1.7.

**Corollary 3.8.** With \( V \) and \( L \) as above, let \( S \subseteq V (\mathbb{R}) \) be a semi-algebraic set. Assume that \( \text{conv}(\varphi_L (S)) \), the closed convex hull in \( \mathbb{A}^n (\mathbb{R}) = L^* \), is a spectrahedral shadow. Then there exists a morphism \( \phi: X \rightarrow V \) of affine \( \mathbb{R} \)-varieties, together with a finite-dimensional linear subspace \( U \subseteq \mathbb{R} [X] \), such that \( S \subseteq \phi (X (\mathbb{R})) \) and the following holds:

For every real closed field \( R \supseteq \mathbb{R} \) and every \( f \in L_R + R 1 \subseteq R [V] \) with \( f \geq 0 \) on \( S_R \), the pull-back \( \phi^*_R (f) \) under \( \phi_R: X_R \rightarrow V_R \) is a sum of squares of elements from \( U \otimes R \) in \( \mathbb{R} [X] \otimes R = R [X] \).

(The converse is true as well, covered by Proposition 2.4.)

**Proof.** By 3.4 there exists a morphism \( \psi: Y \rightarrow \mathbb{A}^L \) of affine \( \mathbb{R} \)-varieties, together with a finite-dimensional subspace \( W \subseteq \mathbb{R} [Y] \), such that, for every \( f \in R 1 + L \) with \( f \geq 0 \) on \( S \), the pull-back \( \psi^* (f) \in \mathbb{R} [Y] \) is a sum of squares of elements from \( W \). Let \( X \) be the fibered product of \( V \) and \( Y \) over \( \mathbb{A}^L \), let \( \phi: X \rightarrow V \) be the canonical morphism, and let \( U \subseteq \mathbb{R} [X] \) be the pull-back of \( W \) under \( X \rightarrow Y \). Then the condition in 3.8 is satisfied for \( R = \mathbb{R} \). By Tarski-Seidenberg, the condition holds over any real closed extension \( R \) as well (see also Remark 3.7).

It may not be obvious immediately, but the necessary condition for semidefinite representability found in Theorems 3.3 resp. 3.4 is quite restrictive. In the next section we’ll elaborate on this in more detail.

4. Constructing examples

We use properties of smooth morphisms of algebraic varieties, together with a weak version of generic smoothness, to construct examples of convex sets that have no semidefinite representation.

**4.1.** Let \( k \) be a field. Recall that a morphism \( \phi: X \rightarrow Y \) of algebraic \( k \)-varieties is smooth at \( x \in X \) if there exist affine open sets \( U = \text{Spec}(A) \subseteq X \) and \( V = \text{Spec}(B) \subseteq Y \) with \( x \in U \) and \( \phi(U) \subseteq V \) such that \( A \) is (via \( \phi \)) \( B \)-isomorphic to \( B[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \), where \( m \leq n \) and \( \det(\partial f_i / \partial x_j)_{1 \leq i, j \leq m} \) is a unit in \( \mathcal{O}_X (x) \). It is equivalent that \( \phi \) is flat at \( x \) and that the fibre \( \phi^{-1} (\phi (x)) \) is geometrically regular at \( x \) over the residue field of \( \phi (x) \), see [8, 17.5.1]. The smooth locus of \( \phi \), i.e. the set of points \( x \in X \) at which \( \phi \) is smooth, is Zariski open in \( X \).

We will use the following weak version of generic smoothness (compare [10, Lemma III.10.5]):
Lemma 4.6. Assume that we could have assumed $V$ sets $\mathcal{S}$ the following observation: $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}^{k} \phi$. We shall present two constructions. Each will give us concrete examples of $A$ is isomorphic over $\mathcal{P}$ -varieties where $\mathcal{P}$ sets.

Proposition 4.3. Let $\phi: X \rightarrow Y$ be a morphism of algebraic $k$-varieties. Let $\xi \in X(k)$, and write $A = \mathcal{O}_{X, \xi}$, $B = \mathcal{O}_{Y, \phi(\xi)}$. Then $\phi$ is smooth at $\xi$ if and only if $A$ is isomorphic over $B$ to a power series algebra $\hat{B}[t_{1}, \ldots, t_{m}]$.

(Here, of course, hat denotes completion of a local ring.) From [4, 3] we deduce the following observation:

Lemma 4.4. Let $\phi: X \rightarrow Y$ be a morphism of algebraic $k$-varieties, and assume that $\phi$ is smooth at $\xi \in X(k)$. If $f \in \mathcal{O}_{Y, \phi(\xi)}$ is such that $\phi^{*}(f)$ is a sum of squares in $\mathcal{O}_{X, \xi}$, then $f$ is a sum of squares in $\mathcal{O}_{Y, \phi(\xi)}$.

Proof. Indeed, if an element of a ring $B$ becomes a sum of squares in $B[[t_{1}, \ldots, t_{m}]]$, it was already a sum of squares in $B$. $\square$

4.5. We shall present two constructions. Each will give us concrete examples of convex semi-algebraic sets without semidefinite representation. For both, the reasoning will be based on the following technical lemma. We will repeatedly assume that data is given as follows:

(*) $V$ is an affine $\mathbb{R}$-variety, $\mathcal{L} \subseteq \mathbb{R}[V]$ is a finite-dimensional linear subspace, $\phi_{L}: V \rightarrow \mathbb{A}_{L} \cong \mathbb{A}^{n}$ ($n = \dim(L)$) is the associated morphism (see [1.7]), and $\mathcal{S} \subseteq V(\mathbb{R})$ is a semi-algebraic set. Moreover, $V'$ is an irreducible component of $V$ and $S' \subseteq S \cap V'(\mathbb{R})$ is a semi-algebraic set, Zariski-dense in $V'$.

(Nota: some of the technicalities in (*) and in 4.6 arise since we want to cover sets $S$ as well whose Zariski closure has several irreducible components. Otherwise we could have assumed $V' = V$ and $S' = S$.)

Lemma 4.6. Assume that (*) as in 4.3 is given. If $\text{conv}(\phi_{L}(S))$ is a spectrahedral shadow in $\mathbb{A}_{L}(\mathbb{R})$, there exists a morphism $\psi: W \rightarrow V'$ of affine $\mathbb{R}$-varieties, together with $\xi \in W(\mathbb{R})$, such that the following hold:

1. $W(\mathbb{R})$ is Zariski dense in $W$,
2. $\psi(\xi) \in S'$,
3. $\psi$ is smooth at $\xi$,
4. for every real closed field $R \supseteq \mathbb{R}$ and every $f \in L_{R} + R_{1} \subseteq R[V]$ with $f \geq 0$ on $S_{R}$, the pull-back $\psi_{R}^{*}(f) \in R[W]$ is a sum of squares in $R[W]$.

In (4) we have written $L_{R} = L \otimes R$, which is a finite-dimensional $R$-linear subspace of $\mathbb{R}[V] \otimes R = R[V]$.

Proof. By Corollary 3.3 there exists a morphism $\phi: X \rightarrow V$ of affine $\mathbb{R}$-varieties with $S \subseteq \phi(X(\mathbb{R}))$ such that, for every real closed $R \supseteq \mathbb{R}$ and every $f \in L_{R} + R_{1} \subseteq R[V]$ with $f \geq 0$ on $S_{R}$, the pull-back $\phi_{R}^{*}(f)$ is a sum of squares in $R[X]$. Using the argument of Remark 3.5 we can find a closed irreducible subvariety $X'$ of $X$ satisfying $\phi(X') \subseteq V'$ and $\dim(X') = V'$, for which $S' \cap \phi(X'(\mathbb{R}))$ is Zariski dense in $V'$. The restriction $\phi' : X' \rightarrow V'$ of $\phi$ is a dominant morphism between irreducible $\mathbb{R}$-varieties of the same dimension. By Proposition 1.2 there is a non-empty open affine subset $W$ of $X'$ such that the restriction $\phi'|_{W}: W \rightarrow V'$ of $\phi'$ is smooth. Writing $Z = X' \setminus W$ we have $\dim(Z) < \dim(V')$, so the set $\phi'(Z(\mathbb{R}))$ is not Zariski...
dense in \( V' \). Therefore \( S' \cap \phi'(W(\mathbb{R})) \) is still Zariski dense in \( V' \). In particular, we can find \( \xi \in W(\mathbb{R}) \) such that \( \eta := \phi'(\xi) \) lies in \( S' \). Then it is clear that (1)–(4) are satisfied for \( \psi := \phi'|_W : W \to V' \) and \( \xi \).

The first construction is very easy and works for convex hulls of suitable sets of dimension \( \geq 3 \). First recall:

**Lemma 4.7.** Let \( A \) be a regular local \( \mathbb{R} \)-algebra, let \( p_1, \ldots, p_d \) be a regular system of parameters of \( A \). If \( f(x_1, \ldots, x_d) \) is a form in \( d \) variables over \( \mathbb{R} \) that is not a sum of squares of forms, then \( f(p_1, \ldots, p_d) \in A \) is not a sum of squares in \( A \).

The proof uses the associated graded ring of \( A \), see [31], proof of Proposition 6.1.

**Proposition 4.8.** Assume that (\( * \)) as in \( 4.9 \) is given. If for every \( \eta \in S' \) there exists \( f \in L + \mathbb{R}1 \subseteq \mathbb{R}[V] \) with \( f|\eta \geq 0 \) such that \( f \) is not a sum of squares in \( \hat{O}_{\eta, \eta} \), then the closed convex hull \( \text{conv}(\varphi_L(S)) \) in \( \mathbb{A}_L(\mathbb{R}) \cong \mathbb{R}^{\dim(L)} \) fails to be a spectrahedral shadow.

**Proof.** Assume that the closed convex hull is a spectrahedral shadow. Then there exists a morphism \( \psi : W \to V' \) together with a point \( \xi \in W(\mathbb{R}) \) as in Lemma 4.6. Let \( \eta = \psi(\xi) \in S' \), and choose \( f \in L + \mathbb{R}1 \) for the given \( \eta \) as in the hypothesis. On the one hand, \( \psi^*(f) \in \mathbb{R}[W] \) should be a sum of squares in \( \mathbb{R}[W] \), by property (\( 4 \)) of \( \psi \) in \( 4.6 \). On the other hand, since \( \psi \) is smooth at \( \xi \), this contradicts Lemma 4.3 by the choice of \( f \).

**Example 4.9.** Let \( x = (x_1, x_2, x_3) \) and put \( L = \{ f \in \mathbb{R}[x] : \deg(f) \leq 6, f(0) = 0 \} \), a linear subspace of \( \mathbb{R}[x] \) with \( \dim(L) = 83 \). For every \( \xi \in \mathbb{R}^3 \) there exists \( f \in L + \mathbb{R}1 \) with \( f \geq 0 \) on \( \mathbb{R}^3 \) such that \( f \) is not a sum of squares in \( \hat{O}_{\mathbb{A}^3, \xi} \) (the ring of formal power series in \( x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3 \)). Indeed, this follows from [17], e.g. by taking \( f = p(x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3) \) where \( p \) is any ternary sextic form that is psd but not a sum of squares (for instance the Motzkin form). Let \( \varphi = \varphi_L : \mathbb{A}^3 \to \mathbb{A}^L \cong \mathbb{A}^{83} \) be the Veronese type embedding associated with \( L \). For any semi-algebraic set \( S \subseteq \mathbb{R}^3 \) with non-empty interior, it follows from Proposition 4.8 that the closed convex hull of \( \varphi(S) \) in \( \mathbb{R}^{83} \) has no semidefinite representation.

**Example 4.10.** Similarly, let \( x = (x_1, x_2, x_3, x_4) \) and \( L = \{ f \in \mathbb{R}[x] : \deg(f) \leq 4, f(0) = 0 \} \). Then \( \dim(L) = 69 \). Using psd, non-sos quartic forms in four variables and proceeding similarly as in 4.9 we find that the closed convex hull of \( \varphi_L(S) \) in \( \mathbb{R}^{69} \) is not a spectrahedral shadow, for any semi-algebraic set \( S \subseteq \mathbb{R}^4 \) with nonempty interior.

**Remark 4.11.** The reasoning used in the preceding examples was still very coarse. With a finer look we arrive at constructions that are considerably more parsimonious. For example, if in 4.9 we work with the Motzkin form \( p \), we can find a linear subspace \( L \subseteq \mathbb{R}[x] \) of dimension \( \dim(L) = 27 \) such that \( p(x - \xi) \in \mathbb{R}1 + L \) for every \( \xi \in \mathbb{R}^3 \), resulting in an embedding \( \mathbb{R}^3 \to \mathbb{R}^{27} \) with the property of 4.9. Similarly, if in 4.10 we work with the Choi-Lam form \( p(x) = x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2 - 4x_1x_2x_3x_4 \), we can find a linear subspace of dimension 19 with the desired property.

**Remark 4.12.** The construction of convex sets without semidefinite representation via Proposition 4.8 did not employ the full strength of the “only if” part of Theorem 3.4. Indeed, it wasn’t used anywhere that pull-backs of non-negative linear polynomials are uniformly sums of squares in \( \mathbb{R}[X] \) (see Remark 3.7). In turn,
the argumentation in [4.9–4.11] applies only to convex hulls of sets of dimension at least three. We now refine the construction. This will provide us with convex hulls of two-dimensional sets without a semidefinite representation.

4.13. Let $R$ be a real closed field containing $\mathbb{R}$. In the sequel, we always denote by $B$ the convex hull of $\mathbb{R}$ in $R$, so $B = \{ a \in R : \exists n \in \mathbb{N} \; -n < a < n \}$. Note that $B$ is a valuation ring (called the canonical valuation ring of $R$), with field of fractions $R$, maximal ideal $m_B = \{ a \in R : \forall n \in \mathbb{N} \; |na| < 1 \}$ and residue field $B/m_B = \mathbb{R}$. The reduction map $B \to B/m_B = \mathbb{R}$ will be denoted $a \mapsto \bar{a}$. Nonzero elements in $m_B$ will be called infinitesimals of $R$.

An example is given by the field $R = \bigcup_{q \geq 1} \mathbb{R}((t^{1/q}))$ of Puiseux series with real coefficients (see [2, Section 2.6]). The sign of $0 \neq f = \sum_{m \geq m_0} c_m t^{m/q} \in R$ with $c_m \in \mathbb{R}$ and $c_m \neq 0$ is the sign of $c_{m_0}$, and the valuation (or order) of $f$ is $\nu(f) = \frac{m_0}{q}$. The valuation ring $B$ resp. its maximal ideal $m_B$ consists of all $f \in R$ with $\nu(f) \geq 0$ resp. $\nu(f) > 0$.

Let $V$ be an affine $\mathbb{R}$-variety. We write $B[V] := \mathbb{R}[V] \otimes B$ (tensor product over $\mathbb{R}$). If $\phi : V \to V$ is a morphism of affine $\mathbb{R}$-varieties, then $\phi^*_R$ (resp. $\phi^*_B$) denotes the induced homomorphism $R[V] \to R[X]$ (resp. $B[V] \to B[X]$). Given $\xi \in V(\mathbb{R})$, let $M_{V, \xi} \subseteq B[V]$ be the kernel of the evaluation map $B[V] \to B$, $f \mapsto f(\xi)$.

We start with several auxiliary results. The following lemma is straightforward:

**Lemma 4.14.** Let $V$ be an affine $\mathbb{R}$-variety, let $\xi \in V(\mathbb{R})$, and let $R$, $B$ as in 4.13. Then for every $N \geq 1$ the natural map

$$(O_{V, \xi} / (m_{V, \xi})^N) \otimes B \to B[V] / (M_{V, \xi})^N$$

of $B$-algebras is an isomorphism. \hfill $\square$

**Lemma 4.15.** Let $R$, $B$ be as in 4.13, and let $X$ be an affine $\mathbb{R}$-variety for which $X(\mathbb{R})$ is Zariski dense in $X$. If $g_1, \ldots, g_r \in R[X]$ are such that $\sum_{i=1}^r g_i^2$ lies in $B[X]$, then $g_i \in B[X]$ for every $i$.

**Proof.** We can assume $g_i \neq 0$ for every $i$. Let $f := \sum_{i=1}^r g_i^2$. There is $0 \neq c \in R$ such that $cg_i \in B[X]$ for every $i$ and $c \sum g_i^2 \neq 0$ in $(B/m_B)[X] = \mathbb{R}[X]$ for at least one index $j$. It follows that $c^2 f \in B[X]$, and moreover $c^2 f = \sum_{i} (cg_i)^2$ is nonzero in $(B/m_B)[X] = \mathbb{R}[X]$, since $X(\mathbb{R})$ is Zariski dense in $X$. Hence $c \notin m_B$, which means that $\frac{1}{c} \in B$, and so indeed $g_i \in B[X]$ for every index $i$. \hfill $\square$

**Lemma 4.16.** Let $R$, $B$ be as in 4.13 and let $\phi : V \to V$ be a morphism of affine $\mathbb{R}$-varieties. Assume that $X(\mathbb{R})$ is Zariski dense in $X$, and that $\phi$ is smooth at $\xi \in X(\mathbb{R})$. If $f \in B[V]$ and $N \geq 1$ are such that $f$ is not a sum of squares in $B[V]$ modulo $(M_{V, \phi(\xi)})^N$, then $\phi^*_R(f) \in R[X]$ is not a sum of squares in $R[X]$.

**Proof.** Write $\eta = \phi(\xi)$. By Proposition 4.3, the smoothness assumption implies that the completed local ring $\hat{O}_{X, \xi}$ is $\hat{O}_{V, \eta}$-isomorphic to a power series ring over $\hat{O}_{V, \eta}$. In particular, this implies that $\phi^* : O_{V, \eta} / (m_{V, \eta})^N \to O_{X, \xi} / (m_{X, \xi})^N$ has a retraction, i.e. there is a homomorphism $\rho : O_{X, \xi} / (m_{X, \xi})^N \to O_{V, \eta} / (m_{V, \eta})^N$ for which the composition $\rho \circ \phi^*$ is the identity on $O_{V, \eta} / (m_{V, \eta})^N$. Tensoring with $B$
and using Lemma 4.14 gives the commutative diagram

\[
\begin{array}{ccc}
B[V] & \xrightarrow{\phi_0^*} & B[X] \\
\downarrow & & \downarrow \\
B[V]/(M_{V,\eta})^N & \xrightarrow{\phi_R^*} & B[X]/(M_{X,\xi})^N
\end{array}
\]

whose bottom map has a retraction. From the hypothesis it therefore follows that \(\phi_R^*(f) \in B[X]\) cannot be a sum of squares in \(B[X]\). By Lemma 4.15, \(\phi_R^*(f)\) is not a sum of squares in \(R[X]\) either.

**Lemma 4.17.** Let \(R, B\) be as in 4.13, let \(V\) be a nonsingular \(\mathbb{R}\)-variety, and let \(\xi \in V(\mathbb{R})\) be an affine \(\mathbb{R}\)-variety. If \(u_1, \ldots, u_d \in \mathbb{R}[V]\) form a regular parameter sequence of \(V\) at \(\xi\), there is an isomorphism

\[
B[V]/(M_{V,\xi})^N \cong B[x_1, \ldots, x_d]/\langle x_1, \ldots, x_d \rangle^N
\]

of \(B\)-algebras which makes the cosets of \(u_i\) and \(x_i\) correspond to each other, for \(i = 1, \ldots, d\).

**Proof.** Clear from the isomorphism \(\mathbb{R}[[x_1, \ldots, x_d]] \to \hat{O}_{V,\xi}\) sending \(x_i\) to \(u_i\), and from Lemma 4.14.

The next result is a key observation. For \(R \neq \mathbb{R}\) a proper real closed field extension of \(\mathbb{R}\), it implies that there exist polynomials \(f \in B[x_1, x_2]\) with \(f \geq 0\) on \(R^2\) such that \(f\) is not a sum of squares in \(B[x_1, x_2]/\langle x_1, x_2 \rangle^N\), for \(N\) sufficiently large. Note that any such \(f\) is a sum of squares in \(\mathbb{R}[x_1, x_2]\), and hence in \(R[x_1, x_2]/\langle x_1, x_2 \rangle^N\) for all \(N \geq 32\).

**Proposition 4.18.** Let \(f \in \mathbb{R}[x_0, x] = \mathbb{R}[x_0, \ldots, x_n]\) be homogeneous of degree \(d\), and assume that \(f\) is not a sum of squares in \(\mathbb{R}[x_0, x]\). Let \(R, B\) be as in 4.13. If \(\epsilon > 0\) is an infinitesimal in \(R\), the polynomial \(f(\epsilon, x) \in B[x]\) is not a sum of squares in \(B[x]/\langle x_1, \ldots, x_n \rangle^d + B[x]\).

**Proof.** Assume we have an identity \(f(\epsilon, x) + g(x) = \sum_j p_j(x)^2\) where \(g(x) \in \langle x \rangle^d + B[x]\) and \(p_j(x) \in B[x]\). Replacing \(x\) by \(\epsilon x\) yields

\[
\epsilon^d f(1, x) + g(\epsilon x) = \sum_j p_j(\epsilon x)^2.
\]

The left hand side is divisible by \(\epsilon^d\) in \(B[x]\). By Lemma 4.15 the polynomial \(q_j(x) := \epsilon^{-d/2} p_j(\epsilon x) \in R[x]\) lies in \(B[x]\) for every \(j\). Putting \(g'(x) = \epsilon^{-(d+1)} g(\epsilon x)\) we have \(g'(x) \in B[x]\), therefore dividing \((\ast)\) by \(\epsilon^d\) gives

\[
f(1, x) + \epsilon g'(x) = \sum_j q_j(x)^2,
\]

an identity in \(B[x]\). Reducing coefficient-wise modulo \(m_B\) implies that \(f(1, x)\) is a sum of squares in \(\mathbb{R}[x]\), contradicting the hypothesis.

**Proposition 4.19.** Assume that \((\ast)\) as in 4.15 is given, and assume that \(R \supseteq \mathbb{R}\), \(R \neq \mathbb{R}\) is a real closed field with canonical valuation ring \(B\) (4.13). For every \(\eta \in S'\), assume that there exists \(f \in L_B + B_1 \subseteq B[V]\) with \(f \geq 0\) on \(\mathbb{R}\) such that \(f\) is not a sum of squares in \(B[V]/(M_{V,\eta})^N\) for some \(N \geq 1\). Then the closed convex hull \(\text{conv}(\varphi_L(S))\) in \(A_L(\mathbb{R}) \cong \mathbb{R}^{\dim(L)}\) is not a spectrahedral shadow.
Theorem 4.23. Let $\hat{N}$.

Proof. Assume that the closed convex hull is a spectrahedral shadow. Then there exists $\psi: W \to V'$ together with $\xi \in W(\mathbb{R})$, as in Lemma 4.10. Let $\eta = \psi(\xi) \in S'$, and choose $f \in L_B + B1$ for the given $\eta$ as in 4.19. On the one hand, $\psi^*_p(f) \in R[W]$ should be a sum of squares in $R[W]$, by property (4) of $\psi$ in 4.6. On the other hand, $\psi^*_p(f)$ is not a sum of squares in $R[W]$ by Lemma 4.16. This contradiction proves Proposition 4.19. $\square$

Example 4.20. Let $x = (x_1, x_2)$, and put $L = \{ f \in \mathbb{R}[x] : \deg(f) \leq 6, f(0) = 0 \}$, a linear subspace of $\mathbb{R}[x]$ of dimension 27. Consider the associated embedding $\phi_L: \mathbb{A}^2 \to \mathbb{A}_L \cong \mathbb{A}^{27}$. If $S \subseteq \mathbb{R}^2$ is any semi-algebraic set with non-empty interior, the closed convex hull of $\varphi_L(S)$ in $\mathbb{R}^{27}$ is not a spectrahedral shadow. Indeed, choose a sextic form $p \in \mathbb{R}[x_0, x_1, x_2]$ that is psd but not a sum of squares, and let $0 \neq \epsilon$ be an infinitesimal of $R$. Given $\xi \in S$, the polynomial $f := p(\epsilon, x_1 - \xi_1, x_2 - \xi_2) \in B[x_1, x_2]$ lies in $L_B + B1$, and $f$ is not a sum of squares in $B[x]/(M_{\hat{B}})$ by Proposition 4.19 and Lemma 4.17. It follows from Proposition 4.19 that $\text{conv}(\varphi_L(S))$ has no semidefinite representation.

Remark 4.21. Similar to Remark 4.11, we can arrive at examples of smaller dimension when we take a finer look. For instance, consider the Motzkin form $p = x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 - 3 x_1 x_2^3 x_2^2$ in $\mathbb{R}[x_0, x_1, x_2]$. This form is psd but not a sum of squares. Let $L \subseteq \mathbb{R}[x, y]$ be the linear space spanned by the 14 monomials $x^i (1 \leq i \leq 6)$, $y^i (1 \leq i \leq 4)$ and $x^i y^j$ $(i, j = 1, 2)$. For any semi-algebraic set $S \subseteq \mathbb{R}^2$ with non-empty interior, the closed convex hull of $\varphi_L(S)$ in $\mathbb{R}^{14}$ fails to be a spectrahedral shadow. Indeed, for any choice of $\epsilon, \xi_1, \xi_2 \in B$, the polynomial $f := p(\epsilon, x_1 - \xi_1, x_2 - \xi_2) \in B[x_1, x_2]$ lies in $L_B + B1$. So we can argue as in 4.20.

A reasoning as in Examples 4.20 or 4.21 can also be applied to affine $\mathbb{R}$-varieties $V$ different from $\mathbb{A}^n$, thanks to the following lemma:

Lemma 4.22. Let $R, B$ be as in 4.13, and assume $R \neq \mathbb{R}$. Let $V$ be an affine $\mathbb{R}$-variety, let $V_{\text{reg}} \subseteq V$ be its smooth locus, let $\eta \in V_{\text{reg}}(R)$, and let $q_1, \ldots, q_n \in B[V]$ be a regular parameter sequence for $\mathcal{O}_{V_{\text{reg}}}$. Moreover let $f \in \mathbb{R}[x_0, \ldots, x_n]$ be a form that is psd but not a sum of squares. If $\epsilon \neq 0$ is any infinitesimal in $R$, then $f(\epsilon, q_1, \ldots, q_n) \in B[V]$ is psd on $V(R)$, but is not a sum of squares in $B[V]/(M_{\hat{B}})^N$ for $N \geq \deg(f) + 1$.

Proof. Put $p := f(\epsilon, q_1, \ldots, q_n)$. It is clear that $p \geq 0$ on $V(R)$. Let $x = (x_1, \ldots, x_n)$. We have an isomorphism $B[x]/\langle x \rangle^N \cong B[V]/(M_{\hat{B}})^N$ for every $N \geq 1$, that sends $x_i$ to $q_i$ for $i = 1, \ldots, n$ (Lemma 4.17). It maps the residue class of $f(\epsilon, x)$ to the residue class of $p$. By Proposition 4.18, this element (in either ring) is not a sum of squares when $N > \deg(f)$.

Summing up, we can conclude:

Theorem 4.23. Let $S \subseteq \mathbb{R}^m$ be any semi-algebraic set with $\dim(S) \geq 2$. Then, for some $k \geq 1$, there exists a polynomial map $\varphi: S \to \mathbb{R}^k$ such that the closed convex hull of $\varphi(S)$ in $\mathbb{R}^k$ has no semidefinite representation.

Proof. Let $V \subseteq \mathbb{A}^m$ be the Zariski closure of $S$. Fix a point $\xi \in S \cap V_{\text{reg}}(\mathbb{R})$ such that $\dim_{\mathbb{C}}(S) \geq 2$ and $S$ contains an open neighborhood of $\xi$ in $V(\mathbb{R})$. Let $p_1, \ldots, p_n \in \mathbb{R}[V]$ $(n \geq 2)$ be a regular sequence of parameters for $\mathcal{O}_{V_{\xi}}$. Let
x = (x1, . . . , xn), y = (y1, . . . , yn) be tuples of variables, let f ∈ R[t, x] be a form in n + 1 variables that is psd but not a sum of squares, and put d = deg(f). We can write

\[ f(t, x + y) = \sum_{i=0}^{d} g_i(x) h_{d-i}(t, y) \]

where \( g_i \in \mathbb{R}[x] \) and \( h_i \in \mathbb{R}[t, y] \) are forms of degree \( i \) \((i = 0, . . . , d)\). There is a Zariski open neighborhood \( U \subseteq V_{reg} \) of \( \xi \) such that, for any \( \eta \in U(\mathbb{R}) \), the sequence \( p_i - p_i(\eta) \) \((i = 1, . . . , n)\) is a regular sequence of parameters for \( \mathcal{O}_{V, \eta} \). Let \( L \subseteq \mathbb{R}[V] \) be a finite-dimensional linear subspace that contains \( g_i(p_1, . . . , p_n) \) for \( i = 1, . . . , d \), and choose a real closed field \( R \) that properly contains \( \mathbb{R} \). For any \( a = (a_0, . . . , a_n) \in B^{n+1} \), the element

\[ q_a := f(a_0, p_1 + a_1, . . . , p_n + a_n) = \sum_{i=0}^{d} g_i(p_1, . . . , p_n) h_{d-i}(a_0, . . . , a_n) \]

lies in \( L_B + B1 \subseteq B[V] \) and satisfies \( q_a \geq 0 \) on \( V(R) \). Let \( \eta \in U(\mathbb{R}) \), and put \( a = (\epsilon, -p_1(\eta), . . . , -p_n(\eta)) \in B^{n+1} \) where \( \epsilon \neq 0 \) is infinitesimal in \( R \). Then \( q_a \in B[V] \) is non-negative on \( V(R) \), and \( q_a \) is not a sum of squares in \( B[V]/(M_{V, \eta})^{n+1} \) by Lemma 4.22. By Proposition 4.19, this shows that \( \text{conv}(\varphi(S)) \) is not a spectrahedral shadow.

The previous examples already indicate that convex hulls of Veronese sets typically fail to be spectrahedral shadows. Specifically, we have:

**Corollary 4.24.** Let \( n, d \) be positive integers with \( n \geq 3 \) and \( d \geq 4 \), or with \( n = 2 \) and \( d \geq 6 \). Let \( m_1, . . . , m_N \) be the non-constant monomials of degree \( \leq d \) in \( (x_1, . . . , x_n) \) (so \( N = \binom{n+d}{d} - 1 \)). Then for any semi-algebraic set \( S \subseteq \mathbb{R}^n \) with non-empty interior, the closed convex hull of

\[ v(S) := \{ (m_1(\xi), . . . , m_N(\xi)) : \xi \in S \} \]

in \( \mathbb{R}^N \) fails to be a spectrahedral shadow.

*Proof.* Hilbert [15] showed that there exists a psd form \( f \) of degree \( d \) in \( n + 1 \) variables. So it suffices to apply Propositions 4.18 and 4.19. □

For positive integers \( n, d \) let \( \Sigma_{n, 2d} \) (resp. \( P_{n, 2d} \)) denote the cone of all degree \( 2d \) forms in \( \mathbb{R}[x_1, . . . , x_n] \) that are sums of squares of forms (resp. that are positive semidefinite).

**Corollary 4.25.** The psd cone \( P_{n, 2d} \) is a spectrahedral shadow only in the cases where \( P_{n, 2d} = \Sigma_{n, 2d} \), i.e. only for \( 2d = 2 \) or \( n = 2 \) or \( (n, 2d) = (3, 4) \).

*Proof.* It is well-known and easy to see that the dual \( \Sigma^*_{n, 2d} \) of the sos cone is a spectrahedral cone. Therefore \( \Sigma_{n, 2d} \), being closed, is a spectrahedral shadow. Let \( n, d \) be such that \( \Sigma_{n, 2d} \neq P_{n, 2d} \). By Hilbert’s theorem [15] quoted before, this happens precisely when \( 2d = 2 \) or \( n = 2 \) or \( (n, 2d) = (3, 4) \). The dual cone \( P^*_{n, 2d} \) can be identified with the convex (or conical) hull of the image of the degree \( 2d \) Veronese map

\[ v_{n, 2d} : \mathbb{R}^n \to \mathbb{R}^N, \quad \xi \mapsto (\xi^*)_{|n|=2d} \]

where \( N = \binom{n+2d-1}{n-1} \) is the number of monomials of degree \( 2d \) in \( (x_1, . . . , x_n) \). By [4,23] a suitable affine hyperplane section of this cone fails to be a spectrahedral
shadow. So $P_{n,2d}^*$ itself cannot be a spectrahedral shadow, and therefore neither can be $P_{n,2d}$.

5. Some open questions

There are many obvious questions that remain open at this point. Here are some that we consider as being particularly natural.

5.1. What is the smallest dimension of a convex semi-algebraic set without semidefinite representation? The smallest dimension that we realize in this paper is 14 (see [4, 21]). A more technical construction gives examples of dimension 11. We expect that the true answer should be much less. Is it three? Recall that the Helton-Nie conjecture has been proved for subsets of $\mathbb{R}^2$ [30].

5.2. Consider the necessary and sufficient condition 3.3 (or 3.4) for semidefinite representability. Although we use it to construct counter-examples to the Helton-Nie conjecture, it seems that in concrete cases, the condition is often hard to decide. For a prominent example, let $C_n \subseteq \text{Sym}_n(\mathbb{R})$ be the copositive cone, consisting of all symmetric matrices $A$ such that $x^tAx \geq 0$ for all $x \in (\mathbb{R}_+)^n$ (see [16] for a recent survey). For $n \geq 5$ it is not known whether $C_n$ is a spectrahedral shadow ([4, p. 135]). We were unable to apply criterion 3.3 to decide this question. Therefore we ask: What are alternative characterizations of spectrahedral shadows that are easier to work with?

5.3. The results of Helton and Nie [11, 12] guarantee the existence of a semidefinite representation in a wide range of cases. Specifically, if a compact convex semi-algebraic set $K \subseteq \mathbb{R}^n$ fails to have a semidefinite representation, their results imply that the boundary of $K$ must have a singular point, or must have zero curvature somewhere ([11], conclusions, p. 790).

The counter-examples to the Helton-Nie conjecture constructed in this paper are typically (closed) convex hulls of low-dimensional sets in high-dimensional euclidean space. In particular, their boundaries have singularities. It seems to be an open question whether there exist counter-examples with smooth boundary.

5.4. The generalized Lax conjecture (see [37] for an overview) asserts that the hyperbolicity cone in $\mathbb{R}^n$ of any hyperbolic form $f(x_1, \ldots, x_n)$ is a spectrahedral cone. For $n = 3$ this is in fact a theorem, proved by Helton-Vinnikov [13] in 2007 in a significantly stronger form. For $n \geq 4$ however, it is not even known in general whether every hyperbolicity cone is a spectrahedral shadow, although this holds when the cone is smooth (Netzer-Sanyal [23]). Can one decide whether hyperbolicity cones are spectrahedral shadows using results of this paper?

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