Black to Negative*
Embedded optionalities in commodities markets

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Abstract

We address the modelling of commodities that are supposed to have positive price but, on account of a possible failure in the physical delivery mechanism, may turn out not to. This is done by explicitly incorporating a ‘delivery liability’ option into the contract. As such it is a simple generalisation of the established Black model.

Introduction

To the consternation of oil traders, WTI crude prices went below zero on 20th April this year, with the May futures contract trading at $-37.63$ USD/bbl just before expiration. This prompted a rethink of the fundamentals of option pricing and risk management.

The standard pricing model is, of course, Black (also known as Black’76 or Black-Scholes), wherein the underlying asset price follows a geometric Brownian motion. As such a model cannot accommodate negative prices, there has predictably been a rush for the use of arithmetic (Bachelier) models instead, but this seems symptomatic of desperately clutching at the nearest simple probabilistic tool to hand, rather than of carefully appreciating the fundamental problem. Further, Black works well most of the time, albeit with the usual difficulties of pricing short-dated out-of-the-money options. It seems unwise simply to discard years of accumulated experience of Black in the following areas: hedging; pricing using volatility surfaces; the concept of convenience yield.

It is obvious from Table 1 that the negative price was caused by a failure in the delivery mechanism. With storage tankers at Cushing full, and transportation tankers also, it was very difficult to store the commodity, and so traders had to pay people to take it off their hands. The deduction: commodities are not simply assets. What do we mean by this?

|        | May | Jun  | Jul  |
|--------|-----|------|------|
| CLK0   | 18.27 | 25.03 | 29.42 |
| CLM0   | -37.63 | 20.43 | 26.28 |
| CLN0   | 10.01 | 11.57 | 18.69 |

Table 1: Closing prices of recent WTI oil contracts just before and after the negative price date. Source: Bloomberg

*The title comes from the convention (at least in the UK) that on a battery, red is the positive terminal and black the negative. In accounting it is the other way round!
In *Rich Dad, Poor Dad*, a series of ‘popular finance’ books published over the last couple of decades, Robert Kiyosaki defines an asset as something that puts money in your pocket, whereas a liability takes it out. He goes on to point out that many things that people consider to be assets are in fact liabilities. A better summary would be: they are pure asset and a pure liability taken together. For example, a house has an intrinsic value, but on the liability side there is the cost of upkeep, and property taxes.

It follows that we should represent a commodity as a hypothetical pure asset $A$ say, whose value is always positive, coupled with a liability whose price is likely to be linked to $A$. Failure of the delivery mechanism can give rise to a large liability. It is attractive, from the perspective of capital markets vernacular, to regard this as a form of intrinsic optionality that generally expires worthless—until one day it doesn’t.

Before making too strong a strong connection between commodities and securities markets, we should point out a fundamental difference between them. In the latter, the spot price of the asset has primacy, and the forward or futures price is linked to it by a simple parity argument based on either buying the asset and delivering it into the forward/futures, or selling it and buying it in the forward/futures market. In oil markets, however, there is no spot contract to speak of; inasmuch as it exists it has different delivery standards and necessarily trades at a basis to each futures contract. There is no notion of arbitrage between spot and futures or between one futures and another, because for such a trade to function one must be able to physically deliver into or receive physical delivery from the contract—the failure of which is the whole crux of the problem, given the complexity and expense of receiving barrels of crude oil rather than a conveniently dematerialised bond or share.

1 Modelling

1.1 Pricing model

See Björk [1, Ch.20] for information about martingale pricing of forward and futures contracts. Throughout $\mathbb{E}$ means expectation under the risk-neutral measure $\mathbb{Q}$ (wherein with the rolled-up money market account as numeraire, discounted expectations are martingales), and $W_t$ is a Brownian motion under $\mathbb{Q}$.

We write $A_t$ for the intrinsic asset price at time $t$. This is the value of the asset if we could store and deliver it at no cost, and is hypothetical. Doing the obvious thing, we make the $A$-dynamics under $\mathbb{Q}$ a geometric Brownian motion, at least in the first instance:

$$dA_t/A_t = (r - y) \, dt + \sigma \, dW_t$$  \hspace{1cm} (1)

where $r$ is the riskfree rate and $y$ is as usual the convenience yield [4, Ch.2,3].

Usually one has for the futures,

$$F_t(T) = \mathbb{E}_t[A_T],$$

with a similar result holding for forwards\footnote{We recall the apocryphal tale of the Russian hotel used as collateral against a bank loan. It was so dilapidated that the local authorities forced the new owner, i.e. the bank, to bring it to state of good repair—which cost more than the final value. Thereby the loan had negative recovery.}, but we are proposing

$$F_t(T) = \mathbb{E}_t[A_T - \psi_-(A_T)],$$  \hspace{1cm} (2)

\footnote{The expectation would be taken under the forward risk-neutral measure. As we are not worrying about stochastic interest rates in this paper, this difference will not detain us.}
where the intrinsic option $\psi_-$ is a convex decreasing function. An obvious idea is a vanilla put option but a better one is a call option on a negative power of $A_T$:

$$\psi_-(A_T) = \ell \hat{A}((\hat{A}/A_T)\lambda - 1)^+.$$  \hfill (3)

There are three parameters, $\hat{A} > 0$, $\ell \geq 0$, $\lambda \geq 0$; the first has the same units as $A_T$ and is interpreted as the price at which ‘things start to go wrong’, and the other two are dimensionless and control the size of the resulting liability. Not only can the futures price go negative: it is also unbounded below as $A \to 0$. This is why we have chosen the above formulation rather than a simple put $(\hat{A} - A_T)^+$. Had we made the intrinsic option a constant, we would have just ended up with a shifted lognormal model, which is a well-known idea in other contexts; its disadvantage is that one never knows how much shifting to do.

We briefly return to a point we made earlier. It is superficially attractive to regard the following construct as a form of spot price:

$$S_t = A_t - \psi_-(A_t)$$

so that $F_t(T) = E_t[S_T]$. But there is a conceptual difficulty with that, as we may well need the parameters of $\psi_-$ to dependent of the futures maturity $T$. This appears to generate a multiplicity of spot prices—but it is legitimate, because in reality the spot contract does not trade.

The prescription (3) has good analytical tractability in the sense that the expectation of $\psi_-(A_T)$ can be evaluated using Black-type formulae. Therefore the futures price is easy to calculate. Indeed,

$$E_t[\psi_-(A_T)] = \ell \hat{A}^{1+\lambda} E_t[A_T^{-\lambda}] P_t^{-\lambda}(A_T < \hat{A}) - \ell \hat{A} P_t(A_T < \hat{A})$$

$$= \ell \hat{A} \left( \frac{\hat{A}e^{(\lambda+1)\sigma^2(T-t)/2}}{F^*} \right)^\lambda \Phi(-\hat{d}_3) - \Phi(-\hat{d}_2)$$  \hfill (4)

with

$$\hat{d}_2 = \frac{\ln(F^*/\hat{A}) - \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}}$$

$$\hat{d}_3 = \hat{d}_2 - \lambda\sigma \sqrt{T-t}$$

$$F^* = A_t e^{(r-y)(T-t)} = E_t[A_T]$$

and $\Phi$ the standard Normal integral. The quantity $F^* = F^*_t(T)$ is interpreted as the futures price on the intrinsic asset, were such a contract to trade; it is always positive even if $F$ is not. The notation $P^\nu$ corresponds to the expectation $E^\nu[Z] = E[A_T^\nu]$, and we use standard arguments about change of numeraire. This formula allows the intrinsic asset price (equivalently $F^*$) to be imputed from the futures price; this also requires the specification of $\psi_-$ and the volatility of $A$.

Valuation of European options on $F_t(T)$ can then be done using compound-option formulae. Let $T_0 \leq T$ be the option maturity. The main thing to note is that for any option strike $K_o$ there is a unique $A^\sharp = A^\sharp(K_o)$ such that

$$E[A_T - \psi_-(A_T)|A_{T_0} = A^\sharp] = K_o$$

and so the exercise decision at time $T_0$ reduces to $A_{T_0} \geq A^\sharp$.

As regards application we note that the exchange-traded options are vanilla options on each futures contract, with the same expiry as the futures ($T_0 = T$). We will only deal with this case...
here. Also the exchange-traded options are American-style but we ignore the value of the early exercise option which, by standard arguments, is technically worthless.

The expected payouts of the call and put are

\[
C_{K_o} = E_t[(A_T - \psi_-(A_T) - K_o)^+] \\
= E_t[(A_T - \psi_-(A_T) - K_o) 1(A_T > A^\sharp)] \\
= E_t[A_T]P_t^+(A_T > A^\sharp) - K_oP_t(A_T > A^\sharp) \\
- \ell A \left[ E_t[(A_t/A_T)^\lambda]P_t^{-\lambda}(A^\sharp < A_T < \hat{A}) - P_t(A^\sharp < A_T < \hat{A}) \right] \\
= F^*\Phi(d_1) - K_o\Phi(d_2) \\
- \ell \hat{A} \times \begin{cases} 
K_o e^{(\lambda + 1)\sigma^2(T-t)/2} F^* \lambda \Phi(-\hat{d}_3) - \Phi(-d_3) - (\Phi(-d_2) - \Phi(-d_2)), & A^\sharp \leq \hat{A} \\
0, & A^\sharp > \hat{A}
\end{cases}
\]

and via the same route

\[
P_{K_o} = E_t[(K_o - A_T + \psi_-(A_T))^+] \\
= E_t[(K_o - A_T + \psi_-(A_T)) 1(A_T < A^\sharp)] \\
= K_o\Phi(-d_2) - F^*\Phi(-d_1) \\
+ \ell \hat{A} \times \begin{cases} 
K_o e^{(\lambda + 1)\sigma^2(T-t)/2} F^* \lambda \Phi(-d_3) - \Phi(-d_2), & A^\sharp \leq \hat{A} \\
K_o e^{(\lambda + 1)\sigma^2(T-t)/2} F^* \lambda \Phi(\hat{d}_3) - \Phi(-d_2), & A^\sharp > \hat{A}
\end{cases}
\]

where

\[
d_2 = \ln(F^*/A^\sharp) - \frac{1}{2}\sigma^2(T-t) \\
d_1 = d_2 + \sigma\sqrt{T-t} \\
d_3 = d_2 - \lambda\sigma\sqrt{T-t}
\]

The prices are these discounted by \( e^{-rT} \). It is easy to see that the put-call parity formula \( C_{K_o} - P_{K_o} = F - K_o \) is obeyed.

If \( T_o < T \) then the conditions \( A_{T_o} < A^\sharp \) and \( A_T < K_o \) refer to the intrinsic asset price at different times, and the expectations require the bivariate Normal integral. We will deal with this in forthcoming work.

1.2 Numerical results

Figure 1 shows results for the June contract CLM0 as of 21-Apr-20, and the same but for the July contract CLN0. The left-hand plot shows prices, and the right-hand plot the equivalent Black volatility (only for positive strikes, of course). It is apparent that the fit for low strikes is excellent—and for the first time ever we can show a Black model with negative strikes!

In both cases the market quotes for the lowest-strike put options indicate positive probability of negative futures price at expiry; though obviously this is greater for the front (June, CLM0) contract. Indeed, attempting to impose that the zero-strike put \( P_0 \) be worthless would lead to a convexity arbitrage in the puts (sell \( 2 \times P_3 \) and buy \( 1 \times P_0 \) and \( 1 \times P_{10} \)). The new model shows that
Figure 1: Price and implied vol for CLM0 and CLN0 on 21-Apr-20: market and model compared. Market data source: Bloomberg.

|       | \( F^* \) | \( \sigma \) | \( A \) | \( \lambda \) | \( \ell \) | \( F^* \) |
|-------|--------|--------|--------|--------|--------|--------|
| CLM0  | 11.57  | 140%   | 19.5   | 0.793  | 2.16   | 19.13  |
| CLN0  | 18.69  | 130%   | 24.7   | 0.310  | 1.33   | 22.54  |

Table 2: Fitted parameters for front two CL contracts on 21-Apr-20.
$P_0$, and indeed some negative-strike puts, should have traded at a positive price, which is entirely reasonable.

Another important point is that the option prices are captured with parameters (Table 2) that have a reasonable physical intuition. While the volatility $\sigma$ is high (around 130%), it retains some sort of plausibility: whereas an implied Black volatility of over 1000%, as is needed to mark the CLM0 5-strike put, is essentially useless. We should point out that different parameter sets give very similar fit, so the model is probably overparametrised: this could be alleviated by fitting to multiple futures at the same time and requiring, for example, that $\sigma$ not vary too much between adjacent contracts. This is a matter for future research.

Finally, we are able to make a deduction about the convenience yield, not from the ratio of adjacent futures prices $F$ (which would fail immediately when prices went negative), but instead from the ‘intrinsic futures’ $F^*$, as
\[
F^*(T_1)/F^*(T_2) = e^{(r-y)(T_1-T_2)}.
\]
As $r$ is context ignorable, and $T_2 - T_1 = \frac{34}{365}$, we have $y \approx -176\%$: a large number indicating extreme contango.

2 Connection with other areas

One thing we specifically set out not to do in this paper is to rewrite from scratch all commodity option pricing models. Quite the reverse: what we propose integrates perfectly well with all branches of the subject, as we now briefly justify.

2.1 Volatility surfaces

As is well known, no single Black volatility prices all options consistently the market—in any asset class. This has led to a convenient visualisation tool: the (Black) volatility surface.

The primary objective of any extension to the Black model is to flatten the volatility surface; ideally the whole surface would be explained by a small number of parameters. For example, Lévy models deal with the problems of short-dated OTM options by using jumps instead of an artificially high volatility.

We do not suggest that the incorporation of an ‘intrinsic put’ of the type herein described will result in perfect pricing—but it does permit negative strikes and captures the ‘smile effect’ for low strikes. For perfect matching of prices we will need to make one parameter vary, and this had better be the volatility $\sigma$. With the Black model the implied volatility is unique (and exists provided the option price does not violate simple arbitrage constraints), because both call and put option prices are increasing functions of volatility. That the same property carries over to the model here is almost too obvious to be worth asking about, but there is a subtlety. When we alter $\sigma$, the futures price will change by $\delta F^*$, and we will no longer match the market unless we alter the intrinsic asset price via a bump $\delta F^*$. Therefore in calibration when we talk about a move $\delta \sigma$ in volatility, we need also to apply a bump $\delta F^*$ so that the futures price is held fixed. The resulting sensitivity to $\sigma$ is not $\partial / \partial \sigma$ but rather
\[
\left( \frac{\partial}{\partial \sigma} \right)_F = \frac{\partial}{\partial \sigma} - \frac{\partial F / \partial \sigma}{\partial F^* / \partial \sigma} \frac{\partial F^*}{\partial \sigma}.
\]
It is clear that for a call option, one has $(\partial C / \partial \sigma)_F > 0$ (to see this, note $\partial C / \partial \sigma > 0$, $\partial C / \partial F^* > 0$, $\partial F / \partial \sigma < 0$, $\partial F / \partial F^* > 0$). The same simple argument cannot be used for the put, but we can just use put-call parity instead. Finding implied volatility is therefore straightforward.
Variation of volatility with maturity is another effect to consider, and there is a natural framework for so doing, which we take next.

2.2 Convenience yield dynamics

This subject has received some attention over the years and an excellent account is given in [2], into which our work integrates completely. A noticeable feature of commodity options is that the longer-dated futures are typically less volatile than the shorter-dated ones, and this suggests that the price is mean-reverting under $Q$. With a nod to interest rate theory and in particular the Hull–White model [6, §17.11], it is obvious to consider the following general linear model, combining lognormal intrinsic asset prices with convenience yield dynamics and, optionally, interest rates (for the present purposes we neglect the last of these effects). Option prices can then be calculated in closed form using the methods of §1.

By GL2, we mean a model in which the logarithm of the intrinsic asset price $x_t = \ln A_t$ and the convenience yield $y_t$ follow a bivariate Gaussian process of the form

$$
\frac{dx_t}{dy_t} = \Lambda \begin{bmatrix} x_t \\ y_t \end{bmatrix} dt + M(t) dt + \begin{bmatrix} \sigma_A dW^A_t \\ \sigma_y dW^y_t \end{bmatrix}
$$

(9)

where the matrix $\Lambda$ is constant, and the drift term $M(t)$ and the volatilities $\sigma$ are not allowed to depend on $x$ or $y$ but may be time-dependent. By (1),

$$
dx_t = (r - y_t - \frac{\sigma_A^2}{2}) dt + \sigma_A dW^A_t.
$$

(10)

The convenience yield follows essentially the Hull–White model, as also suggested by [5], see also [4, Eq.(3.15)], with an important difference in that it can be coupled to the asset price dynamics via a parameter $\beta \geq 0$ thus:

$$
dy_t = \kappa(\alpha(t) + \beta x_t - y_t) dt + \sigma_y dW^y_t.
$$

(11)

When $A_t$ is low, $y_t$ is more likely to be low/negative (contango) and when $A_t$ is high, $y_t$ is more likely to be positive (backwardation). Thereby

$$
\Lambda = \begin{bmatrix} 0 & -1 \\ -\kappa & -\kappa \end{bmatrix}, \quad M(t) = \begin{bmatrix} r - \frac{\sigma_A^2}{2} \\ \kappa \alpha(t) \end{bmatrix}.
$$

The effect of the coupling parameter $\beta$ is to introduce an implicit mean reversion into the asset dynamics, because when the intrinsic asset price is low the futures curve is likely to be in contango and when high, in backwardation. The long-term variance is reduced, so that long-dated futures contracts are less volatile than short-dated ones.

The obvious attraction of GL2 is that many quantities associated with it, principally the mean and variance of $A_T$, can be calculated in closed form. Indeed,

$$
\begin{bmatrix} x_T \\ y_T \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t e^{\Lambda(s-t)} \begin{bmatrix} r - \frac{\sigma_A^2}{2} \\ \kappa \alpha(s) \end{bmatrix} ds + \int_0^t e^{\Lambda(s-t)} \begin{bmatrix} \sigma_A dW^A_s \\ \sigma_y dW^y_s \end{bmatrix}
$$

and so the joint distribution of $x_t$ and $y_t$ is bivariate Normal with mean

$$
\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t e^{\Lambda(s-t)} \begin{bmatrix} r - \frac{\sigma_A^2}{2} \\ \kappa \alpha(s) \end{bmatrix} ds
$$
and covariance matrix
\[ \int_0^t e^{\Lambda(t-s)} \begin{bmatrix} \sigma_A^2 & \rho \sigma_A \sigma_y \\ \rho \sigma_A \sigma_y & \sigma_y^2 \end{bmatrix} e^{\Lambda'(t-s)} ds, \]
with \( \rho \) the correlation between \( dW^A_t \) and \( dW^y_t \). These expressions require matrix exponentiation, achieved by the following lemma 4:
\[ \exp\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} t \right) = e^{(a+d)t/2} \left[ \cosh \frac{bt}{2} + \frac{a-d}{\delta} \sinh \frac{bt}{2} \quad \frac{2b}{\delta} \sinh \frac{bt}{2} \cosh \frac{bt}{2} + \frac{d-a}{\delta} \sinh \frac{bt}{2} \right] \]
with \( \delta^2 = (a - d)^2 + 4bc \).

Contrary to what is implied in [2] there is no requirement that the eigenvalues of \( \Lambda \) be real: all that is necessary is that both have real part \( \leq 0 \), which is automatic provided that \( \beta, \kappa \geq 0 \). In fact, when \( \delta^2 < 0 \), equivalent to \( \beta > \kappa/4 \), we will have oscillatory behaviour with period \( 2\pi i/\delta \), as is observed in autoregressive processes with complex-conjugate poles. It is worth noting that we could delete \( \beta \) and make the mean reversion explicit, by adding a term \( -\kappa x x_t dt \) (where \( \kappa \geq 0 \) is another parameter) into the drift term of (10), replacing 0 with \( -\kappa x \) in the top left-hand element of \( \Lambda \). This has been suggested in e.g. [4, Eq.(3.8)], but has a fundamentally different effect in that the eigenvalues of \( \lambda \) must be real, so no cyclical behaviour can be generated.

The addition of stochastic interest rates via the Hull–White model gives us the GL3 model, and this presents no further difficulties in analysis provided the interest rate dynamics are not in any way driven by \( A \) or \( y \).

In this modelling framework we would fit all futures expiries (and hence their options) at once, as it is a term structure model. This makes the problem higher-dimensional but also imposes rigidity on the structure in the sense that the calibration parameters should not vary too strongly from one maturity to the next. Two restrictions are:

(i) One can no longer choose a different volatility for each maturity. The variation of volatility with \( T \) is determined by \( \sigma_A \) in (10) and the parameters \( \kappa, \beta \).

(ii) As seen in (7), \( F^*_t(T_i)/F^*_t(T_{i+1}) \) gives an estimate of the convenience yield and so there cannot be too wild a variation in adjacent values of \( F^* \).

Work on this area is continuing. Regarding (ii), it is likely that this year’s events will show a wide excursion of the convenience yield from its equilibrium level, adding to the catalogue of real-world examples in which the Ornstein–Uhlenbeck process fails to capture large deviations5.

2.3 Lévy models

As mentioned earlier, these are the ideal tool for dealing with short-dated OTM options (for an introduction see e.g. [9]). We make the elementary point that the work in (11) requires the evaluation of two types of expression: expectations of powers of \( A_T \), and of indicator functions \( 1(A_T > K) \). For nearly all Lévy processes this is straightforward because \( \mathbf{E}_t[A_T^K] \) is given directly by the moment generating function, while \( \mathbf{E}_t[A_T > K] \) is sometimes known in closed form and otherwise has to be done using inverse Fourier integrals. Either way, the implementation of what has been described here does not require new machinery.

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4 Proved by diagonalising the matrix.
5 A discussion of this and possible extensions to the OU model is given in [8].
2.4 Merton model

It might seem strange to drag in structural credit modelling at this point, but there is an obvious parallel. In the Merton framework the equity of the firm is modelled as a call option on its assets and the debt as a riskfree bond minus a put option. The strike relates to the face value of the firm’s debt. In the first instance let the firm value $A_t$ follow a geometric Brownian motion. The SDE followed by the equity $E_t$ is, by Itô’s lemma,

$$dE_t = rE_t dt + \Delta \sigma A_t dW_t$$

$$= rE_t dt + \sigma E_t \left(\frac{E_t}{A_t}\right) dW_t$$

where $\Delta$ is the call option’s delta, and we find that $E_t$ has acquired local volatility given by

$$\sigma_E = \sigma \Delta A_t / E_t.$$ 

Now $E_t$ is a convex function of $A_t$, and so $\Delta > E_t/A_t$, and we conclude that $\sigma_E > \sigma$, the effect being less pronounced when the call option is in-the-money. Thus even if the firm value follows a geometric Brownian motion (which, from the perspective of calibrating to term structure of credit spread, is not a workable model) the equity price still acquires a smile by virtue of the embedded option. A fuller discussion is in [7]. So, in modelling equities one is likely to benefit from modelling the assets and liabilities as separate positive processes rather than trying to model the equity directly. Were the equity not of a limited-liability firm, it could go negative, and then such an approach would be essential.

2.5 Local volatility

The connection with local volatility [3] is that rather than attempting to infer the local volatility surface from traded options, or imposing an arbitrary parametric form upon it, we may do better to instead postulate the existence of certain embedded optionalties, and identify those. The advantage of so doing is that there is a clear financial intuition, and the pricing of options on the asset in question is likely to be straightforward using Black-type formulae, as has been done here. To return to the numerical example here, it is easy to see that to fix up the Black model using local volatility would require an exorbitant level of volatility (even before worrying about the problems of negative price). On the other hand, incorporating an option into the traded asset seems to remove most of the difficulties.

2.6 Other commodity markets

It is not necessarily true that all optionalties have a negative impact on the price. Upward spikes in commodity prices can stem from the opposite kind of difficulty discussed here: low inventories, causing difficulty obtaining the asset. An obvious prescription in the light of this paper is an optionality of the form

$$\psi_+(A) = \ell A \left(\frac{A}{\hat{A}}\right)^\lambda - 1 \right)^+$$

where again $\ell, \lambda \geq 0$. This would in principle explain why in commodity markets implied volatilities often increase for high strikes, sometimes known as the ‘inverse leverage effect’.
3 Conclusions

We have presented an extension of the Black model that incorporates an intrinsic optionality into the commodity price. It avoids the need to switch to the Bachelier model, as some market participants have been forced to do, doubtless causing interpretative difficulties, expense, and loss of historical context. The new model captures a liability caused by failure of the physical delivery process, and can cause negative futures prices. If the intrinsic asset price is well above the price at which this option kicks in then the deformation to the original Black model is negligible and so the years of accumulated experience in handling Black models is preserved. The events of this April are, however, captured with precision and insight.

The opinions expressed in this paper are those of their authors rather than their institutions. Email richard.martin1@imperial.ac.uk, aldous.birchall@trafigura.com

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