Permutation Generation: Two New Permutation Algorithms

JIE GAO and DIANJUN WANG
Tsinghua University, Beijing, China

Abstract. Two completely new algorithms for generating permutations, shift-cursor algorithm and level algorithm, and their efficient implementations are presented in this paper. One implementation of the shift cursor algorithm gives an optimal solution of the permutation generation problem, and one implementation of the level algorithm can be used to generate random permutations.

Key Words and Phrases: permutation, shift cursor algorithm, level algorithm.

1. Introduction

Permutation generation has a history which dates back to the 1650s. After the electronic computer was invented dozens of algorithms have been published which generates all the permutations of \( S_n \) by a computer. In 1977 Robert Sedgewick [3] wrote a survey of this problem and compared nearly all permutation generation methods at that time. As a basic problem of computer science, it is still an interesting problem to be studied.

Sedgewick [3] concluded in his survey that his implementation of Heap’s method was the fastest permutation generation algorithm for most computers, and it can be coded that most permutations are generated using only two store instructions. This is a lower bound of permutation algorithms. Of course it cannot be exceeded but algorithms which run as fast as it may be written.

Two completely new algorithms for generating permutations, shift-cursor algorithm and level algorithm, are presented in this paper and we give some implementations of them. Like Sedgewick’s implementation of Heap’s method, one of the implementations of our first algorithm comes infinitely close to the theoretically optimal solution of the problem, that is, \( 2n! \) store instructions. Also one implementation of the other algorithm can be used to generate random permutations.

2. The Algorithms

Our two algorithms are strongly related. Specifically, each number in a "level" permutation gives the level of the shift cursor in the corresponding permutation.

First, we discuss the shift cursor algorithm which generates the permutations in \( S_n \) beginning with the identity permutation 123...n. The algorithm uses the first number 1 in 123...n to partition the n! permutation in n blocks of \( (n-1)! \) permutations as follows. The algorithm first generates \( (n-1)! \) permutations with cursor 1 in 1st position, the next \( (n-1)! \) permutations with cursor 1 in the 2nd position, and so on, until the cursor 1 is in the \( n^{th} \) position. In other words, the cursor partitions the n! permutations into n blocks, and in each block the cursor is in a fixed position. We define the level of the cursor to be the number of blocks determined by the cursor. Thus the level of cursor 1 is n.
The algorithm, while generating the permutations with the cursor 1 in a fixed position, uses a new cursor to partition each block of \((n-1)\)! permutations into \(n-1\) smaller blocks and in each smaller block the new cursor is in a fixed position. To find the cursor of a smaller block, we first ignore the cursors of the larger blocks from the permutations, and then use the first number of the first permutation in the block as the cursor of the smaller block. For example, all the level \(n-1\) cursors are the first numbers other than cursor 1 in the first permutation of each smaller block. This method can be used until the new blocks contain only one permutation and then all \(n!\) permutations are generated.

Consider, for example, the case \(n = 4\) whose permutation sequence appears in Figure 1. The sequence begins with the permutation 1234. The algorithm first uses 1 as the cursor of all 4! permutations, and we get the following 4 blocks (where * stands for an unknown number):

\[
\begin{array}{ccccc}
1*** & 1*** & 1*** & 1*** \\
1*** & 1*** & 1*** & 1*** \\
1*** & 1*** & 1*** & 1*** \\
1*** & 1*** & 1*** & 1*** \\
\end{array}
\]

A new cursor is then used in each block to partition the 3! = 6 permutations into 3 smaller blocks. For example, in the first larger block the first permutation is 1234, so we use 2 as the cursor of the first block to get the following 3 smaller blocks:

\[
\begin{array}{cc}
12** & 12** \\
1*2* & 1*2* \\
1**2 & 1**2 \\
\end{array}
\]

In the first block above, where the first permutation is 1234, the first number, other than 1 and 2, is 3, so we use 3 as the cursor of the block. Then we get 2 smaller blocks: 123* and 12*3. Both of the blocks contains only one permutation and we put the last number 4 in the permutations to get 1234 and 1243. We continue in this way to obtain the other permutation in the first block ending with the permutation 1432.

In the second block, the first permutation will be 4132. This comes from the previous permutation 1432 (the last permutation in the first block) by interchanging the cursor 1 with the next number in the permutation. Since 4132 is the first permutation in the second block, we use the first number 4 as the cursor of the second block to obtain the following 3 smaller blocks:

\[
\begin{array}{cc}
41** & 41** \\
*1** & *1** \\
*1** & *1** \\
\end{array}
\]

Continuing as above, we get all the permutations in the second block. We then get
all $n! = 4! = 24$ permutations which appear in Figure 1.

Observe that, in each permutation, each number in the permutation is a cursor of a certain block, and in a given permutation the levels of the cursors are different from each other. Thus the cursor levels form a permutation. Our level sequence is obtained from the sequence of cursor level permutations by interchanging 1 and $n$, 2 and $n - 1$, and so on.

Figure 1 shows, for $n = 4$, the shift cursor sequence, the levels of the cursors, and the level sequence. We also include the well-known Johnson-Trotter sequence [2] [4] for reference. ($n = 4$). See Figure 1.

| shift cursor sequence | levels of the cursors | level sequence | Johnson-Trotter sequence |
|-----------------------|-----------------------|----------------|--------------------------|
| 1234                  | 4321                  | 1234           | 1234                     |
| 1243                  | 4312                  | 1243           | 1243                     |
| 1423                  | 4231                  | 1324           | 1423                     |
| 1324                  | 4132                  | 1423           | 4123                     |
| 1342                  | 4213                  | 1342           | 4132                     |
| 1432                  | 4123                  | 1432           | 4312                     |
| 4132                  | 3421                  | 2134           | 1342                     |
| 4123                  | 3412                  | 2143           | 1324                     |
| 2143                  | 2431                  | 3124           | 3124                     |
| 3142                  | 1432                  | 4123           | 3142                     |
| 3124                  | 2413                  | 3142           | 3412                     |
| 2134                  | 1423                  | 4132           | 4312                     |
| 2314                  | 3241                  | 2314           | 4321                     |
| 2413                  | 3142                  | 2413           | 3421                     |
| 4213                  | 2341                  | 3214           | 3241                     |
| 3214                  | 1342                  | 4213           | 3214                     |
| 3412                  | 2143                  | 3412           | 2314                     |
| 4312                  | 1243                  | 4312           | 2341                     |
| 4321                  | 3214                  | 2341           | 2431                     |
| 4231                  | 3124                  | 2431           | 4231                     |
| 2431                  | 2314                  | 3241           | 4213                     |
| 3421                  | 1324                  | 4231           | 2413                     |
| 3241                  | 2134                  | 3421           | 2143                     |
| 2341                  | 1234                  | 4321           | 2134                     |
3. Implementation of the Level Algorithm - Also can be Used to Generate Random Permutations

If \( N \) is so large that we could never hope to generate all permutations of \( N \) elements, it is of interest to study methods for generating "random" permutations of \( N \) elements. The following algorithm implements level algorithm by using a number between 1 to \( N! \) to generate a permutation of level algorithm. Since we can generate a "random" number between 1 to \( N! \) first and use the following algorithm to generate a permutation of level algorithm, the following algorithm can also be used to generate "random" permutations.

Suppose the function divide(int \( a \), int \( b \)) executes \( a/b \) and put the quotient to \( b \), and the remainder to \( a \) and \( 0! = 1 \). We can write implementation of level sequence using the following algorithm:

\[
\text{for } i=1 \text{ to } n! \text{ step 1 do} \\
\text{begin} \\
\text{for } j=1 \text{ to } n \text{ step 1 do} \\
\text{begin} \\
\text{int } m=(n-j)! \\
\text{divide}(i, m) \\
\text{if } i \neq 0 \text{ then} \\
\text{put } j \text{ to the } (m+1)\text{th empty position} \\
\text{else} \\
\text{put } j \text{ to the } m\text{th empty position} \\
\text{end} \\
\text{output one permutation} \\
\text{end} \\
\]

For example, suppose \( n = 4 \), to generate the 15\textsuperscript{th} permutation we first calculate \( 15/3! = 15/6 = 2 \cdots 3 \), the quotient is 2 the remainder is 3, then we put number 1 in the 3\textsuperscript{rd} position, we get **1*. Secondly we calculate \( 3/2! = 3/2 = 1 \cdots 1 \), the quotient is 1 the remainder is 1, then we put number 2 in the 2\textsuperscript{nd} empty position, we get *21*. Third we calculate \( 1/1! = 1 \cdots 0 \), the quotient is 1 the remainder is 0, then the number 3 is in the 1\textsuperscript{st} empty position, we get 321*. Finally we calculate \( 0/0! = 0/1 = 0 \cdots 0 \), the quotient is 0 the remainder is 0, then the number 4 is in the 1\textsuperscript{st} empty position, we get the permutation 3214 at last.

4. Implementation of Shift Cursor Algorithm - an Optimal Solution of Permutation Generation Problem

It is can be noticed that shift cursor sequence can generate next permutation by transposition. And the sequence has the following properties:

**Property 1.** A level \( j \) cursor remains the same position in continuous \((j - 1)!\) permutations.

**Property 2.** If ignore the cursors which level > \( j \) from a sequence of \( n! \) permutations then the sequence turns to \( n!/j! \) blocks with length \( j \), and the rule of change of
each blocks is the same.

We call Property 1 as the local stability of the sequence, and call Property 2 as the rule stability of the sequence.

Property 1 ensures that when generating permutations using shift cursor algorithm, cursors with higher levels can be placed in fixed positions for a certain part of the sequence.

Property 2 ensures that for the certain part of the sequence, if higher level cursors have already been placed in their positions, a predetermined rule can be used to place other cursors in their right positions.

Clearly [1], there are two types of operations needed for generating a new permutation from the previous permutation:

1. Operations needed for actually executing changes from the previous permutation.
2. Operations needed for reaching decisions about which type 1 operations are required.

In theory, any permutation generating algorithm should contain the two types of operations and only the two types of operations, and a theoretically optimal algorithm for the problem should has the following factors:

1. All type 1 operations should be change only two numbers of the permutation.
2. The number of type 2 operations should be limited to reach zero, or much smaller than type 1 operations.

By using the two properties of shift cursor sequence, algorithms which have the above two factors can be written.

Suppose we have already written an algorithm for generating permutations of shift cursor algorithm. For a certain part of the sequence, such as a sequence of continuous \( j! \) permutations, we can place the cursors which level \( > j \) in their places first, and these cursors remain the fixed positions in the \( j! \) permutations (property 1). Then by using a given rule, we can place other cursors directly (property 2), without any type 2 operations.

For example, suppose we are using the algorithm to generate \( 5! \) permutations \((n = 5)\), and suppose \( j = 3 \). Starting from 12345, the level of number 1 is 5, the level of number 2 is 4. Then we fix number 1 in the 1st position, fix number 2 in the 2nd position. Afterwards, by using a given rule, we can place the numbers 3, 4, 5 directly in their positions to generate permutations. The rule to place the three numbers is the same as the rule to generate \( 3! \) permutations \((n = 3)\) using shift cursor algorithm.

When we using the algorithm to generate \( 3! \) permutations \((n = 3)\), the sequence is as the following: 123, 132, 312, 213, 231, 321. In this case we name 3 the No.1 element, name 4 the No.2 element, name 5 the No.3 element. In the above sequence, No.2 element and No.3 element be changed firstly, so we change 4, 5 first, then the next permutation to 12345 is 12354. And No.1 element and No.3 element changed afterwards, so we change 3, 5 secondly, then the next permutation to 12354 is 12534, and so on. And finally we
get the 6th permutation 12543. Some type 2 operations is needed now to determine which cursor’s level > j and where should we place these cursors. After that, we can generate next 3!−1 = 5 permutations by directly change two numbers. This algorithm can be described as following:

Algorithm 1:
1. Starting from 1, 2, 3, ... , n.
2. If the level order of the cursors is 1, 2, 3, ... , n, stop.
3. Determine cursors which level>j and their positions.
4. Output.
5. Generate and output next j!−1 permutations directly.
6. Go to step 2.

When generating permutations using algorithm 1, there are \((j!−1)/j!\) permutations are generated by directly change two numbers without any type 2 operations. And other \(1/j!\) permutations are generated using additional type 2 operations.

Suppose when using algorithm 1, the average execution time of type 1 operation is \(t\) and the average execution time of type 2 operation is \(T\), then the overall execution time of algorithm 1 (\(T1\)) is \(n!*t+n!/j!*T\). Since every type 1 operation of algorithm 1 is change two numbers, the overall execution time of theoretically optimal algorithm of permutation generation (\(T0\)) is \(n!*t\), and \(T1/T0= 1 + T/(j!*t) > 1\). So with \(j\) grows larger \(T1/T0\) could infinitely close to 1, which means algorithm 1 could infinitely close to the theoretically optimal solution of the problem.

Robert Sedgewick’s implementation of Heap’s method is very similar to algorithm 1, the only difference between them is the termination condition (step 2) of the algorithms, and they both have the property 1 and property 2. And it is the two properties make the two algorithms both can infinitely close to the theoretically optimal solution of the problem.

5. Summary and Conclusions

Although there are few applications need to generate all \(n!\) permutations, permutation generation is still an interesting problem to be studied. Further studies are needed on this problem, and these studies may change our understanding of permutations in the future.

Comparing with permutation generation, random permutation generation has more practical meanings. The implementation of level algorithm gives a new method to generate random permutation. And it can be used to generate the permutation which in a given position of the level sequence.

Because of the two important properties, the shift cursor algorithm has an efficient implementation, and in theory both the implementation and Sedgewick’s implementation of Heap’s method could infinitely close to the theoretically optimal solution of the problem, which is \(2n!\) store instructions. Comparing with Heap’s algorithm and another famous algorithm – Johnson-Trotter algorithm, the shift cursor algorithm is easier to understand than Heap’s algorithm and has a more efficient implementation.
than Johnson-Trotter algorithm. And it’s more suitable to be written in textbooks.

**Acknowledgement.** Appreciation must be expressed to Professor Seymour Lipshutz for his valuable comments and suggestions for improving the readability of the manuscript.

References

[1] Gideon Ehrlich. Loopless Algorithms for Generating Permutations, Combinations, and Other Combinatorial Configurations. *Journal of the Association for Computing Machinery*, Vol. 20, No. 3, July 1973, 500-513.

[2] Johnson, S.M. Generation of permutations by adjacent transpositions. *Math. Comput.* 17, 1963, 282-285.

[3] Robert Sedgewick. Permutation Generation Methods. *Computing Surveys*, Vol 9, No 2, June 1977, 137-164.

[4] Trotter, H. F. Algorithm 115, *Perm Communications of the ACM* 5, 8, Aug. 1962, 434-435.

Jie Gao (gj02@mails.tsinghua.edu.cn) is a MSE student of School of Software, Tsinghua University, Beijing, China.

Dianjun Wang (djwang@math.tsinghua.edu.cn) is an associate professor of Department of Mathematical Sciences, Tsinghua University, Beijing, China.