Abstract

The theory of fractional calculus in the complex plane was not built with a specific application in mind. The main obstacle to application was the difficulty with obtaining analytic continuations of fractional derivatives and integrals. It is known that if a function is analytic in a simply connected domain, then its fractional derivatives and integrals are also analytic in this region. In multi-connected domains, in general, this property does not hold. However, for the set of hypergeometric functions $\ _pF_\ _q$, $p \geq 2$, this property is preserved. We propose a new version of fractional calculus, which allows us to reveal the reason for this preservation, and introduce broad classes of functions for which an appropriate theory can be constructed. This allows us to find a meaningful application of our calculus in the asymptotic theory of linear ODE’s with analytic coefficients.
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1 Contents of Part I and Part II

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I. Fractional Calculus in \(\mathbb{C}\).
II. Linear ODE’s with analytic coefficients.

Part 2
III. Borel-type summation.
IV. Error bounds and Stokes’ phenomenon.
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2 Fractional Calculus in \(\mathbb{C}\)

2.1 Introduction

Fractional Calculus has existed for more than 300 years. The encyclopedic
monograph [19] provides an insight into the progress made in this area up to
the early 1990s. In recent years the topic has been rediscovered by scientists
and engineers in an increasing number of fields in which complex analysis is
essential, see, for example, [20–22]. It has been shown, see [28], [29], that
if a function is analytic in a simply connected domain then its fractional
derivatives and integrals are also analytic in this region. However, in the
case of finitely connected domains this property is, in general, not preserved.
We have constructed a version of fractional calculus in the complex plane
that enables us to understand more clearly the reason for the obstacle to
analytical continuation and to overcome the above obstacle for broad classes
of analytic functions suggested by linear ODE’s with analytic coefficients.
We apply then our version of fractional calculus to the asymptotic analysis
of solutions of these ODE’s. In the current paper we extend the results
of [4], which are valid for ODE’s with coefficients given by even analytic
functions, to the general case of the second order linear ODE’s with analytic coefficients.

In the next section, we introduce the two functional linear spaces in which our version of fractional calculus is defined.

2.2 Laplace-Borel dual spaces of analytic functions

Let $H$ be the linear space that consists of all functions $F(t)$ possessing the following properties:

- (i) For some $a > 0$, $F(t)$ is analytic in the region
  \[ \mathcal{D}(a) = \{ t \in \mathbb{C} : \text{dist}\{t,(0, +\infty)\} < a \} \]  
  with boundary
  \[ \gamma(a) = \{ t \in \mathbb{C} : \text{dist}\{t,(0, +\infty)\} = a \}. \]

- (ii) For some $R \geq 0$, $F(t)$ has an exponential growth of type $R$ at infinity of the region $\mathcal{D}(a)$.

Given $F(t) \in H$ set
\[ a(F) = \sup\{a : (i) \text{ is satisfied} \}. \]  
Likewise, we set
\[ R(F) = \inf\{R : (ii) \text{ is satisfied} \}. \]

We denote by $\mathcal{L}$ the Laplace transform operator of the form
\[ P(\zeta) = \mathcal{L}\{F(t)\} = \zeta \int_0^{+\infty} e^{-\zeta t} F(t) \, dt, \]  
where the integral is absolutely convergent for any $\zeta$, $\Re\{\zeta\} > R \geq 0$.

Remark 1. In paper [13], where our version of fractional calculus was first proposed, we denote by $\mathcal{L}$ the standard Laplace transform operator, see equations (44) and (46) with $\alpha = 0$. We note that keeping in mind the application to differential equations with analytic coefficients, it is a little more convenient to use instead of the standard definition. Note that all the definitions and the results obtained here can be easily rewritten in terms of the standard Laplace transform, and vice versa, using the change $P \to \zeta P$.

We set
\[ S = \mathcal{L}H = \{ P : P(\zeta) = \mathcal{L}\{F(t)\} \}, \]  
where $\mathcal{L}$ is given by \[5\], $F(t) \in H$, and the integral is absolutely convergent for any $\zeta$, $\Re\{\zeta\} > R(F)$.

In order to simplify our development, and to provide the space $H$ with a countable normalized topology, we introduce subspaces $H(a, R) \subset H$ and $S(a, R) \subset S$.  

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• (i) If, in the definition of $H$, we fix the values of $a$ and $R$ we obtain the definition of a subspace of $H$, which we denote by $H(a, R)$.

• (ii) Now let $S(a, R) = \mathcal{L}H(a, R)$—a subspace of $S$.

Thus $S(a, R)$ and $H(a, R)$ are Laplace-Borel dual linear spaces and we have

$$H = \cup_{a>0,R\geq 0}H(a, R), S = \cup_{a>0,R\geq 0}S(a, R).$$

It follows immediately that $\mathcal{L} : H \to S$ is a bijection.

Later we will present an intrinsic definition of $S$ and $S(a, R)$ in terms of properties of their elements.

Since $F(t) \in H(a, R)$ is analytic in the circle $|t| < a$, its Taylor series expansion is absolutely convergent inside this circle and we have

$$F(t) = \sum_{k=0}^{\infty} f_k t^k, f_k = \frac{F^{(k)}(0)}{k!}.$$  \hspace{1cm} (6)

Suppose that the function $P(\zeta) = \mathcal{L}\{F(t)\} \in S(a, R)$. Then $P(\zeta)$ is analytic and bounded in any half-plane $\Re\{\zeta\} > r > R = R(F) \geq 0$. Furthermore, setting

$$p_k = f_k k!, k = 0, 1, \ldots,$$  \hspace{1cm} (7)

where $f_k$ is given by (6), we may approximate $P(\zeta)$ by the partial sums $\sum_{k=0}^{n-1} p_k/\zeta^k$ of the power series $\sum_{k=0}^{\infty} p_k/\zeta^k$. We introduce the remainder $P_n(\zeta)$ as

$$P_n(\zeta) = P(\zeta) - \sum_{k=0}^{n-1} \frac{p_k}{\zeta^k}, n = 0, 1, \ldots,$$  \hspace{1cm} (8)

where $P_0(\zeta) = P(\zeta)$. Clearly, $P_n(\zeta) \in S(a, R)$ and it follows from Watson’s lemma that for $\Re\{\zeta\} > R$ the following asymptotic expansion is valid

$$P_n(\zeta) \sim \sum_{k=n}^{\infty} \frac{p_k}{\zeta^k}, \zeta \to \infty,$$  \hspace{1cm} (9)

which means that for $n = 0, 1, \ldots$, we have

$$P_n(\zeta) = O\left(\frac{1}{|\zeta|^n}\right), \zeta \to \infty.$$  \hspace{1cm} (10)

In what follows we will work also with a function $F(t, \alpha) \in H(a, R)$, generated by $F(t)$, which depends on a parameter $\alpha$ such that $F(t, 0) = F(t)$. Then all the formulas (6)–(10) are retained for $F(t, \alpha)$ with the corresponding changes replacing $f_k, P(\zeta), p_k, P_n(\zeta)$ by $f_k(\alpha), P(\zeta, \alpha), p_k(\alpha), P_n(\zeta, \alpha)$, respectively. Our main aim is to find a method of summation of the asymptotic series given by (9) with $p_k = p_k(\alpha)$ that would enable us to represent
the results of summation of the series \( \sum_{k=n}^{\infty} p_k(\alpha) / \zeta^k \) in the form of an integral transformation of \( F(t, \alpha) \). The required function \( F(t, \alpha) \) and the kernel of this transformation have been presented, with the appropriate change of notation, in Section 4 and Section 5 of [13], see equations (55), (100) and (105) where \( s = \alpha + n \). Later we show how to apply this technique to the asymptotic analysis of solutions of linear ODE’s with analytic coefficients.

It can be easily proved that functions \( P(\zeta) \in S \) are uniquely determined by the coefficients \( p_k \) of their asymptotic expansions. In other words, we claim the following statement.

**Theorem 1.** (Uniqueness Theorem) If \( P_1(\zeta), P_2(\zeta) \in S \) and \( p_{1,k} = p_{2,k} \) then \( P_1(\zeta) \equiv P_2(\zeta) \).

Proof. Indeed, suppose \( P_1(\zeta) = \mathcal{L}\{F_1(t)\} \) and \( P_2(\zeta) = \mathcal{L}\{F_2(t)\} \). Setting \( a = \min(a(F_1), a(F_2)) \) and \( R = \max(R(F_1), R(F_2)) \), it follows that \( F_1(t) \) and \( F_2(t) \) belong to \( H(a,R) \). Thus using (7), \( p_{1,k} = p_{2,k} \Rightarrow f_{1,k} = f_{2,k} \Rightarrow F_1(t) \equiv F_2(t) \Rightarrow P_1(\zeta) \equiv P_2(\zeta) \).

In the next section we introduce operators of fractional differentiation and integration in the complex plane in the spaces \( H(a,R) \) and \( S(a,R) \) and explain why we needed new definitions that are different from the classical definitions.

### 2.3 Fractional derivatives and integrals in \( \mathbb{C} \)

For complex \( \alpha, \Re\{\alpha\} > -1 \), we introduce a family of operators \( \mathcal{L}_\alpha \) acting on \( H \):

\[
\mathcal{L}_\alpha \{F(t)\} = \zeta^\alpha \mathcal{L}\{t^\alpha F(t)\}, \tag{11}
\]

For \( \alpha = 0 \) we have \( \mathcal{L}_0 = \mathcal{L} \). The pair of operators \( \mathcal{L} \) and \( \mathcal{L}_\alpha \) generates four linear operators:

\[
\mathcal{D}_\alpha = \mathcal{L}^{-1} \mathcal{L}_\alpha, \quad \mathcal{I}_\alpha = \mathcal{L}_{-1} \mathcal{L}_\alpha, \tag{12}
\]

\[
\hat{\mathcal{D}}_\alpha = \mathcal{L}_\alpha \mathcal{L}^{-1}, \quad \hat{\mathcal{I}}_\alpha = \mathcal{L} \mathcal{L}_\alpha^{-1}. \tag{13}
\]

**Definition 1.** The operators \( \mathcal{D}_\alpha \) and \( \mathcal{I}_\alpha \) are said to be operators of fractional differentiation and integration of order \( \alpha \) acting on the space \( H \). The second pair of operators \( \hat{\mathcal{D}}_\alpha \) and \( \hat{\mathcal{I}}_\alpha \) are the dual operators of fractional differentiation and integration acting on the space \( S \).

**Example 1.** For any \( a > 0 \) and a non-negative integer \( k \), \( F(t) = t^k \in H(a,0) \). Then for \( \Re\{\alpha\} > -1 \)

\[
\mathcal{L}_\alpha \{t^k\} = \zeta^{1+\alpha} \int_0^{+\infty} e^{-\zeta t^\alpha+k} dt = \frac{\Gamma(\alpha+k+1)}{\zeta^k},
\]

so that

\[
\mathcal{D}_\alpha \{t^k\} = \frac{\Gamma(\alpha+k+1)}{k!} t^k. \tag{14}
\]
Since $I_\alpha = D_\alpha^{-1}$,
\begin{equation}
I_\alpha \left\{ t^k \right\} = \frac{k!}{\Gamma(\alpha + k + 1)} t^k.
\end{equation}

Let us introduce the following notation
\begin{equation}
F(t, \alpha) = D_\alpha \{ F(t) \}, \quad F^*(t, \alpha) = I_\alpha \{ F(t) \}.
\end{equation}

\begin{equation}
P(\zeta, \alpha) = \hat{D}_\alpha \{ P(\zeta) \}, \quad P^*(\zeta, \alpha) = \hat{I}_\alpha \{ P(\zeta) \}.
\end{equation}

It follows, using (14) and (15), that Taylor series of $F(t, \alpha)$ and $F^*(t, \alpha)$ can be written as
\begin{equation}
F(t, \alpha) = \sum_{k=0}^{\infty} f_k(\alpha) t^k \quad \text{and} \quad F^*(t, \alpha) = \sum_{k=0}^{\infty} f^*_k(\alpha) t^k,
\end{equation}
where
\begin{equation}
f_k(\alpha) = \frac{f_k \Gamma(\alpha + k + 1)}{k!} \quad \text{and} \quad f^*_k(\alpha) = \frac{f_k k!}{\Gamma(\alpha + k + 1)},
\end{equation}
respectively. Clearly, the expansions in (18) are absolutely convergent in the circle $|t| < a$.

Using integral representations for $F(t, \alpha)$ and $F^*(t, \alpha)$ in terms of $F(t) \in H(a, R)$, which will be demonstrated later, we can claim the following results, valid for all $a > 0$, $R \geq 0$ and every $\alpha$, $\Re\{\alpha\} > -1$:

**Theorem 2.**
- (i) $F(t, \alpha), F^*(t, \alpha) \in H(a, R)$,
- (ii) $D_\alpha H(a, R) = H(a, R), I_\alpha H(a, R) = H(a, R), \quad D_\alpha S(a, R) = S(a, R), \hat{I}_\alpha S(a, R) = S(a, R), \quad D_\alpha H = H, \hat{I}_\alpha H = H, \quad D_\alpha S = S, \hat{I}_\alpha S = S,$

and every operator in (20) – (23) is bijective.

**Remark 2.**
The function $F(t, \alpha) = D_\alpha \{ F(t) \}$ was first introduced in Section 4, Eq.(55) of [13]. The main result of that Section is the derivation of the integral representation for $F(t, \alpha)$ in terms of $F(t)$. Further development of the technique from [13] enables us derive the integral representation for $F^*(t, \alpha)$ in terms of $F(t)$, to prove the validity of (20), and then to study the problem of the analytic continuations of $D_\alpha \{ F(t) \}$ and $I_\alpha \{ F(t) \}$ in the $t$ and $\alpha$-planes.
As far as we know, our definitions of fractional derivatives and integrals given by (12) and (13) are new and can be extended to the cases when the Laplace operators are replaced by the Laplace-Stieltjes operators. This version is different from the classical one. Of the five common standard requirements for different variations of the classical fractional calculus given in [9], our version retains only three. On the other hand, there is a link between our definitions and the classical Liouville-Riemann definitions. There is also a similarity between our version and the theory of Fractional Differ-Integrals by Grunwald-Letnikov.

We note the following fact, which gives some support to the legitimacy of our version of fractional calculus. For the particular case when $F(t) \in H$ and $F(t)/t$ is absolutely integrable on the contour given by (2) the formulas giving the integral representations for $D_\alpha \{F(t)\}$ and $I_\alpha \{F(t)\}$ are much simpler than for the general case. They are given by (12) and (13) in Section 2.5. Similar formulas for a version of fractional derivatives and integrals have appeared earlier in the monograph [19], in the section Fractional calculus in the complex plane, the only difference being that our path of integration given by (2) replaces a circle centered at the origin of radius $a$ used there. We note that our definitions were at that time unknown. We add, in conclusion, that the construction of this version of fractional calculus is not an end in itself for us. This theory proved to be necessary for the implementation of our approach to the asymptotic analysis of linear ODE’s with analytic coefficients.

Let $F(t) \in H(a, R)$ and

$$P(\zeta, \alpha) = L_\alpha \{F(t)\} = \zeta^{1+\alpha} \int_0^{+\infty} e^{-\zeta t} t^\alpha F(t) \, dt,$$  \hfill (24)

where $L_\alpha$ is given by (11). Then result (24) yields the following statement: 

**Theorem 3.**

$$P(\zeta, \alpha) = \zeta \int_0^{+\infty} e^{-\zeta t} D_\alpha \{F(t)\} \, dt,$$  \hfill (25)

and for $\Re\{\zeta\} > R$ the integral (25) is absolutely convergent.

**Proof.** Using relations (18) and (19), it can be verified that the power series expansions in $1/\zeta$ of both the integrals in (24) and (25) coincide. The result (24) then follows from Theorem 1.

**Remark 3.** We note that the Laplace-Mellin transforms of the form (24) have been known for a long time. However, the fact that there is an alternative representation in the form of the Laplace transform (25) was not known except for the cases when $F(t)$ is the hypergeometric function. See, for example, the integral representations of confluent hypergeometric functions.
in [3], [2] and [4] in the form of the Laplace transform and Laplace-Mellin transform of hypergeometric functions.

In the future, for applications to the asymptotic theory of differential equations, we require the following result which follows from Theorem 1, and gives, in fact, alternative definitions of fractional derivatives and integrals in the space \( H(a, R) \).

**Theorem 4.** Let \( F(t) \in H(a, R) \). Then

\[
\zeta \int_0^\infty e^{-\zeta t} F(t) \, dt = \zeta^{1-\alpha} \int_0^\infty e^{-\zeta t} D_\alpha \{F(t)\} \, dt, \tag{26}
\]

\[
\zeta \int_0^\infty e^{-\zeta t} \, dt = \zeta^{1+\alpha} \int_0^\infty e^{-\zeta t} I_\alpha \{F(t)\} \, dt, \tag{27}
\]

and all integrals are absolutely convergent for \( \Re \{\zeta\} > R \).

In what follows we illustrate what has been said in this section by applying the above formula to the case when \( F(t) \) is a generalized hypergeometric function. This case forms an important touchstone for our work.

### 2.4 Hypergeometric Functions

Given \( p = 1, 2, \ldots, \) we consider the hypergeometric series of the form

\[
_{(p+1)F_p}(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; -t)) = \sum_{k=0}^\infty (-1)^k \frac{(a_1)_k \cdots (a_{p+1})_k t^k}{(b_1)_k \cdots (b_p)_k k!}, \tag{28}
\]

It is known, see [2], that this series represents a hypergeometric function \( _{(p+1)F_p}(-t) \) which is analytic in the unit circle of the \( t \)-plane and admits an analytical continuation to the \( t \)-plane cut along the interval \((-\infty, -1)\). Moreover, this function admits further analytic continuation to the extended \( t \)-plane except for the points 0, –1, \( \infty \) which are regular singularities. It follows that for \( p = 0, 1, \ldots, \) the function \( _{(p+1)F_p}(-t) \) belongs to \( H(a, R) \), where \( a = 1, R = 0, \) and using our definitions given by (12) and (13) and relations (18) and (19), we have

\[
d_\alpha \{(p+1)F_p\}(-t)) = \Gamma(\alpha + 1) \frac{1}{(p+2)F_{p+1}(a_1, \ldots, a_{p+1}, a + 1; b_1, \ldots, b_p, 1; -t)), \tag{29}
\]

\[
i_\alpha \{(p+1)F_p\}(-t)) = 1/\Gamma(\alpha + 1) \frac{1}{(p+2)F_{p+1}(a_1, \ldots, a_{p+1}, 1; b_1, \ldots, b_p, a + 1; -t)). \tag{30}
\]

Thus, (29) and (30) show that fractional derivatives and integrals of the function \( _{(p+1)F_p}(-t) \) are also analytic in the \( t \)-plane punctured at the points 0, –1, \( \infty \). Now we call attention to the fact that the hypergeometric function \( _{(p+1)F_p}(-t) \) can be obtained as a result of successive differentiations and integrations of fractional orders of the geometric function \((1 + t)^{-1}\). Indeed,
representing the geometric function in the form \((2F_1 (1, 1; 1; -t))\), and using again the relations (18) and (19), we have

\[
(p+1)F_p (-t) \equiv \frac{\Gamma (a_1) \ldots \Gamma (a_{p+1})}{\Gamma (b_1) \ldots \Gamma (b_p)} \mathcal{D}_{a_1-1} \ldots \mathcal{D}_{a_{p+1}-1} \mathcal{I}_{b_1-1} \ldots \mathcal{I}_{b_{p}-1} (2F_1 (1, 1; 1; -t)).
\]

Thus, the function \(F (t) = (2F_1 (1, 1; 1; -t))\) generates all hypergeometric functions of the form given by (28) and is analytic in the \(t\)-plane except for the pole at \(t = -1\). Applying formulas (29) and (30) for the function \(F (t) = (2F_1 (1, 1; 1; -t))\), we have

\[
\mathcal{D}_\alpha \{F (t)\} = \Gamma (\alpha + 1) \binom{2F_1 (\alpha + 1, 1; 1; -t)}{1}, \quad (31)
\]

\[
\mathcal{I}_\alpha \{F (t)\} = 1/\Gamma (\alpha + 1) \binom{2F_1 (1, 1; \alpha + 1; -t)}{1}. \quad (32)
\]

The relation (31) can be rewritten in the form

\[
F (t, \alpha) = \mathcal{D}_\alpha \{F (t)\} = \Gamma (\alpha + 1) (1 + t)^{-\alpha - 1}, \quad (33)
\]

and if \(\alpha\) is not an integer then \(\mathcal{D}_\alpha \{F (t)\}\) clearly has two singular point at \(t = -1\) and \(t = \infty\). The fractional integral \(\mathcal{I}_\alpha \{F (t)\}\) has three singular points at \(t = -1, t = \infty,\) and \(t = 0\). In order to verify the last statement we consider the following monodromic relation for the hypergeometric function in (32) where \(t^* = 1 - (1 + t) e^{2\pi i}\):

\[
2F_1 (1, 1; \alpha + 1; -t^*) - 2F_1 (1, 1; \alpha + 1; -t) = 2\pi i e^{\pi i \alpha} (1 + t)^{-1}. \quad (34)
\]

This relation can be derived from the Euler linear transformation formula. Letting \(t \to 0\) in (31) shows that \(t = 0\) is a singular point of \(\mathcal{I}_\alpha \{F (t)\}\). In fact, such an appearance of new singular points after fractional differentiation or integration is an obstacle that stopped the creation of the theory of fractional calculus in the complex plane. However, for the current case if we continue this process of fractional differentiation or integration new singularities do not appear. Indeed, applying again (29) or (30) to (31) we have,

\[
\mathcal{D}_\beta \{F (t, \alpha)\} = \Gamma (\beta + 1) \Gamma (\alpha + 1) \binom{2F_1 (\alpha + 1, \beta + 1; 1; -t)}{1}, \quad (35)
\]

\[
\mathcal{I}_\beta \{F (t, \alpha)\} = \Gamma (\alpha + 1) / \Gamma (\beta + 1) \binom{2F_1 (\alpha + 1, 1; \beta + 1; -t)}{1}. \quad (36)
\]

Functions \(\mathcal{D}_\beta \{F (t, \alpha)\}\) and \(\mathcal{I}_\beta \{F (t, \alpha)\}\) are analytic in the extended \(t\)-plane punctured at three points \(t = -1, t = 0\) and \(t = \infty\). Further differentiation and integration of fractional order do not change this set of three singular points. Our aim is to explain the reason for this stabilization and extend this preservation property to wide classes of “hypergeometric” functions generated by the system of linear ODE’s with analytic coefficients in their Laplace-Borel dual complex plane.
Previously the topology of the space \( S \) was determined by the topology of the space \( H \). In the next section we show how to endow separately both the spaces \( H(a, R) \) and \( S(a, R) \) by independent countable normalized topologies.

### 2.5 The Basic Spaces and the Duality Theorem

Given \( R \geq 0 \) and \( a > 0 \), we will endow linear spaces \( H(a, R) \) and \( S(a, R) \), given by definition 3, by countable normalized topologies and we retain the previous notation of the normalized spaces.

Thus, \( H(a, R) \) is the set of all functions \( F(t) \) satisfying the conditions:

- (i) \( F(t) \) is analytic in the region \( \mathcal{D}(a) \) with boundary \( \gamma(a) \) given by (1) and (2), respectively;
- (ii) \( F(t) \) has an exponential growth at infinity;
- (iii) for every \( r > R \geq 0 \), \( 0 < A < a \) the following restriction holds

\[
\|F\|_{r,A} = \int_{\gamma(A)} e^{-r|t|} |F(t)| \, dt < \infty. \tag{37}
\]

\( S(a, R) \) is a set of all functions \( P(\zeta) \), analytic in the half-plane \( \Re\{\zeta\} > R \geq 0 \) and satisfying conditions:

- (i) there exists a sequence \( p_0, p_1, \ldots \), of complex numbers such

\[
P(\zeta) \sim \sum_{k=0}^{\infty} p_k / \zeta^k, \Re\{\zeta\} > R, \zeta \to \infty;
\]

- (ii) the following sequence of inequalities holds for \( 0 < A < a \) and \( R < r < \infty \):

\[
\left| P(\zeta) - \sum_{k=0}^{n-1} \frac{p_k}{\zeta^k} \right| \leq \frac{M_n!}{A^n |\zeta|^n}, , n = 0, 1, \ldots, \tag{38}
\]

where \( 0 < M = M(A, r) < \infty \) is a constant independent of \( n \);

- (iii) for \( 0 < A, R < r < \infty \), \( \Re\{\zeta\} \geq r \) and \( P_n(\zeta) \), defined by (5), the following restriction holds

\[
\|P\|_{r,A} = \sup_{0 \leq n < \infty} \frac{A^n |\zeta|^n}{n!} |P_n(\zeta)| \leq M(A, r). \tag{39}
\]

We claim the following statement (due to F. Nevanlinna).
Theorem 5. (Duality Theorem): \( \mathcal{L}H(a, R) = S(a, R) \) and spaces \( H(a, R) \) and \( S(a, R) \) are isomorphic.

For a further discussion of Nevanlinna’s theorems and their extensions, see [23], [13], and references in these papers.

We introduce also the Banach space \( H^1(a) \) of functions \( F(t) \in H(a, R) \) that satisfy the following restriction

\[
\int_{\gamma(a)} \left| \frac{F(t)}{t} \right| |dt| < \infty. \tag{40}
\]

It follows from (40) that \( F(t) \in H^1(a) \) can be represented by the following Cauchy integral

\[
F(t) = \frac{1}{2\pi i} \int_{\gamma(a)} \left(1 - \frac{t}{\xi}\right)^{-\alpha - 1} \frac{F(\xi)}{\xi} d\xi, \tag{41}
\]

which is absolutely convergent for all \( t \in \mathcal{D}(a) \). Applying operators \( D_\alpha \) and \( I_\alpha \) to both the sides of (41) and using (31) and (32), we have

\[
D_\alpha \{F(t)\} = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{\gamma(a)} \left(1 - \frac{t}{\xi}\right)^{-\alpha - 1} \frac{F(\xi)}{\xi} d\xi, \tag{42}
\]

\[
I_\alpha \{F(t)\} = \frac{1}{\Gamma(\alpha + 1)} \frac{1}{2\pi i} \int_{\gamma(a)} \left(2F_1\left(1, 1; \alpha + 1; \frac{t}{\xi}\right)\right) \frac{F(\xi)}{\xi} d\xi. \tag{43}
\]

The functions \( D_\alpha \{F(t)\}, I_\alpha \{F(t)\} \) don’t belong, in general, to \( H^1(a) \). However, \( D_\alpha \{F(t)\}, I_\alpha \{F(t)\} \in H(a, 0) \).

Applying the Laplace transform operator \( \mathcal{L} \) to both sides of (42) and (43), a straightforward calculations yield the dual integral representations for

\[
P(\zeta, \alpha) = \hat{D}_\alpha \{P(\zeta)\} \tag{44}
\]

\[
P^*(\zeta, \alpha) = \hat{I}_\alpha \{P(\zeta)\} \tag{45}
\]

in terms of \( F(t) \), respectively, with kernels—\( \textit{Dingle-Berry basic kernel} \) and \( \textit{associated kernel} \)—that are given by (44) and (45), respectively, for a particular case \( F(t) = \frac{1}{1+t} \). These kernels will be studied in detail in the beginning of Section III in Part 2.

Remark 4. The dual integral representation for \( P(\zeta, \alpha) \) and \( P^*(\zeta, \alpha) \) in terms of \( F(t) \in H(a, R) \), exponentially growing at infinity, are very important for applications to the asymptotic analysis in general, and especially for the ODE’s with analytic coefficients. This approach was initiated in Section 5 of [13], and is further developed in Part 2.

In conclusion, we note the important properties of Banach spaces \( H^1(a) \):
Lemma 1. For every $R \geq 0$ the closure of $H^1(a)$ in $H(a, R)$ coincides with $H(a, R)$.

In the next sections we will show how to extend the above results given by (42) and (43) for general case of exponentially growing functions of spaces $H(a, R)$.

2.6 Integral representations for fractional derivatives and integrals

2.6.1 Integral representations for $\hat{D}_\alpha \{P(\zeta)\}$ and $\hat{I}_\alpha \{P(\zeta)\}$ in terms of $P(\zeta)$

Let $\Re\{\alpha\} > -1$, and let $P(\zeta) \in S(a, R)$ be given by (42) for some $F(t) \in H(a, R)$. Then using notation (47) we can easily derive the validity of the following integral representation

$$P(\zeta, \alpha) = \hat{D}_\alpha \{P(\zeta)\} = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{r-i\infty}^{r+i\infty} \left(1 - \frac{z}{\zeta}\right)^{-\alpha - 1} \frac{P(z)}{z} dz,$$  \hspace{1cm} (46)

where the integral (46) is absolutely convergent for all $\zeta : 0 \leq R(F) < \Re\{\zeta\} < \infty$, and $R(F)$ is given by (4).

We note that this integral representation can be extended to all functions $P(\zeta)$ satisfying (42). Any additional constraints on the set of functions $F(t)$ are not required. Clearly formula (46) can also be rewritten in the form

$$P(\zeta, \alpha) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{r-i\infty}^{r+i\infty} \left(_2F_1\left(\alpha + 1, 1; 1; \frac{z}{\zeta}\right)\right) \frac{P(z)}{z} dz,$$  \hspace{1cm} (47)

A more sophisticated argument shows that for $P(\zeta) \in S(a, R)$ we have

$$\hat{I}_\alpha \{P(\zeta)\} = \frac{1}{2\pi i \Gamma(\alpha + 1)} \int_{r-i\infty}^{r+i\infty} \left(_2F_1\left(1, 1 + \alpha + 1; \frac{z}{\zeta}\right)\right) \frac{P(z)}{z} dz,$$  \hspace{1cm} (48)

where the integral (48) is also absolutely convergent for $\Re\{\zeta\} > R(F)$. Note that this relation, as opposed to (47), cannot be proved without the additional restrictions on $F(t)$ or $P(\zeta)$. To prove the validity of (48) we must expand both the sides into power series in $1/\zeta$, check that the series are identical and then apply the uniqueness theorem 1.

We note that the above integral representations can not be used for applications. Using them we can not even calculate the coefficients of their asymptotic expansions. In the next section we demonstrate integral representations for $\hat{D}_\alpha \{F(t)\}$ and $\hat{I}_\alpha \{F(t)\}$ in terms of $F(t)$. Their proof is much more complicated, but later we will show how they can be used in asymptotic analysis of the ODE’s to obtain much simpler final results.
2.6.2 Integral representations for $D_\alpha \{F(t)\}$ and $I_\alpha \{F(t)\}$ in terms of $F(t)$

Let $H(a, R)$ and $S(a, R)$ be our basic spaces given by definitions 6 and 7, and let operators $D_\alpha, I_\alpha, \hat{D}_\alpha, \hat{I}_\alpha$ be given by (12) and (13), respectively. Assume that $0 \leq R < \infty$, $0 < A < a < \infty$, and we preserve the notations $F(t, \alpha) = D_\alpha \{F(t)\}$ and $F^*(t, \alpha) = I_\alpha \{F(t)\}$, $\Re\{\alpha\} > -1$, given by (16) and (17), respectively. The following statements hold:

**Theorem 6.** Let $F(t) \in H(a, R)$ and $F(t, \alpha) = D_\alpha \{F(t)\}$. Then (i) For all $r$, $0 \leq R < r < \infty$, and for $t \in \mathfrak{D}(A)$ the following integral representation is valid

$$F(t, \alpha) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + k + 1)}{(k!)^2} (rt)^k f_k(\alpha, r, t),$$

where

$$f_k(\alpha, r, t) = \frac{1}{2\pi i} \int_{\gamma} \left( 2F_1 \left( \alpha + k + 1, 1; k + 1; \frac{t}{\xi} \right) \right) \frac{F(\xi) e^{-r\xi}}{\xi} d\xi,$$

and the path $\gamma = \gamma(A)$ is oriented in the counterclockwise direction.

(ii) The integral of (50) and the sum of (49) are absolutely convergent and

$$F(t, \alpha) \in H(a, R).$$

**Theorem 7.** Let $F(t) \in H(a, R)$ and $F^*(t, \alpha) = I_\alpha \{F(t)\}$. Then

(i) For all $r$, $0 \leq R < r < \infty$, and for $t \in \mathfrak{D}(A), 0 < A < a$, the following integral representation is valid

$$F^*(t, \alpha) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\alpha + k + 1)} (rt)^k f^*_k(\alpha, r, t),$$

where

$$f^*_k(\alpha, r, t) = \frac{1}{2\pi i} \int_{\gamma} \left( 2F_1 \left( k + 1, 1; \alpha + k + 1; \frac{t}{\xi} \right) \right) \frac{F(\xi) e^{-r\xi}}{\xi} d\xi,$$

and the path $\gamma = \gamma(A)$ is oriented in the counterclockwise direction.

(ii) The integral of (53) and the sum of (52) are absolutely convergent and

$$F^*(t, \alpha) \in H(a, R).$$

Theorems 2 and 3 together imply Theorem 1.

Proofs of Theorem 6 is given essentially in [13], and Theorem 7 can be proved using similar technique.
Corollary 1. Assume that $F(t) \in H(a, R)$ admits an analytical continuation to a region of the Riemann surface of $\log t$ that can be represented as a union of overlapping regions of the form $D = \cup e^{i\theta'} D(a')$ for any set of $\theta' \in (-\infty, +\infty)$ and $a' \in (a, \infty)$. Assume further that $F(t)$ retains an exponential growth of the type $R$ in $D$. Then the integral representations given by Theorem 6 and 7 can be extended to the region $D$.

Remark 5. For $\alpha = 0$ both formulas (49) and (52) become trivial identities.

We close this section with the observation that if the function $F(t) \in H^1(a)$ then integral representations in Theorems 6 and 7 simplify considerably. They are given by (42) and (43).

Remark 6. Note that the set of all functions $F(t) \in H^1(a)$ given by (40) can be considered as an analog of the Hardy space $H^1(D_a)$, where $D_a = \{ t : |t| < a \}$.

Upon completion of this work, we found that similar Hardy spaces in the regions $D(a)$, and in the complement of $D(a)$ with respect to $\mathbb{C}$, were studied by Peschansky (1989), see references in [19].

In what follows, we draw attention to the following result on the analytic continuation with respect to a parameter $\alpha$ which has been proved in [13] for the fractional derivatives, and can also be derived for the fractional integrals.

### 2.6.3 Analytic continuation in the $\alpha$-plane

The following statement is valid.

**Theorem 8.** Assume that $F(t)$ satisfies the assumptions of Theorem 3. Given fixed $t \in D(A), 0 < A < a$,

(i) the function $F(t, \alpha) = D_\alpha \{ F(t) \}$ admits an analytic continuation to the whole $\alpha$-plane except for the negative integers;

(ii) the function $F(t, \alpha)$ has a simple pole at every point $\alpha = -n - 1, n = 0, 1, \ldots$, so that the function

$$
\frac{F(t, \alpha)}{\Gamma(\alpha + 1)}
$$

is an entire function of exponential type, and the following relation is valid

$$
\lim_{\alpha \to -n - 1} \frac{F(t, \alpha)}{\Gamma(\alpha + 1)} = \Psi_{n,F}(t),
$$

where $\Psi_{n,F}(t)$ is a polynomial in $t$ of degree $n$, which can be represented in the form

$$
\Psi_{n,F}(t) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f_j t^j,
$$

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with coefficients $f_j = F^{(j)}(0)/j!$, which can be represented as

$$f_j = \frac{1}{2\pi i} \sum_{s=0}^{j} \frac{1}{(j-s)!} \int_{\gamma(A)} \frac{F(\xi)}{\xi^{1+s}} e^{-r\xi} d\xi,$$

(57)

(iii) $F(t, \alpha) = \mathcal{L}_\alpha \{ F(t) \}$ is an entire function of an exponential type.

We note that Theorems 6–8 extend the corresponding results in [28] and [29], that was described also in [19], to our version of fractional calculus.

In the next section we show how to apply the above version of fractional calculus to linear ODE’s with analytic coefficients.

### 3 Linear ODE’s with analytic coefficients

In this section we study linear spaces of vector-valued functions generated by the second order linear ODE’s with analytic coefficients in their Laplace-Borel dual complex plane. We discovered that the elements of these spaces inherit a number of properties of classical hypergeometric functions. In particular, we show how to extend some classical linear transformation formulae for hypergeometric functions (due to Euler, Gauss, Goursat) to the elements of the dual spaces. For the cases of the Kummer or Whittaker differential equations the above elements can be represented in terms of classical hypergeometric function. Even for this special case our formulas are new. In Part II we show how to apply these formulas in asymptotic analysis of the ODE’s.

#### 3.1 Introduction

We develop further an approach to study asymptotic properties of solutions of linear ODE’s with coefficients that are analytic in a neighborhood of infinity first proposed in [4] for the case of even coefficients. The approach is based on the duality between a linear system of monodromic functional equations generated by the ODE and its Laplace-Borel-dual system. Here we consider the case of a second order ODE, which generates a system of two monodromic integral equations of a fractional order for a pair of multi-valued analytic functions: a jump of one function across the cut is expressed in terms of the fractional integral of the other function. Moreover, we stated for this pair of functions a system of Euler and Euler-Goursat type linear transformation formulae. Our approach is based on a version of fractional calculus in the complex plane that was initiated in [13]. It can be extended to wide classes of matrix-valued function.
3.2 A system of functional monodromic equations generated by the ODE in the Laplace-Borel dual complex plane

We consider linear ODE’s of the form

\[ u''(\zeta) + a(\zeta)u'(\zeta) + b(\zeta)u(\zeta) = 0, \]  

where functions \(a(\zeta)\) and \(b(\zeta)\) are analytic at infinity. Set \(a(\infty) = a_0\) and \(b(\infty) = b_0\). If \(a_0 - 4b_0 \neq 0\) then this equation can be reduced to a perturbation of the Whittaker differential equation (pWde):

\[ \frac{d^2u}{d\zeta^2} = \left( \frac{1}{4} - \kappa/\zeta + \left( \mu^2 - 1/4 \right)/\zeta^2 + \left( 1/\zeta^3 \right) \sum_{k=0}^{\infty} \left( \beta_k/\zeta^k \right) \right) u, \]

where \(\kappa, \mu, \beta_k, k = 0, 1, \ldots\) are complex numbers and the series \(\sum_{k=0}^{\infty} \left( \beta_k/\zeta^k \right)\) is absolutely convergent in the exterior of a circle of radius \(R\) centered at the origin for some \(R \geq 0\).

There exists a pair of linearly independent solutions of (59), \(u_1(\zeta)\) and \(u_2(\zeta)\), such that

\[ u_1(\zeta) = e^{-\frac{1}{2} \zeta^2} P_1(\zeta), \]
\[ u_2(\zeta) = e^{\frac{1}{2} \zeta^2} P_2(\zeta), \]

where the phase-amplitudes \(P_1(\zeta)\) and \(P_2(\zeta)\) satisfy the following relations

\[ P_1(\zeta) = 1 + o(1), -\pi \leq \arg \zeta \leq \pi, |\zeta| > R, \]

\[ P_2(\zeta) = 1 + o(1), 0 \leq \arg \zeta \leq 2\pi, |\zeta| > R. \]

We note that the solutions \(u_1(\zeta)\) and \(u_2(\zeta)\) are uniquely determined by the conditions (62) and (63), respectively.

Clearly, \(P_1(\zeta)\) and \(P_2(\zeta)\) are analytic multi-valued functions in the exterior of the circle \(|\zeta| = R\) in the \(\zeta\)-plane, and it can be proved that there exists a pair of complex numbers \(T_1, T_2\) such that the following system of monodromic functional relations is valid for \(\zeta > R\):

\[ P_1(\zeta e^{\pi i}) - P_1(\zeta e^{-\pi i}) = T_1 e^{-\zeta^2} P_2(\zeta e^{\pi i}), \]
\[ P_2(\zeta e^{\pi i}) - P_2(\zeta e^{-\pi i}) = T_2 e^{\zeta^2} P_1(\zeta e^{-\pi i}). \]

Thus, the pWde, with an infinite number of parameters, gives rise to a system of monodromic relations with only three parameters \(T_1, T_2\) and \(\kappa\).

The terminology monodromic means that a jump of one multi-valued function across the cut along any ray \(l_\theta = \{ \zeta : \arg \zeta = \theta, |\zeta| > R \}, -\infty < \theta < \infty\), can be expressed in terms of the other function. The constants \(T_1 = T_1(\kappa, \mu, \beta_0, \ldots), T_2 = T_2(\kappa, \mu, \beta_0, \ldots)\) are invariants of (59) usually referred to as the connection coefficients, or Stokes multipliers. The parameter \(\kappa\), which is the second coefficient in (59), can be expressed in terms of \(T_1\) and \(T_2\).
Definition 2. Using the complex numbers $T_1$, $T_2$ and $\kappa$, we introduce the linear space $S = S(T_1, T_2, \kappa)$ of all vector-valued solutions $(P_1(\zeta), P_2(\zeta))$ of the system of functional equations (64)–(65), which retain the analytic properties of $P_1(\zeta)$ and $P_2(\zeta)$ described above.

3.3 Dual system of monodromic relations

Assume that the vector-valued function $(P_1(\zeta), P_2(\zeta))$ belongs to the linear space $S = S(T_1, T_2, \kappa)$ given by Definition 2. Then there exists a pair of integrable functions $F_1(t)$ and $F_2(t)$ such

$$P_1(\zeta) = \mathcal{L}\{F_1(t)\}, \quad P_2(\zeta) = \mathcal{L}\{F_2(t)\},$$

(67)

where $\mathcal{L}$ is the Laplace transform operator given by (5). Indeed, it follows from (62) that $P_1(\zeta)/\zeta$ is a square integrable function on every line $\Im\{\zeta\} > R \geq 0$. It follows from Parseval’s theorem that $F_1(t)$ is a square integrable function on the interval $(R, +\infty)$. Similar argument can be applied to $P_2(\zeta)$.

Definition 3. We denote by $H(T_1, T_2, \kappa)$ the linear space of all vector-valued functions $(F_1(t), F_2(t))$ given by (67), so that

$$H(T_1, T_2, \kappa) = \mathcal{L}S(T_1, T_2, \kappa),$$

(68)

In what follows we study properties of elements of $H(T_1, T_2, \kappa)$. Applying the Borel transform operator $\mathcal{L}^{-1}$ to the left and right sides of (64)–(65), we show that the jumps of $P_1(\zeta)$ and $P_2(\zeta)$ on the left sides are transformed into the jump of $F_1(t)$ and $F_2(t)$ on the corresponding cut, while the exponential factors and power factors on the right sides are transformed into the shifts and the fractional integrals of orders of $-2\kappa$ and $2\kappa$ of $F_2(t)$ and $F_1(t)$, respectively. The latter fact follows from Theorem 4. This enables us to derive the dual system of monodromic relations for a pair $F_1(t)$ and $F_2(t)$.

Now we demonstrate properties of $F_1(t)$ and $F_2(t)$ that follows from (62) and (63) and (64)–(65).

Assume that $F_1(t)$ and $F_2(t)$ are given by (67). Then the following statements are valid

Theorem 9. \hspace{1em} (i) $F_1(t)$ and $F_2(t)$ are analytic in the $t$-plane cut along the intervals $(-\infty, -1)$ and $(1, +\infty)$, respectively;

(ii) $F_1(t)$ and $F_2(t)$ can be continued analytically further to the Riemann surface of $\log t$, oriented in the counterclockwise direction, except for the points $(-1, 0), (0, 0)$ and $(0, 0), (0, 1)$, respectively;

(iii) In every sectorial region of the form

$$S(\theta_1, \theta_2) = \{t : \theta_1 < \arg t < \theta_2, 0 < |t| < \infty\}$$

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$F_1(t)$ and $F_2(t)$ have an exponential growth of type less or equal to $R$ at infinity.

**Theorem 10.** The following system of monodromic relations

\[
\begin{align*}
F_1(te^{-\pi i}) - F_1(te^{\pi i}) &= T_1(t - 1)^{2\kappa} I_{2\kappa} \{F_2((t - 1)e^{-\pi i})\}, \quad (69) \\
F_2(te^{-\pi i}) - F_2(te^{\pi i}) &= T_2(te^{\pi i} - 1)^{-2\kappa} I_{-2\kappa} \{F_2((te^{\pi i} - 1))\}, \quad (70)
\end{align*}
\]

is valid in the whole $t$-plane punctured in the points $(-1,0), (0,0)$ and $(0,1)$.

The system of relations $(69)-(70)$, which is dual to the system $(64)-(65)$, can be derived from the latter system directly applying to both sides of it the operator $L^{-1}$, using the relation $(24)$ with $\alpha = \pm 2\kappa$.

**Definition 4.** Using the complex numbers $T_1, T_2$ and $\kappa$ of the Definition 3, we introduce the linear space $H(T_1, T_2, \kappa)$ of all vector-valued solutions $(F_1(t), F_2(t))$ of the system of functional equations $(69)-(70)$, where $F_1(t)$ and $F_2(t)$ satisfy the properties (i)-(iii) of Theorem 9.

**Remark 7.** We note that the linear space $H(T_1, T_2, \kappa)$ is essentially wider than the linear space $H(T_1, T_2, \kappa)$ given by $(68)$. This fact was noted in paper [4] where we considered the case when the coefficients of $(58)$ are even function. For this case $T_1 = T_2 = T, \kappa = 0$, and the linear space $S(T_1, T_2, \kappa)$ was denoted by $S_{1,T}$. See then Definition 1 and the last line of the second paragraph after on page 658 of [4]. Therefore it is impossible in general to derive the system $(64)-(65)$ applying the Laplace transform operator to the system $(69)-(70)$ without imposing addition conditions on the functions $F_1(t)$ and $F_2(t)$.

The question arises whether there is a system of functional equations for functions $F_1(t)$ and $F_2(t)$, satisfying the properties (i)-(iii), which is equivalent to the system $(64)-(65)$ and which entails the system $(69)-(70)$?

Now we see that the answer to this question is affirmative.

### 3.4 Euler-Gauss-type system of linear transformation formulas

**Theorem 11.** Let $F_1(t)$ and $F_2(t)$ are given by $(67)$ where $(P_1(\zeta), P_2(\zeta)) \in$. Then the pair $(F_1(t), F_2(t))$, if $2\kappa$ is not integer, satisfies a system of Euler-Gauss-type linear transformation formulae of the form

\[
\begin{align*}
2i \sin(2\kappa\pi) F_1(t) &= -T_1 (1 + t)^{2\kappa} I_{2\kappa} \{F_2(1 + t)\} \\
+ T_2 e^{-2\kappa\pi i} I_{-2\kappa} \{F_1((1 + t)e^{-\pi i})\}, -\pi \leq \arg t \leq \pi, 1 < |t| < \infty, \quad (71)
\end{align*}
\]

\[
\begin{align*}
2i \sin(2\kappa\pi) F_2(t) &= T_2 (t - 1)^{-2\kappa} I_{-2\kappa} \{F_1(t - 1)\} \\
- T_1 I_{2\kappa} \{F_2((t - 1)e^{-\pi i})\}, -2\pi \leq \arg t \leq 0, 1 < |t| < \infty, \quad (72)
\end{align*}
\]

where $I_\alpha$ is the operator of a fractional integration of order $\alpha = \pm 2\kappa$ in our version of fractional calculus in the complex plane, given by $(12)$.  

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We recall that, as follows from Theorem 8 (iii), the function $I_{2\kappa} \{F_j(t)\}, j = 1, 2$, for fixed $t$, is entire function of $\kappa$.

### 3.5 Euler-Goursat-type linear transformation formulas

**Theorem 12.** If $2\kappa \in \mathbb{Z}$ then the system of an Euler-Goursat-type linear transformation formulae can be written in the form

$$F_1(t) = C_1 T_1 (1 + t)^{2\kappa} \log (1 + t) I_{2\kappa} \{F_2(t + 1)\} + \Psi_1(1 + t), -\pi \leq \arg t \leq \pi, 1 < |t| < \infty,$$

$$F_2(t) = C_2 T_2 (t - 1)^{-2\kappa} \log (t - 1) I_{-2\kappa} \{F_1(t - 1)\} + \Psi_2(t - 1), -2\pi \leq \arg t \leq 0, 1 < |t| < \infty,$$

where $C_1$ and $C_2$ are constants, which can be calculated exactly, functions $\Psi_1(t)$ and $\Psi_2(t)$ are analytic in the unit circle of the $t$-plane, and their Taylor coefficients can be found by recurrence following the procedure, which was described in [4] for the case $\kappa = 0, \beta_{2k} = 0, k = 0, 1, \ldots$.

### 3.6 Duality theorem

Our main result, the duality theorem:

**Theorem 13.** For the case when $2\kappa \notin \mathbb{Z}$ the linear space $H(T_1, T_2, \kappa)$ given by (68) coincides with the set of all vector-valued solutions $(F_1(t), F_2(t))$ of the system of functional equations (71)–(72), which satisfy conditions (i) and (ii). For the case when $2\kappa \in \mathbb{Z}$ the linear space $H(T_1, T_2, \kappa)$ coincides with the set of all vector-valued solutions $(F_1(t), F_2(t))$ of the system of functional equations (73)–(74), which satisfy conditions (i)–(iii) of Theorem 9.

### 3.7 The case of the Whittaker equation

Validity of all above result can be verified using the case of the (unperturbed) Whittaker equation for which $F_1(t)$ and $F_2(t)$ are the hypergeometric functions

$$F_1(t) = \left(2F_1 \left(\frac{1}{2} - \kappa - \mu, \frac{1}{2} - \kappa + \mu; 1; -t\right)\right),$$

$$F_2(t) = \left(2F_1 \left(\frac{1}{2} + \kappa - \mu, \frac{1}{2} + \kappa + \mu; 1; (te^{\pi i})\right)\right),$$

and for the case when $2\kappa \notin \mathbb{Z}$ the relations (71)–(72) can be derived from Euler’s linear transformation formulae (see [3], 15.3.6) written separately for two functions $F_1(t)$ and $F_2(t)$ given by (67) and rewritten as (75) and (76). For the case when $2\kappa \in \mathbb{Z}$ the relations (73)–(74) can be derived from the Euler-Goursat linear transformations formulae (see [3], 15.3.10–15.3.14).
3.8 Euler’s linear transformation formula for $(2F_1(a, b; c; t))$

The hypergeometric function $2F_1(a, b; c; t)$ can be introduced as an analytic continuation of the hypergeometric series

$$
\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} t^k, |t| < 1,
$$

(77)

from the unit circle, along any path not crossing the points $t = 0$ and $t = 1$, to the $t$-plane.

Setting

$$A := \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \quad \text{and} \quad B := \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

Euler’s linear transformation formula,

$$2F_1(a, b; c; t) = (1-t)^{c-a-b} A (2F_1(c-a, c-b; c-a-b+1; 1-t)) + B (2F_1(a, b; a+b-c+1; 1-t)),
$$

(78)

for hypergeometric functions is valid if

$$c-a-b \notin \mathbb{Z} \quad \text{and} \quad |\arg(1-t)| \leq \pi.
$$

(79)

This formula is available in all manuals on special functions and still is in the focus of many mathematicians, see recent publications, [31], [32], [30].

Analysis of (78) shows that $2F_1(a, b; c; t)$ given by (77) admits an analytic continuation to the $t$-plane cut along the interval $(1, \infty)$ of the positive ray and restriction $|\arg(1-t)| = \pm \pi$ means that $t$ belongs to the boundary of the cut $t$-plane. In the next section we calculate the jump of the hypergeometric function $(2F_1(a, b; c; t))$ on the cut $(1, +\infty)$. But first, we introduce two transformations

$$\mathcal{K}_1 \{2F_1(a, b; c; t)\} = (2F_1(c-a, c-b; c-a-b+1; 1-t)),
\mathcal{K}_2 \{2F_1(a, b; c; t)\} = (2F_1(a, b; a+b-c+1; 1-t)),
$$

(80)

The relations (78) does not provide any clue of how operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are connected. We answer this question not only for hypergeometric functions given by (75) and (76) but also for wide classes of “hypergeometric” functions that are generated by solutions of the system (64)–(65) in the Laplace-Borel dual complex plane. It turns out that the above transformations are operators of fractional integration of order $\kappa$ and $-\kappa$ in the author’s version of the fractional calculus in the complex plane. In the next Section we explain how to derive the system of linear transformation formulas for the functions (64)–(65) of the form given by (71)–(72) from the formulas (78) written separately for (64)–(65).
3.9 Monodromic relations and connection coefficients for \((2F_1(a, b; c; t))\)

Assume that \(t\) is any point inside the circle \(|1 - t| < 1\), which does not belong to its radius along the interval \((1, 2)\) of the real line. Since \(1 - t^\ast = (1 - t) e^{\pm 2\pi i}\), the point \(t^\ast\) satisfies the same restriction. Let us write Euler’s linear transformation formulas (78) for \(2F_1(a, b; c; t)\) and \(2F_1(a, b; c; t^\ast)\) and note that both the hypergeometric functions in the right hand-side of (78) are coincide at \(t^\ast\) and \(t\). Then subtracting the first relation from the second relation, a straightforward calculation yields the following result

\[
2F_1(a, b; c; t^\ast) - (2F_1(a, b; c; t)) = T^\pm(a, b, c) (1 - t)^{c-a-b} (2F_1(c - a, c - b; c - a - b + 1; 1 - t)),
\]

(81)

where

\[
T^\pm(a, b, c) = \mp 2\pi i e^{\pm \pi i(c-a-b)} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}.
\]

(82)

As oppose to the relation (78), both relations (81) and (82) are valid for the case \(a+b-c \in \mathbb{Z}\), and can be extended for all \(t\) belonging to the closure of the \(t\)-plane cut along the interval \((1, +\infty)\). We refer to the relation (81) and the constant \(T^\pm(a, b, c)\) as the monodromic relation and connection coefficient, respectively.

In the cases when \(a + b - c \in \mathbb{Z}\) the formulas (78) are meaningless while the formulas (81) and (82) remain valid. However, the latter formulas do not allow to recognize the nature of the singularity of the hypergeometric function at \(t = 1\). Goursat discovered that the right hand side of (78) should be replaced by a sum of two terms, where the first term is essentially the same as the first term in (78) but the factor \((1 - t)^{c-a-b}\) is replaced by \((1 - t)^{c-a-b} \log(1 - t)\), while the second term is a function that are analytic in the circle \(|1 - t| < 1\) and its Taylor coefficients were provided with explicit expression. The corresponding formulas are given in [3], 15.3.10–15.3.14. Following ideas of [4], we generalized these formulas for the classes of “hypergeometric” functions, generated by solutions of the system of monodromic relations in the \(t\)-plane that is dual to the system (81)–(85).

We note that the relations (78), (81) and (82) can be rewritten as

\[
2F_1(a, b; c; -t) = (1 + t)^{c-a-b} A (2F_1(c - a, c - b; c - a - b + 1; 1 + t))
\]

\[
+ B (2F_1(a, b; a + b - c + 1; 1 + t)),
\]

(83)

\[
2F_1(a, b; c; -t^\ast) - (2F_1(a, b; c; -t)) = T^\pm(a, b, c) (1 + t)^{c-a-b} (2F_1(c - a, c - b; c - a - b + 1; 1 + t)),
\]

(84)

where \(1 + t^\ast = (1 + t) e^{\pm 2\pi i}\) and

\[
T^\pm(a, b, c) = \mp 2\pi i e^{\pm \pi i(c-a-b)} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}.
\]

(85)
3.10 Conclusion

In conclusion we note that the above systems of linear transformation formulae and monodromic relations can be extended to the more general case of a matrix differential equation of the form \( \frac{du}{d\zeta} = Q(\zeta)u \), where \( u = u(\zeta) \) is an \( n \times n \) matrix-valued function, \( Q(\zeta) = A_0 + \frac{1}{\zeta}A_1 + \ldots \), where \( A_0, A_1, \ldots \) are the constant \( n \times n \) matrices, and \( \det A_0 \neq 0, A_0 = \text{diag} (\lambda_1, \ldots, \lambda_n), \lambda_i \neq \lambda_j, 1 \leq i, j \leq n, A_1 = \text{diag} (\kappa_1, \ldots, \kappa_n) \), and \( Q(\zeta) \) is analytic in a neighborhood of infinity. Then associated system of monodromic relations is expressed in terms of fractional integrals of orders \( \pm (\kappa_1 - \kappa_2, \ldots, \kappa_n - 1 - \kappa_n, \kappa_n - \kappa_1) \).

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