BULK BEHAVIOUR OF GROUND STATES FOR RELATIVISTIC SCHRÖDINGER OPERATORS WITH COMPACTLY SUPPORTED POTENTIALS

GIACOMO ASCIONE AND JÓZSEF LÖRINCZI

Abstract. We propose a probabilistic representation of the ground states of massive and massless Schrödinger operators with a potential well in which the behaviour inside the well is described in terms of the moment generating function of the first exit time from the well, and the outside behaviour in terms of the Laplace transform of the first entrance time into the well. This allows an analysis of their behaviour at short to mid-range from the origin. In a first part we derive precise estimates on these two functionals for stable and relativistic stable processes. Next, by combining scaling properties and heat kernel estimates, we derive explicit local rates of the ground states of the given family of non-local Schrödinger operators both inside and outside the well. We also show how this approach extends to fully supported decaying potentials. By an analysis close-by to the edge of the potential well, we furthermore show that the ground state changes regularity, which depends qualitatively on the fractional power of the non-local operator.

Key-words: massive and massless Schrödinger operator, fractional Laplacian, potential well, Feynman-Kac formula, stable processes, relativistic stable processes, occupation measure, exit time, ground state

2010 MS Classification: Primary 47D08, 60G51; Secondary 47D03, 47G20

CONTENTS

1. Introduction 2
2. Preliminaries 5
2.1. The massive and massless relativistic operators 5
2.2. Feynman-Kac representation and the related random processes 7
2.3. Heat kernel of the killed Feynman-Kac semigroup 8
3. Exit and hitting times estimates 10
3.1. Estimates on the survival probability 10
3.2. Estimates on the moment generating function for the exit time from a ball 14
3.3. Estimates on the Laplace transform of the hitting time for a ball 16
4. Basic qualitative properties of ground states 21
4.1. Martingale representation of ground states 21
4.2. Symmetry properties 22
5. Local estimates 23
5.1. A prime example: Classical Laplacian and Brownian motion 23
5.2. Local behaviour of the ground state 24
5.3. Lack of regularity of \( \varphi_0 \) 30
5.4. Moment estimates of the position in the ground state 32
5.5. Extension to fully supported decaying potentials 34
References 37
1. Introduction

The purpose of this paper is to introduce and explore a relationship between the moment generating functions and Laplace transforms of first hitting times of rotationally symmetric stable and relativistic stable processes, and the ground states of related non-local Schrödinger operators. Making use of this relationship, via precise estimates of these random time functionals we will be able to derive and prove the spatial localization properties of ground states in the bulk, i.e., for short to middle range from the origin.

The (semi-)relativistic Schrödinger operator $H = (-\Delta + m^2)^{1/2} - m + V$ on $L^2(\mathbb{R}^3)$, describing the Hamiltonian of an electrically charged particle with rest mass $m > 0$ moving under a Coulomb potential $V$ is one of the fundamental models of mathematical quantum theory, and it has been studied extensively in the literature. Classic papers include [56, 13, 40] on the square-root Klein-Gordon equation, [58, 31, 20, 21] on the properties of the spectrum, stability of the matter [44, 26, 27, 43], and eigenfunction decay [15]. More recent developments further addressed low-energy scattering theory [49], embedded eigenvalues and Neumann-Wigner type potentials [47], decay rates when magnetic potentials and spin are included [32], a relativistic Kato-inequality [33], Carleman estimates and unique continuation [50, 25], or nonlinear relativistic Schrödinger equations [19, 54, 1].

Given its relationship with random processes with jumps, the $V = 0$ case has received much attention also in potential theory [51, 30, 14].

There are only a very few examples around for which the spectrum and eigenfunctions of relativistic Schrödinger operators are explicitly determined [46, 21], when the potential is confining rather than decaying, and interesting approximations of spectra and eigenfunctions for some other cases have been obtained in [37]. Thus estimates on the eigenfunctions have a special relevance. While eigenfunction decay at infinity for a large class of non-local Schrödinger operators, including the relativistic operator, is now understood to a great detail in function of the asymptotic behaviour of the potential [15, 32, 35, 36], very little is known on their local behaviour, i.e., for small to medium distances from the origin. Some information on local properties of eigenfunctions of non-local Schrödinger operators with Bernstein functions of the Laplacian and general potential wells have been obtained in [9, Sect. 4]. Specifically, these include estimates on the distance of the location of global extrema of eigenfunctions from the edge of the potential well or specific level sets. For domain operators results in a similar spirit have been obtained in [6, 7].

Our goal in this paper is to make up for this hiatus and derive the local behaviour of the ground state of the relativistic operator when $V$ is chosen to be a bounded potential of compact support, and show the extension of our technique to fully supported potentials. Instead of the above operator, we will consider more generally

$$H_{m,\alpha} = (-\Delta + m^{2/\alpha})^{\alpha/2} - m + V$$

on $L^2(\mathbb{R}^d)$, with $0 < \alpha < 2$, $m \geq 0$, and $d \in \mathbb{N}$, and for simplicity we call it in the $m > 0$ case the massive, and for $m = 0$ the massless relativistic Schrödinger operator. In case $V = -v1_K$ with a bounded set $K \subset \mathbb{R}^d$ with non-empty interior, we say that $V$ is a potential well with coupling constant (or depth) $v > 0$. 
The main idea underlying our approach is simple, and it can be highlighted on the case of a spherical potential well \( K = B_a \), where \( B_a \) is a ball of radius \( a \) centered in the origin. When the operator \( H_{m,\alpha} \) has a ground state \( \varphi_0 \) at eigenvalue \( \lambda_0 = \inf \text{Spec} \, H_{m,\alpha} \), a path integral representation gives
\[
e^{-tH_{m,\alpha}}\varphi_0(x) = e^{\lambda_0 t}E^x [e^{-\int_0^t V(X_s)ds} \varphi_0(X_t)], \quad t \geq 0,
\]
for every point \( x \in \mathbb{R}^d \) (see [13]), where now \( \int_0^t V(X_s)ds = -v \int_0^t 1_{B_0}(X_s)ds = -vU_t(a) \) is, apart from the constant prefactor, the occupation measure in the ball of the process \( (X_t)_{t \geq 0} \) starting at \( x \), and \( E^x \) is expectation with respect to its path measure. Clearly, the potential contributes as long as \( X_t \in B_a \) only, thus we may consider the first exit time \( \tau_a = \inf \{ t > 0 : X_t \in B_a^c \} \) when starting from the inside, and the first entrance time \( T_a = \inf \{ t > 0 : X_t \in B_a \} \) when starting from outside of the well. Since, crucially, \( (e^{\lambda_0 t}e^{vU_t(a)}\varphi_0(X_t))_{t \geq 0} \) can be shown to be a martingale, by optional stopping we get
\[
\varphi_0(x) = \begin{cases} 
E^x [e^{(v-|\lambda_0|)\tau_a} \varphi_0(X_{\tau_a})] & \text{if } x \in B_a \\
E^x [e^{-|\lambda_0|T_a} \varphi_0(X_{T_a})] & \text{if } x \in B_a^c.
\end{cases}
\]
(1.1)
When we work with a classical Schrödinger operator having \( -\frac{1}{2} \Delta \) instead of the relativistic operator, so that \( (X_t)_{t \geq 0} = (B_t)_{t \geq 0} \) is Brownian motion, due to path continuity the random variables \( B_{\tau_a} \) and \( B_{T_a} \) are supported on the boundary of \( B_a \), and \( \varphi_0 \) can be determined exactly. (This is shown in full detail in Section 5.1 below.) When we work with \( H_{m,\alpha} \), then \( (X_t)_{t \geq 0} \) is a jump process and now the supports of \( X_{\tau_a} \) and \( X_{T_a} \) spread over the full sets \( B_a \) and \( B_a^c \), respectively. Nevertheless, since \( |X_{\tau_a}| \leq a \) and \( |X_{T_a}| \geq a \), using that \( \varphi_0 \) is (in a spherical potential well, radially) monotone decreasing, the expressions (1.1) yield good approximations. Indeed, our main goal in this paper is to derive precise estimates of these functionals and show how they give tight two-sided bounds on the ground states. We note that while for the classical Schrödinger operator one, though not the only, way to obtain (1.1) is through the actual solution of the eigenvalue equation, which is a PDE, this route for \( H_{m,\alpha} \) is unworkable as the solution of a similar non-local equation is unavailable even for the simplest choices of potential well. Thus the probabilistic alternative which we develop in this paper will prove to be useful in serving this purpose.

To derive bulk estimates of the ground state, we go through these steps systematically leading to the following main results.

1. **Symmetry properties of the ground state.** It is intuitively clear that the ground state should inherit the symmetry properties of the potential well, which is also a technically relevant ingredient in deriving local estimates. In Theorem 4.1 we show rotational symmetry of the ground state when the potential well is a ball, and in Theorem 4.2 reflection symmetry when the potential well has the same symmetry with respect to a hyperplane.

2. **Local estimates of the ground state.** In Theorem 5.1 we prove that, like anticipated above, (1.1) allow to derive two-sided bounds and the ground state of \( H_{m,\alpha} \) with \( m \geq 0 \) can be approximated like
\[
\varphi_0(x) \approx \begin{cases} 
\varphi_0(a)E^x [e^{(v-|\lambda_0|)\tau_a}] & \text{if } x \in B_a \\
\varphi_0(a)E^x [e^{-|\lambda_0|T_a}] & \text{if } x \in B_a^c.
\end{cases}
\]
(1.2)
where \( a = (a,0,\ldots,0) \), and the dependence of the comparability constants on the parameters of the non-local operator, potential well and spatial dimension can be tracked throughout. By deriving
precise two-sided estimates on the moment generating function of $\tau_a$ and the Laplace transform of $T_a$ in Section 3, we can make the expressions more explicit and obtain

$$
\frac{\varphi_0(x)}{\varphi_0(a)} \approx \begin{cases} 
1 + \frac{x - |\lambda|}{\lambda - (a + |\lambda|)} \left( \frac{a - |x|}{a} \right)^{\alpha/2} & \text{if } x \in B_a \\
\bar{j}_{m,\alpha}(|x|) & \text{if } x \in B_a^c
\end{cases}
$$

(1.3)

see Corollary 5.1 where $\bar{j}_{m,\alpha}$ denotes the jump kernel of the operator $L_{m,\alpha}$ (see details in Section 2.1), and $\lambda_a = \lambda_a(m, \alpha)$ is its principal Dirichlet eigenvalue for $B_a$. While the comparability constants depend on $m$, inside the potential well the $x$-dependence is the same for both the massless and massive cases, reflecting the fact that the two processes are locally comparable. Since by using the $L^2$-normalization condition on the ground state the value $\varphi_0(a)$ can further be estimated from both sides (Proposition 5.3), the right hand side above actually provides bounds on $\varphi_0$ itself, with a new proportionality constant (Corollary 5.2). As an application of the information on the local behaviour, in Propositions 5.4-5.5 below we estimate the ground state expectations $\Lambda_a$ with a new proportionality constant (Corollary 5.2). As an application of the information on the local behaviour of (1.2)-(1.3) to bounded decaying potentials supported everywhere in $\mathbb{R}^d$, giving estimates of $\varphi_0$ on appropriate level sets of the potential.

Using all this information, we also get some insight into the mechanisms driving these two regimes of behaviour:

(i) **Inside the potential well.** Since we show that $(a - |x|)^{\frac{\alpha}{2}} \sim \mathbb{E}^x[\tau_a]$, from (1.3) we see that the behaviour of $\varphi_0(x)/\varphi_0(a)$ is essentially determined by the ratio $\mathbb{E}^x[\tau_a]/\mathbb{E}^0[\tau_a]$ of mean exit times. Note that this is different from the case of the classical Schrödinger operator with the same potential well (see Section 5.1 below). For Brownian motion in $\mathbb{R}^d$ it is well known that $\mathbb{E}^x[\tau_a] = \frac{1}{a^2}(a^2 - |x|^2)$ and the moment generating function of $\tau_a$ for $d = 1$ is given by $\mathbb{E}^x[e^{u\tau_a}] = \cos(\sqrt{2ua})/\cos(\sqrt{2ua})$ (and Bessel functions for higher dimensions, see Remark 5.1 below), thus the relation $\varphi_0(x)/\varphi_0(a) \approx \mathbb{E}^x[\tau_a]/\mathbb{E}^0[\tau_a]$ no longer holds and the higher order moments of $\tau_a$ contribute significantly. The reason for this can be appreciated to be that the $\alpha$-stable and relativistically $\alpha$-stable processes related to $L_{m,\alpha}$ and $L_{0,\alpha}$ respectively, have a different nature from Brownian motion. Indeed, we have shown previously that these two processes satisfy the jump-paring property, i.e., that all multiple large jumps are stochastically dominated by single large jumps, while Brownian motion evolves through typically small increments and builds up “backlog events” inflating sojourn times (for the definitions and discussion see [35 Sect. 2.1], [36 Def. 2.1, Rem. 4.4]). Furthermore, it is also seen from (1.3) that the ratio between the maximum $\varphi_0(0)$ of the ground state and $\varphi_0(a)$ is determined by $\frac{\lambda}{\lambda - (a - |\lambda|)}$, i.e., in fact the ratio of the gap between the ground state energy from the minimum value of the potential and the energy necessary to climb and leave the well.

(ii) **Outside the potential well.** The behaviour outside is governed by the Lévy measure which was shown in [36] for large enough $|x|$ and we see here by a different approach that this already sets in from the boundary of the potential well. This is heuristically to be expected due to free motion everywhere outside the well, while to see a “second order” contribution of


non-locality (distinguishing between polynomially vs exponentially decaying jump measures) around the boundary of the well would need more refined tools.

(iii) At the boundary of the potential well. From the profile functions given by (1.3) it can be conjectured that, although the ground state is continuous (see Section 2.2 below), its change of behaviour around the potential well is rather abrupt. Indeed, in Theorem 5.2 and Remark 5.4 we show that at the boundary

$$\varphi_0 \notin C^{\alpha + \delta}_{\text{loc}}(B_{a+\varepsilon} \setminus B_{a-\varepsilon})$$

for every $\delta \in (0, 1 - \alpha)$ whenever $\alpha \in (0, 1)$, and $\varphi_0 \notin C^{1,\alpha + \delta - 1}_{\text{loc}}(B_{a+\varepsilon} \setminus B_{a-\varepsilon})$ for every $\delta \in (0, 2 - \alpha)$ whenever $\alpha \in [1, 2)$, for any small $\varepsilon > 0$. This implies that for the range of small $\alpha$ the ground state cannot be $C^1$ at the boundary, and for values of $\alpha$ starting from 1 it cannot be $C^2$ at the boundary.

(3) Entrance/exit time estimates. All these results depend on precise two-sided estimates on the moment generating function for exit times from balls, and the Laplace transform of hitting times for balls, which we provide here (Section 3). Clearly, these are of independent interest in probabilistic potential theory; for further applications see [24] on crossing times of subordinate Bessel processes.

For the remaining part of the paper, we proceed in Section 2 to a precise description of the operators and processes, and in Section 3 to presenting the details of hitting/exit time estimates. Then in Section 4 we show the martingale property mentioned above and symmetry of the ground state, and in Section 5 derive the local estimates, regularity results and study the moments of the position in the ground states.

2. Preliminaries

2.1. The massive and massless relativistic operators

Let $\alpha \in (0, 2)$, $m \geq 0$, $\Phi_{m,\alpha}(z) = (z + m^{2/\alpha})^{\alpha/2} - m$ for every $z \geq 0$, and denote

$$L_{m,\alpha} = \Phi_{m,\alpha}(-\Delta) = (-\Delta + m^{2/\alpha})^{\alpha/2} - m \quad \text{if } m > 0$$
$$L_{0,\alpha} = \Phi_{0,\alpha}(-\Delta) = (-\Delta)^{\alpha/2} \quad \text{if } m = 0.$$

We will combine the notation into just $L_{m,\alpha}$, $m \geq 0$, when a statement refers to both cases. These operators can be defined in several possible ways. We define them via the Fourier multipliers

$$(\hat{L}_{m,\alpha}f)(y) = \Phi_{m,\alpha}(|y|^2)\hat{f}(y), \quad y \in \mathbb{R}^d, \ f \in \text{Dom}(L_{m,\alpha}),$$

with domain

$$\text{Dom}(L_{m,\alpha}) = \left\{ f \in L^2(\mathbb{R}^d) : \Phi_{m,\alpha}(|\cdot|^2)\hat{f} \in L^2(\mathbb{R}^d) \right\}, \ m \geq 0.$$ 

Then for $f \in C^\infty_c(\mathbb{R}^d)$ the expressions

$$L_{m,\alpha}f(x) = -\lim_{\varepsilon \downarrow 0} \int_{|y-x| > \varepsilon} (f(y) - f(x)) \nu_{m,\alpha}(dy)$$

hold, with the Lévy measures

$$\nu_{m,\alpha}(dx) = j_{m,\alpha}(|x|)dx = \frac{2^{\alpha-d}m^{d/2} \alpha}{\pi^{d/2} \Gamma(1 - \frac{d}{2})} \frac{K_{(d+\alpha)/2}(m^{1/\alpha}|x|)}{|x|^{(d+\alpha)/2}} \ dx, \ x \in \mathbb{R}^d \setminus \{0\},$$
for $m > 0$ (relativistic fractional Laplacian), and
\[
\nu_{0, \alpha}(dx) = j_{0, \alpha}(|x|)dx = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2} |\Gamma\left(-\frac{\alpha}{2}\right)|} \frac{dx}{|x|^{d+\alpha}}, \quad x \in \mathbb{R}^d \setminus \{0\}
\]
for $m = 0$ (fractional Laplacian). Here
\[
K_{\rho}(z) = \frac{1}{2} \left(\frac{z}{2}\right)^{\rho} \int_0^\infty t^{-\rho-1}e^{-t-\frac{z^2}{4t}}dt, \quad z > 0, \quad \rho > -\frac{1}{2},
\]
is the standard modified Bessel function of the third kind. The operator $L_{m, \alpha}$ is positive, and self-adjoint with core $C_c^\infty(\mathbb{R}^d)$, for every $0 < \alpha < 2$ and $m \geq 0$.

The difference of the massive and massless operators is bounded, and the relationship can be made explicit, which will be useful below. For $m, r > 0$ denote
\[
\sigma_{m, \alpha}(r) = \frac{\alpha^2 r^{1-\frac{d+\alpha}{2}}}{\Gamma\left(1-\frac{\alpha}{2}\right) \pi^{\frac{d}{2}}} \left( \frac{2^{\frac{d+\alpha}{2}-1} \Gamma\left(\frac{d+\alpha}{2}\right)}{r^{d+\alpha}} - \frac{m^{\frac{d+\alpha}{2}}}{r^{\frac{d+\alpha}{2}}} K_{\frac{d+\alpha}{2}}\left(m^{1/\alpha} r\right) \right)
\]
and define the measure
\[
\Sigma_{m, \alpha}(A) = \int_A \sigma_{m, \alpha}(|x|)dx,
\]
for all Borel sets $A \subset \mathbb{R}^d$. It can be shown that $\Sigma_{m, \alpha}$ is finite, positive and has full mass $\Sigma_{m, \alpha}(\mathbb{R}^d) = m$. For every function $f \in L^\infty(\mathbb{R}^d)$ consider the operator
\[
G_{m, \alpha} f(x) = \frac{1}{2} \int_{\mathbb{R}^d} (f(x+h) - 2f(x) + f(x-h)) \sigma_{m, \alpha}(|h|)dh.
\]
which is well-defined and $\|G_{m, \alpha} f\|_\infty \leq 2m \|f\|_\infty$ holds. Then the decomposition
\[
j_{0, \alpha}(r) = j_{m, \alpha}(r) + \sigma_{m, \alpha}(r) \quad (2.2)
\]
holds, which implies the formula
\[
L_{m, \alpha} f = L_{0, \alpha} f - G_{m, \alpha} f,
\]
for every function $f$ belonging to the domain of $L_{m, \alpha}$. For the details and proofs we refer to [3, Sect. 2.3.2], see also [51, Lem. 2].

Next consider the multiplication operator $V : \mathbb{R}^d \to \mathbb{R}$ on $L^2(\mathbb{R}^d)$, which plays the role of the potential. In case $V = -v1_K$ with a bounded set $K \subset \mathbb{R}^d$ having a non-empty interior, we say that $V$ is a potential well with coupling constant $v \geq 0$. Since such a potential is relatively bounded with respect to $L_{m, \alpha}$, the operator
\[
H_{m, \alpha} = L_{m, \alpha} - v1_K \quad (2.3)
\]
can be defined by standard perturbation theory as a self-adjoint operator with core $C_c^\infty(\mathbb{R}^d)$. For simplicity, we call $H_{m, \alpha}$ the (massive or massless) relativistic Schrödinger operator with potential well supported in $K$, no matter the value of $\alpha \in (0, 2)$.

Below we will use the following notations. For two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ we write $f(x) \asymp g(x)$ if there exists a constant $C \geq 1$ such that $(1/C)g(x) \leq f(x) \leq Cg(x)$. We denote $f(x) \sim g(x)$ as $|x| \to \infty$ (resp. if $|x| \downarrow 0$) if $\lim_{|x| \to \infty} \frac{f(x)}{g(x)} = 1$ (resp. if $\lim_{|x| \downarrow 0} \frac{f(x)}{g(x)} = 1$). Finally, we denote $f(x) \approx g(x)$ as $|x| \to \infty$ (analogously for $|x| \downarrow 0$) if there exists a constant $C \geq 1$ such that
(1/C) \leq \lim\inf_{|x| \to \infty} f(x)/g(x) \leq \lim\sup_{|x| \to \infty} f(x)/g(x) \leq C. Also, we will use the notation \( B_r(x) \) for a ball of radius \( r \) centered in \( x \in \mathbb{R}^d \), write just \( B_r \) when \( x = 0 \), and \( \omega_d = |B_1| \) for the volume of a \( d \)-dimensional unit ball. Moreover, for a domain \( \mathcal{D} \subset \mathbb{R}^d \) we write \( \mathcal{D}^c \) to denote \( \mathbb{R}^d \setminus \mathcal{D} \). In proofs we number the constants in order to be able to track them, but the counters will be reset in a subsequent statement and proof. Also, in the statements to follow, we will use the default assumptions \( 0 < \alpha < 2 \) and \( m \geq 0 \) implicitly, unless specified otherwise.

2.2. Feynman-Kac representation and the related random processes

The operators \(-L_{m,\alpha}\) are Markov generators and give rise to the following Lévy processes, which can be realised on the space of càdlàg paths (i.e., the space of functions that are continuous from the right with left limits), indexed by the positive semi-axis. To ease the notation, we denote these processes by \((X_t)_{t \geq 0}\) without subscripts, and it will be clear from the context which process it refers to. Also, we denote by \( \mathbb{P}^x \) the probability measure on the space of càdlàg paths, induced by the process \((X_t)_{t \geq 0}\) starting from \( x \in \mathbb{R}^d \), by \( \mathbb{E}^x \) expectation with respect to \( \mathbb{P}^x \), and simplify the notations to \( \mathbb{P} \) and \( \mathbb{E} \) when \( x = 0 \). We will also use the notation \( \mathbb{E}^x[f(X_t); \text{conditions}] \) to mean \( \mathbb{E}[f(X_t)1_{\{\text{conditions}\}}] \).

If \( m > 0 \), the operator \(-L_{m,\alpha}\) generates a rotationally invariant relativistic \( \alpha \)-stable process \((X_t)_{t \geq 0}\); and if \( m = 0 \), the operator \(-L_{0,\alpha}\) generates a rotationally invariant \( \alpha \)-stable process \((X_t)_{t \geq 0}\). Thus in either case

\[
P_tf(x) := (e^{-tL_{m,\alpha}} f)(x) = \mathbb{E}^x[f(X_t)], \quad x \in \mathbb{R}^d, t \geq 0, f \in L^2(\mathbb{R}^d),
\]

holds, giving rise to the Markov semigroup \( \{P_t : t \geq 0\} \). Each \( P_t, t > 0 \), is an integral operator with translation invariant integral kernel \( p(t, x, y) := p_t(x-y), \) i.e., \( P_tf(x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)dy \) for all \( f \in L^p(\mathbb{R}^d), 1 \leq p \leq \infty \). Also,

\[
\mathbb{E}[e^{iu \cdot X_t}] = e^{it\Phi_{m,\alpha}(|u|^2)}, \quad u \in \mathbb{R}^d, m \geq 0,
\]

so that \( \Phi_{m,\alpha}(|u|^2) = (|u|^2 + m^{2/\alpha})^{\alpha/2} - m, m > 0 \), gives the characteristic exponent of the rotationally invariant relativistic \( \alpha \)-stable process, which has the Lévy jump measure \( \nu_{m,\alpha}(dx) \), and \( \Phi_{0,\alpha}(|u|^2) = |u|^\alpha \) gives the characteristic exponent of the rotationally invariant \( \alpha \)-stable process, which has the Lévy jump measure \( \nu_{0,\alpha}(dx) \). From a straightforward analysis it can be seen that for small \( |x| \) the Lévy intensity \( j_{m,\alpha}(x) \) behaves like \( j_{0,\alpha}(x) \), but due to \( K_p(x) \sim C|x|^{-1/2}e^{-|x|} \) as \( |x| \to \infty \) for a suitable constant \( C > 0 \), it decays exponentially, while \( j_{0,\alpha}(x) \) is polynomial. This difference in the behaviours has a strong impact on the properties of the two processes.

The main object of interest in this paper are the ground states \( \varphi_0 \) of the operators \( H_{m,\alpha} \) as given by (2.3), i.e., non-zero solutions of the eigenvalue equation

\[
H_{m,\alpha}\varphi_0 = \lambda_0\varphi_0
\]

corresponding to the lowest eigenvalue, so that \( \varphi_0 \in \text{Dom}(H_{m,\alpha}) \setminus \{0\} \) and \( \lambda_0 = \inf \text{Spec} H_{m,\alpha} \), whenever they exist. Since the potentials \( V = -v1_{\mathcal{K}} \) are relatively compact perturbations of \( H_{m,\alpha} \), the essential spectrum is preserved, and thus \( \text{Spec} H_{m,\alpha} = \text{Spec}_{\text{ess}} H_{m,\alpha} \cup \text{Spec}_{\text{d}} H_{m,\alpha} \), with \( \text{Spec}_{\text{ess}} H_{m,\alpha} = \text{Spec}_{\text{ess}} L_{m,\alpha} = [0, \infty) \). The existence of a discrete component depends on further
details of the potential. Generally, \( \text{Spec}_d H_{m,\alpha} \subset (-\nu, 0) \), and \( \text{Spec}_d H_{m,\alpha} \) consists of a finite set of isolated eigenvalues of finite multiplicity each.

For non-positive compactly supported potentials it is known that \( \text{Spec}_d H_{m,\alpha} \neq \emptyset \) if \( (X_t)_{t \geq 0} \) is a recurrent process \([15, 45] \text{ Th. 4.308}\), i.e., \( H_{m,\alpha} \), \( m > 0 \), does have a ground state \( \varphi_0 \) in every such case for all \( \nu > 0 \). Recall the Chung-Fuchs criterion of recurrence, which says that for a process with characteristic exponent \( \Psi \) the condition \( \int_{|u| < r} \frac{du}{\Psi(u)} < \infty \) for some \( r > 0 \), is equivalent with the transience of the process \([52] \text{ Cor. 37.17}, [45] \text{ Th. 3.84}\). An application to the processes above gives that the relativistic \( \alpha \)-stable process is recurrent whenever \( d = 1 \) or 2, and transient for \( d \geq 3 \), while the \( \alpha \)-stable process is recurrent in case \( d = 1 \) and \( \alpha \geq 1 \), and transient otherwise. In the transient cases, \([4] \text{ Prop. 2.7}\) guarantees that for sufficiently large \( \nu \) (for instance, \( \nu > \lambda_\mathcal{K} \), where \( \lambda_\mathcal{K} \) is the principal Dirichlet eigenvalue of \( L_{m,\alpha} \) over the well \( \mathcal{K} \)) a ground state exists. Furthermore, by \([4] \text{ Lem. 4.5}\) we know that \( \nu + \lambda_0 < \lambda_\mathcal{K} \).

Whenever a ground state \( \varphi_0 \) of the operator \( H_{m,\alpha} \) exists, a Feynman-Kac type representation

\[
e^{-tH_{m,\alpha}} \varphi_0(x) = e^\lambda_0 t \mathbb{E}^x \left[ e^{-\int_0^t V(X_s) \, ds} \varphi_0(X_t) \right] = e^\lambda_0 t \mathbb{E}^x \left[ e^{U_t^\mathcal{K}(X)} \varphi_0(X_t) \right], \quad x \in \mathbb{R}^d, \ t \geq 0 \tag{2.4}\]

holds, where

\[
U_t^\mathcal{K}(X) = \int_0^t 1_{\mathcal{K}(X_s)} \, ds
\]

is the occupation measure of the set \( \mathcal{K} \) by \( (X_t)_{t \geq 0} \). For the details and proofs we refer to \([45] \text{ Sect. 4.6}\). For the non-local Schrödinger operators \( H_{m,\alpha} \) the semigroup \( \{T_t : t \geq 0\} \), \( T_t = e^{-tH_{m,\alpha}} \), is well-defined and strongly continuous. For all \( t > 0 \), every \( T_t \) is a bounded operator on every \( L^p(\mathbb{R}^d) \) space, \( 1 \leq p \leq \infty \). By \([45] \text{ Prop. 4.291}\) the operators \( T_t : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty \), \( T_t : L^p(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d) \) for \( 1 < p \leq \infty \), and \( T_t : L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d) \) are bounded, for all \( t > 0 \). Also, \( T_t \) has a bounded measurable integral kernel \( q(t, x, y) \) for all \( t > 0 \), i.e., \( T_t f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) \, dy \), for all \( f \in L^p(\mathbb{R}^d), \ 1 \leq p \leq \infty \).

Again by \([45] \text{ Prop. 4.291}\), for all \( t > 0 \) and \( f \in L^\infty(\mathbb{R}^d), \ T_t f \) is a bounded continuous function. Thus all the eigenfunctions of \( H_{m,\alpha} \) are bounded and continuous, whenever they exist. Also, they have a pointwise decay to zero at infinity, and the asymptotic behaviour

\[
\varphi_0(x) \approx j_{m,\alpha}(x) \left\{ \begin{array}{ll}
A_{d,\alpha} |x|^{-(d-\alpha)} & \text{for } m = 0 \\
|x|^{-(d+\alpha+1)/2} e^{-m^{1/\alpha}|x|} & \text{for } m > 0
\end{array} \right.
\]

holds, with \( A_{d,\alpha} = \frac{2^{\alpha} \Gamma(d+\alpha)}{\pi^{d/2} \Gamma(1-\frac{d}{2})} \). For further details we refer to \([36]\). Furthermore, it can be shown that if a ground state exists \( \varphi_0 \) for \( H_{m,\alpha} \), then due to the positivity improving property of the Feynman-Kac semigroup \( \varphi_0 \) is unique and has a strictly positive version, which we will choose throughout this paper. For details we refer to \([45] \text{ Sects. 4.3.2, 4.9.1}\).

### 2.3. Heat kernel of the killed Feynman-Kac semigroup

Let \( \mathcal{D} \subset \mathbb{R}^d \) be an open set and consider the first exit time

\[
\tau_\mathcal{D} = \inf \{ t > 0 : X_t \notin \mathcal{D} \}
\]

from \( \mathcal{D} \). When \( \mathcal{D} = B_R \) we simplify the notation to \( \tau_R \), while if \( \mathcal{D} = B_R^c \) we use \( T_R \). (From the context the reader will realise the meanings and not confuse this simple notation with the semigroup
operators $T_i$.) The transition probability densities $p_D(t, x, y)$ of the process killed on exiting $\mathcal{D}$ (or heat kernel of the killed semigroup) are given by the Dynkin-Hunt formula

$$p_D(t, x, y) = p_t(x - y) - \mathbb{E}^x [p_{t - \tau_D} (y - X_{\tau_D}) ; \tau_D < t], \quad x, y \in \mathcal{D}. \tag{2.6}$$

The heat kernel $p_D(t, x, y)$ gives rise to the killed Feynman-Kac semigroup $\{P^D_t : t \geq 0\}$ by $P^D_t f(x) = \int_D p_D(t, x, y) f(y) dy$, for all $x \in \mathcal{D}, t > 0$ and $f \in L^2(\mathbb{R}^d)$. It is known that $\{P^D_t : t \geq 0\}$ is a strongly continuous semigroup of contraction operators on $L^2(\mathcal{D})$ and every operator $P^D_t$, $t > 0$, is self-adjoint.

Below we will make frequent use of the Ikeda-Watanabe formula [34, Th. 1], which says that for every $\eta > 0$ and every bounded or non-negative Borel function $f$ on $\mathbb{R}^d$, the equality

$$\mathbb{E}^x [e^{-\eta \tau_D} f(X_{\tau_D})] = \int_{\mathcal{D}} \int_0^\infty e^{-\eta t} p_D(t, x, y) dt \int_{\mathbb{R}^d} f(z) j_{m, \alpha}(z - y) dz dy, \quad x \in \mathcal{D},$$

holds. The same arguments leading to the above expression also allow the more general formulation (see, for instance, [10, eq. (1.58)] and [34, Th. 2])

$$\mathbb{E}^x [f(\tau_D, X_{\tau_D}, X_{\tau_D})] = \int_{\mathcal{D}} \int_{\mathbb{R}^d} \int_0^\infty p_D(t, x, y) f(t, y, z) j_{m, \alpha}(z - y) dt dz dy, \quad x \in \mathcal{D}, \tag{2.7}$$

which holds for every bounded or non-negative Borel function $f : [0, \infty] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We will keep referring to this as the Ikeda-Watanabe formula.

In what follows we will rely also on some estimates of the heat kernel of the killed semigroup. By (2.6), clearly $p_D(t, x, y) \leq p_t(x - y)$ for all $t > 0$ and $x, y \in \mathcal{D}$. Recall that the semigroup $\{P^D_t : t \geq 0\}$ is said to be intrinsically ultracontractive (IUC) whenever there exists $C^D_0 > 0$ such that $p_D(t, x, y) \leq C^D_0 f_D(x) f_D(y)$, for all $t > 0$ and $x, y \in \mathcal{D}$, where $f_D$ is the principal Dirichlet eigenfunction of the operator $L_{m, \alpha}$ in the domain $\mathcal{D}$. It can be shown that if $\{P^D_t : t \geq 0\}$ is IUC, then a similar lower bound holds with another constant. The following result provides a bound on $p_t(x)$, and will be useful for the IUC property of $\{P^D_t : t \geq 0\}$ for a class of domains $\mathcal{D}$ that we will use below.

**Lemma 2.1.** For every $\delta > 0$ there exists a constant $C_{d, m, \alpha}(\delta)$ such that

$$\sup_{|x| \geq \delta, t > 0} p_t(x) \leq C_{d, m, \alpha}(\delta).$$

**Proof.** Fix $\delta > 0$. By [51, eq. (9)] we know that

$$p_t(x) \leq C^{(1)}_1 e^{m t} \frac{2^\delta \Gamma(\frac{d + \alpha}{2})}{\pi^{d/2} |x|^{d + \alpha}}.$$

Thus for $t \leq 1$ and $|x| \geq \delta$ we obtain

$$p_t(x) \leq C^{(1)}_1 e^{m} \frac{2^\delta \Gamma(\frac{d + \alpha}{2})}{\pi^{d/2} \delta^{d + \alpha}} =: C^{(2)}_{d, m, \alpha}(\delta).$$

For $t \geq 1$ we distinguish two cases. If $m = 0$, we use the estimate (see, for instance, [10])

$$p_t(x) \leq C^{(3)}_{d, \alpha} t^{-\frac{\alpha}{2}} \leq C^{(3)}_{d, \alpha}, \quad t > 1.$$

If $m > 0$, we can use [51, Lem. 3] to conclude that

$$p_t(x) \leq C^{(4)}_{d, m, \alpha} \left( m^{\frac{d}{2} - \frac{d}{2} t^{-\frac{\alpha}{2}} + t^{-\frac{d}{2}} \right) \leq C^{(5)}_{d, m, \alpha}, \quad t > 1.$$
Hence we can define
\[
C_{d,m,\alpha}(\delta) = \begin{cases} 
\max \{ C_{d,0,\alpha}^{(2)}(\delta), C_{d,\alpha}^{(3)} \} & m = 0 \\
\max \{ C_{d,m,\alpha}^{(2)}(\delta), C_{d,m,\alpha}^{(4)} \} & m > 0,
\end{cases}
\]
giving \( \sup_{|x| \geq \delta, t > 0} p_t(x) \leq C_{d,m,\alpha}(\delta) \) for every \( m \geq 0 \).
\[\Box\]

Using that \( \nu_{m,\alpha}(B_r(x)) > 0 \) for every \( x \in \mathbb{R}^d, r > 0 \) and \( m \geq 0 \), we immediately get the following result from the previous lemma and \[29, \text{Th. 3.1}\].

**Corollary 2.1.** Let \( D \) be a bounded Lipschitz domain. The killed semigroup \( \{ P_t^D : t \geq 0 \} \) is IUC.

We will denote the principal Dirichlet eigenfunction of \( L_{m,\alpha} \) by \( f_{R}(x) \) whenever \( D = B_R \). Using IUC and its implication of a similar lower bound, and the continuity of the killed heat kernel, it can be shown \[22, \text{Th. 4.2.5}\] that there exists a large enough \( T > 0 \) such that
\[
\frac{1}{2} e^{-\lambda_R t} f_{R}(x) f_{R}(y) \leq p_{B_R}(t, x, y) \leq \frac{3}{2} e^{-\lambda_R t} f_{R}(x) f_{R}(y),
\]
for all \( t > T \) and \( x, y \in B_R \).

### 3. Exit and hitting times estimates

#### 3.1. Estimates on the survival probability

As we will see below, the local behaviour of ground states depends on a function which can be estimated by using tools of potential theory for the stable and relativistic stable processes. We will denote this by \( \mathcal{V}_{\alpha, m} \) and call it rate function. In this section we derive some key information on this function first. The results contained in this subsection have been obtained in a more general context in \[11\]. Since here we are considering two specific cases, which are widely used in applications, we reconsider some of the proofs in order to identify the values of the involved constants, which are not explicit in the cited work due to the greater generality of the arguments involved.

**Lemma 3.1.** Let \( D \) be a \( C^{1,1} \) bounded open set in \( \mathbb{R}^d \), \( (X^{(0)}_t)_{t \geq 0} \) be an isotropic \( \alpha \)-stable process and \( (X^{(m)}_t)_{t \geq 0} \) be an isotropic relativistic \( \alpha \)-stable process with mass \( m > 0 \). Consider the first exit time \( \tau_D^{(m)} = \inf \{ t > 0 : X^{(m)}_t \notin D \} \), \( m \geq 0 \). Then \( \mathbb{E}^{x}[\tau_D^{(m)}] = \mathbb{E}^{x}[\tau_D^{(0)}] \), for every \( m > 0 \) and the comparability constant is independent of \( D \).

**Proof.** The statement easily follows from \[16, \text{Cor. 1.2}\] and \[17, \text{Th. 1.3}\] due to the comparability of the respective Green functions.
\[\Box\]

As a consequence, we get the following upper bound.

**Corollary 3.1.** We have \( \lambda_R R^\alpha \leq C_{d,m,\alpha} \).

**Proof.** Denote \( s(x) = \mathbb{E}^{x}[\tau_R] \) and \( S = \| s \|_{L^2(B_R)} \). First consider the case \( m = 0 \). Then the explicit formula due to M. Riesz (e.g., \[10\] eq. (1.56))
\[
s(x) = \frac{\pi^{\frac{1+d}{2}} \Gamma \left( \frac{d}{2} \right) \sin \left( \frac{\pi \alpha}{2} \right) \Gamma \left( \frac{-\alpha}{2} \right) \left( R^2 - |x|^2 \right)^{\alpha/2}}{2^\alpha \Gamma \left( \frac{d+\alpha}{2} \right)}, \quad |x| \leq R,
\]
holds. Hence we have
\[ s(x) \geq C_{d,m,\alpha}R^\alpha, \quad |x| \leq \frac{R}{2}. \tag{3.1} \]
Lemma 3.1 guarantees that (3.1) holds even for \( m > 0 \). Thus in general we have \( S^2 \geq C_{d,m,\alpha}R^{2\alpha}|B_{R/2}|. \) By [5 Prop. 2.1] and Schwarz inequality we then obtain
\[
\lambda_R \leq \int_{B_R} \frac{s(x)}{S^2} \, dx \leq \sqrt{\frac{|B_R|}{S^2}} \leq \frac{C_{d,m,\alpha}}{R^\alpha}.
\]

We say that a function \( f : \mathbb{R}^d \to \mathbb{R} \) is \((m, \alpha)\) -harmonic on an open set \( D \subset \mathbb{R}^d \) if for every open set \( U \subset D \) (i.e., \( \overline{U} \subset D \) is compact) the equality \( f(x) = \mathbb{E}^x[f(X_{\tau_D})] \) holds for every \( x \in U \). In the following we come back to the notation by \((X_t)_{t \geq 0}\) meaning either of the processes for the massless and massive cases, as used previously.

**Lemma 3.2.** Let \( d = 1 \) and fix \( r_0 > 0 \). There exist an increasing concave (and thus subadditive) \((m, \alpha)\) -harmonic function \( \mathcal{V}_{m,\alpha}(r) : (0, \infty) \to \mathbb{R}^+ \) and constants \( 0 < C_{m,\alpha,r_0}^{(1)} < C_{m,\alpha,r_0}^{(2)} \) such that
\[
C_{m,\alpha,r_0}^{(1)} r^{\alpha/2} \leq \mathcal{V}_{m,\alpha}(r) \leq C_{m,\alpha,r_0}^{(2)} r^{\alpha/2}, \quad 0 \leq r \leq r_0.
\]

**Proof.** Consider the running supremum \( M_t = \sup_{0 \leq s \leq t} X_s \) and let \( Y_t = M_t - X_t \) be the process obtained by reflecting \( X_t \) on hitting the supremum. Let \( A_t \) be the local time at zero of \( Y_t \), and \( Z_t = \inf \{ \tau > 0 : A_\tau > t \} \) its right-continuous inverse. Also, consider \( H_t = M_{Z_t} \). By [55 eq. (1.8)] there exists a function \( \psi_{m,\alpha} \) such that \( \int_0^\infty \psi_{m,\alpha}(s)f(s)\,ds = \int_0^\infty \mathbb{E}[f(H_s)]\,ds \), for every non-negative Borel function \( f \). Choosing in particular \( f = 1_{[0,r]} \), we define
\[
\mathcal{V}_{m,\alpha}(r) = \int_0^r \psi_{m,\alpha}(\rho)\,d\rho = \int_0^\infty \mathbb{P}(H_\rho \leq r)\,d\rho.
\]

Note that \((H_t)_{t \geq 0}\) is a subordinator (see [8 Lem. VI.2]), different from a Poisson process since \((0, \infty)\) is a regular domain for \((X_t)_{t \geq 0}\). We can define its inverse subordinator \( H_t^{-1} := \inf \{ s > 0 : H_s > t \} \) and observe that \( \mathcal{V}_{m,\alpha}(t) = \mathbb{E}[H_t^{-1}] \), implying subadditivity of \( \mathcal{V}_{m,\alpha} \) (see [8 Ch. III]). The fact that \( \mathcal{V}_{m,\alpha} \) is \((m, \alpha)\)-harmonic in \((0, \infty)\) follows from [55 Th. 2]. The comparability result follows by [38 Prop. 2.2, Ex. 2.3]. Concavity results by [38 Prop. 2.1] and [53 Th. 10.3] as \( \psi_{m,\alpha} = \mathcal{V}_{m,\alpha}' \) is non-increasing.

**Remark 3.1.** In fact, \( \mathcal{V}_{0,\alpha}(r) = r^{\alpha/2} \). Moreover, for \( m > 0 \) again by [38 Prop. 2.2 and Ex. 2.3] we get \( \mathcal{V}_{m,\alpha}(r) \sim r \) as \( r \to \infty \). As a direct consequence of the monotone density theorem, we furthermore have \( \psi_{m,\alpha}(r) \sim r^{\frac{\alpha}{2} - 1} \) as \( r \downarrow 0 \), for all \( m \geq 0 \).

As a consequence, we obtain the following Harnack-type inequality.

**Lemma 3.3.** For every \( 0 < x \leq y \leq z \leq 5x \) we have
\[
\mathcal{V}_{m,\alpha}(z) - \mathcal{V}_{m,\alpha}(y) \leq 5\mathcal{V}_{m,\alpha}'(x)(z - y).
\]

**Proof.** By Lemma 3.2 we know that \( \mathcal{V}_{m,\alpha} \) is concave and thus, in particular, log-concave. Hence the result follows by [11 Lem. 7.1].

Moreover, we can use the function \( \mathcal{V}_{m,\alpha} \) to derive the following estimate.
Corollary 3.2. Let \( d = 1 \) and define \( \tau_{(0,\infty)} = \inf\{t > 0 : X_t \leq 0\} \). There exist constants \( C_{m,\alpha}^{(1)} \) and \( C_{m,\alpha}^{(2)} \) such that
\[
C_{m,\alpha}^{(1)} \left( \frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \leq \mathbb{P}^x(\tau_{(0,\infty)}>t) \leq C_{m,\alpha}^{(2)} \left( \frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \right)
\]
Proof. Immediate by [39, Cor. 3.2] and Lemma 3.2.

Remark 3.2. In the case \( m = 0 \) it is not difficult to determine explicitly the constant given in Corollary 3.1, while it is clear that the upper and lower bounds in Lemma 3.2 are actually identities. Furthermore, the constants obtained in Corollary 3.2 can be computed exactly to be \( C_{m,\alpha}^{(1)} = \frac{1}{2d} \left( \frac{r^{\alpha/2}}{\sqrt{t}} \right)^2 \) and \( C_{m,\alpha}^{(2)} = \frac{e}{\alpha-1} \), which are independent of \( m \) and \( \alpha \). In fact, as observed in [11], these constants are universal for more general unimodal symmetric Lévy processes. The constants given in the following statements can be, at least in the case \( m = 0 \), tracked from the cited results or numerically evaluated via the principal Dirichlet eigenfunction.

As a direct consequence of Lemmas 3.2, 3.3 we obtain the following lower bound.

Proposition 3.1. For every \( R > 0 \) there exist constants \( C_{d,m,\alpha,R}^{(1)}, C_{d,m,\alpha,R}^{(2)} \) such that
\[
\mathbb{P}^x(\tau_R > t) \geq C_{d,m,\alpha,R}^{(1)} \left( \frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \right), \quad t \leq C_{d}^{(2)} \mathbb{V}_{m,\alpha}(R).
\]
Proof. By Lemma 3.3 and [11, Prop. 6.1] we know that there exist constants \( C_{d,R}^{(2)}, C_{d}^{(3)} > 0 \) such that
\[
\mathbb{P}^x(\tau_R > t) \geq C_{d,R}^{(3)} \left( \frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \right), \quad t \leq C_{d}^{(2)} \mathbb{V}_{m,\alpha}(R).
\]
Lemma 3.2 then completes the proof.

Furthermore, we can derive an upper bound on the survival probability \( \tau_R \).

Lemma 3.4. For every \( x \in B_R \) and \( t > 0 \) we have
\[
\mathbb{P}^x(\tau_R > t) \leq 2 \left( \frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \right).
\]
Proof. Since \( (X_t)_{t \geq 0} \) is rotationally symmetric, we may choose \( x = re_1 \) without loss of generality, where \( e_1 = (1,0,\ldots,0) \) and \( r \in (0,R) \). Let \( \mathcal{H}_R := \{ x \in \mathbb{R}^d : x_1 < R \} \) and consider the first exit time \( \tau_R := \inf\{t > 0 : X_t \in (\mathcal{H}_R)^c \} \). Since \( B_R \subseteq \mathcal{H}_R \), we have \( \tau_R \leq \tau_R \) almost surely. With the same notation \( \tau_{(0,\infty)} \) as in Corollary 3.2 it follows that
\[
\mathbb{P}^x(\tau_R > t) \leq \mathbb{P}^x(\tau_R > t) = \mathbb{P}^{(r-R)e_1}(\tau_0 > t) = \mathbb{P}^{r-R}(\tau_{(0,\infty)} > t) \leq 2 \left( \frac{r^{\alpha/2}}{\sqrt{t}} \wedge 1 \right).
\]

Using intrinsic ultracontractivity of the killed semigroup, we can improve these estimates.

Proposition 3.2. For every \( x \in B_R \), we have
\[
\mathbb{P}^x(\tau_R > t) \asymp e^{-\lambda_R t} \left( \frac{r^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}} \wedge 1 \right),
\]

where the comparability constants depend on $d, m, \alpha, R,$ and $\lambda_R$ is the principal Dirichlet eigenvalue of $L_{m,\alpha}$ in the ball $B_R$.

**Proof.** Since we have already recalled Lemma 3.4 and Proposition 3.1, we only need to prove the exponential domination for large values of $t > 0$. Let $f_R$ be the principal Dirichlet eigenfunction of $L_{m,\alpha}$ for the ball $B_R$ and observe that, by [45, Prop. 4.289], $f_R$ is continuous and bounded. Since the killed semigroup is IUC, see Lemma 2.1, we can choose $T > 0$ such that (2.8) holds for every $t \geq 0$ and $x, u \in B_R$. For this fixed $T$, by [16, Th. 1.1] and [17, Th. 1.1], it follows that there exists a constant $C_{d, m,\alpha, R}^{(1)} > 0$ such that for every $t \geq T$ and $x, u \in B_R$

$$
\frac{1}{C_{d, m,\alpha, R}^{(1)}} e^{-\lambda_R t} (R - |x|) \dot{\gamma} (R - |u|) \dot{\gamma} \leq p_{B_R}(t, x, u) \leq \frac{3}{2} e^{-\lambda_R t} (R - |x|) \dot{\gamma} (R - |u|) \dot{\gamma}
$$

holds. Combining (2.8) and (3.2) we have, for all $x, u \in B_R$,

$$
f_R(x) f_R(u) \geq \frac{2}{3C_{d, m,\alpha, R}^{(1)}} (R - |x|) \dot{\gamma} (R - |u|) \dot{\gamma}.
$$

Taking $x = u = 0$, the previous inequality gives

$$
f_R(0)^2 \geq \frac{2}{3C_{d, m,\alpha, R}^{(1)}} R^\alpha > 0. \tag{3.3}
$$

Furthermore, choosing $u = 0$ in (3.3) we get

$$
f_R(x) \geq \frac{2}{3C_{d, m,\alpha, R}^{(1)}} R^\alpha (R - |x|) \dot{\gamma} =: C_{d, m,\alpha, R}^{(2)} (R - |x|) \dot{\gamma}. \tag{3.4}
$$

Finally, by (2.8) and (3.4) we obtain the lower bound

$$
1 \geq \mathbb{P}^x(\tau_R > t) = \int_t^\infty \int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) p_{B_R}(s, x, u) dudz ds
$$

$$
\geq \frac{e^{-\lambda_R t}}{2\lambda_R} \int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) f_R(x) f_R(u) dudz
$$

$$
\geq \frac{C_{d, m,\alpha, R}^{(2)} (R - |x|) \dot{\gamma} e^{-\lambda_R t}}{2\lambda_R} \int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) f_R(u) dudz.
$$

This guarantees that

$$
\int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) f_R(u) dudz < \infty
$$

and, at the same time,

$$
\mathbb{P}^x(\tau_R > t) \geq \frac{C_{d, m,\alpha, R}^{(2)} (R - |x|) \dot{\gamma} e^{-\lambda_R t}}{2\lambda_R} \int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) f_R(u) dudz =: C_{d, m,\alpha, R}^{(3)} (R - |x|) \dot{\gamma} e^{-\lambda_R t},
$$

for every $x \in B_R$ and $t \geq T$. Similarly, we have the estimate from above,

$$
\mathbb{P}^x(\tau_R > t) \leq \frac{3}{2} \int_t^\infty e^{-\lambda_R s} ds \int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) f_R(x) f_R(u) dudz
$$

$$
\leq \frac{3||f_R||_\infty e^{-\lambda_R t}}{2\lambda_R} \int_{B_R} \int_{B_R} j_{m,\alpha}(|z - u|) f_R(u) dudz =: C_{d, m,\alpha, R}^{(4)} e^{-\lambda_R t}.
$$

□
Next we derive an upper bound for the function $\mathbb{P}^x(T_R > t)$. First we need a technical lemma.

**Lemma 3.5.** There exists a constant $C_{d, m, \alpha} > 0$ such that

$$\nu_{m, \alpha}(\mathcal{B}_r^c) \sim C_{d, m, \alpha} r^{-\alpha}, \quad r \downarrow 0.$$  

**Proof.** There is nothing to prove if $m = 0$, thus take $m > 0$ and for all $\varepsilon > 0$ let $t_0(\varepsilon)$ such that

$$(1 - \varepsilon)C_{d, m, \alpha}^{(1)} r^{-\alpha} \leq \nu_{m, \alpha}(\rho) \leq (1 + \varepsilon)C_{d, m, \alpha}^{(1)} r^{-\alpha} \text{ for every } 0 < \rho < t_0(\varepsilon) \text{ (note that this holds by the 0+ asymptotics of the Bessel function).}$$

Consider $r < t_0(\varepsilon)$ and observe that

$$\nu_{m, \alpha}(\mathcal{B}_r^c) = \int_{t_0(\varepsilon)}^{t_0\varepsilon} \rho^{d-1} j_{m, \alpha}(\rho) d\rho + \int_{t_0(\varepsilon)}^{\infty} \rho^{d-1} j_{m, \alpha}(\rho) d\rho =: I_1(\varepsilon, r) + I_2(\varepsilon).$$

Clearly, $I_2(\varepsilon) < \infty$. Since

$$(1 - \varepsilon)C_{d, m, \alpha}^{(1)} r^{-\alpha} - (1 + \varepsilon)C_{d, m, \alpha}^{(3)} r^{-\alpha} \leq I_1(\varepsilon, r) \leq (1 + \varepsilon)C_{d, m, \alpha}^{(1)} r^{-\alpha} - (1 - \varepsilon)C_{d, m, \alpha}^{(3)} r^{-\alpha},$$

the result follows directly. \qed

**Proposition 3.3.** For every $0 < R < R_0$ there exists a constant $C_{d, m, \alpha, R, R_0} > 0$ such that

$$\mathbb{P}^x(T_R > t) \leq C_{d, m, \alpha, R, R_0} \frac{(|x| - R)^{\alpha/2}}{\sqrt{t \wedge R^{\alpha/2}}}, \quad |x| \in [R, R_0).$$  

**Proof.** Consider the function

$$J_{m, \alpha}(R) = \inf_{0 \leq t \leq R} \nu_{m, \alpha}(\mathcal{B}_r^c) V_{m, \alpha}^2 (r).$$

Observe that $\nu_{m, \alpha}(\mathcal{B}_r^c) V_{m, \alpha}^2 (r) > 0$ for every $r > 0$. Moreover, by Lemmas 3.2 and 3.5 we know that $\nu_{m, \alpha}(\mathcal{B}_r^c) V_{m, \alpha}^2 (r) \geq C_{m, \alpha, r_0} > 0$ for $r_0 > 0$ and $r \in (0, r_0)$. This implies $J_{m, \alpha}(R) > 0$. Lemma 3.3 guarantees that [11, Lem. 6.2] applies and we obtain

$$\mathbb{P}^x(T_R > t) \leq \frac{5C_d V_{m, \alpha}(|x| - R)}{(J(R))^2 \sqrt{t \wedge V_{m, \alpha}(R)}}.$$  

Finally, for $|x| \in (R, R_0)$ we can use Lemma 3.2 to complete the proof. \qed

**Remark 3.3.** Note that in case $m = 0$, there exists a constant $C_{d, \alpha} > 0$ such that $J_{0, \alpha}(R) \geq C_{d, \alpha}$ for every $R$. This follows from the asymptotic behaviour of $\nu_{0, \alpha}(\mathcal{B}_r^c)$ as $r \to \infty$ given in [3, Cor. 2.1]. Thus for the massless case (3.5) holds for all $|x| \geq R$, with no dependence on $R_0$. On the other hand, for $m > 0$ we have $\lim_{R \to \infty} J_{m, \alpha}(R) = 0$. This is due to $V_{m, \alpha}(R) \sim R$ as $R \to \infty$, as seen in Remark 3.1 while $\nu_{m, \alpha}(\mathcal{B}_R^c)$ decays exponentially (see [3, Cor. 2.2]).

### 3.2. Estimates on the moment generating function for the exit time from a ball

In view of deriving and using expressions of the type (1.1) in our main analysis below, in this section first we derive estimates of exponentials of exit times of the Lévy processes $(X_t)_{t \geq 0}$ for balls and their complements. Recall (2.5) and denote by

$$g_{tR}(t) = \int_{\mathcal{B}_R^c} \int_{\mathcal{B}_R} j_{m, \alpha}(|z - u|) p_{tR}(t, x, u) du dz, \quad t > 0,$$

the probability density of $\tau_R$. Now we prove the following estimate for the moment generating function of $\tau_R$. 

Theorem 3.1. Fix $R > 0$. Then for every $0 \leq \lambda < \lambda_R$ and $x \in B_R$ we have
\[
\mathbb{E}^x[e^{\lambda \tau_R} - 1] \leq \frac{\lambda}{\lambda_R - \lambda} \left( \frac{R - |x|}{R} \right)^{\alpha/2},
\]
where the comparability constant depends on $d, m, \alpha, R$. Moreover, $\mathbb{E}^x[e^{\lambda \tau_R}] = \infty$ whenever $\lambda \geq \lambda_R$.

Proof. First fix $0 \leq \lambda < \lambda_R$. Using (3.6) and integrating by parts we obtain
\[
\mathbb{E}^x[e^{\lambda \tau_R} - 1] = \int_0^\infty (e^{\lambda t} - 1) g_{\tau_R}(t) dt = -\lim_{s \to \infty} (e^{\lambda s} - 1) \mathbb{P}(\tau_R > s) + \lambda \int_0^\infty e^{\lambda t} \mathbb{P}(\tau_R > t) dt. \quad (3.7)
\]
Note that the limit is zero since by Proposition 3.2
\[
e^{\lambda s} \mathbb{P}(\tau_R > s) \leq C_{d,m,\alpha,R}^1 e^{(\lambda - \lambda_R)s} \left( \frac{R - |x|}{\sqrt{s} \wedge R^{\alpha/2}} \right)^{\alpha/2},
\]
and $\lambda < \lambda_R$.

First we show the lower bound of the remaining integral at the right hand side of (3.7). Using Proposition 3.2 again, we get
\[
\int_0^\infty e^{\lambda t} \mathbb{P}(\tau_R > t) dt \geq C_{d,m,\alpha,R}^2 \frac{R - |x|}{R} \frac{1}{\lambda_R - \lambda} e^{-(\lambda_R - \lambda)t} dt \quad (3.8)
\]
Next note that by Corollary 3.1 we have $\lambda_R R^\alpha \leq C_{d,m,\alpha}^3$ with a constant $C_{d,m,\alpha}^3$, thus $e^{-(\lambda_R - \lambda)t} \geq C_{d,m,\alpha}^4$. Using this lower bound in (3.8) we get
\[
\int_0^\infty e^{\lambda t} \mathbb{P}(\tau_R > t) dt \geq C_{d,m,\alpha,R}^5 \frac{1}{\lambda_R - \lambda} \left( \frac{R - |x|}{R} \right)^{\alpha/2}.
\]
To get the upper bound, we estimate
\[
\int_0^\infty e^{\lambda t} \mathbb{P}(\tau_R > t) dt \leq C_{d,m,\alpha,R}^1 \int_0^\infty e^{-(\lambda_R - \lambda)t} \left( 1 \wedge \frac{(R - |x|)^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}} \right) dt
\]
\[
= C_{d,m,\alpha,R}^1 \left( R^{\alpha/2} \int_0^\infty e^{-(\lambda_R - \lambda)t} dt + \left( \frac{R - |x|}{R} \right)^{\alpha/2} \frac{1}{\lambda_R - \lambda} \right)
\]
\[
\leq C_{d,m,\alpha,R}^1 \left( R^{\alpha/2} e^{-(\lambda_R - \lambda)t} \right. + \left( \frac{R - |x|}{R} \right)^{\alpha/2} \frac{1}{\lambda_R - \lambda} \)
\]
\[
\leq C_{d,m,\alpha,R}^1 \left( (R - |x|)^{\alpha/2} 2 R^{\alpha/2} e^{-(\lambda_R - \lambda)t} + 2R^{\alpha/2} \int_0^\infty \frac{1}{\lambda_R - \lambda} e^{-(\lambda_R - \lambda)t} dt \right)
\]
\[
\leq C_{d,m,\alpha,R}^1 \left( (R - |x|)^{\alpha/2} 2 R^{\alpha/2} e^{-(\lambda_R - \lambda)t} \right. + \left( \frac{R - |x|}{R} \right)^{\alpha/2} \frac{1}{\lambda_R - \lambda} \)
\]
\[
\leq C_{d,m,\alpha,R}^1 \left( (R - |x|)^{\alpha/2} \frac{\lambda_R}{\lambda_R - \lambda} + \left( \frac{R - |x|}{R} \right)^{\alpha/2} \frac{1}{\lambda_R - \lambda} \right) \leq \frac{C_{d,m,\alpha,R}^{(5)}}{\lambda_R - \lambda} \left( \frac{R - |x|}{R} \right)^{\alpha/2},
\]
where we used the bound \( \lambda_R R^\alpha \leq C_{d,m,\alpha}^{(3)} \) again in the last line. This proves the first part of the claim.

To obtain the second statement we only need to prove that \( \mathbb{E}[e^{\lambda T_R}] = \infty \). Notice that by Proposition 3.2

\[
e^{\lambda R s} \mathbb{P}^x(\tau_R > s) \leq C_{d,m,\alpha,R}^{(1)} \frac{(R - |x|)\alpha/2}{\sqrt{s} \wedge R^{\alpha/2}}.
\]

For \( s > R^\alpha \) we get

\[
\mathbb{E}^x[e^{\lambda T_R}] \geq \mathbb{E}^x[e^{\lambda T_R} - 1; \tau_R \leq s] = \int_0^s (e^{\lambda t} - 1) g_{\tau_R}(t) dt
\]

\[
= -(e^{\lambda R s} - 1) \mathbb{P}^x(\tau_R > s) + \lambda R \int_0^s e^{\lambda R t} \mathbb{P}^x(\tau_R > t) dt
\]

\[
\geq -C_{d,m,\alpha,R}^{(1)} \frac{(R - |x|)\alpha/2}{R^{\alpha/2}} + \lambda R \int_{R^\alpha}^\infty e^{\lambda R t} \mathbb{P}^x(\tau_R > t) dt.
\]

Taking the supremum over \( s \) on the right-hand side and using the lower bound in Proposition 3.2 we obtain

\[
\mathbb{E}^x[e^{\lambda T_R}] \geq -C_{d,m,\alpha,R}^{(1)} \frac{(R - |x|)\alpha/2}{R^{\alpha/2}} + \lambda R \int_{R^\alpha}^\infty \frac{(R - |x|)\alpha/2}{R^{\alpha/2}} dt = \infty.
\]

\[\square\]

### 3.3. Estimates on the Laplace transform of the hitting time for a ball

Next we consider \( T_R = \inf\{t > 0 : X_t \in B_R\} \) and derive estimates on the Laplace transform \( \mathbb{E}^x[e^{-\lambda T_R}] \), in which case there is no handy tool such as intrinsic ultracontractivity of the killed semigroup. We start with a lower bound for points in domains of the type \( R \leq |x| \leq R' \), for the remaining choices of domains see Remark 3.4 (2) below.

**Theorem 3.2.** Let \( \lambda, R > 0 \) and \( R_2 > R_1 > R \). There exists a constant \( C_{d,m,\alpha,R_1,R_2,R,\lambda} > 0 \) such that

\[
\mathbb{E}^x[e^{-\lambda T_R}] \geq C_{d,m,\alpha,R_1,R_2,R,\lambda} j_{m,\alpha}(|x|), \quad R_1 \leq |x| \leq R_2.
\]

**Proof.** Define

\[
C_{d,m,\alpha,R_1,R_2}^{(1)} = \min_{R_1 \leq |x| \leq R_2} \frac{j_{m,\alpha}(|x| + \frac{R}{2})}{j_{m,\alpha}(|x|)}.
\]

As before, fix \( x = r\mathbf{e}_1 \) for \( r > 0 \), and define \( A(x) = \{u \in \mathbb{R}^d : |x| + R < |u| < |x| + 2R, \langle u, \mathbf{e}_1 \rangle < 0\} \). Since \( R_1 \leq |x| \leq R_2 \), taking \( D = B_{3R} \setminus \overline{B}_R \) we see that \( x \in D \subset \overline{B}_R^c \). In particular, \( p_{B_R^c}(t,x,u) \geq p_D(t,x,u) \). Since \( D \) is a bounded and open Lipschitz set, the semigroup with kernel \( p_D(t,x,u) \) is IUC and we can apply it to the lower bound (2.8) with some \( T > 0 \), and the principal Dirichlet eigenvalue and eigenfunction \( \lambda_D \) and \( f_D \) of \( L_{m,\alpha} \) on \( D \). Then by using the Ikeda-Watanabe formula
we get
\[ \mathbb{E}^x \left[ e^{-\lambda T_R} \right] \geq \mathbb{E}^x \left[ e^{-\lambda T_R} ; X_{T_R}^+ \in A(x), \ |X_{T_R}^-| < \frac{R}{2}, T_R > T \right] \]
\[ = \int_T^\infty \int_{B_R/2} \int_{A(x)} e^{-\lambda t} j(|u - z|) p_{B_R}^x(t, x, u) dtdzu \]
\[ \geq \frac{R^d \omega_d}{2^{d+1}} \left( |x| + \frac{5 R}{2} \right) \int_T^\infty \int_{A(x)} e^{-\left( \lambda + \lambda_D \right) t} f_D(x) f_D(u) dtdu \]
\[ \geq \frac{C^{(1)}_{d,m,\alpha,R_1,R_2} R^d \omega_d e^{-\left( \lambda + \lambda_D \right) T}}{\left( \lambda + \lambda_D \right)^{d+1}} j(|x|) f_D(x) \int_{A(x)} f_D(u) du. \] (3.9)

Note that since \( \mathcal{D} \) is a bounded \( C^{1,1} \) domain, by [16, Th. 1.1] and [17, Th. 1.1] there exists a constant \( C^{(2)}_{d,m,\alpha,R_1,R_2} > 1 \) such that for every \( t \geq 1 \)
\[ p_D(t, x, u) \leq C^{(2)}_{d,m,\alpha,R_1,R_2} e^{-\lambda_D \delta_D^{\alpha/2}(x) \delta_D^{\alpha/2}(u)}, \]
holds, where \( \delta_D(x) = \text{dist}(x, \partial \mathcal{D}) \). By definition of \( f_D(x) \) we get
\[ f_D(x) = e^{\lambda_D} \int_{\mathcal{D}} p_D(1, x, u) f_D(u) du \]
\[ \leq C^{(2)}_{d,m,\alpha,R_1,R_2} \| f_D \|_{L^\infty(\mathcal{D})} \delta_D^{\alpha/2}(x) \int_{\mathcal{D}} \delta_D^{\alpha/2}(u) du \] (3.10)
\[ \leq C^{(2)}_{d,m,\alpha,R_1,R_2} \| f_D \|_{L^\infty(\mathcal{D})} \left( 6 R_2^{\alpha/2}(3 R_2)^d - R_1^{d} \right) \omega_d \delta_D^{\alpha/2}(x). \]

To obtain a lower bound on \( f_D(x) \), consider \( \tau_D = \inf \{ t > 0 : X_t \in \mathcal{D}^c \} \) and use again (2.7), (3.10) and the fact that \( \delta_D(u) \leq |u - z| \) for all \( z \in \mathcal{D}^c \), giving
\[ \mathbb{P}^x(\tau_D > T) = \int_T^\infty \int_{\mathcal{D}^c} \int_{\mathcal{D}} j_{m,\alpha}(|z - u|) p(t, x, u) dudzdt \]
\[ \leq \frac{3}{2} f_D(x) \int_T^\infty \int_{\mathcal{D}^c} \int_{\mathcal{D}} j_{m,\alpha}(|z - u|) e^{-\lambda_D t} f_D(u) dudzdt \]
\[ \leq \frac{3}{2} f_D(x) C^{(2)}_{d,m,\alpha,R_1,R_2} \| f_D \|_{L^\infty(\mathcal{D})} \left( 6 R_2^{\alpha/2}(3 R_2)^d - R_1^{d} \right) \omega_d \]
\[ \times \int_T^\infty \int_{\mathcal{D}^c} \int_{\mathcal{D}} j_{m,\alpha}(|z - u|) e^{-\lambda_D t} \delta_D(u) dudzdt \]
\[ \leq C^{(4)}_{d,\alpha} \frac{3 e^{-\lambda_D T}}{2 \lambda_D} f_D(x) C^{(2)}_{d,m,\alpha,R_1,R_2} \| f_D \|_{L^\infty(\mathcal{D})} \left( 6 R_2^{\alpha/2}(3 R_2)^d - R_1^{d} \right) \omega_d \]
\[ \times \int_{\mathcal{D}^c} \int_{\mathcal{D}} \frac{dudz}{|z - u|^{d + \frac{\alpha}{2}}} \]
\[ \leq \frac{3}{2} \frac{\text{Per}_\alpha(\mathcal{D})}{2} C^{(3)}_{d,\alpha} C^{(2)}_{d,m,\alpha,R_1,R_2} \| f_D \|_{L^\infty(\mathcal{D})} \left( 6 R_2^{\alpha/2}(3 R_2)^d - R_1^{d} \right) \omega_d e^{-\lambda_D T} f_D(x), \]

where \( \text{Per}_\alpha(\mathcal{D}) = \int_{\mathcal{D}} \int_{\mathcal{D}^c} \frac{dudz}{|z - u|^{d + \frac{\alpha}{2}}} \) is the fractional perimeter of \( \mathcal{D} \) (see, e.g., [28]), and we used that \( j_{m,\alpha}(|z - u|) \leq j_{0,\alpha}(|z - u|) = C^{(3)}_{d,\alpha} |z - u|^{-d - \alpha} \) by [22], see [51, Lem. 2]. Hence \( f_D(x) \geq C^{(4)}_{d,m,\alpha,R_1,R_2} \mathbb{P}^x(\tau_D > T) \), where
\[ C^{(4)}_{d,m,\alpha,R_1,R_2} = \frac{2 \lambda_D e^{\lambda_D T}}{3 C^{(3)}_{d,\alpha} \text{Per}_\alpha(\mathcal{D}) C^{(2)}_{d,m,\alpha,R_1,R_2} \| f_D \|_{L^\infty(\mathcal{D})} \left( 6 R_2^{\alpha/2}(3 R_2)^d - R_1^{d} \right) \omega_d}. \]
Note that $\mathcal{D}$ is a $C^{1,1}$ bounded set with scaling radius $R_3 = (3R_2 + R)/2$. Fix $x \in \mathcal{D}$. Then there exists a point $\bar{x} \in \mathcal{D}$ and a ball $\mathcal{B}_{R_3}(\bar{x})$ such that $x \in \mathcal{B}_{R_3}(\bar{x})$ and $\delta_D(x) = R_3 - |x - \bar{x}|$. By Proposition 3.2 and the fact that $\mathcal{B}_{R_3}(\bar{x}) \subset \mathcal{D}$, we know that there exists a constant $C_{d,m,\alpha,R_1,R_2}$ such that

$$
\mathbb{P}^x(\tau_D > T) \geq \mathbb{P}^x(\tau_{\mathcal{B}_{R_3}^c(\bar{x})} > T) = \mathbb{P}^{x-\bar{x}}(\tau_{R_3} > T) \geq C_{d,m,\alpha,R_1,R_2} e^{-\lambda_D T} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{T} \wedge R_3^{\alpha/2}} \wedge 1 \right),
$$

and then

$$
f_D(x) \geq C_{d,m,\alpha,R_1,R_2}(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{T} \wedge R_3^{\alpha/2}} \wedge 1),
$$

where $C_{d,m,\alpha,R_1,R_2} = C_{d,m,\alpha,R_1,R_2} e^{-\lambda_D T}$. Applying this to (3.3), we have

$$
\mathbb{E}^x[e^{-\lambda_T R}] \geq C_{d,m,\alpha,R_1,R_2,\lambda} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{T} \wedge R_3^{\alpha/2}} \wedge 1 \right) j_m,\alpha(|x|) \int_{A(x)} \left( \frac{\delta_D(u)^{\alpha/2}}{\sqrt{T} \wedge R_3^{\alpha/2}} \wedge 1 \right) du,
$$

where

$$
C_{d,m,\alpha,R_1,R_2,\lambda} = \frac{C_{d,m,\alpha}^{(1)} R_d \omega_d}{(\lambda + \lambda_D) 2^{d+1}} e^{-\lambda_D T} (\frac{C_{d,m,\alpha,R_1,R_2}^{(8)}}{C_{R_1,R_2}^{(9)} R_2} \wedge 1) (\frac{C_{R_1,R_2}^{(9)} R_2}{\sqrt{T} \wedge R_3^{\alpha/2}} \wedge 1) \frac{d R \omega_d}{2} (R_1 + R)^{d-1}.
$$

Recall that $\min_{R_1 \leq |x| \leq R_2} \delta_D(x) = (C_{R_1,R_2}^{(8)} R_2)^{\alpha/2} > 0$ by definition of $\mathcal{D}$. Moreover, $u \in A(x)$ implies $R < R_1 + R \leq |u| \leq R_2 + 2R < 3R_2$, and hence $\min_{u \in A(x)} \delta_D(u) \geq \min_{R_1 + R \leq |u| \leq 2R + R_2} \delta_D(u) = (C_{R_1,R_2}^{(8)} R_2)^{\alpha/2} > 0$. Finally, recall also that $|A(x)| \geq \frac{d R}{2} d(R_1 + R)^{d-1} R$ to conclude that

$$
\mathbb{E}^x[e^{-\lambda_T R}] \geq C_{d,m,\alpha,R_1,R_2,\lambda} \frac{d R \omega_d}{2} (R_1 + R)^{d-1}.
$$

To extend the lower bound up to the boundary of $\mathcal{B}_R$, we need the following result.

**Proposition 3.4.** The following properties hold:

1. There exist $R_{d,m,\alpha,R,\lambda}^{(0)} > R$ and $C_{d,m,\alpha,R,\lambda}$ such that for every $R \leq |x| \leq R_{d,m,\alpha,R,\lambda}^{(0)}$

$$
\mathbb{E}^x[1 - e^{-\lambda_T R}] \leq C_{d,m,\alpha,R,\lambda}(|x| - R)^{\alpha/2}.
$$

2. There exists $\bar{R}_{d,m,\alpha,R,\lambda} > R$ such that for every $R \leq |x| \leq \bar{R}_{d,m,\alpha,R,\lambda}$

$$
\mathbb{E}^x[e^{-\lambda_T R}] \geq \frac{1}{2}.
$$

**Proof.** By Proposition 3.3

$$
\mathbb{P}^x(T_R = \infty) \leq C_{d,m,\alpha,R}^{(1)} \left( 1 \wedge \frac{(|x| - R)^{\alpha/2}}{R^{\alpha/2}} \right),
$$

hence there exists $R_{d,m,\alpha,R,\lambda}^{(0)} > R$ such that, for $R < |x| < R_{d,m,\alpha,R,\lambda}^{(0)}$

$$
\mathbb{P}^x(T_R = \infty) \leq C_{d,m,\alpha,R}^{(1)} \left( \frac{R_{d,m,\alpha,R,\lambda}^{(0)}}{R} \right)^{\alpha/2} R^{\alpha/2} \leq \frac{1}{3}.
$$
so that
\[
\mathbb{P}^x(T_R < \infty) \geq 1 - C^{(1)}_{d,m,\alpha,R} \left( \frac{R^{(0)}_{d,m,\alpha,R,\lambda}}{R} - 1 \right)^{\alpha/2} \triangleq C^{(2)}_{d,m,\alpha,R} > \frac{2}{3}.
\]

Notice that
\[
\mathbb{E}^x[e^{-\lambda T_R}] = \mathbb{E}^x[e^{-\lambda T_R}; T_R < \infty] = \mathbb{E}^x[e^{-\lambda T_R} | T_R < \infty] \mathbb{P}^x(T_R < \infty).
\]

Denote \( \tilde{\mathbb{P}}^x(\cdot) = \mathbb{P}^x(\cdot | T_R < \infty) \). We have
\[
1 - \tilde{\mathbb{E}}^x[e^{-\lambda T_R}] = \int_0^\infty \tilde{\mathbb{P}}^x(1 - e^{-\lambda T_R} > s) ds = \int_0^1 \tilde{\mathbb{P}}^x(1 - e^{-\lambda T_R} > s) ds.
\]

Writing \( s = 1 - e^{-\lambda t} \) we obtain
\[
1 - \tilde{\mathbb{E}}^x[e^{-\lambda T_R}] = \int_0^\infty \lambda e^{-\lambda t} \tilde{\mathbb{P}}^x(1 - e^{-\lambda T_R} > 1 - e^{-\lambda t}) dt = \int_0^\infty \lambda e^{-\lambda t} \mathbb{P}^x(T_R > t) dt
\]
\[
= \frac{\lambda}{\mathbb{P}^x(T_R < \infty)} \int_0^\infty e^{-\lambda t} \mathbb{P}^x(T_R > t, T_R < \infty) dt
\]
\[
\leq \frac{\lambda}{C^{(2)}_{d,m,\alpha,R}} \int_0^\infty e^{-\lambda t} \mathbb{P}^x(T_R > t) dt.
\]

Using Proposition 3.3 gives
\[
\mathbb{P}^x(T_R > t) \leq C^{(1)}_{d,m,\alpha,R} \left( 1 \wedge \frac{|x| - R)^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}} \right), \quad t > 0.
\]

so that, setting \( C^{(3)}_{d,m,\alpha,R} = C^{(1)}_{d,m,\alpha,R}/C^{(2)}_{d,m,\alpha,R} \), we get
\[
1 - \tilde{\mathbb{E}}^x[e^{-\lambda T_R}]
\]
\[
\leq \lambda C^{(3)}_{d,m,\alpha,R} \int_0^\infty e^{-\lambda t} \left( 1 \wedge \frac{|x| - R)^{\alpha/2}}{\sqrt{t} \wedge R^{\alpha/2}} \right) dt
\]
\[
= \lambda C^{(3)}_{d,m,\alpha,R} \left( \int_0^{(|x| - R)^\alpha} e^{-\lambda t} dt + (|x| - R)^{\alpha/2} \left( \int_{(|x| - R)^\alpha} R^{\alpha/2} \frac{e^{-\lambda t}}{\sqrt{t}} dt + \int_{R^{\alpha/2}}^\infty e^{-\lambda t} \frac{R^{\alpha/2}}{\sqrt{t}} dt \right) \right)
\]
\[
= \lambda C^{(3)}_{d,m,\alpha,R} \left( \frac{1 - e^{-\lambda(|x| - R)^\alpha}}{\lambda} + \frac{(|x| - R)^{\alpha/2}}{\lambda R^{\alpha/2}} e^{-\lambda R^{\alpha/2}} + \int_{(|x| - R)^\alpha} R^{\alpha/2} e^{-\lambda t} \frac{|x| - R)^{\alpha/2}}{\sqrt{t}} dt \right).
\]

The last term above can be further estimated as
\[
(|x| - R)^{\alpha/2} \int_{(|x| - R)^\alpha} R^{\alpha/2} e^{-\lambda t} \frac{dt}{\sqrt{t}} = 2(|x| - R)^{\alpha/2} (e^{-\lambda R^{\alpha/2}} R^{\alpha/2} - e^{-\lambda(|x| - R)^\alpha} (|x| - R)^{\alpha/2})
\]
\[
+ 2(|x| - R)^{\alpha/2} \int_{(|x| - R)^\alpha} R^{\alpha/2} e^{-\lambda(|x| - R)^\alpha} (|x| - R)^{\alpha/2} dt
\]
\[
\leq 2(|x| - R)^{\alpha/2} (e^{-\lambda R^{\alpha/2}} R^{\alpha/2} - e^{-\lambda(|x| - R)^\alpha} (|x| - R)^{\alpha/2})
\]
\[
+ 2(|x| - R)^{\alpha/2} R^{\alpha/2} e^{-\lambda(|x| - R)^\alpha} (R^{\alpha/2} - (|x| - R)^{\alpha/2}).
\]
In sum, we obtain
\[ 1 - \mathbb{E}^x[e^{-\lambda T_R}] \leq \lambda C_{d,m,\alpha,R}^{(3)} \left( (|x| - R)^\alpha + \frac{(|x| - R)^{\alpha/2}}{\lambda R^{\alpha/2}} e^{-\lambda R \alpha} + 2(|x| - R)^{\alpha/2} e^{-\lambda(|x| - R)^\alpha} R^{\alpha/2} \right) := C_{d,m,\alpha,R,\lambda}^{(4)}(|x| - R)^\alpha. \]

We can complete the proof of part (1) by observing that
\[ \mathbb{E}^x[1 - e^{-\lambda T_R}] = \mathbb{E}^x[1 - e^{-\lambda T_R}| \mathbb{P}^x(T_R < \infty) + \mathbb{P}^x(T_R = \infty) \leq C_{d,m,\alpha,R,\lambda}(|x| - R)^\alpha, \quad R \leq |x| \leq R_{d,m,\alpha,R,\lambda}' \]
where we made use of (3.11). Part (2) follows from (1) by choosing \( R < \tilde{R}_{d,m,\alpha,R,\lambda} < R_{d,m,\alpha,R,\lambda}' \) so that \( \mathbb{E}^x[1 - e^{-\lambda T_R}] \leq 1/2 \) holds for all \( R \leq |x| \leq \tilde{R}_{d,m,\alpha,R,\lambda} \).

Finally, we can combine Theorem 3.2 with Proposition 3.4 to obtain the following.

**Corollary 3.3.** Let \( R_2 > R \). Then there exists a constant \( C_{d,m,\alpha,R_2,R,\lambda} \) such that
\[ \mathbb{E}^x[e^{-\lambda T_R}] \geq C_{d,m,\alpha,R_2,R,\lambda}|j_m,\alpha(|x|)|, \quad R \leq |x| < R_2. \]

**Proof.** Let \( \tilde{R}_{d,m,\alpha,R,\lambda} \) be defined as in Proposition 3.4. Then we have
\[ \mathbb{E}^x[e^{-\lambda T_R}] \geq \frac{1}{2} = \frac{j_m,\alpha(|x|)}{2j_m,\alpha(|x|)} \geq \frac{j_m,\alpha(|x|)}{2j_m,\alpha(R)}, \quad R \leq |x| < \tilde{R}_{d,m,\alpha,R,\lambda}. \]

Combining this estimate with Theorem 3.2 for \( R_1 = \tilde{R}_{d,m,\alpha,R,\lambda} \) the result follows.

To obtain an upper bound for the same quantities we can make use of [36, Th. 3.3], particularized to the massless and massive relativistic stable processes.

**Theorem 3.3.** Let \( \lambda, R > 0 \). There exists a constant \( C_{d,m,\alpha,R,\lambda} > 0 \) such that
\[ \mathbb{E}^x[e^{-\lambda T_R}] \leq C_{d,m,\alpha,R,\lambda}|j_m,\alpha(|x|)|, \quad |x| \geq R. \]

**Proof.** By [36, Th. 3.3] it follows that there exist constants \( R_{d,a,m,\lambda,R}^{(1)} > R \) and \( C_{d,a,m,\lambda,R}^{(1)} > 0 \) such that
\[ \mathbb{E}^x[e^{-\lambda T_R}] \leq C_{d,a,m,\lambda,R}^{(1)}|j_m,\alpha(|x|)|, \quad |x| \geq R_{d,a,m,\lambda,R}^{(1)}. \]
Let \( R_{d,a,m,\lambda,R}^{(2)} = R_{d,a,m,\lambda,R}^{(1)} + 1 \) and notice that \( j_m,\alpha(|x|) \geq j_m,\alpha(R_{d,a,m,\lambda,R}^{(2)}) \) whenever \( R \leq |x| \leq R_{d,a,m,\lambda,R}^{(2)} \). Hence for every \( R \leq |x| \leq R_{d,a,m,\lambda,R}^{(2)} \) we get
\[ \mathbb{E}^x[e^{-\lambda T_R}] \leq 1 \leq \frac{j_m,\alpha(|x|)}{j_m,\alpha(R_{d,a,m,\lambda,R}^{(2)})}. \]
Setting \( C_{d,m,\alpha,R,\lambda} = \max \left\{ C_{d,a,m,\lambda,R}^{(1)}, \frac{1}{j_m,\alpha(R_{d,a,m,\lambda,R}^{(2)})} \right\} \) completes the proof.

**Remark 3.4.**

1. A similar upper estimate follows by using the Ikeda-Watanabe formula. In this approach we can derive a bound which is uniform with respect to \( \alpha \in [\alpha_0, 2] \) for a suitable \( \alpha_0 > 0 \).
2. Above we obtained a global upper and a local lower bound for \( \mathbb{E}^x[e^{-\lambda T_R}] \). A global lower bound for \( \mathbb{E}^x[e^{-\lambda T_R}] \) outside the well will be obtained as a consequence of the estimates of the ground states.
4. Basic qualitative properties of ground states

4.1. Martingale representation of ground states

For our purposes below it will be useful to consider a variant of the Feynman-Kac representation \([2.4]\) with general stopping times. In order to obtain this, the following martingale property will be important. Define the random process \((M_t^x)_{t \geq 0}\),

\[
M_t^x = e^{\lambda_0 t} e^{- \int_0^t V(X_s + x)\,dr} \varphi_0(X_t + x), \quad x \in \mathbb{R}^d.
\]  

Note that by the eigenvalue equation \(\mathbb{E}[M_t^x] = \varphi_0(x)\), for all \(t \geq 0\) and \(x \in \mathbb{R}^d\). Let \((\mathcal{F}_t^X)_{t \geq 0}\) be the natural filtration of the Lévy process \((X_t)_{t \geq 0}\). A version of the following result dates back at least to Carmona’s work (see [45, Sect. 4.6.3] for a detailed discussion and references), but since it is of fundamental interest in this paper, we provide a proof for a self-contained presentation.

**Lemma 4.1.** \((M_t^x)_{t \geq 0}\) is a martingale with respect to \((\mathcal{F}_t^X)_{t \geq 0}\).

**Proof.** We have

\[
\mathbb{E}[|M_t^x|] = \mathbb{E}[M_t^x] \leq e^{\lambda_0 t} \|\varphi_0\|_{\infty} \mathbb{E}\left[ e^{- \int_0^t V(X_s + x)\,dr} \right] \leq e^{(\nu - |\lambda_0|) t} \|\varphi_0\|_{\infty} < \infty, \quad t \geq 0.
\]

Let \(0 \leq s \leq t\). By the Markov property of \((X_t)_{t \geq 0}\) we have that

\[
\mathbb{E}[M_t^x | \mathcal{F}_s^X] = e^{\lambda_0 t} e^{- \int_0^t V(X_r + x)\,dr} \mathbb{E}[\varphi_0(X_t + x) | \mathcal{F}_s^X] = e^{\lambda_0 s} e^{- \int_0^s V(X_r + x)\,dr} \mathbb{E} X_s [e^{\lambda_0 (t-s)} e^{- \int_s^t V(X_r + x)\,dr} \varphi_0(X_{t-s} + x)] = e^{\lambda_0 s} e^{- \int_0^t V(X_r + x)\,dr} \varphi_0(x + x) = M_s^x.
\]

Hence the lemma follows. \(\square\)

This martingale property easily leads to the following Feynman-Kac type formula for the stopped process.

**Proposition 4.1.** Let \(\tau\) be a \(\mathbb{P}\)-almost surely finite stopping time with respect to the filtration \((\mathcal{F}_t^X)_{t \geq 0}\). Then

\[
\varphi_0(x) = \mathbb{E}^x \left[ e^{- \int_0^\tau (V(X_s) - \lambda_0)\,ds} \varphi_0(X_\tau) \right].
\]

**Proof.** Since \(\varphi_0\) is strictly positive, clearly \(M_t^x\) is almost surely non-negative. Thus by the Feynman-Kac formula

\[
\mathbb{E}[(M_t^x)^+] = \mathbb{E}[M_t^x] = \varphi_0(x) \leq \|\varphi_0\|_{\infty}.
\]

The martingale convergence theorem (see, e.g., [48, Th. 2.10]) implies that \((M_t^x)_{t \geq 0}\) has a final element \(M_\infty^x\) with \(\mathbb{E}[|M_\infty^x|] < \infty\), and the optional stopping theorem (see, e.g., [48, Th. 3.2]) then gives

\[
\varphi_0(x) = \mathbb{E}[M_{\tau}^x] = \mathbb{E}[M_\tau^x] = \mathbb{E}^x \left[ e^{- \int_0^\tau (V(X_s) - \lambda_0)\,ds} \varphi_0(X_\tau) \right].
\]

\(\square\)
4.2. Symmetry properties

Next we discuss some shape properties of ground states, specifically, symmetry and monotonicity, which will be essential ingredients in the study of their local behaviour. First we show radial symmetry of the ground states for rotationally symmetric potential wells. This result can also been obtained by purely analytic methods, see [4] Prop. 4.3.

**Theorem 4.1.** Let $K = B_a$ with a given $a > 0$ and suppose that $H_{m,a}$ has a ground state $\varphi_0$. Then $\varphi_0$ is rotationally symmetric.

**Proof.** First observe that if another function $\tilde{\varphi}_0$ existed satisfying (2.4), $||\tilde{\varphi}_0||_2 = 1$ and $\tilde{\varphi}_0 > 0$, then by the uniqueness of the ground state we would have $\tilde{\varphi}_0 \equiv \varphi_0$ almost surely.

Fix a rotation $R \in SO(d)$ and consider $\tilde{\varphi}_0(x) = \varphi_0(Rx)$. Clearly, since $R$ is an isometry, it is immediate that $||\tilde{\varphi}_0||_2 = 1$, $\tilde{\varphi}_0 > 0$, and $\tilde{\varphi}_0(x) = \mathbb{E}[e^{-\int_0^1 (V(x_s+Rx)-\lambda_0)ds} \varphi_0(x_t + Rx)]$ by (2.4). By rotational invariance of $(X_t)_{t \geq 0}$ we may furthermore write

$$
\tilde{\varphi}_0(x) = \mathbb{E}[e^{-\int_0^1 (V(x_s+Rx)-\lambda_0)ds} \varphi_0(RX_t + Rx)] = \mathbb{E}[e^{-\int_0^1 (V(x_s+x)-\lambda_0)ds} \tilde{\varphi}_0(X_t + x)],
$$

where we used the fact that also $V$ is rotationally invariant and $K = B_a$. Then by the observation above, $\tilde{\varphi}_0 \equiv \varphi_0$ almost surely. Since $R \in SO(d)$ is arbitrary, the claim follows. \qed

We can also prove a reduced symmetry of $\varphi_0$ for cases when $K$ is not spherically symmetric.

**Theorem 4.2.** Let $K$ be reflection symmetric with respect to a hyperplane $H$ such that $0 \in H$, and let $S : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto Sx$, be such that $Sx$ is the reflection of $x$ with respect to $H$. Suppose that $v$ is chosen such that $H_{m,a}$ has a ground state $\varphi_0$. Then $\varphi_0(Sx) = \varphi_0(x)$, for all $x \in \mathbb{R}^d$.

**Proof.** We can argue similarly to Theorem 4.1. Consider $\tilde{\varphi}_0(x) = \varphi_0(Sx)$. By the isometry property of $S$ we have again $||\tilde{\varphi}_0||_2 = 1$, $\tilde{\varphi}_0 > 0$, and $\tilde{\varphi}_0(x) = \mathbb{E}[e^{-\int_0^1 (V(x_s+Sx)-\lambda_0)ds} \varphi_0(X_t + Sx)]$ by (2.4). Since $(X_t)_{t \geq 0}$ is isotropic, we get

$$
\tilde{\varphi}_0(x) = \mathbb{E}[e^{-\int_0^1 (V(x_s+Sx)-\lambda_0)ds} \varphi_0(SX_t + Sx)] = \mathbb{E}[e^{-\int_0^1 (V(x_s+x)-\lambda_0)ds} \tilde{\varphi}_0(X_t + x)],
$$

where we used the fact that if $x \in K$, then also $Sx \in K$. Arguing as before, we obtain $\varphi_0(Sx) = \tilde{\varphi}_0(x) = \varphi_0(x)$ for all $x \in \mathbb{R}^d$. \qed

**Remark 4.1.** We note that Theorems 4.1 and 4.2 hold respectively for any rotationally or reflection symmetric potential $V$ once a ground state exists and is unique. Moreover, they can be seen as converses to [3] Th. 7.1-7.2, be using the expression

$$
V = -\frac{1}{\varphi_0}L_{m,a} \varphi_0 + \lambda_0,
$$

provided $L_{m,a} \varphi_0$ can be defined pointwise.

We fix $K = B_a$ for some $a > 0$ and assume that $H_{m,a}$ has a ground state. Furthermore, we will make extensive use of the following, for a proof see [4].

**Proposition 4.2.** There exists a non-increasing function $\rho_0 : [0, \infty) \to \mathbb{R}$ such that $\varphi_0(x) = \rho_0(|x|)$ for every $x \in \mathbb{R}^d$. 
5. Local estimates

5.1. A prime example: Classical Laplacian and Brownian motion

First we present the case of the classical Schrödinger operator with a potential well, for which not only estimates can be obtained but a full reconstruction of the ground state is possible by using the martingale \((M_t)_{t \geq 0}\) in \([11]\). Alternatively this can be done by an explicit solution of the Schrödinger eigenvalue equation, which in this case is a textbook example, however, our point here is that while the eigenvalue problem cannot in general be solved for non-local cases, the probabilistic approach is a useful alternative and this example shows best how this can be done by using occupation times.

Proposition 5.1. Let

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} - v \mathbf{1}_{\{ |x| \leq a \}}
\]

be given on \(L^2(\mathbb{R})\). Then the normalized ground state of \(H\) is

\[
\varphi_0(x) = A_0 e^{-\sqrt{2|\lambda_0||x|}} \mathbf{1}_{\{ |x| > a \}} + B_0 \cos(\sqrt{2(v - |\lambda_0|)} x) \mathbf{1}_{\{ |x| \leq a \}},
\]

with

\[
A_0 = \sqrt{\frac{\sqrt{2|\lambda_0|}}{1 + a \sqrt{2|\lambda_0|}}} e^{a \sqrt{2|\lambda_0|}} \cos(a \sqrt{2(v - |\lambda_0|)}), \quad B_0 = \sqrt{\frac{\sqrt{2|\lambda_0|}}{1 + a \sqrt{2|\lambda_0|}}}.
\]

Proof. Consider for any \(b, c \in \mathbb{R}\) with \(b < 0 < c\), the first hitting times

\[
T_b = \inf\{ t > 0 : B_t = b \}, \quad T_c = \inf\{ t > 0 : B_t = c \}, \quad \text{and} \quad T_{b,c} = T_b \wedge T_c,
\]

for Brownian motion \((B_t)_{t \geq 0}\) starting at zero, and recall the general formula by Lévy [42]

\[
\mathbb{E}[e^{i u T_{b,c}}] = \frac{e^{(1+i)x\sqrt{u}}}{e^{(1+i)c\sqrt{u}} + e^{(1+i)b\sqrt{u}}} + \frac{e^{-(1+i)c\sqrt{u}}}{e^{-(1+i)b\sqrt{u}} + e^{-(1+i)c\sqrt{u}}},
\]

with \(b < x < c\), and

\[
\mathbb{E}[e^{-u T_b}] = e^{-\sqrt{2u}|b|} \quad \text{and} \quad \mathbb{E}[e^{-u T_{b,c}}] = \frac{\cosh(\sqrt{2u} \frac{x+b}{2})}{\cosh(\sqrt{2u} \frac{c-b}{2})}, \quad u \geq 0. \tag{5.1}
\]

It is well-known that all these hitting times are almost surely finite stopping times with respect to the natural filtration. From \([2,4]\) we have

\[
\varphi_0(x) = \mathbb{E}[e^{-|\lambda_0|t + U_t^x(a)} \varphi_0(B_t + x)],
\]

where we denote

\[
U_t^x(a) = \int_0^t \mathbf{1}_{\{ B_s + x \leq a \}} ds = \int_0^t \mathbf{1}_{\{ -a - x \leq B_s \leq a - x \}} ds.
\]

Then \(U_{T_{-a-x,a-x}}^x(a) = T_{-a-x,a-x}\) whenever \(|x| < a\), and is zero otherwise. Using Proposition \([11]\) we obtain

\[
\varphi_0(x) = \mathbb{E}[e^{-|\lambda_0|T_{-a-x,a-x} + U_{T_{-a-x,a-x}}^x(a)} \varphi_0(B_{T_{-a-x,a-x}} + x)].
\]

Now suppose \(x > a\). By path continuity \(T_{-a-x,a-x} = T_{a-x}\) and thus

\[
\varphi_0(x) = \mathbb{E}[e^{-|\lambda_0|T_{a-x}} \varphi_0(B_{T_{a-x}} + x)] = \varphi_0(a) \mathbb{E}[e^{-|\lambda_0|T_{a-x}}] = \varphi_0(a) e^{-\sqrt{2|\lambda_0|}(x-a)}.
\]
We obtain similarly for \( x < -a \) that \( T_{-a-x,a-x} = T_{-a-x} \) and

\[
\varphi_0(x) = \mathbb{E}[e^{-(\lambda_0)|x|}T_{-a-x}] \varphi_0(B_{T_{-a-x}} + x)] = \varphi_0(-a)e^{-\sqrt{2|\lambda_0|(x-a)}} = \varphi_0(a)e^{\sqrt{2|\lambda_0|(x+a)}}
\]

using \( \varphi_0(-a) = \varphi_0(a) \). When \(-a < x < a\), the two-barrier formula in (5.1) gives

\[
\varphi_0(x) = \mathbb{E}[e^{-(\lambda_0)|x|}T_{-a-x,a-x} \varphi_0(B_{T_{-a-x,a-x}} + x)]
\]

\[
+ \mathbb{E}[e^{-(\lambda_0)|x|}T_{-a-x,a-x} \varphi_0(B_{T_{-a-x,a-x}} + x)1_{\{T_{-a-x} < T_{a-x}\}}]
\]

\[
- \mathbb{E}[e^{-(\lambda_0)|x|}T_{-a-x,a-x} \varphi_0(B_{T_{-a-x,a-x}} + x)1_{\{T_{-a-x} > T_{a-x}\}}]
\]

The constant \( \varphi_0(a) \) can be determined by the normalization condition \( \|\varphi_0\|_2 = 1 \), which then yields the claimed expression of the ground state.

\( \square \)

**Remark 5.1.** The argument can also be extended to higher dimensions. For instance, for \( d \geq 3 \), denote by \( B_r(z) \) a ball of radius \( r \) centered in \( z \), write \( B_r = B_r(0) \), and define the stopping times

\[
T_r = \inf\{t > 0 : X_t \notin B_r\} \quad \text{and} \quad \tau_r = \inf\{t > 0 : X_t \notin B_r\}.
\]

Using the Ciesielski-Taylor formulae (see, e.g., [18, eq. (2.15)] and [12, formula 2.0.1])

\[
\mathbb{E}^x[e^{-uT_r}] = \left( \frac{r}{|x|} \right)^{d-2} \frac{I_{d-2}(|x|\sqrt{2u})}{I_{d-2}(r\sqrt{2u})} \quad \text{and} \quad \mathbb{E}^x[e^{-u\tau_r}] = \left( \frac{r}{|x|} \right)^{d-2} \frac{K_{d-2}(|x|\sqrt{2u})}{K_{d-2}(r\sqrt{2u})},
\]

and the properties of the Bessel function \( J_{(d-2)/2} \) and modified Bessel functions \( I_{(d-2)/2} \) and \( K_{(d-2)/2} \) in standard notation (for properties of the Bessel functions, we refer to [57]), by a similar argument as above for the potential well \(-v1_{B_a}\) we obtain

\[
\varphi_0(x) = A_0 \left( \frac{a}{|x|} \right)^{\frac{d-2}{2}} K_{d-2}((2|\lambda_0|)|x|)1_{\{|x| > a\}} + B_0 \left( \frac{a}{|x|} \right)^{\frac{d-2}{2}} J_{d-2}((2|\lambda_0-\lambda_0|)|x|)1_{\{|x| \leq a\}},
\]

where the constants \( A_0, B_0 \) can be determined from \( L^2 \)-normalization as before. The details are left to the reader.

### 5.2. Local behaviour of the ground state

To come to our main point in this section, we need some scaling estimates on the Lévy measure \( \nu_{m,\alpha} \) of the exterior of a ball.

**Lemma 5.1.** For every \( R > 0 \) there exists a constant \( C_{d,m,\alpha,R} > 1 \) such that

\[
\int_{B_R^c} j_{m,\alpha}(|x-y|)dy \leq \frac{1}{2} \int_{B_R} j_{m,\alpha}(|x-y|)dy.
\]

Moreover, if \( m = 0 \), then \( C_{d,0,\alpha,R} \) does not depend on \( R \).

**Proof.** Since \( j_{m,\alpha} \) is non-increasing, for every \( \theta > 0 \) the set \( \{j_{m,\alpha}(|x|) \geq \theta\} \) is a ball and then \( \nu_{m,\alpha}(dx) \) is unimodal. As a consequence of Anderson’s inequality [2, Th. 1] we get \( \int_{B_R} j_{m,\alpha}(|x-y|)dy \geq \)
Using Lemma 5.1 we thus have

\[
\int_{B_R} j_{m,\alpha}(|y|) dy,
\]
for every \( R > 0 \) and \( x \in B_R \). Taking \( R > 0 \), \( x \in B_R \) and \( k > 2 \), we obtain

\[
\int_{B_{kR}} j_{m,\alpha}(|x-y|) dy \leq \int_{B_{(k-1)R}(x)} j_{m,\alpha}(|x-y|) dy
\]

\[
= \int_{B_{(k-1)R}} j_{m,\alpha}((k-1)|y|) dy = (k-1)^d \int_{B_R} j_{m,\alpha}((k-1)|y|) dy.
\]

First consider \( m = 0 \). We have

\[
\int_{B_R} j_{0,\alpha}((k-1)|y|) dy = \frac{1}{(k-1)^d + \alpha} \int_{B_R} j_{0,\alpha}(|y|) dy,
\]

and thus

\[
\int_{B_{kR}} j_{0,\alpha}(|x-y|) dy \leq \frac{1}{(k-1)^d + \alpha} \int_{B_R} j_{0,\alpha}(|y|) dy \leq \frac{1}{(k-1)^d} \int_{B_R} j_{0,\alpha}(|x-y|) dy.
\]

We can then set \( C_{d,0,\alpha} = 1 + 2^{1/\alpha} \) to complete the proof.

Next consider \( m > 0 \). Using that \( j_{m,\alpha}(r) \sim C_{d,m,\alpha}^2 r^{-\frac{d+\alpha+1}{2}} e^{-m^\alpha r} \) as \( r \to \infty \), we have

\[
j_{m,\alpha}((k-1)|y|) \leq C_{d,m,\alpha}^2 (k-1)^{-\frac{d+\alpha+1}{2}} |y|^{-\alpha} \frac{e^{-m^\alpha (k-1)|y|}}{e^{-m^\alpha |y|}} e^{-m^\alpha |y|}
\]

\[
\leq (C_{d,m,\alpha,R}^2 (k-1)^{-\frac{d+\alpha+1}{2}} e^{-m^\alpha k R} j_{m,\alpha}(|y|),
\]

with some \( C_{d,m,\alpha,R} > 1 \), and hence

\[
\int_{B_{kR}} j_{m,\alpha}(|y|) dy \leq (C_{d,m,\alpha,R}^2 (k-1)^{-\frac{d+\alpha+1}{2}} e^{-m^\alpha k R} \int_{B_R} j_{m,\alpha}(|y|) dy
\]

\[
\leq (C_{d,m,\alpha,R}^2 (k-1)^{-\frac{d+\alpha+1}{2}} e^{-m^\alpha k R} \int_{B_R} j_{m,\alpha}(|x-y|) dy.
\]

Choosing \( C_{d,m,\alpha,R} > 2 \) such that \( (C_{d,m,\alpha,R}^2 (C_{d,m,\alpha,R}^2 - 1)^{-\frac{d+\alpha+1}{2}} e^{-m^\alpha C_{d,m,\alpha,R} R} \leq \frac{1}{2} \) and using it instead of \( k \), the claim follows.

Combining the last estimate with the Ikeda-Watanabe formula, we obtain the following result.

**Lemma 5.2.** For every \( R > 0 \) there exists a constant \( C_{d,m,\alpha,R} > 0 \) such that

\[
\mathbb{E}^x \left[ g(\tau_R); R \leq |X_{\tau_R}| \leq C_{d,m,\alpha,R} R \right] \geq \frac{1}{2} \mathbb{E}^x [g(\tau_R)]
\]

for every non-negative function \( g \) and all \( x \in B_R \).

**Proof.** First consider \( g \in L^\infty(\mathbb{R}^d) \) and let \( C_{d,m,\alpha,R} > 0 \) be defined as in Lemma 5.1. By the Ikeda-Watanabe formula

\[
\mathbb{E}^x [g(\tau_R); |X_{\tau_R}| > C_{d,m,\alpha,R} R] = \int_0^\infty \int_{B_R} g(t) p_{B_R}(t,x,y) \int_{B_{C_{d,m,\alpha,R} R}} j_{m,\alpha}(|y-z|) dz dy dt.
\]

Using Lemma 5.1 we thus have

\[
\mathbb{E}^x [g(\tau_R); |X_{\tau_R}| > C_{d,m,\alpha,R} R] \leq \frac{1}{2} \int_0^\infty \int_{B_R} g(t) p_{B_R}(t,x,y) \int_{B_R} j_{m,\alpha}(|y-z|) dz dy dt = \mathbb{E}^x [g(\tau_R)].
\]

Next suppose that \( g \) is unbounded and let \( g_N(t) = g(t) \wedge N \) for \( N \in \mathbb{N} \). Then \( g_N \uparrow g \) pointwise, moreover

\[
\mathbb{E}^x [g_N(\tau_R); R \leq |X_{\tau_R}| \leq C_{d,m,\alpha,R} R] \geq \frac{1}{2} \mathbb{E}^x [g_N(\tau_R)], \quad N \in \mathbb{N}.
\]
As $N \to \infty$, by monotone convergence we then have

$$\mathbb{E}^x[g(\tau_R)]; R \leq |X_{\tau_R}| \leq C_{d,m,a,R}R \geq \frac{1}{2} \mathbb{E}^x[g(\tau_R)].$$

Now we can turn to local estimates of the ground state. Consider the spherical potential well supported in $K = B_a$ with some $a > 0$.

**Theorem 5.1.** Let $\varphi_0$ be the ground state of $H_{m,a}$ with $V = -v 1_{B_a}$ and denote $a = (a,0,\ldots,0)$. Then the estimates

$$\varphi_0(x) \asymp \varphi_0(a) \begin{cases} \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}] & \text{if } |x| \leq a \\ \mathbb{E}^x[e^{-|\lambda_0|T_a}] & \text{if } |x| \geq a \end{cases}$$

hold, where the comparability constant depends on $d,m,\alpha,a,v,\lambda_0$.

**Proof.** Note that $\varphi_0$ is rotationally symmetric by Theorem 4.1 and non-increasing by Proposition 4.2. We first prove the bound inside and next outside the well.

**Step 1:** First consider $|x| \leq a$. Using Proposition 4.1 with the almost surely finite stopping time $\tau_a$, and that $X_{\tau_a} \in B_a$ and $\varphi_0(X_{\tau_a}) \leq \varphi_0(a)$, we have

$$\varphi_0(x) = \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a} \varphi_0(X_{\tau_a})] \leq \varphi_0(a) \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}]. \quad (5.2)$$

On the other hand, using that $|X_{\tau_a}| \leq C_{d,m,a,a}^{(1)}$, where $C_{d,m,a,a}^{(1)}$ is defined in Lemma 5.2, we furthermore obtain

$$\varphi_0(x) \geq \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a} \varphi_0(X_{\tau_a})]; a \leq |X_{\tau_a}| \leq C_{d,m,a,a}^{(1)}a \geq \varphi_0(C_{d,m,a,a}^{(1)}a) \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}; a \leq |X_{\tau_a}| \leq C_{d,m,a,a}^{(1)}a].$$

Recall that $C_{d,m,a,a}^{(1)} > 1$. Consider $T_a$ and $T_M = T_a \land M$ for any positive integer $M \in \mathbb{N}$. By Proposition 4.1 applied to the almost surely finite stopping time $T_M$, note that

$$\varphi_0(C_{d,m,a,a}^{(1)}a) = \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_M}] \varphi_0(X_{T_M})] \leq \varphi_0(0) \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_M}].$$

By dominated convergence, in the limit $M \to \infty$ we then get

$$0 < \varphi_0(C_{d,m,a,a}^{(1)}a) \leq \varphi_0(0) \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_a}],$$

implying $C_{d,m,a,a}^{(2)} := \mathbb{E}^x[C_{d,m,a,a}^{(1)}a(T_a = \infty)] < 1$. In particular, there exists a constant $C_{d,m,a,a}^{(3)} > 0$ such that $\mathbb{E}^x[C_{d,m,a,a}^{(1)}a(T_a > C_{d,m,a,a}^{(3)})] < C_{d,m,a,a}^{(2)}$. Furthermore, by using Proposition 4.1 again, we get

$$\varphi_0(C_{d,m,a,a}^{(1)}a) = \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_M}] \varphi_0(X_{T_M})] \geq \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_M}] \varphi_0(X_{T_M})] \geq \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_M}] \varphi_0(X_{T_M})].$$

Since on the set $\{T_a \leq C_{d,m,a,a}^{(3)}\}$ the random time $T_M$ is almost surely constant as $M \to \infty$, in the limit

$$\varphi_0(C_{d,m,a,a}^{(1)}a) \geq \mathbb{E}^x[C_{d,m,a,a}^{(1)}a[e^{-|\lambda_0|T_a}] \varphi_0(X_{T_a})]; T_a \leq C_{d,m,a,a}^{(3)} \geq (1 - C_{d,m,a,a}^{(2)}) e^{-|\lambda_0|C_{d,m,a,a}^{(3)}a} \varphi_0(a), \quad (5.3)$$

follows, where we also used Proposition 4.2. On the other hand, by Lemma 5.2 we have

$$\mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}; a \leq |X_{\tau_a}| \leq C_{d,m,a,a}^{(1)}a] \geq \frac{1}{2} \mathbb{E}^x[e^{(v-|\lambda_0|)\tau_a}], \quad (5.4)$$
Combining (5.3)-(5.4) with the above and choosing $C_{d,m,a,a,\lambda_0}^{(4)} = (1 - C_{d,m,a,a}^{(2)}) e^{-|\lambda_0|}$ we obtain

$$\varphi_0(x) \geq \frac{C_{d,m,a,a,\lambda_0}^{(4)}}{2} \varphi_0(a) \mathbb{E}^x [e^{(v-|\lambda_0|)T_a}],$$

thus

$$\varphi_0(x) \asymp \varphi_0(a) \mathbb{E}^x [e^{(v-|\lambda_0|)T_a}], \quad |x| \leq a,$$

where the comparability constant depends on $d, m, \alpha, a, |\lambda_0|$.  

**Step 2:** Next consider $|x| > a$, and let $T_a$ and $T_M$ be defined as before. By Proposition 4.1 we have

$$\varphi_0(x) = \mathbb{E}^x [e^{-|\lambda_0| T_M} \varphi_0(X_{T_M})] \geq \mathbb{E}^x [e^{-|\lambda_0| T_a} \varphi_0(X_{T_a})] \geq \mathbb{E}^x [e^{-|\lambda_0| T_a} \varphi_0(X_{T_M}); T_a < \infty],$$

due to $T_M \leq T_a$. Taking the limit $M \to \infty$ and observing that $T_M$ is a definite constant if $T_a < \infty$, we get

$$\varphi_0(x) \geq \mathbb{E}^x [e^{-|\lambda_0| T_a} \varphi_0(X_{T_a}); T_a < \infty] \geq \varphi_0(a) \mathbb{E}^x [e^{-|\lambda_0| T_a}; T_a < \infty] = \varphi_0(a) \mathbb{E}^x [e^{-|\lambda_0| T_a}]. \quad (5.5)$$

On the other hand, 

$$\varphi_0(x) \leq \varphi_0(0) \mathbb{E}^x [e^{-|\lambda_0| T_M}] \to \varphi_0(0) \mathbb{E}^x [e^{-|\lambda_0| T_a}],$$

as $M \to \infty$, by using dominated convergence. By Step 1, Theorem 3.1 and (5.7) we find a constant $C_{a,\lambda_0}^{(5)}$ such that

$$\varphi_0(0) \leq C_{d,m,a,a,\lambda_0}^{(5)} \varphi_0(a) \left(1 + \frac{v - |\lambda_0|}{\lambda_0 - v + |\lambda_0|}\right) =: C_{d,m,a,a,v,\lambda_0}^{(6)} \varphi_0(a),$$

and thus

$$\varphi_0(x) \leq C_{d,m,a,a,v,\lambda_0}^{(6)} \varphi_0(a) \mathbb{E}^x [e^{-|\lambda_0| T_a}]. \quad (5.6)$$

This leads to

$$\varphi_0(x) \asymp \varphi_0(a) \mathbb{E}^x [e^{-|\lambda_0| T_a}], \quad |x| \geq a,$$

where the comparability constants depend on $d, m, \alpha, a, v, |\lambda_0|$.  

**Remark 5.2.**

1. In fact, along the way we also proved that

$$C_{d,m,a,a,\lambda_0}^{(1)} \varphi_0(a) e^{-C_{d,m,a,a,\lambda_0}^{(2)} \mathbb{E}^x [e^{(v-|\lambda_0|)T_a}]} \leq \varphi_0(x) \leq C_{d,m,a,a,\lambda_0}^{(3)} \varphi_0(a) e^{(v-|\lambda_0|)T_a],}$$

for every $|x| \leq a$, with constants dependent only on $d, m, \alpha, a$ (and independent of $v$ and $\lambda_0$).

2. We point out that we have shown in particular that

$$\mathbb{E}^x [e^{(v-|\lambda_0|)T_a}] \leq \frac{2}{C_{d,m,a,a,\lambda_0}^{(3)}} \varphi_0(a) < \infty.$$  

However, from (3.8) we know that $\mathbb{E}^x [e^{\tau_{\lambda_0}a}]$ is finite if and only if $\lambda < \lambda_a$. Thus we have also shown that

$$v - |\lambda_0| < \lambda_a. \quad (5.7)$$

We note that to prove this only monotonicity of $\varphi_0$ outside the potential well is a required input, which has been proven in [4] without using (5.7) (which is, on the other hand,
indispensable to obtain monotonicity inside the well). Thus this argument provides an alternative, purely probabilistic, proof of [4, Lem. 4.5].

Using the following estimate in conjunction with the estimates in Section 3 we can derive explicit local estimates for the ground states of the massless and massive relativistic operators.

**Corollary 5.1.** With the same notations as in Theorem 5.1 we have

\[ \varphi_0(x) \asymp \varphi_0(a) \begin{cases} 1 + \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{\alpha/2} \quad & \text{if } |x| \leq a, \\ j_{m,\alpha}(|x|) \quad & \text{if } |x| \geq a, \end{cases} \]

where the comparability constant depends on \(d, m, \alpha, a, v, |\lambda_0|\).

**Proof.** For \(|x| \leq a\) the result is immediate by a combination of Theorems 5.1 and 3.1 using (5.7). For \(|x| \geq a\) we distinguish two cases. First, if \(m = 0\), by [36, Cor. 4.1] there exists \(R_{d,0,a,\alpha}\) such that

\[ \varphi_0(x) \geq C_{d,0,a,\alpha}^1 |x|^{-d-\alpha} \geq \frac{C_{d,0,a,\alpha}^1}{|\lambda_a - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{\alpha/2}, \]

where \(C_{d,0,a,\alpha}^1\) is defined in the quoted result and \(C_{d,0,a,\alpha}^1 = C_{d,0,a,\alpha}(\pi^{d/2} \Gamma(-\frac{\alpha}{2}) \frac{\lambda_a}{2})^{1/2}\). Secondly, when \(m > 0\) we use [36, Cor. 4.3(1)] to find that there exists \(R_{d,m,a,\alpha}\) such that

\[ \varphi_0(x) \geq C_{d,m,a,\alpha}^1 |x|^{-d-\alpha} \geq \frac{C_{d,m,a,\alpha}^1}{|\lambda_a - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{\alpha/2} e^{-m^{1/\alpha}|x|}, \]

Moreover, we know that \(j_{m,\alpha}(a) \sim |x|^{-d+\frac{\alpha+1}{2}} e^{-m^{1/\alpha}|x|}\) as \(|x| \to \infty\), hence there exists a constant \(C_{d,m,\alpha}^2\) such that \(\varphi_0(x) \geq C_{d,m,a,\alpha}^2 j_{m,\alpha}(a)\) for \(|x| \geq R_{d,m,a,\alpha}\). Thus by (5.6)

\[ \mathbb{E}^x[e^{-|\lambda_0| T_a}] \geq C_{d,m,a,\alpha}^3 j_{m,\alpha}(a), \quad |x| \geq R_{d,m,a,\alpha}. \]

Combining this with Corollary 3.3 and Theorem 5.3 we obtain

\[ \mathbb{E}^x[e^{-|\lambda_0| T_a}] \asymp j_{m,\alpha}(a), \quad |x| \geq a, \]

where the comparability constants depend on \(d, \alpha, m, a, v, |\lambda_0|\). 

**Remark 5.3.** By Remark 5.2 we have similarly

\[ C_{d,m,a,\alpha}^1 \varphi_0(a) e^{-C_{d,m,a,\alpha}^1|\lambda_0|} \left( 1 + \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{\alpha/2} \right) \]

\[ \leq \varphi_0(x) \leq C_{d,m,a,\alpha}^3 \varphi_0(a) \left( 1 + \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{\alpha/2} \right), \]

for \(|x| \leq a\) it holds and with constants which depend only on \(d, m, \alpha, a\) (and not on \(v\) and \(\lambda_0\)).

The local estimates on \(\varphi_0\) can further be improved to see the behaviour as \(|x| \to a\).

**Proposition 5.2.** There exist \(\varepsilon = \varepsilon_{d,m,a,\alpha,v}, C_{d,m,a,\alpha,v} > 0\) such that for every \(x \in B_{R + \varepsilon} \setminus B_{R - \varepsilon}\)

\[ \left| \frac{\varphi(x)}{\varphi(a)} - 1 \right| \leq C_{d,m,a,\alpha,v} |x - a|^{\alpha/2} \]

holds.
Proof. The estimate is clear once \( x \in \partial \mathcal{B}_a \). Consider first the case \( x \in \mathcal{B}_a \). By (5.2) we have
\[
\varphi(x) - \varphi(a) - 1 \leq \mathbb{E}^x[e^{(v-|\lambda_0|)T_a} - 1] \leq C_{d,m,a,v}(a - |x|)^{\alpha/2},
\]
where we used Theorem 3.1. Taking \( x \in \mathcal{B}_a^c \), we have by (5.5)
\[
1 - \frac{\varphi(x)}{\varphi(a)} \leq \mathbb{E}^x[1 - e^{-|\lambda_0|T_a}].
\]
Choosing \( R^{(0)}_{d,m,a,v} \) as in Proposition 3.1 and defining \( \varepsilon = (R^{(0)}_{d,m,a,v} - a) \wedge a \) the result follows. \( \square \)

By using the normalization condition \( \|\varphi_0\|_2 = 1 \), we are able to provide a two-sided bound on \( \varphi_0(a) \).

**Proposition 5.3.** Denote \( I = \int_1^\infty r^{d-1} j_{m,\alpha}(\frac{r}{a}) \, dr \) and by \( B(x,y) \) the usual Beta-function. Then
\[
\varphi_0(a) \asymp \left( a \right)^{d-2} \int_0^\infty \frac{1}{\kappa} \left( 1 + \frac{\varphi(a - |x|)}{\varphi(a)} \right)^{\frac{\alpha}{2}} \, dx \leq \varphi_0(x) \leq \left( a \right)^{d-2} \int_0^\infty \frac{1}{\kappa} \left( 1 + \frac{\varphi(a - |x|)}{\varphi(a)} \right)^{\frac{\alpha}{2}} \, dx
\]
where the comparability constant is the same as in Corollary 5.1.\( \square \)

Proof. We write \( \kappa = \frac{v-|\lambda_0|}{\lambda_0 - v + |\lambda_0|} \) for a shorthand. Consider \( |x| \leq a \). By Corollary 5.1 we have
\[
\frac{1}{C_{d,m,a,v,|\lambda_0|}} \varphi_0(a) \left( 1 + \kappa \left( \frac{a - |x|}{a} \right)^{\frac{\alpha}{2}} \right) \leq \varphi_0(x) \leq \left( a \right)^{d-2} \int_0^\infty \frac{1}{\kappa} \left( 1 + \frac{\varphi(a - |x|)}{\varphi(a)} \right)^{\frac{\alpha}{2}} \, dx
\]
which gives
\[
\varphi_0(x) \leq \varphi_0(a) \left( 1 + \kappa \left( \frac{a - |x|}{a} \right)^{\frac{\alpha}{2}} \right) \leq C_{d,m,a,v,|\lambda_0|} \varphi_0(x).
\]

Taking the square on both sides and integrating over \( \mathcal{B}_a \), we get
\[
\frac{1}{\left( C_{d,m,a,v,|\lambda_0|} \right)^2} \int_{\mathcal{B}_a} \varphi_0^2(x) \, dx \leq \varphi_0^2(a) \int_{\mathcal{B}_a} \left( 1 + \kappa \left( \frac{a - |x|}{a} \right)^{\frac{\alpha}{2}} \right)^2 \, dx
\]
\[
\leq \left( C_{d,m,a,v,|\lambda_0|} \right)^2 \int_{\mathcal{B}_a} \varphi_0^2(x) \, dx \quad (5.8)
\]

Consider next \( |x| > a \). Proceeding similarly, we have
\[
\frac{1}{\left( C_{d,m,a,v,|\lambda_0|} \right)^2} \int_{\mathcal{B}_a^c} \varphi_0^2(x) \, dx \leq \varphi_0^2(a) \int_{\mathcal{B}_a^c} j_{m,\alpha}^2(|x|) \, dx \leq \left( C_{d,m,a,v,|\lambda_0|} \right)^2 \int_{\mathcal{B}_a^c} \varphi_0^2(x) \, dx. \quad (5.9)
\]

Adding up (5.8)-(5.9) and using that \( \|\varphi_0\|_2 = 1 \), we get
\[
\frac{1}{\left( C_{d,m,a,v,|\lambda_0|} \right)^2} \leq \varphi_0^2(a) \left( \int_{\mathcal{B}_a} \left( 1 + \kappa \left( \frac{a - |x|}{a} \right)^{\frac{\alpha}{2}} \right)^2 \, dx + \int_{\mathcal{B}_a^c} j_{m,\alpha}^2(|x|) \, dx \right) \leq \left( C_{d,m,a,v,|\lambda_0|} \right)^2.
\]

Evaluating the integrals and taking the square root we obtain the desired result. \( \square \)

As a direct consequence, we can rewrite Corollary 5.1 as follows.
Corollary 5.2. With the same notations as in Theorem 5.1 we have

\[ \varphi_0(x) \asymp \begin{cases} 1 + \frac{v - |\lambda_0|}{\lambda_0 - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{\alpha/2} & \text{if } |x| \leq a \\ j_{m,\alpha}(|x|) & \text{if } |x| \geq a, \end{cases} \]

where the comparability constant depends on \( d, m, \alpha, a, v, |\lambda_0| \) and is independent of \( \varphi_0 \).

5.3. Lack of regularity of \( \varphi_0 \)

From a quick asymptotic analysis of the profile functions appearing in the estimates in Corollary 5.1 the difference of the leading terms suggests that, while the regime change around the boundary of the potential well is continuous, it cannot be smooth beyond a degree. To describe this quantitatively, we show next a lack of regularity of the ground state arbitrarily close to the boundary. For a result on Hölder regularity of solutions of related non-local Schrödinger equations see [41].

Lemma 5.3. Consider the operator \( L_{m,\alpha} \) and the following two cases:

1. \( \alpha \in (0, 1) \) and \( f \in C^{\alpha+\delta}_{\text{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) for some \( \delta \in (0, 1 - \alpha) \)
2. \( \alpha \in [1, 2) \) and \( f \in C^{1,\alpha+\delta-1}_{\text{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) for some \( \delta \in (0, 2 - \alpha) \).

In either case above, the function \( \mathbb{R}^d \ni x \mapsto L_{m,\alpha}f(x) \) is continuous.

Proof. Note that under the assumptions above, \( L_{m,\alpha}f \) is well-defined pointwise via the integral representation (2.1). We show the statement for \( m = 0 \) only, for \( m > 0 \) the proof is similar by using the asymptotic behaviour of \( j_{m,\alpha}(r) \) around zero and at infinity.

To prove (1), we use the integral representation (2.1) and claim that in this case

\[ L_{0,\alpha}f(x) = -C^{(1)}_{d,\alpha} \lim_{\varepsilon \downarrow 0} \left( \int_{\varepsilon < |x-y| < 1} + \int_{|x-y| > 1} \right) \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy = -C^{(1)}_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy, \]

with the constant \( C^{(1)}_{d,\alpha} \) entering the definition of the massless operator. Indeed, note that the second integral in the split is independent of \( \varepsilon \), while for the first integral we can use the Hölder inequality giving

\[ \int_{\varepsilon < |x-y| < 1} \frac{|f(y) - f(x)|}{|x-y|^{d+\alpha}} dy \leq C^{(2)}_{d,\alpha} \int_{\varepsilon < |x-y| < 1} \frac{1}{|x-y|^{d-\delta}} \leq dC^{(2)}_{\omega} \int_0^1 \frac{1}{\rho^{1-\delta}} d\rho = \frac{dC^{(2)}_{\omega} \omega d}{\delta}. \]

The claimed right hand side follows then by dominated convergence. Next choosing \( h \in \mathbb{R}^d, |h| < 1 \), we show that \( \lim_{h \to 0} L_{0,\alpha}f(x+h) = L_{0,\alpha}f(x) \). We write

\[ L_{0,\alpha}f(x+h) = -C^{(1)}_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(y) - f(x+h)}{|x+h-y|^{d+\alpha}} dy = -C^{(1)}_{d,\alpha} \left( \int_{B_3(x+h)} + \int_{B_3(x+h)} \right) \frac{f(y) - f(x+h)}{|x+h-y|^{d+\alpha}} dy. \]

To estimate the first integral, note that \( B_3(x+h) \subseteq B_4(x) \) for every \( h \in B_1 \). Let \( C^{(3)} \) be the Hölder constant associated with \( B_4(x) \) and observe that

\[ \int_{B_3(x+h)} \frac{|f(y) - f(x+h)|}{|x+h-y|^{d+\alpha}} dy = \int_{B_3(x+h)} \frac{|f(x+h+y) - f(x+h)|}{|y|^{d+\alpha}} dy \leq C^{(3)} \int_{B_3} \frac{dy}{|y|^{d-\delta}} = \frac{3^\delta C^{(3)} \omega d}{\delta}. \]
Moreover, and thus
\[ B_{\xi} \]

Theorem 5.2. The proof is then completed by dominated convergence. □

By Lagrange’s theorem there exist \( \xi_{\pm}(h) \in [x, x \pm h] \), where \([x, y]\) denotes the segment with endpoints \( x, y \), such that
\[
f(x + h) - 2f(x) + f(x - h) = (\nabla f(\xi_{+}(h)) - \nabla f(\xi_{-}(h)), h)
\]
and thus \(|D_{h}f(x)| \leq \|\nabla f(\xi_{+}(h)) - \nabla f(\xi_{-}(h))\||h||.\) Since \( \xi_{\pm}(h) \in [x, x \pm h] \), in particular \( \xi_{\pm}(h) \in B_{1}(x) \), and we can use the Hölder property of the gradient to conclude that
\[
|\nabla f(\xi_{+}(h)) - \nabla f(\xi_{-}(h))| \leq C^{(1)}(x)|\xi_{+}(h) - \xi_{-}(h)||h|^{\alpha+\delta-1}.
\]
Moreover, \(|\xi_{+}(h) - \xi_{-}(h)| \leq 2\), and thus \(|D_{h}f(x)| \leq 2C^{(1)}(x)|h|^{\alpha+\delta}.\) Using that \( \int_{0}^{1} \frac{1}{\rho^{d-\sigma}} d\rho = \frac{1}{\sigma} \), by an application of [3] Prop. 2.6, Rem. 2.4 we then obtain
\[
L_{0,\alpha}f(x) = -\frac{C^{(2)}_{d,\alpha}}{2} \int_{\mathbb{R}^{d}} D_{h}f(x) \frac{1}{|h|^{d+\alpha}} dh, \quad x \in \mathbb{R}^{d}.
\]
Taking \( k \in B_{1} \), we show that \( \lim_{k \to 0} L_{0,\alpha}f(x + k) = L_{0,\alpha}f(x) \). Write
\[
L_{0,\alpha}f(x + k) = -\frac{C^{(2)}_{d,\alpha}}{2} \int_{B_{3}} D_{h}f(x + k) \frac{1}{|h|^{d+\alpha}} dh - \frac{C^{(2)}_{d,\alpha}}{2} \int_{B_{3}} D_{h}f(x + k) \frac{1}{|h|^{d+\alpha}} dh.
\]
In the first integral we have \( x + k \pm h \in B_{4}(x) \) for every \( k \in B_{1} \) and \( h \in B_{3} \), hence \( |D_{h}f(x + k)| \leq 8C^{(3)}(x)|h|^{\alpha+\delta} \), similarly to in the previous case, where \( C^{(3)}(x) \) is the Hölder constant of \( \nabla f \) in \( B_{4}(x) \). Thus we obtain
\[
\int_{B_{3}} \frac{|D_{h}f(x + k)|}{|h|^{d+\alpha}} dh \leq 8C^{(3)}d\omega^{\delta} \frac{\delta}{\delta} \int_{B_{3}} \frac{dh}{|h|^{d-\sigma}}.
\]
For the second integral, using that \( f \in L^{\infty}(\mathbb{R}^{d}) \) we get
\[
\int_{B_{3}} \frac{|D_{h}f(x + k)|}{|h|^{d+\alpha}} dh \leq 4 \|f\|_{\infty} \int_{B_{3}} \frac{dh}{|h|^{d+\alpha}} < \infty.
\]
The proof is then completed by dominated convergence. □

Theorem 5.2. Let \( \varphi_{0} \) be the ground state of \( H_{m,\alpha} \). The following hold:
\begin{enumerate}
  \item If \( \alpha \in (0, 1) \), then \( \varphi_{0} \notin C^{\alpha+\delta}_{\text{loc}}(\mathbb{R}^{d}) \) for every \( \delta \in (0, 1 - \alpha) \).
  \item If \( \alpha \in [1, 2) \), then \( \varphi_{0} \notin C^{1,\alpha+\delta-1}_{\text{loc}}(\mathbb{R}^{d}) \) for every \( \delta \in (0, 2 - \alpha) \).
\end{enumerate}
Proof. We rewrite the eigenvalue equation like

$$L_{m, \alpha} \varphi_0 = (v \mathbf{1}_{B_a} + \lambda_0) \varphi_0.$$  \hspace{1cm} (5.10)

Suppose that $\alpha \in (0, 1)$ and $\varphi_0 \in C^{\alpha+\delta}_{\text{loc}}(\mathbb{R}^d)$ for some $\delta \in (0, 1 - \alpha)$. Then by (1) of Lemma 5.3 we have that the left-hand side of (5.10) is continuous. On the other hand, take $e_1 = (1, 0, \ldots, 0)$ and notice that

$$\lim_{\varepsilon \downarrow 0} (v \mathbf{1}_{B_a}((a + \varepsilon)e_1) + \lambda_0) \varphi_0((a + \varepsilon)e_1) = \lambda_0 \varphi_0(ae_1)$$

$$\lim_{\varepsilon \downarrow 0} (v \mathbf{1}_{B_a}((a - \varepsilon)e_1) + \lambda_0) \varphi_0((a - \varepsilon)e_1) = (v + \lambda_0) \varphi_0(ae_1),$$

thus the right-hand side is continuous in $ae_1$ if and only if $\varphi_0(ae_1) = 0$, which is in contradiction with the fact that $\varphi_0$ is positive. In particular, the same argument holds for any point $x \in \partial B_a$, thus the right-hand side of (5.10) has a jump discontinuity on $\partial B_a$, which is impossible since the left-hand side is continuous. The same arguments hold for $\alpha \in [1, 2)$ by using part (2) of Lemma 5.3. \hfill \square

**Remark 5.4.**

(1) Instead of using $C^{\alpha+\delta}_{\text{loc}}(\mathbb{R}^d)$ we also can prove part (1) of Lemma 5.3 with $f \in C^{\alpha+\delta}(B_r(x))$ for some $x \in \mathbb{R}^d$, implying that $L_{m, \alpha}f$ is continuous in $x$. With this localization argument we obtain for $\alpha \in (0, 1)$ that $\varphi_0 \notin C^\alpha_{\text{loc}}(B_{a+\varepsilon} \setminus B_{a-\varepsilon})$, for all $\varepsilon \in (0, a)$ and $\delta \in (0, 1 - \alpha)$. In particular, this implies that $\varphi_0$ cannot be $C^1$ on $\partial B_a$. The same arguments apply to part (2) of Lemma 5.3 and the case $\alpha \geq 1$, implying that $\varphi_0$ cannot be $C^2$ on $\partial B_a$. We note that for the classical case the ground state is $C^1$ but fails to be $C^2$ at the boundary of the potential well.

(2) It is reasonable to expect that $\varphi_0$ has at least a $C^{\alpha-\varepsilon}$-regularity, for all $\varepsilon > 0$ small enough, both inside and outside the potential well (away from the boundary). However, this needs different tools and we do not pursue this point here.

### 5.4. Moment estimates of the position in the ground state

As an application of the local estimates of ground states we consider now the behaviour of the following functional. Note that when the ground state is chosen to satisfy $\|\varphi_0\|_2 = 1$, the expression $\varphi_0^2(x)dx$ defines a probability measure on $\mathbb{R}^d$. Let $p \geq 1$ and define

$$\Lambda_p(\varphi_0) = \left( \int_{\mathbb{R}^d} |x|^p \varphi_0^2(x)dx \right)^{1/p},$$

which can then be interpreted as the $p$th moment of an $\mathbb{R}^d$-valued random variable under this probability distribution. In the physics literature the ground state expectation for $p = 2$ is called the size of the ground state.

Let $m \geq 0$, $\alpha \in (0, 2)$, and define

$$p_*(m, \alpha) := \begin{cases} 
    d + 2\alpha & \text{if } m = 0 \\
    \infty & \text{if } m > 0.
\end{cases}$$

**Lemma 5.4.** The following cases occur:
(1) If $0 < p < p_*(m, \alpha)$, then $\Lambda_p(\varphi_0) < \infty$.
(2) If $p \geq p_*(m, \alpha)$, then $\Lambda_p(\varphi_0) = \infty$.

**Proof.** It is a direct consequence of Corollary 5.1 using that $j_{0, \alpha}(r) = C_{d, \alpha} r^{-d+\alpha}$ as $r \to \infty$ if $m > 0$. Indeed, while for $m = 0$ we have $\rho^{d+p laden=12} j_{0, \alpha}(\rho) = C_{d, \alpha} \rho^{(d+1+2\alpha-p)}$, so that it is integrable at infinity if and only if $d+2\alpha > p$. □

**Proposition 5.4.** Let $0 < p < p_*(m, \alpha)$. Then there exist constants $C_{d,m,\alpha,a,p}^{(1)}, C_{d,m,\alpha,a}^{(2)} > 0$ such that

$$\Lambda_p(\varphi_0) \geq C_{d,m,\alpha,a,p}^{(1)} \varphi_0^{2/p}(a) \left( \frac{\lambda_a}{\lambda_a - v + |\lambda_0|} \right)^{2/p} e^{-\frac{\varphi_0(\lambda_a |\lambda_0|)|\lambda_0|}{p}}.$$

**Proof.** By Remark 5.3 we get

$$\varphi_0^2(x) \geq \varphi_0^2(a)(C_{d,m,\alpha,a}^{(3)})^2 \left( 1 + 2 \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^{a/2} \right. + \left. \left( \frac{v - |\lambda_0|}{\lambda_a - v + |\lambda_0|} \right)^{2} \left( \frac{a - |x|}{a} \right)^{a} \right) e^{-2C_{d,m,\alpha,a}^{(2)} |\lambda_0|} \geq \varphi_0^2(a)(C_{d,m,\alpha,a}^{(3)})^2 \left( \frac{\lambda_a}{\lambda_a - v + |\lambda_0|} \right)^{2} \left( \frac{a - |x|}{a} \right)^{a} e^{-2C_{d,m,\alpha,a}^{(2)} |\lambda_0|} |x| \leq a,$$

where the last step follows by the fact that $\frac{a - |x|}{a} \leq 1$. Hence

$$\int_{\mathbb{R}^d} |x|^p \varphi_0^2(x) dx \geq \int_{B_a} |x|^p \varphi_0^2(x) dx \geq \varphi_0^2(a)(C_{d,m,\alpha,a}^{(3)})^2 \left( \frac{\lambda_a}{\lambda_a - v + |\lambda_0|} \right)^{2} e^{-2C_{d,m,\alpha,a}^{(2)} |\lambda_0|} \int_{B_a} |x|^p \left( \frac{a - |x|}{a} \right)^{a} dx.$$

Setting $C_{d,m,\alpha,a,p}^{(1)} = (C_{d,m,\alpha,a}^{(3)})^2 \int_{B_a} |x|^p \left( \frac{a - |x|}{a} \right)^{a} dx$, the result follows. □

**Proposition 5.5.** Let $0 < p < p_*(m, \alpha)$ and $v > \lambda_a + \delta$ for some $\delta > 0$. Then there exists a constant $C_{d,m,\alpha,a,p} > 0$ such that

$$\Lambda_p(\varphi_0) \leq C_{d,m,\alpha,a,p} \varphi_0^{2/p}(a) \left( \frac{\lambda_a}{\lambda_a - v + |\lambda_0|} \right)^{2/p}.$$

**Proof.** As in Theorem 5.1 observe that for $|x| \geq a$ we have by Proposition 4.2

$$\varphi_0(x) \leq \varphi_0(0) \mathbb{E}^x[e^{-|\lambda_0| T_a}]. \quad (5.11)$$

Moreover, by Remark 5.3

$$\varphi_0(0) \leq \frac{C_{d,m,\alpha,a}^{(1)} \lambda_a}{\lambda_a - v + |\lambda_0|} \varphi_0(a). \quad (5.12)$$

On the other hand, from $v - |\lambda_0| < \lambda_a$ we get $|\lambda_0| > v - \lambda_a > \delta$ and then

$$\mathbb{E}^x[e^{-|\lambda_0| T_a}] \leq \mathbb{E}^x[e^{-\delta T_a}] \leq C_{d,m,\alpha,a,j_{m,a}}^{(2)} |x|, \quad |x| \geq a,$$

where we used also Theorem 5.3. Combining (5.12)-(5.13) with (5.11), we obtain

$$\varphi_0(x) \leq \frac{C_{d,m,\alpha,a}^{(3)} \lambda_a}{\lambda_a - v + |\lambda_0|} \varphi_0(a) j_{m,a}(|x|), \quad |x| \geq a.$$
where \( C_{d,m,\alpha,a}^{(3)} = C_{d,m,\alpha,a}^{(1)} C_{d,m,\alpha,a}^{(2)} \). For \( |x| \leq a \) we have directly by Remark 5.3

\[
\varphi_0(x) \leq C_{d,m,\alpha,a}^{(1)} \varphi_0(a) \left( 1 + \frac{v - |\lambda_0|}{\lambda_0 - v + |\lambda_0|} \left( \frac{a - |x|}{a} \right)^2 \right) \leq \frac{C_{d,m,\alpha,a}^{(1)} \lambda_a}{\lambda_a - v + |\lambda_0|} \varphi_0(a),
\]

where again we used that \( \frac{a - |x|}{a} \leq 1 \). Hence by (5.14)-(5.15) we get

\[
\int_{\mathbb{R}^d} |x|^p \varphi_0^2(x) dx = \int_{B_a} |x|^p \varphi_0^2(x) dx + \int_{\mathbb{R}^d \setminus B_a} |x|^p \varphi_0^2(x) dx \\
\leq C_{d,m,\alpha,a}^{(1)} \left( \frac{\lambda_a}{\lambda_a - v + |\lambda_0|} \right)^2 \varphi_0^2(a),
\]

where

\[
C_{d,m,\alpha,a}^{(1)} = \max \left\{ (C_{d,m,\alpha,a}^{(1)})^2 \int_{B_a} |x|^p dx, (C_{d,m,\alpha,a}^{(3)})^2 \int_{\mathbb{R}^d \setminus B_a} |x|^p \varphi_0^2(x) dx \right\}.
\]

\[
\square
\]

\textbf{Remark 5.5.} As discussed in Section 2.2, a ground state exists for all \( v > 0 \) when the process \((X_t)_{t \geq 0}\) is recurrent, and it only exists for \( v > v^* \) with a given \( v^* = v^*(\alpha, m, a, d) > 0 \) when the process is transient. An interesting question is to analyze the blow-up rate of \( \Lambda_\varphi_0(v) \) for some \( p \) as \( v \downarrow v^* \). This would require a good control of the \( v \)-dependence of \( \lambda_0 \) and the comparability constants, however, both appear to be rather involved. An expression of \( \lambda_0 = \lambda_0(v) \) may in principle be expected to follow from the continuity condition \( \varphi_0(a-) = \varphi_0(a+) \), however, this seems to be difficult to obtain in any neat explicit form. In fact, even in the classical Schrödinger eigenvalue problem this is a transcendental equation which can only numerically be solved, and the similar blow-up problem also becomes untractable in terms of closed form expressions.

5.5. Extension to fully supported decaying potentials

Our technique to derive local estimates on the ground state of a non-local Schrödinger operator with a compactly supported potential can be extended to potentials supported everywhere on \( \mathbb{R}^d \). This is of interest since apart from decay rates as \( |x| \to \infty \) (see [30]), there is no information on the behaviour of the ground state from small to mid range.

Consider a potential \( V(x) = -v(|x|) \) with a continuous non-increasing function \( v : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \lim_{r \to \infty} v(r) = 0 \). We assume that \( H_{m,\alpha} \) has a ground state \( \varphi_0 \) with eigenvalue \( \lambda_0 < 0 \). We already know from Remark 5.4 that \( \varphi_0 \) is radially symmetric, thus we can write \( \varphi_0(x) = \varphi_0(|x|) \) with a suitable \( \varphi_0 : \mathbb{R}^+ \to \mathbb{R}^+ \). Also in this case we will suppose the following condition to hold.

\textbf{Assumption 5.1.} The function \( \varphi_0 : [0, \infty) \to \mathbb{R} \) is non-increasing.

A first main result of this section is as follows.

\textbf{Theorem 5.3.} Let \( \varphi_0 \) be the ground state of \( H_{m,\alpha} \) with \( V(x) = -v(|x|) \), \( v : \mathbb{R}^+ \to \mathbb{R}^+ \) non-increasing and continuous. Let Assumption 5.1 hold and consider any \( \gamma > 0 \) such that the level set \( K_\gamma = \{ x \in \mathbb{R}^d : V(x) < -\gamma \} \neq \emptyset \). Then there exists a constant \( C_{d,m,\alpha,\gamma,|\lambda|} \) such that

\[
C_{d,m,\alpha,\gamma,|\lambda|} \varphi_0(x) \xi [e^{\gamma - |\lambda_0|}r_\gamma] \leq \varphi_0(x) \leq \varphi_0(x) \xi [e^{(v(0) - |\lambda_0|)r_\gamma}], \quad x \in K_\gamma,
\]

where \( r_\gamma = \inf \{ t > 0 : X_t \in K_\gamma^c \} \), \( x_\gamma \in \partial K_\gamma \) is arbitrary and \( r_\gamma = |x_\gamma| \).
Also, notice that by the definition of $C_\gamma$.

By Proposition 4.1 and Assumption 5.1 we have

$$
\varphi_0(x) = \mathbb{E}_x[e^{\int_0^{\tau_{r\gamma}} \nu(|X_s|) ds - |\lambda_0| \tau_{r\gamma}} \varphi_0(X_{\tau_{r\gamma}})] \leq \varphi_0(x) \mathbb{E}_x[e^{(\nu(0) - |\lambda_0|) \tau_{r\gamma}}].
$$

Consider $C_{d,m,\alpha,r\gamma}^{(1)} > 1$ defined in Lemma 5.2 and observe that

$$
\varphi_0(x) \geq \mathbb{E}_x[e^{\int_0^{\tau_{r\gamma}} \nu(|X_s|) ds - |\lambda_0| \tau_{r\gamma}} \varphi_0(X_{\tau_{r\gamma}}); r_{\gamma} \leq X_{\tau_{r\gamma}} \leq C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}] \\
\geq \varphi_0(C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}) \mathbb{E}_x[e^{(\gamma - |\lambda_0|) \tau_{r\gamma}}; r_{\gamma} \leq X_{\tau_{r\gamma}} \leq C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}].
$$

Also, notice that by the definition of $C_{d,m,\alpha,r\gamma}^{(1)}$

$$
\mathbb{E}_x[e^{(\gamma - |\lambda_0|) \tau_{r\gamma}}; r_{\gamma} \leq X_{\tau_{r\gamma}} \leq C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}] \geq \frac{1}{2} \mathbb{E}_x[e^{(\gamma - |\lambda_0|) \tau_{r\gamma}}].
$$

On the other hand, arguing as in Theorem 5.1 we have

$$
\varphi_0(C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}) \geq \mathbb{E}_x[C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma} e^{(-|\lambda_0| \tau_{r\gamma})} \varphi_0(X_{\tau_{r\gamma}})],
$$

where $T_{r\gamma} = \inf\{t > 0 : X_t \in \mathcal{K}_\gamma\}$ and we used the fact that $\nu(|x|) \geq 0$ for all $x \in \mathbb{R}^d$. By Assumption 5.1 we have

$$
\varphi_0(C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}) \geq \varphi_0(x_{\gamma}) \mathbb{E}_x[C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma} e^{(-|\lambda_0| \tau_{r\gamma})}] \geq C_{d,m,\alpha,r\gamma}^{(2)} |\lambda_0| \varphi_0(x_{\gamma}),
$$

where

$$
C_{d,m,\alpha,r\gamma}^{(2)} |\lambda_0| := C_{d,m,\alpha,r\gamma}^{(3)} |\lambda_0| \cdot \varphi_0(C_{d,m,\alpha,r\gamma}^{(1)} x_{\gamma}),
$$

$C_{d,m,\alpha,r\gamma}^{(3)} |\lambda_0|$ is defined in Corollary 5.3 by choosing $R_2 > C_{d,m,\alpha,r\gamma}^{(1)} r_{\gamma}$. Combining (5.17)-(5.18) with (5.16) the claim follows.

**Remark 5.6.** We note that when $\nu(0) - |\lambda_0| \geq \lambda_{r\gamma}$, the upper bound is trivial as $\mathbb{E}_x[e^{(\nu(0) - |\lambda_0|) \tau_{r\gamma}}] = \infty$. Also, if $|\lambda_0| \geq \gamma_0$, then the lower bound is trivial since $\mathbb{E}_x[e^{(\gamma - |\lambda_0|) \tau_{r\gamma}}] \leq 1$ and $\varphi_0(x) \geq \varphi_0(0)$ by Assumption 5.1. Furthermore, by a similar argument as in Step 1 of Theorem 5.1 the implication is that $\gamma - |\lambda_0| < \lambda_{r\gamma}$ whenever $\mathcal{K}_\gamma \neq \emptyset$. In particular, due to $\lim_{\gamma \to \nu(0)} \lambda_{r\gamma} = \infty$, there is a constant $\gamma_0 > 0$ such that $\nu(0) - |\lambda_0| < \lambda_{r\gamma}$ for every $\gamma \in (\gamma_0, \nu(0))$.

Exploiting the asymptotic behaviour of the moment generating function involved as above for the spherical potential well, we have the following result.

**Corollary 5.3.** Let $\varphi_0$ be the ground state of $H_{m,\alpha}$ with $V(x) = -\nu(|x|)$, $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous. Let Assumption 5.1 hold, and consider any $\gamma > 0$ such that the set $\mathcal{K}_\gamma = \{x \in \mathbb{R}^d : V(x) < -\gamma\} \neq \emptyset$, $|\lambda_0| < \gamma$ and $\nu(0) - |\lambda_0| < \lambda_{\mathcal{K}_\gamma}$. Then there exists a constant
$C_{d,m,a,\gamma,|\lambda_0|}^{(1)}$ such that

$$C_{d,m,a,\gamma,|\lambda_0|}^{(1)}(x)\left(1 + \frac{\gamma - |\lambda_0|}{\lambda K_\gamma - \gamma + |\lambda_0|} \left(\frac{r_\gamma - |x|}{r_\gamma}\right)^2\right) \leq \varphi_0(x) \leq C_{d,m,a,\gamma,|\lambda_0|}^{(1)}(x)\left(1 + \frac{v(0) - |\lambda_0|}{\lambda K_\gamma - v(0) + |\lambda_0|} \left(\frac{r_\gamma - |x|}{r_\gamma}\right)^2\right),$$

for every $x \in K_\gamma$, where $r_\gamma = \inf\{t > 0 : X_t \in K_\gamma^c\}$, $x_\gamma \in \partial K_\gamma$ is arbitrary, and $r_\gamma = |x_\gamma|$.

**Proof.** Starting from (5.3) and recalling that $K_\gamma = B_{r_\gamma}$, the upper bound follows from the assumption that $v(0) - |\lambda_0| < \lambda_{r_\gamma}$ and Theorem 5.1. The lower bound follows from Remark 5.6 guaranteeing $\gamma - |\lambda_0| < \lambda_{r_\gamma}$, and furthermore by an application of Theorem 3.1. \(\square\)

**Theorem 5.4.** Let $\varphi_0$ be the ground state of $H_{m,a}$ with $V(x) = -v(|x|)$, $v : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous, and let Assumption 5.1 hold. Let $\gamma_1 \leq |\lambda_0|$ and $\gamma_2 \in (\gamma_0, v(0))$, where $\gamma_0$ is defined as in Remark 5.6, such that $\gamma_1 \leq \gamma_2$. Define $K_{\gamma_i} = \{x \in \mathbb{R}^d, V(x) < -\gamma_i\}$, $i = 1, 2$. Then

$$\varphi_0(x_{\gamma_1})E^x[e^{-|\lambda_0|T_{\gamma_1}}] \leq \varphi_0(x) \leq C_{d,m,a,\gamma_2,|\lambda_0|}\varphi_0(x_{\gamma_2})E^x[e^{(\gamma_1 - |\lambda_0|)T_{\gamma_1}}], \quad x \in K_{\gamma_1},$$

where $x_{\gamma_i} \in \partial K_{\gamma_i}$ and $r_\gamma = |x_{\gamma_i}|$, $i = 1, 2$.

**Proof.** By a similar argument as in Theorem 5.3, there exist $r_\gamma$ such that $K_{\gamma_i} = B_{r_\gamma}$, $i = 1, 2$. Moreover, $K_{\gamma_1} \subseteq K_{\gamma_2}$ since $v$ is non-increasing. Let $x \in K_{\gamma_1}$ and observe that, as in Theorem 5.1

$$\varphi_0(x_{\gamma_1})E^x[e^{-|\lambda_0|T_{\gamma_1}}] \leq \varphi_0(x) \leq \varphi_0(0)E^x[e^{(\gamma_1 - |\lambda_0|)T_{\gamma_1}}],$$

where $x_{\gamma_1} \in \partial B_{r_\gamma}$. Using that $0 \in K_{\gamma_2}$, by Corollary 5.3 we get

$$\varphi_0(0) \leq \varphi_0(x_{\gamma_2})\left(1 + \frac{v(0) - |\lambda_0|}{\lambda r_{\gamma_2} - v(0) + |\lambda_0|}\right) = C_{d,m,a,\gamma_2,|\lambda_0|}\varphi_0(x_{\gamma_2}), \quad x_{\gamma_2} \in \partial B_{r_{\gamma_2}}.$$  \(\square\)

Again, by using the asymptotics of the Laplace transform of the hitting times we get the following.

**Corollary 5.4.** Let $\varphi_0$ be the ground state of $H_{m,a}$ with $V(x) = -v(|x|)$, $v : \mathbb{R}^+ \to \mathbb{R}^+$ non-increasing and continuous, and let Assumption 5.1 hold. Choose $\gamma_1 \leq |\lambda_0|$ and $\gamma_2 \in (\gamma_0, v(0))$, where $\gamma_0$ is defined in Remark 5.6, such that $\gamma_1 \leq \gamma_2$. Define $K_{\gamma_i} = \{x \in \mathbb{R}^d, V(x) < -\gamma_i\}$, $i = 1, 2$. Then

$$C_{d,m,a,\gamma_1,|\lambda_0|}^{(1)}\varphi_0(x_{\gamma_2})j_{m,a}(|x|) \leq \varphi_0(x) \leq C_{d,m,a,\gamma_2,|\lambda_0|}^{(2)}\varphi_0(x_{\gamma_2})j_{m,a}(|x|),$$

where $x_{\gamma_i} \in \partial K_{\gamma_i}$ and $r_\gamma = |x_{\gamma_i}|$, $i = 1, 2$.

**Proof.** The upper bound follows directly by Theorems 5.4 and 3.3. For the lower bound first consider the potential well $\tilde{V} = -\tilde{v}1_{K_{\gamma_1}}$, where $\tilde{v}$ is chosen to be large enough to guarantee the existence of a ground state $\tilde{\varphi}_0$. Recall that $K_{\gamma_1}$ is an open ball. By Corollary 5.3 we know that

$$\frac{\tilde{\varphi}_0(x)}{\varphi_0(x_{\gamma_1})} \geq C_{d,m,a,\gamma_1,|\lambda_0|}^{(3)}j_{m,a}(|x|), \quad x \in K_{\gamma_1}^c.$$

On the other hand, by Theorem 5.1 we get

$$E^x[e^{-|\lambda_0|T_{\gamma_1}}] \geq C_{d,m,a,\gamma_1,|\lambda_0|}^{(4)}\frac{\tilde{\varphi}_0(x)}{\varphi_0(x_{\gamma_1})}, \quad x \in K_{\gamma_1}^c.$$
Combining the previous estimates with the lower bound in Theorem 5.4, the statement follows. □

References

[1] V. Ambrosio: The nonlinear fractional relativistic Schrödinger equation: existence, decay and concentration results, *Discr. Cont. Dyn. Syst.* **41**, 5659-5705, 2021
[2] T.W. Anderson: The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities, *Proc. Amer. Math. Soc.* **6**, 170-176, 1955
[3] G. Ascione, J. Lörinczi: Potentials for non-local Schrödinger operators with zero eigenvalues, *J. Diff. Equations* **317**, 264-364, 2022
[4] G. Ascione, J. Lörinczi: Stability of ground state eigenvalues of non-local Schrödinger operators with respect to potentials and applications, [arXiv:2211.10093](https://arxiv.org/abs/2211.10093), 2022
[5] R. Bañuelos, T. Kulczycki: The Cauchy process and the Steklov problem, *J. Funct. Anal.* **211**, 355-423, 2004
[6] R. Bañuelos, T. Kulczycki, P. Méndez-Hernández: On the shape of the ground state eigenfunction for stable processes, *Trans. AMS* **361**, 4871-4900, 2009
[7] T. Beck: Uniform level set estimates for ground state eigenfunctions, *SIAM J. Math. Anal.* **50**, 4483-4502, 2018
[8] J. Bertoin: *Lévy Processes*, Cambridge University Press, 1996
[9] A. Biswas, J. Lörinczi: Universal constraints on the location of extrema of eigenfunctions of non-local Schrödinger operators, *J. Diff. Equations* **267**, 267-306, 2019
[10] K. Bogdan et al.: *Potential Analysis of Stable Processes and its Extensions*, Lecture Notes in Mathematics **1980**, Springer, 2009
[11] K. Bogdan, T. Grzywny, M. Ryznar: Barriers, exit time and survival probability for unimodal Lévy processes, *Probab. Theory Rel. Fields* **162**, 155-198, 2015
[12] A.N. Borodin, P. Salminen: *Handbook of Brownian Motion – Facts and Formulae*, Birkhäuser, 2002
[13] H.J. Briegel, B.G. Englert, M. Michaelis, G. Süssman: Über die Wurzel aus der Klein-Gordon Gleichung als Schrödinger-Gleichung eines relativistischen Spin-0-Teilchens, *Z. Naturforsch.* **46a**, 925-932, 1991
[14] T. Byczkowski, J. Majka, M. Ryznar: Bessel potentials, hitting distributions and Green functions, *Trans. AMS* **361**, 4871-4900, 2009
[15] R. Carmona, W.C. Masters, B. Simon: Relativistic Schrödinger operators: asymptotic behaviour of the eigenfunctions, *J. Funct. Anal.* **91**, 47-114, 1990
[16] Z.Q. Chen, P. Kim, R. Song: Heat kernel estimates for the Dirichlet fractional Laplacian, *J. Eur. Math. Soc.* **12**, 1307-1329, 2010
[17] Z.Q. Chen, P. Kim, R. Song: Sharp heat kernel estimates for relativistic stable processes in open sets, *Ann. Probab.* **40**, 213-244, 2012
[18] Z. Ciesielski, S.J. Taylor: First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path, *Trans. AMS* **103**, 434-450, 1962
[19] V. Coti Zelati, M. Nolasco: Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations, *Commun. Math. Phys.* **228**, 51-72, 2002
[20] I. Daubechies: One-electron molecules with relativistic kinetic energy: properties of the discrete spectrum, *Commun. Math. Phys.* **94**, 523-535, 1984
[21] I. Daubechies, E.H. Lieb: One-electron relativistic molecules with Coulomb interaction, *Commun. Math. Phys.* **90**, 497-510, 1983
[22] E.B. Davies: *Heat Kernels and Spectral Theory*, Cambridge University Press, 1989
[23] E. Di Nezza, G. Palatucci, E. Valdinoci: Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136**, 521-573, 2012
[24] M. D’Ovidio, E. Orsingher: Bessel processes and hyperbolic Brownian motions stopped at different random times, *Stoc. Proc. Appl.* **121**, 441-465, 2011
[25] M.M. Fall, V. Felli: Unique continuation properties for relativistic Schrödinger operators with a singular potential, *Discrete Contin. Dyn. Syst.* **35**, 5827-5867, 2015
[26] C. Fefferman, R. de la Llave: Relativistic stability of matter I, *Rev. Mat. Iberoam.* **2**, 119-213, 1986
[27] R.L. Frank, E.H. Lieb, R. Seiringer: Stability of relativistic matter with magnetic fields for nuclear charges up to the critical value, *Commun. Math. Phys.* **275**, 479-489, 2007
[28] N. Fusco, V. Millot, M. Morini: A quantitative isoperimetric inequality for fractional perimeters, *J. Funct. Anal.* **261**, 697-715, 2011
[29] T. Grzywny: Intrinsic ultracontractivity for Lévy processes, *Probab. Math. Stat.* **28**, 91-106, 2008
[30] T. Grzywny, M. Ryznar: Two-sided optimal bounds for Green functions of half-spaces for relativistic α-stable process, *Potential Anal.* **28**, 201-239, 2008
[31] I.W. Herbst: Spectral theory of the operator \((p^2 + m^2)^{1/2} - Ze^2/r\), *Commun. Math. Phys.* **53**, 285-294, 1977
[32] F. Hiroshima, T. Ichinose, J. Lörinczi: Probabilistic representation and fall-off of bound states of relativistic Schrödinger operators with spin 1/2, *Publ. Res. Inst. Math. Sci.* 49, 189-214, 2013

[33] F. Hiroshima, T. Ichinose, J. Lörinczi: Kato’s inequality for magnetic relativistic Schrödinger operators, *Publ. Res. Inst. Math. Sci.* 53, 79-117, 2017

[34] N. Ikeda, S. Watanabe: On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, *J. Math. Kyoto Univ.* 2-1, 79-95, 1961

[35] K. Kaleta, J. Lörinczi: Pointwise estimates of the eigenfunctions and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes, *Ann. Probab.* 43, 1350-1398, 2015

[36] K. Kaleta, J. Lörinczi: Fall-off of eigenfunctions for non-local Schrödinger operators with decaying potentials, *Potential Anal.* 46, 647-688, 2017

[37] K. Kaleta, J. Malecki, M. Kwasnicki: One-dimensional quasi-relativistic particle in the box, *Rev. Math. Phys.* 25, 1350014, 2013

[38] P. Kim, R. Song, Z. Vondraček: Boundary Harnack principle for subordinate Brownian motions, *Stoch. Proc. Appl.* 119, 1601-1631, 2009

[39] M. Kwasnicki, J. Malecki, M. Ryznar: Suprema of Lévy processes, *Ann. Probab.* 41, 2047-2065, 2013

[40] C. Lämmerzahl: The pseudo-differential operator square root of the Klein-Gordon equation, *J. Math. Phys.* 34, 3918-3932, 1993

[41] M. Lemm: On the Hölder regularity for the fractional Schrödinger equation and its improvement for radial data, *Commun. Partial Diff. Eq.* 41, 1761-1792, 2016

[42] P. Lévy: La mesure de Hausdorff de la courbe du mouvement brownien à n dimensions, *C. R. Acad. Sci. Paris* 233, 600-602, 1951

[43] E.H. Lieb, R. Seiringer: *The Stability of Matter in Quantum Mechanics*, Cambridge University Press, 2010

[44] E.H. Lieb, H.T. Yau: The stability and instability of relativistic matter, *Commun. Math. Phys.* 118, 177-213, 1988

[45] J. Lörinczi, F. Hiroshima, V. Betz: *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space. With Applications to Rigorous Quantum Field Theory*, de Gruyter Studies in Mathematics 34, Walter de Gruyter, 2011; 2nd rev. exp. ed., vol. 1, 2020

[46] J. Lörinczi, J. Malecki: Spectral properties of the massless relativistic harmonic oscillator, *J. Diff. Equations* 253, 2846-2871, 2012

[47] J. Lörinczi, I. Sasaki: Embedded eigenvalues and Neumann-Wigner potentials for relativistic Schrödinger operators, *J. Funct. Anal.* 273, 1548-1575, 2017

[48] D. Revuz, M. Yor: *Continuous Martingales and Brownian Motion*, Springer, 2013

[49] S. Secchi: On some nonlinear fractional equations involving the Bessel operator, *J. Dyn. Diff. Equat.* 29, 1173-1193, 2017

[50] M.L. Silverstein: Classification of coharmonic and coinvariant functions for a Lévy process, *Ann. Probab.* 539-575, 1980

[51] J. Sucher: Relativistic invariance and the square-root Klein-Gordon equation, *J. Math. Phys.* 4, 17-23, 1963

[52] G.N. Watson: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 2nd ed., 1966

[53] R.A. Weder: Spectral properties of one-body relativistic spin-zero Hamiltonians, *Ann. IHP, Sect. A (N.S.)* 20, 211-220, 1974

Giacomo Ascione, Scuola Superiore Meridionale, Università degli Studi di Napoli Federico II, 80126 Napoli, Italy

Email address: giacomo.ascione@unina.it

József Lörinczi, Alfréd Rényi Institute of Mathematics, 1053 Budapest, Hungary

Email address: lorinczi@renyi.hu