Deviation inequalities for martingales with applications to linear regressions and weak invariance principles

Xiequan Fan

Regularity Team, Inria and MAS Laboratory, Ecole Centrale Paris - Grande Voie des Vignes, 92295 Châtenay-Malabry, France.

Abstract

Using changes of probability measure developed by Grama and Haeusler (Stochastic Process. Appl., 2000), we obtain two generalizations of the deviation inequalities of Lanzinger and Stadtmüller (Stochastic Process. Appl., 2000) and Fuk and Nagaev (Theory Probab. Appl., 1971) to the case of martingales. Our inequalities recover the best possible decaying rate of independent case. Applications to linear regressions and weak invariance principles for martingales are provided.

Keywords: Large deviations; Martingales; Deviation inequalities; Linear regressions; Weak invariance principles

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1. Introduction

Assume that \((\xi_i)_{i \geq 1}\) is a sequence of independent and identically distributed (i.i.d.) random variables satisfying the following subexponential condition: for a constant \(\alpha \in (0, 1)\),

\[
K := \mathbb{E}[\xi_1^2 \exp\{(\xi_1^+)^\alpha\}] < \infty,
\]

where \(x^+ = \max\{x, 0\}\). Denote by \(S_n = \sum_{i=1}^n \xi_i\) the partial sums of \((\xi_i)_{i \geq 1}\). Lanzinger and Stadtmüller [17] have obtained the following subexponential inequality: for any \(x, y > 0\),

\[
\mathbb{P}(S_n \geq x) \leq \exp\left\{ -\frac{x}{y^{1-\alpha}} \left(1 - \frac{nK}{2xy^{1-\alpha}}\right) \right\} + \frac{n}{e^y} \mathbb{E}[\exp\{(\xi_1^+)^\alpha\}].
\]

In particular, by taking \(y = x\), inequality (2) implies that for any \(x > 0\),

\[
\limsup_{n \to \infty} \frac{1}{n^\alpha} \log \mathbb{P}(S_n \geq nx) \leq -x^\alpha
\]

and

\[
\mathbb{P}(S_n \geq n) = O\left(\exp\left\{-cn^\alpha\right\}\right), \quad n \to \infty,
\]

where \(c\) is a constant.
where $c > 0$ does not depend on $n$. The last two results (3) and (4) are the best possible under the present condition, since a large deviation principle (LDP) with good rate function $x^\alpha$ can be obtained in situations where some more information on the tail behavior of $\xi_1$ is available; see Nagaev [22]. Under the subexponential condition, more precise estimations on tail probabilities, or large deviation expansions, can be found in Nagaev [22, 23], Saulis and Statulevičius [26] and Borovkov [3, 4].

Our first aim is to give a generalization of (4) for martingales. Let $(\xi_i, F_i)_{i \geq 1}$ be a sequence of martingale differences. Under the Cramér condition sup$_i E[\exp\{\xi_i\}] < \infty$, Lesigne and Volný [18] firstly proved that (4) holds with $\alpha = 1/3$, and that the power $1/3$ is optimal even for the class of stationary and ergodic sequence of martingale differences. Later, Fan, Grama and Liu [9] generalized the result of Lesigne and Volný by proving that (4) holds under the moment condition sup$_i E[\exp\{\xi_i^{2/\alpha}\}] < \infty$, and that the power $\alpha$ in (4) is optimal for the class of stationary sequence of martingale differences. It is obvious that the condition sup$_i E[\exp\{\xi_i^{2/\alpha}\}] < \infty$ is much stronger than condition (4). Thus, the result of Fan, Grama and Liu [9] does not imply (4) in the independent case.

To fill this gap, we consider the case of the martingale differences having bounded conditional subexponential moments. Under this assumption, we can recover the inequalities (2), (3) and (4); see Theorem 2.1. Our first result implies that if

$$u := \max\left\{\left|\sum_{i=1}^n E[\xi_i^2 \exp\{\xi_i^+\alpha\}|F_{i-1}]\right|_\infty + 1 \right\} < \infty,$$

then we have for any $x > 0$,

$$P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq 2 \exp\left\{-\frac{x^2}{2(u + x^{2\alpha})}\right\}.$$  

(5)

To illustrate our result, consider the simple case that $(\xi_i)_{i \geq 1}$ are i.i.d.. Then we have $u = O(n)$ as $n \to \infty$. It is interesting to see that when $0 \leq x = o(n^{1/(2-h)})$, our bound (5) is sub-Gaussian exp\{-$x^2/(2u)$\}, and then it is very tight. When $x = ny$ with $y > 0$ fixed, our bound (5) is subexponential exp \{-$cy^n\$}, where $cy > 0$ does not depend on $n$. This coincides with (4). Moreover, we find that even if $(\xi_i)_{i \geq 1}$ is not stationary, more precisely $u = o(n^{2-h})$, inequality (3) is also true; see Corollary 2.1.

For the methods, an approach for obtaining subexponential bound is to combine the method of Lanzinger and Stadtmüller [17] and the tower property of conditional expectation. This approach has been applied in Fuk [12], Liu and Waterl [20] and Dedecker and Fan [5]. With this approach, one can obtain inequality (2) with

$$nK = \sum_{i=1}^n \left|E[\xi_i^2 \exp\{\xi_i^+\alpha\}|F_{i-1}]\right|_\infty.$$

However, this result is not the best possible in some cases. In this paper, we introduce a better method based on changes of probability measure developed by Grama and Haesler [16] (see also [5]). With this method, we obtain inequality (2) with

$$nK = \left|\sum_{i=1}^n E[\xi_i^2 \exp\{\xi_i^+\alpha\}|F_{i-1}]\right|_\infty.$$
Since the later \( nK \) is less than the former one, i.e.,

\[
\left\| \sum_{i=1}^{n} \mathbb{E}[\xi_i^2 \exp\{((\xi_i)^+)\alpha|\mathcal{F}_{i-1}\}] \right\|_\infty \leq \sum_{i=1}^{n} \left\| \mathbb{E}[\xi_i^2 \exp\{((\xi_i)^+)\alpha|\mathcal{F}_{i-1}\}] \right\|_\infty,
\]

our method has certain significant advantage.

As first example to illustrate this advantage, consider the case of self-normalized deviations. Assume that \((\varepsilon_i)_{i=1,\ldots,n}\) is a sequence of independent, unbounded and symmetric around 0 random variables. Denote by \(\xi_i = \varepsilon_i / \sqrt{\sum_{i=1}^{n} \varepsilon_i^2}\) and \(\mathcal{F}_i = \sigma\{\varepsilon_j, 1 \leq j \leq i, |\varepsilon_k|, 1 \leq k \leq n\}\). Then \((\xi_i, \mathcal{F}_i)_{i=1,\ldots,n}\) is also a sequence of martingale differences. It is easy to see that

\[
\left\| \sum_{i=1}^{n} \mathbb{E}[(\xi_i)^2 \exp\{((\xi_i)^+)\alpha|\mathcal{F}_{i-1}\}] \right\|_\infty \leq \left\| \sum_{i=1}^{n} \frac{\varepsilon_i^2}{\sum_{i=1}^{n} \varepsilon_i^2} \right\|_\infty = e,
\]

and that, by the fact that \((\varepsilon_i)_{i=1,\ldots,n}\) are unbounded,

\[
\sum_{i=1}^{n} \left\| \mathbb{E}[(\xi_i)^2 \exp\{((\xi_i)^+)\alpha|\mathcal{F}_{i-1}\}] \right\|_\infty \geq \sum_{i=1}^{n} \left\| \frac{\varepsilon_i^2}{\sum_{i=1}^{n} \varepsilon_i^2} \right\|_\infty = n.
\]

Second example illustrates this advantage. Assume that \((\varepsilon_i)_{i=1,\ldots,n}\) is a sequence of independent and unbounded random variables, and that \((\varepsilon_i)_{i=1,\ldots,n}\) is independent of \((\xi_i, \mathcal{F}_i)_{i=1,\ldots,n}\). Assume that

\[
\left\| \mathbb{E}[\xi_i^2|\mathcal{F}_{i-1}] \right\|_\infty \geq 1 \quad \text{and} \quad \left\| \mathbb{E}[\xi_i^2 \exp\{\xi_i^\alpha|\mathcal{F}_{i-1}\}] \right\|_\infty \leq D
\]

for a constant \(D\) and all \(i = 1, \ldots, n\). Denote by \(\xi'_i = \xi_i \varepsilon_i / \sqrt{\sum_{i=1}^{n} \varepsilon_i^2}\) and \(\mathcal{F}'_i = \sigma\{\varepsilon_j, 1 \leq j \leq i, |\varepsilon_k|, 1 \leq k \leq n\}\). Then \((\xi'_i, \mathcal{F}'_i)_{i=1,\ldots,n}\) is also a sequence of martingale differences. It is easy to see that

\[
\left\| \sum_{i=1}^{n} \mathbb{E}[(\xi'_i)^2 \exp\{((\xi'_i)^+)\alpha|\mathcal{F}'_{i-1}\}] \right\|_\infty \leq \left\| \sum_{i=1}^{n} \frac{\varepsilon_i^2}{\sum_{i=1}^{n} \varepsilon_i^2} \mathbb{E}[(\xi'_i)^2 \exp\{\xi'_i^\alpha|\mathcal{F}_{i-1}\}] \right\|_\infty \leq D,
\]

and that, by the fact that \((\varepsilon_i)_{i=1,\ldots,n}\) are unbounded,

\[
\sum_{i=1}^{n} \left\| \mathbb{E}[(\xi'_i)^2 \exp\{((\xi'_i)^+)\alpha|\mathcal{F}'_{i-1}\}] \right\|_\infty \geq \sum_{i=1}^{n} \left\| \frac{\varepsilon_i^2}{\sum_{i=1}^{n} \varepsilon_i^2} \mathbb{E}[(\xi'_i)^2|\mathcal{F}_{i-1}] \right\|_\infty \geq \sum_{i=1}^{n} \left\| \frac{\varepsilon_i^2}{\sum_{i=1}^{n} \varepsilon_i^2} \right\|_\infty = n.
\]

Thus the advantage of our method is significant.

With changes of probability measure, we also generalize the following inequality of Fuk for martingales (cf. Corollary 3’ of Fuk [12]; see also Nagaev [23] for independent case): if \(\mathbb{E}[(|\xi_i|^p|\mathcal{F}_{i-1}) < \infty\) for a \(p \geq 2\) and all \(i \in [1, n]\), then for any \(x > 0\),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left\{-\frac{x^2}{2\text{V}^2}\right\} + C_p \frac{x}{\text{V}^2},
\]

(6)
where
\[
\tilde{V}^2 := \frac{1}{4}(p + 2)^2 e^p \sum_{i=1}^{n} \|E[\xi_i^2 | F_{i-1}]\|_{\infty} \quad \text{and} \quad \tilde{C}_p := \left(1 + \frac{2}{p}\right)^p \sum_{i=1}^{n} \|E[|\xi_i|^p | F_{i-1}]\|_{\infty}.
\]

In Corollary 2.2, we prove that (6) holds true when \(\tilde{V}^2 \) and \(\tilde{C}_p \) are replaced by the following two smaller values \(V^2 \) and \(C_p \), respectively, where
\[
V^2 := \frac{1}{4}(p + 2)^2 e^p \sum_{i=1}^{n} \|E[\xi_i^2 | F_{i-1}]\|_{\infty} \quad \text{and} \quad C_p := \left(1 + \frac{2}{p}\right)^p \sum_{i=1}^{n} \|E[|\xi_i|^p | F_{i-1}]\|_{\infty}.
\]

To illustrate the improvement of Corollary 2.2 on Fuk’s inequality (6), consider the following comparison between \(C_p \) and \(\tilde{C}_p \) in the case of random weighted self-normalized deviations. As before, assume that \((\varepsilon_i)_{i=1, \ldots, n}\) is a sequence of independent and unbounded random variables, and that \((\varepsilon_i)_{i=1, \ldots, n}\) is independent of \((\xi_i, F_i)_{i=1, \ldots, n}\). Assume
\[
1 \leq \left\|E[|\xi_i|^p | F_{i-1}]\right\|_{\infty} \leq E
\]
for a constant \(E\) and all \(i = 1, \ldots, n\). Denote by \(\xi_i' = \xi_i \varepsilon_i / (\sum_{i=1}^{n} \varepsilon_i^p)^{1/p} \) and \(F_k' = \sigma\{F_i, |\varepsilon_k|, 1 \leq k \leq n\} \). Then \((\xi_i', F_i')_{i=1, \ldots, n}\) is also a sequence of martingale differences. It is easy to see that
\[
\left\|\sum_{i=1}^{n} E[|\xi_i'|^p | F_{i-1}']\right\|_{\infty} \leq \left\|\sum_{i=1}^{n} \frac{\varepsilon_i^p}{\sum_{i=1}^{n} \varepsilon_i^p} E[|\xi_i|^p | F_{i-1}]\right\|_{\infty} \leq E,
\]
and that, by the fact that \((\varepsilon_i)_{i=1, \ldots, n}\) are unbounded,
\[
\sum_{i=1}^{n} \left\|E[|\xi_i'|^p | F_{i-1}']\right\|_{\infty} \geq \sum_{i=1}^{n} \left\|\frac{\varepsilon_i^p}{\sum_{i=1}^{n} \varepsilon_i^p} E[|\xi_i|^p | F_{i-1}]\right\|_{\infty} \geq n.
\]
Hence \(C_p \) is much smaller than \(\tilde{C}_p \). The improvement of Corollary 2.2 on Fuk’s inequality (6) is significant.

For two positive constants \(\delta \) and \(C\), assume either \(E[|\xi_i|^p + \delta] \leq C \) and \(E[\xi_i^2 | F_{i-1}] \leq C \) a.s. or \(E[|\xi_i|^p | F_{i-1}] \leq C \) a.s. for a \(p \geq 2 \) and all \(i = 1, \ldots, n\). Then we have for any \(\alpha \in (\frac{1}{2}, \infty)\),
\[
P\left(\max_{1 \leq k \leq n} S_k \geq n^\alpha\right) = O\left(\frac{1}{n^{\alpha p - 1}}\right), \quad n \to \infty,
\]
See Theorem 2.3 and Corollary 2.2. Under a stronger condition that \((\xi_i)_{i=1, \ldots, n}\) have bounded conditional moments, inequality (7) improves a result of Lesigne and Volný [18] where Lesigne and Volný proved that if \(E[|\xi_i|^p] \leq C \) for a constant \(C\), then
\[
P\left(S_n \geq n\right) = O\left(\frac{1}{n^{p/2}}\right), \quad n \to \infty,
\]
and that the order \(n^{-p/2}\) of the last inequality is optimal even for the class of stationary and ergodic sequence of martingale differences.

The paper is organized as follows. We present our main results in Section 2 and discuss the applications to linear regressions and weak invariance principle in Section 3. The proofs of theorems are given in Sections 4-8.
2. Main results

Assume that we are given a sequence of real-value martingale differences \((\xi_i, F_i)_{i=0,\ldots,n}\) defined on some probability space \((\Omega, F, P)\), where \(\xi_0 = 0\) and \(\emptyset = F_0 \subseteq \ldots \subseteq F_n \subseteq F\) are increasing \(\sigma\)-fields. So we have \(E[\xi_i|F_{i-1}] = 0, \ i = 1, \ldots, n\), by definition. Set

\[
S_0 = 0 \quad \text{and} \quad S_k = \sum_{i=1}^k \xi_i, \quad k = 1, \ldots, n. \tag{9}
\]

Then \(S := (S_k, F_k)_{k=0,\ldots,n}\) is a martingale. Let \(\langle S \rangle\) be the quadratic characteristic of the martingale \(S\):

\[
\langle S \rangle_0 = 0 \quad \text{and} \quad \langle S \rangle_k = \sum_{i=1}^k E[\xi_i^2|F_{i-1}], \quad k = 1, \ldots, n. \tag{10}
\]

Our first result is the following subexponential inequality on tail probabilities for martingales. A similar inequality for separately Lipschitz functionals has been obtained recently by Dedecker and Fan [5].

**Theorem 2.1.** Assume

\[
C_n := \sum_{i=1}^n E[\xi_i^2 \exp\{(\xi_i^+)^\alpha\}] < \infty
\]

for a constant \(\alpha \in (0, 1)\). Denote by

\[
\Upsilon(S)_k = \sum_{i=1}^k E[\xi_i^2 \exp\{(\xi_i^+)^\alpha\}|F_{i-1}], \quad k \in [1, n]. \tag{11}
\]

Then for all \(x, u > 0\),

\[
P\left(S_k \geq x \ \text{and} \ \Upsilon(S)_k \leq u \ \text{for some} \ k \in [1, n]\right) \leq \begin{cases} 
\exp\left\{ -\frac{x^2}{2u} + C_n \left(\frac{x}{u}\right)^{2(1-\alpha)} \exp\left\{ -\left(\frac{u}{x}\right)^{\alpha/(1-\alpha)} \right\} \right\} & \text{if} \ 0 \leq x < u^{1/(2-\alpha)} \\
\exp\left\{ -x^\alpha \left(1 - \frac{u}{2x^{2-\alpha}}\right) \right\} + C_n \frac{1}{x^2} \exp\left\{ -x^\alpha \right\} & \text{if} \ x \geq u^{1/(2-\alpha)} \end{cases} \tag{12}
\]

It is obvious that

\[
C_n = E[\Upsilon(S)_n] \leq ||\Upsilon(S)_n||_\infty.
\]

Hence, if \(u \geq \max\{||\Upsilon(S)_n||_\infty, 1\}\), then (12) implies the following rough bounds

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \begin{cases} 
2 \exp\left\{ -\frac{x^2}{2u} \right\} & \text{if} \ 0 \leq x < u^{1/(2-\alpha)} \\
2 \exp\left\{ -\frac{1}{2} x^\alpha \right\} & \text{if} \ x \geq u^{1/(2-\alpha)} \end{cases} \tag{13}
\]

\[
\leq 2 \exp\left\{ -\frac{x^2}{2(u + x^{2-\alpha})} \right\}. \tag{14}
\]
Thus for moderate \( x \in (0, u^{1/(2-\alpha)}) \), bound (12) is sub-Gaussian. For all \( x \geq u^{1/(2-\alpha)} \), bound (12) is subexponential, and is of the order \( \exp\left\{ -\frac{1}{2} x^\alpha \right\} \). Moreover, when \( \frac{x}{u^{1/(2-\alpha)}} \to \infty \), by (12), this order can be improved to \( \exp\left\{ -(1 + \varepsilon) x^\alpha \right\} \) for any given \( \varepsilon > 0 \).

Define

\[
\hat{\Upsilon}(S) = \sum_{i=1}^{k} \mathbb{E}[\xi_i^2 \exp\{|\xi_i|^\alpha\} | F_i-1]
\]

for any \( k \in [1, n] \). Then it holds \( \Upsilon(S) \leq \hat{\Upsilon}(S) \). It is obvious that bound (12) is also the upper bound on the tail probabilities

\[
P\left( \pm S_k \geq x \text{ and } \hat{\Upsilon}(S) \leq u \text{ for some } k \in [1, n] \right).
\]

Moreover, if \( \|\hat{\Upsilon}(S)\|_\infty \leq u \), then bound (12) is an upper bound on the partial sums tail probabilities \( P(\pm \max_{1 \leq k \leq n} S_k \geq x) \).

When \( (\xi_i)_{i \geq 1} \) are i.i.d. random variables, then we have \( C_n = \Upsilon(S) = O(n) \) as \( n \to \infty \). In this case, inequality (14) implies the following large deviation inequality: for any \( x > 0 \),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) = O\left( \exp\left\{ -c_x n^\alpha \right\} \right), \quad n \to \infty,
\]

where \( c_x > 0 \) does not depend on \( n \). Moreover, the following LDP result for martingales shows that \( c_x \) in (15) is close to \( x^\alpha \), and that \( \|\Upsilon(S)\|_\infty \) is allowed to tend to infinity in an order larger than \( n \).

**Corollary 2.1.** Assume condition of Theorem 2.1. If

\[
\|\Upsilon(S)\|_\infty = o(n^{2-\alpha}), \quad n \to \infty,
\]

then for any \( x \geq 0 \),

\[
\limsup_{n \to \infty} \frac{1}{n^\alpha} \log P\left( \max_{1 \leq k \leq n} \frac{1}{n} S_k \geq x \right) \leq -x^\alpha.
\]

**Remark 2.1.** This result cannot be improved under the present condition even for the class of i.i.d. random variables; see Nagaev [22]. In fact, if \( (\xi_i)_{i \geq 1} \) are i.i.d. and satisfy the following condition for an integer \( p \geq 2 \) and all \( x \) large enough,

\[
\frac{1}{x^{2p}} \exp\left\{ -x^\alpha \right\} \leq P\left( |\xi_1| \geq x \right) \leq \frac{1}{x^p} \exp\left\{ -x^\alpha \right\},
\]

then we have \( \Upsilon(S) = o(n^{2-\alpha}) \) as \( n \to \infty \) and for any \( x > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log P\left( \max_{1 \leq k \leq n} \frac{1}{n} S_k \geq x \right) = -x^\alpha.
\]

If the martingale differences \( (\xi_i, F_i)_{i=0,\ldots,n} \) have \( p \)-th moments \( (p \geq 2) \), then we have the following inequality, which is similar to the results of Haeusler [13] and [8].
Theorem 2.2. Let $p \geq 2$. Assume $\mathbf{E}[|\xi_i|^p] < \infty$ for all $i \in [1, n]$. Denote by

$$\Xi(S)_k = \sum_{i=1}^k \mathbf{E}[(\xi_i^+)^p|\mathcal{F}_{i-1}], \quad k \in [1, n].$$

Then for all $x, y, v, w > 0$,

$$P\left(S_k \geq x, \langle S \rangle_k \leq v \text{ and } \Xi(S)_k \leq w \text{ for some } k \in [1, n]\right) \leq \exp\left\{-\frac{\alpha^2 x^2}{2 e^p v}\right\} + \exp\left\{-\frac{\beta x}{y} \log\left(1 + \frac{\beta xy^{p-1}}{w}\right)\right\} + P\left(\max_{1 \leq i \leq n} \xi_i > y\right),$$

where

$$\alpha = \frac{2}{p+2} \quad \text{and} \quad \beta = 1 - \alpha. \quad (21)$$

Setting $y = \beta x$, we obtain the following generalization of the Fuk-Nagaev inequality (6).

Corollary 2.2. Let $p \geq 2$. Assume $||\mathbf{E}[|\xi_i|^p|\mathcal{F}_{i-1}]||_{\infty} < \infty$ for all $i \in [1, n]$. It holds for all $x > 0$,

$$P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left\{-\frac{x^2}{2 V^2}\right\} + C_p \frac{x}{x^p}, \quad (22)$$

where

$$V^2 = \frac{1}{4(p+2)^2} \left\|\langle S \rangle_n\right\|_{\infty} \quad \text{and} \quad C_p = \left(1 + \frac{2}{p}\right)^p \left\|\sum_{i=1}^n \mathbf{E}[|\xi_i|^p|\mathcal{F}_{i-1}]\right\|_{\infty}. \quad (23)$$

It is worth noting that if $\mathbf{E}[|\xi_i|^p|\mathcal{F}_{i-1}] \leq C$ for a constant $C$ and all $i \in [1, n]$, then, by Jensen’s inequality, it holds $\mathbf{E}[|\xi_i^{2p}||\mathcal{F}_{i-1}] \leq C^{2/p}$ for all $i \in [1, n]$. Inequality (22) implies the following sub-Gaussian bound for any $x = O\left(\sqrt{n \ln n}/n^\beta\right)$, $n \to \infty$, with $\beta$ satisfying $\beta > 0$ if $p = 2$ and $\beta \in (0, 1/2]$ if $p > 2$,

$$P\left(\max_{1 \leq k \leq n} S_k > n^{\alpha x}\right) = O\left(\exp\left\{-C \frac{x^2}{n}\right\}\right),$$

where $C > 0$ does not depend on $x$ and $n$. The bound (24) is similar to the classical Azuma-Hoeffding inequality, and thus it is tight. Inequality (22) also implies that for any $\alpha \in \left(\frac{1}{2}, \infty\right)$ and any $x > 0$,

$$P\left(\max_{1 \leq k \leq n} S_k > n^{\alpha x}\right) = O\left(\frac{c_x}{n^{\alpha p-1}}\right), \quad n \to \infty,$$  

where $c_x > 0$ does not depend on $n$. Equality (25) is first obtained by Fuk [12] and it is the best possible under the stated condition even for the sums of independent random variables (cf. Fuk and Nagaev [11]).
If the martingale differences \((\xi_i, \mathcal{F}_i)_{i=1,\ldots,n}\) satisfy \(\mathbb{E}[|\xi_i|^p] \leq C\) for a constant \(C\) and all \(i \in [1,n]\), then Lesigne and Volný [18] proved that for any \(x > 0\),

\[
P\left(S_n > nx\right) = O\left(\frac{c_x}{n^{p/2}}\right), \quad n \to \infty, \tag{26}
\]

where \(c_x > 0\) does not depend on \(n\), and that the order \(n^{-p/2}\) is optimal even for the class of stationary and ergodic sequence of martingale differences. When \(\alpha = 1\), equality (25) implies the following large deviation convergence rate for any \(x > 0\),

\[
P\left(\max_{1 \leq k \leq n} S_k > nx\right) = O\left(\frac{c_x}{n^{p-1}}\right), \quad n \to \infty, \tag{27}
\]

where \(c_x > 0\) does not depend on \(n\). When \(p \geq 2\), it holds \(p - 1 \geq p/2\). Thus (27) refines the bound (26) under the stronger assumption that the \(p\)-th conditional moments are uniformly bounded. Moreover, the following proposition of Lesigne and Volný [18] shows that the estimate of (27) cannot be essentially improved even in the i.i.d. case.

**Proposition A.** Let \(p \geq 1\) and \((c_n)_{n \geq 1}\) be a real positive sequence approaching zero. There exists a sequence of i.i.d. random variables \((\xi_i)_{i \geq 1}\) such that \(\mathbb{E}[|\xi_i|^p] < \infty, \mathbb{E}[\xi_i] = 0\) and

\[
\limsup_{n \to \infty} \frac{n^{p-1}}{c_n} P(|S_n| \geq n) = \infty.
\]

When \(\mathbb{E}[|\xi_i|^2|\mathcal{F}_{i-1}]\) and \(\mathbb{E}[|\xi_i|^p]\), for a \(p > 2\) and all \(i = 1,\ldots,n\), are all uniformly bounded (but the condition \(\mathbb{E}[|\xi_i|^p|\mathcal{F}_{i-1}] \leq C\) may be violated for some \(i \in [1,n]\)), we have the following result.

**Theorem 2.3.** Let \(p \geq 2\). Assume \(\mathbb{E}[|\xi_i|^{p+\delta}] < \infty\) for a small \(\delta > 0\) and all \(i \in [1,n]\). Then for all \(x, v > 0\),

\[
P\left(S_k \geq x \text{ and } (S)_k \leq v^2 \text{ for some } k \in [1,n]\right) \leq \exp\left\{-\frac{x^2}{2\left(v^2 + \frac{\delta}{3}x^{(2p+\delta)/(p+\delta)}\right)}\right\} + \frac{1}{x^p} \mathbb{E}\left[|\xi_i|^{p+\delta}1_{\{\xi_i > x^{p/(p+\delta)}\}}\right]. \tag{28}
\]

If \(\mathbb{E}[|\xi_i|^{p+\delta}] \leq C\) and \(\mathbb{E}[\xi_i^2|\mathcal{F}_{i-1}] \leq C\) for a constant \(C\) and all \(i \in [1,n]\), then (28) implies that for any \(\alpha \in \left(\frac{1}{2}, \infty\right)\) and any \(x > 0\),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq n^{\alpha}x\right) \leq \exp\left\{-c_x \min\left\{\frac{\alpha\delta}{n^{p+\delta}}, n^{\alpha - 1}\right\}\right\} + \frac{C/x^p}{n^{\alpha p - 1}} = O\left(\frac{c_x}{n^{\alpha p - 1}}\right), \quad n \to \infty, \tag{29}
\]

where \(c_x > 0\) does not depend on \(n\). Thus (25) and (29) have the same convergence rate.

The different between the conditions of (25) and (29) is that the assumption \(\mathbb{E}[|\xi_i|^p|\mathcal{F}_{i-1}] \leq C\) has been replaced by the two assumptions \(\mathbb{E}[|\xi_i|^{p+\delta}] \leq C\) and \(\mathbb{E}[\xi_i^2|\mathcal{F}_{i-1}] \leq C\) for all \(i \in [1,n]\). Notice that the two assumptions \(\mathbb{E}[|\xi_i|^p|\mathcal{F}_{i-1}] \leq C\) and \(\mathbb{E}[|\xi_i|^{p+\delta}] \leq C\) are not included in each other. Thus Corollary 2.2 and Theorem 2.3 do not imply each other.

From Theorem 2.3, the following corollary is obvious.
Corollary 2.3. Assume the condition of Theorem 2.3. Then for all \( x, v > 0 \),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq \exp \left\{ -\frac{x^2}{2(nv^2 + \frac{1}{2}x(2p+\delta)/(p+\delta))} \right\} + \frac{1}{x^p} \sum_{i=1}^{n} \mathbb{E} \left[ |\xi_i|^{p+\delta} 1_{\{\xi_i > x^p/(p+\delta)\}} \right] + \frac{1}{x^p} \mathbb{E} \left[ \langle S \rangle_n^{(p+\delta)/2} \right].
\] (30)

Moreover, it holds

\[
\mathbb{E} \left[ \langle S \rangle_n^{(p+\delta)/2} \right] \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|\xi_i|^{p+\delta}].
\] (31)

Inequality (31) implies that if \( \sup_i \mathbb{E}[|\xi_i|^p] < \infty \) for a \( p \geq 2 \), then \( \mathbb{E}[\langle S \rangle_n^{p/2}] \) are uniformly bounded for all \( n \).

Compared to Theorem 2.3, Corollary 2.3 is more applicable since it only need the moment of \( \langle S \rangle_n \) instead of the uniform bound of \( \langle S \rangle_n \).

Assume \( \mathbb{E}[|\xi_i|^{p+\delta}] \leq C \) for a \( p \geq 2 \) and all \( i \in [1, n] \) (without any condition on \( \langle S \rangle_n \)). Applying (31) to (30) with \( nv^2 = \frac{2}{3}x(2p+\delta)/(p+\delta) \), we have for all \( x, v > 0 \),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq \exp \left\{ -\frac{1}{2}x^{\delta/(p+\delta)} \right\} + \frac{nC}{x^p} + \left( \frac{3}{2} \right)^{\frac{p+\delta}{2}} \frac{C}{x^{(p+\delta)/2}}.
\] (32)

The last inequality shows that for any \( x > 0 \),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x \right) = O\left( \frac{1}{n^{p/2}} \right), \quad n \to \infty.
\] (33)

Since \( \delta > 0 \) can be any small, equality (33) is closed to the best possible large deviation convergence rate \( n^{-(p+\delta)/2} \) given by Lesigne and Voldý [18] (cf. (26)).

3. Applications

The exponential concentration inequalities for martingales have many applications. McDiarmid [21], Rio [25] and Dedecker and Fan [3] applied such type inequalities to estimate the concentration of separately Lipschitz functions. Liu and Watbled [20] adopted these inequalities to deduce asymptotic properties of the free energy of directed polymers in a random environment. We refer to Bercu and Touati [1] and [10] for more interesting applications of the concentration inequalities for martingales. In the sequel, we discuss how to apply our results to linear regression models and weak invariance principles.

3.1. Linear regressions

Linear regressions can be used to investigate the impact of one variable on the other, or to predict the value of one variable based on the other. For instance, if one wants to see impact of footprint size on height, or predict height according to a certain given value of footprint size. The stochastic linear regression model is given by, for all \( k \in [1, n] \),

\[ X_k = \theta \phi_k + \varepsilon_k, \] (34)
where \((X_k)_{k=1,...,n}\), \((\phi_k)_{k=1,...,n}\) and \((\varepsilon_k)_{k=1,...,n}\) are the observations, the regression variables and the driven noises, respectively. We assume that \((\phi_k)_{k=1,...,n}\) is a sequence of independent random variables, and that \((\varepsilon_k)_{k=1,...,n}\) is a sequence of martingale differences with respect to the natural filtration. Moreover, we suppose that \((\phi_k)_{k=1,...,n}\) and \((\varepsilon_k)_{k=1,...,n}\) are independent. Our interest is to estimate the unknown parameter \(\theta\). The well-known least-squares estimator \(\theta_n\) is given below

\[
\theta_n = \frac{\sum_{k=1}^{n} \phi_k X_k}{\sum_{k=1}^{n} \phi_k^2}.
\]  

(35)

Recently, Bercu and Touati [1] have obtained some very precise exponential bounds on the tail probabilities \(P(\|\theta_n - \theta\| \geq x)\). However, their precise bounds depend on the distribution of input random variables \((\phi_k)_{k=1,...,n}\), which restricts the applications of these bounds when the distributions of input random variables are unknown. When \((\varepsilon_k)_{k=1,...,n}\) are independent normal random variables with a common variation \(\sigma^2 > 0\), Liptser and Spokoiny [19] have established the following estimation: for all \(x \geq 1\),

\[
P(\pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x) \leq \sqrt{\frac{2}{\pi}} \sigma \frac{x}{\sigma} \exp\left\{ - \frac{x^2}{2\sigma^2} \right\}.
\]  

(36)

When \((\varepsilon_k)_{k=1,...,n}\) are conditionally sub-Gaussian, similar estimation is allowed to be obtained in Liptser and Spokoiny [19]. An interesting feature of bound (36) is that the bound does not depend on the distribution of input random variables. Here, we would like to generalize inequality (36) to the case that \((\varepsilon_k)_{k=1,...,n}\) are martingale differences and also non sub-Gaussian.

**Theorem 3.1.** Assume for two constants \(\alpha \in (0, 1)\) and \(D\),

\[
E[\varepsilon_i^2 e^{i\alpha \varepsilon_i} \bigg| \sigma \{\varepsilon_j, j \leq i - 1\}] \leq D
\]

for all \(i \in [1, n]\). Then for any \(u \geq \max\{D, 1\}\) and all \(x > 0\),

\[
P\left(\pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x\right) \leq \begin{cases} 
2 \exp\left\{ - \frac{x^2}{2u} \right\} & \text{if } 0 \leq x < u^{1/(2-\alpha)} \\
2 \exp\left\{ - \frac{1}{2} x^\alpha \right\} & \text{if } x \geq u^{1/(2-\alpha)} 
\end{cases}
\]

(37)

\[
\leq 2 \exp\left\{ - \frac{x^2}{2(u + x^{2-\alpha})} \right\}
\]  

(38)

In particular, it holds for any \(x > 0\),

\[
P\left(\pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq \sqrt{n} x\right) = O\left( \exp\left\{ - c_x n^{\alpha/2} \right\} \right), \quad n \to \infty,
\]

(39)

where \(c_x > 0\) does not depend on \(n\).

If \((\varepsilon_k)_{k=1,...,n}\) have the Weibull distributions and the conditional variances are uniformly bounded, then we have the following inequality which has the same exponentially decaying rate of (39).
\textbf{Theorem 3.2.} Assume for three constants $\alpha \in (0, 1)$, $E$ and $F$,
\[ E \left[ \varepsilon_i^2 \right] \leq E \quad \text{and} \quad E \left[ \exp \left\{ \frac{\varepsilon_i^2}{1 - 2\alpha} \right\} \right] \leq F \]
for all $i \in [1, n]$. Then for all $x > 0$,
\[ P \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \sigma_k^2} \geq x \right) \leq \exp \left\{ - \frac{x^2}{2(E + \frac{1}{3}x^{2-\alpha})} \right\} + nF \exp \left\{ - x^{\alpha} \right\}. \quad (40) \]
In particular, equality \textbf{(39)} holds.

If $(\varepsilon_k)_{k=1}^{n}$ have finite conditional moments, by Corollary 2.2, then we have the following result.

\textbf{Theorem 3.3.} Let $p \geq 2$. Assume for a constant $A$,
\[ E \left[ |\varepsilon_i|^p \right] \leq A \]
for all $i \in [1, n]$. Then for all $x > 0$,
\[ P \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \sigma_k^2} \geq x \right) \leq \exp \left\{ - \frac{x^2}{2V^2} \right\} + \frac{C_p}{x^p}, \quad (41) \]
where
\[ V^2 = \frac{1}{4} (p + 2)^2 e^p A^{2/p} \quad \text{and} \quad C_p = \left( \frac{1}{1 + \frac{2}{p}} \right)^p A. \quad (42) \]
In particular, it holds for any $x > 0$,
\[ P \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \sigma_k^2} \geq \sqrt{n} x \right) = O \left( \frac{c_x}{n^{p/2}} \right), \quad n \to \infty, \quad (43) \]
where $c_x > 0$ does not depend on $n$.

A similar inequality can be obtained by applying the Fuk inequality \textbf{(3)} to the martingale difference sequence (cf. \textbf{(61)} for the definition of $(\xi_i, F_i)_{i=1}^{n}$). The Fuk inequality implies that for all $x > 0$,
\[ P \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \sigma_k^2} \geq x \right) \leq \exp \left\{ - \frac{x^2}{2nV^2} \right\} + \frac{nC_p}{x^p}, \quad (44) \]
where $V^2$ and $C_p$ are defined by \textbf{(42)}. In particular, it implies that for any $x > 0$,
\[ P \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \sigma_k^2} \geq \sqrt{n} x \right) = O \left( \frac{c_x}{n^{p/2-1}} \right), \quad n \to \infty, \quad (45) \]
where $c_x > 0$ does not depend on $n$. The order of \textbf{(43)} is much better than that of \textbf{(45)}. Thus the refinement of \textbf{(41)} on \textbf{(44)} is significant.

If $(\varepsilon_k)_{k=1}^{n}$ have finite moments and uniformly bounded conditional variances, by Theorem 2.3, we obtain the following result which has the same polynomially decaying rate of Theorem 3.3.
Theorem 3.4. Let $p \geq 2$. Assume for two constants $A$ and $B$,
\[
\mathbb{E} \left[ \varepsilon_i^2 \left| \sigma \{ \varepsilon_j, j \leq i - 1 \} \right. \right] \leq A \quad \text{and} \quad \mathbb{E} \left[ |\varepsilon_i|^{p+\delta} \right] \leq B
\]
for a small $\delta > 0$ and all $i \in [1, n]$. Then for all $x > 0$,
\[
\mathbb{P} \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x \right) \leq \exp \left\{ \frac{-x^2}{2 \left( A + \frac{1}{3} x^{(2p+\delta)/(p+\delta)} \right) } \right\} + \frac{B}{x^p}.
\]
In particular, equality (43) holds.

Theorem 3.5. Let $p \in [1, 2]$. Assume for a constant $A$, 
\[
\mathbb{E} \left[ |\varepsilon_i|^p \right] \leq A
\]
for all $i \in [1, n]$. Then for all $x > 0$,
\[
\mathbb{P} \left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x \right) \leq \frac{2A}{x^p}.
\]
In particular, equality (43) holds.

Theorems 3.4 and 3.5 focus on obtaining the large deviation inequalities. These inequalities do not depend on the distribution of input random variables $(\phi_k)_{k=1,...,n}$. Similar bounds are also expected to be obtained via the decoupling techniques of De la Peña [3] and De la Peña and Giné [7]. In particular, if $(\varepsilon_k)_{k=1,...,n}$ are independent (instead of martingale differences), with the method of conditionally independent in De la Peña and Giné [7], more precise bounds, but depend on the distribution of input random variables, are allowed to be established.

Haeusler and Joos [14] proved that if the martingale differences satisfy $\mathbb{E} [ |\xi_i|^{2+\delta} ] < \infty$ for a constant $\delta > 0$ and all $i \in [1, n]$, then there exists a constant $C_\delta$, depending only on $\delta$, such that for all $x \in \mathbb{R}$,
\[
\left| \mathbb{P}(S_n \leq x) - \Phi(x) \right| \leq C_\delta \left( \sum_{i=1}^{n} \mathbb{E} [ |\xi_i|^{2+\delta} ] + \mathbb{E} [ |(S)_n - 1|^{1+\delta/2} ] \right)^{1/(3+\delta)} \frac{1}{1 + |x|^{2+\delta}},
\]
where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-t^2/2\} dt$ is the standard normal distribution; see also Hall and Heyde [15] with the larger factor $\frac{1}{1 + |x|^{4(1+\delta/2)/(3+\delta)}}$ replacing $\frac{1}{1 + |x|^{2+\delta}}$. Using (48), we obtain the following nonuniform Berry-Esseen bound, which depends on the distribution of input random variables.

Theorem 3.6. Let $p > 2$. Assume that $(\varepsilon_i)_{i=1,...,n}$ satisfy $\mathbb{E} [ \varepsilon_i^2 \left| \sigma \{ \varepsilon_j, j \leq i - 1 \} \right. \right] = \sigma^2$ a.s. for a positive constant $\sigma$ and all $i \in [1, n]$. Assume $\mathbb{E} [ |\varepsilon_i|^p ] \leq A$ for a constant $A$ and all $i \in [1, n]$. Then for all $x \in \mathbb{R}$,
\[
\left| \mathbb{P} \left( (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \leq x \sigma \right) - \Phi(x) \right| \leq C_p \left( \sum_{i=1}^{n} \mathbb{E} \left[ \left| \frac{\phi_i}{\sqrt{\sum_{k=1}^{n} \phi_k^2}} \right|^p \right] \right)^{1/(1+p)} \frac{1}{1 + |x|^{p}},
\]
where $C_p$ is a constant depending only on $A, \sigma$ and $p$. 

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Notice that
\[
\sum_{i=1}^{n} E \left[ \frac{\phi_i}{\sqrt{\sum_{k=1}^{n} \phi_k^2}} \right]^p \leq \sum_{i=1}^{n} E \left[ \frac{\phi_i}{\sqrt{\sum_{k=1}^{n} \phi_k^2}} \right]^2 = 1.
\]
Thus (49) implies that the tail probability \( P \left( \left( \theta_n - \theta \right) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x \right) \) has the decaying rate \( x^{-p} \) as \( x \to \infty \), which is coincident with the inequalities (41) and (46).

3.2. Weak invariance principles

In this subsection, let \((\xi_i, F_i)_{i \geq 1}\) be a sequence of stationary martingale differences. We have the following weak invariance principle for martingales.

The following rate of convergence in the central limit theorem (CLT) for martingale difference sequences is due to Ouchti (cf. Corollary 1 of [24]). Assume that there exists a constant \( M > 0 \) such that
\[
E \left[ \left| \xi_i \right|^3 \right|_{F_i-1} \leq M E \left[ \xi_i^2 \right|_{F_i-1} \right] a.s. \text{ for all } i \in \mathbb{N}.
\]
If the series \( \sum_{i=1}^{\infty} E \left[ \xi_i^2 \right|_{F_i-1} \] diverges a.s. and then there is a constant \( C_M > 0 \), depending on \( M \), such that
\[
\sup_{x \in \mathbb{R}} \left| P \left( S_{v(n)} \leq x \sqrt{n} \right) - \Phi(x) \right| \leq \frac{C_M}{n^{1/4}},
\]
where
\[
v(n) = \inf \left\{ k \in \mathbb{N}, \; \langle S \rangle_k \geq n \right\}.
\]
Let
\[
H_n(t) = \frac{1}{\sqrt{n}} S_{v(\lfloor nt \rfloor)} \quad \text{for } 0 \leq t \leq 1.
\]
By Theorem 2.2, we obtain the following weak invariance principle for martingales.

**Theorem 3.7.** Assume that there exists a constant \( M > 0 \) such that \( E \left[ \left| \xi_i \right|^3 \right|_{F_i-1} \leq M E \left[ \xi_i^2 \right|_{F_i-1} \right] a.s. \text{ for all } i \in \mathbb{N} \). If the series \( \sum_{i=1}^{\infty} E \left[ \xi_i^2 \right|_{F_i-1} \] diverges a.s., then the sequence of processes \( \{ H_n(t), 0 \leq t \leq 1 \} \) converges in distribution to the standard Wiener process.

4. Proof of Theorem 2.1

To prove Theorem 2.1, we need the following technical lemma based on a truncation argument.

**Lemma 4.1.** Assume \( E \left[ \xi_i^2 \exp \{ |\xi_i|^\alpha \} \right] < \infty \) for a constant \( \alpha \in (0, 1) \). Set \( \eta_i = \xi_i 1_{\{ \xi_i \leq y \}} \) for \( y > 0 \). Then for all \( \lambda > 0 \),
\[
E \left[ e^{\lambda \eta_i} \big| F_{i-1} \right] \leq 1 + \frac{\lambda^2}{2} E \left[ \eta_i^2 \exp \{ \lambda y^{1-\alpha} (\eta_i^+)^\alpha \} \big| F_{i-1} \right].
\]

The proof of Lemma 4.1 can be found in the proof of Proposition 3.5 in Dedecker and Fan [5]. However, instead of using the tower property of conditional expectation as in Dedecker and Fan [5], we use changes of probability measure in the proof of this theorem. Set \( \eta_i = \xi_i 1_{\{ \xi_i \leq y \}} \) for some \( y > 0 \). The exact value of \( y \) is given later. Then \( (\eta_i, F_i)_{i=1, \ldots, n} \) is a sequence of supermartingale differences,
and it holds $E[\exp \{\lambda \eta_i\}] < \infty$ for all $\lambda \in (0, \infty)$ and all $i$. Define the exponential multiplicative martingale $Z(\lambda) = (Z_k(\lambda), \mathcal{F}_k)_{k=0,\ldots,n}$, where

$$Z_k(\lambda) = \prod_{i=1}^{k} \frac{\exp \{\lambda \eta_i\}}{E[\exp \{\lambda \eta_i\} | \mathcal{F}_{i-1}]} , \quad Z_0(\lambda) = 1.$$ 

If $T$ is a stopping time, then $Z_{T \land k}(\lambda)$ is also a martingale, where

$$Z_{T \land k}(\lambda) = \prod_{i=1}^{T \land k} \frac{\exp \{\lambda \eta_i\}}{E[\exp \{\lambda \eta_i\} | \mathcal{F}_{i-1}]} , \quad Z_0(\lambda) = 1.$$ 

Thus, the random variable $Z_{T \land k}(\lambda)$ is a probability density on $(\Omega, \mathcal{F}, P)$, i.e.

$$\int Z_{T \land k}(\lambda) dP = E[Z_{T \land k}(\lambda)] = 1.$$ 

Define the conjugate probability measure

$$dP_\lambda = Z_{T \land n}(\lambda) dP,$$ 

and denote by $E_\lambda$ the expectation with respect to $P_\lambda$. Since $\xi_i = \eta_i + \xi_i 1_{\{\xi_i > y\}}$, it follows that for any $x, y, u > 0$,

$$P\left( S_k \geq x \text{ and } \Upsilon(S)_k \leq u \text{ for some } k \in [1, n] \right)$$

$$\leq P\left( \sum_{i=1}^{k} \eta_i \geq x \text{ and } \Upsilon(S)_k \leq u \text{ for some } k \in [1, n] \right)$$

$$+ P\left( \sum_{i=1}^{k} \xi_i 1_{\{\xi_i > y\}} > 0 \text{ for some } k \in [1, n] \right)$$

$$= P_1 + P\left( \max_{1 \leq i \leq n} \xi_i > y \right).$$ 

For any $x, u > 0$, define the stopping time

$$T(x, u) = \min \left\{ k \in [1, n] : \sum_{i=1}^{k} \eta_i \geq x \text{ and } \Upsilon(S)_k \leq u \right\},$$

with the convention that $\min \emptyset = 0$. Then

$$1_{\{S_k \geq x \text{ and } \Upsilon(S)_k \leq u \text{ for some } k \in [1, n]\}} = \sum_{k=1}^{n} 1_{\{T(x, u) = k\}}.$$ 

By the change of measure (51), we deduce that for any $x, \lambda, u > 0$,

$$P_1 = E_\lambda \left[ Z_{T \land n}(\lambda)^{-1} 1_{\{S_k \geq x \text{ and } \Upsilon(S)_k \leq u \text{ for some } k \in [1, n]\}} \right]$$

$$= \sum_{k=1}^{n} E_\lambda \left[ \exp \left\{ -\lambda \left( \sum_{i=1}^{k} \eta_i \right) + \Psi_k(\lambda) \right\} 1_{\{T(x, u) = k\}} \right].$$ 

(53)
where

\[ \Psi_k(\lambda) = \sum_{i=1}^{k} \log E \{ \exp \{ \lambda \eta_i \} \mid \mathcal{F}_{i-1} \}. \] (54)

Set \( \lambda = y^{\alpha - 1} \). By Lemma 4.1 and the inequality \( \log(1 + t) \leq t \) for all \( t \geq 0 \), it is easy to see that for any \( x > 0 \),

\[
\Psi_k(\lambda) \leq \sum_{i=1}^{k} \log \left( 1 + \frac{\lambda^2}{2} E[\eta_i^2 \exp\{\lambda y^{1-\alpha}(\eta_i^+)\} \mid \mathcal{F}_{i-1}] \right)
\leq \sum_{i=1}^{k} \frac{\lambda^2}{2} E[\eta_i^2 \exp\{\lambda y^{1-\alpha}(\eta_i^+)\} \mid \mathcal{F}_{i-1}]
\leq \frac{1}{2} y^{2\alpha - 2} \Upsilon(S)_k.
\]

By the fact that \( \sum_{i=1}^{k} \eta_i \geq x \) and \( \Psi_k(\lambda) \leq \frac{1}{2} y^{2\alpha - 2} u \) on the set \( \{T(x, u) = k\} \), we find that for any \( x, u > 0 \),

\[
P_1 \leq \exp \left\{ -\lambda x + \frac{1}{2} y^{2\alpha - 2} u \right\} E_{\lambda} \left[ \sum_{k=1}^{n} 1_{\{T(x, u) = k\}} \right]
\leq \exp \left\{ -y^{\alpha - 1} x + \frac{1}{2} y^{2\alpha - 2} u \right\}.
\]

From (52), it follows that

\[
P \left( S_k \geq x \text{ and } \Upsilon(S)_k \leq u \text{ for some } k \in [1, n] \right)
\leq \exp \left\{ -y^{\alpha - 1} x + \frac{1}{2} y^{2\alpha - 2} u \right\} + P \left( \max_{1 \leq i \leq n} \xi_i > y \right). \tag{55}
\]

By the exponential Markov inequality, we have the following estimation: for any \( x > 0 \),

\[
P \left( \max_{1 \leq i \leq n} \xi_i > y \right) \leq \sum_{i=1}^{n} P (\xi_i > y)
\leq \frac{1}{y^2} \exp\{-y^\alpha\} \sum_{i=1}^{n} E[\xi_i^2 \exp\{((\xi_i^+)^\alpha)\}]
\leq \frac{C_n}{y^2} \exp \{-y^\alpha\}. \tag{56}
\]

Taking

\[
y = \begin{cases} 
\left( \frac{u}{x} \right)^{1/(1-\alpha)} & \text{if } 0 \leq x < u^{1/(2-\alpha)} \\
x & \text{if } x \geq u^{1/(2-\alpha)},
\end{cases}
\]

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from (55) and (56), we obtain the desired inequality. This completes the proof of Theorem 2.1. □

Proof of Corollary 2.1. Set \( u_n = \|\Upsilon(S)_n\|_\infty \). Then \( u_n = o(n^{2-\alpha}), n \to \infty \), by the assumptions of Theorem 2.1. For any \( x \geq 0 \), by Theorem 2.1, we have

\[
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq \exp\left\{ -(nx)^\alpha \left( 1 - \frac{u_n}{2(nx)^{2-\alpha}} \right) \right\} + \frac{C_n}{(nx)^2} \exp\left\{ -(nx)^\alpha \left( 1 - \frac{u_n}{2(nx)^{2-\alpha}} \right) \right\}.
\]

Since \( u_n \geq C_n \), we have \( C_n = o(n^{2-\alpha}), n \to \infty \). Hence it holds

\[
\limsup_{n \to \infty} \frac{1}{n^\alpha} \log P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq -x^\alpha.
\]

This completes the proof of Corollary 2.1. □

Proof of Remark 2.1. Note that

\[
E[\xi_1^2 \exp\{|\xi_1|^\alpha\}] = \int_0^\infty P(|\xi_1| \geq x)\left( 2x + \alpha x^{1+\alpha} \right)e^{x^\alpha} dx < \infty.
\]

Thus

\[
\Upsilon(S)_n \leq n E[\xi_1^2 \exp\{|\xi_1|^\alpha\}] = o(n^{2-\alpha}), \quad n \to \infty.
\]

It is easy to see that for any \( x, \varepsilon > 0 \), we have

\[
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \geq P\left( S_n \geq nx \right) \geq P\left( \sum_{i=2}^n \xi_i \geq -n\varepsilon, \xi_1 \geq n(x + \varepsilon) \right) = P\left( \sum_{i=2}^n \xi_i \geq -n\varepsilon \right) P\left( \xi_1 \geq n(x + \varepsilon) \right).
\]

The first probability on the right-hand side trends to 1 as \( n \to \infty \) due to the law of large numbers. By (18), the second term on the right-hand side has the following lower bound

\[
P\left( \xi_1 \geq n(x + \varepsilon) \right) \geq \left( n(x + \varepsilon) \right)^{-2p} \exp\left\{ -\left( n(x + \varepsilon) \right)^\alpha \right\}
\]

for all \( n \) large enough. Hence

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log P\left( \max_{1 \leq k \leq n} \frac{1}{n} S_k \geq x \right) \geq -(x + \varepsilon)^\alpha.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \log P\left( \max_{1 \leq k \leq n} \frac{1}{n} S_k \geq x \right) \geq -x^\alpha.
\]

Combining this result with Theorem 2.1, we get (19). □
5. Proof of Theorem 2.2

To prove Theorem 2.2, we need the following technical lemma.

**Lemma 5.1.** Let $p \geq 2$. Assume $\mathbb{E}[|\xi_i|^p] < \infty$ for all $i \in [1, n]$. Set $\eta_i = \xi_i1_{\{\xi_i \leq y\}}$ for $y > 0$. Then for all $\lambda > 0$,

$$
\mathbb{E}[e^{\lambda \eta_i} \mid \mathcal{F}_{i-1}] \leq 1 + \frac{1}{2}e^{p\lambda^2}E[\xi_i^2] + f(y)E[(\xi_i^+)p] \mid \mathcal{F}_{i-1},
$$

where the function

$$
f(u) = \frac{e^{\lambda u} - 1 - \lambda u}{u^p}, \quad u > 0. \tag{57}
$$

**Proof.** We argue as in Fuk and Nagaev [11] (see also Fuk [12]). Using a two term Taylor’s expansion, we have for some $\theta \in [0, 1]$,

$$
e^{\lambda \eta_i} \leq 1 + \lambda \eta_i + \frac{\lambda^2}{2} \eta_i^2 1_{\{\lambda \eta_i \leq y\}} e^{\lambda y} + f(\eta_i)(\eta_i^+)p 1_{\{\lambda \eta_i > p\}}.
$$

Remark that the function $f$ is positive and increasing for $\lambda u \geq p$. Since $\mathbb{E}[\eta_i \mid \mathcal{F}_{i-1}] \leq \mathbb{E}[\xi_i \mid \mathcal{F}_{i-1}] = 0$ and $\eta_i \leq y$, it follows that

$$
\mathbb{E}[e^{\lambda \eta_i} \mid \mathcal{F}_{i-1}] \leq 1 + \frac{1}{2}e^{p\lambda^2}\mathbb{E}[\eta_i^2] + f(y)\mathbb{E}[(\eta_i^+)p] \mid \mathcal{F}_{i-1}]
$$

$$
\leq 1 + \frac{1}{2}e^{p\lambda^2}\mathbb{E}[\xi_i^2] + f(y)\mathbb{E}[(\xi_i^+)p] \mid \mathcal{F}_{i-1}],
$$

which gives the desired inequality. \qed

We make use of Lemma 5.1 to prove Theorem 2.2. Set $\eta_i = \xi_i1_{\{\xi_i \leq y\}}$ for $y > 0$. Define the conjugate probability measure $d\mathbb{P}_\lambda$ by (51) and denote by $\mathbb{E}_\lambda$ the expectation with respect to $\mathbb{P}_\lambda$. Since $\xi_i = \eta_i + \xi_i1_{\{\xi_i > y\}}$, it follows that for any $x, y, u, w > 0$,

$$
\mathbb{P}(S_k > x, \langle S \rangle_k \leq v \text{ and } \Xi(S)_k \leq w \text{ for some } k \in [1, n])
$$

$$
\leq \mathbb{P} \left( \sum_{i=1}^{k} \eta_i \geq x, \langle S \rangle_k \leq v \text{ and } \Xi(S)_k \leq w \text{ for some } k \in [1, n] \right)
$$

$$
+ \mathbb{P} \left( \sum_{i=1}^{k} \xi_i1_{\{\xi_i > y\}} > 0 \text{ for some } k \in [1, n] \right)
$$

$$
=: P_2 + \mathbb{P} \left( \max_{1 \leq i \leq n} \xi_i > y \right). \tag{58}
$$

For any $x, v, w > 0$, define the stopping time $T$:

$$
T(x, v, w) = \min \left\{ k \in [1, n] : S_k \geq x, \langle S \rangle_k \leq v \text{ and } \Xi(S)_k \leq w \right\},
$$

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with the convention that \( \min \emptyset = 0 \). Then

\[
\mathbf{1}_{\{S_k > x, \langle S \rangle_k \leq v \text{ and } \Xi(S)_k \leq w \text{ for some } k \in [1,n]\}} = \sum_{k=1}^{n} \mathbf{1}_{\{T = k\}}.
\]

By the change of measure (51), we deduce that for any \( x, y, \lambda, u, w > 0 \),

\[
P_2 = \mathbb{E}_\lambda \left[ Z_{T \wedge n}^{-1} \mathbf{1}_{\{S_k > x, \langle S \rangle_k \leq v \text{ and } \Xi(S)_k \leq w \text{ for some } k \in [1,n]\}} \right]
\]

\[
= \sum_{k=1}^{n} \mathbb{E}_\lambda \left[ \exp \left\{ - \lambda \left( \sum_{i=1}^{k} \eta_i \right) + \Psi_k(\lambda) \right\} \mathbf{1}_{\{T = k\}} \right],
\]

where \( \Psi_k(\lambda) \) is defined by (51). By Lemma 5.1 and the inequality \( \log(1 + t) \leq t \) for \( t \geq 0 \), it is easy to see that for any \( x, y, \lambda, u, w > 0 \),

\[
\Psi_k(\lambda) \leq \sum_{i=1}^{k} \log \left( 1 + \frac{1}{2} e^p \lambda v + f(y) \mathbb{E}[(\xi_i^+)^p | F_{i-1}] \right)
\]

\[
\leq \sum_{i=1}^{k} \left( \frac{1}{2} e^p \lambda v + f(y) \mathbb{E}[(\xi_i^+)^p | F_{i-1}] \right),
\]

where \( f(y) \) is defined by (57). By the fact that \( \sum_{i=1}^{k} \eta_i \geq x \) and \( \Psi_k(\lambda) \leq \frac{1}{2} e^p \lambda v + f(y) w \) on the set \( \{ T = k \} \), we find that for any \( x, y, \lambda, u, w > 0 \),

\[
P_2 \leq \exp \left\{ - \lambda x + \frac{1}{2} e^p \lambda v + f(y)w \right\} \mathbb{E}_\lambda \left[ \sum_{k=1}^{n} \mathbf{1}_{\{T = k\}} \right]
\]

\[
\leq \exp \left\{ - \lambda x + \frac{1}{2} e^p \lambda v + f(y)w \right\}.
\]

Next we carry out an argument as in Fuk and Nagaev [11]. Then

\[
P_2 \leq \exp \left\{ - \alpha^2 x^2 \right\} + \exp \left\{ - \frac{\beta x}{y} \log \left( 1 + \frac{\beta xy^{p-1}}{w} \right) \right\},
\]

where \( \alpha \) and \( \beta \) are defined by (21). Combining the inequalities (58) and (59) together, we obtain the desired inequality. This completes the proof of Theorem 2.2 \( \square \)

**Proof of Corollary 2.2** When \( y = \beta x \), from (20), it is easy to see that for all \( x > 0 \),

\[
\mathbb{P} \left( \max_{1 \leq i \leq n} \xi_i > y \right) \leq \sum_{i=1}^{n} \mathbb{P} \left( \xi_i > \beta x \right) \leq \frac{1}{\beta^p x^p} \sum_{i=1}^{n} \mathbb{E}[|\xi_i|^p] \leq \frac{C_p}{x^p}
\]

and

\[
\exp \left\{ - \frac{\beta x}{y} \log \left( 1 + \frac{\beta xy^{p-1}}{w} \right) \right\} \leq \frac{w}{\beta xy^{p-1} + w} \leq \frac{w}{\beta^p x^p} \leq \frac{C_p}{x^p},
\]

where \( C_p \) is defined by (23). Thus (20) implies (22). \( \square \)
6. Proofs of Theorem 2.3 and Corollary 2.3

To prove Theorem 2.3 we need the following inequality whose proof can be found in Fan, Grama and Liu [8] (cf. Corollary 2.3 and Remark 2.1 therein).

**Lemma 6.1.** Assume $E[\xi_i^2] < \infty$ for all $i \in [1, n]$. Then for all $x, y, v > 0$,

$$P\left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq \exp\left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xy)} \right\} + P\left( \max_{1 \leq i \leq n} \xi_i > y \right).$$

**Proof of Theorem 2.3.** By Lemma 6.1 and the Markov inequality, it follows that for all $x, y, v > 0$,

$$P\left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq \exp\left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xy)} \right\} \sum_{i=1}^{n} P\left( \xi_i > y \right) + \frac{1}{v^{p+\delta}} \sum_{i=1}^{n} E\left[ |\xi_i|^{p+\delta} \mathbf{1}_{\{\xi_i > y\}} \right].$$

Taking $y = \frac{x}{p/(p+\delta)}$ in the last inequality, we obtain the desired inequality. This completes the proof of Theorem 2.3. □

**Proof of Corollary 2.3.** Notice that $p + \delta > 2$. It is easy to see that for any $x, v > 0$,

$$P\left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq P\left( \max_{1 \leq k \leq n} S_k \geq x \text{ and } \langle S \rangle_n \leq nv^2 \right) + P\left( \langle S \rangle_n > nv^2 \right) \leq P\left( S_k \geq x \text{ and } \langle S \rangle_k \leq nv^2 \text{ for some } k \in [1, n] \right) + P\left( \langle S \rangle_n > nv^2 \right) \leq P\left( S_k \geq x \text{ and } \langle S \rangle_k \leq nv^2 \text{ for some } k \in [1, n] \right) + \frac{E[|\langle S \rangle_n|^{(p+\delta)/2}]}{n^{(p+\delta)/2} v^{p+\delta}},$$

which gives the first desired inequality. By the Hölder inequality, it follows that

$$\sum_{i=1}^{n} a_i \leq n^{-2/(p+\delta)} \left( \sum_{i=1}^{n} a_i^{(p+\delta)/2} \right)^{2/(p+\delta)}, \quad a_i \geq 0, \ i = 1, ..., n.$$

Hence

$$\left( \sum_{i=1}^{n} a_i \right)^{(p+\delta)/2} \leq n^{(p-2+\delta)/2} \sum_{i=1}^{n} a_i^{(p+\delta)/2}, \quad a_i \geq 0, \ i = 1, ..., n.$$

Then we have

$$E[|\langle S \rangle_n|^{(p+\delta)/2}] \leq n^{(p-2+\delta)/2} \sum_{i=1}^{n} E[|\xi_i|^2 |\mathcal{F}_{i-1}]^{(p+\delta)/2}.$$
\[
\leq n^{(p-2+\delta)/2} \sum_{i=1}^{n} E \left[ |\xi_i|^{p+\delta} |\mathcal{F}_{i-1} \right] \\
= n^{(p-2+\delta)/2} \sum_{i=1}^{n} E[|\xi_i|^{p+\delta}].
\]

This completes the proof of corollary. \(\square\)

7. Proofs of Theorems 3.1 - 3.6

From (34) and (35), it is easy to see that

\[
\theta_n - \theta = \sum_{k=1}^{n} \frac{\phi_k \varepsilon_k}{\sum_{k=1}^{n} \phi_k^2}. \tag{60}
\]

For any \(i = 1, ..., n\), set

\[
\xi_i = \frac{\phi_i \varepsilon_i}{\sqrt{\sum_{k=1}^{n} \phi_k^2}} \quad \text{and} \quad \mathcal{F}_i = \sigma\{\phi_k, \varepsilon_k, 1 \leq k \leq i, \phi_k^2, 1 \leq k \leq n\}. \tag{61}
\]

Then \((\xi_i, \mathcal{F}_i)_{i=1, ..., n}\) is a sequence of martingale differences, and satisfies

\[
S_n = \sum_{i=1}^{n} \xi_i = (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2}. \tag{62}
\]

**Proof of Theorem 3.1**. Notice that

\[
\Upsilon_n \leq \sum_{i=1}^{n} \frac{\phi_i^2}{\sum_{k=1}^{n} \phi_k^2} E[\varepsilon_i^2 \exp\{\varepsilon_i^a\} |\mathcal{F}_{i-1}] \leq \sum_{i=1}^{n} \frac{\phi_i^2 D}{\sum_{k=1}^{n} \phi_k^2} = D.
\]

Applying Theorem 2.1 to \((\xi_i, \mathcal{F}_i)_{i=1, ..., n}\), we find that (13), with \(u \geq \max\{D, 1\}\), is an upper bound on the tail probabilities \(P \left((\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x\right)\).

Similarly, applying Theorem 2.1 to \((-\xi_i, \mathcal{F}_i)_{i=1, ..., n}\), we find that (13), with \(u \geq \max\{D, 1\}\), is also an upper bound on the tail probabilities \(P \left(-(\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x\right)\). This completes the proof of Theorem 3.1. \(\square\)

**Proof of Theorem 3.2**. By the fact

\[
E[\varepsilon_i^2 |\mathcal{F}_{i-1}] = E[\varepsilon_i^2 |\sigma\{\varepsilon_k, 1 \leq k \leq i - 1\}] \leq E,
\]

it follows that

\[
\langle S \rangle_n \leq \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)} E[\varepsilon_i^2 |\mathcal{F}_{i-1}] \leq E.
\]
Similarly, by the fact \( \mathbb{E}[\exp\{\xi_i^p\}] \leq F \), it is easy to see that

\[
\mathbb{E}[\exp\{\xi_i^p\}] \leq \mathbb{E}[\exp\{\xi_i^{p/2}\}] \leq F.
\]

Applying Theorem 2.2 of Fan, Grama and Liu \([8]\) to \((\pm \xi_i, F_i)_{i=1,\ldots,n}\), we obtain the desired inequality.

\[\square\]

**Proof of Theorem 3.3.** By the fact

\[\mathbb{E}[|\xi_i|^p|F_{i-1}] = \mathbb{E}[|\xi_i|^p|\sigma\{\xi_k, 1 \leq k \leq i-1\}] \leq A,\]

it follows that

\[
\langle S \rangle_n \leq \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)} \mathbb{E}[\xi_i^2|F_{i-1}] \leq \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)} \left( \mathbb{E}[|\xi_i|^p|F_{i-1}] \right)^{2/p} = A^{2/p}
\]

and

\[
\sum_{i=1}^{n} \mathbb{E}[|\xi_i|^p|F_{i-1}] \leq \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)} \mathbb{E}[|\xi_i|^p|F_{i-1}] \leq A.
\]

Applying Corollary 2.2 to \((\pm \xi_i, F_i)_{i=1,\ldots,n}\), we obtain the desired inequality.

\[\square\]

**Proof of Theorem 3.4.** By the fact

\[\mathbb{E}[|\xi_i|^{p+\delta}|F_{i-1}] = \mathbb{E}[|\xi_i|^{p+\delta}|\sigma\{\xi_k, 1 \leq k \leq i-1\}] \leq A,\]

it follows that

\[
\langle S \rangle_n \leq \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)} \mathbb{E}[\xi_i^2|F_{i-1}] \leq A.
\]

Similarly, by the fact \( \mathbb{E}[|\xi_i|^{p+\delta}] \leq B \), it follows that

\[
\sum_{i=1}^{n} \mathbb{E}[|\xi_i|^{p+\delta}] = \sum_{i=1}^{n} \mathbb{E}\left[ \left( \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)} \right)^{p+\delta} \mathbb{E}[|\xi_i|^p] \right] \leq \mathbb{E}\left[ \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)^{p/2}} \right] B = B.
\]

Applying Theorem 2.3 to \((\pm \xi_i, F_i)_{i=1,\ldots,n}\), we obtain the desired inequality.

\[\square\]

**Proof of Theorem 3.5.** Let \( p \in [1, 2] \). By the inequality

\[
\left( \sum_{i=1}^{n} a_i \right)^\alpha \leq \sum_{i=1}^{n} a_i^\alpha, \quad a_i \geq 0 \quad \text{and} \quad \alpha \in (0, 1],
\]

we have

\[
\sum_{i=1}^{n} \mathbb{E}[|\xi_i|^p] = \sum_{i=1}^{n} \mathbb{E}\left[ \left( \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)^{p/2}} \right)^{p/2} \mathbb{E}[|\xi_i|^p] \right] \leq \mathbb{E}\left[ \left( \sum_{i=1}^{n} \frac{\phi_i^2}{(\sum_{k=1}^{n} \phi_k^2)^{p/2}} \right)^{p/2} \right] A = A.
\]

By the inequality of von Bahr and Esseen (cf. Theorem 2 of \([27]\)), we get

\[
\mathbb{E}[|S_n|^p] \leq 2 \sum_{i=1}^{n} \mathbb{E}[|\xi_i|^p] \leq 2A.
\]
Then for all $x > 0$,

$$
P\left( \pm (\theta_n - \theta) \sqrt{\sum_{k=1}^{n} \phi_k^2} \geq x \right) = P\left( \pm S_n \geq x \right) \leq \frac{E[|S_n|^p]}{x^p} \leq \frac{2A}{x^p}.
$$

This completes the proof of theorem. 

\textbf{Proof of Theorem 3.6.} It is obvious that

$$
\frac{\theta_n - \theta}{\sigma} \sqrt{\sum_{k=1}^{n} \phi_k^2} = \sum_{i=1}^{n} \eta_i,
$$

where $\eta_i = \xi_i / \sigma$. Notice that $E[e_{i1}^2 | F_{i-1}] = E[e_{i1}^2 | \sigma \{\varepsilon_j, j \leq i - 1\}] = \sigma^2$ a.s.. Then we have

$$
\sum_{i=1}^{n} E[\eta_i^2 | F_{i-1}] = \frac{\langle S \rangle_n}{\sigma^2} = \sum_{i=1}^{n} \frac{\phi_i^2}{\left(\sum_{k=1}^{n} \phi_k^2\right)} \frac{E[e_{i1}^2 | F_{i-1}]}{\sigma^2} = \sum_{i=1}^{n} \frac{\phi_i^2}{\sum_{k=1}^{n} \phi_k^2} = 1
$$

and

$$
\sum_{i=1}^{n} E[|\eta_i|^p | F_{i-1}] \leq \sum_{i=1}^{n} E\left[\left| \frac{\phi_i}{\sqrt{\sum_{k=1}^{n} \phi_k^2}} \right|^p \right] E[|e_{i1}|^p | F_{i-1}] \leq \frac{A}{\sigma^p} \sum_{i=1}^{n} E\left[\left| \frac{\phi_i}{\sqrt{\sum_{k=1}^{n} \phi_k^2}} \right|^p \right].
$$

Applying inequality (48) to the martingale difference sequence $(\eta_i, F_{i})_{i=1,\ldots,n}$ with $\delta = p - 2$, we obtain the desired inequality.

\textbf{8. Proof of Theorem 3.7.}

The proof is based on the result of Ouchti [24].

\textbf{Proof of Theorem 3.7} By the rate of convergence in the CLT for martingale difference sequences of Ouchti (cf. Corollary 1 of [24]), it suffices to verify the tightness of $H_n$. By Theorem 8.4 of Billingsley [2] for stationary martingale difference sequences, we only need to show that for any $\varepsilon > 0$, there exist a $\lambda$, with $\lambda > 1$, and an integer $n_0$ such that for every $n \geq n_0$,

$$
P\left( \max_{1 \leq i \leq n} |S_i| \geq \lambda \sqrt{n} \right) \leq \frac{\varepsilon}{\lambda^2}.
$$

(63)

Since

$$
E[|\xi_i|^3 | F_{i-1}] \leq M E[\xi_i^2 | F_{i-1}],
$$

we deduce that

$$
(E[\xi_i^2 | F_{i-1}])^{3/2} \leq E[|\xi_i|^3 | F_{i-1}] \leq M E[\xi_i^2 | F_{i-1}].
$$

Thus

$$
E[\xi_i^2 | F_{i-1}] \leq M^2, \quad \langle S \rangle_n \leq nM^2 \quad \text{and} \quad \sum_{i=1}^{n} E[|\xi_i|^3 | F_{i-1}] \leq nM^3.
$$
Applying (20) with \( p = 3, x = 2y = \lambda \sqrt{n} \), we obtain

\[
P\left( \max_{1 \leq i \leq n} |S_i| \geq \lambda \sqrt{n} \right) \leq 2 \exp \left\{ -\frac{2 \lambda^2}{25e^3 M^2} \right\} + 2 \exp \left\{ -\frac{6}{5} \log \left( 1 + \frac{3\lambda^3 \sqrt{n}}{20M^3} \right) \right\}
\]

\[
+ \ P\left( \max_{1 \leq i \leq n} \xi_i > \frac{1}{2} \lambda \sqrt{n} \right) + \ P\left( \max_{1 \leq i \leq n} (-\xi_i) > \frac{1}{2} \lambda \sqrt{n} \right)
\]

\[
\leq 2 \exp \left\{ -\frac{4 \lambda^2}{50e^3 M^2} \right\} + 2 \left( \frac{3\lambda^3 \sqrt{n}}{20M^3} \right)^{6/5}
\]

\[
+ \frac{16}{\lambda^3 \sqrt{n}} E \left[ |\xi_1|^3 \mathbf{1}_{\{|\xi_1| > \frac{1}{2} \lambda \sqrt{n} \}} \right]
\]

\[
\leq \frac{\varepsilon}{\lambda^2} \quad (64)
\]

provided that \( \lambda \) is sufficiently large. This proves (63). \( \square \)

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