A half-normal distribution scheme for generating functions

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This work supplants the extended abstract “A half-normal distribution scheme for generating functions and the unexpected behavior of Motzkin paths” which appeared in the Proceedings of the 27th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2016) Krakow Conference.

Abstract

We present an extension of a theorem by Michael Drmota and Michèle Soria [Images and Preimages in Random Mappings, 1997] which can be used to identify the limiting distribution for a class of combinatorial schemata. This is achieved by determining analytic and algebraic properties of the associated bivariate generating function. We give sufficient conditions implying a half-normal limiting distribution, extending the known conditions leading to either a Rayleigh, a Gaussian, or a convolution of the last two distributions. We conclude with three natural appearances of such a limiting distribution in the domain of lattice paths.

1 Introduction

Generating functions are a powerful tool in combinatorics and probability theory. One of the main reasons of their success is the symbolic method [26], a general correspondence between combinatorial constructions and functional equations. It provides a direct translation of the structural description of a class into an equation on generating functions without the necessity of deriving recurrence relations on the level of counting sequences first. A very powerful aspect of generating functions is the possibility to capture even more information by partitioning the sequences into smaller pieces. In this context one uses multivariate generating functions. The simplest case is the one of bivariate generating functions \( F(z, u) = \sum f_{nk} z^n u^k \). In general, \( n \) is the length or size, and \( k \) is the value of a “marked” parameter.

Schemata on generating functions are general methods which allow to derive results on the counting sequences \( f_{nk} \) by solely analyzing the properties of their bivariate generating functions. Such methods were derived by Drmota and Soria in [21, 22], where in the latter they derived three general theorems which identify the limiting distributions as being Rayleigh,
Table 1: A comparison of the geometric, normal, half-normal, and Rayleigh distribution. We will encounter all four of them in the context of Motzkin walks.

|                         | Geometric | Normal | Half-normal | Rayleigh |
|-------------------------|-----------|--------|-------------|----------|
|                         | $\text{Geom}(p)$ | $\mathcal{N}(\mu, \sigma)$ | $\mathcal{H}(\sigma)$ | $\mathcal{R}(\sigma)$ |
| Support                 | $k \in \{0, 1, \ldots\}$ | $x \in \mathbb{R}$ | $x \in \mathbb{R}_{\geq 0}$ | $x \in \mathbb{R}_{\geq 0}$ |
| PDF                     | $(1-p)^kp$ | $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ | $\sqrt{\frac{2}{\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ | $\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ |
| Mean                    | $\frac{1-p}{p}$ | $\mu$ | $\sigma\sqrt{\frac{2}{\pi}}$ | $\sigma\sqrt{\frac{\pi}{2}}$ |
| Variance                | $\frac{1-p}{p^2}$ | $\sigma^2$ | $\sigma^2 \left(1 - \frac{2}{\pi}\right)$ | $\sigma^2 \left(2 - \frac{\pi}{2}\right)$ |
In Section 4, we apply our result to three properties of walks: the number of returns to zero, the height, and the number of sign changes, where sign changes are only treated in the case of Motzkin walks. In the case of a zero drift a half-normal distribution appears in all cases. In Section 5, we give the proof of our main result, Theorem 2.1. In Section 6, we state a summary of our results and compare the common structure of the generating functions treated in of Section 4.

### 2 The half-normal theorem

In this section we use the notation introduced in [22, Section 3].

Let \( c(z) = \sum_{n \geq 0} c_n z^n \) be a generating function with non-negative real numbers \( c_n \), and \( c(z, u) = \sum_{n,k \geq 0} c_{nk} z^n u^k \) be the corresponding bivariate generating function where a parameter has been marked. These two forms are connected by \( c(z, 1) = c(z) \). For fixed \( n \in \mathbb{N} \) the numbers \( c_{nk} \) implicitly define a (discrete) probability distribution. In particular, we define a sequence of random variables \( X_n, n \geq 1 \), by

\[
\mathbb{P}[X_n = k] := \frac{c_{nk}}{c_n} = \frac{[z^n u^k]c(z, u)}{[z^n]c(z, 1)}. 
\]

We are then interested in the limiting distribution of these random variables, as it gives a qualitative description of the marked parameter for large \( n \). This goal is achieved by a careful analysis of algebraic and analytic properties of \( c(z, u) \). The limiting distribution of \( X_n \) is shown to be either Gaussian, Rayleigh, the convolution of Gaussian and Rayleigh (see [22, Theorems 1–3]), or half-normal (see Theorem 2.1).

The technical conditions for the first three limit laws are stated in [22, Hypothesis [H]]. The Half-normal Theorem also requires Hypothesis [H] to hold, except that we do not need \( h(\rho, 1) > 0 \). We call this weaker form Hypothesis [H’]. Let us state the precise conditions for \( \rho(u) = \text{const} \) for completeness:

**Hypothesis [H’]:** Let \( c(z, u) = \sum_{n,k} c_{nk} z^n u^k \) be a power series in two variables with non-negative coefficients \( c_{nk} \geq 0 \) such that \( c(z, 1) \) has a radius of convergence of \( \rho > 0 \).

We suppose that \( 1/c(z, u) \) has the local representation

\[
\frac{1}{c(z, u)} = g(z, u) + h(z, u) \sqrt{1 - \frac{z}{\rho}},
\]

(1)
for $|u - 1| < \varepsilon$ and $|z - \rho| < \varepsilon$, $\arg(z - \rho) \neq 0$, where $\varepsilon > 0$ is some fixed real number, and $g(z, u)$ and $h(z, u)$ are analytic functions. Furthermore, we have $g(\rho, 1) = 0$.

In addition, $z = \rho$ is the only singularity on the circle of convergence $|z| = \rho$, and $1/c(\rho, 1)$, respectively, $c(z, u)$, can be analytically continued to a region $|z| < \rho + \delta, |u| < 1 + \delta, |u - 1| > \frac{\varepsilon}{2}$ for some $\delta > 0$.

**Theorem 2.1 (Half-normal limit theorem).** Let $c(z, u)$ be a bivariate generating function satisfying $[H']$. If $g_z(\rho, 1) \neq 0$, $h_u(\rho, 1) \neq 0$, and $h(\rho, 1) = g_u(\rho, 1) = g_{uu}(\rho, 1) = 0$, then the sequence of random variables $X_n$ defined by

$$P[X_n = k] = \frac{[z^n u^k]c(z, u)}{[z^n]c(z, 1)},$$

has a half-normal limiting distribution, i.e.,

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}(\sigma),$$

where $\sigma = \sqrt{\frac{h_u(\rho, 1)}{g_z(\rho, 1)}}$, and $\mathcal{H}(\sigma)$ has density $\frac{2}{\sqrt{\pi \sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2}\right)$ for $z \geq 0$. Expected value and variance are given by

$$E[X_n] = \sigma \sqrt{\frac{2}{\pi} n} + O(1) \quad \text{and} \quad \mathbb{V}[X_n] = \sigma^2 \left(1 - \frac{2}{\pi}\right)n + O(\sqrt{n}).$$

Moreover, we have the local law

$$P[X_n = k] = \frac{1}{\sigma} \sqrt{\frac{2}{\pi n}} \exp\left(-\frac{k^2/n}{2\sigma^2}\right) + O(kn^{-3/2}) + O\left(n^{-1}\right),$$

uniformly for all $k \geq 0$.

Consider the example of unweighted unconstrained Motzkin walks given by the jumps $(1, 1), (1, 0), (1, -1)$ with marked returns to zero, i.e. points of altitude $y = 0$. In (5) we will see that their bivariate generating function is given by

$$W(z, u) = \frac{\sqrt{1 + z}}{u \sqrt{1 + z} (1 - 3z) + (1 - u) \sqrt{1 - 3z}}.$$

Due to $W(z, 1) = \frac{1}{1 - 3z}$, we have $\rho = \frac{1}{3}$. The decomposition (1) is valid and we get

$$g(z, u) = u (1 - 3z), \quad \text{and} \quad h(z, u) = \frac{1 - u}{\sqrt{1 + z}}. \quad (2)$$

All other conditions of Hypothesis $[H']$ and Theorem 2.1 are satisfied since $g(z, u)$ and $h(z, u)$ are analytic for $|z| < 1$. Hence, the analytic continuation beyond $z = \frac{1}{3}$ and $u = 1$ is trivial in this case. Note that in general it has to be checked carefully. This proves that the number of returns to zeros of unweighted Motzkin walks satisfies a half-normal limit distribution. In Theorem 4.2 we will see that this holds for general lattice path models with zero drift.
Remark 1. The assumption of a constant singularity in \( z \) given by \( \rho \) can be weakened to a singularity \( \rho(u) = \rho(1) + O((u - 1)^3) \), i.e., \( \rho'(1) = \rho''(1) = 0 \). However, no example is known where \( \rho(u) \) is not constant in a neighborhood of \( u \sim 1 \).

Before we apply these results in the context of lattice paths we need some further definitions and results presented in the next section.

3 Properties of lattice paths

In this section we present needed, known results on directed lattice paths. Readers familiar with the exposition of Banderier and Flajolet [4] or related results may skip this section.

Definition 3.1. A step set \( S \subseteq \mathbb{Z}^2 \) is a fixed, finite set of vectors \( \{(a_1, b_1), \ldots, (a_m, b_m)\} \). An \( n \)-step lattice path or walk is a sequence \( (v_1, \ldots, v_n) \) of vectors, such that \( v_j \in S \). Geometrically, it is a sequence of points \( (\omega_0, \omega_1, \ldots, \omega_n) \) where \( \omega_i \in \mathbb{Z}^2, \omega_0 = (0, 0) \) and \( \omega_i - \omega_{i-1} = v_i \) for \( i = 1, \ldots, n \). The elements of \( S \) are called steps or jumps. The length \( |\omega| \) of a lattice path is its number \( n \) of jumps.

Motzkin paths are a classical example that will occur repeatedly. For more details we refer to [12,17,34].

Definition 3.2. A Motzkin path is a path that starts at the origin and is given by the step set \( S = \{-1, 0, +1\} \).

3.1 Types of lattice paths

We restrict our attention to simple directed paths for which every element in the step set \( S \) is of the form \((1, b)\). In other words, these walks constantly move one step to the right. We introduce the abbreviation \( S = \{b_1, \ldots, b_m\} \) in this case.

Along these restrictions, we work with the following classes (see Table 3): A bridge is a path whose end-point \( \omega_n \) lies on the \( x \)-axis. A meander is a path that lies in the quarter plane \( \mathbb{Z}^2_+ \). An excursion is a path that is at the same time a meander and a bridge. Their generating functions have been fully characterized in [4] by means of analytic combinatorics, see [26].

Remark 2. In the context of Motzkin paths we will refer to Motzkin walks/meanders/bridges/excursions depending on the different restrictions. In common literature Motzkin paths are often defined as Motzkin excursions, e.g. in [17].

In many situations it is useful to associate weights to single steps in order to model different behaviors.

Definition 3.3. For a given step set \( S \), we define the respective system of weights as \( \{p_s : s \in S\} \) where \( p_s > 0 \) is the associated weight to step \( s \in S \). The weight of a path is defined as the product of the weights of its individual steps.

Classical choices for weights are \( p_s = 1 \) for all \( s \in S \), which gives a model where every path has the same weight 1, and \( \sum_{s \in S} p_s = 1 \), which gives a probabilistic model of paths, i.e., step \( s \) is chosen with probability \( p_s \).
Table 3: The four types of paths: walks, bridges, meanders and excursions, and the corresponding generating functions for directed lattice paths [4, Fig. 1].

The following definition is the algebraic link between weights and steps.

**Definition 3.4.** The jump polynomial of $S$ is defined as the polynomial in $u,u^{-1}$ (a Laurent polynomial)

$$P(u) := \sum_{j=1}^{m} p_j u^j.$$  

Let $c = -\min_j s_j$ and $d = \max_j s_j$ be the two extreme jump sizes, and assume throughout $c,d > 0$ to avoid trivial cases. The kernel equation is defined by

$$1 - zP(u) = 0, \quad \text{or equivalently} \quad u^c - z(u^c P(u)) = 0.$$  

The quantity $K(z,u) := u^c - z u^c P(u)$ is called kernel.

**3.2 The kernel method**

The kernel plays a crucial rôle and is name-giving for the kernel method, which is the key tool for characterizing this family of lattice paths. One of the first appearances are in Knuth’s book [32, Exercise 2.2.1–4] and the detailed solution therein. It was later generalized in combinatorics [4,13,14] and possesses many applications. However, it was also independently invented in probability theory [23,33] or statistical mechanics [28,31]. Up to today it is commonly used [35,37]. For a more details on its history the interested reader is referred to [9, Chapter 1].

In the heart of this method lies the observation that the kernel equation is of degree $c+d$ in $u$, and therefore possesses generically $c+d$ roots. These correspond to branches of an algebraic curve given by the kernel equation. From the theory of algebraic curves and Newton-Puiseux series, for $z$ near 0 one obtains $c$ “small branches” that we call $u_1(z), \ldots, u_c(z)$ and $d$ “large
branches” \(v_1(z),\ldots,v_d(z)\). For being well-defined, we restrict ourselves to the complex plane slit along the negative real axis.

They are called “small branches” because they satisfy \(\lim_{z \to 0} u_i(z) = 0\), whereas the “large branches” satisfy \(\lim_{z \to 0}|v_i(z)| = \infty\). Banderier and Flajolet showed, that the generating functions of bridges, excursions and meanders can be expressed in terms of the small branches and the jump polynomial, see Table 3.

We define branch \(u_1(z)\) as the one being real and positive near 0. It is called the principal small branch. It proves to be responsible for the asymptotic behavior of bridges, excursions and meanders, compare [4, Theorems 3 and 4]. The branch \(v_1(z)\) is defined in the same way. It is conjugated to \(u_1(z)\) and called the principal large branch.

3.3 Analytic properties

**Lemma 3.5** ([4, Lemma 1]). Let \(P(u)\) be the jump polynomial associated with the steps of a simple walk. Then there exists a unique number \(\tau\), called the structural constant, such that \(P'(\tau) = 0\). The structural radius is defined by the quantity

\[
\rho := \frac{1}{P(\tau)}.
\]

A walk is called periodic with period \(p\) if there exists a polynomial \(H(u)\) and integers \(b \in \mathbb{Z}\) and \(p \in \mathbb{N}\), \(p > 1\) such that \(P(u) = u^b H(u^p)\). Otherwise it is called aperiodic. Note that generating functions of aperiodic walks possess a unique singularity on the positive real axis [4]. Under the aperiodicity condition, the principal small branch dominates the other branches:

\[
|u_j(z)| < u_1(|z|), \quad \text{for } z \leq \rho, \quad j > 1 \quad \text{and} \quad |u_1(z)| < |v_1(z)|, \quad \text{for } z < \rho.
\]

Furthermore, we know that the principal branches \(u_1(z)\) and \(v_1(z)\) are analytic in the open interval \((0, \rho)\) for an aperiodic step set, and they satisfy the singular expansions

\[
u_1(z) = \frac{1}{\rho} \left(1 - \frac{z}{\rho}\right)^{1/2} + O\left(\frac{1 - z}{\rho}\right), \quad v_1(z) = \tau + C \sqrt{1 - \frac{z}{\rho}} + O\left(1 - \frac{z}{\rho}\right), \quad (3)
\]

for \(z \to \rho^-\), where \(C = \sqrt{2 \frac{P(\tau)}{P''(\tau)}}\).

The previous result is a direct consequence of the implicit function theorem. But one can get even more information with the help of its singular version [26, Lemma VII.3].

**Proposition 3.6.** Let \(u_1(z)\) and \(v_1(z)\) be the principal small and large branches of the kernel equation \(1 - zP(u) = 0\). Then there exists a neighborhood \(\Omega\) such that for \(z \to \rho\) in \(\Omega \setminus (\rho, \infty)\) they have a local representation of the kind

\[
a(z) + b(z)\sqrt{1 - z/\rho},
\]

where \(a(z)\) and \(b(z)\) are analytic functions for every point \(z \in \Omega \setminus (\rho, \infty), z \neq z_0\). We have that \(a(\rho) = \tau, b(\rho) = -C\) for \(u_1(z)\) or \(b(\rho) = C\) for \(v_1(z)\), respectively. The other branches \(u_2(z),\ldots,u_c(z)\) and \(v_2(z),\ldots,v_d(z)\) are analytic in a neighborhood of \(\rho\).
Proof. The branches \( u(z) \), which we use as a shorthand for \( u_1(z) \) and \( v_1(z) \), are implicitly defined by \( \Phi(z, u(z)) = 0 \), where \( \Phi(z, u) = 1 - zP(u) \). We will apply the singular implicit function theorem, see [26, Lemma VII.3]. Firstly, it is easy to check that \( \Phi(\rho, \tau) = 0, \Phi_z(\rho, \tau) = -\rho^{-1} \neq 0, \Phi_u(\rho, \tau) = 0, \) and \( \Phi_{uu}(\rho, \tau) = -\rho P''(\tau) \neq 0 \). Note that the last equation is not equal to 0 because \( P(u) \) is a convex function for real values of \( u \).

The two possible solutions \( y_1(z) \) and \( y_2(z) \) correspond to the principal small branch \( u_1(z) \) and the principal large branch \( v_1(z) \), respectively. Thus, we recovered the asymptotic expansion (3).

Finally, the analytic nature of \( a(z) \) and \( b(z) \) follows from the Weierstrass preparation theorem, see [18,19], for an analytic presentation [26, Theorem B.5], or for an algebraic presentation [1, Chapter 16].

The analytic character of the other small branches, follows from the analytic version of the implicit function theorem: Consider \( \tilde{\Phi}(z, u) := \Phi(z, u) (u - u_1(z))(u - v_1(z)) \). Solving this function for \( u \) gives the solutions of \( \Phi(z, u) = 0 \) not equal to \( u_1(z) \) or \( v_1(z) \). But \( \tilde{\Phi}_u(\rho, \tau) \neq 0 \) and therefore, these solutions are analytic in a neighborhood of \( z_0 \). \( \square \)

4 Applications to lattice path counting

The following examples are motivated by the nice presentation of Feller [24, Chapter III] on one-dimensional symmetric, simple random walks, see also [29, Chapter 12]. Therein, the discrete time stochastic process \((S_n)_{n \geq 0}\) is defined by \( S_0 = 0 \) and \( S_n = \sum_{j=1}^{n} X_j, \ n \geq 1, \) where the \((X_i)_{i \geq 1}\) are iid Bernoulli random variables with \( \mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = \frac{1}{2} \). These results are generalized to the case of aperiodic directed lattice paths. In particular compare [24, Problems 9-10] and [36, Remark of Barton] for returns to zero of symmetric and asymmetric random walks, respectively. Furthermore, see [24, Chapter III.5] for sign changes, and [24, Chapter III.7] for the height. Note that this area is still an active field of research with applications in different fields. See for example [16] on an application of Stein’s method on this parameters in which bounds for the convergence rate in the Kolmogorov and the Wasserstein metric are derived, [11] where the maxima of two random walks are analyzed, and [15] for applications to machine learning.

For the sake of brevity we will only mention the weak convergence law. However, in all cases the local law and the asymptotic expansions for mean and variance hold as well.

4.1 Returns to zero

A return to zero is a point of a walk of altitude 0 except for the starting point, in other words a return to the \( x \)-axis, see Figure 4. In order to count them we consider “minimal” bridges, in the sense that the bridges touch the \( x \)-axis only at the beginning and at the end. We call them arches. As a bridge is a sequence of such arches, we get their generating function in the form of \( A(z) = 1 - \frac{1}{b(z)}. \)
Lemma 4.1. The generating function of arches $A(z)$ is for $z \to \rho$ of the kind
\[ A(z) = a(z) + b(z) \sqrt{1 - z/\rho}, \]
where $a(z)$ and $b(z)$ are analytic functions in a neighborhood $\Omega \setminus (\rho, \infty)$ of $\rho$.

Proof. We know that $B(z) = z \sum_{j=1}^{\infty} \frac{u_j(z)}{u_1(z)}$ is analytic for $|z| < \rho$, see [4, Theorem 3]. Due to the aperiodicity $\rho$ is the only singular point on the circle of convergence. Furthermore, $u_1(z)$ is the only small branch which is singular there, hence
\[ B(z) = \frac{C_1}{\sqrt{1 - z/\rho}} + \mathcal{O}(1), \quad C_1 := \frac{C}{2\tau}, \tag{4} \]
for $z \to \rho$. Then, Proposition 3.6 and (4) imply the desired decomposition. \qed

The number of returns to zero of a bridge is the same as the number of arches it is constructed from. These numbers were analyzed in the more general model of the reflection-absorption model in [10]. The governing limit law behaves like a negative binomial distribution.

Here, we are interested in the number of returns to zero of walks which are unconstrained by definition. Every walk can be decomposed into a maximal initial bridge, and a walk that never returns to the $x$-axis, see Figure 1. Let us denote the generating function of this tail by $T(z)$.

![Figure 1](image-url)

Figure 1: A walk with 9 returns to zero decomposed into a bridge and a tail (path never returns to the $x$-axis anymore). Note that the crossing in the tail are no returns to zero.

As we want to count the number of returns to zero, we mark each arch by an additional parameter $u$ and reconstruct the generating function of walks. This gives
\[ W(z, u) = \frac{1}{1 - u A(z)} T(z) = \frac{W(z)}{u + (1 - u) B(z)}, \quad \text{with} \quad T(z) = \frac{W(z)}{B(z)}. \tag{5} \]

Let us define the random variable $X_n$ to stand for the number of returns to zero of a random walk of length $n$. Thus, $\mathbb{P}[X_n = k] = \frac{[u^k z^n]W(z, u)}{[z^n]W(z, 1)}$.

Theorem 4.2 (Limit law for returns to zero). Let $X_n$ denote the number of returns to zero of an aperiodic walk of length $n$. Let $\delta = P'(1)$ be the drift.
1). For $\delta \neq 0$ we get convergence to a geometric distribution:

$$X_n \xrightarrow{d} \text{Geom} \left( \frac{1}{B(1/P(1))} \right);$$

2). For $\delta = 0$ we get convergence to a half-normal distribution:

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H} \left( \sqrt{\frac{P(1)}{P''(1)}} \right).$$

**Proof.** We see that $[z^n]W(z, 1) = [z^n]W(z) = P(1)^n$. Due to the aperiodicity constraint $B(z)$ is only singular at $\rho$. Obviously, $W(z)$ is singular at $\rho_1 := \frac{1}{B(1)}$.

On the positive real axis the convex nature of $P(u)$ implies that $P(\tau)$ is its unique minimum. Hence, only two cases are possible: $\rho_1 < \rho$, if $\tau \neq 1$; or $\rho_1 = \rho$, if $\tau = 1$. These cases are also characterized by $\delta \neq 0$ or $\delta = 0$, respectively. In the first case $W(z)$ is responsible for the dominant singularity. Then we get (as $B(z)$ is analytic for $|z| < \rho$)

$$[z^n]W(z, u) = \frac{1}{B(\rho_1)} \frac{P(1)^n}{1 - u \left(1 - \frac{1}{B(\rho_1)}\right)} + o(P(1)^n).$$

Hence, the probability that a walk of length $n$ has $k$ returns to zero is for any fixed $k$

$$\Pr[X_n = k] = \frac{1}{B(\rho_1)} \left(1 - \frac{1}{B(\rho_1)}\right)^k + o(1).$$

Thus, the limit distribution is a geometric distribution with parameter $\lambda = \frac{1}{B(\rho_1)}$.

In the second case $\tau = 1$ or $\delta = 0$, we apply Theorem 2.1. By Lemma 4.1 we get that $1/W(z, u)$ has a decomposition of the kind (1). In particular, from (4) we get that

$$\frac{1}{W(z, u)} = \left(1 - \frac{z}{\rho}\right) u + C \frac{z}{2(1 - u)} \sqrt{1 - \frac{z}{\rho}} + O \left(\left(1 - \frac{z}{\rho}\right)(1 - u)\right),$$

for $z \to \rho$ and $u \to 1$, with $g(\rho, 1) = h(\rho, 1) = g_u(\rho, 1) = g_{uu}(\rho, 1) = 0$; and $g_z(\rho, 1) = -P(1)$ and $h_u(\rho, 1) = -\sqrt{\frac{P(1)}{2P''(1)}}$. Hence, Theorem 2.1 yields the result. \qed

### 4.2 Height

For a path of length $n$ we define the **height** as its maximally attained $y$-coordinate, see Figure 2. Formally, let $\omega = (\omega_k)_{k=0}^\infty$ be a walk. Then its height is given by $\max_{k \in \{0, \ldots, n\}} \omega_k$.

In order to analyze the distribution of heights, we define the bivariate generating function $F(z, u) = \sum_{n,h \geq 0} f_{nh} z^n u^h$. The coefficient $f_{nh}$ represents the number of walks of height $h$ among walks of length $n$.

Let $M(z, u) = \sum_{n,h \geq 0} m_{nh} z^n u^h$ be the generating function of meanders, where $m_{nh}$ is the number of meanders of length $n$ ending at final altitude $h$. Banderier and Flajolet derived in [4, Theorem 2] its closed-form as

$$M(z, u) = \prod_{j=1}^c \left(1 - u_{ij}(z)\right) \left(1 - z P(u)\right) = -\frac{1}{P_d(z)} \prod_{\ell=1}^d \frac{1}{u - v_\ell(z)}.$$ (6)
Figure 2: A Motzkin walk of height 2. The relative heights are given at every node.

**Theorem 4.3.** The bivariate generating function of walks (where \( z \) marks the length, and \( u \) marks the height of the walk) is given by

\[
F(z, u) = \frac{W(z)M(z, u)}{M(z)} = -\frac{1}{p_d z} \left( \prod_{j=1}^{c} \frac{1}{1 - u_j(z)} \right) \left( \prod_{t=1}^{d} \frac{1}{u - v_t(z)} \right).
\] (7)

**Proof.** Banderier and Nicodème derived in [6, Theorem 2] the generating function \( F[-\infty, h](z) \) for walks staying always below a wall \( y = h \):

\[
F[-\infty, h](z) = 1 - \sum_{i=1}^{d} \left( \frac{1}{v_i} \right)^{h+1} \prod_{1 \leq j \leq d, j \neq i} \frac{1 - v_j}{v_i - v_j}.
\]

From this we directly get the generating function \( F[h](z) \) for walks that have height exactly \( h \). For \( h \geq 1 \) it equals

\[
F[h](z) = F[-\infty, h](z) - F[-\infty, h-1](z) = \sum_{i=1}^{d} \frac{v_i - 1}{1 - zP(1)} \left( \frac{1}{v_i} \right)^{h+1} \prod_{1 \leq j \leq d, j \neq i} \frac{1 - v_j}{v_i - v_j}.
\]

The last formula also holds for \( h = 0 \). Finally, marking the heights by \( u \) and summing over all possibilities gives

\[
F(z, u) = \sum_{h \geq 0} u^h F[h](z) = \frac{\prod_{j=1}^{d} (1 - v_j)}{1 - zP(1)} \sum_{i=1}^{d} \frac{1}{u - v_i} \prod_{1 \leq j \leq d, j \neq i} \frac{1}{v_i - v_j}.
\]

Note that \( M(z) = -\frac{1}{p_d z} \prod_{j=1}^{c} \frac{1}{1 - u_j} \), see [4, Corollary 1]. Hence, the first factor gives \( \frac{W(z)}{M(z)} \).

What remains is to analyze the sum. Putting everything on a common denominator, we get

\[
\sum_{i=1}^{d} \frac{1}{u - v_i} \prod_{1 \leq j \leq d, j \neq i} \frac{1}{v_i - v_j} = \left( \prod_{i=1}^{d} \frac{1}{u - v_i} \right) \sum_{i=1}^{d} (-1)^{i+1} \left( \prod_{j \neq i} v_j \right) \prod_{k \neq \ell, k, \ell \neq i} (v_k - v_\ell) \prod_{k < \ell} (v_k - v_\ell).
\]

The last fraction is equal to 1, because the numerator is equal to the Vandermonde determinant of the denominator that has been expanded with respect to the first column of all 1s. \( \square \)
We want to remark that the relation given by the first equality in (7) can be interpreted as an instance of what is known in probability theory as the Wiener-Hopf factorization [25, Chapter XII]. Furthermore, this identity is obviously directly related to the kernel equation. Its simple structure suggests a combinatorial interpretation, or even a direct combinatorial proof.

In order to answer this question, we will analyze it in more detail now. Let us start with its factor

\[ \frac{W(z)}{M(z)} = \prod_{j=1}^{c} \frac{1}{1 - u_j(z)}. \]

First, we introduce a new useful dualism. A positive excursion is a “traditional” excursion, i.e., it is required to stay above the \(x\)-axis, whereas a negative excursion is a path which starts at zero, ends on the \(x\)-axis, but is required to stay below the \(x\)-axis.

**Lemma 4.4.** Among all walks of length \(n\), the number of positive excursions is equal to the number of negative excursions.

**Proof.** Mirroring bijectively maps positive excursions to negative ones. \(\square\)

In the same manner, we introduce the notion of negative meanders staying always below the \(x\)-axis and denote their generating function by \(M_-(z)\). Furthermore, let strictly negative meanders be negative meanders that never return to the \(x\)-axis (but start at 0), and denote their generating function by \(M_{<0}(z)\).

**Proposition 4.5.** The generating functions of strictly negative meanders and negative meanders are given by

\[ M_{<0}(z) = \prod_{j=1}^{c} \frac{1}{1 - u_j(z)}, \quad M_-(z) = E(z) M_{<0}(z) = \frac{(-1)^{c-1}}{p_{-c} z} \prod_{j=1}^{c} \frac{u_j(z)}{1 - u_j(z)}. \]

**Proof.** The key idea is that negative meanders are meanders after mirroring the coordinate system along the \(x\)-axis. By doing so, the step polynomial \(P(u) = \sum_{i=-c}^{d} p_i u^i\) changes to the mirrored step polynomial

\[ \tilde{P}(u) = \sum_{i=-d}^{c} p_{-i} u^i. \]

The small branches \(\tilde{u}_i(z)\), which satisfy \(1 - z \tilde{P}(\tilde{u}_i(z)) = 0\) are given by

\[ \tilde{u}_i(z) = \frac{1}{v_i(z)}, \]

where \(v_i(z)\) are the large branches of the original kernel equation \(1 - z P(u) = 0\). Finally, by (6) and because of \(P(1) = \tilde{P}(1)\) we get

\[ M_-(z) = \frac{\prod_{j=1}^{d} (1 - \tilde{u}_j(z))}{1 - z P(1)} = \frac{(-1)^{d-1}}{p_d z} \left( \prod_{j=1}^{d} \frac{1}{v_j(z)} \right) \prod_{j=1}^{c} \frac{1}{1 - u_j(z)}, \]

due to the factorization of the kernel equation. The first factor \(\frac{(-1)^{d-1}}{p_d z} \left( \prod_{j=1}^{d} \frac{1}{v_j(z)} \right)\) is equal to the generating function of excursions \(E(z)\) which can also be expressed in terms of the small branches.

Finally, note that any meander can be uniquely decomposed into an initial negative excursion and a strictly negative meander. \(\square\)
Before we proceed, let us illustrate the previous results for the case of Motzkin walks with step polynomial \( P(u) = \frac{p_1}{u} + p_0 + p_1u \).

**Corollary 4.6.** The bivariate generating function of Motzkin walks with marked height is given by

\[
F_M(z, u) = \frac{2}{(1 + u)(1 - p_0z) - 2z(p_{-1} + up_1) + (1 - u)\sqrt{(1 - p_0z)^2 - 4p_{-1}p_1z^2}}.
\]

This case possesses a simple combinatorial interpretation. From the alternative representation \( E(z) = \frac{1}{zp_1v_1(z)} \) for the generating function of excursions [4, Theorem 2], we get

\[
F_M(z, u) = \frac{1}{1 - p_1zuE(z)} M_-(z).
\]

Figure 3: The first passage decomposition of a Motzkin walks into (negative) excursions and a trailing negative meander.

The above generating function just represents the decomposition of a walk into a sequence of marked blocks, which are (negative) excursions (cf. Lemma 4.4) followed by an up step, and a negative meander at the end, see Figure 3. Note that a similar interpretation exists for other step sets.

We now turn our attention back to the limit laws for the height of walks. Let \( X_n \) be the random variable for the height of a random walk of length \( n \). Thus, \( \Pr[X_n = k] = \left[u^k z^n F(z, u)\right]_{P(z, 1)} = \left[u^k z^n F(z, u)\right]_{P(z, 1)} \).

The following theorem concludes this section with the governing limit laws for the height of walks. Note in particular the different rescaling factors in each case.

**Theorem 4.7** (Limit law for the height). Let \( X_n \) denote the height of a walk of length \( n \). Let \( \delta = P'(1) \) be the drift, and \( \rho_1 = \frac{1}{P'(1)} \).

1). For \( \delta < 0 \) the limit distribution is discrete and characterized in terms of the large branches:

\[
\lim_{n \to \infty} \Pr[X_n = k] = [u^k] \omega(u), \quad \text{where} \quad \omega(u) = \prod_{j=1}^d \frac{1 - v_j(\rho_1)}{u - v_j(\rho_1)}.
\]
2). For $\delta = 0$ the standardized random variable converges to a half-normal distribution:

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}\left(\sqrt{\frac{P''(1)}{P(1)}} \right).$$

3). For $\delta > 0$ the standardized random variable converges to a normal distribution:

$$\frac{X_n - \mu_n}{\sigma \sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \mu = \frac{P'(1)}{P(1)}, \quad \sigma^2 = \frac{P''(1)}{P(1)} + \frac{P'(1)}{P(1)} - \left(\frac{P'(1)}{P(1)}\right)^2.$$

**Proof.** From the structure of the generating function in (7) it is obvious that the result strongly depends on the limit law of the final altitude of meanders. This was analyzed in [4, Theorem 6]. In several cases we will apply the domination property of the small branches [4]:

$$|u_j(z)| < |u_1(z)| \leq \tau \leq |v_1(z)| < |v_\ell(z)|, \quad \text{for } |z| < \rho,$$

and $j = 2, \ldots, c$ as well as $\ell = 2, \ldots, d$.

Let us start with $\delta < 0$. In this case it proves convenient to consider the equivalent representation of (7) given by

$$F(z, u) = \frac{1}{1 - zP(1)} \prod_{\ell=1}^{d} \frac{1 - v_\ell(z)}{u - v_\ell(z)}.$$

In this case we know that $\tau > 1$, implying $\rho > \rho_1$ and that the dominant singularity arises at $z = \rho_1$. The product of the large branches is analytic for $|z| < \rho$ as was already noted in [4]. Hence, by standard methods [26, Theorem VI.12 (Real analysis asymptotics)] we get the asymptotic expansion:

$$[z^n]F(z, u) = P(1)^n \prod_{\ell=1}^{d} \frac{1 - v_\ell(\rho_1)}{u - v_\ell(\rho_1)} + o(P(1)^n).$$

This is the product of several geometric distributions with parameters $v_\ell(\rho_1)$.

In the case of a zero drift, $\delta = 0$, we have $\tau = 1$. Thus, $P(\tau) = P(1)$ and the singularity arises at $\rho = \rho_1 = 1/P(1)$. This means that the singularities of the two factors coincide, and we can apply Theorem 2.1.

Let $\varepsilon > 0$. Then for $|z - \rho| < \varepsilon$, $|u - 1| < \varepsilon$, and $\arg(z - \rho) \neq 0$ we consider

$$\frac{1}{F(z, u)} = -p_2z(1 - u_1(z))(1 - v_1(z)) \left(\prod_{j=2}^{c} (1 - u_j(z))\right) \left(\prod_{\ell=2}^{d} (u - v_\ell(z))\right), \quad =: \tilde{U}_1(z), \quad =: \tilde{V}_1(z, u).$$

The products $\tilde{U}_1(z)$ and $\tilde{V}_1(z, u)$ are analytic for $|z| \leq \rho$. However, the branches $u_1(z)$ and $v_1(z)$ both possess a square root singularity, compare (3). Thus, by Proposition 3.6 we have the desired decomposition

$$\frac{1}{F(z, u)} = g(z, u) + h(z, u)\sqrt{1 - z/\rho},$$
where \( g(z, u) \) and \( h(z, u) \) are analytic functions. In particular, the asymptotic expansion reads as follows
\[
\frac{1}{F(z, u)} = \kappa \left( C(1 - z/\rho) - (u - 1)(1 - z/\rho)^{3/2} \right) + O\left( (u - 1)(1 - z/\rho)^{1/2} \right),
\]
where \( \kappa \) is a non-zero constant. We immediately see that \( g(\rho, 1) = h(\rho, 1) = g_u(\rho, 1) = g_{uu}(\rho, 1) = 0 \), and that \( g_z(\rho, 1) = -\kappa C/\rho \), and \( h_u(\rho, 1) = -\kappa \).

Hence, Theorem 2.1 yields the result with the constant \( \sigma = \sqrt{2h_u(\rho, 1)/p_0 g_z(\rho, 1)} = \sqrt{P''(1)/P(1)}. \)

Finally, in the case \( \delta > 0 \) the same reasoning as in [4] gives the result, as the perturbation by \( M_{\leq 0}(z) \) does not pose any problems. Yet, an alternative proof can be given via the perturbed supercritical sequence scheme [8].

4.3 Sign changes of Motzkin walks

We say that nodes which are strictly above the \( x \)-axis have a positive sign denoted by “+”, whereas nodes which are strictly below the \( x \)-axis have a negative sign denoted by “−”, and nodes on the \( x \)-axis are neutral denoted by “0”. This notion easily transforms a walk \( \omega = (\omega_n)_{n \geq 0} \) into a sequence of signs. In such a sequence a sign change is defined by either the pattern \( +(0)− \) or \( −(0)+ \), where \( (0) \) denotes a non-empty sequence of zeros, see Figure 4.

![Figure 4: A Motzkin walk with 7 returns to zero and 4 sign changes. The positive, neutral or negative signs of the walks are indicated by +, 0, or −, respectively.](image)

The main observation in this context is the non-emptiness of the sequence of zeros. Geometrically this means that a walk has to touch the \( x \)-axis when passing through it. This means that we can count the number of sign changes by counting the number of maximal parts above and below the \( x \)-axis. The idea is to decompose a walk into an alternating sequence of positive (above the \( x \)-axis) and negative (below) excursions terminated by a positive or negative meander.

We define the bivariate generating function \( B(z, u) = b_{nk}z^nu^k \), where \( b_{nk} \) denotes the number of bridges of size \( n \) having \( k \) sign changes. Furthermore, we define
\[
C(z) = \frac{1}{1 - p_0 z}.
\]
as the generating function of \textit{chains}, which are walks constructed solely from the jumps of height 0. Then, the generating function of excursions starting with a \(+1\) jump is

\[
E_1(z) = \frac{E(z)}{C(z)} - 1,
\]

since we need to exclude all excursions which start with a chain or are a chain. Due to Lemma 4.4 this is also the generating function for excursions starting with a \(-1\) jump.

\textbf{Theorem 4.8.} \textit{The bivariate generating function of bridges (where \(z\) marks the length, and \(u\) marks the number of sign changes of the walk) is given by}

\[
B(z, u) = C(z) \left(1 + \frac{2E_1(z)}{1 - uE_1(z)}\right),
\]

\textit{(8)}

\textit{Proof.} A bridge is either a chain, which has zero sign changes, or an alternating sequence of positive and negative excursions, starting with either of them. We decompose it uniquely into such excursions, by requiring that all except the first one start with a non-zero jump. Therefore the first excursion is counted by \(E(z) - C(z)\), whereas all others are counted by \(E_1(z)\). The decomposition is shown in Figure 5. \(\square\)

Figure 5: A bridge is an alternating sequence of positive and negative excursions. Here, it starts with a positive excursion, followed by excursions starting with a non-zero jump.

We start our analysis by locating the dominant singularities of \(B(z, u)\). First we state some inherent structural results of the model which follow from direct computations.

\textbf{Lemma 4.9.} \textit{The structural constant \(\tau\) which is the unique positive root of \(P'(u) = 0\) is}

\[
\tau = \sqrt{\frac{p_{-1}}{p_1}}.
\]

\textit{The structural radius results in \(\rho = \frac{1}{P(\tau)} = \frac{1}{\rho_0 + 2\sqrt{p_{-1}p_1}}\).}

Let \(X_n\) be the random variable for the number of sign changes of a random bridge of length \(n\). Thus, \(\mathbb{P}[X_n = k] = \frac{[u^k z^n]B(z, u)}{[z^n]B(z, 1)}\).

\textbf{Theorem 4.10} (Limit law for sign changes of bridges). \textit{Let \(X_n\) denote the number of sign changes of a Motzkin bridge of length \(n\). Then for \(n \to \infty\) the normalized random variable has a Rayleigh limit distribution}

\[
\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{R}(\sigma) \quad \text{and} \quad \sigma = \frac{\tau}{2} \sqrt{\frac{P''(\tau)}{P(\tau)}},
\]

where \(\tau = \sqrt{\frac{p_{-1}}{p_1}}\) and \(\mathcal{R}(\sigma)\) has the density \(\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\) for \(x \geq 0\).
Proof (Sketch). We will apply the first limit theorem of Drmota and Soria, [22, Theorem 1]. (The conditions of Hypothesis [H] are the same as for Hypothesis [H’] with the additional requirement that \( h(\rho, 1) > 0 \).)

Let us first analyze \( B(z, 1) \). Its dominant singularity is at \( \rho \), as \( \frac{1}{p_0} > \rho = \frac{1}{(p_0 + 2\sqrt{p_1(p_1 + 1)})} \). Next we determine the decomposition at \( z = \rho \) and \( u = 1 \). From Proposition 3.6 it follows that \( E_1(z) \) has a local representation of the kind

\[
E_1(z) = a_E(z) + b_E(z)\sqrt{1-z/\rho},
\]

where \( a_E(z) \) and \( b_E(z) \) are analytic functions around \( z = \rho \) with \( a_E(\rho) = 1 \) and \( b_E(\rho) = -2C/\tau \). From (8) we see that

\[
B(z, u) = C(z)F(E_1(z), u), \quad \text{where} \quad F(y, u) = \frac{1 - uy}{1 - y(u - 2)}.
\]

We can use the Taylor series expansion of

\[
F(y, u)^{-1} = \sum_{n,k \geq 0} f_{nk}(y-1)^n(u-1)^k,
\]

with \( f_{00} = 0 \) and \( f_{10} = -1/2 \) to show the desired decomposition:

\[
B(z, u)^{-1} = C(z)^{-1} F(E_1(z), u)^{-1} = g(z, u) + h(z, u)\sqrt{1-z/\rho}.
\]

We have \( g(\rho, 1) = f_{00} = 0 \), \( h(\rho, 1) = (1 - \rho p_0) f_{10} b_E(\rho) = 2C\rho p_1 > 0 \) and \( g_u(\rho, 1) = (1 - \rho p_0) f_{01} = -\tau \rho p_1 < 0 \). Applying Lemma 4.9 gives the result.

Finally, we consider sign changes of walks. Since we want to count the number of sign changes we need to know whether a bridge ended with a positive or negative sign. Let positive bridges be bridges whose last non-zero signed node was positive, and negative bridges be bridges whose last non-zero signed node was negative. Their generating functions are denoted by \( B_+(z, u) \) and \( B_-(z, u) \), respectively. Figure 5 shows a negative bridge.

Lemma 4.11. The number of positive and negative bridges is the same and given by

\[
B_+(z, u) = \frac{B(z, u) - C(z)}{2} = \frac{E(z) - C(z)}{1 - uE_1(z)}.
\]

Proof. The result is a direct consequence of Lemma 4.4. We have seen that a bridge is a sequence of excursions, see Figure 5. Mapping all positive excursions to negative ones, and vice versa, gives a bijection between positive and negative bridges.

Proposition 4.12. The bivariate generating function of walks \( W(z, u) = \sum_{n,k \geq 0} w_{nk} z^n u^k \) where \( w_{nk} \) is the number of all walks of length \( n \) with \( k \) sign changes, is given by

\[
W(z, u) = B(z, u) \frac{W(z)}{B(z)} + B_+(z, u) \left( \frac{W(z)}{B(z)} - 1 \right) (u - 1), \quad (9)
\]

where \( W(z) = \frac{1}{1-zP_1(1)} \) is the generating function of walks.
Proof. Combinatorially, a walk is either a bridge or a bridge concatenated with a meander that does not return to the $x$-axis again. In the second case an additional sign change appears if the bridge ends with a negative sign and continues with a meander always staying strictly above the $x$-axis, or vice versa. By Lemma 4.11 the desired form follows. 

The last ingredient for the proof of Theorem 4.14 is the following (technical) lemma on the small branch $u_1(z)$. It can also be used to simplify the results on the height from Theorem 4.7 in the case of Motzkin walks because $u_1(z)v_1(z) = \frac{p_1}{p_1}$, see Table 2.

Lemma 4.13. Let $P(u) = p_{-1}u^{-1} + p_0 + p_1u$. Let $u_1(z)$ be the small branch of the kernel equation $1 - zP(u) = 0$ with $\lim_{z \to 0} u_1(z) = 0$, and define $\rho_1 := 1/P(1)$. Then

\[
u_1(\rho_1) = \begin{cases} 1, & \text{for } \delta < 0, \\ \tau^2, & \text{for } \delta > 0, \end{cases} \quad u_1'(\rho_1) = \begin{cases} \frac{P(1)^2}{P'(1)}, & \text{for } \delta < 0, \\ \tau^2 \frac{P(1)^2}{P'(1)}, & \text{for } \delta > 0, \end{cases}
\]

\[
u_1''(\rho_1) = \begin{cases} -\left(\frac{P(1)}{P'(1)}\right)^3 (P(1)P''(1) - 2P'(1)^2), & \text{for } \delta < 0, \\ \tau^2 \left(\frac{P(1)}{P'(1)}\right)^3 (P(1)P''(1) - 2P'(1)^2), & \text{for } \delta > 0. \end{cases}
\]

Proof. In both cases, $\delta < 0$ and $\delta > 0$, $u_1(z)$ is regular at $\rho_1$. As $u_1(z)$ is monotonically increasing we must have $u_1(\rho_1) < u_1(\rho) = \tau = \sqrt{p_{-1}/p_1}$. Then, from the kernel equation $1 - zP(u_1(z)) = 0$ for all $|z| < \rho$, we get the desired result. For the second and third claim one uses the implicit derivative of the kernel equation and the previous results. More details will be discussed in [10].

The next theorem concludes this discussion. Its proof is similar to the one of Theorem 4.2.

Theorem 4.14 (Limit law for sign changes). Let $X_n$ denote the number of sign changes of Motzkin walks of length $n$. Let $\delta = P'(1)$ be the drift.

1). For $\delta \neq 0$ we get convergence to a geometric distribution:

\[X_n \xrightarrow{d} \text{Geom} \left(\lambda\right), \quad \text{with} \quad \lambda = \begin{cases} \frac{p_1}{p_{-1}}, & \text{for } \delta < 0, \\ \frac{p_{-1}}{p_1}, & \text{for } \delta > 0. \end{cases}\]

2). For $\delta = 0$ we get convergence to a half-normal distribution:

\[\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H} \left(\frac{1}{2} \sqrt{\frac{P''(1)}{P(1)}}\right).\]

Proof. Let us start with an analysis of the dominant singularity. The most important term decomposes into

\[\frac{W(z)}{B(z)} = \frac{1}{1 - zP(1)} \frac{u_1(z)}{zu_1'(z)}.\]
The first factor is singular at \( \rho_1 = 1/P(1) \) but the second one is singular at \( \rho = 1/P(\tau) \). As we know, \( P(\tau) \leq P(1) \). Thus, either both are singular at the same time, or only \( W(z) \) is responsible for the singularity.

In the first case, again the key idea is to use the coefficient asymptotics for the product of a singular and an analytic function \([26, \text{Theorem VI.12}]. \) In particular for \( \delta \neq 0 \) only \( W(z) \) is singular at the dominant singularity. Hence, the coefficient asymptotics is given by its asymptotic expansion times the other functions evaluated at \( z = \rho_1 \).

The results from Lemma 4.13 directly give
\[
B(\rho_1) = \begin{cases} -\frac{P(1)}{\delta}, & \text{for } \delta < 0, \\ \frac{P(1)}{\delta}, & \text{for } \delta > 0. \end{cases}
\] (10)

Then some tedious calculations show for \( \delta < 0 \) that
\[
\mathbb{P}[X_n = k] = \frac{[u^k z^n]W(z,u)}{[z^n]W(z)} = [u^k] \left( \frac{-\delta}{p-1} \right) \frac{1}{1 - u \frac{p_1}{p-1}} + o(1).
\]

This is a geometric distribution with parameter \( \lambda = \frac{p_1}{p-1} \). For \( \delta > 0 \) the analogous result holds.

In the second case of \( \delta = 0 \) we also have \( \tau = 1 \) and \( \rho = \rho_1 \). Then, we can apply Theorem 2.1. A reasoning along the lines of Theorem 4.10 shows that
\[ W(z,u)^{-1} = g(z,u) + h(z,u) \sqrt{1 - z/\rho}, \]
where \( g(z,u) \) and \( h(z,u) \) are analytic functions. We omit the tedious calculations and directly derive the asymptotic form for \( z \to \rho \). For the tail we get by (4) the expansion
\[
\frac{W(z)}{B(z)} = \frac{2}{C \sqrt{1 - z/\rho}} + \mathcal{O}(1),
\]
for \( z \to \rho \), where \( C = \sqrt{\frac{2 P(1)}{P'(1)}} \). Thus, we get
\[
W(z,u)^{-1} = \frac{2 C \rho p_{-1}}{\tau^2(u-3)(u+1)} \left( \frac{4 C}{\tau(u-3)} \left( 1 - \frac{z}{\rho} \right) + (u-1) \sqrt{1 - \frac{z}{\rho}} \right)
+ \mathcal{O} \left( \left( 1 - \frac{z}{\rho} \right)^2 \right) + \mathcal{O} \left( \left( 1 - \frac{z}{\rho} \right)(1-u) \right),
\]
for \( |u-1| < \varepsilon, |z-\rho| < \varepsilon \) and \( \arg(z-\rho) \neq 0 \), with \( g(\rho,1) = h(\rho,1) = g_u(\rho,1) = g_{uu}(\rho,1) = 0; \) and \( g_z(\rho,1) = -\frac{C^2 p_{-1}}{\tau^3} \) and \( h_u(\rho,1) = -\frac{C p_{-1}}{2 \tau^2} \). Hence, Theorem 2.1 yields the result with the constant \( \sigma = \sqrt{\frac{h_u(\rho,1) g_z(\rho,1)}{g_{uu}(\rho,1) g_z(\rho,1)}} = \frac{1}{2} \sqrt{\frac{P'(1)}{P(1)}} \).

Using (10) the results of Theorem 4.2 can be simplified, and we get a geometric law with parameter \( \lambda = \frac{|\delta|}{P(1)} = \frac{p_1 p_{-1}}{P(1)} \) for \( \delta \neq 0 \). In Table 2 we will see a comparison of the parameters.
5 Proof of Theorem 2.1

We first list some useful formulæ related to the half-normal distribution. We omit their proofs as they follow the same lines as the ones of the Lemmata 6 and 7 in [22]. For the representation of the characteristic function of the half-normal distribution see [38, Equation (15)].

**Lemma 5.1.** Let $\gamma$ be the Hankel contour starting from $"+e^{2\pi i \infty} "$, passing around 0 and tending to $+\infty$. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^{-s\sqrt{-z-z}}}{\sqrt{-z}} \, dz = \frac{1}{\sqrt{\pi}} e^{-s^2/4} \quad \text{and} \quad \frac{1}{2\pi i} \int_{\gamma} \frac{e^{-z}}{z + is\sqrt{-z}} \, dz = \varphi_{\mathcal{H}}(\sqrt{2} s),$$

where $\varphi_{\mathcal{H}}(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{itx} e^{-x^2/2} \, dx$ is the characteristic function of the half-normal distribution.

**Proof of Theorem 2.1.** The proof follows the same steps as the one of [22, THEOREM 1]. Therefore, we restrict ourselves on only pointing out the main differences.

First, we derive asymptotic expansions for mean and variance. Due to $g(\rho, 1) = h(\rho, 1) = 0$, and $g_z(\rho, 1) \neq 0$ we get from (1) that

$$[z^n]c(z, 1) = -[z^n] \frac{1}{\rho g_z(\rho, 1)} \frac{1}{1 - z/\rho} + O(\sqrt{1 - z/\rho})$$

$$= -\frac{\rho^{-n}}{\rho g_z(\rho, 1)} \left(1 + O(n^{-1/2})\right). \tag{11}$$

Analogously, because of $h_u(\rho, 1) \neq 0$, and $h(\rho, 1) = g_u(\rho, 1) = g_{uu}(\rho, 1) = 0$ we get

$$[z^n]c_u(z, 1) = -\frac{2h_u(\rho, 1)}{(\rho g_z(\rho))^2} \sqrt{\frac{\pi}{\rho}} \left(1 + O(n^{-1/2})\right),$$

$$[z^n]c_{uu}(z, 1) = -\frac{2h_u(\rho, 1)^2}{(\rho g_z(\rho))^3} \rho^{-n} n \left(1 + O(n^{-1/2})\right).$$

Hence,

$$\mathbb{E}[X_n] = \frac{[z^n]c_u(z, 1)}{[z^n]c(z, 1)} = \frac{2h_u(\rho, 1)}{\rho g_z(\rho)} \sqrt{\frac{\pi}{\rho}} \left(1 + O(n^{-1/2})\right),$$

$$\mathbb{V}[X_n] = \frac{[z^n]c_{uu}(z, 1)}{[z^n]c(z, 1)} = 2 \left(\frac{h_u(\rho, 1)}{\rho g_z(\rho)}\right)^2 n + O(n^{1/2}).$$

These results strongly suggest that the underlying limit distribution is a half-normal one. We continue by deriving the asymptotic form of the characteristic function of $X_n/\sqrt{n}$.

Note that the same contour of integration as in [22] sketched in Figure 6 can be used. Therefore, we need the following expansions coming from the substitutions $z = \rho \left(1 + \frac{s}{n}\right)$ and $u = e^{it/\sqrt{n}} = 1 + \frac{it}{\sqrt{n}} + O(n^{-1})$:

$$g(\rho \left(1 + \frac{s}{n}\right), e^{it/\sqrt{n}}) = g_z(\rho, 1) \rho \frac{s}{n} + O\left(\frac{s}{n^{3/2}}\right),$$

$$h(\rho \left(1 + \frac{s}{n}\right), e^{it/\sqrt{n}}) = h_u(\rho, 1) \frac{it}{\sqrt{n}} + O\left(\frac{s}{n}\right), \quad (12)$$
as \( g_u(\rho, 1) = 0 \) and \( h(\rho, 1) = 0 \). We want to emphasize that this behavior is different from the one in [22, Theorem 1], where the differences are \( g_u(\rho, 1) = 1 \) and \( h(\rho, 1) = 0 \). Thus, we get for the Cauchy integral along the contour \( \Gamma_1 \) (see Figure 6)

\[
\frac{1}{2\pi i} \int_{\Gamma_1} c(z, u) \frac{dz}{z^{n+1}} = \frac{\rho^{-n}}{2\pi i} \int_{\gamma'} g_z(\rho, 1) \rho^2_n + h_u(\rho, 1) i t \frac{\sqrt{s}}{n} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right) ds = \frac{\rho^{-n}}{\rho g_z(\rho, 1)} \frac{1}{2\pi i} \int_{\gamma'} \frac{e^{-s}}{s + \sqrt{-s} \frac{h_u(\rho, 1)}{\rho g_z(\rho, 1)}} ds + \mathcal{O}\left(\rho^{-n} n^{-1/2}\right).
\]

(13)

The other computations are again analogous to [22]. By (13) and Lemma 5.1 we get

\[
\frac{1}{2\pi i} \int_{\Gamma_1} c(z, u) \frac{dz}{z^{n+1}} = \frac{\rho^{-n}}{\rho g_z(\rho, 1)} \varphi_H \left( \frac{\sqrt{2} h_u(\rho, 1)}{\rho g_z(\rho, 1)} t \right) + \mathcal{O}\left(\rho^{-n} n^{-1/2}\right).
\]

(14)

What remains is to bound the remaining part of the integral. Using the expansions from (12) we directly get

\[
\left| c\left(\rho \left(1 + \frac{\log^2 n + i}{n}\right), e^{i \frac{\theta}{n}}\right) \right| = \mathcal{O}\left(\frac{n}{\log^3 n}\right).
\]

This proves the weak limit theorem.

In the proof of the local limit theorem we get a different polar singularity of the mapping \( u \mapsto c(z, u) \). For \( z = \rho \left(1 + \frac{\log^2 n}{n}\right) \) and \( u_0 = 1 + \frac{t_0}{\sqrt{n}} \) we get

\[
t_0 = \frac{\rho g_z(\rho, 1)}{h_u(\rho, 1)} \sqrt{-s} + \mathcal{O}\left(\frac{\sqrt{s}}{n}\right),
\]

with residue

\[
\frac{1}{h_u(\rho, 1)} \sqrt{-s} \left(1 + \mathcal{O}\left(\frac{\sqrt{s}}{n}\right)\right).
\]

Then the same steps as in the proof of [22] yield the result. \(\square\)
6 Conclusion

In this paper we presented in Theorem 2.1 a scheme for generating functions yielding a half-normal distribution. This continues the work of Drmota and Soria [22] who presented three schemata leading to three different limiting distributions: Rayleigh, normal, and a convolution of both.

We also showed three different (yet related) instances of this distribution in lattice path theory by applying our scheme. We showed that the number of returns to zero, the height, and the number of sign changes (in the case of a Motzkin-like step set) are for drift 0 half-normally distributed. The necessary generating function relations are summarized in Table 4.

On the one hand, from a technical point of view it is also interesting to ask how Theorem 2.1 behaves in the situation of a singularity \( \rho(u) \) with \( \rho'(1) \neq 0 \) or \( \rho''(1) \neq 0 \). This remains an object for future research.

On the other hand, at this point we also want to comment on the common (technical) link between these applications. In all three examples the decomposition (1) has the common generic structure

\[
\frac{1}{c(z, u)} = \tilde{g}(z, u) \left( 1 - \frac{z}{\rho} \right) + \tilde{h}(z, u)(1 - u)\sqrt{\frac{1 - \frac{z}{\rho}}{\rho}},
\]

with \( \tilde{g}(\rho, 1) \neq 0, \tilde{h}(\rho, 1) \neq 0, \) and \( \tilde{g}(z, u), \tilde{h}(z, u) \) being analytically continuable in the necessary domains for Hypothesis [H'] to hold, see e.g. (2). This special form of \( g(z, u) \) and \( h(z, u) \) guarantees Theorem 2.1 to hold and gives a half-normal distribution. Yet, the scheme does not need such a special factorization and holds in a more general setting.

Thus, with respect to both mentioned extensions, it would be interesting if other “natural” appearances of such situations (and half-normal distributions in general) exist. So far we know of two such appearances. One in number theory [27] and another one in lattice path theory [7]. The latter treats the final altitude of meanders in the reflection-absorption model in the case of zero drift. Chronologically, that was our starting point for the research of this paper.

| Marked Parameter | BGF | Equation |
|------------------|-----|----------|
| Returns to zero  | \( \frac{W(z)}{B(z)} \) | (5) |
| Height           | \( \frac{B(z, u)M(z) - C(z)}{2B(z)} - (\frac{W(z)}{B(z)} - 1)(u - 1) \) | (7) |
| Sign changes (Motzkin) | \( B(z, u) \frac{W(z)}{B(z)} + \frac{B(z, u) - C(z)}{2} \frac{W(z)}{B(z)} - 1 \) | (9) |

Table 4: Relations for the bivariate generating functions of walks with a marked parameter given by \( W(z, u) = \sum_{n,k} w_{nk} z^n u^k \). The functions \( W(z), B(z), M(z) \) are the generating functions of walks, bridges, meanders, respectively, see Table 3. \( M(z, u) \) is the generating function of meanders of length \( n \) with marked final altitude, \( B(z, u) \) is the generating function of bridges of length \( n \) with marked sign changes, and \( C(z) = \frac{1}{1 - \rho_0 z} \).
Acknowledgments

The author is grateful to Cyril Banderier, Bernhard Gittenberger, and Michael Drmota for insightful discussions. Additionally, he wants to thank Svante Janson, Guy Louchard, Hsien-Kuei Hwang, and Philippe Marchal for additional references and helpful suggestions. The author also thanks the two anonymous referees whose suggestions substantially improved the presentation and structure of this work. This work was supported by the Austrian Science Fund (FWF) grant SFB F50-03.

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