Capillary Network Model: Capillary Power and Effective Permeability

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A simple model of two-phase flow in porous media is presented. A connection is made to statistical
mechanics by applying capillary power as a constraint. Stochastic sampling is then used to test the
validity of this approach. Good agreement is found between stochastic sampling and time stepping
for flow-rates above a transition value.

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When one phase displaces another within a porous medium, complex patterns are known to emerge [1]. Less
is known about the flow patterns formed under steady-state conditions. Dynamic effects, i.e., the dependence
of the flow patterns on the total flow-rate also remain poorly understood. This is despite the significant im-
portance to applications such as enhanced oil recovery, groundwater contamination and water transport in fuel
cells [2–4]. Two-phase flow in porous media also holds theoretical interest as a complex system which exhibits
self-organization. Self-organization is evidenced by capillary pressure drops adding up to reduce effective perme-
ability rather than canceling out, even under steady-state conditions.

Steady-state simulations have looked at the transport of disconnected oil ganglia [5] and relations between driv-
ing pressure and fractional flow [6], amongst other things. Experimental data can be classified by the model porous
medium: networks etched in glass [7–11], Hele-Shaw cells [12, 13] and bead packings [14].

Two recent experiments have explored the relationship between applied pressure drop $\Delta P$ and flow-rate [12–14].
They have reported a power law dependence, but have found different exponents. A two-dimensional network
simulator has been used to explore the proposed power law [15, 16]. In the simulator individual menisci are mod-
eled; they are transported according to the flow field, and created and destroyed in a manner which crudely models
snap-off and coalescence. Details are given in [16].

This paper presents a model which captures similar behavior as the simulator without explicitly modeling
menisci. The model consists of a network of capillaries and will be referred to as the Capillary Network Model
(CNM). The capillaries model pore throats, which are the narrow connections between pores. A single variable
$\phi$ is assigned to each throat and a capillary pressure drop $p_c$ is given as a function of $\phi$.

The main control parameter used in simulations and experiments is the capillary number $Ca = \mu Q / \sigma A$, where $\mu$ is viscosity, $Q$ is volumetric flow-rate, $\sigma$ is surface tension, $\phi$ is porosity and $A$ is the cross-sectional
area of the sample. The effective permeability is the sum of the relative permeabilities, and may be defined as
$\kappa_{eff} = Q / Q_0$ where $Q_0$ is the flow-rate obtained by solving Darcy’s law for single-phase flow with pressure drop
$\Delta P$. $\kappa_{eff}$ is found to be less than unity under steady-state conditions, which signifies that the mixture of two
phases results in a larger resistance to flow than if only a single phase is present.

In [15, 16] a modified version of the Young-Laplace relation is used to obtain the capillary pressure drop of
a single meniscus

$$p_c(x) = \frac{2 \sigma}{r} \left[ 1 - \cos\left(\frac{2\pi x}{\ell}\right) \right],$$

where $x$ is the position of the meniscus, $\ell$ is the length of the throat and $r$ measures its width. In the CNM the
explicit menisci are replaced by a single scalar variable for each pore throat. This variable $\phi$ is a coordinate which
can take values between 0 and $2\pi$. A capillary pressure drop function is defined as

$$p_c(\phi) = \frac{4\sigma}{r} \sin(\pi s) \sin \phi,$$

where $s$ is the non-wetting saturation of the throat, given by the length of the droplet divided by the length of the
throat. For simplicity, all throats are assigned equal and
constant \( s \). Eq. (2) is obtained by considering Eq. (1) for the case of two menisci forming a droplet within the pore throat, see Fig. 1. \( \varphi \) gives the position of the droplet. The size of the droplet determines the non-wetting saturation \( S \), and can be removed as an independent parameter by redefining \( \sigma \).

The porous medium is modeled as a 2D square lattice inclined at 45° relative to the main direction of flow. Boundary conditions are bi-periodic, such that the network may be mapped onto a torus. Flow is driven by a pressure drop applied across a cut through the network.

Eq. (2) models pore throats that have a narrowing geometry. In all other respects pore throats are considered to be cylindrical tubes with radius \( r \). Network disorder may be introduced in \( r \), \( \ell \) or both. In this work there is no disorder – all pore throats are equally wide and long. For all results given here, \( r = 0.2 \ell \).

The Hagen-Poiseuille permeability for cylindrical tubes gives the flow-rate \( q \) by the Washburn equation,

\[
q(\Delta p, \varphi) = -\frac{\pi r^4}{8\ell \mu} (\Delta p + p_c(\varphi)), \tag{3}
\]

where \( \mu \) is the effective viscosity of the phases contained in the throat and \( \Delta p \) is the pressure difference between the two pores connected by the throat.

Eqs. (2) and (3) together with the network geometry and Kirchhoff’s circuit laws define the CNM.

Given a configuration consisting of \( \{ \varphi_i \} \) and \( \text{Ca} \), the flow field is obtained by solving a system of linear equations. The coordinates are then updated according to \( \varphi_i = q_i / a \) where \( a = \pi r^2 \). Numerical time stepping is done by the Euler method, which is only first order accurate. The numerical error may be estimated by comparing the capillary power with the total dissipation. This ratio should be zero in the steady-state. It is found to be negligible for \( \text{Ca} > 10^{-3} \), but grows to a few percent for \( \text{Ca} = 10^{-4} \). Below this the Euler method therefore does not provide reliable results.

Fig. 2 shows simulation results for the CNM and a network simulator. In both cases Euler time stepping has been used. The network simulator is the same as that used in [13, 10], but without disorder. Four independent runs underlie each data point. Initialization is done by a random initial configuration and a gradual increase in surface tension (gradually decreasing \( \text{Ca} \)). Saturation is a trivial parameter in the CNM, but not in the simulator. The simulator results are for \( s = 0.4 \). To facilitate comparison between the CNM and the simulator, \( s \) in Eq. (2) was set equal to 0.4. The two models can be seen to produce similar, but not identical results.

Consider Eq. (2). The right-hand side consists of two terms, one proportional to \( \Delta p \) and one to \( p_c \). In obvious notation \( q = q_0 + q_c \). Multiplying with \( q \) and dividing by \( -\pi r^4 / 8\ell \mu \) gives a statement of conservation and conversion of energy, \( d = d_0 + d_c \), where \( d \) is the heat dissipated through viscous shear, always negative by definition, \( -d_0 = -\Delta p q \) is equivalent to the power provided by a pump driving a flow \( q \) with an external pressure drop \( \Delta p \) and \( d_c = p_c q \) is capillary dissipation. Replacing \( -d_0 \) with \( \dot{w} \), where \( w \) represents the energy of the pump, and \( -d_c \) with capillary power \( \dot{w}_c \) gives \( \dot{w} + \dot{w}_c + d = 0 \). This states that the absolute value of heat dissipation within a single throat is equal to the sum of applied power and capillary power.

For the porous medium as a whole the capillary power adds up to \( W_c = \sum \dot{w}_c \). The pump power is \( W = -\Delta P Q \) and the total heat dissipation is \( D = \sum d \). In general, the pump power and the heat dissipation are not equal. The difference must be due to the capillaries, so we have \( W_c = -W - D \) for the total capillary power. If \( W_c \) is positive the capillary power adds to the pump power to increase heat dissipation; it becomes an energy source. If \( W_c \) is negative the capillary power absorbs some of the pump power to decrease heat dissipation; it becomes an energy sink.

Fig. 3 shows the development of \( W, -D \) and \( W_c \) during a single simulation run. Initially, for a completely random configuration, \( W_c \) is positive: the capillaries release energy. \( W_c \) approaches zero as steady-state is approached, after which both pump power and dissipation fluctuate around the same average value.

A thermodynamics of two-phase flow in porous media was first suggested in [18]. There, total dissipation was suggested as being analogous to energy. The preceding discussion implies that instead of total dissipation it is capillary power which is constrained and thus provides a connection to statistical mechanics.

In the following, tools from statistical mechanics usually reserved for classical equilibrium will be applied. The underlying idea is to consider the steady-state as governed by a balance between drive and dissipation – a dissipative equilibrium. \( \langle W_c \rangle = 0 \) states that, on aver-
age, pump power must equal heat dissipation, which is a requirement for the steady-state.

Constructing a space of eigenstates \(\{\varphi_l\}\), where \(l\) indexes eigenstates, gives the number of microscopic configurations for a state \(\{a_l\}\) as

\[
C(\{a_l\}) = \frac{N!}{a_1!a_2!a_3!\ldots},
\]

(4)

where \(a_l\) is the occupancy number of eigenstate \(l\) and \(\sum a_l = N\) is the number of coordinates [19]. In light of the developments so far, and in order to establish the governing principle of steady-state two-phase flow in porous media, the capillary power \(\dot{W}_c\) is applied as a constraint on Eq. (4). Using a Lagrange multiplier \(\lambda\) for the constraint gives the partition function

\[
Z = \sum_l e^{-\lambda \mathcal{H}},
\]

(5)

where the sum runs over the eigenstates of \(\varphi\). \(\mathcal{H}\) is

\[
\mathcal{H} = -\sum_l p_c \bar{q},
\]

(6)

where the sum runs over the number of coordinates, \(p_c = p_c(\varphi_l)\) and \(\bar{q}\) is

\[
\bar{q}(\varphi_l) = \int dq \rho(q|\varphi_l).
\]

(7)

The pdf’s \(\rho\) may be obtained from time stepping at a given \(\text{Ca}\). Capillary power

\[
\dot{W}_c(\lambda) = -Z^{-1} \partial_\lambda Z = -NZ^{-1} \sum_l p_c \bar{q} e^{\lambda p_c \bar{q}},
\]

(8)

is a strictly monotonous function of \(\lambda\), with only a single solution \(\lambda\) for a given value of \(\dot{W}_c\). Fig. 4 gives \(\lambda(\text{Ca})\) obtained by solving Eq. (8) for \(\dot{W}_c = 0\). A transition occurs at \(\text{Ca} \approx 0.1\). This value is in the following referred to as \(\text{Ca}_{\text{up}}\), to emphasize the connection with [16].

The inset of Fig. 4 shows that above \(\text{Ca}_{\text{up}}\), \((1 - \kappa_{\text{eff}})\) scales with \(\text{Ca}\). The best fit to a power law gives \((1 - \kappa_{\text{eff}}) \sim \text{Ca}^{-2.06}\). A further discussion of scaling laws above \(\text{Ca}_{\text{up}}\) is given in connection with the mean-field argument.

It should be noted that \(\dot{W}_c(\lambda) = 0\) implies that \(\partial_\lambda Z = 0\), i.e., dissipative equilibrium implies that the partition function has an extremal value with respect to variations of the constraint. For \(\dot{W}_c(\lambda) = 0\) to be valid, the constraint must apply equally for each configuration within the ensemble. Temporal correlations in the fluctuations of \(\dot{W}_c\) are ignored by this approach.

Using a Metropolis algorithm [20, 21] and the value of \(\lambda\) obtained from time stepping it is possible to generate an ensemble of configurations governed by the constraint of \(\dot{W}_c = 0\). The resulting ensemble will be dominated by the most probable state (according to Eq. (4)) which satisfies the constraint. A comparison with the ensemble obtained by time stepping constitutes a non-trivial test of the validity of the statistical mechanics approach.

Stochastic sampling is done by replacing some randomly chosen coordinates with new, random coordinates. After a trial update, \(\Delta \dot{W}_c\) is calculated as the change in \(\dot{W}_c\) after the update. The new configuration is accepted if \(\Delta \dot{W}_c\) is negative, and with probability \(e^{-\lambda \Delta \dot{W}_c}\) otherwise. This is a standard Metropolis algorithm, where \(\dot{W}_c\) is analogous to energy and \(\lambda\) is analogous to inverse

\[
\begin{align*}
&W_{\mathrm{c}}(\lambda) = -Z^{-1} \partial_\lambda Z = -NZ^{-1} \sum_l p_c \bar{q} e^{\lambda p_c \bar{q}},
\end{align*}
\]
Fig. 5 shows $\kappa_{\text{eff}}(\text{Ca})$ and $1 - \kappa_{\text{eff}}(\text{Ca})$ from both time stepping and stochastic sampling. For Ca above $\text{Ca}_{\text{up}}$ the results are in agreement. Below $\text{Ca}_{\text{up}}$, time stepping and stochastic sampling produce different results.

Inspection of time stepping simulations suggest that above $\text{Ca}_{\text{up}}$, $\bar{q}$ may be approximated by

$$\bar{q}(\varphi) = q_0 - q_c \sin \varphi,$$  \hfill (9)

where $q_0$ is a mean flow-rate and $q_c$ is the maximum perturbation caused by a capillary pressure drop. At $\text{Ca}_{\text{up}}$, $q_c \approx q_0$. Below $\text{Ca}_{\text{up}}$ time stepping simulations show that the dependence of $\bar{q}$ on $\varphi$ is no longer sinusoidal. Eq. 9 represents a mean-field solution: a homogeneous flow field with perturbations that only depend on the local variable.

Applying Eq. 4 to Eq. 5 allows an analytic determination of the scaling of Ca and $(1 - \kappa_{\text{eff}})$ with $\lambda$. To obtain this, the Boltzmann factor is expanded as $e^{-\lambda \mathcal{H}} \approx 1 - \lambda \mathcal{H}$, omitting terms of $\lambda^2$ and higher order. Terms with odd powers of $\sin \varphi$ sum to zero. From $\text{Ca} \sim \langle \bar{q} \rangle$, $\text{Ca} \sim q_0$ is obtained. Considering $\langle \bar{w}_c \rangle = 0$, $\text{Ca} \sim \lambda^{-1/2}$ results. Finally, $\kappa_{\text{eff}} \sim \langle \bar{q}^2 \rangle / \langle \bar{q}^2 \rangle$ gives $(1 - \kappa_{\text{eff}}) \sim \lambda$. This gives $(1 - \kappa_{\text{eff}}) \sim \text{Ca}^{-2}$.

From time stepping and above $\text{Ca}_{\text{up}}$, the CNM gives $\text{Ca} \sim \lambda^{-0.48}$, $(1 - \kappa_{\text{eff}}) \sim \lambda^{0.99}$ and $(1 - \kappa_{\text{eff}}) \sim \text{Ca}^{-2.06}$, see the inset of Fig. 5 and Fig. 6.

In summary, a simple model of two-phase flow in porous media has been presented. It has been found to produce results for effective permeability that are qualitatively similar to a more detailed simulator. Capillary power has been defined and used as a constraint to produce a partition function. At high flow-rates, time stepping and stochastic sampling have been shown to produce the same ensemble, given a constraint of zero capillary power. Scaling exponents obtained from a mean-field theory are in agreement with the numerical results at these high flow-rates. At lower flow-rates, stochastic sampling does not reproduce the results from time stepping.

The main result of this work is to identify zero capillary power as the constraint which governs steady-state two-phase flow in porous media. This is nothing but a statement of conservation of energy.

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\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure5}
\caption{(Color online) Comparison of $\kappa_{\text{eff}}(\text{Ca})$ obtained from time stepping (black squares) and stochastic sampling (blue plusses). Inset: $1 - \kappa_{\text{eff}}$ with the same x-axis as the main figure. The solid red line is a power law with exponent $-2.06$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure6}
\caption{(Color online) $\text{Ca}(\lambda)$. The solid red line is a power law with exponent $-0.48$. Inset: $1 - \kappa_{\text{eff}}$ with the same x-axis as the main figure. The solid red line is a power law with exponent 0.99.}
\end{figure}

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