On Singularly Perturbed Linear Cocycles over Irrational Rotations

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Abstract—We study a linear cocycle over the irrational rotation \( \sigma_\omega(x) = x + \omega \) of the circle \( \mathbb{T}^1 \).
It is supposed that the cocycle is generated by a \( C^2 \)-map \( A_\varepsilon : \mathbb{T}^1 \to SL(2, \mathbb{R}) \) which depends on a small parameter \( \varepsilon \ll 1 \) and has the form of the Poincaré map corresponding to a singularly perturbed Hill equation with quasi-periodic potential. Under the assumption that the norm of the matrix \( A_\varepsilon(x) \) is of order \( \exp(\pm \lambda(x)/\varepsilon) \), where \( \lambda(x) \) is a positive function, we examine the property of the cocycle to possess an exponential dichotomy (ED) with respect to the parameter \( \varepsilon \). We show that in the limit \( \varepsilon \to 0 \) the cocycle “typically” exhibits ED only if it is exponentially close to a constant cocycle. Conversely, if the cocycle is not close to a constant one, it does not possess ED, whereas the Lyapunov exponent is “typically” large.

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1. INTRODUCTION

In this paper we study a skew-product map
\[
F_A : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2
\]
defined for any \((x,v) \in \mathbb{T}^1 \times \mathbb{R}^2\) by
\[
(x,v) \mapsto (\sigma_\omega(x), A(x)v),
\]
where \( \sigma_\omega(x) = x + \omega \) is a rotation of a circle \( \mathbb{T}^1 \) with irrational rotation number \( \omega \) and
\[
A : \mathbb{T}^1 \to SL(2, \mathbb{R})
\]
is a measurable function with respect to the Haar measure. The transformation \( A \) generates a cocycle \( M(x,n) \) by
\[
M(x,n) = A(\sigma_\omega^{n-1}(x)) \ldots A(x), \quad n > 0;
\]
\[
M(x,n) = [A(\sigma_\omega^{-n}(x)) \ldots A(\sigma_\omega^{-1}(x))]^{-1}, \quad n < 0;
\]
\[
M(x,0) = I.
\]

Such discrete dynamical systems and their continuous counterparts are the subject of many papers that have appeared during the last three decades. Among important classes of examples of linear cocycles one needs to mention a discrete ergodic Schrödinger operator and a quasi-periodic Hill’s equation. The main problems stated for this kind of systems include description of an asymptotic behaviour of their trajectories, establishing relations between dynamical characteristics and other (e.g., spectral) properties of a system, studying the genericity of systems with different behaviour in various classes of smoothness. Starting from the fundamental works [9, 15, 18], many different

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methods and techniques were elaborated to investigate these problems [2, 6, 7, 12, 25] (see also [11] and references therein). Nevertheless, there are still many open questions. In particular, for a given family of cocycles the problem of effective description of a set of those parameter values which correspond to some specific property (e.g., hyperbolicity, reducibility, positiveness of the Lyapunov exponent) is far from being solved. The reason is a lack of constructive approaches. One of such approaches is based on the monodromisation method developed in [7, 14]. This method was successfully applied to study the spectrum of the Harper operator and adiabatic perturbation of the 1-dimensional Schrödinger operator. Another constructive approach has been developed by different authors [4, 17, 20, 26] and is based on inductive construction of the so-called “critical” set. We will discuss the details of this method at the end of Section 2.

The paper is organised as follows. In Section 2 we specify the cocycle which is studied and state the problem. Besides, we recall some definitions and constructions. In particular, the idea of the “critical set” method is discussed. We also formulate the main results (Theorems 3 and 4). Section 3 is devoted to number-theoretic conditions which are imposed on the rotation number. In Section 4 we describe the inductive procedure to construct the critical set and study its property. The analysis of the Lyapunov exponent associated to the cocycle is given in Section 5. Finally, Section 6 is devoted to the proofs of Theorems 3 and 4.

2. STATEMENT OF THE PROBLEM

In this paper we consider a one-parameter family of cocycles such that the corresponding transformation $A$ has a special form. Particularly, given a $C^2$-function $f : \mathbb{T}^2 \to \mathbb{R}$ of the two-torus, we define sets
\[
\mathcal{D}_+ = \{ z \in \mathbb{T}^2 : f(z) > 0 \},
\]
\[
\mathcal{D}_- = \{ z \in \mathbb{T}^2 : f(z) < 0 \}
\]
and for any $x \in \mathbb{T}^1$ consider their intersections $S_{\pm}(x) = \mathcal{D}_{\pm} \cap I(x)$ with a segment $I(x) = \{(x + \omega s, s) : s \in \mathbb{T}^1\} \subset \mathbb{T}^2$. We represent $S_{\pm}(x)$ as
\[
S_{\pm}(x) = \bigcup_{k=1}^{K(x)} \Delta_{k}^\pm(x),
\]
where $\Delta_{k}^\pm(x)$ are connected components ordered in a natural way with respect to increase in the parameter $s$ (see Fig. 1). Throughout the paper we will suppose that the number of connected components is bounded, i.e., there exists $K_0$ such that $K(x) < K_0$ for all $x \in \mathbb{T}^1$.

Fig. 1. An example of a two-torus with indicated domains $\mathcal{D}_+$ (red), $\mathcal{D}_-$ (blue) and segments $I(x)$ (yellow) drawn for two different values of $x = \theta_1$. 
Let \( g_\pm : \overline{\mathcal{D}_\pm} \to \mathbb{R}_\pm \) be \( C^2 \)-functions. Then the transformation \( A \) is assumed to be of the form

\[
A(x) = \prod_{k=1}^{\Delta(x)} R(\varphi_k(x)) \cdot Z(\lambda_k(x)),
\]

where

\[
R(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}, \quad Z(\lambda) = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}
\]

and

\[
\varphi_k(x) = \varepsilon^{-1} \int_{\Delta_k(x)} g_\pm \, dl, \quad \lambda_k(x) = \varepsilon^{-1} \int_{\Delta_k(x)} g_\pm \, dl, \quad \varepsilon \ll 1,
\]

where the integrals in (2.2) are understood as the curvilinear integrals with respect to the arc length \( l \). We mention an evident property of the matrices \( R \) and \( Z \):

\[
R(\varphi_1 + \varphi_2) = R(\varphi_1) \cdot R(\varphi_2), \quad Z(\lambda_1 + \lambda_2) = Z(\lambda_1) \cdot Z(\lambda_2), \quad \forall \varphi_1, \varphi_2, \lambda_1, \lambda_2.
\]

To emphasise the dependence of the transformation \( A \) on the parameter \( \varepsilon \), we will denote it as \( A_\varepsilon(x) \).

Such a linear cocycle (1.1) appears as a model in different areas. For instance, it is related to the problem of stability for the singularly perturbed Hill equation or to spectral problems of a one-dimensional singularly perturbed stationary Schrödinger operator with quasi-periodic potential (see, e.g., [23]). Indeed, consider the equation

\[
\varepsilon^2 y'' = f(t)y
\]

and assume the potential \( f \) to be a quasi-periodic function, i.e., \( f(t) = F(\theta_0 + \omega t, t) \), where \( F : \mathbb{T}^2 \to \mathbb{R} \). In the extended phase space this equation can be written as

\[
\varepsilon^2 y'' = F(\theta_1, \theta_2)y, \quad \theta'_1 = \omega, \quad \theta'_2 = 1.
\]

One may consider the Poincaré map associated with this system:

\[
\Phi : \begin{pmatrix} y(0) \\ y'(0) \\ \theta(1) \end{pmatrix} \to \begin{pmatrix} y(1) \\ y'(1) \\ \theta(1) \end{pmatrix},
\]

which is equivalent to

\[
\begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = A_\varepsilon(\theta(1)) \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}
\]

(2.4)

with some \( A_\varepsilon(\theta) \in SL(2, \mathbb{R}) \). Thus, we arrive at a system of the form (1.1). Using the WKB-method, one may show that the matrix \( A_\varepsilon \) in (2.4) can be written in the form (2.1), where the functions \( g_\pm \) are restrictions of \( |f|^{1/2} \) on \( \mathcal{D}_+ \). If the potential \( f \) is periodic, then the stability of the origin for the system (2.3) does not depend on the value \( \theta_1(0) \) and the following theorem holds [16].

**Theorem 1.** Let \( f \) be a periodic \( C^2 \)-function of period \( T \) possessing a finite number of zeroes \( \tau_l, l = 1, \ldots, L \) on the interval \([0, T)\). If multiplicities of \( \tau_l \) are finite for all \( l = 1, \ldots, L \) and the set \( \{ t \in [0, T) : f(t) > 0 \} \) is non-empty, then there exist \( \varepsilon_0 > 0 \) and a subset \( \mathcal{E}_h \subset (0, \varepsilon_0) \) such that

1. for any \( \varepsilon_1 < \varepsilon_0 \) the Lebesgue measure \( \text{leb}((0, \varepsilon_1) \setminus \mathcal{E}_h) = O\left( e^{-c/\varepsilon_1} \right) \) with some positive constant \( c \);
2. for any \( \varepsilon \in \mathcal{E}_h \) the origin is an unstable equilibrium of the system (2.3).
Note also that the instability of the equilibrium is related to the exponential dichotomy possessed by the system (2.3) for \( \varepsilon \in \mathcal{E}_h \) \[16\].

Before stating the results we recall the definition of the exponential dichotomy and some other concepts in cocycle settings.

**Definition 1.** A cocycle \( M \) is said to have an exponential dichotomy (ED) if there are positive constants \( C, \mu \) and a projector-valued function \( P(x) \) continuously dependent on \( x \in \mathbb{T}^1 \) such that

\[
\begin{align*}
\|M(x, m)P(x)M^{-1}(x, n)\| &\leq Ce^{-\mu(m-n)}, \quad m \geq n; \\
\|M(x, m)(I - P(x))M^{-1}(x, n)\| &\leq Ce^{\mu(m-n)}, \quad m \leq n.
\end{align*}
\]

One has to note that the property to possess ED is very robust under perturbation as follows from \[8, 21, 22\].

A concept closely related to that of ED is that of uniform hyperbolicity. Denote by \( \text{Gr}(m, n) \) the Grassmannian of \( m \)-dimensional subspaces of \( \mathbb{R}^n \).

**Definition 2.** A cocycle \( M \) is said to be uniformly hyperbolic (UH) if there exist continuous maps \( E^u, s : \mathbb{T}^1 \to \text{Gr}(1, 2) \) and positive constants \( C, \mu \) such that the subspaces \( E^u, s(x) \) are invariant under the map \((1.1)\) (i.e., \( E^u, s(\sigma(x)) = A(x)E^u, s(x) \)) and \( \forall x \in \mathbb{T}^1, n \geq 0 \)

\[
\begin{align*}
\|M(x, -n)|_{E^u(x)}\| &\leq Ce^{-\mu n}, \\
\|M(x, n)|_{E^s(x)}\| &\leq Ce^{-\mu n}.
\end{align*}
\]

It can be shown (see, e.g., \[2, 13\]) that a cocycle \( M \) possesses ED if and only if it is UH. Moreover \[2\], these properties are equivalent to existence of positive constants \( C, \mu \) such that \( \forall x \in \mathbb{T}^1 \) and \( n \geq 0 \)

\[
\|M(x, n)\| \geq Ce^{\mu n}.
\]

The existence of the invariant subspaces \( E^u, s(x) \) for a.e. \( x \in \mathbb{T}^1 \) is guaranteed by the Oseledets theorem. However, in general, the maps \( E^u, s \) are only measurable. Besides, by Kingman’s sub-additive ergodic theorem (or, in a more general context, by the Oseledets theorem) for a.e. \( x \in \mathbb{T}^1 \) there exists the Lyapunov exponent

\[
\Lambda(x) = \lim_{n \to \infty} \frac{1}{n} \log \|M(x, n)\|.
\]

Note that the rotation \( \sigma_\omega \) is ergodic with respect to Haar measure since \( \omega \) is assumed to be irrational. This implies

\[
\Lambda(x) = \Lambda_0 \quad a.e.,
\]

where \( \Lambda_0 \) is the integrated Lyapunov exponent

\[
\Lambda_0 = \int \Lambda(x) dx = \lim_{n \to \infty} \frac{1}{n} \log \|M(x, n)\| dx.
\]

We also recall the definition of reducibility.

**Definition 3.** A cocycle \( M \) is said to be reducible if it is conjugated to a constant cocycle, i.e., there exists a measurable map \( B : \mathbb{T}^1 \to \text{SL}(2, \mathbb{R}) \) and a constant \( C \in \text{SL}(2, \mathbb{R}) \) such that

\[
B(\sigma_\omega(x))A(x)B^{-1}(x) = C, \quad \forall x \in \mathbb{T}^1.
\]

That is, a cocycle is reducible if it is conjugated to a constant cocycle by a suitable coordinate change.

All these properties are related by the following theorem (see, e.g., \[1, 2\]).

**Theorem 2.** Let \( M \) be a cocycle defined by the skew-product map \((1.1)\). Then the following statements are equivalent:

1. \( M \) possesses ED;
2. \( M \) is \( UH \);

3. \( M \) is reducible and the integrated Lyapunov exponent \( \Lambda_0 > 0 \).

In the present work, using the approach developed in \([4, 17, 20, 26]\), we analyse those values of the parameter \( \varepsilon \) for which the cocycle associated to the map (1.1) is reducible and has a positive Lyapunov exponent. Although the papers \([4, 17, 20, 26]\) are devoted to different objects, they use a common framework. The idea of the method suggested in \([17]\) can be described in the following way. Consider a one-parameter family of skew-product maps \( F_{A,\varepsilon} = (f, A_\varepsilon) \) defined (similarly to (1.1)) on a vector bundle \( V \) over a base \( B \), then the properties of the fibre transformation \( A_\varepsilon \) depend (in general) on a point of the base. Choose those points of the base which correspond to the violation of some specific property (e.g., hyperbolicity) of \( A_\varepsilon \). These points constitute an initial approximation \( C_0 \) of the critical set. Taking a small neighbourhood of \( C_0 \), one may study the dynamics of this set under the map \( f \). Interactions between different parts of \( C_0 \) give rise to the first approximation \( C_1 \) of the critical set. Detuning the parameter \( \varepsilon \), one may put these interactions in “general” position to move the induction forward and to control the properties of \( C_1 \). Continuing the induction, one may hope to construct the critical set as a limit of \( C_n \) and investigate a set \( X_h \) of those points of the base which do not approach this critical set too closely for a sufficiently large period of time. It might happen that the set \( X_h \) is sufficiently large. Then, using the properties of the map \( f \) (e.g., ergodicity), one may extract an additional information on the whole system. We follow this approach in the present paper. The results can be formulated in two theorems:

**Theorem 3.** Let \( \omega \in (0, 1) \) be irrational and \( M \) be a cocycle generated by the transformation (2.1). Assume \( \min_{k,x} \lambda_k(x) = \lambda_0/\varepsilon > 0 \) for any \( k \) \( \varphi_k(x) = \phi_k/\varepsilon \), where \( \phi_k = \text{const} \). Then there exist \( \varepsilon_0 > 0 \), positive constants \( C_\Lambda < 1 \), \( C_0 \) and a subset \( \mathcal{E}_h \subset (0, \varepsilon_0) \) such that

1. for any \( \varepsilon_1 < \varepsilon_0 \) the Lebesgue measure \( \text{leb}((0, \varepsilon_1) \setminus \mathcal{E}_h) = O\left(e^{-C_0/\varepsilon_1}\right) \);

2. for any \( \varepsilon \in \mathcal{E}_h \) the integrated Lyapunov exponent \( \Lambda_0 > e^{C_\Lambda \lambda_0/\varepsilon} \);

3. for any \( \varepsilon \in \mathcal{E}_h \) the cocycle \( M \) possesses ED.

This theorem is an analog of Theorem 1. Indeed, if \( \omega \) is rational and, hence, the function \( F \) associated to (2.3) is periodic, one may always assume without loss of generality that \( F \) does not depend on \( \theta_1 \). It means that the functions \( \varphi_k \), which appear in the representation of the matrix \( A \) in (2.4) in the form (2.1), are also independent of \( \theta_1 \). Note that in this case \( \lambda_k \) become constant too. It will be shown later that the dependence of \( \lambda_k \) on \( x \) is not essential, whereas the inconstancy of \( \varphi_k \) is crucial.

The second theorem describes the opposite case when the functions \( \varphi_k \) are not constant. Let \( p_n/q_n \) denote the rational approximation of order \( n \) to \( \omega \) in its continued fraction expansion.

**Definition 4.** We will say that \( \omega \) satisfies the Brjuno condition with a constant \( C_B \) if

\[
\sum_{n=1}^{\infty} \frac{\log(2q_n+1)}{q_n} = C_B < \infty.
\]

This condition was introduced by A.D. Brjuno in \([5]\) (see also \([24]\)). In this paper we consider an additional condition. To formulate it, denote \( \mathbb{R}_+ = (0, +\infty) \) and consider a set of increasing functions defined on \( \mathbb{R}_+ \) which grow more slowly than \( x \):

\[
\mathcal{H} = \{ h: \mathbb{R}_+ \to \mathbb{R}_+: h(x) > h(y) \ \forall \ x > y; \ \lim_{x \to \infty} h(x)/x = 0 \}.
\]
**Definition 5.** Assume $\omega$ satisfies the Brjuno condition with a constant $C_B$. We will say that $\omega$ satisfies a condition (A) with a function $h \in \mathcal{H}$ if there exist a subsequence $\{q_{n_j}\}_{j=1}^{\infty}$ and positive constants $C_\varepsilon, C_\delta$ such that $C_\varepsilon > C_B, C_\delta < 1$

$$q_{n_j + 1} > q_{n_j} h(q_{n_j}), \ \forall \ j \in \mathbb{N}$$  \hspace{2cm} (2.5)

and for all $k \in \mathbb{N}$ there exists an index $J_k$

$$\frac{1}{q_{n_{j_k}}} \left( \log \left( q_{n_{j_k}} h(q_{n_{j_k}}) \right) + \log C^{-1}_\delta \right) < \frac{\log \left( q_{n_{J_k}} h(q_{n_{J_k}}) \right)}{q_{n_{j_k}}} < C_\varepsilon - C_B.$$  \hspace{2cm} (2.6)

Inequality (2.5) says that for the sequence of denominators $\{q_n\}_{n=1}^{\infty}$ the ratio $q_{n+1}/q_n$ becomes sufficiently large infinitely many times, while inequality (2.6) guarantees that such events of growth of the denominators occur in a regular way.

Finally, denote by $\mathcal{D}_k$ a domain of the function $\varphi_k$ and consider the following system of equations:

$$y \varphi'_k(x) = x \varphi'_l(y), \ \varphi''_k(x) = \varphi''_l(y), \ x \in \mathcal{D}_k, \ y \in \mathcal{D}_l.$$ \hspace{2cm} (2.7)

**Definition 6.** We will say that a collection of the functions $\varphi_k$ is non-degenerate if the system (2.7) does not possess any solutions except trivial $x = y, k = l$.

One may remark that due to the specific dependence of the functions $\varphi_k$ on the parameter $\varepsilon$ the system (2.7) does not contain $\varepsilon$.

**Theorem 4.** Let $\omega \in (0, 1)$ be irrational and $M$ be a cocycle generated by the transformation (2.1). Assume $\min_{k,x} \lambda_k(x) = \lambda_0/\varepsilon > 0$, for any $k$ the function $\varphi_k$ does not have degenerate critical points and the collection of the functions $\varphi_k$ is non-degenerate. If $\omega$ satisfies the Brjuno condition with a constant $C_B$ and the condition (A) with a function $h(x) = C_\omega x^\gamma, C_\omega > 0, 0 < \gamma < 1$, then there exist $\varepsilon_0 > 0$, positive constants $C_\Lambda < 1, C_0$ and a subset $\mathcal{E}_h \subset (0, \varepsilon_0)$ such that

1. for any $\varepsilon_1 < \varepsilon_0$ the Lebesgue measure $\text{leb} \left( ((0, \varepsilon_1) \setminus \mathcal{E}_h) \right) = O \left( e^{-C_0/\varepsilon_1} \right)$;
2. for any $\varepsilon \in \mathcal{E}_h$ the integrated Lyapunov exponent $\Lambda_0 > e^{C_\Lambda \lambda_0/\varepsilon}$;
3. for any $\varepsilon \in \mathcal{E}_h$ the cocycle $M$ does NOT possess ED.

**Remark.** One has to remark that Theorems 3 and 4 are formulated in terms of the functions $\varphi_k, \lambda_k$ only. What is essential is their dependence on the parameter $\varepsilon$, but the representation via integrals (see (2.2)) is not essential. Thus, instead of this representation we may consider the following one: $\varphi_k = \varepsilon^{-1} \hat{\varphi}_k, \lambda_k = \varepsilon^{-1} \hat{\lambda}_k$, with $\hat{\varphi}_k, \hat{\lambda}_k$ being real-valued $C^2$-functions defined on a segment $I_k \subset \mathbb{T}^1$. However, we prefer the definition (2.1), since it reveals the nature of the problem.

### 3. CONDITIONS ON THE PARAMETER $\omega$

Before discussing the Brjuno condition and condition (A), we collect some basic facts from continued fraction theory (see, e.g., [19] for details).

Each irrational number $\omega \in (0, 1)$ has a unique representation as a continued fraction

$$\omega = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}, \ a_n \in \mathbb{N} \ \forall n \in \mathbb{N}$$

or, using the list form,

$$\omega = [a_1, a_2, \ldots, a_n, \ldots].$$
Then the rational number \( p_n/q_n = [a_1, a_2, \ldots, a_n] \) is called the \( n \)th convergent to \( \omega \) and gives the minimum for \(|q\omega - p|\) over all rationals \( p/q \) with \( q \leq q_n \). The \( n \)th convergents satisfy an estimate

\[
\frac{1}{2q_nq_{n+1}} < \frac{1}{q_n(q_n + q_{n+1})} < \frac{\omega - p_n}{q_n} < \frac{1}{q_nq_{n+1}} < \frac{1}{q_n^2}.
\]

Consider those \( \omega \) which are ill-approximated by rational numbers. One class of numbers which satisfy this condition are Diophantine numbers. By definition, an irrational number \( \omega \) is said to be a Diophantine number of order \( \alpha \) if there exists a positive constant \( C \) such that

\[
|\omega - \frac{p}{q}| > \frac{C}{q^{2+\alpha}}, \quad \forall \frac{p}{q} \in \mathbb{Q}.
\]

For any \( \alpha > 0 \) the set of Diophantine numbers of order \( \alpha \) (denoted by \( D(\alpha) \)) has full measure. In the present study we impose on \( \omega \) a weaker condition. In particular, we assume that \( \omega \) satisfies the Brjuno condition (see Definition 4). This condition naturally appears in Sections 4 and 5 of the paper (see also [26]). Denoting the set of Brjuno numbers by \( B \), we have

\[
D(\alpha) \subset B, \quad \forall \alpha > 0
\]

and, thus, \( B \) also has full measure.

To describe condition (A), we first consider inequality (2.5) for arbitrary \( h \in \mathcal{H} \). Since for any \( \omega \in (0, 1) \) there is a relation

\[
\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_n} = [a_n; a_{n-1}, \ldots, a_1],
\]

(2.5) is equivalent to

\[
a_{n_j+1} > h(q_{n_j}), \quad \forall j \in \mathbb{N}.
\]

Let \( \{g(n)\} \) be a sequence of natural numbers. One may show (see [19]) that for almost all \( \omega \in (0, 1) \) the inequality \( a_n > g(n) \) holds infinitely many times if and only if the series \( \sum_{n=1}^{\infty} 1/g(n) \) diverges.

We also mention two well-known assertions of continued fraction theory ([19]): 1. for any irrational \( \omega \in (0, 1) \)

\[
q_n > 2^{\frac{a_n-1}{2}}, \quad \forall n \in \mathbb{N},
\]

and 2. there exists a constant \( C_L \) such that for almost all \( \omega \in (0, 1) \)

\[
q_n < e^{C_Ln}, \quad \forall n \in \mathbb{N}.
\]

(3.1)\hspace{1cm}(3.2)

Denote by \( A_h^{(1)} \) the set of those \( \omega \in (0, 1) \) which possess a subsequence \( \{q_{n_j}\}_{j=1}^{\infty} \) satisfying (2.5). Then one concludes that \( A_h^{(1)} \) is of full measure if the series \( \sum_{n=1}^{\infty} 1/h(e^{C_Ln}) \) diverges. Particularly, the function \( h(x) = \max\{\log(x), 1\} \) provides an example of such a function. Conversely, if the series \( \sum_{n=1}^{\infty} 1/h(2^{\frac{a_n-1}{2}}) \) converges, \( A_h^{(1)} \) has measure zero.

Note that in the case \( \omega \in A_h^{(1)} \) we may always extract a subsequence from \( \{q_{n_j}\}_{j=1}^{\infty} \) such that the left-hand side inequality of (2.6) holds for this new subsequence. Thus, each \( \omega \in A_h^{(1)} \) possesses a subsequence \( \{q_{n_j}\}_{j=1}^{\infty} \) satisfying (2.5) and the left-hand side inequality of (2.6).

Take \( h \in \mathcal{H} \) such that \( \sum_{n=1}^{\infty} 1/h(e^{C_Ln}) \) diverges and denote by \( A_h^{(2)} \) the set of those \( \omega \in A_h^{(1)} \) which possess a subsequence \( \{q_{n_j}\}_{j=1}^{\infty} \) satisfying (2.5) and (2.6).

**Lemma 1.** \( A_h^{(2)} \) has full measure.
Proof. For $C_K > 0$ introduce a set $\hat{A}_{h,C_K}^{(2)} \subset A_h^{(1)}$ consisting of those $\omega$ which satisfy

$$a_k \leq h(e^{C_l k}), \forall n_j < k < n_{j+1}; \ a_{n_j} > h(e^{C_l n_j}); \ n_{j+1} < C_K^{\frac{n_j}{2}}.$$  

Due to (3.1) and (3.2) it is sufficient to prove that $\hat{A}_{h,C_K}^{(2)}$ has full measure for sufficiently large $C_K$. Define for $M \in \mathbb{N}$ a set

$$E_{M,C_K} = \{\omega \in (0,1) : a_k \leq h(e^{C_l k}), k = M, \ldots, L, \ L = \left[ C_K^{\frac{n_j}{2}} \right]\},$$

where $[x]$ stands for the integer part of $x$. The Lebesgue measure of $E_{M,C_K}$ is estimated as

$$\text{leb}(E_{M,C_K}) \leq \prod_{k=M}^{L} \left( 1 - \frac{1}{3(1+h(e^{C_l k}))} \right).$$

Hence, the divergence of $\sum_{n=1}^{\infty} 1/h(e^{C_l n})$ yields $\text{leb}(E_{M,C_K}) \to 0$ as $M \to \infty$. Finally, if $\omega \in A_h^{(1)} \setminus \hat{A}_{h,C_K}^{(2)}$ there exists a sequence $M_i \to +\infty$ as $i \to \infty$ such that $\omega \in E_{M_i,C_K}$ $\forall i \in \mathbb{N}$. Thus, the set $A_h^{(1)} \setminus \hat{A}_{h,C_K}^{(2)}$ has measure zero.

Summarising, we conclude that, in the case $h \in \mathcal{H}$ whose series $\sum_{n=1}^{\infty} 1/h(e^{C_l n})$ is divergent, the set $B \cap A_{h}^{(2)}$ has full measure.

We point out that condition (A) requires only the existence of the subsequence $\{q_{n_j}\}$, but does not possess any restrictions on the initial value of index $n_0$ which the subsequence starts from. Taking any irrational $\omega \in (0,1)$, one may modify the elements $a_n$ of its continued fraction expansion starting from arbitrarily large index $n_0$ in the following way. For any constant $C_K > 0$ define a sequence of indices $n_{j+1} = C_K^{\frac{n_j}{2}} - 1$ and numbers $\hat{a}_{n_j+1} = \left[ h(e^{C_l n_j}) \right] + 1$. Then $\hat{\omega} = [\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{n_j}, \ldots]$ with $\hat{a}_k = a_k$ if $k \neq n_j + 1$ satisfies condition (A) and $|\omega - \hat{\omega}|$ can be made arbitrarily small as $n_0 \to \infty$. Thus, the set $\hat{A}_{h,C_K}^{(2)}$ is dense in $(0,1)$ for any $h \in \mathcal{H}$.

4. CRITICAL SET

In this section we describe an inductive procedure for the construction of a critical set consisting of those points $x \in \mathbb{T}^1$ where the cocycle $M$ exhibits “bad” hyperbolic behaviour.

Given two sequences of positive numbers $\{\varphi_k\}, \{\lambda_k\}_{k=1}^{\infty}$, we introduce $A_k = R(\varphi_k) \cdot Z(\lambda_k)$ and $A^k = \prod_{j=1}^{k} A_j$. Then the transformation (2.1) takes the form $A_x(x) = A^k(x)(x)$, where the sequences $\{\varphi_k\}, \{\lambda_k\}$ are defined by (2.2). Note that, although $\|A_k\| = \lambda_k$, it may happen that $\|A_k \cdot A_j\| \ll \lambda_k \lambda_j$. Nevertheless, one may prove the following lemma.

**Lemma 2.** Let $\{\varphi_k\}, \{\lambda_k\}_{k=1}^{\infty}$ be two sequences of positive numbers such that for any $k$

$$|\cos \varphi_k| \geq \delta, \ \lambda_k \geq \lambda_0 > 0, \ \delta e^{\lambda_0} \gg 1.$$  

Then

$$\|A^k\| \geq \left( C_A \delta e^{\lambda_0} \right)^k, \forall k,$$

where the constant $C_A = 1 + O(\delta e^{\lambda_0})^{-2}$. Moreover, $A^k$ can be represented as

$$A^k = R(\theta_k + \chi_k)Z(\mu_k)R(-\chi_k)$$

with

$$\mu_k \geq k(\lambda_0 + \ln \delta - \ln C_A), \ \chi_k = O\left( \delta^{-1} e^{-2\lambda_0} \right).$$
Proof. Let \( \alpha_1, \alpha_2, \varphi \) be real numbers such that \( \alpha_j > 1, j = 1, 2 \). Then, using the polar decomposition, one may represent

\[
Z(\log \alpha_2)R(\varphi), Z(\log \alpha_1) = R(\theta + \chi)Z(\mu)R(-\chi),
\]

where

\[
cos \theta = \frac{\cot \varphi}{\left( \cot^2 \varphi + \frac{\alpha_1^{-2} + \alpha_2^{-2}}{1 + \alpha_1^{-2} \alpha_2^{-2}} \right)^{1/2}}, \quad \sin \theta = \frac{\alpha_1^{-2} + \alpha_2^{-2}}{1 + \alpha_1^{-2} \alpha_2^{-2}} 2^{1/2},
\]

\[
e^\mu = \frac{a}{2} \left( 1 + c + \left( (1 + c)^2 - 4a^{-2} \right)^{1/2} \right), \quad a = \alpha_1 \alpha_2 \cos \varphi \cos \theta \left( 1 + \alpha_2^{-2} \frac{\alpha_1^{-2} + \alpha_2^{-2}}{1 + \alpha_1^{-2} \alpha_2^{-2}} \tan^2 \varphi \right),
\]

\[
b = \alpha_1^{-2} \tan \varphi \frac{1 - \alpha_2^{-2}}{1 + \alpha_1^{-2} \alpha_2^{-2} + \alpha_2^{-2} (\alpha_1^{-2} + \alpha_2^{-2}) \tan^2 \varphi}, \quad c = b^2 + a^{-2},
\]

\[
\cos(2\chi) = \frac{1 - c}{(1 - c)^2 + 4b^2}, \quad \sin(2\chi) = -\frac{2b}{(1 - c)^2 + 4b^2}.
\]

Taking this into account, we obtain clearly for \( k = 1 \) that \( A^1 = A_1 = R(\theta + \chi)Z(\mu)R(-\chi) \) with

\[
\theta_1 = \varphi_1, \quad \chi_1 = 0, \quad \mu_1 \geq \lambda_0.
\]

Applying consecutively (4.1), we get for \( k = n > 1 \)

\[
|\theta_n - \varphi_n| \leq 2\delta^{-1} e^{-2\lambda_0},
\]

\[
|\chi_n - \chi_{n-1}| \leq \delta^{-2n+3} e^{-2(n-1)\lambda_0} \left( 1 - 2\delta^{-2} e^{-4\lambda_0} \right)^{-2(n-2)},
\]

\[
\mu_n - \mu_{n-1} \geq \lambda_0 + \log \delta - 2\delta^{-2} e^{-4\lambda_0}.
\]

Summation over \( n \) finishes the proof. \( \Box \)

This lemma suggests that one can define the initial approximation of the critical set \( C_0(\epsilon) \) as a union of all solutions of equations \( \cos \varphi_k(x) = 0 \), i.e.,

\[
C_0(\epsilon) = \left\{ x \in \mathbb{T}^1 : \exists (k,i) : \varphi_k(x) = \frac{\pi}{2} (1 + 2i) \right\}.
\]

Here \( k \geq 1, i \geq 0 \). If the functions \( \varphi_k \) have no degenerate critical points, this set consists of a finite number \( N(\epsilon) \) of points, which we denote by \( c_j^{(0)}(\epsilon) \). Thus,

\[
C_0(\epsilon) = \bigcup_{j=1}^{N(\epsilon)} \{ c_j^{(0)}(\epsilon) \}.
\]

Let \( \rho_{j,j'}(\epsilon) = \text{dist}(c_j^{(0)}(\epsilon), c_{j'}^{(0)}(\epsilon)) \), where \( \text{dist}(x_1, x_2) \) stands for the distance between \( x_1 \) and \( x_2 \). The assumption that the collections of functions \( \varphi_k \) are non-degenerate (see Definition 6) yields non-degeneracy of critical points for \( \rho_{j,j'}, j \neq j' \) considered as functions of the parameter \( \epsilon \).

We consider

\[
\mathcal{E}_{p,0} = \{ \epsilon > 0 : N(\epsilon) = p \} = \bigcup_{p=1}^{N(\epsilon)} \{ \epsilon_{p,1}, \epsilon_{p,0} \},
\]

and for \( \epsilon \in \mathcal{E}_{p,0} \) we introduce a set which will be called (following [20]) a layer

\[
LC_0 = \bigcup_{j=1}^{N(\epsilon)} (c_j^{(0)} - \delta_0, c_j^{(0)} + \delta_0) = \bigcup_{j=1}^{N(\epsilon)} I_j, \quad \delta_0 = e^{-\lambda_0 \kappa_0 / \epsilon_{p,0}},
\]

where \( \kappa_0 \) will be chosen later.
Fix a point $x \in \mathbb{T}^1$ and denote its finite trajectory under the map $\sigma_\omega$ by $x_k = \sigma_\omega^k(x)$, $k = 0, \ldots, m$. Note that, if this trajectory does not fall into $LC_0$, one may apply Lemma 2 to the cocycle $M(x, k)$ and obtain the following estimate:

$$M(x, k) = R(\theta_k(x) + \chi_k(x))Z(\mu_k(x))R(-\chi_k(x))$$

with

$$\mu_k(x) \geq \frac{k}{\varepsilon}(x_0 - \kappa_0 - \log CA), \quad \chi_k(x) = O\left(\varepsilon e^{-(2-\kappa_0)\lambda_0/\varepsilon}\right), \quad \forall 0 \leq k \leq m. \quad (4.3)$$

However, due to irrationality of $\omega$ for any $x \in \mathbb{T}^1$ there exists a time $\tau(x)$ such that $\sigma_\omega^{\tau(x)}(x) \in LC_0$. Consider the dynamics of the layer $LC_0$ itself.

**Definition 7.** Let $\tau_{j,j'}^{(0)}$ be the minimum of $k \geq 1$ such that

$$\sigma_\omega^k(I_{j,0}) \cap I_{j',0} \neq \emptyset.$$ 

We call such an event a collision of the zeroth order and $\tau_{j,j'}^{(0)}$ the time of collision. A collision is called primary if $j = j'$ and secondary if $j \neq j'$.

One may note that due to ergodicity of $\sigma_\omega$ the only possibility of avoiding collisions occurs in a trivial case when the set $C_0(\varepsilon)$ is empty. If the functions $\varphi_k$ are not constant, the number of critical points corresponding to some particular index $k_0$ can be estimated from below by the integer part of $\frac{1}{\varepsilon}Var(\varphi_{k_0})$, where $Var$ denotes the variance of a function. Hence, inconstancy of the functions $\varphi_k$ implies $C_0(\varepsilon) \neq \emptyset$ for sufficiently small $\varepsilon$.

Throughout the rest of this section we will assume that $\varphi_k$ are not constant and $\varepsilon$ is sufficiently small such that $C_0(\varepsilon) \neq \emptyset$. In this case we cannot avoid collisions and our goal is to make them as rare as possible. It is evident that primary and secondary collisions behave in a different manner with respect to varying the parameter $\varepsilon$. Indeed, the times of primary collisions depend on $\varepsilon$ via $\delta_0$ only and are characterised essentially by the rotation number $\omega$. Whereas under the assumption of non-degeneracy for the collections of functions $\varphi_k$ one may hope to detune $\varepsilon$ so that the times of secondary collisions become greater than the times of primary ones. We also remark that for all $j$ the times $\tau_{j,j}^{(0)}$ are equal, and denote them by $\tau_0$.

Define

$$\mathcal{E}_{p,0} = \{\varepsilon \in \mathcal{E}_{p,0} : \exists (j, j') : \tau_{j,j'}^{(0)} < \tau_0\}$$

and estimate the Lebesgue measure of this set. If $\{q_n\}$ denote the denominators of the best rational approximations for $\omega$, then $q_1, q_2, \ldots$ are exactly the times when $\sigma_\omega^n(x)$ approximate $x$ better than ever before. Besides, the following inequality holds:

$$\frac{1}{2q_{n+1}} < \text{dist}(\sigma_\omega^n(x), x) < \frac{1}{q_{n+1}}, \quad \forall n. \quad (4.4)$$

For the rest of this section we will suppose that $\omega$ satisfies the Brjuno condition and condition (A). Under this assumption we choose a sufficiently large $J_0$ and $\kappa_0 < 2/3$ such that

$$\frac{1}{q_{k,J_0+1}} < \delta_0 = e^{-\lambda_0\kappa_0/\varepsilon} = \frac{1}{C_\omega q_{k,J_0}^{1+\gamma}}. \quad (4.5)$$

This choice and (4.4) imply

$$\text{dist}(\sigma_\omega^m(x), x) > \frac{1}{2q_{n,J_0}} \gg \frac{1}{C_\omega q_{n,J_0}^{1+\gamma}} = \delta_0, \quad 0 \leq m \leq q_{n,J_0} - 1,$$

$$\text{dist}(\sigma_\omega^{q_{n,J_0}}(x), x) < \frac{1}{q_{n,J_0}+1} < \frac{1}{C_\omega q_{n,J_0}^{1+\gamma}} = \delta_0. \quad (4.6)$$

Hence, one may conclude that $\tau_0 = q_{n,J_0}$.
To estimate \( \text{leb}(\mathcal{E}_{p,0}^{'}) \), we note that

\[
N(\varepsilon) = O(\varepsilon^{-1}), \quad \varepsilon_{p,0} - \varepsilon_{p,1} = O(\varepsilon_{p,0}^2).
\]

Besides, the non-degeneracy of the collections of the function \( \varphi_k \) leads to the following estimate on \( \Delta \rho = \max \rho_{j,j'}(\varepsilon) - \min \rho_{j,j'}(\varepsilon) \) (where max and min are taken over \( \varepsilon \in (\varepsilon_{p,1}, \varepsilon_{p,0}] \) and all \( (j,j') \)):

\[
\Delta \rho = O(\varepsilon_{p,0}^2).
\]

Notice that the number of collisions is bounded from above by \( 2\Delta \rho \left( \frac{2}{p} \right) \tau_0 \). This implies

\[
\text{leb}(\mathcal{E}_{p,0}^{'}) \leq C_\rho \Delta \rho \left( \frac{2}{p} \right) \tau_0 \delta_0.
\]

Substituting (4.5), (4.7) and (4.8) into (4.9), we arrive at the following lemma.

**Lemma 3.** The Lebesgue measure of \( \mathcal{E}_{p,0}^{'}, 0 \) is estimated as

\[
\text{leb}(\mathcal{E}_{p,0}^{'}) = O \left( e^{-\lambda_0 \kappa_0 \gamma/(1+\gamma) \varepsilon_{p,0}} \right).
\]

We eliminate this “bad” set and introduce \( \hat{\mathcal{E}}_{p,1} = \mathcal{E}_{p,0} \setminus \mathcal{E}_{p,0}^{'} \).

For \( \varepsilon \in \hat{\mathcal{E}}_{p,1} \) and \( x \in I_{j,0} \) we consider \( M(x, \tau_0) \). Applying Lemma 2 to \( M(A(x, \varepsilon), \tau_0 - 1) \) and using (4.3), we represent it as

\[
M(A(x, \varepsilon), \tau_0 - 1) = R(\theta_{j,0}(x) + \chi_{j,0}(x)) Z(\mu_{j,0}(x)) R(- \chi_{j,0}(x)),
\]

with

\[
\chi_{j,0}(x) = O(\varepsilon e^{(2-\kappa_0)\lambda_0/\varepsilon_{p,0}}), \quad \mu_{j,0}(x) \geq \frac{\tau_0 - 1}{\varepsilon} (\lambda_0 - \kappa_0 - \log C_A).
\]

We substitute (4.10) into the expression for \( M(x, \tau_0) \) to get

\[
M(x, \tau_0) = R(\theta_{j,0}(x) + \chi_{j,0}(x)) Z(\mu_{j,0}(x)) R(\varphi_k(x) - \chi_{j,0}(x)) Z(\lambda_k(x)),
\]

with \( k \) defined by

\[
\cos \left( \varphi_k(c_j^{(0)}) \right) = 0.
\]

Then we consider the equation

\[
\cos \left( \varphi_k(x) - \chi_{j,0}(x) \right) = 0, \quad x \in I_{j,0},
\]

where \( k \) satisfies (4.11). One notes that, if a point \( x \) is close to a solution of (4.12), \( M(x, \tau_0) \) loses the hyperbolicity.

Expanding (4.12) around \( x = c_j^{(0)} \), we obtain an equation for \( \Delta_j = x - c_j^{(0)} \)

\[
A_j \Delta_j^2 + 2B_j \Delta_j + C_j = O(\Delta_j^3),
\]

where

\[
A_j = \frac{1}{2} \left[ \varphi_k''(c_j^{(0)}) - \chi_{j,0}''(c_j^{(0)}) + \tan \left( \chi_{j,0}(c_j^{(0)}) \right) \left( \varphi_k'(c_j^{(0)}) - \chi_{j,0}'(c_j^{(0)}) \right)^2 \right],
\]

\[
B_j = \frac{1}{2} \left[ \varphi_k'(c_j^{(0)}) - \chi_{j,0}'(c_j^{(0)}) \right], \quad C_j = - \tan \left( \chi_{j,0}(c_j^{(0)}) \right).
\]

Hence, if the conditions

\[
|A_j C_j / B_j^2| < C_{\Delta} \ll 1, \quad |B_j / A_j| > \delta_0 / (1 - C_{\Delta}), \quad |C_j / 2B_j| < \delta_0 (1 + C_{\Delta})
\]

are fulfilled for some positive constant \( C_{\Delta} \), eq. (4.13) possesses a unique solution on the interval \((-\delta_0, \delta_0)\).
Lemma 4. For any $x \in I_{j,0}$ the derivatives of the function $\chi_{j,0}$ defined by (4.10) admit the following estimates:

$$
\chi'_{j,0}(x) = O(e^{-2(1-\kappa_0)\lambda_0/\varepsilon_{p,0}}), \quad \chi''_{j,0}(x) = O(e^{-1-2(2-\kappa_0)\lambda_0/\varepsilon_{p,0}}).
$$

Proof. Let $\{\hat{\varphi}_k\}, \{\hat{\lambda}_k\}_{k=1}^\infty$ be two sequences of $C^2$-functions of a variable $x \in (a, b)$ such that $\hat{\varphi}_k, \hat{\lambda}_k$ are uniformly bounded and for any $x$ the sequences $\{\hat{\varphi}_k(x)\}, \{\hat{\lambda}_k(x)\}_{k=1}^\infty$ satisfy the conditions of Lemma 2. Then by Lemma 2 one may represent

$$
A^k(x) = R(\theta_k(x) + \chi_k(x)) Z(\mu_k(x)) R( - \chi_k(x)).
$$

Differentiating (4.1) with respect to $x$ and using (4.2) yield

$$
\chi_k(x) = O \left( \delta^2 e^{-2\lambda_0} \right), \quad \chi''_k(x) = O \left( \delta^3 e^{-2\lambda_0} \right), \quad (4.15)
$$

We apply (4.15) to $M(A(x, \varepsilon), \tau_0 - 1)$ for $x \in I_{j,0}$. Then taking into account the specific dependence of functions $\varphi_k, \lambda_k$ on the parameter $\varepsilon$ (see (2.2)), one notes that $\varphi'_k(0) = O(\varepsilon^{-1}), \varphi''_k(0) = O(\varepsilon^{-2})$. Together with the definition of $\delta_0$ this finishes the proof.

The non-degeneracy condition on the critical points of the function $\varphi_k$ and Lemma 4 imply the existence of positive constants $C_1, C_2$ such that

$$
\left| \frac{A_j C_j}{B_j^2} \right| \leq C_1 \left| \frac{\varphi'_k(c_j^{(0)}) - \chi''_{j,0}(c_j^{(0)})}{\lambda_k(0) - \chi''_{j,0}(0)} \right|, \quad \left| \frac{B_j}{4A_j \delta_0} \right| \geq C_2 e^{\kappa_0 \lambda_0/\varepsilon} \left| \varphi'_k(c_j^{(0)}) - \chi'_{j,0}(c_j^{(0)}) \right|, \quad (4.16)
$$

Hence, if

$$
\left| \varphi'_k(c_j^{(0)}) \right| \geq e^{-\lambda_0 \kappa_0 \gamma/(1+\gamma) \varepsilon_{p,0}}, \quad (4.17)
$$

we apply inequalities (2.6), (4.16) and (4.5) to conclude that condition (4.14) is fulfilled and Eq. (4.12) possesses a unique solution on the interval $I_{j,0}$. We denote by $\mathcal{E}_{p,0}$ the set of those values of the parameter $\varepsilon$ for which condition (4.17) is violated. This set is a small neighbourhood of one or both end points of the interval $(\varepsilon_{p,1}, \varepsilon_{p,0}]$ and due to the non-degeneracy of critical points of the functions $\varphi_k$ its Lebesgue measure is estimated as

$$
\text{leb} \left( \mathcal{E}_{p,0} \right) = O \left( e^{-\lambda_0 \kappa_0 \gamma/(1+\gamma) \varepsilon_{p,0}} \right).
$$

Introduce $\mathcal{E}_{p,1} = \hat{\mathcal{E}}_{p,1} \setminus \mathcal{E}_{p,0}$. Then for any $\varepsilon \in \mathcal{E}_{p,1}$ there exists a unique solution of Eq. (4.12). Moreover, this solution, denoted by $c_j^{(1)}(\varepsilon)$, satisfies the following estimate:

$$
\text{dist} \left( c_j^{(1)}, c_j^{(0)} \right) = O \left( e^{-2 - \frac{\lambda_k(0)}{1+\gamma} \varepsilon_{p,0}} \right).
$$

These points constitute a set

$$
\mathcal{C}_1(\varepsilon) = \bigcup_{j=1}^{N(\varepsilon)} \left\{ c_j^{(1)}(\varepsilon) \right\},
$$

which we call the first approximation of the critical set and define the layer of order 1 as

$$
\mathcal{L} \mathcal{C}_1 = \bigcup_{j=1}^{N(\varepsilon)} (c_j^{(1)} - \delta_1, c_j^{(1)} + \delta_1) = \bigcup_{j=1}^{N(\varepsilon)} I_{j,1}, \quad \delta_1 = e^{-\lambda_0 \kappa_1 \gamma \varepsilon_{p,0}},
$$

with some $\kappa_1$ which will be fixed later. Then, in the same manner as for $\mathcal{L} \mathcal{C}_0$, we can define collisions of the first order, their times $\tau_{j,j}^{(1)}$ and

$$
\mathcal{E}_{p,1}^{(1)} = \{ \varepsilon \in \mathcal{E}_{p,1} : \exists (j, j') : \tau_{j,j'}^{(1)} < \tau_1 \},
$$

where $\tau_1$ stands for the time of primary collisions of the first order.
Take the return time \( q_{n_{J_1}} \) from the definition of condition (A) and define \( \kappa_1 \) as

\[
\frac{1}{q_{k_{J_1}+1}} < \delta_1 = \frac{1}{C_\omega q_{k_{J_1}}} = e^{-\lambda_0 \kappa_1 / \varepsilon}. \]

As at the previous step, this choice implies

\[
\text{dist} (\sigma^m_\omega (x), x) > \frac{1}{2q_{n_{J_1}}} \gg \frac{1}{C_\omega q_{n_{J_1}+1}} = \delta_1, \quad 0 \leq m \leq q_{n_{J_1}} - 1,
\]

\[
\text{dist} (\sigma^{q_{n_{J_1}}} \omega (x), x) < \frac{1}{q_{n_{J_1}+1}} < \frac{1}{C_\omega q_{n_{J_1}}} = \delta_1
\]

and, hence, \( \tau_1 = q_{k_{J_1}} \). An analog of Lemma 3 gives

\[
\text{leb}(E'_{p,1}) = O \left( e^{-(\lambda_0 \kappa_1 \gamma / (1 + \gamma) \varepsilon \beta_0, 0)} \right).
\]

We exclude \( E'_{p,1} \) to get \( E_{p,2} = E_{p,1} \setminus E'_{p,1} \). One may note that, if \( x \in I_{j,0} \setminus I_{j,1} \), then

\[
\|M(x, \tau_1)\| \geq \left( C_A e^{(1 - \kappa_0) \lambda_0 / \varepsilon} \right)^{q_{n_{J_1}}-1} \left( C_A e^{(1 - \kappa_1) \lambda_0 / \varepsilon} \right) = C_A^{q_{n_{J_1}} \lambda_0 - (\tau_1-1) \lambda_0 / \varepsilon}.
\]

Let \( \varepsilon \in E_{p,2} \) and \( x \in I_{j,1} \), then one notes that for \( 1 \leq m \leq \tau_1 - 1 \) the finite trajectory \( \{ \sigma^m_\omega (x) \} \) cannot fall into \( I_{j,1} \), but only into \( I_{j,0} \setminus I_{j,1} \). Taking this into account, we apply successively (4.1), (4.2), (4.3) together with the first inequality (4.4) to represent \( M(A(x, \varepsilon), \tau_1 - 1) \) as

\[
M(A(x, \varepsilon), \tau_1 - 1) = R(\theta_{j,1}(x) + \chi_{j,1}(x)) Z(m_{j,1}(x)) R(- \chi_{j,1}(x)),
\]

where

\[
\left| \chi_{j,1}(x) - \chi_{j,0}(x) \right| \leq \varepsilon e^{-2(\tau_1-1)(1 - \kappa_0) \lambda_0 / \varepsilon + \log(2q_{n_{J_0}+1})},
\]

\[
\mu_{j,1}(x) \geq \frac{1}{\varepsilon} \left( \left( \tau_1 - 1 \right) \lambda_0 - \kappa_0 \log C_A - \sum_{k=n_{J_0}}^{n_{J_1}} \log(2q_{k+1}) \right).
\] (4.18)

Finally, due to the Brjuno condition one gets

\[
\left| \chi_{j,1}(x) - \chi_{j,0}(x) \right| = O(\varepsilon e^{-2(\tau_1-1)(1 - \kappa_0) \lambda_0 / \varepsilon - C_B / 2}),
\]

\[
\mu_{j,1}(x) \geq \frac{\tau_1 - 1}{\varepsilon} \left( \lambda_0 - \kappa_0 \log C_A - C_B \right).
\]

Similarly to (4.12), we consider the equation

\[
\cos \left( \varphi_k (x) - \chi_{j,1}(x) \right) = 0, \quad x \in I_{j,1},
\] (4.19)

with the same \( k \) defined by (4.11). Since (4.17) holds for \( \varepsilon \in E_{p,2} \) and due to inequality (2.6), this equation has a unique solution in a small vicinity of \( c_j^{(1)} \). We denote this solution by \( c_j^{(2)}(\varepsilon) \). Then it satisfies the following estimate:

\[
\text{dist} \left( c_j^{(2)}, c_j^{(1)} \right) = O \left( \varepsilon e^{-2(\tau_1-1)(1 - \kappa_0) \lambda_0 / \varepsilon - C_B / 2 + \lambda_0 \kappa_0 \gamma / (1 + \gamma) \varepsilon} \right).
\]

We proceed in the same manner. Assuming that the \( n \)th approximation of the critical set

\[
C_n(\varepsilon) = \bigcup_{j=1}^{N(\varepsilon)} \{ c_j^{(n)}(\varepsilon) \}
\]

is constructed, one may define the layer of order \( n \)

\[
LC_n = \bigcup_{j=1}^{N(\varepsilon)} (c_j^{(n)} - \delta_n, c_j^{(n)} + \delta_n) = \bigcup_{j=1}^{N(\varepsilon)} I_{j,n}, \quad \delta_n = e^{-\lambda_0 \kappa_n / \varepsilon \beta_0, 0},
\]

\[
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\]
Lemma 4. The existence and uniqueness of this solution follows from inequalities (2.6), (4.17), (4) and

using the Brjuno condition, we obtain

where \( C_j \) is defined as

This leads to definition of \( c_j^{(n)} \) as a unique solution of the equation

\[
\cos(\varphi_n(x) - \chi_{j,n}(x)) = 0, \quad x \in I_{j,n},
\]

which satisfies the bound

\[
dist\left(c_j^{(n+1)}, c_j^{(n)}\right) = O\left(\varepsilon e^{-2(\tau_{n-1} - \tau_{n-2} - 1)(1 - \kappa_0) \lambda_0 / \varepsilon} - C_B/2\right),
\]

where \( \tau_n \) stands for the time of primary collisions of the \( n \)th order. A corresponding modification of Lemma 3 gives

\[
\text{leb}(\mathcal{E}'_{p,n}) = O\left(e^{-\lambda_0 \kappa_0 \gamma / (1 + \gamma) \varepsilon_{p,0}}\right).
\]

Eliminating \( \mathcal{E}'_{p,n} \), we obtain the \((n + 1)\)th approximation of a set of “good” values of the parameter \( \varepsilon \), namely, \( \mathcal{E}_{p,n+1} = \mathcal{E}_{p,n} \setminus \mathcal{E}'_{p,n} \). For \( \varepsilon \in \mathcal{E}_{p,n+1} \) and \( x \in I_{j,n} \), by similar arguments used to get (4.18), one may represent

\[
M(A(x, \varepsilon), \tau_n - 1) = R(\theta_{j,n}(x) + \chi_{j,n}(x)) Z(\mu_{j,n}(x)) R\left(-\chi_{j,n}(x)\right),
\]

where

\[
\begin{align*}
|\chi_{j,n}(x) - \chi_{j,n-1}(x)| & \leq \frac{\varepsilon}{e^{-2(\tau_{n-1} - \tau_{n-2} - 1)(1 - \kappa_0) \lambda_0 / \varepsilon}} \prod_{k=n_{j,n-2}+1}^{n_{j,n-1}} 2q_{k+1}, \\
\mu_{j,n}(x) & \geq \frac{1}{\varepsilon}\left((\tau_n - 1)(\lambda_0 - \kappa_0 - \log C_A) - \sum_{k=n_{j,n-1}}^{n_{j,n-1}} \log(2q_{k+1})\right).
\end{align*}
\]

Using the Brjuno condition, we obtain

\[
|\chi_{j,n}(x) - \chi_{j,n-1}(x)| = O\left(\varepsilon e^{-2(\tau_{n-1} - \tau_{n-2} - 1)(1 - \kappa_0) \lambda_0 / \varepsilon} - C_B/2\right),
\]

\[
\mu_{j,n}(x) \geq \frac{\tau_n - 1}{\varepsilon}\left(\lambda_0 - \kappa_0 - \log C_A - C_B\right).
\]

This leads to definition of \( c_j^{(n+1)} \) as a unique solution of the equation

Finally, passing to the limit \( n \to \infty \), one may note that the estimates (4.20) and (4.22) yield the following lemma.

Lemma 5. There exists a limit set

\[
\mathcal{E}_{p,\infty} = \mathcal{E}_{p,0} \setminus \bigcup_{n=0}^{\infty} \mathcal{E}'_{p,n}
\]

such that the Lebesgue measure of \( \mathcal{E}_{p,0} \setminus \mathcal{E}_{p,\infty} \) admits an estimate

\[
\text{leb}(\mathcal{E}_{p,0} \setminus \mathcal{E}_{p,\infty}) = O\left(e^{-\lambda_0 \kappa_0 \gamma / (1 + \gamma) \varepsilon_{p,0}}\right).
\]

Moreover, for any \( \varepsilon \in \mathcal{E}_{p,\infty} \) there exists a limit set

\[
C_{\infty}(\varepsilon) = \bigcup_{j=1}^{N(\varepsilon)} \{c_j^{(\infty)}(\varepsilon)\}, \quad c_j^{(\infty)}(\varepsilon) = \lim_{n \to \infty} c_j^{(n)}(\varepsilon).
\]
5. LOWER BOUND FOR THE LYAPUNOV EXPONENT

In this subsection we always suppose that $\varepsilon \in \mathcal{E}_{p,\infty}$.

**Definition 8.** We say that $x \in \mathbb{T}^1$ has property $(H)$ if

$$
\begin{cases}
\sigma^m_\omega(x) \not\in LC_0, \quad &\forall 0 \leq m < \tau_0, \\
\sigma^m_\omega(x) \not\in LC_n, \quad &\forall \tau_n-1 \leq m < \tau_n.
\end{cases}
$$

Denote the set of all $x \in \mathbb{T}^1$ which possess the property $(H)$ by $X_h$ and its complement by $X_e = \mathbb{T}^1 \setminus X_h$. Then the definition of $\delta_n$, together with the property $(A)$, implies

$$
\text{leb} (X_e) \leq 2p \sum_{n=0}^{\infty} \tau_n \delta_n \leq 2p \sum_{n=0}^{\infty} C^\frac{1}{\gamma} \delta_n^n \leq 2p C^\frac{1}{\gamma} \sum_{n=0}^{\infty} \left( \frac{\gamma}{C_\delta} \right)^n \delta_0^n = \frac{2p C^\frac{1}{\gamma}}{1 - C_\delta} e^{-\lambda_0 \kappa_0 / (1+\gamma) \varepsilon,0}.
$$

Taking into account that $p = O(\varepsilon^{-1})$, one concludes

$$
\text{leb}(X_h) = 1 - O \left( \varepsilon^{-1} e^{-\lambda_0 \kappa_0 / (1+\gamma) \varepsilon,0} \right).
$$

Hence, the Lebesgue measure of $X_h$ is positive. Since the Lyapunov exponent exists and is constant a.e. (provided that $\omega$ is irrational), it is sufficient to estimate $\Lambda(x)$ for $x \in X_h$.

Consider $x \in X_h$ and its trajectory $x_m = \sigma^m_\omega(x), m = 0,1,\ldots$. Denote by $m_1$ the first time when $x_m \in LC_{n_1} \setminus LC_{n_1+1}$. By construction, $m_{i+1} - m_i \geq \tau_i$. Let $m_k \leq m < m_{k+1}$. Then, using the property $(H)$ and the Brjuno condition, one gets

$$
\frac{1}{m} \log \|M(x,m)\| \geq \frac{1}{m} \log \left( C_A e^{(1-\kappa_0) \lambda_0 / \varepsilon} \right)^m \prod_{i=1}^{m_k} \delta_{n_{i+1}} \geq (1 - \kappa_0) \lambda_0 / \varepsilon + \log C_A - \frac{1}{m} \sum_{i=1}^{m_k} \log 2^{\kappa_{n_{i+1}}} \geq (1 - \kappa_0) \lambda_0 / \varepsilon + \log C_A - C_B.
$$

Using (5.2), one concludes that for any $x \in X_h$ the Lyapunov exponent $\Lambda(x)$ of the cocycle $M$ is positive. Since the Lebesgue measure of the set $X_h$ is positive and the rotation $\sigma_\omega$ is ergodic, we arrive at the following lemma.

**Lemma 6.** If $\varepsilon \in \mathcal{E}_{p,\infty}$, the integrated Lyapunov exponent $\Lambda_0$ of the cocycle $M$ is positive and satisfies

$$
\Lambda_0 \geq (1 - \kappa_0) \lambda_0 / \varepsilon + \log C_A - C_B.
$$

6. PROOFS OF THEOREMS 3 AND 4

**Proof (of Theorem 3).** Assume that for each $k$ the function $\varphi_k(x)$ has the form $\varphi_k(x) = \phi_k / \varepsilon$ where $\phi_k$ is a constant. Since the set $\mathcal{C}_0$ consists of those points $x \in \mathbb{T}^1$ which solve the equation $\cos \varphi_k(x) = 0$ for some $k$, it is possible to exclude a countable set of values of the parameter $\varepsilon$ to get $\mathcal{C}_0(\varepsilon) = \emptyset$. We exclude a larger set to apply Lemma 2. Namely, define

$$
\varepsilon_{k,j} = \pi^{-1} \phi_k (1/2 + j)^{-1}, \quad \delta_{k,j} = \varepsilon_{k,j} \phi^{-1}_k e^{-\lambda_0 / 2\varepsilon_{k,j}}
$$

and

$$
\mathcal{E}_e = \bigcup_{k,j} (\varepsilon_{k,j} - \delta_{k,j}, \varepsilon_{k,j} + \delta_{k,j}).
$$
Then for any \( \varepsilon_0 > 0 \) the Lebesgue measure

\[
\text{leb}(E_c \cap (0, \varepsilon_0)) = O\left(e^{-\lambda_0/2\varepsilon_0}\right)
\]

and for any \( \varepsilon \in E_h = (0, \varepsilon_0) \setminus E_e \) one has \( |\cos \varphi_k(x)| \geq e^{-\lambda_0/2\varepsilon_0} \) for all \( x \in \mathbb{T}^1 \). Thus, applying Lemma 2 to the cocycle \( M \), corresponding to \( \varepsilon \in E_h \), we obtain a uniform estimate

\[
\|M(x, n)\| \geq e^{C_{\Lambda_0} n/2\varepsilon}, \quad C_{\Lambda} = 1 + O\left(\varepsilon e^{-\lambda_0/2\varepsilon}\right).
\]

This finishes the proof of Theorem 3.

Note that, if \( \varphi_k \) are not constant, but their variances are sufficiently small, then the set \( C_0(\varepsilon) \) might be empty for intermediate values of \( \varepsilon \). Particularly, one gets the following corollary of Theorem 3.

**Corollary 1.** Let functions \( \lambda_k \), constants \( \phi_k \), \( C_{\Lambda} \), \( C_0 \) and the parameter \( \varepsilon_0 \) satisfy the conditions of Theorem 3. Then for a cocycle defined by the transformation (2.1) associated to the set of functions \( \lambda_k(x) \) and \( \varphi_k(x) = (\phi_k + \psi_k(x))/\varepsilon \) such that

\[
\text{Var}(\psi_k) \ll e^{-\lambda_0/2\varepsilon_0}, \quad \forall k
\]

there exist \( \varepsilon_1 \ll \varepsilon_0 \) and a subset \( E_h \subset (\varepsilon_1, \varepsilon_0) \) such that

1. for any \( \varepsilon_2 \in (\varepsilon_1, \varepsilon_0) \) the Lebesgue measure \( \text{leb}\left((\varepsilon_1, \varepsilon_2) \setminus E_h\right) = O\left(e^{-C_{\Lambda_0}/2} \right) \);
2. for any \( \varepsilon \in E_h \) the Lyapunov exponent \( \Lambda_0 > e^{C_{\Lambda_0}/\varepsilon} \);
3. for any \( \varepsilon \in E_h \) the cocycle \( M \) possesses ED.

**Proof (of Theorem 4).** Under the assumptions of Theorem 4 we apply Lemma 5 to construct the limit set \( E_{p, \infty} \subset E_{p, 0} \) satisfying (4.23). Moreover, for any \( \varepsilon \in E_{p, \infty} \) there exist a critical set \( C_{\infty}(\varepsilon) \) and a subset \( X_h \subset \mathbb{T}^1 \) such that each point \( x \in X_h \) has property (H). The Lebesgue measure of \( X_h \) is positive and satisfies the estimate (5.1). Hence, Lemma 6 guarantees that the integrated Lyapunov exponent of the cocycle \( M \) corresponding to \( \varepsilon \in E_{p, \infty} \) is positive. Summation over all \( p \) together with (5.2) proves the first and the second assertion of Theorem 4.

To prove the third statement of the theorem, we note that, if one takes a point \( x \in \mathbb{T}^1 \) sufficiently close to the critical set \( C_{\infty}(\varepsilon) \), the Lyapunov exponent \( \Lambda(x) \) can be made arbitrarily small. In particular, for any \( x \in C_{\infty}(\varepsilon) \) one has

\[
\|M(x, n)\| < e^{Cn},
\]

for any constant \( C > 0 \). Thus, in the case of non-constant functions \( \varphi_k \) the cocycle cannot be reducible for a sufficiently large set of the parameters and, hence, does not possess the exponential dichotomy. \( \square \)

**Remark.** We emphasise that the Brjuno condition and condition (4) play different roles in this paper. While the Brjuno condition is mainly used to get the lower bound for the Lyapunov exponent, condition (4) is applied to avoid the secondary collisions and to estimate the Lebesgue measure of the set \( E_{p, n}' \). Omitting condition (4) (or a condition similar to it) leads to growth of \( \text{leb}(E_{p, n}') \). In this case the measure of \( E_{p, n}' \) will not be exponentially small; however, it tends to zero with decreasing \( \varepsilon \) [26].

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