A globally well-behaved simultaneity connection for stationary frames in the weak field limit

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Local simultaneity conventions are mathematically represented by connections on the bundle of timelike curves that defines the frame. Those simultaneity conventions having an holonomy proportional to the Riemann tensor have a special interest since they exhibit a good global behavior in the weak field limit. By requiring that the simultaneity convention depends on the acceleration, vorticity vector and the angle between them we are able to restrict considerably the allowed simultaneity conventions. In particular, we focus our study on the simplest among the allowed conventions. It should be preferred over Einstein’s in flat spacetime and in general if the tidal force between neighboring particles is weaker than the centrifugal force. It reduces to the Einstein convention if the vectorial product between the acceleration and the vorticity vector vanishes. We finally show how to use this convention in practice.

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I. INTRODUCTION

In 1904 Poincaré [19, 38, 39] defined a synchronization convention for distant clocks that is nowadays know as Einstein synchronization (simultaneity) convention [20]. Given two space points $A$ and $B$ Poincaré defined that the clocks at $A$ and $B$ are “synchronized” if the time it takes light to go from $A$ to $B$ is the same of that needed by light to go from $B$ to $A$. It was later shown [36, 46] that this definition is also transitive between space points if the speed of light over closed paths is a universal constant $c$. In this case it is possible to construct a global time variable $t$ and define as simultaneous those events that correspond to the same value of $t$. These are actually some steps needed to develop special relativity on a firm ground. With the introduction of Minkowski spacetime the previous definition of simultaneity acquired a geometrical interpretation: two events of coordinates $a^\mu$ and $b^\mu$ are simultaneous with respect to the observer (inertial reference frame) of 4-velocity $u^\mu$ if and only if $u^\mu (a^\nu - b^\nu) = 0$, that is, the simultaneity slices belong to the ker of $u^\mu dx^\mu$. This was a signal that something new was happening in the study and application of the simultaneity concept.

Actually, in the second half of the nineteenth century there was a lot of confusion about the diffusion of time. Many cities had their own local time and, as reported by Kern [28], in France there was even a delay of five minutes between the time on the trains and the time in the railway stations, so as to give to the passengers five more minutes to get on the trains. After the introduction of Greenwich Mean Time (GMT) [12, 26], the synchronization of clocks was related to GMT through the definition of time zones. However, as any general relativist knows, that procedure is meaningless without giving a well defined notion of GMT in points different from Greenwich. One can try to use the Einstein convention in order to achieve a worldwide synchronization but, unfortunately, Sagnac discovered in 1913 an effect that prevents the Einstein synchronization convention from being transitive on rotating frames like the earth [3, 8, 42]. In fact, a worldwide synchronization would imply a “time ambiguity” of the order of one hundred of nanoseconds. It means that clocks can be synchronized using the Einstein convention but statements regarding the comparison of distant clocks for time intervals below or of the order of one hundred of nanoseconds are meaningless. Indeed at that scale time intervals strongly depend on the actual succession of clock synchronizations that have been followed to coordinate clocks. So, we can say that event $A$ happened 200 nanoseconds after $B$ only if events $A$ and $B$ happen at the same point on the earth. The synchronization convention is not transitive; therefore, there is no exact definition of Einstein global time. However, one can still use Einstein synchronization being aware of the limits of the procedure.

As the clocks became more accurate there was the need of a much more accurate definition of coordinate time. The
Greenwich Mean Time was replaced by what is now known as Coordinated Universal Time (UTC) and both special relativistic effects, like the one by Sagnac, and general relativistic effect, like time dilation due to the gravitational field, were taken into account. In particular the clock synchronization is achieved using the GPS (global positioning system) using a procedure known as “common view” (complemented by other less accurate methods, see tables C1 and C2 of Ref. 2 or Ref. 13). It should be said that the synchronization using this system starts from a model of spacetime that ignores the general relativistic effect of dragging of inertial frames 6, 7, 12. That is, it is assumed the existence of a timelike Killing vector field having zero vorticity. This Killing vector field defines the inertial frame at rest with respect to the fixed stars, the so called Earth-Centered Inertial (ECI) frame, and since there is no vorticity, the Einstein convention becomes transitive in that frame. The GPS aims at reproducing a coordinate time for the rotating earth so that the simultaneity coincides with the ECI frame simultaneity 6, 7. We know, however, that there is no Killing vector field having the assumed properties since, as the Kerr metric tells us, the Killing field that remains timelike at infinity has a non-vanishing vorticity 22. Of course, this is a small effect that turns out to be negligible with respect to the actual order of accuracy required by practical applications, but it shows that, also in the synchronization through the GPS there is an intrinsic time ambiguity (using the Kerr metric it turns out to be of the order of 10^{-16}s 45) since there is no such exact thing as Einstein time in the inertial frame. As a consequence, as we do in the present paper, it is convenient to define the coordinate time as a function having an intrinsic indeterminacy, its value being based on a synchronization procedure which is path dependent. The coordinate time is then defined through the corresponding synchronization procedure that in general would be only approximately transitive (small time ambiguity, i.e small holonomy of the related connection, see 10).

We use the following notations: The spacetime metric has signature (+ - - -) and the wedge product is fixed by $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$. The Levi-Civita tensor satisfies, in Minkowski spacetime, $\varepsilon_{0123} = 1$. Sometimes, if no ambiguity arises, we will denote vectors (or 1-forms) like 4-velocity $u^\alpha$, with the short notation $u$.

One can try to extend the simultaneity concept to general relativity. First of all, note that Poincaré defined the simultaneity starting from a well defined notion of space. Indeed, in general, simultaneity is a concept that should be related with respect to a frame. In general relativity this is done introducing a congruence of timelike curves generated by a unitary timelike vector field $u(x)$ (threading). A particle is then defined to be “at rest” in the frame if its worldline belongs to the congruence of timelike curves 27, 32. The space $S$ is the set of timelike curves generated by $u(x)$ and the, time dependent, space metric is

$$ds^2 = (1 + 2(\Phi - \Phi_0))dt^2 - (1 - 2V)[d\theta^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2)] - 2\omega r^2 \sin^2 \theta d\phi dt$$

(2)

where $V$ is the gravitational potential, $\Phi$ is the gravitational plus the centrifugal potential, and $\Phi_0$ is the value of $\Phi$ at the surface of the earth geoid. It is introduced so that the variable $t$ represents the proper time of clocks at rest in the earth surface. The frame here is the one generated by $\partial_t$ and the question is how to assign in practice the label $t$ of the previous formula to the events. As we said, the Einstein simultaneity convention does not work accurately because of the Sagnac effect. The GPS was conceived to do this, but it has some disadvantages that we wish to point out. Unlike the Einstein convention, the GPS is a non local method of obtaining a global simultaneity convention. In fact, it depends on global information like the shape of the geoid, its potential and the position of the satellites with respect to the earth. One has to take into account all of these aspects and their influence on the rate of clocks in the satellites. The Einstein convention is conceptually easier since it treats on the same footing all the points at rest in the frame and does not require any global information in order to be applied. It does not require the spacetime to have particular symmetries, or the spacetime metric to be a solution of Einstein equations. Unfortunately, it does not work accurately in the large but it gives us the hint that these kind of local simultaneity conventions are worth studying. By local simultaneity convention we mean a definition of simultaneity that applies locally, like a rule between neighboring observers. Good local simultaneity conventions are those that are “almost integrable” in the sense that they give rise globally to a distribution of horizontal planes which, in the region under
consideration and if the accuracy of experiments is taken into account, has a negligible holonomy. We thus come to the problem addressed in this paper: finding a local simultaneity convention that exhibits a good global behavior. We shall not solve this problem completely but we will make a first significant step in that direction.

We mention that some authors [1] have argued that the classical 1+3 (threading) approach, taken up here, runs into problems because the Einstein convention, in general, is not transitive. It is for this reason that they propose an alternative 3+1 approach typical of Hamiltonian field theories. The problem, however, is not with the threading approach but rather with the Einstein convention. It is possible to consider local conventions different from Einstein’s as we do in the present paper. Moreover, the threading approach is always in direct contact to observables, a feature that makes it really appealing. This line of research has not yet been explored; indeed, the very definition of local simultaneity convention is given here for the first time. Ultimately, this operational approach to coordinate time may provide new insights into the problem of building coordinates in curved spacetimes for both positioning and navigational purposes [4, 11, 24, 31], particularly in the general case of arbitrary spacetimes.

A word should also be said on the role of constant (or zero) mean curvature spacelike hypersurfaces. Their study arose naturally in numerical relativity [15] since the Cauchy problem simplifies starting from initial data on this kind of hypersurfaces. Moreover, their evolution was also shown to have very useful properties like that of avoiding a class of crushing singularities [16]. Also a number of quite strong results on the existence of such hypersurfaces and foliations appeared in the literature [10, 21, 27] so that it would be natural to consider them as a candidate for an operative definition of simultaneity. However, these hypersurfaces are not derived from a congruence of timelike curves and therefore it is not clear how they relate to the frame concept and whether this relation could be local. For instance, one could try to define the congruence of timelike curves from a foliation of constant mean curvature hypersurfaces instead of deriving from a congruence of timelike curves a spacelike foliation. This approach could have some advantage but faces with the problem that the opposite is needed in practical applications and, in general, only a restricted set of congruences of timelike curves would be generated in that way. In fact, suppose a constant mean curvature foliation is given and define the corresponding congruence of timelike curves as the one generated by the normals to the slices, then only vortex-free congruences would be generated. This gives no hint on how to assign a foliation to congruences with non-vanishing vorticity which is exactly the problem that already the Einstein convention left unsolved. It is for these reasons that constant mean curvature slices will not play a significative role in our study.

Finally, we stress that the determination of a useful simultaneity concept, i.e. of an operative definition of time function, has no consequence at the level of physics laws. This follows from the principle of general covariance which assures that the choice of a suitable time function can, especially in spacetimes with symmetries, simplify the physics equations without however altering their ultimate physical content. Although we shall advocate that a certain local simultaneity convention should be preferred, we shall mean by this only that such convention is more useful in order to assign a “time label” to events while at the level of physics laws nothing changes.

II. MATHEMATICAL FRAMEWORK

In this work we adopt a fully covariant point of view. We will make use of the hydrodynamic formalism developed in [16] (see also [18]) and of some results of gauge theories on fiber bundles [24, 38].

Let $\pi : M \to S$ be the differential projection that to any event $x$ associates the timelike curve generated by $u$, $s = \pi(x)$, passing through it. In this way $M$ acquires a local structure of fibre bundle over $S$. Locally the simultaneity is represented by a projector field,

$$C(x) : TM_x \to VM_x$$

(3)

on the vertical space at $x$ (i.e. the one-dimensional space spanned by $u$). The ker of $C(x)$ selects a horizontal space $HV_x$, $TM_x = VM_x \oplus HM_x$. By definition, we shall say that, events on a small neighborhood of $x$ that lie on the exponential map of $HM_x$ are $C$-simultaneous with respect to the observer of 4-velocity $u$. It can be shown [32] that $C$ is a generalized connection in the formalism of generalized gauge theories [32, 37]. For more on the gauge interpretation of simultaneity see [32].

Def. Local simultaneity convention. A local simultaneity convention is a projector field $C(x) : TM_x \to VM_x$ that depends solely on tensors built from $u^\alpha$, $g_{\alpha\beta}$, $\varepsilon_{\alpha\beta\gamma\delta}$, and whose horizontal planes are spacelike.

This definition follows from the fact that a convention must depend on measurable quantities, and the dependence must be the same everywhere. In this way, an observer investigating the motion of neighboring observers and other local spacetime properties can find all the information required in order to apply the simultaneity convention. In general the simultaneity connection $C$ will therefore be built using the 4-velocity vector, the shear tensor, the expansion...
scalar, the vorticity tensor, the metric, the Riemann tensor and the Levi-Civita antisymmetric tensor. More exotic tensors can, in principle, enter the construction but they should have a clear meaning from an operational point of view.

In this framework the Einstein simultaneity convention is $C^\mu_\nu = P^\mu_\nu \equiv u^\mu u_\nu$. Its operational meaning has been discussed in the introduction.

It is convenient to write $C = \omega \otimes u$, with $\omega$ 1-form over $M$, $\omega(u) = 1$ and $u_\alpha \omega^\alpha > 0$. Then $v \in HM \Leftrightarrow \omega(v) = 0$. The curvature $\hat{R}$ of $C$, is a vertical valued two-form $[\mathbb{R}, \mathbb{R}]$,

$$2\hat{R} = -[C, C]_{FN},$$

where $[,]_{FN}$ is the Frölicher-Nijenhuis bracket. After some calculations, one has more simply

$$\hat{R} = \Omega \otimes u \quad \text{with} \quad \Omega = (d\omega)C^\perp.$$

That is, $\hat{R}_\alpha^\beta = u^\delta(\omega_{\nu;\alpha} - \omega_{\nu;\beta}) (\delta^\alpha_\delta - \omega_\alpha u^\delta)(\delta^\nu_\beta - \omega_\beta u^\nu)$. If $\hat{R} = 0$ the distribution of horizontal planes is integrable and the local simultaneity convention is globally well behaved. Let us introduce

$$v^\eta = -\frac{1}{4} h^\eta_{\nu,\varepsilon} v^\beta \alpha \gamma \omega_\beta \Omega_{\alpha\gamma} = \frac{1}{2} h^\eta_{\nu,\varepsilon} v^\beta \alpha \gamma \omega_\beta \omega_\alpha v^\gamma,$$

it is a kind of generalized vorticity vector for the present non-time orthogonal connection. The inverse formula is

$$\Omega_{\alpha\beta} = -2\varepsilon_{\alpha\beta\gamma} u^\gamma v^\delta.$$

The conditions $\hat{R} = 0$, $\Omega = 0$ and $v = 0$ all state that the distribution of horizontal planes is integrable. Note that the Einstein convention is not globally well behaved since $v$ equals the vorticity vector. What is worse, the non-integrability does not arise from constraints imposed by the spacetime geometry. A kinematical property alone as rotation prevents the simultaneity convention from working in the large. This fact makes it clear that a better convention would be welcome. Observe that the condition $v = 0$, that is $h^\nu_{\nu,\varepsilon} v^\beta \alpha \gamma \omega_\beta \Omega_{\alpha\gamma} = 0$ is equivalent to the more usual Frobenius condition $\varepsilon_{\nu,\beta,\alpha} \omega_\beta \Omega_{\alpha\gamma} = 0$; indeed $z^\nu = \varepsilon_{\nu,\beta,\alpha} \omega_\beta \Omega_{\alpha\gamma}$ is orthogonal to $\omega_\nu$ and since $\omega_\nu$ is timelike, $z^\nu$ is spacelike and then $z^\nu = 0$ iff $h^\nu_{\nu,\varepsilon} v^\beta = 0$.

The ker of $\omega$ coincides with the ker of

$$u_\alpha' = (\omega^\beta \omega_\beta)^{-1/2} \omega_\alpha,$$

which can be interpreted as a 4-velocity field $u'^\alpha u'^\alpha = 1$. Hence the $C$-simultaneity convention is equivalent to the Einstein convention with respect to observers moving with a 4-velocity $u'$.

**A. Stationary case**

If the congruence of timelike curves is generated by a Killing vector field, another way to introduce a curvature follows from the mathematical formulation of gauge theories as connections on principal fiber bundles. Indeed over $M$ acts the one parameter group $T_1$ of translations $\phi_t$ generated by the Killing vector field $k$, $u = k/\sqrt{k \cdot k}$. Locally one recovers the structure of a principal fiber bundle, and the base $S$ can be interpreted as the quotient of $M$ over $T_1$. Let $\chi = \sqrt{k \cdot k}$ then, given a local simultaneity convention, consider the 1-form

$$\sigma = \chi^{-1}\omega. \quad (9)$$

It is clear that (a) $\sigma(k) = \chi^{-1} \omega(k) = 1$, and that (b) $L_k \sigma = 0$. Indeed, in order to prove (b) note that $L_k \chi = 0$ and $L_k \omega = 0$ since $\omega$ depend on $u^\alpha$, $g_{\alpha\beta}$, $\varepsilon_{\alpha\beta\gamma\delta}$, and these tensors have a vanishing Lie derivative. Any 1-form field over the fiber bundle $M$ that satisfies (a) and (b) is by definition a connection in the sense of principal bundle gauge theories $^{26}$. It is only a slightly different way of defining a connection with respect to that of generalized gauge theories $^{32, 33, 37}$. Whatever definition is used, the connection always defines a distribution of horizontal planes given by ker $\sigma = \ker \omega$. The curvature is defined by

$$\Sigma_{\mu\nu} = (d\sigma)_{\mu\nu}. \quad (10)$$

We use the formula $L_k = di_k + i_k d$ on $\sigma$

$$L_k \sigma = d\sigma(k) + i_k d\sigma, \quad (11)$$
but the first two terms vanish and therefore $\sum_{\mu\nu} u^\nu = 0$. We search a relation between the two definitions of curvature $\Sigma$ and $\Omega$. Taking the exterior derivative of $\chi^{-1}\omega$ and projecting on the horizontal plane we obtain
\[
(d\sigma)C_{\perp} = (d\chi^{-1})C_{\perp} \wedge \omega C_{\perp} + \chi^{-1}d\omega C_{\perp},
\]
but $\omega C_{\perp} = 0$ and since $i_u d\sigma = 0$, we have $(d\sigma)C_{\perp} = d\sigma$, and finally,
\[
\Omega = \chi \Sigma.
\]
$\Sigma = d\sigma = 0$ is another integrability condition for the distribution of horizontal planes. It is particularly convenient in this form since it implies that (this is always true locally and holds globally in a simply connected spacetime) there is a time function $t : M \to \mathbb{R}$ such that $\sigma = dt$. We shall call this time, $C$-Killing time, since on a worldline of the reference frame the rate of this time equals the one given by the Killing vector field ($dt(k) = 1$) and the hypersurfaces of simultaneity are those of the C-simultaneity ($\ker dt = \ker \omega$).

### III. STATIONARY FRAMES

There is little hope of finding a globally well behaved simultaneity convention that works in every case. In the present paper we tackle the problem introducing some restrictions

- **Stationarity.** We assume this property mainly for technical reasons: it allows us to use repeatedly the Killing vector lemma and to simplify the expressions. Thus, we shall assume that the congruence of timelike curves is generated by a timelike Killing vector field $k$. This assumption is not, however, so restrictive. Firstly, we are mostly interested in simultaneity conventions to be applied at the exterior of massive objects. Most of them are already studied in the stationary case i.e. when the dynamical effects have come to an equilibrium. Secondly, the concept of frame is particularly sharp and clear in the stationary case since there, even a time independent space metric can be defined. It is, therefore, a natural starting point for more complicated investigations.

- $v^\alpha \neq 0$. We relax the integrability condition $v^\alpha = 0$ since, in practice, we need conventions that lead to distributions of horizontal planes which are approximately integrable at least over the scales where a global time variable is required. To this end we weaken that condition looking for conventions having a curvature proportional to the Riemann tensor
\[
v^\alpha \propto R^{\gamma}_{\delta\mu\nu}.
\]
In this way at least in a weak field limit the distribution of horizontal planes is integrable. The holonomy of a connection is proportional to the area enclosed by a closed curve and the curvature. Over a length scale $L$ it is given by
\[
\Delta \sim L^2 R T,
\]
where $R$ is the typical value of the non-zero Riemann coefficients and $T$ is the typical value for the tensor which contracts with the Riemann tensor to give the curvature $v$. This should be compared with the relation that follows from the Einstein convention
\[
\Delta \sim L^2 w.
\]
Thus we are looking for a connection that, at least in the weak field limit $R \to 0$, is much more well behaved than Einstein’s. Moreover, the lack of integrability here is caused by the spacetime geometry (Riemann tensor) and not by the kinematical properties of the reference frame (vorticity). In this sense the lack of integrability is much more natural and acceptable.

We recall some basic definitions that will be needed in the statement of a theorem. Let $k$ be the timelike Killing vector field. The Killing vector lemma states that the following relation holds
\[
k_{\alpha;\beta;\gamma} = -R^\gamma_{\delta\alpha\beta\gamma} k^\delta.
\]
For a timelike Killing vector field the 4-velocity is $u^\alpha = k^\alpha/\sqrt{k \cdot k}$, the acceleration is
\[
a^\alpha = u^\beta_{,\beta} u^\alpha = -(\ln \chi)_{,\beta} h^{\beta\alpha} = -(\ln \chi)^{;\alpha},
\]
and the shear tensor and the expansion scalar vanish. Let us define (this is Eq. (19) for the Einstein convention), 
\[ \sigma^e = k / (k \cdot k), \]
the vorticity tensor can be written
\[ w_{\alpha \beta} = u_{[\gamma \delta]} h^\gamma h^\delta = \frac{k_{\alpha \beta}}{k} + [u_{\alpha} a_{\beta} - u_{\beta} a_{\alpha}] = \chi^\gamma_{[\alpha} a_{\beta]}, \]
and the vorticity is defined as \( w^\alpha = \frac{2}{\sqrt{k}} e^{\alpha \gamma \delta} u_{\beta} w_{\gamma \delta} \) having inverse formula \( w_{\alpha \beta} = \epsilon_{\alpha \beta \gamma \delta} u^\gamma u^\delta. \) Finally, we define the 1-form field \( m_\alpha = w_{\alpha \beta} a^\beta. \) With the short notation \( m^2 \) we denote the positive number \( a^2 = -a^\alpha a_\alpha, \) and analogously for \( w^a \) and \( m_\alpha. \) The vector \( m^a \) is perpendicular to \( u^a, w^a \) and \( \alpha^a. \) The tetrad \( \{w^a, w^\alpha, \alpha^a, m^n\} \) is a base in those events where \( m^a \neq 0. \) Note that \( m^a \neq 0 \Rightarrow m^2 \neq 0 \) where \( m^2 = a^2 w^2 \sin^2 \theta \) and \( \theta \) is the angle between the acceleration and the vorticity vector. In the same events any local simultaneity convention takes the form
\[ \omega_\alpha = u_\alpha + \psi^m(x) m_\alpha + \psi^a(x) a_\alpha + \psi^w(x) w_\alpha, \]  
(19)
for suitable functions \( \psi^m, \psi^a, \psi^w. \) From the definition of local simultaneity convention it follows moreover that \( \psi^m, \psi^a, \psi^w, \) depend on the acceleration \( a, \) the vorticity \( w, \) the angle \( \theta, \) and on more exotic scalars (note that in a stationary frame the possibilities are reduced since the shear and the expansion vanish). The following theorem explores the consequences of a dependence on \( a, w \) and \( \theta. \)

**Theorem.** In a stationary spacetime let \( k \) be a timelike Killing vector field and set \( u = k / \sqrt{k \cdot k}. \) Let \( U \) be the open set \( U = \{x : m(x) > 0 \text{ and } a(x) \neq w(x)\}. \) Consider in \( U \) the 1-form
\[ \omega_\alpha = u_\alpha + \psi^m(x) m_\alpha + \psi^a(x) a_\alpha + \psi^w(x) w_\alpha. \]  
(20)
(i) Let \( \psi^m, \psi^a, \psi^w, \) be \( C^1 \) functions dependent only on \( a, w \) and \( \theta. \) Then, regardless of the stationary spacetime considered, the connection is timelike in \( U \) (and hence it is a simultaneity connection in \( U \)) and has a curvature proportional to the Riemann tensor in \( U \) if and only if
\[ \psi^m = \tilde{\psi}^m = \frac{a^2 + w^2 - \sqrt{(a^2 + w^2)^2 - 4 m^2}}{2 m^2}, \]  
(21)
and there is a \( C^1 \) function \( b : \mathbb{R} \rightarrow \mathbb{R} \) of the variable \((aw \cos \theta)/(a^2 - w^2))\) such that for \( m > 0 \) and \( a \neq w \) the following inequality holds
\[ \frac{\sqrt{(a^2 + w^2)^2 - 4 m^2}}{2 w^2} - \frac{(a^2 - w^2)}{2 b^2} + \frac{m^2}{w^2} \psi^a^2 + m^2 \tilde{\psi}^m < 1, \]  
(22)
and
\[ \psi^w = -\frac{a \cos \theta}{w} \psi^a + \frac{b}{w^2} \left( \frac{\sqrt{(a^2 + w^2)^2 - 4 m^2}}{2} - \frac{(a^2 - w^2)}{2} \right)^{1/2}. \]  
(23)
Thus, in order to have a simultaneity connection on \( U \) having a curvature proportional to the Riemann tensor, it suffices to choose a pair \((\psi^a, b)\) so as to satisfy the inequality \(22. \) Indeed \( \psi^m \) is fixed by \(21\) while \( \psi^w \) follows from \( b \) and \( \psi^a.\)

(ii) Consider in \( U \) the local simultaneity convention defined by \( \tilde{\psi}^a = \psi^w = 0 \)
\[ \tilde{\omega}_\alpha = u_\alpha + \frac{a^2 + w^2 - \sqrt{(a^2 + w^2)^2 - 4 m^2}}{2 m^2} m_\alpha, \]  
(24)
It is timelike, i.e. satisfies \(22, \) and can be extended by continuity to the set \( C = A - B \) where, \( A = \{x : a^2 + w^2 > 0\}, B = \{x : a = w \neq 0 \text{ and } \theta = \pi/2\}, \) by defining, \( \tilde{\omega}_\alpha = u_\alpha, \) in those points where \( m = 0. \) Its curvature is
\begin{align}
\tilde{\omega}^\mu &= \frac{\tilde{\psi}^m}{2} h_{\nu}^{\gamma} e^{\nu \beta \gamma \delta} u_{\beta} R_{\gamma \delta \mu \sigma} \left( \frac{m_\alpha}{\sqrt{(a^2 + w^2)^2 - 4 m^2}} (u^\delta u^\mu a^\sigma) + 2 \tilde{\psi}^m (u^\delta u^\mu a^\sigma) \right) \\
&+ m_\alpha \left( \frac{2 \tilde{\psi}^m}{\sqrt{(a^2 + w^2)^2 - 4 m^2}} (u^\delta m^\mu a^\sigma + u^\delta w^\mu w^\sigma_m^\tau + u^\delta w^\mu w^\alpha - \delta^\mu_\alpha u^\delta a^\sigma) \right). 
\end{align}  
(25)
Motivated by (ii) we shall focus our study on the following simultaneity convention

**Def.** The $\bar{C}$-simultaneity convention is the convention defined on the set $C = A - B$.

Note that the conditions imposed on the sets in the statement of the theorem are not restrictive and arise only for technical reasons. For instance, the set $B$ is empty in most cases and does not have any particular physical interpretation. In practical applications, the $\bar{C}$-simultaneity will be defined on the whole set $A = \{ x : a^2 + w^2 > 0 \}$. Moreover, in most cases, the set $A$ will be the entire spacetime. Indeed, for spacetimes having $w = 0$ on open sets the search for an integrable simultaneity convention does not make much sense since the Einstein simultaneity convention is already integrable. In the theorem there are therefore only four physical assumptions: the spacetime is stationary; the connection scalars depend only on $a$, $w$ and $\theta$ (it means that in order to apply the convention only the acceleration and the vorticity should be locally measured); the connection has spacelike horizontal planes (otherwise it would not be a simultaneity connection); its curvature is proportional to the Riemann tensor (it allows to ignore the holonomy in the weak field limit).

Remarkably, $\bar{C}$-simultaneity follows almost uniquely from these geometrical requirements. While the observers have the freedom to choose a local simultaneity convention, the requirement of being almost integrable in the weak field limit imposes strong constraints on its actual expression. The geometry tells the observers what simultaneity have the freedom to choose in the weak field limit. In the theorem there are therefore only four physical assumptions: the spacetime is stationary; the connection scalars depend only on $a$, $w$ and $\theta$ (it means that in order to apply the convention only the acceleration and the vorticity should be locally measured); the connection has spacelike horizontal planes (otherwise it would not be a simultaneity connection); its curvature is proportional to the Riemann tensor (it allows to ignore the holonomy in the weak field limit).

By construction the curvature is proportional to the Riemann tensor. In order to understand the physical implications of Eq. some comments are in order. It is not difficult to see that the following inequalities hold in $C$

$$m\bar{\psi}^m < 1,$$

$$\left(a^2 + w^2\right)\bar{\psi}^m < 2.$$  \hfill (27)

Using them and defining $v^2 = -\bar{v}^\eta \nu_\eta$, from the expression for $\bar{v}^\eta$ we find that

$$\bar{v} \lesssim 10 \mathcal{R} \sqrt{a^2 + w^2 \over \left( a^2 - w^2 \right)},$$  \hfill (28)

where $\mathcal{R}$ is the value of the greatest non vanishing component of the Riemann tensor in a orthonormal base. Restoring $c$ one sees that $a^2$ is suppressed by an additional factor $c^2$ with respect to $w^2$. In typical practical situations having a non-vanishing vorticity, it therefore happens that $w^2 \gg a^2$ and the previous equation reads

$$\bar{v} \lesssim 10 {\mathcal{R} \over w}.$$  \hfill (29)

We recall that $v = w$ for the Einstein convention, thus in this case the $\bar{C}$-simultaneity connection is better than Einstein’s if

$$\mathcal{R} \ll w^2.$$  \hfill (30)

Now, the force experienced between two points at rest in the frame is expected to have a tidal component of the form $\mathcal{R}\delta x$, where $\delta x$ is the displacement, and a centrifugal component of the form $w^2\delta x$. Thus, roughly speaking, the $\bar{C}$-simultaneity convention is better than Einstein’s in those spacetime regions where tidal forces are weaker than centrifugal forces.

We immediately see that it is not convenient to use the $\bar{C}$-simultaneity at the surface of the earth or at the surface of a spinning planet. In fact $\mathcal{R}$ is, in geometrized units, of the order of $M/R^3$, where $R$ is the planet radius. Consider a body on the surface, at a distance $d \sim R \sin \theta$ from the axis of rotation. The component of the gravitational force directed towards the axis is greater than the centrifugal force. The classical equilibrium of forces gives

$$\frac{M}{R^2} \sin \theta > w^2d,$$  \hfill (31)

otherwise the body would not stay on the surface of the planet. This last equation implies $w^2 < \mathcal{R}$ and therefore the Einstein convention should still be preferred on the surface of a spinning planet. The $\bar{C}$-simultaneity appears to be a very good convention but it should be applied only in regions where the gravitational field is particularly weak. It is convenient to use the $\bar{C}$-simultaneity at a radius greater than that of geostationary orbits. In that region, as the radius increases, the holonomy over a path $\gamma$ of constant radius around the planet decreases. Indeed, the holonomy...
scales as \((M/(wR^3))S\), where \(S\) is the area of a surface such that \(\partial S = \gamma\), but \(S \sim 2\pi R^2\) and hence the holonomy scales as \(M/(wR)\). On the contrary, the holonomy of the Einstein convention increases since it scales as \(wR^2\).

We have calculated the curvature with the assumption of stationarity but any local simultaneity convention makes sense even without that assumption. Thus we can consider \(\bar{C}\)-simultaneity even in non-stationary cases. It remains well defined and timelike as long as \(w, a,\) and \(\theta\) define a point in the set \(C\). In flat spacetime it should be preferred over Einstein’s. In fact, if the curvature of the Einstein convention vanishes in a open set \(O, w = 0\), the same happens for the convention \(\bar{C}\), since in \(O\) the two conventions coincide. On the contrary, there are cases like the rotating platform such that the curvature of \(\bar{C}\) vanishes but \(w \neq 0\).

IV. A SIMPLE EXAMPLE: THE ROTATING PLATFORM

The theorem implies that in flat spacetime the horizontal planes of \(\bar{C}\) are exactly integrable. As a consequence stationary frames in flat spacetime bring an intrinsic privileged global simultaneity convention. This feature is quite surprising. In the gauge interpretation of simultaneity any global simultaneity convention is a section of the principal bundle and therefore we expect global simultaneity conventions to be equivalent since we can change from one to the other by gauge transformations. However, it should be taken into account that here we have a special ingredient bundle and therefore we expect global simultaneity conventions to be equivalent since we can change from one to the stationary frames in flat spacetime bring an intrinsic privileged global simultaneity convention. This feature is quite sense even without that assumption. Thus we can consider \(\bar{C}\)-simultaneity in the rotating platform case. The rotating platform has always provided a natural theoretical laboratory for the study of rotating spacetimes. There is, for instance, a wide array of literature on the definition of space metric and on the problem of clock synchronization for the rotating platform metric. The rotating platform is also interesting in practice since the Sagnac effect can be recognized as a universal effect of rotating frames starting from an intrinsic description of the effect in the rotating platform case. It is nothing but the manifestation of the non-vanishing holonomy for the Einstein simultaneity convention. Our task here is to show that observers on a rotating platform using the \(\bar{C}\)-simultaneity convention succeed in constructing a global coordinate time. As will be explained in detail in section VII, the operational procedure associated to a local simultaneity connection, as the name suggests, is local and therefore the observers on the rotating platform may even ignore their being on a rotating platform. This does not prevent the application of \(\bar{C}\)-simultaneity convention from succeeding.

The metric is obtained from the flat metric in cylindric coordinates,

\[
ds^2 = dt^2 - d\rho^2 - \rho^2 d\theta^2 - dz^2,
\]

after the change of coordinates, \(\theta \to \theta + \omega t\), where \(c = 1\) and \(\omega\) is a constant

\[
ds^2 = (1 - \rho^2 \omega^2)dt^2 - d\rho^2 - \rho^2 d\theta^2 - 2\rho^2 \omega d\theta dt - dz^2,
\]

or

\[
ds^2 = (1 - \rho^2 \omega^2)(dt - \frac{\rho^2 \omega}{1 - \rho^2 \omega^2} d\theta)^2 - d\rho^2 - \frac{\rho^2 d\theta^2}{1 - \rho^2 \omega^2} - dz^2.
\]

Worldlines of equation \(x^i = const\). belong to the congruence of timelike curves, and \(\partial_t\) is the timelike Killing vector field. We shall limit ourselves to the open set \(\rho \omega < 1\), since in this set the Killing vector field is timelike.

The scalar field is \(\chi^2 = (1 - \rho^2 \omega^2)\), and the acceleration is

\[
a_\alpha dx^\alpha = -\frac{1}{2}d\ln \chi^2 = \frac{\omega^2 \rho}{1 - \rho^2 \omega^2} d\rho,
\]

its norm being \(a^2 = \omega^4 \rho^2 / (1 - \rho^2 \omega^2)^2\). From the metric we find

\[
\sigma_\alpha dx^\alpha = dt - \frac{\rho^2 \omega}{1 - \rho^2 \omega^2} d\theta,
\]

the vorticity tensor is

\[
w_{\alpha\beta} dx^\alpha \otimes dx^\beta = \frac{\chi}{2} d\sigma = \frac{\omega \rho}{(1 - \rho^2 \omega^2)^{3/2}} d\rho \wedge d\theta,
\]
and using \( g^{\theta\theta} = (1 - \rho^2 \omega^2)/\rho^2 \), we obtain \( w^2 = \omega^2/(1 - \rho^2 \omega^2)^2 \). The 1-form \( m_\alpha \) is

\[
m_\alpha dx^\alpha = \frac{\omega^2 \rho^2}{(1 - \rho^2 \omega^2)^{5/2}} d\theta, \tag{39}
\]

its norm being \( m^2 = \omega^2 \rho^2/(1 - \rho^2 \omega^2)^4 \). We see immediately that \( m^2 = a^2 w^2 \) hence the angle between \( a \) and \( w \) is \( \pi/2 \). In this case the formula for \( \bar{\psi}^m \) simplifies to

\[
\bar{\psi}^m = \frac{\min(a^2, w^2)}{m^2}, \tag{41}
\]

but since in the region under consideration \( \rho \omega < 1 \) we have \( \min(a^2, w^2) = a^2 \). Using these formulas we obtain finally,

\[
\bar{\omega}_\alpha dx^\alpha = u_\alpha dx^\alpha + \bar{\psi}^m m_\alpha dx^\alpha = \chi dt, \tag{42}
\]
or

\[
\bar{\sigma} = dt. \tag{43}
\]

Thus \( \bar{C} \)-simultaneity coincides with Einstein simultaneity of the inertial frame we started with and the \( \bar{C} \)-Killing time is \( t \). The hypersurfaces of simultaneity have equation \( t = \text{const.} \) From a local simultaneity convention we have been able to recover the most natural global simultaneity convention for the rotating platform.

V. ROTATING PSEUDOCYLINDRICAL COORDINATES

In this section we consider a non trivial example in flat spacetime. An integral curve of a Killing vector field can be described, without reference to the neighboring integral curves, using the Serret-Frenet equation. Because of the translational invariance of the spacetime, the acceleration, torsion and hypertorsion must be constant in absolute value. Curves having this property have been classified, for the case of flat spacetime, by Synge [44] and Letaw [30]. Then Letaw and Pfautsch [31] found, for any Killing vector field in flat spacetime, an adapted coordinate system in the sense that \( k = \partial_t \). As it is well known, a gauge transformation transformation \( t \to t + \phi(x^i) \) preserves the relation \( k = \partial_t \). Letaw and Pfautsch fix the gauge so as to simplify the expression of the metric. Their choice of coordinate time implies a simultaneity convention, thus their simultaneity convention arises from a criteria of simplicity of the metric element. The less trivial Killing vector field having \( m \neq 0 \) that they consider leads to the rotating pseudocylindrical coordinates. The Minkowski metric in these coordinates is

\[
ds^2 = (\xi^2 - r^2 \Omega^2) dt^2 - d\xi^2 - dr^2 - r^2 d\phi^2 - 2r^2 \Omega d\phi dr, \tag{44}
\]

and \( k = \partial_r \). It can be rewritten in the form

\[
ds^2 = (\xi^2 - r^2 \Omega^2)(dt - r^2 \Omega \xi d\phi)^2 - d\xi^2 - dr^2 - (r^2 + \frac{r^4 \Omega^2}{\xi^2 - r^2 \Omega^2}) d\phi^2. \tag{45}
\]

We restrict ourselves to the open set where \( \chi^2 = \xi^2 - r^2 \Omega^2 > 0 \), since there the Killing vector field is timelike. The acceleration is

\[
a_{\alpha} dx^\alpha = -\frac{1}{2} d \ln \chi^2 = \frac{\Omega^2 r}{\xi^2 - r^2 \Omega^2} dr - \frac{\xi}{\xi^2 - r^2 \Omega^2} d\xi, \tag{46}
\]

its norm being \( a^2 = (\Omega^4 r^2 + \chi^2)/(\xi^2 - r^2 \Omega^2)^2 \). From the metric we also easily see that

\[
\sigma^\alpha_{\alpha} dx^\alpha = dr - \frac{r^2 \Omega}{\xi^2 - r^2 \Omega^2} d\phi. \tag{47}
\]

The vorticity tensor is

\[
w_{\alpha \beta} dx^\alpha \otimes dx^\beta = -\frac{\chi}{2} d\sigma = \frac{\Omega r \xi^2}{(\xi^2 - r^2 \Omega^2)^{3/2}} dr \wedge d\phi - \frac{\Omega r^2 \xi}{(\xi^2 - r^2 \Omega^2)^{3/2}} d\xi \wedge d\phi, \tag{48}
\]
and taking into account that \( g^{\phi\phi} = - (\xi^2 - r^2 \Omega^2)/(r^2 \xi^2) \), we find
\[
w^2 = \Omega^2 (\xi^2 + r^2)/(\xi^2 - r^2 \Omega^2)^2.
\]
The only non-vanishing component of \( m_\alpha \) is
\[
m_\phi = \frac{\Omega r^2 \xi^2}{(\xi^2 - r^2 \Omega^2)^{5/2}(\Omega^2 + 1),}
\]
and its norm is
\[
m^2 = \Omega^2 r^2 \xi^2 (\Omega^2 + 1)^2/(\xi^2 - r^2 \Omega^2)^4.
\]
Plugging these results into the expression for \( \tilde{\psi}^m \) we find
\[
\tilde{\psi}^m m_\phi = \chi^{-1} \Omega r^2
\]
and finally
\[
\tilde{\omega}_\alpha dx^\alpha = u_\alpha dx^\alpha + \tilde{\psi}^m m_\alpha dx^\alpha = \chi dr.
\]
Hence, the \( \bar{C} \)-Killing time is the parameter \( \tau \) chosen by Letaw and Pfautsch. This coincidence arises from the fact that the \( \bar{C} \)-simultaneity convention is a natural one. It follows solely from the metric and the Killing vector field, so it is expected to simplify the metric in a coordinate system adapted to \( k \) whose time coordinate induces the \( \bar{C} \)-simultaneity convention. Since Letaw and Pfautsch looked for a gauge that simplifies the metric it is not surprising that they chose the \( \bar{C} \) gauge.

VI. SPACE-TIME SPLITTING AND GAUGE FIXING

Usually stationary metrics are written in the form
\[
ds^2 = \chi^2 (dt + A_i(x^i)dx^i)^2 - dl^2,
\]
where \( dl^2 \) is the space metric \( \mathbf{I} \), \( k = \partial_t \), and \( x^i \) are coordinates on \( S \). The field \( A_i \) is not uniquely determined because of the gauge transformation
\[
t \rightarrow t + \phi(x^i), \quad A_i \rightarrow A_i - \partial_i \phi.
\]
In a region having a small Riemann tensor we suggest to write the metric in the following form
\[
ds^2 = (u_\alpha dx^\alpha)^2 - dl^2
\]
\[
= \chi^2 (\bar{\sigma} - \chi^{-1} \tilde{\psi}^m m_i dx^i)^2 - dl^2.
\]
If the holonomy is sufficiently small we can write, with abuse of notation, \( \bar{\sigma} = dt \), where \( t \) is the \( \bar{C} \)-Killing time. Of course, this equation is understood as an approximation as long as the Riemann tensor differs from zero. Finally,
\[
ds^2 = \chi^2 (dt - \chi^{-1} \tilde{\psi}^m m_i dx^i)^2 - dl^2, \quad \text{(Approximation)}.
\]
Hence, we have obtained a gauge fixing of the simultaneity freedom. The gauge potential becomes
\[
A_i(x^i) = - \chi^{-1} \tilde{\psi}^m m_i.
\]
The advantage of Eq. \[55\] is that \( t \), being the \( \bar{C} \)-Killing time, is defined through a local simultaneity convention and therefore has a clear operational meaning. In theory, any global simultaneity convention can be used by the observers: the change between different global simultaneity convention being given by Eq. \[52\]. However, the observers should also study how to characterize a global simultaneity convention operationally. We argued that convenient procedures make use of an almost integrable local simultaneity convention. This select a global simultaneity convention among the many in principle available thus leading to the gauge fixing studied in this section. We shall see in the next section how to find, in practice, the value of \( C \)-Killing time for a given event of spacetime.

VII. COORDINATE TIME IN PRACTICE

In this section we describe how to build a coordinate time in a stationary spacetime using a \( C \)-simultaneity connection, the letter \( C \) standing for the convention adopted. The procedure will be entirely local, so we have only to describe what an observer \( O \) at rest in the frame has to do, and which information \( O \) needs from neighboring observers in order to assign the correct coordinate time to the events happening in his/her worldline. We assume that in the spacetime region under consideration the Riemann tensor is sufficiently small so that the holonomy becomes
negligible. The coordinate time we wish to build with the local procedure is therefore the C-Killing time. We also assume \( m \neq 0 \) in the region under consideration. The 1-form defining the C-simultaneity convention can be written

\[ \omega_\alpha = u_\alpha + \psi n_\alpha \]  

(57)

where both the scalar field \( \psi(x) \) and the normalized spacelike vector field \( n_\alpha(x) \) are observables depending on the acceleration, vorticity, Riemann tensor and on other local observables. Note that the Killing vector field \( k = \partial_t \) is defined only up to a constant factor. As a consequence, the C-Killing time \( t \) we wish to recover from local measurement is defined only up to an affine transformation \( t \to at + b \). To remove this arbitrariness, we fix its value in a point \( p \) of space by saying that in that point C-Killing time coincide with proper time. Of course, the choice of \( p \) is arbitrary. Consider a light beam sent from \( p \) to a point \( q \) of \( S \). The observer in \( q \) detects a redshift \( \chi(p) = 1 \)

\[ 1 + z(p, q) = \chi(q). \]  

(58)

Thus the C-Killing time of an event happening in \( q \)'s worldline is related the the proper time of a clock in \( q \) by

\[ t = \frac{\tau + b(q)}{1 + z(p, q)} \]  

(59)

the constant \( b \) depending on the synchronization of \( q \)'s clock. Let \( q' \in S \) be another point in a small neighborhood of \( q \). Let \( Z^\alpha \) be their spacelike infinitesimal displacement \( (Z \cdot u = 0 \) and \( q' \) is the exponential map of \( Z \) of base point \( q \). Since the frame is stationary, \( L_k Z = 0 \). Now suppose the C-simultaneity is almost integrable so that we can ignore the non-vanishing holonomy if the sensitivity of the measurements apparatus is taken into account. It means that there is a natural gauge fixing for this convention analogous the the one studied for the C-simultaneity connection in the previous section

\[ ds^2 = \chi^2(dt - \chi^{-1}\psi n_i dx^i)^2 - dl^2. \]  

(Approximation).

(60)

Here \( dl \) is the radar distance and its value \( D = \sqrt{dl(Z, Z)} \) between \( q \) and \( q' \) can be easily measured in the usual way by sending a light beam in a round trip between the two points, and by taking half the round trip (proper) time. Now consider a light beam sent from \( q \) to \( q' \). From the previous equation we find that the C-Killing interval time the light beam needs to reach \( q' \) is

\[ \delta t = \frac{D + \psi n_i Z^i}{\chi(q)}. \]  

(61)

In terms of proper time \( (b = 0) \)

\[ \delta \tau = \frac{\delta \tau}{\chi} - \frac{\tau}{\chi^2} \delta \chi \]  

(62)

thus

\[ \delta \tau = D + \frac{\tau}{\chi} \delta \chi + \psi n_i Z^i. \]  

(63)

Let us introduce the normalized vector \( \hat{z} \) such that \( Z = D \hat{z} \), and call \( \alpha \) the angle between \( n \) and \( \hat{z} \), i.e. \( -\hat{z}_i n^i = \cos \alpha \). Note, moreover, that \( \delta \chi/\chi \) is the redshift \( z(q, q') \) between \( q \) and \( q' \). We have therefore

\[ \delta \tau = D + \tau z(q, q') - \psi D \cos \alpha \]  

(64)

If we consider a light beam sent from \( q' \) to \( q \) the last two terms change sign

\[ \delta \tau' = D - \tau z(q, q') + \psi D \cos \alpha \]  

(65)

In order to relate the proper time of clocks to C-Killing time, we first define the proper setting of clocks.\[\text{Def.}\]

Two neighboring clocks placed in \( q \) and \( q' \) have been properly set according to C-simultaneity if the proper time it takes light to go from \( q \) to \( q' \) equals the proper time it takes light to go from \( q' \) to \( q \) plus a quantity \( 2\tau z(q, q') = 2\psi D \cos \alpha \) where \( D \) is the radar distance between the two points, \( z(q, q') \) is the redshift from \( q \) to \( q' \), \( \psi \) is the observable scalar and \( \alpha \) is the angle between the direction \( qq' \) and the observable \( n \).
By "proper time" we mean here the difference of proper times at the events of arrival and departure of the light beam as measured by the local clocks. The above definition is automatically transitive in the hypothesis of small holonomy because of Eq. \ref{eq:holonomy}. For $C$-simultaneity it reads

**Def.** Two neighboring clocks placed in $q$ and $q'$ have been properly set according to $C$-simultaneity if the proper time it takes light to go from $q$ to $q'$ equals the proper time it takes light to go from $q'$ to $q$ plus a quantity $2\tau z(q, q') - 2\psi^m D \cos \alpha$ where $D$ is the radar distance between the two points, $z(q, q')$ is the redshift from $q$ to $q'$, $\psi^m$ is the observable given by Eq. \ref{eq:psi} and $\alpha$ is the angle between the direction $qq'$ and $m$.

If the clocks are properly set then $C$-Killing time of an event in $q$'s worldline is related to the proper time of a clock placed at $q$ by the simple relation $t = \tau / (1 + z(p, q))$. Changing $p$ changes $t$ only by a global factor. Note that the hypersurfaces of $C$-simultaneity are given by $t = \text{const.}$ and not by $\tau = \text{const.}$, moreover note that the term $2\tau z(q, q')$ vanishes if $q$ and $q'$ lie on the same equipotential slice as it is the case at the surface of a spinning planet.

**VIII. CONCLUSIONS**

In this paper we have introduced the concept of local simultaneity convention and have began the study of conventions alternative to the one by Einstein and Poincaré. We have focused the study to conventions whose curvature is proportional to the Riemann tensor, as they are expected to be useful in the weak field limit. Remarkably, the simultaneity convention was almost completely determined by imposing the condition that $\omega_a$ should be timelike and well defined in a sufficiently large domain. Indeed, $\psi^m$ was completely determined, while $\psi^a$ and $\psi^m$ were constrained. It then became very natural to consider the $C$-simultaneity convention which was obtained imposing $\psi^a = \psi^m = 0$. It shares some good feature as its domain of definition includes events where $m = 0$ and reduces to the Einstein convention in those events. Moreover, it differs only slightly from the Einstein convention but it is much more well behaved in flat spacetime or when the Riemann tensor is particularly weak.

We have seen two examples in flat spacetime where $C$-Killing time have turned out to coincide with the time coordinate chosen by Letaw and Pfautsch \cite{31} on the basis of elegance and simplicity.

Unfortunately, it is of no advantage in the most important application, the synchronization at the surface of a rotating planet, since the Riemann tensor is not sufficiently weak. In this respect we note that we are just at the beginning of the study of local simultaneity conventions. It could be that, in introducing a non vanishing $\psi^a$ some components of the curvature of the connection could be removed and a new convenient simultaneity convention could be discovered. We believe that our approach shares a number of features that makes it preferable over other approaches:

(a) It is coordinate independent as we always use a transparent covariant notation. Moreover, it is local and operationally clear as quantities like the acceleration or the vorticity have a transparent operational meaning.

(b) It does not rely on Einstein equations or their exact solutions indeed, contrary to other approaches, we have no need to assume a special form of the metric in order to apply a local simultaneity convention, and we do not need to develop a different method of synchronization for any different spacetime considered. Moreover, the coordinate time found is universal as the operational definition of $C$-Killing time makes it possible to consider $C$-Killing time in different spacetimes.

(c) It does not rely on experimental models of the distribution of matter (the geoid in the earth case). This is very important since classical, relativistic or general relativistic effects are already taken into account in the local simultaneity convention. We do not have to change the method of synchronization if we discover, for instance, that the metric is not the one which was supposed to be. Once we have proved theoretically that a local simultaneity convention works, it works independently of the global information, like the metric, that we may or may not have.

(d) Finally, it does not require particular spacetime symmetries apart from that of stationarity (local simultaneity conventions make sense also for non-stationary spacetimes but the study of this general case appears to be difficult).

In the end we wish to comment on the theorem. We have argued in the introduction that it would be preferable to obtain a global simultaneity convention starting from a local simultaneity convention. The message of the theorem is that globally well-behaved local simultaneity conventions are not arbitrary but are severely constrained by the spacetime geometry. Remarkably, it seems that Nature tells the observers what should be the convention for the diffusion of time. In principle, global simultaneity conventions are equivalent as they are related by gauge transformations; however, if one tries to characterize them operationally through a local simultaneity convention, only a subset of them survives. In particular, from our definition of local simultaneity convention it follows that in flat spacetime and for an inertial frame only the Einstein simultaneity convention survives since there is only one form $\omega_a$ that can be built using $u_\alpha$, $\eta_{\alpha\beta}$ and $\epsilon_{\alpha\beta\gamma\delta}$ and it is just $\omega_\alpha = u_\alpha$. 


We wonder whether there is some formula which generalizes $\tilde{c}$-simultaneity so as to enlarge the application domain of local simultaneity conventions. We believe this issue to be worth studying particularly along the lines introduced in the present paper.

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Appendix: proof of the theorem

We collect here some useful formulas that will be needed later. They are just for reference and can be proved straightforwardly after some calculations. We recall that we are considering a stationary spacetime.

The covariant derivative of the 4-velocity is

$$u_{\alpha;\beta} = w_{\alpha\beta} + a_{\alpha} u_{\beta}$$  \hspace{1cm} (66)

Recalling that $a_{\alpha} = -\ln \chi_{\alpha} = -k^\gamma k_{\gamma\alpha}/\chi^2$ the covariant derivative of the acceleration can be written (note that $a_{[\alpha;\beta]} = 0$)

$$a_{\alpha;\beta} = -\frac{k^\gamma k_{\gamma\alpha}}{\chi^2} \frac{2}{\chi^4} k^\gamma k_{\gamma\alpha} k^\delta k^\beta_{\delta;\beta} + \frac{R_{\delta\beta\gamma\alpha} k^\gamma k^\delta}{\chi^2}$$

$$= a_{\alpha} a_{\beta} + m_{\alpha} u_{\beta} + m_{\beta} u_{\alpha} + (a^2 + w^2) u_{\alpha} u_{\beta} - w^2 g_{\alpha\beta} - w_{\alpha} w_{\beta} + R_{\beta\delta\gamma\alpha} u^\gamma u^\delta.$$  \hspace{1cm} (67)

The covariant derivative of the vorticity tensor is

$$w_{\alpha\beta;\gamma} = \frac{k_{\alpha;\beta;\gamma}}{\chi} + [u_{\alpha;\gamma} a_{\beta} + u_{\alpha} a_{\beta;\gamma} - u_{\beta;\gamma} a_{\alpha} - u_{\beta} a_{\alpha;\gamma}] - \frac{k_{\alpha;\beta}}{\chi^2}.$$  \hspace{1cm} (68)

while for the vorticity vector we have

$$w_{\gamma} = \frac{1}{2} \epsilon^{\sigma\sigma\beta} w_{\sigma\gamma} w_{\alpha\beta} + \frac{1}{2} \epsilon^{\sigma\sigma\beta} a_{\sigma} u_{\alpha} w_{\beta} - w_{\alpha} w_{\beta} - \epsilon^{\beta\sigma\alpha} a_{\sigma} u_{\alpha} w_{\gamma} - \frac{1}{2} \epsilon^{\sigma\sigma\beta} u_{\sigma} R_{\gamma\delta\alpha\beta} u^\delta.$$  \hspace{1cm} (69)

The covariant derivative of $m_{\alpha}$ is

$$m_{\alpha;\gamma} = w_{\alpha;\beta;\gamma} a^\beta + w_{\alpha\beta} a_{\gamma}$$

$$= -R_{\gamma\delta\alpha\beta} u^\delta a^\beta + u_{\alpha} R_{\gamma\delta\alpha\beta} u^\eta u^\delta a^\beta + R_{\gamma\delta\alpha\beta} u^\eta u^\delta w_{\alpha} - u_{\alpha} w_{\beta} m^\beta + + 2 m_{\alpha} a_{\beta} + w_{\alpha} a_{\beta} w_{\gamma} + w_{\alpha} m^\beta u_{\gamma} + w_{\alpha} a_{\gamma} + 2 a_{\alpha} m^\gamma +$$

$$= -R_{\gamma\delta\alpha\beta} u^\delta a^\beta + u_{\alpha} R_{\gamma\delta\alpha\beta} u^\eta u^\delta a^\beta + R_{\gamma\delta\alpha\beta} u^\eta u^\delta w_{\alpha} + + 2 m_{\alpha} a_{\beta} - w_{\gamma} (a^2 + w^2) + m_{\gamma} a_{\alpha},$$  \hspace{1cm} (70)

where we have used $w_{\alpha\beta} w^\beta w_\gamma = -w^2 w_{\alpha\gamma}$. Finally, from Eqs. (67), (68), (69), (70), we write down the covariant derivatives of the most relevant scalars

$$w^2 = 2w^2 a_{\gamma} - 2w_{\gamma} a_{\alpha} w_{\alpha} - R_{\gamma\delta\alpha\beta} u^\delta w_{\alpha} a_{\beta},$$  \hspace{1cm} (71)

$$a^2 = 2a^2 a_{\gamma} - 2w_{\gamma} a_{\alpha} m_{\alpha} - 2R_{\gamma\delta\alpha\beta} u^\delta a_{\alpha} a_{\beta},$$  \hspace{1cm} (72)

$$m^2 = 4m^2 a_{\gamma} - 2(a^2 + w^2) w_{\gamma} a_{\alpha} m_{\alpha} + 2R_{\gamma\delta\alpha\beta} u^\delta m_{\alpha} a_{\beta} + 2R_{\gamma\delta\alpha\beta} u^\delta w_{\alpha} w_{\beta} a_{\alpha},$$  \hspace{1cm} (73)

$$a^\delta w^\gamma = 2(a^\delta w^\gamma) a_{\gamma} + R_{\gamma\delta\alpha\beta} w_{\alpha} w^\delta a_{\beta} - \frac{1}{2} \epsilon^{\eta\sigma\alpha} a_{\eta} a_{\sigma} R_{\gamma\delta\alpha\beta} u^\delta.$$  \hspace{1cm} (74)
Since \{u, w, a, m\} is a base (not an orthogonal one) on those events where \( m \neq 0 \), any simultaneity convention can be written in the same events in the form

\[
\omega_\alpha = u_\alpha + \psi^m m_\alpha + \psi^a a_\alpha + \psi^w w_\alpha,
\]

where the fields \( \psi \) depend on local measurable quantities. We restrict ourselves to fields \( \psi \) which depend on \( w, a \) and \( \theta \). Thus, we do not consider simultaneity conventions dependent on the Riemann tensor. We split part (i) of the theorem in two lemmas.

**Lemma (a).** In a stationary spacetime let \( U \) be the open set \( U = \{ x : m(x) > 0 \text{ and } a(x) \neq w(x) \} \). Let the functions \( \psi^m, \psi^a, \psi^w, \) depend only on \( a, w \) and \( \theta \). Let us introduce the invertible transformation

\[
\begin{align*}
x_1 &= -a_3 w^3 = aw \cos \theta, \\
x_2 &= a^2 + w^2, \\
x_3 &= a^2 - w^2,
\end{align*}
\]

and consider the functions \( \psi^m, \psi^a, \psi^w \), as dependent on the variables \( x_1, x_2, x_3 \). Assume that the functions \( \psi^m, \psi^a, \psi^w \) are \( C^1 \). Then the curvature of \( \psi^m, \psi^a, \psi^w \) is proportional to the Riemann tensor in \( U \), regardless of the stationary spacetime considered, if and only if

\[
\begin{align*}
\left( \frac{x_2^2 - x_3^2}{2} - 2x_1 \right) \frac{\partial \psi^m}{\partial x_2} &= 1 - x_2 \psi^m, \\
\psi^w + (x_2 - x_3) \frac{\partial \psi^w}{\partial x_2} &= -2x_1 \frac{\partial \psi^a}{\partial x_2}.
\end{align*}
\]

**Proof.** We introduce the vector valued symmetric operator which acts on pairs of fields \( b, c \), given by

\[
\langle b, c \rangle^\eta = \frac{1}{4} h^\eta_\gamma \epsilon^{\nu\beta\alpha\gamma} (b_\beta c_\alpha - c_\beta b_\alpha),
\]

then, omitting the indices,

\[
v^\eta = (u + \psi^m m + \psi^a a + \psi^w w, u + \psi^m m + \psi^a a + \psi^w w)^\eta.
\]

Let us consider the different pairings. The covariant derivative of \( a_3 w^3 \), \( a^2 \) or \( w^2 \) is proportional to terms in \( w^\alpha \) and \( a^\alpha \) plus terms proportional to the Riemann tensor. As a consequence, the same property holds for any function \( \psi \) of the variables \( \cos \theta, a \) and \( w \). This fact, along with the calculation of the covariant derivatives given above, leads us to table 1. The result of the pairing is a linear combination of the fields shown in the last column, apart from terms proportional to the Riemann tensor. We are looking for a connection \( \omega \) such that the sum of the different pairings vanishes apart from terms proportional to the Riemann tensor.

The table shows that the only way to remove the term proportional to \( w \), due to the pairing of \( u \) with itself, is to have \( \psi^m \neq 0 \). However it is a non trivial fact that this can actually be done without adding a term proportional to \( a \).

We shall see below that this happens because \( \psi^m \) must satisfy a dimensional constraint. Indeed \( u \) is dimensionless whereas \( m \) has dimension \([L^{-1}]^2\) (we shall say \( d = 2 \)), thus \( \psi^m \) must have \( d = -2 \).

When we have removed \( w \) from the curvature we can stop or consider more complicated local simultaneity conventions adding terms proportional to \( a \) and \( w \). In general, we expect to need both since if one adds a term to the curvature proportional to \( m \), the other must remove it. Let us denote by, \( \simeq \), equivalence up to terms proportional to the Riemann tensor. The simultaneity convention has a curvature proportional to the Riemann tensor iff

\[
\begin{align*}
\langle u, u \rangle + 2 \langle u, \psi^m m \rangle &\simeq 0, \\
\langle u, \psi^a a \rangle + \langle u, \psi^w w \rangle &\simeq 0.
\end{align*}
\]
The proof of (a) is completed. It is convenient to consider on $M$ the variables $x_i$ have $d_i = 2$. Since the transformation given by (76), (77), (78), is invertible the functions $\psi$ will be considered as functions of $x_1$, $x_2$ and $x_3$. The covariant derivatives of these scalars are

\[
\begin{align*}
x_{1;\gamma} &= d_1 x_1 a_\gamma - R_{\gamma\delta\alpha\beta} u^\delta u^\alpha + \frac{1}{2} \eta_{\eta\sigma} a_\eta u_\sigma R_{\gamma\delta\alpha\beta} u^\delta, \\
x_{2;\gamma} &= d_2 x_2 a_\gamma - 4w_{\gamma\alpha} m^\alpha - 2R_{\gamma\delta\alpha\beta} u^\delta a_\alpha - R_{\gamma\delta\alpha\beta} u^\delta a_\beta, \\
x_{3;\gamma} &= d_3 x_3 a_\gamma - 2R_{\gamma\delta\alpha\beta} u^\delta a_\alpha + R_{\gamma\delta\alpha\beta} u^\delta a_\beta.
\end{align*}
\] (85)

Since the functions $\psi$ depend on dimensional variables they have to satisfy a scaling relation of the form

\[
\psi(\lambda^i x_1, \lambda^i x_2, \lambda^i x_3) = \lambda^\delta \psi(x_1, x_2, x_3).
\] (88)

In particular, $d_{\psi_m} = -2$, $d_{\psi_w} = -1$, $d_{\psi_w} = -1$. Differentiating with respect to $\lambda$ we obtain (Euler's theorem), $\sum_{i=1}^3 d_i x_i \psi_i = d_{\psi}$. Using the covariant derivatives of $x_i$ we obtain

\[
\psi_{\gamma} = \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i} x_i \psi_{\gamma} \simeq \left( \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i} d_i x_i \right) - 4 \frac{\partial \psi}{\partial x_2} w_{\gamma\alpha} m^\alpha,
\] (89)

and finally

\[
\psi_{\gamma} \simeq d_{\psi} \psi a_\gamma - 4 \frac{\partial \psi}{\partial x_2} w_{\gamma\alpha} m^\alpha.
\] (90)

We are ready to solve the system (83), (84). We have

\[
4(u, u)^n = 4 w^n,
\] (91)

\[
4(u, \psi^m m)^n = h^m_{\nu\epsilon} \psi^{\mu\gamma} (u^{\beta} m^\alpha \psi^n_{\gamma}) + \psi^m u_{\beta} m_{\alpha} \psi^n_{\gamma} + \psi^m m_{\beta} a_\alpha u_{\gamma} \simeq -w^n [2(a^2 + w^2) \psi^m + 4m^2 \frac{\partial \psi^m}{\partial x_2}],
\] (92)

\[
4(u, \psi^a a)^n = h^a_{\nu\epsilon} \psi^{\mu\gamma} u_{\beta} a_\alpha \psi^n_{\gamma} \simeq -4(w^a a_\delta) \frac{\partial \psi^a}{\partial x_2} m^n,
\] (93)

\[
4(u, \psi^w w)^n \simeq h^w_{\nu\epsilon} \psi^{\mu\gamma} (u^{\beta} w^\alpha \psi^n_{\gamma}) + 2\psi^w u^\beta w_{\alpha} a_\gamma + \psi^w w_{\beta} a_\alpha u_{\gamma} = (2\psi^w + 4w^2 \frac{\partial \psi^w}{\partial x_2}) m^n,
\] (94)

thus the curvature is proportional to the Riemann tensor if

\[
2m^2 \frac{\partial \psi^m}{\partial x_2} = 1 - (a^2 + w^2) \psi^m,
\] (95)

\[
\frac{\psi^w}{2} + w^2 \frac{\partial \psi^w}{\partial x_2} = (w^a a_\delta) \frac{\partial \psi^a}{\partial x_2},
\] (96)

where $m^2$, $a^2 w^2$ and $w^a a_\delta$ are understood as functions of $x_1$, $x_2$ and $x_3$.

Let us shows that these conditions are also necessary. Let

\[
H(x_1, x_2, x_3) = 2m^2 \frac{\partial \psi^m}{\partial x_2} - 1 + (a^2 + w^2) \psi^m,
\] (97)

\[
K(x_1, x_2, x_3) = \frac{\psi^w}{2} + w^2 \frac{\partial \psi^w}{\partial x_2} - (w^a a_\delta) \frac{\partial \psi^a}{\partial x_2},
\] (98)

and consider a flat spacetime $M$. Since the Riemann tensor is null, the curvature $v$ vanishes only if, for any $y \in M$

\[
H(x_1(y), x_2(y), x_3(y)) = 0,
\] (99)

\[
K(x_1(y), x_2(y), x_3(y)) = 0.
\] (100)

If we are able to show that the Killing vector field and the event $y$ can be chosen so as to obtain any triple $(x_1, x_2, x_3)$ the proof of (a) is completed. It is convenient to consider on $M$ the pseudocylindrical coordinates

\[
ds^2 = ((\xi^2 - r^2 \Omega^2) d\tau^2 - d\xi^2 - dr^2 - r^2 d\phi^2 - 2r^2 \Omega d\phi d\tau),
\] (101)
and the timelike Killing vector field $k = \partial_r$, $(\xi^2 - r^2\Omega^2 > 0)$. $\Omega$ is a parameter that satisfies $\Omega^2 \neq 1$. After some algebra we obtain

$$x_1 = \frac{\Omega}{\xi^2 - r^2\Omega^2}, \quad (102)$$

$$x_2 = (1 + \Omega^2) \frac{\xi^2 + \Omega^2}{(\xi^2 - r^2\Omega^2)^2}, \quad (103)$$

$$x_3 = \frac{1 - \Omega^2}{\xi^2 - r^2\Omega^2}, \quad (104)$$

that fortunately can be inverted to give

$$\Omega = -x_3 + \sqrt{x_3^2 + 4x_1}, \quad (105)$$

$$\xi = \frac{\sqrt{\beta_1 + \beta_2}}{2}, \quad (106)$$

$$r = \frac{1}{\Omega} \sqrt{\frac{\beta_1 - \beta_2}{2}}. \quad (107)$$

with

$$\beta_1 = \frac{x_2}{2x_1} \left( \frac{\sqrt{x_3^2 + 4x_1^2} - x_3}{\sqrt{x_3^2 + 4x_4^2}} \right), \quad (108)$$

$$\beta_2 = \frac{\sqrt{x_3^2 + 4x_1^2} - x_3}{2x_1}. \quad (109)$$

Thus given $(x_1, x_2, x_3)$ we choose on flat spacetime a pseudocylindric coordinate system having a $\Omega$ given by Eq. (105), then in the events characterized by the coordinates $\xi$ and $r$ given by Eqs. (106), (107), the kinematical quantities $x_1$, $x_2$ and $x_3$ assume the required values. Lemma (a) is proved.

**Lemma (b).** Let the functions $\psi^m, \psi^n, \psi^w$ be $C^1$ in $x_1, x_2$ and $x_3$. They satisfy Eqs. (79), (80), and the associated connection (76) is timelike in $U$ (and hence it is a simultaneity connection in $U$) if and only if

$$\psi^m = \psi^m \equiv \frac{a^2 + w^2}{2} - \sqrt{(a^2 + w^2)^2 - 4m^2}, \quad (110)$$

and there is a $C^1$ function $b : \mathbb{R} \rightarrow \mathbb{R}$ of the variable $x_1/x_3 = -a_3 w^2/(a^2 - w^2)$ such that for $m > 0$ and $a \neq w$ the following inequality holds

$$\frac{\sqrt{(a^2 + w^2)^2 - 4m^2} - (a^2 - w^2) b^2}{2w^2} + \frac{m^2}{w^2} \psi^a + \frac{m^2}{w^2} \psi^m < 1, \quad (111)$$

and

$$\psi^w = \frac{a_3 w^2}{w^2} \psi^a + \frac{b}{w^2} \left( \frac{\sqrt{(a^2 + w^2)^2 - 4m^2} - (a^2 - w^2)}{2} \right)^{1/2}. \quad (112)$$

**Proof.** Let us focus on the first differential equation (79) and set $\psi^m = \phi/m^2$. The equation greatly simplifies to give

$$\frac{\partial \phi}{\partial x_2} = \frac{1}{2} \Rightarrow \phi = \frac{x_2}{2} + f(x_1, x_3). \quad (113)$$

But $2f/x_3$ is dimensionless and therefore there is a function $g$ such that $2f/x_3 = g(x_1/x_3)$ and

$$\psi^m = \frac{a^2 + w^2 + (a^2 - w^2) g \left( \frac{a w^2}{m - w^2} \right)}{2m^2}. \quad (114)$$

Let us introduce $\rho = w/a$ and $\xi = x_1/x_3 = \rho \cos \theta/(1 - \rho^2)$, then

$$m \psi^m = \frac{1 + \rho^2 + (1 - \rho^2) g(\xi)}{2 \rho \sin \theta}. \quad (115)$$
Consider the numerator \(N(\rho, \theta)\) of \(m\psi^m\) for \(\theta = 0\),

\[
N(\rho, 0) = 1 + \rho^2 + (1 - \rho^2)g(\frac{\rho}{\rho - 1}),
\]

and assume there is a value \(\bar{\rho} \neq 1\) such that \(N(\bar{\rho}, 0) \neq 0\). By continuity

\[
\lim_{\theta \to 0} N(\bar{\rho}, \theta)^2 = N(\bar{\rho}, 0)^2 > 0,
\]

\[
\lim_{\theta \to 0} |m\psi^m(\bar{\rho}, \theta) = +\infty.
\]

From the hypothesis it follows that the condition \(|m\psi^m| < 1\) is not satisfied in \(U\) and therefore we conclude that the connection is timelike in \(U\) only if \(\forall \rho \neq 1, N(\rho, 0) = 0\). The condition \(N(\rho, 0) = 0\) reads

\[
g(\frac{\rho}{\rho - 1}) = \frac{1 + \rho^2}{\rho^2 - 1}.
\]

Let us introduce \(\lambda = \rho/(1 - \rho^2)\) and \(s = \text{sign}(1 - \rho^2) = \text{sign}(\lambda)\). The inverse formula is \(\rho = (-1 + s\sqrt{1 + 4\lambda^2})/(2\lambda)\) and plugging this in Eq. 119

\[
g(\lambda) = -s\sqrt{1 + 4\lambda^2}.
\]

Finally,

\[
m\psi^m = \frac{1 + \rho^2 - \sqrt{(1 - \rho^2)^2 + 4\rho^2\cos^2 \theta}}{2\rho \sin \theta},
\]

and using \(\rho = w/a\) we obtain Eq. 211. Let us come to Eq. 80. It can be rewritten as

\[
\frac{\partial}{\partial x_2} \left(\frac{x_2 - x_3}{2} \psi^a + x_1 \psi^a\right) = 0.
\]

Let \(p\) be the quantity between brackets. Its square is independent of \(x_2\),

\[
p^2 = w^2(w^2\psi^w + a^2\psi^a - 2a^2 w^8 \psi^a \psi^w) - m^2\psi^a.
\]

Let \(h^2\) be the positive term between brackets. From Eq. 15 we see that a connection is timelike iff

\[
\omega^a \omega_a = 1 - (m\psi^m)^2 - h^2 > 0.
\]

Dividing \(p^2\) by \(x_1\) we obtain a dimensionless function independent of \(x_2\), thus there is a function \(r(x_1/x_3)\) such that

\[
\frac{p^2}{x_1} = \frac{w^2 h^2 - m^2 \psi^a}{x_1} = r(x_1/x_3),
\]

from which it follows that

\[
h^2 = \frac{x_1 r(x_1/x_3) + m^2 \psi^a}{w^2} = \frac{\rho \cos \theta \, r(\frac{\rho \cos \theta}{1 - \rho^2}) + \rho^2 \sin^2 \theta ( \rho \psi^a)^2}{\rho^2}.
\]

Now, let \(\theta = 0\), and introduce again \(\lambda = \rho/(1 - \rho^2)\).

\[
h^2(\rho, \theta = 0) = \frac{1}{\rho} r(\frac{\rho}{1 - \rho^2})
\]

We define \(b^2(\lambda) = h^2(\rho(\lambda),0)\) and find \(r(\lambda) = \rho(\lambda)b^2(\lambda)\). Using the formula for \(\rho(\lambda)\) and Eq. 120 we obtain

\[
h^2 = \frac{\sqrt{(1 - \rho^2)^2 + 4\rho^2 \cos^2 \theta} - (1 - \rho^2)^2 b^2(\frac{\rho \cos \theta}{1 - \rho^2}) + \sin^2 \theta (\rho \psi^a)^2}{2\rho^2}.
\]
Recalling the expression of \( \psi \)

\[
\psi^w = \frac{a_\delta w^\delta}{w^2} \psi^a \pm \frac{|b|}{w^2 \sqrt{2}} \sqrt{(1 - \rho^2)^2 + 4 \rho^2 \cos^2 \theta - (1 - \rho^2)^{1/2}},
\]

(129)

and defining the sign of \( b \) with \( b = \pm |b| \) we obtain Eq. (23), while Eq. (22) follows from Eq. (124). The proof of lemma (b) is complete. Finally point (i) of the theorem follows easily from lemmas (a) and (b).

**Proof of (ii).** The expression of \( \tilde{\psi}^m \) makes clearly sense also for those points where \( m > 0 \) but \( a = w \). Let \( U' = \{ x : m > 0, x \notin B \} \). In the open set \( U' \) the variable \( x = (a^2 + w^2)/(2|m|) \) satisfies \( x > 1 \). Moreover we can write in \( U' \subset C \)

\[
|m\tilde{\psi}^m| = x - \sqrt{x^2 - 1}.
\]

(130)

Let \( p \in C - U' \), \( q \in C \) and consider the limit \( q \rightarrow p \). Since \( a^2 + w^2 > 0 \) and \( m = 0 \) in \( p \) we have that \( x(q) \rightarrow + \infty \) as \( q \rightarrow p \). But in this limit \( |m\tilde{\psi}^m| \sim 1/(2x) \) and hence we can define by continuity \( \tilde{\psi}^m = 1/(a^2 + w^2) \) in \( C - U' \). Of course this implies that in these points \( \bar{\omega}_\alpha = u_\alpha \) since \( m^\alpha = 0 \) and hence \( \bar{\omega}_\alpha \bar{\omega}^\alpha = 1 \). Moreover, \( \bar{\omega}_\alpha \bar{\omega}^\alpha > 0 \) in \( U' \) as it follows from Eq. (130).

The curvature is

\[
\bar{\nabla}^\eta = \langle u, u \rangle^\eta + 2\langle u, \bar{\psi}^m m \rangle^\eta + \bar{\psi}^{m2}(m, m).
\]

(131)

The last term vanishes

\[
2\langle m, m \rangle^\eta = h_{\nu\rho}^\alpha \bar{\psi}^{\rho \gamma} m_\beta m^{\alpha \gamma} = h_{\nu\rho}^\alpha \bar{\psi}^{\rho \gamma} m_\beta (-R_{\gamma\delta\alpha\mu} u^\delta a^\mu + u_\alpha R_{\gamma\delta\mu\rho} u^\mu a^\sigma + R_{\gamma\delta\mu\sigma} u^\mu u^\delta w_\alpha) = -h_{\nu\rho}^\alpha \bar{\psi}^{\rho \gamma} m_\beta R_{\gamma\delta\mu\rho} u^\mu a^\sigma h^{\mu \alpha} = 0.
\]

(132)

By construction the curvature is proportional to the Riemann tensor. The calculation of the remaining terms is straightforward but rather lengthy. We give here only a formula, that follows from Eqs. (71 - 73), that can help the reader in recovering the final formula (24).

\[
\bar{\psi}^m_{\gamma\gamma} = -2a_\gamma \bar{\psi}^m - 4w_{\gamma\alpha} m^\alpha \frac{\partial}{\partial x_2} \bar{\psi}^m(x_1, x_2, x_3)
+ \frac{\bar{\psi}^m}{\sqrt{(a^2 + w^2)^2 - 4m^2}} \left\{ R_{\gamma\delta\alpha\beta} u^\delta w^{\alpha \beta} + 2R_{\gamma\delta\alpha\beta} u^\delta a^\beta \right\}
+ \frac{2\bar{\psi}^{m2}}{\sqrt{(a^2 + w^2)^2 - 4m^2}} \left\{ R_{\gamma\delta\alpha\beta} u^\delta m^{\alpha \beta} + R_{\gamma\delta\eta\beta} u^\delta w^{\beta \alpha} m^\alpha \right\}.
\]

(133)

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The curvature of the simultaneity connection has a hat in order to be distinguished from the Riemann tensor. Note that properly set clocks are not in general synchronized if $\chi$ varies in space.