HAAR-TYPE STOCHASTIC GALERKIN FORMULATIONS FOR HYPERBOLIC SYSTEMS WITH LIPSCHITZ CONTINUOUS FLUX FUNCTION

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Abstract. This work is devoted to the Galerkin projection of highly nonlinear random quantities. The dependency on a random input is described by Haar-type wavelet systems. The classical Haar sequence has been used by Pettersson, Iaccarino, Nordström (2014) for a hyperbolic stochastic Galerkin formulation of the one-dimensional Euler equations. This work generalizes their approach to several multi-dimensional systems with Lipschitz continuous and non-polynomial flux functions. Theoretical results are illustrated numerically by a genuinely multidimensional CWENO reconstruction.

1. Introduction

There are close relations between stochastic processes and orthogonal functions [44]. Following early works by Wiener [51], where Hermite polynomials and homogeneous chaos has been used in the integration theory for Brownian motion, the representation of random quantities by orthogonal functions has received increasing attention [4, 52, 36, 40].

In the context of intrusive stochastic Galerkin methods, the functional dependence on the stochastic input is described a priori by a polynomial chaos (gPC) expansion and a Galerkin projection is used to obtain deterministic evolution equations for the coefficients in the series, called gPC modes. Then, all envolved mathematical operations, e.g. products and norms, must be adopted and applied to the variables in the governing equations. This causes serious theoretical and numerical challenges [14]. For instance, the accuracy of the truncated gPC representations affects the representation of positive quantities, e.g. the density of a gas, and nonlinearities may require a high gPC order, which may be too computationally expensive.

To this end, wavelet-based gPC expansions have been introduced [37, 39]. These are motivated by a robust expansion for solutions that depend on the stochastic input in a non-smooth way and are used for stochastic multiresolution as well as adaptivity [1, 31, 50, 2].

The stochastic Galerkin method has been successfully applied to diffusion [53, 16, 2], kinetic [46, 27, 5, 54, 30, 22] and Hamilton-Jacobi equations [28, 3]. In general, results for hyperbolic systems are not available [15, 36], since desired properties like hyperbolicity and the existence of entropies are not transferred to the intrusive gPC formulation. A problem is posed by the fact that the deterministic Jacobian of the projected system differs from the random Jacobian of the original system and, therefore, not even real eigenvalues, which are necessary for hyperbolicity, are guaranteed in general.

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There are many examples that show a loss of hyperbolicity, when the stochastic Galerkin approach is applied directly to hyperbolic systems. To this end, auxiliary variables have been introduced to establish wellposedness results. For instance, entropy-entropy flux pairs can be obtained by an expansion in entropy variables, i.e. the gradient of the deterministic entropy [15, 19, 32, 33, 34]. Roe variables, which include the square root of the density, preserve hyperbolicity for Euler equations [43, 39, 21, 20]. The drawback of introducing auxiliary variables is an additional computational overhead that arises from an optimization problem, which is required to calculate the auxiliary variables. Furthermore, the transforms may involve integrals and derivatives that have to be calculated numerically [19, Sec. 3.3]. But in this case, the formulation is no more truly intrusive in the sense that all integrals are exactly computed in a precomputation step and not during a simulation. Another problem is the connection between the hyperbolicity of the Galerkin and the original system. Namely, the required optimization problems are well-posed only in a possibly small domain [21, Sec. 4] when the solvability of the transform is guaranteed only locally by the implicit function theorem.

Recently, a hyperbolic stochastic Galerkin formulation for the shallow water equations has been presented that neither requires auxiliary variables nor any transform, since the Jacobian is shown to be similar to a symmetric matrix [12, 13]. Then, positivity of the water height at a finite number of stochastic quadrature points is sufficient to preserve hyperbolicity of the stochastic Galerkin formulation.

These results exploit quadratic relationships that are expressed efficiently by the Galerkin product. Extensions of the classical polynomial chaos expansions to general nonlinearities, e.g. with Legendre or Hermite polynomials, however, are not straightforward. Inspired by the application of the Haar sequence to solve Euler equations [39], we express nonlinearities, which occur for instance in isentropic Euler and level-set equations, by wavelet families that are generated by Haar-type matrices [42]. Those are widely used in signal processing [41, 42], e.g. for the Hadamard, Walsh, Chebyshev matrices and for the discrete cosine transform.

The aim of the paper is to show that many theoretical and numerical challenges that occur for polynomial expansions vanish when these Haar-type expansions are employed. In particular, wellposedness results are achieved for hyperbolic conservation laws with Lipschitz continuous flux function. Since, all expressions are stated in closed form without any optimization problem, the connection between the hyperbolicity of the Galerkin and the original system is established. Namely, deterministic wellposedness properties of the original system at stochastic quadrature points transfer wellposedness results to the stochastic Galerkin formulation. Furthermore, all eigenvalue decompositions are stated in closed form which allows for an efficient implementation also in higher dimensions. The presented results are not restricted to a dimensional splitting. Namely, the characteristic speeds are real with respect to all spatial dimensions. This enables genuinely multidimensional high order methods based on the CWENO reconstruction in space [45, 7] that also allows for discretizations of balance laws.

This paper is structured as follows. Stochastic Galerkin matrices are introduced in Section 2 for general polynomial chaos expansions. In particular, the representation of higher moments and roots is addressed in Lemma 2.2, which involve numerically expensive optimization problems. The main Theorem 3.3 in Section 3 states gPC approximations for several non-polynomial functions. Those are used in Section 4 for stochastic Galerkin formulations to hyperbolic conservation laws. Section 5 presents numerical experiments that confirm the theoretical findings.
2. Representation of nonlinear quantities by polynomial chaos expansions

We introduce a random variable \( \xi : \Omega \rightarrow \mathbb{R} \), which is defined on a probability space \((\Omega, \mathcal{F}(\Omega), \mathbb{P})\), and associated orthogonal basis functions \( \{\phi_k(\xi)\}_{k \in \mathbb{N}_0} \) satisfying

\[
\mathbb{E}\left[\phi_i(\xi)\phi_j(\xi)\right] = \int \phi_i(\xi)\phi_j(\xi) \, d\mathbb{P} =: \langle \phi_i, \phi_j \rangle_{\mathbb{P}} = \delta_{i,j},
\]

where \( \delta_{i,j} \) denotes the Kronecker delta. Then, for each random state \( u(\xi) \in L^2(\Omega, \mathbb{P}) \) there exist gPC modes \( \hat{u} \in \mathbb{R}^{K+1} \) such that the truncated series

\[
(2.1) \quad \Pi_K[\hat{u}](\xi) := \sum_{k=0}^{K} \hat{u}_k \phi_k(\xi) \quad \text{satisfies} \quad \left\| \Pi_K[\hat{u}](\xi) - u(\xi) \right\|_{\mathbb{P}} \rightarrow 0 \quad \text{for} \quad K \rightarrow \infty.
\]

**Stochastic Galerkin matrices** [14, 36, 48, 17] which are defined by

\[
\mathcal{P}(\hat{u}) := \sum_{k=0}^{K} \hat{u}_k \mathcal{M}_k \quad \text{and} \quad \mathcal{M}_k := \left( \langle \phi_k, \phi_j \rangle_{\mathbb{P}} \right)_{i,j=0,...,K}
\]

arise in a Galerkin projection of a random product \( u(\xi)q(\xi) \in L^2(\Omega, \mathbb{P}) \), called Galerkin product, that is denoted by

\[
\hat{u} \ast \hat{q} := \mathcal{P}(\hat{u})\hat{q} \quad \text{and satisfies} \quad \left\| \Pi_K[\hat{u} \ast \hat{q}](\xi) - u(\xi)q(\xi) \right\|_{\mathbb{P}} \rightarrow 0 \quad \text{for} \quad K \rightarrow \infty.
\]

The matrix \( \mathcal{P}(\hat{u}) \) defines a linear operator. Hence, the following sets are convex.

\[
(2.2) \quad \mathbb{H}^+ := \left\{ \hat{u} \in \mathbb{R}^{K+1} \left| \mathcal{P}(\hat{u}) \right. \text{is strictly positive definite} \right\},
\]

\[
\mathbb{H}^+_0 := \left\{ \hat{u} \in \mathbb{R}^{K+1} \left| \mathcal{P}(\hat{u}) \right. \text{is positive semidefinite} \right\}.
\]

These positive definiteness assumptions reflect the representation of positive quantities in the truncated expansion (2.1). More precisely, the matrix \( \mathcal{P}(\hat{u}) \) is strictly positive definite provided that all realizations \( \Pi_K[\hat{u}](\xi(\omega)) > 0 \) are positive [47, 24]. The stochastic Galerkin matrices are symmetric. Hence, they admit an orthogonal eigenvalue decomposition that is denoted by \( \mathcal{P}(\hat{u}) = \mathcal{V}(\hat{u})\mathcal{D}(\hat{u})\mathcal{V}^T(\hat{u}) \) and satisfies \( \mathcal{V}^T(\hat{u}) = \mathcal{V}^{-1}(\hat{u}) \). Furthermore, the Galerkin product is symmetric, i.e. \( \hat{u} \ast \hat{q} = \hat{q} \ast \hat{u} \). For general basis functions we have the following lemma.

**Lemma 2.1.** Consider the orthogonal eigenvalue decomposition \( \mathcal{P}(\hat{u}) = \mathcal{V}(\hat{u})\mathcal{D}(\hat{u})\mathcal{V}^T(\hat{u}) \) with eigenvectors \( \mathcal{V} = (v_0 \cdots v_K) \) and eigenvalues \( \mathcal{D} = \text{diag}\{d_0, \ldots, d_K\} \) that satisfy the equality \( \mathcal{P}(\hat{u})v_k(\hat{u}) = d_k(\hat{u})v_k(\hat{u}) \). Then, we have the equality

\[
\mathcal{D}(\hat{u})|_{\hat{u} = \hat{q}}^{\mathcal{V}^T(\hat{u})\hat{q}} = \mathcal{V}^T(\hat{u})\mathcal{P}(\hat{q}) \quad \text{for all} \quad \hat{u}, \hat{q} \in \mathbb{R}^{K+1}.
\]

**Proof.** Due to the orthonormal eigenvalue decomposition, i.e. \( v_k^T(\hat{u})v_\ell(\hat{u}) = \delta_{k,\ell} \), the equality

\[
0 = \frac{1}{2} \partial_{\hat{u}_j} \left[ v_k^T(\hat{u})v_k(\hat{u}) \right] = v_k^T(\hat{u})\partial_{\hat{u}_j}v_k(\hat{u})
\]

holds, which implies

\[
(2.3) \quad v_k^T(\hat{u})\mathcal{P}(\hat{u})\partial_{\hat{u}_j}v_k(\hat{u}) = v_k^T(\hat{u})d_k(\hat{u})\partial_{\hat{u}_j}v_k(\hat{u}) = 0.
\]
Due to \( \partial_{u_j} \mathcal{P}(\hat{u}) = \mathcal{M}_j \), we have
\[
0 = v_k^T(\hat{u}) \partial_{u_j} \left[ \mathcal{P}(\hat{u}) v_k(\hat{u}) - d_k(\hat{u})v_k(\hat{u}) \right] \\
= v_k^T(\hat{u}) \mathcal{M}_j v_k(\hat{u}) + v_k^T(\hat{u}) \partial_{u_j} \mathcal{M}_j v_k(\hat{u}) - v_k^T(\hat{u}) \partial_{u_j} d_k(\hat{u}) v_k(\hat{u}) - v_k^T(\hat{u}) d_k(\hat{u}) \partial_{u_j} v_k(\hat{u}).
\]
Equations (2.3) and (2.4) yield
\[
\partial_{u_j} d_k(\hat{u}) = v_k^T(\hat{u}) \mathcal{M}_j v_k(\hat{u}).
\]
Then, the identity \( \mathcal{P}(\hat{u}) = [\mathcal{M}_0 \hat{u}] \cdots [\mathcal{M}_K \hat{u}] \) implies
\[
\partial_{u_j} \mathcal{D}(\hat{u}) \hat{w} = \begin{pmatrix} v_0^T(\hat{u}) \mathcal{M}_j v_0(\hat{u}) \hat{w} \\ \vdots \\ v_K^T(\hat{u}) \mathcal{M}_j v_K(\hat{u}) \hat{w} \end{pmatrix} = \mathcal{V}^T(\hat{u}) \mathcal{M}_j \mathcal{V}(\hat{u}) \hat{w},
\]
\[
D_{\hat{a}} \mathcal{D}(\hat{a}) \|_{\hat{a}=\hat{u}} \mathcal{V}^T(\hat{u}) \hat{w} = \begin{bmatrix} \mathcal{V}^T(\hat{u}) \mathcal{M}_0 \hat{w} & \cdots & \mathcal{V}^T(\hat{u}) \mathcal{M}_K \hat{w} \end{bmatrix} = \mathcal{V}^T(\hat{u}) \mathcal{P}(\hat{w}).
\]

It has been emphasised in [14] that the representation of non-polynomial quantities by polynomial chaos expansions is challenging. To elaborate this point further, we introduce the following lemma.

**Lemma 2.2.** Denote the orthonormal eigenvalue decomposition by \( \mathcal{P}(\hat{u}) = \mathcal{V}(\hat{u}) \mathcal{D}(\hat{u}) \mathcal{V}^T(\hat{u}) \), the unit vector by \( \hat{e}_1 := (1, 1, \ldots, 0)^T \) and assume a gPC approximation \( \mathcal{G}_K[\hat{u}] (\xi) > 0 \) that is bounded away from zero. Then, the following statements hold:

(i) The choice \( \hat{p}(\hat{u}) := \mathcal{P}^n(\hat{u}) \hat{e}_1 \) is a consistent approximation of the moment \( p(\hat{u}) = \mathcal{u}^n \), i.e.
\[
\left\| \mathcal{u}^n(\xi) - \Pi_K \hat{p}(\hat{u}) (\xi) \right\|_p \to 0 \quad \text{for} \quad K \to \infty.
\]

(ii) Let a state \( \hat{u} \in \mathbb{H}^+ \) be given such that there is \( \hat{\alpha} \in \mathbb{H}^+ \) satisfying \( \mathcal{P}^n(\hat{\alpha}) \hat{e}_1 = \hat{u} \) and
\[
(\mathcal{A}2) \quad \mathcal{P}(\hat{\alpha}) \text{ is strictly positive definite.}
\]

Then, there is an open subset \( \mathbb{H}^+(\hat{u}) \subseteq \mathbb{H}^+ \), containing \( \hat{u} \), such that for all admissible states \( \tilde{u} = \mathcal{P}^n(\hat{\alpha}) \hat{e}_1 \) with \( \hat{\alpha} \in \mathbb{H}^+(\hat{u}) \) the unique minimum
\[
\hat{\mathcal{V}}(\hat{u}) := \arg\min_{\hat{a} \in \mathbb{H}^+(\hat{u})} \left\{ \eta^{(n)}_{\hat{a}}(\hat{\alpha}) \right\} \quad \text{of the convex function} \quad \eta^{(n)}_{\hat{a}}(\hat{\alpha}) := \frac{\hat{e}_1^T \mathcal{P}^{n+1}(\hat{\alpha}) \hat{e}_1}{n+1} - \hat{u}^T \hat{\alpha}
\]
is a consistent gPC approximation of the \( n \)-th root \( \sqrt[n]{} \hat{u} \), i.e.
\[
\left\| \sqrt[n]{} \hat{u}(\xi) - \Pi_K \left( \hat{\mathcal{V}}(\hat{u}) (\xi) \right) \right\|_p \to 0 \quad \text{for} \quad K \to \infty.
\]
In particular, the gradient reads as
\[
\nabla_{\hat{\alpha}} \eta^{(n)}_{\hat{a}}(\hat{\alpha}) = \mathcal{P}^n(\hat{\alpha}) \hat{e}_1 - \hat{u} + \frac{\mathcal{E}_{n+1}^{(n)}(\hat{\alpha}) \hat{e}_1}{n+1}
\]
with
\[
\mathcal{E}_{n+1}^{(n)}(\hat{\alpha}) := D_{\hat{\alpha}} \mathcal{V}(\hat{\alpha}) \|_{\hat{a}=\hat{u}} \mathcal{P}^{n+1}(\hat{\alpha}) \mathcal{V}^T(\hat{\alpha}) \hat{e}_1 + \mathcal{V}(\hat{\alpha}) \mathcal{D}^{n+1}(\hat{\alpha}) D_{\hat{\alpha}} \mathcal{V}^T(\hat{\alpha}) \|_{\hat{a}=\hat{u}} \hat{e}_1
\]
satisfying \( \| \mathcal{E}_{n+1}^{(n)}(\hat{\alpha}) \|_2 \to 0 \) for \( K \to \infty \).
Proof. Statement (i): We obtain the identity matrix by \( \mathcal{P}(\hat{e}_1) = M_0 = I \in \mathbb{R}^{(K+1) \times (K+1)} \). Hence, the first statement follows from a recursive application of the Galerkin product, i.e.
\[
\hat{\rho}(\hat{u}) = \mathcal{P}^n(\hat{u}) \hat{e}_1 = \mathcal{P}^n-1(\hat{u}) \mathcal{P}(\hat{e}_1) \hat{u} = \hat{u} \ast \ldots \ast (\hat{u} \ast (\hat{u} \ast \hat{u})).
\]

Statement (ii): The implicit function theorem guarantees that the matrix \( D^{+1}(\hat{\alpha}) V^{T}(\hat{\alpha}) \) is invertible for all admissible states \( \hat{\alpha} \in \mathbb{H}^+(\hat{\alpha}) \). Hence, the function
\[
\eta_u(\hat{\alpha}) := \frac{\hat{c}_1^{T} V(\hat{\alpha}) D^{+1}(\hat{\alpha}) V^{T}(\hat{\alpha})}{n+1} \hat{c}_1 - \hat{u}^{T} \hat{\alpha} + \frac{\|D^{+1}(\hat{\alpha}) V^{T}(\hat{\alpha}) \hat{c}_1\|^2}{n+1} - \hat{u}^{T} \hat{\alpha}
\]
is strictly convex as the composition of the convex spectral norm \( \|\cdot\| \) with an affine function and it has a unique minimum on the set \( \mathbb{H}^+(\hat{\alpha}) \). Lemma 2.1 implies the Jacobian
\[
D_{\hat{\alpha}} \left[ \hat{c}_1^{T} \mathcal{P}^{n+1}(\hat{\alpha}) \hat{c}_1 \right] = \hat{c}_1^{T} V(\hat{\alpha}) D_{\hat{\alpha}} D^{+1}(\hat{\alpha}) |_{\hat{\alpha} = \hat{\alpha}} V^{T}(\hat{\alpha}) \hat{c}_1 + \hat{c}_1^{T} E_{n+1}(\hat{\alpha})
\]
(2.6)\[= (n+1) \hat{c}_1^{T} \mathcal{P}^{n}(\hat{\alpha}) + \hat{c}_1^{T} E_{n+1}(\hat{\alpha}).\]
Due to \( \hat{\alpha}^{*n} = \mathcal{P}^{n}(\hat{\alpha}) \hat{c}_1, D_{\hat{\alpha}} \mathcal{P}(\hat{\alpha}) |_{\hat{\alpha} = \hat{\alpha}^{*n}} = \mathcal{P}(\hat{\alpha}^{*n}) \) and the product rule for matrices, we have
\[
D_{\hat{\alpha}} \left[ \mathcal{P}^{n+1}(\hat{\alpha}) \hat{c}_1 \right] = \sum_{k=0}^{n} \mathcal{P}^{k}(\hat{\alpha}) D_{\hat{\alpha}} \mathcal{P}(\hat{\alpha}) |_{\hat{\alpha} = \hat{\alpha}^{*n-k}} = \sum_{k=0}^{n} \mathcal{P}^{k}(\hat{\alpha}) \mathcal{P} \left( \hat{\alpha}^{*n-k} \right).
\]
Hence, equation (2.6) and (2.7) imply the consistency
\[
\|E_{n+1}(\hat{\alpha}) \hat{c}_1\|_2 \leq \sum_{k=0}^{n} \left\| \mathcal{P} \left( \hat{\alpha}^{*n-k} \right) \mathcal{P}^{k}(\hat{\alpha}) \hat{c}_1 - \mathcal{P}^{n}(\hat{\alpha}) \hat{c}_1 \right\|_2
\]
(2.8)\[= \sum_{k=0}^{n} \left\| \hat{\alpha}^{*n-k} \ast \hat{\alpha}^{*k} - \hat{\alpha}^{*n} \right\|_2 \rightarrow 0 \quad \text{for} \quad K \rightarrow \infty.
\]

We note from relation (2.8) that the term \( E_{n+1}(\hat{\alpha}) \hat{c}_1 = \sum_{k=0}^{n} \left[ \hat{\alpha}^{*n-k} \ast \hat{\alpha}^{*k} - \hat{\alpha}^{*n} \right] \) describes projection errors in the repeated Galerkin products that occur, since the Galerkin product is in general not associative [36] as the matrices \( \mathcal{P}(\hat{\alpha}) \) and \( \mathcal{P}(\hat{\hat{w}}) \) generally do not commute. There are two special cases:

(i) The symmetries \( \mathcal{P}(\hat{\alpha}^{*n}) \hat{c}_1 = \mathcal{P}(\hat{e}_1) \hat{\alpha}^{*n} = \hat{\alpha}^{*n} \) and \( \mathcal{P}(\hat{\alpha}^{*2}) \hat{\alpha} = \mathcal{P}(\hat{\alpha}) \hat{\alpha}^{*2} = \hat{\alpha}^{*3} \) imply for the choices \( n = 2,3 \) the equalities
\[
E_3(\hat{\alpha}) \hat{c}_1 = \left[ \mathcal{P}(\hat{\alpha}^{*2}) \hat{c}_1 + \mathcal{P}(\hat{\alpha}) \hat{\alpha} + \hat{\alpha}^{*2} \right] - 3\hat{\alpha}^{*2} = 0,
\]
\[
E_4(\hat{\alpha}) \hat{c}_1 = \left[ \mathcal{P}(\hat{\alpha}^{*3}) \hat{c}_1 + \mathcal{P}(\hat{\alpha}) \hat{\alpha}^{*2} + \hat{\alpha}^{*3} \right] - 4\hat{\alpha}^{*3} = 0.
\]
This is in accordance to [20, Lem. 3.1], where the standard Galerkin product with \( n = 2 \) is investigsted.

(ii) If the matrix \( \mathcal{P}(\hat{\alpha}) = \mathcal{H}D(\hat{\alpha}) \mathcal{H}^{T} \) has constant eigenvectors \( \mathcal{H} = V(\hat{\alpha}) \), we have \( E_n(\hat{\alpha}) \hat{c}_1 = 0 \) for all \( n \in \mathbb{N} \). This reflects associativity of the Galerkin product for those gPC expansions that result in such an eigenvalue decomposition. More precisely, the stochastic Galerkin matrices commute, i.e. we have
\[
\mathcal{P}(\hat{\alpha}) \mathcal{P}(\hat{\hat{w}}) = \mathcal{H} [D(\hat{\alpha}) D(\hat{\hat{w}})] \mathcal{H}^{T} = \mathcal{H} [D(\hat{\hat{w}}) D(\hat{\alpha})] \mathcal{H}^{T} = \mathcal{P}(\hat{\hat{w}}) \mathcal{P}(\hat{\alpha}).
\]
The derivative \( \partial_{\alpha_i} \mathcal{V}(\hat{\alpha}) = \partial_{\alpha_i} \left[ v_0(\hat{\alpha}) \cdots v_K(\hat{\alpha}) \right] \) can be expressed under the restrictive assumption of states \( \hat{\alpha} \in \mathbb{H}^+(\hat{\alpha}) \) that have distinct eigenvalues \( \mathcal{D}_i(\hat{\alpha}) \neq \mathcal{D}_j(\hat{\alpha}) \) for \( i \neq j \). Using the relation \( \partial_k \mathcal{P}(\hat{\alpha}) = \mathcal{M}_k \), we have
\[
\partial_{\alpha_i} v_i(\hat{\alpha}) = \sum_{j \neq i} v_j^T(\hat{\alpha}) \mathcal{M}_k v_j(\hat{\alpha}) / (\mathcal{D}_j(\hat{\alpha}) - \mathcal{D}_j(\hat{\alpha})) v_j(\hat{\alpha}) \quad \text{for all} \quad i, j, k = 0, \ldots, K.
\]
In the other case, the derivatives need to be computed numerically.

3. Haar-type stochastic Galerkin formulations

The expressions in Lemma 2.2 simplify if eigenvectors are constant, i.e. \( \mathcal{V}(\hat{\alpha}) = \mathcal{H} \). Then, we have \( \mathcal{E}(\hat{\alpha}) = 0 \). It has been shown in [39, Appendix B] that the Haar sequence [26] yields a stochastic Galerkin matrix with constant eigenvectors. More precisely, it is defined on a level \( J \in \mathbb{N}_0 \) and generates a wavelet system with \( K = 2^{J+1} \) elements by
\[
\mathbb{W}[\mathcal{H}_J] := \left\{ \chi_{[0,1]}(\xi), \psi_{k}(\xi) \mid k \in \mathbb{L}_J \right\} \quad \text{with} \quad \mathbb{L}_J := \left\{ 0 \right\} \cup \left\{ (k, \ell) \mid k = 0, \ldots, 2^J - 1, j = 1, \ldots, J, \right\},
\]
\[
\psi_k(\xi) := 2^{j/2} \psi(2^j \xi - k) \quad \text{for} \quad k \in \mathbb{L}_J \setminus \{0\} \quad \text{and} \quad \psi_0(\xi) := \begin{cases} 1 & \text{if} \quad 0 \leq \xi < 1/2, \\ 0 & \text{else}, \\ -1 & \text{if} \quad 1/2 \leq \xi < 1, \\ 0 & \text{else}, \end{cases}
\]
where \( \chi_{[0,1]} \) denotes the indicator function on \([0,1]\). Using a lexicographical order for the bases functions, the resulting stochastic Galerkin matrix reads as
\[
\mathcal{P}_J(\hat{u}) := \sum_{\ell \in \mathbb{L}_J} \hat{u}_\ell \mathcal{M}_\ell \quad \text{and} \quad \mathcal{M}_\ell := \left( \langle \psi_\ell, \phi_{\ell'} \rangle \right)_{\ell, \ell' = 0}^J.
\]
According to [39, 41], it admits the orthogonal eigenvalue decomposition \( \mathcal{P}_J(\hat{u}) = \mathcal{H}_J \mathcal{D}_J(\hat{u}) \mathcal{H}_J^T \) with the classical Haar matrix
\[
\mathcal{H}_J = \begin{pmatrix} \mathcal{H}_{J-1} \otimes (1,1) \\ \mathbb{1} \otimes (-1,-1) \end{pmatrix} \quad \text{for} \quad \mathcal{H}_0 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]
where \( \otimes \) denotes the Kronecker product and \( \mathbb{1} := \text{diag}\{1, \ldots, 1\} \) the identity matrix. More generally, we consider Haar-type matrices
\[
\mathcal{H} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0^T & \mathcal{O}_\mathcal{H} \end{pmatrix} \mathcal{H}_c \quad \text{for} \quad \mathcal{H}_c = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ h_1^d & h_2^r & \cdots & h_K^r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} h_i^d \\ s_i \end{pmatrix} := -(K + 1 - i),
\]
with \( \bar{0} := (0, \ldots, 0)^T \). Here, \( \mathcal{H}_c \in \mathbb{R}^{(K+1) \times (K+1)} \) is the canonical Haar matrix [42, Th. 4.1, Cor. 4.4] and \( \mathcal{O}_\mathcal{H} \in \mathbb{R}^{K \times K} \) is an orthogonal matrix. All of these Haar-type matrices \( \mathcal{H} \) generate in turn a wavelet system
\[
\mathbb{W}[\mathcal{H}] := \left\{ \chi_{[0,1]}(\xi), \phi_1(\xi), \ldots, \phi_K(\xi) \right\} \quad \text{for} \quad \phi_k(\xi) := \sum_{\ell = 0}^{K} \mathcal{H}_{k,\ell} \chi_{[0,1]}((K + 1)\xi - \ell)
\]
that allows to approximate any square-integrable function [42, Th. 5.1, Th. 5.2]. Many classical transforms used in signal processing are Haar matrices [41, Ch. 8]. These include Hadamard, Walsh,
Chebyshev matrices and the discrete cosine transform (DCT) matrix [42, Ex. 4.7] with entries

\[
(H_{\cos})_{i,j} = \begin{cases} 
1 & \text{if } i = 1, \\
\sqrt{2} \cos \left( \frac{(i+1)(2j-1)}{2(N+1)} \right) & \text{if } i > 1.
\end{cases}
\]

Wether a given basis leads to an eigenvalue decomposition with constant eigenvectors can be verified numerically by checking if all precomputed matrices \( \mathcal{M}_k \) commute [20, Lem. 4.1]. This allows also for an extension to piecewise linear functions. More precisely, the domain \( \Xi := [0,1] \) can be partitioned into \( N \in \mathbb{N} \) subdomains

\[
\Xi_k := (\xi_k, \xi_{k+1}) \quad \text{with} \quad \xi_k := k/N \quad \text{and} \quad k \in \mathbb{J} := \{0, \ldots, N - 1\}.
\]

The \( L^2(\Xi) \)-scaled orthogonal piecewise functions are

\[
\mathcal{W}_\Xi = \bigcup_{k \in \mathbb{J}} \mathcal{W}_{\Xi_k} \quad \text{for} \quad \mathcal{W}_{\Xi_k} := \left\{ \psi_{k,0}(\xi) := \chi_{[\xi_k, \xi_{k+1}]}(\xi) \sqrt{N}, \; \psi_{k,1}(\xi) := \chi_{[\xi_k, \xi_{k+1}]}(\xi) \sqrt{3N} \xi \right\}.
\]

Since the supports of \( \psi_{k,i} \) and \( \psi_{\ell,i} \) are disjoint for \( k \neq \ell \), the stochastic Galerkin matrix has the block diagonal structure

\[
\mathcal{P}_\Xi(\hat{\mathbf{u}}) := \sum_{\ell \in \mathbb{J}_\Xi} \hat{\mathbf{u}}_{\ell} \mathcal{M}_\ell = \text{diag} \left\{ \sum_{i=0,1} \hat{\mathbf{u}}_{0,i} \mathcal{M}^{(i)}, \ldots, \sum_{i=0,1} \hat{\mathbf{u}}_{N,i} \mathcal{M}^{(i)} \right\}
\]

for \( \mathcal{M}_\ell := \left( (\psi_{\ell,i}, \psi_{\ell,j})_\mathcal{P} \right)_{i,j \in \mathbb{J}_\Xi} \) and \( \mathcal{M}^{(i)} := \left( (\psi_{k,m}, \psi_{k,n}, \psi_{k,i})_\mathcal{P} \right)_{m,n \in \{0,1\}} \) with index set \( \mathbb{J}_\Xi := \{(k,i) \mid k \in \mathbb{J}, \; i = 0, 1\} \). Hence, the matrices \( \mathcal{M}_\ell \in \mathbb{R}^{2N \times 2N} \) commute, since the matrix \( \mathcal{M}^{(1)} \in \mathbb{R}^{2 \times 2} \) commutes with the identity matrix \( \mathcal{M}^{(0)} = I \in \mathbb{R}^{2 \times 2} \).

Lemma 2.2 is extended for these **Haar-type expansions** as follows.

**Lemma 3.1.** Given a Haar-type expansion the following statements hold.

(i) The equality \( \mathcal{P}(\mathcal{P}(\hat{\mathbf{u}})\hat{\mathbf{w}}) = \mathcal{P}(\hat{\mathbf{u}})\mathcal{P}(\hat{\mathbf{w}}) \) is satisfied for all \( \hat{\mathbf{u}}, \hat{\mathbf{w}} \in \mathbb{R}^{K+1} \).

(ii) The mapping \( T_m : E_0^+ \rightarrow E_0^+ \), \( \hat{\mathbf{u}} \mapsto \mathcal{P}^m(\hat{\mathbf{u}}) \) is bijective.

**Proof.** Equation (2.9) and the symmetry of the Galerkin product imply the first statement, i.e.

\[
\mathcal{P}(\mathcal{P}(\hat{\mathbf{u}})\hat{\mathbf{w}}) = \mathcal{P}(\mathcal{P}(\hat{\mathbf{u}})\hat{\mathbf{w}}) = \mathcal{P}(\hat{\mathbf{u}})\mathcal{P}(\hat{\mathbf{w}}) = \mathcal{P}(\hat{\mathbf{u}})\mathcal{P}(\hat{\mathbf{w}}) = \mathcal{P}(\hat{\mathbf{u}})\mathcal{P}(\hat{\mathbf{w}}) \quad \text{for all} \quad \hat{\mathbf{y}} \in \mathbb{R}^{K+1}.
\]

Statement (i) implies

\[
\hat{\mathbf{y}} := \mathcal{P}^n(\hat{\mathbf{u}}) \hat{\mathbf{w}} \Rightarrow \mathcal{P}(\hat{\mathbf{y}}) = \mathcal{P}(\mathcal{P}^n(\hat{\mathbf{u}})\hat{\mathbf{w}}) = \mathcal{P}(\mathcal{P}(\hat{\mathbf{u}})\hat{\mathbf{w}}) = \mathcal{P}(\hat{\mathbf{u}})\mathcal{P}(\hat{\mathbf{w}}) \quad \text{for all} \quad \hat{\mathbf{y}} \in \mathbb{R}^{K+1}.
\]

Hence, the mapping is bijective and the \( n \)-th root, stated in Lemma 2.2, is obtained by

\[
\hat{\mathbf{y}} = \mathcal{P}^n(\hat{\mathbf{u}}) \hat{\mathbf{w}} = \mathcal{P}^{n-1}(\hat{\mathbf{u}}) \mathcal{P} \hat{\mathbf{w}} = H D^{n-1}(\hat{\mathbf{u}}) H^T \hat{\mathbf{w}} = H D^{n-1}(\hat{\mathbf{u}}) H^T \hat{\mathbf{w}} \quad \text{for all} \quad \hat{\mathbf{y}} \in \mathbb{R}^{K+1}.
\]

**Remark 3.2.** Lemma 2.2 states an approach to approximate in principle any non-polynomial function of the form \( \rho^\infty(\xi) \) with \( m, n \in \mathbb{N} \) by

\[
\rho^\infty(\hat{\mathbf{y}}) := \mathcal{P}^m(\mathcal{P}(\hat{\mathbf{u}})) \hat{\mathbf{w}} \quad \text{satisfying} \quad \left\| \rho^\infty(\xi) - \Pi_K \left[ \rho^\infty(\hat{\mathbf{y}}) \right](\xi) \right\|_p \to 0 \quad \text{for} \quad K \to \infty.
\]
As emphasized in [14], projections in the repeated Galerkin multiplications are essentially truncations, which introduce additional approximation errors, since each one reduces a projection of order 2K to K. To achieve a desired accuracy, the order K must be sufficiently large and the computational complexity heavily depends on the choices m, n ∈ ℤ. Furthermore, an optimization problem is involved which may be numerically challenging, especially due to the derivatives ∂α, V(α) and the existence of roots is only locally guaranteed in a possibly small domain ℋ⁺.

In contrast, Lemma 3.1 is based on the solution of this optimization problem that is stated in closed form. It exists and it is unique for all admissible states and the existence of roots is only locally guaranteed in a possibly small domain ℋ⁺.

Let a Haar-type expansion be given. Then, the following statements hold.

(i) The gPC modes of the function \( p(u) := u^\gamma \) for the exponent \( \gamma \geq 1 \) and with the positive expansion \( Π_K[\hat{u}](\xi(\omega)) \in ℜ_+^d \) read as

\[
\hat{p}(\hat{u}) = \mathcal{H}D(\hat{u})^T \hat{c}_1 \quad \text{with Jacobian} \quad \text{D}_u\hat{p}(\hat{u}) = \gamma \mathcal{H}D(\hat{u})^{\gamma-1} \mathcal{H}^T.
\]

(ii) The gPC modes for the signum function of the real-valued expansion \( Π_K[\hat{u}](\xi(\omega)) \in ℜ \) are

\[
\hat{s}(\hat{u}) := \mathcal{H}\text{sign}(D(\hat{u}))^T \hat{c}_1.
\]

(iii) The gPC modes of the absolute value of the real-valued expansion \( Π_K[\hat{u}](\xi(\omega)) \in ℜ \) read as

\[
|\hat{u}| = \hat{s}(\hat{u}) * \hat{u} \quad \text{with Jacobian} \quad \text{D}_u|\hat{u}| = \mathcal{H}\text{sign}(D(\hat{u}))^T \hat{c}_1.
\]

(iv) The gPC modes of the p-norm for a vector-valued expansion \( Π_K[\hat{u}](\xi(\omega)) \in ℜ^d \) with gPC coefficients \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_d)^T \) are

\[
|\hat{u}|_p = \sum_{i=1}^d |\hat{u}_i|^{p-1} \quad \text{with} \quad |\hat{u}_i|^{p-1} = \mathcal{H}|D(\hat{u}_i)|^{p-1} \mathcal{H}^T \hat{c}_1.
\]

The Jacobian reads as

\[
\text{D}_u|\hat{u}|_p = \mathcal{H}\left[ \mathcal{D}^{\frac{1}{p}}(\hat{u})(\mathcal{H}(\hat{u})^T) \right] \mathcal{H}^T.
\]

Proof. Statement (i): According to [47, Th. 2] and [24, Th. 2.1], it holds \( \hat{u} \in ℜ_+^d \), i.e. \( D(\hat{u}) \geq 0 \), provided that all realizations \( u(\xi(\omega)) \in ℜ_+^d \) are non-negative. Due to Lemma 2.1 and Lemma 3.1, which implies \( \text{D}_uD(\hat{u})^T \hat{c}_1 = \mathcal{H}^T \), we have for \( \gamma = m/n \) and arbitrary \( m, n \in ℤ \) the gPC modes

\[
\hat{p}(\hat{u}) = T_m\left[ T_n[\hat{u}] \right] = \mathcal{H}D(\hat{u})^T \hat{c}_1,
\]

with Jacobian \( \text{D}_u\hat{p}(\hat{u}) = \gamma \mathcal{H}D(\hat{u})^{\gamma-1} \mathcal{H}^T \hat{c}_1 = \mathcal{H}D(\hat{u})^{\gamma-1} \mathcal{H}^T \).

Statement (ii): Due to \( T_2[\hat{u}] \in ℜ^*_+ \) for all \( \hat{u} \in ℜ^K \), the gPC modes for the absolute value read

\[
|\hat{u}| = T_2^{-1}\left[ T_2[\hat{u}] \right] = \mathcal{H}\sqrt{\mathcal{D}^2(\hat{u})}^T \hat{c}_1 = \mathcal{H}|D(\hat{u})|^T \hat{c}_1.
\]
Statement (ii): This statement follows from
\[
\hat{\mathbf{u}} = \mathcal{H} \left[ \text{sign} \{ \mathcal{D} (\hat{\mathbf{u}}) \} \mathcal{D} (\hat{\mathbf{u}}) \right] \mathcal{H}^T \hat{c}_1 = \hat{s} (\hat{\mathbf{u}}) \ast \hat{\mathbf{u}}
\]
\[
\| \Pi_K [ \hat{s} (\hat{\mathbf{u}}) \ast \hat{\mathbf{u}}] - \text{sign} \{ \mathbf{u} (\xi) \} \mathbf{u} (\xi) \|_p \to 0 \quad \text{for} \quad K \to \infty.
\]

Statement (iv): For a \( d \)-dimensional random quantity \( \mathbf{u} (\xi) = (\mathbf{u}_1 (\xi), \ldots, \mathbf{u}_d (\xi))^T \), we obtain the gPC modes for the \( p \)-norm by
\[
\| \hat{\mathbf{u}} \|_p = T_p^{-1} \left[ \sum_{i=1}^{d} \| \hat{u}_i \|^{*p} \right] \quad \text{with} \quad \| \hat{u}_i \|^{*p} = \mathcal{P}^p \left( \| \hat{u}_i \| \right) \hat{c}_1 = \mathcal{H} [ \mathcal{D} (\hat{\mathbf{u}}) ]^{p} \mathcal{H}^T \hat{c}_1 \in \mathbb{H}_0^+.
\]

Lemma 2.1 yields \( \mathcal{D}_u \hat{\mathbf{u}} = \mathcal{H} \mathcal{D} (\hat{\mathbf{u}}) \|_{\hat{\mathbf{u}} = \hat{\mathbf{u}}} \mathcal{H}^T \hat{s} (\hat{\mathbf{u}}) = \mathcal{P} (\hat{s} (\hat{\mathbf{u}})) \). Hence, we have
\[
\mathcal{D}_u \hat{\mathbf{u}} \|_*^p = p \mathcal{H} [ \mathcal{D} (\hat{\mathbf{u}}) ]^{p-1} \mathcal{D}_u [ \mathcal{D} (\hat{\mathbf{u}}) ] \mathcal{H}^T \hat{c}_1 = p \mathcal{H} [ \mathcal{D} (\hat{\mathbf{u}}) ]^{p-1} \mathcal{H}^T \mathcal{D}_u \hat{\mathbf{u}} = \mathcal{H} \left[ p \mathcal{D} (\hat{\mathbf{u}}) \|^{p-1} \mathcal{D} (\hat{s} (\hat{\mathbf{u}})) \right] \mathcal{H}^T.
\]

Then, the chain rule implies
\[
\mathcal{D}_u \| \hat{\mathbf{u}} \|_*^p = \mathcal{H} \mathcal{D}_c \left[ \mathcal{D}^\frac{p}{2} (\hat{\mathbf{c}} (\hat{\mathbf{u}})) \right] \mathcal{H} \mathcal{D}_u \| \hat{\mathbf{u}} \|_*^p = \mathcal{H} \left[ \mathcal{D}^\frac{p}{2} (\hat{\mathbf{c}} (\hat{\mathbf{u}})) \mathcal{D} (\hat{\mathbf{u}}) \|^{p-1} \mathcal{D} (\hat{s} (\hat{\mathbf{u}})) \right] \mathcal{H}^T.
\]

We emphasize that all gPC modes, which are presented in Theorem 3.3, are stated in closed form. More precisely, they only depend on the eigenvalues \( \mathcal{D} (\hat{\mathbf{u}}) = \mathcal{H}^T \mathcal{D} (\hat{\mathbf{u}}) \mathcal{H} \) that are directly obtained, since the Haar-type matrix \( \mathcal{H} \) and the linear operator \( \mathcal{P} (\hat{\mathbf{u}}) \) are known.

4. Applications to hyperbolic conservation laws

This section presents eigenvalue decompositions for hyperbolic systems with Lipschitz continuous flux functions that involve the non-polynomial expressions derived in Theorem 3.3. Furthermore, the domain, where the solution is defined, is stated precisely in terms of the convex sets (2.2).

4.1. Scalar conservation law with Lipschitz continuous flux. We consider the example [25, Ex. 3.1], where the scalar conservation law
\[
\partial_t \mathbf{u} (t, x, \xi) + \partial_x \left( \mathbf{u}^2 (t, x, \xi) + \left| \mathbf{u} (t, x, \xi) \right| \right) = 0 \quad \text{with} \quad \mathbf{u}_0 (x) = \text{sign} (\hat{x} (x, \xi))
\]

and initial discontinuity at the point \( \hat{x} (x, \xi (\omega)) := x - (\xi (\omega) - \frac{1}{2}) \) is analyzed. According to [25], the pointwise entropy solution is
\[
\mathbf{u}^{\text{ref}} (t, x, \xi) = \begin{cases} 
-1 & \text{if} \quad \frac{\hat{x} (x, \xi)}{t} \in (-\infty, -3), \\
\frac{1}{2} \left( \frac{\hat{x} (x, \xi)}{t} + 1 \right) & \text{if} \quad \frac{\hat{x} (x, \xi)}{t} \in [-3, -1), \\
0 & \text{if} \quad \frac{\hat{x} (x, \xi)}{t} \in [-1, 1), \\
\frac{1}{2} \left( \frac{\hat{x} (x, \xi)}{t} - 1 \right) & \text{if} \quad \frac{\hat{x} (x, \xi)}{t} \in [1, 3), \\
1 & \text{if} \quad \frac{\hat{x} (x, \xi)}{t} \in [3, \infty).
\end{cases}
\]

We note that there is an intermediate constant state due to a discontinuity in the characteristic speed. The following corollary states the intrusive form.
Corollary 1. Haar-type stochastic Galerkin formulations to the conservation law (4.1) read as
\[
\partial_t \tilde{u} + \partial_x \left( \tilde{u} \tilde{x}^2 + \tilde{u} \circ \tilde{s}(\tilde{u}) \right) = 0 \quad \text{with initial values} \quad \tilde{u}(0, x) = \left( \langle \text{sign}(x \xi), \phi_k(\xi) \rangle_{L^2} \right)_{k=0, \ldots, K},
\]
and generalized Jacobian \( \partial_{\tilde{u}} \tilde{f}(\tilde{u}) = 2P(\tilde{u}) + \tilde{s}(\tilde{u}) = \mathcal{H} \left[ 2D(\tilde{u}) + \text{sign} \left( D(\tilde{u}) \right) \right] \mathcal{H}^T \).

It is defined for all states \( \tilde{u} \in \mathbb{R}^{K+1} \).

Proof. The expressions for the flux function and its Jacobian follow directly from Theorem 3.3. □

4.2. Level set equations. Two-dimensional level-set equations in Hamilton-Jacobi form read as
\[
\partial_t \varphi(t, x) + v(x) \| \nabla \varphi(t, x) \|_2 = 0 \quad \text{for} \quad \varphi(0, x) = \varphi(x) \in \mathbb{R} \quad \text{and} \quad x = (x_1, x_2)^T.
\]
In the sense of viscosity solutions [6, 10], the scalar equation (4.3) is equivalent to the hyperbolic system
\[
\partial_t u(t, x) + \partial_{x_1} f_1(u(t, x)) + \partial_{x_2} f_2(u(t, x)) = 0
\]
with flux functions \( f_1(u, v) = \left( \| u \|_2, 0 \right), \quad f_2(u, v) = \left( 0, \| u \|_2 \right) \)

for \( u := (u_1, u_2)^T \) and initial conditions \( u(0, x) = \nabla_x \varphi_0(x) \in \mathbb{R}^2 \). The two-dimensional system (4.4) is hyperbolic in the sense that for all unit vectors \( \tilde{n} = (n_1, n_2)^T \) the matrix
\[
n_1 D_{\tilde{u}} f_1(u, v) + n_2 D_{\tilde{u}} f_2(u, v) = \frac{v}{\| u \|_2} \left( \begin{array}{cc} n_1 u_1 & n_1 u_2 \\ n_2 u_1 & n_2 u_2 \end{array} \right)
\]
is diagonalizable with real eigenvalues and a complete set of eigenvectors. Note that the Jacobian (4.5) reduces in the spatially one-dimensional case to \( D_{\tilde{u}} f(u, v) = v \partial_{\tilde{u}} |u| \). Here, the derivative of the norm is understood as the generalized gradient [8, 18, 9]. The following theorem states the corresponding stochastic Galerkin formulation and the generalized real characteristic speeds.

Corollary 2. Assume a Haar-type expansion with gPC modes \( \tilde{v} \) that account for random velocities \( v(\xi) \). Then, a stochastic Galerkin formulations for the two-dimensional level-set equations (4.4) read as
\[
\partial_t \tilde{u}(t, x) + \partial_{x_1} \tilde{f}_1(\tilde{u}(t, x), \tilde{v}(x)) + \partial_{x_2} \tilde{f}_2(\tilde{u}(t, x), \tilde{v}(x)) = 0
\]
with flux functions \( \tilde{f}_1(\tilde{u}, \tilde{v}) = \left( \tilde{v} \ast \| u \|_2, 0 \right), \quad \tilde{f}_2(\tilde{u}, \tilde{v}) = \left( 0, \tilde{v} \ast \| u \|_2 \right) \)

and real spectrum\(^*\)
\[ \sigma \left\{ \tilde{\tilde{n}} \cdot D_{\tilde{u}} \tilde{f}(\tilde{u}, \tilde{v}) \right\} = \left\{ D(\tilde{v}) D \left( \| u \|_2 \right)^{-1} \left( n_1 D(\tilde{u}_1) + n_2 D(\tilde{u}_2) \right), \emptyset \right\}, \]

where eigenvalues read as \( D \left( \| u \|_2 \right) = \sqrt{D(\tilde{u}_1)^2 + D(\tilde{u}_2)^2} \). It is defined for all states \( \tilde{u} \in \mathbb{R}^{2K} \)

and satisfies in the sense of generalized gradients the relation
\[ \sigma \left\{ \tilde{\tilde{n}} \cdot D_{\tilde{u}} \tilde{f}(\tilde{u}, \tilde{v}) \right\} = \left\{ D(\tilde{v}) \text{sign} \left( D(\tilde{u}_1) \right), \emptyset \right\} \quad \text{for} \quad \tilde{u}_j = 0. \]

\(^*\)With a slight abuse of notation the spectrum is stated in terms of diagonal matrices.
Proof. According to Theorem 3.3, it holds \( D_{\tilde{u}} \parallel u \parallel_2 = \mathcal{H} D( \parallel u \parallel)^{-1} D(\tilde{u}_i) \mathcal{H}^T \). Defining the block diagonal matrix \( \hat{\mathcal{H}} := \text{diag}\{1, 1\} \otimes \mathcal{H} \), we have the Jacobian

\[
\hat{J}_{\tilde{u}}(\tilde{u}, \tilde{v}) = \hat{\mathcal{H}} \left( \begin{array}{ccc}
\text{diag}\{1, 1\} \otimes \left( D(\tilde{v}) D( \parallel u \parallel)^{-1} \right) \end{array} \right) \left( \begin{array}{cc}
\frac{n_1}{n_2} D(\tilde{u}_i) & \frac{n_1}{n_2} D(\tilde{u}_2) \end{array} \right) \hat{\mathcal{H}}^T.
\]

Due to the sparse structure, the real spectrum is obtained. In particular for \( u_i = 0 \), we have

\[
D(\tilde{v}) D( \parallel u \parallel)^{-1} \left( \begin{array}{cc}
\frac{n_1}{n_2} D(\tilde{u}_i) & \frac{n_1}{n_2} D(\tilde{u}_2) \end{array} \right) = n_i D(\tilde{v}) D(\tilde{u}_i)^{-1} D(\tilde{u}_i) = n_i D(\tilde{v}) \text{sign}\{D(\tilde{u}_i)\}. 
\]

\[
□
\]

### 4.3. Gas flow with Lipschitz continuous pressure law

We consider a model [9, Sec. 5] related to the \( p \)-system. The unknowns \( u := (u, v)^T \) are the velocity \( u(t, x) \) and the specific volume \( v(t, x) \) of a fluid. A Lipschitz continuous pressure is stated in terms of the specific volume that satisfies \( p-(v^*) < p+(v^*) \) and \( p'(v) < 0, p''(v) > 0 \) for \( v \neq v^* \). The hyperbolic system reads as

\[
\partial_t u + \partial_x f(u) = 0
\]

with flux \( f(u) = \left( \frac{p(v)}{-u} \right) \) and pressure \( p(v) := \left\{ \begin{array}{cc} v^{-\gamma_1} & \text{if } v < v_*, \\
\gamma_1 v^{-\gamma_1} + \Delta v_* & \text{if } v > v_* \end{array} \right. \)

where the value \( \Delta v_* := v_*^{\gamma_1} - v_*^{\gamma_2} \) is chosen such that the pressure law is Lipschitz continuous. The following theorem states the stochastic Galerkin formulation that accounts for a Lipschitz continuous, possibly random pressure law.

**Corollary 3.** Assume a Haar-type expansion with gPC modes \( \tilde{v}_* \) and \( \Delta \tilde{v}_* \) that accounts for a random state \( v_i(\xi) \). Then, a stochastic Galerkin formulation to the system (4.6), describing the gPC modes \( \tilde{u} := (\tilde{u}, \tilde{v})^T \) for \( \tilde{u} \in \mathbb{R}^{K+1} \) and \( \tilde{v} \in \mathbb{H}^+ \), reads as

\[
\partial_t \tilde{u}(t, x) + \partial_x \tilde{f}(\tilde{u}(t, x)) = 0 \quad \text{with flux function } \tilde{f}(\tilde{u}) = \left( \frac{\tilde{\rho}(\tilde{v})}{\tilde{u}} \right)
\]

and pressure law \( \tilde{\rho}(\tilde{v}) = -\frac{1}{2} \left[ \tilde{s}(\tilde{v} - \tilde{v}_*) - \tilde{e}_1 \right] * v^{-\gamma_1} + \frac{1}{2} \left[ \tilde{s}(\tilde{v} - \tilde{v}_*) + \tilde{e}_1 \right] * (v^{-\gamma_2} + \Delta \tilde{v}_*). \)

Furthermore, the generalized spectrum is \( \sigma\{D_{\tilde{u}} \tilde{f}(\tilde{u})\} = \{ \pm D_{\tilde{p}/\tilde{v}}(\tilde{v}) \} \) with

\[
D_{\tilde{p}/\tilde{v}}(\tilde{v}) := \sqrt{\frac{\gamma_1}{2}} \left[ D(\tilde{s}(\tilde{v} - \tilde{v}_*)) - 1 \right] D^{-\frac{\gamma_1+1}{2}}(\tilde{v}) - \sqrt{\frac{\gamma_2}{2}} \left[ D(\tilde{s}(\tilde{v} - \tilde{v}_*)) + 1 \right] D^{-\frac{\gamma_2+1}{2}}(\tilde{v}).
\]

**Proof.** We write the Lipschitz continuous pressure (4.1) as

\[
p(v) = -\frac{1}{2} \left[ \text{sign}(v - v^*) - 1 \right] v^{-\gamma_1} + \frac{1}{2} \left[ \text{sign}(v - v^*) + 1 \right] (v^{-\gamma_2} + \Delta v_*).
\]

Theorem 3.3 yields the projected pressure law \( \tilde{\rho}(\tilde{v}) \). Furthermore, Theorem 3.3 implies the generalized Jacobian

\[
D_{\tilde{u}} \tilde{f}(\tilde{u}) = \left( \begin{array}{cc} 0 & D_{\tilde{p}/\tilde{v}}(\tilde{v}) \\
1 & 0 \end{array} \right) \quad \text{with } D_{\tilde{v}} \tilde{\rho}(\tilde{v}) = \frac{\gamma_1}{2} D(\tilde{s}(\tilde{v} - \tilde{v}_*) - \tilde{e}_1) D^{-\gamma_1+1}(\tilde{v}) \mathcal{H}^T
\]

\[
- \frac{\gamma_2}{2} D(\tilde{s}(\tilde{v} - \tilde{v}_*) + \tilde{e}_1) D^{-\gamma_2+1}(\tilde{v}) \mathcal{H}^T.
\]
Hence, we have the eigenvalue decomposition
\[
D_u \hat{f}(\hat{u}) = \hat{T}(\hat{v}) \text{diag} \left\{ -D_{\nu^1}(\hat{v}), D_{\nu^2}(\hat{v}) \right\} \hat{T}(\hat{v})^{-1} \quad \text{for} \quad \hat{T}(\hat{v}) := \begin{pmatrix} H & (D_{\nu^1}(\hat{v}) \ D_{\nu^2}(\hat{v})) \end{pmatrix}.
\]

**4.4. Isentropic Euler equations.** Following [35, Sec. 18.3], we consider two-dimensional isentropic Euler equations that describe the density \( \rho \) of a gas and the mass flux \( q_i \) in \( x_i \)-direction. The flux functions, describing the temporal propagation of the unknown \( u = (\rho, q_1, q_2)^T \), read as
\[
(4.7) \quad f_1(u) = \begin{pmatrix} q_1 \\ \frac{q_1}{\rho} + p(\rho) \end{pmatrix} \quad \text{and} \quad f_2(u) = \begin{pmatrix} q_2 \\ \frac{q_2}{\rho} + p(\rho) \end{pmatrix} \quad \text{with pressure law} \quad \rho^\gamma > 0.
\]

It has been proposed in [39] to use Roe variables \( \alpha := \sqrt{\rho} \) and \( \beta_i(u) = q_i/\alpha \) as auxiliary variables that yield the stochastic Galerkin formulations
\[
\hat{f}_1(\hat{u}) = \begin{pmatrix} \hat{\beta}_1(\hat{u}) * \beta_1(\hat{u}) + \hat{\rho}(\hat{\rho}) \\ \beta_1(\hat{u}) * \beta_2(\hat{u}) \end{pmatrix} \quad \text{and} \quad \hat{f}_2(\hat{u}) = \begin{pmatrix} \hat{\beta}_2(\hat{u}) * \beta_2(\hat{u}) + \hat{\rho}(\hat{\rho}) \\ \hat{\beta}_1(\hat{u}) * \beta_2(\hat{u}) \end{pmatrix},
\]

(4.8) for \( \sqrt{(\hat{\rho}(\hat{\rho}) := \arg\min_{\alpha \in \mathbb{R}^+} \{ \eta^{(2)}(\hat{\alpha}) \} \quad \text{and} \quad \hat{\beta}_i(\hat{u}) := P(\sqrt{\hat{\rho}(\hat{\rho}))\hat{q}_i.} \)

Furthermore, this choice leads to a hyperbolic formulation of one-dimensional isothermal Euler equations for any gPCE expansions [21]. Here, we consider the Haar-type formulations that allow for an extension to non-polynomial pressure laws and multiple space dimensions without optimization problems.

**Corollary 4.** Define the auxiliary variables \( \nu_i(\hat{u}) := HD_{\nu^i}(\hat{u})H^T \epsilon_1 \) for \( D_{\nu^i}(\hat{u}) := D(\hat{q}_i)D(\hat{\rho})^{-1} \).

Then, a Haar-type stochastic Galerkin formulation, which is defined for all states \( \hat{u} := (\hat{\rho}, \hat{q}_1)^T \) satisfying \( \hat{\rho} \in \mathbb{R}^+ \), to the two-dimensional isentropic Euler equations (4.7) reads as
\[
\partial_t \hat{u}(t, x) + \partial_{x_1} f_1(\hat{u}(t, x), \hat{v}(x)) + \partial_{x_2} f_2(\hat{u}(t, x), \hat{v}(x)) = 0
\]

with flux functions \( \hat{f}_1(\hat{u}, \hat{v}) = \begin{pmatrix} \nu_1(\hat{u})^* \beta_1(\hat{u}) + \hat{\rho}(\hat{\rho}) \\ \nu_1(\hat{u})^* \nu_2(\hat{u}) \end{pmatrix} \quad \text{and} \quad \hat{f}_2(\hat{u}, \hat{v}) = \begin{pmatrix} \nu_2(\hat{u})^* \beta_2(\hat{u}) + \hat{\rho}(\hat{\rho}) \\ \nu_2(\hat{u})^* \nu_1(\hat{u}) \end{pmatrix},
\]

and real spectrum \( \sigma \{ \hat{n} \cdot D_u \hat{f}(\hat{u}, \hat{v}) \} = \{ \hat{n} \cdot D_{\nu^i}(\hat{u}) \pm \sqrt{\gamma} D^{1/2}(\hat{\rho}) \}, \quad \hat{n} \cdot D_{\nu^i}(\hat{u}) \}
\]

for the projected pressure law \( \hat{\rho}(\hat{\rho}) := HD(\hat{\rho})^T \epsilon_1 \) and \( \hat{n} \cdot D_{\nu^i}(\hat{u}) := n_1 D_{\nu^i}(\hat{u}) + n_2 D_{\nu^i}(\hat{u}). \)

**Proof.** The solution to the optimization problem is obtained by Theorem 3.3 which yields
\[
\hat{\beta}_i(\hat{u}) = HD(\hat{q}_i)D(\hat{\rho})^{-1/2}H^T \epsilon_1 \quad \text{and} \quad \hat{\beta}_i(\hat{u})^* \beta_j(\hat{u}) = HD(\hat{q}_i)D(\hat{q}_j)D(\hat{\rho})^{-1}H^T \epsilon_1.
\]

Lemma 2.1 yields \( D_{\hat{\rho}}[D(\hat{\rho})V^T \epsilon_1] = 1 \). Hence, we have the Jacobian
\[
D_{\hat{\rho}}[\hat{\beta}_i(\hat{u})^* \beta_j(\hat{u})] = HD(\hat{q}_i)D(\hat{q}_j)D_{\hat{\rho}}[D(\hat{\rho})^{-1}H^T \epsilon_1] = -HD_{\nu^i}(\hat{u})D_{\nu^j}(\hat{u})H^T.
\]
Due to Theorem 3.3, we have the expression $D_\rho \hat{\rho}(\rho) = \mathcal{H}[\gamma \mathcal{D}(\rho)^{\gamma-1}] \mathcal{H}^T$ and the Jacobian of the flux function reads as

$$\vec{n} \cdot D_{\vec{u}} \hat{f}_1(\vec{u}) = n_1 D_{\vec{u}} \hat{f}_1(\vec{u}) + n_2 D_{\vec{u}} \hat{f}_2(\vec{u}) \quad \text{for} \quad \hat{\mathcal{H}} := \text{diag}\{\mathcal{H}, \mathcal{H}, \mathcal{H}\} \quad \text{and}
$$

$$D_{\vec{u}} \hat{f}_1(\vec{u}) = \hat{\mathcal{H}} \begin{pmatrix} \gamma \mathcal{D}(\rho)^{\gamma-1} - \mathcal{D}_{\nu_1}(\vec{u})^2 & 2\mathcal{D}_{\nu_1}(\vec{u}) & 0 \\ -\mathcal{D}_{\nu_1}(\vec{u})\mathcal{D}_{\nu_2}(\vec{u}) & \mathcal{D}_{\nu_2}(\vec{u}) & \mathcal{D}_{\nu_1}(\vec{u}) \\ 0 & 1 & 0 \end{pmatrix} \hat{\mathcal{H}}^T,$n$$

$$D_{\vec{u}} \hat{f}_2(\vec{u}) = \hat{\mathcal{H}} \begin{pmatrix} 0 & 1 & 0 \\ -\mathcal{D}_{\nu_1}(\vec{u})\mathcal{D}_{\nu_2}(\vec{u}) & \mathcal{D}_{\nu_2}(\vec{u}) & \mathcal{D}_{\nu_1}(\vec{u}) \\ \gamma \mathcal{D}(\rho)^{\gamma-1} - \mathcal{D}_{\nu_2}(\vec{u})^2 & 0 & 2\mathcal{D}_{\nu_2}(\vec{u}) \end{pmatrix} \hat{\mathcal{H}}^T.$n

Due to the sparse block diagonal structure, the real spectrum is obtained.

Finally, we remark that all involved matrices can be exactly precomputed at a finite number of quadrature points $\xi^{(1)}, \ldots, \xi^{(Q)}$. According to [21, Sec. 5.1], the positivity of the gPC expansions $\Pi_K[\hat{\rho}](\xi^{(q)}) > 0$ at all points $q = 1, \ldots, Q$ implies $\hat{\rho} \in \mathbb{H}^+$, which guarantees hyperbolicity of the stochastic Galerkin formulation in Theorem 4. Hence, the connection between the hyperbolicity of the Galerkin and the original system is established.

5. Numerical results

All numerical experiments are performed with a globally third order scheme applied on the deterministic Galerkin system. We construct the numerical scheme applying the method of lines and the local Lax-Friedrichs flux. The scheme employs the classical third order strong stability preserving (SSP) Runge-Kutta (RK) with three stages [29] for the time discretization and third-order CWENO reconstructions for the high-order spatial discretization. In particular, for the one dimensional problems we consider the CWENO method of [11], whereas for the two dimensional problems we use the truly 2D reconstruction described in [45, 7] which avoids dimensional splitting. The use of CWENO reconstructions is also suited for the high-order numerical treatment of balance laws where the source terms are integrated with a Gaussian quadrature formula matching the order of the scheme. The scheme is implemented in the finite volume framework SteFVi [23]. The computational grid is set up with uniform cells and corresponding ghost cells taking into account the boundary conditions. The CFL number is always set to 0.45.

Remark 5.1. Since the flux functions of the derived systems are not smooth, i.e. the systems are not in the common strictly hyperbolic setting, nonclassical wave phenomena occur. Thus, numerical solvers usually based on approximations of solutions to Riemann problems might fail to recover nonclassical wave patterns. However, it has been shown in [38] that most solvers are able to correctly approximate solutions to non-classical hyperbolic systems of PDEs provided that the time step is small enough.

The subdifferential of the absolute value at zero is the whole interval $[-1, 1]$ and the generalized spectra derived in Section 4 enter the local Lax-Friedrichs flux as the maximum of the characteristic speeds. This ensures sufficient numerical viscosity and, as our computations show, the SSPRK-CWENO schemes approximate the non-classical waves correctly.

5.1. Scalar conservation law with Lipschitz continuous flux. We illustrate the theoretical results that are based on Haar-type expansions by means of the random initial value problem (4.1) in Subsection 4.1. In particular, we consider the wavelet systems $\mathbb{W}[\mathcal{H}_{\cos}]$ and $\mathbb{W}[\mathcal{H}_J]$ that are
generated by the discrete cosine and Haar transform. Initial values for the stochastic Galerkin formulation presented in Corollary 1 are obtained by the orthogonal projection of the random, space-dependent signum function \( \text{sign}(\hat{x}(x, \xi(\omega))) \). The first mode, i.e. the mean \( \langle \text{sign}(\hat{x}(x, \xi)), \phi_0(\xi) \rangle_p = \mathbb{E} \left[ \text{sign}(\hat{x}(x, \xi)) \right] \), yields the scaling function that is illustrated in Figure 1 as black line with the scale shown at the left axis. The detail functions are obtained by the projections \( \langle \text{sign}(\hat{x}(x, \xi)), \phi_k(\xi) \rangle_p \) for \( k \geq 1 \) and shown at the right axis. We observe that the details are “smoother” if the cosine transform (left panel) is used in comparison to the Haar transform (right panel).

![cosine transform](image1.png)

**Figure 1.** Projection of random initial values \( \langle \text{sign}(\hat{x}(x, \xi)), \phi_k(\xi) \rangle_p \). The left panel accounts for the cosine transform (3.4), the right panel for the Haar transform (3.2). The colorbar states realizations \( \text{sign}(\hat{x}(x, \xi(\omega))) \in \{-1, 1\} \).

![Haar transform](image2.png)

**Figure 2.** Realizations \( \Pi_K[\hat{u}(0, x)](\xi(\omega)) \) that result from the cosine transform (left) and Haar transform (right) applied to the random initial values \( \text{sign}(\hat{x}(x, \xi)) \).
Vice versa, the obtained scaling and detail functions define the approximation $\Pi_K \hat{u}(0,x) \hat{\xi}$ by the wavelet system (3.3), which is shown in Figure 2. The realizations are stated by the colorbar. In comparison to Figure 1, where the realizations sign$(\hat{x}(x, \xi(x))) \in \{-1, 1\}$ define two areas, which are separated by a straight line given by the scaling function, a piecewise approximation is observed in Figure 2, which results from a compression of input data.

The gPC modes $\hat{u}(t, x)$ are propagated over time by the hyperbolic system, derived in Corollary 1. The solution in turn defines again an approximation $\Pi_K \hat{u}(t, x) \hat{\xi}$, which is illustrated in the upper
panels of Figure 3 for both the cosine and Haar transform. In particular, the approximated solution at time \( t = 0 \) is shown in the left, lower panel for the cosine transform with level \( J = 2 \). The mean squared error

\[
\text{MSE}\left[ \hat{u}, u^{\text{ref}} \right](t) := \int \mathbb{E}\left[ (\Pi_K \hat{u}(t, x, \xi) - u^{\text{ref}}(t, x, \xi))^2 \right] \, dx
\]

between the approximation \( \Pi_K [\hat{u}(0.2, x)](\xi) \) and the reference solution (4.2) for the Haar transform is shown in the lower, right panel with respect to the left axis. Furthermore, the difference of the mean squared error of the solution that is obtained by the Haar and cosine transform is stated in terms of the ratio \( \text{MSE}_{\text{Haar}}/\text{MSE}_{\text{cos}} \), i.e. the ratio between the mean squared errors (5.1) that are obtained by the Haar and cosine transform. As expected, we observe a decreasing MSE for an increasing number of basis functions. In this particular example, the Haar expansion results in a lower MSE than the cosine transform. Furthermore, the MSE of the solution may be lower than the MSE of the initial values, since the solution comes along with a smoother dependency on the stochastic input than the initial values.

5.2. **Level set equations.** We consider the two-dimensional level set equations (4.4) with uncertain velocity \( v \sim \mathcal{U}[1/2, 1] \). Initial values are deterministic and read as

\[
\mathbf{u}(0, x) = \begin{cases} 
(1, 0)^T & \text{if } x \in [-2, 2]^2, \\
(-1, 0)^T & \text{else.}
\end{cases}
\]

The left panel of Figure 4 shows the mean for the unknown \( \mathbf{u}_1 \) and the right panel the standard deviation that are obtained by the intrusive formulation in Corollary 2.

**Figure 4.** Left: Mean \( \mathbf{u}_1 \) at time \( t = 1 \) to the intrusive formulation in Corollary 2. Right: Standard deviation given by \( \left\| (\mathbf{u}_1, \ldots, (\mathbf{u}_1)_K) \right\|_2(1, x) \).

5.3. **Gas flow with Lipschitz continuous pressure law.** This subsection is devoted to the \( p \)-system with Lipschitz continuous flux function (4.6) and the corresponding stochastic Galerkin formulation that is introduced in Corollary 3. In particular, we consider the random pressure law

\[
p(v; \xi) := \begin{cases} 
v^{-\gamma_1} & \text{if } v < v_*(\xi), \\
v^{-\gamma_2} + \Delta v_*(\xi) & \text{if } v > v_*(\xi)
\end{cases}
\]

with \( v_*(\xi) \sim \mathcal{U}[1, 1.5] \).
Deterministic initial values for the Riemann problem read as $u(0,x) = (1,0)^T$ for $x < 0$ and $u(0,x) = (3,0)^T$ for $x > 0$, respectively. As in the previous subsections, we consider the wavelet systems $W[H_{\cos}]$ and $W[H_J]$ that are generated by the discrete cosine and Haar transform.

The upper panels of Figure 5 show the exact solution to the relative volume $v$ that has been derived by Correia, Le Floch and Thanh in [9, Sec. 5]. A rarefaction wave is connected by an intermediate state (purble) to a shock (transparent). More precisely, there is an intermediate constant state in the rarefaction wave, which results from a discontinuity in the characteristic speeds. This constant state is highlighted in the right upper panel of Figure 5.
Figure 6. The upper panel shows a zoom on the intermediate state in the rarefaction wave for the levels $J = 2, \ldots, 5$, where the cosine transform is used, as well as the reference solution. The middle (cosine transform) and lower (Haar transform) panels show the absolute error between the stochastic Galerkin approximation in Corollary 3 and the reference solution according to [9, Sec. 5].
The lower, left panel shows the gPC modes $\hat{v}(1,x)$ at time $t = 1$ that are obtained by solving the intrusive system stated in Corollary 3 with level $J = 2$. The first mode $\hat{v}_0$, describing the mean of the random solution, is plotted with respect to the left axis. The details determine the variance and are stated at the right axis. The lower, right panel shows the realizations $\Pi_K \hat{v}(t,x)(\xi(\omega))$. In particular, a stepwise approximation of the intermediate state in the rarefaction wave is observed.

Figure 6 is devoted to a convergence analysis for increasing levels. Therein, a zoom in the area of the rarefaction wave reveals the regularity and the accuracy of the stochastic Galerkin formulation in comparison to the reference solution, given on the right. The lower panels show the corresponding absolute error between the Galerkin formulations for the cosine transform (middle panel) with the levels $J = 2, \ldots, 5$, the Haar transform (lower panel) and the reference solution, also plotted with respect to the right colorbar in the upper, right panel of Figure 5.

5.4. Isentropic Euler equations. The two-dimensional system (4.7) is considered with the choice $\gamma = 4/3$. We use the stochastic Galerkin formulation that is derived in Corollary 4 with the random initial values

$$u(0,x;\xi) = \begin{cases} (ho_0(\xi),0)^T & \text{for } x \in [-1,1]^2, \\ (1,0)^T & \text{else} \end{cases}$$

and $\rho_0(\xi) \sim U[2,3]$.

The left panel of Figure 7 shows the mean of the random density. The profile for $x_2 = 0$ is highlighted by a green line and analyzed in the right panel. Therein, a reference solution (blue, dotted) is shown that is obtained by a Monte Carlo simulation, where the solution to the random Riemann problem is exactly given in [49, Sec. 4]. Furthermore, the bounds of the confidence region, where all exact random solutions occur, are shown as black lines. The grey region is the confidence region that is obtained by the intrusive formulation in Corollary 4.

**Figure 7.** Left: Mean $\hat{\rho}_0(0.5,x)$ at time $t = 0.5$ to the intrusive formulation in Corollary 4. Right: Reference solution to the mean, which is illustrated by the green profile in the left panel.
6. Summary

Stochastic Galerkin formulations have been investigated that are generated by Haar-type matrices. These matrices arise in a number of classical transforms, including the Haar and cosine transform, and generate a wavelet system, which states the functional dependence on a random state. This allows to express also non-polynomial quantities. In particular, stochastic Galerkin formulations for higher moments, the signum function and norms have been established.

Furthermore, theoretical and numerical challenges are discussed that occur when classical polynomials are used in the polynomial chaos expansions. So far, it has been a major restriction in many applications to hyperbolic systems, when stochastic Galerkin projections not preserve deterministic wellposedness results unless numerically expensive transforms are introduced. In contrast, the presented Haar-type formulations are stated in closed form and do not require any additional transforms. This allows for applications to a wide class of hyperbolic conservation laws. In particular, eigenvalue decompositions for several hyperbolic systems, including multi-dimensional level-set equations and isentropic Euler equations with Lipschitz continuous flux, have been analyzed. The theoretical findings have been also numerically investigated by integrating the Galerkin system with globally third order one-dimensional and truly two-dimensional schemes which are based on CWENO reconstructions.

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