Gauge Theories Coupled to Fermions in Generation

Hiromi KASE, Katsusada MORITA* and Yoshitaka OKUMURA**

Department of Physics, Daido Institute of Technology, Nagoya 457-0811, Japan
*Department of Physics, Nagoya University, Nagoya 464-8602, Japan
**Department of Natural Sciences, Chubu University, Kasugai 487-0027, Japan

(Received February 19, 1998)

Gauge theories coupled to fermions in generation are reformulated in a modified version of extended differential geometry with the symbol $X$. After discussing several toy models, we reformulate in our framework the standard model based on Connes’ real structure. It is shown that for the most general bosonic lagrangian which also works in reconstructing the scalar potential in $N = 2$ super Yang-Mills theory, the Higgs mechanism operates only for more than one generation, as first pointed out by Connes and Lott.

§1. Introduction

An ingenious way of ‘geometrizing’ the Higgs mechanism, namely, unifying the gauge and Higgs fields via a Dirac operator on a product space of Minkowski space-time with a two-point internal space, was proposed by Connes, who applied his noncommutative geometry (NCG) to reconstruct the standard model. The gauge group is defined to be the unitary group of a noncommutative algebra which is represented on a Hilbert space of fermions. The algebra representation naturally leads to the concept of generation. It is also remarkable that a pure Yang-Mills action functional in NCG automatically contains a Higgs potential.

Introduction of the NCG approach initiated by Connes has been followed by many works in the same or similar veins. Among these, Sitarz first pointed out that NCG admits a differential geometric formulation in terms of the symbol $X$, which is forced to satisfy curious properties. For instance, unlike $dx$, $X$ is not closed, $dxX = 0$, and the exterior product is symmetric, $X \wedge X = 0$. In fact, it is possible to define a consistent noncommutative differential geometry with $d_X X = 0$ for model building provided $X \wedge X \neq 0$.

On the other hand, it was shown by Chamseddine that NCG is capable of reconstructing $N = 2, 4$ super Yang-Mills theories. We have noted, however, that to describe such theories using $X$ we must abandon the symmetry $X \wedge X \neq 0$ and adopt the new relation $X \wedge X = 0$ with the field strength redefined. This allows $X$ to become more similar to the usual one-form basis.

Connes also introduced real structure into NCG. This concept has been discussed in more details in several related works. This means that flavor and color symmetries are to be implemented in the bimodule structure of the Hilbert space.

*) The symbol $X$ constitutes the fifth one-form basis in addition to the usual four one-form basis.
The purpose of this paper is first to present a modified version of extended differential geometry, next to apply it to broken gauge theories coupled to fermions in generation, and third to discuss the standard model in our framework relating the real structure with the factorization property of the standard model gauge transformations.

The next section defines a modified form of an extended differential geometry. Toy models will be discussed in §3. The standard model will be treated in §4. Section 5 is devoted to discussion. Some calculational details are postponed to the Appendix.

§2. Extended differential geometry

Following Connes, we shall consider an involutive noncommutative algebra \( A \) together with its \(*\)-preserving representation \( \rho \) acting on the Hilbert space \( \mathcal{H} \) of fermions such that

\[
\rho(ab) = \rho(a)\rho(b), \quad \rho(a+b) = \rho(a) + \rho(b), \quad \rho(a^*) = \rho(a)^\dagger,
\]

where \( a, b \in A \), \( a \mapsto a^* \) is the involution, and \( ^\dagger \) denotes hermitian conjugation. In NCG \( \rho(A) \) is the set of multiplicative operators on the fermion Hilbert space \( \mathcal{H} \), \( (a, \psi) \mapsto \rho(a)\psi, a \in A, \psi \in \mathcal{H} \).

For simplicity we omit the symbol \( \rho \) in our notation in the rest of this section. The massless Dirac lagrangian takes the form

\[
\mathcal{L}_D = i\bar{\psi}\gamma \cdot \partial \psi = i(\bar{\psi}, d\psi),
\]

\[
\bar{\psi} = \gamma_\mu \psi d\hat{x}^\mu, \quad (d\hat{x}^\mu, d\hat{x}^\nu) = g^{\mu\nu}, \quad g^{\mu\nu} = \text{diag}(1,-1,-1,-1),
\]

with \( d \) denoting the usual exterior derivative in ordinary differential geometry, \( d\psi = \partial_\mu \psi d\hat{x}^\mu \), and the 'hat' indicating dimensionlessness. Clearly this is not invariant under the gauge transformation

\[
\psi \rightarrow g\psi = g\psi, \quad g \in \mathcal{U}(A),
\]

where the unitary group of the algebra \( A^* \)

\[
\mathcal{U}(A) = \{ g \in A; gg^* = g^*g = 1 \}
\]

defines the gauge group. The lack of gauge invariance is remedied by replacing \( i(\bar{\psi}, d\psi) \) with

\[
\mathcal{L}_D = i\sum_i (a^i\bar{\psi}, d(b^i\psi)), \quad a^i, b^i \in A
\]

and assuming in addition to Eq. (2·3) that the gauge transformation**

\[
a^i \rightarrow ga^i, \quad b^i \rightarrow b^ig^*, \quad g \in \mathcal{U}(A).
\]

---

* The algebra \( A \) contains as a subset the commutative algebra \( C^\infty(M_4) \) defined over Minkowski space-time \( M_4 \).

** We make use of the equivalence relation \( (ag^*, g\psi) \sim (a, \psi), g \in \mathcal{U}(A) \).
Gauge Theories Coupled to Fermions in Generation

We assume that \( \sum_i (a^i \bar{\psi} b^i \psi) = \langle \bar{\psi}, \psi \rangle \), which implies the relation \( \sum_i a^i b^i = 1 \). Equation (2.5) is re-written as

\[
L_D = i \langle \bar{\psi}, (d + A) \psi \rangle, \tag{2.7}
\]

where we have introduced the antihermitian, differential one-form

\[
A = \sum_i a^i d b^i = A_\mu d \hat{x}^\mu, \quad a^i, b^i \in A. \tag{2.8}
\]

Equation (2.8) defines the Yang-Mills gauge field coupled to fermions. Because of the linearity condition \( \rho(a + b) = \rho(a) + \rho(b) \), the fermions in our theory turn out to exist in generation. Thanks to \( \sum_i a^i b^i = 1 \), the inhomogeneous gauge transformation

\[
\delta A = g A g^* + g d g^* \tag{2.9}
\]

is induced by Eq. (2.6).

One may now ask what happens if fermions are massive. To generalize Eq. (2.2) to this case, we define an extra differential to formally produce the mass term using the chiral decomposition of the spinor \( \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \in \mathcal{H} \) by

\[
d_\chi \psi = M \psi \chi, \quad M = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix} = M^*, \tag{2.10}
\]

where \( M \) is fermion mass matrix. The extra one-form basis \( \chi \) combines with the usual ones \( \{ d^\mu \}_\mu=0,1,2,3 \) to form 5-dimensional one-form basis:

\[
d \hat{x}^\mu \wedge d \hat{x}^\nu = -d \hat{x}^\nu \wedge d \hat{x}^\mu, \quad d \hat{x}^\mu \wedge \chi = -\chi \wedge d \hat{x}^\mu, \quad \chi \wedge \chi = 0. \tag{2.11}
\]

Further, assuming

\[
\langle d \hat{x}^\mu, \chi \rangle = \langle \chi, d \hat{x}^\mu \rangle = 0, \quad \langle \chi, \chi \rangle = -1, \tag{2.12}
\]

we obtain the Dirac lagrangian for a massive fermion through

\[
L_D = i \langle \bar{\psi}, d \psi \rangle, \quad \bar{\psi} = \gamma_\mu \psi d \hat{x}^\mu + i \psi \chi, \tag{2.13}
\]

where \( d = d + d_\chi \). As in the massless case, Eq. (2.13) is not invariant under Eq. (2.3). But the recipe is now clear. The sum

\[
L_D = i \sum_i \langle a^i \bar{\psi}, d(b^i \psi) \rangle, \quad a^i, b^i \in A \tag{2.14}
\]

becomes gauge invariant under Eqs. (2.3) and (2.6). Setting

\[
A = \sum_i a^i d b^i = A_\mu d \hat{x}^\mu + \Phi \chi, \tag{2.15}
\]

*) This requires a (finite) sum in Eq. (2.5).
we rewrite Eq. (2·14) as
\[ \mathcal{L}_D = i\langle \bar{\psi}, (\mathbf{d} + \mathbf{A})\psi \rangle. \] (2·16)

Consequently, Eq. (2·14) introduces a gauge-invariant coupling of scalars \( \Phi, \chi \) being assumed to be a Lorentz scalar, to fermions.

The gauge transformation property of \( \mathbf{A} \) under Eq. (2·6) determines that of \( \Phi \) in addition to Eq. (2·9):
\[ g\Phi \chi = g\Phi g^* \chi + gd_\chi g^*, \] (2·17)
where use has been made of the relation \( \sum_i a_i b_i = 1 \). Unless the inhomogeneous term in Eq. (2·17) vanishes, \( \Phi \) is not a physical field.

In writing Eqs. (2·14)~(2·16) we have implicitly assumed the Leibniz rule,
\[ d_\chi (f\psi) = (d_\chi f)\psi + f(d_\chi \psi), \quad f \in \mathcal{A}, \quad \chi \psi = \psi \chi. \] (2·18)
Note that with the chiral decomposition of the spinor we must write elements of \( \mathcal{A} \) as 2x2 matrices in block form like \( f = \begin{pmatrix} f_1 & g_1 \\ g_2 & f_2 \end{pmatrix} \in \mathcal{A} \). It follows that*)
\[ d_\chi f = (M f - \frac{fM}{2})\chi, \quad f \in \mathcal{A}, \quad d_\chi M = 0, \] (2·19)
which in turn implies the Leibniz rule
\[ d_\chi (fg) = (d_\chi f)g + f(d_\chi g), \quad f, g \in \mathcal{A}, \quad \chi g = g\chi. \] (2·20)
The extended operator \( \mathbf{d} \) is nilpotent,
\[ d^2 = 0, \] (2·21)
since \( d_\chi \chi = 0 \)12,13 and \( d^2 = d_\chi^2 = dd_\chi + d_\chi d = 0 \). We need to prove the last equality:
\[ (dd_\chi + d_\chi d)f = d(M f - \frac{fM}{2})\chi + d_\chi df = (M\partial_\mu f - \partial_\mu f M)(d\tilde{\phi}^\mu \wedge \chi + \chi \wedge d\tilde{\phi}^\mu) = 0, \]
\[ (dd_\chi + d_\chi d)\psi = d(M\psi)\chi + d_\chi (d\psi) = M\partial_\mu \psi (d\tilde{\phi}^\mu \wedge \chi + \chi \wedge d\tilde{\phi}^\mu) = 0. \] (2·22)

According to Eq. (2·19), the inhomogeneous term on the right-hand side of Eq. (2·17) vanishes if the mass term is gauge invariant, \( [M, g] = 0 \). Then, \( \Phi \) is a physical field irrelevant to the generation of fermion masses. As an example, one may remark that, in this case, there is no need to make the sum (2·14), but
\[ \mathcal{L}_D = i\sum_i \langle a_i^* \bar{\psi}, d(b_i^i \psi) \rangle + i\langle \bar{\psi}, d_\chi \psi \rangle, \quad a_i^i, b_i^i \in \mathcal{A} \] (2·23)
is already gauge invariant. Hence we are led back to simply add a gauge invariant mass term to Eq. (2·7), leading to a situation in which no scalars are coupled to fermions.

*) Strictly speaking, the Leibniz rule (2·18) is to be required only for a block-diagonal matrix \( f \). For a block-off-diagonal matrix \( g \), there is another option \( d_\chi (g\psi) = (d_\chi g)\psi - g(d_\chi \psi) \) with \( d_\chi M = 2M^2 \chi \). In a previous paper10 we considered this case. The present modification seems to be more natural.
The decomposition (2·15) of the generalized one-form $A$ into gauge fields $A_{\mu}$ and shifted Higgs fields $\Phi$ corresponds to Connes’ prescription of NCG to unify gauge and Higgs fields, where the mass term in the Dirac operator yields the shifted Higgs field. Likewise, our $d_{\chi}$ gives rise to $\Phi$.

We now go on to define the field strength corresponding to $A$. It will yield the bosonic lagrangian involving solely the bosons coupled to fermions. We employ the definition by the Clifford product, as proposed in Ref. 16):

$$G = d \wedge A + A \wedge A = F + F_0,$$

$$F = d \wedge A + A \wedge A \equiv dA + A^2, \quad dA = \sum_i da^i \wedge db^i,$$

$$F_0 = \langle d, A \rangle + \langle A, A \rangle, \quad \langle d, A \rangle = \sum_i (da^i, db^i). \quad (2·24)$$

The field strength $G$ is inhomogeneous in the rank of differential forms. In other words, it is given by the sum of the two-form $F$ and zero-form $F_0$. Both $F$ and $F_0$ are gauge covariant so is $G$. Explicitly, we have

$$F = F + DH \wedge \chi, \quad F = dA + A^2, \quad DH = dH + AH - HA, \quad H = \Phi + M,$$

$$F_0 = X - (H^2 - M^2 - Y), \quad X = -\sum_i a^i \partial^2 b^i + \partial^\mu A_{\mu} + A^\mu A_{\mu} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

$$A_{\mu} = \sum_i a^i \partial_\mu b^i, \quad \Phi = \sum_i a^i[M, b^i], \quad Y = \sum_i a^i[M^2, b^i] = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}. \quad (2·25)$$

The system is governed by the lagrangian

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_B, \quad (2·26)$$

where the fermionic sector is given by Eq. (2·16), while the bosonic sector is obtained by

$$\mathcal{L}_B = -\frac{1}{4} \langle z^2 G, G \rangle = -\frac{1}{4} \langle z^2 F, F \rangle - V, \quad V = \frac{1}{4} \langle z^2 F_0, F_0 \rangle, \quad (2·27)$$

where $z^2$ is a positive matrix commuting with the gauge group, and we have used the fact that the two-form and zero-form are orthogonal.

The lagrangian depends on the model. In the next section we shall discuss toy models.

*) Similarly, Connes’ field strength consists of three terms, $\theta = \theta_0 + \theta_1 + \theta_2$ corresponding to a two-form $\theta_0 \propto [\gamma^\mu, \gamma^\nu]$, three form $\theta_1 \propto \gamma^\mu \gamma_5$, and zero-form $\theta_2$ containing no Dirac matrices, $\gamma^\mu$ being regarded as one-form basis.

**) The fact that both $F$ and $F_0$ are separately gauge covariant implies that the most general field strength in this scheme contains an arbitrary real parameter $\kappa$: $G = F + \kappa F_0$. Then the arbitrary parameter $\kappa^2$ would appear in $V$. For simplicity in this paper we put $\kappa = 1$ so that the field strength is given by the Clifford product.

***) In evaluating Eq. (2·27) we use the inner product

$$\langle d\tilde{x}^\mu \wedge d\tilde{x}^{\nu}, d\tilde{x}^p \wedge d\tilde{x}^{\sigma} \rangle = g^{\mu p} g^{\nu \sigma} - g^{\mu \sigma} g^{\nu p}, \quad (d\tilde{x}^\mu \wedge \chi, d\tilde{x}^{\nu} \wedge \chi) = -g^{\mu \nu}.$$

Downloaded from https://academic.oup.com/ptp/article-abstract/100/1/147/1939452 by guest on 27 July 2018
§3. Toy models

First of all we shall consider $SU(2)$ gauge theory. As is well known, there exists an infinite tower of unitary irreducible representations (IR’s) of $SU(2)$ with dimensions $2j + 1, j = 0, \frac{1}{2}, 1, \cdots$. $SU(2)$ gauge theory is defined for the spinor belonging to any one of them.

Now local $SU(2)$ is defined by the set of unitary elements of the algebra $\mathcal{A} = C^\infty(M_4) \otimes \mathbb{H}$. The real quaternion $\mathbb{H}$ has only one IR, $(3\cdot1)$

$$H \ni a \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}. \quad (3\cdot1)$$

Hence one can write $SU(2) = \{ g \in H; gg^* = g^*g = 1, g^* = g \}$, where $\bar{g}$ is quaternion conjugation. Using the IR $(3\cdot1)$, $H$ is represented by

$$H \ni a \mapsto \rho(a) = \text{diag}(a, \cdots, a^*, \cdots), \quad a^* = \text{c.c. of } a \quad (3\cdot2)$$

which acts on a repetition of fundamental representations of $SU(2)$. Consequently, fermions exist in generation and must be non-chiral so that $M$ commutes with $SU(2)$ and $\Phi = \sum a^i [M, b^j] = 0$. Thus $H = M$ is constant, commuting with $A$, and hence $DH = 0$. We then conclude that $\mathcal{L}_B$ contains only a Yang-Mills term, because $Y = 0$, and $V$ vanishes due to the equation of motion. That is, our algebraic recipe leads to $SU(2)$ gauge theory without scalars, where fermion doublets exist in generation.

The Higgs mechanism can be incorporated into our scheme by enlarging the algebra to $\mathcal{A} = C^\infty(M_4) \otimes (\mathbb{H} \oplus \mathbb{C})$. The unitary group is $U(C^\infty(M_4) \otimes (\mathbb{H} \oplus \mathbb{C})) = \text{Map}(M_4, SU(2) \times U(1))$. Writing

$$\mathcal{A} = \sum_i \rho(a^i_1, b^i_1) d\rho(a^i_2, b^i_2) \quad (3\cdot3)$$

for Eq. (2.15), we assume the following representation of $\mathcal{A}$ on the spinor with chiral components

$$\rho(a, b) = \begin{pmatrix} a \otimes 1_N & 0 \\ 0 & b \otimes 1_N \end{pmatrix}, \quad a \in \mathbb{H}, \ b \in \mathbb{C} \quad (3\cdot4)$$

so that left-handed and right-handed spinors belong to the doublet and singlet, respectively. Except for generation indices, $\psi_L$ has two-components like $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, while $\psi_R$ has only one component like $e_R$. Here and hereafter we omit the infinite-dimensional part $C^\infty(M_4)$ of the algebra and recall that, say, $a$ in Eq. $(3\cdot4)$ is $\mathbb{H}$-valued local functions represented by Eq. $(3\cdot1)$. Choosing the mass matrix as

$$M = \begin{pmatrix} 0 & M_1 \\ M_1^t & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 \\ m \end{pmatrix}, \quad (3\cdot5)$$

*) There is no reason to assume the same $N_g$ in the doublet and singlet sectors. This assumption is only for later convenience. The $N \times N$ unit matrix is denoted by $1_N$.\)
where $m$ is the $N_g \times N_g$ mass matrix, we find from the definition of $\Phi$ in Eq. (2.25),

$$\Phi = \begin{pmatrix} 0 & \varphi M_1 \\ M_1^\dagger \varphi^t & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_0^* & \varphi^+ \\ -\varphi^- & \varphi_0 \end{pmatrix}. \quad (3.6)$$

Note that $\varphi$ is an $H$-valued scalar function. Also,

$$H = \Phi + M = \begin{pmatrix} 0 & hM_1 \\ M_1^\dagger h^t & 0 \end{pmatrix},$$

$$h = \varphi + 1_2 = \begin{pmatrix} \phi_0^* & \phi^+ \\ -\phi^- & \phi_0 \end{pmatrix}, \quad \phi_0 = \varphi_0 + 1, \quad \phi_\pm = \varphi_\pm. \quad (3.7)$$

It can be shown that Eq. (2.16) leads to

$$L_D = \bar{\psi} \gamma^\mu \left( \partial_\mu - \frac{ig_2}{2} A_\mu^{(2)} \right) \gamma^\mu \psi_L + \bar{\psi} \gamma^\mu \left( \partial_\mu - \frac{ig_1}{2} A_\mu^{(1)} \right) \psi_R$$

$$- \bar{\psi} L m \phi \psi_R - \bar{\psi} R \phi^\dagger m^\dagger \psi_L, \quad (3.8)$$

where

$$\phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}, \quad \langle \phi \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.9)$$

is the normalized Higgs field, and we set

$$A = \sum_i \rho(a^i_1, b^i_1) d \rho(a^i_2, b^i_2) = \begin{pmatrix} -\frac{ig_2}{2} A_\mu^{(2)} d \hat{x}^\mu \otimes 1_{N_g} & 0 \\ 0 & -\frac{ig_1}{2} A_\mu^{(1)} d \hat{x}^\mu \otimes 1_{N_g} \end{pmatrix}. \quad (3.10)$$

Here, $A_\mu^{(2)}$ is an $SU(2)$ gauge field, while $A_\mu^{(1)}$ is a $U(1)$ gauge field. Both are hermitian. Also, $g_2$ and $g_1$ are the respective gauge coupling constants.

Let us now evaluate the bosonic lagrangian (2.27), using the formulae (2.25).

Writing

$$z^2 = \frac{4}{N_g} \begin{pmatrix} g_2^2 1_{2g} \otimes 1_{N_g} & 0 \\ 0 & 2g_1^{-2} 1_{N_g} \end{pmatrix}, \quad (3.11)$$

we have for the two-form piece,

$$-\frac{1}{4} \langle z^2 F, F \rangle = -\frac{1}{8} \text{tr} F_\mu^{(2)} F^{(2)\mu \nu} - \frac{1}{4} F_\mu^{(1)} F^{(1)\mu \nu} + L (D_\mu \phi)^\dagger (D_\mu \phi), \quad (3.12)$$

where the $F_\mu^{(i)}$ are the field strengths of the gauge fields, $A_\mu^{(i)}, i = 1, 2, D_\mu \phi = (\partial_\mu - \frac{ig_2}{2} A_\mu^{(2)} + \frac{ig_1}{2} A_\mu^{(1)}) \phi$, and we define

$$L = \frac{2}{N_g} \left( \frac{1}{g_2^2} + \frac{2}{g_1^2} \right) \text{tr}_g (m^\dagger m), \quad (3.13)$$

with $\text{tr}_g$ denoting the trace in the generation space. Next we compute the zero form piece. Since $Y_2 = 0$ in Eq. (2.25), $X_1, X_2$ and $Y_1$ are auxiliary fields to be eliminated.
from the lagrangian using the equation of motion. We obtain

\[ V = K(\phi^\dagger \phi - 1)^2, \quad K = \frac{2}{N_g g_1^2} \left( \text{tr}_g (M_1^\dagger M_1)^2 - \frac{1}{N_g} (\text{tr}_g M_1^\dagger M_1)^2 \right). \quad (3.14) \]

Finally, we have

\[ \mathcal{L}_B = -\frac{1}{8} \text{tr} F^{(2)}_{\mu \nu} F^{(2)\mu \nu} - \frac{1}{4} F^{(1)}_{\mu \nu} F^{(1)\mu \nu} + (D^\mu \phi)^\dagger (D_\mu \phi) - \frac{\lambda}{4} \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2, \quad (3.15) \]

where

\[ \lambda = \frac{4K}{L^2}, \quad v^2 = 2L. \quad (3.16) \]

Since \( \lambda \) should be positive, the theory is consistently defined only if \( N_g > 1 \).

The rescaling \( \phi \rightarrow \frac{1}{\sqrt{L}} \phi = \frac{\sqrt{2}}{v} \phi \) to obtain the above \( \mathcal{L}_B \) renders the mass term in Eq. (3.8) multiplied by \( \frac{\sqrt{2}}{v} \).

There is a representation of the same algebra distinct from (3.4). This is given by

\[ \rho(a, b) = \left( \begin{array}{cc} a \otimes 1_{N_g} & 0 \\ 0 & B \otimes 1_{N_g} \end{array} \right), \quad B = \left( \begin{array}{cc} b & 0 \\ 0 & b^* \end{array} \right), \quad (3.17) \]

which is suitable for a 'massive' neutrino. This representation acts on a doublet like \( \psi_L = \left( \begin{array}{c} \nu \\ e \end{array} \right)_L \) and singlets like \( \psi_R = \left( \begin{array}{c} \nu_R \\ e_R \end{array} \right) \) with generation indices omitted. The mass matrix is given by

\[ M_1 = \left( \begin{array}{cc} m_1 & 0 \\ 0 & m_2 \end{array} \right), \quad m_{1,2} : N_g \times N_g. \quad (3.18) \]

The Higgs field is given by Eqs. (3.6) and (3.7) with \( M_1 \) being of the form (3.18). Setting

\[ A = \sum_i \rho(a_i^1, b_i^1) d \rho(a_i^2, b_i^2) \]

\[ = \left( \begin{array}{cc} \frac{i g_2}{2} A_\mu^{(2)} d\bar{\chi}_\mu \otimes 1_{N_g} & 0 \\ 0 & \frac{i g_1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) A_\mu^{(1)} d\bar{\chi}_\mu \otimes 1_{N_g} \end{array} \right), \quad (3.19) \]

we find

\[ \mathcal{L}_D = \bar{\psi}_L i \gamma^\mu \left( \partial_\mu - \frac{i g_2}{2} A_\mu^{(2)} \right) \psi_L + \bar{\psi}_R i \gamma^\mu \left( \partial_\mu - \frac{i g_1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) A_\mu^{(1)} \right) \psi_R \]

\[ -\bar{\psi}_L (m_1 \bar{\phi}, m_2 \phi) \psi_R - \bar{\psi}_R \left( \begin{array}{c} \phi^\dagger m_1^1 \\ \phi^\dagger m_2 \end{array} \right) \psi_L, \quad \bar{\phi} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \quad (3.20) \]
The bosonic lagrangian takes the same form as (3·15) with $L$ of Eq. (3·16) given by

$$L = \frac{2}{N_g} \left( \frac{1}{g_2^2} + \frac{1}{g_1^2} \right) \text{tr}_g(M_1^\dagger M_1),$$

provided we set

$$z^2 = \frac{4}{N_g} \begin{pmatrix} g_2^{-2} 1_2 \otimes 1_{N_g} & 0 \\ 0 & g_1^{-2} 1_2 \otimes 1_{N_g} \end{pmatrix}.$$  

Next, consider the case $N_g=1$ and let $\psi^a (a = 1, 2, 3)$ belong to the adjoint representation of $SU(2)$ with the matrix field

$$\psi = \tau_a \psi^a,$$

where $\tau = (\tau_1, \tau_2, \tau_3)$ is the Pauli matrix. Choosing the algebra $A = C^\infty(M_4) \otimes M_2(C)$, whose unitary group is $U(C^\infty(M_4) \otimes M_2(C)) = \text{Map}(M_4, SU(2) \times U(1))$ (although the $U(1)$ factor is automatically eliminated in the reconstruction below), we represent the algebra by

$$\rho(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in M_2(C).$$

The action of $d + A$ on $\psi$ is defined by the commutator

$$(d + A)\psi \equiv [d + A, \psi] = [d + A, \psi] + [d_\chi + \Phi, \psi],$$

$$[d + A, \psi] \equiv d\psi + \begin{pmatrix} [A_1, \psi_L] \\ [A_2, \psi_R] \end{pmatrix}, \ A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

$$[d_\chi, \psi] = [M, \psi] \equiv \begin{pmatrix} [M_1, \psi_R] \\ [M_1^\dagger, \psi_L] \end{pmatrix}, \ [\Phi, \psi] \equiv \begin{pmatrix} [\psi, \psi_R] \\ [\psi^\dagger, \psi_L] \end{pmatrix},$$

where

$$\Phi = \sum_i \rho(a^i)[M, \rho(b^i)] = \begin{pmatrix} 0 & \varphi \\ \varphi^\dagger & 0 \end{pmatrix}, \ a^i, b^i \in A.$$  

By assumption, $A_1 = A_2$ and we put $A_1^\mu = -igA_\mu$, where $A_\mu$ is a hermitian $SU(2)$ gauge field due to the commutator (3·25). Hence the gauge group is $SU(2)$, not $SU(2) \times U(1)$. It should also be noted here that $\Phi$ does not take the form (3·6) but has three complex components in view of the commutator (3·25):

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & -\varphi_{11} \end{pmatrix}.$$  

$H = \Phi + M$ is given by Eq. (3·26) with $\varphi \to h = \varphi + M_1$, and we thus put

$$-\frac{1}{g} h \equiv \phi = \tau_a \phi^a \equiv S - iP.$$  

(3·28)
It is now easy to evaluate\textsuperscript{15}) the lagrangian
\[
\mathcal{L}_B = -\frac{1}{4g^2} \langle G, G \rangle = -\frac{1}{4g^2} \langle F, F \rangle - V, \quad V = \frac{1}{4g^2} \langle F_0, F_0 \rangle,
\] (3.29)
where we put \( z^2 = (1/g^2) z \) in Eq. (2.27) to take into account supersymmetry. The result is
\[
\mathcal{L}_B = -\frac{1}{8} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \text{tr} [D_{\mu}, \phi] [D_{\mu}, \phi] - \frac{g^2}{8} \text{tr} [\phi, \phi]^2,
\] (3.30)
where \( D_{\mu} = \partial_{\mu} - igA_{\mu} \), provided \( M_1 \) is a normal matrix, \( M_1 M_1^\dagger = M_1^\dagger M_1 \).

In the fermionic sector we simply find from Eqs. (2.16), (3.25), (3.26) and (3.28)
\[
\mathcal{L}_D = \text{tr} (\bar{\psi} i\gamma^\mu [D_{\mu}, \psi] + g\bar{\psi} [S + i\gamma_5 P, \psi]).
\] (3.31)
As is well known, the sum of Eqs. (3.30) and (3.31) defines the \( N = 2 \) super Yang-Mills theory.\textsuperscript{20)}

\section*{4. Standard model of elementary particles}

To reconstruct the standard model of elementary particles in the present scheme we take into account Connes real structure \( J \). To define \( J \) as simply as possible we follow Connes and consider doubled spinors using the charge conjugate spinor \( \psi^c \) as
\[
\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},
\] (4.1)
upon which gauge transformation acts like
\[
g\Psi = U\Psi = U_1 U_2 \Psi, \quad U^\dagger U = U_1^\dagger U_1 = 1
\] (4.2)
if
\[
U = \begin{pmatrix} u_1 u_2 & 0 \\ 0 & u_1^* u_2 \end{pmatrix}, \quad u_i^* = \text{c.c. of } u_i,
\] (4.3)
where
\[
U_1 = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad U_2 = \begin{pmatrix} u_2^* & 0 \\ 0 & u_1^* \end{pmatrix},
\] (4.4)
with \( u_1 u_2 = u_2 u_1^* \), or \( U_1 U_2 = U_2 U_1 \). In the language of Connes mathematics, obtaining \( U_2 \) from \( U_1 \) is accomplished by means of the real structure \( J, U_2 = JU_1 J^\dagger \), provided \( U_1 \) is a representation of a noncommutative algebra.\textsuperscript{**}) Since Abelian charges cancel among fermions and anti-fermions, \( \det U = 1 \). This condition is satisfied if
\[
\det u_1 u_2^* = 1,
\] (4.5)
\textsuperscript{*)} This implies that \( Y = 0 \) in Eq. (2.25). Hence there must be an \( X \) term in \( F_0 \) to obtain the scalar potential.

\textsuperscript{**}) The operator \( J \) is an anti-linear isometry in the total Hilbert space \( \mathcal{H} \) which satisfies two conditions.\textsuperscript{4, 5}) Then Eq. (4.2) reads \( g\Psi = U_1 \Psi U_1^* \equiv U_1 JU_1 J^* \Psi \).
which is highly nontrivial. We shall refer to Eq. (4-5) as ‘Connes unimodularity condition’.

In the standard model we choose the basis using the mass eigenstates

\[
\psi = \begin{pmatrix}
q_L \\
l_L \\
u_R \\
d_R \\
\nu_e_R \\
e_R
\end{pmatrix}, \quad q_L = \begin{pmatrix} u \\ U_q d \end{pmatrix}_L, \quad l_L = \begin{pmatrix} \nu_e \\ U_l e \end{pmatrix}_L,
\]

(4-6)

where we omit generation indices on leptons and quarks as well as color indices on quarks and similarly anti-fermions. The \(N_g \times N_g\) Kobayashi-Maskawa matrices are denoted by \(U_{l,q}\) in the lepton and quark sectors, respectively. Experiment indicates \(N_g = 3\) up to the present energy, but we let \(N_g\) be a free parameter. We assume massive neutrinos using the representation (3-17) in the lepton sector.

To recover the standard gauge group \(SU(3) \times SU(2)_L \times U(1)_Y\) from \(U(\mathcal{A})\) we make Connes choice \(^4\), \(^5\)

\[
\mathcal{A} = C^\infty(M_4) \otimes (H \oplus C \oplus M_3(C)),
\]

(4-7)

whose the set of unitary elements is \(\text{Map}(M_4, U(3) \times SU(2) \times U(1))\). We shall see below that the unimodularity condition (4-5) reduces \(U(3) \times SU(2) \times U(1)\) to \(SU(3) \times SU(2)_L \times U(1)_Y\) with the correct hypercharge assignment.

In the basis (4-6) Eqs. (4-2) \~ (4-4) turn out to be given by the unitary restriction of the representation of Eq. (4-7): \(u_1 = \rho_w(a, b)\) and \(u_2 = \rho_s(b, c)\) for \((a, b, c) \in U(\mathcal{A})\), where, for general \(a \in H, b \in C, c \in M_3(C), \mathcal{A}\) is represented in Eq. (4-1) by \(^4\), \(^5\), \(^17\) \~ \(^19\)

\[
\rho(a, b, c) = \begin{pmatrix}
\rho_w(a, b) & 0 \\
0 & \rho_s(b, c)
\end{pmatrix},
\]

\[
\rho_w(a, b) = \begin{pmatrix}
\rho_1(a) & 0 \\
0 & \rho_2(b)
\end{pmatrix}, \quad \rho_s(b, c) = \begin{pmatrix}
\rho_3(b, c) & 0 \\
0 & \rho_4(b, c)
\end{pmatrix},
\]

\[
\rho_1(a) = \begin{pmatrix}
a \otimes 1_3 \otimes 1_{N_g} & 0 \\
0 & a \otimes 1_{N_g}
\end{pmatrix}, \quad \rho_2(b) = \rho_1(B),
\]

\[
\rho_3(b, c) = \begin{pmatrix}
1_2 \otimes c^* \otimes 1_{N_g} & 0 \\
0 & b_1 2 \otimes 1_{N_g}
\end{pmatrix} = \rho_4(b, c),
\]

(4-8)

where \(B\) is given by Eq. (3-17). By \(U_1\) flavor acts only in the particle sector, whereas color operates only in the anti-particle sector.\(^*)

\(^*)\) Before introduction of the real structure, Connes considered\(^3\) the bimodule \(\mathcal{A} \oplus \mathcal{B}, \mathcal{A} = H \oplus C, \mathcal{B} = C \oplus M_3(C)\) with the left action by flavor and right action by color. Thanks to the real structure \(J\), the bimodule structure of \(\mathcal{H}\) is obtained with the single algebra (4-7) simply because there exist matter and anti-matter in quantum field theory.
The well-known fermionic sector of the standard model lagrangian is given using the doubled spinor (4·1) by

$$\mathcal{L}_D = \frac{i}{2} \langle \bar{\Psi} (\mathbf{d} + \mathbf{A} + J \mathbf{A} \mathbf{J}^\dagger) \Psi \rangle,$$

where

$$d_\chi \Psi = \mathcal{M} \Psi \chi, \quad \mathcal{M} = \mathcal{M}_1 + J \mathcal{M}_1 \mathbf{J}^\dagger, \quad \mathcal{M}_1 = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix},$$

$$\mathbf{A} = \sum_i \rho(a_i^1, b_i^1, c_i^1) \rho_0(a_i^2, b_i^2, c_i^2) = A + \Phi_\chi,$$

$$A = \begin{pmatrix} A_w & 0 \\ 0 & A_s \end{pmatrix}, \quad A_w = \sum_i \rho_w(a_i^1, b_i^1, c_i^1) \rho_0(a_i^2, b_i^2, c_i^2), \quad A_s = \sum_i \rho_s(b_i^1, c_i^1) \rho_0(b_i^2, c_i^2),$$

$$\Phi = \sum_i \rho(a_i^1, b_i^1, c_i^1)[\mathcal{M}, \rho(a_i^2, b_i^2, c_i^2)] = \begin{pmatrix} 0 & \rho_1(\varphi) M_1 \\ M_1^\dagger \rho_1(\varphi) M_1 & 0 \end{pmatrix},$$

$$H = \Phi + \mathcal{M}_1 = \begin{pmatrix} 0 & \rho_1(\varphi) M_1 \\ M_1^\dagger \rho_1(\varphi) M_1 & 0 \end{pmatrix},$$

with

$$J \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} J^\dagger = \begin{pmatrix} q^* & 0 \\ 0 & p^* \end{pmatrix}.$$}

Here $\varphi$ is defined by Eq. (3·6), $h$ by Eq. (3·7), and $\rho_1(\alpha), \alpha \in \mathbf{H}$ by Eq. (4·8). Note that in the basis (4·6) we have the following mass matrix:

$$M_1 = \begin{pmatrix} M_q \otimes 13 & 0 \\ 0 & M_l \end{pmatrix}, \quad M_q = \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix}, \quad M_l = \begin{pmatrix} M_\nu & 0 \\ 0 & M_e \end{pmatrix},$$

$$M_u = \text{diag}(m_u, m_c, m_t, \cdots), \quad M_d = U_q \text{diag}(m_d, m_s, m_b, \cdots),$$

$$M_\nu = \text{diag}(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}, \cdots), \quad M_e = U_l \text{diag}(m_e, m_{\mu}, m_{\tau}, \cdots),$$

with obvious notations for fermion masses. KM matrices $U_f(f = l, q)$ disappear due to the corresponding factors in Eq. (4·6). Recall that $M_f(f = q, l)$ are assumed to be $2N_g \times 2N_g$ matrices.

In Eq. (4·9) we have two generalized gauge potentials, $\mathbf{A}$ and $J \mathbf{A} \mathbf{J}^\dagger$. They correspond to two factors, $U_1$ and $U_2$ in Eq. (4·2). In fact, we see from the definition (4·10) and (4·11)

$$g \mathbf{A} = U_1 A \mathbf{U}_1^\dagger + U_1 d \mathbf{U}_1^\dagger, \quad U_2 A \mathbf{U}_2^\dagger = \mathbf{A},$$

$$J g \mathbf{A} \mathbf{J}^\dagger = U_2 (J \mathbf{A} \mathbf{J}^\dagger) U_2^\dagger + U_2 d \mathbf{U}_2^\dagger, \quad U_1 (J \mathbf{A} \mathbf{J}^\dagger) U_1^\dagger = J \mathbf{A} \mathbf{J}^\dagger,$$

$$g \mathbf{A} + J g \mathbf{A} \mathbf{J}^\dagger = U (\mathbf{A} + J \mathbf{A} \mathbf{J}^\dagger) U^\dagger + U d U^\dagger,$$
Gauge Theories Coupled to Fermions in Generation

where we have used the relation $U_1(U_2dU_2^\dagger)U_1^\dagger = U_2dU_2^\dagger$. Hence the fermionic lagrangian (4·9) is gauge invariant under (4·2). What remains to be proven is that the unimodularity condition (4·5) determines the correct hypercharge assignment. To see what is happening in the theory, we set

$$A_w = \left( \begin{array}{cc} A_w^L & 0 \\ 0 & A_w^R \end{array} \right), \quad A_s = \left( \begin{array}{cc} A_s^L & 0 \\ 0 & A_s^R \end{array} \right),$$

$$A_w^L = \left( \begin{array}{cc} W \otimes 1_3 \otimes 1_{N_g} & 0 \\ 0 & W \otimes 1_{N_g} \end{array} \right),$$

$$A_w^R = W \rightarrow \left( \begin{array}{cc} B^* & 0 \\ 0 & B \end{array} \right) \text{ in } A_w^L,$$

$$A_s^L = \left( \begin{array}{cc} 1_2 \otimes G^* \otimes 1_{N_g} & 0 \\ 0 & \left( \begin{array}{cc} B^* & 0 \\ 0 & B^* \end{array} \right) \otimes 1_{N_g} \end{array} \right) = A_s^R, \quad (4·14)$$

where

$$W = \sum_i a_i d_i a_i, \quad B^* = \sum_i b_i d_i b_i, \quad G = \sum_i c_i d_i c_i \quad (4·15)$$

are flavor, Abelian and “color” gauge fields. The last still contains the Abelian part

$$G = G' + \frac{1}{3}1_3\text{tr}G, \quad \text{tr}G' = 0. \quad (4·16)$$

Now the unimodularity condition (4·5) means that $\text{tr}(A_w + A_s^*) = 0$. Noting that $W$ and $G'$ are traceless and that gauge fields are chosen as antihermitian, we find

$$B + \text{tr}G = 0, \quad (4·17)$$

indicating that there is only one Abelian gauge field, which we take to be a $U(1)_Y$ gauge field $B$. If $B$ couples to a lepton doublet with strength proportional to $-y$, we are led to the hypercharge assignment

$$Y_L^l = \left( \begin{array}{cc} -y & 0 \\ 0 & -y \end{array} \right), \quad Y_L^q = \left( \begin{array}{cc} \frac{1}{3}y & 0 \\ 0 & \frac{1}{3}y \end{array} \right) \otimes 1_3,$$

$$Y_R^l = \left( \begin{array}{cc} 0 & 0 \\ 0 & -2y \end{array} \right), \quad Y_R^q = \left( \begin{array}{cc} y + \frac{1}{3}y & 0 \\ 0 & -y + \frac{1}{3}y \end{array} \right) \otimes 1_3. \quad (4·18)$$

By rescaling

$$B \rightarrow -\frac{i g'}{2}B, \quad W \rightarrow -\frac{i g}{2}W, \quad G' \rightarrow -\frac{i g_s}{2}G' \quad (4·19)$$

and putting $y = +1$ we are able to recover the fermionic lagrangian of the standard model from Eq. (4·9) with a normalized Higgs field.
Next we turn to the bosonic sector, where we make use of only $A$ in Eq. (4-10) to define the field strength $G$. The formula (2-27) gives

$$\mathcal{L}_B = \mathcal{L}_{YM} + \mathcal{L}_H^{KE} - V,$$

(4.20)

where, putting

$$z^2 = \begin{pmatrix} z_p^2 & 0 \\ 0 & z_a^2 \end{pmatrix},$$

$$z^2 = \begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix},$$

$$z_a^2 = \begin{pmatrix} z_2^2 \\ 0 \\ z_3^2 \end{pmatrix},$$

$$z_2^2 = \begin{pmatrix} z_2^2 \\ 0 \end{pmatrix},$$

$$z_3^2 = \begin{pmatrix} z_3^2 \\ 0 \end{pmatrix},$$

with the conditions

$$(3z_1^2 + z_1'^2)g_1^2 N_g = 1,$$

$$(3z_2^2 + z_2'^2)g_2^2 N_g + (z_3^2 + z_3'^2 + 3(z_3'^2 + z_4'^2))g_3^2 N_g = 2,$$

$$(z_3^2 + z_4^2)g_3^2 N_g = 1,$$

(4.22)

Yang-Mills term becomes

$$\mathcal{L}_{YM} = -\frac{1}{8} \text{tr}(W_{\mu
u}W^{\mu\nu} + G'_{\mu\nu}G'^{\mu\nu}) - \frac{1}{4} B_{\mu\nu}B^{\mu\nu}$$

(4.23)

in terms of $SU(2)_L, SU(3), U(1)_Y$ gauge field strengths, $W_{\mu\nu}, G'_{\mu\nu}, B_{\mu\nu}$, respectively, and the Higgs kinetic energy term is given by

$$\mathcal{L}_H^{KE} = L(D^\mu \phi)^\dagger(D_\mu \phi),$$

(4.24)

with

$$L = \frac{1}{4} \text{tr}_g\left[3(z_1^2 + z_2^2)(M_u M_u^\dagger + M_d M_d^\dagger) + (z_1'^2 + z_2'^2)(M_u M_u^\dagger + M_e M_e^\dagger)\right].$$

(4.25)

Note that $\phi$ in (4.24) is normalized such that $\langle \phi \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. On the other hand, the potential $V$ still contains auxiliary fields to be eliminated to take the following form:

$$V = K(\phi^\dagger \phi - 1)^2,$$

(4.26)

with

$$K = \frac{1}{4} \left(3z_2^2 \sum_{q=u,d} \left[\text{tr}_g(M_q M_q^\dagger) - \frac{1}{N_g} (\text{tr}_g M_q M_q^\dagger)^2\right] \right. + z_2'^2 \sum_{l=\nu,e} \left[\text{tr}_g(M_l M_l^\dagger) - \frac{1}{N_g} (\text{tr}_g M_l M_l^\dagger)^2\right].$$

(4.27)
The final form of the bosonic lagrangian after rescaling $\phi \to -\sqrt{2}\phi$ reads

$$L_B = -\frac{1}{8} \text{tr}(W^2 + G^2) - \frac{1}{4} B^2 + (D^\mu \phi)^\dagger (D_\mu \phi) - \frac{\lambda}{4} \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2,$$

(4·28)

where $W^2 = W_{\mu\nu}W^{\mu\nu}$ and so on in Yang-Mills sector, with

$$\lambda = \frac{4K}{L^2}, \quad v^2 = 2L.$$

(4·29)

The present reconstruction of the standard model requires $N_g > 1$ in order to have $\lambda > 0$. That the restriction $N_g > 1$ is obtained in NCG was first observed by Connes-Lott.\(^1\) It should be noted that the $X$ term in Eq. (2·25) is necessary to deduce this conclusion, though both $X$ and the other terms in Eq. (2·25) are separately gauge covariant. (See the first footnote on p. 156 in regard to $N = 2$ super Yang-Mills theory.) Therefore, our conclusion $N_g > 1$ depends on the assumption that the present formalism works also for $N = 2$ super Yang-Mills theory. For instance, we never come across the same conclusion if we modify $G \to G' = F + F'_0$, where $F'_0 = \sum_i (d_\chi a^i, d_\chi b^i) + (\Phi_\chi, \Phi_\chi)$. However, this modification spoils the previous success in reconstructing $N = 2$ super Yang-Mills theory in contrast with NCG consideration.\(^14\) The detailed derivation of Eqs. (4·25) and (4·27) appears in the Appendix.

§5. Discussion

Using the relation $m_W^2 = (1/4)g^2v^2 = (1/2)g^2L$ (see Eq. (4·29)) and estimating $L$ by the top quark mass dominance for $N_g = 3$, we find

$$m_W = m_t \sqrt{\frac{z_1^2 + z_2^2}{2(z_1^2 + z_1^2)}},$$

If $z_1^2 = z_2^2$, it follows\(^17\) that

$$m_t \geq \sqrt{3}m_W \simeq 139 \text{ GeV}.$$

Hence, our treatment is not so unreasonable, although, in fact, we get no constraints among the parameters due to the most general form of the matrix $z^2$ in Eq. (4·21).

Nevertheless, one may assume that the NCG-based bosonic lagrangian with $z^2$ proportional to the unit matrix $1_{32N_g}$ should become exact at some large energy scale $\Lambda$ for an unknown reason. Then at $\Lambda$ we obtain the GUT relations $g_s^2 = g^2 = \frac{5}{3}g_f^2$, and the top mass dominance even at such large energy scale would lead to the mass relations $m_t = 2m_W$ and $m_H = \sqrt{2}m_t$.\(^1,11\),\(^*\) We do not pursue quantitative analysis\(^17\),\(^21\) in terms of renormalization group equations of these or similar constraints.

\(^*\) If we introduce the parameter $\kappa^2$ in the expression for $V$ of Eq. (2·27), the mass relation becomes $m_H = \sqrt{2}m_t$, which excludes the possibility of predicting the Higgs mass by means of renormalization group equations.
in this paper.

Following Connes, we are led to the representation (4-8) by experiment if \( N_g = 3 \). We do not yet understand, however, what determines the representation (4-8),\(^*\) including the precise value of \( N_g \) except for the theoretical restriction \( N_g > 1 \). Nonetheless, the concept of generation is introduced into the theory in a very natural way. This is accompanied by promoting the Higgs field to a kind of gauge field, as in the more rigorous treatment of Connes NCG. In this sense, the present approach, or the more general approach of NCG, offers a new insight into the Yang-Mills-Higgs theory. It should also be mentioned\(^{17, 21}\) that the Higgs mass is estimated to be below or around the weak scale \( v = 247 \text{ GeV} \) upon quantum corrections.

Some authors\(^{18}\) pointed out that the fact that the correct hypercharge assignment (4-18) comes only from the Connes unimodularity condition (4-5) indicates that the NCG-based standard model should be unified with gravity at a large energy scale,\(^{22}\) because Eq. (4-15) or (4-18) assures the absence of graviton-graviton-\( U(1) \) anomaly, \( \text{tr} Y = 0 \). Our next task is to construct a standard-model-gravity-coupled model based on the present formalism.

**Appendix**

Let us first derive the Higgs kinetic energy term (4-24). By definition we have

\[
L_H^{KE} = \frac{1}{4} \text{tr} z^2 (D_\mu H)^\dagger (D^\mu H),
\]

where

\[
D_\mu H = \begin{pmatrix}
0 & D_\mu \rho_1 (h) M_1 \\
M_1^\dagger (D_\mu \rho_1 (h))^\dagger & 0
\end{pmatrix}
\]

with

\[
D_\mu \rho_1 (h) = (D_\mu \tilde{\phi}, D_\mu \phi) \otimes 1_{4N_g},
\]

\[
D_\mu \phi = \left( \partial_\mu - \frac{ig}{2} W_\mu - \frac{ig'}{2} \text{tr} Y_{12B_\mu} \right) \phi,
\]

\[
D_\mu \tilde{\phi} = \left( \partial_\mu - \frac{ig}{2} W_\mu + \frac{ig'}{2} \text{tr} Y_{12B_\mu} \right) \tilde{\phi}.
\]

We then compute (A-1) using the relation

\[
\text{tr} (D_\mu \tilde{\phi}, D_\mu \phi) (M_1 M_1^\dagger) \left( \frac{(D_\mu \tilde{\phi})^\dagger}{(D_\mu \phi)^\dagger} \right) = \text{tr} M_1^\dagger \left( \frac{(D_\mu \tilde{\phi})^\dagger}{(D_\mu \phi)^\dagger} \right) (D_\mu \tilde{\phi}, D_\mu \phi) M_1
\]

\[
= \text{tr} \left[ 3 (M_u M_u^\dagger + M_d M_d^\dagger) + (M_\nu M_\nu^\dagger + M_e M_e^\dagger) \right] \left( D_\mu \phi \right)^\dagger (D_\mu \phi)
\]

\(^*\) We may argue that the representation (4-8) is determined by the requirements that 1) left-handed fermions be doublets, 2) right-handed fermions be singlets, 3) color nonchirality, 4) \( u_R \) and \( d_R \) have different Abelian charges, and 5) \( \nu_R \) be neutral under \( U(3) \times SU(2) \times U(1) \) in each generation. It should be noted, however, that these requirements are all guided by experiment.
and (4·21) to obtain Eqs. (4·24) and (4·25).

Next we evaluate Eq. (4·27). According to Eq. (2·25) we have

\[
X = \begin{pmatrix}
X_1 & 0 \\
0 & X_2
\end{pmatrix} = \begin{pmatrix}
x_1 & 0 \\
0 & y_1
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

\[
H^2 = \begin{pmatrix}
\Theta & 0 \\
0 & \Theta'
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

\[
M^2 = \begin{pmatrix}
M_1M_1^\dagger & 0 \\
0 & M_1^\dagger M_1
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
Y_1 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
Y_2
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

(A·5)

with

\[
\Theta = \rho_1(h)M_1M_1^\dagger \rho_1^\dagger(h), \quad \Theta' = M_1^\dagger \rho_1^\dagger(h)\rho_1(h)M_1,
\]

\[
Y_1 = \sum_i \rho_1(a_i^i, b_i^i)[M_1M_1^\dagger, \rho_1(a_2^i, b_2^i)], \quad Y_2 = 0.
\]

(A·6)

It is now easy to eliminate the auxiliary fields, \(x_1 + Y_1, x_2\) and \(y_2\) from the expression for the potential \(V\) and determine \(y_1\) in the quark and lepton sectors from the equation of motion:

\[
y_1^q = \frac{1}{N_g}(\phi^\dagger \phi - 1)
\begin{pmatrix}
\text{tr}_gM_u^\dagger M_u & 0 \\
0 & \text{tr}_gM_d^\dagger M_d
\end{pmatrix}
\otimes 1_3 \otimes 1_{N_g},
\]

\[
y_1^l = \frac{1}{N_g}(\phi^\dagger \phi - 1)
\begin{pmatrix}
\text{tr}_gM_u^\dagger M_u & 0 \\
0 & \text{tr}_gM_e^\dagger M_e
\end{pmatrix}
\otimes 1_{N_g}.
\]

(A·7)

Consequently, we arrive at the final form of the Higgs potential (4·26) with the coefficient (4·27).

References

1) A. Connes, p. 9 in The Interface of Mathematics and Particle Physics (Clarendon Press, Oxford, 1990).
2) A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. 18B (1990), 29.
3) A. Connes, Noncommutative Geometry (Academic Press, New York, 1994).
4) A. Connes, J. Math. Phys. 36 (1995), 6194.
5) D. Kastler, Reviews in Math. Phys. 5 (1990), 477.
6) J. C. Várilly and J. M. Gracia-Bondía, J. Geom. Phys. 12 (1993), 223.
7) B. Iochum and T. Schücker, Commun. Math. Phys. 178 (1996), 1.
6) R. Coquereaux, G. Esposito-Farese and J. Fröhlich, Nucl. Phys. B353 (1991), 689.
R. Coquereaux, R. Häfling, N. A. Papadopoulos and F. Scheck, Int. J. Mod. Phys. A7 (1992), 2809.
K. Morita, Prog. Theor. Phys. 90 (1993), 219.
S. Naka and E. Umezawa, Prog. Theor. Phys. 92 (1994), 189.
7) A. H. Chamseddine, G. Felder and J. Fröhlich, Phys. Lett. B296 (1992), 109; Nucl. Phys.
B395 (1993), 672; Commun. Math. Phys. 155 (1993), 205.
A. H. Chamseddine and J. Fröhlich, Phys. Lett. B314 (1993), 308; Phys. Rev. D50 (1994),
2893.
A. H. Chamseddine, Phys. Lett. B373 (1996), 61.
8) T. Schücker and J.-M. Zylinski, J. Geom. Phys. 16 (1995), 207.
9) A. Sitarz, Phys. Lett. B308 (1993), 311.
H.-G. Ding, H.-Y. Guo, J.-M. Li and K. Wu, Z. Phys. C64 (1994), 521.
B. Chen, T. Saito, H.-B. Teng, K. Uehara and K. Wu, Prog. Theor. Phys. 95 (1996), 1173.
10) E. Alvarez, J. M. Gracia-Bondía and C. P. Martin, Phys. Lett. 364 (1995), 33.
11) I. S. Sogami, Prog. Theor. Phys. 94 (1995), 117; 95 (1996), 637.
12) K. Morita and Y. Okumura, Prog. Theor. Phys. 91 (1994), 959; Phys. Rev. D50 (1994),
1016.
13) Y. Okumura, Phys. Rev. D50 (1994), 1026; Prog. Theor. Phys. 96 (1996), 1021.
14) A. H. Chamseddine, Phys. Lett. 332 (1994), 349.
15) K. Morita, Prog. Theor. Phys. 96 (1996), 787.
16) K. Morita, Prog. Theor. Phys. 96 (1996), 801.
17) C. P. Martin, J. M. Gracia-Bondía and J. C. Várilly, hep-th/9605001.
18) B. Asquith, Phys. Lett. 366 (1996), 220.
19) G. Landi, An Introduction to Noncommutative Spaces and Their Geometry, Lecture Notes
in Physics (Springer Verlag, 1997).
20) L. Brink, J. H. Schwarz and J. Sherk, Nucl. Phys. B121 (1977), 77.
F. Gliozzi, J. Sherk and D. Olive, Nucl. Phys. B122 (1977), 253.
21) E. Alvarez, J. M. Gracia-Bondía and C. P. Martin, Phys. Lett. 306 (1993), 55; 329 (1994),
259.
T. Shinohara, K. Nishida, H. Tanaka and I. Sogami, Prog. Theor. Phys. 96 (1996), 1179.
Y. Okumura, Prog. Theor. Phys. 98 (1997), 1333; hep-th/9707350.
22) A. H. Chamseddine and A. Connes, Commun. Math. Phys. 182 (1996), 155, hep-
th/9606001.
L. Calminati, B. Iochum and T. Schöker, hep-th/9706105.