N = 4 Supersymmetric Yang-Mills Multiplet in Non-Adjoint Representations

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Abstract

We formulate a theory for $N = 4$ supersymmetric Yang-Mills multiplet in a non-adjoint representation $R$ of $SO(N)$, as an important application of our recently-proposed model for $N = 1$ supersymmetry. This system is obtained by dimensional reduction from an $N = 1$ supersymmetric Yang-Mills multiplet in non-adjoint representation in ten dimensions. The consistency with supersymmetry requires that the non-adjoint representation $R$ with the indices $i, j, \ldots$ satisfy the three conditions $\eta^{ij} = \delta^{ij}$, $(T_I)^{ij} = -(T_I)^{ji}$ and $(T_I)^{[ij}(T_I)^{k]l} = 0$ for the metric $\eta^{ij}$ and the generators $T^I$, which are the same as the $N = 1$ case.

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1. Introduction

The importance of $N = 4$ extended supersymmetry in four-dimensions (4D) [1] is associated with its all-order finiteness [2], and also its natural link with superstring theories in 4D [3]. Moreover, there is an important duality between $N = 4$ supersymmetric Yang-Mills theory in 4D and IIB string theory in 10D compactified on $\text{AdS}_5 \times S^5$ [4].

In the conventional formulation of $N = 1$ supersymmetry in 4D, a vector multiplet is supposed to be in the adjoint representation, such as $(A_\mu^I, \lambda^I)$ carrying the common adjoint index $I$ [1]. However, we have recently shown [6] that this is not necessarily the case, by constructing an explicit $N = 1$ Yang-Mills multiplet in a non-adjoint representation. We have shown that the multiplet $(B_\mu^i, \chi^i)$ with the non-adjoint real representation index $i$ can consistently couple to the conventional Yang-Mills multiplet $(A_\mu^I, \lambda^I)$. Such a non-adjoint real representation $R$ should satisfy certain conditions [6] for the system to be consistent with supersymmetry (Cf. (2.1) below).

In this paper, we show that the $N = 1$ formulation in [6] can be further generalized to extended $N = 4$ supersymmetry. In addition to the conventional $N = 4$ supersymmetric Yang-Mills multiplet $(A_\mu^I, \lambda^I, A_\alpha^I, \bar{A}_\alpha^I)$ ($\alpha = 1, 2, 3; (i) = 1, 2, 3, 4$), we can consider the additional vector multiplet $(B_\mu^i, \chi^i, B_\alpha^i, \bar{B}_\alpha^i)$ carrying the index $i$ for a particular non-adjoint real representation $R$. As explained in the case of $N = 1$ [6], we have to maintain the conventional Yang-Mills in the adjoint representation, once we introduce the extra vector multiplet in the non-adjoint representation $R$.

It seems to be a prevailing notion that $N = 4$ supersymmetric Yang-Mills theory in 4D has the ‘unique’ field content all in the adjoint representation of a certain gauge group. For example, the first sentence of section 3 in [5] states that “The Lagrangian for the $N = 4$ super-Yang Mills theory is unique”. In our present paper, we establish a counter-example against the prevailing notion of the ‘uniqueness’ of $N = 4$ supersymmetric Yang-Mills theory in 4D.

2. The Lagrangian

As has been mentioned, our system has two $N = 4$ vector multiplets $(A_\mu^I, \lambda^I, A_\alpha^I, \bar{A}_\alpha^I)$ and $(B_\mu^i, \chi^i, B_\alpha^i, \bar{B}_\alpha^i)$. The indices $(i),(j), \cdots = 1, 2, 3, 4$ are for $N = 4$ supersymmetry, while the indices $\alpha, \beta, \cdots = 1, 2, 3$ are used for the three scalars and three pseudo-scalars [1].

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3) For reviews, see, e.g., ref. [5].
The former multiplet is the conventional $N = 4$ supersymmetric Yang-Mills vector multiplet [1] with the adjoint index $i$ of the gauge group $\text{SO}(\mathcal{N})$. The latter multiplet is our new vector multiplet carrying the indices $i, j, \ldots$ for the non-adjoint representation $R$ of $\text{SO}(\mathcal{N})$, which satisfies the conditions

\begin{equation}
\eta^{ij} = \delta^{ij}, \quad (T^I)^{ij} = -(T^I)^{ji}, \quad (T^I)^{[ij}(T^I)^{k]l} \equiv 0,
\end{equation}

where $\eta^{ij}$ and $(T^I)^{ij}$ are the metric of the representation $R$, and the representation matrix of the generators of $\text{SO}(\mathcal{N})$, respectively.

We can obtain our lagrangian for $N = 4$ supersymmetry for these two multiplets by the simple dimensional reduction [7] of $N = 1$ supersymmetric Yang-Mills in 10D in the non-adjoint representation, as outlined in [6]. Our lagrangian thus obtained in 4D is

\begin{equation}
\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^I)^2 - \frac{1}{4}(G_{\mu\nu}^i)^2 + \frac{1}{2}(\bar{\chi}^I D^I \chi^i) + \frac{1}{2}(\bar{\chi}^i D^i \chi^i)
\end{equation}

\begin{equation}
- \frac{1}{2}(D_\mu A_\alpha^I)^2 - \frac{1}{2}(D_\mu \bar{A}_\alpha^I)^2 - \frac{1}{2}(D_\mu B_\alpha^i)^2 - \frac{1}{2}(D_\mu \bar{B}_\alpha^i)^2
\end{equation}

\begin{equation}
- ig f^{IJK}(\bar{\chi}^I \alpha \chi^J) A_\alpha^K - \frac{1}{2} g f^{IJK}(\bar{\chi}^I \gamma_5 \beta \chi^J) \bar{A}_\alpha^K
\end{equation}

\begin{equation}
+ ig (T^I)^{ij}(\bar{\chi}^I \alpha \chi^j) B_\alpha^j + g (T^I)^{ij}(\bar{\chi}^I \gamma_5 \beta \chi^j) \bar{B}_\alpha^j
\end{equation}

\begin{equation}
+ \frac{i}{2} (T^I)^{ij}(\bar{\chi}^I \alpha \chi^j) A_\alpha^I + \frac{i}{2} (T^I)^{ij}(\bar{\chi}^I \gamma_5 \beta \chi^j) \bar{A}_\alpha^I
\end{equation}

\begin{equation}
- \frac{i}{4} g^2 \left[ f^{IJK} A_\alpha^J A_\beta^K - (T^I)^{ij} B_\alpha^i B_\beta^j \right]^2 - \frac{i}{4} g^2 \left[ f^{IJK} \bar{A}_\alpha^J \bar{A}_\beta^K - (T^I)^{ij} \bar{B}_\alpha^i \bar{B}_\beta^j \right]^2
\end{equation}

\begin{equation}
- \frac{1}{4} g^2 \left[ (T^I)^{ij} (A_\alpha^I B_\beta^j - A_\beta^I B_\alpha^j) \right]^2 - \frac{1}{4} g^2 \left[ (T^I)^{ij} (\bar{A}_\alpha^I \bar{B}_\beta^j - \bar{A}_\beta^I \bar{B}_\alpha^j) \right]^2
\end{equation}

\begin{equation}
- \frac{1}{4} g^2 \left[ (T^I)^{ij} (A_\alpha^I \bar{B}_\beta^j - \bar{A}_\beta^I B_\alpha^j) \right]^2.
\end{equation}

The $4 \times 4$ antisymmetric matrices $\alpha$ and $\beta$ satisfy the conditions for $\alpha, \beta, \ldots = 1, 2, 3$:

\begin{equation}
\alpha_\alpha \alpha_\beta = \delta_\alpha_\beta + i \epsilon_\alpha_\beta_\gamma \alpha_\gamma, \quad \beta_\alpha \beta_\beta = \delta_\alpha_\beta + i \epsilon_\alpha_\beta_\gamma \beta_\gamma, \quad [\alpha_\alpha, \beta_\beta] = 0,
\end{equation}

which are $\text{SO}(3)$ matrices, and used for the global $\text{SO}(4) \approx \text{SO}(3) \times \text{SO}(3)$ [1]. Similarly to [6], our field strengths and covariant derivatives are defined by

\begin{equation}
F_{\mu\nu}^I \equiv 2 \partial_{[\mu} A_{\nu]}^I + g f^{IJK} A_{\mu}^J A_{\nu}^K - g (T^I)^{ij} B_\mu^i B_\nu^j,
\end{equation}

\begin{equation}
G_{\mu\nu}^i \equiv 2 \partial_{[\mu} B_{\nu]}^i + 2 g (T^I)^{ij} A_{[\mu}^I B_{\nu]}^j,
\end{equation}

\begin{equation}
D_\mu \chi^i \equiv \partial_\mu \chi^i + g (T^I)^{ij} A_{\mu}^I \chi^j - g (T^I)^{ij} B_\mu^i \lambda^j,
\end{equation}
\begin{align}
\mathcal{D}_\mu \lambda^I & \equiv \partial_\mu \lambda^I + g f^{IJK} A_{\mu}^J \lambda^K - g (T^I)^{ij} B_{\mu}^i \chi^j , \\
\mathcal{D}_\mu A_{\alpha}^I & \equiv \partial_\mu A_{\alpha}^I + g f^{IJK} A_{\mu}^J A_{\alpha}^K - g (T^I)^{ij} B_{\mu}^i B_{\alpha}^j , \label{2.4c} \\
\mathcal{D}_\mu \tilde{A}_{\alpha}^I & \equiv \partial_\mu \tilde{A}_{\alpha}^I + g f^{IJK} A_{\mu}^J \tilde{A}_{\alpha}^K - g (T^I)^{ij} B_{\mu}^i \tilde{B}_{\alpha}^j , \\
\mathcal{D}_\mu B_{\alpha}^i & \equiv \partial_\mu B_{\alpha}^i + g (T^I)^{ij} A_{\mu}^j B_{\alpha}^j - g (T^I)^{ij} A_{\alpha}^j B_{\mu}^i , \label{2.4d} \\
\mathcal{D}_\mu \tilde{B}_{\alpha}^i & \equiv \partial_\mu \tilde{B}_{\alpha}^i + g (T^I)^{ij} A_{\mu}^j \tilde{B}_{\alpha}^j - g (T^I)^{ij} \tilde{A}_{\alpha}^j B_{\mu}^i . \label{2.4e}
\end{align}

Our action \( I \equiv \int d^4 x \mathcal{L} \) is invariant under supersymmetry

\begin{align}
\delta_Q A_{\mu}^I & = + (\bar{\tau} \gamma_\mu \lambda^I) , \quad \delta_Q B_{\mu}^i = + (\bar{\tau} \gamma_\mu \chi^i) , \\
\delta_Q A_{\alpha}^I & = + i (\bar{\tau} \sigma_\alpha \lambda^I) , \quad \delta_Q B_{\alpha}^i = + i (\bar{\tau} \sigma_\alpha \chi^i) , \\
\delta_Q \tilde{A}_{\alpha}^I & = + (\bar{\tau} \gamma_5 \sigma_\alpha \lambda^I) , \quad \delta_Q \tilde{B}_{\alpha}^i = + (\bar{\tau} \gamma_5 \sigma_\alpha \chi^i) , \\
\delta_Q \lambda^I & = + \frac{1}{2} (\gamma^{\mu\nu} \epsilon) F_{\mu\nu}^I + i (\alpha_\gamma \gamma^\mu) \mathcal{D}_\mu A_{\alpha}^I - (\beta_\gamma \gamma^\mu \epsilon) \mathcal{D}_\mu \tilde{A}_{\alpha}^I \\
& \quad + \frac{1}{2} g \epsilon_{\beta\gamma} \gamma_{\mu} \left[ f^{IJK} A_{\mu}^J A_{\beta}^K - (T^I)^{ij} B_{\mu}^i B_{\beta}^j \right] \\
& \quad + \frac{1}{2} g \epsilon_{\beta\gamma} \gamma_{\mu} \left[ f^{IJK} \tilde{A}_{\mu}^J \tilde{A}_{\beta}^K - (T^I)^{ij} \tilde{B}_{\mu}^i \tilde{B}_{\beta}^j \right] \\
& \quad - i g (\alpha_\beta \gamma_{\mu}) \left[ f^{IJK} A_{\mu}^J \tilde{A}_{\beta}^K - (T^I)^{ij} B_{\mu}^i \tilde{B}_{\beta}^j \right] , \\
\delta_Q \chi^i & = + \frac{1}{2} (\gamma^{\mu\nu} \epsilon) G_{\mu\nu}^i + i (\alpha_\gamma \gamma^\mu) \mathcal{D}_\mu B_{\alpha}^i - (\beta_\gamma \gamma^\mu \epsilon) \mathcal{D}_\mu \tilde{B}_{\alpha}^i \\
& \quad + i g \epsilon_{\alpha_\gamma} (T^I)^{ij} (\alpha_\gamma) A_{\mu}^j B_{\beta}^j + i g \epsilon_{\alpha_\gamma} (T^I)^{ij} (\beta_\gamma) \tilde{A}_{\alpha}^j \tilde{B}_{\beta}^j \\
& \quad - i g (\alpha_\beta \gamma_{\mu}) (T^I)^{ij} (\alpha_\beta) A_{\mu}^j \tilde{B}_{\gamma}^j - \tilde{A}_{\beta}^j B_{\mu}^i \chi^i . \label{2.5e}
\end{align}

The supersymmetric invariance of our action \( \delta_Q I = 0 \) can be confirmed in the usual way. The crucial relationships are the conditions \((2.1)\), as well as the Bianchi identities

\begin{align}
\mathcal{D}_{[\mu} F_{\nu\rho]}^I & \equiv \partial_{[\mu} F_{\nu\rho]}^I + g f^{IJK} A_{[\mu}^J F_{\nu\rho]}^K - g (T^I)^{ij} B_{[\mu}^j G_{\nu\rho]}^i j \equiv 0 , \label{2.6a} \\
\mathcal{D}_{[\mu} G_{\nu\rho]}^i & \equiv \partial_{[\mu} G_{\nu\rho]}^i + g (T^I)^{ij} A_{[\mu}^J G_{\nu\rho]}^j - g (T^I)^{ij} B_{[\mu}^j F_{\nu\rho]}^i \equiv 0 . \label{2.6b}
\end{align}

At the cubic-order level in \( \delta_Q \mathcal{L} \), we need the Fierz identity

\begin{equation}
\left[ (\gamma_\mu)_{AB} (\gamma^\mu)_{CD} \delta_{(i)} (\gamma^\mu)_{(j)} \delta_{(k)} (\gamma^\mu)_{(l)} - C_{AB} C_{CD} (\alpha_\alpha)_{(i)} (\gamma^\mu)_{(j)} (\alpha_\alpha)_{(k)} (\gamma^\mu)_{(l)} \\
\quad + (\gamma_5)_{AB} (\gamma_5)_{CD} (\beta_\alpha)_{(i)} (\gamma^\mu)_{(j)} (\beta_\alpha)_{(k)} (\gamma^\mu)_{(l)} \right] + (2 \text{ perm.}) \equiv 0 , \label{2.7}
\end{equation}

where \( A, B, \ldots = 1, \ldots, 4 \) are for the Majorana spinor components in 4D, while ‘2 perm.’ stands for the two more sets of terms for the cyclic permutations of

\begin{align}
B_{(j)} \rightarrow C_{(k)}, \quad C_{(k)} \rightarrow D_{(l)}, \quad D_{(l)} \rightarrow B_{(j)},
\end{align}
so that the whole expression is totally symmetric with respect to these three pairs of indices. This identity is used both for the \( g\lambda\chi^2 \) and the \( g\lambda^3 \)-terms in \( \delta_Q \mathcal{L} \). The key ingredient at the quartic-order level is the usage of the condition (2.1b) in the sector \( g^2\chi B_\alpha B_\beta \bar{B}_\gamma \) in the variation \( \delta_Q \mathcal{L} \).

Our peculiar vector multiplet \((B_\mu^i, \lambda^i, B_\alpha^i, \bar{B}_\alpha^i)\) carrying the indices \( i \) of the representation \( R \) of \( SO(N) \) must satisfy the conditions in (2.1). A necessary conditions of (2.1b) is

\[
\frac{2dI_2(R)}{N(N-1)} - 2I_2(R) + N - 2 = 0 ,
\]

where \( d \equiv \text{dim} (R) \), while the second index \( I_2(R) \) is defined by \((T^IT^I)_{ij} = -2I_2(R)\delta_{ij}\), and accordingly \((T^IT^I)_{ii} = -4dI_2(R)\delta_{II}/N(N-1)\). As long as these conditions are satisfied, the representation \( R \) can be any real representation of \( SO(N) \). A trivial example is the \( \mathcal{N} \) of \( SO(N) \), but this system has the hidden local symmetry, i.e., the system is equivalent to a supersymmetric Yang-Mills theory for the local \( SO(N+1) \) \[6\]. Non-trivial examples are the \( 8_C \) and \( 8_S \)-representations of \( SO(8) \) different from the usual \( 8_V \)-representation.

3. Summary and Concluding Remarks

In this paper, we have constructed the system of \( N = 4 \) supersymmetric Yang-Mills multiplet in a non-adjoint real representation \( R \). As long as the conditions in (2.1) are satisfied for the representation \( R \), we have found the extra vector multiplet \((B_\mu^i, \lambda^i, B_\alpha^i, \bar{B}_\alpha^i)\) can be coupled to the conventional vector multiplet \((A_\mu^I, \lambda^i_I, A_\alpha^I, \bar{A}_\alpha^I)\) consistently with supersymmetry. As in the \( N = 1 \) case, we need at least the conventional \( N = 4 \) supersymmetric Yang-Mills multiplet in the adjoint representation, once we introduce the vector multiplet in the non-adjoint representation \( R \). The non-trivial examples of such representations are the \( 8_S \) and \( 8_C \) of \( SO(8) \).

According to the prevailing notion, since the \( N = 4 \) supersymmetry is the maximal extended global supersymmetry in 4D, there is no other outside multiplet that can be coupled to the basic \( 8 + 8 \) multiplet \((A_\mu^I, \lambda^i_I, A_\alpha^I, \bar{A}_\alpha^I)\). In that sense, the field content of \( N = 4 \) supersymmetric theory is supposed to be unique \[5\]. However, we already know one counter-example against this wisdom, namely a vector multiplet gauging scale symmetry presented in \[8\]. Our theory in this paper has established another counter-example now with the \( N = 4 \) vector multiplet in the non-adjoint representation \( R \) consistently coupled to the conventional \( N = 4 \) Yang-Mills multiplet. It is amazing that such a tight \( N = 4 \) maximally
extended supermultiplet with $8 + 8$ degrees of freedom can be further coupled to an extra vector multiplet with additional physical degrees of freedom.

As the conventional $N = 4$ supersymmetric Yang-Mills theory is finite [2], so may well be our theory to all orders.

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