ON REFLEXIVE GROUPS AND FUNCTION SPACES WITH A MACKEY GROUP TOPOLOGY

S. GABRIYELYAN

Abstract. We prove that every reflexive abelian group $G$ such that its dual group $G^\wedge$ has the $qc$-Glicksberg property is a Mackey group. We show that a reflexive abelian group of finite exponent is a Mackey group. We prove that, for a realcompact space $X$, the space $C_b(X)$ is barrelled if and only if it is a Mackey group.

1. Introduction

For an abelian topological group $(G, \tau)$ we denote by $\widehat{G}$ the group of all continuous characters of $(G, \tau)$. If $\widehat{G}$ separates the points of $G$, the group $G$ is called maximally almost periodic (MAP for short). The class $\mathcal{LQC}$ of all locally quasi-convex groups is the most important subclass of the class $\mathcal{MAP}$ of all MAP abelian groups (all relevant definitions see Section 2).

Two topologies $\tau$ and $\nu$ on an abelian group $G$ are said to be compatible if $(\widehat{G}, \tau) = (\widehat{G}, \nu)$. Being motivated by the classical Mackey–Arens theorem the following notion was introduced and studied in [7]: a locally quasi-convex abelian group $(G, \tau)$ is called a Mackey group in $\mathcal{LQC}$ or simply a Mackey group if for every compatible locally quasi-convex group topology $\nu$ on $G$ it follows that $\nu \leq \tau$. Every barrelled locally convex space is a Mackey group by [7]. Since every reflexive locally convex space $E$ is barrelled by [17, Proposition 11.4.2], we obtain that $E$ is a Mackey group. This result motivates the following question:

Question 1.1. Which reflexive abelian topological groups are Mackey groups?

In Section 2 we obtain a sufficient condition on a reflexive group to be a Mackey group, see Theorem 2.4. Using Theorem 2.4 we obtain a complete answer to Question 1.1 for reflexive groups of finite exponent.

Theorem 1.2. Any reflexive abelian group $(G, \tau)$ of finite exponent is a Mackey group.

Note that any metrizable precompact abelian group of finite exponent is a Mackey group, see [5, Example 4.4]. So there are non-reflexive Mackey groups of finite exponent. If $G$ is a metrizable reflexive group, then $G$ must be complete by [6, Corollary 2]. So $G$ is a Mackey group by [7, Theorem 4.2]. On the other hand, there are reflexive non-complete groups $G$ of finite exponent, see [13]. Such groups $G$ are also Mackey by Theorem 1.2.

For a Tychonoff space $X$ let $C_k(X)$ be the space of all continuous real-valued functions on $X$ endowed with the compact-open topology. The relations between locally convex properties of $C_k(X)$ and topological properties of $X$ are illustrated by the following classical results, see [17, Theorem 11.7.5].

Theorem 1.3 (Nachbin–Shirota). For a Tychonoff space $X$ the space $C_k(X)$ is barrelled if and only if every functionally bounded subset of $X$ has compact closure.
This theorem motivates the following question posed in [12]: For which Tychonoff spaces $X$ the space $C_k(X)$ is a Mackey group? In the next theorem we obtain a partial answer to this question.

**Theorem 1.4.** For a realcompact space $X$, the space $C_k(X)$ is barrelled if and only if it is a Mackey group.

It is well-known (see [17, Theorem 13.6.1]) that a Tychonoff space $X$ is realcompact if and only if the space $C_k(X)$ is bornological. This result and Theorem 1.4 imply

**Corollary 1.5.** A bornological space $C_k(X)$ is barrelled if and only if it is a Mackey group.

We prove Theorem 1.4 in Section 3.

2. Proof of Theorem 1.4

Denote by $S$ the unit circle group and set $S_+ := \{ z \in S : \text{Re}(z) \geq 0 \}$. Let $G$ be an abelian topological group. If $\chi \in \hat{G}$, it is considered as a homomorphism from $G$ into $S$. A subset $A$ of $G$ is called quasi-convex if for every $g \in G \setminus A$ there exists $\chi \in \hat{G}$ such that $\chi(x) \notin S_+$ and $\chi(A) \subseteq S_+$. If $A \subseteq G$ and $B \subseteq \hat{G}$ set

$$A^\circ := \{ \chi \in \hat{G} : \chi(A) \subseteq S_+ \}, \quad B^\circ := \{ g \in G : \chi(g) \in S_+ \ \forall \chi \in B \}.$$  

Then $A$ is quasi-convex if and only if $A^{\circ \circ} = A$. An abelian topological group $G$ is called locally quasi-convex if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. The dual group $\hat{G}$ of $G$ endowed with the compact-open topology is denoted by $G^\wedge$. The homomorphism $\alpha_G : G \to G^\wedge, g \mapsto (\chi \mapsto \chi(g))$, is called the canonical homomorphism. If $\alpha_G$ is a topological isomorphism the group $G$ is called reflexive.

If $G$ is a MAP abelian group, we denote by $\sigma(G, \hat{G})$ the weak topology on $G$, i.e., the smallest group topology on $G$ for which the elements of $\hat{G}$ are continuous. In the dual group $\hat{G}$, we denote by $\sigma(\hat{G}, G)$ the topology of pointwise convergence.

We use the next fact, see Proposition 1.5 of [3].

**Fact 2.1.** Let $U$ be a neighborhood of zero of an abelian topological group $G$. Then $U^\circ$ is a compact subset of $(\hat{G}, \sigma(\hat{G}, G))$.

Let $G$ be a MAP abelian group and $P$ a topological property. Denote by $P(G)$ the set of all subspaces of $G$ with $P$. Following [19], $G$ respects $P$ if $P(G) = P(G, \sigma(G, \hat{G}))$. Below we define weak versions of respected properties. For a MAP abelian group $G$, we denote by $P_{qc}(G)$ the set of all quasi-convex subspaces of $G$ with $P$.

**Definition 2.2.** Let $(G, \tau)$ be a MAP abelian group. We say that

(i) $(G, \tau)$ respects $P_{qc}$ if $P_{qc}(G) = P_{qc}(G, \sigma(G, \hat{G}))$;

(ii) $(G, \tau)^\wedge$ weak$^*$ respects $P$ if $P(G^\wedge) = P(\hat{G}, \sigma(\hat{G}, G))$;

(iii) $(G, \tau)^\wedge$ weak$^*$ respects $P_{qc}$ if $P_{qc}(G^\wedge) = P_{qc}(\hat{G}, \sigma(\hat{G}, G))$.

In the case $P$ is the property $C$ to be a compact subset and a MAP abelian group $(G, \tau)$ (or $G^\wedge$) (weak$^*$) respects $P$ or $P_{qc}$, we shall say that the group $G$ (or $G^\wedge$) has the (weak$^*$) Glicksberg property or $qc$-Glicksberg property, respectively. So $G$ has the Glicksberg property or respects compactness if any weakly compact subset of $G$ is also compact in the original topology $\tau$. By a famous result of Glicksberg, any abelian locally compact group respects compactness. Clearly, if a MAP abelian group $(G, \tau)$ has the Glicksberg property, then it also has the $qc$-Glicksberg property, and if $(G, \tau)^\wedge$ has the weak$^*$ Glicksberg property, then it has also the weak$^*$ $qc$-Glicksberg property.

**Proposition 2.3.** Let $(G, \tau)$ be a locally quasi-convex group such that the canonical homomorphism $\alpha_G$ is continuous. If $(G, \tau)^\wedge$ has the weak$^*$ $qc$-Glicksberg property, then $(G, \tau)$ is a Mackey group.
Proof. Let \( \nu \) be a locally quasi-convex topology on \( G \) compatible with \( \tau \) and let \( U \) be a quasi-convex \( \nu \)-neighborhood of zero. Fact 2.1 implies that the quasi-convex subset \( K := U^\circ \) of \( \hat{G} \) is \( \sigma(\hat{G}, G) \)-compact, and hence \( K \) is a compact subset of \( G^\wedge \) by the weak\(^*\) qc-Glicksberg property. Note that, by definition, \( K^\circ \) is a neighborhood of zero in \( G^\wedge \). As \( \alpha_G \) is continuous, \( U = K^\circ = \alpha_G^{-1}(K^\circ) \) is a neighborhood of zero in \( G \). Hence \( \nu \leq \tau \). Thus \((G, \tau)\) is a Mackey group. \( \square \)

The following theorem gives a partial answer to Question 1.1.

**Theorem 2.4.** Let \((G, \tau)\) be a reflexive abelian group. If \((G, \tau)^\wedge\) has the qc-Glicksberg property (in particular, the Glicksberg property), then \((G, \tau)\) is a Mackey group.

**Proof.** Since \( G \) is a reflexive group, the weak\(^*\) qc-Glicksberg property coincides with the qc-Glicksberg property, and Proposition 2.5 applies. \( \square \)

**Remark 2.5.** In Theorem 2.4 the reflexivity of \( G \) is essential. Indeed, let \( G \) be a proper dense subgroup of a compact metrizable abelian group \( X \). Then \( G^\wedge = X^\wedge \) (see [1, 6]), and hence the discrete group \( G^\wedge \) has the Glicksberg property. Now set \( c_0(\mathbb{S}) := \{(z_n) \in \mathbb{S}^\mathbb{N} : z_n \to 1\} \). Denote by \( \mathfrak{p}_0 \) the product topology on \( c_0(\mathbb{S}) \) induced from \( \mathbb{S}^\mathbb{N} \), and let \( \mathfrak{u}_0 \) be the uniform topology on \( c_0(\mathbb{S}) \) induced by the metric \( d((z_n^1), (z_n^2)) = \sup\{|z_n^1 - z_n^2|, n \in \mathbb{N}\} \). Then, by [9, Theorem 1], \( \mathfrak{p}_0 \) and \( \mathfrak{u}_0 \) are locally quasi-convex and compatible topologies on \( c_0(\mathbb{S}) \) such that \( \mathfrak{p}_0 < \mathfrak{u}_0 \). Thus the group \( G := (c_0(\mathbb{S}), \mathfrak{p}_0) \) is a precompact arc-connected metrizable group such that \( G^\wedge \) has the Glicksberg property, but \( G \) is not a Mackey group.

Theorem 2.4 motivates the following question: For which (reflexive) abelian groups \( G \) the dual group \( G^\wedge \) has the (weak\(^*\), weak\(^*\) qc-, qc-) Glicksberg property? Below in Propositions 2.6 and 2.9 we give some sufficient conditions on \( G \) for which \( G^\wedge \) has the Glicksberg property.

Recall (see [7]) that a MAP abelian group \( G \) is called \( g \)-barrelled if any \( \sigma(\hat{G}, G) \)-compact subset of \( \hat{G} \) is equicontinuous. Every barrelled locally convex space \( E \) is a \( g \)-barrelled group, but the converse does not hold in general, see [7]. Every locally quasi-convex \( g \)-barrelled abelian group \( G \) is a Mackey group by Theorem 4.2 of [4].

**Proposition 2.6.** If \( G \) is a \( g \)-barrelled group, then \( G^\wedge \) has the Glicksberg property.

**Proof.** Let \( K \) be a \( \sigma(\hat{G}, G^\wedge) \)-compact subset of \( \hat{G} \). Then \( K \) is \( \sigma(\hat{G}, G) \)-compact as well, so \( K \) is equicontinuous. Hence there is a neighborhood \( U \) of zero in \( G \) such that \( K \subseteq U^\circ \), see Corollary 1.3 of [7]. The set \( U^\circ \) is a compact subset of \( G^\wedge \) by Fact 2.1. As \( K \) is also a closed subset of \( G^\wedge \), we obtain that \( K \) is compact in \( G^\wedge \). Thus \( G^\wedge \) has the Glicksberg property. \( \square \)

For reflexive groups this proposition can be reversed.

**Proposition 2.7.** If \( G \) is a reflexive group, then \( G \) is \( g \)-barrelled if and only if \( G^\wedge \) has the Glicksberg property.

**Proof.** Assume that \( G^\wedge \) has the Glicksberg property and \( K \) is a \( \sigma(\hat{G}, G^\wedge) \)-compact subset of \( \hat{G} \). By the reflexivity of \( G \), \( K \) is also \( \sigma(\hat{G}, G^\wedge) \)-compact. So \( K \) is compact in \( G^\wedge \) by the Glicksberg property. Therefore \( K^\circ \) is a neighborhood of zero in \( G^\wedge \). So, by the reflexivity of \( G \), \( K^\circ = \alpha_G^{-1}(K^\circ) \) is a neighborhood of zero in \( G \). Since \( K \subseteq K^\circ \), we obtain that \( K \) is equicontinuous, see Corollary 1.3 of [7]. Thus \( G \) is \( g \)-barrelled. The converse assertion follows from Proposition 2.6. \( \square \)

Recall that a topological group \( X \) is said to have a subgroup topology if it has a base at the identity consisting of subgroups. For the definition and properties of nuclear groups, see [3].

**Lemma 2.8.** Let \( G \) be an abelian topological group with a subgroup topology. Then \( G \) is a locally quasi-convex nuclear group and has the Glicksberg property.
Proof. By Proposition 2.2 of [2], G embeds into a product of discrete groups. Therefore G is a locally quasi-convex nuclear group by Propositions 7.5 and 7.6 and Theorem 8.5 of [3]. Finally, the group \( G \) has the Glicksberg property by [4]. □

To prove Theorem 1.2 we need the following proposition.

Proposition 2.9. Let \((G, \tau)\) be a locally quasi-convex abelian group of finite exponent. Then \((G, \tau)\) and hence also \((G, \tau)^\wedge\) have the Glicksberg property.

Proof. Propositions 2.1 of [2] implies that the topologies of the groups \((G, \tau)\) and \((G, \tau)^\wedge\) are subgroup topologies, and Lemma 2.8 applies. □

Proof of Theorem 1.2. Since \((G, \tau)\) is locally quasi-convex, Proposition 2.9 implies that \((G, \tau)^\wedge\) has the Glicksberg property. Thus \((G, \tau)\) is a Mackey group by Theorem 2.4. □

For Tychonoff spaces \(X\) and \(Y\) we denote by \(C_k(X, Y)\) the space of all continuous functions from \(X\) into \(Y\) endowed with the compact-open topology. R. Pol and F. Smente k [18] proved that the group \(C_k(X, D)\) is reflexive for every finitely generated discrete group \(D\) and each zero-dimensional realcompact \(k\)-space \(X\). This result and Theorem 1.2 immediately imply

Corollary 2.10. Let \(X\) be a zero-dimensional realcompact \(k\)-space and \(F\) be a finite abelian group. Then \(C_k(X, F)\) is a Mackey group.

We end this section with the following two questions. We do not know whether the converse in Theorem 2.4 is true.

Question 2.11. Let \(G\) be a reflexive Mackey group. Does \(G^\wedge\) have the qc-Glicksberg property?

Set \(\mathfrak{F}_0(S) := (c_0(S), u_0)\), see Remark 2.5. Then the group \(\mathfrak{F}_0(S)\) is reflexive [9] and does not have the Glicksberg property by [10]. These results motivate the following question.

Question 2.12. Does \(\mathfrak{F}_0(S)\) have the qc-Glicksberg property?

This question is of importance because the dual group \(\mathfrak{F}_0(S)^\wedge\) is the free abelian topological group \(A(s)\) over a convergent sequence \(s\), see [9]. So the positive answer to this question together with Theorem 2.4 would imply: (1) the group \(A(s)\) is a Mackey group, answering in the affirmative a question posed in [14], and (2) there are locally quasi-convex (even reflexive and Polish) abelian groups with the qc-Glicksberg property but without the Glicksberg property. On the other hand, under the assumptions that Question 2.12 has a negative answer and the group \(A(s)\) is Mackey, we obtain a negative answer to Question 2.11.

3. Proof of Theorem 1.4

Let \(E\) be a nontrivial locally convex space and denote by \(E'\) the topological dual space of \(E\). Clearly, \(E\) is also an abelian topological group. Therefore we can consider the group \(\hat{E}\) of all continuous characters of \(E\). The next important result is proved in [16, 20], see also [15, 23.32].

Fact 3.1. Let \(E\) be a locally convex space. Then the mapping \(p : E' \to \hat{E}\), defined by the equality \(p(f) = \exp\{2\pi if\}\), for all \(f \in E'\), is a group isomorphism between \(E'\) and \(\hat{E}\).

Recall that the dual space of \(C_k(X)\) is the space \(M_c(X)\) of all Borel measures \(\mu\) on \(X\) with compact support \(\text{supp}(\mu)\), see [17, Corollary 7.6.5]. For a point \(x \in X\) we denote by \(\delta_x\) the point measure associated with the linear form \(f \mapsto f(x)\).

The next lemma is crucial for our proof of Theorem 1.4.
Lemma 3.2. Let $A$ be a functionally bounded subset of a Tychonoff space $X$. If there is a discrete family $U = \{U_n\}_{n \in \mathbb{N}}$ of open subsets of $X$ such that $U_n \cap A \neq \emptyset$ for every $n \in \mathbb{N}$, then $C_k(X)$ is not a Mackey group.

Proof. For every $n \in \mathbb{N}$, take arbitrarily $x_n \in U_n \cap A$ and set $\chi_n := (1/n)\delta_{x_n}$. Since $A$ is functionally bounded, we obtain that $\chi_n \to 0$ in the weak* topology on $M_c(X)$. Denote by $Q : c_0 \to \mathcal{F}_0(S)$ the quotient map, so $Q((x_n)_{n \in \mathbb{N}}) = ((q(x_n))_{n \in \mathbb{N}})$, where $q : \mathbb{R} \to S$ is defined by $q(x) = \exp\{2\pi i x\}$. Now we can define the linear injective operator $F : C(X) \to C_k(X) \times c_0$ and the monomorphism $F_0 : C(X) \to C_k(X) \times \mathcal{F}_0(S)$ setting ($\forall f \in C(X)$)

$$F(f) := (f, R(f)),$$

where $R(f) := (\chi_n(f)) \in c_0$,

$$F_0(f) := (f, R_0(f)),$$

where $R_0(f) := Q \circ R(f) = \{\exp\{2\pi i \chi_n(f)\}\} \in \mathcal{F}_0(S)$.

Denote by $\mathcal{T}$ and $\mathcal{T}_0$ the topology on $C(X)$ induced from $C_k(X) \times c_0$ and $C_k(X) \times \mathcal{F}_0(S)$, respectively. So $\mathcal{T}$ is a locally convex vector topology on $C(X)$ and $\mathcal{T}_0$ is a locally quasi-convex group topology on $C(X)$ (since the group $\mathcal{F}_0(S)$ is locally quasi-convex, and a subgroup of a product of locally quasi-convex groups is clearly locally quasi-convex). Denote by $\tau_k$ the compact-open topology on $C(X)$. Then, by construction, $\tau_k \leq \mathcal{T}_0 \leq \mathcal{T}$, so taking into account Fact 3.1 we obtain

$$p(M_c(X)) \subseteq (C(X), \mathcal{T}_0) \subseteq p((C(X), \mathcal{T})).$$

Let us show that the topologies $\tau_k$ and $\mathcal{T}_0$ are compatible. By [3.1], it is enough to show that each character of $(C(X), \mathcal{T}_0)$ belongs to $p(M_c(X))$. Fix $\chi \in (C(X), \mathcal{T}_0)$. Then (3.1) and the Hahn–Banach extension theorem imply that $\chi = p(\eta)$ for some $\eta = (\nu, (c_n)) \in M_c(X) \times \ell_1$, where $\nu \in M_c(X)$ and $(c_n) \in \ell_1$, and

$$\eta(f) = \nu(f) + \sum_{n \in \mathbb{N}} c_n \chi_n(f), \ \forall f \in C(X).$$

To prove that $\chi \in p(M_c(X))$ it is enough to show that $c_n = 0$ for almost all indices $n$. Suppose for a contradiction that $|c_n| > 0$ for infinitely many indices $n$. Take a neighborhood $U$ of zero in $\mathcal{T}_0$ such that (see Fact 3.1)

$$\eta(U) \subseteq \left(\frac{-1}{10}, \frac{1}{10}\right) + \mathbb{Z}.$$ 

We can assume that $U$ has a canonical form

$$U = F_0^{-1}\left(\left\{ f \in C(X) : f(K) \subseteq (-\varepsilon, \varepsilon) \right\} \times \left(V^n \cap c_0(S)\right) \cap F_0(C(X))\right),$$

for some compact set $K \subseteq X$, $\varepsilon > 0$ and a neighborhood $V$ of the identity of $S$. For every $n \in \mathbb{N}$, choose a continuous function $g_n : X \to [0, 1]$ such that $g_n(x_n) = 1$ and $g_n(X \setminus U_n) = \{0\}$. So, by the discreteness of $U$, we obtain

$$\chi_n(g_n) = \frac{1}{n}, \ \text{and} \ \chi_m(g_n) = 0 \ \text{for every distinct} \ n, m \in \mathbb{N}.$$

Let $C = \text{supp}(\nu)$, so $C$ is a compact subset of $X$. Then the discreteness of the family $U$ implies that there is $n_0 \in \mathbb{N}$ such that $U_n \cap (K \cup C) = \emptyset$ for every $n > n_0$. Since $|c_n| > 0$ for infinitely many indices, we can find an index $\alpha > n_0$ such that $0 < |c_\alpha| < 1/100$ (recall that $(c_n) \in \ell_1$). Set

$$h(x) = \left[\frac{1}{4c_\alpha}\right] \cdot \alpha \cdot g_\alpha(x),$$

were $[x]$ is the integral part of a real number $x$. Then [3.3] implies that

$$\chi_n(h) = 0 \text{ if } n \neq \alpha, \ \text{and} \ \chi_\alpha(h) = \left[\frac{1}{4c_\alpha}\right] \cdot \alpha = \left[\frac{1}{4c_\alpha}\right] \in \mathbb{Z}.$$
Therefore $R_0(h)$ is the identity of $\mathcal{G}_0(S_0)$. Since also $h \in \{f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon)\}$ we obtain that $h \in U$. Noting that $\nu(h) = 0$ and setting $r_\alpha := \frac{1}{4c_\alpha} - \left[ \frac{1}{4c_\alpha} \right]$ (and hence $0 \leq r_\alpha < 1$), (3.4) implies
\[
\frac{1}{4} - \frac{1}{100} < \eta(h) = c_\alpha \chi_\alpha(h) = c_\alpha \left[ \frac{1}{4c_\alpha} \right] = c_\alpha \left( \frac{1}{4c_\alpha} - r_\alpha \right) = \frac{1}{4} - c_\alpha r_\alpha < \frac{1}{4} + \frac{1}{100}.
\]
But these inequalities contradict the inclusion (3.2). This contradiction shows that $c_\alpha = 0$ for almost all indices $n$, and hence $\eta \in M_k(X)$. Thus $\tau_k$ and $\mathcal{T}_0$ are compatible.

To complete the proof we have to show that the topology $\mathcal{T}_0$ is strictly finer than $\tau_k$. First we note that $(n/2)g_n \to 0$ in $\tau_k$. Indeed, let $K_0$ be a compact subset of $X$ and $\varepsilon > 0$. Since the family $\mathcal{U}$ is discrete, there is $N \in \mathbb{N}$ such that $U_n \cap K_0 = \emptyset$ for every $n > N$. Then $(n/2)g_n \in \{f \in C(X) : f(K_0) \subset (-\varepsilon, \varepsilon)\}$ for $n > N$. On the other hand,
\[
F_0((n/2)g_n) = \left( (n/2)g_n, \exp\{2\pi i\chi_k((n/2)g_n)\} \right) = \left( (n/2)g_n, (0, \ldots, 0, -1, 0, \ldots) \right),
\]
where $-1$ is placed in position $n$. So $(n/2)g_n \not\to 0$ in $\mathcal{T}_0$. Thus $\mathcal{T}_0$ is strictly finer than $\tau_k$. \hfill \Box

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4** Assume that $C_k(X)$ is a Mackey group. Let us show that every functionally bounded subset of $X$ has compact closure. Suppose for a contradiction that there is a closed functionally bounded subset $A$ of $X$ which is not compact. Since $A$ is a closed subset of a realcompact space, there is a continuous real-valued function $f$ on $X$ such that $f|_A$ is unbounded, see [14, Problem 8E.1]. So there exists a discrete sequence of open sets $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ intersecting $A$. Therefore $C_k(X)$ is not a Mackey group by Lemma 3.2. This contradiction shows that every functionally bounded subset of $X$ has compact closure. By the Nachbin–Shirota theorem 1.3, the space $C_k(X)$ is barrelled.

Conversely, if $C_k(X)$ is a barrelled locally convex space, then it is a Mackey group by Proposition 5.4 of [7]. \hfill \Box

We do not know whether the assumption to be a realcompact space can be omitted in Theorem 1.4.

**Question 3.3.** Let $X$ be a Tychonoff space. Is it true that $C_k(X)$ is barrelled if and only if it is a Mackey group?

**References**

1. L. Außenhofer, *Contributions to the duality theory of Abelian topological groups and to the theory of nuclear groups*, Dissertation, Tübingen, 1998, Dissertations Math. (Rozprawy Mat.) 384 (1999), 113p.
2. L. Außenhofer, S. Gabriyelyan, On reflexive group topologies on abelian groups of finite exponent, Arch. Math. 99 (2012), 583–588.
3. W. Banaszczyk, *Additive subgroups of topological vector spaces*, LNM 1466, Berlin-Heidelberg-New York 1991.
4. W. Banaszczyk and E. Martín-Peinador, Weakly pseudocompact subsets of nuclear groups, J. Pure Appl. Algebra 138 (1999), 99–106.
5. F. G. Bonales, F. J. Trigos-Arrieta, R. Vera Mendoza, A Mackey–Arens theorem for topological abelian groups, Bol. Soc. Mat. Mexicana 9 (2003), 79–88.
6. M.J. Chasco, Pontryagin duality for metrizable groups, Arch. Math. 70 (1998) 22-28.
7. M. J. Chasco, E. Martín-Peinador, V. Tarieladze, On Mackey topology for groups, Studia Math. 132 (1999), 257–284.
8. R. Engelking, *General topology*, Panstwowe Wydawnictwo Naukowe, 1985.
9. S. Gabriyelyan, Groups of quasi-invariance and the Pontryagin duality, Topology Appl. 157 (2010), 2786–2802.
10. S. Gabriyelyan, On topological properties of the group of the null sequences valued in an Abelian topological group, available in arXiv: 1306.511.
11. S. Gabriyelyan, On the Mackey topology for abelian topological groups and locally convex spaces, preprint.
12. S. Gabriyelyan, A characterization of barrelledness of $C_0(X)$, available in ArXiv: 1512.00788.
13. J. Galindo, L. Recoder-Núñez, M. Tkachenko, Reflexivity of prodiscrete topological groups, J. Math. Anal. Appl. 384 (2011), 320-330. 
14. L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand, New York, 1960. 
15. E. Hewitt, K. A. Ross, Abstract Harmonic Analysis, Vol. I, 2nd ed. Springer-Verlag, Berlin, 1979. 
16. E. Hewitt, H. Zuckerman, A group-theoretic method in approximation theory, Ann. Math. 52 (1950), 557–567. 
17. H. Jarchow, Locally Convex Spaces, B.G. Teubner, Stuttgart, 1981. 
18. R. Pol, F. Smentek, Note on reflexivity of some spaces of integer-valued functions, J. Math. Anal. Appl. 395 (2012), 251–257. 
19. D. Remus and F. J. Trigos-Arrieta, The Bohr topology of Moore groups, Topology Appl. 97 (1999), 85–98. 
20. M. F. Smith, The Pontrjagin duality theorem in linear spaces, Ann. Math. 56 (1952), 248–253.

DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER-SHEVA, P.O. 653, ISRAEL
E-mail address: saak@math.bgu.ac.il