FLAME: Differentially Private Federated Learning in the Shuffle Model

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Abstract—Federated Learning (FL) is a promising machine learning paradigm that enables the analyzer to train a model without collecting users’ raw data. To ensure users’ privacy, differentially private federated learning has been intensively studied. The existing works are mainly based on the curator model or local model of differential privacy. However, both of them have pros and cons. The curator model allows greater accuracy but requires a trusted analyzer. In the local model where users randomize local data before sending them to the analyzer, a trusted analyzer is not required but the accuracy is limited. In this work, by leveraging the privacy amplification effect in the recently proposed shuffle model of differential privacy, we achieve the best of two worlds, i.e., accuracy in the curator model and strong privacy without relying on any trusted party. We first propose an FL framework in the shuffle model and a simple protocol (SS-Simple) extended from existing work. We find that SS-Simple only provides an insufficient privacy amplification effect in FL, since the dimension of the model parameter is quite large. To solve this challenge, we propose an enhanced protocol (SS-Double) to increase the privacy amplification effect by subsampling. Furthermore, for boosting the utility when the model size is greater than the user population, we propose an advanced protocol (SS-Topk) with gradient sparsification techniques. We also provide theoretical analysis and numerical evaluations of the privacy amplification of the proposed protocols. Experiments on real-world datasets validate that SS-Topk improves the testing accuracy by 60.7% than the local model based FL. We highlight the observation that SS-Topk even can improve by 33.94% accuracy than the curator model based FL without any trusted party. Compared with non-private FL, our protocol SS-Topk only lose 1.48% accuracy under (4.606, 10⁻⁵)-DP.

INTRODUCTION

Federated Learning (FL) [1, 10] is a promising machine learning paradigm that enables the analyzer to train a central model without collecting users’ raw data but only local updates. However, it has been shown that sharing raw local updates may compromise users’ privacy [2, 3, 4]. To this end, differentially private federated learning has been widely studied to provide formal privacy. The existing works are mainly based on the curator model (DP) or local model (LDP) of differential privacy. The curator model based FL (DP-FL) [5, 6] allows better learning accuracy but relies on a trusted analyzer to collect raw local updates. The local model based FL (LDP-FL) [7] preserves strong local privacy since the users randomize local updates before sending them to an untrusted analyzer; but it suffers a low utility. Specifically, for a basic bit summation task over n users with privacy budget $\epsilon$, the error of DP can achieve $O(1/\epsilon)$; whereas, the error of LDP is bounded by $\Omega(\sqrt{n})$ [8].

The recently proposed secure shuffle model (SS) can achieve the best of two worlds, i.e., accuracy in the curator model and strong privacy in the local model. The shuffle model introduces a shuffler sitting between users and the analyzer (as shown in Figure 1(b)) permutes the locally randomized data from users before sending them to the analyzer. The accuracy gain of the shuffle model is obtained from the privacy amplification effect [9], which indicates that the shuffled (i.e., anonymized) outputs of local randomizers provide a stronger (amplified) privacy in the central view of differential privacy than the one without a shuffler. Accordingly, less local noise is needed in the shuffle model to preserve the same level of privacy against the untrusted analyzer.

However, it is not clear how to employ the shuffle model in federated learning. Although a few works have investigated basic tasks such as the bit/real summation [10] and histogram [11], the existing protocols may not be viable for the multi-dimensional aggregation FL.

The dimension of the update vector aggravates the error caused by local noises with a dimensional factor [8]. Moreover, since the number of users participating in one iteration is typically a few thousands, the aggregation escalates into a high-dimensional task. We solve the above challenges by making the following contributions:

- For the first time, we propose FLAME, a federated learning framework in the shuffle model such that the users enjoy strong privacy and the analyzer enjoys the accuracy of the model. We first formalize our privacy goal in FLAME by clarifying trust boundary and a fine-grained trust separation (Table I), then we propose SS-Simple protocol by extending a one-dimensional task [10].

  We find that, although the privacy amplification is achievable by SS-Simple, the magnitude of amplification diminishes with the dimension size of the local updates (Corollary 2).

- To alleviate this challenge, we propose SS-Double protocol to enhance the privacy amplification by subsampling. As we notice that the amplification by subsampling may not be composable with shuffling, we propose a

1A larger privacy budget leads to better utility and less privacy.
novel dummy padding method to bridge two kinds of amplification effects with formal proofs (Theorems[3] and [4]). We demonstrate that the protocol SS-Double enjoys dozens times privacy amplification comparing with the protocol SS-Simple (Figure 5).

- A problem of SS-Double protocol is that the random subsampling treats all dimensions equally and thus may discard “important” dimensions. To further boost the utility in a high-dimensional case, we design an advanced protocol called SS-Topk, which is based on the idea of gradient sparsification. A challenge is that the indexes of Top-k elements in the local update vector may reveal sensitive information to the shuffler since the selection is data-dependent. We quantify this privacy threat by formalizing index privacy and design a method to flexibly trade off between index privacy and utility. We note that the index privacy will not impair the privacy guarantee against the analyzer.

- Finally, we conduct experiments on the real-world dataset to validate the effectiveness of the proposed protocols. It turns out that the proposed double amplification effect in SS-Double and private dimension selection in SS-Topk significantly improve the learning accuracy. We observe a 33.94% accuracy improvement of SS-Topk than the curator model based FL without relying on any trusted party. Composed with non-private FL, SS-Topk only lose 1.48% accuracy under (4.696, 10^{-5})-DP.

PRELIMINARIES

We present the necessary background of our work and defer more details about related works to the appendix. We also list major notations used in the paper in the appendix.

The Curator and Local Models

In the curator model, a trusted analyzer collects users’ raw data (e.g., local updates) and executes private mechanism to ensure differentially private outputs. The privacy goal is to achieve indistinguishability for any outputs w.r.t. two neighboring datasets which differ by replacing one user’s data, denoted as $X \approx_{\tau} X'$. We have the following definition:

**Definition 1:** [Differential Privacy (DP)] A mechanism $M : X^n \rightarrow Z$ satisfies $(\epsilon, \delta)$-differentially privacy if for any two neighboring datasets $X \approx_{\tau} X' \in X^n$ and any subsets $S \subseteq Z$, $\Pr[M(X) \in S] \leq e^{\epsilon} \Pr[M(X') \in S] + \delta$.

However, the curator model assumes the availability of a trusted analyzer to collect raw data. Local differential privacy in Definition 2 does not rely on any trusted party because users send randomized data to the server. If $R$ satisfies $(\epsilon, \delta)$-LDP, observing collected results $(y_1, \cdots, y_n)$ or its summation implies $(\epsilon, \delta)$-DP.

**Definition 2:** [Local Differential Privacy (LDP)] A mechanism $R : X \rightarrow Y$ satisfies $(\epsilon, \delta)$-locally differentially privacy if for any two inputs $x, x' \in X$ and any output $y \in Y$, $\Pr[R(x) = y] \leq e^{\epsilon} \Pr[R(x') = y] + \delta$.

The Shuffle Model

The protocol of a shuffle model consists of three components: $P = A \circ S \circ R^n$, as shown in Figure 1(b). Existing works [13], [11], [14], [15], [10] focus on the basic task where each user holds a one-dimensional data $x \in X$. We denote $n$ users’ data as the dataset $X = (x_1, \cdots, x_n) \in X^n$. Each user runs a randomizer $R : X \rightarrow Y^m$ to perturb the local data into $m$ messages that satisfy $\epsilon_l$-LDP. W.l.o.g. we focus on the single-message protocol where $m = 1$. The shuffler executes $S : Y^m \rightarrow Y^*$ with a uniformly random permutation $\pi$ over received messages. The analyzing function $A : Y^* \rightarrow Z$ takes the shuffled messages as input and outputs the analyzing result.

The privacy goal in shuffle model is to ensure $M = S \circ R^n$ satisfies $(\epsilon_e, \delta_e)$-DP, because $A$ is executed by an untrusted analyzer, which is not obliged to protect users’ privacy. By the post-processing property [12], the protocol $P$ achieves the same privacy level as $M$. Hence, we focus on analyzing the indistinguishability for $M(X)$ and $M(X')$. [9] proved that the privacy of $M$ can be “amplified”. In other words, when each
user applies the local privacy budget $\epsilon_i$ in $\mathcal{R}$, $\mathcal{M}$ can achieve a stronger privacy of $(\epsilon_c, \delta_c)$-DP with $\epsilon_c < \epsilon_i$. Compared with the local model, the shuffle model needs less noise to achieve the same privacy level.

Among existing works, the privacy blanket [10] provides an optimal amplification bound for the single-message protocol. The analyzing intuition is to linearly decompose the output distribution into a real data distribution and a uniform random “privacy blanket” distribution. $\gamma$ denotes the probability for $\mathcal{R}$ to output an element from the blanket distribution. With the local randomizer $\mathcal{R}_{\gamma,b}$ in Algorithm 1, where $\gamma = \frac{1}{b}$, the input value $x$ is encoded into a discrete domain $[b]$ and then randomized. After $\mathcal{S}$ runs a permutation, $\mathcal{A}$ aggregates shuffled results $\hat{z} \leftarrow \frac{1}{b} \sum_{i=1}^{n} y_i$ and de-bias with

$$z \leftarrow (\hat{z} - n\gamma/2)/(1 - \gamma).$$

The privacy amplification bound for Algorithm 1 is shown in Lemma 1 and the effect for generic randomizer is distilled in Corollary 1. For a randomizer with Laplace Mechanism on the domain $[0, 1]$, $\gamma = e^{-(c^2)/2}$. A tighter bound (i.e., a greater amplification) can be accessed with numerical evaluations.

Algorithm 1 $\mathcal{R}_{\gamma,b} : [0, 1] \rightarrow [b]$ [10]

Input: input scalar $x \in [0, 1]$
Output: perturbed value $y \in [b]$
1: $x \leftarrow [\lceil b \rceil] + \text{Ber}(x - [\lceil b \rceil])$
2: Sample $r \leftarrow \text{Ber}(\gamma)$
3: $y = \begin{cases} \bar{x} & \text{if } r = 0, \\ \text{Unif}(\{1, \ldots, b\}) & \text{else}. \end{cases}$

Lemma 1: [Privacy Amplification by Subsampling] [10]
For $\sqrt{\frac{14 \log(2/b\delta_c)(b-1)}{n-1}} < \epsilon_c \leq 1$, if $\mathcal{R}_{\gamma,b}$ satisfies $\epsilon_l$-LDP, we have $(\epsilon_c, \delta_c)$ for $\mathcal{S} \circ \mathcal{R}^n$, where $\epsilon_c = \sqrt{\frac{14 \log(2/b\delta_c)(e^{(c^2)/2})}{n-1}}$.

Corollary 1: In the shuffle model, if $\mathcal{R}$ is $\epsilon_l$-LDP, where $\epsilon_l \leq \log(n/\log(1/\delta_c))/2$, $\mathcal{M}$ satisfies $(\epsilon_c, \delta_c)$-DP with:

$$\epsilon_c = O((1 + \epsilon_l) e^{c^2} \sqrt{\log(1/\delta_c)}/n).$$

Composition and Subsampling Properties.
The composition properties [17] are generic for both the curator and the local model of differential privacy.

Lemma 2: $\forall \epsilon, \delta \geq 0, t \in \mathbb{N}$, the family of $\epsilon$-DP mechanism satisfies $(t\epsilon)$-DP under $t$-fold adaptive composition.

Lemma 3: $\forall \epsilon, \delta \geq 0, t \in \mathbb{N}$, the family of $(\epsilon, \delta)$-DP mechanism satisfies $(\sqrt{2t \ln(1/\delta')} \epsilon + t \epsilon(\epsilon^2 - 1))$, $k\delta + \delta'$-fold adaptive composition.

A mechanism $\mathcal{K}$ that randomly subsamples elements without replacement from a database leads to a privacy amplification by subsampling by Lemma 4.

Lemma 4: [Privacy Amplification by Subsampling] [18]
If $\mathcal{M} : \mathbb{X}^n \rightarrow \mathbb{Y}$ satisfies $(\epsilon, \delta)$-DP with respect to the replacement relationship $\preceq_r$ on sets of size $m$, $\mathcal{M}' : \mathbb{X}^n \rightarrow \mathbb{Y}$ satisfies $(\log(1 + (m/n)(e^c - 1)), (m/n)\delta)$-DP.

FLAME FRAMEWORK
In this section, we formalize the framework of Federated Learning in the Shuffle Model, which we call FLAME.

TABLE I: Separation of Trust between Shuffler and Analyzer.

| Parties | Shuffler $\mathcal{S}$ | Analyzer $\mathcal{A}$ | Observer $\mathcal{O}$ |
|---------|------------------------|------------------------|------------------------|
| gradient vector | ID | gradient vector | ID | query model |
| DP-FL | N/A | $\vee$ | $\vee$ | $\vee$ | $(\epsilon_c, \delta_c)$-DP |
| LDP-FL | $\times$ | $\times$ | $\vee$ | $\times$ | $(\epsilon_c, \delta_c)$-DP |
| FLAME | $\times$ | $\times$ | $\vee$ | $\times$ | $(\epsilon_c, \delta_c)$-DP |

Parties: We present the FLAME architecture in Figure 2 with three parties: 1) $n$ users, each of which owns a $d$-dimensional local update vector $x_i$ and runs a local randomizer $\mathcal{R}$ with output $y_i$. 2) The shuffler $\mathcal{S}$, the server who can perfectly shuffle received messages and send them to the analyzer. 3) The analyzer $\mathcal{A}$, the server that estimate the mean of shuffled messages into $z$ and updates the global model at round $t$ by $\theta^t \leftarrow \theta^{t-1} + z$.

Trust Boundaries: We first clarify the trust boundary in FLAME. We denote the observer as $\mathcal{O}$ which could be any curious party who can observe the global model parameters. In the curator model (DP-FL), the trust boundary lies between $\mathcal{A}$ and $\mathcal{O}$. In the local model (LDP-FL), the trust boundary lies between each individual user and the rest parties. By introducing a shuffler $\mathcal{S}$, FLAME avoids placing full trust in any single party as DP-FL and meanwhile be able to achieve better utility than LDP-FL.

Trust Separation: Further, we clarify what are the private information and who can touch them in our framework FLAME. We design a fine-grained scheme of trust separation for FLAME and compare with DP-FL and LDP-FL in Table I. Specifically, we separate the information of each local update into: the indexes, the values of the corresponding indexes and the user identity (i.e., ID in Figure 2). It should be noted that the indexes could be sensitive when the indexes are selected and sent to the shuffler $\mathcal{S}$ in a value-dependent manner. Thus, our privacy goal in FLAME is to make the indexes are selected in an data-oblivious way and the true values of gradients are invisible to $\mathcal{S}$. But $\mathcal{S}$ should know users’ implicit identity in order to distribute local model parameters and receive local messages. The shuffled messages from $\mathcal{S}$ do not reveal the user identity and satisfy $(\epsilon_c, \delta_c)$-DP against $\mathcal{A}$. For $\mathcal{O}$, $(\epsilon_c, \delta_c)$-DP holds by the post-processing property [12]. In LDP-FL, a $(\epsilon_c, \delta_c)$-LDP $\mathcal{R}$ is required by each user for this goal.
**FLAME framework.** We present the our framework in Algorithm 2 with three building processes: encoding $E$, shuffling $S$ and analyzing $A$. In Line 8, $C$ is the threshold for clipping the vector. We denotes the local privacy budget for each local vector by $\epsilon_l$. We use $pk_a$ and $sk_a$ to represent the public and secret key held by the analyzer, respectively. Different protocols below are designed by implement functions Randomize(·) in Line 10 and Shuffle(·) in Line 15 with different strategies. It should be noted that $\mathcal{R}, \mathcal{A}$ in Algorithm 1 can be applied in Randomize(·) as a basic randomizer, which accords to estimating with equation (1) for line (18). Generic randomizers (e.g. Laplace Mechanism) can also be applied, which does not affect our theorems later.

**Security Assumptions.** We assume that the shuffler and the analyzer are not colluded (otherwise, FLAME is reduced to LDP-FL). We also assume that the cryptographic primitives are safe and adversaries have computational difficulty to learn any information from the cipher text.

**Simple Protocol (SS-Simple).** We first propose SS-Simple $\mathcal{P} = \mathcal{A} \circ \mathcal{S} \circ \mathcal{R}^n$ for $d$-dimensional aggregation under the FLAME framework. In a nutshell, we extend the one-dimensional protocol [10] by conducting its randomization and aggregation for each dimension. With the composition property in Lemma 2, $\mathcal{R}$ should satisfy $\epsilon_{id}$-LDP, where $\epsilon_{id} = \epsilon_l/d$. We instantiate Randomizer(·) in Algorithm 2 for SS-Simple with $\{idx, i\} \leftarrow \{(1, \ldots, d) \}$ and $\{y_1, \ldots, y_n\}$ represent the analyzer are not colluded (otherwise, FLAME is reduced to LDP-FL). We assume that the shuffler and the analyzer are not colluded (otherwise, FLAME is reduced to LDP-FL). We also assume that the cryptographic primitives are safe and adversaries have computational difficulty to learn any information from the cipher text.

**Corollary 2.** From the view of privacy, the amplification effect $e_{cd}$ is diminished with a large $d$. We use $\epsilon_{cd} = \epsilon_l/d$ for each dimension to represent the public and secret key held by the analyzer, respectively. Different protocols below are designed by implement functions Randomize(·) in Line 10 and Shuffle(·) in Line 15 with different strategies. It should be noted that $\mathcal{R}, \mathcal{A}$ in Algorithm 1 can be applied in Randomize(·) as a basic randomizer, which accords to estimating with equation (1) for line (18). Generic randomizers (e.g. Laplace Mechanism) can also be applied, which does not affect our theorems later.

**Theorem 1.** For any neighboring datasets $X \succeq_r X'$ which differ in one user’s $d$-dimensional local vector, $\mathcal{M} = \mathcal{S} \circ \mathcal{R}^n$ in SS-Simple satisfies $(\epsilon_c, \delta_c)$-DP, where:

$$
epsilon_c = d\epsilon_{cd} \wedge (2d\log(1/\delta_{cd}) + d\epsilon_{cd}(e^{\epsilon_{cd}} - 1)), \quad \delta_c = \delta_{cd}(d + 1).$$

**Corollary 2.** For SS-Simple, with $\epsilon_l \leq d \cdot \log(n)/\log((d + 1)/\delta_c)/2$, the amplified central privacy is $e_{cd} = \mathcal{O}(1 \wedge e^{\epsilon_l/d} \log(d/\delta_c) \sqrt{d/n}).$

**Limitation of SS-Simple.** Observing the Corollary 2, we find that the central DP level depends on the dimension $d$. Intuitively, from the view of privacy, the amplification effect is diminished with a large $d$. From the view of utility, randomizing the value $y_{i,j}$ for each dimension with a negligible privacy budget $\epsilon_{ik} = \epsilon_l/d$ will inject large noises.

**Double Amplification (SS-Doublle).**

**Intuition.** For strengthening the privacy amplification effect, we propose an improved protocol SS-Doublle. Instead of perturbing every dimension with a small $\epsilon_{id} = \epsilon_l/d$, we only sample and perturb $k$ dimensions $k \ll d$. As a result, each dimension can benefit from a larger privacy budget $\epsilon_{ik} = \epsilon_l/k$. Furthermore, we notice that the privacy amplification can be further magnified by subsampling, which we call the double amplification. Intuitively, if the privacy amplification is strengthened, we could inject fewer noises under the same central privacy level.

**Challenge.** However, the privacy amplification by subsampling may not be composable with shuffling due to multidimensional vectors. We first show how to compose the privacy amplification of shuffling and subsampling in one dimension with $\mathcal{R}_{\gamma,b}$. Suppose $\mathcal{K}_\gamma^n$ samples $n_s$ users from $n$ users with $\beta = n_s/n$. The shuffler only receives encoded message from sampled users. Applying Lemma 1 and Lemma 4 we derive Theorem 2. In addition, we should ensure $\delta_{cd} \leq \beta$ for a positive logarithm, which is reasonable to achieve since $\delta_{cd} \ll \frac{1}{\epsilon_{cd}}$ is negligible by standard.

**Theorem 2.** With $\gamma = \frac{\epsilon_l}{e^{\epsilon_l/d} \log(n)/\log((d + 1)/\delta_c)}$, $\delta_{cd} < \beta$, $\mathcal{M} = \mathcal{S} \circ \mathcal{R}_{\gamma,b} \circ \mathcal{K}_\gamma^n$ satisfies $(\epsilon_{cd}, \delta_{cd})$-DP, where:

$$e_{cd} = \log(1 + \beta (e^{14 \log(\frac{2d\log(n)}{\delta_{cd}}} - 1)$$

However, we cannot derive a similar theorem with $\mathcal{R}_{\gamma,b}$ in a multi-dimensional case. Intuitively, it is because the proof of privacy amplification by shuffling relies on bounded-size neighboring datasets, while subsampling may lead to two neighboring datasets with distinct size. Due to space limitation, we elaborate this with a formal proof in Appendix.

**Dummy Padding.** To solve the composition issue, we propose the method of dummy padding: let the shuffer pad each dimension into the same size of $n_p$. Denote the number of padding values for one dimension as $n_{p} = n_{p} - n_s$, and $n_{n}$ elements of $\mathcal{R}(0)$ are shuffled with all other messages received from users. Thus, the SS-Doublle is $\mathcal{P} = \mathcal{A} \circ \mathcal{S} \circ \mathcal{R}^n_p \circ \mathcal{K}_\gamma^n$, where $\mathcal{S}_p$ consists of padding and shuffling by the shuffler and $\mathcal{K}_\gamma^n$ denotes randomly subsampling by each user. To instantiate

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**Algorithm 2 FLAME: Encoding, Shuffling, Analyzing.**

**Input:** $A, S, E, n, T, \epsilon_l, pk_a, sk_a$  
**Output:** $\theta^2$

1: Analyzer publishes $pk_a$  
2: for each iteration $t = 1, \ldots, T$ do  
3: Analyzer sends $\theta^{-1}$ to Shuffler  
4: Shuffler distributes $\theta^{-1}$ to a batch of $n$ users  
5: for each user $i \in [n]$ do  
6: $x_i \leftarrow$ Randomize($\theta^{-1}$)  
7: $\triangleright$ Encoding $E$ by each user  
8: $z_i \leftarrow$ Clip($x_i, -C, C$)  
9: $z_i \leftarrow$ $(\hat{x}_i + C) / (2C)$  
10: $\langle idx_i, y_i \rangle \leftarrow$ Randomize($z_i, \epsilon_l$)  
11: $c_i \leftarrow$ Enc$_{pk_a}(y_i)$  
12: user $i$ sends $m_i = \langle idx_i, c_i \rangle$ to Shuffler  
13: $\triangleright$ end for

14: $\triangleright$ Shuffling $S$ by the Shuffler

15: Shuffler sends $\langle m_i[e_i[n]] \rangle$ to Analyzer

16: $\triangleright$ Analyzing $A$ by the Analyzer

17: decrypts values $y_{i(e_i)}[e_i[n]] \leftarrow$ Dec$_{pk_a}(c_{i(e_i)}[e_i[n]])$  

18: estimates mean $z \leftarrow \frac{1}{n} \sum_i \langle idx_i, y_i \rangle$  

19: normalizes $z \leftarrow C \cdot (2z - 1)$  
20: updates model $\theta^2 \leftarrow \theta^2 - 1 + z$  
21: $\triangleright$ end for
SS-Double in our framework, the Randomize(·) works as follows. First, it randomly samples \( k \) indexes \( \{idx_i\} \) for each index \( j \in [d] \). For each index \( j \in [d] \), the perturbed value in \( y_i \) is \( y_{i,j} \leftarrow \mathcal{R}(|x_{i,j}|) \). The procedures of Shuffle(·) in FLAME are listed in Algorithm 3.

**Algorithm 3: Shuffle(·)=\( S_p \) for SS-Double**

1. for \( j \in [d] \) do
2. \( n_{s,j} \leftarrow \sum_s^n \{r|r \in idx_i, r = j\} \)
3. \( n_{a,j} \leftarrow n_p - n_{s,j} \)
4. end for
5. the number of dummy vectors \( v \leftarrow \sum_d n_{s,j} / k \)
6. for \( u \in [v] \) do
7. \( S_{dummy} \leftarrow \{j|n_{a,j} \neq 0\} \)
8. \( idx_{n+u} \leftarrow \{j|j \in S_{dummy}\} \)
9. \( y_{n+u} \leftarrow \{(0)\}^{k} \)
10. \( m_{n+u} \leftarrow (idx_{n+u}, Enc_{p_u}(y_{n+u})) \)
11. end for
12. generates a permutation \( \pi \) over \([ml + v]\)
13. shuffles and sends \( \{m_{n(1)}, \ldots, m_{n(ml+v)}\} \) to analyzer

Privacy and Utility Analysis.: We show the overall privacy amplification bound for \( \mathcal{R}_{\gamma,b} \) in Theorem 3. We notice that a larger \( n_p \) leads to a smaller \( \epsilon_l \), which implies a greater privacy amplification. Correspondingly, a larger \( n_p \) implies more noises injected as shown in Proposition 1.

**Theorem 3:** With \( \gamma = \frac{b_2 \beta}{\epsilon l + b_1 - 1} \) and \( \sigma_{cd} < 2\beta \), \( \mathcal{M} = S_p \circ \mathcal{R}^{n_p} \circ \mathcal{K}_{\beta}^p \) satisfies \((\epsilon_{cd}, \delta_{cd})\)-DP, where:

\[
\epsilon_{ck} = \frac{14 \log (\frac{2d}{\delta_{cd}})(e^{\epsilon_{lk}} + b - 1)}{n_p - 1},
\]

\[
\epsilon_{cd} = \log (1 + \beta (e^{\epsilon_{lk}} - 1)).
\]

**Proposition 1:** The standard deviation of the estimated mean in each dimension from \( P = A \circ S_p \circ \mathcal{R}^{n_p} \circ \mathcal{K}_{\beta}^p \) is \( O((\frac{n_p}{\delta c})^{1/2}(1/\delta_{cd})) \).

Since all dimension-level datasets are padded into the same size and \((\epsilon_{ck}, \delta_{ck})\)-DP holds from the amplification in Theorem 3 we show the vector-level DP composition in Theorem 4. The sample rate is denoted as \( \beta = k/d \). We distill the amplification effect from \( \epsilon_l \) to \( \epsilon_c \) in Corollary 3.

**Theorem 4:** For any neighboring datasets \( X \approx_r X' \) which differ in one user’s local vector, the \( d \)-dimensional vector aggregation protocol \( \mathcal{M} = S_p \circ \mathcal{R} \circ \mathcal{K}_{\beta}^p \) in SS-Double satisfies \((\epsilon_c, \delta_c)\)-DP, where:

\[
\epsilon_c = 2\beta d \delta_{cd} \wedge (\epsilon_{cd} \sqrt{4\beta d \log (1/\delta_{cd})} + 2\beta d \delta_{cd}(e^{\epsilon_{cd}} - 1)),
\]

\[
\delta_c = \delta_{cd}(2\beta + 1).
\]

**Corollary 3:** For SS-Double, with \( \epsilon l \leq \beta d \log (n_p / ((2\beta d + \beta) / \delta_{cd})) / 2 \), the amplified central privacy is:

\[
\epsilon_c = O((1 + e^{\epsilon_{lk}}) e^{\epsilon_{lk}} \beta^{1.5} \sqrt{\frac{d}{n_p}} \log (\frac{\beta d}{\delta_c}) (\log (\frac{\beta d}{\delta_c}))).
\]

**Simulation of Privacy Amplification.:** To compare the privacy amplification effect of SS-Simple and SS-Double, we visualize the magnified privacy w.r.t. \( \mathcal{R}_{\gamma,b} \) in Figure 3.

**Utility Boosting with Top-k (SS-Topk)**

**Intuition.:** A problem of SS-Double protocol is that the random subsampling treats all dimensions equally and thus may discard “important” dimensions. For the high-dimensional case \((d > n)\), random sampling a small fraction \( \beta \) of values from a vector slows down the convergence rate of training. In light of the efficient gradient sparsification technique, we are motivated to adopt the magnitude-based selection \([\mathcal{B}]\) for boosting the convergence rate. However, selecting Top-\( k \) indexes with greatest absolute magnitudes over the vector is data-dependent and thus compromise user privacy. The challenge is how to preserve and qualify the index privacy while maintaining the utility as far as possible.

**Index Privacy.:** According to our trust setting in Table 1 the prime adversary that threats index privacy is the shuffler \( S \) because only \( S \) knows which user sends which indexes (note that the perturbed value is encrypted). Our goal is to bound the shuffler’s success prediction about whether or not the index uploaded by a user ranks Top-\( k \) elements of the local vector. By random guessing, the adversary’s success rate for predicting a dimension as Top-\( k \) is \( k/d \). After observing the privatized selected indexes, the success rate cannot be enlarged by more than \( \epsilon \) times.

Thus, we are motivated to qualify and preserve the index privacy with an anonymity-based metric in Definition 3. We denote whether the magnitude of dimension \( j \in [d] \) ranks in
Top-k ($\beta = k/d$) by $I_1 \in \{0, 1\}$, and whether the index $j$ is selected by $K_\nu^b(j)$. The first inequality bounds the adversary’s success rate with $\nu$ while the second inequality ensures the probability is no greater than 1. Intuitively, the strongest index privacy stands when $\nu = 1$, because observing $K_\nu^b(j)$ does not increase the adversarial success rate.

**Definition 3**: A mechanism $K_\nu^b$ provides $\nu$-index privacy for a $d$-dimensional vector, if and only if for any $j \in [d]$, $\nu \geq 1$, we have: $\Pr[I_j = 1|K_\nu^b(j)] \leq \nu \cdot \Pr[I_j = 1]$ and $\Pr[I_j = 0|K_\nu^b(j)] \geq \Pr[I_j = 0].$

To achieve $\nu$-index privacy, we need to control the probability of $\Pr[I_j = 1] = \beta$. Given the prior knowledge $\Pr[I_j = 1] = \beta$, $\Pr[I_j = 0] = 1 - \beta$, if each user reports $lk$ dimensions to the shuffler of which only $k$ indexes are real Top-k, we have $\Pr[I_j = 1|K_\nu^b(j)] = \frac{1}{\nu}$. Put into the definition, we have: $\Pr[I_j = 1|K_\nu^b(j)] \leq \min(\nu \cdot \beta, \frac{\nu}{\nu+1})$. Thus, $K_\nu^b$ satisfies $\nu$-top index privacy when we set $l \geq \max\{\frac{1}{\nu - 1}, \frac{\nu}{\nu+1}\}$.

**SS-Topk Protocol**: Denote the SS-Topk protocol as $P = A \circ S_p \circ R_\nu^b \circ K_\nu^b$. Each user locally runs $R \circ K_\nu^b$. The shuffler executes $S_p$. The analyzer runs the aggregation $A$. Compared with SS-Double, SS-Topk has the same shuffling and analyzing procedure but differs in the dimension selection $K_\nu^b$ in Randomize() which is shown in Algorithm 4.

By applying $K_\nu^b$ on the processed vector $\tilde{x}_i$, $k$ Top-k indexes are sampled as the set $S_{top}$ while $k(l-1)$ non-top dimensions are randomly sampled from the rest as $S_{non}$. As $k$ true value are perturbed, the privacy budget split for each dimension is $\epsilon_{lk} = \epsilon_l/k$. Then each value of real Top-k dimensions is perturbed as $y_{ij} \leftarrow \mathcal{R}_{\epsilon_{lk}}(\tilde{x}_{i,j})$. Each non-top dimension is padded with value $\mathcal{R}_{\epsilon_{lk}}(0)$. Then $kl$ dimensions are permuted into the list of $idx_i$ and $y_i$, which will be encrypted as $l$ messages by Line 2 in Algorithm 2.

**Privacy Analysis**: First, we show the relationship of $\nu$ and $l$ in Proposition 2 and Figure 4. The strongest index privacy $\nu = 1$ is achieved when $l = \lfloor \frac{1}{\beta} \rfloor$. For SS-Double, $\nu$ naturally holds with the random sampling mechanism $K_\nu^b$. To the other extreme case of no index privacy, SS-Topk still provides a strong privacy guarantee since the shuffler knows nothing except for Top-k indexes. An state-of-the-art work has shown the privacy attack’s availability is significantly impaired even knowing both Top-k indexes as well as their values.

**Proposition 2**: The range of $\nu$-top index privacy is $1 \leq \nu \leq \frac{1}{\beta}$, where the strongest index privacy $\nu = 1$ is achieved when $l = \lfloor \frac{1}{\beta} \rfloor$ and no index privacy is achieved when $l = 1$.

**Proposition 3**: Then, we clarify the compatibility of our index privacy against the shuffler and $(\epsilon, \delta_\epsilon)$-DP against the analyzer. With $S_p$ in Algorithm 3, the analyzer only gets padded results for each dimension with the same size $n_p$. Thus, the $\nu$-index privacy against the shuffler does not affect the amplified privacy $(\epsilon, \delta_\epsilon)$-DP against the analyzer. SS-Topk shares the double amplification effect in SS-Double with the same $n_p$.

Lastly, we discuss the trade-off between the index privacy and the communication costs as well as the utility. Given a desired $\nu$-top index privacy, each user can choose a valid $l$ in Figure 4 for achieving it. The bandwidth of each user depends on $O(|lk|)$. As Proposition 1 implies, the estimation utility depends on the dummy padding size $n_p$. Thus, given $n_p$, $\nu$ and $\beta$, $\nu$ does not affect the accuracy of the mean estimation because the number of dummy values is fixed. We show the strongest index privacy under given parameters in Theorem 5.

**Theorem 5**: Given a protocol with $K_\nu^b$, $n_p$, the strongest index privacy it allows for each user is $\nu = \max\{1, \frac{1}{n_p^{\beta}}\}$.

**Evaluations**

We evaluate the learning performance on MNIST dataset and logistic regression model with $d = 7850$, $n = 1000$. Baselines include non-private Federated Averaging (NP-FL) [11], DP-FL [20] with Gaussian Mechanism in which we double the sensitivity for the comparison under bounded DP definition, LDP-FL with Gaussian Mechanism [21] for $(\epsilon, \delta)$-LDP and Laplace Mechanism for $\epsilon$-LDP and our three protocols of SS-Simple, SS-Double, SS-Topk. Implementation details are presented in our appendix.

**Comparison of Our Protocols:** For SS-Simple, SS-Double, SS-Topk, we apply the Laplace Mechanism as the basic randomizer $R$ for each dimension. Given $\epsilon_l = 78.5$, the split privacy budget of each dimension is $\epsilon_{ld} = 0.01$ for
Fig. 7: Impact of $n/n_p$.  

Fig. 8: Impact of $\beta = k/d$.  

Fig. 9: Under various $\epsilon_l$.  

Fig. 10: Under various $d$.

SS-Simple and $\epsilon_{lk} = 0.5$ for SS-Double and SS-Topk. The double amplification effect boosts the amplified privacy against $A$ for one epoch from $(0.91, 5 \times 10^{-6})$-DP (SS-Simple) to $(0.24, 5 \times 10^{-6})$-DP (SS-Double/Topk). Compared with SS-Simple, SS-Double improves the testing accuracy by 4.07% under a stronger central privacy. Compared with SS-Double, SS-Topk significantly boosts the utility by 55.5% under the same central privacy. With Theorem $5$ and $n/n_p = 3$ in Figure $5$ the maximum index privacy that the protocol allows for each user is $\nu = 3.125$ with $l = 16$.

Comparison with DP-FL/LDP:. It is obvious that even the baseline SS-Simple performs better than LDP-FL. Then we observe that under the same central privacy $(0.24, 5 \times 10^{-6})$-DP, SS-Topk even achieves a dramatic higher accuracy than DP-FL by 33.94%. This is a key observation of our work, because traditional works for the one-dimensional task claim that the shuffle model only stands in the middle-ground between LDP and DP.

We analyze the reasons as follows: 1) The effect of Top-$k$: as SS-Double cannot exceed the performance of DP-FL $(0.24, 5 \times 10^{-6})$-DP but SS-Topk can, it is obvious that Top-$k$ boosts the utility. 2) The effect of the double amplification: If we only counts the amplification from shuffling for SS-Topk without the amplification from subsampling, we have $\epsilon_\nu = 20.53, \delta_\nu = 5 \times 10^{-6}$. Hence, we introduce another baseline of DP-FL $(20.53, 5 \times 10^{-6})$-DP. We observe that this line approaches the non-private version and has higher testing accuracy than SS-Topk. In other words, SS-Topk cannot perform better than DP-FL if only amplification of shuffling is counted. Thus, we validate the effect of proposed double sampling for such nontrivial utility boosting. Thus, we conclude that both double amplification and Top-$k$ boosting are necessary to performs better than DP-FL.

Comparison under variant Parameters:. We then evaluate the impacts of other hyper-parameters of $n_p, \beta$ and privacy budget $\epsilon_l$ in Figures $6$ and $8$. 1) It is obvious in Figure $6$ that a larger local privacy budget for each dimension leads to higher testing accuracy. 2) In Figure $7$, the higher ratio of $n/n_p$ implies less additional noises injected by dummy padding. This validates our utility analysis in Proposition $4$. Thus, we can conclude that $n_p$ is the key knob to tune the privacy and utility trade-off. 3) In Figure $8$ we can observe that larger $\beta$ implies better utility. With $\beta = 1/10$ we can provide $(4.696, 10^{-5})$-DP within 2 epochs and only 1.48% loss compared with NP-FL in Figure $5$.

Privacy Amplification:. We illustrate the overall amplification in previous evaluations with the Bennett inequality for the Laplace Mechanism in Figure $9$ and Figure $10$. This validates our Theorems $\text{[1] [4]}$ and Corollaries $\text{[2] [3]}$ are generic for any local randomizer, as long as the amplification bound can be derived by a close-form solution or numerical evaluations $\text{[10]}$.

CONCLUSION

To conclude, we propose the first differentially private federated learning framework FLAME in the shuffle model for better utility without relying on any trusted server. Our privacy amplification effect and private Top-$k$ selection mechanism significantly boosts the testing accuracy under the high-dimensional setting.

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