Quantum criticality of topological phase transitions in three-dimensional interacting electronic systems

Bohm-Jung Yang\textsuperscript{1,*}, Eun-Gook Moon\textsuperscript{2}, Hiroki Isebe\textsuperscript{3} and Naoto Nagaosa\textsuperscript{1,3,*}

Topological phase transitions in condensed matter systems accompany emerging singularities of the electronic wavefunction, often manifested by gap-closing points in momentum space. In conventional topological insulators in three dimensions, the low-energy theory near the gap-closing point can be described by relativistic Dirac fermions coupled to the long-range Coulomb interaction; hence, the quantum critical point of topological phase transitions provides a promising platform to test the intriguing predictions of quantum electrodynamics. Here we discover a class of quantum critical phenomena in topological materials for which either the inversion symmetry or time-reversal symmetry can be broken. At the quantum critical point, the emerging low-energy fermions, dubbed the anisotropic Weyl fermions, show both relativistic and Newtonian dynamics simultaneously. The interplay between the anisotropic dispersion and the Coulomb interaction brings about a screening phenomenon distinct from the conventional Thomas-Fermi screening in metals and logarithmic screening in Dirac fermions.

The main finding of this work is as follows. Most importantly, we find that the anisotropic dispersion of the AWF induces exotic screening effects. Namely, significant screening appears along the direction with linear dispersion of electrons in the sense that the momentum dependence of the Coulomb potential is strongly modified from the bare value along that direction. On the other hand, along the direction with quadratic dispersion of electrons, the Coulomb potential is just weakly modified. We call this the anisotropic partial screening phenomenon. Strikingly, such a non-trivial screening effect makes the Coulomb interaction irrelevant in the low-energy limit. Thus, the critical theory becomes

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\textsuperscript{1}RIKEN Center for Emergent Matter Science (CEMS), Wako, Saitama 351-0198, Japan, \textsuperscript{2}Department of Physics, University of California, Santa Barbara, California 93106, USA, \textsuperscript{3}Department of Applied Physics, University of Tokyo, Tokyo 113-8656, Japan. \textsuperscript{*}e-mail: biyang@riken.jp; nagaosa@riken.jp
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To describe the dynamics of the AWF coupled to the long-range Coulomb interaction, we use the Euclidean path integral formalism and write the effective action as follows:

\[ S = \int d^4x \left[ \bar{\psi} \left( i \gamma \cdot \partial + ig \phi \right) \psi + \nu (\gamma_\tau \partial_\tau + \gamma_2 \partial_\delta) - A \delta^2 \right] \bar{\psi} + \frac{1}{2 \sqrt{\eta}} \left\{ (\partial \phi)^2 + (\partial \phi)^2 \right\} + \frac{\sqrt{\eta}}{2} (\partial \phi)^2 \]

As usual, the momentum cutoff (A) is implicitly assumed. The matrices \( \gamma_{1,2,3} \), satisfying \( \gamma_3 \gamma_i = \delta_i^3 \), are defined as \( \gamma_0 = \gamma_1, \gamma_2 = \gamma_3, \gamma_3 = -1 \). \( \psi(\phi) \) represents an electron (boson) field operator and \( \bar{\psi} = \psi^\dagger \gamma_0 \). The Hubbard-Stratonovich transformation introduces the bosonic field \( \phi \) for the instantaneous Coulomb interaction. The dimensionless parameter \( \eta \) is used to capture the possible anisotropy of the Coulomb potential induced by the anisotropic fermion dispersion. The electrons are coupled to the boson through the coupling constant \( g^2 = \epsilon^2 / \varepsilon \), where \( \epsilon \) is the electric charge and \( \varepsilon \) is the dielectric constant.

Considering the anisotropic dispersion of electrons, the coupling constants in the action naturally lead to three characteristic energy scales, \( E_{kin} = A - v^2 \), \( E_0 = A - v^2 g^2 \) and \( E_\phi = \Lambda / \varepsilon \), which describe the electron kinetic energy (\( E_{kin} \)) and Coulomb potential (\( E_\phi \)) and the cutoff energy (\( E_0 \)) related to the linear dispersion of electrons, respectively. The associated dimensionless coupling constants are

\[ \alpha = E_0 / E_{kin}, \quad \beta = \sqrt{\eta} E_0 / E_\phi, \quad \gamma = \alpha / \beta \]

Although only two of them are independent (\( \gamma \sim \alpha^2 / \beta \)), for notational convenience, we use the three coupling constants to perform the renormalization group analysis.

The anisotropic fermion dispersion gives rise to unusual scaling properties of the system. Let us first consider the non-interacting fermions with the following scale transformation

\[ \bar{t} = b^{-1} t, \quad \bar{x}_{i} = b^{-1} x_{i}, \quad \bar{x}_{i} = b^{-1} x_{i} \]

where the rescaled variables are indicated by tildes. Namely, the scaling dimensions of \( t, x_{i} \) and \( x_{i} \) are given by \( [t] = -1, [x_{i}] = -1, [x_{i}] = -1 \). It is worth noting that because the dispersion of the fermions is anisotropic whereas that of the boson is isotropic, the scale invariance of the whole system requires the introduction of a general scaling dimension \( z_{\phi} \). However, the fundamental nature of the quantum critical point is independent of \( z_{\phi} \), as shown below.

To reveal the intriguing nature of quantum fluctuations, let us first compute the screened Coulomb interaction using random phase approximation. The polarization function due to the AWF is given by \( \Pi(q) = -B_{c} q_{i} q_{i} / 2 - B_{c} q_{i} q_{i} \) in the leading order with the positive constants \( B_{c} \) and \( B_{c} \), which leads to the screened Coulomb potential

\[ V_{c}(q) \sim \frac{1}{q_{i}^{2n} + q \eta} \]

(See Supplementary Note 2 for details.) This shows that the Coulomb potential receives anisotropic partial screening through the interaction with electrons. Namely, a non-trivial correction of \( V_{c}(q) \) occurs in the \( q_{i} \) direction whereas \( V_{c}(q) \) maintains the same momentum dependence in the \( q_{i} \) direction as the bare Coulomb potential. In real space, \( V_{c} \) depends on the spatial coordinate \( \mathbf{r} = (x, z) \) anisotropically as \( V_{c}(r_{x}, z = 0) \propto r_{x}^{-n} \) and \( V_{c}(z = 0, z) \propto |z|^{-5/3} \). As the screened Coulomb potential is slightly short-ranged as compared with the bare Coulomb potential \( V(r) \sim 1/r \), the effective Coulomb interaction between electrons becomes weaker than the bare interaction.
Renormalization group analysis. To understand the interplay between the quantum fluctuation of fermions and that of bosons more systematically, we perform a momentum-shell renormalization group analysis of \( S \) at the one-loop level. The renormalization group flows of the three coupling constants \( \alpha, \beta, \gamma \) describe the quantum corrections from the interaction as represented by the two Feynman diagrams in Fig. 2a. Instead of showing the details, we present only the beta functions, leaving technical calculations to the Methods and Supplementary Note 3.

\[
\frac{d\alpha}{d\ell} = \alpha \left( -\frac{1}{16\pi} \alpha - \frac{1}{2} \beta - \frac{1}{2} \gamma \right), \quad \frac{d\beta}{d\ell} = \beta \left( 1 - \frac{1}{8\pi} \alpha - \beta \right),
\]

\[
\frac{d\gamma}{d\ell} = \gamma \left( -1 - \gamma \right)
\]

where \( \ell = \ln(\Lambda/\mu) \) is the usual renormalization group parameter and \( \Lambda (\mu) \) denotes the ultraviolet (infrared) momentum cutoff. The renormalization group flow is illustrated in Fig. 2b, which shows two fixed points at \((\alpha, \beta) = (0, 0)\) and \((\alpha, \beta) = (0, 1)\), respectively. At both fixed points, \( \gamma = 0 \). The flow emerges from the former (unstable) fixed point and terminates at the latter (stable) one. Interestingly, the fine-structure constant \( \alpha \) is zero at both fixed points. The linear \( \alpha \) dependence of the electron self-energy,

\[
\Sigma_i \sim \left[ \nu(k_i\tau_1 + k_i\tau_2) + 2Ak_i^2 \tau_1 \right] \left( \frac{\alpha}{16\pi} \ell \right) + O(\ell^2)
\]

manifestly shows that the electrons become free from interaction in the low-energy limit.

The distinct natures of the two fixed points are clearly revealed in the screening effect of the Coulomb potential. The inverse of the boson propagator including the self-energy correction \((\Pi(q))\) becomes

\[
q_i^2 + q_i^2 - \Pi(q) = q_i^2(1 + \beta \ell) + q_i^2(1 + \gamma \ell) + O(\ell^2)
\]

Here the unimportant \( \eta \) dependence is neglected. At the unstable fixed point, \((\alpha, \beta, \gamma) = (0, 0, 0)\), the Coulomb potential decouples from electrons. On the other hand, at the stable fixed point, \((\alpha, \beta, \gamma) = (0, 1, 0)\), the Coulomb potential receives a non-trivial correction in the \( q_3 \) direction whereas there is no screening in the \( q_1 \) direction. This can be understood as the strong screening effect with the anomalous dimension. Considering the different scaling between the \( q_1 \) and \( q_3 \) directions, we obtain

\[
q_i^2 + q_i^2 - \Pi(q) \sim q_1^{2-1/\eta} + q_3^2
\]

at the stable fixed point. Here \( z_1 \) can be fixed from the condition that \( \nu \) and \( A \) do not run at the stable fixed point, which leads to \( z = z_1 = 2 \). Therefore, the renormalized Coulomb interaction becomes \( V_c(q) \sim 1/(q_1^{1/2} + q_3^2) \) at the fixed point, confirming the anisotropic partial screening phenomenon.\(^2\)

Note that the random phase approximation and the one-loop renormalization group calculations match perfectly; thus, we believe that the results of the renormalization group analysis are robust. Besides the fact that two different methods give the same momentum dependence of \( \Pi(q) \), we further consider possible corrections by performing the simplified large \( N_f \) expansion by introducing \( N_f \) different copies of fermions. As shown in Supplementary Note 4, it is explicitly proved that the quantum fluctuations near the stable fixed point do not cause any divergent corrections, demonstrating the robustness of the conclusions from the renormalization group analysis.

**Figure 2** | Feynman diagram for quantum corrections and the resulting renormalization group flow. **a**, Feynman diagrams for electron self-energy (top) and boson self-energy (bottom). The plain (wavy) line is for electrons (bosons). **b**, The renormalization group flow of the coupling constants \((\alpha, \beta)\). The renormalization group flow is obtained by solving the coupled renormalization group equations shown in Supplementary Note 3. The stable fixed point, marked by a black filled dot, is located at \((\alpha, \beta) = (0, 1)\), and the unstable point (grey dot) is located at \((0, 0)\).

**Screening of a charged impurity.** The intriguing quantum effect in the interacting AWF can induce unusual dielectric responses of the system. Here we consider the screening problem of a Coulomb impurity with the electric charge \( Ze (e > 0) \) by the AWF. The Hamiltonian for the non-interacting AWF coupled to a single charged impurity is given by

\[
H = -\imath v_0 \rho_i \sigma_x - \imath \nu \rho_j \sigma_y - \Lambda \delta^2 \sigma_z - \frac{Zg_0^2}{4\pi r}
\]

where \( g_0^2 \) is the coupling constant of the non-interacting system. In fact, because of the anisotropic momentum dependence of the polarization function that can be summarized as \( \Pi(q) = -B_1 q_1^{1/2} - B_2 q_3^2 \), the screening charges distribute in a highly unusual way even in the non-interacting system. According to the linear response theory, the induced charge density is given by \( \rho_{\text{ind}}(q) = ZzV(q)\Pi(q) \), where \( V(q) = g_0^2/q^2 \) (ref. 23). As shown in Fig. 3, the peculiar structure of \( \Pi(q) \) gives rise to strong spatial anisotropy in the distribution of the induced charges in real space, \( \rho_{\text{ind}}(r) \sim \int(\text{d}^2q)/(2\pi)^2 e^{\text{i}qr}\rho_{\text{ind}}(q) \). In particular, the partially integrated charge densities \( Q_\perp(z) = \int d^2r. \rho_{\text{ind}}(r) \) and \( Q_{\parallel}(z) = \int dz. \rho_{\text{ind}}(z) \) show characteristic coordinate dependence such as \( Q_{\perp}(z) \sim 1/r_1^2 \), \( Q_{\parallel}(z) \sim \delta(z) \).

Note that the partially integrated charge densities distribute in a highly unusual way, that is, \( Q_{\parallel}(z) \) is strongly localized near the impurity whereas \( Q_{\perp}(z) \) shows a slow decay with a power-law tail. The detailed behaviour of \( \rho_{\text{ind}}(r) \) is shown in Supplementary Note 5.

Surprisingly, the strong Coulomb interaction between AWFs completely modifies the screening charge distribution. As all coupling constants are irrelevant at the stable fixed point, the induced charge can be computed to the leading order by using \( V_{\text{screen}}(q) = g^2/(q_1^{1/2} + q_3^2) \) and \( \Pi(q) = -B_1 q_1^{1/2} - B_2 q_3^2 \). As the amount of the total induced charge is fixed, that is, \( \int d^2r. Q_\perp(r) = \int dz. Q_{\parallel}(z) \), \( B_1/B_2 \) is fixed to be \( \eta_1 \), which gives rise to \( \rho_{\text{ind}}(r) = -Zg_0^2B_2^2 \delta(r) \). In contrast to the case of the non-interacting system, the screening charge is strictly localized near the impurity site. A similar localized distribution of the induced charge is predicted in graphene.\(^24\)

In graphene, it is shown that the Coulomb interaction can induce small spreading of the screening charge in subleading order. However, in our case such an additional spreading of the screening charge can play only a minor role because the Coulomb interaction is irrelevant at the fixed point whereas it is marginal in graphene.\(^22,25\).
The distribution of the screening charge induced by a charged impurity in the non-interacting anisotropic Weyl fermion. The distribution of the induced charge $\rho_{\text{ind}}(\mathbf{r}) = \rho_{\text{ind}}(r_\perp, z)$ in real space when the charged impurity is at the origin. Here $r_\perp$ and $z$ are measured in the unit of $(B^2_0/B^2_\perp)$ and it is assumed that $g^2 B^2_0/B^2_\perp = 1$. \textbf{a}, The intensity plot of $\rho_{\text{ind}}$ in the $(r_\perp, z)$ plane. Here the white dashed line indicates the boundary on which $\rho_{\text{ind}} = 0$, $\rho_{\text{ind}} > 0$ $(\rho_{\text{ind}} < 0)$ in the red (blue) region on the left-hand (right-hand) side of the dashed line. The colour is darker when the magnitude of the charge density is higher. \textbf{b}, $r_\perp$ dependence of $\rho_{\text{ind}}$ when $z$ is fixed to 0, 0.1, 0.2 and 0.3, respectively. This behaviour will be totally changed when the Coulomb interaction between electrons is considered, where the induced charge density is strictly localized near the impurity site.

Discussion

Up to now we have considered the system having a single AWF. However, in real materials, several AWFs can simultaneously appear at the Fermi level owing to the time-reversal or crystalline symmetries. In this case, to describe the renormalized Coulomb interaction, it is necessary to add the contribution from all AWFs to the polarization. In other words, the polarization function should be given by $\Pi_{\text{real}} = \sum_i \Pi_i$, where $\Pi_i$ indicates the polarization due to the $i$th AWF. Here it is assumed that the polarization induced by the coupling between neighbouring AWFs can be neglected, which is valid if the distance between AWFs is far enough. The important point is that the direction along which the AWF shows the quadratic dispersion can vary for different AWFs in general. Therefore, in the system with multiple AWFs, the total polarization should behave as $\sum_i \Pi_i(q) \propto q^{1/2}$ on average with $q = |\mathbf{q}|$. In this case, the renormalized Coulomb interaction should have the following form of $V_C(q) \propto 1/q^{1/2}$. Equivalently, it means that the long-range Coulomb interaction shows the unusual power law given by $V_C(r) \propto 1/r^{1/2}$ in real space owing to the screening effect. If we introduce a charged impurity to this system, the induced charge spreads in according to $\rho(r) \propto 1/r^{1/2}$ when the electron–electron interaction is neglected. On the other hand, once the electron–electron interaction is considered, the induced charge is again strictly localized near the impurity site, that is, $\rho(r) = Z' e\delta(r)$ with a constant $Z'$. As the low-energy excitation of the interacting system is described by free fermions, measurable physical quantities can be computed precisely. Owing to the anisotropic dispersion, the density of states of the system becomes $D(\epsilon) \propto \epsilon^{3/2}$, which is clearly distinct from that of the 3D Weyl semimetal with $D(\epsilon) \propto \epsilon^{2}$. The unusual power law of $D(\epsilon)$ gives rise to an unconventional temperature dependence of various physical quantities such as the specific heat ($C_v \sim T^{3/2}$), the compressibility ($\kappa \sim T^{3/2}$) and the diamagnetic susceptibility ($\chi_D \sim T^{-3/2}$); ref. 16). Therefore, such an unusual temperature dependence of various physical quantities can provide strong experimental evidence for the intriguing quantum criticality. In this respect, it is worth noting the recent experiment on the pyrochlore iridate Nd$_3$(Ir$_{1-x}$Rh$_x$)$_2$O$_7$ where an insulator–metal transition has been achieved by replacing Ir by Rh (ref. 6). In particular, the observed linear dependence of the optical conductivity near the insulator–metal transition boundary, which is compatible with emerging Weyl semimetal states, suggests that Nd$_3$(Ir$_{1-x}$Rh$_x$)$_2$O$_7$ is a promising candidate to probe the new quantum criticality.

To conclude, we have rigorously demonstrated the new quantum critical phenomena of the AWF. We emphasize that the anisotropic partial screening is clearly distinct from the conventional screening phenomena in usual metallic or semimetallic systems, which is concisely summarized in Table 1. For example, the Thomas–Fermi screening in conventional metals forces the renormalized Coulomb interaction to be short-ranged and the fermionic excitations coupled by the short-range repulsive interaction gives rise to the Fermi liquid state. On the other hand, in the AWF, the interplay between the Coulomb interaction and fermions induces the anisotropic partial screening, which allows the screened Coulomb interaction to maintain the long-ranged nature and the fermions to be free from the Coulomb interaction in the low-energy limit. Therefore, the anisotropic partial screening can be considered as the intermediate screening phenomena distinguished from both the Thomas–Fermi screening in metals and the logarithmic screening in Dirac fermions.


Table 1 | Physical properties of interacting metallic or semimetallic systems in three dimensions coupled with the instantaneous Coulomb interaction.

| Metal   | $\frac{1}{q^2 + q^2_{\text{F}}}$ | Marginal | Fermi liquid |
|---------|---------------------------------|----------|--------------|
| AWF     | $\frac{1}{q^2 + q^2_{\text{C}}}$ | Irrelevant | Fermi liquid |
| Weyl semimetal | $\frac{1}{q^2}$ | Marginally irrelevant | (Marginal) | Fermi liquid |

Here, ‘Metal’ indicates the conventional metal with a Fermi surface and ‘Weyl semimetal’ indicates the 3D semimetal with linear dispersion in all spatial directions. $V_0(q)$ (eq. 1) indicates the screened Coulomb potential (renormalized coupling constant) at the stable fixed point. As for the $q$ dependence of $V_0(q), q^2 + q^2_{\text{F}} = q^2_{\text{C}} + q^2_{\text{F}}$ and $1/g_{\text{TF}}$ indicates the Thomas-Fermi screening length. In contrast to the three states in the table, when the electron dispersion is quadratic in all directions, it is shown that the screened Coulomb potential becomes $V(q) = 1/q^2$ and a non-Fermi liquid ground state can be realized.25

Note added in proof: We have noticed an earlier paper23 that analyzed the same Hamiltonian by random phase approximation. As discussed in the present paper, their conclusion is consistent with our analysis.

Methods

Scaling and renormalization group analysis. Here we briefly explain the procedure for the scaling and renormalization group analysis. The invariance of the fermionic part of the action under the scale transformation in equation (2) leads to the following scaling relations at the tree level, $[\phi] = \nu = 1 + 3/2, [\nu] = z = 2, [A] = -2, [g_{\text{TF}}] = z_{\nu} - 2, [\xi] = z_{\nu} = 2, [\mu_{\text{TF}}] = z_{\mu} - 2$. If we assume that $\nu$ is marginal whereas $A$ is either marginal or irrelevant, we obtain $z = z_{\nu} = 2$ and $[g_{\text{TF}}^2] = 0$. On the other hand, if $A$ is marginal whereas $\nu$ is either marginal or irrelevant, $z = z_{\nu} = 2$ and $[\mu_{\text{TF}}^2] = 0$. Therefore, the Coulomb interaction is marginal for $0 < z_{\nu} < 2$ whereas it is irrelevant otherwise; hence, we focus on the case of $0 < z_{\nu} < 2$ in the following. One natural choice of the scaling is to make the velocity ($\nu$) and the inverse effective mass ($A$) invariant under the scaling, that is, $z_{\nu} = z_{\mu} = 2$, because they are physical observables. Indeed, the renormalization group calculation shows the irrelevance of all the coupling constants at the stable fixed point, which indicates that the above scaling choice is legitimate.

We perform a momentum-shell renormalization group analysis of $S$ at the one-loop level. Owing to the anisotropic dispersion, it is natural to consider two different momentum cutoffs, that is, $\Lambda_{\nu}$ for the dispersion in the plane of $(k_x,k_z)$ and $\Lambda_{\mu}$ for the $k_z$ dispersion. For convenience, we set $\Lambda_{\mu} = \infty$ and perform the momentum-shell integration in the following way,

$$\int (dq_1 2\pi)^{3/2} f_{\nu}(q_1) f_{\mu}(q_1) dq_1, \text{ where } \Lambda = \Lambda_{\mu} = \Lambda_{\nu},$$

In principle, three Feynman diagrams need to be considered for the renormalization group calculation at one-loop order, that is, the fermion self-energy, the boson self-energy and the vertex correction. As the fermion self-energy is frequency independent for the instantaneous bare Coulomb interaction, the vertex correction vanishes, consistent with the Ward identity in the one-loop calculation. Therefore, the two Feynman diagrams shown in Fig. 2a determine the renormalization group flow. The straightforward computation of these two Feynman diagrams leads to the renormalization group equations in equation (3). The details of the computation are presented in Supplementary Note 3.

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Author contributions

N.N. conceived the original ideas. B-J.Y., E-G.M. and H.I. performed the calculations. All authors analysed the data and wrote the manuscript.

Additional information

Supplementary information is available in the online version of the paper. Reprints and permissions information is available online at www.nature.com/reprints. Correspondence and requests for materials should be addressed to B-J.Y. or N.N.

Competing financial interests

The authors declare no competing financial interests.