GROUP-LIKE ALGEBRAS AND HADAMARD MATRICES

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Abstract. We give a description in terms of square matrices of the family of group-like algebras with $S * id = id * S = u \varepsilon$. In the case that $S = id$ and $char k$ is not 2 and does not divide the dimension of the algebra, this translation take us to Hadamard matrices and, particularly, to examples of biFrobenius algebras satisfying $S * id = id * S = u \varepsilon$ and that are not Hopf algebras. Finally, we generalize some known results on separability and coseparability valid for finite dimensional Hopf algebras to this special class of biFrobenius algebras with $S * id = id * S = u \varepsilon$, presenting a version of Maschke’s theorem for this family.

Introduction.

BiFrobenius algebras generalize finite dimensional Hopf algebras, in the following sense: a biFrobenius algebra has a structure of Frobenius algebra and coFrobenius coalgebra, linked by a condition weaker than the pentagonal axiom, namely one assumes the existence of a (bijective) antimorphism of algebras and coalgebras $S : A \rightarrow A$. BiFrobenius algebras were introduced in 2000 by Y. Doi and M. Takeuchi ([DT]).

In general, $S$ is not the convolution inverse of the identity. This is true in the particular situation of Hopf algebras. A natural question is whether this additional condition implies or not that the algebra is Hopf; in other words: does the fact that $S$ is the convolution inverse of the identity forces $A$ to be a bialgebra?

In this paper, we show that this is not true by constructing a family of biFrobenius algebras where $S$ is the convolution inverse of the identity that are not Hopf algebras.

Once we know there is an intermediate class between biFrobenius algebras and finite dimensional Hopf algebras, it could be relevant to study which properties of finite dimensional Hopf algebras still hold for this class. In this direction, we give some first steps, generalizing some known results concerning separability, semisimplicity and its dual notions.

In section 1 we recall the definition and some of the basic properties of biFrobenius algebras.

In section 2 we present the example of group-like algebras. Group-like algebras are the natural generalization of group algebras to the context of biFrobenius algebras. We also study additional conditions for a group-like algebra to satisfy $S * id = id * S = u \varepsilon$ and to be a Hopf algebra.

In section 3 we describe, in terms of a family of square matrices, the algebra structure of a group-like algebra with $S * id = id * S = u \varepsilon$. As the coalgebra structure

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of a group-like algebra is trivial, this matrix approach will give us a complete
description of the family of group-like algebras with $S \ast id = id \ast S = u \varepsilon$.
In section 4 we briefly present Hadamard matrices. These matrices will be used in
section 5 to construct the counterexamples mentioned above.
In section 5 we treat the cases in which $S = id$ and char$k$ does not divide the
dimension of the algebra, showing how the family of square matrices mentioned
before, gives rise, in this particular case, to a Hadamard matrix. We construct, via
Hadamard matrices, a family, of unbounded dimension, of commutative group-like
algebras with $S \ast id = id \ast S = u \varepsilon$ and that are not Hopf algebras.
Finally, in section 6 we prove a version of Maschke’s theorem for this special biFrobe-
nius algebras.

1. BiFrobenius algebras.

All along this paper we use Sweedler’s notation for the coproduct of a coalgebra.

Let $A$ be an algebra over $k$.
We consider $A^*$ with the structure of right $A$-module via $(f \leftarrow a)(x) = f(ax), \forall f \in A^*, a,x \in A$. For every $f \in A^*$, the induced map $f \leftarrow: A \rightarrow A^*$ is a morphism of
right $A$-modules.
If $C$ is a coalgebra over $k$, $C$ has a natural structure of right $C^*$-module, via the
right action $c \leftarrow f = \sum f(c_1)c_2, \forall c \in C, f \in C^*$. For every $c \in C$, the induced map $c \leftarrow: C^* \rightarrow C$ is a morphism of right $C^*$-modules.

Definition 1. Let $k$ be a field. A Frobenius algebra is a pair $(A, \phi)$ where $A$ is
a $k$-algebra and $\phi \in A^*$ is such that the morphism

$$\phi \leftarrow: A \rightarrow A^*,$$

$$x \mapsto \phi \leftarrow x$$

is bijective.
Dually, a coFrobenius coalgebra is a pair $(C, t)$ where $C$ is a $k$-coalgebra and
t $\in C$ is such that the morphism

$$t \leftarrow: C^* \rightarrow C,$$

$$f \mapsto t \leftarrow f$$

is bijective.

Remark 1. Clearly, Frobenius algebras and coFrobenius coalgebras are finite di-

dimensional.

We recall now some results valid for the augmented and coaugmented cases.

Suppose $A$ is a $k$-algebra that admits an algebra morphism $\varepsilon: A \rightarrow k$. In this
case, the algebra is said to be augmented and we say that $s \in A$ is a right integral
if $sx = \varepsilon(x)s, \forall x \in A$.

Proposition 1. If $(A, \phi, \varepsilon)$ is an augmented Frobenius algebra and $s \in A$ is (the
unique element) such that $\phi \leftarrow s = \varepsilon$, then $s$ is an right integral in $A$. 
Dually, suppose \( C \) is a \( k \)-coalgebra with a group-like element \( 1 \in C \), i.e. \( 1 \in C \) satisfies \( \Delta(1) = 1 \otimes 1 \). In this case, we say that the coalgebra is coaugmented and that \( \psi \in C^* \) is a right cointegral for \( C \) if \( \sum \psi(x_1)x_2 = \psi(x)1, \forall x \in C \).

**Proposition 2.** If \((C, t, 1)\) is a coaugmented coFrobenius algebra and \( \psi \in C^* \) is (the unique element) such that \( t↼\psi = 1 \), then \( \psi \) is a right cointegral.

The space of right integrals of an augmented Frobenius algebra \( A \) is one dimensional, since it is \( I_r(A) = A^A \cong (A^*)^A = k\varepsilon \).

Dually, the space of right cointegrals of a coaugmented coFrobenius coalgebra is one dimensional.

Now we can give the definition of a biFrobenius algebra.

**Definition 2.** Let \( k \) be a field. The data \((A, m, 1, \Delta, \varepsilon, \phi, t)\) is said to be a biFrobenius algebra if the following conditions hold:

- \((A, m, 1, \phi)\) is a Frobenius algebra,
- \((A, \Delta, \varepsilon, t)\) is a coFrobenius coalgebra,
- \( \varepsilon \) is an algebra map,
- \( 1 \in C \) is a group-like element,
- The map \( S : A \to A \) defined by \( S(x) = \sum \phi(t_1)x_2 \) is an antimorphism of algebras and coalgebras.

In [DT] it is proved that \( S \) is necessarily bijective. \( S \) is called the antipode of the biFrobenius algebra \( A \).

Notice that propositions 1 and 2 imply that \( \phi \) is a cointegral and \( t \) is an integral for \( A \). Indeed,

\[
S(1) = 1, \varepsilon \circ S = \varepsilon,
\]

means that \( \sum \phi(t_1)x_2 = 1 \) and that \( \phi(tx) = \varepsilon(x), \forall x \in A \); in other words,

\[
t↼\phi = 1, \quad \text{and} \quad \phi↼t = \varepsilon.
\]

**Definition 3.** Let \( A \) be a biFrobenius algebra, \( s, \psi \) respectively a right integral and a right cointegral for \( A \). Once can easily show that

\[
s↼\psi = 1 ⇔ \psi(x) = 1 ⇔ \psi↼s = \varepsilon.
\]

We say that \( (s, \psi) \) is a biFrobenius pair if one of these (equivalent) conditions holds.

Using the fact that the space of right integrals and the space of right cointegrals are one dimensional, it is easy to see that:

**Remark 2.**

- Given a non-zero right integral \( s \in A \), there is one and only one \( \psi \in A^* \) such that \( (s, \psi) \) is a biFrobenius pair.
- Given a non-zero right cointegral \( \psi \in A^* \), there is one and only one \( s \in A \) such that \( (s, \psi) \) is a biFrobenius pair.
- If \( (t, \phi) \) is a biFrobenius pair then any other biFrobenius pair is of the form \((\lambda t, \frac{1}{\lambda} \phi), \lambda \in k, \lambda \neq 0 \).
- If \( (s, \psi) \) is a biFrobenius pair, then \( \sum \psi(s_1)x_2 = S(x) \).

**Examples.**

It is well known that every finite dimensional Hopf algebra is biFrobenius.
In the next section we present the coalgebraically trivial example of biFrobenius algebras. See [DT], [F] and [WZ] for other examples.

**Remark 3.** If $A$ is a biFrobenius algebra, in the finite dimensional algebra $\text{Hom}_k(A, A)$ the condition $S \ast \text{id} = u \varepsilon$ is equivalent to $\text{id} \ast S = u \varepsilon$.

**Proposition 3.** Let $A$ be a biFrobenius algebra over a field $k$. If $A$ is also a bialgebra then $S \ast \text{id} = u \varepsilon$. Hence $A$ is a Hopf algebra.

**Proof:** As $A$ is a bialgebra if and only if $\Delta$ is an algebra morphism, then

$$(S \ast \text{id})(x) = \sum S(x_1)x_2 = \sum \phi(t_1x_1)t_2x_2 = \sum \phi((tx)_1)(tx)_2 = \phi(tx)1 = (\phi \leftarrow t)(x)1 = \varepsilon(x)1.$$\[\square\]

We will show in section 4 that the converse is not true, namely that we can find biFrobenius algebras with the property $S \ast \text{id} = u \varepsilon$ that are not Hopf algebras, in the sense that the multiplication and the comultiplication are not compatible.

**Remark 4.** Notice that, in order to prove proposition 3 we have used in fact a condition that is weaker than the compatibility between product and coproduct in a bialgebra. Namely the condition

$$\varepsilon(x) \sum t_1 \otimes t_2 = \sum (tx)_1 \otimes (tx)_2 = \sum t_1x_1 \otimes t_2x_2 \quad (\ast)$$

implies that $S \ast \text{id} = u \varepsilon$. It is relevant to know that condition (\ast) is in fact weaker than the compatibility condition for a bialgebra. See the end of section 5.

2. Group-Like Algebras.

We present here a family of examples of biFrobenius algebras, introduced in [D] by Y.Doi. It generalizes the group algebras in the sense that the coalgebraic structure of the group-like algebras is trivial, as it is for group algebras.

We will present this example by adding, step by step, the minimal conditions required to have a biFrobenius structure, coalgebraically trivial, in a finite dimensional $k$-vector space.

All along the rest of the paper we set

$$I = \{0, 1, 2, \ldots, n\}.$$ We will use 1 for both $1_k$ and $1_A$, since it will be clear from the context to which of them we are referring.

We start by giving the minimal data we need in order to have the simplest coFrobenius coalgebra structure in a finite dimensional vector space. We leave the proof of the next proposition to the reader.

**Proposition 4.** Let $k$ be a field and $A$ be a finite dimensional $k$-vector space with basis $B = \{b_0, b_1, b_2, \ldots, b_n\}$. Let $\varepsilon : B \to k$ be such that $\varepsilon(b_i) \neq 0$, $\forall i \in I$.

1. If we define

$$\Delta(b_i) = \frac{1}{\varepsilon(b_i)} b_i \otimes b_i, \forall i \in I, \quad t = b_0 + b_1 + b_2 \cdots + b_n$$
and we extend $\Delta$ and $\varepsilon$ to $A$ by linearity, then $(A, \Delta, \varepsilon, t)$ is a coFrobenius coalgebra.

(2) Assume moreover that $\varepsilon(b_0) = 1$. Then, with respect to the structure $\Delta$ given above, $b_0$ is a group-like element of $A$ and the preimage $\phi \in A^*$ of $b_0$ by $t \leftarrow$ satisfies $\phi(b_i) = \delta_{i,0}, \forall i \in I$.

Assume moreover that we have an algebra structure in the vector space $A$ given, in terms of the structure constants, by

$$b_ib_j = \sum_{k \in I} p^k_{ij} b_k, \forall i, j \in I, \quad 1 = b_0.$$ 

Notice that the fact that $b_0 = 1$ forces the cointegral $\phi \in A^*$ to be as defined in proposition (2). Next proposition presents necessary and sufficient conditions on the coefficients $p^k_{ij}$ in order to make the map $S : A \to A$ defined by the formula

$$S(x) = \sum \phi(t_1)x_1 \phi(t_2)$$

an antimorphism of algebras and of coalgebras. Notice that

$$S(b_j) = \sum_{i \in I} \frac{1}{\varepsilon(b_i)} \phi(b_i b_j) b_i = \sum_{i \in I} \frac{1}{\varepsilon(b_i)} p^0_{ij} b_i.$$

If $\sigma : I \to I$ is a permutation, we say that $S$ extends $\sigma$ if $S(b_i) = b_{\sigma(i)}, \forall i \in I$.

**Proposition 5.** Let $k$ be a field and $A$ be a finite dimensional $k$-algebra with basis $B = \{b_0, b_1, \cdots, b_n\}$ and $b_0 = 1$. Let $p^k_{ij}, i, j, k \in I$ be the structure constants associated to the algebra $A$ in the basis $B$. Take $\varepsilon : \mathcal{B} \to k$ to be a function that is never 0 and $\sigma : I \to I$ a permutation of $I$ and endow $A$ with the comultiplication defined before. Then:

1. $S$ extends $\sigma$ if and only if $p^0_{ij} = \varepsilon(b_i) \delta_{i, \sigma(j)}$.
2. Suppose 1 holds. Then:
   
   a) $S$ is an antimorphism of coalgebras if and only if $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_i), \forall i \in I$.
   
   b) $S$ is an antimorphism of algebras if and only if

$$\sigma(0) = 0, \text{ and } p^k_{ij} = p^k_{\sigma(j) \sigma(i)}.$$

3. If $S$ is an antimorphism of algebras and coalgebras that extends $\sigma$, then $\sigma^2 = \text{id}$.

**Proof:**

1. We know that $S(b_j) = \sum_{i \in I} \frac{1}{\varepsilon(b_i)} p^0_{ij} b_i$. As $B$ is linearly independent, the condition $S(b_j) = b_{\sigma(j)}, \forall j \in I$ is satisfied if and only if $\frac{1}{\varepsilon(b_i)} p^0_{ij} = \delta_{i, \sigma(j)}$, which is equivalent to $p^0_{ij} = \varepsilon(b_i) \delta_{i, \sigma(j)}$.

2. We assume now that $S(b_i) = b_{\sigma(i)}, \forall i \in I$.
   
   a) We have that $\frac{1}{\varepsilon(b_{\sigma(i)})} b_{\sigma(i)} \otimes b_{\sigma(i)} = \Delta(b_{\sigma(i)}) = \Delta(S(b_i)) = S(\otimes S)(\Delta(b_i)) = \frac{1}{\varepsilon(b_i)} b_{\sigma(i)} \otimes b_{\sigma(i)}$. Then $S$ is anticommutes with $\Delta$ if and only if $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_i), \forall i \in I$. Notice that this implies that $S$ is counital, i.e. that $\varepsilon \circ S = \varepsilon$.

   b) $S$ is an antimorphism of algebras if and only if $S(1) = 1$ and $S(xy) = S(y)S(x), \forall x, y \in A$. But $S(1) = 1$ if and only if $S(b_0) = b_0$, which means, in terms of $\sigma$, that $b_{\sigma(0)} = b_0$, i.e. $\sigma(0) = 0$.

On the other hand $S(xy) = S(y)S(x), \forall x, y \in A$ if and only if $S(b_i b_j) =$
S(b_j)S(b_i), \forall i, j \in I$. Writing this equality in terms of the structure constants, we obtain $S(\sum_{k \in I} p_{ij}^k b_k) = b_{\sigma(j)} b_{\sigma(i)}, \forall i, j \in I$ and then $\sum_{k \in I} p_{ij}^k b_{\sigma(k)} = \sum_{k \in I} p_{\sigma(j) \sigma(i)}^k b_k, \forall i, j \in I$. Then $\sum_{k \in I} p_{ij}^k b_{\sigma(k)} = \sum_{k \in I} p_{\sigma(j) \sigma(i)}^k b_{\sigma(k)}, \forall i, j \in I$ i.e. $p_{ij}^k = p_{\sigma(j) \sigma(i)}^k, \forall i, j, k \in I$.

(3) Assume that $S$ is an antimorphism of algebras and coalgebras that extends $\sigma$. Observe that, using (1) and (2), we have that $\forall i \in I : \varepsilon(b_{\sigma(i)}) = \varepsilon(b_{\sigma(i)}) \delta_{\sigma(i), \sigma(i)} = p_{\sigma(i) \sigma(i)}^0 = p_{\sigma(i) \sigma^2(i)}^0 = \varepsilon(b_{\sigma(i)}) \delta_{\sigma(i), \sigma^2(i)}; \text{ hence} \sigma^3(i) = \sigma(i), \forall i \in I$ and, as $\sigma$ is bijective, $\sigma^2(i) = i, \forall i \in I$.

Now we are able to summarize all the conditions required in order to have in the vector space $A$ a structure of biFrobenius algebra. The following theorem will follow directly from propositions 4 and 5.

**Theorem 1.** Let $\mathbb{k}$ be a field and $A$ be a finite dimensional $\mathbb{k}$-vector space with basis $B = \{b_0, b_1, \cdots, b_n\}$. Let $p : A \otimes A \to A$ be a linear map with structure constants $\{p_{ij}^k | i, j, k \in I\}$ with respect to $B$. Let $\varepsilon : B \to \mathbb{k}$ be a function and $\sigma : I \to I$ be a permutation, where $I = \{0, 1, 2, \cdots, n\}$. If we have that

1. $p$ is associative, $p$ has neutral element $b_0 = 1$,
2. $\varepsilon(b_i) \neq 0, \forall i \in I$, $\varepsilon(b_{\sigma(i)}) = \varepsilon(b_i), \forall i \in I$,
3. $\varepsilon(b_i) \neq 0, \forall i \in I$,
4. $p_{ij}^k = \delta_{\sigma(i), \sigma(j)} \varepsilon(b_i)$,
5. $\varepsilon(b_i) \neq 0, \forall i \in I$,
6. $\sigma(0) = 0$,
7. $\varepsilon(1) = 1, \text{ hence } \Delta(1) = \Delta(b_0) = \frac{1}{\varepsilon(b_0)} b_0 \otimes b_0 = 1 \otimes 1$ and then 1 is a group-like element for $A$.

then $A$ has a structure of biFrobenius algebra with $\Delta, \phi$ and $t$ defined as before.

**Proof:** Notice that the conditions (GL1) to (GL8) guarantee, following propositions 4 and 5, that $A$ is a Frobenius algebra, $A$ is a coFrobenius coalgebra and that the induced map $S$ is an antimorphism of algebras and coalgebras. Notice that $\varepsilon : A \to \mathbb{k}$ is a morphism of algebras (condition (GL8)). Moreover, if (GL8) holds, then $\varepsilon(1) = 1$, hence $\Delta(1) = \Delta(b_0) = \frac{1}{\varepsilon(b_0)} b_0 \otimes b_0 = 1 \otimes 1$ and then 1 is a group-like element for $A$.

Now we present the definition of a group-like algebra (see [D]).

**Definition 4.** A $\mathbb{k}$-group-like algebra is a 5-tuple $(A, \mathcal{B}, p, \varepsilon, \sigma)$ such that $A$ is a $\mathbb{k}$-algebra with basis $B = \{b_0, b_1, \cdots, b_n\}$, $p : A \otimes A \to A$ is a linear map, $\varepsilon : B \to \mathbb{k}$ is a function and $\sigma : I \to I$ is a bijection, such that conditions (GL1) to (GL8) are satisfied.

**Corollary 1.** Every group-like algebra is a cocommutative biFrobenius algebra with $S^2 = \text{id}$.

Next we will exhibit the additional conditions needed in order to have a Hopf algebra structure in $A$.

**Proposition 6.** Let $A = (A, \mathcal{B}, p, \varepsilon, \sigma)$ be a $\mathbb{k}$-group-like algebra. Then:

1. $S \ast \text{id} = \varepsilon$ if and only if $\mathcal{B}$ is closed under the operation of taking inverses with respect to $p$.
2. The following conditions for $A$ are equivalent:
   (i) $A$ is a Hopf algebra,
Proof:

(1) If $S \ast id = u\varepsilon$, then $(S \ast id)(b_i) = \varepsilon(b_i)b_0, \forall i \in I$. Then we have
\[
\frac{1}{\varepsilon(b_i)}b_{\sigma(i)}b_i = \varepsilon(b_i)b_0,
\]
and \[\frac{1}{\varepsilon(b_i)}p_{\sigma(i)i} = \varepsilon(b_i)\]. Then $p_{\sigma(i)i} = (\varepsilon(b_i))^2$. But conditions (GL5) and (GL4) give $p_{\sigma(i)i} = \varepsilon(b_{\sigma(i)}) = \varepsilon(b_i)$, so using condition (GL3) we get that $\forall i \in I : \varepsilon(b_i) = 1$. Then we have that $b_{\sigma(i)}b_i = b_0 = 1$ (and also that $b_i b_{\sigma(i)} = b_0$ since $A$ is finite dimensional), which means $b_i^{-1} = b_{\sigma(i)}$.

For the converse direction, assume $b_i$ has an inverse in $B$. From the condition (GL5) we deduce that the inverse of $b_i$ has to be $b_{\sigma(i)}$. From the equality $b_{\sigma(i)}b_i = b_0$ and conditions (GL8) and (GL4) we get $\varepsilon(b_i)^2 = \varepsilon(b_0) = 1$, hence $\varepsilon(b_i) \in \{1, -1\}$, which implies $\frac{1}{\varepsilon(b_i)} = \varepsilon(b_i)$. We get that
\[
\frac{1}{\varepsilon(b_i)}b_{\sigma(i)}b_i = \varepsilon(b_i)b_0 = \varepsilon(b_i)1, \forall i \in I
\]
and hence, the equality $S \ast id = u\varepsilon$ holds in the basis $B$ and then in $A$.

(2) The fact that $A$ is a Hopf algebra if and only if it is a bialgebra, follows directly from proposition $\mathbb{K}$ Hence $(i) \leftrightarrow (ii)$. Suppose now that $A$ is a bialgebra. We know, by proposition $\mathbb{K}$ that $S \ast id = u\varepsilon$. So, by the proof of (1), we get that $\varepsilon(b_i) = 1, \forall i \in I$.

As $A$ is a bialgebra, $\forall i, j \in I$, we have that $\Delta(b_i b_j) = \Delta(b_i)\Delta(b_j)$, which means, in terms of the structure constants,
\[
\sum_{k \in I} p_{ij}^k (b_k \otimes b_k) = \sum_{k,l \in I} p_{kl}^i p_{ij}^k (b_k \otimes b_l), \forall i, j \in I.
\]

Then, $p_{ij}^k = 0$ for each pair $(k,l)$ such that $k \neq l$, which means that given $i,j \in I$ there is $k_0 \in I$ such that $p_{ij}^{k_0} = 0, \forall k \neq k_0$. Moreover, using the bialgebra condition above; for $k = l = k_0$, we get that $p_{ij}^{k_0} = (p_{ij}^{k_0})^2$, so that $p_{ij}^{k_0} \in \{0,1\}$. If $p_{ij}^{k_0} = 0$, we would have $b_i b_j = 0$ and so, using condition (GL8), $\varepsilon(b_i)\varepsilon(b_j) = 0$ which contradicts condition (GL3).

We conclude that $p_{ij}^{k_0} = 1$ and then $b_i b_j = b_{k_0}$.

Conversely, suppose $B$ is closed under $p$. Then for each pair $(i,j) \in I \times I$ there is $k \in I$ such that $b_i b_j = b_k$. Applying $\Delta$ we obtain that
\[
\Delta(b_i b_j) = \Delta(b_k) = \frac{1}{\varepsilon(b_k)}b_k \otimes b_k = \frac{1}{\varepsilon(b_i)\varepsilon(b_j)}(b_i b_j \otimes b_i b_j) = 
\]
and then we have proved $(ii) \leftrightarrow (iii)$ (Notice we have used condition (GL8)). Finally, as $A$ is a Hopf algebra if and only if $A$ is a bialgebra and $S \ast id = u\varepsilon$, and these two conditions hold if and only if $(B,p,1)$ is a monoid closed under taking inverses i.e. if $(B,p,1)$ is a group, we have proved $(i) \leftrightarrow (iv)$.

(3) It is clear from the proof of (1) and (2) that in both cases $\varepsilon(b_i) = 1, \forall i \in I$. 

In particular, we have the following result.

**Corollary 2.** If a group-like algebra $A = (A, B, p, ε, σ)$ is Hopf, then $(B, p, 1)$ is a group, $σ$ is the inverse in $B$ and $A$ is the group algebra $kB$.

### 3. The Matricial Approach.

In the next theorem we give a description in terms of square matrices of the family of group-like algebras with the property that $S * id = uε$. In order to do this, we need some previous definitions. From now on, we denote by $w$ both a vector in $k_{n+1}$ and its transpose. Also $\{e_0, e_1, \cdots, e_n\}$ is the canonical basis of $k_{n+1}$.

**Definition 5.** Let $σ : I → I$ be a permutation and $v = (1, 1, \ldots, 1) ∈ k^{n+1}$.

- A $(I, σ)$-system is a family of invertible matrices $F = \{F_i \mid i ∈ I\} ⊆ M_{n+1}(k)$ satisfying:

  $F_{σ(i)} = F_i^t = F_i^{-1}, \forall i ∈ I, \quad F_i v = v, \forall i ∈ I, \quad F_i e_0 = e_i, \forall i ∈ I.$

- If $F$ and $G$ are two systems, we say that they are compatible if $F_i G_j = G_j F_i, \forall i, j ∈ I$.

**Lemma 1.** If $F$ and $G$ are compatible $(I, σ)$-systems, then $F_0 = G_0 = Id$ and $(F_i)_{kj} = (G_j)_{ki}$.

**Proof:** Observe that $F_0 e_i = F_0 G_i e_0 = G_i F_0 e_0 = G_i e_0 = e_i$.

On the other hand, we have that $F_i e_j = F_i G_j e_0 = G_j F_i e_0 = G_j e_i$. Therefore

$$(F_i)_{kj} = (F_i e_j)_k = (G_j e_i)_k = (G_j)_{ki}$$

and the proof is finished.

Notice that we can consider, without loss of generality, group-like algebras whose supporting vector space is $k_{n+1}$. We will do so from now on, letting $I = \{0, 1, 2, \cdots, n\}$, $C = \{e_0, e_1, \cdots, e_n\}$ be the canonical basis of $k_{n+1}$ and for each $i, j, k ∈ I$, $p_{ij}^k ∈ k$.

What follows is the framework needed in order to formulate next theorem. We define

- $p : k_{n+1} ⊗ k_{n+1} → k_{n+1}, \quad p(e_i ⊗ e_j) = \sum k \in I p_{ij}^k e_k$,
- $1 : k_{n+1} → k, \quad 1(e_i) = 1, \forall i ∈ I$,
- $H = \{H_i \mid i ∈ I\}$, where $H_i ∈ M_{n+1}(k)$ is defined by $(H_i)_{kj} = p_{ij}^k, \forall i, j, k ∈ I$,
- $V = \{V_i \mid i ∈ I\}$, where $V_i ∈ M_{n+1}(k)$ is defined by $(V_i)_{kj} = p_{ij}^k, \forall i, j, k ∈ I$.

**Remark 5.** Notice that:

- $H_i$ is the matrix associated to the map $e_i : A → A$ in the canonical basis,
- $V_j$ is the matrix associated to the map $e_j : A → A$ in the canonical basis.

**Theorem 2.** In the context defined above, if $σ : I → I$ is an arbitrary permutation, we have that $(k_{n+1}, C, p, 1, σ)$ is a group-like algebra with $S * id = uε$ if and only if $H$ and $V$ are compatible $(I, σ)$-systems and $σ^2 = id$.
Proof: Assume first that \((k^{n+1}, C, p, 1, \sigma)\) is a group-like algebra. We know that 
\(\sigma^2 = id\) (corollary 2). By remark 5, it is clear that 
\(H_i e_0 = V_i e_0 = e_i, \forall i \in I\) and also that \(H_{\sigma(i)} = H_i^{-1}\) and \(V_{\sigma(i)} = V_i^{-1}\) (since \(p\) is associative and \(S * id = u \varepsilon\)
implies that \(e_{\sigma(i)} e_i = e_0 = e_i e_{\sigma(i)}, \forall i \in I\).
As we have that
\[
\phi \left( e_{\sigma(k)} (e_i e_j) \right) = \phi \left( (e_{\sigma(k)} e_i) e_j \right), \forall i, j, k \in I,
\]
we deduce that \(\sum_{i \in I} p_{\sigma(k)}^0 p_{ij}^1 = \sum_{i \in I} p_{\sigma(k)}^1 p_{ij}^0\), which implies, by condition \((GL5)\)
(and the fact that \(\sigma^2 = id\), that
\[
p_{ij}^k = p_{\sigma(k)}^{\sigma(i)} p_{ij}^\sigma, \forall i, j, k \in I.
\]
Using condition \((GL6)\), we get that
\[
p_{ij}^k = p_{\sigma(k)}^{\sigma(i)} p_{ij}^\sigma, \forall i, j, k \in I,
\]
which means that \(H_i^t = H_{\sigma(i)}\). Similarly, we prove that \(V_i^t = V_{\sigma(i)}\).

Now,
\[
H_i v = \left( \sum_{j \in I} p_{ij}^0, \sum_{j \in I} p_{ij}^1, \ldots, \sum_{j \in I} p_{ij}^n \right) = \left( \sum_{j \in I} p_{\sigma(i)0}^j, \sum_{j \in I} p_{\sigma(i)1}^j, \ldots, \sum_{j \in I} p_{\sigma(i)n}^j \right) = v,
\]
where last equality follows from \(\varepsilon(b_{\sigma(i)} e_0) = \varepsilon(b_{\sigma(i)} e_0) = 1\) (condition \((GL8)\)).
In a similar way we prove that \(V_i v = v, \forall i \in I\).
We have already proved that the families \(H\) and \(V\) are \((I, \sigma)\)-systems. We still
have to prove that they are compatible. For this, we use the associativity of \(p\), i.e.
\((e_i e_k) e_j = e_i (e_k e_j)\). If we put this equality in terms of the associated matrices we get
\[
H_i V_j = V_j H_i,
\]
and we are done.

Let us prove the converse, so assume that \(H\) and \(V\) are \((I, \sigma)\)-compatible systems.
We have to check conditions \((GL1)\) to \((GL8)\) of theorem 4 and also that \(S * id = u \varepsilon\).
The associativity follows from \(V_j H_i = H_i V_j, \forall i, j \in I\) so we have \((GL1)\).
By lemma 1 we have that \(H_0 = V_0 = Id\) and this implies, by remark 5, that \(e_0 = 1\)
is the neutral element of the algebra, i.e. \((GL2)\).
Conditions \((GL3)\) and \((GL4)\) hold trivially, since \(\varepsilon = 1\).
As \(H_{\sigma(i)} = H_i^t\), we have that
\[
p_{ij}^0 = (H_i)_{ij} = (H_{\sigma(i)})_{ij} = p_{\sigma(i)0}^j = (V_0)_{j\sigma(i)} = \delta_{\sigma(i), j},
\]
so condition \((GL5)\) holds (notice we have used \(\sigma^2 = id\) and \(\varepsilon = 1\)).
Now, using first that \(H_i^t = H_{\sigma(i)}\), then that \(V_i^t = V_{\sigma(i)}\), and finally that \(H_i^t = H_{\sigma(i)}\),
we deduce that
\[
p_{\sigma(i)\sigma(i)} = p_{\sigma(k)}^{\sigma(i)} p_{ij}^\sigma = p_{ij}^\sigma
\]
i.e. condition \((GL6)\).
Let us prove condition \((GL7)\), i.e. \(\sigma(0) = 0\). We have that \(H_{\sigma(0)} = H_0^t = Id\), but
notice that \(H_i \neq Id, \forall i \neq 0\), since \(H_i e_0 = e_i\), so we get \(\sigma(0) = 0\).
To prove that \(\varepsilon = 1\) is a morphism of algebras, it is enough to verify that \(\forall i, j \in I\),
\[ \varepsilon(e_i e_j) = \varepsilon(e_i) \varepsilon(e_j). \] But \( H_{\sigma(i)} = H_i^1 \) and \( H_i v = v, \forall i \in I \) imply that
\[
\sum_{k \in I} p_{ij}^k = \sum_{k \in I} p_{\sigma(i)k}^j = 1, \forall i, j \in I,
\]
i.e. \( \varepsilon(e_i e_j) = \sum_{k \in I} p_{ij}^k = 1 = \varepsilon(e_i) \varepsilon(e_j). \)

It remains to prove that \( S \circ \text{id} = u \varepsilon \). This can be deduced directly from associativity and the fact that \( H_{\sigma(i)} = H_i^{-1} \).

\[ \square \]

**Corollary 3.** Let \((I, \sigma)\) be as in definition 2 and \( C \) be the canonical basis of \( k^{n+1} \).
1. There is a one to one correspondence between group-like algebras of the form \( A = (k^{n+1}, C, p, \varepsilon, \sigma) \) satisfying \( S \circ \text{id} = u \varepsilon \) and pairs of compatible \((I, \sigma)\)-systems.
2. Let \( A \) be as described in (1) and \((\mathcal{H}, V)\) its corresponding pair of compatible \((I, \sigma)\)-systems. Then \( A \) is a Hopf algebra if and only if the family \( \mathcal{H} \) is a group with the usual matrix product (and this happens if and only if the family \( V \) is a group with the usual matrix product).

**Proof:**
1. We recall that in proposition 6 we proved that in group-like algebras with \( S \circ \text{id} = u \varepsilon \) it holds that \( \varepsilon = 1 \). It follows from theorem 2 that we can define correspondences
\[
(k^{n+1}, C, p, 1, \sigma) \mapsto (\mathcal{H}, V) \text{ with } (H_i)_{kj} = p_{ij}^k = (V_j)_{ki},
\]
\[(\mathcal{H}, V) \mapsto (k^{n+1}, C, p, 1, \sigma), \text{ where } p: k^{n+1} \otimes k^{n+1} \to k^{n+1} \text{ is given by } p_{ij}^k = (H_i)_{kj}. \]

Lemma 1 guarantees that these correspondences induce a bijection (more specifically, that the first map is surjective).

2. By proposition 6 \( A \) is a Hopf algebra if and only if \( B \) is closed under \( p \) if and only if \( \forall i, j \in I \), there is \( k \in I \) such that \( e_i e_j = e_k \) if and only if \( \forall i, j \in I \), there is \( k \in I \) such that \( H_i H_j = H_k \) (or \( V_i V_j = V_k \)) and we are done.

\[ \square \]

### 4. Hadamard Matrices

The Hadamard maximal determinant problem is the following: find the matrices of a given size with entries \(+1\) and \(-1\) with the largest (in absolute value) possible determinant. Despite well over a century of work, beginning with Sylvester’s results of 1867, the general question remains unanswered. However, it is known that Hadamard matrices are solutions to Hadamard’s maximal \( n^2 \)-determinant problem.

**Definition 6.** A Hadamard matrix of size \( n \) is a square matrix (of size \( n \in \mathbb{N} \)) with coefficients in \( \{1, -1\} \) that is orthogonal (with the usual inner product of \( \mathbb{C}^n \)) and such that the first row and the first column are both the vector \( v = (1, 1, \ldots, 1) \).

**Remark 6.** Sometimes the condition concerning the first column and the first row (called the normalization condition) is not required in the definition of a Hadamard matrix. However, if we multiply by \(-1\) any row or any column of an orthogonal
matrix with coefficients in \{-1,1\}, we still get an orthogonal matrix. In this sense, any "generalized" Hadamard matrix is equivalent to a normalized one.

It is easy to check that if \( P \) is a Hadamard matrix of size \( n \) then \( n = 1, 2 \) or a positive multiple of \( 4 \). Also, it is clear that defining

\[
P_1 = (1), \quad P_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_k = \begin{pmatrix} P_{k-1} & P_{k-1} \\ P_{k-1} & -P_{k-1} \end{pmatrix}
\]

we obtain Hadamard matrices of size \( 2^k \).

For the purposes of the applications to group-like algebras, it will be relevant whether there exist or not Hadamard matrices whose size is not a power of \( 2 \). The answer is negative: it has been proved that there are Hadamard matrices of unbounded size \( n \) where \( n \) is not a power of \( 2 \). Indeed, Paley’s construction (see [W]) guarantees that there exist Hadamard matrices of size \( n \), for every \( n \) divisible by \( 4 \) and of the form \( 2^e(p^n+1) \), with \( m,e,p \in \mathbb{N}, m \neq 0, p \) prime , \( p \neq 2 \). We give below an example of a Hadamard matrix of size 12 (corresponding to \( p = 5, e = m = 1 \)).

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
\end{pmatrix}
\]

5. The case \( S = \text{id} \).

In this section, we consider the particular case of group-like algebras supported in \( \mathbb{K}^{n+1} \) with \( S * \text{id} = u \text{id} \), where \( S = \text{id} \). They are of the form

\[
\mathcal{A} = (\mathbb{K}^{n+1}, C, p, \mathbf{1}, \text{id}), \quad \text{satisfying also that } c_i^2 = 1, \forall i \in I.
\]

This special type of group-like algebras are commutative, since \( \text{id} : \mathcal{A} \to \mathcal{A} \) is an antimorphism of algebras.

This section is divided in three main parts:
First we consider the correspondence described in corollary \$3\$ for this particular case. In other words, we describe all possible pairs of \((I, id)\)-compatible systems.

Second, we show that this family of all pairs of \((I, id)\)-compatible systems is in one to one correspondence with the family of Hadamard matrices.

Finally, we use some results on Hadamard matrices (mentioned in section 4) to find examples of group-like algebras satisfying \(S * id = u \varepsilon\) that are not Hopf algebras.

Suppose we have a group-like algebra \(A = (k^{n+1}, C, p, 1, id)\) with \(e^2_i = 1, \forall i \in \mathcal{I}\) and consider the bijection described in corollary \$3\$. Observe that, as \(A\) is commutative, we have that \(p_{ij}^k = p_{ji}^k, \forall i, j, k \in \mathcal{I}\) and therefore

\[ H_i = V_i, \forall i \in \mathcal{I}, \]

meaning that the compatible pair associated to \(A\) is \((\mathcal{H}, \mathcal{H})\), so it can be thought as a single \((I, id)\)-system, with the additional condition that

\[ H_i H_j = H_j H_i, \forall i, j \in \mathcal{I}. \]

This can be summarized as follows:

**Definition 7.** Let \(\mathcal{H}\) be a \((I, id)\)-system. We say that \(\mathcal{H}\) is \textbf{self-compatible} if

\[ H_i H_j = H_j H_i, \forall i, j \in \mathcal{I}. \]

Explicitely the set \(\{H_0, H_1, \ldots, H_n\}\) satisfies

\[ H_i = H_i^{-1} = H_i^{t}, \quad H_i H_j = H_j H_i, \quad H_i v = v, \quad H_i e_0 = e_i, \]

where \(i, j \in \mathcal{I}, v = (1, 1, \ldots, 1)\).

**Theorem 3.** Let \(k\) be a field. There is a one to one correspondence between group-like algebras of the form \((k^{n+1}, C, p, 1, id)\) satisfying also that \(e^2_i = 1, \forall i \in \mathcal{I}\) and self-compatible \((I, id)\)-systems.

Next we describe, provided some minor restrictions on \(\text{char } k\), a bijective correspondence between \((I, id)\)-self-compatible systems and Hadamard matrices.

In the rest of the paper all our systems will be taken over \((I, id)\).

**Theorem 4.** Let \(k\) be a field and \(n \in \mathbb{N}\). Then:

1. Assume that \(\text{char } k \neq 2\). Then the matrices of a self-compatible system \(\mathcal{H} = \{H_0, H_1, \ldots, H_n\}\) admit a basis of common eigenvectors \(\{v^0, v^1, \ldots, v^n\}\) such that the matrix

\[ P = (v^0 | v^1 | \cdots | v^n) \]

is a Hadamard matrix.

Moreover, we have that

\[ H_i = PD_i P^{-1}, \]

where \(D_i\) is a diagonal matrix whose diagonal is the \(i^{th}\)-row of \(P\). (Explicitely \((D_i)_{rs} = \delta_{rs}P_{ir}\).

2. Conversely, assume now that \(\text{char } k\) does not divide \(n + 1\). Then, given a Hadamard matrix \(P\), we can construct a self-compatible system \(\mathcal{H} = \{H_i | \quad i \in \mathcal{I}\}\) such that the columns of \(P\) form a common basis of eigenvectors for the matrices of \(\mathcal{H}\).

**Proof:**
(1) First notice that, as $H_i^2 = Id$, each $H_i$ is equivalent to a diagonal matrix with diagonal entries in $\{1, -1\}$. Denote as $(,)$ the standard inner product in $k^{n+1}$. We have also that $H_i H_j = H_j H_i$ $\forall i, j \in I$ and that $H_i^t = H_i$, i.e. each $H_i$, $i \in I$ is symmetric. Therefore, we can obtain for the matrices in $\cal H$ a common basis $\cal B = \{v^0, v^1, \ldots, v^n\}$ of orthogonal eigenvectors.

As $v = (1, 1, \ldots, 1)$ is fixed for all $H_i$, $i \in I$, we can choose $v^0 = v$.

Let $w = (w_0, w_1, \ldots, w_n) \in k^{n+1}$ be any common non-zero eigenvector. Then, for each $i \in I$, $H_i w = w$ or $H_i w = -w$. But $H_i w = \sum_{k \in I} w_k (H_i e_k)$ and $(H_i e_k)_i = (e_i, H_i e_k) = (H_i e_i, e_k) = (H_i^{-1} e_i, e_k) = (e_0, e_k) = \delta_{0,k}$ hence $(H_i w)_i = w_0$ and therefore, as $(H_i w)_i = \pm w_i$,

$$\forall i \in I : w_i = w_0 \text{ or } w_i = -w_0.$$

Take $w$ such that $w_0 = 1$. Writing $H_i w = \lambda_i w$ and computing the $i$-th coordinate we deduce that $1 = w_0 = (H_i w)_i = \lambda_i w_i$ and therefore $\lambda_i = w_i$.

In other words, for each $i$ of $H_i$, the eigenvalue of $H_i$ associated to $w$ is $w_i$, i.e. $H_i w = w_i$. We can assume $(v^i)_0 = 1$ and still have an orthogonal basis of eigenvectors.

Call $P$ the matrix $P = (v^0 | v^1 | \ldots | v^n)$.

Then $P$ has coefficients in $\{1, -1\}$ and

$$H_i = P D_i P^{-1},$$

where $D_i$ is a diagonal matrix formed by the (ordered) eigenvalues of $H_i$.

As the eigenvalue associated to $v^i$ the $i^{\text{th}}$-row of $P$ gives all the eigenvalues, meaning the principal diagonal in $D_i$ is the $i^{\text{th}}$-row of $P$.

Moreover, $P$ is an orthogonal matrix with coefficients in $\{1, -1\}$ whose first row and first column are $v = (1, 1, \ldots, 1)$, i.e. $P$ is a Hadamard matrix.

(2) Take a Hadamard matrix $P$ of size $n+1$.

It is clear that $P$ is invertible since $PP^t = (n+1)Id$ and $\text{char} k$ does not divide $n+1$. Let $r_i$ be the $i^{\text{th}}$-row of $P$. Consider $D_i$ the diagonal matrix where the principal diagonal is $r_i$ and take

$$H_i = P D_i P^{-1}.$$

We have to show that $\cal H = \{H_i \mid i \in I\}$ is a self-compatible system. As $H_i = \frac{1}{n+1} P D_i P^t$, then

$$(H_i)^t = H_i, \quad H_i H_j^t = H_j^t = H_j^t = \frac{1}{(n+1)^2} P D_i P^t D_j P^t = \frac{1}{(n+1)^2} P (D_i D_j) P^t = \frac{1}{(n+1)^2} P P^t = Id$$

Now, as $P e_0 = P e_0 = v$ and $P^t v = (n+1) e_0$, we have that, $\forall i \in I$,

$$H_i v = \frac{1}{n+1} P D_i P^t v = \frac{1}{n+1} P D_i (n+1) e_0 = P e_0 = v, \quad H_i e_0 = \frac{1}{n+1} P D_i P^t e_0 = \frac{1}{n+1} P D_i v = \frac{1}{n+1} P r_i = e_i.$$

We have proved that $\cal H$ is a system. It remains to prove that it is self-compatible, i.e. that the matrices in $\cal H$ commute with each other, but this follows from the fact that they are equivalent to a diagonal matrix via the same matrix $P$. 


For the next theorem we will consider in \( k^{n+1} \) the so called point-wise product
\[
w \cdot w' = (w_0 w'_0, w_1 w'_1, \ldots, w_n w'_n).\]

**Remark 7.** If \( P \) is a Hadamard matrix, the rows in \( P \) are self-inverses under the point-wise product.
On the other hand, the matrices in a \((I, id)\)-compatible system are self-inverses under the usual matrix product.

Next theorem gives a stronger link between these two products.

**Theorem 5.** Let \( H \) be a self-compatible system and \( P \) a Hadamard matrix of common eigenvectors for \( H \). Then the matrices in \( H \) form a group (with the usual matrix product) if and only if the rows in \( P \) form a group (with the point-wise product).

**Proof:** In view of remark 7, we only have to prove that \( H \) is closed under the usual matrix product if and only if the set of rows in \( P \) is closed under the point-wise product. But this follows directly from
\[
H_i H_j = H_k \text{ if and only if } D_i D_j = D_k \text{ if and only if } r_i r_j = r_k,
\]
where \( r_i \) is the \( i \)th-row of \( P \).

**Remark 8.** The existence of an invertible Hadamard matrix of size \( n+1 \) implies that \( \text{char } k \) does not divide \( n+1 \) (since the matrix is invertible) and in particular that \( \text{char } k \neq 2 \) (since the existence of a Hadamard matrix implies that \( n+1 \) is even).

**Corollary 4.** There are biFrobenius algebras of unbounded dimension satisfying \( S \ast id = u \varepsilon \) that are not Hopf algebras.

**Proof:** Take \( k \) such that \( \text{char } k = 0 \). Let \( k \in \mathbb{N} \). As we observed before, it is known that there is a Hadamard matrix of size \( n+1 \) bigger than \( k \), with \( n+1 \) not a power of 2.

Let \( H \) be the self-compatible system constructed from \( P \) as in theorem 4 and let \( A \) be its associated group-like algebra as in corollary 3. We claim that \( A \) is not a Hopf algebra.

If \( A \) were Hopf algebra, then, by corollary 3, \( H \) is a group with the usual matrix-product and therefore, by theorem 4 the rows of \( P \) would form a group. Then we would have an abelian finite group all whose elements have order two and whose size is not a power of 2, and this contradicts the structure theorem of finite abelian groups. Therefore \( A \) is not a Hopf algebra.

We finish this section by proving that the condition \((*)\) we considered in remark 4 is weaker than the compatibility condition for bialgebras.

**Proposition 7.** Let \( A = (A, B, p, 1, \sigma) \) be a group-like algebra and \( t \neq 0 \) be a right integral for \( A \). Then \( S \ast id = u \varepsilon \) if and only if
\[
\forall x \in A, \epsilon(x)t = \Delta(tx) = \sum t_1 x_1 \otimes t_2 x_2. \quad (*)
\]
Proof: The converse implication was proved in proposition 8.

For the direct implication, notice that, as \( S \circ \text{id} = u \varepsilon \), we have \( \varepsilon = 1 \), and then \( tb_j = t, \forall j \in I \). Therefore \( \Delta(tb_j) = \Delta(t), \forall j \in I \) and condition \((*)\) becomes

\[
\Delta(t) = \sum t_1 b_j \otimes t_2 b_j, \forall j \in I
\]

i.e., \( \sum_{k,l \in I} b_k \otimes b_k = \sum_{i,k,l \in I} p_{ij}^{kl} b_k \otimes b_l, \forall j \in I \). In other words, we have to prove that

\[
\sum_{i \in I} p_{ij}^{kl} = \delta_{k,l}, \forall j, k, l \in I.
\]

But \( S \circ \text{id} = u \varepsilon \) implies \( b_{\sigma(i)}(b_j b_{\sigma(l)}) = b_{\sigma(i)}, \forall j, l \in I \) and therefore

\[
\sum_{i,u \in I} p_{\sigma(i)j}^{\sigma(u)l} b_u = b_{\sigma(i)}, \forall i, j, l \in I.
\]

For \( u = \sigma(k) \) we have \( \sum_{i \in I} p_{\sigma(k)j}^{\sigma(i)l} = \delta_{\sigma(k), \sigma(l)} \) or equivalently,

\[
\sum_{i \in I} p_{\sigma(k)j}^{\sigma(i)l} = \delta_{k,l}.
\]

But \( p_{\sigma(j)\sigma(i)}^{\sigma(k)} = p_{ij}^{kl} \) (from condition (GL6)) and \( p_{\sigma(j)l}^{\sigma(i)} = p_{ij}^{kl} \) (from condition (GL6) and from the fact that \( V_{\sigma(j)} = V_j^I \)).

Therefore, we have that \( \sum_{i \in I} p_{ij}^{kl} = \delta_{k,l} \) and we are done.

The following result follows directly from corollary 6 and proposition 7.

Corollary 5. There are biFrobenius algebras of unbounded dimension satisfying

\[
\Delta(tx) = \sum t_1 x_1 \otimes t_2 x_2 \text{ for a right integral } t \neq 0 \text{ and that are not bialgebras.}
\]

6. ON SEPARABILITY AND COSEPARABILITY OF BI-FROBENIUS ALGEBRAS.

The notion of separability is classical in ring theory. Every separable algebra is semisimple. Indeed, a \( k \)-algebra \( A \) is separable if and only if for any field extension \( E \supseteq k \), the \( E \)-algebra \( A^E = A \otimes_k E \) is semisimple.

We use an alternative definition -see Definition 8- of a separable algebra and give also the (dual) notion of a coseparable coalgebra (see [DMI] and [T]).

In this section, if \( A \) is an algebra and \( C \) is a coalgebra, we consider

- \( A \) with the usual structure of \( A \times A \) bimodule,
- \( A \otimes A \) with the structure of \( A \times A \) bimodule given by \( a(x \otimes y)b = ax \otimes yb \),
- \( C \) with the usual structure of \( C \times C \) bimodule,
- \( C \otimes C \) with the structure of \( C \times C \) bimodule given by \( \sum (x \otimes y)_{-1} \otimes (x \otimes y)_0 \otimes (x \otimes y) = \sum x_1 \otimes x_2 \otimes y_1 \otimes y_2 \).

It is evident that the product \( m : A \otimes A \rightarrow A \) is an epimorphism of \( A \times A \) bimodules and that the coproduct \( \Delta : C \rightarrow C \otimes C \) is a monomorphism of \( C \times C \) bimodules.

Definition 8. (1) A \( k \)-algebra \( A \) is said to be separable if the product \( m : A \otimes A \rightarrow A \) splits in the category of \( A \times A \) bimodules.

(2) A \( k \)-coalgebra \( C \) is said to be coseparable if the coproduct \( \Delta : C \rightarrow C \otimes C \) splits in the category of \( C \times C \) bimodules.

The following result is well known and can be found for example in [S].
**Theorem 6.** Let \( H \) be a finite dimensional Hopf algebra and \( t \in H \) be a non-zero right or left integral. Then the following statements are equivalent.

(i) \( H \) is semisimple.

(ii) \( H \) is separable.

(iii) \( \varepsilon(t) \not= 0. \)

We give a proof of this result in the context of biFrobenius algebras satisfying \( S \ast id = u \varepsilon \). We refer to [D] for a slightly different proof of \( (c) \rightarrow (b) \). We also state its dual version.

**Theorem 7.** Let \((A, m, 1, \Delta, \varepsilon, t, \phi, S)\) be a biFrobenius algebra such that \( S \ast id = id \ast S = u \varepsilon \). Then:

1. The following assertions are equivalent:
   (a) \( A \) is semisimple.
   (b) \( A \) is separable.
   (c) \( \varepsilon(t) \not= 0. \)

2. The following assertions are equivalent:
   (a') \( A \) is cosemisimple.
   (b') \( A \) is coseparable.
   (c') \( \phi(1) \not= 0. \)

**Proof:**

1. For (a) implies (c) take ker\( \varepsilon \) \( \subseteq A \) as a right \( A \)-submodule. As \( A \) is semisimple, there is a right submodule \( I \subseteq A \) of dimension 1 such that ker\( \varepsilon \) \( \oplus I = A \). We write \( 1 = a + s \), with \( \varepsilon(a) = 0 \) and \( s \in I \), then \( I = k s \) and \( \varepsilon(s) = 1 \). If \( x \in A \), we have \( sx \in I \) and then \( sx = \lambda s \), for some \( \lambda \in k \). Applying \( \varepsilon \) we deduce that \( \varepsilon(x) = \lambda \) and therefore \( \varepsilon(x) s \), which means that \( s \) is a right integral for \( A \). Hence, \( t = \mu s \in I \), for some \( \mu \in k, \mu \not= 0 \), which implies \( \varepsilon(t) \not= 0. \)

For (c) implies (b), take \( S \) the composite inverse of the identity, \( s = \frac{1}{\varepsilon(t)} t \) and \( \psi \in A^* \) such that \( (s, \psi) \) is a biFrobenius pair. Consider \( \delta : A \rightarrow A \otimes A, \delta(x) = \sum x \bar{S}(s_2) \otimes s_1. \) From the equality \( \sum x \bar{S}(s_2)s_1 = x \), it follows that \( \delta \) splits \( m \). We have to prove that \( \delta \) is a morphism of \( A - A \) bimodules; in other words that \( \sum a x b \bar{S}(s_2) \otimes s_1 = \sum a x \bar{S}(s_2) \otimes s_1 b. \) In order to do this, it is enough to show that

\[
\sum b \bar{S}(s_2) \otimes s_1 = \sum \bar{S}(s_2) \otimes s_1 b,
\]

which is equivalent, by applying the morphism \( id \otimes \psi \rightarrow : A \otimes A \rightarrow A \otimes A^* \) to the equality \( b \bar{S}(s_2)p(s_1z) = \bar{S}(s_2)\psi(s_1bz), \forall z \in A, \) or \( b \bar{S}(s) = S(S(bz)), \forall z \in A. \) This last equality is obviously true.

Finally, let us prove that (b) implies (a). Take \( \delta \) the map that splits the product and put \( \delta(1) = \sum_{i=1}^l r_i \otimes r_i. \) We have \( \sum r_i r_i = 1 \) and \( \sum a r_i \otimes r_i = a \delta(1) = \delta(a) = \sum r_i \otimes r_i a. \)

Given \( N \subseteq M \) an inclusion of \( A \)-modules, consider a \( k \)-linear map \( p : M \rightarrow N \) such that \( p_N = id_N. \) The map

\[
\pi : M \rightarrow N,
\pi(x) = \sum_{i=1}^l p(x r_i) r_i.
\]

is a morphism of \( A \)-modules that splits the projection. Indeed, for \( x \in N, \)
\( x r_i \in N, \forall i = 1, 2, \cdots, l \) and therefore \( \pi(x) = \sum x r_i r_i = x \sum r_i r_i = x. \)
Moreover $\pi$ is a morphism of right $A$-modules, since
\[ \pi(xa) = \sum p(xar_i)r^i = \sum p(xr_i)r^i a = \pi(x)a. \]

(2) The proof of this equivalence is obtained by dualizing the methods described above. We only present a sketch in order to obtain explicit expressions for the morphisms involved.

The proof that (a') implies (c') is obtained from (1), working in $A^*$. In the proof that (c') implies (b'), in order to split $\Delta$ in the category of $A-A$ bicomodules, we take the map $*: A \otimes A \to A$ given by $x* y = \sum \psi(Sxy_1)y_2$, where $\psi \in A^*$ is a right cointegral such that $\psi(1) = 1$.

Finally, to prove that (b') implies (c') we take a map $*: A \otimes A \to A$ that splits $\Delta$. If $N \subseteq M$ is a right $A$-subcomodule and $p: M \to N$ is a linear map that splits the inclusion, we consider $\pi: M \to N, \pi(x) = \sum p(m_0) \otimes \varepsilon (p(m_0)1 \otimes m_1)$.

□

For the particular case of group-like algebras, the following version of Maschke’s theorem can be easily deduced.

**Corollary 6.** Every group-like algebra is coseparable.

A group-like algebra is separable if and only if $\sum_{i=0}^n \varepsilon(b_i) = 0$.

In particular a group-like algebra with $S \ast id = \varepsilon$ is separable if and only if $\text{char} k$ does not divide its dimension.

**References**

[A] E.Abe, *Hopf Algebras*, Cambridge University Press, 1980, Translated from the Japanese by H. Kinoshita and H. Tanaka.

[DN] S.Dascalescu, C.Nastasescu, S.Raianu, *Hopf algebras: an introduction*, Marcel Dekker ed., New York, 2001.

[D] Y.Doi, *Bi-Frobenius algebras and Group-Like algebras*, Lecture Notes in Pure and Applied Mathematics, Vol.237, pp 143-145, ed. J.Bergen, S.Catoiu, W.Chin, Marcel Dekker, 2001.

[DMI] F.DeMeyer, E.Ingraham, *Separable Algebras over commutative rings*, Springer-Verlag, Berlin, New York, 1971.

[DT] Y.Doi and M.Takeuchi, *Bi-Frobenius algebras*, Contemporary Mathematics, Vol.267, pp 67-97, ed. N.Andruskiewitsch, W.Ferrer Santos, H.-J Schneider, Amer. Math. Soc., 2000.

[F] W.Ferrer Santos, *Fourier Theory for coalgebras, bicointegrals and injectivity for bicomodules*, Lecture Notes in Pure and Applied Mathematics, Vol. 237, pp 225-240, ed. J.Bergen, S.Catoiu, W.Chin, Marcel Dekker, 2001.

[S] H.-J.Schneider, *Lectures on Hopf Algebras*, notes by S. Natale, Trabajos de Matemática, Vol. 31/95, FaMAF, 1995.

[Sl] N.Sloane, *A Library of Hadamard Matrix*, AT&T Shannon Lab, [http://www.research.att.com/~njas/hadamard/](http://www.research.att.com/~njas/hadamard/)

[T] B.Torrecillas, F.Van oystaeyen and Y.H.Zhang, *The Brauer Group of a Cocommutative Coalgebra*, Journal of Algebra, Vol. 177, pp 536-568, 1995.

[W] E.Weisstein, *Hadamard Matrix*, MathWorld-A Wolfram Web Resource, [http://mathworld.wolfram.com/HadamardMatrix.html](http://mathworld.wolfram.com/HadamardMatrix.html)

[WZ] Y.Whang, P.Zhang, *Construct biFrobenius algebras via quivers*, Tsukuba Journal of Mathematics, Vol 28, 1, pp 215-227, 2004.
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