STRONG CONVERGENCE
OF TWO-DIMENSIONAL WALSH–FOURIER SERIES

We prove that certain means of quadratic partial sums of the two-dimensional Walsh–Fourier series are uniformly bounded operators acting from the Hardy space $H_p$ to the space $L_p$ for $0 < p < 1$.

1. Introduction. It is known [7, p. 125] that the Walsh–Paley system is not a Schauder basis in $L_1(G)$. Moreover (see [8]), there exists a function in the dyadic Hardy space $H_1(G)$, the partial sums of which are not bounded in $L_1(G)$. However, in Simon [9] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the $k$th partial sum of the Walsh–Fourier series of $f$ (for the trigonometric analogue see Smith [11], for the Vilenkin system see Gáat [1]).

Simon [10] proved that there is an absolute constant $c_p$, depends only $p$, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p,$$

for all $f \in H_p$, where $0 < p < 1$.

The author [13] proved that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ in inequality (1) is important.

For the two-dimensional Walsh–Fourier series Weisz [16] generalized the result of Simon and proved that if $\alpha \geq 0$ and $f \in H_p(G \times G)$, then

$$\sup_{n,m \geq 2} \left( \frac{1}{\log n \log m} \right)^{[p]} \sum_{2^{-n} \leq k,l \leq 2^{\alpha}, (k,l) \leq (n,m)} \frac{\|S_{k,l} f\|_p^p}{(kl)^{2-p}} \leq c \|f\|_{H_p}^p,$$

where $0 < p < 1$ and $[p]$ denotes the integer part of $p$.

Goginava and Gogoladze [5] proved that the following result is true:

**Theorem G.** Let $f \in H_1(G \times G)$. Then there exists absolute constant $c$, such that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n} f\|_1}{n \log^2 n} \leq c \|f\|_{H_1}.$$

For two-dimensional trigonometric system analogue theorem was proved in [6].

Convergence of quadratical partial sums of two-dimensional Walsh–Fourier series was investigated in details by Weisz [15], Goginava [4], Gáat, Goginava, Nagy [2], Gáat, Goginava, Tkebuchava [3].

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The main aim of this paper is to prove (see Theorem 1) that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}} \leq c_p \|f\|_{H_p}^p,$$  

for all $f \in H_p(G \times G)$, where $0 < p < 1$. We also proved that sequence $\{1/n^{3-2p}\}_{n=1}^{\infty}$ in inequality (2) is important (see Theorem 2).

2. Definitions and notations. Let $P$ denote the set of positive integers, $N := P \cup \{0\}$. Denote by $Z_2$ the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $Z_2$ is given such that the measure of a singleton is 1/2. Let $G$ be the complete direct product of the countable infinite copies of the compact groups $Z_2$. The elements of $G$ are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$, $k \in N$. The group operation on $G$ is the coordinate-wise addition, the measure (denote by $\mu$) and the topology are the product measure and topology. The compact Abelian group $G$ is called the Walsh group. A base for the neighborhoods of $G$ can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots) \}, \quad x \in G, \ n \in N.$$

These sets are called the dyadic intervals. Let $0 = (0: i \in N) \in G$ denote the null element of $G$, $I_n := I_n(0), n \in N$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the $n$th coordinate of which is 1 and the rest are zeros ($n \in N$). Let $T_n := G \setminus I_n$.

If $n \in N$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}, i \in N$, i.e., $n$ is expressed in the number system of base 2. Denote $|n| := \max\{j \in N : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|}+1$.

It is easy to show that for every odd number $n_0 = 1$ and we can write $n = 1 + \sum_{i=1}^{|n|} n_j 2^i$, where $n_j \in \{0, 1\}, j \in P$.

For $k \in N$ and $x \in G$ let as denote by

$$r_k(x) := (-1)^{x_k}, \quad x \in G, \ k \in N,$$

the $k$th Rademacher function.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k}, \quad x \in G, \ n \in P.$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [8, p. 7])

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\[ D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & x \in \overline{I}_n. \end{cases} \]  

(3)

Furthermore, the following representation holds for the \( D_n \)'s. Let \( n \in \mathbb{N} \) and \( n = \sum_{i=0}^{\infty} n_i 2^i \).

Then

\[ D_n(x) = w_n(x) \sum_{j=0}^{\infty} n_j D_{2^j}(x). \]  

(4)

The rectangular partial sums of the 2-dimensional Walsh–Fourier series of function \( f \in L^2(G \times G) \) are defined as follows:

\[ S_{M,N} f(x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i,j) w_i(x) w_j(y), \]

where the numbers

\[ \hat{f}(i,j) = \int_{G \times G} f(x,y) w_i(x) w_j(y) \, d\mu(x,y) \]

is said to be the \((i,j)\)th Walsh–Fourier coefficient of the function \( f \).

Denote

\[ S^{(1)}_{M} f(x,y) := \int_{G} f(s,y) D_M(x+s) \, d\mu(s) \]

and

\[ S^{(2)}_{N} f(x,y) := \int_{G} f(x,t) D_N(y+t) \, d\mu(t). \]

The norm (or quasinorm) of the space \( L^p(G \times G) \) is defined by

\[ \| f \|_p := \left( \int_{G \times G} |f|^p \, d\mu \right)^{1/p}, \quad 0 < p < \infty. \]

The space \( \text{weak} - L^p(G \times G) \) consists of all measurable functions \( f \) for which

\[ \| f \|_{\text{weak} - L^p(G \times G)} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty. \]

The \( \sigma \)-algebra generated by the dyadic 2-dimensional \( I_n(x) \times I_n(y) \) square of measure \( 2^{-n} \times 2^{-n} \) will be denoted by \( F_{n,n}, \, n \in \mathbb{N} \). Denote by \( f = (f_{n,n}, \, n \in \mathbb{N}) \) one-parameter martingale with respect to \( F_{n,n}, \, n \in \mathbb{N} \).

The expectation operator and the conditional expectation operator relative to the \( F_{n,n}, \, n \in \mathbb{N} \), are denoted by \( E \) and \( E_{n,n} \), respectively.

The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in \mathbb{N}} |f_{n,n}|. \]

Let \( f \in L^1(G \times G) \). Then the dyadic maximal function is given by
f^*(x, y) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x) \times I_n(y))} \left| \int_{I_n(x) \times I_n(y)} f(s, t) \, d\mu(s, t) \right|, \quad (x, y) \in G \times G.

The dyadic Hardy space \( H_p(G \times G) (0 < p < \infty) \) consists of all functions for which
\[
\|f\|_{H_p} := \|f^*\|_p < \infty.
\]

If \( f \in L_1(G \times G) \), then it is easy to show that the sequence \( (S_{2^n, 2^n} f): n \in \mathbb{N} \) is a martingale. If \( f = (f_{n,n}, n \in \mathbb{N}) \) is a martingale, then the Walsh–Fourier coefficients must be defined in a slightly different manner:
\[
\hat{f}(i, j) := \lim_{k \to \infty} \int_G f_{k,k}(x, y) w_i(x) w_j(y) \, d\mu(x, y).
\]

It is known [12] that that Fourier coefficients of \( f \in H_p(G \times G) \) are not bounded when \( 0 < p < 1 \).

The Walsh–Fourier coefficients of \( f \in L_1(G \times G) \) are the same as those of the martingale \( (S_{2^n, 2^n} f): n \in \mathbb{N} \) obtained from \( f \).

A bounded measurable function \( a \) is a \( p \)-atom, if there exists a dyadic 2-dimensional cube \( I \times I \), such that

\begin{enumerate}
\item \( \int_I a \, d\mu = 0 \),
\item \( \|a\|_\infty \leq \mu(I \times I)^{-1/p} \),
\item \( \text{supp}(a) \subset I \times I \).
\end{enumerate}

3. Formulation of main results.

Theorem 1. Let \( 0 < p < 1 \) and \( f \in H_p(G \times G) \). Then
\[
\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}} \leq c_p \|f\|_{H_p}^p.
\]

Theorem 2. Let \( 0 < p < 1 \) and \( \Phi: \mathbb{N} \to [1, \infty) \) is any nondecreasing function, satisfying the condition \( \lim_{n \to \infty} \Phi(n) = +\infty \). Then there exists a martingale \( f \in H_p(G \times G) \) such that
\[
\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_{\text{weak}-L_p}^p \Phi(n)}{n^{3-2p}} = \infty.
\]

4. Auxiliary propositions.

Lemma 1 [14]. A martingale \( f \in L_p(G \times G) \) is in \( H_p(G \times G) \), \( 0 < p \leq 1 \), if and only if there exist a sequence \( (a_k, k \in \mathbb{N}) \) of \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{N}) \) of a real numbers such that
\[
\sum_{k=0}^{\infty} \mu_k E_{n,n} a_k = f_{n,n}
\]
and
\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover, \( \|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \), where the infimum is taken over all decomposition of \( f \) of the form (5).
5. Proof of the theorems. Proof of Theorem 1. If we apply Lemma 1 we only have to prove that
\[ \sum_{n=1}^{\infty} \frac{\| S_{n,n}a \|^p}{n^{3-2p}} \leq c_p < \infty, \]
for every \( p \) atom \( a \).

Let \( a \) be an arbitrary \( p \)-atom with support \( I_N (z') \times I_N (z'') \) and \( \mu (I_N) = \mu (I_N) = 2^{-N} \). We can suppose that \( z' = z'' = 0 \).

Let \( (x, y) \in \mathcal{I}_N \times \mathcal{I}_N \). In this case \( D_{2i} (x + s) 1_{I_N} (s) = 0 \) and \( D_{2i} (y + t) 1_{I_N} (t) = 0 \) for \( i \geq N \). Recall that \( w_{2j} (x + t) = w_{2j} (x) \) for \( t \in I_N \) and \( j < N \). Consequently, from (4) we obtain
\[
S_{n,n}a (x, y) =
\]
\[
= \int_{G \times G} a (s, t) D_n (x + s) D_n (y + t) d\mu (s, t) =
\]
\[
= \int_{I_N \times I_N} a (s, t) D_n (x + s) D_n (y + t) d\mu (s, t) =
\]
\[
= \int_{I_N \times I_N} a (s, t) w_n (x + s + y + t) \sum_{i=0}^{N-1} n_i w_{2i} (x + s) D_{2i} (x + s) \times
\]
\[
\times \sum_{j=0}^{N-1} n_j w_{2j} (y + t) D_{2j} (y + t) d\mu (s, t) =
\]
\[
= w_n (x) \sum_{i=0}^{N-1} n_i w_{2i} (x) D_{2i} (x) w_n (y) \sum_{j=0}^{N-1} n_j w_{2j} (y) D_{2j} (y) \times
\]
\[
\times \int_{I_N \times I_N} a (s, t) w_n (s + t) d\mu (s, t) =
\]
\[
= w_n (x + y) \sum_{i=0}^{N-1} n_i w_{2i} (x) D_{2i} (x) \sum_{j=0}^{N-1} n_j w_{2j} (y) D_{2j} (y) \times
\]
\[
\times \int_{I_N} \left( \int_{I_N} a (t + \tau, t) d\mu (t) \right) w_n (\tau) d\mu (\tau) =
\]
\[
= w_n (x + y) \sum_{i=0}^{N-1} n_i w_{2i} (x) D_{2i} (x) \sum_{j=0}^{N-1} n_j w_{2j} (y) D_{2j} (y) \int_{I_N} \Phi (\tau) w_n (\tau) d\mu (\tau) =
\]

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\[ w_n (x + y) \sum_{i=0}^{N-1} n_i w_{2i} (x) D_{2i} (x) \sum_{j=0}^{N-1} n_j w_{2j} (y) D_{2j} (y) \hat{\Phi} (n) , \]

where

\[ \Phi (\tau) = \int_{I_N} a (t + \tau, t) \, d\mu (t) . \]

Let \( x \in I_s \setminus I_{s+1} \). Using (3) we get

\[ \sum_{i=0}^{N-1} D_{2i} (x) \leq c 2^s . \]

Since

\[ I_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1} \]

we obtain

\[ \int_{T_N} \left( \sum_{i=0}^{N-1} D_{2i} (x) \right)^p \, d\mu (x) \leq c_p \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} 2^{ps} \, d\mu (x) \leq c_p \sum_{s=0}^{\infty} 2^{(p-1)s} < c_p < \infty , \quad 0 < p < 1, \] (7)

applying (7) we can write

\[ \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{T_N \times I_N} |S_{n,n} a(x,y)|^p \, d\mu(x,y) \leq \]

\[ \leq \int_{I_N} \left( \sum_{i=0}^{N-1} D_{2i} (x) \right)^p \, d\mu (x) \leq \]

\[ \leq c_p \sum_{n=1}^{\infty} \left| \hat{\Phi} (n) \right|^p . \]

Let \( n < 2^N \). Since \( w_n (\tau) = 1, \) for \( \tau \in I_N \) we have

\[ \hat{\Phi} (n) = \int_{I_N} \Phi (\tau) w_n (\tau) \, d\mu (\tau) = \]

\[ = \int_{I_N} \left( \int_{I_N} a (t + \tau, t) \, d\mu (t) \right) w_n (\tau) \, d\mu (\tau) = \]

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= \int_{I_N \times I_N} a(s, t) \, d\mu(s, t) = 0.

Hence, we can suppose that \( n \geq 2^N \). By Hölder inequality we obtain

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \left| \Phi(n) \right|^p \leq \left( \sum_{n=2^N}^{\infty} \left| \Phi(n) \right|^2 \right)^{p/2} \left( \sum_{n=2^N}^{\infty} \frac{1}{n(3-2p)/(2(2-p))} \right)^{(2-p)/2} \leq \left( \frac{1}{2^N(2(3-2p)/(2-p)-1)} \right)^{(2-p)/2} \left( \int_G \Phi(\tau)^2 \, d\mu(\tau) \right)^{p/2} \leq \frac{c_p}{2^{N(4-3p)/2}} \left( \int_{I_N \times I_N} a(t+\tau, t) \, d\mu(t) \right)^{p/2} \leq \frac{c_p}{2^{N(4-3p)/2}} \left( \int_{I_N \times I_N} |\Phi(\tau)|^2 \, d\mu(\tau) \right)^{p/2} \leq \frac{c_p}{2^{N(4-3p)/2}} \frac{1}{2^Np/2} \frac{1}{2^Np} \leq \frac{c_p}{2^{N(4-3p)/2}} 2^{2N} \frac{1}{2^{Np/2}} < c_p < \infty.
\]

Let \((x, y) \in \bar{T}_N \times I_N\). Then we have

\[
S_{n,n}a(x, y) = w_n(x) \sum_{j=0}^{N-1} n_j w_{2j}(x) D_{2j}(x) \times \\
G \times G \\
= w_n(x) \sum_{j=0}^{N-1} n_j w_{2j}(x) D_{2j}(x) \int_G S_{n}^{(2)}(a(s, y) w_n(s) \, d\mu(s) = \\
= w_n(x) \sum_{j=0}^{N-1} n_j w_{2j}(x) D_{2j}(x) \tilde{S}_{n}^{(2)}(a(n, y).
\]

Using (7) we get

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{\bar{T}_N \times I_N} |S_{n,n}a(x, y)|^p \, d\mu(x, y) \leq \\
\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{\bar{T}_N \times I_N} \left( \sum_{j=0}^{N-1} D_{2j}(x) \left| \tilde{S}_{n}^{(2)}(a(n, y) \right| \right)^p \, d\mu(x, y) \leq
\]

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\[ \leq \sum_{n=1}^{\infty} \frac{1}{2^{3-2p}} \int_{I_N} \left( \sum_{i=0}^{N-1} D_{2^i} (x) \right)^p \cdot \int_{I_N} \left| \hat{S}_{n}^{(2)} a (n, y) \right|^p d\mu (y) \leq \]

\[ \leq \sum_{n=1}^{\infty} \frac{1}{2^{3-2p}} \int_{I_N} \left| \hat{S}_{n}^{(2)} a (n, y) \right|^p d\mu (y). \]

Let \( n < 2^N \). Then by the definition of the atom we have

\[ \hat{S}_{n}^{(2)} a (n, y) = \int_{G} \int_{G} a (s, t) D_{n} (y + t) d\mu (t) w_{n} (s) d\mu (s) = \]

\[ = D_{n} (y) \int_{I_N \times I_N} a (s, t) d\mu (s, t) = 0. \]

Therefore, we can suppose that \( n \geq 2^N \). Hence

\[ \sum_{n=1}^{\infty} \frac{1}{2^{3-2p}} \int_{T_N \times I_N} |S_{n,n} a(x,y)|^p d\mu(x,y) \leq \]

\[ \leq \sum_{n=2^N}^{\infty} \frac{1}{2^{3-2p}} \int_{I_N} \left| \hat{S}_{n}^{(2)} a (n, y) \right|^p d\mu (y). \]

Since

\[ \left\| S_{n}^{(2)} a (n, y) \right\|_2 \leq c \| a \|_2 \]

from Hölder inequality we can write

\[ \int_{I_N} \left| \hat{S}_{n}^{(2)} a (n, y) \right|^p d\mu (y) \leq \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \left| \hat{S}_{n}^{(2)} a (n, y) \right| d\mu (y) \right)^p = \]

\[ = \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \int_{I_N} S_{n}^{(2)} a (s, y) w_{n} (s) d\mu (s) d\mu (y) \right)^p = \]

\[ = \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \int_{I_N} a (s, t) D_{n} (y + t) d\mu (t) w_{n} (s) d\mu (s) d\mu (y) \right)^p \leq \]

\[ \leq \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \int_{I_N} a (s, t) D_{n} (y + t) d\mu (t) d\mu (y) d\mu (s) \right)^p \leq \]
\[
\leq \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \left( \int_{I_N} |a(s,t) D_n(y + t) d\mu(t)|^2 d\mu(y) \right)^{1/2} d\mu(s) \right)^p \leq \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} |a(s,t)|^2 d\mu(t) \right)^{1/2} \leq \frac{c_p}{2^{N(1-p)}} \left( \frac{2^{2N/\mu}}{2^{2N}} \right)^p \leq c_p 2^{N(1-p)}.
\]

Consequently,
\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n}a(x, y)| d\mu(x, y) \leq c_p \sum_{n=2^N}^{\infty} \frac{1}{n^{3-2p}} 2^{N(1-p)} \leq \frac{c_p}{2^{N(1-p)}} \leq c_p < \infty.
\]

(9)

Analogously, we can prove that
\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n}a(x, y)|^p d\mu(x, y) \leq c_p < \infty.
\]

(10)

Let \((x, y) \in I_N \times I_N\). Then by the definition of the atom we can write
\[
\int_{I_N \times I_N} |S_{n,n}a(x, y)|^p d\mu(x, y) \leq
\]
\[
\leq \frac{1}{2^{N(2-p)}} \left( \int_{I_N \times I_N} |S_{n,n}a(x, y)|^2 d\mu(x, y) \right)^{p/2} \leq \frac{1}{2^{N(2-p)}} \left( \int_{I_N \times I_N} |a(x, y)|^2 d\mu(x, y) \right)^{p/2} \leq \frac{\|a\|_\infty^p}{2^{N(2-p)}} \leq c_p \frac{1}{2^{N(2-p)}} 2^{N} \frac{1}{2^{Np}} \leq c_p < \infty.
\]

(9)

It follows that...
\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n}a(x,y)| \ d\mu(x,y) \leq c_p \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \leq c_p < \infty. \tag{11}
\]

Combining (6) – (11) we complete the proof of Theorem 1.

**Proof of Theorem 2.** Let \( 0 < p < 1 \) and \( \Phi(n) \) is any nondecreasing, nonnegative function, satisfying condition

\[
\lim_{n \to \infty} \Phi(n) = \infty.
\]

For this function \( \Phi(n) \), there exists an increasing sequence of the positive integers \( \{ \alpha_k : k \geq 0 \} \) such that:

\[
\alpha_0 \geq 2
\]

and

\[
\sum_{k=0}^{\infty} \frac{1}{\Phi^{p/4}(2^{\alpha_k})} < \infty. \tag{12}
\]

Let

\[
f_{A,A}(x,y) = \sum_{\{k : \alpha_k < A\}} \lambda_k a_k,
\]

where

\[
\lambda_k = \frac{1}{\Phi^{1/4}(2^{\alpha_k})}
\]

and

\[
a_k(x,y) = 2^{\alpha_k(2/p-2)} (D_{2^{\alpha_k+1}}(x) - D_{2^{\alpha_k}}(x)) (D_{2^{\alpha_k+1}}(y) - D_{2^{\alpha_k}}(y)).
\]

It is easy to show that the martingale \( f = (f_{1,1}, f_{2,2}, \ldots, f_{A,A}, \ldots) \in H_p \).

Indeed, since

\[
S_{2^{\alpha_k}}a_k(x,y) = \begin{cases} 
    a_k(x,y), & \alpha_k < A, \\
    0, & \alpha_k \geq A,
\end{cases}
\]

\[
supp(a_k) = I_{\alpha_k},
\]

\[
\int_{I_{\alpha_k}} a_k d\mu = 0
\]

and

\[
\|a_k\|_\infty \leq 2^{\alpha_k(2/p-2)} 2^{2\alpha_k} \leq 2^{2\alpha_k/p} = (supp(a_k))^{-1/p}
\]

from Lemma 1 and (12) we conclude that \( f \in H_p \).

It is easy to show that

\[
\tilde{f}(i,j) = \begin{cases} 
    2^{\alpha_k(2/p-2)} \Phi^{1/4}(2^{\alpha_k}), & \text{if } (i,j) \in \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\} \times \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\}, \\
    0, & \text{if } (i,j) \notin \bigcup_{k=0}^{\infty} \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\} \times \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\}.
\end{cases}
\]

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Let $2^{\alpha_k} < n < 2^{\alpha_k+1}$. From (14) we have

\[
S_{n,n}f(x, y) = \sum_{i=0}^{2^{\alpha_k-1}-1} \sum_{j=0}^{2^{\alpha_k-1}-1} \hat{f}(i, j) w_i (x) w_j (y) + \\
+ \sum_{i=2^{\alpha_k}}^{n-1} \sum_{j=2^{\alpha_k}}^{n-1} \hat{f}(i, j) w_i (x) w_j (y) = \\
= \sum_{\eta=0}^{k-1} \sum_{i=2^{\alpha\eta}}^{2^{\alpha\eta+1}-1} \sum_{j=2^{\alpha\eta}}^{2^{\alpha\eta+1}-1} 2^{\alpha\eta}(2/p-2) \frac{\Phi^{1/4}(2^{\alpha\eta})}{\Phi} w_i (x) w_j (y) + \\
+ \sum_{i=2^{\alpha_k}}^{n-1} \sum_{j=2^{\alpha_k}}^{n-1} 2^{\alpha_k}(2/p-2) \frac{\Phi^{1/4}(2^{\alpha_k})}{\Phi} w_i (x) w_j (y) = \\
= \sum_{\eta=0}^{k-1} 2^{\alpha\eta}(2/p-2) \frac{\Phi^{1/4}(2^{\alpha\eta})}{\Phi} (D_{2^{\alpha\eta+1}} (x) - D_{2^{\alpha\eta}} (x)) (D_{2^{\alpha\eta+1}} (y) - D_{2^{\alpha\eta}} (y)) + \\
+ \frac{2^{\alpha_k}(2/p-2)}{\Phi^{1/4}(2^{\alpha_k})} (D_n (x) - D_{2^{\alpha_k}} (x)) (D_n (y) - D_{2^{\alpha_k}} (y)) = \\
= I + II. \quad (15)
\]

Let $(x, y) \in (G \setminus I_1) \times (G \setminus I_1)$ and $n$ is odd number. Since $n - 2^{\alpha_k}$ is odd number too and

\[
D_{n+2^{\alpha_k}} (x) = D_{2^{\alpha_k}} (x) + w_{2^{\alpha_k}} (x) D_n (x), \quad \text{when} \quad n < 2^{\alpha_k},
\]

from (3) and (4) we can write

\[
|II| = \frac{2^{\alpha_k}(2/p-2)}{\Phi^{1/4}(2^{\alpha_k})} |w_{2^{\alpha_k}} (x) D_{n-2^{\alpha_k}} (x) w_{2^{\alpha_k}} (y) D_{n-2^{\alpha_k}} (y)| = \\
= \frac{2^{\alpha_k}(2/p-2)}{\Phi^{1/4}(2^{\alpha_k})} |w_{2^{\alpha_k}} (x) w_{n-2^{\alpha_k}} (x) D_1 (x) w_{2^{\alpha_k}} (y) w_{n-2^{\alpha_k}} (y) D_1 (y)| = \\
= \frac{2^{\alpha_k}(2/p-2)}{\Phi^{1/4}(2^{\alpha_k})}. \quad (16)
\]
Applying (3) and condition $\alpha_n \geq 2 \ (n \in \mathbb{N})$ for $I$ we have

$$I = \sum_{y=0}^{k-1} \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})} (D_{2^{\alpha_y+1}}(x) - D_{2^{\alpha_y}}(x))(D_{2^{\alpha_y+1}}(y) - D_{2^{\alpha_y}}(y)) = 0.$$  \hspace{1cm} (17)

Hence

$$\|S_{n,n}f(x,y)\|_{\text{weak-}L_p} \geq \frac{2^{\alpha_k(2/p-2)}}{2\Phi^{1/4}(2^{\alpha_k})} \mu\left\{(x,y) \in (G \setminus I_1) \times (G \setminus I_1) : |S_{n,n}f(x,y)| \geq \frac{2^{\alpha_k(2/p-2)}}{2\Phi^{1/4}(2^{\alpha_k})}\right\}^{1/p} \geq \frac{2^{\alpha_k(2/p-2)}}{2\Phi^{1/4}(2^{\alpha_k})}|(G \setminus I_1) \times (G \setminus I_1)| \geq c_p \frac{2^{\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{\alpha_k})}. \hspace{1cm} (18)$$

Using (18) we have

$$\sum_{n=1}^{2^{\alpha_k}+1-1} \frac{n^{3-2p}}{\Phi(n)} \leq \sum_{n=2^{\alpha_k}+1}^{2^{\alpha_k}+1-1} \frac{n^{3-2p}}{\Phi(n)} \geq c_p \Phi^{3/4}(2^{\alpha_k}) \rightarrow \infty, \quad \text{when} \quad k \rightarrow \infty. \hspace{1cm} (19)$$

Combining (12)–(19) we complete the proof of Theorem 2.

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