INVERSE-CLOSED BANACH SUBALGEBRAS OF HIGHER-DIMENSIONAL NONCOMMUTATIVE TORI

KARLHEINZ GRÖCHENIG AND MICHAEL LEINERT

Abstract. We give a systematic construction of inverse-closed (Banach) subalgebras in general higher-dimensional non-commutative tori.

1. Introduction

Let \( A \subseteq B \) be two algebras with common identity. Then \( A \) is called inverse-closed in \( B \), if \( a \in A \) and \( a^{-1} \in B \) implies that \( a^{-1} \in A \). This property is a generalization of Wiener’s Lemma for absolutely convergent Fourier series and occurs abundantly in many branches of mathematical analysis. The range of applications covers numerical analysis, pseudodifferential operators, frame theory, and Banach algebras occur in numerical analysis last not least non-commutative tori. See [7] for a survey of many versions of Wiener’s Lemma and applications of inverse-closedness.

The main result concerning inverse-closed subalgebras of non-commutative tori is the density theorem. It states that the \( K \)-groups of a non-commutative torus and of all its inverse-closed subalgebras are isomorphic. Similarly the stable rank of an inverse-closed subalgebra coincides with the stable rank of the large algebra [2]. Usually the existence of an inverse-closed subalgebra is taken for granted and is the starting point for the theory. Also, mostly Fréchet subalgebras are considered rather than Banach subalgebras (because Fréchet algebras model “smooth” noncommutative tori).

In this paper we investigate a systematic construction of Banach subalgebras of non-commutative tori in higher dimensions. We will characterize all inverse-closed Banach subalgebras of the form \( \ell_v^1 \), where \( v \) is a weight function on the non-commutative torus. By choosing weights of subexponential growth, we even construct a Banach subalgebra that is contained in the ordinary smooth noncommutative torus. For certain noncommutative tori with an even number of generators these results were already obtained in [8]. This work is motivated by a question of N. C. Phillips at the workshop on “Operator Methods, Fractals, and Wavelets” in Banff 2006. He asked whether the results hold for arbitrary non-commutative tori in higher dimensions.

2000 Mathematics Subject Classification. 46L85,22E25,43A20.

Key words and phrases. Non-commutative torus, twisted convolution, GRS condition, inverse-closed, spectral invariance, enveloping \( C^* \)-algebra.

K. G. was supported in part by the project P22746-N13 of the Austrian Science Foundation (FWF) and by the Marie-Curie Excellence Grant MEXT-CT 2004-517154.
Our methods are drawn from abstract harmonic analysis, in particular the investigation of projective representations and twisted convolution algebras in the school of Leptin and Ludwig. Our arguments are simple, but based on rather deep results in harmonic analysis.

Let us mention that in some areas an inverse-closed subalgebra is also called a spectral subalgebra, a local subalgebra, or a full algebra. If \( A \) is inverse-closed in \( B \), then \( A \) is called spectrally invariant in \( B \) or invariant under holomorphic calculus; \( (A,B) \) is called a Wiener pair.

2. \textbf{Higher-Dimensional Non-Commutative Tori}

We first give a description of non-commutative tori in higher dimensions and explain the link to harmonic analysis.

Let \( T \) denote the unit circle. Let \( U_1, \ldots, U_n \) be unitary symbols satisfying the commutation relations

\[
U_j U_k = \theta_{jk} U_k U_j,
\]

where \( \theta_{jk} \in \mathbb{T} \). Since \( U_j U_k = \theta_{jk} U_k U_j = \theta_{jk} \theta_{kj} U_j U_k \), we have \( \theta_{kj} = \overline{\theta_{jk}} \) and thus the matrix \( \theta = (\theta_{jk})_{k,j=1,\ldots,n} \) is hermitean.

The non-commutative torus \( \mathbb{C}^\ast(\theta) \) is the universal \( \mathbb{C}^\ast \)-algebra generated by the unitaries \( U_j, j = 1, \ldots, n \). To obtain a concrete and workable representation, we interpret \( \mathbb{C}^\ast(\theta) \) as a twisted group \( \mathbb{C}^\ast \)-algebra of \( \mathbb{Z}^n \).

Using multi-index notation with \( U^l = U_1^{l_1} \cdots U_n^{l_n} \) for \( l \in \mathbb{Z}^n \), we obtain

\[
U^l U^m = \sigma(l,m) U^{l+m} \quad \text{for} \quad l, m \in \mathbb{Z}^n,
\]

where \( \sigma(l,m) \in \mathbb{T} \). In fact, the repeated application of the commutation rules yields the expression

\[
\sigma(l,m) = \left( \prod_{j=1}^{n-1} \theta_{n,j}^{m_j} \right)^{l_n} \left( \prod_{j=1}^{n-2} \theta_{n-1,j}^{m_j} \right)^{l_{n-1}} \cdots \left( \prod_{j=1}^{1} \theta_{2,j}^{m_j} \right)^{l_2}.
\]

Since we require the multiplication to be associative, we have

\[
\sigma(l,m) \sigma(l+m,p) = \sigma(l,m+p) \sigma(m,p) \quad \text{for} \quad l, m, p \in \mathbb{Z}^n.
\]

For \( f \) and \( g \in \ell^1(\mathbb{Z}^n) \) we define the twisted convolution \( f \ast_\theta g \) or simply \( f \ast g \) by

\[
f \ast_\theta g (x) = \sum_{y \in \mathbb{Z}^n} f(y) g(x-y) \overline{\sigma(y,x-y)} \quad x \in \mathbb{Z}^n.
\]

The involution is defined \( f^\ast \) by \( f^\ast(x) = \sigma(x,-x) \overline{f(-x)} \) for \( x \in \mathbb{Z}^n \). Then \( (\ell^1(\mathbb{Z}^n), \ast_\theta) \) is a Banach \(*\)-algebra, which we denote by \( \ell^1(\mathbb{Z}^n, \theta) \). This fact can be checked directly, but also follows from the reasoning below.

Following \[19\] and \[15\], we define a central extension \( G \) of \( T \) by \( \mathbb{Z}^n \) as follows. Let \( G = \{(x, \xi) : x \in \mathbb{Z}^n, \xi \in \mathbb{T} \} \) with multiplication \((x, \xi)(y, \eta) = (x + y, \sigma(x,y) \xi \eta)\). Then \( G \) is a nilpotent group with neutral element \( e = (0,1) \) and inverse \((x, \xi)^{-1} = (-x, \overline{\sigma(x,-x)} \xi)\). The Haar measure on \( G \) is \( \int_G f(a) \, da = \sum_{x \in \mathbb{Z}^n} \int_{\mathbb{T}} f(x, \xi) \, d\xi \), and the group convolution \(*\) on \( G \) is defined with respect to this measure.
For \( f \in \ell^1(\mathbb{Z}^n) \) we define \( f^\circ \in L^1(G) \) by \( f^\circ(x, \xi) = f(x)\xi \). This extension satisfies the following properties.

**Lemma 1.** The mapping \( \circ : \ell^1(\mathbb{Z}^n) \to L^1(G) \) is an isometric \(*\)-homomorphism from \( \ell^1(\mathbb{Z}^n, \theta) \) into \( L^1(G) \).

**Proof.** We have

\[
\|f^\circ\|_1 = \int_G |f^\circ(a)| da = \sum_{x \in \mathbb{Z}^n} \int_T |f(x)\xi| d\xi = \sum_{x \in \mathbb{Z}^n} |f(x)| = \|f\|_1,
\]

so \( f \mapsto f^\circ \) is an isometry. This map is compatible with the involution, since

\[
(f^\ast)^\circ(x, \xi) = f^\ast(x)\xi = \sigma(x, -x)\overline{f(-x)}\xi = \overline{f(-x)\sigma(x, -x)}\xi
\]

\[
= \overline{f^\circ((x, \xi)^{-1})} = (f^\circ)^\ast(x, \xi).
\]

For the homomorphism property we first write

\[
(f \circ g)^\circ(x, \xi) = (f \circ g)(x) \xi = \sum_{y \in \mathbb{Z}^n} f(y)g(x - y)\overline{\sigma(y, x - y)}\xi \cdot \int_T \eta \overline{\eta} d\eta = \sum_{y \in \mathbb{Z}^n} \int_T f^\circ(y, \eta)g^\circ(x - y, \overline{\sigma(y, x - y)}\eta) d\eta.
\]

Using (3) and \( \sigma(e, x) = 1 \), we have

\[
(x - y, \overline{\sigma(y, x - y)}\eta) = (-y, \overline{\sigma(y, -y)}\eta)(x, \xi\sigma(y, -y)\overline{\sigma(-y, x}\sigma(y, x - y))
\]

\[
= (-y, \sigma(y, -y)\eta)(x, \xi).
\]

So we arrive at

\[
(f \circ g)^\circ(x, \xi) = \sum_y \int_T f^\circ(y, \eta)g^\circ((y, \eta)^{-1}(x, \xi)) d\eta = \int_G f^\circ(a)g^\circ(a^{-1}(x, \xi)) da
\]

(where \( da \) is the Haar measure on \( G \))

\[
= (f^\circ \ast g^\circ)(x, \xi).
\]

We therefore may think of \( \ell^1(\mathbb{Z}^n, \theta) \) as a closed \(*\)-subalgebra of \( L^1(G) \). In particular, it is a Banach \(*\)-algebra. Its enveloping \( C^* \)-algebra is the non-commutative torus \( C^*(\theta) \).

To obtain a concrete realization of the non-commutative torus \( C^*(\theta) \), we consider the regular representation \( \lambda \) of \( \ell^1(\mathbb{Z}^n, \theta) \) on \( \ell^2(\mathbb{Z}) \) defined by

\[
\lambda(f)g = f \circ g \quad \text{for } f \in \ell^1(\mathbb{Z}^n), g \in \ell^2(\mathbb{Z}^n).
\]

The regular representation \( \lambda \) is faithful, and the closure of \( \lambda(\ell^1) \) with respect to the operator norm is a \( C^* \)-algebra \( \mathcal{C} \). By a special case of Satz 6 in [11], \( C^*(\theta) \)
is isometrically isomorphic to $C$. From now on, we will therefore not distinguish between the abstract algebra $C^*(\theta)$ and its concrete realization $C$.

3. INVERSE-CLOSED SUBALGEBRAS OF $C^*(\theta)$

Next we construct a family of inverse-closed Banach subalgebras of the non-commutative torus $C^*(\theta)$. This construction relies on two important results in Banach algebra theory and abstract harmonic analysis.

First recall that a Banach $*$-algebra $A$ is symmetric, if the spectrum of every positive element is positive, i.e., $\sigma(a^*a) \subseteq [0, \infty)$ for all $a \in A$. The connection between symmetry and inverse-closedness is folklore and implicit in many proofs of symmetry \cite{9,10,12,13}. The following proposition is contained in Palmer’s book \cite[Thm. 11.4.1]{16}. (Since the regular representation of $\ell^1(\mathbb{Z}^n, \theta)$ is faithful, $\ell^1(\mathbb{Z}^n, \theta)$ is semisimple, and we may quote a formulation that is already adapted to semisimple Banach algebras.)

Proposition 2. A unital semisimple Banach $*$-algebra $A$ is symmetric, if and only if it is inverse-closed in its enveloping $C^*$-algebra.

Our second ingredient is a fundamental result of Ludwig \cite{13}.

Proposition 3. If $G$ is a nilpotent group, then $L^1(G)$ is symmetric.

By combining the explicit construction of non-commutative tori with these results, we obtain a fundamental inverse-closed subalgebra of the non-commutative torus $C^*(\theta)$.

Theorem 4. The Banach $*$-algebra $\ell^1(\mathbb{Z}^n, \theta)$ is inverse-closed in the non-commutative torus $C^*(\theta)$.

Proof. By construction, the central extension $G$ of $\mathbb{Z}^n$ is nilpotent, and consequently $L^1(G)$ is symmetric by Ludwig’s result. Lemma \cite{1} identifies $\ell^1(\mathbb{Z}^n, \theta)$ with a closed $*$-subalgebra of $L^1(G)$, and thus $\ell^1(\mathbb{Z}^n, \theta)$ is also a symmetric Banach $*$-algebra. By Proposition 2 this means that $\ell^1(\mathbb{Z}^n, \theta)$ is inverse-closed in $C^*(\theta)$, as claimed.

REMARK: For even dimension and a special representation of the generators of $C^*(\theta)$ by phase-space shifts, Theorem 4 was proved in \cite{8} for the solution of a problem in time-frequency analysis. An earlier result is contained in \cite{1}. See also \cite{14} and \cite[Ch. 13]{6} for the connections to time-frequency analysis.

To generate more examples of inverse-closed subalgebras of $C^*(\theta)$, we introduce weighted $\ell^1$-algebras.

Let $v$ be a submultiplicative and symmetric weight function on $\mathbb{Z}^n$, i.e., $v$ satisfies the conditions

$$v(x + y) \leq C v(x)v(y) \quad \text{and} \quad v(-x) = v(x) \quad \text{for all} \ x, y \in \mathbb{Z}^n,$$

and let $\ell^1_v(\mathbb{Z}^n)$ be the corresponding weighted $\ell^1$-space with norm $\|f\|_{\ell^1_v} = \|fv\|_1$. The pointwise inequality $|\langle f \upharpoonright \theta g \rangle(x)| \leq (|f| \ast |g|)(x)$ for all $x \in \mathbb{Z}^n$ shows that
A nonspectral subalgebra of the irrational rotation algebra \((\ell_1^1(Z^n), \varnothing, \ast)\) is a Banach \(*\)-algebra, which we denote \(\ell_1^1(Z^n, \theta)\). Since \(v\) is symmetric, \(\ell_1^1(Z^n, \theta)\) is a \(*\)–subalgebra of \(\ell_1^1(Z^n, \theta)\).

The next proposition characterizes completely all submultiplicative symmetric weights such that \(\ell_1^1(Z^n, \theta)\) is inverse-closed in the non-commutative torus \(C^*(\theta)\).

**Proposition 5.** The Banach algebra \(\ell_1^1(Z^n, \theta)\) is inverse-closed in \(C^*(\theta)\), if and only if \(v\) satisfies the Gelfand-Raikov-Shilov condition (GRS-condition)

\[
\lim_{n \to \infty} v(nx)^{1/n} = 1 \quad \text{for all} \ x \in \mathbb{Z}^n.
\]

**Proof.** Assume first that \(v\) satisfies the GRS-condition. Then we may extend \(v\) to a weight on \(G\) by setting \(\omega(x, \xi) = v(x)\) for all \(x \in \mathbb{Z}^n, \xi \in \mathbb{T}\). The extended weight \(\omega\) satisfies the GRS-condition on \(G\), so the weighted version of Ludwig’s theorem, as proved in [5], Theorems 1.3 and 3.4, implies that \(L_1^1(G)\) is symmetric. Since obviously \(\|f\|_{L_1(G)} = \|f\|_{\ell_1}\), Lemma [1] implies that \(\ell_1^1(Z^n, \theta)\) can be identified with a closed subalgebra of \(L_1^1(G)\) and thus is also symmetric. Consequently, by Proposition 2 \(\ell_1^1(Z^n, \theta)\) is inverse-closed in its enveloping \(C^*\)-algebra. But since \(\ell_1^1(Z^n, \theta)\) is a dense subalgebra of \(\ell_1^1(Z^n, \theta)\), this means that \(\ell_1^1(Z^n, \theta)\) is inverse-closed in the non-commutative torus \(C^*(\theta)\).

Next assume that \(v\) violates the GRS-condition. This means that there exists an \(x \in \mathbb{Z}^n\), such that \(\lim_{n \to \infty} v(nx)^{1/n} > 1\). Since by [4] the \(n\)-th power of \(\delta_x\) is of the form \(c_n\delta_{nx}\) with \(|c_n| = 1\), the spectral radius of \(\delta_x\) in \(\ell_1^1(Z^n, \theta)\) is

\[
r_{\ell_1^1(Z^n, \theta)}(\delta_x) = \lim_{n \to \infty} \|c_n\delta_{nx}\|_{\ell_1^1(Z^n, \theta)}^{1/n} = \lim_{n \to \infty} v(nx)^{1/n} > 1.
\]

On the other hand, since \(\delta_x\) is a product of unitary elements, \(\delta_x\) is also unitary in \(C^*(\theta)\). Consequently, the spectral radius of \(\delta_x\) in \(C^*(\theta)\) is 1. Therefore the spectrum of \(\delta_x\) in \(\ell_1^1(Z^n, \theta)\) cannot be equal to the spectrum of \(\delta_x\) in \(C^*(\theta)\), and so \(\ell_1^1(Z^n, \theta)\) is not inverse-closed in \(C^*(\theta)\).

**REMARK:** A nonspectral subalgebra of the irrational rotation algebra (the non-commutative torus with two generators) and its simplicity were first discussed by Schweitzer [20].

Proposition 5 provides an abundance of examples of inverse-closed Banach subalgebras of a non-commutative torus in higher dimensions. By taking intersections of weighted \(\ell_1\)-algebras, one may now construct inverse-closed Fréchet subalgebras of \(C^*(\theta)\). In particular, fix \(v(x) = 1 + |x|\) for some norm \(|\cdot|\) on \(\mathbb{Z}^n\) and set

\[
(5) \quad s(\mathbb{Z}^n, \theta) = \bigcap_{s \geq 0} \ell_1^1(\mathbb{Z}^n, \theta) = \{ f \in \ell_1(\mathbb{Z}^n) : |f(x)| = O(|x|^{-s}) \forall s \geq 0 \}.
\]

Then \(s(\mathbb{Z}^n, \theta)\) consists of all rapidly decreasing sequences and coincides with the usual smooth non-commutative torus. Since an arbitrary intersection of inverse-closed subalgebras is again inverse-closed, \(s(\mathbb{Z}^n, \theta)\) is an inverse-closed Fréchet algebra of the non-commutative torus \(C^*(\theta)\). This result goes back to Connes [3].

Proposition 5 yields inverse-closed subalgebras of \(C^*(\theta)\) that are even smaller than \(s(\mathbb{Z}^n, \theta)\). For this fix a subexponential weight \(v(x) = e^{a|x|^b}\) for \(a > 0\) and \(0 < b < 1\). Then \(v\) satisfies the GRS-condition, and thus \(\ell_1^1(\mathbb{Z}^n, \theta)\) is inverse-closed.
in $C^*(\theta)$. On the other hand, $\ell^1_v(\mathbb{Z}^n, \theta)$ is a Banach subalgebra of the smooth non-commutative torus $s(\mathbb{Z}^n, \theta)$. In the language of non-commutative geometry, one might say that $\ell^1_v(\mathbb{Z}^n, \theta)$ consists of “ultra-smooth” elements of $C^*(\theta)$.

4. Simplicity

The construction of inverse-closed subalgebras of non-commutative tori is completely independent of the fine structure of these tori. In particular, the simplicity of $C^*(\theta)$ is not related to its spectral properties.

In this section we treat the question when the twisted $\ell^1$-algebra $\ell^1(\mathbb{Z}^n, \theta)$ is simple. Making use of the symmetry of $\ell^1(\mathbb{Z}^n, \theta)$, one can derive the “if” part of Theorem 6 below from the characterization of the simplicity of higher-dimensional non-commutative tori $C^*(\theta)$ in [17], but one has to go back to [21] and [4] for its proof. We offer a simplified proof that works directly for $\ell^1(\mathbb{Z}^n, \theta)$, from which the known results about $C^*(\theta)$ follow. Our proof for the twisted $\ell^1$-algebras is fairly elementary, but its idea is probably old.

We first recall Rieffel’s observation that the cocycle $\sigma$ defining $C^*(\theta)$ may be assumed to be a hermitean bicharacter, i.e., a bicharacter satisfying $\sigma(x, y) = \overline{\sigma(y, x)}$. The bicharacter in (2) is not hermitean in general. If $\vartheta = (\vartheta_{jk})_{j,k=1,...,n}$ is a (non-unique) skew-symmetric real-valued matrix such that $\theta_{jk} = e^{2\pi i \vartheta_{jk}}$, then we have

$$\sigma(x, y) = e^{2\pi i (x, \vartheta y)},$$

where $\vartheta_{jk} = \vartheta_{kj}$ for $j > k$ and $\vartheta_{jk} = 0$ for $j \leq k$. So $\sigma$ is hermitean only in the trivial case when $\theta$ is the identity matrix and $\ell^1(\mathbb{Z}^n, \theta)$ is commutative. As observed by Rieffel [18, p. 283–285], the cocycle

$$\sigma'(x, y) = e^{2\pi i (x, \frac{1}{2} \vartheta y)}$$

is a suitable choice for a hermitean bicharacter defining $\ell^1(\mathbb{Z}^n, \theta)$. Since $\frac{1}{2} \vartheta = \frac{1}{2} (\vartheta^0 - (\vartheta^0)^t)$ is the skew-symmetric part and $\vartheta^0_{\text{sym}} = \frac{1}{2} (\vartheta^0 + (\vartheta^0)^t)$ the symmetric part of $\vartheta^0$, we have

$$\langle x, (\vartheta^0 - \frac{1}{2} \vartheta) y \rangle = \langle x, \vartheta^0_{\text{sym}} y \rangle = r(x + y) - r(x) - r(y)$$

where $r(z) = \langle z, \frac{1}{2} \vartheta^0 z \rangle$ is the quadratic form defined by $\frac{1}{2} \vartheta^0$. Thus the cocycles $\sigma(x, y) = e^{2\pi i (x, \vartheta y)}$ and $\sigma'(x, y) = e^{2\pi i (x, \frac{1}{2} \vartheta y)}$ are cohomologous. Therefore the corresponding twisted algebras $\ell^1_{\sigma}(\mathbb{Z}^n, \theta)$ and $\ell^1_{\sigma'}(\mathbb{Z}^n, \theta)$ are isometrically isomorphic by the map $V : \ell^1_{\sigma'} \to \ell^1_{\sigma}, (V f)(x) = r(x) f(x)$.

A hermitean bicharacter $\tau$ is called degenerate, if there exists a non-zero $m \in \mathbb{Z}^n$, such that $\tau(l, m) = \pm 1$ for all $l \in \mathbb{Z}^n$. Otherwise $\tau$ is called nondegenerate.

From now on $\delta_l$ denotes the twisted convolution defined by using the hermitean bicharacter $\sigma'$ in (3). This yields $\delta_l \circ \delta_m = \sigma'(l, m) \delta_{l+m}$ for $l, m \in \mathbb{Z}^n$. Since $\sigma'$ is hermitean, we have $\delta_m \circ \delta_l = \sigma'(m, l) \delta_{l+m} = \sigma'(l, m) \delta_{l+m}$. Therefore the bicharacter $\sigma'$ is degenerate, if and only if there exists a non-zero $m \in \mathbb{Z}^n$ such that $\delta_m$ is central in $\ell^1(\mathbb{Z}^n, \theta)$.

The following theorem characterizes the simplicity of the twisted $\ell^1$-algebras.

6 KARLHEINZ GRÖCHENIG AND MICHAEL LEINERT
Theorem 6. The algebra $\ell^1(\mathbb{Z}^n, \theta)$ is simple, if and only if the bicharacter $\sigma'$ is nondegenerate.

Proof. For $j = 1, \ldots, n$ we denote by $\delta_j$ the “Dirac” function of the point $(0, \ldots, 0, 1, 0, \ldots)$ where 1 is in the $j$-th coordinate.

Suppose first that $\sigma'$ is nondegenerate. Then $\delta_j$ is unitary, and its adjoint is $\delta_{-j}$.

For arbitrary $x \in \mathbb{Z}^n$ and $k \in \mathbb{N}$ we have

$$(\delta_j^*)^k \circ \delta_x \circ \delta_j^k = \beta_x \delta_x$$

for some $\beta_x \in \mathbb{T}$. More precisely, $\beta_x = 1$, if and only if $\delta_x$ commutes with $\delta_j$. We denote the centralizer of $\delta_j$ by

$$C_j = \{ y \in \mathbb{Z}^n : \delta_y \circ \delta_j = \delta_j \circ \delta_y \}.$$ 

Now let $I$ be a (closed) two-sided ideal of $\ell^1(\mathbb{Z}^n, \theta)$ and $f = \sum_{x \in \mathbb{Z}^n} \alpha_x \delta_x \in I \subseteq \ell^1(\mathbb{Z}^n, \theta)$. We consider the behavior of the averages

$$J_m(f) = \frac{1}{m} \sum_{k=1}^{m} (\delta_j^*)^k \circ f \circ \delta_j^k = \sum_{x \in \mathbb{Z}^n} \alpha_x \left( \frac{1}{m} \sum_{k=1}^{m} \beta_x^k \right) \delta_x.$$ 

If $x \in C_j$, then $\frac{1}{m} \sum_{k=1}^{m} \beta_x^k = 1$; if $x \notin C_j$, then the average $\frac{1}{m} \sum_{k=1}^{m} \beta_x^k$ converges to zero for $m \to \infty$. Using dominated convergence, we conclude that

$$\lim_{m \to \infty} J_m(f) = \sum_{x \in C_j} \alpha_x \delta_x = f \chi_{C_j}$$

with convergence in the $\ell^1$-norm. For $f \in I$, this means that also $f \chi_{C_j} \in I$. Since this is true for all $j = 1, \ldots, n$, we obtain that $f(\prod_{j=1}^{n} \chi_{C_j}) = f \chi_{\bigcap_{j=1}^{n} C_j} \in I$.

Since $\sigma'$ is nondegenerate, we must have $\bigcap_{j=1}^{n} C_j = \{0\}$ and thus $f(0)\delta_0 \in I$. Either $I = \ell^1(\mathbb{Z}^n, \theta)$ or $I$ is a proper ideal and $f(0) = 0$. By applying the argument to $\delta_j \circ f \in I$ for every $x \in \mathbb{Z}^n$, we obtain that $(\delta_j \circ f)(0) = \sigma'(x, -x) f(-x) = 0$, so $f(x) = 0$ for all $x \in \mathbb{Z}^n$. Consequently either $I = \ell^1(\mathbb{Z}^n, \theta)$ or $I = \{0\}$, and thus $\ell^1(\mathbb{Z}^n, \theta)$ is simple.

For the converse, assume that $\sigma'$ is degenerate. Then there is $m \in \mathbb{Z}^n$ such that $\delta_m$ is a central element in $\ell^1(\mathbb{Z}^n, \theta)$. It follows that $\delta_l$ is central in $\ell^1(\mathbb{Z}^n, \theta)$ for every $l$ in the subgroup $M \subseteq \mathbb{Z}^n$ generated by $m$. So $\ell^1(M, \theta)$ is central in $\ell^1(\mathbb{Z}^n, \theta)$. Now there exists a complementing subgroup $U$ of $\mathbb{Z}^n$, such that $\mathbb{Z}^n = U \oplus M$. As a Banach space, $\ell^1(\mathbb{Z}^n, \theta)$ can be written in the form $\ell^1(U, \ell^1(M))$, the space of all $\ell^1(M)$-valued integrable functions on $U$. Using the canonical bijection $(u, m) \to u + m$ from $U \times M$ onto $\mathbb{Z}^n$, the twisted product $\circ$ can be written as

$$\tag{7} (f \circ g)(u) = \sum_{v \in U} (f(v) *_M g(u - v)) \sigma'(v, u - v),$$

where now $f(v)$ and $g(u - v) \in \ell^1(M)$ and the product $f(v) *_M g(u - v)$ is the ordinary commutative convolution on $M$. Now, if $I$ is a closed proper (two-sided) ideal in $\ell^1(M)$, then by \cite{17} $\ell^1(U, I)$ is a closed proper non-zero two-sided ideal in $\ell^1(\mathbb{Z}^n, \theta)$. Therefore $\ell^1(\mathbb{Z}^n, \theta)$ cannot be simple. \hfill \blacksquare
REMARKS: 1. If one does not want to use the existence of a complementing subgroup $\mathcal{U}$, one could alternatively consider the quotient $\mathbb{Z}^n/M$ instead, then choose a cross-section $c : \mathbb{Z}^n/M \to \mathbb{Z}^n$, and use the canonical bijection $(\dot{x}, m) \mapsto c(\dot{x}) + m$. To write $\natural$ in terms of $\ell^1(\mathbb{Z}^n/M, \ell^1(M))$, it has to be modified by an additional translation with a cocycle determined by the cross-section $c$.  

2. If $v$ is a submultiplicative weight on $\mathbb{Z}^n$, then the nondegeneracy is still sufficient for the simplicity of $\ell^1_v(\mathbb{Z}^n, \theta)$, since the inner automorphism defined by $\delta_j$ still has norm 1, although $\|\delta_j\|_{\ell^1_v}$ can be large.  

3. We may also obtain an alternative proof of the well-known $C^*$-analogue of Theorem 6. 

(i) The “if”-part works as for $\ell^1(\mathbb{Z}^n, \theta)$, because the finitely supported functions are dense in $C^*(\theta)$ and the inner automorphisms are also isometric in the $C^*(\theta)$-norm.  

(ii) If $\sigma'$ is degenerate, then the “only if”-part of Theorem 6 asserts that $\ell^1(\mathbb{Z}^n, \theta)$ is not simple and thus contains a non-trivial closed two-sided ideal $I$. Since $\ell^1(\mathbb{Z}^n, \theta)$ is symmetric, there exists a non-zero positive linear functional $\phi$ on $\ell^1(\mathbb{Z}^n, \theta)$ vanishing on $I$. Now let $\pi_\phi$ be the $*$-representation obtained from $\phi$ by the GNS-construction. Then we have $\pi_\phi(I) = \{0\}$. So the kernel of $\pi_\phi$ in $C^*(\theta)$ is a non-zero proper closed, two-sided ideal in $C^*(\theta)$, since it contains $I$. Therefore $C^*(\theta)$ is not simple.

References

[1] W. Arveson. Discretized CCR algebras. J. Operator Theory, 26(2):225–239, 1991.
[2] C. Badea. The stable rank of topological algebras and a problem of R. G. Swan. J. Funct. Anal., 160(1):42–78, 1998.
[3] A. Connes. $C^*$ algèbres et géométrie différentielle. C. R. Acad. Sci. Paris Sér. A-B, 290(13):A599–A604, 1980.
[4] G. A. Elliott. On the K-theory of the $C^*$-algebra generated by a projective representation of a torsion-free discrete abelian group. In Operator algebras and group representations, Vol. I (Neptun, 1980), volume 17 of Monogr. Stud. Math., pages 157–184. Pitman, Boston, MA, 1984.
[5] G. Fendler, K. Gröchenig, and M. Leinert. Symmetry of weighted $L^1$-algebras and the GRS-condition. Bull. London Math. Soc., 38(4):625–635, 2006.
[6] K. Gröchenig. Foundations of time-frequency analysis. Birkhäuser Boston Inc., Boston, MA, 2001.
[7] K. Gröchenig. Wiener’s lemma: Theme and variations. an introduction to spectral invariance. In B. Forster and P. Massopust, editors, Four Short Courses on Harmonic Analysis, Appl. Num. Harm. Anal. Birkhäuser, Boston, 2010.
[8] K. Gröchenig and M. Leinert. Wiener’s lemma for twisted convolution and Gabor frames. J. Amer. Math. Soc., 17:1–18, 2004.
[9] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. Trans. Amer. Math. Soc., 358(6):2695–2711 (electronic), 2006.
[10] A. Hulanicki. On the spectrum of convolution operators on groups with polynomial growth. Invent. Math., 17:135–142, 1972.
[11] H. Leptin. Darstellungen verallgemeinerter $L^1$-Algebren. Invent. Math., 5:192–215, 1968.
[12] H. Leptin. The structure of $L^1(G)$ for locally compact groups. In *Operator algebras and group representations, Vol. II* (Neptun, 1980), volume 18 of *Monogr. Stud. Math.*, pages 48–61. Pitman, Boston, MA, 1984.

[13] J. Ludwig. A class of symmetric and a class of Wiener group algebras. *J. Funct. Anal.*, 31(2):187–194, 1979.

[14] F. Luef. Projective modules over noncommutative tori are multi-window Gabor frames for modulation spaces. *J. Funct. Anal.*, 257(6):1921–1946, 2009.

[15] G. W. Mackey. Unitary representations of group extensions. I. *Acta Math.*, 99:265–311, 1958.

[16] T. W. Palmer. *Banach algebras and the general theory of $*$-algebras. Vol. 2*, volume 79 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001. $*$-algebras.

[17] N. C. Phillips. Every simple higher dimensional noncommutative torus is an at algebra. 2006, arXiv:math/0609783v1.

[18] M. A. Rieffel. Projective modules over higher-dimensional noncommutative tori. *Canad. J. Math.*, 40(2):257–338, 1988.

[19] O. Schreier. Über die Erweiterung von Gruppen I. *Monatsh. Math. Phys.*, 34(1):165–180, 1926.

[20] L. B. Schweitzer. A nonspectral dense Banach subalgebra of the irrational rotation algebra. *Proc. Amer. Math. Soc.*, 120(3):811–813, 1994.

[21] J. Slawny. On factor representations and the $C^*$-algebra of canonical commutation relations. *Comm. Math. Phys.*, 24:151–170, 1972.

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria

E-mail address: karlheinz.groechenig@univie.ac.at

Institut für Angewandte Mathematik, Fakultät für Mathematik, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany

E-mail address: leinert@math.uni-heidelberg.de