Wave transport in stellar radiation zone influenced by the Coriolis acceleration

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Abstract. Internal gravity waves constitute an efficient process for angular momentum
transport over large distances. They are now seen as an important ingredient in understanding
the evolution of rotation, and could explain the Sun’s quasi-flat rotation profile. Because the
Sun’s rotation frequency is of the same order as that of the waves, it is necessary to refine our
description of wave propagation and to take into account the action of the Coriolis acceleration in
a coherent way. To achieve this aim, we adopt the Traditional Approximation which is verified
in stellar radiation zones. We present the modified transport equations and their numerical
evaluation in a parameter range that is significant for the Sun. The effectiveness of gravity
waves, which become gravito-inertial waves, is reduced while new type of waves, namely the
Rossby, the Yanai and the Kelvin waves appear with their associated transport.

1. Introduction and context
In standard models of stellar interiors, radiation zones which are convectively stable are
postulated to be without motion other than rotation. But various observational results (e.g.
surface abundances of light elements, helioseismology) show that these regions are the seat of
transport and of mild mixing. The most likely cause of such mixing is stellar rotation. First,
it causes thermal imbalance that drives a large scale meridional circulation. Second, since in
general the star does not rotate as a solid body, shear instabilities may appear (for a review
of these processes, see e.g. Talon 2007). Series of models have been built which include a self-
consistent evolution of the internal rotation profile, and for massive stars, they agree rather well
with the observations (see Maeder & Meynet 2000).

The case of the Sun is somewhat different: like all other stars that have a deep surface
convection zone, it has been spun down during its infancy. When only the meridional circulation
and the “classical” hydrodynamic instabilities are invoked, models predict a Sun with a core
rotating much faster than the surface with a gradient of angular velocity in the radiative zone
(Pinsonneault et al. 1989, Chaboyer et al. 1995, Talon 1997, Matias & Zahn 1998) which is
not compatible with helioseismology (cf. Turck-Chièze et al. 2004 and García et al. 2007);
one must conclude therefore that another, more powerful process is operating, at least in stars with a surface convection zone. The most plausible candidates are magnetic torquing (Brun & Zahn 2006 and references therein) and momentum transport by internal gravity waves (hereafter IGWs, Schatzman 1993, Talon et al. 2002, Talon & Charbonnel 2005).

Concerning the waves, their current treatment presents two major weaknesses. The first one is our crude description of their generation by turbulent convection. The second one is that the action of rotation on the waves is not accounted for. In this work, we have undertaken to improve the modelling of the transport by internal waves by introducing the effects of the Coriolis acceleration. Indeed, the low-frequency internal waves which are responsible for the deposit or the extraction of angular momentum deep within the star (cf. Talon et al. 2002) are strongly influenced by the rotation because the frequencies of the waves are of the same order that the inertial frequency, 2Ω. Thus, internal waves become gravito-inertial waves (cf. Berthomieu et al. 1978, Lee & Saio 1997) and we have to treat the action of the rotation on the waves and vice versa.

2. Structure of low-frequency waves influenced by the Coriolis acceleration

In stellar radiative zones, the transport of angular momentum is dominated by low-frequency waves with \( \sigma \ll N \), where \( \sigma \) and \( N \) are respectively the wave frequency and the Brunt-Väisälä frequency, a measure of buoyancy. In a (differentially) rotating stellar radiative zone, we should also consider the Coriolis acceleration, which is characterized by the inertial frequency, 2Ω, with \( \Omega \) being the star’s angular velocity. One then has to quantify the relative importance of each restoring force in the waves dynamics. In particular, we would like to know when the effects of the Coriolis acceleration can be treated in a perturbative way.

In the Sun, the answer is very clear for the acoustic waves which have frequencies much greater than \( \Omega_\odot \). Then, the effects of the Coriolis acceleration can be treated as a perturbation (e.g. rotationally split frequencies). However, in the case of low-frequency IGWs which have frequencies around 1 \( \mu \)Hz, the spin parameter \( \nu = \frac{2\Omega}{\sigma} = R_o^{-1} \), which measures the relative importance of rotation and stratification, is of the order of the unity, \( R_o \) being the Rossby’s number. In this case, Coriolis effects cannot be treated as a perturbation.

Here, we examine how taking into account the effect of the Coriolis acceleration modifies the spatial structure of IGWs, and consequently, angular momentum extraction from the solar interior by IGWs. We first recall the main assumptions in our mathematical model and give the corresponding dynamical equations. The derivation of these equations has been presented in Mathis (2005) and Pantillon, Talon & Charbonnel (2007).

2.1. Main assumptions

**Rotation law**

We consider here a “shellular” rotation \( \bar{\Omega}[r(P)] \) (in other words, the angular velocity is constant on an isobar) due to anisotropic turbulence in a highly stratified star; \( r(P) \) is the radius of the isobar which is the generalization of the equipotential for the case of differential rotation while \( \bar{\Omega} \) refers to horizontal averages. Next, we split this “shellular” rotation law into a solid body rotation, \( \Omega_s \), and a (small) differential rotation fluctuation, \( \delta \bar{\Omega}(r) \). This hypothesis will allow to separate the variables in the treatment of the dynamical equations. The formalism presented here remains valid only in the case of “reasonable” values of the fluctuations of the angular velocity \( \delta \bar{\Omega} \) around its mean value \( \bar{\Omega}_s \) and of the radial gradient of \( \Omega \). Thus, we write:

\[
\Omega(r, \theta) = \bar{\Omega}(r) = \Omega_s + \delta \bar{\Omega}(r) \quad \text{where} \quad \delta \bar{\Omega}(r) \ll \bar{\Omega}_s.
\]  

(1)

\( \bar{\Omega}_s \) will be taken into account for the calculation of the structure of the low-frequency adiabatic waves while \( \delta \bar{\Omega} \) will be accounted for only in the treatment of the damping due to dissipative
processes. This is the “weak differential rotation case”.

**Traditional Approximation**

In the largest part of stellar radiation zones, we are in a regime where $2\Omega \ll N$. Since we are here interested in low-frequency waves where $\sigma \ll N$, the Traditional Approximation which consists in neglecting the horizontal component of the rotation vector $\Omega \hat{e}_z$ in the momentum equation can be adopted (see e.g. Eckart 1960; for a modern description in a stellar context see Lee & Saio 1997, Talon 1997). In this approximation, variable separation in radial and horizontal components remains possible.

**Cowling Approximation**

The gravitational potential fluctuations associated with the waves are neglected as well as the effects of the centrifugal force since we are interested here in situations where the gravity dominates.

### 2.2. Wave velocity and pressure fields with Coriolis acceleration

Following Zahn et al. (1997), Mathis (2005) and Pantillon et al. (2007), the velocity and pressure fields of the low-frequency waves are derived using three supplementary approximations.

The first one is the JWKB approximation, which is adopted for the radial dependency of pressure fluctuations and all three velocity components; it is justified because, in the low-frequency regime where $\sigma \ll N$, $k_Vr$, the vertical wave vector, represents a length scale much smaller than the characteristic spatial length of the medium. The second approximation is that dissipative processes such as thermal and turbulent diffusions are treated assuming quasi-adiabaticity (cf. Press 1981, Zahn et al. 1997). Finally, the anelastic approximation for sonic waves filtering is adopted.

The velocity field associated to a full wave spectrum obeys:

$$\vec{u} = \sum_{m,k} \vec{u}_{k,m}(r, \theta, \varphi, t)$$  \hspace{1cm} (2)

where the monochromatic radial, latitudinal and azimuthal components are respectively given by:

$$u_{r,k,m}(r, \theta, \varphi, t) = \mathcal{E}_{k,m}(r) \cos[\Phi_{k,m}(r, \varphi, t)] H_{k,m}(\cos \theta; \nu_s) \exp\left[-\tau_{k,m}(r, \delta \Omega(r); \nu_s) / 2 \right],$$ \hspace{1cm} (3)

$$u_{\theta,k,m}(r, \theta, \varphi, t) = \frac{r k_v \mathcal{E}_{k,m}(r) \sin[\Phi_{k,m}(r, \varphi, t)]}{\Lambda_{k,m}(\nu_s)} \times \exp\left[-\tau_{k,m}(r, \delta \Omega(r); \nu_s) / 2 \right],$$ \hspace{1cm} (4)

$$u_{\varphi,k,m}(r, \theta, \varphi, t) = \frac{r k_v \mathcal{E}_{k,m}(r) \sin[\Phi_{k,m}(r, \varphi, t)]}{\Lambda_{k,m}(\nu_s)} H_{k,m}^\varphi(\cos \theta; \nu_s) \times \exp\left[-\tau_{k,m}(r, \delta \Omega(r); \nu_s) / 2 \right].$$ \hspace{1cm} (5)

The Hough functions $\Theta_{k,m}$, $\mathcal{H}_{k,m}^\theta$, and $\mathcal{H}_{k,m}^\varphi$ will be discussed later in this section, and the attenuation $\tau_{k,m}$ will be explicited in Eq. (11). For the problem of angular momentum transport by IGWs, we are interested in propagative waves. The phase function $\Phi_{k,m}$ is given in the JWKB regime by:

$$\Phi_{k,m}(r, \varphi, t) = \sigma_0 t + \int_r^{r_f} k_v \mathcal{E}_{k,m} \, dr' + m\varphi,$$ \hspace{1cm} (6)
Here, as in Pantillon et al. (2007), we follow the convention of Unno et al. (1989) and take the sign of \( m \) such that the prograde waves have an \( m < 0 \) while the retrograde waves have an \( m > 0 \).

The JWKB amplitude function \( E_{k,m}(r) = A_{k,m} r^{-2} \rho^{-1/2} \left( \frac{N^2}{\sigma^2} - 1 \right)^{-1/2} \) is obtained, where the amplitude of the wave, \( A_{k,m} \), must be determined from the boundary conditions. On the other hand, the Brunt-Väisälä frequency takes into account the effects of both the thermal and the chemical composition with:

\[
\nu \equiv \frac{1}{\rho} \frac{d}{dr} \left( \frac{\sigma H}{\rho} \right)
\]

Let us remark that this separation of variables has been allowed thanks to the Traditional Approximation and to the “weak differential rotation approximation”. The horizontal dependance for \( \nu_{r,k,m} \), denoted here \( \Theta_{k,m} \), obeys:

\[
\left[ \frac{d}{dx} \left( \frac{1 - x^2}{1 - \nu_s x^2} \frac{d}{dx} \right) - \frac{1}{1 - \nu_s x^2} \left( \frac{\nu_s^2 x^2}{1 - x^2} + m \nu_s \frac{1 + \nu_s^2 x^2}{1 - \nu_s x^2} \right) \right] \Theta_{k,m}(x; \nu_s) = \mathcal{L}_{\nu_{r,k,m}} \Theta_{k,m}(x; \nu_s) = -\Lambda_{k,m}(\nu_s) \Theta_{k,m}(x; \nu_s)
\]  

where \( x = \cos \theta; \nu_s = 2 \overline{\Omega}_s/\sigma \) being the spin parameter associated to \( \overline{\Omega}_s \). Eq. (8) is the so-called Laplace equation (cf. Laplace 1799) and the \( \Theta_{k,m} \) are the Hough functions (cf. Hough 1898). In the non-rotating case, the Laplace operator \( \mathcal{L}_{\nu_{r,m}} \) restricts to the classical horizontal spherical Laplacian. The Hough functions could thus be seen as the generalization of the associated Legendre polynomials which are used in the non-rotating case (cf. Lee & Saio 1997). Hough functions have two major properties for angular momentum transport. First, for high values of the spin parameter, they are concentrated around the equator (\( x = 0 \)), a featured referred to as equatorial trapping (see e.g. Lee & Saio 1997, Pantillon et al. 2007, Fig. 1). Next, since \( \mathcal{L}_{\nu_{r,m}} \) depends expicely on \( m \), we have \( \Lambda_{k,-m} \neq \Lambda_{k,m} \) and \( \Theta_{k,-m} \neq \Theta_{k,m} \). In other words, for a given \( k \), prograde and retrograde waves have a different horizontal spatial structure. This is crucial for the transport of angular momentum which depends on the subtle balance between prograde and retrograde waves (cf. Talon & Charbonnel 2005 and references therein). Moreover, as it will be showed later, the transmission of the kinetic energy flux of the turbulent motions which are at the origin of the generation of the waves at the interface between the convective envelope and the radiative core will depend in the sign of \( m \). In the more general case where \( \Omega (r, \theta) = \overline{\Omega}(r) \), \( \nu = 2 \overline{\Omega}(r)/\sigma \) depends on \( r \) as well as \( \mathcal{L}_{\nu_{r,m}} \) and the variables do not separate anymore.

By using the latitudinal and azimuthal components of the momentum equation, the respective angular functions for \( u_{\theta,k,m} \) and \( u_{\varphi,k,m} \), \( \mathcal{H}_{k,m}^\theta(x; \nu_s) \) and \( \mathcal{H}_{k,m}^\varphi(x; \nu_s) \) are obtained:

\[
\mathcal{H}_{k,m}^\theta(x; \nu_s) = \frac{1}{(1 - x^2 \nu_s^2) \sqrt{1 - x^2}} \left[ -(1 - x^2) \frac{d}{dx} + m \nu_s x \right] \Theta_{k,m}(x; \nu_s)
\]

and

\[
\mathcal{H}_{k,m}^\varphi(x; \nu_s) = \frac{1}{(1 - x^2 \nu_s^2) \sqrt{1 - x^2}} \left[ -\nu_s x (1 - x^2) \frac{d}{dx} + m \right] \Theta_{k,m}(x; \nu_s)
\]

We also have \( \mathcal{H}_{k,-m}^\theta \neq \mathcal{H}_{k,m}^\theta \) and \( \mathcal{H}_{k,-m}^\varphi \neq \mathcal{H}_{k,m}^\varphi \).
Eigenfunctions $\Theta_{k,m}$, $\mathcal{H}_{k,m}^0$, $\mathcal{H}_{k,m}^\nu$ and their associated eigenvalues $\Lambda_{k,m}$ are given in Figs. 1 and 2, for prograde and retrograde waves and three different values of $\nu_s$.

Invoking the quasi-adiabatic approximation, the radiative damping conserves the form it has without the Coriolis acceleration (Zahn et al. 1997)

$$\tau_{k,m}(r; \Omega(r); \nu_s) = \Lambda_{k,m}^{3/2} \nu_s \int_r^{\infty} K \frac{NN^2}{\sigma^4} \sqrt{\frac{N^2 - \sigma^2}{r^3}} \, \mathrm{d}r',$$

where $\Lambda_{k,m}^{3/2} \nu_s$ is the horizontal wave number.

Following Pantillon et al. (2007), we define a horizontal wave number:

$$k_{\chi,k,m} = \frac{\lambda_{k,m}^{1/2} \nu_s}{r} \quad \text{where} \quad \lambda_{k,m}^{2} \nu_s = \frac{\langle |r^2 \nabla^2 H \Theta_{k,m}(\cos \theta, \nu_s)|^2 \rangle_{\theta}}{\langle |\Theta_{k,m}(\cos \theta, \nu_s)|^2 \rangle_{\theta}},$$

where $\langle \cdots \rangle_\theta = \frac{1}{2} \int_0^{\pi} \cdots \sin \theta \, \mathrm{d}\theta$ and $\nabla^2 H$ is the horizontal spherical Laplacian. In the absence of rotation, we recover $\lambda_{k,m}(\nu_s = 0) = \Lambda_{k,m}(\nu_s = 0) = l (l + 1)$.

### 2.3. Wave types

When the Coriolis acceleration is taken into account, low-frequency internal gravity waves are modified and new types of waves appear. In a rotating stably stratified radiation region, four types of waves are then identified (see Townsend 2003 for more details):

- **gravito-inertial waves**: they are internal gravity waves that are modified by the Coriolis acceleration; rotation increases their eigenvalues $\Lambda_{k,m}$, and hence their radial wave number and their damping (cf. Fig. 2). These waves are thus deposited much closer to their excitation region, than when the Coriolis acceleration is ignored.

- **Rossby waves**: they are purely retrograde waves ($m > 0$) which exist only in the case of rapid rotation. Their dynamics is driven by the conservation of specific vorticity combined with the effects of curvature. In the slowly rotating Sun, they are excited only with frequencies that are too large to have a significant effect on angular momentum transport.

- **Yanai waves**: they are mixed gravity and Rossby waves. $m \leq 0$ waves exist in the absence of rotation. $m > 0$ appear when $\nu = m + 1$ with small eigenvalues while their horizontal eigenfunctions are $\Theta_{k,m}(\nu = m+1; x) = P_{m+1}^m(x)$. When they appear and have small eigenvalues, they behave mostly like Rossby waves; $m \leq 0$ and $m > 0$ waves with large eigenvalues behave rather like gravity waves. Their eigenvalues are quite smaller than those of gravito-inertial waves. Thus, they will be damped much farther from the bottom of the convective zone and over a more larger portion of the radiation region.

- **Kelvin waves**: they are purely prograde waves ($m < 0$) whose characteristics change little with rotation, their displacement in the $\theta$ direction being very small. Like Rossby waves, their dynamics is driven by the conservation of specific vorticity, but here it is combined with the stratification effects; their eigenvalue are smaller than those of both gravito-inertial and Yanai waves. Hence, they are damped much deeper in the radiation zone where they deposit their positive angular momentum.

We define a new index $s$ given by

$$s = \ell - m + 1 \quad \text{for} \quad m > 0 \quad \text{or} \quad s = \ell + m - 1 \quad \text{for} \quad m \leq 0.$$  \hfill (13)

Kelvin waves have $s = -1$, Yanai waves, $s = 0$ and gravito-inertial and Rossby waves have $s = 1, 2, 3, \ldots$

In the presence of the Coriolis acceleration, the low-frequency waves’ spatial structure and damping are modified while transport associated to Yanai and Kelvin waves has to be accounted for. We shall now introduce expressions for the mean energy and momentum fluxes associated with these waves.
Figure 1. Hough functions (continuous lines) \( \nu = 0, m = 2 \); (dotted lines) \( m = 2 \) (dashed lines) \( m = -2 \) (left) \( \sigma = 0.5 \mu \text{Hz}, \nu = 1.72 \) (right) \( \sigma = 1 \mu \text{Hz}, \nu = 0.86 \).

Figure 2. (top) Eigenvalues \( \Lambda \) of Laplace’s tidal equation in the presence of rotation, in the range that is relevant for a solar modal. In the absence of rotation, one has \( \Lambda = \ell (\ell + 1) \).

(bottom) Equivalent horizontal eigenvalue \( \lambda \) (see text for details). (left) Gravitoinertial waves with \( 1 \leq \ell \leq 6 \), and \( m = -\ell + 2 \) (continuous line), \( m = 0 \) (dotted line), \( m = \ell \) (dashed line).

(middle) Yanai waves (\( s = 0 \)) with negatives values of \( m \) (are shown \( m = 0, ..., 5 \)) are present in the relevant range of \( \nu \). The only Yanai wave with \( m > 0 \) (it is the \( m = 1 \) mode) appears at \( \nu \simeq 2 \). (right) Kelvin waves (\( s = -1 \)) have indices \( m = -1, ..., -6 \). No Rossby waves are shown because they would appear for too large values of \( \nu \) (or equivalently, for too low frequencies).

3. Transport of angular momentum
Waves deposit their angular momentum inside the star as they are damped. The deposition of angular momentum is then given by the radial derivative of the vertical action of angular momentum (cf. Goldreich & Nicholson 1989), \( \mathcal{L}_V^{\text{AM}} (r) \). The evolution of angular momentum by
waves then follows:

\[
\frac{d}{dt} \left[ r^2 \Pi \right] = \pm \frac{3}{8 \pi r^2} \Theta_r \left[ \mathcal{L}^\text{AM} (r) \right]
\]

\hspace{1cm} (Talon & Zahn 1998). The “+” (“−”) sign in front of the action of angular momentum corresponds to a wave traveling inward (outward). Waves deposit their angular momentum inside the star where they are damped. Locally, the total action of angular momentum is the sum of the contribution of all waves, each one being damped separately.

\[
\mathcal{L}^\text{AM} (r) = \sum_{\sigma,m,k} -4\pi r^2 c \left[ \frac{2}{3} m' (\nu) \mathcal{F}^\text{K} (r_c) \right] \exp \left[ -\tau_{k,m} (r, \delta \Pi (r); \nu_s) \right]
\]

with

\[
m' (\nu_s) = \Lambda_{k,m} (\nu_s) \frac{\mathcal{J}_{I,k,m} (\nu_s) - \nu_s \mathcal{J}_{H,k,m} (\nu_s)}{\mathcal{J}_{H,k,m} (\nu_s)}
\]

where

\[
\begin{align*}
\mathcal{J}_{I,k,m} (\nu_s) &= \left\langle \Theta_{k,m} (\cos \theta; \nu_s) \mathcal{H}^2_{k,m} (\cos \theta; \nu_s) \sin \theta \right\rangle_{\theta} \hfill (17) \\
\mathcal{J}_{H,k,m} (\nu_s) &= \left\langle \Theta_{k,m} (\cos \theta; \nu_s) \mathcal{H}^2_{k,m} (\cos \theta; \nu_s) \sin \theta \right\rangle_{\theta} \\
\mathcal{J}_{H,k,m} (\nu_s) &= \left\langle \mathcal{H}^2_{k,m} (\cos \theta; \nu_s) \right\rangle_{\theta} + \left\langle \mathcal{H}^2_{k,m} (\cos \theta; \nu_s) \right\rangle_{\theta}
\end{align*}
\]

\[
m' (\nu_s) \text{ links the mean flux of angular momentum carried by a monochromatic wave on an isobar, } \mathcal{F}^\text{AM} = r^{-2} \mathcal{L}^\text{AM}, \text{ to the associated mean kinetic energy flux, } \mathcal{F}^\text{K}. \text{ In the non-rotating case, we retrieve } m' (\nu_s = 0) = m.
\]

Let us first take a look at the damping integral given in Eq. (11) assuming that both prograde and retrograde waves are excited with the same amplitude and have the same eigenvalue \( \Lambda_{k,m} \). In solid body rotation, both waves are equally dissipated when traveling inward and there is no impact on the distribution of angular momentum. In the presence of differential rotation, the situation is different. If the interior is rotating faster than the convection zone, the local frequency of prograde waves diminishes, which enhances their dissipation; the corresponding retrograde waves are then dissipated further inside. This produces an increase of the local differential rotation, and creates a double-peaked shear layer\(^1\). In the presence of shear turbulence, this layer oscillates, producing a “shear layer oscillation” or SLO (cf. Ringot 1998 and Kumar, Talon, & Zahn 1999). This is the first important feature of wave-mean flow interaction. This SLO acts as a filter, through which most low-frequency waves cannot pass. However, if the core is rotating faster than the surface, this filter is not quite symmetric, and retrograde waves will be favored. As a result, a net negative flux of angular momentum will result, and produce a spin down of the core (Talon et al. 2002). This is the key to obtain a quasi-flat rotation profile in the actual Sun under the action of IGWs (Charbonnel & Talon 2005).

Let us now come on the specific action of the Coriolis acceleration through \( m' \). In the case of gravito-inertial waves, \( m' (\nu_s) \) is smaller compared to the non-rotating case. Heuristically, this is a consequence of the lesser horizontal extent of the eigenfunctions. There is a saturation in this reduction, and the effective index \( m' \) tends toward \( m' (\nu_s) = m/3 \). Hence, the effectiveness to transport angular momentum is decreased compared to the non-rotating case. However, we can see that the symmetry between prograde and retrograde waves is conserved so that, if the hypothesis that prograde and retrograde waves are equally excited is conserved, we could expect that the damping of these waves could produce a Shear Layer Oscillation (SLO) similar to the

\(^1\) Because local shears are amplified by waves, even a small perturbation will trigger this.
Figure 3. Ratio $m' = -\frac{\sigma^2 F_{AM} V_F}{F_F}$ for various modes. 1st and 2nd: Gravito-inertial waves corresponding to $\ell = 5, 6$ (continuous lines) $\ell = 3, 4$ (dotted lines). 3rd: Yanai waves of order $m = -5, \ldots, +1$. 4th: Kelvin waves of order $m \geq -6$.

one obtained in the case where the Coriolis acceleration is not taken into account (cf. Talon & Charbonnel 2005). In the case of Yanai waves, we obtain the same behaviour, but with a slower convergence rate. Finally, in the case of Kelvin waves, $m'(\nu_s)$ varies only slightly with rotation and remains close to $m$, their angular momentum being always positive so that they could induce a deposit of angular momentum where they are damped (Pantillon et al. 2007).

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