Forward-backward asymmetry in $e^+e^-$ annihilation into pions or kaons revisited

A.B. Arbuzov$^{a,b}$, T.V. Kopylova$^b$, G.A. Seilkhanova$^a$

$^a$Bogoliubov Laboratory for Theoretical Physics, JINR, Dubna, 141980 Russia
$^b$Department of Higher Mathematics, Dubna State University, 141980 Dubna, Russia

Abstract

Forward-backward (charge) asymmetry in the processes of $e^+e^-$ annihilation into a pair of charged pseudoscalar mesons is recalculated in the one-loop approximation. The exact dependence on the meson masses is taken into account. Results known in the literature are partially corrected. The bulk of the charge asymmetry appears due to double-photon exchange in the $s$ channel. Experimental studies of the asymmetry can be used to verify the point-like approximation used in calculation of radiative corrections due to emission of photons by pions or kaons.

Keywords: electron-positron annihilation; forward-backward asymmetry; radiative corrections

1. Introduction and Preliminaries

Let us consider the process of electron-positron annihilation into a pair of charged pions

$$e^+(p_+) + e^-(p_-) \rightarrow \pi^+(q_+) + \pi^-(q_-) \quad (1)$$

This process gives one of the most important contributions to the evaluation of the hadronic vacuum polarization effect extracted from experimental data on electron-positron annihilation into hadrons.\(^1\) The hadronic contribution to vacuum polarization is then used in theoretical description of various phenomena in particle physics, including the electron and muon anomalous magnetic momenta, Bhabha scattering, Drell-Yan processes etc. This gives us the motivation to revisit the two-pion annihilation channel. The annihilation into kaons can be considered in parallel on the same footing. These
processes are studied experimentally with high precision and modern $e^+e^-$ colliders, see e.g. Refs. 2, 3, 4.

The Born level cross section of process (1) has the form

\[
\frac{d\sigma^\text{Born}}{dc} = \frac{\alpha^2 \beta^3 \pi}{4s}(1 - c^2)|F_\pi(s)|^2, \quad \beta = \sqrt{1 - 4m^2_\pi/s},
\]

\[s = (p_+ + p_-)^2, \quad \theta = \overrightarrow{p_+ q_-}, \quad c \equiv \cos \theta, \quad (2)\]

where $F_\pi(s)$ is the pion form factor, $\alpha$ is the fine structure constant, and $m_\pi$ is the pion mass.\footnote{We systematically drop terms suppressed by the factor $m_e^2/s$.}

Note that the Born cross section is even in the cosine $\theta$. But the observed cross section exhibits a certain asymmetry in $c$ which leads to the so-called charge or forward-backward asymmetry

\[
\eta(c) = \frac{\frac{d\sigma}{dc}(c) - \frac{d\sigma}{dc}(-c)}{\frac{d\sigma}{dc}(c) + \frac{d\sigma}{dc}(-c)}. \quad (3)
\]

In scalar electrodynamics (sQED) with point-like pions (or kaons) this asymmetry comes from virtual (loop) and bremsstrahlung corrections. So studies of this effect potentially allow to verify the applicability of sQED to this process and therefore check the validity of the theoretical approximations used in such calculations. One can note also that in the $O(\alpha)$ the asymmetry comes from a particular set of Feynman diagrams which describe interference of photon emission from the initial and final state particles, see Figs. 1, 2.

\[\footnote{2}\]

2. One-loop contributions

Following the traditional technique of QED radiative correction calculations, we separate the one-loop contributions to the charge-odd part of the cross section into three parts:

\[
\frac{d\sigma_{\text{odd}}}{dc} = \frac{d\sigma^\text{Born}}{dc} \delta_{\text{odd}}^\text{Virt}(\lambda) + \frac{d\sigma^\text{Born}}{dc} \delta_{\text{odd}}^\text{Soft}(\lambda, \Delta) + \frac{d\sigma_{\text{odd}}^\text{Hard}}{dc}(\Delta), \quad (4)
\]

where $\lambda$ is a fictitious photon mass which is used to regularize infrared singularities, $\lambda \ll m_e$. Parameter $\Delta$ defines separation of soft and hard photons:
a photon with energy below $\sqrt{s}/2$ is called “soft” (and “hard” above), $\Delta \ll 1$.

If the final state mesons are treated as point-like particles, one-loop calculations are performed in a straightforward way in sQED. The only complication is due to the necessity to keep the exact dependence on the meson mass since we are interested in a wide energy range including the threshold region.

2.1. Soft photon emission contribution

We take the $\mathcal{O}(\alpha)$ result for charge-odd soft photon contribution from the Bremsstrahlung modules of the SANC system. It can be cast in the form

$$s_{\text{odd}}^{\text{Soft}} = \frac{\alpha}{\pi} \left\{ 2 \ln \frac{2\Delta}{\lambda} \ln \left( \frac{1 - \beta c}{1 + \beta c} \right) - \ln^2 \left( \frac{\beta(1 + c)}{1 - \beta c} \right) + 2 \ln \left( \frac{\beta(1 + c)}{1 - \beta c} \right) \right. \times \ln \left( \frac{1 - \beta}{1 + \beta} \right) + 2 \ln \left( \frac{1 - 2\beta c + \beta^2}{(1 - \beta c)^2} \right) \ln \left( \frac{\beta^2(1 - c)^2}{1 - 2\beta c + \beta^2} \right) \right.$$ 

$$+ \ln^2 \left( \frac{1 - 2\beta c + \beta^2}{(1 - \beta c)^2} \right) + 2 \text{Li}_2 \left( \frac{\beta(1 + c)(1 - \beta)}{(1 - \beta c)(1 + \beta)} \right)$$

$$- 2 \text{Li}_2 \left( \frac{(1 - \beta c)^2}{1 - 2\beta c + \beta^2} \right) - 2 \text{Li}_2 \left( \frac{- (1 - \beta)(1 - \beta c)}{\beta(1 + \beta)(1 - c)} \right) \right\} - (c \leftrightarrow -c), \quad (5)$$

where $\text{Li}_2(x)$ is the dilogarithm function.

To verify the SANC analytic result for the soft photon contribution we performed a numerical test. Numerical results received with the help of the formulae extracted from the SANC system were compared with the corresponding results from the direct 3-dimensional numerical integration of the relevant part of the matrix element. The test was performed for several centre-of-mass energy values including points close to the threshold. An agreement within insignificant uncertainties of numerical integration was observed.

In Table 1 we show the comparisons between results for soft photon contributions from paper (upper lines) and the SANC system (lower lines). One can see that a certain difference appears at energies close to the threshold and it goes down at higher energies. One can note also that the difference is not sensitive to the value of the soft-hard separator $\Delta$. 

3
2.2. Hard photon emission contribution

The Feynman diagrams for the process of $e^+e^-$ annihilation into a pair of charged scalar mesons accompanied by emission of a real photon are shown in Fig. 1. The filled circles in the diagrams denote the pion form factor. Note that we do not introduce form factors in the vertexes describing real photon emission, i.e. we adapt the approximations of point-like pions (or kaons) in the description of the bremsstrahlung contribution.

Using the standard techniques of spinor and scalar QED, we reproduced the full analytic result for the differential hard photon bremsstrahlung contribution given by Eqs. (30) and (31) in paper. The charge-odd contribution to annihilation cross sections comes from the interference of the initial and final state radiation. So, the interference of the Feynman amplitudes represented in Fig. 1 by diagrams (a) and (b) with the (c), (d), and (e) ones is relevant. The corresponding contribution is proportional to the product of pion form factors of different arguments: \( \text{Re}(F_\pi(s)F_\pi^*(s_1)) \) where \( s_1 = (q_+ + q_-)^2 = (p_+ + p_- - k)^2 \).

2.3. Virtual loop contribution

We compute the virtual loop diagrams in the point-like pion approximation without introduction of meson form factors. In order to match the contribution of virtual corrections with the other ones, we cast it in the form

\[
\frac{d\sigma_{\text{odd}}^{\text{Virt}}}{dc} = \frac{d\sigma_{\text{odd}}^{\text{Born}}}{dc} \delta_{\text{odd}}^{\text{Virt}}(\lambda),
\]

where the pion form factor is restored in the factorized Born level cross section.
The result of our calculations is

\[ \delta_{\text{odd}}^{\text{Virt}}(\lambda) = \frac{\alpha}{\pi} \left\{ 2 \ln \frac{\sqrt{s}}{\lambda} \ln \left( \frac{1 + \beta c}{1 - \beta c} \right) + \frac{1}{\beta^2 (1 - c^2)} \left\{ (1 - \beta c) \left[ -l_-^2 + 2 \rho L_- + 2 l_- L_- - 2 \text{Li}_2 \left( \frac{1 - \beta^2}{2 (1 - \beta c)} \right) - \frac{(1 - \beta)^2}{2 \beta} \left( \frac{\rho^2}{2} + \frac{\pi^2}{6} \right) \right] \right. \right. \]

\[ + \left. \left. \frac{1 + \beta^2}{\beta} \left( \rho \ln \frac{2}{1 + \beta} + \text{Li}_2 \left( -\frac{1 - \beta}{1 + \beta} \right) + 2 \text{Li}_2 \left( \frac{1 - \beta}{2} \right) \right) \right\} - (c \leftrightarrow -c) \right\} , \]

\[ l_- = \ln \left( \frac{1 - \beta c}{2} \right), \quad L_- = \ln \left( 1 - \frac{1 - \beta^2}{2 (1 - \beta c)} \right), \quad \rho = \ln \frac{s}{m^2}. \]  (7)

One can see that term with the logarithm of the auxiliary photon mass completely cancels out with the corresponding term in the soft photon con-
Note also that the virtual (and soft) photon contributions are free from the so-called large logarithms \( L \equiv \ln(s/m_e^2) \) in spite of the fact that such logs do appear in individual loop integrals. Formula (7) coincides with Eq. (1.8) from Ref.\(^6\) except the sign before the following dilogarithm \( \text{Li}_2 \left( -\frac{1-\beta}{1+\beta} \right) \). One can see that the improper sign before this term in Ref.\(^6\) is a misprint since in the same paper one can find the opposite sign in front of this dilogarithm in the relevant loop integral \( F_Q \) given in Eq. (1.5). Unfortunately the incorrect sign was reproduced in Refs.\(^8,7\).

### 3. Discussion and Conclusions

It is interesting to note that the odd part of the virtual contribution \(^7\) contains terms proportional to \( 1/\beta^2 \). They appear due to the forced factorization of the Born cross section. Nevertheless numerical estimates show that this double pole behavior is completely cancelled out so that the sum of the virtual and soft odd contributions \( \delta_{\text{odd}}^{\text{virt}} + \delta_{\text{odd}}^{\text{soft}} \) behaves as \( \beta \) to the first power in the threshold region.\(^2\) Remind that for the even part of the same cross section we have the \( 1/\beta \) effect in radiative corrections at the threshold due to the Coulomb final state interactions.\(^10\)

In this way we revisited the charge asymmetry in the processes of electron-positron annihilation into a pair of charged pseudoscalar particles (pions or kaons) at low energies. Certain corrections to the earlier calculations\(^6\) of this quantity are found. These corrections are already implemented in the updated version of the MCGPJ event generator.\(^7\)

There is an obvious disagreement of our result with the charge-odd part of the sum of virtual and soft photon contributions given by formula (50) in Ref.\(^11\) In particular, one can see there a logarithmic singularity in the electron mass, which is not common in an interference of ISR and FRS amplitudes.

Since the asymmetry is a one-loop effect proportional to \( \alpha \sim 1/137 \) and since it doesn’t contain large logarithms, the typical magnitude of the effect doesn’t exceed the percent level. Moreover, the asymmetry is suppressed in the threshold region by the meson relative velocity to the first power. Nevertheless as we noted in the introduction, experimental studies of the asymmetry at modern and future \( e^+e^- \) colliders would be useful verify the

\(^2\)Before the correction of the bug in the formula for the virtual loop contribution, the improper \( 1/\beta^2 \) behavior has been apparent in numerical results.
applicability of the point-like point (kaon) approximation. Note that the box-type loop amplitudes as well as the initial-final state interference represent the so-called double-photon exchange effect. Here it is in the s channel. Verification of different approximations to describe such double-photon exchange processes is important for a better understanding of off-mass-shell hadron propagator behavior. So we advise the experimental community to pay attention to measurements of the charge asymmetry.

Acknowledgments

We are grateful to G.V. Fedotovich, and F.V. Ignatov for critical remarks and stimulating discussions. This work was supported by RFBR grant 20-02-00441.

References

[1] S. Actis et al. [Working Group on Radiative Corrections and Monte Carlo Generators for Low Energies], *Eur. Phys. J. C* **66**, 585 (2010).

[2] R. R. Akhmetshin et al. [CMD-2 Collaboration], *Phys. Lett. B* **527**, 161 (2002).

[3] G. V. Fedotovich et al., *EPJ Web Conf.* **199**, 02027 (2019).

[4] E. A. Kozyrev et al., *Phys. Lett. B* **779**, 64 (2018).

[5] A. Andonov, A. Arbuzov, D. Bardin, S. Bondarenko, P. Christova, L. Kalinovskaya, V. Kolesnikov, and R. Sadykov, *Comput. Phys. Commun.* **181**, 305 (2010).

[6] E. A. Kuraev, S. N. Panov, Budker Institute of Nuclear Physics, Preprint No. INP 91-26, 1991 (unpublished).

[7] A. B. Arbuzov, G. V. Fedotovich, F. V. Ignatov, E. A. Kuraev and A. L. Sibidanov, *Eur. Phys. J. C* **46**, 689 (2006).

[8] A. B. Arbuzov, V. A. Astakhov, A. V. Fedorov, G. V. Fedotovich, E. A. Kuraev and N. P. Merenkov, *JHEP* **9710**, 006 (1997).

[9] A. B. Arbuzov, G. V. Fedotovich, E. A. Kuraev, N. P. Merenkov, V. D. Rushai and L. Trentadue, *JHEP* **9710**, 001 (1997).
[10] A.B. Arbuzov and T.V. Kopylova, *JHEP* **1204**, 009 (2012).

[11] A. Hoefer, J. Gluza and F. Jegerlehner, *Eur. Phys. J. C* **24**, 51 (2002).