Scalar Electrodynamics and Higgs Mechanism in the Unfolded Dynamics Approach

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Abstract

We put forward a novel method of constructing unfolded formulations of field theories, which is based on initial fixation of the form of an unfolded field and subsequent looking for the corresponding unfolded equation as an identity that this field satisfies. Making use of this method, we find an unfolded formulation for 4d scalar electrodynamics. By considering a symmetry-breaking scalar potential, we study the implementation of the Higgs mechanism within the framework of the unfolded dynamics approach. We explore a deformation of unfolded modules in the symmetry-broken phase and identify a non-invertible unfolded-field redefinition that diagonalizes the higgsed system.
1 Introduction

Symmetries play the central role in modern theories of fundamental interactions. One of the most complete implementations of this principle is given by higher-spin (HS) gravity theories (for a partial review of the recent literature on the topic see [1]). These are theories containing interacting massless fields of all spins, that leads to the emergence of infinite-dimensional HS gauge symmetry [2]. This makes HS gravities promising candidates for the role of quantum gravity theory. It is also suggested that HS gravities may represent a symmetric "Coulombian" phase of string theory at trans-Planckian energies [3–5].

In order to keep HS symmetry manifest, a special formalism for operating with HS gravity has been developed, called unfolded dynamics approach [6–10]. Within the unfolded framework, a field theory is formulated as a set of first-order differential equations on unfolded fields, being exterior forms. These unfolded fields encode all d.o.f. of the theory, so usually a spectrum of unfolded fields is infinite or, equivalently, unfolded fields are defined in some larger space, equipped with additional coordinates besides space-time ones. This is the price to pay for having a coordinate-independent manifestly gauge-invariant first-order formulation.

It is of natural interest to try to apply this formalism to various models beyond HS gravities of [7, 8, 11, 12]. Up to now, very few such unfolded nonlinear theories are available [10, 13, 14]. The reason behind this is that a general consistency analysis, which is a standard tool for unfolding, becomes drastically more complicated in the nonlinear case.

In this paper we put forward a novel method of constructing unfolded formulations for theories, which is based on postulating a concrete form of an unfolded field and further looking for corresponding unfolded equations as for identities satisfied by this field. We successfully apply this method to 4d scalar electrodynamics, obtaining its unfolded formulation. We then study the spontaneous symmetry breaking in this theory within the unfolded framework, which is of particular interest in light of recent studies of symmetry breaking in HS gravity [15].

The paper is organized as follows. In Section 2, we give a flash review of the unfolded dynamics approach and present a new method of unfolding with the example of a 4d self-
interacting scalar model. In Section 3, we make use of this method in order to construct an unfolded formulation of $4d$ scalar electrodynamics. Then in Section 4, we analyze Higgs mechanism in the unfolded system we built. In Section 5, we present our conclusions.

2 Unfolded Dynamics Approach

In this Section, we briefly discuss a general construction of the unfolded dynamics approach and consider two relevant examples: an unfolded non-dynamical Minkowski background and an unfolded self-interacting scalar field.

2.1 General construction

Unfolded dynamics approach [6–10] to a field theory consists in representing it in the form of "unfolded" first-order equations

$$dW^A(x) + G^A(W) = 0,$$

where $d$ is the exterior derivative on a space-time manifold $M^d$ and unfolded fields $W^A(x)$ are exterior forms on $M^d$, with $A$ standing for all indices of the field. $G^A(W)$ is built from exterior products of unfolded fields (the wedge symbol is omitted throughout the paper). There is one and only one unfolded equation (2.1) for every unfolded field $W^A$.

The identity $d^2 \equiv 0$ imposes a consistency condition on $G$

$$G^B \frac{\delta G^A}{\delta W^B} \equiv 0.$$ (2.2)

Unfolded equations (2.1) are manifestly invariant under infinitesimal gauge transformations

$$\delta W^A = d\varepsilon^A(x) - \varepsilon^B \frac{\delta G^A}{\delta W^B}$$ (2.3)

with a gauge parameter $\varepsilon^A(x)$ being a $(n-1)$-form for a $n$-form $W^A$. A spectrum of unfolded fields is usually infinite, because it corresponds to all d.o.f. of the theory. Typically, there is some grading bounded from below on the space of unfolded fields. Then equations (2.1) relate higher-grade fields to the space-time derivatives of the lower-grade ones and, at the same time, impose dynamical constraints on the lowest-grade fields. For this reason, lowest-grade unfolded fields are referred to as primary fields, while the higher-grade ones are referred to as descendants.

An unfolded formulation provides a manifestly coordinate-independent and gauge-invariant description of a theory. Both of these features are of critical importance for HS gravity. The first-order nature of the formalism can potentially help in studying integrability (in particular, the problem of looking for conserved charges within the unfolded framework becomes the cohomology problem for some operator determined by (2.1) [10]). All this makes it prominent to apply the unfolded dynamics approach to the field theories beyond the realm of HS gravity. However, constructing unfolded formulations (especially for nonlinear theories) is not an easy task. In this paper we put forward a novel method of unfolding, which allows us to construct an unfolded formulation for scalar electrodynamics with an arbitrary scalar potential. But first, in this Section we demonstrate this method using the example of a self-interacting scalar theory.
2.2 Unfolded Minkowski background

The background geometry of $M^d$ is expressed via imposing the Maurer–Cartan equation on a 1-form connection $\Omega$ taking values in the Lie algebra of symmetries of $M^d$

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0,$$

(2.4)

where square brackets stand for the Lie-algebra commutator. Global symmetries of $M^d$ arise as a residual symmetry (2.3), which is left over after choosing some particular $\Omega_0$ that solves (2.4),

$$d\varepsilon_0 + [\Omega_0, \varepsilon_0] = 0.$$

(2.5)

In this paper, we deal with 4d Minkowski space, so the Lie algebra in question is $iso(1,3)$ and the corresponding connection is

$$\Omega = e^{\alpha\beta} P_{\alpha\beta} + \omega^{\alpha\beta} M_{\alpha\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}},$$

(2.6)

with $P_{\alpha\dot{\alpha}}, M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ being generators of translations and rotations of $\mathbb{R}^{1,3}$, $e^{\alpha\beta}$ and $\omega^{\alpha\beta}$ ($\bar{\omega}^{\dot{\alpha}\dot{\beta}}$) being 1-forms of vierbein and Lorentz connection. Greek indices correspond to two-component (Weyl) spinor representations. They are moved by an antisymmetric Lorentz-invariant spinor metric

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(2.7)

as

$$v_{\alpha} = \epsilon_{\beta\alpha} v^{\beta}, \quad v^{\alpha} = \epsilon^{\alpha\beta} v_{\beta}, \quad \bar{v}_{\dot{\alpha}} = \epsilon_{\dot{\beta}\dot{\alpha}} \bar{v}^{\dot{\beta}}, \quad \bar{v}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{v}_{\beta}.$$  

(2.8)

The simplest (non-degenerate) solution to (2.4) with $\Omega$ being (2.6) is provided by global Cartesian coordinates

$$e_m^{\alpha\beta} = (\bar{\sigma}_m)^{\beta\alpha}, \quad \omega_m^{\alpha\beta} = 0, \quad \bar{\omega}_m^{\dot{\alpha}\dot{\beta}} = 0.$$  

(2.9)

Then the general solution to (2.5) determines 10 parameters of global Poincaré transformations.

Here the spectrum of unfolded field is finite, containing only the 1-form $\Omega$, because this system is non-dynamical.

2.3 Example: unfolded self-interacting scalar

Now let us consider two unfolding procedures for the theory of a 4d self-interacting scalar field: the standard one, implemented in [14], and a novel one, which is simpler and more convenient, allowing the application to scalar electrodynamics.

A standard strategy is based on studying the consistency condition (2.2): one assumes some spectrum of unfolded fields, then writes down an appropriate ansatz for unfolded equations (2.1) and finally tries to fix it by solving for the consistency condition (2.2) (which, in its turn, may force one to modify the initially assumed field spectrum and, accordingly, the ansatz). This can be performed for linear theories (see e.g. [16–23]), but for nonlinear models this method is not particularly productive. The consistency equation is of higher order in fields than the initial ansatz, so one ends up with a complicated system of entangled equations on coefficients in the ansatz. Thus, in general it is uneasy to constrain sufficiently the form of the ansatz so that the procedure is practicable.
As an example, consider unfolding a nonlinear scalar theory
\[(\Box + m^2)\phi + U'(\phi) = 0,\] (2.10)
where \(U'\) is the variation of the scalar potential.

This has been unfolded in [14] by analyzing consistency (in fact, a more general off-shell case has been solved there, but that does not interest us here). There an ansatz was guessed, that was simple enough to allow for direct analysis. Let us sketch that derivation.

A spectrum of unfolded fields represents a set of 0-forms that can be collected into a single unfolded master-field
\[\Phi(Y|x) = \sum_{n=0}^{\infty} \Phi_{\alpha(n),\bar{\alpha}(n)}(x) y^{\alpha_1}...y^{\alpha_n} \bar{y}^{{\bar{\alpha}}_1}...\bar{y}^{{\bar{\alpha}}_n},\] (2.11)
where the condensed index notations are used
\[f_{\alpha(n)} := f_{\alpha_1...\alpha_n},\] (2.12)
and a pair of auxiliary commuting Weyl spinors \(Y = (y^{\alpha}, \bar{y}^{{\bar{\alpha}}})\) is introduced for the convenience of operating with symmetric spinor-tensors. Due to their commutativity, \(Y\) are null with respect to the spinor metric
\[y^{\alpha} y^{\beta} \varepsilon_{\alpha\beta} = 0, \quad \bar{y}^{{\bar{\alpha}}} \bar{y}^{{\bar{\beta}}} \varepsilon_{{\bar{\alpha}}{\bar{\beta}}} = 0.\] (2.13)
The master-field (2.11) corresponds to an infinite set of symmetric traceless Lorentz tensors of all ranks, as can be seen from contracting all spinor indices of \(\Phi_{\alpha(n),\bar{\alpha}(n)}\) with \(\sigma\)-matrices
\[\Phi_{a_1a_2...a_n} = (\sigma_{a_1})^{{\dot{\alpha}}_1}...({\sigma}_{a_n})^{{\dot{\alpha}}_n} \Phi_{a_{\alpha(n)},\bar{\alpha}(n)}, \quad \eta^{a_1a_2} \Phi_{a_{\alpha(n)},\bar{\alpha}(n)} = 0.\] (2.14)
This is the unfolded spectrum of a free scalar field [16] and a scalar field of HS gravity [6], so it is natural to take it for the problem in question.

We also introduce spinorial derivatives
\[\partial_{\alpha} y^{\beta} = \delta^{\beta}_{\alpha}, \quad \bar{\partial}_{\dot{\alpha}} \bar{y}^{{\dot{\beta}}} = \delta^{{\dot{\beta}}}_{\dot{\alpha}}\] (2.15)
and spinorial Euler operators
\[N = y^{\alpha} \partial_{\alpha}, \quad \bar{N} = \frac{1}{2} \bar{y}^{{\dot{\alpha}}} \bar{\partial}_{\dot{\alpha}}.\] (2.16)
For a scalar master-field (2.10) two Euler operators in fact coincide
\[N \Phi = \bar{N} \Phi.\] (2.17)

The Euler operator can be taken as a grading operator discussed in Subsection 2.1. We will see that \(Y\)-dependent components of the master-field (2.11) are differential descendants of the primary scalar field, which is a \(Y\)-independent component
\[\phi(x) = \Phi(Y = 0|x).\] (2.18)

In [14], the following ansatz for an unfolded equation was proposed
\[d_L \Phi - a_N e \partial \bar{\partial} \Phi + b_N e y \bar{y} m^2 \Phi + c_N e y \bar{y} U'(f_N \Phi) = 0,\] (2.19)
where
\[ e\bar{\partial} := e^{\alpha\bar{\delta}} \partial_\alpha \bar{\delta}_\bar{\beta}, \quad e y \bar{y} := e^{\alpha\bar{\delta}} y_\alpha \bar{y}_\bar{\beta}; \quad \tag{2.20} \]
coefficients \( a_N, b_N, c_N, f_N \) depend on Euler operator \( N \) (2.16), every \( f_N \) in \( U' \) acts on a single factor of \( \Phi \) and \( d_L \) is the Lorentz-covariant derivative
\[ d_L f(Y|x) := \left( d + \omega^{\alpha\beta} y_\alpha \partial_\beta + \bar{\omega}^{\bar{\alpha}\bar{\beta}} \bar{y}_{\bar{\alpha}} \bar{\partial}_{\bar{\beta}} \right) f(Y|x), \quad \tag{2.21} \]
which in Cartesian coordinates (2.9) comes down to the exterior derivative.

A general solution (up to constant rescaling of \( \Phi, m^2 \) and \( U \)) to the corresponding consistency condition (2.2) for the equation (2.19) can be formulated in terms of dependence of coefficients on arbitrary (but necessarily non-zero) \( a_N \)
\[ b_N = \frac{1}{N(N+1)a_{N-1}}, \quad c_N = \frac{1}{(N+1)! \prod_{i=0}^{N-1} a_i}, \quad f_N = N! \prod_{i=0}^{N-1} a_i. \quad \tag{2.22} \]
Then \( Y \)-dependence of \( \Phi \) is manifestly resolved as
\[ \Phi(Y|x) = \sum_{n=0}^{\infty} \frac{(y^{\alpha\bar{\delta}} \nabla_\alpha \bar{\delta})^n}{(n!)^2 \prod_{i=0}^{n-1} a_i} \phi(x), \quad \tag{2.23} \]
where a 0-form derivative \( \nabla_\alpha \bar{\delta} \) is introduced via
\[ d_L = e^{\alpha\bar{\delta}} \nabla_\alpha \bar{\delta} \quad \tag{2.24} \]
(in Cartesian coordinates, \( \nabla_\alpha \bar{\delta} \) comes down to the usual partial derivative) and the primary scalar \( \phi(x) \) is subjected to the nonlinear Klein–Gordon equation (2.10) with
\[ \Box := \frac{1}{2} \nabla_\alpha \bar{\delta} \nabla^{\alpha\bar{\delta}}. \quad \tag{2.25} \]
For the sake of simplicity, we assume that \( \nabla_\alpha \bar{\delta} \) commutes with \( Y \), because one can always choose Cartesian coordinates. Then the covariance of the final formulas can be achieved by supplementing all partial \( x \)-derivatives with \( \omega \)-terms, since this is the only way the Lorentz-connection can enter the equations.

The dependence (2.23) can be found by acting with
\[ y^{\alpha\bar{\delta}} \bar{y}_\bar{\beta} \frac{\delta}{\delta e^{\alpha\bar{\delta}}}, \quad \tag{2.26} \]
on (2.19). Then (2.10) result from acting on (2.19) with
\[ (\nabla^{\alpha\bar{\delta}} + a_N \partial^\alpha \bar{\delta}) \frac{\delta}{\delta e^{\alpha\bar{\delta}}}. \quad \tag{2.27} \]

Thus, the unfolded system (2.19) with coefficients obeying (2.22) indeed describes the theory of a self-interacting scalar. \( Y \)-dependent components of the unfolded master-field \( \Phi \) (2.11) represent descendants (traceless space-time derivatives) of the primary (with respect to the \( N \)-grading) scalar field \( \phi(x) \), subjected to the nonlinear Klein–Gordon equation (2.10).
In this paper, we put forward a different method of unfolding a field theory: one should first postulate some concrete form of an unfolded master-field and then look for a corresponding unfolded equation, identically satisfied by this master-field. This last step in practice consists of expressing an action of $\epsilon^{\alpha\beta} \partial_\alpha \bar{\partial}_\beta$ on the master-field in terms of $d_L$ and other spinor operators acting on unfolded fields. Let us demonstrate how this works for the problem of unfolding the nonlinear scalar.

We start by postulating that an unfolded scalar master-field is

$$\Phi(Y|x) = e^{y^\alpha \bar{y}^\alpha \nabla_\alpha \phi(x)}, \quad (2.28)$$

with the primary field $\phi(x)$ subjected to (2.10). To simplify the appearance of formulas, we further omit spinor indices contracted between spinors and the derivative

$$\nabla y\bar{y} := y^\alpha \bar{y}^\dot{\alpha} \nabla_\dot{\alpha}. \quad (2.29)$$

Now we need to find an appropriate unfolded equation, whose solution is (2.28). To this end, we use the following identity, which holds for an arbitrary function $f$,

$$\partial_\alpha \bar{\partial}_\dot{\alpha} f(z_{\dot{\alpha}\dot{\beta}} y^{\dot{\beta}} \bar{y}^{\dot{\beta}}) = z_{\alpha\dot{\alpha}}(N+1)f' - \frac{1}{2} y_{\alpha\dot{\alpha}} z_{\dot{\alpha}\dot{\beta}} z^{\beta\dot{\beta}} f'' , \quad (2.30)$$

where the prime denotes a derivative of $f$ with respect to its full argument. Then we have

$$\partial_\alpha \bar{\partial}_\dot{\alpha} \Phi(Y|x) = (N+1) \nabla_{\alpha\dot{\alpha}} e^{\nabla y\bar{y} \phi(x)} - y_{\alpha\dot{\alpha}} \Box e^{\nabla y\bar{y} \phi(x)}, \quad (2.31)$$

and hence

$$\partial_\alpha \bar{\partial}_\dot{\alpha} \Phi = (N+1) \nabla_{\alpha\dot{\alpha}} \Phi + y_{\alpha\dot{\alpha}}(m^2 \Phi + U'(\Phi)). \quad (2.32)$$

Contracting this with the vierbein, we get the desired unfolded equation

$$d_L \Phi - \frac{1}{N+1} \epsilon \partial \bar{\partial} \Phi + \frac{1}{N+1} \epsilon y\bar{y} (m^2 \Phi + U'(\Phi)) = 0, \quad (2.33)$$

which is the particular case of the general solution (2.22), corresponding to the choice

$$a_N = \frac{1}{N+1}. \quad (2.34)$$

Note that in our analysis the resulting unfolded equation (2.33) is consistent by construction, because it arises as the identity satisfied by the unfolded field (2.28) that we started with. This also means that this equation, being nonlinear (the form of the potential $U(\phi)$ is unrestricted), is manifestly integrable (in the sense of restoring the $Y$-dependence) by construction, as we know its solution. This, in its turn, allows us to generate all other particular unfolded systems (2.19) and/or immediately obtain their solutions.

Let us illustrate the last claim. Suppose one wants to generate an unfolded system of the type (2.19) with some given $a_N$. Having the equation (2.33) with its solution (2.28) in hand, there is no need to repeat the analysis again. One just needs to redefine the unfolded master-field with a for-now arbitrary coefficient $\rho_N$

$$\Phi(Y|x) = \rho_N \tilde{\Phi}(Y|x). \quad (2.35)$$
Substituting this into (2.33) yields
\[ d_L \tilde{\Phi} - \frac{\rho_{N+1}}{\rho_N (N + 1)} e \bar{\partial} \partial \tilde{\Phi} + \frac{\rho_{N-1}}{\rho_N (N + 1)} e \bar{y} \bar{m}^2 \tilde{\Phi} + \frac{1}{\rho_N (N + 1)} e \bar{y} \bar{y} U'(\rho_N \tilde{\Phi}) = 0. \] (2.36)

Now, demanding
\[ \frac{\rho_{N+1}}{\rho_N (N + 1)} = a_N, \] (2.37)
on one finds
\[ \rho_N = \rho_0 \cdot N! \prod_{i=0}^{N-1} a_i \] (2.38)
with arbitrary (non-zero) \( \rho_0 \).

Thus, one gets a consistent unfolded system of the required form (2.36) with coefficients determined by (2.38) and with the solution
\[ \tilde{\Phi}(Y | x) = \frac{1}{\rho_N} e^{\nabla \bar{y} \bar{y}} \phi(x), \] (2.39)
as follows from (2.35). For example, if one takes \( a_N = 1 \), which is a common choice in the HS literature, this gives
\[ d_L \tilde{\Phi} - e \partial \bar{\partial} \tilde{\Phi} + \frac{1}{N(N + 1)} e \bar{y} \bar{m}^2 \tilde{\Phi} + \frac{1}{(N + 1)!} e \bar{y} \bar{y} U'(N! \tilde{\Phi}) = 0, \] (2.40)
\[ \tilde{\Phi}(Y | x) = \sum_{n=0}^{\infty} \frac{(\nabla \bar{y} \bar{y})^n}{(n!)^2} \phi(x) = _0 F_1(; 1; \nabla \bar{y} \bar{y}) \phi(x), \] (2.41)
so in this case an unfolding operator is a confluent hypergeometric limit function [14].

This also allows one to immediately write down an unfolded system for a given form of the master-field. To this end one just needs to use (2.39) in order to find corresponding \( \rho_N \), then the required equation is (2.36). Suppose that one needs an unfolded system that leads to the master-field of the form
\[ \tilde{\Phi}(Y | x) = \sum_{n=0}^{\infty} k_n (\nabla \bar{y} \bar{y})^n \phi(x) \] (2.42)
with all \( k_n \) being non-zero. Comparing with (2.39), one finds corresponding \( \rho_N \) to be
\[ \rho_N = \frac{1}{k_N \cdot N!}. \] (2.43)

Thus, although the proposed method is aimed at constructing a particular unfolded formulation, in effect it allows one to easily reproduce all the results of the general consistency analysis. For a nonlinear gauge theory considered in the paper, the consistency analysis is in fact impracticable, so the new method remains the only available tool.

Let us also note that, although we were considering the nonlinear theory, all unfoldings (2.23) and (2.28) were linear. It means that, in general, interactions do not deform the unfolding map. In fact, it is gauge interactions that do. Scalar electrodynamics, considered in the next Section, provides an example. But even here one can see that introducing dynamical gravity would make the unfolding nonlinear, since the operator \( \nabla_{\alpha \dot{\alpha}} \) is defined in terms of the vierbein (2.24).
Summing up all these observations, we can determine the level of generality of the ansatz (2.19). We see that it covers all unfoldings $\phi(x) \to \Phi(Y|x)$ of the form (2.42), i.e. all unfoldings which are regular and linear. Let us discuss both of these properties.

The form of the master-field (2.42) implies that the primary $\phi(x)$ is infinitely differentiable (at least up to d’Alembertians), otherwise the master-field does not exist. However, the unfolded equation (2.19) is of first order and local in $x$ and thus allows for singular solutions, which means that the representation (2.42) may be valid only locally. Solutions which are singular in $x$ and/or in $Y$ are important in HS gravity, in particular in the context of HS black holes [24–28].

Another point is that one may consider nonlinear unfoldings like e.g.

$$\Phi_{\text{nonlin}}(Y|x) = \sum_{n=0}^{\infty} k_n (\nabla y \bar{y})^n \phi(x) + \sum_{n=1}^{\infty} \ell_n (\nabla y \bar{y})^n \phi^2(x),$$

or more complicated ones. They still have the same primary component $\Phi_{\text{nonlin}}(Y = 0|x) = \phi(x)$ (this is why the second sum starts with $n = 1$), but the relation between $Y$-variables and $x$-derivatives of $\phi(x)$ is highly sophisticated now. So the method set out here is not directly applicable to such unfoldings. Perhaps, this may serve as a general guiding principle: one should look for unfoldings which are linear in non-gauge fields (even for interacting theories), since they allow for a direct local space-time interpretation of auxiliary $Y$-variables.

Finally, let us mention that in principle there is a different route to unfold dynamical nonlinear theories. Namely, one can generate corresponding unfolded equations by quotienting the space of off-shell (i.e. non-dynamical) unfolded fields by an invariant subspace, spanned by the differential descendants that are put to zero by dynamical equations. Let us illustrate this by deriving a dynamical nonlinear scalar field theory from a linear off-shell one.

We consider an off-shell unfolded master-field

$$\Psi(Y, \tau|x) = e^{\tau \Box + y \bar{y} \nabla} \phi(x),$$

with the primary field $\phi(x)$ completely unconstrained. A new auxiliary scalar variable $\tau$ encodes an expansion in d’Alembertians of $\phi(x)$, which are now unfixed [21]. The corresponding unfolded equation, which can be deduced analogously to the dynamical case considered above, is [14]

$$\frac{1}{N+1} e \partial \bar{\partial} \Psi - \frac{1}{N+1} e y \bar{y} \frac{\partial}{\partial \tau} \Psi = 0.$$  

(2.46)

This unfolded system just encodes an infinite set of constraints that express all descendants in terms of the primary $\phi(x)$ leaving it unconstrained. So, in a sense, all off-shell systems with the same spectrum of primary fields are formally equivalent, differing only in the specific way in which descendants are parameterized (which, however, can affect such important points as regularity, locality etc.).

Now we want to impose a nonlinear dynamical equation (2.10) on $\phi(x)$. Considering (2.45), we see that it is equivalent to the following nonlinear constraint on $\Psi$

$$\left. \frac{\partial}{\partial \tau} \Psi + m^2 \Psi + U'(\Psi) \right|_{\tau=0} = 0.$$  

(2.47)

Thus, the linear off-shell unfolded system (2.46) endowed with the nonlinear "initial condition" (2.47) for $\tau$-evolution, describes a dynamical scalar field subjected to (2.10). The presence of
the nonlinear constraint, which cannot be manifestly resolved, complicates the analysis of this system. But if one considers another, nonlinear, off-shell scalar system, the constraint becomes linear and can be easily resolved, leading directly to (2.33) [14]. This demonstrates that, despite the formal equivalence of all off-shell formulations, practical analysis requires the choice of a very specific one.

This way of generating a dynamical unfolded system from a non-dynamical one is equivalent to the well-known method used for $d$-dimensional unfolded theories, where unfolded fields are tensors. There, on-shell factorization is implemented by imposing some trace constraints on off-shell tensor fields [10, 16, 29]. These constraints may be very complicated: there are examples of nonlinear off-shell HS theories [10, 12], whose explicit on-shell reductions are still not available.

### 3 Unfolded Scalar Electrodynamics

In this Section, we construct an unfolded formulation for a self-interacting complex scalar, minimally interacting with an electromagnetic field. In the standard formulation, this corresponds to Lagrangian e.o.m.

\[
\frac{1}{2}D_{\alpha\dot{\alpha}}D^{\alpha\dot{\alpha}}\phi + (m^2 + U'(\phi^*))\phi = 0, \quad \frac{1}{2}D^*_{\alpha\dot{\alpha}}D^{*\alpha\dot{\alpha}}\phi^* + (m^2 + U'(\phi^*))\phi^* = 0,
\]

\[
\nabla_{\beta\dot{\alpha}}F^\beta_{\alpha} = iq(\phi D^*_{\alpha\dot{\alpha}}\phi^* - \phi^* D_{\alpha\dot{\alpha}}\phi), \quad \nabla_{\alpha\beta}\bar{F}^\dot{\alpha}_{\dot{\alpha}} = iq(\phi D^*_{\alpha\dot{\alpha}}\phi^* - \phi^* D_{\alpha\dot{\alpha}}\phi),
\]

where $q$ is an electric charge and covariant derivatives are defined as

\[
D_{\alpha\dot{\alpha}} := \nabla_{\alpha\dot{\alpha}} - iqA_{\alpha\dot{\alpha}}, \quad D^*_{\alpha\dot{\alpha}} := \nabla_{\alpha\dot{\alpha}} + iqA_{\alpha\dot{\alpha}},
\]

\[
[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] = -iq\epsilon_{\alpha\beta}\bar{F}^\dot{\alpha}_{\dot{\alpha}} - iq\epsilon_{\alpha\beta}F_{\alpha\beta}, \quad [D^*_{\alpha\dot{\alpha}}, D^*_{\beta\dot{\beta}}] = iq\epsilon_{\alpha\beta}\bar{F}^\dot{\alpha}_{\dot{\alpha}} + iq\epsilon_{\alpha\beta}F_{\alpha\beta}.
\]

Our strategy goes as follows. We start with unfolding a conserved electric current of a general form. Then we couple this current to the unfolded Maxwell field. Finally, we unfold equations for a charged complex scalar field coupled to the Maxwell field and express the electric current that we started with in terms of this unfolded scalar, thus closing the system.

#### 3.1 Electric current

We are to find an unfolded system describing a conserved electric current of a general form, i.e. a vector field $j_{\alpha\dot{\alpha}}(x)$ subjected to the conservation condition

\[
\nabla_{\alpha\dot{\alpha}}j^\alpha_{\dot{\alpha}} = 0.
\]

We begin with postulating the form of the corresponding unfolded field

\[
J(Y|x) := e^{\nabla_{\alpha\dot{\alpha}}j_{\alpha\dot{\alpha}}y^\alpha\bar{y}^\dot{\alpha}}.
\]

Next, we need to calculate $\partial_{\alpha}\bar{\partial}_{\alpha}J$ and express it in terms of $\nabla_{\alpha\dot{\alpha}}J$ and spinorial operators acting on unfolded fields. We have

\[
\partial_{\alpha}J = \nabla_{\alpha\dot{\beta}}\bar{y}^\dot{\beta}J + e^{\nabla_{\alpha\dot{\beta}}j_{\alpha\dot{\beta}}y^\beta\bar{y}^\dot{\beta}},
\]

\[
\partial_{\alpha}\bar{\partial}_{\alpha}J = (N + 2)\nabla_{\alpha\dot{\alpha}}J - y_{\alpha\dot{\alpha}}\square J + e^{\nabla_{\alpha\dot{\beta}}j_{\alpha\dot{\beta}}y^\beta\bar{y}^\dot{\beta}} + e^{\nabla_{\alpha\dot{\beta}}y_{\alpha\dot{\beta}}\nabla_{\alpha\dot{\beta}}j_{\alpha\dot{\beta}}y^\beta\bar{y}^\dot{\beta}}.
\]
The third term on the r.h.s. can be rewritten as

\[ e^\nabla y \bar{a} = j_\alpha + \frac{1}{N} (\nabla y \bar{a}) e^\nabla y \bar{a} = j_\alpha + \frac{1}{N} \nabla_a J + \frac{1}{2N} e^\nabla y (\bar{y}_a \partial_\alpha (\nabla_{\beta\dot{\beta}} \bar{y}^\beta \dot{y}^\beta) + y_a \partial_\alpha (\nabla_{\dot{\beta}\dot{\beta}} \bar{y}^\dot{\beta} \dot{y}^\dot{\beta})) \]

(3.9)

where we have taken into account that

\[ \nabla_\alpha j_\beta = \nabla_\beta j_\alpha \]

(3.10)
due to the conservation condition (3.5). Next,

\[ e^\nabla y \alpha \partial_\alpha (\nabla_{\beta\dot{\beta}} \bar{y}^\beta \dot{y}^\beta) = \bar{y}_a \partial_\alpha J + \bar{y}_a \nabla_\alpha J^+, \]

(3.11)

where a new unfolded field \( J^+ \) is introduced as

\[ J^+(Y|x) := e^\nabla y \nabla_{\dot{\beta}\dot{\beta}} \bar{y}^\dot{\beta} \dot{y}^\dot{\beta}, \]

(3.12)

with a complex conjugate unfolded field \( J^- \) being

\[ J^-(Y|x) := e^\nabla y \nabla_{\beta\dot{\beta}} \bar{y}^\beta \dot{y}^\beta. \]

(3.13)

The reason for introducing additional unfolded fields is that there are new sequences of differential descendants of \( j_\alpha \), that neither fit into the sequence of symmetrized traceless derivatives contained in \( J \) nor are fixed by the differential constraint (3.5). They arise from \( \nabla_{\dot{\beta}\dot{\beta}} \bar{y}^\dot{\beta} \dot{y}^\dot{\beta} \) and \( \nabla_{\beta\dot{\beta}} \bar{y}^\beta \dot{y}^\beta \), which generate \( J^+ \) and \( J^- \) respectively.

For these new unfolded fields, we need unfolded equations as well. We have

\[ \partial_\alpha J^+ = \nabla_{\beta\dot{\beta}} y^\beta \dot{J}^+, \]

(3.14)

\[ \partial_\alpha \bar{\partial}_a J^+ = (N + 1) \nabla_\alpha J^+ - y_a \bar{y}_a \square J^+ - 2y_a \partial_\alpha \bar{\partial}_a J + 2y_a \nabla_\beta \bar{y}^\beta \square J. \]

(3.15)

From here we get, after contracting with \( \bar{y}^\alpha \),

\[ \nabla_\alpha \bar{y}^\alpha J^+ = \frac{\bar{N}}{N+2} \partial_\alpha J^+ + \frac{2}{N+2} y_a \square J. \]

(3.16)

This allows us to close the expression for \( \partial_\alpha \bar{\partial}_a J \) (3.8), except for the terms with \( j_\alpha\bar{a} \) and \( \square J \),

\[ \partial_\alpha \bar{\partial}_a J = j_\alpha\bar{a} + \frac{(N+1)^2}{N} \nabla_\alpha J - \frac{(N+2)}{N} y_a \bar{y}_a \square J + \frac{1}{N} y_a \partial_\alpha J^+ + \frac{1}{N} y_a \bar{\partial}_a J^-. \]

(3.17)

Multiplying by \( \frac{N}{(N+1)^2} \) and contracting with the vierbein, an unfolded equation for \( J \) arises

\[ d_L J = \frac{N}{(N+1)^2} e \partial \bar{\partial} J - \frac{(N+2)}{(N+1)^2} e y \partial J \]

\[ + \frac{1}{(N+1)^2} e \bar{y} \partial J^+ + \frac{1}{(N+1)^2} e \bar{y} \partial J^- = 0, \]

(3.18)

where

\[ e \bar{y} \partial := e^\alpha \beta \bar{y}_\beta \partial_\alpha, \quad e y \partial := e^\alpha \beta y_\alpha \bar{\partial}_\beta. \]

(3.19)

Note that \( j_\alpha\bar{a} \) drops out of (3.18) because \( N j_\alpha\bar{a} = 0 \).
Now, to close the $J^+$-equation (3.15), we contract (3.17) with $y^\alpha(N)\bar{y}_\alpha J$, that produces
\[
\nabla^\beta\bar{\partial}_\alpha y^\beta J = \frac{(N - 1)}{N} \bar{\partial}_\alpha J - \frac{1}{N} \bar{y}_\alpha J^+,
\]
so that
\[
\bar{\partial}_\alpha \bar{\partial}_\alpha J^+ = (N + 1)\nabla_{\alpha \bar{\alpha}} J^+ - \frac{(N + 1)}{(N - 1)} y_\alpha \bar{y}_\alpha \Box J^+ - 2 \frac{1}{(N - 1)} y_\alpha \bar{\partial}_\alpha \Box J,
\]
and the unfolded equation for $J^+$ is
\[
d_L J^+ - \frac{1}{(N + 1)} e \bar{\partial} \partial J^+ - \frac{1}{(N + 1)} e y \bar{y} \Box J^+ - \frac{2}{(N + 1)(N + 1)} e y \bar{\partial} \Box J = 0.
\]
The equation (3.20) also leads to a simple relation between $J$ and $J^+$
\[
J^+ = y^\alpha \bar{\partial}_\alpha \nabla_{\alpha \bar{\alpha}} J.
\]
Conjugation gives an unfolded equation for $J^-$
\[
d_L J^- - \frac{1}{(N + 1)} e \partial \bar{\partial} J^- - \frac{1}{(N + 1)} e y \bar{y} \Box J^- - \frac{2}{(N + 1)(N + 1)} e y \partial \Box J = 0
\]
and
\[
J^- = \bar{y}^\bar{\alpha} \partial^\alpha \nabla_{\alpha \bar{\alpha}} J.
\]
This solves the problem: the system of three unfolded equations (3.18), (3.22) and (3.24) represents an unfolded formulation for the electric current, with the solution being (3.6), (3.23), (3.25) and with the primary field obeying (3.5). Strictly speaking, in order to have a completely unfolded system, one also needs to process $\Box$-terms, because the spacetime derivative is allowed to appear in an unfolded equation only through the exterior derivative. This can be done either by introducing one more auxiliary (scalar) variable on top of spinors $Y$ that encodes new unfolded descendants [21], if the electric current is off-shell, or by expressing them in terms of $J$, if the current is built out of some dynamical on-shell fields. But here we leave the equations as they are for now, since this system plays only an intermediate role in our analysis.

The system can be made more compact via defining a united unfolded master-field as
\[
\mathcal{J}(Y|x) := J + J^+ + J^-.
\]
Then, introducing an averaged Euler operator
\[
\nu := \frac{N + \bar{N}}{2},
\]
three equations (3.18), (3.22), (3.24) can be combined into one
\[
d_L \mathcal{J} - \frac{1}{(N + 1)(\nu + 1)} \left\{ \nu e \partial \bar{\partial} \mathcal{J} + (\nu + 2) e y \bar{y} \Box \mathcal{J} - e y \bar{\partial} (J^+ - 2 \Box \mathcal{J}) - e y \partial (J^- - 2 \Box \mathcal{J}) \right\} = 0.
\]
3.2 Maxwell equations

The second step is to construct an unfolded system for the electromagnetic field sourced by the unfolded electric current.

We need to unfold Maxwell equations

\[ \nabla^\beta \dot{F}_\alpha^\beta = q_j^\alpha, \quad \nabla_\alpha \bar{F}_\dot{\alpha}^\beta = q_j^\alpha, \quad (3.29) \]

where the (anti-)selfdual Maxwell tensor is determined by the gauge field \( A_{\alpha\dot{\alpha}}(x) \) as

\[ F_{\alpha\dot{\alpha}} := \nabla^\beta A_{\alpha\dot{\beta}}, \quad \bar{F}_{\dot{\alpha}\dot{\alpha}} := \nabla^\dot{\beta} A^{\dot{\beta}\dot{\alpha}}. \quad (3.30) \]

To make a gauge symmetry manifest, we treat \( A_{\alpha\dot{\alpha}}(x) \) as the vierbein expansion of an unfolded 1-form \( A(x) = e^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}} \),

that will generate unfolded \( U(1) \) gauge symmetry in the final system according to the general formula (2.3).

Now we postulate following expressions for the unfolded 0-forms of the Maxwell tensors

\[ F(Y|x) = e^y y^\alpha y^\alpha, \quad \bar{F}(Y|x) = e^\bar{y} \bar{y}^\dot{\alpha} \bar{y}^\dot{\alpha}. \quad (3.32) \]

Then (3.30) can be written in the unfolded form as

\[ \text{d}A = \frac{1}{4} e_{\alpha\dot{\beta}} e^{\alpha\dot{\beta}} \partial_\alpha \partial_\dot{\alpha} F|_{\bar{y}=0} + \frac{1}{4} e_{\dot{\beta}} e_{\alpha\dot{\beta}} \bar{\partial}_\dot{\alpha} \bar{\partial}_\alpha F|_{y=0}. \quad (3.33) \]

The next task is to calculate and process \( \partial_\alpha \bar{\partial}_\dot{\alpha} F \). We get

\[ \bar{\partial}_\dot{\alpha} F = \nabla_{\beta\dot{\alpha}} y^\beta F, \quad \partial_\alpha \bar{\partial}_\dot{\alpha} F = (N+1) \nabla_{\alpha\dot{\alpha}} F - y_\alpha \bar{y}_\dot{\alpha} \Box F + 2 q y_\alpha e^y y^\beta j_{\beta\dot{\alpha}} y^\dot{\beta}, \quad (3.35) \]

where we made use of Maxwell equations (3.29). They also yield

\[ \Box F_{\alpha\beta} = -q \nabla_{\alpha\dot{\beta}} j^{\dot{\beta}}, \quad (3.36) \]

so that, taking into account (3.12),

\[ \Box F = -q J^+. \quad (3.37) \]

The last term in (3.35) can be re-expressed as

\[ y_\alpha e^\bar{y} y^\beta j_{\beta\dot{\alpha}} = y_\alpha \bar{\partial}_\dot{\alpha} J - y_\alpha y^\beta \nabla_{\beta\dot{\alpha}} J. \quad (3.38) \]

The equation (3.17) leads to

\[ y^\beta \nabla_{\beta\dot{\alpha}} J = \frac{(N-1)}{(N+1)} \bar{\partial}_\dot{\alpha} J - \frac{1}{(N+1)} \bar{y}_\dot{\alpha} J^+, \quad (3.39) \]

hence

\[ \partial_\alpha \bar{\partial}_\dot{\alpha} F = (N+1) \nabla_{\alpha\dot{\alpha}} F + q \frac{(N+1)}{(N+1)} y_\alpha \bar{y}_\dot{\alpha} J^+ + \frac{2q}{(N+1)} y_\alpha \bar{\partial}_\dot{\alpha} J, \quad (3.40) \]
which solves the problem. Contracting this with the vierbein, we obtain the required unfolded equation
\[ d_L F - \frac{1}{(N+1)(N+1)} \{ \nu \varepsilon \partial \bar{\varepsilon} F - q(\nu + 2) e y \bar{y} J^- - 2 q e y \bar{y} J \} = 0. \]  
(3.41)
Conjugation gives
\[ d_L \bar{F} - \frac{1}{(N+1)(N+1)} \{ \nu \varepsilon \partial \bar{\varepsilon} \bar{F} - q(\nu + 2) e y \bar{y} J^- - 2 q e y \bar{y} J \} = 0. \]  
(3.42)
From (3.40) also a simple expression for \( J \) follows
\[ J = - \frac{1}{2 \bar{q}} y \bar{y} \partial \bar{y} \nabla \alpha F. \]  
(3.43)
This implies that, from a formally-mathematical point of view, the unfolded electric current \( J \) represents just a subsequence of unfolded descendants of \( A \). Indeed, Maxwell equations (3.29) can be interpreted as a constraint, which expresses the descendant field \( j_{\alpha \dot{\alpha}} \) in terms of the primary field \( F_{\alpha \alpha} \). This simple observation plays an important role in the quantization of unfolded field theories [14].

### 3.3 Charged scalar field

Now we move to the main part of the problem: constructing an unfolded system for the charged scalar field, interacting with the electromagnetic one. As was said above, while the cases of electric current and Maxwell field, representing linear models, can be solved via direct consistency consideration [21], this looks almost unfeasible for the problem at hand.

We need to unfold equations (3.1). We postulate unfolded scalar fields to be
\[ \Phi(Y|x) = e^{y^\beta \bar{y}^\beta D_{\beta \bar{\beta}} \phi(x)}, \quad \Phi^*(Y|x) = e^{y^\beta \bar{y}^\beta D^*_{\beta \bar{\beta}} \phi^*(x)}. \]  
(3.44)
This unfolding is strongly nonlinear, containing the exponent of the dynamical gauge field \( A_{\alpha \dot{\alpha}} \), as opposite to all unfoldings considered before. The reason behind this is that we want to keep the gauge invariance manifest. This means that the unfolded fields should transform covariantly, which forces one to replace ordinary derivatives in the exponent, present in the non-gauge case (2.28), with the covariant ones. Then \( \partial_{\alpha} \bar{\partial}_{\dot{\alpha}} \Phi \) generates a covariant term \( D_{\alpha \dot{\alpha}} \Phi \), which gives rise to \( U(1) \) gauge symmetry according to the general formula (2.3).

From (3.44) a useful relation immediately follows
\[ y^\beta \bar{y}^\beta D_{\beta \bar{\beta}} \Phi = N \Phi = \bar{N} \Phi. \]  
(3.45)
We simplify the appearance of formulas by writing
\[ D y \bar{y} := y^\beta \bar{y}^\beta D_{\beta \bar{\beta}}, \quad D^* y \bar{y} := y^\beta \bar{y}^\beta D^*_{\beta \bar{\beta}}, \quad D^2 := \frac{1}{2} D_{\beta \bar{\beta}} D_{\beta \bar{\beta}}, \quad D^* 2 := \frac{1}{2} D^*_{\beta \bar{\beta}} D^*_{\beta \bar{\beta}}. \]  
(3.46)
Now we start calculating \( \partial_{\alpha} \bar{\partial}_{\dot{\alpha}} \Phi \). Combining a relation for a commutator
\[ [\hat{A}, e^{\hat{D}}] = \int_0^1 dt e^{t \hat{D}} [\hat{A}, \hat{D}] e^{-t \hat{D}} e^{\hat{D}} \]  
(3.47)
with an Euler-operator representation of a homotopy integral
\[
\int_0^1 dt t^k F(tz) = \frac{1}{z \frac{\partial}{\partial z} + 1 + k} F(z),
\] (3.48)
we get
\[
\partial_\alpha \Phi = D_\alpha \bar{y}^\beta \Phi - y_\alpha \Phi \frac{iq}{(N + 1)(N + 2)} \bar{F},
\] (3.49)
\[
\partial_\alpha \bar{\partial}_\alpha \Phi = (N + 1)D_{\alpha \alpha} \Phi - y_\alpha \bar{\partial}_\alpha (\Phi \frac{iq}{(N + 1)(N + 2)} \bar{F}) - \frac{iq}{2} \Phi \bar{y}_\alpha \partial_\alpha F_{\beta \beta} y^\beta y^\beta - y_\alpha \bar{y}_\alpha D^2 F - y_\alpha \bar{y}_\alpha \frac{iq}{N(N + 1)} y^\beta \nabla_{\alpha \beta} F - \bar{y}_\alpha (\frac{iq}{(N + 1)(N + 2)} F)(\partial_\alpha \Phi + y_\alpha \Phi \frac{iq}{(N + 1)(N + 2)} F).
\] (3.50)
From (3.40) one finds
\[
\nabla_{\alpha \beta} y^\beta F = \partial_\alpha \frac{N}{N} F - \frac{2q}{(N + 1)} y_\alpha J,
\] (3.51)
so that
\[
\partial_\alpha \bar{\partial}_\alpha \Phi = (N + 1)D_{\alpha \alpha} \Phi - y_\alpha \bar{\partial}_\alpha (\Phi \frac{iq}{(N + 1)(N + 2)} \bar{F}) - \frac{iq}{2} \Phi \bar{y}_\alpha \partial_\alpha F_{\beta \beta} y^\beta y^\beta + \frac{1}{2} y_\alpha \bar{y}_\alpha \frac{iq}{(N + 1)(N + 2)} F + y_\alpha \bar{y}_\alpha \frac{2iq}{N(N + 1)(N + 2)} J - y_\alpha \bar{y}_\alpha D^2 \Phi.
\] (3.52)
This might seem like a solution to the problem. However, the last term is not of acceptable form. The situation here is different from the non-gauge case (2.31), since the covariant box \(D^2\) does not commute with the unfolding exponent \(\exp(Dy\bar{y})\) and one cannot directly apply e.o.m. for the primary field (3.1).

By means of the same combination of the commutator (3.47) and the homotopy integral (3.48), we can write an unfolded generalization for (3.1) as
\[
D^2 \Phi = -(m^2 + U'(\Phi \Phi^*)) \Phi + \left( \frac{1}{N} e^{Dy\bar{y}[D^2, Dy\bar{y}]} e^{-Dy\bar{y}} \right) \Phi.
\] (3.53)
The commutator in (3.53) can be expanded as
\[
[D^2, Dy\bar{y}] = \frac{1}{2} [D_{\beta \beta}, Dy\bar{y}] D^{\beta \beta} + \frac{1}{2} D^{\beta \beta} [D_{\beta \beta}, Dy\bar{y}] = iq(y_\beta \bar{F}_{\alpha \beta} y^\alpha + \bar{y}_\beta F_{\alpha \beta} y^\alpha) D^{\beta \beta} + iq^2 j_{\alpha \alpha} y^\alpha y^\alpha,
\] (3.54)
where (3.4) and (3.29) are used, so that
\[
\frac{1}{N} e^{Dy\bar{y}[D^2, Dy\bar{y}]} e^{-Dy\bar{y}} = iq^2 \frac{1}{N} J + iq \frac{1}{N} e^{Dy\bar{y}} (\bar{y}_\beta F_{\alpha \beta} y^\alpha + y_\beta \bar{F}_{\alpha \beta} y^\alpha) D^{\beta \beta} e^{-Dy\bar{y}}.
\] (3.55)
Next,
\[
e^{Dy\bar{y}} \bar{y}_\beta F_{\alpha \beta} y^\alpha D^{\beta \beta} e^{-Dy\bar{y}} = (e^{\nabla y\bar{y}} (\frac{1}{2} \partial_\beta F_{\alpha \alpha} y^\alpha y^\alpha) e^{-\nabla y\bar{y}})(e^{Dy\bar{y}} \bar{y}_\beta D^{\beta \beta} e^{-Dy\bar{y}}),
\] (3.56)
and two factors can be rewritten as
\[ e^{\nabla y\bar{y}}(\partial_{\beta}F_{\alpha\alpha}y^\alpha y^\beta)e^{-\nabla y\bar{y}} = \frac{2}{N+1}\partial_{\beta}F + \frac{2q}{(N+1)}y_{\beta}J, \] (3.57)
\[ e^{D_{y\bar{y}}\bar{y}\beta}D^{\beta\beta}e^{-D_{y\bar{y}}} = \bar{y}_{\beta}D^{\beta\beta} + iq\frac{1}{N}y^\beta\bar{F}. \] (3.58)

This way we get
\[ D^2\Phi = -(m^2 + U'(\Phi\Phi^*))\Phi + iq^2\Phi \frac{1}{N} + J + \left(\frac{2iq^2}{(N+1)(N+2)}\right) y^\alpha \bar{y}D_{\alpha\alpha}\Phi - \bar{\Phi} \frac{iq}{N} \left(\frac{1}{N+1} - \frac{1}{F} + \frac{1}{F} N\Phi - \Phi \bar{\Phi} \frac{1}{N+1} - \frac{1}{F} \bar{\Phi} y_{a}D_{\alpha\alpha}\Phi, \right. \] (3.59)

and, making use of (3.45) and (3.49) (and its conjugate), arrive at the required representation
\[ D^2\Phi = -(m^2 + U'(\Phi\Phi^*))\Phi + \Phi \frac{iq^2}{N} + J + \left(\frac{2iq^2}{(N+1)(N+2)}\right) \bar{\Phi} \frac{1}{N} - \bar{\Phi} \frac{iq}{N} \left(\frac{1}{N+1} - \frac{1}{F} + \frac{1}{F} \bar{\Phi} y_{a}D_{\alpha\alpha}\Phi, \right. \] (3.60)

Substituting this into (3.52) yields \( \partial_{\alpha}\bar{\partial}_{\alpha}\Phi \) in a suitable form
\[ \partial_{\alpha}\bar{\partial}_{\alpha}\Phi = (N+1)D_{\alpha\alpha}\Phi + y_{a}\bar{y}_{a}(m^2 + U'(\Phi\Phi^*))\Phi - iq y_{a}\bar{\partial}_{\alpha}\Phi \frac{1}{N} + iq \bar{\partial}_{\alpha} N\Phi + iq \frac{1}{N+1} \bar{\Phi} y_{a}\bar{\partial}_{\alpha}\Phi - \bar{\Phi} \frac{iq}{N} \left(\frac{1}{N+1} \bar{\Phi} y_{a}\bar{\partial}_{\alpha}\Phi + \frac{1}{N+1} \bar{\Phi} y_{a}\bar{\partial}_{\alpha} \Phi \right) - \frac{1}{N+1} \bar{\Phi} y_{a}\bar{\partial}_{\alpha} \Phi \frac{1}{N} + \frac{1}{N+1} \bar{\Phi} y_{a}\bar{\partial}_{\alpha} \Phi \frac{1}{N+1} \bar{\Phi} y_{a}\bar{\partial}_{\alpha} \Phi \] (3.61)

\section{3.4 Unfolded equations of scalar electrodynamics}

In (3.61) and in (3.41)-(3.42) \( J \) is an arbitrary (e.g. some external) conserved electric current for now. But in scalar electrodynamics (3.1)-(3.2) it is made up of the dynamical scalar field and describes its back-reaction,
\[ \bar{j}_{\alpha\alpha} = i(\phi\Phi_{\alpha\alpha}\Phi - \phi\Phi_{\alpha\alpha}^*\phi^*), \quad \nabla_{\alpha\alpha} j_{\alpha\alpha} = 0. \] (3.62)

Applying unfolding (3.44) to this expression and comparing with the unfolding of the current (3.6), one finds a surprisingly simple expression for an unfolded electric current of the scalar field
\[ J = i(\Phi\Phi^* N\Phi - \Phi N\Phi^*). \] (3.63)
Then from (3.23), (3.25) it follows that
\[ J^+ = -i\partial_\alpha \Phi \cdot \partial^\alpha \Phi^* - (\bar{N} + 2)(\Phi \Phi^* \frac{2q}{(N + 2)} F), \quad J^- = -i\partial_\alpha \Phi \cdot \partial^\alpha \Phi^* - (N + 2)(\Phi \Phi^* \frac{2q}{(N + 2)} F). \]

Let us stress that this unfolded electric current is conserved on nonlinear equations for the constituent scalar fields, which distinguishes it from unfolded spin-1 conserved currents considered (together with HS currents) e.g. in [30–33], where they are built up from free constituent fields.

Now, substituting (3.63)-(3.64) into (3.41) and (3.42), contracting (3.61) with the vierbein and introducing 1-forms of covariant derivatives as
\[ D := e^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} = d_L - iqA, \quad D^* := e^{\alpha \dot{\alpha}} D^*_{\alpha \dot{\alpha}} = d_L + iqA, \]
one finds after some algebraic simplifications in (3.61)
\[ dA = \frac{1}{4} e_\beta e^{\alpha \dot{\alpha}} \partial_\alpha \partial_{\dot{\alpha}} F \bigg|_{y=0} + \frac{1}{4} e_{\dot{\alpha}} e^{\alpha \dot{\beta}} \partial_{\alpha} \partial_{\dot{\beta}} \bar{F} \bigg|_{y=0}, \]
\[ d_L F - \frac{1}{(N + 1)(N + 1)} \{ \nu e \partial \bar{\partial} F - q(\nu + 2)e \bar{y} \bar{y}(i\partial_\alpha \Phi \cdot \partial^\alpha \Phi^* + (\bar{N} + 2)(\Phi \Phi^* \frac{2q}{(N + 2)} F) - \]
\[ -2iqe \bar{y} \bar{\partial}(\Phi^* N \Phi - \Phi N \Phi^*) \} = 0, \]
\[ (N + 1)D \Phi - e \partial \partial \Phi + ey\bar{y}(m^2 + U^\prime(\Phi \Phi^*)) \Phi + q^2 e \bar{y} \bar{y} \Phi \cdot \frac{1}{(N + 2)} F \cdot \frac{1}{(N + 2)} \bar{F} + \]
\[ + q^2 e \bar{y} \bar{y} \left( \Phi \frac{1}{(N + 2)} (\Phi^* N \Phi - \Phi N \Phi^*) + (N + 1) \Phi \cdot \frac{2}{(N + 1)(N + 2)} (\Phi^* N \Phi - \Phi N \Phi^*) \right) + \]
\[ + iq \left( \frac{1}{(N + 2)} F \cdot e \bar{y} \partial \Phi - (\bar{N} + 1) \Phi \cdot e \bar{y} \partial \frac{1}{(N + 1)(N + 2)} F \right) + \]
\[ + iq \left( \frac{1}{(N + 2)} \bar{F} \cdot e \bar{y} \bar{\partial} \Phi - (N + 1) \Phi \cdot e \bar{y} \bar{\partial} \frac{1}{(N + 1)(N + 2)} \bar{F} \right) = 0, \]
plus an equation for \( F \) resulting from exchanging barred and unbarred objects in (3.67) and an equation for \( \Phi^* \) resulting from exchanging \( \Phi \) and \( \Phi^* \) and changing the sign of \( q \) in (3.68). These equations form an unfolded formulation of scalar electrodynamics. By construction, their solutions (at least locally) are (3.30), (3.32), (3.44) with primary fields subjected to (3.1)-(3.2).

It is interesting that in (3.68) there are cubic terms in the second line that consist of the scalar field only. Traced back to (3.60), these terms might seem like charged-current interactions. However, they are not real vertices, as these terms do not contribute to the e.o.m. of primary fields (3.1)-(3.2). Instead, their appearance must be attributed to the non-linearity of the unfolding (3.44), because it is differentiation of the unfolding exponent, containing covariant derivatives, that gives rise to them. The only scalar self-interaction is stored in the potential \( U(\Phi \Phi^*) \).

Making use of the general formula (2.3), one obtains manifest \( U(1) \) gauge symmetry of the unfolded system (3.66)-(3.68) with a gauge parameter \( \varepsilon(x) \)
\[ \delta A = d\varepsilon, \quad \delta F = 0, \quad \delta \bar{F} = 0, \quad \delta \Phi = iq\varepsilon \Phi, \quad \delta \Phi^* = -iq\varepsilon \Phi^*. \]
4 Higgs Mechanism

Having the system (3.66)-(3.68) in hand, we can study a realization of spontaneous $U(1)$ symmetry breaking in the unfolded dynamics approach.

In the standard Lagrangian approach, one usually considers the massless case with the "Mexican hat" scalar potential

$$U(\phi \phi^*) = -\mu^2 \phi \phi^* + \frac{\lambda}{2} (\phi \phi^*)^2, \quad m^2 = 0. \quad (4.1)$$

Then a continuum of classical vacuums is determined by

$$|\phi_0| = \left(\frac{\mu^2}{\lambda}\right)^{1/2}. \quad (4.2)$$

Considering some particular solution, e.g.

$$\phi_0 = \frac{\mu}{\sqrt{\lambda}}, \quad (4.3)$$

and analyzing linear fluctuations over it in terms of two real scalar fields

$$\phi(x) = \phi_0 + \chi(x) + i\theta(x), \quad (4.4)$$

one finds that the spectrum consists of one real massive scalar field (Higgs boson) and a massive vector field, with the imaginary scalar component $\theta$ eaten by the photon in order to gain weight.

Let us perform this analysis in terms of the unfolded system (3.66)-(3.68). We consider linearization in terms of

$$\Phi = \phi_0 + X + i\Theta, \quad (4.5)$$

with the vacuum (4.3), so that

$$\nabla_{a\dot{a}} \phi_0 = N\phi_0 = \bar{N}\phi_0 = 0. \quad (4.6)$$

Substituting this into the unfolded equations (3.67)-(3.68) gives in the linear limit

$$d_L F - \frac{1}{(N+1)(N+1)} \left\{ n_e \partial \bar{\partial} F - (n + 2)2q^2 \phi_0^2 e y \bar{y} F + 4q\phi_0 e y \partial N \Theta \right\} = 0, \quad (4.7)$$

$$(N + 1)d_L X - e \partial \bar{\partial} X + 2\mu^2 e y \bar{y} X = 0, \quad (4.8)$$

$$(N + 1)d_L \Theta - q\phi_0 A - e \partial \bar{\partial} \Theta + 2q^2 \phi_0^2 e y \bar{y} \frac{N(N + 3)}{(N + 1)(N + 2)} \Theta - \frac{1}{(N + 1)(N + 2)} F - q\phi_0 e y \partial \frac{1}{(N + 1)(N + 2)} \bar{F} = 0. \quad (4.9)$$

Equation (4.8) indeed describes the Higgs boson with the correct mass value

$$m_X^2 = 2\mu^2. \quad (4.10)$$

To see that the rest of equations describe a massive vector, one has to find an appropriate unfolded field redefinition. But first we have to learn how an unfolded system for a massive vector field looks like.
Using Fierz–Pauli formulation, a massive spin-1 field corresponds to the system

\[(\Box + m^2)b_{\alpha\dot{\alpha}}(x) = 0, \quad \nabla^{\alpha\dot{\alpha}}b_{\alpha\dot{\alpha}}(x) = 0. \quad (4.11)\]

We see that, in order to describe a massive spin-1 field, one can simply take the equation (3.28) for \(J\) and put \(\Box J = -m^2 J\) [22]. Then an unfolded equation for a massive vector field is

\[d_L J - \frac{1}{(N+1)(N+1)} \left\{ \nu e \bar{\partial} \bar{\partial} J - (\nu + 2) e y \bar{y} m^2 J - e \bar{y} \partial (J + 2m^2 J) - e y \bar{\partial} (J - 2m^2 J) \right\} = 0. \quad (4.12)\]

Now let us analyze (4.9) more closely. A ground equation at \(Y = 0\) is the only equation that the term with \(A\) contributes to,

\[d_L \Theta(Y = 0) - q\phi_0 A - e \bar{\partial} \partial \Theta(Y = 0) = 0. \quad (4.13)\]

Introducing \(Y\)-expansion of \(\Theta\) as

\[\Theta(Y|x) = \sum_{n=0}^{\infty} \theta_{\alpha(n)\dot{\alpha}(n)}(x) y^{\alpha_1} \ldots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \ldots \bar{y}^{\dot{\alpha}_n}, \quad (4.14)\]

the equation (4.13) gives

\[\theta_{\alpha\dot{\alpha}} = \nabla_{\alpha\dot{\alpha}}\theta - q\phi_0 A_{\alpha\dot{\alpha}}. \quad (4.15)\]

Using the linearized gauge symmetry (3.71), one can gauge away \(\theta(x)\), then (4.15) identifies \(\theta_{\alpha\dot{\alpha}}\) with \(A_{\alpha\dot{\alpha}}\) (in this gauge). Note that \(\theta_{\alpha\dot{\alpha}}\) itself is gauge-invariant, as well as all higher descendants.

Then we introduce a new unfolded gauge-invariant master-field

\[B := -\frac{1}{q\phi_0} e^{\overline{y}y} \theta_{\alpha\dot{\alpha}} y^{\alpha} \bar{y}^{\dot{\alpha}} = -\frac{N}{q\phi_0} \Theta. \quad (4.16)\]

Substituting this back into (4.7) and (4.9) yields

\[d_L F - \frac{1}{(N+1)(N+1)} \left\{ \nu e \bar{\partial} \bar{\partial} F - (\nu + 2) m_\psi^2 e y \bar{y} F - 2m_\psi^2 e y \bar{\partial} B \right\} = 0, \quad (4.17)\]

\[d_L B - \frac{1}{(N+1)(N+1)} \left\{ \nu e \bar{\partial} \bar{\partial} B - (\nu + 2) m_\psi^2 e y \bar{y} F - e \bar{y} \partial F - e y \bar{\partial} \bar{F} \right\} = 0, \quad (4.18)\]

where

\[m_\psi^2 = 2 q^2 \phi_0^2. \quad (4.19)\]

We see that equations (4.17)-(4.18) (plus a conjugate equation for \(\bar{F}\)) are indeed equivalent to the unfolded system (4.12) for a massive vector. Equations for two helicities of the massless photon receive linear corrections that couple them to the equation for the imaginary part of the scalar field, which after performing unfolded field redefinition (4.16) becomes an equation for the longitudinal polarization of the massive vector. Note that this field redefinition is non-invertible, because the Euler operator \(N\) amputates \(Y\)-independent primary component \(\theta(x)\), thus drastically changing the structure of the unfolded module: now the module describes a vector instead of a scalar. This is not inconsistent, because \(U(1)\) gauge symmetry turns \(\theta(x)\) to
the pure gauge over higgsed $\phi_0$-vacuum, while all other fields in the module are gauge-invariant. Finally, the equation (3.66) now becomes just a particular consequence of the equations (4.15), (4.16) (4.18).

Thus, we revealed the picture of the Higgs mechanism within the framework of the unfolded dynamics approach. It includes a changeover of unfolded modules caused by the gauge symmetry, whose action is modified by a nontrivial vacuum. To improve our understanding of unfolded spontaneous symmetry breaking is especially important from the point of view of HS symmetry breaking in HS gravity, presumably resulting in the emergence of string theory as a symmetry-broken phase.

5 Conclusion

In the paper, we put forward a novel method of unfolding field theories, based on postulating a specific form of an unfolded field and the subsequent search for the corresponding unfolded equation as an identity that this field satisfies. We successfully apply this method to the problem of unfolding scalar electrodynamics, where unfolding map is strongly nonlinear due to the presence of gauge interaction. This way we end up with a system of nonlinear unfolded equations for which we have a manifest solution, which is a representation for unfolded field that we started with. A curious feature of this system of unfolded equations is that it contains cubic terms made up of scalar fields solely. These terms might look like charged-current interactions, but in fact they do not correspond to any real vertices and represent just artifacts of the nonlinearity of unfolding that underlies the system.

Considering a particular form of a scalar potential, we are able to study the spontaneous symmetry breaking in this system. We identify an appropriate non-invertible unfolded field redefinition that allows us to reproduce the correct spectrum of the symmetry-broken phase and study the concomitant deformation of unfolded modules. This is interesting in the context of recent research on spontaneous symmetry breaking in HS gravity [15].

It would be interesting to apply the proposed method of unfolding to more complicated theories like e.g. Yang-Mills theory or gravity. Another topical question is to study the problem of integrability of unfolded systems: the presented nonlinear unfolded system has a manifest solution that reconstructs the dependence on auxiliary spinor variables of a given solution of the space-time equation. The problem is to develop an algorithm that would allow one to reconstruct the space-time dependence from the spinorial one, i.e. to generate solutions to classical e.o.m. starting from Cauchy data.

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References

[1] X. Bekaert, N. Boulanger, A. Campoleoni, M. Chiodaroli, D. Francia, M. Grigoriev, E. Sezgin, E. Skvortsov, *Snowmass White Paper: Higher Spin Gravity and Higher Spin Symmetry*, [arXiv:2205.01567].

[2] E.S. Fradkin, M.A. Vasiliev, *Annals Phys.* **177** (1987) 63.

[3] D.J. Gross, P.F. Mende, *Phys.Lett.B* **197** (1987) 129-134.

[4] D.J. Gross, *Phys.Rev.Lett.* **60** (1988) 1229.

[5] Bo Sundborg, *Nucl.Phys.B Proc.Suppl.* **102** (2001) 113-119 [hep-th/0103247].

[6] M.A. Vasiliev, *Annals Phys.* **190** (1989) 59-106.

[7] M.A. Vasiliev, *Phys.Lett.B* **243** (1990) 378-382.

[8] M.A. Vasiliev, *Phys.Lett.B* **285** (1992) 225-234.

[9] M.A. Vasiliev, *Class.Quant.Grav.* **11** (1994) 649-664.

[10] M.A. Vasiliev, *Int.J.Geom.Meth.Mod.Phys.* **3** (2006) 37-80 [hep-th/0504090].

[11] V.E. Didenko, *JHEP* **10** (2022) 191 [arXiv:2209.01966].

[12] V.E. Didenko, A.V. Korybut, *Phys.Rev.D* **108** (2023) 8, 086031 [arXiv:2304.08850].

[13] E. Joung, M. Kim, Y. Kim, *JHEP* **12** (2021) 092 [arXiv:2108.05535].

[14] N. Misuna, *JHEP* **12** (2023) 119 [arXiv:2208.04306].

[15] V.E. Didenko, A.V. Korybut, *Phys.Rev.D* **110** (2024) 2, 026007 [arXiv:2312.11096].

[16] O.V. Shaynkman, M.A. Vasiliev, *Theor.Math.Phys.* **123** (2000) 683-700, *Teor.Mat.Fiz.* **123** (2000) 323-344 [hep-th/0003123].

[17] D.S. Ponomarev, M.A. Vasiliev, *JHEP* **1201** (2012) 152 [arXiv:1012.2903].

[18] N.G. Misuna, M.A. Vasiliev, *JHEP* **05** (2014) 140 [arXiv:1301.2230].

[19] M.V. Khabarov, Yu.M. Zinoviev, *Nucl.Phys.B* **953** (2020) 114959 [arXiv:2001.07903].

[20] I.L. Buchbinder, T.V. Snegirev, Yu.M. Zinoviev, *JHEP* **08** (2016) 075 [arXiv:1606.02475].

[21] N. Misuna, *Phys.Lett.B* **798** (2019) 134956 [arXiv:1905.06925].

[22] N.G. Misuna, *JHEP* **12** (2021) 172 [arXiv:2012.06570].

[23] N.G. Misuna, *Phys.Lett.B* **840** (2023) 137845 [arXiv:2201.01674].

[24] V.E. Didenko, M.A. Vasiliev, *Phys.Lett.B* **682** (2009) 305-315, *Phys.Lett.B* **722** (2013) 389 (erratum) [arXiv:0906.3898].
[25] C. Iazeolla, P. Sundell, *JHEP* **12** (2011) 084 [arXiv:1107.1217].

[26] R. Aros, C. Iazeolla, P. Sundell, Y. Yin, *JHEP* **08** (2019) 171 [arXiv:1903.01399].

[27] D. De Filippis, C. Iazeolla, P. Sundell, *JHEP* **10** (2019) 215 [arXiv:1905.06325].

[28] C. Iazeolla, *PoS CORFU2019* (2020) 181 [arXiv:2004.14903].

[29] S.F. Prokushkin, M.A. Vasiliev, *Nucl.Phys.B* **545** (1999) 385 [hep-th/9806236].

[30] O.A. Gelfond, E.D. Skvortsov, M.A. Vasiliev, *Theor.Math.Phys.* **154** (2008) 294-302 [hep-th/0601106].

[31] O.A. Gelfond, M.A. Vasiliev, *J.Exp.Theor.Phys.* **120** (2015) 3, 484-508 [arXiv:1012.3143].

[32] M.A. Vasiliev, *JHEP* **10** (2017) 111 [arXiv:1605.02662].

[33] O.A. Gelfond, M.A. Vasiliev, *Nucl.Phys.B* **931** (2018) 383-417 [arXiv:1706.03718].