GRADED FROBENIUS RINGS

S. DĂSCĂLESCU¹, C. NÂSTĂSESCU² AND L. NÂSTĂSESCU²

ABSTRACT. In order to study graded Frobenius algebras from a ring theoretical perspective, we introduce graded quasi-Frobenius rings, graded Frobenius rings and a shift-version of the latter ones, and we investigate the structure and representations of such objects. We need to revisit graded simple graded left Artinian rings, graded semisimple rings, and to provide graded versions of certain results concerning the Jacobson radical, the singular radical, and their connection to finiteness conditions and injectivity. We prove a structure result for (shift-)graded Frobenius rings.

2010 MSC: 16W50, 16D50, 16E50, 16G10, 16L60, 16S50.
Key words: graded algebra, quasi-Frobenius algebra, Frobenius algebra, graded division algebra, graded semisimple algebra.

1. INTRODUCTION AND PRELIMINARIES

Originated in the work of Frobenius on group representations, Frobenius algebras and their relatives, quasi-Frobenius algebras, have been objects of intense study after the influential work of Brauer, Nesbitt and Nakayama around 1940. The initial interest was algebraic, but Frobenius algebras occurred, sometimes unexpectedly, in topology, differential geometry, knot theory, homological algebra, topological quantum field theory, Hopf algebra theory, etc. A step towards a deeper understanding of Frobenius algebras from a ring theoretical perspective was the study of (quasi-)Frobenius rings. A presentation of the basic theory of (quasi-)Frobenius rings and their connection to (quasi-)Frobenius algebras can be found in [12].

There are certain Frobenius algebras equipped with more structure that occur in a natural way, for example Frobenius algebras endowed with a grading. Inspired by an equivalent characterization of Frobenius algebras in [1], one can consider Frobenius algebras in an arbitrary monoidal category as algebras \( A \), endowed with a coalgebra structure whose comultiplication is a morphism of \( A \)-bimodules. In particular, one can look at Frobenius algebras in the monoidal category of \( G \)-graded vector spaces, where \( G \) is a group; these are called graded Frobenius algebras, and they were investigated in [2], as well as a version modified by a shift, called \( \sigma \)-graded Frobenius algebras. Such objects occur in noncommutative geometry, where certain connected graded algebras are \( n \)-graded Frobenius for a positive integer \( n \). For example, if \( A \) is a connected Noetherian graded algebra which is Artin-Schelter regular and Koszul, of global dimension \( n \), then the Koszul dual algebra \( A^! \) of \( A \) is \( n \)-graded Frobenius, see [16]. Following the point of view that Calabi-Yau algebras are related to non-commutative potentials, see [5], it is showed in [8] that \( n \)-graded Frobenius connected algebras generated in degree 1 can be constructed from twisted superpotentials. The structure and representation theory of graded Frobenius algebras have been used to classification results for certain algebras playing a role in non-commutative geometry [13], and for proving a non-commutative Bernstein-Gelfand-Gelfand correspondence [9].

In order to understand the structure of graded Frobenius algebras, our initial aim was to fill in a missing piece of the Frobenius puzzle, by defining and investigating graded quasi-Frobenius algebras. In developing the theory, we realized that it is interesting to consider ring theoretical versions of the concepts. The aim of the paper is to introduce graded quasi-Frobenius rings and (\( \sigma \))-graded Frobenius rings, and to investigate them and their representations. As expected,
a finite dimensional graded algebra turns out to be \((\sigma\text{-})\text{graded Frobenius}\) if and only if it is \((\sigma\text{-})\text{graded Frobenius}\) as a ring.

Some results about graded rings and graded modules may give the impression that graded theory is a simple extension of the un-graded one. This is true up to a point, and a reason is that the category of modules over a ring and the category of graded \(R\)-modules over a graded ring \(R\) are both Grothendieck categories. However, the category of graded \(R\)-modules is equipped with a family of category isomorphisms, the shifts by group elements, and this adds an extra level of complexity to the structure of this category and its objects. As an example in support of this idea, we mention the theory of the graded Grothendieck group of an algebra graded by an abelian group, developed in [7]. On the other hand, even in the case where the category of graded \(R\)-modules is equivalent to the category of modules over a ring \(A\), this ring has usually a much more complicated structure than \(R\). For example in the case where the grading group is finite, \(A\) is the smash product \(R\#G^*\), see [14, Chapter 7].

In Section 2 we discuss the structure of a graded ring \(A = M_n(\Delta)(g_1, \ldots, g_n)\) associated with a graded division ring \(\Delta\), and some group elements \(g_1, \ldots, g_n\). \(A\) is graded simple and graded Artinian, so any two graded simple left \(A\)-modules are isomorphic up to a shift. We count the isomorphism types of graded simple left \(A\)-modules, and how many of them embed into \(A\). In Section 3 we consider the graded versions of the Jacobson radical and the singular radical, and we prove some of their properties related to finiteness conditions and to injectivity. We also give an alternative proof of the structure theorem for graded simple graded left Artinian rings, which says that any such ring is isomorphic to \(M_n(\Delta)(g_1, \ldots, g_n)\) for some \(n, \Delta\) and \(g_1, \ldots, g_n\).

In Section 4 we consider the decomposition of a graded left Artinian ring into a sum of graded indecomposable left modules, and obtain some consequences on the graded simple modules when we factor by the graded Jacobson radical. A structure result for projective objects in the category of graded modules is derived. In Section 5 we define graded quasi-Frobenius rings by proving several equivalent characterizations. In the case of a graded ring \(R\) of finite support, we show that \(R\) is graded quasi-Frobenius if and only if it is quasi-Frobenius. More properties of graded quasi-Frobenius rings are investigated in Section 6 where we also associate a certain set of data with a graded quasi-Frobenius ring, including a version of the Nakayama permutation. This set of data is used in Section 7 to introduce graded Frobenius rings and to give equivalent characterizations. In fact, we define the more general version of a \(\sigma\)-graded Frobenius ring, which matches with the shift-modified version of graded Frobenius algebra mentioned above. At this point it will be clear that there is a higher degree of complexity of the concept, compared to the un-graded one. In the un-graded case, the Nakayama permutation and the multiplicities of the isomorphism types of principal indecomposable modules is all that we need for deciding whether a quasi-Frobenius ring is Frobenius, while in the graded case it turns out that one needs more information, related to the inertia groups of the graded simple modules and certain shifts. We note that for developing the theory of graded \((\text{quasi-})\text{Frobenius}\) rings, we need many times to work not with isomorphism types of graded modules, but with isoshift types, see the definition below.

Let \(G\) be a group with neutral element \(\varepsilon\). A ring \(R\) is \(G\)-graded if it has a decomposition \(R = \oplus_{g \in G} R_g\) as a direct sum of additive subgroups such that \(R_g R_h \subset R_{gh}\) for any \(g, h \in G\); in particular, \(R_\varepsilon\) is a subring of \(R\). A graded left \(R\)-module is a left \(R\)-module \(M\) with a decomposition \(M = \oplus_{g \in G} M_g\) of additive subgroups, such that \(R_g M_h \subset M_{gh}\) for any \(g, h \in G\).

We consider the category \(R - gr\) of graded left \(R\)-modules, where a morphism \(f : M \to N\) of graded \(R\)-modules is an \(R\)-module morphism such that \(f(M_g) \subset N_g\) for any \(g \in G\). If \(M \in R - gr\) and \(\sigma \in G\), the \(\sigma\)-shift of \(M\) is the graded \(R\)-module \(M(\sigma)\) which coincides with \(M\) as an \(R\)-module, and has the grading given by \(M(\sigma)_g = M_{g\sigma}\) for any \(g \in G\). If \(M, N \in R - gr\) and \(\sigma \in G\), a morphism of degree \(\sigma\) from \(M\) to \(N\) is a morphism \(f : M \to N\) of \(R\)-modules such that \(f(M_g) \subset N_{g\sigma}\) for any \(g \in G\), i.e., \(f\) is a morphism in \(R - gr\) from \(M\) to \(N(\sigma)\). The category \(R - gr\) is a locally finite Grothendieck category; a family of generators is \((R(\sigma))_{\sigma \in G}\).
We consider the equivalence relation \( \sim \), which we call the isoshift equivalence, defined as follows: if \( M, N \in R - gr \), then \( M \sim N \) if and only if there exists \( \sigma \in G \) such that \( M \) is isomorphic to \( N(\sigma) \). The equivalence classes with respect to \( \sim \) will be called the isoshift types of graded left \( R \)-modules. Similarly we can define the category \( gr - R \) of graded right \( R \)-modules, whose objects are right \( R \)-modules \( M \) with a decomposition \( M = \bigoplus_{g \in G} M_g \) such that \( M_g R_h \subseteq M_{gh} \) for any \( g, h \in G \). For such an object and \( \sigma \in G \), the \( \sigma \)-shift \((\sigma)M\) is defined by \((\sigma)M_g = M_{\sigma g}\), and we can also consider isoshift types of graded right \( R \)-modules.

### 2. The structure of graded simple rings

Let \( R \) be a \( G \)-graded ring. If \( S \) is a graded simple left submodule of \( R \), i.e., a minimal graded left ideal, let \( U \) be the sum of all graded left submodules of \( R \) which are isomorphic to a shift of \( S \). Then \( U \) is a two-sided graded ideal of \( R \). Indeed, write \( U = \sum_i S_i \), where \((S_i)\) is the family of all graded left submodules of \( R \) isomorphic to some shift of \( S \). If \( g \in G \) and \( a \in R_g \), then the map \( \varphi: R \to R, \varphi(x) = xa \), is a morphism of degree \( g \) of graded left modules, and \( Ua = \varphi(U) = \sum_i \varphi(S_i) \). Since \( S_i \) is graded simple, then \( \varphi(S_i) = 0 \) or \( \varphi(S_i) \cong S_i(g) \). Now \( S_i(g) \) is also isomorphic to a shift of \( S \), so it must be one of the \( S_j \)'s. Hence \( Ua \subseteq U \), so \( U \) is also a right ideal of \( R \). We call \( U \) the (left) isoshift component of \( R \) corresponding to \( S \). The socle \( soc^g \ell(R) \) of the graded left \( R \)-module \( R \) is the direct sum of all left isoshift components. Similarly, the right isoshift components of \( R \) are graded ideals, and their direct sum is the right graded socle \( soc^g \ell(R) \).

A graded ring \( R \) is called graded semisimple if \( R \) is a sum of minimal graded left ideals, i.e., \( R = soc^g \ell(R) \). This is equivalent to the fact that the category of graded left \( R \)-modules is semisimple, i.e., any graded left \( R \)-module is a sum of graded simple modules. If moreover, \( R \) is a sum of minimal graded ideals such that any two of them are isomorphic up to a shift, then \( R \) is called graded simple in \([14]\) page 55; in order to avoid confusion, we will call such an \( R \) a graded simple and graded left Artinian ring. These properties (graded semisimple, and graded simple and graded left Artinian) turn out to be left-right symmetric. If \( R \) is graded simple and graded left Artinian, then any two graded simple modules are isomorphic up to a shift, so there is just one isoshift type, and the only isoshift component is equal to the whole of \( R \). If \( R \) is graded semisimple, then \( R \) is the direct sum of the left isoshift components, which are finitely many. This decomposition shows that a graded semisimple ring is isomorphic to a finite product of graded simple and graded Artinian rings; see \([14]\) Section 2.9. A graded semisimple ring has the same number of isoshift types to the left and to the right, and this is just the number of factors in the decomposition of \( R \) as a product of graded simple and graded left Artinian rings. We note that if \( R \) is graded semisimple, a graded simple left \( R \)-module does not necessarily embed into \( R \), however at least one of its shifts does.

Graded left Artinian graded rings \( R \) whose only two-sided ideals are 0 and \( R \) were considered in \([4]\) Section 2.1. We will explain in Section 3 that these are the same objects as the graded simple and graded left Artinian rings discussed above.

Let \( \Delta = \bigoplus_{\sigma \in G} \Delta_\sigma \) be a \( G \)-graded division ring, i.e., \( \Delta \) is a \( G \)-graded ring whose all non-zero homogeneous elements are invertible. Let \( n \) be a positive integer, and \( g_1, \ldots, g_n \in G \). We consider the \( G \)-graded ring \( A = M_n(\Delta)(g_1, \ldots, g_n) \), which is just \( M_n(\Delta) \) as a ring, and has a \( G \)-grading with the homogeneous component of degree \( \sigma \in G \) given by

\[
A_\sigma = \begin{pmatrix}
\Delta_{g_1 \sigma g_1^{-1}} & \Delta_{g_1 \sigma g_2^{-1}} & \cdots & \Delta_{g_1 \sigma g_n^{-1}} \\
\Delta_{g_2 \sigma g_1^{-1}} & \Delta_{g_2 \sigma g_2^{-1}} & \cdots & \Delta_{g_2 \sigma g_n^{-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\Delta_{g_n \sigma g_1^{-1}} & \Delta_{g_n \sigma g_2^{-1}} & \cdots & \Delta_{g_n \sigma g_n^{-1}}
\end{pmatrix}
\]

If we denote by \( e_{ij} \) the usual matrix units in \( A \), then \( e_{ij} \) is homogeneous of degree \( g_i^{-1}g_j \) for any \( i, j \).
It is proved in [13] Corollary 4.6.7 and in [1] Theorem 2.6] that a graded simple and graded left Artinian ring is necessarily isomorphic to a graded ring of the form \( A = M_n(\Delta)(g_1, \ldots, g_n) \) as above. In both cited references the proof uses a version of the Jacobson density theorem for graded simple modules. We will present an alternative proof in Section 3. It is also indicated in [13] page 31 that any graded ring \( A = M_n(\Delta)(g_1, \ldots, g_n) \) of this type is graded simple and graded left Artinian. Therefore there is just one isomorphism type of graded simple \( A \)-modules, and how many of them embed into \( A \).

For any \( 1 \leq j \leq n \), let \( \Sigma_j \) be the left \( A \)-module \( (\Delta \ldots \Delta) \) with a structure of a graded left \( A \)-module given by

\[
(\Sigma_j)_\sigma = \begin{pmatrix} \Delta_{g_1^{\sigma}g_j^{-1}} & \Delta_{g_2^{\sigma}g_j^{-1}} & \cdots & \Delta_{g_n^{\sigma}g_j^{-1}} \\
\end{pmatrix}
\]

We have that \( A \simeq \bigoplus_{1 \leq j \leq n} \Sigma_j \).

**Proposition 2.1.** \( \Sigma_j \) is a graded simple module for any \( 1 \leq j \leq n \).

**Proof.** Let \( 0 \neq z = \begin{pmatrix} u_1 \\
\vdots \\
u_n \end{pmatrix} \in (\Sigma_j)_\sigma \), thus \( u_i \in \Delta_{g_i^{\sigma}g_j^{-1}} \) for any \( 1 \leq i \leq n \). Pick \( i \) such that \( u_i \neq 0 \). Then for any \( y = \begin{pmatrix} v_1 \\
\vdots \\
v_n \end{pmatrix} \in \Sigma_j \) we have

\[
y = (v_1 u_i^{-1} e_{1i}) z + (v_2 u_i^{-1} e_{2i}) z + \ldots + (v_n u_i^{-1} e_{ni}) z \in Az
\]

so \( Az = \Sigma_j \), and this shows that \( \Sigma_j \) is graded simple. \( \square \)

We see that for any \( \sigma \in G \), \( (\Sigma_j(g_1^{-1} g_j))_\sigma = (\Sigma_j)_\sigma g_j^{-1} \), and this has \( \Delta_{g_i^{\sigma}g_j^{-1}} g_j^{-1} = \Delta_{g_i^{\sigma}g_j} \) on the \( i \)th row, so then \( \Sigma_j(g_1^{-1} g_j) = \Sigma_1 \), or \( \Sigma_j = \Sigma_1(g_j^{-1} g_1) \). As a consequence we obtain that

\[
A \simeq \bigoplus_{1 \leq j \leq n} \Sigma_1(g_j^{-1} g_1),
\]

thus \( A \) is a graded simple and graded left Artinian ring.

We denote by \( \text{supp}(\Delta) = \{ \sigma \in G \mid \Delta_\sigma \neq 0 \} \) the support of \( \Delta \), which is a subgroup of \( G \).

**Proposition 2.2.** Let \( \tau \in G \). Then \( \Sigma_1(\tau) \simeq \Sigma_1 \) if and only if \( \tau \in g_1^{-1} \text{supp}(\Delta)g_1 \).

**Proof.** If \( \Sigma \) is a graded simple \( A \)-module, and \( x \in \Sigma_g \setminus \{ 0 \} \), then \( \varphi : A(g^{-1}) \to \Sigma, \varphi(a) = ax \), is a surjective morphism of graded \( A \)-modules, so then \( \Sigma \simeq A(g^{-1})/\text{ann}_A(x) \). This shows that if \( \Gamma \) is another graded simple \( A \)-module, then \( \Sigma \simeq \Gamma \) if and only if there exists \( u \in \Gamma_g \) such that \( \text{ann}_A(u) = \text{ann}_A(x) \).

We apply this fact for \( \Sigma = \Sigma_1 \), \( x = \begin{pmatrix} 1 \\
0 \\
0 \end{pmatrix} \in (\Sigma_1)_e \), and \( \Gamma = \Sigma_1(\tau) \). Clearly, \( \text{ann}_A(x) = \begin{pmatrix} 0 & \Delta & \cdots & \Delta \\
0 & \Delta & \cdots & \Delta \\
0 & \Delta & \cdots & \Delta \\
\end{pmatrix} \), and then \( \Sigma_1(\tau) \simeq \Sigma_1 \) if and only if there exists

\[
u = \begin{pmatrix} u_1 \\
u_2 \\
\vdots \\
u_n \end{pmatrix} \in \Sigma_1(\tau)_e = (\Sigma_1)_\tau = \begin{pmatrix} \Delta_{g_1^{\tau}g_1^{-1}} \\
\Delta_{g_2^{\tau}g_2^{-1}} \\
\vdots \\
\Delta_{g_n^{\tau}g_n^{-1}} \end{pmatrix}
\]

with \( \text{ann}_A(u) = \text{ann}_A(x) \). But this forces \( u_2, \ldots, u_n \) to be all zero, and \( u_1 \) to be non-zero, and so the existence of such a \( u \) is equivalent to \( \Delta_{g_1^{\tau}g_1^{-1}} \neq 0 \). This is the same with \( g_1^\tau g_1^{-1} \in \text{supp}(\Delta) \), or \( \tau \in g_1^{-1} \text{supp}(\Delta)g_1 \). \( \square \)
Corollary 2.3. (i) The number of isomorphism types of graded simple left $A$-modules is $[G : \text{supp}(\Delta)]$.

(ii) There is a bijective correspondence between the set of isomorphism types of graded simple left $A$-modules that embed into $A$ and the set of the right $\text{supp}(\Delta)$-cosets of $G$ containing at least one $g_i$. Moreover, the multiplicity in $A$ of one of these graded simples is the number of the $g_i$'s lying in the corresponding coset.

(iii) $A$ is $gr$-uniform simple, i.e., all the simple graded left submodules of $A$ are isomorphic, if and only if all $g_1, \ldots, g_n$ lie in the same right $\text{supp}(\Delta)$-coset of $G$.

Proof. (i) We know that any graded simple left $A$-module is isomorphic to $\Sigma_1(g)$ for some $g \in G$. If $g, h \in G$, then $\Sigma_1(g) \simeq \Sigma_1(h)$ if and only if $g$ and $h$ lie in the same left $g_1^{-1}\text{supp}(\Delta)g_1$-coset of $G$. Thus the number of isomorphism types of graded simple left $A$-modules is $[G : g_1^{-1}\text{supp}(\Delta)g_1] = [G : \text{supp}(\Delta)]$.

(ii) We have that $\Sigma_1(g_j^{-1}g_1) \simeq \Sigma_1(g_p^{-1}g_1)$ if and only if $g_1^{-1}g_jg_p^{-1}g_1 \in g_1^{-1}\text{supp}(\Delta)g_1$, which is the same to $g_jg_p^{-1} \in \text{supp}(\Delta)$. Now everything is clear.

(iii) It follows from (ii). \hfill $\Box$

In a similar way we can describe the graded simple right $A$-modules. Thus for any $1 \leq i \leq n$ let $\Gamma_i = (\Delta \ldots \Delta)$, regarded as a matrix of type $1 \times n$ with entries in $\Delta$. $\Gamma_i$ is a graded right $A$-module with action given by usual matrix multiplication, and $G$-grading given such that the homogeneous component of degree $\sigma$ is the set of all elements of $\Gamma_i$ with elements from $\Delta_{g_i\sigma g_j^{-1}}$ on the $j$th spot. Then each $\Gamma_i$ is a graded simple right $A$-module and $A \simeq \bigoplus_{1 \leq i \leq n} \Gamma_i$. Moreover $\Gamma_i \simeq (g_i^{-1}g_1)\Gamma_1$ for any $i$. Also $(\tau)\Gamma_1 \simeq \Gamma_1$ if and only if $\tau \in g_1^{-1}\text{supp}(\Delta)g_1$. As a consequence, the number of isomorphism types of graded simple right $A$-modules is $[G : \text{supp}(\Delta)]$, and on the other hand, $\Gamma_i \simeq \Gamma_j$ if and only if $g_j \in \text{supp}(\Delta)g_i$, so the number of isomorphism types of graded simple right $A$-modules that embed into $A$ is the number of the right $\text{supp}(\Delta)$-cosets of $G$ containing at least one $g_i$.

We note that $A$ has the same number of isomorphism types of graded simple left modules as the number of isomorphism types of graded simple right modules, and then the same fact is true for any graded semisimple ring, which is a finite product of such $A$'s.

3. Graded Jacobson Radical and Graded Singular Radical

Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring. The graded Jacobson radical of $R$ is

$$J^{gr}(R) = \bigcap \{ I \mid I \text{ is a maximal graded left ideal of } R \}.$$ 

It turns out that $J^{gr}(R)$ is a graded ideal of $R$, and the same thing is obtained by taking the intersection of all maximal graded right ideals of $R$. One sees that $J^{gr}(R)$ is the intersection of the annihilators of all graded simple left (right) $R$-modules. For any $g \in G$, the homogeneous component of degree $g$ of $J^{gr}(R)$ consists of all elements $y \in R_g$ such that $1-xy$ is invertible for any $x \in R_{g^{-1}}$. We note that $J^{gr}(R) \cap R_e = J(R_e)$ and $J^{gr}(R)$ is the largest graded ideal of $R$ whose intersection with $R_e$ is $J(R_e)$, see [13] Section 2.9]. As in the un-graded case, a graded ring $R$ is graded semisimple if and only if it is graded left Artinian and $J^{gr}(R) = 0$; in particular $R/J^{gr}(R)$ is graded semisimple for any graded left Artinian ring $R$.

Remark 3.1. For later use, we note that if $R$ is graded left Artinian, then for any graded left $R$-module $M$, the socle $\text{soc}^{gr}(M)$ of $M$, i.e., the sum of all graded simple submodules of $M$, is given by $\text{soc}^{gr}(M) = \{ m \in M \mid J^{gr}(R)m = 0 \}$.

If $M$ is a graded left $R$-module and $N$ is a graded submodule of $M$, it is known that $N$ is essential in $M$ as an $R$-submodule (i.e., $N$ intersects non-trivially any non-zero $R$-submodule of $M$) if and only if $N \cap N' \neq 0$ for any non-zero graded submodule $N'$ of $M$, see [13] Proposition 2.3.5]. This is equivalent to the fact that for any non-zero homogeneous element $m \in M$, there
exists a homogeneous $r \in R$ such that $0 \neq rm \in N$. Thus we will be able to check that a graded submodule is essential working only with homogeneous elements.

For any $g \in G$ define

$$Z^{gr}(R)_g = \{ x \in R_g \mid \text{ann}_g(x) \text{ is essential in } R_g \},$$

which is obviously an additive subgroup of $R_g$, and let $Z^{gr}(R) = \sum_{g \in G} Z^{gr}(R)_g$.

If $x_g \in Z^{gr}(R)_g$ and $r_h \in R_h$, then $r_h x_g \in Z^{gr}(R(R))_g$. Indeed, if $p \in G$ and $a_p \in R_p \setminus \text{ann}_p(r_h x_g)$, then $a_p r_h x_g \neq 0$, so $a_p r_h \in R_{ph} \setminus \text{ann}_p(x_g)$. Since $\text{ann}_p(x_g)$ is essential in $R_p$, there exists $q \in G$ and $b_q \in R_q$ with $0 \neq b_q a_p r_h \in \text{ann}_q(x_g)$. Then $b_q a_p r_h x_g = 0$, so $b_q a_p \in \text{ann}_p(r_h x_g)$, and $b_q a_p \neq 0$. This shows that $\text{ann}_p(r_h x_g)$ is essential in $R_p$.

Thus $Z^{gr}(R)$ is a graded right ideal of $R$, and it is clear that it is also a graded left ideal, since $\text{ann}_g(x) \subseteq \text{ann}_g(xr)$ for any $x, r \in R$.

**Proposition 3.2.** If $R$ is graded left Noetherian then $Z^{gr}(R)$ is nilpotent.

**Proof.** Denote $I = Z^{gr}(R)$. Since $R$ is graded left Noetherian, there is a positive integer $m$ such that $\text{ann}_m(I^m) = \text{ann}_m(I^{m+1}) = \ldots$. We show that $I^m = 0$. Indeed, otherwise the family of graded left ideals $\{ \text{ann}_m(z) \mid z \notin \text{ann}_m(I^m) \}$ and $z$ is homogeneous has a maximal element $\text{ann}_m(x_g)$ for some $g \in G$ and $x_g \in R_g \setminus \text{ann}_m(I^m)$.

If $a_h \in I \cap R_h$, then $\text{ann}_m(a_h)$ is essential in $R$, so $\text{ann}_m(a_h) \cap Rx_g \neq 0$. Pick $0 \neq y_p x_g \in \text{ann}_m(a_h)$. Now $y_p \notin \text{ann}_m(x_g)$ and $y_p \in \text{ann}_m(x_g a_h)$, so $\text{ann}_m(x_g) \not\subset \text{ann}_m(x_g a_h)$. The maximality of $\text{ann}_m(x_g)$ shows that $x_g a_h \notin \text{ann}_m(I^m)$, so $x_g a_h I^m = 0$. As this happens for any homogeneous $a_h$ in $I$, we get that $x_g I^{m+1} = 0$, so then $x_g \in \text{ann}_m(I^{m+1}) = \text{ann}_m(I^m)$, a contradiction. □

At this point we need a result that will be used several times in the sequel.

**Theorem 3.3.** (The graded version of Baer’s Theorem, [14 Corollary 2.4.8]) Let $M$ be a graded left $R$-module. Then $M$ is graded injective, i.e., it is an injective object in the category $R - gr$, if and only if for any graded left ideal $I$ of $R$, any $\sigma \in G$ and any morphism $f : I \to M$ of degree $\sigma$ of graded left $R$-modules, there exists $m_\sigma \in M_\sigma$ such that $f(r) = rm_\sigma$ for any $r \in I$.

We say that the graded ring $R$ is graded left injective if it is graded injective when regarded as a left graded $R$-module. On the other hand, $R$ is called graded von Neumann regular if for any homogeneous element $a$ of $R$, there exists $b \in R$ (which can be supposed to be homogeneous) such that $a = aba$.

**Proposition 3.4.** Let $R$ be a graded ring which is graded left injective. Then $Z^{gr}(R) = J^{gr}(R)$ and $R/J^{gr}(R)$ is graded von Neumann regular.

**Proof.** Let $x_g \in J^{gr}(R)_g$. We prove that $\text{ann}_g(x_g)$ is essential in $R_g$, and this will show that $J^{gr}(R) \subset Z^{gr}(R)$. Indeed, if $I$ is a graded left ideal of $R$ with $\text{ann}_g(x_g) \cap I = 0$, then the map $\varphi : I \to R(g)$, $\varphi(a) = ax_g$, is an injective morphism of graded left $R$-modules. Let $i : I \to R$ be the inclusion map. Since $R$ is injective in $R - gr$, there is a morphism $\psi : R(g) \to R$ of graded left $R$-modules such that $\psi \varphi = i$. Then $a = \psi \varphi(a) = \psi(ax_g) = ax_g \psi(1)$, or $a(1 - x_g \psi(1)) = 0$ for any $a \in I$. As $\psi(1) \in R_{g-1}$ and $x_g \in J^{gr}(R)_g$, we see that $1 - x_g \psi(1)$ is invertible, so $a = 0$. Thus $I = 0$.

Now we show that $Z^{gr}(R) \cap R_\epsilon = J(R_\epsilon)$, and the maximality of $J^{gr}(R)$ among all graded ideals I with the property that $I \cap R_\epsilon = J(R_\epsilon)$ implies that $Z^{gr}(R) \subset J^{gr}(R)$. Clearly $J(R_\epsilon) = J^{gr}(R) \cap R_\epsilon \subset Z^{gr}(R) \cap R_\epsilon$. Now let $x \in Z^{gr}(R) \cap R_\epsilon$. Then $\text{ann}_\epsilon(x)$ is essential in $R_\epsilon$ and clearly $\text{ann}_\epsilon(x) \cap \text{ann}_\epsilon(1 - x) = 0$, so $\text{ann}_\epsilon(1 - x) = 0$. Then $1 - x$ is invertible, and $I = 0$ is an isomorphism in $R - gr$. Since $R$ is graded left injective, and $\varphi^{-1} : R(1 - x) \to R$ is a morphism in $R - gr$, there exists $y \in R_\epsilon$ such that $\varphi^{-1}$ is the right multiplication by $y$. Then $r = \varphi^{-1}(r(1 - x)) = r(1 - x)y$ for any $r \in R_\epsilon$, so $(1 - x)y = 1$. Thus $1 - x$ is right invertible for any $x$ in the ideal $Z^{gr}(R) \cap R_\epsilon$ of $R_\epsilon$, so $Z^{gr}(R) \cap R_\epsilon \subset J(R_\epsilon)$.

Next we show that $R/J^{gr}(R)$ is graded von Neumann regular. Let $a \in R_\epsilon$, and let $H$ be a graded left ideal of $R$ maximal with the property that $H \cap \text{ann}_\epsilon(a) = 0$. Then $H + \text{ann}_\epsilon(a)$ is
essential in $R R$, and the map $\varphi : H \to Ha, \varphi(x) = xa$, is a bijective morphism of degree $g$ of graded left $R$-modules. If $j : H \to R$ is the inclusion map, then $j \varphi^{-1} : Ha \to R$ is a morphism of degree $g^{-1}$, and the injectivity of $R$ in $R - gr$ shows that $j \varphi^{-1}$ is the right multiplication by some $b \in R_{g-1}$. Hence $x = \varphi^{-1}(xa) = xab$, and then $x(a - aba) = (x - xab)a = 0$ for any $x \in H$, showing that $H(a - aba) = 0$. Then clearly $(H + \text{ann}(a))(a - aba) = 0$, i.e. $H + \text{ann}(a) \subset \text{ann}(a - aba)$, showing that $a - aba \in Z^{gr}(R)g = J^{gr}(R)g \subset J^{gr}(R)$. We conclude that $a = a \hat{a} b a$ in $R/J^{gr}(R)$.

**Lemma 3.5.** Let $a \in R_g$. Then $Ra$ is a direct summand as a graded left submodule of $R$ if and only if there exists $b \in R_{g-1}$ such that $a = aba$.

**Proof.** If $a = aba$ for some $b \in R_{g-1}$, then $ba$ is an idempotent in $R_\varepsilon$ and $Ra = Rba$, a direct summand of the graded left $R$-module $R$.

Conversely, if $R = Ra \oplus I$ for some graded left ideal $I$, let $1 = u + v$ with $u, v$ idempotents of degree $\varepsilon$, $u \in Ra, v \in I$. Then $u = ba$ for some $b \in R_{g-1}$. Since $a = au + av, au = aba \in Ra$ and $av \in I$, we get that $a = au = aba$.

**Corollary 3.6.** If $R$ is graded semisimple then it is graded von Neumann regular.

Using Lemma 3.5 one can follow the same approach as in the non-graded case (for example as in [10]) to obtain the following.

**Proposition 3.7.** ([13] Proposition 1) Let $R$ be a graded von Neumann regular ring. Then any finitely generated graded left ideal of $R$ is a direct summand of $R$ in $R - gr$.

**Corollary 3.8.** A grading ring $R$ is graded semisimple if and only if it is graded left Noetherian and graded von Neumann regular.

**Proposition 3.9.** Let $R$ be a grading ring such that one of the following two conditions is satisfied:

(a) Any minimal graded left ideal is a direct summand of $R$ in $R - gr$ (in particular if $R$ is graded regular von Neumann).

(b) $R$ is graded left injective.

Then for any minimal graded left ideals $S$ and $S'$ of $R$ such that $S' \simeq S(g)$ for some $g \in G$, there exists a homogeneous $r \in R_g$ such that $S' =Sr$. As a consequence, the graded ideal generated by any minimal graded ideal of $R$ is the whole corresponding isoshift component of $R$.

**Proof.** If (a) holds, let $S = Ru$ and $S' = Rv$ for some idempotents $u, v \in R_\varepsilon$, and let $\varphi : S' \to S$ be an isomorphism of degree $g$. Then $a = \varphi(v) \in S_g = R_gu$, so $a = ru$ for some $r \in R_g$. We see that $au = ru^2 = ru = a$. Let $b = \varphi^{-1}(u) \in R_{g-1}$. Then $v = \varphi^{-1}(a) = \varphi^{-1}(au) = a\varphi^{-1}(u) = ab$. On the other hand $b = \varphi^{-1}(u) \in (Rv)_{g-1}$, so $b = sv$ for some $s \in R_{g-1}$, and then $bv = sv^2 = sv = b$. We obtain $u = \varphi(b) = \varphi(bv) = b(\varphi(v)) = ba$. These also show that $aba = au = a$ and $bab = bv = b$. Now $S' = Rv = Rba \supset Rbab = Sb$. Since $S'$ is a minimal left graded ideal, we must have $Sb = S'$ or $Sb = 0$. In the first case we are done. In the second one we get $Rbab = 0$, so $Rb = 0$, showing that $b = 0$, which is impossible, since $v = ab \neq 0$.

If (b) holds, let $\varphi : S \to S'$ be an isomorphism of degree $g$ of graded left modules. We may regard $\varphi$ as a morphism of degree $g$ from $S$ to $R$, and then the injectivity of $R$ shows that this morphism is the right multiplication by some $r \in R_g$. We get $S' = Im(\varphi) = Sr$.

At this point we can explain why the concept of a graded simple ring of [14], i.e., a grading ring which is a direct sum of minimal graded left ideals, any two of them graded isomorphic up to a shift, is equivalent to the one of a graded left Artinian ring whose only graded two-sided ideals are 0 and the whole ring, used in [4]. Indeed, if $R$ is isomorphic to a sum of minimal graded left ideals, any two of them graded isomorphic up to a shift, then it is graded semisimple, and there is just one isoshift component, which is the whole of $R$. If $I$ is a non-zero graded ideal of $R$, then $I$ contains a minimal graded left ideal $S$. Then $I$ contains the whole isoshift component.
associated to $S$, thus $I = R$. Obviously, $R$ is graded left Artinian, as a finite direct sum of minimal graded left ideals. Conversely, if the only graded simple ideals of $R$ are 0 and $R$, and $R$ is graded left Artinian, then $J^{gr}(R)$, a proper graded ideal, is 0, so $R$ is graded semisimple. Since $R$ is the direct sum of its isoshift components, each of them being a graded ideal, we see that there is just one such component, so $R$ is a sum of minimal graded left ideals, any of them isomorphic up to a shift.

Now we can give another proof for the structure theorem for graded simple rings which are graded left Artinian. Instead of using a density result for graded simple modules, as it is done in the proofs provided in [3], we use an argument inspired by [12] Theorem 3.11.

We first recall some constructions with graded modules. If $M, N \in R_{-gr}$, for any $\sigma \in G$ we denote by $HOM_R(M, N)_\sigma$ the set of all morphisms of degree $\sigma$ from $M$ to $N$, and we denote $HOM_R(M, N) = \sum_{\sigma \in G} HOM_R(M, N)_\sigma$, a direct sum of additive subgroups of $Hom_R(M, N)$.

In general, this sum may not be the whole $Hom_R(M, N)$, but under certain conditions we have $HOM_R(M, N) = Hom_R(M, N)$, for instance if $M$ is finitely generated ([14] Corollary 2.4.4]), or if both $M$ and $N$ have finite support ([14] Corollary 2.4.5]), in particular in the case where the grading group $G$ is finite. If $N = M$, then we denote $HOM_R(M, M)$ by $END_R(M)$, and this is a $G$-graded ring with multiplication the opposite map composition. Moreover, $M$ is a graded right $END_R(M)$-module with action $m \cdot f = f(m)$ for any $m \in M$ and $f \in END_R(M)$. Similar considerations can be done for graded right modules, in which case the multiplication of the endomorphism ring is just the usual map composition.

Theorem 3.10. Let $R$ be a $G$-graded ring which is graded simple and graded left Artinian. Then there exist a graded division ring $\Delta$, a positive integer $n$, and $g_1, \ldots, g_n \in G$ such that $R$ is isomorphic to $M_n(\Delta)(g_1, \ldots, g_n)$.

Proof. Since $R$ is graded left Artinian, it contains a minimal graded left ideal $V$. Then $\Delta = END_R(V)$ is a graded division ring (the multiplication is the opposite map composition), and $V$ is a graded right $\Delta$-module. Then we can consider the graded ring $E = END(V_{\Delta})$, and the map $\varphi : R \to E$, $\varphi(r)(a) = ra$ for any $r \in R$ and $a \in V$, is a morphism of $G$-graded rings. Since $R$ is graded simple, $\varphi$ is injective.

Now let $a \in V$, $\delta \in E$ and $v \in V_g$ for some $g \in G$. Then the map $f_v : V \to V$, $f_v(x) = xv$ lies in $\Delta_g$, and then

$$
(\delta \varphi(a))(v) = \delta(av) = \delta(a \cdot f_v) = \delta(a) \cdot f_v = f_v(\delta(a)) = \delta(a)v = \varphi(\delta(a))(v)
$$

As this holds for any homogeneous $v \in V$, we obtain that $\delta \varphi(a) = \varphi(\delta(a))$, so $E \varphi(V) \subset \varphi(V)$. Now $VR$ is a non-zero graded ideal of $R$, so $VR = R$, and then $\varphi(R) = \varphi(V) \varphi(R)$. This shows that $E \varphi(R) = E \varphi(V) \varphi(R) \subset \varphi(V) \varphi(R) = \varphi(R)$, so $\varphi(R)$ is a left ideal of $E$ which contains the identity element of $E$. We conclude that $\varphi(R) = E$, so $\varphi$ is an isomorphism.

Since $\Delta$ is a graded division ring, any graded right $\Delta$-module is free and has a basis consisting of homogeneous elements; moreover, any two homogeneous bases have the same cardinality. We show that a homogeneous basis of $V$ as a graded right $\Delta$-module is finite. Indeed, otherwise we can consider for any $\sigma \in G$ the set

$$I_\sigma = \{ \delta \in END(V_{\Delta})_\sigma \mid \text{Im}\delta \text{ has a finite homogeneous basis as a graded right } \Delta - \text{module} \},$$

and one can easily check that $I = \bigoplus_{\sigma \in G} I_\sigma$ is a non-proper graded ideal of $E$. This is a contradiction, since $E \simeq R$ is graded simple. We conclude that $V$ has a finite homogeneous basis,
say with \( n \) elements, so \( E = \text{END}(V_\Delta) = \text{End}(V_\Delta) \simeq M_n(\Delta) \), a ring isomorphism. Moreover, by [14 Proposition 2.10.5] or [4 pages 30-31], \( E = \text{END}(V_\Delta) \simeq M_n(\Delta)(g_1, \ldots, g_n) \), an isomorphism of graded rings, where \( g_1, \ldots, g_n \) are the degrees of the basis elements of \( V \). \( \square \)

4. Projective objects in the category of graded modules over a graded Artinian ring

If \( R \) is a graded left Artinian ring, then \( R/\mathfrak{J}^{\mathfrak{g}}(R) \) is graded semisimple, and the isomorphism types of graded simple left (right) \( R \)-modules are in bijection with the the isomorphism types of graded simple left (right) \( R/\mathfrak{J}^{\mathfrak{g}}(R) \)-modules. Moreover, this bijection preserves the isoshift equivalence, i.e., if \( S_1, S_2 \) are graded simple \( R \)-modules and \( \sigma \in G \), then \( S_1 \simeq S_2(\sigma) \) as graded left \( R \)-modules if and only if \( S_1 \simeq S_2(\sigma) \) as graded left \( R/\mathfrak{J}^{\mathfrak{g}}(R) \)-modules. As a consequence, \( R \) has the same number of isoshift types of graded simple modules to the left and to the right.

The following result shows which isoshift types can be found inside \( R \).

**Lemma 4.1.** Let \( M \) be a maximal graded left ideal in the \( G \)-graded algebra \( R \). The following are equivalent.

1. There exists \( g \in G \) such that \( (R/M)(g) \) embeds into \( R \).
2. \( M = \text{ann}_R(x) \) for a homogeneous element \( x \in R \).
3. \( \text{ann}_R(M) \neq 0 \).
4. \( M = \text{ann}_R(\text{ann}_R(M)) \).

**Proof.** 
(1) \( \Rightarrow \) (2) Let \( f : (R/M)(g) \to R \) be an injective morphism in \( R - gr \). Then \( x = f(\hat{1}) \) is a homogeneous element of degree \( g^{-1} \) in \( R \), and since \( f(\hat{r}) = rx \) for any \( r \in R \), we see that \( \text{ann}_R(x) = M \). Here \( \hat{r} \) denotes the class of \( r \) in \( R/M \).

(2) \( \Rightarrow \) (3) Since \( \text{ann}_R(x) = M \neq R \), \( x \) must be non-zero. Now \( \text{ann}_R(M) \) is non-zero since it contains \( x \).

(3) \( \Rightarrow \) (4) The inclusion \( M \subset \text{ann}_R(\text{ann}_R(M)) \) holds for any subset \( M \) of \( R \). If this inclusion is not an equality, then \( \text{ann}_R(\text{ann}_R(M)) = R \), showing that \( \text{ann}_R(M) = 0 \), a contradiction. Thus \( M = \text{ann}_R(\text{ann}_R(M)) \).

(4) \( \Rightarrow \) (1) Since \( M \neq R \), we have \( \text{ann}_R(M) \neq 0 \), so there are \( g \in G \) and \( 0 \neq x \in \text{ann}_R(M)_{g^{-1}} \). Then \( M = \text{ann}_R(\text{ann}_R(M)) \subset \text{ann}_R(x) \neq R \), so \( M = \text{ann}_R(x) \), and this implies that the map \( f : (R/M)(g) \to R \), \( f(\hat{r}) = rx \) for any \( r \in R \), is an injective morphism in \( R - gr \). \( \square \)

The following gives graded versions of fundamental structure results for Artinian rings, see [14, Corollary 2.9.7]. The second part is the graded version of Hopkins-Levitzki Theorem (in a slightly more general form).

**Theorem 4.2.**
1. Let \( R \) be a graded left Artinian ring. Then \( J^{\mathfrak{g}}(R) \) is nilpotent.
2. Let \( R \) be a graded ring such that \( J^{\mathfrak{g}}(R) \) is nilpotent and \( R/\mathfrak{J}^{\mathfrak{g}}(R) \) is graded semisimple. Then a graded left \( R \)-module \( M \) is graded Noetherian if and only if \( M \) is graded Artinian. In particular, a graded left Artinian ring is graded left Noetherian.

As a first consequence, we have the following.

**Proposition 4.3.** Let \( R \) be a graded right Artinian ring. Then any non-zero graded ideal of \( R \) contains a minimal graded left ideal.

**Proof.** Let \( I \) be a non-zero graded ideal of \( R \). We show that \( I \cap \text{ann}_R(J^{\mathfrak{g}}(R)) \neq 0 \). Indeed, otherwise let \( x \in I \setminus \{ 0 \} \). Then there exists \( a_1 \in J^{\mathfrak{g}}(R) \) with \( a_1x \neq 0 \). As \( a_1x \in I \), there exists \( a_2 \in J^{\mathfrak{g}}(R) \) such that \( a_2a_1x \neq 0 \). We continue recurrently and find \( a_1, a_2, \ldots \in J^{\mathfrak{g}}(R) \) such that \( a_m \ldots a_1x \neq 0 \) for any positive integer \( m \). This is in contradiction to the fact that \( J^{\mathfrak{g}}(R) \) is nilpotent.

Thus \( I \cap \text{ann}_R(J^{\mathfrak{g}}(R)) \) is a non-zero graded ideal of \( R \), and \( J^{\mathfrak{g}}(R)(I \cap \text{ann}_R(J^{\mathfrak{g}}(R))) = 0 \), so then \( I \cap \text{ann}_R(J^{\mathfrak{g}}(R)) \) is a non-zero graded left \( R/\mathfrak{J}^{\mathfrak{g}}(R) \)-module, thus a graded semisimple one, since \( R/\mathfrak{J}^{\mathfrak{g}}(R) \) is graded semisimple. We conclude that \( I \cap \text{ann}_R(J^{\mathfrak{g}}(R)) \) contains a graded
simple left $R/J^g(R)$-module, thus also a graded simple graded $R$-submodule. This is obviously a minimal graded left ideal contained in $I$. \hfill $\Box$

In the rest of this section we follow the approach in Sections 6.2 and 6.3 in [15], adapted to the graded case. Let $R$ be a $G$-graded ring which is graded left Artinian.

If $P$ is a graded left $R$-module, then $P/J^g(R)P$ is a graded left $R/J^g(R)$-module. Let $\pi_P : P \to P/J^g(R)P$ be the natural projection. If $Q$ is another graded left $R$-module, then for any morphism $u : P \to Q$ in $R - gr$, there exists a unique morphism $\overline{u} : P/J^g(R)P \to Q/J^g(R)Q$ in $R - gr$, and also in $R/J^g(R) - gr$, such that $\pi_P u = \overline{u} \pi_P$. This defines a linear map

$$
\theta_{P,Q} : \text{Hom}_{R - gr}(P,Q) \to \text{Hom}_{R/J^g(R) - gr}(P/J^g(R)P, Q/J^g(R)Q), \quad \theta_{P,Q}(u) = \overline{u},
$$

which is surjective in the case where $P$ is a projective object in $R - gr$, and a ring morphism in the case where $Q = P$.

**Proposition 4.4.** Let $R$ be a graded left Artinian ring, and let $P$ be a projective object in the category $R - gr$. Then $\text{Ker} \theta_{P,P} = J(\text{End}_{R - gr}(P))$, thus $\theta_{P,P}$ induces a ring isomorphism $\text{End}_{R - gr}(P)/J(\text{End}_{R - gr}(P)) \cong \text{End}_{R/J^g(R)}(P/J^g(R)P)$.

**Proof.** Let $u \in \text{Ker} \theta_{P,P}$. Then $\overline{u} = 0$, so $u(P) \subset J^g(R)P$. Then $u$ is nilpotent since so is $J^g(R)$. This shows that $\text{Ker} \theta_{P,P} \subset J(\text{End}_{R - gr}(P))$. Then $J(\text{End}_{R - gr}(P)/ \text{Ker} \theta_{P,P}) = J(\text{End}_{R - gr}(P))/ \text{Ker} \theta_{P,P}$.

Now $\text{End}_{R - gr}(P)/ \text{Ker} \theta_{P,P} \cong \text{End}_{R/J^g(R)}(P/J^g(R)P)$, and since $P/J^g(R)P$ is a semisimple graded left $R/J^g(R)$-module, we have that $J(\text{End}_{R/J^g(R)}(P/J^g(R)P)) = 0$. We conclude that $J(\text{End}_{R - gr}(P))/ \text{Ker} \theta_{P,P} = 0$, and then $\text{Ker} \theta_{P,P} = J(\text{End}_{R - gr}(P))$. \hfill $\Box$

**Corollary 4.5.** Let $P$ and $Q$ be graded projective left modules over the graded left Artinian ring $R$. Then $P \cong Q$ in $R - gr$ if and only if $P/J^g(R)P \cong Q/J^g(R)Q$ in $R/J^g(R) - gr$.

**Proof.** If $u \in \text{Hom}_{R - gr}(P,Q)$ is an isomorphism, then clearly

$$
\overline{u} = \theta_{P,Q}(u) \in \text{Hom}_{R/J^g(R) - gr}(P/J^g(R)P, Q/J^g(R)Q)
$$

is an isomorphism.

Conversely, let $\varphi \in \text{Hom}_{R/J^g(R) - gr}(P/J^g(R)P, Q/J^g(R)Q)$ be an isomorphism, with inverse $\psi$. Since $\theta_{P,Q}$ and $\theta_{Q,P}$ are surjective, there are $u \in \text{Hom}_{R - gr}(P,Q)$ and $v \in \text{Hom}_{R - gr}(Q,P)$ such that $\varphi = \overline{u}$ and $\psi = \overline{v}$. Since $\psi \varphi = Id$, we get $\overline{uv} = \overline{v} = \psi \varphi = Id = \overline{1P}$ (note that we used the same overline symbol in several Hom-spaces, but there is no danger of confusion), so $1P - uv \in \text{Ker} \theta_{P,P} = J(\text{End}_{R - gr}(P))$. Then $vu = 1P - (1P - uv)$ is invertible in $\text{End}_{R - gr}(P)$, so then $u$ has a left inverse as a graded morphism. Similarly, by $\varphi \psi = Id$, we get that $uv$ is invertible in $\text{End}_{R - gr}(Q)$, and then $u$ has a right inverse as a graded morphism. We conclude that $u$ is an isomorphism. \hfill $\Box$

If $R$ is a graded left Artinian ring, then we have a decomposition $R = P_1 \oplus \ldots \oplus P_n$, where $P_1, \ldots, P_n$ are graded indecomposable left modules. By the graded version of the Krull-Schmidt Theorem, this decomposition is unique (up to isomorphism and permutation of the factors). The factors $P_1, \ldots, P_n$ are called the graded principal indecomposable left $R$-modules. We are interested not only in their isomorphism types, but also in their isoshift types.

**Proposition 4.6.** Let $R$ be a graded left artinian ring. Then the mapping $P \mapsto P/J^g(R)P$ defines a bijective correspondence between the isomorphism types of principal graded indecomposable left $R$-modules and the isomorphism types of graded simple left $R/J^g(R)$-submodules that embed into $R/J^g(R)$. Moreover, the same mapping induces a bijective correspondence between the isoshift types of principal graded indecomposable left $R$-modules and the isoshift types of graded simple left $R/J^g(R)$-modules, and the latter are just the isoshift types of graded simple left $R$-modules.
Proof. Let $P$ be principal graded indecomposable left $R$-module. Since $R$ is graded left artinian, it is also graded left noetherian, thus a graded $R$-module of finite length, and then so is $P$. Since $P$ is indecomposable in $R - gr$, $\text{End}_{R-gr}(P)$ is a local ring, thus $\text{End}_{R/J^g(R)}(P/J^g(R)P) \simeq \text{End}_{R-gr}(P)/J(\text{End}_{R-gr}(P))$ is a division ring. Then $P/J^g(R)P$ is a graded semisimple $R/J^g(R)$-module with a division ring endomorphism ring, so it must be a graded simple module.

Now if $R = P_1 \oplus \ldots \oplus P_n$ is a decomposition with $P_1, \ldots, P_n$ graded indecomposable modules, we see that $R/J^g(R) \simeq P_1/J^g(R)P_1 \oplus \ldots \oplus P_n/J^g(R)P_n$ is a sum of graded simple $R/J^g(R)$-modules. Then any isomorphism type of a graded simple submodule of $R/J^g(R)$ is isomorphic to some $P_i/J^g(R)P_i$. Moreover, the correspondence $P \mapsto P/J^g(R)P$ is injective (as isomorphism types) by Corollary 14.5 and we have proved the first bijective correspondence.

The second bijective correspondence follows immediately if we use Corollary 4.5 for $P$ and $Q(\sigma)$, where $P$ and $Q$ are principal graded indecomposable left $R$-modules and $\sigma \in G$, and the fact that any graded simple left $R/J^g(R)$-module is isomorphic to a shift of a graded simple left submodule of $R/J^g(R)$.

**Theorem 4.7.** Let $R$ be a graded left artinian ring. Then any graded projective left $R$-module is isomorphic to a direct sum of shifts of principal graded indecomposable left $R$-modules, and this representation is unique up to permutation and isomorphism of the terms.

**Proof.** Let $U$ be a projective object in the category $R - gr$. Since $R/J^g(R)$ is graded semisimple, the graded left $R/J^g(R)$-module $U/J^g(R)U$ is a direct sum of graded simple $R/J^g(R)$-modules, and we have seen in Proposition 4.6 that any such simple is isomorphic to $(P/J^g(R)P)\sigma$ for some principal graded indecomposable $R$-module $P$ and some $\sigma \in G$. Thus

$$U/J^g(R)U \simeq \bigoplus_{i \in I}(P_i/J^g(R)P_i)(\sigma_i) \simeq \bigoplus_{i \in I}P_i(\sigma_i)/J^g(R)(\bigoplus_{i \in I}P_i(\sigma_i))$$

for some family $(P_i)_{i \in I}$ of graded indecomposable $R$-modules, and some family $(\sigma_i)_{i \in I}$ of elements of $G$. By Corollary 4.5 we get $U \simeq \bigoplus_{i \in I}P_i(\sigma_i)$.

For the uniqueness part, if $\bigoplus_{i \in I}U_i \simeq \bigoplus_{j \in J}V_j$, where all $U_i$’s and $V_j$’s are shifts of principal graded indecomposable $R$-modules, we have

$$(\bigoplus_{i \in I}U_i)/J^g(R)(\bigoplus_{i \in I}U_i) \simeq (\bigoplus_{j \in J}V_j)/J^g(R)(\bigoplus_{j \in J}V_j),$$

and then $\bigoplus_{i \in I}U_i/J^g(R)U_i \simeq \bigoplus_{j \in J}V_j/J^g(R)V_j$ as $R/J^g(R)$-modules. Both sides are direct sums of graded simple modules, so the terms of the right side are isomorphic in pairs, up to a permutation, to the ones in the left side. Using again Corollary 4.5 we get that the family $(U_i)_{i \in I}$ is just $(V_j)_{j \in J}$ up to a permutation (in fact a bijection from $I$ to $J$), and isomorphisms of graded $R$-modules.

As in the un-graded case, idempotents will play a prominent role in the study of graded Artinian rings, in particular when investigating graded principal indecomposables. Thus, if $R = P_1 \oplus \ldots \oplus P_n$ is a decomposition of the graded ring $R$ into a sum of graded indecomposable left modules, then $P_i = Re_1, \ldots, P_n = Re_n$ for a complete set $e_1, \ldots, e_n$ of primitive idempotents of $R_e$. Moreover, in this case $R = e_1 R \oplus \ldots \oplus e_n R$ is a decomposition of $R$ into a direct sum of graded indecomposable right modules.

In order to study the isoshift types of graded principal indecomposables, it is useful to note that if $e$ and $f$ are idempotents in $R_e$ and $\sigma \in G$, then $Re \simeq Rf(\sigma)$ as graded left $R$-modules if and only if $eR \simeq (\sigma^{-1})(fR)$ as graded right $R$-modules, see for example [19, Lemma 1.2].

5. Graded quasi-Frobenius rings

We recall that a ring $R$ is called quasi-Frobenius if it satisfies any of the following equivalent conditions (see [12, Theorem 15.1]): (1) $R$ is two-sided Artinian and it satisfies the double annihilator condition for right ideals, i.e., $\text{ann}_R(\text{ann}_RA) = A$ for any right ideal $A$ of $R$, and for left ideals, i.e., $\text{ann}_R(\text{ann}_RU) = U$ for any left ideal $U$ of $R$; (2) $R$ is left Noetherian and satisfies
the double annihilator condition for right ideals and for left ideals; (3) \( R \) is left Noetherian and injective as a left \( R \)-module; (4) \( R \) is right Noetherian and injective as a left \( R \)-module.

The aim of this section is to introduce graded quasi-Frobenius rings, by proving a graded version of the above mentioned theorem. We mainly follow the approach in the ungraded case from [12, Section 15A]. Several steps are similar to the ungraded case, however new aspects dictated by the presence of shifts occur at some other ones.

**Lemma 5.1.** If the graded ring \( R \) is graded left injective, then the following hold.
(i) \( \text{ann}_r(U) + \text{ann}_r(V) = \text{ann}_r(U \cap V) \) for any graded left ideals \( U \) and \( V \) of \( R \).
(ii) \( \text{ann}_r(\text{ann}_rA) = A \) for any finitely generated graded right ideal \( A \) of \( R \).

**Proof.** (i) Let \( a \) be a homogeneous element of degree \( g \) in \( \text{ann}_r(U \cap V) \). Define a map \( f : U + V \to R \) by \( f(u + v) = va \) for any \( u \in U, v \in V \). This is well defined: indeed, if \( u + v = u' + v' \) with \( u, u' \in U, v, v' \in V \), then \( u - u' = v' - v \in U \cap V \), so \( (v' - v)a = 0 \), therefore \( va = v'a \). It is clear that \( f \) is a morphism of degree \( g \) of graded left \( R \)-modules. Since \( R \) is graded left injective, \( f \) is the right multiplication by some \( b \in R_g \). Thus \( va = ub + vb \) for any \( u \in U, v \in V \). For \( v = 0 \), this shows that \( ub = 0 \) for any \( u \in U \), thus \( b \in \text{ann}_r(U) \), while for \( u = 0 \) we get \( v(a - b) = 0 \) for any \( v \in V \), so \( a - b \in \text{ann}_r(V) \). Then \( a = b + (a - b) \in \text{ann}_r(U) + \text{ann}_r(V) \). This shows that \( \text{ann}_r(U \cap V) \subset \text{ann}_r(U) + \text{ann}_r(V) \). The converse is obvious.

(ii) We first prove in the case where \( A = aR \) is a cyclic right ideal generated by a homogeneous element \( a \) of degree \( g \). Pick some homogeneous element \( b \) of degree \( h \) in the graded right ideal \( \text{ann}_r(\text{ann}_rA) \), and define the map \( f : Ra \to R \) by \( f(ra) = rb \) for any \( r \in R \). First of all, \( f \) is well defined, since \( ra = r'a \) implies that \( r - r' \in \text{ann}_rA \), so \( (r - r')b = 0 \). Moreover, \( f \) is a morphism of degree \( g^{-1}h \) of graded left \( R \)-modules, and the injectivity of \( R \) shows that \( f \) is the right multiplication by a homogeneous element \( c \) of degree \( g^{-1}h \). Then \( b = f(a) = ac \), so \( b \in aR \). Thus \( \text{ann}_r(\text{ann}_rA) \subset A \), and we have equality since the converse always holds.

Now if \( A \) is an arbitrary finitely generated graded right ideal, let \( a_1, \ldots, a_n \) be a family of homogeneous generators of \( A \). Then

\[
\text{ann}_r(\text{ann}_rA) = \text{ann}_r \left( \bigcap_{1 \leq i \leq n} \text{ann}_r(a_iR) \right) \\
= \sum_{1 \leq i \leq n} \text{ann}_r(\text{ann}_r(a_iR)) \quad \text{(by (i))} \\
= \sum_{1 \leq i \leq n} a_iR \quad \text{(by the cyclic case above)} \\
= A
\]

We say that a graded ring \( R \) is graded Artinian if it is graded left Artinian and graded right Artinian.

**Theorem 5.2.** Let \( R \) be a \( G \)-graded ring. The following assertions are equivalent.
(1) \( R \) is graded Artinian and it satisfies the double annihilator condition for graded right ideals, i.e., \( \text{ann}_r(\text{ann}_rA) = A \) for any graded right ideal \( A \) of \( R \), and for graded left ideals, i.e., \( \text{ann}_r(\text{ann}_rU) = U \) for any graded left ideal \( U \) of \( R \).
(2) \( R \) is graded left Noetherian and satisfies the double annihilator condition for graded right ideals and for graded left ideals.
(3) \( R \) is graded left Noetherian and graded left injective.
(4) \( R \) is graded right Noetherian and graded left injective.
Proof. (1) ⇒ (2) follows from Theorem 4.2.

(2) ⇒ (3) We first see that if $A$ and $B$ are left graded ideals of $R$, then

$$
\text{ann}_\ell(\text{ann}_r A + \text{ann}_r B) = \text{ann}_\ell(\text{ann}_r A) \cap \text{ann}_\ell(\text{ann}_r B) = A \cap B
$$

so then

$$
\text{ann}_r (A \cap B) = \text{ann}_r (\text{ann}_\ell(\text{ann}_r A + \text{ann}_r B)) = \text{ann}_r A + \text{ann}_r B
$$

We show that $R$ is injective in the category $R - \text{gr}$ by using the graded version of Baer’s
Theorem. Let $I$ be a graded left ideal of $R$, and let $f : I \to R$ be a morphism of degree $g \in G$.
Since $R$ is graded left Noetherian, we have $I = \sum_{i=1,n} Rc_i$ for some homogeneous elements
$c_1, \ldots, c_n \in R$. We show by induction on $n$ that there exists $y \in R_g$ such that $f(a) = ay$ for any $a \in I$.

For $n = 1$, let $d = f(c_1)$. Then $\text{ann}_\ell(c_1)d = 0$ since $rc_1 = 0$ implies that $rd = rf(c_1) = f(rc_1) = f(0) = 0$. It follows that $d \in \text{ann}_r(\text{ann}_\ell(c_1)) = \text{ann}_r(\text{ann}_\ell(c_1)R) = c_1 R$, so $d = c_1y$ for some $y \in R_g$. Since $c_1$ and $d$ are homogeneous elements and $\deg(d) = \deg(c_1)g$, we can choose $y$ to be homogeneous of degree $g$. Now $f(rc_1) = rf(c_1) = rd = r c_1 y$ for any $r \in R$, and we are done.

Assume the statement holds true for $n - 1$, and we prove it for $n$. Let $I = \sum_{i=1,n} Rc_i$
with homogeneous $c_1, \ldots, c_n \in R$, and let $J = \sum_{i=2,n} Rc_i$. By the induction hypothesis, the
restriction of $f$ to $J$ is the right multiplication with some $x \in R_g$. By the case $n = 1$, the
restriction of $f$ to $Rc_1$ is the right multiplication to some $y \in R_g$. Now if $a \in Rc_1 \cap J$, then
$ay = f(a)$ and $ax = f(a)$, so $a(x - y) = 0$. Thus $x - y \in \text{ann}_r(Rc_1 \cap J) = \text{ann}_r(Rc_1) + \text{ann}_r(J)$. Moreover, $x - y$ is homogeneous of degree $g$, so then there exist $y' \in \text{ann}_r(Rc_1)$ and $x' \in \text{ann}_r(J)$, both homogeneous of degree $g$, such that $x - y = x' - y'$. We show that $f$ is the right multiplication by $z = x - x' = y - y'$, and then we are done. Indeed, let $c \in I$, and write $c = a + b$ with $a \in Rc_1$ and $b \in J$. Then

$$
f(c) = f(a) + f(b) = ay + bx = a(y - y') + ay' + b(x - x') + bx' = az + bz = cz
$$

Note that we used that $ay' = 0$ and $bx' = 0$.

(3) ⇒ (4) Since $R$ is graded left injective, we have by Proposition 3.4 that $Z^{gr}(R R) = J^{gr}(R)$
and $R/J^{gr}(R)$ is graded von Neumann regular. As $R/J^{gr}(R)$ is also graded left Noetherian,
Corollary 3.2 shows that it is graded semisimple. By Proposition 3.2, $Z^{gr}(R R)$ is nilpotent,
and then so is $J^{gr}(R)$. Now Theorem 4.2 shows that any graded left Noetherian module is also
graded left Artinian; in particular $R$ is graded left Artinian.

Now we show that $R$ is graded right Noetherian. Indeed, if we assume it is not like this,
let $I_1 \subsetneq I_2 \subsetneq \ldots$ be an infinite chain of graded right ideals. Pick some homogeneous elements
$a_1 \in I_1, a_2 \in I_2 \setminus I_1, \ldots$, and denote $U_1 = a_1 R, U_2 = a_1 R + a_2 R, \ldots$. We get an infinite chain
of finitely generated graded right ideals $U_1 \subsetneq U_2 \subsetneq \ldots$. Then $\text{ann}_r(U_1) \supset \text{ann}_r(U_1) \supset \ldots$
is a sequence of graded left ideals, so it terminates since $R$ is graded left Artinian. Now $\text{ann}_r(U_m) = \text{ann}_r(U_{m+1}) = \ldots$
implies that $\text{ann}_r(\text{ann}_r(U_m)) = \text{ann}_r(\text{ann}_r(U_{m+1})) = \ldots$, so by Lemma 5.1 we get $U_m = U_{m+1} = \ldots$, a contradiction.

(4) ⇒ (1) Since any graded right ideal of $R$ is finitely generated, we get from Lemma 5.1 that
the double annihilator condition is satisfied for graded right ideals.
The ascending sequence of graded ideals \( \text{ann}_r(J^{gr}(R)) \subset \text{ann}_r((J^{gr}(R))^2) \subset \ldots \) terminates since \( R \) is graded right Noetherian, so \( \text{ann}_r((J^{gr}(R))^n) = \text{ann}_r((J^{gr}(R))^{n+1}) = \ldots \) for some \( n \). Using the double annihilator condition for graded right ideals we see that \( (J^{gr}(R))^n = (J^{gr}(R))^{n+1} \). Now the graded version of Nakayama’s Lemma (see [14, Corollary 2.9.2]) shows that the finitely generated graded right \( R \)-module \( (J^{gr}(R))^n \) must be zero.

As in (3)\(\Rightarrow\) (4), the injectivity of \( R \) as a graded left module implies that \( R/J^{gr}(R) \) is graded von Neumann regular. As \( R/J^{gr}(R) \) is also graded right Noetherian, the version of Corollary 3.8 to the right shows that it is graded semisimple. Since \( R \) is graded right Noetherian, we get that it is also graded right Artinian by the version of Theorem 4.12 to the right.

Now Lemma 3.1 shows that all isoshift types of graded simple right modules can be found inside \( R \), since the double annihilator condition holds for maximal graded right ideals. Let \( U_1, \ldots, U_n \) be the isoshift type components of the right graded \( R \)-module \( R \). By Proposition 4.3, each \( U_i \) contains a minimal graded left ideal \( \Sigma_i \). We claim that \( \Sigma_1, \ldots, \Sigma_n \) lie in different isoshift classes. Indeed, if \( \Sigma_i \simeq \Sigma_j(g) \) for some \( i \neq j \) and some \( g \in G \), then by Proposition 3.9 we get that \( \Sigma_j = \Sigma_i r \) for some \( r \in R_g \). But then \( \Sigma_j \subset U_i \cap U_j \). As \( \Sigma_j \subset U_j \), this provides a contradiction. Since \( R \) is graded right Artinian, the number of isoshift types of graded simple left \( R \)-modules is also \( n \), so then \( \Sigma_1, \ldots, \Sigma_n \) is a system of representatives for all these isoshift types. We conclude that all isoshift types of graded simple left modules can be found inside \( R \).

Next we show that for any non-zero graded left \( R \)-module \( M \), there exist some \( g \in G \) and a non-zero morphism of degree \( g \) of graded left \( R \)-modules \( f : M \rightarrow R \). Indeed, since \( J^{gr}(R) \) is nilpotent, we have that \( J^{gr}(R)M \neq M \). Then \( M/J^{gr}(R)M \) is a non-zero graded left \( R/J^{gr}(R) \)-module. Moreover, \( M/J^{gr}(R)M \) is a sum of graded simple left \( R/J^{gr}(R) \)-modules, since \( R/J^{gr}(R) \) is graded semisimple. In particular, there is a graded simple left \( R/J^{gr}(R) \)-module \( S \) and a surjective morphism \( \phi : M \rightarrow S \) of graded left \( R/J^{gr}(R) \)-modules. But this is also a surjective morphism of graded right \( R \)-modules, and \( S \) is also simple as a graded \( R \)-module. Moreover, since the isoshift type \( S \) lies inside \( R \), there is an injective morphism of degree \( g \) of graded left \( R \)-modules \( \psi : S \rightarrow R \) for some \( g \in G \). If \( \pi : M \rightarrow M/J^{gr}(R)M \) is the natural projection, then \( \psi \phi \pi : M \rightarrow R \) is a non-zero morphism of degree \( g \) of graded left \( R \)-modules.

Now we show that the double annihilator condition holds for graded left ideals. Indeed, if we assume that for a graded left ideal \( U \) we have \( \text{ann}_r(\text{ann}_r U) \neq U \), let \( M = \text{ann}_r(\text{ann}_r U)/U \), a non-zero graded left \( R \)-module, and let \( \pi : \text{ann}_r(\text{ann}_r U) \rightarrow M \) be the natural projection. We showed above that there exists a non-zero morphism \( f : M \rightarrow R \) of degree \( g \) of graded left \( R \)-modules for some \( g \in G \). Then \( f \pi : \text{ann}_r(\text{ann}_r U) \rightarrow R \) is a morphism of degree \( g \) of graded left \( R \)-modules, and the injectivity of \( R \) shows that \( f \pi \) is the right multiplication by some \( r \in R_g \).

As \( f \pi(U) = 0 \), we see that \( UR = 0 \), so \( r \in \text{ann}_r U \). Therefore for any \( x \in \text{ann}_r(\text{ann}_r U) \) we have \( 0 = xr = (f \pi)(x) \). Thus \( f \pi = 0 \), and so \( f = 0 \), a contradiction.

Finally, we show that \( R \) is also graded left Artinian. Indeed, let \( U_1 \supset U_2 \supset \ldots \) be a descending chain of graded left ideals of \( R \). Then \( \text{ann}_r(U_1) \subset \text{ann}_r(U_2) \subset \ldots \) is an ascending chain of graded right ideals, so \( \text{ann}_r(U_m) \subset \text{ann}_r(U_{m+1}) = \ldots \) for some \( m \). Then \( \text{ann}_r(\text{ann}_r U_m) = \text{ann}_r(\text{ann}_r U_{m+1}) = \ldots \), and the double annihilator condition shows that \( U_m = U_{m+1} = \ldots \), which ends the proof.

**Remark 5.3.** A graded ring \( R \) is called graded quasi-Frobenius if it satisfies the equivalent conditions of Theorem 5.2. We note that condition (1) in the Theorem is left-right symmetric, so we can add more equivalent conditions to the theorem: any condition saying that \( R \) is graded Noetherian at one side and graded injective at one side, also the condition saying that \( R \) is graded right Noetherian and it satisfies the two double annihilator conditions.

We note that if a graded ring \( R \) is quasi-Frobenius as a ring, then it is graded quasi-Frobenius. Indeed, since \( R \) is injective as a left \( R \)-module, it follows that \( R \) is graded left injective by [14, Corollary 2.3.2], and since \( R \) is left Noetherian, we clearly have that \( R \) is graded left Noetherian. The converse does not hold in general, thus a graded ring \( R \) may be graded quasi-Frobenius without being quasi-Frobenius. For example, let \( R = k[X, X^{-1}] \) be the Laurent polynomial ring.
over a field $k$, with its usual $\mathbb{Z}$-graded ring structure. Then $R$ is a graded division ring, so it is graded injective in view of Theorem 3.3. Obviously, it is also graded Noetherian. Thus $R$ is graded quasi-Frobenius. However, $R$ is not a quasi-Frobenius ring, since it is not injective as an $R$-module, see for example [14, Remark 2.3.3].

In the case where the graded ring $R$ has finite support, i.e., only finitely many homogeneous components $R_g$ are non-zero, in particular when $G$ is a finite group, the graded quasi-Frobenius and quasi-Frobenius conditions on $R$ are equivalent, as the following shows.

**Proposition 5.4.** Let $R$ be a graded ring of finite support. Then $R$ is graded quasi-Frobenius if and only if it is a quasi-Frobenius ring.

**Proof.** If $R$ is a graded ring of finite support which is graded left injective, then $R$ is injective as a left $R$-module by [3] Theorem 3.9. On the other hand, if $R$ is graded left Noetherian, then it is easy to see that each $R_g$ is a Noetherian left $R_e$-module. Since $R$ has finite support, we get that $R$ is a Noetherian left $R_e$-module, so it is Noetherian as a left $R$-module, too. \(\square\)

At the end of this section we list some properties of graded quasi-Frobenius rings that follow from the proof of Theorem 5.2. Thus let $R$ be a graded quasi-Frobenius ring. Then:

- Any isoshift type of graded simple module to the left and to the right can be found inside $R$. Thus for any graded simple left (respectively right) $R$-module $\Sigma$ there exists $g \in G$ such that $\Sigma(\ell g)$ (respectively $(g)\Sigma$) embeds into $R$.

- The number of isoshift types of graded simple left $R$-modules is equal to the number of isoshift types of graded simple right $R$-modules. Moreover, the left isoshift components of $R$ coincide with the right isoshift components.

- As a consequence, $soc^{gr}_I(R) = soc^{gr}_I(R)$. Since $R$ is graded left Artinian, we use Remark 3.1 to see that $soc^{gr}_I(R) = \text{ann}_R(J^{gr}_I(R))$ and $soc^{gr}_I(R) = \text{ann}_R(J^{gr}_I(R))$. We conclude that $soc^{gr}_I(R) = soc^{gr}_I(R) = \text{ann}_R(J^{gr}_I(R)) = \text{ann}_I(J^{gr}_I(R))$.

6. More properties of graded quasi-Frobenius rings

The following result is a graded version of [12, Theorem 15.9] and the proof is inspired by it. However, some general results about Grothendieck categories are needed.

**Theorem 6.1.** Let $R$ be graded ring. The following are equivalent:

1. $R$ is graded quasi-Frobenius.
2. The classes of injective objects and projective objects coincide in the category $R - gr$.

**Proof.** A key result that we need is the following graded version of Bass-Matlis-Papp Theorem: if $R$ is a graded ring, then the following are equivalent: (a) $R$ is graded left Noetherian; (b) Any direct sum of injective objects in $R - gr$ is graded injective; (c) Any injective object in $R - gr$ is a direct sum of indecomposable injective objects in $R - gr$. This follows from the more general result [17, Theorem 3], formulated for certain Grothendieck categories.

Assume now that $R$ is graded quasi-Frobenius. If $P$ is a projective object in $R - gr$, then $P$ is a direct summand in $\bigoplus_{i \in I} R(\sigma_i)$ for some family $(\sigma_i)_{i \in I}$ of elements of $G$. Each $R(\sigma_i)$ is graded injective since so is $R$, and then $\bigoplus_{i \in I} R(\sigma_i)$ is graded injective by (a)$\Rightarrow$(b) in the general result mentioned above. We obtain that $P$ is graded injective. On the other hand, if $Q$ is an injective object in $R - gr$, we use (a)$\Rightarrow$(c) in the general result to see that $Q = I \bigoplus_{i \in I} Q_i$, a direct sum of injective indecomposable objects in $R - gr$. Since $R$ is graded left Artinian, each $Q_i$ contains a graded simple module $S_i$, and then $Q_i = E^{gr}(S_i)$, the injective envelope of $S_i$ in $R - gr$. On the other hand, $S_i$ embeds into $R(\sigma_i)$ for some $\sigma_i \in G$. As $R(\sigma_i)$ is graded injective, we get that $Q_i$ embeds into $R(\sigma_i)$, hence it is a direct summand. This shows that each $Q_i$ is graded projective, and then so is $Q = \bigoplus_{i \in I} Q_i$.

Conversely, assume that injectives and projectives are the same in $R - gr$. Then $R$ is injective in $R - gr$, since it is obviously projective. On the other hand, any direct sum of injectives
(=projectives) in $R - gr$ is projective (=injective), and then $R$ is graded left Noetherian by (b) $\Rightarrow$ (a) in the general result. We obtain that $R$ is graded quasi-Frobenius.

Let $M \in R - gr$. We denote by $M^\sim$ the graded right $R$-module $\text{HOM}_R(M, R)$, which is a submodule of $\text{Hom}_R(M, R)$. As we explained before Theorem 6.1, $M^\sim = \text{Hom}_R(M, R)$ whenever $M$ is finitely generated. Similarly, we consider the graded left $R$-module $N^\sim = \text{HOM}_R(N, R)$ for any graded right $R$-module $N$. The natural map $\varphi_M : M \to (M^\sim)^\sim, \varphi(m)(f) = f(m)$ for any $m \in M$ and $f \in M^\sim$, is a morphism of graded left $R$-modules. We will use the simpler notation $M^{\sim^\sim}$ for $(M^\sim)^\sim$.

**Lemma 6.2.** Let $M \in R - gr$ and let $\tau \in G$. Then $M(\tau)^\sim = (\tau^{-1})M^\sim$.

**Proof.** Recall that the homogeneous component of degree $\sigma$ of $M^\sim$ consists of all morphisms of $R$-modules $f : M \to R$ such that $f(M_g) \subset R_{g\sigma}$ for any $g \in G$. Then the homogeneous component of degree $\sigma$ of $M(\tau)^\sim$ consists of all morphisms of $R$-modules $f : M \to R$ such that $f(M_{g\tau}) \subset R_{g\sigma}$ for any $g \in G$, and this is equivalent to $f(M_h) \subset R_{h\tau^{-1}\sigma}$ for any $h \in G$, which means that $f \in (M^\sim)(\tau^{-1}\sigma) = ((\tau^{-1})M^\sim)_\sigma$. We conclude that $M(\tau)^\sim = (\tau^{-1})M^\sim$.

The proof of the following result works as in the un-graded case, see [12, Theorem 15.11, 15.12, and Corollary 15.13].

**Proposition 6.3.** Let $R$ be a graded quasi-Frobenius ring. Then the following assertions hold.

1. $\varphi_M$ is an isomorphism for any finitely generated graded left $R$-module $M$.
2. A graded left $R$-module $M$ is finitely generated if and only if the graded right $R$-module $M^\sim$ is finitely generated.
3. The functor associating $M^\sim$ to a graded left module $M$ is a duality between the category of finitely generated graded left $R$-modules and the category of finitely generated graded right $R$-modules.
4. If $M$ is a graded left $R$-module, then $M$ is graded simple if and only if $M^\sim$ is a graded simple right $R$-module.

We add the following characterization of graded quasi-Frobenius algebras, whose proof goes word by word as in the un-graded case, see [12, Theorem 16.2], working with graded objects.

**Theorem 6.4.** Let $R$ be a graded Artinian ring. Then the following are equivalent.

1. $R$ is graded quasi-Frobenius;
2. The dual $S^\sim$ of any graded simple left (right) $R$-module $S$ is a graded simple right (left) $R$-module.
3. The dual $S^\sim$ of any graded simple left (right) $R$-module $S$ is either 0 or a graded simple right (left) $R$-module.

The following result is a graded version of [12, Theorem 16.4]. Moreover, its proof is on the same line as in the ungraded case, however some new aspects occur in the graded case. Since these will play a key role in defining graded Frobenius rings, we sketch the proof and emphasize the parts specific to the graded situation.

**Theorem 6.5.** A graded Artinian ring $R$ is graded quasi-Frobenius if and only if the following two conditions are satisfied:

1. Any graded simple left (right) $R$-module embeds up to a shift into $R$.
2. Any principal graded indecomposable left (right) $R$-module has just one graded simple submodule.

**Proof.** Assume that conditions (i) and (ii) are satisfied. As in the ungraded case, one shows that $\text{soc}^g_P(R) = \text{soc}^g_S(R)$.

Now we show that $S^\Lambda$ is either 0 or a graded simple right $R$-module for any graded simple left $R$-module $S$. By Lemma 6.2 this is equivalent to proving it for a shift of $S$, so by (i), we may assume that $S$ embeds into $R$. Thus $S = \text{soc}^g_P$ for some principal graded indecomposable
left $R$-module $P$. Write $P = Re$ for a homogeneous idempotent $e$ of trivial degree. Then the map
\[ \phi : eR \to S^\wedge, \phi(er)(x) = xer \] for any $r \in R, x \in S$,
is a morphism of right graded $R$-modules. Moreover,
\[ \phi(eJ^g(R)) = SeJ^g(R) \subseteq S^g(R) \subseteq soc^g_1(R)J^g(R) = soc^g_1(R)J^g(R) = 0, \]
so $\phi$ induces a morphism \( \overline{\phi} : eR/eJ^g(R) \to S^\wedge \).

Since $S$ is graded simple, there is an isomorphism $\gamma : (Re/J^g(R)e')(\sigma) \to S$ of graded left $R$-modules for some homogeneous idempotent $e'$ of trivial degree such that $Re'$ is a principal graded indecomposable left $R$-module. As $\overline{\gamma}$ (the class is modulo $J^g(R)e'$) has degree $\sigma^{-1}$ in $(Re/J^g(R)e')(\sigma)$, we see that $\gamma(\overline{e'}) \in S_\sigma^{-1}$. Clearly, $\gamma(\overline{e'}) = \gamma(e'e) = \gamma(\overline{e'})$, so then $\gamma(\overline{e'}) \in e'R \cap S \subseteq e'R \cap soc^g_1(R) = e'R \cap soc^g_1(R) = soc^g(e'R)$.

Now let $u$ be a nonzero homogeneous element of $S^\wedge$. Then $u(S)$ is isomorphic to a shift of $S$, thus it is graded simple, and working as above for $\overline{\gamma}(\overline{e'})$, we get that $u(\gamma(\overline{e'})) \in e'R \cap u(S) \subseteq soc^g_1(e'R)$. Since $soc^g(e'R)$ is simple, we must have $\gamma(\overline{e'})R = u(\gamma(\overline{e'}))R = soc^g_1(e'R)$, so $u(\gamma(\overline{e'})) = \gamma(\overline{e'})r$ for some homogeneous $r \in R$. Then for any $x \in R$
\[ u(x\gamma(\overline{e'})) = x\gamma(\overline{e'})r = x\gamma(\overline{e'})er \quad \text{(since $\gamma(\overline{e'}) \in S \subseteq Re$)} \]
\[ = \overline{\phi}(x\gamma(\overline{e'})), \]
which shows that $u = \overline{\phi}(x\gamma(\overline{e'}))$. We conclude that $\overline{\phi}$ is surjective. Since $eR/eJ^g(R)$ is simple, this shows that $S^\wedge$ is either 0 or isomorphic to $eR/eJ^g(R)$, thus simple. Now $R$ is graded quasi-Frobenius by Theorem 6.7.4.

A consequence of the proof of the previous theorem is the following.

**Corollary 6.6.** Let $R$ be a graded quasi-Frobenius ring. Let $e$ and $e'$ be two primitive idempotents in $R_1$ such that $soc^g(Re) \simeq (Re/J^g(R)e')(\sigma)$ as graded left $R$-modules for some $\sigma \in G$. Then $soc^g(e'R) \simeq (\sigma)(eR/eJ^g(R))$ as graded right $R$-modules.

**Proof.** Denote by $S = soc^g(Re)$, thus $S \simeq (Re/J^g(R)e')(\sigma)$. We have seen in the proof of Theorem 6.6.5 that $S^\wedge \simeq eR/eJ^g(R)$. Thus $soc^g(Re)^\wedge \simeq eR/eJ^g(R)$. Taking the duals, we get $soc^g(Re)^\wedge \simeq (eR/eJ^g(R))^\wedge$. Proceeding in a similar way with $e'$ to the right, we see that $soc^g(e'R)^\wedge \simeq (eR/eJ^g(R))^\wedge$. Using Lemma 6.7.2 we obtain that
\[ soc^g(e'R) \simeq (S(\sigma^{-1}))^\wedge \simeq (\sigma)S^\wedge \simeq (\sigma)(eR/eJ^g(R)) \]

**Corollary 6.7.** Let $R$ be a graded Artinian ring, and let $Re_1, \ldots, Re_t$ be a system of representatives for the isoshift types of principal graded indecomposable left $R$-modules. Let $S_i = Re_i/J^g(R)e_i$ and $S_i' = e_iR/e_iJ^g(R)$ for any $1 \leq i \leq t$. Then the following are equivalent.
(1) $R$ is graded quasi-Frobenius.
(2) There exist a permutation $\pi \in S(\{1, \ldots, t\})$ and some $\sigma_1, \ldots, \sigma_t \in G$ such that $soc^g(Re_i) \simeq S_\pi(\sigma_i)$ and $soc^g(e_\pi(R)) \simeq (\sigma_i)S_i'$ for any $1 \leq i \leq t$.

**Proof.** (1)⇒(2) If $R$ is graded quasi-Frobenius, then we know by Corollary 6.6.6 that there is a map $\pi : \{1, \ldots, t\} \to \{1, \ldots, t\}$ such that
\[ soc^g(Re_i) \simeq (Re_\pi/J^g(R)e_\pi(\sigma_i)) \text{ and } soc^g(e_\pi(R)) \simeq (\sigma_i)e_iR/e_iJ^g(R). \]
We show that $\pi$ is injective, thus also bijective. Indeed, if $\pi(i) = \pi(j)$ for some $i, j$, then since $Re_i$ is the injective envelope of $soc^g(Re_i)$, we have $Re_i \simeq E^g(Re_\pi/J^g(R)e_\pi(\sigma_i))$. Similarly
\[ Re_j \simeq E(Re_\pi/J^g(R)e_\pi(\sigma_j)) \]
\[ \simeq E(Re_\pi/J^g(R)e_\pi(\sigma_j)) \]
\[ \simeq (Re_i)(\sigma_j\sigma_i^{-1}) \]
so $Re_i$ and $Re_j$ have the same isoshift type, i.e., $i = j$.

(2)⇒(1) Since $\pi$ is bijective, we see that any left (right) graded simple module embeds into a principal graded indecomposable left (right) $R$-module, thus also into $R$. Moreover, the socle of any principal graded indecomposable is graded simple by (2). We get that $R$ is graded quasi-Frobenius by Theorem 6.5.

We conclude this section by summarizing that to a graded quasi-Frobenius ring $R$ we associate

- a positive integer $t$, the number if isoshift types of graded principal indecomposable left $R$-module; choose some system of representatives $P_1, \ldots, P_t$ for these isoshift types, such that $P_1, \ldots, P_t$ embed into $R$. If $S_i = P_i/J^{gr}(R)P_i$ for $1 \leq i \leq t$, we know that $S_1, \ldots, S_t$ are the isoshift types of graded simple left $R$-modules.

- some positive integers $n_1, \ldots, n_t$, indicating the multiplicities of the isoshift types $P_1, \ldots, P_t$ in a decomposition of $R$.

- a set $g_1, \ldots, g_{n_t}$ of elements of $G$, such that the isoshift component of type $P_i$ of $R$ (in a decomposition into a direct sum of graded indecomposable left modules) is $P_i(g_1) \oplus \cdots \oplus P_i(g_{n_t})$.

- a permutation $\pi \in S\{(1, \ldots, t)\}$, called the Nakayama permutation, and some elements $\sigma_1, \ldots, \sigma_t \in G$ such that $soc(P_i) \simeq S_{\pi(i)}(\sigma_i)$ for any $1 \leq i \leq t$.

Thus we associated a set of data $(t, (n_i)_{1 \leq i \leq t}, (g_{ij})_{1 \leq i \leq t}, (\pi, (\sigma_i)_{1 \leq i \leq t})$ to the graded quasi-Frobenius ring $R$. Clearly, this set of data depends on the choices we make: the order of the isoshift types and the choice of each $P_i$.

By the discussion at the end of Section 1 we see that if we consider $P_1 = Re_1, \ldots, P_t = Re_t$ for some idempotents $e_1, \ldots, e_t \in R_e$, then $P_i = e_i R, \ldots, P_t = e_t R$ is a system of representatives for the isoshift types of graded principal indecomposable right $R$-modules, and $S_i' = P_i/J^{gr}(R), \ldots, S_t' = P_t/J^{gr}(R)$ is a system of representatives for the isoshift types of graded simple right $R$-modules. Moreover, the isoshift component of type $P_i$ of $R$ (as a graded right $R$-module) is $(g_{i1}^{-1})P_i \oplus \cdots \oplus (g_{in_t}^{-1})P_n$ for each $i$.

If $M$ is a graded left module over a $G$-graded ring $R$, we denote by $(\Sigma(M) = \{g \in G | M(g) \simeq M\}$, which is a subgroup of $G$, called the inertia group of $M$. Similarly, the inertia group of a graded right module $M$ consists of all $g$ such that $(g)M \simeq M$.

**Lemma 6.8.** Let $R$ be a graded quasi-Frobenius ring. With notations as above, we have:

1. $\Sigma(S_i) = \Sigma(P_i) = \Sigma(P_i') = \Sigma(S_i')$ for any $1 \leq i \leq t$;
2. $\Sigma(S_i) = \sigma_i \Sigma(S_{\pi(i)} \sigma_i^{-1}$ for any $1 \leq i \leq t$.

*Proof.* (1) If we apply Corollary 1.5 for $P = P_i$ and $Q = P_i(\sigma)$, where $\sigma \in G$, we see that $P_i(\sigma) \simeq P_i$ if and only if $S_i(\sigma) \simeq S_i$. This shows that $\Sigma(P_i) = \Sigma(S_i)$. Similarly $\Sigma(P_i') = \Sigma(S_i')$.

Now write $P_i = eR$ and $P_i' = eR$ for some idempotent $e \in R_e$. If $\sigma \in G$, then $\Sigma(e) \Sigma(P_i)$ if and only if $Re \simeq Re(\sigma)$ in $R - \text{gr}$. By [19] Lemma 1.2, this is equivalent to $eR \simeq (\sigma^{-1})eR$, which means that $\sigma^{-1} \in \Sigma(P_i')$. Thus the subgroups $\Sigma(P_i')$ of $G$ are equal.

(2) Let $S = \text{soc}^{gr}(P_i)$. We know that $S \simeq S_{\pi(i)}(\sigma_i)$. Since $P_i$ is graded injective, we have that $P_i = E^{gr}(S)$, the injective envelope of $S$ in $R - \text{gr}$. If $\sigma \in G$ is such that $P_i \simeq P_i(\sigma)$, then $\text{soc}^{gr}(P_i) \simeq \text{soc}^{gr}(P_i(\sigma))$, so $S \simeq S(\sigma)$. Conversely, if $\sigma \in G$ is such that $S \simeq S(\sigma)$, then $E^{gr}(S) \simeq E^{gr}(S(\sigma))$, so $P_i \simeq P_i(\sigma)$. Thus $\Sigma(P_i) = \Sigma(S) = \Sigma(S_{\pi(i)}(\sigma_i))) = \sigma_i \Sigma(S_{\pi(i)}(\sigma_i))^{-1}$, and the result follows since $\Sigma(S_i) = \Sigma(P_i)$.

7. **Graded Frobenius rings**

We recall (see [12] Theorem 16.14 and Corollary 16.16) that a two-sided Artinian ring $R$ is called Frobenius if it satisfies one of the following equivalent conditions: (1) $R$ is quasi-Frobenius and $\text{soc}(R) \simeq R/J(R)$ as left $R$-modules; (2) $R$ is quasi-Frobenius and $\text{soc}(R) \simeq R/J(R)$ as right $R$-modules; (3) $R$ is quasi-Frobenius and $n_i = n_{\pi(i)}$ for any $1 \leq i \leq t$, where $n_1, \ldots, n_t$
are the multiplicities of the isomorphism types of principal indecomposable modules, and \( \pi \) is the Nakayama permutation (which is just the one we described in Section 6 when we regard \( R \) as a graded ring with trivial grading); (4) \( \text{soc}_i(R) \simeq R/J(R) \) as left \( R \)-modules, and \( \text{soc}_i(R) \simeq R/J(R) \) as right \( R \)-modules; (5) \( R \) is quasi-Frobenius and \( R/J(R) \simeq \text{Hom}_{R-}(R/J(R), R) \) as left \( R \)-modules; (6) \( R \) is quasi-Frobenius and \( R/J(R) \simeq \text{Hom}_{R-}(R/J(R), R) \) as right \( R \)-modules.

We will need the following simple fact.

**Lemma 7.1.** If \( R \) is a \( G \)-graded ring, \( S \) is a graded simple left \( R \)-module, \( S' \) is a graded simple right \( R \)-module, and \( g_1, \ldots, g_n, h_1, \ldots, h_n \in G \), then:

1. \( S(g_1) \oplus \cdots \oplus S(g_n) \simeq S(h_1) \oplus \cdots \oplus S(h_n) \) in \( R \)-gr if and only if the sequence of left \( \Sigma(S) \)-cosets \( g_1 \Sigma(S), \ldots, g_n \Sigma(S) \) is a permutation of \( h_1 \Sigma(S), \ldots, h_n \Sigma(S) \).
2. \( (g_1)S' \oplus \cdots \oplus (g_n)S' \simeq (h_1)S' \oplus \cdots \oplus (h_n)S' \) in \( gr-R \) if and only if the sequence of right cosets \( \Sigma(S')g_1, \ldots, \Sigma(S')g_n \) is a permutation of \( \Sigma(S')h_1, \ldots, \Sigma(S')h_n \).

**Proof.** (1) follows from the fact that \( S(g) \simeq S(h) \) if and only if \( g^{-1}h \in \Sigma(S) \), i.e., \( g \Sigma(S) = h \Sigma(S) \). (2) is similar. \( \square \)

Let \( R \) be a graded Artinian ring. Let \( P_1, \ldots, P_t \) be a system of representatives for the isomorphism types of principal graded indecomposable left \( R \)-modules, and say that the principal graded indecomposable left \( R \)-modules of isomorphism type \( P_i \) that occur in a decomposition of \( R \) are isomorphic to \( P_i(g_{i1}), \ldots, P_i(g_{in_i}) \) for any \( 1 \leq i \leq t \).

Then \( S_i = P_i/J^g(R)P_i \), \( 1 \leq i \leq t \), is a system of representatives for the isomorphism types of graded simple left \( R/J^g(R) \)-modules, thus also for the isomorphism types of graded simple left \( R \)-modules, and a decomposition of \( R/J^g(R) \) as a sum of graded simple left \( R \)-modules is

\[
R/J^g(R) = \bigoplus_{1 \leq i \leq t} \bigoplus_{1 \leq j \leq n_i} S_i(g_{ij}).
\]

If moreover \( R \) is graded quasi-Frobenius, let \( \sigma_1, \ldots, \sigma_t \in G \) be such that \( \text{soc}^g(P_i) \simeq S_{\pi(i)}(\sigma_i) \) for any \( i \), where \( \pi \in S(\{1, \ldots, t\}) \) is the Nakayama permutation associated with \( R \).

Recall that if \( R \) is a graded quasi-Frobenius ring, then \( \text{soc}^g(R) = \text{soc}^{gr}(R) \), and we simply denote this graded ideal of \( R \) by \( \text{soc}^g(R) \).

**Theorem 7.2.** Let \( R \) be a graded Artinian ring, and let \( \sigma \in G \). The following are equivalent.

1. \( R \) is graded quasi-Frobenius and \( \text{soc}^g(R)(\sigma) \simeq R/J^g(R) \) in \( R \)-gr.
2. \( R \) is graded quasi-Frobenius and \( (\sigma) \text{soc}^g(R) \simeq R/J^g(R) \) in \( gr-R \).
3. \( R \) is graded quasi-Frobenius and for any \( 1 \leq i \leq t \) we have \( n_i = n_{\pi(i)} \), and the sequence of left cosets \( \sigma g_{i1} \Sigma(S_{\pi(i)}), \ldots, \sigma g_{in_i} \Sigma(S_{\pi(i)}) \) is a permutation of \( g_{1i} \Sigma(S_{\pi(i)}), \ldots, g_{ni} \Sigma(S_{\pi(i)}) \).
4. \( \text{soc}^g(R)(\sigma) \simeq R/J^g(R) \) in \( R \)-gr and \( (\sigma) \text{soc}^g(R) \simeq R/J^g(R) \) in \( gr-R \).
5. \( R \) is graded quasi-Frobenius and \( (R/J^g(R))(\sigma) \simeq R/J^g(R) \) in \( gr-R \) (where \( R/J^g(R) \) is regarded as a graded right \( R \)-module in the left hand side, and as a graded left \( R \)-module in the right hand side).
6. \( R \) is graded quasi-Frobenius and \( (\sigma)(R/J^g(R))(\sigma) \simeq R/J^g(R) \) in \( gr-R \) (where \( R/J^g(R) \) is regarded as a graded left \( R \)-module in the left hand side, and as a graded right \( R \)-module in the right hand side).

**Proof.** (1)\(\Leftrightarrow\)(3) The decomposition \( R = \bigoplus_{1 \leq i \leq t} \bigoplus_{1 \leq j \leq n_i} P_{ij} \) shows that

\[
\text{soc}^g(R) \simeq \bigoplus_{1 \leq i \leq t} \bigoplus_{1 \leq j \leq n_i} \text{soc}^g(P_{ij})
\]

\[
\simeq \bigoplus_{1 \leq i \leq t} \bigoplus_{1 \leq j \leq n_i} (S_{\pi(i)}(\sigma_i))
\]

\[
\simeq \bigoplus_{1 \leq i \leq t} \bigoplus_{1 \leq j \leq n_i} (S_{\pi(i)}(g_{ij} \sigma_i))
\]

On the other hand, \( R/J^g(R) = \bigoplus_{1 \leq i \leq t} \bigoplus_{1 \leq j \leq n_i} S_i(g_{ij}) \), so then \( \text{soc}^g(R)(\sigma) \simeq R/J^g(R) \) if and only if the components of the same isomorphism type in these two graded \( R \)-modules are isomorphic, i.e.,

\[
\bigoplus_{1 \leq j \leq n_i} (S_{\pi(i)}(g_{ij} \sigma_i)) \simeq \bigoplus_{1 \leq j \leq n_{\pi(i)}} S_{\pi(i)}(g_{\pi(i)j})
\]
for any $i$. Using Lemma 6.1, this is equivalent to the fact that for any $i$ we have $n_i = n_{\pi(i)}$, and the sequence of left cosets $\sigma g_{i1} \sigma \Sigma(S_{\pi(i)}), \ldots, \sigma g_{in} \sigma \Sigma(S_{\pi(i)})$ is a permutation of $g_{\pi(i)} \Sigma(S_{\pi(i)}), \ldots, g_{\pi(i) n} \Sigma(S_{\pi(i)})$.

(2)$\leftrightarrow$(3) As in (1)$\leftrightarrow$(3), the decomposition $R = \oplus_{1 \leq r \leq t} \oplus_{1 \leq j \leq n_r} P'_{rj}$ shows that

$\text{soc}^{gr}(R) \simeq \oplus_{1 \leq r \leq t} \oplus_{1 \leq j \leq n_r} ((\sigma g_{\pi(i)r}^{-1}) S'_{\pi^{-1}(r)})$

Hence $(\sigma) \text{soc}^{gr}(R) \simeq R/J^{gr}(R)$ if and only if

$\oplus_{1 \leq j \leq n_r} ((\sigma g_{\pi(i)r}^{-1}) S_{\pi^{-1}(r)}) \simeq \oplus_{1 \leq j \leq n_r} ((g_{\pi(i)r}^{-1}) S'_{\pi^{-1}(r)})$

for any $r$. Denoting $i = \pi^{-1}(r)$, this means that $n_i = n_{\pi(i)}$ and, using Lemma 6.1, that

$\Sigma(S'_i) g_{i1}^{-1}, \ldots, \Sigma(S'_i) g_{in}^{-1}$

is a permutation of $\Sigma(S'_i) \sigma g_{\pi(i)1}, \ldots, \Sigma(S'_i) \sigma g_{\pi(i)n}$, and $\sigma$. Passing to left cosets and taking into account that $\Sigma(S'_i) = \Sigma(S_i)$, this rewrites that $g_{i1} \Sigma(S_i), \ldots, g_{in} \Sigma(S_i)$ is a permutation of $\sigma^{-1} g_{\pi(i)1} \sigma^{-1} \Sigma(S_i), \ldots, \sigma^{-1} g_{\pi(i)n} \sigma^{-1} \Sigma(S_i)$. By Lemma 6.2, we know that

$g_{i1} \sigma \Sigma(S_{\pi(i)}), \ldots, g_{in} \sigma \Sigma(S_{\pi(i)})$

is a permutation of

$\sigma^{-1} g_{\pi(i)1} \sigma^{-1}, \ldots, \sigma^{-1} g_{\pi(i)n} \sigma^{-1}$

which after right multiplication by $\sigma$ and left multiplication by $\sigma$ becomes just the condition in (3).

It is obvious that if (1) holds (thus so does (2)), then (4) holds, too. Assume now that (4) holds and we prove (1). In fact, we just need to show that $R$ is graded quasi-Frobenius. Since each isoshift type of graded simple left $R$-modules occurs inside $R/J^{gr}(R)$, it also occurs inside $\text{soc}^{gr}(R)$, thus it embeds into $R$, too. On the other hand, if we decompose $R = \oplus_{1 \leq p \leq m} Q_p$ into a sum of indecomposable graded left $R$-modules, then $R/J^{gr}(R) \simeq \oplus_{1 \leq p \leq m} Q_p / J^{gr}(R) Q_p$, a direct sum of $m$ graded simple modules, so then $\text{soc}^{gr}(R) = \oplus_{1 \leq p \leq m} \text{soc}^{gr}(Q_p)(\sigma)$ is also a direct sum of $m$ graded simple modules. As each $\text{soc}^{gr}(Q_p)$ is non-zero, we get that it must be graded simple. We proceed the same to the right and Theorem 6.3 shows that $R$ is graded quasi-Frobenius.

(1)$\leftrightarrow$(5) Let us first note that if $R$ is graded quasi-Frobenius, then the injectivity of $R$ shows that for any morphism $f : \text{soc}^{gr}(R) \rightarrow R$ of degree $\sigma$ of graded left $R$-modules, there exists $a \in R_\sigma$ such that $f(r) = ra$ for any $r \in \text{soc}^{gr}(R)$; denote by $f_a$ the morphism associated with $a$ in this way. Clearly, $f_a = f_b$ if and only if $a - b \in \text{ann}_r(\text{soc}^{gr}(R)) = J^{gr}(R)$. This induces an isomorphism of graded right $R$-modules $R/J^{gr}(R) \simeq \text{soc}^{gr}(R)^\sim$, which associates $f_a$ to $\bar{a} \in R/J^{gr}(R)$ ($\bar{a}$ is the class of a modulo $J^{gr}(R)$). Therefore $(R/J^{gr}(R))^{\sim} \simeq \text{soc}^{gr}(R)^{\sim \sim} \simeq \text{soc}^{gr}(R)$ as graded left $R$-modules. Now the equivalence of the two conditions is clear.

(2)$\leftrightarrow$(6) is similar to (1)$\leftrightarrow$(5), working the opposite side.

A graded Artinian ring $R$ satisfying the equivalent conditions in Theorem 6.1 will be called a $\sigma$-graded Frobenius ring. An $\varepsilon$-graded Frobenius ring will be simply called a graded Frobenius ring.

The following gives some examples of graded Frobenius rings and a method of constructing new graded Frobenius rings from known ones.

**Proposition 7.3.** (1) If $R_1, \ldots, R_n$ are $G$-graded rings, then $R_1 \times \ldots \times R_n$ is graded quasi-Frobenius if and only if $R_1, \ldots, R_n$ are graded quasi-Frobenius.

(2) If $R_1, \ldots, R_n$ are $G$-graded rings and $\sigma \in G$, then $R_1 \times \ldots \times R_n$ is $\sigma$-graded Frobenius if and only if $R_1, \ldots, R_n$ are $\sigma$-graded Frobenius.

(3) A graded semisimple ring $R$ is graded Frobenius. In particular, graded division rings are graded Frobenius.
Proposition 7.4. or equivalently, if \( A \sigma \in R \) be a finite dimensional \( G \)-graded \( k \)-algebra, where \( k \) is a field. Then the linear dual space \( A^* = \text{Hom}_k(A,k) \) is a \( G \)-graded vector space, whose homogeneous component of degree \( g \) is \( (A^*)_g = \{ f \in A^* \mid f(A_h) = 0 \text{ for any } h \neq g^{-1} \}. \) Moreover, when regarded with the \( A \)-bimodule structure induced by the \( A \)-bimodule structure of \( A, A^* \) becomes a graded left \( A \)-module and a graded right \( A \)-module. Then \( A \) is called a \( \sigma \)-graded Frobenius algebra if \( A(\sigma) \simeq A^* \text{ in } A - gr \), or equivalently, if \( (\sigma)A \simeq A^* \text{ in } gr - A, \) see \([2\text{, Section 3}].\)

\[ \text{Lemma 7.5. Let } R \text{ be a finite dimensional } G\text{-graded } k\text{-algebra, where } k \text{ is a field, and let } \sigma \in G. \text{ Then } R \text{ is a } \sigma\text{-graded Frobenius algebra if and only if it is a } \sigma\text{-graded Frobenius ring.} \]

\[ \text{Proof. Regard } R^* \text{ as a graded left } R\text{-module. Since } R \text{ is left Artinian, we use Remark 3.1 to see that } \]

\[ \text{soc}^{gr}(R R^*) = \{ r^* \in R^* \mid J^{gr}(R)r^* = 0 \} = \{ r^* \in R^* \mid r^*(J^{gr}(R)) = 0 \}. \]

This shows that there is an isomorphism \( \text{soc}^{gr}(R R^*) \simeq (R/J^{gr}(R))^* \) of graded left \( R \)-modules. Now \( R/J^{gr}(R) \) is a finite dimensional graded semisimple \( k \)-algebra, so by \([2\text{, Corollary 4.5}]\) it is graded Frobenius. Thus \( (R/J^{gr}(R))^* \simeq R/J^{gr}(R) \) as graded left \( R/J^{gr}(R) \)-modules, and then also as graded left \( R \)-modules. We conclude that \( \text{soc}^{gr}(R R^*) \simeq R/J^{gr}(R) \) as graded left \( R \)-modules.

Let us also note that \( R^* \) is an injective left \( R \)-module (since \( R \) is a projective right \( R \)-module), and then so is as a graded left \( R \)-module.

Now assume that \( R \) is a \( \sigma \)-graded Frobenius algebra, i.e., \( R(\sigma) \simeq R^* \text{ in } R - gr \). Thus \( R(\sigma) \) is injective as a graded left \( R \)-module, and then so is \( R \), showing that \( R \) is graded quasi-Frobenius. Moreover, taking the socles, we have that \( \text{soc}^{gr}_R(R(\sigma)) \simeq \text{soc}^{gr}(R R^*) \simeq R/J^{gr}(R) \) as graded left \( R \)-modules. This shows that \( R \) is a \( \sigma \)-graded Frobenius ring.

Conversely, if \( R \) is a \( \sigma \)-graded Frobenius ring, then \( R \) is graded quasi-Frobenius, so \( R \) is an injective graded left \( R \)-module. Since \( R \) is finite dimensional, \( \text{soc}^{gr}_R(R) \) is an essential left submodule of \( R \), so then \( R = E^{gr}(\text{soc}^{gr}_R(R)) \). On the other hand, \( \text{soc}^{gr}(R R^*) \) is essential in \( R^* \) (since \( R^* \) is finite dimensional), and \( R R^* \) is injective, so \( R^* = E^{gr}(\text{soc}^{gr}(R R^*)) \simeq E^{gr}(R/J^{gr}(R)). \)

As \( \text{soc}^{gr}_R(R(\sigma)) \simeq R/J^{gr}(R) \), we get that \( R(\sigma) \simeq R^* \text{ in } R - gr \), so \( R \) is a graded Frobenius algebra. \( \square \)

We will give a structure result for \( \sigma \)-graded Frobenius rings. We recall that if \( M \) is a graded left \( R \)-module, and \( \sigma \in G \), then \( M \) is called \( \sigma \)-faithful if \( X_\sigma \neq 0 \) for any non-zero graded submodule \( X \) of \( M \); this is equivalent to \( R_{\sigma g_1} m_g \neq 0 \) for any non-zero homogeneous element \( m_g \in M_g \). The \( \sigma \)-faithful condition can be defined similarly for graded right modules. The following is obvious.

\[ \text{Lemma 7.5. Let } M \text{ be a graded left } R\text{-module, and let } U \text{ be a graded submodule of } M. \text{ The following hold.} \]

(i) If \( M \) is \( \sigma \)-faithful, then so is \( U \).
(ii) If \( U \) is essential in \( M \) and \( U \) is \( \sigma \)-faithful, then \( M \) is \( \sigma \)-faithful.
(iii) If \( U \) is essential in \( M \), and \( U \) is graded semisimple, then \( U = \text{soc}^{gr}(M) \).

We say that the graded ring \( R \) is \( \sigma \)-faithful to the left (right) if it is \( \sigma \)-faithful as a graded left (right) \( R \)-module.
Lemma 7.6. Let $R$ be a graded left Artinian graded ring which is $\sigma$-faithful to the left. Then $\text{soc}_R^\text{gr}(R)_{\sigma} = \text{soc}(R_\varepsilon R_\sigma)$.

Proof. "⊂" Let $\Sigma$ be a graded simple left submodule of $R$. Then $\Sigma_\sigma$ is either 0 or a simple $R_\varepsilon$-module. In either case $\Sigma_\sigma \subset \text{soc}(R_\varepsilon R_\sigma)$.

"⊃" Let $S$ be a simple $R_\varepsilon$-submodule of $R_\sigma$. Then $RS$ is a non-zero graded left submodule of $R$. As $R$ is graded left Artinian, $U = \text{soc}^\text{gr}_R(RS) \neq 0$. Since $R$ is $\sigma$-faithful to the left, $U_\sigma \neq 0$. But $U_\sigma \subset (RS)_\sigma = S$, so $U_\sigma = S$. Then $S = U_\sigma \subset \text{soc}(R_\varepsilon R_\sigma)$. □

Lemma 7.7. Let $R$ be a graded left Artinian ring. Then the graded left $R$-module $R/J^\text{gr}(R)$ is $\varepsilon$-faithful.

Proof. The graded $R$-submodules of $R/J^\text{gr}(R)$ are the same as the graded $R/J^\text{gr}(R)$-submodules of $R/J^\text{gr}(R)$. Now the result follows since $R/J^\text{gr}(R)$ is a graded semisimple ring, and any graded semisimple ring $A$ is $\varepsilon$-faithful to the left (and to the right) by [14] Proposition 2.9.6. □

We also recall the definition of the coinduced functor from [14] Section 2.5. If $N$ is a left $R_\varepsilon$-module, denote $\text{Coind}(N)_g = \{ f \in \text{Hom}_{R_\varepsilon}(R, N) | f(R_h) = 0 \text{ for any } h \neq g^{-1} \}$ for each $g \in G$. Then $\sum_{g \in G} \text{Coind}(N)_g$ is a direct sum inside $\text{Hom}_{R_\varepsilon}(R, N)$, which we denote by $\text{Coind}(N)$. Then $\text{Coind}(N)$ is an $R$-submodule of $\text{Hom}_{R_\varepsilon}(R, N)$, moreover, it is a graded $R$-module with the decomposition given by the sum above. If $M$ is a graded left $R$-module, and $\sigma \in G$, then the map

$$\nu_M : M \to \text{Coind}(M_\sigma)(\sigma^{-1}), \nu_M(m_g)(a) = a_{\sigma g^{-1}}m_g$$ for any $m_g \in M_g, a \in R$

is a morphism of graded $R$-modules, see [14] page 39. Moreover, $\text{Im} \nu_M$ is an essential submodule of $\text{Coind}(M_\sigma)(\sigma^{-1})$ ([14] Proposition 2.6.2]), and $\nu_M$ is injective if $M$ is $\sigma$-faithful ([14] Proposition 2.6.3]).

Theorem 7.8. Let $R$ be a graded Artinian ring, and let $\sigma \in G$. The following are equivalent.

(1) $R$ is a $\sigma$-graded Frobenius ring.

(2) $R$ is $\sigma$-faithful to the left and to the right, $\text{soc}(R_\varepsilon R_\sigma) \simeq R_\varepsilon/J(R_\varepsilon)$ as left $R_\varepsilon$-modules, and $\text{soc}((R_\varepsilon R_\sigma)_R) \simeq R_\varepsilon/J(R_\varepsilon)$ as right $R_\varepsilon$-modules.

Proof. (1)⇒(2) Since $R$ is graded left Artinian, $\text{soc}_R^\text{gr}(R)$ is essential in $R$ as a graded left submodule. As $R$ is graded injective, we have $E^\text{gr}_\varepsilon(\text{soc}_R^\text{gr}(R)) = R$. Now $\text{soc}_R^\text{gr}(R) \simeq (R/J^\text{gr}(R))(\sigma^{-1})$, and we get $R = E^\text{gr}_\varepsilon(\text{soc}_R^\text{gr}(R)) \simeq E^\text{gr}(R/J^\text{gr}(R))(\sigma^{-1})$. By Lemma 7.7, $R/J^\text{gr}(R)$ is $\varepsilon$-faithful as a graded left $R$-module, then so is $E^\text{gr}(R/J^\text{gr}(R))$ by Lemma 7.5(ii). Therefore its shift $E^\text{gr}(R/J^\text{gr}(R))(\sigma^{-1})$ is $\sigma$-faithful, showing that $R$ is $\sigma$-faithful to the left.

The isomorphism $\text{soc}_R^\text{gr}(R)(\sigma) \simeq R/J^\text{gr}(R)$ of graded left $R$-modules, induces an isomorphism of left $R_\varepsilon$-modules between the homogeneous components of degree $\varepsilon$. We have $(\text{soc}_R^\text{gr}(R)(\sigma))_\varepsilon = \text{soc}_R^\text{gr}(R)_{\varepsilon \sigma} = \text{soc}(R_\varepsilon R_\sigma)$ (the last equality following from Lemma 7.6), and $(R/J^\text{gr}(R))_\varepsilon = R_\varepsilon/J(R_\varepsilon)$ since $J^\text{gr}(R) \subset R_\varepsilon = J(R_\varepsilon)$ (see [14] Corollary 2.9.3]). We get $\text{soc}(R_\varepsilon R_\sigma) \simeq R_\varepsilon/J(R_\varepsilon)$ as left $R_\varepsilon$-modules. Working similarly to the right, we obtain that $R$ is $\varepsilon$-faithful to the right and $\text{soc}((R_\varepsilon R_\sigma)_R) \simeq R_\varepsilon/J(R_\varepsilon)$ as right $R_\varepsilon$-modules.

(2)⇒(1) Since $R$ is $\sigma$-faithful to the left, $\text{soc}_R^\text{gr}(R)$ is also $\sigma$-faithful. Therefore, taking into account the above considerations and Lemma 7.6, we see that

$$\nu_{\text{soc}_R^\text{gr}(R)} : \text{soc}_R^\text{gr}(R) \to \text{Coind}(\text{soc}_R^\text{gr}(R)(\sigma^{-1})) = \text{Coind}(\text{soc}(R_\varepsilon R_\sigma))(\sigma^{-1})$$

is an essential injective morphism in $R - \text{gr}$. As $\text{soc}_R^\text{gr}(R)$ is graded semisimple, it follows by Lemma 7.5(iii) that $\text{soc}_R^\text{gr}(R)(\sigma) \simeq (\text{Im} \nu_{\text{soc}_R^\text{gr}(R)})(\sigma) = \text{soc}(\text{Coind}(\text{soc}(R_\varepsilon R_\sigma)))$.

On the other hand, $R/J^\text{gr}(R)$ is $\varepsilon$-faithful as a graded left $R$-module by Lemma 7.7, so

$$\nu_{R/J^\text{gr}(R)} : R/J^\text{gr}(R) \to \text{Coind}((R/J^\text{gr}(R))_\varepsilon) = \text{Coind}(R_\varepsilon/J(R_\varepsilon))$$
is an essential injective morphism in $R - gr$. Since $R/J^{gr}(R)$ is a semisimple graded $R$-module, we see again by Lemma 7.5(iii) that

$$R/J^{gr}(R) \simeq \text{Im} \nu_{R/J^{gr}(R)} = \text{soc}^{gr}(\text{Coind}(R_{\varepsilon}/J(R_{\varepsilon}))).$$

Now $\text{soc}(R_{\varepsilon}/J) \simeq R_{\varepsilon}/J(R_{\varepsilon})$ as left $R_{\varepsilon}$-modules, therefore $\text{Coind}(\text{soc}(R_{\varepsilon}/J)) \simeq \text{Coind}(R_{\varepsilon}/J(R_{\varepsilon}))$ are isomorphic as graded left $R$-modules, and then so are their socles. We conclude that $\text{soc}^{gr}(R)(\sigma) \simeq R/J^{gr}(R)$. Working similarly to the right (with the adapted version of the coinduced functor), we obtain $(\sigma)\text{soc}^{gr}(R) \simeq R/J^{gr}(R)$ in $gr - R$. These show that $R$ is $\sigma$-graded Frobenius.

\[\square\]

**Corollary 7.9.** Let $R$ be a graded Artinian ring. Then $R$ is graded Frobenius if and only if it is $\varepsilon$-faithful to the left and to the right and $R_{\varepsilon}$ is a Frobenius ring.

**Remark 7.10.** The previous Corollary shows that if $R$ is a graded Frobenius ring, then its homogeneous component $R_{\varepsilon}$ of trivial degree is a Frobenius ring. We note that a similar transfer does not hold for the quasi-Frobenius property, more precisely, $R$ may be graded quasi-Frobenius, such that $R_{\varepsilon}$ is not quasi-Frobenius.

We first recall from [12] Example 16.60 that if $A$ is a finite dimensional algebra over a field $k$, then $\mathcal{E}(A) = A \oplus A^*$ has a $k$-algebra structure with multiplication defined by $(a, a^*)(b, b^*) = (ab, ab^* + a^*b)$ for any $a, b \in A$ and $a^*, b^* \in A^*$; here we regard $A^*$ as an $A$-bimodule in the usual way. $\mathcal{E}(A)$ is called the trivial extension of $A$, and it is always a Frobenius algebra (even a symmetric algebra). Moreover, $\mathcal{E}(A)$ has a grading by the cyclic group $C_2 = \{\varepsilon, c\}$ of order 2, with $\mathcal{E}(A)_\varepsilon = A \oplus 0$ and $\mathcal{E}(A)_c = 0 \oplus A^*$. One can easily see that $\mathcal{E}(A)$ is not $\varepsilon$-faithful to the left, but it is $c$-faithful to the left.

Now we see that if $A$ is a finite dimensional $k$-algebra which is not quasi-Frobenius, then $\mathcal{E}(A)$ is a Frobenius ring (since it is a Frobenius algebra), so it is a quasi-Frobenius ring, thus also a graded quasi-Frobenius ring. On the other hand, $\mathcal{E}(A)_\varepsilon \simeq A$ is not a quasi-Frobenius ring.

Now if we take a Frobenius finite dimensional $k$-algebra $A$, then we see that $\mathcal{E}(A)_\varepsilon$ is a Frobenius ring, but $\mathcal{E}(A)$ is not a graded Frobenius ring, since it is not $\varepsilon$-faithful to the left. Thus the "if" implication in Corollary 7.9 does not hold anymore if we omit the $\varepsilon$-faithful condition.

The next result shows that these connections work better for strongly graded rings.

**Proposition 7.11.** Let $R$ be a strongly graded ring. The following assertions hold.

1. $R$ is a graded quasi-Frobenius ring if and only if $R_{\varepsilon}$ is a quasi-Frobenius ring.

2. $R$ is a graded Frobenius ring if and only if $R_{\varepsilon}$ is a Frobenius ring.

**Proof.** As a consequence of Dade’s Theorem, which says that the induced functor $R \otimes R_{\varepsilon} - : R_{\varepsilon} - \text{mod} \rightarrow R - gr$ is an equivalence of categories (see [14] Theorem 3.1.1), we have that $R$ is graded left Artinian (Noetherian) if and only if $R_{\varepsilon}$ is left Artinian (Noetherian), and the same fact is true to the right. Also, $R$ is injective in $R - gr$ if and only if $R_{\varepsilon}$ is injective in $R_{\varepsilon}$-mod. Now (1) is clear by using Theorem 5.2.

On the other hand, $R$ is $\varepsilon$-faithful to the left (and to the right). Indeed, let $g \in G$ and $r_g \in R_g$ such that $R_{g^{-1}}r_g = 0$. Then $R_gr_{g^{-1}}r_g = 0$. But $R$ is strongly graded, so $R_g R_{g^{-1}} = R_{\varepsilon}$, so $R_{\varepsilon}r_g = 0$, showing that $r_g = 0$. Now (2) follows directly from Corollary 7.9. □

**Remark 7.12.** If $R$ is a finite dimensional graded algebra over a field, it is obvious that if $R$ is graded Frobenius, then $R$ is a Frobenius algebra; indeed, we just regard an isomorphism of graded left $R$-modules between $R$ and $R^*$ just as an isomorphism of $R$-modules.

If $R$ is a $G$-graded ring which is graded Frobenius, then $R$ is not necessarily a Frobenius ring. Indeed, we can use the example after Remark 5.3: $R = k[X, X^{-1}]$ is a graded division ring, so it is clearly a graded Frobenius ring, while it is not even quasi-Frobenius.

We do not know whether for a finite group $G$, a graded Frobenius ring $R$ is also a Frobenius ring. This is true in the case when the order of $G$ is invertible in $R$. Indeed, under this condition,
a consequence of graded Clifford theory is that $J^{gr}(R) = J(R)$ and $soc^{gr}(R) = soc(R)$, see [14, Corollary 4.4.5 and Proposition 4.4.10]. Since $R$ is graded Frobenius, it is graded quasi-Frobenius, thus also quasi-Frobenius, and then the result is clear taking into account Theorem 7.2 (1).

References

[1] L. Abrams, Modules, comodules, and cotensor products over Frobenius algebras, J. Algebra 219 (1999), 201-213.
[2] S. Dăscălescu, C. Năstăsescu and L. Năstăsescu, Frobenius algebras of corepresentations and group graded vector spaces, J. Algebra 406 (2014), 226-250.
[3] S. Dăscălescu, C. Năstăsescu, A. Del Rio, F. Van Oystaeyen, Gradings of finite support. Application to injective objects, J. Pure Appl. Algebra 107 (1996), 193-206.
[4] A. Elduque and M. Kochetov, Gradings on simple Lie algebras, Math. Surveys and Monographs 189 (2013), AMS.
[5] V. Ginzburg, Calabi-Yau algebras, arXiv:math/0612139v3.
[6] R. Hazrat, Leavitt path algebras are graded von Neumann regular rings, J. Algebra 401 (2014), 220-233.
[7] R. Hazrat, Graded rings and graded Grothendieck groups, London Math. Soc. Lecture Note Series 435 (2016), Cambridge Univ. Press.
[8] J. W. He and X. J. Xia, Constructions of graded Frobenius algebras, J. Algebra Appl. 19 (2020), 2050081.
[9] P. Jørgensen, A noncommutative BGG correspondence, Pacific J. Math. 218 (2005), 357-377.
[10] K. R. Goodearl, Von Neumann Regular Rings, 2nd ed., Krieger Publishing Co., Malabar, FL, 1991.
[11] T. Y. Lam, A first course in noncommutative rings, GTM 131, Second Edition, Springer Verlag, 2001.
[12] T. Y. Lam, Lectures on modules and rings, GTM 189, Springer Verlag, 1999.
[13] D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, Regular algebras of dimension 4 and their $A_\infty$-Ext-algebras, Duke Math. J. 137 (2007), 537-584.
[14] C. Năstăsescu and F. van Oystaeyen, Methods of graded rings, Lecture Notes in Math., vol. 1836 (2004), Springer Verlag.
[15] R. S. Pierce, Associative algebras, Graduate Texts in Mathematics 88 (1982), Springer Verlag
[16] S. P. Smith, Some finite dimensional algebras related elliptic curves, Proceedings Workshop Representation Theory Mexico, 1998, 315-348.
[17] B. Stenström, Direct sum decompositions in Grothendieck categories, Ark. Mat. 7 (1968), 427-432.
[18] B. Stenström, Rings of quotients. An introduction to the methods of ring theory, Die Grundlehren der mathematischen Wissenschaften. Band 217. Springer-Verlag. (1975).
[19] L. Vaš, Graded cancellation properties of graded rings and graded unit-regular Leavitt path algebras, Algebr. Represent. Theory 24 (2021), 625-649.

1 University of Bucharest, Faculty of Mathematics and Computer Science, Str. Academiei 14, Bucharest 1, RO-010014, Romania

2 Institute of Mathematics of the Romanian Academy, PO-Box 1-764, RO-014700, Bucharest, Romania

E-mail: sdascal@fmi.unibuc.ro, Constantin_nastasescu@yahoo.com, lauranastasescu@gmail.com