RELATIVE NORTHCOTT NUMBERS FOR THE WEIGHTED WEIL HEIGHTS

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ABSTRACT. It is fundamental in number theory to calculate lower bounds for height functions. Grizzard studied lower bounds for the Weil height in a relative setting. Vidaux and Videla introduced the Northcott number for a set $A \subset \mathbb{Q}$. It bounds the Weil height on $A$ from below, outside the zero-height points and the finitely many small-height points. Pazuki, Technau, and Widmer introduced the weighted Weil heights. These heights generalize both the absolute and relative Weil heights. In this paper, we introduce a relative version of the Northcott number related to the weighted Weil height. We also give a field extension whose Northcott number equals a given positive number. The work is a relative version of the previous work of the author and Sano on the Northcott numbers for the weighted Weil heights.

1. Introduction

Let $h : \overline{\mathbb{Q}} \to \mathbb{R}_{\geq 0}$ be the absolute logarithmic Weil height (see, e.g., [BG06, p.16]). For $\gamma \in \mathbb{R}$ and $a \in \overline{\mathbb{Q}}$, we set

$$h_\gamma(a) := \deg(a)\gamma h(a),$$

where $\deg(a) := [\mathbb{Q}(a) : \mathbb{Q}]$. The function $h_\gamma$ is called the $\gamma$-weighted Weil height, introduced in [PTW21]. We note that $h_0$ (resp. $h_1$) is the absolute (resp. relative) logarithmic Weil height.

**Definition 1.1** ([BZ01, Section 1] or [PTW21, Section 1]). We say that a subset $A \subset \overline{\mathbb{Q}}$ has the $\gamma$-Bogomolov property (resp. $\gamma$-Northcott property) if the set $\{a \in A \mid 0 < h_\gamma(a) < C\}$ (resp. $\{a \in A \mid h_\gamma(a) < C\}$) is finite for some (resp. for all) $C > 0$. We denote $\gamma$-Bogomolov property (resp. $\gamma$-Northcott property) by $\gamma$-(B) (resp. $\gamma$-(N)) for short.

**Remark 1.2.** An algebraic number $a \in \overline{\mathbb{Q}}$ satisfies that $h_\gamma(a) = 0$ if and only if $a$ is 0 or a root of unity (see, e.g., [BG06, p.17, Theorem 1.5.9]).

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Remark 1.3. For a subset $A \subset \overline{Q}$, the property $\gamma$-(B) of $A$ immediately follows from $\gamma$-(N) of $A$. Hence if we set
\[
I_B(A) := \{ \gamma \in \mathbb{R} \mid A \text{ has } \gamma-(B) \}
\]
\[
I_N(A) := \{ \gamma \in \mathbb{R} \mid A \text{ has } \gamma-(N) \}
\]
then $I_B(A) \supset I_N(A)$ holds. By definition, we know that both $I_B(A)$ and $I_N(A \setminus \{\text{root of unity}\})$ are interval $(\gamma, \infty)$ or $[\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$. As we will see in Lemma 2.6, the equality $\inf I_B(A) = \inf I_N(A \setminus \{\text{root of unity}\})$ holds.

That $A \subset \overline{Q}$ has $\gamma$-(B) means that the values of $h_\gamma$ on $A$ are bounded from below by an absolute positive constant, outside the zero-height points. Any number field has $\gamma$-(N) for all $\gamma \in \mathbb{R}$. This is an immediate consequence of the Northcott theorem (see, e.g., [BG06, p.25, Theorem 1.6.8]). On the one hand, there are two previous works on infinite extensions of $\mathbb{Q}$ having $\gamma$-(B) or $\gamma$-(N) (see [OS22] and [PTW21]). However, there are many examples of infinite extensions having 0-(B) or 0-(N) (see, e.g., [AD00], [BZ01], [Hab13], [Sch73], or [Wid11]). Seeing the definition of $\gamma$-(B), we want to set
\[
\text{Nor}_\gamma(A) := \inf \{ C > 0 \mid \# \{ a \in A \mid h_\gamma(a) < C \} = \infty \}.
\]
The non-negative number $\text{Nor}_\gamma(A)$ is called the $\gamma$-Northcott number of $A$, introduced in [OS22] and [VV16]. Note that $A$ has $\gamma$-(B) (resp. $\gamma$-(N)) if and only if $\text{Nor}_\gamma(A \setminus \{\text{root of unity}\}) > 0$ (resp. $\text{Nor}_\gamma(A) = \infty$). It is natural to try to construct a field whose $\gamma$-Northcott number equals a given positive number. The problem was first dealt with in [PTW21, Theorem 3] and later solved in [OS22, Theorem 4.1]. In this paper, we focus on the problem in a “relative” setting. We introduce the following:

Definition 1.4. Let $L/K$ be an extension of subfields of $\overline{Q}$. We say that $L/K$ is a relative $\gamma$-Bogomolov extension (resp. relative $\gamma$-Northcott extension) if the set $L \setminus K$ has $\gamma$-(B) (resp. $\gamma$-(N)). We denote relative $\gamma$-Bogomolov (resp. relative $\gamma$-Northcott) by $\gamma$-(RB) (resp. $\gamma$-(RN)) for short.

Remark 1.5. The 0-(RB) extension is the relative Bogomolov extension introduced in [Gri15].

Remark 1.6. If a field $L \subset \overline{Q}$ has $\gamma$-(B) (resp. $\gamma$-(N)), then $L/K$ is $\gamma$-(RB) (resp. $\gamma$-(RN)) for any field $K \subset L$. We say that a $\gamma$-(RB) (resp. $\gamma$-(RN)) extension $L/K$ is trivial if $L \subset \overline{Q}$ has $\gamma$-(B) (resp. $\gamma$-(N)). By definition, if $K$ has $\gamma$-(B) (resp. $\gamma$-(N)), then any $\gamma$-(RB) (resp. $\gamma$-(RN)) extension $L/K$ is trivial. Thus we should restrict our
attention to the case that $K$ does not have $\gamma$-(B) (resp. $\gamma$-(N)). Here note that the Lehmer conjecture asserts that $Q$ has $1$-(B). On the one hand, we know that the set $Q \{-\text{root of unity}\}$ has $\gamma$-(N) for all $\gamma > 1$ (see [OS22, Remark 2.6]). In addition, for all $\gamma < 0$ (resp. $\gamma \leq 0$), we will see in Section 4 that any $\gamma$-(RB) (resp. $\gamma$-(RN)) extension is trivial. Hence we deal with $\gamma$-(RB) (resp. $\gamma$-(RN)) only in the case $0 \leq \gamma < 1$ (resp. $0 < \gamma \leq 1$).

Now we set $I_B(L/K) := I_B(L \setminus K)$, $I_N(L/K) := I_N(L \setminus K)$, and $\text{Nor}_\gamma(L/K) := \text{Nor}_\gamma(L \setminus K)$. We call $\text{Nor}_\gamma(L/K)$ relative $\gamma$-Northcott number of $L/K$. We also want to construct an extension $L/K$ such that $K$ does not have $\gamma$-(B) and $\text{Nor}_\gamma(L/K)$ equals a given positive number. Hence our result is the following:

**Theorem 1.7.** Let $c \in \mathbb{R}_{>0}$. Then we can construct an extension $L/K$ of subfields of $\overline{Q}$ satisfying the equality $I_N(K) = (1, \infty)$ and the following condition (1), (2), or (3), respectively.

1. $I_B(L/K) = I_N(L/K) = (\gamma, \infty)$ for $\gamma \in [0, 1)$.
2. $I_B(L/K) = [\gamma, \infty) \supset (\gamma, \infty) = I_N(L/K)$ with $\text{Nor}_\gamma(L/K) = c$ for $\gamma \in [0, 1)$.
3. $I_B(L/K) = I_N(L/K) = [\gamma, \infty)$ for $\gamma \in (0, 1]$.

**Remark 1.8.** Regarding Remark 1.6, we note that the condition $I_N(K) = (1, \infty)$ implies that $K$ does not have $\gamma$-(B) (resp. $\gamma$-(N)) for all $\gamma \in \mathbb{R}_{<1}$ (resp. $\gamma \in \mathbb{R}_{\leq 1}$) by Remark 1.3.

## 2. Preparations

This section is devoted to giving technical lemmata to prove Theorem 1.7. The discussions are greatly based on those in [OS22], [PTW21], and [Wid11]. First, we recall (a special case of) Silverman’s inequality.

**Fact 2.1 ([Sil84, Theorem 2]).** Let $K$ be a number field. Assume that $a \in \overline{Q}$ satisfies the inequality $d := [K(a) : K] > 1$. We set $L := K(a)$. Then the inequality

$$h(a) \geq \frac{1}{2(d-1)} \left( \frac{\log(N_{K/Q}(D_{L/K}))}{d[K : Q]} - \log(d) \right)$$

holds, where $N_{K/Q}$ is the usual norm and $D_{L/K}$ is the relative discriminant ideal of the extension $L/K$.

For a number field $K$, we denote by $\mathcal{O}_K$ the ring of integers of $K$. To give a more explicit lower bound for the Weil height by using Fact 2.1 we must estimate $N_{K/Q}(D_{L/K})$ from below. In [Wid11, Proof of Theorem 4], Widmer gave the following nice tool to do it.
Lemma 2.2. Let $p$ be a prime number and $L/K$ be an extension of number fields. Assume that $p$ does not ramify in $K$ and that any prime ideal of $\mathcal{O}_K$ lying above $p$ ramifies totally in $L$. Then we have

$$p^{[K:Q][L:K]-1} | N_{K/Q}(D_{L/K}).$$

Proof. Let $\mathcal{D}_{L/K}$ be the different ideal of $L/K$. We employ the following two facts.

(2.1) Let $\mathfrak{P} \subset \mathcal{O}_L$ be a prime ideal and $e \in \mathbb{N}$ be the ramification index of $\mathfrak{P}$ over $K$. Then it holds that $\mathfrak{P}^{e-1} | \mathcal{D}_{L/K}$ (see, e.g., [Neu99, p.199, (2.6)]).

(2.2) It holds that $D_{L/K} = N_{L/K}(\mathcal{D}_{L/K})$ (see, e.g., [Neu99, p.201, (2.9)]).

By the assumption that $p$ does not ramify in $K$, the prime decomposition of $p\mathcal{O}_K$ in $\mathcal{O}_K$ is a form of

$$p\mathcal{O}_K = p_1 \cdots p_s,$$

where $p_i \neq p_j$ if $i \neq j$. Now for each $1 \leq i \leq s$, there exists a prime ideal $\mathfrak{P}_i \subset \mathcal{O}_L$ such that the prime decomposition of $p_i\mathcal{O}_L$ in $\mathcal{O}_L$ is

$$p_i\mathcal{O}_L = \mathfrak{P}_i^{[L:K]}.$$  

By (2.1), we know that $\mathfrak{P}_i^{[L:K]-1} | \mathcal{D}_{L/K}$ for each $1 \leq i \leq s$. Thus we have

$$ (\mathfrak{P}_1 \cdots \mathfrak{P}_s)^{[L:K]-1} | \mathcal{D}_{L/K}. $$

On the one hand, by (2.2), it holds that

$$N_{K/Q}(D_{L/K}) = N_{K/Q}(N_{L/K}(\mathcal{D}_{L/K})) = N_{L/Q}(\mathcal{D}_{L/K}).$$

Therefore, we have

$$N_{L/Q}((\mathfrak{P}_1 \cdots \mathfrak{P}_s)^{[L:K]-1}) | N_{L/Q}(\mathcal{D}_{L/K}) = N_{K/Q}(D_{L/K}).$$  

(2.5)

Now we know that

$$N_{L/Q}((\mathfrak{P}_1 \cdots \mathfrak{P}_s)^{[L:K]-1}) = \left( N_{L/Q}(\mathfrak{P}_1^{[L:K]} \cdots \mathfrak{P}_s^{[L:K]}) \right)^{[L:K]-1}$$

$$= \left( N_{L/Q}(p\mathcal{O}_L) \right)^{[L:K]-1} \text{ by (2.4)}$$

$$= \left( N_{L/Q}(p\mathcal{O}_L) \right)^{[L:K]-1} \text{ by (2.3)}$$

$$= (p^{[L:K][K:Q]})^{[L:K]-1}$$

$$= p^{K:Q}(L:K)-1.$$  

Combining (2.5), we have completed the proof.  

□
For a number field $K$ and a prime ideal $p \subseteq \mathcal{O}_K$, we say that $X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathcal{O}_K[X]$ is a $p$-Eisenstein polynomial if $a_i \in p$ for all $0 \leq i \leq n - 1$ and $a_0 \notin p^2$.

**Fact 2.3** (*e.g.,* [FT93, p.133, Theorem 24 (a)]). Let $L/K$ be an extension of number fields and $p \in \mathcal{O}_K$ be a prime ideal. Then any $p$-Eisenstein polynomial is irreducible in $K[X]$. Furthermore, the following two conditions are equivalent.

1. $p$ ramifies totally in $L$.
2. $L = K(\rho)$, where $\rho$ is a root of some $p$-Eisenstein polynomial $g(X) \in \mathcal{O}_K[X]$.

**Corollary 2.4.** Let $d$, $p$, and $q$ be prime numbers with $p < q$. Assume that positive integers $e_1, r_1, \ldots, e_j, r_j$ satisfy that $p \not| \prod_{n=1}^{j}(e_n r_n)$ and $q \not| \prod_{n=1}^{j}(e_n r_n)$. We set $K := \mathbb{Q}(r_n^{1/e_n} | 1 \leq n \leq j)$ and $L := K((p/q)^{1/d})$. Then the inequality

$$h(a) > \frac{\log(p)}{d} - \frac{\log(d)}{2(d-1)}$$

holds for all $a \in L \setminus K$.

**Proof.** By [Viv04, Lemma 4.1] and [Hi98, p.97, Theorem 85], we know that $p$ and $q$ do not ramify in $K$. Thus, by Fact 2.3, any prime ideal $p \subseteq \mathcal{O}_K$ lying above $p$ (resp. $q$) ramifies totally in $L$ since $L = K((pqd^{-1})^{1/d})$ (resp. $L = K((p^{-1}q)^{1/d})$) holds and $X^d - pq^{d-1} \in \mathcal{O}_K[X]$ (resp. $X^d - p^{d-1}q \in \mathcal{O}_K[X]$) is a $p$-Eisenstein polynomial. Therefore we have

$$p^{[K:\mathbb{Q}](d-1)} | N_{K/\mathbb{Q}}(D_{L/K})$$

and

$$q^{[K:\mathbb{Q}](d-1)} | N_{K/\mathbb{Q}}(D_{L/K})$$

by Lemma 2.2. Combining the assumption that $p < q$, we get the inequality

$$2\log(p) < \log(pq) \leq \log(N_{K/\mathbb{Q}}(D_{L/K})) $$

$$[K:\mathbb{Q}](d-1)$$

Now note that $K(a) = L$ holds since $a \in L \setminus K$ and

$$[L : K] = \deg(X^d - pq^{d-1})$$

by Fact 2.3

is a prime number. Thus we conclude that

$$h(a) \geq \frac{1}{2(d-1)} \left( \frac{\log(N_{K/\mathbb{Q}}(D_{L/K}))}{d[K:\mathbb{Q}]} - \log(d) \right)$$

by Fact 2.1

$$> \frac{\log(p)}{d} - \frac{\log(d)}{2(d-1)}$$

by (2.6).

This completes the proof. $\square$
Fact 2.5 ([PTW21 Lemma 6]). Let $\gamma \in \mathbb{R}$ and $A \subset \overline{\mathbb{Q}}$. Assume that a nest sequence $A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots$ of subsets of $A$ satisfies that

1. $A_i$ has $\gamma$-(N) for all $i \in \mathbb{Z}_{\geq 0}$ and
2. $A = \bigcup_{i \in \mathbb{Z}_{\geq 0}} A_i$.

Then we have

$$\text{Nor}_\gamma(A) = \liminf_{i \to \infty} \inf h(\gamma(A_i \setminus A_{i-1})).$$

Fact 2.6 ([OS22 Proposition 2.3]). Let $\gamma \in \mathbb{R}$ and $A \subset \overline{\mathbb{Q}}$. If $A$ has $\gamma$-(B), then $A \setminus \{\text{root of unity}\}$ also has $\delta$-(N) for all $\delta > \gamma$.

Throughout the rest of the paper, we denote by $\mathbb{N}$ the set of positive integers.

Lemma 2.7. The field $K := \mathbb{Q}(2^{1/3^j} \mid j \in \mathbb{N})$ satisfies the equality $I_N(K) = (1, \infty)$.

Proof. By Fact 2.6, it is sufficient to prove that $K$ has 1-(B) but not 1-(N). By [Amo16, Theorem 1.3], $K$ has 1-(B). On the one hand, since we have the inequality $h_1(2^{1/3^j}) \leq \log(2)$ for all $j \in \mathbb{N}$, we know that $K$ does not have 1-(N).

3. Proof of Theorem 1.7

In this section, we prove Theorem 1.7. In fact, we prove the following more explicit one.

Theorem 3.1. Let $\gamma \in [0, 1)$, $c \in \mathbb{R}_{>0}$, and $K := \mathbb{Q}(2^{1/3^j} \mid j \in \mathbb{N})$.

(A) We set

$$f(x) := \frac{x \log(x)}{2(x-1)} + cx.$$

Take strictly increasing sequences of prime numbers $(d_i)_{i \in \mathbb{N}}$, $(p_i)_{i \in \mathbb{N}}$, and $(q_i)_{i \in \mathbb{N}}$ satisfying the inequalities $\min\{d_1, p_1\} > 3$,

$$\exp(f(d_i)) \leq p_i < q_i < 2p_i \leq 4 \exp(f(d_i)),$$

and

$$\max\{d_i, 4f(d_i)\} < \exp(f(d_{i+1}))$$

for all $i \in \mathbb{N}$. We set $L := K((p_i/q_i)^{1/p_i} \mid i \in \mathbb{N})$. Then $L/K$ satisfies the conditions (2) in Theorem 1.7 of the case $\gamma = 0$.

(B) We let $f(x)$ be $1/\log(x)$, $c$, or $\log(x)$. Take strictly increasing sequences of prime numbers $(d_i)_{i \in \mathbb{N}}$, $(p_i)_{i \in \mathbb{N}}$, and $(q_i)_{i \in \mathbb{N}}$ satisfying
the inequalities \( \min\{d_1, p_1\} > 3, \)

\[
\frac{d_i \log(d_i)}{2(d_i - 1)} < f(d_i),
\]

\[
\exp(f(d_i)d_i^{1-\gamma}) \leq p_i < q_i < 2p_i \leq 4\exp(f(d_i)d_i^{1-\gamma}), \text{ and}
\]

\[
\max\{d_i, 4\exp(f(d_i)d_i^{1-\gamma})\} < \exp(f(d_{i+1})d_{i+1}^{1-\gamma})
\]

for all \( i \in \mathbb{N} \). We set \( L := K((p_i/q_i)^{1/d_i} \mid i \in \mathbb{N}) \).

(1) If \( f(x) = 1/\log(x) \), then \( L/K \) satisfies the conditions (1) in Theorem 1.7.

(2) If \( f(x) = c \), then \( L/K \) satisfies the conditions (2) in Theorem 1.7 of the cases \( \gamma \in (0, 1) \).

(3) If \( f(x) = \log(x) \), then \( L/K \) satisfies the conditions (3) in Theorem 1.7 of the case \( \gamma = 1 \).

(C) Take strictly increasing sequences of prime numbers \( (p_i)_{i \in \mathbb{N}} \) and \( (q_i)_{i \in \mathbb{N}} \) satisfying the inequalities \( p_1 > 3 \) and

\[
p_i < q_i < 2p_i < p_{i+1}
\]

for all \( i \in \mathbb{N} \). We set \( L := K((p_i/q_i)^{1/p_i} \mid i \in \mathbb{N}) \). Then \( L/K \) satisfies the conditions (3) in Theorem 1.7 of the case \( \gamma = 1 \).

Remark 3.2. We can take the sequences of prime numbers in Theorem 3.1 by the Bertrand–Chebyshev theorem and by letting \( d_i \) be large enough.

Proof of Theorem 3.1. By Lemma 2.7, we know that the equality \( I_N(K) = (1, \infty) \) holds. We set

\[
L_0 := K,
\]

\[
L_i := L_0((p_m/q_m)^{1/d_m} \mid 1 \leq m \leq i),
\]

\[
L_{(0,j)} := \mathbb{Q}(2^{1/2^j}), \text{ and}
\]

\[
L_{(i,j)} := L_{(0,j)}((p_m/q_m)^{1/d_m} \mid 1 \leq m \leq i)
\]

for each \( i \in \mathbb{N} \) and \( j \in \mathbb{Z}_{\geq 0} \) (we consider \( (d_i)_{i \in \mathbb{N}} \) to be \( (p_i)_{i \in \mathbb{N}} \) in (3)). Note that \( L = \bigcup_{i \in \mathbb{Z}_{\geq 0}} L_i \) and \( L_i = \bigcup_{j \in \mathbb{Z}_{\geq 0}} L_{(i,j)} \) hold for each \( i \in \mathbb{Z}_{\geq 0} \).

(A) Take any \( a \in L \setminus K \). We set \( i_0 := \min \{i \in \mathbb{N} \mid a \in L_i\} \). By the assumption that \( a \notin K \), we know that \( i_0 \geq 1 \). Fix \( j_0 \in \mathbb{N} \) such that \( a \in L_{(i_0,j_0)} \). By the definition of \( i_0 \), we know that \( a \in L_{(i_0,j_0)} \setminus L_{(i_0-1,j_0)} \). Applying Corollary 2.4 to the extension
We get the inequalities
\[
\begin{align*}
 h(a) &> \frac{\log(p_{i_0})}{d_{i_0}} - \frac{\log(d_{i_0})}{2(d_{i_0} - 1)} \\
 &\geq \frac{f(d_{i_0})}{d_{i_0}} - \frac{\log(d_{i_0})}{2(d_{i_0} - 1)} \\
 &= \frac{1}{d_{i_0}} \left( \frac{d_{i_0} \log(d_{i_0})}{2(d_{i_0} - 1)} + cd_{i_0} \right) - \frac{\log(d_{i_0})}{2(d_{i_0} - 1)} \\
 &= c.
\end{align*}
\]

This implies that \(\text{Nor}_0(L/K) \geq c\). On the one hand, they hold that \((p_i/q_i)^{1/d_i} \notin K\) for all \(i \in \mathbb{N}\) and that
\[
\begin{align*}
 h((p_i/q_i)^{1/d_i}) &= \frac{\log(q_i)}{d_i} - \frac{\log(4)}{2(d_i - 1)} + \log(d_i) + \frac{c}{d_i} \\
 &\to c
\end{align*}
\]
as \(i \to \infty\). These imply that \(\text{Nor}_0(L/K) \leq c\). Therefore we conclude that \(L/K\) is \(0\)-(RB) with \(\text{Nor}_0(L/K) = c\). By Fact 2.6, the assertion follows.

We will apply Fact 2.5 to \(L \setminus K = \bigcup_{i \in \mathbb{N}} (L_i \setminus K)\). First, we prove that \(L_i \setminus K\) has \(\gamma\)-(N) for all \(i \in \mathbb{N}\). Take any \(a \in L_i \setminus K\). We set \(i_0 := \min \{m \in \{0, 1, \ldots, i\} | a \in L_m\}\). By the same discussion as (1), we get the inequalities
\[
\begin{align*}
 h(a) &> \frac{\log(p_{i_0})}{d_{i_0}} - \frac{\log(d_{i_0})}{2(d_{i_0} - 1)} \\
 &\geq \min_{1 \leq m \leq i} \left\{ \frac{\log(p_m)}{d_m} - \frac{\log(d_m)}{2(d_m - 1)} \right\} \\
 &\geq \min_{1 \leq m \leq i} \left\{ f(d_m)d_m^{-\gamma} - \frac{\log(d_m)}{2(d_m - 1)} \right\} \\
 &= \min_{1 \leq m \leq i} \left\{ d_m^{-\gamma} \left( f(d_m) - \frac{d_m \log(d_m)}{2(d_m - 1)} \right) \right\} =: C_i.
\end{align*}
\]

Note that \(C_i\) is a positive constant depending only on \(i\). Therefore the above inequalities imply that \(L_i \setminus K\) has \(0\)-(B). By Fact 2.6, we know that \(L_i \setminus K\) has \(\gamma\)-(N). Now take any \(\delta \in (0, 1)\). We also take any \(b \in (L_i \setminus K) \setminus (L_{i-1} \setminus K) = (L_i \setminus L_{i-1}) \setminus K\). Again fix \(j_1 \in \mathbb{N}\) such that \(b \in L_{(i,j_1)}\). Since \([L_{(i,j_1)} : L_{(i-1,j_1)}] = d_i\) (recall the proof of Corollary 2.4) is a prime number and \(b \in L_{(i,j_1)} \setminus L_{(i-1,j_1)}\), we know that \(L_{(i-1,j_1)}(b) = L_{(i,j_1)}\). Thus we have the inequality
\[
\deg(b) \geq d_i.
\] (3.1)
Applying Corollary 2.4 to the extension $L_{(i,j_1)}/L_{(i-1,j_1)}$, we get the inequalities

$$h_\delta(b) > d_i^\delta \left( \frac{\log(p_i)}{d_i} - \frac{\log(d_i)}{2(d_i - 1)} \right)$$

by (3.1)

$$\geq d_i^\delta \left( f(d_i)d_i^{-\gamma} \right) - \frac{\log(d_i)}{2(d_i - 1)}$$

$$= d_i^{\delta-\gamma} \left( f(d_i) - \frac{d_i^\gamma \log(d_i)}{2(d_i - 1)} \right)$$

$$\rightarrow \begin{cases} 
0 & (\delta < \gamma) \\
0 & (\delta = \gamma, f(x) = 1/\log(x)) \\
c & (\delta = \gamma, f(x) = c) \\
\infty & (\delta = \gamma, f(x) = \log(x)) \\
\infty & (\delta > \gamma)
\end{cases}$$

as $i \rightarrow \infty$. By Fact 2.5, we have the inequality

$$\text{Nor}_\delta(L/K) \geq \begin{cases} 
0 & (\delta < \gamma) \\
0 & (\delta = \gamma, f(x) = 1/\log(x)) \\
c & (\delta = \gamma, f(x) = c) \\
\infty & (\delta = \gamma, f(x) = \log(x)) \\
\infty & (\delta > \gamma)
\end{cases}$$

(3.2)

On the one hand, they hold that $(p_i/q_i)^{1/d_i} \notin K$ for all $i \in \mathbb{N}$ and that

$$h_\delta((p_i/q_i)^{1/d_i}) = \frac{\log(q_i)}{d_i^{1-\delta}} \leq \frac{\log(4)}{d_i^{1-\delta}} + \frac{f(d_i)}{d_i^{\delta-\gamma}}$$

$$\rightarrow \begin{cases} 
0 & (\delta < \gamma) \\
0 & (\delta = \gamma, f(x) = 1/\log(x)) \\
c & (\delta = \gamma, f(x) = c) \\
\infty & (\delta = \gamma, f(x) = \log(x)) \\
\infty & (\delta > \gamma)
\end{cases}$$

as $i \rightarrow \infty$. These imply the inequality

$$\text{Nor}_\delta(L/K) \leq \begin{cases} 
0 & (\delta < \gamma) \\
0 & (\delta = \gamma, f(x) = 1/\log(x)) \\
c & (\delta = \gamma, f(x) = c) \\
\infty & (\delta = \gamma, f(x) = \log(x)) \\
\infty & (\delta > \gamma)
\end{cases}$$

(3.3)
Therefore the assertions follow from the inequalities (3.2) and (3.3).

\(\Box\) We can samely prove the assertion as (2).

Thus we have completes all the proof.

Remark 3.3. By Lemma 2.2, we may replace the field \(K\) in Theorem 3.1 with such a field \(K'\) that \(I_N(K') = (1, \infty)\) and all \(p_i\) and \(q_i\) do not ramify in any number field included in \(K'\).

4. Comments on non-positive weights

As we mentioned in Section 1, we prove the following proposition regarding the cases of non-positive weights.

Proposition 4.1. Let \(\gamma \in \mathbb{R}\).

1. If \(\gamma < 0\), then any \(\gamma\)-(RB) extension is trivial.
2. If \(\gamma \leq 0\), then any \(\gamma\)-(RN) extension is trivial.

Proposition 4.1 is an immediate consequence of the following:

Lemma 4.2. Let \(L/K\) be an extension of subfields of \(\mathbb{Q}\) and \(\gamma \in \mathbb{R}\).

1. Assume that \(\gamma < 0\). Then \(L/K\) is not \(\gamma\)-(RB) if \(K\) does not have \(\gamma\)-(B).
2. Assume that \(\gamma \leq 0\). Then \(L/K\) is not \(\gamma\)-(RN) if \(K\) does not have \(\gamma\)-(N).

Proof.

(1) Since \(K\) does not have \(\gamma\)-(B), there exists a pairwise distinct sequence \((a_n)_{n \in \mathbb{N}} \subset K\) such that \(0 < h\gamma(a_n) \to 0\) as \(n \to \infty\). First, we prove that \(\lim_{n \to \infty} \deg(a_n) = \infty\) by contradiction. Suppose that there exist \(M > 0\) and strictly increasing sequence of positive integers \((n_i)_{i \in \mathbb{N}}\) such that \(\deg(a_{n_i}) \leq M\) for all \(i \in \mathbb{N}\). Then we have

\[
h(a_{n_i}) \leq \left( \frac{M}{\deg(a_{n_i})} \right)^{-\gamma} h(a_{n_i}) = M^{-\gamma} h\gamma(a_{n_i}) \to 0
\]

as \(i \to \infty\). Thus \(a_{n_i} \in \mathbb{Q}\) is bounded degree and bounded height for all \(i \in \mathbb{N}\). This contradicts the Northcott theorem (see, e.g., [BG06, p.25, Theorem 1.6.8]).

Now fix \(b \in L \setminus K\). Replacing \(b\) with \(2b\) if needed, we may assume that \(ba_n\) is not a root of unity for infinitely many \(n \in \mathbb{N}\). Note that we have the inequalities

\[
\deg(a_n) \leq [\mathbb{Q}(b, ba_n) : \mathbb{Q}] \leq \deg(b) \deg(ba_n).
\]
Thus we get the inequalities
\[
0 < h_\gamma(ba_n) \leq \left( \frac{\deg(a_n)}{\deg(b)} \right)^\gamma h(ba_n) \\
\leq \left( \frac{\deg(a_n)}{\deg(b)} \right)^\gamma (h(b) + h(a_n)) \\
= \frac{1}{\deg(b)^\gamma} \left( \frac{h(b)}{\deg(a_n)^{-\gamma}} + h_\gamma(a_n) \right)
\]
for infinitely many \( n \in \mathbb{N} \). Since \((ba_n)_{n \in \mathbb{N}}\) is a pairwise distinct sequence in \( L \setminus K \), we conclude that the set \( L \setminus K \) does not have \( \gamma \)-\( (B) \) by letting \( n \to \infty \) in the above inequalities.

\( \square \) We can prove the assertion in a similar way to (1).

These complete the proof.

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