C^1-Generic Dynamics: Tame and Wild Behaviour

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Abstract

This paper gives a survey of recent results on the maximal transitive sets of C^1-generic diffeomorphisms.

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1. Introduction

In order to give a global description of a dynamical system (diffeomorphism or flow) on a compact manifold \( M \), the first step consists in characterizing the parts of \( M \) which are, in some sense, indecomposable for the dynamics. This kind of description will be much more satisfactory if these indecomposable sets are finitely many, disjoint, isolated, and not fragile (that is, persistent in some sense under perturbations of the dynamics).

For non-chaotic dynamics, this role can be played by periodic orbits, or by minimal sets. However many (chaotic) dynamical systems have infinitely many periodic orbits and a uncountable number of minimal sets. In order to structure the global dynamics using a smaller number of (larger) sets, we need to relax the notion of indecomposability. A weaker natural notion of topological/dynamical indecomposability is the notion of transitivity.

1.1. Maximal and saturated transitive sets

An invariant compact set \( K \) of a diffeomorphism \( f \) is transitive if there is a point in \( K \) whose positive orbit is dense in \( K \). An equivalent definition is the following: for any open subsets \( U, V \) of \( K \) there is \( n > 0 \) such that \( f^n(U) \cap V \neq \emptyset \).

One easily verifies that the closure of the union of an increasing family of transitive sets is a transitive set. Then Zorn’s Lemma implies that any transitive set is contained in a maximal transitive set (i.e. maximal for the inclusion).

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However, maximal transitive sets are not necessarily disjoint. For this reason, we also consider the stronger notion of saturated transitive sets: a transitive set $K$ is saturated if any transitive set intersecting $K$ is contained in $K$. So two saturated transitive sets are always equal or disjoint.

These notions are motivated by Smale’s approach for hyperbolic dynamics, and more specifically for his spectral decomposition theorem (see [40]):

1.2. Smale’s spectral decomposition theorem

For an Axiom A diffeomorphism $f$ on a compact manifold $M$, the set $\Omega(f)$ of the non-wandering points is the union of finitely many compact disjoint (maximal and saturated) transitive sets $\Lambda_i$, called the basic pieces, which are uniformly hyperbolic.

If furthermore $f$ as no cycles, there is a filtration $\emptyset = M_{k+1} \subset M_k \subset \cdots \subset M_1 = M$ adapted to $f$, that is: the $M_i$ are submanifolds with boundary, having the same dimension as $M$, and are strictly $f$-invariant: $f(M_i)$ is contained in the interior $\overset{\circ}{M_i}$ of $M_i$. Moreover $\Lambda_i$ is the maximal invariant set of $f$ in $M_i \setminus \overset{\circ}{M_{i+1}}$, that is $\Lambda_i = \bigcap_{n \in \mathbb{Z}} f^n(M_i \setminus \overset{\circ}{M_{i+1}})$.

Finally this presentation is robust: the same filtration remains adapted to any diffeomorphism $g$ in a $C^1$-neighborhood of $f$, and the maximal invariant sets $\Lambda_i(g) = \bigcap_{n \in \mathbb{Z}} g^n(M_i \setminus \overset{\circ}{M_{i+1}})$ are the basic pieces of $g$.

1.3. A global picture of $C^1$-generic diffeomorphisms

It is known from the sixties (see [2, 39]) that Axiom A diffeomorphisms are not $C^1$-dense in $Diff^1(M)$ if $\dim(M) > 2$ and, of course, general dynamical systems do not admit such a nice global presentation of their dynamics. One would like to give an analogous description for as large as possible a class of diffeomorphisms.

In this paper I will present a collection of works, trying to give a coherent global picture of the dynamics of $C^1$-generic diffeomorphisms. There are two types of generic behaviours:

- either the manifold contains infinitely many regions having independent dynamical behaviours (we will speak of a wild diffeomorphism, and in Section 4. will give examples of such behaviours),
- or one has a description of the dynamics identical to those given by the spectral decomposition theorem: we speak of a tame diffeomorphism. In this case the role of basic pieces is played by the homoclinic classes (see the definition in Section 2.2.).

An important class of examples of tame dynamics are the robustly transitive dynamics and in Section 3.2. we summarize the known examples of robustly transitive dynamics. The basic pieces of a tame diffeomorphism present a weak form of hyperbolicity called dominated splitting and volume partial hyperbolicity (see Section 3.3. and 3.4.). In Section 3.5. we try to summarize the dynamical consequences of the dominated splittings.
2. Tame and wild dynamics

2.1. $C^1$-generic diffeomorphisms

In this paper, we will consider the set of diffeomorphisms $Diff^1(M)$ endowed with the $C^1$-topology. The choice of the topology comes from the fact that most of the perturbing results (Pugh’s closing Lemma [34], Hayashi’s connecting Lemma and its generalizations [24, 25, 4, 41]) are only known in this topology.

Recall that a property $P$ is generic if it is verified on a residual subset $R$ of $Diff^1(M)$ (i.e. $R$ contains the intersection of a countable family of dense open subsets). In this work we will often use a practical abuse of language; we say:

“Any $C^1$-generic diffeomorphism verifies $P$”

instead of:

“There is a residual subset $R$ of $Diff^1(M)$ such that any $f \in R$ verifies $P$.”

Let me first recall a famous and classical example, relating Pugh’s closing lemma to generic dynamics:

**Theorem** [34] Let $f$ be a diffeomorphism on a compact manifold and $x \in \Omega(f)$ be a non-wandering point. There is $g$ arbitrarily $C^1$-close to $f$ such that $x$ is periodic for $g$.

Using a Kupka-Smale argument (genericity of hyperbolicity of the periodic points and the transversality of invariant manifolds) one get:

**Corollary** The non-wandering set $\Omega(f)$ of a generic diffeomorphism $f$ is the closure of the set of periodic points of $f$, which are all hyperbolic.

2.2. Homoclinic classes

Let $f$ be a diffeomorphism on a compact manifold and $p$ be a hyperbolic periodic point of $f$ of saddle type. Let $W^s(p)$ and $W^u(p)$ denote the invariant manifold of the orbit of $p$. The homoclinic class $H(p, f)$ of $p$ is by definition the closure of the transverse intersection points of its invariant manifold:

$$H(p, f) = \overline{W^s(p, f) \cap W^u(p, f)}.$$

The homoclinic class $H(p, f)$ is a transitive set canonically associated to the orbit of the periodic point $p$.

There is another way to see the homoclinic class of $p$: we tell that a periodic point $q$ of saddle type and of same Morse index as $p$ is homoclinically related to $p$ if $W^u(q)$ cuts transversally $W^s(p)$ in at least one point and reciprocally $W^s(q)$ cuts transversally $W^u(p)$ in at least one point. The $\lambda$-lemma (see [33]) implies that this relation is an equivalence relation and $H(p, f)$ is the closure of the set of periodic points homoclinically related to $p$.

For Axiom A diffeomorphisms, the homoclinic classes are precisely the basic pieces of Smale’s spectral decomposition theorem. However, one easily build examples of diffeomorphisms whose homoclinic classes are not maximal transitive
sets. Moreover, B. Santoro [37] recently build examples of diffeomorphisms on a 3-manifold having periodic points whose homoclinic classes are neither disjoint nor equal.

2.3. Homoclinic classes of generic dynamics

Conjectured during a long time, Hayashi’s connecting lemma allowed the control the perturbations of the invariant manifolds of the periodic points, opening the door for the understanding of generic dynamics.

**Theorem 1** [24] Let $p$ and $q$ be two hyperbolic periodic points of some diffeomorphism $f$. Assume that there is a sequence $x_n$ of points converging to a point $x \in W^u_{loc}(p)$ and positive iterates $y_n = f^{m(n)}(x_n)$, $m(n) \geq 0$, converging to a point $y \in W^s_{loc}(q)$.

Then there is $g$, arbitrarily $C^1$-close to $f$, such that $x$ and $y$ belong to a same heteroclinic orbit of $p$ and $q$; in other words:

$x \in W^u_{loc}(p, g), \ y \in W^s_{loc}(q, g)$ and there is $n > 0$ such that $g^n(x) = y$.

If the periodic points $p$ and $q$ in Theorem 1 belong to a same transitive set, then the sequences $x_n$ and $y_n$ are given by a dense orbit. In [7], using in an essential way Hayashi connecting lemma, we proved:

**Theorem 2** For any $C^1$-generic diffeomorphism, two periodic orbits belong to the same transitive set if and only if their homoclinic classes coincide.

Motivated by this result we conjectured:

The homoclinic classes of a generic diffeomorphism coincide with its maximal transitive sets.

We know now that this conjecture, as stated above, is wrong: in [8] (see Section 4.) we show that any manifold $M$ with dimension $> 2$ admits a non-empty $C^1$-open subset $U \subset Diff^1(M)$ on which generic diffeomorphisms have an uncountable family of maximal (an saturated) transitive sets without periodic points.

However, one part of the conjecture is now proved. Generalizations of Hayashi Connecting Lemma (see [25, 4, 41]) recently allowed to show:

**Theorem 3** For any $C^1$-generic diffeomorphism, the homoclinic class of any periodic point is a maximal (see [4]) and saturated (see [17]) transitive set.

The proof of this theorem is decomposed in two main steps: first, [4] shows that for a generic diffeomorphism $f$ the homoclinic class of a point $p$ coincides with the intersection of its invariant manifolds:

$$H(p, f) \stackrel{def}{=} \bar{W}^s(p, f) \cap W^u(p, f) = W^s(p, f) \cap W^u(p, f).$$

Then [17] shows that for a generic diffeomorphism $f$ the closure $\bar{W}^u(p, f)$ is Lyapunov stable (and so admits a base of invariant neighborhoods) and $W^s(p, f)$ is Lyapunov stable for $f^{-1}$. As a consequence a dense orbit of a transitive set $T$ intersecting $H(p, f)$ is capted in arbitrarily small neighborhoods of $\bar{W}^u(p, f)$ and of $W^u(p, f)$, proving that $T$ is contained in $\bar{W}^s(p, f) \cap \bar{W}^u(p, f)$, finishing the proof of the theorem.
2.4. Tame and wild diffeomorphisms

Using Theorem 2 and the fact that the homoclinic class $H(p, f)$ of a periodic point varies lower semi-continuously with $f$, [1] shows the existence of a $C^1$-residual subset $\mathcal{R}$ of diffeomorphisms (or flows), such that the cardinality of the set of homoclinic classes is locally constant on $\mathcal{R}$: for any Kupka-Smale diffeomorphism $f$ let $n(f) \in \mathbb{N} \cup \{\infty\}$ denote the cardinal of the set of different homoclinic classes $H(p, f)$ where $p$ is an hyperbolic periodic point of $f$; then any $f \in \mathcal{R}$ has a $C^1$-neighborhood $U_f$ such that any $g \in \mathcal{R} \cap U_f$ verifies $n(g) = n(f)$.

This result induces a natural dichotomy the residual set $\mathcal{R}$:

- a diffeomorphism $f \in \mathcal{R}$ is tame if it has finitely many homoclinic classes.
- a diffeomorphism $f \in \mathcal{R}$ is wild if it has infinitely many homoclinic classes.

3. Tame dynamics

3.1. Filtrations, robust transitivity and generic transitivity

[1] shows that the global dynamics of tame diffeomorphisms admit a good reduction to the dynamics of the transitive pieces (up to reduce the residual set $\mathcal{R}$). Let $f \in \mathcal{R}$ be a tame diffeomorphism, then:

1. as in the Axiom A case, the non-wandering set is the union of finitely many disjoint homoclinic classes $H(p_i, f)$.
2. there is a filtration $\emptyset = M_{k+1} \subset M_k \subset \cdots \subset M_1 = M$ adapted to $f$ such that $H(p_i, f)$ is the maximal invariant set in $M_i \setminus M_{i+1}$.
3. moreover (up to reduce the open neighborhood $U_f$ defined above) this filtration holds for any $g \in \mathcal{R} \cap U_f$, and for $g \in \mathcal{R} \cap U_f$ the maximal invariant set of $g$ in $M_i \setminus M_{i+1}$ is the homoclinic class $H(p_i, g, g)$.
4. there is a good notion of attractors: either a homoclinic class is a topological attractor (that is, its local basin contains a neighborhood of it) or its stable manifold has empty interior. Then the union of the basin of the attractors of $f$ is a dense open set of $M$ (see [16]).

The item 3 above shows that the transive sets $H(p_i, f)$ are not fragil. In a previous work, [20] introduced the following notion:

**Definition 1** Let $f$ be a diffeomorphism of some compact manifold $M$. Assume that there is some open set $U \subset M$ and a $C^1$-neighborhood $\mathcal{V}$ of $f$ such that, for any $g \in \mathcal{V}$, the maximal invariant set $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is a compact transitive set contained in $U$.

Then $\Lambda_f$ is called a robustly transitive set of $f$.

In the definition above, if one has $U = M$ (and so $\Lambda_g = M$ for any $g \in \mathcal{V}$), then $f$ is called a robustly transitive diffeomorphism.

This notion is slightly stronger that the property given by the item 3 above; so we have to relax Definition 1: we say that $\Lambda_f$ is generically transitive if, in the
notations of Definition 1, the maximal invariant set $\Lambda_g$ is transitive for $g$ in a residual subset of $V$.

At this moment, there are no known examples of generic transitive sets which are not robustly transitive. So it is natural to ask if this two notions are equivalent:

\[\text{Generic transitivity} \iff \text{robust transitivity}\]

### 3.2. Examples of robust transitivity

The Axiom A dynamics are obvious examples of tame dynamics. On compact surfaces, tame diffeomorphisms are, in fact, Axiom A diffeomorphisms, but there are many non-hyperbolic examples in higher dimensions.

Even if this talk is mostly devoted to diffeomorphisms, let us observe that the most famous robustly transitive non-hyperbolic attractor is the Lorenz attractor (geometric model, see [23, 3]) for flows on 3-manifolds. There are many generalizations of this attractor, called singular attractors, for flow on 3-manifolds, see for instance [30]. See also [12] for robust singular attractors in dimension greater or equal than 4, having a singular point with Morse index (dimension of unstable manifold) greater than 2.

The first example of non-Anosov robustly transitive diffeomorphism is due to Shub [38]: it is a diffeomorphism on the torus $T^3$ which is a skew product over an Anosov map on the torus $T^2$, such that the dynamics on the fibers is dominated by the dynamics on the basis.

Then Mañé [29] built an example of robustly transitive non-hyperbolic diffeomorphism on the torus $T^3$ by considering a bifurcation of an Anosov map $A$ having 3 real positive different eigenvalues $\lambda_1 < 1 < \lambda_2 < \lambda_3$; he performs a saddle node bifurcation creating a (new) hyperbolic saddle of index 1 (breaking the hyperbolicity) in the weak unstable manifold of a fixed point (of index 2) of the Anosov map $A$.

Then [6] shows that a diffeomorphism $f$ admits $C^1$ perturbations which are robustly transitive, if $f$ is:

1. the time one diffeomorphism of any transitive Anosov flow.
2. the product $(A, id)$ where $A$ is some Anosov map and $id$ is the identity map of any compact manifold.

The second case can be easily generalized to any skew product of an Anosov map by rotations of the circle $S^1$. The same technique also allows [6] to build example of robustly transitive attractors, by perturbing product maps of any hyperbolic attractors by the identity map of some compact manifold.

Each of these previous example was partially hyperbolic (see the definitions in Section 3.3.): they admits a splitting $TM = E^s \oplus E^c \oplus E^u$ where $E^s$ is uniformly contracting and $E^u$ is uniformly expanding, and it was conjectured that partial hyperbolicity was a necessary condition for robust transitivity. Then [13] generalizes Mañé example above and exhibits robustly transitive diffeomorphisms on $T^3$ having a uniformly contracting 1-dimensional bundle, but no expanding bundle (there is a splitting $TM = E^s \oplus E^{cu}$), and robustly transitive diffeomorphisms on $T^4$ having

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no hyperbolic subbundles (neither expanding nor contracting): there just admits an invariant dominated splitting $TM = E^{cs} \oplus E^{cu}$.

We do not know what are the manifolds admitting robustly transitive diffeomorphism. For instance:

**Conjecture 1** There is no robustly transitive diffeomorphism on the sphere $S^3$.

This conjecture has been proved in [20] assuming the existence of a codimension 1 (center stable or center unstable) foliation, using Novikov Theorem. Notice that all the known examples of robustly transitive diffeomorphisms on 3-manifolds admits an invariant codimension 1 foliation. However this conjecture remains still open.

### 3.3. Dominated splitting and partial hyperbolicity: definitions

Let $f$ be a $C^1$-diffeomorphism of a compact manifold and let $\mathcal{E}$ be an $f$-invariant compact subset of $M$. Let $TM_x = E_1(x) \oplus \cdots \oplus E_k(x), x \in \mathcal{E}$, be a splitting of the tangent space at any point of $\mathcal{E}$. This splitting is a dominated splitting if it verifies the following properties:

1. For any $i \in \{1, \ldots, k\}$, the dimension $E_i(x)$ is independent of $x \in \mathcal{E}$.
2. The splitting is $f_*$-invariant (where $f_*$ denotes the differential of $f$): $E_i(f(x)) = f_*(E_i(x))$.
3. There is $\ell \in \mathbb{N}$ such that, for any $x \in \mathcal{E}$, for any $1 \leq i < j \leq k$ and any $u \in E_i(x) \setminus \{0\}, v \in E_j(x) \setminus \{0\}$ one has:
   \[
   \frac{\|f^\ell(u)\|}{\|u\|} \leq \frac{\|f^\ell(v)\|}{2\|v\|}.
   \]

**Remark 1** - (Continuity) Any dominated splitting on a set $\mathcal{E}$ is continuous and extend in a unique way to the closure $\bar{\mathcal{E}}$.

- (Extension to a neighborhood) There is a neighborhood $U$ of $\bar{\mathcal{E}}$ on which the maximal invariant set $\Lambda(U, f)$ has a dominated splitting extending those on $\mathcal{E}$.

- (Robust) There is a $C^1$-neighborhood $U_f$ of $f$ such that, for any $g \in U_f$, the maximal invariant set $\Lambda(U, g)$ has a dominated splitting varying continuously with $g$.

- (Unicity) If $\mathcal{E}$ has a dominated splitting, then there is a (unique) dominated splitting $TM|\mathcal{E} = E_1 \oplus \cdots \oplus E_k$, called the finest dominated splitting, such that any other dominated splitting $F_1 \oplus \cdots \oplus F_l$ over $\mathcal{E}$ is obtained by grouping the $E_i$ in packages.

One of the $E_i$ is uniformly contracting if (up to increase $\ell$ in the definition above) $\frac{\|f^\ell(u)\|}{\|u\|} \leq \frac{1}{2}$ for all $x \in \mathcal{E}$ and all $u \in E_i(x) \setminus \{0\}$. In the same way $E_i$ is uniformly expanding if $\frac{\|f^\ell(u)\|}{\|u\|} \geq \frac{1}{2}$ for all $x \in \mathcal{E}$ and all $u \in E_i(x) \setminus \{0\}$.

An $f$-invariant compact set $K$ is hyperbolic if it has a dominated splitting $TM|K = E^s \oplus E^u$ where $E^s$ is uniformly contracting and $E^u$ is uniformly expanding.
The compact $f$-invariant set $K$ is partially hyperbolic if it has a dominated splitting and if at least one of the bundles $E_i$ of its finest dominated splitting is uniformly contracting or expanding. Let $E^s$ and $E^u$ be the sum of the uniformly contracting and expanding subbundles, respectively, and let $E^c$ be the sum of the other sub-bundles. One get a new dominated splitting $E^s \oplus E^c$, $E^c \oplus E^u$ or $E^s \oplus E^c \oplus E^u$, and these bundles are called the stable, central et unstable bundles, respectively.

An $f$-invariant compact set $K$ is called volume hyperbolic if there is a dominated splitting whose extremal bundles $E_1$ and $E_k$ contracts and expands uniformly the volume, respectively. Notice if one of these bundle has dimension 1, it is uniformly contracting or expanding. In particular, a volume hyperbolic set in dimension 2 is a uniformly hyperbolic set, and in dimension 3 it is partially hyperbolic (having at least one uniformly hyperbolic bundle).

### 3.4. Volume hyperbolicity for the robust transitivity

Generalizing previous results by Mañé [28] (in dimension 2) and by [20] in dimension 3, [9] (for robustly transitive set) and [1] for generically transitive sets show:

**Theorem 4** Any robustly (or generically) transitive set is volume hyperbolic.

Then any robustly transitive set in dimension 2 is a hyperbolic basic set (result of Mañé) and in dimension 3 is partially hyperbolic ([20]). In higher dimension, the dominated splitting may have all the subbundles of dimension greater than 2, so the expansion or contraction of the volume does no more imply the hyperbolicity of the bundle, see the example in [13].

The proof of Theorem 4 has two very different steps (as in [28]). The first one consists in showing that the lake of dominated splitting allows to “mixe” the eigenvalues of the periodic orbits, creating an homothecy: a periodic orbit whose differential at the period is an homothecy is (up to a small perturbation) a sink or a source, breaking the transitivity. For that we just perturb the linear cocycle defined by the differential of $f$, and then we use a Lemma of Franks ([22]) for realizing the linear perturbation as a dynamical perturbation. Let state precisely this result:

**Theorem 5** [9] Let $f$ be a diffeomorphism of a compact manifold $M$, and let $p$ be a hyperbolic periodic saddle. Assume that the homoclinic class $H(p,f)$ do not have any dominated splitting. Then, given any $\varepsilon > 0$, there is a periodic point $x$ homoclinically related to $p$, with the following property:

Given any neighborhood $U$ of the orbit of $x$, there is a diffeomorphism $g$, $\varepsilon$-$C^1$-close to $f$, coinciding with $f$ out of $U$ along the orbit of $x$, such that the differential $g^n_x$ is a homothecy, where $n$ is the period of $x$.

The second step consists in proving the uniform contraction and expansion of the volume in the extremal bundles. As in [28], one uses Mañé’s Ergodic Closing Lemma to realize a lake of uniform expansion (or uniform expansion) of the volume in the extremal bundle by a periodic orbit $z$ of a $C^1$-perturbation of $f$: if furthermore, the differential of this point restricted to the corresponding extremal
bundle is an homothecy (as in Theorem 5) one get a sink or a source, breaking the transitivity.

For flows, the existence of singular point lies to additional difficulties. In dimension 3, [31] show that a robustly transitive set \( K \) of a flow on a compact 3-manifold is a uniformly hyperbolic set if it does not contain any singular point. If \( K \) contains a singular point then all the singular points in \( K \) have the same Morse index and \( K \) is a singular attractor if this index is 1 and a singular repellor if this index is 2 (see also [18]).

3.5. Topological description of the dynamics with dominated splittings

The dynamics of diffeomorphisms admitting dominated splitting is already very far to be understood.

In dimension 2, Pujals and Sambarino (see [35, 36]) give a very precise description of \( C^2 \)-diffeomorphism whose non-wandering set admits a dominated splitting.

- the periods of the non-hyperbolic periodic points is upper bounded.
- \( \Omega(f) \) is the union of finitely many normally hyperbolic circles on which a power of \( f \) is a rotation, (maybe infinitely many) periodic points contained in a finite family of periodic normally hyperbolic segments and finitely many pairwise disjoint homoclinic classes, each of them containing at most finitely many non-hyperbolic periodic orbits.

This result is close to Mañé’s result, in dimension 1, for \( C^2 \)-maps far from critical points (see [27]). We hope that this result can be generalized in any dimension, for dynamics having a codimension 1 strong stable bundle:

**Conjecture 2** Let \( f \) be a \( C^2 \)-diffeomorphism and \( K \) be a compact locally maximal invariant set of \( f \) admitting a dominated splitting \( TM|_K = E^s \oplus F \) where \( F \) has dimension 1 and \( E^s \) is uniformly contracting.

Then \( K \) is the union of finitely many normally hyperbolic circles on which a power of \( f \) is a rotation, of periodic points contained in a union of finitely many normally hyperbolic periodic intervals and finitely many pairwise disjoint homoclinic classes each of them containing at most finitely many non-hyperbolic periodic points.

In this direction S. Crovisier [19] obtained some progress in the case where there is a unique non-hyperbolic periodic point.

General dominated splitting cannot avoid wild dynamics: multiplying any diffeomorphism by a uniform contraction and a uniform expansion, we get a normally hyperbolic and partially hyperbolic set. However a dominated splitting give some information of the possible bifurcations and on the index of the periodic point: see [10] which investigate in this direction. In particular a diffeomorphism cannot present any homoclinic tangency if it admits a dominated splitting whose non-hyperbolic bundles are all of dimension 1. We hope that this kind of dominated splitting avoid wild behaviours, but this is unknown, even in dimension 3:
Conjecture 3 Let $M$ be a compact 3-manifold and denote by $\mathcal{PH}(M)$ the $C^1$-open set of partially hyperbolic diffeomorphisms of $M$ admitting a dominated splitting $E^s \oplus E^c \oplus E^u$ where all the bundles have dimension 1.

The open set $\mathcal{PH}(M)$ does not contain any wild diffeomorphism: in other word any generic diffeomorphism in $\mathcal{PH}(M)$ is tame.

For partially hyperbolic diffeomorphism (having a splitting $E^s \oplus E^c \oplus E^u$), Brin and Pesin ([15]) show the existence of unique foliations $F^s$ and $F^u$, $f$-invariant and tangent to $E^s$ and $E^u$ respectively. The dynamics of the strong stable and the strong unstable foliations play an important role for the understanding of the topological and ergodical properties of a partially hyperbolic diffeomorphisms. Let mention two results on these foliations: [21] shows that a dense open subset of partially hyperbolic diffeomorphisms (having strong stable and strong unstable foliations) verify the “accessibility property”, that is, any two points can be joined by a concatenation of pathes tangent successively to the strong stable or the strong unstable foliations. When the center direction has dimension 1, [11] shows the minimality of at least one of the strong stable or strong unstable foliations for a dense open subset of the robustly transitive systems in $\mathcal{PH}(M)$, where $M$ is a compact 3-manifold.

However there is no general result on the existence of invariant foliations tangent to the central bundle even if it has dimension 1. When a partially hyperbolic diffeomorphism presents an invariant foliation $F^c$ tangent to the center bundle $E^c$ and which is plaque expansive, [26] shows that this foliations is structurally stable: any $g$ close to $f$ admits a foliation $F^c_g$ topologically conjugated to $f$ and such that (up to this conjugacy of foliation) $g$ is isotopic to $f$ along the center-leaves. This gives a very strong rigidity of the dynamics. This deep result was a key step for the construction of the examples of robustly transitive examples in [38, 29, 6] (there is now new proofs which do not use the stability of the center foliation (see[5])). So an important problem is:

Problem 1) Does it exist robustly transitive partially hyperbolic diffeomorphisms having an invariant center foliation which is not plaque expansive?

2) If a transitive partially hyperbolic diffeomorphism admits an invariant center foliation, is it dynamically coherent? that is, does it admit invariant center-stable and center unstable foliations which intersect along the center foliation?

3) If the center bundle is 1-dimensional, is there an invariant center foliation?

4. Wild dynamics

Very little is known on wild diffeomorphisms: for surfaces, it is not known whether $C^1$-wild diffeomorphisms exist (recall that the Newhouse phenomenon is a $C^2$-generic phenomenon, see [32]).

In dimension ≥ 3, the known examples are all of them due to the existence of homoclinic classes which do not admit, in a persistent way, any dominated splitting (see the first examples in [7]). Then following the same ideas, [8] present wild diffeomorphisms exhibiting, in a locally generic way, infinitely many hyperbolic and non-hyperbolic non-periodic attractors . The same example will present maximal
transitive sets without any periodic orbits. The rest of this section is devoted to a short presentation of these examples:

Consider an open subset $\mathcal{U} \subset \text{Diff}^1(M)$ such that for any $f$ in $\mathcal{U}$ there is a periodic point $p_f$ depending continuously on $f$ and verifying:

- For all $f \in \mathcal{U}$ the homoclinic class $H(p_f, f)$ contains two periodic points of different Morse indices, and having each of them a complex (non-real) eigenvalue (this eigenvalue is contracting for one point and expanding for the other).
- For all $f \in \mathcal{U}$ there are two periodic points having the same Morse index as $p_f$ and homoclinically related to $p_f$ such that the jacobian of the derivative of $f$ at the period is strictly greater than one for one off this point and strictly less than one for the other point.

First item means that the homoclinic class $H(p_f, f)$ do not have any dominated splitting, and that this property is robust. So Theorem 5 shows that $H(p_f, f)$ admits periodic points whose derivative can be perturbated in order to get an homothecy. Second item above allows to choose this point having a jacobian (at the period) arbitrarily close to 1. Then a new pertubation allows to get a periodic point whose derivative at the period is the identity. Considering then perturbations of the identity map, we get:

**Theorem 6** [8] There is a residual part $\mathcal{R}$ of the open set $\mathcal{U}$ defined above, such that any $f \in \mathcal{R}$ admits an infinite family of periodic disks $D_n$ (let $t_n$ denote the period), whose orbits are pairwise disjoint, and verifying the universal following property:

Given any $C^1$-open set $\mathcal{O}$ of diffeomorphisms from the disk $D^3$ to its interior $\overset{o}{D^3}$, there is $n$ such that the restriction of $f^n$ to the disk $D_n$ is smoothly conjugated to an element of $\mathcal{O}$.

Notice that the set of diffeomorphisms $g: D^3 \rightarrow D^3$ contains an open subset $\mathcal{U}_0$ verifying the property of $\mathcal{U}$ described above, one get some kind of renormalisation process: there is a residual part of $\mathcal{U}$ containing infinitely many periodic disks $D_n$ containing each of them infinitely many periodic subdisks themself containing infinitely many periodic subdisks and so on... In that way one build a tree such that each branch is a sequence (decreasing for the inclusion), of strictly periodic orbits of disks whose periods go to infinity, and whose radius go to zero. The intersection of this sequence is a Lyapunov stable (and so saturated) transitif compact set, conjugated to an adding machine (see for instance [14] for this notion) and so without periodic orbits. The set of the infinite branches of this tree is uncountable, given the following result:

**Theorem 7** [8] Given any compact manifold $M$ of dimension $\geq 3$, there is an open subset $\mathcal{V}$ of $\text{Diff}^1(M)$ and a residual part $\mathcal{W}$ of $\mathcal{V}$, such that any $f \in \mathcal{W}$ admits an uncountable family of saturated transitif sets without periodic orbits.
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