ITERATED INTEGRALS OF MODULAR FORMS
AND NONCOMMUTATIVE MODULAR SYMBOLS

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Abstract. The main goal of this paper is to study properties of the iterated integrals of modular forms in the upper halfplane, eventually multiplied by $z^{s-1}$, along geodesics connecting two cusps. This setting generalizes simultaneously the theory of modular symbols and that of multiple zeta values.

§0. Introduction and summary

This paper was inspired by two sources: theory of multiple zeta values on the one hand (see [Za2]), and theory of modular symbols and periods of cusp forms, on the other ([Ma1], [Ma2], [Sh1]–[Sh3], [Me]). Roughly speaking, it extends the theory of periods of modular forms replacing integration along geodesics in the upper complex half-plane by iterated integration. Here are some details.

0.1. Multiple zeta values. They are the numbers given by the $k$–multiple Dirichlet series

$$
\zeta(m_1, \ldots, m_k) = \sum_{0<n_1<\cdots<n_k} \frac{1}{n_1^{m_1} \cdots n_k^{m_k}}
$$

(0.1)

which converge for all integer $m_i \geq 1$ and $m_k > 1$, or equivalently by the $m$–multiple iterated integrals, $m = m_1 + \cdots + m_k$,

$$
\zeta(m_1, \ldots, m_k) = \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{z_2} \int_0^{z_2} \cdots \int_0^{z_{m_k-1}} \frac{dz_{m_k}}{1 - z_{m_k}} \cdots
$$

(0.2)

where the sequence of differential forms in the iterated integral consists of consecutive subsequences of the form $\frac{dz}{z}, \ldots, \frac{dz}{z}, \frac{dz}{1-z}$ of lengths $m_k, m_k-1, \ldots, m_1$.

Easy combinatorial considerations allow one to express in two different ways products $\zeta(l_1, \ldots, l_j) \cdot \zeta(m_1, \ldots, m_k)$ as linear combinations of multiple zeta values.

If one uses for this the integral representation (0.2), one gets a sum over shuffles which enumerate the simplices of highest dimension occurring in the natural simplicial decomposition of the product of two integration simplices.
If one uses instead (0.1), one gets sums over shuffles with repetitions which enumerate some simplices of lower dimension as well.

These relations and their consequences are called double shuffle relations. Both types of relations can be succinctly written down in terms of formal series on free noncommutating generators. One can include in these relations regularized multiple zeta values for arguments where the convergence of (0.1), (0.2) fails.

For a very clear and systematic exposition of these results, see [De] and [Ra1], [Ra2].

In fact, the formal generating series for (regularized) iterated integrals (0.2) appeared in the famous Drinfeld paper [Dr2], essentially as the Drinfeld associator, and more relations for multiple zeta values were implicitly deduced there. The question about interdependence of (double) shuffle and associator relations does not seem to be settled at the moment of writing this: cf. [Ra3]. The problem of completeness of these systems of relations is equivalent to some difficult transcendence questions.

Multiple zeta values are interesting, because they and their generalizations appear in many different contexts involving mixed Tate motives ([DeGo], [T]), deformation quantization ([Kon]), knot invariants etc.

0.2. Modular symbols and periods of modular forms. Let \( \Gamma \) be a congruence subgroup of the modular group acting upon the union \( \overline{H} \) of the upper complex half–plane \( H \) and the set of cusps \( \mathbb{P}^1(\mathbb{Q}) \).

The quotient \( \Gamma \setminus \overline{H} \) is the modular curve \( X_\Gamma \). Differentials of the first kind on \( X_\Gamma \) lift to the cusp forms of weight 2 on \( H \) (multiplied by \( dz \)).

The modular symbols \( \{\alpha, \beta\}_\Gamma \in H_1(X_\Gamma, \mathbb{Q}) \), where \( \alpha, \beta \in \mathbb{P}^1(\mathbb{Q}) \), were introduced in [Ma1] as linear functionals on the space of differentials of the first kind obtained by lifting and integrating. The fact that one lands in \( H_1(X_\Gamma, \mathbb{Q}) \) and not just \( H_1(X_\Gamma, \mathbb{R}) \) is not obvious. It was proved in [Dr1] by refining a weaker argument given in [Ma1]. This is equivalent to the statement that difference of any two cusps in \( \Gamma \) has finite order in the Jacobian, or else that the mixed Hodge structure on \( H^1(X_\Gamma \setminus \{\text{cusps}\}, \mathbb{Q}) \) is split (cf. [El]). One of the basic new insights of [Ma1] consisted in the realization that studying the action of Hecke operators on modular symbols one gets new arithmetic facts about periods and Fourier coefficients of cusp forms of weight two.

The further generalizations of modular symbols proceeded, in particular, in the following directions.
(a) In [Ma2] it was demonstrated that the same technique applies to the integrals of cusp forms of higher weight, eventually multiplied by polynomials in $z$, producing the similar information about their periods and Fourier coefficients. In principle, such integrals cannot be pushed down to $X_\Gamma$, but they can be pushed down to the appropriate Kuga–Sato varieties over $X_\Gamma$ that is, relative Cartesian powers of the universal elliptic curve. In this way, modular symbols of higher weight can be interpreted as rational homology classes of middle dimension of Kuga–Sato varieties: cf. [Sh1]–[Sh3].

(b) Pushing down an oriented geodesic connecting two cusps in $\mathcal{P}$ to $X_\Gamma$, we get a singular chain with a boundary in cusps, which is a relative cycle modulo cusps with integral coefficients. This is the viewpoint of [Me]. Hence it is more natural to consider the relative/non–compact version of modular symbols, and allow integration of the Eisenstein series, that is, differential forms of the third kind with poles at cusps as well. The same remark applies to the modular symbols of higher weight.

This refinement appears as well in the study of the “noncommutative boundary” of the modular space, that is, the (tower of) space(s) $\Gamma \setminus \mathbf{P}^1(\mathbb{R})$, cf. [MaMar]. Namely, it turns out that the relative 1–homology modulo cusps (and additional groups of similar nature) can be interpreted as (sub)groups of the $K$–theory of the noncommutative boundary.

In this paper I suggest a generalization in the third direction, namely

(c) The study of iterated integrals of cusp forms and Eisenstein series, eventually multiplied by a power of $z$, along geodesics connecting two cusps. Some of these integrals can be pushed down to $X_\Gamma$ and thus produce a de Rham version of modular symbols which assigns iterated (eventually regularized) periods to the elements of the fundamental groupoid of $(X_\Gamma, \{\text{cusps}\})$ instead of its 1-homology group. One may call them noncommutative modular symbols.

Other integrals can only be pushed down to the Kuga–Sato varieties, or preferably, to some (covers of the) moduli spaces $\overline{M}_{1,n}$, in the same vein as it was done for multiple zeta values and $\overline{M}_{0,n}$ in [GoMa]. The related geometry deserves further study, both for integrands related to cusp forms and to Eisenstein series.

Notice in conclusion that the discussion above implicitly referred only to the case of $SL_2$–modular symbols. It would be quite interesting to extend it to groups of higher rank, along the lines of [AB] and [AR].

0.3. Summary of this paper. I recall the basic properties of iterated integrals of holomorphic 1–forms on a simply connected Riemannian surface in §1. The shuf-
file relations for the iterated integrals are reflected directly in terms of a generating function $J$ stating that it is a group–like element with respect to a comultiplication, cf. Proposition 1.3.1.

Then I turn to the main object of study. In §2 I define 1–forms of modular and cusp modular type, introduce and study the iterated and total Mellin transform for families of such forms. The functional equation for the total Mellin transform is deduced which extends the classical functional equation for $L$–series.

Using only critical values of these Mellin transforms, I introduce in 2.6 an iterated modular symbol as a certain noncommutative 1–cohomology class of the relevant subgroup of the modular group.

In §3, I study the representation of such Mellin transforms at integer values of their Mellin arguments in terms of multiple Dirichlet series. The results differ from the classical ones expressed by the identity $(0.1) = (0.2)$ in two essential respects. First, iterated integrals are only linear combinations of certain multiple Dirichlet series. Second, the latter are not of the usual type

$$\sum_{0<n_1<\cdots<n_k} \frac{a_{1,n_1} \cdots a_{n,n_k}}{n_1^{m_1} \cdots n_k^{m_k}},$$

in fact, their coefficients depend on pairwise differences $n_j - n_i$.

In §4, the properties of the multiple Dirichlet series which emerged in §3, are axiomatized, and the shuffle relations for them are deduced. This requires, however, a considerable extension of the initial supply of series; the system of those coming from 1–forms of modular type is not closed.

§5 is dedicated to the iterated analogs of the so called Eichler–Shimura and Manin relations for periods of cusp forms. Whereas the relations of the first type are quite straightforward, the relations of the second type, involving Hecke operators, are not obvious. The results presented here (Theorem 5.3) are preliminary, they clearly allow generalizations and deserve further study.

Finally, in §6 I return to the formalism of §1 and extend it by allowing our integrands to have logarithmic singularities at the boundary. A version of the regularization procedure I use here is the same as in Drinfeld’s paper [Dr2]. It exploits complex analyticity in place of Boutet de Monvel’s technique of [De] and [Ra2].

Using the Manin–Drinfeld theorem on cusps, I suggest a generalization of Drinfeld’s associator and extend to this case a part of the identities satisfied by the
latter. This list includes the group-like property, the duality, and the hexagonal relation, which turn out to have the same source as the Shimura–Eichler relations for the periods of cusp forms. To the contrary, the pentagonal relation seems to be specific for the original Drinfeld’s associator.

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§1. Iterated integrals of holomorphic 1–forms

1.1. Setup. Let \( X \) be a connected Riemann surface, not necessarily compact, \( \mathcal{O}_X \) its structure sheaf of holomorphic functions, \( \Omega^1_X \) the sheaf of holomorphic 1–forms. If \( \omega \) is a (local) 1–form, \( z \in X \) a point, \( \omega(z) \) denotes the value of \( \omega \) at \( z \), i.e. the respective cotangent vector.

Let \( V \) be a finite set which will be used as a set indexing various families. Consider the completed unital semigroup ring freely generated by \( V \). We will write it as the ring of associative formal series \( \mathbb{C}\langle\langle A_V \rangle\rangle \) where \( A_V := (A_v \mid v \in V) \) are noncommuting free formal variables.

More generally, we may consider the ring \( \mathcal{O}_X(U)\langle\langle A_V \rangle\rangle \) where \( \mathcal{O}_X(U) \) is the ring of holomorphic functions on an open subset \( U \subset X \) (\( A_v \) commute with \( \mathcal{O}_X(U) \)), and the bimodule \( \Omega^1_X(U)\langle\langle A_V \rangle\rangle \) over this ring, connected by the differential \( d \) such that \( dA_v = 0 \) for all \( v \in V \). Varying \( U \), we will get two presheaves; the sheaves associated with these presheaves are denoted \( \mathcal{O}_X\langle\langle A_V \rangle\rangle \), resp. \( \Omega^1_X\langle\langle A_V \rangle\rangle \), and \( d \) extends to them, so that \( \text{Ker } d \) is the constant sheaf \( \mathbb{C}\langle\langle A_V \rangle\rangle \).

Let \( \omega_V := (\omega_v \mid v \in V) \) be a family of 1–forms holomorphic in \( U \) and indexed by \( V \). Put

\[
\Omega := \sum_{v \in V} A_v \omega_v .
\] (1.1)

The total iterated integral of this form along a piecewise smooth path \( \gamma : [0, 1] \to U \) is denoted \( J_\gamma(\Omega) \) or \( J_\gamma(\omega_V) \) and is defined by the following formula:

\[
J_\gamma(\Omega) := 1 + \sum_{n=1}^{\infty} \int_0^1 \gamma^*(\Omega)(t_1) \int_0^{t_1} \gamma^*(\Omega)(t_2) \cdots \int_0^{t_{n-1}} \gamma^*(\Omega)(t_n) \in \mathbb{C}\langle\langle A_V \rangle\rangle \] (1.2)

where the integration is taken over the simplex \( 0 < t_n < \cdots < t_1 < 1 \). If \( \gamma, \gamma' \) with the same ends are homotopic, \( J_\gamma(\Omega) = J_{\gamma'}(\Omega) \).
Putting \( z_i = \gamma(t_i) \in X, \ a = \gamma(0), \ z = \gamma(1) \), and considering the whole integral as a function of a variable \( z \) we will also write (1.2) in the form

\[ J^\ast_a(\Omega) = J^\ast_a(\omega_V) = 1 + \sum_{n=1}^\infty \int_a^z \Omega(z_1) \int_a^{z_1} \Omega(z_2) \cdots \int_a^{z_{n-1}} \Omega(z_n). \quad (1.3) \]

If \( U \) is connected and simply connected, this expression is an unambiguously defined element of \( \mathcal{O}_X(U) \langle \langle A_V \rangle \rangle \). Otherwise it is a multivalued function of \( z \) in this domain.

The following result is classical.

1.2. Proposition. (i) \( J^\ast_a(\Omega) \) as a function of \( z \) satisfies the equation

\[ dJ^\ast_a(\Omega) = \Omega(z) J^\ast_a(\Omega). \quad (1.4) \]

In other words, \( J^\ast_a(\Omega) \) is a horizontal (multi)section of the flat connection \( \nabla_\Omega := d - l_\Omega \) on \( \mathcal{O}_X \langle \langle A_V \rangle \rangle \), where \( l_\Omega \) is the operator of left multiplication by \( \Omega \).

(ii) If \( U \) is a simply connected neighborhood of \( a \), \( J^\ast_a(\Omega) \) is the only horizontal section with initial condition \( J^\ast_a = 1 \). Any other horizontal section \( K^\ast \) can be uniquely written in the form \( J^\ast_a(\Omega)C, \ C \in \mathbb{C} \langle \langle A_V \rangle \rangle \). In particular, for any \( b \in U \),

\[ J^\ast_b(\Omega) = J^\ast_a(\Omega)J^a_b(\Omega) \quad (1.5) \]

Proof. (i) follows directly from (1.3). Since \( J^\ast_a(\Omega) \) is an invertible element of the ring \( \mathcal{O}_X(U) \langle \langle A_V \rangle \rangle \), we can form \( J^\ast_a(\Omega)^{-1}K^\ast \) and then directly check that \( d(J^\ast_a(\Omega)^{-1}K^\ast) = 0 \). Hence this element belongs to \( \mathbb{C} \langle \langle A_V \rangle \rangle \), and moreover equals its value at \( z = a \) that is, \( K^a \). Choosing \( K^\ast = J^\ast_b(\Omega) \), we get (1.5).

1.3. \( J^\ast_a(\Omega) \) as a generating series. Clearly, we have

\[ J^\ast_a(\omega_V) = J^\ast_a(\Omega) = 1 + \sum_{n=1}^\infty \sum_{(v_1,\ldots,v_n) \in V^n} A_{v_1} \cdots A_{v_n} I^\ast_a(\omega_{v_1},\ldots,\omega_{v_n}), \quad (1.6) \]

where

\[ I^\ast_a(\omega_{v_1},\ldots,\omega_{v_n}) = \int_a^z \omega_{v_1}(z_1) \int_a^{z_1} \omega_{v_2}(z_2) \cdots \int_a^{z_{n-1}} \omega_{v_n}(z_n) \quad (1.7) \]

are the usual iterated integrals.
In the remaining part of this section, and in the main body of the paper, we will encode various (infinite families of) relations among the iterated integrals (1.7) in the form of relations between the generating functions \( J^z_a(\omega_V) \). Generally, our relations between the generating functions will be (noncommutative) polynomial ones. They may also involve different families \((\omega_V)\), different integration paths, and some linear transformations of the formal variables \( A_v \); cf. especially Theorem 2.3, Proposition 5.1.1, Theorem 5.3, and subsection 6.5 (in the context requiring a regularization).

1.4. Basic relations between total iterated integrals. There are three types of basic relations, which we will call group–like property, cyclicity, and functoriality respectively.

1.4.1. Proposition. Consider the comultiplication

\[
\Delta : \mathbb{C}\langle\langle A_V \rangle\rangle \to \mathbb{C}\langle\langle A_V \rangle\rangle \hat{\otimes} \mathbb{C}\langle\langle A_V \rangle\rangle, \quad \Delta(A_v) = A_v \otimes 1 + 1 \otimes A_v
\]

and extend it to the series with coefficients \( \mathcal{O}_X \) and \( \Omega^1_X \). Then

\[
\Delta(J^z_a(\omega_V)) = J^z_a(\omega_V) \otimes_{\mathcal{O}_X} J^z_a(\omega_V).
\] (1.8)

**Proof.** Both sides of (1.8) satisfy the equation \( dJ = \Delta(\Omega)J \) and have the initial value 1 at \( z = a \).

**NB.** Coefficientwise, (1.8) is a compact version of shuffle relations for iterated integrals (1.7).

1.4.2. Cyclicity. Let \( \gamma \) be a closed oriented contractible contour in \( U \), \( a_1, \ldots, a_n \) points along this contour (cyclically) ordered compatibly with orientation. Then

\[
J^{a_1}_{a_2}(\omega)V_J^{a_2}_{a_3}(\omega)\cdots J^{a_{n-1}}_{a_n}(\omega)V_J^{a_n}(\omega) = 1.
\] (1.9)

This follows from (1.5) by induction.

1.4.3. Functoriality. Consider an automorphism \( g : X \to X \) such that \( g^* \) maps into itself the linear space spanned by \( \omega_v \). In particular, there is a constant matrix \( G = (g_{vu}) \) with rows and columns labeled by \( V \) such that \( g^*(\omega_v) = \sum_u g_{vu} \omega_u \). Define the automorphism \( g_* \) of any of the ring/module of formal series \( \mathbb{C}\langle\langle A_V \rangle\rangle \), \( \mathbb{C}(X)(\langle A_V \rangle) \), \( \Omega^1(X)(\langle A_V \rangle) \) by the formula \( g_*(A_u) = \sum_v A_v g_{vu} \). On coefficients \( g_* \) acts identically.
1.4.4. Claim. We have
\[
J_{g^a}^z(\omega_V) = g_*(J_a^z(\omega_V)). \tag{1.10}
\]

Proof. In fact, both sides coincide with \( J_a^z(g^*(\omega_V)) \). We will give below a calculation which proves a slightly more general statement.

1.5. A variant: multiple lower integration limits. Somewhat more generally, in the simply connected case we can consider a family of points \((a_{\bullet}) := (a_i,v)\) in \(X\) indexed by pairs \(i = 1, 2, 3, \ldots; v \in V\).

Given such a family and \(\omega_V\), we can construct the following formal series in \(C(X)\langle\langle A_V \rangle\rangle\) with constant term 1:
\[
J_{(a_{\bullet})}^z(\omega_V) := \sum_{n=0}^{\infty} \sum_{(v_1, \ldots, v_n) \in V^n} A_{v_1} \cdots A_{v_n} I_{a_1,v_1, \ldots, a_n,v_n}^z(\omega_{v_1}, \ldots, \omega_{v_n}), \tag{1.11}
\]
where
\[
I_{a_1,v_1, \ldots, a_n,v_n}^z(\omega_{v_1}, \ldots, \omega_{v_n}) := \int_{a_1,v_1}^z \omega_{v_1}(z_1) \int_{a_2,v_2}^{z_1} \omega_{v_2}(z_2) \cdots \int_{a_n,v_n}^{z_{n-1}} \omega_{v_n}(z_n). \tag{1.12}
\]
As above, \(z \in X\) denotes a variable point, the argument of our functions. Then we have
\[
dJ_{(a_{\bullet})}^z(\omega_V) = \Omega \cdot J_{(a_{\bullet})}^z(\omega_V) \tag{1.13}
\]
and
\[
J_{(a_{\bullet})}^z(\omega_V) = J_a^z(\omega_V) J_{(a_{\bullet})}^a(\omega_V). \tag{1.14}
\]
The series (1.11) satisfies the following functoriality relation generalizing (1.10):

1.5.1. Claim. We have
\[
J_{(g a_{\bullet})}^z(\omega_V) = g_*(J_{(a_{\bullet})}^z(\omega_V)). \tag{1.15}
\]

Proof. We will check that both sides coincide with \( J_{(a_{\bullet})}^z(g^*(\omega_V)) \). In fact, \(\int_{g u}^v \nu(z) = \int_u^v \nu(gz)\) so that, removing \(g\) step by step from the integration limits, we get
\[
I_{ga_1,v_1, \ldots, ga_n,v_n}^z(\omega_{v_1}, \ldots, \omega_{v_n}) = I_{a_1,v_1, \ldots, a_n,v_n}^z(g^*(\omega_{v_1}), \ldots, g^*(\omega_{v_n})).
\]
Multiplying the l.h.s. by $A_{v_1} \ldots A_{v_n}$ and summing, we get the l.h.s. of (1.15).

On the other hand,

$$
\sum_{v_1, \ldots, v_n \in V^n} A_{v_1} \ldots A_{v_n} I_{a_1, v_1, \ldots, a_n, v_n}^z (g^* (\omega_{v_1}), \ldots, g^* (\omega_{v_n})) =
$$

$$
= \sum_{v_1, \ldots, v_n \in V^n} A_{v_1} g_{v_1, u_1} \ldots A_{v_n} g_{v_n, u_n} I_{a_1, v_1, \ldots, a_n, v_n}^z (\omega_{u_1}, \ldots, \omega_{v_n}) =
$$

$$
= g^* \left( \sum_{v_1, \ldots, v_n \in V^n} A_{v_1} \ldots A_{v_n} I_{a_1, v_1, \ldots, a_n, v_n}^z (\omega_{v_1}, \ldots, \omega_{v_n}) \right).
$$

Summation over $n$ produces the r.h.s. of (1.15), proving the lemma.

**1.6. A variant: nonlinear $\Omega$.** Let now $\Omega \in \Omega^1_X(U)\langle\langle AV \rangle\rangle$ be an arbitrary form without a constant term in $A_v$:

$$
\Omega = \sum_{n=1}^{\infty} \sum_{(v_1, \ldots, v_n) \in V^n} A_{v_1} \ldots A_{v_n} \Omega_{v_1, \ldots, v_n},
$$

where $\Omega_{v_1, \ldots, v_n} \in \Omega^1_X(U)$.

The total iterated integrals $J_{\gamma}^z(\Omega)$ and $J_{\alpha}^z(\Omega)$ are defined by exactly the same formulas (1.2) and (1.3). It is not true anymore that the coefficients of this series are the usual iterated integrals. However, an analog of the Proposition 1.2 and the cyclic identity remain true:

**1.6.1. Proposition.** $J_{\alpha}^z(\Omega)$ as a function of $z$ satisfies the equation

$$
dJ_{\alpha}^z(\Omega) = \Omega(z) J_{\alpha}^z(\Omega).
$$

If $U$ is a simply connected neighborhood of $a$, $J_{\alpha}^z(\Omega)$ is the only horizontal section with initial condition $J_{\alpha}^a = 1$. Any other horizontal section $K^z$ can be uniquely written in the form $J_{\alpha}^z(\Omega) C$, $C \in \mathbb{C}\langle\langle AV \rangle\rangle$. In particular, for any $b \in U$,

$$
J_{\delta}^z(\Omega) = J_{\alpha}^z(\Omega) J_{b}^\delta(\Omega)
$$
1.6.2. Corollary. Let $\gamma$ be a closed oriented contractible contour in $U$, $a_1, \ldots, a_n$ points along this contour (cyclically) ordered compatibly with orientation. Then

$$J_{a_2}(\Omega)J_{a_3}(\Omega) \cdots J_{a_{n-1}}(\Omega)J_{a_1}(\Omega) = 1.$$  \hspace{1cm} (1.19)

Notice in conclusion that the integral formula (0.2) for the multiple zeta values is not quite covered by the formalism reviewed so far because the integrands in (0.2) have logarithmic poles at the boundary. We will return to this situation in §6, to which some readers may prefer to turn right away. However, for applications to the integration of cusp forms in §2 – §5 the regular case treated here suffices.

§2. 1–forms of modular type, iterated Mellin transform, and noncommutative modular symbols

2.1. Setup. In this section, $X$ will be the upper half plane $H$ and $z$ the standard complex coordinate. $H$ is endowed with the metric of constant curvature $-1$: $ds^2 = |dz|^2/(\text{Im } z)^2$.

The limits of integration in our iterated integrals generally lie in $H$, but may be “improper” as well, that is, belong to the set of cusps $\mathbb{Q} \cup \{i\infty\}$. If this is the case, we always assume that the respective integration path in some neighborhood of the cusp coincides with a segment of a geodesic curve.

Our 1–forms generally will have the following structure.

2.1.1. Definition. (i) A 1–form $\omega$ on $H$ is called a form of modular type, if it can be represented as $f(z)z^{s-1}dz$ where $s$ is a complex number, and $f(z)$ is a modular form of some weight with respect to a congruence subgroup of the modular group.

The modular form $f(z)$ is then well defined and called the associated modular form (to $\omega$), and the number $s$ is called the Mellin argument of $\omega$.

(ii) $\omega$ is called a form of cusp modular type if the associated $f(z)$ is a cusp form.

To fix notation, we will recall below some classical facts.

2.1.2. Action of automorphisms. Any matrix $\gamma \in GL^+_2(\mathbb{R})$ defines a holomorphic isometry of $H$, namely $z \mapsto [\gamma]z$ where $[\gamma]$ is the fractional linear transformation corresponding to $\gamma$. We will denote this automorphism also $\gamma$. It induces
the inverse image maps on the sheaves \((\Omega^1_H)^{\otimes r}\) of holomorphic tensor differentials of degree \(r\):

\[
\gamma^*(f(z)\,(dz)^r) = f([\gamma]\,z)\,(d[\gamma]\,z)^r = (\det \gamma)^r f([\gamma]\,z) \frac{(dz)^r}{(c_\gamma z + d_\gamma)^{2r}}
\]

(2.1)

where \((c_\gamma, d_\gamma)\) is the lower row of \(\gamma\).

If one identifies \((\Omega^1_H)^{\otimes r}\) with \(O_H\) by sending \((dz)^r\) to 1, (2.1) turns into the action of weight \(2r\) on functions which is traditionally written as a right action:

\[
f|[\gamma]_{2r}(z) := (\det \gamma)^r f([\gamma]\,z) (c_\gamma z + d_\gamma)^{-2r}.
\]

(2.2)

Assume that \(f(z)\,(dz)^r\) is invariant with respect to \(\gamma\). Then, writing \(f(z)z^{s-1}dz = f(z)(dz)^r \cdot z^{s-1}(dz)^{1-r}\), we see that

\[
\gamma^*(f(z)\,z^{s-1}dz) = (\det \gamma)^{1-r} f(z)(a_\gamma z + b_\gamma)^{s-1}(c_\gamma z + d_\gamma)^{2r-1-s}dz.
\]

(2.3)

where \((a_\gamma, b_\gamma)\) is the upper row of \(\gamma\). In particular, if \(2r \geq 2\) is an integer, \(\gamma^*\) maps into itself the space of 1–forms spanned by

\[
f(z)\,z^{s-1}dz, \quad 1 \leq s \leq 2r - 1, \quad s \in \mathbb{Z}.
\]

(2.4)

More generally, if

\[
\gamma^*(f(z)(dz)^r) = \chi(\gamma)\, f(z)(dz)^r
\]

(2.5)

for some \(\chi(\gamma) \in \mathbb{C}\), then

\[
\gamma^*(f(z)\,z^{s-1}dz) = (\det \gamma)^{1-r} f(z)\chi(\gamma) (a_\gamma z + b_\gamma)^{s-1}(c_\gamma z + d_\gamma)^{2r-1-s}dz,
\]

(2.6)

and the space (2.4) will still remain invariant.

We can apply this formalism to the spaces of modular forms of weight \(2r\) with respect to a congruence subgroup \(\Gamma\) of \(SL_2(\mathbb{Z})\), i.e. to the functions \(f\) in \((dz)^{-r}((\Omega^1_H)^{\otimes r})^\Gamma\). Two special cases will be of particular interest:

(i) For any such \(f\), the space of 1–forms spanned by (2.4) is \(\Gamma\)–invariant.

(ii) Assume that \(\Gamma = \Gamma_0(N)\). This group is normalized by the involution

\[
g = g_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}
\]

(2.7)
Therefore this involution maps into itself the space of $\Gamma_0(N)$-modular forms, and the latter has a basis consisting of forms with

$$
g_N^*(f(z)(dz)^r) = \varepsilon_f f(z)(dz)^r, \quad \varepsilon_f = \pm 1. \tag{2.8}
$$

Applying (2.6) with $\gamma = g_N$ we get, for any complex $s$,

$$
g_N^*(f(z)z^{s-1}dz) = \varepsilon_f N^{r-s}f(z)z^{2r-1-s}dz. \tag{2.9}
$$

### 2.1.3. Geodesics and cusp forms.

The geodesic from 0 to $i\infty$ is the upper half of the pure imaginary line. The unoriented distance of a point $iy$ on it to $i$ is $|\log y|$. The exponential of this distance is thus $y$, if $y > 1$, and $y^{-1}$, if $y < 1$. If we replace $i$ by another reference point, even outside of imaginary axis, the exponential of the distance will behave like $e^{O(1)}y$ (resp. $e^{O(1)}y^{-1}$) as $y \to \infty$ (resp. $y \to 0$.)

Let $f(z)$ be a cusp form of weight $2r$ for a congruence subgroup. Then it can be represented by a Fourier series $f(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i nz/N}$ for some $N \in \mathbb{Z}_+$, whose coefficients are polynomially bounded: $c_n = O(n^C)$ for some $C > 0$. Therefore we have $|f(iy)| = O(e^{-ay})$ for some $a > 0$ as $y \to \infty$. From the previous analysis it follows that more generally, for any cusp form and any geodesic connecting two cusps, $|f(z)| = O(e^{-a y(z)})$ for some $a > 0$ as $z$ tends along the geodesics to one of its ends, where this time $y(z)$ means the exponentiated geodesic distance from $z$ to any reference point in $H$, fixed once and for all.

Let now $\omega(z) = f(z)z^{s-1}dz$ be an 1–form of cusp modular type. Then the estimates above show that the following expected properties indeed hold.

a) As $z_0 \to i\infty$ along the imaginary axis, the family $\int_{z_0}^z \omega$ of holomorphic functions of $z$ in any bounded domain $H$ converges absolutely and uniformly to a holomorphic function of $z$ which is denoted $\int_{i\infty}^z \omega$. The same remains true, if one replaces $i\infty$ by 0.

These integrals are holomorphic functions of the Mellin argument $s$ of $\omega$ as well.

b) The sum $(\int_{i\infty}^z + \int_z^0) \omega$ does not depend on $z$ in $H$ and is denoted $\int_{i\infty}^0 \omega$. As a function of $s$, it is called the classical Mellin transform of $\omega$.

Denote this classical transform $\Lambda(f; s)$. Assume that $f$ satisfies (2.8). Then we have the classical functional equation

$$
\Lambda(f; s) = -\varepsilon_f N^{r-s}\Lambda(f; 2r - s), \tag{2.10}
$$
because in view of (2.9)

\[
\int_{i\infty}^{0} \omega = - \int_{0}^{i\infty} \omega = - \int_{g_{N}(i\infty)}^{g_{N}(0)} \omega = - \int_{i\infty}^{0} g_{N}^{*}(\omega) = - \varepsilon_{f} N^{-s} \int_{i\infty}^{0} f(z)z^{2r-1-s}dz.
\]

Another identity in the same vein uses the fact that \(i/\sqrt{N}\) is the fixed point of \(g_{N}\) so that \(\Lambda(f;s)\) can be written as

\[
\Lambda(f;s) = \int_{i\infty}^{i/\sqrt{N}} \omega - \int_{i\infty}^{i/\sqrt{N}} g_{N}^{*}(\omega).
\] (2.11)

This allows one to use the Fourier expansions of \(f(z)\) and \(f|g_{N}|_{2r}(z)\) in order to deduce series expansions for \(\Lambda(f;s)\) (notice that the Fourier expansions cannot be term–wise integrated near \(z = 0\) because the formal integration produces a divergent series).

Now we can finally write down the analogs of these definitions and results for iterated integrals.

**2.2. Definition.** (i) Let \(f_{1}, \ldots, f_{k}\) be a finite sequence of cusp forms with respect to a congruence subgroup, \(\omega_{j}(z) := f_{j}(z)z^{s_{j}-1}dz\). The iterated Mellin transform of \((f_{j})\) is, by definition,

\[
M(f_{1}, \ldots, f_{k}; s_{1}, \ldots, s_{k}) := I_{i\infty}^{0}(\omega_{1}, \ldots, \omega_{k}) = \\
= \int_{i\infty}^{0} \omega_{1}(z_{1}) \int_{i\infty}^{z_{1}} \omega_{2}(z_{2}) \cdot \cdots \int_{i\infty}^{z_{n-1}} \omega_{n}(z_{n})
\] (2.12)

(ii) Let \(f_{V} = (f_{v} | v \in V)\) be a finite family of cusp forms with respect to a congruence subgroup, \(s_{V} = (s_{v} | v \in V)\) a finite family of complex numbers, \(\omega_{V} = (\omega_{v})\), where \(\omega_{v}(z) := f_{v}(z)z^{s_{v}-1}dz\). The total Mellin transform of \(f_{V}\) is, by definition,

\[
TM(f_{V}; s_{V}) := J_{i\infty}^{0}(\omega_{V}) = \\
= \sum_{n=0}^{\infty} \sum_{(v_{1}, \ldots, v_{n}) \in V^{n}} A_{v_{1}} \ldots A_{v_{n}} M(f_{v_{1}}, \ldots, f_{v_{n}}; s_{v_{1}}, \ldots, s_{v_{n}})
\] (2.13)

(cf. (1.3)).
Below we will assume that the space spanned by all $\omega_v$ is stable with respect to some $g_N^*$. Then as in 1.4.3 denote by $G = (g_{vu})$ the matrix of this action on $(\omega_v)$, and by $g_N*$ the action of the transposed matrix on the formal variables $(A_v)$.

For example, if $(\omega_v)$ and $(f_v(dz)^{r_v})$ respectively can be represented as a union of pairs of forms, corresponding to the left and right hand sides of (2.9), the matrix $G$ consists of two by two antidiagonal blocks each of which after the classical Mellin transform produces a functional equation of the form (2.10).

2.3. **Theorem.** (i) If the space spanned by all $\omega_v$ is stable with respect to some $g_N^*$, we have the following functional equation:

$$J_{i\infty}^0(\omega_V) = g_N^*(J_{i\infty}^0(\omega_V))^{-1}. \tag{2.14}$$

(ii) In the assumptions of the Definition 2.2 (ii), denote the weight of $f_v$ by $2r_v$ and assume that $f_v$ is an eigenvector for $g_N^*$ with eigenvalue $\varepsilon_v$. Then the total Mellin transform (2.13) satisfies

$$TM(f_V; s_V) = g^*(TM(f_V; 2r_V - s_V))^{-1} \tag{2.15}$$

where $g^*$ multiplies each $A_v$ by $\varepsilon_v N^{r_v - s_V}$.

**Proof.** This is a straightforward corollary of the definitions and formulas (1.9) and (1.10) as soon as one has checked that the latter formulas are applicable to the improper iterated integrals of the 1–forms of cusp modular type.

This check is a routine matter, since at each step of an iterated integration we multiply the result of the previous step by a holomorphic function of the type $f(z) z^{s-1}$ which is bounded by $O(e^{-a_y(z)})$ as in 2.1.3 above as $z$ tends to 0 or $i\infty$.

Notice in conclusion that no analog of the functional equation (2.11) can be written for the individual Mellin transforms (2.12), because applying $g_N$ to the integration limits in them we get an expression which is not a Mellin transform in our sense. Only putting them all together produces the necessary environment for replacing the overall minus sign at the r.h.s. of (2.10) by the overall exponent $-1$ at the r.h.s. of (2.15).

A similar reasoning establishes the iterated analog of (2.11):

2.4. **Proposition.** We have

$$TM(f_V; s_V) = (g_N^* J_{i\infty}^{i/\sqrt{N}}(\omega_V))^{-1} J_{i\infty}^{i/\sqrt{N}}(\omega_V). \tag{2.16}$$
2.5. Pushing down iterated integrals. Let $\omega$ be an 1–form of modular type whose associated modular form has weight 2 with respect to a subgroup $\Gamma$ of the modular group, and whose Mellin argument is 1. In this case $\omega$ is $\Gamma$–invariant so that it can be pushed down to an 1–form $\nu$ on $X_\Gamma^\circ := \Gamma \backslash H$. Instead of integrating $\omega$ along a path in $H$, we can integrate $\nu$ along the push–down of this path to $X_\Gamma^\circ$. If all $\omega_v$ have this property, all relevant iterated integrals can be pushed down to $X_\Gamma^\circ$.

This argument admits a partial generalization to higher weights. Assume that the modular form associated with $\omega$ has weight $2r > 2$, whereas its Mellin argument is an integer belonging to the critical strip (2.4), $1 \leq s \leq 2r - 1$. In this case the relevant simple integral along, say, $\{i\infty, 0\}$ can be pushed down to the Kuga–Sato variety $X_\Gamma^{(2r-2)}$ which is the $(2r - 2)$–th fibered power of the universal elliptic curve over $X_\Gamma$ or rather its compactified smooth model. However, on $X_\Gamma^{(2r-2)}$ we obtain an integral of a holomorphic form $\tilde{\omega}$ of degree $2r - 1$ over a relative cycle of the same dimension which is > 1. Therefore iterated “line” integrals of such forms on $H$ cannot be directly translated into integrals of the same type on $X_\Gamma^{(2r-2)}$.

On the other hand, one can generally define Chen’s iterated integrals of forms of arbitrary degree, say, $\tilde{\omega}_v$ on $X_\Gamma^{(2r-2)}$, which take values in the space of differential forms on the path space $PX_\Gamma^{(2r-2)}$ and not just $\mathbb{C}$: cf. reports [Ch] and [Ha], as well as references therein. Studying properties of such iterated integrals in the modular case like presents an interesting challenge.

Here I will restrict myself to explaining how $\tilde{\omega}$ looks like and why its periods coincide with integrals of $\omega$ along geodesics. For more details, see [Sh1], [Sh2], and especially [Sh3].

Denote by $\Gamma^{(r)}$ the semidirect product $\Gamma \ltimes (\mathbb{Z}^{2r-2} \times \mathbb{Z}^{2r-2})$ acting upon $H \times \mathbb{C}^{2r-2}$ via

$$ (\gamma; n, m)(z, \zeta) := ([\gamma]z; (c_\gamma z + d_\gamma)^{-1}(\zeta + zn + m)). $$

Here $n = (n_1, \ldots, n_{2r-2})$, $m = (m_1, \ldots, m_{2r-2})$, $\zeta = (\zeta_1, \ldots, \zeta_{2r-2})$, and $nz = (n_1 z, \ldots, n_{2r-2} z)$.

If $f(z)$ is a holomorphic modular form of weight $2r$, then $f(z)dz \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{2r-2}$ is a $\Gamma^{(r)}$–invariant holomorphic volume form on $H \times \mathbb{C}^{2r-2}$. Hence one can push it down to (a Zariski open subset of) the quotient $\Gamma^{(r)} \backslash (H \times \mathbb{C}^{2r-2})$ which is a Zariski open subset of the respective Kuga–Sato variety. Denote by $\tilde{\omega}$ the image of this
form. Notice that it is common for all 1–forms of modular type $\omega = f(z)z^{s-1}dz$ with different Mellin arguments $s$.

A detailed analysis of singularities performed in [Sh2], [Sh3] shows that the map $f \mapsto \hat{\omega}$ induces an isomorphism of the space of cusp forms of weight $2r$ with the space of holomorphic volume forms on an appropriate smooth projective Kuga–Sato variety. (As I have already remarked in the Introduction, it would be useful to replace it by the base extension $(\overline{M}_{1,2r-2})_{X_{\Gamma}}$.)

The dependence of the period of $\omega$ on the integration path and on the Mellin argument is reflected in the choice of the relative cycle over which we integrate $\hat{\omega}$.

More precisely, let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be two cusps in $\overline{H}$ and let $p$ be a geodesic joining $\alpha$ to $\beta$. Fix $(n_i)$ and $(m_i)$ as above. Construct a cubic singular cell $p \times (0,1)^{2r-2} \to H \times \mathbb{C}^{2r-2}$: $(z,(t_i)) \mapsto (z,(t_i(zn_i + m_i)))$. Take the $S_{2r-2}$-symmetrization of this cell and push down the result to the Kuga–Sato variety. We will get a relative cycle whose homology class is Shokurov’s higher modular symbol $\{\alpha, \beta; n, m\}_\Gamma$. From this construction it is almost obvious that

$$\int_{\alpha}^{\beta} f(z) \sum_{i=1}^{2r-2} (n_i z + m_i) \, dz = \int_{\{\alpha, \beta; n, m\}_\Gamma} \hat{\omega}.$$  

The singular cube $(0,1)^{2r-2}$ may also be replaced by an evident singular simplex. This can be useful for transposing the results of [GoMa] to the genus one moduli spaces.

2.6. Noncommutative modular symbols and continued fractions. I will define in this subsection a generalization of modular symbols involving iterated integrals and allowing a mixture of forms of different weights with respect to the same subgroup $\Gamma$ of $SL(2,\mathbb{Z})$.

Let $(\omega_v)$ be a family of linearly independent 1–forms of cusp modular type whose Mellin arguments are integers lying in the respective critical strip as in (2.4). Let $\Gamma$ be a subgroup of modular group acting on the space spanned by $(\omega_v)$ as in (2.3). Denote by $\Pi$ the multiplicative group of power series in $(A_v)$ with constant term 1. Clearly, the map $J \mapsto g_* J$ (see 1.4.3.) defines the left action of $\Gamma$ on $\Pi$.

2.6.1. Proposition–Definition. (i) For each $a \in \mathbb{P}^1(\mathbb{Q})$, the map $\Gamma \to \Pi : \gamma \mapsto J^a_\gamma(\Omega)$ is a noncommutative 1–cocycle $\zeta_a$ in $Z^1(\Gamma, \Pi)$.

(ii) The cohomology class of $\zeta_a$ in $H^1(\Gamma, \Pi)$ does not depend on the choice of $a$ and is called the noncommutative modular symbol.
**Proof.** We have, omitting Ω for brevity, and using (1.9), (1.10):

\[ J_{\gamma \beta a}^a = J_{\gamma a}^a J_{\gamma \beta a}^x = J_{\gamma a}^a \gamma_a^* (J_{\beta a}^a) \]

which means that ζ_a is an 1–cocycle. Moreover, if b is another cusp,

\[ J_{\gamma a}^a = J_{\gamma b}^b J_{\gamma a}^\gamma b = J_{\gamma b}^b J_{\gamma a}^\gamma b (\gamma_b^*)^{-1} \]

that is, ζ_a and ζ_b are homologous.

**Remark.** Assume that the cusp forms associated with (ω_v) span the sum of all spaces of cusp forms of certain weights, and for each weight and each cusp form, all admissible Mellin arguments actually occur. Then the linear in \( A_v \) term of \( \zeta_a \) encodes all periods of the involved cusp forms along all classical modular symbols corresponding to loops in \( X(\Gamma(\mathbb{C})) \) starting and ending at the cusp \( \Gamma a \).

### 2.6.2. Iterated integrals between arbitrary cusps.

The group \( \Gamma \) generally does not act transitively on cusps, so that the components of cocycles \( \zeta_a \) do not contain iterated integrals along all geodesics connecting two cusps. One can use the technique of continued fractions as in [Ma1], [Ma2] in order to express all such integrals through a finite number of them.

Namely, choose a set of representatives \( C \) of left cosets \( \Gamma \setminus SL_2(\mathbb{Z}) \). Call the iterated integrals of the form \( (J_{\gamma (i \infty)}^g)^{\pm 1}, g \in C, \text{primitive ones.} \) Notice that when \( g \notin \Gamma \) the space spanned by \( (\omega_v) \) is not generally \( g^* \)-stable so that we cannot define \( g^* \).

### 2.6.3. Proposition.

Each \( J_b^a \) can be expressed as a noncommutative monomial in \( \gamma_*(J_c^d) \) where \( \gamma \) runs over \( \Gamma \) and \( J_c^d \) runs over primitive integrals.

**Proof.** First, we can write \( J_b^a = (J_{i \infty}^i)^{-1} J_b^{i \infty} \). So it remains to find a required expression for \( J_b^{i \infty} \). Assume that \( a > 0 \); the case \( a < 0 \) can be treated similarly. Consider the consequent convergents to \( a \):

\[
a = \frac{p_n}{q_n}, \quad \frac{p_{n-1}}{q_{n-1}}, \ldots, \quad \frac{p_0}{q_0} = \frac{p_0}{1}, \quad \frac{p_{-1}}{q_{-1}} := \frac{1}{0}.
\]

Put

\[
g_k := \begin{pmatrix} p_k & (-1)^k - p_{k-1} \\ q_k & (-1)^k - q_{k-1} \end{pmatrix}, \quad k = 0, \ldots, n.
\]
We have \( g_k = g_k(a) \in SL_2(\mathbb{Z}) \). Put \( g_k = \gamma_k c_k \) where \( \gamma_k \in \Gamma \) and \( c_k \in C \) are two sequences of matrices depending on \( a \). Then from cyclicity we get

\[
J_{i \infty}^a = \prod_{k=0}^n J^{p_k/q_k}_{p_{k-1}/q_{k-1}}
\]

and from functoriality we obtain

\[
J^{p_k/q_k}_{p_{k-1}/q_{k-1}} = \gamma_k \ast (J^{c_k(i \infty)}_{c_k(0)}).
\]

§3. Values of iterated Mellin transforms at integer points and multiple Dirichlet series

In this section, we collect some formulas expressing iterated Mellin transforms (2.12) at integer values of their Mellin arguments as linear combinations of “multiple Dirichlet series”.

3.1. Notation. Consider a family of 1–forms \( \omega_v, v \in V \), satisfying the following conditions. First,

\[
\omega_v(z) = \sum_{n=1}^\infty c_{v,n} e^{2\pi i nz} z^{m_v-1} dz, \quad c_{v,n} \in \mathbb{C}, \quad m_v \in \mathbb{Z}, \quad m_v \geq 1.
\]

Moreover, assume that \( c_{v,n} = O(n^C) \) for some \( C \) and each \( v \).

Until a problem of analytic continuation arises, we do not have to assume modularity. The notation \( m_v \) replacing former \( s_v \) is chosen to remind that these Mellin arguments are natural numbers.

We start with introducing some notation.

3.1.1. Functions \( L(z; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, \ldots, j_1) \). Choose \( k \geq 1; v_k, \ldots, v_1 \in V \), and nonnegative integers \( j_k, \ldots, j_1 \); it is convenient to add \( j_0 = 0 \). In our applications, \( j_a \) will satisfy the following restrictions:

\[
j_a \leq m_{v_a} - 1 + j_{a-1}.
\]

Now put

\[
L(z; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, \ldots, j_1) :=
\]
\[(2\pi iz)^{jk} \sum_{n_1,\ldots,n_k \geq 1} \frac{c_{v_1,n_1} \cdots c_{v_k,n_k} e^{2\pi i(n_1+\cdots+n_k)z}}{n_1^{m_{v_1}+\cdots+j_{1-1}}(n_1+n_2)^{m_{v_2}+\cdots+j_{1-1}}(n_1+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k}}. \tag{3.3} \]

Thanks to the presence of exponential terms in (3.2), this series absolutely converges for any \(z\) with \(\text{Im } z > 0\) and defines a holomorphic function in \(H\).

Notice that the enumeration of arguments of \(L\) is reversed in order to get a more natural enumeration of factors in the summands of (3.3).

### 3.1.2 Numbers \(L(0; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, j_{k-1}, \ldots, j_1)\)

If we formally put \(z = 0\) in the expansion for

\[(2\pi iz)^{-jk} L(z; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, \ldots, j_1)\]

we will get the formal series

\[
\sum_{n_1,\ldots,n_k \geq 1} \frac{c_{v_1,n_1} \cdots c_{v_k,n_k}}{n_1^{m_{v_1}+\cdots+j_{1-1}}(n_1+n_2)^{m_{v_2}+\cdots+j_{1-1}}(n_1+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k}}. \tag{3.4} \]

We have

\[c_{v_1,n_1} \cdots c_{v_k,n_k} = O((n_1n_2\cdots n_k)^C) = O((n_1+\cdots+n_k)^{kC}). \tag{3.5} \]

Assume that (3.2) holds. Then the general term of (3.4) is bounded by

\[
\frac{1}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_{k-1})(n_1+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k} + kC}. \]

Hence (3.4) absolutely converges as long as

\[m_{v_k} + j_{k-1} - j_k > 1 + kC. \]

Summarizing, we get three alternatives, describing the possible behavior of

\[L(z; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, j_{k-1}, \ldots, j_1)\]

as \(z \to 0\). We will later identify the respective limit as a component of the total Mellin transform of \((\omega_V)\):

**Case 1:** \(j_k = 0\) and \(m_{v_k} + j_{k-1} > 1 + kC\). Then the limit exists, and equals to the “multiple Dirichlet series”

\[L(0; \omega_{v_k}, \ldots, \omega_{v_1}; 0, j_{k-1}, \ldots, j_1) = \]
\[
\sum_{n_1, \ldots, n_k \geq 1} \frac{c_{v_1, n_1} \cdots c_{v_k, n_k}}{n_1^{m_{v_1} + j_0 - j_1} (n_1 + n_2)^{m_{v_2} + j_1 - j_2} \cdots (n_1 + \cdots + n_k)^{m_{v_k} + j_{k-1} - j_k}}. \tag{3.6}
\]

\((j_0 = j_k = 0\) appear at the r.h.s. only for the uniformity of notation).

**Case 2:** \(j_k > 0\) and \(m_{v_k} + j_{k-1} - j_k > 1 + kC\). Then the limit exists, and vanishes thanks to the factor \((2\pi i)^j_k\):

\[
L(0; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, j_{k-1}, \ldots, j_1) = 0 \tag{3.7}
\]

**Case 3:** \(j_k > 0\) and \(m_{v_k} + j_{k-1} - j_k \leq 1 + kC\). In this case an additional study is needed.

We can now formulate the first main result of this section.

**3.2. Theorem.** For any \(k \geq 1\), \((v_1, \ldots, v_k) \in V^k\), and \(\text{Im } z > 0\) we have

\[
(2\pi i)^{m_{v_1} + \cdots + m_{v_k}} I_\infty^z (\omega_{v_k}, \ldots, \omega_{v_1}) =
\]

\[
= (-1)^{k} \sum_{i=1}^{k} (m_{v_i} - 1) \sum_{j_1=0}^{m_{v_1} - 1} \sum_{j_2=0}^{m_{v_2} - 1 + j_1} \cdots \sum_{j_k=0}^{m_{v_k} - 1 + j_{k-1}} (-1)^{j_k} \times
\]

\[
\times \frac{(m_{v_1} - 1)! (m_{v_2} - 1 + j_1)! \cdots (m_{v_k} - 1 + j_{k-1})!}{j_1! j_2! \cdots j_k!} L(z; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, \ldots, j_1). \tag{3.8}
\]

The proof requires an auxiliary construction.

**3.3. Auxiliary polynomials** \(D_{m_{v_1}, \ldots, m_{v_k}}^{n_1, \ldots, n_k}(t)\). Choose now \(k \geq 1\); \(v_k, \ldots, v_1 \in V\), and positive integers \(n_k, \ldots, n_1\). It is convenient to agree that for \(k = 0\) the respective families are empty.

Define inductively polynomials

\[
D_{m_{v_1}, \ldots, m_{v_k}}^{n_1, \ldots, n_k}(t) \in \mathbb{Q}[t]
\]

putting \(D_0^\emptyset = 1\), and

\[
D_{m_{v_1}, \ldots, m_{v_k}, n_k+1}^{n_1, \ldots, n_k}(t) = (1 + \partial_t)^{-1} (D_{m_{v_1}, \ldots, m_{v_k}}^{n_1, \ldots, n_k} \left( \frac{n_1 + \cdots + n_k}{n_1 + \cdots + n_k + 1} \right) t^{m_{v_k} + 1} - 1). \tag{3.9}
\]
where 

\[(1 + \partial_t)^{-1} := \sum_{k \geq 0} (-1)^k \partial_t^k\]

as a linear operator on polynomials.

For example,

\[D_{m_v}^n (t) = (-1)^{m_v - 1} (m_v - 1)! \sum_{j_1 = 0}^{m_v - 1} \frac{(-1)^j_1 t_j}{j_1!}. \quad (3.10)\]

In particular, \(D_{m_v}^n (0) = (-1)^{m_v - 1} (m_v - 1)!\). Furthermore,

\[D_{m_v, m_{v_2}}^{n_1, n_2} (t) = (-1)^{m_v - 1} (m_v - 1)! \sum_{j_1 = 0}^{m_v - 1} \frac{(-1)^j_1}{j_1!} \frac{(1 + \partial_t)^{-(n_1 + n_2)j_1}}{(n_1 + n_2)j_1} = (-1)^{m_v - 1 + m_{v_2} - 1} (m_v - 1)! (m_{v_2} - 1)! \sum_{j_1 = 0}^{m_v - 1} \frac{(-1)^j_1 n_1^{j_1}}{j_1!(n_1 + n_2)j_1} \sum_{j_2 = 0}^{m_{v_2} - 1 + j_1} \frac{(-1)^j_2 t_{j_2}}{j_2!}. \quad (3.11)\]

In particular,

\[D_{m_v, m_{v_2}}^{n_1, n_2} (0) = (-1)^{m_v - 1 + m_{v_2} - 1} (m_v - 1)! (m_{v_2} - 1)! \sum_{j_1 = 0}^{m_v - 1} \frac{(-1)^j_1 n_1^{j_1}}{j_1!(n_1 + n_2)j_1}.\]

The general formula looks as follows:

**3.3.1. Proposition.** We have for \(k \geq 1\):

\[D_{m_v, \ldots, m_{v_k}}^{n_1, \ldots, n_k} (t) = (-1)^{\sum_{i=1}^k (m_i - 1)} \sum_{j_1 = 0}^{m_v - 1} \sum_{j_2 = 0}^{m_{v_2} - 1 + j_1} \cdots \sum_{j_k = 0}^{m_{v_k} - 1 + j_{k-1}} (-1)^{j_k} \times \]

\[\times \frac{(m_v - 1)! (m_{v_2} - 1 + j_1)! \cdots (m_{v_k} - 1 + j_{k-1})!}{j_1! j_2! \cdots j_k!} \times \]

\[\times \frac{1}{n_1^{-j_1}(n_1 + n_2)j_1 - j_2 \cdots (n_1 + \cdots + n_{k-1})j_{k-2} - j_{k-1}(n_1 + \cdots + n_k)j_{k-1}} t^{j_k}. \quad (3.12)\]
**Proof.** We argue by induction on $k$. Assume that (3.12) holds for $k$ and apply the operator at the right hand side of (3.9) to the right hand side of (3.12). Looking for brevity only at the last line of (3.11), we get:

\[
\frac{1}{n_1^{-j_1}(n_1 + n_2)^{j_1-j_2} \cdots (n_1 + \cdots + n_k)^{j_k-1-j_k}(n_1 + \cdots + n_{k+1})^{j_k}} (1+\partial_t)^{-1} t^{m_{v_{k+1}}-1+j_{k}} = \\
= \frac{1}{n_1^{-j_1}(n_1 + n_2)^{j_1-j_2} \cdots (n_1 + \cdots + n_k)^{j_k-1-j_k}(n_1 + \cdots + n_{k+1})^{j_k}} \times \\
\times \sum_{j_{k+1}=0}^{m_{v_{k+1}}-1+j_{k}} (-1)^{j_{k+1}} (-1)^{m_{v_{k+1}}-1+j_{k}} (m_{v_{k+1}} - 1 + j_{k})! t^{j_{k+1}}.
\]

Combining this with (3.12) for $k$, we get (3.12) for $k + 1$.

### 3.4. Proof of Theorem 3.2.

By induction on $k$ we will prove the following formula:

\[
(2\pi i)^{m_{v_1}+\cdots+m_{v_k}} I_{i\infty}^z (\omega_{v_k}, \ldots, \omega_{v_1}) = \\
= \sum_{n_1, \ldots, n_k \geq 1} c_{v_1, n_1} \cdots c_{v_k, n_k} e^{2\pi i (n_1 + \cdots + n_k) z} D_{m_{v_1}, \ldots, m_{v_k}}^{m_{v_1}, \ldots, m_{v_k}} (2\pi i (n_1 + \cdots + n_k) z).
\]

(3.13)

Combining it with (3.12) and (3.3) we will get (3.8).

For $k = 1$ we check (3.13) directly:

\[
(2\pi i)^{m_{v_1}} I_{i\infty}^z (\omega_{v_1}) = (2\pi i)^{m_{v_1}} \int_{i\infty}^z \omega_{v_1} (z_1) = \\
= (2\pi i)^{m_{v_1}} \sum_{n_1=1}^{\infty} c_{v_1, n_1} \int_{i\infty}^z e^{2\pi i n_1 z_1} z_1^{m_{v_1}-1} dz_1.
\]

Putting in the $n_1$-th summand $t = 2\pi i n_1 z_1$, we can rewrite this as

\[
\sum_{n_1=1}^{\infty} c_{v_1, n_1} \frac{1}{n_1^{m_{v_1}}} \int_{-\infty}^{2\pi i n_1 z_1} e^{t^{m_{v_1}-1}} dt.
\]

(3.14)

Since

\[
\int e^{t} P(t) dt = e^{t} (1 + \partial_t)^{-1} P(t) + \text{const},
\]

(3.15)
this is equivalent to (3.13).

The inductive step from \( k \) to \( k + 1 \) is similar: assuming (3.13)_k, we have

\[
(2\pi i)^{m_{v_k} + \cdots + m_{v_{k+1}}} I_{i\infty}^z (\omega_{v_{k+1}}, \ldots, \omega_{v_1}) =
\]

\[
(2\pi i)^{m_{v_{k+1}}} \sum_{n_{k+1} = 1}^{\infty} c_{v_{k+1}, n_{k+1}} \int_{i\infty}^z e^{2\pi in_{k+1}z_k} z_k^{m_{v_{k+1}} - 1} \times
\]

\[
\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{c_{v_1, n_1} \cdots c_{v_k, n_k} e^{2\pi i(n_1 + \cdots + n_k)z_k}}{n_1^{m_{v_1}} (n_1 + n_2)^{m_{v_1}} \cdots (n_1 + \cdots + n_k)^{m_{v_k}}} D_{m_{v_1}, \ldots, m_{v_k}} (2\pi i(n_1 + \cdots + n_k) z_k) dz_k =
\]

\[
(2\pi i)^{m_{v_k+1}} \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{c_{v_1, n_1} \cdots c_{v_k, n_k} c_{v_{k+1}, n_{k+1}}}{n_1^{m_{v_1}} (n_1 + n_2)^{m_{v_1}} \cdots (n_1 + \cdots + n_k)^{m_{v_k}}} \times
\]

\[
\int_{i\infty}^z e^{2\pi i(n_1 + \cdots + n_k + n_{k+1}) z_k} D_{m_{v_1}, \ldots, m_{v_k}} (2\pi i(n_1 + \cdots + n_k) z_k) z_k^{m_{v_k+1} - 1} dz_k.
\]

Putting here \( t = 2\pi i(n_1 + \cdots + n_{k+1}) z_k \), we can rewrite the last integral as

\[
\frac{1}{(2\pi i)^{m_{v_{k+1}}}} \frac{1}{(n_1 + \cdots + n_{k+1})^{m_{v_{k+1}}}} \int_{i\infty}^z e^t D_{m_{v_1}, \ldots, m_{v_k}} \left( \frac{n_1 + \cdots + n_k}{n_1 + \cdots + n_{k+1}} t \right) t^{m_{v_{k+1}} - 1} dt
\]

that is,

\[
\frac{1}{(2\pi i)^{m_{v_{k+1}}}} \frac{1}{(n_1 + \cdots + n_{k+1})^{m_{v_{k+1}}}} e^{2\pi i(n_1 + \cdots + n_{k+1}) z} \times
\]

\[
(1 + \partial_t)^{-1} D_{m_{v_1}, \ldots, m_{v_k}} \left. \left( \frac{n_1 + \cdots + n_k}{n_1 + \cdots + n_{k+1}} t \right)^{m_{v_{k+1}} - 1} \right|_{t = 2\pi i(n_1 + \cdots + n_{k+1}) z}.
\]

Substituting this into (3.16), we finally obtain (3.13) and (3.12).

**3.5. The limit** \( z \to 0 \). We can try to get an expression for

\[
I_{i\infty}^0 (\omega_{v_k}, \ldots, \omega_{v_1})
\]

as a (linear combination of) multiple Dirichlet series, by formally putting \( z = 0 \) in the r.h.s. of (3.8). However, we will find out that this cannot be done automatically for a certain range of values of \((j_k, j_{k-1})\), namely, for \( j_k \geq m_{v_k} + j_{k-1} - 1 - kC\): cf. Case 3 at the end of subsection 3.1.2.
To solve this problem, we will have for the first time to assume that $z^{1-m} \omega_v(z)$ are of cusp modular type, say, for the group $\Gamma_0(N)$, or for any modular subgroup which is normalized by the involution $z \mapsto gz$,

$$g = g_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$ 

We can then apply Proposition 2.4 which we reproduce and slightly augment:

**3.5.1. Proposition.** Assume that $\omega_V$ as above is a basis of a space of 1-forms invariant with respect to $g_N$. Then

$$J^0_{i\infty}(\omega_V) = (g_N \ast (J_{i\infty}^{\sqrt{N}}(\omega_V)))^{-1} J_{i\infty}^{\sqrt{N}}(\omega_V).$$

Replacing the coefficients of the formal series at the r.h.s of (3.17) by their (convergent) representations via multiple Dirichlet series (3.8), we get such representations for $I^0_{i\infty}(\omega_{v_1}, \ldots, \omega_{v_k})$.

**3.5.2. Main application.** If we fix a modular subgroup normalized by $g_N$, this proposition becomes applicable to any family of cusp forms of the type $\omega_i(z)z^{m-1}$, $m \geq 1$, where $\omega_i(z)$ runs over a basis of the space of forms of a fixed weight $2r$, and $m$ runs over $[1, 2r - 1]$ (cf. (2.4)). Moreover, we can mix different weights, that is, take a finite union of such families.

Passing to a different basis of such a space, we may even assume that $(\omega_v)$ consists of eigenforms for $g_N$: $g^*_N(\omega_v) = \varepsilon_v \omega_v$, $\varepsilon_v = \pm 1$, for all $s \in V$.

Coefficients of $J_{i\infty}^{\sqrt{N}}$ are the series (3.3) at $z = \frac{i}{\sqrt{N}}$.

§4. Shuffle relations between multiple Dirichlet series

**4.1. Notation.** In this section, we will consider (formal) multiple Dirichlet series of a special form generalizing expressions (3.4), and deduce bilinear relations between them generalizing the well known harmonic shuffle relations involving shuffles with repetitions.

Each such series will depend on a set of coefficients data $C$ and several complex or formal arguments $s_i$. Here are precise definitions. Let $k \geq 1$ be a natural number.

**4.1.1. Definition.** (i) Coefficients data $C$ of depth $k$ is a family of numbers $c_{n,m}^{(j,i)}$ indexed by two pairs of integers satisfying $j > i \geq 0$, $j \leq k$, and $n > m \geq 0$. 

The multiple Dirichlet series associated with $C$ and arguments $s_1, \ldots, s_k$ is

$$L_C(s_1, \ldots, s_k) := \sum_{0=u_0<u_1<\cdots<u_k \in \mathbb{Z}} \prod_{k \geq j > i \geq 0} c_{u_j, u_i}^{(j, i)} u_1^{s_1} u_2^{s_2} \cdots u_k^{s_k}$$ (4.1)

4.1.2. Examples. (a) Assume that $c_{n, m}^{(j, i)} = 1$ if $m > 0$ or $i > 0$ and put $c_{n, 0}^{(j, 0)} = a_n^{(j)}$. Then

$$L_C(s_1, \ldots, s_k) = \sum_{0<u_1<\cdots<u_k \in \mathbb{Z}} a_1^{(1)} a_2^{(2)} \cdots a_k^{(k)} u_1^{s_1} u_2^{s_2} \cdots u_k^{s_k}$$ (4.2)

is an usual multiple Dirichlet series.

(b) Define $c_{v, n}$ as in 3.1, and choose $v_1, \ldots, v_k \in V$ as in 3.1.1. Construct the coefficients data $C$ putting

$$c_{v, n}^{(j, j-1)} := c_{v, n-m},$$

and $c_{n, m}^{(j, i)} = 1$ otherwise. Then $L_C(m_{v_1} + j_0 - j_1, \ldots, m_{v_k} + j_{k-1} - j_k)$ becomes the formal series (3.4) if we redenote $u_j = n_1 + \cdots + n_j$.

4.1.3. Shuffles and a composition of the coefficients data. Let $C = (c_{n, m}^{(j, i)})$ and $D = (d_{n, m}^{(j, i)})$ be two data of depths $k$ and $l$ respectively. A $(k, l, p)$-shuffle with repetitions is a pair of strictly increasing maps $\sigma = (\sigma_1, \sigma_2)$,

$$\sigma_1 : [0, k] \to [0, p], \quad \sigma_2 : [0, l] \to [0, p]$$

satisfying the following conditions:

$$\sigma_1(0) = \sigma_2(0) = 0, \quad \sigma_1([0, k]) \cup \sigma_2([0, l]) = [0, p].$$ (4.4)

It follows that max $(k, l) \leq p \leq k + l$. We will say that the $\sigma$-multiplicity of $j \in [0, p]$ is one, if $j \notin \sigma_1([0, k]) \cap \sigma_2([0, l])$. Otherwise the $\sigma$-multiplicity of $j \in [0, p]$ is two. In particular, the $\sigma$-multiplicity of 0 is two.

Given such $C$, $D$ and $\sigma = (\sigma_1, \sigma_2)$, we will define the third coefficients data $E = (e_{n, m}^{(j, i)})$ of depth $p$ which we denote $E = C \ast_{\sigma} D$. Choose $j, i$ with $p \geq j > i \geq 0$. We have the following set of mutually exclusive and exhausting alternatives $(A)_1$, $(A)_2$, (B), (C).
(A) Assume that both $j$ and $i$ have multiplicities one.

(A)\textsubscript{1}. Both $j$ and $i$ belong to the image of one and the same $\sigma_a$ with $a = 1$ or $a = 2$. Then we put

$$e_{n,m}^{(j,i)} := c_{n,m}^{(\sigma^{-1}_a(j), \sigma^{-1}_a(i))} \text{ for } a = 1,$$

and

$$e_{n,m}^{(j,i)} := d_{n,m}^{(\sigma^{-1}_a(j), \sigma^{-1}_a(i))} \text{ for } a = 2.$$

(A)\textsubscript{2}. Assume that $j$, resp. $i$, belongs to the image of $\sigma_a$, resp. $\sigma_b$, with $a \neq b$. Then we put

$$e_{n,m}^{(j,i)} := 1.$$

(B) Assume that exactly one of $j$, $i$ has multiplicity two. Then there exists only one value $a = 1$ or 2 such that $j$ and $i$ belong to the image of $\sigma_a$. We put then as in the case (A)\textsubscript{1}

$$e_{n,m}^{(j,i)} := c_{n,m}^{(\sigma^{-1}_a(j), \sigma^{-1}_a(i))} \text{ for } a = 1,$$

and

$$e_{n,m}^{(j,i)} := d_{n,m}^{(\sigma^{-1}_a(j), \sigma^{-1}_a(i))} \text{ for } a = 2.$$

(C) Assume that both $i$ and $j$ have multiplicities two. Then we put

$$e_{n,m}^{(j,i)} := c_{n,m}^{(\sigma^{-1}_a(j), \sigma^{-1}_a(i))} d_{n,m}^{(\sigma^{-1}_a(j), \sigma^{-1}_a(i))}.$$

4.1.4. Shuffles and a composition of the arguments. Let $s := (s_1, \ldots, s_k)$ and $t := (t_1, \ldots, t_l)$ be arguments for the data $C$ and $D$ as above, and $\sigma$ a $(k, l, p)$–shuffle as above. We define $s +_\sigma t := (r_1, \ldots, r_p)$ as follows.

If $i$ has multiplicity one and is covered by $\sigma_1$, resp. $\sigma_2$, then $r_i := s_{\sigma_1^{-1}(i)}$, resp. $r_i := t_{\sigma_2^{-1}(i)}$.

If $i$ has multiplicity two, then $r_i := s_{\sigma_1^{-1}(i)} + t_{\sigma_2^{-1}(i)}$.

We can now state the main result of this section.

4.2. Theorem. Let $C$, resp. $D$, be some coefficients data of depths $k$, resp. $l$, as above. Then we have

$$L_C(s) \cdot L_D(t) = \sum_\sigma L_{C*\sigma D}(s +_\sigma t) \quad (4.5)$$
where the summation is taken over all \((k, l, p)\)-shuffles with repetitions.

**Proof.** Consider a term of the series (4.1) corresponding to \((u_0 = 0, u_1, u_2, \ldots, u_k)\) and a term of the series \(L_D(t)\) corresponding to, say, \((w_0 = 0, w_1, w_2, \ldots, w_l)\). This pair of terms determines a unique \((k, l, p)\)-shuffle \((\sigma_1, \sigma_2)\), where \(p\) is the cardinality of the union of sets

\[
\{u_1, u_2, \ldots, u_k\} \cup \{w_1, w_2, \ldots, w_l\} := \{q_1, \ldots, q_p\}.
\]

Namely, we may and will assume that \(q_0 = 0 < q_1 < \cdots < q_p\). Then \(\sigma_1(i) = j\) if \(u_i = q_j\), and \(\sigma_2(i) = j\) if \(w_i = q_j\).

Group together all pairwise products corresponding to one and the same shuffle, and denote the resulting sum by \(L_\sigma\).

The denominator of one such a product will obviously be \(q_1^{r_1} \cdots q_p^{r_p}\) where \(r = s +_\sigma t\). Moreover, knowing such a denominator, we uniquely reconstruct the two terms from \(L_C(s)\) and \(L_D(t)\), from which it was produced, at least if \(s, t\) take generic values so that in the family \(\{s_a, t_b, s_a + t_b\}\) all terms are pairwise distinct. Finally, all possible sequences \(q_0 = 0 < q_1 < \cdots < q_p\) will occur.

To prove that \(L_\sigma = L_{C*_{\sigma}D}(s +_\sigma t)\), it remains to check that the numerator of such a product will be as predicted by (4.5), in other words, that

\[
\prod_{p \geq j > i \geq 0} e_{q_j, q_i}^{(j, i)} = (\cdot) \prod_{k \geq j > i \geq 0} e_{u_j, u_i}^{(j, i)} \prod_{l \geq j > i \geq 0} d_{w_j, w_i}^{(j, i)}
\]

if \(e_{n, m}^{(j, i)}\) are defined as in 4.1.3.

This is straightforward, although somewhat tedious.

**4.3. Concluding remarks.** It would be interesting to describe some nontrivial spaces of Dirichlet series containing periods of cusp forms, closed with respect to the series shuffle relations, and consisting entirely of periods in the sense of [KonZa].

Regarding shuffle relations themselves, motivic philosophy predicts that they should be obtainable by standard manipulations with integrals. For the harmonic shuffle relations between multiple zeta values, A. Goncharov established this in the Ch. 9 of [Go6] (for convergent integrals), and in 7.5 of [Go6] elaborating the last page of [Go4] (for regularized integrals). Conversely, integral shuffle relations can be deduced from harmonic ones: see [Go5], Ch. 2.
§5. Iterated Eichler–Shimura and Hecke relations

5.1. Eichler–Shimura relations for iterated integrals. In this subsection, we take for $X$ the upper half–plane $H$ and for $(\omega_V)$ a family of 1–forms of cusp modular type (see 2.1.1) spanning a finite–dimensional linear space stable with respect to the modular transformations

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

5.1.1. Proposition. With these assumptions, we have

$$\sigma_*(J_{i\infty}^1(\omega_V)) \cdot J_{i\infty}^0(\omega_V) = 1.$$  \hspace{1cm} (5.1)

$$\tau_2^*\sigma_*(J_{i\infty}^i(\omega_V)) \cdot \tau_*^\tau(J_{i\infty}^i(\omega_V)) \cdot J_{i\infty}^1(\omega_V) = 1.$$  \hspace{1cm} (5.2)

Proof. From (1.15) we get

$$\sigma_*(J_{i\infty}^0(\omega_V)) = J_{i\sigma(0)}^\sigma(\omega_V) = J_{i\infty}^0(\omega_V) = J_{i\infty}^i(\omega_V)^{-1},$$

which shows (5.1).

Similarly, $\tau$ transforms $(0, i\infty)$ into $(i\infty, 1)$, then to $(1, 0)$, and according to (1.9)

$$J_1^0(\omega_V) \cdot J_1^1(\omega_V) \cdot J_{i\infty}^i(\omega_V) = 1$$  \hspace{1cm} (5.3)

which is the same as (5.2).

Notice that some care is needed in establishing (5.3): the geodesic triangle with vertices $0, i\infty, 1$ should be first replaced by a sequence of geodesic hexagons lying entirely in $H$ and cutting the corners of the triangle, and then it must be checked that in the limit the hexagon relation replacing (5.3) tends to (5.3). This is routine for cusp modular 1–forms, cf. 2.1.3.

We now pass to the relations involving Hecke operators.

5.2. Hecke operators. In this subsection, $V$ denotes a finite set, $\omega_v = f_v(z)dz$ a family of modular forms of weight two, and $p_v$ a family of primes, both indexed by $v \in V$. Moreover, we assume that $T_{p_v}, \omega_v = \lambda_v \omega_v, \lambda_v \in \mathbb{C}$, where $T_{p_v}$ is the Hecke operator

$$T_{p_v} := \begin{pmatrix} p_v & 0 \\ 0 & 1 \end{pmatrix} + \sum_{b=0}^{p_v-1} \begin{pmatrix} 1 & b \\ 0 & p_v \end{pmatrix} = p_v + \sum_{b=0}^{p_v-1} h(p_v, b).$$  \hspace{1cm} (5.4)
Put $U := V \coprod V'$ where $V'$ is another copy of the indexing set $V$, and for $v' \in V'$ corresponding to $v \in V$ put $\omega_{v'} := (p_v)^*(\omega_v)$. Let $\omega_U$ be the family consisting of all $\omega_v$ and $\omega_{v'}$. When we consider formal series of the type $J_0^\infty(\omega_U)$ as in §1, we denote the variables corresponding to $V$, resp. $V'$, by $A_v$, resp. $A_{v'}$.

Denote by $W$ the set of pairs $w = (v, b)$ where $v \in V$ and $b \in [0, p_v - 1]$. Let $\omega_W$ be the family consisting of $\omega_{(v, b)} := h(p_v, b)^*(\omega_v)$. When we consider formal series of the type $J_0^\infty(\omega_W)$, we denote the variables corresponding to $w$, by $B_w$ or $B_{(v, b)}$.

Define the following two continuous homomorphisms of rings of formal series:

\[
\begin{align*}
 l : \mathbb{C}\langle\langle A_U \rangle\rangle &\rightarrow \mathbb{C}\langle\langle A_V \rangle\rangle : \quad l(A_v) := \lambda_v A_v, \quad l(A_{v'}) := -A_v, \\
 r : \mathbb{C}\langle\langle B_W \rangle\rangle &\rightarrow \mathbb{C}\langle\langle A_V \rangle\rangle : \quad r(B_{(v, b)}) := A_v.
\end{align*}
\]

5.3. Theorem. We have

\[ l(J_0^\infty(\omega_U)) = r(J_0^\infty(\omega_W)). \]

Proof. We will check that

\[ l(J_0^\infty(\omega_U)) = J_0^\infty((\lambda_v \omega_v - \omega_{v'})) \]

whereas

\[ r(J_0^\infty(\omega_W)) = J_0^\infty((\sum_{b=0}^{p_v-1} \omega_{(v, b)})) \]

where at the right hand sides we consider both families as indexed by $V$. Since from (5.4) and the definitions above we obtain for each $v \in V$

\[ \lambda_v \omega_v - \omega_{v'} = \sum_{b=0}^{p_v-1} \omega_{(v, b)}, \]

this will prove the theorem.

We have

\[ J_0^\infty(\omega_U) = \sum_{n=0}^{\infty} \sum_{(u_1, \ldots, u_n) \in U^n} A_{u_n} \ldots A_{u_1} J_0^\infty(\omega_{u_n}, \ldots, \omega_{u_1}). \]
Consider one summand in (5.11). In the sequence \((u_1, \ldots, u_n)\) there are several, say \(0 \leq k \leq n\), elements \(v_i \in V\) and the remaining \(n - k\) elements \(v'_j \in V'\). Application of \(l\) eliminates all primes in the subscripts of \(A_{u_n} \ldots A_{u_1}\) and produces a monomial in \(A_v\); besides, it multiplies this monomial by \((-1)^{n-k} \prod \lambda_{v_i}\). Hence the coefficient at any monomial \(A_{v_n} \ldots A_{v_1}\) in \(l(J^0_{i\infty}(\omega_U))\) can be written as
\[
\sum_{S \subseteq \{1, \ldots, n\}} (-1)^{n-|S|} \left( \prod_{i \in S} \lambda_{v_i} \right) I^0_{i\infty}(\pi_S(\omega_{v_n}, \ldots, \omega_{v_1}))
\]  
(5.12)
where the operator \(\pi_S\) replaces \(v_j\) by \(v'_j\) whenever \(j \notin S\).

On the other hand, the similar coefficient in \(J^0_{i\infty}((\lambda_v \omega_v - \omega_{v'}))\) is
\[
I^0_{i\infty}(\lambda_{v_n} \omega_{v_n} - \omega'_{v_n}, \ldots, \lambda_{v_1} \omega_{v_1} - \omega'_{v_1})
\]
which obviously coincides with (5.12) because the iterated integrals are polylinear in \(\omega\). This proves (5.8).

The check of (5.9) is similar. We have
\[
J^0_{i\infty}(\omega_W) = \\
= \sum_{((v_n, b_n), \ldots, (v_1, b_1)) \in W^n} B(v_n, b_n) \cdots B(v_1, b_1) I^0_{i\infty}(\omega(v_n, b_n), \ldots, \omega(v_1, b_1)).
\]
Application of \(r\) produces a series in \((A_{v})\) whose coefficient at \(A_{v_n} \ldots A_{v_1}\) equals
\[
\sum_{(b_1, \ldots, b_n)} I^0_{i\infty}(\omega(v_n, b_n), \ldots, \omega(v_1, b_1)).
\]
This is the same as the respective coefficient at the r.h.s. of (5.9).

§6. Differentials of the third kind
and generalized associators

6.1. Normalized horizontal sections. In this subsection, following [Dr2], we will define and study solutions of the differential equation (1.4), \(dJ^z(\Omega) = \Omega(z)J^z(\Omega)\) in the case when \(\Omega = \sum A_{v} \omega_{v}\) may have a logarithmic singularity at a point \(a\) so that it cannot be normalized by the condition \(J^a(\Omega) = 1\) and in fact cannot be defined by the series (1.3).
We start with a local situation. Put
\[ r_{v,a} := \text{res}_a \omega_v, \quad R_a := \text{res}_a \Omega = \sum_v r_{v,a} A_v. \] (6.1)

The normalized solution will depend on the choice of a local parameter \( t_a \) at \( a \), and a branch of logarithm \( \log t_a \).

Let \( U \) be a disc around \( a \) uniformized by \( t_a \). Denote by \( \log t_a \) the branch of the logarithm in \( U \) which is real on \( \text{Im} t_a = 0, \text{Re} t_a > 0 \). Delete from \( U \) a cut from \( a \) to the boundary which does not intersect the latter interval and denote \( U' \) the remaining domain. Write \( t_a^{R_a} \) for \( e^{R_a \log t_a} \). It is a formal series in \( A_v \) with coefficients which are holomorphic functions near \( a \) in \( U' \). Assume that outside of \( a \), all \( \omega_v \) are regular in \( U' \).

6.1.1. Definition. A \( \nabla\Omega \)-horizontal section \( J \) in \( U' \) is called normalized at \( a \) (with respect to a choice of \( t_a \)) if it is of the form \( J = K \cdot t_a^{R_a} \), where \( K \) can be extended to a holomorphic section in some neighborhood of \( a \) in \( U \) which takes the value \( 1 \) at \( a \).

We will see that this definition produces a version of \( J_a^z(\Omega) \). In fact, we get precisely \( J_a^z(\Omega) \), if \( R_a = 0 \) so that \( t_a^{R_a} = 1 \).

6.1.2. Proposition. For any \( a \) and \( t_a \) as above, there exists a unique normalized at \( a \) local section holomorphic in \( U' \).

Proof. In the course of proof, we will be considering only one point \( a \), so we will omit it in the notation for brevity and write \( R, t, r_v \) etc in place of former \( R_a, t_a, r_{v,a} \).

The equation \( \nabla\Omega(K \cdot t^R) = 0 \) is equivalent to
\[ dK = \Omega' K + t^{-1}[R, K] dt \] (6.2)
where
\[ \Omega' := \Omega - R \frac{dt}{t} = \sum_v r_v \nu_v A_v. \]

We look for a solution to (6.2) of the form
\[ K = 1 + \sum_{n=1}^{\infty} \sum_{(v_1, \ldots, v_n) \in V^n} f_{v_1, \ldots, v_n} A_{v_1} \cdots A_{v_n} \]
where $f_{v_1,\ldots,v_n}$ must be holomorphic functions defined in some common neighborhood of $a$ in $U$ and vanishing at $a$. From (6.2) we find $df_v = \nu_v$ so that the only choice is $f_v(z) := \int_a^z \nu_v$. Notice that all $\nu_v$ are regular in a common neighborhood of $a$ uniformized by $t_a$. We will see that in this neighborhood all other coefficients can be defined as well.

If $f_{v_1,\ldots,v_{n-1}}$ with required properties are defined for some $n-1 \geq 1$, we find from (6.2)

$$df_{v_1,\ldots,v_n} = \nu_{v_1} f_{v_2,\ldots,v_n} + t^{-1}(r_{v_1} f_{v_2,\ldots,v_n} - f_{v_2,\ldots,v_{n-1}} r_{v_n}) dt.$$  

By the inductive assumption, the r.h.s. is well defined and regular at $a$, so integrating it from $a$ to $z$ we get $f_{v_1,\ldots,v_n}(z)$.

6.2. Scattering operators. Let now $a, b$ be two points where $\Omega$ may have logarithmic singularities. Choose $t_a, t_b$ as above, construct the neighborhoods $U_a, U_b$ and neighborhoods with deleted cuts $U'_a, U'_b$ in which we have the holomorphic normalized horizontal sections $J_a, J_b$. Now embed $U'_a, U'_b$ into a connected simply connected domain $W$ to which both $J_a$ and $J_b$ can be analytically extended. Clearly, they are invertible elements of $O_x\langle\langle A_V \rangle\rangle$ at almost all points $x \in W$. Put

$$\tilde{J}_b^a = J_a^{-1} J_b.$$  

As in the proof of Proposition 1.2, one sees that $\tilde{J}_b^a \in C\langle\langle A_V \rangle\rangle$. Borrowing the physics terminology, we can call this transition element the scattering operator.

I added twiddle in the notation in order to remind the reader that $\tilde{J}_b^a$, besides $\Omega$, depends on $t_a$ and $t_b$ as well, if at least one of the residues $R_a, R_b$ does not vanish. This dependence however is pretty mild. Let $J'_a = K'(t'_a)^{R_a}$ be another horizontal section normalized with respect to some $t'_a$. Denote by $\tau_a \in C^*$ the value of $t'_a/t_a$ at $a$, and let $t'_a = T_a \cdot t_a \tau_a$, $T_a(a) = 1$.

6.2.1. Proposition. (i) We have

$$J'_a = J_a \cdot \tau_a^{R_a}, \quad K = K' \cdot T^{R_a}. \quad (6.4)$$

(ii) Therefore, after replacing two uniformizing parameters $t_a, t_b$ by $t'_a, t'_b$, we get

$$\tilde{J}_b'^a = \tau_a^{-R_a} \tilde{J}_b^a \tau_b^{R_b}. \quad (6.5)$$
Proof. (i) We have

\[ J_a' = K'(t_a')^R_a = K' \cdot T_a^R_a \cdot (t_a)^R_a \cdot R_a \]

Since \( R_a \in C \langle \langle A \rangle \rangle \), and this section is invertible, \( K' \cdot T_a^R_a \cdot (t_a)^R_a \) is \( \nabla \Omega \)-horizontal as well, and since \( K' \cdot T_a^R_a \) at \( a \) equals 1, it is normalized, so \( K' \cdot T_a^R_a \cdot (t_a)^R_a = J_a \), which proves (6.4).

(ii) Similarly, from \( J_b' = J_a' \tilde{J}_b^a \) and (6.4) we get

\[ J_b = J_a \tau_a^R_a \tilde{J}_b^a \tau_b^{-R_b} \]

which together with (6.3) proves (6.5).

6.3. Example: Drinfeld’s associator. Let \( X = \mathbb{P}^1(\mathbb{C}) \), \( V = \{0, 1\} \),

\[ \omega_0 = \frac{1}{2\pi i} \frac{dz}{z}, \quad \omega_1 = \frac{1}{2\pi i} \frac{dz}{z-1}. \]

Then

\[ \Omega = A_0 \omega_0 + A_1 \omega_1 \]

has poles at 0, 1, \( \infty \) with residues \( A_0/2\pi i \), \( A_1/2\pi i \), \( -(A_0+A_1)/2\pi i \) respectively. Put \( t_0 = z, t_1 = 1 - z \). Then \( \tilde{J}_0^1 \) in our notation is the Drinfeld associator \( \phi_{KZ}(A_0, A_1) \) from §2 of [Dr2].

6.4. Generalized associators. An essential feature of the the last example is that \( \omega_v \) are global logarithmic differentials on the compact Riemannian surface \( \mathbb{P}^1 \).

Generally, let \( X \) be such a surface, and \( (a_i) \) a finite set of \( N \) points on it. The dimension of the space of global logarithmic differentials with poles in this set, \( \sum c_i \log f_i \), where \( c_i \in \mathbb{C}, f_i \) are meromorphic on \( X \), is bounded by \( N - 1 \). It achieves the maximum value \( N - 1 \) iff the difference of any two points \( a_i - a_j \) is torsion in the divisor class group. I will call such a set \( (a_i) \) logarithmic. The supply of logarithmic set depends on the genus of \( X \).

a) Genus zero. Any finite set of points is logarithmic. The respective iterated integrals in this case include multiple polylogarithms introduced in [Go1]. In fact, as Goncharov remarked, general iterated integrals can be reduced to multiple polylogarithms.

b) Genus one. In this case we can take any finite set of points of finite order, for example, the subgroup of all points of a given order \( M \).
For a general subset of points, Goncharov found a Feynman integral presentation (in the sense of the last section of [Go2]) of the respective real periods. His formulas involve the generalized Kronecker–Eisenstein series and can be considered as an elliptic version of multiple zeta values. A similar problem is addressed in the recent work of A. Levin and G. Racinet.

c) Higher genera. If genus of $X$ is $> 1$, the order of such a logarithmic set is bounded. The most interesting explicitly known examples are modular curves and cusps on them, cf. [Dr1], [El].

Notice, that the initial Drinfeld’s setting is modular as well: $\mathbb{P}^1$ with three marked points “is” the modular curve $\Gamma_0(4) \setminus \mathcal{H}$ together with its cusps.

According to Deligne and Elkik ([El]), a set of points is logarithmic iff the mixed Hodge structure on $H^1(X^\circ, \mathbb{Q})$ (where $X^\circ$ is the complement to ) is split that is, the direct sum of pure Hodge structures.

6.4.1. Definition. Let $X$ be a compact Riemannian surface, and $(a_i)$ a logarithmic set of points on it. Then any scattering operator of the form $\widetilde{J}^{a_i}_{a_j}$ is called a generalized associator.

6.5. Relations between scattering operators. Three types of relations established for $J^z_a(\omega_V)$ in §1 are extended below to the case of the general scattering operators.

6.5.1. Group–like property. We have

$$\Delta(\widetilde{J}^b_a) = \widetilde{J}^b_a \otimes \widetilde{J}^b_a$$

where $\Delta$ is defined in 1.4.1.

To see this, notice that $\Delta(J_a)$, by definition, is the series which is obtained from $J_a$ by replacing each $A_v$ with $B_v := A_v \otimes 1 + 1 \otimes A_v$. Hence $\Delta(R_a) = R_a \otimes 1 + 1 \otimes R_a$ is the residue of $\Delta(\Omega)$ at $a$. Therefore $\Delta(J_a)$ is the normalized $\nabla_{\Delta(\Omega)}$–horizontal section in the ring of formal series in $B_v$, and $\Delta(\widetilde{J}^b_a)$ is the respective scattering operator.

On the other hand, $J_a \otimes J_a$ satisfies the same equation $d(J_a \otimes J_a) = (\Omega \otimes 1 + 1 \otimes \Omega)(J_a \otimes J_a)$ and clearly is as well normalized at $t_a$. Hence the passage from $J_a \otimes J_a$ to $J_b \otimes J_b$ is governed by the same scattering operator, this time represented as the r.h.s. of (6.6).

6.5.2. Cycle identities. Let $\gamma$ be a closed oriented contractible contour in $U$, inside which there are no singularities of $\Omega$. Let $a_1, \ldots, a_n$ be points along this
contour (cyclically) ordered compatibly with orientation. Then

$$\tilde{J}_{a_2}^{a_1} \tilde{J}_{a_3}^{a_2} \ldots \tilde{J}_{a_n}^{a_{n-1}} \tilde{J}_{a_1}^{a_n} = 1. \quad (6.7)$$

Of course, in this statement we assume that at each point \(a_i\) one and the same local parameter \(t_{a_i}\) and one and the same branch of its logarithm is used for the definition of the two relevant normalized sections corresponding to the incoming and outcoming segments of the contour. Otherwise the relevant \(\tau^R\) factors as in (6.5) must be inserted.

6.5.3. Functoriality. Let \(g\) be an automorphism of \(X\) compatibly acting upon all the relevant objects, with the possible exception of parameters \(t_a\), and transforming the space spanned by \(\omega_v\) into itself. Define \(g_*\) as in 1.4. Then

$$\tilde{J}_{g(a)}^{b} = \tau_{a}^{-R_a} g_* (\tilde{J}_{a}^{b}) \tau_{b}^{R_b}. \quad (6.8)$$

where the \(\tau^R\) factors account for the passage from \(t_a\), resp. \(t_b\), to \(g^*(t_a)\), resp. \(g^*(t_a)\).

The proof is essentially the same as in 1.4 and 1.5.

6.6. Example: Drinfeld associator revisited. If we treat Drinfeld’s setup as the \(\Gamma_0(4)\)-modular curve, lift it to \(H\) and apply to the respective family of scattering operators the Eichler–Shimura relations (5.1), (5.2) (following from the cycle identities of lengths 2 and 3, and functoriality), we will get the duality and the hexagonal relations. Of course, this is how they were deduced in the first place, with \(\sigma, \tau\) pushed down to \(\mathbf{P}^1\) rather than everything else lifted to \(H\).

It seems very probable that the somewhat mysterious relationship between the double logarithms at roots of unity and the modular complex discovered in [Go3] can be explained in the same way.

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