The Piecewise Collocation Solution Of Second Kind
Fredholm Integral Equations By Using Quarter-Sweep Iteration

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Abstract. In this paper, a piecewise collocation discretization scheme based on the piecewise constant approximation with the concept of quarter-sweep Jacobi (QSJ) iteration is discussed in solving the linear Fredholm integral equations of second kind. By considering the piecewise approximation equations, the generated linear system has been constructed with its large scale coefficient matrix. The purpose of this quarter-sweep iteration concept is to reduce the computational complexity of the linear system. For the purpose of comparison, the formulation and implementation of full-sweep Jacobi (FSJ), half-sweep Jacobi (HSJ) and QSJ iterative methods are also included. The results of these three proposed methods showed that the QSJ method is better than others Jacobi iteration family.

1. Introduction
In the recent studies, there are many authors that have chosen the Fredholm integral equations as their research study since the integral equations is quite famous among the mathematician. Nevertheless, the integral equations regularly been used in many fields and it particularly can be used in applied mathematics, science, engineering, geophysics, electricity, magnetism and kinetic theory [1]. Based on previous studies, there are three types of linear Fredholm integral equations which are the first kind, second kind and third kind [2-4]. These three types were introduced by a few authors but commonly, the second kind of integral equations is repeatedly used. The following is second kind Fredholm integral equations

\[ U(t) + \lambda \int_{a}^{\phi} K(t, y)U(x)dx = g(t), \quad x \in [a, \phi] \] (1)

As mentioned from the previous paragraph, many researchers have pay attention to solve the Fredholm integral equation of second kind. Referring to equation (1), we discuss briefly about the integral equation of second kind. The function of \( U \) is the unknown function that will be determined in this study, while the function \( g \) is actually the given function [5-6]. Besides these two functions, the parameter of \( \lambda \) plays as a crucial rule in order to get the best accuracy of the solutions and it should be the non-zero values. In this study, the types of kernel need to be alert thoroughly as there are a few types of kernel that can be introduced in the numerical calculation. Based on equation (1), the kernel used in this paper is
a smooth kernel. Smooth kernel was introduced by Jacoby, (1968) which has eliminated the difficulty caused from the discontinuous line of the kernel itself [7].

According to a few studies there are several types of kernel such as Hankel type, weakly singular and potential kernel [8, 9, and 10]. Smooth kernel was introduced by Jacoby, (1968) which has eliminated the difficulty caused from the discontinuous line of its kernel itself.

We proposed the Quarter-Sweep piecewise collocation discretization scheme based on the Quarter-Sweep piecewise constant function for constructing the corresponding approximation equation. Next discussion of this paper is presented with an application of the collocation method that is going to be an approximation equation based on the combination of collocation with piecewise constant polynomial function. Means that, with the concept of the quarter-sweep Jacobi (QSJ) iteration may be used in solving the linear system which is generated from the discretization process over problem (1). It can be observed that the generated linear system has its coefficient matrix which is large scale and dense.

2. Derivative of quarter-sweep piecewise constant function.

This section will convey the explanation of which the quarter-sweep concept is used to construct the approximation equation of Fredholm integral of second kind by using the polynomial piecewise constant collocation method. We will briefly explain the general ideas of the implementation of quarter-sweep iteration concept later.
Figure 1. (a), (b) and (c) are the cell center finite grid of full, half and quarter-sweep cases respectively at $N$.

The limit of the interval is defined as $I = [a, \phi]$, $t \in I$ with the subinterval, $N$. Then we calculate the length of the subinterval with $h = \frac{\phi - a}{N}$. Also that, we define the quarter-sweep collocation node points as follows

$$t_{i-2} = a + (i - 2.0)h, \; i = 4,8,12,...,N$$

(2)

Figure 1 shows three cases on the finite network grid in which each type has its own characteristics. Referring to Figure 1(c), this finite grid can be categorized as the network of QuarterSweep cell-centred in which the node points of type $\bullet$ will be used during the implementation of the Quarter-Sweep iteration. It means that only one quarter of all node points can be considered. In this study, the first step calculates the approximation value of $U_{(j-2)}$, $i = 4,8,12,...,N$ until the convergence test can be used while the remaining node points of different types will be calculated again by using direct method. In addition, this step is also similar with the half-sweep iteration process [11]. Besides that, the special behaviour of Quarter-Sweep is used to reduce the computational complexity and also to increase the speed up of convergence rate [12].

Before generating a linear system, the concept of quarter-sweep iteration needs to be imposed for constructing approximation function. By utilizing the quarter-sweep iteration concept and referring to Figure 1 (c), we introduce the general form of quarter-sweep constant functions as follows [13].

$$T_{i-2}(t_{j-2}) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \; j = 4,8,12,...,N$$

(3)

Referring to equation (3), we proposed the quarter-sweep piecewise constant approximation equation being given as

$$U(t) \approx \sum_{j=4}^{N} U_{j-2} T_{j-2}(t) \text{ where } U_{j-2} = U(t_{j-2}), \; i = 4,8,12,...,N$$

(4)
As provided in Eq. (6), the problem (1) will be replaced with polynomial basis function and formed the piecewise approximate function as below

\[ U(t) + \dot{\lambda} \int a^b K(t,x) \sum_{j=4}^{N-4} U_{j-2} T_{j-2} dx = g(t) \]  \hspace{1cm} (5)

Again, we need to improvise the equation (5) by rewriting this equation into other simple form as follows

\[ U(t) + \dot{\lambda} \sum_{j=4}^{N-4} U_j \int_a^b K(t,x) T_{j-2}(x) dx = g(t) \]  \hspace{1cm} (6)

By taking account of all entire collocation points of quarter-sweep case, \( t = t_{i-2} \), the collocation point of the quarter-sweep \( i = 4, 8, 12, \ldots, N \), we can generate the following equation

\[ U_j + \dot{\lambda} \sum_{j=4}^{N-4} U_j \int_a^b K(s_{i-2}, x) T_{j-2}(x) dx = g_{i-2} \ , \ i = 4, 8, 12, \ldots, N \]  \hspace{1cm} (7)

where,

\[ K_{i-2, j-2} = \int_a^b K(t_{i-2}, x) T_j(x) dx, \]

By considering the approximation equation (7), the linear system has successfully formed by discretizing the Fredholm integral equation of second kind by using the collocation method of a polynomial piecewise constant. As a result, we can construct the following linear system based on the quarter-sweep collocation node points being given by

\[ ZU = G \]  \hspace{1cm} (8)

where

\[ Z = \begin{bmatrix} Z_{2,2} & Z_{6,2} & Z_{10,2} & \cdots & Z_{N-2,2} \\ Z_{2,6} & Z_{6,6} & Z_{10,6} & \cdots & Z_{N-2,6} \\ Z_{2,10} & Z_{6,10} & Z_{10,10} & \cdots & Z_{N-2,10} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{N, n-2} & Z_{N,n-2} & Z_{N,n-2} & \cdots & Z_{N-2,N} \end{bmatrix}, \]

\[ U = \begin{bmatrix} U_2 \\ U_6 \\ U_10 \\ \cdots \\ U_{N-2} \end{bmatrix}, \]

\[ G = \begin{bmatrix} G_2 \\ G_6 \\ G_{10} \\ \cdots \\ G_{N-2} \end{bmatrix}, \]

\[ Z_{i,j} = \begin{cases} 1 - \lambda K_{i,j}, & i = j \\ -\lambda K_{i,j}, & i \neq j \end{cases} \text{ For } i = 2, 6, 10, \ldots, N - 2 \]

Clearly it can be said that the coefficient matrix \( X \), \( Z \) has form of large-scale and dense matrix.
3. Formulation of Jacobi Iteration Family

The Jacobi iterative family is selected in this paper to solve the linear system (8) generated by discretizing problem (1). The Jacobi family has been used to make the comparative analysis based on its own families which is the half-sweep Jacobi (HSJ) and quarter-sweep Jacobi (QSJ) iterative methods. While the full-sweep Jacobi (FSJ) method acts as a control method in this study.

Based on the findings of previous studies [14], it can be related that, its results more accurate compared to the other methods. This Jacobi’s approach has been imposed to the symmetric matrices. In fact, it can be also used for the no symmetrical matrices [15].

Before constructing the formulation of Jacobi iteration family, let the coefficient matrix of $Z$ be decomposed as

$$Z = D + C + V \quad (9)$$

Where matrices $D$, $S$, and $V$ are represent the diagonal matrix, lower matrix and upper matrix respectively, the Based on work done by [15], general scheme formulation of the Jacobi iterative family can be

$$U^{(p+1)} = D^{-1}(G - (C + V))U^p \quad (10)$$

According the algorithm 1, the equation (10), and iteration has explained the implementation of Jacobi iterative family

**Algorithm:** Jacobi method

Step 1. Set $U_i^{(0)} \leftarrow 0, \varepsilon \leftarrow 10^{-10}$

Step 2. Calculate $U^{(p+1)}$ using

$$U^{(p+1)} = D^{-1}(G - (C + V))U^p$$

Step 3. Check the convergence test, $|U_i^{(p+1)} - U_i| \leq \varepsilon = 10^{-10}$. If yes, go to step (4). Otherwise go back to step (2).

Step 4. Display the output.

4. Computational Experiments

There are three examples that have been considered to test the efficiency of Jacobi iteration family by using the quarter-sweep approximation equation. These numerical results have been highlighted on the number of iterations ($W$), execution time ($s$) and maximum abs. error. For the numerical approximations, implementation of these proposed iterative methods has considered with several different values of the size grids such as

Example 1

The following equation is Fredholm integral equation.

$$\text{(11)}$$

The exact solution of the Eq. (11) is

$$\text{(12)}$$
Example 2[16]
The following equation is Fredholm integral equation.

\[ (13) \]

The exact solution of the Eq. (13) can be stated as

\[ U(x) = e^{3x} \]  

(14)

Example 3[17]
The following equation is Fredholm integral equation.

\[ (15) \]

The exact solution of the Eq. (15) is

\[ (16) \]

Table 1: The iteration number, execution times and Max. Abs. Error for the three examples.

| EX. | NUMBER OF ITERATIONS (V) | TIME (Second)(s) | Max. Abs.Error |
|-----|------------------------|-----------------|---------------|
|     | M | FSJ | HSJ | QSJ | FSJ | HSJ | QSJ | FSJ | HSJ | QSJ | FSJ | HSJ | QSJ |
| 512 | 41 | 41 | 41 | 16.26 | 4.15 | 1.06 | 3.768065E-07 | 1.506235E-06 | 1.165411E-05 |
| 1024 | 41 | 41 | 41 | 64.23 | 16.16 | 4.04 | 9.430797E-08 | 3.768065E-07 | 2.912254E-06 |
| 2048 | 41 | 41 | 41 | 257.31 | 64.56 | 16.27 | 2.366581E-08 | 9.430797E-08 | 7.279782E-07 |
| 4096 | 41 | 41 | 41 | 1027.51 | 258.01 | 64.93 | 6.003098E-09 | 2.366586E-08 | 1.820591E-07 |
| 8192 | 41 | 41 | 41 | 4115.2 | 1032.24 | 259.73 | 1.589771E-09 | 6.003393E-09 | 4.559860E-08 |
| 512 | 24 | 24 | 24 | 1.34 | 0.36 | 0.13 | 1.909286e-05 | 7.673838E-05 | 3.044531E-04 |
| 1024 | 24 | 24 | 24 | 5.16 | 1.33 | 0.37 | 4.751341e-06 | 1.905449E-05 | 7.632094E-04 |
| 2048 | 24 | 24 | 24 | 20.42 | 5.09 | 1.31 | 1.185067e-06 | 4.746657E-06 | 1.910585E-04 |
| 4096 | 24 | 24 | 24 | 81.58 | 20.19 | 5.08 | 2.959520e-07 | 1.184533E-06 | 4.779645E-05 |
| 8192 | 24 | 24 | 24 | 326.15 | 80.68 | 20.16 | 7.407930E-08 | 2.958990E-07 | 1.195311E-05 |
| 512 | 35 | 35 | 34 | 2.01 | 0.55 | 0.18 | 1.410000E-06 | 5.651453E-06 | 1.165411E-05 |
| 1024 | 35 | 35 | 35 | 7.81 | 2.08 | 0.58 | 3.527496e-07 | 1.411660E-06 | 2.922540E-06 |
| 2048 | 35 | 35 | 35 | 31.11 | 8.17 | 2.09 | 8.817297e-08 | 3.528010E-07 | 7.279782E-07 |
| 4096 | 35 | 35 | 35 | 124.27 | 32.51 | 8.15 | 2.205607E-08 | 8.822509E-08 | 1.820591E-07 |
| 8192 | 35 | 35 | 35 | 496.83 | 129.9 | 32.54 | 5.551774E-09 | 2.210289E-08 | 4.559860E-08 |
Table 2. The reduction percentage of the comparison between the Half-sweep Jacobi (HSJ) and Quarter-sweep Jacobi (QSJ)

| EXAMPLE | METHODS | NUMBER OF ITERATION (%) | EXECUTION TIME (%) |
|---------|---------|-------------------------|--------------------|
| 1       | HSJ     | 0                       | 74.47-74.91        |
|         | QSJ     | 0                       | 93.48-93.71        |
| 2       | HSJ     | 0                       | 73.13-75.26        |
|         | QSJ     | 0                       | 90.29-93.77        |
| 3       | HSJ     | 0                       | 72.63-73.85        |
|         | QSJ     | 2.86                    | 91.04-93.45        |

Based on Table 1, there are no differences in term of the number of iteration between (HSJ) and (QSJ) iterative methods. Moreover it can be seen in Table 2 in which it showed that the reduction percentage is 0%. But the (QSJ) of example three shows the very small reduction of number iterations which is only 2.89%. Meanwhile, the execution time for all the examples showed with the big reduction of percentage. The percentages for the three examples of (HSJ) are 74.47%-74.91%, 73.13%-75.26%, 72.63%-73.85% respectively. While the reduction percentage for (QSJ) are 93.48%-93.71%, 90.29%-93.77%, 91.04%-93.45% respectively.

5. Conclusion
In this study, the quarter-sweep scheme collocation based on the polynomial piecewise constant had been used to construct the quarter-sweep piecewise constant approximations equations. This approximation equation has been used to form a linear system with a dense coefficient matrix from discretizing the linear Fredholm integral equation. This linear system has been solved by using Jacobi iteration family. By observing the numerical results, the execution time has shown big significant difference between (HSJ) and (QSJ) iterative methods. Whereas, the number of iterations has shown no any difference. Again, the accuracy for the all examples are consistent between the Jacobi iterative families. Overall, we can conclude that the best method can be nominated is the quarter-sweep Jacobi mainly on the execution time.

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