Scalar quantum kinetic theory for spin-1/2 particles: mean field theory

J Zamanian¹, M Marklund and G Brodin
Department of Physics, Umeå University, SE-901 87 Umeå, Sweden
E-mail: jens.zamanian@physics.umu.se, mattias.marklund@physics.umu.se
and gert.brodin@physics.umu.se

New Journal of Physics 12 (2010) 043019 (28pp)
Received 4 January 2010
Published 13 April 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/4/043019

Abstract. Starting from the Pauli Hamiltonian operator, we derive scalar quantum kinetic equations for spin-1/2 systems. Here, the regular Wigner two-state matrix is replaced by a scalar distribution function in extended phase space. Apart from being a formulation of significant interest, such a scalar quantum kinetic equation makes the comparison with classical kinetic theory straightforward and lends itself naturally to currently available numerical Vlasov and Boltzmann schemes. Moreover, while the quasi-distribution is a Wigner function in regular phase space, it is given by a $Q$-function in spin space. As such, nonlinear and dynamical quantum plasma problems are readily handled. Moreover, the issue of gauge invariance is treated.

¹ Author to whom any correspondence should be addressed.
Quantum kinetic theory has a long history. In many respects, it all started with the seminal paper by Wigner in 1932 [1], see also [2, 3], and the later developments of Moyal [4]. While the approach of Wigner has the advantage of being of interest for the interpretation of quantum mechanics [5] and also for the development of quantum optics (for an overview, see e.g. [6]), detailed calculations of material properties in condensed matter systems have relied, to a large extent, on either semiclassical techniques [7], in which the collisional operator in Boltzmann’s equations involves quantum transition probabilities, or Green’s function techniques [8, 9], as well as diagrammatic techniques [10]. The theory of Baym and Kadanoff, as well as the works of Keldysh [11, 12], has been successful in dealing with certain quantum transport phenomena. The theory contains memory effects (non-local terms, both in space and time), has a straightforward interpretation in terms of the different Green’s functions and works well even on timescales shorter than the typical relaxation time of the system in question. However, the gap between classical plasma physics and quantum transport theory does not seem to have been bridged, probably due to reasons of formalism as well as a difference in application of the respective models. Moreover, while the Kadanoff–Baym equation gives a very good description of certain systems, it is not specifically adapted to some of the possible future applications.
of quantum kinetic theories, such as high-intensity laser–plasma interactions [13], high-energy density physics [14] and nonlinear collective quantum problems [15–19].

In particular, the field of quantum plasmas has recently sparked interest in the field of laser plasmas [20–22], where high-density ionized plasmas can be created in the laboratory. Moreover, the advent of nano-devices and technology on sub-micron scales, such as quantum dots [23–25] and plasmonic components [26, 27], has sparked the interest of many researchers in analyzing the dynamic and nonlinear properties of such systems. A recent result is that quantum effects in plasmas can be important for parameter regimes that, for a long time, have been considered to be purely classical [28].

The above discussion is mainly related to the statistical and dispersive behavior of unmagnetized quantum plasmas [29]. However, one intrinsic non-classical property of quantum systems is the spin. The magnetization that follows from the intrinsic spin, as well as that of orbital angular momentum, is of course the foundation for many important material properties [30]. Investigations of such condensed matter systems are often directed toward equilibrium properties, although the nonlinear dynamics of magnetization is sometimes considered interesting and probed using the Landau–Lifshitz–Gilbert equation [31]. There are a variety of different physical systems where the spin can be of importance, such as metal alloys and semiconductor material for memory use [31], cold atom gases [32] and high-density and high-field astrophysical plasmas [14], to mention a few. Collective effects originating in the plasma particle species spin has therefore recently become an active field of research for fully ionized systems (see e.g. [19, 33] and references therein), in particular in the nonlinear regime, where spin solitons [34] and ferromagnetic behavior in plasmas can be found [35]. Many of the studies presented in the literature have so far been of a theoretical nature, but it is not difficult to envision future applications to e.g. plasmonic devices [26] or femtosecond physics [36].

For the purpose of connecting classical plasma physics to the evolution of non-equilibrium quantum systems, utilization of quasi-distributions is of great value. Firstly, the interpretation of the quasi-distribution function using ensemble averages of observables is in direct analogy with the classical case. It is even possible to directly construct a quasi-distribution, such as the Wigner function, from measurements [37] (with the only information loss being the initial phase). Secondly, the quasi-distribution evolution follows from the quantum Liouville equation for the density operator and gives a quantum analogue of the Vlasov or Boltzmann equation. This may also render a quantum kinetic theory for the quasi-distribution function useful for the adoption of classical numerical codes to the quantum regime. There are of course infinitely many ways to construct a quasi-distribution function, giving certain elementary requirements (see the next section). However, a few quasi-distribution functions are more prominent in the literature than others. The best known quasi-distribution function is probably the Wigner distribution [1], but there are many others frequently used. In short, different definitions correspond to different operator orderings; hence, depending on the application, different definitions are natural. For example, when considering optical coherence normally ordered operators occur naturally and hence the Glauber–Sudarshan $P$-distribution [38, 39] is a convenient choice. On the other hand, anti-normal ordered operators, i.e. the $Q$-function or the more general Husimi function [40], are useful when dealing with quantum chaotic systems. For reviews of the subject, see for example [5, 41, 42].

In this paper, we will construct a quasi-distribution function for a particle with spin-$1/2$ as a combination of a Wigner distribution for the position and momenta and the $Q$-function.
for the spin degree of freedom. Moreover, a quantum kinetic equation giving the evolution of this scalar distribution function, in the mean field or Hartree approximation, will be derived and applications to magnetized systems will be presented. The theory presented does not allow for relativistic particle motion and, hence, is not yet well suited for treating e.g. high-intensity laser plasmas. Collisions are also neglected and hence the theory breaks down in the strongly coupled regime. A discussion of possible future applications and research directions will also be given.

The structure of the paper is as follows. In section 2, we give a short overview of different quantum quasi-distributions. In section 3, we consider the evolution equation for a density matrix for a spin-1/2 particle in an external electromagnetic field. In section 4, we proceed to derive a combined transformation for the phase space and spin variable. This transformation then renders an evolution equation for the system in extended phase space \((x, p, \hat{s})\), which is derived in section 5. The extension to the mean field approximation is reviewed in section 6, and in the section that follows we calculate the thermodynamic equilibrium density matrix for a set of \(N\) non-interacting particles. In section 8, we consider the evolution equation in the long scale length limit and compare our results with previous semi-classical kinetic descriptions in the literature. In section 9, we consider the linear solutions to the derived equations. Section 10 is devoted to a discussion of gauge properties and the fully gauge invariant evolution equation is presented. Finally, we summarize the main results and discuss future development and applications in section 11.

2. General requirements of the quasi-probability distribution function

2.1. Historical note

Following the success of the classical theory of non-equilibrium statistical mechanics, it was natural to seek a similar theory for quantum systems in the late 1920s and early 1930s. However, while the classical Liouville equation generates trajectories in phase space as in a classical Hamilton–Jacobi theory, we, in the quantum realm, have to consider the Heisenberg uncertainty principle. This will not allow us to describe, as in classical systems, precise trajectories, but rather ‘smeared out’ paths in what would be the corresponding phase space. Indeed, the attempts by de Broglie, Bohm and others to give a close-to-classical interpretation of the Schrödinger equation by using Hamilton–Jacobi theory show that, if one is inclined to stick to this interpretational scheme and extend this to statistical interpretations, one has to consider the wave function rather as an ensemble of (non-classical \([43, 44]\)) trajectories (a similar conclusion can be drawn from path integral \([45]\) as well as Ehrenfest techniques \([46]\)), satisfying certain initial and boundary conditions. Thus, the introduction by Wigner of a quasi-distribution function (see below) was a natural step in the direction of relating measurements to classical transport theory. This is perhaps most obvious in the field of quantum optics, where phase space techniques since long have been widely used. Three main definitions of quasi-distributions can be found in this field, namely the Wigner function \([1]\), the Husimi (or, equivalently, the \(Q\)-) function \([40]\) and the Glauber–Sudarshan \(P\)-distribution \([38, 39]\). Below, we will give a short summary of some of the properties of the first two types of quasi-distribution functions as these are the two used in this paper.
2.2. Basic requirements

Some basic requirements can be imposed on a quantum probability distribution function in phase space, in order for it to have a reasonable interpretation [44, 47]. We denote the quantum (quasi-)distribution function by \( f(x, p) \) (for the moment, we drop the explicit time dependence for notational convenience) for a given quantum state \( \hat{\rho} \) of the system. Then the marginal distribution functions \( \langle x|\hat{\rho}|x \rangle \) and \( \langle p|\hat{\rho}|p \rangle \) should be related to \( f(x, p) \) according to

\[
  f(x) \equiv \int d^3p \ f(x, p) = \langle x|\hat{\rho}|x \rangle \tag{1}
\]

and

\[
  f(p) \equiv \int d^3x \ f(x, p) = \langle p|\hat{\rho}|p \rangle, \tag{2}
\]

respectively. Moreover, we should require that the distribution function is positive definite, i.e.

\[
  f(x, p) \geq 0. \tag{3}
\]

However, it can be shown that the conditions (1)–(3) are not sufficient to uniquely determine a suitable quantum distribution function in phase space. In fact, Cohen [48] has shown that there are an infinite number of functions \( f(x, p) \) satisfying (1)–(3).

A more complete list of properties that are desirable is found in [42], where besides the above three properties, there are additional properties that the distribution function is real, bilinear in the wave function and that the distribution functions for eigenstates of the Hamiltonian form a complete and orthogonal set. In fact, it can be shown that, in general, one will not be able to find a distribution function that satisfies all of equations (1)–(3) simultaneously if one requires the distribution function to be bilinear in the wave function [75].

Although the above conditions are important when it comes to interpreting the distribution functions, a more important condition is that it should be possible to calculate the expectation value of any operator. This condition is important since it means that all physically relevant information is included. To calculate the expectation value, one first maps the operator to the corresponding phase space function \( \hat{O} = O(\hat{x}, \hat{p}) \rightarrow O(x, p) \), using the Weyl correspondence, and then calculates the phase space average weighted by the distribution function

\[
  \langle \hat{O} \rangle = \int d^3x \ d^3p \ f(x, p) O(x, p). \tag{4}
\]

The mapping from the operator space to phase space depends on which distribution function is used (see [42] for details). Below, we will collect the properties of two distribution functions of interest in our context, namely the Wigner distribution [1] and the Husimi function [40] (or \( Q \)-function [6, 47]). These are also perhaps the most frequently encountered quantum probability distribution functions in the literature (see [6, 47] for further references and other prominent distributions used in quantum optics, such as the \( P \)-distribution of Glauber [38] and Sudarshan [39], and their interrelations).

2.3. The Wigner function

The Wigner function for a quantum state \( \hat{\rho} \) is defined as the Fourier transform of the two-point correlation function (i.e. density matrix). Thus, we accordingly have

\[
  f_w(x, p) = \frac{1}{(2\pi \hbar)^3} \int d^3y \ e^{ip\cdot y/\hbar} \langle x + y/2|\hat{\rho}|x - y/2 \rangle. \tag{5}
\]
Through this definition of the Wigner function, we see that it satisfies the marginal distribution requirements (1) and (2). However, it does not satisfy the positivity criteria (3). The latter property then prevents a probability distribution interpretation. However, the negativity of the Wigner function is limited in the sense that the proper number density in physical space is

\[ n(x) = \int d^3p \ f_W(x, p), \]

which is thus always positive. For a pure state \( \hat{\rho} = |\psi\rangle\langle\psi| \), this definition gives

\[ f_W(x, p) = \frac{1}{(2\pi\hbar)^3} \int d^3y \ e^{ip\cdot y/\hbar} \psi^*(x+y/2)\psi(x - y/2). \]  

(6)

One of the important properties of the Wigner function is that it cannot have too sharp peaks, expressed by

\[ \iint d^3x d^3p \ [f_W(x, p)]^2 \leq \frac{1}{(2\pi\hbar)^3}, \]

(7)
a result of the non-commutativity between coordinate and momentum operators.

The time evolution for the Wigner function in an external (analytic) potential \( V(x, t) \) is given by

\[ \frac{\partial f_W}{\partial t} + \frac{p}{m} \cdot \nabla_x f_W + \frac{2V}{\hbar} \sin \left( \frac{\hbar}{2} \nabla_x \cdot \nabla_p \right) f_W = 0, \]

(8)

where the sine function is defined in terms of its Taylor expansion in the case of analytic potentials, and we have used the indices \( x \) and \( p \) on the \( \nabla \) to denote its operation in phase space. To find the phase space function that corresponds to a given operator, we must first express the operator in Weyl order \( [2] \), i.e. express it in symmetric products of \( \hat{x}_i \) and \( \hat{p}_i, i = 1, 2, 3 \), using the commutation relations, and then substitute \( \hat{x}_i \rightarrow x \) and \( \hat{p}_i \rightarrow p \). For example, when calculating the average of the operator \( \hat{x}_i \hat{p}_j \), we have

\[ \hat{x}_i \hat{p}_j = \frac{1}{2} \left( \hat{x}_i \hat{p}_j + \hat{p}_j \hat{x}_i \right) + \frac{i\hbar}{2} \delta_{ij} \rightarrow x p + \frac{i\hbar}{2} \delta_{ij}, \]

(9)

where \( \delta_{ij} \) denotes the Kronecker delta function, so that

\[ \langle \hat{x}_i \hat{p}_j \rangle = \int d^3x d^3p f_W(x, p) \left( x p + \frac{i\hbar}{2} \delta_{ij} \right). \]

(10)

2.4. The Husimi function

The Husimi function (see (11) below) is based on minimum uncertainty wave packets, and it does not satisfy (1) and (2) but is positive definite (thus satisfying (3)). As will be seen below, this allows probability distribution interpretation of the Husimi function; however, it gives a different, greater uncertainty measure than expected through naive application of the Heisenberg uncertainty relation. These properties can be immediately understood from the following definition. For a given Wigner function, the Husimi function can be obtained through a Gaussian smoothing as

\[ f_H(x, p) = \frac{1}{(\pi\hbar)^3} \int d^3x' d^3p' \exp\left[ -(x' - x)^2/2d^2 - \hbar^2(p' - p)^2/2d^2 \right] f_W(x', p'). \]

(11)

where the parameter \( d \) sets the scale of the smoothing.
While the Husimi function is positive definite and produces the correct expectation values of observables, it satisfies an indeterminacy relation of the form
\[(\Delta x)_H(\Delta p)_H \geq \hbar,\] (12)
as compared to the relation
\[\Delta x \Delta p \geq \hbar/2\] (13)
for a quantum state (the latter being satisfied by the Wigner function). This result is due to the smoothing introduced in the definition of the Husimi function. The Husimi function does not give the probability for the particle to be at a certain phase space position, but rather the probability to find the particle in the minimum uncertainty state centered around the phase space point in question [49]. Introducing minimum uncertainty states \(|x_0, p_0\rangle\) that satisfy \(\Delta x^2 \Delta p^2 = \hbar^2/4\), one can write the Husimi function as
\[f_H(x, p) = \langle x, p | \hat{\rho} | x, p \rangle.\] (14)
However, as mentioned above, it can still be used to calculate any observable, but the operator ordering rule is more complicated than in the Wigner case and we will not consider this further here. The evolution of the Husimi equation can be found from (8) and definition (11). It is fairly complicated (see [42]) and it is more convenient to compute the evolution of the Husimi distribution function by evaluating the Wigner function for all times through (8). This said, we note that although the evolution equation for the Husimi function is more complicated than the corresponding equation for the Wigner function, it is sometimes the convenient choice. One such example is when considering a chaotic system in which the phase space distribution function becomes very complicated. The Husimi function, being a Gaussian average, may then behave more regularly (see, e.g., [50–52]).

2.5. Quasi-distribution functions for spin

Similarly to the case of phase space, it is possible to construct quasi-distribution functions for the spin degree of freedom. This has already been done in the 1950s by Stratonovic [53]. Later on, the spin quasi-distribution functions were further developed and were applied to problems related to calculating the correlation between spins [54–56]. The spin quasi-distribution function has also been discussed in connection with quantum scattering problems [57].

As in the case of the regular phase space variables \(x\) and \(p\), there is no unique way to introduce a spin quasi-distribution function. Scully and Wódkiewicz [58] give a good review of the many different choices that can be considered. There are at least three different methods for defining spin distribution functions: delta distributions, distributions based on coherent states (\(Q\) and \(P\)) and Stratonovic distribution functions. However, the different outcomes of these choices overlap.

In this paper, we will consider only the \(Q\)-function for spin, which is defined as
\[f(\theta, \varphi) = \langle \hat{s} | \hat{\rho} | \hat{s} \rangle,\] (15)
where \(|\hat{s}\rangle\) is the state that has spin-up in the direction of the unit vector \(\hat{s} = \hat{s}(\theta, \varphi)\), often called a spin-coherent state [59, 60]. Note that this is analogous to the definition of the \(Q\)-function in
position/momentum space, the latter given by equation (14). As for the Husimi or \( Q \)-function in phase space, this distribution does not give the correct marginal distributions. This means that integrating over the \( \varphi \) angle does not leave the correct distribution function for the \( \theta \) variable. However, it still contains all the information about the system and it can be used to calculate the expectation value of any observable, just like in the density matrix formalism. The mapping between spin operators and the corresponding spin-space functions will be considered in detail in section 4. The main reason for choosing to work with this particular distribution function for the spin variable is that it is a function on the unit sphere and hence resembles the classical picture of a dipole moment; in fact, the evolution equation in the long scale length limit (see equation (64)) is almost identical to an equation derived previously from a semiclassical treatment of the spin [61]. The \( Q \)-function for the spin is also non-negative, which may be desirable in some cases.

3. The density matrix description

In order to derive our phase space model, we start here from the density matrix description for a spin-\( \frac{1}{2} \) particle. The basis states we will use are \( |x, \alpha\rangle = |x\rangle \otimes |\alpha\rangle \), where \( |x\rangle \) is the state with position definitely at position \( x \) and \( |\alpha\rangle \) is the state with spin-up (\( \alpha = 1 \)) and spin-down (\( \alpha = 2 \)) along the axis of quantization, which we here take to be in the \( z \)-direction. The density matrix in this basis is then

\[
\rho(x, \alpha; y, \beta, t) \equiv \langle x, \alpha | \hat{\rho} | y, \beta \rangle = \sum_i p_i \psi_i(x, \alpha, t) \psi_i^*(y, \beta, t),
\]

where \( p_i \) is the probability to have state \( \psi_i \) and the Greek letters denote the spin indexes. Here \( \psi(x, 1) \) and \( \psi(x, 2) \) give, respectively, the probability amplitudes to have spin-up and spin-down.

The Hamiltonian for a particle in an external electromagnetic field is given by

\[
\hat{H} = \frac{1}{2m} \left[ \hat{p} - qA(\hat{x}, t) \right]^2 + qV(\hat{x}, t) - \mu B(\hat{x}, t) \cdot \sigma,
\]

where \( m \) is the mass of the particle, \( q \) is the charge (for an electron \( q = -e < 0 \), where \( e \) is the elementary charge), \( \mu \) is the magnetic moment of the particle, which for electrons is given by the (signed) Bohr magneton \( \mu_e = -e\hbar/(2m_e) \), \( A \) and \( V \) are the electromagnetic potentials and \( B = \nabla \times A \) is the magnetic field. \( \sigma \) is the vector containing the three Pauli matrices as its components. With the axis of quantization in the \( z \)-direction, they are given by

\[
\sigma^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{(y)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^{(z)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We will use the notation \( \sigma(\alpha, \beta) \equiv (\sigma^{(x)}(\alpha, \beta), \sigma^{(y)}(\alpha, \beta), \sigma^{(z)}(\alpha, \beta)) \), where \( \sigma^{(x)}(\alpha, \beta) \) denotes the component on row \( \alpha \) and column \( \beta \) of \( \sigma^{(x)} \) and similarly for the \( \sigma^{(y)} \) and \( \sigma^{(z)} \) matrices.

The evolution equation for the density matrix can be derived from the Schrödinger equation for the wave function and its complex conjugate, giving the von Neumann equation

\[
\frac{i\hbar}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}].
\]
Using the basis described above and Hamiltonian (17), we obtain

\[
\frac{i\hbar}{\partial t} \rho(x, \alpha; y, \beta, t) = -\frac{\hbar^2}{2m} \left[ \nabla_x^2 - \nabla_y^2 \right] \rho(x, \alpha; y, \beta, t) + q[V(x, t) - V(y, t)]\rho(x, \alpha; y, \beta, t) + \frac{iq}{m} \left[ A(x, t) \cdot \nabla_x + A(y, t) \cdot \nabla_y \right] \rho(x, \alpha; y, \beta, t) + q^2 \left[ A^2(x, t) - A^2(y, t) \right] \rho_{\alpha\beta}(x, y, t) - \mu \sum_{\gamma=1}^2 \left[ B(x, t) \cdot \sigma(\alpha, \gamma) \rho(x, \gamma; y, \beta, t) - B(y, t) \cdot \sigma^*(\beta, \gamma) \rho(x, \alpha; y, \gamma, t) \right],
\]

(20)

where we have used the Coulomb gauge \( \nabla_x \cdot A = 0 \). In general, the evolution equation of the diagonal terms \( \rho(x, \alpha; y, \alpha), \alpha = 1, 2 \) are coupled via the off-diagonal terms. However, for static fields, it is possible to obtain two decoupled equations for the diagonal elements by orienting the axes so that the magnetic field is in the direction of the axis of quantization [62].

4. The Wigner and \( Q \) transformation

The Wigner transformation for a spin-1/2 particle is given by

\[
W(x, p, \alpha, \beta) = \frac{1}{(2\pi\hbar)^3} \int d^3 z \ e^{-ipz/\hbar} \rho(x + z/2, \alpha; x - z/2, \beta),
\]

(21)

where we have emphasized that for a particle with spin, the Wigner transform must be taken for each spin matrix element of the density matrix separately. The Wigner transform of the spin density matrix has been calculated previously [57, 62, 63]. One approach is to consider the different components of the Wigner matrix \( W(x, p, \alpha, \beta) \), for \( \alpha = 1, 2 \) and \( \beta = 1, 2 \) and derive evolution equations for \( W(x, p, 1, 1) \) and \( W(x, p, 2, 2) \), which, as for the density matrix, are, in general, coupled via the off-diagonal terms [62, 64]. Another approach is to define a quasi-distribution function for the spin degree of freedom. This can, as has been discussed above, be done in a variety of different ways [55, 56, 58]. One way that is a direct generalization of the Wigner function is to consider two different spin components in two arbitrary directions \( s_1 \) and \( s_2 \), corresponding to the two operators \( \sigma_1 \) and \( \sigma_2 \), see [55]. Since the two operators, in general, do not commute, the values of \( s_1 \) and \( s_2 \) cannot be known simultaneously. This manifests itself in that the Wigner function \( W(s_1, s_2) \) can take on negative values, analogously to the corresponding case of position and momentum. Another possible choice of distribution function (corresponding to anti-normal operator ordering) is the \( Q \)-function. In the position/momentum space, this distribution function is the Gaussian averaged Wigner function and, due to this, it is positive definite. In optics, the \( Q \)-function can be measured directly [6]. The spin \( Q \)-function [58] gives the probability to measure the spin in a given direction and it is this we will use here to describe the spin degree of freedom.

To derive an evolution equation for the extended phase space distribution function \( f(r, p, \hat{s}) \), where \( \hat{s} \) is a unit vector (not an operator), we impose the following properties:

\[
f(x, \hat{s}) = \int d^3 p \ f(x, p, \hat{s})
\]

(22)
should give the probability density to find the particle at position $r$ with spin-up in the direction of $\hat{s}$ and, similarly,

$$f(p, \hat{s}) = \int d^3 x \ f(x, p, \hat{s})$$  \hspace{1cm} (23)

should give the probability to have momentum $p$ and spin-up in the $\hat{s}$-direction, a direct extension of the marginal distribution conditions (1) and (2). In order to derive the distribution function in the extended phase space, we note that for a state $\psi(x, \alpha)$ we have the probabilities $|\psi(x, 1)|^2$ ($|\psi(x, 2)|^2$) to measure spin-up (spin-down) in the $z$-direction. The corresponding density matrix is given by $\rho(x, \alpha; y, \beta) = \psi(x, \alpha) \psi^\dagger(y, \beta)$. We can then write the probability to measure spin-up in the direction of the unit vector $\hat{s}$ as

$$\text{Tr}(\hat{P}_\uparrow(\hat{s}) \rho) = \sum_{\alpha, \beta=1}^2 \frac{1}{2} \left[ \delta_{\alpha\beta} + \hat{s} \cdot \sigma(\alpha, \beta) \right] \rho(x, \beta; x, \alpha),$$  \hspace{1cm} (24)

where we have defined the (Hermitian) operator

$$\hat{P}_\uparrow(\hat{s}) = \frac{1}{2} \left[ 1 + \hat{s} \cdot \sigma \right]$$  \hspace{1cm} (25)

and where $\delta_{\alpha\beta}$ denotes the Kronecker delta. As an example, we consider the probability to measure the spin in the direction $\hat{s} = -\hat{z}$ and we obtain

$$\sum_{\alpha, \beta=1}^2 \frac{1}{2} \left[ \delta_{\alpha\beta} - \sigma^{(z)}(\alpha, \beta) \right] \rho(x, \beta; x, \alpha) = |\psi_2(x)|^2,$$  \hspace{1cm} (26)

as we expect (note that measuring spin-up in the $-z$-direction is equivalent to measuring spin-down in the $z$-direction). The generalization to a statistical distribution of states is straightforward. Using the Wigner transform for the position and momentum and the spin transform discussed above, we obtain the function

$$f(x, p, \hat{s}) = \sum_{\alpha, \beta=1}^2 \frac{1}{2} \left[ \delta_{\alpha\beta} + \hat{s} \cdot \sigma(\alpha, \beta) \right] W(x, p, \beta, \alpha),$$  \hspace{1cm} (27)

which has the properties (22) and (23) stated above. The function $f$ may also be written as

$$f(x, p, \hat{s}) = \text{Tr}[\hat{P}_\uparrow(\hat{s}) W(x, p)],$$  \hspace{1cm} (28)

where $W$ is the $2 \times 2$ matrix with elements $W(x, p, \alpha, \beta)$.

The normalization of the extended Wigner function is given by

$$\text{Tr} \int d^3 x \ d^3 p \ d^2 \hat{s} \ \frac{1}{2} \left( 1 + \hat{s} \cdot \sigma \right) W = 2\pi.$$  \hspace{1cm} (29)

Hence, we obtain a distribution function that is normalized over the allowed spin values if we redefine the operator in equation (25) as

$$\hat{P}_\uparrow(\hat{s}) \equiv \frac{1}{4\pi} \left( 1 + \hat{s} \cdot \sigma \right).$$  \hspace{1cm} (30)

In the Wigner formalism without spin, the density matrix is transformed into the Wigner function and the operators are transformed into phase space functions. For an operator $\hat{g} = g(\hat{x}, \hat{p})$, the corresponding phase space function is given by

$$g(x, p) = \int d^3 z \ e^{-i/\hbar p z} \left( x + \frac{z}{2} \hat{g} \right) \left| x - \frac{z}{2} \right|. $$  \hspace{1cm} (31)
It can also be obtained by using Weyl ordering as described in section 2.3. With this function, the expectation value of the operator is calculated as a phase space integral

\[ \langle \hat{g} \rangle = \int d^3x \int d^3p \, f(x, p) g(x, p) = \text{Tr}(\hat{\rho} \hat{g}) , \]  

(32)

where \( f(x, p) \) and \( \hat{\rho} \) are related via a Wigner transform. In analogy with this, for a given operator \( \hat{h} \) acting on the spin degree of freedom, we define the corresponding spin-space function

\[ h(\hat{s}) = \text{Tr}\left[ \frac{1}{2} \left( 1 + 3\hat{s} \cdot \sigma \right) \hat{h} \right] = \sum_{\alpha, \beta=1}^{2} \frac{1}{2} \left[ \delta_{\alpha\beta} + 3\hat{s} \cdot \sigma (\alpha, \beta) \right] h(\beta, \alpha) , \]  

(33)

where \( h(\alpha, \beta) \) denotes the \((\alpha, \beta)\) component of the operator \( \hat{h} \). With this definition, the expectation value of the operator is now calculated as an integral over the possible spin directions according to

\[ \text{Tr}(\hat{\rho} \hat{h}) = \int d^2\hat{s} \, f(\hat{s}) h(\hat{s}) , \]  

(34)

where we have used \( \int d^2\hat{s} \, s_a s_b = (4\pi/3)\delta_{ab} \) and \( \sigma^i_{\alpha\beta} \sigma^j_{\gamma\delta} = 2\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta} \). In performing the above calculation, we have also assumed that the general form of a spin-operator is \( \hat{h} = aI + b \cdot \sigma \), where \( a \) and \( b \) may be dependent on position and momenta. Note that the definition of the spin-space function, equation (33), implies that the spin operator \( \sigma \) is related to the spin unit vector \( \hat{s} \) according to

\[ \sigma \rightarrow 3\hat{s} . \]  

(35)

For operators depending on both the position and momentum and the spin degree of freedom, the corresponding extended phase space function is obtained by performing both the transformations (31) and (33).

The operator (30) can also be written as \( \hat{P}(\hat{s}) = |\hat{s}\rangle \langle \hat{s}| \), where \( |\hat{s}\rangle \) is the spin coherent state [59, 60]. The definition (28) is then seen to coincide with the definition of the spin \( Q \)-function, see equation (15). The function \( f(x, p, \hat{s}) \) is hence a combination of a Wigner function in the phase space variables and the \( Q \)-function for the spin.

4.1. Equivalence with the density matrix formalism

The above construction contains the same information as the density matrix, and the distribution function can be used to calculate the expectation value of any observable. A more direct way to see the equivalence is to note that, for a given distribution function \( f(x, p, \hat{s}) \), it is possible to obtain the corresponding Wigner matrix as

\[ \overline{W}(x, p) = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \int d^2\hat{s} \, f(x, p, \hat{s}) \frac{1}{2} \begin{pmatrix} 1 + 3s_x & 3(s_x - is_y) \\ 3(s_x + is_y) & 1 - 3s_z \end{pmatrix} . \]  

(36)

From this, it is the possible to obtain the density matrix by taking the inverse Wigner transform (see, for example, [76]).
5. Evolution equation

To derive the evolution equation for \( f(x, p, \hat{s}) \), the Wigner transform of equation (20) is calculated with the result (assuming that the fields and potentials are analytic functions)

\[
\left( \frac{\partial}{\partial t} + \frac{1}{m} p \cdot \nabla_x \right) W(x, p, \alpha, \beta) = \frac{2q}{\hbar} V(x) \sin \left( \frac{\hbar}{2} \nabla_x \cdot \nabla_p \right) W(x, p, \alpha, \beta) \\
+ \frac{q}{m} \left[ A(x) \cdot \nabla_x \cos \left( \frac{\hbar}{2} \nabla_x \cdot \nabla_p \right) - \frac{2}{\hbar} p \cdot A(x) \sin \left( \frac{\hbar}{2} \nabla_x \cdot \nabla_p \right) \right] W(x, p, \alpha, \beta) \\
+ \frac{i\mu}{\hbar} \sum_{\gamma=1}^{2} B(x) \cdot \left[ \sigma(\alpha, \gamma) \exp \left( \frac{ih}{2} \nabla_x \cdot \nabla_p \right) \right] W(x, p, \gamma, \beta) \\
- \sigma^*(\beta, \gamma) \exp \left( - \frac{ih}{2} \nabla_x \cdot \nabla_p \right) W(x, p, \alpha, \gamma). \tag{37}
\]

where functions of an operator are defined by its formal Taylor expansion and the left (right) arrow above the differential operators indicate that they act on the functions on the left (right). If the potentials have discontinuities the above equation should instead be written explicitly in the form of an integro-differential equation. Next we multiply by \([\delta_{\beta\alpha} + s \cdot \sigma(\beta, \alpha)]/2\) and sum over \(\alpha\) and \(\beta\). The operators acting on the left-hand side and the first four terms on the right-hand side commute with the Pauli matrices and for these we obtain \( W(x, p, \alpha, \beta) \rightarrow f(x, p, \hat{s}) \). For the last two terms, we use the property

\[
\sum_{\gamma=1}^{2} A \cdot \sigma(\alpha, \gamma) B \cdot \sigma(\gamma, \beta) = A \cdot B \delta_{\alpha\beta} + i[\sigma(\alpha, \beta) \cdot (A \times B)] \tag{38}
\]

and also that \( \sigma^*_{\alpha\beta} = \sigma_{\beta\alpha} \). After some straightforward calculations, we obtain the evolution equation for the extended Wigner function

\[
\left( \frac{\partial}{\partial t} + \frac{1}{m} p \cdot \nabla_x \right) f(x, p, \hat{s}) = \left[ \left( -\frac{q}{m} p \cdot A + \frac{q^2}{2m} A^2 + q V \right) - \mu \left( B \cdot \nabla_{\hat{s}} + \hat{s} \cdot B \right) \right] \\
\times \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \nabla_x \cdot \nabla_p \right) + \left[ \frac{q}{m} A \cdot \nabla_x - \frac{2\mu}{\hbar} (\hat{s} \times B) \cdot \nabla_{\hat{s}} \right] \\
\times \cos \left( \frac{\hbar}{2} \nabla_x \cdot \nabla_p \right) \right] f(x, p, \hat{s}). \tag{39}
\]

An advantage of writing the evolution equation in this form is that we may Taylor expand the trigonometric function to sufficient order in \( \hbar \) to obtain the semi-classical limit directly.

Next, we make a variable transformation in the evolution equation. The canonical momentum \( p \) is related to the velocity by \( v = (p - q A)/m \). Changing variables from \( x, p \) and \( t \) to \( x, v, t \), we obtain

\[
\nabla_{xj} \rightarrow \nabla_{vj} - \frac{q}{m} \sum_{j=1}^{3} (\nabla_{xj} A_j) \nabla_{vj}, \tag{40a}
\]
∇_{pi} \rightarrow \frac{1}{m} \nabla_{vi}, \quad (40b)

where \( \nabla_{xi} = \partial/\partial x_i \) and \( \nabla_{vi} = \partial/\partial v_i \). For the time derivative, we obtain \( \partial_t \rightarrow \partial_t - \frac{q}{m} \sum_{i=1}^{3} [\partial_t A_i(\mathbf{x})] \nabla_{vi} \). \( (40c) \)

We can then write the full quantum-kinetic equation \( (39) \) as

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{x} f + \left[ \frac{q}{m} \mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{\mu}{m} \nabla_{x} \left[ (\hat{\mathbf{s}} \times \hat{\mathbf{B}}) \cdot \mathbf{B} \right] \right] \cdot \nabla_{v} f + \frac{2\mu}{\hbar} (\hat{\mathbf{s}} \times \mathbf{B}) \cdot \nabla_{i} f
\]

\[
= \left[ \frac{q}{m} \mathbf{V} - \mathbf{v} \cdot \mathbf{A} - \frac{\mu}{m} \left( \mathbf{B} \cdot \nabla_{x} \hat{\mathbf{s}} + \hat{\mathbf{s}} \cdot \mathbf{B} \right) \right] \cdot \nabla_{v} f
\]

\[
\times \left[ \frac{2m}{\hbar} \sin \left( \frac{\hbar}{2m} \mathbf{v} \cdot \nabla_{x} \mathbf{v} \right) - \mathbf{v} \cdot \mathbf{v} \right] f
\]

\[
+ \left[ \frac{q}{m} \mathbf{A} \cdot \mathbf{v} - \frac{q^{2}}{m^{2}} \left[ (\mathbf{A} \cdot \mathbf{v}) \mathbf{A} \right] \cdot \mathbf{v} - \frac{2\mu}{\hbar} (\hat{\mathbf{s}} \times \mathbf{B}) \cdot \nabla_{i} \hat{\mathbf{s}} \right] \nabla_{v} f
\]

\[
\times \left[ \cos \left( \frac{\hbar}{2m} \mathbf{v} \cdot \nabla_{x} \mathbf{v} \right) - 1 \right] f, \quad (41)
\]

displaying the classical and semiclassical terms more explicitly on the left-hand side of the equation. We note that the terms on the right-hand side are all higher-order derivative corrections.

6. Many-particle evolution equation

So far we have only considered one particle in an external electromagnetic field. To make a straightforward generalization to an \( N \)-body system, we consider the mean field approximation. In order to keep things simple, we neglect effects due to spin statistics (anti-symmetry of the wave function). To a certain degree, such effects can be incorporated by choosing an appropriate background distribution, see the next section. We introduce the many-particle density matrix \( \hat{\rho}_{1...N} \) that satisfies the von Neumann equation

\[
\hat{H}^{(N)} \quad (42)
\]

The \( N \)-body Hamiltonian \( \hat{H}^{(N)} \), in general, includes interactions between the particles

\[
\hat{H}^{(N)} = \sum_{i=1}^{N} \hat{H}_{i} + \sum_{i<j=1}^{N} \hat{H}_{ij}, \quad (43)
\]

where \( \hat{H}_{i} = (\hat{p}_{i} - qA_{0}(\hat{x}_{i}))^{2}/2m + qV_{0}(\hat{x}_{i}) \) is the Hamiltonian for particle \( i \) and contains the kinetic energy and the interaction with an external electromagnetic field \( (V_{0}, A_{0}) \), and \( \hat{H}_{ij} \) is the interaction between particles \( i \) and \( j \), which we assume to be the full electromagnetic interaction between the particles. The interaction is hence obtained by solving Maxwell’s equations. Following Bonitz’s paper \([66]\), we introduce the reduced density matrix in the thermodynamic limit \( (N, V \rightarrow \infty, N/V = n_{0} = \text{const}) \)

\[
\hat{\rho}_{1...s} = V^{s} \text{Tr}_{s+1,....N} \hat{\rho}_{1...N}, \quad (44)
\]
where \( V \) is the volume of the system and the trace includes summing over the spin degree of freedom. The normalization is given by
\[
\frac{1}{V} \text{Tr}_{1,\ldots,s} \hat{\rho}_{1,\ldots,s} = 1.
\] (45)

Note that this means that for the diagonal elements of the one-particle reduced density matrix \( \rho_1(x, x) \) is proportional to the probability density to find any one of the \( N \) particles in position \( x \) independently of the positions of all the other particles. Expectation values of an \( s \)-body operator are given by
\[
\langle \hat{A}_{1,\ldots,s} \rangle = \frac{n_0}{s!} \text{Tr} \hat{A}_{1,\ldots,s} \hat{\rho}_{1,\ldots,s}.
\] (46)

The evolution equation for the reduced density matrix is given by the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy [69]
\[
i\hbar \frac{\partial \hat{\rho}_{1,\ldots,s}}{\partial t} - \left[ \hat{H}^{(s)}, \hat{\rho}_{1,\ldots,s} \right] = n_0 \text{Tr}_{s+1} \sum_{i=1}^{s} \left[ \hat{H}_{i,s+1}, \hat{\rho}_{1,\ldots,s} \right],
\] (47)
where \( \hat{H}^{(s)} \) is obtained by changing \( N \to s \) in equation (43). Considering the first-order equation and introducing the two-particle correlation as \( \hat{\rho}_{12} = \hat{\rho}_1 \hat{\rho}_2 + \hat{\rho}_{12} \), we may write this as
\[
i\hbar \frac{\partial \hat{\rho}_1}{\partial t} - \left[ \hat{H}^{(1)}, \hat{\rho}_1 \right] - \left[ \hat{H}_{\text{MF}}, \hat{\rho}_1 \right] = n_0 \text{Tr}_2 \left[ \hat{H}_{12}, \hat{\rho}_{12} \right],
\] (48)
where \( \hat{H}_{\text{MF}} = \text{Tr}_2 \hat{H}_{12} \hat{\rho}_2 \) is the mean field that is found by solving Maxwell’s equations self-consistently. The effects of particle–particle scattering are included in the correlation operator \( \hat{\rho}_{12} \). This, in turn, satisfies an equation that is coupled to the three-particle correlations and so on. Here we will be mainly interested in the collective effects of the plasma and hence we will neglect the right-hand side of equation (48), i.e. use the Hartree approximation. In order to include self-energy effects and ionization/recombination, it is necessary to keep higher-order correlations.

Comparing equation (48) (with \( \hat{\rho}_{12} \) neglected) with the corresponding equation for a single particle in an external electromagnetic field, equation (19), we note that they are formally the same. Thus, in order to include systems of \( N \)-particles in the Hartree approximation, we hence need to assume that the fields in the evolution equation are the self-consistent fields and then keep in mind that the density matrix is now normalized according to equation (45). However, since we will not pursue further the issue of the quantum BBGKY hierarchy, we may redefine the one-particle distribution function so that it has the normalization
\[
\text{Tr} \hat{\rho}_1 = n_0,
\] (49)
so that, for example, \( \langle x | \hat{\rho}_1 | x \rangle = n(x) \) gives the mean density of particles at position \( x \).

The mean field interaction \( \hat{H}_{\text{MF}} \) is obtained by coupling the equation to Maxwell’s equations. The expressions for the charge and current densities are then
\[
n(x, t) = q \int d^3v \, d^2\hat{s} \, f(x, v, \hat{s}, t),
\] (50)
\[
j(x, t) = j_f(x, t) + j_M(x, t) = q \int d^3v \, d^2\hat{s} \, f(x, v, \hat{s}, t)v + \mu \nabla_v \int d^3v \, d^2\hat{s} \, f(x, v, \hat{s}, t)3\hat{s},
\] (51)
where the second term in the current density is a magnetization current contribution due to the spin (see [67]).

New Journal of Physics 12 (2010) 043019 (http://www.njp.org/)
7. Thermodynamic equilibrium density matrix

As an example, we calculate the extended phase space distribution function for a system of \( N \) non-interacting particles in a constant magnetic field that are in thermodynamic equilibrium at temperature \( T \). Assuming that the magnetic field is \( \mathbf{B} = B_0 \mathbf{\hat{z}} \) and using the Landau gauge \( \mathbf{A} = (-y B_0, 0, 0) \), we can obtain the eigenstates of the Hamiltonian, equation (17),

\[
\psi_{p_x, n, p_z, a}(x, y, z, \alpha) = \frac{e^{i/p_x(p_x + p_z)}}{2\pi \hbar} \phi_n \left( y + \frac{p_x}{q B_0} \right) \chi_a(\alpha),
\]

where \( \phi_n \) is the \( n \)th harmonic oscillator wave function given by

\[
\phi_n(y) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m \omega_c}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m \omega_c}{2\hbar} y^2 \right) H_n \left( \sqrt{\frac{m \omega_c}{\hbar}} y \right),
\]

where \( H_n \) are the Hermite polynomials [68] and where \( \omega_c = q B_0 / m \) is the cyclotron frequency. Furthermore, we have introduced the spinors \( \chi_a(\alpha) \) that satisfy \( \sigma_z \chi_a(\alpha) = a \chi_a(\alpha) \) with \( a = \pm 1 \). The energy levels corresponding to equation (52) are given by

\[
E_{n, p_z, a} = \hbar \omega_c \left( \frac{1}{2} + n \right) + \frac{p_z^2}{2m} - a \mu_B B_0.
\]

Note that the energy is independent of the momentum in the \( x \)-direction so that the energy levels are degenerate. The thermal equilibrium density matrix at temperature \( T \) is given by

\[
\hat{\rho} = \frac{e^{-\hat{H}/k_B T}}{Z},
\]

where \( k_B \) is Boltzmann’s constant and the partition function is given by

\[
Z = \text{Tr} \, e^{-\hat{H}/k_B T}.
\]

Considering the one-particle density matrix, it can be written as

\[
\rho(x, \alpha; y, \beta) = \sum_{p_x, n, p_z, a} p_{p_x, n, p_z, a} \psi_{p_x, n, p_z, a}(x, \alpha) \psi_{p_x, n, p_z, a}^{\ast}(y, \beta) \chi_a(\alpha) \chi_{a'}(\beta),
\]

where the probability for the state with quantum numbers \((p_x, n, p_z, a)\) is given by

\[
p_{p_x, n, p_z, a} = \frac{1}{e^{(E_n + p_z \mu_c)/k_B T} + 1},
\]

where \( \mu_c \) is the chemical potential. The Wigner transform of the density matrix for a harmonic oscillator has been calculated by Schleich [47]. Using Schleich’s result, we can calculate the Wigner transform to be

\[
W(x, p, \alpha, \beta) = \sum_{n, a} \frac{1}{e^{(E_n + p_z \mu_c)/k_B T} + 1} 2(-1)^n \exp \left[ -\frac{2}{\hbar \omega_c} \left( \frac{p_y^2}{2m} + \frac{m \omega_c^2}{2} \left( y + \frac{p_x}{q B} \right)^2 \right) \right]
\times L_n \left[ \frac{4}{\hbar \omega_c} \left( \frac{p_y^2}{2m} + \frac{m \omega_c^2}{2} \left( y + \frac{p_x}{q B} \right)^2 \right) \right] \chi_a(\alpha) \chi_{a'}(\alpha'),
\]
where $L_n$ denotes the Laguerre polynomials [68]. Calculating the spin transform, equation (28), of this and also changing variables to $v = (p - qA)/m$, we finally obtain

$$
f(x, v, \hat{s}) = \sum_{n,a} \frac{n_0(-1)^a}{2\pi(2\pi\hbar)^3} \frac{1 + a \cos \theta_s}{e^{(E_{n,a} - \mu_s)/k_BT} + 1} \exp \left[ - \frac{2}{\hbar\omega_c} \left( \frac{m(v_x^2 + v_y^2)}{2} \right) \right] \times L_n \left[ \frac{4}{\hbar\omega_c} \left( \frac{m(v_x^2 + v_y^2)}{2} \right) \right], \quad (60)
$$

where we have also multiplied by $n_0$ to obtain the chosen normalization (see the previous section). Note that the argument appearing in the exponential and the Laguerre polynomials is just the kinetic energy of the motion perpendicular to the magnetic field. However, as opposed to the classical case, $v_x$ and $v_y$ are non-commuting quantities and cannot be determined simultaneously. To verify that equation (60) is indeed a solution to the Wigner equation, we note that for stationary solutions in the given choice of magnetic field, the equation can be written as

$$(v_x \partial_x + v_y \partial_y) f = 0 \quad (61)$$

and we see that in fact any spatially homogenous function solves this.

The expression (60) contains Landau quantization, spin splitting of energy states and Fermi–Dirac statistics. By using this as the unperturbed background when considering linear waves, the spin statistics is partially taken into account. For cases where the chemical potential $\mu_c$ is large and the difference between nearby Landau levels is smaller than the thermal energy, the velocity distribution approaches the classical Maxwellian as can be seen in figure 1(a). An important quantum mechanical result that remains in this limit is that the probability distribution of the spin-up and -down populations scales as $1 + \cos \theta_s$ and $1 - \cos \theta_s$, respectively. Thus, in the above regime, the distribution can be approximated by

$$f(x, v, \hat{s}) = F_+(v)(1 + \cos \theta_s) + F_-(v)(1 - \cos \theta_s), \quad (62)$$

where $F_\pm$ are Maxwellian distributions. The ratio $F_+/F_-$ in thermodynamic equilibrium is $F_+/F_- = \exp(-2\mu_B B_0)/k_BT$. For a small chemical potential, when Fermi–Dirac statistics is applied to $F_\pm$, we can still have the form (62), but the ratio $F_+/F_-$ is velocity dependent. Actually, even in the absence of thermodynamical equilibrium, equation (62) is the most general time-independent, homogeneous expression for the distribution function in a constant magnetic field. In a semiclassical treatment of the spin, the corresponding distribution is $f \propto \exp[(\mu_B B_0/k_BT)\cos \theta_s]$. For a low to moderate density (i.e. for a large chemical potential), as $\mu_B B_0/k_BT \rightarrow 0$ this distribution and the quantum mechanical distribution, equation (62), become more and more similar, see figures 1(b)–(c). However, the predicted magnetic moment for the two theories becomes more and more different. This is due to the factor 3 in equation (35), which is present in the quantum mechanical treatment but not in the semiclassical theory. For $\mu_B B_0/k_BT \gg 1$, the two distribution functions become more and more different but yield the same magnetic moment.

8. Long scale length limit

To obtain the long scale limit, we Taylor expand the trigonometric operators to order $\hbar$, which applies if the characteristic scale lengths are longer than the thermal de Broglie length. Thus,
we henceforth neglect higher-order terms in ħ, such as \( \frac{\hbar^2}{4} \nabla_x^2 V(x) \cdot \nabla_p^2 f \). The evolution equation then becomes

\[
\left( \frac{\partial}{\partial t} + \frac{1}{m} \mathbf{p} \cdot \nabla_x \right) f = \left( -\frac{q}{m} p_i \nabla_{xj} A_i + \frac{q^2}{2m^2} r \nabla_{xj} A_i^2 + q \nabla_{xj} V \right) \cdot \nabla_p f 
- \mu \left( \nabla_{xj} B_i \nabla_x \hat{s} + s_i \nabla_{xj} B_i \right) \cdot \nabla_p f + \left[ \frac{q}{m} \mathbf{A} \cdot \nabla_x - \frac{2\mu}{\hbar} (\hat{s} \times \mathbf{B}) \cdot \nabla_\hbar \right] f. \tag{63} \]

Making a variable change from \( \mathbf{x}, \mathbf{p} \) and \( t \) to \( \mathbf{x}, \mathbf{v}, t \) according to equation (5), the second term in equation (40a) above will combine with other terms in equation (63) to produce the magnetic field term in the Lorentz force. The last term in the time derivative equation (40c) will combine with the gradient of the scalar potential \( \nabla V \) to produce the electric field \( \mathbf{E} = -\nabla V + \partial_t \mathbf{A} \).
The evolution equation then takes the form
\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \left[ \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{\mu}{m} \nabla_x (\mathbf{s} \cdot \mathbf{B}) \right] \cdot \nabla_v f = 0,
\]
\[
+ \frac{2\mu}{\hbar} (\mathbf{s} \times \mathbf{B}) \cdot \nabla_v f + \frac{\mu}{m} [\nabla_x (\mathbf{B} \cdot \nabla_v)] \cdot \nabla_v f = 0.
\]
(64)

Note that the last term contains derivatives both with respect to the velocity \(\mathbf{v}\) and the spin \(\mathbf{s}\). The above equation has already been studied in [61] with the last term missing due to semi-classical approximations. It is there shown to give rise to new oscillation modes due to the anomalous magnetic moment of the electron. A similar equation has also been studied in [72], where it is investigated whether spin may be of importance in magnetic confined fusion experiments. The difference between equation (64) and the semi-classical case (with the last term missing) is due to the fact that the quantum mechanical probability distribution is always spread out, as follows from equation (30). In order to demonstrate this, we consider the distribution function for a single particle that at a time \(t\) has a given spin state, pointing in the direction \(\hat{e}\), where \(\hat{e}\) is a unit vector. As follows from equation (30), the corresponding distribution function, which is smeared out in spin space, can be written as
\[
f(\mathbf{x}, \mathbf{v}, \hat{s}) = F(\mathbf{x}, \mathbf{v})(1 + \hat{e} \cdot \hat{s})/4\pi.
\]
If we average over all spin directions, the last term combines with the magnetic dipole term according to
\[
\frac{1}{4\pi m} \int d^2 \mathbf{s} \nabla_x \left( \mathbf{B} \cdot \hat{s} + \mathbf{B} \cdot \nabla \mathbf{s} \right) \cdot \nabla_v F(\mathbf{x}, \mathbf{v})(1 + \hat{e} \cdot \hat{s}) = \frac{\mu}{m} \nabla_x \cdot (\mathbf{B} \cdot \hat{e}) \nabla_v F(\mathbf{x}, \mathbf{v}),
\]
(65)

where we stress that 2/3 of the contribution comes from the latter term. The full evolution equation for this reduced distribution function can be written as
\[
\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla_x F + \left[ \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{\mu}{m} \nabla_x \mathbf{B} \right] \cdot \nabla_v F = 0.
\]
(66)

where \(B = |\mathbf{B}|\) and we have allowed the spin direction \(\hat{e}\) to be varied slowly, following the variations of the magnetic field direction, i.e. \(\hat{e} = \hat{B}(\mathbf{x}, t)\). The above equation is useful when the spin state of each particle is conserved for a sufficiently long time.

For a semi-classical treatment of the spin, we would expect that the probability to measure the spin in the direction \(\hat{s}\), given that the spin is in the \(\hat{e}\)-direction, is given by \(f_{cl}(\hat{s}) = \delta(\hat{s} - \hat{e})\). However, as can be seen from the above equation, the classical limit of a particle with spin in the \(\hat{e}\)-direction is not a particle with definite magnetic moment in the \(\hat{e}\)-direction but a statistical distribution of spins in all directions (except \(\hat{s} = -\hat{e}\), which has zero probability). For a magnetized electron plasma, the magnetization is given by \(\nabla \times \mathbf{M} = \nabla \times \mu \langle \mathbf{\sigma} \rangle\), where the expectation value is taken with respect to the spin degree of freedom. In the quantum model developed here, the spin is given by
\[
\langle \mathbf{\sigma} \rangle = \int d^3 v \ d^2 \hat{s} \ 3\hat{s} \ f(\mathbf{x}, \mathbf{v}, \hat{s}, t).
\]
(67)

The factor 3 will account for the fact that the probability to find the spin in a certain direction is smeared out over the whole unit sphere. In a classical treatment of the spin variable, the corresponding integral contains no factor 3, but instead the distribution function is a delta function of the spin and hence the same result can be obtained. The latter is the model used in [61].
9. Examples from linearized theory

9.1. Spin-induced damping of Alfvén waves

As an example of the usefulness of equation (64), we will consider shear Alfvén-like waves in the linear limit. First, we divide the variables as \( f = f_0 + f_1 \) and \( B = B_0 + B_1 \), in which case the linearized electron equation can be written as

\[
\left[ \frac{\partial}{\partial t} + v \cdot \nabla_x + \frac{q_e}{m_e} \left( v \times B_0 \right) \cdot \nabla_v + \frac{2\mu_e}{\hbar} (\hat{s} \times B_0) \cdot \nabla_\phi \right] f_1 = - \left\{ \frac{q_e}{m_e} (E + v \times B_1) + \frac{\mu_e}{m_e} \nabla_s (\hat{s} \cdot B_1) + \frac{\mu_e}{m_e} \nabla_s [(B_1 \cdot \nabla_\phi)] \right\} \cdot \nabla_v f_0 - \frac{2\mu_e}{\hbar} (\hat{s} \times B_1) \nabla_\phi f_0.
\]

(68)

The magnetic moment is given by \( \mu_e = -(g/2)eh/(2m_e) \), where we have explicitly introduced the Landé \( g \)-factor that is exactly 2 within Dirac theory, but from QED we obtain \( g/2 - 1 \approx 0.0016 \). The term \( (v \times B_1) \cdot \nabla_v f_0 \) can be dropped for an isotropic equilibrium distribution, which will be used below. Furthermore, let \( B_0 = B_0\hat{z} \); introducing cylindrical coordinates in velocity space \( (v_\perp, \varphi, v_z) \) and spherical coordinates in spin space \( (\varphi_s, \theta_s) \) and making a plane wave ansatz \( f_1 = \tilde{f}_1 \exp[i(k \cdot r - \omega t)] \), the equation is written as

\[
\left[ i (\omega - k \cdot v) + \omega_{ge} \frac{\partial}{\partial \varphi} + \omega_{ce} \frac{\partial}{\partial \varphi_s} \right] \tilde{f}_1 = \left[ \frac{q_e}{m_e} \bar{E} + \frac{i\mu_e}{m_e} (\hat{s} \cdot \bar{B}_1 + \bar{B}_1 \cdot \nabla_\phi) \hat{k} \right] \cdot \nabla_v f_0 + \frac{2\mu_e}{\hbar} (\hat{s} \times \bar{B}_1) \cdot \nabla_\phi f_0,
\]

(69)

where we have introduced \( \omega_{ge} = 2\mu_e B_0/\hbar \) and \( \omega_{ce} = q_e B_0/m_e \). We note that \( \omega_{ge} = (g/2 - 1)\omega_{ce} \). Following Bordin et al [61], the above equation can be solved by an expansion in the eigenfunctions

\[
\psi_{n_1}(\varphi, v_\perp) = \exp[-i(n_1\varphi - k_1 v_\perp \sin \varphi_s/\omega_c)],
\]

(70)

where we use cylindrical coordinates for the velocity \( v = (v_\perp \cos \varphi, v_\perp \sin \varphi, v_z) \). Thus, we let

\[
\tilde{f}_1 = \sum_{n_1, n_2} g_{n_1 n_2}(v_\perp, v_z, \theta_s) \psi_{n_1}(\varphi, v_\perp) \exp(-in_2 \varphi_s),
\]

(71)

where \( n_1 = 0, \pm 1, \pm 2, \ldots \) and \( n_2 = -1, 0, 1 \), where we have used spherical coordinates for the spin \( \hat{s} = (\cos \theta_s \sin \varphi_s, \sin \theta_s \sin \varphi_s, \cos \theta_s) \). The above expansion could contain any integer \( m \), but it so happens that after integration over \( \varphi_s \), only \( n_2 = -1, 0, 1 \) gets a nonzero contribution, as can be seen below. Using the orthogonality properties

\[
\frac{1}{2\pi} \int_0^{2\pi} \psi_n^* \psi_m^* \, d\varphi_v = \delta_{nm},
\]

we find

\[
i \left( \omega - k_z v_z - n_1 \omega_{ce} - n_2 \omega_{ge} \right) g_{n_1 n_2} = I_{n_1 n_2}(v_\perp, v_z, \theta_s)
\]

(72)
with
\[
I_{\alpha\beta} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left\{ \frac{q}{m} \hat{E} + i\frac{\mu_e}{m} \left( \hat{s} \cdot \hat{B}_1 + \hat{B}_1 \cdot \nabla_k \right) k \right\} \cdot \nabla_v f_0 + \frac{2\mu_e}{\hbar} \left( \hat{s} \times \hat{B}_1 \right) \cdot \nabla f_0 \right| \psi_n \exp(i n \varphi) \, d\varphi_v. \tag{73}
\]
A relation that is useful when trying to write results in a more explicit form is the Bessel expansion
\[
\psi_n(\varphi_v, v_\perp) = \sum_m J_m \left( \frac{k_\perp v_\perp}{\omega_c} \right) \exp[i(m - n)\varphi_v]. \tag{74}
\]
Here it is seen that the results are considerably simplified in the limit where \( k_\perp v_\perp / \omega_c \) is small (where we estimate \( v_\perp \) with the thermal velocity \( v_\text{te} = \sqrt{k_B T_e / m_e} \)), but, in general, the conductivity tensor turns into a sum over various combinations of Bessel functions. The conductivity tensor \( \bar{\sigma}_{ij} \) for each species \( s = e, i \), as defined by
\[
\bar{\sigma}^{ij}_{(s)} = \bar{\sigma}^{ij}_{(s)} E_j, \tag{75}
\]
is found from (equation (51))
\[
j_{(s)} = q_s \int d^3v d^2\hat{s} \hat{s} f_{is} + 3\mu_{gs} \nabla \times \left( \int d^3v d^2\hat{s} \hat{s} f_{is} \right) \tag{76}
\]
by expressing the magnetic field in terms of \( \hat{E} \) and then solving for \( \hat{f}_1 \) in terms of \( \hat{E} \) using the eigenfunctions as outlined above. For the ions, the second term, i.e. the magnetization part, is negligible due to the small magnetic moment of the ions. Similarly, for the ion correspondence of equation (68), all spin terms are neglected and thus the classical Vlasov equation is used. So far the linear theory presented here applies to the general case. Without loss of generality we can let \( k = k_\perp \hat{x} + k_\parallel \hat{z} \). We will now focus on a specific geometry. For the specific case of shear Alfvén waves, we may have the approximate polarization \( \hat{E} = \hat{E}_s \hat{x} \) and \( \hat{B}_1 = \hat{B}_s \hat{y} \), in which case the (approximate) dispersion relation reads
\[
\omega^2 - k_\parallel^2 c^2 + i\omega \varepsilon_0 \sum_j \bar{\sigma}_{ij}^{ss} = 0,
\]
where Ampere’s law has been used. That this polarization is indeed possible must be checked evaluating the full linear theory involving all components of \( \bar{\sigma}^{ij} \) [70]. In a regime without very high temperatures or low temperatures, we may expect the standard classical Vlasov theory to be applicable to a first approximation. However, as is evident from (72), the introduction of spin gives new resonances, which may significantly affect the resonant wave–particle interaction even if the spin terms are otherwise small. For shear Alfvén waves \( \omega \ll \omega_{ci} \), where \( \omega_{ci} \) is the ion cyclotron frequency. Making an expansion in \( \omega / \omega_{ci} \), the standard classical theory shows that it is the \( g_10 \)-term for the ions (recall that ions are always classical with \( n_2 = 0 \), since their magnetic moment is negligible) that gives the dominant contribution to the current, since the electron terms scale as \( \omega / \omega_{ce} \), compared to \( \omega / \omega_{ci} \) for ions. However, the possibility to have spin terms \( g_{1-1} \) and \( g_{-11} \) in the expansion opens up for the electron contribution to be significant even in the regime \( \omega \ll \omega_{ci} \), since the factor \( (\omega - k_\perp v_\perp - n_1 \omega_c - n_2 \omega_{ce}) \) becomes reduced for the cases \( (n_1, n_2) = (1, -1) \) and \( (n_1, n_2) = (-1, 1) \). In particular, the wave particle corresponding to these terms may occur in the bulk of the thermal distribution rather than in the exponentially small tail. Thus, when computing \( \bar{\sigma}^{ss} \) we keep the terms that are dominating classically, which
are \( g_{10} \) and \( g_{-10} \) (given that the classical \( g_{00} \) term does not contribute to \( \tilde{\sigma}_{xx} \), see e.g. [70]), together with the \( g_{1-1} \) and \( g_{-11} \) terms for electrons. After straightforward algebra, assuming \( \omega^2 \ll k_z^2 c^2 \) the result is

\[
0 = k_z^2 c^2 + \sum_{n_1=\pm 1, s=i,e} \omega_{ps}^2 \int d^3 v \, d^2 \hat{s} \frac{\omega}{(\omega - k_z v_z - n_1 \omega_{cs}) \left( \frac{\omega_{cs}}{k_z v_{\perp}} \right)^2} f_{n_1}^2 \left( \frac{k_z v_{\perp}}{\omega_{cs}} \right) \tilde{f}_0
\]

\[
- \frac{3 \omega_{pe}^2 k_z^2 \hbar^2}{8 m_e^2} \sum_{n=\pm 1} \int d^3 v \, d^2 \hat{s} \frac{n \sin^2 \theta_s}{\omega - k_z v_z - n (\omega_c - \omega_{ce})} \left( \frac{k_z v_{\perp}}{\omega_c} \right)^2
\]

\[
\times \left[ \frac{\omega_c}{v_\perp} \frac{\partial \tilde{f}_0}{\partial v_\perp} + n k_z \frac{\partial \tilde{f}_0}{\partial v_z} - \frac{2 m_e}{h} \sin \theta_s \frac{\partial \tilde{f}_0}{\partial \theta_s} \right],
\]

where we have normalized the distribution functions so that \( f_0 = n_0 \tilde{f}_0 \), where \( n_0 \) is the unperturbed number density, and introduced the plasma frequency for each species \( \omega_{ps} = n_0 q_e^2 / \epsilon_0 m_s \). A number of simplifications can be made. Firstly, in the sum over the species, only the ions need to be included. Secondly, for \( k_z v_n / \omega_{cs} \ll 1 \), we may use Taylor expansion of the Bessel functions. Thirdly, for the quantum term only the two pole contributions are kept, and we assume for simplicity that the resonant electron velocity can be approximated as \( v_{res} \equiv (\omega - \Delta \omega_{ce}) / k_z \approx \Delta \omega_{ce} / k_z \), where \( \Delta \omega_{ce} \equiv \omega_{ce} - \omega_{cg} \). Provided that the wave frequency is approximately real, the dispersion relation then simplifies to

\[
k_z^2 c^2 - \omega_{pi}^2 \left[ \frac{\omega_e^2}{\omega_{ci}} + \frac{i \pi \omega}{k_z v_{th}} \exp \left( - \frac{\omega_{ci}^2}{k_z^2 v_{th}^2} \right) \right] + \frac{3 \pi}{4} \frac{k_z \hbar^2 \omega_{pe}^2 k_z^2 \omega}{m_e^2 v_{te} \omega_{ce}^2} \exp \left( - \frac{\Delta \omega_{ce}^2}{k_z^2 v_{th}^2} \right) = 0,
\]

where the first imaginary term is the classical ion contribution and the second imaginary term is the spin contribution from the electrons. Neglecting the damping, we thus have the standard shear Alfvén wave dispersion relation, \( \omega^2 \approx k_z^2 c_A^2 \), with the Alfvén velocity given by \( c_A = c \omega_{ci} / \omega_{pi} \). For parameter values corresponding to typical classical plasmas, the coefficient of the second exponential is much smaller than that of the first exponential term. However, since the quantum term can have a very small exponent, since the resonance may lie in the bulk of the distribution at the same time as the classical resonance lies in the tail, the spin term can be the dominating wave damping mechanism in parts of the wavenumber space. An example of specific plasma parameters is given in figure 2. Here we have introduced the growth rate \( \gamma = \text{Im}(\omega) = \text{Im}(\omega_{ci} + \omega_{sp}) = \gamma_{ci} + \gamma_{sp} \), with the classical and quantum contributions \( \gamma_{ci} \) and \( \gamma_{sp} \) to the growth rate of (78), respectively.

### 9.2. Generalized L and R waves

As a further example in the linear regime, we linearize the full evolution equation, equation (41). To simplify the algebra, we look at waves propagating parallel to a background magnetic field. Following the steps from the last subsection it is straightforward to derive the dispersion relation

\[
\det \left( \begin{bmatrix} \omega^2 - k_z^2 c^2 & 0 & 0 \\ 0 & \omega^2 - k_z^2 c^2 & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} + \frac{i \omega}{\epsilon_0} \tilde{\sigma} \right) = 0,
\]
Figure 2. The dependence of the normalized growth rate $\gamma = (\gamma_{cl} - \gamma_{sp}) / (\gamma_{cl} + \gamma_{sp})$ on the normalized wavenumber $k = k_z v_{thi} / \omega_{ci}$ for $n_0 = 10^{24}$ m$^{-3}$, $B_0 = 10$ T and $k_\perp = 3 \times 10^6$ m$^{-1}$ in a plasma with equal electron and ion temperatures, $T_e = T_i = T$. It is clear that $\gamma \rightarrow -1$ in the spin-dominated damping regime to the left and $\gamma \rightarrow 1$ in the classically dominated regime to the right. Besides depending on the normalized wavenumber, the transition from quantum to classical cyclotron damping depends slightly on the temperature, and the temperatures chosen here are $T = 10^3$, $10^5$ and $10^7$ K. As a specific example we note that for $k = 0.15$ and $T = 10^5$ K, the damping is essentially due to the spin, i.e. $\gamma \approx \gamma_{sp}$, with $\gamma_{sp} = 0.1$ rad s$^{-1}$.

where

$$\tilde{\sigma}_{xx} = \tilde{\sigma}_{yy} = \sum_{\pm} \int d\Omega \frac{1}{\omega \mp \omega_c - kv_z} \left\{ -\frac{iq^2}{2m\omega} (kv_z - \omega) \mp \frac{iq^3 B}{2m^2\omega} \left[ 1 - \cos \left( \frac{i\hbar k}{2m} \partial_{v_z} \right) \right] - \frac{q^2 v^2}{2\hbar \omega} \sin \left( \frac{i\hbar k}{2m} \partial_{v_z} \right) + \frac{3k^2 \mu_B^2}{\hbar \omega} \left[ \cos 2\theta_s \sin \left( \frac{i\hbar k}{2m} \partial_{v_z} \right) \right] \right\} f_0,$$

$$\tilde{\sigma}_{xy} = -\tilde{\sigma}_{yx} = \sum_{\pm} \int d\Omega \frac{1}{\omega \mp \omega_c - kv_z} \left\{ \frac{q^2 v^2}{2m\omega} (kv_z - \omega) + \frac{q^3 B}{2m^2\omega} \left[ 1 - \cos \left( \frac{i\hbar k}{2m} \partial_{v_z} \right) \right] \right\} f_0,$$

$$\tilde{\sigma}_{zz} = -\frac{q^2}{\hbar k} \sum_{\pm} \int d\Omega \frac{1}{\omega - kv_z} \sin \left( \frac{i\hbar k}{2m} \partial_{v_z} \right) f_0,$$

and $\tilde{\sigma}_{xz} = \tilde{\sigma}_{yz} = \tilde{\sigma}_{zx} = \tilde{\sigma}_{xy} = 0$. This dispersion relation reduces to the classical dispersion relation for $L$ and $R$ waves in the limit $\hbar \rightarrow 0$. This dispersion relation clearly shows the contribution from the spin as well as the particle dispersive that becomes significant in the short-wavelength regime.
A thorough discussion of the dispersion relation (79) is beyond the scope of the present paper. However, a few things can be noted on dimensional grounds. Firstly, we see that the higher-order terms in the ‘sin’ and ‘cos’ operators become important for \( k \Lambda_{db} \sim 1 \), where \( \Lambda_{db} \equiv \hbar/(m_e v_{te}) \) is the thermal de Broglie wavelength for the electrons. For collective effects to be significant for such short wavelengths, we need \( \hbar \omega_{pe}/(k_B T_e) \sim 1 \). Secondly, quantum effects associated with the zero-order distribution function tend to be significant either if \( \mu_e B/(k_B T_e) \sim 1 \) (Landau quantization and unsymmetric spin populations [74]) or if \( \hbar^2 n_0^{2/3}/(m_e k_B T_e) \sim 1 \) (Fermi–Dirac rather than Maxwell–Boltzmann statistics). Finally, the spin terms of equation (79) tend to be important in the regime \( \hbar^2 \omega_{pe}^2/mc^2 k_B T_e \sim 1 \). It should be stressed that these estimates may very well have to be revised when a thorough analysis is made, due to e.g. resonance effects.

10. Gauge dependence

The definition of the Wigner function (21) is not gauge invariant since it is a function of the gauge-dependent canonical momentum rather than the gauge-independent kinetic momentum \( m \mathbf{v} = \mathbf{p} - q \mathbf{A}(\mathbf{x}) \). The above theory is hence only valid in the Coulomb gauge. It is possible to modify the definition to obtain a gauge-independent Wigner function [65]. In principle, there is nothing that prevents us from using a gauge-dependent Wigner function as long as care is taken when performing gauge transformations. However, problems may arise when calculating, for example, the third-order moment of the velocity \( \langle \hat{v}_i \hat{v}_j \hat{v}_k \rangle \).

The phase space function that corresponds to the operator \( \hat{v}_i \hat{v}_j \hat{v}_k \) is \( \rho \mathbf{p} = \mathbf{p} - q \mathbf{A}(\mathbf{x}) \) [65]. In order to obtain the correct function it is necessary to first order the operator \( [\hat{p}_i - q A_i(\mathbf{x})][\hat{p}_j - q A_j(\mathbf{x})][\hat{p}_k - q A_k(\mathbf{x})] \) in Weyl ordering [2] and then make the substitution \( \hat{\mathbf{x}} \rightarrow \mathbf{x}, \hat{\mathbf{p}} \rightarrow \mathbf{p} \). This is, in general, difficult to do since the vector potential is a function of \( \mathbf{x} \). However, in the current paper, we have only considered first-order moments of the velocity and the ordering problem will not arise. We may, hence, use our distribution function to calculate, for example, the free charge current

\[
\mathbf{j}_f(\mathbf{x}, t) = \int d^3 \mathbf{v} d^3 \hat{\mathbf{s}} \mathbf{v} f(\mathbf{x}, \mathbf{v}, \hat{\mathbf{s}}, t) = \int d^3 \mathbf{p} d^3 \hat{\mathbf{s}} [\mathbf{p} - q \mathbf{A}(\mathbf{x})] f(\mathbf{x}, \mathbf{p}, \hat{\mathbf{s}}, t)
\]

in agreement with equation (51).

10.1. Gauge invariant distribution function

For completeness, the fully gauge invariant distribution function is given here. Following Serimaa et al [65], the Wigner matrix is defined by

\[
W(\mathbf{x}, \mathbf{v}, \alpha, \beta, t) = \int \frac{d^3 z}{(2\pi \hbar)^3} \exp \left\{ -\frac{i m}{\hbar} \mathbf{v} \cdot \left[ \mathbf{z} + q \int_{-1/2}^{1/2} d\tau \mathbf{A}(\mathbf{x} + \tau \mathbf{z}, t) \right] \right\} \rho \left( \mathbf{x} + \frac{\mathbf{z}}{2}, \alpha; \mathbf{x} - \frac{\mathbf{z}}{2}, \beta \right).
\]

where \( \mathbf{v} \) is the velocity. The explicit dependence of the vector potential in this construction is there to compensate for the phase factor that the wave function acquires under a gauge transformation. Using the spin projection (28), we obtain a fully gauge invariant distribution function. The evolution equation for the gauge invariant distribution function without spin.
was derived in [65]. It is straightforward to generalize this equation to include spin with the result

\[
\frac{\partial f}{\partial t} + (v + \Delta \tilde{v}) \cdot \nabla_x f + \frac{q}{m} \left[ (v + \Delta \tilde{v}) \times \tilde{B} + \tilde{E} \right] \cdot \nabla_v f + \frac{\mu}{m} \nabla_x \left[ (\hat{s} + \nabla_x) \cdot \tilde{B} \right] \cdot \nabla_v f + 2 \frac{\mu}{\hbar} \left[ \hat{s} \times (\tilde{B} + \Delta \tilde{B}) \right] \cdot \nabla_x f = 0, \tag{83}
\]

where we have defined

\[
\tilde{E} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \ E \left( x + \frac{i h \tau}{m} \nabla_v \right) = E(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \cos \left( \frac{\tau \hbar}{m} \nabla_x \cdot \nabla_v \right), \tag{84}
\]

\[
\tilde{B} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \ B \left( x + \frac{i h \tau}{m} \nabla_v \right) = B(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \cos \left( \frac{\tau \hbar}{m} \nabla_x \cdot \nabla_v \right), \tag{85}
\]

\[
\Delta \tilde{v} = -\frac{iq \hbar}{m^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \ \tau B \left( x + \frac{i h \tau}{m} \nabla_v \right) \nabla_v = \frac{q \hbar}{m^2} \left[ B(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \ \tau \sin \left( \frac{\tau \hbar}{m} \nabla_x \cdot \nabla_v \right) \right] \nabla_v, \tag{86}
\]

\[
\Delta \tilde{B} = -\frac{i \hbar}{m} \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \ \tau B \left( x + \frac{i h \tau}{m} \nabla_v \right) \nabla_v \cdot \nabla_v = \frac{\hbar}{m} B(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} \! d\tau \ \tau \sin \left( \frac{\tau \hbar}{m} \nabla_x \cdot \nabla_v \right) \nabla_v \cdot \nabla_v. \tag{87}
\]

Thus, to the lowest order in \( \hbar \), we have

\[
\tilde{E} \approx E(x) \left[ 1 - \frac{\hbar^2}{24m^2} \left( \nabla_x \cdot \nabla_v \right)^2 \right], \tag{88}
\]

\[
\tilde{B} \approx B(x) \left[ 1 - \frac{\hbar^2}{24m^2} \left( \nabla_x \cdot \nabla_v \right)^2 \right], \tag{89}
\]

\[
\Delta \tilde{v} \approx \frac{q \hbar^2}{12m^3} B(x) \times \nabla_v \left( \nabla_x \cdot \nabla_v \right), \tag{90}
\]

\[
\Delta \tilde{B} \approx \frac{\hbar^2}{12m^2} B(x) \left( \nabla_x \cdot \nabla_v \right)^2, \tag{91}
\]

so that (cf equation (40))

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \left[ \frac{q}{m} (E + v \times B) \cdot \nabla_v + \frac{\mu_B}{m} \nabla_x \left[ (\hat{s} + \nabla_x) \cdot B \right] \cdot \nabla_v + \frac{2\mu_B}{\hbar} (\hat{s} \times B) \cdot \nabla_v \right] f
\]

\[
= \frac{\hbar^2}{24m^2} \left\{ \left[ \frac{q}{m} (E + v \times B) \cdot \nabla_v + \frac{\mu_B}{m} \nabla_x \left[ (\hat{s} + \nabla_x) \cdot B \right] \cdot \nabla_v - \frac{2\mu_B}{\hbar} (\hat{s} \times B) \cdot \nabla_v \right] \left( \nabla_x \cdot \nabla_v \right)^2 - 2 \left[ \frac{q}{m} B \times \nabla_v \left( \nabla_x \cdot \nabla_v \right) \right] \left( \frac{q}{m} B \times \nabla_v + \nabla_x \right) \right\} f. \tag{92}
\]
Various plasma regimes in the temperature–density parameter space are illustrated. The dotted line is given by the strong coupling parameter \( \Gamma = E_p / k_B T = 1 \), where \( E_p = e^2 n^{1/3} / 4\pi \varepsilon_0 \) is the potential energy due to the nearest neighbor. For larger densities, this parameter is replaced by \( \Gamma_F = E_p / k_B (T + T_F) \), since the average kinetic energy of the particles is given by the Fermi energy rather than the thermal energy. \( \Gamma_F = 1 \) is illustrated by the dashed curve, and the strong coupling region, which is shaded, occurs below this line. In this region, our model is not directly applicable, since collisions have not been taken into account. For comparison, we have also drawn the line \( \hbar \omega_p / k_B T \) (the dotted gray line) and the line \( T_F / T \) (the dotted–dashed gray line), which measure the importance of wave function dispersion and the Fermi pressure, respectively. As a rough estimate, the quantum regime is below either of these lines. Note, however, that spin effects can sometimes be important even above these lines [28].

The gauge invariant Wigner function has a modified Weyl correspondence, which is well suited for calculating fluid moments. In order to obtain the phase space \( O(x, v) \) function that corresponds to an operator \( O(\hat{x}, \hat{v}) \), all products of the operators \( \hat{x} \) and \( \hat{v} \equiv [\hat{p} - q A(\hat{x})] / m \) are first ordered in a symmetric form using the commutation relation \( \hat{x} \) and \( \hat{p} \) and then the substitution \( \hat{x} \rightarrow x \) and \( \hat{v} \rightarrow v \) is taken (details can be found in [65]).

11. Summary and discussion

In the present paper, we have derived an evolution equation, equation (41), for a quasi-distribution function of electrons, based on a Wigner transformation of the density matrix, together with a spin operator contracting the \( 2 \times 2 \) Wigner matrix to a scalar function \( f(x, p, s) \). The free current and the magnetization can be directly computed from the quasi-distribution function, and hence equation (39) (or the gauge invariant alternative, equation (83)), together with Maxwell’s equations with the sources (50) and (51), form a closed set. The present theory has the advantage of including the full quantum dynamics in a single equation and provides an immediate path between the classical and quantum descriptions. For macroscopic scale lengths longer than the characteristic de Broglie wavelength, the kinetic equation is greatly simplified.
In particular, the semi-classical kinetic theory proposed by Bordin et al [61] is recovered, but with some small but significant deviations, as shown by Manfredi et al (64). The difference between the semi-classical theory and our result follows from the smeared out probability distribution of the spin.

In order to illustrate the theory, examples of linear wave propagation solving the full quantum theory, equation (41), as well as the long wavelength limit, equation (64), are given. An interesting result is that the wave damping can be significantly affected by the non-classical terms even in a supposedly classical temperature and density regime, although the real part of the wave frequency then is always well approximated by the classical Vlasov theory. The reason is that the spin terms give rise to new types of wave particle resonances.

Proper initial conditions for the quasi-distribution function can be found by computing the Wigner transformation of the density matrix in the thermodynamical ground state. The result for the simple but important special case of a magnetic field is given, see equation (60), in which case the energy levels are Landau quantized and split due to the two spin states [74].

The present quantum theory can be used for a broad range of parameters, extending the applicability to regimes of high density, strong magnetic field, low temperature and short scalelength, which are not covered by the classical Vlasov equation. However, there is still considerable room for improvement. In particular, when the strong coupling parameter $\Gamma$ is increased, collisional effects become important [71]. A schematic view of the different plasma regimes is given in figure 3. Furthermore, the present theory does not account for relativistic effects that are crucial in, e.g., laser–plasma interaction. Removal of these restrictions, as well as a more complete evaluation of the present theory, constitutes interesting projects for future research.

Acknowledgments

This research was supported by the European Research Council under contract no. 204059-QPQV and the Swedish Research Council under contract no. 2007-4422.

References

[1] Wigner E 1932 Phys. Rev. 40 749
[2] Weyl H 1931 The Theory of Groups and Quantum Mechanics (New York: Dover)
[3] Groenewold A 1946 Physica 12 405
[4] Moyal J E 1949 Proc. Camb. Phil. Soc. 45 99
[5] Zachos C K, Fairlie D B and Curtright T L (ed) 2005 Quantum Mechanics in Phase Space: An Overview with Selected Papers (Singapore: World Scientific)
[6] Leonhardt U 1997 Measuring the Quantum State of Light (Cambridge: Cambridge University Press)
[7] Haug H and Jauho A-P 2007 Quantum Kinetics in Transport and Optics of Semiconductors (Springer Series in Solid-State Sciences vol 123) (Berlin: Springer)
[8] Baym G and Kadanoff L P 1961 Phys. Rev. 124 287
[9] Kadanoff L P and Baym G 1962 Quantum Statistical Mechanics (New York: Benjamin)
[10] Rammer J 2007 Quantum Field Theory of Non-Equilibrium States (Cambridge: Cambridge University Press)
[11] Keldysh L P 1958 Sov. Phys.—JETP 7 788
[12] Keldysh L P 1965 Sov. Phys.—JETP 20 1018
[13] Eliezer S 2009 Applications of Laser–Plasma Interactions (London: Taylor and Francis)
[14] Drake R P 2006 High-Energy-Density Physics (Berlin: Springer)

New Journal of Physics 12 (2010) 043019 (http://www.njp.org/)
[15] Anderson D, Hall B, Lisak M and Marklund M 2002 Phys. Rev. E 65 046417
[16] Marklund M 2005 Phys. Plasmas 12 082110
[17] Shukla P K, Ali S, Stenflo L and Marklund M 2006 Phys. Plasmas 13 112111
[18] Shukla N, Brodin G, Marklund M, Shukla P K and Stenflo L 2009 Phys. Plasmas 16 072114
[19] Shukla P K and Eliasson B 2009 arXiv:0906.4051
[20] Glenzer S H et al 2007 Phys. Rev. Lett. 98 065002
[21] Kritcher A L et al 2008 Science 322 69
[22] Lee H J et al 2008 Phys. Rev. Lett. 102 115001
[23] Alvisatos A P 1996 Science 271 933
[24] Haas F 2005 Phys. Plasmas 12 062117
[25] Manfredi G and Hervieux P-A 2007 Appl. Phys. Lett. 91 061108
[26] Maier S A 2007 Plasmonics (New York: Springer)
[27] Marklund M, Brodin G, Stenflo L and Liu C S 2008 Europhys. Lett. 84 17006
[28] Brodin G, Marklund M and Manfredi G 2008 Phys. Rev. Lett. 100 175001
[29] Melrose D 2008 Quantum Plasmdynamics (New York: Springer)
[30] Stancil D D and Prabhakar A 2009 Spin Waves (New York: Springer)
[31] Bertotti G, Mayergoyz I and Serpico C 2009 Nonlinear Magnetization Dynamics in Nanosystems (Amsterdam: Elsevier)
[32] Pethick C J and Smith H 2008 Bose–Einstein Condensation in Dilute Gases (Cambridge: Cambridge University Press)
[33] Marklund M and Brodin G 2007 Phys. Rev. Lett. 98 025001
[34] Brodin G and Marklund M 2007 Phys. Plasmas 14 112107
[35] Brodin G and Marklund M 2007 Phys. Rev. E 76 055403
[36] Grossman F 2008 Theoretical Femtosecond Physics (Berlin: Springer)
[37] Kurtz U, Pfau T and Mlynek J 1997 Nature 386 150–3
[38] Glauber R J 1963 Phys. Rev. Lett. 10 84
[39] Sudharshan E C G 1963 Phys. Rev. Lett. 10 277
[40] Husimi K 1940 Proc. Phys.—Math. Soc. Japan 22 264
[41] Hillery H, O’Connell R F, Scully M O and Wigner E P 1984 Phys. Rep. 106 121
[42] Lee H-W 1995 Phys. Rep. 259 147
[43] Styer D F 1996 Am. J. Phys. 64 31
[44] Ballentine L E 1998 Quantum Mechanics: A Modern Development (Singapore: World Scientific)
[45] Daugera D E, Decyk V K and Dawson J M 2005 J. Comput. Phys. 209 559
[46] Ballentine L E, Yang Y and Zibin J P 1994 Phys. Rev. A 50 2854
[47] Schleich W P 2001 Quantum Optics in Phase Space (Berlin: Wiley)
[48] Cohen L 1986 Frontiers of Nonequilibrium Statistical Physics ed G T Moore and M O Scully (New York: Plenum)
[49] Harriman J E 1988 J. Chem. Phys. 88 6399
[50] Lin W A and Ballentine L E 1990 Phys. Rev. Lett. 65 2927
[51] Izrailev F M 1990 Phys. Rep. 196 299
[52] Stöckmann H-J 1999 Quantum Chaos: An Introduction (Cambridge: Cambridge University Press)
[53] Stratonovic R L 1970 Zh. Eksp. Teor. Fiz. 58 1612
[54] Stratonovic R L 1970 Sov. Phys.—JETP 31 864 (Engl. Transl.)
[55] Scully M O 1983 Phys. Rev. D 28 2477
[56] Cohen L and Scully M O 1986 Found. Phys. 16 295
[57] Chandler C, Cohen L, Lee C, Scully M and Wódkiewicz K 1992 Found. Phys. 22 867
[58] Carruthers P and Zachariasen F 1983 Rev. Mod. Phys. 55 1
[59] Scully M O and Wódkiewicz K 1994 Found. Phys. 24 85
[60] Radcliffe J M 1971 J. Phys. A. Math. Gen. 4 313

New Journal of Physics 12 (2010) 043019 (http://www.njp.org/)
[60] Areccchi F T, Courtens E, Gilmore R and Thomas H 1971 Phys. Rev. A 6 2211
[61] Brodin G, Marklund M, Zamanian J, Ericsson Å and Mana P L 2008 Phys. Rev. Lett. 101 245002
[62] Arnold A and Steinarch H 1989 Z. Angew. Math. Phys. 40 6
[63] O’Connell R F and Wigner E P 1984 Phys. Rev. A 30 5
[64] Manfredi G, Hervieux P A, Yin Y and Crouseilles N 2009 Advances in the Atomic-Scale Modeling of Nanosystems and Nanostructured Materials (Lecture Notes in Physics) ed C Massobrio, H Bulou and C Goyenex (Heidelberg: Springer)
[65] Serimaa O T, Javanainen J and Varró S 1986 Phys. Rev. A 33 2913
[66] Bonitz M 1998 Quantum Kinetic Theory (Leipzig: B G Tubner Stuttgart)
[67] De Groot S R and Suttorp L G 1972 Foundations of Electrodynamics (Amsterdam: North-Holland)
[68] Abramowitz M and Stegun I A (ed) 1972 Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (New York: Dover)
[69] Bogolyubov N N 1946 Zh. Eksp. Teor. Fiz. 16 691
   Bogolyubov N N 1946 J. Exp. Theor. Phys. 10 265 (Engl. Transl.)
[70] Swanson D G 2003 Plasma Waves (London: Taylor and Francis)
[71] Fortov V, Iakubov I and Khrapak A 2006 Strongly Coupled Plasma (New York: Oxford University Press)
[72] Cowley S C, Kulsrud R M and Valeo E 1986 Phys. Fluids 29 430
[73] Demircioglu B and Vercin A 2003 Ann. Phys. 305 1
[74] Landau L D and Lifshitz E M 1981 Quantum Mechanics: Non-Relativistic Theory (Oxford: Butterworth-Heinemann)
[75] Wigner E 1971 Perspectives in Quantum Theory ed W Yourgrau and A van der Merwe (Cambridge, MA: MIT Press)
[76] Mizrahi S S 1988 Physica A 150 541