The classical limit of mean-field theories

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Abstract Since the fibers $A_{1/N} = M_{N+1}(C) \cong B(\text{Sym}^N(C^2))$ and $A_0 = C(S^2)$ form a continuous bundle of $C^*$-algebras over the base space $I = \{0\} \cup 1/N \subset [0,1]$, one can define a strict deformation quantization of $A_0$ where quantization is specified by certain quantization maps $Q_{1/N} : \hat{A}_0 \to A_{1/N}$, with $A_0$ a dense Poisson subalgebra of $A_0$. In this paper we prove the existence of the classical limit of mean-field quantum spin chains whose Hamiltonians $H_N$ typically satisfy $H_N|_{\text{Sym}^N(C^2)} = Q_{1/N}(h_0) + \sum_{k=1}^M N^{-k} Q_{1/N}(h_k)$, for some natural number $M$ and real polynomials $h_0$ and $h_k$ on the unit two-sphere $S^2$ ($k = 1, \ldots, M$), where $\text{Sym}^N(C^2)$ denotes the symmetric subspace of the Hilbert space $\bigotimes_{n=1}^N C^2$ on which $H_N$ is originally defined. Under some assumptions we show that if a sequence of eigenvectors $\psi_N$ of $H_N|_{\text{Sym}^N(C^n)}$ is invariant under some group action, then this sequence has a classical limit in the sense that $\omega_0(f) := \lim_{N \to \infty} \langle \psi_N, Q_{1/N}(f) \psi_N \rangle$ exists as a state on $C(S^2)$ given by $\omega_0(f) = \frac{1}{\pi} \sum_{i=1}^n f(\Omega_i)$. We give an application regarding spontaneous symmetry breaking (SSB) and moreover we show that the spectrum of a mean field quantum spin system converges to the range of the polynomial $h_0$, in that $\lim_{N \to \infty} \text{dist}(\sigma(H_N|_{\text{Sym}^N(C^2)}), \text{ran}(h_0)) = 0$, where dist denotes the distance function.
1 Introduction

This paper focusses on the relation between classical (spin) theories viewed as limits of certain (mean-field) quantum theories. A first well-known example is classical mechanics of a particle on the phase space $\mathbb{R}^{2n}$ versus quantum mechanics on the Hilbert space $L^2(\mathbb{R}^n)$. The relation between these both a priori different theories can be described by a continuous bundle of $C^*$-algebras of observables equipped with quantization maps. A modern and rigorous way to do this is based on the concept of strict deformation quantization, i.e., the mathematical formalism that describes the transition from a classical theory to a quantum theory in terms of deformations of (commutative) Poisson algebras (representing the classical theory) into non-commutative $C^*$-algebras characterizing the quantum theory [24, 25]. This machinery allows us to prove the existence of physical phenomena often arising in the classical theory, but which do not exist in the corresponding underlying quantum theory. A typical example of such an emergent feature is spontaneous symmetry breaking (SSB). In this paper we focus on quantum spin systems, i.e. we study the asymptotic relation between classical thermodynamics of a classical spin system and statistical mechanics of a quantum (mean field) spin system on a finite lattice [15], as the number of lattice sites increases to infinity.
1.1 Strict deformation quantization

Before giving the mathematical definitions, let us go back to our first example concerning the quantization of the classical phase space $\mathbb{R}^{2n}$ on which classical mechanics of a particle is described. For convenience we take the simplest functional-analytic setting in which only smooth compactly supported functions $f \in C_c(\mathbb{R}^{2n})$ (with Poisson structure given by the natural symplectic form $\sum_{j=1}^n dp_j \wedge dq_j$) are quantized. In order to relate $C_c(\mathbb{R}^{2n})$ to a quantum theory described on some Hilbert space, one needs to deform $C_c(\mathbb{R}^{2n})$ into non-commutative $C^*$-algebras exploiting a family of quantization maps. In this setting Weyl introduced the maps

$$Q_\hbar : C^\infty(\mathbb{R}^{2n}) \rightarrow B_0(L^2(\mathbb{R}^n));$$

$$Q_\hbar(f) = \int_{\mathbb{R}^{2n}} \frac{d^n p d^n q}{(2\pi \hbar)^n} f(p, q) |\phi_h^{(p,q)}\rangle \langle \phi_h^{(p,q)}|,$$

where $\hbar \in (0, 1]; B_0(H)$ is the $C^*$-algebra of compact operators on the Hilbert space $H = L^2(\mathbb{R}^n)$, and for each point $(p, q) \in \mathbb{R}^{2n}$ the (projection) operator $|\phi_h^{(p,q)}\rangle \langle \phi_h^{(p,q)}| : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is induced by the normalized wavefunctions, where $x \in \mathbb{R},$

$$\phi_h^{(p,q)}(x) = (\pi \hbar)^{-n/4} e^{-ipq/2\hbar} e^{-i}px/\hbar e^{-(x-q)^2/2\hbar}, \quad \phi_h^{(p,q)} \in L^2(\mathbb{R}),$$

defining the well-known (Schrodinger) coherent states. Inspired by Dixmier’s concept of a continuous bundle [8], Rieffel showed that [24, 25]

1. The fibers $A_0 = C_0(\mathbb{R}^{2n})$ and $A_\hbar = B_0(L^2(\mathbb{R}^n)), \ h \in (0, 1]),$ can be combined into a (locally non-trivial) continuous bundle $A$ of $C^*$-algebras over $I = [0, 1];$

2. $\tilde{A}_0 = C^\infty(\mathbb{R}^{2n})$ is a dense Poisson subalgebra of $A_0.$

3. Each quantization map $Q_\hbar : \tilde{A}_0 \rightarrow A_\hbar$ is linear, and if we also define $Q_0 : \tilde{A}_0 \hookrightarrow A_0$ as the inclusion map, then the ensuing family $Q = (Q_\hbar)_{\hbar \in I}$ satisfies:

(a) Each map $Q_\hbar$ is self-adjoint, i.e. $Q_\hbar(f) = Q_\hbar(f)^*$ (where $f^*(x) =$
(b) For each \(f \in \tilde{A}_0\) the following cross-section of the bundle is continuous:

\[
0 \to f; \quad \hbar \to Q_\hbar(f) \quad (\hbar \in I \setminus \{0\}).
\]

\[
0 \to f; \quad \hbar \to Q_\hbar(f) \quad (\hbar \in I \setminus \{0\}).
\]

(c) Each pair \(f, g \in \tilde{A}_0\) satisfies the **Dirac-Groenewold-Rieffel condition**:

\[
\lim_{\hbar \to 0} \left\| \frac{i}{\hbar} [Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\}) \right\|_h = 0.
\]

These observations led to the general concept of a **strict deformation of a Poisson manifold** \(X\), which we here state in the case of interest to us in which \(X\) is compact or in which \(X\) is a manifold with stratified boundary \(\{16, 22\}\). In that case the space \(I\) in which \(\hbar\) takes values cannot be all of \([0, 1]\), but should be a subspace \(I \subset [0, 1]\) thereof that at least contains 0 as an accumulation point. This is assumed in what follows. Furthermore, the Poisson bracket on \(X\) is denoted, as usual, by \(\{\cdot, \cdot\} : C^\infty(X) \times C^\infty(X) \to \mathbb{C}\) (where the smooth space \(C^\infty(X)\) is suitably defined when \(X\) is a more complicated object than a compact smooth manifold as we shall say shortly).

**Definition 1.1.** Let \(I \subset \mathbb{R}\) containing 0 as accumulation point. A **strict deformation quantization** of a compact Poisson manifold \(X\) on \(I\) consists of

- A continuous bundle of unital \(C^*\)-algebras \((A_\hbar)_{\hbar \in I}\) over \(I\) with \(A_0 = C(X)\);

- A dense Poisson subalgebra \(\tilde{A}_0 \subseteq C^\infty(X) \subset A_0\) (on which \(\{\cdot, \cdot\}\) is defined);

- A family \(Q = (Q_\hbar)_{\hbar \in I}\) of linear maps \(Q_\hbar : \tilde{A}_0 \to A_\hbar\) indexed by \(\hbar \in I\) (called **quantization maps**) such that \(Q_0\) is the inclusion map \(\tilde{A}_0 \hookrightarrow A_0\), and the next conditions (a) - (c) hold, as well as \(Q_\hbar(1_X) = 1_{A_\hbar}\) (the unit of \(A_\hbar\))

  1. Each map \(Q_\hbar\) is self-adjoint, i.e. \(Q_\hbar(\overline{f}) = Q_\hbar(f)^*\) (where \(f^*(x) = f(x)\)).
2. For each \( f \in \tilde{A}_0 \) the following cross-section of the bundle is continuous:

\[
0 \rightarrow f; \quad (1.7)
\]
\[
\hbar \rightarrow Q_{\hbar}(f) \quad (\hbar \in I\setminus\{0\}). \quad (1.8)
\]

3. Each pair \( f, g \in \tilde{A}_0 \) satisfies the \textbf{Dirac-Groenewold-Rieffel condition}:

\[
\lim_{\hbar \to 0} \left| \frac{i}{\hbar} [Q_{\hbar}(f), Q_{\hbar}(g)] - Q_{\hbar}\{f,g\} \right|_{\hbar} = 0. \quad (1.9)
\]

\textbf{Remark 1.2.} It follows from the definition of a continuous bundle of \( C^* \)-algebras that continuity properties like

\[
\lim_{\hbar \to 0} \|Q_{\hbar}(f)\|_{\hbar} = \|f\|_{\infty}, \quad (1.10)
\]

and

\[
\lim_{\hbar \to 0} \|Q_{\hbar}(f)Q_{\hbar}(g) - Q_{\hbar}(fg)\|_{\hbar} = 0, \quad (1.11)
\]

hold automatically.

\textit{Mean-field quantum spin systems} fit into this framework. There, the index set \( I \) is given by \((0 \notin \mathbb{N} := \{1, 2, 3, \ldots\})\)

\[
I = \{1/N \mid N \in \mathbb{N}\} \cup \{0\} \equiv \{1/N\} \cup \{0\}, \quad (1.12)
\]

with the topology inherited from \([0, 1]\). That is, we put \( \hbar = 1/N \), where \( N \in \mathbb{N} \) is interpreted as the number of sites of the model; our interest is the limit \( N \to \infty \). In this setting the above definition particularly applies when \( X = S^2 \subset \mathbb{R}^3 \), i.e. the classical phase space for quantum mean-field spin systems. Indeed, the following shows the existence of a strict deformation quantization of \( S^2 \) with Poisson bracket given on \( C^\infty(S^2) \) by

\[
\{f, g\}(x) = \sum_{a,b,c=1}^3 \varepsilon_{acb} \frac{\partial f(x)}{\partial a} \frac{\partial f(x)}{\partial b}, \quad (1.13)
\]

A typical example of a mean field quantum spin system is the Curie–Weiss model (see for example [11, 13, 30, 29] and references therein).
where $\varepsilon_{ab}^{c}$ the Levi-Civita symbol. Let us hereto indicate the algebra of bounded operators on $\text{Sym}^N(\mathbb{C}^2)$ by $B(\text{Sym}^N(\mathbb{C}^2))$. Here $\text{Sym}^N(\mathbb{C}^2) \subseteq \bigotimes_{n=1}^N \mathbb{C}^2$ is the symmetric subspace of dimension $N + 1$. More precisely, $\text{Sym}^N(\mathbb{C}^2)$ is given by the range of the symmetrizer operator $P_N$, defined by linear extension of the following map $P_N$ on elementary tensors ($v_1 \otimes \cdots \otimes v_N$):

$$P_N(v_1 \otimes \cdots \otimes v_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}. \quad (1.14)$$

It is known [15, Theorem 8.1] that

$$A_0 = C(S^2); \quad (1.15)$$

$$A_{1/N} = M_{N+1}(\mathbb{C}) \cong B(\text{Sym}^N(\mathbb{C}^2)), \quad (1.16)$$

are the fibers of a continuous bundle of $C^*$-algebras over base space $I$ defined by

$$I = \{1/N \mid N \in \mathbb{N}\} \cup \{0\} = (1/N) \cup \{0\}. \quad (1.17)$$

The continuous cross-sections are given by all sequences $(a_{1/N})_{N \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} A_{1/N}$ for which $a_0 \in C(S^2)$ and $a_{1/N} \in M_{N+1}(\mathbb{C})$ and such that the sequence $(a_{1/N})_{N \in \mathbb{N}}$ is asymptotically equivalent to $(Q_{1/N}(a_0))_{N \in \mathbb{N}}$, in the sense that

$$\lim_{N \to \infty} ||a_{1/N} - Q_{1/N}(a_0)||_N = 0. \quad (1.18)$$

Here, the symbol $Q_{1/N}$ denotes the quantization maps

$$Q_{1/N} : \tilde{A}_0 \to A_{1/N}, \quad (1.19)$$

where $\tilde{A}_0 \subset C^\infty(S^2) \subset A_0$ is the dense Poisson subalgebra made of polynomials in three real variables restricted to the sphere $S^2$. The maps $Q_{1/N}$ are defined by the integral computed in weak sense

$$Q_{1/N}(p) := \frac{N + 1}{4\pi} \int_{S^2} p(\Omega) |\Omega\rangle \langle \Omega|_{N} d\Omega, \quad (1.20)$$

$^2$The reason we recall the space $\text{Sym}^N(\mathbb{C}^2)$ rather than the isomorphic space $\mathbb{C}^{N+1}$ is because of our application to mean field quantum spin chains, whose Hamiltonians originally defined on the $N$-fold tensor-product of $\mathbb{C}^2$ with itself, typical leave the symmetric subspace $\text{Sym}^N(\mathbb{C}^2)$ given by the symmetric $N$-fold tensor product of $\mathbb{C}^2$ with itself, invariant.

$^3$Equivalent definitions of these quantization maps are used in literature, see e.g. [15, 21].

6
where $p$ denotes an arbitrary polynomial restricted to $S^2$, $d\Omega$ indicates the unique $SO(3)$-invariant Haar measure on $S^2$ with $\int_{S^2} d\Omega = 4\pi$, and $|\Omega\rangle\langle\Omega|_N \in B(\text{Sym}^N(\mathbb{C}^2))$ are the projectors associated to the $N$ coherent spin states (see Appendix 3.3 for a general construction). In particular, if 1 is the constant function $1(\Omega) = 1$ ($\Omega \in S^2$), and $1_N$ is the identity on $A_{1/N} = B(\text{Sym}^N(\mathbb{C}^2))$, the previous definition implies

$$Q_{1/N}(1) = 1_N .$$

(1.21)

Indeed, it can be shown that the quantization maps (1.20) - (1.21) satisfy the axioms of Definition 1.1, which implies the existence of a strict deformation quantization of $S^2$.

These quantization maps, constructed from a family coherent states, also define a so-called pure state Berezin quantization [14] for which (1.22), viz.

$$f(\Omega) = \lim_{N \to \infty} \frac{N + 1}{4\pi} \int_{S^2} d\Omega' f(\Omega')|\langle\Omega', \Omega\rangle_N|^2, \quad (\Omega \in S^2)$$

(1.22)

typically holds as well as positivity, in that $Q_{1/N}(f) \geq 0$ if $f \geq 0$ almost everywhere on $S^2$. Moreover, for all $N \in \mathbb{N}$ one has

$$1 = \frac{N + 1}{4\pi} \int_{S^2} f(\Omega)|\langle\Omega', \Omega\rangle_N|^2 .$$

(1.23)

### 1.2 Mean-field theories and symbol

We consider homogenous mean-field theories, which are defined by a single-site Hilbert space $\mathcal{H}_x = \mathcal{H} = \mathbb{C}^n$ and local Hamiltonians of the type

$$H_\Lambda = |\Lambda|\tilde{h}(T_0^{(A)}, T_1^{(A)}, \cdots, T_{n^2-1}^{(A)}) ,$$

(1.24)

where $\tilde{h}$ is a polynomial on $M_n(\mathbb{C})$, and $\Lambda$ denotes a finite lattice on which $H_\Lambda$ is defined [15]. Here $T_0 = 1_{M_n(\mathbb{C})}$, and the matrices $(T_i)_{i=1}^{n^2-1}$ in $M_n(\mathbb{C})$ form a basis of the real vector space of traceless self-adjoint $n \times n$ matrices; the latter may be identified with $i$ times the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$, so

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4We remark that $S^2$ is a special case of a regular integral coadjoint orbit in the dual of the Lie algebra associated to $SU(2)$, which can be identified with $\mathbb{R}^3$. In fact, this theory can be generalized to arbitrary compact connected Lie groups [14].
that \( (T_0, T_1, ..., T_{n^2-1}) \) is a basis of \( i \) times the Lie algebra \( u(n) \) of the unitary group \( U(n) \) on \( \mathbb{C}^n \). In those terms, we define

\[
T_i^{(A)} = \frac{1}{|A|} \sum_{x \in A} T_i(x).
\] (1.25)

Let us now introduce the notion of a classical symbol, i.e. a function

\[
h_N := \sum_{k=0}^{M} N^{-k} h_k + O(N^{\infty}),
\] (1.26)

for some \( M \in \mathbb{N} \) and where each \( h_k \) is a smooth real-valued function on the manifold \( M \) one considers. The first term \( h_0 \) is called the principal symbol. In the case for mean field quantum theories we will see below that the principal symbol exactly plays the role of the polynomial \( \tilde{h} \) defined above.

We remark that a general mean-field model is defined on a lattice with \( N \) sites. However, as the geometric configuration including its dimension is irrelevant, as is typical for mean-field models \([16, 30]\), we therefore restrict ourselves to mean field quantum spin chains, i.e. we consider tensor products of \( M_2(\mathbb{C}) \). Moreover, their Hamiltonians share the property that they leave the symmetric subspace \( \text{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2 \) invariant. In what follows we consider mean field quantum spin systems whose Hamiltonians \( H_{1/N} \) are restricted to this subspace, since quantum spin systems arising in that way are typically of the form \( Q_{1/N}(h_N) \) for some (\( N \)-dependent) real polynomial function \( h_N \) on \( M = S^2 \) given by \( 1.26 \) and \( Q_{1/N} \) given by \( 1.20 - 1.21 \) (see e.g. \[19\] Theorem 3.1) for an example). Such spin systems are widely studied in (condensed matter) physics, but also in mathematical physics they form an important field of research. One tries to calculate the corresponding partition function, or quantities like the free energy or the entropy of the system in question and considers their thermodynamic limit as the number of sites \( N \) increases to infinity. Therefore, the Bloch sphere \( S^2 \) can be seen as a classical phase space describing a collection of \( N \) two-level atoms corresponding to a spin chain of \( N \) sites, given by a mean field quantum spin Hamiltonian \( H_{1/N}|_{\text{Sym}^N(\mathbb{C}^2)} \).

**Example 1.** Consider the scaled-quantum Curie-Weiss Hamiltonian on a
Here $\sigma_k(j)$ stands for $I_2 \otimes \cdots \otimes \sigma_k \otimes \cdots \otimes I_2$, where $\sigma_k$ occupies the $j$-th slot, and $J, B \in \mathbb{R}$ are given constants defining the strength of the spin-spin coupling and the (transverse) external magnetic field, respectively. Regarding (1.24) it is not difficult to see that

$$\tilde{h}_{CW}^N(T_1, T_2, T_3) = -2(JT_3^2 + BT_1),$$

(1.28)

where $T_i = \frac{1}{2}\sigma_i$. In order to determine the symbol we show that the Hamiltonian is a quantization of the classical Curie-Weiss Hamiltonian, i.e. a polynomial $h_{CW}^0$ on $S^2$ given (in spherical coordinates) by

$$h_{CW}^0(\theta, \phi) = -1/2(J \cos(\theta)^2 + B \sin(\theta) \cos(\phi)).$$

(1.29)

Hereto we recall a result originally obtained by Lieb [17], namely that under the maps $Q_{1/N}$ given by (1.20) - (1.21) one has a correspondence between functions $G$ (also called upper symbol) on the sphere $S^2$ and operators $A_G$ on $\mathbb{C}^{N+1}$ such that they satisfy the relation $A_G = Q_{1/N}(G)$. For some spin operators, the functions $G$ are determined (see Table 1 below). Here, $S_z$ is the total spin operator in the $z$-direction: $S_z = \frac{1}{2} \sum_k \sigma_z(k)$, $S_x$ is the total spin operator in the $x$-direction, and $S_y$ is the total spin operator in the $y$-direction. Using these results, a straightforward computation shows that

$$H_{1/N}^{CW}|_{\text{Sym}^N(\mathbb{C}^2)} = Q_{1/N}(h_0^{CW}) - \frac{3J}{2N}Q_{1/N}(z^2) + \frac{1}{2N}Q_{1/N}(1).$$

(1.30)

We write $h_N := h_0^{CW} + N^{-1}((-3Jz^2 + 1)/2)$, so that by linearity of $Q_{1/N}$ one has $Q_{1/N}(h_N) = H_{1/N}^{CW}|_{\text{Sym}^N(\mathbb{C}^2)}$. The function $h_N : S^2 \to \mathbb{R}$ is the classical symbol associated to the quantum spin operator $H_{1/N}^{CW}|_{\text{Sym}^N(\mathbb{C}^2)}$ with principal symbol $h_0^{CW}$ which indeed has the same form as $\tilde{h}_{CW}^N$ in (1.28).

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5 The scaling factor is given by $1/(N + 2)$. 

9
Table 1: Spin operators on $\text{Sym}^N(\mathbb{C}^2) \cong \mathbb{C}^{N+1}$ and their corresponding upper symbols $G$.  

| Spin Operator | $G(\theta, \phi)$ |
|---------------|-------------------|
| $S_z$         | $\frac{1}{4}(N+2)\cos(\theta)$ |
| $S_x$         | $\frac{1}{4}(N+2)(N+3)\cos(\theta)^2 - \frac{1}{4}(N+2)$ |
| $S_y$         | $\frac{1}{4}(N+2)\sin(\theta)\cos(\phi)$ |
| $S_z^2$       | $\frac{1}{4}(N+2)(N+3)\cos(\phi)\sin(\theta) - \frac{1}{4}(N+2)$ |
| $S_x^2$       | $\frac{1}{4}(N+2)\sin(\theta)\sin(\phi)$ |
| $S_y^2$       | $\frac{1}{4}(N+2)(N+3)\sin(\phi)\sin(\theta) - \frac{1}{4}(N+2)$ |

Example 2. Let us consider the Lipkin-Meshkov-Glick (LMG) model\footnote{It was first proposed to describe phase transitions in atomic nuclei \cite{18, 10}, and more recently it was found that the LMG model is relevant to many other quantum systems, such as cavity QED \cite{20}.}. The (scaled) Hamiltonian of a general LMG model\footnote{Similarly as for the CW model we have rescaled the LMG model by the factor $1/(N+2)$.} is given by

$$H_{1/N}^{LMG} = -\frac{\lambda}{N(N+2)}(S_x^2 + \gamma S_y^2) - \frac{B}{N+2}S_z, \quad (1.31)$$

where as before $S_x = \frac{1}{2} \sum_k \sigma_x(k)$ is the total spin operator in direction $x$ and so on. We are interested in $\lambda > 0$, standing for a ferromagnetic interaction, $\gamma \in (0,1]$ describing the anisotropic in-plane coupling, and $B$ is the magnetic field along $z$ direction with $B \geq 0$. By a similar computation as before, we find

$$H_{1/N}^{LMG}|_{\text{Sym}^N(\mathbb{C}^2)} = Q_{1/N}(h_0^{LMG}) + \frac{1}{N}Q_{1/N}(1) - \frac{3}{2N}Q_{1/N}(x^2 + \gamma y^2), \quad (1.32)$$

where $h_0^{LMG} := -\frac{1}{4}(x^2 + \gamma y^2) - Bz$ denotes the principal symbol of $h_N := h_0^{LMG} + N^{-1}(-\frac{3}{2}(x^2 + \gamma y^2) + 1)$.

1.3 Classical limit

Let us first stress that the limit $N \to \infty$ can be taken in two entirely different ways, which depends on the class of observables one considers, namely either
quasi-local observables or macroscopic observables. The former are the ones traditionally studied for quantum spin systems, but the latter relate these systems to strict deformation quantization, since macroscopic observables are precisely defined by (quasi-)symmetric sequences which form the continuous cross sections of a continuous bundle of \( C^* \)-algebras [16, 19, 15]. In [19, Theorem 2.3] it has been shown that the quantization maps \((1.20) - (1.21)\) define quasi-symmetric sequences, and hence macroscopic observables.

We then consider a sequence of eigenvectors \( \psi_N \in \text{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2 \) of a mean field quantum spin Hamiltonian \( H_N|_{\text{Sym}^N(\mathbb{C}^2)} = Q_{1/N}(h_N) \) with \( h_N \) some classical symbol as in \((1.26)\) for which each \( h_k \) is a real polynomial \( \mathbb{S} \rightarrow \mathbb{R} \). These vectors induce a sequence of pure (vector) states \( (\omega_N) \) of \( \omega_N \), viz. \( \omega_N(\cdot) = \langle \psi_N, (\cdot)\psi_N \rangle \). In section 2.2 we prove that under some conditions the sequence \( (\omega_N) \) admits a classical limit, in that

\[
\omega_0(f) = \lim_{N \to \infty} \omega_N(Q_{1/N}(f)),
\]

exists for all polynomials \( f \in \tilde{A}_0 \) and and defines a state on \( \tilde{A}_0 \), which in turn extends to a state on \( A_0 = C(S^2) \). The state \( \omega_0 \) may be regarded as the classical limit of the family \( \omega_N \). Using Riesz representation theorem \((1.33)\) means that for all \( f \in \tilde{A}_0 \) one has

\[
\mu_0(f) = \lim_{N \to \infty} \int_{S^2} d\mu_{\psi_N}(\Omega)f(\Omega),
\]

where \( \mu_0 \) is the probability measure corresponding to the state \( \omega_0 \) (i.e. \( \omega_0(f) = \int_{S^2} d\mu_0f \) ) and \( \mu_{\psi_N} \) is a probability measure on \( S^2 \) with density \( B_{\psi_N}(\Omega) := |\langle \Omega, \psi_N \rangle|^2 \) called the Husimi density function associated to the unit vector \( \psi_N \). In other words \( \mu_{\psi_N} \) is given by

\[
d\mu_{\psi_N}(\Omega) = d\Omega \frac{N+1}{4\pi}|\langle \Omega, \psi_N \rangle|^2.
\]

The plan of this paper is as follows. In section 2 we state and prove our main results (Theorem 2.1 with Corollary 2.2 and Theorem 2.7). Theorem 2.1 and Corollary 2.2 establish a connection between the spectrum of the quantum mean field spin in question sytems and the principal symbol on the classical Bloch sphere \( S^2 \). Theorem 2.7 instead shows the existence of the
2 Semiclassical properties of mean-field quantum spin systems

2.1 Asymptotic properties of the spectrum

In this subsection we assume that we are given a strict deformation quantization of a compact symplectic manifold in the sense of Berezin, in that the quantization map is defined in terms of a family of coherent states such that in particular properties (1.22) and (1.23) hold. Note that this clearly applies to $S^2$ where the quantization is defined by maps (1.20) - (1.21) as we have seen in the previous section. We now prove a result relating the spectrum of such quantization maps to the range of the function that is quantized.

Theorem 2.1. Given a pure state Berezin strict deformation quantization of a compact symplectic manifold $X$ on base space $I$. Denote by $Q_{1/N}$ the associated quantization maps. Then for any $f \in \tilde{A}_0 = \text{Dom}(Q_{1/N})$

$$\lim_{N \to \infty} \text{dist}\left(\text{ran}(f), \sigma(Q_{1/N}(f))\right) = 0,$$  

(2.1)

where $\sigma(Q_{1/N}(f))$ denotes the spectrum of the operator $Q_{1/N}(f)$, and dist the distance function. In general the distance between a bounded set $X \subset \mathbb{C}$ and a nonempty set $Y \subset \mathbb{C}$ is defined by

$$\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} |x - y|. \quad (2.2)$$

Proof. Let us assume by contradiction that the statement in the theorem is not true. Then there exists $\delta > 0$, a function $f \in \tilde{A}_0$, and a sequence $(\lambda_{N_k})_k$ in $\text{ran}(f)$ such that $\text{dist}(\lambda_{N_k}, \sigma(Q_{1/N_k}(f))) \geq \delta > 0$ for all $k$. Since $X$ is compact and $f$ is continuous also $\text{ran}(f)$ is compact so that we can extract a subsequence $\lambda_{N_k'}$ converging to a point $r \in \text{ran}(f)$. Hence, for all $\epsilon > 0$ there exists a $K_\epsilon$ such that $|r - \lambda_{N_k'}| < \epsilon$ for all $k' \geq K_\epsilon$. This implies that $r \notin \sigma(Q_{1/N_{k'}}(f))$ for $k' \geq K_\epsilon$, which means that the resolvent operator associated to $r$ and denoted by $R_r(Q_{1/N_{k'}}(f))$, exists for all $k' \geq K_\epsilon$.\[12\]
Now, we can find an element $\Omega \in X$ such that $f(\Omega) = r$. By property (1.22), we can always recover $f(\Omega)$ as

$$f(\Omega) = \lim_{N \to \infty} \langle \Omega | Q_1/N(f) | \Omega \rangle,$$  \hspace{1cm} (2.3)

where $|\Omega\rangle$ denote the coherent state induced by the point $\Omega \in X$ on which $f$ is defined. For this $\Omega$ and $k' \geq K$, let us now estimate

$$1 = |\langle \Omega | R_r(Q_1/N'_k(f))Q_1/N'_k(f-r)\Omega \rangle|^2 \leq \| R_r(Q_1/N'_k(f)) \|^2 \cdot \| Q_1/N'_k(f-r)\Omega \|^2,$$  \hspace{1cm} (2.4)

using in the first equality the fact that $Q_1/N'_k(f-r) = Q_1/N'_k(f) - Q_1/N'_k(1_X)r = Q_1/N'_k(f) - r$. We now make the following estimation on

$$\| Q_1/N'_k(f-r)\Omega \|^2 = \langle \Omega, (Q_1/N'_k(f-r))^*Q_1/N'_k(f-r)\Omega \rangle = \langle \Omega, Q_1/N'_k(f-r)^*Q_1/N'_k(f-r)\Omega \rangle,$$  \hspace{1cm} (2.5)

where we used that the quantization maps preserve the adjoint. Then,

$$\left| \langle \Omega, Q_1/N'_k(f-r)^*Q_1/N'_k(f-r)\Omega \rangle - \langle \Omega, Q_1/N'_k(|f-r|^2)\Omega \rangle \right| \leq \left| Q_1/N'_k(f-r)^*Q_1/N'_k(f-r) - Q_1/N'_k(|f-r|^2) \right|,$$  \hspace{1cm} (2.6)

using the Cauchy-Schwarz inequality and the fact that $|\Omega\rangle$ are unit vectors. Since the quantization map is strict, we conclude that the above inequality uniformly converges to zero as $k' \to \infty$ (see the remark under Definition 1.1). This together with (2.4) and (2.5) implies that

$$1 \leq \| R_r(Q_1/N'_k(f)) \|^2 \left( \langle \Omega, Q_1/N'_k(f-r)^*Q_1/N'_k(f-r)\Omega \rangle - \langle \Omega, Q_1/N'_k(|f-r|^2)\Omega \rangle \right)$$

$$+ \| R_r(Q_1/N'_k(f)) \|^2 \langle \Omega, Q_1/N'_k(|f-r|^2)\Omega \rangle \leq$$

$$\| R_r(Q_1/N'_k(f)) \|^2 \cdot \left| Q_1/N'_k(f-r)^*Q_1/N'_k(f-r) - Q_1/N'_k(|f-r|^2) \right|$$

$$+ \| R_r(Q_1/N'_k(f)) \|^2 \langle \Omega, Q_1/N'_k(|f-r|^2)\Omega \rangle.$$

By (2.3) it follows that $\lim_{k' \to \infty} \langle \Omega, Q_1/N'_k(|f-r|^2)\Omega \rangle = |f(\Omega) - r|^2 = 0$. Since also $\lim_{k' \to \infty} \left| Q_1/N'_k(f-r)^*Q_1/N'_k(f-r) - Q_1/N'_k(|f-r|^2) \right| = 0$, it
must follow that $||R_r(Q_{1/N_k'}(f))||^2 \to \infty$ as $k' \to \infty$, which also implies that $||R_r(Q_{1/N_k'}(f))|| \to \infty$ as $k' \to \infty$. In order to conclude we recall that that

$$||R_r(Q_{1/N_k'}(f))||_{N'_k} \leq \frac{1}{\text{dist}(r, \sigma(Q_{1/N_k'}(f)))}.$$ Combining the above inequalities yields for $k'$ large enough the final inequality

$$0 < \delta \leq \text{dist}(\lambda_{N'_k}, \sigma(Q_{1/N_k'}(f))) \leq \frac{1}{||R_r(Q_{1/N_k'}(f))||_{N'_k}}. \quad (2.7)$$

By taking the limit $k' \to \infty$ of the above inequality we clearly arrive at a contradiction, since the right-hand side converges to zero. This proves the theorem.

This brings us the an important corollary, relating the spectrum of a mean field quantum spin system with Hamiltonian $H_{1/N}$ to the range of a polynomial on $S^2$. As already mentioned in the introduction, the operator $H_{1/N}$ leaves $\text{Sym}^N(C^2)$ invariant, and we therefore consider its restriction $H_{1/N}|_{\text{Sym}^N(C^2)}$ which typically assumes the form $Q_{1/N}(h_N)$, where $Q_{1/N}$ is defined by (1.20) - (1.21), $h_N$ is a classical symbol of the type $h = h_0 + \sum_{k=1}^{M} N^{-k}h_k$, with each $h_i$ ($i = 0, ..., M$) a real polynomial on $S^2$. With slightly abuse of notation, unless specified otherwise, let us from now on write $H_{1/N} := H_{1/N}|_{\text{Sym}^N(C^2)}$. Using this notation we prove the following result.

**Corollary 2.2.** Given a mean-field quantum spin system with Hamiltonian $H_{1/N} = Q_{1/N}(h_N)$, where $h_N$ is a classical symbol of the type $h = h_0 + \sum_{k=1}^{M} N^{-k}h_k$, and each $h_i$ ($i = 0, ..., M$) is a real polynomial on $S^2$. Then, the spectrum of $H_{1/N}$ is related to the range of the principal symbol $h_0$ in the following way,

$$\lim_{N \to \infty} \text{dist}(\text{ran}(h_0), \sigma_H) = 0. \quad (2.8)$$

**Proof.** This follows for example from Weyl’s Theorem (see e.g. [31] for details) applied to the hermitian matrices $Q_{1/N}(h_0)$ and $Q_{1/N}(\sum_{k=1}^{M} N^{-k}(h_k)) = \sum_{k=1}^{M} N^{-k}Q_{1/N}(h_k)$, stating that if $\lambda^{(i)}_N$ is the $i^{th}$ eigenvalue of $H_{1/N} = Q_{1/N}(h_0) +$
\[ \sum_{k=1}^{M} N^{-k} Q_{1/N}(h_k), \text{ and } \epsilon_{N}^{(i)} \text{ is the } i^{th} \text{ eigenvalue of } Q_{1/N}(h_0), \text{ then} \]

\[ |\lambda_{N}^{(i)} - \epsilon_{N}^{(i)}| \leq \| Q_{1/N}(\sum_{k=1}^{M} N^{-k}(h_k)) \| \leq \frac{1}{N} \max_{1 \leq k \leq M} \| h_k \|_{\infty} \to 0 \quad (N \to \infty), \]

where in the last step we used that \( \| Q_{1/N}(h_k) \| \leq \| h_k \|_{\infty} \). In particular, we conclude that \( \lim_{N \to \infty} \text{dist}(\sigma(Q_{1/N}(h_0)), \sigma(H_{1/N})) = 0 \). By the triangle inequality applied to the distance function in the previous theorem, the result follows.

**2.2 Classical limit of mean-field theories**

In this subsection we prove the existence of the classical limit. Since this paper focuses on mean-field quantum spin systems we state the results for the case when the manifold equals \( S^2 \). We remark that the main results can be generalized for arbitrary compact quantizable Kahler manifolds \[7\]. We start with a lemma.

**Lemma 2.3.** Given a strict deformation quantization of \( S^2 \) with associated quantization maps \( Q_{1/N}(h_0) \) defined by (1.20) - (1.21), where \( h_0 \) is a real polynomial function on \( S^2 \). Suppose that \( E \in \text{ran}(h_0) \) such that \( E = h_0(\Omega) \) for some finitely many distinct \( \Omega \in S^2 \), then for these \( \Omega \) one has

\[ \lim_{N \to \infty} \| Q_{1/N}(h_0)\Omega - E\Omega \| = 0. \quad (2.9) \]

**Proof.** Consider those \( \Omega \in S^2 \) with \( h_0(\Omega) = E \). Since there are finitely many of such \( \Omega \), say \( n \), we can label them \( \Omega_1, ..., \Omega_n \). By uniform continuity, given \( \epsilon > 0 \), we can find \( \delta > 0 \) such that for all \( \Omega' \in B_\delta(\Omega_i) = \{ \Omega' \in S^2 | d_{S^2}(\Omega', \Omega_i) < \delta \} \) one has \( |h_0(\Omega') - h_0(\Omega_i)| < \epsilon/2 \) for all \( i = 1, ..., n \). Since the \( \Omega_i \) are distinct we can shrink each \( B_\delta(\Omega_i) \) to make them pairwise disjoint,
as is always possible in a Hausdorff space. For any such \( \Omega_i \) we compute

\[
\left\| Q_{1/N}(h_0)\Omega_i - E\Omega_i \right\| = \left\| \frac{N+1}{4\pi} \int_{S^2} d\Omega'(h_0(\Omega') - h_0(\Omega_i))\langle \Omega', \Omega_i \rangle \right\|
\leq \left\| \frac{N+1}{4\pi} \int_{B_\delta(\Omega_i)} d\Omega'(h_0(\Omega') - h_0(\Omega_i))\langle \Omega', \Omega_i \rangle \right\|
+ \left\| \frac{N+1}{4\pi} \int_{S^2 \setminus B_\delta(\Omega_i)} d\Omega'(h_0(\Omega') - h_0(\Omega_i))\langle \Omega', \Omega_i \rangle \right\|
\leq \sup_{\Omega' \in B_\delta(\Omega_i)} |h_0(\Omega') - h_0(\Omega_i)| \left\| \frac{N+1}{4\pi} \int_{B_\delta(\Omega_i)} d\Omega'\langle \Omega', \Omega_i \rangle \right\|
+ \frac{N+1}{4\pi} \int_{S^2 \setminus B_\delta(\Omega_i)} d\Omega' |h_0(\Omega') - h_0(\Omega_i)| \left\| \langle \Omega', \Omega_i \rangle \right\|.
\]

The first term is bounded by \( \epsilon/2 \) using uniform continuity of \( h_0 \), and the fact that \( 1 = \frac{N+1}{4\pi} \int_{S^2} d\Omega|\langle \Omega \rangle \langle \Omega \rangle | \). For the second addendend, we note that the overlap between two coherent states is given by

\[
|\langle \Omega, \Omega' \rangle| = \left( \frac{1 + t}{2} \right)^{N/2},
\]

where \( t \in [-1, 1] \) denotes the cosine of the angle between the both (different) coherent states. For \( \Omega' \in S^2 \setminus U_\delta \), we have \( \frac{1 + t}{2} \leq M_\delta < 1 \) for some constant \( M_\delta \) depending on \( \delta \). For \( N \) sufficiently large, one can estimate

\[
\left( \frac{1 + t}{2} \right)^{N/2} \leq M_\delta^N < \epsilon/2C.
\]

where \( C \) is the integration constant given by \( C := \frac{N+1}{4\pi} \int_{S^2 \setminus B_\delta(\Omega_i)} d\Omega'|h_0(\Omega') - h_0(\Omega)| \). Hence, we conclude that for all \( i = 1, ..., n \) one has \( \lim_{N \to \infty} \|Q_{1/N}(h_0)\Omega_i - E\Omega_i\| = 0 \).

**Remark 2.4.** In particular, if \( E \in \text{ran}(h_0) \), then there is a sequence \( \lambda_N \) of eigenvalues of \( Q_{1/N}(h_0) \) such that \( |E - \lambda_N| \to 0 \) as \( N \to \infty \). The same holds when we replace the principal symbol \( h_0 \) by \( h_N \). Indeed, since such \( h_N \) is typically given by \( h = h_0 + \sum_{k \geq 1} N^{-k} h_k \) for some \( M \in \mathbb{N} \). Then, by linearity \( Q_{1/N}(h_N) = Q_{1/N}(h_0) + \sum_{k \geq 1} N^{-k} Q_{1/N}(h_k) \). Since each \( \|Q_{1/N}(h_k)\| \leq \|h_k\|_\infty \), clearly \( \|\sum_{k \geq 1} N^{-k} Q_{1/N}(h_k)\| \leq \frac{1}{N} \max_{1 \leq k \leq M} \|h_k\|_\infty \), and hence \( \|Q_{1/N}(h_N)\Omega - E\Omega\| \leq \|Q_{1/N}(h_0)\Omega - E\Omega\| + O(1/N) \), so that by the lemma also \( \lim_{N \to \infty} \|Q_{1/N}(h_N)\Omega - E\Omega\| = 0 \).
Before proving our main result regarding the existence of the classical limit of a mean field quantum spin system we assume the following data of the corresponding Hamiltonian.

**Assumption 2.5.** Given a mean-field quantum spin system with Hamiltonian \( H_{1/N} := H_{1/N}|_{\text{Sym}^N(C^2)} = Q_{1/N}(h_N) \) with polynomial symbol \( h_N \) as defined before. We assume that there exists a number of distinct points \( \Omega_i \) \( (i = 1, ..., n) \) such that the principal symbol \( h_0 \) of \( h_N \) satisfies \( h_0(\Omega_i) = E \), where \( E \) is a fixed constant. We moreover assume that there exists a topological group \( G \) of isometries of \( S^2 \) acting continuously on \( S^2 \) and a point \( \Omega_1 \) with \( h_0(\Omega_1) = E \) such that the corresponding orbit \( \mathcal{O}(\Omega_1) \) equals \( \{\Omega_1, ..., \Omega_n\} \).

This action obviously induces an action \( T \) on the space of functions \( \Psi \) on \( C(S^2) \), namely

\[
(T_g \Psi)(\Omega) := \Psi(g^{-1}\Omega) \quad (g \in G, \ \Omega \in S^2),
\]

where the function \( \Omega \mapsto \Psi(\Omega) \) is defined by \( \Psi(\Omega) := \langle \Omega, \psi_N \rangle \) (see Appendix 3.1 for details). We furthermore assume that this action is such that for a particular sequence of normalized eigenvectors \( (\psi_N)_N \) of \( Q_{1/N}(h_N) \) one has

\[
\Psi(g^{-1}\Omega) = \Psi(\Omega) \quad (g \in G, \ \Omega \in S^2).
\]

**Remark 2.6.** This is the case for the Curie-Weiss model, where the symmetry group \( \mathbb{Z}_2 \) consists of two elements and the non-trivial element acts by reflection. Indeed, the principal symbol \( h_0^{CW} \) has exactly two distinct minimal points, the orbit corresponding to any such minimal point generates both minima, and the (unique, up to phase) ground state eigenvector \( \psi_N \) of \( H_{1/N}^{CW} \) is \( \mathbb{Z}_2 \)-invariant for each \( N \in \mathbb{N} \).

Under Assumption 2.5 there exists a group \( G \), \( n \) distinct points \( \Omega_i \in S^2 \) \((i = 1, ..., n)\) with \( h_0(\Omega_i) = E \) such that \( \mathcal{O}(\Omega_i) = \{\Omega_1, ..., \Omega_n\} \) (for some particular point \( \Omega_1 \)), and we can find a sequence of \( G \)-invariant eigenstates \( \psi \equiv (\psi_N)_N \) (with corresponding eigenvalues \( \lambda_N \)) of the operator \( H_{1/N} \). As a result of Remark 2.4 one clearly has \( \lim_{N \to \infty} |\lambda_N - E| = 0 \). For these eigenstates we now prove the following result.

**Theorem 2.7.** Consider the sequence of algebraic states \( (\omega_N)_N \) induced by the eigenstates \( \psi \equiv (\psi_N)_N \) of the mean field operator \( H_{1/N} = Q_{1/N}(h_N) \). Let \( E \in \text{ran}(h_0) \) such that \( |\lambda_N - E| \to 0 \) as \( N \to \infty \), where \( (\lambda_N)_N \) is the
sequence of eigenvalues corresponding to $\psi_N$. Then, the following limit exists and defines a state on $A_0$

$$\omega_0(f) = \lim_{N \to \infty} \omega_N(Q_{1/N}(f)) \ (f \in \tilde{A}_0),$$

(2.14)

where $\omega_0(f) = \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i)$ and $\Omega_i$ are the points in $S^2$ such that $h_0(\Omega_i) = E$.

**Proof.** Clearly the symmetric combination $\frac{1}{n} \sum_{i=1}^{n} f(\Omega_i)$ where the $\Omega_i$ are such that $h_0(\Omega_i) = E$ in the above theorem defines a (mixed) state and hence a probability measure on $S^2$. By uniform continuity of $f$, given $\epsilon > 0$, we can find $\delta > 0$ such that for all $\Omega \in B_\delta(\Omega_i) = \{\Omega \in S^2 \mid d_{S^2}(\Omega, \Omega_i) < \delta\}$ one has $|f(\Omega) - f(\Omega_i)| < \epsilon/2n$, for all $i = 1, ..., n$. Let us fix one such neighborhood $U_1 := B_\delta(\Omega_1)$. By the assumption we can clearly find $g_i \in G \ (i = 1, ..., n)$ so that the sets $U_i := g_i^{-1}U_1$ are neighborhoods of $\Omega_i$, using the fact that the map $\Omega \mapsto g\Omega$ is a homeomorphism. Since the points $\Omega_i$ are distinct the $n$ neighborhoods can be picked to be pairwise disjoint. It now follows that

$$\frac{N+1}{\pi} \int_{U_1} \frac{dz}{(1 + |z|^2)^2} |\langle z, \psi \rangle|^2 = \frac{N+1}{4\pi} \int_{U_1} d\Omega |\Psi(\Omega)|^2 =$$

$$\frac{N+1}{4\pi} \int_{U_2} d\Omega |\Psi(\Omega)|^2 = ... = \frac{N+1}{4\pi} \int_{U_n} d\Omega |\Psi(\Omega)|^2 =$$

(2.15)

where we used the property that for these $g_i \in G \ (i = 1, ..., n)$ and $\Omega \in U_1$, one has $\Psi(g_i^{-1}\Omega) = \Psi(\Omega)$ by assumption. This implies that $\Psi(U_1) = \Psi(g_i^{-1}U_1) = \Psi(U_i)$ for all $i = 1, ..., n$, from which (2.15) follows. Under stereographic projection the neighborhoods $U_i$ correspond to open sets in $\mathbb{C}$ which we will denote by the same name, and the points $S^2 \ni \Omega_i$ will be denoted by $z_i \in \mathbb{C}$. Given now the sequence of normalized eigenstates $\psi \equiv (\psi_N)_N$ of $Q_{1/N}(h_N)$ inducing the pure states $(\omega_N)_N$, we compute $\omega_N(Q_{1/N}(f))$, i.e.

$$\langle \psi, Q_{1/N}(f) \psi \rangle_{C^{N+1}} = \frac{N+1}{4\pi} \int_{S^2} d\Omega f(\Omega) |\langle \psi, \Omega \rangle|^2 =$$

$$\frac{N+1}{\pi} \int_{\mathbb{C}} \frac{dz^2}{(1 + |z|^2)^2} f(z) |\Psi(z)|^2,$$

(2.16)
where as before $\Psi(z) = \langle z, \psi_N \rangle = (1 + |z|^2)^{-N/2} p(z)$. We then split this integral as follows:

$$\int_{\mathcal{U}_i} + \int_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i} \frac{N + 1}{\pi} \frac{dz^2}{(1 + |z|^2)^2} f(z)|\Psi(z)|^2.$$  \hspace{1cm} (2.17)

Moreover, we use the coherent state property, i.e.

$$1 = \langle \psi, \psi \rangle = \frac{N + 1}{\pi} \int_{\mathcal{C}} \frac{dz^2}{(1 + |z|^2)^2} |\langle \psi, z \rangle|^2.$$  \hspace{1cm} (2.18)

We then write

$$\lim_{N \to \infty} \left| \langle \psi, Q_{1/N}(f) \psi \rangle - \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i) \right|$$

$$= \lim_{N \to \infty} \frac{N + 1}{\pi} \int_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i} \frac{dz^2}{(1 + |z|^2)^2} f(z)(|\langle z, \psi \rangle|^2 - \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i)(|\langle z, \psi \rangle|^2)$$

$$+ \frac{N + 1}{2\pi} \int_{\bigcup_i \mathcal{U}_i} \frac{dz^2}{(1 + |z|^2)^2} \left( f(z) - \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i) \right)|\langle z, \psi \rangle|^2.$$  \hspace{1cm} (2.19)

We now estimate the first integral:

$$\left| \frac{N + 1}{\pi} \int_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i} \frac{dz^2}{(1 + |z|^2)^2} \left( f(z) - \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i) \right)|\langle z, \psi \rangle|^2 \right|$$

$$\leq \frac{(n + 1)(N + 1)}{\pi} \int_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i} \frac{dz^2}{(1 + |z|^2)^2} |\Psi(z)|^2$$

$$= (n + 1)||f||_2^2||\psi||_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i}^2.$$  \hspace{1cm} (2.20)

By the Theorem 3.2 the norm $||\psi||_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i}^2$ can be made bounded by $\epsilon/(2(n + 1)||f||_2^2)$ as $N$ becomes sufficiently large, since outside the sets $U_i$ the distance to $\{h_0 = E\}$ is clearly positive. \footnote{By the comments under Theorem 3.2 for $N$ large enough we can estimate the norm $||\psi||_{\mathcal{C}\setminus \bigcup_i \mathcal{U}_i}^2$ by $C/N$, for some constant $C$ that does not depend on $N$.} We are thus done if we can show that

$$\left| \frac{N + 1}{\pi} \int_{\bigcup_i \mathcal{U}_i} \frac{dz^2}{(1 + |z|^2)^2} f(z)(|\langle z, \psi \rangle|^2 - \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i)|\langle z, \psi \rangle|^2) \right| < \epsilon/2,$$
whenever $N$ sufficiently large. Since $U_i \cap U_j = \emptyset$ if $i \neq j$, we can write
\[
\frac{N + 1}{\pi} \int_{\bigcup U_i} \frac{dz^2}{(1 + |z|^2)^2} f(z) |\langle z, \psi \rangle|^2 
= \sum_{i=1}^{n} \frac{N + 1}{\pi} \int_{U_i} \frac{dz^2}{(1 + |z|^2)^2} f(z) |\langle z, \psi \rangle|^2.
\]
Making use of (2.15) we can estimate
\[
\left| \frac{N + 1}{\pi} \int_{\bigcup U_i} \frac{dz^2}{(1 + |z|^2)^2} f(z) |\langle z, \psi \rangle|^2 - \frac{1}{n} \sum_{i=1}^{n} \frac{N + 1}{\pi} \int_{U_i} \frac{dz^2}{(1 + |z|^2)^2} f(z_i) |\langle z, \psi \rangle|^2 \right| 
\leq \sum_{i=1}^{n} \left| \frac{N + 1}{\pi} \int_{U_i} \frac{dz^2}{(1 + |z|^2)^2} (f(z) - f(z_i)) |\langle z, \psi \rangle|^2 \right|. 
\tag{2.20}
\]
By the fact that $\frac{N + 1}{\pi} \int_{\mathbb{C}} \frac{dz^2}{(1 + |z|^2)^2} |\Psi(z)|^2 = 1$, we find that each of the $n$ terms is bounded by $\sup_{z \in U_i} |f(z) - f(z_i)|$. By uniform continuity of $f$, each supremum is bounded by $\epsilon/2n$, so that (2.20) is bounded by $\epsilon/2$. This shows that
\[
\left| \langle \psi, Q_{1/N}(f) \psi \rangle - \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i) \right| < \epsilon, 
\tag{2.21}
\]
whenever $N$ sufficiently large, which concludes the proof of the theorem. \hfill \Box

Let us now say some words about spontaneous symmetry breaking (see also [15, 16] for a more general discussion). Under Assumption 2.5 and the assumptions of Theorem 2.7 we moreover assume that the sequence of eigenvectors $(\psi_N)_N$ corresponds to non-degenerate ground states (i.e. eigenvectors corresponding to the lowest eigenvalue) of the mean field Hamiltonian $H_{1/N}|_{\text{Sym}^N(\mathbb{C}^2)}$. Then, clearly for each $N \in \mathbb{N}$ the corresponding algebraic pure state $\omega_N$ is $G$-invariant, implying there is no symmetry breaking for any finite $N$. However, the classical limiting state which also qualifies as a ground state in the algebraic sense, is invariant but not pure. At least in the

\begin{footnote}
One should notice that (ground state) eigenvectors of the compressed Hamiltonian $H_{1/N}|_{\text{Sym}^N(\mathbb{C}^2)}$ do a priori not correspond to (ground state) eigenvectors of the original Hamiltonian $H_{1/N}$ with the same eigenvalue. This only happens when the (ground state) eigenvector is permutation-invariant, i.e. an element of $\text{Sym}^N(\mathbb{C}^2)$ as is for example the case for the Curie-Weiss model.
\end{footnote}
language of algebraic quantum theory this is the essence of SSB:

*Pure ground states are not invariant, whilst invariant ground states are not pure.*

This mathematically explains that spontaneous symmetry breaking is an emergent feature: it only appears in the classical limit as }$N \to \infty$

In the next corollary we give an illustrating example.

**Corollary 2.8.** The Curie-Weiss model }$H_{1/N}^{\text{CW}}$

defined in (1.27) admits a classical limit which breaks the }$\mathbb{Z}_2$-symmetry in the limit as }$N \to \infty$.

**Proof.** In [16, 30] it has been shown that for any finite }$N$

the ground state eigenvector }$\psi_N$ of }$H_{1/N}^{\text{CW}}$

is unique and lies in the symmetric subspace }$\text{Sym}^N(\mathbb{C}^2)$.

In view of Example 1 in Subsection 1.2 for any finite }$N$

the vector }$\psi_N$ is the ground state of }$Q_{1/N}(h_N) = H_{1/N}^{\text{CW}}|_{\text{Sym}^N(\mathbb{C}^2)}$

where }$h_N = h_0^{\text{CW}} + N^{-1}((-3Jz^2 + 1)/2)$. By Remark 2.6 and Theorem 2.7 this sequence admits a classical limit.

In particular this limit is given by }$\omega_0(f) = \frac{1}{2}f(x_-) + f(x_+)$, where }$x_\pm$ are the two minima of }$h_0^{\text{CW}}$. For the proof that the }$\mathbb{Z}_2$-symmetry is spontaneously broken in the limit }$N \to \infty$ we refer to the remarks in [16, Section 4.1].

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11The question why in nature one of the pure symmetry-breaking states }$\omega_i(f) := f(\Omega_i)$

is found, rather than the mixture }$\omega_0(f) = \frac{1}{n} \sum_{i=1}^{n} f(\Omega_i)$, is answered in [30].

21
3 Appendix

3.1 Toeplitz quantization

The following definition can also be found in [7]. We shall briefly summarize the main parts.

**Definition 3.1.** 1. A compact Kähler manifold \((M, J, \omega)\) is said to be quantizable when the symplectic form \(\omega\) has integer cohomology: there exists a unique Hermitian line bundle \((L, h)\) over \(M\) such that the curvature of \(h\) is \(-2i\pi \omega\). This line bundle is called the prequantum line bundle over \((M, J, \omega)\). The manifold \((M, J, \omega)\) is said to be real-analytic when \(\omega\) (or, equivalently, \(h\)) is real-analytic on the complex manifold \((M, J)\).

2. Let \((M, J, \omega)\) be a quantizable compact Kähler manifold with \((L, h)\) its prequantum bundle and let \(N \in \mathbb{N}\).

The Hilbert space \(L^2(M, L^\otimes N)\) is the closure of \(C^\infty(M, L^\otimes N)\), the space of smooth sections of the \(N\)-th tensor power of \(L\). From the Hermitian metric \(h\) on \(L\), one deduces a Hermitian metric \(h_N\) on \(L^\otimes N\). If \(u, v\) are sections of \(L^\otimes N\), the scalar product is defined as

\[
\langle u, v \rangle_N = \int_M \langle u, v \rangle_{h_N} \omega^{\text{dim}_\mathbb{C} M} (\text{dim}_\mathbb{C} M)!,
\]

where \(\langle u, v \rangle_{h_N} = h_N(u_1 \otimes \cdots \otimes u_N, v_1 \otimes \cdots \otimes v_N) = \prod_{m=1}^{N} h(u(m), v(m))\).

3. The Hardy space is the space of holomorphic sections of \(L^\otimes N\), denoted by \(H_0(M, L^\otimes N)\). It is a finite-dimensional closed subspace of \(L^2(M, L^\otimes N)\). The Bergman projector \(S_N\) is the orthogonal projector from the space of square-integrable sections \(L^2(M, L^\otimes N)\) to \(H_0(M, L^\otimes N)\). If \(f \in C^\infty(M, \mathbb{C})\), then the contravariant Toeplitz operator \(T_N(f) : H_0(M, L^\otimes N) \to H_0(M, L^\otimes N)\) is defined as

\[
T_N(f)u = S_N(fu).
\]
3.2 The case $M = S^2$

Consider the projective complex line $M = \mathbb{CP}^1$ which is a Kahler manifold endowed with the Fubini-Study form $\omega_{FS}$, given by

$$\omega_{FS} = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2},$$

with associated Kähler potential $\phi$ defined by

$$\phi(z, \bar{z}) = \log(1 + |z|^2).$$

We can identify $\mathbb{CP}^1 \cong S^2$, the Riemann sphere $S^2$. Note that the Riemann sphere $S^2$ is canonically embedded in $\mathbb{R}^3$ as the unit 2-sphere: $S^2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$. It is clear that $M$ is compact and quantizable since by a computation one finds that the integral of $\omega_{FS}$ on $\mathbb{CP}^1$ is $2\pi$ times an integer. It turns out that the prequantum line bundle $L$ over $M = \mathbb{CP}^1$ is the dual of the tautological bundle, that is, $L = O(1)$. In this case $L^\otimes N = O(N)$. In particular the space of holomorphic sections of $L^\otimes N$ is identified with the space of holomorphic square integrable functions on $\mathbb{C}$ with respect to the scalar product defined below. Indeed, unfolding the definitions for the Kahler manifold $M$, we find that the following scalar product identifies a Hilbert space structure on $H_0(M, L^\otimes N)$:

$$\langle f, g \rangle = \int_{\mathbb{C}} \frac{f(z)g(z)}{(1 + |z|^2)^{N+2}} d^2z,$$

where $f, g$ are holomorphic functions on $\mathbb{C}$. We should remark that only holomorphic polynomials of degree $\leq N$ have a finite $L^2$-norm for this scalar product. We therefore only consider polynomials. An orthonormal basis consists of the monomials

$$e_k = \sqrt{\frac{N+1}{\pi}} \sqrt{\binom{N}{k}} X^k, \quad (k = 0, \ldots, N).$$

Analogously, one can define an inner product by absorbing the factor $\frac{N+1}{\pi}$, i.e., one considers the basis given by $w_k = \sqrt{\binom{N}{k}} X^k$ and scalar product

$$\langle p_1, p_2 \rangle_{FB^N} := \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{p_1(z)p_2(z)}{(1 + |z|^2)^{N+2}} d^2z.$$
This scalar product induces a \((N + 1)\)-dimensional Hilbert space, called the Fock-Bargmann space \(\mathcal{FB}^N\). Consider now a vector \(\psi \in \mathbb{C}^{N+1}\). To this vector we associate the function

\[
\Psi(z) = \langle z, \psi \rangle, \tag{3.8}
\]

where \(|z\rangle\) is a coherent state in complex coordinate \(z\) (playing the same role as \(|\Omega\rangle\)). It can be shown that the function \(\Psi(z)\) assumes the form

\[
\Psi(z) = (1 + |z|^2)^{-N/2}p(z), \tag{3.9}
\]

where \(p(z)\) is an analytic polynomial of degree \(\leq N\) in the Fock Bargmann space \(\mathcal{FB}^N\). Also the functions \(\Psi(z)\) of the type (3.9) form a finite-dimensional Hilbert space of dimension \(N + 1\), which are square integrable with respect to the scalar product

\[
\langle \Psi, \Phi \rangle_{\mathbb{C}^{N+1}} := \frac{N + 1}{\pi} \int_{\mathbb{C}} \frac{\overline{\Psi(z)}\Phi(z)}{(1 + |z|^2)^2} d^2z. \tag{3.10}
\]

This clearly implies that \(||p||_{\mathcal{FB}^N} = ||\Psi||_{\mathbb{C}^{N+1}}\). In fact, the map \(\mathbb{C}^{N+1} \ni \psi \mapsto \Psi(z) \in \mathcal{FB}^N\) is an isometry. Therefore, for any vectors \(\psi, \phi \in \mathbb{C}^{N+1}\), one has

\[
\langle \psi, \phi \rangle_{\mathbb{C}^{N+1}} = \frac{N + 1}{\pi} \int_{\mathbb{C}} \frac{dz^2}{(1 + |z|^2)^2} \overline{\Psi(z)}\Phi(z) \tag{3.11}
\]

\[
= \frac{N + 1}{4\pi} \int_{S^2} d\Omega \overline{\Psi(\Omega)}\Phi(\Omega), \tag{3.12}
\]

where \(\Psi(z)\) defined above, and \(\Psi(\Omega) = \langle \Omega, \psi \rangle\) and \(\Phi(\Omega) = \langle \Omega, \psi \rangle\). The measure \(d\Omega\) is the usual measure on the 2-sphere. We now state a result obtained by Deleporte [7].

**Theorem 3.2 (Deleporte).** Let \(f\) be a real-analytic, real-valued function on \(S^2\) and \(E \in \mathbb{R}\). Let \((\psi_N)_N\) be a normalized sequence of \((\lambda_N)_N\)-eigenstates of \(T_N(f)\) with \(\lambda_N \to E\) as \(N \to \infty\). Then, for every open set \(V\) at positive distance from \(\{f = E\}\) there exist positive constants \(c, C\) such that, for every \(N \geq 1\), one has

\[
||\psi_N||_{L^2(V)} \leq C \exp(-cN), \tag{3.13}
\]

where the norm is the one induced by the scalar product (3.7). We say informally that, in the forbidden region \(\{f \neq E\}\), the sequence \((\psi_N)_N\) has an exponential decay rate.
Let us now take a function of the form \( f_N = f_0 + \sum_{k=1}^{M} N^{-k} f_k \), where \( f_0 \) and \( f_k \) \((k = 1, \ldots, M)\) are real polynomials on \( S^2 \), and let us consider the operator \( T_N(f_N) \) with \((\psi_N)_N\) a normalized sequence of eigenstates with corresponding eigenvalues \((\lambda_N)_N\), such that \( \lambda_N \to E \) as \( N \to \infty \). Modifying the proof of the above theorem still results in convergence of the sequence of eigenvectors but with another decay rate, i.e. one obtains that for every open set \( V \) at positive distance from \( \{ f_0 = E \} \), there holds \( \| \psi_N \|_{L^2(V)} = O(1/N) \). In order to see this, let us first go back to the case when \( f = f_0 \). Taking a glance at the proof of Theorem 3.2 (applied to \( T_N(f_0) \), with eigenstates \( \phi_N \) and eigenvalues \( \epsilon_N \)) one first defines a function \( a \in C^\infty(S^2, \mathbb{R}_+) \) such that \( \text{supp}(a) \cap \{ f_0 = E \} = \emptyset \), and \( a = 1 \) on \( V \). Moreover one considers a neighborhood \( W \) such that \( \text{supp}(a) \subset W \subset \{ f_0 \neq E \} \). On \( W \) the function \( f_0 - E \) is bounded away from zero and one can consider the analytic covariant symbol \( g \) which is such that covariant Topelitz operator \( T_N^{\text{cov}}(g) \) is the analytic inverse on this neighborhood of \( T_N(f_0 - \lambda_N) \). One can prove that this symbol is well-defined and that \( T_N^{\text{cov}}(g) \) is indeed the mircolocal inverse of \( T_N(f_0 - \lambda_N) \). Then it has been shown that

\[
T_N(a)T_N^{\text{cov}}(g)T_N(f_0 - \lambda_N) = T_N(a) + O(e^{-cN}), \quad (3.14)
\]

for some small \( c > 0 \). This implies that the sequence of eigenstates \( \phi_N \) of \( T_N(f_0) \) satisfies

\[
0 = T_N(a)\phi_N + O(e^{-cN}), \quad (3.15)
\]

so that \( T_N(a)\phi_N = O(e^{-cN}) \). In particular,

\[
\| \phi_N \|^2_{L^2(V)} = \frac{N+1}{4\pi} \int_V |\Phi_N(\Omega)|^2 d\Omega \leq \frac{N+1}{4\pi} \int_{S^2} |\Phi_N(\Omega)|^2 a(\Omega) d\Omega = \langle \phi_N, T_N(a)\phi_N \rangle = O(e^{-cN}). \quad (3.16)
\]

In the general case, where \( f_N = f_0 + \sum_{k=1}^{M} N^{-k} f_k \), clearly \( \| T_N(f_0)\psi_N \| = \| T_N(f_N)\psi_N - \sum_{k=1}^{M} N^{-k} T_N(h_k)\psi_N \| \leq \| T_N(f_N)\psi_N \| + C_1/N \), for some \( C_1 > 0 \). By a similar argument as in the proof of Corollary 2.2 also \( |\lambda_N - \epsilon_N| \leq C_2/N \), for some \( C_2 > 0 \). Then, on the one hand by (3.14) we obtain

\[
T_N(a)\psi_N = T_N(a)T_N^{\text{cov}}(g)T_N(f_0 - \epsilon_N)\psi_N + O(e^{-cN})\psi_N. \quad (3.17)
\]
On the other hand,

\[ ||T_N(a)T_N^{cov}(g)T_N(f_0 - \epsilon_N)\psi_N|| = \]

\[ ||T_N(a)T_N^{cov}(g)T_N(f_N - \sum_{k=1}^{M} N^{-k} f_k - \lambda_N + \lambda_N - \epsilon_N)\psi_N|| \leq \]

\[ ||T_N(a)T_N^{cov}(g)T_N(f_N - \lambda_N)\psi_N|| + \sum_{k=1}^{M} N^{-k}||T_N(a)T_N^{cov}(g)T_N(f_k)\psi_N|| + \]

\[ |\lambda_N - \epsilon_N||T_N(a)T_N^{cov}(g)\psi_N|| \leq \frac{\tilde{C}_1}{N} + \frac{\tilde{C}_2}{N}, \quad (3.18) \]

using that \( T_N(a)T_N^{cov}(g)T_N(f_N - \lambda_N)\psi_N = 0, \sum_{k=1}^{M} N^{-k}||T_N(a)T_N^{cov}(g)T_N(f_k)\psi_N|| \leq \tilde{C}_1/N, \) and similarly that \( ||(\lambda_N - \epsilon_N)T_N(a)T_N^{cov}(g)\psi_N|| \leq C_2/N \) where the constants \( \tilde{C}_1, \tilde{C}_2 \) do not depend on \( N \). Combining the above equations yields

\[ T_N(a)\psi_N = O(1/N). \quad (3.19) \]

Repeating the same argument as in (3.16) gives the estimate \( ||\psi_N||_{L^2(V)}^2 = O(1/N) \).

Furthermore, it can be shown that for any smooth \( f \) on the sphere \( S^2 \), one has \( Q_{1/N}(f) = T_N(f) \), where \( T_N(f) \) is the Toeplitz operator with symbol \( f \) and the quantization maps \( Q_{1/N} \) are defined by (1.20) - (1.21) (see e.g. [26, Prop. 6.8] for a proof and general approach). This means that the above machinery is perfectly applicable to mean field quantum spin systems.

### 3.3 Coherent spin states

If \( |\uparrow\rangle, |\downarrow\rangle \) are the eigenvectors of \( \sigma_3 \) in \( \mathbb{C}^2 \), so that \( \sigma_3|\uparrow\rangle = |\uparrow\rangle \) and \( \sigma_3|\downarrow\rangle = -|\downarrow\rangle \), and where \( \Omega \in S^2 \), with polar angles \( \theta_\Omega \in (0, \pi) \), \( \phi_\Omega \in (-\pi, \pi) \), we then define the unit vector

\[ |\Omega\rangle_1 = \cos \frac{\theta_\Omega}{2}|\uparrow\rangle + e^{i\phi_\Omega} \sin \frac{\theta_\Omega}{2}|\downarrow\rangle. \quad (3.20) \]

If \( N \in \mathbb{N} \), the associated \textbf{N-coherent spin state} \( |\Omega\rangle_N \in \text{Sym}^N(\mathbb{C}^2) \), equipped with the usual scalar product \( \langle \cdot, \cdot \rangle_N \) inherited from \( (\mathbb{C}^2)^N \), is defined
as follows [21]:

\[ |\Omega\rangle_N = |\Omega\rangle_1 \otimes \cdots \otimes |\Omega\rangle_1, \quad (3.21) \]

An important property relevant for our computations was established in [16]

\[ f(\Omega') = \lim_{N \to \infty} \frac{N + 1}{4\pi} \int_{S^2} d\Omega f(\Omega') |\langle \Omega, \Omega' \rangle_N|^2, \quad (f \in C(S^2)). \quad (3.22) \]
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