N term pairwise correlation inequalities, steering and joint measurability

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Chained inequalities involving pairwise correlations of qubit observables in the equatorial plane are constructed based on the positivity of a sequence of moment matrices. When a jointly measurable set of fuzzy POVMs is employed in first measurement of every pair of sequential measurements, the chained pairwise correlations do not violate the classical bound imposed by the moment matrix positivity. We identify that incompatibility of the set of POVMs employed in first measurements is only necessary, but not sufficient, in general, for the violation of the inequality. On the other hand, there exists a one-to-one equivalence between the degree of incompatibility (which quantifies the joint measurability) of the equatorial qubit POVMs and the optimal violation of a non-local steering inequality, proposed by Jones and Wiseman (Phys. Rev. A, 84, 012110 (2011)). To this end, we construct a local analogue of this steering inequality in a single qubit system and show that its violation is a mere reflection of measurement incompatibility of equatorial qubit POVMs, employed in first measurements in the sequential unsharp-sharp scheme.

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I. INTRODUCTION

Conceptual foundations of quantum theory deviate drastically from the classical world view. The prominent counter intuitive features pointing towards the quantum-classical divide are a subject of incessant debate ever since the birth of quantum theory. Pioneering works by Bell [1], Kochen-Specker [2], Leggett-Garg [3] are significant in bringing forth the perplexing features arising within the quantum scenario, in terms of correlation inequalities, constrained to obey classical bounds. Violation of the inequalities sheds light on the non-existence of a joint probability distribution for the measurement outcomes of all the associated observables [4,6].

In fact, non-commutativity of the observables forbids assignment of joint sharp realities to their outcomes in projective valued (PV) measurements. Subsequently, it is not possible to envisage a bona fide joint probability distribution for the outcomes of PV measurements of non-commuting observables. However, the generalized measurement framework [7] goes beyond the conventional PV measurement scenario, where positive operator valued measures (POVMs) are employed. Joint measurability (or compatibility) of a set of POVMs is possible even when they do not commute. For declaring that a set of POVMs are jointly measurable there should exist a global POVM, measurement statistics of which enables one to retrieve that of the set of compatible POVMs. Within the purview of generalized measurements, it is possible to assign fuzzy joint realities (and in turn, a valid joint probability distribution) to the statistical outcomes of non-commuting observables, when the corresponding POVMs are compatible.

In recent years there is a surge of research activity dedicated to explore the notion of measurement incompatibility and its connection with the counter-intuitive quantum notions like non-locality, contextuality and non-macrorealism [8–23]. In particular, it is known that measurement incompatibility plays a key role in bringing to surface the violations of the so-called no-go theorems in the quantum world. Wolf et al. [11] proved that a set of two incompatible dichotomic POVMs are necessary and sufficient to violate the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [24]. However, this result may not hold, in general, for Bell non-locality tests where more than two incompatible POVMs with any number of outcomes are employed i.e., it is possible to identify a set of non-jointly measurable POVMs, which fail to reveal Bell-type non-locality, in general [24]. Interestingly, there exists a one-to-one equivalence [16,18] between measurement incompatibility and quantum steerability (i.e., Alice’s ability to non-locally alter Bob’s states by performing local measurements on her part of the quantum state [24]). More specifically, a set of fuzzy POVMs is said to be incompatible if and only if it can be used to show steering in a quantum state.

From the point of view of an entirely different mathematical perspective, the classical moment problem [22,24,51] addresses the issue of the existence of probability distribution corresponding to a given sequence of statistical moments. Essentially, the classical moment problem points out that a given sequence of real numbers qualify to be the moment sequence of a legitimate
probability distribution if and only if the corresponding moment matrix is positive. In other words, existence of a valid joint probability distribution, consistent with the given sequence of moments, necessitates positivity of the associated moment matrix. Moment matrix constructed in terms of pairwise correlations of observables in the quantum scenario is not necessarily positive \[22, 31, 31\] and thus, one witnesses violation of Bell, Leggett-Garg, non-contextual inequalities (which can be realized to be the positivity constraints on the eigenvalues of the moment matrix). In turn, violation of these inequalities points towards non-existence of joint probabilities corresponding to the measurements of all the observables employed. Moments extracted from measurements of a set of POVMs result in a positive moment matrix, if the degree of incompatibility is restricted to lie within the range specified by the compatibility of the set of POVMs employed \[22\].

In this paper, we construct \(N\) tern chain correlated relation inequalities involving pairwise correlations of \(N\) dichotomic random variables based on the positivity of a sequence of \(4 \times 4\) moment matrices. The bound on the linear combination of pairwise correlations (recognized through the positivity of moment matrices) ensures the existence of joint probabilities for the statistical outcomes. When the dichotomic classical random variables are replaced by qubit observables, one witnesses a violation of the chained correlation inequalities \[32, 33\]. The maximum violation of the inequalities in the quantum scenario (i.e., the corresponding Tsirelson-like bound) has been established in Refs. \[32, 33\]. The dichotomic observables, which result in the maximum quantum violation, correspond to the qubit observables in a plane. Here, we investigate the degree of incompatibility necessary for the joint measurability of the equatorial plane noisy qubit POVMs (i.e., a mixture of qubit observable in the equatorial plane and the identity matrix). Based on this, we identify that the chained inequalities are always satisfied, when the equatorial noisy qubit POVMs, employed in first measurements of sequential pairwise measurements are all jointly measurable. But, incompatible POVMs are, in general, not sufficient for violation of the chained inequalities for \(N > 3\). On the other hand, we show that there is a one-to-one correspondence between the joint measurability of a set of equatorial plane noisy qubit POVMs and the optimal violation of a linear non-local steering inequality proposed by Jones and Wiseman \[34\]. This leads us towards the construction of a local analogue of this steering inequality in a single qubit system — violation of which gives an evidence for the non-joint measurability of the set of equatorial plane qubit POVMs, employed in first of every sequential pair measurements.

We organize the contents of the paper as follows: In Sec. II we outline the notion of compatible POVM. As a specific case, we discuss the compatibility of qubit POVMs in the equatorial plane of the Bloch sphere and obtain the necessary condition for the unsharpness parameter quantifying the degree of incompatibility. Sec. III is devoted towards (i) formulation of chained \(N\) tern pairwise correlation inequalities constructed from the positivity of moment matrices; (ii) optimal violation of the inequalities in the quantum scenario, when qubit observables in the equatorial plane are employed and the connection between the degree of incompatibility of POVMs, used in first of every sequential pair measurements, and the strength of violation of the correlation inequalities. In Sec.IV we show that the steering inequality proposed by Jones and Wiseman \[34\] has a one-to-one correspondence with the joint measurability of equatorial qubit POVMs. A local analogue of this steering inequality for a single qubit system, involving \(N\) settings of sequential unsharp-sharp pairwise correlations is constructed. Sec.V contains a summary of our results and concluding remarks.

II. JOINT MEASURABILITY OF POVMS

In the conventional quantum framework, measurements are described in terms of the spectral projection operators of the corresponding self-adjoint observables. And joint measurability of two commuting observables is ensured because results of a single PV measurement are comprised of those of both the observables. However, non-commuting observables are declared as incompatible under the regime of PV measurements. Introduction of POVMs in 1960’s by Ludwig \[35\] and subsequent investigations on their applicability \[36\], led to a mathematically rigorous generalization of measurement theory. It is the notion of compatibility (the notion of compatibility of a set of POVMs will be defined in the following) — rather than commutativity — which gains importance so as to recognize if a given set of POVMs are jointly measurable or not \[37\].

A POVM is a set \(\mathbb{E}_x = \{E_x(a) = M_x^a(a)\}_{a}\) comprising of positive self-adjoint operators 0 \(\leq E_x(a) \leq 1\), satisfying \(\sum_a E_x(a) = \sum_a M_x^a(a) M_x^a(a) = 1\); \(a\) denotes the outcome of measurement and \(1\) is the identity operator. Under measurement \(\{M_x(a)\}\) a quantum system, prepared in the state \(\rho\), undergoes a positive trace-preserving generalized Lüder’s transformation i.e.,

\[
\rho \mapsto \sum_a M_x(a) \rho M_x^a(a),
\]

and an outcome \(a\) occurs with probability \(p(a|x) = \text{Tr}[\rho M_x^a(a)]= \text{Tr}[\rho E_x(a)]\). Results of PV measurements can be retrieved as a special case, when the POVM \(\{E_x(a)\}\) consists of complete, orthogonal projectors.

A finite collection \(\{\mathbb{E}_{x_1}, \mathbb{E}_{x_2}, \ldots, \mathbb{E}_{x_N}\}\) of \(N\) POVMs is said to be jointly measurable (or compatible), if there exists a grand POVM \(\mathcal{G} = \{G(\lambda); \ 0 \leq G(\lambda) \leq 1, \sum_{\lambda} G(\lambda) = 1\}\), with outcomes denoted by a collective index \(\lambda \equiv \{a_1, a_2, \ldots, a_N\}\), such that the individual
POVMs $\mathbb{E}_x$, can be expressed as its marginals: \[ E_{x_k}(a_k) = \sum_{a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_N} G(\lambda = \{a_1, a_2, \ldots, a_N\}), \]

for all $k = 1, 2, \ldots, N$. From now on, we denote the collective index $\lambda = \{a_1, a_2, \ldots, a_N\}$ characterizing measurement outcomes of global POVM $\mathcal{G}$ by $a = (a_1, a_2, \ldots, a_N)$ for brevity.

When a measurement of the global POVM $\mathcal{G} \equiv \{G(a)\}$ is carried out in an arbitrary quantum state $\rho$, an outcome ‘a’ occurs with probability $\text{Tr}[\rho G(a)] = p(a)$ . Then, the corresponding results $(p(a_k|a_k), a_k)$ (viz., the outcomes $a_k$ and the probabilities $p(a_k|a_k) = \text{Tr}[\rho E_{x_k}(a_k)]$, for all the compatible POVMs $\mathbb{E}_{x_k}$ can be deduced by post-processing the collective measurement data $p(a), \{a\}$ of the global POVM $\mathcal{G}$:

\[ p(a_k|a_k) = \sum_{a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_N} p(a). \] 

A set of POVMs $\{\mathbb{E}_{x_k}\}, k = 1, 2, \ldots, N$ are declared to be compatible if they are marginals of a global POVM $\mathcal{G}$ (as expressed in (2)).

### A. Example of noisy qubit POVMs:

Consider a pair of qubit observables $\sigma_x = \sum_{a_x = \pm 1} a_x \Pi_x(a_x)$ and $\sigma_z = \sum_{a_z = \pm 1} a_z \Pi_z(a_z)$. Sharp PV measurements of these self-adjoint observables $\sigma_x, \sigma_z$ are incorporated in terms of their spectral projectors,

\[ \Pi_x(a_x) = \frac{1}{2} \left( I + a_x \sigma_x \right), \]
\[ \Pi_z(a_z) = \frac{1}{2} \left( I + a_z \sigma_z \right). \]

Within the conventional framework of PV measurements, the non-commuting qubit observables $\sigma_x, \sigma_z$ are not jointly measurable. However, it is possible to consider a particular choice of jointly measurable noisy qubit POVMs $\mathbb{E}_x = \{E_x(a_x)\}, \mathbb{E}_z = \{E_z(a_z)\}$, by mixing white noise to the respective projection operators i.e.,

\[ E_x(a_x) = \eta \Pi_x(a_x) + (1 - \eta) \frac{I}{2} \]
\[ = \frac{1}{2} \left( I + a_x \sigma_x \right) \]
\[ E_z(a_z) = \eta \Pi_z(a_z) + (1 - \eta) \frac{I}{2} \]
\[ = \frac{1}{2} \left( I + a_z \sigma_z \right). \]

where $0 \leq \eta \leq 1$ denotes the unsharpness parameter. When $\eta = 1$, the noisy qubit POVMs reduce to their corresponding sharp PV counterparts. Throughout this paper, we will be focusing on the joint measurability (compatibility) of noisy qubit observables of the form given by (5).

The dichotomic POVMs $\mathbb{E}_x, \mathbb{E}_z$ are jointly measurable if there exists a four outcome global POVM $\mathcal{G} = \{G(a_x, a_z)\}$: $a_x = \pm 1, a_z = \pm 1$, such that

\[ \sum_{a_x, a_z = \pm 1} G(a_x, a_z) = E_x(a_x) \]
\[ \sum_{a_x, a_z = \pm 1} G(a_x, a_z) = E_z(a_z) \]
\[ \sum_{a_x, a_z = \pm 1} G(a_x, a_z) = I, \quad G(a_x, a_z) \geq 0. \] (6)

It has been shown [3, 10] that the POVMs $\mathbb{E}_x, \mathbb{E}_z$ are jointly measurable in the range $0 \leq \eta \leq \frac{1}{\sqrt{2}}$ i.e., it is possible to construct a global POVM $\mathcal{G}$ comprised of the elements $G(a_x, a_z) = \frac{1}{4} (I + \eta a_x \sigma_x + a_z \sigma_z), \quad 0 \leq \eta \leq \frac{1}{\sqrt{2}}$ (7) which obey (6).

Similarly, triple-wise joint measurements of the qubit observables $\sigma_x, \sigma_y, \sigma_z$ could be envisaged by considering the fuzzy POVMs $\mathbb{E}_x, \mathbb{E}_y, \mathbb{E}_z$, elements of which are given respectively by,

\[ E_x(a_x) = \frac{1}{2} \left( I + a_x \sigma_x \right) \]
\[ E_y(a_y) = \frac{1}{2} \left( I + a_y \sigma_y \right) \]
\[ E_z(a_z) = \frac{1}{2} \left( I + a_z \sigma_z \right) \]

in the range $0 \leq \eta \leq \frac{1}{\sqrt{3}}$ of the unsharpness parameter [14, 13].

In general, the necessary condition on the unsharpness parameter such that the qubit POVMs $\{E_{x_k}(a_k = \pm 1) = \frac{1}{2} \left[ I + a_k \vec{\sigma} \cdot \vec{n}_k \right], \quad k = 1, 2, \ldots, N\}$ are jointly measurable is derived in Ref. [13, 38]:

\[ \eta \leq \frac{1}{N} \max_a |\vec{m}_a|, \] (8)

where $\vec{m}_a$ is defined by,

\[ \vec{m}_a = \sum_{k=1}^{N} \hat{n}_k a_k, \quad a_k = \pm 1. \] (9)

The maximization is carried out over all the $2^N$ outcomes $a = (a_1 = \pm 1, a_2 = \pm 1, \ldots, a_N = \pm 1)$. A sufficient condition places the following constraint on the unsharpness parameter (derived in Ref. [13]):

\[ \eta \leq \frac{2^N}{\sum_a |\vec{m}_a|}. \] (10)
In Table I, we list the optimal value $\eta_{opt}$ of the unsharpness parameter (evaluated using the necessary and sufficient conditions \cite{13,38}), below which the joint measurability of the qubit POVMs $\{E_{z_k}(a_k) = \frac{1}{2} (I + \eta a_k \hat{\sigma} \cdot \hat{n}_k)\}$ for different orientations $\hat{n}_k$ are compatible.

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Number of POVMs & Orientation of $\hat{n}_k$ & $\eta_{opt}$ \\
\hline
$N = 3$ & Orthogonal axes & \frac{1}{\sqrt{3}} \\
$\hat{n}_k \cdot \hat{n}_l = 0, \ k \neq l = 1, 2, 3$ & & \\
\hline
$N = 2$ & Trine axes & \frac{1}{\sqrt{2}} \\
$\hat{n}_1 \cdot \hat{n}_2 = 0$ & & \\
\hline
$N = 3$ & & 0.732 \\
$\hat{n}_k \cdot \hat{n}_l = -\frac{1}{2}; \ k \neq l = 1, 2, 3$ & & \\
\hline
$N = 2$ & & \\
$\hat{n}_1 \cdot \hat{n}_2 = -\frac{1}{\sqrt{2}}$ & & \\
\hline
\end{tabular}
\end{center}
\caption{Optimal value $\eta_{opt}$ of the unsharpness parameter (evaluated using the necessary and sufficient conditions \cite{38,40}, below which the joint measurability of the qubit POVMs $\{E_{z_k}(a_k) = \frac{1}{2} (I + \eta a_k \hat{\sigma} \cdot \hat{n}_k)\}$ for different orientations $\hat{n}_k$ are compatible.}
\end{table}

We recognize the following cut-off $\eta \leq \eta_{opt}$ for any $N$:

$$\eta_{opt} = \frac{1}{N} \sqrt{N + 2 \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (N - 2k) \cos \left( \frac{k \pi}{N} \right)}. \quad (12)$$

The values of $\eta_{opt}$ are listed in Table II. In the large $N$ limit, the degree of incompatibility (i.e., the cut-off value of the unsharpness parameter) approaches $\eta_{opt} \to 0.6366$ and thus the POVMs associated with the set of all qubit observables $\sigma_\theta$, $0 \leq \theta \leq \pi$ in the equatorial plane of the Bloch sphere are jointly measurable in the range $0 \leq \eta_{opt} \leq 0.6366$.

More recently \cite{41} Uola et al. investigated incompatibility of some noisY observables in finite dimensional Hilbert spaces by developing a new technique – which they referred to as adaptive strategy. In particular, they independently identified the following sufficient condition for the simultaneous measurements of qubit observables in a plane $\sigma_{\theta_k} = \sigma_x \cos(\theta_k) + \sigma_y \sin(\theta_k)$, $\theta_k = k \pi/N$, $k = 1, 2, \ldots, N$ based on their approach \cite{41}:

$$\eta \leq \frac{2}{N} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \cos \left( \frac{(2k - 1) \pi}{2N} \right) = \frac{1}{N \sin(\pi/2N)}, \quad (13)$$

which too agrees perfectly with the optimal value \cite{12} of the unsharpness parameter (see Table II).

\begin{table}[h]
\begin{center}
\begin{tabular}{|c|c|}
\hline
Number of POVMs & $\eta_{opt}$ \\
\hline
3 & 0.6666 \\
4 & 0.6532 \\
5 & 0.6472 \\
6 & 0.6439 \\
10 & 0.6392 \\
20 & 0.6372 \\
50 & 0.6367 \\
100 & 0.6366 \\
\hline
\end{tabular}
\end{center}
\caption{Optimal value $\eta_{opt}$ of the unsharpness parameter (see \cite{12}) specifying the joint measurability of the equatorial qubit observables $\sigma_{\theta_k} = \sigma_x \cos(k \pi/N) + \sigma_y \sin(k \pi/N)$; $k = 1, 2, \ldots, N$.}
\end{table}

\section{III. CHAINED $N$ TERM CORRELATION INEQUALITIES AND JOINT MEASURABILITY}

The local realistic framework places bounds on correlations between the outcomes of measurements, carried out by spatially separated parties and Bell inequalities formulated in terms of these correlations get violated in the framework of quantum theory. On the other
hand, quantum theory too places a strict limit on the strength of these correlations. The maximum violation of CHSH inequality [23], by non-local quantum correlations, is constrained by the Tsirelson bound [12] $2\sqrt{2}$. The CHSH inequality involves measurements of two pairs of dichotomic observables on a bipartite system (denoted by $(A_1, A_2)$ and $(B_1, B_2)$ which are local observables measured by Alice, Bob respectively) and four correlation terms:

$$\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2 \quad (14)$$

An interesting connection between joint measurability and violation of the CHSH inequalities, within the framework of quantum theory, has been brought out recently by Banik et al. [13]. In general, they showed that, in a no-signaling probabilistic theory the maximum strength of violation of the inequality (14) by any pair of $(A_1^2, A_2^2)$ of quantum dichotomic observables (unsharp counterparts of $(A_1, A_2)$) is essentially determined by the optimal degree of incompatibility $\eta_{\text{opt}}$ which, in quantum theory, is identified to be $\frac{1}{\sqrt{2}}$, i.e.,

$$\langle A_1^2 B_1 \rangle + \langle A_1^2 B_2 \rangle + \langle A_2^2 B_1 \rangle - \langle A_2^2 B_2 \rangle \leq \frac{2}{\eta_{\text{opt}}} = 2\sqrt{2}. \quad (15)$$

In other words, the degree of incompatibility $\eta_{\text{opt}} = \frac{1}{\sqrt{2}}$ of measurements in the quantum framework is shown to place limitations on the maximum strength of violations of the four term CHSH inequality, by retrieving the quantum Tsirelson bound $2\sqrt{2}$.

Does this connection between the degree of incompatibility and the Tsirelson-like bound (maximum strength of violation), hold in general, when more than two incompatible measurements are involved? We explore this question through $N$ term correlation inequalities, which we formulate from positivity of a sequence of moment matrices.

Consider $N$ classical random variables $X_k$, $k = 1, 2, \ldots, N$ with outcomes $a_k = \pm 1$. Let $M_k = (\xi_k \xi_k^T)$, expressed explicitly as,

$$M_k = \begin{pmatrix} 1 & \langle X_1 X_k \rangle & \langle X_k X_{k+1} \rangle & \langle X_1 X_{k+1} \rangle \\ \langle X_1 X_k \rangle & 1 & \langle X_k X_{k+1} \rangle & \langle X_1 X_{k+1} \rangle \\ \langle X_k X_{k+1} \rangle & \langle X_k X_{k+1} \rangle & 1 & \langle X_1 X_k \rangle \\ \langle X_{k+1} X_{k+1} \rangle & \langle X_{k+1} X_{k+1} \rangle & \langle X_1 X_k \rangle & 1 \end{pmatrix}, \quad (16)$$

where $(X_1, X_l)$, $k \neq l$ denote pairwise correlations of the variables $X_k$, $X_l$ (Here $\langle \cdot \rangle$ denotes the expectation value).

In the classical probability setting, the moment matrix is, by construction, real symmetric and positive semidefinite. The eigenvalues $\lambda_i^{(k)}$, $i = 1, 2, 3, 4$ of the moment matrix are given by

$$\lambda_1^{(k)} = 1 + \langle X_1 X_k \rangle - \langle X_k X_{k+1} \rangle - \langle X_1 X_{k+1} \rangle$$
$$\lambda_2^{(k)} = 1 - \langle X_1 X_k \rangle + \langle X_k X_{k+1} \rangle - \langle X_1 X_{k+1} \rangle$$
$$\lambda_3^{(k)} = 1 - \langle X_1 X_k \rangle - \langle X_k X_{k+1} \rangle + \langle X_1 X_{k+1} \rangle$$
$$\lambda_4^{(k)} = 1 + \langle X_1 X_k \rangle + \langle X_k X_{k+1} \rangle + \langle X_1 X_{k+1} \rangle. \quad (17)$$

Replacing classical random variables $X_k$ by quantum dichotomic observables $X_k = \hat{\sigma} \cdot \hat{M}_k$, $k = 1, 2, \ldots, N$ with eigenvalues $\pm 1$, and the classical probability distribution by a density matrix, the moment matrix positivity results in linear constraints on pairwise correlations of the observables measured sequentially.

Based on the positivity of a sequence of $N - 1$ moment matrices $M_2, M_3, \ldots, M_{N-1}$ one obtains the inequalities

$$\sum_{k=2}^{N-1} \lambda_i^{(k)} \geq 0,$$

which correspond to the following chained inequalities involving pairwise correlations:

$$\sum_{k=2}^{N-1} (\langle X_k X_{k+1} \rangle - \langle X_1 X_N \rangle - \langle X_1 X_k \rangle) \leq N - 2 \quad (18)$$
$$2 \sum_{k=2}^{N} (\langle X_k X_{k+1} \rangle - \langle X_1 X_{k+1} \rangle) + \langle X_1 X_k \rangle \leq N - 2 \quad (19)$$
$$\sum_{k=2}^{N-1} (\langle X_k X_{k+1} \rangle - \langle X_1 X_N \rangle) \leq N - 2 \quad (20)$$
$$\sum_{k=2}^{N} (\langle X_k X_{k+1} \rangle + 2 \sum_{k=2}^{N-2} \langle X_1 X_{k+1} \rangle + \langle X_1 X_k \rangle) \leq N - 2. \quad (21)$$

Violation of these inequalities imply at least one of the moment matrix $M^{(k)}$ is not positive – which in turn highlights non-existence of a valid joint probability distribution for the outcomes of all the observables employed. However, it may be identified that by employing unsharp measurements of the observables – within their joint measurability region – one can retrieve positivity of the sequence of moment matrices, and consequently, the chained inequalities (18)-(21) are satisfied.

In particular, (20) is analogous to the $N$-term temporal correlation inequality investigated by Budroni et al. [35]. The pairwise correlations $(X_k X_{k+1})$ arise from the sequential measurements of the observables $X_k$ and $X_{k+1}$ in a single quantum system. Such inequalities involving sequential pairwise correlations of observables in a single quantum system (in contrast to correlations of the outcomes of local measurements at different ends of a spatially separated bipartite system as in [14]) have been well explored to highlight quantum contextuality [48] and non-macroreality [16, 47, 15].

Budroni et al. [35] computed the maximal achievable value (Tsirelson-like bound) of the left hand side of the
chained $N$ term temporal correlation inequality $\langle 20 \rangle$ and obtained

$$S_N^Q = \sum_{k=1}^{N-1} \langle X_k X_{k+1}\rangle)_{seq} - \langle X_1 X_N\rangle)_{seq} \leq N \cos \left( \frac{\pi}{N} \right).$$

(22)

The classical bound $\langle 20 \rangle$ can get violated in the quantum framework and a maximum value of $N \cos \left( \frac{\pi}{N} \right)$ could be achieved by choosing sequential measurements of appropriate observables. In particular, when a single qubit is prepared in a maximally mixed state $\rho = \frac{1}{2}$, sequential PV measurements of the observables $\sigma_k = \sigma_x \cos(\theta_k) + \sigma_y \sin(\theta_k)$, $\theta_k = k \pi/N$, $k = 1, 2, \ldots, N$ lead to pairwise correlations

$$\langle X_k X_{k+1}\rangle)_{seq} = \langle \sigma_{\theta_k} \sigma_{\theta_{k+1}}\rangle)_{seq} = \cos(\theta_{k+1} - \theta_k)$$

(23)

Substituting $\langle 23 \rangle$ in $\langle 22 \rangle$ we obtain the quantum Tsirelson-like bound $\langle 20 \rangle$.

It is pertinent to point out that the observables $\{\sigma_k = \sigma_x \cos(\theta_k) + \sigma_y \sin(\theta_k), \theta_k = k \pi/N, k = 1, 2, \ldots, N\}$ need not, in general, be associated with any particular time evolution; they are considered to be any ordered set of observables. Moreover, the pairs of sequential measurements are performed in independent statistical trials i.e., the input state in every first measurement of the pair is $\rho = \frac{1}{2}$.

A. Degree of incompatibility and violation of the chained correlation inequality $\langle 20 \rangle$

It is seen that the average pairwise correlations $\langle X_k X_{k+1}\rangle)_{seq}$ of qubit observables $X_k \equiv \sigma_{\theta_k}, k = 1, 2, \ldots, N$, evaluated based on the results of sequential sharp PV measurements, lead to maximal violation of the chained correlation inequality $\langle 20 \rangle$. Instead of sharp PV measurements of the observables, we consider here an alternate sequential measurement scheme. We separate the set of observables $\{X_k \equiv \sigma_{\theta_k}, k = 1, 2, \ldots, N\}$ of first measurements of every sequential pair. We ask if the chained inequality $\langle 20 \rangle$ is violated, when measurement of first observables of every pair correlation $\langle X_k X_{k+1}\rangle)_{seq}$ is done using noisy POVMs, while sharp PV measurements are employed for second observables in the sequence. Interestingly, we identify that the chained inequality $\langle 20 \rangle$ is not violated, whenever a compatible set of POVMs $\{E_{\theta_k}, k = 1, 2, \ldots, N\}$ (see $\langle 11 \rangle$) is employed to carry out measurements of first observables of every sequential pair irrespective of the fact that second measurements are all sharp (and hence incompatible). In other words, incompatibility of the set of POVMs, employed in carrying out first measurements in the sequential scheme, is sufficient to witness violation of the chained inequality $\langle 20 \rangle$.

We now proceed to describe the sequential measurement scheme explicitly in the following.

Consider $N$ noisy qubit observables $E_{\theta_k}$ with elements $\{E_{\theta_k}(a_k = \pm 1) = M_{\theta_k}(a_k) M_{\theta_k}(a_k)\}$ given by $\langle 11 \rangle$. From our discussions in Sec. IIB, it is seen that there exists a global qubit POVM ${\mathbb G}$, when $\eta$ lies in the range $0 \leq \eta \leq \eta_{\text{opt}}$ (see $\langle 12 \rangle$, for the values of the parameter $\eta_{\text{opt}}$), such that the POVMs $E_{\theta_k}, k = 1, 2, \ldots, N$ are all jointly measurable.

As before, we consider the initial state of the qubit to be $\rho = \frac{1}{2}$, a maximally mixed state. Carrying out unsharp measurement $M_{\theta_k}(a_k)$, yielding an outcome $a_k$, the initial state gets transformed to

$$\rho \rightarrow \rho_{\text{opt}} = \frac{M_{\theta_k}(a_k) \rho M_{\theta_k}(a_k) \rho}{p(a_k | \theta_k)}$$

(24)

where we have denoted $\text{Tr}[\rho M_{\theta_k}(a_k) M_{\theta_k}(a_k)] = \text{Tr}[\rho E_{\theta_k}(a_k)] = p(a_k | \theta_k)$. Following this with a second PV measurement of $\sigma_{\theta_{k+1}}$, on the state $\rho_{\text{opt}}$ results in the pairwise correlations

$$\langle X_k^{(\eta)} X_{k+1}\rangle)_{seq} = \sum_{a_k} p(a_k | \theta_k) \text{Tr}[\rho_{\text{opt}} \sigma_{\theta_{k+1}}] = \eta \cos(\theta_{k+1} - \theta_k) = \eta \cos(\pi/N).$$

(25)

So, the left hand side of chained correlation inequality $\langle 20 \rangle$ assumes the value,

$$S_N^Q(\eta) = \sum_{k=1}^{N-1} \langle X_k^{(\eta)} X_{k+1}\rangle)_{seq} - \langle X_1^{(\eta)} X_N\rangle)_{seq} = \eta N \cos \left( \frac{\pi}{N} \right),$$

(26)

when pairwise unsharp-sharp measurements of equatorial qubit observables are carried out. Within the joint measurability domain of the set $\{E_{\theta_k}, k = 1, 2, \ldots, N\}$ of first unsharp measurements in this sequential scheme, the sum of pairwise correlations obey

$$S_N^Q(\eta) = \sum_{k=2}^{N-1} \langle X_k^{(\eta)} X_{k+1}\rangle)_{seq} - \langle X_1^{(\eta)} X_N\rangle)_{seq} \leq \eta_{\text{opt}} N \cos \left( \frac{\pi}{N} \right).$$

(27)

Using the optimal values $\eta_{\text{opt}}$ specifying the degree of incompatibility of the equatorial qubit observables (see $\langle 12 \rangle$ and the values listed in Table II), we evaluated the maximum value $S_N^Q(\eta_{\text{opt}}) = \eta_{\text{opt}} N \cos \left( \frac{\pi}{N} \right)$ attainable by the left hand side of the inequality $\langle 27 \rangle$ for different values of $N$; these values are listed together with the corresponding classical and quantum bounds in Table III. It is evident that as the number of measurements $N$ increases, the quantum Tsirelson-like bound approaches the algebraic maximum value $N$, while the maximum achievable
value of \( \eta_{\text{opt}} \) approaches \( S_N^Q(\eta_{\text{opt}}) \rightarrow 0.6366 \times N \) (which is equal to the classical bound \( N - 2 = 1 \) for \( N = 3 \) and is less than \( N - 2 = N > 3 \)). More specifically, the classical bound is always satisfied, when the first measurements in the sequential scheme are carried out by compatible POVMs. However, unlike the situation in the CHSH-Bell inequality \( \ref{eq:chsh} \), the maximum achievable value \( S_N^Q(\eta_{\text{opt}}) \) is not identically equal to the classical bound of \( N - 2 \), except in the case of \( N = 3 \) \( \ref{eq:chsh} \). So, it is evident that incompatible set \( \{ E_\theta; \ \eta > \eta_{\text{opt}}, \ k = 1, 2, \cdots N \} \) of POVMs are necessary, but not sufficient to violate the chained \( N \) term correlation inequality \( \ref{eq:chsh} \). Is it possible to find a steering protocol, for which incompatibility of equatorial qubit measurements is both necessary and sufficient? In the next section we discuss a linear steering inequality involving equatorial qubit observables \( \ref{eq:chsh} \) and unravel how violation of the inequality gets intertwined with measurement incompatibility.

### IV. LINEAR STEERING INEQUALITY AND JOINT MEASURABILITY

Quantum steering (introduced by Schrödinger in 1935 \( \ref{eq:schroedinger} \)) has gained much impetus in recent years. In 1989, Reid \( \ref{eq:reid} \) proposed an experimentally testable steering criterion, which revealed that – apart from Bell-type non-locality – steering is yet another distinct manifestation of Einstein-Podolsky-Rosen (EPR) non-locality in spatially separated composite quantum systems. A conceptually clear formalism of EPR steering (in terms of local hidden state (LHS) model) has been formulated by Wiseman et. al. \( \ref{eq:wiseman} \). They elucidated that steering constitutes a different kind of non-locality, which lies between entanglement and Bell-type non-locality. Several steering inequalities – suitable for the experimen-

| No. of POVMs employed | Classical bound N-2 | Quantum bound \( N \cos \left( \frac{\pi}{N} \right) \) | Maximum achievable value \( S_N^Q(\eta_{\text{opt}}) \) |
|-----------------------|---------------------|-------------------------|----------------------------------|
| 3                     | 1                   | 1.5                     | 1                                |
| 4                     | 2                   | 2.83                    | 1.85                             |
| 5                     | 3                   | 4.05                    | 2.62                             |
| 6                     | 4                   | 5.20                    | 3.35                             |
| 10                    | 8                   | 9.51                    | 6.08                             |
| 20                    | 18                  | 19.75                   | 12.59                            |
| 50                    | 48                  | 49.90                   | 31.77                            |
| 100                   | 98                  | 99.95                   | 63.62                            |

TABLE III. Maximum attainable value \( S_N^Q(\eta_{\text{opt}}) = \eta_{\text{opt}} N \cos \left( \frac{\pi}{N} \right) \) of the left hand side of the \( N \) term temporal correlation inequality \( \ref{eq:chsh} \) when the qubit POVMs employed are jointly measurable (see \( \ref{eq:chsh} \)).

Consider a qubit observable \( \sigma = \sigma_x \cos(\theta) + \sigma_y \sin(\theta) \) denotes an equatorial qubit observable and \( -1 \leq \alpha \leq 1 \). Expectation value of the observable \( S_{\text{plane}} \) is upper bounded by

\[
\langle S_{\text{plane}} \rangle \leq \frac{1}{\pi} \int_0^\pi d\theta \langle \sigma \rangle
\]

\[
= \frac{1}{\pi} \int_0^\pi d\theta \left( \langle \sigma_x \rangle \cos(\theta) + \langle \sigma_y \rangle \sin(\theta) \right)
\]

\[
\Rightarrow \langle S_{\text{plane}} \rangle \leq \frac{2}{\pi}
\]
Suppose Alice and Bob share a two qubit state $\rho_{AB}$; Bob asks Alice to perform measurements of $\sigma^B_\theta$ and communicate the outcome $a_\theta = \pm 1$ of her measurements. After Alice’s measurements, Bob will be left with an ensemble $\{p(a_\theta|\theta), \rho^B_{a_\theta}\}$ where $\rho^B_{a_\theta} = \text{Tr}_A[\Pi_\theta(\rho_{AB}) \otimes I_B]$. Let $p(a_\theta|\theta)$ be the conditional state (here $\Pi_\theta(\rho_{AB}) \otimes I_B$) denote Bob’s conditional states (here $(\Pi_\theta(\rho_{AB}) \otimes I_B)$ denote PV measurements of the observable $\sigma^B_\theta$). At his end, Bob would then measure the observable $\sigma^B_{\theta_k}$. Suppose he gets an outcomes $b_\theta = \pm 1$ with probability $p(b_\theta|a_\theta, \theta) = \text{Tr}[\Pi_\theta(b_\theta) \rho^B_{a_\theta|a_\theta, \theta}]$. He evaluates the conditional expectation value of the observable $\sigma^B_{\theta_k}$ based on the statistical data he obtains as follows:

$$\langle \sigma^B_{\theta_k} \rangle_{a_\theta} = \sum_{b_\theta = \pm 1} b_\theta p(b_\theta|a_\theta, \theta). \quad (30)$$

If the conditional probabilities $p(b_\theta|a_\theta, \theta)$ originate from a LHS model i.e., if

$$p(b_\theta|a_\theta, \theta) = \sum_\lambda p(\lambda|a_\theta, \theta, \lambda) \text{Tr}[\Pi_\theta(b_\theta) \rho^B_{a_\theta|a_\theta, \theta}] = \sum_\lambda p(\lambda|a_\theta, \theta, \lambda) \langle \Pi_\theta(b_\theta) \rangle_\lambda, \quad (31)$$

(where we have denoted $\sum_{b_\theta = \pm 1} b_\theta \langle \Pi_\theta(b_\theta) \rangle_\lambda = \langle \sigma^B_{\theta_k} \rangle_\lambda$, one gets the conditional expectation value in the LHS model as follows:

$$\langle \sigma^B_{\theta_k} \rangle_{a_\theta|\theta} = \sum_\lambda p(\lambda|a_\theta, \theta, \lambda) \text{Tr}[\Pi_\theta(b_\theta) \rho^B_{a_\theta|a_\theta, \theta}] = \sum_\lambda p(\lambda|a_\theta, \theta, \lambda) \langle \sigma^B_{\theta_k} \rangle_\lambda. \quad (32)$$

Whenever the LHS model holds, the inequality

$$\frac{1}{\pi} \int_0^\pi d\theta \alpha_\theta \langle \sigma^B_{\theta_k} \rangle_{a_\theta|\theta} \leq \frac{2}{\pi} \quad (33)$$

is obeyed, for any $-1 \leq \alpha_\theta \leq 1$ in the LHS framework. Now, denoting $\sum_{a_\theta = \pm 1} a_\theta p(a_\theta|\theta) \langle \sigma^B_{\theta_k} \rangle_{a_\theta|a_\theta, \theta} = \langle \sigma^A \sigma^B \rangle$ one obtains the linear steering inequality [34]

$$\frac{1}{\pi} \int_0^\pi d\theta \langle \sigma^A \sigma^B \rangle \leq \frac{2}{\pi}. \quad (34)$$

Violation of the inequality [34] in any bipartite quantum state $\rho_{AB}$ demonstrates non-local EPR steering phenomena (more specifically, violation implies falsification of the LHS model, which confirms that Alice can indeed steer Bob’s state remotely via her local measurements).

Note that implementing infinite number of measurements (i.e., measurement of $\sigma^B_{\theta_k}$ by Bob conditioned by the outcomes of Alice’s measurement of $\sigma^A_{\theta_k}$, in the entire equatorial half plane $0 \leq \theta \leq \pi$, is a tough task in a realistic experimental scenario. So, it would be suitable to consider a finite-setting of $N$ evenly spaced equatorial measurements of $\sigma^A_{\theta_k}$ (such that the successive angular separation is given by $\pi/N$ i.e., $\theta_{k+1} - \theta_k = \pi/N$) by Bob, conditioned by the $\pm 1$ valued outcomes $a_k$ of Alice’s measurements $\sigma^A_{\theta_k}$. This leads to the following linear steering inequality in the finite setting [34]:

$$\frac{1}{N} \sum_{k=1}^N \langle \sigma^A_{\theta_k} \sigma^B_{\theta_k} \rangle \leq f(N) \quad (35)$$

where

$$f(N) = \frac{1}{N} \left( \left| \sin \left( \frac{N \pi}{2} \right) \right| + 2 \sum_{k=1}^{[N/2]} \sin \left( (2k-1) \frac{\pi}{2N} \right) \right), \quad (36)$$

corresponds to the maximum eigenvalue of the observable $\sum_{\theta_k} \sigma^A_{\theta_k}$.

One obtains $f(2) = 1/\sqrt{2}$, $f(3) \approx 0.6666$, $f(4) = 0.6533$, $f(10) \approx 0.6392$ for smaller values of $N$. (Note that there is a striking match between the degree of incompatibility $\eta_q$ listed in Table II and the upper bound $f(N)$ of the inequality [35]). The factor $f(N) \rightarrow 2/\pi \approx 0.6366$ in the limit $N \rightarrow \infty$.

We discuss the violation of the steering inequality [35] when Alice and Bob share a maximally entangled two qubit state.

B. Violation of the linear steering inequality by a two qubit maximally entangled state

Let Alice and Bob share a maximally entangled Bell state $|\psi^-\rangle = (1/\sqrt{2}) |0A, 1B\rangle - |1A, 0B\rangle$. Alice performs PV measurement $\{\Pi_\theta(a_k) = \frac{1}{2} (1 + a_k \sigma^A_{\theta_0})\}$ of one of the equatorial qubit observable $\sigma^A_{\theta_k}$, which results in an outcome $a_k = \pm 1$, leaving Bob’s conditional state in the form:

$$\rho^B_{a_k|\theta_k} = \text{Tr}_A[\Pi_\theta(a_k) \otimes I_B |\psi^-\rangle \langle \psi^-|] / p(a_k|\theta_k) = \Pi_{\theta_k}(a_k). \quad (37)$$

(Alice’s outcomes $a_k = \pm 1$ are totally random and occur with probability $p(a_k|\theta_k) = 1/2$ for any measurement setting $\theta_k$).

Bob then performs sharp measurements $\{\Pi_{\theta_k}(b_k)\}$ on his state and computes the conditional average value of the observable $\sigma^B_{\theta_k}$ to obtain,

$$\langle \sigma^B_{\theta_k} \rangle_{a_k|\theta_k} = \sum_{b_\theta = \pm 1} b_\theta \text{Tr}[\Pi_{\theta_k}(a_k) \Pi_{\theta_k}(b_k)] = a_k \quad (38)$$

Further, evaluating the average of $\langle \sigma^B_{\theta_k} \rangle_{a_k|\theta_k}$ together with Alice’s outcomes $a_k$, one obtains,

$$\langle \sigma^A \sigma^B_{\theta_k} \rangle = \sum_{a_k = \pm 1} a_k p(a_k) \langle \sigma^A \sigma^B_{\theta_k} \rangle_{ak|\theta_k} = 1 \quad \text{for all } \theta_k. \quad (39)$$

Thus, the left hand side of the linear steering inequality [35] may be readily evaluated and it is given by
$$\frac{1}{N} \sum_{k=1}^{N} \langle \sigma^A_k \sigma^B_k \rangle = 1,$$

which is clearly larger than the upper bound \( f(N) \) of the steering inequality (note that \( f(N) \) varies from its largest \( f(2) \approx 0.7071 \) for \( N = 2 \) measurement settings to its limiting value \( f(\infty) = 0.6366 \) when \( N \to \infty \)). In the next subsection we show that the violation of the steering inequality reduces to an inequality \( \eta > \eta_{\text{opt}} \) (i.e., the unsharpness parameter \( \eta \) of Alice’s local equatorial qubit POVMs exceeds the cut-off value \( \eta_{\text{opt}} \) specifying their compatibility) which, in turn, implies that the set of Alice’s measurements are incompatible.

It is pertinent to point out that a modification of the finite setting linear steering inequality \( \eta_{\text{opt}} \) – violation of which has been tested experimentally \[51\]: Including a single nonequatorial measurement of \( \sigma_z \) by Bob, the linear steering inequality \( \eta_{\text{opt}} \) – constructed for a finite set of equatorial observables – gets modified into a non-linear steering inequality \[34\], violation of which is shown to be more feasible for experimental detection, than that of its linear counterpart \[34\]. In an ingenious experimental set up \[51\], where a single photon is split into two ports by a beam-splitter, it has been rigorously demonstrated that a set of six different equatorial measurements in one port (i.e., Alice’s end) can indeed steer the state of the photon in the other port (Bob’s end).

C. Joint measurability condition from linear steering inequality

Now, we proceed to discuss the implications of joint measurability on the linear steering inequality \( \eta_{\text{opt}} \).

If Alice performs unsharp measurement of one of the equatorial qubit POVMs \( E_{\theta_k}(a_k) = \{E_{\theta_k}(a_k) = \frac{1}{2} (\mathbb{I} + \eta \sigma_x \theta_k) \} \) with an outcome \( a_k = \pm 1 \), Bob is left with the following conditional state,

\[ \rho^B_{a_k|\theta_k} = \text{Tr}_A \left[ (E_{\theta_k}(a_k) \otimes \mathbb{I}_B) |\psi^-\rangle\langle\psi^-| \right] / p(a_k|\theta_k) \]

\[ = \frac{1}{4} \left[ (\mathbb{I} - \eta \sigma_x \theta_k) / p(a_k|\theta_k) \right] \]

\[ = E_{\theta_k}(a_k), \]

the probability of Alice’s obtaining the outcome \( a_k \) being \( p(a_k|\theta_k) = 1/2 \). Here, we have denoted the spin-flipped version of the POVM \( \{E_{\theta_k}(a_k) = \frac{1}{2} (\mathbb{I} + \eta \sigma_x \theta_k) \} \) by \( \{E_{\theta_k}(a_k) = \frac{1}{2} (\mathbb{I} - \eta \sigma_x \theta_k) \} \). Following Alice’s measurement, Bob carries out sharp measurements \( \{\Pi_{\theta_k}(b_k) = \frac{1}{2} (\mathbb{I} - \eta \sigma_x \theta_k) \} \) on his state and computes the conditional average value of the observable \( \sigma^B_{\theta_k} \) to obtain,

\[ \langle \sigma^B_{\theta_k}|a_k=\pm 1 \rangle = \sum_{b_k=\pm 1} b_k \text{Tr} \left[ \rho^B_{a_k|\theta_k} \Pi_{\theta_k}(b_k) \right] \]

\[ = \sum_{b_k=\pm 1} b_k \text{Tr} \left[ E_{\theta_k}(a_k) (b_k) \right] \]

\[ = \eta a_k. \]

Averaging the conditional expectation value \( \langle \sigma^B_{\theta_k}|a_k=\pm 1 \rangle \) with Alice’s outcomes \( a_k \), we obtain,

\[ \langle \sigma^B_{\theta_k}|a_k=\pm 1 \rangle = \sum_{a_k=\pm 1} a_k p(a_k|\theta_k) \langle \sigma^B_{\theta_k}|a_k=\pm 1 \rangle = \eta. \]

Thus the finite setting linear steering inequality \( \eta_{\text{opt}} \) reduces to,

\[ \eta \leq f(N). \]

This reduces to the joint measurability condition \( \eta \leq \eta_{\text{opt}} \) for Alice’s local unsharp measurements, as one can identify a striking agreement between the degree of incompatibility \( \eta_{\text{opt}} \) (given by \[41\] and listed in Table. II)) and the upper bound \( f(N) \) (given in \[40\]) of the finite setting linear steering inequality \( \eta_{\text{opt}} \). This is a clear example of the intrinsic connection (established in Refs. \[16\] \[18\]) between steering and measurement incompatibility. Moreover, the equivalence between the degree of incompatibility (as given in \[41\]) and the linear steering inequality in the finite setting (see \[40\]) highlights the relation between a local quantum feature i.e., non-joint measurability and a non-local one viz., steerability. Would it be possible to demonstrate measurement incompatibility, without employing a non-local resource (i.e., an entangled state)? In this direction, it is pertinent to point out that time-like analogues of steering have been formulated recently \[19\] \[52\], and there has been an ongoing research interest towards developing resource theories of measurement incompatibility and non-local steerability \[20\] \[21\] \[23\]. This leads us to formulate (in the next subsection) a local analogue of the linear steering inequality \( \eta_{\text{opt}} \) in a single qubit system – violation of which implies incompatibility of the qubit POVMs employed in first measurements of the sequential pair.

D. Local analogue of the linear steering inequality

As has been discussed in previous subsections, expectation value of the qubit observable \( S_{\text{plane}} = (1/\pi) \int_0^{2\pi} d\theta \sigma_\theta; -1 \leq \sigma_\theta \leq 1 \), is bounded by \( 2/\pi \) (see \[29\]). This bound is not obeyed, in general, if the expectation value of the observable \( \langle \sigma_\theta \rangle \) is replaced by its conditional expectation value \( \langle \sigma_\theta|a_\theta \rangle \), evaluated in a sequential measurement, with the first measurement resulting in an outcome \( a_\theta \). In particular, in the setting where finite number of pairwise sequential measurements of the equatorial qubit observable \( \sigma_\theta \), with same angle \( \theta_\theta \), are carried out \[53\], the analogue of the steering inequality \( \eta_{\text{opt}} \)

\[ \frac{1}{N} \langle \sigma^{(1)}_{\theta_\theta} \sigma^{(2)}_{\theta_\theta} \rangle \leq f(N) \]

could get violated in the single qubit system. Here, we have denoted \( \langle \sigma^{(1)}_{\theta_\theta} \sigma^{(2)}_{\theta_\theta} \rangle = \sum_{a_\theta} a_\theta p(a_\theta|\theta_\theta) \langle \sigma^{(2)}_{\theta_\theta}|a_\theta=\pm 1 \rangle |\theta_\theta \rangle = \sum_{a_\theta} a_\theta b_\theta p(b_\theta|a_\theta, \theta_\theta) \) denotes the conditional expectation value of \( \sigma_\theta \) – given that the first measurement has resulted in an outcome \( a_\theta \) with probability
pairs of unsharp-sharp measurements are carried out se-
ing different manifestations of non-classicality. An inter-
cessant in that it leads to conceptual clarity in understand-
non-locality and measurement incompatibility is signifi-
measurements of the sequential scheme.

The average value \( \langle \sigma^{(1)}_{\theta_k} \sigma^{(2)}_{\theta_k} \rangle \), evaluated using the statis-
tical data of the first measurement results in,

\[
\langle \sigma^{(1)}_{\theta_k} \sigma^{(2)}_{\theta_k} \rangle = \sum_{a_k = \pm 1} a_k p(a_k|\theta_k) \left( \langle \sigma^{(2)}_{\theta_k} \rangle_{a_k} \right)_{a_k|\theta_k}
\]

Thus, the inequality (43) reduces to \( \eta \leq f(N) \), when \( N \) pairs of unsharp-sharp measurements are carried out se-
entially in a single qubit system. Clearly, the inequality is violated, when only sharp PV measurements (with \( \eta = 1 \)) are carried out. On the other hand, the inequality is always obeyed, when the set \( \{ E_{\theta_k}(a_k), k = 1, 2, \ldots N \} \) of all POVMs employed in first measurements of every sequential pair measurements, is jointly meas-
urable. In other words, we have shown that violation of the local analogue of the steering inequality (43) in a single qubit system is a consequence of incompatibility of measurements of the qubit POVMs employed in first measurements of the sequential scheme.

V. CONCLUSIONS

Discerning the intrinsic connection between quantum non-locality and measurement incompatibility is signifi-
cant in that it leads to conceptual clarity in understanding different manifestations of non-classicality. An inter-
esting recent result by Banik et. al. (44), revealed that the degree of measurement incompatibility – quantifying joint measurability of two dichotomic observables – places restrictions on the maximum strength of violation of the CHSH-Bell inequality. A natural question then is whether such a quantitative connection exists in general, when more than two measurement settings are involved. In this paper we have explored the connection between the maximum achievable bound (Tsirelson-like quantum bound) on the violation of \( N \) term pairwise correlation inequality (33) and the degree of measurement incompat-
ibility of \( N \) dichotomic qubit POVMs, employed in car-
ying out first measurement of sequential pair measure-
ments. To this end, we have constructed \( N \) term chained correlation inequalities based on the positivity of a se-
quence of \( 4 \times 4 \) moment matrices in the classical proba-
ability setting. Replacing the classical dichotomic random variables by qubit observables and classical probability distribution by quantum state, we obtain the analogue of chained \( N \) term correlation inequalities in the quantum scenario; in general the correlations do not obey the classical bound – resulting in the violation of the inequalities. Maximum achievable quantum bound (Tsirelson-
like bound) on one of these chained inequalities – in-
volving pairwise correlations of statistical outcomes of dichotomic observables measured sequentially in a sin-
gle quantum system – is known (33); and the dichotomic observables, which result in the maximum quantum vi-
lation of the inequality, correspond to qubit observables, having equal successive angular separation of \( \pi/N \) in a plane. We have shown in this work that the \( N \)-term chained inequality (20) is always obeyed, when the set of all POVMs employed in first measurements of every pair-
wise correlation term, is compatible. However, measure-
ment incompatibility of equatorial qubit POVMs serves, in general, as a necessary condition. For \( N > 3 \), incompat-
bility is not sufficient to result in violation of (20).

To be specific, a tight relation between the degree of in-
compatibility and the maximum strength of quantum vi-
olation of the correlation inequality holds mainly in two special cases: (i) Measurements of a pair of dichotomic observables on one part of a bipartite quantum system are considered. In this case, the degree of incompatibility \( \eta_{\text{opt}} = 1/\sqrt{2} \) – for the pair of dichotomic observables to be jointly measurable – places an upper bound \( 2/\eta_{\text{opt}} = 2\sqrt{2} \) on the maximum achievable quantum bound of CHSH-
Bell inequality (43) (ii) In a three term correlation inequal-
ity (20), with a classical upper bound 1; here, se-
quential pairwise measurements of \( N = 3 \) dichotomic ob-
servables are carried out in a single qubit system prepared initially in a maximally mixed state. The inequality is known to be violated maximally ( quantum upper bound being 3/2), when the three dichotomic observables cor-
respond to qubit orientations, forming trine axis (three axes with equal successive angular separations of \( \pi/3 \) in a plane). In this case, the degree of measurement incompat-
ibility of the three POVMs is given by \( \eta_{\text{opt}} = 2/3 \). When these POVMs are used in first measurements of the sequential pair measurements, the degree of incom-
compatibility places restrictions on the maximum achievable quantum bound \( \frac{\eta}{\eta_{\text{opt}}} \), i.e., \( \frac{1}{\eta_{\text{opt}}} = 3/2 \). In view of the recent research focus on the equivalence between joint measurability and non-local steering \[16-18\], we have explored a linear steering inequality – introduced by Jones and Wiseman \[34\] – which involves measurements of \( N \) equatorial plane qubit POVMs. We have shown that this indeed unfolds a striking connection between the optimal violation of the \( N \) term steering inequality and the degree of incompatibility of equatorial qubit POVMs.

Within the perspective of our study, it appears natural to ask if one can devise a local test (by carrying out a set of sequential measurements on a single quantum system) to infer about measurement incompatibility – than employing a non-local steering protocol (which requires an entangled state)? We have addressed this question – by restricting to the specific example pertaining to \( N \) equatorial qubit observables – and have shown that a local analogue of the linear steering inequality of Ref. \[34\] can be formulated in a single quantum system – involving a linear combination of pairwise conditional correlations, resulting from \( N \) sequentially ordered unsharp-sharp pairwise measurements (performed in independent statistical trials for each pair, with the input state for every first measurement being \( \rho = 1/2 \)) of equatorial qubit observables. Violation of this local steering inequality is shown to be a reflection of measurement incompatibility of POVMs employed in first of the sequential pairwise measurements.

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Joint measurability requires that in every state, there exists underlying joint probabilities corresponding to the measurement outcomes of a set of compatible POVMs; and these joint probabilities yield correct marginal probabilities corresponding to the outcomes of all the individual POVMs – which comprise of a compatible set.

Note that there are $2^N$ possible sets of \( \vec{m}_a = \sum_k \hat{n}_k a_k \), based on all the distinct arrangements of the measurement outcomes \( \{a_1 = \pm 1, a_2 = \pm 1, \ldots \} \).

In the generalized framework, one finds pairwise joint measurability of POVMs does not imply their triple-wise compatibility, even in the case of two dimensional Hilbert space of qubits – which is not the case in the conventional PV measurement scenario \[10, 13, 38\]. Due to the existence of pairwise – but not triplewise – jointly measurable qubit POVMs, violations of a state dependent inequality, which highlights Kochen-Specker measurement contextuality, by a quantum system in the two dimensional Hilbert space of qubits – is not the case in the conventional PV measurement scenario \[10, 13, 38\]. This extended notion of contextuality gains importance due to the fact that the Kochen-Specker non-contextuality \[2\] could only be refuted in quantum systems belonging to a Hilbert space dimension \( d \geq 3 \) – when one is confined to the conventional PV measurement framework.

In order to construct an analogue of the linear steering inequality \[35\] in a single qubit system, sequential measurements of pairs of observables with same angle \( \theta_k \) has to be carried out.