The Poincaré Duality in Quantization of the Norm.

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Abstract

The more important difference between Riemann and pseudo Riemann manifolds is the metric signature and its theoretical consequences. The scientific literature in Riemann manifolds is sound and extensive. However, the practical application for Physics Theories becomes often impossible due to the signature consequences. Eg., some of the rich results in Riemann Geometry and Topology becomes invalid for Physics if they are based on the concept of positive semi definite norm. To avoid this problem, the proof machinery must avoid such assumption and must be based in other tools. This paper is a contribution to provide methodologies for Hodge decomposition and Poincaré duality based on the concept of linear independence instead of positive norm.

As result, the Hodge decomposition and also the norm decomposition are expressed based on continuous and discrete terms. When this results are applied to Classical Electromagnetic Theory, in pseudo Riemann manifolds with minkowskian metric, magnitudes as the field norm and action have one discrete stepwise term. This result of quantization of the norm is a property of the Topology, in special of the Cohomology classes.

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1 Introduction

The Hodge decomposition of any p-form $\phi$ in Riemann manifolds allows a canonical decomposition as:

$$\phi = d\alpha + \delta\beta + \phi_h$$

where $d$ and $\delta$ are the exterior derivative and coderivative\[8\], respectively, $\alpha$ is a $(p-1)$-form, $\beta$ is a $(p+1)$-form and $\phi_h$ is an harmonic form characteristic of the cohomology classes. The decomposition establishes that $d\alpha$, $\delta\beta$ and $\phi_h$ are univocally determined.

In one of the most simple cases of Riemann manifolds –the Euclidean one– a close related and older similarity to the Hodge decomposition has been used, from the Ninth Century, in Theoretical Physics as the Helmholtz Theorem which allows to decompose a vector field, $\mathbf{V}$, in two sub-fields according to its potential and rotational sources as:

$$\mathbf{V} = \nabla\Phi + \nabla \times \gamma$$
This decomposition have been very important in the developing of Classical Electromagnetic Theory involved with fields generated by electrical charges and solenoid sources, eg. vectors fields $\mathbf{E}$ and $\mathbf{B}$. Also, in Fluid Dynamics have allowed the decomposition of the fluid speed field in terms related to potential and vortex sources. A remarkable property of Hodge decomposition is that each one of the terms become null for the operators derivative, $d$, and coderivative $\delta$. Similarly, in the Helmholtz decomposition each one of the term become null for the operators divergence ($\nabla \cdot$) and rotational ($\nabla \times$). Also, it must be highlighted the linear independence between both operators, that is, if:

$$A \nabla \Phi + B \nabla \times \gamma = 0$$

then, must be: $A = B = 0$. This linear independence in more modern field decomposition, as the based on Hodge one, will be extensively exploited because the advantage of Hodge decomposition for modern Field Theory in Riemann manifolds is that allows the analysis by splitting in different canonical components which have specific formal properties.

Some proofs of the Hodge decomposition, eg. Morita [15] (Sec. 4.3) and Jost [12], are based on that in Riemann manifolds the norm is positive semi-definite, it verifies: $(\alpha, \alpha) \geq 0$, also if it is verified that $(\alpha, \alpha) = 0$, then must be $\alpha = 0$. Thus, it is positive definite for all non null forms. This property is central in the theory of harmonic forms, but this concept can be different between Riemann and pseudo-Riemann manifolds as physical examples will be exposed. The use of positive defined norm is an useful tool for Riemannian manifolds, but is not the case for pseudo-Riemannian ones because the norm could not be positive definite.

This report provides an alternative method to study some formal properties of the Hodge Decomposition. Our approach is rather based on the linear independence of canonical differential forms instead of be based on the use of positive norm. In this process, Poincaré duality [10, 3] plays a remarkable role, mainly in the decomposition of the norm including a sum of discrete terms based on the integral of cohomology representative forms. This allows that the norm value is the sum of some discrete stepwise values. This type of discontinuity is named the quantization of the norm and each cohomology class contributes with one quantum.

Both Poincaré and Hodge duality are some type of complement relationship in relation to the unit or volume form, that implies that all the forms are quadratic integrable. However, beyond that resemblance, both dualities are applied to different theoretical objects. But there is a closed relationship between both types when are involved in homological classes. Self Poincaré duals cohomologies in 4$^k$ manifold emerge naturally as relevant solutions.

Remarkably is the result that is achieved when this property of the quantized norm of differential forms is translated to Physical Theories. In the case of Electromagnetism in no simple connected manifolds, the Electromagnetic Action in Laplacian formalism becomes quantized such as to each cohomology class is associated one action quantum. Although the Wormholes approaches are also based in Topology, the approach of this paper is founded on Topology concepts used abstractly as mathematical models, no assumption is included about how the space-time is. However the described picture has in common with wormholes theories the concept of field without elementary particles [13]. But this contribution provided the concept of charges has manifold properties that are collective properties of the set of cohomologies. The study of the properties of such set is open, it have been started only with the simple value of Betti number $\beta_m = 2$.

The plan of this report is as follows, Section 2 presents a summary of theoretical materials used as the cohomological classes and the Poincaré duality. Section 3 presents some concepts of qualitative meta-theory of linear decomposition. Section 4 presents the Hodge decomposition from the viewpoint of linear independent classes and the Poincaré duality in relation to the decomposition. Section 5 presents the decomposition of the norm that includes some discrete, or quantized, terms associated to each cohomology class. The special case of even dimensional
manifolds is presented in Section 6. An application for generalized Electromagnetism is presented in Section 7. Finally, a Conclusions Section is included.

2 Theoretical Framework in Pseudo Riemann Manifolds

Let \((M, g)\) be a \(n\)-dimensional compact, differentiable, oriented and connected manifold \(M\) with a metric \(g\) locally reducible to a diagonal case:

\[
\eta = \text{diag}(1,\ldots,1,-1,\ldots,-1)
\]

where \(r\) and \(s = n - r\) are the number of positive and negative ones respectively. If both \(r\) and \(s\) are non null, it is named a pseudo-Riemann, or semi-Riemann, manifold with indefinite metric, while Riemann manifold with \(s = 0\) is a case that has positive definite metric.

Although there are many similarities between Riemann and pseudo Riemann manifolds, there are some differences in the properties that are consequences of the metric signature; in differential forms mainly in the compute of the Hodge dual and its subsequent uses. Many properties depends on this duality, therefore many of the results in the widespread bibliography in Riemann manifolds must be carefully used.

Let \(A_p(M, \mathbb{R})\) be the set of \(p\)-forms on \(M\) with values in \(\mathbb{R}\). The Hodge duality gets a linear isomorphism between \(A_p\) and \(A_{n-p}\). The Hodge star operator, \(\ast\), defines a linear map \(\ast: A_p \rightarrow A_{n-p}\). The Hodge duality is close related to the unit \(n\)-form \(\omega\), such as is \(\alpha\) is a \(p\)-form and its dual \(\ast\alpha\) verify:

\[
\alpha \wedge \ast\alpha = \langle \alpha, \alpha \rangle \omega \int_M \omega = 1
\]

where \(\langle \alpha, \alpha \rangle\) is their inner product or square of their norm dependent on the metric signature, eg. for a vector \(V\), it is: \(\langle V, V \rangle = V_a V_b g^{ab}\). The Hodge dual of the unit \(n\)-form \(\omega\) is the scalar or 0-form 1; the relationships between both are: \(\ast 1 = \omega\) and \(\ast \omega = (-1)^s\). The double Hodge duality for \(\phi \in A_p\) verifies:

\[
\ast \ast \phi = (-1)^{C(p)} \phi \quad D(p) = p(n-p) + s
\]

It is verified that: \(D(p) = D(n-p)\). The exterior derivative, \(d\), defines a linear map \(d: A_p \rightarrow A_{p+1}\), allows the definition of the coderivative \(\delta: A_p \rightarrow A^{p-1}\) defined as:

\[
\delta \phi = (-1)^{C'(p)} \ast d \ast \phi \quad C'(p) = np + n + 1 + s
\]

The Hodge duality allows to define a bilinear integral in the manifold \(M\) of two \(p\)-forms \(\alpha\) and \(\beta\) as:

\[
(\alpha, \beta) = \int_M \alpha \wedge \ast \beta
\]

The bilinear integral allows the definition of the norm of a differential form as: \((\alpha, \alpha)\). Derivative and coderivative operators have some similar properties eg. derivative verifies that: \(dd = 0\) and coderivative verifies: \(\delta \delta = 0\). It is verified that \(\alpha \wedge \ast \beta = \beta \wedge \ast \alpha\); from this the bilinear integral in Equation (8) has some properties common for Riemann and pseudo Riemann manifolds as: \((A, B) = (B, A)\). Also, other properties are: \((dC, A) = (C, \delta A)\), where \(A\) and \(B\) are \(p\)-form and \(C\) is a \((p-1)\)-form, also the linear property: \((A, B + C) = (A, B) + (A, C)\) [15](Sec. 4.2).
2.1 Strong harmonic forms

Let $\triangle$ be a second order differential operator –called Laplace-Beltrami– that map $\triangle : \mathcal{A}^p \to \mathcal{A}^p$. It is defined as: $\triangle = \delta d + d\delta$. A $p$-form $\phi$ is named harmonic if it verifies: $\triangle \phi = 0$. In Riemann manifolds this implies that $\phi$ is closed, $d\phi = 0$, and also its dual is closed, that is $d^* \phi = 0$. That last condition can be called co-closed or dual closed (its dual is closed) that implies that: $\delta \phi = 0$. If both conditions are verified, it is an harmonic form, also called by Bott and Tu[3, pag. 92] global closed.

However, in pseudo-Riemann manifolds there are some differences concerning this concept. In this paper, a differential form $\phi$ verifying $d\phi = 0$ and also $\delta \phi = 0$ is named strong harmonic to indicate that it is included in the harmonic class but the reciprocal is not true in pseudo-Riemann manifolds as it is explained below.

With positive definite metric, the equation $\triangle \phi = 0$ defines an elliptic second order differential equation. However, with indefinite metric, it defines an hyperbolic second order differential equation, whose general solution is a wave. In Riemann manifolds is verified that[12, Lemma 2.1.5]:

$$(d\phi = 0, \delta \phi = 0) \leftrightarrow \triangle \phi = 0 \quad (9)$$

it involves and if-only-if bi-directional implication. However, in pseudo-Riemann manifolds is asserted that: $(d\phi = 0, \delta \phi = 0) \to \triangle \phi = 0$ involving an unidirectional implication. Instead of a theoretical proof, a real world evidence is more illustrative; this is the existence of Electromagnetic waves in the vacuum.

These can be represented by an harmonic 1-form, $\triangle A = 0$, also a gauge condition can be imposed $\delta A = 0$, but the Electromagnetic Field, $F = dA$, is non null. Therefore, Electromagnetic waves in the vacuum are physical examples of harmonic but non strong harmonic forms. The reason is rather simple, because the space-time –the theoretical framework of physical phenomena– is modelled as a pseudo Riemann manifold with Minkowski metric, instead of a pure Riemannian one. As conclusion, not all results and methodologies coming from Riemannian manifold literature can be directly transferred to the pseudo Riemannian case, mainly if harmonic forms are involved. Eg., the Moritas’s [15, pag.157] Proposition 4.13(iii) the necessary and sufficient conditions on Riemannian must be rewrite as sufficient for pseudo Riemann ones.

In this paper the existence of solution for the equation: $\triangle \theta = \varphi$ is supposed. This has been deeply analysed in hyperbolic differential equations[7]. The solution can be represented by: $\theta = G \circ \varphi$, where the $G = \Delta^{-1}$ operator can be considered a Green map $G : \mathcal{A}^p \to \mathcal{A}^p$.

2.2 Cohomology Classes

Let $Z^p(M)$ be the set of all closed $p$-forms in $M$ and $B^p(M)$ the set of all the exact $p$-forms. The quotient set $H^p(M) = Z^p(M) \setminus B^p(M)$ contains all the closed but non-exact $p$-forms in $M$. The dimensionality of the cohomology $H^p(M)$, which is finite dimensional, is the $p$-dimensional Betti number $\beta_p = \dim H^p(M)$.

A $p$-rank cycle $z$ on $M$ is a closed $p$-dimensional sub-manifold on $M$; that is without boundary, $\partial z = \emptyset$. If a simply connected manifolds $M$ has not non null cycles, $\dim H^p(M) = 0$, then $Z^p(M)$ and $B^p(M)$ are equivalent and all the closed forms are exact ones, this is the Poincaré Lemma. But in no simple connected manifolds not all closed forms $(d\omega = 0)$ are exacts $(\omega \neq d\phi)$. This difference is the theoretical arena of the cohomology classes.

2.2.1 The de Rham Theorem

According to the de Rham Theorem[8](pp 68), the necessary and sufficient condition for a form $\phi \in \mathcal{A}^p(M)$ be exact is that $\int_z \phi = 0$ for any $p$-dimensional cycle $z$ in $M$. Let $[\phi]$ be a representa-
Figure 1: Cycles are closed sub-manifolds $z \subset M$. For the shown surface, Torus genus-2, there are four linear independent 1-cycles associated to the four cohomology classes of $H^1(M)$ grouped in two Poincaré dual pairs.

For any class of the cohomology class to which $\phi$ belongs, it is verified that if $\phi = [\phi] + d\varphi$ then both $\phi$ and $[\phi]$ are cohomologous forms. Any cohomologous form can be chosen as the representative of the class. The integral in a cycle $z$ of any cohomologous form is a characteristic of the class:

$$\int_z \phi = \int_z [\phi] + \int_z d\varphi = \int_z [\phi]$$

(10)

because by applying the Stokes theorem in the closed submanifold $z$, it is verified that: $\int_z d\varphi = \int_{\partial z} \varphi = 0$.

The Hodge Theorem, as shown by Jost [12, Theorem 2.2.1] and Morita [15, pag.159], proof that every cohomology class $H^p(M)$ contains precisely one harmonic form. Hence, as asserted by Morita [15, pp 116], let $z_1^{(p)}, \ldots, z_r^{(p)}; (r = \beta_p)$ be a set of linearly independent $p$-dimensional cycles. Each cycle can be characterized by one homology class; therefore one strong harmonic form $\gamma_a^{(p)} \in \mathcal{A}^p(M, R)$ is chose as the representative of the $a^{th}$ class in $H^p(M)$.

However the proof of this property is highly dependent of the use of definite positive norm, such as if $(\alpha, \alpha) = 0$ implies that $\alpha = 0$. This is not true for pseudo Riemann manifold. To avoid this problem, instead of a set including the unique harmonic form, we will use a set including one representative set. Later, we will proof that the uniqueness is not formally true but practical true because the difference between all the harmonic forms cohomologous ins a class are almost different in one constant form.

**Definition 2.1.** Let $\gamma^{(p)}(M) = \{\gamma_1^{(p)}, \ldots, \gamma_r^{(p)}\}$ be a set of of linear independent $p$-forms representative of the classes in $H^p(M)$. To each cycle corresponds one representative form and vice versa:

$$z_a^{(p)} \leftrightarrow \gamma_a^{(p)} \quad a = 1, \ldots, \beta_p$$

(11)

According with the de Rham theorem [6, 15], there is a set of real numbers: $w_1, \ldots, w_r$ for which there is an unique closed $p$-form $\phi$ which verifies: $\int_{\gamma_a^{(p)}} \phi = w_a$. This form is undefined only in an arbitrary exact form. The following closed $p$-form $\phi$ must verify the assertion of the de Rham theorem.

$$\phi = \sum_{a=1}^{\beta_p} w_a \gamma_a^{(p)}$$

(12)

The integral of a cohomologous form in a cycle must be non null, i.e. $\int_z \gamma \neq 0$, if and only if
\( \gamma \) is the cohomologous correspondent of \( z \). Linear independence of representative forms means that:

\[
\sum_{a=1}^{b_p} w_a \gamma_a^{(p)} = 0 \quad \rightarrow \quad w_a = 0 \quad (13)
\]

In previous Equations the question of normalization of \( \gamma^p \) forms have not been addressed. A set of quadratically integrable forms on \( M \) are used, such as the following two integrals can be suitably renormalized:

\[
\int_{z^M} \gamma_a^{(p)} = c_a \quad \int_M \gamma_a^{(p)} \wedge *\gamma_a^{(p)} = n_a \quad (14)
\]

Neither \( c_a \) nor \( n_a \) have definite sign. The renormalization in the cohomology integral as:

\[
\gamma^{(p)} \rightarrow c_a \gamma_a^{(p)} \quad \text{preserves the sign of the norm, but the renormalization in the norm as:} \quad \gamma_a^{(p)} \rightarrow \sqrt{n_a} \gamma_a^{(p)} \quad \text{is avoided. The first is the choose normalization; the summary of normalizations for the set of representative forms is:}
\]

\[
\int_{z^M} \gamma_a^{(p)} = \delta_{ab} \quad \int_M \gamma_a^{(p)} \wedge *\gamma_a^{(p)} = \lambda_a^{(p)} \quad d\gamma^{(p)} = 0 \quad \delta\gamma^{(p)} = 0 \quad (15)
\]

another significant integral is the extension of the norm: \( (\gamma_a^{(p)}, \gamma_b^{(p)}) = \int_M \gamma_a^{(p)} \wedge *\gamma_b^{(p)} = \lambda_{ab}^{(p)} \), such as the norm is: \( \lambda_a^{(p)} = \lambda_{aa}^{(p)} \). The set \( \gamma(M) \) represents the global properties of the manifold topology, being an intrinsic property that characterize it. It is as basic as the metric \( g \) that represents the differential geometry or local properties of the manifold. Both the metric and the characteristic forms \( (M; g, \gamma) \) must be used to explicit the role of both, the local/differential and global/topological properties of the manifold.

### 2.3 The Poincaré duality

Poincaré duality is, as Hodge duality, a kind of \textit{complementary} relationship. In both cases, for a defined object \( A \) of \( p \)-rank, the goal is to obtain an object \( B \) of \((n-p)\)-rank, such as it complements the first in some way. In differential forms, until (or to fit) a no null \( n \)-form, that is \( A \wedge B = C \omega \), with \( C \neq 0 \); this means that \( \int_M A \wedge B = C \). However, both dualities types differ in the involved object.

The Hodge duality does a map \( \mathcal{A}^p \rightarrow \mathcal{A}^{n-p} \) for every \( \phi \in \mathcal{A}^p \) while Poincaré duality\([10, 3]\) does a map for the cohomology classes as: \( H^p \rightarrow H^{n-p} \). Simplified, Poincaré duality defines a map for each pair of cohomologous forms in \( H^p(M) \) and \( H^{n-p}(M) \) into \( \mathbb{R} \), that is:

\[
H^p(M) \times H^{n-p}(M) \rightarrow \mathbb{R} \quad (16)
\]

From this Equation, the Poincaré Duality Theorem, eg. Morita\([15] \text{ sec. 4.4}\), establishes that for any non null cohomology class \([\alpha] \in H^p(M) \) exists a no null cohomology class \([\beta] \in H^{n-p}(M) \) such as \( \int_M \alpha \wedge \beta \neq 0 \). This duality is an isomorphism, \( H^p \cong H^{n-p} \), therefore exist one and only one class \([\beta] \in H^{n-p}(M) \) for each \([\alpha] \in H^p(M) \) and the reciprocal. Due to the isomorphism between the sets \( \gamma^{(p)} \) and \( \gamma^{(n-p)} \), their dimensionalities or Betti numbers have the same value: \( \beta_p = \beta_{n-p} \).

This implies that the following matrix: \( E^{(p)} = (\varepsilon^{(p)}_{ab}) \in M(\beta_p, \mathbb{R}) \) defined as follows has one and only one non null element for each row and column, therefore it is no singular.

\[
\varepsilon^{(p)}_{ab} = \int_M \gamma_a^{(p)} \wedge *\gamma_b^{(n-p)} \quad (17)
\]
An alternative presentation of Poincaré duality follows Bott and Tu [3]; it focuses in the relationship between $z$ cycle and representative form, such as if $z$ is a closed oriented sub-manifold of dimension $p$ (a $p$-cycle) in an oriented manifold $M$ of dimension $n$, the Poincaré dual of $z$ is the cohomology class of closed $(n-p)$-form $\eta_z \in H^{n-p}(M)$ characterized by the following property:

$$\int_z \phi = \int_M \phi \wedge \eta_z$$  \hspace{1cm} (18)

The Poincaré duality defined in this way is focused on a relationship between sub-manifold $z$ of dimension $p$, and the cohomology classes of $H^{n-p}(M)$. Using the previous definitions and theoretical materials, the Bott and Tu can be rewritten based on $\gamma^{(p)}$ set. Based on Equation (18), it is verified that:

$$\int_{z_a^{(p)}} \gamma_b^{(p)} = \int_M \gamma_b^{(p)} \wedge \eta_a^{(n-p)} = \delta_{ab}$$  \hspace{1cm} (19)

where $\eta_a^{(n-p)}$ is the Poincaré dual of $z_a^{(p)}$. However, $\eta_a^{(n-p)} \in H^{n-p}(M)$ is not necessarily one of the classes $\gamma_a^{(n-p)}$ previously presented. A linear combination can be used:

$$\eta_a^{(n-p)} = \sum_{c=1}^{\beta_{n-p}} \eta_{ac} \gamma_c^{(n-p)} \quad \eta_{ac} \in \mathbb{R}$$  \hspace{1cm} (20)

This implies that:

$$\sum_{c=1}^{\beta_{n-p}} \eta_{ac} \int_M \gamma_b^{(p)} \wedge \gamma_c^{(n-p)} = \delta_{ab}$$  \hspace{1cm} (21)

It can be rewrite based on the matrix $\varepsilon_{ab}^{(p)}$ as: $\sum_{c=1}^{\beta_{n-p}} \eta_{ac} \varepsilon_{bc}^{(p)} = \delta_{ab}$. If one and only one of the row and column values of $\varepsilon_{ab}^{(p)}$ is non null, then one and only one of the values of $\eta_{ab}$ can be chosen as non null. If $\varepsilon_{ab}^{(p)} \neq 0$, then $\eta_{ab} = 1/\varepsilon_{ab}^{(p)}$. That means that the difference between the dual of $z_a^{(p)}$, in the Bott-Tu definition way, and the dual of $\gamma_a^{(p)}$ is a multiplicative constant.

To simplify the expressions dealing Poincaré duality, we will use the symbol $P$ in two ways, as operator: $P : H^k(M) \rightarrow H^{n-k}(M)$; also as index function/modifier, such as we will use $b = P(a)$ to express that the index $b$ corresponds to the Poincaré dual of index $a$. This means that:

$$P\gamma^{(p)}_a = \gamma^{(n-p)}_{P(a)}$$  \hspace{1cm} (22)

If $a$ is a index in $\gamma^{(p)}$ set, then $P(a)$ is a index in set $\gamma^{(n-p)}$ and vice versa. Due to the isomorphism, must be that $P(P(a)) = a$. Also it is verified that:

$$PP\gamma^{(p)}_a = \gamma^{(n-p)}_{P(P(a))} = \gamma^{(p)}_{P(a)}$$  \hspace{1cm} (23)

To add clarity we will use the expression: $\varepsilon_{ab}^{(p)}$ and the comma spaced: $\varepsilon_{a,P(a)}^{(p)}$ when $P()$ is used in indexes. Thus, it must be verified that: $\varepsilon_{a,P(a)}^{(p)} \neq 0$. By using this index definition, the non null value in Equation (17) is that involves to $a$ and $P(a)$ indexes:

$$\varepsilon_{a,P(a)}^{(p)} = \int_M \gamma_a^{(p)} \wedge \gamma_{P(a)}^{(n-p)}$$  \hspace{1cm} (24)

The Bott and Tu [3] definition in Equation (18) becomes:

$$\int_{z_a^{(p)}} \phi = \eta \int_M \phi \wedge \gamma_{P(a)}^{(n-p)} \quad \eta = \left(\varepsilon_{a,P(a)}^{(p)}\right)^{-1}$$  \hspace{1cm} (25)
Remark that in $E^{(p)} = \{ e_{ab} \}^{(p)}$ the $a$ index corresponds to $\gamma_{(p)}$ set while the $b$ index corresponds to $\gamma_{(n-p)}$ set, if also one and only one element in each row and column is non null, then the matrix can be transformed to a diagonal one by using a reordering in the indexes of both classes; in this case, $P(a) = a$. But this reordering is possible if $p \neq (n-p)$ because in these cases both matrices, $E^{(p)}$ and $E^{(n-p)}$, are different and the indexes are in one isomorphism between these two different sets. However, if $p = (n-p)$, that is, $n = 2p$ is even, there is not isomorphism between two sets, rather there is an endomorphism because both matrices and sets are the same. In this case, a diagonal element reveals a self-duality, a more exotic case, which can no be achieved by a simple index reordering.

3 A meta-theory of linear decomposition

The Hodge decomposition is presented in next Section as an application of some general concepts that rather can be called a meta-theory. The elements of the meta-theory for linear decomposition are presented mainly with a high degree of informality.

**Definition 3.1.** Let $Op = \{ A, B, C, \ldots \}$ a set of operators and let $Ob = \{ \alpha, \beta, \gamma, \ldots \}$ be a set of homogeneous objects to which the operators can be applied to: $Op \circ Ob$. It is proposed the verify the following conditions:

1. The set of objects is homogeneous, all its members belong to a common category. Example of homogeneous objects can be 3-vector or $p$-forms.
2. The objects admit a pair of basis binary operations: addition between objects and multiplication by a set of factor: $\{ a, b, c, \ldots \} \in \mathbb{R}$, such as allows linear combinations as:
   \[ \phi = a\alpha + b\beta + c\gamma + \cdots \] (26)
3. The set of operators can be no homogeneous, such as the way how each operator is applied to the object, $A \circ \alpha$, is operator dependent. They verify that: $A \circ (a\alpha) = a(A \circ \alpha)$.
4. For each operator applied to objects, the result is non null only for one of the objects, eg.:
   \[ A \circ \alpha = r_\alpha \neq 0 \quad A \circ \beta = A \circ \gamma = \cdots = 0 \] (27)
   and also there is some possible type of reversible result $(r_\alpha)^{-1}$ that is operator dependent.
Proposition 3.1. The set of objects $\text{Ob} = \{\alpha, \beta, \gamma, \ldots\}$ is linearly independent in relation to the set of operators: $\text{Op} = \{A, B, C, \ldots\}$.

Proof. It must be proved that in the following expression:

$$a\alpha + b\beta + c\gamma + \cdots = 0$$  \hspace{1cm} (28)

the only solution is: $a = b = c = \cdots = 0$. By applying successively each one of the operators, it is achieved that:

$$A \circ (a\alpha + b\beta + c\gamma + \cdots) = aA \circ \alpha = 0 \quad \rightarrow a = 0 \quad \text{(29)}$$

$$B \circ (a\alpha + b\beta + c\gamma + \cdots) = bB \circ \beta = 0 \quad \rightarrow b = 0 \quad \text{(30)}$$

and so on.

Proposition 3.2. The set of objects $\text{Ob} = \{\alpha, \beta, \gamma, \ldots\}$ is complete in relation to the set of operators: $\text{Op} = \{A, B, C, \ldots\}$, such as any object $\phi$ is completely determined except in a residual term $\phi_0$ verifying that:

$$\phi = a\alpha + b\beta + c\gamma + \cdots + \phi_0 \quad A\phi_0 = B\phi_0 = C\phi_0 = \cdots = 0$$  \hspace{1cm} (31)

Proof. By applying successively each one of the operator, eg.:

$$A \circ \phi = aA \circ \alpha \quad \rightarrow a = (r_a)^{-1} \circ [A \circ \phi] \quad \text{(32)}$$

the solution becomes:

$$\phi = [(r_a)^{-1} \circ (A \circ \phi)]\alpha + [(r_b)^{-1} \circ (B \circ \phi)]\beta + \cdots + \phi_0$$  \hspace{1cm} (33)

The lack of completeness of the operator set can be qualitatively evaluate. It depends on the significance, or relevance, of the residual element $\phi_0$.

Definition 3.2. The set: $\text{Sour}(\phi) = \{A \circ \phi, B \circ \phi, \ldots\}$ are the sources of $\phi$ because from they is possible to determine $\phi$, except in the residual term $\phi_0$.

The report presents an application of this abstract concepts to Hodge decomposition and the Poincaré duality involved in it. The objects used in the decomposition are $p$-forms and the operators and sources of $\phi$ are:

$$\text{Op} = \left\{d, \delta, \int_{z_1}^{(p)}, \int_{z_2}^{(p)}, \ldots, \int_{z_\beta}^{(p)} \right\}$$  \hspace{1cm} (34)

$$\text{Sour}(\phi) = \left\{d\phi, \delta\phi, \int_{z_1}^{(p)} \phi, \int_{z_2}^{(p)} \phi, \ldots, \int_{z_\beta}^{(p)} \phi \right\}$$  \hspace{1cm} (35)

The lack of completeness is the residue, a $p$-form $\phi_0$ that verify:

$$d\phi_0 = \delta\phi_0 = 0 \quad \int_{z_\beta}^{(p)} \phi_0 = 0$$  \hspace{1cm} (36)

It is a no cohomologous and constant strong harmonic form.
The Poincaré Duality in the Hodge Decomposition

**Definition 4.1.** Let $Z^p(M)$ be the class of closed form $p$-forms in $M$, $\star Z^{n-p}(M)$ the class of dual closed $p$-forms and $W^p(M)$ the class of strong harmonic forms, being: $W^p(M) = Z^p(M) \cap \star Z^{n-p}(M)$

**Proposition 4.1.** The three classes $Z^p \setminus W^p$, $\star Z^{n-p} \setminus W^p$ and $W^p$ are linearly independent.

**Proof.** If $\phi \in Z^p \setminus W^p$ ($d\phi = 0, \delta\phi \neq 0$), $\psi \in \star Z^{n-p} \setminus W^p$ ($d\psi = 0, \delta\psi = 0$) and $\varphi \in W^p$ ($d\varphi = 0, \delta\varphi = 0$) are non null $p$-forms. It must be proved that in the equation $A\phi + B\psi + C\varphi = 0$, with constant $\{A, B, C\} \subseteq \mathbb{R}$, the only solution must be: $A = B = C = 0$. By using the coderivative: $\delta(A\phi + B\psi + C\varphi) = Bd\psi = 0$, therefore it must be $B = 0$. By using the coderivative: $\delta(A\phi + B\psi + C\varphi) = A\delta\varphi = 0$, therefore it must be $A = 0$ which also implies that $C = 0$. \hfill \square

**Definition 4.2.** Let $H^p \subseteq W^p$ be a $p$-form class based on linear combinations of the $\gamma^p$ set of the $p$-forms representative of the cohomology classes of $H^p(M)$ and $w_a \in \mathbb{R}$:

$$\sum_{a=1}^{\delta_p} w_a \gamma_a^{(p)} \in H^p \subseteq W^p \quad (37)$$

**Definition 4.3.** Let $dA^{p-1} \equiv B^p(M)$ be the class of exact $p$-forms, but not dual exact ones; if $\sigma \in dA^{p-1} \subset A^p$, it is verified that: $d\sigma = 0$ and $\delta\sigma \neq 0$. It can be expressed as: $\sigma = d\phi$ where $\phi \in A^{p-1}$.

**Definition 4.4.** Let $\delta A^{p+1}$ be the class of dual exact $p$-forms, but not exact ones, that is $\sigma = \delta\phi$ where $\sigma \in \delta A^{p+1} \subset A^p$ and $\phi \in A^{p+1}$. It is verified that: $d\sigma \neq 0$ and $\delta\sigma = 0$. They verify: $dA^{p-1} \subseteq Z^p/W^p$, $\delta A^{p+1} \subseteq \star Z^{n-p}/W^p$ and $H^p \subseteq W^p$, and they are linearly independent according the Proposition 4.1.

**Proposition 4.2.** The following integrals are null.

$$\int_{z^{(p)}} d\alpha = 0 \quad \int_{z^{(p)}} \delta\beta = 0 \quad (38)$$

**Proof.** The first is null because the cycles $z$ are closed sub-manifolds of $M$, therefore $\partial z = \emptyset$; it must be: $\int_{z^{(p)}} d\alpha = \int_{\partial z^{(p)}} \alpha = 0$. However, both first and second integrals are null based on the Poincaré duality in Equation [25]:

$$\int_{z^{(p)}} d\alpha = \eta \int_M d\alpha \wedge \gamma^{(n-p)}_{\gamma_{P(a)}} = \eta \int_M \alpha \wedge \delta\gamma^{(n-p)}_{P(a)} = 0 \quad (39)$$

$$\int_{z^{(p)}} \delta\beta = \eta \int_M \delta\beta \wedge \gamma^{(n-p)}_{\gamma_{P(a)}} = \eta \int_M \beta \wedge d\gamma^{(n-p)}_{P(a)} = 0 \quad (40)$$

\hfill \square

**Proposition 4.3.** The objects $p$-forms $\{d\alpha, \delta\beta, w_a \gamma_a^{(p)}\}$ are linearly independent on the operators $\{d, \delta, \int_{z^{(p)}}\}$. The only solution for:

$$Ad\alpha + B\delta\beta + \sum C_a \gamma_a^{(p)} = 0 \quad (41)$$

is: $A = B = C_a = 0$
Table 1: Summary of operators and reverses. The additional gauge conditions: $\delta \alpha = 0$ and $d \beta = 0$ allows the use of a pair of mutually inverse operators: the Laplacian and Green.

| Operator | Object | $d\alpha$ | $\delta \beta$ | $u_a \gamma_a^{(p)}$ | Reverse |
|----------|--------|-----------|----------------|----------------------|---------|
| $d$      |        | 0         | $d\delta \beta = \Delta \beta$ | 0 | $G \circ \Delta \beta$ |
| $\delta$ | $\delta d\alpha = \Delta \alpha$ | 0 | 0 | $G \circ \Delta \alpha$ |
| $\int_{z^{(p)}}$ | 0 | 0 | | $u_a$ | $u_a \gamma_a^{(p)}$ |

The objects $d\alpha \in d\mathcal{A}^{p-1}$, $\delta \beta \in \delta \mathcal{A}^{p+1}$ and $\gamma_a^{(p)} \in W^p$ are in linearly independent classes; this requires that expressions as Equation (41) are no possible with non null coefficients $A$, $B$ and $C_a$. The same result is also valid for:

$$A_\alpha + B_\beta + \sum C_a \star \gamma_a^{(n-p)} = 0 \quad (42)$$

because: $\star \gamma_a^{(n-p)} \in W^p$.

**Theorem 4.1** (Hodge decomposition). **If** $\phi$ **is a p-form, then there is solution for each one of the** $\beta_{p+2}$ **linearly independent terms in the following decomposition:**

$$\phi = d\alpha + \delta \beta + \sum \left(C_a \star \gamma_a^{(n-p)}\right) + \phi_0 = d\alpha + \delta \beta + \sum u_a \gamma_a^{(p)} + \phi_0 \quad (43)$$

**where** $\alpha$ **is a** $(p-1)$-form **verifying** $\delta \alpha = 0$ **and** $\beta$ **is a** $(p+1)$-form **verifying** $d \beta = 0$, $\phi_0$ **is the residue p-form.**

**Proof.** Each one of the terms can be determined by applying suitable operator. By using derivative:

$$d\phi = d\delta \beta = \Delta \beta \quad (44)$$

By applying coderivative:

$$\delta \phi = \delta d\alpha = \Delta \alpha \quad (45)$$

By applying integration in cycle $z_a^{(p)}$:

$$\int_{z_a^{(p)}} \phi = u_a \quad (46)$$

There are solutions for $\alpha = G \circ \delta \phi$ and $\beta = G \circ d\phi$. The forms $d\phi$ and $\delta \phi$ play the role of sources in the Laplace equations. The additional condition $\delta \alpha = 0$ and $d \beta = 0$ are gauge constraints which allow the solutions based on the Laplacian operator and its inverse the Green one. Table 1 summarizes the operators properties. □

**Remark 4.1.** In the Electromagnetic Field Theory, the solution of the Maxwell Equations, that are first order, requires also one residual solution. It is one Electric and Magnetic constant field in all the space-time. That solution has infinite energy and its is discharged based on physical criteria, not in mathematical ones. Similarly, the continuous sources of $\phi$, that is $d\phi$ and $\delta \phi$ are also first order, therefore the residual element is necessary at mathematical level. However, it can be discharged if it is no relevant or compatible in some application.
Jost in Theorem 2.2.1 proved that there exists one and only one harmonic form in each cohomology class. The question of the existence of this is not fully addressed in this report, but the existence condition is presented. The uniqueness is presented without the use of the positive definite norm. Previously, to avoid the question, we have used the concept of representative set, avoiding the concept of set of unique harmonic forms.

**Proposition 4.4.** If a p-cycle, e.g., \( z_a^{(p)} \), of the \( a^{th} \) class of the Cohomology \( H^p(M) \) admits Poincaré dual, then it exists at least one strong harmonic form cohomologous in it.

**Proof.** It is proof that the opposite is not possible: if all strong harmonic \( p \)-form \( \phi \) are not cohomologous on \( z_a^{(p)} \), therefore it must be verified based on Poincaré duality:

\[
\int_{z_a^{(p)}} \phi = \eta \int_M \phi \wedge \gamma^{(n-p)}_{P(a)} = 0
\]

But due to Hodge duality, it exists one unique strong harmonic \( p \)-form that verifies:

\[
\int_M \phi \wedge \gamma^{(n-p)}_{P(a)} \neq 0
\]

It is the \( p \)-form: \( \star \gamma^{(n-p)}_{P(a)} \). Thus, it exists at least one strong harmonic form that is cohomologous on \( z_a^{(p)} \) verifying:

\[
\int_{z_a^{(p)}} \star \gamma^{(n-p)}_{P(a)} = \eta \int_M \star \gamma^{(n-p)}_{P(a)} \wedge \gamma^{(n-p)}_{P(a)} \neq 0
\]

Later, it is proof that \( \star \gamma^{(n-p)} \) must include \( \gamma^{(p)} \).

**Proposition 4.5.** If \( \phi_1 \) and \( \phi_2 \) are two cohomologous strong harmonic forms in \( a^{th} \) class of the Cohomology \( H^p(M) \), then they differ at most in the residue \( \phi_0 \).

**Proof.** Let \( \phi_1 \) and \( \phi_2 \) be strong harmonic \( p \)-forms cohomologous in the class that is cohomologous to \( z_a^{(p)} \). They must verify: 
\( d\phi_1 = d\phi_2 = \delta \phi_1 = \delta \phi_2 = 0 \), also:

\[
\int_{z_a^{(p)}} \phi_1 = \int_{z_a^{(p)}} \phi_2 \neq 0
\]

\[
\int_{z_a^{(p)}} \phi_1 = \int_{z_a^{(p)}} \phi_2 = 0 \quad z_b^{(p)} \neq z_a^{(p)}
\]

The difference \( \phi_1 - \phi_2 \) can be expressed based on the Hodge decomposition as:

\[
\phi_1 - \phi_2 = d\alpha + \delta \beta + \sum_{b=1}^{b=p} w_b \gamma_b^{(p)} + \phi_0
\]

But \( \phi_1 - \phi_2 \in W^p(M) \) and due to linear independence must be \( d\alpha = 0 \) and \( \delta \beta = 0 \). Also, based on Equation [50], it must be: \( w_a = \int_{z_a^{(p)}} (\phi_1 - \phi_2) = 0 \), and based on Equation [51]: 
\( w_b = \int_{z_b^{(p)}} (\phi_1 - \phi_2) = 0 \), for \( z_b^{(p)} \neq z_a^{(p)} \). It is concluded that: \( \phi_1 - \phi_2 = \phi_0 \). Although we have not proof that they are the same, that is: \( \phi_1 - \phi_2 = 0 \), its uniqueness is on the relevance of the residue, but it is a constant no cohomologous.
4.1 Properties of the auxiliary matrices

The properties of some matrices, as \( E(p) \) and other that will be introduced, are presented.

**Proposition 4.6.** The matrix \( E(p) \) verifies that:

\[
E(p) = (-1)^{(n-p)p}E^{(n-p)}T
\]

**Proof.** It follows from the anti-commutative property of the wedge product, from it is verified that:

\[
\int_M \gamma_a^{(p)} \wedge \gamma_b^{(n-p)} = (-1)^{(n-p)p} \int_M \gamma_b^{(n-p)} \wedge \gamma_a^{(p)}
\]

which implies:

\[
\varepsilon_{ab}^{(p)} = (-1)^{(n-p)p} \varepsilon_{ba}^{(n-p)}.
\]

**Proposition 4.7.** The Hodge dual, \(*\phi\), of the \( p \)-form \( \phi \) in Theorem 4.1 is a \((n-p)\)-form, so it admits the following decomposition:

\[
*\phi = d\alpha' + \delta\beta' + \sum_{a=1}^{\beta_{n-p}} v_a \gamma_a^{(n-p)} v_a = \int_{\int^{(n-p)}} *\phi
\]

where \( \alpha' \) is a \((n-p-1)\)-form verifying \( \delta\alpha' = 0 \) and \( \beta \) is a \((n-p+1)\)-form verifying \( d\beta' = 0 \).

**Proof.** Each one of the terms can be determined by applying the same operator than in the previous Proposition. By using derivative:

\[
d*\phi = d\delta\beta' = \Delta\beta'
\]

By applying coderivative:

\[
\delta*\phi = \delta d\alpha' = \Delta\alpha'
\]

By applying integration in cycle \( z_b^{(n-p)} \):

\[
\int_{z_b^{(n-p)}} *\phi = v_b
\]

The forms \( d*\phi \) and \( \delta*\phi \) play the role of sources in the corresponding Laplace equations. \( \square \)

**Remark 4.2.** The \(*\phi\) form decomposed in Equation (54) can be alternatively obtained from Equation (43) by using Hodge duality operator. However, this requires some linear relationship between the sets \(*\gamma^{(p)}\) and \(\gamma^{(n-p)}\). This means that must be a relationship between \(*H^p\) and \(H^{n-p}\).

**Proposition 4.8.** The Hodge dual of the representative form, \(*\gamma_a^{(p)}\), admit the following expression:

\[
*\gamma_a^{(p)} = \sum_{b=1}^{\beta_{n-p}} \tau_{ab}^{(n-p)} \gamma_b^{(n-p)}
\]

where the matrix \( T^{(p)} = (\tau_{ab}^{(p)}) \in M(\beta_p, \mathbb{R}) \) is defined as:

\[
\tau_{ab}^{(n-p)} = \int_{z_b^{(n-p)}} *\gamma_a^{(p)}
\]

**Proof.** It follows from the Proposition 4.7 which generates:

\[
*\gamma_a^{(p)} = d\alpha' + \delta\beta' + \sum_{b=1}^{\beta_{n-p}} v_b \gamma_b^{(n-p)} v_b = \int_{z_b^{(n-p)}} *\gamma_a^{(p)}
\]

But the left side term and last right term belongs both to the class \( W^p(M) \). Due to the linear independence, the terms \( d\alpha' \) and \( \delta\beta' \) must be null. \( \square \)
Proposition 4.9. The product of matrices $T^{(p)}$ and $T^{(n-p)}$ verifies: $T^{(n-p)} \times T^{(p)} = (-1)^{D(p)} I$

Proof. From the Proposition 4.8 are obtained the following two expressions:

$$\star \gamma_a^{(p)} = \sum_{b=1}^{\beta_n} \tau_{ab} \gamma_b^{(n-p)}$$  \hspace{1cm} (61)

$$\star \gamma_b^{(n-p)} = \sum_{c=1}^{\beta_p} \tau_{bc} \gamma_c^{(p)}$$  \hspace{1cm} (62)

By including the second Equation into the Hodge dual of the first one:

$$\gamma_a^{(p)} = \sum_{b=1}^{\beta_n} \sum_{c=1}^{\beta_p} (-1)^{D(p)} \tau_{ab} \gamma_b^{(n-p)} \gamma_c^{(p)}$$  \hspace{1cm} (63)

it is conclude that:

$$\sum_{c=1}^{\beta_p} [\delta_{ac} - \sum_{b=1}^{\beta_n} (-1)^{D(p)} \tau_{ab} \tau_{bc}] \gamma_c^{(p)} = 0$$  \hspace{1cm} (64)

However, the representative forms are linearly independent, therefore if $\sum_a u_a \gamma_a^{(p)} = 0$, then must be $u_a = 0$. From this is obtained that must be:

$$\delta_{ac} - \sum_{b=1}^{\beta_n} (-1)^{D(p)} \tau_{ab} \tau_{bc} = 0$$  \hspace{1cm} (65)

Proposition 4.10. The matrix $E^{(p)}$ and $T^{(n-p)}$ verify:

$$E^{(p)} \times (T^{(n-p)})^T = \Lambda^{(p)}$$  \hspace{1cm} (66)

where $\Lambda^{(p)} \in M(\beta_p, \mathbb{R})$ is the matrix containing the elements $\lambda_{ab}^{(p)} = (\gamma_a^{(p)}, \gamma_b^{(p)})$. The elements of $T^{(n-p)}$ are:

$$\tau_{b, P(a)}^{(n-p)} = \frac{\lambda_{ab}^{(p)}}{\epsilon_{a, P(a)}}$$  \hspace{1cm} (67)

Proof. The bilinear integral $(\gamma_a^{(p)}, \gamma_b^{(p)})$ is defined as:

$$(\gamma_a^{(p)}, \gamma_b^{(p)}) = \int_M \gamma_a^{(p)} \wedge \gamma_b^{(p)} = \sum_{c=1}^{\beta_p} \tau_{bc} (\gamma_a^{(n-p)} \wedge \gamma_c^{(n-p)}) = \sum_{c=1}^{\beta_p} \tau_{bc} (\gamma_a^{(n-p)} \beta_{bc}^{(p)})$$  \hspace{1cm} (68)

The elements no null in matrix $E^{(p)}$ are $\epsilon_{a, P(a)}$, therefore it is obtained that: $(\gamma_a^{(p)}, \gamma_b^{(p)}) = \tau_{b, P(a)}^{(n-p)} \epsilon_{a, P(a)}$. \hfill $\Box$

Theorem 4.2. The necessary an sufficient condition for an isomorphism between $\star H^p$ and $H^{n-p}$ is that the matrix $T^{(n-p)}$ have only a no null element in each row and column. A special case, but not unique, of this condition is when the set of representative forms of the cohomology $H^p(M)$ is orthogonal, defined the orthogonality as: $(\gamma_a^{(p)}, \gamma_b^{(p)}) = \lambda_{ab}^{(p)}$. 

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Proof. If there is the proposed isomorphism, then an one-to-one correspondence is required, so the matrix $T^{(n-p)}$ must have the proposed property. The converse, if the matrix has that property, then there is an one-to-one correspondence. If the set is orthogonal defined from the bilinear integral as:

$$\int_M \gamma_a^{(p)} \wedge \gamma_b^{(p)} = \lambda_a^{(p)} \delta_{ab}$$

then it is verified that the only non null elements of $T^{(n-p)}$ are:

$$\tau_a^{(n-p)} = \frac{\lambda_a^{(p)}}{\varepsilon_a^{(p)} \varepsilon_{a, P(a)}}$$

Thus, it is verified that:

$$\star \gamma_a^{(p)} = \sum_{c=1}^\beta \tau_a^{(n-p)} \gamma_b^{(n-p)} = \frac{\lambda_a^{(p)}}{\varepsilon_a^{(p)} \varepsilon_{a, P(a)}}$$

Thus, it is verified that:

$$\star \gamma_a^{(p)} = \sum_{c=1}^\beta \tau_a^{(n-p)} \gamma_b^{(n-p)} = \frac{\lambda_a^{(p)}}{\varepsilon_a^{(p)} \varepsilon_{a, P(a)}}$$

Hence, Hodge and Poincaré duals of $\gamma_a^{(p)}$ are proportional, and therefore the classes of $\star H^p$ and $H^{n-p}$ are univocally correspondents.

Remark 4.3. It is possible to generate a set of orthogonal $p$-forms $\Phi^{(p)}$ from the $\gamma^{(p)}$ set if it were no orthogonal by defining a linear relationship as: $\Phi_a^{(p)} = \sum_{b=1}^\beta W_{ab} \gamma_b^{(p)}$. To obtain the matrix $W$ the eigenvalues and eigenvectors of matrix $\Lambda^{(p)}$ are required. The problem is the relationship of set $\Phi^{(p)}$ to the cohomology classes, mainly if only one harmonic forms is included in each class, and this is already in $\gamma^{(p)}$ set.

Proposition 4.11. If $\phi$ is a $p$-form, it is verified that:

$$\int_{z_b^p} \phi = \sum_{b=1}^\beta (-1)^{D(b)} \beta_{ba} \int_{z_b^{n-p}} \star \phi$$

$$\int_{z_b^{n-p}} \star \phi = \sum_{b=1}^\beta \beta_{ba} \int_{z_b^p} \phi$$

Proof. From the Theorem 4.1 and 4.7 is obtained that:

$$\phi = d\alpha + \delta \beta$$

$$\star \phi = d\alpha' + \delta \beta'$$

By using Hodge dual to the last Equation and based on the result of Proposition 4.8 it is
obtained in successive steps:

\begin{align}
\phi &= (-1)^{D(p)} \ast d\alpha' + (-1)^{D(p)} \ast \delta \beta' + \sum_{c=1}^{\beta_{n-p}} \left( \int_{z_{(n-p)}} \ast \phi \right) (-1)^{D(p)} \ast \gamma_{c}^{(n-p)} \\
\phi &= \delta \alpha'' + d\beta'' + \sum_{c=1}^{\beta_{n-p}} \left( \int_{z_{(n-p)}} \ast \phi \right) (-1)^{D(p)} \sum_{a=1}^{\beta} \epsilon_{ca}^{(p)} \gamma_{a}^{(p)} \\
\phi &= \delta \alpha'' + d\beta'' + \sum_{a=1}^{\beta} \left( (-1)^{D(p)} \epsilon_{ca}^{(p)} \left( \int_{z_{(n-p)}} \ast \phi \right) \gamma_{a}^{(p)} \right)
\end{align}

where \((-1)^{D(p)} \ast d\alpha' = \delta \alpha'' = \ast d \ast \alpha''\), that is: \((-1)^{D(p)} \ast d\alpha' = \ast \alpha''\). Also: \((-1)^{D(p)} \ast \delta \beta' = d \ast \beta' = d\beta''\), that is: \(\ast \beta' = \beta''\). By comparing Equations (74) and (78) it is obtained the first of proposed expressions. The second expression is achieved based on the result of Proposition 4.9 applied to the first expression.

\section{Canonical Decomposition of the Norm}

The Hodge decomposition theorem allows to express any differential form by decomposed it in several canonical terms with specific formal properties. Similarly, the norm can be also decomposed in some canonical terms.

\begin{proposition}
Let \(\phi\) be a \(p\)-forms with Hodge Decomposition as follows, also for its Hodge dual, \(\ast \phi\):

\begin{align}
\phi &= d\alpha + \delta \beta + \sum_{a=1}^{\beta} u_{a} \gamma_{a}^{(p)} + \phi_{0} \\
\ast \phi &= d\alpha' + \delta \beta' + \sum_{a=1}^{\beta} v_{a} \gamma_{a}^{(n-p)} + \ast \phi_{0}
\end{align}

\end{proposition}

Its norm \((\phi, \phi)\) can be expressed as:

\begin{align}
(\phi, \phi) = (d\alpha, d\alpha) + (\delta \beta, \delta \beta) + (\phi_{0}, \phi_{0}) + (\phi_{0}, \phi_{0})
\end{align}

\begin{proof}
If \(A\) and \(B\) are \(p\)-forms, they verifies: \((A,B) = (B,A)\) and also verifies: \((dC,B) = (C,\delta B)\). From these properties it is follows that all the cross terms: \((d\xi, \delta \pi), (d\xi, \gamma), (\delta \xi, \gamma), (da, \phi_{0})\) and \((\delta \beta, \phi_{0})\) are all null. The terms \((\phi_{0}, \gamma^{(p)})\) are also nulls because \(\ast \gamma^{(p)}\) they can be expressed based on the \(\gamma^{(n-p)}\), but \(\phi_{0}\) is no cohomologous on any class:

\begin{align}
(\phi_{0}, \gamma_{a}^{(p)}) = \sum_{b=1}^{\beta_{n-p}} \gamma_{ab}^{(n-p)} \int_{M} \phi_{0} \wedge \gamma_{b}^{(n-p)} = \sum_{b=1}^{\beta_{n-p}} \gamma_{ab}^{(n-p)} \epsilon_{(p)(b)} \int_{z_{(p)}} \phi_{0} = 0
\end{align}

\end{proof}

\begin{proposition}
It is verified that: \((d\alpha, d\alpha) = (\alpha, \delta \phi)\) and \((\delta \beta, \delta \beta) = (\beta, d\phi)\)
\end{proposition}
Proof. It follows from: \((dC, B) = (C, \delta B)\), \(d\phi = d\delta \beta\) and \(\delta \phi = \delta d\alpha\).

**Proposition 5.3.** It is verified that the cohomologous term, \((\phi_h, \phi_h)\), in Proposition 5.1 can be decomposed as:

\[
(\phi_h, \phi_h) = \sum_{a=1}^{\beta_p} \epsilon^{(p)}_{a, P(a)} u_a v_{P(a)}
\]  
(84)

**Proof.** From its definition:

\[
(\phi_h, \phi_h) = \int_M \phi_h \wedge \star \phi_h = \sum_{a=1}^{\beta_p} \sum_{a=1}^{\beta_p} u_a v_b \int_M \gamma^{(p)}_a \wedge \gamma^{(n-p)}_b = \sum_{a=1}^{\beta_p} \sum_{a=1}^{\beta_p} u_a v_b \epsilon^{(p)}_{ab} = \sum_{a=1}^{\beta_p} \epsilon^{(p)}_{a, P(a)} u_a v_{P(a)}
\]  
(85)

**Theorem 5.1 (Quantization of the Norm).** The norm of a \(p\)-form \(\phi\) in no simply connected manifold can be expressed from two integrals based on its sources, \(d\phi\) and \(\delta \phi\), and from a discrete sum of values corresponding to every one of the \(\beta_p\) cycles of the cohomology classes as follows:

\[
(\phi, \phi) = (\alpha, \delta \phi) + (\beta, d\phi) + \sum_{a=1}^{\beta_p} \epsilon^{(p)}_{a, P(a)} u_a v_{P(a)} + (\phi_0, \phi_0)
\]

\[
u_a = \int_{\gamma^{(p)}_a} \phi \quad v_a = \int_{\gamma^{(n-p)}_b} \star \phi
\]  
(86)

6 Even dimensional pseudo-Riemann manifolds

Some special results can be obtained in even dimensional pseudo-Riemann manifolds, where \(n = 2m\). In this case the Poincaré duality for \(p = m\) defines one endomorphism: \(H^m \to H^m\) where it is verified that for every representative form \(\gamma^{(m)}_a\) exists one and only one other \(\gamma^{(m)}_b\) verifying:

\[
\epsilon^{(m)}_{ab} = \int_M \gamma^{(m)}_a \wedge \gamma^{(m)}_b \neq 0
\]  
(87)

Based on the previous results in Section 4 must be:

\[
\epsilon^{(m)}_{ab} = (-1)^m \epsilon^{(m)}_{ba}
\]  
(88)

Thus, the values of middle dimension \(m\) and signature \(s\), both combined in \(D(m)\), rule many properties of these manifolds. Diagonal elements in the matrix \(E^{(m)}\) are possible only if it is a symmetric matrix, \(m\) is even. However, diagonal elements implies self Poincaré duality, \(P(a) = a\); it implies a possibility for odd Betti number \(\beta_m\). The following list shows some cases for matrix \(E^{(m)}\):

1. **Even** \(m\), the matrix \(E^{(m)}\) is symmetric, Betti number \(\beta_m\) can be odd or even. The odd case happens if self duality is possible. The following are cases with even Betti number, with Poincaré pairs \((1 - 2)\) in the first case and \((1 - 2, 3 - 4)\) in the second one:

\[
\begin{pmatrix}
0 & A \\
A & 0
\end{pmatrix}
\quad
\begin{pmatrix}
0 & A & 0 & 0 \\
A & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & B & 0
\end{pmatrix}
\]  
(89)
the following case is with odd Betti number, the pair are: \((1 - 1, 2 - 3)\):

\[
\begin{pmatrix}
A & 0 & 0 \\
0 & 0 & B \\
0 & B & 0
\end{pmatrix}
\]  
(90)

2. **Odd** \(m\), the matrix \(E^{(m)}\) is antisymmetric, Betti number \(\beta_m\) must be even. The following examples are with Poincaré pairs \((1 - 2)\) in the first case and \((1 - 2, 3 - 4)\) in the second one:

\[
\begin{pmatrix}
0 & A \\
-A & 0
\end{pmatrix}
\]  
\[
\begin{pmatrix}
0 & A & 0 & 0 \\
-A & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & -B & 0
\end{pmatrix}
\]  
(91)

**Remark 6.1.** Self Poincaré duality is possible if the following condition is verified:

\[
\int_M \gamma_a^{(m)} \wedge \gamma_a^{(m)} \neq 0
\]
(92)

Self duality is only possible in even \(m\), that is: \(n = 2m = 4k\). Hence, this connects specifically to the problems concerning 4k manifolds that is a special branch of Topology and Geometry [5].

**Proposition 6.1.** In even dimensional manifolds, \(n = 2m\), for even values of \(\beta_mD(m)\), it can be \(T^{(m)} \in M(\beta_m, \mathbb{R})\), while for odd values it must be \(T^{(m)} \in M(\beta_m, \mathbb{C})\).

**Proof.** The result of Proposition 4.9 applied to \(T^{(m)}\) implies: \(T^{(m)} \times T^{(m)} = (-1)^{D(m)} I\). Thus, its determinant value is: \(|T^{(m)}|^2 = (-1)^{\beta_mD(m)}\). It means that the determinant always verifies: \(|T^{(m)}|^4 = 1\), with \(|T^{(m)}| = \pm 1\) for even \(\beta_mD(m)\) and \(|T^{(m)}| = \pm i\) for odd \(\beta_mD(m)\). With \(T^{(m)} \in M(\beta_m, \mathbb{R})\) the determinant is always real, the complex option can be used but it is not necessary. In real domain, \(T^{(m)}\) belongs to the \(GL(\beta_m, \mathbb{R})\) matrix Lie Group.

**Remark 6.2.** The \(GL(n, \mathbb{R})\) Lie Group has two connected components different on the determinant sign [9] pag.23. For the positive determinant there is the subgroup \(GL^+(n, \mathbb{R})\) while the negative determinant matrices does not form a Lie Group. It fails in the closure property. Additionally, the matrix must verify that its square must be the identity matrix with positive/negative sign. The procedure used later in this report avoid such problems because it do not use Lie Groups.

**Proposition 6.2.** In manifolds with minkowskian metric, \(n = 4\) and odd \(s\), the effective value of factor \(\beta_2D(2)\) is just: \(\beta_2\). Thus, it can be \(T^{(2)} \in M(\beta_2, \mathbb{R})\) for even values of \(\beta_2\) and it must be \(T^{(2)} \in M(\beta_2, \mathbb{C})\) for odd ones. If \(\gamma^{(2)}\) set is real, it must be: \(T^{(2)} \in M(\beta_2, \mathbb{R})\) because are integrals in \(z\) cycles, then only even values for \(\beta_2\) are allowed.

**Proposition 6.3.** The matrices \(E^{(m)}\) and \(\Lambda^{(m)}\) verify the following constraint Equation:

\[
\Lambda^{(m)} \times \left( E^{(m)} \right)^{-1} \times \Lambda^{(m)} = (-1)^{D(m)} E^{(m)}
\]  
(93)

**Proof.** It is a consequence of Propositions 4.9 and 4.10 with:

\[
T^{(m)} = (\Lambda^{(m)})^T \times ((E^{(m)})^{-1})^T
\]  
(94)
Proposition 6.4. In an even \( n = 2m \) pseudo-Riemann manifold, \( \phi \in \mathcal{A}^m \) can be expressed as:

\[
\phi = d\alpha - \star d\beta + \sum_{i=1}^{\beta_m} w_a \gamma_a^{(m)} \quad \delta \alpha = \delta \beta = 0
\]  

(95)

where \( \alpha \in \mathcal{A}^{m-1} \) and also \( \beta \in \mathcal{A}^{m-1} \).

Proof. Form the Hodge decomposition:

\[
\phi = d\alpha + \delta \theta + \sum_{i=1}^{\beta_m} w_a \gamma_a^{(m)} \quad \delta \alpha = \delta \beta = 0
\]  

(96)

where \( \theta \in \mathcal{A}^{m+1} \). From the definition of the operator \( \delta \): as:

\[
\delta \theta = (-1)^{C(m+1)} \star d \star \theta
\]

rewriting:

\[
(-1)^{C(m+1)} \star \theta = -\beta,
\]

therefore the gauge constraint \( d\theta = 0 \) must be rewritten as:

\( \delta \beta = 0 \). The minus sign in the \( \beta \) term is arbitrary, but highly convenient.

Proposition 6.5. The \( m \)-form \( \phi \) and its dual can be expressed as follows:

\[
\phi = d\alpha - \star d\beta + \phi_h
\]

(97)

\[
\star \phi = \star d\alpha + (-1)^{D(m)+1} d\beta + \star \phi_h
\]

(98)

A more compact expression of the previous equations can be achieved by introducing the following matrix representation:

\[
\begin{bmatrix}
\phi \\
\star \phi
\end{bmatrix} = [\sigma_1 d + \sigma_2 (\star d)]
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} + \sum_{a=1}^{\beta_m} u_a v_{aP} \gamma_a^{(m)}
\]

(99)

where: \( u_a = \int_{\mathcal{A}^m} \phi \) and \( v_a = \int_{\mathcal{A}^m} \star \phi \). The 2 \times 2 matrices \( \sigma_1 \) and \( \sigma_2 \) are:

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{D(m)+1} \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

(100)

For odd \( D(m) \) both \( \sigma_1 \) and \( \sigma_2 \) define the rotation matrix in the \( SO(2) \) group: \( R(\xi) = \sigma_1 \cos \xi + \sigma_2 \sin \xi \). The operator coderivative applied to this compact representation verifies:

\[
\delta \begin{bmatrix} \alpha \\
\beta
\end{bmatrix} = 0 \quad \delta \begin{bmatrix} \phi \\
\star \phi
\end{bmatrix} = \sigma_1 \Delta \begin{bmatrix} \alpha \\
\beta
\end{bmatrix}
\]

(101)

Proposition 6.6. The norm of the \( m \)-form \( \phi \) is decomposed as:

\[
(\phi, \phi) = (\alpha, \delta \phi) + (-1)^s (\beta, \delta \star \phi) + \sum_{a=1}^{\beta_m} e_{aP}^{(m)} u_a v_{aP} + (\phi_0, \phi_0)
\]

(102)

Proof. From the Hodge decomposition: \( \phi = d\alpha + \delta \theta + \cdots \), such as its norm is: \( (\phi, \phi) = (\alpha, \delta \phi) + (\theta, d\phi) + \cdots \). The second term has been modified as: \( \delta \theta = -* d\beta \), that is equivalent to:

\( (-1)^{C(m+1)} \star \theta = -\beta \), therefore:

\( \theta = (-1)^{C(m+1)+D(m+1)+1} \star \beta \). But by erasing the even terms:

\[
C(m+1) + D(m+1) + 1 = m^2
\]

(103)

Hence, the second term of the norm must be changed as:

\[
(\theta, d\phi) = (-1)^{m^2} (\star \beta, d\phi) = (-1)^{m^2} (\beta, * d\phi) = (-1)^{m^2+D(m)} (\beta, \delta \star \phi) = (-1)^s (\beta, \delta \star \phi)
\]

(104)

\( \square \)
Proposition 6.7. It is verified the following compact relationship between topology integrals:

\[
\begin{bmatrix}
u_a \\
v_a
\end{bmatrix} = \sum_{b=1}^{\beta_m} \tau_{ba} \begin{bmatrix} 0 & (-1)^{D(m)} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_b \\
v_b
\end{bmatrix}
\]

(105)

where the matrix in the right side verifies:

\[
\begin{bmatrix} 0 & (-1)^{D(m)} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & (-1)^{D(m)} \\ 1 & 0 \end{bmatrix} = (-1)^{D(m)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(106)

Proof. It is based on Proposition 4.11.

Remark 6.3. For odd \(D(m)\) the matrix in right side on the previous Proposition is a rotation of \(\pi/2\) in a plane. The following complex representation can be introduced: \(w = u + vi\), an the previous Equation becomes:

\[
w_a = i \sum_{b=1}^{\beta_m} \tau_{ba} w_b
\]

(109)

Each left side term is in quadrature to the sum in right side in the complex plane as shown in Figure 3. In the case of even \(\beta_m\), the matrices must be complex as it have been presented previously, a extreme case is \(\beta_m = 1\). In that case, must be: \(w_1 = r_1 w_1\), that has solution for: \(r_1 = -1\). However, it must verify: \((r_1) \times (r_1) = (-1)^{D(m)}\), therefore: \((-i)(-i) = -1 = (-1)^{D(m)}\), that means odd \(D(m)\) as it must be verified for this representation in the complex plane. For odd \(D(m)\) the matrices of \(m\)-forms seems be in the SL(\(\beta_m, \mathbb{R}\))SO(2) Group.

6.1 Solving the case \(\beta_m = 2\)

In the \(\mathbb{R}\) domain the solutions with odd \(\beta_m\) are no possible, therefore the simplest solution is for \(\beta_m = 2\). The solutions of matrices will be designed by using a simple coding criteria as \(S\beta_m.X.Y..\), according the sub cases that can be found in the analysis. The case of two cohomologies is split in two sub case; the first contains one pair of mutually dual cohomologies (Group S2.1), while the second has two self dual cohomologies (Group S2.2). The solutions for \(\beta_m = 4\), not analysed in this report, must include the sub cases of: two pair of cohomologies (S4.1), one pair and two self cohomologies (S4.2) and four self cohomologies (S4.3).

Several procedures can be defined based in the relations of matrices \(\Lambda^{(m)}\), \(E^{(m)}\) and \(T^{(m)}\). The initial data must be the matrix \(E^{(m)}\) with the symmetries and Poincaré duality relationship between cohomology classes. Using the result of Proposition 6.3 can be obtained the compatible values for \(\Lambda^{(m)}\) matrix. Other rather simple procedure to solve the matrices for low Betti numbers can be summarized as:

1. In the Equation (94), to code the right hand side based on the defined matrix \(E^{(m)}\) and general \(\Lambda^{(m)}\).
Figure 3: Each $w_a$ is in quadrature to $\sum_{b=1}^{\beta_m} \tau_b \tau_0 w_b$ in the complex plane. At right when two cohomologies have the matrix $T^{(2)}$ with null diagonal, both are in advanced/delayed quadrature in complex plane.

2. Based on the symmetries of $\Lambda^{(m)}$ and symmetries/antisymmetries of $E^{(m)}$, to identify the actual different terms in that side.

3. To code the matrix $T^{(m)}$ by including only the different elements.

4. To solve the Equation in Proposition (4.9); some constraints can be required.

In the case of $\beta_m = 2$, the general Solution S2, the analysis must include the firsts cases of a pair of mutually cohomologous classes, while the second is two self dual classes.

6.1.1 One Pair of Cohomologous Classes. Group S2.1

The corresponding cohomologous indexes in Poincaré duality are $1 - 2$ and the reciprocal $2 - 1$. The $E^{(m)}$ matrix can be symmetric or antisymmetric with elements: $E_{21} = (-1)^m E_{12}$. The matrices are:

$$E^{(m)} = \begin{pmatrix} 0 & E_{12} \\ E_{21} & 0 \end{pmatrix}, \quad \Lambda^{(m)} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

(110)

The right side of Equation (94) is the following:

$$\left(\Lambda^{(m)} \right)^T \times \left(\left(E^{(m)}\right)^{-1}\right)^T = \begin{pmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} \times \begin{pmatrix} 0 & 1/E_{12} \\ 1/E_{21} & 0 \end{pmatrix}$$

(111)

$$= \begin{pmatrix} \lambda_{21}/E_{21} & \lambda_{11}/E_{12} \\ \lambda_{22}/E_{21} & \lambda_{12}/E_{12} \end{pmatrix}$$

(112)

There are thee different terms from the symmetry relationships: $A = \lambda_{12}/E_{12}$, $B = \lambda_{11}/E_{12}$ and $C = \lambda_{22}/E_{12}$. The matrix $T^{(m)}$ is coded as follows:

$$T^{(m)} = \begin{pmatrix} (-1)^m A & B \\ -1)^m C & A \end{pmatrix}$$

(113)
It must be verified that: \( T^{(m)} \times T^{(m)} = (-1)^{D(m)} I \), that generates the following equations:

\[
A^2 + (-1)^m BC = (-1)^{D(m)} \tag{114}
\]
\[
(-1)^m AB + BA = 0 \tag{115}
\]
\[
CA + (-1)^m AC = 0 \tag{116}
\]
\[
(-1)^m CB + A^2 = (-1)^{D(m)} \tag{117}
\]

Several options with different solutions can be proposed generating a taxonomy of cases. The first is the concerning \( m \) values.

1. **Even** \( m \). The solution is obtained from \( AB = 0, AC = 0 \) and \( A^2 + BC = (-1)^s \), that also implies other two possibilities; in both cases it is verified:

\[
E^{(m)} = \begin{pmatrix}
0 & E_{12} \\
E_{12} & 0
\end{pmatrix} \tag{118}
\]

**S2.1.1** \( A = 0 \), that implies: \( BC = (-1)^s \). It means that: \( \lambda_{12} = 0 \) and \( \lambda_{11} \lambda_{22} = (-1)^s (E_{12})^2 \).

The matrix solutions are:

\[
\Lambda^{(m)} = \begin{pmatrix}
\lambda_{11} & 0 \\
0 & \lambda_{22}
\end{pmatrix} \quad T^{(m)} = \begin{pmatrix}
0 & \lambda_{11}/E_{12} \\
\lambda_{22}/E_{12} & 0
\end{pmatrix} \tag{119}
\]

**S2.1.2** \( B = C = 0 \), that implies: \( A^2 = (-1)^s \), only possible in metric with even \( s \). It is verified: \( \lambda_{11} = \lambda_{22} = 0 \) and: \( (\lambda_{12})^2 = (-1)^s (E_{12})^2 \). The matrix solutions are:

\[
\Lambda^{(m)} = \begin{pmatrix}
0 & \lambda_{12} \\
\lambda_{12} & 0
\end{pmatrix} \quad T^{(m)} = \begin{pmatrix}
\lambda_{12}/E_{12} & 0 \\
-\lambda_{12}/E_{12} & \lambda_{11}/E_{12}
\end{pmatrix} \tag{120}
\]

**S2.1.3** , **Odd** \( m \). The following condition must be verified: \( A^2 - BC = (-1)^{s+1} \), that implies: \( (\lambda_{12})^2 - \lambda_{11} \lambda_{22} = (-1)^{s+1} (E_{12})^2 \). The matrices are:

\[
E^{(m)} = \begin{pmatrix}
0 & E_{12} \\
-E_{12} & 0
\end{pmatrix} \quad T^{(m)} = \begin{pmatrix}
\lambda_{12}/E_{12} & \lambda_{11}/E_{12} \\
-\lambda_{12}/E_{12} & \lambda_{11}/E_{12}
\end{pmatrix} \tag{121}
\]

**Remark 6.4.** For \( n = 2m \), even \( m \) and \( \beta_m = 2 \), there is an isomorphism between \( H^m \) and \( \star H^m \) such the Theorem 4.2 is verified in Solution S2.1.; Equation (71) implies:

\[
\star \gamma_1^{(m)} = \frac{\lambda_{11}}{\varepsilon_{12}^{(m)}} \gamma_1^{(m)} \quad \star \gamma_2^{(m)} = \frac{\lambda_{22}}{\varepsilon_{12}^{(m)}} \gamma_1^{(m)} \tag{122}
\]

However, the Solution S2.1.2 is a more unusual with self Hodge duality (but not self Poincaré duality), that only happens in \( 4k \) manifold with even signature:

\[
\star \gamma_1^{(m)} = \frac{\lambda_{12}}{\varepsilon_{12}^{(m)}} \gamma_1^{(m)} \quad \star \gamma_2^{(m)} = \frac{\lambda_{12}}{\varepsilon_{12}^{(m)}} \gamma_2^{(m)} \tag{123}
\]
Table 2: Solution taxonomy for $\beta_m = 2$ with their required conditions. In the Group S2.2, the middle dimension $m$ must be even for all the solutions.

| Group | Solution | $m$ | $s$ | $|T^{(m)}|$ | Description |
|-------|----------|-----|-----|-------------|-------------|
| S2.1  | S2.1.1   | even| even| $(−1)^{s+1}$ | One dual pair |
|       | S2.1.2   | even| even| $+1$        |             |
|       | S2.1.3   | odd |      | $(−1)^s$    |             |
| S2.2  | S2.2.1   | even| even| $±1$        | Two self dual |
|       | S2.2.2   |      |      | $(−1)^{s+1}$|             |

6.1.2 Two Self Cohomologous Classes. Group S2.2

If two self Poincaré dual cohomologies are present, then $m$ must be even and $E^{(m)}$ symmetric because it is diagonal. The analytical procedure is similar that performed in the previous case. The matrices are:

$$E^{(m)} = \begin{pmatrix} E_{11} & 0 \\ 0 & E_{22} \end{pmatrix}, \quad \Lambda^{(m)} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$ (124)

The right side of Equation (94) is the following:

$$(\Lambda^{(m)})^T \times ((E^{(m)})^{-1})^T = \begin{pmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} \times \begin{pmatrix} 1/E_{11} & 0 \\ 0 & 1/E_{22} \end{pmatrix}$$ (125)

$$= \begin{pmatrix} \lambda_{11}/E_{11} & \lambda_{21}/E_{22} \\ \lambda_{12}/E_{11} & \lambda_{22}/E_{22} \end{pmatrix}$$ (126)

Unfortunately in this case there are not reduction in the freedom degrees. The four different terms are: $A = \lambda_{11}/E_{11}$, $B = \lambda_{21}/E_{22}$, $C = \lambda_{12}/E_{11}$ and $D = \lambda_{22}/E_{22}$. The matrix $T^{(m)}$ is coded as follows:

$$T^{(m)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ (127)

The compatibility Equation generates the following:

$$A^2 + BC = (-1)^{D^{(m)}}$$ (128)
$$AB + BD = 0$$ (129)
$$CA + DC = 0$$ (130)
$$CB + D^2 = (-1)^{D^{(m)}}$$ (131)

The different cases are the following with many possible solutions:

S2.2.1 $B = C = 0$, that implies $A^2 = D^2 = (-1)^{D^{(m)}}$, therefore there are solution for even $D^{(m)}$, that implies even $s$. It implies: $\lambda_{12} = 0$ and $(\lambda_{11}/E_{11})^2 = (\lambda_{22}/E_{22})^2 = 1$. The matrices are:

$$\Lambda^{(m)} = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \quad T^{(m)} = \begin{pmatrix} \lambda_{11}/E_{11} & 0 \\ 0 & \lambda_{22}/E_{22} \end{pmatrix}$$ (132)

S2.2.2 $A + D = 0$, that implies: $\lambda_{11}/E_{11} + \lambda_{22}/E_{22} = 0$, and: $A^2 = D^2 = (-1)^s - BC$, that means: $(\lambda_{11}/E_{11})^2 = (\lambda_{22}/E_{22})^2 = (-1)^s - \lambda_{12}^2/(E_{11} E_{22}) \geq 0$. The matrices are:

$$\Lambda^{(m)} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix}, \quad T^{(m)} = \begin{pmatrix} \lambda_{11}/E_{11} & \lambda_{12}/E_{22} \\ \lambda_{12}/E_{11} & -\lambda_{11}/E_{11} \end{pmatrix}$$ (133)
Figure 4: The 2-Torus has two cohomologous 1-cycles, $z_1^{(1)}$ and $z_2^{(1)}$, associated to cycles in coordinates $du$ and $dv$ respectively. Betti number is: $\beta_1 = 2$.

Table 3: Some remarkable surface manifolds $M \subset \mathbb{R}^3$ with dimensionality $n = 2$ and Riemannian metric with signature $s = 0$. The manifold dimension is even, $n = 2m$, and $m = 1$ is odd, therefore it is a Solution S2.1.3. The Euler number is: $\chi = \beta_0 - \beta_1 + \beta_2$.

| Surface          | Euler: $\chi$ | Betti: $\beta_0, \beta_1, \beta_2$ | Poincaré pairs |
|------------------|---------------|-------------------------------------|----------------|
| Sphere           | 2             | 1,0,1                                | 0              |
| 2-Torus          | 0             | 1,2,1                                | 1              |
| 2-Torus g-2      | -2            | 1,4,1                                | 2              |
| 2-Torus g-3      | -4            | 1,6,1                                | 3              |

6.2 The Cohomology Forms and Matrices in the 2-Torus

An application of the theoretical materials in the Subsection 6.1 is presented for the 2-Torus. However, the presentation follows a different analytical way. It starts from the metric and next induces the characteristic forms, computes the matrices and verifies their proposed properties.

The 2-Torus is a surface manifold $M \subset \mathbb{R}^3$ with a Riemannian metric defined as follows, where: $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$

$$ds^2 = (R + r \cos v)^2 du^2 + r^2 dv^2$$  \hspace{1cm} (134)

The metric tensor $g_{ab}$ and its determinant $|g|$ are:

$$g = \begin{pmatrix} (R + r \cos v)^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad \sqrt{|g|} = (R + r \cos v)r$$ \hspace{1cm} (135)

It is a even dimensional manifolds, $n = 2$, $m = 1$, pure Riemannian metric, $s = 0$, $D(1) = 1$. With odd $m$ the matrix $\mathbf{E}^{(1)}$ must be antisymmetric and the value of Betti number $\beta_1$ must be even; as Table 3 shows, its value is $\beta_1 = 2$. The volume form is: $\Omega = \sqrt{|g|} du \wedge dv = (R + r \cos v)r \ du \wedge dv$. Its integral is the manifold volume $\text{vol}(M)$, in this case is the Torus surface area $A$:

$$A = \text{vol}(M) = \int_M \Omega = \int_M \sqrt{|g|} \ du \wedge dv = \int_M (R + r \cos v)r \ du \wedge dv = Rr(2\pi)^2$$ \hspace{1cm} (136)

This is a manifold of finite volume, therefore the unit 2-form $\omega$ can be defined as: $\omega = \frac{1}{A} \Omega$. It can be expressed as:

$$\omega = \frac{1}{A} \Omega = \frac{1}{Rr(2\pi)^2} (R + r \cos v)r \ du \wedge dv = \frac{du}{2\pi} \wedge \frac{dv}{2\pi} + d \left( \frac{r}{2\pi R} \sin v \ du \right)$$ \hspace{1cm} (137)
The second term in the right side is an exact term \( d\xi \), where \( \xi = \frac{r}{2\pi R} \sin v \, du \). This allows to define an effective \( \omega \) without the exact and vanishing term in the integral:

\[
\int_M \omega = \int_M \frac{du}{2\pi} \wedge \frac{dv}{2\pi} + \int_M d\xi = 1 + \int_{\partial M} \xi = 1
\]

because the manifold \( M \) has not boundaries: \( \partial M = \emptyset \). There are two linear independent cycles, two homology classes, therefore there are two cohomology 1-forms, \( \beta_1 = 2 \), so they must be mutually Poincaré duals, that is: \( P(1) = 2 \) and \( P(2) = 1 \). The two representative forms can be chosen as the related to \( du \) and \( dv \) respectively. These 1-forms and its Hodge duals, that fit the unit \( n \) form, are:

\[
\gamma_1^{(1)} = \frac{du}{2\pi} \quad \gamma_2^{(1)} = \frac{dv}{2\pi} \\
\gamma_1^{(1)} = \frac{dv}{2\pi} \quad \gamma_2^{(1)} = -\frac{du}{2\pi}
\]

The \( T^{(1)} \) matrix is:

\[
T^{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
T^{(1)} \times (T^{(1)})^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

The values of the elements of matrix \( E^{(1)} \) are:

\[
\varepsilon_{11}^{(1)} = \int_M \gamma_1^{(1)} \wedge \gamma_1^{(1)} = 0 \\
\varepsilon_{12}^{(1)} = \int_M \gamma_1^{(1)} \wedge \gamma_2^{(1)} = 1 \\
\varepsilon_{21}^{(1)} = \int_M \gamma_2^{(1)} \wedge \gamma_1^{(1)} = -1 \\
\varepsilon_{22}^{(1)} = \int_M \gamma_2^{(1)} \wedge \gamma_2^{(1)} = 0
\]

The matrix \( E^{(1)} \) is:

\[
E^{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
E^{(1)} \times (T^{(1)})^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
The Proposition \[6.3\] becomes verified with \( \Lambda^{(1)} \) diagonal and the inverse of \( E^{(1)} \):

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= (-1)^{D^{(1)}}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

This topology solution for the 2-Torus is in the Group S2.1.3 in Subsection \[6.1\] and Table \[2\]; it has: \( n = 2m \), \( m = 1 \) odd, \( s = 0 \) even and \( \beta_1 = 2 \). The constraint \((\lambda_{12})^2 - \lambda_{11}\lambda_{22} = (-1)^{s+1}(E_{12})^2\) is verified.

7 Application to Electromagnetism in Non Simple Connected Manifolds

An application is presented of the previous theoretical results for the Electromagnetic Field into a no simple connected manifold. Classic Physic, including Relativity and Electromagnetism, is mainly a theory in the simple connected manifold \( \mathbb{R}^4 \). If instead of the classic approach, the theory is developed in a 4-dimensional no simple connected manifold \( M \), then some different results can arise due to the involved topology with its cohomologies. However, this remains as a classic theory, no a quantum one.

Following the Classical Theory \[14\], this field can be represented by using a 2-form \( F \). In this case, it is used a general form, without the restriction of be exact. The elements of the 2-form \( F \) include the 3-dimensional vectors magnetic field, \( B \), and electric field, \( E/c \), where \( c \) is the light speed in the vacuum; it is included to provide physical dimensional compatibility in the SI of units. It verify: \( c^2 = 1/\mu_0\epsilon_0 \). No material media is considered but currents and charge distributions are included.

Let \((M, g, \gamma)\) be a 4-dimensional compact, differentiable, oriented and connected pseudo-Riemann manifold \( M \) with coordinates according MTW conventions \[14\]: \((ct, x, y, z)\) having a metric \( g \) locally reducible to a minkowskian diagonal case with \( s = 1 \), that a space-like metric: \( \eta = \text{diag}(-1, 1, 1, 1) \).

Also, it must be considered \( \gamma \) a set of representative cohomologous forms associated to closed sub-manifolds \( z \). There are \( \beta_1 \) 1-forms \( \gamma^{(1)} \) and \( \beta_2 \) 2-forms \( \gamma^{(2)} \). This manifold is even dimensional, \( n = 4 \), being \( F \) a middle dimensional form, \( m = 2 \). The signature \( s \) is odd, and \( m^2 \) is even, therefore the factor \( D(2) \) is odd and \( D(1) = D(3) \) is even. The double Hodge duality is anti-symmetric for 2-forms and symmetric for 1-forms, the Betti number \( \beta_2 \) is even and the matrix \( (\varepsilon^{(2)}_{ab}) \) is symmetric. The cohomology representative forms are physical no dimensional as well as all the matrices generated from their.

According the MTW conventions \[14\] pag.108, \( F \) and \( \ast F \) components are:

\[
\begin{align*}
F &= \frac{E_1}{c} dx^1 \wedge dx^0 + \cdots + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \\
\ast F &= -B_1 dx^1 \wedge dx^0 + \cdots + \frac{E_1}{c} dx^2 \wedge dx^3 + \frac{E_2}{c} dx^3 \wedge dx^1 + \frac{E_3}{c} dx^1 \wedge dx^2
\end{align*}
\]

The following matrices meet such criteria \[14\] pag.74:

\[
(F)_{ab} = \begin{pmatrix}
0 & -E_1/c & -E_2/c & -E_3/c \\
E_1/c & 0 & B_3 & -B_2 \\
E_2/c & -B_3 & 0 & B_1 \\
E_3/c & B_2 & -B_1 & 0
\end{pmatrix}
\]
\[(\ast F)_{ab} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3/c & -E_2/c \\ -B_2 & -E_3/c & 0 & E_1/c \\ -B_3 & E_2/c & -E_1/c & 0 \end{pmatrix} \] (155)

They verify that:

\[\nabla_b F^0_0 = \nabla \cdot E/c \] (156)

\[\nabla_b F^{ab} = -\frac{\partial E_a}{c \partial t} + (\nabla \times B)_a \] (157)

\[\nabla_b (\ast F)^0_0 = -\nabla \cdot B \] (158)

\[\nabla_b (\ast F)^{ab} = \frac{\partial B_a}{c \partial t} + (\nabla \times E/c)_a \] (159)

The contravariant components of 4-dimensional currents vector \(J^a\) are: \(\{c \rho, J_1, J_2, J_3\}\) and the covariant ones \(J_a\) are: \(\{-c \rho, J_1, J_2, J_3\}\). The two first Equations in the previous system are the Maxwell Equations that can be rewritten as: \(\nabla_b F^{ab} = \mu_0 J^a\), while the last two are: \(\nabla_b (\ast F)^{ab} = 0\).

However, the general 2-form \(F\) admits an Hodge decomposition, according the results of Section 6, but it is required to physically identify its sources. These are the continuous: \(\delta F\), \(\delta \ast F\) and the discrete: \(\int \ast F\).

### 7.1 The Physical meaning of Cohomology Integrals

According to the Classic Electromagnetism the physical meaning of one integral as \(\int_{\gamma(2)} \ast F\) is an electric charge. However, the meaning of \(\int_{\gamma(2)} F\) must be one magnetic charge, that is an unusual or exotic concept in Classical Theory, but have been largely considered in theoretical proposals of extension of the Classical Theory in Electromagnetic Duality\cite{1,11,16}. Both types previously defined are topological charges, that is, they are associated to some integrals that have no null values only in no simple connected manifolds. Thus, they depend mainly on the topological properties of the manifold. It must be remarked that this magnetic charge is not a free particle in the sense of Dirac magnetic monopoles\cite{4}, instead this magnetic charge is only one integral, that is a global property of the manifold, one source of the field.

The magnetic charges are theoretical concept, without experimental evidence. Thus, they have not neither physical dimension nor units nor experimental values for the hypothetical charges. There are some freedom degree in defining some of these properties, as the physical dimensionality of magnetic charges. In this report, using the available freedom degrees, a dimensional relationship as: \([q^{(E)}] = [q^{(M)}]\) is used. Remark that the chosen physical dimension implies that both are measured using the same units, but the numerical values of experimental charges, if they could exist, do not need to be the same that the electric ones.

The Maxwell equations in Classic Electromagnetic Theory for the rationalized International System of Units in differential form, generalization of Equations (156) to (159), can be rewritten as:

\[\delta F = \mu_0 J^{(E)} \quad \delta \ast F = -\mu_0 J^{(M)} \] (160)

where \(J^{(E)}\) and \(J^{(M)}\) are 1-forms that are interpreted as the electric and magnetic relativistic currents. These 1-forms include: \((c \rho, J)\), where its space components, \(J\), are the vector current and the time component is: \(c \rho\), where \(\rho\) the volumetric density of charge. These equations are dimensional consistent and the \(-1\) in the second equation is required to meet some compatibility criteria that is explained in the next paragraphs. From their definitions and the property of double coderivative, \(\delta \delta = 0\), is verified that: \(\delta J^{(E)} = \delta J^{(M)} = 0\).
The equations can be rewritten based on the double Hodge duality: \( \star (\delta F) = d \star F \) and \( \star (\delta \star F) = -dF \), that is symmetric for \( (1,3) \)-forms and is antisymmetric for 2-forms.

\[
d \star F = \mu_0(\star J^{(E)}) \quad dF = \mu_0(\star J^{(M)})
\]  

(161)

The vector expressions in 3-vectors are the following, for the first Maxwell equation:

\[
\nabla \cdot E/c = \mu_0 c \rho^{(E)} \quad \nabla \times B - \frac{\partial E}{c \partial t} = \mu_0 J^{(E)}
\]  

(162)

while that second Maxwell equation [11, pag.273] in \( dF \) is:

\[
-\nabla \cdot B = -\mu_0 c \rho^{(M)} \quad \nabla \times E/c + \frac{\partial B}{c \partial t} = -\mu_0 J^{(M)}
\]  

(163)

The integrals in a 2-dimensional submanifold \( \partial \Omega \) have the following meaning:

\[
\int_{\partial \Omega} \star F = \int_{\partial \Omega} \frac{1}{c} (E_1 \, dx^2 \wedge dx^3 + E_2 \, dx^3 \wedge dx^1 + E_3 \, dx^1 \wedge dx^2) + \cdots
\]

\[
= \frac{1}{c} \int_{\partial \Omega} E \cdot ds + \cdots = \mu_0 c Q^{(E)}
\]  

(164)

\[
\int_{\partial \Omega} F = \int_{\partial \Omega} (B_1 \, dx^2 \wedge dx^3 + B_2 \, dx^3 \wedge dx^1 + B_3 \, dx^1 \wedge dx^2) + \cdots
\]

\[
= \int_{\partial \Omega} B \cdot ds + \cdots = \mu_0 c Q^{(M)}
\]  

(165)

More compactly, the result can be achieved but using the Stokes Theorem:

\[
\int_{\partial \Omega} F = \int_{\Omega} dF = \mu_0 \int_{\Omega} \star J^{(M)} = \mu_0 c Q^{(M)}
\]  

(166)

\[
\int_{\partial \Omega} \star F = \int_{\Omega} d \star F = \mu_0 \int_{\Omega} \star J^{(E)} = \mu_0 c Q^{(E)}
\]  

(167)

The criteria to choose the signs in Equation (160) is to provide positive value for the two previous integrals with meaning of charge. Based on this results, the cohomology integrals must be interpreted as:

\[
\int_{z^{(2)}} F = \mu_0 c q^{(M)}_a \\
\int_{z^{(2)}} \star F = \mu_0 c q^{(E)}_a
\]  

(168)

where \( q^{(M)}_a \) and \( q^{(E)}_a \) have the meaning of magnetic and electric charges respectively associated to the 2-cycles. However, in this context charges do not mean charged virtual (point-like) particles. These charges are generated by the topology properties and are specific to no simple connected manifolds. They are manifold properties and disappear in simple connected manifolds. However, the charges \( Q^{(M)} \) and \( Q^{(E)} \) are common in both simple and no simple connected manifold. Both concepts are formal different and coherently must be physical differentiated in a complete physic theory.
7.2 Double Potential and Quantization of the Norm

In classical Electromagnetic Theory \( F \) is exact, \( F = dA \), where the Potential \( A \) is an 1-form. However, it is not in a general case as is analysed in this report. The main consequence is that there is not an equivalent to the 1-form \( A \). The Hodge decomposition of \( F \) requires two 1-forms as have been presented in Section 6:

\[
F = dA(E) - \star dA(M) + \mu_0 c \sum_{a=1}^{\beta_2} q_a^{(M)} \tau_a^{(2)}
\]

(169)

\[
\star F = dA(M) + \star dA(E) + \mu_0 c \sum_{a=1}^{\beta_2} q_a^{(E)} \tau_a^{(2)}
\]

(170)

with the following definitions:

\[
\delta A(E) = \delta A(M) = 0 \quad \mu_0 c q_a^{(M)} = \int_{\Sigma^{(2)}} F \quad \mu_0 c q_a^{(E)} = \int_{\Sigma^{(2)}} \star F
\]

(171)

where \( A(E) \) and \( A(M) \) are dual closed 1-forms required for the Hodge decomposition of a general 2-form. The first term \( F = dA(E) + \cdots \) resembles the 1-form \( A \) in Classical Theory related to electric charges, this is the reason to name it as: \( A(E) \). The compacted Equation (99) becomes:

\[
\begin{bmatrix}
F \\
\star F
\end{bmatrix} = [\sigma_1 d + \sigma_2 (\star d)] \begin{bmatrix}
A^{(E)} \\
A^{(M)}
\end{bmatrix} + \mu_0 c \sum_{a=1}^{\beta_2} \begin{bmatrix}
q_a^{(M)} \\
q_a^{(E)}
\end{bmatrix} \tau_a^{(2)}
\]

(172)

The continuous sources of \( F \) are [14, pag.569]:

\[
\delta F = \delta dA^{(E)} = \Delta A^{(E)} = \mu_0 J^{(E)} \quad \delta \star F = \delta dA^{(M)} = \Delta A^{(M)} = -\mu_0 J^{(M)}
\]

(173)

It must be remarked that conversely to Classical Electromagnetism, the sources \( \delta F \) and \( \delta \star F \) are not sufficient enough to determine the field \( F \). It is required the additional discrete sources associated to the cohomology classes, that is, the topological charges. These charges \( q_a^{(M)} \) and \( q_a^{(E)} \) are not independent; they are related by the result of Proposition 4.11 concerning both type of topology integrals:

\[
q_a^{(M)} = -\sum_{b=1}^{\beta_2} \tau_{ba}^{(2)} q_b^{(E)}
\]

(174)

\[
q_a^{(E)} = \sum_{b=1}^{\beta_2} \tau_{ba}^{(2)} q_b^{(M)}
\]

(175)

This Equations can be also rewritten as:

\[
\begin{bmatrix}
q_a^{(M)} \\
q_a^{(E)}
\end{bmatrix} = \sum_{b=1}^{\beta_2} \tau_{ba} \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
q_b^{(M)} \\
q_b^{(E)}
\end{bmatrix}
\]

(176)

The electric and magnetic charges of the manifold are not individual properties of each class of cohomology, but appear to be collective properties, so their values are such that they verify certain properties in the set of all classes. If the compact we introduce the complex charge in the
plane (M)-(E) as: \( q_a = q_a^{(M)} + q_a^{(E)} \), then the right side is the rotation: \( e^{i\pi/2}q_b \). The previous Equation becomes in complex representation:

\[
q_a = \sum_{b=1}^{\beta_2} \gamma_{ab} q_b
\]  

(177)

From Remark 6.3, each charge in left side, in the complex plane defined by the (M) and (E) terms, is in quadrature to the sum in right side.

Based on Proposition 6.6, the norm of \( F \) can be decomposed in two continuous terms, that are generalization of the Classical Theory term \((A, J)\) and one discrete/quantized stepwise term with \( \beta_2 \) values:

\[
(F, F) = \mu_0 (A^{(E)}, J^{(E)}) + (-1)^s \mu_0 (A^{(M)}, -J^{(M)}) + (\mu_0 c)^2 \sum_{a=1}^{\beta_2} \varepsilon_{a, P(a)} g_a^{(M)} q_{P(a)}^{(E)}
\]  

(178)

### 7.3 Quantization of the Action

The electromagnetic action, \( S \), in the Lagrangian formalism is:

\[
S = \frac{1}{c} \int_M \mathcal{L} = \frac{1}{c} \int_M \mathcal{L} \ d\Omega = \int_M \mathcal{L} \ dV dt
\]  

(179)

where \( \mathcal{L} \) is the Lagrangian density and \( \mathcal{L} \) is its equivalent 4-form. If the Maxwell equations must be generated from the variational principle over the action \( S \), then the variations must be on each one of the 1-forms \( A^{(E)} \) and \( A^{(M)} \). In Classical Theory only one of the two Maxwell equations set (the related to sources: electric currents and charges) is generated from the variational principle. The other are structural, they are a consequence of the mathematical structure: if \( F \) is exact: \( F = dA \), then \( dF = ddA = 0 \), is a consequence of the mathematical structure; they do not come from any extremal principle. However, if the two Maxwell equations are related to sources, as is the case of a general analysis of the field, both are required to be generated from the variational principle.

The Lagrangian must be function of the field variables and its first derivatives, that is: \( \mathcal{L}(A^{(E)}, A^{(M)}, F) \). The Lagrangian density \( \mathcal{L} \) in tensor expression is the following, a generalization of the classic one:

\[
\mathcal{L} = -\frac{1}{4 \mu_0} F^{ab} F_{ab} - A^{(E)}_a (J^{(E)})^a - A^{(M)}_a (J^{(M)})^a
\]  

(180)

Based on the relationship between tensors, 2-form and 1-forms:

\[
F = \frac{1}{2} F_{ab} \ dx^a \wedge dx^b \quad A^{(E)} = A^{(E)}_a \ dx^a
\]  

(181)

Thus, the Lagrangian 4-form \( \mathcal{L} \) must be:

\[
\mathcal{L} = -\frac{1}{\mu_0} F \wedge \ast F - A^{(E)} \wedge \ast J^{(E)} - A^{(M)} \wedge \ast J^{(M)}
\]  

(182)

The application of variational principle to the Action is presented in Appendix A. The Action becomes:

\[
S = -\frac{1}{\mu_0 c} (F, F) - \frac{1}{c} (A^{(E)}, J^{(E)}) - \frac{1}{c} (A^{(M)}, J^{(M)})
\]  

(183)
Based on Equation (178), the result is the following Action with two continuous and one discrete/quantized term: \( S_d \). The amount of discrete terms is just the number of cohomology cycles, the Betti number \( \beta_2 \):

\[
S = -\frac{2}{c} (A(E), J(E)) - \frac{2}{c} (A(M), J(M)) - \mu_0 c \sum_{a=1}^{\beta_2} \varepsilon_{a,P(a)} q_a^{(M)} q_{P(a)}^{(E)}
\] (184)

### 7.4 Charges and Action in Case \( \beta_2 = 2 \)

An application of the results obtained in Subsection 6.1 for Physics Theories in Electromagnetism in a pseudo Riemann manifold is presented. It is verified that: \( n = 2m \) even dimensional, \( m = 2 \) is even and \( s \) is odd, and \( D(2) \) is odd; the matrix \( E^{(2)} \) is symmetric. Hence, the solutions S2.1.1 and S2.2.2 in Table 2 are possible.

The solution S2.1.1 with a pair of Poincaré dual cohomologies has symmetric matrix \( E^{(2)} \) and verify the condition: \( \lambda_1^{(2)} \lambda_2^{(2)} = -(\varepsilon_{12}^{(2)})^2 \). Thus, the norms \( \lambda_1^{(2)} \) and \( \lambda_2^{(2)} \) are in opposite sign, also it must be an orthogonal case with \( \lambda_1^{(2)} = 0 \). The matrices are:

\[
E^{(2)} = \begin{pmatrix} 0 & \varepsilon_{12}^{(2)} \\ \varepsilon_{12}^{(2)} & 0 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 0 & \lambda_1^{(2)}/\varepsilon_{12}^{(2)} \\ \lambda_2^{(2)}/\varepsilon_{12}^{(2)} & 0 \end{pmatrix} \quad \Lambda^{(2)} = \begin{pmatrix} \lambda_1^{(2)} & 0 \\ 0 & \lambda_2^{(2)} \end{pmatrix}
\] (185)

There is an isomorphism between \( H^2 \) and \( \ast H^2 \) such the Theorem 4.2 is verified with the following 2-forms:

\[
\ast \gamma_1^{(2)} = \frac{\lambda_1^{(2)}}{\varepsilon_{12}^{(2)}} \gamma_1^{(2)} \quad \ast \gamma_2^{(2)} = \frac{\lambda_2^{(2)}}{\varepsilon_{12}^{(2)}} \gamma_1^{(2)}
\] (186)

The relationship between magnetic and electric charges is:

\[
\begin{pmatrix} q_1^{(M)} \\ q_2^{(M)} \end{pmatrix} = -\frac{1}{\varepsilon_{12}^{(2)}} \begin{pmatrix} 0 & \lambda_2^{(2)} \\ \lambda_1^{(2)} & 0 \end{pmatrix} \begin{pmatrix} q_1^{(E)} \\ q_2^{(E)} \end{pmatrix}
\] (187)

The compact representation in Equation (176) is:

\[
\begin{pmatrix} q_1^{(M)} \\ q_1^{(E)} \\ q_2^{(M)} \\ q_2^{(E)} \end{pmatrix} = \frac{\lambda_2^{(2)}}{\varepsilon_{12}^{(2)}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1^{(M)} \\ q_1^{(E)} \\ q_2^{(M)} \\ q_2^{(E)} \end{pmatrix}
\] (188)

that show that the charge relationship is composed of a scale factor and a rotation of \( \pi/2 \) in the plane \( (M) - (E) \), such as the both charges are in quadrature. Due to \( \lambda_1^{(2)} \) and \( \lambda_2^{(2)} \) have different sign, there is a combination of net charge, or monopole, and differential charge, or dipole, for both electric and magnetic components. The monopole or net charge is the sum of charges: \( m = q_1 + q_2 \), while the dipole moment is proportional to the differential charge: \( d = q_1 - q_2 \). By using the following expression for each charges from the monopole and dipole factors:

\[
q_1 = \frac{1}{2} (m + d) \quad q_2 = \frac{1}{2} (m - d)
\] (189)

It is concluded the following:

\[
\varepsilon_{12}^{(2)} \left( m^{(M)} + d^{(M)} \right) = -\lambda_1^{(2)} \left( m^{(E)} - d^{(E)} \right)
\] (190)

\[
\varepsilon_{12}^{(2)} \left( m^{(M)} - d^{(M)} \right) = -\lambda_2^{(2)} \left( m^{(E)} + d^{(E)} \right)
\] (191)
by multiplying both previous equations, it is conclude that:

\[(m^{(M)})^2 - (d^{(M)})^2 + (m^{(E)})^2 - (d^{(E)})^2 = 0\]  

(192)

The quantized action \(S_d\) expressed from both type of charges becomes:

\[
S_d = -\mu_0 c \sum_{a=1}^{b} \varepsilon_{a, P(a)} q_{a}^{(M)} q_{P(a)}^{(E)} = -\mu_0 c \varepsilon_{12}^{(2)} \left( q_1^{(M)} q_2^{(E)} + q_2^{(M)} q_1^{(E)} \right) 
\]

(193)

\[
S_d = \mu_0 c \left( \lambda_1^{(2)} (q_1^{(E)})^2 + \lambda_2^{(2)} (q_2^{(E)})^2 \right) 
\]

(194)

The \(\lambda\) factors have different sign, therefore the Action has not defined sign. Even though there are charge configurations with null discrete Norm and Action. Although from a quantum theory viewpoint, have been argued [1, 4, 16] that the product of electric and magnetic elementary charges is in the order of Plank constant; it must be considered that this depends on the used physical unit systems; the Gaussian one and \(c = 1\) are usually used for such comparative. Equation (193) is consistent to such argument, in the relationship between the charge product and the action, but in SI physical units. Also, a topological magnitude is included: \(\varepsilon_{12}^{(2)}\). However the approach on this Section is fully included in Classical Physic, but in no simple connected manifolds. The physical magnitudes involved in the discrete Action are:

\[
[h] = [\mu_0] [c] [e]^2 [\lambda] \quad [\lambda] = \frac{[h]}{[\mu_0] [c] [e]^2} 
\]

(195)

The \(\lambda\) magnitudes are physical no dimensional. If the values are in the order of the electron charge and one action quantum, then the \(\lambda\) values are in the order of the inverse of the fine structure constant \(\alpha\) defined as:

\[
\alpha = \frac{\mu_0 c e^2}{2h} 
\]

(196)

Thus, the value for \(\lambda\) are in the order of the inverse of this constant:

\[
\lambda \simeq \frac{h}{\mu_0 c [e]^2} = \frac{1}{2\alpha} 
\]

(197)

This result that for an hypothetical study in elementary particles and cohomologies, the following Equation can be an useful starting point, where actions becomes in the order of Plank constant and electric charges are in the order of the electron charge:

\[
\int_M \gamma_a^{(2)} \land \ast \gamma_a^{(2)} = \frac{c e_0}{\alpha} 
\]

(198)

In the solution S2.2.2 the matrices are:

\[
E^{(2)} = \begin{pmatrix} 
\varepsilon_{11}^{(2)} & 0 \\
0 & \varepsilon_{22}^{(2)} 
\end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 
\lambda_1^{(2)}/\varepsilon_{11}^{(2)} & \lambda_1^{(2)}/\varepsilon_{11}^{(2)} \\
\lambda_1^{(2)}/\varepsilon_{11}^{(2)} & \lambda_2^{(2)}/\varepsilon_{22}^{(2)} 
\end{pmatrix} \quad \Lambda^{(2)} = \begin{pmatrix} 
\lambda_1^{(2)} & \lambda_1^{(2)} \\
\lambda_1^{(2)} & \lambda_2^{(2)} 
\end{pmatrix} 
\]

(199)

with conditions: \(\lambda_1^{(2)}/\varepsilon_{11}^{(2)} + \lambda_2^{(2)}/\varepsilon_{22}^{(2)} = 0\), and:

\[
\left(\frac{\lambda_1^{(2)}}{\varepsilon_{11}^{(2)}}\right)^2 - \left(\frac{\lambda_2^{(2)}}{\varepsilon_{22}^{(2)}}\right)^2 = -1 - \left(\frac{\lambda_2^{(2)}}{\varepsilon_{11}^{(2)}}\right)^2 \geq 0 
\]

(200)
That means that $\varepsilon_{11}^{(2)}$ and $\varepsilon_{22}^{(2)}$ must be of different sign. The quantized action $S_d$ is expressed as:

$$S_d = -\mu_0 c \sum_{a=1}^{\beta_2} \varepsilon_{a,P(a)}^{(2)} q_a^{(M)} q^a_{P(a)} = -\mu_0 c \left( \varepsilon_{11}^{(2)} q_1^{(M)} q_1^{(2)} + \varepsilon_{22}^{(2)} q_2^{(M)} q_2^{(2)} \right)$$

(201)

Both Solutions, S2.1.1 and S2.2.2, are instances of a continuous matrix with some free variables $T \in M(2, \mathbb{R})$:

$$T(u, v) = \begin{pmatrix} u & v \\ -(1 + u^2)/v & -u \end{pmatrix} \quad TT = -I$$

(202)

8 Conclusions

Hodge decomposition have been presented as a practical application of decomposition based on linear independent operators. The operators of derivative, coderivative and cohomology integrals are a linear independent operator set. They generate a differential form decomposition in exact, dual exact and cohomology expansion, that have been proof to be complete, except in an constant harmonic no cohomologous form as the residue element of such decomposition.

That methodology is not so norm dependent as most of presented in scientific literature concerning Geometry and Topology. In Riemann manifolds the norm is positive definite and such property is extensively used in many proof; but the more interesting for Physical applications are the pseudo or semi Riemann manifolds without that property. Thus, a methodology not founded on the norm signature is more useful to be applied in both Riemann and pseudo Riemann manifolds.

A set of representative forms of the cohomologies have been used to define the Poincaré duality. Some auxiliary matrices to characterize the relationship between Hodge and Poincaré dualities of the representative forms have been proposed. In some case both dualities becomes isomorphic.

The decomposition of differential forms in canonical terms requires a topology term obtained from the integral in the cohomology cycles. This decomposition has a counterpart in the decomposition of the norm in also canonical terms are generated. The term concerning the cohomologies is a discrete/quantized finite sum whose number is just the Betti number. The special case of even dimensional manifolds have been analysed including examples of representarve forms and the auxiliary matrices.

An application for the Electromagnetic Field in non simple connected manifold have been presented. A phenomenological interpretation of the cohomology integrals is needed. It is concluded that electric and magnetic charges must be required to a correct interpretation of these cohomology integrals. That means that Electromagnetic Duality is necessary in the study of Electromagnetism in no simple connected manifolds. However, The electric and magnetic charges of the manifold are neither free particles nor individual properties of each class of cohomology, but appear to be collective properties of the manifold, so their values are such that they verify certain properties in the set of all classes. It not possible to have electric monopoles without magnetic ones and vice versa.

A significant result is the presence of one discrete/quantized term in the field Norm and Action associated to the Lagrangian formalism. The amount of such discrete values is the Betti number or Poincaré pairs of cohomology classes. A Classic Theory approach has been presented, but as result the Action of Electromagnetic Field includes one quantized term. Even though if no continuous sources are present, there is a topology generated Electromagnetic Field whose Action is quantized in discrete steps. The relation between magnetic monopoles and quantization
of the action in the Electromagnetic Fields, widely suggested in previous theoretical studies in Quantum Physics, is again confirmed but from a point of view of Classical Physics.

A well known problem is Classical Electromagnetism happens when the integral of the field norm is extended to all the space-time; in this case one infinite solution appears. It is an outcome of the point-like model of particles or the lack of a model for finite particles. In the proposed model of Electromagnetism coming from Topology, and in most of wormholes theories family, these integral in all the manifold do not generate infinite values, instead there is a finite sum. The sources of these properties are the cohomologies.

The contradiction between continuous vs. discrete/quantized values was a significant problem in the History of Physics, when the continuous values predict by Classic Theories was in contradiction to discrete values reported by the experiments, eg. the black-body spectrum or the photoelectric effect. From this controversy emerged the Quantum Theory in early twenty century as an inductive framework constructed ex novo to explain the experiment results.

The link between Classic Physic and quantized magnitudes was no possible if the theories were developed in a simple connected manifold. However, that link between Classic Physic and quantized magnitudes comes naturally if it is developed in no simple connected manifolds because continuous and discrete magnitudes coexist consistently, being the discrete magnitudes that involved on the cohomology classes of the manifold topology. Hence, quantization seems be a natural result of Topology.

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A Variational Principle on the Action

The variational principle used in the field equations generation is founded on the hypothesis that the action becomes an extremal on the variation $\delta$ of field variables. The variation $\delta S = 0$, implies that must be: $\int_\Omega \delta L \, d\Omega = 0$. There are two independent field variable: $A^{(E)}$ and $A^{(M)}$ and $F$ depends on the first derivative of both.

$$\delta L = \frac{\partial L}{\partial A^{(E)}_a} \delta A^{(E)}_a + \frac{\partial L}{\partial \nabla_b A^{(E)}_a} \delta \nabla_b A^{(E)}_a + \frac{\partial L}{\partial A^{(M)}_a} \delta A^{(M)}_a + \frac{\partial L}{\partial \nabla_b A^{(M)}_a} \delta \nabla_b A^{(M)}_a$$  \hspace{1cm} (203)

where $\nabla_b$ is the tensor derivative. This variational principle can be also rewritten in differential form as:

$$\delta L = \frac{\partial \mathcal{L}}{\partial A^{(E)}_a} \delta A^{(E)}_a + \frac{\partial \mathcal{L}}{\partial dA^{(E)}_a} \delta dA^{(E)}_a + \frac{\partial \mathcal{L}}{\partial A^{(M)}_a} \delta A^{(M)}_a + \frac{\partial \mathcal{L}}{\partial dA^{(M)}_a} \delta dA^{(M)}_a$$  \hspace{1cm} (204)

The variation $\delta A^{(E)}_a$ generates, in tensor or differential presentation, one of the two sets of Maxwell Equations. If $\delta$ commutes with both the tensor and exterior derivative: $\delta dA = d\delta A$, it is obtained the following pair of Equations for the tensor or differential formulation:

$$\frac{\partial \mathcal{L}}{\partial A^{(E)}_a} - \nabla_b \left( \frac{\partial \mathcal{L}}{\partial \nabla_b A^{(E)}_a} \right) = 0 \quad \frac{\partial \mathcal{L}}{\partial dA^{(E)}_a} - d \left( \frac{\partial \mathcal{L}}{\partial dA^{(E)}_a} \right) = 0$$  \hspace{1cm} (205)

The differential form presentation allows more simplified operations. Based on the definition in Equation (169), it is obtained the following simplified equivalences:

$$\frac{\partial F}{\partial dA^{(E)}_a} = 1 \quad \frac{\partial \star F}{\partial dA^{(E)}_a} = (\star 1) \quad \frac{\partial F}{\partial dA^{(M)}_a} = - (\star 1) \quad \frac{\partial \star F}{\partial dA^{(M)}_a} = 1$$  \hspace{1cm} (206)

Thus, the Lagrangian terms are:

$$\frac{\partial \mathcal{L}}{\partial A^{(E)}_a} = - 1 \wedge J^{(E)} = - \star J^{(E)}$$  \hspace{1cm} (207)

$$\frac{\partial \mathcal{L}}{\partial dA^{(E)}_a} = - \frac{1}{\mu_0} \left( 1 \wedge \star F + \underbrace{F \wedge (\star 1)}_0 \right) = - \frac{1}{\mu_0} \star F$$  \hspace{1cm} (208)

That generates: $d \star F = \mu_0 \star J^{(E)}$. The variation $\delta A^{(M)}_a$ generates in tensor or differential form:

$$\frac{\partial \mathcal{L}}{\partial A^{(M)}_a} - \nabla_b \left( \frac{\partial \mathcal{L}}{\partial \nabla_b A^{(M)}_a} \right) = 0 \quad \frac{\partial \mathcal{L}}{\partial dA^{(M)}_a} - d \left( \frac{\partial \mathcal{L}}{\partial dA^{(M)}_a} \right) = 0$$  \hspace{1cm} (209)
Similarly to the electric case, the Lagrangian terms are:

\[
\frac{\partial \Sigma}{\partial A^{(M)}} = -1 \wedge J^{(M)} = - \star J^{(M)} \tag{210}
\]

\[
\frac{\partial \Sigma}{\partial dA^{(M)}} = - \frac{1}{\mu_0} F \tag{211}
\]

That generates: \( dF = \mu_0 \star J^{(M)} \)