Wavelet regularization of Euclidean QED

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I. INTRODUCTION

This paper was initially conceived as an erratum to the paper [1], where we have found technical errors in the evaluation of one-loop diagrams [Eqs.(34,36)] in wavelet-based quantum electrodynamics. However, it was found later, that a simple model of wavelet-based QED, briefly described in aforementioned paper, can shed some new light on the scale dependence of coupling constance on observation scale in an Abelian gauge theory – starting from completely finite quantum field theory model with no need of renormalization.

In the previous papers [1, 2] the possibility to construct a finite theory of scale-dependent fields $\psi_a(x)$ was developed, where the field $\psi_a(x)$ describes the fluctuations of typical size $a$. In this paper we make simplifying assumption that all measurable quantities can be determined in terms of effective fields $\psi_A \sim \sum_{A \leq a < \infty} \psi_a(x)$ [with the meaning of the sum clarified later in the text], which are the sums of all fluctuations larger than the observation scale $A$. This approach allows us to start with a standard QED Langrangian at large scales, with the "bare" coupling constant understood as a physical electron charge $\frac{e}{4\pi} \approx \frac{1}{137}$. In this sense our approach of integrating from large scales to small scales is opposite to that used in standard RG calculations [3], where the bare charge is formally located at infinitely small scales. The physical results at any finite observation scale of course should not depend on the direction at which we sum the fluctuations of different scales.

The remainder of this paper is organized as follows. In Sec. II we summarize the scale-dependent approach to QED, described in the previous paper [1], and present the results of one-loop calculations performed in Euclidean $\mathbb{R}^4$ space, with two different wavelets, viz. the first and the second derivatives of the Gaussian. Sec. III accounts for the role of gauge invariance and corresponding Ward-Takahashi identities, which stem from this invariance. We have shown by direct calculation that in the theory with local gauge invariance, $\psi(x) \to e^{-ieA(x)}\psi(x)$, defined for point-depended fields, the Ward identity $\partial_\mu \Pi_{\mu \nu} = 0$ is violated for any finite scale $A > 0$. In Conclusions we summarize the reasons for violation of a locally defined gauge invariance by finite-scale wavelet calculations, and propose to substitute it by the scale-dependent gauge invariance, which have been already proposed by different authors [2, 4].

II. WAVELET-BASED REGULARIZATION IN QUANTUM ELECTRODYNAMICS

Quantum electrodynamics (QED) was the first quantum field theory model to face the problem of deriving finite observable quantities – physical charge and physical mass of the electron – from formally divergent Feynman integrals. Formal solution of this problem have been found in terms of renormalization group (RG) formalism [5, 6], which is physically related to the assumption of self-similarity of underlying physical processes [7]. The renormalization procedure consists of two steps. The first step is the regularization – formal subtraction of the divergent parts of Feynman integrals. The second step is the multiplicative renormalization of the fields and the model parameters, so that the theory of new (renormalized) fields becomes finite. Different technical means of regularization have been proposed, see, e.g. [8, 9]. Most of them are essentially based on subtracting infinities from the Green functions defined in a space of square-integrable functions of either Minkowski or Euclidean coordinate.

However, there is an alternative point of view on the divergences in quantum field theory [10]. An attempt to measure any physical field sharp at a point $x$, with an infinite resolution $a \to 0$, inevitably demands an infinite energy injection with a momentum of order $\frac{1}{a}$, which would certainly destroy the system to be measured. This makes the point-wise definition of fields physically meaningless. So, phenomenologically meaningful description
of physical fields should incorporate both the position 
(x) and the resolution (a).

Technical way to the construction of quantum field theory models on the fields \( \psi_n(x) \) that depend on both the coordinate and the scale (resolution) from very beginning is provided by continuous wavelet transform [1, 10]. The scale-dependent Green functions \( \langle \psi_n(x_1) \ldots \psi_n(x_n) \rangle \) are finite by construction.

The simplest way to construct a field theory for the scale-dependent fields \( \psi_n(x) \) is to express the fields \( \psi(x) \in L^2(\mathbb{R}^d) \) in terms of their wavelet transform

\[
\psi(x) = \frac{1}{C_\chi} \int_{\mathbb{R}^d} \frac{1}{a^n} \chi \left( \frac{x-b}{a} \right) \psi_a(b) \frac{d^d b}{a} \tag{1}
\]

in the original model built of the fields \( \psi(x) \). The coefficients

\[
\psi_a(b) := \int_{\mathbb{R}^d} \frac{1}{a^n} \chi \left( \frac{x-b}{a} \right) \psi(x) d^d x \tag{2}
\]

are known as wavelet coefficients of \( \psi \) with respect to the mother wavelet \( \chi \). In fact, the transform (1) is a particular case of the partition of unity with respect to square-integrable representation \( U(g), g \in G \) of a Lie group \( G \):

\[
\int_G U(g)|\chi\rangle d\mu_L(g) \langle \psi|U^\dagger(g),
\]

for the case of \( G \) being the affine group \( G : x' = ax + b, a \in \mathbb{R}_+, b, x \in \mathbb{R}^d \) [11]. Here we have simplified the matter assuming the basic wavelet \( \chi \) to be isotropic, and exclude \( SO(d) \) rotations from the left-invariant measure \( d\mu_L(g) \) on the Lie group \( G \).

For an isotropic wavelet \( \chi \) the sufficient condition to ensure that (1) is identity, is a finite normalization of the basic wavelet \( \chi \) with respect to the group of scale transformations:

\[ C_\chi = \int_0^\infty |\tilde{\chi}(ak)|^2 \frac{da}{a} < \infty. \tag{3} \]

Tilde means the Fourier transform: \( \tilde{\chi}(k) = \int e^{ikx} \chi(x) dx \).

More details on continuous wavelet transform can be found in many monographs, e.g. in [12, 13].

In common quantum field theory models, say in \( \phi^4 \) model, the field function \( \phi(x) \) is a scalar product of the state vector of the field \( |\phi\rangle \), and a state vector which corresponds to localization at the point \( x \): \( \phi(x) := \langle x|\phi\rangle \).

Similarly, in wavelet-based theory

\[
\psi_a(x) = \langle x, a; \chi|\psi\rangle,
\]

where the l.h.s of the scalar product corresponds to the settings of measurement, which can be potentially performed on the field \( \psi \) by a device described by the aperture function \( \chi \) – this is an interpretation borrowed from optics [14]. The reason for the introduction of the parameters of observation \( (\chi, a) \) into the definition of fields is a potential benefit of getting a field theory finite by construction.

Why should we use something instead of the standard basis of plane waves? The basis of plane waves is the simplest basis for analytical calculations in QED, and is phenomenologically adequate to the registration of particles far from reaction domain. However, it is not an ultimate one. For instance, studying an atom in QED microcavity in case of the energy of interlevel transitions being comparable to the inverse microcavity size, we can (at least in principle) use some other basic functions, which fit geometry of the problem, to estimate the vacuum energy effects of the shape and the size of microcavity. In this sense, the mother wavelet \( \chi \) may be referred too as an aperture function. The plane waves do not suit for that: they are based on translational invariance and do not respect localization.

Unfortunately, it still remains practically unfeasible to use a real aperture function of a physical device in analytical calculations. For this reason we have to use some simple localized functions, satisfying the admissibility condition (3), as a mother wavelet in our calculations. The use of (discrete) wavelet transform in gauge theories have been first proposed in the context of QCD [15], but have not succeed for a number of reasons. First, wavelet transform is a linear integral transform. Hence, it respects the linearity of the gradient transform of gauge fields in the Abelian gauge theory, but does not behave so for non-Abelian (i.e., nonlinear) gauge theories. Second, the linearity of wavelet transform imposes a question of whether we can respect the local gauge invariance of the matter fields: \( \psi(x) \to e^{-i\alpha(x)}\psi(x) \). This question is partially discussed in [2]. Third, the introduction of the scale argument into the definition of quantum fields imposes two types of causality conditions: the standard (signal) causality, which provides the time-ordering in Minkowski space, and the causality between the small and the large scales (the part – the whole relations) [16–18]. Of course this does not preclude either to use discrete wavelet transform with the summation over a discrete set of scales [19, 20] or to combine wavelet transform with light-cone variables, which seems better from the standpoint of causality [21, 22].

We skip these difficult questions now (but keep them for future research), and will concentrate on the Euclidean model, where the scale parameter, considered in Euclidean space, is merely the best attainable resolution. In this way we assume that ”physical” fields are sums of all scale components up to the best resolution \( A \):

\[
\psi^{(A)}(x) = \frac{1}{C_\chi} \int_{a \geq A} \chi \left( \frac{x-b}{a} \right) \psi_a(b) \frac{d^d b}{a}. \tag{4}
\]

In this sense wavelet-based regularization in quantum field theory is similar to the momentum cutoff \( A \), but has an advantage of respecting translation invariance and the momentum conservation of each vertex of Feynman diagrams.
We start with the (Euclidean) QED Lagrangian

\[
L_E = \bar{\psi} (x) (i \gamma \cdot \partial + m) \psi (x) + \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{1}{2 \alpha} (\partial_\mu A_\mu)^2, \tag{5}
\]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu + i e A_\mu, \)

with the Euclidean gamma matrices obeying the anti-commutation relation

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2 \delta_{\mu \nu} \tag{6}
\]

in \( d = 4 \) dimensions. Slashed vectors mean the convolution with the Dirac gamma matrices: \( \not\! x \equiv \gamma_\mu D_\mu \).

The generating functional of the quantum field theory model

\[
Z_E[J, \eta, \bar{\eta}] = \int \mathcal{D} A D \bar{\psi} D \psi \exp \left[ - \int L_E d^4 x - \int (J_\mu (x) A_\mu (x) + \bar{\psi} (x) \eta (x) + i \bar{\eta} (x) \psi (x)) d^4 x \right] \tag{7}
\]

can be made into the generating functional for the scale-dependent fields \((A_\mu (x), \bar{\psi}_a (x), \psi_a (x))\) Green functions by the expression of the original fields in terms of (1). This gives:

\[
Z_W [J_a, \eta_a, \bar{\eta}_a] = \int \mathcal{D} A_a D \bar{\psi}_a D \psi_a \exp \left[ - S_W[A_a, \bar{\psi}_a, \psi_a] - \int J_{\mu,a} (x) A_{\mu,a} (x) \frac{d^4 x d a}{C a} \right] + 1 \tag{8}
\]

where the "action functional" \( S_W [A_a, \bar{\psi}_a, \psi_a] \) is a non-local functional obtained by substitution of (1) into Euclidean action \( S_E = \int L_E d^4 x, \) see [1] for details.

This substitution takes the most simple form in Fourier representation, where the convolutions become products. In Fourier space the inverse wavelet transform (1) for any field \( \psi \) becomes:

\[
\psi (x) = \frac{1}{C} \int_0^\infty \frac{d a}{a} \int \frac{d^4 k}{(2 \pi)^d} e^{-i k x} \bar{\chi} (a k) \bar{\psi}_a (k), \tag{9}
\]

where

\[
\bar{\psi}_a (k) = \bar{\chi} (a k) \bar{\psi} (k) \tag{10}
\]

is wavelet image of the field \( \psi \) written in Fourier space.

The relations (9,10) provide a set of simple rules for building Feynman diagrams for scale-dependent fields [10]:

- each field \( \bar{\psi} (k) \) will be substituted by the scale component \( \bar{\psi}_a (k) \to \bar{\psi}_a (k) = \bar{\chi} (a k) \bar{\psi} (k). \)

- each integration in momentum variable is accompanied by corresponding scale integration:

\[
\frac{d^d k}{(2 \pi)^d} \to \frac{d^d k}{(2 \pi)^d} \frac{d a}{a} C \chi
\]

- each interaction vertex is substituted by its wavelet transform: for the \( N \)-th power local interaction vertex this gives multiplication by factor \( \prod_{i=1}^N \bar{\chi} (a_i k_i). \)

This means we have changed the coordinates \( x \) [or \( p \)] on the translation group to the coordinates \((x, a)\) [or \((p, a)\)] on the affine group and we go on with the integration over all scale arguments in infinite limits would certainly drive us back to the common divergent theory.

Here is a point to make some physical assumptions. If we admit, that our hypothetical equipment has a best resolution scale \( A, \) which corresponds to the minimal of all scales of all external lines of a Feynman diagram of a process we are going to measure, – then the integration over all scale arguments in infinite limits would certainly drive us back to the common divergent theory.

In Euclidean QED we have the following elements of Feynman diagrams:

- propagator of the spin-half fermion:

\[
\begin{array}{c}
\begin{array}{c}
\mu
\end{array}
\end{array} = \bar{\chi} (ap) \frac{i (p - m)}{p^2 + m^2} \bar{\chi} (-ap),
\]

- photon propagator (taken in Feynman’s gauge):

\[
\begin{array}{c}
\begin{array}{c}
\mu
\end{array}
\end{array} = \bar{\chi} (ap) \delta_{\alpha \mu} \bar{\chi} (-ap),
\]

- fermion-photon vertex:

\[
\begin{array}{c}
\begin{array}{c}
\mu
\end{array}
\end{array} = -i e e_\mu \prod_{i=1}^3 \bar{\chi} (a_i p_i).
\]

Since each internal line in a Feynman diagram is connected to two vertexes, from the left and from the right, the integration in left and right scale arguments, according the above imposed scale limitation rule, results in

\[
\int_A^\infty \frac{|\bar{\chi} (a_k R_k)|^2}{a_k c_\chi} \cdot da_k \int_A^\infty \frac{|\bar{\chi} (a_k R_k)|^2}{a_k c_\chi} \cdot da_R = f^2 (A), \tag{11}
\]

where

\[
f (x) = \frac{1}{C_\chi} \int_x^\infty \frac{\bar{\chi} (a)}{a} da.
\]
is the wavelet cutoff function, which satisfy an evident condition \( f(0) = 1 \). If we are not interested in how the fields of different scales \( \psi_a(x) \) and \( \psi_{a'}(x') \) talk to each other, but are interested only in the total effect of all fluctuations of scales larger than \( a \), we can merely insert the wavelet cutoff factors in all internal lines of Feynman diagrams.

In our calculations we use different derivatives of the Gaussian as mother wavelets. The admissibility condition (3) is rather loose: practically any well-localized function with the Fourier image vanishing at zero momentum \( \tilde{\chi}(0) = 0 \) obey this requirement. As for the Gaussian functions

\[
\chi_n(x) = (-1)^n e^{-\frac{x^2}{2}}, \quad n > 0,
\]

(13)

they are easy to integrate in Feynman diagrams. The graphs of first two wavelets of the (13) family,

\[
\chi_1(x) = -xe^{-\frac{x^2}{2}}, \quad \chi_2(x) = (1-x^2)e^{-\frac{x^2}{2}},
\]

are shown in Fig. 1. The Fourier images of the (13) family wavelets are

\[
\tilde{\chi}_n(k) = -(ik)^n e^{-\frac{k^2}{2}}.
\]

Respectively, the normalization constants and the wavelet cutoff functions are:

\[
C_{\chi_n} = \frac{\Gamma(n)}{2}, \quad f_{\chi_n}(x) = \frac{\Gamma(n,x^2)}{\Gamma(n)}.
\]

where \( \Gamma(\cdot) \) is the Euler gamma function, and \( \Gamma(\cdot, \cdot) \) is the incomplete gamma function. For the first two wavelets this gives the wavelet cutoff functions:

\[
f_{\chi_1}(x) = e^{-x^2}, \quad f_{\chi_2}(x) = (1 + x^2)e^{-x^2}.
\]

We will now proceed to the calculation of one-loop diagrams in wavelet-based Euclidean QED. These are the fermion self-energy diagram and the vacuum polarization diagram, see Figs. 3, 2.

a. Vacuum polarization diagram. First we calculate the vacuum polarization diagram shown in Fig. 2. For convenience of calculation we symmetrize the loop momenta. The external lines of the diagram are labelled by scale arguments \( a \) and \( a' \). So, according to the assumptions made above, the integration in scale arguments in the fermion loop is limited by the minimal scale \( A = \min(a, a') \). In contrast to the paper [1], intended for calculation of the Green functions of scale-dependent fields \( \langle \psi_{a_1}(x_1) \ldots \psi_{a_n}(x_n) \rangle \), here we do not specify any propagators on external lines, so that the results can be taken as usual diagrams regularized to a scale \( A \). That is why the wavelet factors are omitted in the definitions of 1PI diagrams. Doing so, we get the expression for vacuum polarization diagram:

\[
\Pi^{(A)}_{\mu\nu}(p) = -e^2 \int \text{Sp}(\gamma_{\mu}(\not{q} + \not{p} - m)\gamma_{\nu}(\not{q} - \not{p} - m)) \frac{F_A(p,q)}{(2\pi)^4} d^4q,
\]

(16)

and for the electron self-energy diagram:

\[
\Sigma^{(A)}(p) = -ie^2 \int \gamma_{\mu} \frac{\not{q} - \not{\gamma} - m}{(\not{q} - \not{\gamma})^2 + m^2} \gamma_{\nu} F_A(p,q) \frac{d^4q}{(2\pi)^4},
\]

(17)

FIG. 2. Vacuum polarization diagram in scale-dependent QED
the loop momenta

\[ F_A(p, q) = f^2 \left( A \left( \frac{p}{2} - q \right) \right) f^2 \left( A \left( \frac{p}{2} + q \right) \right). \] (18)

Let us start the calculations with \( \chi_1 \) wavelet. In this case (Eq.(15))

\[ F_A(p, q) = e^{-A^2p^2 - 4A^2q^2}, \]

and we have the integral

\[ \Pi^{(A, \chi_1)}_{\mu\nu} = -\frac{e^2 p^2}{32\pi^3} e^{-A^2p^2} \int_0^\infty dy y e^{-4A^2p^2 y^2} \int_0^\pi d\theta \sin^2 \theta \times \]

\[ \times \left[ \frac{2y^2 - \frac{1}{2} y^2 p^2 + \delta_{\mu\nu} \left( \frac{1}{2} - y^2 - \frac{m^2}{p^2} \right)}{y^2 + \frac{m^2}{p^2} + \cos \theta} \right] \left[ \frac{1+y^2+\frac{m^2}{p^2} - \cos \theta}{y^2 + \frac{m^2}{p^2} - \cos \theta} \right], \] (19)

where we have introduced a dimensionless vector in the direction of loop momentum: \( q = |p|y \), with \( \theta \) being the angle between \( p \) and \( q \). The integral (19) can be evaluated analytically in the limit of high momentum \( p^2 \gg 4m^2 \). This gives:

\[ \Pi_{\mu\nu} = I_D \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + I_L \frac{p_\mu p_\nu}{p^2}, \] (20)

\[ I_D = \frac{e^2 p^2}{48\pi^2 s^3} \left[ (4s^2 - 2s - 1)e^{-2s} + (1 + s - 4s^2)e^{-s} \right. \]

\[ + 4s^3 (E_1(s) - 2E_1(2s)) \] \left[ \right],

\[ I_L = \frac{e^2 p^2}{16\pi^2 s^3} e^{-2s} ((s - 1)e^s + 1) \],

where \( s \equiv A^2 p^2 \) is dimensionless scale argument, \( E_1(z) = \int_1^\infty \frac{e^{-x}}{x} dx \) is exponential integral of the first type. The details of calculations are presented in Appendix. As we can see from (A10,A11), the singular at \( s \to 0 \) parts of these equations, proportional to exponential integrals, cancel each other, which would provide the transversality in case the exponential parts would do the same. This does not happen: the longitudinal part \( I_L \) does not vanish. In this sense the wavelet observation scale \( A \) plays the role of inverse regularising mass in Pauli-Villars regularization [24]. In contrast to dimensional regularization, where \( q_0q_\nu \) and \( 2q^2 \) terms cancel each other in the sense of leading divergences, this does not happen in the theory with a finite scale \( A \) and local gauge invariance. There may be different reasons for that. First, the finite terms, neglected by dimensional regularization turn into the scale-dependent contributions, which can’t be neglected in our case. Second, the scale \( A \) is a scale in Euclidean space and we cannot match it exactly to what is measured in Minkowski space. Third, changing the coordinates from \( x \) to \( (x, a) \) we need to pay an extra attention on what is gauge invariance in scale-dependent settings [2] – the consideration above ignored this completely by making standard assumption of local gauge invariance.

b. Fermion self-energy diagram. The loop integral of the fermion self-energy diagram, shown in Fig. 3, has the form:

\[ \Sigma^{(A)}(p) = -ie^2 \int \frac{d^4q}{(2\pi)^4} F_A(p, q) \gamma_\mu \left[ \frac{q}{2} - \frac{q - m}{2} \right] \gamma_\mu, \] (21)

as in the previous example \( A \) is the minimal scale of all external lines \( A = \min(a, a') \). We will calculate the diagram (21) with the wavelet-cutoff functions \( F_A(p, q) \) for both \( \chi_1 \) and \( \chi_2 \) wavelets (15).

Using the identities for Euclidean gamma matrices, and assuming the high energy limit \( p^2 \gg 4m^2 \) for simplicity of calculations, we rewrite (21) in the form

\[ \Sigma^{(A)}(p) = -ie^2 \int \frac{d^4y}{(2\pi)^4} \frac{(p + 4m - 2|p|y)}{\left[ y^2 + \frac{1}{4} - y \cos \theta - \frac{m^2}{p^2} \right] \left[ y^2 + \frac{1}{4} + y \cos \theta \right]} F_A(p, |p|y) \]

(22)

cutoff function of the type \( \chi_1 \), (Eq.15), we can easily see that

\[ \Sigma^{(A)}_{\chi_1}(p) = -\frac{ie^2 p e^{-s}}{16\pi^4} \int dy y e^{-4sy^2} \sin^2 \theta d\theta = -\frac{ie^2 e^{-s}}{4\pi^3} \delta \]
where the integral $J$ is given by (A5). Thus we get:

$$
\Sigma^{(A)}_{\chi_1}(p) = -\frac{ie^2}{16\pi^2} \left[ 2E_1(2s) - E_1(s) - \frac{e^{-2s}}{s} + \frac{e^{-s}}{s} \right] p. \tag{23}
$$

\[\text{\textit{c. Fermion-photon vertex.}}\text{ The one-loop contribution to the fermion-photon vertex is shown in Fig. 4. Since the bare fermion-photon vertex is } -ie\gamma_\rho, \text{ we similarly normalize the vertex function:}

$$
-ie\Gamma^{(A)}_\rho (p_1, p_2, p_3) = -ie\gamma_\rho + (-ie)^3 \int \frac{d^4f}{(2\pi)^4} \frac{\gamma_\mu ((f_2 + m)\gamma_\rho (f_1 + m)\gamma_\mu)}{(l_2^2 + m^2)(l_2^2 + m^2)^2} f^2(Al_1)f^2(Al_2)f^2(Al_3), \tag{24}
$$

where $f(x)$ is the wavelet-cutoff function given by (12). To get rid of the angle dependence in the wavelet cutoff factors we have symmetrized the loop momenta:

$$
l_1 = f + \frac{p_3 - p_1}{3}, \quad l_2 = f + \frac{p_1 - p_3}{3}, \quad l_3 = f + \frac{p_2 - p_1}{3}.
$$

To calculate the one loop contribution to the vertex let us consider the decay of a photon with momentum $p_3 = \rho$ into the fermion-antifermion pair. This corresponds to the loop momenta

$$
l_1 = f + \frac{p}{2}, \quad l_2 = f - \frac{p}{2}, \quad l_3 = f. \tag{25}
$$

Considering the high energy case $p^2 \gg 4m^2$ we can omit the mass terms. This gives

$$
A_\rho = \gamma_\mu \left( f - \frac{p}{2} \right) \gamma_\rho \left( f + \frac{p}{2} \right) \gamma_\mu
= 2 \left( f + \frac{p}{2} \right) \gamma_\rho \left( f - \frac{p}{2} \right)
$$

The one-loop contribution to the vertex then takes the form

$$
A_\rho \left( -\frac{p}{2}, \frac{p}{2}, p \right) = -e^2 \int \frac{d^4f}{(2\pi)^4} \frac{A_\rho F_A(p, f)}{(f - \frac{p}{2})^2 (f + \frac{p}{2})^2 f^2}. \tag{26}
$$

The vertex wavelet cutoff factor is the product of 3 wavelet cutoff functions

$$
F_A(p, f) = f^2(A(f - \frac{p}{2})f^2(A(f + \frac{p}{2}))f^2(A)). \tag{27}
$$

For the case of $\chi_1$ wavelet, see Eq. 15, we have

$$
F_A(p, f) = \exp(-A^2p^2 - 6A^2 f^2).
$$

The calculation of the integral (26) with this cutoff function, presented in Appendix, gives:

$$
\Lambda_\rho \left( -\frac{p}{2}, \frac{p}{2}, p \right) = e^2 \gamma_\rho \left[ \frac{e^2 E_1(3s)}{3\pi^2} - \frac{e^2 E_1(\frac{3s}{2})}{2} \right]
- \frac{e^{-2s}}{8s} + \frac{e^{-s}}{12s} - \frac{5e^{-s}E_1(\frac{3s}{2})}{16} + \frac{e^{-s}}{36s^2} - \frac{e^{-2s}}{36s^2} \tag{28}
$$

In terms of the fine structure constant $\alpha(s) = e^2(s) / 4\pi$, the one loop contribution to the QED vertex (28) can be cast in the form

$$
\alpha(s) = \alpha \left[ 1 + \frac{4}{3\pi} \alpha R(s) \right]^2, \tag{29}
$$

$$
R(s) = e^2 E_1(3s) - e^2 E_1(\frac{3s}{2}) + \frac{e^{-2s}}{8s} + \frac{5e^{-s}E_1(\frac{3s}{2})}{16} + \frac{e^{-s}}{36s^2} - \frac{e^{-2s}}{36s^2} \tag{30}
$$

The graph of the running coupling constant $\alpha(s)$ calculated according to the formula (29) is shown in Fig. 5 below. Decomposing Eq.(28) is a series for small scales ($s \to 0$)

$$
\Lambda_\rho \approx \gamma_\rho e^2 \left( \frac{3s}{2} - \frac{9s}{16} \gamma + \frac{3}{8} \ln s - \frac{3}{8} \ln 3 + \frac{1}{32} \ln 3 - \ln 3 \right) + O(s),
$$

we get the logarithmic derivative

$$
\frac{\partial e(s)}{\partial \ln s} = -\frac{e^3}{16\pi^2}. \tag{31}
$$

The calculations performed with $\chi_2$ wavelet, presented in Appendix, give similar results.
where the gauge-fixing terms compensate each other. This implies

\[ \delta A_\mu = \partial_\mu \Lambda(x), \quad \delta \psi = -ie\Lambda(x)\psi, \quad \delta \bar{\psi} = ie\Lambda(x)\bar{\psi}. \]

Since the Lagrangian is gauge-invariant by construction, to make the generation functional \( Z[J, \eta, \bar{\eta}] \) gauge invariant, we need to ensure that variations of the source terms and the gauge-fixing terms compensate each other. This implies

\[
\int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \left[ e - \int d^4x (L_E + J_\mu A_\mu + i\bar{\psi} + \psi\eta) \right] \delta \Lambda = 0,
\]

where

\[
\delta \Lambda \equiv \int d^4x \left[ -\frac{1}{\alpha} \partial^2(\partial_\mu A_\mu) + \partial_\mu J_\mu + e(\bar{\psi} \eta - \bar{\eta} \psi) \right] \Lambda(x).
\]

(32)

Considering an infinitesimal transform we can approximate \( e^{\delta \Lambda} \approx 1 + \delta \Lambda \), and hence, in view of arbitrariness of \( \Lambda(x) \), the equality \( \langle \delta \Lambda \rangle = 0 \) can be written in a form of variational equation:

\[
\left[ -\frac{1}{\alpha} \partial^2(\partial_\mu A_\mu) + \partial_\mu J_\mu + e(\bar{\psi} \eta - \bar{\eta} \psi) \right] Z[J, \bar{\eta}, \eta] = 0,
\]

(33)

where \( \psi(x) = i \frac{\delta}{\delta \bar{\eta}(x)}, \bar{\psi}(x) = i \frac{\delta}{\delta \eta(x)} \), \( A_\mu = -\frac{\delta}{\delta J_\mu(x)} \). The Ward-Takahashi identities can be obtained by taking appropriate number of functional derivatives of the equation (33). This is usually done by changing from generating functional \( Z \) to the generating functional for the connected Green’s functions:

\[ Z[J, \bar{\eta}, \eta] = e^{-W[J, \bar{\eta}, \eta]}, \]

and then applying the Legendre transform to get an effective action functional:

\[ \Gamma[A, \psi, \bar{\psi}] = W[J, \bar{\eta}, \eta] - JA - i\bar{\eta} \psi - i\bar{\psi} \eta. \]

The latter enables to work with proper vertices and write the Ward-Takahashi identities generating equation in the form:

\[
\frac{\partial^2}{\alpha} \partial_\mu A_\mu + \partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + ie \left( \psi \frac{\delta \Gamma}{\delta \psi} - \bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} \right) = 0.
\]

(34)

The first derivative of (34) with respect to \( A_\mu \) gives the Ward identity [25], that demands the transversality of the vacuum polarization diagram:

\[ \partial_\mu \Pi_{\mu\nu} = 0. \]

(35)

The integrals in vacuum polarization diagram will satisfy the requirement (35) only in case they are invariant under the shift of loop momenta. This is not always true when a regularization procedure is applied. In QED, the condition (35) is observed by dimensional regularization, but not by momentum cutoff. That is why dimensional regularization has become the most common regularization method in QFT models [26].

To fulfill the Ward-Takahashi identities, a regulator is usually assumed to satisfy the requirement of the type [27, 28]:

\[
\int \frac{d^4l}{(2\pi)^4} \frac{l_{\mu}l_{\nu}}{(l^2 + \Delta)^2} = \frac{\delta_{\mu\nu}}{2} \int \frac{d^4l}{(2\pi)^4} \frac{l_{\mu}l_{\nu}}{l^2 + \Delta}.
\]

(36)

The integration over the Feynman \( x \)-parameter used to get rid of angle integrations is not shown here). This is definitely true for dimensional regularization, but is not true for momentum cutoff and is not true for wavelet regularization we consider in this paper. In our case of finite theory we cannot use a relation like (36) as a "rule", but have to evaluate everything explicitly.

Regardless the undoubted merits of dimensional regularization, it deals only with the main singular parts of Feynman diagrams, and cannot tackle the amplitudes at finite scales. In this respect the finite cutoff regularization and the wavelet regularization have the potential...
advantage of describing what happens at a finite observation scale [10, 29]. The goal of the wavelet cutoff technique, provided by continuous wavelet transform, is to get a capability of calculations at finite observation scale. The respect to the gauge invariance can be also paid in a cutoff-momentum regularization scheme by assuming the gauge transformations to act below the momentum cutoff $\Lambda$ [4]:

$$a_\mu(k) \rightarrow a_\mu(k) - i k_\mu \lambda(k), \quad \text{with} \quad \lambda(|k| > \Lambda) = 0$$  (37)

In the case of continuous wavelet transform regularization there is an alternative – to consider gauge transformations which directly depend on the scale argument $a$

$$\psi_a(x) \rightarrow e^{-\alpha_a(x)} \psi_a(x)$$  (38)

Doing so we get a theory that is gauge invariant separately at each given scale [2].

We do not consider scale-dependent modifications of gauge invariance in this paper, leaving this subject for future studies. Instead, the above considered wavelet cutoff factors of Gaussian type, are rather similar to already proposed exponential modifications of the momentum cutoff [30], based on the Schwinger proper time method [31]. Using wavelet regularization, in the case of small scales $s < 1$, when in the final limit of $s \rightarrow 0$ the integration over all scales $\int_0^\infty \frac{da}{a}$ would definitely restore the symmetries of the original theory, we can use the approximation formulae that follow from Ward identities of the full (non-regularized) theory.

Technically the Ward identities follow from the observation that a proper vertex of the fermion-photon interaction can be associated with the fermion self-energy diagram by inserting a photon line in the internal fermion line of the latter. Ward noticed, that the bare inverse electron propagator

$$S_{(e)}^{-1}(p) = i(p + m),$$

the derivation with respect to the momentum $p_\mu$ gives the fermion-photon interaction vertex

$$\frac{\partial S_{(e)}^{-1}(p)}{\partial p_\mu} = i\gamma_\mu,$$

and proved the same for the inverse full propagator

$$\frac{\partial G_{(e)}^{-1}(p)}{\partial p_\mu} = i\Gamma_\mu,$$  (39)

where $-ie\Gamma_\mu$ and $-ie\gamma_\mu$ are the bare and the full vertex of the fermion-photon interaction. (Here we use the Euclidean notation, in contrast to the original paper of Ward [25], written in Minkowski space.)

More generally, the Ward-Takahashi [32] identity in spinor electrodynamics, written in integral form, relates the vertex function to the difference of fermion propagators:

$$q_\mu \Gamma_\mu(p, -p - q, q) = G^{-1}(p + q) - G^{-1}(p).$$  (40)

Here $G(p)$ is the full fermion propagator. The identity (40) is a helpful constraint which ensures the gauge invariance of the renormalized QED in any order of perturbation theory [25, 32]. The constraint (40) makes the perturbation expansion gauge invariant at the presence of the gauge fixing terms in the QED generating functional.

The most straightforward application of the Ward’s finding is the calculation of the full fermion-photon vertex in the limit of zero photon momentum. In this case

$$\Gamma_\mu(p, -p, 0) = \gamma_\mu + \Lambda_\mu(p, -p, 0).$$  (41)

As it follows from the Dyson equation, the inverse full propagator is equal to

$$G^{-1}(p) = S^{-1}(p) - \Sigma(p),$$  (42)

where $\Sigma(p)$ is the electron self-energy. Taking the derivatives of both sides of (42) by $\frac{\partial}{\partial p_\mu}$ we get

$$\Lambda_\mu(p, -p, 0) = i \frac{\partial \Sigma(p)}{\partial p_\mu}.$$  (43)

The formula (43) can now be applied to our wavelet-regularized calculations of one-loop diagrams. Since we are interested only in the contributions to the vertex proportional to $-ie\gamma_\mu$, it is sufficient to differentiate only the last term in (23): $\frac{\partial p}{\partial p_\mu} = \gamma_\mu$. This gives the one-loop equation for the fermion-photon vertex regularized at scale $A$:

$$-ie \Gamma_{\chi,\mu}^{(A)}(p) = -ie\gamma_\mu \left[ 1 + \frac{e^2}{16\pi^2} R_1(s) \right] + \ldots,$$

$$R_1^{\chi}(s) \equiv 2Ei_1(2s) - Ei_1(s) - \frac{e^{-2s}}{s} + \frac{e^{-s}}{s}.$$  (44)

Since all dependence on scale in our model is contained in function $R_1(s)$, we can now calculate the dependence of scale of the effective charge. The wavelet regularization scheme includes the integration over all scale components from observation scale $A$ to infinity. The equation (44) thus gives the value of the effective charge $e_{eff}(s)$, i.e., the effective charge measured at scale $A$, in terms of physical electron charge measured at infinity $e_0 = e(\infty)$. It is convenient to rewrite it in terms of fine structure constant

$$\alpha(s) = \frac{e^2(s)}{4\pi},$$

the physical value of which $\alpha \approx 1/137.036$ [33, 34]. Then the scale dependence of the effective charge, given in one-loop approximation by the equation (44), is

$$\alpha(s) = \alpha \left( 1 + \frac{\alpha}{4\pi} R_1(s) \right)^2.$$  (45)
Since we use the coordinate scale $a$ as the scale argument the sign will be opposite to that in dimensional regularization $s \frac{R}{\partial} \to -\mu^2 \frac{\partial}{\partial \mu^2}$. The scaling equation for the effective charge – we do not call it RG-equation, since there is no field renormalization in our model – takes the form

$$s \frac{\partial e_{\text{eff}}}{\partial s} = \frac{e_{\text{eff}}}{16\pi^2} \frac{\partial R_1(s)}{\partial s}, \quad (46)$$

$$s \frac{\partial R_1(s)}{\partial s} = \frac{e^{-s}}{s} \left(e^{-s} - 1\right).$$

The scaling equation (46) can be integrated in a usual RG-like form

$$\frac{de_{\text{eff}}}{e_{\text{eff}}} = \frac{ds}{16\pi^2} \frac{e^{-s}}{s} \left(e^{-s} - 1\right) \quad (47)$$

The solution of the equation (47) is given by

$$e_{\text{eff}}^2(s) = \frac{e_0^2}{1 - \frac{1}{2\pi} R_1(s)},$$

which can be casted in terms of the fine structure constant:

$$\alpha(s) = \frac{\alpha}{1 - \frac{1}{2\pi} R_1(s)}. \quad (48)$$

Similar results can be obtained using other wavelets. In this way using $\chi_2$ wavelet cutoff, see Eq.(B3) in Appendix, we get:

$$R_{\chi^2}^2(p) = 2E_1(2s) - E_1(s) - \frac{e^{-2s}(s + 5)}{2s} + \frac{e^{-s}(s^3 + 18s^2 + 134s + 640)}{256s}. \quad (49)$$

The scale dependences of $\alpha(s)$, calculated for both cases of (15) wavelet cutoff functions, are shown in Fig. 6. Since the value of $\alpha$ is very small, the value of $\alpha(s)$ given by Eq. (48) is practically indistinguishable from that given by (44). The Landau singularity in Eq.(48) matters only when $s \sim e^{-\frac{\alpha^2}{2}}$, so that

$$\frac{2\pi}{\alpha} \approx R_1(s) = 1 - \gamma - 2 \ln 2 - \ln s + \frac{3}{2} s + O(s^2), \quad (50)$$

with $\gamma \approx 0.5772$ being the Euler-Mascheroni constant.

**IV. CONCLUSIONS**

The basic symmetries of quantum electrodynamics are the relativistic invariance and the gauge invariance. In standard approach to QED, which assumes the quantum fields to be local square-integrable functions, the calculations of observable quantities may violate both the Lorentz symmetry and the gauge symmetry due to the formal infinities of the calculated Green functions. Different regularization schemes have been used to get rid of divergences. The momentum cutoff regularization was historically the first, being at the same time most physically meaningful – since the cutoff momentum $\Lambda$ may be understood as a maximal momentum available for a given experiment. The dimensional regularization has become the most common, utmost a standard way of regularization, because it does not violate the gauge symmetry; although it is not ubiquitous being incapable of treating supersymmetric theories [35].

Wavelet regularization is different from all above regularizations. It changes the space of functions from the space of square-integrable functions to the space of functions $\psi_\alpha(x)$, depending on both the coordinate $x$, and the resolution $\alpha$. The former is dynamical – it enters the dynamical equations, the latter describes only the settings of observations and does not enter any dynamical equations. In this sense we extend the description of observed physical fields by incorporating the conditions of observation $(\alpha, \chi)$ in the field definition. A physical field per se is then a collection of all physical fields that can be potentially observed: $\Psi = \{\psi_\alpha(x, \cdot)\}_{a,\chi,...}$, and hence cannot keep the perturbation expansion gauge invariant. Since the scale-dependent fields are defined not in a sharp point $x$, but in a region of typical size $a$, there is no need of infinite momentum injection for measuring such fields, and there are no physical reasons for UV divergences.

The key issue in the theory of scale-dependent fields is the problem of how the physical fields interact to each other. The description of physical interactions is determined by the symmetry of the problem. In this way the symmetry with respect to local $U(1)$ transformations de-
terms electromagnetic interaction, the symmetry with respect to $SU(3)$ gauge transformations determines the strong interaction and so on. In present paper we have followed exactly the same way: electrodynamics is understood as a theory with $U(1)$ gauge group, acting on the space of square-integrable local functions. This definition immediately implies that the physically observed fields $\psi_\alpha(x)$ are merely a projections of square-integrable fields $\psi(x)$ performed with the help of a mother wavelet $\chi$. This is a rather strong restriction: it states that gauge interactions take place in the space of local square-integrable functions, and inverse wavelet transform is used to reconstruct local fields from a set of their projections; the interaction then takes place between reconstructed fields. In this sense wavelet regularization is similar to momentum cutoff regularization, and gives the dependence of physical parameters on the observation scale.

Having performed the calculations presented in this paper, we have found out that wavelet regularization can give a qualitatively adequate description of the QED running coupling constant (in one loop approximation), which increases with the logarithm of inverse observation scale. The advantage of wavelet regularization, if compared to dimensional regularization and other methods, is that it does not need any external tools, such as renormalization, which is always demanded by dimensional regularization to get physically interpretable results. The reason is that the wavelet transform itself is already based on the group of scale transformations, similar to renormalization. That is why, instead of the renormalization group equation we have just a logarithmic derivatives of the effective charge on the dimensionless scale argument $s$. This $s = (Ap)^2$ is similar to normalization scale $1/\mu^2$ in dimensional regularization, but has a physical interpretation in terms of the measurement scale. The crucial difference from standard renormalization group approach is that we face no divergences to get rid of and we have no field renormalization. The latter is due to the fact that by extending the space of fields $\psi(x)$ to the space of scale depended fields $\psi_\alpha(x)$ we have already get the collections of all scales rather than a poor man collection of two scales only. At the same time our calculations show that know results such as Landau pole of the form $1/(-\alpha X)$ also take place in a wavelet theory finite by construction, but in the later case they have more mild form of $1 + \alpha X$, where $\alpha X$ is small and there is no threat of pole.

At the same time, we have to admit that our persistence on keeping the standard definition of gauge invariance in the space of local field does not allow to preserve the transversality of vacuum polarization operator at one loop level: $p_\mu \Pi_{\mu
u}(p) \neq 0$. This has long been known for momentum-cutoff and other regularization schemes, and is quite expected for the wavelet regularization of a locally defined gauge theory: having declared the scale-dependent fields to be the physically observed fields we still insist that gauge interaction acts on local fields. It might be more reasonable to define the interaction directly in the space of scale-dependent fields, as is proposed in [2] in the space of QCD, but this is planned for future research.

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Appendix A: Calculations with $\chi_1$ wavelet

1. Vacuum polarization diagram

Substituting the integration measure $d^4q = 4\pi q^2 dq \sin^2 \theta d\theta$ into integral (16) and dividing both the numerator and the denominator by $p^2 q^2$ we arrive at the equation (19):

$$
\Pi^{(A,\chi_1)}_{\mu\nu}(k) = -\frac{e^2}{\pi^3} \epsilon^{\alpha\beta\gamma\delta} \int_0^\infty \int_0^\pi d\theta \sin^2 \theta \int_0^\infty dy y \epsilon_{\alpha\beta\gamma\delta} \left[ \frac{2p_\mu p_\nu - \frac{1}{2} p_\alpha p_\beta}{p^2 - m^2} + \frac{\epsilon_{\mu\nu}(\frac{1}{2} - y^2 - \frac{m^2}{p^2})}{\beta^2(y) - \cos^2 \theta} \right].
$$

(19)

The angle part of integral (19) can be evaluated explicitly. The corresponding integrals have the form:

$$
I_k[\beta(y)] = \int_0^\pi \frac{\sin^2 \theta \cos^2 \theta}{\beta^2 - \cos^2 \theta} d\theta, \quad I_0[\beta(y)] = \pi (1 - \sqrt{1 - \beta^{-2}}), \quad I_1[\beta(y)] = -\frac{\pi}{2} + \beta^2 I_0[\beta],
$$

(A1)

where in our case the constant $\beta$ depends on $y$. To make the calculations analytically feasible, we simplify the matter by considering the "massless" limit $p^2 \gg 4m^2$. In this limit

$$
\beta(y) = y + \frac{1}{4y}.
$$

(A2)

In this approximation the vacuum polarization integral takes the form

$$
\Pi^{(A,\chi_1)}_{\mu\nu}(k) = -\frac{e^2}{\pi^3} \epsilon^{\alpha\beta\gamma\delta} \int_0^\infty \int_0^\pi d\theta \sin^2 \theta \int_0^\infty dy y \epsilon_{\alpha\beta\gamma\delta} \left[ \frac{2p_\mu p_\nu - \frac{1}{2} p_\alpha p_\beta}{\beta^2(y) - \cos^2 \theta} \right].
$$

There are two basic integrals here: the scalar integrals, which do not contain $y_\mu y_\nu$, and the tensor integrals, which contain this term. The scalar integral has the form:

$$
J \equiv \int_0^\infty dy y e^{-4s^2} I_0[\beta(y)].
$$

(A3)

As it follows from Eqs.(A1,A2):

$$
I_0[\beta(y)] = \begin{cases} \frac{6\pi y^2}{1+4y^4}, & 0 \leq y \leq \frac{1}{2} \\ \frac{2\pi y^2}{2+4y^4}, & y \geq \frac{1}{2} \end{cases}, \quad I_1[\beta(y)] = \begin{cases} \frac{2\pi y^2}{8y^4}, & 0 \leq y \leq \frac{1}{2} \\ 0, & y \geq \frac{1}{2} \end{cases},
$$

(A4)

and hence

$$
J = 8\pi \int_0^{\frac{1}{2}} dy y^3 e^{-4s^2} + 2\pi \int_0^\infty dy y e^{-4s^2} = \frac{\pi}{4} \left[ 2E_1(2s)e^s - E_1(s)e^s - \frac{e^{-s} - 1}{s} \right].
$$

(A5)

The other scalar integral is a derivative of $J$:

$$
J_0^0 = \int_0^\infty dy y^3 I_0[\beta(y)] e^{-4s^2} = \frac{1}{4} \frac{dJ}{ds} = -\frac{\pi}{16} \left[ 2E_1(2s)e^s - E_1(s)e^s - \frac{e^{-s} - 1}{s} + \frac{e^{-s} - 1}{s^2} \right].
$$

The terms of the integral, which contain $y_\mu y_\nu$, and which do not can be evaluated separately. Since our problem has only one preferable direction – the direction of vector $p$, we can evaluate the tensor integral

$$
I_{q\mu\nu}^{(2)} = \int y dy \sin^2 \theta d\theta e^{-4s^2} \frac{y_\mu y_\nu}{(y + \frac{1}{4y})^2 - \cos^2 \theta}.
$$

(A6)
by substituting $y_{\mu}y_{\nu} \to B y^2 \delta_{\mu\nu} + C y^2 \frac{p_{\mu}p_{\nu}}{p^2}$ in the integrand; where $B$ and $C$ are the constants to be determined.

The integral to be evaluated takes the form

$$I_{\mu\nu} = \int y dy \sin^2 \theta d\theta e^{-4sy^2} \frac{B y^2 \delta_{\mu\nu} + C y^2 \frac{p_{\mu}p_{\nu}}{p^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta} \tag{A7}$$

The constants $B$ and $C$ are determined from a system of two linear equation

$$\text{Tr} I^{(2)}_{\mu\nu} = \text{Tr} I_{\mu\nu}, \quad I^{(2)}_{\mu\nu} = I_{\mu\nu} P_{\mu\nu} \quad \Leftrightarrow \quad 4B + C = 1, \quad I^0_q = (B + C) I^0_q,$$

where

$$I^0_q = \int_0^\infty y^3 dy e^{-4sy^2} I_1(\beta y) = \frac{\pi}{32} \frac{e^{-s}}{s^3}[e^s - s - 1]$$

Hence we can determine the constants $B$ and $C$ in terms of $\eta \equiv \frac{I^1_q}{I^0_q}$:

$$B = \frac{1 - \eta}{3}, \quad C = \frac{4\eta - 1}{3}, \quad \eta = \frac{1 + s - e^{-s}}{2s^3 e^{2s}(2Ei_1(2s) - Ei_1(s)) + s(s-1)(e^s-1)},$$

and now can evaluate the vacuum polarization diagram

$$\Pi_{\mu\nu} = -\frac{e^2 p^2}{\pi^3} e^{-s} (I_q + I_P + I_L) \equiv I_D \delta_{\mu\nu} + I_P \frac{p_{\mu}p_{\nu}}{p^2}, \tag{A8}$$

$$I_D = \frac{e^2 p^2}{48\pi^2 s^3} \left[ (4s^2 - 2s - 1)e^{-2s} + (1 + s - 4s^2)e^{-s} + 4s^3(Ei_1(s) - 2Ei_1(2s)) \right], \tag{A10}$$

$$I_P = \frac{e^2 p^2}{48\pi^2 s^3} \left[ (-4s^2 + 2s + 4)e^{-2s} + 2(2s^2 + s - 2)e^{-s} - 4s^3(Ei_1(s) - 2Ei_1(2s)) \right], \tag{A11}$$

$$I_L = I_D + I_P = \frac{e^2 p^2}{16\pi^2 s^3} e^{-2s} ((s-1)e^s + 1).$$

Now we can represent the vacuum polarization diagram in a standard form:

$$\Pi_{\mu\nu} = I_D \left( \delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) + I_L \frac{p_{\mu}p_{\nu}}{p^2}. \tag{A9}$$

2. One-loop corrections to the QED vertex with $\chi_1$ wavelet cutoff

To evaluate the (26) integral we take the tensor structure in the form

$$A_\rho = 2f \gamma_\rho f - \frac{1}{2} \gamma_\rho \not{\! p}$$

omitting the terms linear in $f$ and using the massless. Thus the whole integral in (26) takes the form

$$I_\Delta = \int \frac{d^4 y}{y^2 p^2(2\pi)^4} e^{-4Ay^2} \frac{(2p^2 y^2 \gamma_\rho \not{\! p} - \frac{1}{2} \gamma_\rho \not{\! p})}{(y + \frac{1}{4y})^2 - \cos^2 \theta} = 2\gamma_\alpha \gamma_\rho \gamma_\beta I_{\alpha\beta} - \frac{\gamma_\rho \not{\! p} I_{C}}{2p^2}, \tag{A12}$$

$$I_{\alpha\beta} = \frac{1}{4\pi^3} \int \frac{d\theta \sin^2 \theta dy e^{-4As^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta} \frac{y_{\alpha} y_{\beta}}{y^2}, \quad I_{C} = \frac{1}{4\pi^3} \int \frac{d\theta \sin^2 \theta dy e^{-4As^2} y^2}{(y + \frac{1}{4y})^2 - \cos^2 \theta} \frac{1}{y^2}. \tag{A13}$$
where we have used dimensionless momentum: \( f = |p| y \) and changed to integration in the spherical coordinates.

First we evaluate the scalar integral \( I_C \). Integrating over the angle variable \( \theta \) we get

\[
I_C = \frac{1}{4\pi^2} \int_0^\infty \frac{y dy}{y^2} I_0[\beta(y)] e^{-s A^2 p^2}.
\]

Since \( I_0[\beta(y)] \) is a piece-wise defined function (A4), we get

\[
I_C = \frac{1}{4\pi^2} \int_0^1 \frac{dt e^{-\frac{2st}{1+t}}}{1+t} \int_0^\infty \frac{dt e^{-\frac{2st}{t(1+t)}}}{t(1+t)} = \frac{1}{4\pi^2} \left[ \frac{\text{Ei}_1 \left( \frac{3s}{2} \right) \left( 1 + e^{\frac{2y}{s}} \right) - 2\text{Ei}_1(3s) e^{\frac{2y}{s}}}{\beta} \right],
\]

where we have changed to a new variable \( t = 4y^2 \) and used dimensionless scale argument \( s = A^2 p^2 \).

Next we evaluate the tensor integral. Since we have only one preferable direction – that of \( \gamma \) we can find the integral \( f_{\alpha\beta} \) in the form

\[
\int \frac{d\theta \sin^2 \theta d\gamma y e^{-s A^2 p^2 y^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta} \gamma_{\alpha\beta} = \left( B\delta_{\alpha\beta} + C P_{\alpha} P_{\beta} \right) \int \frac{d\theta \sin^2 \theta d\gamma y e^{-s A^2 p^2 y^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta}.
\]

(A14)

where \( B \) and \( C \) are unknown constants to be determined. Tracing of both sides of equality (A14) gives the constraint \( 4B + C = 1 \). Taking the convolution of both sides of (A14) with \( \frac{P_{\alpha} P_{\beta}}{p^2} \) we get another constraint

\[
B + C = I_1 / I_0, \quad I_1 = \int \frac{d\theta \cos^2 \theta \sin^2 \theta d\gamma y e^{-s A^2 p^2 y^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta}, \quad I_0 = \int \frac{d\theta \sin^2 \theta d\gamma y e^{-s A^2 p^2 y^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta}.
\]

(A15)

The integral \( I_0 \) in Eq.(A15) coincides with the integral \( J \) given by (A5) up to the change of scale \( s \to \frac{3}{2}s \). This gives:

\[
I_0 = \frac{3}{4} \left[ 2\text{Ei}_1(3s) e^{\frac{2y}{s}} - \text{Ei}_1 \left( \frac{3s}{2} \right) e^{\frac{2y}{s}} - \frac{2e^{-\frac{2y}{s}}}{3s} + \frac{2}{3s} \right].
\]

(A16)

After the angle integration in \( I_1 \) we get \( I_1 = \int_0^\infty y dy e^{-s y^2} I_1[\beta(y)] \), with \( I_1[\beta(y)] \) given by Eq. (A4), from where we get:

\[
I_1 = \frac{\pi}{16} \int_0^1 dt e^{-\frac{2st}{1+t}} + \frac{\pi}{16} \int_1^\infty \frac{e^{-\frac{2st}{t}}}{t} = \frac{\pi}{16} \left[ \frac{\text{Ei}_1 \left( \frac{3s}{2} \right) + 4}{9s^2} - \frac{4e^{-\frac{2y}{s}}}{9s^2} - \frac{2e^{-\frac{2y}{s}}}{3s} \right],
\]

(A17)

where \( t = 4y^2 \).

We can rewrite (A2) in the form

\[
I_\Delta = 2\gamma_\alpha \gamma_\beta \left( \frac{I_0}{4\pi^3} \right) \left( B\delta_{\alpha\beta} + C P_{\alpha} P_{\beta} \right) - \frac{\gamma_{\alpha} \gamma_{\beta}}{2p^2} I_C = \left( \frac{I_0}{2\pi^3} (2B + C) - I_0 \right) \gamma_\beta + \frac{\gamma_{\beta} p_{\beta}}{p^2} \left( I_C - \frac{C I_0}{\pi^3} \right),
\]

(A18)

where the last term is not proportional to \( \gamma_\rho \) and will be ignored. Now we can substitute the found constants

\[
B = \frac{1 - \eta}{3}, \quad C = \frac{4\eta - 1}{3}, \quad \eta = \frac{I_1}{I_0}
\]

into equation (A18) to obtain the one loop contribution to the fermion-photon vertex:

\[
\Lambda_{\rho}(\pm \frac{p}{2}, \pm \frac{p}{2}, p) = e_0^2 e^{-s} I_\Delta = e_0^2 e^{-s} \left[ \frac{I_0}{6\pi^3} + \frac{I_1}{3\pi^3} - \frac{I_0}{2} \right] \gamma_\rho
\]

\[
= \frac{e_0^2 \gamma_{\rho}}{3\pi^2} \left[ \frac{e^{\frac{2y}{s}} \text{Ei}_1(3s)}{2} - \frac{e^{-\frac{2y}{s}}}{8s} + \frac{e^{-s} \text{Ei}_1 \left( \frac{3s}{2} \right)}{16} - \frac{5e^{-s} \text{Ei}_1 \left( \frac{3s}{2} \right)}{36s^2} - \frac{e^{-\frac{2y}{s}}}{36s^2} \right].
\]
Appendix B: Calculations with $\chi_2$ wavelet

1. Electron self-energy diagram

Similarly to the case of $\chi_1$ wavelet, we make the one-loop calculation with $\chi_2$ wavelet by symmetrizing the loop momenta ($\frac{p}{2} + q, \frac{p}{2} - q$) for both the vacuum polarization and the electron self-energy diagrams. Thus, the wavelet cutoff factor in this diagrams is

$$F_A(p,q) = f^2 \left( A\left( \frac{p}{2} + q \right) \right) f^2 \left( A\left( \frac{p}{2} + q \right) \right),$$

with $f(x)$ for $\chi_2$ given by (15).

$$F_A(p,q) = \frac{\pi}{16} \left[ \left( \frac{4p^2}{\pi^2} + \frac{1}{A^2} \right) - p^2 q^2 \cos^2 \theta \right]^2 e^{-A^2 p^2 - 4A^2 q^2} = s^4 e^{-s} \left[ (y + \frac{1}{4y} + \frac{1}{sy})^2 - \cos^2 \theta \right]^2 e^{-4sy^2} y^4 \ (B1)$$

Now we can substitute $F_A(p,|p|y)$ into the equation for electron self-energy (22). We omit the term $4m - 2|p|y$ in the numerator, since $m$ is small in comparison to $p$ in our approximation, and $y$ does not contribute for symmetry reasons. This gives

$$\Sigma^{(A)}(p) = -ie^2 p s^4 e^{-s} \int \frac{d^4y}{(2\pi)^4} y^2 e^{-4sy^2} \left[ \left( y + \frac{1}{4y} + \frac{1}{sy} \right)^2 - \cos^2 \theta \right]^2.$$

Using the notation $\beta = y + \frac{1}{4y}$ and the integration measure $d^2y = 4\pi \sin^2 \theta d\theta dy dy$ we get

$$\Sigma^{(A)}(p) = -ie^2 p \int \frac{d^2y}{(2\pi)^2} \sin^2 \theta \left[ \left( \frac{\beta^2}{\beta^2 - \cos^2 \theta} + 1 \right) + \frac{4\beta y + 1}{\beta^2} \right]$$

We perform the angle integration first

$$\Sigma^{(A)}(p) = -ie^2 p \int_0^{\infty} dy e^{-4sy^2} \int_0^\pi d\theta \sin^2 \theta \left[ \left( \frac{\beta^2}{\beta^2 - \cos^2 \theta} + 1 \right) + \frac{4\beta y + 1}{\beta^2} \right].$$

Since the angle integral $I_{0}[\beta(y)]$, given by Eq.(A1), is defined piece-wise (A4), we split $\Sigma^{(A)}(p)$ into a sum of two integrals

$$\Sigma^{(A)}(p) \equiv J_1 + J_2,$$

with the second one, which depends on $I_0[\beta(y)]$, should be splited: $j_0^\infty = j_0^{1/2} + j_0^{1/2}$.

$$J_1 = -ie^2 p \frac{1}{32\pi^2} \int_0^\infty \left[ s^4 y^4 \left( \frac{1}{4y^2} + 1 \right) + 16s^3 y^3 \left( y + \frac{1}{4y} \right) + 8s^2 y^2 \right] e^{-4sy^2} y dy = -ie^2 p \frac{s^2 + 18s + 70}{128},$$

$$J_2 = -ie^2 p \frac{1}{4\pi^3} \int_0^\infty I_0[\beta(y)](4s^2 \beta^2 y^2 + 4\beta sy + 1) e^{-4sy^2} y dy$$

$$= -ie^2 p \frac{1}{4\pi^3} \left( \int_0^{1/2} \frac{8y^2}{1+4y^2} (4s^2 \beta^2 y^2 + 4\beta sy + 1) e^{-4sy^2} y dy + \int_{1/2}^\infty \frac{2}{1+4y^2} (4s^2 \beta^2 y^2 + 4\beta sy + 1) e^{-4sy^2} y dy \right)$$

$$= -ie^2 p \frac{1}{64\pi^3} \left( \int_0^t \frac{1}{t+1} (s^2 (t+1)^2 + 4s(t+1) + 4) e^{-st} dt + \int_1^\infty \frac{1}{t+1} (s^2 (t+1)^2 + 4s(t+1) + 4) e^{-st} dt \right)$$

$$= -ie^2 p \frac{1}{64\pi^3} \left( 4e^s (2Ei_1(2s) - Ei_1(s)) + \frac{10}{s} (1 - e^{-s}) + 1 - 2e^{-s} \right)$$
The final result is
\[
\Sigma_{\chi^2}^{(A)}(p) = -\frac{ie^2\beta}{16\pi^2} \left[ 2E_{1i}(2s) - E_{1i}(s) - \frac{s+5}{2s} e^{-2s} + \frac{s^3 + 18s^2 + 134s + 640}{256s} e^{-s} \right]. \tag{B3}
\]

2. Vacuum polarization diagram

Calculation of vacuum polarization diagram for the case of the $\chi_2$ wavelet cutoff function (15) is completely analogous to that performed with $\chi_1$ wavelet cutoff. To simplify analytical calculation here we also assume the massless limit $p^2 \gg 4m^2$ and omit appropriate terms. In this way Eq. (16) becomes
\[
\Pi_{\mu\nu;\chi^2}(p) = -\frac{e^2p^2}{\pi^3} \int_0^\infty dy \int_0^\pi d\theta \sin^2 \theta F_A(p, q) \frac{2y_\mu y_\nu - \delta_{\mu\nu} y^2 + \frac{1}{4} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) - \frac{\delta_{\mu\nu}}{4}}{\beta^2 - \cos^2 \theta}.
\]
The wavelet cutoff function $F_A(p, q)$ is given by Eq. (B1). In dimensionless variables $(s, y)$ it has the form
\[
F_A(y) = e^{-s}e^{-4s^2} y^4 \left( \beta(y) + \frac{1}{sy} \right)^2 - \cos^2 \theta \right]^2.
\]
So, the integral to be evaluated is
\[
\Pi_{\mu\nu;\chi^2}(p) = -\frac{e^2p^2}{\pi^3} \int_0^\infty dy \int_0^\pi d\theta \sin^2 \theta s^4 y^4 \left( \beta(y) + \frac{1}{sy} \right)^2 - \cos^2 \theta \right]^2 \times
\]
\[
\times \frac{2y_\mu y_\nu - \delta_{\mu\nu} y^2 + \frac{1}{4} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) - \frac{\delta_{\mu\nu}}{4}}{\beta^2 - \cos^2 \theta}.
\]
Similar to the evaluation of the self-energy diagram, we expand the polynomial part of the wavelet cutoff function and get:
\[
\Pi_{\mu\nu;\chi^2}(p) = -\frac{e^2p^2}{\pi^3} \int_0^\infty dy \int_0^\pi d\theta \sin^2 \theta \left[ s^4 y^4 \left( \beta^2 - \cos^2 \theta \right) + 4\beta s^3 y^3 +
\right.
\]
\[
+ 2s^2 y^2 \left( \frac{2\beta^2}{\beta^2 - \cos^2 \theta} + 1 \right) + \frac{4\beta sy + 1}{\beta^2 - \cos^2 \theta} \right] \times \left[ 2y_\mu y_\nu - \delta_{\mu\nu} y^2 + \frac{1}{2} \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \frac{\delta_{\mu\nu}}{4}. \tag{B4}
\]
To calculate the vacuum polarization diagram, Eq. (B4), we need two integrals. The scalar integral, and the tensor integral dependent on $y_\mu y_\nu$. They are:
\[
J = \int_0^\infty dy \int_0^\pi d\theta \sin^2 \theta \left[ s^4 y^4 \left( \beta^2 - \cos^2 \theta \right) + 4\beta s^3 y^3 +
\right.
\]
\[
+ 2s^2 y^2 \left( \frac{2\beta^2}{\beta^2 - \cos^2 \theta} + 1 \right) + \frac{4\beta sy + 1}{\beta^2 - \cos^2 \theta} \right] \times \left[ 2y_\mu y_\nu - \delta_{\mu\nu} y^2 + \frac{1}{2} \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \frac{\delta_{\mu\nu}}{4}. \tag{B5}
\]
\[
J^{(2)} = \int_0^\infty dy \int_0^\pi d\theta \sin^2 \theta \left[ s^4 y^4 \left( \beta^2 - \cos^2 \theta \right) + 4\beta s^3 y^3 +
\right.
\]
\[
+ 2s^2 y^2 \left( \frac{2\beta^2}{\beta^2 - \cos^2 \theta} + 1 \right) + \frac{4\beta sy + 1}{\beta^2 - \cos^2 \theta} \right] y_\mu y_\nu. \tag{B6}
\]
The integral $J$ is identical to that calculated for electron self-energy diagram in Eq. (B2). Its value is
\[
J = \frac{\pi}{4} \left[ \frac{s^3 + 18s^2 + 134s + 640}{256s} e^{-s} \right] - \frac{e^{-s}(s+5)}{2s} + e^s \left( 2E_{1i}(2s) - E_{1i}(s) \right). \tag{B7}
\]
The integral (B6) is evaluated by changing $y_\mu y_\nu \rightarrow B y^2 \delta_{\mu\nu} + C y^2 \frac{p_\mu p_\nu}{p^2}$:
\[
I_{\mu\nu} = \int_0^\infty dy \int_0^\pi d\theta \sin^2 \theta \left[ s^4 y^4 \left( \beta^2 - \cos^2 \theta \right) + 4\beta s^3 y^3 +
\right.
\]
\[
+ 2s^2 y^2 \left( \frac{2\beta^2}{\beta^2 - \cos^2 \theta} + 1 \right) + \frac{4\beta sy + 1}{\beta^2 - \cos^2 \theta} \right] \left[ By^2 \delta_{\mu\nu} + Cy^2 \frac{p_\mu p_\nu}{p^2} \right].
\]

The unknown constants $B$ and $C$ are determined from the equality $I_{qμ}^{(2)} = I_{μν}$, exactly in the same way as for $χ_1$ wavelet. Taking the trace of both sides we get the constraint $4B + C = 1$. The other constraint is obtained by convolution of both sides of $I_{qμ}^{(2)} = I_{μν}$ with $\frac{p_μp_ν}{p^2}$. This gives $I_q^1(B + C) = I_q^0$, where

$$ I_q^0 = \int_0^\infty d yy^3 e^{-y^2} \int_0^\pi dθ sin^2 θ \left[ s^4 y^4 (β^2 - cos^2 θ) + 4β s^3 y^3 + 2s^2 y^2 \left( \frac{2β^2}{β^2 - cos^2 θ} + 1 \right) + \frac{4βs y + 1}{β^2 - cos^2 θ} \right] $$

$$ = \int_0^\infty d yy^3 e^{-y^2} \int_0^\pi dθ sin^2 θ \frac{π}{2} \left[ \left( β^2 - 4βs^3 y^3 + 2s^2 y^2 \right) - \frac{π}{8} s^4 y^4 + I_0[β(y)] \left( 4β^2 s^2 y^2 + 4βs y + 1 \right) \right] , $$

$$ I_q^1 = \int_0^\infty d yy^3 e^{-y^2} \int_0^\pi dθ sin^2 θ \left[ s^4 y^4 (β - cos^2 θ) + 4β s^3 y^3 + 2s^2 y^2 \left( \frac{2β^2}{β^2 - cos^2 θ} + 1 \right) + \frac{4βs y + 1}{β^2 - cos^2 θ} \right] $$

$$ = \int_0^\infty d yy^3 e^{-y^2} \int_0^\pi dθ sin^2 θ \frac{π}{8} \left[ \left( s^4 y^4 + 2βs^3 y^3 + 2s^2 y^2 \right) - \frac{π}{16} s^4 y^4 + I_1[β(y)] \left( 4β^2 s^2 y^2 + 4βs y + 1 \right) \right] . $$

The part of the integral, which depends on piecewise defined functions $I_0[β(y)], I_1[β(y)]$, given by Eq.(A4), is integrated in $I_q^{1/2} + f_{1/2}^{∞}$ limits, accordingly. This gives:

$$ I_q^0 = \frac{9π(1 - e^{-s})}{32s^3} + \frac{7π}{512s^3} \left( ρ \right) e^{-s} \left( \frac{3π e^{-s}}{32s^3} + \frac{19π}{2048} \left( s^3 \right) + \frac{π e^{-s}}{32s^3} \right) $$

$$ = \frac{7π(1 - e^{-s})}{32s^3} + \frac{5π}{64s^3} \left( ρ \right) e^{-s} \left( \frac{3π e^{-s}}{32s^3} + \frac{19π}{2048} \left( s^3 \right) + \frac{π e^{-s}}{32s^3} \right) , $$

(B8)

$$ I_q^1 = \frac{7π(1 - e^{-s})}{32s^3} + \frac{5π}{64s^3} \left( ρ \right) e^{-s} \left( \frac{3π e^{-s}}{32s^3} + \frac{19π}{2048} \left( s^3 \right) + \frac{π e^{-s}}{32s^3} \right) , $$

(B9)

Using the expression (A9) we get the equation for vacuum polarization diagram in the form:

$$ Π_{μν}^{(A)χ_2} = I_D δ_{μν} + I_P \frac{p_μp_ν}{p^2}, \quad I_D = -\frac{e^2 p^2 e^{-s}}{π^3} \left[ -\frac{1}{2} I_q^0 - \frac{2}{3} I_q^1 + \frac{J}{4} \right], \quad I_P = -\frac{e^2 p^2 e^{-s}}{π^3} \left[ \frac{2}{3} (4I_q^1 - I_q^0) - \frac{J}{2} \right] . $$

This gives:

$$ I_D = \frac{e^2 p^2}{π^2} \left[ -\frac{1}{12} \left( 2Ei(2s) - Ei(s) \right) \right] $$

$$ + \frac{s^2 e^{-s}}{4096} - \frac{17s e^{-s}}{48s} + \frac{17s e^{-s}}{48s^2} - \frac{7e^{-s}}{48s^3} + \frac{29e^{-s}}{1024} + \frac{e^{-2s}}{48s} + \frac{7e^{-2s}}{24s^2} - \frac{7e^{-2s}}{48s^3} , $$

(B10)

$$ I_P = \frac{e^2 p^2}{π^2} \left[ \frac{1}{12} \left( 2Ei(2s) - Ei(s) \right) \right] $$

$$ + \frac{s^2 e^{-s}}{2048} + \frac{9s e^{-s}}{1024} + \frac{13s e^{-s}}{48s^2} + \frac{e^{-s}}{12s^3} - \frac{7e^{-s}}{12s^3} + \frac{29e^{-2s}}{128} + \frac{e^{-2s}}{24s^2} + \frac{7e^{-2s}}{12s^3} . $$

(B11)

3. **One-loop corrections to the QED vertex with χ_2 wavelet cutoff**

In complete analogy to the calculations performed with χ_1 wavelet, we can express the wavelet cutoff function corresponding to the diagram shown Fig. 4 in the massless limit $p^2 ≫ 4m^2$, in the form

$$ F_A(p, f) = e^{-s} e^{-6sy^2} s^4 y^4 \left[ \left( y + \frac{1}{4y} \right)^2 - cos^2 θ \right]^2 (1 + sy^2)^2, \quad where \ f = y|p|. \quad (B12) $$

The one-loop contribution to the fermion-photon vertex calculated with this cutoff is

$$ \Lambda_p \left( -\frac{p}{2} + \frac{p}{2} p \right) = -e^2 s^4 e^{-s} \int \frac{d^4 y}{(2π)^4} \left( \frac{2y}{y + \frac{1}{4y}} \right)^2 - \frac{p_μp_ν}{2p^2} e^{-6sy^2} \left[ \left( y + \frac{1}{4y} \right)^2 - cos^2 θ \right]^2 (1 + sy^2)^2, \quad (B13) $$

$$ ≡ -e^2 s^4 e^{-s} s^4 I_△ . \quad (B13) $$
The integral (B13) is a sum of two integrals: $I_\Delta = 2\gamma_\alpha\gamma_\rho y_\beta I_{\alpha\beta} + \frac{p_\alpha p_\beta}{p^2} I_C$, where

$$I_C = \int \frac{d^4y}{(2\pi)^4} \frac{e^{-6sy^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta} \left[ \left( y + \frac{1}{4y} + \frac{1}{y} \right)^2 - \cos^2 \theta \right]^2 (1 + sy^2)^2,$$

$$I_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} \frac{y_\alpha y_\beta e^{-6sy^2}}{(y + \frac{1}{4y})^2 - \cos^2 \theta} \left[ \left( y + \frac{1}{4y} + \frac{1}{y} \right)^2 - \cos^2 \theta \right]^2 (1 + sy^2)^2.$$

The evaluation of these integral is identical to the case of $\chi_1$ wavelet, described by Eqs. (A13,A12). Using the variable $\beta(y) = y + \frac{1}{4y}$, we get

$$I_C = \frac{1}{4\pi^3} \int dy dy^3 \sin^2 \theta e^{-6sy^2} \left( \frac{2\beta}{s} + \frac{1}{s^2y^2} \right)^2 (1 + sy^2)^2 \left[ \beta^2 - \cos^2 \theta + \frac{2\beta}{s} \right],$$

$$+ \frac{1}{4\pi^3} \int dy dy^3 \sin^2 \theta e^{-6sy^2} \left( \frac{2\beta}{s} + \frac{1}{s^2y^2} \right)^2 (1 + sy^2)^2 \left[ \beta^2 - \cos^2 \theta + \frac{2\beta}{s} \right],$$

$$= \frac{1}{4\pi^3} \int dy dy^3 e^{-6sy^2} \left( \beta + \frac{1}{4y} \right) + \frac{1}{2\pi^2} \int_0^\infty dy \int_0^\infty \left( \beta - \frac{1}{4y} \right) \left( 2\beta + \frac{1}{s^2y^2} \right) \left( 2\beta + \frac{1}{s^2y^2} \right) \theta e^{-6sy^2} \left( 1 + sy^2 \right)^2 \frac{2\pi}{1 + 4y^2},$$

where we have used the angle integration rule (A4). After the change of variable $t = 4y^2$, the final result is

$$I_C = \frac{211}{6912\pi^2 s^2} + \frac{25}{13824\pi^2 s} + \frac{173}{1296\pi^2 s^3} + \frac{161}{432\pi^2 s^4} + \frac{e^{\frac{\pi}{4} y_1} \left( \frac{\pi}{4} y_1 \right)}{64\pi^2 s^3} - \frac{e^{\frac{\pi}{4} y_1} \left( \frac{\pi}{4} y_1 \right)}{8\pi^2 s^3} \quad \text{(B14)}$$

The tensor integral $I_{\alpha\beta}^\Delta$ can be decomposed with respect to two basic tensors $\delta_{\alpha\beta}$ and $\frac{p_\alpha p_\beta}{p^2}$:

$$I_{\alpha\beta} = \frac{1}{4\pi^3} \int dy dy^3 e^{-6sy^2} \left( \frac{2\beta}{s} + \frac{1}{s^2y^2} \right)^2 \left[ \beta^2 - \cos^2 \theta + \frac{2\beta}{s} \right],$$

$$= \frac{1}{4\pi^3} \int dy dy^3 e^{-6sy^2} \left( \beta + \frac{1}{4y} \right) \left( \beta - \frac{1}{4y} \right) \left( 2\beta + \frac{1}{s^2y^2} \right) \left( 2\beta + \frac{1}{s^2y^2} \right) \theta e^{-6sy^2} \left( 1 + sy^2 \right)^2 \left[ \beta^2 - \cos^2 \theta + \frac{2\beta}{s} \right].$$

where $B$ and $C$ are unknown constants to be determined. Tracing both sides of Eq.(B15) gives the constraint $4B + C = 1$. The other constraint is identical to (A15):

$$I_1 = (B + C)I_0,$$

with

$$I_0 = \int \frac{dy dy^3 e^{-6sy^2}}{\beta^2 - \cos^2 \theta} \left( \beta + \frac{1}{4y} \right)^2 \left( \beta - \frac{1}{4y} \right)^2 \left( 2\beta + \frac{1}{s^2y^2} \right)^2,$$

$$I_1 = \int \frac{dy dy^3 \theta \cos^2 \theta e^{-6sy^2}}{\beta^2 - \cos^2 \theta} \left( \beta + \frac{1}{4y} \right)^2 \left( \beta - \frac{1}{4y} \right)^2 \left( 2\beta + \frac{1}{s^2y^2} \right)^2 \left[ \beta^2 - \cos^2 \theta + \frac{2\beta}{s} \right].$$
The remaining calculations are analogues to previous integrals

\[ I_0 = \int dyd\theta \sin^2 \theta y^5 e^{-6sy^2} (1 + sy^2)^2 (\beta^2 - \cos^2 \theta) + 2 \int dyd\theta \sin^2 \theta y^5 e^{-6sy^2} (1 + sy^2)^2 \left( \frac{2\beta}{sy} + \frac{1}{s^2y^2} \right) \]

\[ + \int \frac{dyd\theta \sin^2 \theta y^5 e^{-6sy^2} (1 + sy^2)^2}{\beta^2 - \cos^2 \theta} \left( \frac{2\beta}{sy} + \frac{1}{s^2y^2} \right)^2 \]

\[ = - \frac{\pi e^{\frac{3s}{2}}}{124416s^3} \left( 14976s + 7056s^2 - 64896e^{\frac{s}{2}} + 432s^3 - 1376se^{\frac{s}{2}} - 3048s^2e^{\frac{s}{2}} + 64896 - 15524s^2/2\right) \]

\[ + 31104e^{3s}Ei_1(3s) - 3888e^{3s}Ei_1(3s) + 1944e^{3s}Ei_1(3s) - 99s^3e^{\frac{3s}{2}}, \]

\[ I_1 = \int dyd\theta \sin^2 \theta \cos^2 \theta y^5 e^{-6sy^2} (1 + sy^2)^2 (\beta^2 - \cos^2 \theta) + 2 \int dyd\theta \sin^2 \theta \cos^2 \theta y^5 e^{-6sy^2} (1 + sy^2)^2 \left( \frac{2\beta}{sy} + \frac{1}{s^2y^2} \right) \]

\[ + \int \frac{dyd\theta \sin^2 \theta \cos^2 \theta y^5 e^{-6sy^2} (1 + sy^2)^2}{\beta^2 - \cos^2 \theta} \left( \frac{2\beta}{sy} + \frac{1}{s^2y^2} \right)^2 \]

\[ = - \frac{\pi e^{\frac{3s}{2}}}{497064s^3} \left( 99s^4 e^{\frac{3s}{2}} + 12608s^3 e^{\frac{3s}{2}} + 1584s^2 e^{\frac{3s}{2}} + 46848se^{\frac{3s}{2}} + 94976e^{\frac{3s}{2}} - 94976 - 189312s - 121536s^2 - 1728s^4 \right. \]

\[ - 28224s^3 + 7776s^4 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) + 31104s^3 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) + 31104s^2 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}), \]

where we have used the angle integration rules (A4). Substituting these integrals into the final equation (B13), and comparing it to (A19), we get

\[ \Lambda^2 \left( -\frac{p}{2} - \frac{p}{2} \right) = \epsilon_0^2 s^4 e^{-s} \left[ \frac{I_0}{6s^3} + \frac{I_1}{3s^3} - \frac{I_C}{2} \right] \gamma_e = \]

\[ = - \frac{\gamma_e \epsilon_0^2 e^{\frac{3s}{2}}}{14992092\pi^2 s^4} \left( 319104s - 126720s^2 - 94976e^{\frac{3s}{2}} - 12096s^3 - 176640se^{\frac{3s}{2}} + 262848s^2 e^{\frac{3s}{2}} \right. \]

\[ + 38880s^3 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) + 155520s^2 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) + 155520s e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) + 94976 + 248832s^2 e^{\frac{3s}{2}} + 248832s^3 Ei_1(\frac{3s}{2}) \]

\[ - 124416s^2 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) + 248832s^3 Ei_1(3s) - 497664s^2 e^{\frac{3s}{2}}Ei_1(3s) + 15552s^4 e^{\frac{3s}{2}}Ei_1(\frac{3s}{2}) \]

\[ - 31104s^4 e^{\frac{3s}{2}}Ei_1(3s) + 1350s^5 e^{\frac{3s}{2}} + 22491s^4 e^{\frac{3s}{2}} + 91968s^3 e^{\frac{3s}{2}} \)