The damping of gravitational waves in dust

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Abstract
We examine a simple model of the interaction of gravitational waves with matter (primarily represented by dust). The aim is to investigate a possible damping effect on the intensity of the gravitational wave when passing through a medium. This might be important for gravitational wave astronomy when the sources are obscured by dust or molecular clouds.

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1. Introduction
At present, we witness an ongoing dedicated search for gravitational waves from various astrophysical sources using ground-based laser interferometers [1]. Since the sensitivity of these instruments is at the upper bound of assumed intensities coming from realistic sources, it might be interesting to estimate the damping effects of various dust formations on the propagating gravitational waves.

In the past, several papers (for a review, see [2]) dealt with the effects of media on gravitational waves but their main concern was the modification of dispersion relation and not the possible influence on the amplitude. Two different models for the medium used most often were: a medium composed of deformable ‘molecules’ with internal structure [2, 3] giving rise to anisotropic pressures or free particles with rare collisions described by kinetic theory [4]. To describe the damping in the second model, the rate of particle collisions has to be addressed giving rise to the imaginary part of the refractive index [5]. The damping was also studied in viscosity approximation [6]. In these treatments, the damping was negligible due to the slow accumulation of the phase shift that is important for the damping effect only at the values close to 2π.

In sections 2 and 3, the damping is derived using a straightforward computation without the assumption regarding particle collisions. The interaction of gravitational waves and matter (represented by a dust cloud) is derived based on the following picture. The incoming gravitational wave produces periodic oscillations within the ‘molecules’ in the dust cloud computed using the geodesic deviation equation (hence, we use the first of the above mentioned models for the medium). In the second step, these oscillations themselves produce gravitational waves derived using multipole approximation. Finally, we compose them with the original waves propagating through the cloud from a distant source. We use the term dust because we assume no interaction between the ‘molecules’. It means that the model cannot describe effects connected with collective phenomena; on the other hand, it can still accommodate self-gravitation of the dust cloud if it can be encoded in the background metric.

In section 2, we assume that the wave is approximately planar at the region of the cloud and the background might be described using the Minkowski metric so the source of the gravitational wave has to be far enough. In section 3, we generalize the procedure to the curved background and treat the example of a spherical shell of dust around a central source of gravitational waves. To arrive at the solution in this case the generalization of quadrupole momentum tensor to this specific problem is given. Section 4 briefly summarizes the results of a previous work [11, 12] and mentions its seemingly unfavorable implications. However, it is shown that the result concerns orders of magnitude that are not decisive for any concrete implications.

2. Derivation for planar waves in the Minkowski background
We start with the simplest case where the geometry in the neighborhood of the cloud is assumed to be flat. First, we recall the behavior of particle separations in the flat background (that we use for simplification in the region of the cloud) under the influence of the gravitational

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wave (determined by geodesic deviation). However, we will
generalize the formula to the situation where the reference
center of the ‘molecule’ is geodesic while the particle
(whose separation from the center we want to determine)
is additionally sitting in a harmonic potential (described by
‘spring constant’ \( k > 0 \)) centered at the initial position and
feeling the dissipative effect of its environment (described by
‘damping constant’ \( b > 0 \)). At the same time, this is the
most straightforward generalization of the second order
differential equation using only linear terms. Generalization
of the geodesic deviation equation to include nongravitational
forces is usually called relative deviation. We suppose that the
particle is at rest before the wave arrives and therefore we
use only the so called ‘steady state’ solution of the driven,
damped oscillator equation that serves for generalization of
the geodesic deviation equation
\[
x^k_B(t) = x^k_B(0)[\delta^k + \frac{1}{2}CH^2 T^k_j(x)|_{A}],
\]
(1)
where \( \omega \) is the frequency of the gravitational wave and \( i \) is a
complex unit. Equation (1) expresses the position of particle B
in the proper reference frame of particle A lying at the origin.
Transverse-traceless (TT) calibration was used to compute the
Riemann tensor components needed in the geodesic deviation
equation. It should be noted that there exists a TT system
that agrees, to the first order of perturbation, with the proper
reference frame. In the following, we will use exactly this
system for further computations.

Next, we need to know how these periodic movements
generate gravitational waves. For this purpose we use the
reduced quadrupole moment
\[
\mathcal{I}_{jk} = \int \rho(x_jx_k - \frac{1}{3}r^2\delta_{jk})d^3x.
\]
(2)
We compute this expression using equation (1). Dropping
the particle labels and assuming that the density \( \rho \) of the
dust stays approximately constant throughout the passage of
the wave (cross-section changes only in the second order of
perturbation), the result computed in linear terms in
perturbation is the following:
\[
\mathcal{I}_{jk} = \rho \int_{\mathcal{V}} [(\tilde{x}_j\tilde{x}_k - \frac{1}{3}r^2\delta_{jk}) + \frac{1}{2}C(\tilde{x}_j\tilde{x}_m h^{TM}_{jk} + \tilde{x}_k\tilde{x}_m h^{TM}_{mj})
- \frac{2}{5}\tilde{x}_m h^{TM}_{jk}\delta_{jk}]d^3\tilde{x},
\]
(3)
where \( \tilde{x} \) denotes the positions before the wave arrival. It
indicates, that generally, we cannot neglect the influence of
matter on passing gravitational radiation, since the generated
intensity depends on second time derivative of reduced
quadrupole moment. We can see that this intensity would be
produced by the terms in the second pair of round brackets in
equation (3).

The generated perturbations \( \rho \) in the TT frame have the
following form [9]:
\[
\rho h_{jk}(t-r) = \frac{2}{r}\mathcal{I}_{jk}(t-r) + O(1/r^2),
\]
(4)
where
\[
\mathcal{I}_{jk} = P^b_{jk}g^{ab}P_{ab} - \frac{1}{2}P_{jk}P^{ab}\mathcal{I}_{ab}
\]
(5)
and \( P_{ab} = \delta_{ab} - n_a n_b \) is the projection operator to the
subspace transverse to unit radial vector \( n_a \).

2.1. Example with cuboid

Let us show the computation for one very simple situation.
Suppose the cubical dust cloud has approximate sizes \( X, Y \)
and \( Z \) (with respect to Cartesian coordinates) in the reference
frame with origin in the clouds center and the \( z \)-axis pointing
to the distant source of gravitational waves on one side and
to the distant observer on the other side. We assume that
the wave from the external source is approximately planar
in the region of the cloud. We compute the second time
derivative of highest order terms for the relevant components
of reduced quadrupole moment in TT gauge (with \( n^a = (\delta^a)^y \)
for a ‘molecule’ (region where the driven, damped oscillator
model described above might be applied) of linear dimension
\( a \), obtaining
\[
\rho h_{xx} = \rho h_{yy} = \frac{1}{8}\rho a^4 T_{tt},
\]
(6)
where \( T_{tt}(z = 0) \sim e^{-\lambda t} \) is a planar perturbation in the
TT gauge from the distant source (we have chosen the
\( \oplus \) polarization mode for simplicity). By superposing the
generated gravitational waves (4) from all molecules in a
given layer of dust cloud in the \( xy \) plane, we obtain a planar
wave (neglecting the effects of finite dimensions of the cloud)
\[
\rho h_{xy} = \frac{4\pi}{io}\sigma h_{xy} = \frac{2\pi C}{3i} \rho a^2 h_{xy},
\]
(7)
where the planar density of ‘molecules’ \( \sigma \) can be computed as
\( a^{-2} \) for a dense medium and equality \( h_{xy} = -\omega^2 h_{tt} \)
was used. Now we add the generated wave (7) to the original
perturbation \( h_{tt} \). This process will happen in subsequent
layers with efficiency described by the parameter \( \kappa = (0.1) \)
(roughly speaking it would happen in every 1/\( \kappa \) slice), which
gives the following exponential decay of the amplitude of
gravitational perturbation as it travels through the cloud:
\[
h_{xy} = h_{xy} e^{-2\pi C/3}\sigma \rho a^2 \mu Z Ma^2,
\]
(8)
where \( \lambda \) is the linear density of molecules in the direction of
propagation and we have omitted coordinate indices, \( h_{xy} \) is the
amplitude before the cloud and \( h_{xy} \) after it. We may define a
volume number density of ‘molecules’ \( \tilde{\rho} = \sigma \lambda \).

Since the exponential factor in (8) is complex the overall
effect is twofold. The imaginary part is responsible for the
phase shift and the real part for the possible damping. The real
part of the factor can be written as \(-2\pi C/3\sigma \rho a^2 \mu Z Ma^2\),
where \( M \) is the mass of the ‘molecule’. This number is
nonpositive and therefore really describes exponential decay
of amplitude.

Although this picture seems to be pretty naive, there is
direct correspondence between the above-mentioned second
time derivatives of quadrupole moment and perturbations of stress energy tensor representing anisotropic stresses that generate gravitational waves via standard linearized Einstein equations (for a detailed discussion, see [2]).

3. ‘Spherical waves’ and the dust shell

Now we are going to examine a more complex example where the approach from the previous section cannot be applied straightforwardly. As a background we use the Vaidya–(anti-)de Sitter metric in stereographic coordinates $(u, r, \eta, \xi)$

$$ds^2 = \left(-1 + \frac{2m(u)}{r} + H^2 r^2\right) du^2 + 2 \, du \, dr + \frac{r^2}{p^2} (d\eta^2 + d\xi^2),$$

where $p(\eta, \xi) = 1 + (1/4)(n^2 + \xi^2)$ and $H = \sqrt{\Lambda / 3}$ (with $\Lambda$ being a cosmological constant). To determine the form of the gravitational waves, we can use the results of the paper [7] where the background is a general Robinson–Trautman spacetime with cosmological constant (describing also the above metric as a special case) and the high-frequency approximation developed by Isaacson [8] is used. The wave vector is $k_\mu = (\phi_\mu, 0, 0, 0)$ and we specialize to the $\oplus$ polarization mode. So the only nonzero elements of the perturbation tensor are the following:

$$h_{\eta\eta} = A \frac{r}{\sqrt{2} p^2} \exp(i\phi(u)) = -h_{\xi\xi},$$

with $A$ being an amplitude and $\phi$ the phase. The effect will be analyzed with respect to the stationary observer near infinity with four-velocity $v = (\dot{u}, 0, 0, 0)$. For zero cosmological constant, this observer is almost geodesic. Once again we will employ the geodesic deviation equation (in a more general form [9])

$$\nabla_\nu \nabla_\eta n^\sigma = -R^\nu_{\rho\sigma\lambda} n^\rho \eta^\sigma \eta^\lambda.
$$

The deviation vector $n^\sigma$ connecting neighboring geodesics has components in the $\partial_\eta$ and $\partial_\xi$ directions only. Assuming that the spherical dust shell that interacts with the gravitational waves starts the free fall from rest ($\partial \eta^\mu / \partial u = 0$) the right-hand side of (11) could be written as $(d^2 n^\mu / du^2) u^\mu$ and $\ddot{u} = (-g_{uu})^{-1/2}$ from four-velocity normalization. The Riemann tensor could be decomposed into background part $R^{(0)}$ and the contribution induced by perturbations $R^{(1)}$, which is the dominant one in the high-frequency approximation [8]

$$R^{(1)}_{\nu\rho\sigma\lambda} = 2 h_{[\nu}[\rho\sigma\lambda]} \Rightarrow R^{(1)}_{\nu\mu\eta\eta} = -2^{-3/2} A \frac{\rho}{p^2} \frac{d^2}{du^2} \exp(i\phi(u)) = -R^{(1)}_{\xi\xi\eta\eta}.$$  

Now we can integrate the simplified equation (11) (assuming that $R^{(0)}$ changes very slowly) with the added effect of nongravitational forces (i.e. when the particle position satisfies $\nabla_\nu \nabla_\eta n = -R(v, n) n - b \nabla_\nu n - k(n - n_0)$, generalizing the relative deviation used in the flat case) to obtain

$$n^\eta = \left(1 + \frac{R^{(0)}_{\eta\mu\eta}}{kg_{\eta\eta}} + 2^{-3/2} A \frac{\rho}{p^2} g^{\gamma\gamma} \exp(i\phi(u))\right) n_0^\eta,$$

where

$$\tilde{C} = \frac{(\phi_\mu)^2 + g_{\mu\nu} k + i \sqrt{-g_{\eta\eta}} b \phi_\mu}{((\phi_\mu)^2 + g_{\mu\nu} k^2) - g_{\eta\eta} b^2}.$$
corresponds to discarding the term related to boundary in the flat case), we obtain
\[ \varepsilon h^\pi_\eta = \frac{2\pi i r_0}{\phi_\mu r} \frac{d^2}{dr^2} F^\pi_\eta, \] (16)
where \( \sigma \) is the surface density of the ‘molecules’. For the covariant metric perturbation, one may then write
\[ h^{\pi}_{\mu\nu}(r_0 + (j + 1)\Delta r) = \left[1 + g_{\mu\nu}r_0H\Delta r \right] h^{\pi}_{\mu\nu}(r_0 + j\Delta r), \] (17)
where
\[ K = -\frac{8\pi i}{3} \left( 1 + \frac{R^{(0)\mu\nu}_{\text{eq}}}{k g_{\mu\nu}} \right) \sigma \rho \tilde{C} a^4 \phi_\mu. \] (18)

Setting \( \Delta r = \lambda^{-1} = (r_1 - r_0/N) \) (with \( N \gg 1 \) being the number of layers and \( r_0, r_1 \) the inner and outer radii of a dust shell, respectively) and assuming that the wave decreases with efficiency \( \kappa \) (as in the planar model), we get for the intensities under the inner radius \( h_B \) and above the outer radius \( h_A \)
\[ h_A \approx h_B \left[ \frac{K \Delta r}{1 + \frac{r_0 K \Delta r}{r_0 - 2m - H^2 r_0^3}} \right] \]
\[ \approx h_B \left[ 1 \right] \left[ 1 + \frac{K (N + 1) r_0 K \Delta r}{r_0 - 2m - H^2 r_0^3} \right] \]
\[ + O(\Delta r^3). \] (19)

We can check that this result has at least the same length dimensionality and similar dependence on frequency as for our cuboid case on Minkowski, which might give some vindication to the calculation used in the curved space.

### 4. Perturbation viewpoint

In 1987, Ehlers et al [11] investigated the propagation of small-amplitude gravitational waves through pressureless matter (‘dust’) using the linearization of the Einstein equations for dust
\[ g_{\mu\nu}, \rho, u_{\mu} \rightarrow g_{\mu\nu} + \varepsilon \delta g_{\mu\nu}, \rho + \varepsilon \delta \rho, u_{\mu} + \varepsilon \delta u_{\mu}. \] (20)

They used the Wentzel–Kramers–Brillouin (WKB) method to study the locally planar, linearized perturbation of an arbitrary background dust spacetime asymptotically for small wavelengths (thus assuming the high-frequency character of the perturbations)
\[ \delta g_{\mu\nu} = 0 \text{[e^{(i/\varepsilon)S(x)} f_{\mu\nu}(x, \varepsilon)]}. \] (21)

Finally, they cast the resulting system of equations into the form where \( \delta g_{\mu\nu} \) serves as a source for generating perturbations \( \delta \rho \) and \( \delta u_\mu \).

However, they arrived at the conclusion that in the leading order no perturbations of density or vorticity are generated
\[ \delta \rho = 0, \quad \delta u_\mu = 0. \] (22)

This makes it impossible to alter the passing gravitational wave in the second step (the medium in this model is unchanged in the first place). This result would not allow the damping of gravitational waves seen in the previous sections. But one should also keep in mind that the model from sections 2 and 3 relies on an internal structure of the ‘molecules’ (although their collection behaves globally like dust) and therefore is not directly comparable with the perturbation picture described above.

Anyway, careful reading of the paper [11] (and of a more recent paper [12]) dealing with the perfect fluid case reveals that, in fact, the order in which the right-hand sides of equations (22) are supposed to vanish is \( \varepsilon^{-2} \). But this is a necessary condition for the perturbation method itself to be consistent and does not restrict the existence of nonzero matter perturbations to the order \( \varepsilon \).

### 5. Conclusion

The above described simple model (presented in sections 2 and 3) might produce a measurable effect on the intensity of detectable gravitational waves. The model assumes certain (simple) interactions of particles inside the ‘molecules’ forming the cloud, so it most closely resembles a molecular cloud. The results described in section 4 were shown not to cause any important restriction on the model from the point of view of the perturbative approach.

To give some notion of the strength of this effect, we can compute the damping for some astrophysically relevant data. We use the flat model (section 2) with \( k = \omega^2 \) (maximal damping effect, see (1)), \( b = \omega/10 \) (to stay safely at underdamped oscillations when the driving force is neglected), \( a = 10^{-10} \text{m} \) (typical small molecule dimension) and efficiency \( \kappa = 10^{-1} \) (to account for limited internal structure). We will compute the length over which the amplitude of gravitational waves is damped by 10%. For a dense medium \( (\rho \sim a^{-3}) \), this length is \( 10^{-12} \text{pc} \) for a frequency of 100 Hz (and \( 10^{-10} \text{pc} \) for 1 Hz). For average interstellar medium densities of our galaxy \( (10^4 \text{m}^{-3}) \), the length is, however, \( 10^{10} \text{pc} \) for 100 Hz \( (10^{12} \text{pc} \text{ at 1 Hz}) \).

The model is too simple and rough at this stage to make any final judgment considering the observational consequences. Some parameters have not been set by clear physically founded estimates. Future work should concentrate on providing a more complex model capturing more precisely the structure of the medium. If one is concerned with molecular structures, the approach should probably utilize quantum mechanics rather than the classical one at the ‘microscopic’ level. Also, the treatment of the curved background should be more consistent (e.g. the model neglects the effects of the curved background in the computation of the perturbation produced by a given layer).

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