A Complexity Preserving Transformation from Jinja Bytecode to Rewrite Systems

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We revisit known transformations from Jinja bytecode to rewrite systems from the viewpoint of runtime complexity. Suitably generalising the constructions proposed in the literature, we define an alternative representation of Jinja bytecode (JBC) executions as computation graphs from which we obtain a novel representation of JBC executions as constrained rewrite systems. We prove non-termination and complexity preservation of the transformation. We restrict to well-formed JBC programs that only make use of non-recursive methods. Our approach allows for simplified correctness proofs and provides a framework for the combination of the computation graph method with standard techniques from static program analysis.

1. Introduction

In recent years research on complexity of rewrite systems has matured and a number of noteworthy results could be established. We give a quantitative assessment based on the annual competition of complexity analysers within TERM COMP\(^1\). With respect to last year’s run of TERM COMP, we see a success rate of 38% in the category Runtime Complexity – Innermost Rewriting. Note that the corresponding testbed is not restricted to polynomial runtime complexity in any way. With respect to a qualitative assessment

\(^1\)http://termcomp.uibk.ac.at/
we want to mention the very recent efforts to apply methods from linear algebra and automata theory to complexity [19]; recent efforts on adaption of the dependency pair method to complexity [11, 12, 22, 13] and the ongoing quest to incorporate compositionality [31, 2]. (See [20] for an overview in methods of complexity analysis of term rewrite systems.)

In this paper we are concerned with the applicability of these results to automated runtime complexity analysis of imperative programs, in particular of Jinja bytecode (JBC) programs. Jinja is a Java-like language that exhibits the core features of Java [29]. Its semantics is clearly defined and machine checked in the theorem prover Isabelle/HOL [15].

We establish a complexity preserving transformation from JBC programs $P$ to constrained term rewrite systems $R$, that is, the runtime complexity function with respect to $P$ is bounded by the runtime complexity function with respect to $R$ (Theorem 6.1). As a simple corollary to this result we obtain that the proposed transformation is non-termination preserving (Corollary 6.1). In our analysis we restrict to well-formed JBC programs that only make use of non-recursive methods. The proposed transformation encompasses two stages. The first stage provides a finite representation of all execution paths of $P$ through a graph, dubbed computation graph (Theorem 5.1). The nodes of the computation graph are abstractions of JVM states and the graph is formed by symbolic execution essentially employing widening akin to those used in abstract interpretations [7]. We develop a new graph-based representation of abstractions of JVM states (Definition 4.4). Furthermore we show that finiteness of the computation graph can always be guaranteed (Lemma 5.1). In the second stage, we encode the (finite) computation graph as constrained term rewrite system (cTRS for short). CTRSs form a special type of rewrite systems that allow the formulation of conditions $C$ over a theory $T$, such that a rule can only be used if the condition $C$ is satisfied in $T$. Constraints are used to express relations on program variables.

We emphasise, that the proposed transformation is not directly automatable, but its implementation asks for a combination with an external shape analysis as presented for example in [27, 25, 33]. This allows the mating of the proposed term-based abstraction technique with more standard concepts from static program analysis. In principle, the established transformation allows for the use of rewriting-based runtime complexity analysis for the resource analysis of JBC programs. However, currently existing methods for complexity analysis do not (yet) extend to cTRSs; this is subject to future work.

1.1. Related Work

Our work was inspired by Panitz and Schmidt-Schauß original observation that term-based abstraction can provide powerful termination analysis [24]. Furthermore, we got inspiration from the ongoing quest to establish non-termination preserving transformations from JBC programs to integer term rewrite system [23, 6, 4]. The approach has been implemented in AProVe² and has shown significant power in comparison to dedicated complexity and termination tools for JBC programs [28, 11]. Comparing our work with

²http://aprove.informatik.rwth-aachen.de/
earlier results reported for the termination graph method \cite{23, 6} we see that a similar transformation from graphs to rewrite systems is employed. On the other hand in Otto et al. \cite{23} (and follow-up work) sharing is dealt with explicitly, while in our context sharing is always allowed if not stated otherwise. Furthermore Otto et al. rely on heuristics to obtain a finite termination graph, while we can prove finiteness of computation graphs.

Termination behaviour and complexity of JBC programs is studied by Albert et al. in \cite{1}. The approach employs program transformations to constrained logic programs and has been successfully implemented in the COSTA$^3$ tool; it often allows precise bounds on the resource usage and is not restricted to runtime complexity. A theoretical limitation of the work is the focus on a path-length analysis of the heap, which does not provide the same detail as the term based abstraction presented here. Zuleger et al. \cite{32} employs size-change abstraction to analyse the runtime complexity of C programs automatically. In connection with pathwise analysis and contextualisation size-change abstraction yields a powerful analysis. The approach has been implemented in the tool LOOPUS. Our approach extends the use of transition systems by cTRSs, which theoretically form a strict extension. Furthermore, as our methods are rooted in rewriting we are not limited to the powers of invariant generation tools. Very recently Hofmann and Rodrigues proposed in \cite{14} an automated resource analysis based on Tarjan’s amortised cost analysis \cite{30} for object-oriented programs. The method is implemented in the prototype RAJA$^4$.

1.2. Structure

This paper is structured as follows. In Sections \ref{sec:preliminaries} and \ref{sec:abstraction} we fix some basic notions to be used in the sequel. In particular, we give an overview over the Jinja programming language. Our notion of abstract states is presented in Section \ref{sec:abstraction} while computation graphs are proposed in Section \ref{sec:cg}. Section \ref{sec:trs} introduces cTRSs and presents the transformation from computation graphs to rewrite systems. In Section \ref{sec:implementation} we briefly mention crucial design choices for our prototype implementation. Finally, in Section \ref{sec:conclusion} we conclude.

2. Preliminaries

Let $f$ be a mapping from $A$ to $B$, denoted $f : A \rightarrow B$, then $\text{dom}(f) = \{x \mid f(x) \in B\}$ and $\text{rg}(f) = \{f(x) \mid x \in A\}$. Let $a \in \text{dom}(f)$. We define:

$$f\{a \mapsto v\}(x) := \begin{cases} v & \text{if } x = a \\ f(x) & \text{otherwise} \end{cases}$$

We compare partial functions with Kleene equality: Two partial functions $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ are equal, denoted $f =_k g$, if for all $n \in \mathbb{N}$ either $f(n)$ and $g(n)$ are defined and $f(n) = g(n)$ or $f(n)$ and $g(n)$ are not defined.

We usually use square brackets to denote a list. Further, ($::$) denotes the cons operator, and ($@$) is used to denote the concatenation of two lists.

\footnotesize
\begin{itemize}
  \item \texttt{http://costa.ls.fi.upm.es/}
  \item \texttt{http://raja.tcs.ifi.lmu.de}
\end{itemize}
Definition 2.1. A directed graph $G = (V_G, \text{Succ}_G, L_G)$ over the set $\mathcal{L}$ of labels is a structure such that $V_G$ is a finite set, the nodes or vertices, $\text{Succ}_G : V_G \rightarrow V_G^*$ is a mapping that associates a node $u$ with an (ordered) sequence of nodes, called the successors of $u$. Note that the sequence of successors of $u$ may be empty: $\text{Succ}_G(u) = \emptyset$. Finally $L_G : V_G \rightarrow \mathcal{L}$ is a mapping that associates each node $u$ with its label $L_G(u)$. Let $u, v$ be nodes in $G$ such that $v \in \text{Succ}_G(u)$, then there is an edge from $u$ to $v$ in $G$; the edge from $u$ to $v$ is denoted as $u \rightarrow v$.

Definition 2.2. A structure $G = (V_G, \text{Succ}_G, L_G, E_G)$ is called directed graph with edge labels if $(V_G, \text{Succ}_G, L_G)$ is a directed graph over the set $\mathcal{L}$ and $E_G : V_G \times V_G \rightarrow \mathcal{L}$ is a mapping that associates each edge $e$ with its label $E_G(e)$. Edges in $G$ are denoted as $u \xrightarrow{l} v$, where $E_G(u \rightarrow v) = l$ and $u, v \in V_G$. We often write $u \rightarrow v$ if the label is either not important or is clear from context.

If not mentioned otherwise, in the following a graph is a directed graph with edge labels. Usually nodes in a graph are denoted by $u, v, \ldots$ possibly followed by subscripts. We drop the reference to the graph $G$ from $V_G$, $\text{Succ}_G$, and $L_G$, i.e., we write $G = (V, \text{Succ}, L)$ if no confusion can arise from this. Further, we also write $u \in G$ instead of $u \in V$.

Let $G = (V, \text{Succ}, L)$ be a graph and let $u \in G$. Consider $\text{Succ}(u) = [u_1, \ldots, u_k]$. We call $u_i \ (1 \leq i \leq k)$ the $i$-th successor of $u$ (denoted as $u \xrightarrow{i} G u_i$). If $u \xrightarrow{i} G v$ for some $i$, then we simply write $u \xrightarrow{1} G v$. A node $v$ is called reachable from $u$ if $u \xrightarrow{1} G v$, where $\xrightarrow{1} G$ denotes the reflexive and transitive closure of $\rightarrow G$. We write $\xrightarrow{1} G$ for $\rightarrow G \circ \xrightarrow{1} G$. A graph $G$ is acyclic if $u \xrightarrow{1} G v$ implies $u \neq v$. We write $G \upharpoonright u$ for the subgraph of $G$ reachable from $u$.

3. Jinja Bytecode

In this section, we give an overview over the Jinja programming language [15]. In particular we inspect the internal state of the Jinja Virtual Machine (JVM). We expect the reader to be familiar with the Java programming language.

Definition 3.1. A Jinja value can be a Boolean of type bool, an (unbounded) integer of type int, the dummy value unit of type void, the null reference null of type nullable, or a reference (or address).

We usually refer to (non-null) references as addresses. The dummy value unit is used for the evaluation of assignments (see [15]) and also used in the JVM to allocate uninitialised local variables. The actual type of addresses is not important and we usually identify the type of an address with the type of the object bounded to the address.

Example 3.1. Figure 1 depicts a program defining a List class with the append method. Deviating from the notation employed by Klein and Nipkow in [15], we present Jinja code in a Java-like syntax.

In preparation for the sequent sections, we reflect the structure and properties of JBC programs and the JVM.
class List{
   List next;
   int val;

   void append(List ys){
      List cur = this;
      while(cur.next != null){
         cur = cur.next
      }
      cur.next = ys;
   }
}

Figure 1: The append program.

Definition 3.2. A JBC program $P$ consists of a set of class declarations. Each class is identified by a class name and further consists of the name of its direct superclass, field declarations and method declarations. The superclass declaration is non-empty, except for a dedicated class termed Object. Moreover, the subclass hierarchy of $P$ is tree-shaped. A field declaration is a pair of field name and field type. A method declaration consists of the method name, a list of parameter types, the result type and the method body. A method body is a triple of $(\text{mxs} \times \text{mxl} \times \text{instructionlist})$, where $\text{mxs}$ and $\text{mxl}$ are natural numbers denoting the maximum size of the operand stack and the number of local variables, not including the this reference and the parameters of the method, while instructionlist gives a sequence of bytecode instructions. The this reference can be conceived as a hidden parameter and references the object that invokes the method.

The set of Jinja bytecode instructions is adapted for our needs and listed in Figure 2. We employ following conventions: Let $n$ denote a natural number, $i$ an integer, $v$ a Jinja value, $cn$ a class name, and $mn$ a method name.

\[
\text{Ins} := \text{Load } n | \text{Store } n | \text{Push } v | \text{Pop} \\
| \text{IAdd} | \text{ISub} | \text{ICmpGt} | \text{CmpEq} | \text{CmpNeq} | \text{BAnd} | \text{BOr} | \text{BNot} \\
| \text{Goto } i | \text{IfFalse } n | \\
| \text{New } cn | \text{Getfield } fn cn | \text{Putfield } fn cn | \text{Checkcast } cn | \\
| \text{Invoke } mn n | \text{Return}
\]

Figure 2: The Jinja bytecode instruction set.

Definition 3.3. A (JVM) state is a pair consisting of the heap and a list of frames. Let $\prec$ denote the strict subclass relation and $\preceq$ its reflexive closure. A heap is a mapping from addresses to objects, where an object is a pair $(cn,fTable)$ such that:

- $cn$ denotes the class name, and
• ftable denotes the fieldtable, i.e., a mapping from \((cn', fn)\) to values, where \(fn\) is a field name and \(cn'\) is a (not necessarily proper) superclass of \(cn\), i.e., \(cn \preceq cn'\).

A frame represents the environment of a method and is a quintuple \((stk, loc, cn, mn, pc)\), such that:

• \(stk\) denotes the operation stack, i.e., an array of values,
• \(loc\) denotes the registers, i.e., an array of values,
• \(cn\) denotes the class name,
• \(mn\) denotes the method name, and
• \(pc\) is the program counter.

Let \(stk(loc)\) denote the operation stack (registers) of a given frame. Typically the structure of \(loc\) is as follows: the 0th register holds the this-pointer, followed by the parameters and the local variables of the method. Uninitialised registers are preallocated with the dummy value unit. We denote the entries of \(stk\) (\(loc\)) by \(stk(i)\), \(loc(i)\) for \(i \in \mathbb{N}\) and write \(\text{dom}(stk)\) (\(\text{dom}(loc)\)) for the set of indices of the array \(stk\) (\(loc\)). The collection of all stack (register) indices of a state is denoted \(Stk(Loc)\). Often there is no need to separate between the local variables of a Jinja program and the registers in a JBC program. Hence we use registers and local variables interchangeably. Observe that the domain of the fieldtable for a given object of class \(cn\) contains all fields declared for \(cn\) together with all fields declared for superclasses of \(cn\). Clearly the domain of the fieldtable is equal for any instance of class \(cn\).

Figure 3 illustrates the one-step execution of the \texttt{IAdd} bytecode instruction. We have extended the original set of instructions by some standard operations on values, taking ideas from Jinja with Threads into account [17, 18]. The semantics of all employed JBC instructions can be found in the Appendix.

\[
\text{IAdd} \quad \frac{(heap, (i_2 :: i_1 :: stk, loc, cn, mn, pc) :: frms)}{(heap, ((i_2 + i_1) :: stk, loc, cn, mn, pc + 1) :: frms)}
\]

Figure 3: The \texttt{IAdd} bytecode instruction.

Example 3.2. Consider the append program from Example 3.1. Figure 4 depicts the corresponding bytecode program, resulting from the compilation rules in [15]. In the following we name the registers 0, 1, and 2 as this, ys, and cur, respectively.

Definition 3.4. We extend the subclass relation to a partial order on types, denoted \(\leq_{\text{type}}\). The types of \(P\) consists of \{bool, int, void, nullable\} together with all classes \(cn\) defined in \(P\). We use \(\text{type}(v)\) to denote the type of value \(v\) and \(\text{types}(P)\) to denote the collection of types in \(P\). Recall that we usually identify the type of an address with the type of the object bound to the address. Let \(t, t', cn, cn'\) be types in \(P\). Then \(t \leq_{\text{type}} t'\) holds if \(t = t'\) or
Class: List

Name: List

Classbody: 00: Load 0
Superclass: Object 01: Store 2
Fields: 02: Push unit
List next 03: Pop
int val 04: Load 2
Methods: 05: Getfield next List
Method: unit append 06: Push null
Parameters: 07: CmpNeq
List ys 08: IfFalse 7
Methodbody: 09: Load 2
MaxStack: 10: Getfield next List
2 11: Store 2
MaxVars: 12: Push unit
1 13: Pop
14: Goto -10
15: Push unit
16: Pop
17: Load 2
18: Load 1
19: PutField next List
20: Push unit
21: Return

Figure 4: The bytecode for the List program.

- \( t = \text{void} \),
- \( t = \text{nullable} \) and \( t' = cn \),
- \( t = cn, t' = cn' \) and \( cn \leq cn' \).

The least common superclass is the least upper bound for a set of classes \( CN \subseteq \text{types}(P) \) and is always defined.

The bytecode verifier established in [15] ensures following properties: All bytecode instructions are provided with arguments of the expected type. No instruction tries to get a value from the empty stack, nor puts more elements on the stack or access more registers than specified in the method. The program counter is always within the code array of the method. All registers except from the register storing this must be first written to before accessed. Furthermore the verifier ensures that for states with equal program counter the size of the stack is of equal length. Moreover, the list of registers is of fixed length. The compiler presented in [15] transforms a well-formed Jinja program into a well-formed JBC program. A JBC program that passes the bytecode verification is again called well-formed.
While the set of instruction used here are a (slight) extension of the minimalistic set considered in [15], this notion of well-formedness is still applicable, as all considered extensions are present in Jinja with Threads [17, 18]. In the following we consider Jinja programs and JBC programs to be well-formed. To ease readability we do not consider exception handling, that is, an exception yields immediate termination of the program. This is not a restriction of our analysis, as it could be easily integrated, but complicates matters without gaining additional insight.

While Definition 3.3 provides a succinct presentation of the state, it is more natural to conceive the heap (and conclusively a state) as a graph. We omit the technical definition here but provide the general idea: Let $s = (heap, frms)$ be a state. We define the state graph of $s$ as $S = (V, Succ, L, E)$. For all non-address values of $s$ we define an unique implicit reference. The idea is that sharing is only induced via references but not implicit references. The nodes of $S$ consists of all stack (register) indices, the references in $heap$ and the implicit references of $s$. The successors of a node indicate the values bound to stack (register) indices and the fields of instances in $heap$, and is an implicit reference if a non-address value is bound and a reference otherwise. The label of a node is either a stack (register) index, the type of an instance $heap(u)$ or a non-address value. The label of an edge indicates the fields $(cn, id)$ for instances $heap(u)$, and is empty otherwise.

In presenting state graphs, we indicate references, but do not depict implicit references. Furthermore, we use representative names for stack (register) indices.

Example 3.3. Recall the append program of example 3.1. Suppose $this$ is initially a list of length one, and $ys$ is null. Figure 5 depicts the state graph after the assignment $cur = this$.

![Figure 5: State graph.](image)

Let $P$ be a program and let $s$ and $t$ be states. Then we denote by $P: s \xrightarrow{jvm} t$ the one-step transition relation of the JVM. If there exists a (normal) evaluation of $s$ to $t$, we write $P: s \xrightarrow{jvm} t$. Let $JS$ denote the set of states. The complete lattice $P(JS) := (P(JS), \subseteq, \cup, \cap, \emptyset, JS)$ denotes the concrete computation domain.

The size of a state is defined on a per-reference basis, which unravels sharing. We explicitly add 1 to the overall construction. This does not affect the results but allows a more convenient relation to the size of its term representation we present later.
Definition 3.5. Let \( s \) be a state and let \( S \) be its state graph. Let \( u, v \) be nodes in \( S \) and \( u \rightarrow_S v \) denote a simple path \( P \) in \( S \) from \( u \) to \( v \). Note that \( P \) does not contain cycles. Then the size of a stack or register index \( u \), denoted as \(|u|\), is defined as follows:

\[
|u| := \sum_{u \rightarrow_S v} |L_S(v)|,
\]

where \( |l| \) is \( \text{abs}(l) \) if \( l \in \mathbb{Z} \), otherwise 1, for \( l \in L_S \). Here, \( \text{abs}(z) \) denotes the absolute value of the integer \( z \). Then the size of \( s \) is the sum of all sizes of stack or register indices in \( S \) plus 1. In the following we use \(|s|\) to denote the size of a state \( s \).

We define the runtime of a JVM for a given normal evaluation \( P: s \xrightarrow{\text{jvm}} t \) as the number of single-step executions in the course of the evaluation from \( s \) to \( t \).

Definition 3.6. Let \( JS \) denote the set of JVM states of \( P \), and \( S \subseteq JS \). We define the runtime complexity with respect to \( P \) as follows:

\[
rc_{\text{jvm}}(n) = k \max \{ m \mid P:\ i \xrightarrow{\text{jvm}} t \text{ holds such that the runtime is } m, \ i \in S \text{ and } |i| \leq n \}.
\]

Note that we adopt a (standard) unit cost model for system calls.

4. Abstract States

In this section, we introduce abstract states as generalisations of JVM states. The intuition being that abstract states represent sets of states in the JVM. The idea of abstracting JVM states in this way is due to Otto et al. [23]. However, our presentation crucially differs from [23] (and also from follow-up work in the literature) as we employ an implicit representation of sharing that makes use of graph morphisms, rather than the explicit sharing information proposed in [23, 6, 5, 4]. Furthermore, abstract states as defined below are a straightforward generalisation of JVM states as defined in [15]. This circumvents an additional transformation step as presented in [6].

Definition 4.1. We extend Jinja expressions by countable many abstract variables \( X_1, X_2, X_3, \ldots \), denoted by \( x, y, z, \ldots \). An abstract variable may either abstract an object, an integer or a Boolean value.

In denoting abstract variables typically the name is of less importance than the type, that is we denote an abstract variable for an object of class \( cn \), simply as \( cn \), while abstract integer or Boolean variables are denoted as \( \text{int} \), and \( \text{bool} \), respectively. The (strict) subclass relation \( \prec \) is extended in the natural way to abstract variables for classes. For brevity we sometimes refer to an abstract variable of integer or Boolean type, as abstract integer or abstract Boolean, respectively.

Definition 4.2. An abstract value is either a Jinja value (cf. Definition 3.1), or an abstract Boolean or integer. In turn a Jinja value is also called a concrete value.
Note that, as in the JVM, only (abstract) objects can be shared. In particular abstract variables for objects are only referenced via the heap. The next definition abstracts the heap of a JVM through the use of abstract variables and values.

**Definition 4.3.** An abstract heap is a mapping from addresses to abstract objects, where an abstract object is either a pair \((cn, ftable)\) or an abstract variable. Abstract frames are defined like frames of the JVM, but registers and operand stack of an abstract frame store abstract values.

We define (partial) projection functions \(cl\) and \(ft\) as follows:

\[
cl(obj):= \begin{cases} 
    cn & \text{if } obj \text{ is an object and } obj = (cn, ftable) \\
    cn & \text{if } obj \text{ is an abstract variable of type } cn \\
    ftable & \text{if } obj \text{ is an object and } obj = (cn, ftable) \\
    \text{undefined} & \text{otherwise}
\end{cases}
\]

Furthermore, we define annotations of addresses in an abstract state \(s\), denoted as \(iu\). Formally, annotations are pairs \(p \neq q\) of addresses, where \(p, q \in heap\) and \(p\) is not \(q\).

**Definition 4.4.** An abstract state \(s = (heap, frms, iu)\) is either a triple consisting of an abstract heap \(heap\), a list of abstract frames \(frms\), and a set of annotations \(iu\), the maximal abstract state, denoted as \(\top\), or the minimal abstract state, denoted as \(\bot\). If \(s = (heap, frms, iu)\), we demand that all addresses in \(heap\) are reachable from local variables or stack entries in the list of frames \(frms\). The set of abstract states is collected in the set \(AS\).

When depicting (abstract) states, we replace stack and register indices by intuitive names, denoted in roman font. Furthermore, we make use of the following conventions: we use an italic font (and lower-case) to describe abstract variables and a sans serif (and upper-case) to depict class names.

**Example 4.1.** Consider the List program from Example 3.1 together with the well-formed JBC program depicted in Figure 4. Consider the state \(A\) depicted below:

\[
04 \epsilon | this = o_1, ys = o_2, cur = o_1 \\
A \quad o_1 = \text{List}(\text{List.val} = \text{int}, \text{List.next} = o_3) \\
\quad o_2 = \text{list}, o_3 = \text{list}
\]

The operation stack in \(A\) is empty. The registers \(this\) and \(cur\) contain the same address \(o_1\) and \(ys\) is mapped to \(o_2\). In the heap \(o_1\) is mapped to an object of type List whose value is abstracted to \(\text{int}\) and whose next element is referenced by \(o_3\). It is not difficult to see that \(A\) forms an abstraction of any JVM state obtained at instruction 04 in the List program (if \(this\) initially references a non-empty list) before any iteration of the while-loop. Furthermore, consider the following state \(B\):

\[
04 \epsilon | this = o_1, ys = o_2, cur = o_3 \\
B \quad o_1 = \text{List}(\text{List.val} = \text{int}, \text{List.next} = o_3) \\
\quad o_2 = \text{list}, o_3 = \text{list}
\]

\[
\quad o_3 = \text{List}(\text{List.val} = \text{int}, \text{List.next} = o_4)
\]
Again it is not difficult to see that $B$ abstracts any JVM state obtained if exactly one iteration of the loop has been performed.

Due to the presence of abstract variables, abstract states can represent sets of states as the variables can be suitably instantiated. The annotation $p \neq q \in iu$ will be used to disallow aliasing of addresses in JVM states represented by the abstract state. Different JVM states can be abstracted to a single abstract state. To make this precise, we will augment $\mathcal{AS}$ with a partial order $\sqsubseteq$, the instance relation (see Definition 4.6). We will extend the partial order $(\mathcal{AS}, \sqsubseteq)$ to a complete lattice $\mathcal{AS} := (\mathcal{AS}, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and show a Galois insertion between $P(\mathcal{JS})$ and $\mathcal{AS}$.

**Definition 4.5.** We define a preorder on abstract values, which are not references, and abstract objects. We extend $\text{type}(v)$ (cf. Definition 3.4) to abstract values the intended way, i.e., $\text{type}(\text{int}) = \text{int}$, $\text{type}(\text{bool}) = \text{bool}$ and $\text{type}(cn) = cn$ for an integer variable $\text{int}$, a Boolean variable $\text{bool}$, and class variable $cn$. Then the preorder $\sqsubseteq$ is defined as follows:

1. $v = w$, or
2. $\text{type}(v) \sqsubseteq \text{type}(w)$ and $w$ is an abstract variable.

We write $w \trianglerighteq v$, if $v \sqsubseteq w$.

Let $|stk|$, $|loc|$ denote the maximum size of the operand stack and the number of variables respectively. We make use of the following abbreviation: $w \trianglerighteq v$ if either $w \trianglerighteq v$ or $v, w$ are references and we have $v = m(w)$, where $m$ denotes a mapping on references.

**Definition 4.6.** Let $s = (heap, frms, iu)$ be a state in $\mathcal{AS} \setminus \{\top, \bot\}$ with $frms = [frm_1, \ldots, frm_k]$ and $frm_i = (stk_i, loc_i, cn_i, mn_i, pc_i)$, and let $t = (heap', frms', iu')$ be a state with $frms' = [frm'_1, \ldots, frm'_k]$ and $frm'_i = (stk'_i, loc'_i, cn'_i, mn'_i, pc'_i)$. Then $s$ is an abstraction of $t$ (denoted as $s \sqsupseteq t$) if the following conditions hold:

1. for all $1 \leq i \leq k$: $pc_i = pc'_i$, $cn_i = cn'_i$, and $mn_i = mn'_i$;
2. for all $1 \leq i \leq k$: $\text{dom}(stk_i) = \text{dom}(stk'_i)$ and $\text{dom}(loc_i) = \text{dom}(loc'_i)$, and
3. there exists a mapping $m: \text{dom}(heap) \rightarrow \text{dom}(heap')$ such that
   - for all $1 \leq i \leq k$, $1 \leq j \leq |stk_i|$: $stk_i(j) \trianglerighteq_m stk'_i(j)$,
   - for all $1 \leq i \leq k$, $1 \leq j \leq |loc_i|$: $loc_i(j) \trianglerighteq_m loc'_i(j)$,
   - for all $a \in \text{dom}(heap)$: $heap(a) \trianglerighteq heap'(m(a))$,
   - for all $a \in \text{dom}(heap)$, such that $\text{ft}(heap(a))$ is defined and for all $1 \leq i \leq \ell$: $f(cn_i, id_i) \trianglerighteq_m f'(cn'_i, id_i)$, where $f := \text{ft}(heap(a))$ with $\text{dom}(f) = \{(cn_1, id_1), \ldots, (cn_\ell, id_\ell)\}$, and $f' := \text{ft}(heap'(m(a)))$ with $\text{dom}(f') = \{(cn_1, id_1), \ldots, (cn_\ell, id_\ell)\}$.
4. finally, we have $iu' \trianglerighteq_m m^*(iu)$.
Here, $m^*$ denotes the lifting of the mapping $m$ to sets: $m(\{isunshared_1, \ldots, isunshared_k\}) = \{m(isunshared_1), \ldots, m(isunshared_k)\}$. Furthermore for all $s \in \mathcal{A}S$: $s \subseteq \top$ and $\bot \subseteq s$.

**Example 4.2.** Consider the states $A$ and $B$ described in Example 4.1. For the state $S$ depicted below we obtain that $A \subseteq S$ and $B \subseteq S$, i.e., $S$ forms an abstraction of both states.

| 04 | $\epsilon | this = o_1, ys = o_2, cur = o_4$ |
|----|-----------------------------------------------|
|    | $o_1 = \text{List}(\text{List.val} = \text{int}, \text{List.next} = o_3)$ |
|    | $o_2 = \text{list}, o_3 = \text{list}, o_5 = \text{list}$ |
|    | $S | o_4 = \text{List}(\text{List.val} = \text{int}, \text{List.next} = o_2)$ |

The definition of state graphs naturally extends to abstract states, when incorporating $isunshared$ and considering abstract values. Furthermore, we use $\top$ to denote the state graph of $\top \in \mathcal{A}S$ and the empty graph to denote $\bot \in \mathcal{A}S$.

**Example 4.3.** Consider the states $A$, $B$, and $S$ presented in Examples 4.1 and 4.2. The state graph of $A$ and $B$ are given in Figure 6 and Figure 7, respectively. The state graph of the abstraction $S$ is depicted in Figure 8.

![Figure 6: Abstract State A](https://example.com/figure6.png)

![Figure 7: Abstract State B](https://example.com/figure7.png)

We introduce **state homomorphisms** that allow an alternative, but equivalent definition of the instance relation $\subseteq$.

**Definition 4.7.** Let $S$ and $T$ be state graphs of states $s$ and $t$, respectively such that $S,T \neq \emptyset$. A **state homomorphism** from $S$ to $T$ (denoted $m: S \rightarrow T$) is a function $m: V_S \rightarrow V_T$ such that

1. for all $u \in S$ and $u \in Stk \cup Loc$, $L_S(u) = L_T(m(u))$,
2. for all $u \in S \setminus (Stk \cup Loc)$, $L_S(u) \supseteq L_T(m(u))$,
3. for all $u \in S$: if $u \xrightarrow{\ell} v$, then $m(u) \xrightarrow{m} m(v)$ and
4. for all $u \xrightarrow{\ell} v \in S$ and $m(u) \xrightarrow{\ell'} m(v) \in T$, $\ell = \ell'$. 

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Lemma 4.1. Let $iu$. Straightforward.

Due to Lemma 4.1 and the composability of morphism it follows that the instance relation $\subseteq$ is transitive. Hence the relation $\subseteq$ is a preorder. Furthermore $\subseteq$ can be lifted to a partial order, if we consider the factorisation of the set of abstract states with respect to the equivalence relation $\sim$. In order to express this fact notationally, we identify isomorphic states and replace $\sim$ by $\equiv$. Conclusively $(\mathcal{AS}, \subseteq)$ is a partial order.

We are left to provide a least upper bound definition of the join of abstract states.

**Definition 4.8.** Let $s$ and $s'$ be states such that there exists an abstraction $t$ of $s$ and $s'$, respectively. We call $t$ the join of $s$ and $s'$, denoted as $s \sqcup s'$, if $t$ is a least upper bound of $\{s, s'\}$ with respect to the preorder $\subseteq$.

The limit cases are handled as usual. If the program locations of $s$ and $s'$ differ, then $s \sqcup s' = \top$. Otherwise, we can identify invariants to construct an upper bound $t \neq \top$ and prove well-definedness of $s \sqcup s'$. Let $S = (V_S, Succ_S, L_S, E_S, iu_S)$ and $S' = (V_{S'}, Succ_{S'}, L_{S'}, E_{S'}, iu_{S'})$ be the two state graphs of state $s$ and $s'$, respectively. Furthermore, let $t$ be an abstraction of $s$ and $s'$, and let $T = (V_T, Succ_T, L_T, E_T, iu_T)$ be its state graph. By definition we have the following properties:

1. Let $Stk$ ($Loc$) collect the stack (register) indices of state $s$. As $s \subseteq t$, $Stk$ ($Loc$) coincides with the set of stack (register) indices of $t$. Similarly for $s'$ and thus $V_T \supseteq Stk \cup Loc$.

2. For any node $u \in T$ there exist uniquely defined nodes $v \in V_S$, $w \in V_{S'}$ such that $L_S(v) \subseteq L_T(u)$, $L_{S'}(w) \subseteq L_T(u)$. We say the nodes $v$ and $w$ correspond to $u$.

3. For any node $u \in T$ and any successor $u'$ of $u$ in $T$ there exists a successor $v'$ ($w'$) in $S$ ($S'$) of the corresponding node $v$ ($w$) in $S$ ($S'$). Furthermore $v'$ and $w'$ correspond to $u'$.

4. For any edge $u \xrightarrow{\ell} u' \in T$ such that $v$ ($w$) corresponds to $u$ in $S$ ($S'$) there is an edge $v \xrightarrow{k} v' \in S$ and an edge $w \xrightarrow{k'} w' \in S'$ such that $\ell = k = k'$.
5. For any annotation \( u \neq u' \in iu_T \) there exists \( v \neq v' \) in \( iu_S \) and \( w \neq w' \) in \( iu_{S'} \), where \( v (v') \) and \( w (w') \) correspond to \( u (u') \).

In order to construct an abstraction \( t \) of \( s \) and \( s' \) we use the above properties as invariants and define its state graph \( T \) by iterated extension. We define \( T^0 \) by setting \( V_{T^0} := Stk \cup Loc \). Due to Property 4 these nodes exist in \( S \) and \( S' \) as well. The labels of stack or register indices trivially coincide in \( S \) and \( S' \), cf. Definition 4.7. Thus we set \( L_{T^0} \) accordingly. Furthermore we set \( Succ_{T^0} = E_{T^0} := \emptyset \). Then \( T^0 \) satisfies Properties 1–5.

Suppose state graph \( T^n \) has already been defined such that the Properties 1–4 are fulfilled. In order to update \( T^n \), let \( u \in V_{T^n} \) such that \( v \) and \( w \) correspond to \( u \). Suppose \( v \xrightarrow{n} v' \in S \) and \( w \xrightarrow{n} w' \in S' \) such that there is no node \( u' \) in \( T^n \) where \( v' \) and \( w' \) correspond to \( u' \). Let \( u' \) denote a node fresh to \( T^n \). We define \( V_{T_{n+1}} := V_{T^n} \cup \{ u' \} \) and establish Property 2 by setting \( L_{T_{n+1}}(u') \) such that \( L_{S}(v') \culc L_{T_{n+1}}(u') \) and \( L_{S'}(w') \culc L_{T_{n+1}}(u') \) where \( L_{T_{n+1}}(u') \) is as concrete as possible. If we succeed, we fix that \( v' \) and \( w' \) correspond to \( u' \). It remains to update \( iu_{T_{n+1}} \) suitably such that Property 5 is fulfilled. If this also succeeds Properties 1–5 are fulfilled for \( T^{n+1} \). On the other hand, if no further update is possible we set \( T := T^n \). By construction \( T \) is an abstraction of \( S \) and \( S' \) and indeed represents \( s \sqcup s' \).

**Example 4.4.** Consider the states \( A, B, \) and \( S \) described in Example 4.3. In Figure 8 an abstraction of \( A \) and \( B \) is given. In particular, abstraction \( S \) results of the construction defined above, ie., \( S = A \sqcup B \).

![Abstraction S](image)

A sequence of states \( (s_i)_{i \geq 0} \) forms an ascending sequence, if \( i < j \) implies \( s_i \sqsubseteq s_j \). An ascending sequence \( (s_i)_{i \geq 0} \) eventually stabilises, if there exits \( i_0 \in \mathbb{N} \) such that for all \( i \geq i_0 \): \( s_i = s_{i_0} \). The next lemma shows that any ascending sequence eventually stabilises.

**Lemma 4.2.** The partial order \( (AS, \sqsubseteq) \) satisfies the ascending chain condition, that is, any ascending chain eventually stabilises.

**Proof.** In order to derive a contradiction we assume the existence of an ascending sequence \( (s_i)_{i \geq 0} \) that never stabilises. By definition for all \( i \geq 0 \): \( |s_i| \geq |s_{i+1}| \). By assumption there exists \( i \in \mathbb{N} \) such that for all \( j > i \): \( |s_i| = |s_j| \) and \( s_i \sqsubset s_j \). The only
possibility for two different states \( s_i, s_j \) of equal size that \( s_i \subseteq s_j \) holds, is that addresses shared in \( s_i \) become unshared in \( s_j \). Clearly this is only possible for a finite amount of cases. Contradiction.

Lemma 4.2 in conjunction with the fact that \((\mathcal{A}S, \sqsubseteq)\) has a least element \( \perp \) and binary least upper bounds implies that \((\mathcal{A}S, \sqsubseteq, \bigcup, \perp, \top)\) is a complete lattice. In particular any set of states \( \mathcal{S} \) has a least upper bound, denoted as \( \bigcup \mathcal{S} \). The meet operation \( \sqcap \) can be expressed by \( \bigcup \), yet in practice we do not need it.

4.1. Correctness

In the remainder of the paper we fix to a concrete JBC program \( P \). Above, we already restricted our attention to well-formed JBC programs \( P \) using the expressions and instructions defined in Section 3. For the proposed static analysis of these programs we additionally restrict to non-recursive methods. Note that the states in \( \mathcal{A}S \) can in principle express recursive methods, but for recursive methods, we cannot use the below proposed construction to obtain finite computation graphs, as the graphs defined in Definition 5.1 cannot handle unbounded list of frames. In the following we use superscript \( \natural \), if we want to distinguish between concrete and abstract states, or between operations on concrete and abstract states.

Let \( s = (heap, frms) \in \mathcal{JS} \), we define a mapping \( \beta: \mathcal{JS} \to \mathcal{A}S \), that injects JVM states into \( \mathcal{A}S \). For that let \( \text{dom}(heap) = \{p_1, \ldots, p_n\} \) and define \( iu \) such that all \( p_i \neq p_j \in iu \) for all different \( i, j \).

Definition 4.9. We define the abstraction function \( \alpha: \mathcal{P}(\mathcal{JS}) \to \mathcal{A}S \) and the concretisation function \( \gamma: \mathcal{A}S \to \mathcal{P}(\mathcal{JS}) \) as follows:

\[
\alpha(S) := \bigcup \{\beta(s) \mid s \in S\}, \\
\gamma(s^\natural) := \{s \in \mathcal{JS} \mid \beta(s) \sqsubseteq s^\natural\}.
\]

We set \( \alpha(s) := \alpha(\{s\}) \).

It is easy to see that \( \mathcal{A}S \) contains redundant states: Consider abstract states \( s^\natural, t^\natural \in \mathcal{A}S \). Let \( s^\natural = (heap, frms, iu) \), \( p, q \in \text{dom}(heap) \) and \( p \neq q \in iu \). Let \( t^\natural \) be defined like \( s^\natural \) but \( p \neq q \notin iu \). Now suppose that the types of \( p \) and \( q \) are not related with respect to the subclass order. Then \( s^\natural \sqsubseteq t^\natural \) and \( \gamma(s^\natural) = \gamma(t^\natural) \). To form a Galois insertion between \( \mathcal{P}(\mathcal{JS}) \) and \( \mathcal{A}S \), we introduce a reduction operator that adds annotations for non-aliasing addresses.

Definition 4.10. Let \( s^\natural = (heap, frms, iu) \) be an abstract state. We define the reduction operator \( \varsigma: \mathcal{A}S \to \mathcal{A}S \) as follows:

\[
\varsigma(s^\natural) := (heap, frms, iu'),
\]

where \( iu' := \{p \neq q \mid p, q \in \text{dom}(heap)\} \setminus \{p \neq q \mid s \in \gamma(s^\natural), m : s^\natural \to \beta(s), m(p) = m(q)\} \). Then \( \varsigma(s^\natural) \sqsubseteq s^\natural \) and \( \gamma(\varsigma(s^\natural)) = \gamma(s^\natural) \).
In practice, we compute the reduction by a unification argument of \( p \) and \( q \) in \( s^\natural \): We try to construct a new state \( t^\natural \subseteq s^\natural \), where \( r = m(p) = m(q) \). Let \( T^\natural \) and \( S^\natural \) be the state graphs of \( t^\natural \) and \( s^\natural \). Suppose \( u, v, w \) represent \( r, p, q \) in \( T^\natural \) and \( S^\natural \). We can use a similar reasoning we used for the join construction, but now require \( L_{T^\natural}(u) \subseteq L_{S^\natural}(v) \) and \( L_{T^\natural}(u) \subseteq L_{S^\natural}(w) \) if \( v \) and \( w \) correspond to \( u \). If the construction succeeds, we can easily find a concrete state from \( t^\natural \) such that \( m(p) = m(q) \). The construction does not succeed if, for example, successors of corresponding nodes have different concrete values; then we add \( p \neq q \).

**Lemma 4.3.** The maps \( \alpha \) and \( \gamma \) define a Galois insertion between the complete lattices \( \mathcal{P}(\mathcal{JS}) \) and \( \zeta^*(\mathcal{AS}) \), where \( \zeta^* \) denotes the set extension of \( \zeta \).

**Proof.** It suffices to prove that \( \gamma \) is injective, i.e., for all \( s^\natural, t^\natural \in \zeta^*(\mathcal{AS}) \) if \( s^\natural \neq t^\natural \) then \( \gamma(s^\natural) \neq \gamma(t^\natural) \). Suppose \( s^\natural \neq t^\natural \) but \( \gamma(s^\natural) = \gamma(t^\natural) \). It is a simple consequence of our morphism definition that \( \gamma(s^\natural) \neq \gamma(t^\natural) \), if the state graphs of \( s^\natural \) and \( t^\natural \) differ. Hence, \( s^\natural \) can only be different from \( t^\natural \) if the annotations of \( s^\natural \) and \( t^\natural \) differ. However, by assumption they are equal. Contradiction.

It follows that the reduction operator defined in Definition 4.10 indeed returns the greatest lower bound that represents the same element in the concrete domain as required. In the following we identify the \( \zeta^*(\mathcal{AS}) \) with \( \mathcal{AS} \).

In order to prove that the abstract domain \( \mathcal{AS} \) correctly approximates the concrete domain \( \mathcal{P}(\mathcal{JS}) \) we need to define a suitable notion of abstract computation on abstract states. Recall that Figure 3 presents the single-step execution of the IAdd instruction on the JVM. Based on these instructions, and actually mimicking them quite closely, we define how abstract states are evaluated symbolically. This is straightforward in most cases, with the exception of Putfield and CmpEq instructions. With respect to the former, we suppose a preliminary analysis on different heap shape properties. In particular our analysis requires may-share, may-reachable, and maybe-cyclic analyses as given, see for example [27, 25, 33].

**Definition 4.11.** Let \( s^\natural \) be an abstract state and \( p, q \) be addresses in the heap of \( s \). We use \( S \) to denote the state graph of \( \beta(s) \) for some concrete state \( s \). We say that:

- \( p \) and \( q \) may-alias, if \( m(p) = m(q) \) for some \( s \in \gamma(s^\natural) \) and morphism \( m: s^\natural \rightarrow \beta(s) \);
- \( p \) may-reaches \( q \), if \( m(p) \rightarrow_{s}^\natural m(q) \) for some \( s \in \gamma(s^\natural) \) and morphism \( m: s^\natural \rightarrow \beta(s) \);
- \( p \) is maybe-cyclic, if \( m(p) \rightarrow_{s}^\natural m(p) \) for some \( s \in \gamma(s^\natural) \) and morphism \( m: s^\natural \rightarrow \beta(s) \);
- \( p \) is acyclic, if \( p \) is not maybe-cyclic.

Note that our representation does not provide a precise approximation of these properties, as abstract variables generally also present cyclic instances.

In Figure 9 we have worked out the cases for the instructions Load\[^2\], IAdd\[^2\], CmpEq\[^2\], IfFalse\[^2\], New\[^1\] and Putfield\[^2\]. We follow the notation used in Figure 3 above. The other cases are left to the reader. In addition to symbolic evaluations, we define refinement
steps on abstract states $s^\natural$ if the information given in $s^\natural$ is not concrete enough to execute a given instruction. It will be a consequence of our definitions that for any refinement $s^\natural_i$ of $s^\natural$, we have $s^\natural_i \sqsubseteq s^\natural$.

In the following assume $s^\natural = (heap, frms, iu)$. Some comments: The symbolic instruction $\text{Load}^\natural n$ loads the value of the $n$th register onto the stack. The only difference to $\text{Load} n$ is that the value may be an integer or Boolean variable. For the $\text{IAdd}^\natural$ instruction, we introduce a new abstract integer $i_3$ and the side-condition $i_1 + i_2 = i_3$, if either $i_1$ or $i_2$ is an integer variable. The $\text{CmpEq}^\natural$ splits into different cases, depending on the status of the compared values. We adapt the instruction to abstract values as follows:

1. Let $val_1$ and $val_2$ be addresses. If the addresses of $val_1$ and $val_2$ are the same then the test evaluates to $\text{true}$. Otherwise, we have to check if $val_1$ and $val_2$ may alias and perform a unsharing refinement (cf. Definition 4.13) if necessary. In the latter case the test returns $\text{false}$.

2. Wlog. let $val_1$ be an address and $val_2$ be $\text{null}$. If $heap(val_1) = obj$ and $cl(obj) = cn$, we perform a instance refinement according to Definition 4.12 on $val_1$ and reconsider the condition.

3. If $val_1$ and $val_2$ are concrete non-address Jinja values, then the test $(val_1 = val_2)$ can be directly executed and the symbolic execution equals the instruction on the JVM.

4. If $val_1$ and $val_2$ are abstract Boolean or integer variables, then we introduce a new Boolean variable $b_3$ and the side condition $(val_1 = val_2) \equiv b_3$. Figure 9 only shows the latter case.

$\text{New}^\natural cn$ allocates a new instance of type $cn$ in the heap and pushes the corresponding address onto the stack. All fields of the fresh created instance are instantiated with the default value. That is, 0 for integer typed fields, $\text{false}$ for Boolean typed fields, and $\text{null}$ otherwise. If the top element of the stack is a concrete value, $\text{IfFalse}^\natural$ can be executed directly. Otherwise we perform a Boolean refinement, replacing the variable with values $\text{true}$ and $\text{false}$. Recall that a class variable $cn$ represents $\text{null}$ as well as instances of $cn$ and its subtypes. Hence, $\text{Putfield}^\natural fn\ cn'$ may require an instance refinement (cf. Definition 4.12). Let $v$ be a value and $p$ be an address such that $heap(p) = (cn'', \text{ftable})$. Due to abstraction there may exist addresses $q \in \text{dom}(heap)$ different from $p$ that alias with $p$. Hence they are affected by the field update. We introduce unsharing refinements (cf. Definition 4.13) for all $q$, where $p \neq q \notin iu$.

**Definition 4.12.** Let $s^\natural = (heap, frms, iu)$ be a state and let $p$ be an address such that $heap(p) = cn'$. Let $cn \in \text{subclasses}(cn')$. Furthermore, suppose $(cn_1, id_1), \ldots, (cn_n, id_n)$ denote fields of $cn$ (together with the defining classes). We perform the following class instance steps, where the second takes care of the case, where address $p$ is replaced by $\text{null}$.

\[
\begin{align*}
(\text{heap}, \text{frms}, iu) & \quad \quad \quad \quad \quad \quad (\text{heap}, \text{frms}, iu) \\
(\text{heap}\{p \mapsto (\text{cn}, \text{ftable}_1)\}, \text{frms}, iu) & \quad \quad \quad \quad \quad \quad (\text{heap}_2, \text{frms}_2, iu).
\end{align*}
\]
Here \( ftable_1((cn_i, id_i)) = v_i \) such that the type of the abstract variable \( v_i \) is defined in correspondence to the type of field \((cn_i, id_i)\), e.g., a fresh int variable for integer fields. On the other hand we set \( heap_2'(frms_2') \) equal to \( heap'(frms)\), but \( p \notin \text{dom}(heap_2)\) and all occurrences of \( p \) are replaced by null.

**Definition 4.13.** Let \( s^b = (heap, frms, iu) \) and let \( p \) and \( q \) denote different addresses in \( heap \) such that \( p \neq q \notin iu \). We perform the following unsharing steps: The first case forces these addresses to be distinct. The second case substitutes all occurrences of \( p \) with \( q \).

\[
\begin{align*}
\text{Load}^b & : (heap, (stk, loc, cn, mn, pc) :: frms, iu) \\
& \quad \quad \quad \quad \quad \rightarrow (heap, (loc(n)) :: stk, loc, cn, mn, pc + 1) :: frms, iu) \\
\text{IAdd}^b & : (heap, (i_2 :: i_1 :: stk, loc, cn, mn, pc) :: frms, iu) \\
& \quad \quad \quad \quad \quad \rightarrow (heap, (i_3 :: stk, loc, cn, mn, pc + 1) :: frms, iu) \\
\text{CmpEq}^b & : (heap, (val_2 :: val_1 :: stk, loc, cn, mn, pc) :: frms, iu) \\
& \quad \quad \quad \quad \quad \rightarrow (heap, (b_3 :: stk, loc, cn, mn, pc + 1) :: frms, iu) \\
\text{IfFalse}^b & : (heap, (false :: stk, loc, cn, mn, pc) :: frms, iu) \\
& \quad \quad \quad \quad \quad \rightarrow (heap, (true :: stk, loc, cn, mn, pc) :: frms, iu) \\
\text{Neu}^c & : (heap, (stk, loc, cn, mn, pc) :: frms, iu) \\
& \quad \quad \quad \quad \quad \rightarrow (heap\{a \mapsto x\}, (a :: stk, loc, cn, mn, pc + 1) :: frms, iu) \\
\text{Putfield}^d & : fn cn' \\
& \quad \quad \quad \quad \quad \rightarrow (heap, (v :: a :: stk, loc, cn, mn, pc) :: frms, iu) \\
& \quad \quad \quad \quad \quad \rightarrow (heap\{a \mapsto (cn''/ftable')\}, (stk, loc, cn, mn, pc + 1) :: frms, iu)
\end{align*}
\]

Figure 9: Symbolic evaluations of Jinja bytecode instructions

**Example 4.5.** In Figure 10 we present an example detailing the need for the given definition of class instantiation. Here class \( B \) overrides method \( m \) inherited from class \( A \). We only know the static type of the parameter when analysing method \( \text{call}(A \ a) \). Method \( \text{call}(A \ a) \) accepts any instances of class \( A \) or any instances of a subclass of \( A \) as parameter. In particular any instance of class \( B \). Due to the overridden method \( \text{call}(A \ a) \) does not terminate for instances of class \( B \).

Let \( s^b, s'^b \) and \( t^b \) be abstract states such that \( s'^b \) is obtained by zero or multiple refinement steps from \( s^b \). Furthermore, suppose \( t^b \) is obtained from \( s'^b \) due to a symbolic evaluation. Then we say \( t^b \) is obtained form \( s^b \) by an abstract computation.

To prove correctness of an symbolic evaluation step, we have to show that \( f^*(\gamma(s^b)) \subseteq \gamma(f^b(s^b)) \). Hence, it is enough to show that for all \( s \in \gamma(s^b) \) and \( P: s \xrightarrow{\text{form}} t \) it follows
class A{
    void m(){unit}
}

class B extends A{
    void m(){while(true)}
}

class C{
    void call(A a){a.m()}
}

void main(){
    C c = new C();
    c.call(new B());
}

Figure 10: All subclasses need to be considered.

that \( t \in \gamma(t^3) \), where \( t^3 \) is obtained from a symbolic evaluation step, i.e., \( t^3 = f^3(s^3) \).
Similarly, to prove correctness of the refinement steps it is enough to show that for all \( s \in \gamma(s^3) \) there exists a state \( s^3 \) obtained by a state refinement of \( s^3 \) such that \( s \in \gamma(s^3) \).
Correctness of an abstract computation step follows from the correctness of refinement and symbolic evaluation steps.

Lemma 4.4. Let \( s^3 \in AS \). Suppose \( s^3_1, \ldots, s^3_n \) is obtained by a state refinement from \( s^3 \).
Then \( s^3 \supseteq s^3_i \) for all \( s^3_i \). Furthermore, \( s \in \gamma(s^3) \) implies that there exists an abstract state \( s^3_i \) such that \( s \in \gamma(s^3_i) \).

Proof. The claim follows easily by the definition of Boolean and class variables, and the fact that two addresses in the heap of \( s^3 \) either alias or not.

Lemma 4.5. Let \( s^3, t^3 \in AS \) such that \( t^3 \) is obtained by a symbolic evaluation from \( s^3 \).
Suppose \( s \in \gamma(s^3) \) and \( P: s \xrightarrow{\text{jvm}} t \). Then \( t \in \gamma(t^3) \).

Proof. The proof is straightforward in most cases; we only treat some informative ones.
Let \( s^3 = (heap^3, frm^3 : frms^3, inu) \) and \( s = (heap, frm : frms) \). By assumption the domain of \( frm^3 : frms^3 \) and \( frm : frms \) coincide.

- Consider Load\(^3 \) \( n \). By assumption \( loc^3(n) \supseteq_m loc(n) \). In the abstract computation step \( loc^3(n) \) is loaded on to the top of the stack. Obviously \( stk_i^3(n) \supseteq_m stk_i(n) \), where \( stk_i \) represents the top of the stack. Then \( t \in \gamma(t^3) \).

- Consider IfAdd\(^3 \). Let \( i_2, i_1 \) denote the first two stack elements of \( s^3 \). Wlog, suppose that \( i_1 \) is abstract. By definition of the symbolic evaluation of IfAdd\(^3 \) we perform the step by introducing a new abstract integer \( i_3 \) and adding the constraint \( i_3 = i_1 + i_2 \). Then \( t \in \gamma(t^3) \), since \( i_3 \supseteq z \) for all numbers \( z \).

- Consider IfFalse\(^3 \) \( i \). Wlog, let \( false \) be the top element of the stack of \( s^3 \). Executing the symbolic step yields a state \( t^3 \), which is an abstraction of \( t \) by assumption on \( s \) and \( s^3 \). Then \( t \in \gamma(t^3) \).

- Consider Putfield\(^3 \) \( fn cn \) on address \( p \). By assumption the instruction can be symbolically evaluated and \( p \) does not alias with some address \( q \in \text{dom}(heap^3) \) different from \( p \). The only interesting case to consider is when \( heap^3(q) \) is a class variable
and there exists $s \in \gamma(s^\sharp)$ such that $m(q) \rightarrow_S r \leftarrow_S m(p)$, where $r \in \text{dom}(\text{heap})$. Then $m(q)$ reaches $m(p)$ via $r$ and is affected by the update instruction. This does not matter, since $\text{heap}^\sharp(q)$ is also a class variable in $t^\sharp$, thus also representing the affected instance. Then $t \in \gamma(t^\sharp)$.

- Consider $\text{CmpEq}^\sharp$. By assumption the instruction can be symbolically executed. That is the necessary refinement steps are already performed. Then $t \in \gamma(t^\sharp)$ follows directly.

The next theorem is an immediate result of the lemma.

**Theorem 4.1.** Let $s$ and $t$ be JVM states, such that $P: s \xrightarrow{\text{jvm}} t$. Suppose $s \in \gamma(s^\sharp)$ for some state $s^\sharp$. Then there exists an abstract computation of $t^\sharp$ from $s^\sharp$ such that $t \in \gamma(t^\sharp)$.

Theorem 4.1 formally proves the correctness of the proposed abstract domain with respect to the operational semantics for Jinja, established by Klein and Nipkow [15]. In order to exploit this abstract domain we require a finite representation of the abstract domain $\mathcal{AS}$ induced by $P$. For that we propose in the next section computation graphs as finite representations of all relevant states in $\mathcal{AS}$, abstracting JVM states in $P$.

### 5. Computation Graphs

In this section, we define *computation graphs* as finite representations of the abstract domain $\mathcal{AS}$ with respect to $P$.

**Definition 5.1.** A *computation graph* $G = (V_G, E_G)$ is a directed graph with edge labels, where $V_G \subset \mathcal{AS}$ and $s^\sharp \xrightarrow{\ell} t^\sharp \in E_G$ if either $t^\sharp$ is obtained from $s^\sharp$ by an abstract computation or $s^\sharp$ is an instance of $t^\sharp$. Furthermore, if there exists a constraint $C$ in the symbolic evaluation, then $\ell := C$. For all other cases $\ell := \emptyset$. We say that $G$ is the computation graph of program $P$ if for all initial states $i$ of $P$ there exists an abstract state $i^\sharp \in G$ such that $i \in \gamma(i^\sharp)$.

We obtain a finite representation of loops, if we suitably exploit the fact that any subset of $\mathcal{AS}$ has a least upper bound. The intuition is best conveyed by an example.

**Example 5.1.** Consider the List program from Example 3.1 together with the well-formed JBC program depicted in Figure 4. Figure 11 illustrates the computation graph of append. For the sake of readability we omit the val field of the list, the unsharing annotations and some intermediate nodes.

Consider the initial node $I$. It is easy to see that $I$ is an abstraction of all concrete initial states, when this is not null. We assume that this is acyclic and initially do not share with ys. Nodes A, B and S correspond to the situation described in Example 4.1 and Example 4.2. That is, node $A$ is obtained after assigning cur to this before any iteration of the loop, node $B$ is obtained after exactly one iteration of the loop and node
Figure 11: The (incomplete) computation graph of `append`.

$S = \bigcup \{ A, B \}$. Intermediate iterations are normally removed. This is indicated by a dashed border for $B$.

After pushing the reference of `cur.next` and `null` onto the operand stack, we reach node $C$. At $pc = 7$ we want to compare the reference of `cur.next` with `null`. But,
cur.next is not concrete. Therefore, a class instance refinement is performed, yielding nodes $C_1$ and $C_2$.

First, we consider that cur.next is not null, but references an arbitrary instance, as illustrated in node $C_1$. The step from $C_1$ to $D$ is trivial. Let $id$ denote the identity function and $m = id(V_S)$. Then $m\{o_1 \mapsto o_5, o_5 \mapsto o_6\}$ is a morphism from $S$ to $D$. Therefore, $D$ is an instance of $S$. Second, we consider the case when cur.next is null, as depicted in node $C_2$. Node $E$ is obtained from $C_2$ after loading registers cur and ys onto the stack. At program counter 19 a Putfield instruction is performed. Therefore we perform a refinement according to Definition 4.13. We obtain nodes $E_1, E_2$ and $E_3$. In $E_1$, this and cur point to the same reference, in $E_2$ this.next and cur point to the same reference, and in $E_3$ the abstracted part from cur is distinct from this, yet this and cur shares. Nodes $F_1, F_2$ and $F_3$ are obtained after performing the Putfield instruction.

To concretise the employed strategy, note that whenever we are about to finish a loop, we attempt to use an instance refinement to the state starting this loop. If this fails, for example in an attempted step from $B$ to $A$ in Example 5.1, we widen the corresponding state. Here we collect all states that need to be abstracted and join them to obtain an abstraction. Complementing the proposed strategy, we restrict the applications of refinements, such that refinement steps are only performed if no other steps are applicable. We say that this strategy is an eager strategy. The next lemma shows that if an eager strategy is followed we are guaranteed to obtain a finite computation graph.

**Lemma 5.1.** Let $G$ be the computation graph of a program $P$ such that in the construction of $G$ an eager strategy is applied. Then $G$ is finite.

**Proof.** We argue indirectly. Suppose the computation graph $G$ of $P$ is infinite. This is only possible if there exists an initial state $i$ of $P$ that is non-terminating, which implies that starting from $i$ we reach a loop in $P$ that is called infinitely often. As $G$ is infinite this implies that the widening operation for this loop gives rise to an infinite sequence of states $(s^k_j)_{j \geq 0}$ such that $s^k_j \subseteq s^k_{j+1}$ for all $j$. However, this is impossible as any ascending chain of abstract states eventually stabilises, cf. Lemma 4.2.

Let $G$ be a computation graph. We write $s^k \rightarrow_G t^k$ to indicate that state $t^k$ is directly reachable in $G$ from $s^k$. Sometimes we want to distinguish whether $t^k$ is obtained by a refinement (denoted as $s^k \rightarrow_{ref} t^k$) or by a symbolic evaluation (denoted as $s^k \rightarrow_{eva} t^k$), or whether $s^k$ is an instance of $t^k$ (denoted as $s^k \rightarrow_{ins} t^k$). If $t^k$ is reachable from $s^k$ in $G$ we write $s^k \rightarrow_G t^k$. If $s^k \neq t^k$ this is denoted by $s^k \rightarrow_G t^k$.

**Lemma 5.2.** Let $s, t \in JS$ such that $P: s \xrightarrow{\text{jump}} t$. Let $G$ denote the computation graph of $P$, and $s^k, t^k \in G$. Suppose $s \in \gamma(s^k)$, then there exists $t^k$ such that $t \in \gamma(t^k)$ and $s^k \rightarrow_{ins} \rightarrow_{ref} \rightarrow_{eva} t^k$.

**Proof.** By construction of $G$ we have to consider two cases: Suppose $t^k$ is obtained by an abstract computation from $s^k$. We employ Lemma 5.2 to conclude that $t \in \gamma(t^k)$. Then $s^k \rightarrow_{ref} \rightarrow_{eva} t^k$. Next, suppose $t^k$ is obtained by an abstract computation from
terms, and substitution $\sigma$. We define the rewrite relation $\rightarrow_{l}$ on the signature $\mathcal{V} \supseteq \mathcal{V}'$. Let $C$ denote the set of sorted variables. Furthermore, let $\gamma$ be a (not necessarily finite) sorted signature, let $\mathcal{C}$ denote a theory over $\mathcal{C}$. Quantifier-free formulas over $\mathcal{C}$ are called constraints. Suppose $\mathcal{F}$ is a sorted signature that extends $\mathcal{C}$ and let $\mathcal{V} \supseteq \mathcal{V}'$ denote an extension of the variables in $\mathcal{V}'$. Let $\mathcal{T}(\mathcal{F}, \mathcal{V})$ denote the set of (sorted) terms over the signature $\mathcal{F}$ and $\mathcal{V}$. Note that the sorted signature is necessary to distinguish between theory variables that are to be interpreted over the theory $\mathcal{T}$ and term variables whose interpretation is free. A constrained rewrite rule, denoted as $l \rightarrow r \left[ \mathcal{C} \right]$, is a triple consisting of terms $l$ and $r$, together with a constraint $C$. We assert that $l \not\in \mathcal{V}$, but do not require that $\text{Var}(l) \supseteq \text{Var}(r) \cup \text{Var}(C)$, where $\text{Var}(t)$ (or $\text{Var}(C)$) denotes the variables occurring in the term $t$ (constraint $C$). A constrained term rewrite system (cTRS) is a finite set of constrained rewrite rules.

Let $\mathcal{R}$ denote a cTRS. A context $D$ is a term with exactly one occurrence of a hole $\square$, and $D[\square]$ denotes the term obtained by replacing the hole $\square$ in $D$ by the term $t$. A substitution $\sigma$ is a function that maps variables to terms, and $t\sigma$ denotes the homomorphic extension of this function to terms. We define the rewrite relation $\rightarrow_{\mathcal{R}}$ as follows. For terms $s$ and $t$, $s \rightarrow_{\mathcal{R}} t$ holds, if there exists a context $D$, a substitution $\sigma$ and a constrained rule $l \rightarrow r \left[ \mathcal{C} \right] \in \mathcal{R}$ such that $s =_{T} D[l] \sigma$ and $t = D[r\sigma]$ with $T \vdash C\sigma$. Here $=_{T}$

We arrive at the main result of this section.

**Theorem 5.1.** Let $i, t \in \mathcal{J}\mathcal{S}$ and suppose $P : i \xrightarrow{j} \mathcal{m} \rightarrow t$, where the runtime of the execution is $m$. Let $G$ denote the computation graph of $P$ obtained from some initial state $i^2$ such that $i \in \gamma(i^2)$. Then there exists an abstraction $i^2 \in G$ and a path $i^2 \rightarrow_{G} t^3$ of length $m'$ such that $m \leq m' \leq K \cdot m$. Here constant $K \in \mathbb{N}$ only depends on $G$.

**Proof.** By induction on $m$ (employing Lemma 5.2), we conclude the existence of state $t^3$ such that $i \rightarrow_{G} t^3$. Hence, the first part of the theorem follows. Furthermore by Lemma 5.2 there exists $m'$ such that $m \leq m' \leq K \cdot m$. □

### 6. Constrained Rewrite Systems

Let $G$ be the computation graph for program $P$ with initial state $i^2$; $G$ is kept fixed for the remainder of the section. In the following we describe the translation from $G$ into a constrained term rewrite system (cTRS for short). Our definition is a variation of cTRSs as for example defined by Falke and Kapur [8, 9] or Sakata et al. [26]. Recently, Kop and Nishida introduced a very general formalism of term rewrite systems with constraints, termed logical constrained term rewrite systems (LCTRSs) [10]. The proposed notion of cTRSs is not directly interchangeable with LCTRSs, yet the rewrite system resulting from the transformation could also be formalised as LCTRS. The here proposed transformation is inspired by [23]. Otto et al. transform termination graphs into integer term rewrite systems (ITRSs for short) [10].

Let $C$ be a (not necessarily finite) sorted signature, let $\mathcal{V}'$ denote a countably infinite set of sorted variables. Furthermore let $T$ denote a theory over $\mathcal{C}$. Let $\mathcal{F}$ be a sorted signature that extends $\mathcal{C}$ and let $\mathcal{V} \supseteq \mathcal{V}'$ denote an extension of the variables in $\mathcal{V}'$. Let $\mathcal{T}(\mathcal{F}, \mathcal{V})$ denote the set of (sorted) terms over the signature $\mathcal{F}$ and $\mathcal{V}$. Note that the sorted signature is necessary to distinguish between theory variables that are to be interpreted over the theory $\mathcal{T}$ and term variables whose interpretation is free. A constrained rewrite rule, denoted as $l \rightarrow r \left[ \mathcal{C} \right]$, is a triple consisting of terms $l$ and $r$, together with a constraint $C$. We assert that $l \not\in \mathcal{V}$, but do not require that $\text{Var}(l) \supseteq \text{Var}(r) \cup \text{Var}(C)$, where $\text{Var}(t)$ (or $\text{Var}(C)$) denotes the variables occurring in the term $t$ (constraint $C$). A constrained term rewrite system (cTRS) is a finite set of constrained rewrite rules.

Let $\mathcal{R}$ denote a cTRS. A context $D$ is a term with exactly one occurrence of a hole $\square$, and $D[\square]$ denotes the term obtained by replacing the hole $\square$ in $D$ by the term $t$. A substitution $\sigma$ is a function that maps variables to terms, and $t\sigma$ denotes the homomorphic extension of this function to terms. We define the rewrite relation $\rightarrow_{\mathcal{R}}$ as follows. For terms $s$ and $t$, $s \rightarrow_{\mathcal{R}} t$ holds, if there exists a context $D$, a substitution $\sigma$ and a constrained rule $l \rightarrow r \left[ \mathcal{C} \right] \in \mathcal{R}$ such that $s =_{T} D[l] \sigma$ and $t = D[r\sigma]$ with $T \vdash C\sigma$. Here $=_{T}$

\[s^3, \text{where } s^3 \subseteq s^2. \text{Hence, we also have } s \in \gamma(s^2). \text{We employ Lemma 4.1 to conclude that } t \in \gamma(t^2). \text{Then } s^3 \rightarrow_{\mathcal{R}} \gamma(s^2) \cdot \rightarrow_{\mathcal{R}} \gamma(s^2) \cdot \rightarrow_{\mathcal{R}} t^3. \text{Since } G \text{ is finite we conclude that } s^3 \rightarrow_{\mathcal{R}} \gamma(s^2) \cdot \rightarrow_{\mathcal{R}} t^3 \text{ has finitely many instance and refinement steps, only depending on } G. \]
denotes unification modulo $T$. For extra variables $x$, possibly occurring in $t$, we demand that $\sigma(x)$ is in normal-form.

We often drop the reference to the cTRS $R$, if no confusion can arise from this. A function symbol in $F$ is called defined if $f$ occurs as the root symbol of $l$, where $l \to r \in C \in R$. Function symbols in $F \setminus C$ that are not defined, are called constructor symbols, and the symbols in $C$ are called theory symbols.

A cTRS $R$ is called terminating, if the relation $\to_R$ is well-founded. For a terminating cTRS $R$, we define its runtime complexity, denoted as $\text{rctrs}_R$. We adapt the runtime complexity with respect to a standard TRS suitable for cTRS $R$. (See [11] for the standard definition.) The derivation height of a term $t$ (with respect to $R$) is defined as the maximal length of a derivation (with respect to $R$) starting in $t$. The derivation height of $t$ is denoted as $\text{dh}(t)$. Note that $\to_R$ is not necessarily finitely branching for finite cTRSs, as fresh variables on the right-hand side of a rule can occur.

**Definition 6.1.** We define the runtime complexity (with respect to $R$) as follows:

$$\text{rctrs}_R(n) = \max \{ \text{dh}(t) \mid t \text{ is basic and } \|t\| \leq n \} ,$$

where a term $t = f(t_1, \ldots, t_k)$ is called basic if $f$ is defined, and the terms $t_i$ are only built over constructor, theory symbols, and variables. We fix the size measure $\|\cdot\|$ below.

In the following we are only interested in cTRS over a specific theory $T$, namely Presburger arithmetic, that is, we have $T \vdash C$, if all ground instances of the constraint $C$ are valid in Presburger arithmetic. Recall, that Presburger arithmetic is decidable. If $T \vdash C$, then $C$ is valid. On the other hand, if there exists a substitution $\sigma$, such that $T \vdash C\sigma$, then $C$ is satisfiable.

To represent the basic operations in the Jinja bytecode instruction set (cf. Figure 3) we collect the following connectives and truth constants in $C$: $\land$, $\lor$, $\neg$, $\text{true}$, and $\text{false}$, together with the following relations and operations: $=, \neq, \geq, +, -$. Furthermore, we add infinitely many constants to represent integers. We often write $l \to r$ instead of $l \to r[\text{true}]$. As expected $C$ makes use of two sorts: $\text{bool}$ and $\text{int}$. We suppose that all abstract variables $X_1, X_2, \ldots$ are present in the set of variables $\mathcal{V}$, where abstract integer (Boolean) variables are assigned sort $\text{int}$ ($\text{bool}$) and all other variables are assigned sort $\text{univ}$. The remaining elements of the signature $\mathcal{F}$ will be defined in the course of this section. As the signature of these function symbols is easily read off from the translation given below, in the following the sort information is left implicit, to simplify the presentation.

The size of a term $t$, denoted as $\|t\|$ is defined as follows:

$$\|t\| := \begin{cases} 1 & \text{if } t \text{ is a variable} \\ \text{abs}(t) & \text{if } t \text{ is an integer} \\ 1 + \sum_{i=1}^n \|t_i\| & \text{if } t = f(t_1, \ldots, t_n) \text{ and } f \text{ is not an integer} . \end{cases}$$

In the next definition, we show how a state becomes representable as term over $\mathcal{F}$. 

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Definition 6.2. Let \( s^i = (heap, frms, iu) \) be a state and let the index sets \( Stk \) and \( Loc \) be defined as above. Suppose \( v \) is a value. Then the value \( v \) is translated as follows:

\[
tval(v) := \begin{cases} 
null & \text{if } v \in \{\text{unit}, \text{null}\} \\
v & \text{if } v \text{ is a non-address value, except unit or null} \\
taddr(v) & \text{if } v \text{ is an address}.
\end{cases}
\]

Let \( a \) be an address. Then \( a \) is translated as follows:

\[
taddr(a) := \begin{cases} 
x & \text{if } a \text{ is maybe-cyclic and } x \text{ is a fresh variable} \\
x & \text{if } heap(a) \text{ denotes an abstract variable } x \\
\text{cn}(tval(v_1), \ldots, tval(v_n)) & \text{if } heap(a) = (\text{cn}, ftable).
\end{cases}
\]

Here we suppose in the last case that \( \text{dom}(ftable) = \{(cn_1, id_1), \ldots, (cn_n, id_n)\} \) and for all \( 1 \leq i \leq n: ftable((cn_i, id_i) = v_i \). Finally, to translate the state \( s \) into a term, it suffices to translate the values of the registers and the operand stacks of all frames in the list \( frms \). Let \( (stk, i, j) \in Stk \) such that \( stk_i(j) \) denotes the \( j \)th value in the operation stack of the \( i \)th frame in \( frms \). Similarly for \( (loc, i', j') \in Loc \). Then we set

\[
ts(s) := [\text{tval}(stk_1(1)), \ldots, \text{tval}(stk_k([stk_k])), \text{tval}(loc_1(1)), \ldots, \text{tval}(loc_k([loc_k]))],
\]

where the list \([\ldots]\), is formalised by an auxiliary binary symbol :: and the constant nil.

Example 6.1. Consider the simplified presentation of state \( C \) in Figure 11. Then \( ts(C) \) yields following term:

\[
ts(C) = [\text{list5}, \text{null}, \text{List(list3)}, \text{list2}, \text{List(List(list5))}].
\]

Note that we can omit the information of the defining classes of the fields, since this is already captured in the symbolic evaluation. Furthermore, observe that our term representation can only fully represent acyclic data. In this sense, the term representation of a state \( s \) is less general, than its graph-based representation. However, we still obtain the following lemma.

Lemma 6.1. Let \( s^i \) and \( t^i \) be abstract states. If \( t^i \subseteq s^i \), then there exists a substitution \( \sigma \) such that \( ts(t^i) = ts(s^i)\sigma \).

Proof. Let \( S^i \) and \( T^i \) be the state graphs of \( s^i \) and \( t^i \), respectively. By assumption there exists a morphism \( m: S^i \rightarrow T^i \). The lemma is a direct consequence of the following observations:

- Consider the terms \( ts(s^i) \) and \( ts(t^i) \). By definition these terms encode the standard term representations of the graphs \( S^i \) and \( T^i \).

- Let \( u \) and \( v \) be nodes in \( S^i \) and \( T^i \) such that \( m(u) = v \). The label of \( u \) (in \( S^i \)) can only be distinct from the label of \( v \) (in \( T^i \)), if \( L_{S^i}(u) \) is an abstract variable or null. In the former case \( tval(L_{S^i}(u)) \) is again a variable and the latter case implies that \( L_{T^i}(v) = \text{unit} \). Thus in both cases, \( tval(L_{S^i}(u)) \) matches \( tval(L_{T^i}(v)) \).
• By correctness of our abstraction, we have \( m(u) \) is maybe-cyclic, if \( v \) is maybe-cyclic. In this case \( \text{tval}(L_{S\uparrow}(u)) \) and \( \text{tval}(L_{T\uparrow}(v)) \) are fresh variables. Hence, \( \text{tval}(L_{S\uparrow}(u)) \) matches \( \text{tval}(L_{T\uparrow}(v)) \).

The next lemma relates the size of a state to its term representation and vice versa.

**Lemma 6.2.** Let \( s = (\text{heap}, \text{frms}) \) be a state such that heap does not admit cyclic data structures. Then \( \| ts(\beta(s)) \| = |s| \).

*Proof.* As a consequence of Definition 6.2 and the above proposed variant of the term complexity we see that \( \| ts(\beta(s)) \| = |s| \) for all states \( s \).

**Lemma 6.3.** Let \( s = (\text{heap}, \text{frms}) \) be a state such that heap may contain cyclic data structures. Then \( \| ts(\beta(s)) \| \leq |s| \) and therefore \( \| ts(\beta(s)) \| \in O(|s|) \).

*Proof.* Follows from the previous lemma and the fact that addresses bounded to cyclic data structures are replaced by fresh variables.

Let \( G \) be a computation graph. For any state \( s^2 \) in \( G \) we introduce a new function symbol \( f^2 \). Suppose \( ts(s^2) = [s^2_1, \ldots, s^2_n] \). To ease presentation we write \( f^2(s^2) \) instead of \( f^2(s^2_1, \ldots, s^2_n) \).

**Definition 6.3.** Let \( G \) be a finite computation graph and \( s^2 = (\text{heap}, \text{frms}, iu) \) and \( t^2 \) be states in \( G \). We define the constrained rule corresponding to the edge \((s^2, t^2)\), denoted by \( \text{rule}(s^2, t^2) \), as follows:

\[
\text{rule}(s^2, t^2) = \begin{cases} 
    f^2_1(s^2) \to f^2_1(t^2) & \text{if } s^2 \subseteq t^2 \\
    f^2_2(s^2) \to f^2_2(t^2) & \text{if } t^2 \text{ is a state refinement of } s^2 \\
    f^2_3(s^2) \to f^2_1(ts^*)(t^2) & \text{the edge is labelled by } C \\
    f^2_4(s^2) \to f^2_1(ts^*)(t^2) & s^2 \text{ corresponds to a Putfield on address } p, \text{ heap}(q) \text{ is variable } cn, \text{ and } q \text{ may-reach } p \\
    f^2_5(s^2) \to f^2_1(ts(t^2)) & \text{otherwise}.
\end{cases}
\]

Here \( \text{tval}(C) \) denotes the standard extension of the mapping \( \text{tval} \) to labels of edges and \( ts^* \) is defined as \( ts \) but employs fresh variables for any reference \( q \) that may-reach the object that is updated. The cTRS obtained from \( G \) consists of rules \( \text{rule}(s^2, t^2) \) for all edges \( s^2 \to t^2 \in G \).

**Example 6.2.** Figure 12 illustrates the cTRS obtained from the computation graph of Example 5.1. We use following conventions: \( L \) denotes the list constructor symbol and \( l \) followed by a number a list variable. In the last rule \( l4 \) is fresh on the right-hand side. This is because we update \( cur \) and have a side-effect on \( this \) that is not directly observable in the abstraction.
need to consider the following four cases. The argument for the omitted fifth case is very
then
Suppose
s
is non-empty, then
The proof proceeds by case analysis on the edge
Proof.
Lemma 6.4.
\[(\text{In the following we show that the rewrite relation of the obtained cTRS safely approx-
\text{m}\text{imates the concrete semantics of the concrete domain. We first argue informally:}]
\begin{itemize}
\item By Lemma 5.2 there exists a path \(s^\circ \xrightarrow{\text{ins}} s^\circ \xrightarrow{\text{ref}} \xrightarrow{\text{eva}} t^\circ\) in \(G\) for \(P\): \(s \xrightarrow{\text{jvm}} t\) such that \(s \in \gamma(s^\circ)\) and \(t \in \gamma(t^\circ)\).
\item Together with Lemma 5.1 we have to show that \(f_{s^\circ}(\text{ts}(\beta(s))) \xrightarrow{R} f_{t}(\text{ts}(\beta(t)))\).
\item We do this by inspecting the rules obtained from the transformation. We will see
that instance steps and refinement steps do not modify the term instance. In case
of evaluation steps the effect is either directly observable in the abstract state, as
it happens for \text{Push}^\circ\) for example, or indirectly by requiring that the substitution is
conform with the constraint. In the case of the \text{Putfield}^\circ\) instructions we have to
find a suitable substitution for fresh variables to accommodate possible side-effects.
\end{itemize}
Lemma 6.4. Let \(s^\circ\) and \(t^\circ\) be states in \(G\) connected by an edge \(s^\circ \xrightarrow{\ell} t^\circ\) from \(s^\circ\) to \(t^\circ\).
Suppose \(s \in JS\) with \(s \in \gamma(s^\circ)\). Suppose further that if the constraint \(\ell\) labelling the edge
is non-empty, then \(s\) satisfies \(\ell\). Moreover, if \(s^\circ \xrightarrow{\ell} t^\circ\) follows due to a refinement step,
then \(s\) is consistent with the chosen refinement. Then there exists \(t \in \gamma(t^\circ)\) such that
\(f_{s^\circ}(\text{ts}(s^\circ)) \xrightarrow{\text{rule}(s^\circ, \ell)} f_{t^\circ}(\text{ts}(t^\circ))\) with \(s' = \beta(s), t' = \beta(t)\).

Proof. The proof proceeds by case analysis on the edge \(s^\circ \xrightarrow{\ell} t^\circ\) in \(G\), where we only
need to consider the following four cases. The argument for the omitted fifth case is very
similar to the third case.

Figure 12: The cTRS of \text{append}.
• Case $s^k \xrightarrow{\ell} t^k$, as $s^2 \sqsubseteq t^2$; $\ell = \emptyset$. By assumption $s' \sqsubseteq s^2 \sqsubseteq t^2$. Hence, $s \in \gamma(t^2)$ by transitivity of the instance relation. By Lemma 6.1 there exists a substitution $\sigma$ such that $ts(s') = ts(s^2)\sigma$. In sum, we obtain:

$$f_{s^2}(ts(s')) = f_{s^2}(ts(s^2))\sigma \rightarrow_{\text{rule}(st,t)} f_{t^2}(ts(s^2))\sigma = f_{t^2}(ts(t')),$$

where we set $t' := s'$.

• Case $s^k \xrightarrow{\ell} t^k$, as $t^2$ is a refinement of $s^2$; $\ell = \emptyset$. By assumption $s' \sqsubseteq s^2$ and $s$ is concrete. Hence, $s' \sqsubseteq t^2$ by definition of $t^2$. Again by Lemma 6.1 there exists a substitution $\sigma$, such that $ts(s') = ts(t^2)\sigma$. In sum, we obtain:

$$f_{s^2}(ts(s')) = f_{s^2}(ts(t^2))\sigma \rightarrow_{\text{rule}(st,t)} f_{t^2}(ts(t^2))\sigma = f_{t^2}(ts(t')),$$

where we again set $t' := s'$.

• Case $s^k \xrightarrow{\ell} t^k$, as $t^2$ is the result of the symbolic evaluation of $s^2$ and $\ell = C \neq \emptyset$. By assumption $s$ satisfies the constraint $C$. More precisely, there exists a substitution $\sigma$ such that $ts(s') = ts(s^2)\sigma$ and $T \vdash \text{val}(C)\sigma$. We obtain:

$$f_{s^2}(ts(s')) = f_{s^2}(ts(s^2))\sigma \rightarrow_{\text{rule}(st,t)} f_{t^2}(ts(t^2))\sigma.$$

Let $t$ be defined such that $P: s \xrightarrow{\text{jvm}1} t$. By Lemma 4.5 we obtain $t' \sqsubseteq t^2$ and by inspection of the proof of Lemma 4.5 we observe that $ts(t') = ts(t^2)\sigma$. In sum, $f_{s^2}(ts(s')) \rightarrow_{\text{rule}(st,t)} f_{t^2}(ts(t'))$.

• Case $s^k \xrightarrow{\ell} t^k$, as $t^2$ is the result of a Putfield command on $p$ and there exists an address $q$ in $s^2$ that may-reaches $p$. By assumption $s' \sqsubseteq s^2$ and thus $ts(s') = ts(s^2)\sigma$ for some substitution $\sigma$. Let $t$ be defined such that $P: s \xrightarrow{\text{jvm}1} t$. Due to Lemma 4.5 we have $t' \subseteq t^2$ and thus there exists a substitution $\tau$ such that $ts(t') = ts^*(t^2)\tau$.

Consider the rule $f_{s^2}(ts(s^2)) \rightarrow f_{t^2}(ts^*(t^2))$. By definition address $q$ points in $s^2$ to an abstract variable $x$ such that $x$ occurs in $ts(s^2)$ and $ts(t^2)$. Furthermore, $x$ is replaced by an extra variable $x'$ in $ts^*(t^2)$. Wlog., we assume that $x'$ is the only extra variable in $ts^*(t^2)$. Let $m$ be a morphism such that $m: s^k \rightarrow s'$ and $m(q) \xrightarrow{\tau} m(p)$. By definition of Putfield command, $m(p)$ and $m(q)$ exist in $t'$ and only the part of the heap reachable from these addresses can differ in $s'$ and $t'$.

In order to show the admissibility of the rewrite step $f_{s^2}(ts(s')) \rightarrow f_{t^2}(ts(t'))$ we define a substitution $\rho$ such that $ts(s^2)\rho = ts(s')$ and $ts^*(t^2)\rho = ts(t')$. We set:

$$\rho(y) := \begin{cases} \tau(x) & \text{if } y = x' \\ \sigma(y) & \text{otherwise} \end{cases}.$$

Then $ts(s^2)\rho = ts(s')$ by definition as $x' \notin \text{Var}(s^2)$. On the other hand $ts^*(t^2)\rho = ts(t')$ follows as the definition of $\rho$ forces the correct instantiation of $x'$ and Lemma 4.5 in conjunction with Lemma 4.1 implies that $\sigma$ and $\tau$ coincide on the portion of the heap that is not changed by the field update.
The next lemma emphasises that any execution step is represented by finitely many but at least one rewrite steps in $R$.

**Lemma 6.5.** Let $s^i \in G$ and $s \in JS$ such that $s \in \gamma(s^i)$. Then $P : s \xrightarrow{jvm} t$ implies that there exists a state $t^i \in G$ such that $t \in \gamma(t^i)$ and $f^i(\text{ts}(\beta(s))) \xrightarrow{\leq K} f^i(\text{ts}(\beta(t)))$. Here $K$ depends only on $G$ and $\xrightarrow{\leq K}$ denotes at least one and at most $K$ many rewrite steps in $R$.

*Proof.* The lemma follows from the proof of Lemma 5.2 and Lemma 6.4.

We arrive at the main result of this thesis.

**Theorem 6.1.** Let $s, t \in JS$. Suppose $P : s \xrightarrow{jvm} t$, where $s$ is reachable in $P$ from some initial state $i$. Set $s' = \beta(s)$, $t' = \beta(t)$. Then there exists $s^i, t^i \in AS$ and a derivation $f^i(\text{ts}(s')) \xrightarrow{\oplus R} f^i(\text{ts}(t'))$ such that $s \in \gamma(s^i)$ and $t \in \gamma(t^i)$. Furthermore, for all $n$:

$$\text{rcjvm}(n) \in O(\text{rctrs}(n)).$$

*Proof.* The existence of $s^i$ follows from the correctness of abstract computation together with the construction of the computation graph. Let $m$ denote the runtime of the execution $P : s \xrightarrow{jvm} t$. Then by induction on $m$ in conjunction with Lemma 6.3 we obtain the existence of a state $t^i$ such that $t' \subseteq t^i$ and a derivation:

$$f^i(\text{ts}(s')) \xrightarrow{\leq K \cdot m} f^i(\text{ts}(t')).$$

(1)

Here the constant $K$ depends only on $G$. In particular we have $f^i(\text{ts}(s')) \xrightarrow{\oplus R} f^i(\text{ts}(t'))$ from which we conclude the first part of the theorem.

To conclude the second part, let $n$ be arbitrary and suppose $m$ denotes the runtime of the execution $P : i \xrightarrow{jvm} t$, where $|i| \leq n$. We set $i' = \beta(i)$. As $G$ is the computation graph of $P$ we obtain $i' \subseteq i^i$. From Lemma 6.3 it follows that $|\text{ts}(\beta(i))| \leq |i|$. Specialising (1) to $i^i$ and $i'$ yields $f^i(\text{ts}(i')) \xrightarrow{\leq K \cdot m} f^i(\text{ts}(t'))$. Thus we obtain

$$\text{rcjvm}(|i|) = m \leq K \cdot m \leq \text{rctrs}(|\text{ts}(\beta(i))|) \leq \text{rctrs}(|i|).$$

It is tempting to think that the precise bound on the number of rewrite steps presented in Lemma 6.5 should translate to a linear simulation between JVM executions and rewrite derivation. Unfortunately this is not the case as the transformation is not termination preserving. For this consider Figure 13. Here the outer loop cuts away the last cell until the initial list consists only of one cell whereas the inner loop is used to iterate through the list. It is easy to see that the main function terminates if the argument is an acyclic list. Since variables $ys$ and $cur$ share during iteration, the proposed transformation introduces a fresh variable for the $\text{next}$ field of the initial argument $ys$ when performing the $\text{Putfield}$ instruction. Termination of the resulting rewrite system cannot be shown any more.

However *non-termination preservation* follows as an easy corollary of Theorem 6.1.
class List{ List next; }

class Main{
    void inits(List ys){
        while(ys.next != null){
            List cur = ys;
            while(cur.next.next != null){
                cur = cur.next
            }
            cur.next = null;
        }
    }
}

Figure 13: The inits program.

Corollary 6.1. The computation graph method, that is the transformation from a given JBC program $P$ to a cTRS $R$ is non-termination preserving.

Proof. Suppose there exists an infinite run in $P$, but $R$ is terminating. Let $i$ be some initial state $i$ of $P$. By Theorem 6.1 there exists a state $t$ such that $P: i \xrightarrow{\text{jvm}} t$ and $f_{\beta}\text{ts}(i') \rightarrow_R f_{\beta}\text{ts}(t')$, where $i \in \gamma(i')$, $i' = \beta(i)$, $t \in \gamma(t')$, and $t' = \beta(t)$. Furthermore, as $R$ is terminating we can assume $f_{\beta}\text{ts}(t')$ is in normalform. However, as $t'$ is non-terminating, there exists a successor, thus Lemma 6.5 implies that $f_{\beta}\text{ts}(t')$ cannot be in normalform. Contradiction.

7. Implementation

A prototype, termed JaT, of the proposed method has been implemented in the Haskell programming language. We use [27, 25, 33] to provide acyclicity and reachability facts.

Example 7.1. Figure 14 depicts a slightly modified version of the motivating example from [23]. The program flatten collects all integers from a list of trees storing integers. The complexity tool TCT is able to show that the rewrite system resulting from our proposed transformation has linear runtime complexity.

Currently TCT only provides limited support for cTRSs. A meaningful experimental evaluation will be provided in the future.

8. Conclusion and Future Work

In this paper we define a representation of JBC executions as computation graphs from which we obtain a representation of JBC executions as constrained rewrite systems. We precise the widening of abstract states so that the representation of JBC executions is
class IntList{
    IntList next;
    int value;
}
class Tree{
    Tree left;
    Tree right;
    int value;
}
class TreeList{
    TreeList next;
    Tree value;
}
class Flatten {
    IntList flatten(TreeList list)
    TreeList cur = list;
    IntList result = null;
    while (cur != null){
        Tree tree = cur.value;
        if (tree != null) {
            IntList oldIntList = result;
            IntList newIntList = new IntList();
            newIntList.value = tree.value;
            newIntList.next = oldIntList;
            result = newIntList;
            TreeList oldCur = cur;
            cur = new TreeList();
            cur.next = oldCur;
            cur.value = tree.left;
            oldCur.value = tree.right;
        } else {
            cur = cur.next;
        }
    }
    return result;
}

Figure 14: The flatten program.

provably finite. Furthermore, we show that the resulting transformation is complexity preserving.

As emphasised above our approach does not directly give rise to an automatable complexity-preserving transformation, but for that requires an extension by annotation or a dedicated shape analysis [21]. However our main result applies to any computable approximation of the transformation and in particular it shows complexity preservation of the transformation proposed by Otto et al. [23]. Moreover, it allows for an easy incorporation of the existing wealth of results on shape analysis present in the literature and thus improves upon the modularity of the proposed transformational approach.

Future work will be dedicated towards new methods for complexity analysis of cTRRs.

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A. Semantics of Jinja Bytecode Instructions

\[
\begin{align*}
\text{Load } n & \quad (\text{heap}, (n :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{loc}(n) :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{Store } n & \quad (\text{heap}, (v :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{stk}, \text{loc}(n \mapsto v), \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{Push } v & \quad (\text{heap}, (v :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
\text{Pop} & \quad (\text{heap}, (\text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms})
\end{align*}
\]

We use \text{BOp} together with \(\otimes = \{+, -, \lor, \land, \geq, \leq, \neq\}\) to define instructions \text{IAdd}, \text{ISub}, \text{BOR}, \text{BAnd}, \text{ICmpGt}, \text{CmpEq} and \text{CmpNeq}.

\[
\begin{align*}
\text{BOp} & \quad (\text{heap}, (v_1 :: v_2 :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (v_2 \otimes v_1 :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{BNot} & \quad (\text{heap}, (b :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (-b :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{IfFalse } i & \quad (\text{heap}, (\text{false} :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{true} :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + i) :: \text{frms}) \\
\text{Goto } i & \quad (\text{heap}, (\text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + i) :: \text{frms})
\end{align*}
\]

\text{New cn'} creates a new instance \text{obj} of class \text{cn'}. The fields of \text{obj} are instantiated with the default values, i.e., 0 for \text{int}, \text{false} for \text{bool} and \text{null} otherwise. Instance \text{obj} is mapped to by a fresh address \(a\) in \text{heap}. \text{Getfield fn cn'} access field \((\text{cn'}, \text{fn})\) of \text{ft}(\text{heap}(a)). \text{Putfield fn cn'} updates field \((\text{cn'}, \text{fn})\) in \((\text{cn''}, \text{ftable}) = \text{heap}(a)\) with value \(v\). \text{Checkcast cn'} fails if \text{cn'} \(\leq\) \text{cn} does not hold. \text{Getfield} and \text{Putfield} fail if \(a\) is null.

\[
\begin{align*}
\text{New cn'} & \quad (\text{heap}, (\text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (a :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{Getfield fn cn'} & \quad (\text{heap}, (\text{a} :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{ftable}([\text{cn'}, \text{fn}]) :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{Putfield fn cn'} & \quad (\text{heap}, (v :: a :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{a} \to (\text{cn''}, \text{ftable}')), (\text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms}) \\
\text{Checkcast cn'} & \quad (\text{heap}, (\text{cn} :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc}) :: \text{frms}) \\
& \quad (\text{heap}, (\text{cn} :: \text{stk}, \text{loc}, \text{cn}, \text{mn}, \text{pc} + 1) :: \text{frms})
\end{align*}
\]

\text{Invoke mn'} \(n\) inspects the type of \text{heap}(a), and performs a bottom-up search (with respect to the subclass hierarchy) for the first method declaration \text{mn'}. The new frame
is $frm' = (\epsilon, loc, cn', mn', 0)$, where $loc$ consists of the $this$ reference (address $a$), parameters $p_0 :: \cdots :: p_{n-1}$ and $mxl$ registers instantiated with $unit$ ($mxl$ is defined in the method declaration), and $cn'$ denotes the class where $mn'$ is declared. The program terminates if $Return$ is executed and $frms$ consists of a single frame. Otherwise, the top frame is dropped and the next frame updated; $frm'$ drops the parameters and the reference and pushes the return value $v$ onto the stack.

$$\begin{align*}
\text{Invoke } mn' n & \quad \frac{(heap, (p_{n-1} :: \cdots :: p_0 :: a :: stk, loc, cn, mn, pc) :: frms)}{(heap, frm' :: (p_{n-1} :: \cdots :: p_0 :: a :: stk, loc, cn, mn, pc) :: frms)} \\
\text{Return} & \quad \frac{(heap, [frm])}{(heap, [])} \quad \frac{(heap, (v :: stk, loc, cn, mn, pc) :: frm :: frms)}{(heap, frm' :: frms)}
\end{align*}$$