Robust peak-to-peak gain analysis
using integral quadratic constraints *

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Abstract: This work provides a framework to compute an upper bound on the robust peak-to-peak gain of discrete-time uncertain linear systems using integral quadratic constraints (IQCs). Such bounds are of particular interest in the computation of reachable sets and the $\ell_1$-norm, as well as when safety-critical constraints need to be satisfied pointwise in time. The use of $\rho$-hard IQCs with a terminal cost enables us to deal with a wide variety of uncertainty classes, for example, we provide $\rho$-hard IQCs with a terminal cost for the class of parametric uncertainties. This approach unifies, generalizes, and significantly improves state-of-the-art methods, which is also demonstrated in a numerical example.

Keywords: Robust control, integral quadratic constraints, peak-to-peak gain, $\ell_1$-norm, reachable set

1. INTRODUCTION

This work provides a framework to guarantee an upper bound on the robust peak-to-peak gain of discrete-time uncertain linear systems. Such bounds are particularly interesting in safety critical systems where one needs to guarantee constraint satisfaction pointwise in time despite persistent disturbances, uncertainties, and noise. The control community has been interested in peak-to-peak gains and the closely related $\ell_1$-norm since the early works of Vidyasagar (1986) and Dahleh and Pearson (1987). Ten years later, Abedor et al. (1996) proposed a computationally scalable (but approximate) solution based on linear matrix inequalities (LMIs). All these works address the nominal problem without model uncertainties. This restriction has been addressed by Ji et al. (2007) and Rieber et al. (2008), where a robust bound on the peak-to-peak gain is computed for linear systems with parametric uncertainties. In comparison, we propose a framework based on integral quadratic constraints (IQCs), which enables us to deal with a wide variety of uncertainty classes. Furthermore, using IQCs we can reduce conservatism by exploiting additional knowledge on the structure or time-invariance of the uncertainty, and we can recover the results from Ji et al. (2007) and Rieber et al. (2008) as special cases.

IQCs were first introduced by Megretski and Rantzer (1997) and have proven to be an efficient tool to analyze uncertain systems (cf. the tutorial by Veenman et al. (2016)). In the literature, a distinction is made between hard IQCs, which have to hold on all finite horizons $[0, T], T \geq 0$, and soft IQCs, which only have to hold over the infinite horizon $T \to \infty$. Veenman and Scherer (2013) already conjectured that hard time-bounds, such as the desired peak-to-peak gain, can be derived by assuming hard IQCs and using a dissipativity-based proof. However, hard IQCs are more restrictive than soft IQCs and thus often come with additional conservatism. Hence, Scherer and Veenman (2018) proposed to relax hard IQCs by using a terminal cost and thereby removing some of the conservatism (see also (Scherer, 2022a) and (Scherer, 2022b)). Moreover, to bound the impact of persistent disturbances of possibly infinite energy, Abedor et al. (1996), Ji et al. (2007), Rieber et al. (2008) used exponential stability bounds. Similarly, exponential stability analysis within the IQC framework can be performed using $\rho$-hard IQCs as proposed by Lessard et al. (2016). Considering the above discussion, we utilize $\rho$-hard IQCs with a terminal cost to analyze the robust peak-to-peak gain in a general setting.

Related work. Abou Jaoude and Farhood (2020) use IQCs to analyze the robust energy-to-peak gain, i.e., pointwise bound on the output given a bound on the energy of the disturbance signal. Similarly, Scherer (2022b) provides pointwise bounds on the output of an uncertain system described with IQCs assuming no disturbance but a norm-bound on the initial condition. Furthermore, outer approximations of the reachable set have been provided by Yin et al. (2020) and Buch and Seiler (2021) using the IQC framework and assuming disturbances of finite energy. All these works, however, do not allow for persistent disturbances. In the context of IQCs, only Abou Jaoude et al. (2021) deal with persistent disturbances and provide an outer approximation of the reachable set. However, their approach requires pointwise IQCs whereas the proposed approach requires the less restrictive $\rho$-hard IQCs with a terminal cost. We demonstrate in Example 15 that this can lead to less conservative results.

The results of this work are in particular relevant for the design of robust model predictive control (MPC) schemes, where

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constraints in terms of pointwise bounds on the output need to be satisfied despite uncertainties in the system and persistent disturbances. In our previous works (Schwenkel et al., 2020) and (Schwenkel et al., 2022), we proposed a robust MPC scheme based on an outer approximation of the reachable set using ρ-hard IQCs. In this work, we improve the approach to obtain outer approximations therein by allowing for ρ-hard IQCs with a terminal cost. Moreover, the proposed robust peak-to-peak gain analysis can be used in MPC to directly compute a suitable constraint tightening to ensure constraint satisfaction. Even when the constraints are directly influenced by the uncertainty.

Outline. After defining the problem setup in Section 2, we present several contributions in this work. In Section 3, we provide a framework to give guaranteed bounds on the robust peak-to-peak gain of an uncertain system where the uncertain components are characterized by ρ-hard IQCs with a terminal cost. In addition, we also show how to compute an outer approximation of the reachable set. In Section 4, we provide ρ-hard IQCs with a terminal cost for the special case of parametric uncertainties and show that existing results are special cases of our approach. Finally, in Section 5, we demonstrate with two examples that our results have little conservatism and can significantly improve state-of-the-art methods.

Notation. We denote the set of eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$ by $\sigma(A) \subseteq \mathbb{C}$. For $x \in \mathbb{R}^n$, denote the infinity norm by $\|x\|_\infty = \max_{i=1}^n |x_i|$ and the Euclidean norm by $\|x\| = \sqrt{x^\top x}$. The set of bounded (sequences) $x : \mathbb{N} \rightarrow \mathbb{R}^n$ is denoted by $l_\infty^\mathbb{N}$ and the ℓ∞-norm of a sequence $x \in l_\infty^\mathbb{N}$ is denoted by $\|x\|_{\ell_\infty} = \sup_{n \in \mathbb{N}} |x_n|$. For matrices $A$, $B$, $C$, $D$ with suitable dimensions we denote the operator (system) that maps input signals $u \in l_2^\mathbb{N}$ to output signal $y \in l_2^\mathbb{N}$ according to $x_{k+1} = Ax_k + Bu_k$, $x_0 = 0$, $y_k = Cx_k + Du_k$ for all $k \geq 0$ by $y = \left[\begin{array}{c} AB \end{array}\right] u$. We denote the set of exponentially stable systems with input dimension $m$ and output dimension $n$ by $\mathbb{RH}_{\infty}^{n \times m} := \left\{ \left[\begin{array}{c} AB \\ CD \end{array}\right] \max_{\lambda \in \sigma(A)} |\lambda| < 1 \right\}$. The set of symmetric matrices $A = A^\top \in \mathbb{R}^{n \times n}$ is denoted by $\mathbb{S}^n$. If $A \in \mathbb{S}^n$ is a positive (semi-)definite matrix, we write $A \succ 0$ ($A \succeq 0$). If $A \in \mathbb{S}^n$ is a negative (semi-)definite matrix, we write $A \prec 0$ ($A \preceq 0$). If $x \in \mathbb{R}^n$ and $A \succ 0$, we write $|x|_A = \sqrt{x^\top Ax}$ to indicate that this is a weighted norm on $\mathbb{R}^n$. For matrices $A \in \mathbb{R}^{n \times m}$ and $P \in \mathbb{S}^m$, we denote $\left[\begin{array}{c} A \end{array}\right] \preceq P \iff A \preceq P \iff \exists \delta \in \mathbb{R}^m : A = \delta \delta^\top$ and call $P$ the inner and $A$ the outer factor. For a vector $x \in \mathbb{R}^n$ we define the diagonal matrix $\text{diag}(x) = \left[\begin{array}{cccc} x_1 & \cdots & x_{n-1} & x_n \end{array}\right]$. We denote the convex hull of a set of points $\delta^1, \ldots, \delta^m \in \mathbb{R}^n$ by $\text{conv}\{\delta^1, \ldots, \delta^m\}$.

2. SETUP

We consider the linear system $G$ given by

$$
G: \begin{aligned}
&x_{k+1} = A_Gx_k + B_G^pu_k + B_G^mw_k \\
&q_k = C_G^px_k + D_G^pw_k + D_G^pw_k \\
&z_k = C_G^qz_k + D_G^pz_k + D_G^qw_k
\end{aligned}
$$

where $k \in \mathbb{N}$ denotes the time index and the system is initialized at $x_0 = 0$. The channel $p \rightarrow q$ in $G$ is called the uncertainty channel and $w \rightarrow z$ the performance channel. The dimensions of the signals are $x \in l_2^\infty$, $p \in l_2^\infty$, $q \in l_2^\infty$, $w \in l_2^\infty$, and $z \in l_2^\infty$ and the system matrices have suitable dimensions. The uncertainty channel of system (1) is in feedback with a causal unknown operator $\Delta : \ell_2^\infty \rightarrow \ell_2^\infty$ that belongs to a known set $\Delta \in \Delta$, i.e.,

$$
p = \Delta(q).
$$

We denote this feedback interconnection by $\Delta \ast G$ and throughout this work, we assume well-posedness, i.e., for all $w \in l_2^\infty$ and $\Delta \in \Delta$ there is a unique solution of (1), (2) that causally depends on $w$. This work analyzes the peak-to-peak gain of the performance channel, i.e.,

$$
\|\Delta \ast G\|_{\text{peak-ind}} = \sup_{\substack{w \in l_2^\infty \setminus \{0\} \atop \Delta \in \Delta}} \frac{\|z\|_{\text{peak}}}{\|w\|_{\text{peak}}}
$$

where the peak-norm is defined by

$$
\|z\|_{\text{peak}} = \sup_{\substack{w \in \mathbb{R}^\infty \setminus \{0\}} \|w\|_{\text{peak}}}
$$

Since $\|\Delta \ast G\|_{\text{peak-ind}}$ depends on the uncertainty $\Delta$, we are particularly interested in desirably small upper bound $\gamma > 0$ on the robust peak-to-peak gain

$$
\sup_{\Delta \in \Delta} \|\Delta \ast G\|_{\text{peak-ind}} \leq \gamma.
$$

Remark 1. As discussed by Rieber et al. (2008), the peak-to-peak gain is closely related to the $\ell_\infty$-to-$\ell_\infty$ gain, also known as $\ell_1$-norm. In particular, for every system $H \in \mathbb{RH}_{\infty}^{n \times m}$ it holds

$$
\frac{1}{\sqrt{n}} \|H\|_{\text{peak-ind}} \leq \|H\|_1 \leq \sqrt{m} \|H\|_{\text{peak-ind}},
$$

such that every bound $\gamma$ on the peak-to-peak gain implies also a bound on the $\ell_1$-norm.

We assume that we can describe the uncertainty set $\Delta$ by a ρ-hard IQCs with a terminal cost.

Definition 2. (ρ-hard IQC with terminal cost). Let $\rho \in (0,1)$, $M \in \mathbb{S}^n$, $X \in \mathbb{S}^m$, and $\Psi \in \mathbb{R}^{n \times (n_q + n_p)}$ with state space realization

$$
\begin{aligned}
\Psi_k &:= A_k \Psi_k + B_k^q q_k + B_k^p p_k \\
&= C_k \Psi_k + D_k^q q_k + D_k^p p_k
\end{aligned}
$$

with $\Psi_0 = 0 \in \mathbb{R}^{n\Psi}$. A causal operator $\Delta : \ell_2^p \rightarrow \ell_2^q$ is said to satisfy the

- ρ-hard IQC defined by $(\rho, \Psi, M, X)$ if for $p = \Delta(q)$ and for all $q \in \ell_2^p$, $x \in \mathbb{N}$ it holds

$$
\sum_{k=0}^{\infty} p^\rho \Psi_k^\top M_{nk} \Psi_{k+1} + \Psi_{k+1}^\top X \Psi_{k+1} \geq 0.
$$

- pointwise IQC defined by $(\Psi, M)$ if for $p = \Delta(q)$ and for all $q \in \ell_2^p$, $x \in \mathbb{N}$ it holds $s_j M_{nk} \geq 0$.

We call $M$ the multiplier, $\Psi$ the filter, $\rho$ the squared exponential decay rate$^1$, $X$ the terminal cost matrix, and $s$ the filter output.

Remark 3. Definition 2 reduces to the standard definition of ρ-hard IQCs (without a terminal cost) by Lessard et al. (2016) if $X = 0$. As noted in Lessard et al. (2016), a pointwise IQC defined by $(\Psi, M)$ implies a ρ-hard IQC defined by $(\rho, \Psi, M, 0)$ for all $\rho \in (0,1)$.

3. ROBUST ANALYSIS

In this section, we present our main results: a guaranteed upper bound on the peak-to-peak gain and a guaranteed outer approximation of the reachable set. For the analysis, we augment the

$^1$ In literature on ρ-hard IQCs, $\rho$ is usually the exponential decay rate and not its square. However, using $\rho$ instead of $\rho^2$ simplifies notation significantly.
system $G$ with the filter $\Psi$ to obtain the following augmented system $\Sigma$ with state $\chi = \frac{w}{x} \in \mathbb{R}^{p_x}$, state space representation

$$\Sigma: \begin{align*}
\dot{x}_{k+1} &= A_\Sigma \chi_k + B_\Sigma^P P_k + B_\Sigma^w w_k, \\
\dot{s}_k &= C_\Sigma \chi_k + D_\Sigma^P P_k + D_\Sigma^w w_k, \\
\dot{z}_k &= C_\Sigma \chi_k + D_\Sigma^P P_k + D_\Sigma^w w_k.
\end{align*} \tag{8a-8c}$$

initial condition $\chi_0 = 0$, and the matrices

$$\begin{bmatrix}
A_\Sigma & B_\Sigma^P & B_\Sigma^w \\
C_\Sigma & D_\Sigma^P & D_\Sigma^w
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0, \tag{9}$$

and

$$\begin{bmatrix}
I & 0 & 0 \\
0 & A_\Sigma & B_\Sigma^P \\
0 & C_\Sigma & D_\Sigma^P
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0. \tag{10}$$

This reformulation allows us to state the following result.

**Theorem 4.** Assume that all $\Delta \in \Delta$ satisfy the $\rho$-hard IQC defined by $(\rho, \Psi, M, X)$. Further, assume that there exist $P \in \mathbb{S}^{p_x}$, $\gamma \geq \mu$, and $\mu \geq 0$ such that

$$\begin{bmatrix}
I & 0 & 0 \\
0 & A_\Sigma & B_\Sigma^P \\
0 & C_\Sigma & D_\Sigma^P
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0, \tag{11}$$

and

$$\begin{bmatrix}
I & 0 & 0 \\
0 & A_\Sigma & B_\Sigma^P \\
0 & C_\Sigma & D_\Sigma^P
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0. \tag{12}$$

respectively. Combining both inequalities as follows

$$\delta_2(t) + \sum_{k=0}^{t-1} \rho^{t-k} \delta_1(k) \leq 0$$

and using the telescoping sum argument

$$\sum_{k=0}^{t-1} \rho^{t-k} \chi_{k+1}^T P_k \chi_{k+1} - \rho \chi_0^T P \chi_0 = \rho \chi_0^T P \chi_0 - \rho \chi_0^T \chi_0 \frac{P \chi_0}{1 - \rho}$$

leads to

$$0 \geq \frac{\rho}{\gamma(1 - \rho)} \|z_k\|^2 - \frac{\rho(\gamma - \mu)}{1 - \rho} \|w_k\|^2 - \mu \sum_{k=0}^{t-1} \rho^{t-k} \|w_k\|^2.$$  

Further, using the geometric sum upper estimate

$$\sum_{k=0}^{t-1} \rho^{t-k} \|w_k\|^2 \leq \rho \|w\|^2_{\text{peak}} \sum_{k=0}^{t-1} \rho^k \leq \frac{\rho}{1 - \rho} \|w\|^2_{\text{peak}}$$ \tag{14}

it follows

$$0 \geq \frac{\rho}{\gamma(1 - \rho)} \|z_k\|^2 - \frac{\rho(\gamma - \mu)}{1 - \rho} \|w_k\|^2 - \mu \frac{\rho}{1 - \rho} \|w\|^2_{\text{peak}}.$$  

Finally, using this inequality divided by $\rho^t > 0$, we obtain

$$0 \geq \frac{1}{\gamma} \|z_k\|^2 - (\gamma - \mu) \|w_k\|^2 - \mu \|w\|^2_{\text{peak}}$$

$$0 \geq \frac{1}{\gamma} \|z_k\|^2 - (\gamma - \mu) \|w\|^2_{\text{peak}} - \mu \|w\|^2_{\text{peak}}$$

$$0 \geq \frac{1}{\gamma} \|z_k\|^2 - \gamma \|w\|^2_{\text{peak}}.$$  

As the above reasoning holds for all $t \in \mathbb{N}$ and $\Delta \in \Delta$, we deduce $\|z\|^2_{\text{peak}} \leq \gamma^2 \|w\|^2_{\text{peak}}$, i.e., $\sup_{\Delta \in \Delta} \|A \Sigma G\|^2_{\text{peak}} \leq \gamma$.

**Remark 5.** For fixed $\rho$, the matrix inequalities (9) and (10) are linear in the decision variables $(P, M, X, \gamma, \mu)$ and can be solved with an LMI solver. Suppose that we have an LMI defining the set $\mathcal{M} \in \mathbb{S}^{p_x} \times \mathbb{S}^{p_y}$ such that for all $(M, X) \in \mathcal{M} (\rho)$, any $\Delta \in \Delta$ satisfies the $\rho$-hard IQC defined by $(\rho, \Psi, M, X)$ (cf. Section 4 for examples of such LMs $\mathcal{M} (\rho)$). Then, we can minimize the upper bound $\gamma(\rho)$ from Theorem 4 with fixed $\rho$ by solving the semi-definite program

$$\gamma^*(\rho) = \min \gamma$$

s.t. $(M, X) \in \mathcal{M} (\rho), \gamma \geq \mu \geq 0, (9), (10).$$

Performing a line search over $\rho \in (0,1)$ to minimize $\gamma^*(\rho)$ yields the best upper bound on the robust peak-to-peak gain we can get using Theorem 4.

If the IQC holds pointwise, we can use a multiplier $M_2 \neq M$ in (10) that is different from the multiplier $M$ in (9), as the following theorem shows. This additional degree of freedom can reduce the conservatism as Example 14 demonstrates.

**Theorem 6.** Assume that there exists a set $\mathcal{M} \subseteq \mathbb{S}^{p_x}$ such that all $\Delta \in \Delta$ satisfy the pointwise IQC defined by $(\Psi, M)$ for all $M \in \mathcal{M}$. Further, assume that there exists $P \in \mathbb{S}^{p_x}, P = P(T)$, $\gamma \geq \mu, \mu \geq 0, M \in \mathcal{M}$, and $M_2 \in \mathcal{M}$ such that (9) and

$$\begin{bmatrix}
I & 0 & 0 \\
0 & A_\Sigma & B_\Sigma^P \\
0 & C_\Sigma & D_\Sigma^P
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0, \tag{11}$$

respectively. Combining both inequalities as follows

$$\delta_2(t) + \sum_{k=0}^{t-1} \rho^{t-k} \delta_1(k) \leq 0$$

and using the telescoping sum argument

$$\begin{bmatrix}
I & 0 & 0 \\
0 & A_\Sigma & B_\Sigma^P \\
0 & C_\Sigma & D_\Sigma^P
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0. \tag{12}$$

respectively. Combining both inequalities as follows

$$\delta_2(t) + \sum_{k=0}^{t-1} \rho^{t-k} \delta_1(k) \leq 0$$

and using the telescoping sum argument

$$\begin{bmatrix}
I & 0 & 0 \\
0 & A_\Sigma & B_\Sigma^P \\
0 & C_\Sigma & D_\Sigma^P
\end{bmatrix} \begin{bmatrix}
P_k & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \preceq 0, \tag{13}$$

to obtain

$$0 \geq \frac{\rho}{\gamma(1 - \rho)} \|z_k\|^2 - \frac{\rho(\gamma - \mu)}{1 - \rho} \|w_k\|^2 - \mu \sum_{k=0}^{t-1} \rho^{t-k} \|w_k\|^2$$

and

$$0 \geq \frac{\rho}{\gamma(1 - \rho)} \|z_k\|^2 - \frac{\rho(\gamma - \mu)}{1 - \rho} \|w_k\|^2 - \mu \sum_{k=0}^{t-1} \rho^{t-k} \|w_k\|^2.$$  

Note that both Theorems 4 and 6 do not require positive definiteness of $P - \tilde{X}$ or $P$. Hence, we obtain an outer approximation of the reachable set of the output $z$ but not of the state $x$ or the extended state $\chi$. Still these results can be used to obtain an ellipsoidal outer approximation of the reachable set. In particular, for a given shape matrix $Q \in \mathbb{S}^{p_y}$, $Q > 0$ we can consider the performance output $z = C_G^x$, where $C_G^x$ is defined by the
Cholesky decomposition \((C_p^T C_p) = Q\), then the Theorems 4 and 6 yield the following ellipsoidal outer approximation of the reachable set: \(\|x_k\|_Q \leq \gamma \|w\|_{\text{peak}}\) for all \(k \in \mathbb{N}\). However, often one does not want to fix the shape matrix \(Q\) beforehand, but rather use it as decision variable and fix \(\gamma\) and \(\mu\) instead. For such cases, we provide the following theorem.

**Theorem 7.** Assume that all \(\Delta \in \Delta\) satisfy the \(\rho\)-hard IQC defined by \((\rho, \Psi, M, X)\). Further, let \(\mu = 1 - \rho\) and assume that there exist \(P \in \mathbb{S}^q\) and \(Q \in \mathbb{S}^q\) such that (9), \(Q > 0\), and \(P \succ X\) hold. Then \(\|x_k\|_Q \leq \|x_k\|_{p-X} \leq \|w\|_{\text{peak}}\) holds for all \(k \in \mathbb{N}\) and all \(\Delta \in \Delta\).

**Proof.** Due to \(P \succ X\) it follows \(\|x_k\|_Q^2 \leq \|x_k\|_{p-X}^2\).

Since (9) implies (11), we compute \(\sum_{k=0}^{t-1} \rho^{t-k-1} \delta_k(k) \leq 0\) and use a telescoping sum argument similar to (13) to obtain \(\chi_t^T P \chi_t + \sum_{k=0}^{t-1} \rho^{t-k-1} \delta_k(k) \|w_k\|_2^2 \leq 0\).

Using the \(\rho\)-hard IQC (7) as well as (14) with \(\mu = 1 - \rho\) yields \(\chi_t^T P \chi_t - \psi_t^T \chi_t \|x\|_{\text{peak}}^2 \leq 0\), hence, \(\|x_k\|_{p-X}^2 = \chi_t^T P \chi_t - \psi_t^T \chi_t \|x\|_{\text{peak}}^2 \leq 0\) holds for all \(t \in \mathbb{N}\).

In the special case of \(\rho\)-hard IQCs with \(X = 0\), Theorem 7 is a corollary of (Schwenkel et al., 2020, Theorem 2).

4. \(\rho\)-HARD IQC FOR PARAMETRIC UNCERTAINTIES

In this section, we provide \(\rho\)-hard IQCs with terminal conditions for the uncertainty class of parametric uncertainties. Thereby, we demonstrate how we can construct rich sets \(\mathbf{MX}(\rho)\) using LMIs such that we can implement the procedure from Remark 5. Furthermore, we show that our approach unifies existing LMI approaches to determine the robust peak-to-peak gain. In particular, we consider the following set of parametric uncertainties \(\Delta_p = \{ \ell_{2e} \to \ell_{2e} : \Delta(q) \Delta = \Delta_k q_k \}\):

- time-varying with coefficients from a polytope

\[\Delta_{p,tv} := \{ \Delta \in \Delta_p \mid \Delta_k = \text{ diag} (\delta_k), \delta_k \in \text{ conv} (\delta^1, \ldots, \delta^m) \} \quad (17)\]

- time-invariant with coefficients from a polytope

\[\Delta_{p, ti} := \{ \Delta \in \Delta_p \mid \Delta_k = \text{ diag} (\delta), \delta \in \text{ conv} (\delta^1, \ldots, \delta^m) \} \quad (18)\]

with \(\delta^1, \ldots, \delta^m \in \mathbb{R}^n\) as in Rieber et al. (2008), time-varying with gain bound \(s(\Delta) \leq \sigma_\delta\) in Ji et al. (2007):

\[\Delta_{p,tv,g} := \{ \Delta \in \Delta_p \mid \Delta_k^2 \Delta_k \leq I \} \quad (19)\]

These sets of uncertainties have been studied within the classical IQC framework leading to well-known IQCs (see, e.g., Megretski and Rantzer (1997) or Veenman et al. (2016)). The class of IQCs (Veenman et al., 2016, Class 4) for \(\Delta_{p,tv}\) is known to hold pointwise and thus we can directly use it without modifications.

**Theorem 8.** (Pointwise IQC for \(\Delta_{p,tv}\).) Define \(\Psi = I_{2n_q} \otimes M\) as the set of all \(M \in \mathbb{S}^{2n_q}\) that satisfy

\[\begin{bmatrix} M & 0 \\ 0 & I_{2n_q} \end{bmatrix} \succeq 0 \quad \text{and} \quad \begin{bmatrix} M \otimes \text{ diag} (\delta) \end{bmatrix} \succeq 0 \quad \forall j = 1, \ldots, m.\]

Then any \(\Delta \in \Delta_{p,tv}\) satisfies the pointwise IQC defined by \((\Psi, M)\) for all \(M \in \mathcal{M}\).

**Proof.** The proof in (Veenman et al., 2016, Class 4) can be analogously transferred to discrete-time.

The class of IQCs (Veenman et al., 2016, Class 6) for \(\Delta_{p,ti}\) on the other hand is a class of soft IQCs and hence, we cannot use it directly but need to restrict it to \(\rho\)-hard IQCs with terminal cost to apply Theorem 4.

**Theorem 9.** (\(\rho\)-hard IQC with terminal cost for \(\Delta_{p,ti}\).) Let \(\Phi = \{ A_{\Psi} \otimes B_{\Psi} \} \in \mathbb{R}^{2n_q \times 1} \) have state dimension \(n_q\), i.e., \(A_{\Psi} \in \mathbb{R}^{n_q \times n_q}\). Define \(\Psi\) as the Kronecker product \(\Psi = I_{2n_q} \otimes \Phi\), i.e.,

\[ \begin{bmatrix} A_{\Psi} \otimes B_{\Psi} \\ B_{\Psi} \otimes A_{\Psi} \end{bmatrix} \quad \begin{bmatrix} I_{n_q} \otimes A_{\Psi} & I_{n_q} \otimes B_{\Psi} \\ I_{n_q} \otimes C_{\Psi} & I_{n_q} \otimes D_{\Psi} \end{bmatrix} \quad \begin{bmatrix} I_{n_q} \otimes B_{\Psi} & I_{n_q} \otimes A_{\Psi} \\ I_{n_q} \otimes C_{\Psi} & I_{n_q} \otimes D_{\Psi} \end{bmatrix} \]

and for \(\rho \in (0, 1)\) define \(\mathbf{MX}(\rho)\) as the set of all \((M, X) \in \mathbb{S}^{2n_q \times 2n_q}\) for which there exist \(Y_j \in \mathbb{S}^{2n_q}\) such that the following matrix inequalities hold for all \(j = 1, \ldots, m:\)

\[ \begin{bmatrix} T & -\rho \mathbf{X}_2 \end{bmatrix} \begin{bmatrix} X_{22} & M_{22} \\ M_{22}^T & \mathbf{I}_{n_q} \end{bmatrix} \begin{bmatrix} I_{n_q} & I_{n_q} \otimes \Phi \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \leq 0 \quad (20) \]

\[ \begin{bmatrix} T & -\rho Y_j \end{bmatrix} \begin{bmatrix} Y_j & \mathbf{C}_q \end{bmatrix} \begin{bmatrix} I_{n_q} & I_{n_q} \otimes \Phi \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \leq 0 \quad (21) \]

Then for all \(\rho \in (0, 1)\) and all \((M, X) \in \mathbf{MX}(\rho)\), any \(\Delta \in \Delta_{p,ti}\) satisfies the \(\rho\)-hard IQC defined by \((\rho, \Psi, M, X)\).

**Proof.** We introduce the notation \(q^{i} = \begin{bmatrix} A_{\Psi} & B_{\Psi} \\ C_{\Psi} & D_{\Psi} \end{bmatrix} q^{i}\) and \(\sigma^i = \Phi q^{i}\) for state and output of \(\Phi\) when the input \(q^{i}\) is the \(i\)th component in \(q\). Further, we define the block diagonal matrices

\[ \hat{q} = \begin{bmatrix} q^{1} \\ \vdots \\ q^{n_q} \end{bmatrix}, \quad \hat{\sigma} = \begin{bmatrix} \sigma^1 \\ \vdots \\ \sigma^n \end{bmatrix}, \quad \hat{\sigma} = \begin{bmatrix} q^{1} \\ \vdots \\ q^{n_q} \end{bmatrix}. \]

For clarity, we explicitly highlight the dependency on \(\delta\) in the state \(\Psi(q) = \psi(q)\) and the output \(s = s(\delta)\) of \(\Psi\) with the input \(\hat{q}\), where \(p = \text{ diag}(\delta)q = \hat{q}\). Note that the special structure

\[ \psi(q) = \begin{bmatrix} \sigma^1 & \sigma^n \end{bmatrix} \]

and from right by its transpose, then we obtain

\[ \begin{bmatrix} T & -\rho Y_j \end{bmatrix} \begin{bmatrix} Y_j & \mathbf{M}_s \end{bmatrix} \begin{bmatrix} \psi(q) \psi(q+1) \end{bmatrix} \geq 0 \quad (22) \]

and thus

\[ -\rho \begin{bmatrix} Y \psi(q) \psi(q+1) \end{bmatrix} + \begin{bmatrix} Y \psi(q+1) \psi(q) \end{bmatrix} \geq 0. \]

After multiplying this inequality by \(\rho^{-j-k}\), summing it from \(k = 0\) to \(k = t\), and using a telescoping sum argument similar to (13), it follows...
Furthermore, let us multiply \( \delta \) to the left and its transpose to the right of (20), then we obtain

\[
-\rho \left[ \begin{array}{c} X_{22} \delta_k + \left[ \begin{array}{c} \psi_k \end{array} \right] X_{22} \delta_k + [\psi] M_{22} \delta_k \end{array} \right] \geq 0
\]

and hence, after multiplying this inequality by \( \delta^T \), summing it from \( k = 0 \) to \( k = t \), and using a telescoping sum argument similar to (13), it follows

\[
\sum_{k=0}^{t} \rho^{-k} \left[ \begin{array}{c} X_{22} \delta_k + \left[ \begin{array}{c} \psi_k \end{array} \right] X_{22} \delta_k + [\psi] M_{22} \delta_k \end{array} \right] \geq 0.
\]  

(26)

Since both \( \psi = \psi(\delta) \) and \( s = s(\delta) \) are linear in \( \delta \), we can write

\[
\sum_{k=0}^{t} \rho^{-k} \left[ \begin{array}{c} X_{22} \delta_k + \left[ \begin{array}{c} \psi_k \end{array} \right] X_{22} \delta_k + [\psi] M_{22} \delta_k \end{array} \right] \geq 0
\]

with suitable \( c_0 \in \mathbb{R} \), \( c_1 \in \mathbb{R}^{n_q} \), and \( c_2 \in \mathbb{R}^{n_q \times n_q} \) which are independent of \( \delta \).

This shows (7) and completes the proof as the arguments hold for all \( \delta \in \text{conv}[\delta^1, \ldots, \delta^n] \), i.e., all \( \Delta \in \Delta \), all \( q \in \mathbb{R}^{n_2} \), all \( t \geq 0 \), and all \( \rho, M, X, Y \) satisfying (20)–(22).

In Remark 10, we use Definition 6 to obtain the IQC approach from (Rieber et al., 2008). In the second example, we perform a reachability analysis and compare Theorem 7 to the approach in (Abou Jaoude et al., 2021).

Example 14. We consider the following MIMO system \( G \) taken from (Rieber et al., 2008)

\[
\begin{align*}
A_G & = \begin{bmatrix} 0.2 & 0.01 & 0.1 & 0.2 \end{bmatrix},
B_G & = \begin{bmatrix} 1.0 \end{bmatrix},
C_G & = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix},
D_G & = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix},
\end{align*}
\]

which is in feedback with \( \Delta \in \Delta_{p,v} \) according to (1), (2), with \( \Delta_{p,v} \) defined in (18) with

\[
\delta^1 = \begin{bmatrix} -0.1 \\ -0.3 \end{bmatrix}, \quad \delta^2 = \begin{bmatrix} -0.1 \\ 0.6 \end{bmatrix}, \quad \delta^3 = \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix}, \quad \delta^4 = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}.
\]

To compute an upper bound on the peak-to-peak gain, we use the analysis in Theorem 4 in combination with the IQC from Theorem 9 with \( \Phi \) from Remark 10 with \( v = 2 \) and \( \lambda = -0.25 \). Then, we solve the semi-definite program (15) with the LMI parser YALMIP (Löfberg, 2004) and the solver MOSEK ApS (2021) and obtain the bound \( \gamma \leq 60.61 \) for \( \rho = 0.18 \). For comparison, we found the lower bound \( \gamma \geq 59.41 \) on the peak-to-peak gain by maximizing the peak of the output signal.
over $\Delta \in \Delta_{p,\text{tv}}$ and $w$ with $\|w\|_{\text{peak}} \leq 1$. This maximization problem is non-convex and hence we cannot expect that the local maximum we found results in a tight lower bound. The local maximum was obtained by using MATLAB’s fmincon and starting the optimization at the initial guess $\delta_i = 0.5$, $\delta_2 = 0.6$ and $w_k = \frac{1}{\sqrt{2}}[1 \ 1]^T$ for all $k \in [0, 0.8]$. If we instead use the result in (Rieber et al., 2008),\(^2\) to compute an upper bound $\gamma$ on the peak-to-peak gain, then we obtain $\gamma \leq 66.93$, which shows that the proposed IQC-based improves the state-of-the-art approach significantly. The reason for this significant improvement lies in the fact, that the approach of (Rieber et al., 2008) is equivalent to using Theorem 6 with the pointwise IQC from Theorem 8 for $\Delta \in \Delta_{p,\text{tv}}$ (cf. Remark 11), i.e., the knowledge about the time-invariance of $\Delta$ is not exploited in their approach. So if we allow the uncertain parameters to be time varying, i.e., $\Delta \in \Delta_{p,\text{tv}}$ as defined in (17), then we obtain with both approaches the same upper bound $\gamma \leq 66.93$. We can also use Theorem 4 in combination with the pointwise IQC from Theorem 8, as pointwise IQCs imply $\rho$-hard IQCs with $X = 0$ for all $\rho \in (0, 1)$. Then, we obtain the best resulting bound $\gamma \leq 67.81$ for $\rho = 0.23$. This shows that Theorem 6 indeed yields less conservative results than Theorem 4 if the IQC holds pointwise. Again, for comparison we computed a lower bound $\gamma \geq 65.66$ using MATLAB’s fmincon and starting the optimization at the initial guess of our previously found time-invariant worst case. All obtained bounds on $\gamma$ are summarized in Table 1.

**Example 15.** We consider the following MIMO system $G$ taken from (Abou Jaoude et al., 2021)

$$
\begin{bmatrix}
A_G & B_{G_1} & B_{G_2} \\
C_{G_2}D_{G_1} & D_G
\end{bmatrix} = \begin{bmatrix}
0.5 & -2 & 3 & 1.2 & 1 & 5.1 \\
1 & 0.8 & 2.5 & -3.4 & -3.7 \\
-2 & 0.5 & -11.3 & -3.2 & -5.2 \\
1 & -5.3 & 1.6 & -5.4 \\
-0.9 & -5.2 & -5.6 & -9 & 3.1
\end{bmatrix},
$$

which is in feedback with $\Delta \in \Delta_{p,\text{tv}}$ with $\delta^1 = [-0.3 \ -0.3]^T$ and $\delta^2 = [0.3 \ 0.3]^T$ and is subject to disturbances $\|w\|_{\infty} \leq 0.5$, which implies $\|w\|_{\text{peak}} \leq \sqrt{0.5}$. Using Theorem 7 with the $\rho$-hard IQC from Theorem 9 and $\Phi$ from Remark 10 with $\nu = 2$ and $\lambda = 0.2$, we can compute $Q \in S^p_{\gamma}$ such that $\|x_k\|_{\infty} \leq 1$ for all $k \in N$ with $Q = \mathcal{Q}(\mu_{\text{peak}})$. We minimize the volume of the ellipsoidal outer approximation by minimizing $-\log\det(Q)$, which yields $\log det(Q) = 7.04$ for $\rho = 0.96$. Abou Jaoude et al. (2021) used an approach based on pointwise IQCs to obtain $-\log\det(Q) = 8.38$. Note that $\|w\|_{\text{peak}} \leq \sqrt{0.5}$ is a conservative bound on $\|w\|_{\infty} \leq 0.5$ and yet, the proposed approach using $\rho$-hard IQCs with a terminal cost and $\|w\|_{\text{peak}} \leq \sqrt{0.5}$ can reduce the conservatism significantly compared to using pointwise IQCs and $\|w\|_{\infty} \leq 0.5$.

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\(^2\) Specifically Theorem 4 in combination with Remark 5

\(^3\) To avoid numerical problems, we rescaled LMI (9) by using $\mu = 1000\mu$ and restored $Q = \mu\mathcal{Q}$.