Instanton-Induced Interactions in Finite Density QCD

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We consider the finite density, zero-temperature behaviour of quark matter in the instanton picture. Since the instanton-induced interactions are attractive in both $\bar{q}q$ and $qq$ channels, a competition ensues between phases of matter with condensation in either or both. It results in chiral symmetry restoration due to the onset of diquark condensation, a ‘colour superconductor’, at finite density.

1. Introduction

Due to a lack of lattice QCD techniques for implementing quark chemical potential, the finite density properties of strongly-interacting matter remain unresolved. To date, model studies suggest not only chiral symmetry restoration but also the possibility of Cooper pairing of quarks at high density, via an attractive $qq$ interaction, similar to superconducting electrons. The analogy has been extended to nomenclature, with the QCD version called colour superconductivity.

It has been known for some years that perturbative, single gluon exchange between quarks is attractive and will generate a pairing gap around the Fermi surface \cite{1}. More recently, it was suggested that colour superconductivity might also arise by non-perturbative means at moderate quark density \cite{2}. Since then, more detailed studies using models inspired by that of Nambu and Jona-Lasinio \cite{3} and instantons \cite{4} have supported this idea. This talk describes how diquark formation restores broken chiral symmetry in the context of the QCD instanton vacuum, an approach which has accounted for many hadronic observables through the use of fundamental degrees of freedom (quarks and gluons) in a microscopic approximation.

2. Quark Effective Action

The derivation of an effective action for chiral quarks in $N_f$ flavours has been discussed in detail in other publications. Here we concentrate on the two flavour case, which is often adequate for low energy phenomenology. Growing quark chemical potential naturally makes the strange quark more relevant, as has been studied in other models \cite{5}. These authors conclude that the two-flavour superconducting state is likely to be present at moderate values of the quark chemical potential for a realistic strange mass.

The basic idea is to replace the partition function of QCD with an effective form which divides the low and high energy contributions. The high momentum part is taken to be
perturbative and as such the gluons here are assumed to be small corrections to the stable, low-energy configurations of the gauge fields – the instantons. Each (anti-) instanton in turn induces a quark zero mode of (right) left chirality, and averaging over all possible instanton backgrounds results in a delocalization of the zero modes which spontaneously breaks chiral symmetry.

This picture of the vacuum is supported by various lattice studies and has a long history of successful phenomenology. Following this procedure, the expected ’t Hooft interaction is obtained and one has an effective quark action which is suitable for practical calculations. We have reformulated this effective action for finite quark chemical potential. The result can be expressed as

$$S[\psi, \psi^\dagger] = - \int d^4 p \psi^\dagger (\slashed{p} + i \mu \gamma_4) \psi + \lambda \left( \int dU \int_0^{N_f} \left[ \frac{(d^4 p_f d^4 k_f)}{(2\pi)^8} \right] (2\pi)^4 \delta^4 \left( \sum (p_f - k_f) \right) \right)$$

$$\prod_{f}^{N_f} \left[ \psi^\dagger_{L f \alpha_i j_f} (p_f) \mathcal{F}(p_f, \mu)^{\alpha_i \beta_i}_{f k_f} U_{\alpha_f \beta_f}^{j_f j_f} U^\dagger_{\alpha_f \beta_f} \epsilon_{\alpha_f \beta_f \gamma_f} \mathcal{F}^\dagger(k_f, -\mu)^{\gamma_f \beta_f \alpha_f} \psi_{L f \beta_f j_f} (k_f) \right] + (L \leftrightarrow R),$$

where the $\psi$ are quark fields and the $\mathcal{F}$ are spin/colour matrix form factors. They involve the quark zero modes and as such have specific dependence on momentum and chemical potential. Unique to this approach, the coupling constant $\lambda$ is not fixed here. One rather integrates over all possible coupling strengths since the constant has been introduced as a Lagrange multiplier. This gives rise to a subtle interdependence between the instanton background and the quark interaction.

### 3. Results for Finite Quark Density

With this effective theory it is straightforward to perturbatively expand in the coupling constant $\lambda$. This amounts to a virial expansion in the instanton density $N/V$, with which $\lambda$ scales non-linearly. We consider only the scalar condensates, which translates into an ansatz of three propagators: two normal and one Gorkov. It allows condensation in the $\bar{3}$ diquark channel and breaks the colour symmetry as $SU(3) \rightarrow SU(2) \times U(1)$. Should this occur, the normal propagators (and ensuing condensates) will lose their colour degeneracy and their separation becomes necessary. Namely, there will be two gapped and one ungapped quark species. These propagators allow for chiral symmetry breaking, as explained in Ref. [4].

This procedure closes a set of Schwinger-Dyson-Gorkov equations, shown in Fig. [1]. The diagrams reduce to three closed loops corresponding to a set of three gap equations which specify the condensates $g_1$, $g_2$, and $f$. The first two are chiral condensates, distinct in the case of colour symmetry breaking, and the third a diquark condensate. They
combine in the physical quantities:

\[ M_1 = (5 - 4/N_c)g_1 + (2N_c - 5 + 2/N_c)g_2, \]
\[ M_2 = 2(2 - 1/N_c)g_1 + 2(N_c - 2)g_2, \]
and \( \Delta = (1 + 1/N_c)f \). The \( M_{1,2} \) are measures of chiral symmetry breaking and act as an effective mass. Meanwhile the diquark loop \( 2\Delta \) plays the role of twice the single-quark energy gap formed around the Fermi surface.

The solution of the gap equations depends on the vertex coupling constant, \( \lambda \), which itself is determined by balancing the instanton background with the condensates through its saddle-point value. This minimization of the partition function leads to

\[ N/V = \lambda \langle Y^+ + Y^- \rangle = 4(N_c^2 - 1) [2g_1M_1 + (N_c - 2)g_2M_2 + 4f\Delta]/\lambda. \] 

(2)

This joins the gap equations to close a system of equations, numerically solvable. Once this is done, the chiral condensate proper may be computed as an integral over the resummed propagator.

For any given chemical potential, multiple solutions can be obtained for the gaps. These correspond to different phases of quark matter, and they are summarized as follows: (0) Free massless quarks: \( g_1 = g_2 = f = 0 \); (1) Pure chiral symmetry breaking: \( g_1 = g_2 \neq 0, f = 0 \); (2) Pure diquark condensation: \( g_1 = g_2 = 0, f \neq 0 \); and (3) Mixed symmetry breaking: \( g_1 \neq g_2 \neq 0, f \neq 0 \). The free energy, calculated to first order in \( \lambda \), is minimized in order to resolve the stable solution. The phase corresponding to the lowest coupling \( \lambda \) is the thermodynamically favoured [4].

No solutions were found matching Phase (0), and the Phase (3) solution obtained disappears at relatively low chemical potential (\( \mu \approx 80 \text{ MeV} \)) and is never thermodynamically competitive [4]. The remaining phase competition is then between Phases (1) and (2). In the vacuum, where \( \mu = 0 \), one finds Phase (1) preferred – this is the standard picture. However, at a critical chemical potential \( \mu_c \), defined by the ratio of superconductive gap to chiral effective masses \( \Delta/M = \sqrt{N_c/8(N_c - 1)} = \sqrt{3}/4 \), a first-order phase transition occurs. With the standard instanton parameters \( N/V = 1 \text{ fm}^{-3} \) and \( \bar{\rho} = 0.33 \text{ fm} \), we find \( \mu_c \approx 340 \text{ MeV} \). The first-order nature of the phase transition is clearly seen in Fig. 2.

Physically, the quark density is more relevant than the chemical potential. As an intermediate step and in order to demonstrate the microscopic differences between the two phases, we have calculated the occupation number density for quarks. This is nontrivial

Figure 2. Condensates for \( N_c = 3 \) as a function of \( \mu \).

Figure 3. The quark density \( n_q \) vs. \( \mu \).
and here we present only numerical results in Figs. 4 and 5. In Phase (1), there is clearly an effective mass brought about by spontaneous symmetry breaking, indicated by the reduced Fermi radius. We stress that, despite the complicated four-momentum dependence of the interaction, the resulting occupation number density appears as a perfect Fermi step function. Cooper pairing, however, smears the Fermi surface, and this is evinced in the second plot. The residual discontinuity at $|\vec{p}| = \mu$ is the contribution from the free, colour-3 quarks which do not participate in the diquark.

Integrating over momenta and recalling the critical chemical potential, the quark density profile as a function of chemical potential is plotted in Fig. 3 for the equilibrium states. We see a discontinuity at the phase transition, where the horizontal line signifies the quark density of stable nuclear matter. The phase transition occurs at an extremely low quark density, which remains a conceptual conundrum.

4. Conclusions

Beginning from the instanton picture of the QCD vacuum, we have extended the model for finite density and found chiral symmetry restoration due to the onset of colour superconductivity. This phase transition is strongly first order and in agreement with other quark-based approaches.

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