Figure Caption

Figure 1: The potential $U(\phi)$ with two relative minima $\phi_{\pm}$. 
Proper fluctuations associated with quantum tunneling in field theory

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Abstract

It is shown that, to the lowest order in $\hbar$, the particle production related to the tunneling that leads to the false vacuum decay is described by the orthogonal part of fluctuation field with respect to the bounce solution. As a simple example the spatially homogeneous tunneling is considered in order to illustrate the consequences coming from such a restriction of the fluctuation field.

1 Introduction

Coupling of tunneling field $\phi$ to the fluctuation one leads to the particle production effect during the tunneling process. To make this more precise, consider the potential $U(\phi)$ shown in Fig.1. There are two minima, $\phi_{\pm}$, both of which correspond to classically stable homogeneous field configurations. The false vacuum corresponds to $\phi = \phi_{+}$ for which we assume $U(\phi_{+}) = 0$. In the WKB approximation the tunneling probability per unit volume per unit time of such a state is of the form

$$P/VT = Ae^{-B/\hbar}(1 + O(\hbar)).$$

(1)

Coleman [1] showed that to find $B$ one has to solve the field equations of the theory in Euclidean space-time subjected to the boundary condition that the field asymptotically approaches its false vacuum value. In addition to the trivial solution $\phi = \phi_{+}$ there is a time-reversal invariant zero-energy solution, $\phi_{b}$, referred to as a bounce one and $B$ is the Euclidean action evaluated at the bounce. The prefactor $A$ contains the first quantum correction to the tunneling process. The problem of particle creation during the tunneling process, leading to the decay of false vacuum in quantum field theory, was first studied in [2]. The further development of the general formalism describing this process is given in [3, 4, 5, 6]. Recently the particle spectrum in the thin-wall approximation has been considered in [7]. It was shown that in this case the particle production is strongly suppressed. The purpose of this brief paper is to point out the existence of a constraint on the fluctuation field in the tunneling process. This constraint arises due to fact that the fluctuation field associated with the tunneling that gives rise to the particle production is the transverse part of total fluctuation field with respect to the bounce solution. This is in fact implicit in [8]. Namely, the one-loop functional determinant obtained in this article is explicitly written as a functional integral over transverse part of fluctuation field. Nevertheless, this fact has not received attention so far in consideration of particle production during the tunneling process in field theory. We briefly notice here that for evaluating the preexponential factor, $A$, Callan and Coleman [10] used an Euclidean version of Hamilton’s action in the path-integral formalism. But this approach has the serious drawbacks considered by Patrascioiu [11]. In Sec.2 we specify the fluctuation field associated with the tunneling process and, in this context, consider an example of spatially homogeneous tunneling. The conclusions are briefly summarized in Sec.3.
2 Proper fluctuations associated with the tunneling

At first we demonstrate that in order to recover the one-loop determinant expression of [8] in the path integral formalism one has to use the Jacobi type action. Because this Jacobi type action corresponds to the Euclidean Hamilton’s action in much the same way that connects Jacobi’s action to Hamilton’s one, we refer to it as the Euclidean Jacobi’s action. The Euclidean Jacobi’s action has the form

$$J_E[\phi] = \int_{-\infty}^{\infty} d\sigma \sqrt{2V[\phi]} \int d^3 y \dot{\phi}^2,$$

where $V[\phi] = \int d^3 y \left(\frac{1}{2} \nabla \phi^2 + U(\phi)\right)$ and $\dot{\phi} \equiv \frac{d\phi}{d\sigma}$. It is obtained from the Jacobi’s action $J$ formally as $\sigma \rightarrow -i\sigma$, $J_E \rightarrow iJ$. We want to estimate the following functional integral in the semiclassical(small-$\hbar$) limit.

$$P \equiv \int_{\phi(-\infty)=\phi_f}^{\phi(\infty)=\phi_f} [D\phi]_{FP} e^{-\frac{i}{\hbar} \int_{-\infty}^{\infty} d\sigma \sqrt{2V[\phi]} \int d^3 y \dot{\phi}^2},$$

where $[D\phi]_{FP}$ is a Faddeev-Popov measure and the integration is over all functions $\phi(\sigma, \vec{x})$ obeying the boundary conditions $\phi(\pm \infty, \vec{x}) = \phi_\perp$. As an essential point for our discussion we want to emphasize that the action (2) is invariant under the reparameterizations of the configuration space-path that preserve the end point values of the parameter. That is, (2) is invariant under the replacements $\sigma \rightarrow f(\sigma)$ and $\phi(\sigma, \vec{x}) \rightarrow \tilde{\phi}(f(\sigma), \vec{x})$ with $f(\pm \infty) = \pm \infty$. Their infinitesimal form is $\sigma \rightarrow \sigma + \epsilon(\sigma)$ and $\phi \rightarrow \phi + \epsilon\phi$ where $\epsilon(\pm \infty) = 0$. Now it is obvious that the proper fluctuations for action (2) are transverse ones $\delta \phi_{\perp}(\vec{x}) = \int d^3 y \Pi_{\perp}(\vec{x}, \vec{y}) \delta \phi(\vec{y})$, where $\Pi_{\perp}(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}) - \phi(\vec{x})\phi(\vec{y})/\int d^3 z \dot{\phi}^2$ is the projection operator onto the subspace of configuration space that is orthogonal to the configuration space-path $\phi(\sigma, \vec{x})$, while the longitudinal fluctuations reproduce a gauge transformation. For $\hbar \rightarrow 0$ the path integral is dominated by the contribution of stationary point of action (2). In imaginary time parameterization, $\sigma(\tau)$, obtained by using the Euclidean zero-energy condition

$$\int d^3 y \frac{1}{2} \left(\frac{d\phi}{d\tau}\right)^2 - V[\phi] = 0,$$

this yields the bounce solution $\phi_b$. Since we are interested in evaluation of that functional integral for $\hbar \rightarrow 0$ it is sufficient to restrict ourselves to the consideration of small fluctuations around the bounce solution. Thus, we can use the Faddeev-Popov method [12] immediately for the infinitesimal gauge transformation. Taking into account that the longitudinal fluctuations reproduce a gauge transformation then following to the Faddeev-Popov procedure one has to integrate the quadratic action over transverse ones, which are proper fluctuations for the action (2). It may be done by splitting of the transverse and longitudinal fluctuations, $D\delta \phi = J D\delta \phi_{\perp} D\delta \phi_{\parallel}$, where $J$ is Jacobian associated with this transformation and integrating the path-integral over (proper) transverse fluctuations with the Faddeev-Popov measure, $[D\delta \phi]_{FP} \equiv JD\delta \phi_{\perp}$. The integration over the gauge degrees of freedom, longitudinal fluctuations, is dropped. Expanding about the bounce solution to quadratic order in fluctuation field the functional integral (3) becomes

$$P = e^{-\frac{i}{\hbar} J_E[\phi_b]} \int JD\delta \phi_{\perp} \exp \left[ -\frac{1}{2\hbar} \int d^4 x \delta \phi_{\perp} \left(-\frac{\partial^2}{\partial \tau^2} - \Delta + U''(\phi_b)\right) \delta \phi_{\perp}\right],$$

(5)
with the boundary conditions $\delta \phi_\perp (\pm \infty, \vec{x}) = 0$. Here $\delta \phi_\perp (\vec{x}) = \int d^3y \Pi_\perp^b (\vec{x}, \vec{y}) \delta \phi (\vec{y})$ and $\Pi_\perp^b (\vec{x}, \vec{y})$ denotes orthogonal projection onto the bounce solution. The integration over the zero modes associated with the space and time translations gives an explicit $VT$ factor in computing the vacuum tunneling amplitude. So, in the appropriate limit our result recovers those obtained in [8]. On the other hand the stationary zero-energy Schrödinger equation describing this tunneling event may be obtained by the quantization of the zero-energy Jacobi’s action,

$$J[\phi] = \int_{-\infty}^{\infty} d\sigma \sqrt{-2V[\phi]} \int d^3y \dot{\phi}^2. \quad (6)$$

The relationship between Jacobi’s and Hamilton’s action principles as well as their quantization is considered in detail in [9]. Because this Lagrangian is homogenous of degree one in the $\dot{\phi}(\vec{x})$, the canonical Hamiltonian vanishes identically. This is of course a well-known feature of the reparametrization invariant theories. Consequently the momenta conjugate to the $\phi(\vec{x})$, $\pi(\vec{x}) = \dot{\phi}(\vec{x}) \sqrt{-2V[\phi]} / \int d^3y \dot{\phi}^2$, gives rise to a constraint

$$\int d^3y \frac{\pi^2(y)}{2} + V[\phi] = 0. \quad (7)$$

There are no secondary constraints and the constraint (7) is then trivially first class. Due to the Dirac’s procedure this constraint is imposed as a quantum operator equation acting on the wave functional. In the coordinate representation $\dot{\phi}(\vec{x}) = \phi(\vec{x})$, $\pi(\vec{x}) = -i\hbar \frac{\delta}{\delta \phi(\vec{x})}$ and the wave functional $\Psi[\phi]$ depends on $\phi(\vec{x})$. The resulting operator-constraint equation is just the stationary zero-energy Schrödinger equation

$$\left( -\frac{\hbar^2}{2} \int d^3y \frac{\delta^2}{\delta \phi(y)^2} + V[\phi] \right) \Psi[\phi] = 0. \quad (8)$$

This equation is obviously obtained as a result of starting with Jacobi’s action, in which time nowhere appears but the energy is fixed $E = 0$. According to the formalism developed by Bitar and Chang [8] the lowest-order WKB approximation to this equation shows that the wave functional $\Psi$ is strongly peaked around a one-parameter family of field configurations $\tilde{\phi}(\sigma, \vec{x})$ which in a classically forbidden region is determined by the stationary point of Euclidean Jacobi’s action [2] and by the stationary point of Jacobi’s action [6] in the classically allowed region. As it was discussed above, the proper fluctuation field about the $\tilde{\phi}$ corresponding to the (Euclidean)Jacobi’s action is the transverse part of fluctuation field with respect to this field configuration. To take into account the effect of fluctuations to this tunneling process one has to split the field into the tunneling one and the fluctuation field around it $\phi \rightarrow \phi + \delta \phi_\perp$. For the combined system of $\phi + \delta \phi_\perp$ one gets again a zero-energy stationary Schrödinger equation from the Jacobi’s action [6]. Thus one simply concludes that the basic equation governing the fluctuation field $\delta \phi$ during the tunneling must be supplemented by the constraint

$$\int d^3x \delta \phi(\tau, \vec{x}) \phi_\perp'/\sqrt{\tau^2 + \vec{x}^2} = 0, \quad (9)$$

where we have taken into account the $O(4)$ symmetry of the bounce solution [11] and the prime denotes differentiation with respect to $\rho = \sqrt{\tau^2 + \vec{x}^2}$. The constraint in the Minkowskian region is obtained from (9) by the analytic continuation $\tau \rightarrow it$. To see the idea let us consider an example of spatially homogeneous tunneling considered previously in [2, 4]. In this case the tunneling configuration denoted by $\phi_b$ is spatially homogeneous, $\phi_b(\tau)$. Following [4] we assume $U''(\phi_b(\tau))$ to be a step function

$$U''(\phi_b(\tau)) = \begin{cases} m_+^2, & \tau < \tilde{\tau}, \\ m_-^2, & \tau > \tilde{\tau}, \end{cases} \quad (10)$$
where $\tilde{\tau} < 0$ with a large absolute value. The equation governing the fluctuation field takes the form
\[
(\partial_\tau^2 + \Delta + m_+^2)\delta\phi = 0 \quad \tau < \tilde{\tau}, \\
(\partial_\tau^2 + \Delta + m_-^2)\delta\phi = 0 \quad \tau > \tilde{\tau}.
\] (11)

Taking the mode expansion $\delta\phi = (2\pi)^{-3/2} \int v_\mathbf{p}(\tau)e^{i\mathbf{p}\cdot\mathbf{x}}d^3p$ one easily finds the (unnormalized) solution to Eq. (11) satisfying the vanishing boundary condition when $\tau \to -\infty$
\[
v_\mathbf{p}(\tau) = \begin{cases} 
\begin{aligned}
& e^{\omega_+ \tau} & \tau < \tilde{\tau}, \\
& a_+(\mathbf{p})e^{\omega_- \tau} + a_-(\mathbf{p})e^{-\omega_- \tau} & \tau > \tilde{\tau},
\end{aligned}
\end{cases}
\] (12)

where $\omega_\pm = \sqrt{\mathbf{p}^2 + m_\pm^2}$ and
\[
a_\pm = \frac{1}{2\omega_-}(\omega_- \pm \omega_+)e^{\mp(\omega_- \mp \omega_+)(\tau - \tilde{\tau})}.
\]

The spectrum of created particles has the form
\[
n(p) = \frac{1}{(\omega_- + \omega_+)^2} e^{-4\omega_- \tau - 1}.
\] (13)

Furthermore, the constraint (9), that the fluctuation field must be orthogonal to the $\phi_b$, implies now that the integral of $\delta\phi$ over all space vanishes,
\[
v_b(\tau) \propto \int d^3x\delta\phi = 0.
\] (14)

So, we arrive at the result that the spatially homogeneous tunneling does not allow the particle production with zero momentum. Correspondingly, for $\mathbf{p} = 0$ one can not use the Eq. (13), which remains valid for $p > 0$.

3 Summary

We have demonstrated that the proper fluctuation field associated with the tunneling process is the transverse part of fluctuation field with respect to the bounce solution. This statement is quite natural since the stationary Schrödinger equation needed to describe the tunneling phenomenon is obtained by the quantization of Jacobi’s action for which the longitudinal part of fluctuation field reproduces the gauge transformation. Correspondingly, we point out that the general formalism describing the particle production in the tunneling process must be supplemented by the constraint (9). The use of this constraint is demonstrated in the case of a spatially homogenous tunneling. Roughly speaking, one must subtract the number of particles created by the longitudinal part of fluctuation field from the particle spectrum obtained by the total fluctuation field. Of course such a subtraction will not affect the conclusion made in [7] that in general the particle production in the thin-wall approximation is exponentially suppressed. It is of interest to explicate the consequences coming from the constraint (9) in the realistic cases.

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