COMPACTNESS RESULTS FOR LINEARLY PERTURBED
YAMABE PROBLEM ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let \((M,g)\) a compact Riemannian \(n\)-dimensional manifold. It is well known that, under certain hypothesis, in the conformal class of \(g\) there are scalar-flat metrics that have \(\partial M\) as a constant mean curvature hypersurface. Also, under certain hypothesis, it is known that these metrics are a compact set. In this paper we prove that, both in the case of umbilic and non-umbilic boundary, if we linearly perturb the mean curvature term \(h_g\) with a negative smooth function \(\alpha\), the set of solutions of Yamabe problem is still a compact set.

1. Introduction. Let \((M,g)\), a smooth, compact Riemannian manifold of dimension \(n \geq 3\) with boundary. In [9] Escobar asked it there exists a conformal metric \(\tilde{g} = u^{\frac{4}{n-2}}g\) for which \(M\) has zero scalar curvature and constant boundary mean curvature.

This problem can be understood as a generalization of the Riemann mapping theorem and it is equivalent to finding a positive solution to the following nonlinear boundary value problem

\[
\begin{align*}
L_g u &= 0 \quad \text{in } M \\
B_g u + \left( n - 2 \right) u \frac{n}{n-2} &= 0 \quad \text{on } \partial M.
\end{align*}
\]

Where \(L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g\) and \(B_g = -\frac{\partial}{\partial \nu} - \frac{n-2}{2} h_g\) are respectively the conformal Laplacian and the conformal boundary operator, \(R_g\) is the scalar curvature of the manifold, \(h_g\) is the mean curvature of the \(\partial M\) and \(\nu\) is the outer normal with respect to \(\partial M\).

The existence of solutions of (1.1) was established by the works of Escobar [9], Marquez [18], Almaraz [1], Chen [6], Mayer and Ndiaye [21].

Solutions of (1.1) are the critical points of the functional quotient

\[
Q(u) := \inf_{u_0 \in H^1 \setminus 0} \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g + \int_{\partial M} \frac{n-2}{2} h_g u^2 d\sigma_g}{\left( \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n}{n-2}}}.
\]

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In [9] Escobar introduced, in analogy of the classical Yamabe problem

\[ Q(M, \partial M) := \inf \{ Q(u) : u \in H^1(M), u \neq 0 \text{ on } \partial M \}. \]

Concerning the compactness of the full set of positive solutions of (1.1), the only interesting case occurs when \( Q > 0 \). Indeed, when \( Q < 0 \) the solution \( u \) is unique while when \( Q = 0 \) the solution is unique up to positive multiplicative constants.

First compactness results have been proven by Felli and Ould Ahmedou [10] for any \( n \geq 3 \), in the case of locally conformally flat manifolds and by Almaraz in [1] for \( n \geq 7 \), in the case of manifolds with nonumbilic boundary.

We recall that the boundary of \( M \) is respectively called umbilic (nonumbilic) if the trace-free second fundamental form of \( \partial M \) is zero (different from zero) everywhere.

If either \( n > 8 \) and the Weyl tensor of \( M \) never vanishes on \( \partial M \) or \( n = 8 \) and the Weyl tensor of \( \partial M \) never vanishes on \( \partial M \), the compactness is still true for manifolds with umbilic boundary [11].

Very recently the compactness was showed for manifold of dimension \( n = 3 \) [4], \( n = 4 \) [16] and -when the boundary is nonumbilic- \( n = 5, 6 \) [16].

An interesting point is the stability problem that is if the compactness is preserved under small perturbations of the equation (1.1).

In particular we consider the linear perturbation problem

\[
\begin{cases}
L_g u = 0 & \text{in } M \\
\frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u + \varepsilon \alpha u = (n-2) u \frac{n}{n-2} & \text{on } \partial M
\end{cases}
\]  

(1.2)

where \( \varepsilon \) is a small positive parameter and \( \alpha : M \to \mathbb{R} \) is a smooth function.

We can prove that the sign of the function \( \alpha \) on \( \partial M \) has an effect on compactness and non compactness of solutions of (1.2): in [13] we proved the existence of blowing up solution of (1.2) when \( \alpha > 0 \) in the case of \( \partial M \) non umbilic and \( n \geq 7 \) and in [12] we proved an analogous result in the case of \( n \geq 11 \) and the Weyl tensor not vanishing on \( \partial M \).

In the following we show that when \( \alpha \) is negative everywhere on \( \partial M \) there are no blowing up solutions for \( \varepsilon \to 0 \), i.e. compactness holds. This is analogous of what happens when perturbing the Scalar curvature term in the classical Yamabe problem (see [7, 8] and the references therein).

Our main results are

**Theorem 1.** Let \((M, g)\) a smooth, \( n \)-dimensional Riemannian manifold of positive type not conformally equivalent to the standard ball with regular umbilic boundary \( \partial M \).

Let \( \alpha : M \to \mathbb{R} \) such that \( \alpha < 0 \) on \( \partial M \). Suppose that \( n > 8 \) and that the Weyl tensor \( W_g \) is not vanishing on \( \partial M \) or suppose that \( n = 8 \) and that the Weyl tensor referred to the boundary \( \bar{W}_g \) is not vanishing on \( \partial M \). Then, given \( \bar{\varepsilon} > 0 \) there exists a positive constant \( C \) such that for any \( \varepsilon \in (0, \bar{\varepsilon}) \) and for any \( u > 0 \) solution of (1.2) it holds

\[ C^{-1} \leq u \leq C \]

and

\[ \|u\|_{C^2, \eta(M)} \leq C \]

for some \( 0 < \eta < 1 \). The constant \( C \) does not depend on \( u, \varepsilon \).

**Theorem 2.** Let \((M, g)\) a smooth, \( n \)-dimensional Riemannian manifold of positive type with non umbilic boundary \( \partial M \), with \( n \geq 7 \).

Let \( \alpha : M \to \mathbb{R} \) such that \( \alpha < 0 \) on \( \partial M \). Then, given \( \bar{\varepsilon} > 0 \) there exists a positive constant \( C \) such that for any \( \varepsilon \in (0, \bar{\varepsilon}) \) and for any \( u > 0 \) solution of (1.2) it holds

\[ C^{-1} \leq u \leq C \]

and

\[ \|u\|_{C^2, \eta(M)} \leq C \]

for some \( 0 < \eta < 1 \). The constant \( C \) does not depend on \( u, \varepsilon \).
for some $0 < \eta < 1$. The constant $C$ does not depend on $u, \varepsilon$.

1.1. Structure of the paper. We will give the proof of Theorem 1 in full detail in Section 8, while in Section 9 we will give only the main ingredients to prove Theorem 2 following the same strategy of Thm 1. In Section 2 we recall a version of Pohozaev identity for Problem (1.2). In Section 3 we choose a suitable metric conform to the given metric and Section 4 collects the definition of blow up points for a sequence of solutions of (1.2) as well as the definitions of isolated and isolated simple blow up points. In Section 5 a careful analysis of the profile of the rescaled solution near an isolated simple blow up point is proven. By this result, in Section 6 we can give an estimate of the sign of the terms of Pohozaev identity near an isolated simple blow up point. By this result, and by a splitting Lemma recalled in Section 7, we prove that only isolated simple blow up points can occur for a sequence of solution of (1.2). Finally in Section 8 we will prove that with the hypothesis of Theorem 1, also the case of an isolated simple blow up point is ruled out, and we prove our main result. This strategy of the proof of these compactness results was firstly introduced by R. Schoen (see [22]) and it is well established in literature, so in this paper we will provide only the proofs of the new results, while we will give references for the other ones.

1.2. Notations and preliminary definitions.

Notation. We will use the indices $1 \leq i, j, k, m, p, r, s \leq n - 1$ and $1 \leq a, b, c, d \leq n$. Moreover we use the Einstein convention on repeated indices. We denote by $g$ the Riemannian metric, by $R_{abcd}$ the full Riemannian curvature tensor, by $R_{ab}$ the Ricci tensor and by $R$ and $h_g$ respectively the scalar curvature of $(M, g)$ and the mean curvature of $\partial M$; moreover the Weyl tensor of $(M, g)$ will be denoted by $W$. The bar over an object (e.g. $\bar{q}$) will means the restriction to this object to the metric of $\partial M$.

Finally, on the half space $\mathbb{R}^n_+ = \{y = (y_1, \ldots, y_{n-1}, y_n) \in \mathbb{R}^n, y_n \geq 0\}$ we set $B_r(y_0) = \{y \in \mathbb{R}^n, |y - y_0| \leq r\}$ and $B_r^+(y_0) = B_r(y_0) \cap \{y_n > 0\}$. When $y_0 = 0$ we will use simply $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$. On the half ball $B_r$ we set $\partial B_r^+ = B_r^+ \cap \partial \mathbb{R}^n_+ = B_r^+ \cap \{y_n = 0\}$ and $\partial^+ B_r^+ = \partial B_r^+ \cap \{y_n > 0\}$. On $\mathbb{R}^n_+$ we will use the following decomposition of coordinates: $(y_1, \ldots, y_{n-1}, y_n) = (\bar{y}, y_n) = (z, t)$ where $\bar{y}, z \in \mathbb{R}^{n-1}$ and $y_n, t \geq 0$.

Fixed a point $q \in \partial M$, we denote by $\psi_q : B_r^+ \to M$ the Fermi coordinates centered at $q$. We denote by $B_r^+(q, r)$ the image of $\psi_q(B_r^+)$. When no ambiguity is possible, we will denote $B_r^+(q, r)$ simply by $B_r^+$, omitting the chart $\psi_q$.

We introduce the following notation for integral quantities which recur often in the paper

$$I_m^\alpha := \int_0^\infty \frac{s^\alpha ds}{(1 + s^2)^m}.$$  

By direct computation (see [1, Lemma 9.4]) it holds

$$I_m^\alpha = \frac{2m}{\alpha + 1} I_{m+1}^{\alpha+2} \text{ for } \alpha + 1 < 2m$$

$$I_m^\alpha = \frac{2m}{2m - \alpha - 1} I_{m+1}^\alpha \text{ for } \alpha + 1 < 2m$$

$$I_m^\alpha = \frac{2m - \alpha - 3}{\alpha + 1} I_{m+2}^\alpha \text{ for } \alpha + 3 < 2m$$
We shortly recall here the well known function $U(y) := \frac{1}{[(1 + y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}}$ which is also called the standard bubble and which is the unique solution, up to translations and rescaling, of the nonlinear critical problem.

\[
\begin{cases}
-\Delta U = 0 & \text{on } \mathbb{R}^n_+; \\
\frac{\partial U}{\partial n} = -(n-2)U\frac{y_l}{[(1 + y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}} & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\] (1.4)

We set

\[
j_l := \partial_l U = -(n-2)\frac{y_l}{[(1 + y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}}
\] (1.5)

\[
\partial_k\partial_l U = (n-2)\begin{pmatrix}
\frac{n y_l y_k}{[(1 + y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}} - \frac{\delta_{kl}}{[(1 + y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}}
\end{pmatrix}
\]

\[
j_n := y^b \partial_b U + \frac{n-2}{2}U = -\frac{n-2}{2}\frac{|\bar{y}|^2 - 1}{[(1 + y_n)^2 + |\bar{y}|^2]^{\frac{n-2}{2}}}. \tag{1.6}
\]

and we recall that $j_1, \ldots, j_n$ are a base of the space of the $H^1$ solutions of the linearized problem

\[
\begin{cases}
-\Delta \phi = 0 & \text{on } \mathbb{R}^n_+; \\
\frac{\partial \phi}{\partial n} + nU\frac{2}{\pi^{\frac{n-2}{2}}} \phi = 0 & \text{on } \partial \mathbb{R}^n_+,
\end{cases}
\] \tag{1.7}

2. A Pohozaev type identity. In the following, we will use this version of a local Pohozaev type identity [1, 11]

**Theorem 3** (Pohozaev Identity). Let $u$ a $C^2$-solution of the following problem

\[
\begin{cases}
L_g u = 0 & \text{in } B^+_q \\
\frac{\partial u}{\partial \nu} + \frac{n-2}{2}h_g u + \varepsilon \alpha u = (n-2)u\frac{2}{\pi^{\frac{n-2}{2}}} & \text{on } \partial B^+_r
\end{cases}
\]

for $B^+_q = \psi^{-1}_q(B^+_g(q, r))$ for $q \in \partial M$, with $\tau = \frac{n}{n-2} - p > 0$. Let us define

\[
P(u, r) := \int_{\partial^+ B^+_r} \left( \frac{n-2}{2} \frac{\partial u}{\partial \bar{\nu}} - \frac{r}{2} \|\nabla u\|^2 + r \left| \frac{\partial u}{\partial \bar{\nu}} \right|^2 \right) d\sigma_r + \frac{r(n-2)^2}{2(n-1)} \int_{\partial \bar{\partial} B^+_r} u^{\frac{2(n-1)}{n-2}} d\sigma_g.
\]

and

\[
\tilde{P}(u, r) := -\int_{B^+_r} \left( y^a \partial_a u + \frac{n-2}{2} u \right) [(L_g - \Delta)u] dy + \frac{n-2}{2} \int_{\partial^+ B^+_r} \left( \tilde{y}^k \partial_k u + \frac{n-2}{2} u \right) h_g \, d\tilde{y}
\]

\[
+ \frac{n-2}{2} \int_{\partial^+ B^+_r} \left( \tilde{y}^k \partial_k u + \frac{n-2}{2} u \right) \alpha u \, d\tilde{y}.
\]

Then $P(u, r) = \tilde{P}(u, r)$.

Here $a = 1, \ldots, n$, $k = 1, \ldots, n-1$ and $y = (\bar{y}, y_n)$, where $\bar{y} \in \mathbb{R}^{n-1}$ and $y_n \geq 0$. 

3. Expansion of the metric. Since the boundary \( \partial M \) of \( M \) is umbilic, given \( q \in \partial M \) there exists a conformally related metric \( g_q = \Lambda^\frac{\nu}{n-2} g \) such that some geometric quantities at \( q \) have a simpler form which will be summarized in this paragraph. We have

\[
\Lambda_q(q) = 1, \quad \frac{\partial \Lambda_q}{\partial y_k}(q) = 0 \text{ for all } k = 1, \ldots, n - 1.
\]

Set \( \tilde{u}_q = \Lambda_q^{-1} u \) and problem (1.2) is equivalent to

\[
\begin{cases}
L_{g_q} \tilde{u}_q = 0 & \text{in } M \\
B_{g_q} \tilde{u}_q + (n - 2)\tilde{u}_q \frac{\nu}{n-2} - \varepsilon \left( \Lambda_q \frac{\nu}{n-2} \alpha \right) \tilde{u}_q = 0 & \text{on } \partial M.
\end{cases}
\]

(3.1)

In the following, in order to simplify notations, we will omit the \( \tilde{\text{tilda}} \) symbol and we will omit \( \psi \), whenever is not needed.

**Remark 4.** In Fermi conformal coordinates around \( q \in \partial M \), it holds (see [18])

\[
|\det g_q(y)| = 1 + O(|y|^n) 
\]

(3.2)

\[
|h_{ij}(y)| = O(|y|^4) \quad |h_{ij}(y)| = O(|y|^4) 
\]

(3.3)

\[
g_{ij}^q(y) = \delta^{ij} + \frac{1}{3} \bar{R}_{ikjl} y_{ij} + R_{ninj} y_{n}^2 
\]

(3.4)

\[
\bar{R}_{ij}(y) = O(|y|^2) \text{ and } \partial^2_{ij} \bar{R}_{ij} = -\frac{1}{6} |\bar{W}|^2
\]

(3.5)

\[
\partial^2_{ij} \bar{R}_{ij} = -2R_{ninj} - 2R_{ninj,ij}
\]

(3.6)

\[
\bar{R}_{ik} = R_{nn} = R_{nk} = R_{nn,kk} = 0
\]

(3.7)

\[
R_{nn,nn} = -2R_{nins}.
\]

(3.8)

All the quantities above are calculate in \( q \in \partial M \), unless otherwise specified.

4. Isolated and isolated simple blow up points. Here we recall the definitions of some type of blow up points, and we give the basic properties about the behavior of these blow up points (see [1, 10, 15, 19]). We will omit the proofs of some well known results.

Let \( \{u_i\}_i \) be a sequence of positive solution to

\[
\begin{cases}
L_{g_i} u = 0 & \text{in } M \\
B_{g_i} u + (n - 2)\bar{u}_i^{\frac{\nu}{n-2}} - \varepsilon_i \alpha_i u = 0 & \text{on } \partial M
\end{cases}
\]

(4.1)

where \( \alpha_i = \Lambda_{x_i}^{-\frac{\nu}{n-2}} \alpha \to \Lambda_{x_0}^{-\frac{\nu}{n-2}} \alpha, \ x_i \to x_0, \ g_i \to g_0 \) in the \( C_0^3 \) topology and \( 0 < \varepsilon_i < \bar{\varepsilon} \).
Definition 5.  1) We say that \( x_0 \in \partial M \) is a blow up point for the sequence \( u_i \) of solutions of (4.1) if there is a sequence \( x_i \in \partial M \) of local maxima of \( u_i|_{\partial M} \) such that \( x_i \to x_0 \) and \( u_i(x_i) \to +\infty \).

   Shortly we say that \( x_i \to x_0 \) is a blow up point for \( \{u_i\}_i \).

   2) We say that \( x_i \to x_0 \) is an isolated blow up point for \( \{u_i\}_i \) if \( x_i \to x_0 \) is a blow up point for \( \{u_i\}_i \) and there exist two constants \( \rho, C > 0 \) such that

   \[
   u_i(x) \leq Cd\bar{g}(x,x_i)^{\frac{2-n}{2}} \quad \text{for all } x \in \partial M \setminus \{x_i\}, \quad d\bar{g}(x,x_i) < \rho.
   \]

   Given \( x_i \to x_0 \) an isolated blow up point for \( \{u_i\}_i \), and given \( \psi_i : B^+_\rho(0) \to M \) the Fermi coordinates centered at \( x_i \), we define the spherical average of \( u_i \) as

   \[
   \bar{u}_i(r) = \frac{2}{\omega_{n-1}r^{n-1}} \int_{B^+_r} u_i \circ \psi_i d\sigma,
   \]

   and

   \[
   w_i(r) := r^{\frac{2-n}{2}} \bar{u}_i(r)
   \]

   for \( 0 < r < \rho \).

   3) We say that \( x_i \to x_0 \) is an isolated simple blow-up point for \( \{u_i\}_i \) solutions of (4.1) if \( x_i \to x_0 \) is an isolated blow up point for \( \{u_i\}_i \) and there exists \( \rho \) such that \( w_i \) has exactly one critical point in the interval \( (0, \rho) \).

   Given \( x_i \to x_0 \) a blow up point for \( \{u_i\}_i \), we set

   \[
   M_i := u_i(x_i) \quad \text{and} \quad \delta_i := M_i^{\frac{2}{n-2}}.
   \]

   Obviously \( M_i \to +\infty \) and \( \delta_i \to 0 \).

   We recall the following results

Proposition 6. Let \( x_i \to x_0 \) is an isolated blow up point for \( \{u_i\}_i \) and \( \rho \) as in Definition 5. We set

   \[
   v_i(y) = M_i^{-\frac{n-2}{2}}(u_i \circ \psi_i)(M_i^\frac{2}{n-2} y), \quad \text{for } y \in B^+_{\rho M_i^{\frac{n-2}{2}}}(0).
   \]

   Then, given \( R_i \to \infty \) and \( \beta_i \to 0 \), up to subsequences, we have

   1. \( |v_i - U|_{C^2(B^+_{R_i}(0))} < \beta_i \);  

   2. \( \lim_{i \to \infty} \frac{R_i}{\log M_i} = 0 \).

Proposition 7. Let \( x_i \to x_0 \) be an isolated simple blow-up point for \( \{u_i\}_i \) and \( \alpha < 0 \). Let \( \eta \) small. Then there exist \( C, \rho > 0 \) such that

   \[
   M_i^\lambda |\nabla^k u_i(\psi_i(y))| \leq C|y|^{2-k-n+\eta}
   \]

   for \( y \in B^+_{\rho}(0) \setminus \{0\} \) and \( k = 0, 1, 2 \). Here \( \lambda_i = \left(\frac{2}{n-2}\right)(n-2-\eta) - 1 \).

   Since \( \alpha < 0 \) the proof of Proposition 7 is analogous of Lemma 2.7 of [10].

Proposition 8. Let \( x_i \to x_0 \) be an isolated simple blow-up point for \( \{u_i\}_i \) and \( \alpha < 0 \). Then \( \varepsilon_i \to 0 \).

Proof. We compute the Pohozaev identity in a ball of radius \( r \) and we set \( \frac{r}{\delta_i} =: R_i \to \infty \). We estimate any term of \( P(u_i, r_i) \) and \( \tilde{P}(u_i, r_i) \).
We set
\[ I_1(u, r) := \int_{\partial^+ B_i^+} \left( \frac{n-2}{2} \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r \]
\[ I_2(u, r) := \frac{r(n-2)^2}{2(n-1)} \int_{\partial^+ B_i^+} u^{2(n-1)} d\sigma_g, \]
so \( P(u, r) = I_1(u, r) + I_2(u, r) \).

By Proposition 7 and by definition of \( v \), we have
\[ I_1(u, r) = M_i^{-2\lambda} I_1(M_1^{\lambda_1} u_i, r) \leq c M_i^{-2\lambda} \int_{\partial^+ B_i^+} |y|^{2(2-n+\eta)} d\sigma_r \leq c \delta_i^{\lambda_1(n-2)} \]
\[ I_2(u, r) \leq c M_i^{-\lambda} \left( \frac{2(n-1)}{n-1} \right)^2 \leq c \delta_i^{\lambda_1(n-2)} \]

Then
\[ P(u, r) \leq \delta_i^{\lambda_1(n-2)}. \tag{4.2} \]

In a similar way we decompose
\[ \hat{P}(u, r) : \]
\[ = - \int_{B_i^+} \left( y^a \partial_u + \frac{n-2}{2} u \right) [(L_g - \Delta)u] d\sigma + \frac{n-2}{2} \int_{\partial^+ B_i^+} \left( y^k \partial_k u + \frac{n-2}{2} u \right) h_{\gamma \bar{\gamma}} u d\bar{\gamma} \]
\[ + \frac{n-2}{2} \int_{\partial^+ B_i^+} \left( y^k \partial_k u + \frac{n-2}{2} u \right) \alpha u d\bar{\gamma} =: I_3(u, r) + I_4(u, r) + I_5(u, r). \]

By Proposition 7 and by definition of \( v_i \) we have
\[ |\nabla^k v_i(s)| \leq M_i^{\eta \frac{2}{k}} |1 + |s||^{2-k-n} = \delta_i^{-\eta} |1 + |s||^{2-k-n}. \]

So, after a change of variables, since \( |h_{\gamma \bar{\gamma}}(\delta_i s)| \leq O(\delta_i^4 |s|^4) \),
\[ |I_4(u, r)| = \frac{n-2}{2} \delta_i \int_{\partial^+ B_i^+} \left( s^k \partial_k v_i + \frac{n-2}{2} v_i \right) h_{\gamma \bar{\gamma}}(\delta_i s) v_i d\bar{s} \leq c \delta_i^{5-2\eta}. \tag{4.3} \]

Analogously
\[ I_5(u, r) = \varepsilon_i \delta_i \int_{\partial^+ B_i^+} \left( s^k \partial_k v_i + \frac{n-2}{2} v_i \right) \alpha_i(\delta_i s) v_i d\bar{s}. \]

Since \( \alpha_i(\delta_i s) = A_i^{\frac{2}{n-1}}(\delta_i s) \alpha_i(\delta_i s) \) and by Claim 1 of Proposition 6 and (5.32) we get
\[ \lim_{i \to \infty} \int_{\partial^+ B_i^+} \left( s^k \partial_k v_i + \frac{n-2}{2} v_i \right) \alpha_i(\delta_i s) v_i d\bar{s} \]
\[ = \alpha(x_0) \int_{\partial^+ B_i^+} \left( s^k \partial_k U + \frac{n-2}{2} U \right) U d\bar{s} \tag{4.4} \]
\[= \frac{n-2}{2} \alpha(x_0) \int_{\mathbb{R}^{n-1}} \frac{1-|\bar{s}|^2}{[1+|\bar{s}|^2]^{n-1}} d\bar{s} =: A > 0.\]

Furthermore we have
\[I_3(u_i, r) = -\int_{B_{\rho}^+} \left( s^a \partial_a v_i + \frac{n-2}{2} v \right) [(L_{\bar{g}} - \Delta) v_i] dy\]
and it holds
\[(L_{\bar{g}} - \Delta) v = (g^{kl}(\delta_i s) - \delta^k l) \partial_{kl} v + \delta_i \partial_k g^{kl}(\delta_i s) \partial_l v - \delta_i^2 \frac{n-2}{4(n-1)} R_g(\delta_i s) v + O(\delta_i^N |s|^{N-1}) \partial_l v\]
we have
\[|I_3(u_i, r)| \leq c\delta_i^{2-2\eta} \tag{4.5}\]

Concluding, by (4.2), (4.5), (4.3), (4.4) we get
\[-c\delta_i^{2-2\eta} + (A + o(1)) \varepsilon_i \delta_i \leq \delta_i^{\lambda_i(n-2)}\]
which is possible only if \(\varepsilon_i \to 0\).

Since \(\varepsilon_i \to 0\) by Prop. 8, the proof of the next proposition is analogous to Prop. 4.3 in [1].

**Proposition 9.** Let \(x_i \to x_0\) be an isolated simple blow-up point for \(\{u_i\}\) and \(\alpha < 0\). Then there exist \(C, \rho > 0\) such that

1. \(M_i u_i(\psi_i(y)) \leq C|y|^{2-n}\) for all \(y \in B_{\rho}^+(0) \setminus \{0\}\);
2. \(M_i u_i(\psi_i(y)) \geq C^{-1} G_i(y)\) for all \(y \in B_{\rho}^+(0) \setminus B_{r_i}(0)\) where \(r_i := R_i M_i^{\frac{2-n}{2n}}\) and \(G_i\) is the Green’s function which solves

\[
\begin{cases}
L_g G_i = 0 & \text{in } B_{\rho}^+(0) \setminus \{0\} \\
G_i = 0 & \text{on } \partial^+ B_{\rho}^+(0) \\
B_{\rho} G_i = 0 & \text{on } \partial' B_{\rho}^+(0) \setminus \{0\}
\end{cases}
\]

and \(|y|^{n-2} G_i(y) \to 1\) as \(|z| \to 0\).

By Proposition 6 and Proposition 9 we have that, if \(x_i \to x_0\) is an isolated simple blow-up point for \(\{u_i\}\), then it holds
\[v_i \leq C U \text{ in } B_{\rho M_i^{\frac{2-n}{2n}}}^+(0).\]

**5. Blowup estimates.** Our aim is to provide a fine estimate for the approximation of the rescaled solution near an isolated simple blow up point.

In the following lemma, given a point \(q \in \partial M\), we introduce the function \(\gamma_q\) which arises from the secondo order term of the expansion of the metric \(g\) on \(M\) (see 3.4). The choice of this function plays a fundamental role in this paper. Using the function \(\gamma_q\) we are able to cancel the term of second order in formula (5.14).

Also, the estimates of Proposition 13 and of Lemma 15 depend on the properties of function \(\gamma_q\).

For the proof of the Lemma we refer to [12, Lemma 3] and [1, Proposition 5.1].
Lemma 10. Assume $n \geq 5$. Given a point $q \in \partial M$, there exists a unique $\gamma_q : \mathbb{R}^n_+ \to \mathbb{R}$ a solution of the linear problem

\[
\begin{cases}
-\Delta \gamma = \left[ \frac{1}{3} R_{ijkl}(q) y_k y_l + R_{nny}(q) y_n^2 \right] \partial_{ij}^2 U \quad & \text{on } \mathbb{R}^n_+ \\
\frac{\partial \gamma}{\partial y_n} = -nU \, \tau_n \gamma & \text{on } \partial \mathbb{R}^n_+
\end{cases}
\]

which is $L^2(\mathbb{R}^n_+)$-orthogonal to the functions $j_1, \ldots, j_n$ defined in (1.5) and (1.6). Moreover it holds

\[
|\nabla^\tau \gamma_q(y)| \leq C(1 + |y|)^{4-\tau-n} \text{ for } \tau = 0, 1, 2.
\]

\[
\int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dy \leq 0,
\]

\[
\int_{\partial \mathbb{R}^n_+} U \tau_n \gamma_q(t, z) dz = 0
\]

\[
\gamma_q(0) = \frac{\partial \gamma_q}{\partial y_1}(0) = \cdots = \frac{\partial \gamma_q}{\partial y_{n-1}}(0) = 0.
\]

Finally the map $q \mapsto \gamma_q$ is $C^2(\partial M)$.

In this section $x_i \to x_0$ is an isolated simple blowup point for a sequence $\{u_i\}_i$ of solutions of (4.1). We will work in the conformal Fermi coordinates in a neighborhood of $x_i$.

Set $\tilde{u}_i = \Lambda^{-1}_x u_i$ and

\[
\delta_i := \tilde{u}_i^{\frac{2}{2-n}}(x_i) = u_i^{\frac{2}{2-n}}(x_i) = M_i^{\frac{2}{2-n}} v_i(y) := \delta_i^{\frac{n-2}{2}} u_i(\delta_i y) \text{ for } y \in B^+_{\delta_i}(0).
\]

Then $v_i$ satisfies

\[
\begin{cases}
L_{\tilde{g}_i} v_i = 0 & \text{in } B^+_{\delta_i}(0) \\
B_{\tilde{g}_i} v_i + (n-2)v_i^{\frac{2}{2-n}} + \varepsilon \alpha_i(\delta, y) v_i = 0 & \text{on } \partial B^+_{\delta_i}(0)
\end{cases}
\]

(5.7)

where $\tilde{g}_i := \tilde{g}(\delta, y) = \Lambda_x^{\frac{4}{2-n}}(\delta, y) g(\delta, y)$, and $\alpha_i(y) = \Lambda_x^{\frac{n-2}{2-n}}(y)\alpha(y)$.

The estimates that follow are similar to the ones of [1, Lemma 6.1] and [11, Section 4], where the main difference is the term containing the linear perturbation $\alpha$. For the sake of self-containedness we sketch the main proofs.

Lemma 11. Assume $n \geq 8$. Let $\gamma_x$ be defined in (5.1). There exist $R, C > 0$ such that

\[
|v_i(y) - U(y) - \delta_i^2 \gamma_x(y)| \leq C \left( \delta_i^3 + \varepsilon_i \delta_i \right)
\]

for $|y| \leq R/\delta_i$.

Proof. Let $y_i$ such that

\[
\mu_i := \max_{|y| \leq R/\delta_i} |v_i(y) - U(y) - \delta_i^2 \gamma_x(y)| = |v_i(y_i) - U(y_i) - \delta_i^2 \gamma_x(y_i)|.
\]

We can assume, without loss of generality, that $|y_i| \leq \frac{R}{2\delta_i}$.

In fact, suppose that there exists $c > 0$ such that $|y_i| > \frac{c}{\delta_i}$ for all $i$. Then, since $v_i(y) \leq CU(y)$, and by (5.2), we get the inequality

\[
|v_i(y_i) - U(y_i) - \delta_i^2 \gamma_x(y_i)| \leq C \left( |y_i|^{2-n} + \delta_i^2 |y_i|^{1-n} \right) \leq C\delta_i^{n-2}
\]

which proves the Lemma. So, in the next we will suppose $|y_i| \leq \frac{R}{2\delta_i}$. This fact will be used later.
By contradiction, suppose that
\[ \max \left\{ \mu_i^{-1} \delta_i^3, \mu_i^{-1} \varepsilon_i \delta_i \right\} \to 0 \text{ when } i \to \infty. \tag{5.8} \]
Defined
\[ w_i(y) := \mu_i^{-1} (v_i(y) - U(y) - \delta_i^2 \gamma_i(y)) \text{ for } |y| \leq R/\delta_i, \]
we have, by direct computation, that \( w_i \) satisfies
\[
\begin{align*}
L_{\bar{g}_i} w_i &= Q_i \quad \text{in } B^+_R(0) \\
B_{\bar{g}_i} w_i + b_i w_i &= F_i \quad \text{on } \partial B^+_R(0)
\end{align*}
\tag{5.9}
\]
where
\[
b_i = (n - 2) \frac{n}{v_i - U - \delta_i^2 \gamma_i} \]
\[
Q_i = - \frac{1}{\mu_i} \left\{ (n - 2)(U + \delta_i^2 \gamma_i) - \frac{n}{2} h_i(U + \delta_i^2 \gamma_i) \right\}
\]
\[
F_i = Q_i + \frac{1}{\mu_i} \left\{ (L_{\bar{g}_i} - \Delta)(U + \delta_i^2 \gamma_i) + \delta_i^2 \Delta \gamma_i \right\}.
\]
We estimate for terms \( b_i, Q_i, F_i \) obtaining that the sequence \( w_i \) converges in \( C^2_{\text{loc}}(\mathbb{R}^n_+) \) to some \( w \) solution of
\[
\begin{align*}
\Delta w &= 0 \quad \text{in } \mathbb{R}^n_+ \\
\frac{\partial}{\partial 
u} w + nU \frac{\partial}{\partial x} w &= 0 \quad \text{on } \partial \mathbb{R}^n_+ \tag{5.10}
\end{align*}
\]
then we will derive a contradiction using (5.8).

Since \( v_i \to U \) in \( C^2_{\text{loc}}(\mathbb{R}^n_+) \) we have, at once,
\[
b_i \to nU \frac{\partial}{\partial x} \text{ in } C^2_{\text{loc}}(\mathbb{R}^n_+) \tag{5.11}
\]
\[
|b_i(y)| \leq (1 + |y|)^{-2} \text{ for } |y| \leq R/\delta_i. \tag{5.12}
\]

We proceed now by estimating \( Q_i \) and \( \bar{Q}_i \). We recall that
\[
[L_{\bar{g}_i} - \Delta]u(y) = \left( g_i^{kl} \delta_i y_k y_l \right) \partial_k \partial_l u + \delta_i \partial_k g_i^{kl} \delta_i y_k \partial_l u - \delta_i^2 \frac{n}{2} R_{\bar{g}_i}(\delta_i y) u
\]
\[ + O(\delta_i^N |y|^{N-1}) \partial_l u \tag{5.13}
\]
where \( N \) can be chosen arbitrarily large. At this point using the definition of the function \( \gamma_i \), (5.13), (3.4) and the decays properties of \( U \) and \( \gamma_i \), we obtain
\[
-\mu_i Q_i = \delta_i^2 \left( \frac{1}{3} R_{\bar{g}_i y_k y_j y_l} + R_{\text{nlk} y_i y_l} \right) \left( \partial_k \partial_l U + \delta_i^2 \partial_k \partial_l \gamma_i \right)
\]
\[ + O(\delta_i^N |y|^3) \left( \partial_k \partial_l U + \delta_i^2 \partial_k \partial_l \gamma_i \right) + \delta_i^2 \left( \frac{1}{3} R_{\bar{g}_i y_k y_j} + \frac{1}{3} R_{\text{nlk} y_i} \right) \left( \partial_l U + \delta_i^2 \partial_l \gamma_i \right)
\]
\[ + O(\delta_i^N |y|^2) \left( \partial_l U + \delta_i^2 \partial_l \gamma_i \right) + O(\delta_i^N |y|^2) \left( U + \delta_i^2 \gamma_i \right) + \delta_i^2 \Delta \gamma_i + O(\delta_i^N |y|^{N-1}) \left( \partial_l U + \delta_i^2 \partial_l \gamma_i \right) \]
\[= O\left(\delta_i^3 (1 + |y|)^{3-n}\right) + O\left(\delta_i^4 (1 + |y|)^{4-n}\right) + O\left(\delta_i^5 (1 + |y|)^{5-n}\right) + O\left(\delta_i^6 (1 + |y|)^{6-n}\right) + O\left(\delta_i^N (1 + |y|)^{N-n}\right) O\left(\delta_i^{N+2} (1 + |y|)^{N+2-n}\right). \] (5.14)

Since \(|y| \leq R/\delta_i\), we have \(\delta_i (1 + |y|) \leq C\), thus
\[Q_i = O\left(\mu_i^{-1} \delta_i^3 (1 + |y|)^{3-n}\right). \] (5.15)

In light of (5.8) we have also \(Q_i \in L^p(B_{R/\delta_i}^+)\) for all \(p \geq 2\).

By Taylor expansion, and proceeding as above we have
\[\int_{B_{r/\delta_i}^+} \hat{\xi} \, d\bar{\sigma}_{\bar{g}_i}(\xi) \leq C \left(\mu_i^{-1} \delta_i^4 (1 + |y|)^{5-n}\right) + O\left(\mu_i^{-1} \delta_i^2 (1 + |y|)^{2-n}\right), \] (5.16)

and \(F_i \in L^p(\partial \partial^+ B_{R/\delta_i}^+)\) for all \(p \geq 2\).

Finally we remark that \(|w_i(y)| \leq 1\), so by (5.8) (5.11), (5.12), (5.15), (5.16) and by standard elliptic estimates we conclude that, up to subsequence, \(\{w_i\}_i\) converges in \(C^2_{\text{loc}}(\mathbb{R}^n_+)^N\) to some \(w\) solution of (5.10).

The next step is to prove that \(|w(y)| \leq C |\xi - y|^{-1}\) for \(y \in \mathbb{R}^n_+\). Consider \(G_i\) the Green function for the conformal Laplacian \(L_{\bar{g}_i}\) defined on \(B_{r/\delta_i}^+\) with boundary conditions \(B_{\bar{g}_i}G_i = 0\) on \(\partial \partial^+ B_{r/\delta_i}^+\) and \(G_i = 0\) on \(\partial \partial^+ B_{r/\delta_i}^+\). It is well known that \(G_i = O(|\xi - y|^{-1})\). By the Green formula and by (5.15) and (5.16) we have
\[w_i(y) = \int_{B_{r/\delta_i}^+} G_i(\xi, y)Q_i(\xi)\, d\mu_{\bar{g}_i}(\xi) - \int_{\partial \partial^+ B_{r/\delta_i}^+} \frac{\partial G_i}{\partial n}(\xi, y)w_i(\xi)\, d\sigma_{\bar{g}_i}(\xi)
+ \int_{\partial \partial^+ B_{r/\delta_i}^+} G_i(\xi, y) (b_i(\xi)w_i(\xi) - F_i(\xi))\, d\sigma_{\bar{g}_i}(\xi),\]

so
\[|w_i(y)| \leq \frac{\delta_i^3}{\mu_i} \int_{B_{r/\delta_i}^+} |\xi - y|^{2-n}(1 + |\xi|)^{3-n}\, d\xi + \int_{\partial \partial^+ B_{r/\delta_i}^+} |\xi - y|^{1-n}w_i(\xi)\, d\sigma(\xi)
+ \int_{\partial \partial^+ B_{r/\delta_i}^+} |\xi - y|^{2-n}\left((1 + |\xi|)^{-2} + \frac{\delta_i^4}{\mu_i} (1 + |\xi|)^{5-n} + \frac{\delta_i^4}{\mu_i} (1 + |\xi|)^{2-n}\right)\, d\xi,\]

Notice that in the second integral we used that \(|y| \leq \frac{R}{2\delta_i}\) to estimate \(|\xi - y| \geq |\xi| - |y| \geq \frac{R}{2}\) on \(\partial \partial^+ B_{R/\delta_i}^+\). Moreover, since \(v_i(\xi) \leq CU(\xi)\), we get
\[|w_i(\xi)| \leq \frac{C}{\mu_i} \left((1 + |\xi|)^{2-n} + \delta_i^{2n-3} (1 + |\xi|)^{4-n}\right) \leq \frac{C\delta_i^{2n-3}}{\mu_i} \text{ on } \partial \partial^+ B_{R/\delta_i}^+; \] (5.17)

hence
\[\int_{\partial \partial^+ B_{r/\delta_i}^+} |\xi - y|^{1-n}w_i(\xi)\, d\sigma(\xi) \leq C \int_{\partial \partial^+ B_{r/\delta_i}^+} \frac{\delta_i^{2n-3}}{\mu_i} \, d\sigma(\xi) \leq C \frac{\delta_i^{2n-3}}{\mu_i}. \] (5.18)
For the other terms we use the following formula (see [1, Lemma 9.2] and [5, 14])

$$\int_{\mathbb{R}^m} |\xi - y|^{3-n} (1 + |\xi|)^{3-n} d\xi \leq C (1 + |y|)^{3-n}$$  \hspace{1cm} (5.19)

where \( y \in \mathbb{R}^{m+k} \supseteq \mathbb{R}^m, \eta, \beta \in \mathbb{N}, 0 < \beta < \eta < m \). We get

$$\frac{\delta^3}{\mu_i} \int_{\mathcal{B}_B^+} \frac{|\xi - y|^{3-n} (1 + |\xi|)^{3-n} d\xi}{\xi^i} \leq C \frac{\delta^3}{\mu_i} (1 + |y|)^{3-n},$$  \hspace{1cm} (5.20)

$$\frac{\delta^4}{\mu_i} \int_{\mathcal{B}_B^+} \frac{|\xi - y|^{2-n} (1 + |\xi|)^{-2} d\xi}{\xi^i} \leq (1 + |y|)^{-1}$$  \hspace{1cm} (5.21)

$$\frac{\delta^4}{\mu_i} \int_{\mathcal{B}_B^+} \frac{|\xi - y|^{2-n} (1 + |\xi|)^{5-n} d\xi}{\xi^i} \leq C \frac{\delta^4}{\mu_i} (1 + |y|)^{6-n}$$  \hspace{1cm} (5.22)

By (5.18), (5.20), (5.21), (5.22) (5.23) we have

$$|w_i(y)| \leq C \left( (1 + |y|)^{-1} + \frac{\delta^3}{\mu_i} (1 + |y|)^{5-n} + \frac{\varepsilon \delta_i}{\mu_i} (1 + |y|)^{3-n} \right)$$  \hspace{1cm} for \(|y| \leq \frac{R}{2 \delta_i}\)  \hspace{1cm} (5.24)

so by assumption (5.8) we prove

$$|w(y)| \leq C (1 + |y|)^{-1} \text{ for } y \in \mathbb{R}_+^n$$  \hspace{1cm} (5.25)

as claimed.

Finally we notice that, since \( v_i \to U \) near 0, and by (5.5) we have \( w_i(0) \to 0 \) as well as \( \frac{\partial w_i}{\partial y_i}(0) \to 0 \) for \( j = 1, \ldots, n - 1 \). This implies that

$$w(0) = \frac{\partial w}{\partial y_1}(0) = \cdots = \frac{\partial w}{\partial y_{n-1}}(0) = 0.$$  \hspace{1cm} (5.26)

We are ready now to prove the contradiction. In fact, it is known (see [1, Lemma 2]) that any solution of (5.10) that decays as (5.25) is a linear combination of \( \frac{\partial U}{\partial y_1}, \ldots, \frac{\partial U}{\partial y_{n-1}}, \frac{n-2}{2} U + y^b \frac{\partial U}{\partial y_n} \). This fact, combined with (5.26), implies that \( w \equiv 0 \).

Now, on one hand \(|y_i| \leq \frac{R}{2 \delta_i}\), so estimate (5.24) holds; on the other hand, since \( w_i(y_i) = 1 \) and \( w \equiv 0 \), we get \(|y_i| \to \infty\), obtaining

$$1 = w_i(y_i) \leq C (1 + |y_i|)^{-1} \to 0$$

which gives us the contradiction. \( \square \)

**Lemma 12.** Assume \( n \geq 8 \) and \( \alpha < 0 \). There exists \( C > 0 \) such that

$$\varepsilon \delta_i \leq C \delta_i^3.$$  \hspace{1cm} (5.27)

**Proof.** We proceed by contradiction, supposing that

$$(\varepsilon \delta_i)^{-1} \delta_i^3 \to 0 \text{ when } i \to \infty.$$  \hspace{1cm} (5.27)

Thus, by Lemma 11, we have

$$|v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)| \leq C \varepsilon_i \delta_i \text{ for } |y| \leq R/\delta_i.$$
We define, similarly to Lemma 11,
\[ w_i(y) := \frac{1}{\varepsilon_i \delta_i} (v_i(y) - U(y) - \delta^2_i \gamma_i(y)) \] for \(|y| \leq R/\delta_i),
and we have that \( w_i \) satisfies (5.9) where
\[ b_i = (n-2) \frac{\varepsilon_i \delta_i}{v_i - U - \delta^2_i \gamma_i}, \]
\[ Q_i = -\frac{1}{\varepsilon_i \delta_i} \left\{ (n-2)(U + \delta^2_i \gamma_i) \frac{\varepsilon_i}{v_i} - (n-2)U \frac{\varepsilon_i}{v_i} - n \delta^2_i U \frac{\varepsilon_i}{v_i} \gamma_i - \frac{n-2}{2} h_{\delta_i}(U + \delta^2_i \gamma_i) \right\} \]
\[ F_i = Q_i + \alpha_i(\delta_i y) v_i(y) \]
As before, \( b_i \) satisfies inequality (5.12) while
\[ Q_i = O \left( (\varepsilon_i \delta_i)^{-1} \delta^3 (1 + |y|)^{3-n} \right) \] (5.28)
\[ Q_i = O \left( (\varepsilon_i \delta_i)^{-1} \delta^4 (1 + |y|)^{5-n} \right) \] (5.29)
\[ F_i = O \left( (\varepsilon_i \delta_i)^{-1} \delta^4 (1 + |y|)^{5-n} \right) + O((1 + |y|)^{2-n}), \] (5.30)
so by classic elliptic estimates we can prove that the sequence \( w_i \) converges in \( C^2_\text{loc}(\mathbb{R}^n_+) \) to some \( w \).
Moreover, we can proceed as in Lemma 11 to deduce that
\[ |w_i(y)| \leq C \left( (1 + |y|)^{-1} + \frac{\delta^3}{\varepsilon_i \delta_i} (1 + |y|)^{5-n} + (1 + |y|)^{3-n} \right) \] (5.31)
\[ \leq C \left( (1 + |y|)^{-1} + \frac{\delta^3}{\varepsilon_i \delta_i} (1 + |y|)^{5-n} \right) \text{ for } |y| \leq \frac{R}{2\delta_i}. \]
Now let \( j_n \) defined as in (1.6). In light of (5.29) easily we get
\[ \lim_{i \to +\infty} \int_{\partial B^+_\varepsilon_i} j_n Q_i d\sigma_{\delta_i} = 0. \]
We recall that \( \alpha_i(\delta_i y) = \Lambda^{-2} (\delta_i y) \alpha(\delta_i y) \), so, by Proposition 6, we have
\[ \alpha_i(\delta_i y) v_i(y) \to \alpha(x_0) U(y) \text{ for } i \to +\infty. \]
So, since \( \alpha < 0 \), we get, by (1.3),
\[ \lim_{i \to +\infty} \int_{\partial B^+_\varepsilon_i} \alpha_i(\delta_i y) v_i(y) j_n(y) = \alpha(x_0) \int_{\mathbb{R}^{n-1}} \frac{1 - |y|^2}{(1 + |y|^2)^{n-1}} \]
\[ = \alpha(x_0) \omega_{n-2} \int_0^{+\infty} \frac{s^{n-2} - s^n}{(1 + s^2)^{n-1}} ds = -\frac{2\omega_{n-2}}{n-1} \alpha(x_0) \int_0^{+\infty} \frac{s^n}{(1 + s^2)^{n-1}} ds > 0, \] (5.32)
where \( \omega_{n-2} \) is the volume element of the \((n - 1)\) unit sphere and where we used (1.3) in the last passage. Thus we have
\[ \lim_{i \to +\infty} \int_{\partial B^+_\varepsilon_i} j_n F_i d\sigma_{\delta_i} > 0, \] (5.33)
and (5.33) leads us to a contradiction. Indeed, since \( w_i \) satisfies (5.9), integrating by parts we obtain

\[
\int_{\partial^+ B^+_{r_i}} j_n F_i d\bar{\sigma}_{i} = \int_{\partial^+ B^+_{r_i}} j_n [B_{\bar{\eta}_i} w_i + b_i w_i] d\bar{\sigma}_{i} = 0,
\]

where \( \eta_i \) is the inward unit normal vector to \( \partial^+ B^+_{r_i} \).

By the decay of \( j_n \) and the decay of \( w_i \), given by (5.31) and by (5.27), we have

\[
\lim_{i \to +\infty} \int_{\partial^+ B^+_{r_i}} \left[ \frac{\partial}{\partial \eta_i} w_i - \frac{\partial}{\partial \eta_i} j_n \right] d\bar{\sigma}_{i} = 0 \tag{5.34}
\]

and by (5.9) and the decay of \( Q_i \) given in (5.28) we have

\[
\lim_{i \to +\infty} \int_{B^+_{r_i}} j_n Q_i d\mu_{\bar{\eta}_i} = 0. \tag{5.35}
\]

Finally, since \( \Delta j_n = 0 \), by (5.13) we get

\[
\lim_{i \to +\infty} \int_{B^+_{r_i}} w_i L_{\bar{\eta}_i} j_n d\mu_{\bar{\eta}_i} = 0, \tag{5.36}
\]

thus by (5.34) (5.35) and (5.36) we have

\[
\lim_{i \to +\infty} \int_{\partial^+ B^+_{r_i}} j_n F_i d\bar{\sigma}_{i} = \lim_{i \to +\infty} \int_{\partial^+ B^+_{r_i}} j_n [B_{\bar{\eta}_i} j_n + b_i j_n] d\bar{\sigma}_{i} = 0 \tag{5.37}
\]

since \( \frac{\partial}{\partial y_n} + nU \frac{\partial}{\partial \bar{y}_n} j_n = 0 \) when \( y_n = 0 \). Comparing (5.33) and (5.37) we get the contradiction.

The above lemmas are the core of the following proposition, in which we iterate the procedure of Lemma 11, to obtain better estimates of the rescaled solution \( v_i \) of (5.7) around the isolated simple blow up point \( x_i \to x_0 \).

**Proposition 13.** Assume \( n \geq 8 \). Let \( \gamma_{x_i} \) be defined in (5.4). There exist \( R, C > 0 \) such that

\[
|\nabla_{\bar{y}} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y))| \leq C \delta_i^2 (1 + |y|)^{5 - \tau - n}
\]

\[
\left| \frac{\partial}{\partial y_n} (v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y)) \right| \leq C \delta_i^2 (1 + |y|)^{5 - \tau - n}
\]

for \( |y| \leq \frac{R}{2 \delta_i} \). Here \( \tau = 0, 1, 2 \) and \( \nabla_{\bar{y}} \) is the differential operator of order \( \tau \) with respect the first \( n - 1 \) variables.
Proof. In analogy with Lemma 11, we set

\[ w_i(y) := v_i(y) - U(y) - \delta_i^2 \gamma_x, (y) \]

and we have that \( w_i \) satisfies (5.9) where

\[
\begin{aligned}
b_i & = -(n-2) \frac{\nu_i^{\frac{2}{n-2}} - (U + \delta_i^2 \gamma_x) \nu_{\gamma_x}}{\nu_i^{\frac{2}{n-2}} - (n-2) U \nu_{\gamma_x} - n\delta_i^2 U \nu_{\gamma_x} - \frac{n-2}{2} h_{\gamma_i}(U + \delta_i^2 \gamma_x)} \\
Q_i & = - \left\{ (n-2)(U + \delta_i^2 \gamma_x) \nu_{\gamma_x} - (n-2)U \nu_{\gamma_x} - n\delta_i^2 U \nu_{\gamma_x} - \frac{n-2}{2} h_{\gamma_i}(U + \delta_i^2 \gamma_x) \right\} \\
F_i & = Q_i + \varepsilon_i \delta_i \alpha_i(\delta_i y) v_i(y) \\
Q_i & = - \left\{ (L_{\gamma_i} - \Delta) (U + \delta_i^2 \gamma_x) + \delta_i^2 \Delta \gamma_x \right\}.
\end{aligned}
\]

As before, \( b_i \) satisfies inequality (5.12) and

\[
|w_i| \leq C \delta_i^3 \text{ on } B^+_R/\delta_i \quad \text{and} \quad |w_i| \leq C \delta_i^{n-2} \text{ on } \partial^+ B^+_R/\delta_i.
\]

Plugging (5.12), (5.38), (5.39) and (5.41) in (5.40) and proceeding as in Lemma 11 we obtain

\[
\begin{aligned}
\int_{B^+_R/\delta_i} |\xi - y|^{2-n} Q_i(\xi) d\xi & \leq C \delta_i^3 (1 + |y|)^{5-n} \\
\int_{\partial^+ B^+_R/\delta_i} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi) & \leq C \delta_i^{n-2} \\
\int_{\partial^+ B^+_R/\delta_i} |\xi - y|^{2-n} b_i(\xi) w_i(\xi) d\xi & \leq \delta_i^3 (1 + |y|)^{-1} \\
\int_{\partial^+ B^+_R/\delta_i} |\xi - y|^{2-n} Q_i(\xi) d\xi & \leq C \delta_i^3 (1 + |y|)^{5-n} \\
\int_{\partial^+ B^+_R/\delta_i} |\xi - y|^{2-n} \varepsilon_i \delta_i \alpha_i(\delta_i \xi) v_i(\xi) d\xi & \leq C \delta_i^3 (1 + |y|)^{5-n}
\end{aligned}
\]

so

\[
|w_i| \leq C \delta_i^3 (1 + |y|)^{-1} \text{ for } |y| \leq \frac{R}{2\delta_i}.
\]

As before, we iterate the procedure until we reach

\[
|w_i| \leq C \delta_i^3 (1 + |y|)^{5-n} \text{ for } |y| \leq \frac{R}{2\delta_i}.
\]
which proves the first claim for $\tau = 0$. The other claims follow as in the previous proofs.

6. Sign estimates of Pohozaev identity terms. In this section, we want to estimate $P(u_i, r)$, where $\{u_i\}$ is a family of solutions of (4.1) which has an isolated simple blow up point $x_i \to x_0$. This estimate, given in the following Proposition 14, is a crucial point for the proof of the vanishing of the Weyl tensor at an isolated simple blow up point.

Since the leading term of $P(u_i, r)$ will be $-\int_{B_{x_i}^+} (y^k \partial_k u + n-2) [(L_{g_i} - \Delta)u] dy$ we set

$$R(u, v) = -\int_{B_{x_i}^+} (y^k \partial_k u + n-2) [(L_{g_i} - \Delta)v] dy.$$  

Proposition 14. Let $x_i \to x_0$ be an isolated simple blow up point for $u_i$ solutions of (4.1). Then, fixed $r$, we have, for $i$ large

$$\hat{P}(u_i, r) \geq \delta^4 \int_{B_{x_i}^+} (n-2) \omega_{n-2} I_{n-2} \left( \frac{(n-2)}{(n-1)(n-3)(n-5)(n-6)} \left[ \frac{(n-2)}{6} (\hat{W}(x_i))^2 + \frac{4(n-8)}{(n-4)} R_{\min}^2(x_i) \right] \right)$$

$$- 2\delta^4 \int_{\mathbb{R}^n} \gamma_i \Delta \gamma_i dy + o(\delta^4).$$

Proof. We recall that $\hat{P}(u_i, r) := -\int_{B_{x_i}^+} (y^k \partial_k u_i + n-2) [(L_{g_i} - \Delta)u_i] dy$

$$+ \frac{n-2}{2} \int_{\partial B_{x_i}^+} (y^k \partial_k u_i + n-2) h_{g_i} u_i d\bar{y}$$

$$+ \frac{n-2}{2} \int_{\partial B_{x_i}^+} (y^k \partial_k u_i + n-2) \bar{\varepsilon} \alpha_i u_i d\bar{y},$$

where $B_{x_i}^+$ is the counter-image of $B_{x_i}^+(x_i, r)$ by $\psi_{x_i}$. Now, set

$$v_i(y) := \delta^\frac{n-2}{2} u_i(\delta y) \quad \text{for} \quad y \in B_{\frac{r}{\delta}}^+(0)$$

After a change of variables we have

$$\int_{\partial B_{x_i}^+} (y^k \partial_k u_i + n-2) \bar{\varepsilon} \alpha_i u_i d\bar{y} = \bar{\varepsilon}_i \int_{\partial B_{x_i}^+} (y^k \partial_k u_i + n-2) v_i d\bar{y}.$$
and, recalling that \( \alpha_i(\delta_i y) \to \alpha(x_0) < 0 \) and proceeding as in (5.32) we get

\[
\lim_{i \to \infty} \int_{\partial' B_{r_i/4}} \left( y^k \partial_k U + \frac{n-2}{2} U \right) \alpha_i(\delta_i y) U d\bar{y} = \frac{n-2}{2} \alpha(x_0) \int_{\mathbb{R}^{n-1}} \frac{1-|\bar{y}|^2}{(1+|\bar{y}|^2)^{n-1}} d\bar{y} > 0.
\]

Thus, for \( i \) sufficiently large we obtain

\[
\hat{P}(u_i, r) \geq -\int_{B_{r_i/4}} \left( y^b \partial_b y + \frac{n-2}{2} v_i \right) [(L_{\tilde{g}_i} - \Delta)v_i] dy + \frac{n-2}{2} \int_{\partial' B_{r_i/4}} \left( y^b \partial_b v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\delta_i y) v_i d\bar{y}.
\]

Since \( h_{g_i}(\delta_i y) = O(\delta_i^5|y|^4) \) we have

\[
\int_{\partial' B_{r_i/4}} \left( y^b \partial_b v_i + \frac{n-2}{2} v_i \right) h_{g_i}(\delta_i y) v_i d\bar{y} = O(\delta_i^5) \int_{\partial' B_{r_i/4}} (1 + |y|)^{4-2n} |y|^4 dy = O(\delta_i^5) \text{ for } n \geq 8.
\]

So

\[
\hat{P}(u_i, r) \geq -\int_{B_{r_i/4}} \left( y^b \partial_b y + \frac{n-2}{2} v_i \right) [(L_{\tilde{g}_i} - \Delta)v_i] dy + O(\delta_i^5)
\]

for \( i \) sufficiently large. Now define, in analogy with Proposition 13,

\[
w_i(y) := v_i(y) - U(y) - \delta_i^2 \gamma_{x_i}(y).
\]

Recalling (6.1), we have

\[
\hat{P}(u_i, r) \geq R(U, U) + R(U, \delta_i^2 \gamma_{x_i}) + R(\delta_i^2 \gamma_{x_i}, U) + R(w_i, U) + R(U, w_i) + R(w_i, w_i) + R(\delta_i^2 \gamma_{q_i}, \delta_i^2 \gamma_{x_i}) + R(\delta_i^2 \gamma_{x_i}, w_i) + R(\delta_i^2, w_i) + O(\delta_i^5)
\]

and, by the following Lemma 15 we conclude

\[
\hat{P}(u_i, r) \geq R(U, U) + R(U, \delta_i^2 \gamma_{x_i}) + R(\delta_i^2 \gamma_{x_i}, U) + O(\delta_i^5)
\]

\[
= \delta_i^4 \left\{ \frac{(n-2) \omega_{n-2} I_n^2}{(n-1)(n-3)(n-5)(n-6)} \left[ (n-2) \frac{1}{6} |\tilde{W}(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{nlnj}(x_i) \right] 
- 2 \delta_i^4 \int_{\mathbb{R}^n_+} \gamma_{x_i} \Delta \gamma_{x_i} dy + o(\delta_i^4) \right\}
\]

and we prove the result. \( \square \)

**Lemma 15.** For \( n \geq 8 \) we have

\[
R(U, U) = \delta_i^4 \left\{ \frac{(n-2) \omega_{n-2} I_n^2}{(n-1)(n-3)(n-5)(n-6)} \left[ (n-2) \frac{1}{6} |\tilde{W}(q)|^2 + \frac{4(n-8)}{(n-4)} R_{nlnj}^2 \right] + o(\delta_i^4) \right\}
\]

\[
R(U, \delta_i^2 \gamma_q) = R(\delta_i^2 \gamma_q, U) = -2 \delta_i^4 \int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dy + o(\delta_i^4)
\]

\[
R(\delta_i^2 \gamma_q, \delta_i^2 \gamma_q) = O(\delta_i^4)
\]

\[
R(w_i, w_i) = O(\delta_i^6)
\]

\[
R(U, w_i) = R(w_i, U) = O(\delta_i^5)
\]

\[
R(\delta_i^2 \gamma_q, w_i) + R(w_i, \delta_i^2 \gamma_q) = O(\delta_i^5)
\]

**Proof.** For the proof we refer to [11]. \( \square \)
Proposition 16. Let $x_i \to x_0$ be an isolated simple blow-up point for $u_i$ solutions of (4.1). Then

1. If $n = 8$ then $|\tilde{W}(x_0)| = 0$.
2. If $n > 8$ then $|W(x_0)| = 0$.

Proof. By Proposition 9 and Proposition 7, and since $M_i = \delta_i^{2-n}$ we have,

\[
P(u_i, r) := \frac{1}{M_i^{2\lambda_i}} \int_{\partial^+ B_i^r} \left( \frac{n-2}{2} M_i^{\lambda_i} u_i \frac{\partial M_i^{\lambda_i} u_i}{\partial r} - \frac{r}{2} |\nabla M_i^{\lambda_i} u_i|^2 + r \left| \frac{\partial M_i^{\lambda_i} u_i}{\partial r} \right|^2 \right) \, d\sigma_r
\]

\[
+ \frac{r(n-2)^2}{(n-1) M_i^{2(\lambda_i-1)}} \int_{\partial(\partial^+ B_i^r)} \left( M_i^{\lambda_i} u_i \right)^{2(n-1)} \, d\bar{\sigma}_g.
\]

\[
\leq \frac{C}{M_i^{\lambda_i-1}} \leq C\delta_i^{(n-1)\lambda_i} \leq C\delta_i^{n-2}.
\]

On the other hand recalling Proposition 14 and Theorem 3 we have

\[
P(u_i, r) = \tilde{P}(u_i, r)
\]

\[
\geq \delta_i^4 \frac{(n-2)\omega_{n-2} P_i^n}{(n-1)(n-3)(n-5)(n-6)} \left[ \frac{n-2}{6} |\tilde{W}(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{\text{nlnj}}^2(x_i) \right] + o(\delta_i^4),
\]

because $\int \gamma_{x_i} \Delta \gamma_{x_i} \leq 0$ (see (5.3) of Lemma 10) so we get $|\tilde{W}(x_i)| \leq \delta_i^2$ if $n = 8$, and $\left[ \frac{(n-2)}{6} |\tilde{W}(x_i)|^2 + \frac{4(n-8)}{(n-4)} R_{\text{nlnj}}^2(x_i) \right] \leq \delta_i^2$ if $n > 8$. For the case $n > 8$ we recall that when the boundary is umbilic $\tilde{W}(q) = 0$ if and only if $\tilde{W}(q) = 0$ and $R_{\text{nlnj}}(q) = 0$ (see [18, page 1618]), and we conclude the proof. \(\square\)

Remark 17. Let $x_i \to x_0$ be an isolated blow up point for $u_i$ solutions of (4.1). We set

\[
P'(u, r) := \int_{\partial^+ B_i^r} \left( \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) \, d\sigma_r,
\]

so

\[
P(u_i, r) = P'(u_i, r) + \frac{r(n-2)^2}{(n-1)} \int_{\partial(\partial^+ B_i^r)} u_i \frac{2(n-1)}{r} \, d\sigma_g
\]

and, keeping in mind that for $i$ large $M_i u_i \leq C|y|^{2-n}$ by Proposition 9, we have

\[
|r| \int_{\partial(\partial^+ B_i^r)} u_i \frac{2(n-1)}{M_i^{\frac{2(n-1)}{2}}} \, d\bar{\sigma}_g \leq \frac{Cr}{M_i^{\frac{2(n-1)}{2}}} \int_{\partial(\partial^+ B_i^r)} y_n = 0 \frac{1}{|y|^{2(n-1)}} \, d\bar{\sigma}_g \leq \frac{C(r)}{M_i^{\frac{2(n-1)}{2}}} = C(r)\delta_i^{n-2}
\]

(6.3)

for $i$ sufficiently large.

Using Proposition 14, (6.3), and since $n \geq 8$ we get

\[
P'(u_i, r) = P(u_i, r) - \frac{r(n-2)^2}{(n-1)} \int_{\partial(\partial^+ B_i^r)} u_i \frac{2(n-1)}{M_i^{\frac{2(n-1)}{2}}} \, d\sigma_g \geq A\delta_i^4 + o(\delta_i^4)
\]

(6.4)

where $A > 0$. 

Proposition 18. Let $x_i \to x_0$ be an isolated blow up point for $u_i$ solutions of (4.1). Assume $n = 8$ and $|W(x_0)| \neq 0$ or $n > 8$ and $|W(x_0)| \neq 0$. Then $x_0$ is isolated simple.

For the proof of this Lemma we refer to [1, 11]

7. A splitting lemma. The first result in this section are analogous to [17, Proposition 5.1], [22, Lemma 3.1], [15, Proposition 1.1] and [1, Proposition 4.2], so the proof will be omitted.

Proposition 19. Given $\beta > 0$ and $R > 0$ there exist two constants $C_0, C_1 > 0$ (depending on $\beta$, $R$ and $(M, g)$) such that if $u$ is a solution of

$$
\begin{align}
\begin{cases}
L_g u &= 0 \\
p_n u + \frac{n-2}{2} h_g u + \epsilon \alpha u &= (n-2) u^{\frac{n}{n-2}} & \text{in } M \\
\partial u &= 0 & \text{on } \partial M
\end{cases}
\end{align}
$$

and $\max_{\partial M} u > C_0$, then $\tau := \frac{n}{n-2} - p > \beta$ and there exist $q_1, \ldots, q_N \in \partial M$, with $N = N(u) \geq 1$ with the following properties: for $j = 1, \ldots, N$

1. set $r_j := R u(q_j)^{1-p}$ then $\{B_{r_j} \cap \partial M\}$ are a disjoint collection;
2. we have $|u(q_j)^{-1} u(y) - U(u(q_j)^{p-1})|_{C^{2}(B_{r_j})} < \beta$ (here $\psi_j$ are the Fermi coordinates at point $q_j$);
3. we have

$$u(x) d_{\bar{g}} (x, (q_1, \ldots, q_n)) \leq C_1 \text{ for all } x \in \partial M$$

$$u(q_j) d_{\bar{g}} (q_j, q_k) \geq C_0 \text{ for any } j \neq k.$$

Here $\bar{g}$ is the geodesic distance on $\partial M$.

Now we prove that only isolated blow up points may occur to a blowing up sequence of solution. For the proof of the next proposition we refer to [11]

Proposition 20. Assume $n \geq 8$. Given $\beta, R > 0$, consider $C_0, C_1$ as in the previous proposition. Assume $W(x) \neq 0$ for any $x \in \partial M$ if $n > 8$ or $W(x) \neq 0$ for any $x \in \partial M$ if $n = 8$. Then there exists $d = d(\beta, R)$ such that for any $u$ solution of (7.1) with $\max_{\partial M} u > C_0$, we have

$$\min_{i \neq j} d_{\bar{g}} (q_i(u), q_j(u)) \geq d,$$

where $q_1(u), \ldots q_N(u)$ and $N = N(u)$ are given in the previous proposition.

8. Proof of the main result.

Proof of Theorem 1. By contradiction, suppose that $x_i \to x_0$ is a blowup point for $u_i$ solutions of (1.2). Let $q_1^i, \ldots q_{N(u_i)}^i$ the sequence of points given by Proposition 19. By Claim 3 of Proposition 19 there exists a sequence of indices $k_i \in 1, \ldots, N$ such that $d_{\bar{g}} (x_i, q_{k_i}^i) \to 0$. Up to relabeling, we say $k_i = 1$ for all $i$. Then also $q_1^i \to x_0$ is a blow up point for $u_i$. By Proposition 20 and Proposition 18 we have that $q_1^i \to x_0$ is an isolated simple blow up point for $u_i$. Then by Proposition 16 we deduce that $W(x_0) = 0$ if $n = 8$ or that $W(x_0) = 0$ if $n > 8$, which contradicts the assumption of this theorem and proves the result.
9. Proof of Theorem 2. In this case the manifold is not umbilic, so we have a different expansion of the metric. Firstly, there exists a metric \( \tilde{g} \), conformal to \( g \), such that \( h \tilde{g} \equiv 0 \) (see [18, Prop. 3.1]). So, we can suppose w.l.o.g. that \( h \equiv 0 \) in the original problem, that is

\[
\begin{align*}
\frac{\partial u}{\partial \nu} + \varepsilon au &= (n - 2)u^{n-2} \\
\text{on } \partial M
\end{align*}
\]

This leads to obvious modification in the Pohozaev identity. The expansion of the metric in this case is

\[
\begin{align*}
|g(y)|^{1/2} &= 1 - \frac{1}{2} \left[ ||\pi||^2 + \text{Ric}(0) \right] y_n^2 - \frac{1}{6} \bar{R}_{ij}(0)y_iy_j + O(|y|^3) \\
g^{ij}(y) &= \delta_{ij} + 2h_{ij}(0)y_n + \frac{1}{3} \bar{R}_{kji}(0)y_ky_l + 2\partial h_{ij}(0)ty_k \\
+ [R_{mnjn}(0) + 3h_{ik}(0)h_{kj}(0)] y_n^2 + O(|y|^3)
\end{align*}
\]

where \( \pi \) is the second fundamental form and \( h_{ij}(0) \) are its coefficients, and \( \text{Ric}(0) = R_{nn}(0) = R_{nini}(0) \) (see [9]).

The main difference with the previous case lies in the second order approximation of the solution near an isolated simple blow up point. We define here, as in [1, Section 5] \( \hat{\gamma}_q : \mathbb{R}^+ \rightarrow \mathbb{R} \) is the unique solution of the problem

\[
\begin{align*}
-\Delta \gamma &= 2h_{ij}(q) \partial^2_{ij}U \quad \text{on } \mathbb{R}^+; \\
\frac{\partial \gamma}{\partial t} + nU \frac{\gamma}{\pi^{n/2}} &= 0 \quad \text{on } \partial \mathbb{R}^+; \\
\end{align*}
\]

such that \( \hat{\gamma}_q \) is \( L^2(\mathbb{R}^+_n) \)-orthogonal to \( j_b \) for all \( b = 1, \ldots, n \). Again, we have that (see [1, Section 5] and [13, Section 2] for the proofs).

\[
|\nabla^r v_q(y)| \leq C(1 + |y|)^{3-r-n} \text{ for } r = 0, 1, 2,
\]

\[
\int_{\partial \mathbb{R}^+} U \frac{\gamma}{\pi^{n/2}} v_q = 0
\]

\[
\int_{\partial \mathbb{R}^+} \Delta v_q v_q dz dt \leq 0,
\]

In this case we will have the following result (see [1, Proposition 6.1]) which replaces Proposition 13

Proposition 21. Assume \( n \geq 7 \). Let \( \hat{\gamma}_q \) be defined in (9.4). There exist \( R, C > 0 \) such that

\[
|\nabla^r v_i(y) - U(y) - \delta_i \hat{\gamma}_q(y)| \leq C\delta^2(1 + |y|)^{4-r-n}
\]

\[
\left| y_n \frac{\partial}{\partial n} (v_i(y) - U(y) - \delta^2 \hat{\gamma}_q(y)) \right| \leq C\delta^2(1 + |y|)^{4-n}
\]

for \( |y| \leq \frac{R}{\delta} \).

By the expansion of the metric, the Pohozaev identity and Proposition 21 we have the following estimate on the sign condition which corresponds to Proposition 14.
Proposition 22. Let $x_i \to x_0$ be an isolated simple blow-up point for $u_i$ solutions of (4.1). Then, fixed $r$, we have, for $i$ large

$$P(u_i, r) \geq \delta_i^2 \left( \frac{(n-6)\omega_{n-2} I_n}{(n-1)(n-2)(n-3)(n-4)} \right) \left[ |h_{ki}(x_i)|^2 \right] + o(\delta_i^2)$$

Proof. As in Proposition 14, we use that $\alpha < 0$ to get that

$$P(u_i, r) \geq -\int_{B^+_{r/\delta_i}} \left( y^b \partial_b y + \frac{n-2}{2} v_i \right) [(\tilde{L}_{g_i} - \Delta) v_i] \, dy.$$ 

Then, by the estimates contained in [1, Theorem 7.1], and in light of (9.7) we get the proof.

At this point we have all the tools to prove Theorem 2 using the same strategy of Section 8.

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