Measuring the quark condensate from the decays $\tau \to 3\pi + \nu_\tau$

L. Girlanda\textsuperscript{a}

\textsuperscript{a} Groupe de Physique Théorique, Institut de Physique Nucléaire, 91406 Orsay Cedex, France.

The possibility of detecting the S-wave of the decays $\tau \to 3\pi + \nu_\tau$ in the threshold region is explored, with emphasis on the sensitivity to the size of the quark antiquark condensate $\langle \bar{q}q \rangle$.

1. INTRODUCTION

Despite a great deal of efforts on both the theoretical and phenomenological side (see Ref. 1 and references therein), the mechanism of the spontaneous breakdown of chiral symmetry (SB$\chi$S) in QCD remains still unknown, in particular for what concerns the size of the simplest order parameter of this breakdown, the quark-antiquark condensate $\langle \bar{q}q \rangle$. If most of (quenched) lattice simulations seem to indicate that chiral phase transitions could happen for a number of massless flavors $N_f$ as small as $N_f = 4$ 3. On one hand some unquenched simulations seem to indicate that chiral phase transitions could happen for a number of massless flavors $N_f$ as small as $N_f = 4$ 3. On the other hand, a recent sum-rule evaluation of the Zweig rule violating low energy constant $L_6$, suggests that the quark condensate in the SU(3) chiral limit could be considerably smaller than the same quantity in the SU(2) chiral limit 4. On the experimental side, on-going (E865 at BNL, DIRAC at CERN) and forthcoming (KLOE at Daöne) facilities, will allow to pin down $\langle \bar{q}q \rangle$ in the SU(2) chiral limit from low energy $\pi\pi$ observables 5. Due to the enormous $\tau$-data sample already accumulated ($\sim 10^7 \tau$ pairs at CLEO) and to the improvements expected in the near future 6 (CLEOIII-CESR, BaBar, . . . ), it is natural to explore the possibility of measuring $\langle \bar{q}q \rangle$ from $\tau$ decays, which would provide an independent cross-check of determinations from $\pi\pi$ scattering.

The sensitivity to $\langle \bar{q}q \rangle$ in the exclusive decays $\tau \to 3\pi + \nu_\tau$ is contained in the S-wave. The latter, being proportional to the light quark mass $\tilde{m} = (m_u + m_d)/2$, is small compared to the P-wave contribution, which, moreover, is enhanced by the resonance $a_1$. However it turns out that, close to threshold, the P-wave is kinematically suppressed relatively to the S-wave, thus allowing the latter to appear through sizeable azimuthal left-right asymmetries.

2. KINEMATICS

There are two different charge modes in the decay $\tau \to 3\pi + \nu_\tau$, the $2\pi^0\pi^-$ mode and the all charged mode, $2\pi^-\pi^+$, whose hadronic matrix elements, $H_{\mu}^{00-}(p_1, p_2, p_3)$ and $H_{\mu}^{-++}(p_1, p_2, p_3)$, are

\begin{equation}
\begin{aligned}
H_{\mu}^{00-} &= \langle \pi^0(p_1)\pi^0(p_2)\pi^-(p_3)|A^-_\mu|0\rangle, \\
H_{\mu}^{-++} &= \langle \pi^-(p_1)\pi^-(p_2)\pi^+(p_3)|A^-_\mu|0\rangle,
\end{aligned}
\end{equation}

with $A^-_\mu = \bar{u}\gamma_\mu\gamma_5d$. In the isospin limit, $m_u = m_d = \tilde{m}$, the most general Lorentz decomposition of each matrix element involves three form factors,

\begin{equation}
\begin{aligned}
H_{\mu}^{\text{hfs}} &= V_{1\mu} F_1^{\text{hfs}}(p_1, p_2, p_3) \\
&+ V_{2\mu} F_2^{\text{hfs}}(p_1, p_2, p_3) + V_{4\mu} F_4^{\text{hfs}}(p_1, p_2, p_3),
\end{aligned}
\end{equation}

where hfs (hadronic final state) stands for 00− or −−+ and

\begin{equation}
\begin{aligned}
V_1^\mu &= p_1^\mu - p_3^\mu - \frac{Q(p_1 - p_2)}{Q^2} Q^\mu, \\
V_2^\mu &= p_2^\mu - p_3^\mu - \frac{Q(p_2 - p_3)}{Q^2} Q^\mu, \\
V_4^\mu &= p_1^\mu + p_2^\mu + p_3^\mu = Q^\mu.
\end{aligned}
\end{equation}
For both the hadronic final states, Bose symmetry requires that $F_2(p_1, p_2, p_3) = F_1(p_2, p_1, p_3)$. The form factors $F_1$ ($F_2$) and $F_4$ correspond respectively to $J = 1$ and $J = 0$, where $J$ is the total angular momentum of the hadronic system in the hadronic rest frame. The differential decay rate is given by

$$d\Gamma(\tau \to 3\pi\nu_\tau) = \frac{(2\pi)^4}{2M_\tau} |\mathcal{M}|^2 d\Phi_4,$$

where $\mathcal{M}$ is the matrix element of the electroweak interaction and $d\Phi_4$ is the invariant phase space of four particles. As shown explicitly by Kuhn and Mirkes [7], the matrix element squared can be expressed in terms of 9 independent leptonic and hadronic structure functions $L_X$ and $W_X$. The hadronic structure functions $W_X$ only depend on the hadronic variables, $Q^2$ and the Dalitz plot variables $s_1 = (p_2 + p_3)^2$ and $s_2 = (p_1 + p_3)^2$, while all the angular dependence is relegated in the leptonic structure functions $L_X$. Four of the hadronic structure functions correspond to the square of the spin 1 part of the hadronic matrix element,

$$W_A = (x_1^2 + x_3^2)|F_1|^2 + (x_2^2 + x_3^2)|F_2|^2 + 2(x_1x_2 - x_3^2)\text{Re}(F_1F_2^*)$$

$$W_C = (x_1^2 - x_3^2)|F_1|^2 + (x_2^2 - x_3^2)|F_2|^2 + 2(x_1x_2 + x_3^2)\text{Re}(F_1F_2^*),$$

$$W_D = 2|x_1x_3|F_1^2 - 2x_3x_2|F_2|^2 + 2x_3(x_2 - x_1)\text{Re}(F_1F_2^*),$$

$$W_E = -2x_3(x_1 + x_2)\text{Im}(F_1F_2^*),$$

one is the square of the spin 0 component

$$W_{SA} = Q^2|F_4|^2,$$

and the remaining ones are the interference between the spin 0 and spin 1 components

$$W_{SB} = 2\sqrt{Q^2x_1}\text{Re}(F_1F_4^*) + (p_1 \leftrightarrow p_2),$$

$$W_{SC} = -2\sqrt{Q^2x_1}\text{Im}(F_1F_4^*) + (p_1 \leftrightarrow p_2),$$

$$W_{SD} = 2\sqrt{Q^2x_3}\text{Re}(F_1F_4^*) - (p_1 \leftrightarrow p_2),$$

$$W_{SE} = -2\sqrt{Q^2x_3}\text{Im}(F_1F_4^*) - (p_1 \leftrightarrow p_2).$$

The $x_i$ are kinematical functions which are linear in the pion three-momenta in the hadronic rest frame$^2$, $x_1 = V_1^x$, $x_2 = V_2^x$ and $x_3 = V_3^y$, they vanish at the production threshold. Their explicit expressions in terms of Lorentz invariant quantities can be found in Ref. [8]. As it is clear from the explicit expressions of the $W_X$, the purely spin 1 structure functions are suppressed, at threshold, by two powers of the $x_i$'s, the interferences of spin 1 and spin 0 by only one power whereas the purely spin 0 structure function $W_{SA}$ is not suppressed at all. This is the reason for expecting a large sensitivity to the S-wave, and then to $\langle \bar{q}q \rangle$ in the threshold region.

3. GENERALIZED CHIRAL PERTURBATION THEORY

The appropriate tool for studying the dependence of the form factors on $\langle \bar{q}q \rangle$ at low energy is the generalized version of chiral perturbation theory (GCHPT) [8]. The results of the explicit calculation of the form factors at one-loop level can be found in Ref. [8]. Taken in the standard yPT limit [10], they reproduce the results of Ref. [11]. It is convenient to parametrize the dependence on $\langle \bar{q}q \rangle$ of the form factors of $\tau \to 3\pi + \nu_\tau$ in terms of two parameters $\alpha$ and $\beta$, related to the elastic $\pi\pi$ scattering amplitude. At leading order they correspond respectively to the amplitude and the slope at the symmetrical point $s = t = u = 4/3M_\pi^2$,

$$A(s|t,u) = \frac{M_\pi^2}{3F_\pi^2} + \frac{\beta}{F_\pi^2} \left(s - \frac{4}{3}M_\pi^2\right) + \ldots$$

In fact $\alpha$ and $\beta$ are correlated quantities: they must satisfy the so-called Morgan-Shaw universal curve, which results from the analysis of Roy equations (see Ref. [12] and references therein). All the dependence on $\langle \bar{q}q \rangle$ is contained in the parameter $\alpha$: at leading order, it varies between 1 and 4 if the quark condensate decreases from its standard value down to zero, while $\beta$ stays always close to 1. The relationship between the quark condensate in the SU(2) chiral limit and $\alpha$

$^2$ This system is oriented in such a way that the $x$-axis is along the direction of $\vec{p}_3$ and all the pions fly in the $x$-$y$ plane.

$^3$ For the definition of $\alpha$ and $\beta$ up to two-loop level see Ref. [13].
Figure 1. The Gell-Mann–Oakes–Renner ratio as function of $\alpha + 2\beta$.

and $\beta$ at one-loop level, with the associated theoretical uncertainties, is shown in Fig. 1 where we plot the Gell-Mann–Oakes–Renner ratio,

$$x_2 = -\frac{2\bar{m}}{F^2 \bar{m}^2} \lim_{\bar{m} \to 0, \bar{m} \neq 0} \langle \bar{q}q \rangle,$$

(9)
as function of $\alpha + 2\beta$. Also shown in the figure, between the dashed lines, is the experimentally allowed range for the combination $\alpha + 2\beta$, as extracted from a fit of the two-loop $\pi\pi$ amplitude to the $K_{e4}$ Rosselet data. The latter fit yields for $\alpha$ and $\beta$ the result,

$$\alpha_{\text{exp}} = 2.16 \pm 0.86, \quad \beta_{\text{exp}} = 1.074 \pm 0.050,$$

(10)
while the predictions of standard $\chi$PT read,

$$\alpha^s = 1.07 \pm 0.01, \quad \beta^s = 1.105 \pm 0.015.$$

(11)

4. AZIMUTHAL ASYMMETRIES

The hadronic structure functions which we are interested in are the interferences between the spin 0 and spin 1 amplitudes. These structure functions contain the dependence on $\langle \bar{q}q \rangle$ through the S-wave component and, at the same time, they are not as small as the purely spin 0 structure function, which is suppressed by two powers of the light quark mass $\bar{m}$. If the $\tau$ rest frame can not be reconstructed (which is always the case except for $\tau$-charm factories), only two of them are measurable, $W_{SB}$ and $W_{SD}$. They are related to the azimuthal asymmetries, obtained by integrating the decay rate over all angles except the azimuthal angle $\gamma$ (for the definitions of all angular variables see Ref. [6]),

$$d\Gamma = \frac{G_F^2 V_{ud}^2}{128M_\tau (2\pi)^3} \left[ \frac{M_\tau^2 - Q^2 \gamma}{Q^2} \right]^2 \frac{M_\tau^2 + 2Q^2}{3M_\tau^2} \times f(Q^2, \gamma) W(Q^2)dQ^2 d\gamma d\tau,$$

(12)
with $W(Q^2)$ defined by

$$W(Q^2) = w_A + \frac{3M_\tau^2}{M_\tau^2 + 2Q^2} w_{SA},$$

(13)
and the azimuthal distribution, normalized to 1,

$$f(Q^2, \gamma) = 1 + \lambda_2 (C'_{LR} \cos 2\gamma + C'_{UD} \sin 2\gamma) + \lambda_1 (C_{LR} \cos \gamma + C_{UD} \sin \gamma).$$

(14)

We have denoted by lowercase letters $w_X$ the corresponding structure functions $W_X$ integrated over the whole Dalitz plot. The asymmetry coefficients $C'_{LR}, C'_{UD}, C_{LR}$ and $C_{UD}$ in Eq. (14) are related to the Kuhn and Mirkes’ structure functions by the relations\footnote{We are neglecting the polarization of the $\tau$'s, which is justified provided that the $Z$ exchange can be neglected (far from the Z peak).}

$$C'_{LR} = \frac{1}{3} \left( 1 - \frac{Q^2}{M_\tau^2} \right) \frac{3M_\tau^2}{M_\tau^2 + 2Q^2} w_{LR},$$

$$C'_{UD} = \frac{1}{3} \left( 1 - \frac{Q^2}{M_\tau^2} \right) \frac{3M_\tau^2}{M_\tau^2 + 2Q^2} w_{UD},$$

$$C_{LR} = \frac{1}{4} \frac{3M_\tau^2}{M_\tau^2 + 2Q^2} w_{SB},$$

$$C_{UD} = \frac{1}{4} \frac{3M_\tau^2}{M_\tau^2 + 2Q^2} w_{SD}.$$

(15)
The coefficients $\lambda_1$ and $\lambda_2$\footnote{We are neglecting the polarization of the $\tau$'s, which is justified provided that the $Z$ exchange can be neglected (far from the Z peak).} are kinematical functions resulting from the integration over the $\tau$-decay angle. They depend on $Q^2$ and on the $\tau$ velocity. In the limit of ultrarelativistic $\tau$ (relevant e.g. for CLEO), they take the form,

$$\lambda_1(Q^2) = \frac{M_\tau^4 - Q^4 + 2M_\tau^2 Q^2 \log \frac{Q^2}{M_\tau^2}}{(M_\tau^2 - Q^2)^2},$$

$$\lambda_2(Q^2) = -2 + 3 \frac{M_\tau^2 + Q^2}{M_\tau^2 - Q^2} \lambda_1(Q^2).$$

(16)
The explicit calculation of the form factors at one-loop level shows that the structure function $w_{SD}$
vanishes at leading order, unlike $w_{SB}$. The latter governs the left-right asymmetry. In Fig. 2 we plot the quantity $\Delta N(Q^2) = |N_L(Q^2) - N_R(Q^2)|$ for a total number of $10^7 \tau$-pairs produced, where $N(Q^2)$ denotes the total number of events, integrated from threshold up to $Q^2$, and the subscript $L$ ($R$) refers to events with $\pi/2 \leq \gamma \leq 3\pi/2$ (or the complementary interval). In order to study the sensitivity to $\langle \bar{q}q \rangle$ this quantity is plotted for three values of $\alpha$ and $\beta$: for each charge mode the lower band corresponds to the standard predictions [11], the middle band to the central “experimental” values [10] and the upper band to $\alpha = 4$, $\beta = 1.16$. For both the charge modes $\Delta N(0.35\text{GeV}^2)$ changes by a factors 3 if $\alpha$ varies between its standard value and 4. Quantitatively the effect is larger for the all charged mode, large enough to be eventually detected with a sample of $10^7 \tau$-pairs. Each band represents the theoretical uncertainties coming from the low energy constants, but not from higher order chiral corrections. The influence of the latters is estimated by studying the convergence of the chiral series. The lines immediately below the bands correspond to the result at tree level. We see that the one-loop corrections amount approximately to 30% at $Q^2 = 0.4 \text{ GeV}^2$. Therefore, supposing a geometrical behavior of the $\chi$PT series, we expect the two-loop corrections to modify the results by no more that 10%.

5. CONCLUSIONS

We have identified the decays $\tau \to 3\pi + \nu_\tau$ in the threshold region as a promising observable for measuring the strength of the quark condensation in QCD. A total statistics of $10^7 \tau$-pairs seems sufficient to provide an interesting cross-check of future determinations of $\langle \bar{q}q \rangle$ from $\pi\pi$ observables.

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