SOME GENUINE SMALL REPRESENTATIONS OF A NONLINEAR DOUBLE COVER

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Abstract. Let $G$ be the real points of a simply connected, semisimple, simply laced complex Lie group, and let $\tilde{G}$ be the nonlinear double cover of $G$. We discuss a set of small genuine irreducible representations of $\tilde{G}$ which can be characterized by the following properties: (a) the infinitesimal character is $\rho/2$; (b) they have maximal $\tau$-invariant; (c) they have a particular associated variety $O$. When $G$ is split, we construct them explicitly. Furthermore, in many cases, there is a one-to-one correspondence between these small representations and the pairs (genuine central characters of $\tilde{G}$, real forms of $O$) via the map $\tilde{\pi} \mapsto (\chi_{\tilde{\pi}}, AV(\tilde{\pi}))$.

1. INTRODUCTION

Assume that $G_C$ is a simply connected, semisimple, simply laced complex Lie group, and $G$ is a real form of $G_C$ with nontrivial fundamental group. Then $G$ has a nonlinear double cover $\tilde{G}$, which is not a matrix group (see [6], Proposition 3.6). For example, this holds when $G$ is a split group, or $G = SU(p,q), Spin(p,q), p, q \geq 1$, and most real forms of the exceptional groups. In fact, most real forms of $G_C$ have a nonlinear double cover (see [2]). The purpose of this paper is to discuss some small genuine representations of $\tilde{G}$ and their properties. By a genuine representation of $\tilde{G}$ we mean that a representation of $\tilde{G}$ which does not factor through $G$.

In Section 2, we first introduce some basic invariants, such as infinitesimal character, $\tau$-invariant, and associated variety, which are used to classify representations. These notions are quite general, and are defined for more general real reductive groups $G$, which can be linear or nonlinear, and are not necessarily simply laced. For each type, we fix an infinitesimal character $\lambda$. If $G$ is simply laced or of type $G_2$ and $F_4$, $\lambda$ is chosen to be $\rho/2$, where $\rho$ is half of the sum of the positive roots. For type $B_n$ and $C_n$, $\lambda$ is defined as in [3], and is listed in Table 1. Then we define a class of representations of $\tilde{G}$ denoted

$$\prod_s(\tilde{G}) = \{\tilde{\pi} \mid \tilde{\pi} \in \hat{\tilde{G}}_{\text{adm},\lambda}, \tilde{\pi} \text{ is genuine and has maximal } \tau\text{-invariant}\},$$

where $\hat{\tilde{G}}_{\text{adm},\lambda}$ is the set of irreducible admissible representations of $\tilde{G}$ with infinitesimal character $\lambda$. Here the superscript $s$ stands for small in the sense that the representations in this set have maximal $\tau$-invariant. There is a unique complex nilpotent orbit $O$ which is the complex associated variety of every $\tilde{\pi}$ from $\prod_s(\tilde{G})$, i.e. $AV(I_{\tilde{\pi}}) = \overline{O}$ (see the notation in Section 2.3). We calculate this orbit $O$ explicitly for all types and list them in Table 1.

Denote

$$\prod^O(\tilde{G}) = \{\tilde{\pi} \mid \tilde{\pi} \in \hat{\tilde{G}}_{\text{adm},\lambda}, \tilde{\pi} \text{ is genuine and } AV(I_{\tilde{\pi}}) = \overline{O}\}.$$
Then we have

**Theorem 1.1.** $\prod^s G = \prod^O G$.

The proof of the theorem is based on truncated induction of representations of Weyl groups and the Springer correspondence.

This set of representations $\prod^s G = \prod^O G$ is what we are going to discuss throughout this paper. First of all, we can attach to each $\bar{\pi} \in \prod^s G$ a pair $(\chi_{\bar{\pi}}, O_{\bar{\pi}})$, where $\chi_{\bar{\pi}}$ is the central character of $\bar{\pi}$ and $O_{\bar{\pi}}$ is the real associated variety of $\bar{\pi}$, denoted $AV(\bar{\pi}) = \bar{O}_{\pi}$. Here, $O_{\pi}$ is one of the real forms of $O$, and in Section 4, we will see that there are not many real groups which have nonempty intersection with $O$ and the number of real forms of $O$ is tiny as well. The notions of real associated variety and genuine central character will be discussed in more detail in Sections 4 and 5.

In Section 6, we restrict our attention to simply laced split groups, and hence $\lambda = \rho/2$. For split groups, there is a well-understood family of representations, called the Shimura representations (see [3]). Starting with these, we construct other genuine representations in $\prod^s_{\rho/2} G$. There are standard ways to get new representations from old ones: the theory of cross actions and Cayley transforms.

In our setting these are non-standard, because they involve half-integral roots. It is possible to start with a Shimura representation, and apply some cross actions and Cayley transforms to it, to obtain other representations in $\prod^s_{\rho/2} G$. The conditions which need to be satisfied are very rigid, and we get a small number of representations in $\prod^s_{\rho/2} G$. Let $\prod^s_{R O}(G)$ denote the set of representations obtained this way. In the last part of Section 6, furthermore, by counting the elements in $\prod^s_{\rho/2}(G)$ using a Weyl group calculation, we show that $\prod^s_{\rho/2}(G) = \prod^s_{R O}(G)$ for type $A_{n-1}$ and $D_n$. We conjecture this is true for type $E$.

In Section 7, we denote $P_{O}(G) = \{ (\chi_i, O_j) | \chi_i \in \prod^g(Z(G)), O_j \text{ is a real form of } O \}$, where $\prod^g(Z(G))$ is the set of genuine central characters of $G$. The fact is that $|\prod^s_{\rho/2}(G)| = |P_{O}(G)|$ if $G$ has type $A_{n-1}$ or $D_n$. Furthermore, the map $\bar{\pi} \mapsto (\chi_{\bar{\pi}}, AV(\bar{\pi}))$ gives a bijection between $\prod^s_{\rho/2}(G)$ and $P_{O}(G)$ in many cases. Here is the main theorem.

**Theorem 1.2.** Let $G$ be the split real form of a simply connected, semisimple, complex Lie group and let $\bar{G}$ be the nonlinear double cover of $G$. Consider the map 

$\xi : \bar{\pi} \mapsto (\chi_{\bar{\pi}}, AV(\bar{\pi}))$

from $\prod^s_{\rho/2}(G)$ to $P_{O}(\bar{G})$. Then, (a) $\xi$ is bijective if $G$ has type $A_{n-1}$, unless $n$ is divisible by 4; (b) $\xi$ is bijective if $G$ has type $D_n$; (c) $\xi$ is surjective if $G$ has type $E_6$ or $E_8$.

We conjecture that $\xi$ is bijective for type $E$.

This paper is part of my Ph.D. thesis. In my thesis, we introduce the Kazhdan-Patterson lifting, which is an operator taking stable representations of $G$ to 0 or virtual genuine representations of $\bar{G}$ (see [3]). We claim that when $G$ is simply laced and split, the small representations $\prod^s_{\rho/2}(G)$ can be obtained from lifting of the trivial representation (see [5] for the result of the case when $G = GL(n, \mathbb{R})$). Since the lifting operator is worked on the level of global characters, we compute
the character formulas for these small representations, and expect to prove the
unitarity of these representations as in the case of $GL(n, \mathbb{R})$. This part is to appear
in a future paper.

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2. Invariants of a representation

Before introducing the set of representations of interest, some notions are needed.
Let us get started with the setting. Let $G$ be a connected real Lie group, and
suppose that the complexified Lie algebra of $G$, denoted $\mathfrak{g}$, is reductive. Here
$G$ is allowed to be nonlinear, which means it cannot be embedded into any $GL(n, \mathbb{C})$
(see [4], [6], for example). We fix a Cartan involution $\theta$ of $G$ and let $K = G^\theta$ be
the corresponding maximal compact subgroup. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$,
and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the root system and
$W$ be the Weyl group of $\mathfrak{g}$.

Let $HC(\mathfrak{g}, K)$ be the set of Harish-Chandra modules and let $\hat{G}_{adm}$ denote the set
of equivalence classes of irreducible admissible representations of $G$. Then $\hat{G}_{adm}$
can be viewed as a subset of $HC(\mathfrak{g}, K)$ by sending an irreducible admissible rep-
resentation $\pi \in \hat{G}_{adm}$ to its space $V_\pi$ of $K$-finite vectors and then the latter can be regarded as an irreducible $(\mathfrak{g}, K)$-module. What we are going to do is to attach
certain invariants to the representations in $\hat{G}_{adm}$.

2.1. Infinitesimal Characters. The most basic invariant is the infinitesimal char-
acter of a representation. The center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ can be identified with the $W$-
invariant polynomials on $\mathfrak{h}$ via the Harish-Chandra homomorphism $\zeta : Z(\mathfrak{g}) \to
U(\mathfrak{h})^W$. In this way, we have a map $\text{infchar} : \hat{G}_{adm} \to \mathfrak{h}^*/W$, and the infinitesimal
character of $\pi \in \hat{G}_{adm}$ is identified with a weight $\lambda \in \mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*/W$, we denote
by

$$\hat{G}_{adm, \lambda} = \{ \pi \in \hat{G}_{adm} | \text{infchar}(\pi) = \lambda \}$$

and refer to the representations in $\hat{G}_{adm, \lambda}$ as the irreducible admissible representations with infinitesimal character $\lambda$. Similarly, let $HC(\mathfrak{g}, K)_\lambda$ denote the set of Harish-Chandra modules with infinitesimal character $\lambda$.

2.2. Primitive Ideals. Many invariants to be considered are actually invariants attached to the primitive ideals in $U(\mathfrak{g})$, though there are some invariants attached directly to an irreducible Harish-Chandra module. Thus let us first define

Definition 2.1. Let $V$ be an irreducible $U(\mathfrak{g})$-module. The annihilator of $V$ in
$U(\mathfrak{g})$ is

$$\text{Ann}(V) := \{ X \in U(\mathfrak{g}) | Xv = 0, \forall v \in V \},$$

which is a two-sided ideal in $U(\mathfrak{g})$. It is called the primitive ideal in $U(\mathfrak{g})$ attached
to $V$.

If two $U(\mathfrak{g})$-modules have the same primitive ideals, then their infinitesimal characters are the same, and hence it makes sense to talk about the primitive ideals with infinitesimal character $\lambda$. We set $\text{Prim}(\mathfrak{g})_\lambda$ to be the set of primitive ideals in $U(\mathfrak{g})$ with infinitesimal character $\lambda$. For any $\pi \in \hat{G}_{adm, \lambda}$, let $V_\pi$ be the
Let $\gamma$ be the highest weight of $U$, be regarded as a left $U$-module.

Proof. Choose a good filtration $\{V_i\}$ on $V$, then we obtain a good filtration $\{V_i \otimes F\}$ on $V \otimes F$. With these filtrations, $\text{gr}(V \otimes F)$ as a $S(\mathfrak{g})$-module is a sum of copies of $\text{gr}(V)$. Hence the lemma follows. $\Box$

Now suppose $\pi \in \bar{G}_{\text{adm}}$, $\pi$ be the primitive ideal attached to $\pi$, which can be regarded as a left $U(\mathfrak{g})$-module, and hence we define $\text{AV}(I_\pi)$ and $\text{GKdim}(I_\pi)$, $\text{GKdim}(\pi)$ in usual sense; whereas $\text{AV}(\pi)$ will be defined upon a $K$-invariant filtration, and we won’t talk about this until Section 4. By Kostant’s theory of harmonics, $\text{AV}(I_\pi)$ consists of nilpotent elements in $\mathfrak{g}^*$, and hence is a union of finite number of closures of nilpotent coadjoint orbits. In fact, it is a single orbit. Let us record some remarkable facts as follows.

**Theorem 2.3.** (1) (Borho, Brylinski, see [9]) There exists a unique (complex) nilpotent coadjoint orbit $\mathcal{O}$ such that $\text{AV}(I_\pi) = \overline{\mathcal{O}}$.

(2) (See [18]) $2\text{GKdim}(\pi) = \text{GKdim}(I_\pi) = \text{dim}_\mathbb{C} \overline{\mathcal{O}}$, where $\overline{\mathcal{O}} = \text{AV}(I_\pi)$ is obtained from (1).

**Remark 2.4.** $\text{AV}(I_\pi)$, which it the closure of a complex nilpotent orbit, is called the complex associated variety of $\pi$, while $\text{AV}(\pi)$ (which will be discussed in detail later in Section 4) is the real associated variety of $\pi$.

2.4. $\tau$-invariant. Given $I \in \text{Prim}(\mathfrak{g})$. Put $\Delta(\lambda) = \{ \alpha \in \Delta | (\lambda, \alpha^\vee) \in \mathbb{Z} \}$, the integral root system for $\lambda$, and let $W_\lambda$ denote the Weyl group for $\Delta(\lambda)$. Choose $\Delta^+(\lambda) \subseteq \Delta(\lambda)$, a positive system making $\lambda$ dominant. Write $\prod(\lambda) \subseteq \Delta^+(\lambda)$ for the set of simple roots. There is the Borho-Jantzen-Duflo $\tau$-invariant attached to $I$, which is a subset of $\prod(\lambda)$ (see [23], [24]), denoted $\tau(I)$.

Since $G_C$ is simply connected, we have an alternative definition for $\tau$-invariant. Let $\pi \in HC(\mathfrak{g}, K)$ and $F_\gamma$ be the finite-dimensional representation of $G$ with highest weight $\gamma$. Also let $\Delta(F_\gamma)$ denote the set of all weights of $F_\gamma$. Consider the
Zuckerman translation functor \( \psi^+_{\lambda}(\pi) = P_{\lambda\gamma}(\pi \otimes F_{\gamma}) \), where \( P_{\lambda\gamma} \) by definition is the projection on the representations with infinitesimal character \( \lambda + \gamma \), and hence \( \psi^+_{\lambda}(\pi) \) is a functor that projects \( \pi \otimes F_{\gamma} \) on representations with infinitesimal character \( \lambda + \gamma \). Let \( \alpha \in \prod \{\lambda\} \), and \( \lambda_\alpha \) be singular with respect to \( \alpha \) and \( \lambda - \lambda_\alpha \) is a sum of roots. Define \( \psi_\alpha(\pi) := \psi^+_{\lambda}(\pi) \) be the translation functor of \( \pi \) to the \( \alpha \)-wall. Then we define

\[
\tau(\pi) = \{ \alpha \in \prod \{\lambda\} | \psi_\alpha(\pi) = 0 \}.
\]

It turns out that \( \tau \)-invariant is a measure of size of \( \pi \): the bigger the \( \tau \)-invariant, the smaller the representation.

**Definition 2.5.** We say that \( \pi \) has maximal \( \tau \)-invariant if \( \tau(\pi) = \prod \{\lambda\} \), or equivalently, \( \psi_\alpha(\pi) = 0 \) for all \( \alpha \in \prod \{\lambda\} \).

**Lemma 2.6.** Let \( F \) be a finite dimensional representation. Then \( \psi_\alpha(F) = 0 \) for every root \( \alpha \) and hence \( F \) has maximal \( \tau \)-invariant.

**Proof.** Note that the infinitesimal character of every finite dimensional representation is regular.

Assume the setting in the Lemma. We have \( \psi_\alpha(F) = P_{\lambda}(F \otimes F') = 0 \), where \( \lambda' \) is singular for \( \alpha \) and \( F' \) is a finite dimensional representation, since \( F \otimes F' \) is a virtual finite dimensional representation and each constituent has regular infinitesimal character.

**Definition 2.7.** We call a representation small if it has maximal \( \tau \)-invariant.

The Gelfand-Kirillov dimension of an irreducible representation is a measure of the growth of \( K \)-types. Here is the proposition connecting these two measures.

**Proposition 2.8.** ([23]) Let \( \pi \in \widehat{G}_{\text{adm},\lambda} \). If \( I_\pi \) has max \( \tau \)-invariant, then

\[
\text{GKdim}(\pi) = |\Delta^+| - |\Delta^+(\lambda)|.
\]

### 2.5. Weyl Group Representations

There are some details of Weyl group representations that can be found in various places, for instance, [10], [17], [18], and [21]. We recall some of the useful facts as follows.

In [14], Joseph has attached to \( I \in \text{Prim}(g) \lambda \) a representation \( \sigma_I \in \widehat{W}_\lambda \). In fact, the map from \( I \in \text{Prim}(g) \lambda \) to \( \sigma_I \) is surjective onto the set of special representations of \( W_\lambda \) (see [10] for definition of a special Weyl group representation).

On the other hand, Springer provides a method for producing a representation of \( W \) from a nilpotent orbit \( O \), which is the well-known Springer correspondence. We write \( \text{sp}(O) \) for the irreducible representation of \( W \) attached to \( O \). There is an algorithm to calculate the \( \text{sp}(O) \) if given \( O \) by use of symbols (see [18]). Note that the map \( O \to \text{sp}(O) \) is injective, but not surjective usually.

Let \( W' \) be any subgroup of \( W \) generated by reflections. There is an operation called truncated induction \( j^W_{W'} \) (see [10], [17]), taking irreducible representations of \( W' \) to those of \( W \). It is a fact that \( j^W_{W'} \) is an injective map.

The following proposition summarizes and connects all concepts stated above.

**Proposition 2.9.** Let \( \pi \in \widehat{G}_{\text{adm},\lambda} \), \( I = I_\pi \), \( W_\lambda \) be the integral Weyl group for \( \lambda \). Then there is a unique nilpotent orbit \( O \) such that \( \sigma = \text{sp}(O) \). Furthermore, this \( O \) is dense in \( AV(I) \), that is, \( AV(I) = \overline{O} \). Thus, we have a commutative diagram:
3. Some Small Representations of $\tilde{G}$

In this section we assume that $G$ is a real form of a simply connected, semisimple complex Lie group, and $\tilde{G}$ is the nonlinear double cover of $G$. First, we identify the kernel of the covering map $p : \tilde{G} \to G$ with $\pm 1$ and write $\tilde{H}$ for the inverse image in $\tilde{G}$ of a subgroup $H$ of $G$. We define

**Definition 3.1.** A representation $\tilde{\pi}$ of $\tilde{H}$ is called genuine if $\tilde{\pi}(-1) = -I$. If $\tilde{\pi}$ is irreducible, then $\tilde{\pi}$ is genuine if and only if $\tilde{\pi}$ does not factor through $H$.

We focus on the genuine representations with a particular infinitesimal character $\lambda$. If $G$ is simply laced or of type $G_2$, $F_4$, $\lambda$ is chosen to be $\rho/2$, where $\rho$ is half sum of the positive roots. For type $B_n$ and $C_n$, $\lambda$ is defined as in [3], and is listed in Table 1. We are interested in a special category of representations with certain properties, defined as follows.

Denote

\[ \prod^\lambda_{\tilde{G}} = \{ \tilde{\pi} | \tilde{\pi} \in \tilde{G}_{adm, \lambda}, \tilde{\pi} \text{ is genuine and has maximal } \tau\text{-invariant} \} \]

The following is the key Lemma.

**Lemma 3.2.** There is a unique complex nilpotent orbit $O$ such that $AV(I_{\tilde{\pi}}) = \overline{O}$ for every $\tilde{\pi} \in \prod^\lambda_{\tilde{G}}$. This $O$ can be computed explicitly (case by case) and $\dim(O) = 2GKdim(\tilde{\pi}) = 2(|\Delta^+| - |\Delta^+ (\lambda)|)$, where $\Delta$ and $\Delta(\lambda)$ are the root system and integral root system, respectively.

**Proof.** Let $\pi \in \prod^\lambda_{\tilde{G}}$. Since $\tilde{\pi}$ has maximal $\tau$-invariant, $\sigma_{I_{\tilde{\pi}}} = sgn_{W_\lambda}$, the sign representation of the integral Weyl group for $\lambda$. Then the truncated induction takes $sgn_{W_\lambda}$ to a special representation of $W$, denoted $j(sgn) = j^W_{W_\lambda}(sgn)$, since $sgn_{W_\lambda}$ is a special representation of $W_\lambda$. Hence $j(sgn)$ defines a nilpotent orbit $O$ of $g$ through the Springer correspondence, i.e. $sp(O) = j(sgn)$, and this $O$ is dense in the associated variety of $I_{\tilde{\pi}}$, which means $AV(I_{\tilde{\pi}}) = \overline{O}$. The uniqueness of this $O$ follows from either Theorem 2.3 (1) or the injectivity of the Springer correspondence.

From [23], for a representation at infinitesimal character $\lambda$ with maximal $\tau$-invariant, $GKdim(\tilde{\pi}) = |\Delta^+| - |\Delta^+ (\lambda)|$, and hence $\dim(O) = 2(|\Delta^+| - |\Delta^+ (\lambda)|)$ by Theorem 2.3 (2). For exceptional groups, there is a unique complex nilpotent orbit of this dimension (see [11]), so it is exactly the one that we are looking for.

For classical types, there is an algorithm to calculate $j(sgn)$ and the corresponding $O$ explicitly via the Springer correspondence (see [11]). The parametrization sets of nilpotent orbits are partitions of $n$ for type $A_{n-1}$, and are partitions of $2n(2n+1)$, resp.) which even (odd, resp.) parts occur with even multiplicity for type $B_n$ and $D_n$ ($C_n$, resp.) (see [11] and [11]). All of the nilpotent orbits and the corresponding Weyl group representations are listed in Table 1. □
Table 1.

| Type    | n     | $\lambda$ | $\Delta(\lambda)$ | $\dim O$ | $\mathfrak{O}$ | $j(\text{sgn}_{W_{\lambda}})$ |
|---------|-------|-----------|---------------------|----------|----------------|-------------------------------|
| $A_{n-1}$ | $2m$ | $\rho/2$ | $A_{m-1} \times A_{m-1}$ | $\frac{n^2}{2}$ | $[2^m]$ | $[2^m]$ |
|         | $2m + 1$ | $\rho/2$ | $A_{m-1} \times A_{m-1}$ | $\frac{n^2-1}{2}$ | $[2^m]$ | $[2^m]$ |
| $B_n$   | $2m$   | $\rho(C_n)/2$ | $B_m \times B_m$ | $n^2$ | $[2^m]$ | $(\phi; [2^m])$ |
|         | $2m + 1$ | $\rho(C_n)/2$ | $B_{m+1} \times B_m$ | $n^2 - 1$ | $[2^{n-1} 2^m]$ | $(\phi; [2^m])$ |
| $C_n$   | $2m$   | $\rho(B_n)$ | $D_n$ | $2n$ | $[2^{2n-2}]$ | $(1^n; \phi)$ |
|         | $2m + 1$ | $\rho(B_n)$ | $D_{m+1} \times D_m$ | $n^2 - 1$ | $[3 2^{n-3} 1^3]$ | $(\phi; [2^m])$ |
| $E_6$   | $\rho/2$ | $A_1 \times A_5$ | $40$ | $3A_1$ | $\phi_{15,16}$ |
| $E_7$   | $\rho/2$ | $A_7$ | $70$ | $4A_1$ | $\phi_{15,28}$ |
| $E_8$   | $\rho/2$ | $D_8$ | $128$ | $4A_1$ | $\phi_{31,36}$ |
| $F_4$   | $\rho/2$ | $B_4$ | $16$ | $A_1$ | $\phi_{2,16}$ |
| $G_2$   | $\rho/2$ | $A_1 \times A_1$ | $8$ | $A_1$ | $\phi_{2,2}$ |

Because of Lemma 3.2 let $O$ denote the complex nilpotent orbit such that $\text{AV}(I_\bar{\pi}) = \mathfrak{O}$ for $\bar{\pi} \in \prod_{\lambda}^\circ(G)$, and define

$$\prod_{\lambda}^O(G) = \{ \bar{\pi} \mid \bar{\pi} \in \hat{G}_{\text{adm},\lambda}, \bar{\pi} \text{ is genuine and } \text{AV}(I_\bar{\pi}) = \mathfrak{O} \}.$$

Then here is the main theorem of this section.

**Theorem 3.3.** $\prod_{\lambda}^*(\hat{G}) = \prod_{\lambda}^O(\hat{G})$.

**Proof.** It is clear that $\prod_{\lambda}^*(\hat{G}) \subseteq \prod_{\lambda}^O(\hat{G})$ due to Lemma 3.2. Conversely, given a representation $\bar{\pi} \in \prod_{\lambda}^O(\hat{G})$, we need to show that $\bar{\pi}$ has maximal $\tau$-invariant, that is, to show that $\sigma_{I_\bar{\pi}} = \text{sgn}_{W_{\lambda}}$. This simply follows from the injectivity of the truncated induction. $\square$

4. REAL ASSOCIATED VARIETY

In the previous section, given $\pi \in \hat{G}_{\text{adm}}$, we defined its complex associated variety $\text{AV}(I_\pi)$. Now we want to attach nilpotent orbits directly to $\pi$. Notice that these notions are quite general and they can be defined for linear and nonlinear groups.

Suppose $(\pi, V)$ is the given finitely-generated $(\mathfrak{g}, K)$-module. As in Section 2.3 suppose $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ is a good filtration, and furthermore suppose this is a $K_\mathbb{C}$-invariant filtration since $V$ is also a $K_\mathbb{C}$-module. Hence the associated variety of $(\pi, V)$, denoted $AV(\pi) = AV(V)$, is a closed subvariety of $(\mathfrak{g}/t)^*$. Since $V$ is also a $K_\mathbb{C}$-module, $AV(\pi)$ is actually a $K_\mathbb{C}$-invariant subset of $(\mathfrak{g}/t)^*$. Similarly, $AV(\pi)$ consists of nilpotent elements, say, $AV(\pi) \subseteq N(\mathfrak{g}/t)^* := N(\mathfrak{g}^*) \cap (\mathfrak{g}/t)^*$, where $N(\mathfrak{g}^*)$ denotes the nilpotent cone of $\mathfrak{g}^*$. By a theorem of Kostant-Rallis, there are finitely many $K$ orbits on $N(\mathfrak{g}/t)^*$, and hence we may write

$$AV(\pi) = \overline{O_{1}^{K_\mathbb{C}}} \cup \cdots \cup \overline{O_{j}^{K_\mathbb{C}}},$$

for orbits $O_{i}^{K_\mathbb{C}}$ of $K_\mathbb{C}$ on $N(\mathfrak{g}/t)^*$.

The next result of Vogan relates $AV(I_\pi)$ and $AV(\pi)$.

**Theorem 4.1.** (see [22], for example) Suppose $\pi \in \hat{G}_{\text{adm}}$. Write

$$AV(\pi) = \overline{O_{1}^{K_\mathbb{C}}} \cup \cdots \cup \overline{O_{j}^{K_\mathbb{C}}},$$

and $AV(I_\pi) = \overline{O}$. 

Then each \( O_i^{K_C} \) is a Lagrangian submanifold of the canonical symplectic structure of \( O \). In particular, for each \( i \), we have
\[
G \cdot O_i^{K_C} = O \quad \text{and} \quad G\text{Kdim}(\pi) = \text{dim}(O_i^{K_C}).
\]

Next we introduce the Sekiguchi correspondence (see [11], chapter 9, for example).

**Theorem 4.2.** (Sekiguchi) There is a natural one-to-one correspondence between nilpotent \( G \)-orbits in \( g_R \) and nilpotent \( K_C \)-orbits in \( (g/\mathfrak{k}) \).

Thus, by the Sekiguchi correspondence, \( \text{AV}(\pi) \) can be viewed as \( O_1 \cup \cdots \cup O_j \), where each \( O_i \) is a \( G \)-orbit in \( g_R \) corresponding to \( O_i^{K_C} \) via the Sekiguchi correspondence, and hence \( \text{AV}(\pi) \) is called the real associated variety of \( \pi \). Moreover, if \( AV(I_\pi) = \overline{O} \), then we have \( G_C \cdot O_i = O \), and hence we say that each \( O_i \) is a real form of \( O \). Equivalently, we say \( \{O_i\}_{i=1}^l \) is the set of real forms of \( O \) if \( O \cap g_R = O_1 \cup \cdots \cup O_l \).

### 4.1. Real associated variety of representations in \( \prod_{\lambda}(\tilde{G}) \).

Resuming the setting of \( G \) and \( \tilde{G} \) in Section 3, recall that we defined a set of representations \( \prod_{\lambda}(\tilde{G}) \), and the complex associated variety of each representation in this set is the closure of a particular \( O \) (see Table 1). In Table 2, we list all real groups \( G \) such that \( O \cap g_R \) is nonempty, as well as the number of real forms of \( O \), denoted \#\( O_i \), with respect to each \( G \) (see [11] for the parametrization of real nilpotent orbits).

**Remark 4.3.** It can be observed from Table 2 that there are not many real groups which have nonempty intersection with \( O \). More precisely, if \( G \) is not listed in Table 2, then \( O \cap g_R = \phi \).

#### Table 2.

| Type   | \( A_{n-1} \) (\( g = \mathfrak{s}l_n \)) | \( B_n \) (\( g = \mathfrak{so}_{2n+1} \)) |
|--------|----------------------------------------|----------------------------------------|
| \( n \) | 2m, 2m + 1, 2m | 2m, 2m + 1, 2m + 1 | 2m, 2m + 1, 2m + 1 |
| \( G \) | SL(\( n, \mathbb{R} \)), SU(\( m, m \)), SU(\( m + 1, m \)) | Spin(\( n + 1, n \)), Spin(\( n + 2, n - 1 \)) |
| \( \#O_i \) | 2, 1, 1 | 2, 1, 1 |

| Type   | \( C_n \) (\( g = \mathfrak{sp}_{2n} \)) | \( D_n \) (\( g = \mathfrak{so}_{2n} \)) |
|--------|----------------------------------------|----------------------------------------|
| \( n \) | 2m, 2m + 1, 2m | 2m, 2m + 1, 2m + 1 |
| \( G \) | Sp(\( 2n, \mathbb{R} \)), Sp(\( 2p, 2q \)) | Spin(\( n, n \)), Spin(\( n + 1, n - 1 \)), Spin(\( n + 2, n - 2 \)) |
| \( \#O_i \) | 2, 1, 1 | 2, 1, 1 |

| Type   | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|--------|-----------|-----------|-----------|-----------|-----------|
| \( G \) | \( E_6(A_1 \times A_5) \), \( E_6(C_4) \) | \( E_7(A_7) \), \( E_8(D_8) \) | \( F_4(B_4) \) | \( G_2(A_1 \times A_1) \) |
| \( \#O_i \) | 2, 1, 2, 1 | 1, 1, 1 |

We have the following proposition saying that we can attach to each small representation defined in Section 3 a single real nilpotent orbit.
Proposition 4.4. We resume the setting and notations in Section 3. Suppose $G_C$ is a simply connected, semisimple complex Lie group, $G$ is a real form of $G_C$ and $\tilde{G}$ is the nonlinear double cover of $G$. For each $\pi \in \prod_i \lambda_i(\tilde{G}) = \prod_i \lambda_i(G)$, there is a unique real nilpotent orbit $O_\pi$ such that $AV(\pi) = O_\pi$. This $O_\pi$ is one of the real forms of $O$.

Proof. By a result of Vogan (see [20]), if $O_l$ is a real orbit of maximal dimension in $AV(\pi)$, and the complement of $O_l$ has codimension at least two in $\overline{O_l}$, then $AV(\pi) = \overline{O_l}$. Since $\dim_{\mathbb{R}} O_l = \dim C O$ for each real form $O_l$ of $O$, we just need to pick a complex nilpotent orbit $O'$, which is one step down smaller than $O$, and see if the difference of $\dim O$ and $\dim O'$ is at least 2. This can be checked case by case. □

5. Genuine Central Characters

5.1. Regular Characters. The following material can be found in several places, for example, [6], [20], and [27]. Again, $G$ is a real form of a simply connected, semisimple complex Lie group and $\tilde{G}$ is the nonlinear double cover of $G$. Let $\pi \in \tilde{G}_{adm, \lambda}$, where $\lambda$ is a regular infinitesimal character. Then $\pi$ can be specified by a parameter, which is called a $\lambda$-regular character, $\gamma = (H, \Gamma, \gamma)$, where $H$ is a $\theta$-stable Cartan subgroup of $G$, $\Gamma$ is a character of $H$, and $\gamma$ is an element in $\mathfrak{h}^*$ which defines the same infinitesimal character as $\lambda$, and there are certain compatibility conditions between $\gamma$ and $\Gamma$. More precisely, $\pi = J(\gamma)$, the unique irreducible quotient of a standard representation $I(\gamma)$, which is parametrized by $\gamma$ from a $K$-conjugacy class of regular characters for $\lambda$. Here the standard module $I(\gamma)$ is defined as follows. Write $H = TA$, where $T = H^0$ and $A$ is the identity component of $\{h \in H|\theta(h) = h^{-1}\}$. Let $M = \text{Cent}_G(A)$ and choose a parabolic subgroup $P = MN$, then we define $I(\gamma) = \text{Ind}_{P_M}^{G}(\sigma_M)$, where $\sigma_M$ is some relative discrete series of $M$ (see [11] for details).

Recall that (see [11], for instance) when $\lambda$ is a regular infinitesimal character, $\mathcal{HC}(g, K)_\lambda$ is parametrized by the set $P_\lambda$ of $K$-conjugacy classes of $\lambda$-regular characters. Furthermore, the following two sets are bases of the Grothendieck group:

$$\{[J(\gamma)]_{\gamma \in P_\lambda}\} \text{ and } \{[I(\gamma)]_{\gamma \in P_\lambda}\}.$$ 

We have the following definition.

Definition 5.1. Define the change of basis matrix

$$[J(\delta)] = \sum_{\gamma \in P_\lambda} M(\gamma, \delta)[I(\gamma)]$$

and the inverse matrix

$$[I(\delta)] = \sum_{\gamma \in P_\lambda} m(\gamma, \delta)[J(\gamma)].$$

Here $M(\gamma, \delta)$ and $m(\gamma, \delta)$ are integers and $M(\gamma, \delta)$ are computed by the Kazhdan-Lusztig-Vogan algorithm when $G$ is linear.

The above notions can also be defined for nonlinear groups. More specifically, let $\lambda$ be the regular infinitesimal character defined in Section 3. In this case, suppose that $\tilde{\pi}$ is an irreducible genuine representation from $G_{adm, \lambda}$. Then $\tilde{\pi}$ is parametrized by a genuine $\lambda$-regular character $\gamma = (\tilde{H}, \Gamma, \gamma)$, where $\Gamma$ is an irreducible genuine representation of $\tilde{H} = p^{-1}(H)$. Note that in this case $\Gamma$ can be replaced by a
character of $Z(\tilde{H})$, a central character of $\tilde{H}$, because of the following proposition (see \[3\]).

**Notation 5.2.** Let $G'$ be a subgroup of $\tilde{G}$. We write $Z(G')$ for the center of $G'$ and $\prod_g(G')$ for equivalence classes of irreducible genuine representations of $G'$.

**Proposition 5.3.** Let $\gamma = (\tilde{H}, \Gamma, \tau)$ be a genuine regular character defined as above. Let $n = |H/Z(H)|^2$. For every $\chi \in \prod_g(Z(H))$ there is a unique representation $\Gamma = \Gamma(\chi) \in \prod_g(\tilde{H})$ for which $\Gamma|_{Z(\tilde{H})}$ is a multiple of $\chi$. The map $\chi \to \Gamma(\chi)$ is a bijection between $\prod_g(Z(H))$ and $\prod_g(\tilde{H})$. The dimension of $\Gamma(\chi)$ is $n$, and $\text{Ind}_{Z(\tilde{H})}^\tilde{H}(\chi) = n^2$.  

When $G$ is simply laced, a genuine representation of $\tilde{H}$ is determined by the infinitesimal character and its restriction to $Z(\tilde{G})$. We record the properties as follows (see \[6\]).

**Proposition 5.4.** All the setting is as before, and also suppose $G$ is simply laced, $H$ is a Cartan subgroup of $G$, and $H^0$ is the identity component of $H$. Then

1. $Z(\tilde{H}) = Z(\tilde{G})H^0$. In particular, a genuine character of $Z(\tilde{H})$ is determined by its restriction to $Z(\tilde{G})$ and its differential;
2. A genuine regular character $\gamma = (\tilde{H}, \Gamma, \tau)$ of $\tilde{G}$ is determined by $\tau$ and the restriction of $\Gamma$ to $Z(\tilde{G})$, and so is $\tilde{\gamma} = J(\gamma)$.

The second part of this proposition is basically a corollary of the first part. Consequently $\tilde{G}$ typically has few genuine irreducible representations, denoted $\prod_g(\tilde{G})$.

### 5.2. Action of $\text{Aut}(G)$ on $\prod_g(\tilde{G})$

In this section we want to see how an automorphism of $G$ acts on $\prod_g(\tilde{G})$. Let $\text{Aut}(G)$ denote the automorphism group of $G$, and

$$\text{Int}(G) = \{\tau \in \text{Aut}(G) | \tau = \tau_x \text{ for some } x \in G\},$$

where $\tau_x(g) = xgx^{-1}$ for $g \in G$.

$$\text{Out}(G) = \text{Aut}(G)/\text{Int}(G).$$

**Lemma 5.5.** There is a natural map from $\text{Out}(G)$ to $\text{Aut}(Z(\tilde{G}))$, which sends each $\tau \in \text{Aut}(G)$ to $\tilde{\tau} \in \text{Aut}(Z(\tilde{G}))$. When $G$ is simply laced, This map is an embedding.

**Proof.** The map $\tau \in \text{Aut}(G) \to \tilde{\tau} \in \text{Aut}(Z(\tilde{G}))$ is defined as follows. Every $\tau \in \text{Aut}(G)$ can be lifted to an automorphism $\tilde{\tau}$ of $\tilde{G}$. Then by restricting $\tilde{\tau}$ to $Z(\tilde{G})$, we get an automorphism of $Z(\tilde{G})$, which is also denoted by $\tilde{\tau}$. This map is well-defined since if $\tilde{\tau} \in \text{Int}(G)$, say, $\tau = \tau_x$ for some $x \in G$, $\tilde{\tau}(\tilde{z}) = \tilde{x}\tilde{z}\tilde{x}^{-1} = \tilde{z}$, for $\tilde{z} \in Z(\tilde{G})$.

The proof of the second assertion can be found in \[4\].

Let $\tau \in \text{Aut}(G)$. Define an action of $\tau$ on $\prod_g(Z(\tilde{G}))$ as follows. Let $\chi \in \prod_g(Z(\tilde{G}))$, define $\chi^\tau(z) := \chi(\tilde{\tau}(z))$, $z \in Z(\tilde{G})$. When $G$ is simply laced, we have an action of $\text{Aut}(G)$ on $\prod_g(\tilde{G})$. Due to Proposition 5.4 (2), every $\pi = J(\gamma)$, where $\gamma = (\tilde{H}, \Gamma, \tau)$, is determined by $\tau$ and $\chi := \Gamma|_{Z(\tilde{G})}$. Then we can define $\pi^\tau := J(\gamma^\tau)$, where $\gamma^\tau$ is a regular character determined by $\tau$ and $\chi^\tau$.

The following is a corollary of Lemma 5.5.
There is a bijection between $\prod_{s}(\tilde{G})$, and $\tau \in \text{Out}(G)$, then $\tilde{\pi}$ and $\tilde{\pi}^\tau$ are inequivalent representations in $\prod_{s}(\tilde{G})$.

6. $\prod_{s}(\tilde{G})$ – Split Case

In this section, the setting is as in Section 3 and furthermore we assume that $G$ is simply laced and split, and hence $\lambda = \rho/2$ from now on. Let $H$ denote the split Cartan subgroup of $G$ with Lie algebra $\mathfrak{h}$ (then $\tilde{H} = p^{-1}(H)$ is the split Cartan subgroup of $\tilde{G}$). It follows from Proposition 4.4 (see [8]) that there is a unique minimal principal representation of $G$ coming from $\tilde{H}$ if we fix a genuine central character and infinitesimal character. In [3], the unique irreducible quotients of them are called genuine pseudospherical representations, or Shimura representations. We will show in this section that we can get more representations in $\prod_{s/2}(\tilde{G})$ by applying Cayley transforms to the Shimura representations.

6.1. Shimura Representations. In [3], there is a set of irreducible genuine representations denoted $S\mathcal{H}$, called Shimura representations, each of which is the unique irreducible quotient of a genuine pseudospherical principal series. We enumerate them as $S\mathcal{H} = \{Sh_i\}_{i=1}^k$, where $k = 1, 2, or 4$. We have the following important properties for Shimura representations.

Proposition 6.1. Suppose we have the same setting as in Section 3 and suppose that $G$ is simply laced and split. Then,

1. $S\mathcal{H} \subset \prod_{s/2}(\tilde{G}) = \prod_{s/2}(\tilde{G})$, where $\mathcal{O}$ is defined in Section 3.

2. There is a bijection between $S\mathcal{H}$ and $\prod_{s}(Z(\tilde{G}))$. Furthermore, there is a bijection between this and $P/R$, where $P$ and $R$ are the weight lattice and root lattice, respectively. Therefore, the central character of a Shimura representation can be specified by an element in $P/R$.

3. There is a bijection between $\prod_{s}(Z(\tilde{G}))$ and $\{\mathcal{O}_i\}$, where $\{\mathcal{O}_i\}$ is the set of real forms of $\mathcal{O}$, the complex associated variety of every Shimura representation.

Proof. The infinitesimal character of each Shimura representation is $\rho/2$ due to construction. For the parameter of a Shimura representation, all roots are real roots and it is shown in [20] that all real integral roots are nonparity real roots, and hence all simple integral roots are in the $\tau$-invariant (cf. [28], Theorem 4.12), which shows part (1).

Let $M$ be defined as in the beginning of Section 5.1 corresponding to $H$. It is proved in [3] that $Z(M) = Z(\tilde{G})$ and there is a one-to-one correspondence between $\prod_{s}(Z(M))$ and $[2P' \cap R']/2R'$, which is isomorphic to $P/R$ when $G$ is simply laced and hence part (2) follows.

It can be observed from Table 1 in [3] and Table 2 that when $G$ is simply laced and split, $\#(\mathcal{O}_i) = |\prod_{s}(Z(\tilde{G}))|$, and hence part (3) follows.

Remark 6.2. Because of Proposition 4.4 and 6.1 we can attach to each $Sh_i$ a pair $(\chi_i, \mathcal{O}_i)$, where $\chi_i$ is the central character of $Sh_i$, and $\mathcal{O}_i = \text{AV}(Sh_i)$. Then for each $\tau \in \text{Out}(G)$, $Sh_i^\tau$ (see the notation defined in Section 5.2) is associated to the pair $(\chi_i^\tau, \mathcal{O}_i)$, (i.e. as $\tau$ permutes the central characters, it also permutes the real associated varieties).
6.2. Translation functors across a nonintegral wall. We recall some basic tools: cross-actions, Cayley and inverse Cayley transforms before starting to construct new representations. Most of the material in this section can be found in [28, 6] and [20]. Fix an infinitesimal character $\lambda$. (In our case, $\lambda = \rho/2$). In order to compute characters for nonlinear groups, we need a family of infinitesimal characters containing $\lambda$, denoted $\mathcal{F}(\lambda)$, which is defined as follows. We define

$$\Delta^+ = \{\alpha \in \Delta | (\lambda, \alpha^\vee) > 0\},$$

$$P = \{\mu \in \mathfrak{h}^* | (\mu, \alpha^\vee) \in \mathbb{Z} \text{ for } \alpha \in \Delta\},$$

the integral weight lattice,

$$W_p(\lambda) = \{w \in W \mid w\lambda - \lambda \in P\}.$$

Let $\mathcal{F}(\lambda)$ be a family of representatives of $(W \cdot \lambda + P)/P$ containing $\lambda$, and hence it is clear that $\mathcal{F}(\lambda)$ is indexed by $W/W_p(\lambda)$: if $\gamma \in \mathcal{F}(\lambda)$, then $\gamma = y\lambda$ modulo $P$ for some $y \in W$ which is unique modulo $W_p(\lambda)$. So we can write

$$\mathcal{F}(\lambda) = \{\gamma_y = y\lambda \mid y \in W/W_p(\lambda)\}.$$ 

In particular, $\lambda = \gamma_1$. There is an obvious action of $W$ on $\mathcal{F}(\lambda)$: $w * \gamma_y := w^{-1}(\gamma_y + \mu(y, w)) = \gamma_{yw}$, by picking some $\mu(y, w) \in P$. We fix once and for all integral weights $\mu(y, w) \in P$ satisfying the above conditions and we want to use them to define the following. First let $\alpha$ be a nonintegral simple root in $\Delta^+$, $s_0$ be the corresponding simple reflection. Then we define:

(a) the nonintegral wall-crossing functors $\psi_\alpha$ and $\phi_\alpha$, where $\psi_\alpha(X) := \psi_{\gamma_{\alpha}}^\gamma\alpha(X)$, a functor realizes an equivalence of categories between $\mathcal{H}\mathcal{C}(g, K)_{\gamma_{\alpha}}$ and $\mathcal{H}\mathcal{C}(g, K)_{\gamma_{\alpha}}$: its inverse is $\phi_\alpha$ (see [24]);

(b) the cross action of $W$: let $\gamma = (H', \Gamma, \overline{\gamma})$ be a $(\gamma_{\alpha})$-regular character, $w \in W$, then the regular character $w \times \gamma = (H', w \times \Gamma, w \times \overline{\gamma})$ is defined by $w \times \overline{\gamma} = \overline{\gamma} + \mu(y, w)$ and $w \times \Gamma = \Gamma \otimes \mu(y, w) \otimes \partial p(w)$, where $\partial p(w) := w \cdot (\rho_i - 2\rho_c) - (\rho_i - 2\rho_c)$, $\rho_i$ (resp. $\rho_c$) denotes the half-sum of positive imaginary (resp. compact imaginary) roots that make $\overline{\gamma}$ dominant. Note that $w \times \overline{\gamma}$ defines the same infinitesimal character as $\gamma_{yw}$.

**Remark 6.3.** Let $\alpha_1, \ldots, \alpha_p$ be simple roots and $s_1, \ldots, s_p$ be the corresponding reflections. If $w = s_p \cdots s_1 \in W_p(\lambda)$, we can define $\mu(1, w) = w\lambda - \lambda$, which is equal to $\mu(s_1, s_2) + \mu(s_2, s_3) + \cdots + \mu(s_{p-1}, s_p)$. Thus, $w \times \overline{\gamma} = \overline{\gamma}$ if and only if $w \in W_p(\lambda)$, where $\overline{\gamma} \sim \lambda$.

We also need some basic facts about Cayley and inverse Cayley transforms. The related concepts can be found in various references (e.g. [20, 27]). Here we just introduce some notation and quote some important facts.

Let $\gamma = (H, \Gamma, \overline{\gamma})$ be a $\lambda$-regular character. Assume $\alpha$ is a nonintegral root, then we can define Cayley (or inverse Cayley) transform on $\gamma$ (see Section 5 of [20]) through $\alpha$ if $\alpha$ is noncompact imaginary (real, resp.) and this action is denoted by $e_\alpha^\gamma(\gamma) = \gamma^\alpha$ (or $c_\alpha(\gamma) = \gamma_\alpha$, resp.) Note that after Cayley (inverse Cayley, resp.) transform, we get a new $\lambda$-regular character, say, $\gamma^\alpha = (H^\alpha, \Gamma^\alpha, \overline{\gamma^\alpha})$ (or $\gamma_\alpha = (H_\alpha, \Gamma_\alpha, \overline{\gamma_\alpha})$, resp.), which has infinitesimal character $\lambda$ and $I(\gamma^\alpha)$ (or $I(\gamma_\alpha)$, resp.) has the same central character as the original representation $I(\gamma)$. For convenience, we call both operators $e_\alpha$ and $c_\alpha$ Cayley transforms through the root $\alpha$.

Now we are ready to state the result of Vogan describing translation functors across a nonintegral wall.

**Theorem 6.4.** ([20]) Let $\gamma$ be a genuine $\lambda$-regular character of $G$. Suppose $\alpha$ is a nonintegral simple root in $\Delta^+(\overline{\gamma})$. Then, with the translation functor $\psi_\alpha$ defined
by the weight \( \mu_\alpha \) fixed above, we have:

\[
\psi_\alpha(J(\gamma)) = J(\langle \gamma + \mu_\alpha \rangle) = J((s_\alpha \times \gamma)') \quad \text{if } \alpha \text{ is noncompact imaginary,}
\]

\[
\psi_\alpha(J(\gamma)) = J(\langle \gamma + \mu_\alpha \rangle) = J((s_\alpha \times \gamma)_\alpha) \quad \text{if } \alpha \text{ is real satisfying the parity condition,}
\]

\[
\psi_\alpha(J(\gamma)) = J(\gamma + \mu_\alpha) = J(s_\alpha \times \gamma) \quad \text{otherwise.}
\]

Remark 6.5. It can be observed from Theorem 6.3 that \( \psi_\alpha(J(\gamma)) \) and \( J(s_\alpha \times \gamma) \) have the same infinitesimal character and central character.

6.3. Construction related to Dynkin Diagrams. Now we are ready to construct representations in \( \prod_{\rho/2}(\tilde{G}) \). As described in [19], to the Dynkin diagram \( \tilde{D} \) of \( \tilde{G} \), we attach a finite abelian group denoted by \( R_D \) as follows. Let \( \prod \) be the set of simple roots. Define

\[
R_D = \{ S \subseteq \prod \mid S \text{ is strongly orthogonal, so that any } \beta \notin S \text{ is adjacent to an even number of elements in } S \}. 
\]

In Table 3, we list out the elements in \( R_D \) for simply laced groups using Dynkin diagrams. Note that the root is in the element in \( R_D \) if and only if the corresponding node is filled.

Lemma 6.6. There is a one-to-one correspondence between \( R_D \) and \( (Z(G_C))^2 \), the characters of elements in \( Z(G_C) \) of order 2. The latter is isomorphic to \( P/(2P + R) \) as group, and hence \( R_D \) can be parametrized by the elements in \( P/(2P + R) \).

Since \( G \) is simply laced, this is simply \( P/R \).

Proof. Denote \( Z = Z(G_C) \) and \( Z_2 = Z(G_C)/2 \).

From the exact sequence

\[
1 \to Z_2 \to Z \to Z/Z_2 \to 1,
\]

we have another exact sequence

\[
1 \to (Z/Z_2)^\wedge \to Z^\wedge \to Z_2^\wedge \to 1.
\]

Notice that \( Z^\wedge \cong P/R \). Write \( Z_2 = \{ \exp(2\pi i \tau^\wedge) \mid \tau^\wedge \in X_\ast \otimes \mathbb{C}, \exp(2\pi i(2\tau^\wedge)) = 1 \} \).

Then \( (Z/Z_2)^\wedge \cong (2P + R)/R \), since \( \gamma \in P \) such that \( \gamma|_{Z_2} = 1 \) (i.e. \( \gamma(\exp(2\pi i \tau^\wedge)) = \exp(2\pi i(\gamma(\tau^\wedge)) = 1 \) if and only if \( \langle \gamma, \tau^\wedge \rangle = 1 \), and only if \( \gamma \in 2P + R \). Therefore, \( Z_2^\wedge \cong P/(2P + R) \) from the above exact sequence.

Associate to each \( S = \{ \alpha_1, \cdots, \alpha_p \} \in R_D \) an element \( w_S = s_{\alpha_1} \cdots s_{\alpha_p} \in W \), then we have a map sending elements in \( R_D \) to \( P/(2P + R) \) by \( S \mapsto w_S(\rho/2) - \rho/2 \).

This is a bijection by counting the elements in \( R_D \) and \( P/(2P + R) \) case by case.

We will show that we can get a subset of representations in \( \prod_{\rho/2}(\tilde{G}) \) from each Shimura representation \( Sh_i \) by a sequence of Cayley transforms or wall-crossings through the simple roots in \( S \in R_D \).

Associate to each \( S = \{ \alpha_1, \cdots, \alpha_p \} \in R_D \) an element \( w_S = s_{\alpha_1} \cdots s_{\alpha_p} \in W \), and let \( c_S = c_{\alpha_1} \cdots c_{\alpha_p} \) and \( \psi_S = \psi_{\alpha_1} \cdots \psi_{\alpha_p} \) be the corresponding Cayley transform and wall-crossing functor respectively.

Lemma 6.7. For every \( w_S, S \in R_D \), \( w_S \in W_p(\rho/2) \), and hence \( w_S \times \tau = \tau \), where \( \gamma \sim \rho/2 \), by Remark 5.2.1.

Proof. Let \( S = \{ \alpha_1, \cdots, \alpha_p \} \in R_D \). Then \( w_S(\rho/2) - \rho/2 = s_{\alpha_1} \cdots s_{\alpha_p}(\rho/2) - \rho/2 = -\langle \rho/2, \alpha_1^\wedge \rangle \alpha_1 - \cdots - \langle \rho/2, \alpha_p^\wedge \rangle \alpha_p \).

For each simple root \( \beta \notin S \), \( \beta \) is adjacent to even numbers of \( \alpha_i \)'s, and hence
Table 3.

| Type | \( n \) | \( R_D \) | \(|R_D|\) |
|------|--------|----------|--------|
| \( A_{n-1} \) | \( 2m \) | \( \begin{array}{c} \cdot \cdot \cdot \end{array} \) | \( \{\phi\} \) | 2 |
| \( A_{n-1} \) | \( 2m+1 \) | \( \begin{array}{c} \cdot \cdot \cdot \end{array} \) | \( \{\phi\} \) | 1 |
| \( D_n \) | \( 2m \) | \( \begin{array}{c} \cdot \cdot \cdot \end{array} \) | \( \{\phi\} \) | 4 |
| \( D_n \) | \( 2m+1 \) | \( \begin{array}{c} \cdot \cdot \cdot \end{array} \) | \( \{\phi\} \) | 2 |
| \( E_6 \) | \( \begin{array}{c} \cdot \cdot \cdot \\cdot \end{array} \) | \( \{\phi\} \) | 1 |
| \( E_7 \) | \( \begin{array}{c} \cdot \cdot \cdot \\cdot \end{array} \) | \( \{\phi\} \) | 2 |
| \( E_8 \) | \( \begin{array}{c} \cdot \cdot \cdot \\cdot \end{array} \) | \( \{\phi\} \) | 1 |

\[ \langle w_S(\rho/2) - \rho/2, \beta^\vee \rangle \in \mathbb{Z}. \] For \( \beta = \alpha_i \) some \( i \), \( \langle w_S(\rho/2) - \rho/2, \beta^\vee \rangle = -\langle \rho/2, \alpha_i^\vee \rangle \langle \alpha_i, \alpha_i^\vee \rangle \in \mathbb{Z}. \] Therefore, \( w_S \in W(p/2) \). \( \square \)

**Lemma 6.8.** (1) \( \psi_S(Sh_i) \) is in \( \prod_{\rho/2}(\tilde{G}) \) for every \( S \in R_D \) and \( Sh_i \in \mathcal{S}H \);
(2) \( \bigcup_{Sh_i \in \mathcal{S}H} \{\psi_S(Sh_i) | S \in R_D\} = \bigcup_{Sh_i \in \mathcal{S}H} \{c_S(Sh_i) | S \in R_D\} \).
Proof. Let $S \in R_D$ and $Sh_i \in SH$. We apply $\psi_S$, a series of nonintegral wall-crossings $\psi_n$'s, to $Sh_i$, and in each step, $\psi_n(Sh_i) = P^\gamma_{ys_n}(Sh_i \otimes F_{\mu(y,s_n)})$, the projection of $Sh_i \otimes F_{\mu(y,s_n)}$ on to the Harish-Chandra modules with infinitesimal character $\gamma_{ys_n}$, where $\gamma_n$, $\gamma_{ys_n}$, and $\mu(y,s_n)$ are described in Section 6.2. Note that $Sh_i \in \prod^*_2(G) = \prod^0_2(G)$, we have $AV(I_{Sh_i}) = \bigotimes$ and hence $AV(I_{Sh_i} \otimes F) = \bigotimes$ for any finite dimensional $F$ by Lemma 2.2. Therefore, $AV(I_{\psi_n(Sh_i)}) = \bigotimes$ and also $AV(I_{\psi_S(Sh_i)}) = \bigotimes$. By Remark 6.3 and Lemma 6.7, $\psi_S(Sh_i)$ has infinitesimal character $\rho/2$ and we conclude that $\psi_S(Sh_i) \in \prod^0_2(G) = \prod^*_2(G)$.

For the second part of the proof, first we observe that all representations on both sides have infinitesimal character $\rho/2$ due to (1) and the fact that the infinitesimal character doesn’t change by Cayley transforms. Fix $S \in R_D$ and suppose that $S \neq \phi$. According to Theorem 6.4, each step of the wall-crossings in $\psi_S$ goes through the same Cayley transform as in $c_S$, and hence $\psi_S(Sh_i)$ and $c_S(Sh_i)$ are specified by the same Cartan subgroup for every $Sh_i \in SH$.

It remains to show that there is a $Sh_j \in SH$, $j \neq i$, such that $\psi_S(Sh_i)$ and $c_S(Sh_j)$ have the same central character. Let $\gamma$ be the regular character of $Sh_i$. The central character of $J(w_S \times \gamma)$ (as well as that of $\psi_S(Sh_i)$) can be specified by a nontrivial element $w_S(\rho/2) - \rho/2 \in P/R$ by Lemma 6.6. On the other hand, $P/R$ also parametrizes $\prod_2(Z(\tilde{G}))$ and also the set $\{c_S(Sh_i) | Sh_k \in SH\}$, which means that there exists $j \neq i$ such that the central character of $c_S(Sh_j)$ is parametrized by $w_S(\rho/2) - \rho/2$ and hence $c_S(Sh_j) = \psi_S(Sh_i)$. \qed

Denote the set we construct by

$$\prod_{R_D} \tilde{G} = \bigcup_{Sh_i \in SH} \{c_S(Sh_i) | S \in R_D\},$$

then the following Theorem follows immediately from Lemma 6.8.

**Theorem 6.9.** $\prod_{R_D} \tilde{G} \leq \prod^*_2(G)$.

**Remark 6.10.** If $|\prod_2(Z(\tilde{G})))| = p$, then $|\prod_{R_D} \tilde{G}| = p^2$.

### 6.4 Exhaustion of $\prod^*_2(G)$

We have shown that $\prod_{R_D} \tilde{G} \subseteq \prod^*_2(G)$. In this section we will show by counting the elements in $\prod^*_2(G)$ that this is in fact an equality when $G$ has type $A_{n-1}$ or $D_n$.

Fix a central character $\chi$ of $G$. Let $\prod^*_2(\tilde{G})_{\chi}$ be the subset of representations in $\prod^*_2(\tilde{G})$ with central character $\chi$. The goal is to count $|\prod^*_2(\tilde{G})_{\chi}|$. Take a block $B$ of representations with central character $\chi$ and infinitesimal character $\rho/2$, and consider $\prod(\rho/2)$, $\Delta(\rho/2)$, $W(\rho/2)$, the simple roots of the integral root system, the integral root system for $\rho/2$, and the integral Weyl group, respectively. Let $\mathbb{Z}[B]$ be the $\mathbb{Z}$-span of the set of standard modules $I(\gamma)$, where each $\gamma$ is a $\rho/2$-regular character in $B$. Then $W(\rho/2)$ acts on $\mathbb{Z}[B]$ by the coherent continuation action (see [29] or [30]) and this action is denoted by $w \cdot I(\gamma)$, or simply $w \cdot \gamma$ for $w \in W(\rho/2)$ and $\gamma \in B$.

Consider $\{J(\gamma) | \gamma \in B\}$, the set of irreducible quotients of $\{I(\gamma) | \gamma \in B\}$, as another basis of $\mathbb{Z}[B]$, we have

**Lemma 6.11.** Let $\alpha \in \prod(\rho/2)$, $\gamma \in B$, then $s_\alpha \cdot J(\gamma) = -J(\gamma)$ if and only if $\alpha \in \tau(J(\gamma))$. 

Proof. Let $\alpha \in \prod (\rho/2)$ and let $\lambda$ be an infinitesimal character which is singular for $\alpha$. Define a coherent family with $\pi(\rho/2) = J(\gamma)$. Then we have the identity

$$\pi(\rho/2) + \pi(s_\alpha(\rho/2)) = \psi^{\rho/2}_\lambda \circ \psi^\lambda_{\rho/2}(\pi(\rho/2)).$$

Notice that $\alpha \in \tau(J(\gamma))$ if and only if $\psi^\lambda_{\rho/2}(\pi(\rho/2)) = 0$, which is equivalent to $\psi^{\rho/2}_\lambda \circ \psi^\lambda_{\rho/2}(\pi(\rho/2)) = 0$, since the functor of push-to or push-off walls is injective. We conclude that $\alpha \in \tau(J(\gamma))$ if and only if $J(\gamma) = \pi(\rho/2) = -\pi(s_\alpha(\rho/2)) = -s_\alpha \cdot (J(\gamma))$ (the last equality holds by the definition of coherent continuation action). 

\begin{lemma}
\label{lem:6.12}
$|\prod^{\circ}(\tilde{G})_x| = \dim \operatorname{Hom}_{W(\rho/2)}(\operatorname{sgn}, Z[B])$
\end{lemma}

Proof. Let $\pi = J(\gamma), \gamma \in B$. Then by Lemma \ref{lem:6.11}, $\pi \in \prod^{\circ}(\tilde{G})_x$ if and only if $s_\alpha \cdot \pi = -\pi$ for all $\alpha \in \prod (\rho/2)$, which is equivalent to saying that $W(\rho/2)$ acts on $\pi$ as the sign representation. Thus the lemma follows.

Therefore, to count the left hand side, we just need to count the right hand side in Lemma \ref{lem:6.12}. More precisely, we need to analyze the $W(\rho/2)$-representation $Z[B]$ in order to count the right hand side.

The first observation is that it makes counting more convenient if we consider a special block $D$, which is equivalent to $B$, instead of $B$, but with an infinitesimal character other than $\rho/2$, and in a different chamber.

The following lemma tells what this block is.

\begin{lemma}
\label{lem:6.13}
Let $\rho = (\rho^+, \rho^-)$, and $\lambda_0 = \rho/2$. Let $\prod$ be the simple roots in the chamber of $\rho/2$. Then we can find $w \in W$ with the following properties. Let $(\Delta^+ = w\Delta^+$ and $\prod^w = \prod \{\alpha_i\}$. There is some $\alpha_k \in \prod^w$ such that if setting $\lambda = w\rho - \frac{1}{2}\lambda_k^w((\Delta^+)^-)$, where $\lambda_k^w$ is the fundamental weight for $\alpha_k$, we have (1) $\langle \lambda, \alpha_k^w \rangle = 1/2$ and $\langle \lambda, \alpha_i^w \rangle = 1$ elsewhere; (2) $\Delta(\lambda) = \Delta(\lambda_0)$, (and $\prod(\lambda) = \prod(\lambda_0)$ as well). Therefore, for type $A_{n-1}$, we can always move to a block $D$ (through nonintegral wall-crossing equivalence) with infinitesimal character $\lambda_D = \lambda$, such that every root in $\prod(\lambda)$ is simple for the entire root system; for type $D_{n}$, $n \geq 4, E_6, E_7, E_8$, we can move to a block $D$ with infinitesimal character $\lambda$, such that every root in $\prod(\lambda)$ is simple for the entire root system but one, say, $\alpha$.
\end{lemma}

Proof. This lemma is proved by a case by case calculation. 

For example, consider type $D_4$, $G = \text{Spin}(4,4)$. In this case, $\rho/2 = (\frac{1}{2}, 1, \frac{1}{2}, 0)$, $\prod = \{e_1 - e_2, e_2 - e_3, e_3 + e_4\}$, and $\prod(\rho/2) = \{e_1 \pm e_3, e_2 \pm e_4\}$. The infinitesimal character $\lambda$ to be chosen in Lemma \ref{lem:6.13} is $\lambda = (\frac{1}{2}, 1, \frac{1}{2}, 0)$ and $\prod^w$, the simple roots in this chamber, is $\{e_1 - e_3, e_3 - e_2, e_2 \pm e_4\}$, and it can be easily seen that $\prod(\lambda) = \prod(\rho/2)$ and the only simple integral root that is not simple in the entire root system is $\alpha = e_1 + e_3$.

Due to the equivalence of block $B$ and block $D$, we’ll focus on analyzing $Z[D]$ from now on and then count $\dim_{W(\lambda)}(\text{sgn}, Z[D])$.

Now take a closer look at the coherent continuation action of $W(\lambda)$ on $Z[D]$. The formulas of the coherent continuation action can be derived from the formula of the Hecke operators (see \cite{20}, Definition 9.4). More precisely, let $\beta$ be a simple root, consider $T_{s_\beta}$, which we denote $T_\beta$ for simplicity, an operator acting on $Z[q, q^{-1}][D]$.
(the formulas are given in \[20\]), and the coherent continuation action of \(s_\beta \in W\) can be defined as
\[
(6.1) \quad s_\beta \cdot \gamma = -T_\beta(1)(\gamma) \quad \text{with each term } \delta \text{ multiplied by } (-1)^{l(\gamma) - l(\delta)},
\]
where \(\gamma \in \mathcal{D}\) and \(l\) is a length function defined on parameters and it can be looked up in \[30\]. Notice that in \[6.1\], \(\beta\) could be nonintegral and hence \(s_\beta \cdot \gamma\) is possibly outside of \(\mathbb{Z}[\mathcal{D}]\). But if \(\beta\) is integral, \(s_\beta \in W(\lambda)\) and the following proposition gives explicit formulas for the action of \(W(\lambda)\) on \(\mathbb{Z}[\mathcal{D}]\), which can be derived from Definition 9.4 of \[20\] and \(6.1\).

**Proposition 6.14.** Fix \(\gamma \in \mathcal{D}\) and \(\beta \in \prod(\lambda)\). Furthermore, suppose \(\beta\) is simple in the whole root system. Let \(s := s_\beta \in W(\lambda)\).

(a) If \(\beta\) is complex or real for \(\gamma\), then \(s \cdot \gamma = s \times \gamma\).

(b) If \(\beta\) is compact imaginary for \(\gamma\), then \(s \cdot \gamma = -\gamma\).

(c) If \(\beta\) is noncompact imaginary for \(\gamma\), then \(s \cdot \gamma = -s \times \gamma + c_\beta(\gamma)\).

The formulas in Proposition 6.14 are the same as those for the linear case (or say, when the infinitesimal character is integral) (see Theorem 4.12 in \[28\]). Notice that in Proposition 6.14, \(\beta\) is simple for the whole root system; if \(\beta = \alpha\), the only simple integral root which is not simple for the whole root system (see Lemma 6.13), we will need an additional formula for \(s_\alpha \cdot \gamma\) and this will be discussed later.

From Proposition 6.14, we can see that the coherent continuation action is closely related to the cross action, so we also consider the cross action of \(W(\lambda)\) on \(\mathcal{D}\). Notice that two \(\lambda\)-regular characters \(\gamma_i = (\tilde{H}_i, \Gamma_i, \overline{\gamma_i})\) and \(\gamma_j = (\tilde{H}_j, \Gamma_j, \overline{\gamma_j})\) from \(\mathcal{D}\) are in the same cross action orbit if and only if \(\tilde{H}_i = \tilde{H}_j\). Indeed, if \(\tilde{H}_i = \tilde{H}_j = \tilde{H}\), then \(\Gamma_i\) and \(\Gamma_j\) agree on \(Z(\tilde{G})\), since \(Z(\tilde{H}) = Z(\tilde{G})\tilde{H}_0\) (by Proposition 5.4 (1)). Since \(\gamma_i\) and \(\gamma_j\) are in the same block, \(\overline{\gamma_i}\) and \(\overline{\gamma_j}\) define the same infinitesimal character, say, \(\overline{\gamma_i} \sim \overline{\gamma_j} \sim \lambda\) and hence \(\gamma_j = w \times \gamma_i\) for some \(w \in W(\lambda)\). Enumerate the Cartan subgroups of \(\tilde{G}\) as \(\{\tilde{H}_1, \cdots, \tilde{H}_r\}\), and pick a \(\lambda\)-regular character \(\gamma_j\) specified by \(\tilde{H}_j\), then \(\{\gamma_1, \cdots, \gamma_l\}\) is a set of representatives of the cross action orbits of \(W(\lambda)\) on \(\mathbb{Z}[\mathcal{D}]\).

Let \(W_{\gamma_j} = \{w \in W(\lambda) | w \times \gamma_j = \gamma_j\}\) be the cross stabilizer of \(\gamma_j\) in \(W(\lambda)\). Then we have the following proposition.

**Proposition 6.15.** \(\mathbb{Z}[\mathcal{D}] \simeq \bigoplus_j \text{Ind}_{W_{\gamma_j}}^{W(\lambda)}(\epsilon_j)\), where \(\epsilon_j\) is a one-dimensional representation of \(W_{\gamma_j}\) such that for \(w \in W_{\gamma_j}\), \(w \cdot \gamma_j = \epsilon_j(w) \gamma_j + \text{other terms from more split Cartan subgroups.}\)

**Proof.** The proof is similar to the linear case (cf. \[8\]) by using formulas in Definition 9.4 of \[20\] and \(6.1\). 

By Proposition 6.15 and Frobenius reciprocity, the multiplicity of \(\text{sgn}_{W(\lambda)}\) in \(\mathbb{Z}[\mathcal{D}]\) is \([\text{sgn}_{W(\lambda)} : \mathbb{Z}[\mathcal{D}]] = [\text{sgn}_{W(\lambda)}|_{W_{\gamma_j}} : \epsilon_j]\), which is equal to 0 or 1, since \(\text{sgn}_{W(\lambda)}|_{W_{\gamma_j}}\) is one-dimensional. This means that we have reduced our goal to count the number of \(\gamma_j\)'s making \([\text{sgn}_{W(\lambda)}|_{W_{\gamma_j}} : \epsilon_j] = 1\), which is called condition (\(\ast\)). Equivalently, \(\gamma_j\) satisfies condition (\(\ast\)) if
\[
\text{sgn}_{W(\lambda)}|_{W_{\gamma_j}} = \epsilon_j
\]

To reach this goal, we need to analyze \(\epsilon_j\) for each \(j\). By \[6\],
\[
W_{\gamma_j} = W^C(\overline{\gamma_j})^\theta \times (W^i(\overline{\gamma_j}) \times W^r(\overline{\gamma_j})).
\]
So we can decompose $\epsilon_j = \epsilon_j^C \otimes \epsilon_j^I \otimes \epsilon_j^\gamma$, where $\epsilon_j^C, \epsilon_j^I, \epsilon_j^\gamma$ are characters of $W^C(\tau_j^\gamma)$, $W^I(\tau_j^\gamma)$, $W^\gamma(\tau_j^\gamma)$ respectively. Notice that in the linear case (or say, when the block considered has integral infinitesimal character), we have $\epsilon_j = sgn_i$ for all $j$ (see [3]), where $sgn_i = sgn_{W^I(\tau_j^\gamma)}$. When $\widetilde{G}$ has type $A_{n-1}$, we are in a chamber where every $\beta \in \prod(\lambda)$ is simple, and then using formulas in Proposition 6.14 an argument analogous to the linear case shows that $\epsilon_j = sgn_i$ for every $j$. In fact, there is no compact imaginary root in the case of type $A_{n-1}$, and hence the factor $W^I(\tau_j^\gamma)$ in $W_{\gamma_j}$ is trivial and therefore $sgn_i = 1$. The following proposition follows.

**Proposition 6.16.** If $\widetilde{G}$ has type $A_n$, $\epsilon_j = 1$ for all $\gamma_j$.

The following lemma is an easy result of Proposition 6.16.

**Lemma 6.17.** For type $A_{n-1}$, $\gamma_j$ satisfies condition $(\ast)$ if and only if there are no real integral roots for $\gamma_j$.

Analyzing $\epsilon_j$ for type $D_n$, $n \geq 4$, requires more work. The formulas in Proposition 6.14 are not enough since there is an integral root $\alpha \in \prod(\lambda)$ which is not simple. So the first goal is to calculate $s_\alpha \cdot \gamma, \gamma \in D$. Notice that

\[
\prod(\lambda) = \prod(\rho/2) = \{ e_i - e_{i+2}, 1 \leq i \leq n-2, e_{n-3} + e_{n-1}, e_{n-2} + e_n \}
\]

\[
\prod = \begin{cases} \{ e_i - e_{i+2}, 1 \leq i \leq n-2, e_{n-1} + e_{n-2} + e_n \} & \text{if } n \text{ is even} \\
\{ e_i - e_{i+2}, 1 \leq i \leq n-2, e_{n-2} + e_{n-3} + e_n \} & \text{if } n \text{ is odd} \end{cases}
\]

Therefore, $\alpha = \begin{cases} e_{n-3} + e_{n-1} & \text{if } n \text{ is even} \\
\text{e}_{n-2} + e_n & \text{if } n \text{ is odd} \end{cases}$

We decompose $s_\alpha = s_{\alpha_1}s_{\alpha_2} \cdots s_{\alpha_m}$, a product of simple reflections.

\[
s_\alpha = \begin{cases} s_{n-1,n-2}s_{n-2,n-3}s_{n-1,n-3}s_{n-2,n-4} \cdots s_{2,n-1,n-2} & \text{if } n \text{ is even} \\
s_{n-2,n-3}s_{n-2,n-4} \cdots s_{2,n-1,n-2} & \text{if } n \text{ is odd} \end{cases}
\]

where $s_{i,j} = s_{e_i} - s_{e_j}$ and $s_{i,j} = s_{e_i} + s_{e_j}$.

Given $\gamma \in D$. By (6.1), to calculate $s_\alpha \cdot \gamma$, we have to calculate $T_\alpha(\gamma)$ first. Consider the decomposition $T_\alpha(\gamma) = T_{\alpha_1}T_{\alpha_2} \cdots T_{\alpha_m}(\gamma)$. Note that on the right hand side, the Hecke operation is calculated step by step. In each step, we have to deal with some $T_{\alpha_k}(\delta)$, where $\delta$ is the parameter of a standard module not necessarily belonging to block $D$. In fact, this is an "abstract" Hecke operation, and it should be denoted by $T_{\phi_{\alpha_k}}(\delta)$. Taking an inner automorphism $\phi_k$ of $g$ sending $(\lambda, h^\ast)$ to $(\overline{\delta}, h^\ast_{\delta})$, we define $T_{\phi_{\alpha_k}}(\delta) = T_{\phi_k(\alpha_k)}(\delta)$. Here $\phi_k(\alpha_k)$ is a simple root in the chamber of $\delta$, and hence we can use the formulas in Definition 9.4 of [20] to calculate $T_{\phi_k(\alpha_k)}(\delta)$ in each step.

\[
T_\alpha(\gamma) = T_{\alpha_1} \cdot \alpha_1 T_{\alpha_2} \cdot \alpha_2 \cdots T_{\alpha_m}(\gamma)
\]

\[
= (T_{\phi_1(\alpha_1)}(T_{\phi_2(\alpha_2)}(\cdots (T_{\phi_m(\alpha_m)}(\gamma)))) \cdots )
\]

\[
= p_1(q) \cdots p_m(q)\phi_1(\alpha_1) \times (\phi_2(\alpha_2 \times \cdots (\phi_m(\alpha_m) \times \gamma))
\]

+ (terms from more split Cartan subgroups),

where $p_j(q) \in \mathbb{Z}[q, q^{-1}]$. According to (6.1) and (30), it turns out that at each step, we may define

\[
s_{\alpha_k} \cdot \delta = -T_{\phi_k(\alpha_k)}(1)(\delta), \text{ if } \phi_k(\alpha_k) \text{ is real or imaginary for } \delta
\]
and

\[ s_\alpha \cdot \delta = T_{\phi_k}(\alpha_k)(1)(\delta), \]

if \( \phi_k(\alpha_k) \) is complex for \( \delta \).

Let \( t_\gamma \) be the number of occurrences imaginary roots in \( \{\phi_j(\alpha_j), 1 \leq j \leq m\} \). An easy calculation shows that

\[ (6.3) \quad s_\alpha \cdot \gamma = (-1)^{t_\gamma} s_\alpha \times \gamma + \text{(terms from more split Cartan subgroups)}. \]

Notice that \( \{\alpha_j\}_{j=1}^m \) can be read off from (6.2).

When \( n \) is even, \( m = 9 \) and

\[ \{\phi_j(\alpha_j)\} = \{e_{n-3} + e_{n-2}, e_{n-3} + e_n, e_{n-3} - e_n, e_{n-2} + e_{n-1}, e_{n-3} + e_{n-1}, e_{n-3} - e_{n-2}, e_{n-1} + e_n, e_{n-1} - e_n, e_{n-2} - e_{n-1}\}. \]

When \( n \) is odd, \( m = 11 \) and

\[ \{\phi_j(\alpha_j)\} = \{e_n - e_{n-2}, e_{n-3} + e_n, e_{n-3} + e_{n-2}, e_{n-3} - e_n, e_{n-1} - e_n, e_{n-2} + e_n, e_{n-2} - e_{n-1}, e_n - e_{n-1}, e_{n-2} + e_{n-3}, e_{n-2} - e_{n-3}, e_{n-2} - e_n\}. \]

**Example 6.18.** Consider type \( D_4 \) and \( \tilde{G} = \text{Spin}(4, 4) \). In this case, \( \lambda = (\frac{3}{2}, 1, \frac{3}{2}, 0), \) \( \alpha = e_1 + e_3 \). According to Section 6.2, we need a family of infinitesimal characters to define the cross action of \( W \) on \( D \) and it could be chosen to be \( F(\lambda) = \{(\frac{3}{2}, 1, \frac{3}{2}, 0), (\frac{1}{2}, 1, \frac{3}{2}, 0), (\frac{3}{2}, 2, \frac{1}{2}, 0)\} \). Let \( \gamma \in D \) be parametrized by \( (\frac{3}{2}, 1, \frac{3}{2}, 0) \), meaning that \( e_1 - e_2 \) is imaginary for \( \gamma \) and \( e_1 + e_2, e_3 \pm e_4 \) are real for \( \gamma \). We apply a sequence of cross actions to \( \gamma \) through the roots \( \{\phi_0(\alpha_0), \ldots, \phi_1(\alpha_1)\} \):

\[
\begin{align*}
\gamma &= \begin{pmatrix} \frac{5}{2} & 1 & 3 & 2 \end{pmatrix}, 0 \xrightarrow{s_{e_2-e_3}} \begin{pmatrix} 3 & 1 & 2 & 1 \end{pmatrix}, 0 \xrightarrow{s_{e_3-e_4}} \begin{pmatrix} 3 & 2 & 0 & 1 \end{pmatrix}, 0 \xrightarrow{s_{e_3+e_4}} \begin{pmatrix} 3 & 1 & 1 & 2 \end{pmatrix}, 0 \xrightarrow{s_{e_1-e_2}} \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix}, 0 \xrightarrow{s_{e_1+e_2}} \begin{pmatrix} 3 & 2 & 0 & 0 \end{pmatrix}, 0 \xrightarrow{s_{e_1+e_4}} \begin{pmatrix} 1 & 2 & 3 & 0 \end{pmatrix}, 0 \xrightarrow{s_{e_2+e_3}} \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}, 0 \xrightarrow{s_{e_1-e_4}} \begin{pmatrix} 0 & 3 & 0 & 0 \end{pmatrix}, 0 \xrightarrow{s_{e_2+e_4}} \begin{pmatrix} 0 & 3 & 1 & 2 \end{pmatrix}, 0 \xrightarrow{s_{e_1-e_4}} \begin{pmatrix} 0 & 3 & 2 & 1 \end{pmatrix}, 0 \xrightarrow{s_{e_1-e_2}} \begin{pmatrix} 0 & 3 & 2 & 1 \end{pmatrix}, 0 = s_\alpha \times \gamma
\end{align*}
\]

Notice that we vary the chambers and fix the types of roots at each step. It can be seen that \( \phi_0(\alpha_0) = e_1 - e_2 \) is the only imaginary root for \( \gamma \) among \( \{\phi_j(\alpha_j)\} \) and hence \( t_\gamma = 1 \). We conclude that

\[ s_\alpha \cdot \gamma = -(s_\alpha \times \gamma) + \text{(terms from more split Cartan subgroups)} \]

and hence \( e(s_\alpha) = -1 \).

Due to the remark above Proposition 6.15, we can choose each \( \gamma_j \) properly and calculate the \( \epsilon_j \)'s according to the chosen \( \gamma_j \)'s. In fact, our goal is to rule out \( \gamma_j \)'s satisfying either of the following conditions:

- (R) If there is a real integral root, then choose \( \gamma_j \) such that \( \alpha \) is real for \( \gamma_j \).
- (C) If there are no real integral roots, and there is an orthogonal set of \( 4 \) nonintegral roots of the form \( \{e_p \pm e_q, e_r \pm e_s\} \), where \( e_p \pm e_q \) are both imaginary, or both real, and one of \( \{e_r \pm e_s\} \) is real, whereas the other is imaginary, then choose \( \gamma_j \) such that \( \{e_n-3 \pm e_{n-2}, e_{n-1} \pm e_n\} \) is the desired quadruple. In this case \( \alpha \) is a complex root.

**Remark 6.19.** The \( \gamma \) chosen in Example 6.18 is a parameter satisfying condition (C).

With the setting, we have the following key lemma.

**Lemma 6.20.** Suppose that \( \tilde{G} \) has type \( D_n \), \( n \geq 4 \). If \( \gamma_j \) satisfies condition (R) or (C) then \( \gamma_j \) doesn't satisfy condition (*).
Proof. To show that the chosen $\gamma_j$ fails to satisfy condition (*), we will pick a $w \in W_{\gamma_j}$ and show that $\epsilon_j(w)$ and $\text{sgn}(w)$ do not coincide. In either case, we have to calculate $\epsilon_j(s_\alpha)$ for $\gamma_j$. By (6.3), we just need to count the number $t_{\gamma_j}$ for the chosen $\gamma_j$.

Suppose that $n$ is even. If $\gamma_j$ satisfies condition (R), $e_{n-3} - e_{n-1}$ is also a real integral root, and the roots $e_{n-2} \pm e_{n-1}, e_{n-2} \pm e_{n-3}, e_n \pm e_{n-1}, e_n \pm e_{n-3}$ can be arranged so that each of them is either real or complex. Therefore, $t_{\gamma_j} = 0$, and hence $\epsilon_j(s_\alpha) = 1$. This result follows for the odd case by applying the same argument. Since $s_\alpha \in W_{\gamma_j}$ and $\text{sgn}(s_\alpha) = -1$, $\gamma_j$ fails to satisfy condition (*).

Now suppose that $\gamma_j$ does not satisfy condition (R), but satisfies condition (C). It can be counted that $t_{\gamma_j} = 1$ or $3$ (see Example 6.18 for the calculation), which implies $\epsilon_j(s_\alpha) = -1$. Let $w = s_{e_{n-3}}s_{e_{n-1}}s_{e_{n-2}}s_{e_{n-3}}s_{e_{n-2}}s_{e_{n-1}}e_{n}$. Notice that $w \in W_{\gamma_j}$. When $n$ is even (odd, respectively), we have $e_{n-3} - e_{n-1}, e_{n-2} \pm e_{n}$ ($e_{n-3} \pm e_{n-1}, e_{n-2} - e_{n}$, respectively) are simple and complex, so $\epsilon_j(s_\beta) = 1$ for every $\beta$ from these three roots, and hence $\epsilon_j(w) = \epsilon_j(s_\alpha) = -1$. But it is obvious that $\text{sgn}(w) = 1$. Therefore $\gamma_j$ fails to satisfy condition (*). □

**Theorem 6.21.** For type $A_{n-1}$ and $D_n$, $n \geq 4$, we have $\prod_{D}(G) = \prod_{R_D}(G)$.

Proof. It suffices to count the number of $\gamma_j$'s in Proposition 6.15 satisfying condition (*). Fixing a genuine central character, we shall show that this number is 2 (1, respectively) for type $A_{n-1}$, when $n$ is even (odd, respectively), and it is 4 (2, respectively) for type $D_n$, when $n$ is even (odd, respectively). We will give the proof for the even case only, and a similar argument applies for the odd case.

For type $A_{n-1}$, we claim that if the real rank of the Cartan subgroup $H_j$ is at least $n/2$ (when $n$ is even) or $(n-1)/2$ (when $n$ is odd), then there exists a real integral root for $\gamma_j$, and hence such $\gamma_j$ can be ruled out by Lemma 6.17.

When $n$ is even, we enumerate all Cartan subgroups (on the level of linear groups) as $\{H_{n/2-1}, H_{n/2}, \cdots, H_{n-2}, H_{n-1}\}$, where the real rank of $H_j$ is $j$. Let $\gamma_{n-1}$ be the parameter of the principal series, and $e_k = e_{2k-1} - e_{2k}, 1 \leq k \leq n/2$, then we pick $\gamma_{n-1-k} = c_{\alpha_k} \cdots c_{\alpha_1} (\gamma_{n-1})$ to be the representative of the cross action orbit specified by $H_{n-1-k}, 1 \leq k \leq n/2$. Notice that when $k \leq n/2 - 2$, $e_{n-2} - e_n$ is a real integral root for $\gamma_{n-1-k}$, which means that we can rule out $\gamma_j$, for $n/2 + 1 \leq j \leq n - 1$. Only $\gamma_{n/2-1}$ and $\gamma_{n/2}$ are not ruled out, and they are exactly the $\gamma_j$'s satisfying condition (*) since the number of $\prod_{R_D}(G)$ with a fixed central character is also 2. Hence the theorem follows for type $A_{n-1}$, when $n$ is even. When $n$ is odd, it can be shown by a similar argument that the only $\gamma_j$ satisfying condition (*) comes from the fundamental Cartan.

For type $D_n$, when $n$ is even, we enumerate all Cartan subgroups (on the level of linear groups) as $\{H^d_j, 0 \leq j \leq n\}$, where the real rank of $H^d_j$ is $j$, and we use the superscript $d$ to distinguish Cartan subgroups of the same real rank but not conjugate to each other. For example, when $n = 4$, there are three Cartan subgroups of real rank 2, and they are labeled by $H^d_1, H^d_2, H^d_3$, all of which are isomorphic to $\mathbb{R}^x \times S^1 \times \mathbb{C}^\times$.

Let $\gamma_n$ be the parameter of the principal series. We start with a set of orthogonal nonintegral real roots $R(\gamma_n) = \{\alpha_k, \beta_k, 1 \leq k \leq n/2\}$ of $\gamma_n$, where $\alpha_k = e_{2k-1} - e_{2k}, \beta_k = e_{2k-1} + e_{2k}$, and obtain $\gamma^d_j$ by taking Cayley transforms through the roots in $R(\gamma_n)$. We attach to each $\gamma^d_j$ a set of real roots $R(\gamma^d_j) = \{\beta \in R(\gamma_n) | \beta \text{ is real for } \gamma^d_j\}$. Now let $\gamma_0$ be the parameter of the discrete series with
$R(\gamma_0) = \phi$, $\gamma_1^2$ be the parameter two steps up from $\gamma_0$, with $R(\gamma_1^2) = \{\alpha_{n/2}, \beta_{n/2}\}$, $\gamma_{n/2}^2$ be the parameter with $R(\gamma_{n/2}^2) = \{\beta_1, \cdots, \beta_{n/2}\}$, and $\gamma_{n/2}^3$ be the parameter with $R(\gamma_{n/2}^3) = \{\beta_1, \cdots, \beta_{n/2-1}, \alpha_{n/2}\}$. Observe that when $n = 4$, $\gamma_1^2$ is the representative from the middle Cartan subgroup $H_1^2$; when $n > 4$, choose $\gamma_{n/2}^1$ to be the representative from $H_{n/2}^1$ with $R(\gamma_{n/2}^1) = \{\beta_2, \cdots, \beta_{n/2}, \alpha_{n/2}\}$. Note that it is possible that there exists $\gamma_{n/2}^2$, $d > 3$. In this case, choose $\gamma_{n/2}^d$ such that $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\} \subseteq R(\gamma_{n/2}^d)$.

Now we claim that if $\gamma_j^d$ is not one of these four, then it satisfies either condition (R) or (C), and hence can be ruled out by Lemma 6.20.

When $j \geq n/2+2$, $\gamma_j^d$ can be chosen so that $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\} \subseteq R(\gamma_j^d)$ and hence $e_{n-2} \pm e_n$ are real integral roots of $\gamma_j^d$.

Now suppose $n > 4$. We observe that $\gamma_{n/2}^1$ satisfies condition (C) since there is an $n$-tuple $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\}$, where $\alpha_{n/2}, \beta_{n/2}, \beta_{n/2-1}$ are real, and $\alpha_{n/2-1}$ is imaginary for $\gamma_{n/2}^1$. For $d > 3$, since $\{\alpha_{n/2-1}, \alpha_{n/2}, \beta_{n/2-1}, \beta_{n/2}\} \subseteq R(\gamma_{n/2}^d)$, $e_{n-2} \pm e_n$ are real imaginary roots of $\gamma_{n/2}^1$, and hence $\gamma_{n/2}^1$ satisfies condition (R).

Any $\gamma_{n/2+1}^d$ is obtained from some $\gamma_{n/2}^d$ by an inverse Cayley transform, that is, we can choose $\gamma_{n/2+1}^d$ such that $R(\gamma_{n/2+1}^d)$ is obtained from $R(\gamma_{n/2}^d)$ by adding a root. But adding a root to $R(\gamma_{n/2}^d)$ would result in either a real integral roots or a quadruple as described in condition (C) for $\gamma_{n/2+1}^d$.

Finally we observe $\gamma_j^d$, $j > n/2$. Every $\gamma_{n/2}^d$ can be obtained from some $\gamma_{n/2}^d$ by a sequence of Cayley transforms through roots in $R(\gamma_{n/2}^d)$, that is, $\gamma_j^d$ is chosen such that $R_j^d$ is obtained by removing roots from $R(\gamma_{n/2}^d)$. It turns out that when $j > n/2$, there would be a quadruple as described in condition (C) for all $\gamma_{n/2}^d$, except $\gamma_0$ and $\gamma_1^2$. We conclude that $\gamma_{n/2}^1$, $\gamma_{n/2}^2$, $\gamma_{n/2}^3$ are the $\gamma_j$’s satisfying condition $(\ast)$ since the number of $\prod_{R_D(G)}(\tilde{G})$ with a fixed central character is exactly 4. Hence the theorem follows for type $D_n$, $n \geq 4$, when $n$ is even.

When $n$ is odd, it can be shown by a similar argument that the two $\gamma_j$’s satisfying condition $(\ast)$ come from the compact Cartan subgroup and a Cartan subgroup with real rank 2.

We would like to do the same thing for type $E$, parallel to the case of type $D_n$. Like in type $D_n$, we can also move to a block $D$ where all integral simple roots are simple but one, say $\alpha$. Then there come some difficulties. First, to decompose $s_\alpha$ into a product of simple reflections is never easy, and after having done that, we have to keep track of a sequence of inner automorphisms when trying to calculate the coherent continuation action $s_\alpha \cdot \gamma$, where $\gamma$ is a standard module parameter. Even though this complication has not been solved yet, we strongly believe that the counting $|\prod_{\rho/2}(\tilde{G})| = |\prod_{R_D}(\tilde{G})|$ in Theorem 6.21 holds for type $E_6$, $E_7$ and $E_8$.

We conjecture that when we count the number of $\gamma_j$ satisfying condition $(\ast)$, those satisfying condition (R) or (C) should be ruled out and hence Theorem 6.21 is true for type $E$, that is, Conjecture 6.23 would be true if Conjecture 6.22 is true.
Conjecture 6.22. For type $E_6$, $E_7$, and $E_8$, $\gamma_j$ does not satisfy condition (\$) if it satisfy condition (R) or (C).

Conjecture 6.23. For type $E_6$, $E_7$, and $E_8$, we have $\prod_s^* (G) = \prod_{R_0}^* (G)$.

7. Relation to the pairs $(\chi, \mathcal{O}_R)$

In this section, the setting is the same as in Section 6, that is, $G$ is simply laced and split. We define

\[ P_0(G) = \{ (\chi_i, O_J) | \chi_i \in \prod_s (Z(G)), O_J \text{ is a real form of } O \} . \]

Due to Proposition 6.1(3) and Remark 6.10, $|P_0(G)| = |\prod_{R_0}^* (G)|$, and this number is equal to $\prod_{\rho/2}^* (G)$ when $G$ has type $A_{n-1}$ or $D_n$. We will take a look at the correspondence between $\prod_{\rho/2}^* (G)$ and $P_0(G)$ in the cases of type $A$ and $D$.

7.1. Type $A_{n-1}$. When $n$ is odd, $\prod_{\rho/2}^* (G) = \mathcal{SH}$, which contains a single Shimura representation, and hence $|P_0(G)| = 1$. The mapping $\xi$ in (7.1) is nothing but a one-to-one correspondence.

When $n$ is even, say, $n = 2m$, $\prod_{\rho/2}^* (G) = \{ Sh_1, Sh_2, \pi_1, \pi_2 \}$, where $\pi_1$ and $\pi_2$ are constructed in Section 6.3 and representations with the same subscript have the same central character. The $K$-types of these representations are calculated in 15, and we list them as follows. By convention, the $K$-types of $Sh_1$ and $Sh_2$ are

\[ \left( \frac{1}{2} + 2a_1, \cdots, \frac{1}{2} + 2a_m \right), a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, a_i \in \mathbb{Z} \] and

\[ \left( \frac{1}{2} + 2a_1, \cdots, \frac{1}{2} + 2a_m-1, \left( \frac{1}{2} + 2a_m \right) \right), a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, a_i \in \mathbb{Z}, \]

respectively. Because $\pi_i$ and $Sh_i$ have the same central character, when $m$ is even, the $K$-types of $\pi_1$ and $\pi_2$ are

\[ \left( \frac{3}{2} + 2a_1, \cdots, \frac{3}{2} + 2a_m, a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, a_i \in \mathbb{Z} \] and

\[ \left( \frac{3}{2} + 2a_1, \cdots, \frac{3}{2} + 2a_m-1, \left( \frac{3}{2} + 2a_m \right) \right), a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, a_i \in \mathbb{Z}, \]

respectively; when $m$ is odd, the $K$-types of $\pi_1$ and $\pi_2$ are

\[ \left( \frac{3}{2} + 2a_1, \cdots, \frac{3}{2} + 2a_m-1, \left( \frac{3}{2} + 2a_m \right) \right), a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, a_i \in \mathbb{Z} \] and

\[ \left( \frac{3}{2} + 2a_1, \cdots, \frac{3}{2} + 2a_m, a_1 \geq a_2 \geq \cdots \geq a_m \geq 0, a_i \in \mathbb{Z}, \]

respectively. Observe that $Sh_i$ and $\pi_i, i = 1, 2$, have the same asymptotic $K$-types when $m$ is odd; $Sh_1$ and $\pi_2$ have the same asymptotic $K$-types and $Sh_2$ and $\pi_1$ have the same asymptotic $K$-types when $m$ is even. By the convention made in Remark 6.2 we have Table 5 and 6 to illustrate the correspondence $\xi$.

| Table 5. $A_{2m-1}$, $m$ even |
|-----------------------------|
| $\chi_1$ | $\mathcal{O}_1$ | $\mathcal{O}_2$ |
| $\chi_1$ | $Sh_1, \pi_1$ | N/A |
| $\chi_2$ | N/A | $Sh_2, \pi_2$ |
Table 6. \(A_{2m-1}, m \text{ odd}\)

| \(\gamma\)  | \(\mathcal{O}_1\) | \(\mathcal{O}_2\) |
|-----------|-----------------|-----------------|
| \(\chi_1\) | \(\frac{3}{2}, \cdots, \frac{3}{2}\) | \(\pi_1\) |
| \(\pi_1\) | \(\frac{1}{2}, \cdots, \frac{1}{2}\) | \(\pi_1\) |
| \(\chi_2\) | \(\frac{1}{2}, \cdots, \frac{1}{2}\) | \(\pi_2\) |
| \(\pi_2\) | \(\frac{1}{2}, \cdots, \frac{1}{2}\) | \(\pi_1\) |

We conclude that \(\xi\) is a bijection if only if \(n \equiv 1, 2, 3 \pmod{4}\).

7.2. Type \(D_n\). In this case, the double cover \(\tilde{G} = \widetilde{\text{Spin}}(n, n)\). In [16], there are some small representations defined for a bigger group \(\tilde{G}' = \text{Spin}(n+1, n)\) with maximal compact subgroup \(\tilde{K}' = \text{Spin}(n+1) \times \text{Spin}(n)\). When \(n = 2m\), there are four of them, denoted \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\); when \(n = 2m+1\), there are two of them, denoted \(\Gamma_1, \Gamma_2\). Their \(K\)-types are calculated in [10].

Let \(\tilde{K} = \text{Spin}(2n) \times \text{Spin}(2n)\), the maximal compact subgroup of \(\tilde{G}\). Suppose \(n = 2m\). Then \(\text{Out}(\tilde{G})\) is generated by two elements \(\sigma\) and \(\gamma\) and the action of them on \(K\)-types are as follows.

\[
\begin{align*}
\sigma(\lambda_1, \cdots, \lambda_m; \lambda_{m+1}, \cdots, \lambda_n) &= (\lambda_1, \cdots, -\lambda_m; \lambda_{m+1}, \cdots, -\lambda_n), \\
\gamma(\lambda_1, \cdots, \lambda_m; \lambda_{m+1}, \cdots, \lambda_n) &= (\lambda_{m+1}, \cdots, \lambda_n; \lambda_1, \cdots, \lambda_m)
\end{align*}
\]

The \(K\)-types parametrized by \((\lambda; \chi')\) and \((\lambda'; \lambda)\) represent different representations when restricted to \(\tilde{K}\) since \(\gamma\) is in \(\text{Out}(\tilde{G})\), and hence restricting these \(\Gamma_i\)’s to \(\tilde{G}\), we get 16 representations (see Table 9 and 10 for the labeling for these representations and their \(K\)-types), and they are the small representations in \(\prod_{s/2}(\tilde{G})\). Notice that in Table 9 and 10, \(\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n\) and \(\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n\) are decreasing sequences of nonnegative integers. By \(\gamma < \lambda\) we mean that \(\lambda_1 \geq \gamma_1 \geq \cdots \geq \lambda_n \geq \gamma_n \geq -\lambda_n\). Also, \(0 = (0, \cdots, 0), 1 = (1, \cdots, 1), \frac{1}{2} = (\frac{1}{2}, \cdots, \frac{1}{2})\).

Suppose \(n = 2m+1\). Then \(\text{Out}(\tilde{G})\) is generated by \(\sigma\) (defined as above). Restricting \(\Gamma_i\)’s to \(\tilde{G}\) gives four representations and they are the ones in \(\prod_{s/2}(\tilde{G})\) (see Table 11 for their \(K\)-types). Notice that in Table 11, \(\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n\), \(\lambda' = (\lambda_1, \cdots, \lambda_n, 0) \in \mathbb{Z}^{n+1}\) and \(\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n\) are decreasing sequences of nonnegative integers. By \(\gamma < \lambda'\) we mean that \(\lambda_1 \geq \gamma_1 \geq \cdots \geq \lambda_n \geq \gamma_n \geq 0\).

Observing the asymptotic \(K\)-types of the representations in Table 9, 10 and 11, it is illustrated in Table 7 and 8 that \(\xi\) is a bijection.

The following theorem follows immediately from the above discussion.

**Theorem 7.1.** Assume the setting in this section. If \(G\) has type \(A_{n-1}\), then \(\xi\) is a one-to-one correspondence between \(\prod_{s/2}(\tilde{G})\) and \(P_\mathcal{O}(\tilde{G})\).

Notice that for type \(E_6\) and \(E_8\), \(\xi\) is a one-to-one correspondence between \(\prod_{R_0}(\tilde{G})\) and \(P_\mathcal{O}(\tilde{G})\) since both of them contain a single Shimura representation, and hence \(\xi\) is surjective map from \(\prod_{s/2}(\tilde{G})\) to \(P_\mathcal{O}(\tilde{G})\). Because of Conjecture 6.23 we conjecture that Theorem 7.1 is true for type \(E\).

**Conjecture 7.2.** Assume the setting in this section. If \(G\) has type \(E_6, E_7\) or \(E_8\), then \(\xi\) is a one-to-one correspondence between \(\prod_{s/2}(\tilde{G})\) and \(P_\mathcal{O}(\tilde{G})\).
Table 7. $D_{2m}$

| $\chi_1$ | $\psi_1$ | $\varphi_1$ | $\delta_1$ | $\tau_1$ |
|----------|----------|----------|----------|----------|
| $(\frac{1}{2}, \ldots, \frac{1}{2}, 1, \ldots, 1)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 1, \ldots, 1)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 1, \ldots, 1)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 1, \ldots, 1)$ |

| $\chi_2$ | $\varphi_2$ | $\delta_2$ | $\tau_2$ | $\psi_2$ |
|----------|----------|----------|----------|----------|
| $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ |

| $\chi_3$ | $\delta_3$ | $\tau_3$ | $\psi_3$ | $\varphi_3$ |
|----------|----------|----------|----------|----------|
| $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ |

| $\chi_4$ | $\varphi_4$ | $\delta_4$ | $\tau_4$ | $\psi_4$ |
|----------|----------|----------|----------|----------|
| $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ | $(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2})$ |

Table 8. $D_{2m+1}$

| $\chi_1$ | $\psi_1$ | $\varphi_1$ | $\delta_1$ | $\tau_1$ |
|----------|----------|----------|----------|----------|
| $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ | $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ |

Table 9. All $K$-types of representations in $\prod_{n/2}(G)$ when $G = Spin(n, n), n = 2m (1)$

| $G'$-rep. | $K'$-type | $L, K', T$ | restriction to $K$ | $L, K, T$ | $G$-rep |
|-----------|-----------|------------|-------------------|------------|---------|
| $\Gamma_1$ | $V^+ = \bigoplus_{\lambda} (\sigma(\lambda + \frac{1}{2}); \lambda)$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $Sh_3$ |
| $\Gamma_1$ | $V^+ = \bigoplus_{\lambda} (\sigma(\lambda + \frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $Sh_3$ |
| $\Gamma_2$ | $V^- = \bigoplus_{\lambda} (\sigma(\lambda + \frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $Sh_3$ |
| $\Gamma_2$ | $V^- = \bigoplus_{\lambda} (\sigma(\lambda + \frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $(0, \sigma(\frac{1}{2}))$ | $Sh_3$ |
| $G'$-rep | $K'$-type | $L.K'.T$ | restriction to $K$ | $L.K'.T$ | $G$-rep |
|---|---|---|---|---|---|
| $\Gamma_3$ | $V_0^+ = \bigotimes_{\lambda} (\lambda + 1; \lambda + 1)$ | $\frac{1}{2}$: 1 | $\bigotimes_{\lambda < \lambda} (\gamma + \frac{1}{2}; \lambda + 1), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | $\delta_1(m \text{ even})/\delta_2(m \text{ odd})$ | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\gamma + \frac{1}{2}); \lambda + 1), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\gamma + \frac{1}{2}; \lambda + 1), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\gamma + \frac{1}{2}); \lambda + 1, \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | | | |
| $\Gamma_3$ | $V_0^- = \bigotimes_{\lambda} (\lambda + 1; \lambda + 1)$ | $\frac{1}{2}$: 2 | $\bigotimes_{\lambda < \lambda} (\lambda + 1 + \frac{1}{2}, \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | $\delta_3(m \text{ even})/\delta_4(m \text{ odd})$ | |
| | | | $\bigotimes_{\lambda < \lambda} (\lambda + 1, \sigma(\gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\lambda + 1; \gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\lambda + 1; \sigma(\gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | | | |
| $\Gamma_4$ | $V_0^- = \bigotimes_{\lambda} (\lambda + 1; \sigma(\lambda + 1))$ | $\frac{1}{2}$: 1 | $\bigotimes_{\lambda < \lambda} (\gamma + \frac{1}{2}; \sigma(\lambda + 1), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | $\delta_5(m \text{ even})/\delta_6(m \text{ odd})$ | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\gamma + \frac{1}{2}; \sigma(\lambda + 1)), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\gamma + \frac{1}{2}; \sigma(\lambda + 1), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\gamma + \frac{1}{2}; \sigma(\lambda + 1)), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | | | |
| $\Gamma_4$ | $V_0^- = \bigotimes_{\lambda} (\sigma(\lambda + 1); \lambda + 1)$ | $\frac{1}{2}$: 2 | $\bigotimes_{\lambda < \lambda} (\sigma(\lambda + 1); \gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | $\delta_7(m \text{ even})/\delta_8(m \text{ odd})$ | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\lambda + 1); \sigma(\gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\lambda + 1); \sigma(\gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | | | |
| | | | $\bigotimes_{\lambda < \lambda} (\sigma(\lambda + 1); \sigma(\gamma + \frac{1}{2}), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | | | |
Table 11. All $K$-types of representations in $\prod_{\rho/2}(\tilde{G})$, when $G = Spin(n,n)$, $n = 2m + 1$

| $G$-rep | $K$-type | restriction to $K$ | $L.K:T$ | $G$-rep |
|---------|----------|-------------------|---------|---------|
| $\Gamma$ | $V = \bigoplus_{\lambda} (\lambda; \lambda + \frac{1}{2})$ | | $(0, \frac{1}{2}, 0)$ | $Sh_2$ |
| $\Gamma$ | $V = \bigoplus_{\lambda} (\lambda; \lambda + \frac{1}{2})$ | $\bigoplus_{\lambda \lambda' \gamma} (\lambda; \lambda + \frac{1}{2}, \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | $(0, \frac{1}{2}, 0)$ | $\pi_1$ |
| | | $\bigoplus_{\lambda \lambda' \gamma} (\lambda; \lambda + \frac{1}{2}, \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | $(0, \frac{1}{2}, 0)$ | $Sh_1$ |
| | | $\bigoplus_{\lambda \lambda' \gamma} (\lambda; \lambda + \frac{1}{2}, \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z}$ | $(\frac{1}{2}, 0)$ | $\pi_2$ |
| | | $\bigoplus_{\lambda \lambda' \gamma} (\lambda; \lambda + \frac{1}{2}, \gamma), \Sigma(\lambda_i + \gamma_i) \in 2\mathbb{Z} + 1$ | $(\frac{1}{2}, 0)$ | |

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