WHEN IS THE FRAME OF NUCLEI SPATIAL: A NEW APPROACH

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Dedicated to the memory of Harold Simmons

Abstract. For a frame $L$, let $X_L$ be the Esakia space of $L$. We identify a special subset $Y_L$ of $X_L$ consisting of nuclear points of $X_L$, and prove the following results:

- $L$ is spatial iff $Y_L$ is dense in $X_L$.
- If $L$ is spatial, then $N(L)$ is spatial iff $Y_L$ is weakly scattered.
- If $L$ is spatial, then $N(L)$ is boolean iff $Y_L$ is scattered.

As a consequence, we derive the well-known results of Beazer and Macnab [1], Simmons [22], Niefield and Rosenthal [13], and Isbell [10].

1. Introduction

Nuclei play an important role in pointfree topology as they are in 1-1 correspondence with onto frame homomorphisms, and hence describe sublocales of locales [6, 9]. For each frame $L$, let $N(L)$ be the set of nuclei on $L$. There is a natural order on $N(L)$ given by

$$ j \leq k \text{ iff } ja \leq ka \text{ for each } a \in L. $$

With this order $N(L)$ is also a frame [9, 11, 11], which we will refer to as the frame of nuclei or the assembly of $L$. The complicated structure of $N(L)$ has been investigated by many authors; see for example [6, 9, 21, 11, 11, 13, 10, 24, 15, 16, 5, 8, 23, 4].

To describe some of the landmark results about $N(L)$, we recall that a frame $L$ is spatial if it is isomorphic to the frame $\mathcal{O}S$ of open sets of a topological space $S$. For a subspace $T$ of $S$, a point $x \in T$ is isolated in $T$ if \{x\} is an open subset of $T$, and weakly isolated in $T$ if there is an open subset $U$ of $S$ such that $x \in T \cap U \subseteq \{x\}$. It is clear that an isolated point of $T$ is weakly isolated in $T$.

The space $S$ is scattered if each nonempty subspace of $S$ contains an isolated point. It is easy to see that $S$ is scattered iff each nonempty closed subspace of $S$ has an isolated point. The space $S$ is weakly scattered if each nonempty closed subspace has a weakly isolated point. It is well known that $S$ is scattered iff $S$ is weakly scattered and $T_D$, where $S$ is $T_D$ if each point is the intersection of an open and a closed set (see, e.g., [14, Sec. VI.8.1]).

We next describe some of the landmark results about $N(L)$. 

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Key words and phrases. Frame, nucleus, spatial frame, booleanization, Priestley space, Esakia space, scattered space, weakly scattered space.
• Beazer and Macnab [1] proved that if \( L \) is boolean, then \( N(L) \) is isomorphic to \( L \), and gave a necessary and sufficient condition for \( N(L) \) to be boolean.

• Simmons [22] proved that if \( S \) is a \( T_0 \)-space, then \( N(\emptyset S) \) is boolean iff \( S \) is scattered; and that dropping the \( T_0 \) assumption results in the following more general statement: \( N(\emptyset S) \) is boolean iff \( S \) is dispersed (see Section 7 for the definition).

• Simmons [22, Thm. 4.4] also gave a necessary and sufficient condition for \( S \) to be weakly scattered. This result of Simmons is sometimes stated erroneously as follows: \( N(\emptyset S) \) is spatial iff \( S \) is weakly scattered (see, e.g., [13, p. 267]). While this formulation is false (see Example 7.8), Isbell [10] proved that if \( S \) is a sober space, then indeed \( N(\emptyset S) \) is spatial iff \( S \) is weakly scattered.

• Niefield and Rosenthal [13] gave necessary and sufficient conditions for \( N(L) \) to be spatial, and derived that if \( N(L) \) is spatial, then so is \( L \).

Nuclei on \( L \) can be studied by utilizing Priestley duality [17] for distributive lattices and Esakia duality [7] for Heyting algebras. This was done independently in [20] and [5]. While the authors of [20] did not use Esakia duality, it was utilized in [5] where it was shown that nuclei on a Heyting algebra \( L \) correspond to special closed subsets of the Esakia space \( X_L \) of \( L \) (see Section 4 for the definition). In this paper we term such closed subsets nuclear. In [20] these were called \( L \)-sets. If \( N(X_L) \) denotes all nuclear subsets of \( X_L \), then we utilize the dual isomorphism between \( N(L) \) and \( N(X_L) \) to give an alternate proof of the results mentioned in the previous paragraph. We single out a subset \( Y_L \) of \( X_L \) consisting of nuclear points of \( X_L \) and show that \( L \) is spatial iff \( Y_L \) is dense in \( X_L \) (see also [20, Sec. 2.11]). We prove that join-prime elements of \( N(X_L) \) are exactly the singletons \( \{y\} \) where \( y \in Y_L \). From this we derive that there is a bijection between the points of \( L \) and the points of \( N(L) \). We also obtain a characterization of when \( N(L) \) is spatial in terms of \( X_L \), which yields an alternate proof of the results of Niefield and Rosenthal [13].

We prove that \( N(L) \) is boolean iff the set of maximal points of each clopen downset of \( X_L \) is clopen. From this we derive an alternate proof of the result of Beazer and Macnab [1]. We next turn to the setting of \( L = \emptyset S \) for some topological space \( S \). We show that \( Y_{\emptyset S} \) is homeomorphic to the soberification of \( S \), and prove that \( N(\emptyset S) \) is spatial iff \( Y_{\emptyset S} \) is weakly scattered. This implies that \( N(\emptyset S) \) is spatial iff the soberification of \( S \) is weakly scattered. As a corollary we obtain the result of Isbell [10] that if \( S \) is sober, then \( N(\emptyset S) \) is spatial iff \( S \) is weakly scattered. We give an example showing that this result is false if \( S \) is not assumed to be sober.

We finally turn to the results of Simmons [22]. One of Simmons’ main tools is the use of the front topology. We show that if \( S \) is \( T_0 \), then \( X_L \) is a compactification of \( S \) with respect to the front topology on \( S \). From this, by utilizing the \( T_0 \)-reflection, we derive Simmons’ characterization [22, Thm. 4.4] of arbitrary (not necessarily \( T_0 \)) weakly scattered spaces. In addition, we prove that \( N(L) \) is boolean iff \( Y_L \) is scattered. From this we derive Simmons’ theorem that if \( S \) is \( T_0 \), then \( N(\emptyset S) \) is boolean iff \( S \) is scattered. We generalize this result
to an arbitrary space by showing that $S$ is dispersed iff its $T_0$-reflection is scattered. This yields the general form of Simmons’ theorem that $N(\mathcal{O}S)$ is boolean iff $S$ is dispersed.

2. Frames, spaces, and nuclei

In this section we recall basic facts about frames. We use [21, 11, 14] as our basic references. A frame is a complete lattice $L$ satisfying the join-infinite distributive law
\[ a \land \bigvee S = \bigvee \{ a \land s \mid s \in S \}. \]

Frames are complete Heyting algebras where the implication is defined by
\[ a \rightarrow b = \bigvee \{ x \in L \mid a \land x \leq b \}. \]

We set $\neg a = a \rightarrow 0$.

A frame homomorphism is a map $h : L \rightarrow K$ preserving finite meets and arbitrary joins. As usual, we denote by Frm the category of frames and frame homomorphisms. Let also Top be the category of topological space and continuous maps. There is a contravariant functor $\mathcal{O} : \text{Top} \rightarrow \text{Frm}$ sending each topological space $S$ to the frame $\mathcal{O}S$ of opens of $S$, and each continuous map $f : S \rightarrow T$ to the frame homomorphism $f^{-1} : \mathcal{O}T \rightarrow \mathcal{O}S$.

To define a contravariant functor in the other direction, we recall that a point of a frame $L$ is a frame homomorphism $p : L \rightarrow 2$ where $2 = \{0, 1\}$ is the two-element frame. It is well known that there is a one-to-one correspondence between points of $L$, meet-prime elements of $L$, and completely prime filters of $L$ (see, e.g., [11, Sec. II.1.3]). We will mainly think of points as completely prime filters of $L$, but at times it will also be convenient to think of them as meet-prime elements of $L$.

Let $\text{pt}(L)$ be the set of points of $L$. For $a \in L$, we set
\[ \eta(a) = \{ x \in \text{pt}(L) \mid a \in x \}. \]

Then $\{ \eta(a) \mid a \in L \}$ is a topology on $\text{pt}(L)$, and $\eta : L \rightarrow \mathcal{O}(\text{pt}L)$ is an onto frame homomorphism. If $h : L \rightarrow K$ is a frame homomorphism, then $\text{pt}(h) : \text{pt}(K) \rightarrow \text{pt}(L)$ given by $\text{pt}(h)(y) = h^{-1}(y)$ is a continuous map. This defines a contravariant functor $\text{pt} : \text{Frm} \rightarrow \text{Top}$.

The functors $\text{pt}, \mathcal{O}$ yield a contravariant adjunction between Frm and Top. The unit of the adjunction is given by the frame homomorphism $\eta : L \rightarrow \mathcal{O}(\text{pt}L)$, and the counit by the continuous map $\varepsilon : S \rightarrow \text{pt}(\mathcal{O}S)$ where $\varepsilon(s) = \{ U \in \mathcal{O}S \mid s \in U \}$.

We call a frame $L$ spatial if $\eta$ is an isomorphism and a space $S$ sober if $\varepsilon$ is a homeomorphism. It is well known that $L$ is spatial iff whenever $a, b \in L$ with $a \not\leq b$, there is a point $x$ with $a \in x$ and $b \notin x$ (see, e.g., [11 Sec. II.1.5]); that $S$ is sober iff each irreducible closed set is the closure of a unique point (see, e.g., [11 Sec. II.1.6]); and that the contravariant adjunction $(\text{pt}, \mathcal{O})$ restricts to a dual equivalence between the category $S\text{Frm}$ of spatial frames and the category $S\text{ob}$ of sober spaces (see, e.g., [11 Sec. II.1.7]).

Definition 2.1. [21 Def. 1] A nucleus on a frame $L$ is a map $j : L \rightarrow L$ satisfying

(1) $a \leq ja$;
(2) \( jja \leq ja \);
(3) \( j(a \land b) = ja \land jb \).

Nuclei play an important role in pointfree topology as they characterize quotients of frames: If \( h : L \to K \) is a frame homomorphism and \( r : K \to L \) is its right adjoint, then \( rh \) is a nucleus on \( L \); conversely, if \( j \) is a nucleus on \( L \), then the fixpoints

\[
L_j := \{ a \in L \mid a = ja \} = \{ ja \mid a \in L \}
\]

form a frame where finite meets are the same as in \( L \) and the joins are defined by

\[
\bigcup S = j \left( \bigvee S \right)
\]

for each \( S \subseteq L_j \). This establishes a one-to-one correspondence between onto frame homomorphisms and nuclei on \( L \) (see, e.g., [14, Prop. III.5.3.2]).

Let \( N(L) \) be the set of all nuclei on \( L \). Define a partial order \( \leq \) on \( N(L) \) by

\[
j \leq k \text{ iff } ja \leq ka \text{ for each } a \in L.
\]

As we pointed out in the introduction, it is well known that \( N(L) \) is a frame with respect to \( \leq \). Finite meets are defined in \( N(L) \) componentwise, the bottom \( \bot \) is the identity nucleus, and the top \( \top \) is the nucleus sending every element of \( L \) to 1. Calculating joins in \( N(L) \) is more involved (see, e.g., [11, Sec. II.2.5]). The frame \( N(L) \) is often referred to as the assembly of \( L \).

The following nuclei play an important role:

\[
u_a(x) = a \lor x;
v_a(x) = a \to x;
w_a(x) = (x \to a) \to a.
\]

It is well known (see, e.g., [21, Lem. 7(ii)]) that each nucleus \( j \) can be written as

\[
j = \bigwedge \{ w_a \mid a = ja \} = \bigwedge \{ w_{ja} \mid a \in L \}.
\]

Moreover, sending \( a \in L \) to \( u_a \) defines a frame embedding of \( L \) into \( N(L) \), and we can form the tower of assemblies:

\[
L \hookrightarrow N(L) \hookrightarrow N^2(L) \hookrightarrow \cdots
\]

The booleanization of a frame \( L \) is defined as the fixpoints of \( w_0 \):

\[
B(L) := \{ w_0(a) \mid a \in L \} = \{ \neg \neg a \mid a \in L \}.
\]

It is well known that \( B(L) \) is a boolean frame (a complete boolean algebra). In fact, the embedding \( L \hookrightarrow N(L) \) factors through \( B(N(L)) \) since \( u_a, v_a \) are complemented elements of \( N(L) \), hence belong to \( B(L) \) (see, e.g., [11, Sec. II.2.6]).
3. Priestley and Esakia dualities

In this section we recall Priestley duality for bounded distributive lattices, and Esakia duality for Heyting algebras. We use [17, 18, 7] as our basic references. A subset of a topological space $X$ is *clopen* if it is both closed and open. If $\leq$ is a partial order on $X$ and $S \subseteq X$, then

$$\uparrow S := \{x \in X \mid s \leq x \text{ for some } s \in S\} \text{ and } \downarrow S := \{x \in X \mid x \leq s \text{ for some } s \in S\}.$$ 

If $S = \{s\}$, then we simply write $\uparrow s$ and $\downarrow s$. We call $S$ an *upset* if $S = \uparrow S$, and a *downset* if $S = \downarrow S$.

**Definition 3.1.** [17] A *Priestley space* is a pair $(X, \leq)$ where $X$ is a compact space, $\leq$ is a partial order on $X$, and the Priestley separation axiom holds:

From $x \not\leq y$ it follows that there is a clopen upset $U$ containing $x$ and missing $y$.

Let $\mathbf{Pries}$ be the category of Priestley spaces and continuous order preserving maps, and let $\mathbf{Dist}$ be the category of bounded distributive lattices and bounded lattice homomorphisms.

**Theorem 3.2** (Priestley duality). $\mathbf{Pries}$ is dually equivalent to $\mathbf{Dist}$.

The contravariant functor $\mathcal{X} : \mathbf{Dist} \to \mathbf{Pries}$ sends a bounded distributive lattice $L$ to the Priestley space $X_L$ of prime filters of $L$ ordered by inclusion. The topology on $X_L$ is given by the basis

$$\{\varphi(a) \setminus \varphi(b) \mid a, b \in L\}$$

where

$$\varphi(a) = \{x \in X_L \mid a \in x\}.$$ 

If $h : L \to K$ is a bounded lattice homomorphism, then $\mathcal{X}(h) : X_K \to X_L$ is given by $\mathcal{X}(h)(x) = h^{-1}[x]$.

The contravariant functor $\mathcal{U} : \mathbf{Pries} \to \mathbf{Dist}$ sends a Priestley space $X$ to the bounded distributive lattice $\mathcal{U}(X)$ of clopen upsets of $X$, and a morphism $f : X \to Y$ to the bounded lattice homomorphism $\mathcal{U}(f) : \mathcal{U}(Y) \to \mathcal{U}(X)$ given by $\mathcal{U}(f)(U) = f^{-1}(U)$.

The unit of this dual equivalence is given by the isomorphism $\varphi : L \to \mathcal{U}(X_L)$ in $\mathbf{Dist}$, and the counit by the isomorphism $\xi : X \to X_{\mathcal{U}(X)}$ in $\mathbf{Pries}$ given by

$$\xi(x) = \{U \in \mathcal{U}(X) \mid x \in U\}.$$ 

**Notation 3.3.** For a Priestley space $X$, we denote by $\pi$ the topology on $X$, by $\pi_u$ the topology of open upsets, and by $\pi_d$ the topology of open downsets. We then have that $\pi = \pi_u \vee \pi_d$. We use $\text{cl}_\pi$ and $\text{int}_\pi$ to denote the closure and interior in $(X, \pi)$.

**Definition 3.4.** [7] A Priestley space $X$ is an *Esakia space* if $U$ clopen in $X$ implies that $\downarrow U$ is clopen.
Let $\text{Esa}$ be the category of Esakia spaces and continuous maps $f : X \to Y$ satisfying $\uparrow f(x) = f[\uparrow x]$. Such maps are known as bounded morphisms or $p$-morphisms. Each such map is order preserving, thus $\text{Esa}$ is a non-full subcategory of $\text{Pries}$. Let $\text{Heyt}$ be the category of Heyting algebras and Heyting algebra homomorphisms. Then $\text{Heyt}$ is a non-full subcategory of $\text{Dist}$.

**Theorem 3.5 (Esakia duality).** $\text{Esa}$ is dually equivalent to $\text{Heyt}$.

The dual equivalence is established by the same functors $\mathcal{X}$ and $\mathcal{U}$. The additional condition on Esakia spaces guarantees that $\mathcal{U}(X)$ is a Heyting algebra, where for $U, V \in \mathcal{U}(X)$,

$$U \to V = X \setminus \downarrow(U \setminus V).$$

Then the unit $\varphi : L \to \mathcal{U}(X_L)$ is a Heyting algebra isomorphism, and so for $a, b \in L$,

$$\varphi(a \to b) = X_L \setminus \downarrow(\varphi(a) \setminus \varphi(b)).$$

Also, being a bounded morphism yields that $\mathcal{U}(f)$ is a Heyting algebra homomorphism.

Since frames are complete Heyting algebras, their dual Esakia spaces satisfy an additional condition, which is an order-topological version of extremal disconnectedness.

**Definition 3.6.** An Esakia space is *extremally order-disconnected* if the closure of each open upset is clopen.

**Theorem 3.7.** ([19, Thm. 2.3], [2, Thm. 2A(2)]) A Heyting algebra $L$ is a frame iff the Esakia space $X_L$ is extremally order-disconnected.

We will mostly work with extremally order-disconnected Esakia spaces since they are Esakia spaces of frames. We will frequently use the following well-known facts about Esakia spaces (see, e.g., [3]).

**Lemma 3.8.** Let $X$ be an Esakia space.

1. The order $\leq$ is closed in the product $X \times X$, so $\uparrow F, \downarrow F$ are closed for each closed $F \subseteq X$.
2. If $F$ is a closed upset and $D$ a closed downset of $X$ with $F \cap D = \emptyset$, then there is a clopen upset $U$ with $F \subseteq U$ and $U \cap D = \emptyset$.
3. Each open upset is a union of clopen upsets and each open downset is a union of clopen downsets.
4. Let $F$ be a closed subset of $X$ and let $\max(F)$ be the set of maximal points of $F$. Then $\max(F) = \max(\downarrow F)$ and $F \subseteq \downarrow \max(F)$.
5. If $F$ is a closed subset of $X$, then $\max(F)$ is closed.

We point out that the first four items of Lemma 3.8 hold for any Priestley space.
4. Nuclear subsets of Esakia spaces

Let $L$ be a Heyting algebra and $X_L$ its Esakia space. It was shown in [3] that nuclei on $L$ are characterized as special closed subsets of $X_L$. Further results in this direction were obtained in [3, 4].

**Definition 4.1.** Let $X$ be an Esakia space.

1. We call a closed subset $F$ of $X$ a **nuclear subset** provided for each clopen set $U$ in $X$, the set $\downarrow(U \cap F)$ is clopen in $X$.
2. We call $x \in X$ a **nuclear point** if $\{x\}$ is a nuclear subset of $X$.
3. Let $N(X)$ be the set of all nuclear subsets of $X$ ordered by inclusion.

**Remark 4.2.** In [5] nuclear subsets of $X$ were called subframes of $X$ because of their connection to subframe logics. In [20] nuclear sets were called $L$-sets and nuclear points $L$-points. For our purposes the adjective nuclear appears more appropriate.

**Theorem 4.3.** [3 Thm. 30] Let $L$ be a Heyting algebra and $X_L$ its Esakia space. Then $N(L)$ is dually isomorphic to $N(X_L)$.

The dual isomorphism is obtained as follows: If $j$ is a nucleus on $L$, then $N_j := \{x \in X_L \mid j^{-1}(x) = x\}$ is a nuclear subset of $X_L$; if $N$ is a nuclear subset of $X_L$, then $j_N : L \to L$ given by $\varphi(j_Na) = X_L \setminus \downarrow(N \setminus \varphi(a))$ is a nucleus on $L$; and this correspondence is a dual isomorphism. Therefore, if $L$ is a frame, and hence $X_L$ is an extremally order-disconnected Esakia space, then $N(X_L)$ is a coframe; that is, a complete lattice satisfying the meet-infinite distributive law

\[
a \lor \bigwedge S = \bigwedge \{a \lor s \mid s \in S\}.
\]

Under the dual isomorphism, the bottom of $N(L)$ corresponds to $X_L$, the top of $N(L)$ to $\emptyset$, and $N_j \land N_k = N_j \cup N_k$ [5 Cor. 31]. Moreover, we have:

**Theorem 4.4.** [5 Thm. 34]

1. $N_{ua} = X_L \setminus \varphi(a)$;
2. $N_{va} = \varphi(a)$;
3. $N_{wa} = \max(X_L \setminus \varphi(a))$.

The following is a quick corollary of Theorem [4, 3].

**Corollary 4.5.** Let $X$ be an Esakia space.

1. If $D$ is a clopen downset of $X$, then $\max(D) \in N(X)$.
2. If $U$ is a clopen subset of $X$, then $\max(U) \in N(X)$.
3. If $N \in N(X)$, then $\max(N) \in N(X)$. 
Proof. (1). This is an immediate consequence of Theorem 4.4(3).

(2). This follows from (1) since \( \max(U) = \max(\downarrow U) \) (see Lemma 3.8(4)).

(3). This follows from (1) since \( \downarrow N \) is a clopen downset.

We will also use the following basic facts about nuclear sets of extremally order-disconnected Esakia spaces.

Lemma 4.6. Let \( X \) be an extremally order-disconnected Esakia space.

1. If \( U \) is a clopen subset of \( X \), then \( U \in N(X) \).
2. If \( F \) is a regular closed subset of \( X \), then \( F \in N(X) \).
3. If \( U \) is clopen and \( N \) is nuclear, then \( U \cap N \in N(X) \).

Proof. (1). This is immediate since \( X \) is an Esakia space.

(2). See [3, Prop. 4.11].

(3). Let \( V \) be clopen in \( X \). Then \( V \cap U \) is clopen, so \( \downarrow (V \cap U \cap N) \) is clopen since \( N \in N(X) \). Thus, \( U \cap N \in N(X) \). \( \square \)

In contrast to Lemma 4.6(3), it is not the case that the intersection of two nuclear sets is nuclear as the following simple example shows.

Example 4.7. Let \( \mathbb{N} \) be the set of natural numbers. We view \( \mathbb{N} \) as a discrete space, and let \( X \) be the one-point compactification of \( \mathbb{N} \) with order as in the following picture.

\[ \begin{array}{c}
0 \\
| \\
1 \\
| \\
\vdots \\
| \\
\downarrow \infty
\end{array} \]

If \( A \) and \( B \) are the sets of even and odd numbers in \( \mathbb{N} \), respectively, it is easy to see that \( A \cup \{\infty\} \) and \( B \cup \{\infty\} \) are both nuclear sets, but their intersection \( \{\infty\} \) is not nuclear.

We conclude this section by showing how to calculate meets in \( N(X_L) \). This provides a dual description of joins in \( N(L) \). For this we require the following lemma (see [20, Prop. 2.5]).

Lemma 4.8. Let \( X \) be an extremally order-disconnected Esakia space and let \( \{N_{\alpha} \mid \alpha \in \Gamma\} \) be a family of nuclear subsets of \( X \). Then \( \text{cl}_{\pi}(\bigcup\{N_{\alpha} \mid \alpha \in \Gamma\}) \) is a nuclear subset of \( X \).

Proof. Let \( F = \text{cl}_{\pi}(\bigcup\{N_{\alpha} \mid \alpha \in \Gamma\}) \). Then \( F \) is closed. Since \( X \) is a Priestley space, \( \downarrow F \) is closed. Therefore, \( U := X \setminus \downarrow F \) is an open upset. As \( X \) is an extremally order-disconnected Esakia space, \( \text{cl}_{\pi}(U) \) is a clopen upset. We show that \( U = \text{cl}_{\pi}(U) \). Clearly \( N_{\alpha} \cap U = \emptyset \) for each \( \alpha \), so \( \downarrow N_{\alpha} \cap U = \emptyset \) since \( U \) is an upset. Because \( N_{\alpha} \) is nuclear, \( \downarrow N_{\alpha} \) is clopen, and hence \( \downarrow N_{\alpha} \cap \text{cl}_{\pi}(U) = \emptyset \). Thus, \( N_{\alpha} \cap \text{cl}_{\pi}(U) = \emptyset \) for each \( \alpha \), and so \( F \cap \text{cl}_{\pi}(U) = \emptyset \). Finally,
Since each

hence

Thus,

Theorem 4.9. Let \( X \) be an extremally order-disconnected Esakia space, \( \{N_\alpha \mid \alpha \in \Gamma\} \) a family of nuclear subsets of \( X \), and \( N = \bigcap \{N_\alpha \mid \alpha \in \Gamma\} \). Then the meet of \( \{N_\alpha \mid \alpha \in \Gamma\} \) in \( N(X) \) is calculated by the formula:

\[
\bigwedge \{N_\alpha \mid \alpha \in \Gamma\} = \text{cl}_\pi \left( \bigcup \{F \in N(X) \mid F \subseteq N\} \right).
\]

Proof. By Lemma 4.8, \( \text{cl}_\pi \left( \bigcup \{F \in N(X) \mid F \subseteq N\} \right) \) is a nuclear subset of \( X \), hence it follows from the definition that \( \text{cl}_\pi \left( \bigcup \{F \in N(X) \mid F \subseteq N\} \right) \) is the greatest lower bound of \( \{N_\alpha \mid \alpha \in \Gamma\} \) in \( N(X) \). Thus, \( \text{cl}_\pi \left( \bigcup \{F \in N(X) \mid F \subseteq N\} \right) = \bigwedge \{N_\alpha \mid \alpha \in \Gamma\} \). □

Corollary 4.10. Let \( L \) be a frame, \( X_L \) its Esakia space, and \( \{j_\alpha \mid \alpha \in \Gamma\} \) a family of nuclei on \( L \). Then

\[ N_{\bigvee j_\alpha} = \bigwedge N_{j_\alpha}. \]

5. When are \( L \) and \( N(L) \) spatial?

Let \( L \) be a frame and let \( X_L \) be its Esakia space. In this section we characterize in terms of \( X_L \) when \( L \) and \( N(L) \) are spatial, and show how to derive the results of Niefield and Rosenthal [13] from our characterizations. For this we require the following definition.

Definition 5.1. For a frame \( L \), let \( Y_L = \{y \in X_L \mid \{y\} \in N(X_L)\} \). Thus, \( Y_L \) is the set of nuclear points of \( X_L \).

Remark 5.2. By [4, Lem. 5.1], \( y \in Y_L \) iff \( y \) is a completely prime filter of \( L \). Thus, points of \( Y_L \) correspond to points of \( L \). As we will see in Proposition 5.4, this correspondence is a homeomorphism.

We recall that the specialization order of a topological space \( S \) is defined by

\[ x \leq y \text{ iff } x \in \overline{\{y\}}, \]

where \( \overline{\{y\}} \) denotes the closure of \( \{y\} \) in \( S \). Then it is easy to see that \( \downarrow y = \overline{\{y\}} \). We also recall from Section 3 that \( \pi \) is the original topology on \( X_L \), and \( \pi_u \) is the topology of open upsets. To simplify notation, we use the same \( \pi \) for the subspace topology on \( Y_L \), and let \( \tau \) be the subspace topology on \( Y_L \) arising from \( \pi_u \). We use \( \text{cl}_\pi \) for closure in \( (Y_L, \tau) \).

Lemma 5.3. Let \( L \) be a frame and \( X_L \) its Esakia space.

1. \( V \subseteq Y_L \) is open in \( (Y_L, \tau) \) iff \( V = U \cap Y_L \) for some clopen upset \( U \) of \( X_L \).
(2) $F \subseteq Y_L$ is closed in $(Y_L, \tau)$ iff $F = D \cap Y_L$ for some clopen downset $D$ of $X_L$.

(3) The restriction to $Y_L$ of the order $\leq$ on $X_L$ is the specialization order of $(Y_L, \tau)$. Thus, $\text{cl}_\tau(\{y\}) = \down y \cap Y_L$ for each $y \in Y_L$.

Proof. (1) The right-to-left implication follows from the definition of $\tau$. For the left-to-right implication, let $V$ be open in $(Y_L, \tau)$. By definition, $V = U \cap Y_L$ for some open upset $U$ of $X_L$. By Lemma 5.3(3), $U$ is a union of clopen upsets. Therefore, there is $S \subseteq L$ such that $V = \bigcup \{ \varphi(s) \cap Y_L \mid s \in S \}$. Let $a = \vee S$. Then $V \subseteq \varphi(a) \cap Y_L$. To see the reverse inclusion, let $y \in \varphi(a) \cap Y_L$. Since $y \in Y_L$, we have $\down y$ is clopen. Therefore, by [4, Lem. 5.1], $y$ is a completely prime filter of $L$. Since $a \in y$, there is $s \in S$ with $s \in y$. Thus, $y \in \varphi(s) \cap Y_L \subseteq V$. This shows $V = \varphi(a) \cap Y_L$, completing the proof of (1).

(2) This follows from (1).

(3) Let $\leq_\tau$ be the specialization order of $(Y_L, \tau)$. Suppose $x, y \in Y_L$. If $x \leq y$, then for every open upset $V$ of $X_L$, from $x \in V$ it follows that $y \in V$. Therefore, by the definition of $\tau$, if $U$ is an open set in $(Y_L, \tau)$, then $x \in U$ implies $y \in U$. Thus, $x \leq_\tau y$. Conversely, if $x \not\leq y$, then there is a clopen upset $V$ of $X_L$ with $x \in V$ and $y \notin V$. Therefore, $V \cap Y_L$ is open in $(Y_L, \tau)$ containing $x$ and missing $y$. Thus, $x \not\leq_\tau y$, and hence $\leq_\tau$ is the restriction of $\leq$ to $Y_L$. □

Proposition 5.4. The space $(Y_L, \tau)$ is (homeomorphic to) the space $\text{pt}(L)$ of points of $L$.

Proof. If we view points of $L$ as completely prime filters, then $Y_L = \text{pt}(L)$ by [4, Lem. 5.1]. The open sets of $\text{pt}(L)$ are the sets of the form $\{ y \in \text{pt}(L) \mid a \in y \}$ as $a$ ranges over the elements of $L$. These are exactly $\varphi(a) \cap Y_L$. Therefore, by Lemma 5.3(1), these are precisely the open sets of $(Y_L, \tau)$. Thus, $(Y_L, \tau)$ is (homeomorphic to) the space $\text{pt}(L)$ of points of $L$. □

We next characterize when $L$ is spatial (see also [20, Sec. 2.11]).

Theorem 5.5. A frame $L$ is spatial iff $Y_L$ is dense in $(X_L, \pi)$.

Proof. Suppose that $L$ is spatial. Let $U$ be a nonempty open subset of $(X_L, \pi)$. Then there are $a, b \in L$ with $\emptyset \neq \varphi(a) \setminus \varphi(b) \subseteq U$. Therefore, $a \not\leq b$. Since $L$ is spatial, there is $y \in Y_L$ with $a \in y$ and $b \notin y$. Then $y \in \varphi(a) \setminus \varphi(b)$, so $Y_L \cap U \neq \emptyset$. Thus, $Y_L$ is dense in $(X_L, \pi)$.

Conversely, suppose that $Y_L$ is dense in $(X_L, \pi)$. Let $a, b \in L$ with $a \not\leq b$. Then $\varphi(a) \setminus \varphi(b)$ is a nonempty open subset of $X_L$. Since $Y_L$ is dense in $(X_L, \pi)$, there is $y \in Y_L \cap (\varphi(a) \setminus \varphi(b))$. Therefore, $a \in y$ and $b \notin y$. Thus, $L$ is spatial. □

We now turn to the question of spatiality of $N(L)$. For a topological space $S$, we let $\mathcal{F}(S)$ be the closed sets of $S$. If we need to specify the topology $\pi$, we write $\mathcal{F}_\pi(S)$.

Lemma 5.6. For a frame $L$, the map $\gamma: N(X_L) \to \mathcal{F}_\pi(Y_L)$, defined by $\gamma(N) = N \cap Y_L$, is an onto coframe homomorphism.
Proof. Clearly $\gamma$ preserves finite joins since finite joins in both $N(X_L)$ and $\mathcal{F}_\pi(Y_L)$ are finite unions. To see that $\gamma$ preserves arbitrary meets, let $\{N_\alpha\}$ be a family of nuclear sets of $X_L$. If $N$ is their meet, then $N \subseteq \bigcap N_\alpha$, so

$$\gamma(N) = N \cap Y_L \subseteq \left( \bigcap N_\alpha \right) \cap Y_L = \bigcap (N_\alpha \cap Y_L) = \bigcap \gamma(N_\alpha).$$

For the reverse inclusion, let $y \in \bigcap \gamma(N_\alpha)$. Then $y \in \bigcap (N_\alpha \cap Y_L)$. So $\{y\} \subseteq N(X_L)$ and $y \in N_\alpha$ for each $\alpha$. Therefore, $y \in \bigwedge N_\alpha$ by Theorem 4.9. Thus, $\gamma(\bigwedge N_\alpha) = \bigcap \gamma(N_\alpha)$, and hence $\gamma$ is a coframe homomorphism. To see that $\gamma$ is onto, let $F \in \mathcal{F}_\pi(Y_L)$. Then $F = D \cap Y_L$ for some $D \in \mathcal{F}_\pi(X_L)$. Since $(X_L, \pi)$ is a Stone space, $D$ is the intersection of clopen sets $U$ containing it. Because clopen sets are nuclear (see Lemma 4.6(1)), we have

$$F = D \cap Y_L = \bigcap \{U \cap Y_L \mid D \subseteq U\} = \bigcap \{\gamma(U) \mid D \subseteq U\} = \gamma \left( \bigwedge \{U \mid D \subseteq U\} \right).$$

Thus, $\gamma$ is onto. \qed

We will prove that $N(L)$ is spatial iff $\gamma$ is 1-1. For this we require the following lemma.

**Lemma 5.7.** The join-prime elements of $N(X_L)$ are precisely the singletons $\{y\}$ with $y \in Y_L$.

Proof. Since binary join in $N(X_L)$ is union, it is clear that $\{y\}$ is join-prime in $N(X_L)$ for each $y \in Y_L$. Conversely, suppose that $N$ is join-prime in $N(X_L)$. If there are $x \neq y$ in $N$, then there is a clopen set $U$ containing $x$ but not $y$. Since $U \cup (X_L \setminus U) = X_L$, we see that $N \subseteq U \cup (X_L \setminus U)$. But $N$ is not contained in either. This is a contradiction to $N$ being join-prime in $N(X_L)$ since $U$ and $X_L \setminus U$ are clopen, hence belong to $N(X_L)$. Thus, $N$ is a singleton $\{y\}$ with $y \in Y_L$. \qed

Lemma 5.7 has several useful consequences. First we show that there is a bijection between the points of $L$ and the points of $N(L)$.

**Proposition 5.8.** There is a bijection between $\text{pt}(L)$ and $\text{pt}(N(L))$.

Proof. Let $Z_L$ be the set of join-prime elements of $N(X_L)$. By Lemma 5.7, $Z_L$ is in 1-1 correspondence with $Y_L$. Since $N(L)$ is dually isomorphic to $N(X_L)$, the join-prime elements of $N(X_L)$ are in 1-1 correspondence with the meet-prime elements of $N(L)$. But the meet-prime elements of $N(L)$ are in 1-1 correspondence with the points of $N(L)$. Therefore, there is a 1-1 correspondence between $Y_L$ and $\text{pt}(N(L))$. Thus, by Proposition 5.4, there is a 1-1 correspondence between $\text{pt}(L)$ and $\text{pt}(N(L))$. \qed

We next derive the following well-known theorem.

**Theorem 5.9.** [13, Cor. 3.5] If $N(L)$ is spatial, then $L$ is spatial.

Proof. Suppose that $N(L)$ is spatial. Since points are in 1-1 correspondence with meet-prime elements, each element of $N(L)$ is a meet of meet-prime elements of $N(L)$. As $N(L)$ is dually isomorphic to $N(X_L)$, every element of $N(X_L)$ is a join of join-prime elements of $N(X_L)$. To see that $L$ is spatial, let $a, b \in L$ with $a \nleq b$. Then $\varphi(a) \nleq \varphi(b)$. Since these sets are
clopen, they are in \( N(X_L) \). Therefore, there is a join-prime element \( P \) of \( N(X_L) \) such that \( P \subseteq \varphi(a) \) and \( P \not\subseteq \varphi(b) \). By Lemma \ref{lemma5.7}, there is \( y \in Y_L \) with \( y \in \varphi(a) \) and \( y \notin \varphi(b) \). Thus, \( a \in y \) and \( b \notin y \). Since \( y \in pt(L) \), we conclude that \( L \) is spatial. \( \square \)

Finally, we use Lemmas \ref{lemma5.6} and \ref{lemma5.7} to obtain the following characterization of spatiality of \( N(L) \).

**Theorem 5.10.** For a frame \( L \), the following conditions are equivalent.

1. The frame \( N(L) \) is spatial.
2. If \( N \in N(X_L) \) is nonempty, then so is \( N \cap Y_L \).
3. \( \gamma : N(X_L) \to \mathcal{F}_\pi(Y_L) \) is 1-1.
4. \( \gamma : N(X_L) \to \mathcal{F}_\pi(Y_L) \) is an isomorphism.
5. \( N(L) \) is isomorphic to \( \mathcal{O}_\pi(Y_L) \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose \( N \neq \varnothing \). Since \( N(L) \) is spatial, each element of \( N(L) \) is a meet of meet-prime elements, and hence each element of \( N(X_L) \) is a join of join-prime elements. Therefore, \( N \neq \varnothing \) implies there is a join-prime \( P \in N(X_L) \) with \( P \subseteq N \). But then \( N \cap Y_L \neq \varnothing \) by Lemma \ref{lemma5.7}.

(2) \( \Rightarrow \) (3). It is sufficient to show that for \( N, M \in N(X_L) \) with \( N \not\subseteq M \), there is \( y \in Y_L \) with \( y \in N \setminus M \). Suppose that \( N \not\subseteq M \). Since \( X_L \) is a Stone space and \( M \) is closed, there is a clopen set \( U \) with \( M \subseteq U \) and \( N \not\subseteq U \). Therefore, \( N \setminus U \) is a nonempty nuclear set by Lemma \ref{lemma1.6}(3). By (2), there is \( y \in Y_L \) with \( y \in N \setminus U \). Thus, \( y \in N \setminus M \).

(3) \( \Rightarrow \) (4). By Lemma \ref{lemma5.6} \( \gamma : N(X_L) \to \mathcal{F}_\pi(Y_L) \) is an onto coframe homomorphism. By (3), \( \gamma \) is 1-1. Thus, \( \gamma \) is an isomorphism.

(4) \( \Rightarrow \) (5). This is obvious since \( N(L) \) is dually isomorphic to \( N(X_L) \) and \( \mathcal{O}_\pi(Y_L) \) is dually isomorphic to \( \mathcal{F}_\pi(Y_L) \).

(5) \( \Rightarrow \) (1). This is clear. \( \square \)

**Remark 5.11.** When both \( L \) and \( N(L) \) are spatial, then it is a consequence of Proposition \ref{proposition5.4} and Theorem \ref{theorem5.10} that \( L \) is isomorphic to the frame of opens of \((Y_L, \tau)\), while \( N(L) \) is isomorphic to the frame of opens of \((Y_L, \pi)\).

We conclude this section by deriving the characterization of when \( N(L) \) is spatial given in \ref{subsubsection13} in terms of essential primes. Let \( L \) be a frame and \( a \in L \). We recall that a meet-prime element \( p \geq a \) is a minimal prime (with respect to \( a \)) if \( p \) is minimal among meet-primes \( q \geq a \). Let \( \operatorname{Min}(a) \) be the set of all minimal primes (with respect to \( a \)). If \( a = \bigwedge \operatorname{Min}(a) \), then \( p \in \operatorname{Min}(a) \) is an essential prime provided \( a \neq \bigwedge \{ q \in \operatorname{Min}(a) \mid p \neq q \} \).

We next characterize minimal primes and essential primes in terms of \( X_L \) and \( Y_L \). For this we require the following lemma.

**Lemma 5.12.** Let \( L \) be a frame, \( X_L \) its Esakia space, \( a \in L \), and \( S \subseteq L \).

1. \( a \) is meet prime iff \( \varphi(a) = X_L \setminus \downarrow y \) for some \( y \in Y_L \).
2. \( a = \bigwedge S \) iff \( \varphi(a) = X_L \setminus \downarrow (X_L \setminus \operatorname{int}_\pi(\bigcap\{ \varphi(s) \mid s \in S \})) \).
(3) \( a = \bigwedge S \text{ iff } X_L \setminus \varphi(a) = \downarrow \text{cl}_\pi \bigcup \{X_L \setminus \varphi(s) \mid s \in S\} \).

Proof. (1). A characterization of join primes of \( L \) is given in [2] Thm. 2.7(1). Dualizing the proof yields the result.

(2). See [2] Lem. 2.3(3)].

(3). This is immediate from (2). \( \square \)

In order to distinguish between minimal primes and essential primes of \( L \), for a clopen downset \( D \) of \( X_L \), we will look at \( \max(D) \cap Y_L \) and \( \max(D \cap Y_L) \). It is clear that \( \max(D) \cap Y_L \subseteq \max(D \cap Y_L) \). The reverse inclusion does not always hold, as the following example shows.

**Example 5.13.** Let \( \beta(\mathbb{N}) \) be the Stone-Čech compactification of the discrete space \( \mathbb{N} \), and let \( X \) be the disjoint union of \( \beta(\mathbb{N}) \) and the two-element discrete space \( \{x, y\} \). Let \( \mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N} \) and define \( \leq \) to be the least partial order on \( X \) satisfying \( z \leq y \) for each \( z \in X \) and \( x \leq z \) iff \( z = x \) or \( z \in \mathbb{N}^* \).

Since \( \beta(\mathbb{N}) \) is extremally disconnected, it is clear that so is \( X \). It is also easy to see that \( X \) is an Esakia space. Therefore, \( X \) is an extremally order-disconnected Esakia space. Thus, the clopen upsets of \( X \) form a frame \( L \), and we identify \( X \) with \( X_L \). Then \( Y_L \) is identified with \( \mathbb{N} \cup \{x, y\} \).

Let \( D = X \setminus \{y\} \). Then \( D \) is a clopen downset, \( \max(D) = \beta(\mathbb{N}) \), \( \max(D) \cap Y_L = \mathbb{N} \), and \( \max(D \cap Y_L) = \mathbb{N} \cup \{x\} \). Therefore, \( \max(D) \cap Y_L \) is a proper subset of \( \max(D \cap Y_L) \).

**Proposition 5.14.** Let \( L \) be a frame, \( X_L \) its Esakia space, \( a \in L \), and \( p \geq a \) a meet-prime element of \( L \).

(1) \( p \) is a minimal prime iff \( \varphi(p) = X_L \setminus \downarrow y \) for some \( y \in \max[(X_L \setminus \varphi(a)) \cap Y_L] \).

(2) \( p \) is an essential prime iff \( \varphi(p) = X_L \setminus \downarrow y \) for some \( y \in \max(X_L \setminus \varphi(a)) \cap Y_L \).

Proof. (1). First suppose that \( \varphi(p) = X_L \setminus \downarrow y \) for some \( y \in \max[(X_L \setminus \varphi(a)) \cap Y_L] \). Let \( q \geq a \) be a meet-prime with \( q \leq p \). Then \( \varphi(q) = X_L \setminus \downarrow z \) for some \( z \in Y_L \) by Lemma 5.12(1). Since \( a \leq q \), we have \( z \in X_L \setminus \varphi(a) \), so \( z \in (X_L \setminus \varphi(a)) \cap Y_L \). From \( q \leq p \) it follows that \( \downarrow y \subseteq \downarrow z \), so \( y \leq z \). Since \( y \in \max[(X_L \setminus U) \cap Y_L] \), we conclude that \( y = z \), and hence \( q = p \). Thus, \( p \) is a minimal prime.

Conversely, suppose \( p \) is a minimal prime. Then \( \varphi(p) = X_L \setminus \downarrow y \) for some \( y \in (X_L \setminus \varphi(a)) \cap Y_L \). Let \( y \leq z \) with \( z \in (X_L \setminus \varphi(a)) \cap Y_L \). There is a meet-prime \( q \in L \) such that...
\[\varphi(g) = X_L \setminus \downarrow z\] and \[\varphi(a) \subseteq \varphi(g) \subseteq \varphi(p).\] Therefore, \(a \leq q \leq p\), and since \(p\) is a minimal prime, \(q = p\). Thus, \(z = y\), proving that \(y \in \max((X_L \setminus \varphi(a)) \cap Y_L)\).

(2) Suppose \(a = \bigwedge \text{Min}(a)\). By Lemma \ref{lemma5.12}(3),
\[
X_L \setminus \varphi(a) = \downarrow \text{cl}_\pi \bigcup \{X_L \setminus \varphi(p) \mid p \in \text{Min}(a)\}.
\]
By (1),
\[
X_L \setminus \varphi(a) = \downarrow \text{cl}_\pi \bigcup \{\downarrow y \mid y \in \max((X_L \setminus \varphi(a)) \cap Y_L)\} = \downarrow \text{cl}_\pi \max((X_L \setminus \varphi(a)) \cap Y_L) = \downarrow \text{cl}_\pi \max([X_L \setminus \varphi(a)](Y_L)].
\]
Now suppose \(\varphi(p) = X_L \setminus \downarrow y\) for some \(y \in \max(X_L \setminus \varphi(a)) \cap Y_L\). Set
\[
T = \max((X_L \setminus \varphi(a)) \cap Y_L) \setminus \{y\}.
\]
If \(\varphi(a) = X_L \setminus \downarrow y\) for some \(y \in \max((X_L \setminus \varphi(a)) \cap Y_L)\). As above set \(T = \max((X_L \setminus \varphi(a)) \cap Y_L) \setminus \{y\}\). If \(X_L \setminus \varphi(a) = \downarrow \text{cl}_\pi(T)\), then \(X_L \setminus \downarrow \text{cl}(T) \subseteq X \setminus \downarrow y\), so \(\downarrow y \subseteq \downarrow \text{cl}(T)\). Since \(y \in \max(X_L \setminus \varphi(a))\) and \(\text{cl}(T) \subseteq X \setminus \varphi(a)\), it follows that \(y \in \text{cl}(T)\). But \(\downarrow y\) is a clopen set containing \(y\) and \(\downarrow y \cap T = \emptyset\) by definition of \(T\). Thus, \(y \notin \text{cl}(T)\). The obtained contradiction proves that \(X_L \setminus \varphi(a) \neq \downarrow \text{cl}(T)\), so \(p\) is an essential prime.

Conversely, suppose \(p\) is an essential prime. We have \(\varphi(p) = X_L \setminus \downarrow y\) for some \(y \in \max((X_L \setminus \varphi(a)) \cap Y_L)\). As above set \(T = \max((X_L \setminus \varphi(a)) \cap Y_L) \setminus \{y\}\). If \(X_L \setminus \varphi(a) = \downarrow \text{cl}_\pi(T)\), then \(a = \bigwedge \text{Min}(a) \setminus \{p\}\) by Lemma \ref{lemma5.12}(3), which is false since \(p\) is an essential prime. Therefore, \(X_L \setminus \varphi(a) \not\subseteq \downarrow \text{cl}_\pi(S)\). Thus, \(\max(X_L \setminus \varphi(a)) \not\subseteq \text{cl}_\pi(S)\). Let \(x \in \max(X_L \setminus \varphi(a))\) with \(x \notin \text{cl}_\pi(S)\). Then there is a clopen neighborhood \(U\) of \(x\) with \(U \cap S = \emptyset\). Let \(V\) be a clopen upset containing \(x\). Since \(x \in \max(X_L \setminus \varphi(a))\), by \ref{lemma17}, \(x \in \text{cl}_\pi(\max((X_L \setminus \varphi(a)) \cap Y_L))\).

The set \(U \cap V\) is an open neighborhood of \(x\) contained in \(U\), so
\[
U \cap V \cap \max((X_L \setminus \varphi(a)) \cap Y_L) \neq \emptyset\text{ and }U \cap V \cap S = \emptyset.
\]
Therefore, \(y \in V\). Since this is true for all clopen upsets containing \(x\), it follows that \(x \leq y\). Thus, \(x = y\) as \(x \in \max(X_L \setminus \varphi(a))\). Consequently, \(y \in \max(X_L \setminus \varphi(a)) \cap Y_L\).

We are ready to give an alternate proof of the results of \ref{c13}.

**Theorem 5.15.** Let \(L\) be a frame.

1. \ref{c13} p. 264] \(L\) is spatial iff \(a = \bigwedge \text{Min}(a)\) for each \(a \in L\).
2. \ref{c13} Thm. 3.4] \(N(L)\) is spatial iff each \(1 \neq a \in L\) has an essential prime.

**Proof.** (1) First suppose that \(L\) is spatial. By Theorem \ref{c5.5} \(Y_L\) is dense in \((X_L, \pi)\). Let \(a \in L\). To see that \(a = \bigwedge \text{Min}(a)\), we need to show that \(X_L \setminus \varphi(a) = \downarrow \text{cl}_\pi(\max((X_L \setminus \varphi(a)) \cap Y_L))\). For this it is sufficient to show that \(\max(X_L \setminus \varphi(a)) \subseteq \text{cl}_\pi(\max((X_L \setminus \varphi(a)) \cap Y_L))\). Let \(x \in \max(X_L \setminus \varphi(a))\) and let \(U\) be a clopen neighborhood of \(x\). Then \(U \setminus \varphi(a) \neq \emptyset\). Since \(Y_L\) is dense, \((U \setminus \varphi(a)) \cap Y_L \neq \emptyset\). Thus, \(x \in \text{cl}_\pi(\max((X_L \setminus \varphi(a)) \cap Y_L))\).

Conversely, suppose that \(a = \bigwedge \text{Min}(a)\) for each \(a \in L\). By Theorem \ref{c5.5} it is sufficient to show that \(Y_L\) is dense in \((X_L, \pi)\). Let \(U\) be nonempty clopen. Then \(\downarrow U\) is a clopen
downset, so there is \( a \in L \) with \( \varphi(a) = X_L \setminus \downarrow U \). Since \( a = \bigwedge \operatorname{Min}(a) \), by Lemma 5.12(3) and Proposition 5.14(1), \( \downarrow U = \downarrow \operatorname{cl}_{\pi} \max(\downarrow U \cap Y_L) \). Because \( \max(\downarrow U) = \max(U) \), we have \( \max(U) \subseteq \operatorname{cl}_{\pi} \max(\downarrow U \cap Y_L) \). Therefore, \( U \cap \operatorname{cl}_{\pi} \max(\downarrow U \cap Y_L) \neq \emptyset \). Since \( U \) is clopen, \( U \cap \max(\downarrow U \cap Y_L) \neq \emptyset \), so \( U \cap Y_L \neq \emptyset \). Thus, \( Y_L \) is dense in \( (X_L, \pi) \).

(2) First suppose that \( N(L) \) is spatial. Let \( a \neq 1 \). Then \( \max(X_L \setminus \varphi(a)) \) is a nonempty nuclear set by Theorem 4.4(3). Therefore, \( \max(X_L \setminus \varphi(a)) \cap Y_L \neq \emptyset \) by Theorem 5.10. If \( y \in \max(X_L \setminus U) \cap Y_L \), then the \( p \) such that \( \varphi(p) = X_L \setminus \downarrow y \) is an essential prime by Proposition 5.14(2).

Conversely, suppose that each \( 1 \neq a \in L \) has an essential prime. Let \( N \) be a nonempty nuclear set. Then \( \downarrow N \) is a nonempty clopen downset, so there is \( a \neq 1 \) such that \( \varphi(a) = X_L \setminus \downarrow N \). Therefore, \( a \) has an essential prime \( p \). By Proposition 5.14(2), there is \( y \in \max(X_L \setminus \varphi(a)) \cap Y_L \) with \( \varphi(p) = X_L \setminus \downarrow y \), so \( \max(X_L \setminus \varphi(a)) \cap Y_L \neq \emptyset \). Thus, \( \max(\downarrow N) \cap Y_L = \max(N) \cap Y_L \) is nonempty, and so \( N \cap Y_L \) is nonempty. Consequently, \( N(L) \) is spatial by Theorem 5.10.

6. When is \( N(L) \) boolean?

In [1] Beazer and Macnab proved that if \( L \) is boolean, then \( N(L) \cong L \). To see this in terms of \( X_L \), if \( L \) is boolean, then the order of \( X_L \) is equality, and \( L \) is isomorphic to the clopens of \( X_L \). Since the order of \( X_L \) is equality, nuclear sets of \( X_L \) are exactly clopen sets of \( X_L \). Thus, \( N(L) \) is isomorphic to \( L \), yielding [1 Cor. 1].

Beazer and Macnab also gave a characterization of when \( N(L) \) is boolean [1 Thm. 2]. Let \( L \) be a frame. Recall that \( d \in L \) is dense if \( d = 0 \). If \( a \in L \), then \( \uparrow a \) is a frame, and \( d \geq a \) is dense in \( \uparrow a \) iff \( d \rightarrow a = a \). Following [15 Sec. 1.4] and [41 Sec. 6], we call \( L \) scattered if for each \( a \in L \) the principal upset \( \uparrow a \) has a smallest dense element. Using this definition, [41 Thm. 2] can be phrased as \( N(L) \) is boolean iff \( L \) is scattered. In this section we give a characterization of when \( N(L) \) is boolean in terms of \( X_L \), from which we derive [1 Thm. 2].

For a topological space \( S \), let \( \text{RC}(S) \) be the boolean frame of regular closed sets of \( S \). By [4 Prop. 4.12], if \( L \) is a frame, then its booleanization \( B(N(L)) \) is dually isomorphic to \( \text{RC}(X_L) \).

**Theorem 6.1.** Let \( L \) be a frame and \( X_L \) its Esakia space. Then the following conditions are equivalent.

1. \( N(L) \) is boolean;
2. \( N(X_L) = \text{RC}(X_L) \);
3. \( \max(D) \) is clopen for each clopen downset \( D \) of \( X_L \);
4. \( L \) is scattered.

**Proof.** (1) \( \Leftrightarrow \) (2). See [4 Thm. 4.14].

(2) \( \Rightarrow \) (3). Let \( D \) be a clopen downset. Then \( D \) is nuclear by Lemma 4.6(1), so \( \max(D) \) is nuclear by Corollary 4.3(1). Let \( V = D \setminus \max(D) \). Then \( V \) is open, so \( \operatorname{cl}_{\pi}(V) \) is regular closed, and hence nuclear by Lemma 4.6(2). Clearly \( V \) is a downset. We show that \( \operatorname{cl}_{\pi}(V) \)
is a downset. Let \( x \leq y \in \text{cl}_\pi(V) \). Since \( \text{cl}_\pi(V) \subseteq D \), we have \( x \in D \). If \( x \notin \text{max}(D) \), then \( x \in V \subseteq \text{cl}_\pi(V) \). If \( x \in \text{max}(D) \), then \( x = y \), so again \( x \in \text{cl}_\pi(V) \). Thus, \( \text{cl}_\pi(V) \) is a downset. Since \( \text{cl}_\pi(V) \) is nuclear, it is then a clopen downset, so \( F := \text{cl}_\pi(V) \cap \text{max}(D) \) is nuclear by Lemma 4.6(3), and hence regular closed by (2). This implies that \( F = \text{cl}_\pi(V) \setminus V \). Since \( \text{int}_\pi(\text{cl}_\pi(V) \setminus V) = \emptyset \) and \( F \) is regular closed, this forces \( F = \emptyset \). Thus, \( \text{cl}_\pi(V) = V \), so \( V \) is clopen, and hence \( \text{max}(D) = D \setminus V \) is clopen.

(3) \( \Rightarrow \) (2). By Lemma 4.6(2), it suffices to prove that each nuclear subset of \( X_L \) is regular closed. Let \( N \subseteq \text{cl}_\pi(\text{int}_\pi(N)) \). Let \( x \in N \). For each clopen \( U \) containing \( x \) we have \( U \cap N \) is a nonempty nuclear set, so \( \downarrow(U \cap N) \) is clopen. Thus, \( \text{max} \downarrow(U \cap N) \) is clopen. But \( \text{max} \downarrow(U \cap N) = \text{max}(U \cap N) \), so \( \text{max}(U \cap N) \) is clopen, and hence it is contained in \( \text{int}_\pi(N) \). Therefore, \( U \cap \text{int}_\pi(N) \neq \emptyset \), which proves \( x \in \text{cl}_\pi(\text{int}_\pi(N)) \). Thus, \( N \) is regular closed.

(3) \( \Leftrightarrow \) (4). See \([4, \text{Thm. 6.6}]\). \( \square \)

7. Soberification and the theorems of Simmons and Isbell

Let \( S \) be a topological space. In this section we show how to derive the results of Simmons [22] and Isbell [10] relating \( S \) being weakly scattered and scattered to that of \( N(\mathcal{G}S) \) being spatial and boolean. For this we recall that the \emph{soberification} of \( S \) is the space \( \text{pt}(\mathcal{G}S) \).

Viewing points of \( \mathcal{G}S \) as completely prime filters, we have the mapping \( \varepsilon : S \to \text{pt}(\mathcal{G}S) \) sending \( s \) to the completely prime filter \( \{U \in \mathcal{G}S \mid s \in U\} \). It is well known (see, e.g., [11, Sec. II.1]) that \( \varepsilon \) is continuous, is an embedding iff \( S \) is \( T_0 \), and induces an isomorphism of the frames of open sets.

**Proposition 7.1.** For a topological space \( S \), the space \( (Y_{\mathcal{G}S}, \tau) \) is (homeomorphic to) the soberification of \( S \).

**Proof.** By Proposition 5.3 if \( L \) is a frame, then \( (Y_L, \tau) \) is (homeomorphic to) \( \text{pt}(L) \). Thus, \( (Y_{\mathcal{G}S}, \tau) \) is (homeomorphic to) \( \text{pt}(\mathcal{G}S) \), and hence is (homeomorphic to) the soberification of \( S \). \( \square \)

Let \( S \) be a topological space and \( T \) a subspace of \( S \). We recall from the introduction that a point \( x \in T \) is \emph{weakly isolated} in \( T \) if there is an open subset \( U \) of \( S \) such that \( x \in T \cap U \subseteq \{x\} \); and that \( S \) is \emph{weakly scattered} if each nonempty closed subspace has a weakly isolated point.

**Lemma 7.2.** Let \( L \) be a spatial frame and \( D \) a clopen downset of \( (X_L, \pi) \). If \( y \in D \cap Y_L \) is weakly isolated in \( D \cap Y_L \) viewed as a subspace of \( (Y_L, \tau) \), then \( y \in \text{max}(D) \).

**Proof.** If \( y \in D \cap Y_L \) is weakly isolated, then there is an open subset \( U \) of \( (Y_L, \tau) \) such that \( y \in D \cap Y_L \cap U \subseteq \text{cl}_\tau(\{y\}) = \downarrow y \cap Y_L \), where the equality follows from Lemma 5.3(3). By Lemma 5.3(1), we may write \( U = V \cap Y_L \) for some clopen upset \( V \) of \( X_L \). Therefore, \( D \cap V \cap Y_L \subseteq \downarrow y \). This implies \( D \cap V \cap (X \setminus \downarrow y) \cap Y_L = \emptyset \), so \( D \cap V \cap (X \setminus \downarrow y) = \emptyset \) as
$Y_L$ is dense in $(X_L, \pi)$ by Theorem 5.5. Thus, $D \cap V \subseteq \downarrow y$. If $y \leq z$ for some $z \in D$, then $z \in D \cap V$ since $V$ is an upset. Therefore, $z \in \downarrow y$, which forces $z = y$. Thus, $y \in \max(D)$. \qed

**Theorem 7.3.** Let $L$ be a spatial frame. Then $N(L)$ is spatial iff $(Y_L, \tau)$ is weakly scattered.

**Proof.** First suppose that $N(L)$ is spatial. To show $(Y_L, \tau)$ is weakly scattered, let $F \in \mathcal{F}_r(Y_L)$ be nonempty. By Lemma 5.3(2), $F = D \cap Y_L$ for some clopen downset $D$. Since $\max(D)$ is nuclear (see Corollary 4.5(1)) and nonempty (as $\downarrow \max(D) = D$), we have $\max(D) \cap Y_L \neq \emptyset$ by Theorem 5.10. Let $x \in \max(D) \cap Y_L$. We show that $x$ is weakly isolated in $F$. Since $x \in Y_L$, the downset $\downarrow x$ is clopen. Set

$$N = \text{cl}_\pi \left( \bigcup \{ \downarrow y \setminus \downarrow x \mid y \in F, y \not\leq x \} \right).$$

Since each $\downarrow y \setminus \downarrow x$ is clopen, hence nuclear, $N$ is nuclear by Lemma 4.8. If $y \in F$, then $\downarrow y \setminus \downarrow x \subseteq D$ since $D$ is a downset. Therefore, the union is in $D$, and so $N \subseteq D$. Moreover, $\downarrow N \subseteq \downarrow D$ since $D$ is a downset. We see that $x \not\in N$; for, $\downarrow x$ is a clopen neighborhood of $x$ and $\downarrow x \cap (\downarrow y \setminus \downarrow x) = \emptyset$ for each $y \not\leq x$ with $y \in Y_L$. Thus, $\downarrow x$ misses the union, and so $x \not\in N$. Since $x \in \max(D) \setminus N$ and $\downarrow N \subseteq D$, it follows that $x \not\in \downarrow N$. Hence $X_L \setminus \downarrow N$ is a clopen upset containing $x$. We have $x \in (X \setminus \downarrow N) \cap F \subseteq \downarrow x$ since if $y \in F$ with $y \not\leq x$, then $y \in N$. This shows that $x$ is weakly isolated in $F$. Consequently, $Y_L$ is weakly scattered.

Conversely, suppose that $Y_L$ is weakly scattered. By Theorem 5.10, it is sufficient to show that if $N$ is a nonempty nuclear subset of $X_L$, then $N \cap Y_L \neq \emptyset$. Since $L$ is spatial, $Y_L$ is dense in $(X_L, \pi)$ by Theorem 5.5. Therefore, $\downarrow N \cap Y_L \neq \emptyset$ since $\downarrow N$ is a nonempty clopen. As $\downarrow N \cap Y_L$ is a nonempty closed set in $(Y_L, \tau)$, since $Y_L$ is weakly scattered, there is a weakly isolated point $y$ of $N \cap Y_L$. By Lemma 7.2, $y \in \max(\downarrow N) = \max(N)$. Thus, $N \cap Y_L \neq \emptyset$. \qed

**Example 7.4.** The following example shows that the hypothesis of Theorem 7.3 that $L$ is spatial is necessary. Let $L$ be the frame of regular open sets of the space $S = [0, 1] \cup \{2\}$ with the usual Euclidean topology. Since $L$ is boolean, points of $L$ correspond to isolated points of $S$ (see, e.g., [14, Sec. II.5.4]). Consequently, $Y_L$ is a singleton, and so $Y_L$ is weakly scattered. As $L$ is boolean, $L = N(L)$. But $L$ is not spatial since it is not atomic. Thus, $N(L)$ is not spatial.

We next show how to derive Isbell’s theorem [10, Thm. 7] as a consequence of Theorem 7.3. For this we require the following simple lemma.

**Lemma 7.5.** If $S$ is a weakly scattered $T_0$-space, then $S$ is sober.

**Proof.** Let $F$ be a closed irreducible subset of $S$. Then $F$ is nonempty, so there is a weakly isolated point $x \in F$. Therefore, there is an open set $U$ of $S$ such that $x \in U \cap F \subseteq \{x\}$. Thus, $F \setminus U$ is a proper closed subset of $F$ and $F = (F \setminus U) \cup \{x\}$. Since $F$ is irreducible and $F \setminus U \neq F$, we must have $F = \{x\}$. As $S$ is $T_0$, we conclude that $S$ is sober. \qed

**Theorem 7.6.** Let $S$ be a topological space.

1. $N(O\mathcal{S})$ is spatial iff the soberification of $S$ is weakly scattered.
(2) [10] Thm. 7] If $S$ is $T_0$, then $S$ is sober and $N(\mathcal{O}S)$ is spatial iff $S$ is weakly scattered. In particular, if $S$ is sober, then $N(\mathcal{O}S)$ is spatial iff $S$ is weakly scattered.

Proof. (1). By Proposition 7.1 view $(Y_{\mathcal{O}S}, \tau)$ as the soberification of $S$ and apply Theorem 7.3.

(2). Let $S$ be a $T_0$-space. If $S$ is sober, then $S$ is homeomorphic to $(Y_{\mathcal{O}S}, \tau)$. Hence, $N(\mathcal{O}S)$ spatial implies that $S$ is weakly scattered by Theorem 7.3. Conversely, if $S$ is weakly scattered, then $S$ is sober by Lemma 7.2 so $S$ is homeomorphic to $(Y_{\mathcal{O}S}, \tau)$. Applying Theorem 7.3 then yields that $N(\mathcal{O}S)$ is spatial. □

As we pointed in the introduction, the assumption that $S$ is sober cannot be dropped from Theorem 7.6(2), as the following example shows. For this we recall the following definition.

**Definition 7.7.** [22, p. 24] The front topology on a topological space $S$ is the topology $\tau_F$ generated by $\{U \setminus V \mid U, V \in \mathcal{O}S\}$.

**Example 7.8.** Let $S$ be the set of natural numbers with the usual order. We put the Alexandroff topology on $S$, so $U$ is open in $S$ iff $U$ is an upset. We then have the following picture, where $n$ is the prime filter of $\mathcal{O}S$ consisting of all open sets containing $n$ and $\infty$ is the prime filter $\mathcal{O}S \setminus \{\emptyset\}$.

The space $(X_{\mathcal{O}S}, \pi)$ is the one-point compactification of $(S, \tau_F)$, which is a discrete space, and the order on $X_{\mathcal{O}S}$ is described in the picture. Since the downset of each $x \in X_{\mathcal{O}S}$ is clopen, we see that $Y_{\mathcal{O}S} = X_{\mathcal{O}S}$ and $N(X_{\mathcal{O}S}) = \mathcal{P}_x(Y_{\mathcal{O}S})$, yielding that $N(\mathcal{O}S)$ is spatial. On the other hand, $S$ is not weakly scattered since there are no weakly isolated points in $S$. Indeed, if there was a weakly isolated point $s \in S$, then there would exist an open set $U$ with $s \in U \cap S = U \subseteq \downarrow s$. But $\downarrow s$ is finite for each $s \in S$, while nonempty open subsets of $S$ are infinite. This shows that $s$ is not weakly isolated in $S$, and hence $S$ is not weakly scattered.

We next turn to Simmons’ results. One of Simmons’ main tools was to use the front topology on a topological space $S$. We denote the frame of open sets of the front topology by $\mathcal{O}_F(S)$. Since Simmons did not assume that $S$ is $T_0$, we recall that the $T_0$-reflection of $S$ is defined as the quotient space $S_0$ by the equivalence relation $\sim$ given by $x \sim y$ iff $\{x\} = \{y\}$. Clearly $S_0$ is a $T_0$-space, and the canonical map $\rho : S \to S_0$ is both an open and a closed...
map since each open set and hence each closed set of $S$ is saturated with respect to $\sim$. We denote the equivalence class of $x \in S$ by $[x]$. In the next lemma we show how the front topology is connected to the space $(X_{\theta S}, \pi)$.

**Lemma 7.9.** The map $\varepsilon : S \to X_{\theta S}$ factors through the natural map $\rho : S \to S_0$. If $\varepsilon' : S_0 \to X_{\theta S}$ is the induced map, then the pair $((X_{\theta S}, \pi), \varepsilon')$ is a compactification of $(S_0, \tau_F)$. In particular, if $S$ is $T_0$, then $((X_{\theta S}, \pi), \varepsilon)$ is a compactification of $(S, \tau_F)$.

![Diagram](image)

**Proof.** Since $\varepsilon$ is continuous with respect to the open upset topology on $X_{\theta S}$, which is a $T_0$ topology, $\varepsilon'$ factors through $\rho$.

A basic open set of $(X_{\theta S}, \pi)$ has the form $\varphi(U) \setminus \varphi(V)$ for some $U, V$ open in $S$. Therefore, $\varepsilon^{-1}(\varphi(U) \setminus \varphi(V)) = U \setminus V$. Thus, $\varepsilon$ is continuous with respect to the front topology on $S$. Because $(\varepsilon')^{-1}(\varphi(U) \setminus \varphi(V)) = \rho(\varepsilon^{-1}(\varphi(U) \setminus \varphi(V)))$ and $\rho$ is an open map, we see that $\varepsilon'$ is continuous with respect to the front topology on $S_0$. It is a homeomorphism onto its image since for $U, V \in \partial S$, we have $\varepsilon'(\rho(U) \setminus \rho(V)) = (\varphi(U) \setminus \varphi(V)) \cap \varepsilon[S]$. To see that the image is dense, let $\varphi(U) \setminus \varphi(V)$ be nonempty. Then $U \setminus V \neq \emptyset$, which shows that $(\varphi(U) \setminus \varphi(V)) \cap \varepsilon[S] \neq \emptyset$. Therefore, $\varepsilon'[S_0] = \varepsilon[S]$ is dense in $(X_{\theta S}, \pi)$. Thus, $\varepsilon' : S_0 \to X_{\theta S}$ is a compactification with respect to the front topology on $S_0$.

Let $A$ be a subset of $S$. Following [22, Def. 1.2], we call $x \in A$ detached in $A$ if there is an open set $U$ of $S$ with $x \in U \cap A \subseteq [x]$. The space $S$ is dispersed if each nonempty closed subset of $S$ has a detached point (see [22, Def. 1.8, Thm. 1.9]).

**Proposition 7.10.** Let $S$ be a topological space and let $S_0$ be its $T_0$-reflection.

1. $S$ is weakly scattered iff $S_0$ is weakly scattered.
2. $S$ is dispersed iff $S_0$ is scattered.

**Proof.** (1). Suppose that $S$ is weakly scattered. Let $F$ be a nonempty closed subset of $S_0$. Then $\rho^{-1}(F)$ is a nonempty closed subset of $S$, so there is $x \in S$ and an open set $U$ of $S$ with $x \in U \cap \rho^{-1}(F) \subseteq [x]$. Therefore, $\rho(x) \in \rho(U) \cap F \subseteq \rho([x]) = [\rho(x)]$. Since $\rho(U)$ is open in $S_0$, this shows $\rho(x)$ is weakly isolated in $F$. Thus, $S_0$ is weakly scattered.

Conversely, suppose that $S_0$ is weakly scattered. Let $F$ be a nonempty closed subset of $S$. Then $\rho(F)$ is closed in $S_0$ and nonempty. Therefore, there is $x \in S$ and an open set $V$ of $S_0$ with $\rho(x) \in V \cap \rho(F) \subseteq \{\rho(x)\}$. Since $F = \rho^{-1}(\rho(F))$, this yields $x \in \rho^{-1}(V) \cap F \subseteq \rho^{-1}(\{\rho(x)\}) = [x]$. Thus, $x$ is weakly isolated in $F$, and hence $S$ is weakly scattered.

(2). First suppose that $S$ is dispersed. Let $F$ be a nonempty closed subset of $S_0$. Then $\rho^{-1}(F)$ is a nonempty closed subset of $S$. Since $S$ is dispersed, there is a detached point $x$ of $\rho^{-1}(F)$. Therefore, there is an open set $U$ of $S$ with $x \in U \cap \rho^{-1}(F) \subseteq [x]$. Thus, $\rho(x) \in \rho(U) \cap F \subseteq \{\rho(x)\}$, which shows $\rho(x)$ is isolated in $F$. Consequently, $S_0$ is scattered.
Conversely, suppose that $S_0$ is scattered. Let $F$ be a nonempty closed subset of $S$. Then $\rho(F)$ is a nonempty closed subset of $S_0$. Since $S_0$ is scattered, $\rho(F)$ has an isolated point $\rho(x)$. But then $x$ is a detached point of $F$. Therefore, $F$ has a detached point, and hence $S$ is dispersed.

For a topological space $S$, let $\mathcal{F}_F(S)$ denote the coframe of closed sets of the front topology on $S$. Define $\delta: N(X_{\mathcal{OS}}) \to \mathcal{F}_F(S)$ by $\delta(N) = \varepsilon^{-1}(N)$ for each $N \in N(X_{\mathcal{OS}})$. Then $\delta$ is the composition of $\gamma$ with the pullback map $\mathcal{F}_\pi(Y_{\mathcal{OS}}) \to \mathcal{F}_F(S)$. Therefore, the following is a consequence of Lemma 5.6.

**Lemma 7.11.** $\delta: N(X_{\mathcal{OS}}) \to \mathcal{F}_F(S)$ is an onto coframe homomorphism.

**Theorem 7.12.** For a topological space $S$, the following are equivalent.

1. $S$ is weakly scattered.
2. If $N \in N(X_{\mathcal{OS}})$ is nonempty, then so is $\delta(N)$.
3. $\delta$ is 1-1.
4. $\delta$ is an isomorphism.

**Proof.** (1) $\Rightarrow$ (2). Suppose $S$ is weakly scattered. Then $S_0$ is weakly scattered by Proposition 7.10(1). Since $S_0$ is $T_0$, we see that $S_0$ is sober by Lemma 7.5. Thus, $\varepsilon: S_0 \to Y_{\mathcal{OS}}$ is a homeomorphism.

Therefore, $Y_{\mathcal{OS}}$ is weakly scattered, so $N(\mathcal{OS})$ is spatial by Theorem 7.3. Let $\varnothing \neq N \in N(X_{\mathcal{OS}})$. Then $\gamma(N) \neq \varnothing$ by Theorem 5.10. Because $\varepsilon$ is a homeomorphism, the pullback map $\varepsilon^{-1}: \mathcal{F}_\pi(Y_{\mathcal{OS}}) \to \mathcal{F}_F(S)$ is an isomorphism. Thus, $\delta(N) = \varepsilon^{-1}(\gamma(N)) \neq \varnothing$.

(2) $\Rightarrow$ (3). Suppose $N, M \in N(X_{\mathcal{OS}})$ with $N \not\subseteq M$. Since $M$ is closed, there is a clopen set $U$ with $M \subseteq U$ and $N \not\subseteq U$. Then $N \setminus U$ is a nonempty nuclear set, so $\delta(N \setminus U) \neq \varnothing$ by (2). Therefore, there is $s \in S$ with $\varepsilon(s) \in N \setminus U$. Thus, $\varepsilon(s) \in N$ but $\varepsilon(s) \notin M$. Consequently, $\delta(N) = \varepsilon^{-1}(N) \not\subseteq \varepsilon^{-1}(M) = \delta(M)$, and so $\delta$ is 1-1.

(3) $\Rightarrow$ (4). This is clear since $\delta$ is an onto coframe homomorphism by Lemma 7.11.

(4) $\Rightarrow$ (1). Suppose that $\delta$ is an isomorphism. Then $\gamma$ is 1-1, so $N(\mathcal{OS})$ is spatial by Theorem 5.10. Therefore, $Y_{\mathcal{OS}}$ is weakly scattered by Theorem 7.3. If $y \in Y_{\mathcal{OS}}$, then $\{y\} \in N(X_{\mathcal{OS}})$, so $\delta(\{y\}) \neq \varnothing$ since $\delta$ is an isomorphism. This implies $y \in \varepsilon[S]$. Thus, $\varepsilon$ is onto, and so $Y_{\mathcal{OS}}$ is homeomorphic to $S_0$. Consequently, $S_0$ is weakly scattered, and hence $S$ is weakly scattered by Proposition 7.10(1).

We can now recover one of the main results of [22]. For this we recall [22] Def. 3.2] that $\sigma: N(\mathcal{OS}) \to \mathcal{O}_F(S)$ is defined by $\sigma(j) = \bigcup \{j(U) \setminus U \mid U \in \mathcal{OS}\}$ for each $j \in N(\mathcal{OS})$. By [22] Thm. 3.6], $\sigma$ is an onto frame homomorphism.

**Corollary 7.13.** [22] Thm. 4.4] A topological space $S$ is weakly scattered iff $\sigma: N(\mathcal{OS}) \to \mathcal{O}_F(S)$ is 1-1 (and hence an isomorphism).
Proof. By Theorem 7.12 it suffices to prove that \( \sigma \) is 1-1 iff \( \delta \) is 1-1. Let \( j \in \mathcal{N}(\mathcal{O}S) \) and let \( N_j \) be the corresponding nuclear set. We show \( \sigma(j) = S \setminus \delta(N_j) \). Let \( U \in \mathcal{O}S \). We have

\[
N_j \subseteq (N_j \setminus \varphi(U)) \cup \varphi(U) \subseteq \downarrow(N_j \setminus \varphi(U)) \cup \varphi(U),
\]

so

\[
[X_{\mathcal{O}S} \setminus \downarrow(N_j \setminus \varphi(U))] \setminus \varphi(U) \subseteq X_{\mathcal{O}S} \setminus N_j.
\]

Recalling that \( j \) satisfies

\[
\varphi(j(U)) = X_{\mathcal{O}S} \setminus \downarrow(N_j \setminus \varphi(U)),
\]

the last inclusion implies that \( \varphi(j(U)) \subseteq X_{\mathcal{O}S} \setminus N_j \), and \( \delta(\varphi(j(U))) \setminus \varphi(U) \subseteq \delta(X_{\mathcal{O}S} \setminus N_j) \). Therefore, \( j(U) \setminus U \subseteq S \setminus \delta(N_j) \). Thus, \( \sigma(j) \subseteq S \setminus \delta(N_j) \).

For the reverse inclusion, let \( s \in S \setminus \delta(N_j) \). Then \( \varepsilon(s) \notin N_j \). Therefore, there are \( U, V \in \mathcal{O}S \) with \( \varepsilon(s) \in \varphi(U) \setminus \varphi(V) \) and \( N_j \cap (\varphi(U) \setminus \varphi(V)) = \emptyset \). Thus, \( \varepsilon(s) \notin \varphi(V) \) and \( (N_j \setminus \varphi(V)) \setminus \varphi(U) = \emptyset \). Since \( \varphi(U) \) is an upset, \( \varphi(U) \cap \downarrow(N \setminus \varphi(V)) = \emptyset \), which implies \( \varepsilon(s) \notin \downarrow(N_j \setminus \varphi(U)) \) because \( \varepsilon(s) \in \varphi(U) \). This yields \( \varepsilon(s) \in \varphi(j(V)) \setminus \varphi(V) \), so \( s \in j(V) \setminus V \). Consequently, \( S \setminus \delta(N_j) \subseteq \sigma(j) \), and hence \( \sigma(j) = S \setminus N_j \). From this it follows that \( \sigma \) is 1-1 iff \( \delta \) is 1-1, completing the proof. \( \square \)

We conclude the paper by determining when \( \mathcal{N}(\mathcal{O}S) \) is boolean, and recovering another main result of Simmons [22, Thm. 4.5].

**Theorem 7.14.** For a spatial frame \( L \), the following conditions are equivalent.

1. \( N(L) \) is boolean.
2. \( N(L) \) is a complete and atomic boolean algebra.
3. \( (Y_L, \tau) \) is scattered.

Proof. (1) \( \Rightarrow \) (2). Since \( N(L) \) is boolean, \( N(X_L) = \text{RC}(X_L) \) (see Theorem 6.1). To see that \( N(L) \) is atomic, let \( N \in N(X_L) \) be nonempty. As \( N \) is regular closed, \( N = \text{cl}_\pi(\text{int}_\pi(N)) \). Because \( L \) is spatial, \( Y_L \) is dense in \( X_L \) by Theorem 5.5. Therefore, \( N = \text{cl}_\pi(\text{int}_\pi(N) \cap Y_L) \). Since \( N \) is nonempty, so is \( \text{int}_\pi(N) \cap Y_L \). Thus, each \( y \in \text{int}_\pi(N) \cap Y_L \) gives rise to the atom \( \{y\} \) of \( N(L) \) underneath \( N \). This yields that \( N(X_L) \) is an atomic boolean algebra. Consequently, \( N(L) \) is a complete and atomic boolean algebra.

(2) \( \Rightarrow \) (3). Let \( F \neq \emptyset \) be a closed subset of \( (Y_L, \tau) \). By Lemma 5.3(2), \( F = D \cap Y_L \) for some clopen downset \( D \) of \( (X_L, \pi) \). Since \( N(L) \) is boolean, \( \text{max}(D) \) is clopen by Theorem 6.1. As \( \text{max}(D) \) is nonempty, \( \text{max}(D) \cap Y_L \neq \emptyset \) because \( Y_L \) is dense in \( X_L \) by Theorem 5.5. Let \( y \in \text{max}(D) \cap Y_L \), and set \( U = X_L \setminus (D \setminus \{y\}) \). Since \( y \in Y_L \), the singleton \( \{y\} \) is nuclear, hence regular closed because \( N(L) \) is boolean. This implies \( y \) is an isolated point of \( X_L \), so \( U \) is clopen, and it is an upset as \( y \in \text{max}(D) \) and \( D \) is a downset. Clearly \( U \cap D = \{y\} \). Thus, \( U \cap D \cap Y_L = \{y\} \), and so \( y \) is an isolated point of \( F = D \cap Y_L \), proving that \( Y_L \) is scattered.

(3) \( \Rightarrow \) (1). It is sufficient to show that \( N(X_L) = \text{RC}(X_L) \). Let \( N \in N(X_L) \). Then \( \text{cl}_\pi(N \cap Y_L) \) is nuclear by Lemma 4.8 and is contained in \( N \). We have \( N \cap Y_L = \text{cl}_\pi(N \cap Y_L \cap Y_L) \). By Theorem 7.12 \( \delta \) is 1-1, so \( \gamma \) is 1-1, which yields \( N = \text{cl}_\pi(N \cap Y_L) \). We show that each
$y \in Y_L$ is isolated in $X_L$. Let $y \in Y_L$. Then $\downarrow y \cap Y_L \in \mathcal{F}_\tau(Y_L)$. Since $(Y_L, \tau)$ is scattered, there is an isolated point $z$ of $\downarrow y \cap Y_L$. By Lemma 5.3(1), there is a clopen upset $U$ of $X_L$ with \{z\} = $U \cap \downarrow y \cap Y_L$. Since $z \in \downarrow y$ we have $z \leq y$. Therefore, $y \in U$, and so $y \in U \cap \downarrow y \cap Y_L$.

This implies $z = y$. The set $U \cap \downarrow y$ is clopen in $X_L$, so $(U \cap \downarrow y) \setminus \{y\}$ is open in $X_L$. Since $Y_L$ is dense in $X_L$ and misses $(U \cap \downarrow y) \setminus \{y\}$, it follows that \{y\} = $U \cap \downarrow y$, hence $y$ is an isolated point of $X_L$. From this we conclude that $N \cap Y_L \subseteq \text{int}_\pi(N)$, so $N = \text{cl}_\pi(\text{int}_\pi(N))$. Therefore, $N$ is regular closed. Thus, $N(X_L) \subseteq \text{RC}(X_L)$, hence $N(X_L) = \text{RC}(X_L)$ by Lemma 4.6(2). □

**Corollary 7.15.** Let $L$ be a spatial frame. Then $L$ is scattered iff $Y_L$ is scattered.

**Proof.** Apply Theorems 6.1 and 7.14. □

As another consequence of Theorems 6.1 and 7.14, we obtain Simmons’ theorem [22] for $T_0$-spaces.

**Corollary 7.16.** For a $T_0$-space $S$, the following are equivalent.

1. $N(S)$ is boolean.
2. $S$ is scattered.
3. $\mathcal{O}S$ is scattered.

**Proof.** (1) ⇒ (2). First suppose that $N(S)$ is boolean. By Theorem 7.14 $(Y_{\mathcal{O}S}, \tau)$ is scattered. Since $S$ is a $T_0$-space, $\varepsilon : S \to Y_{\mathcal{O}S}$ is a topological embedding. Thus, $S$ is scattered.

(2) ⇒ (1). Conversely, suppose $S$ is scattered. Since it is $T_0$, it is sober by Lemma 7.5. Thus, $S$ is homeomorphic to $(Y_{\mathcal{O}S}, \tau)$, and hence $(Y_{\mathcal{O}S}, \tau)$ is scattered. Applying Theorem 7.14 then yields that $N(\mathcal{O}S)$ is boolean.

(1) ⇔ (3). Apply Theorem 6.1. □

Since $\mathcal{O}S$ and $\mathcal{O}(S_0)$ are isomorphic frames, as an immediate consequence of Corollary 7.16 and Proposition 7.10(2), we arrive at the general form of Simmons’ theorem.

**Corollary 7.17.** [22] Thm. 4.5] For an arbitrary topological space $S$, the following are equivalent.

1. $N(S)$ is boolean.
2. $S$ is dispersed.
3. $\mathcal{O}S$ is scattered.

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