Enriched weakness

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Definition. Let $\mathcal{E}$ be a class of morphisms in a symmetric monoidal closed category $\mathcal{V}$. Let $f : A \to B$ be a morphism in a $\mathcal{V}$-category $\mathcal{K}$. We say that an object $C$ from $\mathcal{K}$ is $f$-injective over $\mathcal{E}$ when the induced morphism

$$\mathcal{K}(f, C) : \mathcal{K}(B, C) \to \mathcal{K}(A, C)$$

is in $\mathcal{E}$.

Given a class $\mathcal{F}$ of morphisms in $\mathcal{K}$, $C$ is $\mathcal{F}$-injective over $\mathcal{E}$ if it is $f$-injective for all $f \in \mathcal{F}$. $\mathcal{F}$-$\text{Inj}$ will denote the full subcategory of $\mathcal{K}$ consisting of $\mathcal{F}$-injective objects.

Examples. (1) For $\mathcal{E} =$ isomorphisms, one gets the classical enriched orthogonality.
(2) For $\mathcal{V} = \text{Set}$ and $\mathcal{E} =$ surjections, one gets the classical injectivity.
**Proposition 1.** $\mathcal{F}\text{-}\text{Inj}$ is closed in $\mathcal{K}$ under any class $\Phi$ of limits for which $\mathcal{E}$ is closed in $\mathcal{V}^2$ under $\Phi$-limits.

**Proposition 2.** $\mathcal{F}\text{-}\text{Inj}$ is closed in $\mathcal{K}$ under any class $\Phi$ of colimits for which

1. $\mathcal{E}$ is closed under $\Phi$-colimits;

2. $\mathcal{K}(A, -)$ preserves $\Phi$-colimits for any object $A$ which is the domain or the codomain of a morphism in $\mathcal{F}$. 
Let $G : \mathcal{K} \to \mathcal{L}$ be $\mathcal{V}$-functor. We say that a family of morphisms

$$(\eta_L : L \to UFL)_{L \in \mathcal{L}}$$

makes $F$ a weak left adjoint to $G$ if the induced morphisms

$$\mathcal{K}(FL, K) \xrightarrow{G} \mathcal{K}(GFL, GK) \xrightarrow{\mathcal{K}(\eta_L, GK)} \mathcal{K}(L, GK)$$

are in $\mathcal{E}$.

Of course, $F$ does not need to be a functor.

Given $\mathcal{V}$-functors $D : \mathcal{D} \to \mathcal{K}$ and $H : \mathcal{D}^{\text{op}} \to \mathcal{V}$, $H \ast_w D$ is a weak colimit of $D$ weighted by $H$ if the induced morphism

$$\mathcal{K}(H \ast_w D, K) \to [\mathcal{D}^{\text{op}}, \mathcal{V}](H, \mathcal{K}(D, K))$$

is in $\mathcal{E}$. 
The right choice for enriching classical injectivity is 
\[ \mathcal{E} = \text{pure epimorphisms}. \]
The reason is that the latter are precisely filtered colimits of split epimorphisms. One has to assume that \( \mathcal{V} \) is locally finitely presentable as a closed category. This means that the underlying ordinary category \( \mathcal{V}_0 \) is locally finitely presentable and the full subcategory of finitely presentable objects is closed under the monoidal structure.

**Theorem 1.** The following conditions are equivalent for a full subcategory \( \mathcal{A} \) of a locally presentable \( \mathcal{V} \)-category:

1. \( \mathcal{A} = \mathcal{F}-\text{Inj} \) for a set \( \mathcal{F} \);
2. \( \mathcal{A} \) is accessible, accessibly embedded, and closed under products and finite cotensors;
3. \( \mathcal{A} \) is accessibly embedded and weakly reflective.
Theorem 2. The following are equivalent for a \( \mathcal{V} \)-category \( \mathcal{A} \):

1. \( \mathcal{A} \) is accessible and weakly cocomplete;
2. \( \mathcal{A} \) is accessible and has products and finite cotensors;
3. \( \mathcal{A} \) is a small injectivity class in some locally presentable \( \mathcal{V} \)-category;
4. \( \mathcal{A} \) is weakly reflective, accessibly embedded subcategory of \([\mathcal{C}, \mathcal{V}]\) for some small \( \mathcal{V} \)-category \( \mathcal{C} \);
5. \( \mathcal{A} \) is equivalent to the category of models of a (limit, \( \mathcal{E} \))-sketch.

A \( \mathcal{V} \)-functor is a model of a (limit, \( \mathcal{E} \))-sketch if it preserves specified limits and sends specified morphisms to pure epimorphisms.
Now, take $\mathcal{V} = \textbf{Cat}$ and $\mathcal{E} = \text{equivalences}.$

**Theorem 3.** The following conditions are equivalent for a full subcategory $\mathcal{A}$ of a locally presentable 2-category:

1. $\mathcal{A} = \mathcal{F}\text{-}\text{Inj}$ for a set $\mathcal{F};$

2. $\mathcal{A}$ is accessible, accessibly embedded, closed under flexible limits and $2$-replete;

3. $\mathcal{A}$ is accessible, accessibly embedded, weakly reflective and $2$-replete.

Flexible limit $\{H, D\}$ is a limit weighted by a retract $H : \mathcal{D} \to \textbf{Cat}$ of some $G'$ where $'$ denotes left adjoint to the inclusion

$$[\mathcal{D}, \textbf{Cat}] \to \text{Psd}[\mathcal{D}, \textbf{Cat}]$$

where $\text{Psd}[\mathcal{D}, \textbf{Cat}]$ denotes the 2-category of 2-functors, pseudonatural transformations and modifications.

If $\mathcal{K}$ has all flexible limits then it has all pseudolimits:

$$\{H, D\}_p \cong \{H', D\}$$
Theorem 4. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:

1. $\mathcal{A}$ is a small injectivity class in some locally presentable 2-category;

2. $\mathcal{A}$ is equivalent to the category of models of a $(\text{limit}, \mathcal{E})$-sketch.

One does not have the analogy of Theorem 2 here: the full subcategory of $\text{Cat}$ consisting of the terminal category and the free-living isomorphism is accessible, accessible embedded and weakly reflective but does not have flexible limits.

But one has the analogies of Theorems 1 and 2 for the choice of $\mathcal{V} = \text{Cat}$ and $\mathcal{E} =$ retract equivalences. Products and finite cotensors are replaced by flexible limits.

Notice that both pure epimorphisms and retract equivalences are right parts of weak factorization systems in $\text{Cat}$ and that retracts equivalences are pure epimorphisms.