RIESZ POTENTIALS, BESSEL POTENTIALS AND FRACTIONAL DERIVATIVES ON BESOV-LIPSCHITZ SPACES FOR THE GAUSSIAN MEASURE.

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Abstract. In [5] Gaussian Lipschitz spaces $\text{Lip}_\alpha(\gamma_d)$ were considered and then the boundedness properties of Riesz Potentials, Bessel potentials and Fractional Derivatives were studied in detail. In this paper we will study the boundedness of those operators on Gaussian Besov-Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$. Also these results can be extended to the case of Laguerre or Jacobi expansions and even further to the general framework of diffusions semigroups.

1. Introduction

On $\mathbb{R}^d$ let us consider the Gaussian measure

$$\gamma_d(x) = \frac{e^{-|x|^2}}{\pi^{d/2}} dx, \; x \in \mathbb{R}^d$$

and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle.$$ 

Let $\nu = (\nu_1, \ldots, \nu_d)$ be a multi-index such that $\nu_i \geq 0, i = 1, \ldots, d$, let $\nu! = \prod_{i=1}^d \nu_i!, \; |\nu| = \sum_{i=1}^d \nu_i, \; \partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_1^{\nu_1} \cdots \partial_d^{\nu_d}$, consider the normalized Hermite polynomials of order $\nu$ in $d$ variables,

$$h_\nu(x) = \frac{1}{(2|\nu|!)^{1/2}} \prod_{i=1}^d (-1)^{\nu_i} e^{x_i^2} \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}} (e^{-x_i^2}),$$

it is well known, that the Hermite polynomials are eigenfunctions of the operator $L$,

$$Lh_\nu(x) = -|\nu| \; h_\nu(x).$$

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Given a function $f \in L^1(\gamma_d)$ its $\nu$-Fourier-Hermite coefficient is defined by

$$\hat{f}(\nu) = \langle f, h_\nu \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_\nu(x) \gamma_d(dx).$$

Let $C_n$ be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\{h_\nu : |\nu| = n\}$. By the orthogonality of the Hermite polynomials with respect to $\gamma_d$ it is easy to see that $\{C_n\}$ is an orthogonal decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n,$$

this decomposition is called the Wiener chaos.

Let $J_n$ be the orthogonal projection of $L^2(\gamma_d)$ onto $C_n$, then if $f \in L^2(\gamma_d)$

$$J_n f = \sum_{|\beta|=n} \hat{f}(\nu) h_\nu.$$

Let us define the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t \geq 0}$ as

$$T_t f(x) = \frac{1}{(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-2t(|x|^2 + |y|^2) - 2t\langle x, y \rangle} f(y) \gamma_d(dy)$$

(1.5)

$$= \frac{1}{\pi^{d/2}(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-tx|^2}{1-e^{-2t}}} f(y) dy$$

The family $\{T_t\}_{t \geq 0}$ is a strongly continuous Markov semigroup on $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $L$. Also, by a change of variable we can write,

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_t^{1/4u} f(x) du$$

(1.6)

Now, by Bochner subordination formula, see Stein [11] page 61, we define the Poisson-Hermite semigroup $\{P_t\}_{t \geq 0}$ as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_t^{1/4u} f(x) du$$

(1.7)

From (1.5) we obtain, after the change of variable $r = e^{-t^2/4u}$,

$$P_t f(x) = \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 \frac{t \exp(t^2/4 \log r) \exp(-|y-rx|^2/1-r^2)}{(-\log r)^{3/2} (1-r^2)^{d/2}} dr f(y) dy$$

(1.8)
with

\[ p(t, x, y) = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 t \exp \left( \frac{t^2}{4} \log r \right) \exp \left( \frac{-|y-rx|^2}{1-r^2} \right) \frac{dr}{(1-r^2)^{d/2}}. \]

Also by the change of variable \( s = \frac{t^2}{4}u \) we have,

\[ P_tf(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}f(x)du = \int_0^\infty T_{su}f(x) \mu_t^{(1/2)}(ds), \]

where the measure

\[ \mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds, \]

is called the one-side stable measure on \((0, \infty)\) of order \(1/2\).

The family \( \{P_t\}_{t \geq 0} \) is also a strongly continuous semigroup on \( L^p(\gamma_d) \), \( 1 \leq p < \infty \), with infinitesimal generator \(-(-L)^{1/2}\). In what follows, often we are going to use the notation

\[ u(x, t) = P_tf(x), \]

and

\[ u^{(k)}(x, t) = \frac{\partial^k}{\partial t^k} P_t f(x). \]

Observe that by (1.4) we have that

\[ T_{t}h_{\nu}(x) = e^{-t|\nu|}h_{\nu}(x), \]

and

\[ P_{t}h_{\nu}(x) = e^{-t\sqrt{\nu}}h_{\nu}(x), \]

i.e. the Hermite polynomials are eigenfunctions of \( T_t \) and \( P_t \) for any \( t \geq 0 \).

The operators that we are going to consider in this paper are the following:

- For \( \beta > 0 \), the Fractional Integral or Riesz potential of order \( \beta \), \( I_{\beta} \), with respect to the Gaussian measure is defined formally as

\[ I_{\beta} = (-L)^{-\beta/2} \Pi_0, \]

where, \( \Pi_0 f = f - \int_{\mathbb{R}^d} f(y) \gamma_d(dy) \), for \( f \in L^2(\gamma_d) \). That means that for the Hermite polynomials \( \{h_{\beta}\} \), for \( |\beta| > 0 \),

\[ I_{\beta}h_{\nu}(x) = \frac{1}{|\nu|^{\beta/2}} h_{\nu}(x), \]
and for $\nu = 0$, $I_\beta(h_\nu) = 0$. Then by linearity can be extended to any polynomial.

Now, it is easy to see that if $f$ is a polynomial,

$$I_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1}(P_t f(x) - P_\infty f(x)) \, dt. \quad (1.16)$$

Moreover by P. A. Meyer’s multiplier theorem, see [7] or [17], $I_\alpha$ admits a continuous extension to $L^p(\gamma_d)$, $1 < p < \infty$, and then (1.16) can be extended for $f \in L^p(\gamma_d)$.

- The Bessel Potential of order $\beta > 0$, $\mathcal{J}_\beta$, associated to the Gaussian measure is defined formally as

$$\mathcal{J}_\beta = (I + \sqrt{-L})^{-\beta}, \quad (1.17)$$

meaning that for the Hermite polynomials we have,

$$\mathcal{J}_\beta h_\nu(x) = \frac{1}{(1 + |\nu|)^\beta} h_\nu(x).$$

Again by linearity can be extended to any polynomial and Meyer’s theorem allows us to extend Bessel Potentials to a continuous operator on $L^p(\gamma_d)$, $1 < p < \infty$. It can be proved that the Bessel potentials can be represented as

$$\mathcal{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1}e^{-t}P_t f(x) \frac{dt}{t}. \quad (1.18)$$

Moreover $\{\mathcal{J}_\beta\}_\beta$ is a strongly continuous semigroup on $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $\frac{1}{2} \log(I - L)$, see [3].

- The Riesz fractional derivate of order $\alpha > 0$ with respect to the Gaussian measure $D^\alpha$, is defined formally as

$$D^\alpha = (-L)^{\alpha/2}, \quad (1.19)$$

meaning that for the Hermite polynomials, we have

$$D^\alpha (x) = |\nu|^{\alpha/2} h_\nu(x), \quad (1.20)$$

thus by linearity can be extended to any polynomial.

The Riesz fractional derivate $D^\beta$ with respect to the Gaussian measure was first introduced in [6]. For more detail we refer to that article. Also see [8] for improved and simpler proofs of some results contained there. In the case of $0 < \beta < 1$ we have the following integral representation,

$$D^\beta f = \frac{1}{c_\beta} \int_0^\infty t^{-\beta-1}(P_t - I) f \, dt, \quad (1.21)$$
where \( c_\beta = \int_0^\infty u^{-\beta-1}(e^{-u}-1)du \). Moreover for \( f \in C^2_\beta(\mathbb{R}^d) \), i.e. the set of two times continuously differentiable functions with bounded derivatives, then it can be proved using integration by parts (for details see [6]), that

\[
D^\beta f = \frac{1}{\beta c_\beta} \int_0^\infty t^{-\beta} \frac{\partial}{\partial t} P_t f dt.
\]

Moreover, if \( f \in C^2_\beta(\mathbb{R}^d) \), i.e. \( k \) be the smallest integer greater than \( \beta \) i.e. \( k-1 \leq \beta < k \), then the fractional derivative \( D^\beta \) can be represented as

\[
D^\beta f = \frac{1}{c_k^\beta} \int_0^\infty t^{-\beta-1}(P_t - I)^k f dt,
\]

where \( c_k^\beta = \int_0^\infty u^{-\beta-1}(e^{-u}-1)^k du \). Now, if \( f \) is a polynomial, by the linearity of the operators \( I_\beta \) and \( D^\beta \), we get

\[
\Pi_0 f = I_\beta(D^\beta f) = D^\beta(I_\beta f).
\]

• We can also consider a Bessel fractional derivative \( D^\beta \), defined formally as

\[
D^\beta = (I + \sqrt{-L})^\beta,
\]

which means that for the Hermite polynomials, we have

\[
D^\beta h_\nu(x) = (1 + \sqrt{\nu})^\beta h_\nu(x),
\]

In the case of \( 0 < \beta < 1 \) we have the following integral representation,

\[
D^\beta f = \frac{1}{c_\beta} \int_0^\infty t^{-\beta-1}(e^{-t}P_t - I) f dt,
\]

where, as before, \( c_\beta = \int_0^\infty u^{-\beta-1}(e^{-u}-1)du \). Moreover, if \( \beta > 1 \) let \( k \) be the smallest integer greater than \( \beta \) i.e. \( k-1 \leq \beta < k \), then the fractional derivative \( D^\beta \) can be represented as

\[
D^\beta f = \frac{1}{c_k^\beta} \int_0^\infty t^{-\beta-1}(e^{-t}P_t - I)^k f dt,
\]

where \( c_k^\beta = \int_0^\infty u^{-\beta-1}(e^{-u}-1)^k du \).

The Gaussian Besov-Lipschitz \( B^\alpha_{p,q}(\gamma_d) \) spaces were introduced in [9], see also [8], as follows.
Definition 1.1. Let $\alpha > 0$, $k$ be the smallest integer greater than $\alpha$, and $1 \leq p, q \leq \infty$. For $1 \leq q < \infty$ the Gaussian Besov-Lipschitz space $B^\alpha_{p,q}(\gamma_d)$ are defined as the set of functions $f \in L^p(\gamma_d)$ for which

\begin{equation}
(\int_0^{+\infty}(t^{k-\alpha}\|\frac{\partial^k P_tf}{\partial t^k}\|_{p,\gamma_d})^q \frac{dt}{t})^{1/q} < \infty.
\end{equation}

The norm of $f \in B^\alpha_{p,q}(\gamma_d)$ is defined as

\begin{equation}
\|f\|_{B^\alpha_{p,q}} := \|f\|_{p,\gamma_d} + \left(\int_0^{+\infty}(t^{k-\alpha}\|\frac{\partial^k P_tf}{\partial t^k}\|_{p,\gamma_d})^q \frac{dt}{t}\right)^{1/q}
\end{equation}

For $q = \infty$ the Gaussian Besov-Lipschitz space $B^\alpha_{p,\infty}(\gamma_d)$ are defined as the set of functions $f \in L^p(\gamma_d)$ for which exists a constant $A$ such that

$$\|\frac{\partial^k P_tf}{\partial t^k}\|_{p,\gamma_d} \leq At^{-k+\alpha}$$

and then the norm of $f \in B^\alpha_{p,\infty}(\gamma_d)$ is defined as

\begin{equation}
\|f\|_{B^\alpha_{p,\infty}} := \|f\|_{p,\gamma_d} + A_k(f),
\end{equation}

where $A_k(f)$ is the smallest constant $A$ appearing in the above inequality. In particular, the space $B^\alpha_{\infty,\infty}(\gamma_d)$ is the Gaussian Lipschitz space $Lip_\alpha(\gamma_d)$.

The definition of $B^\alpha_{p,q}(\gamma_d)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent, for the proof of this result and other properties of these spaces see [9].

In what follows, we need the following technical result about $L^p(\gamma_d)$-norms of the derivatives of the Poisson-Hermite semigroup, see [9], Lemma 2.2

Lemma 1.1. Suppose $f \in L^p(\gamma_d)$, $1 \leq p < \infty$ then for any integer $k$ the function $\|\frac{\partial^k P_tf}{\partial t^k}\|_{p,\gamma_d}$ is a non-increasing function of $t$, for $0 < t < +\infty$. Moreover,

\begin{equation}
\|\frac{\partial^k P_tf}{\partial t^k}\|_{p,\gamma_d} \leq C\|f\|_{p,\gamma_d}t^{-k}, t > 0
\end{equation}

Also we will need some inclusion relations among the Gaussian Besov-Lipschitz spaces, see [9],

Proposition 1.1. The inclusion $B^\alpha_{p,q_1}(\gamma_d) \subset B^\alpha_{p,q_2}(\gamma_d)$ holds if either:

i) $\alpha_1 > \alpha_2 > 0$ where $q_1$ and $q_2$ need not to be related, or
ii) If $\alpha_1 = \alpha_2$ and $q_1 \leq q_2$. 
In [5] Gaussian Lipschitz spaces Lip$_{\alpha}(\gamma_d)$ were considered and the boundedness of Riesz Potentials, Bessel potentials and Fractional Derivatives on them were study. In the next section, we are going to extend those results for Gaussian Besov-Lipschitz spaces, but not including them. Thus, the main purpose of this paper is to study the boundedness of Gaussian fractional integrals and derivatives associated to Hermite polynomial expansions on Gaussian Besov-Lipschitz spaces $B_{p,q}^\alpha(\gamma_d)$ . To get these results we introduce formulas for these operators in terms of the Hermite-Poisson semigroup as well as the Gaussian Besov-Lipschitz spaces. This approach was originally developed for the classical Poisson integral, see Stein [11], Chapter V Section 5. These proofs can also be extended to the case of Laguerre and Jacobi expansions. These results can be also obtained using abstract interpolation theory on the the Poisson-Hermite semigroup, see [14].

As usual in what follows $C$ represents a constant that is not necessarily the same in each occurrence.

2. Main results

In the case of the Lipschitz spaces only a truncated version of the Riesz potentials is bounded from Lip$_{\alpha}(\gamma_d)$ to Lip$_{\alpha+\beta}(\gamma_d)$, see [5] Theorem 3.2. Now, we wil study the boundedness properties of the Riesz potentials on Besov-Lipschitz spaces, and we will see that in this case the results are actually better.

**Theorem 2.1.** Let $\alpha \geq 0, \beta > 0, 1 < p < \infty, 1 \leq q \leq \infty$ then $I_\beta$ is bounded from $B_{p,q}^\alpha(\gamma_d)$ into $B_{p,q}^{\alpha+\beta}(\gamma_d)$.

**Proof.**

Let $k > \alpha + \beta$ a fixed integer, $f \in B_{p,q}^\alpha(\gamma_d)$, using the integral representation of Riesz Potentials (1.16), the semigroup property of $\{P_t\}$ and the fact that $P_\infty f(x)$ is a constant and the semigroup is conservative, we get

$$P_t I_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} P_t (P_sf(x) - P_\infty f(x)) ds$$

$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} (P_{t+s}f(x) - P_\infty f(x)) ds.$$  (2.1)
Now using again that $P_\infty f(x)$ is a constant and the chain rule,
\[
\frac{\partial^k}{\partial t^k} (P_\infty f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \frac{\partial^k}{\partial t^k} (P_{t+s} f(x) - P_\infty f(x)) \, ds
\]
(2.2)
\[
= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} u^{(k)}(x, t) \, ds.
\]

Now, by Minkowski's integral inequality
\[
\| \frac{\partial^k}{\partial t^k} P_\infty f \|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \| u^{(k)}(\cdot, t+s) \|_{p,\gamma} \, ds.
\]
(2.3)

Then, if $1 \leq q < \infty$,
\[
\left( \int_0^{+\infty} \left( t^{-(\alpha+\beta)} \left\| \frac{\partial^k}{\partial t^k} (P_\infty f) \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
\leq \frac{1}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{-(\alpha+\beta)q} \left( \int_0^{+\infty} s^{\beta-1} \| u^{(k)}(\cdot, t+s) \|_{p,\gamma} \, ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
\leq C_\beta \left( \int_0^{+\infty} t^{-(\alpha+\beta)q} \left( \int_0^{+\infty} s^{\beta-1} \| u^{(k)}(\cdot, t+s) \|_{p,\gamma} \, ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
+ C_\beta \left( \int_0^{+\infty} t^{-(\alpha+\beta)q} \left( \int_t^{+\infty} s^{\beta-1} \| u^{(k)}(\cdot, t+s) \|_{p,\gamma} \, ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
= (I) + (II).
\]

Now, as $\beta > 0$ using Lemma 1.1 as $t+s > t$,

(I) \leq C_\beta \left( \int_0^{+\infty} t^{-(\alpha+\beta)q} \left( \int_0^t s^{\beta-1} \| u^{(k)}(\cdot, t) \|_{p,\gamma} \, ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
= C_\beta \left( \int_0^{+\infty} t^{-(\alpha+\beta)q} \left\| \frac{\partial^k}{\partial t^k} P_\infty f \right\|_{p,\gamma} \left( \frac{t^\beta}{\beta} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
= C_\beta' \left( \int_0^{+\infty} (t^{-(\alpha+\beta)q} \left\| \frac{\partial^k}{\partial t^k} P_\infty f \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty,
\]
since $f \in B_\alpha^0(\gamma_d)$.

On the other hand, as $k > \alpha + \beta$ using again Lemma 1.1 since $t+s > s$, and Hardy’s inequality (2.3), we obtain

(II) \leq C_\beta \left( \int_0^{+\infty} t^{-(\alpha+\beta)q} \left( \int_t^{+\infty} s^{\beta} \| u^{(k)}(\cdot, s) \|_{p,\gamma} \, ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]
\[
\leq \frac{C_\beta}{k-(\alpha+\beta)} \int_0^{+\infty} \left( s^{-(\alpha+\beta)} \left\| \frac{\partial^k}{\partial s^k} P_\infty f \right\|_{p,\gamma} \right)^q \frac{ds}{s} \right)^{\frac{1}{q}} < +\infty,
\]
since \( f \in B^\alpha_{p,q}(\gamma_d) \). Therefore \( I_\beta f \in B^\alpha_{p,q}(\gamma_d) \) and moreover,

\[
\|I_\beta f\|_{B^\alpha_{p,q}} = \|I_\beta f\|_{p,\gamma} + \left( \int_0^{+\infty} (t^{k-(\alpha+\beta)} \|\frac{\partial^k}{\partial t^k}(P_tI_\beta f)\|_{p,\gamma})^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\leq C\|f\|_{p,\gamma} + C_{\alpha,\beta} \left( \int_0^{+\infty} (t^{k-\alpha} \|\frac{\partial^k}{\partial t^k}(P_t f)\|_{p,\gamma})^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\leq C\|f\|_{B^\alpha_{p,q}}.
\]

Now if \( q = \infty \), (2.3) can be written as

\[
\|\frac{\partial^k}{\partial t^k} P_t I_\beta f\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds
\]

\[
= \frac{1}{\Gamma(\beta)} \int_0^t s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_t^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds
\]

\[
= (I) + (II).
\]

Now, using that \( \beta > 0 \), Lemma 1.1 as \( t+s > t \) and since \( f \in B^\alpha_{p,\infty}(\gamma_d) \),

\[
(I) \leq \frac{1}{\Gamma(\beta)} \|\frac{\partial^k}{\partial t^k} f\|_{p,\gamma} \int_0^t s^{\beta-1} ds \leq \frac{1}{\Gamma(\beta)} \frac{t^\beta}{\beta} A_k(f)t^{-k+\alpha}
\]

\[
= C_\beta A_k(f) t^{-k+\alpha+\beta}.
\]

On the other hand, since \( k > \alpha + \beta \), using Lemma 1.1 as \( t+s > s \) and since \( f \in B^\alpha_{p,\infty}(\gamma_d) \), we get

\[
(II) \leq \frac{1}{\Gamma(\beta)} \int_t^{+\infty} s^{\beta-1} \|\frac{\partial^k}{\partial s^k} P_s f\|_{p,\gamma} ds \leq \frac{A_k(f)}{\Gamma(\beta)} \int_t^{+\infty} s^{-k+\alpha+\beta-1} ds
\]

\[
= \frac{A_k(f)}{\Gamma(\beta)} \frac{t^{-k+\alpha+\beta}}{k-(\alpha+\beta)} = C_{k,\alpha,\beta} t^{-k+\alpha+\beta}.
\]

Therefore

\[
\|\frac{\partial^k}{\partial t^k} P_t I_\beta f\|_{p,\gamma} \leq CA_k(f)t^{-k+\alpha+\beta}, \quad t > 0,
\]

and this implies that \( I_\beta f \in B^\alpha_{p,\infty}(\gamma_d) \) and \( A_k(I_\beta f) \leq CA_k(f) \).

Moreover, as \( I_\beta \) is bounded operator on \( L^p(\gamma_d) \), \( 1 < p < \infty \),

\[
\|I_\beta f\|_{B^\alpha_{p,\infty}} = \|I_\beta f\|_{p,\gamma} + A_k(I_\beta f)
\]

\[
\leq \|f\|_{p,\gamma} + CA_k(f) \leq C\|f\|_{B^\alpha_{p,\infty}}.
\]

\[\square\]
Now we want to study the boundedness properties of the Bessel potentials on Besov-Lipschitz spaces. In [5], Theorem 3.1, the following result was proved,

**Theorem 2.2.** Let \( \alpha \geq 0, \beta > 0 \) then \( J_\beta \) is bounded from \( \text{Lip}_\alpha(\gamma_d) \) into \( \text{Lip}_{\alpha+\beta}(\gamma_d) \).

Also in [9], Theorem 2.4, it was proved that

**Theorem 2.3.** Let \( \alpha \geq 0, \beta > 0 \) then for \( 1 \leq p, q < \infty \) \( J_\beta \) is bounded from \( B^\alpha_{p,q}(\gamma_d) \) into \( B^{\alpha+\beta}_{p,q}(\gamma_d) \).

Therefore the following result is the only case that was missing,

**Theorem 2.4.** Let \( \alpha \geq 0, \beta > 0 \) then for \( 1 \leq p < \infty \) \( J_\beta \) is bounded from \( B^\alpha_{p,\infty}(\gamma_d) \) into \( B^{\alpha+\beta}_{p,\infty}(\gamma_d) \).

**Proof.**

Let \( k > \alpha + \beta \) a fixed integer, \( f \in B^\alpha_{p,\infty}(\gamma_d) \), by using the representation of Bessel potential (1.18), we get

\[
P_t(J_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} P_t (f)(x) \frac{ds}{s},
\]

thus using the chain rule, we obtain

\[
\frac{\partial^k}{\partial t^k} P_t(J_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} u^{(k)}(x, t + s) \frac{ds}{s},
\]

this implies, using Minkowski’s integral inequality,

\[
\| \frac{\partial^k}{\partial t^k} P_t(J_\beta f) \|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^\beta e^{-s} \| u^{(k)}(\cdot, t + s) \|_{p,\gamma} \frac{ds}{s} \\
= \frac{1}{\Gamma(\beta)} \int_0^{t} s^\beta e^{-s} \| u^{(k)}(\cdot, t + s) \|_{p,\gamma} \frac{ds}{s} \\
+ \frac{1}{\Gamma(\beta)} \int_t^{\infty} s^\beta e^{-s} \| u^{(k)}(\cdot, t + s) \|_{p,\gamma} \frac{ds}{s} \\
= (I) + (II).
\]

Now, as \( \beta > 0 \), using Lemma [11] (as \( t + s > t \) and since \( f \in B^\alpha_{p,\infty}(\gamma_d) \),

\[
(I) \leq \frac{1}{\Gamma(\beta)} \int_0^t s^\beta e^{-s} \frac{ds}{s} \leq \frac{1}{\Gamma(\beta)} \| \frac{\partial^k}{\partial t^k} f \|_{p,\gamma} \int_0^t s^{\beta-1} ds \\
\leq \frac{1}{\Gamma(\beta)} \beta^{\beta} A_k(f) t^{-k+\alpha} = C_\beta A_k(f) t^{-k+\alpha+\beta}.
\]
On the other hand, as \( k > \alpha + \beta \) using Lemma 1.1 as \( t + s > s \), and since \( f \in B_{p,\infty}^{\alpha}(\gamma_d) \)

\[
\begin{align*}
(II) & \leq \frac{1}{\Gamma(\beta)} \int_t^\infty s^\beta e^{-s} \| \partial^k P_s f \|_{p,\gamma} ds \\
& \leq \frac{A_k(f)}{\Gamma(\beta)} \int_t^\infty s^{-k+\alpha+\beta-1} ds = \frac{A_k(f)}{\Gamma(\beta)} \frac{t^{-k+\alpha+\beta}}{k-(\alpha+\beta)} = C_{k,\alpha,\beta} A_k(f) t^{-k+\alpha+\beta}.
\end{align*}
\]

Therefore

\[
\left\| \partial^k \partial_t^k P_t (J^{\beta} f) \right\|_{p,\gamma} \leq C A_k(f) t^{-k+\alpha+\beta},
\]

then \( J^{\beta} f \in B^{\alpha+\beta}_{p,\infty}(\gamma_d) \) and \( A_k(J^{\beta} f) \leq C A_k(f) \). Thus,

\[
\left\| J^{\beta} f \right\|_{B^{\alpha+\beta}_{p,\infty}} = \left\| J^{\beta} f \right\|_{p,\gamma} + A_k(J^{\beta} f) \\
\leq \left\| f \right\|_{p,\gamma} + C A_k(f) \leq C \left\| f \right\|_{B^{\alpha}_{p,\infty}}.
\]

In what follows we will need Hardy’s inequalities, so for completeness we will write them here, see \[11\] page 272,

\[
(2.4) \quad \int_0^{+\infty} \left( \int_0^{+\infty} f(y) dy \right)^p x^{-r-1} dx \leq \frac{p}{r} \int_0^{+\infty} (y f(y))^p y^{-r-1} dy,
\]

and

\[
(2.5) \quad \int_0^{+\infty} \left( \int_x^{+\infty} f(y) dy \right)^p x^{-r-1} dx \leq \frac{p}{r} \int_0^{+\infty} (y f(y))^p y^{-r-1} dy,
\]

where \( f \geq 0, p \geq 1 \) and \( r > 0 \).

Now, we will study now the boundedness of the (Riesz) fractional derivative \( D^\beta \) on Besov-Lipschitz spaces. We will use the representation \[1.20\] of the fractional derivative and Hardy’s inequalities.

**Theorem 2.5.** Let \( 0 < \beta < \alpha < 1, 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \) then \( D^\beta \) is bounded from \( B^{\alpha}_{p,q}(\gamma_d) \) into \( B^{\alpha-\beta}_{p,q}(\gamma_d) \).

**Proof.**
Let \( f \in B^{\alpha}_{p,q}(\gamma_d) \), using Hardy’s inequality (2.4), with \( p = 1 \), and the Fundamental Theorem of Calculus,

\[
|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |P_s f(x) - f(x)| ds \\
\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \frac{\partial}{\partial r} P_r f(x) |dr ds \\
\leq \frac{1}{c_\beta^2} \int_0^{+\infty} r^{1-\beta} \left| \frac{\partial}{\partial r} P_r f(x) \right| \frac{dr}{r}.
\]

(2.6)

Thus, using Minkowski’s integral inequality

\[
\|D^\beta f\|_{p,\gamma} \leq C_\beta \int_0^{+\infty} r^{1-\beta} \left\| \frac{\partial}{\partial r} P_r f \right\|_{p,\gamma} \frac{dr}{r} < \infty,
\]

(2.7)

since \( f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\beta}_{p,1}(\gamma_d), 1 \leq q \leq \infty \) as \( \alpha > \beta \), i.e., \( D^\beta f \in L^p(\gamma_d) \).

Now, by analogous argument,

\[
\frac{\partial}{\partial t} P_t (D^\beta f)(x) = \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} [\frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x)] ds \\
= \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} u^{(2)}(x, r) dr ds
\]

and again, by Minkowski’s integral inequality

\[
\|\frac{\partial}{\partial t} P_t (D^\beta f)\| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot, r)\|_p dr ds
\]

(2.8)

Then, if \( 1 \leq q < \infty \), by (2.8)

\[
\int_0^{\infty} \left( t^{1-(\alpha-\beta)} \left\| \frac{\partial}{\partial t} P_t (D^\beta f) \right\|_{p,\gamma} \right)^q \frac{dt}{t} \\
\leq C_\beta \int_0^{\infty} \left( t^{1-(\alpha-\beta)} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot, r)\|_p dr ds \right)^q \frac{dt}{t} \\
= C_\beta \int_0^{\infty} \left( t^{1-(\alpha-\beta)} \int_0^t s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot, r)\|_p dr ds \right)^q \frac{dt}{t} \\
+ C_\beta \int_0^{\infty} \left( t^{1-(\alpha-\beta)} \int_t^{+\infty} s^{-\beta-1} \int_t^{t+s} \|u^{(2)}(\cdot, r)\|_p dr ds \right)^q \frac{dt}{t} \\
= (I) + (II).
\]
Now, since $r > t$ using Lemma 1.1 and the fact that $0 < \beta < 1$,

\[
(I) \leq C_\beta \int_0^\infty (t^{1-(\alpha-\beta)} \int_0^t s^{-\beta} ds \|u^{(2)}(\cdot, r)\|_{p,\gamma})^q \frac{dt}{t} = C_{\beta,q} \int_0^\infty (t^{2-\alpha} \|\frac{\partial^2}{\partial r^2} P_r f\|_{p,\gamma})^q \frac{dt}{t}.
\]

On the other hand, as $r > t$ using Hardy’s inequality (2.5), since $(1-\alpha)q > 0$, we get

\[
(II) \leq C_\beta \int_0^\infty t^{(1-(\alpha-\beta))q} \left( \int_t^\infty s^{-\beta-1} ds \int_0^\infty \|u^{(2)}(\cdot, r)\|_{p,\gamma} dr \right)^q \frac{dt}{t} = C'_\beta \int_0^\infty t^{(1-\alpha)q} \left( \int_t^\infty \|u^{(2)}(\cdot, r)\|_{p,\gamma} dr \right)^q \frac{dt}{t} \leq \frac{C'_\beta}{(1-\alpha)} \int_0^\infty (r^{2-\alpha} \|\frac{\partial^2}{\partial r^2} P_r f\|_{p,\gamma})^q \frac{dr}{r}.
\]

Thus,

\[
\left( \int_0^\infty \left( t^{1-\alpha+\beta} \frac{\partial}{\partial t} P_t D_\beta f \|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \leq C \left( \int_0^\infty \left( t^{2-\alpha} \|\frac{\partial^2}{\partial t^2} P_t f\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} < \infty,
\]

as $f \in B^\alpha_{p,q}(\gamma_d)$. Then, $D_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$ and

\[
\|D_\beta f\|_{B^{\alpha-\beta}_{p,q}} = \|D_\beta f\|_{p,\gamma} + \left( \int_0^\infty \left( t^{1-\alpha+\beta} \frac{\partial}{\partial t} P_t D_\beta f \|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \leq C_1 f\|_{B^\alpha_{p,q}} + C_2 \left( \int_0^\infty \left( t^{2-\alpha} \|\frac{\partial^2}{\partial t^2} P_t f\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \leq C \|f\|_{B^\alpha_{p,q}}.
\]

Therefore $D_\beta f : B^\alpha_{p,q} \to B^{\alpha-\beta}_{p,q}$ is bounded.

Now if $q = \infty$, inequality (2.8) can be written as

\[
\|\frac{\partial}{\partial t} P_t (D_\beta f)\|_{p,\gamma} \leq \frac{1}{c_\beta} \int_0^\infty \int_t^t s^{-\beta-1} ds \|\frac{\partial^2}{\partial r^2} P_r f\|_{p} dr \int_0^s \frac{\partial^2}{\partial t^2} P_t f\|_{p} dr ds = \frac{1}{c_\beta} \int_0^\infty \int_t^t s^{-\beta-1} ds \|\frac{\partial^2}{\partial r^2} P_r f\|_{p} dr \int_0^s \frac{\partial^2}{\partial t^2} P_t f\|_{p} dr ds = (I) + (II).
\]
Now, by Lemma 1.1 since $r > t$

\[ (I) \leq \frac{1}{c_\beta} \int_0^t s^{-\beta} \| \frac{\partial^2}{\partial t^2} P_t f \|_p \, ds = C_\beta \| \frac{\partial^2}{\partial t^2} P_t f \|_{p, \gamma} t^{1-\beta} \]

\[ \leq C_\beta A(f) t^{-2+\alpha - \beta} = C_\beta A(f) t^{-1+\alpha - \beta}, \]

and by Lemma 1.1 since $r > t$, and the fact that $f \in B_{p, \infty}^{\alpha}$,

\[ (II) \leq \frac{1}{c_\beta} \int_t^{+\infty} s^{-\beta - 1} \int_t^{s} \| \frac{\partial^2}{\partial r^2} P_r f \|_p \, dr \, ds \]

\[ \leq C_\beta t^{-\beta} \int_t^{\infty} \| \frac{\partial^2}{\partial r^2} P_r f \|_{p, \gamma} dr \leq C_\beta A(f) t^{-\beta} \int_t^{\infty} r^{-2+\alpha} dr \]

\[ = C_{\alpha, \beta} A(f) t^{-1+\alpha - \beta}. \]

Thus,

\[ \| \frac{\partial}{\partial t} P_t (D_\beta f) \|_{p, \gamma} \leq C A(f) t^{-1+\alpha - \beta}, \quad t > 0. \]

i.e., $D_\beta f \in B_{p, \infty}^{\alpha-\beta}(\gamma_d)$ then $A(D_\beta f) \leq CA(f)$, and

\[ \| D_\beta f \|_{B_{p, \infty}^{\alpha-\beta}} = \| D_\beta f \|_{p, \gamma} + A(D_\beta f) \]

\[ \leq C_1 \| f \|_{B_{p, \infty}^\alpha} + C_2 A(f) \leq C \| f \|_{B_{p, \infty}^\alpha}. \]

Therefore $D_\beta : B_{p, \infty}^\alpha \to B_{p, \infty}^{\alpha-\beta}$ is bounded.

Now we will study now the boundedness of the Bessel fractional derivative on Besov-Lipschitz spaces, for $0 < \beta < \alpha < 1$

**Theorem 2.6.** Let $0 < \beta < \alpha < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$ then $D_\beta$ is bounded from $B_{p,q}^\alpha(\gamma_d)$ into $B_{p,q}^{\alpha-\beta}(\gamma_d)$.

**Proof.**
Let $f \in L^p(\gamma_d)$, using the Fundamental Theorem of Calculus we can write,

$$|D_\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} - 1 |f(x)| ds$$

$$\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} - 1 |f(x)| ds$$

$$\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} P_r f(x)| dr ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} - 1 |f(x)| ds$$

$$= \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} P_r f(x)| dr ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr ds$$

Now, using Hardy’s inequality \[2.4\], with $p = 1$, in both integrals, we have

$$|D_\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} P_r f(x)| dr ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr ds$$

$$\leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{\beta} |\frac{\partial}{\partial r} P_r f(x)| r^{-\beta-1} dr + \frac{1}{\beta c_\beta} \int_0^{+\infty} e^{-r} r^{-\beta-1} dr$$

$$= \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f(x)| \frac{dr}{r} + \frac{1}{\beta c_\beta} |f(x)| \int_0^{+\infty} r^{(1-\beta)-1} e^{-r} dr$$

Therefore, using the Minkowski’s integral inequality

$$\|D_\beta f\|_p \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} \|\frac{\partial}{\partial r} P_r f\|_p \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) \|f\|_p < C_1 \|f\|_{B_{p,q}^\alpha} < \infty,$$

since $f \in B_{p,q}^\alpha(\gamma_d) \subset B_{p,1}^\beta(\gamma_d)$, $1 \leq q \leq \infty$ as $\alpha > \beta$, i.e. $D_\beta f \in L^p(\gamma_d)$.

On the other hand, using the Fundamental Theorem of Calculus and using Hardy’s inequality \[2.4\], with $p = 1$, in the second integral we
have,
\[
\left| \frac{\partial}{\partial t} P_t(D_\beta f)(x) \right| \leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} |e^{-s} \frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x)| ds \\
\leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} e^{-s} \left| \frac{\partial^2}{\partial r^2} P_r f(x) \right| dr ds \\
+ \frac{1}{c_\beta} \left| \frac{\partial}{\partial t} P_t f(x) \right| \int_0^\infty s^{-\beta-1} e^{-s} dr ds,
\]
\[
\leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} \int_t^{t+s} \left| \frac{\partial^2}{\partial r^2} P_r f(x) \right| dr ds \\
+ \frac{1}{\beta c_\beta} \left| \frac{\partial}{\partial t} P_t f(x) \right| \int_0^\infty r(1-\beta)^{-1} e^{-r} dr
\]
\[
= \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} \int_t^{t+s} \left| \frac{\partial^2}{\partial r^2} P_r f(x) \right| dr ds + \frac{\Gamma(1-\beta)}{\beta c_\beta} \left| \frac{\partial}{\partial t} P_t f(x) \right|.
\]
Therefore, by Minkowski’s integral inequality (2.9)
\[
\left\| \frac{\partial}{\partial t} P_t(D_\beta f) \right\|_{p,\gamma} \leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} \int_t^{t+s} \left\| \frac{\partial^2}{\partial r^2} P_r f \right\|_{p,\gamma} dr ds + \frac{\Gamma(1-\beta)}{\beta c_\beta} \left\| \frac{\partial}{\partial t} P_t f \right\|_{p,\gamma}.
\]

Then, if \(1 \leq q < \infty\), by (2.9) Minkowski’s integral inequality, we get
\[
\left( \int_0^\infty \left( t^{1-(\alpha-\beta)} \left\| \frac{\partial}{\partial t} P_t D_\beta f \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q}
\]
\[
\leq \frac{1}{c_\beta} \left( \int_0^\infty \left( t^{1-(\alpha-\beta)} \int_0^\infty s^{-\beta-1} \int_t^{t+s} \left\| \frac{\partial^2}{\partial r^2} P_r f \right\|_{p,\gamma} dr ds \right)^q \frac{dt}{t} \right)^{1/q}
\]
\[
+ \frac{\Gamma(1-\beta)}{\beta c_\beta} \left( \int_0^\infty \left( t^{1-(\alpha-\beta)} \left\| \frac{\partial}{\partial t} P_t f \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q}
\]
\[
= (I) + (II).
\]
Now, the first term is the same as the one considered in the second part of the proof of Theorem 2.5, thus by the same argument
\[
(I) \leq C_\beta \left( \int_0^\infty \left( t^{2-\alpha} \left\| \frac{\partial^2}{\partial t^2} P_t f \right\|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} < \| f \|_{B^\alpha_{p,q}} < \infty,
\]
since \(f \in B^\alpha_{p,q} \gamma_d\), and for the second term trivially
\[
(II) \leq C \| f \|_{B^\alpha_{p,q}} \leq C \| f \|_{B^\alpha_{p,q}}
\]
since $\alpha > \alpha - \beta$ and the inclusion relation, Proposition [1.1]

Thus if $1 \leq q < \infty$,
$$
(\int_0^\infty (t^{1-(\alpha-\beta)} \| \frac{\partial}{\partial t} P_t(D_\beta f) \|_{p,\gamma})^q \frac{dt}{t})^{1/q} \leq C_2 \| f \|_{B^\alpha p,q}
$$
i.e. $D_\beta f \in B^\alpha_{p,q}(\gamma_d)$ and moreover

$$
\| D_\beta f \|_{B^\alpha_{p,q}} = \| D_\beta f \|_{p,\gamma} + (\int_0^\infty (t^{1-\alpha+\beta} \| \frac{\partial}{\partial t} P_t D_\beta f \|_{p,\gamma})^q \frac{dt}{t})^{1/q}
\leq C_1 \| f \|_{B^\alpha_{p,q}} + C_2 (\int_0^\infty (t^{2-\alpha} \| \frac{\partial^2}{\partial t^2} P_t f \|_{p,\gamma})^q \frac{dt}{t})^{1/q}
\leq C \| f \|_{B^\alpha_{p,q}}.
$$

If $q = \infty$, using the same argument as in Theorem 2.5, inequality (2.9) can be written as

$$
\| \frac{\partial}{\partial t} P_t D_\beta f \|_{p,\gamma} \leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} \int_t^{t+s} \| \frac{\partial^2}{\partial r^2} P_r f \|_{p,\gamma} dr ds + \frac{\Gamma(1-\beta)}{\beta c_\beta} \| \frac{\partial}{\partial t} P_t f \|_{p,\gamma}
\leq C_{\alpha,\beta} A(f) t^{-1+\alpha-\beta} + \frac{\Gamma(1-\beta)}{\beta c_\beta} A(f) t^{-1+\alpha-\beta}
\leq C_{\alpha,\beta} A(f) t^{-1+\alpha-\beta}, \quad t > 0
$$
i.e. $D_\beta f \in B^\alpha_{p,\infty}(\gamma_d)$ and $A(D_\beta f) \leq C_{\alpha,\beta} A(f)$, thus

$$
\| D_\beta f \|_{B^\alpha_{p,\infty}} = \| D_\beta f \|_{p,\gamma} + A(D_\beta f)
\leq C_1 \| f \|_{B^\alpha_{p,\infty}} + C_2 A(f) \leq C \| f \|_{B^\alpha_{p,\infty}}.
$$

□

Now we will consider the general case for fractional derivatives, removing the condition that the indexes must be less than 1. We need to consider forward differences. Remember for a given function $f$, the $k$-th order forward difference of $f$ starting at $t$ with increment $s$ is defined as,

$$
\Delta_s^k(f, t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t + (k - j)s).
$$
The forward differences have the following properties (see Appendix in [5]) we will need the following technical result

**Lemma 2.1.** For any positive integer $k$

1) $\Delta_s^k(f, t) = \Delta_s^{k-1}(\Delta_s(f, \cdot), t) = \Delta_s(\Delta_s^{k-1}(f, \cdot), t)$
Theorem 2.7. Let
\[ \Delta^k_s(f, t) = \int_t^{t+s} \int_{v_1}^{v_1+s} \ldots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)dv_kdv_{k-1}\ldots dv_1 dv_1 \]
For any positive integer \( k \),
\[ \frac{\partial}{\partial s}(\Delta^k_s(f, t)) = k \Delta^{k-1}_s(f', t + s), \]
and for any integer \( j > 0 \),
\[ \frac{\partial^j}{\partial t^j}(\Delta^k_s(f, t)) = \Delta^k_s(f^{(j)}, t). \]

Observe that, using the Binomial Theorem and the semigroup property of \( \{P_t\} \), we have
\[ (P_t - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} P_t^{k-j}(-I)^j f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_t^{k-j} f(x) \]
\[ = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{t^{k-j}} f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j u(x, (k - j)t) \]
\[ = \Delta^k_t(u(x, \cdot), 0), \]
where as usual, \( u(x, t) = P_t f(x) \).

Additionally, we will need in what follows the following result,

Lemma 2.2. Let \( f \in L^p(\gamma_d), 1 \leq p < \infty \) and \( k, n \in \mathbb{N} \) then
\[ \| \Delta^k_s(u^{(n)}, t) \|_{p, \gamma_d} \leq s^k \| u^{(k+n)}(\cdot, t) \|_{p, \gamma_d} \]

Proof. From ii) of Lemma 2.1 we have
\[ \Delta^k_s(u^{(n)}(x, \cdot), t) = \int_t^{t+s} \int_{v_1}^{v_1+s} \ldots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} u^{(k+n)}(x, v_k)dv_kdv_{k-1}\ldots dv_1 dv_1, \]
then, using Minkowski’s integral inequality \( k \)-times and Lemma [1,1]
\[ \| \Delta^k_s(u^{(n)}, t) \|_{p, \gamma_d} \leq \int_t^{t+s} \int_{v_1}^{v_1+s} \ldots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} \| u^{(k+n)}(\cdot, v_k) \|_{p, \gamma_d} dv_kdv_{k-1}\ldots dv_1 dv_1 \]
\[ \leq s^k \| u^{(k+n)}(\cdot, t) \|_{p, \gamma_d} = s^k \| \frac{\partial^{k+n}}{\partial t^{k+n}} u(\cdot, t) \|_{p, \gamma_d}. \]

Let us start with the case of the Riesz derivative,

Theorem 2.7. Let \( 0 < \beta < \alpha, 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \) then \( D^\beta \) is bounded from \( B^\alpha_{p,q}(\gamma_d) \) into \( B^{\alpha-\beta}_{p,q}(\gamma_d) \).
Proof.
Let $f \in B^\alpha_{p,q}(\gamma_d)$, using (2.12), Hardy’s inequality (2.4), $p = 1$, the Fundamental Theorem of Calculus and iii) of Lemma 2.1, we get

$$|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1}\left|\Delta^k_s(u(x, \cdot), 0)\right| ds$$

$$\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \left|\frac{\partial}{\partial r}\Delta^k_r(u(x, \cdot), 0)\right| dr \ ds$$

$$\leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} \left|\frac{\partial}{\partial r}\Delta^k_r(u(x, \cdot), 0)\right| dr$$

$$= \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} \left|\Delta^k-1_s(u'(x, \cdot), r)\right| dr.$$ 

Now, using Minkowski’s integral inequality and Lemma 2.2

$$\|D^\beta f\|_{p,\gamma} \leq \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} \|\Delta^k-1_s(u', r)\|_{p,\gamma} dr$$

$$\leq \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} \|\frac{\partial}{\partial r}P^k_f\|_{p,\gamma} \frac{dr}{r} < \infty,$$

since $f \in B^\alpha_{p,q}(\gamma_d) \subset B^\beta_{p,1}(\gamma_d)$, as $\alpha > \beta$. Therefore, $D^\beta f \in L^p(\gamma_d)$.

On the other hand,

$$P_t[(P_s - I)^k f(x)] = P_t(\Delta^k_s(u(x, \cdot), 0)) = P_t\left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{(k-j)s} f(x)\right)$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{(k-j)s} f(x) = \Delta^k_s(u(x, \cdot), t).$$

Thus, if $n$ be the smaller integer greater than $\alpha$, i.e. $n - 1 \leq \alpha < n$, then by Lemma 2.1 iv),

$$\frac{\partial^n}{\partial t^n} P_t(D^\beta f)(x) = \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \frac{\partial^n}{\partial t^n}(\Delta^k_s(u(x, \cdot), t))$$

$$= \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1}\Delta^k_s(u^{(n)}(x, \cdot), t) ds.$$ 

and therefore, by Minkowski’s integral inequality

$$\|\frac{\partial^n}{\partial t^n} P_t(D^\beta f)\|_{p,\gamma} \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1}\|\Delta^k_s(u^{(n)}(x, \cdot), t)\|_{p,\gamma} ds.$$
Now if $1 \leq q < \infty$, by \((2.13)\),
\[
\left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \| \frac{\partial^n}{\partial t^n} P_t(D_\beta f) \|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{c_{\beta}} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_0^{+\infty} s^{-\beta-1} \| \Delta_s^k (u^{(n)}, t) \|_{p,\gamma} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{c_{\beta}} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{-\beta-1} \| \Delta_s^k (u^{(n)}, t) \|_{p,\gamma} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
+ \frac{1}{c_{\beta}} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{-\beta-1} \| \Delta_s^k (u^{(n)}, t) \|_{p,\gamma} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
= (I) + (II).
\]

Then, by Lemma \(2.2\)
\[
(I) \leq \frac{1}{c_{\beta}} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \| \frac{\partial^{n+k}}{\partial t^{n+k}} P_t f \|_{p,\gamma} \int_0^t s^{-\beta-1} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
= \frac{1}{c_{\beta}(k-\beta)} \left( \int_0^\infty \left( t^{n+k-\alpha} \| u^{(n+k)}(\cdot, t) \|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} < \infty,
\]
since \(f \in B_{p,q}^\alpha(\gamma_d)\), and by Lemma \(1.1\)
\[
(II) \leq \frac{1}{c_{\beta}} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{-\beta-1} \left( \sum_{j=0}^k \left( \begin{array}{c} k \\ j \end{array} \right) \| u^{(n)}(\cdot, t+(k-j)s) \|_{p,\gamma} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{c_{\beta}} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{-\beta-1} \left( \sum_{j=0}^k \left( \begin{array}{c} k \\ j \end{array} \right) \| u^{(n)}(\cdot, t) \|_{p,\gamma} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
= \frac{2^k}{c_{\beta}^2} \left( \int_0^\infty \left( t^{n-\alpha} \| \frac{\partial^n}{\partial t^n} P_t f \|_{p,\gamma} \int_t^{+\infty} s^{-\beta-1} ds \right)^q \frac{dt}{t} \right)^{1/q} \\
= \frac{2^k}{c_{\beta}^2} \left( \int_0^\infty \left( t^{n-\alpha} \| \frac{\partial^n}{\partial t^n} P_t f \|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} < \infty,
\]
since \(f \in B_{p,q}^\alpha(\gamma_d)\). Therefore, if \(1 \leq q < \infty\), \(D_\beta f \in B_{p,q}^{\alpha-\beta}(\gamma_d)\), and moreover,
\[
\| D_\beta f \|_{B_{p,q}^{\alpha-\beta}} = \| D_\beta f \|_{p,\gamma} + \left( \int_0^\infty \left( t^{n-\alpha+\beta} \| \frac{\partial^n}{\partial t^n} P_t (D_\beta f) \|_{p,\gamma} \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq C_1 \| f \|_{B_{p,q}^\alpha} + C_2 \| f \|_{B_{p,q}^\alpha} \leq C \| f \|_{B_{p,q}^\alpha}
\]
Thus, \(D_\beta f : B_{p,q}^\alpha \to B_{p,q}^{\alpha-\beta}\) is bounded.
If \( q = \infty \), inequality (2.13) can be written as

\[
\| \frac{\partial^n}{\partial t^n} P_t(D\beta f) \|_{p,\gamma} \leq \frac{1}{c_\beta} \int_0^t s^{-\beta-1} \| \Delta_k^s(u^{(n)}(\cdot), t) \|_{p,\gamma} ds + \frac{1}{c_\beta} \int_t^{+\infty} s^{-\beta-1} \| \Delta_k^s(u^{(n)}(\cdot), t) \|_{p,\gamma} ds
\]

\[
= (I) + (II)
\]

and then as \( f \in B^\alpha_{p,\infty} \), by Lemma 2.2,

\[
(I) \leq \frac{1}{c_\beta} \int_0^t s^{-\beta-1}s^k \| u^{(n+k)} \|_{p,\gamma} ds = C_\beta \| \frac{\partial^{n+k}}{\partial t^{n+k}} P_t f \|_{p,\gamma} t^{k-\beta}
\]

\[
\leq C_\beta A(f) t^{-n-k+\alpha} t^{-\beta} = C_\beta A(f) t^{-n+\alpha-\beta},
\]

and as above, by Lemma 1.1,

\[
(II) \leq \frac{1}{c_\beta} \int_t^{+\infty} s^{-\beta-1} \left( \sum_{j=0}^k \binom{k}{j} \| u^{(n)}(\cdot, t + (k-j)s) \|_{p,\gamma} \right) ds
\]

\[
\leq C_\beta \int_t^{+\infty} s^{-\beta-1} \left( \sum_{j=0}^k \binom{k}{j} \| u^{(n)}(\cdot, t) \|_{p,\gamma} \right) ds = C_\beta t^{-\beta} \| \frac{\partial^n}{\partial t^n} P_t f \|_{p,\gamma}
\]

\[
\leq C_\beta A(f) t^{-n+\alpha} t^{-\beta} = C_\beta A(f) t^{-n+\alpha-\beta}.
\]

\[\square\]

There is an alternative proof of the fact that \( D\beta f \in L^p(\gamma_d) \) without using Hardy’s inequality following the same scheme as in the proof of i) Theorem 3.5 in [5], using the inclusion \( B^\alpha_{p,q} \subset B^{\beta+\epsilon}_{p,\infty} \) with \( \beta + \epsilon < k \).

**Theorem 2.8.** Let \( 0 < \beta < \alpha, 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \) then \( D\beta \) is bounded from \( B^\alpha_{p,q}(\gamma_d) \) into \( B^{\alpha-\beta}_{p,q}(\gamma_d) \).

**Proof.**

Let \( f \in B^\alpha_{p,q}(\gamma_d) \), and set \( v(x,t) = e^{-t}u(x,t) \) then using the Hardy’s inequality (2.4), the Fundamental Theorem of Calculus and iii) of Lemma 2.1

\[
|D\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta_k^s(v(x,\cdot), 0)| ds
\]

\[
\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} \Delta_k^r(v(x,\cdot), 0)| dr ds
\]

\[
\leq \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} |\Delta_{k-1}^r(v'(x,\cdot), r)| dr
\]
and this implies by Minkowski’s integral inequality

$$\| \mathcal{D}_\beta f \|_{p,\gamma} \leq \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} \| \Delta_{r}^{k-1}(v', r) \|_{p,\gamma} dr.$$ 

Now, using Lemma 2.1

$$\| \Delta_{r}^{k-1}(v', r) \|_{p,\gamma} \leq \int_r^{2r} \int_{v_1}^{v_1+r} \cdots \int_{v_{k-2}}^{v_{k-2}+r} \| v^{(k)}(\cdot, v_{k-1}) \|_{p,\gamma} dv_{k-1} dv_{k-2} \cdots dv_1$$

and by Leibnitz’s differentiation rule for the product

$$\| v^{(k)}(\cdot, v_{k-1}) \|_{p,\gamma} = \| \sum_{j=0}^{k} \binom{k}{j} (e^{-v_{k-1}})^{(j)} u^{(k-j)}(\cdot, v_{k-1}) \|_{p,\gamma} \leq \sum_{j=0}^{k} \binom{k}{j} e^{-v_{k-1}} \| u^{(k-j)}(\cdot, v_{k-1}) \|_{p,\gamma}.$$ 

Then

$$\| \Delta_{r}^{k-1}(v', r) \|_{p,\gamma} \leq \sum_{j=0}^{k} \binom{k}{j} r^{-\beta} \| u^{(k-j)}(\cdot, v_{k-1}) \|_{p,\gamma} dv_{k-1} dv_{k-2} \cdots dv_1$$

Therefore

$$\| \mathcal{D}_\beta f \|_{p,\gamma} \leq \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| u^{(k-j)}(\cdot, r) \|_{p,\gamma} dr$$

$$= \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{(k-j)-(\beta-j)-1} e^{-r} \| \partial_{r}^{k-j} P_{r} f \|_{p,\gamma} dr$$

$$+ \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| P_{r} f \|_{p,\gamma} dr$$

$$\leq \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{(k-j)-(\beta-j)-1} \| \partial_{r}^{k-j} P_{r} f \|_{p,\gamma} dr$$

$$+ \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| f \|_{p,\gamma} dr$$
Thus
\[
\|D_\beta f\|_{p, \gamma} \leq \frac{k}{\beta c_\beta} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{k-j-(\beta-j)} \frac{\partial^{k-j}}{\partial r^{k-j}} \|P_f\|_{p, \gamma} \frac{dr}{r} + \frac{k \Gamma(k-\beta)}{\beta c_\beta} \|f\|_{p, \gamma} < \infty,
\]

since \( f \in B_{p,q}^\alpha(\gamma_d) \subset B_{p,1}^{\beta-j}(\gamma_d) \) as \( \alpha > \beta > \beta - j \geq 0 \), for \( j \in \{0, \ldots, k-1\} \), then \( D_\beta f \in L^p(\gamma_d) \).

On the other hand,
\[
P_t(e^{-s}P_s-I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-s(k-j)} u(x, t + (k-j)s).
\]

Let \( n \) be the smaller integer greater than \( \alpha \), i.e. \( n - 1 \leq \alpha < n \), we have
\[
\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x) = \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-s(k-j)} u^{(n)}(x, t + (k-j)s) ds
\]
\[
= \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-(t+s(k-j))} u^{(n)}(x, t + (k-j)s) ds
\]
\[
= \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \Delta_s^k(w(x, \cdot), t) ds,
\]

where \( w(x, t) = e^{-t} u^{(n)}(x, t) \). Now using the Fundamental Theorem of Calculus,
\[
\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x) = \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \Delta_s^k(w(x, \cdot), t) ds
\]
\[
= \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^{s} \frac{\partial}{\partial r} \Delta_s^k(w(x, \cdot), t) dr ds.
\]

Then, using Hardy’s inequality (2.4), and iii) of Lemma 2.1
\[
\left| \frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x) \right| \leq \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^{s} \left| \frac{\partial}{\partial r} \Delta_s^k(w(x, \cdot), t) \right| dr ds
\]
\[
\leq \frac{e^t}{c_\beta \beta} \int_0^{+\infty} r^{-\beta} \Delta_s^{k-1}(w(x, \cdot), t + r) dr
\]
\[
= \frac{ke^t}{c_\beta \beta} \int_0^{+\infty} r^{-\beta} \Delta_s^{k-1}(w(x, \cdot), t + r) dr.
\]
and by Minkowski’s integral inequality we get
\[ \| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \|_{p,\gamma} \leq \frac{k\epsilon}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} \| \Delta_{\eta}^{k-1}(w', t + r) \|_{p,\gamma} dr. \]

Now, by analogous argument as above, Lemma 2.1 and Leibnitz’s product rule give us
\[ \| \Delta_{\eta}^{k-1}(w', t + r) \|_{p,\gamma} \leq \sum_{j=0}^k \binom{k}{j} r^{k-1} e^{-(t+r)} \| u^{(k+n-j)}(\cdot, t + r) \|_{p,\gamma}, \]
and this implies that
\[ \| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \|_{p,\gamma} \leq \frac{e^r}{c_\beta} \sum_{j=0}^k \binom{k}{j} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| u^{(k+n-j)}(\cdot, t + r) \|_{p,\gamma} dr. \]

Thus
\[(2.14)\]
\[ \| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \|_{p,\gamma} \leq \frac{k}{c_\beta} \sum_{j=0}^k \binom{k}{j} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| u^{(k+n-j)}(\cdot, t + r) \|_{p,\gamma} dr. \]

Now if \( 1 \leq q < \infty \), using (2.14) we have,
\[ \left( \int_0^{+\infty} \left( t^{n-\alpha-\beta} \| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \|_{p,\gamma} \right)^q dt \right)^{1/q} \leq \frac{k}{c_\beta} \sum_{j=0}^k \binom{k}{j} \left( \int_0^{+\infty} r^{n-(\alpha-\beta)} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| u^{(k+n-j)}(\cdot, t + r) \|_{p,\gamma} dr \right)^{q dt/t} \left( \frac{dt}{t} \right)^{1/q}. \]

For each \( 1 \leq j \leq k, 0 < \alpha - \beta + k - j \leq \alpha \) and by Lemma [14]
\[ \left( \int_0^{+\infty} \left( t^{n-(\alpha-\beta)} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \| u^{(k+n-j)}(\cdot, t + r) \|_{p,\gamma} dr \right)^q dt \right)^{1/q} \leq \left( \int_0^{+\infty} t^{n-(\alpha-\beta)} \| u^{(n+k-j)}(\cdot, t) \|_{p,\gamma} \right. \int_0^{+\infty} r^{k-\beta-1} e^{-r} dr \right)^{q dt/t} \left( \frac{dt}{t} \right)^{1/q} \]
\[ = \Gamma(k - \beta) \left( \int_0^{+\infty} t^{n+(k-j)-(\alpha-\beta+k-j)} \| u^{(n+k-j)}(\cdot, t) \|_{p,\gamma} dr \right)^{q dt/t} \left( \frac{dt}{t} \right)^{1/q} < \infty, \]
as \( f \in B^\alpha_{p,q}(\gamma_d) \subset B^\alpha_{p,q}(\gamma_d) \) for any \( 0 \leq j \leq k \).
Now, for the case \( j = 0 \),
\[
\left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \|u^{(n+k)}(\cdot, t+r)\|_{p,\gamma} dr \right) \frac{dt}{t} \right)^{1/q}
\]
\[
\leq \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_0^t r^{k-\beta-1} e^{-r} \|u^{(n+k)}(\cdot, t+r)\|_{p,\gamma} dr \right) \frac{dt}{t} \right)^{1/q} + \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_t^{+\infty} r^{k-\beta-1} e^{-r} \|u^{(n+k)}(\cdot, t+r)\|_{p,\gamma} dr \right) \frac{dt}{t} \right)^{1/q}
\]
\[
= (I) + (II).
\]
Using Lemma 1.1 and \( k > \beta \),
\[
(I) \leq \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_0^t r^{k-\beta-1} \|u^{(n+k)}(\cdot, t)\|_{p,\gamma} dr \right) \frac{dt}{t} \right)^{1/q}
\]
\[
= \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \|u^{(n+k)}(\cdot, t)\|_{p,\gamma} \right) \frac{dt}{t} \right)^{1/q}
\]
\[
= \frac{1}{k-\beta} \left( \int_0^\infty \left( t^{n+k-\alpha} \|u^{(n+k)}(\cdot, t)\|_{p,\gamma} \right) \frac{dt}{t} \right)^{1/q} < \infty,
\]
since \( f \in B^\alpha_{p,q}(\gamma_d) \) and \( n+k > \alpha \) and for the second term, using Lemma 1.1 and Hardy’s inequality (2.5)
\[
(II) \leq \frac{1}{n-(\alpha-\beta)} \left( \int_0^\infty \left( r^{n+k-\alpha} \|u^{(n+k)}(\cdot, r)\|_{p,\gamma} \right) \frac{dr}{r} \right)^{1/q} < \infty
\]
since \( f \in B^\alpha_{p,q}(\gamma_d) \).

Therefore \( D_\beta f \in B^\alpha_{p,q}(\gamma_d) \). Moreover
\[
\|D_\beta f\|_{B^\alpha_{p,q}} = \|D_\beta f\|_{p,\gamma} + \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \|\frac{\partial}{\partial t^n} P_t D_\beta f\|_{p,\gamma} \right) \frac{dt}{t} \right)^{1/q}
\]
\[
\leq C_1 \|f\|_{p,\gamma} + \frac{k}{c_{\beta}} \sum_{j=0}^k \left( \begin{array}{c} k \\ j \end{array} \right) C_2 \left( \int_0^\infty \left( r^{n-\alpha} \|\frac{\partial}{\partial r^n} P_r f\|_{p,\gamma} \right) \frac{dr}{r} \right)^{1/q}
\]
\[
\leq C \|f\|_{B^\alpha_{p,q}}
\]
Finally, if \( q = \infty \), from the inequality (2.14)
\[
\|\frac{\partial}{\partial t^n} P_t (D_\beta f)\|_{p,\gamma} \leq \frac{k}{c_{\beta}} \sum_{j=0}^k \left( \begin{array}{c} k \\ j \end{array} \right) \int_0^{+\infty} r^{k-\beta-1} e^{-r} \|u^{(k+n-j)}(\cdot, t+r)\|_{p,\gamma} dr,
\]
and then, the argument is essentially similar to the previous case, as we did in the last part of the proof of Theorem 2.7.
Observation Let us observe that if instead of considering the Ornstein-Uhlenbeck operator (1.2) and the Poisson-Hermite semigroup (1.7) we consider the Laguerre differential operator in $\mathbb{R}^d_+$,

\[
\mathcal{L}^\alpha = \sum_{i=1}^{d} \left[ x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right],
\]

and the corresponding Poisson-Laguerre semigroup, or if we consider the Jacobi differential operator in $(-1,1)^d$,

\[
\mathcal{L}^{\alpha,\beta} = -\sum_{i=1}^{d} \left[ (1 - x_i^2) \frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial}{\partial x_i} \right],
\]

and the corresponding Poisson-Jacobi semigroup (for details we refer to [15]), the arguments are completely analogous. That is to say, we can defined in analogous manner Laguerre-Besov-Lipschitz spaces, and Jacobi-Besov-Lipschitz spaces then prove that the corresponding notions of Fractional Integrals and Fractional Derivatives behave similarly. In order to see this it is more convenient to use the representation (1.7) of $P_t$ in terms of the one-sided stable measure $\mu_t^{(1/2)}(ds)$, see [9].

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