Quantum X waves with orbital angular momentum in nonlinear dispersive media

Marco Ornigotti1, Claudio Conti2,3 and Alexander Szameit1

1 Institut für Physik, Universität Rostock, Albert-Einstein-Straße 23, D-18059 Rostock, Germany
2 Institute for Complex Systems (ISC-CNR), Via dei Taurini 19, I-00185, Rome, Italy
3 Department of Physics, University Sapienza, Piazzale Aldo Moro 5, I-00185, Rome, Italy

E-mail: marco.ornigotti@uni-rostock.de

Received 30 November 2017, revised 11 April 2018
Accepted for publication 18 April 2018
Published 8 May 2018

Abstract
We present a complete and consistent quantum theory of generalised X waves with orbital angular momentum in dispersive media. We show that the resulting quantised light pulses are affected by neither dispersion nor diffraction and are therefore resilient against external perturbations. The nonlinear interaction of quantised X waves in quadratic and Kerr nonlinear media is also presented and studied in detail.

Keywords: quantum optics, singular optics, X waves, nonlinear optics

(Some figures may appear in colour only in the online journal)

1. Introduction

Electromagnetic waves are usually subject to diffraction and dispersion, i.e. a progressive broadening during propagation of the wave in both space and time, respectively. Ultimately, these effects are connected with the bounded nature of the wave spectrum and, therefore, to its finite energy content [1]. Maxwell’s equations, however, admit diffraction- and dispersion-free solutions, the so-called localised waves [2]. An example of such solutions in the monochromatic domain are the well known Bessel beams [3]. In the pulsed domain, the most famous representatives of localised waves are X waves. Firstly introduced in acoustics by Lu and Greenleaf in 1992 [4, 5], they have been the subject of an extensive study in different areas of physics, such as nonlinear [6, 7] and quantum [8] optics, condensed matter physics [9], integrated optics [10, 11] and optical communications [12], to name a few. A comprehensive review of the topic can be found in [2, 13].

Traditionally, X waves are understood as superpositions of Bessel beams of zero order, therefore neglecting their possible orbital angular momentum (OAM) content. The latter, in fact, is known to be related to the twisted phase front of higher order Bessel beams [14]. Generalisation of the traditional X waves to the case of OAM-carrying X waves has been only recently investigated [15].

Despite the great amount of work that has been done in the subject in the last few decades, however, investigations of quantum properties of X waves are very few and limited to the case of no OAM content [8, 16, 17]. Very recently, the effect of the OAM content of X waves in squeezing processes has been analysed in detail, highlighting new features and possibilities [18]. However, a comprehensive quantum theory of X wave and a complete analysis of their properties and dynamics at the quantum level has only be sketched in [18] and not fully developed.

In this work, therefore, we present a complete and comprehensive quantum theory of X waves with OAM. Although the quantisation of the electromagnetic field carrying angular momentum has already been carried out in both the context of quantum field theory [19] and for generalised Gaussian-Airy wave packets [20], the approach presented here constitutes a more general framework, where dispersion effects and nonlinearities are automatically accounted for. In particular, we discuss in detail the dynamics of quantum X waves in media exhibiting $\chi^{(2)}$-and $\chi^{(3)}$-nonlinearities. Our findings reveal that photon pairs generated via parametric down conversion present continuous variable entanglement in their velocity (i.e. Bessel cone angle) degree of freedom, while their OAM content plays an active role in the determination of the properties of photons propagating in Kerr media. In the latter case, the Kerr nonlinearity introduces a coupling between different OAM states, whose strength is...
2. Wave equation in dispersive media

As a starting point of our analysis, let us consider the propagation of a scalar electromagnetic field in a linear, dispersive medium, characterised by the refractive index \( n = n(\omega) \)

\[
\left( \nabla^2 - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right) E(\mathbf{r}, t) = 0.
\]

Moreover, let us assume that the paraxial approximation applies and that we can write the electric field as

\[
E(\mathbf{r}, t) = \left( \frac{\varepsilon_0 n^2}{2} \right)^{-1/2} A(\mathbf{r}, t) e^{i(kz - \omega t)},
\]

where \( A(\mathbf{r}, t) \) varies slowly with \( z, k = n\omega/c, \) and the normalisation constant has been chosen such that the total energy of the field (i.e. the field intensity) can be written as

\[
\mathcal{E} = \frac{\varepsilon_0 n^2}{2} \int d^3r |E(\mathbf{r}, t)|^2 = \int d^3r |A(\mathbf{r}, t)|^2.
\]

Direct substitution of this Ansatz into equation (1) results in the conventional Fock–Leontovich equation, namely [13]

\[
2i \frac{\partial A}{\partial \zeta} + 2i \frac{n^2 \omega}{c^2} \frac{\partial A}{\partial t} + \nabla_z^2 A - \frac{n^2}{c^2} \frac{\partial^2 A}{\partial t^2} = 0.
\]

The above equation describes a propagating wave along the \( z \)-direction. However, this problem can be equivalently casted in terms of a time evolution problem, thus allowing a generalisation of the results of this investigation to all problems that admit evolution equations that can be casted in terms of Schrödinger-like equations. To do that, we introduce the reference frame \( \{ \zeta = z - (c/n)t, \tau = t \} \), co-moving with the envelope itself, and rewrite equation (4) as follows:

\[
2i \frac{n^2 \omega}{c^2} \frac{\partial A}{\partial \tau} + \nabla_\tau^2 A - \frac{n^2}{c^2} \frac{\partial^2 A}{\partial \tau^2} = \frac{2i}{c} \frac{\partial^2 A}{\partial \tau \partial \zeta}.
\]

Then, we apply the slowly varying envelope approximation to the envelope function \( A(\mathbf{r}, \zeta) \), so that the right-hand side of the above equation can be neglected [21]. If we now transform back into the reference frame \( \{ x, y, z, t \} \), and recall that

\[
\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z},
\]

\[
\frac{\partial}{\partial \tau} = \frac{c}{n} \frac{\partial}{\partial z} + \frac{\partial}{\partial t},
\]

we obtain the following equation:

\[
\frac{1}{2} \frac{\partial A}{\partial t} + i \omega_\ell \frac{\partial A}{\partial \zeta} - \frac{\omega''}{2} \frac{\partial^2 A}{\partial \zeta^2} + \frac{\omega'}{2k} \nabla_z^2 A = 0,
\]

where we have defined, for later convenience, \( \omega' = c/n \) and \( \omega'' = c^2/(n^2\omega) \). The above equation is now describing the evolution in time of the field envelope \( A(\mathbf{r}, t) \), as desired. Before going any further, it is worth noticing that the above equation describes the propagation of a quasi-monochromatic field in a non-dispersive medium, and that, for this particular case, \( \omega'/(2k) = \omega''/2 = c^2/(2n^2\omega) \) holds. In particular, one can assume that the refractive index \( n(\omega) \) is approximately constant over the very narrow spectrum of \( A(\mathbf{r}, t) \), so that no derivatives of the refractive index are necessary, and therefore the wave experiences no dispersion, due to its quasi-monochromatic nature. However, equation (7) also describes the propagation of an electromagnetic field in a dispersive medium, provided that \( \omega' \) and \( \omega'' \) are understood as the first and second order dispersion, respectively, i.e. \( \omega' = c^2 dk/d\omega, \) and \( \omega'' = c^4 dk^2/d\omega^2, \) where \( k = k_0 n(\omega) \). To understand why this is true, one could follow the standard approach for deriving the evolution equation of the electromagnetic field in dispersive media described, for example, in [22]. First, write the time-dependent electric field \( E(\mathbf{r}, t) \) in Fourier space as

\[
E(\mathbf{r}, t) = A(\mathbf{r}, t) e^{ik_0 z} \propto \int d\omega \tilde{A}(\mathbf{r}, \omega)e^{-iz_0\omega}.
\]

If we substitute this Ansatz into equation (1), we get a term proportional to \( -k_0^2 A(\mathbf{r}, \omega) \), which comes from the \( z \)-component of the Laplace operator (i.e. \( \nabla^2 = \nabla_z^2 + \partial_t^2 \)), while the second derivative with respect to time gives a term proportional to \( (\omega^2/k^2)(c^2) \tilde{A}(\mathbf{r}, \omega) \equiv k^2(\omega^2) \tilde{A}(\mathbf{r}, \omega) \). If we now further assume that the spectrum \( \tilde{A}(\mathbf{r}, \omega) \) is peaked around a central frequency \( \omega_0 \) and Taylor expand the term \( k^2/\omega - k^2_0 \) around \( \omega_0 \), we obtain (with second order precision)

\[
E(\mathbf{r}, t) \propto e^{ik_0 z - \omega_0 t} \int d\Omega \left[ \frac{dk}{d\omega} \left| \Omega_0 + \frac{1}{2} \frac{dk}{d\omega} \right|^2 \left| \Omega_0 \right|^2 \right] \tilde{A}(\mathbf{r}, \Omega)e^{-i\Omega t},
\]

where \( \omega = \omega_0 + \Omega (\Omega \ll 1) \) has been assumed. Next, we can transform powers of \( \Omega \) in time derivatives with partial integration, and define \( \omega' = dk/d\omega \) and \( \omega'' = d^2k/d\omega^2 \). With this result, and recalling that we are considering a paraxial field with carrier frequency \( \omega_0 \), i.e. \( E(\mathbf{r}, t) \propto A(\mathbf{r}, t)e^{i(k_0 z - \omega_0 t)} \), it is not difficult, using Maxwell’s equations, to arrive to a wave equation for \( A(\mathbf{r}, t) \), which is exactly equation (7).
Equation (7) constitutes then the starting point of our investigation and will be used in the remaining of this manuscript to study the propagation of an electromagnetic field in a dispersive medium characterised by a refractive index \( n(\omega) \) and first and second order dispersion \( \omega' \) and \( \omega'' \), respectively.

### 2.1. Generalised X wave expansion

We now look for solutions of equation (7) as superposition of X waves. To do that, let us first write the envelope function \( A(\mathbf{r}, t) \) in terms of its 3D Fourier transform, namely

\[
  A(\mathbf{r}, t) = \int \! dk \, \hat{A}(k) \, e^{i k \cdot \mathbf{r}} e^{-i \hat{\Omega}(k)t} \, , \tag{10}
\]

where \( \hat{\Omega}(k) \) reflects the fact that the frequency of the field envelope experiences dispersion while the field propagates in the medium. If we now introduce the cylindrical coordinates \( \{ R, \theta, z \} \) and \( \{ k_z, \varphi, k_r \} \), the previous integral becomes

\[
  A(\mathbf{r}, t) = \int \! dk_z \, \int \! dk_r \, \hat{A}(k) \, e^{i k_r R \cos(\varphi-\theta)} e^{-i \hat{\Omega}(k)t} \, e^{i k_z z} \, , \tag{11}
\]

where \( d^2 k = k_z \, dk_z \, dk_r \). Following [2], we choose the following explicit form for the Fourier spectrum \( \hat{A}(k) \) to represent the solution \( A(\mathbf{r}, t) \) as a superposition of X waves

\[
  \hat{A}(k) = \sum_{m=-\infty}^{\infty} d_m S(k_z, k_r) e^{i m \varphi} \, , \tag{12}
\]

where \( S(k_z, k_r) \) is an arbitrary spectrum, whose explicit form will be given below. Moreover, we also assume that \( \partial \hat{\Omega}(k) / \partial \varphi = 0 \). Substituting the above ansatz into equation (11) and using the integral definition of the Bessel function [23], after a straightforward manipulation we arrive at the following form for the field envelope:

\[
  A(\mathbf{r}, t) = \sum_{m} d_m \int \! dk_z \, e^{i k_z z} \int_{0}^{\infty} \! dk_r \, S(k_z, k_r) \times J_m(k_r R) \, e^{i m \varphi - i \hat{\Omega}(k)t} \, . \tag{13}
\]

The above result constitutes an exact solution of equation (7) if \( \hat{\Omega}(k) \) has the following expression:

\[
  \hat{\Omega}(k) = \omega' k_z - \frac{\omega''}{2} k_z^2 + \frac{\omega'}{2} k_r^2 \, . \tag{14}
\]

This can be simply checked by substituting equation (13) into equation (7) and solving for \( \hat{\Omega}(k) \). Moreover, the above result allows us to introduce the co-moving coordinate \( Z = z - \alpha t \) and the new frequency \( \Omega = -(\omega' k_z^2)/2 + (\omega'' k_z^2)/2k_z \) such that \( k_z \theta - \hat{\Omega}(k)t = k_z Z - \hat{\Omega}t \). If we now introduce the normalised transverse wave vector \( \alpha \) and the X wave velocity\(^4\) \( v \)

\[
  \left\{ \begin{array}{l}
    k_z = \alpha \frac{\omega''}{\omega'}, \\
    k_r = \alpha \frac{\omega'}{\omega''} - \frac{v}{\omega''} \, .
  \end{array} \tag{15}\right.
\]

Equation (13) can be rewritten in the following form:

\[
  A(\mathbf{r}, t) = \sum_{m} d_m e^{im\varphi} \int \! dv \int_{0}^{\infty} \! \! dv \, X(\alpha, v) J_m \left( \sqrt{\frac{\omega'^2}{\omega''} \frac{v^2}{\omega''}} \alpha R \right) \times e^{i \left( \frac{\omega''}{\omega'} \frac{v^2}{\omega''} - \frac{v}{\omega''} \right) \varphi} \, , \tag{16}
\]

where

\[
  X(\alpha, v) = k \alpha \left( \frac{\omega'' k_z}{\omega'} \alpha - \frac{v}{\omega''} \right) \tag{17}
\]

is the so-called X wave transform [24]. A suitable choice for the spectrum \( X(\alpha, v) \) is represented by the generalised X wave spectrum presented in [25], namely

\[
  X(\alpha, v) = \sum_{p=0}^{\infty} \tilde{C}_p(v) f_p(\alpha) \, , \tag{18}
\]

where \( \tilde{C}_p(v) \) are some arbitrarily expansion coefficients and

\[
  f_p(\alpha) = \sqrt{\frac{k \Delta}{2\mu^2(1 + p)}} \left( \frac{\omega''}{\omega'} \alpha \right) L_p^{(1)}(2\alpha \Delta) e^{-\alpha \Delta} \, , \tag{19}
\]

being \( L_p^{(1)} \) the generalised Laguerre functions of the first kind of index \( p \) [23] and \( \Delta \) is a normalisation length related to the spatial extension of the spectrum. Substituting this result into equation (16) and introducing the generalised OAM-carrying X wave of order \( p \) and velocity \( v \) as

\[
  \psi^{(p)}_{m,p}(R, \zeta) = \int_{0}^{\infty} \! dv \, f_p(\alpha) \, J_m \left( \sqrt{\frac{\omega'^2}{\omega''} \frac{v^2}{\omega''}} \alpha R \right) e^{i \left( \frac{\omega''}{\omega'} \frac{v^2}{\omega''} - \frac{v}{\omega''} \right) \varphi} \, , \tag{20}
\]

where \( R = \sqrt{x^2 + y^2}, \, \zeta = Z - vt \) is the usual co-moving coordinate associated to X waves [2] and \( C_{m,p}(v) \equiv d_m \tilde{C}_p(v) \), we can rewrite equation (16) in the following, inspiring, form:

\[
  A(\mathbf{r}, t) = \sum_{m,p} \int \! dv \, C_{m,p}(v) e^{i \frac{v}{\omega''} \varphi} \psi^{(p)}_{m,p}(R, \zeta) \, . \tag{21}
\]

This is the first result of our work. The propagation of a paraxial electromagnetic field in a dispersive medium can be described by expanding the field envelope as a superposition of generalised X waves of order \( p \) and velocity \( v \), carrying \( m \) units of OAM. To emphasise this, in the above equations we introduce the notation \( \psi^{(p)}_{m,p}(R, \zeta) \) to highlight the three degrees of freedom that later upon quantisation will be promoted to quantum numbers, namely the OAM index \( m \), the spectral order \( p \) and the X wave velocity \( v \). Note, moreover, that while \( m \) and \( p \) are discrete quantum numbers (i.e. the correspondent operators have discrete spectra), the X wave velocity is a continuous quantum number. This result, moreover, is corroborated by the fact that \( \psi^{(p)}_{m,p}(R, \zeta) \) are a complete and orthogonal set of functions and can be therefore used as a basis to represent any arbitrary field envelope \( A(\mathbf{r}, t) \). A proof of the orthogonality of generalised OAM-carrying X waves is given in appendix A. The transverse and longitudinal profiles of the mode functions \( \psi^{(p)}_{m,p}(R, \zeta) \) are reported in figures 1 and 2 for some values of \( m \) and \( p \).
2.2. Total energy of the field

In preparation for the quantisation of the field $A(r, t)$, it is instructive to calculate the total energy carried by the field $E(r, t)$ or, equivalently, the Hamiltonian function corresponding to $E(r, t)$. As already pointed out in equation (2), the total energy $\mathcal{E}$ carried by the field is simply obtained by integrating the quantity $|A(r, t)|^2$ over the whole space. If we use the expression for $A(r, t)$ given by equation (21) and substitute it into equation (3), it is not difficult to show that $\mathcal{E}$ has the following well known expression [1]:

$$\mathcal{E} = \int d^3r |A(r, t)|^2 = \sum_{p,m} \int dv |C_{m,p}(v)|^2.$$  

(22)

We can rewrite the above expression in a more inspiring form by noticing that we can define the following time-dependent expansion coefficients from equation (21), namely

$$C_{m,p}(v, t) = C_{m,p}(v)e^{-i\frac{v^2}{2\omega^2}},$$  

(23)

and observe that they are solution to the harmonic oscillator differential equation, i.e.

$$\frac{d^2C_{m,p}(v, t)}{dt^2} = -\omega_{m,p}(v)^2 C_{m,p}(v, t),$$  

(24)

with

$$\omega_{m,p}(v) = \frac{v^2}{2\omega^2}.$$  

(25)

With this result at hand, we can rewrite equation (21) as

$$A(r, t) = \sum_{p,m} \int dv C_{m,p}(v, t) \psi_{m,p}(r, \zeta).$$  

(26)

Figure 1. Transverse (left column) and longitudinal (right column) intensity distribution of the mode function $|\psi_{m,p}(R, \zeta)|^2$ for different values of $m$ and $p$, as a function of the scaled transverse coordinates $X = (\sqrt{\omega^2/k}}/\omega^2)x$ and $Y = (\sqrt{\omega^2/k}}/\omega^2)y$, and the co-moving coordinate $\zeta = z - vt$. (a) Transverse structure of the mode $\psi_{m,p}(R, \zeta)$ for $m = 0, p = 0$ (the so-called fundamental X wave) at $\zeta = 0$. (b) Cut along the plane $Y = 0$ of the fundamental X wave defined in (a). The characteristic ‘X’-shape is clearly visible. (c) Same as a = panel (a), but for $p = 2$. Notice that, for $p = 0$, a non-trivial radial structure appear (in analogy with Laguerre–Gaussian modes). (d) Same as panel (b), but for $p = 2$. Notice how sidebands appear because of the fact that $p \neq 0$. Notice that, since the velocity $v$ only appears as a global phase term, it plays no role in the determination of the mode function $\psi_{m,p}(R, \zeta)$, when represented in the co-moving frame $\{X, Y, \zeta\}$. 


If we then notice that $M = \left| \frac{1}{\omega} \right| (0, 0)$, we can also rewrite the total energy of the field as

$$E = \sum_{p,m} \int \mathrm{d}v \left| C_{m,p}(v, t) \right|^2.$$  

Equations (26) and (27) have a very simple interpretation: the field envelope $A(r, t)$ (and, consequently, the electric field itself) can be viewed as a collection (integral sum) of harmonic oscillators, each of them with complex amplitude $C_{m,p}(v)$ and associated to a travelling invariant wave with a $v$-dependent resonant frequency $\omega_{m,p}(v)$, corresponding to the kinetic energy of free particles.

The result above is similar to the case of a field expanded onto the normal modes of an optical cavity, with the difference that in this case the modes are continuously distributed and rigidly moving, instead of being standing waves as in the traditional case. This allows us to adapt the ordinary quantisation techniques, as the one described for example in [26], to the case of X waves in dispersive media.

3. Quantisation of X waves

To quantise the field given by equation (26), we employ the standard technique of expressing the total energy $E$ as a collection of harmonic oscillators and then associate creation and annihilation operators to the field itself [26]. As discussed in the previous section, according to equation (27) the total energy of the field is already written as a (continuous) collection of harmonic oscillators, and therefore the quantisation is immediate. To make this more explicit, however, we introduce the two real quantities $Q_{m,p}(v)$ and $P_{m,p}(v)$ and write the complex amplitudes $C_{m,p}(v)$ as

$$C_{m,p}(v) = \frac{1}{\sqrt{2}} \left[ \omega_{m,p}(v) Q_{m,p}(v) + i P_{m,p}(v) \right],$$

such that

$$|C_{m,p}(v)|^2 = \frac{1}{2} \left[ P_{m,p}^2(v) + \omega_{m,p}^2(v) Q_{m,p}^2(v) \right].$$

It is therefore not difficult to interpret $Q_{m,p}(v)$ and $P_{m,p}(v)$ as the position and momentum of the field. We can then promote these...
quantities to operators and introduce the creation and annihilation operators of the field in the traditional way as follows:
\[
\hat{Q}_{m,p}(v) = \sqrt{\frac{\hbar}{2\omega_{m,p}(v)}} [\hat{a}_{m,p}^+(v) + \hat{a}_{m,p}(v)],
\]
(30a)
\[
\hat{H}_{m,p}(v) = \sqrt{\frac{\hbar}{2\omega_{m,p}(v)}} [\hat{a}_{m,p}^+(v) - \hat{a}_{m,p}(v)],
\]
(30b)
with the usual bosonic commutation relations\(^5\)
\[
[\hat{a}_{m,p}(v), \hat{a}_{m,q}^+(u)] = \delta_{m,q} \delta_{p,q} \delta(u - v),
\]
(31a)
\[
[\hat{a}_{m,p}(v), \hat{a}_{n,q}(u)] = 0 = [\hat{a}_{m,p}^+(v), \hat{a}_{n,q}(u)].
\]
(31b)
Substituting equations (28) and (30) into equation (26) brings to the following expression for the quantised field
\[
\hat{A}(\mathbf{r}, t) = \sum_{m,p} \int d^3v \ e^{i \left( \frac{\mathbf{r} \cdot \mathbf{k}}{\hbar} \right)} \sqrt{\hbar \omega_{m,p}(v)} \psi_{m,p}^{(0)}(\mathbf{r}, \zeta) \hat{a}_{m,p}(v)
+ \text{h.c.},
\]
(32)
where h.c. stands for Hermitian conjugate. The expression for the Hamilton operator can be instead found by substituting equations (28) and (30) into equation (27), thus obtaining
\[
\hat{H} = \sum_{m,p} \int d^3v \ h_\omega_{m,p}(v) \left\{ \frac{\hat{a}_{m,p}^+(v) \hat{a}_{m,p}(v)}{2} + \frac{1}{2} \right\}.
\]
(33)
Moreover, if we introduce the ‘mass of the X wave’ as \( M \equiv \hbar / \omega^n \), the above equation can be rewritten as follows:
\[
\hat{H} = \sum_{m,p} \int d^3v \ \frac{Mv^2}{2} \left\{ \hat{a}_{m,p}^+(v) \hat{a}_{m,p}(v) + \frac{1}{2} \right\}.
\]
(34)
Equations (32)–(34) are the second main result of our work and represent quantisation of generalised OAM-carrying X waves. In particular, equation (34) is the analogue of a quantised Hamiltonian of a 1D gas of weakly interacting bosons, with mass \( M \) and velocity \( v \) [27]. This analogy is quite important, as it allows us to treat quantum X waves, which are intrinsically 3 + 1-dimensional fields, as one-dimensional particle-like objects parametrised only by their own velocities \( v \). This feature is similar to the particle-like nature of quantum solitons [28] and it is ultimately due to the non-differentiating nature of X waves.

### 4. Second order nonlinearity

We now turn our attention to quantum nonlinear processes involving X waves, and in particular to \( \chi^{(2)} \)-processes such as optical parametric amplification and Kerr nonlinearity. To start with, let us then consider three fields, namely a pump field, which, for the sake of simplicity, will be treated as a bright coherent state, characterised by the frequency \( \omega^n \), and a signal and idler fields characterised by the frequencies \( \omega_1 \) and \( \omega_2 \), respectively. Moreover, we assume that signal and idler have different group velocities, i.e. \( \omega_1' \neq \omega_2' \), but the group velocity dispersion of both fields is the same, namely \( \omega_1'' = \omega_2'' = \Omega'' \). The field operators for the signal and idler fields are obtained from equation (32) with the substitutions \( \omega \rightarrow \omega_{1,2} \) and \( \zeta \rightarrow \zeta_{1,2} \), respectively. The total (time-dependent) Hamiltonian of the system in presence of second order nonlinearity is then given as follows:
\[
\hat{H}(t) = \hat{H}_0(t) + \lambda \hat{H}_1(t),
\]
(35)
where \( \hat{H}_0 \) is the free field Hamiltonian given by equation (33). \( \lambda \ll 1 \) is the interaction parameter (assumed very small) and \( \hat{H}_1 \) is the quantised version of the classical \( \chi^{(2)} \)-interaction Hamiltonian [21], namely
\[
\hat{H}_1 = \chi \langle A_1 | A_2^* \rangle + \chi^* \langle A_2 | A_1^* \rangle.
\]
(36)
Before considering on the nonlinear quantum dynamics of the system described by equation (35), we must find a quantised expression for the classical interaction Hamiltonian. To do so, let us consider the first term in the above equation (the second is then obtained by simply taking the Hermitian conjugate) and uses equation (32) to obtain
\[
\chi \langle A_1 | A_2^* \rangle = \chi \int d^3r \ \hat{A}_1^*(\mathbf{r}, t) \hat{A}_2^*(\mathbf{r}, t)

= \chi \sum_{m,p,n,q} \int du \ e^{i \left( \frac{\mathbf{r} \cdot \mathbf{k}_1}{\hbar} \right)} \sqrt{\hbar \omega_{m,p}(u)} \psi_{m,p}^{(0)}(\mathbf{r}, \zeta_1) \hat{a}_{m,p}(u) \hat{a}_{n,q}(u)

\times \int d^3v \psi_{m,p}^{(0)*}(\mathbf{r}, \zeta_2) \psi_{n,q}^{(0)*}(\mathbf{r}, \zeta_2) \hat{a}_{m,p}^+(v) \hat{a}_{n,q}^+(v).
\]
(37)
where the subscript 1 refers to the signal field and 2 to the idler field, respectively. We can now use the above result to write the quantised form of the interaction Hamiltonian for second order nonlinearity. To do this, however, it is useful to define the quantity \( \rho = \sqrt{k_1 \omega_2' / (k_2 \omega_1')} \) and introduce the interaction function
\[
\chi_{mpq}(x) = \frac{(-1)^n 4\pi^2 \rho \mu f_{p,1}}{k_1 \sqrt{\omega_2'}} \left( \frac{x}{\omega''(1 + \rho)} \right)

\times f_{q,2} \left( \frac{\rho x}{\omega''(1 + \rho)} \right) \Theta(x),
\]
(38)
where \( \Theta(x) \) is the Heaviside step function [23] and \( f_{p,1}, f_{q,2} \) are defined according to equation (19). Moreover, we also define the function
\[
F(u, v) = u^2 + v^2 - 2u^2 + (v - \rho u)(u - v + \omega_1' - \omega_2') / \omega''(1 + \rho).
\]
(39)
If we now substitute equation (37) (and its complex conjugate) into equation (36) and perform the spatial integration using the orthogonality relation between X waves modes given by

---

\(^5\) In this context, the bosonic nature of light has been implicitly assumed. However, in general, one could as well define a theory of quantum X waves for fermionic systems. In this case, then, the fermionic, rather than the bosonic, commutation relation must be used to correctly quantise the correspondent harmonic oscillator.
equation (103), we can write the interaction Hamiltonian in the following form:

\[
\hat{H}_I = \hbar \sum_{m,p,q} \int \mathrm{d}u \mathrm{d}v \chi_{m,p,q}(u + v) \sqrt{\omega_{m,p}(u) \omega_{-m,q}(v)} \times e^{iF(u,v)\tau} \hat{a}_{m,p}^\dagger(u) \hat{a}_{-m,q}(v) + \text{h.c.}
\]

(40)

We are now in the position to calculate the state of the electromagnetic field after the nonlinear interaction with the medium. To do so, we work in the interaction picture, where both the state and the operators are time-dependent, and the evolution is governed by the interaction Hamiltonian. Then, since \( \lambda \ll 1 \), we can treat the nonlinearity as perturbative and use the Schwinger–Dyson expansion of the propagator \( \exp[-i\hat{H}_I(t)/\hbar] \) [29], namely

\[
e^{-\frac{i}{\hbar}\hat{H}_I(t)} = 1 + \sum_{n=1}^\infty \left(-\frac{i}{\hbar}\right)^n \int_0^\tau \mathrm{d}\tau_1 \cdots \int_0^\tau \mathrm{d}\tau_n \cdots \int_0^\tau \mathrm{d}\tau_{n-1} \cdots \hat{H}_I(\tau) \ldots \hat{H}_I(\tau_1) \ldots = \left[ \int_0^\tau \mathrm{d}\tau \hat{H}_I(\tau) \right]_0^\tau \ldots \ldots \ldots,
\]

(41)

to write the expression of the final state as follows:

\[
|\psi(t)\rangle &= |0\rangle - \frac{i}{\hbar} \int_0^\tau \mathrm{d}\tau \hat{H}_I(\tau)|0\rangle + \frac{1}{\hbar^2} \int_0^\tau \int_0^\tau \mathrm{d}\tau' \hat{H}_I(\tau') \hat{H}_I(\tau)|0\rangle + \cdots
\]

\[= |\psi(0)\rangle + |\psi^{(1)}(t)\rangle + |\psi^{(2)}(t)\rangle + \cdots,
\]

(42)

where \( |\psi(0)\rangle = |0\rangle \) has been implicitly assumed. For the purposes of our analysis, we can truncate this perturbative series \(\sum_{n=1}^\infty \left(-\frac{i}{\hbar}\right)^n \int_0^\tau \mathrm{d}\tau_1 \cdots \int_0^\tau \mathrm{d}\tau_n \cdots \int_0^\tau \mathrm{d}\tau_{n-1} \cdots \hat{H}_I(\tau) \ldots \hat{H}_I(\tau_1) \ldots \) to the first order, thus obtaining the following expression for the state of the field after the nonlinear interaction:

\[
|\psi^{(1)}(t)\rangle = -\frac{i}{\hbar} \int_0^\tau \mathrm{d}\tau \hat{H}_I(\tau)|0\rangle,
\]

(43)

If we now substitute the expression of the interaction Hamiltonian given by equation (40), we have

\[
-\frac{i}{\hbar} \int_0^\tau \mathrm{d}\tau \hat{H}_I(\tau)
\]

\[=-\frac{i}{\hbar} \sum_{m,p,q} \int \mathrm{d}u \mathrm{d}v \chi_{m,p,q}(u + v) \sqrt{\omega_{m,p}(u) \omega_{-m,q}(v)} \times \int_0^\tau \mathrm{d}\tau' e^{i\tilde{F}(u,v)\tau'} \hat{a}_{m,p}^\dagger(u) \hat{a}_{-m,q}(v).
\]

(44)

If we now introduce the quantities

\[
K(u, v) = \frac{(v - mu)(u - v + \omega'_1 - \omega'_2)}{2 \omega' (1 + \rho)},
\]

\[
G(u, v, t) = -\frac{2i}{\tilde{F}(u, v)} \sin \left( \frac{\tilde{F}(u, v)t}{2} \right),
\]

(45)

and introduce, for convenience of notation, the quantity

\[
G_{m,p,q}(u, v, t) = \sqrt{\omega_{m,p}(u) \omega_{-m,q}(v)} G(u, v, t) \times e^{iK(u,v)\tau} \chi_{m,p,q}(u + v),
\]

(46)

and the two particle state

\[
|m, p, u; -m, q, v\rangle \equiv \hat{a}_{m,p}^\dagger(u) \hat{a}_{-m,q}(v)|0\rangle,
\]

(47)

we can write the perturbed state \( |\psi^{(1)}(t)\rangle \) as follows:

\[
|\psi^{(1)}(t)\rangle = \sum_{m,p,q} \int \mathrm{d}u \mathrm{d}v \ G_{m,p,q}(u, v, t)|m, p, u; -m, q, v\rangle.
\]

(48)

The above state represents the superposition of two particles, corresponding to the two modes \( \omega'_1 \) and \( \omega'_2 \) (generated by the nonlinear process) travelling with velocities \( u \) and \( v \), respectively. Moreover, since the function \( G_{m,p,q}(u, v, t) \) is in general non-separable in the variables \( u \) and \( v \), the above state represents a continuous variable entangled state in the (continuous valued) velocities \( u \) and \( v \) of the two particles.

4.1. Transition probability

We can now use the explicit, analytic expression for the state after the nonlinear interaction to calculate the probability for the field to be in such a state after the interaction with the \( \chi^{(2)} \)-nonlinearity of the medium. We then get

\[
P(t) = \sum_{m,p,q} \sum_{n,r,s} \int \mathrm{d}u' \mathrm{d}v' \ G_{n,r,s}(u', v') G_{m,p,q}(u, v, t)
\]

\[\times \langle m, p, u; -m, q, v|n, r, s|u', v'\rangle|\chi_{m,p,q}(u + v)|^2
\]

\[= \sum_{m,p,q} \int \mathrm{d}u \mathrm{d}v \ P_{m,p,q}(u, v, t),
\]

(49)

where

\[
P_{m,p,q}(u, v, t) = \omega_{m,p}(u) \omega_{-m,q}(v)|\chi_{m,p,q}(u + v)|^2
\]

\[\times \sin^2 \left( \frac{F(u, v)t}{2} \right)^2.
\]

(50)

As it can be seen, the transition probability is, in general, non-separable in \( \{u, v\} \) due to the inseparability of \( \chi_{m,p,q}(u + v) \) with respect to \( u \) and \( v \). Although the above form represents an exact (within the perturbative limit) expression for the transition probability \( P(t) \), its form is quite difficult to handle, and further approximations are needed to fully understand its properties. In particular, here we employ two different approximations. First, we consider the state of the system for \( t \to \infty \), meaning that we look at the system far away (in time) from the moment of interaction. If we do that, \( P_{m,p,q}(u, v, t) \) tends to a Dirac delta function peaked at \( F(u, v) = 0 \), since

\[
\lim_{t \to \infty} \sin^2 \left( \frac{F(u, v)t}{2} \right) = \delta(F(u, v)).
\]

(52)

We then have

\[
P_{m,p,q}(u, v; t \to \infty) = \omega_{m,p}(u) \omega_{-m,q}(v)|\chi_{m,p,q}(u + v)|^2
\]

\[\times \delta(F(u, v))^2.
\]

(53)

For large times, therefore, the entangled particles with velocities \( u \) and \( v \) are associated to points in the \( (u, v) \)-plane which are constrained on the parabolic surface \( F(u, v) = 0 \).
To further simplify this result, we can look at the so-called low velocity limit, which corresponds, in the theory of interacting quantum gases [27], to the small momentum approximation. In this limit, we neglect the quadratic contributions in $F(u, v)$. This allows us to rewrite the function $F(u, v)$ as follows

$$F(u, v) \simeq \frac{\omega^2_1 - \omega^2_2}{2\omega}(v - \rho u),$$  \hspace{1cm} \text{(54)}$$

and therefore the Dirac delta above can be rewritten as

$$\delta(F(u, v)) \simeq \frac{2\omega^2(1 + \rho)}{[\omega^2_1 - \omega^2_2]} \delta(v - \rho u).$$  \hspace{1cm} \text{(55)}$$

Inserting this approximation in equation (53) brings to the following result

$$P_{m,p,q}(u, v; \infty) \simeq \frac{\omega_{m,q}(u)\omega_{-m,q}(v)}{\chi_{m,p,q}(u + v)^2} \times \frac{2\omega^2(1 + \rho)}{[\omega^2_1 - \omega^2_2]} \delta(v - \rho u) t^2.$$  \hspace{1cm} \text{(56)}$$

For practical cases, $\rho \approx 1$ and therefore the above expression constrain the two particles to travel at the same speed $u = v$. In this case, the transition probability (50) assumes the following explicit form

$$P(t) = \frac{4\omega^2 t^2}{[\omega^2_1 - \omega^2_2]} \sum_{m,p,q} \int du \omega_{m,p}(u)\omega_{-m,q}(u) |\chi_{m,p,q}(2u)|^2.$$  \hspace{1cm} \text{(57)}$$

The above integral over $u$ can be performed analytically if we use the definition of generalised Laguerre polynomials [30] and the integral formula [31]

$$\int_0^\infty dx x^k e^{-bx} = \frac{k!}{b^{k+1}},$$  \hspace{1cm} \text{(58)}$$

thus obtaining

$$C_{p,q} = \int_0^\infty du \omega_{m,p}(u)\omega_{-m,q}(u) |\chi_{m,p,q}(2u)|^2$$

$$= \left[ \frac{\pi^2}{\omega^2_1(1 + p)(1 + q)} \right] \sum_{j=0}^p \sum_{i=0}^q \left[ \begin{array}{c} p + 1 \\downarrow \begin{array}{c} q + 1 \\downarrow \end{array} \\ p - s \\downarrow \begin{array}{c} q - t \\downarrow \end{array} \end{array} \right]^2$$

$$\times \left( 2s + 2t + 1 \right)! \left( 4s^2 + 4t^2 + 4st \right)^2.$$  \hspace{1cm} \text{(59)}$$

The transition probability in the large times, low velocity limit thus reads (with $C = \sum_{p,q} C_{p,q}$)

$$P(t) = \left( \frac{4C\omega^2}{[\omega^2_1 - \omega^2_2]} \right)^2.$$  \hspace{1cm} \text{(60)}$$

Therefore, in the large times and low velocity limit, the transition probability scales with the square of the interaction time. This result, anyway, is not unexpected, as in our case the large times limit corresponds to the infinite thickness, perfect phase matching limit for standard quantum optics in bulk crystals [21].

4.2. Accounting for the finite length of the nonlinear crystal

The discussion in the previous section assumes that the nonlinear crystal is infinitely long. Although this approximation can well describe many different situations in a very good way, it is also important to account for effects deriving from a nonlinear crystal of finite length. To account for it in a correct way, one could simply numerically solve the dynamics of a system governed by the Hamiltonian (35) applying the correct boundary conditions (in space and time) to account for the finiteness of the nonlinear crystal, as well as the finite time required for the measurement process. In this section, however, we will take a different approach, which will result in the possibility of obtaining some approximate analytical expressions for the transition probability as a function of the crystal length.

First of all, let us fix some convention: we assume that the nonlinear interaction takes place at time $t = 0$, while the initial and final states of the system are accessed at time $t = -\infty$ and $t = \infty$, respectively. This allows us to use a scattering matrix approach to the nonlinear interaction, similar to what it is done in quantum field theory [27]. In our case, this allows us to first take the time integral of the interaction Hamiltonian (40) and then integrate over the crystal length. Intuitively, this approach is complementary to the one presented in the previous section, where we have considered the crystal infinitely long but restricted ourselves in considering a finite interaction time.

To obtain the final form of the interaction Hamiltonian within this approximation, we start again from equation (36) and we take its integral over time, namely

$$\hat{H}_I(z) = \int dt \left[ \chi \langle A_1 | A_2^\dagger \rangle + \chi^* \langle A_2 | A_1^\dagger \rangle \right].$$  \hspace{1cm} \text{(61)}$$

A similar calculation than the one carried out in the previous section, together with the assumption that in practical cases $\rho \approx 1$, brings to the following result:

$$\hat{H}_I(z) = \hbar \sum_{m,p,q} \int dv \Xi_{mp,q}(u, v)$$

$$\times \left[ \hat{\omega}_{m,p}(u)\hat{\omega}_{-m,q}(v) e^{i\Lambda(u,v)} \right] \hat{d}_{m,p}(v)\hat{d}_{-m,q}(v) + h.c.,$$  \hspace{1cm} \text{(62)}$$

where $\Lambda(u, v) = 2\nu/[\omega^2(u + v)]$ and the modified vertex function $\Xi_{mp,q}(u, v)$ is given by

$$\Xi_{mp,q}(u, v) = (-1)^{m+n}8\pi^2 \chi \left[ \frac{\omega^2_1}{k_1k_2} \frac{u + v}{u^2 + v^2} \right]$$

$$\times f_p(X(u, v))f_q(X(u, v)),$$  \hspace{1cm} \text{(63)}$$

with $X(u, v) = (u^2 + v^2)/[2\omega^2(u + v)]$. To calculate the state of the system at the output facet of the crystal, assumed to be of length $L$, we essentially apply again the Schwinger–Dyson expansion to the propagator $\exp[-\hat{H}_I(z)/\hbar]$ and consider only the first order term, thus obtaining

$$|\psi^{(1)}(L)\rangle = -\frac{i}{\hbar} \int_0^L dz \hat{H}_I(z) |0\rangle.$$  \hspace{1cm} \text{(64)}$$
If we substitute the expression for $\hat{H}(z)$ as given above, we arrive at the following result, for the state of the field at the output of the crystal:  

$$|\psi^{(1)}(L)| = \sum_{m,p,q} \int du \, dv \, L_{m,p,q}(u,v) \times \mathcal{F}(u,v;L)|m,p,u; -m,q,v\rangle,$$  

where  

$$\mathcal{F}(u,v;L) = e^{i\Lambda(u,v)L} = 1$$  

and  

$$L_{mpq}(u,v) = -\frac{i}{2} \sqrt{\omega_{mp}(\omega_{mq}(v)} \times \left[ \frac{\omega''(u + v)}{uv} \right] \Xi_{mpq}(u,v).$$  

The probability for the field to be in the state $|\psi^{(1)}(L)|$ at the end facet of the crystal is then given by  

$$P(L) = \sum_{p,q} \int du \, dv \, |L_{mpq}(u,v)|^2 |\mathcal{F}(u,v;L)|^2.$$  

Before introducing further approximations to derive an analytical expression for $P(L)$ in the limit of small velocities, let us take a closer look at the above equation. In particular, let us focus on the term $\mathcal{F}(u,v;L)$. First, let us write the expression of $|\mathcal{F}(u,v;L)|^2$ explicitly, i.e.  

$$|\mathcal{F}(u,v;L)|^2 = 2[1 - \cos(\Lambda(u,v)L)].$$  

Then, let us note that the above expression reaches its maximum when the condition  

$$\Lambda(u,v)L = (2n + 1)\pi,$$  

is fulfilled.

Together with regular phase matching (that here is implicitly assumed, given the scalar nature of our theory), this condition imposes a new ‘phase matching’ condition (which from here onwards we will call velocity matching condition) over the velocities of the two X wave modes representing the signal and idler photons, respectively. In particular, this condition requires that  

$$u = \frac{\kappa_n v}{v - \kappa_n},$$  

where $\kappa_n = (n + 1/2)\pi \omega''/L$. The velocity matching condition, therefore, fixes either the length $L$ of the crystal, or the relative velocity of the two X wave modes involved in the nonlinear process. In both cases, fulfilling the velocity matching condition results in maximum efficiency for the nonlinear process. In the small approximation limit, $v \ll 1$, and the above velocity matching condition reduces to  

$$u \simeq -v\left(1 + \frac{v}{\kappa_n}\right) + O(v^3).$$  

This is the most important result of this section: considering a nonlinear crystal of finite length introduces a matching condition between the velocities of the signal and idler photons, similar to the phase matching condition typical of nonlinear optics [21]. While in traditional nonlinear optics, phase matching is achieved independently on the crystal length, however, in this case the length of the crystal plays an active role in determining the matching condition, as $\kappa_n$ depends on $L$. This is intuitively easy to understand, as the longer the crystal gets, the more time is needed to the photon to traverse it and therefore it might adjust its velocity accordingly to be matched with its twin partner.

The above integral, in general, admits no analytical solution and must be evaluated numerically. However, in the small velocity limit, $v \ll 1$ and the above expression can be simplified and solved analytically. To calculate the explicit expression of $P(L)$ in this limit, first we note that for at the lowest order in $v$, $L_{mpq}(\kappa_n v/(v - \kappa_n), v)$ assumes the following form  

$$L_{mpq}\left(\frac{\kappa_n v}{v - \kappa_n}, v\right) \simeq \frac{i(-1)^m \Delta \kappa_n}{2(\omega'')^2 \sqrt{(1 + p)(1 + q)}} \cdot \frac{L^{(1)}_p}{L^{(1)}_p} \cdot \left(\frac{-2\kappa_n \Delta}{\omega''}\right) \times L^{(1)}_q \cdot e^{2\kappa_n \Delta / \omega''} + O(v^2).$$  

In the case of velocity matching, $|\mathcal{F}(u,v;L)|^2 = 2$ and therefore, the expression of the transition probability $P(L)$ in the small velocity limit (assuming velocity matching)
reduces to
\[ P(L) \approx 2 \sum_{p,q} \int \mathrm{d}v \left| \mathcal{L}_{mpq}(u, v) \right|^2 \]
\[ = \frac{1}{2(\omega v)^2} \frac{2}{L^2} I(v) e^{i\kappa_v/L} \sum_{p,q} \frac{\mathcal{P}_{pq}(L)}{1 + p} (1 + q), \] (75)
where \( s_n = (n + 1/2)|\Delta| \) and
\[ I(v) = \int \mathrm{d}v \left[ 1 + O(v^2) \right], \] (76a)
\[ \mathcal{P}_{pq}(L) = L_p^{(1)} \left( \frac{2s_n}{L} \right) L_q^{(1)} \left( \frac{2s_n}{L} \right). \] (76b)

It is important to point out, however, that in order to calculate \( P(L) \) from equation (76), or from its low velocity approximation (75), one needs to fix a maximum velocity \( v_{\text{max}} = V \), to guarantee the convergence of the integral \( I(v) \).

Moreover, one also needs to fix a maximum spectral order \( p_{\text{max}} \), so that the summation in equation (75) remains finite. Experimentally, this is perfectly admissible, since one has only access to a finite set of X wave velocities and spectral orders. Moreover, the limits imposed by the maximum velocity \( V \) and the maximum spectral order \( p_{\text{max}} \), reflect both the paraxial approximation and the finite bandwidth of the pulse. Experimentally, in fact, the X wave velocity is connected with the numerical aperture of the system that generates it, and therefore the maximum velocity \( V \) corresponds to the maximum angular aperture, which still guarantees the paraxial approximation to hold. Concerning the maximum spectral order, on the other hand, an experimentally feasible optical pulse will always have a bounded spectrum, and therefore the spectrum \( f_p(\alpha) \) must comply to this constraints. This means, ultimately, that the summation over \( p \) in the expressions above, is truncated to a certain maximum value \( p_{\text{max}} \), which depends on the (finite) form of the spectrum. As a last remark, it is instructive to consider the limiting expressions for \( P(L) \) when the crystal length approaches infinity and zero. In the first case, if we use notice that
\[ \lim_{L \to \infty} P_{pq}(L) = (1 + p)(1 + q), \] then, we have
\[ P(L \to \infty) = \frac{1}{2(\omega v)^2} \frac{2}{L^2} I(v). \] (78)

When \( L \to 0 \), instead, the low velocity approximation gives \( P(L \to 0) \to \infty \). Clearly, this means that this approximation is not good anymore for small crystal lengths \( L \). When \( L \) becomes very small, in fact, \( \kappa_v \to \infty \) and the small velocity approximation breaks down. To correctly estimate the transition probability in the limit of small crystal length, therefore, one should take the exact form of \( P(L) \) given by equation (68) and expand perturbatively around \( L = 0 \). By doing so \( |J_f(u, v; L)|^2 \approx L^2 \) and therefore the transition probability becomes
\[ P(L \to 0) \approx L^2 \sum_{p,q} \int \mathrm{d}u \mathrm{d}v \left| \mathcal{L}(u, v) \Lambda(u, v) \right|^2, \] (79)
A closer look to the above equation also reveals, that in the limit of small \( L \), the transition probability resembles the traditional \( L^2 \)-dependence typical of perfectly phase matched systems [21]. Moreover, the above expression for the transition probability for small lengths \( L \) correctly vanishes when \( L = 0 \).

5. Kerr effect

We now turn our attention to the case of Kerr nonlinearity and study the dynamics of OAM-carrying X waves in presence of such nonlinearity. To do that, we first derive, in the low velocity limit, an expression for the interaction Hamiltonian and then calculate the state of the system after the interaction, assuming that the initial state of the field is the vacuum state. Afterwards, we show that the Kerr dynamics can be split into two parts: a classical evolution of the envelope function, which obeys a wave equation, and a quantum evolution for the field operators, which is described by a nonlinear Schrödinger equation, whose potential depends on the OAM content of the system.

5.1. Interaction Hamiltonian

The quantised form of the interaction Hamiltonian in the case of Kerr nonlinearity can be derived from its classical counterpart [21] and reads as follows:
\[ \hat{H}_I = \frac{\lambda}{2} \int \mathrm{d}^3 \vec{r} \hat{A}^\dagger(\vec{r}, t) \hat{A}(\vec{r}, t) \hat{A}(\vec{r}, t) \hat{A}^\dagger(\vec{r}, t). \] (80)

If we substitute the expressions for the field operators as given by equation (32) and its Hermitian conjugate, the above equation can be written as follows:
\[ \hat{H}_I = \frac{\hbar^2 \lambda}{2} \sum_{\{m\}} \int \mathrm{d}u \mathrm{d}^3 \vec{r} \sum_{\{pq\}} \int \mathrm{d}v \left[ \psi_{mpq}(\vec{r}, \psi_{pq}(\vec{r}, v)) \psi_{pq}(\vec{r}, \psi_{pq}(\vec{r}, v)) \right] \]
\[ \times \mathcal{S}_m(\vec{u}, \vec{R}, \vec{\zeta}) \hat{d}^\dagger_{m,p}(\vec{u}) \hat{d}_{n,q}(\vec{v}) \hat{d}_{n,q}(\vec{u}) \hat{d}^\dagger_{m,p}(\vec{v}). \] (81)

where we have introduced the shorthand notation \( \{m\} = \{m, n, l, s, p, q, r, t\} \) (with \( \{m, n, l, s\} \subset (-\infty, \infty) \) and \( \{p, q, r, t\} \subset [0, \infty) \)) and \( u = \{u, v, \tilde{u}, \tilde{v}\} \). The quantity \( \mathcal{S}_m(\vec{u}, \vec{R}, \vec{\zeta}) \) is defined as \( \mathcal{S}_m(\vec{u}, \vec{R}, \vec{\zeta}) = \psi_{mpq}(\vec{R}, \vec{\zeta}) \psi_{pq}(\vec{R}, \vec{\zeta}) \psi_{pq}(\vec{R}, \vec{\zeta}) \psi_{mpq}(\vec{R}, \vec{\zeta}) \), where \( \psi = \sqrt{\omega_{mpq}(v) \omega_{pq}(v)} \omega_{pq}(v) \omega_{pq}(v) \).

If we now assume \( t = 0 \), perform the spatial integration and introduce the vertex function \( \chi(\{u\}, \{v\}) \) (see appendix B for the details), after some manipulation we can rewrite the above interaction Hamiltonian in the following compact form:
\[ \hat{H}_I = \frac{1}{2} \sum_{\{m\}} \int \mathrm{d}^4 \chi(\{v\} - v_4 + v_1 + v_2) \]
\[ \times \left[ \sqrt{\omega_{mpq}(v_1) \omega_{pq}(v_2) \omega_{pq}(v_3) \omega_{pq}(v_4)} \right] \]
\[ \times \hat{d}^\dagger_{m,p}(v_1) \hat{d}^\dagger_{n,q}(v_2) \hat{d}_{n,q}(v_3) \hat{d}^\dagger_{m,p}(v_4), \] (82)
where \( \chi(\{u\}, \{v\}) \) is the vertex function, as defined in appendix B.
Although the above expression for the Kerr Hamiltonian is exact, it cannot be treated analytically any further, thus limiting the amount of insight one can get about the effect of Kerr nonlinearity on OAM-carrying X waves. To this aim, we now introduce the so-called low velocity approximation, which corresponds to neglect quadratic terms in the expression of the eigenfrequencies $\omega_m(v)$. Moreover, we introduce the Fourier transform of the vertex function (see appendix B)

$$\chi_{\{m\}}(p) = \int dp |\psi_{\{m\}}(p)| e^{-ipq/\omega_m},$$  

(83)

and the Fourier representation for the creation and annihilation operators

$$\hat{a}_{m,p}(v) = \int dv \hat{c}_{m,v}(v) e^{i\eta v/\omega_m},$$  

(84a)

$$\hat{\phi}_{mp}(\eta) = \int dv \hat{a}_{m,p}(v) e^{i\eta v/\omega_m}.$$  

(84b)

Using the quantities above, the Kerr Hamiltonian in the low velocity limit (and for $t = 0$) assumes the following form:

$$\hat{H}_l = \sum_{\{m\}} \delta_{m+n+\ell+\delta} \int dv \Sigma_{\{m\}}(\eta) \hat{c}^{\dagger}_{m,v}(\eta) \hat{c}_{n,v}(\eta) \hat{c}_{\ell,v}(\hat{\phi}_{v}) \hat{\phi}_{v}(\eta).$$  

(85)

The explicit expression of $\Sigma_{\{m\}}(\eta)$ (which essentially comprises $\sigma_{\{m\}}(\eta)$ and the result of the integration over $d^4v$) is given in appendix C. A closer inspection to the above form of the Kerr Hamiltonian reveals that it has the same form as the standard quantum Kerr Hamiltonian used, for example, to derive quantum solitons in optical fibres [28].

5.2. Time evolution

The evolution of a system under the action of the interaction Hamiltonian described by equation (85) can be easily calculated in the Heisenberg representation, where the time evolution is applied to the field operators, while the state does not evolve in time [29]. First, we write the field operators (84) as

$$\hat{a}_{m,p}(v) \rightarrow \hat{a}_{m,p}(v, t) = a_{m,p}(v)e^{-i\eta v/\omega_m t},$$  

(86a)

$$\hat{\phi}_{mp}(\eta) \rightarrow \hat{\phi}_{mp}(\eta, t) = \hat{\phi}_{mp}(\eta)e^{i\eta v/\omega_m t}.$$  

(86b)

Then, we use the above relations to rewrite the field operators $\hat{A}(r, t)$ and $\hat{A}(r, t)$ as a function of $\hat{\phi}_{mp}(\eta, t)$ and $\hat{\phi}_{mp}(\eta, t)$, thus obtaining

$$\hat{A}(r, t) = \sum_{m,p} \int dv \xi_{mp}(\eta, r, t) \hat{\phi}_{mp}(\eta),$$  

(87)

where

$$\xi_{mp}(\eta, r, t) = \int dv \sqrt{\hbar \omega_{mp}(v)} e^{i\eta v/2 \omega_m} \psi_{mp}(R, \xi).$$  

(88)

Notice that the form of the field operator given by equation (87) is very similar to the one of a field operator of an optical beam, where the function $\xi_{mp}(\eta, r, t)$ plays the role of the mode function [32]. In particular, it is not difficult to see that the mode function $\xi_{mp}(\eta, r, t)$ is a solution of the initial wave equation (7). This result is quite important, as it states that the mode function $\xi_{mp}(\eta, r, t)$ contains information only on the classical (i.e. deterministic) evolution of the system under the action of the Kerr nonlinearity and it is somehow decoupled from the quantum evolution, which only affects $\hat{\phi}_{mp}(\eta)$.

To study the quantum evolution of the system, therefore, we impose that the operator $\hat{\phi}_{mp}(\eta, t)$ obeys the Heisenberg equation of motion

$$\frac{d\hat{\phi}_{mp}(\eta, t)}{dt} = [\hat{H}_l, \hat{\phi}_{mp}(\eta, t)],$$  

(89)

where $\hat{H}_l = \hat{H}_0 + \hat{H}_l$. Using a bit of algebra, it is not difficult to show that

$$[\hat{H}_0, \hat{\phi}_{mp}(\eta)] = -\pi \omega^2 \frac{\partial^2 \hat{\phi}_{mp}(\eta)}{\partial \eta^2},$$  

(90)

and

$$[\hat{H}_l, \hat{\phi}_{mp}(\eta)] = -\sum_{\{a\}} \delta_{a+m,e+g} \Sigma_{\{a\}}(\eta) \hat{\phi}_{mp}(\eta) \hat{\phi}_{eg}(\eta),$$  

(91)

where $\{a\} = \{a, b, e, f, g, h\}$ and $\{b\} = \{a, m, e, g, b, p, f, h\}$, with $\{a, m, e, g\} \in \mathbb{Z}$ and $\{b, p, f, h\} \in [0, \infty]$. The equation of motion for the field operator $\hat{\phi}_{mp}(\eta)$ can be then written in the following form:

$$\frac{i \hbar}{\partial t} \frac{\partial \hat{\phi}_{mp}(\eta)}{\partial \eta} = -\frac{\hbar^2}{2M} \frac{\partial^2 \hat{\phi}_{mp}(\eta)}{\partial \eta^2} + \sum_{\{a\}} V_{\{a\}}(\eta) \hat{\phi}_{mp}(\eta) \hat{\phi}_{eg}(\eta),$$  

(92)

where

$$V_{\{a\}}(\eta) = -\hbar \delta_{a+m,e+g} \Sigma_{\{a\}}(\eta).$$  

(93)

The field operator $\hat{\phi}_{mp}(\eta, t)$ is then a solution of a set of coupled nonlinear Schrödinger equation. As can be seen, the coupling is essentially given by the coupling of the different OAM modes that define the nonlinear potential $V_{\{a\}}(\eta)$. This is the main effect of the Kerr nonlinearity, namely to introduce a coupling between different OAM values of different X wave states. This coupling, moreover, depends essentially from $\Sigma_{\{a\}}(\eta)$, i.e. from the overlap integral between the four modes involved in the nonlinear process. The Kroncker delta in the definition of the nonlinear potential $V_{\{a\}}(\eta)$, moreover, does not fix univocally a relation between the four OAM states involved in the process, but it imposes only the conservation of angular momentum between the states involved in the interaction by defining a family of possible sets of OAM states that can be generated via Kerr effect. This, in principle, could be used to generate single photon X wave states with high OAM content.
6. Summary and conclusions

In this work, we have presented a rigorous theory of quantum X waves with OAM in dispersive media. In particular, we have shown that quantised OAM-carrying X waves are formally analogue to a 1D gas of interacting bosons, characterised by a mass $M$ and a velocity $v$ (see equation (34)), which are, respectively, proportional to the group velocity dispersion and the Bessel cone angle of the X wave. We have then used these results to investigate the dynamics of quantum X waves in media exhibiting $\chi^{(2)}$, as well as $\chi^{(3)}$-nonlinearities. For the case of quadratic nonlinearities, we have shown that a continuous variable entanglement between the X wave velocities can be realised (see equation (49)). For the case of Kerr nonlinearity, instead, we have shown that the dynamics of the X wave can be splitted into a classical (deterministic) and a quantum part. The classical mode function $\xi_{mp}(q, r, t)$ evolves accordingly to the wave equation in dispersive media (equation (4)), while the quantum part evolves according to a system of coupled nonlinear Schrödinger equations (equation (92)), with a potential that depends on the coupling between the various OAM states involved in the interaction. For the case of second order nonlinearity, moreover, we treated in detail the effect of the finite size of the non-linear crystal, deriving an exact expression for the transition probability as a function of the crystal length $L$ and discussing in detail its limits in the case of small velocity and small crystal length.

In conclusion, our work presents a complete theoretical toolkit for the handling of non-diffracting quantum states of light and we envisage that it could be useful for the realisation of a new generation of quantum communication and quantum information protocols based on non-diffracting optical pulses. The natural resilience of X waves to external perturbations, in fact, makes them the idea candidate for the realisation of free space quantum communication channels. The fact that they carry OAM, moreover, gives the possibility to increase the amount of information that can be transferred in a (virtually) undistorted way through the atmosphere.

Acknowledgments

CC acknowledges support from the Templeton foundation, grant number 58277. The authors thank the Deutsche Forschungsgemeinschaft (grants BL 547/13-1, SZ 276/9-1 and SZ 276/12-1) for financial support.

Appendix A. Orthogonality of generalised OAM-carrying X waves

The aim of this appendix is to show that generalised OAM-carrying X waves of order $p$ and velocity $v\psi_{mp}(R, \zeta)$ as given by equation (20), constitute an orthogonal set of functions.

Let us consider the following scalar product between two generalised OAM-carrying X waves, one of order $p$ and velocity $v$, namely $\psi_{mp}(R, \zeta)$, and the other one of order $q$ and velocity $u$, i.e. $\psi_{pq}(R, \zeta)$

$$s \equiv \int d^3 r \psi_{pq}^*(R, \zeta) \psi_{mp}(R, \zeta).$$

This quantity can be written explicitly by using equation (20) to obtain

$$s = \frac{k\Delta/(2\pi^2\omega)}{\sqrt{(1 + p)(1 + q)}} \int dz \int_0^\infty d\alpha (\alpha) L_q^{(1)}(2\alpha\Delta)e^{-\alpha\Delta} \times \int_0^\infty d\beta L_p^{(1)}(2\beta\Delta)e^{-\beta\Delta} \int_0^{2\pi} d\theta e^{i(m-l)\theta} \times \int_0^\infty dR R J_l\left[(\omega k/\omega')^\frac{1}{2}\beta R\right]J_m\left[(\omega k/\omega')^\frac{1}{2}\alpha R\right] e^{i\left[(\alpha - \frac{\omega}{\omega'})^\frac{1}{2}(\beta - \frac{\omega}{\omega'})^\frac{1}{2}\right](Z-z)}.$$  

First, we solve the azimuthal integral, which gives

$$\int_0^{2\pi} d\theta e^{i(m-l)\theta} = 2\pi\delta_{m,l}.\quad (96)$$

Then, by substituting this result in the expression above and introducing the scaled radial coordinate $\rho = R\sqrt{\omega'^2k/\omega}$, we have

$$s = \frac{2}{\omega''}\Delta^2\delta_{m,l} \int dz \int_0^\infty d\alpha \times \int_0^\infty d\beta \alpha \beta e^{-\Delta(\alpha + \beta)} \times L_q^{(1)}(2\alpha\Delta)L_p^{(1)}(2\beta\Delta) e^{i\left[(\alpha - \frac{\omega}{\omega'})^\frac{1}{2}(\beta - \frac{\omega}{\omega'})^\frac{1}{2}\right](Z-z)} \int_0^\infty d\rho \rho J_m(\alpha\rho) J_m(\beta\rho).\quad (97)$$

We now use the orthogonality of Bessel functions [23]

$$\int_0^\infty d\rho \rho J_m(\alpha\rho) J_m(\beta\rho) = \frac{1}{\beta}\delta(\alpha - \beta),\quad (98)$$

and we rewrite (using also the above result) the exponential function in equation (97) that contains $Z$ and $t$ as follows:

$$e^{i\left[(\alpha - \frac{\omega}{\omega'})^\frac{1}{2}(\beta - \frac{\omega}{\omega'})^\frac{1}{2}\right](Z-z)} = e^{-\alpha(\omega''-\omega)z^2/2}e^{-\beta(\omega''-\omega)z^2/2}\times e^{-i\omega''\frac{z^2}{2}}.\quad (99)$$

Now, since $dz = dz$, we can perform the z-integration in equation (97) obtaining

$$\int dZ e^{-i\omega''\frac{z^2}{2}} = 2\pi|\omega''|\delta(u - v).\quad (100)$$

Substituting this result into equation (97) and using moreover the orthogonality relation for generalised Laguerre
polynomials [23]

$$\int_0^\infty d\alpha \Delta \alpha L_p^{(1)}(2\alpha \Delta)L_q^{(1)}(2\alpha \Delta)e^{-2\Delta \alpha} = (p + 1)\delta_{p,q},$$

we obtain the final result

$$s = \delta_{m,n}\delta_{p,q}(u - v).$$

(102)

Therefore,

$$\int d^3r \psi^{(r)}_m(R, \zeta)^\dagger \psi^{(r)}_n(R, \zeta) = \delta_{m,n}\delta_{p,q}(u - v).$$

(103)

Then, generalised OAM-carrying X waves are an orthogonal set. Moreover, from the above expression it can also be noted that generalised OAM-carrying X waves carry (unsurprisingly) infinite energy, as their norm is infinite, like for plane waves.

### Appendix B. Particle states, coherent states, and normalisation

#### B.1. Particle states

We can now look for the eigenstates of the Hamilton operator, represented by either equation (33) or (34). In particular, in analogy to the traditional case [26], we can introduce the N-particle states \( |m, p, v, N\rangle \) as states containing N field excitation with velocity v in the travelling mode (i.e. the X wave) \( \psi^{(r)}_m(r, t) \) with energy \( MV^2/2 \). These states are mutually orthogonal

$$\langle n, q, v, N|m, p, v, N\rangle = \delta_{m,n}\delta_{p,q}(u - v),$$

(104)

complete and they can be obtained from the vacuum state \( |0\rangle \) by successive applications of the creation operator, i.e.

$$\langle m, p, v, N \rangle = \frac{1}{\sqrt{N!}}|\hat{a}_m^\dagger(v)|^N\langle 0\rangle.$$  

(105)

Moreover, it is not difficult to prove that the usual results concerning Fock states [26] are also valid in this case, namely the expectation value of the electric field operator is zero on the particle eigenstates, i.e.

$$\langle m, p, v, N |\hat{A}(r, t)|m, p, v, N\rangle = 0.$$  

(106)

However, the Fock states \( |m, p, v, N\rangle \) are not normalisable, since their representation in configuration state \( \langle r, t|m, p, v, N\rangle = \psi^{(r)}_m(r, t) \) carries infinite energy. In fact, if we compute the expectation value of the Hamilton operator over the Fock states, we get the following result

$$\langle m, p, v, N |\hat{H}|m, p, v, N\rangle = \frac{NMV^2}{2}\delta(v - v).$$

(107)

The Dirac delta appearing above, comes from the normalisation condition (104). This is not surprising, since also in the classical case, X waves are non-normalisable solutions of Maxwell’s equations (exactly as plane waves), due to the fact that they carry infinite energy.

To solve this issue, there are different approaches possible. First, one could introduce a finite quantisation volume and carry out the field quantisation is such finite volume [26]. In doing this, however, one should be careful to introduce a finite cavity with the desired symmetry, to use X waves as eigenstates for the field expansion. Another possibility would be to redefine the normalisation condition given by equation (104) by imposing that it only makes sense when calculated for X states with the same velocity, namely

$$\langle n, q, v, N|m, p, v, N\rangle = \delta_{m,n}\delta_{p,q}.$$

(108)

In this case, however, all the observable quantities will be finite if defined per unit volume, while the integrated quantities will be infinite, as in the standard case for X waves [2]. In particular, the Hamiltonian density in this case would be simply given by

$$\hat{H} = \hat{A}^\dagger(r, t)\hat{A}(r, t),$$

(109)

and its expectation value over Fock states (\( \hat{H} \)) is

$$\langle \hat{H} \rangle = N\hbar\omega_{m,p}(v)\langle \psi^{(r)}_m(r, \zeta) \rangle^2,$$

(110)

which is finite. However,

$$\int d^3r \langle \hat{H} \rangle = N\hbar\omega_{m,p}(v)\int d^3r \langle \psi^{(r)}_m(r, \zeta) \rangle^2 = \infty,$$  

(111)

since X waves carry infinite energy. Moreover, in analogy with what is done for monochromatic paraxial beams [33], all the other observables can be normalised using the energy density, thus obtaining finite, well defined quantities per unit volume.

Another possible way to overcome the problem in equation (107) is to consider X waves with finite energy (such as Bessel-X pulses, for example [2]). In this case, we can define a new set of Fock states (to differentiate them from the set of Fock states corresponding to infinite energy X waves) \( |X_{m,p}; n\rangle \), which are normalisable, since

$$\int d^3r |X_{m,p}(r, t)\rangle^2 < \infty,$$  

(112)

where \( X_{m,p}(r, t) = \langle r, t|X_{m,p}; n\rangle \). We therefore have

$$\langle X_{m,p}; k|X_{m,p}; r\rangle = \delta_{m,n}\delta_{p,q}\delta_{k,r},$$  

(113)

and the expectation value of \( \hat{H} \) over such states gives now a finite value.

Having clarified this point, in the remaining of the manuscript, we will actually employ (unless otherwise explicitly specified) equation (104) as ‘normalisation condition’ for the X waves particle states, implicitly remembering that they are associated with waves carrying infinite energy.

#### B.2. Coherent states

In analogy with the case of single mode quantum optics [26], we can also define coherent X wave states as follows:

$$|\alpha_{m,p}(v)\rangle = e^{-|\alpha_{m,p}(v)|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_{m,p}(v)^n}{\sqrt{n!}}|m, p, v, n\rangle,$$

(114)
Appendix C. Kerr Hamiltonian and vertex function

In this appendix, we calculate explicitly the spatial integral appearing in equation (81) and define the correspondent vertex function. In particular, we will first rewrite the integral appearing in equation (81) in a simpler form, then define the vertex function and calculate its Fourier transform, which will allow us to give a compact insightful form to the vertex function \( \chi_{|m|}(x) \). First of all, let us write the spatial integral explicitly (remembering that for \( \theta = 0 \), \( \zeta = X \)):

\[
I \equiv \int d^3 r \psi_{m,p}^{(i)}(R, z) \psi_{n,q}^{(j)}(R, z) \psi_{l,s}^{(k)}(R, z) \psi_{l,t}^{(k)}(R, z) \\
= \int d^3 r \int_0^\infty d^4 \alpha f_p(\alpha_1) f_q(\alpha_2) f_s(\alpha_3) f_t(\alpha_4) \\
\times J_{m}(\xi \alpha_1 r) J_{l}(\xi \alpha_2 r) J_{l}(\xi \alpha_3 r) J_{l}(\xi \alpha_4 r) e^{i(\ell_3 - m - n - s) \phi} \\
\times e^{\left[-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \frac{P - \omega}{\omega}ight] z} \\
= I_2 + I_3,
\]

where \( d^4 \alpha = d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \) and \( \xi = \sqrt{\omega^2 - \mathbf{k}^2} \). Moreover, in the last line we used the notation \( I_2 \) and \( I_3 \) to indicate the integrals with respect to \( z \) and \( \phi \), which can be solved immediately, thus leading to

\[
I_2 = 2\pi \delta \left( \alpha_3 + \alpha_4 - \alpha_1 - \alpha_2 - \frac{u + v - \bar{u} - \bar{v}}{\omega} \right),
\]

and

\[
I_3 = 2\pi \delta_{l + s, l + m + n}.
\]

respectively. Substituting these results in the expression for \( I \) above, we can define the vertex function as

\[
\chi_{|m|}(P) = \left( h^2 / 2 \right) I(P),
\]

leading to

\[
\chi_{|m|}(P) = 2\pi^2 h^2 \delta_{l + s, l + m + n} \int_0^\infty \mathbf{d}^4 \alpha F_{pqrt}(\alpha) \\
\times J_{m}(\xi \alpha_1 r) J_{l}(\xi \alpha_2 r) J_{l}(\xi \alpha_3 r) J_{l}(\xi \alpha_4 r) \\
\times \left( \alpha_3 + \alpha_4 - \alpha_1 - \alpha_2 - \frac{P - \omega}{\omega} \right),
\]

where \( F_{pqrt}(\alpha) = f_p(\alpha_1)f_q(\alpha_2)f_s(\alpha_3)f_t(\alpha_4) \). For later convenience, it is instructive to calculate the Fourier transform of the above expression. To that, we first define the function

\[
K_{m,n}(r, \alpha, \xi) = J_{m}(\xi \alpha_1 r) J_{l}(\xi \alpha_2 r) J_{l}(\xi \alpha_3 r) J_{l}(\xi \alpha_4 r),
\]

so that we can define

\[
K_{mn}(\xi, \alpha) = \int_0^\infty dr \, r K_{m,n}(r, \alpha, \xi).
\]

Then, the Fourier transform \( \sigma_{|m|}(X) \) of the vertex function \( \chi_{|m|}(P) \) can be written as

\[
\sigma_{|m|}(X) = \frac{1}{2\pi \omega} \int \mathbf{d} \alpha F_{pqrt}(\alpha) e^{i\mathbf{P} \cdot \mathbf{X} / \omega} \\
= \frac{\pi h^2 \delta_{m+n+l} \int_0^\infty \mathbf{d} \alpha K_{mn}(\xi, \alpha) F_{pqrt}(\alpha)}{\omega} \\
\times e^{i(\ell_3 - m - n - s) X}.
\]

The \( \alpha \)-integral in the above expression can be further simplified by noticing that each \( \alpha_k \)-integral can be solved individually and it just amounts to the definition of generalised OAM-carrying X wave given by equation (20), evaluated at \( \theta = 0 \), \( \zeta = X \) and with \( v = 0 \), namely

\[
\int_0^\infty \mathbf{d} \alpha_k f_p(\alpha_k) J_{l}(\xi \alpha_k r) e^{i\mathbf{P} \cdot \mathbf{X} / \omega} = \psi^0_{l,m}(r, X).
\]

Using this fact, we can then rewrite the Fourier transform of the vertex function as

\[
\sigma_{|m|}(X) = \frac{\pi h^2}{\omega} \delta_{m+n+l} \int_0^\infty \mathbf{d} r \psi^0_{m,p}(r, X) \\
\times \psi^0_{n,q}(r, X) \psi^0_{l,s}(r, X) \psi^0_{l,t}(r, X).
\]

The vertex function can be therefore written in terms of its Fourier transform as follows:

\[
\chi_{|m|}(P) = \int \mathbf{d} \eta \sigma_{|m|}(\eta) e^{-i\mathbf{P} \cdot \mathbf{\eta} / \omega}.
\]

Appendix D. Derivation of equation (70)

By applying the low velocity limit to equation (82), we have that \( \sqrt{\omega_{mp}^2 (\nu \omega_{mq} (v_2 \omega_{lq} (v_1 \omega_{pq} (v)) \propto \omega^2 \propto \omega^2} \). Then, after having substituted the creation and annihilation operators \( \hat{a}_{m,p}^{\dagger} \) and \( \hat{a}_{m,p}(v) \) with their Fourier transforms as prescribed by equations (84), the Kerr Hamiltonian has the following form

\[
\hat{H}_1 = \frac{\omega^2}{2} \sum_{|m|} \int d^4 \eta \left( \hat{\phi}_{mp}^{\dagger}(\eta) \hat{\phi}_{mq}^{\dagger}(\eta) \hat{\phi}_{lq}^{\dagger}(\eta) \hat{\phi}_{pq}(\eta) \right) \\
\times \int d^4 v \chi_{|m|}(v) e^{i\mathbf{v} \cdot \mathbf{\eta} / \omega},
\]

where \( \mathbf{v} \cdot \mathbf{\eta} = -v_1 \eta_1 - v_2 \eta_2 + v_3 \eta_3 + v_4 \eta_4 \) and \( \nu = -v_1 - v_2 + v_3 + v_4 \). In this appendix, we will calculate explicitly the integral over \( d^4 \nu \) and show, that the final expression for the Kerr Hamiltonian is given by equation (85). To do that, we use the expression for the vertex function \( \chi_{|m|}(v) \) as given by equation (126) and notice that, by doing so, the above
integral in $d^4v$ can be written as the product of four independent integrals in the four variables $v_k$ ($k \in \{1, 2, 3, 4\}$). Moreover, these integrals are all of the form

$$\int dv \: e^{iv_\eta/\omega'} = 2\pi \delta(\eta).$$

(128)

Using this result in the above equation for each of the $v_k$-integrals leads to the following definition for the quantity $\Sigma_{l|m}(\eta)$:

$$\Sigma_{l|m}(\eta) = (2\hbar \omega'' \pi)^2 \chi \times \int_{0}^{\infty} dr \: r v_{m,p}^{(0)}(r, X) \psi_{n,q}^{(0)}(r, X) \psi_{l,r}^{(0)}(r, X).$$

(129)

Direct substitution of this quantity into the above expression for $\hat{H}_0$ gives exactly equation (85).

**ORCID iDs**

Marco Ornigotti \(\text{https://orcid.org/0000-0002-3814-7871}\)

Claudio Conti \(\text{https://orcid.org/0000-0003-2583-3415}\)

**References**

[1] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York: Wiley)
[2] Hernandez-Figueroa H E, Zamboni-Rached M and Recami E (ed) 2008 *Localized Waves* (New York: Wiley)
[3] Dumin J, Miceli J J and Eberly J H 1987 *Phys. Rev. Lett* 58 1499
[4] Lu J and Greenleaf J F 1992 *IEEE Trans. Ultrason.* *Ferroelectr. Freq. Control* 39 19
[5] Lu J and Greenleaf J F 1992 *IEEE Trans. Ultrason.* *Ferroelectr. Freq. Control* 39 441
[6] Con C, Trillo S, Di Trapani P, Valilius G, Piskarskas A, Jedrkiewicz O and Trull J 2003 *Phys. Rev. Lett.* 90 170406
[7] Valilius G, Kilius J, Jedrkiewicz O, Bramati A, Minardi S, Conti C, Trillo S, Piskarskas A and Di Trapani P 2001 *Quantum Electronics and Laser Science Conf. (Trends in Optics and Photonics* vol 57) (Optical Society of America)
[8] Ciattoni A and Conti C 2007 *J. Opt. Soc. Am. B* 24 2195
[9] Conti C and Trillo S 2004 *Phys. Rev. Lett.* 92 120404
[10] Lahini Y, Fruncker E, Silberberg Y, Drouillas S, Hizanidis K, Morandotti R and Christodoulides D N 2007 *Phys. Rev. Lett.* 98 023901
[11] Heinrich M, Szameit A, Dreisow F, Keil R, Minardi S, Pertsch T, Nolte S, Tünnermann A and Lederer F 2009 *Phys. Rev. Lett.* 103 113903
[12] Lu J Y and He S 1999 *Opt. Commun.* 161 187
[13] Hernandez Figueroa H E, Recami E and Zamboni-Rached M 2013 *Nondiffracting Waves* (New York: Wiley)
[14] Andrews D L and Babiker M 2013 *The Angular Momentum of Light* (Cambridge: Cambridge University Press)
[15] Ornigotti M, Conti C and Szameit A 2015 *Phys. Rev. Lett.* 115 100401
[16] Conti C 2003 arXiv:quant-ph/0309069
[17] Conti C 2004 arXiv:quant-ph/0409130v1
[18] Ornigotti M, Di Mauro Villari L, Szameit A and Conti C 2017 *Phys. Rev. A* 95 011802(R)
[19] Calvo G F, Picon A and Bagan E 2006 *Phys. Rev. A* 73 013805
[20] Karlovets D V 2015 *Phys. Rev. A* 91 013847
[21] Boyd R 2008 *Nonlinear Optics* 3rd edn (New York: Academic)
[22] Butcher P N and Cotter D 1990 *The Elements of Nonlinear Optics Cambridge Studies in Modern Physics* vol 9 (Cambridge: Cambridge University Press)
[23] Oliver F W J, Lozier D W, Boisvert F R and Clark C W 2010 *NIST Handbook of Mathematical Functions* (Cambridge: Cambridge University Press)
[24] Lu J and Liu A 2000 *IEEE Trans. Ultrason.* *Ferroelectr. Freq. Control* 47 1472
[25] Salo J and Saloman M M 2001 *J. Phys. A: Math. Gen.* 34 9319
[26] Loudon R 1997 *The Quantum Theory of Light* 3rd edn (Oxford: Oxford University Press)
[27] Fetter A L and Walecka J D 2012 *Quantum Theory of Many-Particle Systems* (New York: Dover)
[28] Lai Y and Haus H A 1989 *Phys. Rev. A* 40 844
[29] Messiah A 2014 *Quantum Mechanics* (New York: Dover)
[30] Lebedev N N and Silverman R A 2012 *Special Functions and Their Applications* (New York: Dover)
[31] Gradsteyn I S and Ryzhiz M I 2006 *Table of Integrals, Series and Products* (New York: Academic)
[32] Chiao R and Garrison J 2014 *Quantum Optics* (Oxford: Oxford University Press)
[33] Haus H A and Pan J L 1993 *Am. J. Phys.* 61 818–21