Stable analytic bounce in non-local Einstein–Gauss–Bonnet cosmology

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Received 14 March 2013, in final form 8 May 2013
Published 24 June 2013
Online at stacks.iop.org/CQG/30/155001

Abstract
We consider the most general quadratic in curvature stringy-motivated non-local action for the modified Einstein’s gravity. We present exact analytic cosmological solutions including the bouncing ones and develop the relevant techniques for the further study of these types of models. We also elaborate on the perturbation formalism and argue that the found bouncing solution is stable during the bounce phase.

PACS numbers: 98.80.Bp, 11.10.Lm

1. Introduction

Recent observations [1] strongly support the fact that primordial inflation is the theoretical explanation of how the currently observed Universe was formed at the early stages. Along with the observations, theoretical approaches also show how nice inflation can be connected to the nucleosynthesis and subsequent appearance of the particle standard model. A number of inflationary scenarios are reviewed in [2] and references therein, for instance.

Even though inflation is a great model for many reasons, it has still problems: one of which is the lack of the UV completion. To be more precise, it is not UV-complete in the framework of Einstein’s General Relativity (GR), since geodesics are not past-complete. This is a general statement and it is known as the Big Bang singularity elaborated in [3–5]. One can find that alternatives to Big Bang such as ‘emergent’ Universe or bouncing Universe [6] hit the singularity theorem by Hawking and Penrose [7] as long as we are in GR and the spacetime is of the Friedmann–Lemaître–Robertson–Walker (FLRW) type.

One of the possible resolutions is a modification of GR. This can be done in general in a number of ways and one may have an insight into this using the review paper [8] and references therein, for example. It is inevitable that any modification of gravity introduces higher derivatives and only special structures like Gauss–Bonnet term or Lovelock terms in general [8, 9] preserve the second order of the equations of motion, but this is applicable only...
in more than four dimensions. On the other hand, finite higher derivatives lead to ghosts due to the Ostrogradski theorem [10]. Having all orders of higher derivatives may open a way to evade the Ostrogradski statement and a successful attempt in this direction was made considering a special class of gravity modifications where higher curvature corrections are accompanied with non-local operators [11, 12]. Analysis in those papers shows how one can construct a ghost-free and asymptotically free modification of GR featuring a non-singular bouncing solution, which resembles GR in the infrared limit. Note that similar approaches involving non-local models were used in other cosmological and gravity contexts in the literature [13–17]. A further study of this model [18] has shown that the model features the expected perturbative spectrum at late times and is stable with respect to small isotropic inhomogeneous perturbations during the bounce phase, and in parallel a more general model [19] (see also [20]) was proposed and considered in the Minkowski background.

Absolutely, the non-local operators are what makes these models novel and the operators in question are of the type of analytic functions of the covariant d’Alembertian operator $\Box$, i.e. $F(\Box)$. Note that other theoretically motivated operators such as $1/\Box$ were considered, for instance, in [21, 22] and references therein. It is important that the initial inspiration for introducing the analytic $F(\Box)$ operators comes from string field theory (SFT) models because SFT as the whole theory is a UV-complete non-perturbative description of strings. We refer the reader to more stringy oriented literature [23–28] for the more comprehensive overview of this aspect. A decent progress was achieved in studying non-local scalar field models derived from SFT in the cosmological context [29–32] as well as the exploration of other aspects of these types of models including their thermodynamics [33–35]. The major question of the rigorous derivation of the modified GR action involving the non-local operators of interest from the scratch, i.e. from the closed SFT, is still awaiting for its resolution and this is beyond the scope of our present study.

This paper is aimed at extending and continuing both papers [18, 19]. In [18], a lot of technical issues were solved for a model which contains the scalar curvature squared non-local term. The main focus there is perturbations around a bouncing solution. In [19], non-local terms containing the Ricci and Riemann tensors squared were added but only analyzed around the Minkowski background. We want to join the efforts of those works in this paper. We confine ourselves to this by considering the FLRW type of the metric and positive cosmological term $\Lambda$. Having these in mind, we focus on deriving the full equations of motion and bringing them to a form one can use in the future study. Here our main goal is to present the classical solutions and develop the perturbation technique at least in some regimes or around some backgrounds. This is crucial for claiming whether the model is viable or not for the purpose of describing the non-singular bounce. Bouncing or cyclic models of the Universe [36] moreover try to establish a connection between the perturbative spectrum during or prior the bounce phase and the spectrum observed in CMB. In order to do the same in stringy motivated non-local models, an appropriate technique must be developed. This provides us with the additional strong motivation for this study.

The paper is organized as follows. In section 2, we formulate the model and introduce the relevant notations. Also, we write down the full equations of motion relevant for the FLRW-type metric. In section 3, we demonstrate explicitly that one of the known bouncing solutions, namely the cosine hyperbolic, is the solution to our extended equations of motion as well. In section 4, we make an attempt to construct more solutions and develop relevant techniques. In section 5, we build the road to analyze perturbation. We draw the stability statement about our exact bouncing solution from general arguments as well as sketch the derivation of the closed system of perturbation equations for the de Sitter asymptotic. In section 6, we summarize what is done and formulate open questions for the future study.
2. Action and equations of motion

We focus on the model described by the following non-local action:

\[ S = \int d^4x \sqrt{-g} \left( \frac{M_p^2}{2} R + \frac{\lambda}{2} R_2 - \Lambda \right), \tag{1} \]

where

\[ R_2 = R F_1(\Box) R + R_{\mu} F_2(\Box) R^\mu + C_{\mu\nu\alpha\beta} F_4(\Box) C^{\mu\nu\alpha\beta} \]

and hence we limit ourselves with \( O(R^2) \) corrections. Here, the dimensionality is manifest and in the following all the formulae are written having four dimensions in mind. \( M_p \)
the Planckian mass, \( \Lambda \) is a cosmological constant and \( \lambda \) is a dimensionless parameter measuring the effect of the \( O(R^2) \) corrections. The most novel and crucial ingredients for our analysis are the functions of the covariant \( d'\)Alembertian operator \( F_I \). For simplicity to avoid extra complications, we assume that these functions are analytic with real coefficients \( f_I \) in the Taylor series expansion \( F_I = \sum_{n \geq 0} f_I n! \Box^n / M_p^{2n} \). The new mass scale determines the characteristic scale of the gravity modification. We assume it universal for all \( F_I \) and refer the reader to [11] for a detailed discussion of this new physics parameter. Also, apart from the canonical usage of the Riemann tensor, we use the Weyl tensor because the characteristic scale of the gravity modification. We assume it universal for all \( F_I \) and only becomes relevant in perturbations. Moreover, even in perturbations the only non-vanishing contribution is the one where both Weyl tensors are perturbed and the non-local functions \( F_4 \) take its background form.

This action appears in [19] (equation (27)) and is the most general covariant non-local action up to the square in curvature terms and analytic operator functions \( F_I \). We have however less terms because possible contractions of the covariant derivatives with the Ricci tensor can be eliminated thanks to the Bianchi identity or converted to the higher order in curvature terms using the commutation relations for the covariant derivatives. Also, the questions of a ghost-free gravity modification are addressed in [37] using different action still using non-local terms using the commutation relations for the covariant derivatives. Furthermore, we note that working in four dimensions we can assume \( f_{20} = 0 \) because using that the Gauss–Bonnet scalar is a total derivative and combining this with the Ricci decomposition, we can have only \( R^2 \) when no \( d'\)Alembertian operators are in between. In other words, among the terms without \( d'\)Alembertian operator insertions, only \( R^2 \) survives on the FLRW backgrounds. This does not work for non-constant terms in \( F_I \) though.

In this formula, we use slightly unusual position of indices which is useful in performing further computations. We use the Weyl tensor because \( C_{\alpha\nu\beta} = 0 \) on a conformally flat manifold which is the case for the FLRW metric. We are focused on the FLRW cosmologies and thus benefit out of this. Indeed, it means that the Weyl tensor squared does not show up in the perturbations. Moreover, even in perturbations the only non-vanishing contribution is the one where both Weyl tensors are perturbed and the non-local functions \( F_4 \) take its background form.

\[\begin{align*}
C_{\alpha\nu\beta} &= R_{\alpha\nu\beta} - \frac{1}{2} \left( R_{\mu\nu} \delta^\mu_\alpha \delta^\beta_\beta - \delta_{\alpha\beta} R_{\mu\nu} \delta^\mu_\mu - R_{\alpha\nu} \delta^\mu_\beta \delta^\beta_\mu + R_{\alpha\beta} \delta^\mu_\mu \delta^\nu_\nu \right) + \frac{R}{6} \left( \delta_{\alpha\beta} \delta_{\mu\nu} - \delta_{\alpha\mu} \delta_{\beta\nu} \right).
\end{align*}\]

Equations of motion for action (1) can be derived by a straightforward variation and they are as follows:

\[\begin{align*}
\left[M_p^2 + 2\lambda F_1(\Box) R\right] G^\mu_\nu &= T^\mu_\nu - \Lambda \delta^\mu_\nu - \frac{\lambda}{2} R F_1(\Box) R^\mu_\nu + 2\lambda \left( \nabla_\mu \partial_\nu - \delta^\mu_\nu \Box \right) F_1(\Box) R \\
& - 2\lambda R_{\mu} F_2(\Box) R^\mu_\nu + \frac{\lambda}{2} \delta^\mu_\nu R^\rho_\rho F_2(\Box) R^\mu_\nu + 2\lambda \left( \nabla_\mu \nabla_\nu F_2(\Box) R^\rho_\rho - \frac{1}{2} \Box F_2(\Box) R^\mu_\nu \right).
\end{align*}\]
and
\[4\] of terms simplifying the subsequent analysis considerably. Known solutions (apart from recursively. If, however, the above ansatz is satisfied, we effectively cut all the infinite series somehow know the metric, we can compute the curvature scalar in our notations.

\[\Box R = r_1 R + r_2\] 

where we have defined

\[K_{1,\mu} = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} \partial^\mu R^{(l)}_{\mu \rho} R^{(n-l-1)}_{\rho}, \quad \hat{\cal K}_1 = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} R^{(l)} R^{(n-l)},\]

\[K_{2,\mu} = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla^\mu R^{(l)}_{\mu \beta} R^{(n-l-1)}_{\beta}, \quad \hat{\cal K}_2 = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} R^{(l)} R^{(n-l)} R^{(n-l-1)}_{\nu} \nabla \nu R^{(l)} R^{(n-l-1)}_{\nu} \]

\[\Delta_{\mu} = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla^\mu \left[R^{(l)} R^{(n-l-1)}_{\nu} \nabla \nu R^{(l)} R^{(n-l)}_{\nu} - \nabla \nu R^{(l)} R^{(n-l-1)}_{\nu} \nabla \nu R^{(l)} R^{(n-l)}_{\nu}\right]\]

and \(G_{\mu}^\nu\) is the part coming from the variation of the Weyl tensor squared piece. As we are mostly concerned about FLRW cosmologies, we simply neglect this contribution in the background as it is zero. We shall return to it later during the consideration of perturbations. Here, \(G_{\mu}^\nu\) is the Einstein tensor, \(R^{(n)} = \Box^\mu R, R^{(n)}_{\mu} = \Box^\mu R_{\mu}^n\) and \(T_{\mu}^\nu\) is the matter stress tensor if one is present in the system.

One would benefit considering first the trace equation

\[-M^2 R = T - 4\Lambda + 6 \lambda \Box F_1(\Box) R - \lambda (K_1 + 2\hat{\cal K}_1) - \lambda \Box F_2(\Box) R - 2\lambda \nabla^\mu \nabla_{\mu} F_2(\Box) R^{(n)} + \lambda (K_2 + 2\hat{\cal K}_2) + 2\lambda \Delta + 2\lambda \hat{\cal C},\] 

where quantities without indices denote the trace and significant simplifications are obvious.

3. Exact solutions based on a simplifying ansatz

3.1. The ansatz

In what follows in this section we use quite intensively ideas and the knowledge accumulated in [18] (see also [12]) where this model without \(F_2\) and \(\bar{\cal F}_2\) pieces was considered and solutions (and their construction) were discussed. We therefore omit extensive citing of these papers and do refer the reader to those manuscripts for a detailed analysis of a model which is \(F_2 = \bar{\cal F}_2 = 0\) in our notations.

Clearly, those equations have the Minkowski (if \(\Lambda = 0\)) and de Sitter (if \(\Lambda = const > 0\)) backgrounds as the solutions. The question is however: whether we can find other solutions especially focusing on the possible bouncing backgrounds? Equations are definitely highly non-trivial, but a decent progress was achieved in [11] by employing the ansatz

\[\Box R = r_1 R + r_2\]

in the absence of \(F_2\) and \(\bar{\cal F}_2\). This makes the system much simpler. Indeed, provided we somehow know the metric, we can compute the curvature scalar \(R\) and all the tower \(R_{\nu}^{(n)}\) recursively. If, however, the above ansatz is satisfied, we effectively cut all the infinite series of terms simplifying the subsequent analysis considerably. Known solutions (apart from \(M_1\) and \(dS_1\)) to the above ansatz include

\[a(t) = a_0 \cosh(\sigma t)\] 

found in [11] 

and

\[a(t) = a_0 \exp\left(\frac{\sigma}{2} t^2\right)\]

found in [38],
where $a(t)$ is the scale factor of the FLRW metric $g_{\mu\nu} = \text{diag}(-1, a(t)^2, a(t)^2, a(t)^2)$.

For the Ricci tensor term, we suggest the following relation as a quite general ansatz:

$$\Box R^{(n)}_{\mu\nu} = \sum_{m=0}^{n} \left( r_{1m} R^{(m)}_{\mu\nu} + r_{2m} g_{\mu\nu} R^{(m)} \right) + r g_{\mu\nu}, \quad (7)$$

where $r_{1,2,m}$ and $r$ are constants. Surely, we require $n$ to be finite. This would manifest a linear relation between different degrees of boxes and would give us a chance to have this as an exact solution to equations (2). Needless to say that smaller $n$ makes the succeeding analysis simpler.

The explanation why finding solution to such a relation is almost enough is based on the following arguments. One should start with the trace equation (3). If there is no matter or only radiation (which is conformal and traceless) is present, then the trace equation only contains the gravitational sector of the model. It can be solved simply by imposing null conditions on coefficients in front of the different linearly independent functions appeared upon acting by the $\Box$ operator on the curvature tensors. These are some conditions on the model parameters, but clearly the restrictions are very weak as far as $n$ is finite. Further, we must check one of the component equations (we shall do for $(00)$ component) in (2) whether it is satisfied or not. But thanks to the Bianchi identity constraint, any discrepancy we can meet would be of the form of the radiation energy density, i.e. $\sim a(t)^{-4}$. Hence, the only question would be: how much radiation we must inject in the system and would it have good or a ghost sign of the energy density?

3.2. Exact analytic bounce

Of course, the above proposed ansatz is of extreme generality, and in order to be more specific, we examine its following simplified version:

$$\Box^n \bar{G}_\mu^\nu = \alpha_n \Box \bar{G}_\mu^\nu + \beta_n \bar{G}_\mu^\nu, \quad n \geq 2, \text{ where } \bar{G}_\mu^\nu = R_\mu^\nu - \frac{1}{4} s_0^\rho \bar{R}. \quad (8)$$

These relations assume that all the powers of the $\Box$ operators greater than 1 are in fact linear combinations of the tensors $\bar{G}_\mu^\nu$ and $\Box \bar{G}_\mu^\nu$. Then, we can solve

$$\alpha_n = c_+ s_+^n + c_- s_-^n, \quad s_{\pm} = \frac{a_2 \pm \sqrt{a_2^2 + 4 \beta_2}}{2}, \quad \beta_n = \alpha_{n-1} \beta_2 \quad (9)$$

and $c_{\pm}$ are to be found examining explicit values of $a_3$ and $\beta_3$. However, there is a limiting value $n = 4$. $n \leq 4$ gives a hypothetical chance to solve the latter relation, while greater $n$ will result, in general, in the algebraic equations of a degree higher than 4.

This choice of the simplified ansatz is motivated by the fact that for the scale factor (5), we have after some simplifications

$$\Box^2 \bar{G}_\mu^\nu = 14 a^2 \Box \bar{G}_\mu^\nu - 40 a^4 \bar{G}_\mu^\nu. \quad (10)$$

It means as explained in the previous subsection that the scale factor (5) must be a solution to the equations of motion in our model. The only question is: what are the conditions on the model parameters and can one guarantee that the possible additional radiation is not a ghost?

Let us show explicitly that our claim is correct. Below we study the particular solution (5) meaning that we have specific $s_+ \pm$ and $c_{\pm}$. One can, surely, track all the steps considering (8) in general without specifying particular values for $a_2$ and $\beta_2$. This is interesting in the case where more solutions satisfying this ansatz can be found. In general, one would come to a bit more complicated expressions than we will. We do not do this at the moment for the sake of...
clarity. Moreover, (8) is already the simplified version of (7) which is worth analyzing if a real general scenario is of interest. We thus focus mostly on analyzing the obtained solution (5).

In solution (5), we find

$$\alpha_n = \frac{1}{6\sigma^2} (s_1^n - s_2^n), \quad \beta_n = -\frac{40\sigma^2}{6} (s_1^{n-1} - s_2^{n-1}), \quad s_1 = 10\sigma^2, \quad s_2 = 4\sigma^2.$$ 

Note that these relations can be used for $n = 1$ and $n = 0$ as they generate correct values $\alpha_1 = 1$, $\beta_1 = 0$ and $\alpha_0 = 0$, $\beta_0 = 1$, respectively, which are reasonable. Even though $f_{20} = 0$ as explained in the previous section, we shall use this property formally to simplify the evaluation of the sums in the equations of motion. Apart from this, the first ansatz (4) is also satisfied with $r_1 = 2\sigma^2$ and $r_2 = -24\sigma^2$. We thus have

$$\Box R = r_1^\mu (R + r_2/r_1), \quad \Box \tilde{G}^\mu_{\nu} = \frac{s_1^n}{6\sigma^2} (\Box \tilde{G}^\mu_{\nu} - s_2 \tilde{G}^\mu_{\nu}) - \frac{s_1^n}{6\sigma^2} (\Box \tilde{G}^\mu_{\nu} - s_1 \tilde{G}^\mu_{\nu}).$$

These relations yield

$$K_{1,\mu} = F_1^{(1)} (r_1) \tilde{\tilde{R}}^\mu_{\rho} R_{\rho},$$

$$K_{2,\mu} = F_2^{(1)} (s_1) \nabla^\mu S_{1,\rho} S_{1,\mu} + F_2^{(1)} (s_2) \nabla^\mu S_{2,\rho} S_{2,\mu} + \nabla^\mu S_{2,\rho} S_{1,\mu} + \frac{1}{4} F_2^{(1)} (r_1) \tilde{\tilde{R}}^\mu_{\rho} R_{\rho},$$

$$\Delta_\nu = \nabla_\rho \left[ F_2^{(1)} (s_1) (S_{1,\nu}^\mu - S_{1,\rho} S_{1,\nu}^\rho) + F_2^{(1)} (s_2) (S_{2,\nu}^\mu - S_{2,\rho} S_{2,\nu}^\rho - \nabla^\mu S_{2,\rho} S_{2,\nu}^\rho) \right] - \frac{F_2^{(1)} (s_1) - F_2^{(1)} (s_2)}{s_1 - s_2} \left( S_{1,\nu}^\mu - S_{1,\rho} S_{1,\nu}^\rho \right) - \frac{F_2^{(1)} (r_1) - F_2^{(1)} (r_2)}{r_1 - r_2} \left( F_2^{(1)} (s_1) - F_2^{(1)} (s_2) \right) \left( s_1 + s_2 \right) S_{2,\rho} S_{2,\mu}^\rho,$$

$$\mathcal{F}_1 (\Box) = F_1 (r_1) R + \frac{F_1 (r_1) - f_{10}}{r_1} r_1^\mu, \quad \mathcal{F}_2 (\Box) = F_2 (r_1) R + 
abla_\rho \left\{ F_2 (s_1) S_{1,\nu}^\mu - F_2 (s_2) S_{2,\nu}^\mu + \frac{1}{4} \delta^\mu_\nu F_2 (\Box) R \right\},$$

where we have defined

$$S_{1,\nu}^\mu = \frac{\Box \tilde{G}^\mu_{\nu} - s_2 \tilde{G}^\mu_{\nu}}{6\sigma^2}, \quad S_{2,\nu}^\mu = \frac{\Box \tilde{G}^\mu_{\nu} - s_1 \tilde{G}^\mu_{\nu}}{6\sigma^2}.$$
simplifications, we have for the trace of the Einstein equations

\[ -M_p^2 R = T - 4\Lambda - 6\lambda (F_1(r_1) + \frac{1}{4} F_2(r_1))(r_1 R + r_2) + \frac{\lambda}{2} s_2 \mathcal{F}_2(s_1) - s_1 \mathcal{F}_2(s_2) (r_1 R + r_2) \]

\[ + 2\lambda (F_1(r_1) + \frac{1}{4} F_2(r_1) - f_{10}) \frac{r_2}{r_1} (R + r_2/r_1) + \lambda \mathcal{F}_2^{(1)}(s_1) E_1 + \lambda \mathcal{F}_2^{(1)}(s_2) E_2 \]

\[ + \lambda \mathcal{F}_2(s_1) - \mathcal{F}_2(s_2) \frac{E_3}{s_1 - s_2} \] (11)

where

\[ E_1 = -\nabla^\mu S_1^\mu \nabla_\mu S_1^\nu - 2s_1 S_1^\mu S_1^\nu + 2\nabla_\mu (S_1^\mu \nabla_\mu S_1^\nu - \nabla_\mu S_1^\mu S_1^\nu) \]

\[ E_2 = -\nabla^\mu S_2^\mu \nabla_\mu S_2^\nu - 2s_2 S_2^\mu S_2^\nu + 2\nabla_\mu (S_2^\mu \nabla_\mu S_2^\nu - \nabla_\mu S_2^\mu S_2^\nu) \]

\[ E_3 = -2\nabla_\mu \nabla_\nu \Box \tilde{G}^{\mu \nu} + 2\nabla_\mu S_1^\mu \nabla_\nu S_1^\nu + 2(s_1 + s_2) S_1^\mu S_1^\nu \]

\[ -2\nabla_\mu (S_1^\mu \nabla_\nu S_1^\nu - \nabla_\nu S_1^\mu S_1^\nu) + S_2^\mu \nabla_\nu S_2^\nu - \nabla_\nu S_2^\mu S_2^\nu \]

\[ E_4 = -\partial^\mu \partial^\nu R - 2r_1 R^2 - 4r \sigma R - 2r^2 R_1 . \]

By imposing the following conditions

\[ \mathcal{F}_2^{(1)}(s_1) = \mathcal{F}_2^{(1)}(s_2) = \mathcal{F}_2(s_1) - \mathcal{F}_2(s_2) = 0, \]

we cancel structures containing \( \Box \tilde{G}^{\mu \nu} \) and \( (\Box \tilde{G}^{\mu \nu})^2 \), i.e. \( E_{1,2,3} \) in the last equation and by imposing one more condition

\[ \mathcal{F}_1^{(1)}(r_1) + \frac{1}{4} \mathcal{F}_2^{(1)}(r_1) = 0, \]

we cancel structures containing \( \partial_\mu R^2 \) and \( R^2 \), i.e. \( E_4 \) as well. All the expressions are simplified considerably. In particular, we have

\[ \mathcal{F}_2(\Box) R^\mu = \mathcal{F}_2(s_1) \left( G^\mu + \frac{1}{4} \delta^\mu R \right) + \frac{1}{4} \delta^\mu \mathcal{F}_2(r_1)(R + r_2/r_1) \]

or equivalently

\[ \mathcal{F}_2(\Box) \tilde{G}^\mu = \mathcal{F}_2(s_1) \tilde{G}^\mu . \]

Regarding equation (11) only two upper lines survive there. After further minor simplifications, we have for the trace of the Einstein equations

\[ -M_p^2 R = T - 4\Lambda - 6\lambda (F_1(r_1) + \frac{1}{4} F_2(r_1))(r_1 R + r_2) \]

\[ + 2\lambda (F_1(r_1) + \frac{1}{4} F_2(r_1) - f_{10}) \frac{r_2}{r_1} (R + r_2/r_1) \]

\[ + \lambda \mathcal{F}_2(r_1) + \frac{1}{4} \mathcal{F}_2(r_1) f_{10} \frac{r_2}{r_1} (R + r_2/r_1) . \]

and we solve it (assuming \( T = 0 \) by imposing

\[ \frac{M_p^2}{r_1} - \frac{\lambda}{2} \mathcal{F}_2(s_1) - 6\lambda (F_1(r_1) + \frac{1}{4} F_2(r_1)) + 2\lambda (F_1(r_1) + \frac{1}{4} F_2(r_1) - f_{10}) \frac{r_2}{r_1} = 0 \]

\[ - \frac{M_p^2 r_2}{4} r_1 = \Lambda . \]

These relations in terms of \( \sigma \) are as follows:

\[ \frac{M_p^2}{2\sigma^2} = \frac{\lambda}{2} \mathcal{F}_2(4\sigma^2) - 18\lambda F_1(2\sigma^2) + \frac{9}{2} \lambda \mathcal{F}_2(2\sigma^2) + 12\lambda f_{10} = 0, \quad 3M_p^2\sigma^2 = \Lambda . \]

Even though some tuning is required for functions \( F_1,2 \), the solution to the above conditions is really ambiguous.

Substituting the obtained results into the \((00)\) component of (2) and performing a number of manipulations, we find that only terms proportional to a constant \( 1/cosh(\sigma t)^2 \)
and $1/\cosh(\sigma t)^4$ are present and, moreover, upon applying the above conditions (16), one is left with the following combination:

$$\rho_r = -\frac{54\lambda \sigma^4}{\cosh(\sigma t)^4} \left( F_1(r_1) + \frac{1}{4} F_2(r_1) + \frac{1}{12} F_2(s_2) \right),$$

(17)

where we clearly see that we are really left with terms resembling the radiation (i.e. $\sim a(t)^{-4}$). What is remarkable here is that there are two possible resolutions to have the equations solved completely.

The first way is to avoid extra radiation, requiring

$$F_1(r_1) + \frac{1}{4} F_2(r_1) + \frac{1}{12} F_2(s_2) = 0.$$  (18)

This means that extra parameter adjustment is important which however does not seem to be unnatural. Equations do not trivialize in this case and we can analyze such a configuration consistently. Note that it is not the case if $F_2 = 0$ (see [18] for details).

The second way is to admit some amount of radiation with a weaker requirement

$$F_1(r_1) + \frac{1}{4} F_2(r_1) + \frac{1}{12} F_2(s_2) < 0,$$  (19)

so that this radiation has the positive energy and is not a ghost.

We therefore have proven that the scale factor (5) is a solution in our model provided conditions (12) and (13) are met. This solution features a number of very nice properties: (i) it has non-singular bounce, (ii) it comes at large time to the de Sitter phase rather than eternal superinflation, (iii) it requires just few fine tunings in the system which still leave us with an enormous freedom and (iv) it does not have to be supported by extra sources if we impose an extra condition (18) (to avoid this extra tuning we would assume (19) to avoid ghosts in the system).

One comment is in order here. As was already mentioned, the scale factor (5) is a solution also when $F_2 = F_4 = 0$. It is however important to say that in that case the presence of radiation is inevitable. The present extended model, however, gives a way to avoid an extra radiation to be included. It does not look that surprising since we have added extra structures releasing some constraints, but on the other hand represents an important step forward since we can now construct the bouncing scenario using a purely gravitational sector. Technically, this means that the $F_2$ term behaves as an effective radiation in the background on this solution.

4. More analytic cosmological solutions?

In this section, we provide several solutions which unfortunately in most cases do not contribute to the question of the non-singular bounce but are nevertheless interesting for the future developments in these types of theories.

4.1. Closely related solutions

Apart from solution (5) analyzed in the previous section, one can find after simple analysis that the following are also analytic solutions:

$$a(t) = a_0 \sinh(\sigma t),$$  (20)

$$a(t) = a_0 \cos(\sigma t).$$  (21)
They can be quite simply obtained using the already known one and unfortunately none of them contributes to our main subject of a non-singular bounce. Indeed,

- solution (20) is clearly not a bouncing solution and we mention it here just for the sake of completeness,
- solution (21) is just our already explored solution (5) with an imaginary parameter $\sigma$. It has periodic zeros for the scale factor. It can be used as a test-bed for discussing the possible stages of the Universe evolution if they obey $\cos(\sigma t)$ behavior.

Obviously, there should be other solutions in the system. At first, let us mention that (6) does not obey some simplifying ansatz for the Ricci tensor terms. One could also think that just finding a solution other than presented in the previous section to the ansatz conditions is not an easy task as it comes to solving at least the third-order nonlinear differential equation. As of now, it is an open question how to construct more solutions in this model using some ansatz (see [39] for various proposal regarding solving the ansatz condition).

So, on one hand, we absolutely understand that taming all the non-localities by virtue of an ansatz is the good technical point of the all the above considered solutions. This also must help in studying perturbations around those particular backgrounds. On the other hand, however, this significantly reduces the possible solutions. We thus want to go beyond ansatz relations and to find a more general approach to the problem of solving equations of motion.

4.2. Model reformulation using $\tilde{G}^\mu_\nu$

The first useful technical step is a passage from the Ricci tensor to the traceless analogue of the Einstein tensor $\tilde{G}^\mu_\nu$. We mention in this regard that the appearance of a combination $F_1 + \frac{1}{4} F_2$ is not spontaneous because we can rewrite the initial action (1) in terms of $\tilde{G}^\mu_\nu$ as follows:

$$
S = \int d^4x \sqrt{-g} \left( \frac{M_\text{Pl}^2}{2} R + \frac{\lambda}{2} \tilde{R}_2 - \Lambda \right),
$$

(22)

where

$$
\tilde{R}_2 = R \tilde{F}_1(\Box) R + \tilde{G}^\mu_\nu F_2(\Box) \tilde{G}^\mu_\nu + C_{\mu_1 \nu_1} F_4(\Box) C^{\mu_1 \nu_1},
$$

and we have used that $\tilde{G}^\mu_\nu$ is traceless and have defined $F_1(\Box) + \frac{1}{4} F_2(\Box) = \tilde{F}_1(\Box)$. Equations of motion for action (22) can be derived by substituting $R^\mu_\nu = \tilde{G}^\mu_\nu + \frac{1}{2} \delta^\mu_\nu R$ into (2):

$$
M_\text{Pl}^2 \tilde{G}^\mu_\nu = T^\mu_\nu - \Lambda \delta^\mu_\nu - 2 \lambda \tilde{G}^\mu_\nu \tilde{F}_1(\Box) R + 2 \lambda (\nabla^\mu \partial_\nu - \delta^\mu_\nu \Box) \tilde{F}_1(\Box) R - \frac{1}{2} \lambda R \tilde{F}_2(\Box) \tilde{G}^\mu_\nu
$$

$$
- 2 \lambda \tilde{G}^\mu_\nu F_2(\Box) \tilde{G}^\nu_\rho + \frac{\lambda}{2} \delta^\mu_\nu \tilde{G}^\rho_\sigma F_2(\Box) \tilde{G}^\mu_\rho + 2 \lambda \left( \nabla_\rho \nabla_\nu F_2(\Box) \tilde{G}^\mu_\rho \right)
$$

$$
- \frac{1}{2} \Box F_2(\Box) \tilde{G}^\mu_\nu - \frac{1}{2} \delta^\mu_\nu \nabla_\sigma \nabla_\rho F_2(\Box) \tilde{G}^\sigma_\rho + \lambda \tilde{L}_1^\mu_\nu - \frac{\lambda}{2} \delta^\mu_\nu (\tilde{L}_1^\sigma_\sigma + \tilde{L}_1)
$$

$$
+ \lambda \tilde{L}_2^\mu_\nu - \frac{\lambda}{2} \delta^\mu_\nu (\tilde{L}_2^\sigma_\sigma + \tilde{L}_2) + 2 \lambda \tilde{\Delta}^\mu_\nu + 2 \lambda \tilde{\Delta}_\mu_\nu,
$$

(23)

where we have defined
\[ \mathcal{L}_1^\mu = \sum_{n=0}^{\infty} \tilde{f}_n \sum_{l=0}^{n-1} \delta^\mu_{(l)} \partial_{\alpha} R^{(n-l-1)}, \quad \tilde{L}_1 = \sum_{n=1}^{\infty} \tilde{f}_n \sum_{l=0}^{n-1} R^{(l)} R^{(n-l)}. \]

\[ \mathcal{L}_2^\mu = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla^\mu \tilde{G}^{\mu\beta\gamma\delta}_{\beta\gamma} \tilde{G}^{(n-l-1)\beta\gamma}_{\alpha}, \quad \tilde{L}_2 = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \tilde{G}^{(l)}_{\alpha\beta} \tilde{G}^{(n-l)}_{\alpha\beta}. \]

\[ \tilde{\Lambda}^\mu = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla^\mu \tilde{G}^{(l)}_{\gamma\beta} \nabla^\mu \tilde{G}^{(n-l-1)}_{\gamma\beta} - \nabla^\mu \tilde{G}^{(l)}_{\gamma\beta} \tilde{G}^{(n-l-1)}_{\gamma\beta} \]

and \( \tilde{f}_{1n} \) are coefficients of the Taylor expansion of function \( \tilde{F}_1 \). The Weyl-tensor-related part may have an impact now since we are going to consider perturbations. One can find the relevant part of it is

\[ C_\nu^\mu = (R_{\nu\beta} + 2 \nabla_{\nu} \nabla_{\beta}) \mathcal{F}_3 (\square) C_{\nu}^{\mu}. \]

Saying relevant we mean only the piece which is obtained by the variation of one of the Weyl tensor factors in the action. Then, we are left with only one Weyl tensor as it is obvious from the latter formula and further perturbation of this remaining a Weyl tensor factor may produce a non-zero contribution to the perturbation equations.

The trace equation becomes

\[ -M_2^2 R = T - 4 \Lambda - 6 \lambda \Box \tilde{F}_1 (\square) R - \lambda (L_1 + 2 \tilde{L}_1) - 2 \lambda \nabla_{\nu} \nabla_{\mu} \mathcal{F}_2 (\square) \tilde{G}^{\mu\rho} - \lambda (L_2 + 2 \tilde{L}_2) + 2 \lambda \tilde{\Lambda} \]

(24)

and the Weyl-tensor-related term \( C \) turns out to be zero thanks to the full tracelessness of the Weyl tensor.

This form of action and the equations of motion also turn out to be beneficial for studying perturbations as we will see in the following section.

4.3. Avoiding ansatz relations

Surely, we are still sticking to the FLRW type of metric, but have kept up to some extent the scale factor general. This means that we cannot say anything specific about functions \( R^{(n)} \) which appear after the box operator acts on the scalar curvature \( n \) times. Fortunately, we now can have a progress with the second-rank tensor \( \tilde{G}^\mu_{\mu} \). It is traceless and the action of the box operator does not break this property because the box commutes with the metric. It means that \( \tilde{G}^{(n)}_{\mu\nu} = 0 \) for any \( n \) and thus we can introduce

\[ \Box^\mu \tilde{G}^\nu_{\mu} = b_\alpha(t) \xi^\mu, \quad \text{where} \quad \xi^\mu = \text{diag}(3, -1, -1, -1), \]

(25)

where one can compute that

\[ b_{n+1}(t) = (\Box + 8 \dot{H}) b_n(t) \quad \text{and} \quad b_0(t) = \dot{H}/2. \]

(26)

Here, as usual \( H = \dot{a}/a \) is the Hubble function and the dot is the derivative with respect to the cosmic time \( t \).

This allows us to simplify all the tensor structures in the trace equation (24). As was explained in the previous section, this is the cornerstone equation since a solution to it is almost automatically a solution to all the Einstein equations modulo perhaps some radiation (assuming no other matter is present in the system). Careful substitution gives
If neither $R_n$ nor $b_n$ appear to obey some ansatz, then all the coefficients $f_{1,2,n}$ are in the play. Again they can be either trivialized or fine tuned. Here, however, another type of fine tuning may arise when instead of fixing all the coefficients we do fix only some relations in between $f_1$s and $f_2$s. This is still an infinite fine tuning from the point of view of individual coefficients, but from the point of view of the operator functions $\mathcal{F}$,
this may be just one simple relation. To be more precise, provided all terms cancel if $f_{1n}/f_{2n} = \text{const}$ which does not depend on $n$, then it is equivalent to have $\tilde{F}_1 \sim F_2$.

4.4. Other solutions

Solutions of the first type. The solution belonging to the first type when both the scalar curvature and the Ricci tensor obey an ansatz, apart from (5), is

$$a(t) = a_0 t.$$  \hspace{1cm} (28)

One can find that in this solution

$$R = 6/\ell^2, \quad \Box R = 0, \quad b_0 = -\frac{1}{2\ell^2}, \quad b_1 = -\frac{4}{\ell^2}, \quad b_2 = 0,$$

and it is easy to show that the equations of motion can be satisfied. It is not a bouncing solution, but may be interesting in analyzing the regime when the scale factor grows linearly.

Configurations of the second type. Known configuration of the second type is

$$a(t) = a_0 \sqrt{t}.$$  \hspace{1cm} (29)

For this scale factor, we have

$$R = 0, \quad b_0 = -\frac{1}{4\ell^2}, \quad b_1 = \frac{1}{4\ell^2}, \quad b_2 = -\frac{3}{\ell^2}, \ldots,$$

and the sequence of $b$s does not stop. Also, as mentioned above, the scale factor (6) has an infinite series of $b_n$ functions.

Note that the above configurations of the second type are the solutions with $F_2 = 0$.

Configurations of the third type. The third type of configurations is extremely generic and the only known scale factor which makes us able to track the equation up to some reasonable extent is

$$a = a_0 t^p$$  \hspace{1cm} (30)

for a generic $p$. For this scale factor, we have

$$R^{(l)} = \frac{r_l}{\ell^{2l+2}}, \quad b_l = \frac{\beta_l}{\ell^{2l+2}},$$

where constant coefficients may be expressed through $p$ explicitly. Note that for odd $p$, the series for $R^l$ stops at a certain $l$ and the solutions fall back in the previous subclass when one simplifying ansatz exists. Now, it is a matter of a straightforward computation to see that one can satisfy the Einstein equations by imposing algebraic relations on coefficients $f_{1n}$ and $f_{2n}$. This is because internal summations over $l$ in expressions for $L$s do not involve the time variable $t$.

Note also that all the above solutions are still solutions with an arbitrary $F_4$. Even though it is not important for the background, one would change the perturbative picture playing with the $F_4$ parameter.

5. A road-map to perturbations

Needless to say that even writing down the perturbation equations may be an unfeasible task in the above formulation of the model. Below in this section, we outline the general approach and emphasize what shall be developed shortly in subsequent papers since the full detailed analysis is expected to be rather cumbersome and deserves a separate paper.
5.1. Bianchi identity

We know that the Einstein equations are constrained with the Bianchi identity which says $\nabla_\mu G^\mu_\nu \equiv 0$. In our case, we have more than just Einstein–Hilbert Lagrangian, but all the additional ingredients we have are covariant terms. This guarantees that the Bianchi identity holds trivially without imposing any extra condition. On the other hand, this implies thanks to the arbitrariness of coefficients $f_{I n}$ that each separate term does covariantly conserve. Indeed, $f_{I n}$ is a coefficient in front of some covariant structure in the Einstein equations, say $\tau^{\mu}_\nu$. Assuming that only one of $f$ coefficients is non-zero, we come to a conclusion that the corresponding structure must covariantly conserve due to Bianchi identities, i.e. $\nabla_\mu \tau^{\mu}_\nu \equiv 0$. In other words, it resembles a conserving perfect fluid stress–energy tensor. The same argument is applicable to all the $f$ coefficients as well as their arbitrary combinations.

The above arguments imply that, thanks to the Bianchi identity, the parts with different $F_I$ covariantly conserve separately. To make use of this, we define

$$T_0^\mu_\nu = T^{\mu}_\nu,$$

$$T_1^\mu_\nu = -2\lambda \tilde{G}^\mu_\nu \tilde{F}_1 (\Box) R + 2\lambda (\nabla^\mu \partial_\nu - \delta^\mu_\nu \Box) \tilde{F}_1 (\Box) R + \lambda \mathcal{L}_\mu^\nu - \frac{\lambda}{2} \delta^\mu_\nu (\mathcal{L}_\sigma^\sigma + \tilde{L}_1),$$

$$T_2^\mu_\nu = -\frac{1}{2} \lambda R \mathcal{F}_2 (\Box) \tilde{G}^\mu_\nu - 2\lambda \tilde{G}^\mu_\rho \mathcal{F}_2 (\Box) \tilde{G}^\rho_\nu + \frac{\lambda}{2} \delta^\mu_\nu \tilde{G}^\rho_\rho \mathcal{F}_2 (\Box) \tilde{G}^\rho_\rho + 2\lambda \left( \nabla_\rho \nabla_\sigma \mathcal{F}_2 (\Box) \tilde{G}^{\rho\sigma} - \frac{1}{2} \nabla_\nu (\mathcal{L}_\xi^\xi + \tilde{L}_2) + 2\lambda \tilde{L}_\mu^\nu \right),$$

$$T_4^\mu_\nu = 2\lambda (R_{\alpha\beta} + 2\nabla_\alpha \nabla_\beta) \mathcal{F}_4 (\Box) C^\alpha_\beta^\mu_\nu.$$ 

Now, Einstein equations can be written in an extremely concise form

$$M^2 G^\mu_\nu = \sum_I T_I^\mu_\nu - \Lambda \delta^\mu_\nu,$$  \hspace{1cm} (31)

and, moreover, we have

$$\nabla_\mu T_I^\mu_\nu = 0 \text{ (for any } I).$$

One recognizes here the system of minimally coupled perfect fluids minimally coupled to gravity. The perturbation technique is known in general (see [41, 32] for instance), but it is not obvious that it can be applicable ‘as is’ to our model as we will see shortly.

5.2. Subtleties of the non-localities

Having brought Einstein equations in our model to an already studied structure does not mean that we are easily able, if at all, to solve the perturbation equations.

It looks promising that we could clearly separate the whole equations into minimally coupled terms. But we have to stress here that each term is not a canonical perfect fluid, i.e. it cannot be written just in terms of energy density $\rho$, pressure density $p$ and 4-fluid velocity $u_\mu$ as discussed in [42]. This results in the presence of anisotropic stresses as well as entropic perturbations for our fluids at the level of perturbations. This in turn does not allow us to use the already known techniques straightforwardly, since the system of perturbation equations (even non-local ones) will not be closed.

On the other hand, all $T_I$ are not independent external stress–energy tensors, but rather some structures built on the metric. We therefore must be able in principle to have the corresponding perturbative quantities in terms of the metric perturbations.
Also, one must not go straight with $T_{4}$ term since it represents a contribution which is absent in the background. It means that both energy and pressure for $T_{4}$ are zero. Thus, it is not obvious how one can define the equation of state parameter $w = p/\rho$ and the speed of sound $c_{s}^{2} = \rho/\dot{\rho}$.

It is, however, the simplest term for performing actual calculations at the perturbative level. This is because it is zero on the background and completely traceless. The first point means that

$$\delta T_{4}^{\mu \nu} = 2\lambda (R_{\alpha \beta} + 2\nabla_{\alpha} \nabla_{\beta}) F_{4}(\Box) \delta C_{\alpha \beta}^{\mu \nu}. $$

The second point means that for isotropic perturbations

$$\delta C_{\alpha \beta}^{\mu \nu} = c(\eta) e^{i \vec{k} \cdot \vec{x}} P_{\alpha \beta}^{\mu \nu},$$

where $\eta$ is the conformal time, $k$ is the comoving wave-vector and $P_{\alpha \beta}^{\mu \nu}$ is a constant tensor which keeps all the symmetry properties of the Weyl tensor. We can thus use the same idea as in the previous section for the traceless tensor $\tilde{G}_{\mu \nu}$ and compute the action of some differential operator $\mathcal{D}$ on $\delta C_{\alpha \beta}^{\mu \nu}$ as

$$\mathcal{D} \delta C_{\alpha \beta}^{\mu \nu} = (\mathcal{D}(\eta)) e^{i \vec{k} \cdot \vec{x}} P_{\alpha \beta}^{\mu \nu}$$

with a simple algebraic relation between $\mathcal{D}$ and $\mathcal{D}(\eta)$.

We can push these arguments further and draw some conclusions regarding when $R$ or $\tilde{G}_{\mu \nu}$ are perturbed in $T_{1}$ and $T_{2}$, respectively. We are then able to implicitly compute the action of the non-local operators on those perturbed quantities. However, the real problem comes when one would perturb the non-local operators themselves and compute the action of the perturbed operators on unperturbed background quantities. This is where the presence of an ansatz is crucial. This was the key point why we could end up with the final results in [18]. It is still an open question: how would one proceed in a general case when there are no additional relations in the model? We hope to address this issue in a forthcoming research [43].

We nevertheless can come up with a conclusion that the above discussed solution (5) is an example of a stable bounce configuration. Even though this particular solution satisfies the simplifying ansatz and in principle one would much easily write down the system of perturbation equations, we can claim that the bounce phase is stable without tedious calculations. This stability is guaranteed because the solution has analytic dependence on coefficients $f$.

Indeed, the stability of this solution with $F_{2} = F_{4} = 0$ was explicitly demonstrated in [18] and conditions on the model parameters were formulated. Denoting the metric perturbations $h_{\mu \nu}$, we can then write schematically the equation governing the perturbations

$$\mathcal{P}(\Box) h_{\mu \nu} = 0,$$

where $\mathcal{P}(\Box)$ is some non-local operator acting on the metric perturbations. It can be presented as a Taylor series in $\Box$:

$$\mathcal{P}(\Box) = \sum_{n \geq 0} p_{n} \Box^{n},$$

where $p_{n}$ may depend on the background quantities, say $R$, and only on coefficients $f_{1, n}$ since other functions are zero. Now, turning on $F_{2}$ and/or $F_{4}$, we essentially change the coefficients $p_{n}$. Moreover, as long as the dependence on $f_{2, 4, n}$ is analytic small new coefficients would create small changes to $p_{n}$.

In other words, it is important to stress that including the new terms in the action we do not add new physical degrees of freedom for $h_{\mu \nu}$. Furthermore, all these degrees of freedom are present when $F_{2, 4} = 0$, and we thus preserve this characteristic of the system. Also, technically
by construction, equations for $h_{\mu\nu}$ are always linear and this simplifies the understanding of the possible changes.

As a trivial example, one can think about a free massive field with a mass $m$. As long as the mass squared is positive in the action, the system is stable. Including new terms, one effectively changes the mass squared of this toy field. This toy model remains stable however, provided changes are small. Referring to this toy model, there is a critical situation when an initially massless scalar field acquires a non-zero mass squared which may appear to be negative. Fortunately, such a potentially dangerous scenario is not relevant for our discussion, since the obtained operator $\mathcal{P}(\Box)$ is quite general, i.e. one does not have to require some coefficients $p_n$ to be zero. Then small changes to $p_n$ will not break down the system.

Therefore, considering non-zero $\mathcal{F}_2$ and/or $\mathcal{F}_4$, we must have the configuration stable up to some range of coefficients $f_{2,4n}$. Of course, this argument is not enough to find the allowed domain for new coefficients and this is the primary goal for the forthcoming paper [43].

5.3. de Sitter limit

The de Sitter limit is perhaps the most simple configuration for the analysis of perturbations (apart from Minkowskian spacetime, of course). The de Sitter Universe is described by the scale factor

$$a = a_0 e^{Ht}$$

and the corresponding Hubble function is just the constant $H$. This is where we see why the traceless tensor $\tilde{G}^\mu_\nu$ is convenient, since it is identically zero in such a metric. Moreover, the scalar curvature is a constant $R = 12H^2$. One can easily check that this is a solution to our equations. But the most important part of equations which may be relevant for perturbations is much shorter, since we can drop terms quadratic in $\delta R$ as well as some others. Then the relevant equations become

$$M^2_0 \delta G^\mu_\nu = \delta T^{\mu}_{\nu} - 2\lambda \delta G^\mu_\nu \tilde{f}_{10} R + 2\lambda (\nabla^\mu \partial_\nu - \delta^\mu_\nu \Box) \tilde{f}_1 (\Box) \delta R - \frac{1}{2} \lambda R \mathcal{F}_2 (\Box) \delta G^\mu_\nu$$

$$+ 2\lambda \left( \nabla_\mu \nabla_\nu \mathcal{F}_2 (\Box) \delta G^{\mu\nu} - \frac{1}{2} \Box \mathcal{F}_2 (\Box) \delta G^\mu_\nu - \frac{1}{2} \delta^{\mu}_{\nu} \nabla_\mu \nabla_\rho \mathcal{F}_2 (\Box) \delta G^{\rho\sigma} \right)$$

$$- \frac{\lambda}{2} \delta^\mu_\nu R (\tilde{f}_1 (\Box) - \tilde{f}_{10}) \delta R + 4\lambda \nabla_\mu \nabla_\nu \mathcal{F}_4 (\Box) \delta C_\alpha^{\mu\rho\sigma\nu}$$

(33)

with the trace

$$- M^2_0 \delta R = \delta T - 6\lambda \tilde{f}_1 (\Box) \delta R - 2\lambda R (\tilde{f}_1 (\Box) - \tilde{f}_{10}) \delta R - 2\lambda \nabla_\mu \nabla_\nu \mathcal{F}_2 (\Box) \delta G^{\mu\nu}. \tag{34}$$

These can be analyzed up to the end explicitly for some configurations. To highlight the track toward solutions to the perturbation equations for scalar perturbations, we note that $\delta R$ has an explicit expression in terms of the Bardeen potentials $\Phi$ and $\Psi$, the two gauge invariant scalar degrees of freedom [44]. Then, one can show that $i \neq j$ component of equation (33) is a non-local equation containing $\delta R$ and $\Phi - \Psi$, where $i$, $j$ are the spatial indices. Assuming further that the matter is the radiation without anisotropic stresses and computing explicitly $\nabla_\mu \nabla_\nu \mathcal{F}_2 (\Box) \delta G^{\mu\nu}$ which is feasible thanks to the tracelessness of the $\tilde{G}^{\mu\nu}$ tensor, we must end up with two non-local equations on $\Phi$ and $\Psi$.

Even though the next steps in this directions are just technical, we are going to postpone this discussion because there are many other serious and unrevealed questions regarding the de
Sitter background in these types of models such as the ghost-free condition. These unexplored matters deserve a separate study which will include the detailed analysis of perturbations as well [43].

6. Summary and outlook

We have considered in this paper the most general extension of GR based on the inclusion of stringy motivated non-local operators and keeping the quadratic in curvature terms. The primary goal was to find out an exact bouncing solution, and it is for the first time up to the best of our knowledge when this task is accomplished when all the general quadratic terms with scalar curvature, Ricci and Riemann tensors are in the action. It is intriguing that the bouncing configuration is the cosine hyperbolic which is still a solution provided the operator functions $F_2$ and $F_4$ which control the presence of extra terms are trivial. This indicates the presence of some symmetry one would have to find and so far is one of the open questions. A good point of our bouncing solution (5) is that it may be a solution in our model without extra matter which was not the case without $F_2$ and $F_4$. This solution requires just a couple of conditions which are very general and are easy to satisfy. Moreover, we argue that this solution must be stable at least for some domain of small coefficients $f_{2,4n}$.

Also, we have presented several other solutions which are not all bouncing solutions but anyway widen our understanding of the model. Practically, we divided possible solutions into three groups based on the fact whether they obey some simplifying relations or not when the scalar curvature and the Ricci tensor are being acted by the powers of the d’Alembertian operator.

On the way to these new solutions, we were able to recast the initially difficult problem of dealing with the Ricci tensor as a problem of many scalar functions $b_n$. This is a general construction which does not rely in any way on the properties of the metric and can be applied to further investigations in these types of models.

One more interesting point of this model is that the last term with the Weyl tensor squared in the Lagrangian does not contribute to the background at all. It, however, shows up in perturbations and it is interesting to see whether the presence of this term can be efficiently used to tackle the problem of solving the perturbation equations.

Also, we were able to bring the model formulation to such a form that was already analyzed at the perturbative level for other configurations. The perturbations can be analyzed in full at least numerically in the de Sitter limit and the clear way toward this was outlined. As was mentioned in the main part of the paper, however, we postpone the more detailed and technical study of this question, since it is more natural to join it with other unrevealed problems one can put for this model in the de Sitter space. Namely, ghost-free condition and subsequent quantization are the issues to be fully studied as well and results should appear shortly in [43].

The full analysis of perturbations is expected to be very hard in general. At the moment, it is not even obvious when one will be able to close the system of perturbation equations for scalar perturbations. Tensor perturbations may become simpler, but one would still face the difficulty that the effective anisotropic stresses are present in the system. This is tough and open question at the moment.

Furthermore, one question which we kept aside is the anisotropic perturbations during the contraction phase since this question is under investigation in the parallel project [45]. The generic expectation is that such perturbations must grow during the contraction phase and the main problem is to formulate the domains of parameters of the model allowing the
bounce to happen smoothly. Approaches to study anisotropic perturbations in non-local as well as stringy-inspired models can be found in [22, 46] and references therein.

Acknowledgments

AK is supported by an ‘FWO-Vlaanderen’ postdoctoral fellowship and also supported in part by Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P7/37, the ‘FWO-Vlaanderen’ through the project G.0114.10N and the RFBR grant 11-01-00894.

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