Some dual definite integrals for Bessel functions

Howard S. COHL,* Sean J. NAIR,† and Rebekah M. PALMER‡

* Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, MD, 20899-8910, USA
E-mail: howard.cohl@nist.gov
URL: http://www.nist.gov/itl/math/msg/howard-s-cohl.cfm

† Mathematics, Science, and Computer Science Magnet Program, Montgomery Blair High School, Silver Spring, MD, 20901, USA
E-mail: s.j.nair2@gmail.com

‡ Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail: rmaepalmer4@gmail.com

Received XX July 2012 in final form ????; Published online ????
doi:10.3842/JOURNAL.201*.*

Abstract. Based on known definite integrals of Bessel functions of the first kind, we obtain exact solutions to unknown definite integrals using the method of integral transforms from Hankel’s transform.

Key words: Definite integrals; Bessel functions; Associated Legendre functions; Hypergeometric functions; Struve functions; Chebyshev polynomials; Jacobi polynomials

2010 Mathematics Subject Classification: 26A42; 33C05; 33C10; 33C45; 35A08

1 Introduction

In Cohl (2012) [1], orthogonality and Hankel’s transform are used to generate solutions to new definite integrals based on known integrals. In this paper, we use the method of integral transforms to create new integrals from a variety of known integrals containing Bessel functions of the first kind \( J_\nu \). In this method, we use the closure relation for Bessel functions of the first kind to generate a guess for a function to use in Hankel’s transform. This guess may be incorrect if it does not satisfy the first condition for Hankel’s transform, in which case a new definite integral is not generated. If the guess satisfies the condition, restrictions on \( \nu \) must then be adjusted to satisfy the other condition of Hankel’s transform. For the functions used in this paper, superscripts and subscripts refer to lists of parameters. Any exceptions to this will be clear from context.

As far as we are aware, the 40 definite integrals over Bessel functions of the first kind that we present in this manuscript, do not currently appear in the literature. An extension of the survey presented in this manuscript can be used to mechanically compute new definite integrals from pre-existing definite integrals over Bessel functions of the first kind.

1.1 Application of Hankel’s transform

We use the following result where for \( x \in (0, \infty) \) we define

\[ F(r \pm 0) := \lim_{x \to r \pm} F(x); \]

see Watson (1944) [7] p. 456];
Theorem 1. Let \( F : (0, \infty) \to \mathbb{C} \) be such that
\[
\int_0^\infty \sqrt{x} |F(x)| \, dx < \infty, \tag{1}
\]
and let \( \nu \geq -\frac{1}{2} \). Then
\[
\frac{1}{2} (F(r + 0) + F(r - 0)) = \int_0^\infty u J_\nu(ux) \int_0^\infty x F(x) J_\nu(ux) \, dx \, du \tag{2}
\]
provided that the positive number \( r \) lies inside an interval in which \( F(x) \) has finite variation.

The effort described in this paper was motivated by the large collection of Bessel function definite integrals which exist in the book “Table of Integrals, Series, and Products” [4]. Not counting Theorem 2 (which stands alone), the method of integral transforms was applied to the Bessel function definite integrals appearing in Sections 6.51 and 6.52 of [4]. This method can be applied to many definite integrals appearing in the remainder of sections appearing in Sections 6.5-6.7 of [4].

For the definite integrals presented in this manuscript, we have directly verified that (1) is satisfied. This is easily accomplished by analyzing the behavior of the integrands in a small neighborhood of the endpoints \( \{0, \infty\} \). For this paper, this technique produced 30 theorems including 40 definite integrals which are given below. The method of integral transforms does not always succeed in producing new definite integrals because the conditions on the Hankel transform (1) is not satisfied. Some cases of this are shown in Section 7.

2 Polynomial, rational, algebraic, and power functions

Theorem 2. Let \( b,c > 0, \nu > -\frac{1}{2}, t \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty (\Delta(a,b,c))^{2\nu-1} a^{1-\nu} J_\nu(at) \, da = 2^{1-\nu} \sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right) \left( \frac{b}{c} \right)^\nu J_\nu(bt) J_\nu(ct). \tag{3}
\]
\( \Delta : [0, \infty)^3 \to [0, \infty) \) (Heron’s formula [5]), defined by
\[
\Delta(a,b,c) := \sqrt{s(s-a)(s-b)(s-c)},
\]
where \( s = (a + b + c)/2 \), is the area of a triangle with sides of length \( a, b, \) and \( c \).

Proof. We apply Theorem 1 to the function \( F_{b,c} : (0, \infty) \to \mathbb{C} \) defined by
\[
F_{b,c}(t) := 2^{1-\nu} \sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right) \left( \frac{b}{c} \right)^\nu J_\nu(bt) J_\nu(ct),
\]
where \( \Gamma : \mathbb{C} \setminus -\mathbb{N}_0 \to \mathbb{C} \) is Euler’s gamma function is defined in [3 (5.2.1)], and \( J_\nu : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} \), (order) \( \nu \in \mathbb{C} \), is the Bessel function of the first kind defined in [3 (10.2.2)]. The desired result is obtained from Sonine’s formula [4]
\[
\int_0^\infty J_\nu(at) J_\nu(bt) J_\nu(ct) t^{1-\nu} \, dt = \frac{2^{\nu-1}(\Delta(a,b,c))^{2\nu-1}}{\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right) (abc)^\nu},
\]
where \( \text{Re}a > 0, \text{Re}c > 0, \text{Re} \nu > -\frac{1}{2} \).
Theorem 3. Let \( \nu > \frac{1}{2}, \mu > 0, \alpha, \beta > 0, z \in \mathbb{C} \setminus (-\infty, 0] \). Then

\[
\int_{0}^{\alpha} b^\nu J_{\nu-1}(bz) \, db = \alpha^\nu z^{-1} J_{\nu}(\alpha z), \tag{4}
\]

\[
\int_{\beta}^{\infty} a^{1-\mu} J_{\mu}(az) \, da = \beta^{1-\mu} z^{-1} J_{\mu-1}(\beta z). \tag{5}
\]

**Proof.** By applying Theorem 1 to the functions \( F_{\alpha}^{\nu} : (0, \infty) \to \mathbb{C} \) and \( G_{\beta}^{\mu} : (0, \infty) \to \mathbb{C} \) defined by \( F_{\alpha}^{\nu}(x) := \alpha^\nu x^{-1} J_{\nu}(\alpha x) \), \( G_{\beta}^{\mu}(x) := \beta^{1-\mu} x^{-1} J_{\mu-1}(\beta x) \), we obtain the desired results from the known integral \[4, (6.512.3)\]

\[
\int_{0}^{\infty} J_{\nu}(\alpha x) J_{\nu-1}(\beta x) \, dx = \begin{cases} 
\alpha^{-\nu} \beta^{\nu-1} & \text{if } \beta < \alpha, \\
(2\beta)^{-1} & \text{if } \beta = \alpha, \\
0 & \text{if } \beta > \alpha,
\end{cases}
\]

where \( \text{Re} \nu > 0 \). ■

Theorem 4. Let \( \nu \geq -\frac{1}{2}, z \in \mathbb{C} \setminus (-\infty, 0] \). Then

\[
\int_{0}^{\infty} \frac{e^{\nu+1}}{1 + c^2} J_{\nu}(cz) \, dc = K_{\nu}(z). \tag{6}
\]

**Proof.** We are given the integral \[4, (6.521.2)\]

\[
\int_{0}^{\infty} xK_{\nu}(ax) J_{\nu}(bx) \, dx = \frac{b^\nu}{a^\nu(b^2 + a^2)},
\]

where \( \text{Re } a > 0, b > 0, \text{Re } \nu > -1 \). By applying Theorem 1 to the function \( F_{\nu}^{\alpha} : (0, \infty) \to \mathbb{C} \) defined by \( F_{\nu}^{\alpha}(x) := a^\nu K_{\nu}(ax) \), we obtain the following integral

\[
\int_{0}^{\infty} \frac{b^{\nu+1}}{b^2 + a^2} J_{\nu}(bx) \, db = a^\nu K_{\nu}(ax),
\]

where \( \text{Re } a > 0, \nu \geq -\frac{1}{2}, x \in \mathbb{C} \setminus (-\infty, 0] \). With the substitutions \( z = \alpha x \) and \( c = \beta/a \), we obtain the desired result. ■

Note that when the method of integral transforms is applied to \( F_{a}(x) := aK_{1}(ax) \) given the integral \[4, (6.521.7)\]

\[
\int_{0}^{\infty} xK_{1}(ax) J_{1}(bx) \, dx = \frac{b}{a(a^2 + b^2)},
\]

where \( a > 0, b > 0 \), we obtain the integral generated from \[4, (6.521.2)\] when \( \nu = 1 \).

Theorem 5. Let \( z \in \mathbb{C} \setminus (-\infty, 0] \). Then

\[
\int_{0}^{\infty} \frac{c}{(1 + c^2)^2} J_{0}(cz) \, dc = \frac{z}{2} K_{1}(z). \tag{7}
\]
Proof. We are given the integral \[4\] (6.521.12)
\[
\int_0^\infty x^2K_1(ax)J_0(bx) = \frac{2a}{(a^2 + b^2)^2},
\]
where \(a > b > 0\). By applying Theorem \[1\] to the function \(F_a : (0, \infty) \to \mathbb{C}\) defined by \(F_a(x) := \frac{x}{a}K_1(ax)\), we obtain the following integral
\[
\int_0^\infty \frac{bJ_0(bx)}{(a^2 + b^2)^2}db = \frac{x}{2a}K_1(ax),
\]
where \(a > 0, x \in \mathbb{C} \setminus (-\infty, 0]\). With the substitutions \(z = ax\) and \(c = b/a\), we obtain the desired result. \(\blacksquare\)

**Theorem 6.** Let \(z \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty \frac{c^2}{(1 + c^2)^2}J_1(cz)dc = \frac{z}{2}K_0(z).
\] (8)

**Proof.** We are given the integral \[4\] (6.521.12)
\[
\int_0^\infty x^2K_0(ax)J_1(bx)dx = \frac{2b}{(a^2 + b^2)^2},
\]
where \(a,b > 0\). By applying Theorem \[1\] to the function \(F_a : (0, \infty) \to \mathbb{C}\) defined by \(F_a(x) := \frac{x}{2}K_0(ax)\), we obtain the following integral
\[
\int_0^\infty \frac{b^2J_1(bx)}{(a^2 + b^2)^2}db = \frac{x}{2}K_0(ax),
\]
where \(a > 0, x \in \mathbb{C} \setminus (-\infty, 0]\). With the substitutions \(z = ax\) and \(c = b/a\), we obtain the desired result. \(\blacksquare\)

**Theorem 7.** Let \(\gamma > 0, \nu \geq -\frac{1}{2}, \Re \alpha > |\Im \beta|, z \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty bJ_\nu(bz)\frac{l_\nu'}{l_\nu^2(l_\frac{3}{2} - l_\frac{1}{2})}db = K_0(\alpha z)J_\nu(\gamma z),
\] (9)
\[
\int_0^\infty cJ_\nu(cz)\frac{l_\nu'}{l_\nu^2(l_\frac{3}{2} - l_\frac{1}{2})}dc = K_0(\alpha z)J_\nu(\beta z),
\] (10)
where \(l_1\) and \(l_2\) are defined as
\[
l_1 = \frac{1}{2} \left[ \sqrt{(b + c)^2 + a^2} - \sqrt{(b - c)^2 + a^2} \right],
\] (11)
\[
l_2 = \frac{1}{2} \left[ \sqrt{(b + c)^2 + a^2} + \sqrt{(b - c)^2 + a^2} \right].
\] (12)

**Proof.** By applying Theorem \[1\] to the function \(F_{\nu}^{a,c} : (0, \infty) \to \mathbb{C}\) and \(G_{\nu}^{a,b} : (0, \infty) \to \mathbb{C}\) defined by \(F_{\nu}^{a,c}(x) := K_0(ax)J_\nu(cx), G_{\nu}^{a,b}(x) := K_0(ax)J_\nu(bx)\), we obtain the desired result from the known integral \[4\] (6.522.12)
\[
\int_0^\infty xK_0(ax)J_\nu(bx)J_\nu(cx)dx = \frac{l_\nu'}{l_\nu^2(l_\frac{3}{2} - l_\frac{1}{2})},
\]
where \(c > 0, \Re \nu > -1, \Re a > |\Im b|\). \(\blacksquare\)
Theorem 8. Let \( Re \, b > Re \, a, \, z \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty cJ_0(cz)(a^4 + b^4 + c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2)^{-1/2}dc = I_0(az)K_0(bz).
\] (13)

Let \( Re \, b > Re \, c, \, z \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty \frac{aJ_0(az)}{l_2^2 - l_1^2}da = I_0(cz)K_0(bz).
\] (14)

where \( l_1 \) and \( l_2 \) are defined in (11) and (12) respectively.

Proof. By applying Theorem 1 to the function \( F^b_a : (0, \infty) \rightarrow \mathbb{C} \) and \( G^b_c : (0, \infty) \rightarrow \mathbb{C} \) defined by \( F^b_a(x) := I_0(ax)K_0(bx) \), \( G^b_c(x) := I_0(cx)K_0(bx) \), we obtain the desired results from the known integrals (see [1] (6.522.4))
\[
\int_0^\infty xI_0(ax)K_0(bx)J_0(cx)dx = (a^4 + b^4 + c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2)^{-1/2},
\]
where \( Re \, b > Re \, a, \, c > 0 \), and
\[
\int_0^\infty xI_0(cx)K_0(bx)J_0(ax)dx = \frac{1}{l_2^2 - l_1^2},
\]
where \( Re \, b > Re \, c, \, a > 0 \), respectively. \( \blacksquare \)

Theorem 9. Let \( \gamma > 0, \, \nu > -\frac{1}{2}, \, Re \, \alpha > |Im \, \beta|, \, z \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty \frac{b^{\nu+1}J_\nu(bz)}{(l_2^2 - l_1^2)^{2\nu+1}}db = \frac{z^{\nu}(a\gamma)^{-\nu}\sqrt{\pi}}{2^{2\nu} \Gamma(\nu + \frac{1}{2})}K_\nu(\alpha z)J_\nu(\gamma z),
\] (15)
\[
\int_0^\infty \frac{c^{\nu+1}J_\nu(cz)}{(l_2^2 - l_1^2)^{2\nu+1}}dc = \frac{z^{\nu}(\alpha\beta)^{-\nu}\sqrt{\pi}}{2^{2\nu} \Gamma(\nu + \frac{1}{2})}K_\nu(\alpha z)J_\nu(\beta z),
\] (16)

where \( l_1 \) and \( l_2 \) are defined in (11) and (12), respectively.

Proof. By applying Theorem 1 to the functions \( F^{a,c}_\nu : (0, \infty) \rightarrow \mathbb{C}, \, G^{a,b}_\nu : (0, \infty) \rightarrow \mathbb{C} \), defined by
\[
F^{a,c}_\nu(x) := \frac{x^{\nu}(ac)^{-\nu}\sqrt{\pi}}{2^{2\nu} \Gamma(\nu + \frac{1}{2})}K_\nu(ax)J_\nu(cx),
\]
\[
G^{a,b}_\nu(x) := \frac{x^{\nu}(ab)^{-\nu}\sqrt{\pi}}{2^{2\nu} \Gamma(\nu + \frac{1}{2})}K_\nu(ax)J_\nu(bx),
\]
we obtain the desired results from the known integral [4] (6.522.15)
\[
\int_0^\infty x^{\nu+1}J_\nu(bx)K_\nu(ax)J_\nu(cx)dx = \frac{2^{3\nu}(abc)^{\nu} \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi} (l_2^2 - l_1^2)^{2\nu+1}},
\]
where \( Re \, a > |Im \, b|, \, c > 0 \). \( \blacksquare \)
Theorem 10. Let $\gamma > 0$, $\Re \beta \geq |\Im \alpha|$, $\Re \alpha > 0$, $\Re p > |\Re q|$, $\Re q > 0$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Then

$$
\int_{0}^{\infty} 2a^2 J_1(az)(a^2 + \beta^2 - \gamma^2) \left[(a^2 + \beta^2 + \gamma^2)^2 - 4a^2 \gamma^2\right]^{-3/2} da = zK_0(\beta z)J_0(\gamma z),
$$

(17)

$$
\int_{0}^{\infty} cJ_0(cz)(\alpha^2 + \beta^2 - c^2) \left[(\alpha^2 + \beta^2 + c^2)^2 - 4\alpha^2 c^2\right]^{-3/2} dc = \frac{z}{2\alpha} J_1(\alpha z)K_0(\beta z).
$$

(18)

$$
\int_{0}^{\infty} J_1(bz)\frac{2b^2(p^2 + b^2 - \gamma^2)}{(l_2 - l_1)^3} db = zK_0(pz)J_0(\gamma z),
$$

(19)

$$
\int_{0}^{\infty} J_0(cz)\frac{c(p^2 + q^2 - c^2)}{(l_2 - l_1)^3} dc = \frac{z}{2q} J_1(qz)K_0(pz),
$$

(20)

where $l_1$ and $l_2$ are defined in (11) and (12), respectively.

Proof. By applying Theorem 11 to the functions $F^b_\nu : (0, \infty) \to \mathbb{C}$, $G^a_\nu : (0, \infty) \to \mathbb{C}$, $H^b_\nu : (0, \infty) \to \mathbb{C}$, $I^a_\nu : (0, \infty) \to \mathbb{C}$ defined by $F^b_\nu(x) := xK_0(bx)J_0(cx)$, $G^a_\nu(x) := \frac{x}{2a} J_1(ax)K_0(bx)$, $H^b_\nu(x) := xK_0(ax)J_0(cx)$, $I^a_\nu(x) := \frac{x}{2a} J_1(ax)K_0(ax)$, we obtain the desired results from the known integrals (see [4] (6.525.1))

$$
\int_{0}^{\infty} x^2 J_1(ax)K_0(bx)J_0(cx)dx = 2a(a^2 + b^2 - c^2) \left[(a^2 + b^2 + c^2)^2 - 4a^2 c^2\right]^{-3/2},
$$

where $c > 0$, $\Re b \geq |\Re a|$, $\Re a > 0$,

$$
\int_{0}^{\infty} x^2 J_1(bx)K_0(ax)J_0(cx)dx = \frac{2b(a^2 + b^2 - c^2)}{(l_2 - l_1)^3},
$$

where $c > 0$, $\Re a > |\Im b|$, $\Re b > 0$.

Theorem 11. Let $\Re a > 0$, $\nu \geq -\frac{1}{2}$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Then

$$
\int_{0}^{\infty} \frac{J_\nu(bz)}{\sqrt{b^2 + 4a^2}} db = I_{\nu/2}(az)K_{\nu/2}(az).
$$

(21)

Proof. By applying Theorem 11 to the function $F^a_\nu : (0, \infty) \to \mathbb{C}$ defined by

$$
F^a_\nu(x) := I_{\nu/2}(ax)K_{\nu/2}(ax),
$$

we obtain the desired result from the known integral [4] (6.522.9)

$$
\int_{0}^{\infty} xI_{\nu/2}(ax)K_{\nu/2}(ax)J_\nu(bx)dx = b^{-1}(b^2 + 4a^2)^{-1/2},
$$

where $b > 0$, $\Re a > 0$, $\Re \nu > -1$.

Theorem 12. Let $a > 0$, $\nu \geq -\frac{1}{2}$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Then

$$
\int_{2a}^{\infty} \frac{J_\nu(bz)}{\sqrt{b^2 - 4a^2}} db = -\frac{\pi}{2} J_{\nu/2}(az)Y_{\nu/2}(az).
$$

(22)
Theorem 14. Let \( \Re a > 0, \nu > -\frac{1}{2}, \Re \mu \geq \frac{2}{7}, \ z \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty \frac{J_\nu(bz)}{\sqrt{b^2 + 4a^2}} \left[ b + (b^2 + 4a^2)^{1/2} \right]^\mu \, db = 2^\mu a^\mu I_{(\nu - \mu)/2}(az)K_{(\nu + \mu)/2}(az). \tag{23}
\]

Proof. By applying Theorem \([1]\) to the function \( F_{\nu}^\alpha : (0, \infty) \rightarrow \mathbb{C} \) defined by
\[
F_{\nu}^\alpha(x) := 2^\nu a^\nu I_{(\nu - \mu)/2}(ax)K_{(\nu + \mu)/2}(ax),
\]
we obtain the desired result from the known integral \([1] (6.522.10)\)
\[
\int_0^\infty xI_{(\nu - \mu)/2}(ax)K_{(\nu + \mu)/2}(ax)J_\nu(bx) \, dx = 2^{-\mu} a^{-\mu} b^{-1} (b^2 + 4a^2)^{-1/2} \left[ b + (b^2 + 4a^2)^{1/2} \right]^\mu,
\]
where \( \Re \nu > -1 \).

Theorem 15. Let \( \Re a > 0, \nu > 0, \ z \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty cJ_0(cx)(b^2 + c^2 - a^2) \left[ (a^2 + b^2 + c^2)^2 - 4a^2b^2 \right]^{-3/2} \, dc = \frac{x}{2b} I_0(ax)K_1(bx). \tag{24}
\]

Proof. By applying Theorem \([1]\) to the function \( F_{a}^b : (0, \infty) \rightarrow \mathbb{C} \) defined by
\[
F_{a}^b(x) := \frac{1}{2a} I_0(ax)K_1(bx),
\]
we obtain the desired result from the known integral \([1] (6.525.2)\)
\[
\int_0^\infty x^2I_0(ax)K_1(bx)J_0(cx) \, dx = 2b(b^2 + c^2 - a^2) \left[ (a^2 + b^2 + c^2)^2 - 4a^2b^2 \right]^{-3/2},
\]
where \( \Re b > |\Re a|, \ c > 0 \).

3 Bessel and Struve functions

Theorem 16. \( \nu > 0, z \in \mathbb{C} \setminus (-\infty, 0] \). Then
\[
\int_0^\infty J_\nu(cz)J_{2\nu}(2\sqrt{c}) \, dc = \frac{1}{z} J_\nu \left( \frac{1}{z} \right). \tag{25}
\]

Proof. We are given the integral \([1] (6.514.1)\)
\[
\int_0^\infty J_\nu \left( \frac{x}{2} \right) J_\nu(bx) \, dx = b^{-1} J_{2\nu}(2\sqrt{ab}),
\]
where \( \Re \nu > 0, a, b > 0 \). By applying Theorem \([1]\) to the function \( F_{\nu}^a : (0, \infty) \rightarrow \mathbb{C} \) defined by
\[
F_{\nu}^a(x) := x^{-1}J_\nu(ax^{-1}),
\]
we obtain the following integral
\[
\int_0^\infty J_\nu(bx)J_{2\nu}(2\sqrt{ab}) \, db = x^{-1} J_\nu \left( \frac{a}{x} \right),
\]
where \( \nu > 0, a > 0, x \in \mathbb{C} \setminus (-\infty, 0] \). By making the substitutions \( x = az, c = ba \), we obtain the desired result.
Theorem 16. Let \(-\frac{1}{2} \leq \nu < \frac{5}{4}\), \(z \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty c J_\nu(cz) \left[ e^{i(\nu+1)\pi/2} K_{2\nu} \left( 2e^{i\pi/4}\sqrt{c} \right) + e^{-i(\nu+1)\pi/2} K_{2\nu} \left( 2e^{-i\pi/4}\sqrt{c} \right) \right] dc
= \frac{1}{z^3} K_{\nu} \left( \frac{1}{z} \right).
\] (26)

Proof. We are given the integral \([4] (6.514.3)\)
\[
\int_0^\infty J_\nu \left( \frac{a}{x} \right) K_{\nu}(bx) dx = b^{-1} e^{i(\nu+1)\pi/2} K_{2\nu} \left[ 2e^{i\pi/4}\sqrt{ab} \right] + b^{-1} e^{-i(\nu+1)\pi/2} K_{2\nu} \left[ 2e^{-i\pi/4}\sqrt{ab} \right],
\]
where \(a > 0\), \(\text{Re} b > 0\), \(|\text{Re} \nu| < \frac{5}{2}\). By applying Theorem [1] to the function \(F_{\nu}^b : (0, \infty) \rightarrow \mathbb{C}\) defined by \(F_{\nu}^b(x) := bx^{-3} K_{\nu}(bx^{-1})\), we obtain the following integral
\[
\int_0^\infty a J_\nu(ax) \left[ e^{i(\nu+1)\pi/2} K_{2\nu} \left( 2e^{i\pi/4}\sqrt{ab} \right) + e^{-i(\nu+1)\pi/2} K_{2\nu} \left( 2e^{-i\pi/4}\sqrt{ab} \right) \right] da = \frac{b}{x^3} K_{\nu} \left( \frac{b}{x} \right),
\]
where \(\text{Re} b > 0\), \(-\frac{1}{2} \leq \nu < \frac{5}{2}\), \(x \in \mathbb{C} \setminus (-\infty, 0]\). With the substitutions \(x = bz\), \(c = ba\), we obtain the desired result. ■

Theorem 17. Let \(|\nu| < \frac{1}{2}\), \(z \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty J_\nu(cz) \left[ K_{2\nu} \left( 2\sqrt{c} \right) - \frac{\pi}{2} Y_{2\nu} \left( 2\sqrt{c} \right) \right] dc = -\frac{\pi}{2z} Y_{\nu} \left( \frac{1}{z} \right).
\] (27)

Proof. We are given the integral \([4] (6.514.4)\)
\[
\int_0^\infty Y_{\nu} \left( \frac{a}{x} \right) J_\nu(bx) dx = -\frac{2b^{-1}}{\pi} \left[ K_{2\nu} \left( 2\sqrt{ab} \right) - \frac{\pi}{2} Y_{2\nu} \left( 2\sqrt{ab} \right) \right],
\]
where \(a,b > 0\), \(|\text{Re} \nu| < \frac{1}{2}\). By applying Theorem [1] to the function \(F_{\nu}^a : (0, \infty) \rightarrow \mathbb{C}\) defined by
\[
F_{\nu}^a(x) := -\frac{\pi}{2x} Y_{\nu} \left( \frac{a}{x} \right),
\]
we obtain the following integral
\[
\int_0^\infty J_\nu(bx) \left[ K_{2\nu} \left( 2\sqrt{ab} \right) - \frac{\pi}{2} Y_{2\nu} \left( 2\sqrt{ab} \right) \right] db = -\frac{\pi}{2x} Y_{\nu} \left( \frac{a}{x} \right),
\]
where \(a > 0\), \(|\nu| < \frac{1}{2}\), \(x \in \mathbb{C} \setminus (-\infty, 0]\). With the substitutions \(x = az\), \(c = ab\), we obtain the desired result. ■

Theorem 18. Let \(\nu \geq -\frac{1}{4}\), \(\mu > -\frac{1}{2}\), \(z \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty J_{2\nu}(cz) J_\nu \left( \frac{c^2}{4} \right) c dc = 2J_\nu \left( z^2 \right),
\] (28)
\[
\int_0^\infty J_{\mu}(cz) J_\mu \left( \frac{1}{4c} \right) dc = z^{-1} J_{2\mu} \left( \sqrt{z} \right).
\] (29)
Proof. We are given the integral \[4, (6.516.1)\]
\[
\int_0^\infty J_{2\nu}(a\sqrt{x}) J_\nu(bx) \, dx = b^{-1} J_\nu \left( \frac{a^2}{4b} \right),
\]
where \(\Re \nu > -\frac{1}{2}, \, a, b > 0\). By applying Theorem \[4\] to the functions \(F_b^\nu : (0, \infty) \to \mathbb{C}\) and \(G_\mu^a : (0, \infty) \to \mathbb{C}\) defined by \(F_b^\nu(x) := 2b J_\nu \left( bx^2 \right)\), \(G_\mu^a(x) := x^{-1} J_{2\mu} \left( a\sqrt{x} \right)\), we obtain the following integrals
\[
\int_0^\infty a J_{2\nu}(ax) J_\nu \left( \frac{a^2}{4\beta} \right) \, da = 2\beta J_\nu \left( \beta x^2 \right),
\]
\[
\int_0^\infty J_\mu(bx) J_\mu \left( \frac{a^2}{4b} \right) \, db = x^{-1} J_{2\mu} \left( \alpha\sqrt{x} \right),
\]
where \(\alpha, \beta > 0, \, \nu \geq -\frac{1}{4}, \, \mu > -\frac{1}{2}, \, z \in \mathbb{C} \setminus (-\infty, 0]\). With the substitutions \(z^2 = bx^2, \, c = a/\sqrt{b}\), and \(\sqrt{z} = a\sqrt{x}, \, c = b/a^2\) respectively, we obtain the desired results.

Theorem 19. Let \(\nu > -1, \, \mu \geq -\frac{1}{2}, \, z \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty J_{\nu/2}(cz) J_{\nu/2} \left( \frac{1}{4c} \right) \, dc = z^{-1} J_\nu \left( \sqrt{z} \right),
\]
\[
\int_0^\infty c J_\mu(cz) J_\mu \left( \frac{2}{4} \right) \, dc = 2J_{\mu/2}(z^2).
\]
Proof. We are given the integral \[4, (6.526.1)\]
\[
\int_0^\infty x J_{\nu/2}(ax^2) J_\nu(bx) \, dx = \frac{1}{2a} J_{\nu/2} \left( \frac{b^2}{4a} \right),
\]
where \(a, b > 0, \, \Re \nu > -1\). By applying Theorem \[4\] to the function \(F_b^\nu : (0, \infty) \to \mathbb{C}\) defined by \(F_b^\nu(x) := x^{-1} J_\nu \left( b\sqrt{x} \right)\), we obtain the following integrals
\[
\int_0^\infty J_{\nu/2}(ax) J_{\nu/2} \left( \frac{\beta^2}{4a} \right) \, da = x^{-1} J_\nu \left( \beta \sqrt{x} \right),
\]
\[
\int_0^\infty b J_\mu(bx) J_\mu \left( \frac{b^2}{4\alpha} \right) \, db = 2\alpha J_{\mu/2}(\alpha x^2),
\]
where \(\alpha, \beta > 0, \, \nu > -1, \, \mu \geq -\frac{1}{2}, \, x \in \mathbb{C} \setminus (-\infty, 0]\). With the substitutions \(\sqrt{z} = \beta \sqrt{x}, \, c = a/\beta^2\), and \(z^2 = \alpha x^2, \, x = z\sqrt{\alpha}\), we obtain the desired results.

Theorem 20. Let \(\nu \geq -\frac{1}{4}, \, x \in \mathbb{C} \setminus (-\infty, 0]\). Then
\[
\int_0^\infty a^2 J_{2\nu}(ax) J_{\nu+1/2}(a^2) \, da = \frac{x}{4} J_{\nu-1/2} \left( \frac{2}{4} \right).
\]
Proof. By applying Theorem \[4\] to the function \(F_\nu : (0, \infty) \to \mathbb{C}\) defined by \(F_\nu(x) := \frac{x}{4} J_{\nu-1/2} \left( \frac{2}{4} \right)\), we obtain the desired result from the known integral \[4, (6.527.1)\]
\[
\int_0^\infty J_{2\nu}(2ax) J_{\nu-1/2}(x^2) \, dx = \frac{1}{2} a J_{\nu+1/2}(a^2),
\]
where \(a > 0, \, \Re \nu > -\frac{1}{2}\).
Theorem 21. Let \( ν ≥ -\frac{1}{2}, \; x ∈ \mathbb{C} \setminus (-∞, 0] \). Then

\[
\int_0^∞ a^2 J_{2ν}(ax)J_{ν-1/2}(a^2)da = \frac{x}{4} J_{ν+1/2}\left(\frac{x^2}{4}\right).
\]  \tag{33}

Proof. By applying Theorem 1 to the function \( F_ν : (0, ∞) → \mathbb{C} \) defined by \( F_ν(x) := \frac{x}{4} J_{ν+1/2}\left(\frac{x^2}{4}\right) \), we obtain the desired result from the known integral \( \int_0^∞ J_{2ν}(2ax)J_{ν+1/2}(x^2)dx = \frac{1}{2} a J_{ν-1/2}(a^2) \), where \( a > 0, \; \text{Re} \; ν > -2 \).

Theorem 22. Let \( ν ≥ -\frac{1}{2}, \; z ∈ \mathbb{C} \setminus (-∞, 0] \). Then

\[
\int_0^∞ c J_ν(cz)H_{ν/2}\left(\frac{c^2}{4}\right)dc = -2Y_{ν/2}(z^2).
\]  \tag{34}

Proof. We are given the integral \( \int_0^∞ x Y_{ν/2}(ax^2)J_ν(bx)dx = \frac{1}{2a} H_{ν/2}\left(\frac{b^2}{4a}\right) \), where \( a > 0, \; \text{Re} \; b > 0, \; \text{Re} \; ν > -1 \) and \( H_{ν} : \mathbb{C} → \mathbb{C} \), for \( ν ∈ \mathbb{N}_0 \), is the Struve function defined in \( \int_0^∞ b J_ν(bx)H_{ν/2}\left(\frac{b^2}{4a}\right)db = -2aY_{ν/2}(ax^2) \), where \( a > 0, \; ν ≥ -\frac{1}{2}, \; x ∈ \mathbb{C} \setminus (-∞, 0] \). With the substitutions \( z^2 = ax^2, \; c = b/\sqrt{a} \), we obtain the desired result.

4 Exponential, logarithmic and inverse trigonometric functions

Theorem 23. Let \( ν ≥ -\frac{1}{2}, \; z ∈ \mathbb{C} \setminus (-∞, 0] \). Then

\[
\int_0^∞ J_ν(cz)e^{-2/c}c^{-1}dc = 2J_ν(2\sqrt{z})K_ν(2\sqrt{z}).
\]  \tag{35}

Proof. We are given the integral \( \int_0^∞ x J_ν(2\sqrt{ax})K_ν(2\sqrt{ax})J_ν(bx)dx = \frac{1}{2} b^{-2}e^{-2a/b} \), where \( \text{Re} \; a > 0, \; b > 0, \; \text{Re} \; ν > -1 \). By applying Theorem 1 to the function \( F_ν^a : (0, ∞) → \mathbb{C} \) defined by \( F_ν^a(x) := 2J_ν(2\sqrt{ax})K_ν(2\sqrt{ax}) \), we obtain the following integral

\[
\int_0^∞ b^{-1} J_ν(bx)e^{-2a/b}db = 2J_ν(2\sqrt{ax})K_ν(2\sqrt{ax}),
\]  

where \( \text{Re} \; a > 0, \; ν ≥ -\frac{1}{2}, \; x ∈ \mathbb{C} \setminus (-∞, 0] \). With the substitutions \( z = ax, \; c = b/a \), we obtain the desired result.
Theorem 24. Let $a > 0$, $z \in \mathbb{C} \setminus (-\infty, 0)$. Then

$$
\int_0^a J_1(bz) \ln \left(1 - \frac{b^2}{a^2}\right) \, db = -\pi z^{-1} Y_0(az). \tag{36}
$$

Proof. We apply Theorem 1 to the function $F_a : (0, \infty) \to \mathbb{C}$ defined by $F_a(x) := -\pi x^{-1} Y_0(ax)$, where $Y_\nu : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, (order) $\nu \in \mathbb{C}$, is the Bessel function of the second kind defined in [3, (10.2.3)]. We obtain the desired result from the known integral [4, (6.512.6)]

$$
\int_0^\infty J_1(bx) Y_0(ax) \, dx = -\frac{b^{-1}}{\pi} \ln \left(1 - \frac{b^2}{a^2}\right),
$$

where $0 < b < a$. ■

Theorem 25. Let $z \in \mathbb{C} \setminus (-\infty, 0)$. Then

$$
\int_0^\infty J_1(cz) \ln(1 + c^2) \, dc = 2z^{-1} K_0(z). \tag{37}
$$

Proof. We are given the integral [4, (6.512.9)]

$$
\int_0^\infty K_0(ax) J_1(bx) \, dx = \frac{1}{2b} \ln \left(1 + \frac{b^2}{a^2}\right),
$$

where $a, b > 0$ and $K_\nu : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, (order) $\nu \in \mathbb{C}$, is the modified Bessel function of the second kind defined in [3, (10.25.3)]. We apply Theorem 1 to the function $F_a : (0, \infty) \to \mathbb{C}$ defined by $F_a(x) := 2x^{-1} K_0(ax)$, and obtain the following integral

$$
\int_0^\infty J_1(bx) \ln \left(1 + \frac{b^2}{a^2}\right) \, db = 2x^{-1} K_0(ax),
$$

where $a > 0$, $x \in \mathbb{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$ and $c = b/a$, we obtain the desired result. ■

Theorem 26. Let $a > 0$, $z \in \mathbb{C} \setminus (-\infty, 0)$. Then

$$
\int_0^{2a} \sin^{-1} \left(\frac{b}{2a}\right) J_1(bz) \, db = \frac{\pi}{2z} [J_0^2(ax) - J_0(2az)]. \tag{38}
$$

Proof. By applying Theorem 1 to the function $F_a : (0, \infty) \to \mathbb{C}$ defined by

$$
F_a(x) := \frac{\pi}{2x} J_0^2(ax),
$$

we obtain the desired result from the known integral [4, (6.513.9)]

$$
\int_0^\infty J_0^2(ax) J_1(bx) \, dx = \left\{ \begin{array}{ll}
\frac{b^{-1}}{2b} \sin^{-1} \left(\frac{b}{2a}\right) & \text{if } 0 < 2a < b, \\
\frac{2}{\pi b} \sin^{-1} \left(\frac{b}{2a}\right) & \text{if } 0 < b < 2a.
\end{array} \right.
$$

■
5 Hypergeometric and Legendre functions

Theorem 27. Let \( n \in \mathbb{N}_0, \mu > 0, \nu > \frac{1}{2}, t \in \mathbb{C} \setminus (-\infty, 0]. \) Then

\[
\int_0^\alpha J_{\nu-n-1}(bt) \binom{\nu-n}{\nu-n} \frac{b^2}{\alpha^2} b^{\nu-n} \, db = \frac{n! \alpha^{\nu-n} \Gamma(\nu-n) J_{\nu+n}(at)}{t \Gamma(\nu)}, \tag{39}
\]

\[
\int_0^\infty J_{\mu+n}(at) \binom{\mu-n}{\mu-n} \frac{\beta^2}{\alpha^2} a^{-\mu+n+1} \, da = \frac{n! \beta^{-\mu+n+1} \Gamma(\mu-n) J_{\mu-n-1}(\beta t)}{t \Gamma(\mu)}. \tag{40}
\]

Proof. By applying Theorem 4 to the functions \( G_{\nu}^{\nu,a} : (0, \infty) \to \mathbb{C} \) and \( H_{\mu}^{\mu,b} : (0, \infty) \to \mathbb{C} \) defined by

\[
G_{\nu}^{\nu,a}(t) := \frac{n! \alpha^{\nu-n} \Gamma(\nu-n) J_{\nu+n}(at)}{t \Gamma(\nu)},
\]

\[
H_{\mu}^{\mu,b}(t) := \frac{n! \beta^{-\mu+n+1} \Gamma(\mu-n) J_{\mu-n-1}(\beta t)}{t \Gamma(\mu)},
\]

we obtain the desired results from the known integral [4 (6.512.2)]

\[
\int_0^\infty J_{\nu+n}(at) J_{\nu-n-1}(\beta t) dt = \begin{cases} \frac{\beta^{-\nu+n-1} \Gamma(\nu)}{\alpha^{\nu-n} \Gamma(\nu-n)} 2F1 \left( \begin{array}{c} \nu-n \\ \nu-n \end{array} ; \frac{\beta^2}{\alpha^2} \right) & \text{if } 0 < \beta < \alpha, \\ (-1)^n (2\alpha)^{-1} & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta > \alpha, \end{cases}
\]

where \( \text{Re} \nu > 0 \) and \( 2F1 : \mathbb{C}^2 \times (\mathbb{C} \setminus \mathbb{N}_0) \times (\mathbb{C} \setminus [1, \infty)) \to \mathbb{C} \) is the hypergeometric function defined in [3 (15.2.1)].

Theorem 28. Let \( \nu \geq -\frac{1}{2}, \nu > -2 \text{Re} \mu - 1, z \in \mathbb{C} \setminus (-\infty, 0]. \) Then

\[
\int_0^\infty P_{-1/2+\nu/2}^{-\mu} \left( \sqrt{1 + \frac{4}{c^2}} \right) Q_{-1/2+\nu/2}^{-\mu} \left( \sqrt{1 + \frac{4}{c^2}} \right) J_{\nu}(cz) dc = \frac{e^{-\mu \pi i} \Gamma \left( \frac{\nu-2\mu+1}{2} \right)}{z \Gamma \left( \frac{\nu+2\mu+1}{2} \right)} I_{\nu}(z) K_{\mu}(z). \tag{41}
\]

Proof. We are given the integral [4 (6.513.3)]

\[
\int_0^\infty I_{\mu}(ax) K_{\mu}(ax) J_{\nu}(bx) dx = \frac{e^{\mu \pi i} \Gamma \left( \frac{\nu+2\mu+1}{2} \right) b \Gamma \left( \frac{\nu-2\mu+1}{2} \right)}{P_{-1/2+\nu/2}^{-\mu} \left( \sqrt{1 + \frac{4a^2}{b^2}} \right) Q_{-1/2+\nu/2}^{-\mu} \left( \sqrt{1 + \frac{4a^2}{b^2}} \right)},
\]

where \( \text{Re} a > 0, b > 0, \text{Re} \nu > -1, \text{Re} \nu + 2\mu > -1 \) and \( P_{\nu}^{\mu} : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}, \) for \( \nu + \mu \notin \mathbb{N} \) with degree \( \nu \) and order \( \mu, \) and \( Q_{\nu}^{\mu} : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}, \) for \( \nu + \mu \notin \mathbb{N} \) with degree \( \nu \) and order \( \mu, \) are the associated Legendre functions of the first [3 (14.3.6)] and second [3 (14.3.7)] kind respectively. By applying Theorem 4 to the function \( F_{\nu}^{\mu,a} : (0, \infty) \to \mathbb{C} \) defined by

\[
F_{\nu}^{\mu,a}(x) := \frac{e^{-\mu \pi i} \Gamma \left( \frac{\nu-2\mu+1}{2} \right)}{x \Gamma \left( \frac{\nu+2\mu+1}{2} \right)} I_{\mu}(ax) K_{\mu}(ax),
\]
Theorem 29. Let 
\[ F(x) = \int \frac{1 + \frac{4a^2}{b^2}}{1 + \frac{4a^2}{b^2}} \left( \frac{a}{\sqrt{1 + \frac{4a^2}{b^2}}} \right) J_\nu(ax) \, dx = \frac{e^{-a \pi i}}{a} \Gamma \left( \frac{1 + \nu}{2} \right) J_\nu(ax), \]

where \( \text{Re} a > 0, \nu \geq -\frac{1}{2}, \nu > -2\mu - 1, x \in \mathbb{C} \setminus (-\infty, 0] \). By making the substitutions \( z = ax, c = b/a \), we obtain the desired result. \qed

Theorem 30. Let \( \nu > \text{Re} \mu - 1, \nu \geq -\frac{1}{2}, z \in \mathbb{C} \setminus (-\infty, 0] \). Then

\[ \int_0^\infty J_\nu(cz) \left[ Q_-^{\mu} \left( \frac{1 + \frac{4a^2}{b^2}}{1 + \frac{4a^2}{b^2}} \right)^2 \right] dc = \frac{e^{-2\pi i \mu}}{z \Gamma \left( \frac{1 + \nu + 2\mu}{2} \right)} [K_\mu(z)]^2. \]

Proof. We are given the integral \( [4] (6.513.5) \)

\[ \int_0^\infty [K_\mu(ax)]^2 J_\nu(bx) \, dx = \frac{e^{-2\pi i \mu}}{b \Gamma \left( \frac{1 + \nu + 2\mu}{2} \right)} \left[ Q_-^{\mu} \left( \frac{1 + \frac{4a^2}{b^2}}{1 + \frac{4a^2}{b^2}} \right)^2 \right], \]

where \( \text{Re} a > 0, b > 0, \text{Re} (\nu \pm \mu) > -\frac{1}{2} \). By applying Theorem 1 to the function \( F_{\nu}^{\mu,a} : (0, \infty) \rightarrow \mathbb{C} \) defined by

\[ F_{\nu}^{\mu,a}(x) := \frac{e^{-2\pi i \mu}}{x \Gamma \left( \frac{1 + \nu + 2\mu}{2} \right)} [K_\mu(ax)]^2, \]

we obtain the following integral

\[ \int_0^\infty J_\nu(bx) \left[ Q_-^{\mu} \left( \frac{1 + \frac{4a^2}{b^2}}{1 + \frac{4a^2}{b^2}} \right)^2 \right] db = \frac{e^{-2\pi i \mu}}{x \Gamma \left( \frac{1 + \nu + 2\mu}{2} \right)} [K_\mu(ax)]^2, \]

where \( \text{Re} a > 0, \nu > \text{Re} \mu - 1, \nu \geq -\frac{1}{2}, x \in \mathbb{C} \setminus (-\infty, 0] \). With the substitutions \( z = ax, c = b/a \), we obtain the desired result. \qed

6 Jacobi polynomials and Chebyshev polynomials of the first kind

Theorem 30. Let \( n \in \mathbb{N}_0, \nu > -n - 1, \alpha, \beta > 0, z \in \mathbb{C} \setminus (-\infty, 0] \). Then

\[ \int_{-1}^{\infty} P_{\nu}^{(n,0)} \left( 1 - \frac{2\beta^2}{a^2} \right) J_{\nu+2n+1}(az)a^{-\nu} \, da = z^{-1} \beta^{-\nu} J_\nu(\beta z), \]

\[ \int_0^\infty P_{\nu}^{(n,0)} \left( 1 - \frac{2\beta^2}{a^2} \right) J_{\nu}(bz)b^{\nu+1} \, db = z^{-1} \alpha^{\nu+1} J_{\nu+2n+1}(az). \]

Proof. By applying Theorem 1 to the functions \( F_{\nu}^{b} : (0, \infty) \rightarrow \mathbb{C} \) and \( G_{\nu}^{a,n} : (0, \infty) \rightarrow \mathbb{C} \) defined by \( F_{\nu}^{b}(x) := x^{-1} b^{-\nu} J_{\nu}(bx) \), \( G_{\nu}^{a,n}(x) := x^{-1} a^{\nu+1} J_{\nu+2n+1}(ax) \), we obtain the desired results from the known integral \( [4] (6.512.4) \)

\[ \int_0^\infty J_{\nu+2n+1}(ax)J_{\nu}(bx) \, dx = \begin{cases} b^\nu a^{-\nu-1} P_{n}^{(\nu,0)} \left( 1 - \frac{2a^{-2}b^2}{a^2} \right) & \text{if } 0 < b < a, \\ 0 & \text{if } 0 < a < b, \end{cases} \]

where \( \text{Re} \nu > -n - 1 \) and \( P_{n}^{(\alpha,\beta)} : \mathbb{C} \rightarrow \mathbb{C} \) is the Jacobi polynomial defined in \( [3] (18.3.1) \). \qed
Theorem 31. Let $a > 0$, $\nu \geq -\frac{1}{2}$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Then

$$\int_0^{2a} \frac{J_\nu(bz)}{\sqrt{4a^2 - b^2}} T_n \left( \frac{b}{2a} \right) db = \frac{\pi}{2} J_{(\nu+n)/2}(az)J_{(\nu-n)/2}(az). \tag{45}$$

Proof. By applying Theorem 1 to the function $F_\nu^{n,a}: (0, \infty) \to \mathbb{C}$ defined by

$$F_\nu^{n,a}(x) = \frac{\pi}{2} J_{(\nu+n)/2}(ax)J_{(\nu-n)/2}(ax),$$

we obtain the desired result from the known integral [4, (6.522.11)]

$$\int_0^\infty x J_{(\nu+n)/2}(ax)J_{(\nu-n)/2}(ax)J_\nu(bx)dx = \begin{cases} 2\pi^{-1}b^{-1}(4a^2 - b^2)^{-1/2} T_n \left( \frac{b}{2a} \right) & \text{if } 0 < b < 2a, \\ 0 & \text{if } 2a < b, \end{cases}$$

where $\Re \nu > -1$ and $T_n: \mathbb{C} \to \mathbb{C}$, for $n \in \mathbb{N}_0$, is the Chebyshev polynomial of the first kind found in [3, (18.3.1)].

7 Examples where the Hankel transforms fails

In the following examples, the potential use of the method given by Theorem 1 fails because the condition (1) can not be satisfied. This has been verified by analyzing the well-understood behavior of the integrands in a small neighborhood of the endpoints $\{0, \infty\}$.

- The definite integral [4, (6.512.1)] with $G_\nu^{b,c}(x) := \alpha(\nu, \mu)\Gamma(\nu+1)b^{-\nu}x^{-1}J_\nu(bx)$, $H_\nu^{b,c}(x) := \alpha(\nu, \mu)\Gamma(\nu+1)a^{\nu+1}x^{-1}J_\nu(ax)$, where $\alpha(\nu, \mu) := \Gamma \left( \frac{\mu+\nu+1}{2} \right) / \Gamma \left( \frac{\mu+\nu+1}{2} \right)$.
- The definite integral [4, (6.514.1)] with $G_\nu^b(x) := bx^{-3}J_\nu(bx^{-1})$.
- The definite integral [4, (6.514.2)] with $F_\nu^b(x) := bx^{-3}Y_\nu(bx^{-1})$.
- The definite integrals [4, (6.516.2)], [4, (6.516.3)], [4, (6.516.4)], and [4, (6.516.7)] with respectively $F_\nu^b(x) := -2bY_\nu(bx^2)$, $F_\nu^a(x) := 4b\pi^{-1}K_\nu(bx^2)$, $F_\nu^a(x) := x^{-1}Y_\nu(a\sqrt{x})$, and $F_\nu^b(x) := \frac{4b^{-1}x^{-1}}{\sec(\nu\pi)} K_\nu(a\sqrt{x})$.
- The definite integral [4, (6.522.2)] with $F_\nu^{\mu,a}(x) := \frac{1}{2}e^{-2\mu i\pi b(\nu, \mu)[K_\mu(ax)]^2}$, where $\beta(\nu, \mu) := \Gamma(\frac{\mu}{2} - \mu) / \Gamma(1 + \frac{\mu}{2} + \mu)$.
- The definite integrals [4, (6.522.6)] and [4, (6.522.8)] with $F_\nu^a(x) := -\frac{b}{2}J_0(2ax)Y_0(ax)$, and $G_\nu^{\mu,a}(x) := \frac{1}{2}e^{-2\mu i\pi b(\nu, \mu)K_{\nu-1/2}(ax)K_{\nu+1/2}(ax)}$, respectively.
- The definite integral [4, (6.522.16)] with $F_\nu^{b,c}(x) := \sqrt{\pi}x^\nu \gamma(\nu)J_\nu(ax)K_\nu(bx)$, where $\gamma(\nu) := (8bc)^{-\nu} / \Gamma \left( \nu + \frac{1}{2} \right)$.
- The definite integrals [4, (6.526.2)] and [4, (6.526.3)] with $F_\nu^b(x) := 2x^{-1}Y_\nu(b\sqrt{x})$, $G_\nu^b(x) := \cos(\frac{\nu \pi}{2})K_\nu(b\sqrt{x})/(2\pi x)$, respectively.
- The definite integral [4, (6.526.6)] with $F_\nu^a(x) := 4b\pi^{-1}K_{\nu/2}(ax^2)$.
- The definite integral [4, (6.527.3)] with $F_\nu(x) := -xY_{\nu-1/2}(x^2/4)/4$. 
References

[1] H. S. Cohl and H. Volkmer. Definite integrals using orthogonality and integral transforms. *Symmetry, Integrability and Geometry: Methods and Applications*, 8, 2012.

[2] NIST Digital Library of Mathematical Functions. Release 1.0.9 of 2014-08-29. Online companion to [3].

[3] F. W. J. Olver and D. W. Lozier and R. F. Boisvert and C. W. Clark, editor. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [2].

[4] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[5] A. Ostermann and G. Wanner. *Geometry by its History*. Undergraduate Texts in Mathematics. Readings in Mathematics. Springer, Heidelberg, 2012.

[6] K. Stempak. A new proof of Sonine’s formula. *Proceedings of the American Mathematical Society*, 104(2):453–457, 1988.

[7] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1944.