Symmetry energies for $A = 24$ and $48$ and the USD and KB3 shell model Hamiltonians

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Calculations in the $sd$ and $pf$ shells reported some time ago by Satula et al. [Phys. Lett. B 407, 103 (1997)] are redone for an extended analysis of the results. As in the original work, we do calculations for one mass number in each shell and consider in each case the sequence of lowest energies for isospins 0, 2, and 4, briefly the symmetry spectrum. Following further the original work we study how this spectrum changes when parts of the two-nucleon interaction are turned off. The variation of its width is explored in detail. A differential combination $\epsilon_W$ of the three energies was taken in the original work as a measure of the so-called Wigner term in semi-empirical mass formulas, and it was found to decrease drastically when the two-nucleon interaction in the channel of zero isospin is turned off. Our analysis shows that the width of the symmetry spectrum experiences an equally drastic decrease, which can be explained qualitatively in terms of schematic approximations. We therefore suggest that the decrease of $\epsilon_W$ be seen mainly as a side effect of a narrowing of the symmetry spectrum rather than an independent manifestation of the two-nucleon interaction in the channel of zero isospin.

I. INTRODUCTION

The article by Satula et al. was pivotal in discussions around the turn of millennium of the roles of different parts of the effective two-nucleon interaction in the formation of patterns of binding energies of nuclei with approximately equally many neutrons and protons. The authors calculate ground state energies in the $sd$ and $pf$ shells from the isobarically invariant Hamiltonians USD and KB3, respectively. For the analysis of their results they use two different differential combinations of the ground state energies of doubly even nuclei with pairs of the neutron number $N$ and the proton number $Z$ in the vicinity of a given pair with $N = Z$. One combination involves three nuclei with the same $A = N + Z$, and one involves eight nuclei spanning five different $A$. Our present study focuses on the first type of analysis, which displays the pattern of ground state energies along a single isobaric chain.

For a given $A$ we denote by $E(T)$ the calculated lowest energy of a state with isospin $T$, which is also the ground state energy when $N$ and $Z$ are even and $|N - Z|/2 = T$. Two parameters $\epsilon_W$ and $\kappa$ are defined by

$$E(T) = E(0) + \epsilon_W + \frac{1}{2} \kappa T^2 \quad \text{for } T = 2 \text{ and } 4.$$ (1)

Satula et al. consider the parameter $\epsilon_W$, which they take as a measure of the so-called Wigner term included by Myers and Swiatecki in their semi-empirical mass formula. For $A = 24$ and 48 they calculate $\epsilon_W$ both for the full Hamiltonian and for variants where components of the two-nucleon interaction in the isospin $T' = 0$ channel are turned off successively. (We denote in this article the two-nucleon isospin by $T'$ to distinguish it from the isospin $T$ of the nucleus.) They find that when the entire $T' = 0$ interaction is turned off, $\epsilon_W$ vanishes completely or almost completely. They conclude that the $T' = 0$ interaction is responsible for the Wigner term.

An analysis of the sequence of energies $E(T)$, $T = 0, 2, 4$, briefly the symmetry spectrum, in terms of the single parameter $\epsilon_W$ is incomplete. To complete the analysis and examine in particular the behavior of the parameter $\kappa$ in Eq. (1) we redid the calculations of Ref. [1]. The results are presented in Secs. II and III. In Sec. II the dependence of $E(4) - E(0) = \epsilon_W + 8\kappa$ on the set of components included in the interaction is explored. It is found to vary as drastically as $\epsilon_W$. Most of this variation is shown to have straightforward explanations in terms of schematic approximations. The parameter $\epsilon_W$ is then discussed in Secs. III. Its variation follows mostly that of $E(4) - E(0)$, which leads to the suggestion that the reduction of $\epsilon_W$ observed in Ref. [1] be seen mainly as a side effect of a narrowing of the symmetry spectrum rather than an independent manifestation of the $T' = 0$ interaction.

Previously, Bentley et al. analyzed along the same lines results from a calculation for $A = 48$ using the $1f_{7/2}$ valence space. In Sec. IV we briefly summarize their results and present them in a form compatible with the presentation of our present ones. They are found mainly representative for the KB3 results, which corroborates that the $1f_{7/2}$ shell seems to dominate these results. Section V concludes the article.

II. WIDTH OF THE SYMMETRY SPECTRUM

Calculations for $A = 24$ with the Hamiltonian USD and $A = 48$ with the Hamiltonian KB3 were done with the program NUSHELLX@MSU for $T = 0, 2$, and 4. In each case the lowest energy was calculated from the full Hamiltonian and with components of the two-nucleon interaction successively turned off. First the components with $T' = 0$ and $J = 1, 2, \ldots, J_{\text{max}}$ were turned off in this order, and then the components with $T' = 1$ and $J = J_{\text{max}} - 1, J_{\text{max}} - 2, \ldots, 1$ in this order, where $J_{\text{max}}$ is the maximal two-nucleon angular momentum of the valence space. In the $sd$ space, $J_{\text{max}} = 5$, and in the $pf$ space, $J_{\text{max}} = 7$. Note that in these valence spaces,
Both \( J_{\text{cut}} = J_{\text{max}} \) and \( J'_{\text{cut}} = J_{\text{max}} \) mean that the entire \( T' = 0 \) interaction is turned off, so when this case is referred to in our figures and tables, the label may be understood as a value of \( J_{\text{cut}} \) or a value of \( J'_{\text{cut}} \); this makes no difference. In the calculations, first \( J_{\text{cut}} \) ascends from 0 to \( J_{\text{max}} \), when it equals \( J'_{\text{cut}} \); thence \( J'_{\text{cut}} \) descends to 0.

The calculated energies \( E(T) \) are shown in Figs. 1 and 2 as functions of \( J_{\text{cut}} \) or \( J'_{\text{cut}} \). The energy \( E(4) \) is constant as long as only \( T' = 0 \) interactions are turned off. This is because to make \( T = 4 \) with eight valence nucleons, every pair must have \( T'i = 1 \), so these states do not feel the \( T' = 0 \) interaction. In every case of \( A, J_{\text{cut}} \), and \( J'_{\text{cut}} \), the energy \( E(T) \) increases with \( T \). The difference \( E(T) - E(0) \) is the symmetry energy. For \( A = 48 \) the symmetry spectrum shrinks to degeneracy in the absence of interactions because the system of eight \( 1f_{7/2} \) nucleons can have \( T = 4 \). For \( A = 24 \), one has in this case \( E(0) = E(2) \) while \( E(4) - E(2) \) equals twice the spacing of the \( 1d_{5/2} \) and \( 2s_{1/2} \) levels.

### A. \( A = 48 \)

We begin our discussion with Fig. 2 because the case \( A = 24 \) is slightly more complicated. First notice that the symmetry spectrum almost does not change when components of the interaction with even \( T' + J \) are turned off. This is a natural consequence of the absence of pairs with these quantum numbers from the \( 1f_{7/2} \) shell. Thus, in a calculation of the energy by perturbation theory starting from a configuration in the \( 1f_{7/2} \) shell, terms involving the said components would not appear before the third order in the interaction.

#### 1. Monopole interaction

One can attempt to simulate the variation of the symmetry spectrum by schematic interactions. First assume that the two-nucleon interaction \( v \) depends on nothing but \( T' \), that is,

\[
v = v_m = - \sum_{T'} G_{T'} P_{T'},
\]

where \( G_{T'} \) are constants, and \( P_{T'} \) projects on isospin \( T' \). Using

\[
P_{T'} = \begin{cases} \frac{1}{2} - t_1 \cdot t_2, & T' = 0, \\ \frac{3}{2} + t_1 \cdot t_2, & T' = 1, \end{cases}
\]

where \( t_i \) is the nucleonic isospin, one gets by an easy calculation (given, for example, in Ref. [7]) that in a state with isospin \( T \), the interaction term in the Hamiltonian has expectation value

\[
E_{\text{int}} = -\frac{n}{8}[(n + 2)G_0 + 3(n - 2)G_1] + \frac{1}{2}(G_0 - G_1)T(T + 1) := -\sum_{T'} G_{T'} n_{T'},
\]
where \( n = 8 \) is the number of valence nucleons. The factors \( n_{T'} \) in the last expression are the numbers of bonds with isospin \( T' \) in the system with \( n \) nucleons and isospin \( T \). In any state of this system, the two-nucleon density matrix \( \rho_2 \) satisfies

\[
\text{tr} \rho_2 P_{T'} = n_{T'} \tag{5}
\]

and

\[
E_{\text{int}} = \text{tr} \rho_2 \nu. \tag{6}
\]

It follows that Eq. (4) is exact for lowest-energy states with isospin \( T \) provided

\[
-G_{T'} = \frac{\text{tr} \rho_2(T) P_{T'} \nu}{\text{tr} \rho_2(T) P_T}, \tag{7}
\]

where \( \rho_2(T) \) is the two-nucleon density matrix in one such state. By the charge independence of \( \nu \), the centroid (7) does not depend on which particular substate of the lowest-energy isospin multiplet generates \( \rho_2(T) \).

The expectation value of the total Hamiltonian is

\[
E = \text{tr} \rho_1 h + E_{\text{int}}, \tag{8}
\]

where \( \rho_1 \) and \( h \) are the density matrix and Hamiltonian of a single nucleon. Provided the first term in this expression is constant, the symmetry spectrum is thus given by this constant plus the expression (4) in so far as the centroids (7) do not depend on \( T \). From a deviation of the symmetry spectrum from this form, and from the \((T+1)\) proportionality of the symmetry energy, in particular, on can infer, conversely, that either the variation of \( T \) redistributes the nucleons on the single-nucleon levels, or the centroid (7) depends on \( T \), or both.

The interaction (2) is seen from Eq. (6) to be a monopole interaction. It was shown by Caurier et al. [8] that the monopole part of the interaction in the pair of orbits with angular momenta \( j_1 \) and \( j_2 \) is given by

\[
-G_{T'} = \frac{\sum_{j_1} j(2J + 1) V_{T' j(j_1 j_2 j_1 j_2)}}{\sum_{j_1} j(2J + 1)}, \tag{9}
\]

where \( V_{T' j(j_1 j_2 j_1 j_2)} \) is the interaction matrix element in the notation of, for example, Eq. (17.46) of Ref. [9]. The summations in Eq. (9) run over the \( J \) that are compatible with \( j_1 \), \( j_2 \), and \( T' \), and the weight \( 2J+1 \) is proportional to the dimension of the two-nucleon space with quantum numbers \( j_1 \), \( j_2 \), \( T' \), and \( J \). Centroids covering a part of the valence space can also be defined,

\[
-G_{T'} = \frac{\sum_{j_1, j_2 \in S, j_1 \leq j_2} j(2J + 1) V_{T' j(j_1 j_2 j_1 j_2)}}{\sum_{j_1, j_2 \in S, j_1 \leq j_2} j(2J + 1)}, \tag{10}
\]

where \( S \) is some set of orbits. In the configuration of the fully occupied space \( S \) (which has \( T = 0 \), the two-nucleon density matrix \( \rho_2 \) is the unit matrix. The centroids (7) and (10) then coincide, so one gets the interaction energy \( E_{\text{int}} \) in this configuration by inserting the expressions (10) into the expression (4). We shall explore what may be learned from assuming identity of these centroids also in the present case of an open \( 1f_{7/2} \) shell.

We now describe the scheme of calculation employed for this purpose in its general form because it will be used also in the case of \( A = 24 \). The energy \( E(T) \) is calculated by Eq. (5), where \( \text{tr} \rho_1 h \) is taken to be constant. For the interaction energy \( E_{\text{int}} \) we use the expression (4) with \( -G_{T'} \) given by Eq. (10). We denote the energy \( E(T) \) obtained in this approximation, which we call the monopole approximation, by \( E_m(0) \) to distinguish it from the energy given by the full Hamiltonian. For \( A = 48 \), we take the energy \( E_m(0) \) and \( G_0 - G_1 \) for the \( 1f_{7/2} \) orbit have zero energy in the KB3 Hamiltonian. The results for two different \( S \) are shown in Table I. One set consists of only the orbit \( 1f_{7/2} \), and one set includes the entire \( pf \) shell. Shown in the table are \( E_m(0) \) and \( G_0 - G_1 \), which is the coefficient of \( \frac{1}{2} T(T+1) \) in Eq. (4).

The calculation with \( 1f_{7/2} \) centroids is seen to reproduce quite well the energies \( E(0) \) in Fig. 2 except for a general upwards shift by roughly 10 MeV. The difference \( E_m(4) - E(4) \) is about 1 MeV almost independently of \( J_{\text{cut}} \) and \( J'_{\text{cut}} \). Necessarily, it does not depend on \( J_{\text{cut}} \). Thus, roughly, \( E_m(T) - E(T) \) depends on \( T \) alone for \( T = 0 \) and 4. More precisely, \( E_m(0) - E(0) \) is 13.7 MeV for \( J_{\text{cut}} = 0 \), and 8.3 MeV for \( J_{\text{cut}} = 1 \). This decrease may be understood by observing that in the diagonalization, the system seeks to utilize the most attractive part of the interaction rather than its average, and the full Hamiltonian offers more opportunities for this optimization than the Hamiltonian with only the \( J = 0 \) interaction.

| \( J_{\text{cut}} \) or \( J'_{\text{cut}} \) | \( E_m(0) \) | \( G_0 - G_1 \) | \( E_m(0) \) | \( G_0 - G_1 \) |
|-----------------|-----------|-----------|-----------|-----------|
| 0               | -19.36    | 1.26      | -14.01    | 1.41      |
| 1               | -18.38    | 1.16      | -12.40    | 1.25      |
| 2               | -10.77    |           | 0.80      |           |
| 3               | -16.71    | 1.00      | -7.92     | 0.61      |
| 4               | -14.54    | 0.78      | -3.27     | 0.33      |
| 5               | -4.33     | -0.24     | 0.04      | 0.00      |
| 6               | -5.84     | -0.32     | 0.55      | 0.03      |
| 7               | -4.74     | -0.26     | -0.79     | -0.04     |
| 8               | -1.23     | -0.07     | -0.36     | -0.02     |
| 9               | -0.40     |           | -0.40     | -0.02     |
Even the pattern of changes of \( E(0) \), including, in particular, a decrease when the \( T' = 1, J = 6 \) interaction is turned off, and the approximate size of each shift, is reproduced in detail. It follows from Eqs. (3), (4), and (9) that each shift of \( E_m(0) \) is proportional to only one interaction matrix element, namely the \( 1f_{7/2} \) shell matrix element with the \( T' \) and \( J = 0 \) that are turned off. The coefficient of proportionality is negative and depends on \( T' \) and \( J \).

The calculation with \( pf \) shell centroids is much less successful. The general slope of increase of \( E_m(0) \) is further below that of \( E(0) \) than in the \( 1f_{7/2} \) case, and, at variance with the KB3 results, fairly large shifts occur when components of the interaction that are not active in the \( 1f_{7/2} \) shell are turned off. The shift when the \( T' = 0, J = 7 \) interaction is turned off is much smaller than in the KB3 results and the calculation with \( 1f_{7/2} \) centroids. The deviation from the latter is the natural result of the \( 1f_{7/2} \) matrix element being the only one with these quantum numbers. This single matrix element indeed has a much smaller relative weight in the \( pf \) shell centroid than in the corresponding \( 1f_{7/2} \) centroid. Our analysis thus suggests a dominance of the \( 1f_{7/2} \) shell in the KB3 results.

2. Pairing

As is evident from Fig. 2 and Table I, even with \( 1f_{7/2} \) centroids, the monopole approximation fails badly in more than one respects to reproduce the KB3 results. We mentioned already the discrepancy \( E_m(0) - E(0) \approx 10 \text{ MeV} \). Moreover, the monopole approximation gives decreasing symmetry spectra after all \( T' = 0 \) interactions have been turned off. This is, in fact, a direct consequence of \( E_{\text{int}}(0) \) and \( G_0 - G_1 \) having equal signs by the expression (11) when \( G_0 = 0 \). In particular \( E_m(0) < 0 \) implies \( G_0 > G_1 < 0 \). A path to simultaneous remedy of both failures is suggested by the observation that the \( J = 0 \) part of the KB3 interaction is similar to the pairing interaction \( v_p \), which, in a single oscillator shell, has matrix elements

\[
V_{T'J}(j_1j_2j'_1j'_2) = -\frac{1}{2}\delta_{j0}G\sqrt{(2j_1+1)(2j'_1+1)}
\]

with a constant \( G \). (Note that in a single oscillator shell, \( \delta_{j0} \) vanishes unless \( j_1 = j_2, j'_1 = j'_2, \) and \( T' = 1 \).) Indeed every KB3 matrix element \( V_{10}(jj'j') \) is negative, and the ratio of the largest and the smallest \( G \) extracted from these matrix elements by means of Eq. (11) is 4.6. The value \( G = 0.48 \text{ MeV} \) extracted from the \( 1f_{7/2} \) matrix element is close to both the arithmetic and the geometric mean.

For \( G > 0 \) the lowest eigenvalue of \( v_p \) with a given \( T \) is

\[
E_p(T) = \frac{1}{2}G\left[ n\left(\frac{n}{4} - \frac{\Omega}{2} - \frac{3}{2}\right) + T(T+1) \right],
\]

where \( 4\Omega \) is the valence space dimension. We notice in passing that the fact that the expression (12) cannot be written in the form of the expression (1) entails that in the lowest-energy eigenstates of \( v_p \), the centroid (4) varies with \( T \). In the pure monopole approximation, every state with the same \( T \) has the same energy. Therefore, if one sets \( v = v_m + v_p \), one gets just \( E(T) = E_m(T) + E_p(T) \). In this approximation, \( -E_p(T) \) should thus account for the discrepancies \( E_m(T) - E(T) \). In particular, this difference should not depend on \( J_{\text{cut}} \) and \( J'_{\text{cut}} \), which is roughly what was observed for \( T = 0 \) and 4.

When the \( J = 0 \) interaction is approximated by \( v_p \), it must be excluded from the centroids (10) for consistency. This amounts to subtracting the last row of results in Table I from every row. In the following, every reference to results in the monopole approximation is therefore to the \( 1f_{7/2} \) results in the table thus modified. In the monopole plus pairing approximation, the coefficient of \( \frac{1}{2}E(T+1) \) is \( G_0 - G_1 + G \) by Eqs. (4) and (12). This is seen to be always positive when \( G = 0.48 \text{ MeV} \). We now have a correlation between the variations of \( E(0) \) and the width \( E(4) - E(0) = 10(G_0 - G_1 + G) \) of the symmetry spectrum: When an attractive component with \( J > 0 \) is turned off, either \( G_0 \) or \( G_1 \) decreases and, accordingly, \( E(0) \) increases. On the other hand, the effect on \( E(4) - E(0) \) depends on whether the component that is turned off has \( T' = 0 \) or 1. If it has \( T' = 0 \), then \( G_0 \), and therefore \( E(4) - E(0) \), decreases, as well (as required when \( E(0) \) approaches the constant \( E(4) \) from below). If it has \( T' = 1 \), then \( G_1 \) decreases, so \( E(4) - E(0) \) increases. The changes are opposite when the component that is turned off is repulsive. Exactly this pattern is found in Fig. 3.

Now consider the case when every interaction but the one in the \( J = 0 \) channel has been turned off. In our approximation, the width of the symmetry spectrum then is \( E_p(4) - E_p(0) = 10G = 4.8 \text{ MeV} \). This fairly agrees with \( E(4) - E(0) = 6.1 \text{ MeV} \) in the KB3 calculation. With \( \Omega = 10 \), corresponding to the \( pf \) shell, Eq. (12) gives \(-E_p(0) = 18.2 \text{ MeV} \), which is much larger than \(-E(0) = 9.7 \text{ MeV} \) from KB3. This must be due to the spread of the single-nucleon levels, which makes the higher orbits participate less effectively in the pair correlations. Taking, on the other hand, \( \Omega = 4 \), corresponding to the \( 1f_{7/2} \) shell alone, gives \(-E_p(0) = 6.7 \text{ MeV} \), which is too small. The \( J = 0 \) component of the KB3 interactions then causes some scattering out of the \( 1f_{7/2} \) shell, so the dominance of the \( 1f_{7/2} \) shell does not mean that the nucleons stay there. Despite the scattering out of it by the \( J = 0 \) interaction, the contribution to the total energy of the rest of the interaction seems, however, mainly determined by its \( 1f_{7/2} \) part.

On a final note, we point out that by Eqs. (4) and (12), the symmetry energy is proportional to \( T(T+1) \) in the monopole plus pairing approximation.
results similar to those between the calculation with pf shell centroids and the KB3 results. The present analysis thus suggests a dominance of the 1d_{5/2} + 2s_{1/2} valence subspace in the USD results.

With 1d_{5/2} centroids, we get $E_m(0) - E(0) = 14.9$ MeV and $E_m(4) - E(4) = -7.8$ MeV for $J_{cut} = 0$, and $E_m(0) - E(0) = 10.4$ MeV and $E_m(4) - E(4) = -1.1$ MeV for $J_{cut} = 1$. (The increase of $E_m(4) - E(4)$ mainly occurs between $J_{cut} = 4$ and 3, where, as seen from Fig. 4, $E(4)$ drops even more than $E(0)$ due to the removal of the repulsive $T' = 1, J = 3$ interaction in the configuration 1d_{5/2}2s_{1/2} while $E_m(4)$ stays constant.) As these changes are relatively small compared to the total increase of $E(0)$, we conclude, like for $A = 48$, that with 1d_{5/2} centroids, the difference $E_m(T) - E(T)$ is roughly independent of $J_{cut}$ and $J_{cut}'$ for $T = 0$ and 4.

The USD matrix elements $V_{10}(jj'jj')$ are negative and the ratio of the largest and smallest $G$ extracted from them by means of Eq. (11) is 2.1. The value $G = 0.94$ MeV given by the 1d_{5/2} diagonal matrix element is close to the arithmetic and geometric means. It renders $G_0 - G_1 + G$ positive for all $G_0 - G_1$ in Table II after subtraction of the last row of results from every row. The pattern of narrowing and widening of the symmetry spectrum then suggested by the calculation with 1d_{5/2} + 2s_{1/2} centroids materializes in Fig. 1. Again there is one exception: When the $T' = 1, J = 4$ interaction is turned off, the spectrum narrows slightly while it is predicted to widen. We have no explanation of this. It may be related to another unexplained anomaly for $J_{cut}' = 5$ and 4 discussed in Sec. III.

The pairing energies $E_p(0)$ calculated from Eq. (12) with $\Omega = 6, 4, 3$, corresponding to the full $sd$ shell and the 1d_{5/2} + 2s_{1/2} and 1d_{5/2} valence spaces are 20.7, 13.2, and 9.4 MeV as compared to an additional binding energy of 13.7 MeV induced by turning on the $J = 0$ component of the USD interaction. This suggests that mainly the 1d_{5/2} and 2s_{1/2} shells take part in pair correlations. The resulting width of the symmetry spectrum is 9.6 MeV as compared to 10G = 9.4 MeV.

### III. Term $\epsilon_W$

Figures 3 and 4 shows the term $\epsilon_W$ in Eq. (11) given by the energies in Figs. 1 and 2. Readers may verify that the left hand parts of these plots coincide with those in Fig. 2 of Ref. 1 when $\epsilon_W, J_{cut} = 0$ is normalized to 1. Also shown in Figs. 3 and 4 is the ratio

$$r_W = \frac{\epsilon_W}{E(4) - E(0)}.$$  

(13)

For both $A$ one gets $r_W = 0.113$ for $J_{cut} = 0$ and $r_W = 0.063$ for $J_{cut}' = 1$. The latter value is close to what results when the symmetry energy is proportional to $T(T + 1)$, namely $r_W = 1/15 = 0.067$. Thus, despite the single-nucleon spectra being non-degenerate and the $J = 0$ interactions being not exact pairing interactions,
the $J'_{\text{cut}} = 1$ Hamiltonians are sufficiently close to the pairing Hamiltonian with degenerate single-nucleon levels to almost maintain the $T(T + 1)$ proportionality of the symmetry energy.

There is a one-to-one correspondence between the ratio $r_W$ and the parameter $x$ in

$$E(T) = E(0) + \frac{T(T + x)}{2\theta} \quad \text{for } T = 2 \text{ and } 4,$$

the precise relation being

$$x = \frac{12r_W}{1 - 3r_W}.$$  \hfill (15)

With $r_W = 0.113$, this gives $x = 2.05$, which is close to the values extracted by Bentley and Frauendorf from the measured binding energies in the $A = 24$ and 48 isobaric chains \cite{11}. This is natural because the USD and KB3 Hamiltonians were fitted partly to these binding energies. In the compilation by Bentley and Frauendorf, $x$ has, as a function of $A$, local maxima at $A = 24$ and 48. These local maxima result in Nilsson-Strutinski\j calculations by Bentley et al, which reasonably reproduce the data, from large deformations of $^{24}\text{Mg}$ and $^{48}\text{Cr}$ \cite{12}. It is not evident how this translates to interacting nucleons in a single oscillator shell.

The variation of $r_W$ from $J'_{\text{cut}} = 0$ to $J'_{\text{cut}} = 1$ is markedly different in the two cases of $A = 24$ and 48. For $A = 48$ it is gentle and almost monotonic while for $A = 24$ it proceeds violently from $J'_{\text{cut}} = 4$ to $J'_{\text{cut}} = 2$. If one neglects, however, the three cases of $J'_{\text{cut}} = 5, 4$, and 3, the variation for $A = 24$ is similar to the one for $A = 48$. We have no explanation of the apparent anomaly pertaining to these three $J'_{\text{cut}}$, but whatever its reason, it is evidently a non-generalizable feature of $A = 24$ and the USD Hamiltonian. It may be noticed that $r_W$ measures the partition of the interval from $E(0)$ to $E(4)$ by the energy $E(2)$. Thus,

$$r_W = \frac{4r_2 - 1}{3}, \quad r_2 = \frac{E(2) - E(0)}{E(4) - E(0)}. \quad (16)$$

Because of the factor $4/3$ in this relation, fairly small variations of $r_2$ result in variations of $r_W$ which are large in comparison with the typical $r_W \approx 0.1$.

In a gross picture, $r_W$ thus varies with $J'_{\text{cut}}$ gently within narrow limits. This suggests that the drastic decrease of $\epsilon_W$ observed in Ref. \cite{1} when the $T' = 1$ interaction is turned off be seen mainly as a side effect of the equally drastic decrease of $E(4) - E(0)$ rather than an independent manifestation of the $T' = 0$ interaction.

IV. OTHER HAMILTONIANS

We mention for completeness that we applied the analysis above also to some more recent Hamiltonians: in the $sd$ shell the Hamiltonians USDA and USDB \cite{12}, and in the $pf$ shell FPD6 \cite{13} and GXP1 \cite{14}. The results given by these other Hamiltonians are only marginally different from those of USD and KB3, and we shall not discuss them.

For $A = 48$, Bentley et al, made \cite{5} the analogous investigation using the $1f_{7/2}$ valence space and the interaction model I of Zamick and Robinson \cite{15}. As noted already, in this valence space, even $J$ is equivalent to $T' = 1$, and odd $J$ to $T' = 0$. The matrix elements of Zamick and Robinson are relative ones with the $J = 0$ matrix element normalized to zero. To make the interaction attractive, Bentley et al, subtracted from all the matrix elements the highest one, which is the $J = 6$ matrix element. By construction the interaction then is negative semidefinite and the $J'_{\text{cut}} = 7$ and 6 Hamiltonians are identical. Because the model has only one single-nucleon level, its energy plays no role. We name this Hamiltonian ZRI.

Figure 4 shows the plots similar to Figs. 3 and 2 derived from the results in Table I of Ref. \cite{5}. The ratio $r_W$ of the full Hamiltonian is smaller than that of KB3 and therefore also below the empirical value. Otherwise, the features of the KB3 results discussed above are well represented by this simpler model. This once again corroborates that the $1f_{7/2}$ shell seems to dominates these
As the $J = 0$ part of a single-$j$-shell interaction is an exact pairing interaction, it gives (with zero $1f_{7/2}$ level) energies in the form of Eq. (12), and $r_W = 1/15$ in particular.

V. CONCLUSIONS

For mass numbers $A = 24$ and 48 we examined with standard isobaric invariant Hamiltonians for the $sd$ and $pf$ shells the variation of the sequence of lowest energies for isospins $T = 0, 2,$ and 4, briefly the symmetry spectrum, when the two-nucleon interaction is turned off successively in channels with definite isospin and angular momentum. By definition, these energies form the symmetry spectrum, and the energy in excess of the $T = 0$ energy is the symmetry energy. When only the angular momentum $J = 0$ component of the two-nucleon interaction remains, the symmetry spectrum is close to that of the pairing Hamiltonian with degenerate single-nucleon energies, whose symmetry energy is proportional to $T(T + 1)$. Turning on components with $J > 0$ and negative diagonal matrix element in the $1d_{5/2}^2$ or $1f_{7/2}^2$ configuration decreases the $T = 0$ energy. It widens the symmetry spectrum when the component that is turned off has isospin $T' = 0$, and narrows it when $T' = 1$. The effects are opposite if the matrix element is positive. This behavior was explained in terms of schematic approximations. A modification for $A = 24$ of this general behavior caused by the interaction in the $1d_{5/2}2s_{1/2}$ configurations was discussed.

By and large, the ratio $r_W$ of the term $\epsilon_W$ in Eq. (1), which was taken in Ref. [1] as a measure of the so-called Wigner term in mass formulas, to the width of the symmetry spectrum varies gently within narrow limits when components of the two-nucleon are turned off. This suggests that the decrease of $\epsilon_W$ observed in Ref. [1] when the $T' = 0$ interactions are turned off should be seen mainly as a side effect of a narrowing of the symmetry spectrum rather than an independent manifestation of the $T' = 0$ interaction. This narrowing is well understood by the aforesaid analysis. An extraordinary violent variation of $r_W$ when some components are turned off for $A = 24$ remains unexplained but is certainly a non-generalizable feature of this particular case.

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