A differential neural network learns stochastic differential equations and the Black-Scholes equation for pricing multi-asset options

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Abstract

Neural networks with sufficiently smooth activation functions can approximate values and derivatives of any smooth function, and they are differentiable themselves. We improve the approximation capability of neural networks by utilizing the differentiability of neural networks; the gradient and Hessian of neural networks are used to train the neural networks to satisfy the differential equations of the problems of interest. Several activation functions are also compared in terms of effective differentiation of neural networks. We apply the differential neural networks to the pricing of financial options, where stochastic differential equations and the Black-Scholes partial differential equation represent the relation of price of option and underlying assets, and the first and second derivatives, Greeks, of option play important roles in financial engineering. The proposed neural network learns – (a) the sample paths of option prices generated by stochastic differential equations and (b) the Black-Scholes equation at each time and asset price. Option pricing experiments were performed on multi-asset options such as exchange and basket options. Experimental results show that the proposed method gives accurate option values and Greeks; sufficiently smooth activation functions and the constraint of Black-Scholes equation contribute significantly for accurate option pricing.

Keywords: differential neural networks, stochastic differential equations, the Black-Scholes equation, option pricing, smooth activation functions.
1. Introduction

Neural networks (NNs) have found successful applications such as image understanding, handwriting recognition, speech recognition, and drug discovery [Krizhevsky et al. (2012); Le et al. (2012); Graves and Schmidhuber (2009); Deng and Yu (2014); Ramsundar et al. (2015)]. This success is due to the universal approximation capability of NNs, approximating the function accurately without assuming any mathematical relationship between the input and the output of a function. Whereas no mathematical relationships are found for the foregoing applications, strong governing differential equations are presented in many fields such as finance, fluid dynamics, quantum mechanics, and diffusion phenomena. For these problems, the approximation capabilities of NNs can be improved when the differential relationships are included in the modeling.

Neural networks with sufficiently smooth activation functions can approximate a function and its derivatives [Cybenko (1989); Hornik et al. (1990); Gallant and White (1992)]; they are also differentiable because the composition of differentiable functions is differentiable. Thus, we can obtain the gradient and Hessian of differentiable NNs, and use these derivatives to make the NNs themselves satisfy the differential equations of the problems of interest. To obtain accurate derivatives of NNs, we calculate the gradient and Hessian of NNs using automatic differentiation [Nocedal and Wright (2006); Abadi et al. (2015); Raissi (2018)], which is an exact method based on the chain rule for differentiating compositions of functions. Since the differentiation of NNs plays a key role in the present work, activation functions should be sufficiently smooth as well as effective for our applications. Although any smooth activation function makes NNs differentiable, we need the activation functions which show good practical performance; we compare several activation functions in terms of estimation accuracy for the value, gradient, and Hessian.

We need suitable benchmark problems to examine the effectiveness of our differential NN. We consider financial option pricing because the option pricing problem has a widely known mathematical relationship, the Black-Scholes equa-
Black and Scholes (1973), and the derivatives of option price, Greeks, are extensively studied since they have many practical applications such as hedging or speculating the future asset price Hull (2018); Shreve (2004); Glasserman (2004). Hence, the performance of the proposed differential NNs can be easily tested using the experiments on option pricing.

Most multidimensional options have no general closed-form solution for pricing and they are priced by numerical approximation techniques. Various numerical methods have therefore been developed to solve this problem such as finite differences methods (FDMs), Fourier methods, and Monte Carlo (MC) simulations Hull (2018); Shreve (2004); Carr et al. (2001); Glasserman (2004). FDMs approximate the discrete version of Black-Scholes partial differential equation (PDE); MC methods use the generated sequence of stock prices governed by the stochastic differential equation (SDE). The present work combines the information of the PDE and SDE, and trains NNs using – (a) the option price generated by the SDE and (b) the constraint of the Black-Scholes equation. As a result, the proposed method becomes mesh-free method like MC simulations, and does not suffer from the curse of dimensionality associated with high-dimensional FDMs. Furthermore, Greeks are easily computed by differentiating the neural network because the proposed method represents the solution of Black-Scholes equation in the form of a neural network.

NNs have also been applied for option pricing problems Gencay and Min Qi (2001); Kohler et al. (2010); Sirignano and Spiliopoulos (2018); E et al. (2017); Becker et al. (2019). Our approach relates to methods of differentiating NNs used in Sirignano and Spiliopoulos (2018); Raissi (2018). We use automatic differentiation method to compute the gradient and Hessian of NNs, while Sirignano and Spiliopoulos (2018) use a MC method to approximate the Hessian; automatic differentiation is an exact operation and not a kind of approximation methods used in MC methods and FDMs Nocedal and Wright (2006); Abadi et al. (2015). The present work generates training data using the SDE which gives more realistic evolution of asset prices while Sirignano and Spiliopoulos (2018) use random samples from the region where the function is defined. Raissi
uses automatic differentiation method to compute the gradient of NNs and SDE to generate training data; but, the loss function for training NN does not contain PDE and the Hessian of NNs are not used. In addition, we adopt sufficiently smooth activation function softplus \cite{Glorot11} for the efficient differentiation of NNs based on the performance comparison experiments, while \cite{Sirignano18} use tanh and \cite{Raissi18} uses sine as an activation function.

The remainder of the paper is organized as follows. We discuss the related works on multidimensional option pricing and numerical approximation methods in Section 2. The proposed method is developed for learning stochastic differential equations and the Black-Scholes equation for option pricing in Section 3. The validity of the proposed method is demonstrated in Section 4, and conclusions are drawn in Section 5.

2. Price and Greeks of multidimensional options

A financial option is a contract between the seller and the buyer (holder of the option). A European call option gives the holder the right but no obligation to buy the risky assets at expiration time \( T \) for a price that is agreed on now, the strike price \( K \). The option itself has a price because it give the holder a right. Let the stock price \( S_i(t) \) be a geometric Brownian motion,

\[
dS_i(t) = rS_i(t)dt + \sigma_iS_i(t)d\tilde{W}_i(t),
\]

where \( t \) denotes time, \( r \) risk-free interest rate, \( \sigma_i \) the volatility of the asset price, and \( \tilde{W}_i \) a Brownian motion under a risk-neutral probability measure with covariance \( \text{Cov}(\tilde{W}_i(t), \tilde{W}_j(t)) = \rho_{ij}t \). The correlated Brownian motion \( \tilde{W}_i(t) \) can be expressed as \( \sqrt{t}L_iZ \) where \( Z \) is a standard \( n \)-dimensional normally distributed vector and \( L_i \) is the \( i \)th row of \( L \); \( L \) represents any matrix such that \( \Sigma = LL' \) where the \((i, j)\) component of \( \Sigma \) is \( \rho_{ij} \). We choose \( L \) to be the Cholesky factor of \( \Sigma \). Then the stock prices between time \( t_1 \) and \( t_2 \) are related by

\[
S_i(t_2) = S_i(t_1) \exp((r - \frac{1}{2}\sigma_i^2)(t_2 - t_1) + \sigma_i\sqrt{t_2 - t_1}L_iZ) \quad (2)
\]
2.1. Exchange option

An exchange option gives the holder the right to exchange one asset for another in given time in the future and it is commonly seen in the energy market. The payoff of exchange option is

$max(S_1(T) - S_2(T), 0). \quad (3)$

Margrabe (1978) developed an analytical solution formula for the price of this option. The price of exchange option at time $t$ is given by

$$u(t, S_1, S_2) = S_1 N(d_1) - S_2 N(d_2), \quad (4)$$

where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_1\sigma_1\sigma_2}$, $d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S_1}{S_2} \right) + \frac{\sigma^2}{2}(T-t) \right]$, $d_2 = d_1 - \sigma \sqrt{T-t}$, and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} z^2} dz$.

Greeks are the sensitivities of the option price to the movement of various parameters. They are used for risk management; the risk in a short position in an option is offset by holding delta units of each underlying asset, where the delta is the partial derivative of the option price with respect to the current price of that underlying asset. Sensitivities with respect to other parameters are also widely used to measure and manage risk. The Greeks of exchange option is obtained by differentiating Eq.(4). Let $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right)$, differentiating Eq.(4) by $S_1$ and using the property $\phi(d_1)/\phi(d_2) = S_2/S_1$ gives the delta

$$\Delta = \frac{\partial u}{\partial S_1} = N(d_1). \quad (5)$$

In some other hedging strategy, we need to hedge away the risk due to the changes of underlying asset’s delta; the gamma is defined by

$$\Gamma = \frac{\partial \Delta}{\partial S_1} = \frac{\partial^2 u}{\partial S_1^2} = \frac{\phi(d_1)}{S_1 \sigma \sqrt{T-t}}. \quad (6)$$

The time decay of the value for an option is called theta, given by

$$\Theta = \frac{\partial u}{\partial t} = -\frac{\sigma}{4\sqrt{T-t}} \left[ S_1 \phi(d_1) + S_2 \phi(d_2) \right]. \quad (7)$$
2.2. Basket option

A European basket call option gives the holder to buy a group of underlying assets at the same time. The price of the option at maturity are given by

\[
C(T, S_1(T), S_2(T), \ldots, S_n(T)) = \max(\sum_{i=1}^{n} w_i S_i(T) - K, 0)
\]  

(8)

where \( w_i \) denotes the quantity of \( i \)th asset in basket option contracts. Since this option has no exact formula for price and Greeks, these values are approximated by numerical methods such as finite difference methods (FDMs) and Monte Carlo (MC) simulations [Hull (2018); Shreve (2004); Glasserman (2004)].

While FDMs are accurate in low dimensional problems, they become infeasible in higher dimensions due to increased number of grid points and numerical instability. MC methods are versatile for handling general type of options, multi-asset, and path dependent problems. But, the MC method converges slow at rate \( O(1/\sqrt{M}) \), where \( M \) denotes the number of random samples.

FDMs approximate the price of option, \( u(t, S_1, S_2, \ldots, S_n) \), on the basis of the Black-Scholes equation [Hull (2018); Shreve (2004)],

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{i,j} S_i S_j \frac{\partial^2 u}{\partial S_i \partial S_j} + r \sum_{i=1}^{n} S_i \frac{\partial u}{\partial S_i} = r u,
\]

(9)

where \( r \) denotes risk-free interest rate, and \( (\sigma_i \sigma_j \rho_{i,j})_{1\leq i,j\leq n} \) denotes the covariance matrix of the stock prices. FDMs discretize the stock price and time dimensions and use the option values at the time of expiration. We use central difference equation to obtain the first and second derivatives with respect to stock price, and forward difference equation to obtain time derivatives. The Greeks at time zero are obtained using the solved \( u(\cdot) \). The delta is given by

\[
\Delta_i = \frac{u(0, S_1, \ldots, S_i + \delta_s, \ldots, S_n) - u(0, S_1, \ldots, S_i - \delta_s, \ldots, S_n)}{2\delta_s},
\]

(10)

where \( \delta_s \) denotes a stock price discretization unit. The gamma is given by

\[
\Gamma_i = \frac{u(0, \ldots, S_i + \delta_s, \ldots) - 2u(0, \ldots, S_i, \ldots) + u(0, \ldots, S_i - \delta_s, \ldots)}{\delta_s^2}
\]

(11)

The theta is given by

\[
\Theta = \frac{u(\delta_t, S_1, \ldots, S_n) - u(0, S_1, \ldots, S_n)}{\delta_t},
\]

(12)
Greeks | Expectation of the following formulas
---|---
**Exchange option**
\[ \Delta = e^{-rT} \frac{S_1(T)}{S_1(0)} \mathbb{1}\{S_1(T) > S_2(T)\} \]
\[ \Theta = e^{-rT} \left[ S_1(T) \left( \frac{\sigma_1 L_1 Z}{\sqrt{T}} - \frac{\sigma_2 L_2 Z}{\sqrt{T}} \right) - S_2(T) \left( \frac{\sigma_2 L_2 Z}{\sqrt{T}} - \frac{\sigma_2 L_2 Z}{\sqrt{T}} \right) \right] \mathbb{1}\{S_1(T) > S_2(T)\} \]
\[ \Gamma = e^{-rT} \left[ \frac{Z'}{\sqrt{T}} \frac{L_1 A^{-1}}{S_1(0)} \right] \mathbb{1}\{S_1(T) > S_2(T)\} \]

**Basket option**
\[ \Delta_i = e^{-rT} \frac{S_i(T)}{S_i(0)} w_i \mathbb{1}\{\sum_{i=1}^n w_i S_i(T) > K\} \]
\[ \Theta = e^{-rT} \left[ -rK + \sum_{i=1}^n w_i S_i(T) \left( \frac{\sigma_i L_i Z}{\sqrt{T}} - \frac{\sigma_i L_i Z}{\sqrt{T}} \right) \right] \mathbb{1}\{\sum_{i=1}^n w_i S_i(T) > K\} \]
\[ \Gamma_i = e^{-rT} \left[ \frac{Z'}{\sqrt{T}} \frac{L_i A^{-1}}{S_i^2(0)} \right] \left( w_i S_i(T) - \sum_{i=1}^n w_i S_i(T) + K \right) \times \mathbb{1}\{\sum_{i=1}^n w_i S_i(T) > K\} \]

Table 1: Greeks estimation by MC simulation.

where \( \delta_t \) denotes a time discretization unit.

MC methods generate multiple simulated paths of stock price by Eq. (2). The price of options is obtained by averaging the payoff as follows.

\[ V(t) = E_{Q}[e^{-r(T-t)}V(T)|\mathcal{F}(t)], \tag{13} \]

where \( V(T) \) is the payoff function, \( \mathcal{F}(t) \) is a filtration for the Brownian motion \( \tilde{W}(t) \) in Eq. (1) and \( Q \) is the risk-neutral measure \( \text{Hull} (2018); \text{Shreve} (2004); \text{Glasserman} (2004) \). The Greeks estimation is a practical challenge in MC methods. In the present work, the pathwise derivative estimate is used for calculating the delta and theta, and combination of likelihood ratio method and pathwise derivative estimate is used for calculating gamma. The Greeks at time zero are given by averaging the formulas in Table 1. Derivations of these formulas are described in Appendix.
3. Learning stochastic differential equations and the Black-Scholes equation

3.1. Learning stochastic differential equations

We approximate the price of option \( u(t, S_1, S_2, \ldots, S_n) \) in Eq. (9) using a neural network. Expiry time \( T \) is divided into \( N (= 200) \) equal intervals \( t_0 = 0 < t_1 < t_2 < \cdots < t_N = T \). Multiple stock price \( S(t_k) = (S_1(t_k), S_2(t_k), \ldots, S_n(t_k)) \) is generated by Eq. (2)

\[
S_i(t_k) = S_i(t_{k-1}) \exp((-\frac{1}{2} \sigma_i^2)(t_k - t_{k-1}) + \sigma_i d\tilde{W}_i(t_k))
\] (14)

with an initial stock price \( S(t_0) = S(0) \) and \( d\tilde{W}_i(t_k) = \sqrt{t_k - t_{k-1}} L_i Z \). The price of option \( u(t_k, S(t_k)) \) is approximated by the neural network \( N(t, S) \)

\[
\begin{align*}
N^1(t_k, S(t_k)) &= f[w^1(t_k, S(t_k)) + b^1], \\
N^2(t_k, S(t_k)) &= f[w^2 N^1(t_k, S(t_k)) + b^2], \\
& \vdots, \\
N(t_k, S(t_k)) &= N^l(t_k, S(t_k)) = w^l N^{l-1}(t_k, S(t_k)) + b^l,
\end{align*}
\] (15)

where \( f \) is an activation function, \( w^l \) and \( b^l \) parameters of \( l \)th layer of the neural network. In the present work, the number of layers \( l \) is five, the number nodes 35 for layer \( l = 1, 2, 3, 4 \), and one in the last layer \( l = 5 \).

To train the neural network \( N(t, S) \), we need to know the target \( u(t, S) \). Although the \( u(\cdot) \) is of course unknown at \( t < T \), this function satisfies the following SDE when the stock price follows Eq. (1) [Hull (2018); Shreve (2004); Glasserman (2004)].

\[
du = (u_t + \sum_{i=1}^{n} r S_i u S_i + \frac{1}{2} S' H S) dt + \sum_{i=1}^{n} \sigma_i S_i u S_i d\tilde{W}_i,
\] (16)

where \( u_t = \frac{\partial u}{\partial t} \), \( S' = (S_1, S_2, \ldots, S_n) \), \( H_{ij} = \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 u}{\partial S_i \partial S_j} \) and \( u S_i = \frac{\partial u}{\partial S_i} \).

Based on this SDE and the derivatives of the neural network \( N(t, S) \), we generate the following \( \tilde{u}(t_k, S(t_k)) \) that approximates \( u(t_k, S(t_k)) \).
\( \tilde{u}(t_k, S(t_k)) = \mathcal{N}(t_{k-1}, S(t_{k-1})) + \{\mathcal{N}_t(t_{k-1}, S(t_{k-1})) + \sum_{i=1}^{n} rS_i(t_{k-1})\mathcal{N}_{S_i}(t_{k-1}, S_i(t_{k-1})) + \frac{1}{2} S(t_{k-1})'H_{k-1}S(t_{k-1})\}dt_k + \sum_{i=1}^{n} \sigma_i S_i(t_{k-1})\mathcal{N}_{S_i}(t_{k-1}, S_i(t_{k-1}))d\tilde{W}_i(t_k), \) \hspace{1cm} (17)

where \( \mathcal{N}_t \) and \( \mathcal{N}_{S_i} \) respectively denote partial derivatives of the neural network \( \mathcal{N}(t, S(t)) \) with respect to \( t \) and \( S_i \), 

\( (H_{k-1})_{ij} = \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 \mathcal{N}(t_{k-1}, S(t_{k-1}))}{\partial S_i \partial S_j} \), 

\( dt_k = t_k - t_{k-1} \) and 

\( d\tilde{W}_i(t_k) = \sqrt{t_k - t_{k-1}}L_iZ. \) 

This \( \tilde{u}(t_k, S(t_k)) \) is used to train \( \mathcal{N}(t_k, S(t_k)) \) with the following loss function which is minimized during training the neural network.

\( L_{SDE} = \sum_{k=1}^{N} [\tilde{u}(t_k, S(t_k)) - \mathcal{N}(t_k, S(t_k))]^2. \) \hspace{1cm} (18)

3.2. Learning the Black-Scholes equation

We enforce the neural network \( \mathcal{N}(t, S) \) to satisfy the Black-Scholes equation in Eq. (9) by minimizing the following representation of Black-Scholes equation in the form of a neural network

\( L_{BS} = \sum_{k=1}^{m} [\mathcal{N}_t(t_k, S(t_k)) + \sum_{i=1}^{n} rS_i(t_k)\mathcal{N}_{S_i}(t_k, S_i(t_k)) + \frac{1}{2} S(t_k)'H_kS(t_k) - \tau \mathcal{N}(t_k, S(t_k))]^2. \) \hspace{1cm} (19)

We need to differentiate the neural network with respect to variables \( t_k \) and \( S(t_k) \) for the calculation of Eq. (17) and Eq. (19). This differentiation is performed by using automatic differentiation. Automatic differentiation is an exact differentiation method, not an approximation method, so that it gives accurate differential results. It applies the chain rule for differentiating compositions of a set of elementary functions for which derivatives are known exactly Nocedal and Wright (2006); Abadi et al. (2015); Raissi (2018). Since software libraries like Tensorflow Abadi et al. (2015) already provide operations for automatic differentiation, we used the tensorflow.GradientTape operation in TensorFlow to calculate the gradient of neural network; tensorflow.GradientTape
operation is performed to the gradient of NN for the calculation of the Hessian of NN.

At expiration time $T$, true $u(T, S_1, S_2, \ldots, S_n)$ is given by the payoff function. To match the option value at maturity time $T$, the third loss function is

$$L_T = [N(T, S(T)) - h(T)]^2 \ast w_T,$$

(20)

where $h(T)$ denotes the payoff function such as Eq.(3) and Eq.(8), and $w_T$ denotes the weight for accurate approximation at $T$, $w_T = N/20(= 10.)$ as a default weight in this work. Finally, we use the following loss function that penalizes the neural network output deviations from the stochastic differential equation, the neural network differential structure deviations from Black-Scholes equation, and the neural network output deviations from the payoff function at expiration time $T$,

$$Loss = L_{SDE} + L_{BS} + L_T.$$  

(21)

3.3. Training and estimation procedures of the neural network

We call our neural network SDBS, because it is based on the stochastic differential equation and the Black-Scholes equation. We summarize the training procedure of SDBS in Algorithm 1. For updating the parameters of SDBS (line 12 in Algorithm 1), we use a stochastic gradient descent method, adam, that is based on adaptive estimation of first-order and second-order moments Kingma and Ba (2014). It is commonly observed that a monotonically decreasing learning rate results in a better performing model. The present work uses a PolynomialDecay schedule in Tensorflow and the learning rate is linearly decayed from $10^{-3}$ to $10^{-7}$. As the final trained model, we choose the model which has the minimum $Loss$ during training iterations.

The estimation procedure is similar to the training procedure. There are three differences: (1) in addition to the input of training procedure, the parameters of trained SDBS is loaded, (2) the learning rate is fixed by $10^{-7}$ in line 12 of Algorithm 1, (3) the line 13, 14, and 15 are replace by output $Price = N(t, S(t))$, $\Delta_i = N_{S_i}(t, S(t))$, $\Gamma_i = N_{S_iS_i}(t, S(t))$, $\Theta = N_i(t, S(t))$. 

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Algorithm 1 Training of SDBS.

1: **Input:** \( S(t_0), T, \sigma_i, r, \rho_{ij}, N, n\text{Epoch} \)

2: Calculate \( L \) representing \( \Sigma = LL', t_0 = 0 \)

3: **for** epoch=1, 2, \cdots, nEpoch **do**

4: Calculate \( N, N_t, N_{S_i}, H \) at \((t_0, S(t_0))\)

5: **for** \( k = 1, 2, \cdots, N \) **do**

6: \( t_k = t_{k-1} + T/N \)

7: Calculate \( \tilde{u}(t_k, S(t_k)) \) by Eq. (17)

8: Calculate \( S(t_k) \) by Eq. (14) and

9: Calculate \( N, N_t, N_{S_i}, H \) at \((t_k, S(t_k))\)

10: **end for**

11: Calculate \( \text{Loss} \) by Eq. (21)

12: Update the parameters of neural network by minimizing \( \text{Loss} \)

13: **if** \( \text{Loss} \) is reduced comparing with that in the previous epoch **then**

14: \hspace{1cm} Save parameters of SDBS

15: **end if**

16: **end for**
| Name  | Equation                                      | Range     | Order of continuity |
|-------|-----------------------------------------------|-----------|---------------------|
| sigmoid | \(1/(1 + e^{-x})\)                          | \((0, 1)\) | \(C^\infty\)       |
| tanh  | \((e^x - e^{-x})/(e^x + e^{-x})\)            | \((-1, 1)\) | \(C^\infty\)       |
| sin    | \(\sin(x)\)                                 | \((-1, 1)\) | \(C^\infty\)       |
| relu   | \(\begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x > 0. \end{cases}\) | \((0, \infty)\) | \(C^0\)            |
| elu    | \(\begin{cases} \alpha(e^x - 1) & \text{if } x \leq 0, \\ x & \text{if } x > 0. \end{cases}\) | \((-\alpha, \infty)\) | \(\begin{cases} C^1 & \text{if } \alpha = 1, \\ C^0 & \text{otherwise.} \end{cases}\) |
| selu   | \(\lambda \begin{cases} \alpha(e^x - 1) & \text{if } x \leq 0, \\ x & \text{if } x > 0. \end{cases}\) | \((-\lambda \alpha, \infty)\) | \(C^0\)            |
| softplus | \(\ln(1 + e^x)\)                         | \((0, \infty)\) | \(C^\infty\)       |

Table 2: Classification of activation functions

3.4. Activation functions for differentiable SDBS

Activation functions determine the output of a node and the characteristics of NNs. They have various shape, range, order of continuity, and magnitude of gradient. Every function in Table 2 has nonlinear shape because nonlinearity of the activation functions allows NNs to be universal function approximators \(\text{Cybenko (1989); Hornik et al. (1990)}\). The range of function is finite for sigmoid, tanh, and sin while infinite for relu \(\text{Nair and Hinton (2010), elu Clevert et al. (2015), selu Klambauer et al. (2017), softplus Glorot et al. (2011)}\). The boundedness and continuity of partial derivatives of the activation functions up to order \(m\) are required to approximate the functions with all partial derivatives up to \(m\) are continuous and bounded \(\text{Hornik et al. (1990); Gallant and White (1992)}\). Hence, the order of continuity of activation functions critically affects the performance of the proposed method in which the first and second derivatives of a neural network play a key role. This consideration suggests that the following sufficiently smooth activation functions are suitable for the present work.

\[ C^\infty \text{ functions } = \{ \text{sigmoid, tanh, sin, softplus} \}. \] (22)
The magnitude of gradient of activation function should not vanish for the update of parameters of NNs during training. When the sigmoid and tanh are either too high or too low, the gradient vanishing is observed. The following relu-like functions have been widely used in many neural network applications because they show high performance and their gradients are non-zero for all positive values.

\[
\text{relu-like functions} = \{\text{relu, elu, selu, softplus}\}, \tag{23}
\]

where \(\alpha = 1\) is used for elu and pre-defined constants \(\alpha = 1.67326324\) and \(\lambda = 1.05070098\) in Tensorflow are used for selu.

In section 4.4, we compare the performance of activation functions in Table 2 in terms of estimation accuracy for value, the first and second derivatives of NNs.

4. Numerical experiments

In this section, numerical experiments are performed to demonstrate the effectiveness of the proposed method for pricing multidimensional options such as exchange and basket options.

4.1. Estimation of price and Greeks of exchange options

We can compare the performance of numerical methods using an exchange option because this option has an exact solution. Margrabe (1978) provided the formula for the exchange option price in Eq. (4). By differentiating the Margrabe formula, we obtained the Greeks in Eq. (5)-(7). In the same notation in section 2.1, initial stock prices \(S_1(0) = 60\) and \(S_2(0) = 60\), time to maturity \(T = 1\), volatility of first asset \(\sigma_1 = 0.4\), volatility of second asset \(\sigma_2 = 0.2\), interest rate \(r = 0.1\), and correlation coefficient \(\rho_{12} = 0.4\) are parameters of the first experiment. We define the error between exact solution and the estimate by relative error

\[
r_{\text{Error}} = \left| \frac{\text{exact} - \text{estimate}}{\text{exact}} \right|. \tag{24}
\]
|          | Exact       | FDM1        | FDM2        | MC1         | MC2         | MC3         |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| Price    | 8.777591    | 8.765359    | 8.776234    | 8.784203    | 8.776402    | 8.777109    |
| rError   | 0           | 1.39e-03    | 1.55e-04    | 7.53e-04    | 1.35e-04    | 5.49e-05    |
| Δ        | 0.573140    | 0.572740    | 0.573102    | 0.573611    | 0.573094    | 0.57313     |
| rError   | 0           | 7.09e-04    | 7.80e-05    | 8.10e-04    | 9.24e-05    | 2.97e-05    |
| Γ        | 0.017726    | 0.017728    | 0.017726    | 0.017738    | 0.017724    | 0.017725    |
| rError   | 0           | 1.15e-04    | 1.07e-05    | 6.82e-04    | 9.47e-05    | 4.06e-05    |
| Θ        | -4.339281   | -4.344155   | -4.339812   | -4.343678   | -4.338946   | -4.339065   |
| rError   | 0           | 1.12e-03    | 1.22e-04    | 1.01e-03    | 7.72e-05    | 4.99e-05    |

Table 3: Numerical estimate of the price and Greeks of the exchange option. The best results are highlighted in bold face.

| nEpoch   | Exact       | 25,000      | 50,000      | 100,000     | 200,000     | 400,000     |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| Price    | 8.777591    | 8.777736    | 8.777681    | 8.777660    | 8.777587    | 8.777516    |
| rError   | 0           | 1.65e-05    | 1.02e-05    | 7.85e-06    | 4.85e-07    | 8.51e-06    |
| Δ        | 0.573140    | 0.573155    | 0.573158    | 0.573152    | 0.573139    | 0.573152    |
| rError   | 0           | 1.40e-05    | 1.84e-05    | 9.44e-06    | 1.45e-05    | 9.37e-06    |
| Γ        | 0.017726    | 0.017692    | 0.017653    | 0.017699    | 0.017708    | 0.017701    |
| rError   | 0           | 1.90e-03    | 4.12e-03    | 1.52e-03    | 1.01e-03    | 1.38e-03    |
| Θ        | -4.339281   | -4.323961   | -4.311028   | -4.330462   | -4.330742   | -4.331632   |
| rError   | 0           | 3.53e-03    | 6.51e-03    | 2.03e-03    | 1.97e-03    | 1.76e-03    |

Table 4: SDBS estimate of the price and Greeks of the exchange option using several training epochs. The best results are highlighted in bold face.
In Table 3, FDMs discretize the domain of stock price \([0, 300]\) with 100 uniform intervals for FDM1 and 300 for FDM2, and the time domain \([0, T]\) with 5,000 uniform intervals for FDM1 and 50,000 for FDM2. MC simulations calculate the price and Greeks of options given by Eq. (13) and Table 1. MC1 uses \(10^7\) simulated paths, \(10^8\) for MC2, and \(10^9\) for MC3. As can be seen in Table 3, FDMs and MC methods give accurate estimation. Furthermore, the finer discretization makes FDMs more accurate; MC methods estimate more accurately with more simulated paths.

Five independent SDBS models are trained for several nEpochs using the Algorithm 1. The batch size 10,000 of simulated stock price \(S(t)\) was used for training and estimation procedure. The price and Greeks are generated 1000 times for each trained NN. The average of 5,000 (5 models x 1,000 times) values is used as the final estimated values. Comparing with the results in Table 3, the SDBS in Table 4 gives comparable performance to the classical FDM and MC methods. In particular, the estimation of price for nEpoch 200,000 is significantly accurate. These experiments show that the differential neural network based SDBS performs accurately which estimates Price = \(N(t_0, S(t_0))\), \(\Delta_i = N_{S_i}(t_0, S(t_0))\), \(\Gamma_i = N_{S_iS_i}(t_0, S(t_0))\), and \(\Theta = N_t(t_0, S(t_0))\).

Table 5 shows the estimates of price and Greeks of exchange options with different initial stock price pairs \((S_1(0), S_2(0))\), where the SDBS uses the same experimental configuration as in Table 4 with 100,000 nEpoch. In the estimation of Price and \(\Delta\), the SDBS shows good performance; five bests and five second bests out of total ten combinations of initial stock price pairs \((S_1(0), S_2(0))\); but, the performance of SDBS is degraded for the estimation of \(\Gamma\) and \(\Theta\).

From these experiments, we find that the differential neural network, SDBS, accurately estimates the value, the first and second derivatives of solution function. But, the SDBS consumes large memory for storing many simulated paths and model parameters of NNs; our simulation configuration needs 8GB memory of NVIDIA GTX 1080. The SDBS takes about 4.5 hours for 10,000 training nEpoch. Thus, the current SDBS cannot be a practical substitute of MC simulations for the pricing of multidimensional European options.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
& \multicolumn{5}{|c|}{$S_1(0), S_2(0)$} \\
\cline{2-6}
& 20,60 & 40,60 & 60,60 & 60,40 & 60,20 \\
\hline
\multirow{3}{*}{\text{Price}} & FDM2 & 5.64e-02 & 5.55e-04 & 1.55e-04 & 4.08e-07 & 1.81e-08 \\
& MC2 & 3.13e-03 & 2.36e-05 & 1.35e-04 & 9.42e-05 & 6.75e-05 \\
& SDBS & 1.25e-02 & 6.22e-05 & 7.85e-06 & 1.31e-06 & 2.90e-07 \\
\hline
\multirow{3}{*}{$\Delta$} & FDM2 & 4.67e-02 & 2.15e-04 & 7.80e-05 & 1.03e-04 & 3.28e-06 \\
& MC2 & 1.52e-03 & 2.91e-04 & 9.24e-05 & 1.20e-04 & 4.21e-05 \\
& SDBS & 2.22e-03 & 6.63e-05 & 9.44e-06 & 1.78e-05 & 2.25e-06 \\
\hline
\multirow{3}{*}{$\Gamma$} & FDM2 & 4.06e-02 & 1.70e-04 & 1.07e-05 & 2.48e-04 & 2.07e-04 \\
& MC2 & 1.74e-03 & 2.55e-04 & 9.47e-05 & 5.14e-05 & 2.06e-02 \\
& SDBS & 1.36e-03 & 2.11e-03 & 1.52e-03 & 4.14e-03 & 8.42e-02 \\
\hline
\multirow{3}{*}{$\Theta$} & FDM2 & 8.95e-02 & 6.51e-06 & 1.22e-04 & 2.82e-05 & 2.07e-04 \\
& MC2 & 1.15e-03 & 1.55e-04 & 7.72e-05 & 2.71e-04 & 2.39e-02 \\
& SDBS & 8.80e-04 & 3.73e-03 & 2.03e-03 & 6.84e-03 & 1.09e-01 \\
\hline
\end{tabular}
\caption{rErrors of numerical methods for the exchange options with several initial stock prices. The best results are highlighted in bold face.}
\end{table}

### 4.2. Estimation of price and Greeks of basket call options

We consider a call option on a basket with four independent stocks. The parameters are as in Korn and Zeytun (2013): $T = 0.5, r = 0.06, (S_1, S_2, S_3, S_4) = (40, 50, 60, 70), (w_1, w_2, w_3, w_4) = (0.25, 0.25, 0.25, 0.25)$, and various volatilities. Since there is no exact solution for the basket option, we need a reliable numerical method to give us the benchmark value. In Table 6, we used MC simulation based benchmark values in Korn and Zeytun (2013) where the number of MC simulations paths is $10^6$. We compare our method, SDBS used in Table 5 with LN (the log-normal approximation of Levy (1992)), RG (the reciprocal gamma approximation of Milevsky and Posner (1998)), SLN (the shifted log-normal approximation of Korn and Zeytun (2013)), JU (the Taylor expansion approximation of Ju (2002)), and MC2 which is the MC method with $10^8$ simulated paths used in Table 3. We rounded up to four decimals to be consistent with previous results in Korn and Zeytun (2013).
| K  | MC  | LN  | RG  | SLN | JU  | MC2 | SDBS |
|----|-----|-----|-----|-----|-----|-----|------|
|    |     |     |     |     |     |     |      |
|    | \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) = (0.2, 0.2, 0.2, 0.2) |     |     |     |     |     |      |
| 50 | 6.5355 | 6.5412 | 6.5340 | 6.5653 | 6.5404 | 6.5407 | 6.5404 |
|    | 0 | 8.72e-04 | **2.30e-04** | 4.56e-03 | 7.50e-04 | 7.96e-04 | 7.50e-04 |
| 55 | 2.5063 | 2.5104 | 2.5010 | 2.5343 | 2.5092 | 2.5094 | 2.5092 |
|    | 0 | 1.64e-03 | 2.11e-03 | 1.12e-02 | **1.16e-03** | 1.24e-03 | **1.16e-03** |
| 60 | 0.5041 | 0.5037 | 0.5133 | 0.4719 | 0.5049 | 0.5049 | 0.5049 |
|    | 0 | **7.93e-04** | 1.83e-02 | 6.39e-02 | 1.59e-03 | 1.59e-03 | 1.59e-03 |
|    | \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) = (0.5, 0.5, 0.5, 0.5) |     |     |     |     |     |      |
| 55 | 4.8324 | 4.8499 | 4.7920 | 4.9492 | 4.8384 | 4.8382 | 4.8377 |
|    | 0 | 3.62e-03 | 8.36e-03 | 2.42e-02 | 1.24e-03 | 1.20e-03 | **1.10e-03** |
| 60 | 2.7402 | 2.7463 | 2.7444 | 2.6729 | 2.7450 | 2.7441 | 2.7436 |
|    | 0 | 2.23e-03 | 1.53e-03 | 2.46e-02 | 1.75e-03 | 1.42e-03 | **1.25e-03** |
| 65 | 1.4468 | 1.4413 | 1.4831 | 1.2550 | 1.4488 | 1.4479 | 1.4476 |
|    | 0 | 3.80e-03 | 2.51e-02 | 1.33e-01 | 1.38e-03 | 7.60e-04 | **5.48e-04** |
|    | \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) = (0.8, 0.8, 0.8, 0.8) |     |     |     |     |     |      |
| 60 | 5.3401 | 5.3897 | 5.2725 | 5.3819 | 5.3563 | 5.3468 | 5.3457 |
|    | 0 | 9.29e-03 | 1.27e-02 | 7.83e-03 | 3.03e-03 | 1.25e-03 | **1.05e-03** |
| 65 | 3.8179 | 3.8418 | 3.8123 | 3.5776 | 3.8336 | 3.8230 | 3.8215 |
|    | 0 | 6.26e-03 | 1.47e-03 | 6.29e-02 | 4.11e-03 | 1.34e-03 | **9.45e-04** |
| 70 | 2.7011 | 2.7003 | 2.7430 | 2.2590 | 2.7135 | 2.7025 | 2.7019 |
|    | 0 | **2.96e-04** | 1.55e-02 | 1.64e-01 | 4.59e-03 | 5.18e-04 | 2.98e-04 |
|    | \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) = (0.6, 1.2, 0.3, 0.9) |     |     |     |     |     |      |
| 60 | 5.5569 | 5.9128 | 5.7558 | 5.9371 | 5.5922 | 5.5635 | 5.5619 |
|    | 0 | 6.40e-02 | 3.58e-02 | 6.84e-02 | 6.35e-03 | 1.19e-03 | **8.98e-04** |
| 65 | 4.1555 | 4.3459 | 4.2836 | 4.0874 | 4.1973 | 4.1604 | 4.1588 |
|    | 0 | 4.58e-02 | 3.08e-02 | 1.64e-02 | 1.01e-02 | 1.18e-03 | **8.03e-04** |
| 70 | 3.1196 | 3.1607 | 3.1798 | 2.6941 | 3.1710 | 3.1222 | 3.1207 |
|    | 0 | 1.32e-02 | 1.93e-02 | 1.36e-01 | 1.65e-02 | 8.33e-04 | **3.62e-04** |

Table 6: Basket call option prices and rErrors (the upper and lower values in each cell). The best results are highlighted in bold face.
Table 6 shows that SDBS gives the best accurate estimation. Out of 12 combinations of \( K \) and \((\sigma_1, \sigma_2, \sigma_3, \sigma_4)\), SDBS gives 8 bests and 1 equal best, LN 2 bests, RG 1 best, and JU 1 equal best performances. SDBS gives good performance for all levels of strike prices; it gives more accurate price of options with increasing volatilities and non-uniform volatilities. The simulation based methods such as MC2 and SDBS can effectively process the options with large and non-uniform volatilities accurately, while the analytic approximation methods like LN, RG, SLN, and JU are degraded with such volatilities. This trend is consistent with the results in Korn and Zeytun (2013).

Table 7 shows the relative errors of Greeks estimation obtained with the methods JU, MC2 and SDBS used in Table 6. Since benchmark values are required for the calculation of the relative errors of Greeks estimations, large number of MC simulated paths (=10^{11}) were generated and Greeks were estimated with the formulas shown in Table 6. In \( \Delta \) estimation experiments, MC2 gives 11 best and 1 equal best, and SDBS 10 best and 1 equal best, JU 2 best performances in 24 combinations of \( K \) and \((\sigma_1, \sigma_2, \sigma_3, \sigma_4)\). In estimation experiments of \( \Gamma \) and \( \Theta \), MC2 gives 26 best, SDBS 3 best, and JU 1 best performances in 30 combinations of \( K \) and \((\sigma_1, \sigma_2, \sigma_3, \sigma_4)\). The estimation performance of SDBS is high for \( \Delta \), but low for \( \Gamma \) and \( \Theta \); this trend is consistent with the results on exchange options in Table 5.

As can be seen in Table 6 and 7, the SDBS estimates accurately for price and \( \Delta \) of basket option with four assets; for the estimation of \( \Gamma \) and \( \Theta \), the SDBS is worse than MC simulation methods but better than analytic approximation methods.

4.3. Effect of the loss function of the Black-Scholes equation

We expect that the loss \( L_{BS} \) in Eq. (19) constrains the SDBS to satisfy the Black-Scholes partial differential equation; the derivatives of SDBS, Greeks, are estimated more accurately with \( L_{BS} \) than without it. To validate the effectiveness of this constraint, we test several weights \((w = 10^{-3}, 10^{-2}, 10^{-1}, 1, 10^1, \ldots)\).
Table 7: rErrors of Greeks estimation of basket options. The best results are highlighted in bold face.

| (σ₁, σ₂, σ₃, σ₄) | (0.5, 0.5, 0.5, 0.5) | (0.6, 1.2, 0.3, 0.9) |
|-------------------|----------------------|----------------------|
| \( \Delta_1 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 2.15e-04             | \textbf{1.17e-04}    | 2.33e-04              |
| 60                | 2.33e-04             | 1.08e-04             | 1.58e-04              |
| 65                | 5.73e-03             | \textbf{8.17e-06}    | 1.69e-04              |
| \( \Delta_2 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 9.75e-05             | \textbf{1.88e-05}    | 2.13e-04              |
| 60                | 2.11e-04             | 8.27e-03             | 7.92e-03              |
| 65                | 7.49e-05             | 4.44e-03             | 7.32e-05              |
| \( \Delta_3 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 1.28e-04             | \textbf{2.31e-05}    | 4.09e-04              |
| 60                | 1.08e-06             | 9.81e-05             | 5.51e-05              |
| 65                | 8.49e-04             | 1.98e-04             | 1.27e-04              |
| \( \Delta_4 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 5.34e-04             | \textbf{8.03e-05}    | 2.37e-04              |
| 60                | 2.93e-04             | 1.08e-06             | 1.27e-04              |
| 65                | 5.15e-04             | 6.89e-04             | 7.48e-04              |
| \( \Gamma_1 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 5.79e-03             | \textbf{3.34e-04}    | 2.98e-03              |
| 60                | 7.61e-03             | 2.89e-05             | 7.64e-04              |
| 65                | 2.29e-03             | 9.91e-03             | 6.70e-03              |
| \( \Gamma_2 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 2.96e-03             | \textbf{9.01e-05}    | 3.81e-03              |
| 60                | 4.10e-03             | 3.43e-04             | 3.58e-04              |
| 65                | 1.38e-01             | \textbf{2.95e-04}    | 1.96e-04              |
| \( \Gamma_3 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 1.25e-04             | \textbf{1.86e-05}    | 8.40e-04              |
| 60                | 8.70e-04             | 1.96e-04             | 5.93e-04              |
| 65                | 1.03e-02             | 6.38e-03             | 6.54e-03              |
| \( \Gamma_4 \)    |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 1.16e-03             | \textbf{2.02e-04}    | 6.97e-04              |
| 60                | 6.16e-03             | 1.94e-04             | 2.16e-04              |
| 65                | 3.64e-03             | 2.16e-04             | 2.89e-04              |
| \( \Theta \)      |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 1.44e+00             | \textbf{1.35e-04}    | 1.79e-04              |
| 60                | 8.76e-01             | 3.20e-04             | 3.26e-04              |
| 65                | 5.12e-01             | 4.16e-04             | 3.71e-04              |
| \( \text{MD} \)   |                      |                      |
| K                 | JU                   | MC2                  | SDBS                  |
| 55                | 1.59e-03             | \textbf{1.59e-03}    | 1.95e-03              |
| 60                | 1.99e-04             | 3.71e-04             | 3.71e-04              |
\( L_w = L_{SDE} + w \cdot L_{BS} + L_T, \) \hspace{1cm} (25)

where \( w = 1 \) corresponds to the loss in Eq. (21); small \( w \) means small contribution of PDE constraint and relatively large contribution of the SDE, and large \( w \) means large contribution of PDE constraint and relatively small contribution of the SDE. We perform the same experiments as in Table 5 with \((S_1, S_2) = (20, 60), (60, 60), \text{ and } (60, 20)\) except that this experiment uses the weighted loss \( L_w \) instead of the original loss in Eq. (21).

Table 8 shows that this version of SDBS performs accurately for a wide range of \( w \), but it degrades when the \( w \) deviates from unity. This means that suitable constraint of Black-Scholes equation enhances the performance of option pricing. Price estimation performs better with \( w \leq 1 \) while \( \Gamma \) and \( \Theta \) estimation perform...
better with $w \geq 1$. This is because weighting more to the loss of differential equation $L_{BS}$ gives accurate estimation for derivatives while weighting more to the loss of value $L_{SDE}$ gives accurate estimation for values. When $L_{BS}$ is removed completely, no convergent value is obtained. These observations reveal that the loss $L_{BS}$ is essential for the accurate estimation of gradient and Hessian of NN, and both loss $L_{BS}$ and $L_{SDE}$ play a key role in making the SDBS suitable for option pricing problems.

4.4. Effect of the smoothness of activation functions

In section 3.4, we described the mathematical conditions of activation functions for the approximation of the functions with all partial derivatives; the boundedness and continuity of all its partial derivatives up to order $m$ are required to approximate the functions with all partial derivatives up to $m$ are continuous and bounded \cite{Hornik1990, Gallant1992}. The activation functions such as sigmoid, tanh, and sin satisfy these conditions. But, the practical performance can differ from one activation function to another. We compare the performance of several activation functions in terms of their suitability for option pricing problems.

Using the same experimental setting used for SDBS in Table 5 with nEpoch 100,000, we examine the rErrors of estimation with several activation functions. As can be seen in Table 9, the accuracy differs substantially from activation function to activation function. Sigmoid and softplus show significantly better accuracy than other methods for estimating price and $\Delta$; but, sigmoid degraded for $(S_1, S_2) = (20, 60)$. The relu-like function such as relu, elu, selu, softplus have given good performance in most applications of NNs. But, the insufficient smoothness of relu, elu, and selu degrade the estimation accuracy of the $\Gamma$ and $\Theta$. These accuracies are exactly proportional to the smoothness of the relu-like functions. Non-differentiable $C^0$ functions such as relu and selu give the lowest performance, especially large errors($\sim e^{-01}$ and $e^{+00}$). Another relu-like $C^1$ function elu gives intermediate performance, while relu-like infinitely differentiable function softplus shows high performance. This is because the smoothness
The accuracy of price estimation by relu-like functions in Eq. (23) is comparable to that by $C^\infty$ functions in Eq. (22); for example, elu shows good performance for estimating price comparing with $C^\infty$ functions. This is because the vanishing gradient problem is reduced for relu-like functions.

The above considerations suggest that relu-like functions are required for the estimation of accurate price and $C^\infty$ functions are required for accurate Greeks estimation. Hence, the intersection of relu-like functions in Eq. (23) and $C^\infty$ functions in Eq. (22), softplus, gives the best performance for the present work.
5. Conclusions

We improve the approximation capability of NNs by making NNs have the differential relationship of the problem we wish to solve. Following this approach, the proposed method learns the Black-Scholes equation of option price. The sufficient smoothness of activation functions is an essential factor for the proposed method which relies heavily on the differentiability of NNs; The softplus activation function is suitable for our method.

The SDBS is trained using more realistic asset price paths generated by SDEs. Hence, the SDBS can accurately model option price and does not suffer from the curse of dimensionality associated with high dimensional FDMs. Since the SDBS utilizes the exact differential relationship along a single simulation path, it can easily use parallel and distributed computing. We can also use this single-path dependency for backward recursion in American option pricing which will be discussed in a separate paper.

Further improvements can be made using sample paths generated by advanced MC simulation methods such as variance reduction techniques and low-discrepancy sequences. Although the proposed method was applied to a typical Black-Scholes model of option pricing in this work, it can be modified to suit different models and option types. In addition, differential neural networks can be applied to solve problems related to partial differential equations such as inverse problems, the Navier-Stokes equation, and the Schrödinger equation.

Appendix

In this section we derive the calculation formulas of Greeks in Table 1. Although the exchange option has the exact solution, we prepare the MC approximation of Greeks for comparing with other numerical methods. Pathwise derivative estimate (PW) method first differentiates $e^{-r(T-t)}V(T)$ in Eq. (13) and then expectation is performed Glasserman (2004), i.e., differentiation and expectation is interchanged. For the delta of exchange option, $e^{-r(T-t)}[S_1(T)-$
\( S_2(T) \mathbb{1}\{S_1(T) > S_2(T)\} \) at \( t = 0 \) is differentiated with respect to \( S_1(0) \),

\[
\frac{d}{dS_1(0)} e^{-rT} V(T) = e^{-rT} \frac{dS_1(T)}{dS_1(0)} \frac{dV(T)}{dS_1(T)}
\]

\[
= e^{-rT} \frac{S_1(T)}{S_1(0)} \mathbb{1}\{S_1(T) > S_2(T)\},
\]

where \( S_1(T) \) is presented in Eq. (2) and \( \mathbb{1} \) denotes a indicator function.

Similarly to the calculation of delta, the theta of exchange option is obtained by the expectation of the following,

\[
\frac{\partial e^{-r(T-t)} V(T-t)}{\partial t} \bigg|_{t=0} = \frac{\partial e^{-r(T-t)} V(T-t)}{\partial t} \bigg|_{t=0} = e^{-rT} \left[ V(T) + e^{-rT} \frac{\partial V(T-t)}{\partial t} \right] \bigg|_{t=0}
\]

\[
= e^{-rT} \left[ r[S_1(T) - S_2(T)] - (r - \frac{\sigma^2}{2}) L_1 Z \frac{\sqrt{T}}{2} \mathbb{1}\{S_1(T) > S_2(T)\} \right]
\]

\[
= e^{-rT} [S_1(T) \left( r \frac{\sigma^2}{2} L_1 Z \frac{\sqrt{T}}{2} \right) - S_2(T) \left( r - \frac{\sigma^2}{2} L_1 Z \frac{\sqrt{T}}{2} \right)] \mathbb{1}\{S_1(T) > S_2(T)\} \quad (27)
\]

where \( L_i \) and \( Z \) are described in (2).

We also use the PW method for pricing basket call option. The delta of basket option is calculated with \( V(T) \) in Eq.(8). The delta is given by the expectation of

\[
\frac{d e^{-rT} V(T)}{dS_i(0)} = e^{-rT} \frac{S_i(T)}{S_i(0)} w_i \mathbb{1}\{\sum_{i=1}^{n} w_i S_i(T) > K\} \quad (28)
\]

The theta of basket call option is given by the expectation of

\[
\frac{\partial e^{-r(T-t)} V(T-t)}{\partial t} \bigg|_{t=0} = e^{-rT} \left[ -r K + \sum_{i=1}^{n} w_i S_i(T) \left( r \frac{\sigma^2}{2} L_i Z \frac{\sqrt{T}}{2} \right) \right] \mathbb{1}\{\sum_{i=1}^{n} w_i S_i(T) > K\} \quad (29)
\]

For the calculation of the gamma using PW method, the first derivatives of payoff functions are required to be differentiable functions of the parameter \( S_i(0) \). But, the first derivatives, the delta, are not differentiable for exchange and basket options and PW cannot be applied for these options. Likelihood ratio method (LR), an alternate method, assumes that the assets \( S_i \) has a probability
density $g_\alpha$ and that $\alpha$ denotes a parameter of this density. To emphasize that the expectation is computed with respect to $g_\alpha$, $E_\alpha$ is used. LR interchange the order of differentiation and integration to derive a Greeks

$$
\frac{d}{d\alpha} E_\alpha[V(T)] = \int_{\mathbb{R}^n} V(T) \frac{d}{d\alpha} g_\alpha(x) dx
= \int_{\mathbb{R}^n} V(T) \frac{\dot{g}_\alpha(x)}{g_\alpha(x)} g_\alpha(x) dx = E_\alpha[V(T) \frac{\dot{g}_\alpha(x)}{g_\alpha(x)}] \quad (30)
$$

In the present work, Eq.(2) suggests that $S_i(T)$ for $i = 1, \ldots, n$, has the distribution $\exp(Y_i)$. The vector $Y = (Y_1, Y_2, \ldots, Y_n)'$ follows the normal distribution $N(\mu(\alpha), \tilde{\Sigma}(\alpha))$, where $\mu(\alpha; \cdot)$ denotes a vector with elements $\mu_i = \ln(S_i(0)) + (r - \frac{1}{2}\sigma_i^2)T$, $\tilde{\Sigma}(\alpha) = T A \Sigma A'$ ($A = \text{diag}(\sigma_1, \ldots, \sigma_n)$), and $\alpha = (S_1(0), \ldots, S_n(0))$. The probability density function of $Y$ could be written by

$$
g_\alpha(Y) = \frac{1}{\sqrt{(2\pi)^n |\tilde{\Sigma}(\alpha)|}} \exp\left[-\frac{1}{2} (y - \mu(\alpha))' \tilde{\Sigma}(\alpha)^{-1} (y - \mu(\alpha))\right]. \quad (31)
$$

We generate the sample paths $Y$ by $\mu + \sqrt{T} ALZ$, where $LL' = \Sigma$, and $Z$ denotes a standard normal distribution. Hence,

$$
\frac{\dot{g}_{S_i(0)}}{g_{S_i(0)}} = \frac{d}{dS_i(0)} \log(g_{S_i(0)}(Y)) = (Y - \mu)' \tilde{\Sigma}^{-1} \frac{d\mu}{dS_i(0)} = (\sqrt{T} ALZ)' \tilde{\Sigma}^{-1} \frac{d\mu}{dS_i(0)}
= (\sqrt{T} ALZ)' (T A \Sigma A')^{-1} (0, \ldots, 1_{S_i(0)}, \ldots, 0)' = \frac{(Z' L^{-1} A^{-1})_i}{\sqrt{T S_i(0)}} \quad (32)
$$

where $\frac{(Z' L^{-1} A^{-1})_i}{\sqrt{T S_i(0)}}$, denotes the $i$th component of the row vector $Z' L^{-1} A^{-1}$. For calculating gamma in the present study, LR method is first applied to the delta calculation and this delta is differentiated and averaged by the PW method. This LR-PW method for gamma calculation gives superior performance compared with the pure LR method in Glasserman (2004). For the exchange option, the gamma is given by the expectation of the
following expression
\[
\frac{d}{dS_1(0)} (e^{-rT} \max[S_1(T) - S_2(T), 0] \frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_1(0)})
\]
\[
= e^{-rT} \left( \frac{d}{dS_1(0)} (e^{-rT} \max[S_1(T) - S_2(T), 0]) \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_1(0)}\right) \right)
\]
\[
+ e^{-rT} \max[S_1(T) - S_2(T), 0] \left(\frac{d}{dS_1(0)} \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_1(0)}\right)\right)
\]
\[
= e^{-rT} \mathbb{1}\{ (S_1(T) > S_2(T) ) \frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_1(0)} \}
\]
\[
+ e^{-rT} \max[S_1(T) - S_2(T), 0] \left(\frac{d}{dS_1(0)} \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_1(0)}\right)\right)
\]
\[
= e^{-rT} \frac{(Z'L^{-1}A^{-1})}{\sqrt{T}} \mathbb{1}\{ (S_1(T) > S_2(T) ) \}. \quad (33)
\]

For the basket call option, the gamma is given by the expectation of the following expression
\[
\frac{d}{dS_i(0)} (e^{-rT} \max[\sum_{i=1}^{n} w_i S_i(T) - K, 0] \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_i(0)}\right))
\]
\[
= e^{-rT} \frac{d}{dS_i(0)} \left(\max[\sum_{i=1}^{n} w_i S_i(T) - K, 0] \right) \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_i(0)}\right)
\]
\[
+ e^{-rT} \max[\sum_{i=1}^{n} w_i S_i(T) - K, 0] \left(\frac{d}{dS_i(0)} \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_i(0)}\right)\right)
\]
\[
= e^{-rT} w_i \mathbb{1}\{ \sum_{i=1}^{n} w_i S_i(T) > K \} \frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_i(0)}
\]
\[
+ e^{-rT} \max[\sum_{i=1}^{n} w_i S_i(T) - K, 0] \left(\frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_i(0)}\right)
\]
\[
= e^{-rT} \frac{(Z'L^{-1}A^{-1})}{\sqrt{T}S_i^2(0)} \mathbb{1}\{ w_i S_i(T) - \sum_{i=1}^{n} w_i S_i(T) + K \} \mathbb{1}\{ \sum_{i=1}^{n} w_i S_i(T) > K \}. \quad (34)
\]

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