Differential equations for the closed geometric crystal chains

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Abstract. We present two types of systems of differential equations that can be derived from a set of discrete integrable systems which we call the closed geometric crystal chains. One is a kind of extended Lotka-Volterra systems, and the other seems to be generally new but reduces to a previously known system in a special case. Both equations have Lax representations associated with what are known as the loop elementary symmetric functions, which were originally introduced to describe products of affine type $A$ geometric crystals for symmetric tensor representations. Examples of the derivations of the continuous time Lax equations from a discrete time one are described in detail, where a novel method of taking a continuum limit by assuming asymptotic behaviors of the eigenvalues of the Lax matrix in Puiseux series expansions is used.
1. Introduction

In modern mathematical physics, studies on the relations between continuous, discrete, and ultra-discrete integrable dynamical systems have been developed extensively [6, 8, 7, 9, 24, 27]. Motivated by many studies in this field, and in particular by those based on the theory of crystals [13, 14] and that of geometric crystals [1], the author and T. Yoshikawa [26] constructed a new class of discrete integrable systems that can be viewed as a geometric lifting of a class of integrable cellular automata known as the periodic box-ball systems [17, 18, 19, 31, 32]. Since they are related to a realization of type $A_{n-1}$ geometric crystals and geometric $R$-matrices by G. Frieden [4, 5], we called the new integrable systems **closed geometric crystal chains**.

From a conventional viewpoint, the rank of the affine Lie algebra $A_{n-1}$ is related to the number of solitons in the periodic box-ball system. In this viewpoint, the geometric lifting of the system has already known and identified with the discrete periodic Toda chain [8, 15]. On the other hand, this rank is related to the number of degrees of freedom in each site variables in our construction. Therefore, a closed geometric crystal chain is not equivalent to the discrete periodic Toda chain. As a result, while the discrete Toda chain yields the Toda equation in a continuum limit, it is unclear what kind of differential equations will be derived from the closed geometric crystal chains in such a limit, except for the case of $n = 2$ that was studied in §2.3.2 of reference [26].

The purpose of this paper is to give an outlook for extending this result to the case of general $n$. We present two types of systems of differential equations that can be derived from the chains. One is equation (1), which is a kind of extended Lotka-Volterra systems [2, 11, 21]. The other is equation (24) with a function $e^{(\alpha)}_{L-1}$ in (3), which seems to be generally new but reduces to a previously known system studied in [23] in a special case (36). An important point here is that both equations are related to Lax equations associated with the **loop elementary symmetric functions**, which were originally introduced in reference [20] for a description of products of affine type $A$ geometric crystals for symmetric tensor representations. (We shall use its variation in reference [10].) Examples of the derivations of the continuous time Lax equations from a discrete time one are described in detail, where a novel method of taking a continuum limit by assuming asymptotic behaviors of the eigenvalues of the Lax matrix in Puiseux series expansions is used. In those examples, the method of imposing periodic boundary conditions on the discrete integrable systems in [26], which uses Perron-Frobenius theorem, turns out to be related to an unexpected way of taking the continuum limit. We expect that it will provide a new technique for studying the above mentioned relations between various types of integrable dynamical systems. In this paper we restrict ourselves to the case of $n = 4$ for such explicit derivations. Discussions for more general cases will be explored in a separate publication.
The remaining part of this paper is organized as follows. In §2 we study a system of differential equations that can be viewed as a kind of the extended Lotka-Volterra systems. We introduce the loop elementary symmetric functions in §2.1 and define a matrix $L$ with elements given by them. We present Theorem 1 which is the first result of this paper, showing that the system of differential equations leads to a Lax equation satisfied by $L$ with a companion matrix $Y$. A proof of this theorem is shown in §2.2, where the Lax equation is decomposed into a set of Lax triads [25]. In §2.3, we study several properties that can be derived from the Lax equation. In §3 we introduce another system of differential equations related to the loop elementary symmetric functions. The second result of this paper, Theorem 4 is presented in §3.1 to show that the system of differential equations leads to a Lax equation satisfied by $L$ with another companion matrix $Z$. A proof of this theorem is shown in §3.2, where the Lax equation is decomposed into another set of Lax triads. In §3.3, we study several properties that can be derived from the Lax equation. In §4 we present a connection of the systems of differential equations and the closed geometric crystal chains. We give a brief review of the latter system in §4.1 mainly for the case that can be described by $4 \times 4$ matrices as an example. In §4.2, §4.3, and §4.4, we give discussions for deriving the two types of differential equations from the closed geometric crystal chains in the case of $n = 4$, where Theorems 13 and 14 are presented as the third and fourth results of this paper. Several concluding remarks are given in §5 and a few detailed calculations are shown in Appendix A and Appendix B.

2. Type I differential equations

2.1. Definitions and the first result

Let $L, n$ be a pair of coprime integers. Then there is a unique integer $0 \leq p < n$ such that the condition $Lp \equiv 1 \pmod{n}$ is satisfied. Let $t \in \mathbb{R}$ be the time variable and $u_i^{(\alpha)}$ be a set of dependent variables labeled by $(\alpha, i) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z})$. Suppose that the system of differential equations

$$\frac{du_i^{(\alpha)}}{dt} = u_i^{(\alpha)} \left( \sum_{j=1}^{\min(Lp-1,L(n-p))} (u_{i-j}^{(\alpha+j)} - u_{i+j}^{(\alpha-j)}) \right),$$

(1)

is satisfied by them.

For any $(\alpha, i) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z})$, there is a unique integer $0 \leq m < n$ such that the condition $1 - i - \alpha \equiv Lm \pmod{n}$ is satisfied. Set $f(\alpha, i) := i + Lm$, which can be viewed as an element of $\mathbb{Z}/(nL)\mathbb{Z}$. By the Chinese remainder theorem, the map sending $(\alpha, i)$ to the $f(\alpha, i)$ gives an isomorphism of rings $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z}) \simeq \mathbb{Z}/(nL)\mathbb{Z}$, for which the inverse map is given by sending $f$ to $((1-f)(\pmod{n}), f(1-f))$. 


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Thus if we write $P_{f(a,i)} = u_i^{(a)}$, then equation (1) is written as $\dot{P}_f = P_f \left( \sum_{g=1}^{\min(Lp-1,L(n-p))} (P_{f-g} - P_{f+g}) \right)$, which is a kind of the extended Lotka-Volterra systems.

For the set of variables $u_i^{(a)}$ and an integer $m$, let $e_m^{(a)} (\alpha \in \mathbb{Z}/n\mathbb{Z})$ be the $m$-th loop elementary symmetric functions defined by

$$e_m^{(a)} = \sum_{1 \leq j_1 < j_2 < \ldots < j_m \leq L} u_{j_1}^{(a+1-j_1)} u_{j_2}^{(a+2-j_2)} \ldots u_{j_m}^{(a+m-j_m)},$$

and $e_0^{(a)} = 1$, $e_m^{(a)} = 0$ ($m < 0$ or $m > L$) [10, 20]. In particular, we have $e_1^{(a)} = u_1^{(a)} + u_2^{(a-1)} + \ldots + u_L^{(a_L+1)}$, $e_L^{(a)} = u_1^{(a)} u_2^{(a)} \ldots u_L^{(a)}$, and

$$e_{L-1}^{(a)} = \sum_{i=1}^{L} \left( \prod_{j=1}^{i-1} u_j^{(a)} \prod_{k=i+1}^{L} u_k^{(a-1)} \right).$$

Let $\lambda$ be an indeterminate and $\mathcal{L}$ be the $n \times n$ matrix defined by

$$\mathcal{L} = (\mathcal{L}_{ij})_{1 \leq i, j \leq n}, \quad \mathcal{L}_{ij} = \sum_{m=0}^{\infty} \begin{pmatrix} e_{j-i + L - mn}^{(i)} \lambda^m \end{pmatrix},$$

which we call a Lax matrix. We define

$$y^{(a)} = \sum_{j=0}^{p-1} e_1^{(a-jL)} - (p/n) \sum_{r=1}^{n} e_1^{(r)},$$

and let $\mathcal{Y}$ be the $n \times n$ matrix

$$\mathcal{Y} = \begin{pmatrix} y^{(1)} & \lambda \\ 1 & y^{(2)} \\ & \ddots \\ & & \ddots \\ & & & 1 & y^{(n)} \end{pmatrix}.$$  

Through the variables $u_i^{(a)}$, the elements of these matrices are functions of the time variable $t$.

**Theorem 1** Suppose that the variables $u_i^{(a)}$ are satisfying the system of differential equations (1). Then the Lax matrix satisfies the equation

$$\frac{d\mathcal{L}}{dt} = [\mathcal{L}, \mathcal{Y}].$$

This result implies that conserved quantities of the dynamical system represented by equation (1) can be given by the coefficients of the characteristic polynomial of the Lax matrix $\mathcal{L}$, or equivalently by $\text{Tr} \mathcal{L}^m / m$ ($m = 1, \ldots, n$).
2.2. Proof of Theorem 1

We define
\[ U_i = (u_i^{(1)}, \ldots, u_i^{(n)}), \quad U = (U_1, U_2, \ldots, U_L), \] (8)
and \( \sigma U = (U_2, \ldots, U_L, U_1) \), where \( \sigma \) denotes the cyclic shift to the left. For the \( y^{(\alpha)} \) defined in (5), we write its dependence on the variable \( U \) as \( y^{(\alpha)}(U) \), and define
\[ y_i^{(\alpha)} = y^{(\alpha)}(\sigma^{i-1}U), \] (9)
for any \( i \in \mathbb{Z}/L\mathbb{Z} \). Let \( Y, M_i \) denote the \( n \times n \) matrices defined by
\[ Y_i = \begin{pmatrix} 1 & y_i^{(1)} & \cdots & 0 \\ \vdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \\ \end{pmatrix}, \quad M_i = \begin{pmatrix} 1 & y_i^{(1)} & \cdots & 0 \\ \vdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \\ \end{pmatrix}. \] (10)

Using the identities \( L = M_1 \cdots M_L \) (Lemma 6.1 of [10]) and \( Y = Y_1 \), one sees that the assertion of Theorem 1 follows from:

**Proposition 2** The system of differential equations (11) is equivalent to
\[ \frac{dM_i}{dt} = M_i Y_{i+1} - Y_i M_i. \] (11)

**Proof.** The matrix elements of equation (11) are explicitly written as
\[ y_{i+1}^{(\alpha)} - y_i^{(\alpha+1)} = u_i^{(\alpha)} - u_i^{(\alpha+1)}, \] (12)
\[ \frac{d u_i^{(\alpha)}}{dt} = u_i^{(\alpha)} (y_{i+1}^{(\alpha)} - y_i^{(\alpha)}). \] (13)

Thus the assertion of the proposition is a consequence of the following two lemmas, which are satisfied by any set of variables \( u_i^{(\alpha)} \) for \( (\alpha, i) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z}) \). □

**Lemma 3** Let \( y_i^{(\alpha)} \) be the one defined in (9). Then the relation (12) holds.

**Proof.** Using the expression \( e_1^{(\alpha)} = \sum_{k=0}^{L-1} u_1^{(\alpha-k)} \), we can write \( e_1^{(\alpha-jL)} = \sum_{k=0}^{L-1} u_1^{(\alpha-jL-k)} = \sum_{k=0}^{L-1} u_{1+k+jL}^{(\alpha-jL-k)} \). Therefore
\[ y^{(\alpha)}(U) = \sum_{j=0}^{p-1} \sum_{k=0}^{L-1} u_{1+k+jL}^{(\alpha-jL-k)} - p c_0 = \sum_{j=0}^{Lp-1} u_1^{(\alpha-jL)} - p c_0, \] (14)
where \( c_0 := (1/n) \sum_{r=1}^{n} e_r^{(p)} \). Hence we have \( y_i^{(\alpha)} - y_i^{(\alpha+1)} = y^{(\alpha)}(\sigma^j U) - y^{(\alpha+1)}(\sigma^{i-1}U) = \sum_{j=0}^{Lp-1} (u_1^{(\alpha-j)} - u_1^{(\alpha-j-1)}) = u_1^{(\alpha-Lp)} - u_1^{(\alpha+1)} = u_1^{(\alpha)} - u_1^{(\alpha+1)} \), where the condition \( Lp \equiv 1 \pmod{n} \) is used in the last step. □
Lemma 4 Let \( y^{(α)}_i \) be the one defined in (9). Then the following relation holds:

\[
y^{(α)}_{i+1} - y^{(α)}_i = \sum_{j=1}^{\min(Lp-1,L(n-p))} \left( u^{(α+j)}_{i-j} - u^{(α-j)}_{i+j} \right).
\]

Proof. First we suppose \( p \leq \frac{n}{2} \). Then \( \min(Lp - 1, L(n-p)) = Lp - 1 \). Using (14) and the relation \( u^{(α-Lp+1)}_{i+Lp} = u^{(α)}_{i} \), we have

\[
y^{(α)}_{i+1} - y^{(α)}_i = y^{(α)}(σ^iU) - y^{(α)}(σ^{i-1}U) = \sum_{j=0}^{Lp-1} \left( u^{(α-j)}_{1+i+j} - u^{(α-j)}_{i+j} \right) = \sum_{j=1}^{Lp-1} \left( u^{(α-j+1)}_{i+j} - u^{(α-j)}_{i+j} \right).
\]

Replacing the index \( j \) by \( Lp - j \) and using the condition \( Lp \equiv 1 \pmod{n} \), we have
\[
\sum_{j=1}^{Lp-1} u^{(α-j+1)}_{i+j} = \sum_{j=1}^{Lp-1} u^{(α-Lp+j+1)}_{i+Lp-j} = \sum_{j=1}^{Lp-1} u^{(α+j)}_{i-j} ,
\]
so that (15) follows.

Second we suppose \( p > \frac{n}{2} \). Then \( \min(Lp - 1, L(n-p)) = L(n-p) \). Since \( L \) and \( n \) are coprime, we have \( \sum_{j \in \{n/2\}} e_{1}^{(α-jL)} = \sum_{α=1}^{n} e_{1}^{(α)} = nc_{0} \). This implies that

\[
y^{(α)}(U) = \sum_{j=0}^{p-1} e_{1}^{(α-jL)} - pc_{0} = (n - p)c_{0} - \sum_{j=1}^{n-p} e_{1}^{(α+jL)}.
\]

Using the expression \( e_{1}^{(α+L)} = \sum_{k=0}^{L-1} u_{1+k}^{(α+L-k)} = \sum_{k=1}^{L} u_{1-k}^{(α+k)} \), one can rewrite the summation in the second term of the right hand side of equation (17) as

\[
\sum_{j=1}^{n-p} e_{1}^{(α+jL)} = \sum_{j=1}^{L(n-p)} u_{1-j}^{(α+j)} = \sum_{j=1}^{L(n-p)} u_{j}^{(α-j)},
\]

where the last expression is derived from the second one by replacing \( j \) by \( L(n-p)+1-j \).

Hence we have \( y^{(α)}(σ^{i-1}U) = (n - p)c_{0} - \sum_{j=1}^{L(n-p)} u_{i-j}^{(α+j)} \) and \( y^{(α)}(σ^iU) = (n - p)c_{0} - \sum_{j=1}^{L(n-p)} u_{i+j}^{(α-j)} \), so that (15) follows.

\( \square \)

Remark 5 Let the loop elementary symmetric functions \( e_{1}^{(α)} \) be denoted by \( e_{1}^{(α)}(U) \) for showing their dependence on the variable \( U \). Suppose \( p = 1 \) or the condition \( L \equiv 1 \pmod{n} \) is satisfied. Then the set of differential equations (1) is written as

\[
\frac{d u^{(α)}_{i}}{dt} = u^{(α)}_{i} \left( e_{1}^{(α)}(σ^iU) - e_{1}^{(α)}(σ^{i-1}U) \right).
\]

In particular, consider the case of \( n = 2 \). By the reason that will be shown in Remark 6, we can set \( u^{(1)}_{i} = 1 \). So if we define \( u_{i} = u^{(1)}_{i} \), then \( u^{(2)}_{i} = 1/u_{i} \). In this case one has \( e_{1}^{(1)}(σ^{i-1}U) = \sum_{j=0}^{L-1}(u_{i+j})^{(-1)^{j}} = u_{i} + \sum_{j=1}^{L-1}(1/u_{i+j})^{(-1)^{j-1}} \) and
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\[ e_1^{(1)}(\sigma^i U) = \sum_{j=0}^{L-1}(u_{i+j+1})^{(-1)^j} = \sum_{j=1}^L(u_{i+j})^{(-1)^{j-1}} = u_i + \sum_{j=1}^{L-1}(u_{i+j})^{(-1)^{j-1}}, \]

because \( L \) is odd. Therefore equation (18) is written as

\[
\frac{du_i}{dt} = u_i \sum_{j=1}^{L-1}(-1)^{j-1}\left(u_{i+j} - \frac{1}{u_{i+j}}\right). \tag{19}
\]

This is the system of differential equations that we obtained in §2.3.2 of reference [20].

2.3. Properties derived from the Lax equation

In this section, we study equation (7) without assuming any explicit expression for \( y^{(\alpha)} \) to find several conserved quantities. As a byproduct, we are going to see that this equation on the Lax matrix \( L \) and a matrix of the form \( Y \) in (6) basically determines the expression for \( y^{(\alpha)} \) uniquely.

Equation (7) with a matrix of the form \( Y \) in (6) is also equivalent to the set of differential equations

\[
\frac{de_i^{(\alpha)}}{dt} = e_i^{(\alpha)}(y^{(i+\alpha-L)} - y^{(\alpha)}) + e_l^{(\alpha)} - e_{l+1}^{(\alpha-1)}. \tag{20}
\]

Letting \( i = L \), we see that \( e_L^{(\alpha)}(= \prod_{i=1}^L u_i^{(\alpha)}) \) is a conserved quantity, because \( e_{L+1}^{(\alpha)} = 0 \) for any \( \alpha \). On the other hand, if we let \( i = 0 \) in equation (20), then

\[
0 = \frac{de_0^{(\alpha)}}{dt} = (y^{(\alpha-L)} - y^{(\alpha)}) + e_1^{(\alpha)} - e_1^{(\alpha-1)},
\]

because \( e_0^{(\alpha)} = 1 \). Since \( e_1^{(\alpha-1)} = e_1^{(\alpha-L)} \), this identity is equivalent to \( y^{(\alpha-L)} + e_1^{(\alpha)} = y^{(\alpha)} + e_1^{(\alpha-L)} \), which determines an expression for \( y^{(\alpha)} \) as

\[
y^{(\alpha)} = \sum_{j=0}^{p-1} e_1^{(\alpha-jL)} + C, \tag{21}
\]

where \( C \) is a term not depending on \( \alpha \). Now by letting \( i = 1 \) in equation (20) we have

\[
\frac{de_1^{(\alpha)}}{dt} = e_1^{(\alpha)}(y^{(1+\alpha-L)} - y^{(\alpha)}) + e_2^{(\alpha)} - e_2^{(\alpha-1)}. \tag{22}
\]

Using (21) and the condition \( Lp \equiv 1 \text{ (mod } n) \), one sees that

\[
y^{(1+\alpha-L)} = y^{(\alpha+L(p-1))} = \sum_{j=0}^{p-1} e_1^{(\alpha+L(p-1-j))} + C = \sum_{j=0}^{p-1} e_1^{(\alpha+jL)} + C. \tag{23}
\]

Then by using the identities (21), (22), (23), and \( \sum_{\alpha=1}^n e_1^{(\alpha)} e_1^{(\alpha+jL)} = \sum_{\alpha=1}^n e_1^{(\alpha-jL)} e_1^{(\alpha)} \), we have \( d(\sum_{\alpha=1}^n e_1^{(\alpha)})/dt = 0 \), so that \( \sum_{\alpha=1}^n e_1^{(\alpha)} \) is a conserved quantity. Therefore, our choice of \( C = -(p/n) \sum_{\alpha=1}^n e_1^{(\alpha)} \) for the \( y^{(\alpha)} \) above equation (18) is so chosen as to make it a conserved quantity, and to impose the condition \( \sum_{\alpha=1}^n y^{(\alpha)} = 0 \).
Remark 6 In the case of \( p \leq \frac{n}{2} \), an expression for \( y_{i+1}^{(\alpha)} - y_i^{(\alpha)} \) was given by (16). A similar expression for the case of \( p > \frac{n}{2} \) can also be derived as

\[
y_{i+1}^{(\alpha)} - y_i^{(\alpha)} = \sum_{j=0}^{L(n-p)-1} \left( u_i^{(\alpha-j-1)} - u_i^{(\alpha-j)} \right) + u_i^{(\alpha-1)} - u_i^{(\alpha+1)}.
\]

Using these expressions, one sees that the relation

\[
\sum_{\alpha=1}^{n} (y_{i+1}^{(\alpha)} - y_i^{(\alpha)}) = 0
\]

is satisfied for any \( i \). Therefore, by using equation (13), we have

\[
\frac{d}{dt} \log \prod_{\alpha=1}^{n} u_i^{(\alpha)} = 0,
\]

so that \( \prod_{\alpha=1}^{n} u_i^{(\alpha)} \) is a conserved quantity for any \( i \).

3. Type II differential equations

3.1. Definitions and the second result

As in \( \S 2.1 \) let \( t \in \mathbb{R} \) be the time variable and \( u_i^{(\alpha)} \) be a set of dependent variables labeled by \( (\alpha, i) \in (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/L\mathbb{Z}) \), but now the integers \( L \) and \( n \) are not necessarily coprime. Suppose that the system of differential equations

\[
\frac{d u_i^{(\alpha)}}{dt} = u_i^{(\alpha)} \left( \frac{1}{e_{L-1}^{(\alpha)}} \prod_{l=1}^{i-1} u_l^{(\alpha)} \prod_{k=i+1}^{L} u_k^{(\alpha-1)} - \frac{1}{e_{L-1}^{(\alpha+1)}} \prod_{l=1}^{i-1} u_l^{(\alpha+1)} \prod_{k=i+1}^{L} u_k^{(\alpha)} \right),
\]

which is satisfied by them.

Comparing with the definition (3), one sees that part of the product in the first term of the right hand side of equation (24) equals to the \( i \)-th term of the summation in the definition of \( e_{L-1}^{(\alpha)} \). Therefore, by using this equation and the definitions of \( e_{L}^{(\alpha)} \) and \( e_{L-1}^{(\alpha)} \), we have

\[
\frac{d \log e_{L}^{(\alpha)}}{dt} = \sum_{i=1}^{L} \frac{1}{u_i^{(\alpha)}} \frac{d u_i^{(\alpha)}}{dt} = 1 - 1 = 0.
\]

Hence \( e_{L}^{(\alpha)} \) is a conserved quantity for any \( \alpha \). Recall that the Lax matrix \( \mathcal{L} \) was defined by equation (4), and let \( \mathcal{Z} \) be the \( n \times n \) matrix

\[
\mathcal{Z} = \begin{pmatrix}
0 & z^{(2)} & z^{(3)} \\
0 & \ddots & \ddots \\
z^{(1)}/\lambda & \ddots & z^{(n)} \\
z^{(1)}/\lambda & 0
\end{pmatrix},
\]

where

\[
z^{(\alpha)} = 1/e_{L-1}^{(\alpha)}.
\]
Theorem 7 Suppose that the variables $u_i^{(α)}$ are satisfying the system of differential equations \(24\), and that the condition $e_L^{(α)} = 1$ is satisfied for any $α$. Then the Lax matrix $L$ satisfies the equation
\[
\frac{dL}{dt} = [L, Z].
\] \(27\)

By Theorems 1 and 7, we see that the dynamical systems represented by equations \(1\) and \(24\) share a common Lax matrix defined by \(4\) in some cases. As a result, they share a common set of conserved quantities in such cases.

3.2. Proof of Theorem 7

For the $z^{(α)}$ defined in \(26\), we write its dependence on the variable $U$ in \(8\) as $z^{(α)}(U)$, and define
\[
z_i^{(α)} = z^{(α)}(σ^{i-1}U),
\] \(28\)
for any $i ∈ \mathbb{Z}/L\mathbb{Z}$. Let $M_i$ be the matrix in \(10\) and define the matrix $Z_i$ as
\[
Z_i = \begin{pmatrix}
0 & z_i^{(2)} & 0 & \cdots & 0 \\
-1 & 0 & z_i^{(3)} & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & 0 \\
\end{pmatrix},
\] \(29\)

Proposition 8 The system of differential equations \(24\) with the condition $e_L^{(α)} = 1$ is equivalent to
\[
\frac{dM_i}{dt} = M_i Z_{i+1} - Z_i M_i.
\] \(30\)
This together with the relations $L = M_1 \cdots M_L$ and $Z = Z_1$ yields the assertion of Theorem 7.

Proof. The matrix elements of equation \(30\) are explicitly written as
\[
u_i^{(α-1)} z_i^{(α)} = u_i^{(α)} z_i^{(α)},
\] \(31\)
\[
\frac{d\nu_i^{(α)}}{dt} = z_i^{(α)} z_{i+1} - z_i^{(α+1)}.
\] \(32\)
Thus the assertion of the proposition is a consequence of the following two lemmas, which are satisfied by any set of variables $u_i^{(α)}$ for $(α, i) ∈ (\mathbb{Z}/n\mathbb{Z}) × (\mathbb{Z}/L\mathbb{Z})$ under the condition $e_L^{(α)} = 1$. □

Lemma 9 If $e_L^{(α)}$ does not depend on $α$, then the relation \(31\) or equivalently the following relation holds:
\[
e_L^{(α)}(σ^{i}U) \frac{u_i^{(α)}}{u_i^{(α-1)}} = e_L^{(α)}(σ^{i-1}U).
\] \(33\)
Proof. Suppose $e^{(\alpha)}_L = C$ for any $\alpha$. Then

$$
e_{L-1}^{(\alpha)}(\sigma^i U) \frac{u^{(\alpha)}_i}{u^{(\alpha-1)}_i} = \sum_{l=1}^{L-1} \left( \prod_{j=1}^{l-1} u^{(\alpha)}_{j+i} \prod_{k=l+1}^{L} u^{(\alpha-1)}_k \right) \frac{u^{(\alpha)}_i}{u^{(\alpha-1)}_i} + \frac{C}{u^{(\alpha-1)}_i}$$

$$= \sum_{l=1}^{L-1} \left( \prod_{j=1}^{l-1} u^{(\alpha)}_{j+i} \prod_{k=l+1}^{L} u^{(\alpha-1)}_k \right) + \prod_{k=1}^{L} u^{(\alpha-1)}_k$$

$$= \sum_{l=2}^{L-1} \left( \prod_{j=1}^{l-1} u^{(\alpha)}_{j+i-1} \prod_{k=l+1}^{L} u^{(\alpha-1)}_k \right) + \prod_{k=2}^{L} u^{(\alpha-1)}_k$$

$$= \sum_{l=1}^{L} \left( \prod_{j=1}^{l-1} u^{(\alpha)}_{j+i-1} \prod_{k=l+1}^{L} u^{(\alpha-1)}_k \right) = e^{(\alpha)}_{L-1}(\sigma^{-1} U).$$

In the second line, we wrote the term for $l = L$ separately. In the third line, the extra factor is absorbed in the parenthesized expression, and we used $C = e^{(\alpha-1)}_L$ in the separated term. In the fourth line, we replaced $(j, k, l)$ by $(j - 1, k - 1, l - 1)$. In the last line, we included the separated term into the summation as the term for $l = 1$. □

Lemma 10 Suppose $e^{(\alpha)}_L = 1$ for any $\alpha$, and let $z^{(\alpha)}_i$ be the one defined in (28). Then

$$z^{(\alpha)}_{i+1} - z^{(\alpha+1)}_i = u^{(\alpha)}_i \left( \frac{1}{e^{(\alpha)}_{L-1}} \prod_{l=1}^{i-1} u^{(\alpha)}_l \prod_{k=i+1}^{L} u^{(\alpha-1)}_k - \frac{1}{e^{(\alpha+1)}_{L-1}} \prod_{l=1}^{i-1} u^{(\alpha+1)}_l \prod_{k=i+1}^{L} u^{(\alpha)}_k \right). \quad (34)$$

Proof. By definition, we have

$$z^{(\alpha)}_{i+1} - z^{(\alpha+1)}_i = \frac{1}{e^{(\alpha)}_{L-1}(\sigma^i U)} - \frac{1}{e^{(\alpha+1)}_{L-1}(\sigma^{-1} U)}$$

$$= u^{(\alpha)}_i \left( \frac{1}{u^{(\alpha)}_i e^{(\alpha)}_{L-1}(\sigma^i U)} - \frac{1}{u^{(\alpha+1)}_i e^{(\alpha+1)}_{L-1}(\sigma^i U)} \right),$$

where we used Lemma [9] in the second line. Therefore it suffices to show that

$$u^{(\alpha)}_i e^{(\alpha)}_{L-1}(\sigma^i U) = \left( \prod_{l=1}^{i-1} u^{(\alpha)}_l \prod_{k=i+1}^{L} u^{(\alpha-1)}_k \right)^{-1} e^{(\alpha)}_{L-1}(U),$$

for any $\alpha \in \mathbb{Z}/(n\mathbb{Z})$ and $1 \leq i \leq L$. This relation is proved by applying Lemma [9] on
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the left hand side repeatedly as,

\[ u^{(\alpha)}_i e^{(\alpha)}_{L-1}(\sigma^i U) = u^{(\alpha-1)}_i e^{(\alpha)}_{L-1}(\sigma^{i-1} U) = \frac{u^{(\alpha-1)}_i u^{(\alpha-1)}_{i-1} \cdots u^{(\alpha-1)}_1}{u^{(\alpha)}_i \cdots u^{(\alpha)}_1} e^{(\alpha)}_{L-1}(U) = \left( \prod_{l=1}^{i-1} u^{(\alpha)}_l \prod_{k=i+1}^{L} u^{(\alpha-1)}_k \right)^{-1} e^{(\alpha)}_{L-1}(U), \]

where we used \( \prod_{k=1}^{L} u^{(\alpha-1)}_k = e^{(\alpha-1)}_L = 1 \). The proof is completed.

\[ \square \]

Remark 11 There is a simple expression for the system of differential equations (24) that can be compared with equation (18) in the type I case. In fact, from the above considerations we see that equation (24) is written as

\[ \frac{du^{(\alpha)}}{dt} = \frac{1}{e^{(\alpha)}_{L-1}(\sigma^i U)} - \frac{1}{e^{(\alpha+1)}_{L-1}(\sigma^{i-1} U)}. \]  

(35)

In particular, consider the case of \( L = 2 \). If we define \( u^{(\alpha)} = u^{(\alpha)}_1 \), then we have \( u^{(\alpha)}_2 = 1/u^{(\alpha)} \) because we set \( e^{(\alpha)}_L = 1 \). Therefore one has

\[ \frac{du^{(\alpha)}}{dt} = \frac{u^{(\alpha)}}{u^{(\alpha)}u^{(\alpha-1)} + 1} - \frac{u^{(\alpha)}}{u^{(\alpha)}u^{(\alpha+1)} + 1}. \]

(36)

This is basically the same differential equation that was appeared in reference [23] as equation (13.58), where its discretization was referred to as a lattice KdV equation.

3.3. Properties derived from the Lax equation

In this section, we study equation (27) without assuming any explicit expression for \( z^{(\alpha)} \) and find several conserved quantities. As a byproduct, we are going to see that this equation on the Lax matrix \( \mathcal{L} \) and a matrix of the form \( \mathcal{Z} \) in (25) basically determines the expression for \( z^{(\alpha)} \) uniquely.

In view of the definition (4), one sees that equation (27) with a matrix of the form \( \mathcal{Z} \) in (25) is equivalent to the system of differential equations

\[ \frac{de^{(\alpha)}_i}{dt} = z^{(i+\alpha-L)} e^{(\alpha)}_{i-1} - z^{(\alpha+1)} e^{(\alpha+1)}_{i-1}. \]  

(37)

By letting \( i = L+1 \) we see that \( e^{(\alpha)}_L = e^{(\alpha+1)}_L \), because \( e^{(\alpha)}_{L+1} = 0 \) for any \( \alpha \). On the other hand, the summation \( \sum_{\alpha=1}^n e^{(\alpha)}_L \) is a conserved quantity, because the conservation of
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\[ \text{Tr } \mathcal{L} = \sum_{m \geq 0} \sum_{\alpha=1}^{n} e^{(\alpha)}_{L-m} \lambda^m \]

is derived from equation (27). This forces each \( e^{(\alpha)}_L \) to be a conserved quantity. Then by letting \( i = L \) in equation (37) we have

\[ 0 = \frac{d e^{(\alpha)}_L}{dt} = z^{(\alpha)} e^{(\alpha)}_{L-1} - z^{(\alpha+1)} e^{(\alpha+1)}_{L-1}. \]

This identity determines an expression for \( z^{(\alpha)} \) as \( z^{(\alpha)} = C/e^{(\alpha)}_{L-1} \), where \( C \) is a factor not depending on \( \alpha \).

Now letting \( i = L - 1 \) in equation (37) and dividing it by \( e^{(\alpha)}_{L-1} \), we have

\[ \frac{1}{e^{(\alpha)}_{L-1}} \frac{d e^{(\alpha)}_L}{dt} = C \left( \frac{e^{(\alpha)}_{L-2}}{e^{(\alpha)}_{L-1}} - \frac{e^{(\alpha+1)}_{L-2}}{e^{(\alpha+1)}_{L-1}} \right). \]

This implies that

\[ \frac{d}{dt} \log(\prod_{\alpha=1}^{n} e^{(\alpha)}_{L-1}) = \sum_{\alpha=1}^{n} \frac{1}{e^{(\alpha)}_{L-1}} \frac{d e^{(\alpha)}_L}{dt} = 0. \]

Hence \( \prod_{\alpha=1}^{n} e^{(\alpha)}_{L-1} \) is a conserved quantity. Moreover, by using equation (33) we have \( \prod_{\alpha=1}^{n} e^{(\alpha)}_{L-1}(\sigma^i U) = \prod_{\alpha=1}^{n} e^{(\alpha)}_{L-1} \) for any \( i \).

**Remark 12** We have shown the conservation of \( e^{(\alpha)}_L = \prod_{i=1}^{L} u^{(\alpha)}_i \) for any \( 1 \leq \alpha \leq n \) by using equation (24). In the same way, we can show that

\[ \frac{d}{dt} \log(\prod_{\alpha=1}^{n} u^{(\alpha)}_i) = \sum_{\alpha=1}^{n} \frac{1}{u^{(\alpha)}_i} \frac{d u^{(\alpha)}_i}{dt} = 0. \]

Hence \( \prod_{\alpha=1}^{n} u^{(\alpha)}_i \) is also a conserved quantity for any \( 1 \leq i \leq L \).

4. Connection to the closed geometric crystal chains: A case study for \( n = 4 \)

4.1. A review of the closed geometric crystal chains

We briefly review on the closed geometric crystal chains for the totally one-row tableaux case (26, §3.1). Based on the notion of **rational rectangles** in (4, 5), we introduce the set \( \mathbb{Y}_1 = (\mathbb{R}_{>0})^{n-1} \times \mathbb{R}_{>0} \). Let \((x, s)\) denote an element of \( \mathbb{Y}_1 \) with \( x = (x^{(1)}, \ldots, x^{(n-1)}) \), and let \( x^{(n)} := s/(x^{(1)} \cdots x^{(n-1)}) \). Furthermore, we define \( x^{(i)} \) for arbitrary \( i \in \mathbb{Z} \) to be a variable determined from \( x \) by the relation \( x^{(i)} = x^{(i+n)} \). In what follows, we set
\[ n = 4. \text{ Given } (x, s) \in (\mathbb{R}_{>0})^3 \times \mathbb{R}_{>0}, \text{ we define the matrices } g^* \text{ and } g \text{ by} \]

\[
g^*(x, s; \lambda) = \begin{pmatrix}
  x^{(1)} & x^{(2)} & x^{(3)} & \lambda \\
  x^{(1)} & x^{(2)} & x^{(1)} & x^{(3)} & \lambda \\
  x^{(1)} & x^{(2)} & x^{(1)} & x^{(3)} & \lambda \\
  1 & x^{(4)} & x^{(4)} & x^{(4)} & x^{(4)} \\
\end{pmatrix}, \quad (38)
\]

\[
g(x, s; \lambda) = \begin{pmatrix}
  x^{(1)} & 0 & 0 & \lambda \\
  1 & x^{(2)} & 0 & 0 \\
  0 & 1 & x^{(3)} & 0 \\
  0 & 0 & 1 & x^{(4)} \\
\end{pmatrix}. \quad (39)
\]

Actually, any element of the matrix \( g^*(x, s; \lambda) \) is so defined as to be an order 3 minor of the matrix \( g(x, s; \lambda) \). For instance, the top-left element of the former is equal to the determinant of the top-left 3 \times 3 \text{ submatrix of the latter.}

By Theorem 16 of [26], we see that for any \( s, l \in \mathbb{R}_{>0} \) and \((b_1, \ldots, b_L) \in (\mathbb{R}_{>0})^{3L}\), there is a unique positive real solution \((v, b_1', \ldots, b_L') \in (\mathbb{R}_{>0})^{3(L+1)}\) to the equation

\[ g(b_1, s; \lambda) \cdots g(b_L, s; \lambda)g(v, l; \lambda) = g(v, l; \lambda)g(b_1', s; \lambda) \cdots g(b_L', s; \lambda). \quad (40) \]

For any \(|b| = (b_1, \ldots, b_L)\), let \( \mathcal{L}(|b|; \lambda) \) be the matrix

\[ \mathcal{L}(|b|; \lambda) = g(b_1, s; \lambda) \cdots g(b_L, s; \lambda), \quad (41) \]

which is called a Lax matrix, and let \( \mathcal{M}_0^{(1)}(|b|) \) be the matrix

\[ \mathcal{M}_0^{(1)}(|b|) = g^*(b_1, s; l) \cdots g^*(b_L, s; l), \quad (42) \]

which we call the monodromy matrix of the Lax matrix \( \mathcal{L}(|b|; \lambda) \) with \( \lambda = l \). Note that every matrix element of \( \mathcal{M}_0^{(1)}(|b|) \) is an order 3 minor of \( \mathcal{L}(|b|; l) \). Note also that, if the eigenvalues of \( \mathcal{L}(|b|; l) \) are given by \( \mu_1, \mu_2, \mu_3, \mu_4 \), then the eigenvalues of \( \mathcal{M}_0^{(1)}(|b|) \) are given by \( \mu_1 \mu_2 \mu_3, \mu_1 \mu_2 \mu_4, \mu_1 \mu_3 \mu_4, \mu_2 \mu_3 \mu_4 \), where the multiplicity of the eigenvalues has been taken into account (Corollary 29 of [26]).

Let \( E \) be the largest eigenvalue in absolute value of matrix \( \mathcal{M}_0^{(1)}(|b|) \), and \( \vec{P} = (P_1, P_2, P_3, 1)^t \) be an eigenvector corresponding to \( E \). By the Perron-Frobenius theorem, \( E \) is real positive, \( \vec{P} \) is uniquely determined, and \( P_1, P_2, P_3 \) are all positive. Then, the solution \( v \in (\mathbb{R}_{>0})^3 \) of equation (40) is given by \( v = (P_3, P_2/P_3, P_1/P_2) \) (Proposition 15 of [26]). This unique solution \( v \) allows us to define \( T_1^{(1)} : (\mathbb{R}_{>0})^{3L} \to (\mathbb{R}_{>0})^{3L} \) to be the map given by

\[ T_1^{(1)}(b_1, \ldots, b_L) = (b_1', \ldots, b_L'), \quad (43) \]

which is called a time evolution. Due to equation (40), this time evolution (43) is governed by a discrete time analogue of the Lax equation

\[ \mathcal{L}(T_1^{(1)}|b|; \lambda) = g(v, l; \lambda)^{-1} \mathcal{L}(|b|; \lambda)g(v, l; \lambda). \quad (44) \]
In what follows, we are going to show that this equation reduces to the continuous time Lax equation (7) in the limit \( l \to \infty \), or to equation (27) in the limit \( l \to 0 \), by making several reasonable assumptions.

4.2. Limit for the type I Lax equation

4.2.1. The case of \( p=1 \). We assume \( L \equiv 1 \pmod{4} \), and set \( L = 4\kappa + 1 \). Recall that \(|b\rangle = (b_1, \ldots, b_L)\) and \( b_i = (b_i^{(1)}, b_i^{(2)}, b_i^{(3)})\). By the reason explained above Proposition 2 the Lax matrix (41) can be identified with the matrix \( L \) defined by (4), in which the loop elementary symmetric functions are defined by (2) but with the substitution \( u_\alpha^{(a)} = b_i^{(a)} \). Using the explicit expression (4) and the condition \( L = 4\kappa + 1 \), we can obtain the asymptotic form of the Lax matrix \( L(|b\rangle; l) \) under the limit \( l \to \infty \) as

\[
L(|b\rangle; l) \approx \begin{pmatrix}
    e_1^{(1)} l^\kappa & e_2^{(1)} l^\kappa & e_3^{(1)} l^\kappa & \cdots \\
    e_1^{(2)} l^\kappa & e_2^{(2)} l^\kappa & e_3^{(2)} l^\kappa & \cdots \\
    e_1^{(3)} l^{\kappa-1} & e_2^{(3)} l^{\kappa-1} & e_3^{(3)} l^{\kappa-1} & \cdots \\
    e_1^{(4)} l^{\kappa-1} & e_2^{(4)} l^{\kappa-1} & e_3^{(4)} l^{\kappa-1} & \cdots
\end{pmatrix}.
\]

We assume that the eigenvalues \( \eta_q (q \in \mathbb{Z}/4\mathbb{Z}) \) of the Lax matrix \( L(|b\rangle; l) \) for sufficiently large \( l \)'s are given by the Puiseux series expansion

\[
\eta_q = l^\kappa \sum_{m=-1}^{\infty} c_m \exp \left( \frac{\pi \sqrt{-1} m q}{2} \right) l^{-m/4},
\]

where \( c_{-1} = 1 \) and \( c_0 = (\sum_{\alpha=1}^{4} e_1^{(\alpha)})/4 \). This assumption is consistent with the relations

\[
(s - l)^L = \det L(|b\rangle; l) = \prod_{q=1}^{4} \eta_q = -l^{4\kappa+1} + \mathcal{O}(l^{4\kappa}),
\]

\[
(\sum_{\alpha=1}^{4} e_1^{(\alpha)}) l^\kappa + \mathcal{O}(l^{\kappa-1}) = \text{Tr} L(|b\rangle; l) = \sum_{q=1}^{4} \eta_q = 4l^{\kappa} c_0 + \mathcal{O}(l^{\kappa-1}),
\]

where \( \mathcal{O} \) denotes Landau’s symbol, and we used the identity \( \det g(b_i, s; l) = s - l \) for \( 1 \leq i \leq L \) and the asymptotic form (45). For reference, we show an explicit derivation of such series expansions for a simpler case of \( n = 2 \) in \textbf{Appendix B}. Then, the asymptotic form of the largest eigenvalue of the monodromy matrix \( M_1^{(1)}(|b\rangle) \) is given by

\[
E = \eta_1 \eta_3 \eta_4 = l^{3\kappa + \frac{3}{4}} + c_0 l^{3\kappa + \frac{3}{4}} + \ldots.
\]

(47)
In view of (45), we see that the asymptotic form of the monodromy matrix \( M^{(l)}(\mathbf{b}) \) under the limit \( l \to \infty \) is given by

\[
M^{(l)}(\mathbf{b}) \approx \begin{pmatrix}
O(l^{3\kappa}) & t^{3\kappa+1} e_1^{(3)} & O(l^{3\kappa+1}) \\
O(l^{3\kappa}) & O(l^{3\kappa}) & t^{3\kappa+1} e_1^{(2)} \\
e_1^{(1)} l^{3\kappa} & \frac{e_1^{(2)}}{l^{3\kappa}} & e_1^{(2)} l^{3\kappa+1}
\end{pmatrix},
\]

where the last row is omitted because we do not need it. Let \( M_{ij} \) denote the \( ij \) element of \( M^{(l)}(\mathbf{b}) \). Then, the eigenvector \( \vec{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, 1)^t \) corresponding to the eigenvalue \( E \) is determined by the linear equation

\[
\begin{pmatrix}
M_{11} - E & M_{12} & M_{13} \\
M_{21} & M_{22} - E & M_{23} \\
M_{31} & M_{32} & M_{33} - E
\end{pmatrix}
\begin{pmatrix}
\mathcal{P}_1 \\
\mathcal{P}_2 \\
\mathcal{P}_3
\end{pmatrix}
= -
\begin{pmatrix}
M_{14} \\
M_{24} \\
M_{34}
\end{pmatrix}.
\]

Let \( l = 1/\delta^4 \). By expressing the solution of this equation by the Cramer formula and then substituting the asymptotic forms (47) and (48) into it, we obtain the expressions

\[
\mathcal{P}_1 = \frac{1}{\delta^3} + (e_1^{(1)} + e_1^{(2)} + e_1^{(3)} - 3c_0) \frac{1}{\delta^2} + \mathcal{O}\left(\frac{1}{\delta}\right),
\]

\[
\mathcal{P}_2 = \frac{1}{\delta^2} + (e_1^{(1)} + e_1^{(2)} - 2c_0) \frac{1}{\delta} + \mathcal{O}\left(1\right),
\]

\[
\mathcal{P}_3 = \frac{1}{\delta} + (e_1^{(1)} - c_0) + \mathcal{O}\left(\delta\right).
\]

We show a detailed derivation of this result in Appendix A.1. Therefore, the asymptotic form of the matrix \( g(\mathbf{v}, l; \lambda) \) with \( \mathbf{v} = (\mathcal{P}_3, \mathcal{P}_2/\mathcal{P}_3, \mathcal{P}_1/\mathcal{P}_2) \) and \( l = 1/\delta^4 \) is expressed as

\[
\delta \cdot g(\mathbf{v}, 1/\delta^4; \lambda) = \delta \cdot 
\begin{pmatrix}
\mathcal{P}_3 \\
\frac{1}{\mathcal{P}_3} \mathcal{P}_2/\mathcal{P}_3 \\
\frac{1}{\mathcal{P}_1/\mathcal{P}_2} \frac{1}{\mathcal{P}_1}
\end{pmatrix}
\begin{pmatrix}
\lambda \\
1 \\
1/(\delta^4 \mathcal{P}_1)
\end{pmatrix}
= \mathbb{I}_4 + \delta \cdot \mathcal{Y} + \mathcal{O}\left(\delta^2\right),
\]

where \( \mathcal{Y} \) is a matrix of the form (6) with \( n = 4, p = 1 \).

To summarize, we state the result of the above arguments in the case of \( n = 4, p = 1 \) as:

**Theorem 13** Let \( \mathcal{L}(\mathbf{b}; \lambda) \) and \( \mathcal{L}(T^{(1)}(\mathbf{b}); \lambda) \) be denoted by \( \mathcal{L}(l) \) and \( \mathcal{L}(l + \delta) \), respectively, where \( l = 1/\delta^4 \). Assume that the eigenvalues of the matrix \( \mathcal{L}(\mathbf{b}; l) \) for sufficiently large \( l \)’s are given the expansion (46). Then, the discrete time Lax equation (44) reduces to the continuous time Lax equation (7) in the limit \( \delta \to 0 \).

**Proof.** By using the expression (51), we obtain

\[
\mathcal{L}(l + \delta) = (\delta \cdot g(\mathbf{v}, 1/\delta^4; \lambda))^{-1} \mathcal{L}(l)(\delta \cdot g(\mathbf{v}, 1/\delta^4; \lambda)) = \mathcal{L}(l) + \delta \cdot [\mathcal{L}(l), \mathcal{Y}] + \mathcal{O}\left(\delta^2\right),
\]

from equation (44). Hence the assertion of the theorem follows. \( \square \)
4.2.2. The case of \( p=3 \). We assume \( L \equiv 3 \pmod{4} \), and set \( L = 4\kappa + 3 \). As in the previous case, we can obtain the asymptotic form of the Lax matrix \( \mathcal{L}(|b|; l) \) under the limit \( l \to \infty \) as

\[
\mathcal{L}(|b|; l) \approx \begin{pmatrix}
    e_3^{(1)} l^\kappa & e_3^{(1)} l^{\kappa+1} & e_2^{(1)} l^{\kappa+1} & e_2^{(1)} l^{\kappa+1} \\
    e_2^{(2)} l^\kappa & e_3^{(2)} l^\kappa & e_3^{(2)} l^\kappa & e_3^{(2)} l^\kappa \\
    e_3^{(1)} l^\kappa & e_3^{(1)} l^\kappa & e_2^{(3)} l^\kappa & e_2^{(3)} l^\kappa \\
    l^\kappa & l^\kappa & l^\kappa & l^\kappa
\end{pmatrix}.
\]

(52)

We assume that the eigenvalues \( \eta_q \) \((q \in \mathbb{Z}/4\mathbb{Z})\) of the Lax matrix \( \mathcal{L}(|b|; l) \) for sufficiently large \( l \)'s are given by the Puiseux series expansion

\[
\eta_q = l^{\kappa + \frac{1}{4}} \sum_{m=-1}^{\infty} c_m \exp\left( \frac{\pi \sqrt{-1}(m - 2)q}{2} \right) l^{-m/4},
\]

(53)

where \( c_{-1} = 1 \) and \( c_2 = \left( \sum_{\alpha=1}^{4} e_3^{(\alpha)} \right)/4 \). This assumption is consistent with the relations

\[
(s - l)^L = \det \mathcal{L}(|b|; l) = \prod_{q=1}^{4} \eta_q = -l^{4\kappa+3} + \mathcal{O}(l^{4\kappa+2}),
\]

\[
\left( \sum_{\alpha=1}^{4} e_3^{(\alpha)} \right) l^\kappa + \mathcal{O}(l^{\kappa-1}) = \text{Tr} \mathcal{L}(|b|; l) = \sum_{q=1}^{4} \eta_q = 4l^\kappa c_2 + \mathcal{O}(l^{\kappa-1}),
\]

where we used the asymptotic form (52). Then, the asymptotic form of the largest eigenvalue of the monodromy matrix \( M_l^{(1)}(|b|) \) is given by

\[
E = \eta_1 \eta_3 \eta_4 = l^{3\kappa + \frac{2}{5}} + c_0 l^{3\kappa + \frac{2}{5}} + \ldots.
\]

(54)

In view of (52), we see that the asymptotic form of the monodromy matrix \( M_l^{(1)}(|b|) \) under the limit \( l \to \infty \) is given by

\[
M_l^{(1)}(|b|) \approx \begin{pmatrix}
    e_3^{(3)} l^{3\kappa+2} & \mathcal{O}(l^{3\kappa+2}) & \mathcal{O}(l^{3\kappa+2}) & l^{3\kappa+3} \\
    l^{3\kappa+2} & e_3^{(2)} l^{3\kappa+2} & \mathcal{O}(l^{3\kappa+2}) & \mathcal{O}(l^{3\kappa+2}) \\
    \mathcal{O}(l^{3\kappa+1}) & l^{3\kappa+2} & e_1^{(1)} l^{3\kappa+2} & \mathcal{O}(l^{3\kappa+2}) \\
    \mathcal{O}(l^{3\kappa+1}) & \mathcal{O}(l^{3\kappa+1}) & l^{3\kappa+2} & e_1^{(4)} l^{3\kappa+2}
\end{pmatrix}.
\]

(55)

From (54) and (55) one sees that

\[
0 = \det(M_l^{(1)}(|b|)) - E l^4 = (4c_0 - (e_1^{(1)} + e_1^{(2)} + e_3^{(3)} + e_4^{(4)})) l^{12\kappa+8+\frac{4}{5}} + \mathcal{O}(l^{12\kappa+8+\frac{4}{5}}),
\]

which forces us to set \( c_0 = \left( \sum_{\alpha=1}^{4} e_1^{(\alpha)} \right)/4 \). Let \( l = 1/\delta^4 \). By expressing the solution of equation (49) by the Cramer formula and then substituting the asymptotic forms (54)
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and (55) into it, we obtain the expressions

\[ \mathcal{P}_1 = \frac{1}{\delta^4} + (e_1^{(3)} - c_0)\frac{1}{\delta^2} + \mathcal{O}\left(\frac{1}{\delta}\right), \]

\[ \mathcal{P}_2 = \frac{1}{\delta^2} + (e_1^{(2)} + e_1^{(3)} - 2c_0)\frac{1}{\delta} + \mathcal{O}\left(1\right), \]

\[ \mathcal{P}_3 = \frac{1}{\delta} + (\epsilon_1^{(1)} + \epsilon_1^{(2)} + \epsilon_1^{(3)} - 3c_0) + \mathcal{O}\left(\delta\right). \]  

(56)

From this result and equation (17), one sees that the asymptotic form of the matrix \( g(v, l; \lambda) \) with \( v = (\mathcal{P}_3, \mathcal{P}_2/\mathcal{P}_3, \mathcal{P}_1/\mathcal{P}_2) \) and \( l = 1/\delta^4 \) is expressed as equation (51), in which \( \mathcal{Y} \) is a matrix of the form (6) with \( n = 4, p = 3 \). Hence we obtain the continuous time Lax equation (7) in the limit \( \delta \to 0 \), in the same way as for the case of \( p = 1 \).

4.3. Limit for the type II Lax equation

As in the previous subsection, we assume that \( n = 4 \) but now let \( L(\geq 2) \) be an arbitrary integer. We define

\[ U_\lambda = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}. \]  

(57)

Consider the matrix (41) with \( s = 1 \) and rewrite equation (41) as

\[ \mathcal{L}(T_l^{(1)}b; \lambda) = (U_\lambda^{-1}g(v, l; \lambda))^{-1} \cdot (U_\lambda^{-1}\mathcal{L}(|b\rangle; \lambda)U_\lambda) \cdot (U_\lambda^{-1}g(v, l; \lambda)). \]  

(58)

Accordingly, we adopt the interpretation

\[ \mathcal{L}(t) = U_\lambda^{-1}\mathcal{L}(|b\rangle; \lambda)U_\lambda, \quad \mathcal{L}(t + d\delta) = \mathcal{L}(T_l^{(1)}b; \lambda), \]  

(59)

where \( d \) is a parameter that will be determined later. Let them to be identified with the matrix \( \mathcal{L} \) given by (1), but in which the set of variables for the loop elementary symmetric functions (2) should be interpreted as

\[ u_i^{(\alpha - 1)}(t) = b_i^{(\alpha)} \quad \text{and} \quad u_i^{(\alpha)}(t + d\delta) = (b_i')^{(\alpha)}, \]  

(60)

respectively. Here we used the notations defined by \( T_l^{(1)}b = (b_1', \ldots, b_L') \) and \( b_i' = ((b_1')^{(1)}, (b_1')^{(2)}, (b_1')^{(3)}). \) The first equation of (60) implies that in the matrix \( \mathcal{L}(|b\rangle; \lambda) \) we must replace \( e_m^{(\alpha)} \) by \( e_m^{(\alpha - 1)} \) when we change its variables from the \( b \)'s to the \( u \)'s. After this replacement in (1) and with the condition \( e_L^{(\alpha)} = 1 \) for all \( \alpha \), the asymptotic form of the Lax matrix \( \mathcal{L}(|b\rangle; l) \) under the limit \( l \to 0 \) is given by

\[ \mathcal{L}(|b\rangle; l) \approx \begin{pmatrix} 1 & e_L^{(4)} & e_L^{(4)} & e_L^{(4)} \\ e_L^{(1)} & 1 & e_L^{(1)} & e_L^{(1)} \\ e_L^{(2)} & e_L^{(2)} & 1 & e_L^{(2)} \\ e_L^{(3)} & e_L^{(3)} & e_L^{(3)} & 1 \end{pmatrix}. \]  

(61)
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Note that the condition $e_L^{(a)} = 1$ is preserved under the time evolution $T_l^{(1)}$, which can be obtained from equation (40) with $\lambda = 0$. We assume that the eigenvalues $\xi_q (q \in \mathbb{Z}/4\mathbb{Z})$ of the Lax matrix $L(\{|b\}; l)$ under the condition $s = 1$ and for sufficiently small $l$’s are given by the Puiseux series expansion

$$\xi_q = \sum_{m=0}^{\infty} d_m \exp \left( \frac{\pi \sqrt{-1}mq}{2} \right) l^{m/4},$$

where $d_0 = 1$. This assumption is consistent with the relations

$$(1-l)^L = \det L(\{|b\}; l) = \prod_{q=1}^{4} \xi_q = (d_0)^4 + O(l),$$

$$4 + O(l) = \text{Tr} L(\{|b\}; l) = \sum_{q=1}^{4} \xi_q = 4d_0 + O(l),$$

where we used the identity $\det g(b_i, 1; l) = 1 - l$ for $1 \leq i \leq L$, and the expression (61). Then, the asymptotic form of the largest eigenvalue of the monodromy matrix $M_l^{(1)}(\{|b\})$ is given by

$$E = \xi_1 \xi_3 \xi_4 = 1 + d_1 l^{1/4} + \cdots .$$

From the expression (61), we see that the asymptotic form of the monodromy matrix $M_l^{(1)}(\{|b\})$ under the limit $l \to 0$ is given by

$$M_l^{(1)}(\{|b\}) \approx \begin{pmatrix} 1 & O(l) & O(l) & e_L^{(4)}(l) \\ e_{L-1}^{(3)} & 1 & O(l) & O(l) \\ O(1) & e_{L-1}^{(2)} & 1 & O(l) \\ O(1) & O(1) & e_{L-1}^{(1)} & 1 \end{pmatrix}.$$

From (63) and (64) one sees that

$$0 = \det(M_l^{(1)}(\{|b\}) - E \mathbb{1}_4) = ((d_1)^4 - e_{L-1}^{(1)} e_{L-1}^{(2)} e_{L-1}^{(3)} e_{L-1}^{(4)}) l + O(l^{5/2}),$$

which forces us to set $d_1 = (\prod_{a=1}^{4} e_{L-1}^{(a)})^{1/4}$. Let $l = \delta^4$. Again by using the solution of equation (19) expressed by the Cramer formula and then now substituting the asymptotic forms (63) and (64) into it, we obtain the expressions

$$P_1 = \frac{1}{d_1} e_{L-1}^{(4)} \delta^3 + O(\delta^4),$$

$$P_2 = \frac{1}{(d_1)^2} e_{L-1}^{(4)} e_{L-1}^{(3)} \delta^2 + O(\delta^3),$$

$$P_3 = \frac{1}{(d_1)^3} e_{L-1}^{(4)} e_{L-1}^{(3)} e_{L-1}^{(2)} \delta + O(\delta^2) = \frac{d_1}{e_{L-1}^{(1)}} \delta + O(\delta^2).$$
We show a detailed derivation of this result in Appendix A.2. Therefore, the asymptotic form of the matrix \( U^{-1}_\lambda g(v, \delta^4; \lambda) \) with \( v = (P_3/P_2, P_1/P_2) \) and \( l = \delta^4 \) is expressed as

\[
U^{-1}_\lambda g(v, \delta^4; \lambda) = \begin{pmatrix}
1 & P_2/P_3 & 1 & \delta^4/P_1 \\
P_3/\lambda & 1 & \delta^4/P_1 & 1 \\
\end{pmatrix} = \mathbb{I}_4 + d_1 \delta \cdot Z + \mathcal{O}(\delta^2), \tag{66}
\]

where \( Z \) is a matrix of the form \( (25) \) with \( n = 4 \).

To summarize, we state the result of the above arguments in the case of \( n = 4, s = 1 \) as:

**Theorem 14** Let \( U^{-1}_\lambda \mathcal{L}(b; \lambda) U_\lambda \) and \( \mathcal{L}(T^{-1}_l(b; \lambda) \) be denoted by \( \mathcal{L}(t) \) and \( \mathcal{L}(t + d\delta) \), respectively, where \( d = d_1 = (\prod_{\alpha=1}^{l} e_{\alpha}^{(\alpha)-1}/4 \) and \( l = \delta^4 \). Assume that the condition \( e_{\alpha}^{(\alpha)} = 1 \) is satisfied for all \( \alpha \), and that the eigenvalues of the matrix \( \mathcal{L}(b; \lambda) \) for sufficiently small \( l \)'s are given the expansion \( (62) \). Then, the discrete time Lax equation \( (14) \) reduces to the continuous time Lax equation \( (27) \) in the limit \( \delta \to 0 \).

**Proof.** By using the expression \( (66) \), we obtain

\[
\mathcal{L}(t + d\delta) = (U^{-1}_\lambda g(v, \delta^4; \lambda))^{-1} \mathcal{L}(t)(U^{-1}_\lambda g(v, \delta^4; \lambda)) = \mathcal{L}(t) + d\delta \cdot [\mathcal{L}(t), Z] + \mathcal{O}(\delta^2),
\]

from equation \( (58) \), which was equivalent to equation \( (14) \). Hence the assertion of the theorem follows. \( \square \)

4.4. Limits for the type I and II differential equations

The discussions for deriving the Lax equations in the previous two subsections can be generalized to those for derivations of equations \( (11) \) and \( (24) \). Consider the following matrix equation

\[
g(b, s; \lambda)g(a, l; \lambda) = g(a', l; \lambda)g(b', s; \lambda). \tag{67}
\]

For any \( s, l \in \mathbb{R}_{>0} \) and \( (a, b) \in (\mathbb{R}_{>0})^6 \), there is a unique solution \( (a', b') \in (\mathbb{R}_{>0})^6 \) to this matrix equation. Let \( R^{(s,l)} : (\mathbb{R}_{>0})^6 \to (\mathbb{R}_{>0})^6 \) be a rational map given by \( R^{(s,l)} : (b, a) \mapsto (a', b') \). This is the geometric R-matrix in the present case. An explicit expression for the rational map is written as

\[
a'(1) = a^{(1)}(a^{(2)}a^{(3)}a^{(4)} + a^{(2)}a^{(3)}b^{(4)} + a^{(2)}b^{(4)}b^{(1)} + b^{(3)}b^{(4)}b^{(1)}),
\]

\[
a'(2) = a^{(2)}(a^{(3)}a^{(4)}a^{(1)} + a^{(3)}a^{(4)}b^{(2)} + a^{(3)}b^{(2)}b^{(1)} + b^{(4)}b^{(1)}b^{(2)}),
\]

\[
a'(3) = a^{(3)}(a^{(4)}a^{(1)}a^{(2)} + a^{(4)}a^{(1)}b^{(3)} + a^{(4)}b^{(2)}b^{(3)} + b^{(1)}b^{(2)}b^{(3)}),
\]

\[
a'(4) = a^{(4)}a^{(1)}a^{(2)} + a^{(4)}a^{(1)}b^{(3)} + a^{(4)}b^{(2)}b^{(3)} + b^{(1)}b^{(2)}b^{(3)}),
\]

\[
a'(5) = a^{(5)}a^{(1)}a^{(2)} + a^{(5)}a^{(1)}b^{(3)} + a^{(5)}b^{(2)}b^{(3)} + b^{(1)}b^{(2)}b^{(3)}),
\]

\[
a'(6) = a^{(6)}a^{(1)}a^{(2)} + a^{(6)}a^{(1)}b^{(3)} + a^{(6)}b^{(2)}b^{(3)} + b^{(1)}b^{(2)}b^{(3)}).
\]
and \(b'^{(i)} = a^{(i)}b^{(i)}/a^{(i)}\) [29]. Let the map \(R^{(s,l)}: (b, a) \mapsto (a', b')\) be depicted as

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\hline
\text{a'} \\
\text{b'}
\end{array}
\]

Then the solution of equation (40) satisfies the relations in the diagram

\[
v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_{L-1} \quad v_L \quad v
\]

where \(v_i\)'s are defined by the downward recursion relation \(R^{(s,l)}(b_i, v_{i+1}) = (v_i, b'_i)\) with the initial condition \(v_{L+1} = v\), where \(v\) is determined by the Perron-Frobenius eigenvector of the monodromy matrix \(M^{(1)}_i([b])\) as described in §4.1. As in §2.2 let \(\sigma\) denote the cyclic shift to the left, so we have \(\sigma |b) = (b_2, \ldots, b_L, b_1)\). It is easy to see that an obvious generalization of equation (44) is

\[
\mathcal{L}(T^{(1)}_l(\sigma^{i-1}|b); \lambda) = g(v_i, l; \lambda)^{-1}\mathcal{L}(\sigma^{i-1}|b); \lambda)g(v_i, l; \lambda). \tag{69}
\]

This equation implies that \(v_i\) can also be obtained in the same way as for \(v\) in §4.1 by simply replacing \(M^{(1)}_i([b])\) by \(M^{(1)}_i(\sigma^{i-1}|b))\). Based on this fact, one can generalize equation (51) as

\[
\delta \cdot g(v_i, 1/\delta^4; \lambda) = \mathbb{I}_4 + \delta \cdot \mathcal{Y}_i + \mathcal{O}(\delta^2), \tag{70}
\]

where \(\mathcal{Y}_i\) is a matrix of the form (10) with \(n = 4, p = 1\), and equation (69) as

\[
U_{\lambda}^{-1}g(v_i, \delta^4; \lambda) = \mathbb{I}_4 + d_i \delta \cdot \mathcal{Z}_i + \mathcal{O}(\delta^2), \tag{71}
\]

where \(\mathcal{Z}_i\) is a matrix of the form (29) with \(n = 4\).

First we consider type I case, where \(l = 1/\delta^4\). Around each vertex in the diagram (68) we have the relation

\[
g(b'_i, s; \lambda) = (\delta \cdot g(v_i, l; \lambda))^{-1}g(b_i, s; \lambda)(\delta \cdot g(v_{i+1}, l; \lambda)).
\]

Then by setting

\[
\mathcal{M}_i(t + \delta) = g(b'_i, s; \lambda), \quad \mathcal{M}_i(t) = g(b_i, s; \lambda),
\]

and using (70), we can derive equation (11) in the limit \(\delta \to 0\).

Second we consider type II case, where \(l = \delta^4, s = 1\), and with the condition \(e^{(\alpha)}_L = 1\) for all \(\alpha\). As in the previous case, we have the relation

\[
g(b'_i, 1; \lambda) = (U_{\lambda}^{-1}g(v_i, l; \lambda))^{-1}(U_{\lambda}^{-1}g(b_i, 1; \lambda)U_{\lambda})(U_{\lambda}^{-1}g(v_{i+1}, l; \lambda)).
\]
Then by setting
\[ \mathcal{M}_i(t + d_1 \delta) = g(b_i^t, 1; \lambda), \quad \mathcal{M}_i(t) = U_{-1}^{-1}g(b_i, 1; \lambda)U_{\lambda}, \]
and using (71), we can derive equation (30) in the limit \( \delta \to 0 \).

Finally, we conclude that equations (11) for \( p = 1 \) and (24) are derived from the closed geometric crystal chains by the above arguments and using Propositions 2 and 8 respectively.

5. Concluding remarks

5.1. On a related discrete dynamical system

In order to avoid a potential confusion, we present a discussion to illustrate the difference between the closed geometric crystal chain and another discrete dynamical system in reference [12], which is also defined by using the geometric \( R \) matrix.

First we consider the closed geometric crystal chain. Let \( R^{(s,l)} : (\mathbb{R}_{>0})^{2(n-1)} \to (\mathbb{R}_{>0})^{2(n-1)} \) be a rational map that is essentially the same one in § 4.4 but the condition \( n = 4 \) has been generalized to for an arbitrary positive integer \( n \geq 2 \). Let \( R^{(s,l)}_i \) be a map from \( (\mathbb{R}_{>0})^{(L+1)(n-1)} = (\mathbb{R}_{>0})^{n-1} \times \cdots \times (\mathbb{R}_{>0})^{n-1} \) to itself, which acts as the map \( R^{(s,l)} \) on factors \( i \) and \( i + 1 \), and as the identity on the other factors. We define \( R^{(s,l)} = R^{(s,l)}_1 \circ \cdots \circ R^{(s,l)}_L \), which are maps from \( (\mathbb{R}_{>0})^{(L+1)(n-1)} \) to itself. Then the diagram (68) for general \( n \) implies that we have the relation \( R^{(s,l)}(b_1, \ldots, b_L, v) = (v, b_1', \ldots, b_L') \), which enables us to define the time evolution \( T^{(1)}_i : (\mathbb{R}_{>0})^{(n-1)L} \to (\mathbb{R}_{>0})^{(n-1)L} \) and to obtain its Lax representation, in the same way as in (43) and (44), respectively. The above \( v \) is determined by the Perron-Frobenius eigenvector of the monodromy matrix \( M^{(1)}_i(b) \) as in the case of \( n = 4 \) in § 4.4 The commutativity of the time evolutions \( T^{(1)}_{l_1} \circ T^{(1)}_{l_2} = T^{(1)}_{l_2} \circ T^{(1)}_{l_1} \) is satisfied for any \( l_1, l_2 \in \mathbb{R}_{>0} \), as a consequence of the fact that the maps \( R^{(s,l)}_i \) are obeying the Yang-Baxter relation \( R^{(s,l_2)}_i R^{(s,l_1)}_{i+1} R^{(l_2,l_1)}_{i+1} = R^{(l_2,l_1)}_i R^{(s,l_1)}_{i+1} R^{(s,l_2)}_i \) and the involution \( R^{(L+1)}_{L+1} \circ R^{(l_1,l_2)}_{L+1} = \text{Id} \). The Lax equation (44) implies that the characteristic polynomial of the matrix \( \mathcal{L}(b; \lambda) \) in (44) is invariant under the actions of the time evolutions \( T^{(1)}_i \) for any \( l \in \mathbb{R}_{>0} \), hence its coefficients are the conserved quantities of the closed geometric crystal chain.

Second we consider the discrete dynamical system in [12], but slightly modified to be consistent with the notations used in this paper. In this case, we choose the values of the parameter \( s \) attached to each site unequally. So let \( |s\rangle = (s_1, \ldots, s_L) \) and define \( \mathcal{L}(b; s; \lambda) \) to be the matrix
\[ \mathcal{L}(b, s; \lambda) = g(b_1, s_1; \lambda) \cdots g(b_L, s_L; \lambda), \] (72)
which is called a Lax matrix. As in the previous paragraph, let \( R_{i}^{(s_{i}, s_{i+1})} \) be a map from \( (\mathbb{R}_{>0})^{(L+1)(n-1)} = (\mathbb{R}_{>0})^{n-1} \times \cdots \times (\mathbb{R}_{>0})^{n-1} \) to itself, which acts as the map \( R^{(s_{i}, s_{i+1})} \) on factors \( i \) and \( i+1 \), and as the identity on the other factors. More precisely, the map \( R_{i}^{(s_{i}, s_{i+1})} \) acts on the set \( (\mathbb{R}_{1})^{L} \). Applied to \( (|b\rangle, |s\rangle) \in (\mathbb{R}_{1})^{L} \), it sends \( |b\rangle = (b_{1}, \ldots, b_{i}, b_{i+1}, \ldots, b_{L}) \) to \( (b_{1}, \ldots, b'_{i+1}, b'_{i}, \ldots, b_{L}) \) where \( (b'_{i+1}, b'_{i}) = R_{i}^{(s_{i}, s_{i+1})}(b_{i}, b_{i+1}) \), and \( |s\rangle = (s_{1}, \ldots, s_{i}, s_{i+1}, \ldots, s_{L}) \) to \( (s_{1}, \ldots, s_{i+1}, s_{i}, \ldots, s_{L}) \). Let \( Y_{i} (1 \leq i \leq L) \) be the map defined by

\[
Y_{i} = R_{i}^{(s_{i+1}, s_{i})} \circ \cdots \circ R_{L-1}^{(s_{L}, s_{L-1})} \circ \sigma \circ R_{1}^{(s_{1}, s_{1})} \circ \cdots \circ R_{i-1}^{(s_{i-1}, s_{i})},
\]

where \( \sigma \) denotes the cyclic shift to the left. If viewed as a map on the set \( (\mathbb{R}_{1})^{L} \), it does not change \( |s\rangle = (s_{1}, \ldots, s_{L}) \). So it can be simply regarded as a map on the set \( (\mathbb{R}_{>0})^{(L+1)(n-1)} \) and the relation \( Y_{i}(|b\rangle) = |b\rangle \) can be described by the following diagram

\[
\begin{array}{cccccc}
& b_{1} & \cdots & b_{i} & b_{i+1} & b_{L} \\
\vdots & v_{1} & \cdots & v_{i} & v_{i+1} & v_{L} \\
& b'_{1} & \cdots & b'_{i} & b'_{i+1} & b'_{L} \\
\end{array}
\]

The commutativity of the rational maps \( Y_{i} \circ Y_{j} = Y_{j} \circ Y_{i} \) is satisfied for any \( 1 \leq i, j \leq L \). It is proved by using the property of the maps \( R_{i}^{(s_{i}, s_{j})} \) obeying the Yang-Baxter relation, and it also reflects the fact that the \( Y_{i} \)s for \( 1 \leq i \leq L - 1 \) can be interpreted as generators of the translation subgroup of the extended affine Weyl group \( \hat{W}(A_{L-1}^{(1)}) \) [12]. The commutativity of this type of transfer maps defined as a composition of general Yang-Baxter maps was discussed in [28], where its original idea was attributed to a result in the classical paper [30] by C. N. Yang. Following this historical background, we shall call the discrete dynamical system described by this set of rational maps \( Y_{i} \) a birational Yang’s system, because its tropical counterpart was referred to as a combinatorial Yang’s system [16] by the same reason.

In order to see a Lax representation of this system, we consider the map \( Y_{L} \). Let \( Y_{L}(|b\rangle) = |b\rangle = (b_{1}, \ldots, b_{L}) \). Then we have

\[
g(b_{1}, s_{1}; \lambda) \cdots g(b_{L}, s_{L}; \lambda) = g(b_{L}, s_{L}; \lambda)g(b_{1}, s_{1}; \lambda) \cdots g(b_{L-1}, s_{L-1}; \lambda).
\]

Comparing this with equation (72), we obtain the relation

\[
L(Y_{L}(|b\rangle, |s\rangle; \lambda) = g(b_{L}, s_{L}; \lambda)^{-1}L(|b\rangle, |s\rangle; \lambda)g(b_{L}, s_{L}; \lambda),
\]

which can be regarded as a Lax equation. By applying the cyclic shift operator \( \sigma \) repeatedly, one can obtain Lax representations for all the other time evolutions \( Y_{i} \). Therefore, the characteristic polynomial of the matrix \( L(|b\rangle, |s\rangle; \lambda) \) is invariant under the actions of the time evolutions \( Y_{i} \)s for any \( 1 \leq i \leq L \), because it is common to that of \( L(\sigma^{i}|b\rangle, \sigma^{i}|s\rangle; \lambda) \) for all \( i \)s. Hence its coefficients are the conserved quantities of the birational Yang’s system.
For the closed geometric crystal chain, the Lax matrix $L(b; \lambda)$ is independent of the parameter $l$, which is in the companion matrix $g(x, l; \lambda)$ of the Lax pair in equation (44). Therefore, without changing the Lax matrix, one can consider the limit for taking $l \to \infty$ and that for $l \to 0$ in equation (44) to obtain the continuous time Lax equations. As a result, we were able to obtain integrable differential equations that share a Lax matrix in common with a discrete dynamical system, at least for the cases treated in §4. In contrast, for the birational Yang’s system, the Lax matrix $L(b, s; \lambda)$ depends on the same parameter $s_L$ in common with the companion matrix $g(\tilde{b}_L, s_L; \lambda)$ of the Lax pair in equation (74). Therefore, one can not apply the same method to take the continuum limits. However, it seems reasonable to expect that a generalization of the symmetric form of the Painlevé equations $P_{IV}$ and $P_V$ in [22], which is also regarded as a variation of extended Lotka-Volterra systems, can be obtained from this discrete dynamical system by taking a continuum limit through a similar method.

5.2. An outlook for generalizations

In §4 we restricted ourselves to the case of $n = 4$ for deriving the differential equations. It seems that the above derivations can also be applied to the case of general $n$, but a remark on the largest eigenvalue of the monodromy matrix may be in order. In this generalization, the matrices corresponding to (38) and (39) are connected by the relation where elements of the matrix $g^*(x, s; \lambda)$ are so defined as to be order $n - 1$ minors of the matrix $g(x, s; (-1)^n \lambda)$ (See §3.1.1 of reference [26]). Consider the type I case with $L = n\kappa + 1$. As in equation (46), suppose that the eigenvalues $\eta_q (q \in \mathbb{Z}/n\mathbb{Z})$ of the Lax matrix $L(b; l)$ for sufficiently large $l$’s are given by the Puiseux series expansion

$$\eta_q = l^\kappa \sum_{m=-1}^{\infty} c_m \exp\left(\frac{2\pi \sqrt{-1}mq}{n}\right) l^{-m/n},$$

where $c_{-1} = 1$ and $c_0 = (\sum_{a=1}^{n} e_1^{(a)})/n$. This implies that the eigenvalues $\eta'_q (q \in \mathbb{Z}/n\mathbb{Z})$ of the matrix $L(b; -l)$ are given by

$$\eta'_q = (-l)^\kappa \sum_{m=-1}^{\infty} c_m \exp\left(\frac{\pi \sqrt{-1}m(2q - 1)}{n}\right) l^{-m/n}.$$ 

By the above relation between $g^*(x, s; \lambda)$ and $g(x, s; (-1)^n \lambda)$, we see that the largest eigenvalue of the monodromy matrix $M^{(1)}_l(b)$, which is defined similarly by equation (12) for general $n$, is given by $E = (\prod_{q=1}^{n} \eta_q)/\eta_{n/2}$ for the case of even $n$, or by $E = (\prod_{q=1}^{n} \eta'_q)/\eta'_{(n+1)/2}$ for the case of odd $n$. In both cases, the asymptotic form of $E$ is given by

$$E = l^{(n-1)\kappa + \frac{n-1}{n}} + c_0 l^{(n-1)\kappa + \frac{n-2}{n}} + \cdots.$$
To summarize, depending on whether \( n \) is even or odd, we have to pay an attention to the choice of the sign of the loop parameter in the matrix \( \mathcal{L}(|b\rangle; (-1)^n l) \) for obtaining the largest eigenvalue of the monodromy matrix \( M^{(1)}_l(|b\rangle) \).

It also seems that the derivation is not restricted to the totally one-row tableaux case but can be generalized to the rectangular tableaux cases in \( \S 3.2 \) of [26], which uses the description of the geometric R-matrices in reference [5]. In this case, to define another time evolution \( T^{(k)}_l \) we use the monodromy matrix \( M^{(k)}_l(|b\rangle) \) for a \( k \)-row tableau “carrier”, and the entries of this matrix are order \( n-k \) minors of the Lax matrix \( \mathcal{L}(|b\rangle; (-1)^{n-k-1} l) \) with such a prescribed sign of the loop parameter (\( \S 3.2.5 \) of [26]). Therefore, an additional attention to the choice of the sign of the loop parameter must be payed for considering its largest eigenvalue \( E = E^{(k)}_l \).

For instance, in the case of \( k = 2 \), the largest eigenvalue of the monodromy matrix \( M^{(2)}_l(|b\rangle) \) is given by \( E = (\prod_{q=1}^n \eta_q)/(\eta_{n/2}^2 \eta_{n/2+1}) \) for the case of even \( n \), or by \( E = (\prod_{q=1}^n \eta_q)/(\eta_{n-1/2} \eta_{n+1/2}) \) for the case of odd \( n \).

Appendix A. Derivations of the asymptotic forms of the largest eigenvector of the monodromy matrix

Appendix A.1. Type I case

Here we show a derivation for the case of \( p = 1 \). (The other case for \( p = 3 \) is analogous.) Using the asymptotic expressions (48) for the monodromy matrix \( M^{(1)}_l(|b\rangle) \) and (17) for its largest eigenvalue \( E \), we obtain

\[
\det \begin{pmatrix}
M_{11} - E & M_{12} & M_{13} \\
M_{21} & M_{22} - E & M_{23} \\
M_{31} & M_{32} & M_{33} - E
\end{pmatrix} \approx (M_{11} - E)(M_{22} - E)(M_{33} - E) + M_{12}M_{23}M_{31}
\]

\[
= -l^{9\lambda + \frac{3}{4}} - (3c_0 - c_1)l^{9\lambda + \frac{5}{4}} + \ldots.
\]
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In the same way, we have

\[
det \begin{pmatrix}
M_{11} - E & M_{12} & -M_{14} \\
M_{21} & M_{22} - E & -M_{24} \\
M_{31} & M_{32} & -M_{34}
\end{pmatrix} \approx (M_{11} - E)(M_{22} - E)(-M_{34})
= -l^{9\kappa+\frac{14}{3}} - 2c_0 l^{9\kappa+\frac{2}{3}} + \ldots,
\]

\[
det \begin{pmatrix}
M_{11} - E & -M_{14} & M_{13} \\
M_{21} & -M_{24} & M_{23} \\
M_{31} & -M_{34} & M_{33} - E
\end{pmatrix} \approx (M_{11} - E)(-M_{24})(M_{33} - E) + (M_{11} - E)M_{23}M_{34}
= -l^{9\kappa+\frac{14}{3}} - (c_0 + e_1^{(2)}) l^{9\kappa+\frac{2}{3}} + \ldots,
\]

\[
det \begin{pmatrix}
-M_{14} & M_{12} & M_{13} \\
-M_{34} & M_{22} - E & M_{23} \\
-M_{34} & M_{32} & M_{33} - E
\end{pmatrix} \approx M_{12}(M_{24}(M_{33} - E) - M_{23}M_{34}) + M_{13}(M_{22} - E)M_{34}
= -l^{9\kappa+\frac{12}{3}} - (e_1^{(2)} + e_1^{(3)}) l^{9\kappa+\frac{4}{3}} + \ldots.
\]

Therefore

\[
\mathcal{P}_1 = \frac{l^{9\kappa+\frac{14}{3}} + (e_1^{(2)} + e_1^{(3)}) l^{9\kappa+\frac{2}{3}} + \ldots}{l^{9\kappa+\frac{2}{3}} + (3c_0 - e_1^{(1)}) l^{9\kappa+\frac{4}{3}} + \ldots}
= l^{\frac{3}{4}} \frac{1 + (e_1^{(2)} + e_1^{(3)}) l^{-\frac{1}{4}} + \ldots}{1 + (3c_0 - e_1^{(1)}) l^{-\frac{1}{4}} + \ldots},
\]

\[
\mathcal{P}_2 = \frac{l^{9\kappa+\frac{14}{3}} + (c_0 + e_1^{(2)}) l^{9\kappa+\frac{2}{3}} + \ldots}{l^{9\kappa+\frac{2}{3}} + (3c_0 - e_1^{(1)}) l^{9\kappa+\frac{4}{3}} + \ldots}
= l^{\frac{3}{4}} \frac{1 + (c_0 + e_1^{(2)}) l^{-\frac{1}{4}} + \ldots}{1 + (3c_0 - e_1^{(1)}) l^{-\frac{1}{4}} + \ldots},
\]

\[
\mathcal{P}_3 = \frac{l^{9\kappa+\frac{10}{3}} + 2c_0 l^{9\kappa+\frac{2}{3}} + \ldots}{l^{9\kappa+\frac{2}{3}} + (3c_0 - e_1^{(1)}) l^{9\kappa+\frac{4}{3}} + \ldots}
= l^{\frac{3}{4}} \frac{1 + 2c_0 l^{-\frac{1}{4}} + \ldots}{1 + (3c_0 - e_1^{(1)}) l^{-\frac{1}{4}} + \ldots},
\]

Hence we obtained the result \[50\] in §4.4.

**Appendix A.2. Type II case**

Using the asymptotic expressions \[61\] for the monodromy matrix \(M_{1}^{(1)}(\mathbf{b})\) and \[63\] for its largest eigenvalue \(E\), we obtain

\[
det \begin{pmatrix}
M_{11} - E & M_{12} & M_{13} \\
M_{21} & M_{22} - E & M_{23} \\
M_{31} & M_{32} & M_{33} - E
\end{pmatrix} \approx (M_{11} - E)(M_{22} - E)(M_{33} - E)
= -d_1^{3} l^{\frac{2}{3}} + \ldots.
\]
In the same way, we have
\[
\begin{vmatrix}
M_{11} - E & M_{12} & -M_{14} \\
M_{21} & M_{22} - E & -M_{24} \\
M_{31} & M_{32} & -M_{34}
\end{vmatrix}
\approx -M_{14}M_{21}M_{32} = -e_{L-1}^{(4)}e_{L-1}^{(3)}e_{L-1}^{(2)}l + \ldots,
\]
\[
\begin{vmatrix}
M_{11} - E & -M_{14} & M_{13} \\
M_{21} & -M_{24} & M_{23} \\
M_{31} & -M_{34} & M_{33} - E
\end{vmatrix}
\approx M_{14}M_{21}(M_{33} - E) = -d_1e_{L-1}^{(4)}e_{L-1}^{(3)}l^{\frac{3}{2}} + \ldots,
\]
\[
\begin{vmatrix}
-M_{14} & M_{12} & M_{13} \\
-M_{24} & M_{22} - E & M_{23} \\
-M_{34} & M_{32} & M_{33} - E
\end{vmatrix}
\approx -M_{14}(M_{22} - E)(M_{33} - E) = -d_1^2e_{L-1}^{(4)}l^6 + \ldots.
\]

Therefore
\[
\mathcal{P}_1 = \frac{d_1^2e_{L-1}^{(4)}l^{\frac{3}{2}} + \ldots}{d_1^{\frac{3}{2}}l^{\frac{3}{2}} + \ldots} = \frac{1}{d_1}e_{L-1}^{(4)}l^{\frac{3}{2}} + \ldots,
\]
\[
\mathcal{P}_2 = \frac{d_1e_{L-1}^{(3)}e_{L-1}^{(2)}l^{\frac{3}{2}} + \ldots}{d_1^{\frac{3}{2}}l^{\frac{3}{2}} + \ldots} = \frac{1}{(d_1)^2}e_{L-1}^{(3)}e_{L-1}^{(2)}l^{\frac{3}{2}} + \ldots,
\]
\[
\mathcal{P}_3 = \frac{e_{L-1}^{(4)}e_{L-1}^{(3)}e_{L-1}^{(2)}l + \ldots}{d_1l^{\frac{3}{2}} + \ldots} = \frac{1}{(d_1)^3}e_{L-1}^{(4)}e_{L-1}^{(3)}e_{L-1}^{(2)}l^{\frac{3}{2}} + \ldots.
\]

Hence we obtained the result (65) in §4.2.

Appendix B. Explicit derivations of the Puiseux series expansions of the eigenvalues of the Lax matrix for \( n = 2 \)

First we consider type I case. In the case of \( n = 2 \), using the explicit expression (4) and the condition \( L = 2\kappa + 1 \), we can obtain the asymptotic form of the Lax matrix \( \mathcal{L}(|b\rangle;l) \) under the limit \( l \to \infty \) as

\[
\mathcal{L}(|b\rangle;l) = \begin{pmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{pmatrix}
\approx \begin{pmatrix}
e_1^{(1)}l^\kappa & l^{\kappa+1} \\
l^\kappa & e_1^{(2)}l^\kappa
\end{pmatrix}.
\]

The eigenvalues of the matrix \( \mathcal{L}(|b\rangle;l) \) are explicitly written as

\[
\eta_\pm = \frac{L_{11} + L_{22} \mp \sqrt{(L_{11} - L_{22})^2 + 4L_{12}L_{21}}}{2}.
\]

Since \((L_{11} - L_{22})^2 + 4L_{12}L_{21} = 4l^{2\kappa+1} + O(l^{2\kappa})\), one has the expansion of the form

\[
\sqrt{(L_{11} - L_{22})^2 + 4L_{12}L_{21}} = 2l^{\kappa+\frac{1}{2}} \left( 1 + \sum_{m=1}^{\infty} c_{2m-1}l^{-m} \right).
\]
Also we have \( L_{11} + L_{22} = (e_1^{(1)} + e_1^{(2)}) l^\kappa + \mathcal{O}(l^{\kappa-1}) \), hence the expansion of the form

\[
L_{11} + L_{22} = 2l^\kappa \left( \sum_{m=0}^{\infty} c_{2m} l^{-m} \right),
\]

where \( c_{2m} = 0 \) for sufficiently large \( m \)'s. Therefore, we have the Puiseux series expansion of the eigenvalues \( \eta_1 = \eta_- \) and \( \eta_2 = \eta_+ \) for sufficiently large \( l \) as

\[
\eta_q = l^\kappa \sum_{m=-1}^{\infty} c_m \exp \left( \pi \sqrt{-1}mq \right) l^{-m/2},
\]

where \( c_{-1} = 1 \) and \( c_0 = (e_1^{(1)} + e_1^{(2)})/2 \).

Second we consider type II case. In the case of \( n = 2 \), using the explicit expression (4) and with the condition \( e_L^{(\alpha)} L = 1 \), we can obtain the asymptotic form of the Lax matrix \( L(|b|; l) \) under the limit \( l \to 0 \) as

\[
L(|b|; l) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \approx \begin{pmatrix} 1 & e_L^{(2)} l \\ e_L^{(1)} & 1 \end{pmatrix}.
\]

Since \( (L_{11} - L_{22})^2 + 4L_{12}L_{21} = 4L_{L-1}e_L^{(2)} + \mathcal{O}(l^2) \), one has the expansion of the form

\[
\sqrt{(L_{11} - L_{22})^2 + 4L_{12}L_{21}} = 2l^{1/2} \left( \sqrt{e_L^{(1)}e_L^{(2)} + \sum_{m=1}^{\infty} d_{2m-1} l^m} \right).
\]

Also we have \( L_{11} + L_{22} = 2 + \mathcal{O}(l) \), hence the expansion of the form

\[
L_{11} + L_{22} = 2 \left( \sum_{m=0}^{\infty} d_{2m} l^m \right),
\]

where \( d_{2m} = 0 \) for sufficiently large \( m \)'s. Therefore, we have the Puiseux series expansion of the eigenvalues \( \xi_1 = \eta_- \) and \( \xi_2 = \eta_+ \) for sufficiently small \( l \) as

\[
\xi_q = \sum_{m=0}^{\infty} d_m \exp \left( \pi \sqrt{-1}mq \right) l^{m/2},
\]

where \( d_0 = 1 \) and \( d_1 = \sqrt{e_L^{(1)}e_L^{(2)}} \).

Acknowledgements

The author thanks Masatoshi Noumi and Yasuhiko Yamada for discussions and drawing his attention to the difference between the two discrete dynamical systems discussed in §5. He also thanks Kohei Motegi for the invitation of giving a talk at the online
workshop Combinatorial Representation Theory and Connections with Related Fields at RIMS, Kyoto University in November 2021, which motivated him to initiate this work.

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