CARLESON EMBEDDING THEOREM FOR AN EXPONENTIAL BERGMAN SPACE ON THE UNIT BALL

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Abstract. We characterize the Carleson measures for an exponential Bergman space on the unit ball of $\mathbb{C}^n$ in terms of the ball induced by the complex Hessian of the logarithm of the weight function. The boundedness (or compactness) of integral operators, Cesàro operators and Toeplitz operators, is given using the Carleson measure (or vanishing Carleson measure) characterization.

1. Introduction

Let $\mathbb{C}^n$ denote the cartesian product of $n$ copies of the complex field $\mathbb{C}$ for positive integer $n$. For $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, the inner product is $\langle z, w \rangle = \sum_{j=1}^n z_j w_j$ and the associated norm is $|z|^2 = \langle z, z \rangle$. The open unit ball of $\mathbb{C}^n$ is denoted by $B_n = \{ z \in \mathbb{C}^n; |z| < 1 \}$ and $\mathbb{D} := B_1$. Let $d\mu$ be a Borel measure. For a Borel set $E \subset B_n$, $\mu(E) := \int_E d\mu$.

The purpose of this article is to demonstrate the Carleson measure characterization for a weighted Bergman space with a particular weight $e^{-\psi}$ on $B_n$ with

$$\psi(z) := \frac{1}{1-|z|^2}.$$ 

Let $1 \leq p < \infty$. Let $\mathcal{O}(B_n)$ be the space of all holomorphic functions on $B_n$, and let $L^p_\psi(B_n) := L^p(B_n, e^{-\psi} dv)$, where $dv$ denotes the ordinary Lebesgue measure on $\mathbb{C}^n$. The exponential Bergman space $A^p_{\psi}(B_n) := \mathcal{O}(B_n) \cap L^p_\psi(B_n)$ is the space of holomorphic functions whose $L^p$-norm with the measure $e^{-\psi} dv$ is bounded, namely,

$$\|f\|_{p,\psi} := \left\{ \int_{B_n} |f(z)|^p e^{-\psi(z)} dv(z) \right\}^{1/p} < \infty.$$ 

The exponential Bergman space $A^p_{\psi}(B_n)$ is a closed subspace of $L^p_\psi(B_n)$ by Lemma 2.13.

When $p = 2$, the space $A^2_{\psi}(B_n)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\psi} := \int_{B_n} f(z) \overline{g(z)} e^{-\psi(z)} dv(z)$$

for $f, g \in A^2_{\psi}(B_n)$. Lemma 2.13 guarantees that each point evaluation $L_z f = f(z)$ is bounded on $A^2_{\psi}(B_n)$. By the Riesz representation theorem, there is a holomorphic function $K_z \in A^2_{\psi}(B_n)$ satisfying $f(z) = \langle f, K_z \rangle_{\psi}$. We call $K_{\psi}(z, w) := \overline{K_z(w)}$ the
Bergman kernel for $A^2_{\psi}(B_n)$, i.e.,

$$f(z) = \int_{\mathbb{B}_n} f(w) K_{\psi}(z, w) e^{-\psi(w)} \, dv(w).$$

For a Borel measure $d\mu$, if there is a constant $C > 0$ satisfying

$$\int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} \, d\mu(z) \leq C \int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} \, dv(z),$$

then we call the measure $d\mu$ a Carleson measure for $A^p_{\psi}(B_n)$. It means that the inclusion operator $i_p : A^p_{\psi}(B_n) \to L^p(B_n, e^{-\psi} \, d\mu)$ is bounded, i.e., $i_p$ is embedding.

Carleson [11] gave the embedding theorem on the Hardy space for solving the Corona problem on the unit disk $\mathbb{D}$. Results on Carleson measure for the Bergman space on $\mathbb{D}$ was given by Hastings [9]. Carleson type measures for exponential type weighted Bergman spaces on the unit disk $\mathbb{D}$ was introduced in [14], and has since been studied in numerous exponential type weighted $L^p$ analytic function spaces (see [13, 10] for $\mathbb{D}$; [4] for $C$; [20, 22] for $\mathbb{C}^n$).

We are going to focus on some results on the unit ball $\mathbb{B}_n$. The Carleson measure theorem for the standard weighted and unweighted Bergman spaces on $\mathbb{B}_n$ was proved due to Cima and Wogen [3]. Luecking [14] suggested a new method which gives simple proofs of the results. Pau and Zhao [17] gave equivalent conditions of $(p,q)$-Carleson measures for standard weighted Bergman spaces. Besides weighted Bergman spaces with the Lebesgue measure, Schuster and Varolin considered the Möbius invariant measure with generalized weights including $(1 - |z|^2)^{n+c}$ for some $c > 0$ (see Theorem 5.8 of [24]). Now, we consider the exponential Bergman space $A^p_{\psi}(B_n)$ with $\psi(z) = \frac{1}{1-|z|^2}$, which had not been dealt with in any results we mentioned. The weight is rapidly decreasing compared to others.

We introduce the Carleson type embedding theorem for the exponential Bergman space $A^p_{\psi}(B_n)$. First of all, we define the function $\hat{\mu}_p$ for $p \geq 1$ as

$$\hat{\mu}_p(z) := \frac{1}{\|\Phi_{p,z}\|_{p,\psi}} \int_{\mathbb{B}_n} |\Phi_{p,z}(w)|^p e^{-\psi(w)} \, d\mu(w)$$

where $\Phi_{p,z}(w) := e^{\frac{2}{p} \frac{1}{1-|z|^2} - \frac{1}{p} \frac{1}{1-|z|^2}}$ is the test function in Lemma 2.12.

**Theorem 1.** Let $d\mu$ be a positive Borel measure. The following statements are equivalent:

(a) The measure $d\mu$ is a Carleson measure for $A^p_{\psi}(B_n)$;
(b) $\hat{\mu}_p$ is a bounded function on $\mathbb{B}_n$;
(c) For $z \in \mathbb{B}_n$ and sufficiently small $r > 0$, there is a constant $C > 0$ satisfying

$$\mu(B_H(z,r)) \leq C \psi(B_H(z,r)),$$

where $B_H(z,r)$ is the $\psi$-Hessian ball centered at $z$ with radius $r$ (it is defined in Section 2.1).

For details and the proof of Theorem 1 see Theorem 3. We also provide the theorem for vanishing Carleson measure in Section 3.

**Remark 1.1.** (1) The statement (c) does NOT depend on the number $p$. It shows that if $d\mu$ is a Carleson measure for $A^p_{\psi}(B_n)$ for some $p$, then it also holds for every $p \geq 1$. The fact is analogous to the result on the standard Bergman space.
(2) The implication \((b) \Rightarrow (a)\) means that if inequality (1.1) holds for all test functions, then it also holds for every function in \(A^p_\psi(\mathbb{B}_n)\).

**Remark 1.2.** The \(\psi\)-Hessian ball (cf. [20, 22, 21, 6]), which is induced by the metric \(\left(\frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k}\right)_{n \times n}\), plays an important role for the proof of Theorem 1.

(1) With the help of \(\psi\)-Hessian balls, not only covering lemma (Lemma 2.10), but also estimates for the test function \(\Phi_{p,z}\) (as in the estimate (2.13) in Lemma 2.12) are obtained.

(2) Lemma 2.13 is crucial for the proof of Theorem 1. It is a weighted sub-mean-value property on the \(\psi\)-Hessian ball \(B_H(z,r)\).

We should note that using the \(\psi\)-Hessian ball is suitable for investigating exponential type weighted Bergman spaces on the unit ball rather than using the ball with \((\Delta \psi)^{-\frac{1}{2}}\). Actually, the ball with \((\Delta \psi)^{-\frac{1}{2}}\) is helpful tool for studying function spaces with exponential weight on \(\mathbb{D}[13, 16]\) and \(\mathbb{C}^n[5]\). But it is not proper in the case of the unit ball. For example, Lemma 2.13 with the reproducing property and comparable property implies the following estimate for the Bergman kernel on diagonal:

\[
K_\psi(z,z) \leq C e^{2\psi(z)} \frac{e^{2\psi(z)}}{(1-|z|^2)^{2n+1}}. \quad (1.2)
\]

The estimate is same as the result which can be obtained from Theorem 3.3 in [2] using series expansion. However, one could get only

\[
K_\psi(z,z) \leq C e^{2\psi(z)} \frac{e^{2\psi(z)}}{(1-|z|^2)^{3n}} \quad (1.3)
\]

if one use the ball induced by a radius function \((\Delta \psi)^{-\frac{1}{2}}\) instead of the \(\psi\)-Hessian ball \(B_H(z,r)\). The estimate (1.2) is sharper than (1.3) when \(n > 1\).

We study integral operators on the exponential Bergman space using the Carleson embedding theorem. For Cesàro operators on \(A^p_\psi(\mathbb{B}_n)\) with holomorphic symbols, and Toeplitz operators on \(A^2_\psi(\mathbb{B}_n)\) with symbols in \(L^2_\psi(\mathbb{B}_n)\), equivalent conditions of boundedness and compactness are presented in Section 3. Note that we get the results on Toeplitz operators only in the case of \(p = 2\). We need more properties about the Bergman kernel for the result of \(p \neq 2\). We have not yet acquired appropriate estimates of off-diagonal of the Bergman kernel which can give boundedness of the Bergman projection of \(A^2_\psi(\mathbb{B}_n)\) (cf. [12, 20]).

For studying the Toeplitz operator, the test functions have essential roles. The Toeplitz operator with a symbol function \(u \in L^2_\psi(\mathbb{B}_n)\), is defined

\[
T_u f(z) = \int_{\mathbb{B}_n} K_\psi(z,w) f(w) u(w) e^{-\psi(w)} \, dv(w)
\]

for \(f \in A^2_\psi(\mathbb{B}_n)\). Let \(d\mu = u \, dv\). Then we have

\[
\hat{\mu}_z(z) = \frac{1}{\|\Phi_{2,z}\|_{L^2_\psi}} \langle T_u \Phi_{2,z}, \Phi_{2,z} \rangle_\psi,
\]

and its boundedness is equivalent to boundedness of the Toeplitz operator (see (4.7) and Theorem 7). The function \(\hat{\mu}_z\) behaves like the Berezin transform, which is defined with the Bergman kernel. Precisely, the test function \(\Phi_{2,z}\) is used instead of the Bergman kernel function in typical methods.
Throughout this paper, $C$ will be a symbol of a positive constant. The value of the constant can be changed often. The expression $A \lesssim B$ indicates $A \leq CB$, and $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

2.1. The ball induced by $H_\psi$. Let

$$\psi(z) = \frac{1}{1 - |z|^2},$$

then $\psi$ is strictly plurisubharmonic. The complex Hessian of $\psi$ is defined by

$$H_\psi := \left( \frac{\partial^2 \psi}{\partial z_j \partial z_k} \right)_{n \times n}.$$  

Lemma 2.1. For $\psi(z) = \frac{1}{1 - |z|^2}$, the complex Hessian $H_\psi$ has the following properties:

(a) $H_\psi(z) = \frac{1}{1 - |z|^2} \left( (1 - |z|^2)I_{n \times n} + 2A(z) \right)$, where $A(z) = (\overline{z}_j z_k)_{jk}$;

(b) $H_\psi(z)^{-1} = (1 - |z|^2)^2 \left( I_{n \times n} - \frac{2}{1 - |z|^2}A(z) \right)$;

(c) $\det H_\psi(z) = \frac{1 + |z|^2}{(1 - |z|^2)^2}$;

(d) $H_\psi(z) = \frac{1 + |z|^2}{(1 - |z|^2)^2} P_z + \frac{1}{1 - |z|^2} Q_z$, where $P_z \zeta = \langle \overline{z}, \zeta \rangle z$ for $z \in \mathbb{B}_n - \{0\}$, $P_0 = 0$, and $Q_z = I - P_z$.

Proof. Statements (a) and (d) are given by simple calculations. Besides, the facts $A(z) = |z|^2 P_z$ and $P_z + Q_z = I$ are used for the case of (d).

Let $u = (\overline{z}_1, \cdots, \overline{z}_n)$, $v = (z_1, \cdots, z_n)$ be column vectors, then $uv^T = (\overline{z}_j z_k)_{jk} = A(z)$ and $v^T u = |z|^2$. From (a), it is obtained

$$H_\psi(z) = \frac{2}{(1 - |z|^2)^2} \left( (1 - |z|^2)I_{n \times n} + uv^T \right) = \frac{2}{(1 - |z|^2)^2} \left( B(z) + uv^T \right) \quad (2.1)$$

where $B(z) = \frac{(1 - |z|^2)}{2} I_{n \times n}$. The Sherman-Morrison formula \[23\] gives

$$(B(z) + uv^T)^{-1} = B(z)^{-1} - \frac{B(z)^{-1}uv^TB(z)^{-1}}{1 + v^TB(z)^{-1}u}$$

$$= \frac{2}{1 - |z|^2}I_{n \times n} - \frac{4}{(1 + |z|^2)(1 - |z|^2)}A(z)$$

which shows (b).

The matrix determinant lemma \[8\] gives

$$\det \left( B(z) + uv^T \right) = (1 + v^TB(z)^{-1}u) \det B(z)$$

$$= \frac{(1 + |z|^2)(1 - |z|^2)^{n-1}}{2n}$$

which provides (c) from (2.1). \[

For a piecewise $C^1$ curve $\gamma : [0,1] \to \mathbb{B}_n$, the length induced by Hessian metric is defined by

$$\ell_\psi(\gamma) := \int_0^1 \sqrt{\langle H_\psi(\gamma(t))\gamma'(t), \gamma'(t) \rangle} \, dt.$$
For \( z, w \in \mathbb{B}_n \), the distance induced by Hessian metric is
\[
\sigma(z, w) := \inf_{\gamma} \ell_{\psi}(\gamma)
\]
where \( \gamma \) is a parametrized curve from \( z \) to \( w \) in \( \mathbb{B}_n \).

The \( \psi \)-Hessian ball centered at \( z \) with radius \( r > 0 \) is defined by the associated ball with \( \sigma \), namely,
\[
B_H(z, r) := \{ w : \sigma(z, w) < r \}.
\]

Since \( P_z \) is the orthogonal projection of \( \mathbb{C}^n \) onto the subspace \( [z] \) generated by \( z \), and \( Q_z = I - P_z \) is the projection onto the orthogonal complement of \( [z] \), the statement (d) in Lemma 2.1 means that for \( n \geq 2 \) and \( z \neq 0 \), the matrix \( H_\psi(z) \) has two eigenvalues, namely, \( \frac{1 + |z|^2}{|1 - z|^2} \) with eigenspace \( [z] \), and \( -\frac{1}{|1 - z|^2} \) with eigenspace \( \mathbb{C}^n \ominus [z] \). That brings the definition of another region:
\[
D_\psi(z, r) := \left\{ w : |z - P_z w| < r \left(1 - |z|^2\right)^\frac{3}{2}, |Q_z w| < r \left(1 - |z|^2\right) \right\}.
\]

We also denote \( B(z, r) \) for the Euclidean ball centered at \( z \) with radius \( r > 0 \). The following lemma is well known.

**Lemma 2.2 (\( \text{[16]} \)).** Let \( f : \mathbb{B}_n \to \mathbb{R} \) be a positive function. If Lipschitz norm of the function \( f \) is bounded, i.e.,
\[
\|f\|_L := \sup_{z, w \in \mathbb{B}_n, z \neq w} \frac{|f(z) - f(w)|}{|z - w|} < \infty,
\]
then there is a constant \( C > 0 \) satisfying
\[
\frac{1}{2} f(z) \leq f(w) \leq 2 f(z) \quad \text{for} \quad w \in B(z, \kappa f(z))
\]
when \( 0 < \kappa \leq \rho_f \) and \( \rho_f = \frac{1}{2} \min \left\{ 1, \frac{1}{\|f\|_L} \right\} \).

**Corollary 2.3.** Let \( f(z) = 1 - |z|^2 \). Then \( \|f\|_L = 2 \).

**Proof.** Because \( |f(z) - f(w)| = ||z|^2 - |w|^2| \leq (|z| + |w|)|z - w| \), we have
\[
\|f\|_L = \sup_{z, w \in \mathbb{B}_n, z \neq w} \frac{|f(z) - f(w)|}{|z - w|} \leq \sup_{z, w \in \mathbb{B}_n, z \neq w} (|z| + |w|) \leq 2.
\]

For \( \varepsilon > 0 \), let \( z_\varepsilon = (1 - \varepsilon, 0, \cdots, 0) \) and \( w_\varepsilon = (1 - 2\varepsilon, 0, \cdots, 0) \). Then
\[
\|f\|_L \geq \frac{|z_\varepsilon|^2 - |w_\varepsilon|^2|}{|z_\varepsilon - w_\varepsilon|} = 2 - 3\varepsilon.
\]
Since \( \varepsilon \) is arbitrary, \( \|f\|_L \geq 2 \). Hence \( \|f\|_L = 2 \). \( \square \)

**Lemma 2.4.** Let \( 0 < r \leq \frac{C_0}{2} \) where \( C_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\|1 - |z|^2\|_L} \right\} = \frac{1}{4} \). Then
\[
1 - |z|^2 \simeq 1 - |w|^2
\]
whenever \( w \in D_\psi(z, r) \) for \( z, w \in \mathbb{B}_n \).

**Proof.** Let \( w \in D_\psi(z, r) \). Then \( |z - w| \leq |z - P_z w| + |Q_z w| < 2r(1 - |z|^2) \) which means \( w \) belongs \( B(z, 2r(1 - |z|^2)) \). It implies \( 1 - |z|^2 \simeq 1 - |w|^2 \) provided \( 2r < C_0 \) by Lemma 2.2. \( \square \)
We can obtain a geometric description of the ψ-Hessian ball $B_H(z, r)$ with $D_\psi(z, r)$ (see Section 7 in [22]). For the completeness, we give the proof in detail.

**Proposition 2.5.** Let $0 < r \leq \frac{C_0}{20}$. Then

$$D_\psi \left( z, \frac{r}{10} \right) \subset B_H(z, r)$$

where $C_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\|1 - |z|^2\|} \right\}$.

**Proof.** Recall that the distance induced by the metric $H_\psi$ is

$$\sigma(z, w) = \inf_\gamma \ell_\psi(\gamma)$$

where the infimum is taken over all piecewise smooth curve $\gamma : [0, 1] \rightarrow \mathbb{B}_n$ with $\gamma(0) = z$ and $\gamma(1) = w$ and the length induced by Hessian metric is

$$\ell_\psi(\gamma) = \int_0^1 \left\{ \frac{2}{(1 - |\gamma(t)|^2)^3} |\langle \gamma(t), \gamma'(t) \rangle|^2 + \frac{1}{(1 - |\gamma(t)|^2)^2} |\gamma'(t)|^2 \right\} \frac{1}{2} dt$$

for each curve $\gamma$.

Suppose that $w$ belongs to $D_\psi(z, m)$ with $0 < m < 1$. We assume $w \neq z$ without loss of generality. We have $1 - |z|^2 \geq 1 - |w|^2$ when $2m < C_0$ by Lemma 2.4. Let $\gamma_1$ be a line segment from $z$ to $P_zw$ and $\gamma_2$ be a line segment from $P_zw$ to $w$, precisely,

$$\gamma_1(t) = (P_zw - z)t + z$$

and

$$\gamma_2(t) = (w - P_zw)t + P_zw = (Q_zw)t + P_zw.$$

Let $\tilde{\gamma}$ be a parametrized curve for $\tilde{\gamma}_1 + \tilde{\gamma}_2$; then

$$\ell_\psi(\tilde{\gamma}) = \ell_\psi(\tilde{\gamma}_1) + \ell_\psi(\tilde{\gamma}_2).$$

We have

$$\ell_\psi(\tilde{\gamma}_1) \leq \int_0^1 \left\{ \frac{\sqrt{2}}{(1 - |\tilde{\gamma}_1(t)|^2)^{3/2}} |\langle \tilde{\gamma}_1(t), \tilde{\gamma}_1'(t) \rangle| + \frac{1}{(1 - |\tilde{\gamma}_1(t)|^2)^2} |\tilde{\gamma}_1'(t)| \right\} \frac{1}{2} dt$$

$$\leq \frac{4}{(1 - |z|^2)^{3/2}} \int_0^1 |\tilde{\gamma}_1(t), \tilde{\gamma}_1'(t)| dt + \frac{2}{1 - |z|^2} \int_0^1 |\tilde{\gamma}_1'(t)| dt$$

by Lemma 2.4. Note that $\tilde{\gamma}_1(t) = P_zw - z$. The Cauchy-Schwarz inequality yields that

$$\ell_\psi(\gamma_1) \leq \frac{4}{(1 - |z|^2)^{3/2}} \int_0^1 |\gamma_1(t)||\tilde{\gamma}_1'(t)| dt + 2 \frac{|z - P_zw|}{1 - |z|^2}$$

$$\leq \frac{4}{(1 - |z|^2)^{3/2}} \sup_t |\gamma_1(t)| \int_0^1 |\tilde{\gamma}_1'(t)| dt + 2 \frac{|z - P_zw|}{1 - |z|^2}.$$

The fact $\sup_t |\gamma_1(t)| \leq 1$ gives

$$\ell_\psi(\tilde{\gamma}_1) \leq \frac{4}{(1 - |z|^2)^{3/2}} |z - P_zw| + 2 \frac{|z - P_zw|}{1 - |z|^2} < 6m$$

when $w \in D_\psi(z, m)$. 


For the length induced by $H_\psi$ of $\hat{\gamma}_2$, we also have

$$\ell_\psi(\hat{\gamma}_2) \leq \frac{4}{(1 - |z|^2)^2} \int_0^1 |(\hat{\gamma}_2(t), \hat{\gamma}'_2(t))| \, dt + \frac{2}{1 - |z|^2} \int_0^1 |\hat{\gamma}'_2(t)| \, dt$$

by Lemma 2.4. Note that $\hat{\gamma}'_2(t) = Qzw$. The fact that $Pzw$ and $Qzw$ are perpendicular asserts

$$\langle \hat{\gamma}_2(t), \hat{\gamma}'_2(t) \rangle = \langle (Qzw)t + Pzw, Qzw \rangle = t|Qzw|^2.$$

Then,

$$\ell_\psi(\hat{\gamma}_2) \leq \frac{4}{(1 - |z|^2)^2} |Qzw|^2 \int_0^1 t \, dt + \frac{2}{1 - |z|^2} |Qzw| \int_0^1 \langle Qzw \rangle \, dt + 2 \frac{|Qzw|}{1 - |z|^2} < 4m$$

when $w \in D_\psi(z, m)$. As a result, we get

$$\sigma(z, w) \leq \ell_\psi(\hat{\gamma}) < 10m$$

which implies $w \in B_H(z, 10m)$. By putting $m = \frac{r}{10}$, it is obtained

$$D_\psi \left( z, \frac{r}{10} \right) \subset B_H(z, r).$$

\[\square\]

**Proposition 2.6.** Let $0 < r \leq \frac{C_0}{20}$. Then

$$B_H(z, r) \subset D_\psi(z, 18r)$$

where $C_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\|1 - |z|^2 \|^2} \right\} = \frac{1}{4}$.

**Proof.** Suppose that $w$ belongs to $B_H(z, r)$. We assume $w \neq z$ without loss of generality. The proof is divided into three steps.

**Step 1.** We will show that $\sigma(z, w) < r$ implies $|z - w| \leq 2r(1 - |z|^2)$ and $|Qz(w)| \leq 2r(1 - |z|^2)$.

Suppose $\sigma(z, w) < r$. As in Proposition 5 in [5], let

$$s = \frac{|z - w|}{1 - |z|^2}.$$ 

For any piecewise $C^1$ curve $\gamma$ from $z$ to $w$, let $T_0 \in (0, 1]$ be the minimum of $t$ satisfying

$$|z - \gamma(t)| = \min \left\{ s, \frac{C_0}{10} \right\} (1 - |z|^2)$$

where $C_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\|1 - |z|^2 \|^2} \right\}$. It gives

$$\frac{1}{2}(1 - |z|^2) \leq 1 - |\gamma(t)|^2 \leq 2(1 - |z|^2) \quad \text{for} \quad t \in [0, T_0]$$
by Lemma \ref{lem:2.2} we have
\begin{align*}
\ell_\psi(\gamma) &\geq \int_0^1 \frac{1}{1 - |\gamma(t)|^2} |\gamma'(t)| \, dt \\
&\geq \int_0^{T_0} \frac{1}{1 - |\gamma(t)|^2} |\gamma'(t)| \, dt \\
&\geq \frac{1}{2(1 - |z|^2)} \int_0^{T_0} |\gamma'(t)| \, dt.
\end{align*}
(2.2)

Since \(\int_0^{T_0} |\gamma'(t)| \, dt\) is the Euclidean length of the curve \(\gamma\) from 0 to \(T_0\),

\[\ell_\psi(\gamma) \geq \frac{1}{2} \min \left\{ s, \frac{C_0}{10} \right\}.\]

Because \(\ref{lem:2.2}\) holds for arbitrary \(\gamma\) connecting \(z\) and \(w\),

\[\sigma(z, w) \geq \frac{1}{2} \min \left\{ s, \frac{C_0}{10} \right\}.\]

By the assumption \(w \in B_H(z, r)\), it is obtained

\[\frac{C_0}{20} \geq r > \sigma(z, w) \geq \frac{1}{2} \min \left\{ s, \frac{C_0}{10} \right\},\]

which is a contradiction whenever \(s \geq \frac{C_0}{10}\). Hence we have \(s < \frac{C_0}{10}\) when \(w \in B_H(z, r)\). It gives

\[r > \min \left\{ s, \frac{C_0}{10} \right\} = \frac{1}{2}s,\]

which asserts

\[|z - w| < 2r(1 - |z|^2) \quad \text{for} \quad w \in B_H(z, r).\] (2.3)

Also, \(\ref{lem:2.3}\) yields

\[|Q_z w| \leq |z - w| < 2r(1 - |z|^2) \quad \text{for} \quad w \in B_H(z, r)\] (2.4)

since \(z - P_z w, z - w,\) and \(Q_z w\) construct a right triangle with hypotenuse \(z - w\).

Now, we will show that \(\sigma(z, w) < r\) implies \(|z - P_z w| \lesssim r(1 - |z|^2)^{\frac{3}{2}}\). It divides into the cases of \(|z| \leq \frac{1}{2}\) and \(|z| > \frac{1}{2}\).

**Step 2.** We assume that \(|z| \leq \frac{1}{2}\). \(\ref{lem:2.3}\) in Step 1 gives

\[B_H(z, r) \subset B(z, 2r)\]

where \(B(z, r)\) is the Euclidean ball centered at \(z\) with radius \(r > 0\). Since \(|z| \leq \frac{1}{2}\) means \(\frac{1}{4} \leq 1 - |z|^2 \leq 1\), it is obtained

\[|z - w| < 2r = \frac{16}{\sqrt{27}} r \left( \frac{3}{4} \right)^{\frac{3}{2}} \leq \frac{16}{\sqrt{27}} r(1 - |z|^2)^{\frac{3}{2}} < 4r(1 - |z|^2)^{\frac{3}{2}}.\]

Because \(z - P_z w, z - w,\) and \(Q_z w\) construct a right triangle with hypotenuse \(z - w\), we have \(|z - P_z w| \leq |z - w|\).

Therefore,

\[|z - P_z w| < 4r(1 - |z|^2)^{\frac{3}{2}} \quad \text{for} \quad w \in B_H(z, r)\] (2.5)

when \(|z| \leq \frac{1}{2}\).
Step 3. We assume that \(|z| > \frac{1}{2}\). Suppose that \(\sigma(z, w) < r\). We also assume \(z \neq P_z w\) without loss of generality. Hereinafter, we consider only the curves \(\gamma\) connecting \(z\) and \(w\) satisfying
\[
\ell_\psi(\gamma) \leq 2\sigma(z, w).
\]
For each curve \(\gamma\), define \(\gamma_1(t) = P_z(\gamma(t))\) and \(\gamma_2(t) = \gamma(t) - \gamma_1(t) = Q_z(\gamma(t))\). Let \(t_0 \in [0, 1]\) be the minimum of \(t\) such that
\[
|z - \gamma_1(t)| = |z - P_z w|.
\]
Similar to Lemma 2.2, we have comparable property in the ball \(B_H(z, r)\) by (2.3), precisely,
\[
\frac{1}{2}(1 - |z|^2) \leq 1 - |w|^2 \leq 2(1 - |z|^2) \quad \text{for} \quad w \in B_H(z, r).
\]
Thus, we have
\[
\ell_\psi(\gamma) \geq \int_0^1 \frac{\sqrt{2}}{(1 - |\gamma(t)|^2)^{\frac{3}{2}}} |\langle \gamma(t), \gamma'(t) \rangle| \, dt
\]
\[
\geq \frac{1}{2} \int_0^{t_0} |\langle \gamma(t), \gamma'(t) \rangle| \, dt
\]
\[
\geq \frac{1}{2} \int_0^{t_0} \{|\langle \gamma_1(t), \gamma_1'(t) \rangle| - |\langle \gamma_2(t), \gamma_2'(t) \rangle|\} \, dt.
\]
Since \(\gamma_1\) and \(\gamma_1'\) are parallel,
\[
\int_0^{t_0} |\langle \gamma_1(t), \gamma_1'(t) \rangle| \, dt = \int_0^{t_0} |\gamma_1(t)||\gamma_1'(t)| \, dt \geq \inf_{0 \leq t \leq t_0} |\gamma_1(t)| \int_0^{t_0} |\gamma_1'(t)| \, dt.
\]
The hypothesis \(\ell_\psi(\gamma) \leq 2\sigma(z, w)\) gives
\[
\sigma(z, \gamma_1(t)) < \sigma(z, \gamma(t)) < 2r.
\]
Inequality (2.3) and \(0 < 1 - |z|^2 < \frac{1}{2}\) yield
\[
\gamma_1(t) \in B_H(z, 2r) \subset B(z, 4r(1 - |z|^2)) \subset B(z, 3r).
\]
It gives
\[
\inf_{0 \leq t \leq t_0} |\gamma_1(t)| \geq |z| - 3r > \frac{1}{2} - \frac{3}{80} = \frac{37}{80}
\]
with \(0 < r \leq \frac{C_0}{20} = \frac{1}{80}\). Hence, we have
\[
\int_0^{t_0} |\langle \gamma_1(t), \gamma_1'(t) \rangle| \, dt \geq \frac{37}{80}|z - P_z w|
\]
from (2.7) and \(\int_0^{t_0} |\gamma_1'(t)| \, dt \geq |z - P_z w|\).
By Cauchy-Schwartz inequality,
\[
\int_0^{t_0} |\langle \gamma_2(t), \gamma_2'(t) \rangle| \, dt \leq \int_0^{t_0} |\gamma_2(t)||\gamma_2'(t)| \, dt \leq \sup_{0 \leq t \leq t_0} |\gamma_2(t)| \int_0^{t_0} |\gamma_2'(t)| \, dt.
\]
Let \(t^* \in [0, t_0]\) satisfy
\[
\sup_{0 \leq t \leq t_0} |\gamma_2(t)| = |\gamma_2(t^*)| = |Q_z(\gamma(t^*))|.
\]
Since \( z - \gamma_1(t^*) = z - P_z(\gamma(t^*)) \), \( z - \gamma(t^*) \) and \( \gamma_2(t^*) = Q_z(\gamma(t^*)) \) construct a right triangle with hypotenuse \( z - \gamma(t^*) \), we have

\[
\sup_{0 \leq t \leq t_0} |\gamma_2(t)| = |Q_z(\gamma(t^*))| \leq |z - \gamma(t^*)| \leq \int_0^{t_0} |\gamma'(t)| \, dt \leq \int_0^{t_0} |\gamma'(t)| \, dt.
\]

Then, we get

\[
\int_0^{t_0} |(\gamma_2(t), \gamma_4(t))| \, dt \leq \left\{ \int_0^{t_0} |\gamma'(t)| \, dt \right\}^2 \leq 4(1 - |z|^2)^2 \ell_\psi(\gamma)^2
\]

since \( \int_0^{t_0} |\gamma'(t)| \, dt \leq 2(1 - |z|^2) \ell_\psi(\gamma) \) as in (2.2).

Therefore, we obtain

\[
\ell_\psi(\gamma) \geq \frac{37}{160} \frac{1}{(1 - |z|^2)^{\frac{1}{2}}} |z - P_z w| - 2 \left( 1 - |z|^2 \right)^{\frac{9}{2}} \ell_\psi(\gamma)^2
\]

which implies

\[
\frac{37}{160} \frac{1}{(1 - |z|^2)^{\frac{1}{2}}} |z - P_z w| \leq 2 \ell_\psi(\gamma) \leq 4 \sigma(z, w) < 4r.
\]

It shows

\[
|z - P_z w| < 18r \left( 1 - |z|^2 \right)^{\frac{9}{2}} \quad \text{for} \quad w \in B_H(z, r)
\]

(2.8)

when \( |z| > \frac{1}{2} \).

Finally, we get the desired result

\[ B_H(z, r) \subset D_\psi(z, 18r) \]

by gathering with \( (2.4), (2.5) \), and \( (2.8) \).

From Propositions 2.5 and 2.6 we have the following theorem.

**Theorem 2.** For \( 0 < r \leq \frac{C_0}{20} \), there is a constant \( \delta > 1 \) depending only on \( C_0 \) satisfying

\[ D_\psi(z, \delta^{-1} r) \subset B_H(z, r) \subset D_\psi(z, \delta r) \]

where \( C_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\|1 - |z|^2\|_z} \right\} = \frac{1}{4} \).

**Corollary 2.7.** For \( 0 < r_1, r_2 \leq \frac{C_0}{20} \), we have

\[ v(B_H(z, r_1)) \simeq v(D_\psi(z, r_2)) \simeq \left( 1 - |z|^2 \right)^{2n + 1} \]

where \( C_0 = \frac{1}{2} \min \left\{ 1, \frac{1}{\|1 - |z|^2\|_z} \right\} = \frac{1}{4} \).

By triangle inequality of the distance \( \sigma \), we get the following lemma.

**Lemma 2.8.** Given \( r > 0 \), there are \( r_1, r_2 > 0 \) such that

\[ B_H(w, r_1) \subset B_H(z, r) \subset B_H(w, r_2) \]

whenever \( \sigma(z, w) < r \).

We have covering lemmas for \( \psi \)-Hessian balls by the same way in Lemmas 2.22 and 2.23 in [24] using Lemma 2.8.

**Lemma 2.9.** Let \( R \) be a positive number and \( m \) be a positive integer. Then there exists a positive integer \( N \) such that every ball \( B_H(z, r) \) with \( r \leq R \) can be covered by \( N \) balls \( B_H(a_k, \frac{r}{m}) \).
Lemma 2.10. There is a positive integer \( N \) such that for any \( 0 < r \leq 1 \) we can find a sequence \( \{a_k\} \) in \( \mathbb{B}_n \) with the following properties:

1. \( \mathbb{B}_n = \cup_k B_H(a_k, r) \).
2. The sets \( B_H(a_k, r/4) \) are mutually disjoint.
3. Each point \( z \in \mathbb{B}_n \) belongs to at most \( N \) of the sets \( B_H(a_k, 4r) \).

We say that a sequence \( \{a_k\} \) is a \( H_\psi \)-lattice with \( r \) when \( \{a_k\} \) is a sequence satisfying the properties in Lemma 2.10.

2.2. Test functions. For \( z \in \mathbb{B}_n \), the involutive automorphisms on \( \mathbb{B}_n \) are defined as

\[
\varphi_z(w) := \frac{z - P_z w - \sqrt{1 - |z|^2 Q_z w}}{1 - \langle w, z \rangle}.
\]

It has the following property:

\[
1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{1 - \langle w, z \rangle^2}.
\] (2.9)

For more details of the automorphisms of \( \mathbb{B}_n \), see page 23 of [19] or page 3 of [24].

Due to the definition of \( D_\psi(z, r) \), we can get the following inequality which is essential for proving the first estimate of the test functions in Lemma 2.12.

Lemma 2.11. For \( z \in \mathbb{B}_n \) and small \( r > 0 \), there is a constant \( C \) depending only on the radius \( r \) satisfying

\[
|2\text{Re} \left( \frac{1}{1 - \langle w, z \rangle} \right) - \frac{1}{1 - |z|^2} - \frac{1}{1 - |w|^2} | \leq C
\]

for \( w \in D_\psi(z, r) \).

Proof. Using (2.9), we get the reformulation:

\[
2\text{Re} \left( \frac{1}{1 - \langle w, z \rangle} \right) - \frac{1}{1 - |z|^2} - \frac{1}{1 - |w|^2} = \frac{|z - w|^2}{|1 - \langle w, z \rangle|^2} - |\varphi_z(w)|^2 \left( \frac{1}{1 - |z|^2} + \frac{1}{1 - |w|^2} \right),
\] (2.10)

which indicates

\[
- |\varphi_z(w)|^2 \left( \frac{1}{1 - |z|^2} + \frac{1}{1 - |w|^2} \right) \leq \text{LHS of (2.10)} \leq \frac{|z - w|^2}{|1 - \langle w, z \rangle|^2}.
\] (2.11)

First, we show that \( \frac{|z - w|^2}{|1 - \langle w, z \rangle|^2} \) is dominated with some constant independent of \( z \) and \( w \), which means the LHS of (2.10) has an upper bound \( C_r \). For \( w \in D_\psi(z, r) \), we have \( |z - w|^2 < 4r^2(1 - |z|^2)^2 \) since \( |z - w| \leq |z - P_z(w)| + |Q_zw| \).

By Lemma 2.4, we have \( 1 - |z|^2 \approx 1 - |w|^2 \) for \( w \in D_\psi(z, r) \) for small \( r > 0 \). Hence there exists \( C_r > 0 \) such that

\[
\frac{|z - w|^2}{|1 - \langle w, z \rangle|^2} < \frac{4r^2(1 - |z|^2)^2}{|1 - \langle w, z \rangle|^2} \leq C_r \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle w, z \rangle|^2}.
\] (2.12)

The RHS of (2.12) is equal to

\[
C_r (1 - |\varphi_z(w)|^2)
\]

by (2.9). It is dominated by \( C_r \) since \( \varphi_z(w) \) belongs to \( \mathbb{B}_n \).
Next, we show that the LHS of (2.11) has a lower bound \(-C'_r\). Since \(z - P_z(w)\) and \(Q_z(w)\) are perpendicular, we have

\[
|\varphi_z(w)|^2 = \frac{|z - P_z(w)|^2 + (1 - |z|^2)|Q_z(w)|^2}{|1 - \langle w, z \rangle|^2}.
\]

The definition of \(D_\psi(z, r)\) yields

\[
|\varphi_z(w)|^2 < \frac{2r^2(1 - |z|^2)^3}{|1 - \langle w, z \rangle|^2},
\]

for \(w \in D_\psi(z, r)\). It implies

\[
|\varphi_z(w)|^2 \left(\frac{1}{1 - |z|^2} + \frac{1}{1 - |w|^2}\right) < \frac{2r^2(1 - |z|^2)^3}{|1 - \langle w, z \rangle|^2} \left(\frac{1}{1 - |z|^2} + \frac{1}{1 - |w|^2}\right)
\]

\[
\leq C_r' \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle w, z \rangle|^2} \leq C'_r
\]

with Lemma 2.4. The proof is done by getting \(C = \max \{C_r, C'_r\}\).

By the previous inequality, we can get the following lemma for test functions. It will be used for proving the weighted sub-mean-value property and the results in Section 3 and Section 4.

**Lemma 2.12.** For \(z \in \mathbb{B}_n\), let \(\Phi_{p,z}(w) := e^{\frac{2r}{1 - |w|^2} - \frac{2}{1 - |z|^2}}\). The holomorphic function \(\Phi_{p,z}\) has following properties:

\[
|\Phi_{p,z}(w)|^p e^{-\frac{1}{1 - |w|^2}} \simeq 1 \quad \text{when} \quad w \in D_\psi(z, r)
\]  
(2.13)

and

\[
\|\Phi_{p,z}\|_{p,\psi}^p \simeq (1 - |z|^2)^{2n + 1}.
\]  
(2.14)

**Proof.** By Lemma 2.11, we can show that for \(z \in \mathbb{B}_n\) and small \(r > 0\), there is a constant \(C\) depending only \(r\) satisfying

\[
C^{-1} e^{-\frac{1}{1 - |w|^2}} \leq |e^{-\frac{2r}{1 - |w|^2} - \frac{2}{1 - |z|^2}}| \leq Ce^{-\frac{1}{1 - |w|^2} - \frac{1}{1 - |z|^2}}
\]

for \(w \in D_\psi(z, r)\). It gives (2.13). By Lemma 3 in [7], (2.14) is obtained.

### 2.3. Mean value inequality with exponential weight

**Lemma 2.13.** Let \(f\) be a holomorphic function on \(\mathbb{B}_n\) and \(s \in \mathbb{R}\). For \(z \in \mathbb{B}_n\) and a sufficiently small radius \(r > 0\), there is a constant \(C\) depending on \(s\) and \(r\) satisfying

\[
|f(z)|^p e^{-\frac{1}{1 - |z|^2}} \leq \frac{C}{(1 - |z|^2)^{2n + 1}} \int_{B_H(z, r)} |f(\zeta)|^p e^{-\frac{1}{1 - |\zeta|^2}} \, dv(\zeta).
\]

**Proof.** Since the function \(\Phi_{p,z}\) is non-vanishing, \(\Phi_{p,z}(\zeta)^{-s}\) with a principle branch is holomorphic. Subharmonicity of \(|f(\zeta)|\Phi_{p,z}(\zeta)^{-s}\) gives that for \(\delta > 1\),

\[
|f(z)\Phi_{p,z}(z)^{-s}|^p \leq \frac{C}{v(D_\psi(z, \delta^{-1}r))} \int_{D_\psi(z, \delta^{-1}r)} |f(\zeta)|^p |\Phi_{p,z}(\zeta)^{-s}|^s \, dv(\zeta)
\]

\[
\leq \frac{C}{v(D_\psi(z, \delta^{-1}r))} \int_{D_\psi(z, \delta^{-1}r)} |f(\zeta)|^p e^{-\frac{1}{1 - |\zeta|^2}} \, dv(\zeta)
\]
Definition 3.1. For a Borel measure $d\mu$, we call the measure $d\mu$ a Carleson measure for $A^p_\psi(\mathbb{B}_n)$ if there is a constant $C > 0$ satisfying
\[ \int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} d\mu(z) \leq C \int_{\mathbb{B}_n} |f(z)|^p d\mu(z). \]

We denote
\[ \|f\|_{p,\mu} := \left( \int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} d\mu(z) \right)^{\frac{1}{p}} \]
as the norm of $f$ which belong to $L^p(\mathbb{B}_n, e^{-\psi} d\mu)$.

Let $p \geq 1$. For $z \in \mathbb{B}_n$, let
\[ \tilde{\Phi}_{p,z}(w) := \frac{\Phi_{p,z}(w)}{\|\Phi_{p,z}\|_{p,\psi}} \]
be the normalized test function in Lemma 2.12. Then $\tilde{\Phi}_{p,z} \in A^p_\psi(\mathbb{B}_n)$ and
\[ \tilde{\Phi}_{p,z}(w) \simeq \frac{\Phi_{p,z}(w)}{\|\Phi_{p,z}\|_{p,\psi}} e^{\frac{\psi}{2} \frac{1}{1-|w|} - \frac{1}{2} \frac{1}{1-|z|}} \]
by (2.14). We note that $\tilde{\Phi}_{p,z}(w)$ converges to 0 uniformly on compact subsets of $\mathbb{B}_n$ as $|z| \to 1^-$.

For a finite positive Borel measure $\mu$, we define a function $\tilde{\mu}_p$ with the normalized test functions in $A^p_\psi(\mathbb{B}_n)$;
\[ \tilde{\mu}_p(z) := \int_{\mathbb{B}_n} |\tilde{\Phi}_{p,z}(w)|^p e^{-\psi(w)} d\mu(w) = \frac{1}{\|\Phi_{p,z}\|_{p,\psi}} \int_{\mathbb{B}_n} |\Phi_{p,z}(w)|^p e^{-\psi(w)} d\mu(w). \]

Theorem 3. Let $d\mu$ be a positive Borel measure. The following statements are equivalent:
(a) The measure $d\mu$ is a Carleson measure for $A^p_\psi(\mathbb{B}_n)$;
(b) $\tilde{\mu}_p$ is a bounded function on $\mathbb{B}_n$;
(c) For $z \in \mathbb{B}_n$ and sufficiently small $r > 0$, there is a constant $C > 0$ satisfying
\[ \mu(\mathbb{B}_H(z,r)) \leq C \nu(\mathbb{B}_H(z,r)); \]
Corollary 2.7 and (2.14) give therefore, it is obtained that which shows (b) implies (c).

Proof. Suppose \(d\mu\) is a Carleson measure for \(A^p(\mathbb{B}_n)\). There is \(C > 0\) satisfying
\[
\int_{\mathbb{B}_n} |\Phi_{p,z}(w)|^p e^{-\psi(w)} \, d\mu(w) \leq C \int_{\mathbb{B}_n} |\Phi_{p,z}(w)|^p e^{-\psi(w)} \, dv(w) = C \|\Phi_{p,z}\|_{p,\psi}^p
\]
for \(z \in \mathbb{B}_n\). It gives (a) implies (b).

Using Theorem 2 and Lemma 2.12 we have
\[
\mu(B_H(z, r)) = \int_{B_H(z,r)} d\mu(w) \simeq \int_{B_H(z,r)} |\Phi_{p,z}(w)|^p e^{-\psi(w)} \, d\mu(w).
\]

Corollary 2.7 and Lemma 2.12 give
\[
v(B_H(z, r)) \simeq (1 - |z|^2)^{2n+1} \simeq \|\Phi_{p,z}\|_{p,\psi}^p.
\]

Therefore,
\[
\frac{\mu(B_H(z, r))}{v(B_H(z, r))} \lesssim \tilde{\mu}_p(z) \quad (3.2)
\]
which shows (b) implies (c).

The implication (c) \(\Rightarrow\) (d) is trivial.

Suppose (d) holds. By Lemma 2.10 we have
\[
\int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} \, d\mu(z) \leq \sum_{k=1}^\infty \int_{B_H(a_k, r)} |f(z)|^p e^{-\psi(z)} \, d\mu(z).
\]  

(3.3)

Since Lemma 2.12 asserts
\[
\sup_{z \in B_H(a_k, r)} |f(z)|^p e^{-\psi(z)} \lesssim \frac{1}{v(B_H(a_k, r))} \int_{B_H(a_k, 2r)} |f(z)|^p e^{-\psi(w)} \, dv(w),
\]

it is obtained that
\[
\text{LHS of (3.3)} \lesssim \sum_{k=1}^\infty \int_{B_H(a_k, r)} \frac{1}{v(B_H(a_k, r))} \int_{B_H(a_k, 2r)} |f(z)|^p e^{-\psi(w)} \, dv(w) \, d\mu(z)
\]
\[
= \sum_{k=1}^\infty \frac{1}{v(B_H(a_k, r))} \int_{B_H(a_k, r)} d\mu(z) \int_{B_H(a_k, 2r)} |f(z)|^p e^{-\psi(w)} \, dv(w)
\]
\[
= \sum_{k=1}^\infty \frac{\mu(B_H(a_k, r))}{v(B_H(a_k, r))} \int_{B_H(a_k, 2r)} |f(z)|^p e^{-\psi(w)} \, dv(w).
\]

By the hypothesis, there is \(C > 0\) such that
\[
\text{LHS of (3.3)} \leq C \sum_{k=1}^\infty \int_{B_H(a_k, 2r)} |f(z)|^p e^{-\psi(w)} \, dv(w).
\]

Lemma 2.10 yields
\[
\int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} \, d\mu(z) \leq CN \int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(z)} \, dv(z)
\]
which implies (a).

\[\square\]

**Corollary 3.2.** For \(p \geq 1\) and a positive Borel measure \(d\mu\), the following quantities are equivalent:
(a) \( \| i_p \|^p \) where \( \| i_p \| = \sup \{ \| f \|_{p, \mu} \mid \| f \|_{p, \psi} = 1 \} \);
(b) \( \| \mu_i \|_{\infty} \);
(c) For small \( r > 0 \), \( \sup_{z \in B_n} \frac{\mu(B_{H}(z, r))}{v(B_{H}(z, r))} \); 
(d) For any \( H_{\psi} \)-lattice \( \{ a_k \} \), \( \sup_{k=1,2,...} \frac{\mu(B_{H}(a_k, r))}{v(B_{H}(a_k, r))} \).

**Definition 3.3.** For a Borel measure \( d\mu \), we call the measure \( d\mu \) is a vanishing Carleson measure for \( A_{\psi}^p(B_n) \) if
\[
\lim_{j \to \infty} \int_{B_n} |f_j(z)|^p e^{-\psi(z)} \, d\mu(z) \to 0,
\]
whenever \( \{ f_j \} \) is a bounded sequence in \( A_{\psi}^p(B_n) \) which converges to 0 uniformly on compact subsets.

**Theorem 4.** Let \( d\mu \) be a positive Borel measure. The following statements are equivalent:
(a) The measure \( d\mu \) is a vanishing Carleson measure for \( A_{\psi}^p(B_n) \);
(b) \( \mu_i(z) \to 0 \) as \( z \to \partial B_n \);
(c) For \( z \in B_n \) and sufficiently small \( r > 0 \),
\[
\frac{\mu(B_H(z, r))}{v(B_H(z, r))} \to 0 \quad \text{as} \quad z \to \partial B_n ;
\]
(d) There is a \( H_{\psi} \)-lattice \( \{ a_k \} \) such that
\[
\frac{\mu(B_H(a_k, r))}{v(B_H(a_k, r))} \to 0 \quad \text{as} \quad k \to \infty.
\]

**Proof.** Suppose \( \mu \) is a vanishing Carleson measure for \( A_{\psi}^p(B_n) \). Since \( \tilde{\Phi}_{p, z} \in A_{\psi}^p(B_n) \) with \( \| \tilde{\Phi}_{p, z} \|_{p, \psi} = 1 \) and
\[
\tilde{\Phi}_{p, z}(w) \simeq \frac{\Phi_{p, z}(w)}{(1 - |z|^2)^{\frac{1}{p}}} = \frac{e^{-\psi(w)}}{(1 - |z|^2)^{\frac{1}{p}}} \frac{1}{\tilde{\Phi}_{p, z}(w)}
\]
converges to 0 uniformly on compact subsets of \( B_n \) as \( |z| \to 1^- \), we have (b).

Relation (3.2) yields the implication (b) \( \Rightarrow \) (c).

Suppose the statement (c). For a \( H_{\psi} \)-lattice, \( a_k \) goes to the boundary of \( B_n \) as \( k \to +\infty \) which gives (d).

Let \( \{ f_j \} \) be a bounded sequence in \( A_{\psi}^p(B_n) \) which converges to 0 uniformly on compact subsets, and let
\[
I_j = \int_{B_n} |f_j(z)|^p e^{-\psi(z)} \, d\mu(z).
\]
By Lemma 2.13 and Lemma 2.10 (the same way to Theorem 3), we obtain
\[
I_j \leq \sum_{k=1}^{\infty} \frac{\mu(B_{H}(a_k, r))}{v(B_{H}(a_k, r))} \int_{B_{H}(a_k, 2r)} |f_j(w)|^p e^{-\psi(w)} \, dv(w).
\]
Since \( \frac{\mu(B_{H}(a_k, r))}{v(B_{H}(a_k, r))} \to 0 \) as \( k \to +\infty \), for any \( \varepsilon > 0 \), there is a positive integer \( M \) such that for every \( k > M \), we have
\[
\frac{\mu(B_{H}(a_k, r))}{v(B_{H}(a_k, r))} \leq \varepsilon.
\]
It gives that
\[ I_j \leq C \sum_{k=1}^{M} \int_{B(a_k, 2r)} |f_j(w)|^p e^{-\psi(w)} \, dv(w) \]
\[ + \varepsilon \sum_{k=M+1}^{+\infty} \int_{B(a_k, 2r)} |f_j(w)|^p e^{-\psi(w)} \, dv(w). \]

Because the sequence \( \{f_j\} \) converges to 0 uniformly on \( \overline{B(a_k, 2r)} \), the first summation also converges to 0 as \( j \to +\infty \). The second summation is dominated by the norm of the function \( f_j \) by Lemma 2.10, namely,
\[ \sum_{k=M+1}^{+\infty} \int_{B(a_k, 2r)} |f_j(w)|^p e^{-\psi(w)} \, dv(w) \leq N \|f_j\|_{p, \psi}^p. \]

Therefore, we have
\[ \limsup_{j \to +\infty} I_j \leq \varepsilon N \sup_j \|f_j\|_{p, \psi}^p. \]
Because \( \varepsilon \) is arbitrary, the limit of \( I_j \) is zero. This completes the proof. \( \Box \)

As we can see from statements (c) and (d) in Theorem 3 and Theorem 4, the property of (vanishing) Carleson measure does not depend on \( p \). When the indication of \( p \) is not necessary, we will call it a (vanishing) \( \psi \)-Carleson measure instead of a (vanishing) Carleson measure for \( A^p_\psi(B_n) \).

4. Applications

4.1. Boundedness and compactness of Cesàro operators. Originally, the extended Cesàro operator is defined on analytic function spaces on the unit disk:
\[ V_g f(z) = \int_0^z f(t)g'(t) \, dt, \quad z \in \mathbb{D}. \] (4.1)

In 1977, Pommerenke [18] defined \( V_g \) and studied on the boundedness of the operator on Hardy space \( H^2(\mathbb{D}) \). In \( n \)-dimensional case, Hu [10] introduced the extended Cesàro operator \( V_g \) on the unit ball by means of radial derivative. The following is the definition of the operator \( V_g \) for \( n \)-dimensional spaces:

**Definition 4.1.** For \( g \in O(B_n) \),
\[ V_g f(z) := \int_0^1 f(tz)\mathcal{R}g(tz) \frac{dt}{t}, \quad z \in B_n, \] (4.2)

where \( \mathcal{R}g(z) := \sum_{j=1}^{n} z_j \frac{\partial g}{\partial z_j}(z) \).

One can see (4.2) is same as (4.1) when \( n = 1 \).

**Remark 4.2.** Let \( f \) belong to \( O(B_n) \). Following [2], we can get
\[ \int_{B_n} |f(z)|^p e^{-\psi(z)} \, dv(z) \simeq |f(0)|^p + \int_{B_n} |\mathcal{R}f(z)|^p (1 - |z|^2)^2 e^{-\psi(z)} \, dv(z). \] (4.3)

It gives
\[ f \in A^p_\psi(B_n) \iff (1 - |z|^2)^2 \mathcal{R}f(z) \in L^p(B_n, e^{-\psi} \, dv), \]
which has an essential role of the proof of Theorem 5 and Theorem 6.
Theorem 5. Let $g \in \mathcal{O}(\mathbb{B}_n)$. The following statements are equivalent:

(a) $V_g$ is bounded on $A^p_{\psi}(\mathbb{B}_n)$;
(b) $|R_g(z)|^p(1 - |z|^2)^{2p} d\mu(z)$ is a Carleson measure for $A^p_{\psi}(\mathbb{B}_n)$;
(c) $|R_g(z)|(1 - |z|^2)^2$ is bounded.

Proof. By the relation (4.3) and the fact $V_g f(0) = 0$ and $\mathcal{R}V_g f(z) = f(z)R_g(z)$ as in [10], we get

$$\|V_g f\|^p_{p,\psi} \simeq |V_g f(0)|^p + \int_{\mathbb{B}_n} |\mathcal{R}V_g f(z)|^p(1 - |z|^2)^{2p}e^{-\psi(z)} d\nu(z)$$

$$= \int_{\mathbb{B}_n} |f(z)|^p|R_g(z)|^p(1 - |z|^2)^{2p}e^{-\psi(z)} d\nu(z),$$

which means

$$\|f\|^p_{p,\mu} \simeq \|V_g f\|^p_{p,\psi},$$

(4.4)

where $d\mu(z) = |R_g(z)|^p(1 - |z|^2)^{2p} d\nu(z)$. It asserts that (a) implies (b).

Next, suppose $d\mu(z) = |R_g(z)|^p(1 - |z|^2)^{2p} d\nu(z)$ is a Carleson measure for $A^p_{\psi}(\mathbb{B}_n)$. Lemma 2.13 and (2.6) yield

$$|R_g(z)|^p(1 - |z|^2)^{2p} \leq (1 - |z|^2)^{2p} \frac{1}{v(B_H(z,r))} \int_{B_H(z,r)} \mathcal{R}g(w)|^p(1 - |w|^2)^{2p}e^{-\psi(w)} d\nu(w)$$

$$\simeq \frac{1}{v(B_H(z,r))} \int_{B_H(z,r)} \mathcal{R}g(w)|^p(1 - |w|^2)^{2p}e^{-\psi(w)} d\nu(w),$$

i.e.,

$$|R_g(z)|^p(1 - |z|^2)^{2p} \lesssim \frac{\mu(B_H(z,r))}{v(B_H(z,r))},$$

(4.5)

The last term is dominated by some constant with aid of Theorem 3. It shows (b) implies (c).

Suppose (c) holds, then

$$\|V_g f\|^p_{p,\psi} \simeq \int_{\mathbb{B}_n} |f(z)|^p|R_g(z)|^p(1 - |z|^2)^{2p}e^{-\psi(z)} d\nu(z)$$

$$\leq \sup_{\mathbb{B}_n} \{ |R_g(z)|^p(1 - |z|^2)^{2p} \} \int_{\mathbb{B}_n} |f(z)|^p e^{-\psi(w)} d\nu(z).$$

(4.6)

It gives (c) implies (a).

\[\square\]

Theorem 6. Let $g \in \mathcal{O}(\mathbb{B}_n)$. The following statements are equivalent:

(a) $V_g$ is compact on $A^p_{\psi}(\mathbb{B}_n)$;
(b) $|R_g(z)|^p(1 - |z|^2)^{2p} d\mu(z)$ is a vanishing Carleson measure for $A^p_{\psi}(\mathbb{B}_n)$;
(c) $|R_g(z)|(1 - |z|^2)^2 \to 0$ as $|z| \to 1^-$.

Proof. Similar to the proof of Theorem 5 [10, 4.3, 4.5], and (4.6) yield the implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a), respectively.  \[\square\]
4.2. Boundedness and compactness of Toeplitz operators.

**Definition 4.3.** The Toeplitz operator with symbol $u$ is

$$T_u f(z) = \int_{\mathbb{B}_n} K(z, w)f(w)u(w)e^{-\psi(w)}\,dv(w)$$

for $A^2_\psi(\mathbb{B}_n)$.

Throughout this section, we consider $d\mu(z) = u(z)\,dv(z)$ for positive function $u$ and define $\hat{u}(z) := \hat{\mu}_2(z)$. Then

$$\hat{u}(z) = \int_{\mathbb{B}_n} |\overline{\Phi}_2(z,w)|^2 e^{-\psi(w)}u(w)\,dv(w)$$

and

$$\hat{u}(z) = \langle T_u \overline{\Phi}_2, \overline{\Phi}_2 \rangle_\psi$$

by the reproducing property of the Bergman kernel. We can see $\hat{u}(z)$ plays a similar role to Berezin transform which defined with a normalized Bergman kernel.

**Theorem 7.** Let $u$ be a positive function in $L^2_\psi(\mathbb{B}_n)$. The following statements are equivalent:

(a) $T_u$ is bounded on $A^2_\psi(\mathbb{B}_n)$;
(b) $\hat{u}$ is a bounded function on $\mathbb{B}_n$;
(c) $u\,dv$ is a $\psi$-Carleson measure.

**Proof.** Suppose that $T_u$ is bounded on $A^2_\psi(\mathbb{B}_n)$. We have

$$|\hat{u}(z)| = |\langle T_u \overline{\Phi}_2, \overline{\Phi}_2 \rangle_\psi| \leq \|T_u \overline{\Phi}_2\|_{L^2_\psi} \leq \|T_u\|.$$ (4.8)

Since $\|T_u \overline{\Phi}_2\|_{L^2_\psi} \leq \|T_u\|$, (a) implies (b).

Let $d\mu(z) = u(z)\,dv(z)$. We will show that (b) implies $\sup \left\{ \frac{\mu(B_H(z,r))}{\psi(B_H(z,r))} : z \in \mathbb{B}_n \right\} < +\infty$, which is equivalent that $d\mu$ is a $\psi$-Carleson measure. For any $z \in \mathbb{B}_n$ and a sufficiently small $r > 0$,

$$\hat{u}(z) = \int_{\mathbb{B}_n} |\overline{\Phi}_2(z,\zeta)|^2 e^{-\psi(\zeta)}d\mu(\zeta)$$

$$\geq \int_{B_H(z,r)} |\overline{\Phi}_2(z,\zeta)|^2 e^{-\psi(\zeta)}d\mu(\zeta)$$

$$\simeq \int_{B_H(z,r)} \frac{1}{(1-|z|^2)^{2n+1}} |\zeta|^2 e^{-\frac{r^2}{1-|z|^2}} e^{-\frac{1}{1-|\zeta|^2}}d\mu(\zeta)$$

by (8.3). It is obtained

$$\hat{u}(z) \geq \frac{1}{(1-|z|^2)^{2n+1}} \int_{B_H(z,r)} d\mu(\zeta) \simeq \frac{\mu(B_H(z,r))}{\psi(B_H(z,r))}$$ (4.9)

by (2.13) in Lemma 2.12 and Corollary 2.17.

Fubini’s theorem and the reproducing property of the Bergman kernel give that

$$\langle T_u f, g \rangle_\psi = \int_{\mathbb{B}_n} T_u f(w)g(w)e^{-\psi(w)}\,dv(w)$$

$$= \int_{\mathbb{B}_n} f(\zeta)\overline{g}(-\zeta)e^{-\psi(\zeta)}\,d\mu(\zeta).$$
By Hölder’s inequality,
\begin{equation}
|\langle Tu f, g \rangle_\psi| \leq \|f\|_{2,\mu} \|g\|_{2,\mu}.
\end{equation}
(4.10)

Since \(d\mu\) is a \(\psi\)-Carleson measure for \(A^2_\psi(B_n)\), we have
\[ |\langle Tu f, g \rangle_\psi| \leq C \|f\|_{2,\psi} \|g\|_{2,\psi}.\]

Therefore,
\begin{equation}
\|Tu f\|_{2,\psi} = \sup_{\|g\|_{2,\psi}=1} |\langle Tu f, g \rangle_\psi| \leq C \|f\|_{2,\psi}.
\end{equation}

It shows (c) implies (a). □

**Theorem 8.** Let \(u\) be a positive function in \(L^2_\psi(B_n)\). The following statements are equivalent:

(a) \(T_u\) is compact on \(A^2_\psi(B_n)\);

(b) \(|\hat{u}(z)| \to 0\) as \(|z| \to 1^-\);

(c) \(u d\nu\) is a vanishing \(\psi\)-Carleson measure.

**Proof.** Inequalities (4.8) and (4.9) assert the implications (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (c), respectively.

Suppose \(d\mu = u d\nu\) is a vanishing \(\psi\)-Carleson measure. Since \(d\mu\) is also a \(\psi\)-Carleson measure, (4.10) implies
\begin{equation}
\|Tu f\|_{2,\psi} = \sup_{\|g\|_{2,\psi}=1} |\langle Tu f, g \rangle_\psi| \leq C \|f\|_{2,\psi}.
\end{equation}

Since \(d\mu\) is a vanishing \(\psi\)-Carleson measure,
\[ \|Tu f\|_{2,\psi} \leq C \|f\|_{2,\mu} \to 0, \]
whenever \(\{f_j\}\) is a bounded sequence in \(A^p_\psi(B_n)\) which converges to 0 uniformly on compact subsets. It completes the proof. □

When the symbol function \(u\) is subharmonic, we get further results on \(T_u\).

**Corollary 4.4.** Let \(u\) be a positive function in \(L^2_\psi(B_n)\). If the symbol \(u\) is subharmonic, then the following statements are equivalent:

(a) \(T_u\) is bounded on \(A^2_\psi(B_n)\);

(b) \(u\) is a bounded function on \(B_n\).

**Proof.** If the symbol function \(u\) is bounded, then the Toeplitz operator \(T_u\) is bounded.

Since the symbol \(u\) is subharmonic, Lemma 2.13 with \(p = 1\) gives
\[ |u(z)| \lesssim \frac{1}{\psi(B_H(z,r))} \int_{B_H(z,r)} |u(w)| e^{-\psi(w)} d\nu(w) \]
for some small \(r > 0\). Boundedness of the operator \(T_u\) implies \(\mu(B_H(z,r)) \leq C\) where \(d\mu = u d\nu\) by Theorem 7. Hence, \(u\) is bounded on \(B_n\). □

**Corollary 4.5.** Let \(u\) be a positive function in \(L^2_\psi(B_n)\). If the symbol \(u\) is subharmonic, then the following statements are equivalent:

(a) \(T_u\) is compact on \(A^2_\psi(B_n)\);

(b) \(u(z) \to 0\) when \(z \to \partial B_n\).

**Proof.** Similar to the proof of Corollary 4.4, Lemma 2.13 and Theorem 8 give the result. □
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