On the chiral limit in lattice gauge theories with Wilson fermions

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Abstract

The chiral limit \( \kappa \simeq \kappa_c(\beta) \) in lattice gauge theories with Wilson fermions and problems related to near–to–zero (‘exceptional’) eigenvalues of the fermionic matrix are studied. For this purpose we employ compact lattice QED in the confinement phase. A new estimator \( \tilde{m}_\pi \) for the calculation of the pseudoscalar mass \( m_\pi \) is proposed which does not suffer from ‘divergent’ contributions at \( \kappa \simeq \kappa_c(\beta) \). We conclude that the main contribution to the pion mass comes from larger modes, and ‘exceptional’ eigenvalues play no physical role. The behaviour of the subtracted chiral condensate \( \langle \bar{\psi}\psi \rangle_{\text{subt}} \) near \( \kappa_c(\beta) \) is determined. We observe a comparatively large value of \( \langle \bar{\psi}\psi \rangle_{\text{subt}} \cdot Z_{\text{P}}^{-1} \), which could be interpreted as a possible effect of the quenched approximation.

1 Introduction

As it is well known, chiral symmetry is broken explicitly in lattice gauge theories with Wilson fermions as in QCD and QED. Presumably, it can be restored by fine–tuning the parameters in the continuum limit. If so, one can approach the continuum limit and chiral symmetry restoration along a ‘critical’ line \( \kappa_c(\beta) \). It is another question, whether on this line the chiral symmetry becomes explicitly realized or spontaneously broken. In the continuum limit the lowest–lying state in the spectrum of the Wilson fermionic matrix \( \mathcal{M}(\kappa; U) \) should have eigenvalue zero for \( \kappa \to \kappa_c(\beta) \).

For nonzero lattice spacing \( a \neq 0 \) the chiral symmetry cannot be restored exactly. At the same time in the confinement phase the pion mass \( m_\pi \) tends to zero in the limit \( \kappa \to \kappa_c(\beta) \) (the so–called partial symmetry restoration). The mechanism of this partial symmetry restoration is still not well determined. For

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example, one can not exclude that it is connected with the transition to a parity violating phase where $\langle \overline{\psi} \gamma_5 \psi \rangle \neq 0$ \cite{4}. If this transition is of second order then there will be a massless pion at $\kappa = \kappa_c(\beta)$.

Computations near the chiral transition line $\kappa_c(\beta)$ are notoriously difficult. The first problem is that the matrix inversion becomes very slow when $\kappa$ approaches $\kappa_c(\beta)$ and most of the known inversion methods fail very close to $\kappa_c(\beta)$. Nevertheless, the conjugate gradient method \cite{3, 4} appears to be reliable even in the ‘critical’ region. Another problem which arises in simulations of lattice gauge theories with Wilson fermions is that near the chiral transition very small (‘exceptional’) eigenvalues $\{\lambda_i \sim 0\}$ of the fermionic matrix $\mathcal{M}(\kappa; U)$ appear which make it practically impossible to approach $\kappa_c(\beta)$ without an enormous increase of statistics. The usual way to bypass this problem is to carry out calculations at $\kappa$–values sufficiently below $\kappa_c(\beta)$ and then to extrapolate the observables, e.g., hadron masses, to the ‘critical’ value $\kappa_c(\beta)$.

Apart from practical considerations the zero mode problem is connected with the question of the mechanism of the chiral transition in lattice theories with Wilson fermions. As it was already mentioned, zero eigenvalues are expected to exist in the chiral limit in the continuum. Within this context the question has to be answered, whether the near–to–zero eigenvalues of $\mathcal{M}(\kappa; U)$ observed on the lattice in the ‘critical’ region in finite volumes have physical relevance or rather arise as an effect of the lattice discretization.

Configurations with extremely small eigenvalues $\{\lambda_i\}$ were first discovered in \cite{5, 6}, and were called ‘exceptional’ configurations. In fact, very close to the transition point $\kappa \simeq \kappa_c(\beta)$ such configurations appear to be very ‘normal’ (see, e.g., \cite{7, 8, 9, 10}). In practice the appearance of these configurations can be used as an indicator for approaching $\kappa_c(\beta)$.

It is important to point out that the appearance of near–to–zero eigenvalues $\{\lambda_i\}$ at $\kappa \sim \kappa_c(\beta)$ is not a disease of the quenched approximation. The fermionic determinant decreases the spread of small eigenvalues but does not eliminate them totally \cite{3, 4, 5}.

It is the aim of this work to study the properties of a lattice gauge theory with Wilson fermions very close to the chiral transition line $\kappa_c(\beta)$. In the confinement phase we study the behaviour of the pseudoscalar pion mass $m_\pi$ and the subtracted chiral condensate $\langle \overline{\psi} \psi \rangle_{\text{subt}}$, which for Wilson fermions is an order parameter of chiral symmetry breaking in the continuum limit, near $\kappa_c(\beta)$. We employ compact lattice QED in the quenched approximation. Compact lattice QED with Wilson fermions in the confinement phase possesses similar features near the chiral transition $\kappa \sim \kappa_c(\beta)$ as nonabelian theories, and thus can serve as a comparatively simple model of QCD in this study. The cost of numerical calculations is much less than for QCD. We expect that our analysis is applicable to QCD, as well.
2 Action and observables

The standard Wilson lattice action $S_{WA}(U, \bar{\psi}, \psi)$ for 4d compact $U(1)$ gauge theory (QED) is

$$S_{WA} = S_G(U) + S_F(U, \bar{\psi}, \psi).$$

In eq. (1) $S_G(U)$ is the plaquette (Wilson) action for the pure gauge $U(1)$ theory

$$S_G(U) = \beta \cdot \sum_P (1 - \cos \theta_P),$$

where $\beta = 1/g_{\text{bare}}^2$, and $U_{x\mu} = \exp(i \theta_{x\mu})$, $\theta_{x\mu} \in (-\pi, \pi]$ are the field variables defined on the links $l = (x, \mu)$. Plaquette angles $\theta_P \equiv \theta_{x;\mu} + \theta_{x+\hat{\mu};\nu} - \theta_{x;\nu}$. The fermionic part of the action $S_F(U, \bar{\psi}, \psi)$ is

$$S_F = \sum_{x,y,s,s'} \bar{\psi}_x \mathcal{M}^{ss'}_{xy} \psi_{y} \equiv \bar{\psi} \mathcal{M} \psi,$$

$$\mathcal{M} \equiv \hat{1} - \kappa \cdot Q(U),$$

$$Q^{ss'}_{xy} = \sum_\mu \left[ \delta_{y,x+\hat{\mu}} \cdot (\hat{1} - \gamma_\mu)_{ss'} \cdot U_{x\mu} + \delta_{y,x-\hat{\mu}} \cdot (\hat{1} + \gamma_\mu)_{ss'} \cdot U_{x+\hat{\mu},\mu} \right],$$

where $\mathcal{M}$ is Wilson’s fermionic matrix, and $\kappa$ is the hopping parameter.

It is known, that at least in the strong coupling region for the standard Wilson action (in the confinement phase) the ordinary mass term and the Wilson mass term cancel within the pseudoscalar mass $m_\pi$ at some $\kappa = \kappa_c(\beta)$, so that quadratic terms in the effective potential vanish for the pseudoscalar field, and $\kappa_c \sim 0.25$ at $\beta = 0$ [11, 12].

In the weak coupling range perturbative calculations indicate that the mass of the fermion becomes equal to zero along the line $\kappa_c(\beta)$ (for the free field theory $\kappa_c = 0.125$) [11].

We calculated the following fermionic observables

$$\langle \bar{\psi} \psi \rangle = \frac{1}{4V} \cdot \langle \text{Tr}(\mathcal{M}^{-1}) \rangle_G; \quad \langle \bar{\psi} \gamma_5 \psi \rangle = \frac{1}{4V} \cdot \langle \text{Tr}(\gamma_5 \mathcal{M}^{-1}) \rangle_G;$$

$$\langle \Pi \rangle = \frac{1}{4V} \cdot \langle \text{Tr}(\mathcal{M}^{-1}\gamma_5 \mathcal{M}^{-1}\gamma_5) \rangle_G,$$

where $\langle \rangle_G$ stands for averaging over gauge field configurations, and $V = N_x \cdot N_s^3$ is the number of sites. Pseudoscalar zero–momentum correlators $\Gamma(\tau)$ are defined as follows
\[ \Gamma(\tau) = -\frac{1}{N^6_s} \cdot \sum_{\vec{x}, \vec{y}} \langle \bar{\psi} \gamma_5 \psi(\tau, \vec{x}) \cdot \bar{\psi} \gamma_5 \psi(0, \vec{y}) \rangle \]

\[ \equiv \frac{1}{N^6_s} \cdot \sum_{\vec{x}, \vec{y}} \langle \left\{ \text{Sp} (\mathcal{M}^{-1}_{xy} \gamma_5 \mathcal{M}^{-1}_{yx} \gamma_5) - \text{Sp} (\mathcal{M}^{-1}_{xx} \gamma_5) \cdot \text{Sp} (\mathcal{M}^{-1}_{yy} \gamma_5) \right\} \rangle_G \]  

(5)

where \( \text{Sp} \) means the trace with respect to the Dirac indices. These correlators as well as the pion norm \( \Pi \) appear to be very sensitive observables in the 'critical' region. This can be understood by considering the spectral representation of the fermionic order parameters. Let \( f_n \equiv f_n(s, x) \) be the eigenvectors of \( \mathcal{M} \) with eigenvalues \( \lambda_n \), and \( g_n \equiv g_n(s, x) \) be the eigenvectors of \( \gamma_5 \mathcal{M} \) with eigenvalues \( \mu_n \):

\[ \mathcal{M} f_n = \lambda_n \cdot f_n \, , \quad \gamma_5 \mathcal{M} g_n = \mu_n \cdot g_n \, . \]  

(6)

Then one can easily obtain a spectral representation of the fermionic order parameters:

\[ \langle \bar{\psi} \psi \rangle = \frac{1}{4V} \langle \sum_n \frac{1}{\lambda_n} \rangle_G \, , \quad \langle \bar{\psi} \gamma_5 \psi \rangle = \frac{1}{4V} \langle \sum_n \frac{1}{\mu_n} \rangle_G \, , \]

\[ \langle \Pi \rangle = \frac{1}{4V} \langle \sum_n \frac{1}{\mu_n^2} \rangle_G \, . \]  

(7)

Evidently, an eigenstate of \( \mathcal{M} \) with eigenvalue zero is also an eigenstate of \( \gamma_5 \mathcal{M} \). So, the presence of configurations which belong to zero eigenvalues of \( \mathcal{M} \) also gives rise to poles in \( \Pi \). For correlators one obtains

\[ \sum_{\vec{x}} \text{Sp} (\mathcal{M}^{-1}_{x0} \gamma_5 \mathcal{M}^{-1}_{0x} \gamma_5) |_{x_4=\tau} = \sum_{nn'} \frac{1}{\mu_n} \cdot b_{nn'}(\tau) \cdot \frac{1}{\mu_{n'}} \cdot b_{n'n}(0) \, , \]  

(8)

where

\[ b_{nn'}(\tau) \equiv \sum_{\vec{x}, \vec{s}} g_n^*(s, \vec{x}, \tau) \cdot g_{n'}(s, \vec{x}, \tau) \, . \]  

(9)

For further discussion on properties of the fermionic matrix see, e.g., \([13, 14]\) and references therein.

3 The pion mass in the chiral limit
3.1 Pseudoscalar correlators and the standard estimator

\( m_\pi \) near \( \kappa_c(\beta) \)

Transfer matrix arguments suggest the following form of the pseudoscalar correlator \( \Gamma(\tau) \) (at least, for \( \tau > 0 \)) \[10\]

\[
\Gamma(\tau) = A_\pi \cdot \left[ e^{-m_\pi \tau} + e^{-m_\pi (N_\tau - \tau)} \right] + \text{higher–energy states}.
\]

The standard choice of the estimator for the effective mass of the pseudoscalar particle \( m^{eff}_\pi(\tau) \equiv m_\pi(\tau) \) is

\[
\frac{\cosh m_\pi(\tau) \left( \frac{N_\tau}{2} - \tau - 1 \right)}{\cosh m_\pi(\tau) \left( \frac{N_\tau}{2} - \tau \right)} = \frac{\hat{\Gamma}(\tau + 1)}{\hat{\Gamma}(\tau)} = e^{-m_\pi(\tau)} \quad \text{if} \quad m_\pi N_\tau \gg 1,
\]

where \( \hat{\Gamma}(\tau) \) is the estimator of the pseudoscalar zero–momentum correlator

\[
\hat{\Gamma}(\tau) = \frac{1}{n} \sum_{i=1}^{n} \Gamma_i(\tau),
\]

and \( n \) is the number of measurements. The \( \tau \)–dependence of the effective mass stems from the contribution of higher–energy eigenstates of the transfer matrix. With increasing \( \tau \) the contribution of these higher–energy states is expected to be suppressed, and the resulting plateau in the \( \tau \)–dependence of the effective mass gives the true mass \( m_\pi \).

However this approach fails when one comes close to the chiral limit, i.e., when \( \kappa \rightarrow \kappa_c(\beta) \). The well–known problem in both QED and QCD with the calculation of \( m_\pi \) (and other fermionic observables) is connected with extremely small eigenvalues of the fermionic matrix \( \mathcal{M} \) (and correspondingly \( \gamma_5 \mathcal{M} \)) which appear on a finite lattice at some \( \kappa'(\beta; N_s; N_\tau) \lesssim \kappa_c(\beta) \).

To illustrate the problem we show in Figs.1a–c the dependence of the observables \( \langle \bar{\psi} \psi \rangle \), \( \langle \bar{\psi} \gamma_5 \psi \rangle \) and \( \langle \Pi \rangle \) on \( \kappa \) at \( \beta = 0.8 \) on a \( 8^4 \) lattice. Well below the transition point, i.e., when \( \kappa < \kappa_c'(\beta; N_\tau; N_s) \) , these averages are statistically well-defined. The increase of the number of measurements from, say, \( n = 100 \) to \( n = 200 \) produces just a slight change of the averages, and statistical errors decrease as \( \sim 1/\sqrt{n} \). The situation however changes significantly at \( \kappa \gtrsim \kappa_c'(\beta; N_\tau; N_s) \) (to the right of the vertical dashed line in Figs.1a–c). The averages begin to fluctuate drastically with increasing \( n \) (compare circles, crosses and squares in Figs.1a–c), and the errorbars become dramatically large. It is worthwhile to note that the increase of statistics does not necessarily entail the diminishing of the errorbars (compare, e.g., circles and squares in Fig.1c). This means in fact that these averages and errors, which were calculated using the jackknife procedure, make not very much sense and the accumulation of measurements will not change the state of affairs.
Figure 1: The dependence on $\kappa$ of $\langle \bar{\psi}\psi \rangle$ (a), $\langle \bar{\psi}\gamma_5\psi \rangle$ (b) and $\langle \Pi \rangle$ (c) at $\beta = 0.8$ on a $8^4$ lattice. Dotted lines have been added to guide the eye.

Note, that $\kappa_c'(\beta; N_\tau; N_s)$ in Fig.1c is a little bit smaller than in Figs.1a,b, since $\langle \Pi \rangle$ is the more sensitive observable with respect to small eigenvalues of $\gamma_5\mathcal{M}$ (and $\mathcal{M}$ respectively).

The behaviour of $\langle \bar{\psi}\gamma_5\psi \rangle$ in Fig.1b deserves maybe some additional comment. It was suggested [4] that at $\kappa = \kappa_c(\beta)$ there is a transition to a parity-violating phase, in which $\langle \bar{\psi}\gamma_5\psi \rangle \neq 0$ (see also [11]). At first sight our data for $\langle \bar{\psi}\gamma_5\psi \rangle$ in Fig.1b look like the confirmation of this hypothesis, but in fact they are not, because all averages and statistical errors to the right of the vertical dashed line are unreliable, and no definite conclusion can be drawn.

It is interesting to mention that on ‘exceptional’ configurations the number of conjugate gradient iterations $N_{cg}$ behaves differently from the ‘non–exceptional’ case where the convergence of the CG method is mainly determined by the condition
Figure 2: The time history of $\Pi$ and $N_{cg}$ at $\beta = 0.8$ and $\kappa = 0.22$ on an $8^4$–lattice.

number $\xi \equiv \lambda_{\text{max}}/\lambda_{\text{min}}$ of the to be inverted matrix. Moreover, when $\lambda_{\text{min}}$ is sufficiently small then $N_{cg}$ behaves as

$$N_{cg} \sim \lambda_{\text{min}}^{-\frac{1}{2}}$$

and therefore can be used as an indicator for approaching the chiral limit (see, e.g., [8, 14]). However, it is generally not known for which classes of distributions of eigenvalues eq.(13) holds. In a special study of this problem using hermitian matrices with some predefined (but continuous) distributions of eigenvalues we could confirm the commonly accepted results [10]. Now, in case of 'exceptional' small eigenvalues the behaviour of $N_{cg}$ does not follow eq.(13). This can be seen from Fig.2, where we display the time histories of the pion norm $\Pi$ and $N_{cg}$ on a $8^4$ lattice slightly above $\kappa_c(\beta)$. While the pion norm $\Pi$ develops huge spikes

\[\text{Fig. 2: The time history of } \Pi \text{ and } N_{cg} \text{ at } \beta = 0.8 \text{ and } \kappa = 0.22 \text{ on an } 8^4 \text{-lattice.}\]
(amplitudes up to $\sim 10^5$) for some configurations, the values of $N_{cg}$ vary within a $\sim 10\%$ corridor around the average value. Thus the CG convergence behaviour seems to be not only determined by $\xi$ but may also considerably depend on the eigenvalue distribution of the matrix to be inverted.

Figure 3: The $\tau$–dependence of the standard estimator $m_\pi(\tau)$ at $\beta = 0$; the lattice size is $12 \cdot 4^3$. Lines are to guide the eye.

In Fig.3 we present the $\tau$–dependence of the standard estimator $m_\pi(\tau)$ eq. for different $\kappa$’s at $\beta = 0$ on a $12 \cdot 4^3$ lattice. The value of $\kappa_c$ is expected to be $\sim 1/4$ at this value of $\beta$ (at least in the infinite volume). For the $12 \cdot 4^3$ lattice the ‘critical’ region starts at $\kappa_c'(\beta = 0) \sim 0.24$. With increasing the lattice size this ‘critical’ region shrinks: $\kappa_c'(\beta; N_T, N_s) \rightarrow \kappa_c(\beta)$ for $N_T, N_s \rightarrow \infty$. Nevertheless, on a finite lattice the problem connected with the approach to the chiral limit remains, and the usual ‘safe’ choice of $\kappa$ in QCD calculations, i.e., the choice of $\kappa$ well below $\kappa_c(\beta)$, entails an unrealistically large ratio $m_\pi/m_\rho$. 

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Sufficiently below the ‘critical’ value of $\kappa$ the standard estimator of the pion mass eq.(11) yields well-defined results. The contributions of higher excited states die out with increasing $\tau$, leaving a nice plateau in the $\tau$–dependence of the effective mass $m_\pi(\tau)$, which determines the actual mass at these values of couplings. For the reasons given above, this estimator fails to give precise results in the region $\kappa \geq \kappa'_c(\beta; N_\tau; N_s)$, i.e. when approaching the chiral limit. (All data points of this figure represent averages of at least 5000 measurements).

### 3.2 Another estimator for $m_\pi$

One possible way to bypass this problem is based on the observation which we call the factorization of the contribution of near–to–zero eigenvalues. In Figs.4a–b we display the values of $\Gamma_i(\tau)$ on individual configurations $i = 1, \ldots, 900$ on a $16 \times 8^3$ lattice at $\beta = 0$ and $\kappa = 0.25$ for $\tau = 3$ (Fig.4a) and $\tau = 4$ (Fig.4b). The huge spikes (up to $\sim 10^4$) in the time histories are due to the existence of very small eigenvalues in the spectrum of $\gamma_5 \mathcal{M}$ (respectively $\mathcal{M}$). These spikes appear with a certain frequency and just reflect the content of the spectrum of $\gamma_5 \mathcal{M}$ for this value of $\kappa$ which is close to $\kappa'_c(\beta)$ in a finite system, i.e. the lowest eigenvalues are allowed to fluctuate from configuration to configuration.

The important observation is the absence of such peaks in the ratio of the individual correlators

$$g_i(\tau) = \frac{\Gamma_i(\tau + 1)}{\Gamma_i(\tau)} \quad (14)$$

which means the factorization of the ‘divergent’ contributions originating from near–to–zero eigenvalues. The statistical fluctuations of this ratio are very small even at $\kappa \gtrsim \kappa'_c(\beta)$ (see Fig.4c) as compared with fluctuations of $\Gamma_i(\tau)$ (Figs.4a,b).

Therefore, we propose to use the ratio eq.(14) for the extraction of the ‘pion’ mass near the chiral transition. Then a new estimator for the pseudoscalar mass, which we denote by $\tilde{m}_\pi(\tau)$, can be obtained from the following expression

$$\frac{\cosh \tilde{m}_\pi(\tau) \left( \frac{N_\tau}{2} - \tau - 1 \right)}{\cosh \tilde{m}_\pi(\tau) \left( \frac{N_\tau}{2} - \tau \right)} = \frac{1}{n} \sum_{i=1}^{n} g_i(\tau) \quad (15)$$

Therefore, the relation to the standard estimator $m_\pi(\tau)$ in eq.(11) is given by

$$\frac{\cosh m_\pi(\tau) \left( \frac{N_\tau}{2} - \tau - 1 \right)}{\cosh m_\pi(\tau) \left( \frac{N_\tau}{2} - \tau \right)} = \frac{\cosh \tilde{m}_\pi(\tau) \left( \frac{N_\tau}{2} - \tau - 1 \right)}{\cosh \tilde{m}_\pi(\tau) \left( \frac{N_\tau}{2} - \tau \right)} + \frac{1}{n} \sum_{i=1}^{n} g_i(\tau) \cdot \frac{\delta \Gamma_i(\tau)}{\Gamma(\tau)} \quad (16)$$

with

$$\delta \Gamma_i(\tau) = \Gamma_i(\tau) - \hat{\Gamma}(\tau) \quad \sum_{i}^{n} \delta \Gamma_i(\tau) \equiv 0.$$
In general, of course, this is not the correct way to calculate masses. However, under certain circumstances it is justified to do so.

Assume that $x$ and $y$ are two correlated random variables with some distribution $P(x; y)$, where $\int dx dy P(x; y) = 1$. Then the average of any functional $O(x; y)$ is

$$\langle O \rangle = \int dx dy O \cdot P(x; y). \quad (17)$$

One can define the conditional average $\bar{y}(x)$ as

$$\bar{y}(x) = \int dy y P(x; y) / \int dy P(x; y). \quad (18)$$

If the distribution $P(x; y)$ is such that
Figure 5: $\Gamma_i(\tau = 2)$ as a function of $\Gamma_i(\tau = 1)$ at $\beta = 0$ and $\kappa = 0.25$ on a $16 \cdot 8^3$ lattice at different scales. Single points represent individual configurations. Straight lines correspond to the linear fit according to the least squares method.
\[ y(x) = C \cdot x, \quad C = \text{const}, \]  

then it is easy to see that

\[ \frac{\langle y \rangle}{\langle x \rangle} = \frac{\langle y \rangle}{\langle x \rangle}. \]  

In our case \( x = \Gamma_i(\tau) \) and \( y = \Gamma_i(\tau + 1) \). In principle, one can use also the ratios \( \Gamma_i(\tau+k)/\Gamma_i(\tau) \) with \( k \geq 2 \), though the signal–to–noise ratio decreases with increasing \( k \).

In Fig.5 we show \( \Gamma_i(2) \) as a function of \( \Gamma_i(1) \) at \( \beta = 0 \) and \( \kappa = 0.25 \) on a \( 16 \times 8^3 \) lattice. For illustrative purposes we have shown our data of \( \sim 2000 \) measurements with four different choices of the scale. The distribution gives a clear indication in favour of eq.(19). The correlation coefficient \( \rho \) defined in a standard way \[17\] is very close to unity, and varies between \( \sim 0.98 \) and \( \sim 0.995 \) for different subsets of measurements shown in Fig.5a ÷ Fig.5d.

To provide a proof that \( \tilde{m}_\pi(\tau) \) can serve as a reliable estimator of the pion mass \( m_\pi(\tau) \) we will compare the properties of both estimators. We’ll show that for \( \kappa \)–values sufficiently below \( \kappa_c(\beta) \), i.e., where the standard estimator of \( m_\pi \) can be reliably defined, both estimators are in a very good agreement. For values of \( \kappa \) very close to \( \kappa_c(\beta) \), where the standard estimator fails to work, while \( \tilde{m}_\pi^2 \) fits

the same straight line.

We have run simulations in the confinement phase at \( \beta = 0 \) and \( \beta = 0.8 \) using lattices with \( N_\tau = 12; 16; 20 \) and \( N_s = 4; 8 \). For the matrix inversion we used the standard conjugate gradient method [3, 4] with even–odd decomposition \[18, 19\] which guaranties convergence even at \( \kappa \lesssim \kappa_c(\beta) \).

The \( \tau \)–dependence of the effective mass \( \tilde{m}_\pi(\tau) \) appears to be more complicated than that of the standard estimator \( m_\pi(\tau) \). In Fig.6 we have plotted the dependence of \( \tilde{m}_\pi(\tau) \) on \( \tau \) at different \( \beta \)'s and \( \kappa \)'s on lattices with \( N_s = 4 \) and \( N_s = 12; 16; 20 \). At \( \beta = 0 \) the value \( \kappa = 0.23 \) is sufficiently below the transition point \( \kappa_c(0) \sim 0.25 \) (no appearance of 'exceptional' small eigenvalues of \( \mathcal{M} \) and \( \gamma_5\mathcal{M} \) respectively) and therefore \( \tilde{m}_\pi(\tau) \) can be compared with the corresponding \( m_\pi(\tau) \) shown in Fig.3. The standard estimator \( m_\pi(\tau) \) exhibits a nice plateau up to \( \tau = 6 \) on a \( 12 \times 4^3 \) lattice, while the corresponding plateau for \( \tilde{m}_\pi(\tau) \) is to be seen only for \( \tau \lesssim 5 \). In this case both estimators \( (m_\pi(\tau) \) and \( \tilde{m}_\pi(\tau) \) ) are in a very good agreement as long as \( \tau \) is not too close to \( N_\tau/2 \). These deviations from the standard \( m_\pi(\tau) \) at \( \tau \sim N_\tau/2 \) are a general feature of the new estimator which presumably stem from the way of averaging the corresponding observables on a finite lattice (but which fade with increasing \( N_s \), see below). The increasing of \( N_\tau \) entails the extension of the plateau. The same effect, i.e. the enlarging of the plateau in the \( \tau \)–dependence of \( \tilde{m}_\pi(\tau) \) occurs for other \( \beta \)'s and \( \kappa \)'s, including values \( \kappa \lesssim \kappa_c(\beta) \) where the standard estimator is not well–defined (e.g., \( \kappa = 0.25; \beta = 0 \) in Fig.6 ). It is important to notice that increasing \( N_\tau \) does not change the position of the plateau, influencing only its extension.
Figure 6: The $\tau$–dependence of $\tilde{m}_\pi(\tau)$ at $\beta = 0$ and $\beta = 0.8$. Lattice sizes are $12 \cdot 4^3$, $16 \cdot 4^3$ and $20 \cdot 4^3$. Lines are to guide the eye.

Finite volume effects, i.e., the dependence of $\tilde{m}_\pi(\tau)$ on $N_s$ tend to increase slightly with increasing coupling $\beta$ – this could be interpreted as an extension of the 'critical' zone in $\kappa$ with rising $\beta$. In Fig.7a we compare the $\tau$–dependence of both estimators $m_\pi(\tau)$ and $\tilde{m}_\pi(\tau)$ at $\beta = 0$ and $\beta = 0.8$ for two lattice sizes: $16 \cdot 4^3$ and $16 \cdot 8^3$. The values of the corresponding $\kappa$’s are chosen to be sufficiently far from the transition point, ensuring the applicability of the standard definition of the pion mass. At $\beta = 0$ both estimators are in an excellent agreement even on the smaller lattice with $N_s = 4$. At $\beta = 0.8$ $\tilde{m}_\pi(\tau)$ on the smaller lattice gives an $\approx 3\%$ overestimated value. However, on the lattice with $N_s = 8$ the agreement between both estimators becomes very good. Note, that at these $\kappa$–values the deviations of $\tilde{m}_\pi(\tau)$ from the plateau for $\tau \rightarrow N_s/2$ as described in Fig.6 decreased substantially.

In Fig.7b and Fig.7c we compare the values of $\tilde{m}_\pi(\tau)$ at the same $\beta$’s as in Fig.7a but now at $\kappa$’s chosen near $\kappa_c(\beta)$. The finite volume dependence becomes
stronger when $\kappa$ becomes close to $\kappa_c(\beta)$. The estimate of $\tilde{m}_\pi(\tau)$ is lowered by increasing the spatial extension $N_s$. The quality of the plateau is good enough to determine $\tilde{m}_\pi(\tau)$ with good accuracy.

In Fig. 8 we show the $\kappa$-dependence of $\tilde{m}_\pi^2$ for $\beta = 0$ and $\beta = 0.8$ on a $16 \cdot 8^3$ lattice. Different symbols correspond to different values of $\tau$. At both values of $\beta$ the new estimator $\tilde{m}_\pi^2$ behaves in a similar way. Even at $\kappa > \kappa_c(\beta)$ the values of $\tilde{m}_\pi^2$ at different $\tau$ and $\beta$ are in a good agreement. For both $\beta$-values $\tilde{m}_\pi^2$ shows a linear dependence on $\kappa$ in the vicinity of $\kappa_c(\beta)$. This leads to the behaviour

$$\tilde{m}_\pi^2 = B_\pi(\beta) \cdot m_q; \quad m_q \to 0, \quad (21)$$

where the dimensionless bare fermion mass $m_q$ is defined as
Figure 8: The $\kappa$–dependence of $\tilde{m}_{\pi}^2(\tau; \kappa)$ at $\beta = 0$ and $\beta = 0.8$ on a $16 \cdot 8^3$ lattice for several values of $\tau$. The errorbars are smaller than the symbol size. Lines drawn are to guide the eye.

$\tilde{m}_{\pi}^2(\tau; \kappa) = \frac{1}{2\kappa} - \frac{1}{2\kappa_c(\beta)}$.  

At $\beta = 0$ the straight line extrapolation (broken line in Fig.8) predicts for the 'critical' value $\kappa_c(0) \simeq 0.2502(1)$ which is in a very good agreement with the strong coupling results [11]. With increasing $\beta$ the value $\kappa_c(\beta)$ decreases: $\kappa_c(\beta = 0.8) \simeq 0.2171(1)$ \textbf{(errorbars correspond to one standard deviation in $\chi^2/n_{d.o.f.}$)}.

The corresponding slopes are: $B_\pi(0) = 4.91(4)$ and $B_\pi(0.8) = 3.42(3)$.

For $\kappa$–values sufficiently below $\kappa_c(\beta)$, i.e., for $\kappa < \kappa'(\beta)$, $\tilde{m}_{\pi}^2$ is just the same as the standard estimator $m_{\pi}^2$. For $\kappa$'s very close to $\kappa_c(\beta)$ the standard estimator can’t be applied, still $\tilde{m}_{\pi}^2$ fits very well the same straight line having a very small statistical error. Our data at $\beta = 0$ suggest, that for $\kappa \gtrsim 0.246$ ($m_q \lesssim 0.034$) the standard estimator of $m_{\pi}^2$ deviates from the straight–line behaviour in Fig.8. For
instance, at $\kappa \geq 0.248$ ($m_q \sim 0.0176$) the standard estimator was not applicable any more. Numbers change when considering the case $\beta = 0.8$. Here the deviation of $m^2_\pi$ from the straight-line behaviour starts already at $m_q \simeq 0.078$, which corresponds to $\kappa = 0.21$. This demonstrates the advantage of $\bar{m}_\pi$ which gives the possibility to approach the chiral limit much closer than the standard estimator of $m_\pi$ would allow. Especially, systematic errors induced by the finite volume can be investigated in the 'critical' region by means of $\bar{m}_\pi$ with good accuracy, which in turn should allow a more precise extrapolation to the thermodynamic limit.

4 The subtracted chiral condensate

One can use the Goldstone theorem to define the 'physical' (subtracted) chiral condensate $\langle \bar{\psi}\psi \rangle_{\text{subt}}$. Following [20] we define $\langle \bar{\psi}\psi \rangle_{\text{subt}}$ which is supposed to serve as an order parameter for the chiral symmetry breaking (at least, in the continuum limit)

$$\langle \bar{\psi}\psi \rangle_{\text{subt}} = \lim_{\kappa \to \kappa_c} \langle \bar{\psi}\psi \rangle_{\text{subt}}(\kappa);$$

$$\langle \bar{\psi}\psi \rangle_{\text{subt}}(\kappa) = \left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right) \cdot Z_P \sum_x \langle \bar{\psi}_x \gamma_5 \psi \rangle \cdot \langle \bar{\psi}_0 \gamma_5 \psi \rangle,$$  \hspace{1cm} (23)

where $Z_P$ is the renormalization constant of the pseudoscalar quark density and the sum in the r.h.s. of eq.(23) is connected with the pion norm $\langle \Pi \rangle$. At nonzero lattice spacing the pion mode in the limit $\kappa \to \kappa_c(\beta)$ should result in a constant contribution to the condensate $\langle \bar{\psi}\psi \rangle_{\text{subt}}$ while the contribution of the higher energy (massive) modes should decrease as $\sim (\kappa_c - \kappa)$.

Fig.9 shows the dependence of $\langle \bar{\psi}\psi \rangle_{\text{subt}}(\kappa) \cdot Z_P^{-1}$ on $\kappa$ for $\beta = 0$ and $\beta = 0.8$. Here we have chosen $\kappa$ values in the region where near-to-zero 'exceptional' modes not yet show up. The pion mass $m_\pi$ reached already the asymptotic regime shown in Fig.8. For these $\kappa$'s the values of the condensate are on the (expected) straight line which permits to make a reasonable extrapolation to $\kappa \to \kappa_c(\beta)$. The extrapolated values of $\langle \bar{\psi}\psi \rangle_{\text{subt}}(\kappa)$ are nonzero, of course (at least, because of the finite spacing).

The main observation here is that the values of the condensate are comparatively large, comparing with what one could expect from the naive definition $\langle \bar{\psi}\psi \rangle_{\text{bare}} - \langle \bar{\psi}\psi \rangle_{\text{pert}}$ with $\langle \bar{\psi}\psi \rangle_{\text{bare}} \leq 1$ (although one cannot exclude that the renormalization constant $Z_P$ is responsible for this effect).

A possible interpretation of this effect of amplification of $\langle \bar{\psi}\psi \rangle_{\text{subt}} \cdot Z_P^{-1}$ is due to the unphysical contributions in the quenched approximation. In the dynamical fermion case the fermionic determinant tends to decrease the pion norm substantially comparing with that in the quenched approximation [10] (at least in the confinement phase). Therefore, we expect that in the dynamical fermion case
Figure 9: The $m_q$–dependence of the subtracted condensate $Z_P^{-1} \cdot \langle \overline{\psi} \psi \rangle_{\text{subt}}(\kappa)$ at $\beta = 0$ and $\beta = 0.8$ on a $16 \cdot 8^3$–lattice. Lines are to guide the eye.

the condensate should be much smaller. From another point of view the unphysical contributions in the quenched approximation were discussed also in [21, 22, 23].

At $\kappa \sim \kappa_c(\beta)$ the behaviour of $\langle \overline{\psi} \psi \rangle_{\text{subt}}(\kappa)$ should be mainly controlled by the value of the preexponential factor $A_\pi$ in the pseudoscalar correlator $\Gamma(\tau)$ in eq.(10). The existence of the pion requires $A_\pi$ to behave as $A_\pi \sim 1/m_\pi$ when $m_\pi \to 0$. Together with eq.(21) this implies

$$A_\pi \sim \frac{1}{\sqrt{m_q}}, \quad m_q \to 0.$$  \hspace{1cm} (24)

In Fig.10 we present the value $A_\pi^2 \cdot m_q$ as a function of $m_q$ obtained from our data on a $16 \cdot 8^3$ lattice. For bare masses $m_q < 0.06$ we are not able to indicate $A_\pi^2 \cdot m_q$, since the extracted $A_\pi$’s become statistically unreliable. At $\beta = 0$ the
value $A^2_{\pi} \cdot m_q$ shows a gradual increase with decreasing $m_q$, which, however, is not in contradiction to eq. (24), taking into account the comparatively large values of $m_q$. Remarkably, we observe $A^2_{\pi} \cdot m_q$ to fit a constant value at $\beta = 0.8$ with good accuracy in agreement with eq. (24).

Figure 10: $A^2_{\pi} \cdot m_q$ vs. $m_q$ at $\beta = 0$ and $\beta = 0.8$ on a $16 \times 8^3$ lattice. Added lines are to guide the eye.

5 Conclusions

In this work we have studied the chiral limit of a lattice gauge theory with Wilson fermions within the quenched approximation, employing compact lattice QED in the confinement phase.

We observed that ratios of the pseudoscalar correlators $\Gamma_1(\tau + 1)/\Gamma_1(\tau)$ do not suffer from near-to-zero ('exceptional') eigenmodes of the fermionic matrix.
This means that for every given configuration all 'divergent' contributions to the correlators are factorized.

Making use of this observation we propose another estimator \( \tilde{m}_\pi \) of the pseudoscalar mass, which is well defined near the chiral transition line \( \kappa_c(\beta) \) and even for \( \kappa > \kappa_c(\beta) \) in contrast to the standard estimator \( m_\pi \).

It should be stressed that the possibility to introduce this estimator is based on the special kind of correlations between \( \Gamma_i(\tau) \) with different \( \tau \), i.e., such that eq. (19) is fulfilled. This is not a universal property, of course.

For \( \kappa \)-values sufficiently below \( \kappa_c(\beta) \), i.e., where the standard estimator of \( m_\pi \) can be reliably defined, both estimators are proved to be in a very good agreement. By approaching \( \kappa_c(\beta) \) we observe a linear dependence of \( m^2_\pi \) on \( \kappa \):

\[
m^2_\pi \sim \left( \kappa - \kappa_c(\beta) \right).
\]

For values of \( \kappa \) very close to \( \kappa_c(\beta) \) the standard estimator fails to work, while \( \tilde{m}^2_\pi \) still fits the same straight line having a very small statistical error.

This study leads us to the conclusion that the main contribution to the pion mass comes from larger modes, and the 'exceptional' near–to–zero eigenvalues of the fermionic matrix \( \gamma_5 \mathcal{M} \) (and \( \mathcal{M} \) respectively) play no physical role. This conclusion is in agreement with [24].

The new estimator \( \tilde{m}_\pi \) gives the possibility to approach the chiral limit much closer than the standard estimator of \( m_\pi \) would allow. Especially, systematic errors induced by the finite volume can be investigated in the 'critical' region by means of \( \tilde{m}_\pi \) with good accuracy, which in turn should allow a more precise extrapolation to the thermodynamic limit.

The 'critical' value \( \kappa_c(\beta) \) can be determined with high accuracy. Our \( \kappa_c(0) \) is in very good agreement with strong coupling predictions [11].

We investigated the subtracted chiral condensate \( \langle \bar{\psi}\psi \rangle_{\text{subt}} \) following the definition in [24] near the chiral transition line \( \kappa_c(\beta) \). We obtained a comparatively large value of \( \langle \bar{\psi}\psi \rangle_{\text{subt}} \cdot Z_p^{-1} \) which we interpret as a possible effect of the quenched approximation.

It would be very interesting to know in which way the fermionic sea –dynamical fermions– could influence the behaviour of the pseudoscalar mass \( m_\pi \) and the condensate \( \langle \bar{\psi}\psi \rangle_{\text{subt}} \). This work is in progress.
References

[1] K. Wilson, Phys. Rev. D10 (1974) 2445; in New phenomena in subnuclear physics, ed. A. Zichichi (Plenum, New York, 1977).

[2] S. Aoki, Phys. Rev. D30 (1984) 2653; Phys. Rev. Lett. 57 (1986) 3136; Phys. Lett. B190 (1987) 140.

[3] M. R. Hestenes and E. Stiefel, Journal of Research of the NBS 49 (1952) 409.

[4] M. Engeli et al., Mitteilungen aus dem Institut für angewandte Mathematik (Birkhäuser Verlag, Berlin, 1959), Vol. 8, p. 24.

[5] Ph. De Forcrand, A. König, K.-H. Mütter, K. Schilling and R. Sommer, in: Proc. Intern. Symp. on Lattice gauge theory (Brookhaven, 1986), (Plenum, New York, 1987).

[6] Ph. De Forcrand, R. Gupta, S. Güsken, K.-H. Mütter, A. Patel, K. Schilling and R. Sommer, Phys. Lett. 200B (1988) 143.

[7] K. Bitar, A. D. Kennedy and P. Rossi, Phys. Lett. B234 (1990) 333.

[8] I. Barbour, E. Laermann, Th. Lippert and K. Schilling, Phys. Rev. D46 (1992) 3618.

[9] A. Hoferichter, V.K. Mitrjushkin, M. Müller-Preussker and Th. Neuhaus, Nucl. Phys.B (Proc.Suppl.) 34 (1994) 537.

[10] A. Hoferichter, V.K. Mitrjushkin, M. Müller-Preussker, Th. Neuhaus and H. Stüben, Nucl.Phys. B434 (1995) 358.

[11] N. Kawamoto, Nucl. Phys. B190 (1981) 617.

[12] N. Kawamoto and J. Smit, Nucl. Phys. B192 (1981) 100.

[13] S. Itoh, Y. Iwasaki and T. Yoshié, Phys. Rev. D36 (1987) 527.

[14] A. Ukawa, CERN-TH-5245/88 (1988).

[15] M. Lüscher, Commun. Math. Phys. 54 (1977) 283.

[16] A. Nakamura and R. Sinclair, Phys. Lett. 243B (1990) 396.

[17] B.L. Van Der Vaerden, Mathematische Statistik, Springer Verlag (1957).

[18] T.A. DeGrand, Comp.Phys.Comm. 52 (1988) 161.

[19] T.A. DeGrand and P. Rossi, Comp.Phys.Comm. 60 (1990) 211.
[20] M. Bochicchio, L. Maiani, G. Martinelli, G. Rossi and M. Testa, Nucl. Phys. B262 (1985) 331.

[21] C. Bernard and M. Golterman, Phys. Rev. D46 (1992) 853.

[22] S. Sharpe, Phys. Rev. D46 (1992) 3146.

[23] R. Gupta, Nucl. Phys. B (Proc. Suppl.) 42 (1994) 85.

[24] B. Bunk, K. Jansen, B. Jegerlehner, M. Lüscher, H. Simma and R. Sommer, Nucl. Phys. B (Proc. Suppl.) 42 (1994) 49.