A Proof On Arnold Chord Conjecture

Renyi Ma  
Department of Mathematical Sciences  
Tsinghua University  
Beijing, 100084  
People’s Republic of China  
rma@math.tsinghua.edu.cn

Abstract

In this article, we first give a proof on the Arnold chord conjecture which states that every Reeb flow has at least as many Reeb chords as a smooth function on the Legendre submanifold has critical points on contact manifold. Second, we prove that every Reeb flow has at least as many close Reeb orbits as a smooth round function on the close contact manifold has critical circles on contact manifold. This also implies a proof on the fact that there exists at least number $n$ close Reeb orbits on close $(2n - 1)$-dimensional convex hypersurface in $R^{2n}$ conjectured by Ekeland.

Keywords Contact manifold, Reeb Chord, Periodic orbits.

2000 MR Subject Classification 32Q65, 53D35,53D12

1 Introduction and results

Let $\Sigma$ be a smooth closed oriented manifold of dimension $2n - 1$. A contact form on $\Sigma$ is a 1-form such that $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on $\Sigma$. Associated to $\lambda$ there are two important structures. First of all the so-called Reeb vectorfield $\dot{x} = X$ defined by

$$i_X \lambda \equiv 1, \quad i_X d\lambda \equiv 0;$$
and secondly the contact structure $\xi = \xi_\lambda \mapsto \Sigma$ given by

$$\xi_\lambda = \ker(\lambda) \subset T\Sigma.$$  

By a result of Gray, [7], the contact structure is very stable. In fact, if $(\lambda_t)_{t \in [0,1]}$ is a smooth arc of contact forms inducing the arc of contact structures $(\xi_t)_{t \in [0,1]}$, there exists a smooth arc $(\psi_t)_{t \in [0,1]}$ of diffeomorphisms with $\psi_0 = \text{Id}$, such that

$$T\Psi_t(\xi_0) = \xi_t$$  

(1.1)

where it is important that $\Sigma$ is compact. From (1.1) and the fact that $\Psi_0 = \text{Id}$ it follows immediately that there exists a smooth family of maps $[0,1] \times \Sigma \mapsto (0,\infty) : (t,m) \mapsto f_t(m)$ such that

$$\Psi_t^*\lambda_t = f_t\lambda_0$$  

(1.2)

In contrast to the contact structure the dynamics of the Reeb vectorfield changes drastically under small perturbation and in general the flows associated to $X_t$ and $X_s$ for $t \neq s$ will not be conjugated.

Let $(\Sigma, \lambda)$ be a contact manifold with contact form $\lambda$ of dimension $2n - 1$, then a Legendre submanifold is a submanifold $L$ of $\Sigma$, which is $(n-1)$-dimensional and everywhere tangent to the contact structure $\ker \lambda$. Then a characteristic chord for $(\lambda, L)$ is a smooth path

$$x : [0, T] \to M, T > 0$$

with

$$\dot{x}(t) = X_\lambda(x(t)) \text{ for } t \in (0, T),$$

$$x(0), x(T) \in L$$

The main results of this paper is following:

**Theorem 1.1** Let $(\Sigma, \lambda)$ be a contact manifold with contact form $\lambda$, $X_\lambda$ its Reeb vector field, $L$ a closed Legendre submanifold. Then there exists at least as many Reeb characteristic chords for $(X_\lambda, L)$ as a smooth function on the Legendre submanifold $L$ has critical points.
Theorem 1.1 is asked in [1]. In [9], we have proved there exists at least one Reeb chord by Gromov’s $J$–holomorphic curves. Partial results is obtained in [12, 13].

We recall that a round function $f$ on $M$ is the one whose critical sets consist of smooth circles $\{C_i, i = 1, \ldots, k\}$ (see[3]). One can define Round Morse-Bott function.

**Theorem 1.2** Let $(\Sigma, \lambda)$ be a contact manifold with contact form $\lambda$, $X_\lambda$ its Reeb vector field. Then there exists at least as many close Reeb characteristic orbits for $X_\lambda$ as a smooth round function on the $\Sigma$ has critical circles.

In [10], we have proved there exists at least one close Reeb orbit by Gromov’s $J$–holomorphic curves. Theorem 1.2 is related to the Arnold-Ginzburg question on magnetic field and partial results was obtained in [1, 4, 6].

**Corollary 1.1** Let $(\Sigma, \lambda)$ be a close $(2n-1)$–dimensional star-shaped hypersurface in $\mathbb{R}^{2n}$ with contact form $\lambda$, $X_\lambda$ its Reeb vector field. Then there exists at least number $n$ close Reeb orbits.

This implies that the Ekeland conjecture holds in [5].

The proofs of Theorem 1.1-1.2 are the extension of the methods in [11].

## 2 Proof of Theorem 1.1

**Proof of Theorem 1.1:** Let $(\Sigma, \lambda)$ be a contact manifold with the contact form $\lambda$. By Whitney’s embedding theorem, we first embed $\Sigma$ in $\mathbb{R}^N$, then by considering the cotangent bundles, the symplectizations and contactizations, one can embed $(\Sigma, \lambda)$ into $(S^{2N+1}, \lambda_0)$ with $\lambda = f \lambda_0$, $f$ is positive function on $\Sigma$. By the contact tubular neighbourhood theorem, the neighbourhood $U(\Sigma)$ is contactomorphic to the symplectic vector bundle $E$ on $\Sigma$ with symplectic fibre $(\mathbb{R}^{2N-2n+2}, \omega_0)$. By our construction, it is easy to see that there exists Lagrangian sub-bundle $L$ in $E$ with Lagrangian fibre $\mathbb{R}^{N-n+1}$. So, we can extend the contact form $\lambda$ on $\Sigma$ to the neighbourhood $U(\Sigma)$ as $\bar{\lambda}$ such that the Reeb vector fields $X_\lambda|\Sigma = X_\lambda$.

We extend $f$ positively to whole $S^{2N+1}$. So the contact form $\bar{\lambda}$ on $U(\Sigma)$ is extended to whole $S^{2N+1}$ as $f \lambda_0$. 
Let $\lambda_s = (sf + 1 − s)\lambda_0$ be the one parameter family of contact forms on $S^{2N+1}$. Let $X_{\lambda_s}$ its Reeb vector field.

Let $\mathcal{L}$ be a close Legendre submanifold contained in $\Sigma$, i.e., $T\mathcal{L} \subset \xi$. Let $\bar{\mathcal{L}}$ be a Legendre submanifold contained in $U(\Sigma)$ which is fibred on $\mathcal{L}$ and $T\bar{\mathcal{L}} \subset \xi_0$, here $\xi_0$ is the standard contact structure on $(S^{2N+1}, \lambda_0)$.

Let $S$ be a smooth hypersurface in $\Sigma$ which contains $\mathcal{L}$. We can assume that the Reeb vector fields $X_{\lambda_s}$ is transversal to $S$. Let $\bar{S}$ be a smooth hypersurface in $S$ and extends $\mathcal{L}$ and extends $S$. We can assume that the Reeb vector fields $X_{\lambda_s}$ is transversal to $\bar{S}$.

Let $\eta_{s,t}$ be the Reeb flow generated by the Reeb vector field $X_{\lambda_s}$. Let $f_s : \bar{\mathcal{L}}_s \to \bar{S}$ be the arrival time function of the Reeb flow $\eta_{s,t}$ from the parts $\bar{\mathcal{L}}_s$ of $\bar{\mathcal{L}}$ to $\bar{S}$. Then $\varphi_s = \eta_{s,f_s} : \bar{\mathcal{L}}_s \to \bar{S}$ is an exact Lagrange embedding for symplectic form $d\lambda_s|\bar{S}$.

One observes that $\varphi_s(\bar{\mathcal{L}}_s) \cap \bar{\mathcal{L}}$ corresponds to the Reeb chords of $X_{\lambda_s}$.

Now we extend the family of functions $f_s$ on $\bar{\mathcal{L}}_s$ to the whole $\bar{\mathcal{L}}$. This extends $\varphi_s = \eta_{s,f_s} : \bar{\mathcal{L}} \to S^{2N+1}$ as an exact isotropic embedding for symplectic form $d\lambda_s$ on $S^{2N+1}$. Then, we obtain $\bar{\varphi}_s = \eta_{s,f_s,t} : \bar{\mathcal{L}} \times [-\varepsilon, \varepsilon] \to S^{2N+1}$ as an exact Lagrangian embedding for symplectic form $d(e^t\lambda_s)$ on $R \times S^{2N+1}$. By Moser’s stability theorem, this defines an exact Lagrangian isomorphism $\bar{\varphi}_s : \bar{\mathcal{L}} \times [-\varepsilon, \varepsilon] \to (R \times S^{2N+1}, d(e^t\lambda_0))$. It is well known that an exact Lagrangian isomorphism is determined by Hamilton isotopy with Hamilton function $h_s$ on $R \times S^{2N+1}$. Let $X_{h_s}$ be the Hamilton vector field induced by the Hamilton function $h_s$.

We first assume that near the Lagrangian submanifold $\mathcal{L}$ in $S$ the form $\lambda$ is the standard Liouville form in $T^*\mathcal{L}$. Also, We assume that near the Lagrangian submanifold $\bar{\mathcal{L}} \times [-\varepsilon, \varepsilon]$ the form $e^t\lambda_0$ is the standard Liouville form in $T^*(\bar{\mathcal{L}} \times [-\varepsilon, \varepsilon])$. Consider the Lie derivative $L_{X_{h_s}} e^t\lambda_0 = e^t\lambda_0$ on $\bar{\varphi}_s(\mathcal{L} \times [-\varepsilon, \varepsilon])$. Moreover, $L_{X_{h_s}} e^t\lambda_0 = dH_s$, here $H_s$ is defined on $R \times S^{2N+1}$. The level sets of $H_s$ defines a foliation $\mathcal{F}_s$ on $R \times S^{2N+1}$. For $s$ is small enough, the Lagrangian submanifold $\varphi_s(\mathcal{L})$ in $S$ is transversal to the foliation $\mathcal{F}_s|S$ except the Lagrangian submanifold $\mathcal{L}$. So, we can perturb the Lagrangian isotopy $\varphi_s$ such that the intersection points sets $\bar{\varphi}_s(\mathcal{L}) \cap \mathcal{L}$ is invariant and the isotropic submanifold $\varphi_s(\mathcal{L})$ is transversal to the foliation $\mathcal{F}_s$ except the intersection points set $\bar{\varphi}_s(\mathcal{L}) \cap \mathcal{L}$.

This shows that the critical points of $H_1|\varphi_1(\mathcal{L})$ are in the intersection
points set $\varphi_1(\mathcal{L}) \cap \mathcal{L}$.

This yields Theorem 1.1.

3 Proof of Theorem 1.2

Proof of Theorem 1.2: Let $(\Sigma, \lambda)$ be a close contact manifold with contact form $\lambda$ and $X_\lambda$ its Reeb vector field, then $X_\lambda$ integrates to a Reeb flow $\eta_t$ for $t \in \mathbb{R}$. Consider the form $d(-e^a\lambda)$ at the point $(a, \sigma)$ on the manifold $(\mathbb{R} \times \Sigma)$, then one can check that $d(-e^a\lambda)$ is a symplectic form on $\mathbb{R} \times \Sigma$. Let

$$(\Sigma', \lambda') = ((\mathbb{R} \times \Sigma) \times \Sigma, (-e^a\lambda) \oplus \lambda).$$

Then $(\Sigma', \lambda')$ is a non-compact contact manifold. But the Reeb flow $\eta'_s = (id, \eta_s)$ of Reeb vector field $X_{\lambda'}$ behaves as close contact manifold, especially the compact set $([-a, a] \times \Sigma) \times \Sigma$ is invariant under Reeb flow. Let

$$\mathcal{L} = \{((0, \sigma), \sigma) | \sigma \in \Sigma\}.$$ 

Then, $\mathcal{L}$ is a close Legendre submanifold in $(\Sigma', \lambda')$.

Then, by considering the round function and round intersection, the method of proving Theorem 1.1 yields Theorem 1.2. e.q.d.

References

[1] Arnold, V. I., First steps in symplectic topology, Russian Math. Surveys 41(1986),1-21.

[2] Arnold, V.& Givental, A., Symplectic Geometry, in: Dynamical Systems IV, edited by V. I. Arnold and S. P. Novikov, Springer-Verlag, 1985.

[3] Asimov, D., Round handles and non-singular Morse-Smale flows, Ann.Math.(1)102,pp41-51,1975.

[4] Banyaga A., A Note On Weinstein Conjecture, Proceeding of AMS, 3901-3906, Vol123,No12,1995.

[5] Ekeland,I., Convexity Methods in Hamiltonian Mechanics. Springer. Berlin. (1990).
[6] Ginzburg, V.L., New generalization of Poincare’s Geometric theorem, Funct.Anal.Appl.21(1987),100-107.

[7] Gray, J.W., Some global properties of contact structures. Ann. of Math., 2(69): 421-450, 1959.

[8] Gromov, M., Pseudoholomorphic Curves in Symplectic manifolds. Inv. Math. 82(1985), 307-347.

[9] Ma, R., J−holomorphic Curves, Legendrian submanifolds And Reeb Chords, math/0004038.

[10] Ma, R., J−holomorphic Curves And Periodic Reeb Orbits, math/0004037.

[11] Ma, R., Proofs On Arnold Conjectures, math/arXiv:0808.0613.

[12] Ma, R., Legendrian submanifolds and A Proof on Chord Conjecture, Boundary Value Problems, Integral Equations and Related Problems, edited by J K Lu & G C Wen, World Scientific, 135-142,2000.

[13] Mohnke, K.: Holomorphic Disks and the Chord Conjecture, Annals of Math., (2001), 154:219-222.