DRY TEN MARTINI PROBLEM FOR THE NON-SELF-DUAL EXTENDED
HARPER’S MODEL

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Abstract. In this paper we prove the dry version of the Ten Martini problem: Cantor spectrum
with all gaps open, for the extended Harper’s model in the non self-dual region for Diophantine
frequencies.

1. Introduction

The study of independent electrons on a two-dimensional lattice exposed to a perpendicular
magnetic field and periodic potentials can be reduced via an appropriate choice of gauge field to the
study of discrete one-dimensional quasiperiodic Jacobi matrices. The most extensively studied case
is the almost Mathieu operator (AMO) acting on $l^2(\mathbb{Z})$ defined by

$$(H_{\lambda,\alpha,\theta} u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n.$$  

This is a one-dimensional tight-binding model with anisotropic nearest neighbor couplings in general. 
A more general model, called the extended Harper’s model (EHM), is the operator acting on $l^2(\mathbb{Z})$ defined by:

$$(H_{\lambda,\alpha,\theta} u)_n = c(\theta + n\alpha)u_{n+1} + \tilde{c}(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n.$$  

where $c(\theta) = \lambda_1 e^{-2\pi i(\theta + \alpha/2)} + \lambda_2 + \lambda_3 e^{2\pi i(\theta + \alpha/2)}$ and $\tilde{c}(\theta) = \lambda_1 e^{2\pi i(\theta + \alpha/2)} + \lambda_2 + \lambda_3 e^{-2\pi i(\theta + \alpha/2)}$. It is obtained when both the nearest neighbor coupling (expressed through $\lambda_2$) and the next-nearest couplings (expressed through $\lambda_1$ and $\lambda_3$) are included. This model includes AMO as a special case (when $\lambda_1 = \lambda_3 = 0$).

For the AMO, it was proved in [5] that the spectrum is a Cantor set for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda \neq 0$. This is the Ten Martini Problem dubbed by Barry Simon, after an offer of Mark Kac. A much more difficult problem, known as the dry version of the Ten Martini Problem, is to prove that the spectrum is not only a Cantor set, but that all gaps predicted by the Gap-Labeling theorem [10], [15] are open. The first result was obtained for Liouvillean $\alpha$ [12], and later it was proved for a set of $(\lambda, \alpha)$ of positive Lebesgue measure [16]. The most recent result is [6], in which they were able to deal with all Diophantine frequencies and $\lambda \neq 1$. A solution for all irrational frequencies and $\lambda \neq 1$ was also recently announced in [9].

Recently, there have been several important advances on the spectral theory of the EHM: purely
point spectrum for Diophantine $\alpha$ and a.e.$\theta$ in the positive Lyapunov exponent region [13]; the exact
formula for Lyapunov exponent for all coupling constants [14]; the spectral decomposition for a.e.$\alpha$
[7]. However the results that study the spectrum as a set have not been obtained for the EHM.

For EHM, depending on the values of the parameters $\lambda_1, \lambda_2, \lambda_3$, we could divide the parameter
space into three regions as shown in the picture below:
\[ \begin{align*}
\text{region I} & : 0 < \max(\lambda_1 + \lambda_3, \lambda_2) < 1, \\
\text{region II} & : 0 < \max(\lambda_1 + \lambda_3, 1) < \lambda_2, \\
\text{region III} & : 0 < \max(1, \lambda_2) < \lambda_1 + \lambda_3.
\end{align*} \]

According to the action of the duality transformation \( \sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \rightarrow \hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2}) \), region I and region II are dual to each other and region III is a self-dual region. Region I is the positive Lyapunov exponent region, which is a natural extension of the segment \( \{\lambda_1 + \lambda_3 = 0, 0 < \lambda_2 < 1\} \) corresponding to the case \( \lambda > 1 \) in the AMO. Region II is the subcritical region, which is an extension of the segment \( \{\lambda_1 + \lambda_3 = 0, 1 < \lambda_2\} \) corresponding to the case \( \lambda < 1 \) in the AMO.

In this paper we prove the dry version of the Ten Martini Problem in region I and region II under the Diophantine condition.

Let \( p_n/q_n \) be the continued fraction approximants of \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Let
\[
\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.
\]

If \( \beta(\alpha) = 0 \), we say \( \alpha \) satisfies the Diophantine condition, denoted by \( \alpha \in \text{DC} \). It is easily seen that such \( \alpha \) form a full measure subset of \( T \).

It is known that when \( E \) is in the closure of a spectral gap, the integrated density of state (IDS) \( N(E) \in \alpha \mathbb{Z} + \mathbb{Z} \) (refer to (2.5) for the definition of IDS) \( [10, 15] \). Here we prove the inverse is true.

**Theorem 1.1.** If \( \alpha \in \text{DC} \) and \( \lambda \) belongs to region I or region II, all possible spectral gaps are open.

**Remark 1.1.** We note the Dry Ten Martini problem has not yet been solved for the self-dual AMO. In the self-dual region III, Cantor spectrum is known in the isotropic case (when \( \lambda_1 = \lambda_3 \)), see Fact 2.1 in \([7]\). In fact one could prove the operator has zero Lebesgue measure spectrum for all frequencies.

**Remark 1.2.** In region I and II, for Liouvillean \( \alpha \) (where \( \beta(\alpha) \) is large), it is not clear whether even the Cantor spectrum holds. The proof may require a non-trivial adjustment of the proof for AMO in \([12]\).
We first establish almost localization (see section 3.1) in region I, then a quantitative version of Aubry duality to obtain almost reducibility (see section 3.2) in region II which enables us to deal with all energies whose rotation numbers are $\alpha$-rational.

Thus the strategy follows that of [6], but we need to extend the almost localization and quantitative duality, as well as the final argument to our Jacobi setting, which is non-trivial on a technical level. At the same time unlike [6], we only deal with a short-range dual operator, leading to a significant streamlining of some arguments of [6].

We organize the paper as follows: in section 2 we present some preliminaries, in section 3 we state our main results about almost localization and almost reducibility, relying on which we provide a proof of Theorem [1.1]. In section 4 and 5 we prove the main results that we present in section 3.

2. PRELIMINARIES

2.1. Cocycles. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \in C^0(\mathbb{T}, M_2(\mathbb{C}))$ measurable with $\log\|A(x)\| \in L^1(\mathbb{T})$. The quasi-periodic cocycle $(\alpha, A)$ is the dynamical system on $\mathbb{T} \times \mathbb{C}^2$ defined by $(\alpha, A)(x, v) = (x + \alpha, A(x)v)$. The Lyapunov exponent is defined by

\[ L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int_\mathbb{T} \log\|A_n(x)\| \, dx = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_\mathbb{T} \log\|A_n(x)\| \, dx. \]

where

\[
\begin{cases}
A_n(x) = A(x + (n - 1)\alpha) \cdots A(x) & \text{for } n \geq 0, \\
A_n(x) = A^{-1}(x + n\alpha) \cdots A^{-1}(x - \alpha) & \text{for } n < 0.
\end{cases}
\]

Lemma 2.1. (c.g.[6]) Let $(\alpha, A)$ be a continuous cocycle, then for any $\delta > 0$ there exists $C_0 > 0$ such that for any $n \in \mathbb{N}$ and $\theta \in \mathbb{T}$ we have

\[ \|A_n(\theta)\| \leq C_0 e^{(L(\alpha, A) + \delta)n}. \]

We say that $(\alpha, A)$ is uniformly hyperbolic if there exists continuous splitting $\mathbb{C}^2 = E^s(x) \bigoplus E^u(x)$, $x \in \mathbb{T}$ such that for some constant $C, \eta > 0$ and all $n \geq 0$, $\|A_n(x)v\| \leq Ce^{-\eta n}\|v\|$ for $v \in E^s(x)$ and $\|A_{-n}(x)v\| \leq Ce^{-\eta n}\|v\|$ for $v \in E^u(x)$.

Given two complex cocycles $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$, we say they are complex conjugate to each other if there is $M \in C^0(\mathbb{T}, SL(2, \mathbb{C}))$ such that

\[ M^{-1}(x + \alpha)A^{(1)}(x)M(x) = A^{(2)}(x). \]

We assume now that $A$ is a real cocycle, $A \in C^0(\mathbb{T}, SL(2, \mathbb{R}))$. The notation of real conjugacy (between real cocycles) is the same as before, except that we look for $M \in C^0(\mathbb{T}, PSL(2, \mathbb{R}))$. A reason why we look for $M \in C^0(\mathbb{T}, PSL(2, \mathbb{R}))$ instead of $M \in C^0(\mathbb{T}, SL(2, \mathbb{R}))$ is given by the following well-known result.

Theorem 2.2. Let $(\alpha, A)$ be uniformly hyperbolic, assume $\alpha \in \text{DC}$ and $A$ analytic, then there exists $M \in C^\infty(\mathbb{T}, PSL(2, \mathbb{R}))$ such that $M^{-1}(x + \alpha)A(x)M(x)$ is constant.

We say $(\alpha, A)$ is (analytically) reducible if it is real conjugate to a constant cocycle by an analytic conjugacy.

Let

\[ R_\theta = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix}. \]

Any $A \in C^0(\mathbb{T}, PSL(2, \mathbb{R}))$ is homotopic to $x \to R_{\theta x}$ for some $k \in \mathbb{Z}$ called the degree of $A$, denoted by $\deg A = k$.\footnote{In general one cannot take $M \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$.}
Assume now that \( A \in C^0(T, SL(2, \mathbb{R})) \) is homotopic to identity. Then there exists \( \phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) and \( v : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+ \) such that
\[
A(x) \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = v(x, y) \begin{pmatrix} \cos 2\pi(y + \phi(x, y)) \\ \sin 2\pi(y + \phi(x, y)) \end{pmatrix}.
\]
The function \( \phi \) is called a lift of \( A \). Let \( \mu \) be any probability on \( \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \) which is invariant under the continuous map \( T : (x, y) \mapsto (x + \alpha, y + \phi(x, y)) \), projecting over Lebesgue measure on the first coordinate. Then the number
\[
\rho(\alpha, A) = \int \phi \, d\mu \mod \mathbb{Z}
\]
is independent of the choices of \( \phi \) and \( \mu \), and is called the fibered rotation number of \( (\alpha, A) \).

It can be proved directly by the definition that
\[
|\rho(\alpha, A) - \theta| < C\|A - R_\theta\|_0.
\]
(2.1)

If \( (\alpha, A^{(1)}) \) and \( (\alpha, A^{(2)}) \) are real conjugate, \( M^{-1}(x + \alpha)A^{(2)}(x)M(x) = A^{(1)}(x) \), and \( M : \mathbb{R}/\mathbb{Z} \to PSL(2, \mathbb{R}) \) has degree \( k \) then
\[
\rho(\alpha, A^{(1)}) = \rho(\alpha, A^{(2)}) - ka/2.
\]
(2.2)

For uniformly hyperbolic cocycles there is the following well-known result.

**Theorem 2.3.** Let \( (\alpha, A) \) be a uniformly hyperbolic cocycle, with \( \alpha \in \mathbb{R}\setminus\mathbb{Q} \). Then \( 2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z} \).

### 2.2. Extended Harper’s model

We consider the extended Harper’s model \( \{ H_{\lambda, \theta} \}_{\theta \in T} \). The formal solution to \( H_{\lambda, \theta}u = Eu \) can be reconstructed via the following equation
\[
\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{\lambda, E}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},
\]
where \( A_{\lambda, E}(\theta) = \frac{1}{c(\theta)} \begin{pmatrix} E - 2\cos 2\pi\theta & -\bar{c}(\theta - \alpha) \\ c(\theta) & 0 \end{pmatrix} \). Notice that since \( A_{\lambda, E}(\theta) \notin SL(2, \mathbb{R}) \), we introduce the following matrix (see Lemma A.2)
\[
\tilde{A}_{\lambda, E}(\theta) = \frac{1}{\sqrt{|c(\theta)||c(\theta - \alpha)|}} \begin{pmatrix} E - 2\cos 2\pi\theta & -|c(\theta - \alpha)| \\ |c(\theta)| & 0 \end{pmatrix} = Q_\lambda(\theta + \alpha)A_{\lambda, E}(\theta)Q_\lambda^{-1}(\theta),
\]
where \( |c(\theta)| = \sqrt{c(\theta)\bar{c}(\theta)} \) (which is not the same as \( |c(\theta)| = \sqrt{c(\theta)\bar{c}(\theta)} \) when \( \theta \notin T \)) and \( Q_\lambda(\theta) \) is analytic on \( \{|\text{Im}\theta| \leq \frac{\pi}{2}\} \).

The spectrum of \( H_{\lambda, \theta} \) denoted by \( \Sigma_\lambda \), does not depend on \( \theta \) \( \mathbb{R} \), and it is the set of \( E \) such that \( (\alpha, \tilde{A}_{\lambda, E}) \) is not uniformly hyperbolic.

The Lyapunov exponent is defined by \( L_\lambda(E) = L(\alpha, A_{\lambda, E}) = L(\alpha, \tilde{A}_{\lambda, E}) \).

For a matrix-valued function \( M(\theta) \), let \( M_\epsilon(\theta) = M(\theta + i\epsilon) \) be the phase-complexified matrix.

In \( \mathbb{H} \), Avila divides all the energies in the spectrum into three categories: super-critical, namely the energy with positive Lyapunov exponent; subcritical, namely the energy whose Lyapunov exponent of the phase-complexified cocycle is identically equal to zero in a neighborhood of \( \epsilon = 0 \); critical, otherwise.

The following theorem is shown in \( \mathbb{H} \) (see also the appendix):

**Theorem 2.4.** Extended Harper’s model is super-critical in region I and sub-critical in region II. Indeed
\[
\bullet \text{ when } \lambda \text{ belongs to region II, } L_\lambda(E) = L(\alpha, A_{\lambda, E, \epsilon}) = L(\alpha, \tilde{A}_{\lambda, E, \epsilon}) = 0 \text{ on } |c| \leq \frac{1}{2\pi}c_1(\lambda),
\]
• when \( \lambda \) belongs to region II, we have \( \hat{\lambda} = \left( \frac{\lambda_1}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2} \right) \) belongs to region I and

\[
L_{\hat{\lambda}}(E) = c_1(\lambda),
\]

where

\[
e_1(\lambda) = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{\max(\lambda_1 + \lambda_3, 1) + \sqrt{\max(\lambda_1 + \lambda_3, 1)^2 - 4\lambda_1\lambda_3}} > 0.
\]

Fix a \( \theta \) and \( f \in l^2(\mathbb{Z}) \). Let \( \mu_{\lambda,\theta}^f \) be the spectral measure of \( H_{\lambda,\theta} \) corresponding to \( f \),

\[
\langle (H_{\lambda,\theta} - z)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{E - z} d\mu_{\lambda,\theta}^f(E).
\]

for \( z \) in the resolvent set \( \mathbb{C} \setminus \Sigma_\lambda \).

The integrated density of states (IDS) is the function \( N_\lambda : \mathbb{R} \rightarrow [0, 1] \) defined by

\[
N_\lambda(E) = \int_{\mathbb{T}} \mu_{\lambda,\theta}^f(-\infty, E) d\theta,
\]

where \( f \in l^2(\mathbb{Z}) \) is such that \( \|f\|_1 = 1 \). It is a continuous non-decreasing surjective function.

Notice that \( A_{\lambda,E}(\theta) \in SL(2, \mathbb{R}) \) is homotopic to identity in \( C^0(\mathbb{T}, SL(2, \mathbb{R})) \), in fact just consider \( H_\lambda(\lambda, E, \theta) = \frac{1}{\sqrt{|c|(|\theta|)|c|(|\theta - t\alpha|)}} \begin{pmatrix} t(E - v(\theta)) & -|c|(|\theta - t\alpha|) \\ |c|(|\theta|) & 0 \end{pmatrix} \).

which establishes a homotopy of \( \hat{A}_{\lambda,E}(\theta) \) to \( R_{\frac{\lambda}{\lambda_2}} \) and hence to the identity. Therefore we can define the rotation number \( \rho(\alpha, \hat{A}_{\lambda,E}) \). Let \( \rho_\lambda(E) = \rho(\alpha, \hat{A}_{\lambda,E}) \). Notice that \( \rho_\lambda(E) \) is associated to the operator

\[
(\hat{H}_{\lambda,\theta} u_n) = |c|(|\theta + n\alpha|)u_{n+1} + |c|(|\theta + (n - 1)\alpha|)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n.
\]

It is easily seen that for each \( \theta \), \( \hat{H}_{\lambda,\theta} \) and \( H_{\lambda,\theta} \) differ by a unitary operator, thus they share the same spectrum and integrated density of states, \( \hat{N}_{\lambda}(E) = N_{\lambda}(E) \). The relation between the integrated density of states and rotation number of \( \hat{H}_{\lambda,\theta} \) yields the following

\[
N_\lambda(E) = \hat{N}_{\lambda}(E) = 1 - 2\rho_\lambda(E).
\]

2.3. The dual model. It turns out the spectrum \( \Sigma_\lambda \) of \( H_{\lambda,\theta} \) is related to the spectrum \( \Sigma_{\hat{\lambda}} \) of \( H_{\hat{\lambda},\theta} \) in the following way

\[
\Sigma_\lambda = \lambda_2 \Sigma_{\hat{\lambda}}
\]

by Aubry duality. This map \( \sigma : \lambda \rightarrow \hat{\lambda} \) establishes the duality between region I and region II. The IDS \( N_{\lambda}(E) \) of \( H_{\lambda,\theta} \) coincide with the IDS \( N_{\hat{\lambda}}(E/\lambda_2) \) of \( H_{\hat{\lambda},\theta} \). Since \( \Sigma_\lambda = \lambda_2 \Sigma_{\hat{\lambda}} \), we have the following

**Theorem 2.5.** [11, 17] For any \( \lambda, \theta \), there exists a dense set of \( E \in \Sigma_\lambda \) such that there exists a non-zero solution of \( H_{\lambda,\theta} u = \lambda E u \) with \( |u_k| \leq 1 + |k| \).

2.4. Bounded eigenfunction for every energy. The next result from [6] allows us to pass from a statement of every \( \theta \) to every \( E \).

**Theorem 2.6.** [6] If \( E \in \Sigma_\lambda \) then there exists \( \theta(E) \in \mathbb{T} \) and a bounded solution of \( H_{\lambda,\alpha,\theta} u = \frac{E}{\lambda_2} u \) with \( u_0 = 1 \) and \( |u_k| \leq 1 \).
2.5. Localization and reducibility.

Theorem 2.7. Given $\alpha$ irrational, $\theta \in \mathbb{R}$ and $\lambda$ in region $H$, fix $E \in \Sigma_{\lambda}$, and suppose $H_{\lambda, \theta} u = \frac{E}{x^2} u$ has a non-zero exponentially decaying eigenfunction $u = \{u_k\}_{k \in \mathbb{Z}}$, $|u_k| \leq e^{-\varepsilon|k|}$ for $k$ large enough. Then the following hold:

- (A) If $2\theta \notin \alpha \mathbb{Z} + \mathbb{Z}$, then there exists $M : \mathbb{R}/\mathbb{Z} \to SL(2, \mathbb{R})$ analytic, such that
  \begin{equation*}
  M^{-1}(x + \alpha) \tilde{A}_{\lambda, E}(x) M(x) = R_{\perp \theta}.
  \end{equation*}
  In this case $\rho(\alpha, \tilde{A}_{\lambda, E}) = \pm \theta + \frac{\pi}{2} \alpha \mod \mathbb{Z}$, where $m = \deg M$ (here since $M \in SL(2, \mathbb{R})$, we have that $m$ is an even number) and $2\rho(\alpha, \tilde{A}_{\lambda, E}) \notin \alpha \mathbb{Z} + \mathbb{Z}$.

- (B) If $2\theta \in \alpha \mathbb{Z} + \mathbb{Z}$ and $\alpha \in DC$, then there exists $M : \mathbb{R}/\mathbb{Z} \to PSL(2, \mathbb{R})$ analytic, such that
  \begin{equation*}
  M^{-1}(x + \alpha) \tilde{A}_{\lambda, E}(x) M(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}
  \end{equation*}
  with $a \neq 0$. In this case $\rho(\alpha, \tilde{A}_{\lambda, E}) = \frac{\pi}{2} \alpha \mod \mathbb{Z}$, where $m = \deg M$, i.e. $2\rho(\alpha, \tilde{A}_{\lambda, E}) \in \alpha \mathbb{Z} + \mathbb{Z}$.

Proof. Let $u(x) = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{2\pi i k x}$, $U(x) = \begin{pmatrix} e^{2\pi i \theta} u(x) \\ u(x - \alpha) \end{pmatrix}$. Then
  \begin{equation*}
  A_{\lambda, E}(x) U(x) = e^{2\pi i \theta} U(x + \alpha),
  \end{equation*}
  \begin{equation*}
  \tilde{A}_{\lambda, E}(x) \tilde{U}(x) = e^{2\pi i \theta} \tilde{U}(x + \alpha).
  \end{equation*}

Notice $\tilde{U}(x) = Q_{\lambda}(x) U(x)$ is analytic in $|\text{Im} x| < \frac{\pi}{\sqrt{c}}$, where $\hat{c} = \min (\epsilon_1, c)$, $\epsilon_1$ as in $[2, 4]$ and $Q_{\lambda}$ as in $[4, 7]$. Define $\tilde{U}(x)$ to be the complex conjugate of $\tilde{U}(x)$ on $T$ and its analytic extension to $|\text{Im} x| < \frac{\pi}{\sqrt{c}}$. Let $M(x)$ be the matrix with columns $\tilde{U}(x)$ and $\tilde{U}(x)$, then
  \begin{equation*}
  \tilde{A}_{\lambda, E}(x) M(x) = M(x + \alpha) \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix} \text{ on } T.
  \end{equation*}

Then since $\det M(x + \alpha) = \det M(x)$, we know $\det M(x)$ is a constant on $T$.

Case 1. If $\det M(x) \neq 0$, then let $M(x) = \tilde{M}(x) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$.

  \begin{equation*}
  \tilde{M}^{-1}(x + \alpha) \tilde{A}_{\lambda, E}(x) \tilde{M}(x) = R_{\theta} = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix}.
  \end{equation*}

Case 2. If $\det M(x) = 0$, then if we denote $\tilde{U}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$, then $\det M(x) = 0$ means there exists $\eta(x)$ such that $u_1(x) = \eta(x) u_1(x)$ and $u_2(x) = \eta(x) u_2(x)$. This implies that $\eta(x) \in \mathbb{C}^c(T, \mathbb{C})$, and $|\eta(x)| = 1$ on $T$. Therefore there exists $\phi(x) \in \mathbb{C}^c(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ such that $\phi^2(x) = \eta(x)$ and $|\phi(x)| = 1$. It is easy to see $\phi(x) u_1(x) = \phi(x) u_1(x)$ and $\phi(x) u_2(x) = \phi(x) u_2(x)$. Then we define $W(x) = \begin{pmatrix} \phi(x) u_1(x) \\ \phi(x) u_2(x) \end{pmatrix}$, it is a real vector on $\mathbb{R}/2\mathbb{Z}$ with $W(x + 1) = \pm W(x)$, and $\tilde{U}(x) = \phi(x) W(x)$. Now let us define $\tilde{M}(x)$ to be the matrix with columns $W(x)$ and $\begin{pmatrix} 1 \\ \|W(x)\| \end{pmatrix} R_{\perp \theta} W(x)$, then $\det \tilde{M}(x) = 1$ and $\tilde{M}(x) \in PSL(2, \mathbb{R})$. Since
  \begin{equation*}
  \tilde{A}_{\lambda, E}(x) W(x) = \frac{e^{2\pi i \theta} \phi(x + \alpha)}{\phi(x)} W(x + \alpha).
  \end{equation*}
There exists $\psi \in \mathbb{C}^\omega(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$. Let $0 = \alpha$ of Definition 3.1. We have

$$\tilde{M}(x) = M(x + \alpha) \begin{pmatrix} d(x) & \tau(x) \\ 0 & d(x)^{-1} \end{pmatrix}$$

where $d(x) = e^{2\pi i \alpha(x+\alpha)}/\phi(x)$, $|d(x)| = 1$ and $d(x)$ being real number, therefore $d(x) = \pm 1$. Also $\tau(x) \in \mathbb{C}^\omega(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$. But in fact $M^{-1}(x + \alpha)\tilde{M}(x)M(x)$ is well-defined on $T$. Therefore $\tau(x) \in \mathbb{C}^\omega(T, \mathbb{C})$. Now since we assumed $\alpha \in DC$, we can further reduce $\tau(x)$ to the constant $\tau = \int_T \tau(x)dx$. In fact there exists $\psi(x) \in \mathbb{C}^\omega(T, \mathbb{C})$ such that $-\psi(x + \alpha) + \psi(x) + \tau(x) = \int_T \tau(x)dx$. This implies

$$\begin{pmatrix} 1 & -\psi(x + \alpha) \\ 0 & 1 \end{pmatrix} M^{-1}(x + \alpha)\tilde{M}(x)M(x) \begin{pmatrix} 1 & \psi(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pm 1 & \tau \\ 0 & \pm 1 \end{pmatrix}.$$ 

In fact if $\det M(x) = 0$, then $e^{2\pi i \alpha(x+\alpha)}/\phi(x) = \pm 1$, which implies that $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$. Therefore if $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$, we must be in case (A). If on the other hand, $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, $2\theta = \kappa\alpha + n$, suppose $M^{-1}(x + \alpha)\tilde{M}(x)M(x) = R_\alpha$ then $R_{-\frac{1}{2}\kappa\alpha}M^{-1}(x + \alpha)\tilde{M}(x)M(x)R_{\frac{1}{2}\kappa\alpha} = R_\alpha = \pm I$ leading to a contradiction. Therefore if $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, we must be in case (B).$

2.6. Continued fractions. Let $\{q_n\}$ be the denominators of the continued fraction approximants of $\alpha$. We recall the following properties:

$$\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} = \inf_{1 \leq |k| \leq q_{n+1}-1} \|k\alpha\|_{\mathbb{R}/\mathbb{Z}},$$

$$\frac{1}{2q_{n+1}} \leq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{q_{n+1}}.$$ 

Recall that the Diophantine condition of $\alpha$ is $\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n} = 0$. Thus for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq C_\xi e^{-\xi |k|} \quad \text{for any } k \neq 0.\n
\textbf{Lemma 2.8.} \ [5] \text{Let } \alpha \in \mathbb{R}\setminus\mathbb{Q}, x \in \mathbb{R} \text{ and } 0 \leq l_0 \leq q_n - 1 \text{ be such that } |\sin \pi(x + l_0\alpha)| = \inf_{0 \leq l \leq q_n - 1} |\sin \pi(x + l\alpha)|, \text{ then for some absolute constant } C_1 > 0,

$$-C_1 \ln q_n \leq \sum_{0 \leq l \leq q_n - 1, l \neq l_0} \ln |\sin \pi(x + l\alpha)| + (q_n - 1) \ln 2 \leq C_1 \ln q_n$$ 

\textbf{Lemma 2.9.} \ [6] Let $1 \leq r \leq [q_{n+1}/q_n]$. If $p(x)$ has essential degree at most $k = rq_{n+1} - 1$ and $x_0 \in \mathbb{R}/\mathbb{Z}$, then for some absolute constant $C_2$,

$$\|p(x)\|_0 \leq C_2q_{n+1}^{C_2r} \sup_{0 \leq j \leq k} |p(x_0 + j\alpha)|.$$ 

3. Main estimates and proof of Theorem 1.1

3.1. Almost localization for every $\theta$.

\textbf{Definition 3.1.} Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\theta \in \mathbb{R}$, $\epsilon_0 > 0$. We say that $k$ is an $\epsilon_0$–resonance of $\theta$ if $\|2\theta - k\alpha\| \leq e^{-\epsilon_0|k|}$ and $\|2\theta - k\alpha\| = \min_{|l| \leq |k|} \|2\theta - l\alpha\|$. 

\textbf{Definition 3.2.} Let $0 = |n_0| < |n_1| < \ldots$ be the $\epsilon_0$–resonances of $\theta$. If this sequence is infinite, we say $\theta$ is $\epsilon_0$–resonant, otherwise we say it is $\epsilon_0$–non-resonant.
Definition 3.3. We say the extended Harper’s model \( \{ H_{\lambda,\alpha,\theta} \} \) exhibits almost localization if there exists \( C_0, C_3, \epsilon_0, \epsilon_0 > 0 \), such that for every solution \( \phi \) to \( H_{\lambda,\alpha,\theta} \phi = E \phi \) satisfying \( \phi(0) = 1 \) and \( |\phi(m)| \leq 1 + |m| \), and for every \( C_0(1 + |n_j|) < |k| < C_0^{-1}|n_{j+1}| \), we have \( |\phi(k)| \leq C_0 e^{-\epsilon_0 |k|} \) (where \( n_j \) are the \( \epsilon_0 \)-resonances of \( \theta \)).

Theorem 3.1. If \( \lambda \) belongs to region II, \( \{ H_{\lambda,\alpha,\theta} \} \) is almost localized for every \( \alpha \in DC \).

Remark 3.1. It is clear from Theorem 3.1 that almost localization implies localization for non-resonant \( \theta \).

We will actually prove the following explicit lemma:

Lemma 3.2. Let \( \theta \) be in region II. Let \( C_4 \) be the absolute constant in Lemma 3.3, \( \epsilon_1 = \epsilon_1(\lambda) \) be as in 2.4, then for any \( 0 < \epsilon_0 < \frac{\epsilon_1}{100 C_4} \), there exists constant \( C_3 > 0 \), which depends on \( \lambda, \alpha \) and \( \epsilon_0 \), so that for every solution \( u \) of \( H_{\lambda,\alpha,\theta} u = E u \) satisfying \( u(0) = 1 \) and \( |u_k| \leq 1 + |k| \), if \( 3(|n_j| + 1) < |k| < \frac{1}{3}|n_{j+1}| \), then \( |u_k| \leq C_3 e^{-\frac{\epsilon_0}{3} |k|} \), where \( \{ n_j \} \) are the \( \epsilon_0 \)-resonances of \( \theta \).

The proof of Lemma 3.2 (and thus of Theorem 3.1) is given in Section 4.

3.2. Almost reducibility.

Let \( \lambda \) be in region II. For every \( E \in \Sigma_\lambda \), let \( \theta(E) \in \mathbb{T} \) be given in Theorem 2.6. Let \( 0 < \epsilon_0 < \frac{\epsilon_1}{100 C_4} \) and \( \{ n_j \} \) be the set of \( \epsilon_0 \)-resonances of \( \theta(E) \). Then for some positive constants \( N_0, C \) and \( c \), independent of \( E \) and \( \theta \), we have the following theorem:

Theorem 3.3. For any fixed \( j \), with \( N_0 < n = |n_j| + 1 < \infty \), let \( N = |n_{j+1}| \), \( L^{-1} = \| 2\theta - n_j \alpha \| \). Then there exists \( W: \mathbb{T} \to SL(2, \mathbb{R}) \) analytic such that \( \| \text{deg} W \| \leq CN \), \( \| W \|_0 \leq CLC \) and \( \| W^{-1}(x + \alpha) \tilde{\lambda},E \tilde{\lambda},E(x)W(x) - L \| \leq C e^{-cN} \).

Remark 3.2. Notice that this theorem requires \( n > N_0 \), which is not always ensured when \( \theta(E) \) is non-resonant, however in that case we have localization for \( H_{\lambda,\alpha,\theta} \) instead of almost localization. We will prove Theorem 3.3 in Section 5.

3.3. Spectral consequences of Almost reducibility.

Let \( \epsilon_1 = \epsilon_1(\lambda) \) and \( C_4 \) be as in Lemma 3.2.

Theorem 3.4. Assume \( \alpha \in DC \). For \( \lambda \) in region II, fix \( E \in \Sigma_\lambda \). Assume \( \theta(E) \in \mathbb{T} \) is such that \( H_{\lambda,\alpha,\theta} u = \hat{E}u \) has solution satisfying \( u_0 = 1 \) and \( |u_k| \leq 1 \). Let \( C \) be the constant in Theorem 3.3. Then \( \theta(E) \) and \( \rho(\alpha, \tilde{\lambda},E) \) have the following relation:

- (A) If \( \theta \) is \( \epsilon_0 \)-non-resonant for some \( \frac{\epsilon_1}{100 C_4} > \epsilon_0 > 0 \), then \( 2\theta \in \mathbb{Z} \alpha + \mathbb{Z} \) if and only if \( 2\rho(\alpha, \tilde{\lambda},E) \in \mathbb{Z} \alpha + \mathbb{Z} \).
- (B) If \( \theta \) is \( \epsilon_0 \)-resonant for some \( \frac{\epsilon_1}{100 C_4} > \epsilon_0 > 0 \), then \( \rho(\alpha, \tilde{\lambda},E) \) is \( \frac{\epsilon_0}{100 C_4} \)-resonant.

Proof.

(A): When \( \theta \) is \( \epsilon_0 \)-non-resonant for some \( \frac{\epsilon_1}{100 C_4} > \epsilon_0 > 0 \), Theorem 3.1 implies \( H_{\lambda,\alpha,\theta} \) has exponentially decaying eigenfunction. Then applying Theorem 2.7 we get \( 2\theta \in \mathbb{Z} \alpha + \mathbb{Z} \) if and only if \( 2\rho(\alpha, \tilde{\lambda},E) \in \mathbb{Z} \alpha + \mathbb{Z} \).

(B): Assume \( \theta \) is \( \epsilon_0 \)-resonant for some \( \frac{\epsilon_1}{100 C_4} > \epsilon_0 > 0 \). Fix any \( \xi < \frac{\epsilon_0}{2C+2} \), then there exists \( C_\xi > 0 \) such that for any \( k \neq 0 \) we have \( \| k \alpha \| \geq C_\xi e^{-\xi |k|} \). Now take an \( \epsilon_0 \)-resonance \( n_j \) of \( \theta \) such that \( n = |n_j| \geq \frac{1}{C_\xi(\epsilon_0(2C+2)^{-1})} N_0 \). Then there exists \( |m| \leq Cn \) such that \( 2\rho(\alpha, \tilde{\lambda},E) - m \alpha = -2\theta \). Then

\[
\| 2\rho(\alpha, \tilde{\lambda},E) - (m - n_j) \alpha \| = \| 2\theta - n_j \alpha \| < e^{-\epsilon_0 n} \leq e^{-\epsilon_0 |m-n_j|}.
\]
Take any $|l| \leq |m - n_j|$, $l \neq m - n_j$. Then
\[ ||(l - (m - n_j))\alpha|| \geq C_\ell e^{-2\ell|m-n_j|} > 2 e^{-\ell\alpha n} > 2\|2\rho(\alpha, \tilde{A}_E) - (m - l_0)\alpha||.\]
Thus $\|2\rho(\alpha, \tilde{A}_E) - l_0\| > \|2\rho(\alpha, \tilde{A}_E) - (m - n_j)\alpha\|$ for any $|l| \leq |m - n_j|$, $l \neq m - n_j$. This by definition means $\rho(\alpha, \tilde{A}_E, \mathbb{Z})$ is $\tilde{\alpha}$-resonant.

Now based on Theorem 1.1, we can complete the proof of the dry version of Ten Martini Problem for extended Harper’s model in regions I and II.

**Proof of Theorem 1.1**

It is enough to consider $\lambda$ in region II. Let $E \in \Sigma_\lambda$ be such that $N_{\lambda}(E) \in \mathbb{Z}\alpha + \mathbb{Z}$. We are going to show $E$ belongs to the boundary of a component of $\mathbb{R} \setminus \Sigma_\lambda$. Now by (2.10) we have $2\rho(\alpha, \tilde{A}_E, \mathbb{Z}) \in \alpha\mathbb{Z} + \mathbb{Z}$. By Theorem 2.7, this means there exist $M(x) \in C_0^1(\mathbb{T}, PSL(2, \mathbb{R}))$ such that $M^{-1}(x + \alpha)\tilde{A}_E(x)M(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$. Without loss of generality, we assume $M^{-1}(x + \alpha)\tilde{A}_E(x)M(x) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Let $\tilde{M}(x) = \frac{M(x)}{\sqrt{\rho(x, \alpha)}}$, then

\[
\tilde{M}^{-1}(x + \alpha) \begin{pmatrix} 2 - vc(x) \\ 0 \end{pmatrix} \tilde{M}(x) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}
\]

Now let $\tilde{M}(x) = \begin{pmatrix} M_{11}(x) & M_{12}(x) \\ M_{21}(x) & M_{22}(x) \end{pmatrix}$. Then $M_{21}(x) = M_{11}(x - \alpha)$ and $M_{22}(x) = M_{12}(x - \alpha) - aM_{11}(x - \alpha)$ and

\[
\tilde{M}^{-1}(x + \alpha) \begin{pmatrix} 2 - vc(x) \\ 0 \end{pmatrix} \tilde{M}(x) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}
\]

Now we look for $Z_\epsilon(x)$ of the form $e^{\epsilon Y(x)}$ such that

\[
Z_\epsilon^{-1}(x + \alpha)(M_0 + \epsilon M_1(x))Z_\epsilon(x) = M_0 + \epsilon[M_1] + O(\epsilon^2).
\]

We then just need to solve the equation:

\[
(I - \epsilon Y(x + \alpha) + O(\epsilon^2))(M_0 + \epsilon M_1(x))((I + \epsilon Y(x) + O(\epsilon^2)) = M_0 + \epsilon[M_1] + O(\epsilon^2).
\]

It is sufficient to solve the cohomological equation:

\[
Y(x + \alpha)M_0 - M_0 Y(x) = M_1(x) - [M_1],
\]

which is guaranteed by the Diophantine condition on $\alpha$. Thus

\[
(M(x + \alpha)Z_\epsilon(x + \alpha))^{-1}\tilde{A}_E(x)(M(x)Z_\epsilon(x)) = \begin{pmatrix} 1 + \epsilon[M_{11}] & a\epsilon[M_{12}] - a\epsilon[M_{11}]M_{12} \\ -\epsilon[M_{12}] & 1 - \epsilon[M_{11}]M_{12} \end{pmatrix} + O(\epsilon^2)
\]

Notice that $\tilde{A}_E$ is uniformly hyperbolic iff $\text{Trace}(M_\epsilon) > 2$ which is fulfilled when $-a\epsilon[M_{11}^2] > 0$. Thus for $\epsilon$ small, satisfying $-a\epsilon[M_{11}^2] > 0$, $E + \epsilon \notin \Sigma_\lambda$, which means this spectral gap is open. \(\square\)
In this section we will prove Lemma 4.2. For fixed \( \lambda \) in region II and \( E \), let \( D_{\lambda,E}(\theta) = c_{1}(\theta)A_{\hat{\lambda},E}(\theta) \), where \( c_{1}(\theta) = \frac{1}{x_{2}} e^{-2\pi i (\theta + \hat{\theta})} + \frac{1}{x_{2}} e^{2\pi i (\theta + \hat{\theta})} \). Regarding the Lyapunov exponent, we recall the following result in [14],

\[
L(\alpha, A_{\hat{\lambda},E}) = L(\alpha, D_{\hat{\lambda},E}) - \int_{\mathbb{T}} \ln |c_{\hat{\lambda}}(\theta)| d\theta \equiv \tilde{L} - \int \ln |c_{\hat{\lambda}}| > 0,
\]

where \( \tilde{L} = \ln \frac{x_{2} - \hat{\lambda}x_{1}}{x_{2}} \) and \( \int \ln |c_{\hat{\lambda}}| = \ln \max(\lambda_{1} + \lambda_{3}, 1) + \sqrt{\max(\lambda_{1} + \lambda_{3}, 1)^{2} - 4\lambda_{1}\lambda_{3}} \).

**Proof of Lemma 4.2**

Suppose \( u \) is a solution satisfying the condition of Lemma 4.2. For an interval \( I = [x_{1}, x_{2}] \), let \( \Gamma_{I} \) be the coupling operator between \( I \) and \( \mathbb{Z} \setminus I \):

\[
\Gamma_{I}(i,j) = \begin{cases} 
\tilde{c}(\theta + (x_{1} - 1)\alpha), & (i,j) = (x_{1}, x_{1} - 1) \\
\tilde{c}(\theta) + (x_{1} - 1)\alpha), & (i,j) = (x_{1} - 1, x_{1}) \\
\tilde{c}(\theta + x_{2}\alpha), & (i,j) = (x_{2} + 1, x_{2}) \\
\tilde{c}(\theta + x_{2}\alpha), & (i,j) = (x_{2}, x_{2} + 1) \\
\end{cases}
\]

Let us denote \( P_{k}(\theta) = \det (E - H_{[0,k-1]}(\theta)) \). Then the \( k \)-step matrix \( D_{\lambda,E,k}(\theta) \) satisfies:

\[
D_{\lambda,E,k}(\theta) = \begin{pmatrix} P_{k}(\theta) & -\tilde{c}(\theta - \alpha)P_{k-1}(\theta + \alpha) \\
\tilde{c}(\theta + (k-1)\alpha)P_{k-1}(\theta) & \tilde{c}(\theta - \alpha)c(\theta + (k-1)\alpha)P_{k-2}(\theta + \alpha) \end{pmatrix}.
\]

This relation between \( P_{k}(\theta) \) and \( D_{\lambda,E,k}(\theta) \) gives a general upper bound of \( P_{k}(\theta) \) in terms of \( \tilde{L} \). Indeed by Lemma 2.1 for any \( \varepsilon > 0 \) there exists \( C(\varepsilon) > 0 \) so that

\[
|P_{n}(\theta)| \leq C(\varepsilon)e^{(\tilde{L} + \varepsilon)n} \quad \text{for any } n \in \mathbb{N}.
\]

By Cramer’s rule:

\[
|G_{I}(x_{1}, y)| = \prod_{j=x_{1}}^{y-1} |c(\theta + j\alpha)||\frac{\det (E - H_{[y+1,y+2]}(\theta))}{\det (E - H_{1}(\theta))}| = \prod_{j=x_{1}}^{y-1} |c(\theta + j\alpha)||\frac{P_{y-x_{1}}(\theta + y\alpha)}{P_{k}(\theta + x_{1}\alpha)}|,
\]

\[
|G_{I}(y, x_{2})| = \prod_{y+1}^{x_{2}} |c(\theta + j\alpha)||\frac{\det (E - H_{[x_{2},x_{2}+1]}(\theta))}{\det (E - H_{1}(\theta))}| = \prod_{y+1}^{x_{2}} |c(\theta + j\alpha)||\frac{P_{y-x_{1}}(\theta + x_{1}\alpha)}{P_{k}(\theta + x_{1}\alpha)}|.
\]

Notice that \( P_{k}(\theta) \) is an even function about \( \theta + \frac{k-1}{2}\alpha \), it can be written as a polynomial of degree \( k \) in \( \cos 2\pi(\theta + \frac{k-1}{2}\alpha) \). Let \( P_{k}(\theta) = Q_{k}(\cos 2\pi(\theta + \frac{k-1}{2}\alpha)) \). Let \( M_{k,r} = \{ \theta \in \mathbb{T}, \ |Q_{k}(\cos 2\pi \theta)| \leq e^{(k+1)r} \} \).

**Definition 4.1.** Fix \( m > 0 \). A point \( y \in \mathbb{Z} \) is called \( (k, m) \)-regular if there exists an interval \( [x_{1}, x_{2}] \) containing \( y \), where \( x_{2} = x_{1} + k - 1 \) such that

\[
|G_{I}(y, x_{1})| \leq e^{-m|y-x_{1}|} \text{ and } \text{dist}(y, x_{i}) \geq \frac{1}{3} m \text{ for } i = 1, 2,
\]
otherwise $y$ is called $(k, m)$--singular.

**Lemma 4.1.** Suppose $y \in \mathbb{Z}$ is $(k, \bar{L} - \int \ln |c_\alpha| - \rho)$--singular. Then for any $\epsilon > 0$ and any $x \in \mathbb{Z}$ satisfying $y - \frac{k}{3} \leq x \leq y - \frac{1}{3}k$, we have $\theta + (x + \frac{1}{2}(k-1)) \alpha$ belongs to $M_{k, \bar{L} - \frac{k}{3} \rho + \epsilon}$ for $k > k(\lambda, \epsilon, \rho)$.

**Proof.** Suppose there exists $\epsilon > 0$ and $x_1$: $y - (1 - \delta)k \leq x_1 \leq y - \delta k$, such that $\theta + (x_1 + \frac{1}{2}(k-1)) \alpha$ does not belong to $M_{k, \bar{L} - \frac{k}{3} \rho + \epsilon}$, that is $|P_k(\theta + x_1 \alpha)| > e^{(k+1)(\bar{L} - \rho \delta + \epsilon)}$,

$$|G_1(x_1, y)| \leq \prod_{j=x_1}^{y-1} |c_\alpha(\theta + j \alpha)| e^{(k-|x_1-y|)(\bar{L} + \epsilon)} e^{-(k+1)(\bar{L} - \frac{k}{3} \rho + \epsilon)} < e^{-(\bar{L} - \int \ln |c_\alpha| - \rho)|y-x_1|} \text{ for } k > k(\lambda, \epsilon, \rho).$$

Similarly

$$|G_1(x_2, y)| \leq e^{-(\bar{L} - \int \ln |c_\alpha| - \rho)|y-x_2|}.$$

\[\begin{array}{c|c|c|c|c}
\hline
x & x + \frac{1}{2}(k-1) & y & y + (\frac{1}{2} - \delta)k \\
\hline
y - (1 - \delta)k & y - \delta k & y - (\frac{1}{2} - \delta)k & y \\
\hline
\end{array}\]

\[\blacksquare\]

**Definition 4.2.** We say that the set $\{\theta_1, ..., \theta_{k+1}\}$ is $\gamma$--uniform if

$$\max_{x \in [-1, 1]} \max_{i = 1, ..., k+1} \prod_{j=1, j \neq i}^{k+1} \frac{|x - \cos 2\pi \theta_j|}{|\cos 2\pi \theta_i - \cos 2\pi \theta_j|} < e^{k\gamma}$$

**Lemma 4.2.** Let $\gamma_1 < \gamma$. If $\theta_1, ..., \theta_{k+1} \in M_{k, \bar{L} - \gamma}$, then $\{\theta_1, ..., \theta_{k+1}\}$ is not $\gamma_1$--uniform for $k > k(\gamma, \gamma_1)$.

**Proof.** Otherwise, using Lagrange interpolation form we can get $|Q_k(x)| < e^{k\bar{L}}$ for all $x \in [-1, 1]$. This implies $|P_k(x)| < e^{k\bar{L}}$ for all $x$. But by Herman’s subharmonic function argument, $\int_{\mathbb{R}/\mathbb{Z}} \ln |P_k(x)| dx \geq k\bar{L}$. This is impossible. \[\blacksquare\]

Now take $\xi$ and $\epsilon_0$ such that $0 < 1000 \xi < \epsilon_0$. Then for $|n_{j+1}| > N(\xi)$ we have

$$2e^{-4|n_{j+1}|} \leq C_\xi e^{-2|n_{j+1}|} \leq ||(n_{j+1} - n_j)\alpha|| \leq ||(n_{j+1} + 1 - 2\theta - 2\theta - n_j \alpha)|| \leq 2||2\theta - n_j \alpha|| < 2e^{-\epsilon_0|n_j|}$$

which yields that

$$|n_{j+1}| > \frac{\epsilon_0}{4\xi} |n_j| > 250|n_j|.$$  \hspace{1cm} (4.1)

Without loss of generality, assume $3(|n_j| + 1) < y < \frac{|n_{j+1}|}{3}$ and $y > N(\xi)$. Select $n$ such that $q_n \leq \frac{y}{6} < q_{n+1}$ and let $s$ be the largest positive integer satisfying $sq_n \leq \frac{y}{6}$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$I_1 = [1 - 2sq_n, 0]$ and $I_2 = [y - 2sq_n + 1, y + 2sq_n]$, if $n_j < 0$

$I_1 = [0, 2sq_n - 1]$ and $I_2 = [y - 2sq_n + 1, y + 2sq_n]$, if $n_j \geq 0$

**Lemma 4.3.** Let $\theta = \theta + j \alpha$, then set $\{\theta_j\}_{j \in I_1 \cup I_2}$ is $C_4 \epsilon_0 + C_4 \xi$--uniform for some absolute constant $C_4$ and $y > y(\alpha, \epsilon_0, \xi)$. 
Lemma 2.1 reduces this problem to estimating the minimal terms.

First we estimate \( \sum_1 \):

\[
\sum_1 = \sum_{j \in I_1 \cup I_2 \neq i} \ln |\cos 2\pi\alpha - \cos 2\pi\theta_j| \\
= \sum_{j \in I_1 \cup I_2 \neq i} \ln |\sin (\alpha + \theta_j)| + \sum_{j \in I_1 \cup I_2 \neq i} \ln |\sin (\alpha - \theta_j)| + (6s_qn - 1) \ln 2 \\
\leq \sum_{1,+} \sum_{1,-} + (6s_qn - 1) \ln 2.
\]

We cut \( \sum_{1,+} \) or \( \sum_{1,-} \) into 6 sums and then apply Lemma 2.1 we get that for some absolute constant \( C_1 \):

\[
\sum_1 \leq -6s_qn \ln 2 + C_1 s \ln q_n.
\]

Next, we estimate \( \sum_2 \):

\[
\sum_2 = \sum_{j \in I_1 \cup I_2 \neq i} \ln |\cos 2\pi\theta_j - \cos 2\pi\alpha| \\
= \sum_{j \in I_1 \cup I_2 \neq i} \ln |\sin (2\theta + (i + j)\alpha)| + \sum_{j \in I_1 \cup I_2 \neq i} \ln |\sin (i - j)\alpha| + (6s_qn - 1) \ln 2 \\
\leq \sum_{2,+} \sum_{2,-} + (6s_qn - 1) \ln 2.
\]

We need to carefully estimate the minimal terms. For \( \sum_{2,+} \), we use the property of resonant set; and for \( \sum_{2,-} \), we use the Diophantine condition on \( \alpha \).

For any \( 0 < |j| < n_{j+1} \), we have \( |j\alpha| \geq |q_n\alpha| \geq C_\alpha e^{-\xi q_n} \). Therefore

\[
\max(\ln |\sin x|, \ln |\sin (x + \pi j\alpha)|) \geq -2\xi q_n \text{ for } y > y(\alpha, \xi).
\]

This means in any interval of length \( s_qn \), there can be at most one term which is less than \( -2\xi q_n \). Then there can be at most 6 such terms in total.

For the part \( \sum_{2,-} \), since \( |(i - j)\alpha| \geq C_\xi e^{-\xi |i-j|} \geq e^{-20\xi s_qn} \), these 6 smallest terms must be bounded by \(-20\xi s_qn\) from below. Hence \( \sum_{2,-} \geq -6s_qn \ln 2 - C\xi s_qn - Cs \ln q_n \) for \( y > y(\xi) \) and some absolute constant \( C \).

For the part \( \sum_{2,+} \), notice \( |i + j| \leq 2y + 4s_qn < 3y < |n_{j+1}| \) and \( i + j > 0 > -n_j \). Suppose \( \|2\theta + k_0\alpha\| = \min_{j \in I_1 \cup I_2} \|2\theta + (i + j)\alpha\| < e^{-100e_0s_qn} < e^{-e_0|k_0|} \). Then for any \( |k| \leq |k_0| \leq 40s_qn \) (including \( |n_j| \),

\[
\|2\theta - k\alpha\| \geq \|(k + k_0)\alpha\| - \|2\theta + k_0\alpha\| > \|2\theta + k_0\alpha\| \text{ for } y > y(\alpha, e_0, \xi)
\]

This means \( -k_0 \) must be a \( e_0 \)-resonance, therefore \( |k_0| \leq |n_{j-1}| \). Then

\[
\|2\theta - n_j\alpha\| \geq \|(n_j + k_0)\alpha\| - \|2\theta + k_0\alpha\| \geq C_\xi e^{-12\xi s_qn} - e^{-100e_0s_qn} > e^{-100e_0s_qn} \geq \|2\theta + k_0\alpha\|
\]

leads to a contradiction. Thus the smallest terms must be greater than \(-100e_0s_qn\). We can bound \( \sum_{2,+} \) by \(-6s_qn \ln 2 - 600e_0s_qn - 12\xi s_qn - Cs \ln q_n \) from below. Therefore \( \sum_2 \geq -6s_qn \ln 2 -
Theorem 5.1. For some $C > 0$ depending on $\lambda$ and $\alpha$,

$$\|\tilde{A}_k(x)\|_T \leq C(1 + |k|)^C.$$
Proof.

Let $\tilde{U}(x) = Q(x)U(x)$, $\tilde{G}(x) = Q(x + \alpha)G(x)$, where $Q = Q_\lambda$ is given in (A.2). Since

$$\max(||Q(x)||_{\frac{1}{n}}, ||Q^{-1}(x)||_{\frac{1}{n}}) \leq C,$$

we have

$$\tilde{A}(x)\tilde{U}(x) = e^{2\pi i \theta} \tilde{U}(x + \alpha) + \tilde{G}(x),$$

where $||\tilde{G}(x)||_{\frac{1}{n}} \leq Ce^{-\frac{1}{20}}n$.

**Lemma 5.2.** Let $C_2$ be the constant from Lemma 2.3 then for any $\delta$, $2C_2 \xi < \delta < \frac{40}{n}$, we have

$$\inf_{\text{Im}(x) \leq \frac{200}{n}} ||\tilde{U}(x)|| \geq e^{-2\delta n},$$

for $n > n(\alpha, \delta)$.

**Proof.** We will prove the statement by contradiction. Suppose for some $x_0 \in \{\text{Im}(x) \leq \frac{200}{n}\}$ we have $||\tilde{U}(x_0)|| < e^{-2\delta n}$. Notice that for any $l \in \mathbb{N}$,

$$e^{2\pi i \theta} \tilde{U}(x_0 + l\alpha) = \tilde{A}(x_0)\tilde{U}(x_0) - \sum_{m=1}^{l} e^{2\pi i (m-1)\theta} \tilde{A}_{l-m}(x_0 + m\alpha)\tilde{G}(x_0 + (m-1)\alpha).$$

This implies for $n > n(\delta)$ large enough and for any $0 \leq l \leq n$, $||\tilde{U}(x_0 + l\alpha)|| \leq e^{-\delta n}$, thus $||u(x_0 + l\alpha)|| \leq C_\delta e^{-\delta n}$. By Lemma 2.3 $||u(x + i\text{Im}(x_0))||_{\mathbb{T}} \leq C_2 C_\delta e^{C_2 \xi n} e^{-\delta n} \leq e^{-\frac{1}{2}n}$. This contradicts with $\int_{\mathbb{T}} u(x + i\text{Im}(x_0)) dx = u_0 = 1$. \qed

**Lemma 5.3.** [3] Let $V : \mathbb{T} \to \mathbb{C}^2$ be analytic in $|\text{Im}(x)| < \eta$. Assume that $\delta_1 < ||V(x)|| < \delta_2^{-1}$ holds on $|\text{Im}(x)| < \eta$. Then there exists $M : \mathbb{T} \to SL(2, \mathbb{C})$ analytic on $|\text{Im}(x)| < \eta$ with first column $V$ and $||M||_{\eta} \leq C_\delta_1^{-2} \delta_2^{-1} (1 - \ln(\delta_1 \delta_2))$.

Applying Lemma 5.3 let $M(x)$ be the matrix with first column $\tilde{U}(x)$. Then $e^{-2\delta n} \leq ||\tilde{U}(x)|| \leq e^{\delta n}$ and hence $||M(x)|| \leq C e^{\delta n}$. Therefore

$$M^{-1}(x + \alpha)\tilde{A}(x)M(x) = \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_3(x) & \beta_4(x) \end{pmatrix},$$

where $||\beta_1(x)|| \leq ||\beta_3(x)|| \leq ||\beta_4(x)|| \leq Ce^{-\frac{1}{20}}n$, and $||b(x)|| \leq Ce^{13\delta n}$. Let

$$\Phi(x) = M(x) \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix}.$$

Then we would have:

$$\Phi(x + \alpha)^{-1}\tilde{A}(x)\Phi(x) = \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix} + H(x),$$

where $||H(x)|| \leq Ce^{-\frac{1}{20}}n$, and $||\Phi(x)|| \leq Ce^{\frac{1}{10}n}$. Thus

$$\sup_{0 \leq s \leq e^{\frac{1}{10}n}} ||\tilde{A}(x)|| \leq e^{\frac{1}{10}n}$$
for \( n \geq n(\lambda, \alpha) \) satisfying (5.1). For \( s \) large, there always exists \( 9|n_j| < n < \frac{1}{4}\|n_{j+1}\| \) satisfying (5.1) such that \( cn \leq \frac{20a}{e_1} \ln s \leq n \) with some absolute constant \( c \). Thus there exists \( C \) depending on \( \lambda \) and \( \alpha \) such that \( \|A_k(x)\|_T \leq C(1 + |k|)^C \).

Now we come back to the proof of Theorem 5.3. Fix some \( n = |n_j| \), and \( N = |n_{j+1}| \). Let \( u(x) = u^2(x) \) with \( I_2 = [-\frac{N}{3}, \frac{N}{3}] \) and \( U(x) = \begin{pmatrix} e^{2\pi i \theta} u(x) \\ u(x - \alpha) \end{pmatrix} \).

Then
\[
A(x)U(x) = e^{2\pi i \theta} U(x + \alpha) + G(x) \quad \text{with} \quad \|G(x)\|_{\frac{1}{20}} \leq Ce^{-\frac{1}{20}N}.
\]

Define \( U_0(x) = e^{n_{1j}x^2} U(x) \). Notice that if \( n_j \) is even, then \( U_0(x) \) is well-defined on \( \mathbb{T} \), otherwise \( U_0(x + 1) = -U_0(x) \).

$$
\tilde{A}(x)U_0(x) = e^{2\pi i \tilde{\theta}} U_0(x + \alpha) + H(x),
$$
where \( \tilde{\theta} = \theta - \frac{n_j}{2} \alpha \), \( \tilde{U}_0(x) = Q(x)U_0(x) \) and \( \|H(x)\|_{\frac{1}{20}} \leq Ce^{-\frac{1}{20}N} \). Consider the matrix \( W(x) \) with \( \tilde{U}_0(x) \) and \( \tilde{U}_0(x) \) being its two columns. Then
\[
\tilde{A}(x)W(x) = W(x + \alpha) \begin{pmatrix} e^{2\pi i \tilde{\theta}} & 0 \\ 0 & e^{-2\pi i \tilde{\theta}} \end{pmatrix} + \tilde{H}(x).
\]

**Theorem 5.4.** Let \( L^{-1} = \|2\theta - n_j \alpha\| \). Then for \( n > N(\lambda, \alpha) \) we have
\[
|\det W(x)| \geq L^{-4C} \quad \text{for any} \quad x \in \mathbb{T},
\]
where \( C \) is the constant appeared in Theorem 5.1.

**Proof.** First, we fix \( \xi_1 < \frac{e^{\epsilon_0}}{1000} \) so that \( \|k\alpha\| \geq C \xi_1 e^{-\xi_1|k|} \) for any \( k \neq 0 \). We have the following estimate about \( L \):

**Lemma 5.5.** \( e^{\epsilon_0 n} \leq L \leq e^{4\xi_1 N} \).

\[
e^{-2\xi_1 N} \leq \|(n_{j+1} - n_j) \alpha\| \leq 2\|n_j \alpha - 2\theta\| = 2L^{-1} \leq 2e^{-\epsilon_0 n} \quad \text{for} \quad n \geq N(\xi_1).
\]

Now we prove by contradiction. Suppose there exists \( \kappa \) and \( x_0 \in \mathbb{T} \) such that \( \|\tilde{U}_0(x_0) - \kappa \tilde{U}_0(x_0)\| < L^{-4C} \). Then
\[
\|\tilde{U}_0(x_0 + l\alpha) e^{2\pi i \tilde{\theta}} - \kappa \tilde{U}_0(x_0 + l\alpha) e^{-2\pi i \tilde{\theta}}\|
\]
\[
\leq \|\sum_{m=0}^{l-1} \tilde{A}_{l-m}(x_0 + m\alpha) H(x_0 + m\alpha) - \kappa \sum_{m=0}^{l-1} \tilde{A}_{l-m}(x_0 + m\alpha) H(x_0 + m\alpha)\| + \|A_l(x_0)\| L^{-4C}
\]
\[
\leq CL^{2C} e^{-\frac{1}{20}N} + CL^{-2C} < L^{-C}.
\]

for \( 0 \leq |l| \leq L^2 \). If we take \( j = \frac{L}{4} \), then
\[
\|\tilde{U}_0(x_0 + L^4 \alpha) + \kappa \tilde{U}_0(x_0 + L^4 \alpha)\| < L^{-1}.
\]

Next since \( \|U_0(x)\|_T \leq n \), we have \( \|\tilde{U}_0(x)\|_T \leq Cn \). Thus
\[
\|\tilde{U}_0(x_0 + l\alpha) - \kappa \tilde{U}_0(x_0 + l\alpha)\| < L^{-\frac{1}{2}} \quad \text{for} \quad 0 \leq |l| \leq \frac{L}{2}.
\]
For any analytic function \( f(x) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{2\pi i k x} \), define \( \hat{f}_{[-m,m]}(x) = \sum_{|k| \leq m} \hat{f}_k e^{2\pi i k x} \). For any column vector \( V(x) = \begin{pmatrix} v^{(1)}(x) \\ v^{(2)}(x) \end{pmatrix} \), let \( V_{[-m,m]}(x) = \begin{pmatrix} v^{(1)}_{[-m,m]}(x) \\ v^{(2)}_{[-m,m]}(x) \end{pmatrix} \). Now let us define \( \tilde{U}_0^{[9n]}(x) = Q(x)e^{\pi in x}U_{[-9n,9n]}(x) \). Then
\[
\|\tilde{U}_0^{[9n]}(x) - \hat{U}_0(x)\|_T \leq C e^{-\frac{5}{4} n}.
\]
Consider \(|e^{-\pi in x}\tilde{U}_0^{[9n]}(x)|^{-18n,18n}(x)e^{\pi in x}| \). This function differs from a polynomial with essential degree 36n only by a multiple of \( e^{\pi in x} \). Notice that \( Q(x) \) is analytic in \( \{x : |\text{Im}(x)| \leq \frac{N}{100}\} \), thus \(|Q(k)| \leq C e^{-\frac{5}{4} |k|} \). Then
\[
|e^{-\pi in x}\tilde{U}_0^{[9n]}(k)| \leq \sum_{|m| \leq 9n} |Q(k-m)\hat{U}(m)| \leq C n e^{-\frac{4}{5} |k| - 9n} \quad \text{for } |k| \geq 18n.
\]
Thus
\[
\|e^{-\pi in x}\tilde{U}_0^{[9n]}(x) - [e^{-\pi in x}\tilde{U}_0^{[9n]}]|^{-18n,18n}(x)\|_T \leq e^{-4 \epsilon_n},
\]
\[
\|\tilde{U}_0(x) - [e^{-\pi in x}\tilde{U}_0^{[9n]}]|^{-18n,18n}(x)e^{\pi in x}\|_T \leq e^{-4 \epsilon_n}.
\]
Hence
\[
\|\tilde{U}_0(x) - [e^{-\pi in x}\tilde{U}_0^{[9n]}]|^{-18n,18n}(x0 + l\alpha)e^{2\pi in x(x0 + l\alpha) - \kappa[e^{-\pi in x}\tilde{U}_0^{[9n]}]|^{-18n,18n}(x0 + l\alpha)\|_T \leq 2L^{-\frac{4}{5}} + e^{-4 \epsilon_n},
\]
for \(|l| \leq L^\frac{1}{2} \). Notice that
\[
[e^{-\pi in x}\tilde{U}_0^{[9n]}]|^{-18n,18n}(x)e^{2\pi in x - \kappa[e^{-\pi in x}\tilde{U}_0^{[9n]}]|^{-18n,18n}(x)e^{\pi in x}\]
\]
is a polynomial whose essential degree is at most 37n. Thus by Lemma 2.9 we would have
\[
\|\tilde{U}_0(x) - \kappa\hat{U}_0(x)\|_T \leq L^{-\frac{1}{2}} + 2e^{-2 \epsilon_n}. \quad \text{But combining with (9.1) we would get } \|\tilde{U}_0(x) + \frac{1}{2}\kappa(x)\|_T < 2L^{-\frac{1}{2}} + 2e^{-2 \epsilon_n}, \text{ but this contradicts with inf}_{x \in T} \tilde{U}_0(x) > e^{-2 \delta n} \text{ since } \delta < \frac{100}{50}.
\]

Now for \( n > N_0(\lambda, \alpha) \), take \( S(x) = \text{Re}\hat{U}_0(x) \) and \( T(x) = \text{Im}\hat{U}_0(x) \). Let \( W_1(x) \) be the matrix with columns \( S(x) \) and \( T(x) \). Notice that \( \det W_1(x) \) is well-defined on \( T \) and \( \det W_1(x) \neq 0 \) on \( T \), hence without loss of generality we could assume \( \det W_1(x) > 0 \) on \( T \), otherwise we simply take \( W_1(x) \) to be the matrix with columns \( S(x) \) and \(-T(x) \). Then
\[
\|\tilde{A}(x)W_1(x) - W_1(x + \alpha)R_{-\delta}\|_T \leq C e^{-\frac{5}{4} N}.
\]
By taking determinant, we get
\[
\det W_1(x) = \det W_1(x + \alpha) + O(e^{-\frac{5}{4} N}) \quad \text{on } T.
\]
Since \( \det W_1(x) \) is analytic on \(|\text{Im} x| \leq \frac{N}{100} \), by considering the Fourier coefficients we could get
\[
\det W_1(x) = w_0 + O(e^{-\frac{5}{4} N}) \quad \text{on } T,
\]
where \( w_0 \geq L^{-5c} \). Thus \( \det W_1(x) \) is almost a positive constant.

Define \( W_2(x) = \det W_1(x)^{-\frac{1}{2}}W_1(x) \). Then \( W_2(x) \in C^\infty(T) \) and \( \det W_2(x) = 1 \). We have
\[
W_2^{-1}(x + \alpha)\tilde{A}(x)W_2(x) = \frac{\det W_1(x + \alpha)^{-\frac{1}{2}}}{\det W_1(x)^{-\frac{1}{2}}} R_{-\delta} + O(e^{-\frac{5}{4} N}) \quad \text{on } T,
\]
Thus $R_\vartheta$ appeared in Theorem 5.

Then since $\vartheta$ for $36\pi$, we say $\deg M = k$ if $M$ is homotopic to $(\cos k\pi x \sin k\pi x)$. For some constant $c > 0$, we obviously have

$$\int_T ||S(x)|| \, dx + \int_T ||T(x)|| \, dx \geq \int_T ||S(x) + iT(x)|| \, dx = \int_T ||\tilde{U}_0(x)|| \, dx \geq c.$$ 

Without loss of generality we could assume $\int_T ||S(x)|| \, dx > \frac{c}{2}$. Also

$$\tilde{A}(x)S(x) = S(x + \alpha) + O(L^{-\frac{1}{2}}) \text{ on } \mathbb{T}.$$ 

Then since $\|2\tilde{\vartheta}\| = L^{-1}$,

$$\tilde{A}(x)S(x) = S(x + \alpha) + O(L^{-\frac{1}{2}}) \text{ on } \mathbb{T}.$$ 

First we prove $\inf_{x \in \mathbb{T}} ||S(x)|| \geq e^{-2\varepsilon_1 n}$. Suppose otherwise. Then there exists $x_0 \in \mathbb{T}$, so that $||S(x_0)|| < e^{-2\varepsilon_1 n}$. Then $||\text{Re}\tilde{U}_0(x_0 + t\alpha)|| < e^{-\frac{t\varepsilon_1 n}{2}}$ for $|t| < e^{\frac{\varepsilon_1 n}{2}}$, where $C$ is the constant that appeared in Theorem 5.1. We have already shown that

$$\|\tilde{U}_0(x) - [e^{-\pi in_j x} \tilde{U}_0^{[n]}][-18n,18n]e^{\pi in_j x}\|_T < e^{-4\varepsilon_1 n}.$$ 

Thus

$$\|\text{Re}[e^{-\pi in_j x} \tilde{U}_0^{[n]}][-18n,18n](x_0 + t\alpha)|| < e^{-\frac{t\varepsilon_1 n}{2}}$$

for $|t| < e^{\frac{\varepsilon_1 n}{2}}$. However $\text{Re}[e^{-\pi in_j x} \tilde{U}_0^{[n]}][-18n,18n]$ is a polynomial with essential degree at most $36n$. Using Lemma 2.4 we are able to get $\|\text{Re}[e^{-\pi in_j x} \tilde{U}_0^{[n]}][-18n,18n]e^{\pi in_j x}\|_T < e^{-\frac{\varepsilon_1 n}{2}}$, and thus $||\text{Re}\tilde{U}_0(x)||_T < e^{-\frac{\varepsilon_1 n}{2}}$ which is a contradiction to $\int_T ||\text{Re}\tilde{U}_0(x)|| \, dx > \frac{c}{2}$. At the meantime, we also get $||S(x) - \text{Re}[e^{-\pi in_j x} \tilde{U}_0^{[n]}][-18n,18n](x)e^{\pi in_j x}\|_T \triangleq ||S(x) - h(x)||_T \leq e^{-4\varepsilon_1 n}$. The first column of $W_2(x)$ is $\det W_1(x)\frac{1}{2} S(x)$. We have

$$\|\frac{S(x)}{\det W_1(x)\frac{1}{2}} - \frac{h(x)}{w_0\frac{1}{2}}\|$$

$$\leq \frac{1}{|\det W_1(x)\frac{1}{2}|} ||S(x) - h(x) + (1 - \frac{\det W_1(x)\frac{1}{2}}{w_0\frac{1}{2}})h(x)||$$

$$\leq L^2C(e^{-4\varepsilon_1 n} + L^8C e^{-\frac{\varepsilon_1 n}{2}})$$

$$\leq e^{-3\varepsilon_1 n} < \|\frac{S(x)}{\det W_1(x)\frac{1}{2}}\| \text{ on } \mathbb{T}.$$ 

Thus by Rouche’s theorem $|\deg W_2(x)| = |\deg h(x)| \leq 19n$. Notice that

$$|\rho(\alpha, W_2^{-1}\tilde{A}W_2) + \tilde{\vartheta}| < Ce^{-\frac{\varepsilon_1 n}{2}}.$$ 

Then, by 2.2 for some $|m| \leq 19n$:

$$|\rho(\alpha, \tilde{A}) - \frac{m}{2}\alpha + \tilde{\vartheta}| < Ce^{-\frac{\varepsilon_1 n}{2}}.$$
APPENDIX A.

When \( \lambda \) belongs to region II, let \( \epsilon_2 = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{\lambda_1 + \lambda_3 + \sqrt{(\lambda_1 + \lambda_3)^2 - 4\lambda_1 \lambda_3}} > \epsilon_1 \). Then \( c(x) \) is analytic and nonzero on \( |\text{Im}(x)| < \frac{\epsilon_2}{2\pi} \). Furthermore, the winding number of \( c(\cdot + i \epsilon) \) is equal to zero when \( |\epsilon| < \frac{\epsilon_2}{2\pi} \).

**Lemma A.1.** When \( \lambda \) belongs to region II, we can find an analytic function \( f(x) \) on \( |\text{Im}(x)| \leq \frac{\epsilon_2}{2\pi} \) such that \( c(x) = |c(x)|e^{f(x+\alpha)-f(x)} \) and \( \tilde{c}(x) = |c(x)|e^{-f(x+\alpha)+f(x)} \).

**Proof.** Since the winding numbers of \( c(x) \) and \( \tilde{c}(x) \) are 0 on \( |\text{Im}(x)| \leq \frac{\epsilon_2}{2\pi} \), there exist analytic functions \( g_1(x) \) and \( g_2(x) \) on \( |\text{Im}(x)| \leq \frac{\epsilon_2}{2\pi} \), such that \( c(x) = e^{g_1(x)} \) and \( \tilde{c}(x) = e^{g_2(x)} \). Notice that

\[
\int_{\mathbb{T}} \ln |c(x)| \, dx = \int_{\mathbb{T}} \ln |\tilde{c}(x)| \, dx
\]

\[
\int_{\mathbb{T}} \arg c(x) \, dx = \int_{\mathbb{T}} \arg \tilde{c}(x) \, dx,
\]

so there exists an analytic function \( f(x) \) such that \( 2f(x + \alpha) - 2f(x) = g_1(x) - g_2(x) \). Then \( c(x) = |c(x)|e^{f(x+\alpha)-f(x)} \).

**Lemma A.2.** When \( \lambda \) belongs to region II, there exists an analytic matrix \( Q_{\lambda}(x) \) defined on \( |\text{Im}(x)| \leq \frac{\epsilon_2}{2\pi} \) such that

\[
Q_{\lambda}^{-1}(x + \alpha) A_{\lambda,E}(x) Q_{\lambda}(x) = A_{\lambda,E}(x).
\]

**Proof.**

\[
A_{\lambda,E}(x) = \frac{1}{|c(x)|^2} \begin{pmatrix}
1 & 0 & E - v(x) & -\tilde{c}(x) - \alpha \\
0 & \sqrt{\frac{c(x)}{c(x)}} & c(x) & 0 \\
0 & 0 & c(x) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & e^{f(x+\alpha)} & 0 \\
0 & \sqrt{\frac{c(x)}{c(x)}} & A(x) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}^{-1} = Q_{\lambda}(x + \alpha) A_{\lambda,E}(x) Q_{\lambda}^{-1}(x).
\]

**Lemma A.3.** If \( \alpha \) is irrational, \( \lambda \) belongs to region II, \( E \in \Sigma(\lambda) \), then \( L(\alpha, A_{\lambda,E}(\cdot + i \epsilon)) = L(\alpha, A_{\lambda,E}(\cdot + i \epsilon)) = 0 \) for \( |\epsilon| < \frac{\epsilon_2}{2\pi} \).

**Proof.**

\[
L(A(\cdot + i \epsilon)) = L(D(\cdot + i \epsilon)) - \int \ln |c(x + i \epsilon)| \, dx
\]

\[
D(x + i \epsilon) = \left( \begin{array}{ccc}
E - e^{2\pi i(x+\epsilon)} & -e^{-2\pi i(x+\epsilon)} & -\lambda_1 e^{2\pi i(x+\epsilon)} - \lambda_2 - \lambda_3 e^{-2\pi i(x-\epsilon)} \\
\lambda_1 e^{-2\pi i(x+\epsilon)} + \lambda_2 + \lambda_3 e^{2\pi i(x+\epsilon)} & 0 & 0 \\
0 & -e^{2\pi i\epsilon} + o(1) & -\lambda_3 e^{-2\pi i(x-\epsilon)} + o(1) \\
\lambda_1 e^{-2\pi i(x+\epsilon/2)} + o(1) & 0 & 0
\end{array} \right).
\]
Thus the asymptotic behaviour of $L(D(\cdot + i\epsilon))$ is:

$$L(D(\cdot + i\epsilon)) = \ln \left| 1 + \frac{\sqrt{1 - 4\lambda_1 \lambda_3}}{2} \right| 2\pi \epsilon \text{ when } \epsilon \to \infty,$$

$$L(D(\cdot + i\epsilon)) = \ln \left| 1 + \frac{\sqrt{1 - 4\lambda_1 \lambda_3}}{2} \right| - 2\pi \epsilon \text{ when } \epsilon \to -\infty.$$ 

Then it suffices to calculate $\int \ln |c(x + i\epsilon)| dx$ in region II. We have

$$\int \ln |c(x + i\epsilon)| dx = \ln \lambda_3 - 2\pi \epsilon + \int \ln |e^{2\pi ix} - y_{1,\epsilon}| + \int \ln |e^{2\pi ix} - y_{2,\epsilon}|,$$

where $y_{1,\epsilon} = -\frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3} e^{2\pi \epsilon}$ and $y_{2,\epsilon} = -\frac{\lambda_2 - \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3} e^{2\pi \epsilon}$.

$$\int \ln |c(x + i\epsilon)| dx = \begin{cases} 
2\pi \epsilon + \ln \lambda_1 & \epsilon > \frac{1}{2\pi} \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3}, \\
\ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3} & \frac{1}{2\pi} \ln \frac{\lambda_2 - \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3} \leq \epsilon \leq \frac{1}{2\pi} \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3}, \\
-2\pi \epsilon + \ln \lambda_3 & \epsilon < \frac{1}{2\pi} \ln \frac{\lambda_2 - \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{2\lambda_3}. 
\end{cases}$$

Thus $L(A(\cdot + i\epsilon)) = 0$ when $|\epsilon| \leq \frac{1}{2\pi} \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1 \lambda_3}}{\max (\lambda_1 + \lambda_3) + \sqrt{\max (\lambda_1 + \lambda_3)^2 - 4\lambda_1 \lambda_3}}$.

Since $\tilde{A}_{\lambda,E}(x + i\epsilon) = Q_{\lambda}(x + \alpha + i\epsilon) A_{\lambda,E}(x + i\epsilon) Q_{\lambda}^{-1}(x + i\epsilon)$, the statement about $\tilde{A}_{\lambda,E}$ is also true.

\[\Box\]

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