GROUP FORMATION WITH NETWORK CONSTRAINTS

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Abstract. In this paper, I illustrate the importance of both dynamics and network constraints in the group formation process. I introduce a class of games called sequential group formation games, in which players make their group membership decisions sequentially over time, and show that the dynamics act as an equilibrium refinement. However, the resulting equilibrium is highly suboptimal—groups tend to be much too large, relative to the social optimum. I then introduce a network constraint, which limits a player’s action set to those groups that she is connected to on an exogenous network. The network constraint mitigates the tendency for groups to get too large and social welfare is higher when the network is sparse and highly ordered. This result has the surprising implication that informational, institutional, and geographic barriers to group membership may actually improve social welfare by restricting groups from becoming too large.

1. Introduction

In a group formation game, each player can be a member of one and only one group, and individual payoffs depend, directly or indirectly, on group structure. Many difficult and pressing economic problems fall into this category, including rent-seeking, resource management, contract bidding, volunteer organization, problem solving, and political lobbying. Depending on the application, the collection of individuals may be called a group, club, or coalition. However, apart from this basic semantic difference, all of these problems share the same basic characteristics—they all create incentives, such as economies of scale, risk-sharing, skill aggregation, and social capital accumulation, that make working within a group (coalition or club) more attractive than working alone. In this paper, I combine two recent areas of interest—dynamic coalition formation games and social network constraints—to explore how groups form when individuals move dynamically and face social, spatial, and institutional constraints on group membership.

Traditionally, group formation has been modeled statically—that is, players make their group membership decisions simultaneously—see, for instance, Hart and Kurz (1983), Nitzen (1991), Yi and Shin (1997), Konishi and Weber (1997), and Heintzelman et al. (2006). These models are advantageous because of their analytical simplicity and clarity. However, as I will show in Section 2.1, static group formation games often have multiple equilibria, suggesting that there are potentially large gains to clarifying the process by which they form. The presence of multiple equilibria raises a new, more complicated set of questions. Which of these equilibria can we realistically expect to reach given a dynamic group formation process? Will the equilibrium outcome reached be efficient? What characteristics of the problem...
affect that outcome? These questions cannot be addressed using a static model, and thus researchers have increasingly turned towards dynamic models of the coalition formation process. Recent work spans many different subfields, including industrial organization, political economy, rent-seeking, and local public good provision and includes (among many others) Bloch (1996), Yi (2000), Arnold and Schwalbe (2002), Konishi and Ray (2003), Macho-Stadler et al (2004), Arnold and Woorders (2005), and Page and Woorders (2007).

However, these dynamic group formation models largely still assume that players are unencumbered by social, spatial, or institutional barriers to group membership. That is, the players interact freely with all other individuals in the game and can join any of the groups in the game without regard to the composition of the group. This is a reasonable assumption in some contexts—however, in many other cases, individuals face substantial barriers in making their group membership decisions. The nature of these barriers will differ depending on the context of the specific problem being considered. Barriers to group membership may be social (eg: an individual can only join a group that contains someone he knows) or spatial (eg: an individual can only join a group with close neighbors). Some of these barriers are explicit (eg: a requirement that a current member “vouch” for the applicant) but others are implicit (eg: a social norm against attending a party composed only of strangers). The barriers may either limit actions (an individual is unable to join a particular group) or information (an individual does not know about the group). However, by modeling these constraints explicitly, we can look beyond the more superficial of these differences and ask a whole new set of questions. How do characteristics of the underlying constraint affect the eventual group structure? Are individuals better or worse off when they are constrained more heavily? How do constraints of different types affect social welfare?

In this paper, I model the constraints faced by individuals via a network of connections— a player can only join a group if she is connected to a current member. This method allows me to use machinery from the burgeoning networks literature, which explores how social, spatial and institutional networks affect individual behavior. This literature encompasses a wide range of subfields that (as noted in Jackson (2005)) have only recently started to interact. One branch of the literature has developed tools used to identify communities within existing social networks (see, for instance, Girvan and Newman (2002), Newman and Girvan (2004), and Copic et al (2007)); another branch examines how limiting interactions between individuals can affect strategic behavior (see, for instance, Galeotti et al (2007) and Charness and Jackson (2006)); and a third branch explores the dynamics of how social networks form (see Jackson (2005) for a survey of this work).

By combining elements of these two emerging literatures, I am able to illustrate the importance of both dynamics and network constraints in the group formation process. As a baseline for comparison, I start with a static game in which individuals are completely unconstrained in their choice of groups and show that this game has multiple equilibria. I then allow individuals to move sequentially, and solve explicitly for the set of Nash Equilibria of this game. I show that the dynamics act as an equilibrium refinement. However, the equilibrium reached in the dynamic game is highly suboptimal—the negative externality imposed by entering individuals drives groups to be much too large, relative to the social optimum.
I then compare the grouping behavior of the unconstrained individuals to the behavior of individuals constrained by an exogenous network of connections. The network limits a player’s action set to those groups containing individuals she is connected to. I show that the network constraint mitigates the tendency for groups to get too large. The efficiency of the outcome depends on the topological characteristics of the network constraint—social welfare is higher when the network is sparse and highly ordered. This result has the surprising implication that informational, institutional, and geographic barriers to group membership may actually improve social welfare by restricting groups from becoming too large.

Finally, I consider optimal institutional design and show that the optimal membership rule also depends on network topology—when a network is dense or random, the Exclusive Membership rule (which allows a group to reject members who do not improve the group’s welfare) is always optimal. However, when the network is sparse or highly ordered, the Exclusive Membership rule can lead to highly suboptimal results.

The structure of the paper is as follows. In Section 2, I introduce a static coalition formation model, in which individuals choose their group membership simultaneously. I characterize the set of Nash Equilibria of that game, and show that only one is optimal. In Section 3, I transform the static model by allowing individuals to make their group membership decisions sequentially over time. This defines a dynamic game similar to that of Arnold and Wooders (2005). I characterize the set of Nash Equilibria of this game, and show that a single, highly suboptimal equilibrium survives. In Section 4, I introduce the network constraint. I first characterize the set of Nash Equilibria of the constrained static game. I then move to the dynamic game and show how network topology affects social welfare. In Section 5, I consider optimal institutional design and show that the optimal membership rule depends on the topology of the network constraint. In Section 6, I conclude and discuss extensions to the model.

2. Basic Model Elements and the Static Game

Before considering the behavior of individuals who face a constraint, I will first consider a game in which individuals are unconstrained. This game is actually a special case of the constrained game (ie: one in which all individuals are connected) and thus provides a good baseline for comparison between this game and the existing unconstrained literature.

Consider a group formation game with \(N\) homogeneous individuals, \(I = \{1, 2, \ldots, N\}\). An individual can be a member of one and only one group—thus, the group structure at time \(t\) is a partition of \(I\), \(\pi(t) = \{G_1G_2\ldots G_{J(t)}\}\), where \(G_j\) denotes the set of individuals in group \(j\). Note that the number of groups is determined endogenously, and thus \(J(t)\) may vary from one period to the next. The set of all such partitions of the players into groups is denoted \(\Pi\).

Although all of the games defined in this paper could be played using a generalized payoff function, in the following I will assume that the players have identical payoff functions that depend only on the size of the player’s own group. Thus, individual \(i\)'s payoffs are given by \(f(g_i)\) where \(i \in G_j\) and \(g_j = |G_j|\) is the size of group \(j\). I also assume that \(f(g)\) is single-peaked with maximum value \(g^*\). The assumption that payoffs depend only on own group size obviously does not allow for externalities between groups. Nor does it allow players to have preferences over
group composition. However, this is an appropriately simple starting point for dynamic analysis—
to the extent that inter-coalition externalities muddy behavior, they are best left to future extensions.

I will also assume that payoffs are single-peaked in group size. This assumption is useful because individual and social preferences are aligned—the individuals all want to be in groups of size $g^*$ and social welfare is highest when this occurs. I will show that the equilibrium reached is suboptimal, despite this alignment. The assumption that payoffs are single-peaked also covers nearly all cases that we might encounter—generalizing further would add considerable complication without yielding much useful insight. However, extensions to more general payoff forms are obviously important, and are of interest for future studies. Section 6 includes a discussion of these generalizations.

Define $\bar{g}$ to be the smallest $g$ such that $f(g + 1) < f(2)$. That is, $\bar{g}$ is the largest group that will form before an individual forms a new group of size 2. If $f(N) > f(2)$, then a new group will never form, and for convenience, I will define $\bar{g} = N$ in these cases. Figure 1 illustrates an example of $\bar{g}$ with $\bar{g} < N$.

![Figure 1](image)

**Figure 1.** An illustration of $\bar{g}$—the smallest $g$ such that $f(g + 1) < f(2)$.

Note that since individuals in this game are homogeneous, the exact arrangement of the players in the groups is not as important as the sizes of the groups. Thus, I will often find it convenient to refer to the vector of group sizes resulting from a particular partition of the individuals into coalitions, rather than referring to the partition itself. To that end, define the group size vector of a partition $\pi(t) = \{G_1...G_J\}$ by $\langle g_1...g_J \rangle$. Note that the mapping from partitions to group size vectors is many-to-one, and thus the mapping from equilibrium partitions to equilibrium group size vectors will be as well.

2.1. **Static Group Formation Game.** Ultimately, the process of group formation is a dynamic one—individuals join, leave, and form new groups over time. However,
the assumption that moves are made simultaneously may be accurate in some instances and since dynamics add a good deal of analytical complication to the model, it is reasonable to ask whether making the model dynamic adds to our understanding of the problem. To that end, I will first examine a static group formation model. I show that when payoffs are single peaked in group size, there are often multiple Nash equilibria. In Section 3 I allow individuals to choose their group membership sequentially, and show that the dynamics of the game refine the set of equilibria, leaving a single equilibrium group size vector. This indicates that adding model dynamics can yield insights beyond those gained from static models.

Consider a static group formation game with $N$ players and payoffs, $f(g)$, single-peaked in group size. Individuals choose their group membership simultaneously. We can think of the individuals as choosing a “location”, and all of the individuals who jump to the same location are then members of the same group—thus, an individual’s behavior strategy consists of a choice of coalition: $\beta \in \{1, 2,...,N\}$. The pair $(N, f(g))$ defines the static coalition formation game. A Nash equilibrium of this game is a partition of the players into coalitions, such that no individual wishes to deviate unilaterally. Let $\Omega (N, f(g))$ be the set of partitions of the individuals into coalitions such that no individuals wishes to move unilaterally—that is, $\Omega (N, f(g)) = \{ \pi = \{G_1,...G_J\} \in \Pi| f(|G_i|) \geq f(|G_k| + 1) \forall G_j and G_k \in \pi \}$. Let $\varepsilon (N, f(g))$ denote the set of Nash Equilibrium coalition size vectors induced by those equilibrium partitions.

In the following, I characterize $\varepsilon (N, f(g))$. This characterization reveals several interesting aspects of group formation with single-peaked utility and also establishes the need for equilibrium refinement. Lemmas 2.1 and 2.2 establish several characteristics that an equilibrium of the static game will have: 1) the coalitions will mostly be larger than the social optimum (at most one will be smaller) and 2) all of the groups larger than the optimum will be approximately the same size. Theorem 2.3 assembles these conditions into a complete characterization of $\varepsilon (N, f(g))$. Finally, Theorem 2.4 puts a lower bound on the number of equilibria in the set $\varepsilon (N, f(g))$, showing that this static game will often have multiple equilibria.

Lemma 2.1 states that in equilibrium, most groups will be larger than the social optimum—at most one group can be too small.

**Lemma 2.1.** Let $(N, f(g))$ be a static group formation game with $f(g)$ single-peaked. If players are unconstrained in their choice of group, then there $\exists$ no equilibrium $(g_1...g_J) \in \varepsilon (N, f(g))$ such that $g_i \leq g_2 < g^*$, $i \neq j$. That is, in equilibrium at most one group will be smaller than the social optimum.

**Proof.** Towards a contradiction, suppose $\exists (g_1...g_J) \in \varepsilon (N, f(g))$ such that $g_1 \leq g_2 < g^*$. $f(.)$ is strictly increasing in that range, so $f(g_1) < f(g_2 + 1)$. But then players in group 1 have an incentive to move to group 2, so $(g_1...g_J)$ cannot be an equilibrium.

Lemma 2.1 implies that in characterizing $\varepsilon (N, f(g))$, we need consider only two cases: either all of the groups are larger than the socially optimal size $(g_1...g_k \geq g)$, or exactly one group is small $(g_1 < g^* and g_2...g_k \geq g^*)$. The following two Lemmas address the sizes of the groups in these two different cases. Lemma 2.2 shows that in any equilibrium where all groups are larger than the social optimum, the groups...
must be approximately the same size. Lemma 2.2 sets a more restrictive condition in the case where one group is smaller than the social optimum.

**Lemma 2.2.** Let \((N, f(g))\) be a static group formation game with \(f(g)\) single-peaked. If players are unconstrained in their choice of group, then for all \((g_1, \ldots, g_J) \in \varepsilon(N, f(g))\), \(|g_i - g_j| \leq 1\) \(\forall i, j, g_j \geq g^*\). That is, in equilibrium, all groups larger than the social optimum must be the same size, up to integer constraints.

**Proof.** Towards a contradiction, suppose \(\exists (g_1, \ldots, g_J) \in \varepsilon(N, f(g))\) such that \(g_1 > g_2 \geq g^*\) and \(g_1 - g_2 > 1\). \(f(.)\) is strictly decreasing in this range, so \(f(g_1) < f(g_2 + 1)\). But then players in group 1 have an incentive to move to group 2, so \((g_1, \ldots, g_J)\) cannot be an equilibrium.

Note that this result extends a result in Nitzen (1991) to the case of single-peaked utility. Arnold and Wooders (2005) prove a similar result for a sequential game. The following lemma extends that result to the case where one group is smaller than the social optimum. The Nash Equilibrium requires a slightly stronger restriction on the size of the groups.

**Lemma 2.3.** Let \((N, f(g))\) be a static group formation game with \(f(g)\) single-peaked. If players are unconstrained in their choice of group, then for all \((g_1, \ldots, g_J) \in \varepsilon(N, f(g))\) such that \(g_1 < g^*\), both of the following must be true:

1. \(f(g_j) \geq f(g_1 + 1) \geq f(g_1) \geq f(g_j + 1) \forall j > 1\)
2. \(g_j = g_k \forall j, k \neq 1\)

**Proof.** Let \((g_1, \ldots, g_J) \in \varepsilon(N, f(g))\) such that \(g_1 < g^*\).

Part 1: Consider group 1 (the small coalition) and an arbitrary group \(j\), such that \(g_j \geq g^*\). Note that \(f(g_1) < f(g_1 + 1)\) and \(f(g_j) > f(g_j + 1)\) if \(f(g_1) < f(g_j + 1)\), then players in group 1 would move to group \(j\). Similarly, if \(f(g_j) < f(g_1 + 1)\), then players in group \(k\) would move to group 1. Together, these three inequalities imply \(f(g_j) \geq f(g_1 + 1) \geq f(g_1) \geq f(g_j + 1) \forall j > 1\).

Part 2: Consider two arbitrary groups, \(j\) and \(k\), such that \(g_k \geq g_j \geq g^*\). Lemma 2.2 indicates that \(g_k - g_j \leq 1\). Towards a contradiction, suppose \(g_k - g_j = 1\), so that \(g_k = g_j + 1\). By Part 1, \(f(g_j + 1) \leq f(g_1)\). Since we assumed \(g_k = g_j + 1\), this implies that \(f(g_k) \leq f(g_1)\). But since \(f(g_1) < f(g_1 + 1)\) to the left of the optimum, \(f(g_k) < f(g_1 + 1)\), meaning that players in coalition \(j\) would move to coalition 1. Thus, it must be that \(g_j = g_k\) exactly.

Theorem 2.4 combines the insights from Lemmas 2.1 and 2.3 to fully characterize \(\varepsilon(N, f(g))\).

**Theorem 2.4.** Let \((N, f(g))\) be a static group formation game with single-peaked payoff function \(f(g)\). If the individuals are unconstrained in their choice of group, the set of Nash Equilibria of that game, \(\varepsilon(N, f(g))\), is the union of two sets:

1. \(\{ (g_1, \ldots, g_J) | \forall j, g_j \geq g^* \land |g_j - g_k| \leq 1 \forall j, k \}\)
2. \(\{ (g_1, \ldots, g_J) | g_1 < g^* , g_j \geq g^* \forall j \neq 1 , g_j = g_k \forall j, k \neq 1 , \land f(g_k) \geq f(g_1 + 1) \geq f(g_1) \geq f(g_k + 1) \}\)

**Example 2.5.** A Static Group Formation Game with Logistic Utility

\(^3\)By Lemma 2.3, this implies \(g_l \geq g^* \forall l \neq k\)
The implications of Lemmas 2.1-2.3 and Theorem 2.4 can best be illustrated through a specific example. Consider a static group formation game with 100 players and a logistic payoff function \( f(g) = g(20 - g) \). This function is single-peaked with maximum \( g^* = 10 \) and \( \bar{g} = 17 \). It is illustrated in Figure 2.

**Figure 2.** Individual payoff function for Example 2.5. Note that the players enjoy the highest payoff in a coalition of size 10.

In broad terms, this game could represent any of a number of different applications. It might, for example, represent a simple rent-seeking game, in which players compete for a single rent. There are many incentives for individuals to pool their efforts and compete as a group: risk averse individuals may be willing to trade some rent in expectation for a more consistent income stream; a complicated rent seeking task may require a range of skills; and economies of scale may make larger groups more likely to win. However, individuals must weigh these advantages against the disadvantages of higher maintenance costs, free-rider problems, and the division of rents. Many of these incentives come down to a decision over different size groups. Thus, we might think of this example as one in which individuals have determined that a rent-seeking group of size 10 maximizes their expected utilities by mitigating risk and taking advantage of economies of scale. Beyond that ideal size, the losses from managing a larger size group, free-riding off of others efforts, and division of rents between a larger number of people reduce the expected utility of the individuals in those groups. However, they still prefer that larger group to pursuing the rents alone. When a group has more than 17 members, the costs of maintaining that group make pursuing rents alone more attractive than staying in the group.

Lemma 2.1 indicates that in any Nash equilibrium of this game, at most one coalition will be smaller than the socially optimal group size, \( g^* = 10 \). Lemma 2.2 indicates that all of the groups larger than the social optimum will be approximately the same size. Using these two facts, one can show that there are 5 Nash Equilibria of this static game: \( \langle 10, 10, 10, 10, 10, 10, 10, 10, 10, 10 \rangle, \langle 11, 11, 11, 11, 11, 11, 11, 11, 11, 11 \rangle, \langle 12, 12, 12, 12, 12, 12, 12, 12, 12, 12 \rangle, \langle 13, 13, 13, 13, 13, 13, 13, 13, 13, 13 \rangle, \langle 14, 14, 14, 14, 14, 14, 14, 14, 14, 14 \rangle, \) \( \) and \( \langle 16, 16, 16, 16, 16, 16, 16, 16, 16, 16 \rangle \). Note that \( \langle 20, 20, 20, 20, 20 \rangle \) is not an equilibrium, because an individual in a group of size 20 is better off striking out as an individual.
This example illustrates several difficulties with the static game. First of all, the set of stable coalition configurations is highly sensitive to the particular parameters used. For example, with $N = 100$ individuals, the game illustrated above does not have an equilibrium with a small group. However, if we change the game slightly, so that there are $N = 101$ individuals, there will be an “odd-sized” equilibrium: $\langle 16, 16, 16, 16, 16, 16, 16, 16, 16, 5 \rangle$.

Secondly, most games will have multiple stable group size configurations. In fact, it is possible to put a lower bound on the number of equilibria for a given game. Theorem 2.6 does just that.

**Theorem 2.6.** Let $(N, f (g))$ be a static group formation game. Then $|\varepsilon (N, f (g))| \geq \frac{N}{g^*} - \frac{N}{\bar{g}} - 1$.

**Proof.** I will set the lower bound by enumerating the equilibria in which all groups are larger than the social optimum (i.e., the first set in Theorem 2.4). Note that since all groups are approximately the same size, each equilibrium with all large groups is entirely characterized by the number of groups. The largest possible group is $\bar{g}$ and the smallest possible group is $g^*$. Thus, there should be one equilibrium for each integer in the interval $\left[ \frac{N}{\bar{g}}, \frac{N}{g^*} \right]$ or $\frac{N}{g^*} - \frac{N}{\bar{g}} - 1$.

Since the lower bound in Theorem 2.6 is usually greater than 1, the static game will usually have multiple equilibria. However, the static game provides no insight into which of those equilibria is most likely to occur. Are they all equally likely, or is there a distribution of equilibria? Does that distribution depend in a predictable way on the observable elements of a particular problem? These questions are particularly important because some of the equilibria lead to considerably higher social welfare than others.

The following section shows that a more dynamic model of the group formation process, in which individuals make their decisions sequentially over time, provides an equilibrium refinement. As I will show, not all of the equilibria characterized in Theorem 2.4 are attainable when players start the game as individuals. Surprisingly, the surviving equilibrium group size vector is the worst possible of the static equilibria.

3. **Sequential Group Formation Game--The Unconstrained Case**

While a static group formation game $(N, f (g))$ will typically have more than one equilibrium, only one is efficient. This obviously begs the question—will individuals moving sequentially reach the efficient coalition arrangement, or will they reach an inefficient outcome? Will that outcome be unique or are several outcomes possible? In this section, I show that allowing the players to move sequentially refines the set of equilibria from the static game. When the players start the game as individuals, they will always reach an equilibrium group structure with the same coalition size vector. Furthermore, it is not the socially optimal one—when individuals make their group formation decisions dynamically, they end up in groups that are much too large, despite a clear alignment between individual and social welfare.

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4This is actually also a lower bound on the number of equilibria with all large groups. There could be more, depending on whether $g^*$ and $\bar{g}$ divide $N$ evenly, but including that complication only adds more equilibria, keeping the lower bound accurate (albeit a bit lower than is strictly necessary).
3.1. Sequential Coalition Formation Game. In the unconstrained sequential group formation game, individuals are able to join any currently existing group \( (G_j \in \pi(t)) \), or alternatively they can strike off as an individual, forming a group of size 1. Thus, individual \( i \)'s action set at time \( t \) can be denoted by \( A_i(t) = \pi(t) \cup \emptyset \), where \( \emptyset \) denotes the action of striking out as an individual. In the following, I will assume that individuals make their group membership decisions myopically—that is, they decide which group will maximize their return, given only the current group structure. This defines a behavior strategy that simply maps the current group partition, \( \pi(t) \), to the individual's action set as defined above: \( \beta(\pi(t)) \in A_i(t) \).

This myopic behavior strategy is identical to that used in Arnold and Schwalbe (2002) and Arnold and Woooders (2005). The myopia assumption is convenient because it makes the analysis more tractable. However, in the case of sequential coalition formation games, it is also behaviorally more reasonable than perfect foresight. The sequential nature of this game induces an explosion in number of possible states, making the sequential coalition formation game more like Chess or Go than Tic-tac-toe. Moreover, as I will mention later, one can show that myopia is not the sole cause of the observed behavior, making the assumption relatively innocuous.

\((N, f(g), \phi)\) defines an unconstrained sequential coalition formation game, where \( \phi \) is a particular order of motion for the players. An equilibrium of this dynamic game is a partition of the players into groups, \( \pi = \{G_1 \ldots G_J\} \) such that \( f(g_j) \geq f(g_{k+1}) \forall G_j, G_k \in \pi \)—that is, a group configuration is an equilibrium if no individual wishes to deviate unilaterally. Let \( \varepsilon(N, f(g), \phi) \) represent the set of equilibrium coalition size vectors resulting from those partitions.

Note that any Nash Equilibrium of the dynamic game must be a stable group configuration in the static game, and therefore \( \varepsilon(N, f(g), \phi) \subseteq \varepsilon(N, f(g)) \). Theorem 3.1 shows that the sequential game has a unique equilibrium up to the symmetry of the players. Furthermore, Theorem 3.2 shows that when players start the game as individuals, this equilibrium is always the worst possible stable group configuration from the standpoint of social welfare, despite the alignment between social and individual preferences.

**Theorem 3.1.** Let \( (N, f(g), \phi) \) be an unconstrained sequential group formation game with \( f(g) \) single-peaked. Then there is a unique Nash Equilibrium group size vector, \( \gamma(N, f(g)) = (g_1 \ldots g_J) \), which is a function of the number of players and the payoff function alone.

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5 This game was first introduced by Arnold and Woooders (2005). However, whereas Arnold and Woooders consider a Nash Club Equilibrium (a group structure which is stable to deviations by coalitions of individuals within a particular group) and a k-remainder Nash Club Equilibrium (which is stable to deviations when k individuals are dropped from the system) I use a Nash Equilibrium. See the later text for a discussion of deviations by groups of \( n \) individuals. It is worth noting that I obtain dramatically different results using the Nash Equilibrium than Arnold and Woooders do using the Nash Club and k-remainder Equilibria.

6 Obviously the equilibrium reached will depend on the initial condition. Starting the game with the individuals acting alone seems very natural, and also mimics the spirit of the static game. A full characterization of the basins of attraction for the different equilibria of this game is beyond the scope of the current work. However, I will note that the results that follow are unchanged if the individuals start the game in a grand coalition.
Theorem 3.2. Let \((N, f(g), \phi)\) be an unconstrained sequential group formation game with \(f(g)\) single-peaked. Let \((N, f(g))\) be the static coalition formation game with the same number of players and payoff function. If \((g_1, \ldots, g_J)\) is the (unique) Nash Equilibrium of \((N, f(g), \phi)\) then \((g_1, \ldots, g_J) = \arg \min_{\varepsilon(N, f(g))} \sum_{i=1}^{n} f(g_i).\) That is, the Nash Equilibrium of the sequential game is the element of \(\varepsilon(N, f(g))\) that minimizes social welfare.

Proof. Let \((N, f(g), \phi)\) be an arbitrary sequential group formation game. Let \((N, f(g))\) be the corresponding static group formation game. The equilibrium of the static game \((N, f(g))\) that yields the lowest social welfare is the equilibrium with groups of the largest size—or conversely, the equilibrium with the smallest number of groups. I will show that regardless of the order of motion, \(\phi\), the players always reach a configuration with the minimum number of groups, and thus the lowest possible social welfare value.\(^7\)

If \(f(g)\) is strictly increasing or decreasing, then the result follows trivially. So suppose \(f(g)\) is unimodal. \(f(g)\) unimodal implies \(f(1) < f(2)\). Thus, the first individual will always want to start a new group. Since \(f(g + 1) > f(2)\) \(\forall g < \bar{g}\), subsequent individuals will prefer to join the existing group to forming a new group of size 2. In fact, it is only worthwhile to create a second group of size 2 when the existing group is size \(\bar{g}\). If \(\bar{g} \geq N\), then a second group never forms—the individuals ultimately form one large group of size \(N\) and the result follows trivially. So suppose \(\bar{g} < N\).

For the sake of clarity, let \(\bar{r} = N \mod \bar{g} > 0\).\(^8\) Thus, the equilibrium in \(\varepsilon(N, f(g))\) with the lowest social welfare is the equilibrium with \(\frac{N - \bar{r}}{g} + 1\) groups. Regardless of the order of motion, a new group forms only if all existing groups have reached size \(\bar{g}\). Thus, the final group forms only once there are \(\frac{N - \bar{r}}{g}\) groups of size \(\bar{g}\). This implies that the unique equilibrium of the sequential game will have \(\frac{N - \bar{r}}{g} + 1\) groups.

The basic insights of the proof can best be appreciated via a specific example.

Example 3.3. An Unconstrained Sequential Group Formation Game with Logistic Utility

Note that while a static group formation game, \((N, f(g))\), often has multiple equilibrium group size vectors, only one is efficient. Recall from Example 2.5 that while the game \((100, g(20 - g))\) has 5 Nash equilibria—\((10, 10, 10, 10, 10), (11, 11, 11, 11, 11, 11, 11, 11, 11, 12), (12, 12, 12, 12, 13, 13, 13, 13), (14, 14, 14, 14, 15, 15, 15), and (16, 16, 17, 17, 17)\)—only the equilibrium with groups of size 10 is efficient.

Now, consider the sequential group formation game with the same parameters and an arbitrary order of play: \((100, g(20 - g), \phi)\). According to Theorem 3.1, this sequential game has a unique equilibrium. Moreover, Theorem 3.2 indicates that the equilibrium will be the stable group structure with the lowest possible social welfare—in this case, the configuration with coalitions of size 16 and 17. Note that this equilibrium is inefficient, because all players are better off in groups of size

\(^7\)Note that an “odd-size” equilibrium (one with a single small group) will always have higher social welfare than the equilibrium with the smallest possible number of groups. However, it should become clear in the following that such an equilibrium could never arise through sequential movement with the given initial condition. Therefore, I will not address it explicitly here.

\(^8\)The same result holds for \(r = 0\), but this assumption simplifies the exposition.
The following analysis shows how players wind up in this suboptimal group structure.

The players start the game as individuals, so the first player to move faces a choice between remaining as an individual and forming a group of size 2. The player is myopic, so she chooses the group of size 2 because it gives her higher utility in the next period (Figure 3). The second player to move faces a similar choice–she must decide whether to join the existing large group to form a group of 3, or join another individual to form a second group of 2. The group of 3 gives her higher utility, so she joins that group (Figure 4).

A new group only forms when \( f(2) \geq f(g + 1) \) where \( g \) is the size of the existing large group. The smallest such \( g \) is obviously \( \bar{g} \)– in this case, a group of 17 (Figure 5). This is true regardless of how many “large” groups (groups with more than one
Figure 5. A new group forms when the large group is size $\bar{g} = 17$ because the individual is better off in a new group of size 2.

individual) there are. Thus, the second group forms when there are 83 individuals and one group of 17, the third group forms when there are 69 individuals and two groups of size 17, and so on. The last group forms when there are 15 individuals and five groups of size 17.

This sixth group is the final group that will ever form. Individuals may (and indeed, will) move between the existing groups, but no new group will ever form. The individuals will stop moving when all six groups are approximately the same size—namely, in the configuration with two groups of size 16 and four groups of size 17: $\langle 16, 16, 17, 17, 17, 17 \rangle$. As predicted by Theorem 3.2, this is the stable group arrangement with the lowest possible social welfare value. Note also that at no point did we specify the order of play—thus, the players will reach the arrangement $\langle 16, 16, 17, 17, 17, 17 \rangle$ regardless of their order of motion.

3.2. Discussion—Externalities in Coalition Membership. One might be tempted to attribute the behavior detailed in Theorems 3.1 and 3.2 to the players’ myopia. However, it is possible to show that even perfectly forward-looking players will form groups that are larger than the socially optimal size. Since even forward-looking agents reach a suboptimal equilibrium, it is clear that myopia is not all that is at work in this result.

The cause of the observed behavior is the externality that joining a group imposes on existing group members. When the group is smaller than the social optimum, that externality is positive. However, when the group is at the optimal size, the externality is a negative one. The entering member is obviously made better off by the change (otherwise, he would not move), but the rest of the group is made worse off. The negative externality causes individuals to enter a group that does not benefit from the extra member, which then drives groups to become too large.

Obviously, in the real world, groups of individuals will sometimes make their membership decisions together. If we allow a subgroup of up to $n$ individuals to move as a group, then any equilibrium that exists will necessarily have smaller groups. However, the set of equilibria that are stable to such coalitional deviations

\footnote{An example with six players is available from the author upon request.}
are largely empty (see Arnold and Wooders (2005) for a discussion of this problem). More importantly, when we move on to games with a network constraint, as in the following section, it becomes less clear what is meant by a configuration that is stable to “coalitional deviations”. Analysis of more complicated, network-specific equilibrium concepts are obviously venues for future work.

Additionally, there is empirical evidence that groups tend to be too large. Many institutions exist that constrain the size of coalitions, a measure that would be unnecessary if individuals found themselves in groups of ideal size. In the following section, I consider the effects of social and spatial constraints on individual behavior and show that such constraints can improve total social welfare. Section 5 continues this discussion by exploring the effect of the network constraint on optimal institutional design.

4. Sequential Group Formation with a Network Constraint

The analysis of the previous section (as well as much of the existing literature) assumes that individuals are free to join any existing group, regardless of its current composition. However, the cases where individuals are completely unconstrained are relatively few—in most instances, individuals face social, spatial, and information constraints when making their membership decisions. Consider, for example, a set of farmers forming water management groups along the banks of a river. Although it is conceivable that the farmers would organize into groups at random, they are more likely to join farmers who are adjacent to them on the river than those in a distant location. Similarly, research groups are more likely to be composed of colleagues than strangers, and an individual is unlikely to attend a party unless he already knows someone who is attending.

The most natural way of modeling these constraints is via a network of connections. I give each individual an exogenous network of connections to other people. An individual can only join a group if it contains a person she is connected to on the network. More formally, \((N, f(g), \phi, C)\) defines a particular sequential group formation game with a network constraint, where \(C\) is an exogenous, unchanging matrix of connections between individuals—that is, \(C_{ij} = 1\) if \(i\) is connected to \(j\) on the network and 0 otherwise. In the constrained game, an individual’s action set is restricted to include only those groups she is connected to: \(A_i(t) = \{G | C_{ij} = 1 \text{ for some } j \in G \} \cup \emptyset \subseteq \pi(t) \cup \emptyset\). Clearly such a matrix of connections can model any set of constraints faced by individual agents, making the network formulation of this problem extremely general.

However, there is an additional advantage of using a network constraint—namely, it allows us to draw conclusions about general “classes” of constraints that seem similar, without getting caught up in the details of a particular case. For instance, we might want to determine how individuals on a spatial network behave differently

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10One obvious extension is to allow the network structure to evolve over time. This could provide insights into network formation. Jackson (2005) provides some background and a survey of the network formation literature.

11Note that this differs significantly from the use of networks in Page and Wooders (2007), which uses a bipartite network to illustrate the partition of individuals into groups—ie: each individual is linked to the group to which it is a member. Thus, this network contains no information about how individuals are connected. In this paper, the network links individuals to one another, restricting an individual’s choice of groups. Although we could use a second, bipartite network to denote the division of players into groups, it adds no insights in the current context.
than individuals on a social network, without getting tied up in the details of a particular network structure. Fortunately, by varying only a few parameters, we can obtain a natural spectrum of network structures that correspond nicely to the types of networks that we would observe in the real world. For my analysis of network topology I will use a Watts-Strogatz network, which has only two parameters. The first is average degree, $d$, which enumerates the average number of connections each individual in the network has. The second is the Watts-Strogatz parameter, $p \in (0, 1)$, which allows us to examine a spectrum of different network types—when $p = 0$, the network is regular and approximates a spatial network; when $p = 1$, the individuals are connected at random; for values of $p$ between 0 and 1, the network has a “small world” structure, which approximates that of a social network. A pair $(d, p)$ describes a family of networks with similar topological characteristics.

Note that when the network is fully connected, every player knows someone in every group and therefore $A_i(t) = \pi(t) \cup \emptyset$. Thus, the unconstrained static and sequential games considered previously are a special case of the constrained game—namely one where the average degree is at a maximum: $d = N - 1$. I will first use the static game to illustrate the effects of the network constraints on individual behavior given a particular network and then I will show how social welfare is affected by the network constraint.

4.1. The Static Game–Network Constraints and Variable Group Size. As in the unconstrained case, I will start by looking at individual behavior when players make their group membership decisions simultaneously. This section generalizes the results of Section 2.1 to the constrained case.

An equilibrium of the static game with a network constraint is a partition of the players into groups that is both feasible and individually rational.

Definition 4.1. A Nash equilibrium of the game $(N, f(g), C)$ is a partition of the players into groups, $\{G_1...G_J\}$, such that $\forall i, i \in G_j$ implies:

1. $f(g_j) \geq f(g_k + 1) \forall G_k \in \{G_1...G_J\}$
2. $C_{ij} = 1$ for some $j \in G_j$

One result of adding a network constraint is that the analogue to Lemma 2.2 need not be true. That is, when players are constrained, there may exist stable group structures in which groups are different sizes.

Claim 4.2. For a given static group formation game $(N, f(g), C)$, there may exist a Nash equilibrium group structure, $\{G_1...G_J\}$, such that $|g_j - g_k| > 1$ for some $g_j, g_k > g^*$.

As an illustration of this claim, consider a game with 12 players on a ring, as pictured in Figure 6. Further suppose $g^* = 2$ and $\bar{g} = 6$, so that all individuals want to be in a group of size 2, and will never form a group larger than size 6. Figure 7 illustrates a stable coalition structure of the static game $(N, f(g), C)$ with uneven group sizes.

It is obvious from Figure 7 how the ring affects the stability of this configuration. The individuals in group C would like to join group A, but they are unable to because they are not connected to that group on the social network. If the network were fully connected, the individuals in group C would like to move to group A, and

\[^{12}\text{Watts and Strogatz (1998)}\]
the configuration would not be stable. Note that the constraint of the ring could represent either a constraint on actions (the players would move if they could) or information (the players would move if they knew). It could also equally well represent an explicit constraint (a legal constraint), an implicit constraint (a social norm), or a functional constraint (a geographic coincidence).

This result is significant because while existing models predict that groups will be the same size in equilibrium, real-world groups are seldom identical in size. This analysis indicates that if individuals are constrained, group sizes need not be the same. By exploiting the fact that any two connected individuals form a fully connected subgraph, we can extend the results in Theorem 2.4 to the current case. To do so, we need one final definition: given a network constraint \( C \), I call two groups, \( G_j \) and \( G_k \), connected if \( \exists h \in G_j \) and \( i \in G_k \) such that \( C_{hi} = 1 \).

**Theorem 4.3.** Let \((N, f(g), C)\) be a static group formation game with single-peaked payoff function \( f(g) \) and network constraint \( C \). \( \{G_1...G_J\} \in \varepsilon(N, f(g), C) \) if for all connected groups, \( G_j \) and \( G_k \), either

\[(1) \ g_j, g_k \geq g^* \text{ and } |g_j - g_k| \leq 1\]
or

(2) \( g_j < g^* \leq g_k \) and \( f(g_k) \geq f(g_j + 1) \geq f(g_j) \geq f(g_k + 1) \)

Proof. Simply note that any pair of connected groups contains a pair of connected agents, who form a fully connected subgraph of the original graph. The result above follows immediately from Theorem 2.4. \( \square \)

4.2. The Sequential Game–Network Topology and Efficiency. As in the special case of a fully connected network, the set of equilibria of the dynamic game is a subset of the equilibria of the static game: \( \varepsilon (N, f(g), \phi, C) \subseteq \varepsilon (N, f(g), C) \). However, claim 4.4 indicates that Theorem 3.1 need not be true—there will often be multiple equilibrium group size configurations and the set of equilibria may depend on the order of play.

Claim 4.4. A sequential coalition formation game \((N, f(g), \phi, C)\) need not have a unique equilibrium. Furthermore, the set of equilibria of this game may depend on the order of play, \( \phi \).

Examples illustrating this claim can be found in Appendix A.

Since the outcome of the constrained group formation game may depend on the order of play and random moves and there is no strong theoretical foundation for a particular order of play or set of random choices, I must somehow deal with this multiplicity of equilibria. One method would be to determine the distribution of outcomes combinatorially and calculate the expected social welfare exactly. However, this method would yield results that are overly narrow, applying only to the specific networks considered. As discussed early, I would like to draw conclusions about a “class” of networks with similar topologies.

To that end, I will rely on the computational version of the combinatorial argument above. I will average social welfare over a large number of similar games. In this case, I use a Watts-Strogatz network, which is constructed as follows. We start with a regular network of degree \( d \)—this is a network in which every individual is connected to her \( \frac{d}{2} \) nearest neighbors on each side. We then rewire each of the links in the regular graph with probability \( p \). A link is rewired by disconnecting one end and reconnecting it to a different, random node in the network. Thus, the pair \((d, p)\) describes a family of networks with a similar topology.

As an illustration of the effect of varying these two parameters, consider a coalition formation game with 12 players. Figure 8 depicts four networks with different degree. In the first panel, every player is connected to every other player—\( d = 11 \). This is called a fully-connected graph, and represents the special case examined in Sections 2.1 and 3. The subsequent panels depict the same network with random links removed. Obviously, as the degree of the network decreases, the individuals within that network have fewer choices of groups to join (the size of an individual’s action set is bounded above by the number of neighbors she has). This parameter potentially has different meaning in different types of networks—in a spatial network, the average degree specifies how far an individuals can “see” in all directions, whereas in a social network, the average degree varies inversely with the “familiarity” required for membership.

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13For instance, in the case of the farmers on the river, it might indicate how many upstream and downstream neighbors an individual can interact with.

14Consider, for example, an individual deciding to join a club. Membership in a college activity may simply require having an acquaintance in the club. This low threshold of familiarity implies...
Figure 8. Four different networks connecting 12 players. Degree of the network decreases from left to right. The first panel depicts a fully connected network. As edges are removed at random, average degree declines and players potentially become more constrained in their choice of groups.

Figure 9. Three different networks connecting 12 players. In the first panel, the players are connected to their two nearest neighbors on each side. This is called a regular network, and is often used to represent arrangements of individuals in space. In the second panel, a small number of the links in the regular network are rewired at random. The result is called a small world network, and is a simple model of a social network. In the last panel, all of the links in the original network are rewired at random. The result is a random network, similar to those depicted in Figure 8. Random networks are easily analyzed, but a poor approximation of social connections.

As noted earlier, the parameter $p$ allows us to examine networks with different topologies. When $p = 0$, none of the links are rewired, and we have a regular network such as that pictured in the first panel of Figure 9. These networks have a high clustering coefficient (average probability that two of a node's neighbors are connected) and a high network diameter (largest minimum path length between two nodes), and are a good model for spatial networks. On the other hand, when $p = 1$, all of the links in the network are rewired at random. The result is a completely random network, pictured in the last panel of Figure 9. These networks have a low clustering coefficient and low diameter. Although easy to work with statistically, random networks are unfortunately relatively rare empirically. By rewiring a small, densely connected network constraint. Membership in a secret society, on the other hand, may require an applicant to have a very strong social tie to a current member. This network constraint would be much more sparse.
but non-zero fraction of the links, we obtain a small world network, pictured the second panel of Figure 9. A small world network has a high clustering coefficient but low diameter, and is a reasonable first-order approximation of a social network.

In the following analysis, I average social welfare over 100 games with random order of play and networks with the same pair \((d, p)\). Note that with the exception of the regular networks \((p = 0)\), the network structure will differ from one run to the next, even as the parameters remain the same. This allows me to average over a number of networks with the same parameters, which gives the results greater generality. Since I hope to isolate the effects of network topology on outcomes, the non-network elements of the game remain the same. All results in this section use a game with 100 players and logistic utility function \(f(g) = g(20 - g)\).

![Average Efficiency--Random Graphs of Different Degree](image)

**Figure 10.** Define efficiency to be ratio of actual social welfare to the maximum possible social welfare. This plot shows average efficiency over 100 runs of a sequential coalition formation game with \(N = 100\) and \(f(g) = g(20 - g)\). The network constraints are random \((p = 1)\). As the degree of the network constraint decreases, social welfare increases. Social welfare increases because the constraint binds more heavily, mitigating the tendency for groups to get too large.

Figure 10 shows that holding the Watts-Strogatz parameter constant, social welfare declines in the degree of the network constraint. Since the size of an individual’s action set is bounded above by her degree on the network, degree provides a rough measure of how constraining the network is on individual behavior. As the degree of the network decreases, the individuals are more constrained in their

\(^{15}\)For Figure 10, I used a random graph \((p = 1)\). The results are qualitatively similar for other values of \(p\).
choice of groups, which mitigates the tendency for groups to get too large. The fact that social welfare increases as individuals are more constrained is consistent with the hypothesis that groups are too large because of a negative externality.

Figure 11. Define efficiency to be ratio of actual social welfare to the maximum possible social welfare. This plot shows average efficiency for 100 runs of a sequential coalition formation game. For all runs, $N = 100$ and $f(g) = g(20 - g)$. Holding degree constant (at 2, 4, 6) average social welfare declines in the Watts-Strogatz parameter—that is, social welfare is higher when the network is ordered than when it is random.

Figure 11 shows the effects of the Watts-Strogatz parameter on social welfare. As the graph moves from regular, to small world, to random, social welfare declines. One possible reason for this trend is that as the Watts-Strogatz parameter increases, the clustering coefficient decreases. The clustering coefficient is the probability that two of a nodes neighbors are connected. As the clustering coefficient decreases, the probability that an individual knows more than one person in a group decreases, and the expected size of the action set increases. Thus, as the clustering coefficient decreases, the network becomes less constraining and average social welfare declines.

For Figure 11 I used networks of degree 2, 4, and 6. The results are qualitatively similar for networks of different degree. Obviously, as degree increases, the drop in social welfare from the regular graph to the random graph becomes less dramatic.
5. The Effects of Network Topology on Optimal Institutional Design

There is considerable evidence that real-world groups do tend to be too large. The most compelling evidence is that groups have developed institutions to artificially restrict membership—a measure that I argue would not be required if individuals self-organized optimally. In this section, I examine how network topology affects the optimal choice of membership rule.

When individuals are homogeneous, there are only two possible membership rules: the “Open Membership Rule” (no restriction on group membership) and the “Exclusive Membership Rule” (groups can reject a member). When individuals are unconstrained, the Exclusive Membership Rule is always preferable to the Open Membership Rule. In other words, the coalition members should never allow the group to get larger than $g^*$.

![Figure 12.](image)

One might think that the Exclusive Membership Rule would always be preferable. However, Figure 12 illustrates that when individuals are restricted in their choice of groups via a social network, the Exclusive Membership Rule can sometimes result in group configurations with extremely low social welfare values. An even simpler example uses a ring network. Suppose 20 individuals are arranged on a ring as shown in Figure 13. If $g^* = 3$, then the Exclusive Membership Rule may cause individuals to be “isolated” between groups of the ideal size. Both of these examples highlight why the Exclusive Membership Rule is less beneficial when individuals are constrained in their choice of groups. In the unconstrained case, all of the individuals who were excluded from other groups could band together. When

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17 Because players are homogeneous, all current members will have the same opinion on whether or not to admit a new member. Therefore, we need only consider two rules: one where the group and the individual need to agree, and one where the individual can act as a dictator. Charness and Jackson (2006) examines how the game play of entire groups depends on the voting rule used to make decisions within groups. In his case, the individuals may have heterogeneous opinions on the strategic decision, and thus he must consider different types of voting rules. Any extension of this work to include heterogeneous players would have to make more careful consideration of how a group decides to admit or reject a member.

18 This terminology was introduced in Yi (2000)
individuals are restricted, they no longer have that option, and there is a much greater chance of individuals being forced into low utility outcomes.

\[ g^* = 3. \]

Because the large groups can prevent them from joining, the isolated individuals must accept a lower payoff.

Once again, more general results can be obtained by averaging over many runs on topologically similar networks. Figure 14 shows that when degree is low, the Exclusive Membership Rule is no longer the clear optimal choice. Similarly, Figure 15 shows that when the graph is very ordered, the Exclusive Membership Rule is, on average, less beneficial. Given the possibility of extremely poor outcomes, such as those pictured in Figures 12 and 13, the Open Membership rule may be more desirable when the network constraint is low degree or highly ordered.

\[ \text{Average Efficiency--Veto Rules} \]

![Average Efficiency--Veto Rules](image)

**Figure 14.**
6. Extensions and conclusions

The model I have introduced in this paper lends itself to numerous interesting extensions. Although I have chosen a very simple payoff function for this initial work, the basic game structure of the model is easily generalized to model any problem in which payoffs depend on coalition structure. For instance, by making payoffs a function of the entire coalition size vector \( f(g_1, g_2, ..., g_J) \), we can include inter-coalitional externalities. This would allow us to model (among other things) a sequential version of the traditional 2-period rent-seeking game (such as that in Nitzen (1991)) or resource-allocation game (such as that in Heintzelman et al (2006)) and ask whether inter-coalitional externalities are affected by the structure of the underlying network constraint. For example, we might ask whether social welfare is higher when resource management groups are formed according to geography (eg: water management groups on a river) or social/family ties (eg: fisheries on a bay).

If we allow heterogeneity of players, then the payoff function might depend on the composition of the groups, as well as their size \( f(G_1, G_2, ..., G_J) \), which would allow us to explore another set of problems. If individuals are heterogeneous in ability or tool sets, then we can ask where self-organized teams contain an optimal level of diversity for problem-solving. If individuals are heterogeneous in ideology, we can explore the formation of lobbyist organizations and political parties. More abstractly, if individuals are heterogeneous in an arbitrary characteristic, we can model discriminatory behavior.
Finally, making the network structure endogenous may add some insight into network formation (See Jackson (2005) for an excellent survey of this growing literature). This might be accomplished either by making links weighted, or allowing links to be added and lost over time.

The process of group formation is one that has attracted increasing interest in the past decade. In this paper, I use a simple extension of a static coalition formation game to illustrate the importance of dynamics and membership constraints in the coalition formation process. I show that in the sequential coalition formation game, individuals tend to form groups that are too large, especially when players are unconstrained. Real world groups often implement membership restrictions, indicating that without such restrictions, the groups would tend to be too large. I also show that if individuals are constrained by a network, they tend to form groups that are closer to the ideal size, without the addition of membership restrictions. In fact, constraining group membership by requiring a social or spatial connection can effectively substitute for the institutional constraint of a membership rule.

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I will illustrate the second half of this claim first—that the order of play can affect the set of Nash Equilibria. Consider a game with 12 players arranged in a ring, as shown in Figure 6. Further, suppose the payoff function is $f(g)$ such that $g^* = 2$ and $\bar{g} = 6$.

For the first case, suppose that the players proceed in order around the ring—that is, $\phi_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$. Figure 16 shows game play leading to an equilibrium coalition structure with two groups of size six. Because of the order of play, the individuals are always choosing between joining an existing large group, forming a new group of two, or remaining as an individual. This choice is much the same as the choice players face in the unconstrained game with the same payoff function—pictured Figure 17. Thus, it should be unsurprising that the players reach

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{In this game, 12 individuals are arranged in a ring. The payoff function, $f(g)$, has maximum $g^* = 2$ and $\bar{g} = 6$. The individuals move in order around the ring—$\phi_1 = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$—and wind up in two groups of size 6. In fact, $(6, 6)$ is the only equilibrium group size configuration of the game $(12, f(g), \phi_1)$. Figure 18 shows the same game with a different order of play.}
\end{figure}
the same equilibrium coalition structure as they would in the unconstrained game: \( \langle 6, 6 \rangle \). In fact, this is the only equilibrium coalition size vector possible in this particular coalition formation game.

Now consider a second game with the same number of players, network constraint, and payoff function, but a different order of play \( \phi_2 = (2, 3, 5, 6, 8, 9, 11, 12, 1, 7, 4, 10) \). Figure 18 shows one possible sequence of game play, given \( \phi_2 \). Because the first few players to move are separated from the existing large groups, they are unable to
impose on the groups that have already formed, as they did in the previous exam-
ple. The result is an equilibrium coalition structure with four groups of the ideal
size: \(\langle 3, 3, 3, 3 \rangle\). Since \(\langle 3, 3, 3, 3 \rangle\) is in \(\varepsilon (N, f(g), \phi_2, C)\) but not in \(\varepsilon (N, f(g), \phi_1, C)\),
it is clear that the order of motion does affect the set of equilibria.

Of course, the outcome pictured in Figure 18 is not the only possible equilib-
rium of the game with order of play \(\phi_2\). Many players in this game are forced to make
random choices. Figure 19 shows that if some of those players make different
choices, then the players will find themselves in a different configuration—in this
case, \(\langle 4, 4, 4 \rangle\). This is an illustration of the first half of Claim 4.4 which states

\[ \text{Figure 19. This game is identical to that presented in Figure 18. Note, in particular, that the order of play is the same: } \phi_2 = 2, 3, 5, 6, 8, 9, 11, 12, 1, 7, 4, 10. \text{ However, the players have made different random choices, leading to a different equilibrium outcome: } \langle 4, 4, 4 \rangle. \text{ This shows that when players are sufficiently constrained, there need not be a unique equilibrium coalition size configuration.} \]

that a coalition formation game with a network constraint need not have a unique
equilibrium coalition size configuration.