Levels of discontinuity, limit-computability, and jump operators

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We develop a general theory of jump operators, which is intended to provide an abstraction of the notion of “limit-computability” on represented spaces. Jump operators also provide a framework with a strong categorical flavor for investigating degrees of discontinuity of functions and hierarchies of sets on represented spaces. We will provide a thorough investigation within this framework of a hierarchy of $\Delta^0_2$-measurable functions between arbitrary countably based $T_0$-spaces, which captures the notion of computing with ordinal mind-change bounds. Our abstract approach not only raises new questions but also sheds new light on previous results. For example, we introduce a notion of “higher order” descriptive set theoretical objects, we generalize a recent characterization of the computability theoretic notion of “lowness” in terms of adjoint functors, and we show that our framework encompasses ordinal quantifications of the non-constructiveness of Hilbert’s finite basis theorem.

1 Introduction

This paper is concerned with two relatively new developments in the field of descriptive set theory. The first development is the extension of the classical descriptive set theory for metrizable spaces to more general topological spaces and mathematical structures. Although it is not uncommon, particularly in measure theory, to define the Borel algebra for an arbitrary topological space, detailed analysis of the Borel hierarchy has been mainly restricted to the class of metrizable spaces, or possibly Hausdorff spaces on rare occasion. However, relatively recent work by V. Selivanov [37, 35, 36, 38, 39], D. Scott [34], A. Tang [45, 46], and the author [11, 12], have demonstrated that a significant portion of the descriptive set theory of metrizable spaces generalizes naturally to countably based $T_0$-spaces. This development opens up the possibility of finding new applications of descriptive set theory to mathematical fields heavily relying on non-Hausdorff topological spaces, such as theoretical computer science (e.g., $\omega$-continuous domains) and modern algebraic geometry (e.g., the Zariski topology on the prime spectrum of a countable commutative ring). These generalizations can also shed new light on old results. For example, although the Gandy-Harrington space (a non-metrizable space that plays an important role in effective descriptive set theory) cannot be topologically embedded into any Polish space, it can be embedded as a co-analytic set into a quasi-Polish space [11].

The second development is a shift from a focus on the complexity of subsets of a space to a focus on the complexity of functions between spaces. Certainly Baire’s hierarchy of discontinuous functions has a long history, but it is fair to say that Borel’s hierarchy of sets has played a more prominent role in the development of the theory. However, recently there has been growing interest within the field of computable analysis concerning the relationship between hierarchies of discontinuous functions, Turing degrees, and limit-computability, in particular by researchers such as V. Brattka, P. Hertling, A. Pauly, M. Ziegler, T. Kihara, and the author [6, 10, 20, 7, 48, 25, 12]. Furthermore, recent extensions of the Wadge game by researchers such as A. Andretta, L. Motto Ros, and B. Semmes [2, 29, 27, 28, 40] have provided new classifications of discontinuous functions and new methods to generalize classical results like the Jayne-Rogers theorem [22]. V. Selivanov has made contributions in this area as well, for example by generalizing the Hausdorff-Kuratowski theorem for the difference hierarchy to a hierarchy of $\Delta^0_2$-measurable functions into finite discrete spaces [38, 39].

These two developments should not be considered independent. For example, if we simply add to our framework the two-point non-trivial non-metrizable space $S$, known as the Sierpinski space, then we can obtain an elegant bijective correspondence between the family of $\Sigma^0_2$-subsets of a space $X$ and the family of $\Sigma^0_0$-measurable functions from $X$ to $S$. This is a natural generalization of the known bijection between open subsets of a space and continuous functions into the Sierpinski space, and is also similar to the role of the subobject classifier in a topos. Domain theory teaches us that the mathematical object $\Sigma^0_0(X)$, now viewed as a family of functions into $S$, will certainly not be metrizable, even if we can hope for it to be a topological space at all.

Continuing a little more our analogy with a topos, if we wish to work within a single category then we are faced with a dilemma if we have only one “sub-object” classifier $S$ but want a hierarchy of classes of subobjects such as $\Sigma^0_1(X) \subseteq \Sigma^0_2(X) \subseteq \cdots$. One natural solution is to abandon the idea of having a single subobject classifier, and instead have a sequence $S_1, S_2, \ldots$ of subobject classifiers that respectively classify the families $\Sigma^0_1, \Sigma^0_2, \ldots$. Such a theory would be very unwieldy if the subobject classifiers were all unrelated, but we might have some hope for the theory if the subobject classifiers $S_1, S_2, \ldots$ are defined as the iterates of a single

∗This paper is dedicated to Victor Selivanov in celebration of his 60th birthday and his valuable contributions to the descriptive set theory of general topological spaces. The author thanks Arno Pauly, Luca Motto Ros, Vasco Brattka, and Takayuki Kihara for valuable discussions and comments on earlier drafts of this paper.
endofunctor $F$ applied to a single subobject classifier $S$. We now only have to worry about which functors $F$ to consider, and what the “base” subobject classifier $S$ should be.

This abstract view is closely related to recent work initiated by A. Pauly on synthetic descriptive set theory [31, 32]. Ultimately, an axiomatic approach in the same spirit as topos theory would be most desirable, as it might help expose connections between the descriptive set theory of general topological spaces and the descriptive complexity of finite structures [11]. However, it seems a little premature to attempt that now, and instead we develop these ideas within the category of represented spaces and continuously realizable functions [4]. In this context, we introduce (topological) “jump operators”, which modulate the representation of a space and in effect play the role of the endofunctors $F$ described above.

The concept of a jump operator that we present here has its roots in the work of M. Ziegler [48] and V. Brattka [6, 10], where numerous connections are made between levels of discontinuity, limit-computability, and the representation of a function’s output space. The hierarchy of discontinuity that jump operators characterize turns out to be a subset of the strong Weihrauch degrees [9, 8], but we believe that the categorical framework that jump operators provide has much to offer.

This paper is organized into five major sections. After this Introduction, we will develop the general theory of (topological) jump operators. The third major section will investigate a lower portion of the jump operator hierarchy consisting of $\Delta^0_\alpha$-measurable functions. Our main contribution here is to extend some previous results concerning functions between metrizable spaces to functions between arbitrary countably based $T_0$-spaces. The results in this section are also important because they demonstrate that the jump operator framework is powerful enough to characterize functions as finely as P. Hertling’s hierarchy of discontinuity levels [21, 20].

The fourth major section presents several examples and applications, such as connections with the difference hierarchy, a quantification of the non-constructiveness of Hilbert’s basis theorem in terms of the ordinal $\omega^\omega$ (essentially due to S. Simpson [41] and F. Stephan and Y. Ventson [44]), and show some applications to the Jayne-Rogers theorem. It is our attempt to find a common thread between the results in this section that should be considered new, more so than the results themselves, so in several cases we omit proofs. We conclude in the fifth major section.

We will expect that the reader is familiar with classical descriptive set theory [24] and domain theory [17]. The reader should consult [11] for additional results on quasi-Polish spaces.

**Definition 1.** Let $X$ be a topological space. For each ordinal $\alpha (1 \leq \alpha < \omega_1)$ we define $\Sigma^0_\alpha(X)$ inductively as follows.

1. $\Sigma^0_1(X)$ is the set of all open subsets of $X$.

2. For $\alpha > 1$, $\Sigma^0_\alpha(X)$ is the set of all subsets $A$ of $X$ which can be expressed in the form

$$A = \bigcup_{i \in \omega} B_i \setminus B'_i,$$

where for each $i$, $B_i$ and $B'_i$ are in $\Sigma^0_\beta(X)$ for some $\beta_i < \alpha$.

We define $\Pi^0_\alpha(X) = \{ X \setminus A | A \in \Sigma^0_\alpha(X) \}$ and $\Delta^0_\alpha(X) = \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X)$.

The above definition is equivalent to the classical definition of the Borel hierarchy for metrizable spaces, but it differs for more general spaces.

A function $f : X \to Y$ is $\Sigma^0_\alpha$-measurable if and only if $f^{-1}(U) \in \Sigma^0_\alpha(X)$ for every open subset $U$ of $Y$. We will also be interested in $\Delta^0_\alpha$-measurability, which requires the preimage of every open set to be a $\Delta^0_\alpha$-set.

Later in the paper we will present some results specific to quasi-Polish spaces, which are defined as the countably based spaces that admit a Smyth-complete quasi-metric. Polish spaces and $\omega$-continuous domains are examples of quasi-Polish spaces. A space is quasi-Polish if and only if it is homeomorphic to a $\Pi^0_\omega$-subset of $\mathcal{P}(\omega)$, the power set of $\omega$ with the Scott-topology. The reader should consult [11] for additional results on quasi-Polish spaces.

## 2 Jump operators

A represented space is a pair $\langle X, \rho \rangle$ where $X$ is a set and $\rho : \subseteq \omega^\omega \to X$ is a surjective partial function. If $\langle X, \rho_X \rangle$ and $\langle Y, \rho_Y \rangle$ are represented spaces and $f : \subseteq X \to Y$ is a partial function, then a function $F : \subseteq \omega^\omega \to \omega^\omega$ realizes $f$, denoted $F \vdash f$, if and only if $f \circ \rho_X = \rho_Y \circ F$. If there exists a continuous realizer for $f$ then we say that $f$ is continuously realizable and write $\vdash f$.

Note that if $F \vdash f$ and $G \vdash g$, then $G \circ F \vdash g \circ f$, assuming the composition $g \circ f$ makes sense.
In some cases, a function \( f : X \to Y \) between represented spaces may fail to be continuously realizable, but will become continuously realizable if we strengthen the information content of the representation of \( X \) or weaken the information content of the representation of \( Y \). The notion of “limit computability” is a common example of weakening the output representation. The motivation for the following definition is to create an abstract framework to investigate in a uniform manner how modifications of the information content of a representation affects the realizability of functions.

**Definition 2.** A (topological) jump operator is a partial surjective function \( j : \subseteq \omega^\omega \to \omega^\omega \) such that for every partial continuous \( F : \subseteq \omega^\omega \to \omega^\omega \), there is partial continuous \( F' : \subseteq \omega^\omega \to \omega^\omega \) such that \( F \circ j = j \circ F' \). □

The identity function \( id : \omega^\omega \to \omega^\omega \) is a trivial example of a jump operator. Let \( f : X \to Y \) be a function between represented spaces. A \( j \)-realizer of \( f \) is a function \( F : \subseteq \omega^\omega \to \omega^\omega \) such that \( j \circ F \vdash f \). We use the notation \( \vdash f \) to denote that \( F \) is a \( j \)-realizer for \( f \). If there exists a continuous \( j \)-realizer for \( f \) then we will say that \( f \) is \( j \)-realizable and write \( \vdash_j f \).

The definition of “jump operator” given above is appropriate for the category of represented spaces and continuously realizable functions. Given a represented space \( \langle X, \rho_X \rangle \) and a jump operator \( j \), we can write \( j(X) \) to denote the represented space \( \langle X, \rho_X \circ j \rangle \). For each function \( f : X \to Y \) between represented spaces, we define \( j(f) \) to be the same function as \( f \) but now interpreted as being between the represented spaces \( j(X) \) and \( j(Y) \). It is now clear that the definition of a jump operator is precisely what is needed to guarantee that \( j(\cdot) \) is a well-defined endofunctor on the category of represented spaces.

If working in the category of represented spaces and computably realizable functions, then the appropriate definition of a (computability theoretic) jump operator would be to require that for every computable \( F : \subseteq \omega^\omega \to \omega^\omega \), there is computable \( F' : \subseteq \omega^\omega \to \omega^\omega \) such that \( F \circ j = j \circ F' \). The definition of \( j \)-realizability would also be modified in a similar manner. These modifications are necessary because, for example, the operator \( \mathcal{L} \) introduced in [7] to characterize low computability is a computability theoretic jump operator but it is not a topological jump operator.

In this paper, unless explicitly mentioned otherwise we will assume the topological jump operator definition given above and shall drop the term “topological”. However, much of the theory we develop will also apply to the computability theoretic jump operators as well.

Examples 3, 4, and 5 below provide typical examples of jump operators. In the following, \( (\cdots)_{n \in \omega} : (\omega^\omega)^\omega \to \omega^\omega \) is some fixed (computable) encoding of countable sequences of elements of \( \omega^\omega \) as single elements of \( \omega^\omega \).

**Example 3.** Define \( j_{\mathcal{S}_2^2} : \subseteq \omega^\omega \to \omega^\omega \) as:

\[
\langle \xi_n \rangle_{n \in \omega} \in \text{dom}(j_{\mathcal{S}_2^2}) \Leftrightarrow \xi_0, \xi_1, \ldots \text{ converges in } \omega^\omega
\]

\[
j_{\mathcal{S}_2^2}(\langle \xi_n \rangle_{n \in \omega}) = \lim_{n \in \omega} \xi_n
\]

□

**Example 4.** Define \( j_{\Delta} : \subseteq \omega^\omega \to \omega^\omega \) as:

\[
\langle \xi_n \rangle_{n \in \omega} \in \text{dom}(j_{\Delta}) \Leftrightarrow (\exists n)(\forall m \geq n)[\xi_m = \xi_n]
\]

\[
j_{\Delta}(\langle \xi_n \rangle_{n \in \omega}) = \lim_{n \in \omega} \xi_n
\]

□

**Example 5.** For each countable ordinal \( \alpha \), define \( j_\alpha : \subseteq \omega^\omega \to \omega^\omega \) as:

\[
\langle \langle \beta_n \rangle_\alpha \circ \xi_n \rangle_{n \in \omega} \in \text{dom}(j_\alpha) \Leftrightarrow (\forall n)(\alpha > \beta_n \geq \beta_{n+1}) \text{ and } (\forall n)(\xi_n \neq \xi_{n+1} \Rightarrow \beta_n \neq \beta_{n+1})
\]

\[
j_\alpha(\langle \langle \beta_n \rangle_\alpha \circ \xi_n \rangle_{n \in \omega}) = \lim_{n \in \omega} \xi_n
\]

where \( \langle \cdot \rangle_\alpha : \alpha \to \omega \) is some fixed encoding of ordinals less than \( \alpha \) as natural numbers, and \( \langle \beta \rangle_\alpha \circ \xi \) is the element of \( \omega^\omega \) obtained by prepending the encoding of \( \beta \) to the beginning of \( \xi \). □

The jump operators \( j_{\mathcal{S}_2^2} \) and \( j_{\Delta} \) are also computability theoretic jump operators. If \( \alpha < \omega_1^{CK} \) then \( j_\alpha \) is a computability theoretic jump operator assuming that the encoding \( \langle \cdot \rangle_\alpha \) is effective.

Intuitively, \( j_{\mathcal{S}_2^2} \)-realizing a function only requires the realizer to output a sequence of “guesses” which is guaranteed to converge to the correct answer. Each guess is an infinite sequence in \( \omega^\omega \), and convergence means that each finite prefix of the guess can be modified only a finite number of times. The jump operator \( j_{\mathcal{S}_2^2} \) and its connections with limit computability have been extensively studied in the field of computable analysis, for example by V. Brattka [6,10] and M. Ziegler [48]. In the context of Wadge-like games and reducibilities, the jump operator \( j_{\mathcal{S}_2^2} \) essentially captures the notion of an “eraser” game (see [27,28,40] and the references
hence change complexity in the field of inductive inference (see [16, 26, 13, 14]), and to the Hausdorff difference hierarchy [24]. We will show later in this paper that a function between countably based ordinal bound on the number of times it will change its guess in the future. For example, when

It follows from the Jayne-Rogers theorem ([22], see also [29, 23, 43, 25]) that a total function on the natural numbers. In the context of Wadge-like games, investigated by M. Ziegler [48] in terms of finite revising computation, and was shown in [7] by V. Brattka, A. Pauly, and the author if it is

Jump operators are (quasi-)ordered by

Theorem 7.

The next two theorems show that the above definitions are in fact (topological) jump operators corresponding to the supremum

In the following, given

The author is indebted to A. Pauly for pointing out that the proofs in this section only apply to topological jump operators and may fail to hold for computability theoretic jump operators in general.

In the following, given

Definition 6. Let \((j_i)_{i \in \omega}\) be a countable sequence of jump operators. Define \(\bigvee j_i : \subseteq \omega^\omega \to \omega^\omega\) and \(\bigwedge j_i : \subseteq \omega^\omega \to \omega^\omega\) by

1. \((\bigvee j_i)((i) \circ \xi) = j_i(\xi)\), where \(\text{dom}(\bigvee j_i) = \{i \circ \xi \in \omega^\omega | \xi \in \text{dom}(j_i)\}\).

2. \((\bigwedge j_i)((\xi_n)_{n \in \omega}) = j_0(\xi_0)\), where \(\text{dom}(\bigwedge j_i) = \{\{\xi_n\}_{n \in \omega} \in \omega^\omega | \forall i, k : j_i(\xi_i) = j_k(\xi_k)\}\).

The next two theorems show that the above definitions are in fact (topological) jump operators corresponding to the supremum and infimum of \((j_i)_{i \in \omega}\).

Theorem 7. \(\bigvee j_i\) is a jump-operator and is the supremum of \((j_i)_{i \in \omega}\).

Proof. Assume \(f : \subseteq \omega^\omega \to \omega^\omega\) is continuous. For \(i \in \omega\) there is continuous \(g_i : \subseteq \omega^\omega \to \omega^\omega\) such that \(f \circ j_i = j_i \circ g_i\). Define \(g : \subseteq \omega^\omega \to \omega^\omega\) as \(g((i) \circ \xi) = (i) \circ g_i(\xi)\). Clearly \(g\) is continuous and

\[
\begin{align*}
f((\bigvee j_i)((i) \circ \xi)) &= f(j_i(\xi)) \\
&= j_i(g_i(\xi)) \\
&= (\bigvee j_i)((i) \circ g_i(\xi)) \\
&= (\bigvee j_i)(g((i) \circ \xi)),
\end{align*}
\]

hence \(f \circ \bigvee j_i = \bigvee j_i \circ g\). Therefore \(\bigvee j_i\) is a jump operator.
Next, for $i \in \omega$ define $f_i(\xi) = \langle i \rangle \circ \xi$. Then $f_i$ is continuous and $j_i = (\bigvee j_i) \circ f_i$. Therefore $j_i \leq \bigvee j_i$ for all $i \in \omega$.

Finally, assume $p: \omega^\omega \to \omega^\omega$ is such that $j_i = p \circ q_i$ for some continuous $q_i: \omega^\omega \to \omega^\omega$ (for all $i \in \omega$). Define $q: \omega^\omega \to \omega^\omega$ so that $q(\langle i \rangle \circ \xi) = q_i(\xi)$. Then $q$ is continuous and

\[
(\bigvee j_i)(\langle i \rangle \circ \xi) = j_i(\xi) = p(q_i(\xi)) = p(q(\langle i \rangle \circ \xi))
\]

hence $\bigvee j_i = p \circ q$. Therefore $\bigvee j_i \leq p$. It follows that $\bigvee j_i$ is the supremum of $(j_i)_{i \in \omega}$. \hfill \Box

**Theorem 8.** $\bigwedge j_i$ is a jump-operator and is the infimum of $(j_i)_{i \in \omega}$.

**Proof.** Assume $f: \subseteq \omega^\omega \to \omega^\omega$ is continuous. For $i \in \omega$ there is continuous $g_i: \subseteq \omega^\omega \to \omega^\omega$ such that $f \circ j_i = j_i \circ g_i$. Define $g: \subseteq \omega^\omega \to \omega^\omega$ as

\[
g(\langle \xi_n \rangle_{n \in \omega}) = \langle g_n(\xi_n) \rangle_{n \in \omega}.
\]

Clearly $g$ is continuous and if $\langle \xi_n \rangle_{n \in \omega} \in \text{dom}(\bigwedge j_i)$ then for all $i, k \in \omega$, $j_i(\xi_i) = j_k(\xi_k)$ hence $j_i \circ g_i(\xi_i) = f \circ j_k(\xi_k) = f \circ j_k(\xi_k) = f \circ j_k(\xi_k)$, and it follows that $\langle g_n(\xi_n) \rangle_{n \in \omega} \in \text{dom}(\bigwedge j_i)$. So for $\langle \xi_n \rangle_{n \in \omega} \in \text{dom}(\bigwedge j_i)$ we have

\[
f((\bigwedge j_i)(\langle \xi_n \rangle_{n \in \omega})) = f(j_0(\xi_0)) = j_0(g_0(\xi_0)) = (\bigwedge j_i)(\langle g_n(\xi_n) \rangle_{n \in \omega}),
\]

hence $f \circ \bigwedge j_i = \bigwedge j_i \circ g$. Therefore $\bigwedge j_i$ is a jump operator.

Next, define $\pi_i(\langle \xi_n \rangle_n \in \omega) = \xi_i$, which is clearly continuous. Then

\[
(\bigwedge j_i)(\langle \xi_n \rangle_{n \in \omega}) = j_0(\xi_0) = j_0(\xi_i) = j_i(\pi_i(\langle \xi_n \rangle_{n \in \omega}),
\]

hence $\bigwedge j_i \leq j_i$ for all $i \in \omega$.

Assume $p: \omega^\omega \to \omega^\omega$ is such that $p = j_i \circ q_i$ for some continuous $q_i: \omega^\omega \to \omega^\omega$ (for all $i \in \omega$). Define $q: \omega^\omega \to \omega^\omega$ so that $q(\xi) = \langle q_n(\xi) \rangle_{n \in \omega}$. Clearly $q$ is continuous. If $\xi \in \text{dom}(g)$ then $p(\xi) = j_0(q_i(\xi))$ for all $i \in \omega$, so $\langle q_n(\xi) \rangle_{n \in \omega} \in \text{dom}(\bigwedge j_i)$ and

\[
p(\xi) = j_0(q_i(\xi)) = (\bigwedge j_i)(\langle q_n(\xi) \rangle_{n \in \omega}) = (\bigwedge j_i)(q(\xi)),
\]

hence $p \leq \bigwedge j_i$. It follows that $\bigwedge j_i$ is the infimum of $(j_i)_{i \in \omega}$. \hfill \Box

For example, let $(j_{\alpha_i})_{i \in \omega}$ be a sequence of jump operators from Example 8. Then it is straightforward to verify that $\bigvee j_{\alpha_i} = j_{\bigvee \alpha_i}$ and $\bigwedge j_{\alpha_i} = j_{\bigwedge \alpha_i}$.

### 2.2 The “jump” of a representation

Let $\mathcal{J}$ be the lattice of jump operators. It is easy to show that if $j$ and $k$ are jump operators, then so is $j \circ k$. Furthermore, if $j_1 \leq j_2$, then $j_1 \circ k \leq j_2 \circ k$. Thus, every jump operator $k$ defines a monotonic function on $\mathcal{J}$, which we call the $k$-jump, that maps $j$ to $j \circ k$. This notion of iterating “jumps” can be found in [48] and [10] for the case of $\Sigma^0_2$.

A jump operator $j$ is **extensive** if the identity function $\text{id}: \omega^\omega \to \omega^\omega$ is $j$-realizable. Currently the author is unaware of any topological jump operators that are not extensive, but the non-extensive computability theoretic jump operators have a non-trivial structure (for example, the inverse of the Turing jump, or integral, in [10] and [7] is non-extensive).

A jump operator $j$ is **idempotent** if $j \circ j = j$. The jump operator $j_{\Delta}$ is idempotent, but $j_{\Sigma^0_2}$ is not.

We will say that $j$-realizability is **closed under compositions** if for every pair of $j$-realizable functions $f: X \to Y$ and $g: Y \to Z$ we have that $g \circ f$ is also $j$-realizable.

**Theorem 9.** If $j$ is an extensive jump operator, then $j$-realizability is closed under composition if and only if $j$ is idempotent.
Proof. Assume \( j \) is extensive and closed under compositions. Clearly, \( j \) is \( j \)-realizable, so \( j \circ j \) must be \( j \)-realizable, hence \( j \circ j \leq j \).

On the other hand, since \( \text{id} \leq j \) it follows by the monotonicity of the \( j \)-jump that \( j \leq j \circ j \). Therefore, \( j \circ j \equiv j \).

For the converse, assume \( j \) is extensive and idempotent, and assume \( F \vdash_j f \) and \( G \vdash_j g \) and the composition \( g \circ f \) is possible. Then \( j \circ F \vdash f \) and \( j \circ G \vdash g \), and composition gives \( j \circ G \circ j \circ F \vdash g \circ f \). Since \( j \) is a jump operator, there is continuous \( G' \) such that \( j \circ j \circ G' \circ F \vdash g \circ f \). Now using the idempotent property of \( j \) we obtain \( G' \circ F \vdash_j g \circ f \).

Recall that a closure operator on a partially ordered set is a function which is monotonic, extensive, and idempotent. The above theorem can be reworded as follows.

**Corollary 10.** If \( j \) is an extensive jump operator, then \( j \)-realizability is closed under composition if and only if the \( j \)-jump is a closure operator on \( J \).

It is easy to see that if the \( j \)-jump is a closure operator on \( J \), then \( j \) is the least fixed point of the \( j \)-jump above \( \text{id} \). In particular, the \( j_\Delta \)-jump of \( j_\alpha \) is equivalent to \( j_\Delta \) for each \( \alpha < \omega_1 \). It turns out that \( j_{\Sigma^0_0} \) is a fixed point of the \( j_\Delta \)-jump for each \( \alpha < \omega_1 \) because \( j_{\Sigma^0_0} \circ j_\alpha \) is \( \Sigma^0_\alpha \)-measurable.

### 2.3 Strong Weihrauch Degrees

In this section we will compare jump operators with the notion of strong Weihrauch reducibility (see [9, 8, 30, 7] for more on Weihrauch reducibility). We only consider the topological version of reducibility for the case of single valued functions.

**Definition 11.** Let \( f : X \to Y \) and \( g : W \to Z \) be functions between represented spaces. Define \( f \leq_{\text{SW}} g \) if and only if there are continuous functions \( K, H : \subseteq \omega^\omega \to \omega^\omega \) satisfying \( K \circ G \circ H \vdash f \) whenever \( G \vdash g \).

**Theorem 12.** Let \( f : X \to Y \) be a function between represented spaces, and let \( j \) be a jump operator. Then \( f \leq_{\text{SW}} j \) if and only if \( f \) is \( j \)-realizable.

**Proof.** Assume \( f \leq_{\text{SW}} j \) and let \( K \) and \( H \) be the relevant continuous functions. Since \( \omega^\omega \) is represented by the identity function, it follows that \( K \circ j \circ H \vdash f \). Using the fact that \( j \) is a jump operator, there is continuous \( K' : \subseteq \omega^\omega \to \omega^\omega \) such that \( j \circ K' \circ H \vdash f \). Therefore, the continuous function \( K' \circ H \) \( j \)-realizes \( f \).

For the converse, assume \( F : \omega^\omega \to \omega^\omega \) is a continuous \( j \)-realizer of \( f \). Then \( j \circ F \vdash f \) by definition. Again, since \( \omega^\omega \) is represented by the identity function we have \( J \vdash j \) if and only if \( J = j \). Thus, taking \( K \) as the identity function and \( H = F \) demonstrates that \( f \leq_{\text{SW}} j \).

The above theorem shows that jump operators form a subset of the strong Weihrauch degrees. However, this inclusion is strict, in the sense that there are strong Weihrauch degrees that do not correspond to any jump operator. For example, a constant function on \( \omega^\omega \) is not strong Weihrauch equivalent to any jump operator because jump operators are surjective.

### 2.4 Adjoint

This section actually applies more to computability theoretic jump operators than topological jump operators, but the basic definitions and immediate results are the same in both cases. This section mainly consists of generalizations of results found in [10] and [7].

Let \( j \) and \( k \) be jump operators and let \( \text{id} : \omega^\omega \to \omega^\omega \) be the identity function. We say that \( j \) is left adjoint to \( k \) or that \( k \) is right adjoint to \( j \), and write \( j \vdash k \), if and only if \( k \circ j \leq \text{id} \leq j \circ k \). This is equivalent to stating that the \( j \)-jump on \( J \) is left adjoint to the \( k \)-jump, and it also implies that the associated endofunctors are adjoint.

**Example 13** (see [10] and [7]). Let \( (U_n)_{n \in \omega} \) be a standard enumeration of the computably enumerable open subsets of \( \omega^\omega \). Define \( J : \omega^\omega \to \omega^\omega \) by \( J(\xi)(n) = 1 \) if \( \xi \in U_n \) and \( J(\xi)(n) = 0 \), otherwise. Then \( J^{-1} \), the inverse of \( J \), is a computability theoretic jump operator and \( J^{-1} \vdash j_{\Sigma^0_1} \).

**Proposition 14.** If \( j \vdash k \) then the \( (j \circ k) \)-jump is a closure operator. In particular, \( (j \circ k) \)-realizable functions are closed under composition.

**Proof.** This is a well known property of adjoints. Since \( k \circ j \leq \text{id} \) it follows that \( j \circ k \circ j \circ k \leq j \circ \text{id} \circ \text{id} \circ k \equiv j \circ k \). Furthermore, \( \text{id} \leq j \circ k \) implies \( j \circ k \equiv \text{id} \circ j \circ k \leq j \circ k \circ j \circ k \), and it follows that \( (j \circ k) \) is idempotent. Therefore, the \( (j \circ k) \)-jump is a closure operator.

**Note:** However, that \( J^{-1} \) is not a topological jump operator [7].
The low-jump-operator is defined as $\mathcal{L} = J^{-1} \circ j_{\mathbb{2}}$. It is shown in [7] that $\mathcal{L}$-realizability captures the notion of “lowness” from computability theory. It immediately follows from the above proposition that $\mathcal{L}$-realizable functions are closed under composition.

The general theory of adjoints provides much information about $j$ and $k$ when it is known that $j \vdash k$. For example, the $j$-jump preserves joins on $\mathcal{F}$ and the $k$-jump preserves meets. Viewed as functors, $j$ preserves colimits and $k$ preserves limits. This means, in particular, that $k(X) \times k(Y)$ will be isomorphic to $k(X \times Y)$ for every pair of represented spaces $X$ and $Y$.

Although so far we have been investigating the effects of weakening the output representation, it is also interesting to investigate the effects of strengthening the input representation. Given jump operators $j$ and $k$, represented spaces $(X, \rho_X)$ and $(Y, \rho_Y)$, and a function $f : X \to Y$, we will say that a function $F : \omega^\omega \to \omega^\omega$ (j, k)-realizes $f$ if and only if $f \circ \rho_X = j \circ \rho_Y \circ k \circ F$. This simply means that $F$ realizes $f$ reinterpreted as a function between $j(X)$ and $k(Y)$. We will say that a function is $(j, k)$-realizable if and only if it has a continuous $(j, k)$-realizer. Clearly, $(j, k)$-realizability as defined earlier corresponds to $(\langle id, j \rangle, k)$-realizability.

The following theorem shows that if $j \vdash k$, then strengthening the input representation by $j$ is equivalent to weakening the output representation by $k$.

**Theorem 15.** If $j$ and $k$ are jump operators and $j \vdash k$, then $(\langle id, j \rangle, k)$-realizability is equivalent to $(\langle id, k \rangle, j)$-realizability.

**Proof.** Assume $j \vdash k$ and that $f : X \to Y$ is $(\langle id, j \rangle, k)$-realizable. Let $F_j$ be any continuous $(\langle id, j \rangle, k)$-realizer for $f$. Since $k$ is a jump operator there is a partial continuous $F_j'$ that $(k, k)$-realizes $F_j$, hence $F_j'$ is a $(j \circ k, k)$-realizer of $f$. If we let $I$ be a continuous function reducing $id$ to $j \circ k$, then $F_j' \circ I$ is a continuous $(\langle id, k \rangle, j)$-realizer for $f$. Therefore, $f$ is $(\langle id, k \rangle, j)$-realizable.

Proving that $(\langle id, k \rangle, j)$-realizability implies $(\langle id, j \rangle, k)$-realizability is done similarly. □

Finally, the following proposition shows that it is easy to create new pairs of adjoint jump operators from a given pair of adjoint operators. We leave the proof as an easy exercise.

**Proposition 16.** If $j \vdash k$, then $k \circ k \circ j \circ j \leq k \circ j \circ j \circ k \circ k$. In particular, we have $j \circ j \vdash k \circ k$. □

2.5 Additional properties

In our final section on the general theory of jump operators, we would like to emphasize how they can contribute to the development of a categorical framework for descriptive set theory. The observations in this section are closely related to recent work initiated by A. Pauly on synthetic descriptive set theory [31, 32].

Let $\mathcal{S} = \{\bot, \top\}$ be the Sierpinski space and let $\mathbb{2} = \{0, 1\}$ be the discrete two point space. It is well known that there is a bijection between the open (resp., clopen) subsets of a topological space $X$ and the continuous functions from $X$ to $\mathcal{S}$ (resp., $\mathbb{2}$). In the same manner, there is an obvious bijection between $\Sigma^0_2(X)$ and the set of $j_{\Sigma^0_2}$-realizable functions $\chi : X \to \mathcal{S}$. Furthermore, $\Delta^0_2(X)$ is in bijective correspondence with the set of $j_{\Sigma^0_2}$-realizable functions $\chi : X \to \mathbb{2}$.

In general, given an arbitrary jump operator $j$ and a represented space $X$, we can define $\Sigma_j(X)$ to be the set of $j$-realizable functions from $X$ into $\mathcal{S}$, and define $\Delta_j(X)$ to be the set of $j$-realizable functions from $X$ into $\mathbb{2}$. Thus, each jump operator $j$ determines a “$j$-decidable” class $\Delta_j(X)$ of subsets of $X$ and a “$j$-semi-decidable” class $\Sigma_j(X)$.

It is well known that the category of represented spaces and continuously realizable (total) functions is cartesian closed (see [4], for example). Given a represented space $Y$ and a jump operator $j$, recall that $j(Y)$ denotes the represented space obtained by composing the representation with $j$ (this is the image of $Y$ under the endofunctor determined by $j$). Then for any pair of represented spaces $X$ and $Y$, the exponential object $j(Y)^X$ is the natural candidate for the represented space of $j$-realizable functions from $X$ to $Y$. In particular, $j(S)^X$ corresponds to $\Sigma_j(X)$ and $j(2)^X$ corresponds to $\Delta_j(X)$.

We can therefore define notions such as “$\Sigma^0_2$-set” on an arbitrary represented space $X$, and we can interpret the set of $\Sigma^0_2$-sets as a new represented space. This can be done even when it is impossible to interpret $X$ as a topological space in any natural way.

What kind of a space is $\Sigma^0_2(\Sigma^0_2(X))$? Note that $j_{\Sigma^0_2}(\mathcal{S})$ is isomorphic to the Sierpinski space with the total representation $\rho : \omega^\omega \to \mathcal{S}$ sending $\xi \in \omega^\omega$ to $\top$ if and only if $(\exists m)(\forall n)[\xi(n, m) = 1]$. Thus, $\Sigma^0_2(\mathcal{S})$ represents in a sense the family of $\Sigma^0_2$-predicates on $\mathbb{2}$, and $\Sigma^0_2(\Sigma^0_2(X))$ is the second-order object corresponding to the family of $\Sigma^0_2$-predicates on the $\Sigma^0_2$-predicates on $X$. This connection between $\Sigma^0_2(X)$ and $\Sigma^0_2$-predicates can easily extend to $n > 2$. It is the topic of future research to determine what kind of general “topological” information can be extracted from spaces like $\Sigma^0_2(\Sigma^0_2(X))$.

3 Levels of discontinuity

The next part of this paper will be dedicated to characterizing $j_{\Delta^*}$ and $j_{\alpha^*}$-realizability ($1 \leq \alpha < \omega_1$) for functions between arbitrary countably based $T_0$-spaces.

\footnote{Note that $\Delta_j$ and $\Sigma_j$ will completely coincide for some jump operators, such as $j_{\Delta^*}$.}
A characterization of $j_\Delta$-realizability for functions on $\omega^\omega$ has already been given by A. Andretta [2]. In addition, L. Motto Ros [27] has independently investigated a notion related to $j_{a^1}$-realizability on metric spaces. However, the extension of the theory to arbitrary countably based $T_0$-spaces that we provide here appears to be new.

In the following sections, we will assume that all represented spaces are countably based $T_0$-topological spaces with admissible representations. Recall from [33, 27] that a representation $\rho$: $\omega^\omega \rightarrow X$ to a topological space $X$ is admissible if $\rho$ is continuous and for any continuous $f$: $\subseteq \omega^\omega \rightarrow X$ there is continuous $F$: $\omega^\omega \rightarrow \omega^\omega$ such that $f = \rho \circ F$. It is well known that a function $f$: $X \rightarrow Y$ between admissibly represented spaces is continuously realizable if and only if it is continuous.

### 3.1 Characterization of $j_\Delta$-realizability

A total function $f$: $X \rightarrow Y$ is $\Delta^0_2$-piecewise continuous if and only if there is a family $\{A_i\}_{i \in \omega}$ of sets in $\Delta^0_2(X)$ such that $X = \bigcup_{i \in \omega} A_i$ and $f|_{A_i}: A_i \rightarrow Y$, the restriction of $f$ to $A_i$, is continuous for all $i \in \omega$.

Let $\omega_\infty$ be the one point compactification of the natural numbers, with $\infty$ the point at infinity. Recall that a function $\xi$: $\omega_\infty \rightarrow X$ is continuous if and only if the sequence $(\xi(i))_{i \in \omega}$ converges to $\xi(\infty)$ in $X$. Given a continuous function $\xi$: $\omega_\infty \rightarrow X$ and $S \subseteq X$, we say that $\xi$ is eventually in $S$ if and only if $\xi(\infty) \in S$ and $\xi(m) \in S$ for all but finitely many $m \in \omega$. We will say that $\xi$ is eventually equal to $x$ for some $x \in X$ if $\xi$ is eventually in the singleton set $\{x\}$, and in this case we will also say that $\xi$ is eventually constant.

Assuming, as we do, that $X$ and $Y$ are countably based, a function $f$: $X \rightarrow Y$ is $\Delta^0_2$-piecewise continuous if and only if there is a $\Delta^0_2$-measurable function $\iota$: $X \rightarrow \omega$ such that for any continuous function $\xi$: $\omega_\infty \rightarrow X$, if $\iota \circ \xi$ is eventually constant then $f \circ \xi$ is continuous. Converting from this definition to the above definition only requires the equivalence between continuity and sequential continuity for countably based spaces, and the generalized $\Sigma^0_2$-reduction principle which allows us to convert a $\Sigma^0_2$-partitioning into a $\Sigma^0_2$-partitioning. We will call the function $\iota$: $X \rightarrow \omega$ above a $\Delta^0_2$-indexing function for $f$. For example, the function $\iota_\Delta$: $\text{dom}(\Delta) \rightarrow \omega$ that maps each $(\xi_n)_{n \in \omega} \in \text{dom}(\Delta)$ to the least $n \in \omega$ satisfying $(\forall m \geq n)[\xi_m = \xi_n]$ is a $\Delta^0_2$-indexing function for $\Delta$.

The next theorem generalizes a result by A. Andretta [2].

**Theorem 17.** Let $f$: $X \rightarrow Y$ be a function between admissibly represented countably based $T_0$-spaces. Then $f$ is $j_\Delta$-realizable if and only if $f$ is $\Delta^0_2$-piecewise continuous.

**Proof.** Let $\rho_X$ be the admissible representation for $X$ and $\rho_Y$ the admissible representation for $Y$. We can assume without loss of generality that $\rho_X$ is an open map and has Polish fibers (i.e., $\rho_X^{-1}(x)$ is Polish for each $x \in X$), and similarly for $\rho_Y$.

Assume $F$: $\subseteq \omega^\omega \rightarrow \omega^\omega$ $j_\Delta$-realizes $f$. Then $\iota' = \iota_\Delta \circ F$ is a $\Delta^0_2$-indexing function for $f \circ \rho_X = \rho_Y \circ \iota_\Delta \circ F$. Since $\iota'$ is $\Delta^0_2$-measurable, we can write $\iota'^{-1}(n) = \bigcup_{i \in \omega} A_i^n$ for suitably chosen closed sets $A_i^n$. Let $\{B_i\}_{i \in \omega}$ be a countable basis for $\omega^\omega$, and define $U^k_{n,i} = \rho_X(B_k) \cap \overline{A_i^n}$ and $V^k_{n,i} = \rho_X(B_k \setminus A_i^n)$. Note that $U^k_{n,i}$ and $V^k_{n,i}$ are open subsets of $X$ by our assumption that $\rho_X$ is an open map.

We first show that each $x \in X$ is in $U^k_{n,i} \setminus V^k_{n,i}$ for some choice of $k, n, i \in \omega$. Since $\rho_X^{-1}(x) \subseteq \bigcup_{n,i \in \omega} A_i^n$, the Baire category theorem implies some $A_i^n$ must have non-empty interior in $\rho_X^{-1}(x)$. Thus there is some $k \in \omega$ such that $B_k \cap \rho_X^{-1}(x) \neq \emptyset$ and $B_k \cap \rho_X^{-1}(x) \subseteq A_i^n \cap \rho_X^{-1}(x)$. It follows that $x \in U^k_{n,i} \setminus V^k_{n,i}$.

Let $\langle \iota(i), \iota' \rangle: \omega^\omega \rightarrow \omega$ be a bijection, and define $\iota$: $X \rightarrow \omega$ so that $\iota(x) = \langle k, n, i \rangle$, where $\langle k, n, i \rangle$ is the least number satisfying $x \in U^k_{n,i} \setminus V^k_{n,i}$. It is immediate that $\iota$ is $\Delta^0_2$-measurable.

Let $\xi$: $\omega_\infty \rightarrow X$ be a continuous function such that $\iota \circ \xi$ is eventually constant. The admissibility of $\rho_X$ implies there is continuous $\xi'$: $\omega_\infty \rightarrow \omega^\omega$ such that $\xi = \rho_X \circ \xi'$. Assume $(\iota \circ \xi)(\infty) = \langle k, n, i \rangle$. Then $\xi$ is eventually in $\rho_X(B_k) \setminus \rho_X(B_k \setminus A_i^n)$, hence $\xi'$ is eventually in $B_k \cap A_i^n$, and it follows that $\iota' \circ \xi'$ is eventually equal to $n$. Since $\iota'$ is a $\Delta^0_2$-indexing function for $f \circ \rho_X$, it follows that $f \circ \xi = f \circ \rho_X \circ \xi'$ is continuous. Therefore, $\iota$ is a $\Delta^0_2$-indexing function for $f$, and we have proven that $f$ is $\Delta^0_2$-piecewise continuous.

For the converse, let $\iota$: $X \rightarrow \omega$ be a $\Delta^0_2$-indexing function for $f$. Then $\iota' = \iota \circ \rho_X$ is a $\Delta^0_2$-indexing function for $f \circ \rho_X$. We can write $\iota'^{-1}(n) = \bigcup_{i \in \omega} A_i^n$ for suitably chosen closed sets $A_i^n$, and we have that $f \circ \rho_X$ restricted to $A_i^n$ is continuous. By the admissibility of $\rho_Y$, there is continuous $F_i^n$: $\subseteq \omega^\omega \rightarrow \omega^\omega$ that realizes the restriction of $f \circ \rho_X$ to $A_i^n$. By relabeling, we can assume that $\{A_i\}_{i \in \omega}$ is a family of closed sets covering the domain of $f \circ \rho_X$, and $F_i$ is a continuous realizor for the restriction of $f \circ \rho_X$ to $A_i$.

The most intuitive way to explain how to \"glue\" together the continuous realizers $F_i$ into a single $j_{\Delta}$-realizer $F$, is to define an algorithm for a Type Two Turing Machine that computes $F$ (possibly with access to some oracle). This description will also help clarify the connections between limit computing with finite mind changes and the $j_\Delta$ jump operators. The reader should consult [47] for more on Type Two Turing Machines, and [48] for an intuitive description of computing with finite mind changes.

The realizor $F$ first initializes a pointer $p := 0$, and begins reading in the input $\xi \in \omega^\omega$. While reading in the input, $F$ attempts to write to its output tape (an encoding of) an infinite sequence of copies of the output of $F_p(\xi)$. In parallel, $F$ will try to determine whether or not $\xi$ really is in $A_p$. If $\xi$ is not in $A_p$, then this will be observed after reading in some finite prefix of $\xi$ because $A_p$ is a
Theorem 20. Let have a strictly decreasing transfinite sequence of closed sets in particular, if and only if

Definition 18. Let be the sequence of elements of extended with infinitely many zeros. This guarantees that realizes a valid encoding of an infinite sequence of elements of as output.

Since covers the domain of after a finite number of “mind changes” the pointer will never be modified again afterwards. Since realizes the restriction of to , we see that the output of converges after a finite number of modifications to the desired output.

3.2 Characterization of \( j_\alpha \)-realizability

In this section we will characterize \( j_\alpha \)-realizability in terms of a hierarchy of discontinuity levels introduced by P. Hertling [21, 20].

Recall that a function \( f: X \to Y \) is continuous at \( x \in X \) iff for any neighborhood \( V \) of \( f(x) \) there is an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq V \). If \( f \) is not continuous at \( x \) then \( f \) is discontinuous at \( x \).

Definition 18 (P. Hertling [21, 20]). Let \( c(\cdot) \) be the closure operator on \( X \) and let \( f: X \to Y \) be a function. For each ordinal \( \alpha \), define \( L_\alpha(f) \) recursively as follows:

1. \( L_0(f) = X \)
2. \( L_{\alpha+1}(f) = c(\{ x \in L_\alpha(f) \mid f|_{L_\alpha(f)} \text{ is discontinuous at } x \}) \)
3. If \( \alpha \) is a limit ordinal, then \( L_\alpha(f) = \bigcap_{\beta < \alpha} L_\beta(f) \).

The level of \( f \), denoted \( Lev(f) \), is defined as \( Lev(f) = \min\{ \alpha \mid L_\alpha(f) = \emptyset \} \) if there exists \( \alpha \) such that \( L_\alpha(f) = \emptyset \), and \( Lev(f) = \infty \), otherwise.

Note that, assuming that \( X \) is countably based, there is some \( \alpha < \omega_1 \) such that \( L_\alpha(f) = L_{\alpha+1}(f) \). This is because we cannot have a strictly decreasing transfinite sequence of closed sets in \( X \) with non-countable order type (see Theorem 6.9 in [24]). In particular, if \( Lev(f) \neq \infty \) then \( Lev(f) < \omega_1 \) when the domain is countably based.

The next definition will provide a convenient characterization of \( Lev(\cdot) \) in terms of “piecewise continuity”.

Definition 19. For each ordinal \( \alpha \) \( (1 \leq \alpha < \omega_1) \), a total function \( f: X \to Y \) is \( D_\alpha \)-piecewise continuous if and only if there is a family \( \{ U_\beta \}_{\beta < \alpha} \) of open subsets of \( X \) such that \( X = \bigcup_{\beta < \alpha} D_\alpha(U_\beta) \) and \( f|_{D_\alpha(U_\beta)}: D_\alpha(U_\beta) \to Y \) is continuous for all \( \beta < \alpha \), where we define \( D_\alpha(U_\beta) = U_\beta \setminus \bigcup_{\gamma < \beta} U_\gamma \).

The following theorem shows that Definitions [18] and [19] describe equivalent hierarchies of discontinuity.

Theorem 20. Let \( X \) and \( Y \) be non-empty countably based \( T_0 \) spaces, and \( f: X \to Y \) a function. Then \( Lev(f) = \alpha \) \((\alpha \neq \infty)\) if and only if \( f \) is \( D_\alpha \)-piecewise continuous and \( f \) is not \( D_\beta \)-piecewise continuous for any \( \beta < \alpha \).

Proof. We divide the proof into two parts. In Part 1, we show that if \( Lev(f) = \alpha \) then \( f \) is \( D_\alpha \)-piecewise continuous. In Part 2, we show that if \( f \) is \( D_\alpha \)-piecewise continuous then \( Lev(f) \leq \alpha \). The theorem clearly follows from these two claims.

(Part 1): First assume that \( Lev(f) = \alpha \). Clearly, \( \alpha \geq 1 \) because \( L_0(f) = X \neq \emptyset \). For \( \beta < \alpha \), define \( U_\beta = X \setminus L_{\beta+1}(f) \). Clearly \( U_\beta \) is open. Note that

\[
\bigcup_{\gamma < \beta} U_\gamma = \bigcup_{\gamma < \beta} (X \setminus L_{\gamma+1}(f)) = X \setminus \bigcap_{\gamma < \beta} L_{\gamma+1}(f) = X \setminus \bigcup_{\gamma < \beta} \bigl( X \setminus L_{\gamma+1}(f) \bigr) \subseteq \bigcup_{\gamma < \beta} U_\gamma \]

where the last equality holds when \( \beta \) is a limit ordinal by definition of \( L_\beta(f) \) and holds when \( \beta \) is a successor by the fact that \( \{ L_{\gamma+1} \}_{\gamma < \beta} \) is a decreasing sequence that ends with \( L_\beta \). It follows that

\[
D_\alpha(U_\beta) = U_\beta \setminus \bigcup_{\gamma < \beta} U_\gamma = (X \setminus L_{\beta+1}(f)) \setminus (X \setminus L_\beta(f)) = L_\beta(f) \setminus L_{\beta+1}(f).
\]
We first show that \( X = \bigcup_{\beta < \alpha} D_\alpha(U_\beta) \). For \( x \in X \), let \( \beta_x = \min \{ \beta \mid x \notin L_\beta(f) \} \). Since \( L_\alpha(f) = \emptyset \) by assumption, we have that \( \beta_x \) is defined and \( \beta_x \leq \alpha \). It is also clear that \( \beta_x \) is a successor ordinal, because if \( \beta_x \) was a limit ordinal then \( x \in L_\gamma(f) \) for all \( \gamma < \beta \) (by our choice of minimal \( \beta_x \)) hence \( x \in L_\beta(f) \) (by definition of \( L_\beta \), for limit \( \beta_x \)), a contradiction. Therefore, \( \beta_x = \gamma_x + 1 \) for some ordinal \( \gamma_x < \alpha \). It follows that \( x \in D_\alpha(U_{\gamma_x}) \), hence \( X = \bigcup_{\beta < \alpha} D_\alpha(U_\beta) \).

It only remains to show that \( f|_{D_\alpha(U_\beta)} \) is continuous for all \( \beta < \alpha \). Assume for a contradiction that \( f|_{D_\alpha(U_\beta)} \) is discontinuous at some point \( x \). Since \( D_\alpha(U_\beta) \) is a subspace of \( L_\beta(f) \), it must be the case that \( f|_{L_\beta(f)} \) is also discontinuous at \( x \). Therefore, \( x \in L_{\beta + 1}(f) \), contradicting \( x \in D_\alpha(U_\beta) \). Thus, \( f|_{D_\alpha(U_\beta)} \) is continuous for all \( \beta < \alpha \).

(Part 2): Let \( \{ U_\gamma \}_{\gamma < \alpha} \) be open subsets of \( X \) such that \( X = \bigcup_{\gamma < \alpha} D_\alpha(U_\gamma) \) and \( f|_{D_\alpha(U_\beta)} : D_\alpha(U_\beta) \to Y \) is continuous for all \( \beta < \alpha \). We can assume without loss of generality that \( \bigcup_{\gamma < \alpha} U_\gamma \subseteq U_\beta \) for all \( \beta < \alpha \).

We claim that \( L_{\beta + 1}(f) \subseteq X \setminus U_\beta \) for all \( \beta < \alpha \). The case \( \beta = 0 \) is easy, so assume that \( \beta > 0 \) and the claim holds for all \( \gamma < \beta \). First note that, since \( f|_{D_\alpha(U_\beta)} = f|_{U_\alpha \setminus \gamma < \beta U_\gamma} \) is continuous by assumption, and since \( U_\beta \setminus \gamma < \beta U_\gamma \) is an open subspace of \( X \setminus \gamma < \beta U_\gamma \), \( f|_{X \setminus \gamma < \beta U_\gamma} \) is continuous at every \( x \in U_\beta \). Now assume that \( f|_{L_{\beta + 1}(f)} \) is discontinuous at \( x \). By induction hypothesis,

\[
L_{\beta}(f) = \bigcap_{\gamma < \beta} L_{\gamma + 1}(f) \subseteq \bigcap_{\gamma < \beta} X \setminus U_\gamma = X \setminus \bigcup_{\gamma < \beta} U_\gamma
\]

so it follows that \( f|_{X \setminus \gamma < \beta U_\gamma} \) is discontinuous at \( x \). Therefore, \( x \notin U_\beta \).

\[
\{ x \in L_{\beta}(f) \mid f|_{L_{\beta}(f)} \text{ is discontinuous at } x \} \subseteq X \setminus U_\beta
\]

It follows that \( L_{\beta + 1}(f) \subseteq X \setminus U_\beta \) or \( f|_{L_{\beta + 1}(f)} \) is closed. This concludes the proof of the claim.

Since \( \{ X \setminus U_\beta \}_{\beta < \alpha} \) is a decreasing sequence of closed sets and \( \bigcap_{\beta < \alpha} (X \setminus U_\beta) = \emptyset \), the claim implies that \( L_\alpha(f) = \emptyset \), hence \( \text{Lev}(f) \leq \alpha \).

For each countable ordinal \( \alpha \), we let \( \alpha^{op} \) denote the topological space whose points are the ordinals less than \( \alpha \) and whose open sets are generated from the sets \( \downarrow \beta = \{ \gamma \mid \gamma \leq \beta \} \) for each \( \beta < \alpha \).

Let \( f : X \to Y \) be a function between countably based spaces, and let \( \alpha \) be a countable ordinal. An \( \alpha \)-indexing function for \( f \) is a continuous function \( \iota : X \to \alpha^{op} \) such that for any continuous function \( \xi : \omega_{\omega} \to X \), if \( \iota \circ \xi \) is eventually constant then \( f \circ \xi \) is continuous.

The existence of an \( \alpha \)-indexing function is a necessary and sufficient condition for a function to be \( D_\alpha \)-piecewise continuous. If \( f \) is \( D_\alpha \)-piecewise continuous, then the function \( \iota \) mapping \( D_\alpha(U_\beta) \) to \( \beta \) is an \( \alpha \)-indexing function for \( f \). Conversely, if \( \iota \) is an \( \alpha \)-indexing function for \( f \), then defining \( U_\beta = \iota^{-1}(\downarrow \beta) \) for \( \beta < \alpha \) clearly, \( U_\beta \) is an open subset of \( X \) because \( \rho_X \) is an open map. Finally, define \( \iota : X \to \alpha^{op} \) so that \( x \mapsto \min \{ \beta \mid x \in U_\beta \} \). It is easy to see that \( \iota \) is a well-defined total function. For each ordinal \( \beta < \alpha \), \( \iota^{-1}(\downarrow \beta) = \bigcup_{\gamma \leq \beta} U_\gamma \), hence \( \iota \) is continuous.

Let \( \xi : \omega_{\omega} \to X \) be an open function such that \( \iota \circ \xi \) is eventually constant. The admissibility of \( \rho_X \) implies that \( \rho_X \) is continuous such that \( \xi = \rho_X \circ \iota \). Assume \( \iota(\xi(\omega_{\omega})) = \beta \). Then \( \xi \) is eventually in \( D_\alpha(U_\beta) \), hence \( \xi' \) is eventually in \( \iota^{-1}(\downarrow \beta) \cup \bigcup_{\gamma < \beta} \iota^{-1}(\downarrow \gamma) \). It follows that \( \iota \circ \xi' \) is eventually equal to \( \beta \). Since \( \iota \) is an \( \alpha \)-indexing function for \( f \circ \xi \), it follows that \( f \circ \xi = f \circ \rho_X \circ \iota' \) is continuous. Therefore, \( \iota \) is an \( \alpha \)-indexing function for \( f \), and we have proven that \( f \) is \( D_\alpha \)-piecewise continuous.

For the converse, we will use the same method as in the proof of Theorem[17] and define an oracle Type Two Turing Machine that computes a \( j_\alpha \)-realizer for \( f \). Let \( \iota : X \to \omega \) be an \( \alpha \)-indexing function for \( f \). Then \( \iota' = \iota \circ \rho_X \) is an \( \alpha \)-indexing function for \( f \circ \rho_X \). Let \( U_\beta = \iota'^{-1}(\downarrow \beta) \) for \( \beta < \alpha \), and we have that \( f \circ \rho_X \) restricted to \( D_\alpha(U_\beta) \) is continuous. By the admissibility of \( \rho_Y \), there is continuous \( F_\beta : \omega_{\omega} \to \omega_{\omega} \) that realizes the restriction of \( f \circ \rho_X \) to \( D_\alpha(U_\beta) \).

Our algorithm is as follows. Begin reading in the input \( \xi \in \omega_{\omega} \), and search in parallel for \( \beta < \alpha \) such that \( \xi \in U_\beta \). Such a \( \beta \) can be found after reading in a finite prefix of \( \xi \) because each \( U_\beta \) is open and the \( U_\beta \) cover the domain of \( f \circ \rho_X \). The algorithm then initializes an ordinal counter \( \beta := \beta \) and attempts to write to the output tape an infinite sequence of copies of the element \( \langle \beta \rangle_\alpha \circ F_\beta(\xi) \). While outputting the copies of \( \langle \beta \rangle_\alpha \circ F_\beta(\xi) \) the algorithm continues to search for some \( \gamma < \beta \) such that \( \xi \in U_\gamma \). If
such a γ is ever found, then the algorithm sets \( \hat{\beta} := \gamma \) and begins outputting an infinite sequence of copies of \( \langle \hat{\beta} \rangle_\alpha \odot F_{\hat{\beta}}(\xi) \) for the new value of \( \hat{\beta} \). It is easy to see that such an algorithm computes a \( j_\alpha \)-realizer for \( f \). \( \square \)

4 Examples and applications

In this last section of this paper we provide a few examples and applications of \( j_\Delta \) and \( j_\alpha \)-realizability.

4.1 The Difference Hierarchy

Given a jump operator \( j \) and a represented space \( X \), recall that \( \Delta_j(X) \) is the set of \( j \)-realizable functions from \( X \) into the discrete two point space \( 2 = \{0, 1\} \). In this section, we will show that \( \Delta_{j_\alpha}(X) (1 \leq \alpha < \omega_1) \) correspond to the ambiguous levels of the difference hierarchy when \( X \) is a countably based space.

Definition 22. Any ordinal \( \alpha \) can be expressed as \( \alpha = \beta + n \), where \( \beta \) is a limit ordinal or 0, and \( n < \omega \). We say that \( \alpha \) is even if \( n \) is even, and odd, otherwise. For any ordinal \( \alpha \), let \( r(\alpha) = 0 \) if \( \alpha \) is even, and \( r(\alpha) = 1 \), otherwise. For any ordinal \( \alpha \), define

\[
D_\alpha(\{A_\beta\}_{\beta < \alpha}) = \bigcup \{A_\beta \setminus (\bigcup_{\gamma < \beta} A_\gamma) \mid \beta < \alpha, r(\beta) \neq r(\alpha)\},
\]

where \( \{A_\beta\}_{\beta < \alpha} \) is a sequence of sets such that \( A_\gamma \subseteq A_\beta \) for all \( \gamma < \beta < \alpha \).

For any topological space \( X \) and ordinal \( \alpha \), define \( \Sigma^{-1}_\alpha(X) \) to be the set of all sets \( D_\alpha(\{U_\beta\}_{\beta < \alpha}) \), where \( \{U_\beta\}_{\beta < \alpha} \) is an increasing sequence of open subsets of \( X \).

The following connection with the difference hierarchy has already been observed by both P. Hertling and V. Selivanov, so we omit the proof.

Proposition 23 (see [38]). If \( X \) is a countably based \( T_0 \)-space and \( 1 \leq \alpha < \omega_1 \), then a total function \( f : X \to 2 \) is \( j_\alpha \)-realizable if and only if both \( f^{-1}(1) \) and \( f^{-1}(0) \) are in \( \Sigma^{-1}_\alpha(X) \). \( \square \)

4.2 Cantor-Bendixson Rank

A limit point of a topological space is a point that is not isolated, i.e. a point \( x \) such that for every open \( U \) containing \( x \) there is \( y \in U \) distinct from \( x \). A space is perfect if all of its points are limit points.

Definition 24 (see [24]). For any topological space \( X \), let

\[
X' = \{x \in X \mid x \text{ is a limit point of } X\}.
\]

For ordinal \( \alpha \), define \( X^{(\alpha)} \) recursively as follows:

1. \( X^{(0)} = X \),
2. \( X^{(\alpha+1)} = (X^{(\alpha)})' \),
3. If \( \alpha \) is a limit ordinal, then \( X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)} \).

If \( X \) is countably based, then there is a least countable ordinal \( \alpha_0 \) such that \( X^{(\alpha)} = X^{(\alpha_0)} \) for all \( \alpha \geq \alpha_0 \). Such \( \alpha_0 \) is called the Cantor-Bendixson rank of \( X \), and is denoted \( |X|_{CB} \). We also let \( X^\infty = X^{(|X|_{CB})} \), which is a perfect subset of \( X \). \( \square \)

Assuming \( X \) is countably based, \( X \setminus X^\infty \) must be countable. This is because for every \( x \in (X \setminus X') \) there must be a (basic) open \( U \) containing \( x \) and no other elements of \( X \), so \( (X \setminus X') \setminus X' \) must be countable.

Let \( \omega_\perp = \omega \cup \{\perp\} \) be such that \( \{n\} \) is open for \( n \in \omega \) and the only open set containing \( \perp \) is \( \omega_\perp \) itself. Given countably based \( X \), define \( p : X \to \omega_\perp \) so that \( p(x) = \perp \) for \( x \in X^\infty \) and \( p \) restricted to the elements of \( X \setminus X^\infty \) is injective into \( \omega \).

The following is a generalization of a result by Luo and Schulte [26] concerning ordinal mind-change complexity of inductive inference (see also [13,14]).

Proposition 25. For any countably based space \( X \), \( p : X \to \omega_\perp \) is \( j_\alpha \)-realizable, where

1. \( \alpha = |X|_{CB} \) if \( X^\infty = \emptyset \) or \( |X|_{CB} \) is a successor ordinal
2. \( \alpha = |X|_{CB} + 1 \) if \( X^\infty \neq \emptyset \) and \( |X|_{CB} \) is a limit ordinal
Proof. If \( X^\infty = \emptyset \), then define \( U_\beta = X \setminus (X^{(\beta)})' \) for \( \beta < |X|_CB \). Letting \( \alpha = |X|_CB \), we see that \( D_\alpha(U_\beta) = X^{(\beta)} \setminus (X^{(\beta)})' \), which is the set of isolated points of \( X^{(\beta)} \), hence a discrete subspace of \( X \).

If \( X^\infty \neq \emptyset \) but \( |X|_CB = \gamma + 1 \), then set \( U_\beta = X \setminus (X^{(\beta)})' \) for \( \beta < \gamma \) and \( U_\gamma = X \). Letting \( \alpha = |X|_CB \), \( D_\alpha(U_\beta) = X^{(\beta)} \setminus (X^{(\beta)})' \) for \( \beta < \gamma \) and \( D_\alpha(U_\gamma) = X^\infty \).

If \( X^\infty \neq \emptyset \) and \( |X|_CB \) is a limit ordinal, then set \( U_\beta = X \setminus (X^{(\beta)})' \) for \( \beta < |X|_CB \) and \( U_\gamma = X \). Letting \( \alpha = |X|_CB + 1 \), \( D_\alpha(U_\beta) = X^{(\beta)} \setminus (X^{(\beta)})' \) for \( \beta < |X|_CB \) and \( D_\alpha(U_{|X|_CB}) = X^\infty \).

In all three of the above cases, it is easy to see that \( U_\beta (\beta < \alpha) \) is open, \( X = \bigcup_{\beta < \alpha} D_\alpha(U_\beta) \), and the corresponding restrictions of \( p \) are continuous. \( \square \)

### 4.3 Hilbert’s basis theorem

Let \( (R, +, \cdot) \) be a commutative ring. A subset \( I \subseteq R \) is an ideal if and only if \( (I, +) \) is a subgroup of \( (R, +) \) and \( (\forall x \in I)(\exists r \in R)[x \cdot r \in I] \). A ring is Noetherian if and only if it does not have an infinite strictly ascending chain of ideals.

Given a ring \( R \), we let \( R[x_1, \ldots, x_n] \) denote the ring of polynomials with coefficients in \( R \) and \( n \) indeterminates \( x_1, \ldots, x_n \).

A famous theorem by David Hilbert states that if \( R \) is a Noetherian ring then \( R[x_1, \ldots, x_n] \) is also Noetherian. Hilbert’s proof was non-constructive, and was initially criticized by Paul Gordan with the famous quote “Das ist nicht Mathematik. Das ist Theologie.”

Here we quantify one aspect of the “non-constructiveness” of the basis theorem in terms of the level of discontinuity of converting an enumeration of an ideal into a Gröbner basis for ideals. The approach is much in the same spirit as V. Brattka’s project to quantify the non-computability of mathematical theorems in terms of their Weihrauch degrees (see, for example, \( \{9\} \{8\} \{7\} \)). Our contribution is only in the way that we formalize the problem, and our main result is essentially a reformulation of results on Hilbert’s basis theorem by S. Simpson \( \{41\} \) in the context of reverse mathematics, and by F. Stephan and Y. Ventsov \( \{44\} \) in the context of inductive inference.

Let \( \mathbb{Q} \) be the ring of rational numbers, and let \( \mathcal{I}_n \) be the set of ideals of the polynomial ring \( \mathbb{Q}[x_1, \ldots, x_n] \). If we encode the elements of \( \mathbb{Q}[x_1, \ldots, x_n] \) as elements of \( \omega \), then each element of \( \omega^{\omega} \) can be interpreted as an infinite sequence of elements of \( \mathbb{Q}[x_1, \ldots, x_n] \). We will interpret \( \mathcal{I}_n \) as a representable space with the representation \( \rho : \omega^{\omega} \to \mathcal{I}_n \) that maps each enumeration of an ideal \( I \in \mathcal{I}_n \) to \( I \). This representation is admissible with respect to the topology on \( \mathcal{I}_n \) generated by \( \uparrow (r) = \{ I \in \mathcal{I}_n \mid r \in I \} \), where \( r \) varies over elements of \( \mathbb{Q}[x_1, \ldots, x_n] \). Note that this topology is very far from being Hausdorff.

Let \( \mathcal{G}_n \) be the set of finite subsets of \( \mathbb{Q}[x_1, \ldots, x_n] \). We will think of \( \mathcal{G}_n \) as the set of Gröbner bases for ideals in \( \mathcal{I}_n \) (for some predefined monomial order). We think of each Gröbner basis in \( \mathcal{G}_n \) as being represented by a finite terminated string, hence \( \mathcal{G}_n \) carries the discrete topology.

Let \( f_n : \mathcal{I}_n \to \mathcal{G}_n \) be the function that maps each \( I \in \mathcal{I}_n \) to its unique Gröbner basis. Intuitively, \( f_n \) embodies the problem of converting an infinite enumeration of an ideal of \( \mathbb{Q}[x_1, \ldots, x_n] \) into a finite Gröbner basis for the ideal.

The next theorem immediately follows from work by S. Simpson \( \{41\} \) and F. Stephan and Y. Ventsov \( \{44\} \), so we omit the proof.

**Theorem 26.** The functions \( f_n : \mathcal{I}_n \to \mathcal{G}_n \) are \( j_{\omega^n} \)-realizable for each \( n \in \omega \). In fact, \( \text{Lev}(f_n) = \omega^n \).

Hilbert’s basis theorem holds for all \( n \in \omega \), so it is natural to consider the function \( \forall_n f_n \) corresponding to universal quantification over \( \omega \). The most natural interpretation for such a function is to simply take the disjoint union of all of the \( f_n \). Then \( \forall_n f_n \) essentially takes some \( n \in \omega \) as initial input, and then operates like \( f_n \) thereafter. It is easy to see that \( \text{Lev}(\forall_n f_n) = \omega^{\omega} \), which is consistent with S. Simpson’s \( \{41\} \) characterization of Hilbert’s basis theorem.

### 4.4 \( \Delta^0_2 \)-measurable functions and the Jayne-Rogers theorem

Recall that a function \( f : X \to Y \) is \( \Delta^0_2 \)-measurable if and only if \( f^{-1}(U) \in \Delta^0_2(X) \) for each open \( U \subseteq Y \). Note that this is equivalent to requiring that \( f^{-1}(A) \in \Sigma^0_2(X) \) for each \( A \in \Sigma^0_2(Y) \).

The following is a slight generalization of a theorem by J. E. Jayne and C. A. Rogers \( \{22\} \). A much simpler proof of the original theorem was given by L. Motto Ros and B. Semmes \( \{29\} \{23\} \). The original version of the Jayne-Rogers theorem only applied to functions that had a metrizable domain.

In the following, an analytic space is a topological space that has an admissible representation with analytic domain. For countably based \( T_0 \)-spaces, this is easily seen to be equivalent to the space being homeomorphic to an analytic subset of a quasi-Polish space \( \{11\} \).

**Theorem 27 (Jayne and Rogers).** Assume \( X \) is an analytic countably based \( T_0 \)-space and \( Y \) is a separable metrizable space. Then a function \( f : X \to Y \) is \( \Delta^0_2 \)-measurable if and only if it is \( \Delta^0_2 \)-piecewise continuous.

**Proof.** Let \( \rho_X \) be an admissible representation of \( X \) with analytic domain. Then \( f \circ \rho_X \) is a \( \Delta^0_2 \)-measurable function from an analytic metrizable space into a metrizable space, hence \( f \circ \rho_X \) is \( \Delta^0_2 \)-piecewise continuous by the original Jayne-Rogers theorem.
Example 28. We consider two topologies on the ordinal \( \omega + 1 = \{0, 1, 2, \ldots, \omega \} \). The first topology, \( \tau_1 \), is the Scott-topology, and is generated by the open sets \( \uparrow n = \{ \beta \in \omega + 1 | n \leq \beta \} \) for \( n < \omega \). The second topology, \( \tau_2 \), is defined so that a non-empty subset \( U \subseteq \omega + 1 \) is open if and only if \( \omega \in U \) and all but finitely many \( n \in \omega \) are in \( U \). The topological space \( (\omega + 1, \tau_1) \) is one of the simplest examples of an infinite \( \omega \)-continuous domain \([17]\). The topological space \( (\omega + 1, \tau_2) \) is homeomorphic to the Zariski topology on the prime spectrum of the ring of integers. Both \( \tau_1 \) and \( \tau_2 \) are quasi-Polish topologies on \( \omega + 1 \). Furthermore, in both of these spaces the singleton set \( \{ \omega \} \) is \( \Pi^0_2 \) but not \( \Sigma^0_2 \), hence these spaces fail the \( T_\Delta \)-separation axiom of Aull and Thron \([3]\) (see also \([11, 14]\) for more on the \( T_\Delta \)-axiom).

The function \( f : (\omega + 1, \tau_1) \to (\omega + 1, \tau_2) \), defined to behave as the identity on \( \omega + 1 \), is a \( \Delta^0_1 \)-measurable function that is not \( \Delta^0_2 \)-piecewise continuous. Since \( (\omega + 1, \tau_1) \) is quasi-Polish, it follows from \([11]\) that it has a total admissible representation \( \rho : \omega^\omega \to (\omega + 1, \tau_1) \). As any \( j_\Delta \)-realizer of \( f \circ \rho \) would be a \( j_\Delta \)-realizer of \( f \), we see that the function \( f \circ \rho : \omega^\omega \to (\omega + 1, \tau_2) \) is an example of a function with Polish domain which is \( \Delta^0_2 \)-measurable but not \( \Delta^0_2 \)-piecewise continuous.

Let \( F \) be a class of functions between (admissibly represented) topological spaces. We will say that a jump operator \( j \) captures the class \( F \) if it holds that \( f \in F \) if and only if \( f \) is \( j \)-realizable. Note that such a \( j \) must be in \( F \) because \( j \) is trivially \( j \)-realizable.

If we let \( F \) be the class of \( \Delta^0_2 \)-measurable functions with (countably based) analytic domain and metrizable codomain, then the Jayne-Rogers theorem states that \( j_\Delta \) captures \( F \). However, the example above shows that \( j_\Delta \) does not capture the class of \( \Delta^0_2 \)-measurable functions between arbitrary countably based \( T_0 \)-spaces. One might wonder if some other jump operator might capture this larger class, but the following result shows that this is not possible.

Proposition 29. There is no jump operator that captures the entire class of \( \Delta^0_2 \)-measurable functions between countably based \( T_\Delta \)-spaces.

Proof. Assume for a contradiction that \( j \) captures the entire class of \( \Delta^0_2 \)-measurable functions between countably based \( T_\Delta \)-spaces. Let \( f : (\omega + 1, \tau_1) \to (\omega + 1, \tau_2) \) be the \( \Delta^0_2 \)-measurable function from Example 28. We can assume that \( (\omega + 1, \tau_1) \) has a total admissible representation, hence there is a total continuous \( F : \omega^\omega \to \omega^\omega \) that \( j \)-realizes \( f \). The Jayne-Rogers theorem now implies that the \( \Delta^0_2 \)-measurable function \( j \circ F : \omega^\omega \to \omega^\omega \) has a continuous \( j_\Delta \)-realizer \( F' : \omega^\omega \to \omega^\omega \). However, \( F' \) would then be a continuous \( j_\Delta \)-realizer of \( f \), which is a contradiction.

By applying the same argument to the function \( f \circ \rho \) from Example 28 it can be seen that the above proposition holds true even if we further restrict to \( \Delta^0_2 \)-measurable functions with Polish domain.

4.5 A generalization of the Hausdorff-Kuratowski theorem

The Hausdorff-Kuratowski theorem (see \([24]\)) states that the difference hierarchy on a Polish space exhausts all of the \( \Delta^0_2 \)-sets. The full version of the theorem actually applies to all levels of the Borel hierarchy. It was observed by V. Selivanov \([36, 39]\) that the Hausdorff-Kuratowski theorem holds for some important non-metrizable spaces such as \( \omega \)-continuous domains. Later it was shown that the full version of the Hausdorff-Kuratowski theorem holds for all quasi-Polish spaces \([11]\).

In addition to extending the Hausdorff-Kuratowski theorem to a more general class of spaces, V. Selivanov has also generalized the theorem from being a classification of sets to a classification of functions \([36]\). In particular, it was observed in \([36]\) that each \( \Delta^0_2 \)-measurable function \( f \) from a Polish space into a finite discrete space will satisfy \( \text{Lev}(f) = \alpha \) for some \( \alpha < \omega_1 \). In this section, we will extend this result to show that any \( \Delta^0_2 \)-measurable function \( f \) from a quasi-Polish space to a separable metrizable space will satisfy \( \text{Lev}(f) = \alpha \) for some \( \alpha < \omega_1 \). Given the connections between P. Hertling’s levels of discontinuity and the difference hierarchy, our result is a very broad generalization of the Hausdorff-Kuratowski theorem restricted to \( \Delta^0_2 \)-sets. L. Motto Ros \([27]\) has independently made a similar observation for \( \Delta^0_2 \)-measurable functions on metrizable spaces.

As in the original proof of the Hausdorff-Kuratowski theorem, the Baire category theorem plays an important role in our generalized result as well. One version of the Baire category theorem states that if a Polish space is equal to the union of a countable family of closed sets, then one of the closed sets must have non-empty interior. Clearly, the same statement holds for Polish spaces if we replace “closed” by either “\( F_\alpha \)” or “\( \Sigma^0_\alpha \)”. However, since the equivalence between \( F_\alpha \)-sets and \( \Sigma^0_\alpha \)-sets breaks down for

\( ^2 \)Luca Motto Ros has pointed out to the author that the proof in \([23]\) suggests regularity is a sufficient criterion on the codomain for Theorem 27 to hold, even in the absence of separability and metrizability.
non-metrizable spaces, the version of the Baire category theorem presented in the following lemma is more appropriate in general. This generalization of the Baire category theorem has already been investigated by R. Heckmann [19] and by V. Becher and S. Grigorieff [5].

**Lemma 30.** Assume $X$ is quasi-Polish and $\{A_i\}_{i \in \omega}$ is a family of sets from $\Sigma^0_2(X)$ such that $X = \bigcup_{i \in \omega} A_i$. Then there is $i \in \omega$ such that $A_i$ has non-empty interior. Equivalently, the intersection of a countable family of dense $\Pi^0_2$-subsets of a quasi-Polish space is a dense $\Pi^0_2$-set.

**Proof.** Let $f: \omega^\omega \to X$ be an open continuous surjection (see [11]) and let $B_i = f^{-1}(A_i)$. Each $B_i$ is a $\Sigma^0_2$-subset of a metrizable space hence equal to a countable union of closed sets. Since $\omega^\omega = \bigcup_{i \in \omega} B_i$, the Baire category theorem for Polish spaces implies there is $i \in \omega$ such that $B_i$ has non-empty interior. It follows that $A_i$ has non-empty interior because $f$ is an open map. $\square$

**Theorem 31.** If $X$ is quasi-Polish and $Y$ is a countably based $T_0$-space, then $f: X \to Y$ is $j_\Delta$-realizable if and only if it is $j_\alpha$-realizable for some $\alpha < \omega_1$.

**Proof.** Assume $f: X \to Y$ is $j_\Delta$-realizable. Let $\alpha < \omega_1$ be the least ordinal such that $L_\alpha(f) = L_{\alpha+1}(f)$, which exists because $X$ is countably based. Assume for a contradiction that $L_\alpha(f) \neq \emptyset$. Clearly, $f|\{L_\alpha(f)\}$ is $j_\Delta$-realizable, hence there is a $\Delta^0_2$-partitioning $\{A_i\}_{i \in \omega}$ of $L_\alpha(f)$ such that the restriction of $f$ to $A_i$ is continuous for each $i \in \omega$. Note that $L_\alpha(f)$ is quasi-Polish because it is a closed subset of the quasi-Polish space $X$. Therefore, Lemma 30 applies and there is $i \in \omega$ such that $A_i$ has non-empty interior relative to $L_\alpha(f)$. But then $f|\{L_\alpha(f)\}$ is continuous on a non-empty open subset of $L_\alpha(f)$, contradicting our assumption that $L_\alpha(f) = L_{\alpha+1}(f)$. Thus, $L_\alpha(f)$ is empty and it follows that $L_{\alpha+1}(f) = \alpha < \omega_1$.

The converse holds for all represented spaces because $j_\alpha \leq j_\Delta$. $\square$

Combining the above result with the Jayne-Rogers theorem yields the following generalization of the Hausdorff-Kuratowski theorem.

**Theorem 32.** If $X$ is quasi-Polish and $Y$ is a separable metrizable space, then $f: X \to Y$ is $\Delta^0_2$-measurable if and only if it is $j_\alpha$-realizable for some $\alpha < \omega_1$. $\square$

5 Conclusions

Although much of classical descriptive set theory has been extended to arbitrary countably based $T_0$-spaces, it is still a major open problem to understand how descriptive set theory should work for non-countably based topological spaces and more general represented spaces. This is a very strange realm, where even singleton sets can have complexity of arbitrarily high rank in the projective hierarchy.

The approach we have taken here with jump operators provides a general framework, with a nice categorical flavor, for which to extend descriptive set theory to more general mathematical structures. In particular, it raises natural questions concerning the structure and applications of “higher-order” descriptive set theoretical objects such as $\Sigma^0_2(\Sigma^0_2(X))$.

There is also a strong need for a refined analysis of the categorical logic of the category of represented spaces and realizability functions with closer attention to the “level” of the represented spaces. For example, the “naive Cauchy” representation of the real numbers in [4], which is obtained via a kind of double negation of the standard Cauchy representation of the reals, happens to be equivalent to the $j_{\Sigma^0_2}$-jump of the standard Cauchy representation of reals [49]. S. Hayashi [18] has also investigated connections between limit-computability and non-constructive principles such as double negation elimination and the excluded middle restricted to certain subclasses of formulae. It would be very interesting to see how these concepts are connected.

References

[1] J. Addison, *Tarski’s theory of definability: common themes in descriptive set theory, recursive function theory, classical pure logic, and finite-universe logic*, Annals of Pure and Applied Logic 126 (2004), 77–92.

[2] A. Andretta, *More on Wadge determinacy*, Annals of Pure and Applied Logic 144 (2006), 2–32.

[3] C. E. Aull and W. J. Thron, *Separation axioms between T₀ and T₁*, Indag. Math. 24 (1963), 26–37.

[4] A. Bauer, *Realizability as the connection between computable and constructive mathematics*, Second International Conference on Computability and Complexity in Analysis (CCA 2005), 2005.

[5] V. Becher and S. Grigorieff, *Borel and Hausdorff hierarchies in topological spaces of Choquet games and their effectiviation*, (submitted), 2012.
[6] V. Brattka, *Effective Borel measurability and reducibility of functions*, Mathematical Logic Quarterly **51** (2005), 19–44.

[7] V. Brattka, M. de Brecht, and A. Pauly, *Closed choice and a uniform low basis theorem*, Annals of Pure and Applied Logic **163**, no. 8.

[8] V. Brattka and G. Gherardi, *Effective choice and boundedness principles in computable analysis*, The Bulletin of Symbolic Logic **17** (2011), 73–117.

[9] V. Brattka and M. Makananise, *Limit computable functions and subsets*, (unpublished notes).

[10] M. de Brecht, *Quasi-Polish spaces*, Annals of Pure and Applied Logic **164** (2013), no. 3, 356 – 381.

[11] V. Brattka and A. Pauly, *Closed choice and a uniform low basis theorem*, Annals of Pure and Applied Logic **163**, no. 8.

[12] V. Brattka and G. Gherardi, *Effective choice and boundedness principles in computable analysis*, The Bulletin of Symbolic Logic **17** (2011), 73–117.

[13] M. Schröder, *Extended admissibility*, Theoretical Computer Science **284** (2002), 519–538.
[34] D. S. Scott, *Data types as lattices*, SIAM Journal on Computing 5 (1976), 522–587.

[35] V. Selivanov, *Index sets in the hyperarithmetical hierarchy*, Siberian Mathematical Journal 25 (1984), 474–488.

[36] ______, *Difference hierarchy in \( \phi \)-spaces*, Algebra and Logic 43 (2004), 238–248.

[37] ______, *Towards a descriptive set theory for domain-like structures*, Theoretical Computer Science 365 (2006), 258–282.

[38] ______, *Hierarchies of \( \Delta^0_2 \)-measurable \( k \)-partitions*, Math. Logic Quarterly 53 (2007), 446–461.

[39] ______, *On the difference hierarchy in countably based \( T_0 \)-spaces*, Electronic Notes in Theoretical Computer Science 221 (2008), 257–269.

[40] B. Semmes, *A game for the Borel functions*, Ph.D. thesis, Universiteit van Amsterdam, 2006.

[41] S. Simpson, *Ordinal numbers and the Hilbert basis theorem*, Journal of Symbolic Logic 53 (1988), 961–974.

[42] S. Solecki, *Covering analytic sets by families of closed sets*, The Journal of Symbolic Logic 59 (1994), no. 3, 1022–1031.

[43] ______, *Decomposing Borel sets and functions and the structure of Baire class 1 functions*, Journal of the Amer. Math. Soc. 11 (1998), no. 3, 521–550.

[44] F. Stephan and Y. Ventsov, *Learning algebraic structures from text*, Theoretical Computer Science 268 (2001), 221–273.

[45] A. Tang, *Chain properties in \( P(\omega) \)*, Theoretical Computer Science 9 (1979), 153–172.

[46] ______, *Wadge reducibility and Hausdorff difference hierarchy in \( P(\omega) \)*, Lecture Notes in Mathematics 871 (1981), 360–371.

[47] K. Weihrauch, *Computable analysis*, Springer-Verlag, 2000.

[48] M. Ziegler, *Revising type-2 computation and degrees of discontinuity*, Electronic Notes in Theoretical Computer Science 167 (2007), 255–274.