A Deep Conditioning Treatment of Neural Networks

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Abstract

We study the role of depth in training randomly initialized overparameterized neural networks. We give the first general result showing that depth improves trainability of neural networks by improving the conditioning of certain kernel matrices of the input data. This result holds for arbitrary non-linear activation functions, and we provide a characterization of the improvement in conditioning as a function of the degree of non-linearity and the depth of the network. We provide versions of the result that hold for training just the top layer of the neural network, as well as for training all layers, via the neural tangent kernel. As applications of these general results, we provide a generalization of the results of Das et al. (2019) showing that learnability of deep random neural networks with arbitrary non-linear activations (under mild assumptions) degrades exponentially with depth. Additionally, we show how benign overfitting can occur in deep neural networks via the results of Bartlett et al. (2019b).

1. Introduction

Deep neural networks have enjoyed tremendous empirical success, and theory is starting to emerge which attempts to explain this success. A sequence of papers has recently shown the benefits of overparametrization via large width for training neural networks: see, for example, (Li and Liang, 2018; Du et al., 2019; Allen-Zhu et al., 2019; Zou and Gu, 2019) and the references therein. These papers show that with sufficiently large width, starting from a random initialization of the network weights, gradient descent provably finds a global minimizer of the loss function on the training set.

While several of the aforementioned papers do analyze deep neural networks, to our knowledge, there is no prior work that provably demonstrates the benefits of depth for training neural networks in general settings. Prevailing wisdom is that while depth enables the neural network to express more complicated functions (see, for example, (Eldan and Shamir, 2016; Telgarsky, 2016; Raghu et al., 2017; Lee et al., 2017; Daniely, 2017a) and the references therein), it hinders efficient training, which is the primary concern in this paper. Indeed, the papers mentioned earlier showing convergence of gradient descent either assume very shallow (one hidden layer) networks, or expend considerable effort to show that depth doesn’t degrade training by more than a polynomial factor. A few exceptions are the papers (Arora et al., 2018b, 2019a) which do show that depth helps in training neural networks, but are restricted to very specific problems with linear activations. See Section A for an in-depth discussion of these and other related works.

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In this paper, we provide general results showing how depth improves trainability of neural networks by improving the conditioning of certain kernel matrices of the input data. Recent developments (Jacot et al., 2018; Yang, 2019; Arora et al., 2019c) have shown that training wide, randomly initialized neural networks is effectively a kernel method, and thus a convex optimization problem. It is well-known that the rate of convergence of gradient descent depends crucially on the condition number (or related quantities, such as smoothness or strong convexity) of the function being minimized, and in the case of kernel methods, these quantities are directly related to the eigenvalues of the kernel matrix of the input data.

Our main result is that for a randomly initialized neural network with an arbitrary non-linear activation function, the condition number of the appropriate kernel matrices tend to the best possible value, 1, exponentially fast in the depth of the network. This result holds under very mild conditions on the input data, and a suitable normalization of the activation function. The rate at which the condition number tends to 1 is determined by a coefficient of non-linearity of the activation function, a concept that we define in this paper.

We then apply our main result to show that when training large width neural networks of sufficient depth, gradient descent with square loss approaches $\epsilon$ training error at a $\log(1/\epsilon)$ rate, regardless of the initial conditioning of the data. This is in contrast to prior works (Arora et al., 2019c; Allen-Zhu et al., 2018) and demonstrates the optimization benefits of using deeper networks. This result holds for either training just the top layer of the neural network, or all layers of the network with a sufficiently small learning rate (the so-called lazy training regime). In particular, when training just the top layer with the popular ReLU activations, we show that the width of the network only needs to grow logarithmically in the initial conditioning of the input data. These results are established by using our main result to show that the smoothness and the strong convexity of the loss function improve exponentially with depth.

At the core of our work is an analysis for the case where the network has infinite width. We establish conditioning for the infinite width kernel and its neural tangent counterpart. Building on the analysis for infinite-width networks, the extension to finite width follows typically by applying standard concentration inequalities. Our optimization results then follow from the standard paradigm of choosing a suitably small step-size allowing for little movement of the underlying kernel. The generality of our results also leads to multiple applications beyond optimization.

As an application of our conditioning results, we extend the recent work of Das et al. (2019) on learnability of randomly initialized deep neural networks under the statistical query (Kearns, 1998) framework of learning. More specifically we show that learning a target function that is a sufficiently deep, and randomly initialized neural network with a general class of activations (including sign, ReLU and tanh), requires exponentially (in depth) many queries in the statistical query model of learning. As another application, we extend the work of Bartlett et al. (2019b) on interpolating classifiers and show that randomly initialized and sufficiently deep neural networks can not only fit the training data, but in fact, the minimum norm (in the appropriate RKHS) interpolating solution achieves non-trivial excess risk guarantees in some settings as well.

2. Notation and preliminaries
For two vectors $x$ and $x'$ of like dimension, we denote their inner product by $x \cdot x'$. Unless otherwise specified, $\| \cdot \|$ denotes the Euclidean norm for vectors and the spectral norm for matrices. For a symmetric positive definite matrix $M$, the condition number $\kappa(M)$ is defined to be the ratio
is a standard non-degeneracy assumption made in the literature. For a positive integer $n$, define $[n] = \{1, 2, \ldots, n\}$.

We are given a training set of $n$ examples: $S = \{(x_i, y_i) \in \mathbb{R}^d \times \mathcal{Y}\}_{i=1}^n$, where $\mathcal{Y}$ is the output space. We assume, as is standard in related literature, that for all $i$ we have $\|x_i\| = 1$. Let $K \in \mathbb{R}^{n \times n}$ be the Gram matrix of the training data, i.e. $K_{ij} = x_i \cdot x_j$. We make one of the following two assumptions on the input data:

**Assumption 1** For all $i, j \in [n]$ with $i \neq j$, we have $|x_i \cdot x_j| \leq 1 - \delta$.

**Assumption 2** $\lambda_{\min}(K) \geq \delta$.

Assumption 1 is a standard non-degeneracy assumption made in the literature. Assumption 2 is a stronger assumption than Assumption 1 but still quite benign. In particular we show in Theorem 30 (in Appendix F) that for ReLU activations, the representations derived after passing a dataset satisfying Assumption 1 through one layer, satisfy Assumption 2. This statement can also be made for more general activations, see Theorem 32.

To keep the presentation as clean as possible, we assume a very simple architecture of the neural network:\textsuperscript{1} it has $L$ hidden fully-connected layers, each of width $m$, and takes $x \in \mathbb{R}^d$ as input and outputs $y \in \mathbb{R}$, with activation function $\sigma : \mathbb{R} \to \mathbb{R}$ to $\mathbb{R}^m$ by entry-wise application. The network can thus be defined as the following function\textsuperscript{2} $f_{\bar{W}} : \mathbb{R}^d \to \mathbb{R}$:

$$f_{\bar{W}}(x) = v \cdot \sqrt{m}\sigma(W_L \sqrt{m}\sigma(W_{L-1} \cdots \sqrt{m}\sigma(W_1 x) \cdots)),$$

where $W_1 \in \mathbb{R}^{m \times d}$, $W_2, \ldots, W_L \in \mathbb{R}^{m \times m}$ denote the weight matrices for the hidden layers, $v \in \mathbb{R}^m$ denotes the weight vector of the output layer, $\bar{W} \in \mathbb{R}^{dm + (L-1)m^2 + m}$ denotes a vector obtained by concatenating vectorizations of the weight matrices. We use the notation $\mathcal{N}(\mu, \Sigma)$ for the normal distribution with mean $\mu$ and covariance $\Sigma$. All weights are initialized to independent, standard normal variables (i.e. drawn i.i.d. from $\mathcal{N}(0, 1)$).

We assume that the activation function $\sigma : \mathbb{R} \to \mathbb{R}$ is normalized (via translation and scaling) to satisfy the following conditions:

$$\mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma(X)] = 0 \quad \text{and} \quad \text{Var}_{X \sim \mathcal{N}(0, 1)}[\sigma(X)] = \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma^2(X)] = 1. \quad (1)$$

The first condition is somewhat non-standard and is crucial to our conditioning analysis. In Section 7 we discuss how the commonly used BatchNorm operation makes it possible for us to assume this condition without loss of generality. Throughout the paper, statements of the type “If $q = \Theta(r)$ then [consequence].” should be taken to mean that there exist universal constants $c_1, c_2$ such that if $c_1 \cdot r \leq q \leq c_2 \cdot r$ then [consequence] follows. We use $O(\cdot)$ and $\Omega(\cdot)$ notation in a similar manner. Similarly, statements of the type “If $q = \text{poly}(\cdot)$ then [consequence].” should be taken to mean that there exists a polynomial of bounded degree in the arguments such that if $q$ equals that polynomial then [consequence] follows.

\textsuperscript{1} Extending our analysis to layers of different sizes and outputs of length greater than 1 poses no mathematical difficulty and is omitted for the sake of clarity of notation.

\textsuperscript{2} Note that we’re using the so-called neural tangent kernel parameterization (Jacot et al., 2018) instead of the standard parameterization here.
3. Main results on conditioning of kernel matrices

3.1. Top layer kernel matrix.

The first kernel matrix we study is the one defined by (random) feature mapping generated at the top layer by the lower layer weights, i.e.\(^3\)

\[
\Phi_W(x) := \frac{1}{\sqrt m} \sigma(W_L \frac{1}{\sqrt m} \sigma(W_{L-1} \cdots \frac{1}{\sqrt m} \sigma(W_1 x) \cdots)).
\]

The feature mapping \(\Phi_W\) defines a kernel function \(k\) and the associated \(n \times n\) kernel matrix \(K\) on a training set \(S\) as,

\[
k(x, x') := \Phi_W(x) \cdot \Phi_W(x'), \quad K_{ij} := k(x_i, x_j)
\]

The main results on conditioning in this paper are cleanest to express in the limit of infinite width neural networks, i.e. \(m \to \infty\). In this limit, the kernel function \(k\) and the kernel matrix \(K\), tend almost surely to deterministic limits (Daniely et al., 2016), denoted as \(\bar{k}\) and \(\bar{K}\) respectively. We study the conditioning of \(\bar{K}\) next. The rate at which the condition number of \(\bar{K}\) improves with depth depends on the following notion of degree of non-linearity of the activation function \(\sigma\):

**Definition 1** The coefficient of non-linearity of the activation function \(\sigma\) is defined to be \(\mu := 1 - \left(\mathbb{E}_{X \sim \mathcal{N}(0,1)}[X \sigma(X)]\right)^2\).

The normalization (1) of the activation function implies via Lemma 22 (in Appendix B, where all missing proofs of results in this section can be found) that for any non-linear activation function \(\sigma\), we have \(0 < \mu \leq 1\). To state our main result, it is convenient to define the following quantities: for \(\delta \in (0, 1)\) and a positive integer \(L\), let \(L_0(\delta) = \max\left\{\frac{\log(\frac{1}{\delta})}{\log(1 + \frac{\mu}{2})}, 0\right\} = O\left(\frac{\log(1/\delta)}{\mu}\right)\), and define

\[
B(L, \delta) := \begin{cases} 
1 - \delta(1 + \frac{\mu}{2})^L & \text{if } L \leq L_0(\delta) \\
\frac{1}{2}(1 - \frac{\mu}{2})^{L-L_0(\delta)} & \text{if } L > L_0(\delta).
\end{cases}
\]

We are now ready to state our main result on conditioning of the kernel matrix:

**Theorem 2** The following bounds hold:

1. Under Assumption 1, we have \(|\bar{K}_{ij}| \leq B(L, \delta)\) for all \(i, j \in [n]\) with \(i \neq j\).

2. Under Assumption 2, we have \(\lambda_{\min}(\bar{K}) \geq 1 - B(L, \delta)\).

The following corollary is immediate, showing that the condition number of the kernel matrix \(\bar{K}\) approaches the smallest possible value, 1, exponentially fast as depth increases.

**Corollary 3** The following bounds on \(\kappa(\bar{K})\) hold:

1. Under Assumption 1, if \(L \geq L_1(\delta) := \left[\frac{\log(n)}{\log(1 - \frac{\mu}{2})}\right] + L_0(\delta)\), then \(\kappa(\bar{K}) \leq 1 + 2n(1 - \frac{\mu}{2})^{L-L_1(\delta)}\).

2. Under Assumption 2, we have \(\kappa(\bar{K}) \leq 1 + \frac{n}{\mu} (1 + \frac{\mu}{2})^{-L}\).

3. Note that \(\Phi_W\) does not depend on the \(v\) component of \(W\); this notation is chosen for simplicity.
3.1.1. Proof of Theorem 2.

For the conditioning analysis, we need a key concept from Daniely et al. (2016), viz. the notion of the dual activation $\hat{\sigma}$ for the activation $\sigma$:

**Definition 4** For $\rho \in [-1, 1]$, define matrix $\Sigma_\rho = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. Define the conjugate activation function $\hat{\sigma} : [-1, 1] \to [-1, 1]$ as $\hat{\sigma}(\rho) := \mathbb{E}_{(X,X') \sim N(0,\Sigma_\rho)}[\sigma(X)\sigma(X')]$.

The random initialization of the neural network induces a feature representation of the input vectors at every depth $l$ in the neural network: $\Phi_W^{(l)}(x) := \frac{1}{\sqrt{m}}\sigma(W_l\frac{1}{\sqrt{m}}\sigma(W_{l-1}...\frac{1}{\sqrt{m}}\sigma(W_1x)...))$.

This feature representation naturally yields a kernel function $k_W^{(l)}(x,x') := \Phi_W^{(l)}(x) \cdot \Phi_W^{(l)}(x')$. The results of Daniely et al. (2016) imply that $\hat{\sigma}$ describes the behavior of kernels through the layers:

**Lemma 5** Suppose $\|x\|_2 = \|x'\|_2 = 1$. Then for any depth $l$, as $m \to \infty$, $k_W^{(l)}(x,x') \xrightarrow{a.s.} \hat{\sigma}^{(l)}(x \cdot x')$, where $\hat{\sigma}^{(l)}$ denotes the $l$-fold composition of $\hat{\sigma}$ with itself.

By analyzing the Hermite expansion of $\sigma$ (see Appendix B for details), we have the following key lemma which shows that $\hat{\sigma}^{(l)}$ decays to 0 rapidly as we move up the layers:

**Lemma 6 (Correlation decay lemma)** Suppose $|\rho| \leq 1 - \delta$ for some $\delta \in (0, 1]$. Then

$$|\hat{\sigma}^{(L)}(\rho)| \leq B(L, \delta).$$

The proof of this lemma relies crucially on the normalization (1) of the activation $\sigma$, and its non-linearity. The normalization implies that each application of $\hat{\sigma}$ decreases pairwise inner products of the input feature representations, at a rate governed by the coefficient of non-linearity. Using this fact repeatedly leads to the stated bound.

The final technical ingredient we need is the following linear-algebraic lemma which gives a lower bound on the smallest eigenvalue of a matrix obtained by the application of a given function to all entries of another positive definite matrix:

**Lemma 7 (Eigenvalue lower bound lemma)** Let $f : [-1, 1] \to \mathbb{R}$ be an arbitrary function whose power series $f(\rho) = \sum_{i=0}^{\infty} a_i\rho^i$ converges everywhere in $[-1, 1]$ and has non-negative coefficients $a_i \geq 0$. Let $K \in \mathbb{R}^{n \times n}$ be a positive definite matrix with $K \geq \delta I_n$ for some $\delta > 0$, and all diagonal entries equal to 1. Let $f[K]$ be matrix obtained by entrywise application of $f$. Then we have

$$f[K] \succeq (f(1) - f(1 - \delta))I_n.$$
3.2. Neural tangent kernel matrix.

The second kernel matrix we study arises from the *neural tangent kernel*, which was introduced by Jacot et al. (2018). This kernel matrix naturally arises when all the layers of the neural network are trained via gradient descent. For a given set of network weights \( \hat{W} \), the neural tangent kernel matrix \( \mathcal{K} \in \mathbb{R}^{n \times n} \) is defined as

\[
\mathcal{K}_{ij} = (\partial_{\hat{W}} f_{\hat{W}}(x_i)) \cdot (\partial_{\hat{W}} f_{\hat{W}}(x_j)).
\]

As in the previous section, as the width \( m \) of the hidden layers tends to infinity, the random \( \mathcal{K} \) tends to a deterministic limit, \( \bar{\mathcal{K}} \). For this infinite width limit, we have the following theorem analogous to part 1 of Theorem 2:

**Theorem 8** The diagonal entries of \( \bar{\mathcal{K}} \) are all equal. Furthermore, the following bounds hold if \( L \geq 2L_0(\delta) \):

1. Under Assumption 1, we have \( |\bar{\mathcal{K}}_{ij}| \leq 2B(L/2, \delta) \cdot \bar{\mathcal{K}}_{11} \) for all \( i, j \in [n] \) with \( i \neq j \).
2. Under Assumption 2, we have \( \lambda_{\text{min}}(K) \geq (1 - 2B(L/2, \delta)) \bar{\mathcal{K}}_{11} \).

The following corollary, analogous to Corollary 3, is immediate:

**Corollary 9** The following bounds on the condition number \( \kappa(\bar{\mathcal{K}}) \) hold:

1. Under Assumption 1, if \( L \geq L_2(\delta) := \left\lfloor \frac{2\log(2n)}{-\log(1 - \frac{\mu}{2})} \right\rfloor + 2L_0(\delta) \), then \( \kappa(\bar{\mathcal{K}}) \leq 1 + 4n(1 - \frac{\mu}{2})^{L_2(\delta)} \).
2. Under Assumption 2, if \( L \geq 4L_0(\delta) \), then \( \kappa(\bar{\mathcal{K}}) \leq 1 + \frac{2n}{\delta}(1 + \frac{\mu}{2})^{-L_2(\delta)} \).

3.2.1. Proof of Theorem 8

We use the following formula for the NTK given by Arora et al. (2019c): defining \( \rho := x_i \cdot x_j \) and \( \hat{\sigma} \) to be the derivative of \( \sigma \), we have

\[
\tilde{\mathcal{K}}_{ij} = \sum_{h=1}^{L+1} \hat{\sigma}^{(h-1)}(\rho) \left( \prod_{h'=h}^{L} \hat{\sigma}(\sigma^{(h')}(\rho)) \right)^{i-j}.
\]  

(2)

We need the following bound in our analysis which follows via the Hermite expansion of \( \sigma \):

**Lemma 10** For any \( \rho \in [-1, 1] \), we have \( \frac{\hat{\sigma}(\rho)}{\hat{\sigma}(1)} \leq 1 - \mu(1 - |\rho|) \).

We can now prove Theorem 8:

**Proof** (Theorem 8) First, we show that all diagonal values of \( \bar{\mathcal{K}} \) are equal. For every \( i \), we have \( x_i \cdot x_i = 1 \), and since \( \hat{\sigma}^{(h)}(1) = 1 \) for any \( h \), we have from (2),

\[
\tilde{\mathcal{K}}_{ii} = \sum_{h=1}^{L+1} \hat{\sigma}^{(h-1)}(1) \left( \prod_{h'=h}^{L} \hat{\sigma}(\sigma^{(h')}(1)) \right) = \sum_{h=1}^{L+1} \left( \prod_{h'=h}^{L} \hat{\sigma}(1) \right) = \frac{\hat{\sigma}(1)^{L+1} - 1}{\hat{\sigma}(1) - 1},
\]

which is a fixed constant.

†. We assume the convention that \( \prod_{a}^{b} (\cdot) = 1 \) if \( a > b \).
To prove part 1, let \( \rho := x_i \cdot x_j \). It is easy to show (say, via the Hermite expansion of \( \sigma \)) that \( \hat{\sigma}(1) > 0 \). Thus, we have

\[
\frac{\hat{K}_{ij}}{\hat{K}_{11}} = \frac{\sum_{h=1}^{L} \hat{\sigma}^{(h-1)}(\rho) \left( \prod_{h'=h}^{L} \hat{\sigma}(h')(\rho) \right)}{\sum_{h=1}^{L} \left( \prod_{h'=h}^{L} \hat{\sigma}(1) \right)} \leq \max_{h \in [L+1]} \hat{\sigma}^{(h-1)}(\rho) \cdot \prod_{h'=h}^{L} \hat{\sigma}(h')(\rho) \\
\leq \max_{h \in [L+1]} \left( \hat{\sigma}^{(h-1)}(\rho) \right) \cdot \prod_{h'=h}^{L} (1 - \mu(1 - |\hat{\sigma}(h')(\rho)|)) \leq \max_{h \in [L+1]} B(h - 1, \delta) \cdot \prod_{h'=h}^{L} (1 - \mu(1 - B(h', \delta))),
\]

where the penultimate inequality follows Lemma 10 and the final one from Lemma 6. We now show that since \( L \geq 2L_0(\delta) \), for any any \( h \in [L + 1] \), we have

\[
B(h - 1, \delta) \cdot \prod_{h'=h}^{L} (1 - \mu(1 - B(h', \delta))) \leq 2B(L/2, \delta),
\]

which gives the bound of part 1. We do this in two cases: if \( h - 1 \geq L/2 \), then \( B(h - 1, \delta) \leq B(L/2, \delta) \), which gives the required bound since all terms in the product are at most 1. Otherwise, if \( h - 1 < L/2 \), then there are at least \( L/2 - L_0(\delta) \) values of \( h' \) in \{\( h, h + 1, \ldots, L \} \) which are larger than \( L_0(\delta) \), and for these values of \( h' \), we have \( B(h', \delta) \leq \frac{1}{2} \), so \( 1 - \mu(1 - B(h', \delta)) \leq 1 - \frac{\mu}{2} \). The product of these terms is therefore at most \((1 - \mu/2)^{L/2 - L_0(\delta)} = 2B(L/2, \delta) \), which gives the required bound in this case.

To prove part 2, define \( f : [-1, 1] \to \mathbb{R} \) as \( f(\rho) = \sum_{h=1}^{L} \hat{\sigma}^{(h-1)}(\rho) \left( \prod_{h'=h}^{L} \hat{\sigma}(h')(\rho) \right) \). Equation (2) shows that this defines a kernel on the unit sphere, and so by Schoenberg’s theorem (Schoenberg, 1942), its power series expansion has only non-negative coefficients. Thus, applying Lemma 7 to \( \hat{K} = f[K] \), we conclude that

\[
\hat{K} \succeq (f(1) - f(1 - \delta)) I_n \succeq (1 - 2B(L/2, \delta)) f(1) I_n,
\]

using the calculations in part 1. Since \( f(1) = \hat{K}_{11} \), the bound of part 2 follows.

\[\] 4. Implications for optimization

Suppose we train the network using gradient descent on a loss function \( \ell : \mathbb{R} \times \mathcal{Y} \to \mathbb{R} \), which defines the empirical loss function

\[
\mathcal{L}(\hat{W}) := \frac{1}{n} \sum_{i=1}^{n} \ell(\hat{W}(x_i), y_i).
\]

For the rest of this section we will assume that the loss function \( \ell \) is the square loss, i.e. \( \ell(\hat{y}, y) = (\hat{y} - y)^2 \). The results presented can appropriately be extended to the setting where the loss function is smooth and strongly convex. Training a finite-width neural network necessitates the study of the conditioning of the finite-width kernel matrices \( K \) and \( \mathcal{K} \), rather than their infinite-width counterparts. In such settings optimization results typically follow from a simple 2-step modular analysis:

- **Step 1. [Initial Stability]** Standard concentration inequalities imply that if the width is large enough, conditioning of the infinite-width kernel matrices transfers to their finite-width counterparts at initialization.
• **Step 2. [Training Stability]** Standard optimization theory implies that conditioning in finite-width kernel matrices leads to fast training. In the case of training only the top layer this is sufficient. When training all layers, a much more careful analysis is needed to show that the NTK stays "close" to initialization, leading to conditioning throughout the training process.

We now provide a couple of representative optimization results that follow from this type of analysis. Our goal here is to merely provide representative examples of typical optimization scenarios and highlight what benefits conditioning can lead to. Indeed, we believe extensions and improvements can be derived with significantly better bounds.

### 4.1. Training only the top layer

We consider a mode of training where only the top layer weight vector, \(v\), is updated, while keeping \(W_1, W_2, \ldots, W_L\) frozen at their randomly initialized values. To highlight this we introduce the notation \(\vec{W}_{1:L} = \{W_1 \ldots W_L\}\). Let \(\eta > 0\) be a step size, the update rule at iteration \(t\) is given by

\[
v_{t+1} = v_t - \eta \cdot \partial_v \mathcal{L}(\{v_t, \vec{W}_{1:L}\}) = v_t - \eta \cdot \frac{1}{n} \sum_{i=1}^{n} 2(v_t \cdot \Phi_{\vec{W}}(x_i) - y_i)\Phi_{\vec{W}}(x_i).
\]

Note that in this mode of training, the associated optimization problem is convex in \(v\). To implement Step 1 of the modular analysis, we appeal to the results of Daniely et al. (2016). They show that when the activations are suitably bounded (see Definition 6 in their paper for \(C\)-bounded activations) and the width is large enough, then with high probability, each entry in the kernel matrix \(K\) is close to the corresponding entry in \(\bar{K}\). Specifically, via Theorems 2 and 3 in their paper, we have the following version of Theorem 2 for finite width neural networks:

**Lemma 11 (Via Theorem 2 in Daniely et al. (2016))** For any \(\gamma > 0\), suppose that either

- The activation \(\sigma\) is \(C\)-bounded and \(m = \Omega(\frac{(4C)\log(n)}{\gamma^2})\), or
- The activation is ReLU, \(\gamma = O(\frac{1}{L})\) and \(m = \Omega(\frac{L^2\log(n)}{\gamma^2})\).

Then with high probability, we have that for all \(i, j\), \(|K_{ij} - \bar{K}_{ij}| \leq \gamma\).

Step 2 follows by using standard convex optimization theory (Nesterov, 2014), which tells us that the convergence rate of gradient descent for this problem depends on the condition number of \(\bar{K}\). Specifically, we have the following result:

**Theorem 12** Suppose \(L = \Theta\left(\frac{\log(n/\delta)}{\mu}\right)\). Then,

- If \(\sigma\) is \(C\)-bounded and the width \(m = \text{poly}(n, \frac{1}{\delta})\), or
- If \(\sigma\) is ReLU and the width \(m = \Omega(n^2 \log^3(n/\delta))\).

Then setting \(\eta = \Theta\left(\frac{1}{\lambda_{\max}(K)}\right)\), we get that with high probability over the initialization,

\[
\mathcal{L}(\{v_t, \vec{W}_{1:L}\}) \leq e^{-\Omega(\frac{\mu}{\eta})} \cdot \mathcal{L}(\{v_0, \vec{W}_{1:L}\})
\]

Alternatively, in order to find a point that is \(\epsilon\) sub-optimal, gradient descent needs \(O(\log(\frac{1}{\epsilon}))\) steps.
Similarly, one can also derive a linear convergence theorem for stochastic gradient descent:

**Theorem 13** With the same choice of parameters as in Theorem 12, appropriate choice of $\eta$ and with high probability over the initialization, stochastic gradient descent finds a point that is $\epsilon$-suboptimal in expectation in at most $O\left( \frac{\log(\frac{1}{\delta})}{\eta} \right)$ steps.

**Remark 14** The rate in the exponent in the theorem above naturally depends upon the condition number of the kernel matrix $K$. For simplicity, we choose to state the theorem for a depth at which the condition number is $O(1)$. Precise rates depending on $L$, can be derived from Corollary 3.

4.2. Training All The Layers Together

In this section we provide a representative result for the training dynamics when all the layers are trained together with a fixed common learning rate. The dynamics are given by

$$\tilde{W}(t + 1) = \tilde{W}(t) - \eta \nabla_{\tilde{W}} \mathcal{L}(\tilde{W}(t))$$

Now since the bottom layers also move the kernel changes at every step. The standard analysis in this setting follows from carefully establishing that the NTK does not change too much during the training procedure allowing for the rest of the analysis to go through. The following theorem from Lee et al. (2019) summarizes one such setting for smooth activation functions.

**Theorem 15 (Theorem G.4 in Lee et al. (2019))** Suppose that the activation $\sigma$ and its derivative $\sigma'$ further satisfies the properties that there exists a constant $c$, such that for all $x, x'$

$$|\sigma(x)|, |\sigma'(x)|, \left| \frac{\sigma'(x) - \sigma'(x')}{x - x'} \right| \leq c.$$

Then there exists a constant $N$ (depending on $L, n, \delta$) such that for width $m > N$ and setting the learning rate $\eta = 2(\lambda_{\min}(\bar{K}) + \lambda_{\max}(\bar{K}))^{-1}$, with high probability over the initialization the following is satisfied for gradient descent for all $t$,

$$\mathcal{L}(\tilde{W}(t)) \leq e^{-\Omega\left( \frac{t}{\kappa(K)} \right) \mathcal{L}(\tilde{W}(0))}$$

The following corollary is now a simple application of the above theorem and Corollary 9.

**Corollary 16** Suppose the conditions in Theorem 15 are satisfied and the width is taken to be a large enough constant (depending on $L, n, \delta$) and further $L = \Theta\left( \frac{\log(n/\delta)}{\mu} \right)$, then gradient descent with high probability finds an $\epsilon$ suboptimal point in total time $O\left( \log(1/\delta) \right)$.

**Remark 17** As stated in Theorem 15 the width required could be a very large constant. However, note that we require the depth to be logarithmic in $\frac{1}{\delta}$ for achieving constant condition number. Therefore the exponential in $L$ factors accrued in the analysis of Theorem 15 are actually polynomial in $\frac{1}{\delta}$. Therefore, merging results from Arora et al. (2019c), we can derive a polynomial in $\frac{1}{\delta}$ upper bound on the width of the network. This matches the best known bounds on the overparameterization while improving the optimization rates exponentially (in $\frac{1}{\delta}$). Further we believe similar results can also be derived for ReLU activations following techniques in Allen-Zhu et al. (2018).

The proofs from this section follow easily from our established results and standard arguments from optimization theory. We have included the proofs in Appendix E for completeness.
5. SQ Learnability of Random Deep Neural Nets

In this section we show that our main result in Theorem 2 leads to a generalization of the recent result of Das et al. (2019) regarding learnability of random neural networks. The work of Das et al. (2019) studied randomly initialized deep neural networks with sign activations at hidden units. Motivated from the perspective of complexity of learning, they studied learnability of random neural networks in the popular statistical query learning (SQ) framework (Kearns, 1998; Bshouty and Feldman, 2002). Their main result establishes that any algorithm for learning a function that is a randomly initialized deep network with sign activations, requires exponential (in depth) many SQ queries in the worst case. Here we extend their result for arbitrary activations under mild assumptions and show that randomly initialized deep neural networks with arbitrary activations are hard to learn under the SQ model. Specifically, we prove our result under the assumption that the (normalized) activation $\sigma$ is subgaussian with constant subgaussian norm. In particular we assume that

$$\mathbb{E}_{X \sim \mathcal{N}(0,1)}[e^{\lambda \sigma(X)}] \leq e^{\lambda^2 \alpha^2/2},$$

(3)

for a constant $\alpha > 0$. Many activations such as the sign, ReLU and tanh satisfy this assumption.

A key component in establishing SQ hardness of learning is to show that given two non-collinear unit length vectors, a randomly initialized network of depth $h$ and sufficiently large width width makes, in expectation, the pair nearly orthogonal. In other words, the magnitude of the expected dot product between any pair decreases exponentially with depth. While Das et al. (2019) proved the result for sign activations, we prove the statement for more general activations and then use it to establish SQ hardness of learning. We will work with networks that will normalize the output of each layer to unit length via the operation $\Pi : \mathbb{R}^m \rightarrow \mathbb{R}^m$. Then we have the following theorem:

**Theorem 18** Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a non linear activation with $\mu$ being the coefficient of non-linearity as in Definition 1 and satisfying (3). Let $x_i, x_j \in \mathbb{R}^d$ be unit length vectors such that $|x_i \cdot x_j| \leq 1 - \delta$. Define $\Phi_W(x) := \frac{1}{\sqrt{m}} \Pi(\sigma(W_L \sqrt{\frac{1}{m}} \Pi(\sigma(W_{L-1} \cdots \sqrt{\frac{1}{m}} \Pi(\sigma(W_1 x) \cdots)),$ where each column of $W_1$ is sampled from $\mathcal{N}(0, I_{d \times d})$ and each column of $W_i$ is sampled from $\mathcal{N}(0, I_{m \times m})$ for $i > 1$. Furthermore, the operation $\Pi$ normalizes the output of each layer to unit length. Let $m > c_1 \frac{L}{\mu^2 \delta^2}$ for a universal constant $c_1 > 0$ and for $h \in [1, L]$ define $\rho_h$ be the dot product obtained by taking the representation of $x_i, x_j$ at depth $h$ of the network defined above. Then for any $h > 1$, it holds that

$$\left| \mathbb{E}[\rho_{L_0(\delta) + h}] \right| \leq e^{-\Omega(h)} + L e^{-\Omega(L)},$$

where $L_0(\delta) = c_1 \frac{\log(\delta)}{\mu}$ and $c_1 > 0$ is a universal constant.

While the above theorem is not a black box application of our main result (Theorem 2) since careful concentration arguments are required due to finite width, the calculations are of a similar flavor.

We now show how the above theorem can be used to generalize the SQ lower bound of Das et al. (2019). Before describing our results, we recall that in the SQ model (Kearns, 1998) the learning algorithm does not have access to a labeled training set. Instead, for a given target function $f$ and a distribution $D$ over $\mathbb{R}^d$, the algorithm has access to a query oracle $SQ_{f,D}(\psi, \tau)$. The oracle takes as input a query function $\psi$, and outputs a value $v$ such that $|\mathbb{E}_D[\psi(x, f(x))] - v| \leq \tau$. The goal of the algorithm is to use the query algorithm to output a function $g$ that $\epsilon$ approximates $f$, i.e., $P_{TD}[g(x)f(x)] \geq \epsilon$, for a given $\epsilon > 0$.  

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...conditioning in deep neural networks...

(Das et al., 2019) established an SQ learnability lower bound for a subclass \( \mathcal{F} \) of neural networks with the property that a randomly initialized neural network falls in \( \mathcal{F} \) with high probability. This however only establishes that the class \( \mathcal{F} \) is hard to SQ learn as opposed to showing that a randomly initialized neural network is hard to learn. Furthermore, the lower bound only applies to networks with sign activations. We now show how to generalize their result in two ways: (a) we allow arbitrary activations satisfying (3), and (b) our lower bound shows that a randomly initialized network is hard to learn in the SQ model with constant probability. We achieve the stronger lower bound by carefully adapting the lower bound technique of Bshouty and Feldman (2002).

In our context we will fix a non linear activation \( \sigma: \mathbb{R} \to \mathbb{R} \) and let the target be of the form

\[
\hat{f}_W(x) = \left( v \cdot \frac{1}{\sqrt{m}} \prod_{i=1}^{L} \sigma(W_L \frac{1}{\sqrt{m}} \prod_{i=1}^{L-1} \sigma(W_{L-i+1} \frac{1}{\sqrt{m}} \prod_{i=1}^{L} \sigma(W_1 x) \ldots)) \right)
\]

where each column of \( W_i \) is sampled from \( \mathcal{N}(0, I_{d \times d}) \) and \( v \) and each column of \( W_i \) is sampled from \( \mathcal{N}(0, I_{m \times m}) \) for \( i > 1 \). Furthermore we will use the depth \( L \) and the dimensionality \( d \) to parameterize the bit complexity of the network description. We say that an algorithm \( (p(d, L), r(d, L), q(d, L)) \)-SQ learns \( \hat{f}_W(x) \) if with probability at least 1/2 over the randomness in \( \hat{W} \), the algorithm makes at most \( p(d, L) \) queries to the SQ oracle for \( \hat{f}_W(x) \), receives responses from the oracle up to tolerance \( \tau = 1/r(d, L) \) and outputs a \( g \) that \( \epsilon = 1/q(d, L) \)-approximates \( f \). Furthermore it is the case that each query function \( \psi \) used by the algorithm can be evaluated in time \( \psi(d, L) \).

Then we have the following lower bound extending the result of Das et al. (2019). The proofs of this section can be found in Appendix C.

**Theorem 19** Fix any non linear activation \( \sigma \) with the coefficient of non-linearity being \( \mu \) that satisfies (3). Any algorithm that \((p(d, L), \text{poly}(d, L), \text{poly}(d, L))\)-SQ learns the random depth \( L \) networks as defined above with width \( m = \Omega(L^{\frac{L^2}{\mu^2}}) \) must satisfy \( p(d, L) \geq e^{\Omega(L)} \).

6. Benign Overfitting in Deep Neural Networks

In this section, we give an application of our conditioning results showing how interpolating classifiers (i.e. classifiers achieving perfect training accuracy) can generalize well in the context of deep neural networks. Specifically, building on the work of Bartlett et al. (2019b), we consider the problem of linear regression with square loss where the feature representation is obtained via a randomly initialized deep network, and an interpolating linear predictor is obtained by training only the top layer (i.e. the \( v \) vector). Since there are infinitely many interpolating linear predictors in the overparameterized setting we consider, we focus our attention on the minimum norm predictor.

In this setting, the input space is the \( d \) dimensional unit sphere \( S^{d-1} \), the output space \( \mathcal{Y} = [-1, 1] \), and samples \((x, y) \in S^{d-1} \times [-1, 1] \) are drawn from an unknown distribution \( D \). The training set is \( \mathcal{S} = \{(x_i, y_i) \in S^{d-1} \times [-1, 1]\}_{i=1}^{n} \). To simplify the presentation, we work in the infinite width setting, i.e. we learn the minimum norm linear predictor in the RKHS \( \mathcal{H} \) corresponding to the kernel function \( \bar{k} \) for a deep neural network as defined in Section 3.1. The number of hidden layers in the neural network, \( L \), depends on the the sample size \( n \) in our results.

Following the notational conventions in (Bartlett et al., 2019b), for \( v, v' \in \mathcal{H} \), we denote by \( v^\top v' \) their inner product. Let \( \Phi: S^{d-1} \to \mathcal{H} \) be the feature map corresponding to \( \bar{k} \). We denote by the infinite matrix \( X \) the linear map from \( \mathcal{H} \to \mathbb{R}^n \) corresponding to the inputs \( x_1, x_2, \ldots, x_n \), so that for any \( v \in \mathcal{H} \), \( Xv \in \mathbb{R}^n \) has \( i \)th component \( v^\top \Phi(x_i) \). Note that \( XX^\top = \bar{K} \), the kernel
Assumption 1 and its applications clearly demonstrate the joint benefit of
Du et al., 2019
Corollary 16
holds for a randomly drawn sample set of size
2018
that when training all layers of a deep
We denote by 
With this definition, we have the following excess risk bound (proof in
A few caveats about the theorem are in order. Note that the number of layers, 
i = [n], and indeed, is the minimum norm interpolating linear predictor. Our goal is to bound the excess risk of 
A key quantity of interest is the function
A key quantity of interest is the function \( \Delta : \mathbb{N} \times [0, 1] \rightarrow [0, 1] \) defined as follows: \( \Delta(m, \gamma) \) is the largest value of \( \delta \) for which Assumption 1 holds for a randomly drawn sample set of size \( m \) with probability at least \( 1 - \gamma \). Specifically, if \( T = \{x_1', x_2', \ldots, x'_m\} \) denotes a sample set of size \( m \) drawn i.i.d. from the marginal distribution of \( D \) over the \( x \)-coordinate, then
With this definition, we have the following excess risk bound (proof in Appendix D):
\[
\Delta(m, \gamma) := \sup \left\{ \delta : \Pr \left[ \max_{i,j \in [m]: i \neq j} |x'_i \cdot x'_j| \leq 1 - \delta \right] \geq 1 - \gamma \right\}.
\]
For any \( \gamma \in (0, 1/2) \), let
\[
L = \left\lceil \frac{\log(n^2)}{- \log(1 - \gamma)} \right\rceil + L_0(\Delta(n^2, \gamma)).
\]
Then, with probability at least \( 1 - \gamma \) over the choice of \( S \), there exists an interpolating linear predictor, and we have
\[
\mathbb{E}_{(x, y)} \left[ (y - v_S^* \Phi(x))^2 \right] - \mathbb{E}_{(x, y)} \left[ (y - v^* \Phi(x))^2 \right] \leq O \left( \frac{\log(n/\gamma)}{n} \|v^*\|^2 \right).
\]
A few caveats about the theorem are in order. Note that the number of layers, \( L \), and therefore \( \mathcal{H} \) and the optimal linear predictor \( v^* \) depends on the sample size \( n \). Thus, the excess risk goes to 0 when \( n \) increases if \( \|v^*\| = o(\sqrt{n}) \).

7. Discussion
Our main result stated in Theorem 2 and its applications clearly demonstrate the joint benefit of using deeper networks with non-linear activations from an optimization and generalization perspective. Our Assumption 1 is standard and has been used in prior works on optimization of neural networks via stochastic gradient descent (Allen-Zhu et al., 2018; Zou and Gu, 2019; Du et al., 2019). Assumption 2 is stronger but is easily satisfied by randomly initialized one layer networks with popular activations such as the ReLU. We establish this in Theorem 30.
Using our conditioning analysis we obtain in Corollary 16 that when training all layers of a deep enough network via gradient descent, the iteration complexity is independent of \( n \) and the initial separation \( \delta \), thereby clearly demonstrating the benefit of depth. This is in contrast to prior works where the iteration complexity depends polynomially in the depth \( L \) and \( 1/\delta \) (Allen-Zhu et al., 2018; Zou and Gu, 2019; Du et al., 2019). For the case of training all the layers with ReLU activations, our improved analysis implies that (see Theorem 12) the width requirement only has a logarithmic dependence in \( 1/\delta \), as opposed to polynomial in \( 1/\delta \) in prior works.
Finally, note that all our theorems and their implications hold for the case of normalized activations as defined in (1). As discussed earlier, the only somewhat non-standard part of the normalization is the requirement that the activation is centered so that its expectation on standard normal inputs is 0. This requirement is not simply a limitation of our analysis but is inherently necessary
since when working with uncentered activations, a similar analysis to the one in this paper shows all pairwise dot products approach 1 (rather than 0) at an exponential rate.

Beyond this fact, we note that the commonly used batch normalization (BatchNorm) operation (Ioffe and Szegedy, 2015) makes it possible to assume that activations are centered without loss of generality. BatchNorm is an essential operation for efficient training of deep networks: in this operation the input $x$ to a given layer $k$ is processed to obtain $x'$ where $x' = (x - \mu_B)/\sigma_B$ and $\mu_B$ and $\sigma_B$ are the mean and the standard deviation of the inputs for given batch $B$ of examples. It is evident from the definition of BatchNorm that the output of the operation is invariant to translation of the activation by a fixed constant. Thus, without loss of generality we can assume that the activation is centered. In this light, our results can be viewed as providing a theoretical justification for the superior optimization performance of BatchNorm that has been observed in practice.

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Appendix A. Related Work

Representational Benefits of Depth. Analogous to depth hierarchy theorems in circuit complexity, many recent works have aimed to characterize the representational power of deep neural networks when compared to their shallow counterparts. The work of Delalleau and Bengio (2011) studies sum-product networks and constructs examples of functions that can be efficiently represented by depth 4 or higher networks and require exponentially many neurons for representation with depth one networks. The works of Martens and Medabalimi (2014) and Kane and Williams (2016) study networks of linear threshold gates and provide similar separation results. Eldan and Shamir (2016) show that for many popular activations such as sigmoid, ReLU etc. there are simple functions that can be computed by depth 3 feed forward networks but require exponentially (in the input dimensionality) many neurons to represent using two layer feed forward networks. Telgarsky (2016) generalizes this to construct, for any integer \( k \), a family of functions that can be approximated by \( \Theta(k^3) \) layers and \( \Theta(k^3) \) size and require exponential in \( k \) neurons to represent with \( O(k) \) depth.
Optimization Benefits of Depth. While the benefits of depth are well understood in terms of the representation power using a small number of neurons, the question of whether increasing depth helps with optimization is currently poorly understood. The recent work of Arora et al. (2018b) aims to understand this question for the special case of linear neural networks. For the case of $\ell_p$ regression, they show that gradient descent updates on a depth 2 linear network correspond to accelerated gradient descent type updates on the original weight vector. Similarly, they derive the form of the weight updates for a general over parameterized deep linear neural network and show that these updates can be viewed as performing gradient descent on the original network but with a preconditioning operation applied to the gradient at each step. Empirically this leads to faster convergence. The works of Bartlett et al. (2019a) and Arora et al. (2018a) study the convergence of gradient descent on linear regression problems when solved via an over parameterized deep linear network. These works establish that under suitable assumptions on the initialization, gradient descent on the over parameterized deep linear networks enjoys the same rate of convergence as performing linear regression in the original parameter space which is a smooth and strongly convex problem.

In a similar vein, the recent work of Arora et al. (2019b) analyzes over parameterized deep linear networks for solving matrix factorization, and shows that the solution to the gradient flow equations approaches the minimum nuclear norm solution at a rate that increases with the depth of the network. The recent work of Malach and Shalev-Shwartz (2019) studies depth separation between shallow and deeper networks over distributions that have a certain fractal structure. In certain regimes of the parameters of the distribution the authors show that, surprisingly, the stronger the depth separation is, the harder it becomes to learn the distribution via a deep network using gradient based algorithms.

Optimization of Neural Networks via Gradient Descent In recent years there has been a large body of work in analyzing the convergence of gradient descent and stochastic gradient descent (SGD) on over parameterized neural networks. The work of Andoni et al. (2014) shows that depth one neural networks with quadratic activations can efficiently represent low degree polynomials and performing gradient descent on the network starting with random initialization can efficiently learn such classes. The work of Li and Yuan (2017) shows convergence of gradient descent on the population loss and under Gaussian input distribution, of a two layer feed forward network with relu activations and the identity mapping mimicking the ResNet architecture. Under similar assumptions the work of Soltanolkotabi et al. (2018) analyzes SGD for two layer neural networks with quadratic activations. The work of Li and Liang (2018) extends these results to more realistic data distributions.

Building upon the work of Daniely et al. (2016), Daniely (2017b) shows that SGD when run on over parameterized neural networks achieves at most $\epsilon$ excess loss (on the training set) over the best predictor in the conjugate kernel class at the rate that depends on $1/\epsilon^2$ and $M$, the norm of the best predictor. This result is extended in the work of Du et al. (2019) showing that by running SGD on a randomly initialized two layer over parameterized networks with relu activations, one can get $\epsilon$ loss on the training data at the rate that depends on $\log(1/\epsilon)$ and the smallest eigenvalue of a certain kernel matrix. While the authors show that this eigenvalue is positive, no explicit bound is provided. These results are extended to higher depth in (Du et al., 2018) at the expense of an exponential dependence on the depth on the amount of over parameterization needed. In (Allen-Zhu et al., 2018) the authors provide an alternate analysis under the weaker Assumption 1
and at the same time obtain convergence rates that depend on $\log(1/\epsilon)$ and only polynomially in the depth of the network. The recent work of Zou and Gu (2019) provides an improved analysis with better dependence on the parameters. We would like to point out that all the above works fail to explain the optimization benefits of depth, and in fact the resulting bounds degrade as the network gets deeper.

The work of Jacot et al. (2018) proposed the Neural Tangent Kernel (NTK) that is associated with a randomly initialized neural network in the infinite width regime. The authors show that in this regime performing gradient descent on the parameters of the network is equivalent to kernel regression using the NTK. The work of Lee et al. (2019) and Yang (2019) generalizes this result and the recent work of Arora et al. (2019c) provides a non-asymptotic analysis and an algorithm for exact computation of the NTK for feed forward and convolutional neural networks. There have also been works analyzing the mean field dynamics of SGD on infinite width neural networks (Mei et al., 2018; Chizat and Bach, 2018; Rotskoff and Vanden-Eijnden, 2018; Sirignano and Spiliopoulos, 2018) as well as works designing provable learning algorithms for shallow neural networks under certain assumptions (Arora et al., 2016; Ge et al., 2017; Goel and Klivans, 2017; Ge et al., 2018; Goel et al., 2018; Bakshi et al., 2018; Vempala and Wilmes, 2018). Recent works have also explored the question of providing sample complexity based separation between training via the NTK vs. training all the layers (Wei et al., 2019; Allen-Zhu and Li, 2020).

**SQ Learnability of Neural Networks.** Several recent works have studied the statistical query (SQ) framework of Kearns (1998) to provide lower bounds on the number of queries needed to learn neural networks with a certain structure (Song et al., 2017; Vempala and Wilmes, 2018; Das et al., 2019). The closest to us is the recent work of Das et al. (2019) that shows that learning a function that is a randomly initialized deep neural network with sign activations requires exponential in depth many statistical queries. A crucial part of their analysis requires showing that for randomly initialized neural networks with sign activations, the pairwise (normalized) dot products decrease exponentially fast with depth. Our main result in Theorem 2 strictly generalizes this result for arbitrary non-linear activations (under mild assumptions) thereby implying exponential SQ lower bounds for networks with arbitrary non-linear activations. In particular, we show any algorithm that works in the statistical query framework, and learns (with high probability) a sufficiently deep randomly initialized network with an arbitrary non-linear activation, must necessarily use exponentially (in depth) many queries in the worst case. The only requirement we impose on the non-linear activations is that they satisfy subgaussianity (see Section C), a condition satisfied by popular activations such as relu, sign, and tanh.

**Generalization in Neural Networks.** It has been observed repeatedly that modern deep neural networks have sufficient capacity to perfectly memorize the training data, yet generalize to test data very well (see, e.g., (Zhang et al., 2017)). This observation flies in the face of conventional statistical learning theory which indicates that such overfitting should lead to poor generalization. Since then there has been a line of work providing generalization bounds for neural networks that depend on compressibility of the network (Arora et al., 2018c), norm based bounds (Neyshabur et al., 2015; Bartlett et al., 2017), bounds via PAC-bayes analysis (Neyshabur et al., 2017; Dziugaite and Roy, 2017; Nagarajan and Kolter, 2019) and bounds that depend on the distance to initialization (Long and Sedghi, 2019). Since randomly initialized neural networks are interpolating classifiers, i.e., they achieve zero error on the training set, there have also been recent works (e.g. (Belkin et al., 2018, 2019b; Liang and Rakhlin, 2018; Liang et al., 2019; Bartlett et al., 2019a; Dai et al., 2019b).
Belkin et al. (2019b; Belkin et al., 2019a; Hastie et al., 2019)) that study the generalization phenomenon in the context of specific interpolating methods (i.e. methods which perfectly fit the training data) and show how the obtained predictors can generalize well.

Appendix B. Conditioning Analysis

Recall the notion of the dual activation \( \hat{\sigma} \) for the activation \( \sigma \):

**Definition 21** For \( \rho \in [-1, 1] \), define matrix \( \Sigma_\rho = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \). Define the conjugate activation function \( \hat{\sigma} : [-1, 1] \to [-1, 1] \) as follows:

\[
\hat{\sigma}(\rho) := \mathbb{E}_{(X, X') \sim \mathcal{N}(0, \Sigma_\rho)} [\sigma(X)\sigma(X')].
\]

The following facts can be found in Daniely et al. (2016):

1. Let \( x, x' \in \mathbb{R}^d \) such that \( \|x\| = \|x'\| = 1 \). Then
   \[
   \mathbb{E}_{w \sim \mathcal{N}(0, I_d)} [\sigma(w \cdot x)\sigma(w \cdot x')] = \hat{\sigma}(x \cdot x').
   \]

2. Since \( \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma^2(X)] = 1 \), \( \sigma \) is square integrable w.r.t. the Gaussian measure. The (probabilist’s) Hermite polynomials \( h_0, h_1, \ldots \) form an orthogonal basis for the Hilbert space of square integrable functions w.r.t. the Gaussian measure, and hence \( \sigma \) can be written as \( \sigma(u) = \sum_{i=0}^{\infty} a_i h_i(u) \), where \( a_i = \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma(X)h_i(X)] \). This expansion is known as the Hermite expansion for \( \sigma \).

3. We have \( \hat{\sigma}(\rho) = \sum_{i=0}^{\infty} a_i^2 \rho^i \).

4. The normalization (1) has the following consequences. Since \( \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma(X)] = 0 \), we have \( a_0 = 0 \), and since \( \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma^2(X)] = 1 \) we have \( \sum_{i=1}^{\infty} a_i^2 = 1 \).

5. If \( \hat{\sigma} \) denotes the derivative of \( \sigma \), then \( \hat{\sigma} = \hat{\sigma} \).

The above facts imply the following simple bound on the coefficient of non-linearity \( \mu \):

**Lemma 22** For any normalized non-linear activation function \( \sigma \), we have \( 0 < \mu \leq 1 \).

**Proof** The degree 1 Hermite polynomial is \( h_1(u) = u \), so \( a_1 = \mathbb{E}_{X \sim \mathcal{N}(0, 1)}[\sigma(X)X] \). Since \( \sigma \) is non-linear, for at least one \( i \neq 1 \), we have \( a_i 
eq 0 \). This, coupled with the fact that \( \sum_{i=1}^{\infty} a_i^2 = 1 \) implies that \( a_1 \in (-1, 1) \), which implies that \( \mu = 1 - a_1^2 \in (0, 1] \). 

The random initialization of the neural network induces a feature representation of the input vectors at every depth \( l \) in the neural network: \( \Phi^{(l)}_W(x) := \frac{1}{\sqrt{m}} \sigma(W_l \cdot \frac{1}{\sqrt{m}} \sigma(W_{l-1} \cdots \frac{1}{\sqrt{m}} \sigma(W_1 x) \cdots)) \). This feature representation naturally yields a kernel function \( k^{(l)}(x, x') := \Phi^{(l)}_W(x) \cdot \Phi^{(l)}_W(x') \). In particular, after the first layer, the kernel function \( k^{(1)}(x, x') = \frac{1}{m} \sigma(W_1 x) \cdot \sigma(W_1 x') \). The central limit theorem implies that as the width \( m \) goes to infinity, this kernel function tends to a deterministic value, viz. its expectation, which is \( \mathbb{E}_{w \sim \mathcal{N}(0, I_d)}[\sigma(w \cdot x)\sigma(w \cdot x')] \), which equals \( \hat{\sigma}(x \cdot x') \) if \( x \) and
$x'$ are unit vectors. Furthermore, the normalization $\|\mathbb{E}_{X \sim \mathcal{N}(0,1)}[\sigma^2(X)]\|_2 = 1$ implies that the feature representation is itself normalized in the sense for any unit vector $x$, that as $m \to \infty$, we have $\|\mathcal{G}_{W}(x)\|_2 = 1$. Applying these observations recursively, we get Lemma 5, which was also proved by Daniely et al. (2016).

The following technical lemma shows how one application of $\sigma$ behaves:

**Lemma 23** Let $\delta \in [0, 1]$. Then

$$|\sigma(-(1-\delta))| \leq \sigma(1-\delta) \leq \begin{cases} 1 - (1 + \frac{\delta}{2})\delta & \text{if } \delta \leq \frac{1}{2} \\ (1 - \frac{\delta}{2})(1-\delta) & \text{if } \delta > \frac{1}{2}. \end{cases}$$

**Proof** The fact that $|\sigma(-(1-\delta))| \leq \sigma(1-\delta)$ follows from the fact that the power series $\sigma(\rho) = \sum_{i=1}^{\infty} a_i^2 \rho^i$ has only non-negative coefficients. Next, we have

$$\sigma(1-\delta) = \sum_{i=1}^{\infty} a_i^2 (1-\delta)^i \leq a_1^2 (1-\delta) + \sum_{i=2}^{\infty} a_i^2 (1-\delta)^2 = a_1^2 (1-\delta) + (1 - a_1^2)(1-\delta)^2 = (1-\delta)(1-\mu \delta).$$

Now if $\delta > \frac{1}{2}$, we have $(1-\delta)(1-\mu \delta) \leq (1 - \frac{\delta}{2})(1-\delta)$. If $\delta \leq \frac{1}{2}$, we have $(1-\delta)(1-\mu \delta) = 1 - (1 + \mu \delta + \mu \delta^2 \leq 1 - (1 + \frac{\delta}{2})\delta$. $\blacksquare$

Recall the definition of $B(L, \delta)$: let $L_0(\delta) = \max \left\{ \left\lfloor \frac{\log(\frac{\delta}{1+L})}{\log(1+\frac{\delta}{2})} \right\rfloor, 0 \right\} = O\left( \frac{\log(1/\delta)}{\mu} \right)$, and define

$$B(L, \delta) := \begin{cases} 1 - \delta(1 + \frac{\delta}{2})^L & \text{if } L \leq L_0(\delta) \\ \frac{1}{2} \left(1 - \frac{\delta}{2}\right)^{L - L_0(\delta)} & \text{if } L > L_0(\delta). \end{cases}$$

Lemma 6 is an immediate consequence via repeated application of Lemma 23.

Finally, we restate and prove Lemma 7:

**Lemma 24** Let $f : [-1, 1] \to \mathbb{R}$ be an arbitrary function whose power series $f(\rho) = \sum_{i=0}^{\infty} a_i \rho^i$ converges everywhere in $[-1, 1]$ and has non-negative coefficients $a_i \geq 0$. Let $K \in \mathbb{R}^{n \times n}$ be a positive definite matrix with $K \geq \delta I_n$ for some $\delta > 0$, and all diagonal entries equal to 1. Let $f[K]$ be matrix obtained by entrywise application of $f$. Then we have

$$f[K] \succeq (f(1) - f(1-\delta))I_n.$$ 

**Proof** We have $f[K] = \sum_{i=1}^{\infty} a_i K^{\otimes i}$, where $K^{\otimes i}$ denotes the $i$-fold Hadamard (i.e., entrywise) product of $K$ with itself. Since all diagonal entries of $K$ equal 1, we can also write $K^{\otimes i}$ as

$$K^{\otimes i} = (K - \delta I_n)^{\otimes i} + (1 - (1-\delta)^i)I_n.$$ 

By assumption, $K - \delta I_n \succeq 0$. Since the Hadamard product of positive semidefinite matrices is also positive semidefinite, we have $(K - \delta I_n)^{\otimes i} \succeq 0$. Thus, $K^{\otimes i} \succeq (1 - (1-\delta)^i)I_n$. Thus, we have

$$f[K] = \sum_{i=0}^{\infty} a_i K^{\otimes i} \succeq \sum_{i=0}^{\infty} a_i (1 - (1-\delta)^i)I_n = (f(1) - f(1-\delta))I_n,$$

as required. $\blacksquare$
B.1. Top Layer Kernel Matrix

Corollary 3 follows easily from Theorem 2:

**Proof [Corollary 3]** To prove part 1, note that the normalization (1) implies that \( \tilde{K}_{ii} = 1 \) for all \( i \in [n] \). This fact, coupled with Theorem 2 (part 1) and the Gershgorin circle theorem implies the following bounds on the largest and smallest eigenvalues of \( \tilde{K} \): we have \( \lambda_{\text{max}}(\tilde{K}) \leq 1 + (n - 1)B(L, \delta) \) and \( \lambda_{\text{min}}(\tilde{K}) \geq 1 - (n - 1)B(L, \delta) \), which implies that \( \kappa(\tilde{K}) \leq \frac{1 + (n - 1)B(L, \delta)}{1 - (n - 1)B(L, \delta)} \). Since \( L \geq L_1(\delta) = \left[ \frac{-\log(n)}{-\log(1 - \frac{B(L, \delta)}{n})} \right] + L_0(\delta) \), we have \( (n - 1)B(L, \delta) \leq \frac{1}{2} \), and then using the inequality \( \frac{1 + x}{1 - x} \leq 1 + 4x \) for \( x \in [0, \frac{1}{2}] \), the bound on the condition number follows.

As for part 2, using Theorem 2 (part 2) and the bound \( \lambda_{\text{max}}(\tilde{K}) \leq 1 + (n - 1)B(L, \delta) \), we have \( \kappa(\tilde{K}) \leq \frac{1 + (n - 1)B(L, \delta)}{1 - B(L, \delta)} \). Now if \( L \leq L_0(\delta) \), using the definition of \( B(L, \delta) \), we have

\[
\kappa(\tilde{K}) - 1 \leq \frac{n(1 + \delta(1 + \frac{B}{2})L)}{1 - (1 - \delta(1 + \frac{B}{2})L)} \leq \frac{n}{\lambda} (1 + \frac{B}{2})^{-L}.
\]

If \( L > L_0(\delta) \), then we have

\[
\kappa(\tilde{K}) - 1 \leq \frac{n}{\lambda} (1 - \frac{B}{2})^{-L} \leq n(1 - \frac{B}{2})^{-L} \leq n(1 + \frac{B}{2})^{-L} \leq \frac{n}{\lambda} (1 + \frac{B}{2})^{-L},
\]

as required. \( \square \)

B.2. Neural Tangent Kernel Matrix

Recall the formula (2) for the NTK given by Arora et al. (2019c): defining \( \rho := x_i \cdot x_j \), we have

\[
\tilde{K}_{ij} = \sum_{h=1}^{L+1} \sigma^{(h-1)}(\rho) \left( \prod_{h'=h}^{L} \hat{\sigma}(\sigma^{(h')}(\rho)) \right).
\]

Using this formula, we now restate and prove Lemma 10:

**Lemma 25** For any \( \rho \in [-1, 1] \), we have \( \frac{\hat{\sigma}(\rho)}{\sigma(1)} \leq 1 - \mu(1 - |\rho|) \).

**Proof** We have \( \hat{\sigma}(\rho) = \hat{\sigma}(\rho) = \sum_{i=1}^{\infty} i \cdot a_i^2 \cdot \rho^{i-1} \) This implies that \( \hat{\sigma}(\rho) \leq \hat{\sigma}(|\rho|) \), so it suffices to prove the bound for \( \rho \geq 0 \). Also note that by definition \( \hat{\sigma}(\rho) \) is non-negative for all \( \rho \geq 0 \) as well as an increasing function over \( \rho \geq 0 \). Thus, using the fact that \( \sum_{i=2}^{\infty} a_i^2 = 1 - a_1^2 = \mu \), we have

\[
\frac{\hat{\sigma}(\rho)}{\hat{\sigma}(1)} = \frac{a_1^2 + \sum_{i=2}^{\infty} i \cdot a_i^2 \cdot \rho^{i-1}}{a_1^2 + \sum_{i=2}^{\infty} i \cdot a_i^2} \leq \frac{a_1^2 + \sum_{i=2}^{\infty} a_i^2 \cdot \rho}{a_1^2 + \sum_{i=2}^{\infty} a_i^2} = 1 - \left( \frac{\sum_{i=2}^{\infty} i \cdot a_i^2}{a_1^2 + \sum_{i=2}^{\infty} i \cdot a_i^2} \right) (1 - \rho) \leq 1 - \mu(1 - \rho),
\]

as required. \( \square \)

Corollary 9 follows easily from Theorem 8:

**Proof [Corollary 9]** To prove part 1, note that Theorem 8 (part 1) and the Gershgorin circle theorem implies the following bounds on the largest and smallest eigenvalues of \( \tilde{K} \): we have \( \lambda_{\text{max}}(\tilde{K}) \leq \frac{1 + (n - 1)B(L, \delta)}{1 - (n - 1)B(L, \delta)} \).
(1 + 2(n - 1)B(L/2, δ)) \hat{K}_{11} \text{ and } \lambda_{\min}(\hat{K}) \geq (1 - 2(n - 1)B(L/2, δ)) \hat{K}_{11}, \text{ which implies that } \kappa(\hat{K}) \leq \frac{1 + 2(n - 1)B(L/2, δ)}{1 - 2(n - 1)B(L/2, δ)}.

Since \( L \geq L_2(\delta) = \left[ \frac{2\log(2n)}{\log(1 - \frac{x}{2})} \right] + 2L_0(\delta) \), we have \( 2(n - 1)B(L/2, δ) \leq \frac{1}{2} \), and then using the inequality \( \frac{1 + x}{1 - x} \leq 1 + 4x \) for \( x \in [0, \frac{1}{2}] \), the bound on the condition number follows.

As for part 2, using Theorem 8 (part 2) and the bound \( \lambda_{\max}(\hat{K}) \leq (1 + 2(n - 1)B(L/2, δ)) \hat{K}_{11} \), we have \( \kappa(\hat{K}) \leq \frac{1 + 2(n - 1)B(L/2, δ)}{1 - 2B(L/2, δ)} \). Thus,

\[
\kappa(\hat{K}) - 1 \leq \frac{n(1 - \frac{\mu}{2})L/2 - L_0(\delta)}{1 - (1 - \frac{\mu}{2})L/2 - L_0(\delta)} \leq 2n(1 - \frac{\mu}{2})L/2 - L_0(\delta) \leq 2n(1 + \frac{\mu}{2})^{-L/2}(1 + \frac{\mu}{2})L_0(\delta) \leq \frac{2n}{\delta} (1 + \frac{\mu}{2})^{-L/2},
\]

the second inequality follows since \( L/2 - L_0(\delta) \geq L_0(\delta) \), and so \( (1 - \frac{\mu}{2})L/2 - L_0(\delta) \leq (1 - \frac{\mu}{2})L_0(\delta) \leq \frac{1}{2} \).

\[\blacksquare\]

Appendix C. Proofs on SQ Learnability of Random Deep Neural Nets

Proof [Proof of Theorem 18] We use the following notation in the proof. Given input \( x_i \), we denote \( x_i^{(h)} \) to be the representation obtained at depth \( h \) of the network and \( \hat{x}_i^{(h)} \) to be the corresponding normalized input. Recall that we are normalizing the output of each layer to be unit length. Similarly, given \( x_i, x_j \), we denote by \( \hat{p}_h = \hat{x}_i^{(h)} \cdot \hat{x}_j^{(h)} \) and \( p_h = x_i^{(h)} \cdot x_j^{(h)} \). Next we have that conditioned on \( x_i^{(h-1)} \) and \( x_j^{(h-1)} \),

\[
\|x_i^{(h)}\|^2 = \frac{1}{m} \sum_{j=1}^{m} \sigma^2(w_j \cdot \hat{x}^{(h-1)})
\]

where \( w_j \sim N(0, I) \) and \( \mathbb{E}[\|x_i^{(h)}\|^2] = 1 \). Furthermore, since \( \sigma(w_j \cdot \hat{x}^{(h-1)}) \) is a subgaussian random variable with constant subgaussian norm, \( \|x_i^{(h)}\|^2 \) is a sum of subexponential random variables. By Bernstein’s inequality for subexponential random variables (Vershynin, 2018) we have that for a universal constant \( c > 0 \),

\[
Pr(\|x_i^{(h)}\|^2 - 1 > t) \leq 2e^{-c\min(mt^2,mt)}.
\]

(4)

Similarly we have that

\[
\rho_h = \frac{1}{m} \sum_{j=1}^{m} \sigma(w_j \cdot \hat{x}_i^{(h-1)})\sigma(w_j \cdot \hat{x}_j^{(h-1)})
\]

with \( \mathbb{E}[\rho_h] = \hat{\sigma}(\rho_{h-1}) \). Noting that product of subgaussian random variables is subexponential and again applying Bernstein’s inequality for subexponential random variables we get that

\[
Pr(|\rho_h - \hat{\sigma}(\rho_{h-1})| > t) \leq 2e^{-c\min(mt^2,mt)}.
\]

(5)
Next, we will use (4) and (5) to argue that with high probability \( \hat{\rho}_h \) remains close to \( \hat{\sigma}(\hat{\rho}_{h-1}) \). For suitable constant \( \epsilon < 1 \) to be chosen later, we have that

\[
Pr(|\hat{\rho}_h - \hat{\sigma}(\hat{\rho}_{h-1})| > t) = Pr\left(\left|\frac{\rho_h}{\|x_i^{(h)}\|\|x_j^{(h)}\|} - \hat{\sigma}(\hat{\rho}_{h-1})\right| > t\right)
\]

\[
\leq Pr\left(\left|\frac{\rho_h - \hat{\sigma}(\hat{\rho}_{h-1})}{\|x_i^{(h)}\|\|x_j^{(h)}\|}\right| > t\right) + Pr\left(\left|\hat{\sigma}(\hat{\rho}_{h-1})\left(\frac{1}{\|x_i^{(h)}\|\|x_j^{(h)}\|} - 1\right)\right| > t\right)
\]

\[
\leq Pr\left(\|x_i^{(h)}\|\|x_j^{(h)}\| > (1 + \epsilon)^2\right) + Pr\left(\|\rho_h - \hat{\sigma}(\hat{\rho}_{h-1})\| > t(1 + \epsilon)^2\right)
\]

\[
+ Pr\left(\left|\|x_i^{(h)}\|\|x_j^{(h)}\| - 1\right| > \frac{2t}{\hat{\sigma}(\hat{\rho}_{h-1})}\right).
\]

Noticing that \( \hat{\sigma}(\hat{\rho}_{h-1}) \leq 1 - (1 + \frac{1}{2})\delta \), and using (4) and (5), we get that

\[
Pr(|\hat{\rho}_h - \hat{\sigma}(\hat{\rho}_{h-1})| > t) \leq 2\left(e^{-cm\epsilon^2} + e^{-cm(1+\epsilon)^4t^2} + e^{-\frac{m^2}{4t(1+(1+\epsilon)/2)^2}\delta}\right).
\]

Setting \( t = \delta \mu/4 \) and \( \epsilon \) to be a small enough constant we get that

\[
Pr(|\hat{\rho}_h - \hat{\sigma}(\hat{\rho}_{h-1})| > \frac{\delta \mu}{4}) \leq 2\left(e^{-\Omega(m)} + e^{-\Omega(m\mu^2\delta^2)} + e^{-\Omega(m\delta\mu)}\right).
\]

Setting \( m \geq c_1\frac{L}{\mu^2\delta^2} \) and using a union bound over all layers we get that with probability at least \( 1 - Le^{-\Omega(L)} \), the updates of \( \hat{\rho}_h \) will approximately satisfy the ideal updates from Theorem 2 and as a result, for a constant \( c_1 > 0 \), after \( L_0(\delta) = c_1 \frac{\log(1/\delta)}{\mu^2} \) depth, with high probability, \( \hat{\rho}_h \) (and \( \rho_h \)) will fall below \( 1/4 \) and will continue to be below \( 1/2 \) for all \( L \). Define \( G \) to be the intersection of above good event and that \( \rho_h \in [(1-\epsilon)\hat{\rho}_h, (1+\epsilon)\hat{\rho}_h] \) for all \( h \in [L] \). Then we know that \( P(G) \geq 1 - 2Le^{-\Omega(L)} \). Conditioned on this good event and using Lemma 23 we have that for \( h > L_0(\delta) \),

\[
|E[\rho_h | G, \rho_{h-1}]| = |\hat{\sigma}(\hat{\rho}_{h-1})| \leq (1 - \frac{\mu}{2})|\hat{\rho}_{h-1}|
\]

\[
\leq (1 + \epsilon)(1 - \frac{\mu}{2})|\rho_{h-1}|
\]

\[
\leq (1 - \frac{\mu}{4})|\rho_{h-1}|
\]

for a small enough constant \( \epsilon \). Hence we get that for \( h > L_0(\delta) \), \(|E[\rho_h | G]| \leq e^{-\Omega(h)} \). Finally notice that

\[
E[\rho_h] = P(G)E[\rho_h | G] + P(\bar{G})E[\rho_h | \bar{G}]
\]

Combined with the probability of the good event and noticing that \( \hat{\sigma}(\hat{\rho}_h) \) is always bounded, we get that

\[
|E[\rho_{L_0(\delta)+h}]| \leq e^{-\Omega(h)} + Le^{-\Omega(L)}.
\]
Proof [Proof of Theorem 19] As mentioned in Section 5 we will consider a randomly initialized deep neural network defined as

$$f_W(x) = v \cdot \Pi(\sigma(W_L \frac{1}{\sqrt{m}} \Pi(\sigma(W_{L-1} \cdots \frac{1}{\sqrt{m}} \Pi(\sigma(W_1 x)\ldots))))))$$

We consider a distribution $D$ that is the uniform distribution over a set $S$ that consists of half of the inputs in $\{+1, -1\}^d$ thereby ensuring that no two inputs are collinear. In particular, one can take the set of all $2^{d-1}$ inputs that fall on one side of a fixed halfspace. The first step in the analysis is to show that $f$ is uncorrelated with any fixed function $g: \mathbb{R}^d \to [-1, 1]$. In particular we have that

$$\mathbb{E}[\mathbb{E}[g(x) f(x)^2]] = \frac{1}{|S|^2} \left( \sum_{x \in S} \mathbb{E}[g^2(x) f_W^2(x)] + 2 \sum_{x \neq y} \mathbb{E}[g(x)g(y) f_W(x)f_W(y)] \right)$$

$$= \frac{1}{|S|^2} \left( \sum_{x \in S} \mathbb{E}[g^2(x)] + 2 \sum_{x \neq y} \mathbb{E}[f_W(x)f_W(y)] \right)$$

$$= \frac{1}{|S|^2} \left( \sum_{x \in S} \mathbb{E}[g^2(x)] + 2 \sum_{x \neq y} \mathbb{E}[\Phi_W(x) \cdot \Phi_W(y)] \right)$$

Next, Theorem 18 implies that

$$|\Phi_W(x) \cdot \Phi_W(y)| \leq Le^{-\Omega(L)}$$

Substituting above and noticing that $g(x) \in [-1, 1]$ we have

$$\mathbb{E}[\mathbb{E}[g(x) f(x)^2]] \leq \frac{1}{|S|^2} \left( \sum_{x \in S} 1 + 2 \sum_{x \neq y} Le^{-\Omega(L)} \right)$$

$$\leq Le^{-\Omega(L)}$$

(6)

Next assume that there exists an algorithm $A$ that makes $p(d, L)$ queries of tolerance $r(d, L)$ to an SQ oracle for a random function $f_W(x)$ as defined above, and with probability at least half (over the randomness of the algorithm and the random draw of the function), outputs a function $g$ such that

$$\mathbb{E}_D[f_W(x)h(x)] \geq \frac{1}{q(d, L)}$$

Here we assume that both $r(d, L)$ and $q(d, L)$ are polynomial in $d$ and $L$. To get a contradiction we will use the technique from the work of Bshouty and Feldman (2002) (see Theorem 31). As a first step, since we are in the case of learning with respect to a fixed distribution, from the work of Bshouty and Feldman (2002) it follows that we can, without loss of generality, assume that the statistical queries $\psi(x, f_W(x))$ used by the algorithm are correlation queries, i.e. $\psi(x, f_W(x)) = \mathbb{E}_D[f_W(x)g(x)]$. Next we simulate the algorithm $A$ and each time the algorithm makes a statistical query $\mathbb{E}_D[f_W(x)g_i(x)]$, we add $g_i$ to a set $\mathcal{H}$. Finally, if the algorithm outputs a hypothesis $h$ at the end, we add $h$ to $\mathcal{H}$ as well. Notice that if $A$ makes $p(d, L)$ queries then $|\mathcal{H}| \leq p(d, L) + 1$. Next from (6) and a union bound over $\mathcal{H}$ we can say that

$$\mathbb{P}_W \left[ \exists h \in \mathcal{H}: \mathbb{E}_D[(f_W(x)h(x))^2] > \frac{1}{\max(q^2(d, L), r^2(d, L))} \right] \leq O\left( \max(q^2(d, L), r^2(d, L))(p(d, L) + 1)e^{-\Omega(L)} \right).$$
Since the correlation of each function in $\mathcal{H}$ with $f_W(x)$ is at most $1/r(d, L)$, a zero answer to every query asked by the algorithm is a valid output of the SQ oracle, and hence with probability at least $1/2$, the algorithm must output a function in $\mathcal{H}$ that is correlated with $f_W(x)$. In other words, we have that

$$\Pr_{W}[\exists h \in \mathcal{H} : \mathbb{E}_{D}[(f_W(x)h(x))]^2 > \frac{1}{\max(q^2(d, L), r^2(d, L))}] \geq \frac{1}{2}.$$ 

From the above we get that

$$\max(q^2(d, L), r^2(d, L))(p(d, L) + 1) \geq e^{O(L)}.$$

\[\square\]

**Appendix D. Interpolation analysis**

The first step in the analysis is the following bound on $\|\Sigma\|_2$:

**Lemma 26** For any positive integer $N$ and $\gamma \in (0, 1/2)$, if the number of hidden layers $L \geq \left\lceil \frac{\log (N)}{-\log (1-\gamma)} \right\rceil + L_0(\Delta(N, \gamma))$, then $\|\Sigma\| \leq \frac{8\ln(N)}{N}$.

**Proof** Let $T = \{x_1', x_2', \ldots, x_N'\}$ be a sample set of size $N$ drawn i.i.d. from the marginal distribution of $\mathcal{D}$ over the $x$-coordinate. Let $X'$ denote by the infinite matrix corresponding to the linear map from $\mathcal{H} \to \mathbb{R}^N$ such that for any $v \in \mathcal{H}$, $X'v \in \mathbb{R}^m$ has $i$th component $v^\top \Phi(x_i')$. Let $X'X'^\top = \bar{K}'$, the top layer kernel matrix for the training data defined by $\bar{k}$.

Then by the definition of $\Delta$, with probability at least $1 - \gamma$, Assumption 1 holds for $T$ with $\delta = \Delta(N, \gamma)$. Conditioned on this Assumption 1 holding, Theorem 2 (part 1) implies that $\|\bar{K}'\| \leq 2$ since $L \geq \left\lceil \frac{\log (N)}{-\log (1-\gamma)} \right\rceil + L_0(\Delta(N, \gamma))$. Thus, $\|X'^\top X'\| = \|X'X'^\top\| \leq 2$, which implies that $\Pr_T[\|X'^\top X'\| \leq 2] \geq 1 - \gamma$.

Note that for any $x \in \mathbb{S}^{d-1}$, we have $\|\Phi(x)\|^2 = \bar{k}(x, x) = \sigma(L)(1) = 1$. Thus $\|\Phi(x)\Phi(x)^\top\| = 1$, $\text{Tr}(\Sigma) \leq 1$, $\text{intdim}(\Sigma) := \frac{\text{Tr}(\Sigma)}{\|\Sigma\|} \leq \frac{1}{\|\Sigma\|}$, and $(\Phi(x)\Phi(x)^\top)^2 = \Phi(x)\Phi(x)^\top$. Thus, Theorem 7.7.1 in (Tropp, 2015) and some simple calculations imply that for $\ell := \ln(8/|\Sigma|)$, we have

$$\Pr_T \left[ \left\| \frac{1}{N}X'^\top X' - \Sigma \right\| > \frac{2\ell}{N} + \sqrt{\frac{2\ell\|\Sigma\|}{N}} \right] \leq \frac{1}{2}.$$ 

By a union bound, we have

$$\Pr_T \left[ \|X'^\top X'\| \leq 2 \text{ and } \left\| \frac{1}{N}X'^\top X' - \Sigma \right\| \leq \frac{2\ell}{N} + \sqrt{\frac{2\ell\|\Sigma\|}{N}} \right] \geq \frac{1}{2} - \gamma > 0.$$ 

This implies that $\|\Sigma\| \leq \frac{2\ell}{N} + \sqrt{\frac{2\ell\|\Sigma\|}{N}} \leq \frac{2\ell}{N} + \frac{2\ell(N + |\Sigma|)}{2N} \Rightarrow \|\Sigma\| \leq \frac{6\ell}{N} \Rightarrow \|\Sigma\| \leq \frac{8\ln(N)}{N}$ using the fact that $\ell = \ln(8/|\Sigma|)$.

The following lemma is a standard calculation that is a slight generalization of a similar statement\(^4\) in (Bartlett et al., 2019b):

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\(^4\) Here, we don’t need the $\mathbb{E}[y/\Phi(x)]$ to be a linear function of $\Phi(x)$. 

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Lemma 27  The excess risk of the minimum norm estimator satisfies
\[
\mathbb{E}_{(x,y)} \left[ (y - v_S^T \Phi(x))^2 \right] - \mathbb{E}_{(x,y)} \left[ (y - v^* \Phi(x))^2 \right] \leq 2v^T B v^* + 2\epsilon^T C \epsilon,
\]
where \( \epsilon = y - X v^* \), \( B = (I - X^T \tilde{K}^{-1} X) \Sigma (I - X^T \tilde{K}^{-1} X) \), and \( C = \tilde{K}^{-1} X \Sigma X^T \tilde{K}^{-1} \).

Proof Since \( v^* \) is a minimizer of \( \mathbb{E}_{(x,y)} \left[ (y - v^T \Phi(x))^2 \right] \), we have \( \nabla_v \mathbb{E}_{(x,y)} \left[ (y - v^T \Phi(x))^2 \right] = 0 \), which implies that \( \mathbb{E}_{(x,y)} \left[ (y - v^T \Phi(x)) \Phi(x) \right] = 0 \). Using this fact, we have
\[
\mathbb{E}_{(x,y)} \left[ (y - v_S^T \Phi(x))^2 \right] = \mathbb{E}_{(x,y)} \left[ (y - v^* \Phi(x) + (v_S - v^*) \Phi(x))^2 \right]
= \mathbb{E}_{(x,y)} \left[ (y - v^* \Phi(x))^2 \right] + 2 \mathbb{E}_{(x,y)} \left[ (v_S - v^*) \Phi(x) \right] + \mathbb{E}_{(x,y)} \left[ ((v_S - v^*) \Phi(x))^2 \right]
\]
Using this fact, and that \( v_S = X^T \tilde{K}^{-1} y = X^T \tilde{K}^{-1} (X v^* + \epsilon) \), we get that the excess risk equals
\[
\mathbb{E}_{(x,y)} \left[ ((v_S - v^*) \Phi(x))^2 \right] = \mathbb{E}_{(x,y)} \left[ (X^T \tilde{K}^{-1} (X v^* + \epsilon) - v^*) \Phi(x) \right]^2
= \mathbb{E}_{(x,y)} \left[ (X^T \tilde{K}^{-1} X - I) v^* + X^T \tilde{K}^{-1} \epsilon \right] \Phi(x))^2 \right]
\leq 2 \mathbb{E}_{(x,y)} \left[ (X^T \tilde{K}^{-1} X - I) v^* \right] \Phi(x))^2 \right] + 2 \mathbb{E}_{(x,y)} \left[ (X^T \tilde{K}^{-1} \epsilon \right] \Phi(x))^2 \right]
= 2v^T B v^* + 2\epsilon^T C \epsilon.
\]
The last equality uses the fact that for any \( v \in \mathcal{H} \), we have
\[
\mathbb{E}_{(x,y)} \left[ (v^T \Phi(x))^2 \right] = \mathbb{E}_{(x,y)} \left[ v^T \Phi(x) \Phi(x)^T v \right] = v^T \Sigma v.
\]

We can now prove Theorem 20:

Proof (Theorem 20) First, as in the proof of Lemma 26, by the definition of \( \Delta \), with probability at least \( 1 - \gamma \), Assumption 1 holds for \( S \) with \( \delta = \Delta(n^2, \gamma) \). Conditioned on this Assumption 1 holding, Theorem 2 (part 1) implies that \( \lambda_{\min}(\tilde{K}) \geq \frac{1}{2} \) since \( L = \left[ \frac{-\log(n^2)}{\log(1 - \Delta)} \right] + L_0(\Delta(n^2, \gamma)) \). Thus, with probability at least \( 1 - \gamma \) over the choice of \( S, \tilde{K} \) is non-singular, and hence there exists an interpolating linear predictor.

We now bound the excess risk via Lemma 27. We first analyze the \( v^T B v^* \) part of the bound. Note that \( (I - X^T \tilde{K}^{-1} X) \) is the matrix corresponding to the projection on to the orthogonal complement of the row space of \( X \), and so \( \|I - X^T \tilde{K}^{-1} X\| \leq 1 \). Thus, \( \|B\| \leq \|(I - X^T \tilde{K}^{-1} X)\| \|\Sigma\| \|(I - X^T \tilde{K}^{-1} X)\| \leq \|\Sigma\| \), and so
\[
v^T B v^* \leq \|\Sigma\| v^* v^* \|
\]
Next, we turn to bounding the $\epsilon^T C \epsilon$ part. We have

$$\epsilon^T C \epsilon \leq \|C\| \|\epsilon\|^2 = \|C\| \|y - X v^*\|^2 \leq n(1 + \|v^*\|^2) \|C\|,$$

since for all $i$, $y_i \in [-1, 1]$ and $\|\Phi(x_i)\| = 1$. So now we need to bound $\|C\|$. We have $C \preceq \bar{K}^{-1} X (\Sigma \Sigma^T) X^T \bar{K}^{-1} = \|\Sigma\| \bar{K}^{-1}$, so $\|C\| \leq \frac{\|\Sigma\|}{\lambda_{\min}(\bar{K})}$. As described in the beginning of this proof, we have $\Pr_S [\lambda_{\min}(\bar{K}) \geq 1/2] \geq 1 - \gamma$. This implies that with probability at least $1 - \gamma$ over the choice of $S$, we have

$$\epsilon^T C \epsilon \leq 2n(1 + \|v^*\|^2) \|\Sigma\|. \quad (8)$$

Finally, note that the setting $L = \left[ \frac{\log(n^2)}{-\log(1 - \frac{\mu}{2})} \right] + L_0(\Delta(n^2, \gamma))$ implies that Lemma 26 holds for $N = n^2$. So, $\|\Sigma\| \leq \frac{16 \ln(n)}{n^2}$. Plugging this bound into (7) and (8), and using Lemma 27, we get the bound stated in the theorem. \(\blacksquare\)

Appendix E. Optimization Proofs

We begin by proving simple well-known theorems regarding gradient descent and stochastic gradient descent for linear regression. Consider the following problem

$$\mathcal{L}(w) = \frac{1}{n} \sum_{i=1}^{n} \|a_i^T w - y\|^2$$

Let $A$ be the matrix whose rows are $a_i$. We will assume that $\lambda_{\min}(A^T A) > 0$, which in particular implies that $\min_w \mathcal{L}(w) = 0$. Let’s first consider gradient descent, i.e.

$$w_{t+1} = w_t - \eta \nabla \mathcal{L}(w_t)$$

We have the following well known guarantee (Nesterov, 2014).

**Theorem 28** For gradient descent we have that

$$\mathcal{L}(w_t) \leq e^{-\frac{t}{\lambda_{\min}(A^T A)}} \mathcal{L}(w_0)$$

Next we consider the stochastic gradient descent algorithm,

$$w_{t+1} = w_t - \eta \tilde{\nabla} \mathcal{L}(w_t),$$

where $\tilde{\nabla} \mathcal{L}(w_t)$ is a gradient over a single $a_i$, which is uniformly randomly sampled. Since we have assumed that the loss is 0, even SGD is known to have linear convergence in this setting. Since we did not find a concise proof of this fact anywhere we include it here for completeness.

**Theorem 29** Let $\|a_i\|^2 \leq \beta$ for all $i$, then stochastic gradient descent produces an $\epsilon$-sub-optimal point in expectation in total number steps bounded by

$$\frac{8n \beta}{\lambda_{\min}(A^T A)} \cdot \log(1/\epsilon).$$
Lemma 11

Theorem 29

we get the required result.

The theorem follows by noticing that

Theorem 29

Corollary 3

. To this end note that, using

Theorem 13

im-

Lemma 11

W e wish to invoke

Theorem 28

Corollary 3

implies that at that

Corollary 9

implies that the finite-width kernel also has a constant condition number. The statement then follows

Proof [Proof of Theorem 12] The statement follows by noticing that at that setting of depth, Corollary 3 implies that the infinite-width kernel has constant condition number. Now invoking Lemma 11 implies that the finite-width kernel also has a constant condition number. The statement then follows from Theorem 28.

Proof [Proof of Theorem 13] We wish to invoke Theorem 29. To this end note that, using Lemma 11 and the fact that the diagonal entries are 1 in $K$, we get that $\beta \leq 2$ w.h.p. Similarly using Corollary 3, we can derive that $\lambda_{\text{min}} = \Omega(n)$ w.h.p. Therefore using Theorem 29 we get the required result.

Proof [Proof of Corollary 16] The theorem follows by noticing that Corollary 9 implies that at that depth, the condition number of the infinite-width NTK is constant. The statement now follows from Theorem 15.

Appendix F. Conditioning for One Layer ReLU Networks

In this section we establish that given a set of non-collinear points in $\mathbb{R}^d$, a sufficiently wide one layer neural network with ReLU activations leads to a non-singular gram matrix at the output layer. This property crucially relies on the property of the ReLU activations and we in addition show that one cannot hope for such a statement to hold for a general non-linear activation function. We state our main theorem below.
**Theorem 30** Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of $n$ vectors in $\mathbb{R}^d$ such that each $x_i$ is a unit length vector and for each $i \neq j$, it holds that $|x_i \cdot x_j| \leq 1 - \delta$. Let $w_1, w_2, \ldots, w_m$ be vectors drawn i.i.d. from $\mathcal{N}(0, I_{d \times d})$ and consider the feature mapping $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as

$$\Phi(x) = \frac{1}{\sqrt{m}}(\sigma(w_1 \cdot x), \sigma(w_2 \cdot x), \ldots, \sigma(w_m \cdot x)),$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is the ReLU activation defined as $\sigma(x) = \max(x, 0)$. Let $\Phi(X)$ be the corresponding $m \times n$ data matrix obtained by applying $\Phi$ to points in $S$, i.e., column $i$ of $\Phi(X)$ equals $\Phi(x_i)$. There exists a universal constant $c > 0$, such that if $m \geq c \frac{n^8 \log(n/\delta)}{\delta^3}$, then w.p. at least $1 - 1/poly(n)$, we have that $\sigma_{\min}(\Phi(X)^T \Phi(X)) \geq \Omega(\frac{\delta^{3/2}}{n})$.

**Proof** The proof is a modification of the gradient lower bound argument as detailed in the proof of Lemma 9.3 of Allen-Zhu et al. (2018). We will show that $\sigma_{\min}(\Phi(X)) \geq \frac{\delta^{3/2}}{9000n^{3/2}}$. This will imply the claim of the Theorem. We will first show that for a fixed $\alpha \in \mathbb{R}^n$, such that $\|\alpha\| = 1$, $\|\Phi(X)\alpha\|$ is large. Then we will complete the argument using a union bound over an appropriate net for unit length vectors in $\mathbb{R}^n$. We have that

$$\Phi(X)\alpha = \sum_{i=1}^{n} \alpha_i \Phi(x_i)$$

and hence

$$\|\Phi(X)\alpha\|^2 = \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \alpha_i h_j(x_i) \right)^2,$$

where $h_j(x_i) = \sigma(w_j \cdot x_i)$. Next, fix a particular $j \in [m]$ and let

$$T_j = \sum_{i=1}^{n} \alpha_i h_j(x_i).$$

We will first show that with non-trivial probability $T_j$ is large. Let $i^* \in [n]$ be such that $|\alpha_{i^*}| \geq \frac{1}{\sqrt{n}}$, with ties broken arbitrarily. Next, we will write

$$w_j = \left( \sqrt{1 - \theta^2} z_1 x_{i^*} + g \right) + \theta z_2 x_{i^*}$$

$$= w_{j,1} + w_{j,2}$$

Here we pick $\theta = \frac{\sqrt{3}}{5n}$ and $z_1, z_2$ are independent $\mathcal{N}(0, 1)$ Gaussians and $g$ is a standard $d$ dimensional Gaussians orthogonal to $x_{i^*}$. Next, define $G_j$ to be the following good event

$$G_j = 1\left( |w_{j,1} \cdot x_{i^*}| \leq \frac{\sqrt{3}}{10n} \land \forall i \neq i^*, |w_{j,1} \cdot x_i| > \frac{\sqrt{3}}{4n} \right).$$

Next, we have that $P_{w_{j,1}}(G_j) \geq \frac{\sqrt{3}}{90n}$. This is established in Lemma 31 at the end of the section. Conditioning on $G_j$, i.e., fixing the randomness in $w_{j,1}$, we notice that

$$|w_{j,2} \cdot x_{i^*}| = \theta |z_2|$$
and for any \( i \neq i^* \),
\[
|w_{j,2} \cdot x_i| \leq \theta|z_2|.
\]

Since \( \theta z_2 \) is a standard Gaussian with variance \( \theta^2 \), we have that the event \( E : \{ \theta|z_2| \in \left[ \frac{\sqrt{\delta}}{3m}, \frac{\sqrt{\delta}}{m} \right] \} \) holds with constant probability, i.e.,
\[
P(E) \geq 0.2.
\]

Now conditioned on \( G_j \cap E \), we have that fixing the randomness in \( w_{j,1} \) fixes the sign of \( \sum_{i \neq i^*} \alpha_i \sigma(w_{j,1} \cdot x_i) \). Furthermore, after fixing the randomness in \( w_{j,1} \), there is still a probability of 0.5 over the randomness in \( w_{j,2} \) that \( \alpha_i \sigma(w_{j,2} \cdot x_{i^*}) \) matches that of \( \alpha_i \sigma(w_{j,2} \cdot x_{i^*}) \). Combining everything, we get that with probability at least \( \frac{\sqrt{\delta}}{300n} \) (over \( w_{j,2} \)), it holds that \( T_j^2 \geq \frac{\delta}{81n^2} \).

Next define \( B \) to be the event that for all \( j \), \( |T_j| \leq 100\sqrt{n} \sqrt{\log n \log m} \). It is easy to see that \( B \) holds with probability at least \( 1 - \Theta(n \log n \log m) \), and hence when conditioned on \( B \), we also have that \( |T_j^2| \geq \frac{\delta}{81n^2} \) with probability at least \( \frac{\sqrt{\delta}}{300n} \). Next we will argue that when conditioned on \( B \), \( \| \Phi(X) \alpha \| ^2 \) is large except with exponentially small probability. Combined with the fact that \( B \) happens with high probability, this will imply that over the randomness in \( w_1, \ldots, w_m \), \( \| \Phi(X) \alpha \| \) is large with high probability.

When conditioned on \( B \), \( \| \Phi(X) \alpha \| ^2 \) is an average of \( m \) independent random variables, each bounded in \( [0, 100^2 n \log n \log m] \) and that
\[
\mathbb{E}[\| \Phi(X) \alpha \| ^2] \geq \frac{\delta^{3/2}}{81000n^2}.
\]

Hence from Chernoff bound and the fact that \( B \) holds with high probability we get that with probability at least \( 1 - \epsilon^{-\Omega(\frac{\delta^3 m}{n \log n \log m})} \), \( \| X \alpha \| ^2 \geq \Omega(\frac{\delta^{3/2}}{n^{m/2}}) \).

Having argued the bound for a fixed \( \alpha \), we now consider an appropriate net over unit length vectors in \( \mathbb{R}^n \) to argue that over all \( \alpha \), \( \| \Phi(X) \alpha \| \) is large. In particular, consider an \( \epsilon \)-net of the unit sphere with \( \epsilon = \frac{\delta^{3/4}}{2000n^{3/2} \sqrt{m \log n \log m}} \). The size of such a net is at most \( (3/\epsilon)^n \). Hence, we get that with probability at least \( 1 - \epsilon^{-\Omega(\frac{\delta^3 m}{n \log n \log m})} \) \( e^{n \log(3/\epsilon)} \), for any vector \( \alpha \) in the net, we have \( \| \Phi(X) \alpha \| \geq \frac{\delta^{3/4}}{1000n^{3/2}} \). This in turn implies that with the same probability, for any \( \alpha \) on the unit sphere with \( \hat{\alpha} \) being its closest vector in the net, we have
\[
\| \Phi(X) \alpha \| \geq \| \Phi(X) \hat{\alpha} \| - \epsilon \| \Phi(X) \| \geq \frac{\delta^{3/4}}{1000n^{3/2}} - \epsilon \| \Phi(X) \| .
\]

The bound then follows from noticing that via standard Gaussian concentration we have that with probability at least \( 1 - \frac{1}{\Theta(n \log n \log m)} \), \( \| \Phi(X) \| \leq O(\sqrt{mn \log m \log n}) \).

**Lemma 31** Let \( x_1, x_2, \ldots, x_n \) be unit length vectors in \( \mathbb{R}^m \), where \( |x_i \cdot x_j| \leq 1 - \delta \) for \( i \neq j \). Let
\[
w = \sqrt{1 - \theta^2} x_1 + g
\]
where \( z \) is \( \mathcal{N}(0, 1) \), \( g \) is a standard \( d \) dimensional Gaussian orthogonal to \( x_1 \), and \( \theta = \delta^{1/4}/(5n) \). Define \( G \) to be the event

\[
G = 1 \left( |w \cdot x_1| \leq \frac{\sqrt{\delta}}{10n} \land \forall i \neq 1, |w \cdot x_i| > \frac{\sqrt{\delta}}{4n} \right).
\]

Then it holds that

\[
P_w(G_j) \geq \frac{\sqrt{\delta}}{50n}.
\]

**Proof** We have that \( w \cdot x_1 \) is \( \mathcal{N}(0, (1 - \theta^2)) \) and also that \( \theta \leq 1/5 \). Hence, we have that

\[
P(|w \cdot x_1| < \frac{\sqrt{\delta}}{10n}) \geq \frac{\sqrt{\delta}}{25n}.
\]

For a fixed \( i \neq 1 \), we have that

\[
w \cdot x_i = \sqrt{1 - \theta^2} z(x_1 \cdot x_i) + g.x_i.
\]

Conditioning on the fact that \( |w \cdot x_1| \) is at most \( \frac{\sqrt{\delta}}{10n} \), \( w \cdot x_i \) is a Gaussian with mean at most \( \frac{\sqrt{\delta}}{10n} \) and variance at least \( \frac{\delta^2}{4} \) (since projection of \( x_i \) on \( g \) is at least \( \delta/\sqrt{2} \)). Hence, with probability at least \( 1 - 1/8n \), we have that \( |w \cdot x_i| > \frac{\sqrt{\delta}}{4n} \). Using a union bound we get that, conditioned on \( z \) being small, with probability at least 0.5, all \( i \neq 1 \) satisfies \( |w \cdot x_i| > \sqrt{\delta}/(4n) \). Combining with (9) we get the claim. \( \blacksquare \)

Theorem 1(b) in (Gneiting, 2013) provides a generalization of Theorem 30 to a large class of activations \( \sigma \), although it doesn’t prove a quantitative lower bound on the smallest eigenvalue. For completeness, we reformulate that theorem in our language here:

**Theorem 32** Suppose the Hermite expansion of \( \sigma \) has infinitely many even and infinitely many odd coefficients. If the inputs satisfy Assumption 1, then the kernel matrix \( \bar{K} \) for a 1-hidden layer neural network is non-singular.