A translation of Henri Joris’ “Le chasseur perdu dans la forêt” (1980)

Steven Finch

October 1, 2019

Abstract. This is an English translation of Henri Joris’ article “Le chasseur perdu dans la forêt (Un problème de géométrie plane)” that appeared in Elemente der Mathematik v. 35 (1980) n. 1, 1–14. Given a point $P$ and a line $L$ in the plane, what is the shortest search path to find $L$, given its distance but not its direction from $P$? The shortest search path was described by Isbell (1957), but a complete and detailed proof was not published until Joris (1980). I am thankful to Natalya Pluzhnikov for her dedicated work and to the Swiss Mathematical Society for permission to post this translation on the arXiv.

1. A hunter wandered into the woods and became disoriented. After awhile, he found a sign with information about a road that passes exactly 1 kilometer away. Unfortunately, the tree on which the sign was attached had fallen to the ground, and the hunter possessed no idea in what direction he should travel to reach the road. He decided to walk 1 kilometer straight in an arbitrary direction and then continue along a circle with center at the sign location. Therefore, he is certain to find the road after walking at most $1 + 2\pi$ kilometers. 

This situation is described in a problem presented at a mathematical competition for high school students, which are so popular in the United States. The question is: Suppose the road is straight. What is the shortest search path that the hunter would have taken, had he thought about everything a little more carefully? The answer probably is: “Instead of traversing the full circumference of the circle, he would replace the last quarter by a straight path of length 1 km tangent to the circle (Figure 1), saving $\pi/2 - 1 \approx 0.57$ km”.

However, he can make his way even shorter if he first continues to walk in the direction of the radius beyond the circle and then return to the circle tangentially (Figure 2). This will give the path $ABCDEF$ of length

$$\frac{1}{\cos \alpha} + \tan \alpha + \omega + 1 = \frac{1}{\cos \alpha} + \tan \alpha + \frac{3}{2\pi} - 2\alpha + 1 = \ell(\alpha).$$

$\ell(\alpha)$ attains its minimum at $\alpha = 30^\circ = \pi/6$,

$$\ell \left( \frac{\pi}{6} \right) = \frac{7}{6\pi} + 1 + \sqrt{3}. \quad (1)$$
2. If we disassociate from the hunter and the forest, and rephrase in terms of plane geometry, we come to the following problem (P):

Given a circle of radius 1, find the shortest path that starts at the center of the circle and intersects all the tangents to the circle.

The following theorem holds:

**Theorem 1.** A solution of (P) is the path of Figure 2 with $\alpha = \pi/6$. Any other solution is obtained from it by rotations and reflections of the plane that leave the circle in its place.

If we remove from (P) the condition about the starting point, we obtain

**Theorem 2.** The shortest path that intersects all the tangents of a given circle consists of a semicircle (subset of the given circle) extended on each side by a segment of the tangent of length of the radius (Figure 3).

(If the radius is of length 1, the length of the path is $\pi + 2$.)

Although these theorems, especially the second, look quite elementary and plausible, I could not find simple and totally elementary proofs of them. This is due to the fact that one must take into consideration all continuous and rectifiable curves that satisfy the conditions, and also that the solutions must be curves composed of straight line segments and circular arcs.

3. In what follows, we shall prove these two theorems. The idea is to show that each minimal curve is one of those indicated in the theorems. First we prove the existence of minimal curves.
Let $m$ be the infimum of the lengths of the curves that meet all the tangents, and $C_n$ a minimal sequence of such curves; then $\ell(C_n) = (\text{length of } C_n) \to m$. Each $C_n$ can be parametrized by $f_n : [0, 1] \to E_2$, the parameter being proportional to the arc length. The $\ell(C_n)$ are bounded; therefore, all the $C_n$ are contained in a closed square, and
\[
\text{distance } (f_n(t_2), f_n(t_1)) \leq |t_1 - t_2| \cdot \ell(C_n) \leq |t_1 - t_2| \cdot M
\]
for all $t_1, t_2, n$, where $M$ is a constant. Hence one can apply the Arzela-Ascoli theorem: there is a path $C$ given by $f : [0, 1] \to E_2$ such that $f_n \to f$ uniformly. Then $\ell(C) \leq m$, and $C$ meets every tangent. Indeed, assume the contrary. If $C$ does not meet $t$, then it is at a positive distance from $t$, and in view of the uniform convergence $f_n \to f$, $C_n$ will be at a positive distance from $t$ for $n \geq N$, which is impossible. Consequently, there is a minimum for Theorem 2. As for Theorem 1, each $C_n$ starts at the center of the circle, and hence so does $C$.

4. Notation. If $A, B, C, D$ are points of the plane, then $AB$ will denote the straight segment between $A$ and $B$, $|AB|$ the distance between $A$ and $B$, and $ABCD \ldots$ the path $AB \cup BC \cup CD \cup \ldots$. If $A \neq B$, let $d(AB)$ be the straight line that connects $A$ and $B$, and $r(AB)$ the half-line of $d(AB)$ that starts at $A$ and passes through $B$. If $A \neq B \neq C$ and $r(BA)$ is not the half-line opposite to $r(BC)$, then $a(ABC)$ will be the closed convex angular region bounded by $r(BA)$ and $r(BC)$, and $\angle(ABC)$ the angular measure of $a(ABC)$. Therefore, $0 \leq \angle(ABC) < \pi$ always.

The convex envelope of a set $S$ will be denoted by $k(S)$. In particular, if $A, B, C, \ldots$ are points, $k(ABC \ldots)$ will be the smallest convex polygon that contains $A, B, C, \ldots$; $k(ABC)$ will be the triangle with vertices $A, B, C$. 

**Figure 3**
The circle under consideration will be $K$, and its interior and exterior, $\text{int}(K)$ and $\text{ext}(K)$.

If a path is given by $f : [a, b] \to E_2$ and $X = f(t), Y = f(s)$, we write $X < Y$ if $t < s$.

5. First, let $\mathcal{C}$ be an arbitrary path given by $c : [a, b] \to E_2$. The interval $[a, b]$ is a union of $c^{-1}(\text{int}(K)), c^{-1}(\text{int}(K))$ and $c^{-1}(\text{ext}(K))$; as we know from topology, $c^{-1}(\text{int}(K))$ and $c^{-1}(\text{ext}(K))$ are composed of open intervals in $[a, b], c^{-1}(K)$ is composed of closed intervals in $[a, b]$, and additionally, if there are infinitely many such intervals, of the accumulation points. I will denote the paths corresponding to the intervals of $c^{-1}(K)$, $c^{-1}(\text{int}(K))$ and $c^{-1}(\text{ext}(K))$ by $j_1, j_2, j_3$ respectively, after having assigned to them their initial and terminal points if necessary.

6. Consider now a $j_1$ of a minimal curve $\mathcal{C}$. A $j_1$ is an arc of $K$, but not all of $K$. If the ends of the arc are not the initial and terminal point of $j_1$, then a part of the arc is traversed twice, which allows for a shortening by a chord. Therefore, for a minimal $\mathcal{C}$ the $j_1$ are arcs of $K$ traversed once. The $j_2$, being in the interior of the circle, where there are no tangents, are obviously chords or segments of chords.

7. It remains to consider the $j_3$. First let $X \in \text{ext}(K)$; let $t_+$ and $t_-$ be the two tangents of $K$ from $X$ such that, when viewed from $X$, $K$ is on the left of $t_+$. Let $P_+(X)$ and $P_-(X)$ be the points at which $t_+$ and $t_-$ touch $K$ (Figure 4). For $X \in K$, we write $P_+(X) = P_-(X) = X$. Thus $P_+$ and $P_-$ are continuous maps of $K \cup \text{ext}(K)$ onto $K$. Each $P_+(j_3)$ and $P_-(j_3)$ is a closed arc on $K$, and if $j_3$ has a point on $K$, then $P_+(j_3) \cup P_-(j_3)$ is a closed arc on $K$. Now let $X_0 \in \mathcal{C} \cap \text{ext}(K)$ and let $t$ be a tangent that properly separates $K$ from $X_0$ so that $X_0 \notin t$. Suppose for simplicity that $X_0$ is not initial or terminal on $\mathcal{C}$. There are $Y, Z$ on $\mathcal{C}, Y < X_0 < Z$, such that for $Y \leq X \leq Z$, $X$ is also properly separated from $K$ by $t$. Let $X_1, X_2, X_3, X_4$ be four points, $Y \leq X_1 \leq X_2 \leq X_3 \leq X_4 \leq Z$, for which the minimum and maximum of $P_+$ and $P_-$ on the path $\{X \in \mathcal{C} : Y \leq X \leq Z\}$ are attained. Then the polygonal path $YX_1X_2X_3X_4Z$ meets the same tangents and is strictly shorter than any other path that contains $Y, X_1, X_2, X_3, X_4, Z$ in the same order. It follows that each $j_3$ is a succession of segments. We call $AB$ a maximal segment of $\mathcal{C}$ if $AB$ is not part of a segment $A'B'$ of $\mathcal{C}$ that properly contains $AB$.

8. It is useful to consider the problem in a different way. Let $D$ be a convex compact domain on the plane $E_2$. The support lines of $D$ are the lines $\ell$ that intersect $D$ in such a way that $D$ is contained completely in one of the two half-planes defined by $\ell$. Let $S$ be a connected set. Then $D$ belongs to $k(S)$ if and only if $S$ intersects all the support lines of $D$. In particular, if $D$ is bounded by a smooth curve, then the support lines are the tangents of the boundary curve.
Thus, our problem reduces to finding the shortest curve \( C \) such that \( K \subseteq k(C) \).

We recall that \( k(S_1 \cup S_2) \) is a union of segments \( XY \) with \( X \in k(S_1) \) and \( Y \in k(S_2) \).

We say that an arc \( \mathcal{S} \) of a path \( C \) is repeating if either \( K \subseteq k(C \setminus \mathcal{S}) \) or \( C \setminus \mathcal{S} \) intersects all the tangents to \( K \). In particular, \( \mathcal{S} \) is repeating if \( \mathcal{S} \subseteq k(C \setminus \mathcal{S}) \). A repeating arc is always a straight segment traversed once.

9. Let \( P \in K \cap \mathcal{C} \), let \( t \) be a tangent at \( P \), and let \( Q \in \mathcal{C} \) be properly separated from \( K \) by \( t \), that is, \( Q \) belongs to the interior of the half-plane defined by \( t \) that does not contain \( K \). Then \( \mathcal{C} \) is a straight segment near \( P \). Indeed, if \( \mathcal{S} \) is an arc of \( \mathcal{C} \) that contains \( P \) and is contained in the interior of the triangle \( k(P_+(Q)QP_-(Q)) \) (Figure 5), then there must be points \( X, Y \) in the dashed regions with \( X, Y \in k(\mathcal{C} \setminus \mathcal{S}) \). Therefore, \( \mathcal{S} \subseteq k(YQX) \subseteq k(\mathcal{C} \setminus \mathcal{S}) \) hence \( \mathcal{S} \) is repeating, and hence a segment.

10. The arguments of §8 allow us to exclude the following impossible situations:

Imp. a): Two noncollinear segments \( AB \) and \( CD \) that intersect at an interior point of at least one of the segments \( AB \) and \( CD \). Indeed, in Figure 6, \( k(AB \cup DC) = k(DB \cup AC) \). We obtain a shortening by replacing \( AB \) and \( CD \) with \( CA \) and \( DB \) and by changing the direction of a part of \( \mathcal{C} \).

Imp. b): Two consecutive segments \( ABC \) form an angle such that the opposite angular region contains a point \( X \) of \( K \) or of \( k(\mathcal{C} \setminus ABC) \), \( X \not\in B \). Indeed, in Figure 7, let \( S = (\mathcal{C} \setminus ABC) \cup A \cup C \), \( \mathcal{C} = ABC \cup S \). There exists \( Q \in k(S) \) such that \( X \in PQ \), \( P \in k(ABC) \); then \( ABC \subseteq k(QAC) \), \( ABC \) is repeating and must be replaced with the segment \( AC \), which is shorter than \( ABC \).
Imp. c): A segment $AC$ such that $C$ is the initial or terminal point of $\mathcal{C}$ and such that $r(AC)$ intersects $K$ beyond $C$. The only exception: $C$ is the initial point imposed by Problem (P).

Imp. d): Two coinciding segments at least one of which is followed by a segment, in a different direction (Figure 8).

Imp. e): Two consecutive segments $A'B'C''$ such that $K$ is contained in $a(A'B'C'')$. Indeed, if $A$ and $C$ are in the interior of $A'B$ and $C'B$ respectively, sufficiently close to $B$, then it is easily seen that $d(AC)$ properly separates $B$ from $K$ and that $ABC$ can be replaced with $AC$, which is shorter.

Imp. f): Two consecutive segments $ABC$ such that $K$ is contained in one of the angular regions supplementary to $a(ABC)$, respectively, the half-planes defined by $d(AB)$ if $r(BA) = r(BC)$. The only exception is the case that $A = C \in K$ and $d(AB)$ is a tangent to $K$. We consider here the nondegenerate case where $r(AB) \neq r(BC)$ and $ABC \subseteq \text{ext}(K)$ (Figure 9). Suppose that $BA$ and $BC$ are maximal and that $d(BA)$ separates $K$ from $BC$. Let the tangents $t_1$ and $t_2$ from $A$ and $B$ be as in the figure. For $C$ we have the possibilities $C_0, C_1, C_2$. $BC_2$ is repeating, hence $C \neq C_0$. If $C = C_1$, then $BA'$ is repeating, and hence $BA$ is not maximal. Therefore, $C = C_2$. If $C$ is the terminal point of $\mathcal{C}$, then $BC$ is superfluous (because the only imposed terminal point of $\mathcal{C}$ is the center of $K$). Consider the continuation of $\mathcal{C}$ beyond $C$. If this is a segment $CD_1$ with $D_1$ properly separated from $K$ by $t_1$, then $BA$ cannot be maximal. If, on the other hand, the continuation is $CD_2$ with $D_2$ in the same closed half-plane (determined by $t_1$) as $K$, then $BCD_2$ is repeating, which is impossible.
11. We are now ready to determine what form \( j_3 \) can have. Let \( AB \) be a maximal segment of \( j_3 \) such that \( d(AB) \) does not intersect \( K \). In view of Imp. b), e), and f), either \( B \) is the terminal point or \( C \) passes after \( B \) along a segment directed towards \( K \). The same is true for \( A \). If \( AB \subseteq j_3 \) is such that \( d(AB) \) intersects \( K \) [say, \( r(AB) \) intersects \( K \)], then by Imp. c) and b), \( B \) must be on \( K \). If, moreover, \( d(AB) \) is not the tangent to \( K \) from \( B \), then the segment \( AB \) must continue to \( \text{int}(K) \) (in view of §9), beyond \( B \). \( A \) can be terminal. If \( A \) is not terminal, \( C \) continues beyond \( A \) along \( AC \), \( C \in K \), or along \( AC \) such that \( d(AC) \subseteq \text{ext}(K) \), which is the case considered above.

Thus, we found for \( j_3 \) the seven possibilities indicated in Figure 10.

The continuations to the interior of \( K \) are indicated. Identical angles are marked by identical letters. Also the right angles are indicated.

For example, the equality of the two angles \( \beta \) in case V is due to the following elementary fact: if \( X, Y \) are two points on the same side of a straight line \( d \), then the shortest path between \( X \) and \( Y \) that passes through \( d \) consists of two segments \( XZ, YZ \) with \( Z \in d \) that form the same angle with \( d \). (The explanation of the right angles is even simpler.) In case IV we obtain \( \varepsilon = 0 \) as a limit case. It is easily seen that

\[
\alpha \leq \beta + \gamma, \quad \beta \leq \alpha + \gamma.
\]

(2)

It can also be shown that \( 2\beta + 2\alpha + \varepsilon \geq \pi \), but we will not use this inequality.

12. We have shown that \( j_1, j_2, \) and \( j_3 \) are the simplest arcs of \( \mathcal{C} \); it remains to
show that complicated configurations cannot be produced by accumulations of \(j_1, j_2, j_3\) on \(K \cap \mathcal{C}\). First we prove the following.

If \(AB\) is a \(j_2\) of \(\mathcal{C}\), that is, a chord or part of a chord of \(K\), and \(B \in K\), then \(B\) is not the terminal point of \(\mathcal{C}\), and \(\mathcal{C}\) continues to a segment \(BC\); \(C \in \text{ext}(K)\), \(B \in AC\), that is, \(\mathcal{C}\) continues straight ahead to \(d(AB)\). First of all, if \(B\) were terminal, let \(A' \in AB \cap \text{int}(K)\), \(\mathcal{S} = (\mathcal{C} \setminus A'B) \cup A'\). \(\mathcal{S}\) meets all the tangents, except perhaps those from \(B\). But since \(\mathcal{S}\) is closed, it meets also those from \(B\) by continuity, and hence \(A'B\) would be superfluous. In the same way, the rest of \(\mathcal{C}\) is not formed by another \(j_2\). The arguments that follow refer to Figure 11: \(t\) is the tangent from \(B\) that we take to be horizontal. If there is a \(Y \in \mathcal{C}\) that is properly separated from \(K\) by \(t\), then the assertion follows from \(\S\)\(9\). Otherwise there will be a \(Z > B\) such that \(\mathcal{S} = \{X \in \mathcal{C} : Z \geq X \geq B\}\) lies in the rectangle \(k(PQRW)\), where \(R, W \in K\) and \(P, Q \in t\), with small \(|QB|\) and \(|PB|\). Suppose \(\angle(A'BP) \leq \pi/2\). To pass from \(k(QBR)\) to \(k(PBTW)\), \(\mathcal{S}\) must pass through \(B\), since \(\mathcal{S}\) cannot cross \(A'B\) inside, in view of Imp. a). If \(\mathcal{S}\) passes the two sides of \(A'B\) infinitely many times, it must have infinitely many loops \(b\) departing from and arriving at \(B\), in \(k(PBTW)\). Let \(b\) be such a loop, with horizontal elongation \(|BS|\). For a sufficiently small \(BP\), the path \(A'SB\) is shorter than \(A'B\) and \(b\) together, since

\[
|A'S| + |BS| \leq \frac{|A'B|}{\cos \gamma} + |BS| \leq \frac{|A'B|}{\cos \gamma} - |BS| + \text{length}(b)
\]

\[
\leq \frac{|A'B|}{\cos \gamma} - |A'B| \sin \gamma + \text{length}(b) < |A'B| + \text{length}(b)
\]

if \(\gamma\) is sufficiently small. Therefore, if \(b\) is a loop with maximum horizontal elongation, then we can replace \(A'B\) with \(A'SB\) and omit all the loops in \(k(PBTW)\). Therefore, we may assume that \(\mathcal{S}\) is entirely on one side of \(A'B\), say in \(k(QBR)\). However, in this case \(\mathcal{S} \subseteq k(QBR)\). Indeed, if \(X\) is a point in the triangle \(k(BTR)\) without \(RB\), then \(X\) is on a chord \(j_2\) that must intersect \(RW\) or \(BT\) in the interior, unless \(\mathcal{S} \subseteq j_2\), the case which has already been excluded. Now let \(U \in \mathcal{S}\) be the leftmost for all of \(\mathcal{S}\). We replace \(A'B \cup \{X \in \mathcal{S} : U \geq X \geq B\}\) with \(A'VU\), which meets all the tangents met by \(A'B \cup \{X \in \mathcal{S} : U \geq X \geq B\}\) and has the length \(\geq |A'B| + |BV|\), whereas for \(\omega = \angle(A'BV)\),

\[
|AA'V| + |VU| = |A'B| \cos \delta + |VB| \cos(\pi - \omega - \delta) + |VU|
\]

\[
\leq |A'B| \cos \delta + |VB| |\cos(\omega + \delta)| + |VB| \tan \alpha < |A'B| + |VB|
\]

if \(\alpha\) and \(\delta\) are sufficiently small. This proves the assertion.

13. Each \(j = j_1\) or \(j_3\) \emph{"covers"} a set of tangents whose intersection points with \(K\) form an arc \(P_+(j) \cup P_-(j)\) on the circle \(K\). We denote this arc \(P(j)\). We claim that
$P(j)$ does not have common interior points with $P(j')$ if $j \neq j'$. This is clear if $j$ or $j'$ is a $j_1$, and hence an arc of $K$, since a part of this $j_1$ can be replaced by a shorter chord. Thus, assume that $j$ and $j'$ are $j_3$. Obviously, $P(j) \not\subseteq P(j')$; otherwise $j$ is repeating. Therefore, we are in a situation of Figure 12. If $j$ is a $j_3$ of type V (Figure 10), we can shorten the path by cutting through the angle formed by $\mathcal{C}$ at $C$. If $j_3$ is of type IV (Figure 10), we can go down from $C$ along the left tangent, which will ensure a shortening unless $\beta = \alpha + \gamma$, and so forth. We arrive at the following situation as the only possibility (Figure 13): $j$ contains the segment $b = BD$, with $\beta \leq \pi/2$, and $j'$ contains the segment $a = AC$ with $\alpha \leq \pi/2$. In the same way as in the proofs of §10, if $S = \mathcal{C} \setminus a \setminus b$, we find points $X, Y, W \in k(S)$, in the indicated angular regions for which $R, Q, P$, respectively, are in $k(\mathcal{C})$. But then $a, b \subseteq k(ABWXY)$, and $a \setminus A$ and $b \setminus B$ are repeating, as well as their continuations up to the tangents $t_1$ and $t_2$, respectively, which would enforce a forbidden crossing. We note that this is the first time we used the fact that $K$ is a circle, or rather that the normals to the tangents at $Q$ and $P$ meet at $Z \in \text{int}(K)$. Up until this moment, everything was applicable for smooth and convex $K$s that do not contain straight segments.

14. It is easily seen from the above that if $j = j_1$ or $j_3$, then $\mathcal{C} \setminus j$ is entirely on one side of $K$ with respect to each tangent that touches $K$ at $P(j)$, that is, in the non-dashed part of the plane shown in Figure 14. It follows that if $AB \subseteq \mathcal{C}$ is a maximal segment that contains a chord $PQ$ of $K$, then neither $A$ nor $B$ is terminal and the continuations of $\mathcal{C}$ from $A$ and from $B$ do not go in the same half-plane determined by $d(AB)$. In other words, we have the situation of Figure 15.
The intersection of $C$ with the tangent $t \parallel AB$ must be in the dashed region, which $C$ can reach neither from $X$ nor from $Y$ without crossing $AB$.

15. We show that $C$ consists of finitely many $j_1, j_2, j_3$. Assume the contrary. Then there exists an $X \in K \cap C$ such that for all $Y, Z \in C$, $Y < X < Z$, there are infinitely many $j_1, j_2, j_3$ between $Y$ and $Z$, say between $Y$ and $X$. Suppose there are no chords $j_2$ among them. Then the $j_3$ must be of type II' in Figure 10. But each of those has length $2$, and so there are only finitely many of them; therefore, if $Y$ is close enough to $X$, there are only $j_1$’s, whence $\{W \in C : Y \leq W \leq X\}$ is an arc on $K$ and belongs to a single $j_1$. Consequently, there are infinitely many chords $j_2$ approaching $X$. In view of §14, they must zigzag, as in Figure 16. The $j_2$ have exactly the same direction as the tangent $t$ at $X$. To each $j_2$ we attach a $j_3$ of the form III, IV, or V (Figure 10).

For a $j_3$ of the form V we have, in Figure 17, $2\pi = \varepsilon + \delta + \pi - \alpha + \pi - \delta$; therefore $\alpha + \beta = \varepsilon + \delta$. But $\alpha + \beta = \pi - \delta$, and hence $\delta = \pi/2 - \varepsilon/2$; for $|\varepsilon|$ small, $\delta \approx \pi/2$, and $P(j_3)$ has a length $\approx \pi/2$, which is too large. In the same way we have contradictions for $j_3$ of the types III and IV. Thus we have found that $C$ is a finite union of paths $j_1, j_2, j_3$.

16. We can now prove Theorems 1 and 2. In Theorem 1 we have one free terminal point; in Theorem 2, both terminal points are free. We see that an arc $j_1$ on $K$ cannot be a terminal path of $C$. We already know that the $j_2$ are not terminal except for the fixed terminal point in Theorem 1. In view of §14, the $j_3$ of type I are not terminal. Therefore, only the $j_3$ of type II or VI remain candidates for a free terminal path.
Suppose that the free terminal segment is a $j_3$ of type II, and hence a tangent of length 1. It may be followed by a $j_3$ of type III or by an arc $j_1$. In the first case we have Figure 18. But here we see that $ABC$ can be replaced with $BAC$, which is shorter and has the same convex envelope. The same argument is valid if $C$ ends with a $j_3$ of type VI. Consequently, the only possibility is a tangent of length 1 followed by an arc. We show that $C$ does not admit entire chords. Indeed, assume the contrary and take the first such chord after the arc. We obtain Figure 19. Let $\alpha + \omega \leq \pi/2$. If

$$|RE| < \frac{4}{\pi - 2} - \frac{\pi - 2}{4},$$

we verify that the circle centered at $E$ of radius $|ER| + (\pi/2) - 1$ passes through $V$ (on the diameter $RM$), and hence

$$|ED| > |ER| + \frac{\pi}{2} - 1.$$  \hspace{1cm} (3)

This enables us to replace the path $ABCDE$ (where $BC$ is an arc) with $DCBRE$, where $CBR$ is an arc on $K$. The length of the first is $1 + \omega + \tan \alpha + |DE| > 1 + \omega + \tan \alpha + |ER| + (\pi/2) - 1 = \omega + \tan \alpha + (\pi/2) + |ER|$ which is the length of the second. If

$$|RE| > \frac{4}{\pi - 2} - \frac{\pi - 2}{4},$$
we consider Theorem 2 first. The length of the path will be at least

\[ |AB| + |BE| > 2 + |RE| \geq 2 + \frac{4}{\pi - 2} - \frac{\pi - 2}{4} > \pi + 2, \]

and hence it is longer than the path in the theorem.

For Theorem 1, the curve must return to the center from \( E \). Therefore, the path will be longer than\( |AB| + |BE| + |EZ| > 2 (1 + |RE|) > 2 + 2\pi \), and hence longer than the path of Theorem 1.

Now let \( \omega + \alpha > \pi/2 \). For Theorem 1, we see that \( DE \) is an obstacle on the way of \( \mathcal{C} \) back to the center. For Theorem 2, we start from the other terminal point, but we must have \( \omega' + \alpha' < \pi/2 \) in view of §14, which reduces to the previous case.

Thus, we see that the single \( j_2 \) can only be the semichord leading to the center, for Theorem 1, and so we cannot obtain anything different from the paths described in the theorems.

**Remarks**

(a) The problem considered can be largely generalized by replacing, for example, \( K \) with an arbitrary convex domain, or the tangent lines with tangent circles, etc. The general statement of the problem is this: In an arcwise connected metric space, given a family \( \mathcal{F} \) of closed \( F \) and a connected compact set \( K \) that intersects each \( F \in \mathcal{F} \), find the shortest path that intersects all the \( F \).

(b) In higher dimensions, if \( E_n \) is \( n \)-dimensional Euclidean space, \( S_{n-1} \) the unit sphere, and \( \ell_n \) the length of the shortest path that meets all hyperplanes tangent to
$S_{n-1}$, or the shortest path $C$ with $S_{n-1} \subseteq k(C)$, then by induction

$$\ell_3 \geq \sqrt{\left(2 + \sqrt{3} + \frac{7}{6}\pi\right)^2 + 4} \approx 7.6628,$$

$$\ell_n \geq \text{const} + 2n.$$

By constructing particular paths we find that

$$\ell_n \leq \text{const} \cdot n^{3/2},$$

$$\ell_3 \leq 4 + \frac{1}{2} \sqrt{2} \cdot 3 \cdot \pi \approx 10.6643.$$

The upper bounds seem to me to be the closest to reality.

(c) In an infinite-dimensional Hilbert space, say $H = l^2$, the circumstances are slightly different. If $S$ is the unit sphere, there is no path of finite length, nor a compact path $C$ such that $S \subseteq k(C)$, since $k(C)$ is compact, but $S$ is not. Therefore, we must consider compact convex sets $K \subseteq H$. In this case there surely exists a compact path $C$ (which can be easily constructed) such that $k(C) \supseteq K$. [In view of a theorem by Hahn and Mazurkiewicz, there even exists a continuous map $f : [0, 1] \to H$ with $\text{Im} \, f = K$ (see [1]).] However, this curve in general is not of finite length. If

$$K = \left\{ (x_1, x_2, \ldots) : \sum_{j=1}^{\infty} j \, x_j^2 \leq 1 \right\},$$

then $K$ is compact and convex, but any curve $C$ with $k(C) \supseteq K$ has infinite length. If

$$K = \left\{ (x_1, x_2, \ldots) : \sum_{j=1}^{\infty} j^{100} \, x_j^2 \leq 1 \right\},$$

then there exists $C$ of finite length with $k(C) \supseteq K$. Moreover, it can be shown that if $K$ is compact and convex and $C$ has finite length and $k(C) \supseteq K$, then there exists a minimal curve. This can be done as in §3. It must simply be shown that if $\{C_n\}$ is a minimal chain of curves, then $\bigcup_{n=1}^{\infty} C_n$ is relatively compact.

Written by Henri Joris, Genève
Translated by Natalya Pluzhnikov

Addendum
The only citation to the literature given in Joris (1980) is [1].
I have included [2, 3, 4, 5, 6, 7, 8, 9, 10] for the sake of completeness.
John Wetzel provided valuable comments as I prepared this draft for posting.
A translation of Henri Joris’ “Le chasseur perdu dans la forêt” (1980)

References

[1] M. H. A. Newman, Elements of the Topology of Plane Sets of Points, Cambridge Univ. Press, 1964, pp. 89–92; MR0044820.

[2] R. Bellman, Minimization problem, Bull. Amer. Math. Soc. 62 (1956) 270.

[3] J. R. Isbell, An optimal search pattern, Naval Res. Logist. Quart. 4 (1957) 357–359; MR0090474.

[4] Z. A. Melzak, Companion to Concrete Mathematics: Mathematical Techniques and Various Applications, Wiley, 1973, pp. 150–153; MR0462824.

[5] H. G. Eggleston, The maximal inradius of the convex cover of a plane connected set of given length, Proc. London Math. Soc. 45 (1982) 456–478; MR0675417.

[6] V. Faber, J. Mycielski and P. Pedersen, On the shortest curve which meets all the lines which meet a circle, Annales Polon. Math. 44 (1984) 249–266; MR0817799.

[7] V. Faber and J. Mycielski, The shortest curve that meets all the lines that meet a convex body, Amer. Math. Monthly 93 (1986) 796–801; MR0867106.

[8] S. R. Finch, Beam detection constant, Mathematical Constants, Cambridge Univ. Press, 2003, pp. 515–519; MR2003519.

[9] S. R. Finch and J. E. Wetzel, Lost in a forest, Amer. Math. Monthly 111 (2004) 645–654; MR2091541.

[10] M. Ghomi, The length, width, and inradius of space curves, Geom. Dedicata 196 (2018) 123–143; MR3853631.

Steven Finch
MIT Sloan School of Management
Cambridge, MA, USA
steven_finch@harvard.edu