Abstract

We consider regular Calabi-Yau hypersurfaces in \( N \)-dimensional smooth toric varieties. On such a hypersurface in the neighborhood of the large complex structure limit point we construct a fibration over a sphere \( S^{N-1} \) whose generic fibers are tori \( T^{N-1} \). Also for certain one-parameter families of such hypersurfaces we show that the monodromy transformation is induced by a translation of the \( T^{N-1} \) fibration by a section. Finally we construct a dual fibration and provide some evidence that it describes the mirror family.

1 Introduction

Strominger, Yau and Zaslow [SYZ] conjectured that any Calabi-Yau manifold \( X \) having a mirror partner \( X^\vee \) should admit a special Lagrangian fibration \( \pi : X \to B \) (a mathematical account of their construction can be found in [M]). If so, the mirror manifold \( X^\vee \) is obtained by finding some suitable compactification of the moduli space of flat \( U(1) \)-bundles along the nonsingular fibers, which restricts the fibers to be tori. More precisely, if \( B_0 \subseteq B \) is the largest set such that \( \pi_0 = \pi|_{\pi^{-1}(B_0)} \) is smooth, then \( X^\vee \) should be a compactification of the dual fibration \( R^1\pi_0^*(\mathbb{R}/\mathbb{Z}) \to B_0 \).

The conjecture is trivial in the elliptic curve case. On a K3 surface the hyperkähler structure translates the theory of special Lagrangian \( T^2 \)-fibrations to the standard theory of elliptic fibrations in another complex structure. However, very little progress has been made in higher dimensions so far, though Gross and Wilson have worked out some aspects of the conjecture for the Voisin-Borcea 3-folds of the form \( (K3 \times T^2)/\mathbb{Z}_2 \) [GW]. But the general question of finding special Lagrangian fibrations on Calabi-Yau’s still remains open.

We restrict our attention to the case of regular anticanonical hypersurfaces in smooth toric varieties. The main result of our this paper is that such a hypersurface in a neighborhood of the large complex structure admits a torus fibration over a
sphere. Unfortunately, we were unable to control the fibers to be special Lagrangian. However, we will argue that on some open patches the fibers do possess some calibration property.

Batyrev [B] showed that toric varieties $X_\Delta$ with ample anticanonical bundles are given by reflexive polyhedra. Such a polyhedron $\Delta$ contains a unique integral interior point $\{0\}$. A Calabi-Yau hypersurface $Y \subset X_\Delta$ is defined by an equation in the form $\sum_{\omega \in \Delta(\mathbb{Z})} a_\omega x^\omega = 0$, where $\omega$ runs over the integral points in $\Delta$. The image of $Y$ under the moment map $\mu : X_\Delta \to \Delta$ has the shape of an amoeba (cf. [GKZ], Ch.6), a blob with holes around some lattice points $\omega$ in $\Delta(\mathbb{Z})$. The sizes of the holes are determined by the corresponding coefficients $a_\omega$. If $Y$ is near the large complex structure, $a_{\{0\}}$ is large, and $\mu(Y)$ is homeomorphic to $S^{N-1} \times I$. The idea is to choose the right trivialization of this product, so that for general $s \in S^{N-1} \simeq \partial \Delta$ the preimage of the interval $\mu^{-1}(\{s\} \times I)$ would be an $(N-1)$-dimensional torus $T^{N-1}$. Singular fibers come from the intervals $I_s := \{s\} \times I$ which intersect the $(N-2)$-dimensional skeleton of $\Delta$. For example, consider $Y$, a $K3$ surface given by a quartic in $\mathbb{CP}^3 = X_\Delta$, where $\Delta$ is the integral 3-simplex in $\mathbb{Z}^3$ with the vertices $(-1,-1,-1)$, $(-1,-1,3)$, $(-1,3,-1)$, $(3,-1,-1)$. There are exactly 4 points in $\mu(Y)$ on each of the 6 one-dimensional edges of $\Delta$, corresponding to 4 points of intersection of $Y$ with the projective line determined by this edge. Altogether they give 24 singular fibers.

But instead of pursuing this idea we will modify the moment map and deform the original hypersurface. An explicit parameterization of the fibers will allow us to analyze the action of the monodromy for one-parameter families of hypersurfaces and construct a dual fibration. We will speculate that this dual fibration represents the mirror Calabi-Yau.

Let us demonstrate almost all essential ideas by a simple example. Consider a family of elliptic curves $E_t$ in $\mathbb{CP}^2$ given by the equations:

$$txyz + x^3 + y^3 + z^3 = 0,$$

where $t$ plays the rôle of a parameter. As $t \to \infty$ the curve $E_t$ degenerates to 3 lines with normal crossings. The main idea is roughly to consider the asymptotic behavior of $E_t$ up to the next order to keep the curve smooth.

There are 6 regions in $E_t$ according to its image under the moment map (see Fig.1). In each of them there are different terms in addition to $txyz$ which are dominant. For instance, in $U_z$ the elliptic curve $E_t$ for large $t$ is approximated by $txyz + z^3 = 0$, and in $U_{xz}$ by $txyz + x^3 + z^3 = 0$, etc. It is easy to introduce a coordinate on a curve (which is still smooth in the corresponding region) defined by the abbreviated equation. In $U_z$ either $x/z$ or $y/z$ is a coordinate, in $U_{xz}$ we can use $x/z$ or $z/x$ and similar in the other regions. The circle fibration is provided by fixing an absolute value of the coordinate. The set of all possible absolute values in all 6 regions clearly forms a circle, the base of the fibration. The partition of unity technique of gluing these 6 pieces into one curve which approximates the original elliptic curve constitutes section 3.
To compute the monodromy as $\arg(t) \mapsto \arg(t) + 2\pi i$, we need to understand how the identification of the fibres changes in the overlaps. Nothing happens in $U_{zx}$, $U_{xy}$ or $U_{yz}$, as the parameterization of fibers changes by reversing an orientation of the circles, which does not depend on $t$ at all. Monodromy is nontrivial only in $U_z$, $U_y$ or $U_x$. Consider, e.g. $U_z$, where two coordinates can be used $y/z$ or $x/z$, which are related by the equation $t(x/z)(y/z) = -1$. As $\arg(t) \mapsto \arg(t) + 2\pi i$, the circles parameterized by $\arg(x/z)$ and by $\arg(y/z)$ are twisted by $2\pi i$ with respect to each other. Combining all together we get a triple Dehn twist. More careful monodromy calculations are performed in section 4.

Section 5 is devoted to dual fibrations and mirror symmetry. Leung and Vafa [LV] describe the idea of the mirror construction as follows. $T$-duality interchanges small circle fibers in the corner regions of one family with large circle fibers in the facet regions of the mirror family. More precisely, let us consider the polytope $\Delta^\vee$ dual to the above family of elliptic curves. In this case it is just the polar polytope $\Delta^D$, hence again reflexive. The associated toric variety $X_{\Delta^D}$ is singular but the elliptic curve in $X_{\Delta^D}$ is smooth because it misses all singular points (the vertices of the triangle).
$\Delta^D$ is combinatorially dual to $\Delta$ and the dual elliptic curve also breaks into 6 regions according to its image under the moment map (see Fig.2). We will speculate that this dual curve also admits similar fibration by circles and the fibers in the corresponding regions $U_\alpha$ and $V_\alpha$ are naturally dual circles.

This example captures all the features except for the fact that in higher dimensions there will always be singular fibers.

Acknowledgments. I am very grateful to Ron Donagi, my thesis advisor, for suggesting this problem to me and constant supervising my work on it, and to Tony Pantev for illuminating discussions.

2 Hypersurfaces in toric varieties

In this section we review some basic constructions in the theory of toric varieties and set up the notations. For more details see, e.g. [Cx] and references provided there.

Let $M \cong \mathbb{R}^N$ be an $N$-dimensional real affine space and $\mathbb{Z}^N$ an integral lattice in $M$. We will choose an integral point $\{0\}$ in $\mathbb{Z}^N$ to be the origin. This endows $M$ with a vector space structure and $\mathbb{Z}^N$ becomes a free abelian group.

An $N$-dimensional convex integral polyhedron $\Delta$ in $M$, is called reflexive if it contains the origin $\{0\}$ as an interior point and if its polar polyhedron $\Delta^\ast = \{u \in M^\ast : \langle u, m - \{0\} \rangle \geq -1 \text{ for all } m \in \Delta\} \subset M^\ast$ is also integral. We will denote by $\Delta(\mathbb{Z})$ the lattice points in the polyhedron $\Delta$, and by $\partial \Delta$ its boundary. It follows that each $(N-1)$-dimensional face $\Sigma$ (which we will call a facet in the future) of $\Delta$ is defined by an equation $\langle u_{\Sigma}, m \rangle = -1$ for some $u_{\Sigma} \in M^\ast$. This easily implies that $\{0\}$ is the unique integer interior point. For an arbitrary face $\Theta \subseteq \partial \Delta$ we will denote by $\Theta_{\mathbb{Z}}$ the affine sublattice of $\mathbb{Z}^N$ generated by the integer points of $\Theta$.

Also we assume that $\Delta$ is nonsingular. This mean that every vertex of $\Delta$ is $N$-valent, that is exactly $N$ edges $e_1, \ldots, e_N$ emanate from it, and the integer points of these edges $(e_i)_\mathbb{Z}, i = 1, \ldots, N$, generate the lattice $\mathbb{Z}^N$. Because $\Delta$ is reflexive, i.e. the integral distance from the origin $\{0\}$ to any facet $\Sigma \subset \Delta$ is 1, the lattice $\mathbb{Z}^n$ is as well generated by $\Sigma_\mathbb{Z}$ together with $\{0\}$.

Let us denote by $X_\Delta$ the projective toric variety corresponding to $\Delta$. The normal projective embedding is given, e.g., by the closure of the image of the map $(\mathbb{C}^*)^N \hookrightarrow \mathbb{P}^{\Delta(\mathbb{Z})^\ast - 1}, x \mapsto \{x^{\omega_1} : x^{\omega_2} : \ldots : x^{\omega_{\Delta(\mathbb{Z})}}\}$. Because $\Delta$ is nonsingular, the toric variety $X_\Delta$ is smooth. We will use $\{x^\omega\}$ as projective coordinates on $X_\Delta$. One may want to restrict the set of monomials $\{x^\omega\}$ to a subsystem of the anticanonical linear system. In this case we allow $\omega$ vary among $A \subset \Delta(\mathbb{Z})$, a subset of integer points in $\Delta$, such that $\{0\} \in A$ and $\Delta$ is the convex hull of $A$.

The moment map $X_\Delta \rightarrow \Delta$ is given by

$$\mu(x) = \frac{\sum_{\omega \in A} |x^\omega| \cdot \omega}{\sum_{\omega \in A} |x^\omega|}.$$
It is a well-defined function on $X_\Delta$, because both the top and bottom are polynomials of the same homogeneous degree.

A **triangulation** of a convex polyhedron is a decomposition of it into a finite number of simplices such that the intersection of any two of these simplices is a common face of them both (maybe empty). By a triangulation $T$ of $(\Delta, A)$ we simply mean a triangulation of $\Delta$ with vertices in $A$. Note that we do not require every element of $A$ to appear as a vertex of a simplex. A continuous function $\psi : \Delta \to \mathbb{R}$ is called **$T$-piece-wise linear** if it is affine-linear on every simplex of $T$. Such a function $\psi$ is **convex** if for any $x, y \in \Delta$, we have $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y), 0 \leq t \leq 1$. We call it **strictly convex** if the (maximal-dimensional) simplices of $T$ are the maximal domains of linearity of $\psi$.

A triangulation $T$ of $(\Delta, A)$ is called **coherent** (some authors call it regular or projective) if there exists a strictly convex $T$-piece-wise linear function. Call a coherent triangulation $T$ **central** if $\{0\}$ is a vertex in every (maximal dimensional) simplex of $T$. Let $\partial T$ denote the collection of simplices of $T$ lying in $\partial \Delta$. We will use $\sigma$ to denote maximal dimensional simplices in $\partial T$ and $\tau$ for arbitrary simplices of $\partial T$. Denote by $C_\tau \subset \Delta$ the corresponding simplex in $T$ of one dimension higher with the base $\tau$ and the vertex $\{0\}$. We will denote by $\tau_\mathbb{Z}$ the sublattice of $\mathbb{Z}^N$ generated by the integer points in $\tau$. $\Lambda_\tau$ will denote the sublattice in $\tau_\mathbb{Z}$ of index $(\dim \tau)! \cdot \text{vol}(\tau)$ generated by the vertices of $\tau$.

Given a triangulation $T$ of $(\Delta, A)$, every function $\lambda_\mathbb{R} : A \to \mathbb{R}$ defines a **characteristic** function $\psi_{\lambda_\mathbb{R}} : \Delta \to \mathbb{R}$, a $T$-piece-wise linear function, by its values on the vertices of $T$. Denote by $C(T) \subset \mathbb{R}^{[A]}$ a subset of such functions $\lambda_\mathbb{R}$, whose corresponding characteristic functions $\psi_{\lambda_\mathbb{R}}$ are convex and $\psi_{\lambda_\mathbb{R}}(\omega) \geq \lambda_\mathbb{R}(\omega)$ for any $\omega \in A$. A person familiar with toric varieties immediately recognizes a secondary cone, the normal cone to the secondary polyhedron at the vertex corresponding to the triangulation $T$.

In particular $C(T)$ has non-empty interior if $T$ is a coherent triangulation. The last piece of data we will need is an integral vector $\lambda$ in the interior of $C_T \subset \mathbb{R}^{[A]}$. This means that the characteristic function $\psi_{\lambda} : \Delta \to \mathbb{Z}$ is strictly convex with respect to the triangulation $T$ and $\psi_{\lambda}(\omega) \geq \lambda(\omega)$, with equality holding exactly for the vertices of $T$.

From now on we fix the following data: a nonsingular reflexive integral polyhedron $\Delta$, a subset of its integral points $A$, a central coherent triangulation $T$ and an integral vector $\lambda \in C(T)$. Given such data we can define the 1-parameter family of Calabi-Yau hypersurfaces $F_t$ by the equations

$$t^{\lambda(0)}x^{\{0\}} - \sum_{\omega \in \Lambda \cap \partial \Delta} t^{\lambda(\omega)}x^\omega = 0.$$

To any hypersurface given by an equation in the form $\sum_{\omega \in A} a_\omega x^\omega = 0$ we can associate the vector $\lambda_a := \{\log |a_{\omega_1}|, \ldots, \log |a_{\omega_d}|\} \in \mathbb{R}^{[A]}$. If this vector $\lambda_a$ is sufficiently far from the walls of the secondary fan then the corresponding hypersurface in $X_\Delta$ is nonsingular. Note that for $|t| \gg 0$, $\log |t^{\lambda(\omega)}| = \lambda \cdot \log |t|$ lies deeply inside $C_T$, so that the hypersurface $F_t$ is smooth (cf. [GKZ], Ch 10). The main object of
study will be the behavior of this family as $|t| \to \infty$. In the limit we approach the large complex structure limit point.

It may sometimes be convenient to have local coordinates on affine subsets of $X_\Delta$. We define $\tau$ - associated coordinates for any $k$-dimensional simplex $\tau \in \partial T$ in the triangulation. Choosing coordinates on the open orbit $(\mathbb{C}^*)^N \subset X_\Delta$ is equivalent to choosing an affine basis $\{\Omega, \omega_1, ..., \omega_N\}$ for the lattice $\mathbb{Z}^N$ together with a marked reference point $\Omega$. Consider the minimal face $\Theta \subset \Delta$ containing $\tau$. Let $l$ be its dimension, $k \leq l \leq N - 1$. We use the fact that $\Delta$ is reflexive and nonsingular to choose $\{\Omega, \omega_1, ..., \omega_N\}$ such that the reference point $\Omega$ is a vertex of $\tau$, the first $k + 1$ vertices $\{\Omega, \omega_1, ..., \omega_k\}$ generate the lattice $\tau_\mathbb{Z}$, the first $l + 1$ vertices $\{\Omega, \omega_1, ..., \omega_l\}$ generate the lattice $(\Theta_\tau)_\mathbb{Z}$. Moreover, because of the unit integral distance from the facets of $\Delta$ intersecting at $\Theta_\tau$ to the origin $\{0\}$, we can have the rest of the basis satisfy $\{0\} - \Omega = (\omega_{l+1} - \Omega) + ... + (\omega_N - \Omega)$. Notice that the last $N - l$ points are uniquely determined modulo $(\Theta_\tau)_\mathbb{Z}$. Then $\{y_1, ..., y_N\} := \{x^{\omega_1 - \Omega}, ..., x^{\omega_N - \Omega}\}$ give local coordinates on $(\mathbb{C}^*)^N$. In fact, they extend to coordinates on the affine subset of $X_\Delta$ obtained by removing the divisors which correspond to the facets of $\Delta$ not containing $\Omega$.

It may also be useful to identify $\Delta$ with the closure of the positive real part $(X_\Delta)_{>0}$ of $X_\Delta$. The homeomorphism $(X_\Delta)_{>0} \simeq \Delta$ is provided by the restriction of the moment map. In particular, $\{|y_1|, ..., |y_N|\}$ will provide coordinates in $\Delta$ with facets not containing $\Omega$ excluded as above.

Inspired by the proof of Viro’s theorem (see [GKZ], Ch. 11) we will use the weighted moment map $\mu_t : X_\Delta \to \Delta$ defined as

$$\mu_t(x) = \frac{\sum_{\omega \in A} |t^\lambda(\omega)| \cdot |x^\omega| \cdot \omega}{\sum_{\omega \in A} |t^\lambda(\omega)| \cdot |x^\omega|}.$$  

The right way to think about this weighted moment map is the following. Add one extra dimension to $M \cong \mathbb{R}^N$ and extend the lattice to $\mathbb{Z}^{N+1} \subset \mathbb{R}^{N+1}$. Let $P$ be the convex hull of $\{(\omega, \lambda(\omega))\}_{\omega \in A}$ in $\mathbb{R}^{N+1}$ (see fig. 3). Then we can think of the whole family $\{F_t\}$ as a hypersurface in $X_P$ ($t$ is considered as a coordinate). The vertical projection $p : P \to \Delta$ splits the boundary of $P$ into two pieces $\partial P = \partial_+ P \cup \partial_- P$. In fact $\partial_+ P$ is exactly the graph of the characteristic function $\psi_\lambda$ and the projection $p : \partial_+ P \to \Delta$ identifies the faces of $\partial_+ P$ with the simplices in the triangulation $T$. The weighted moment map $\mu_{t_0}$ will be just the composition of the restriction $\mu_{t_0}^{(P)} := \mu^{(P)}|_{t=t_0} : X_P|_{t=t_0} \to P$ of the ordinary moment map $\mu^{(P)}$ to the hypersurface $H_{t_0} := \{t = t_0\}$ in $X_P$ with the vertical projection $p : P \to \Delta$.

The image of $\mu^{(P)}(H_{t_0}) \subset P$ is the graph of a function $\Psi_{t_0} : \Delta \to \mathbb{R}$ with the following crucial property (see [GKZ], Ch. 11):

**Proposition 2.1.** As $|t_0| \to \infty$ the function $\Psi_{t_0} : \Delta \to \mathbb{R}$ is continuous and smooth outside $\partial \Delta$, converging uniformly to the characteristic function $\psi_\lambda$, whose graph is $\partial_+ P$. 

6
The extended polyhedron $P$ for the family $txyz - t^2x^3 - y^3 - z^3 = 0$ in $\mathbb{CP}^2$ and the image of the weighted moment map as $t \to \infty$.

The image of the hypersurface $\{F_t\} \subset X_P$ under the moment map $\mu^{(P)}$ misses all the vertices of $P$ (cf. [GKZ], Ch.6) and, in particular, (because $\psi_\lambda$ is convex) it misses some neighborhood of $({\{0\}}, \lambda(0)) \in P$. Applying the above proposition we see that for sufficiently large $|t|$ the image of the weighted moment map $\mu_t(F_t) \subset \Delta$ misses some ball $B$ around $\{0\} \in \Delta$. Denote by $\Delta^0 := \Delta \setminus B$ our polytope with that ball removed. Clearly $\Delta^0$ is homeomorphic to $S^{N-1} \times I$ and the next step will be to find a good trivialization of this product.

Let us conclude this section with a list of notations used throughout the rest of the paper.

- $\Delta \subset M$ a convex nonsingular integral reflexive polyhedron, $\partial \Delta$ its boundary;
- $\Delta(\mathbb{Z})$ the set of integral points in $\Delta$;
- $\{0\} \in \Delta(\mathbb{Z})$ the unique integral interior point;
- $A$ a subset of $\Delta(\mathbb{Z})$ containing $\{0\}$ and all vertices of $\Delta$;
- $\Theta$ a face of $\Delta$, $\Sigma$ a maximal dimensional face of $\Delta$;
- $T$ a central coherent triangulation of $(\Delta, A)$;
- $\lambda \in C(T)$ an integral vector in the interior of the secondary cone at $T$;
- $\partial T$ the induced triangulation of $\partial \Delta$;
- $\tau$ a $k$-dimensional simplex in $\partial T$, $\sigma$ a maximal dimensional simplex in $\partial T$;
- $\Theta_\tau$ the minimal face of $\Delta$ containing $\tau$;
- $C_\tau$ the $(k+1)$-dimensional simplex in $T$ over $\tau$ with the vertex $\{0\}$;
- $O(\tau)$ the center of a simplex $\tau$;
- $\Theta_\tau$ or $\tau_\mathbb{Z}$ the affine sublattices of $\mathbb{Z}^N$ generated by the integral points in $\Theta$ or $\tau$;
- $\Lambda_\tau$ the sublattice of $\tau_\mathbb{Z}$ generated by the vertices of $\tau$;
- $X_\Delta$ the toric variety associated to $\Delta$;
- $F_t$ the family of the Calabi-Yau hypersurfaces in $X_\Delta$;
- $\mu_t$ the weighted moment map;
- $\Delta^0$ the polyhedron $\Delta$ with a small ball around $\{0\}$ removed, $X_\Delta^0 := \mu_t^{-1}(\Delta^0)$.
3 Torus fibrations

At this moment unfortunately we must leave the realm of beautiful algebraic geometry and employ some analysis techniques like partitions of unity and transversality theory. Let $\text{Bar}(\partial T)$ be the first barycentric subdivision of the triangulation $\partial T$. The vertices in this subdivision are the centers $O(\tau)$ of the simplices $\tau$ in $\partial T$. Consider the subdivision $\{V^0_\tau\}_{\tau \subset \partial T}$ dual to $\text{Bar}(\partial T)$. Namely, take the second barycentric subdivision $\text{Bar}^{(2)}(\partial T)$ of $\partial T$ and define $V^0_\tau$ to be the union of all simplices in $\text{Bar}^{(2)}(\partial T)$ having $O(\tau)$ as a vertex. Every $V^0_\tau$ contains the point $O(\tau)$ for a unique $\tau$ and is labeled correspondingly (see Fig.4). For each $V^0_\tau$ we take its small open neighborhood $V_\tau$ to get an open cover of $\partial \Delta$. By construction $V_\tau$ and $V_{\tau'}$ intersect iff either $\tau \subset \tau'$ or $\tau' \subset \tau$. So every point in $\partial \Delta$ lies in at most $N$ different $V_{\tau_i}$’s for $\{\tau_i\}$ forming a nested sequence.

Define the following subsets of $\partial \Delta$:

$$U_\tau := V_\tau - \bigcup_{\tau' \neq \tau} V_{\tau'}, \quad W_\tau := \overline{V_\tau} - \bigcup_{\tau' \supset \tau} V_{\tau'},$$

(set-theoretic difference),

where $\overline{V_\tau} \subset \partial \Delta$ denotes the closure of $V_\tau$.

The collection of cells $\{W_\tau\}_{\tau \in \partial T}$ provides a CW-decomposition of $\partial \Delta$ homeomorphic to that given by $\{V^0_\tau\}$ (see fig.5). $U_\tau$ are the “pure” open subsets in $W_\tau \subseteq \overline{V_\tau}$.
Now we will construct a trivialization of $\Delta^0 \simeq \partial \Delta \times I$. This will provide $X_\Delta^0 := \mu_\tau^{-1}(\Delta^0)$ with the structure of a fibration via the composition of the maps

$$X_\Delta^0 \xrightarrow{\mu_\tau} \Delta^0 \simeq \partial \Delta \times I \xrightarrow{pr_1} \partial \Delta.$$ 

Inside the small central ball $B^0$ choose a concentric mini-copy $\Delta'$ of $\Delta$ with the induced triangulation $\partial T'$ of $\partial \Delta'$. To construct a trivialization we need to connect the two boundaries $\partial \Delta$ and $\partial \Delta'$ by non-intersecting intervals. To do this we need a rule how to choose which pairs of points $s \in \partial \Delta$ and $s' \in \partial \Delta'$ are to be connected and an interval connecting them. After that we must make sure that these intervals do provide a trivialization of $\Delta^0$. We will denote by $\Theta' \subset \partial \Delta'$ and $\tau' \in \partial T'$ the mini-copies of $\Theta$ and $\tau$ correspondingly.

Let $s'$ be an interior point of an $l$-dimensional face $\Theta' \subset \partial \Delta'$. We choose $\tau$-associated coordinates $\{y_1, \ldots, y_N\}$ for some $\tau$, such that $\Theta_\tau$ is the minimal face containing $\tau$. Then $\{|y_1|, \ldots, |y_N|\}$ will provide coordinates in the open subset of $\Delta$ corresponding to $\tau$ (see the previous section) by means of the weighted moment map. Let $m_i = |y_i|(m)$ be the coordinates of a point $m \in \Delta$. Define the curved normal cone to $\Theta'$ at the point $s' \in \Theta'$ by

$$n(s') = \{m \in \Delta : m_i = s'_i, \; i = 1, \ldots, l, \text{ and } m_i \leq s'_i, \; i = l + 1, \ldots, N\}.$$ 

Note that the definition does not depend on the choice of $\tau$-coordinates. Combined all together the curved normal cones form a fat curved normal fan to the polyhedron $\Delta'$. Namely, to every $k$-dimensional face $\Theta' \subset \partial \Delta'$ we can associate an $(N-l) \times l$-dimensional fat curved normal cone $N(\Theta') \simeq n(\Theta') \times \Theta'$, where $n(\Theta')$ is the curved normal cone to $\Theta'$ at any point $s' \in \Theta'$. This fat fan provides a fat cone decomposition of $\Delta$ with $\Delta'$ deleted.

![Fig.6](image_url) The fat cone decomposition and $I_s$-fibration of $\Delta^0$ for $X_\Delta = \mathbb{CP}^1 \times \mathbb{CP}^1$.

The trivialization of $\Delta^0 \simeq \partial \Delta \times I$ is achieved by connecting any point $s' \in \partial \Delta'$ with all the points $s \in \partial \Delta$ lying in the curved normal cone $n(s')$. The connecting intervals are given by the unique line (given by affine linear equations in the remaining $(N-l)$ coordinates) in $n(s')$ passing through both points $s$ and $s'$ (see fig.6). Because every point $s \in \partial \Delta$ belongs to a unique interval, we will denote this interval by $I_s$. Notice that the interval $I_s$ does not depend on the choice of $\tau$-coordinates, for a
change of coordinates does not affect the affineness of the defining equations in the last $N - l$ coordinates. But $I_s$ does depend on the value of $t$ through the dependence of the weighted moment map on $t$ and this $t$-dependence will be crucial in the next proposition.

For any $k$-dimensional simplex $\tau \in \partial T$ we will refer to $s \in \partial \Delta$ as a $\tau$-point if in any $\tau$-associated coordinate system the first $k$ coordinates $\{|y_1|,...,|y_k|\}$ are constant along the interval $I_s$. In other words, $I_s$ lies in a fat cone $N(\Theta')$ for some $\Theta' \supset \Theta'_\tau$. Note again that this does not depend on the choice of the $\tau$-associated coordinates. We will say that a collection $\{I_s\}_{s \in \partial \Delta}$ is $W$-supported if for any $\tau \in \partial T$ all the points in the cell $W_\tau$ are $\tau$-points.

**Proposition 3.1.** Given a cell decomposition $\{W_\tau\}$ there exists a positive real number $R$, such that for $|t| \geq R$ the collection $\{I_s\}_{s \in \partial \Delta}$ providing a trivialization of $\Delta^0 \simeq S^{N-1} \times I$ is $W$-supported.

*Proof.* We claim that in the limit $t \to \infty$ the intervals $I_s$ will provide a bijective correspondence between points in $\tau$ and points in $\tau'$ for all $\tau \in \partial T$. In particular, this means that all interior points in any simplex $\tau$ eventually become $\tau$-points. Assuming this claim the proposition follows immediately from the following observation. $W_\tau$ is compact and is contained in the interior of the union of the facets of the polytope $\Delta$ containing the face $\Theta_\tau$, and every point in this interior becomes a $\tau$-point for large $|t|$.

To show the claim we consider the extended polytope $\tilde{P}$. Keeping the first $k$ $\tau$-associated coordinates fixed is equivalent to fixing $(k + 1)$ projective coordinates $\{|t^{\lambda}(\omega)\omega^\alpha|\}_{\omega \in \tau}$. The restriction of these equations to $p^{-1}(C_\tau) \subset \partial_+ P$, where $p : \partial_+ P \to \Delta$ is the vertical projection, defines a ray $R_\tau$ from the origin ($\{0\}, \lambda(0)$) to a point in $p^{-1}(\tau)$. Thus we see that every point in the interior of $p^{-1}(\tau')$ connects to a unique point in the interior of $p^{-1}(\tau)$. But because the function $\Psi_{\tau_0} : \Delta \to \mathbb{R}$ converges uniformly to the characteristic function $\psi_\lambda$ as $|t| \to \infty$, $I_s$ would be given in the limit by the projection of the ray $R_\tau$ to $\Delta$. \hfill $\square$

**Remark.** If one wants a fibration of $X^0_\lambda$ over a smooth sphere as opposed to just topological, then some care should be taken to smooth out $I_s$ near the boundaries of the fat cone decomposition of $\Delta^0$. From now on we will assume that $\partial \Delta \simeq S^{N-1}$ is endowed with a smooth structure compatible with the fibration.

For any subset $S \subset \partial \Delta$ we will denote by $I_S \simeq S \times I$ the union of $I_s$ for $s \in S$, and let $\tilde{S} := \mu^{-1}(I_S) \subset X^0_\lambda$. In particular, we will be using $\tilde{U}_\tau$ and $\tilde{W}_\tau$.

Next we choose a partition of unity $\{\rho^0_\tau\}$ subordinate to the cover $V_\tau$, and define the cut-off functions $\rho_\omega : \partial \Delta \to [0,1]$ for $\omega \in \partial T$, a vertex of the triangulation, by $\rho_\omega := \sum_{\tau \ni \omega} \rho^0_\tau$. In particular, it is clear that

$$\rho_\omega(s) = \begin{cases} 0, & \text{unless } s \in V_\tau \text{ for some } \tau \text{ containing } \omega, \\ 1, & \text{if } s \in V_\tau \text{ only for those } \tau \text{ which contain } \omega. \end{cases}$$
To uniformize the notation let $\rho_{\{0\}} \equiv 1$ and $\rho_\omega \equiv 0$ for those $\omega \in A$ which are not vertices in the triangulation. Extend $\rho$'s to the entire $\Delta^0$ by setting $\rho_\omega(I_s) := \rho_\omega(s)$ for the entire interval $I_s$. We will denote by $\rho_\omega$ also the pull back of the cut-off functions to $X_\Delta^0 := \mu^{-1}(\Delta^0)$ via the moment map.

Now we are in position to define the auxiliary object $H_t$, a real (non-analytic) codimension 2 submanifold in $X_\Delta$, which we will still call a hypersurface. It is defined by the following equation:

$$ t^{\lambda(0)} x^{\{0\}} - \sum_{\omega \in A \cap \partial \Delta} \rho_\omega t^{\lambda(\omega)} x^\omega = 0. $$

The restriction of $H_t$ to $\widetilde{U_\tau}$ defines an open set (in complex topology) of an algebraic subvariety in $X_\Delta$ given by the equation

$$ t^{\lambda(0)} x^{\{0\}} - \sum_{\omega \in \tau} t^{\lambda(\omega)} x^\omega = 0. $$

And the cut-off functions $\rho$ are designed to connect these pieces together.

We showed that $\mu_l(F_t)$ misses a ball around $\{0\} \in \Delta$. A similar argument applies to show that $\mu_l(H_t)$ also lies in $\Delta^0$. Namely, every point of $H_t$ can be thought as a point in an algebraic hypersurface in $X_\Delta$ defined by the same equation as $H_t$ but with constant $\rho$'s. Every such algebraic hypersurface clearly misses a ball around $\{0\} \in \Delta$. Using the compactness of $H_t$ we get the claim.

The structure of the rest of this section is the following. The fibration $X_\Delta^0 \rightarrow \partial \Delta$, when restricted to the auxiliary hypersurface, induces a torus fibration $h_t : H_t \rightarrow \partial \Delta$. Then we will show that the auxiliary hypersurface is, in fact, diffeomorphic to the original one. Moreover this diffeomorphism extends to a diffeomorphism between the entire families $\{F_t\}$ and $\{H_t\}$. This will provide a torus fibration of the original Calabi-Yau hypersurface.

The first step is to show that the collection $\{I_s\}_{s \in \partial \Delta}$ defines a torus fibration $h_t : H_t \rightarrow \partial \Delta$. Of course, some of the fibers are degenerate and we will try to describe them as explicitly as possible. Let $\tau$ be a $k$-dimensional simplex in $\partial T$, and $\Theta_\tau \subset \Delta$ the minimal face containing $\tau$, let $l := \dim \Theta_\tau$. Let $\{y_1, ..., y_l\}$ be the first $l$ of $\tau$-associated coordinates. A $\tau$-point $s$ defines an $l$-dimensional torus $T^l \subset (\mathbb{C}^*)^l$ by setting $|y_i| = s_i$. Let $P(y_1, ..., y_k) = 0$ be the equation

$$ \sum_{\omega \in \tau} \rho_\omega t^{\lambda(\omega)} x^\omega = 0, $$

written in the local coordinates. Denote by $D_{\tau,s} \subset T^l$ the zero locus of the polynomial $P(y_1, ..., y_k)$. To get some idea what $D_{\tau,s}$ looks like we consider a $k$-dimensional torus $T^k \subset (\mathbb{C}^*)^k = \text{Spec} [z_1^{\pm 1}, ..., z_k^{\pm 1}]$, defined by fixing $|z|$. Denote by $D_{\tau,z}^0$ the intersection of $T^k$ with a plane $\rho_0 + \sum_{i=1}^k \rho_i z_i = 0$. For generic $|z|$ and $\rho$, $D_{\tau,z}^0$ will be either empty or a $(k-1)$-torus. In exceptional cases $D_{\tau,z}^0$ can be a single (real) point. The relation between $D_{\tau,s}$ and $D_{\tau,z}^0$ can be described as follows. The substitution
\(z_i = t^{\lambda(\omega_i) -\lambda(\Omega)} \omega_i - \Omega\), where \(\{\Omega, \omega_1, ..., \omega_k\}\) are the vertices of the simplex \(\tau\), defines a covering map \(\pi_{\tau} : T^k \to T^k\). The degree of \(\pi_{\tau}\) is equal to the index of the sublattice \(\Lambda_{\tau}\) inside the lattice \(\tau\mathbb{Z}\), which is given by \(k! \cdot \text{vol}(\tau)\). Then \(D_{\tau,s} \subset T^l\) is a pull back of \(D^0_{\tau,|z|} \subset T^k\) under the composition of maps

\[
T^l \xrightarrow{pr} T^k \xrightarrow{\pi_{\tau}} T^k,
\]

where \(pr : \{y_1, ..., y_l\} \to \{y_1, ..., y_k\}\) is a projection onto the first \(k\) coordinates.

**Proposition 3.2.** Let \(s \in W_{\tau}\) as above be a point in the interior of an \(L\)-dimensional face of \(\Delta\), \(L \geq l\). Then the fiber \(T_s := \mu_{l}^{-1}(I_s)\) itself has a structure of a fibration \(p_s : T_s \to T^l\) with a generic fiber \(T^{N-1-l}\) and fibers \(T^{l-1}\) over the discriminant locus \(D_{\tau,s}\). Thus, \(T_s\) is homeomorphic to \(T^{l-1} \times (T^l \times T^{N-1-l})/\sim\), where \((d,t_1) \sim (d,t_2)\), if \(d \in D_{\tau,s}\).

In particular, if the point \(s\) is in the interior of an \((N-1)\) dimensional face, then \(T_s\) is a smooth \((N-1)\)-torus.

**Proof.** We choose \(\tau\)-associated coordinates \(\{y_1, ..., y_N\}\). In particular, they provide coordinates in \(\tilde{W}_{\tau}\). The equation of the auxiliary hypersurface restricted to \(T_s\) becomes:

\[y_{l+1}y_{l+2}...y_N = P_t(y_1, ..., y_k),\]

where \(P_t\) is a polynomial (all \(\rho\) are constants on \(T_s\)). Because \(s\) is a \(\tau\)-point, \(|y_i|\) are fixed and non zero for \(i = 1, ..., l\). This gives a projection \(p_s : T_s \to T^l\). A fiber of this projection is determined by fixing a point \(Y := \{y_i\}, i = 1, ..., l\), on the base. After that we are left with the equation \(y_{l+1}...y_N = P_t(Y) = \text{const.}\). At this point we must remember that \(I_s\) is given by a line in chosen coordinates, hence it intersects the hyperbola \(|y_{l+1}|...|y_N| = |P_t(Y)|\) in exactly one point in \(\Delta\). Thus it determines the remaining \(|y_i|\) uniquely.

We see that a generic fiber of the projection \(p_s : T_s \to T^l\) is an \((N-l-1)\)-torus. The dimension drops to \((L-l)\), if \(P_t(Y) = 0\), i.e. exactly if the point \(Y\) is in the discriminant locus \(D_{\tau,s}\). \(\square\)

We want to say some words about the discriminant locus \(D(H_t)\) of the fibration \(h_t : H_t \to \partial\Delta\). The above proposition says that \(D(H_t)\) consists of all points in \(\partial\partial\Delta\), the \((N-2)\)-dimensional skeleton of \(\partial\Delta\), which are in the image of the moment map \(\mu_t\). Thus \(D(H_t)\) is homeomorphic to \(\partial\partial\Delta\) with some neighborhoods of its vertices removed, which has a homotopy type of \(\text{Sk}_{T}^{(N-3)}\), the \((N-3)\)-skeleton of the subdivision dual to the triangulation of \(\partial\partial\Delta \subset \partial\Delta\). Moreover, with an appropriate choice of the \(V\)-subdivision (\(W_{\tau}\) should have small volume for all \(\tau\) with \(1 \leq \dim \tau \leq N-2\)) the discriminant locus \(D(H_t)\) will lie in an arbitrarily small neighborhood of \(\text{Sk}_{T}^{(N-3)}\). We will refer to this limit as the right \(W\)-decomposition limit.

To deform the auxiliary hypersurface \(H_t\) and to use transversality theory we must make sure that it is smooth.
Lemma 3.3. For a generic choice of \( \{ \rho \} \) the hypersurface \( H_t \) is smooth.

Proof. It is enough to show that \( H_t \) is smooth in \( \tilde{W}_\tau := \mu_t^{-1}(I_{W_\tau}) \) for every \( \tau \subset \partial T \). We will use \( \tau \)-associated coordinates \( \{ y_1, ..., y_N \} \) in \( \tilde{W}_\tau \). Let \( G_t \) be the defining equation of the auxiliary hypersurface in \( \tilde{W}_\tau \) (see the previous proposition)

\[
G_t := y_{l+1} y_{l+2} ... y_N - b \Omega \rho_\Omega(|y|) - \sum_{i=1}^k b_i \rho_i(|y|) y^{\alpha_i} = 0,
\]

where \( t \)-dependence is encoded into \( b_i \)’s.

Denote by \( \mathcal{R} \) the family of \( \{ \rho_\omega \} \) constructed from the family of partitions of unity subordinate to \( \{ V_\tau \} \). Consider the map \( G_t^\rho : \tilde{W}_\tau \rightarrow \mathbb{C} \). The statement that \( G_t^\rho = 0 \) is smooth in this language translates as \( G_t^\rho \) is transversally regular to \( 0 \in \mathbb{C} \). We want to show that there are enough functions in \( \mathcal{R} \), so that a generic \( G_t^\rho \) is transversally regular to \( 0 \in \mathbb{C} \). According to the restricted transversality theory, it is enough to show that the map of the entire family \( \tilde{G}_t : \tilde{W}_\tau \times \mathcal{R} \rightarrow \mathbb{C} \) is transversally regular to \( 0 \in \mathbb{C} \) (cf, e.g. [DNF]). For any point \( \tilde{x} = (x, \rho) \in \tilde{G}_t^{-1}(0) \) we need to show that the tangent space at \( \tilde{x} \) maps onto \( \mathbb{C} \). Consider the restriction of \( \tilde{G}_t \) to the slice in a small neighborhood of \( \tilde{x} \), given by \( \rho = \text{const.} \). The function \( \tilde{G}_t|_{\rho=\text{const.}} \) becomes algebraic and it is a straightforward calculation to show that the tangent space to that slice is transversal to \( 0 \in \mathbb{C} \). Just notice that all \( y_i, i = 1, ..., k \) are non zero and at least one of the \( \rho_i \) is non zero too. Hence a generic choice of \( \rho \) will provide a smooth preimage of \( 0 \in \mathbb{C} \). This completes the proof. \( \Box \)

The next step is to find a small deformation of \( H_t \) and a diffeomorphism of \( X_\Delta^0 := \mu_t^{-1}(\Delta^0) \) inside \( X_\Delta \), which transforms the deformed equation for \( H_t \) into the equation of a genuine hypersurface \( F_{(\Gamma, t)} \) for some real number \( \Gamma \). For this we need a technical lemma.

Lemma 3.4. There exists a function \( \chi(s, \gamma, \omega) : \partial \Delta \times [0, \infty) \times A \rightarrow \mathbb{R} \), smooth with respect to \( (s, \gamma) \) and affine linear with respect to \( \omega \), and satisfying:

- \( \chi(s, 0, \omega) \equiv 0 \) and \( \chi(s, \gamma, \{ 0 \}) \equiv \gamma \cdot \lambda(0) \).
- As \( \gamma \rightarrow \infty \) the function \( e^{\gamma \lambda(\omega)} - \chi(s, \gamma, \omega) \) converges (uniformly) to \( \rho_\omega(s) \) for every \( \omega \in A \).

Proof. First, for every \( \tau \subset \partial T \) in the triangulation we choose an affine linear function \( \chi_\tau : A \rightarrow \mathbb{R} \) with the following property: \( \chi_\tau(\omega) \geq \lambda(\omega) \) with equality holding exactly for \( \omega \in \tau \cup \{ 0 \} \). Note that for \( \sigma \), an \( (N-1) \)-dimensional simplex, \( \chi_\sigma \) is uniquely determined by \( \psi_\lambda|_C \) and the inequality condition is satisfied because \( \psi_\lambda \) is a strictly convex function. By the same reason we can satisfy the inequality for the simplices of smaller dimension. For every \( (N-1) \)-dimensional simplex \( \sigma \) we define the function

\[
\chi_\sigma(s, \gamma, \omega) := -\log(\sum_{\tau \in \partial T} \rho_\tau^0 e^{-\gamma \chi_\tau}),
\]

where...
The function $\chi(s, \gamma, \omega)$ is constructed by gluing the functions $\chi_\sigma(s, \gamma, \omega)$ together in the following way. The collection of maximal dimensional simplices $\sigma \in \partial T$ provides a triangulation of $\partial \Delta$. We take small open neighborhoods of each $\sigma$ to get an open cover $\{\tilde{\sigma}\}$ of $\partial \Delta$ and choose a partition of unity $\{\alpha_\sigma\}$ subordinate to it. We require the open enlargements of $\sigma$’s to be small enough, so that $\tilde{\sigma} \subset \bigcup_{\tau \subset \sigma} W_\tau$. In particular, $\rho_\tau^0|_{\tilde{\sigma}} \equiv 0$ unless $\tau \subset \sigma$. Now we can define the desired function

$$\chi(s, \gamma, \omega) := \sum_{\sigma \in \partial T} \alpha_\sigma(s) \chi_\sigma(s, \gamma, \omega),$$

which is smooth with respect to $(s, \gamma)$ and affine linear with respect to $\omega$ by construction. It is also clear that

$$\chi_\sigma(s, 0, \omega) = -\log\left(\sum_{\tau \in \partial T} \rho_\tau^0\right) = 0$$

and

$$\chi_\sigma(s, \gamma, \{0\}) = -\log\left(\sum_{\tau \in \partial T} \rho_\tau^0 e^{-\gamma \cdot \lambda(0)}\right) = \gamma \cdot \lambda(0)$$

for all $\sigma$. Hence $\chi(s, 0, \omega) \equiv 0$, and $\chi(s, \gamma, \{0\}) \equiv \gamma \cdot \lambda(0)$.

The last thing to check is the behavior of $\chi(s, \gamma, \omega)$ as $\gamma \to \infty$. Fix a point $s \in W_{\tau_0}$, and first consider vertices $\omega \in \tau_0$. The partition functions $\alpha_\sigma(s) = 0$ unless $\tau_0 \subset \sigma$, hence $\omega \in \sigma$ and $\chi(s, \gamma, \omega) = -\log\left(\sum_{\tau \in \partial T} \rho_\tau^0 e^{-\gamma \cdot \chi_\tau(\omega)}\right)$.

$$\lim_{\gamma \to \infty} e^{-\chi(s, \gamma, \omega)} e^{\gamma \cdot \lambda(\omega)} = \lim_{\gamma \to \infty} \sum_{\tau \in \partial T} \rho_\tau^0 e^{\gamma \cdot \lambda(\omega) - \chi_\tau(\omega)} = \sum_{\tau \ni \omega} \rho_\tau^0 = \rho_\omega,$$

because $-\chi_\tau(\omega) \leq 0$ with equality holding exactly for $\omega \in \tau \cup \{0\}$.

If $\omega \notin \tau_0$, then to show that $\lim_{\gamma \to \infty} e^{-\chi(s, \gamma, \omega)} e^{\gamma \cdot \lambda(\omega)} = \rho_\omega(s) = 0$ we look at the asymptotics of the affine functions $\chi_\sigma(s, \gamma, \omega)$ as $\gamma \to \infty$. Let $Q$ be the collection of simplices $\tau$ with the minimal value of $\chi_\tau(\omega)$ among those with nonzero $\rho_\tau^0(s)$. Note that if $\rho_\tau^0(s) \neq 0$, then $\tau \subset \tau_0$, and hence $\chi_\tau(\omega) > \lambda(\omega)$ as $\omega \notin \tau$. So we see that

$$\chi_\sigma(s, \gamma, \omega) \sim -\log\left(\sum_{\tau \in Q} \rho_\tau^0\right) + \gamma \cdot \chi_\tau(\omega).$$

Combining these together for all $\sigma$’s we get

$$\chi(s, \gamma, \omega) \sim -\log\left(\sum_{\tau \in Q} \rho_\tau^0\right) + \gamma \cdot \chi_\tau(\omega),$$

and hence

$$e^{-\chi(s, \gamma, \omega)} e^{\gamma \cdot \lambda(\omega)} \sim \sum_{\tau \in Q} \rho_\tau^0 e^{\gamma \cdot \lambda(\omega) - \chi_\tau(\omega)} \to 0$$

as $\gamma \to \infty$. This completes the proof.
Remark. In the proof of the above lemma the crucial fact we used was that the characteristic function \( \psi_\lambda \) is strictly convex.

Just as we did for \( \rho_\omega \), we will use the same notation for both \( \chi(\gamma, \omega) \) and its pull back to \( X_\Delta^0 \) via the moment map. Now we have all the tools to prove the main theorem. Let \( R \) be the positive real number as in proposition 3.1. Denote by \( H_R \) the one-(real) parameter family of the auxiliary hypersurfaces \( \{ H_t, |t| = R \} \). Let \( F_{\Gamma_0 R} \) be the one-(complex) parameter family of the original Calabi-Yau hypersurfaces \( \{ F_t, |t| \geq \Gamma_0 R \} \).

**Theorem 3.5.** There is a positive real number \( \Gamma_0 \), such that there exists a diffeomorphism between the families \( \{ H_R \} \times (\Gamma_0, +\infty) \) and \( F_{\Gamma_0 R} \), which specializes to a diffeomorphism between the hypersurfaces \( (H_t, \Gamma) \) and \( F_{\Gamma t} \).

**Proof.** First we define a hypersurface \( H_\varepsilon \subset X_\Delta^0 \) by the equation

\[
t^\lambda(0)x(0) - \sum_{\omega \in A \cap \partial \Delta} (\rho_\omega + \varepsilon_\omega)t^\lambda(\omega)x^\omega = 0, \quad \text{where } \varepsilon_\omega := e^{\gamma \lambda(\omega) - \chi(\gamma, \omega)} - \rho_\omega.
\]

According to lemma 3.4, all \( \varepsilon_\omega \) uniformly vanish as \( \gamma \to \infty \), that is for \( \gamma \geq \gamma_0 \) \( H_\varepsilon \) is indeed a small deformation of \( H_t \) and hence diffeomorphic to it. Using the substitution \( x' : = x^\omega e^{-\chi(\gamma, \omega)} \) we get

\[
e^{\gamma \lambda(0)} t^\lambda(0)x(0) - \sum_{\omega \in A \cap \partial \Delta} e^{\gamma \lambda(\omega)} t^\lambda(\omega)x^\omega = 0,
\]

which is exactly the equation of the hypersurface \( F_{(\Gamma, t)} \) for \( \Gamma = e^\gamma \). A priori this substitution defines a map \( X_\Delta^0 \to \mathbb{P}^{\Delta(\mathbb{Z})|\Lambda|} \). But because \( \chi(\omega) \) is an affine function, the image of this map, in fact, lies in \( X_\Delta \). Hence the above equation indeed defines a hypersurface in \( X_\Delta \).

Notice that this construction works for the entire families, because the deformation diffeomorphisms clearly form a trivial system, and the substitutions depend only on the absolute values of the parameters of the families.

According to [GKZ], Ch. 10, all hypersurfaces which lie inside a translated cone \( C(T) + b \), where \( b \) is some vector in the interior of \( C(T) \), are smooth and hence diffeomorphic to each other. Combining the above theorem with proposition 3.2, we get the main result of the paper.

**Corollary 3.6.** A Calabi-Yau hypersurface in \( X_\Delta \), which is sufficiently far away from the walls of the secondary fan to \( \Delta \) and sufficiently close to the large complex structure, admits a fibration over a sphere \( S^{N-1} \) with generic fibers \( (N-1) \)-dimensional tori.

Remark. It should be possible to remove the smoothness requirement. In this case one has to be more careful with deforming the equation of a non smooth hypersurface. For a \( \Delta \)-regular hypersurface in the translated cone \( C(T) + b \) all singularities come from the singularities of \( X_\Delta \). There is a natural stratification of \( X_\Delta \) by \( \mu^{-1}(\Theta) \), as \( \Theta \) runs over open parts of the faces of \( \Delta \), which induces a stratification of \( F_t \). All diffeomorphisms should then be understood in this stratified sense (see, e.g. [GM].)
The ultimate goal would be, of course, to construct a special Lagrangian fibration. Our construction, unfortunately, leaves this problem open. But there are some features which may be worth mentioning. For instance, our fibration is quite special in the following sense. It tends to concentrate the singularities of the fibers into a smaller number of fibers with worse degenerations. As an example let us consider the family of quartic K3 surfaces in $\mathbb{C}P^3$ given by the equations

$$t \cdot x^{(0)} + \sum_{\Omega \text{ vertex of } \Delta} x^\Omega + O(t^{-1}) = 0.$$ 

A generic fibration is expected to have 24 degenerate fibers, and each one of them is homeomorphic to the standard $I_1$ degenerate elliptic curve. In our fibration the terms $O(t^{-1})$ don’t matter and we get just 6 singular fibers of type $I_4$.

There is a local special Lagrangian structure on the algebraic pieces of the auxiliary hypersurface. However for this we should have defined the cutoff functions $\rho_\omega$ slightly differently (we didn’t do so in the first place because it would have spoiled the uniformness of the definition). Namely, let $\rho_\omega := \sum_\tau \rho_\tau^\omega$, where $\tau$ runs over the simplices in $\partial T$ containing $\omega$, but of dimension at most $(N - 2)$. This reduces the support of $\rho_\omega$ to a neighborhood of the $(N - 2)$-skeleton of $\partial \Delta$. Then for a maximal dimensional simplex $\sigma \subset \partial T$ the equation of $H_t$ in $\tilde{U}_\tau$ would be just $x^{(0)} = 0$, which defines some open subset of the corresponding to $\sigma$ toric divisor. Notice that $\mu_t(T_s)$ is just one point in $\Delta$ (which in this case lies on the boundary $\partial \Delta$), so with respect to the standard symplectic form on the toric variety $X_\Delta$, $T_s$ is clearly Lagrangian. Moreover, if we define the top holomorphic form in $\tilde{U}_\sigma$ according to the equation (every hypersurface is locally Calabi-Yau), then it restricts to a volume form on each fiber. It is easy to check that the same is true in $\tilde{U}_\omega$ for any vertex in the triangulation, where the local equation is $y_1...y_N = const$. The fibration is given by fixing $|y_i|, 1 \leq i \leq N$, which is clearly Lagrangian. A top holomorphic form can be written as $\frac{dy_1}{y_1} \wedge ... \wedge \frac{dy_{N-1}}{y_{N-1}}$, which restricts to a volume form on $T_s$.

So that in the right $W$-decomposition limit every $T_s$ becomes Lagrangian with respect to the deformed symplectic structure except for $s$ in the singular locus. Unfortunately we cannot say the same thing about special Lagrangian property. Although the local holomorphic forms on the auxiliary hypersurface do give the volume forms when restricted to the fibers, it is not at all clear what are their pull-backs to the original hypersurface and how to patch them together in the transition regions.

### 4 Monodromy

In this section we want to show an application of the constructed fibration to the monodromy calculations. Gross has made a conjecture [G] about the monodromy transformation in a family of Calabi-Yau manifolds $\mathcal{X} \to S$. Let $X = \mathcal{X}_t \to B$, $t \in S$ be a torus fibration with a section $\delta_0$. Then $X^t$, the complement of the critical locus of $f$, has a structure of a fiber space of abelian groups with the zero section $\delta_0$. 


Given another section $\delta$ one obtains a diffeomorphism $T_\delta : X^\sharp \to X^\sharp$ given by $x \mapsto x + \delta(f(x))$, which extends to a diffeomorphism of the entire $X$. Given a degeneration divisor in $S$ passing through the large complex structure point, and a loop around this divisor we can consider the monodromy transformation on cohomology. The conjecture says that this monodromy is induced by $T_\delta$ for some section $\delta$.

We are going to construct such a section for our family of hypersurfaces $F_t$. Because theorem 3.5 establishes the diffeomorphism between the entire families, the monodromy question is identical for the auxiliary family $H_R$. Without loss of generality we may assume that the base point in the family is given by a hypersurface $H_{t_0}$ with $t_0 = R$, a real positive number. The monodromy loop is parameterized by $t = t_0 e^{2\pi i \gamma}$, $0 \leq \gamma \leq 1$. But first of all we need a zero section.

**Lemma 4.1.** The fibration $h_{t_0} : H_{t_0} \to \partial \Delta$ has a section $\delta_0$, which misses all singular points of the fibers.

**Proof.** The section is given by the set of all real positive points of $H_{t_0}$. Let $s \in W_\tau$ be a $\tau$-point. We just have to show that the interval $I_s$ of positive real points in $X^0_\Delta$ has a unique solution to the equation

$$t^{\lambda(0)} x^{\{0\}} = \sum_{\omega \in \tau} \rho_\omega t^{\lambda(\omega)} x^\omega.$$ 

But with the identification of the real positive points of $X^0_\Delta$ with $\Delta^0$, the real positive points satisfying this equation form a hypersurface separating $\{0\}$ from those $\omega$ for which $\rho_\omega \neq 0$ (cf. [GKZ], Ch.11). In particular, it has a unique point of intersection with the line $I_s$, which, moreover, lies in the interior of $\Delta^0$. But all the singular points of the fiber $T_s$ are mapped to the boundary of $\Delta$. Thus we get the desired section $\delta_0 : \partial \Delta \to H_{t_0}$. \hfill $\square$

To construct the other section $\delta$ we consider a Delzant type polytope $\Delta^\vee_\gamma \in M^*$ defined by the inequalities:

$$\langle m, \omega - \{0\} \rangle \geq \gamma \cdot (\lambda(\omega) - \lambda(0)),$$

where $\omega$ runs over the vertices in the triangulation $T$. This is a convex polytope with non empty interior for $\gamma > 0$, because of strict convexity of the characteristic function $\psi_\lambda$. By the same reason it is combinatorially dual to $(\Delta, T)$, namely to each $k$-dimensional simplex $\tau$ in $\partial \Delta$, there corresponds an $(N - 1 - k)$-dimensional face $\tau^\vee$ in $\partial \Delta^\vee_\gamma$ with the reverse incidence relation.

The bijective correspondence between the centers of the dual pairs, $\tau$ and $\tau^\vee$, gives rise to a simplicial map $\nu_\gamma : \text{Bar}(\partial T) \to \text{Bar}(\partial \Delta^\vee_\gamma)$ between the first barycentric subdivisions. Considering the $W$-decompositions, which are dual to the barycentric ones, we get a homeomorphism $\nu_\gamma : \partial \Delta \to \partial \Delta^\vee_\gamma$ satisfying $\nu_\gamma(W_\tau) = W_{\tau^\vee}$, where $\{W_{\tau^\vee}\}$ provide a CW-decomposition of $\partial \Delta^\vee_\gamma$. Because each $W_{\tau^\vee}$ contains the center of the simplex $\tau^\vee$, there is a map $\nu_\gamma : \partial \Delta \to \partial \Delta^\vee_\gamma$, homotopic to $\nu_\gamma$, with the property that $\nu_\gamma(W_\tau) \subset \tau^\vee$. \hfill 17
We will use the same notation for both $\nu'_\gamma$ and its pull back to $X^0_\Delta$. Let us define a diffeomorphism $D_\gamma : X^0_\Delta \to X^0_\Delta$ by

$$x^\omega \mapsto x^\omega e^{2\pi i \langle \nu'_\gamma(x), \omega - \{0\} \rangle}.$$ 

This is well defined as $\nu'_\gamma(x)$ is a linear functional with respect to $\omega$. In fact, this diffeomorphism is equivariant with respect to the toric action, i.e. $\mu_t \circ D_\gamma = \mu_t$. For convenience we will drop the index $\gamma$ in all notations whenever $\gamma = 1$. The desired section $\delta : \partial \Delta \to H_{t_0}$ is given by applying the diffeomorphism $D_\gamma$ to the zero section. Thus we define $\delta := D_{\delta_0}$.

**Theorem 4.2.** The section $\delta : \partial \Delta \to H_{t_0}$ is well defined and induces the monodromy transformation on $H_{t_0}$.

**Proof.** The correctness of the definition follows easily from the following observation. For $s \in W_\tau$ notice that $\langle \nu'_\gamma(s), \omega - \{0\} \rangle = \gamma(\lambda(\omega) - \lambda(0))$ for all $\omega \in \tau$. This means that for $\gamma = 1$ the diffeomorphism $D_\gamma$ has no effect on any monomial $x^\omega$ for $\omega \in \tau$, as it gets multiplied by the factor of $e^{2\pi i (\lambda(\omega) - \lambda(0))}$. So that the equation of $H_{t_0}$ in $\tilde{W}_\tau$

$$t_0^\lambda(0) x^{\{0\}} - \sum_{\omega \in \tau} \rho_\omega t_0^\lambda(\omega) x^\omega = 0$$

is still satisfied for $\delta = D(\delta_0)$. Notice that the diffeomorphism $D$ respects the fibration $h_{t_0} : H_{t_0} \to \partial \Delta$, hence the action in a fiber $T_s$ is just the translation by $\delta(s) - \delta_0(s)$. This action is also well defined on singular fibers. Indeed, a singular fiber is itself a fibration $p_s : T_s \to T^{\dim \tau}$, according to the proposition 3.2. And the action on $T_s$ translates points along the fibers of $p_s$, which generically are $(N - 1 - \dim \tau)$-dimensional tori.

To see that the diffeomorphism $D$ induces the monodromy transformation we notice that $D_\gamma$ provides a diffeomorphism of $H_{t_0}$ with $H_{t_0 e^{2\pi i \gamma}}$. Indeed, in $\tilde{W}_\tau$ the defining equation of $H_{t_0}$ translates into

$$t_0^\lambda(0) x^{\{0\}} - \sum_{\omega \in \tau} \rho_\omega t_0^\lambda(\omega) x^\omega e^{2\pi i \gamma (\lambda(\omega) - \lambda(0))} = 0,$$

which is exactly the defining equation for $H_{t_0 e^{2\pi i \gamma}}$. As $\gamma$ runs from 0 to 1 the family of the diffeomorphisms $D_\gamma$ provides the monodromy along the loop $t_0 e^{2\pi i \gamma}$. This completes the proof. 

\[\square\]

5 Dual fibrations and mirror symmetry

This section is rather speculative in character but it is impossible to overlook a connection of our construction with the mirror symmetry. A triple $(\Delta, T, \lambda)$ defines a family of complex structures on a Calabi-Yau hypersurface. For simplicity we assume that the subset $A \subset \mathbb{Z}^N$ coincides with the set of vertices of the triangulation $T$. On
the mirror side we want to get a family of Kähler structures on some other Calabi-Yau. This family is provided by the monomial-divisor map [AGM]. To construct it we consider the polytopes $\Delta^\vee$ defined in the previous section.

Let $N(\Delta^\vee)$ be the normal fan to $\Delta^\vee$. $N(\Delta^\vee)$ is a rational convex polyhedral fan and the corresponding toric variety $X_{\Delta^\vee}$ is a blow-up of the variety $X_{\Delta}\otimes$ for the polar polytope $\Delta^D$. The one-dimensional cones in $N(\Delta^\vee)$ are in one-to-one correspondence with the vertices of the triangulation $\partial T$. The exceptional divisors are labeled by those of them which are not the vertices of $\Delta$ and $N(\Delta^\vee)$ is a simplicial cone subdivision of $N(\Delta^D)$ by means of the triangulation $T$. The vector $\lambda$ lies in the interior of the Kähler cone of $X_{\Delta^\vee}$ and defines the Kähler class (in the orbifold sense) by the linear combination of the toric divisors $[\omega^\vee]$ corresponding to the facets $\omega^\vee$ in $X_{\Delta^\vee}$:

$$[\kappa_\gamma] := - \sum_{\omega \in \partial T} \gamma(\lambda(\omega) - \lambda(0))[\omega^\vee].$$

We will consider the family $\{X_{\Delta^\vee}\}$, where $\gamma$ runs over positive real numbers. The symplectic form $\kappa_\gamma$ defines the moment map $\mu^\vee: X_{\Delta^\vee} \to \Delta^\vee$ (cf. [Gu]). Now we choose a regular anti-canonical hypersurface $Z^D$ in $X_{\Delta}\otimes$ with large complex structure (e.g., with large central coefficient). Denote by $Z \subset X_{\Delta^\vee}$ its proper transform induced by the blow-up $X_{\Delta^\vee} \to X_{\Delta}\otimes$. $Z$ is a Calabi-Yau hypersurface (cf. [B]) endowed with a Kähler structure by restriction from $X_{\Delta^\vee}$ in the orbifold sense. This is the mirror family. We will let $\gamma = 1$ for the future consideration as the behaviour of the family changes by a simple rescaling for other $\gamma$.

To study the geometry of $Z$ we will again use the moment map $\mu^\vee: X_{\Delta^\vee} \to \Delta^\vee$. At this point we will make an assumption that $Z$ possesses a torus fibration analogous to that of a smooth hypersurface. All singularities of $Z$ are mapped by $\mu^\vee$ to the $(N - 2)$-skeleton of $\Delta^\vee$. So that a generic fiber is still a smooth $T^{N-1}$. But degenerations in singular tori may give rise to the singularities in the total space.

First, let us introduce some notations. We will use $\tau_\mathbb{Z}$ also to denote the subgroup of $\mathbb{Z}^N$ modeled on the affine sublattice $\tau_\mathbb{R}$. In other words, $\tau_\mathbb{R} := \tau_\mathbb{Z} \otimes \mathbb{R}$ is the $k$-dimensional vector subspace parallel to $\tau$ and passing through $\{0\}$. Denote by $\tau_\mathbb{R}^*$ the quotient of $(\mathbb{Z}^N)^*$ dual to $\tau_\mathbb{Z} \subset \mathbb{Z}^N$, and let $\tau_\mathbb{R}^* := \tau_\mathbb{Z} \otimes \mathbb{R}$ be the corresponding quotient of $M^*$. Now we consider an explicit parameterization of nonsingular fibers in the original family. Let $s \in W_\tau$, for $\tau \subset \partial T$, be a point in $\partial \Delta$. According to proposition 3.2, a fiber $T_s$ is a fibration itself $p_s: T_s \to T^k$, and $T^k$ is naturally isomorphic to the torus $\tau_\mathbb{R}^*/\tau_\mathbb{Z}^*$. Choosing a point in $T^k$ determines the phases not only of $x^\omega$, $\omega \in \tau$, but also of $x^{(0)}$. Considering the fact that $\tau_\mathbb{R}^*$ is defined by the equations $\langle u, \omega - \{0\} \rangle = 0$, $\omega \in \tau$, we conclude that an $(N - k - 1)$-dimensional fiber $T^{N-k-1}$ can be identified with $\tau_\mathbb{R}^*/\tau_\mathbb{Z}^*$. Hence the fiber $T_s$ is isomorphic to the torus $\tau_\mathbb{R}^*/\tau_\mathbb{Z}^* \oplus \tau_\mathbb{Z}^*/\tau_\mathbb{Z}^*$, though the splitting into the direct sum is not natural. The dual fiber $T_{s^\vee}$, where $s^\vee = \nu(s)$ is a point in $W_{s^\vee} \subset \partial \Delta^\vee$, will be isomorphic to $\tau_\mathbb{R}^*/\tau_\mathbb{Z}^* \oplus (\tau_\mathbb{Z}^*/(\tau_\mathbb{Z}^*))^*$. To conclude the picture we need to take into consideration the singular tori. Remember that the discriminant locus $D(H_t)$ is homotopy equivalent (and, in fact,
can be made arbitrarily close in the appropriate $W$-decomposition limit) to $Sk_T^{(N-3)}$, the $(N-3)$-skeleton of the subdivision dual to the triangulation of $\partial \Delta$. $Sk_T^{(N-3)}$ is a simplicial complex consisting of the simplices $O(\tau_{i_1}), ..., O(\tau_{i_k})$, with vertices $O(\tau_{i_j})$, the centers of $\tau_{i_j}$, and $\tau_{i_1} \subset ... \subset \tau_{i_k}$ running over all nested chains of simplices in $\partial \Delta$ of positive dimension.

On the mirror side the discriminant locus is again homotopy equivalent to a simplicial complex $(Sk_T^{(N-3)})^\vee$ with the vertices $O(\tau^\vee)$ and the simplices labeled by the nested chains of $\tau^\vee$’s. However the simplices $\tau^\vee \subset \partial \Delta^\vee$ which have appeared as a result of the blow up $X_\Delta^\vee \to X_\Delta$ do not contain any points in the image of the moment map $\mu^\vee(Z)$, hence should be excluded from the discriminant locus. They correspond exactly to those simplices $\tau \in \partial T$, for which the minimal face $\Theta_{\tau}$ is a facet of $\Delta$, i.e. to those which are not in $\partial \Delta$.

The simplicial map $\nu: \partial \Delta \to \partial \Delta^\vee$ provides a one-to-one correspondence between the points $O(\tau)$ and $O(\tau^\vee)$, and thus establishes the simplicial isomorphism between $Sk_T^{(N-3)}$ and $(Sk_T^{(N-3)})^\vee$. So that in the right limit we get the identification between the two discriminant loci. This suggests to consider the corresponding singular fibers $T_s$ and $T^\nu_{\nu(s)}$ to be dual to each other.

References

[AGM] P. Aspinwall, B. Greene and D. Morrison, The Monomial-Divisor Mirror Map, [alg-geom 9309007].

[B] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Algebraic Geometry 3 (1994), 493-535.

[Cx] D. Cox, Recent Developments in Toric Geometry, [alg-geom 9606016].

[DNF] B. Dubrovin, S. Novikov, A. Fomenko, Modern Geometry, Nauka, Moscow, 1986.

[GKZ] I. Gelfand, M. Kapranov and A. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhauser, Boston Basel Berlin, 1994.

[GM] M. Goresky and R. MacPherson, Stratified Morse Theory, Springer-Verlag, Berlin Heidelberg, 1988.

[G] M. Gross, Special Lagrangian Fibrations I: Topology, [alg-geom 9710006].

[GW] M. Gross and P. M. H. Wilson, Mirror Symmetry via 3-tori for a class of Calabi-Yau Threefolds, [alg-geom 9608004].

[Gu] V. Guillemin, Kaehler Structures on Toric Varieties, J. Diff. Geometry 40 (1994), 285-309.

[LV] N. Leung and C. Vafa, Branes and Toric Geometry, [hep-th 9711013].
| Ref | Author            | Title                                      | ArXiv Code  |
|-----|------------------|--------------------------------------------|-------------|
| M   | D. Morrison      | *The Geometry Underlying Mirror Symmetry* | alg-geom 9608006 |
| SYZ | A. Strominger, S. T. Yau, E. Zaslow | *Mirror Symmetry is T-Duality* | hep-th 9606040 |