Hypercyclic deformation of the strip-equation and the accessory parameters for the torus

Pietro Menotti
Dipartimento di Fisica, Università di Pisa and
INFN, Sezione di Pisa, Largo B. Pontecorvo 3, I-56127
e-mail: menotti@df.unipi.it

July 2013

Abstract

By applying an hyperbolic deformation to the uniformization problem for the infinite strip, we give a method for computing the accessory parameter for the torus with one source as an expansion in the modular parameter $q$. At $O(q^0)$ we obtain the same equation for the accessory parameter and the same value of the semiclassical action as the one obtained from the $b \to 0$ limit of the quantum one point function. The procedure can be carried over to the full $O(q^2)$ or even higher order corrections although the procedure becomes somewhat complicated. Here we compute to order $q^2$ the correction to the weight parameter intervening in the conformal factor and it is shown that the unwanted contribution $O(q)$ to the accessory parameter equation cancel exactly.
1 Introduction

The accessory parameters related to punctured Riemann surfaces play an important role in conformal field theories. Not only they give an explicit solution to the uniformization problem but through the Polyakov relation they provide the dependence of the action, i.e. of the semiclassical limit of the quantum correlation functions, on the position of the singularities. A lot of work has been devoted to the determination of such accessory parameters which turned out to be a highly transcendental problem.

Several conjectures and proposals \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} for the computation of such accessory parameters have been put forward e.g. taking the semiclassical limit of the quantum correlation functions in conformal theories, the 4-point function on the sphere and the 1-point function on the torus. More recently a renewed interest in conformal blocks, which are related to the accessory parameters, arose in connection with the AGT conjecture \cite{12, 13} that Liouville theory on a Riemann surface of genus $g$ is related to a certain class of $N = 2$, four-dimensional gauge theories and the conjecture has been supported by extensive tests on genera 0 and 1 \cite{13, 14, 15} and proven in a class of cases \cite{4, 16}.

A better understanding of the classical counterpart would shed light on several problems concerning the conformal blocks e.g. the convergence region of the expansion in the invariant cross ratio, or in the “nome” $q = e^{i\pi\tau}$ for the torus and the validity of the exponentiation hypothesis \cite{1}.

Most of the literature, in particular the mathematical literature \cite{17, 18, 19, 20, 21, 22, 23} is concerned with parabolic singularities i.e. punctures. The reason lies in the fact that in this case results from the fuchsian mapping theory can be applied. On the other hand in Liouville theory the case of elliptic singularities is of most interest.

From the general viewpoint it was proven in \cite{9} that the accessory parameter for the torus with one elliptic singularity is an analytic function of the coupling in the whole physical range except at most a finite number of points and that the accessory parameter is a real-analytic (not analytic) function of the complex modulus $\tau$ except at most a zero measure set in the fundamental region for $\tau$, extending results of \cite{24, 25}.

In addition it was found that the perturbative series in the source strength converges in a finite region and a rigorous lower bound on the convergence radius given \cite{9}. Moreover the first \cite{7} and second order expression \cite{9} for the accessory parameter was explicitly written in terms of elliptic functions and related integrals.

In conformal theories another type of expansion has been brought to attention i.e. the expansion in the modular parameter which appears in the formal expression of conformal
blocks.

For the four point problem on the sphere the expansion of the conformal blocks is done in the invariant cross ratio $x$ which corresponds to an expansion in the distance of a pair of singularities after the others have been fixed in standard way. The conformal blocks are defined as a formal power expansion in such parameter but little is known about the convergence of such a series even though numerical investigations give good support for a convergence region common to the parameters $x$ and $1 - x$ where the validity of the conformal bootstrap can be numerically verified [1], [26].

Similar problems occur for the torus with one source. Important results regarding the four-point conformal correlation functions and their relation to the one-point function on the torus have been obtained in [6] and [4].

The dependence of the accessory parameter from the distance, in the limit of coalescing singularities has been studied rigorously for parabolic singularities in simplified mathematical models [19, 20]. At the semiclassical level it was suggested that one can reach the accessory parameters through the ansatz of the exponentiation of the conformal blocks in the classical limit [1, 5] and a saddle point procedure in such a limit.

The present paper examines the dependence of the accessory parameter for the torus with a single source for small values of $|q|$ directly from the classical Liouville theory. The final aim is to obtain a direct comparison with the procedures described above, which start from the quantum correlation function.

In papers [7, 8, 9] it was proven that in addition to the case of the square and the equianharmonic case i.e. the torus with $\tau = e^{i\pi/3}$, the limit case of the infinite strip is soluble. This correspond to $q = e^{i\pi\tau} = 0$. The idea developed in the present paper is to treat large but finite values of $\text{Im} \tau$ starting form the soluble infinite strip problem through an appropriate deformation or the relative equations. This is reached by an expansion in $q^2$ of the kernel given by the Weierstrass $\wp(z)$ function whose first two terms in fact represent the problem for the strip. The procedure is to compute the three fundamental monodromies and to impose on them the $SU(1, 1)$ nature which is the necessary and sufficient condition for the single valuedness of the conformal factor. The developed treatment although perturbative in $q$ is completely non perturbative in the source strength $\eta$ and thus can be applied to the whole physical range $0 < \eta \leq 1/2$.

The paper is structured as follows. In section 2 we give the general description of the problem and summarize some results on the treatment of the strip given in [9] which will be necessary for the subsequent developments. In section 3 we examine all possible deformations of the equation for the strip proving that even though elliptic deformations can satisfy the requirement for the monodromy around the source and along the short
cycle, they cannot satisfy the $SU(1,1)$ requirements on the long cycle. The hyperbolic deformation on the other hand is consistent and provides, working to order $O(q^0)$, an implicit equation for the accessory parameter. Integrating an equation in $\eta$ one obtains also the value of the action i.e. of the semiclassical 1-point function. We show in the same section that the derived equation for the accessory parameter coincides with the one obtained from the saddle point treatment of the quantum 1-point function and that the derived action equals the quantum action in the semiclassical limit.

In section 4 we perform the first iteration of the Volterra integral equation in which the $q^2$ term in the expansion of the kernel is taken into account. The change due to this term of the weight parameter appearing in the conformal factor which is necessary to impose the consistency of the monodromies, is computed.

On the other hand due to the exponential behavior of the kernel for large imaginary values of the coordinate it is shown that the first term in the expansion of the kernel which is formally $O(q^2)$ contributes actually $O(q)$ to the computation of the monodromy on the long cycle. It is shown that this does not contrast with the structure of the quantum correlation function, as such contribution cancel exactly with the term of the same order appearing in the expansion of the unperturbed function. However due to this fact to reach the full $O(q^2)$ order correction to the equation for the accessory parameter one has to perform two iterations of the integral equation taking into account also the second term in the expansion of the kernel. This is a rather lengthy process which will be pursued in an other work. In general to reach order $q^{2n}$ one has to perform not $n$ but $2n$ iterations in the integral equation.

2 General setting and the infinite strip problem

Liouville equation with one generic source for the torus \cite{17}

\begin{equation}
-\partial_z \partial_{\bar{z}} \phi + e^\phi = 2\pi \eta \delta^2(z)
\end{equation}

can be translated into the ordinary differential equation in the complex plane

\begin{equation}
f'^n(z) + \epsilon(\phi(z) + \beta)f^n(z) = 0
\end{equation}

with

\begin{equation}
e^{-\phi/2} = \frac{1}{\sqrt{2 |w_{12}|}} \left[ \kappa^{-2} f_1^2(z) f_1'(z) - \kappa^2 f_1(z) f_2'(z) \right]
\end{equation}

being $f_1(z)$ and $f_2(z)$ two independent solutions of eq.\cite{17}, $w_{12} = f_1'(z) f_2'(z) - f_1(z) f_2''(z)$. $\kappa$ is a real weight parameter and $\beta$ is the accessory parameter to be determined as to
satisfy the torus periodic boundary conditions. The approach pursued in the present paper is to exploit the convergent expansion for \( \wp(z) \) [27]

\[
\wp(z) = -\frac{\zeta(\omega_1)}{\omega_1} + \frac{\pi^2}{4\omega_1^2} \frac{1}{\sin^2\left(\frac{\pi}{2\omega_1}z\right)} - \frac{2\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \cos \left(\frac{n\pi z}{\omega_1}\right)
\]  

(4)

where \( q = e^{i\pi \tau} \) is the “nome” of the torus, \( \tau \) its modulus and \( \omega_1 \) the first half-period.

The interest for such an expansion is that the problem where only the first two terms in eq.(4) are taken into account is exactly soluble [9] and thus the remaining terms can be treated as a perturbation. The unperturbed problem is the infinite strip problem.

In [9] the conformal factor for the infinite strip with one source was explicitly given in terms of hypergeometric functions. Due to the relevance of this result for the following treatment, we report here the main formulas and notations.

The infinite vertical strip is a degenerate case of the torus [28] reached with the parameters \( e_1 = 2a, e_2 = e_3 = -a \). The Weierstrass \( \wp \) and \( \zeta \) functions associated with the torus degenerate to

\[
u = \wp(z) = -a + \frac{3}{\sin^2(\sqrt{3}a z)}, \quad \zeta(z) = az + \sqrt{3a} \frac{\cos(\sqrt{3}a z)}{\sin(\sqrt{3}a z)}
\]  

(5)

with

\[
\omega_1 = \frac{\pi}{2\sqrt{3}a}, \quad \omega_2 = i\infty.
\]  

(6)

In the following we shall normalize \( a = 1 \). The differential equation associated with the uniformization problem in presence of a source at the origin \( z = 0 \) is

\[
y''(u) + Q(u)y(u) = 0
\]  

(7)

with

\[
Q(u) = \frac{1 - \lambda^2}{16} \frac{u + \beta}{(u+1)^2(u-2)} + \frac{3}{16} \frac{u^2 - 2u + 9}{(u-2)^2(u+1)^2}
\]  

(8)

and

\[
\eta = \frac{1 - \lambda}{2}, \quad \epsilon = \frac{1 - \lambda^2}{4}.
\]  

(9)

The limit of an infinite rectangle requires \( \beta = 1 \) [9]. The first and second order values of \( \beta \) as a power expansion in \( \eta \) for the general torus were given in [7] [8] [9]. To first order we have

\[
\beta = \frac{\zeta(\omega_2)\bar{\omega}_1 - \zeta(\omega_1)\bar{\omega}_2}{\omega_2\bar{\omega}_1 - \omega_1\bar{\omega}_2}.
\]  

(10)

It is of interest that using (5) one gets \( \beta = 1 \) already to first order

\[
\lim_{\omega_2 \to i\infty} \beta = 1.
\]  

(11)
Two independent solutions of the differential equation (7) are
\[ y_1 = (-v)^{1/4}(1-v)^{1/2} \, _2F_1\left(\frac{1}{4}, \frac{1+\lambda}{4}; \frac{1}{2}; v\right) \]  (12)
\[ y_2 = 2(-v)^{3/4}(1-v)^{1/2} \, _2F_1\left(\frac{3}{4}, \frac{3+\lambda}{4}; \frac{3}{2}; v\right) \]  (13)
with
\[ v = \frac{2 - u}{3}. \]  (14)
Such a pair of solutions is canonical at \( v = 0 \), thus assuring monodromy at the point \( v = 0 \) which corresponds to \( z = \omega_1 \). The functions \( f_1(z) \) and \( f_2(z) \) relative to the original problem are obtained through the standard transformation, taking into account their nature of order \(-1/2\)-forms [29], using
\[ \left(\frac{dv}{dz}\right)^{-\frac{1}{2}} = \frac{e^{\frac{i\pi}{4} \sqrt{2} \frac{1}{4}}}{\sqrt{2} (1-v)^{\frac{1}{4}}} \]  (15)
while \( \kappa \) is obtained by imposing the \( SU(1, 1) \) nature of the monodromy at \( u = \infty \) which corresponds to \( z = 0 \) to obtain [9]
\[ \kappa^4 = \left[ \gamma\left(\frac{3-\lambda}{4}\right)\gamma\left(\frac{3+\lambda}{4}\right) \right]^2 \]  (16)
where as usual
\[ \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \]  (17)
The quantity \( X_z \) appearing in the expansion
\[ \phi(z) = -2\eta \log |z|^2 + X_z + o(z) \]  (18)
is given by
\[ X_z = \log 6 + 2\eta \log \frac{4}{3} - 2 \log \gamma(\eta) - 2 \log \gamma\left(\frac{1}{2} - \eta\right). \]  (19)
Integrating the equation
\[ \frac{1}{2\pi} \frac{\partial S_z(\text{strip})}{\partial \eta} = -X_z \]  (20)
we have for the action
\[ \frac{1}{2\pi} S_z(\text{strip}) = -\eta \log 6 - \eta^2 \log \frac{4}{3} + 2F(\eta) - 2F\left(\frac{1}{2} - \eta\right) - 2F(0) \]  (21)
where
\[ F(x) = \int_{\frac{1}{2}}^{x} \log \gamma(x') dx'. \]  (22)
3 Deformation of the equation for the strip

We shall denote by $C_1$ the cycle encircling the origin $z = 0$ which is the location of the source, with $C_2$ the cycle obtained by identifying $z$ with $z - 2\omega_1$ (the short cycle) while the cycle $C_3$ (the long cycle) is the one obtained by identifying the line $\text{Im } z = \omega_1 \tau_2$ with the line $\text{Im } z = -\omega_1 \tau_2$. The procedure we shall employ to determine the accessory parameter for the elongated rectangle is to impose in addition to the monodromy conditions on the cycle $C_1$ and $C_2$ the monodromy on the cycle $C_3$ i.e. the periodic conditions at $\text{Im}(z) = \pm \omega_1 \tau_2$.

This, as we shall see, is consistent only with values of the accessory parameter $\beta \neq 1$, and here we examine the nature of such a deformation.

Separating the soluble part of eq.(2) from the remainder we can translate the original differential equation to the Volterra type integral equation

$$f_k(z) = f_k(z) + \epsilon \int_{\omega_1}^{z} K(z')G(z, z')f_k(z')dz', \quad k = 1, 2$$

(23)

where

$$G(z, z') = \frac{1}{w_{12}}(f_1(z)f_2(z') - f_2(z)f_1(z'))$$

(24)

and

$$K(z') = -\frac{2\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos \left(\frac{n\pi z'}{\omega_1}\right) \equiv \sum_{n=1}^{\infty} K_n(z')$$

(25)

and $f_k$ are the solutions for $q = 0$. Eq.(23) can be solved by the standard convergent iteration procedure. Notice that the chosen lower integration bound assures that the solutions $f_k(z)$ of eq.(23) are still canonical at $z = \omega_1$ thus assuring to all orders the single valued behavior of the conformal factor around $\omega_1$.

It is of interest that the terms $\cos(n\pi z/\omega_1)$ in the expansion (25) formally contribute starting from order $O(q^{2n})$. It will not so in practice due to exponential behavior of the cosine for large imaginary values of $z$ thus contributing $O(q^n)$ to the evaluation of the monodromy $C_3$. This is uncomfortable as to reach the precision $O(q^{2n})$ we have to take into account $2n$ terms in the expansion (25) and accordingly $2n$ iterations of (23). This will be discussed in detail in section 4.

On the other hand in the evaluation of the monodromies $C_1$ and $C_2$ the $n$–th term contributes starting from the order $q^{2n}$ as one would naively expect.

We saw that the problem for the strip is solved by $\beta = 1$. For $\beta = 1 + \beta_1$, $\beta_1 \neq 0$ a pair of solution $f_k(z)$ is given by

$$f_1(z) = (1 - v)^\Lambda \ _2F_1\left(\frac{1 - \lambda}{4}, \frac{1 + \lambda}{4}; \frac{1}{2}; v\right)$$

(26)
\[
f_2(z) = 2 \left(1 - v \right)^{\frac{1}{2}} \left(1 - v \right)^{-\Lambda} 2 F_1 \left(\frac{3 - \lambda}{4}, -\Lambda, \frac{3 + \lambda}{4}, -\Lambda, \frac{3}{2}; v \right) \tag{27}\]

where

\[
\Lambda = \frac{1}{4} \sqrt{\left(1 - \lambda^2\right) \beta_1}. \tag{28}\]

In the rest of this section we shall work to level \(O(q^0)\).

In order to compute the monodromy along \(C_1\) we need the behavior of \(f_1, f_2\) at \(v = \infty\). This is given by

\[
f_1 = B_1^{(1)}(-v)^{-\frac{1-i\lambda}{4}} + B_2^{(1)}(-v)^{-\frac{1+i\lambda}{4}} \tag{29}\]

\[
f_2 = B_1^{(2)}(-v)^{-\frac{1-i\lambda}{4}} + B_2^{(2)}(-v)^{-\frac{1+i\lambda}{4}} \tag{30}\]

with

\[
B_1^{(1)} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{4} + \frac{1}{4} - \frac{\lambda}{4}\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{4} - \frac{\lambda}{4}\right) - \Lambda \Gamma\left(\frac{1}{4} + \frac{1}{4} + \Lambda\right)}, \quad B_2^{(1)} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{2} + \frac{1}{4} - \frac{\lambda}{4}\right) - \Lambda \Gamma\left(-\frac{1}{2} + \frac{1}{4} + \Lambda\right)} \tag{31}\]

\[
B_1^{(2)} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{2} + \frac{1}{4} - \frac{\lambda}{4}\right) - \Lambda \Gamma\left(-\frac{1}{2} + \frac{1}{4} + \Lambda\right)}, \quad B_2^{(2)} = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{2} + \frac{1}{4} - \frac{\lambda}{4}\right) - \Lambda \Gamma\left(-\frac{1}{2} + \frac{1}{4} + \Lambda\right). \tag{32}\]

The monodromy matrix for a complete turn in \(z\) is

\[
M(C_1) = -\Lambda \begin{pmatrix} (B_1^{(1)} B_2^{(2)} e^{i\pi(1+\lambda)} - B_2^{(1)} B_1^{(2)} e^{i\pi(1+\lambda)}) & 2 B_1^{(1)} B_2^{(2)} (e^{i\pi(1+\lambda)} - e^{i\pi(1-\lambda)}) / 2 \\ 2 B_1^{(2)} B_2^{(2)} (e^{i\pi(1-\lambda)} - e^{i\pi(1+\lambda)}) & (B_1^{(1)} B_2^{(2)} e^{i\pi(1+\lambda)} - B_2^{(1)} B_1^{(2)} e^{i\pi(1-\lambda)}) \end{pmatrix} \tag{33}\]

from which the value of the parameter \(\kappa^4\) appearing in eq.(3) is derived using \(\kappa^4 = M_{12}(C_1)/M_{21}(C_1)\)

\[
\kappa^4 = \frac{\Gamma\left(\frac{3}{4} - \frac{1}{4} - \frac{\lambda}{4}\right) - \Lambda \Gamma\left(\frac{3}{4} + \frac{1}{4} + \Lambda\right) \Gamma\left(\frac{3}{4} + \frac{1}{4} + \Lambda\right)}{\Gamma\left(\frac{1}{4} - \frac{1}{4} - \frac{\lambda}{4}\right) - \Lambda \Gamma\left(\frac{1}{4} + \frac{1}{4} + \Lambda\right) \Gamma\left(\frac{1}{4} + \frac{1}{4} + \Lambda\right)}. \tag{34}\]

We recall that in order to have a single valued conformal factor such \(\kappa^4\) should be real and positive. From the previous expression we see that reality is achieved only for \(\Lambda\) real i.e. elliptic deformation, or \(\Lambda\) pure imaginary i.e. hyperbolic deformation. For \(\Lambda\) real, \(\kappa^4\) is positive only for small values of \(\left|\Lambda\right|\). On the other hand for \(\Lambda\) pure imaginary \(\kappa^4\) is always real and positive. Moreover due to the reality of the \(B_k^{(j)}\) for real or imaginary \(\Lambda\), we see from eq.(33) that \(M_{22}(C_1) = M_{11}(C_1)\) thus assuring that the matrix \(M(C_1)\) transformed by \(\text{diag}(\kappa^{-1}, \kappa)\) belongs to \(SU(1,1)\). Thus we have single valuedness of the conformal factor around the source for any imaginary \(\Lambda\) and not too large real \(\Lambda\). We shall see later that real values of \(\Lambda\) are excluded by an other reason. For a rotation of \(\pi\) in \(z\) and from the invariance of eq.(2) we have a \(U(1,1)\) monodromy matrix which assures that the conformal factor is invariant under such a transformation.

We come now to the imposition of the monodromy condition along the cycle \(C_2\), obtained by identifying \(\text{Re } z = \omega_1\) with \(\text{Re } z = -\omega_1\). To deal with this problem it
is useful to rewrite the functions \( f_1(z) \) and \( f_2(z) \) using standard transformations of the hypergeometric functions in the following symmetric form

\[
\begin{align*}
f_1(z) &= \frac{\pi^{3/2}}{\sin 2\pi \Lambda} (-a_1 g_1(z) + b_1 g_2(z)) \\
f_2(z) &= -i \frac{\pi^{3/2}}{\sin 2\pi \Lambda} (-a_2 g_1(z) + b_2 g_2(z))
\end{align*}
\]

where

\[
\begin{align*}g_1(z) &= (1 - v)^\Lambda \, _2F_1\left(\frac{1 - \lambda}{4}, \frac{1 + \lambda}{4} ; \Lambda, 2\Lambda + 1, 1 - v\right) \\
g_2(z) &= (1 - v)^{-\Lambda} \, _2F_1\left(\frac{1 - \lambda}{4}, \frac{1 + \lambda}{4} ; -\Lambda, -2\Lambda + 1, 1 - v\right)
\end{align*}
\]

and

\[
\begin{align*}a_1 &= \frac{1}{\Gamma(1 + 2\Lambda)\Gamma\left(\frac{1 - \lambda}{4} - \Lambda\right)\Gamma\left(\frac{1 + \lambda}{4} - \Lambda\right)} ,
\quad b_1 &= \frac{1}{\Gamma(1 - 2\Lambda)\Gamma\left(\frac{1 - \lambda}{4} + \Lambda\right)\Gamma\left(\frac{1 + \lambda}{4} + \Lambda\right)} \\
a_2 &= \frac{1}{\Gamma(1 + 2\Lambda)\Gamma\left(\frac{1 - \lambda}{4} - \Lambda\right)\Gamma\left(\frac{1 + \lambda}{4} - \Lambda\right)},
\quad b_2 &= \frac{1}{\Gamma(1 - 2\Lambda)\Gamma\left(\frac{1 - \lambda}{4} + \Lambda\right)\Gamma\left(\frac{1 + \lambda}{4} + \Lambda\right)}.
\end{align*}
\]

Under the cycle \( C_2 \), using eqs. (35,36), we have for \( \Lambda = iB \) the monodromy matrix for \( (f_1, if_2) \)

\[
\begin{pmatrix}
-1
-a_1 b_2 e^{2\pi B} + a_2 b_1 e^{-2\pi B} \\
a_2 b_2 (e^{2\pi B} - e^{-2\pi B})
\end{pmatrix}
\]

with \( b_1 = \bar{a}_1, \ b_2 = \bar{a}_2 \) and thus

\[
\kappa^4 = \frac{a_1 b_1}{a_2 b_2} = \frac{a_1 \bar{a}_1}{a_2 \bar{a}_2} > 0
\]

which agrees with the value found from \( C_1 \). Being \(-a_1 b_2 + a_2 b_1\) pure imaginary we have that the rescaled matrix belongs to \( SU(1,1) \).

For \( \Lambda \) real we have for \( M(C_2) \)

\[
\begin{pmatrix}
-1
-a_1 b_2 e^{-2\pi \Lambda} + a_2 b_1 e^{2\pi \Lambda} \\
-a_2 b_2 (-e^{-2\pi \Lambda} + e^{2\pi \Lambda})
\end{pmatrix}
\]

agreeing with (31) and being now \(-a_1 b_2 + a_2 b_1\) real we have again \( M_{22}(C_2) = \bar{M}_{11}(C_2) \) and single valuedness of \( \phi \).

In conclusion for any \( \Lambda \) imaginary we always have single valuedness of the conformal factor along the cycles \( C_1 \) and \( C_2 \) and the same holds for real but not too large \( \Lambda \). We see from the above formulas that while the monodromy along \( C_1 \) is always elliptic with
using \(\phi\) to get \(\kappa\) ensuing conformal factor obeys following function which will play a key role in the sequel.

We can also compute the finite part \(X_z\) of the conformal factor at the origin for \(B \neq 0\) to get

\[-X_z(\eta, B) = -\log 6 - 2\eta \log \frac{4}{3} + \log \gamma(\eta + 2iB) + \log \gamma(\eta - 2iB) + 2\log \gamma(\frac{1}{2} - \eta)\]  

(45)

which integrated in \(\eta\) at fixed \(B\), with boundary condition \(G(0, B) = 0\) provides the following function which will play a key role in the sequel

\[-G(\eta, B) = -\eta \log 6 - \eta^2 \log \frac{4}{3} + F(\eta + 2iB) + F(\eta - 2iB) - F(2iB) - F(-2iB) - 2F(\frac{1}{2} - \eta)\]  

(46)

The condition determining the value of the accessory parameter can be obtained in two different ways. We saw that the reality of the parameter \(\kappa\) derived from the monodromy \(M(C_1)\) requires \(\Lambda\) either real or pure imaginary and from the results after eq. (34) the ensuing conformal factor obeys \(\phi(z) = \phi(\bar{z}) = \phi(-z)\).

We can satisfy torus boundary conditions by computing the monodromy of the transformation \(\tilde{f}_j(z) \equiv f_j(z - 2\omega_2)\), with \(\omega_2\) pure imaginary, i.e. the \(C_3\) cycle (the long cycle) using [9]

\[M_{12}(C_3) = -\tilde{f}_1(z)f_1'(z) + \tilde{f}_1'(z)f_1(z), \quad M_{21}(C_3) = \tilde{f}_2(z)f_2'(z) - \tilde{f}_2'(z)f_2(z)\]  

(47)

and imposing

\[\frac{M_{12}(C_3)}{M_{21}(C_3)} = \kappa^4\]  

(48)

being \(\kappa^4\) given by eq. (34). An alternative amounts to imposing that for \(z = x + iy\) at \(y = -i\omega_2\) the derivative of \(e^{-\phi/2}\) w.r.t. \(y\) vanishes i.e. given

\[e^{-\phi/2} = \text{const} \left[\kappa^{-2}\tilde{f}_1(\bar{z})f_1(z) - \kappa^2\tilde{f}_2(\bar{z})f_2(z)\right]\]  

(49)

\[0 = \kappa^{-2}\left(\tilde{f}_1'(\bar{z})f_1(z) - f_1'(z)f_1(\bar{z})\right) - \kappa^2\left(\tilde{f}_2'(\bar{z})f_2(z) - f_2(z)f_2'(\bar{z})\right)\]  

(50)

thus allowing for a solution \(\phi(z)\) periodic in \(y\). The two methods, as expected give the same result. In fact the functions \(f_1, f_2\) are real analytic functions of \(z\). This is apparent for \(\Lambda\) real. For \(\Lambda\) pure imaginary if follows from the fact that they satisfy eq. (24) with \(q^2 = 0\) with a real \(\beta\) and real boundary conditions at \(z = \omega_1, f_1(\omega_1) = 1, f_1'(\omega_1) = 0, f_2(\omega_1) = 0, f_2'(\omega_1) = \pi/\omega_1\). Then we have at \(y = -i\omega_2\)

\[f_1(z)f_1'(z)|_{z=\bar{z}} - f_1'(z)f_1(z)|_{z=\bar{z}} = f_1(z)f_1'(\bar{z}) - f_1'(z)f_1(\bar{z})\]  

(51)
It is useful to introduce the variable $\zeta$ given by $z = \omega_1(1 + \zeta)$, $\zeta = t_1 + it$ and write $T = e^{i\pi \zeta}$; for large and positive $t = \text{Im } \zeta$, $T$ tends to zero. Recalling that $1 - v = \lfloor \cos(\pi \zeta/2) \rfloor^{-2}$, for $g_1$ and $g_2$ we have the following expansion convergent in $\text{Im } \zeta > 0$

$$g_1 = 4^{\Lambda} T^\Lambda \left(1 + \frac{1}{1 + 2\Lambda} T - \frac{1}{2 (1 + \Lambda)} T^2 + e^2 \frac{1}{4(1 + \Lambda)(1 + 2\Lambda)} T^2 + O(T^3)\right)$$

(52)

$$g_2 = 4^{-\Lambda} T^{-\Lambda} \left(1 + \frac{1}{1 - 2\Lambda} T - \frac{1}{2 (1 - \Lambda)} T^2 + e^2 \frac{1}{4(1 - \Lambda)(1 - 2\Lambda)} T^2 + O(T^3)\right)$$

(53)

where $g_2$ is simply obtained from $g_1$ by sending $\Lambda$ into $-\Lambda$.

The above given expressions are very useful to compute the large $\text{Im } \zeta$ behavior of the $f_k$. As in this section we work $O(q^0)$ only the first term in the expansion intervenes. For $\Lambda$ real we obtain from eq. (50)

$$\kappa^{-2}(a_1^2 4^{2\Lambda} e^{i\pi \Lambda (\zeta - \bar{\zeta})} - b_1^2 4^{-2\Lambda} e^{-i\pi \Lambda (\zeta - \bar{\zeta})}) = \kappa^{-2}(a_2^2 4^{2\Lambda} e^{i\pi \Lambda (\zeta - \bar{\zeta})} - b_2^2 4^{-2\Lambda} e^{-i\pi \Lambda (\zeta - \bar{\zeta})})$$

(54)

which using eq. (51) gives

$$4^{4\Lambda} e^{-4\pi \Lambda t} = \frac{b_1 b_2}{a_1 a_2}.$$  

(55)

As we are interested in the limit of large $t$, such an equation should be solvable for small $\Lambda$ but this is not possible as the left hand side is always positive while the right hand side for $\Lambda \to 0$ tends to $-1$. We conclude that an elliptic deformation even if it can satisfy the monodromy condition along the cycles $C_1$ and $C_2$ it cannot satisfy the monodromy condition along the cycle $C_3$.

We examine now the case of imaginary $\Lambda$, $\Lambda \equiv iB$, i.e. the hyperbolic deformation. In this case we find

$$\kappa^{-2}(a_1^2 4^{2iB} e^{-\pi B (\zeta - \bar{\zeta})} - b_1^2 4^{-2iB} e^{\pi B (\zeta - \bar{\zeta})}) = \kappa^{-2}(a_2^2 4^{2iB} e^{-\pi B (\zeta - \bar{\zeta})} - b_2^2 4^{-2iB} e^{\pi B (\zeta - \bar{\zeta})})$$

(56)

which using eq. (52) gives

$$4^{4iB} e^{-4i\pi B t} = -\frac{a_1 a_2}{b_1 b_2}.$$  

(57)

From the expressions (39,40) for the $a_j, b_j$ and Legendre duplication formula $\Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + 1/2)/\sqrt{\pi}$ the r.h.s. of equation (57) can be written as

$$- \frac{\Gamma^2(1 + 2iB) \gamma(\frac{1 + \Lambda}{2} - 2iB)}{\Gamma^2(1 - 2iB) \gamma(\frac{1 - \Lambda}{2} + 2iB)} e^{4it \log 4}$$

(58)

thus obtaining the equation for $B$ as a function of $\omega_2 = it \omega_1 = i\tau_2 \omega_1$

$$4\pi B t = i \log \left[- \frac{\Gamma^2(1 + 2iB) \gamma(\frac{1 + \Lambda}{2} - 2iB)}{\Gamma^2(1 - 2iB) \gamma(\frac{1 - \Lambda}{2} + 2iB)}\right] = \pi + i \log \frac{\Gamma^2(1 + 2iB) \gamma(\eta - 2iB)}{\Gamma^2(1 - 2iB) \gamma(\eta + 2iB)}.$$  

(59)
The \(\pi\) appearing on the r.h.s. of eq.\(^{(59)}\), originates from the minus sign in eq.\(^{(58)}\), \(i \log(-1) = \pi\). We could have also chosen \(-\pi\); then the solution of eq.\(^{(59)}\) will simply be minus the solution of the previous equation. This is due to the fact that the solution of the problem i.e. the conformal factor, is invariant under \(B \rightarrow -B\). Higher values of \(i \log(-1)\) like \(3\pi\) correspond to solutions of eq.\(^{(50)}\) lying beyond the first hyperbolic horizon. From the value of \(B\) we can also compute the trace of the \(C_3\) monodromy. For large \(t\) we find
\[
M_{11}(C_3) = M_{22}(C_3) = \frac{1}{2\pi B} \cos \frac{\pi \lambda}{2},
\]
where according to eq.\(^{(59)}\) \(B = 1/4t + O(1/t^2)\) and as such it is an hyperbolic monodromy. Given the value of \(B\) extracted from eq.\(^{(59)}\) one can compute the action to order \(O(q^0)\) i.e. keeping into account all logarithmic corrections by using the relation
\[
\frac{1}{2\pi} S_z(\eta) = -\int_0^\eta X_z(\eta, B(\eta)) d\eta
\]
where \(X_z\) is the finite part of the conformal factor \(\phi\) at the origin
\[
\phi = -2\eta \log |z|^2 + X_z + o(z).
\]
The action \(^{(61)}\) is really the action on the torus because if one limits the integration of \(\frac{1}{2} \partial_z \phi \partial_{\bar{z}} \phi + e^\phi\) to the periodicity region \(-\omega_1 < \text{Re} \, z < \omega_1, -\tau_2 \omega_1 < \text{Im} \, z < \tau_2 \omega_1\) we have no boundary terms in the action and the only source of variation of \(S_z\) is just the contribution at the origin \(X_z\). The derivative of the function \(G\) eq.\(^{(46)}\) with respect to \(B\) gives
\[
\frac{\partial G}{\partial B} = 2i \log \left( -\frac{\Gamma(1 + 2iB)^2 \gamma(\eta - 2iB)}{\Gamma(1 - 2iB)^2 \gamma(\eta + 2iB)} \right)
\]
which is twice the r.h.s. of eq.\(^{(59)}\). Then we see that the \(S_z/2\pi\) of eq.\(^{(61)}\) coincides with the value of the expression
\[
4\pi B^2 t - G(\eta, B)
\]
computed at the value of \(B(\eta)\) which realizes the minimum of eq.\(^{(64)}\). In fact we have for the total derivative of eq.\(^{(64)}\) with respect to \(\eta\)
\[
\left( 8\pi B(\eta) t - \frac{\partial G(\eta, B(\eta))}{\partial B} \right) dB/d\eta - \frac{\partial G(\eta, B(\eta))}{\partial \eta} = -X_z(\eta, B(\eta))
\]
We give below the comparison with the semiclassical limit of the quantum one-point function. The primary fields in Liouville theory are given by \(e^{2\alpha \phi(z, \bar{z})}\). Following the notation of [26], but replacing \(\lambda\) with \(l\) not to create confusion with the \(\lambda\) introduced in the previous sections of the present paper, we have for the dimension \(\Delta\) of \(V_{l,l} = e^{2\alpha \phi(z, \bar{z})}\)
\[
\Delta = \alpha(Q - \alpha) = \frac{1}{4} (Q^2 - l^2) \quad \text{where} \quad \alpha = \frac{Q}{2} + \frac{l}{2}.
\]
The central charge in Liouville theory is given by
\[ c = 1 + 6Q^2, \quad Q = \frac{1}{b} + b. \] (67)

The torus one-point function is given by [26, 14, 30]
\[ \langle V_{i,l} \rangle = \text{Tr}(e^{-\tau_1 H + i\tau_2 P} V_{i,l}(1, 1)) = (\tilde{q} \tilde{q})^{-\frac{1}{2}} \text{Tr}(q^{L_0} \tilde{q}^{\tilde{L}_0} V_{i,l}(1, 1)) \] (68)
where \( \tilde{q} = e^{2\pi i r} \). The trace has to be computed on the Verma module
\[ \nu_{\Delta, M} = L_{-M} \nu_{\Delta} = L_{-m} \ldots L_{-m} \nu_{\Delta}, \quad \bar{\nu}_{\Delta, N} = \bar{L}_{-N} \bar{\nu}_{\Delta} = \bar{L}_{-n} \ldots \bar{L}_{-n} \bar{\nu}_{\Delta} \] (69)
with [30]
\[ \langle V_1 | V_b(z, \bar{z}) | V_2 \rangle = \lim_{w \to \infty} w^{2\Delta_1} \bar{w}^{2\Delta_1} \langle V_1(w, \bar{w}) V_3(z, \bar{z}) V_2(0, 0) \rangle. \] (70)

Only matrix elements with the same dimensions appear in the computation of the conformal block. In Liouville we have real \( \Delta \) and \( \Delta = \bar{\Delta} \). The fundamental matrix element (70)
\[ \langle \nu_{\Delta'}, \bar{\nu}_{\Delta'}, | V_{i,l}(1, 1) | \nu_{\Delta}, \bar{\nu}_{\Delta} \rangle = C(\frac{Q}{2} - \frac{l'}{2}, \frac{Q}{2} + \frac{l}{2}, \frac{Q}{2} + \frac{l'}{2}) \] (71)
is provided by the DOZZ [31, 1, 32] structure constant. Due to the continuum spectrum of Liouville theory the general formal expression [26] for the trace (68) \([B_{c,\Delta}^n]^{MN}[\bar{B}_{c,\bar{\Delta}}^m]^{\bar{M}\bar{N}}\) are the inverses of the Kac matrices)
\[ \langle V_{i,l} \rangle = (\tilde{q} \tilde{q})^{-\frac{1}{2}} \sum_{\Delta, \bar{\Delta}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{q}^{\Delta + n} \bar{\tilde{q}}^{\bar{\Delta} + m} \sum_{n=|M|=|N|, m=|\bar{M}|=|\bar{N}|} \frac{[B_{c,\Delta}^n]^{MN}[\bar{B}_{c,\bar{\Delta}}^m]^{\bar{M}\bar{N}}}{\nu_{\Delta, M} \nu_{\bar{\Delta}, \bar{N}} | V_{i,l}(1, 1) | \nu_{\Delta}, \bar{\nu}_{\bar{\Delta}, \bar{N}}} \] (72)
go over to
\[ \langle V_{i,l} \rangle = \int_0^{\infty} \frac{dl'}{l} \mathcal{F}_{c,\Delta}^l(\tilde{q}) \bar{\mathcal{F}}_{c,\bar{\Delta}}^{l'}(\tilde{q}) C_{c,\Delta, \Delta'}^{l,l'} = \int_0^{\infty} \frac{dl'}{l} |\mathcal{F}_{c,\Delta}^l(\tilde{q})|^2 C(\frac{Q}{2} - l', \frac{Q}{2} + \frac{l}{2}, \frac{Q}{2} + \frac{l'}{2}) \] (73)
with
\[ \mathcal{F}_{c,\Delta}^l(\tilde{q}) = \tilde{q}^{\Delta - \frac{l}{2}} \sum_{n=0}^{\infty} q^n F_{c,\Delta}^{l,n} \] (74)
the conformal block. Such 1-point torus conformal blocks are not yet known in general form [14, 33].

The semiclassical limit is obtained for \( b \to 0 \). Using \( l' = \tilde{l}_p / b, \tilde{l}_p = ip \) and \( l = \tilde{l} / b \) we have
\[ \Delta' = \frac{1}{4} (Q^2 - t'^2) \to \frac{1}{b^2} \frac{1}{4} (1 - \tilde{t}_p^2) = \frac{1}{b^2} \frac{1}{4} (1 + p^2) \] (75)
\[
C\left(\frac{Q}{2} - l', \frac{Q}{2} + \frac{l}{2}, \frac{Q}{2} + l' \right) \to e^{-\frac{\text{g}(cl)}{2\pi\nu}}
\]  

(76)

\[
\frac{S^{(cl)}}{2\pi} = F(\eta) + F(\eta + ip) + F(\eta - ip) + F(1 - \eta) - F(1 + ip) - F(2\eta) - F(1 - ip) - \eta \log 2
\]  

(77)

where \(\eta = (1 + l)/2 = (1 - \lambda)/2\). The equation for the saddle point to order \(q^0\) is

\[
-4\pi \tau_2 \frac{l}{2} - \frac{1}{2\pi} \frac{\partial S^{(cl)}}{\partial p} = 0
\]  

(78)

where recalling the definition of the function \(F(\eta)\) eq.(22)

\[
\frac{1}{2\pi} \frac{\partial S^{(cl)}}{\partial p} = i \left( \log \gamma(\eta + ip) - \log \gamma(\eta - ip) - \log \gamma(1 + ip) + \log \gamma(1 - ip) \right)
\]  

(79)

and the equation for \(p\) becomes

\[
2\pi \tau_2 p = i \log \left( -\frac{\Gamma^2(1 + ip)\gamma(\eta - ip)}{\Gamma^2(1 - ip)\gamma(\eta + ip)} \right)
\]  

(80)

which is the same as eq.(59). One sees from eqs.(59) and (80) that the correspondence between the classical accessory parameter \(B\) and the saddle point value \(p_s\) of \(p\) is \(p_s = 2B\).

With regard to the value of the action computed from the saddle point we have from eq.(77)

\[
\frac{1}{2\pi} \frac{\partial S^{(cl)}}{\partial \eta} = \log \gamma(\eta) - \log \gamma(1 - \eta) - 2\log \gamma(2\eta) + \log \gamma(\eta + ip) + \log \gamma(\eta - ip) - \log 2
\]

\[
= -X_z - (2\eta - 1) \log 12
\]  

(81)

The term \((2\eta - 1) \log 12\) is due to our choice \(\omega_1 = \pi/(2\sqrt{3})\), see section 2 instead of the standard one \(\omega_1 = \pi\), and the scaling behavior of the 1-point function whose classical dimension is given by \(\eta(1 - \eta)\). Thus also the action computed from the saddle point agrees with action computed from the imposition of the monodromy along the cycle \(C_3\).

4 Discussion of higher order calculations

As we mentioned already in the introduction even if the terms \(K_n\) appearing in the expansion of \(K\) eq.(25) are formally of order \(O(q^{2n})\) due to the exponential behavior of \(\cos(n\pi z/\omega_1)\) at large imaginary values of \(z\), such term contributes in the corrections to the \(f_j\) to order \(O(q^n)\). In particular for \(n = 1\) we shall have \(O(q)\) contributions. This appears in contrast with the structure of the quantum one-point function (73,74) where only \(q^2 \equiv \tilde{q}\) appears. We show in the following that such \(O(q)\) contributions cancel exactly
in the equation determining the value of the accessory parameter. The corrections $O(q^2)$ have several origins. In imposing the $SU(1, 1)$ nature of the monodromy along the cycle $C_3$ or equivalently in imposing eq. (50), the functions $f_k(z)$ at $\text{Im}z = \omega_1 t$ will have to be computed to order $T$ and $T^2$ included expanding the hypergeometric function appearing in (37, 38) according to eqs. (52, 53). Secondly to reach order $O(q^2)$ two iterations in the solution of the integral equation (23) have to be performed.

First we must examine the change in the parameter $\kappa^4$ due to the kernel appearing in eq. (23). At the origin $z \approx 0$ the corrected solutions $f^{(1)}_k$ behave, in matrix form, as

$$f^{(1)} = (1 + \Delta)f$$

where

$$\Delta = -2\pi e q^2 \begin{pmatrix} I_{12} & -I_{11} \\ I_{22} & -I_{12} \end{pmatrix}$$

being $I_{jk}$ the numbers

$$I_{jk} = \int_0^{\pi} \cos \pi \zeta \ f_j(\zeta) \ f_k(\zeta) \ d\zeta.$$  \hspace{1cm} (84)

The monodromy matrix $M(C_1)$ goes over to

$$M^{(1)}(C_1) = M(C_1) + [\Delta, M(C_1)].$$ \hspace{1cm} (85)

The $I_{jk}$ are all real and this assures through the nature of the $M(C_1)$ matrix that $M^{(1)}_{22}(C_1) = M^{(1)}_{11}(C_1)$ and that the ratio $M^{(1)}_{12}(C_1)/M^{(1)}_{21}(C_1)$ is real and positive. Such a ratio provides the value of the corrected $\kappa^4$

$$\kappa^4 \left( 1 - 2\pi e q^2 \left[ 4I_{12} + (M_{11}(C_1) - M_{22}(C_1))(\frac{I_{11}}{M_{12}(C_1)} + \frac{I_{22}}{M_{21}(C_1)}) \right] \right).$$ \hspace{1cm} (86)

In addition it is easy to prove from eq. (85) applied to the half-turn monodromy matrix, that the new corrected conformal factor still satisfies $\phi(-z) = \phi(z)$.

The monodromy is still elliptic with trace $-2\cos(\pi \lambda)$. There is an alternative way to compute such a correction, i.e. by imposing the $SU(1, 1)$ nature of the monodromy along the cycle $C_2$. This is important as it enlightens the structure of the functions $g_k$ and their $O(q^2)$ corrections for large values of $\text{Im} \zeta$ which is the region of interest in computing the monodromy along the cycle $C_3$, i.e. the long cycle and sets a relation between the $I_{jk}$ and other integration constants.

To this end we start from the two functions

$$f_1(z) = \frac{\pi^{3/2}}{\sin 2\pi \Lambda} \left(-a_1 g_1(z) + b_1 g_2(z)\right)$$ \hspace{1cm} (87)
\[ if_2(z) = \frac{n^{3/2}}{\sin 2\pi\Lambda}(-a_2g_1(z) + b_2g_2(z)) \] (88)

which are canonical at \( z = \omega_1 \) and compute the corrections on the \( g_k \). We have this time, in matrix form,

\[ g^{(1)}(\zeta) = (1 + R(\zeta))g(\zeta) \] (89)

where taking into account the wronskian factor

\[ R(\zeta) = \frac{\pi\epsilon q^2}{B} \begin{pmatrix} J_{12}(\zeta) & -J_{11}(\zeta) \\ J_{22}(\zeta) & -J_{12}(\zeta) \end{pmatrix} \] (90)

with

\[ J_{jk}(\zeta) = \int_0^{\zeta} \cos \pi\zeta' g_j(\zeta') g_k(\zeta') \, d\zeta' \]. (91)

The structure of the \( J_{kl}(\zeta) \) is

\[ J_{12}(\zeta) = s_{12}(e^{i\pi\zeta}) + \frac{\epsilon\zeta}{1 + 4B^2} + r_{12} \] (92)

\[ J_{11}(\zeta) = 4^{2iB}e^{-2\pi Bc} s_{11}(e^{i\pi\zeta}) + r_{11} \] (93)

\[ J_{22}(\zeta) = 4^{-2iB}e^{2\pi Bc} s_{22}(e^{i\pi\zeta}) + r_{22} \] (94)

where the \( s_{jk} \) being functions of \( e^{i\pi\zeta} \) are invariant under \( \zeta \to \zeta - 2 \) and the numbers \( r_{jk} \) are the contributions due to the lower integration limit in eq.(91). From eq.(52,53) we have \( r_{12} \) pure imaginary and \( r_{22} = -\bar{r}_{11} \). Thus we can write to order \( q^2 \) the corrected \( g_1 \) as

\[ g_1(\zeta) \approx \frac{\epsilon^2 q^2}{e^{2i\pi B} + e^{-2i\pi B}} \left( 1 + \frac{\epsilon\pi q^2}{B} \left[ s_{12}(e^{i\pi\zeta}) + r_{12} - 4^{2iB}e^{-2\pi Bc}s_{11}(e^{i\pi\zeta})g_2(\zeta) \right] \right) - \frac{\epsilon^2 q^2}{B} r_{11} g_2(\zeta) \equiv \tilde{g}_1(\zeta) - \frac{\epsilon^2 q^2}{B} r_{11} g_2(\zeta) \] (95)

and similarly for \( g_2 \). Under the cycle \( C_2 \) i.e. \( \zeta \to \zeta - 2 \) (the \( C_2 \) monodromy can be computed at any value of \( \text{Im}\zeta \)) the \( \tilde{g}_j \) transform like \( \tilde{g}_1 \to e^{2\pi B'}\tilde{g}_1 \) and \( \tilde{g}_2 \to e^{-2\pi B'}\tilde{g}_2 \) with

\[ B' = B - \frac{\epsilon^2 q^2}{B(1 + 4B^2)} \]. (96)

Denoting by \( B' \), \( R \) and \( A \) the matrices

\[ B' = \begin{pmatrix} e^{2\pi B'} & 0 \\ 0 & e^{-2\pi B'} \end{pmatrix}, \quad R = \frac{\epsilon\pi q^2}{B} \begin{pmatrix} 0 & -r_{11} \\ r_{22} & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -a_1 & b_1 \\ -a_2 & b_2 \end{pmatrix} \] (97)

we have for the \( C_2 \) monodromy to order \( q^2 \)

\[ M^{(1)}(C_2) = AB'A^{-1} + A[R, B]A^{-1} \] (98)
where $B$ is the matrix $B'$ with $B'$ replaced by $B$. Due to the diagonal nature of $B'$, the ratio of the off-diagonal elements of $AB'A^{-1}$ is the same as those of the unperturbed monodromy eq. (91). For the second term we find

$$
2\epsilon\pi q^2 \sinh(2\pi B) \begin{pmatrix}
-a_1a_2r_{11} - \bar{a}_1\bar{a}_2\bar{r}_{11} & a_1^2r_{11} + \bar{a}_1^2\bar{r}_{11} \\
-a_2^2r_{11} - \bar{a}_2^2\bar{r}_{11} & a_1a_2r_{11} + \bar{a}_1\bar{a}_2\bar{r}_{11}
\end{pmatrix}
$$

where $\det A = -a_1\bar{a}_2 + \bar{a}_1a_2$. Again taking the ratio $M_{12}^{(1)}(C_2)/M_{21}^{(1)}(C_2)$ we have the corrected value of $\kappa^4$

$$
\kappa^4 \left(1 + 2\frac{\pi\epsilon q^2 \sinh(2\pi B)}{B \det A} \left[ \frac{a_1^2r_{11} + \bar{a}_1^2\bar{r}_{11}}{M_{12}(C_2)} - \frac{a_2^2r_{11} + \bar{a}_2^2\bar{r}_{11}}{M_{21}(C_2)} \right] \right).
$$

Moreover from eq. (98) we read for the trace of the $C_2$ monodromy the value $\text{Tr} M^{(1)}(C_2) = 2\cosh(\pi B')$, correct to order $O(q^2)$ included.

The two values of $\kappa^4$ eq. (86) and eq. (100) have to agree due to the fact that the invariance of the conformal factor under inversion $z \rightarrow -z$ and under reflections $\text{Im} z \rightarrow -\text{Im} z$ imply monodromy under $C_2$. This sets a relation between the value of $r_{11}$ arising from the lower integration limit in eq. (91) and the integrals $I_{jk}$ of product of hypergeometric functions. We verified the validity of such a relation with high precision numerical tests.

We come now to the resolution of the apparent contradiction of the presence of $O(q)$ corrections for the equation determining the accessory parameter $B$.

In deriving the equation for the accessory parameter one needs to compute the functions $f_k$ i.e. the $g_k$ for $\text{Im} \zeta = \tau_2$ i.e. for large imaginary values of $\zeta$. The explicit form of such corrections due to the first iteration with $K_1$ in equation (23) is given using eq. (89) by

$$
\delta g_1 = K_1 g_1
$$

$$
= \epsilon q^2 4iB e^{-\pi B\zeta} \left( \frac{e^{-i\pi \zeta}}{1 - 2iB} + \epsilon \frac{\pi \zeta}{B(1 + 4B^2)} + \text{const} \right) - \frac{\pi \epsilon q^2}{B} 4^{-iB} r_{11} e^{\pi B \zeta} + O(e^{-\pi \text{Im}(\zeta)}),
$$

and similarly

$$
\delta g_2 = K_1 g_2
$$

$$
= \epsilon q^2 4^{-iB} e^{\pi B \zeta} \left( \frac{e^{-i\pi \zeta}}{1 + 2iB} - \epsilon \frac{\pi \zeta}{B(1 + 4B^2)} + \text{const} \right) + \frac{\pi \epsilon q^2}{B} 4^{iB} r_{22} e^{-\pi B \zeta} + O(e^{-\pi \text{Im}(\zeta)}).
$$

In the computation of the $C_3$ monodromy after taking the derivatives in eq. (50) one has to set $\text{Im} \zeta = \tau_2$. Thus we see from the presence of the factor $e^{-i\pi \zeta}$ that the first terms in eq. (101,102) contribute $q^2 q^{-1} = q$ and this seems to contrast with the structure of the
quantum one-point function where only \( \tilde{q} \equiv q^2 \) appears. However the second term in the expansion of \( g_1 \) in (52)

\[
\epsilon 4^{iB} e^{-\pi B \zeta} \frac{e^{i\pi \zeta}}{1+2iB}
\]

is also of order \( O(q) \) and the two contributions cancel exactly in the computation of the \( C_3 \) monodromy thus leaving only \( q^2 \) terms in the problem. However the complete evaluation of the \( q^2 \) corrections to the monodromy implies the computation of the iterated \( K_1 \times K_1 \) contribution and of \( K_2 \) which is a rather lengthy process.

5 Conclusions

In the present paper we developed a method for computing the accessory parameter for the torus with one generic source in the regime of high values of the imaginary part of the modulus. This is the region which is of interest in the usual formulation of the conformal blocks in quantum conformal theories. The main idea is to use an expansion of Weierstrass \( \wp \) function in the parameter \( q = e^{i\pi \tau} \). The advantage in that the zero order problem is related to a soluble one i.e. the infinite strip, and actually turns out to be an hyperbolic deformation of it, which is also soluble. In this way we reproduce to \( O(q^0) \) the same equation for the accessory parameter and the same value for the action as those obtained from the saddle point method applied to the semiclassical limit of the quantum one-point function. In principle the procedure can be carried over to all orders in \( q \).

We also have given the full \( O(q^2) \) contribution to the change of the weight parameter \( \kappa^4 \) necessary to extract the equation for the accessory parameter to \( O(q^2) \), and performed the first iteration of the Volterra equation. As discussed in the text the procedure generates also terms \( O(q) \) which are absent in the expression of the quantum one-point function but it is shown that they give contributions which cancel exactly. Due to the behavior of the kernel of the Volterra equation for large imaginary values of the coordinate, to reach the order \( O(q^{2n}) \) one needs not \( n \) but \( 2n \) iterations of the integral equation. Thus a complete computation of the \( O(q^2) \) terms is already a lengthy procedure as it involves the second iteration of the Volterra with the kernel \( K_1 \) and one iteration with the kernel \( K_2 \) as both give \( O(q^2) \) contributions to the equation for \( B \) and such computation will be attempted elsewhere.

Acknowledgments

The author is grateful to Massimo Porrati for correspondence.
References

[1] A.B. Zamolodchikov, Al.B. Zamolodchikov, *Conformal bootstrap in Liouville field theory*, Nucl. Phys. B477 (1996) 577, arXiv:hep-th/9506136

[2] L. Hadasz, Z. Jaskolski, *Classical geometry from the quantum Liouville theory*, Nucl.Phys. B724 (2005) 529, arXiv:hep-th/0504204

[3] L. Hadasz, Z. Jaskolski, *Liouville theory and uniformization of four-punctured sphere*, J.Math.Phys. 47 (2006) 082304, arXiv:hep-th/0604187

[4] L. Hadasz, Z. Jaskolski *Modular bootstrap in Liouville field theory*, Phys.Lett. B685 (2010) 79, arXiv:0911.4296 [hep-th]

[5] F. Ferrari, M. Piatek, *Liouville theory, N=2 gauge theories and accessory parameters*, JHEP 1205 (2012) 025, arXiv:1202.2149 [hep-th]

[6] V.A. Fateev, A.V. Litvinov, A. Neveu, E. Onofri, *Differential equation for four-point correlation function in Liouville field theory and elliptic four-point conformal blocks*, J.Phys. A42:304011 (2009), arXiv:0902.1331 [hep-th]

[7] P. Menotti, *Riemann-Hilbert treatment of Liouville theory on the torus*, J.Phys. A44 (2011) 115403, arXiv:1010.4946 [hep-th]

[8] P. Menotti, *Riemann-Hilbert treatment of Liouville theory on the torus: The general case*, J.Phys. A44 (2011) 335401, arXiv:1104.3210 [hep-th]

[9] P. Menotti, *Accessory parameters for Liouville theory on the torus*, JHEP 1212 (2012) 001, arXiv:1207.6884 [hep-th]

[10] R. Poghossian, *Recursion relations in CFT and N=2 SYM theory*, JHEP 0912:038 (2009), arXiv:0909.3412 [hep-th]

[11] Amir-Kian Kashani-Poor, J. Troost, *The toroidal block and the genus expansion*, JHEP 1303 (2013) 133, arXiv:1212.0722 [hep-th]

[12] D. Gaiotto, *N=2 dualities*, JHEP 1208 (2012) 034, arXiv:0904.2715 [hep-th]

[13] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. 91 (2010) 167, arXiv:0906.3219 [hep-th]
[14] V. Alba, A. Morozov, Non-conformal limit of AGT relation from the 1-point torus conformal block, JETP Lett. 90 (2009) 708, arXiv:0911.0363 [hep-th]

[15] N. Drukker, J. Gomis, T. Okuda and J. Teschner, Gauge Theory Loop Operators and Liouville Theory, JHEP 1002 (2010) 057, arXiv:0909.1105 [hep-th]

[16] L. Hadasz, Z. Jaskolski, P. Suchanek, Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals JHEP 1006 (2010) 046, arXiv:1004.1841 [hep-th]

[17] L. Keen, H.E. Rauch and A.T. Vasquez, Moduli of punctured tori and the accessory parameter of Lamé equation, Trans. Am. Math. Soc. 255 (1979) 201

[18] I. Kra, Accessory parameters for punctured spheres, Trans. Am.Math.Soc. 313 (1989) 589

[19] S.J. Smith, J.A. Hempel, The accessory parameter problem for the uniformization of the twice-punctured disc, J. London Math. Soc. (2) 40 (1989) 269

[20] J.A. Hempel, S.J. Smith, Uniformization of the twice-punctured disk- Problems of confluence, Bull. London Math. Soc. 39 (1989) 369;

[21] D.A. Hejhal, On Schottky and Koebe-like uniformizations, Acta Mathematica 135 (1975) 1

[22] J.A. Hempel, On the Uniformization of the n-Punctured Sphere, Bull. London Math. Soc. 20 (1988) 97;

[23] J.A. Hempel, S.J. Smith, Hyperbolic length of geodesics surrounding two punctures, Proc. American Math. Soc. 103, 2, 513 (1988)

[24] L. Cantini, P. Menotti, D. Seminara, Proof of Polyakov conjecture for general elliptic singularities, Phys.Lett. B517 (2001) 203, arXiv:hep-th/0105081

[25] L. Cantini, P. Menotti, D. Seminara, Liouville theory, accessory parameters and (2+1)-dimensional gravity, Nucl.Phys. B638 (2002) 351, arXiv:hep-th/0203103

[26] L. Hadasz, Z. Jaskolski, P. Suchanek, Recursive representation of the torus 1-point conformal block, JHEP 1001 (2010) 063, arXiv:0911.2353 [hep-th]

[27] Digital Library of Mathematical Functions, NIST project http://dlmf.nist.gov/

[28] A. Erdelyi (Ed.) Higher Transcendental Functions, vol.II McGraw-Hill, New York, 1953
[29] N. S. Hawley and M Schiffer, *Half-order differentials on Riemann surfaces*, Acta Math. 115 199, 1966

[30] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal field Theory*, 1997 Springer Verlag New York, Inc.

[31] H. Dorn, H.J. Otto, *Two and three point functions in Liouville theory*, Nucl.Phys. B429 (1994), arXiv:hep-th/9403141

[32] J. Teschner, *On the Liouville three point function*, Phys.Lett. B363 (1995) 65, arXiv:hep-th/9507109

[33] A. Mironov, S. Mironov, A. Morozov, A. Morozov, *CFT exercises for the needs of AGT*, arXiv:0908.2064 [hep-th]