ON THE VANISHING OF RELATIVE NEGATIVE K-THEORY

VIVEK SADHU

Abstract. In this article, we study the relative negative K-groups $K_{-n}(f)$ of a map $f : X \to S$ of schemes. We prove a relative version of the Weibel conjecture i.e. if $f : X \to S$ is a smooth affine map of noetherian schemes with dim $S = d$ then $K_{-n}(f) = 0$ for $n > d + 1$ and the natural map $K_{-n}(f) \to K_{-n}(f \times \mathbb{A}^r)$ is an isomorphism for all $r > 0$ and $n > d$. We also prove a vanishing result for relative negative K-groups of a subintegral map.

1. Introduction

In 1980, Weibel conjectured that for a $d$-dimensional noetherian scheme $X$, the negative $K$-groups should vanish after the dimension and the natural map $K_{-d}(X) \to K_{-d}(X \times \mathbb{A}^r)$ for all $r > 0$ should be an isomorphism i.e. $X$ should be $K_{-d}$-regular (see Question 2.9 of [21]). Significant progress related to this conjecture has been made in the articles [3], [4], [5], [6], [7], [9], [10], [23] by various authors. Very recently, a complete answer is given in [8] by Kerz-Strunk-Tamme.

Let $f : X \to S$ be a morphism of schemes. By definition, the $n$-th relative K-group $K_n(f)$ is $\pi_n K(f)$, where $n \in \mathbb{Z}$ and $K(f)$ is the homotopy fiber of $K(S) \to K(X)$. Here and throughout, $K(X)$ denotes the non-connective Bass $K$-theory spectrum of the scheme $X$. Similarly, by replacing $K$ by $KH$, we get the $n$-th relative homotopy $K$-group $KH_n(f)$. We say that $f : X \to S$ is $K_n$-regular if the natural map $K_n(f) \to K_n(f \times \mathbb{A}^r)$ is an isomorphism for all $r > 0$. In this article, we are considering Weibel conjecture in the relative setting. More precisely, we are interested in investigating the condition on $f$ under which an analogous vanishing and regularity result holds for the relative negative $K$-groups.

Firstly, we consider the case when $f$ is a smooth affine map. We discuss such a case in Section 3. Using the technique of [7] and [8], we prove the following

Theorem 1.1. Let $f : X \to S$ be a smooth, affine map of noetherian schemes. Assume that dim $S = d$. Then $K_{-n}(f) = 0$ for $n > d + 1$ and $f$ is $K_{-n}$-regular for $n > d$.

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Secondly, we consider the case when \( f \) is smooth, but may not be affine. In this situation, we are able to prove a vanishing result for relative negative homotopy \( K \)-groups assuming the resolution of singularities. We prove such a result in Section 4. In Section 4, all the schemes are defined over a field \( k \) and we assume that the resolution of singularities holds over \( k \). Here is our result

**Theorem 1.2.** Let \( f : X \rightarrow S \) be a smooth and surjective map of noetherian schemes over a field \( k \). Assume that \( \dim S = d \). Then \( KH_{n}(f) = 0 \) for \( n > d + 1 \) and \( H^{d}_{cdh}(S, K^{f, cdh}_{-1}) = KH_{-d-1}(f) \). Here \( K^{f, cdh}_{n} \) is the cdh-sheafification of the presheaf \( U \mapsto K_{n}(U, f^{-1}U) \).

However, we notice that the surjectivity of \( f \) can be dropped in the above result when \( f \) is an étale map (see Remark 4.8 and Theorem 4.9). Using the Theorem 1.2, we show that \( KH_{n}(\mathbb{P}^{t}_{X}) = 0 \) for \( t \geq 0 \) and \( n > d \), where \( X \) is a \( d \)-dimensional noetherian scheme over \( k \) (see Corollary 4.6).

Next, we discuss the situation when the map \( f : X \rightarrow S \) may not be smooth. In particular, we consider subintegral maps. In [17], the author and Weibel have shown that if \( f : X = \text{Spec}(B) \rightarrow S = \text{Spec}(A) \) is a subintegral map (i.e. \( A \hookrightarrow B \) is subintegral) then \( K_{-n}(f) = 0 \) for \( n > 0 \) (see Proposition 2.5 of [17]). It has also been observed in [17, Example 6.6] that if \( S \) is not affine then the above mentioned result may fail. For example, consider \( S = \mathbb{P}^{1}_{k} \) and \( X = \text{Spec} \left( O_{B} \right) \) where \( O_{B} = O_{S} \oplus O(-2) \) with \( O(-2) \) is a square zero ideal. In this situation, \( K_{-1}(f) \neq 0 \). This suggests that the relative negative \( K \)-groups may be nonzero at the dimension (i.e. \( K_{-\dim S} \)) in the non affine situation. So it is natural to wonder what the groups \( K_{-n}(f) \) are for subintegral morphisms with non affine base. This is answered in Section 5 by proving the following theorem.

**Theorem 1.3.** Let \( f : X \rightarrow S \) be a subintegral morphism of noetherian schemes. Assume that \( \dim S = d \). Then

1. \( K_{-n}(f) = 0 \) for \( n > d \),
2. \( H^{d}_{zar}(S, f_{*}O_{X}^{\times} / O_{S}^{\times}) = K_{-d}(f) \),
3. If \( X \) and \( S \) are \( \mathbb{Q} \)-schemes then \( H^{d}_{et}(S, f_{*}O_{X}^{\times} / O_{S}^{\times}) = K_{-d}(f) \).

As a corollary, we obtain \( K_{-n}(X) \cong K_{-n}(X_{sn}) \) for \( n > d \) and \( K_{-d}(X) \rightarrow K_{-d}(X_{sn}) \) is surjective, where \( X \) is a \( d \)-dimensional noetherian scheme and \( X_{sn} \) is the seminormalization of \( X \) (see Corollary 5.7).

In Section 6, we prove a relative version of Vorst regularity result i.e. \( K_{n} \)-regularity implies \( K_{n-1} \)-regularity. More precisely, we prove

**Theorem 1.4.** If \( f : X = \text{Spec}(B) \rightarrow S = \text{Spec}(A) \) is \( K_{n} \)-regular then \( f \) is \( K_{n-1} \)-regular.
As a consequence, we show that if \( f : A \hookrightarrow B \) is subintegral ring extension then \( f \) cannot be \( K_n \)-regular and \( K_n(f) \neq 0 \) for \( n \geq 0 \) (see Proposition 6.1). Finally, we conclude this article with the following theorem (see Section 7).

**Theorem 1.5.** Let \( S \) be a noetherian scheme of dimension \( d \). The the following are equivalent

1. \( K_{-n}(f) = 0 \) for \( n > d + 1 \) and \( f \) is \( K_{-n} \)-regular for \( n > d \), for every smooth affine map \( f : X \to S \) of noetherian schemes.
2. (Weibel Conjecture) \( K_{-n}(S) = 0 \) for \( n > d \) and \( S \) is \( K_{-n} \)-regular for \( n \geq d \).

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### 2. Preliminaries

**Subintegral and Seminormal extension.** Let \( A \hookrightarrow B \) be a commutative ring extension. This extension \( A \hookrightarrow B \) is subintegral if \( B \) is integral over \( A \) and \( \text{Spec}(B) \to \text{Spec}(A) \) is a bijection inducing isomorphisms on all residue fields. We say that \( A \hookrightarrow B \) is seminormal (or \( A \) is seminormal in \( B \)) if whenever \( b \in B \) and \( b^2, b^3 \in A \) then \( b \in A \). More details can be found in [13], [18].

**Relative K-groups.** Given a map \( f : X \to S \) of schemes, \( K_n(f) = \pi_n K(f) \), where \( K(f) \) is the homotopy fiber of \( K(S) \to K(X) \). These relative K-groups fit into the following exact sequence \( \text{Seq}(K_n, f) \)

\[
\cdots \to K_n(f) \to K_n(S) \to K_n(X) \to K_{n-1}(f) \to K_{n-1}(S) \to \cdots
\]

For details see [2], [24].

**Relative Picard groups.** We also have a notion of relative Picard group \( \text{Pic}(f) \) for a map \( f : X \to S \) of schemes. The relative \( \text{Pic}(f) \) is the abelian group generated by \([L_1, \alpha, L_2]\), where the \( L_i \) are line bundles on \( S \) and \( \alpha : f^* L_1 \to f^* L_2 \) is an isomorphism. The relations are:

1. \([L_1, \alpha, L_2] + [L_1', \alpha', L_2'] = [L_1 \otimes L_1', \alpha \otimes \alpha', L_2 \otimes L_2']\);
2. \([L_1, \alpha, L_2] + [L_2, \beta, L_3] = [L_1, \beta \alpha, L_3]\);
3. \([L_1, \alpha, L_2] = 0 \) if \( \alpha = f^*(\alpha_0) \) for some \( \alpha_0 : L_1 \cong L_2 \).

This relative Picard group \( \text{Pic}(f) \) fits into the following exact sequence

\[
\mathcal{O}^\times(S) \to \mathcal{O}^\times(X) \to \text{Pic}(f) \to \text{Pic}(S) \to \text{Pic}(X).
\]

Some relevant details and basic properties can be found in [1], [16], [17].
Relative Homotopy K-groups. Let $\Delta^n = \text{Spec}(\mathbb{Z}[t_1, t_2, \ldots, t_n]/(t_1 + t_2 + \cdots + t_n - 1))$. Then the $n$-th homotopy K-group of a scheme $X$ is $KH_n(X) = \pi_n(KH(X))$, where $n \in \mathbb{Z}$ and $KH(X) = \text{hocolim}_j K(X \times \Delta^j)$. For a map of schemes $f : X \to S$, let $KH(f)$ be the homotopy fiber of $KH(S) \to KH(X)$. In fact, $KH(f) = \text{hocolim}_j K(f \times \Delta^j)$ by Lemma 5.19 of [19]. Then for $n \in \mathbb{Z}$, the $n$-th relative homotopy K-group of $f$ is $KH_n(f) = \pi_n(KH(f))$. The relative homotopy K-groups fit into the following exact sequence $\text{Seq}(KH_n,f)$

\[(2.3) \quad \cdots \to KH_n(f) \to KH_n(S) \to KH_n(X) \to KH_{n-1}(f) \to KH_{n-1}(S) \to \cdots.\]

For more details, we refer [22] and Chapter IV.12 of [24].

Remark 2.1. For a scheme $X$, there is a natural map $K(X) \to KH(X)$. Therefore, we get a natural map $K(f) \to KH(f)$ for any map $f : X \to S$ of schemes. In particular, there are natural maps $K_n(f) \to KH_n(f)$ for all $n$. For every scheme $X$, $KH_n(X) \cong KH_n(X \times \mathbb{A}^t)$ for all $n$ and $t \geq 0$. It is also well known that for a regular scheme $X$, $K_n(X) \cong KH_n(X)$ for all $n$. Using the exact sequences (2.1) and (2.3), the following facts are easy to check

1. If $X$ and $S$ are regular schemes then $K_n(f) \cong KH_n(f)$ for all $n$.
2. (Homotopy Invariance) $KH_n(f) \cong KH_n(f \times \mathbb{A}^t)$ for all $n$.

3. Relative negative K-theory of smooth, affine maps

In this section, we prove Theorem 1.1 which is a vanishing and regularity result for relative negative K-groups of a smooth, affine map. To prove this, we need some preparations. Let us begin with the following observation.

Lemma 3.1. Let $f : X \to S$ be a map of noetherian schemes with $\dim S = d$. Then the following are true:

1. for $n > d$, $K_{-n}(X) = 0$ if and only if $K_{-n-1}(f) = 0$.
2. for $n \geq d$, $X$ is $K_{-n}$-regular if and only if $f$ is $K_{-n-1}$-regular.

Proof. By Theorem B of [8], $K_{-n}(S) = 0$ for $n > d$ and $S$ is $K_{-n}$-regular for $n \geq d$. Now the first assertion follows from the long exact sequence (2.1). For the second assertion, apply $N^i$ to the sequence (2.1) and use the fact $S$ is $K_{-n}$-regular for $n \geq d$.

Lemma 3.2. Let $f : X \to S$ be a smooth map of noetherian schemes with $\dim S = 0$. Assume that $S$ is reduced. Then $K_{-n}(f) = 0$ for $n > 1$ and $f$ is $K_{-n}$-regular for $n > 0$.

Proof. First we claim that $S = \{s_1, s_2, \ldots, s_n\}$, where $s_i = \text{Spec}(k_i)$ with $k_i$ a field. Let $T$ be an irreducible component of $S$. The topological space $S$ carries the discrete topology because it is a zero dimensional noetherian space. So, $T$ is open in $S$ and $\dim T = 0$. 

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Since $S$ is reduced, $T$ is also reduced. Then $T$ is a zero dimensional noetherian integral scheme. We know that every zero dimensional noetherian scheme is a disjoint union of spectra of artinian local rings. Therefore, $T = \text{Spec}(A)$, where $A$ is an artinian domain. Since $S$ has a finite number of irreducible components and every artinian domain is a field, we get the claim.

Since $f$ is smooth, each fiber $f^{-1}(s) = X_s$ is regular for $s \in S$. By the above claim, $S = \text{Spec}(k_1) \sqcup \cdots \sqcup \text{Spec}(k_n)$ and for each $i$, $\text{Spec}(k_i) \to S$ is an open immersion. Then $X_{s_i} \to X$ is also an open immersion for each $i$. Now, we can write $X$ as a finite disjoint union of open subschemes $X_{s_i}$ which are regular. Hence $X$ is regular. Then $X$ is $K_n$-regular for all $n$ and $K_n(X) = 0$ for $n > 0$. We also have $K_n(S) = 0$ for $n > 0$ and $S$ is $K_n$-regular for $n \geq 0$. Therefore by (2.1), $K_n(f) = 0$ for $n > 1$ and $f$ is $K_n$-regular for $n > 0$.

For a morphism of schemes $f : X \to S$, let $\mathcal{K}(f)$ be the presheaf of spectra on $S$, defined as

$$\mathcal{K}(f)(U) = \text{hofib}[\mathcal{K}(S)(U) \to \mathcal{K}(X)(X \times_S U)],$$

where $\mathcal{K}(X)$ is the presheaf of spectra on $X$ (resp. $\mathcal{K}(S)$ on $S$). Similarly, we can define the nil presheaf of spectra $N^i\mathcal{K}(f)$ on $S$ for $i > 0$.

**Lemma 3.3.** $\mathcal{K}(f)$ and $N^i\mathcal{K}(f)$ satisfy Zariski descent.

**Proof.** We have a sequence of presheaves of spectra on $S$,

$$\mathcal{K}(f) \to \mathcal{K}(S) \to f_*\mathcal{K}(X).$$

It is easy to check that if $\mathcal{K}(X)$ satisfies Zariski descent then $f_*\mathcal{K}(X)$ does too. By Corollary V.7.10 of [24], $\mathcal{K}(X)$ satisfies Zariski descent. Then $\mathcal{K}(f)$ satisfies Zariski descent (see Exercise V.10.1 of [24]). By a similar argument, $N^i\mathcal{K}(f)$ satisfies Zariski descent. \(\square\)

For a morphism of schemes $f : X \to S$, let $\mathcal{K}'_n$ be the Zariski sheafification of the presheaf $U \mapsto K_n(U, f^{-1}U)$.

**Lemma 3.4.** Let $f : X \to S$ be an affine map of noetherian schemes with $\dim S = d$. Then the canonical map $K_n(f) \to K_n(f \times_S S_{\text{red}})$ is an isomorphism for $n > d$, where $f \times_S S_{\text{red}} : X \times_S S_{\text{red}} \to S_{\text{red}}$.

**Proof.** Write $\tilde{f}$ for $f \times_S S_{\text{red}}$. First we suppose that $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$. Then $X \times_S S_{\text{red}} = \text{Spec}(B \otimes_A \text{nil}(A))$. Note that $\text{nil}(A)B$ is a nil ideal of $B$. Then by comparing sequences (see [2.1]) $\text{Seq}(K_n, f)$ and $\text{Seq}(K_n, \tilde{f})$, we get $K_n(f) \cong K_n(\tilde{f})$ for $n > 0$ because for any ring $R$, $K_n(R) \cong K_n(R/I)$ for $n \geq 0$ with $I$ a nil ideal. Now by looking at the stalk level it is easy to see that $K_n(\tilde{f}) \cong K_n(f)$ for all $n < 0$ as a Zariski sheaf.
on $S$. There is a canonical map of Zariski descent spectral sequence for $S$ (Theorem 10.3 of [20]),

$$E_2^{p,q} = H^p(S, K_{-q}) \Rightarrow K_{-p-q}(f)$$

to for $S_{\text{red}}$

$$E_2^{p,q} = H^p(S_{\text{red}}, K_{-q}) \Rightarrow K_{-p-q}(\tilde{f})$$

which is an isomorphism on $E_2^{p,q}$ page for $q > 0$. Moreover, Zariski cohomological dimension is at most $d$. Hence the result. □

**Lemma 3.5.** Let $f : X \to S$ be a map of noetherian schemes. Suppose $\dim S = d$. Write $f_s$ for the map $X_\times S \to O_{S,s}$, $s \in S$. Then the following are true:

(1) If $K_{-n}(f_s) = 0$ for all $s \in S$ with $n > \dim O_{S,s} + 1$ then $K_{-n}(f) = 0$ for $n > d + 1$.

(2) If $N^iK_{-n}(f_s) = 0$ for all $s \in S$ with $n > \dim O_{S,s}$ and $i > 0$ then $N^iK_{-n}(f) = 0$ for $n > d$ and $i > 0$.

**Proof.** The result is clear by Lemma 3.3 and Proposition 6.1 of [8]. More precisely, apply Proposition 6.1 of [8] to the presheaves of spectra $\mathcal{K}(f)[-1]$ and $N^i\mathcal{K}(f)$.

□

We are now ready to prove Theorem 1.1

**Proof of Theorem 1.1:** By Lemma 3.3 we can assume that $S$ is affine. We can also assume that $S$ is reduced by Lemma 3.4. We prove by using induction on $\dim S$. If $\dim S = 0$ then the assertion is clear by Lemma 3.2. Suppose $d > 0$. Assume that for every smooth, affine map $X \to S$ with $\dim S < d$, we have $K_{-n}(X) = 0$ for $n > \dim S$ (see Lemma 3.1). Let $i < -d$ and consider an element $\xi$ in $K_{i}(X)$. Here $f$ is smooth and quasi-projective. Apply Proposition 5 of [7] to the map $f : X \to S$. Then there exist a projective birational map $p : S' \to S$ such that $\tilde{p}^\ast\xi = 0$ where $\tilde{p} : X' = X_\times S' \to X$. We can choose a nowhere dense closed subset $Y \hookrightarrow S$ such that $p$ is an isomorphism outside $Y$. Then we obtain the following abstract blow-up squares

\[
\begin{array}{ccc}
Y' & \longrightarrow & S' \\
\downarrow & & \downarrow p \\
Y & \longrightarrow & S
\end{array}
\]

and

\[
\begin{array}{ccc}
X \times_S Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \tilde{p} \\
X \times_S Y & \longrightarrow & X
\end{array}
\]

By applying Theorem A of [8], we get a long exact sequence

$$\cdots \to \lim_{n} K_{i+1}(X \times_S Y_n') \to K_i(X) \to \lim_{n} K_i(X \times_S Y_n) \oplus K_i(X') \to \cdots$$

of pro-groups. Here $Y_n$ (resp. $Y_n'$) is the $n$-th infinitesimal thickening of $Y$ (resp. $Y'$) in $S$ (resp. $S'$). Observe that $X \times_S Y \to Y$ and $X \times_S Y' \to Y$ are smooth, affine with $\dim Y < d$ and $\dim Y' < d$. Then by induction hypothesis, the pro-groups involving
Hence $\xi K_i(X \times_S Y_n)$ vanish. Therefore, $\tilde{p}^*: K_i(X) \to K_i(X')$ is injective and hence $\xi = 0$. This proves the first part.

In the second part, we can assume that $S$ is affine by Lemma 3.3. Then $X$ is affine. Now by the proof of Lemma 3.4, we can also assume that $S$ is reduced. Again, we use induction on the dimension of $S$. If $\dim S = 0$ then the assertion is clear by Lemma 3.2. Suppose $d > 0$. Assume that for every smooth, affine map $X \to S$ with $\dim S < d$, we have $N^iK_{-n}(X) = 0$ for $n \geq \dim S$ and $i > 0$ (see Lemma 3.1). Let $n \geq d$ and $r > 0$.

For each $r$, we can argue the inductive step separately. Consider $\xi \in K_{-n}(\mathbb{A}^n_X)$. Apply Proposition 5 of [7], to the map $\mathbb{A}^r_X \to \mathbb{A}^r_S \to S$, which is smooth and quasi-projective. Then there exist a projective birational map $p : S' \to S$ such that $\tilde{p}^*\xi = 0$ where $\tilde{p} : \mathbb{A}^r_{X'} \to \mathbb{A}^r_X$ and $X' = X \times_S S'$. We can choose a nowhere dense closed subset $Y \hookrightarrow S$ such that $p$ is an isomorphism outside $Y$. Now we have the following commutative diagram

$$
\begin{array}{ccccccc}
\longrightarrow & \text{``lim''} & K_{i+1}(X \times_S Y_n) & \longrightarrow & K_i(X) & \longrightarrow & \text{``lim''} & K_i(X \times_S Y_n) \oplus K_i(X') & \longrightarrow & \ldots \\
\beta^*_Y \downarrow & & \beta^* \downarrow & & \beta^* \oplus \beta^*_Y \downarrow & & \\
\longrightarrow & \text{``lim''} & K_{i+1}(\mathbb{A}^r_X \times_S Y_n) & \longrightarrow & K_i(\mathbb{A}^r_X) & \longrightarrow & \text{``lim''} & K_i(\mathbb{A}^r_X \times_S Y_n) \oplus K_i(\mathbb{A}^r_{X'}) & \longrightarrow & \ldots,
\end{array}
$$

where the horizontal sequence is exact by Theorem A of [8]. Here $\beta$ is the projection map $\mathbb{A}^r_X \to X$ and $\beta^*$ is the induced morphism. Since $\dim Y < d$ and $\dim Y' < d$, $\beta^*_Y$, $\beta^*_{Y'}$ are isomorphism by induction hypothesis. By the first part the pro-groups in the upper horizontal sequence involving $K_i(X \times_S Y_n)$ vanishes. Now a simple diagram chase gives that $\beta^*$ is surjective. Since $\beta^*$ is always injective, we get the result. □

4. Relative negative homotopy K-theory of smooth, surjective maps

In this section, all the schemes are defined over a field $k$ and we assume that the resolution of singularities holds over $k$. The main goal is to prove Theorem 1.2 which is a vanishing result for relative negative homotopy K-groups of a smooth, surjective map.

**Lemma 4.1.** Let $f : X \to S$ be a map of schemes over a field $k$ with $S$ smooth. Suppose $f$ factors into $X \twoheadrightarrow \mathbb{A}^n_S \to S$ with $g$ étale. Then $K_{-n}(f) \cong K_{-n}(g)$ for all $n$ and $K_{-n}(g) = 0$ for $n > 1$.

**Proof.** Since $S$ is smooth, $K_n(\mathbb{A}^n_S) \cong K_n(S)$ by $K_n$-regularity for all $n$. Now by comparing the exact sequence (2.1) for the maps $f$ and $g$, we get the first assertion. Note that $\mathbb{A}^n_S$ is regular. Then $X$ is regular by Proposition I.3.17(c) of [11], because $g$ is étale. It is well known that the negative absolute K-theory of regular scheme vanish. Hence the second assertion by the exact sequence (2.1). □
Lemma 4.2. Let $f : X \to S$ be a map of schemes. Let $x \in X$. Let $V \subset S$ be an affine open nbd of $f(x)$. If $f$ is smooth at $x$, then there exists an integer $d \geq 0$ and affine open $U \subset X$ with $x \in X$ and $f(U) \subset V$ such that there exists a commutative diagram

$$
\begin{array}{ccc}
X & \leftarrow & U \\
\downarrow & & \downarrow \pi \\
Y & \leftarrow & V,
\end{array}
$$

where $\pi$ is étale.

Proof. See Lemma 34.20 of [14].

Let $K_{n,cdh}^f$ be the cdh-sheafification of the presheaf $U \mapsto K_n(U, f^{-1}U)$. By replacing $K$ by $KH$, we get $K_{n,cdh}^f$.

Lemma 4.3. Let $f : X \to S$ be a smooth and surjective map of schemes over a field $k$. Then $K_{n,cdh}^f \cong K_{n,cdh}^f$ as a cdh sheaf on $S$. Moreover, $K_{n,cdh}^f$ is zero for $n < -1$.

Proof. Pick any $s \in S$. Since $f$ is surjective, $f(x) = s$ for some $x \in X$. Let $V \subset S$ be an affine open nbd of $f(x)$. Now by Lemma [12] there exists an integer $d \geq 0$ and affine open $U \subset X$ with $x \in X$ and $f(U) \subset V$ such that locally $f$ factor as $U = X \times_S V \to \mathbb{A}_V^d \to V$. In cdh topology, schemes are locally smooth. So at stalk, $K_{n,cdh}^f$ (resp. $K_{n,cdh}^f$) is $K_n(X \times_S R \to \mathbb{A}_R^d \to R)$ (resp. $KH_n$) where $R$ is a regular local ring. By Proposition I.3.17(c) of [11], $X \times_S R$ is regular because $\mathbb{A}_R^d$ is regular. Then we get $K_n(X \times_S R \to \mathbb{A}_R^d \to R) \cong KH_n(X \times_S R \to \mathbb{A}_R^d \to R)$ by Remark [21]. Therefore, at stalk level $K_{n}^f$ and $K_{n,cdh}^f$ are isomorphic. Also, $K_n(X \times_S R \to \mathbb{A}_R^d) = 0$ for $n < -1$ by Lemma [11]. Hence the assertion.

For a morphism of schemes $f : X \to S$, let $KH(f)$ be the presheaf of spectra on $S$, defined as

$$KH(f)(U) = \text{hofib}[KH(S)(U) \to KH(X)(X \times_S U)],$$

where $KH(X)$ is the presheaf of spectra on $X$ (resp. $KH(S)$ on $S$).

Lemma 4.4. $KH(f)$ satisfies cdh descent.

Proof. By Theorem 3.9 of [3], $KH(X)$ satisfies cdh descent. The rest of the arguments are similar to Lemma [3.3].

Proof of Theorem 1.2. By Lemma 4.4, $KH(f)$ satisfies cdh descent. Therefore, we have a descent spectral sequence (by Theorem 3.4 of [4] and Theorem 2.8 of [6])

$$E_2^{p,q} = H_{cdh}^p(S, KH_{-q,cdh}^f) \Rightarrow KH_{-p-q}(f).$$

Also by Lemma 4.3 $KH_{q,cdh}^f \cong KH_{q,cdh}^f$ as a cdh sheaf on $S$ and for $q < -1$, $KH_{q,cdh}^f$ is zero. Moreover, the cdh cohomological dimension is at most $d$. Hence, we get $KH_{-n}(f) = 0$ for $n > d + 1$ and $H_{cdh}^d(S, KH_{-1,cdh}^f) = KH_{-d-1}(f)$.
Some explicit calculations in lower dimensions are given in the following corollary. Write $\mathcal{K}_q^f$ for $\mathcal{K}_{q,cdh}^f$.

**Corollary 4.5.** Let $f$ be as in Theorem 1.2. Then

1. If $\dim S = 0$ then $KH_{-1}(f) \cong H^0_{cdh}(S, \mathcal{K}_q^f)$ and $KH_{-n}(f) = 0$ for $n > 1$.
2. If $\dim S = 1$ then the sequence

$$0 \to H^1_{cdh}(S, \mathcal{K}_0^f) \to KH_{-1}(f) \to H^0_{cdh}(S, \mathcal{K}_-1^f) \to 0$$

is exact, $KH_{-2}(f) \cong H^1_{cdh}(S, \mathcal{K}_-1^f)$ and $KH_{-n}(f) = 0$ for $n > 2$.
3. If $\dim S = 2$ then the sequence

$$0 \to H^1_{cdh}(S, \mathcal{K}_0^f) \to KH_{-1}(f) \to H^0_{cdh}(S, \mathcal{K}_-1^f) \to H^2_{cdh}(S, \mathcal{K}_-1^f) \to KH_{-2}(f) \to H^2_{cdh}(S, \mathcal{K}_-2^f) \to H^0_{cdh}(S, \mathcal{K}_0^f)$$

is exact, $KH_{-3}(f) \cong H^2_{cdh}(S, \mathcal{K}_-1^f)$ and $KH_{-n}(f) = 0$ for $n > 3$.

**Proof.** The assertions are clear from the following seven term exact sequence

$$0 \to H^1_{cdh}(S, \mathcal{K}_0^f) \to KH_{-1}(f) \to H^0_{cdh}(S, \mathcal{K}_-1^f) \to H^2_{cdh}(S, \mathcal{K}_-1^f) \to KH_{-2}(f) \to H^2_{cdh}(S, \mathcal{K}_-2^f) \to H^0_{cdh}(S, \mathcal{K}_0^f) \to 0$$

The next result is well known, but we are including it here as an application of Theorem 1.2.

**Corollary 4.6.** Let $X$ be a $d$-dimensional noetherian scheme over a field $k$. Then for all $t \geq 0$, $KH_{-n}(\mathbb{P}^t_X) = 0$ for $n > d$.

**Proof.** Note that the projection $\pi : \mathbb{P}^t_X \to X$ is a smooth and surjective map. By Theorem 1.2 $KH_{-n}(\pi) = 0$ for $n > d+1$. We have $KH_{-n}(X) = 0$ for $n > d$ by Theorem 1 of [7]. Then the exact sequence (2.3) implies that $KH_{-n}(\mathbb{P}^t_X) = 0$ for $n > d$.

**Remark 4.7.** The vanishing of $KH_{-n}(f)$ remains valid after the finite base change. More precisely, let $f : X \to S$ be as in Theorem 1.2. Let $h : S' \to S$ be a finite map. Then $\dim S' \leq d$. Since smooth and surjective maps are stable under base change, $f' : X \times_S S' \to S'$ is smooth and surjective. Hence $KH_{-n}(f') = 0$ for $n > d + 1$.

**Remark 4.8.** We do not know whether Lemma 4.3 is true without surjective assumption on $f$. But, if $f : X \to S$ is just an étale map, then the Lemma 4.3 holds without $f$ being surjective. Indeed, at stalk $KH_{q,cdh}^f$ is $K_q(X \times_S R \to R)$, where $R$ is a regular local ring. Then $X \times_S R$ is regular by Proposition I.3.17(c) of [11]. By Remark 2.1 at stalk level $KH_{q}^f$ and $KH_{q,cdh}^f$ are isomorphic and for $q < -1$, $KH_{q,cdh}^f$ is zero. Therefore, the Theorem 1.2 is also true for an étale map.
In view of above remark, we are in situation to write the following

**Theorem 4.9.** Let \( f : X \to S \) be an étale map of noetherian schemes over a field \( k \).
Assume that \( \dim S = d \). Then \( KH_{-n}(f) = 0 \) for \( n > d + 1 \) and \( H_{cdh}^{d}(S, K_{-1, cdh}) = KH_{-d-1}(f) \).

5. Relative negative K-theory of subintegral maps

In this section, we study the relative negative \( K \)-groups of a subintegral map of schemes. In particular, we prove Theorem 1.3. We begin with the following definition.

**Definition 5.1.** Let \( f : X \to S \) be a faithful affine morphism, i.e, affine and the structure map \( \mathcal{O}_{S} \to f_{*}\mathcal{O}_{X} \) is injective. We say that \( f \) is subintegral if \( \mathcal{O}_{S}(U) \to f_{*}\mathcal{O}_{X}(U) \) is subintegral for all affine open subsets \( U \) of \( S \).

Let \( A \hookrightarrow B \) be a ring extension. The Roberts-Singh group \( I(A, B) \) (or \( I(f) \), where \( f : \text{Spec}(B) \to \text{Spec}(A) \)) is the multiplicative group of invertible \( A \)-submodules of \( B \). We refer section 2 of [13] for details. There is a natural group homomorphism \( \psi : I(f) \to \text{Pic}(f) \), \( I \mapsto [I, \alpha, A] \), where \( \alpha : I \otimes_{A} B \cong B \).

If \( \mathbb{Q} \subset A \) then a natural group homomorphism \( \xi_{B/A} : B/A \to I(f) \) is constructed in [13] as follows: for \( b \in B \), let \( I_{B/A}(b) = B[t] \cap A[[t]]e^{ht} \), where \( t \) is an indeterminate. By Theorem 4.17 and Corollary 4.3 of [13], \( I_{B/A}(b) \in I(f[t]) \). Here \( f[t] \) is \( \text{Spec}(B[t]) \to \text{Spec}(A[t]) \). Let \( \tau : I(f[t]) \to I(f) \). Then the homomorphism \( \xi_{B/A} \) is given by \( \xi_{B/A}(b) = \tau(I_{B/A}(b)) \), where \( b \in B/A \) with representative \( b \in B \).

**Lemma 5.2.** Let \( f : A \hookrightarrow B \) be a subintegral extension of \( \mathbb{Q} \)-algebras. Then the natural map \( \psi \circ \xi_{B/A} : B/A \to \text{Pic}(f) \) is an isomorphism.

**Proof.** By Lemma 1.2 of [17], \( \psi \) is an isomorphism. The isomorphism of \( \xi_{B/A} \) was proven in Theorem 5.6 of [13] and Theorem 2.3 of [12]. Hence the lemma. \( \square \)

The following Proposition generalizes the above result for schemes.

**Proposition 5.3.** Let \( f : X \to S \) be a subintegral morphism of \( \mathbb{Q} \)-schemes. Then \( \text{Pic}(f) \cong H^{0}_{zar}(S, f_{*}\mathcal{O}_{X}/\mathcal{O}_{S}) \).

**Proof.** Let \( s \in S \). Then \( (f_{*}\mathcal{O}_{X}/\mathcal{O}_{S})_{s} \cong B_{s}/A_{s} \cong I(A_{s}, B_{s}) \cong (f_{*}\mathcal{O}_{X}^{\times}/\mathcal{O}_{S}^{\times})_{s} \), where the second isomorphism by Lemma 5.2 and the third isomorphism by the exact sequence (2.2). This implies that \( f_{*}\mathcal{O}_{X}/\mathcal{O}_{S} \cong f_{*}\mathcal{O}_{X}^{\times}/\mathcal{O}_{S}^{\times} \) as a sheaves on \( S \). Now the result follows from Lemma 5.4 of [16]. \( \square \)

**Proposition 5.4.** Let \( f : X \to S \) be a subintegral morphism of noetherian \( \mathbb{Q} \)-schemes. Then the following are true:
(1) $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ is a quasi-coherent sheaf.

(2) $H^i_{\text{zar}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) = H^i_{\text{et}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)$.

(3) If $S$ is affine and $f$ is finite then for $i > 1$, $H^i_{\tau}(S, \mathcal{O}_S^\times) \cong H^i_{\tau}(X, \mathcal{O}_X^\times)$, where $
\tau = \{\text{zar, et}\}$.

Proof. (1) Since $f$ is affine, $f_*\mathcal{O}_X$ is quasi-coherent. Then the quotient $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ is also quasi-coherent. Therefore, we get the result by using the fact that $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times \cong f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ (see the proof of Proposition 5.3).

(2) This follows from the fact that Zariski and ´etale cohomology coincide for a quasi-coherent sheaf (see Remark 3.8 of [11]).

(3) Consider the long exact cohomology sequence associated to the following exact sequence of sheaves on $S$,

$$1 \to \mathcal{O}_S^\times \to f_*\mathcal{O}_X^\times \to f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times \to 1.$$ 

Since $f$ is finite, $H^i_{\tau}(S, f_*\mathcal{O}_X^\times) \cong H^i_{\tau}(X, \mathcal{O}_X^\times)$. By (1), $H^i_{\text{zar}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) = 0$ for $i > 0$, because $S$ is affine. Hence the assertion. □

Remark 5.5. The statement (3) of the above Proposition may fail for $i = 1$. For example, consider $A = \mathbb{Q}[t^2, t^3]$ and $B = \mathbb{Q}[t]$. In this case, $\text{Pic}(A) \cong \mathbb{Q}$ and $\text{Pic}(B) = 0$.

Lemma 5.6. If $f$ is subintegral then $K^f_{-q} = 0$ for $q > 0$.

Proof. Each stalk of $K^f_{-q}$ is $K_{-q}(A, B)$, where $A \subset B$ is a subintegral extension of local rings. By Proposition 2.5 of [17], $K_{-q}(A, B) = 0$ for $q > 0$. Hence the result. □

Proof of Theorem 1.3: We have a descent spectral sequence

$$E^2_p = H^p_{\text{zar}}(S, K^f_{-q}) \Rightarrow K_{-p-q}(f).$$

By Lemma 5.6, $K^f_{-q} = 0$ for $q > 0$. Moreover, $K^f_0 \cong f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ by the exact sequence (2.1) of [17]. Since the Zariski cohomological dimension is at most $d$, we get the first two assertions. The last assertion follows from Proposition 5.3(2). □

Let $X$ be a scheme. The seminormalization of $X$ can be obtained by mimicking the normalization process. For each affine open subset $U = \text{Spec}(A)$ of $X$, let $\bar{A}$ be the subintegral closure (or seminormalization) of $A$ in its total quotient ring. Let $\bar{U} = \text{Spec}(\bar{A})$. Now by gluing together such schemes $\bar{U}$ we get $X_{\text{sn}}$, which we call the seminormalization of $X$. Clearly, then $X_{\text{sn}} \to X$ is a subintegral morphism.

Corollary 5.7. Let $X$ be a $d$-dimensional noetherian scheme. Let $X_{\text{sn}}$ be the seminormalization of $X$. Then $K_{-n}(X) \cong K_{-n}(X_{\text{sn}})$ for $n > d$ and $K_{-d}(X) \to K_{-d}(X_{\text{sn}})$ is surjective.

Proof. Clear from Theorem 1.3(1) and the long exact sequence (2.1). □
6. On Regularity

A theorem of Vorst says that if a ring $A$ is $K_n$-regular then it is $K_{n-1}$-regular (see V. 8.6 of [24]). Now we prove Theorem 1.4 which is a relative version of Vorst’s result.

**Proof of Theorem 1.4.** First we suppose that $K_n(f) \cong K_n(f[s,t])$. Then $NK_n(f[s]) = 0$. The goal is to show that $NK_{n-1}(f) = 0$. Applying $N$ to the exact sequence (2.1) for $f[s]: \text{Spec}(B[s]) \to \text{Spec}(A[s])$, we get the following exact sequence $\text{Seq}(NK_n, f[s])$

$$\cdots \to NK_n(f[s]) \to NK_n(A[s]) \to NK_n(B[s]) \to NK_{n-1}(f[s]) \to NK_{n-1}(A[s]) \to \cdots$$

By Theorem 5.6 of [17], $NK_n(f[s])$ is a $W(A[s])$-module for all $n$. We know that $NK_n(A[s])$ is also a $W(A[s])$-module (see IV.6.7 of [24]). Then $NK_n(B[s])$ is a $W(A[s])$-module via the map $W(A[s]) \to W(B[s])$. In fact (6.1) is a sequence of $W(A[s])$-modules. Consider the multiplicative closed set $T = \{[s]^q\}_{q \geq 0}$ in $W(A[s])$, where $[s]$ is the Teichmüller representative of $s$ in $W(A[s])$. So after localization, we get the sequence $\text{Seq}((NK_n)_s, f[s])$, more precisely the terms are $NK_n(A[s])_s$, $NK_n(f[s])_s$ etc. Then we have a natural map $\text{Seq}((NK_n)_s, f[s]) \to \text{Seq}(NK_n, f[s, 1/s])$ of $T^{-1}W(A[s])$-module. By Lemma V.8.5 of [24], $NK_n(A[s])_s \cong NK_n(A[s, 1/s])$. Now a diagram chase implies that $NK_n(f[s])_s = NK_n(f[s, 1/s])$. Applying the Bass fundamental theorem (see Example 5.1 of [17]) on $NK_n(f[s, 1/s])$, we get

$$NK_n(f[s, 1/s]) = NK_n(f) \oplus N^2K_n(f) \oplus N^2K_n(f) \oplus NK_{n-1}(f).$$

But $NK_n(f[s, 1/s]) = 0$ because $NK_n(f[s]) = 0$. Hence, $NK_{n-1}(f) = 0$.

Similarly, we can show that $NK_n(f[s,t])_t = NK_n(f[s,t,1/t])$. Again by the Bass fundamental theorem, $N^2K_{n-1}(f) = 0$. Therefore, by repeating the same arguments we get $N^iK_{n-1}(f) = 0$ for all $i > 0$. \hfill $\Box$

**Proposition 6.1.** If $f$ is a subintegral map of affine schemes then $f$ can not be $K_n$-regular for $n \geq 0$. Moreover, $K_n(f) \neq 0$ for $n \geq 0$.

**Proof.** Since $f$ is subintegral, $K_0(f) \cong \text{Pic}(f)$ by Proposition 2.5 of [17]. Note that $NK_0(f) \cong NPic(f)$. By Theorem 1.5 of [15], $NPic(f) = 0$ if and only if $f$ is seminormal. Therefore, $NK_0(f) \neq 0$ and hence $f$ is not $K_0$-regular. Now Theorem 1.4 implies that $f$ can not be $K_n$-regular for $n \geq 0$.

Suppose $K_n(f) = 0$ for some $n \geq 0$. Since $f$ is subintegral, so is $f[t_1, t_2, \ldots, t_i]$. Then $K_n(f[t_1, \ldots, t_i]) = 0$. This shows that $f$ is $K_n$-regular, which is a contradiction by the first part. Hence, $K_n(f) \neq 0$ for $n \geq 0$. \hfill $\Box$

**Remark 6.2.** The converse of Theorem 1.4 does not hold. Because, if $f$ is a subintegral map of affine schemes then $K_n(f) = K_n(f[t_1, t_2, \ldots, t_i]) = 0$ for $n < 0$ by Proposition 2.5.
of [17]. Hence \( f \) is \( K_n \)-regular for \( n < 0 \). But \( f \) is not \( K_0 \)-regular by Proposition 6.1. In particular, consider \( f : \text{Spec}(\mathbb{Q}[x]/(x^2)) \to \text{Spec}(\mathbb{Q}) \). Here \( f \) is subintegral, \( K_{-1} \)-regular but not \( K_0 \)-regular.

7. Relative vs Absolute

In this section, we prove Theorem 1.5. Recall that \( KH(f) = \hocolim_j K(f \times \Delta^j) \). Then, there is a right half-plane spectral sequence (see Proposition 5.17 of [19]),

\[
E_1^{p,q} = K_q(f \times \Delta^p) \Rightarrow KH_{p+q}(f),
\]

for any \( f : X \to S \) map of schemes. This is the standard Bousfield-Kan spectral sequence of simplicial spectrum. If \( f \) is \( K_{-n} \)-regular for \( n > d \), then the spectral sequence (7.1) implies that \( K_{-n}(f) = KH_{-n}(f) \) for \( n > d \).

Proof of Theorem 1.5:

(1) \( \Rightarrow \) (2) For a fix \( t \), consider \( f : \mathbb{A}_S^t \to S \). Then \( K_{-n}(f) \cong KH_{-n}(f) \) for all \( n > d \). Since \( KH \) is homotopy invariant, \( KH_{-n}(f) = 0 \) for \( n > d \). Thus, \( K_{-n}(f) = 0 \) for \( n > d \). Now by the exact sequence (2.1), \( K_{-n}(S) \cong K_{-n}(\mathbb{A}_S^t) \) for \( n \geq d \). We can argue for each \( t \) separately, hence \( S \) is \( K_{-n} \)-regular for \( n \geq d \).

For the second assertion, consider the spectral sequence \( K_q(S \times \Delta^p) \Rightarrow KH_{p+q}(S) \). By the first part \( S \) is \( K_{-n} \)-regular for \( n \geq d \). Therefore, \( K_{-n}(S) \cong KH_{-n}(S) \) for \( n \geq d \). By Theorem 1 of [7], \( KH_{-n}(S) = 0 \) for \( n > d \). Hence the result.

(2) \( \Rightarrow \) (1) This follows from Lemma 3.1 and Theorem 1.1. \( \square \)

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School of Mathematics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Colaba, Mumbai 400005, India

*E-mail address: sadhu@math.tifr.res.in, viveksadhu@gmail.com*