Blow up of Solutions to Semilinear Wave Equations with variable coefficients and boundary

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Abstract
This paper is devoted to studying the following two initial-boundary value problems for semilinear wave equations with variable coefficients on exterior domain with subcritical exponent in n space dimensions:

\[ u_{tt} - \partial_i (a_{ij}(x) \partial_j u) = |u|^p, \quad (x, t) \in \Omega^c \times (0, +\infty), \quad n \geq 3 \]  \hspace{1cm} (0.1)

and

\[ u_{tt} - \partial_i (a_{ij}(x) \partial_j u) = |u_t|^p, \quad (x, t) \in \Omega^c \times (0, +\infty), \quad n \geq 1. \]  \hspace{1cm} (0.2)

where \( u = u(x, t) \) is a real-valued scalar unknown function in \( \Omega^c \times [0, +\infty) \), here \( \Omega \) is a smooth compact obstacle in \( \mathbb{R}^n \), \( \Omega^c \) is its complement, \( n \geq 3 \) for (0.1) and \( n \geq 1 \) for (0.2), here \( \{a_{ij}(x)\}_{i,j=1}^n \) denotes a matrix valued smooth function of the variable \( x \in \Omega^c \), which takes values in the real, symmetric, \( n \times n \) matrices, such that for some \( C > 0 \),

\[ C^{-1} |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \quad x \in \Omega^c, \]

here and in the sequence, a repeated sum on an index is never indicated, and

\[ a_{ij}(x) = \delta_{ij}, \quad \text{when} \quad |x| \geq R, \]

where \( \delta_{ij} \) stands for the Kronecker delta function.

The exponents \( p \) satisfies \( 1 < p < p_1(n) \) in (0.1), and \( p \leq p_2(n) \) in (0.2), where \( p_1(n) \) is the larger root of the quadratic equation \((n-1)p^2 - (n+1)p - 2 = 0\), and \( p_2(n) = \frac{2}{n-1} + 1 \), respectively. It is well-known that the number \( p_1(n) \) is the critical exponent of the semilinear wave equation (0.1), while \( p_2(n) \) is the critical exponent of (0.2).

We will establish two blowup results for the above two initial-boundary value problems, it is proved that there can be no global solutions no matter how small the initial data are, and also we give the lifespan estimate of solutions for above problems.

Keywords: Semilinear wave equation; Critical exponent; Initial-boundary value problem; Blow up

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1 Introduction

In this paper, we will consider the blow up of solutions of the initial-boundary value problems for the following two semilinear wave equations on exterior domain:

\[
\begin{align*}
\left\{ \begin{array}{l}
  u_t - \partial_i(a_{ij}(x) \partial_j u) = |u|^p, \quad (x,t) \in \Omega^c \times (0, +\infty), \quad n \geq 3, \\
  u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x), \quad x \in \Omega^c, \\
  u(t,x)|_{\partial \Omega} = 0, \quad \text{for } t \geq 0,
\end{array} \right.
\]  

(1.1)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
  u_t - \partial_i(a_{ij}(x) \partial_j u) = |u|^p, \quad (x,t) \in \Omega^c \times (0, +\infty), \quad n \geq 1, \\
  u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x), \quad x \in \Omega^c, \\
  u(t,x)|_{\partial \Omega} = 0, \quad \text{for } t \geq 0,
\end{array} \right.
\]  

(1.2)

where \(A(x) = \{a_{ij}(x)\}_{i,j=1}^n\) denotes a matrix valued smooth function of the variable \(x \in \Omega^c\), which takes values in the real, symmetric, \(n \times n\) matrices, such that for some \(C > 0\),

\[C^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq C|\xi|^2, \forall \xi \in R^n, \quad x \in \Omega^c,\]

here and in the sequence, a repeated sum on an index is never indicated, and

\[a_{ij}(x) = \delta_{ij}, \text{when } |x| \geq R,\]

where \(\delta_{ij}\) stands for the Kronecker delta function. \(\Omega\) is a smooth compact obstacle in \(R^n\), \(\Omega^c\) is its complement, \(n \geq 3\) for (1.1) and \(n \geq 1\) for (1.2). Without loss of generality, we assume that \(0 \in \Omega \subset B_R\), where \(B_R\) is a ball of radius \(R\) centered at the origin and \(\text{supp}\{f, g\} \subset B_R\). We consider dimensions \(n \geq 3\) and exponents \(p \in (1, p_1(n))\) for problem (1.1), and dimensions \(n \geq 1\) and exponents \(p \leq p_2(n)\) for problem (1.2), where \(p_1(n)\) is the larger root of the quadratic equation \((n - 1)p^2 - (n + 1)p - 2 = 0\), and \(p_2(n) = \frac{2}{n-1} + 1\), respectively. The number \(p_1(n)\) is known as the critical exponent of the semilinear wave equation (1.1) (see, e.g., [23]) and the number \(p_2(n)\) is known as the critical exponent of the semilinear wave equation (1.2) (see, e.g., [33]). And we consider compactly supported nonnegative data \((f, g) \in H^1(\Omega^c) \times L^2(\Omega^c)\) for problem (1.1) and \(f, g \in C_0^\infty(\Omega^c)\) for problem (1.2).

If \(a_{ij} = \delta_{ij}\), we say problems (1.1), (1.2) are of constant coefficients. In the case of cauchy problems of subcritical semilinear wave equation with constant coefficients, there is an extensive literature which we shall review briefly, for details, see [3, 4, 5, 8, 10, 11, 12, 16, 19, 22, 23, 24, 25, 27, 29, 30, 31, 32, 33].

For the problem (1.1) with constant coefficients, the case \(n = 3\) was first done by F. John [8] in 1979, he showed that when \(n = 3\) global solutions always exist if \(p > p_1(3) = 1 + \sqrt{2}\) and initial data are suitably small, and moreover, the global solutions do not exist if \(1 < p < p_1(3) = 1 + \sqrt{2}\) for any nontrivial choice of \(f\) and \(g\). The number \(p_1(3) = 1 + \sqrt{2}\) appears to have first arisen in Strauss’ work on low energy scattering for the nonlinear Klein-Gordon equation [22]. This led him to conjecture that when \(n \geq 2\) global solutions of (1.1) should always exist if initial data are sufficiently small and \(p\) is greater than a critical power \(p_1(n)\).
The conjecture was verified when $n = 2$ by R. T. Glassey [5]. In higher space dimensions, the case $n = 4$ was proved by Y. Zhou [32] and V. Georgiev, H. Lindblad and C. Sogge [3] showed that when $n \geq 4$ and $p_1(n) < p \leq \frac{n+3}{n-1}$, (1.1) has global solutions for small initial values (see also [13] and [26]). Later, a simple proof was given by Tataru [26] in the case $p > p_1(n)$ and $n \geq 4$. R. T. Glassey [4] and T. C. Sideris [19] showed the blow-up result of $1 < p < p_1(n)$ for $n = 2$ and all $n \geq 4$, respectively. Sideris’ proof of the blow up result is quite delicate, using sophisticated computation involving spherical harmonics and other special functions. His proof was simplified by Rammaha [15] and Jiao and Zhou [7]. In 2005, the proof was further simplified by Yordanov and Zhang [27] by using a simple test function, also, more importantly they use their method to establish blowup phenomenon for wave equations (1.1) with constant coefficients and a potential. On the other hand, for the critical case $p = p_1(n)$, it was shown by Schaeffer [16] that the critical power also belongs to the blowup case for small data when $n = 2, 3$ (see also [24, 30, 31]). B. Yordanov, and Q. S. Zhang [28] and Y. Zhou [34] independently have extended Sideris’ blowup result to $p = p_1(n)$ for all $n \geq 4$ by different methods respectively.

For the problem (1.2) with constant coefficients, the blowup part was first proved by F. John [9] and the global existence part was first obtained by T.C. Sideris [20] in the case $n = 3$, and both by J. Schaeffer [17] in the case $n = 5$. The blow-up part in the case $n = 2$ was proved by Schaeffer [18] for $p = p_2(2)$. Later, R. Agemi [1] proved it for $1 < p \leq p_2(2)$ by different method from [18]. The case $n = 1$ is essentially due to K. Masuda [14] who proved the blowup result in the case $n = 1, 2, 3$ and $p = 2$. In higher space dimensions, M. A. Rammaha [15] proved the blow-up part of $n \geq 4$ in the case where $p = p_2(n)$ for odd $n$ and $1 < p < p_2(n)$ for even $n$. A simple proof of blowup part was later given by Y. Zhou [33].

Recently, K. Hidano et. al [6] has established global existence for problem (1.1) with $p > p_1(n)$ and $n = 3, 4$. For related result, one can see Sogge and Wang’s work [21]. However, to the best of our knowledge, there are no blowup results concerning initial-boundary value problems for semilinear wave equations with variable coefficients on exterior domain. In this paper, we shall establish blowup results for the initial-boundary value problem for subcritical values of $p$. We shall also estimate the lifespan $T(\varepsilon)$ for small initial data of size $\varepsilon$. Our result is complement to the global existence result of K. Hidano et. al [6]. For the problem (1.1), we obtain our result by constructing two test functions $\phi_0$ and $\psi_1$ (see Section 2), which is motivated by the work of Yordanov and Zhang [27]. For the problem (1.2), we still use the test function $\psi_1$ and by introducing an auxiliary function $G_0(t)$ (see Section 4), we reduced the problem to a Ricatti equation. This proof is new even in the constant coefficients case.

We are interested in showing the ”blow up” of solutions to problems (1.1) and (1.2). For that, we require

$$1 < p < p_1(n) \text{ for (1.1), and } p \leq p_2(n) \text{ for (1.2)},$$

where $p_1(n)$ is the larger root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$, and $p_2(n) = \frac{2}{n-1} + 1$. We are also interested in estimating the time when ”blow up” occurs. For
initial data of the form
\[ u(0, x) = \varepsilon f(x), \quad u_t(0, x) = \varepsilon g(x), \quad (1.4) \]
with constant \(0 < \varepsilon \leq 1\), smallness can be measured conveniently by the size of \(\varepsilon\) for fixed \(f, g\). We define "life span" \(T(\varepsilon)\) of the solutions of \((1.1)\) or \((1.2)\) to be the largest value such that solutions exist for \(x \in \Omega^c\), \(0 \leq t < T(\varepsilon)\).

For problem \((1.1)\), we consider compactly supported nonnegative data \((f, g) \in H^1(\Omega^c) \times L^2(\Omega^c)\), \(n \geq 3\) and satisfy
\[ f(x) \geq 0, \quad g(x) \geq 0 \quad \text{a.e.}, \quad f(x) = g(x) = 0, \quad \text{for } |x| > R, \quad \text{and } f(x) \neq 0. \quad (1.5) \]
We establish the following theorem for \((1.1)\):

**Theorem 1.1.** Let \((f, g) \in H^1(\Omega^c) \times L^2(\Omega^c)\) and satisfy \((1.5)\), \(\partial \Omega\) is smooth, and \(\Omega\) satisfies the exterior ball conditions, space dimensions \(n \geq 3\). Suppose that problem \((1.1)\) has a solution \((u, u_t) \in C([0, T), H^1(\Omega^c) \times L^2(\Omega^c))\) such that
\[ \text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + R\} \cap (\Omega^c \times R^+). \]
If \(1 < p < p_1(n)\), then \(T < \infty\), and there exists a positive constant \(A_1\) which is independent of \(\varepsilon\) such that
\[ T(\varepsilon) \leq A_1 \varepsilon^{- \frac{2p(p-1)}{2+(n+1)p-(n-1)p}}. \quad (1.6) \]

**Remark 1.1.** Exterior ball condition may not be necessary, but in certain point of our proof, we use strong maximum principle for the elliptic equation, so this condition is needed technically.

For problem \((1.2)\), we consider compactly supported nonnegative data \(f, g \in C^\infty_0(\Omega^c)\), \(n \geq 1\) and satisfy
\[ f(x) \geq 0, \quad g(x) \geq 0, \quad f(x) = g(x) = 0, \quad \text{for } |x| > R \quad \text{and} \quad f(x) \neq 0. \quad (1.7) \]
Similarly, we establish the following theorem for \((1.2)\):

**Theorem 1.2.** Let \(f, g\) are smooth functions with compact support \(f, g \in C^\infty_0(\Omega^c)\) and satisfy \((1.7)\), space dimensions \(n \geq 1\). Suppose that problem \((1.2)\) has a solution \((u, u_t) \in C([0, T), H^1(\Omega^c) \times L^q(\Omega^c))\), where \(q = \max(2, p)\) such that
\[ \text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + R\} \cap (\Omega^c \times R^+). \]
If \(p \leq p_2(n)\), then \(T < \infty\), moreover, we have the following estimates for the life span \(T(\varepsilon)\) of solutions of \((1.2)\):
(i) If \((n-1)(p-1) < 2\), then there exists a positive constant \(A_2\) which is independent of \(\varepsilon\) such that
\[ T(\varepsilon) \leq A_2 \varepsilon^{- \frac{2}{(n-1)(p-1)/2}}. \quad (1.8) \]
(ii) If \((n-1)(p-1) = 2\), then there exist a positive constant \(B_2\) which is independent of \(\varepsilon\) such that
\[ T(\varepsilon) \leq \exp(B_2 \varepsilon^{-(p-1)}). \quad (1.9) \]
The rest of the paper is arranged as follows. We state several preliminary propositions in Section 2, Section 3 is devoted to the blowup proof for our Theorem 1.1 and we prove the Theorem 1.2 in Section 4.

2 Preliminaries

To prove the main results in this paper, we will employ the following important ODE result:

Lemma 2.1. (see [19]) Let \( p > 1, \ a \geq 1, \ (p - 1)a > q - 2 \). If \( F \in C^2([0, T]) \) satisfies

1. \( F(t) \geq \delta(t + R)^a \),
2. \( \frac{d^2 F(t)}{dt^2} \geq k(t + R)^{-q}[F(t)]^p \),

with some positive constants \( \delta, \ k, \ \text{and} \ R \), then \( F(t) \) will blow up in finite time, \( T < \infty \). Furthermore, we have the following estimate for the life span \( T(\delta) \) of \( F(t) \):

\[
T(\delta) \leq \frac{c}{\delta^{\frac{(p-1)}{(p-1)a-q+2}}},
\]

(2.1)

where \( c \) is a positive constant depending on \( k \) and \( R \) but independent of \( \delta \).

Proof. For the proof of blow up result part see Sideris [19]. We only prove the estimate of the life span of \( F(t) \) as following:

Let us make a translation \( \tau = t\delta^{\frac{(p-1)}{(p-1)a-q+2}} \) and define

\[
H(\tau) = \delta^{\frac{(q-2)}{(p-1)a-q+2}} F(t) = \delta^{\frac{(q-2)}{(p-1)a-q+2}} F(\tau \delta^{\frac{(p-1)}{(p-1)a-q+2}}),
\]

then we have

\[
\begin{cases}
H(\tau) \geq (\delta^{\frac{(p-1)}{(p-1)a-q+2}} + \tau)^a, \\
H''(\tau) \geq c(\delta^{\frac{(p-1)}{(p-1)a-q+2}} + \tau)^{-q}H^p(\tau),
\end{cases}
\]

(2.2)

where \( c \) is a positive constant.

So when \( \delta \leq R^{\frac{(p-1)a-q+2}{p-1}} \), easy computation shows that

\[
\begin{cases}
H(\tau) \geq \tau^a, \\
H''(\tau) \geq c(R + \tau)^{-q}H^p(\tau).
\end{cases}
\]

(2.3)

So \( H(\tau) \) will blow up in finite time and the life span of \( F(t) \) satisfies (2.1). This completes the proof. \( \square \)

Lemma 2.2. There exists function \( \phi_0(x) \in C^2(\Omega^c) \), space dimensions \( n \geq 3 \), satisfying the following boundary value problem:

\[
\begin{cases}
\partial_i(a_{ij}\partial_j\phi_0(x)) = 0, \ \text{in} \ \Omega^c, \ n \geq 3, \\
\phi_0|_{\partial \Omega} = 0, \\
|x| \to \infty, \ \phi_0(x) \to 1.
\end{cases}
\]

(2.4)

Moreover, \( \phi_0(x) \) satisfies: for \( \forall \ x \in \Omega^c, \ 0 < \phi_0(x) < 1. \)
Proof. To solve \( \phi_0(x) \), let \( \tilde{\phi}_0 \) be solution for the following boundary value problem on exterior domain:

\[
\begin{aligned}
\partial_i (a_{ij} \partial_j \tilde{\phi}_0(x)) &= 0, \quad \text{in } \Omega^c, \quad n \geq 3, \\
\tilde{\phi}_0|_{\partial \Omega} &= -1, \\
|x| \to \infty, \quad &\tilde{\phi}_0(x) \to 0, 
\end{aligned}
\]

this problem is well-posed, it has unique solution \( \tilde{\phi}_0(x) \), and by maximum principle, we can easily obtain \(-1 < \tilde{\phi}_0(x) < 0\), for \( \forall x \in \Omega^c \), then we can easily check that \( \phi_0(x) = 1 + \tilde{\phi}_0(x) \) satisfy the boundary value problem \( (2.6) \). This proves the existence of \( \phi_0 \) in \((2.4)\) and satisfies \( 0 < \phi_0(x) < 1 \) for \( \forall x \in \Omega^c, \ n \geq 3 \). The proof is complete. \( \square \)

Similarly, we have the following:

**Lemma 2.3.** There exists a function \( \phi_1(x) \in C^2(\Omega^c) \), space dimensions \( n \geq 1 \), satisfying the following boundary value problem:

\[
\begin{aligned}
\partial_i (a_{ij} \partial_j \phi_1(x)) &= \phi_1, \quad \text{in } \Omega^c, \quad n \geq 1, \\
\phi_1|_{\partial \Omega} &= 0, \\
|x| \to \infty, \quad &\phi_1(x) \to \int_{S^{n-1}} e^{\varepsilon \omega} d\omega.
\end{aligned}
\]

Moreover, \( \phi_1(x) \) satisfies: there exists positive constant \( C_1 \), for \( \forall x \in \Omega^c, \ 0 < \phi_1(x) \leq C_1(1 + |x|)^{-n/2} \cdot e|x| \).

**Proof.** To solve \( \phi_1(x) \), let \( \tilde{\phi}_1 \) be solution for the following boundary value problem on exterior domain:

\[
\begin{aligned}
\partial_i (a_{ij} \partial_j \tilde{\phi}_1(x)) &= \tilde{\phi}_1(x) - w(x), \quad \text{in } \Omega^c, \quad n \geq 1, \\
\tilde{\phi}_1|_{\partial \Omega} &= -h(x)|_{\partial \Omega}, \\
|x| \to \infty, \quad &\tilde{\phi}_1(x) \to 0,
\end{aligned}
\]

where \( h(x) = \int_{S^{n-1}} e^{\varepsilon \omega} d\omega, \ w(x) = \partial_i ((a_{ij} - \delta_{ij}) \partial_j h) \), since the function \( h \) satisfies \( \Delta h = h \), so by the condition of \( a_{ij}(x) \), we get \( w(x) \in C^\infty(\Omega^c) \), so by the theory of second order elliptic partial differential equation, the problem \( (2.7) \) is well-posed, it has unique solution \( \tilde{\phi}_1(x) \), then we can easily check that \( \phi_1(x) = h(x) + \tilde{\phi}_1(x) \) satisfies the boundary value problem \( (2.6) \), this proves the existence of \( \phi_1 \) in \((2.6)\). To derive the estimate of \( \phi_1(x) \) in \( \Omega^c \), we rewrite the boundary value problem \( (2.6) \) as the following form:

\[
\begin{aligned}
-\partial_i (a_{ij} \partial_j \phi_1(x)) + \phi_1(x) &= 0, \quad \text{in } \Omega^c, \quad n \geq 1, \\
\phi_1|_{\partial \Omega} &= 0, \\
|x| \to \infty, \quad &\phi_1(x) \to h(x).
\end{aligned}
\]
So by maximum principle, we can easily get
\[ \phi_1(x) > 0, \quad \text{for } \forall \, x \in \Omega^c. \] (2.9)
Next we analyze \( \tilde{\phi}_1(x) \) in order to get the estimation of \( \phi_1(x) \), we will prove that \( \tilde{\phi}_1(x) \) is bounded by some positive constant \( C \), that is, \( |\tilde{\phi}_1(x)| \le C \) for \( \forall \, x \in \Omega^c \). Here and hereafter, we shall denote by \( C \) (or \( c \)) a positive constant in the estimates, and the meaning of \( C \) (or \( c \)) may change from line to line.

For this purpose, we rewrite problem (2.7) as follows:
\[
\begin{align*}
-\partial_t(a_{ij}\partial_j\tilde{\phi}_1(x)) + \tilde{\phi}_1(x) &= w(x), \quad \text{in } \Omega^c, \quad n \ge 1, \\
\tilde{\phi}_1|_{\partial\Omega} &= -h(x)|_{\partial\Omega}, \\
|x| \to \infty, \quad \tilde{\phi}_1(x) \to 0.
\end{align*}
\] (2.10)
For the purpose of employing the maximum principle, we denote \( C = \max_{x \in \partial\Omega^c}|h(x)| + \max_{x \in \Omega^c}|w(x)| > 0 \), because the function \( w(x) \) is compactly supported function in \( \Omega^c \), so the above expression \( C \) is well defined. By the maximum principle, we can get the upper bound of \( \tilde{\phi}_1(x) \) as follows:

We rewrite the equation of \( \tilde{\phi}_1(x) \) as following:
\[
\begin{align*}
-\partial_t(a_{ij}\partial_j(\tilde{\phi}_1(x) - C)) + (\tilde{\phi}_1(x) - C) &= w(x) - C \le 0, \quad \text{in } \Omega^c, \quad n \ge 1, \\
(\tilde{\phi}_1 - C)|_{\partial\Omega} &= (-h(x) - C)|_{\partial\Omega} \le 0, \\
|x| \to \infty, \quad (\tilde{\phi}_1(x) - C) \to -C \le 0.
\end{align*}
\] (2.11)
So we apply maximum principle to \( (\tilde{\phi}_1(x) - C) \), we can obtain for \( \forall \, x \in \Omega^c, \quad \tilde{\phi}_1(x) - C \le 0 \), that is, \( \tilde{\phi}_1(x) \le C \), in \( \Omega^c \).

In a similar way, we can get \( -\tilde{\phi}_1(x) \le C \), in \( \Omega^c \).

Thus we conclude that \( |\tilde{\phi}_1(x)| \le C \) for any \( x \in \Omega^c \).
Hence we have for \( \forall \, x \in \Omega^c \),
\[ \phi_1(x) = \tilde{\phi}_1(x) + h(x) \le C + h(x) \le C'h(x) \le C_1(1 + |x|)^{-\left(n-1\right)/2 \cdot e^{[x]}}. \] (2.12)
This together with (2.10) implies that \( \phi_1(x) \) satisfies
\[ 0 < \phi_1(x) \le C_1(1 + |x|)^{-\left(n-1\right)/2 \cdot e^{[x]}}, \quad \text{in } \Omega^c, \quad n \ge 1. \] (2.13)
This proves Lemma 2.3. \( \square \)

In order to describe the following lemmas, we define the following test function
\[ \psi_1(x,t) = \phi_1(x)e^{-t}, \quad \forall \, x \in \Omega^c, \quad t \ge 0. \] (2.14)
We have

Lemma 2.4. Let \( p > 1 \). Assume that \( \phi_1 \) satisfy the conditions in Lemma 2.3, \( \psi_1(x,t) \) is as in (2.14). Then for \( \forall \, t \ge 0 \),
\[ \int_{\Omega^c \cap \{|x| \le t+R\}} \left|\psi_1(x,t)\right|^{p/(p-1)} \, dx \le C(t + R)^{n-1-(n-1)p'/2}, \]
where \( p' = p/(p - 1) \) and \( C \) is a positive constant.
Proof. Let \( I(t) \) be the integral in Lemma 2.4 by the property of \( \phi_1(x) \), we have

\[
I(t) = \int_{\Omega \cap \{ |x| \leq t + R \}} \left[ \psi_1(x, t) \right]^{p/(p-1)} \, dx = \int_{\Omega \cap \{ |x| \leq t + R \}} \left[ \phi_1(x) e^{-t} \right]^{p/(p-1)} \, dx
\]

\[
\leq \int_{\Omega \cap \{ |x| \leq t + R \}} \left[ C_1 (1 + |x|)^{-(n-1)/2} \cdot e^{\frac{1}{p} |x|} \right]^{p/(p-1)} \cdot e^{-tp'} \, dx
\]

\[
\leq \int_{\{ |x| \leq t + R \}} \left[ C_1 (1 + |x|)^{-(n-1)/2} \cdot e^{\frac{1}{p} |x|} \right]^{p/(p-1)} \cdot e^{-tp'} \, dx
\]

\[
= \text{area}(S^{n-1}) C_1^{p/(p-1)} \int_0^{t+R} (1 + r)^{-(n-1)/2} \cdot e^{r^p} \cdot e^{\frac{1}{p} r^p} \, dr,
\]

where \( p' = p/(p-1) \) and \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). It is sufficient to show that

\[
I(t) \leq C e^{-tp'} \int_0^{t+R} (1 + r)^{n-1-(n-1)p'/2} \cdot e^{rp} \, dr \leq C(t + R)^{n-1-(n-1)p'/2}.
\]

(2.16)

This estimate is evident after splitting the last integral into two parts, that is,

\[
\int_0^{t+R} (1 + r)^{n-1-(n-1)p'/2} \cdot e^{rp} \, dr = \left[ \int_0^{(t+R)/2} + \int_{(t+R)/2}^{t+R} \right] (1 + r)^{n-1-(n-1)p'/2} \cdot e^{rp} \, dr.
\]

(2.17)

\[
\int_0^{(t+R)/2} (1 + r)^{n-1-(n-1)p'/2} \cdot e^{rp} \, dr \leq (1 + t + R)^qn \int_0^{(t+R)/2} e^{rp} \, dr
\]

\[
= (1 + t + R)^qn \cdot \frac{1}{p} \left( e^{p'((t+R)/2)} - 1 \right)
\]

\[
\leq (1 + t + R)^qn \cdot \frac{1}{p} e^{p'((t+R)/2)} = \frac{e^{p'R/2}}{p'} (1 + t + R)^n e^{p't/2},
\]

where \( q_1 = \max(0, n - 1 - (n - 1)p'/2) \), and

\[
\int_{(t+R)/2}^{t+R} (1 + r)^{n-1-(n-1)p'/2} \cdot e^{rp} \, dr \leq 2^{-q_2} (1 + t + R)^{n-1-(n-1)p'/2} \int_{(t+R)/2}^{t+R} e^{rp} \, dr
\]

\[
= 2^{-q_2} (1 + t + R)^{n-1-(n-1)p'/2} \cdot \frac{1}{p'} \left( e^{p'((t+R)/2)} - e^{p'((t+R)/2)} \right)
\]

\[
\leq \frac{2^{-q_2} e^{p'R}}{p'} \cdot (1 + t + R)^{n-1-(n-1)p'/2} e^{p't},
\]

where \( q_2 = \min(0, n - 1 - (n - 1)p'/2) \).

This proves Lemma 2.4. \qed
Lemma 2.5. Let $p > 1$. Assume that $\phi_0$ and $\phi_1$ satisfy the conditions in Lemma 2.3 and Lemma 2.4, respectively. $\psi(x,t)$ is as in (2.14), $\partial \Omega$ and $\Omega$ satisfies the conditions in Theorem 1.1. Then for $\forall \ t \geq 0$, 

\[
\int_{|x| \leq t+R} |\phi_0(x)|^{-1/(p-1)} \cdot [\psi_1(x,t)]^{p/(p-1)} \, dx \leq C(t + R)^{n-1-(n-1)p'/2}, \tag{2.18}
\]

where $p' = p/(p - 1)$ and $C$ is a positive constant.

Proof. To estimate the integral in Lemma 2.5, we split it into two parts as follows

\[
\int_{\Omega \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x,t)]^{p/(p-1)} \, dx
\]

\[
= \int_{\Omega \cap B_R} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x,t)]^{p/(p-1)} \, dx + \int_{B_R \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x,t)]^{p/(p-1)} \, dx
\]

\[
= I_1(t) + I_2(t). \tag{2.19}
\]

We will estimate $I_1(t)$ and $I_2(t)$ separately.

First let us estimate $I_2(t)$. Since for $\forall \ x \in \Omega^c, \ 0 < \phi_0(x) < 1$, we remark that there exists a constant $c \in (0,1)$, such that when $x \in B_R^c \cap \{|x| \leq t + R\}$, $\phi_0(x) \geq c$. By Lemma 2.4 we have

\[
I_2(t) = \int_{B_R^c \cap \{|x| \leq t+R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x,t)]^{p/(p-1)} \, dx
\]

\[
\leq \int_{B_R^c \cap \{|x| \leq t+R\}} c^{-1/(p-1)} \cdot [\psi_1(x,t)]^{p/(p-1)} \, dx
\]

\[
\leq c^{-1/(p-1)} \int_{\Omega \cap \{|x| \leq t+R\}} [\psi_1(x,t)]^{p/(p-1)} \, dx \tag{2.20}
\]

\[
\leq c^{-1/(p-1)} C(t + R)^{n-1-(n-1)p'/2},
\]

\[
= C_2(t + R)^{n-1-(n-1)p'/2}.
\]

Next we estimate $I_1(t)$. On the one hand, because of smoothness of $\phi_1(x)$, the first derivative of $\phi_1(x)$ is bounded in $\Omega^c \cap B_R$, this lead to $\phi_1(x) = \phi_1(x) - \phi_1(y) \leq C_3 |x - y|$, for $\forall \ y \in \partial \Omega$. Therefore by taking the infimum on $\partial \Omega$ we have,

\[
|\phi_1(x)| \leq C_3 \text{dist}(x, \partial \Omega).
\]

On the other hand, $\phi_0(x)$ obeys the maximum (minimum) principle, and assumes its minimum value (zero) on $\partial \Omega$, since $\Omega$ satisfies exterior ball condition, so by [2] Hopf’s Lemma, p. 330, it follows that, for any $y \in \partial \Omega$, there exists an open ball $B \subset \Omega^c$ with $y \in \partial B$, then we have, for any $y \in \partial \Omega$,

\[
\frac{\partial \phi_0}{\partial \nu}(y) > 0, \tag{2.21}
\]
where $\nu$ is the inner unit normal to $\Omega^c$ at $y$. By the compactness of $\partial\Omega$, we have, for $\forall \ y \in \partial\Omega$, we have
\[
\frac{\partial \phi_0(y)}{\partial \nu} \geq C_* > 0,
\]
where $C_*$ is a positive constant.

For $\forall \ x \in \Omega^c \cap B_R$, there exists a $y \in \partial\Omega$ such that $(x - y) / \|x - y\| = \nu(y)$, i.e., $\frac{(x-y)}{|x-y|} = \nu(y)$, $\nu(y)$ is the outer unit normal to $\partial\Omega$ at $y$. So we have
\[
\partial \phi_0 \partial \nu (y) = \frac{\partial \phi_0}{\partial \nu} (y) \geq C^* > 0,
\]
where $C^*$ is a positive constant.

For $\forall \ x \in \Omega^c \cap B_R$, there exists a $y \in \partial\Omega$ such that $(x - y) / \|x - y\| = \nu(y)$, i.e., $(x-y) / \|x-y\| = \nu(y)$, $\nu(y)$ is the outer unit normal to $\partial\Omega$ at $y$. So we have
\[
\nabla \phi_0 (y) \cdot \frac{(x-y)}{|x-y|} = \frac{\partial \phi_0}{\partial \nu} (y) \geq C_* > 0,
\]
(2.22)

by the continuity, for $\forall \ x \in \Omega^c \cap B_R$ and $|x - y| \ll 1$, we know that $(sx + (1-s)y)$ is sufficiently close to $y$, so we can guarantee that
\[
\nabla \phi_0 (sx + (1-s)y) \cdot \frac{(x-y)}{|x-y|} \geq \frac{1}{2} C_* > 0.
\]

So there exists a positive constant $\varepsilon_0 > 0$ such that the above expression holds for $\forall \ x \in \Omega^c \cap B_R$ and $dist(x, \partial\Omega) < \varepsilon_0$.

We discuss in the following in two cases respectively:

One case is that for $\forall \ x \in \Omega^c \cap B_R$, and $dist(x, \partial\Omega) < \varepsilon_0$, we have
\[
|\phi_0(x)| \geq \frac{1}{2} C_* |x-y| \geq \frac{1}{2} C_* \text{dist}(x, \partial\Omega).
\]
(2.23)

The other case is that when $\forall \ x \in \Omega^c \cap B_R$, and $dist(x, \partial\Omega) \geq \varepsilon_0$, on the one hand, by the property of the function $\phi_0(x)$, there is a positive constant $c_1 \in (0, 1)$, such that
\[
\phi_0(x) \geq c_1 > 0,
\]
on the other hand, for $\forall \ x \in \Omega^c \cap B_R$, there definitely exists a positive constant $c' > 0$ such that $dist(x, \partial\Omega) \leq c'$, so we have
\[
\frac{\phi_0(x)}{dist(x, \partial\Omega)} \geq \frac{\phi_0(x)}{c'} \geq \frac{c_1}{c'} = c'' > 0, \quad \text{for } x \in \Omega^c \cap B_R, \text{ and } dist(x, \partial\Omega) \geq \varepsilon_0,
\]
(2.24)

that is
\[
\phi_0(x) \geq c'' \text{dist}(x, \partial\Omega),
\]
where $c''$ is a positive constant.

So combining the above two cases, for $\forall \ x \in \Omega^c \cap B_R$, we have
\[
\phi_0(x) \geq C^* \text{dist}(x, \partial\Omega),
\]
(2.25)
where $C_{**}$ is a positive constant. Hence, we have

$$I_1(t) = \int_{\Omega \cap B_R} \left[ \phi_0(x) \right]^{1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx$$

$$\leq \int_{\Omega \cap B_R} \left[ C_{**} \right]^{1/(p-1)} \left[ \text{dist}(x, \partial \Omega) \right]^{1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx$$

$$= \int_{\Omega \cap B_R} \left[ C_{**} \right]^{1/(p-1)} \left[ \text{dist}(x, \partial \Omega) \right]^{1/(p-1)} \cdot e^{-t p'} \left[ \phi_1(x) \right]^{p/(p-1)} dx$$

$$\leq \int_{\Omega \cap B_R} \left[ C_{**} \right]^{1/(p-1)} \left[ \text{dist}(x, \partial \Omega) \right]^{1/(p-1)} \cdot e^{-t p'} C_3^{p/(p-1)} \left[ \text{dist}(x, \partial \Omega) \right]^{p/(p-1)} dx$$

$$= e^{-t p'} \int_{\Omega \cap B_R} \left[ C_{**} \right]^{1/(p-1)} C_3^{p/(p-1)} \text{dist}(x, \partial \Omega) dx$$

$$= Ce^{-t p'} \int_{\Omega \cap B_R} \text{dist}(x, \partial \Omega) dx \leq Ce^{-t p'},$$

(2.25)

where $p' = p/(p - 1)$.

So we conclude that

$$\int_{\Omega \cap \{|x| \leq t + R\}} \left[ \phi_0(x) \right]^{1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx$$

$$= I_1(t) + I_2(t)$$

$$\leq C_4 e^{-t p'} + C_2 (t + R)^{n-1-(n-1)p'/2}$$

$$\leq C_5 (t + R)^{n-1-(n-1)p'/2},$$

(2.26)

where $C_5$ is a positive constant. The proof is complete. \(\square\)

**Lemma 2.6.** Let $p > 1$. Assume that $\phi_1$ satisfies the conditions in Lemma 2.3, $\psi_1(x, t)$ is as in (2.14). Then for all $t \geq 0$,

$$\int_{\Omega \cap \{|x| \leq t + R\}} \psi_1 dx \leq C(t + R)^{(n-1)/2},$$

(2.27)

where $C$ is a positive constant.

**Proof.** We note that for all $t \geq 0$, $\psi_1(x, t) = e^{-t \phi_1(x)}$, and since for all $x \in \Omega$, $0 < \phi_1(x) \leq C_1 (1 + |x|)^{-(n-1)/2} e^{||x||}$, we can get that there exists a positive constant $C_0$ such that $0 < \phi_1(x) \leq C_0 |x|^{-(n-1)/2} e^{||x||}$ for any $x \in \Omega$.\[11\]
So we have
\[
\int_{\Omega \cap \{|x| \leq t+R\}} \psi_1 dx = \int_{\Omega \cap \{|x| \leq t+R\}} e^{-t} \phi_1(x) dx
\]
\[
\leq \int_{\Omega \cap \{|x| \leq t+R\}} e^{-t} \cdot C_6 |x|^{-(n-1)/2} e^{|x|} dx
\]
\[
\leq \int_{\{|x| \leq t+R\}} e^{-t} \cdot C_6 |x|^{-(n-1)/2} e^{|x|} dx
\]
\[
= C_6 e^{-t} \int_0^{t+R} r^{-(n-1)/2} e^r \cdot r^{n-1} dr \int_{S^{n-1}} d\omega = C_7 e^{-t} \int_0^{t+R} e^r \cdot r^{(n-1)/2} dr
\]
\[
= C_7 e^{-t} e^t R(t + R)^{(n-1)/2} = C_7 e^R(t + R)^{(n-1)/2} = C_8 (t + R)^{(n-1)/2}.
\]
(2.28)

This completes the proof. \(\square\)

3 The proof of Theorem 1.1

Theorem 1.1 is a consequence of the lower bound and the blowup result about nonlinear differential inequalities in Lemma 2.1.

To outline the method, we will introduce the following functions:

\[
\begin{align*}
F_0(t) &= \int_{\Omega^c} u(x, t) \phi_0(x) dx, \\
F_1(t) &= \int_{\Omega^c} u(x, t) \psi_1(x, t) dx, \quad \psi_1(x, t) = \phi_1(x) e^{-t},
\end{align*}
\]
(3.1)

here \(\phi_0(x)\) and \(\phi_1(x)\) are as in Lemma 2.2 and Lemma 2.3. The assumptions on \(u\) imply that \(F_0(t)\) and \(F_1(t)\) are well-defined \(C^2\)-functions for all \(t\). By a standard procedure, we derive a nonlinear differential inequality for \(F_0(t)\). We also derive a linear differential inequality for \(F_1(t)\) and combine these to obtain a polynomial lower bound on \(F_0(t)\) as \(t \to \infty\).

To this end, we first establish the following lemma:

Lemma 3.1. Let \((f, g)\) satisfy (1.5). Suppose that problem (1.1) has a solution \((u, u_i) \in C([0, T), H^1(\Omega^c) \times L^2(\Omega^c))\), such that

\[
\text{supp}(u, u_i) \subset \{(x, t) : |x| \leq t + R\} \cap (\Omega^c \times R^+).
\]

Then for all \(t \geq 0\),

\[
F_1(t) \geq \frac{1}{2} (1 - e^{-2t}) \varepsilon \int_{\Omega^c} |f(x) + g(x)| \phi_1(x) dx + e^{-2t} \varepsilon \int_{\Omega^c} f(x) \phi_1(x) dx \geq \varepsilon c_0 > 0.
\]
Proof. We multiply (1.1) by the test function \( \psi \in C^2(\Omega^c \times R) \) and integrate over \( \Omega^c \times [0, t] \), then we use integration by parts and Lemma 2.3.

First,
\[
\int_0^t \int_{\Omega^c} \psi_t (a_{ij}(x) \partial_j u) - u_t + |u|^p \, dx \, d\tau = 0.
\]

By the expression \( \psi_1(x, t) = \phi_1(x) e^{-t} \) and Lemma 2.3 we have
\[
\int_0^t \int_{\Omega^c} \psi_1 \partial_i (a_{ij}(x) \partial_j u) \, dx \, d\tau = \int_0^t \left[ \int_{\partial \Omega^c} \psi_1 a_{ij}(x) \partial_j u \cdot n_i \, dS - \int_{\Omega^c} (a_{ij}(x) \partial_i \psi_1) \partial_j u \, dx \right] \, d\tau
\]
\[
= \int_0^t \left[ \int_{\partial \Omega^c} a_{ij}(x) \partial_i \psi_1 \cdot u \cdot n_j \, dS - \int_{\Omega^c} \partial_j (a_{ij}(x) \partial_i \psi_1) u \, dx \right] \, d\tau
\]
\[
= \int_0^t \int_{\Omega^c} \psi_1 u \, dx \, d\tau,
\]
by the expression of \( \psi_1(x, t) \), we get \( (\psi_1)_t = -\psi_1 \), \( (\psi_1)_u = \psi_1 \). So we have
\[
\int_0^t \int_{\Omega^c} \psi_1 u_t \, dx \, d\tau = \int_0^t \int_{\Omega^c} [\partial_r (\psi_1 u_r) - (\psi_1)_r u_r] \, dx \, d\tau
\]
\[
= \int_{\Omega^c} \psi_1 u_r \, dx |_{r=t} - \int_{\Omega^c} \psi_1 u_r \, dx |_{r=0} + \int_0^t \int_{\Omega^c} \psi_1 u_r \, dx \, d\tau
\]
\[
= \int_{\Omega^c} \psi_1 u_r \, dx |_{r=t} - \int_{\Omega^c} \psi_1 u_r \, dx |_{r=0} + \int_0^t \int_{\Omega^c} [\partial_r (\psi_1 u) - (\psi_1)_r u] \, dx \, d\tau
\]
\[
= \int_{\Omega^c} \psi_1 u_r \, dx |_{r=t} - \int_{\Omega^c} \psi_1 u_r \, dx |_{r=0} + \int_{\Omega^c} \psi_1 u \, dx |_{r=t} - \int_{\Omega^c} \psi_1 u \, dx |_{r=0} + \int_0^t \int_{\Omega^c} \psi_1 u \, dx \, d\tau
\]
\[
= \int_{\Omega^c} (\psi_1 u_t + u \psi_1) \, dx - \varepsilon \int_{\Omega^c} \phi_1(x) g(x) \, dx - \varepsilon \int_{\Omega^c} \phi_1(x) f(x) \, dx + \int_0^t \int_{\Omega^c} \psi_1 u \, dx \, d\tau.
\]

Combining the above equalities, we have
\[
\int_0^t \int_{\Omega^c} \psi_1 |u|^p \, dx \, d\tau = \int_{\Omega^c} (\psi_1 u_t + \psi_1 u) \, dx - \varepsilon \int_{\Omega^c} \phi_1(x) [f(x) + g(x)] \, dx.
\]

We notice that
\[
\int_{\Omega^c} (\psi_1 u_t + \psi_1 u) \, dx = \frac{d}{dt} \int_{\Omega^c} (\psi_1 u) \, dx - \int_{\Omega^c} (\psi_1)_t u \, dx + \int_{\Omega^c} \psi_1 u \, dx
\]
\[
= \frac{d}{dt} \int_{\Omega^c} (\psi_1 u) \, dx + 2 \int_{\Omega^c} \psi_1 u \, dx
\]
\[
= \frac{dF_1(t)}{dt} + 2F_1(t).
\]

So by \( \psi_1 > 0 \), we have
\[
\frac{dF_1(t)}{dt} + 2F_1(t) = \int_0^t \int_{\Omega^c} |u|^p \psi_1(x, \tau) \, dx \, d\tau + \varepsilon \int_{\Omega^c} \phi_1(x) [f(x) + g(x)] \, dx
\]
\[
\geq \varepsilon \int_{\Omega^c} [f(x) + g(x)] \phi_1(x) \, dx.
\]
Multiplying the above expression by $e^{2t}$, we obtain
\[
\frac{d(e^{2t}F_1(t))}{dt} \geq e^{2t} \varepsilon \int_{\Omega^c} [f(x) + g(x)] \phi_1(x)dx,
\]
and integrating the above differential inequality over $[0,t]$, we get
\[
e^{2t}F_1(t) - F_1(0) \geq \frac{1}{2}(e^{2t} - 1) \varepsilon \int_{\Omega^c} [f(x) + g(x)] \phi_1(x)dx.
\]
Observing $F_1(0) = \int_{\Omega^c} u(x,0)\psi_1(x,0)dx = \varepsilon \int_{\Omega^c} f(x)\phi_1(x)dx$. So, by the property of the function $f(x)$ and $\phi_1(x)$, we arrive at
\[
F_1(t) \geq \frac{1}{2}(1 - e^{-2t}) \varepsilon \int_{\Omega^c} [f(x) + g(x)] \phi_1(x)dx + e^{-2t} \varepsilon \int_{\Omega^c} f(x)\phi_1(x)dx \geq \varepsilon c_0 > 0.
\]
Thus we obtain the lower bound in Lemma 3.1.

Next we shall show that $F_0(t)$ satisfies the differential inequalities in Lemma 2.1 for suitable $a$, $q$. For this purpose, we multiply (1.1) by $\phi_0$ and integrate over $\Omega^c$. We note that for a fixed $t$, $u(\cdot, t) \in H_0^1(D_t)$ where $D_t$ is the support of $u(\cdot, t)$. Hence we can use integration by parts and Lemma 2.2.

First,
\[
\int_{\Omega^c} [\phi_0 \partial_i(a_{ij}(x) \partial_j u) - \phi_0 u_t + |u|^p \phi_0] dx = 0.
\]
Since
\[
\int_{\Omega^c} \phi_0 \partial_i(a_{ij}(x) \partial_j u) dx = \int_{\partial \Omega} \phi_0 a_{ij}(x) \partial_i u \cdot n_i dS - \int_{\Omega^c} \partial_i \phi_0 a_{ij}(x) \partial_j u dx
\]
\[
= - \left( \int_{\partial \Omega} a_{ij}(x) \partial_i \phi_0 u \cdot n_j dS - \int_{\Omega^c} \partial_j (a_{ij}(x) \partial_i \phi_0) u dx \right)
\]
\[
= \int_{\Omega^c} \partial_j (a_{ij}(x) \partial_i \phi_0) u dx = 0.
\]
So we get
\[
\frac{d^2 F_0(t)}{dt^2} = \int_{\Omega^c} |u(x,t)|^p \phi_0(x)dx.
\]
Estimating the right side of the above equality by the H"{o}lder inequality, we have
\[
\left| \int_{\Omega^c} u(x,t)\phi_0(x)dx \right|
\]
\[
= \left| \int_{\Omega^c \cap \{|x| \leq t+R\}} u(x,t)[\phi_0(x)]^{1/p}[\phi_0(x)]^{(p-1)/p} dx \right|
\]
\[
\leq \left( \int_{\Omega^c \cap \{|x| \leq t+R\}} |u(x,t)|^{1/p}^p dx \right)^{1/p} \cdot \left( \int_{\Omega^c \cap \{|x| \leq t+R\}} ||\phi_0(x)||^{(p-1)/p}^{p'} dx \right)^{1/p'}
\]

14
where \( p' = p/(p - 1) \), this implies that
\[
\left| \int_{\Omega^c} u(x, t) \phi_0(x) \, dx \right|^p \leq \left( \int_{\{ |x| \leq t + R \} \cap \Omega^c} |u(x, t)|^p \phi_0(x) \, dx \right) \left( \int_{\{ |x| \leq t + R \} \cap \Omega^c} \phi_0(x) \, dx \right)^{p-1} \\
\leq \left( \int_{\Omega^c} |u(x, t)|^p \phi_0(x) \, dx \right) \left( \int_{\{ |x| \leq t + R \} \cap \Omega^c} \phi_0(x) \, dx \right)^{p-1}.
\]

So we have
\[
\int_{\Omega^c} |u(x, t)|^p \phi_0(x) \, dx \geq \frac{\int_{\Omega^c} u(x, t) \phi_0(x) \, dx}{\left( \int_{\{ |x| \leq t + R \} \cap \Omega^c} \phi_0(x) \, dx \right)^{p-1}}.
\]

By Lemma 2.2, we have
\[
\int_{\{ |x| \leq t + R \} \cap \Omega^c} \phi_0(x) \, dx \leq \int_{\{ |x| \leq t + R \}} 1 \, dx \leq Vol\{ x : |x| \leq t + R \} = Vol(\mathbb{B}^n)(t + R)^n.
\]

Therefore
\[
\int_{\Omega^c} |u(x, t)|^p \phi_0(x) \, dx \geq \frac{\left| \int_{\Omega^c} u(x, t) \phi_0(x) \, dx \right|^p}{Vol(\mathbb{B}^n)(t + R)^n} \geq \frac{|F_0(t)|^p}{Vol(\mathbb{B}^n)^{p-1} \cdot (t + R)^{n(p-1)}}.
\]

Thus
\[
\frac{d^2 F_0(t)}{dt^2} \geq k(t + R)^{-n(p-1)} \cdot |F_0(t)|^p,
\]
where \( k = [Vol(\mathbb{B}^n)]^{-p} > 0 \). So \( F_0 \) satisfies the differential inequality (2) in Lemma 2.1.

To show that \( F_0 \) admits the lower bound (1) in Lemma 2.1, we relate \( d^2 F_0(t)/dt^2 \) to \( F_1 \) using again (1.1) and the Hölder inequality.

Since
\[
\left| \int_{\Omega^c} u(x, t) \psi_1(x, t) \, dx \right|
\]
\[
= \left| \int_{\Omega^c \cap \{ |x| \leq t + R \}} u(x, t) \left[ \phi_0(x) \right]^{1/p} \cdot \left[ \phi_0(x) \right]^{-1/p} \cdot \psi_1(x, t) \, dx \right|
\]
\[
\leq \left( \int_{\Omega^c \cap \{ |x| \leq t + R \}} |u(x, t)|^{p/p'} \cdot \phi_0(x) \, dx \right)^{1/p'} \cdot \left( \int_{\Omega^c \cap \{ |x| \leq t + R \}} \left[ \phi_0(x) \right]^{-1/p} \cdot \psi_1(x, t) \, dx \right)^{1/p'}
\]
\[
\leq \left( \int_{\Omega^c} |u(x, t)|^{p/p'} \cdot \phi_0(x) \, dx \right)^{1/p'} \cdot \left( \int_{\Omega^c \cap \{ |x| \leq t + R \}} \left[ \phi_0(x) \right]^{(p-1)/p} \cdot \left[ \psi_1(x, t) \right]^{p/(p-1)} \, dx \right)^{(p-1)/p},
\]
where \( p' = p/(p - 1) \), this implies that
\[
\left| \int_{\Omega^c} u(x, t) \psi_1(x, t) \, dx \right|^p
\]
\[
\leq \left( \int_{\Omega^c} |u(x, t)|^{p/p'} \cdot \phi_0(x) \, dx \right) \cdot \left( \int_{\Omega^c \cap \{ |x| \leq t + R \}} \left[ \phi_0(x) \right]^{(p-1)/p} \cdot \left[ \psi_1(x, t) \right]^{p/(p-1)} \, dx \right)^{p-1}.
\]

15
By \((3.1)\), the above becomes

\[
\frac{d^2 F_0(t)}{dt^2} = \int_{\Omega^c} |u(x, t)|^p \phi_0(x) dx \geq \frac{\int_{\Omega^c} u(x, t) \psi_1(x, t) dx}{\left( \int_{\Omega^c \cap \{|x| \leq t + R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx \right)^{p-1}} = \frac{|F_1(t)|^p}{\left( \int_{\Omega^c \cap \{|x| \leq t + R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx \right)^{p-1}}.
\]

In the following, we will estimate the numerator and denominator, respectively, and provide a lower bound on \(d^2 F_0/dt^2\).

By the Lemma \(3.1\), we have

\[
|F_1(t)|^p \geq \varepsilon^p (c_0)^p > 0. \tag{3.4}
\]

Also, by the Lemma \(2.5\) we know that

\[
\int_{\Omega^c \cap \{|x| \leq t + R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx \leq C_5(t + R)^{n-1-(n-1)p'/2}, \tag{3.5}
\]

where \(p' = p/(p - 1)\) and \(C_5\) is a positive constant.

So by combining (3.4) and (3.5), we obtain

\[
\frac{d^2 F_0(t)}{dt^2} \geq \frac{|F_1(t)|^p}{\left( \int_{\Omega^c \cap \{|x| \leq t + R\}} [\phi_0(x)]^{-1/(p-1)} \cdot [\psi_1(x, t)]^{p/(p-1)} dx \right)^{p-1}} \geq \frac{\varepsilon^{p(0)}}{[C_5(t + R)^{n-1-(n-1)p'/2}]^{p-1}} \geq L(t + R)^{-(n-1)(p/2-1)},
\]

where \(L = \varepsilon^{p_0} c_0 C_5^{-(p-1)} > 0\). Integrating twice, we have the final estimate

\[
F_0(t) \geq \delta(t + R)^{n+1-(n-1)p/2} + \frac{dF_0(0)}{dt} t + F_0(0),
\]

with constant

\[
\delta = \frac{L}{|n - \frac{p}{2}(n - 1)p + 1||n - \frac{1}{2}(n - 1)p|} = \frac{\varepsilon^{p_0} p C_5^{-(p-1)}}{|n - \frac{p}{2}(n - 1)p + 1||n - \frac{1}{2}(n - 1)p|} > 0.
\]

When \(1 < p < p_1(n)\), it is easy to check that \(n + 1 - (n-1)p/2 > 1\). Hence the following estimate is valid when \(t\) is sufficiently large:

\[
F_0(t) \geq \frac{1}{2} \delta(t + R)^{n+1-(n-1)p/2}. \tag{3.6}
\]
Estimates (3.3) together with (3.6) and Lemma 2.1 with parameters
\[ a \equiv n + 1 - (n - 1)p/2, \quad \text{and} \quad q \equiv n(p - 1) \]
imply Theorem 1.1 for all exponents \( p \) such that
\[ (p - 1)(n + 1 - (n - 1)p/2) > n(p - 1) - 2 \quad \text{and} \quad p > 1. \]
It is easy to see that the solution set is \( p \in (1, p_1) \), so by Lemma 2.1, all solutions of problem (1.1) with nontrivial nonnegative initial values must blow up in finite time.

Also, recall from Lemma 2.1, we have the following estimate for the life span \( T(\varepsilon) \) of solutions of (1.1) as follows:
\[
T(\varepsilon) \leq c(\frac{1}{2})^\frac{(p-1)(p-1)}{2p + (p-1)^2},
\]
(3.7)
where \( A_1 \) is a positive constant which is independent of \( \varepsilon \). The proof of Theorem 1.1 is complete.

4 The proof of Theorem 1.2

By the expression \( \psi_1(x, t) = e^{-t}\phi_1(x) \geq 0 \), we have \( (\psi_1)_t = -\psi_1 \), and \( \partial_i(a_{ij}\partial_j\psi_1(x, t)) = \psi_1(x, t), \quad \text{in} \quad \Omega^c \times (0, +\infty). \) So \( \psi_1 \) satisfies

\[
\begin{align*}
\partial_i(a_{ij}\partial_j\psi_1(x, t)) &= \psi_1(x, t), \quad \text{in} \quad \Omega^c \times (0, +\infty), \\
\psi_1|_{\partial\Omega \times (0, +\infty)} &= 0, \\
|x| \to \infty, \quad \psi_1(x, t) &\to e^{-t} \int_{S^{n-1}} e^{x\cdot\omega}d\omega, \quad \text{for} \quad t \geq 0. 
\end{align*}
\]
(4.1)

We multiply (1.2) by function \( \psi_1 \), and integrate over \( \Omega^c \), then we use integration by parts and Lemma 2.3. First,
\[
\int_{\Omega^c} \psi_1(\partial_t u - \partial_i(a_{ij}(x)\partial_j u))dx = \int_{\Omega^c} \psi_1|u_t|^pdx.
\]
Note that for a fixed \( t \), \( u(\cdot, t) \in H^1_0(D_t) \), where \( D_t \) is the support of \( u(\cdot, t) \). Hence by
Combining the above two identities, we conclude

\[
\int_{\Omega} \psi_1 \partial_i(a_{ij}(x)\partial_j u)dx = \int_{\Omega} (\partial_i[\psi_1a_{ij}(x)\partial_j u] - \partial_t \psi_1a_{ij}(x)\partial_j u)dx = \int_{\partial\Omega} \psi_1a_{ij}(x)\partial_j u \cdot n_i dS - \int_{\Omega} (a_{ij}(x)\partial_t \psi_1) \cdot \partial_j udx
\]

Adding two expressions (4.2) and (4.6), we have

\[
\int_{\Omega} \psi_1 a_{ij}(x)\partial_j \psi_1 \cdot u \cdot n_j dS + \int_{\Omega} \partial_j(a_{ij}(x)\partial_t \psi_1)udx = \int_{\Omega} \partial_j(a_{ij}(x)\partial_t \psi_1) \cdot udx = \int_{\Omega} \psi_1 \cdot udx.
\]

So we have

\[
\int_{\Omega} \psi_1 u_t - \int_{\Omega} \psi_1 \cdot u dx = \int_{\Omega} \psi_1 \cdot |u_t|^p dx. \tag{4.2}
\]

Notice that

\[
\frac{d}{dt} \int_{\Omega} \psi_1 u_t dx = \int_{\Omega} (\psi_1 \cdot u_t - u_t \psi_1) dx, \tag{4.3}
\]

Adding up the above two expression, we obtain the following

\[
\frac{d}{dt} \int_{\Omega} (\psi_1 u_t + \psi_1 u) dx = \int_{\Omega} (\psi_1 \cdot u_t - u \cdot \psi_1) dx = \int_{\Omega} \psi_1 \cdot |u_t|^p dx. \tag{4.5}
\]

So we have

\[
\int_{\Omega} (\psi_1 u_t + \psi_1 u) dx = \int_{\Omega} (\psi_1 u_t + \psi_1 u) dx|_{t=0} + \int_{0}^{t} \int_{\Omega} \psi_1 \cdot |u_t|^p dx d\tau
\]

\[
= \int_{\Omega} \varepsilon \phi_1(x)[f(x) + g(x)] dx + \int_{0}^{t} \int_{\Omega} \psi_1 \cdot |u_t|^p dx d\tau \tag{4.6}
\]

Adding two expressions (4.2) and (4.6), we have

\[
\int_{\Omega} (\psi_1 u_t + \psi_1 u_t) dx \geq \int_{\Omega} \psi_1 \cdot |u_t|^p dx + \int_{0}^{t} \int_{\Omega} \psi_1 \cdot |u_t|^p dx d\tau + \varepsilon \int_{\Omega} \phi_1(x)g(x) dx. \tag{4.7}
\]

Also, we know that

\[
\frac{d}{dt} \int_{\Omega} \psi_1 u_t dx + 2 \int_{\Omega} \psi_1 \cdot u_t dx = \int_{\Omega} [\psi_1 u_t + u_t(\psi_1)_t + 2\psi_1 u_t] dx \tag{4.8}
\]

\[
= \int_{\Omega} (\psi_1 u_t + \psi_1 u_t) dx.
\]
So we have

\[
\frac{d}{dt} \int_{\Omega^c} \psi_1 u_t \, dx + 2 \int_{\Omega^c} \psi_1 \cdot u_t \, dx \geq \int_{\Omega^c} \psi_1 \cdot |u_t|^p \, dx + \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau + \varepsilon \int_{\Omega^c} \phi_1(x)g(x) \, dx. 
\] (4.9)

To show the blowup property, we define the following auxiliary function

\[
G_0(t) = \int_{\Omega^c} \psi_1 u_t \, dx - \frac{1}{2} \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau - \frac{\varepsilon}{2} \int_{\Omega^c} \phi_1(x)g(x) \, dx. 
\] (4.10)

We note that, when \( t = 0, \)

\[
G_0(0) = \varepsilon \int_{\Omega^c} \phi_1(x)g(x) \, dx - \frac{\varepsilon}{2} \int_{\Omega^c} \phi_1(x)g(x) \, dx = \frac{\varepsilon}{2} \int_{\Omega^c} \phi_1(x)g(x) \, dx \geq 0, 
\]

and we have

\[
\frac{d}{dt} G_0(t) = \int_{\Omega^c} (\psi_1 u_{tt} - u_t \psi_1) \, dx - \frac{1}{2} \int_{\Omega^c} \psi_1 \cdot |u_t|^p \, dx. 
\] (4.11)

Hence, we conclude that

\[
\frac{d}{dt} G_0(t) + 2G_0(t) \\
= \int_{\Omega^c} (\psi_1 u_{tt} + u_t \psi_1) \, dx - \frac{1}{2} \int_{\Omega^c} \psi_1 \cdot |u_t|^p \, dx - \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau - \varepsilon \int_{\Omega^c} \phi_1(x)g(x) \, dx \\
\geq \int_{\Omega^c} \psi_1 \cdot |u_t|^p \, dx + \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau + \varepsilon \int_{\Omega^c} \phi_1(x)g(x) \, dx - \frac{1}{2} \int_{\Omega^c} \psi_1 |u_t|^p \, dx \\
- \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau - \varepsilon \int_{\Omega^c} \phi_1(x)g(x) \, dx \\
= \frac{1}{2} \int_{\Omega^c} \psi_1 \cdot |u_t|^p \, dx \geq 0. 
\] (4.12)

Multiplying the above differential inequality by \( e^{2t} \), we get the following expression

\[
\frac{d}{dt} (e^{2t} G_0(t)) \geq 0. 
\]

So for \( \forall \ t \geq 0 \), we have \( e^{2t} G_0(t) \geq G_0(0) \), that is \( G_0(t) \geq e^{-2t} G_0(0) \geq 0 \).

By (4.10), we have for \( \forall \ t \geq 0, \)

\[
\int_{\Omega^c} \psi_1 u_t \, dx \geq \frac{1}{2} \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau + \frac{\varepsilon}{2} \int_{\Omega^c} \phi_1(x)g(x) \, dx. 
\] (4.13)

Let

\[
F(t) = \frac{1}{2} \int_{0}^{t} \int_{\Omega^c} \psi_1 \cdot |u_{\tau}|^p \, dx \, d\tau + \frac{\varepsilon}{2} \int_{\Omega^c} \phi_1(x)g(x) \, dx, \quad t \geq 0. 
\] (4.14)

Then we have

\[
\int_{\Omega^c} \psi_1 u_t \, dx \geq F(t), \quad \forall \ t \geq 0. 
\] (4.15)

Next we only need to prove that \( F(t) \) blow up.

From the expression of \( F(t) \), we get for \( \forall \ t \geq 0, \ F(t) \geq 0, \) and \( F'(t) = \frac{1}{2} \int_{\Omega^c} \psi_1 \cdot |u_t|^p \, dx. \)

Estimating the right side of \( F'(t) \) by the Holder inequality, we have
\[
\int_{\Omega^c} |u_t(x,t)|^p \psi_1 dx \geq \left( \int_{\Omega^c} u_t(x,t) \psi_1 dx \right)^p \left( \int_{\Omega \cap \{ |x| \leq t+R \}} \psi_1 dx \right)^{p-1},
\]

By Lemma 2.6, we know that for \( \forall \ t \geq 0, \)
\[
\int_{\Omega \cap \{ |x| \leq t+R \}} \psi_1 dx \leq C_8(t+R)^{(n-1)/2},
\]
where \( C_8 \) is a positive constant.
Therefore we conclude that
\[
F'(t) = \frac{1}{2} \int_{\Omega^c} \psi_1 \cdot |u_t|^p dx \geq \frac{1}{2} \left( \int_{\Omega^c} u_t(x,t) \psi_1 dx \right)^p \left( \int_{\Omega \cap \{ |x| \leq t+R \}} \psi_1 dx \right)^{p-1}
\]
\[
\geq C_9 \left( \int_{\Omega^c} u_t(x,t) \psi_1 dx \right)^p \geq C_9 \frac{|F(t)|^p}{(t+R)^{(n-1)(p-1)/2}}.
\]

By the property of Ricatti equation, we know that when \((n-1)(p-1)/2 \leq 1,\)
the solution of the initial-boundary value problem (1.2) blow up.
In detail, let
\[
M = \frac{1}{2} \int_{\Omega^c} \phi_1(x) g(x) dx.
\]
Then \( F(t) \) satisfies the following problem
\[
\begin{align*}
F'(t) &\geq C_9 \frac{|F(t)|^p}{(t+R)^{(n-1)(p-1)/2}}, \\
F(0) &= M \varepsilon.
\end{align*}
\]
(4.18)
Now we introduce a function \( v(t) \) satisfying the following Ricatti equation
\[
\begin{align*}
v'(t) &= C_9 \frac{|v(t)|^p}{(t+R)^{(n-1)(p-1)/2}}, \\
v(0) &= M \varepsilon.
\end{align*}
\]
(4.19)
So the life span of \( F \) is less than that of \( v \) which will be the upper bound of \( T(\varepsilon). \)
Thus, in the case \((n-1)(p-1) < 2,\) integrating (4.19), we get
\[
v(t) = \left( (M \varepsilon)^{-(p-1)} + C'(t+R)^{-(n-1)(p-1)/2} - C'(t+R)^{1-(n-1)(p-1)/2} \right)^{-\frac{1}{p-1}},
\]
(4.20)
where
\[
C' = \frac{C_9(p-1)}{1 - (n-1)(p-1)/2}.
\]
Thus
\[
T(\varepsilon) \leq A_2 \varepsilon^{\frac{p-1}{1-(n-1)(p-1)/2}},
\]
20
where \( A_2 \) is a positive constant which is independent of \( \varepsilon \).

When \((n-1)(p-1)=2\), integrating (4.19), we get

\[
v(t) = \left[ (M\varepsilon)^{-(p-1)} - C'' \ln \left( \frac{t+R}{R} \right) \right]^{\frac{1}{(p-1)}},
\]

(4.21)

where

\[
C'' = C_9(p-1),
\]

\[
T(\varepsilon) \leq \exp\left(B_2 \varepsilon^{-(p-1)}\right),
\]

where \( B_2 \) is a positive constant which is independent of \( \varepsilon \). This ends the proof of Theorem 1.2.

Acknowledgments.

This work is supported by the National Natural Science Foundation of China (10728101), the 973 Project of the Ministry of the Science and Technology of China, the Doctoral Foundation of the Ministry of Education of China and the ‘111’ Project (B08018) and SGST 09DZ2272900.

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