Limit load in the problem on penetration of wedge-shaped tool in anisotropic medium

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Abstract. The mathematical model of plastic deformation in an initially anisotropic medium assumes that principal directions of strain in elasticity remain principal directions of strain in plasticity and fracture as well. The modeling considers penetration of a bit represented by a cylindrical tool with a wedge-shaped tip. The maximal penetration depth of the bit is calculated at the preset initial rate of penetration. The influence of the anisotropy of the medium and the bit parameters on the bit penetration depth is studied.

1. Introduction
Many studies address the problem of penetration of a stiff toll in a plastic medium [1–10]. Such problems allow determining indentation hardness of materials (Brinnel test), stability of foundations for various structures, and penetration depths of hammering tools in soil and rocks. In the meanwhile, these studies analyze plasticity of media which are initially isotropic than anisotropic. This fact is governed by the advancement of the plasticity theory without any relationship with elasticity. The plastic theory arose from observations over the surfaces of plastically deformable bodies. Plastic deformation generated slide lines on the body surfaces, similar to planes of maximal shear stresses (Chernov–Luders lines). The Tresca criterion was invented in associations with those planes, then the plasticity theories were originated to reflect the gradient influence of plastic strains relative to the maximal shear sites. Von Mises condition was the approximation of the Tresca condition for the initially anisotropic media and gave rise to the quadratic condition for the initially isotropic media that time (the early 20th century).

For the first time, the randomness of the mentioned assumption and the disconnectedness of the plasticity and elasticity were pointed out by the authors of [11, 12]. These researchers emphasized that the principal directions of strains in elasticity remain the principal directions of strains in plasticity. This fact is particularly pronounced in the theories of plasticity and creep of initially isotropic media where the principal directions are the spherical tensor and deviator. On the other hand, the ellipsoids based on the elastic constants and ultimate strength limits of initially anisotropic media are similar [13].

2. Plastic model of initially anisotropic medium
We proceed from the solution of problem on indentation of a press-tool in an initially anisotropic medium [11, 12]. Let the deformation conditions be plane. The Hooke law in the coordinate system xOy is given by:
here, \( a_{ij} \) are the yields \( a_{ij} > 0 \). The tensor basis is selected to be:

\[
T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

This basis is orthonormal and orthogonal \([14]\).

In basis (2) the Hooke law is presented in the form of a matrix:

\[
\begin{pmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & 0 \\
a_{12} & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix},
\]

where \( \Omega_i = (T_x, T_i), \ S_i = (T_x, T_i), \ \Omega_1 = \epsilon_{xy}, \ \Omega_2 = \epsilon_{yx}, \ \Omega_3 = \sqrt{2} \epsilon_{xy}, \ldots \).

With respect to (3), we transform basis (2) so that (3) is arranged diagonally. To this end, we need to solve the equation \( \lambda E \), where \( A \) is the matrix determined from (3). The roots of this equation are:

\[
\lambda_1 = \frac{a_{11} + a_{22}}{2} + \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2}, \quad \lambda_2 = \frac{a_{11} + a_{22}}{2} - \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2}, \quad \lambda_3 = a_{33}.
\]

For roots (4) we find principal vectors:

\[
\tilde{b}_1 = (\cos \beta, -\sin \beta, 0), \quad \tilde{b}_2 = (-\sin \beta, \cos \beta, 0), \quad \tilde{b}_3 = (0, 0, 1).
\]

Finally, principal tensors (1) have the form:

\[
\tilde{T}_1 = \cos \beta T_1 - \sin \beta T_2, \quad \tilde{T}_2 = \sin \beta T_1 + \cos \beta T_2, \quad \tilde{T}_3 = T_3
\]

or

\[
\tilde{T}_1 = \begin{pmatrix} \cos \beta & 0 \\ 0 & -\sin \beta \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} \sin \beta & 0 \\ 0 & \cos \beta \end{pmatrix}, \quad \tilde{T}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

For the plasticity theory, we need to analyze multiplicity of roots (4). For the initially isotropic medium, \( \lambda_1 = \lambda_3 \). Let this case be taken as the basis. For this case, the plasticity condition is written as [12]:

\[
S_1^2 + S_3^2 = k^2 \quad \text{or} \quad S_1 = k \cos \varphi, \quad S_3 = k \sin \varphi,
\]

where \( k \) is the elastic limit; \( \varphi \) is the polar angle in the plane of the variables \( S_1, S_3 \). In other words, we introduce the relations:

\[
\sigma_x \cos \beta - \sigma_y \sin \beta = k \cos \varphi, \quad \sqrt{2} \tau_{xy} = k \sin \varphi.
\]

Here:

\[
\tan 2\beta = \frac{2a_{12}}{(a_{11} - a_{22})}.
\]

The connection of stresses and strains along the ort \( \tilde{T}_2 \) is assumed as elastic, and plasticity develops in the plane formed by the orts \( \tilde{T}_1, \tilde{T}_3 \). Then, we disregard all elastic strains and place Eq. (5) in the equation of equilibrium:
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0.
\] (7)

For determining functions \(\varphi(x, y), \sigma_y(x, y)\), we obtain the system of equations:
\[
\begin{cases}
-\frac{k \sin \varphi}{\cos \beta} \frac{\partial \varphi}{\partial x} + \sqrt{2} \frac{\partial \sigma_y}{\partial x} + \tan \beta \frac{\partial \sigma_y}{\partial x} = 0, \\
\sqrt{2} \frac{\partial \sigma_y}{\partial x} + k \cos \varphi \frac{\partial \varphi}{\partial y} = 0.
\end{cases}
\] (8)

Characteristics (8) are found in the form:
\[
\begin{align*}
\left( \frac{dy}{dx} \right)_1 &= \frac{\tan \varphi + \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta}, \\
\left( \frac{dy}{dx} \right)_2 &= \frac{\tan \varphi - \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta}.
\end{align*}
\] (9)

The relations on these characteristics are:
\[
d \sigma_y \frac{dy}{dx} + k \cos \varphi \cot \beta \frac{d \varphi}{d \sigma_y} = 0
\]
or, with regard to (9):
\[
d \sigma_y - k \frac{2 \sin \beta}{\sin \varphi} \left[ \frac{\sin \varphi}{\sqrt{2} \sin \beta} \right] \left[ \frac{\sin \varphi + \sqrt{\sin 2 \beta + (1 - \sin 2 \beta) \sin^2 \varphi}}{1 - \sin^2 \varphi} \right] d \varphi = 0.
\] (10)

The integrals for (10) are:
\[
\sigma_y + k \left( \frac{\cos \varphi - \cos \varphi_0}{2 \sin \beta} \right) + k \frac{2 \sin \beta}{\sin \varphi} \int_{\varphi_0}^{\varphi} \left[ 1 - \sin^2 \left( \frac{\varphi - \pi}{2} \right) \right] d \varphi = C,
\] (11)

where \(N^2 = 1 - \sin 2 \beta\); \(C\) is the integration constant. For the problem solution, we need to find the spherically symmetric field.

Let the point \(A\) be the pole. In term of the polar coordinates \(\rho\) and \(\theta\), the coordinates of any point can be given by the formulas:
\[
y = y_A + \rho \sin \theta, \quad x = x_A + \rho \cos \theta.
\] (12)

Substitution of (12) in (9) yields:
\[
\begin{align*}
\frac{d \rho}{\sin \theta - \frac{\tan \varphi + \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta} \cos \theta} + \rho \left[ \frac{\cos \varphi + \frac{\tan \varphi + \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta} \sin \varphi}{\sin \theta} \right] d \theta &= 0, \\
\frac{d \rho}{\sin \theta - \frac{\tan \varphi - \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta} \cos \theta} + \rho \left[ \frac{\cos \varphi - \frac{\tan \varphi - \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta} \sin \varphi}{\sin \theta} \right] d \theta &= 0.
\end{align*}
\] (13)

Let in the second equation in (13), \(\theta = \text{const}\) and \(d \theta = 0\). One more condition is:
\[
\tan \theta = \frac{\frac{\tan \varphi - \sqrt{\tan^2 \varphi + \sin^2 2\beta}}{\sqrt{2} \sin \beta}}{\sin \theta}.
\] (14)

These two conditions meant that the characteristics of the second family in (9) turn into rays \(\theta = \text{const}\), where the extreme values \(\theta\) are determined in terms of the extreme values of the angle \(\varphi\) using (14). Regarding the characteristics of the first family in (9), placement of (14) in the first equation of (13) yields:
\[
\frac{d \rho}{\rho} = \frac{\sin \beta - \cos \beta}{\sqrt{2} \sqrt{\tan^2 \varphi + \sin^2 2\beta}}.
\] (15)
With regard to (14), we find:
\[
\sqrt{\tan^2 \varphi + \sin 2\beta} = \tan \varphi - \tan \theta \sqrt{2} \sin \beta .
\] (16)

After raising (16) to the second power, we have:
\[
\tan \varphi = \frac{\tan^2 \theta \sin \beta - \cos \beta}{\sqrt{2} \tan \theta} .
\] (17)

Placement of (17) in (16) gives:
\[
\sqrt{\tan^2 \varphi + \sin 2\beta} = -\frac{\tan^2 \theta \sin \beta + \cos \beta}{\sqrt{2} \tan \theta} .
\] (18)

It follows from (18) and (15) that:
\[
\frac{d\rho}{\rho} = \frac{(1 - \tan \beta) \tan \theta}{1 + \tan \beta \tan^2 \theta} d\theta .
\] (19)

We integrate (19) and have:
\[
\ln \rho - \ln c = -\frac{1}{2} \ln (\cos^2 \theta + \tan \beta \sin^2 \theta)
\]
or
\[
\rho^2 \cos^2 \theta + \tan \beta \rho^2 \sin^2 \theta = C^2 .
\] (20)

It is apparent that condition (20) defines an ellipse with half-axes
\[
a = |C| , \ b = \sqrt{\cot \beta} |C| ,
\]
where \( C \) is the integration constant. Since \( b / a = \sqrt{\cot \beta} \), we have a family of similar ellipses.

### 3. Problem on penetration of rigid wedge in anisotropic medium

Let there be an anisotropic medium with elasticity law (1) and condition \( \lambda_1 = \lambda_3 \), where \( \lambda_1, \lambda_2, \lambda_3 \) is determined from (4). Let the plasticity condition \( S^2_i + S^2_i = k^2 \) represented by (5) lead to characteristics (9) with relations (10) and (11) on them. This medium is penetrated with a rigid nondeformable wedge with a nose angle \( 2\gamma \) (Figure 1). It is assumed that the wedge moves at an initial velocity \( v_0 \). Using the scheme of a rigid-plastic body, we are going to find the maximum penetration depth of the wedge.

**Figure 1.** Wedge \( AEF \) penetrates rock mass. The domains \( ABC \) and \( ADE \) are the uniform stress state zones, the domain \( ADC \) is a centered field formed by the beams emerged from the point \( A \) and by the arcs of similar ellipses.
We construct the problem solution from the boundary $AB$. This boundary is free of stresses for $\sigma_y = \tau_{xy} = 0$. It follows from (5) that $\varphi = \pi$, since at such value of $\varphi$:

$$\sigma_x = -k / \cos \beta < 0.$$  

At the boundary $AE$ $\tau_n = 0$, or:

$$\tau_n = \frac{\sigma_x - \sigma_y}{2} \sin 2\gamma + \tau_{xy} \cos 2\gamma = 0. \quad (21)$$

Let us express the stress $\sigma_y$ from (21) using (5). From (5) we find:

$$\sigma_x = k \frac{\cos \varphi + \sigma_y \sin \beta}{\cos \beta}, \quad \tau_{xy} = \frac{k \sin \varphi}{\sqrt{2}}. \quad (22)$$

We insert (22) in (21) and have:

$$\sigma_y = k \frac{\cos \varphi \sin 2\gamma + \sqrt{2} \sin \varphi \cos 2\gamma \cos \beta}{\sin 2\gamma (\cos \beta - \sin \beta)}. \quad (23)$$

We denote the unknown value $\varphi$ in (22) and (23) as $\varphi_*$. Then, we turn to characteristics (10). Along the first characteristic (9), relation (1) is fulfilled with the upper sign before radical. We integrate it from the point $P$, where $\sigma_y = 0$, $\varphi = \pi$, to the point $Q$, where $\sigma_y$ is calculated from (23) and $\varphi = \varphi_*$. As a result, we obtain a transcendental equation to find $\varphi_*$ at the boundary $EA$:

$$\frac{\cos \varphi_* \sin 2\gamma + \sqrt{2} \sin \varphi_* \cos 2\gamma \cos \beta}{\sin 2\gamma (\cos \beta - \sin \beta)} + \frac{\cos \varphi_* + 1}{2 \sin \beta} - \frac{1}{2 \sin \beta} \int_{\pi}^{\varphi_*} \sqrt{1 - \sin^2 (\varphi - \pi / 2)} d\varphi = 0. \quad (24)$$

The normal stress $\sigma_n$ is recovered from the formula:

$$\sigma_n = \sigma_x \cos^2 \gamma - 2 \tau_{xy} \cos \gamma \sin \gamma + \sigma_y \sin^2 \gamma,$$  

where $\sigma_x, \sigma_y, \tau_{xy}$ are expressed in terms of (22), (23) at $\varphi = \varphi_*$. With $\sigma_n$ known from (25), we find force on the wedge side $AE$ by multiplying $\sigma_n$ by the length of $AE$. The same force affects the side $FE$. After that, we find the vertical force which prevents downward movement of the wedge. Newton’s equation of motion is integrated at the initial condition $v = v_0$, $t = t_0$. The time at which the velocity turns to zero is determined, and using this time, the maximal penetration depth of the wedge is found.

**Table 1.** Stresses at smooth wedge for different angles $\gamma$

| $\gamma$, deg | $\sigma_x / k$ | $\sigma_y / k$ | $\tau_{xy} / k$ | $\varphi_*$, rad |
|---------------|---------------|---------------|-----------------|------------------|
| In anisotropic rock mass |
| 10            | $-0.282$      | $-1.208$      | $0.169$         | $2.901$          |
| 15            | $-0.421$      | $-1.242$      | $0.237$         | $2.780$          |
| 20            | $-0.559$      | $-1.268$      | $0.298$         | $2.707$          |
| 25            | $-0.698$      | $-1.29$       | $0.352$         | $2.620$          |
| 30            | $-0.842$      | $-1.308$      | $0.403$         | $2.535$          |
| In isotropic rock mass |
| 10            | $-0.289$      | $-1.616$      | $0.241$         | $2.793$          |
| 15            | $-0.465$      | $-1.687$      | $0.353$         | $2.619$          |
| 20            | $-0.658$      | $-1.74$       | $0.454$         | $2.445$          |
| 25            | $-0.869$      | $-1.776$      | $0.541$         | $2.271$          |
| 30            | $-1.093$      | $-1.799$      | $0.612$         | $2.096$          |
Table 1 compiles the calculation results for quartz rock crystal at $\beta = 0.442$ (elastic characteristics are taken from [15], MPa$^{-1}$: $a_{11} = 19.66 \times 10^{-6}$, $a_{22} = 12.73 \times 10^{-6}$, $a_{12} = 4.23 \times 10^{-6}$) and for isotropic medium, MPa$^{-1}$ ($\beta = 0.785$, $a_{11} = 19.66 \times 10^{-6}$, $a_{22} = 19.60 \times 10^{-6}$, $a_{12} = 4.23 \times 10^{-6}$), where the values of $\sigma_y/k$ are proportional to the expression $-1 - 2\gamma + \cos 2\gamma$.

4. Conclusions
The authors have constructed the mathematical model of plastic deformation in an initially anisotropic medium with multiple roots of the characteristic equation. The authors use the model to solve the problem on penetration of a rigid wedge in the initially anisotropic medium.

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