EIGENREC: An Efficient and Scalable Latent Factor Family for Top-N Recommendation

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Abstract

Sparsity presents one of the major challenges of Collaborative Filtering. Graph-based methods are known to alleviate its effects, however their use is often computationally prohibitive; Latent-Factor methods, on the other hand, present a reasonable and viable alternative. In this paper, we introduce EIGENREC; a versatile and efficient Latent-Factor framework for Top-N Recommendations, that generalizes the well-known PureSVD algorithm (a) providing intuition about its inner structure, (b) paving the path towards improving its efficacy and, at the same time, (c) reducing its complexity. One of our central goals in this work is to ensure the applicability of our method in realistic big-data scenarios. To this end, we propose building our model using a computationally efficient Lanczos-based procedure, we discuss its Parallel Implementation in distributed computing environments, and we verify its favourable performance using real-world datasets. Furthermore, from a qualitative point of view, a comprehensive set of experiments on the MovieLens and the Yahoo!R2Music datasets based on widely applied performance metrics, indicate that EIGENREC outperforms several state-of-the-art algorithms, in terms of Standard and Long-Tail recommendation accuracy, exhibiting low susceptibility to sparsity, even in its most extreme manifestations – the Cold-Start problems.

Index Terms

Recommender Systems, Collaborative Filtering, Dimensionality Reduction, Sparsity, Long-Tail Recommendation, Cold-Start Recommendation, Lanczos Method, Distributed Computing

1 INTRODUCTION

Recommender Systems (RS) are information filtering tools that have been vastly adopted in the past decade, by the majority of e-commerce sites. Generally, RS focus at producing intelligent personalized suggestions in order to help users navigate through an ever-growing ocean of options regarding products [1], [2], services as well as sources of information and content [3], [4] that are becoming increasingly available as we enter the big data era. The utilization of RS technology enhances user experience and it is known to increase user fidelity to the system [5]; from an economic perspective, their use is known to assist in building bigger, and more loyal customer bases, as well as to drive significant increases in the volume of product sales [6], [7], [8].

Out of the several different approaches to building recommender systems, Collaborative Filtering (CF) is widely regarded as one of the most successful. Given a set of users, a set of items and – implicitly or explicitly – stated opinions about how much a user likes or dislikes the items he has already seen, traditional CF techniques try to build “neighborhoods”, based on the similarities between users (user-oriented CF) or items (item-oriented CF) as depicted from the data, in order to predict preference scores for the unknown user-item pairs (prediction-based recommendation [9], [10], [11]), or provide a list of items that the user might find preferable (ranking-based or top-N recommendation [12], [13], [14], [15], [16]).

Despite their success in real application settings, CF methods suffer a number of problems that remain to be resolved. One of the most significant such problems arises from the inadequacy of available data and is usually referred to as the Sparsity problem [9]. Sparsity is known to impose severe limitations to the quality of recommendations [17], and to decrease substantially the diversity and the effectiveness of CF methods – especially in recommending unpopular items (Long-Tail problem) [18]. Unfortunately, sparsity is an innate characteristic of recommender systems since in the majority of realistic applications, users interact with only a small percentage of the available items, with the problem being aggravated even more, by the fact that newcomers with no ratings at all, are frequently added to the system (Cold-Start problem [17], [19]).

Although traditional CF techniques such as neighborhood models are very susceptible to sparsity, Graph-Based methods manage to cope a lot better [9]. The fundamental characteristic that makes the methods of this family particularly suited for alleviating problems related to limited coverage and sparsity is that – unlike the standard CF methods – they allow elements of the dataset that are not directly connected, to “influence” each other by propagating information along the edges of
the graph [9]. Then, the transitive relations captured in this way, can be used to recommend items either by estimating measures of proximity between the corresponding nodes [20], [21], [22] or by computing node similarity scores between them [21], [23], [24].

**Latent Factor Methods and the PureSVD algorithm.** While promising in dealing with sparsity problems, the unprecedented growth of the number of users and items in modern e-commerce applications make many graph-based CF techniques suffer serious computational and scalability issues. Latent Factor methods, on the other hand, present a more feasible alternative [9], [25], [26], [27], [28]. The fundamental hypothesis behind using latent factor models for building recommender systems is that users’ preferences are influenced by a set of “hidden taste factors” that are usually very specific to the domain of recommendation [10], [26]. These factors are generally not obvious and might not necessarily be intuitively understandable. Latent Factor algorithms, however, can infer those factors by the user’s feedback as depicted in the rating data. Generally speaking, the methods of this family work by projecting the elements of the recommender database into a denser subspace that captures their most meaningful features, giving them the power to relate previously unrelated elements, and thus making them less vulnerable to sparsity [9].

Although the majority of latent factor related work was done for prediction-based recommendations, one very successful example of latent factor algorithms for top-N recommendation is PureSVD, proposed by Cremonesi et al. in [13]. This algorithm considers all missing values in the user-item rating matrix, \( R \), as zeros, and produces recommendations by estimating \( \hat{R} \) by the factorization

\[
\hat{R} = U_f \Sigma_f Q_f^T,
\]

where, \( U_f \) is an \( n \times f \) orthonormal matrix, \( Q_f \) is an \( m \times f \) orthonormal matrix, and \( \Sigma_f \) is an \( f \times f \) diagonal matrix containing the \( f \) largest singular values. The rows of matrix \( \hat{R} \) contain the recommendation vectors for every user in the system. Cremonesi et al. [13], after evaluating PureSVD’s performance against various latent factor-based algorithms and neighbourhood models, found that it was able to produce better top-N recommendations compared to sophisticated matrix factorization methods [11], [28] and other popular CF techniques.

However, despite showing promising qualitative results and being fairly simple to apply, PureSVD as presented in [13], does not lend itself into fertile generalizations, nor does it leave room for qualitative improvements. The method behaves like an enigmatic “black box” that takes as an input the rating matrix, and outputs its low-rank estimate based on the truncated singular value decomposition; almost like a “happy accident” that effectively perturbs the previously zero values of \( R \) into something meaningful. But is there a more fruitful way to look at PureSVD? A way that can give more intuition about how it works and how it can be improved?

In this work, we try to shed more light to these questions by exploring the algorithmic inner workings of the method; our aim is to expose the modeling “scaffoldings” behind its mathematical structure, while building PureSVD from scratch. Interestingly, as we will see, our approach provides an illustrative reformulation of the model that paves the path towards a straightforward generalization to a whole family of related methods – which we denote \( \text{EIGENREC} \) – that can lead to qualitatively superior, and computationally attractive top-N recommendation schemes.

**Summary of Contributions** The main contribution of this work is the proposal of \( \text{EIGENREC} \); a latent factor based collaborative filtering family for top-N recommendations that contains PureSVD as a special case.

- \( \text{EIGENREC} \) works by building a low dimensional subspace of a novel inter-element proximity matrix, the intuition behind which lays at the heart of PureSVD. This matrix is designed to be simple, easily handleable from a computational point of view, and at the same time able to allow the necessary modeling freedom for achieving significantly better qualitative outcomes with respect to the standard PureSVD.

- One of the primary concerns of this work pertains to the computability of our method in realistic big data recommendation scenarios. Our new modeling approach implies immediate computational gains with respect to PureSVD, since it reduces the computation of a truncated singular value decomposition to the solution of a simple symmetric eigenvalue problem applied to a significantly lower dimensional matrix. Our latent-factor model is built using a computationally efficient Krylov subspace procedure – namely the Lanczos Method. We discuss in detail its parallel implementation\(^1\) in a distributed environment and we perform extensive computational experiments using real-world datasets that verify the efficiency of our approach, ensuring its applicability in the large-scale.

- We conduct a comprehensive set of qualitative experiments on the MovieLens and Yahoo datasets and we show our method produces recommendations that outperform several state-of-the-art methods, in widely used metrics, achieving high-quality results even in the considerably harder task of recommending Long-Tail items. \( \text{EIGENREC} \) displays low sensitivity to the sparsity of the underlying space and shows promising potential in alleviating a number of related problems that occur commonly in recommender systems. This is true both in the very interesting case where sparsity is localized in a small part of the dataset – as in the New Users problem, and in the case where extreme levels of sparsity are found throughout the data – as in the New Community problem.

The rest of the paper is organized as follows: In Section 2, we revisit PureSVD and we “rediscover” it under different modeling lenses in order to set the intuitive grounds behind the \( \text{EIGENREC} \) approach (Section 2.1); its mathematical details are presented rigorously in Section 2.2. In Section 3 we present the \( \text{EIGENREC} \) algorithm, we comment on its computational

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1. We make our parallel implementation code available in https://github.com/nikolakopoulos/EigenRec
complexity and we delve into the details behind its parallelization (Section 3.3). Our computational tests are presented in Section 3.4. The qualitative evaluation of EIGENRec, including experimental methodology, metrics definition, a detailed discussion of the competing recommendation methods, as well as top-N recommendation results in standard, long-tail and cold-start scenarios are presented in Section 4. Finally, Section 5 comments briefly on related literature and Section 6 concludes this work.

2 EIGENRec Recommendation Framework

Notation: All vectors are denoted by bold lower-case letters and they are column vectors (e.g., \( \mathbf{v} \)). All matrices are represented by bold upper-case letters (e.g., \( \mathbf{Q} \)). The \( j^{th} \) column and the \( i^{th} \) row of matrix \( \mathbf{Q} \) are denoted \( \mathbf{q}_i \) and \( \mathbf{q}_j^\top \), respectively. The \( ij^{th} \) element of matrix \( \mathbf{Q} \) is denoted \( Q_{ij} \). We use \( \text{diag}(\mathbf{Q}) \) to refer to the matrix that has the same diagonal with matrix \( \mathbf{Q} \) and zeros elsewhere, and \( \text{diag}(\mathbf{v}) \) to denote the matrix having vector \( \mathbf{v} \) on its diagonal, and zeros elsewhere. Furthermore, \( \| \cdot \| \) denotes a norm that – unless stated otherwise – is assumed to be Euclidean. We use calligraphic uppercase letters to denote sets (e.g., \( \mathcal{U}, \mathcal{V} \)). Finally, symbol \( \pm \) is used in definition statements.

2.1 From PureSVD to EIGENRec

Let us consider the full singular value decomposition of the Ratings matrix \( \mathbf{R} \):

\[
\mathbf{R} = \mathbf{U} \Sigma \mathbf{Q}^\top,
\]

(2)

where, \( \mathbf{U} \) is an \( n \times n \) orthonormal matrix, \( \mathbf{Q} \) is an \( m \times m \) orthonormal matrix, and \( \Sigma \) is an \( n \times m \) rectangular diagonal matrix containing the singular values. The columns of matrix \( \mathbf{U} \) are eigenvectors of \( \mathbf{R} \mathbf{R}^\top \), and the columns of matrix \( \mathbf{Q} \) are eigenvectors of \( \mathbf{R}^\top \mathbf{R} \). The min\((n, m)\) singular values on the diagonal of \( \Sigma \) are the square roots of the nonzero eigenvalues of both \( \mathbf{R} \mathbf{R}^\top \) and \( \mathbf{R}^\top \mathbf{R} \), in descending order.

Let us now multiply equation (2) from the right with matrix \( \mathbf{Q} \),

\[
\mathbf{RQ} = \mathbf{U} \Sigma \mathbf{Q}^\top \mathbf{Q}
\]

\[
\mathbf{RQ} = \mathbf{U} \Sigma \mathbf{Q}
\]

(3)

since by the orthonormality of \( \mathbf{Q} \), we have \( \mathbf{Q}^\top \mathbf{Q} = \mathbf{I} \).

Now if we use \( \mathbf{I}_f \) to denote the \( f \times f \) identity matrix and we multiply again from the right with the \( m \times m \) matrix \( \begin{pmatrix} 1_f & 0 \\ 0 & 0 \end{pmatrix} \), we get

\[
\mathbf{R} \begin{bmatrix} \mathbf{Q}_f & 0 \end{bmatrix} = \mathbf{U} \begin{bmatrix} \Sigma_f & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
\Rightarrow \mathbf{RQ}_f = \mathbf{U}_f \Sigma_f.
\]

(4)

Substituting equation (4) in (1) gives

\[
\hat{\mathbf{R}} = \mathbf{RQ}_f \mathbf{Q}^\top_f.
\]

(5)

Similarly, if we multiply equation (2) from the left with matrix \( \mathbf{U}^\top \), we get

\[
\mathbf{U}^\top \mathbf{R} = \mathbf{U}^\top \mathbf{U} \Sigma \mathbf{Q}^\top
\]

\[
\mathbf{U}^\top \mathbf{R} = \Sigma \mathbf{Q}^\top
\]

(6)

since by the orthonormality of \( \mathbf{U} \), we have \( \mathbf{U}^\top \mathbf{U} = \mathbf{I} \).

Now if we multiply this time from the left with the \( n \times n \) matrix \( \begin{pmatrix} 1_f & 0 \\ 0 & 0 \end{pmatrix} \) we get

\[
\begin{bmatrix} \mathbf{U}_f^\top & 0 \end{bmatrix} \mathbf{R} = \begin{bmatrix} \Sigma_f & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Q}^\top
\]

\[
\Rightarrow \mathbf{U}_f^\top \mathbf{R} = \Sigma_f \mathbf{Q}_f^\top.
\]

(7)

Substituting equation (7) in (1) gives

\[
\hat{\mathbf{R}} = \mathbf{U}_f \mathbf{U}_f^\top \mathbf{R}.
\]

(8)

Relations (5) and (8) show that the recommendation matrix of PureSVD can be expressed only in terms of the ratings matrix and either matrix \( \mathbf{Q}_f \) alone, or matrix \( \mathbf{U}_f \) alone. As we discussed earlier, matrix \( \mathbf{Q}_f \) contains the orthonormal set of eigenvectors that correspond to the \( f \) principal eigenvalues of the symmetric matrix \( \mathbf{R}^\top \mathbf{R} \), and matrix \( \mathbf{U}_f \) contains the orthonormal set of eigenvectors that correspond to the \( f \) principal eigenvalues of the symmetric matrix \( \mathbf{R} \mathbf{R}^\top \). A closer look
at matrices $R^T R$ and $RR^T$ reveals that their elements have a very intuitive interpretation in the recommender systems parlance. In particular,

$$R^T R = \begin{bmatrix} \text{users} & -r_i^T \\ \& \\ \& -\end{bmatrix} \times \begin{bmatrix} \text{items} \\ r_j \end{bmatrix}$$

$$= \begin{bmatrix} \text{items} \\ \& \\ \& \cdot \end{bmatrix} \parallel \begin{bmatrix} r_i \parallel r_j \end{bmatrix} \cos \theta_{ij}. \quad \text{scaling} \quad \text{similarity}$$

Thus, the $ij^{th}$ element of $R^T R$ can be interpreted as the traditional cosine-based inter-item similarity score, scaled up by a factor related to the popularity of the items $i, j$ as expressed in the ratings matrix. Similarly,

$$RR^T = \begin{bmatrix} \text{users} & -u_i^T \\ \& \\ \& -\end{bmatrix} \times \begin{bmatrix} \text{items} \\ u_j \end{bmatrix}$$

$$= \begin{bmatrix} \text{users} \\ \& \\ \& \cdot \end{bmatrix} \parallel \begin{bmatrix} u_i \parallel u_j \end{bmatrix} \cos \phi_{ij}, \quad \text{scaling} \quad \text{similarity}$$

where the $ij^{th}$ element of $RR^T$ can be interpreted as the traditional cosine-based inter-user similarity score, scaled up by a factor related to the past behaviour of the users $i, j$ as expressed by the number and the magnitude of their ratings.

Therefore, the latent factor model of PureSVD is essentially built from the eigendecomposition of a scaled cosine-based inter-user or inter-item similarity matrix.

From a purely computational perspective, this observation reduces the extraction of PureSVD’s recommendation matrix to the calculation of the $f$ principal eigenvectors of a $\min(m, n) \times \min(m, n)$ symmetric matrix; a fact that decreases significantly the overall computational and storage needs (Section 3.2). From a modeling perspective, the above observation places PureSVD in the center of a family of latent factor methods that can readily be obtained using inter-element proximity matrices that allow (a) different similarity functions and (b) different scaling functions. We denote this family EIGENREC, and we will show that it can lead to high-quality results even in challenging recommendation scenarios (Section 4).

In the following section we proceed to the rigorous definition of EIGENREC and its constituent parts. Since in the majority of realistic applications the number of items is much smaller than the number of the users (and it also scales better), we choose to focus to the item-based view. Of course, an exactly analogous approach can result in a user-based application of our method when preferable.

### 2.2 EIGENREC Model Definition

**Definitions:** Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of users and $V = \{v_1, v_2, \ldots, v_m\}$ a set of items. Let $R$ be a set of tuples $t_{ij} = (u_i, v_j, r_{ij})$, where $r_{ij}$ is a nonnegative number referred to as the rating given by user $u_i$ to the item $v_j$, and let $R \in \mathbb{R}^{n \times m}$ be a matrix whose $ij^{th}$ element contains the rating $r_{ij}$ if the tuple $t_{ij}$ belongs in $R$, and zero otherwise. These ratings can either come from the explicit feedback of the user or inferred by the user’s behavior and interaction with the system.

**Inter-Item Proximity Matrix A.** We first need to define a matrix that captures the relations between the elements of the item space. Following the discussion in the previous section, we define a symmetric matrix $A \in \mathbb{R}^{m \times m}$ whose $ij^{th}$ element is given by:

$$A_{ij} \triangleq \xi(i, j) \cdot \kappa(i, j), \quad \text{(9)}$$

where $\xi(\cdot, \cdot) : V \times V \mapsto [0, \infty)$ is a symmetric scaling function and $\kappa(\cdot, \cdot) : V \times V \mapsto \mathbb{R}$ is a symmetric similarity function.

**Recommendation Matrix II.** The final recommendation matrix contains the recommendation vectors for each user in the system. Concretely, for each user $u_i$ the corresponding personalized recommendation vector is given by:

$$\pi_i \triangleq r_i^T V V^T, \quad \text{(10)}$$

where $r_i^T$ the ratings of user $u_i$ and $V \in \mathbb{R}^{m \times f}$ is the matrix whose columns contain the $f$ principal orthonormal eigenvectors of the inter-item proximity matrix $A$. Notice that since matrix $A$ is real and symmetric, we can always choose its eigenvectors such that they are real, orthogonal to each other and have unity norm. The computational aspects of building matrix $V$ efficiently are discussed in detail in Section 3. Below we discuss possible choices for the scaling and the similarity components of the inter-item proximity matrix $A$. 

2.2.1 Scaling Component

The definition of the scaling function can be done in many different ways, subject to various aspects of the recommendation problem at hand. In this work, we are utilizing this function as an easy way to control how much the inter-item proximity scores are affected by the prior popularity of the items involved. This was found to be very important for the overall recommendation quality as we will see in the experimental section of our paper. In particular, for the scaling function \( \xi(\cdot, \cdot) \), we use the simple symmetric function

\[
\xi(i, j) \triangleq (||r_i|| ||r_j||)^d.
\]

Notice that the definition of the scaling function allows the final inter-item proximity matrix to be written in the following form:

\[
A = SKS
\]

where

\[
S \triangleq \text{diag}\{||r_1|| \ldots ||r_m||\}^d,
\]

and where matrix \( K \) (the \( ij \)th element of which is defined to be \( \kappa(i, j) \)), denotes the pure similarity matrix.

2.2.2 Similarity Component

The definition of the similarity matrix \( K \) can be approached in several different ways, depending on the particular nature of the recommendation task, possible computational restrictions etc. Note that the final offline computational cost of the method may depend significantly on the choice of matrix \( K \). For example, given that the inter-item proximity matrices can be relatively dense, in applications where the number of the items is extremely large there may arise storage as well as computational issues. Furthermore, the use of involved statistical techniques e.g. to learn sophisticated kernels or incorporate content-based meta-information etc. implies additional burden, which may prove prohibitively large for many realistic big data recommendation scenarios.

Having this in mind, in this work we propose using three simple similarity matrices that can be applied easily and are able to achieve high-quality results, namely (a) the Cosine Similarity (as in the standard PureSVD), (b) the Pearson-Correlation Similarity and finally (c) the Jaccard Similarity.

**Cosine Similarity** \( \text{K}_{cos} \). In this case, the similarity function \( \kappa(\cdot, \cdot) \) is defined to be the cosine of the angle between the vector representation of the items \( v_i, v_j \)

\[
K_{ij} \triangleq \cos(v_i, v_j).
\]

One of the particularly useful properties of \( \text{K}_{cos} \) is that it can be expressed as a product of extremely sparse matrices, ensuring significant advantages in terms of both storage and computability. In particular, its \( ij \)th element can be expressed as follows:

\[
K_{ij} \triangleq \cos(v_i, v_j) = r_i^T r_j (||r_i|| ||r_j||)^{-1} = r_i^T r_j S_i^{-1} S_j^{-1},
\]

which allows the final inter-item proximity matrix \( A \) to be written as:

\[
A = SK_{cos}S = W^TW,
\]

where

\[
W \triangleq R \text{diag}\{||r_1|| \ldots ||r_m||\}^{-1}.
\]

**Pearson Similarity** \( \text{K}_{rc} \) The similarity score between two items \( v_i \) and \( v_j \) is defined as the \( ij \)th element of matrix \( \text{K}_{rc} \) which is given by

\[
K_{ij} \triangleq \frac{C_{ij}}{\sqrt{C_{ii}C_{jj}}},
\]

with \( C_{ij} \) denoting the covariance between the vector representation of the items \( v_i, v_j \). For the purposes of computing the latent space this matrix will be accessed only through MV products, thus, it is useful to express \( \text{K}_{rc} \) in terms of sparse components. In particular, taking advantage of the properties of the covariance – and after a little bit of algebra – the final matrix \( \text{K}_{rc} \) can be written as:

\[
\text{K}_{rc} \triangleq D(R^TR - n\mu\mu^T)D,
\]

where

\[
D \triangleq \text{diag}\{R^TR - n\mu\mu^T\}^{-1/2},
\]

and \( \mu \) the vector containing the mean ratings of the items.

**Jaccard Similarity** \( \text{K}_{jac} \). The Jaccard similarity index between two items is defined as the ratio of the number of users that have rated both items to the number of users that have rated at least one of them. Concretely,

\[
K_{ij} \triangleq \frac{|R_i \cap R_j|}{|R_i \cup R_j|},
\]
where \( R_i \) the set of users that have rated item \( i \). As with the two previous similarity matrices, \( K_{\text{JAC}} \) can also be brought in a more useful matrix form, which allows us not to compute and store it explicitly. Concretely, if we use \( Y \) to denote the matrix whose \( ij^{\text{th}} \) element is 1 if and only if \( R_{ij} > 0 \), zero otherwise, matrix \( K_{\text{JAC}} \) can be expressed as follows

\[
K_{\text{JAC}} \triangleq Y^T Y \odot (ey^T + ye^T - Y^T Y),
\]

where \( y \) denotes a vector containing the number of ratings of each item, and \( \odot \), the Hadamard division operator.

Having defined the basic components of our model, we are now ready to proceed to the discussion of the computational aspects behind building the reduced dimensional subspace.

3 Buidling The Latent Space

The specific properties of our model (symmetry and sparsity), make Krylov subspace methods [29] ideal candidates for building the latent space and producing the recommendation lists efficiently. In particular, we employ the Lanczos algorithm [30], an iterative Krylov subspace method for solving symmetric eigenvalue problems. In the subsections to follow we delve into the details behind the application of the Lanczos method, we discuss its parallel implementation in distributed computing environments and we evaluate experimentally its computational performance.

3.1 The Algorithm

Given a matrix \( A \in \mathbb{R}^{m \times m} \) and an initial unit vector \( q \), the corresponding Krylov subspace of size \( \ell \) is given by \( K_\ell(A, q) \triangleq \text{range}\{q, Aq, A^2q, \ldots, A^{\ell-1}q\} \). By forming an orthogonal basis for \( K_\ell \), Krylov subspace methods can be used to solve several types of numerical problems. In this section we describe the application of Lanczos algorithm for building our latent factor subspace, \( V \).

The algorithm starts by choosing a random starting vector \( q \) and builds up an orthogonal basis \( Q_j \) of the Krylov subspace, one column at a time. In the new orthogonal basis \( Q_j \) the operator \( A \) is represented by a real symmetric tridiagonal matrix,

\[
T_j = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \ddots \\
& \ddots & \ddots & \beta_{j-1} \\
& & \beta_{j-1} & \alpha_j
\end{bmatrix},
\]

which is also built up one row and column at a time [31], using the recurrence,

\[
AQ_j = Q_j T_j + r_j e_j^T \quad \text{with} \quad Q_j^T r_j = 0.
\]

In exact arithmetic, orthogonality of the Krylov subspace is performed implicitly by the elegant three-term recurrence (23). However, in most real-case applications of Lanczos, orthogonality of the Krylov subspace is maintained explicitly by re-orthogonalization. Since our latent model is based on the few top eigenvectors of \( A \) (similarly to the PureSVD approach), these can be approximated by computing at any step \( j \), the eigendecomposition of \( T_j \),

\[
T_j = \Xi \Theta \Xi^T.
\]

The Ritz value \( \theta_i \) and its Ritz vector \( Q_j \xi_i \), will give a good approximation to an eigenpair of \( A \) if the residual \( \delta_i \) has small norm. It can be shown that the residual of the \( ij^{\text{th}} \) eigenpair at the \( j^{\text{th}} \) Lanczos step satisfies

\[
\delta_i^{(j)} = |\beta_j \xi_{ji}|, \quad i = 1, \ldots, j
\]

so it suffices to monitor only the subdiagonal element \( \beta_j \) of \( T \) and the last row of \( \Xi \) to get an estimate of the norm of the residual [31]. The complete algorithm for the computation of \( V \) and the final recommendation matrix \( \Pi \) for the whole set of users is given below:

**Computational Cost:** The most expensive operations of the symmetric Lanczos algorithm are the MV product and the orthogonalization phase. The total cost introduced by the MV products in \( j \) Lanczos steps amounts to \( O(j \cdot \text{nnz}) \), with \( \text{nnz} \) denoting the number of non-zero entries in matrix \( A \), whereas making the \( j^{\text{th}} \) Krylov vector orthogonal to the previous ones costs \( O jm \).

3.2 Faster PureSVD Computation

As we discussed in Section 2.1, from a mathematical point of view, the final recommendation matrix of PureSVD coincides with that produced by our method, using the similarity matrix \( K_{\text{cos}} \) and the standard scaling matrix \( S \) with parameter \( d = 1 \). From a computational point of view however, there can be significant differences. In particular, a direct application of the standard truncated singular value decomposition is also obtaining a set of the \( f \) principal eigenvectors of \( \mathbf{R R}^T \) – the
Algorithm 1 EIGEN REC

Input: Inter-Item proximity matrix $A \in \mathbb{R}^{m \times m}$. Rating Matrix $R \in \mathbb{R}^{n \times m}$. Latent Factors $f$.
Output: Matrix $\Pi \in \mathbb{R}^{n \times m}$ whose rows are the recommendation vectors for every user.

1: $q_j = 0$, set $r \leftarrow q$ as a random vector
2: $\beta_0 \leftarrow ||r||_2$
3: for $j \leftarrow 1, 2, ..., $ do
4: $q_j \leftarrow r / \beta_{j-1}$
5: $r \leftarrow Aq_j$
6: $r \leftarrow r - q_{j-1} \beta_{j-1}$
7: $\alpha_j \leftarrow q_j^T r$
8: $r \leftarrow r - q_j \alpha_j$
9: $r \leftarrow (I - Q_j Q_j^T) r$ $\triangleright$ where $Q_j = [q_1, ..., q_j]$
10: $\beta_j \leftarrow ||r||_2$
11: Solve the tridiagonal problem $(Q_j^T A Q_j) \Xi_j = \Theta_j \Xi_j$
12: Form the $j$ approximate eigenvectors $Q_j \Xi_j$ of $A$
13: If the $f$ top eigenvectors have converged, stop.
14: end for
15: Compute latent factors $V = Q_f \Xi$
16: return $\Pi \leftarrow R V V^T$

first $f$ columns of the $n \times f$ matrix $U$ – which implies additional computational and storage burden. More specifically, the standard application of PureSVD would be to use the Lanczos algorithm on matrix

$$\hat{A} = \begin{bmatrix} 0 & R^T \\ R & 0 \end{bmatrix},$$

and solve the associated symmetric eigenvalue problem. The (non-zero) eigenvalues of $\hat{A}$ are equal to $\pm \sigma_i$, with the corresponding eigenvectors being

$$\begin{bmatrix} q_i \\ u_i \end{bmatrix}, \begin{bmatrix} -q_i \\ u_i \end{bmatrix}.$$

If the singular values of smallest magnitude are of interest, and/or the highest possible accuracy is needed, operating with $\hat{A}$ is more robust from a numerical point-of-view. However, in our case, there are good reasons to proceed as in Algorithm 1. First, we are interested in the top singular vectors of $R$ and using the $R^T R$ operator actually magnifies the spectral gap of the dominant eigenvalues in $R$. Thus, convergence of Lanczos algorithm is improved. Second, and perhaps most importantly, Algorithm 1 operates on a matrix of size $m \times m$ and thus all operations including the orthogonalization, the inner products, as well as the vector update operations, are performed on vectors of length $m$, instead of $n + m$, where the latter is the size of matrix $\hat{A}$. To quantify the time difference that arises from the exploitation of EIGENREC’s equivalent formulation for computing PureSVD, we run the two algorithms in Matlab, on the MovieLens10M and MovieLens20M datasets [32], using in both cases Matlab’s native functions and the same convergence criterion for fair comparison.

![Speed Up](image1)

![Speed Up](image2)

Fig. 1. PureSVD’s computation speed up using EIGENREC’s formulation for latent factors in the range $[50, ..., 500]$.

Figure 1 reports the results. As predicted, with our approach, PureSVD is computed significantly faster, with the speedup increasing with the dimension of the desired latent subspace ranging from 6, to 16 times faster computation for latent factors in the range $[50, ..., 500]$. 
3.3 Parallel Implementation for Large-Scale Scenarios

In this subsection we consider the practical application of the EIGENREC approach in the context of very large datasets. To tackle large-scale computations, we consider the implementation of EIGENREC in distributed computing environments.

While Lanczos algorithm is an inherently serial procedure – in the sense that the next iteration starts only after the previous one is completed – we can speed-up its application by performing all inner computations, such as the Matrix-Vector products with \( A \), the inner products, as well as the re-orthogonalization of the Krylov basis, in a distributed manner.

All of the discussion to follow adopts a distributed environment in which communication is achieved by means of the Message Passing Interface (MPI) standard [33]. We will assume that the number of available processors is \( P \) and they are organized in an 1-D grid (vector of processors). Then, both a matrix \( A \in \mathbb{R}^{m \times m} \) and a random vector \( q \in \mathbb{R}^m \) can be written as

\[
A = \begin{bmatrix}
- A_1 & - \\
- A_2 & - \\
\vdots & - \\
- A_P & - 
\end{bmatrix}, \quad q = \begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_P
\end{bmatrix},
\]

where \( A_1 \) and \( q_1 \) are formed by consecutive rows of \( A \) and \( q \) respectively, and are local in each processor. Below, for reasons of simplicity, we discuss the parallel implementation for the latent space construction of an inter-item proximity matrix \( A \) that can be written in a simple product form, \( A = W^T W \), as in the case of Cosine similarity matrix.

**The MV product:** The MV product between \( A \) and a vector \( q \) can be achieved by a two-stage procedure where we first perform \( q = (Wq) \) followed by \( y = W^T q \). Assuming that \( W^T \) is distributed row-wise (thus \( W \) is distributed column-wise), the only need for communication appears when performing \( q = (Wq) \) and consists of a simple \texttt{mpi_allreduce} operation (of length \( n \)) to sum the local contributions of each process.

**The inner products:** The inner product is the second operation of Lanczos which demands communication among the processors. It is a two-stage procedure where in the first stage each processor computes its local part of the global inner product, while in the second stage the local inner products (a scalar value per processor) are summed by \texttt{mpi_allreduce} and the final value is distributed to all processors.

Performing the re-orthogonalization step: Similarly to the above computations, the full re-orthogonalization step during the \( j^\text{th} \) Lanczos iteration, to get an improved vector \( q_{j+1} \) is given by

\[
q_{j+1} = q_{j+1} - Q_j Q_j^T q_{j+1},
\]

and it is performed by a two-stage procedure where we first compute \( \hat{y} = Q_j^T q_{j+1} \) followed by \( q'_{j+1} = q_{j+1} - Q_j \hat{y} \). Again, the only need for communication among the different processors appears when performing \( \hat{y} = Q_j^T q_{j+1} \) and is of the \texttt{mpi_allreduce} form.

**The vector updates:** The vector updates (also known as DAXPY operations) are trivially parallel.

**Remark:** Similar approaches can be followed if matrix \( A \) is given by more general expressions as in the case of \( K_{jc} \) or \( K_{jac} \). In general, as it is the case with all Krylov subspace methods, in Lanczos method the actual matrix need not be formed; only a routine that is able to perform the MV product between \( A \) and a vector is necessary. Furthermore, having the inter-item proximity matrix explicitly formed is not only impractical but also not advised since any explicit formation will probably be much more dense; resulting in an unnecessary raise of the computational time spent on Lanczos compared to the same run using the product form.

Scalability: The scaling as we increase the number of computing cores will be limited by two key factors: a) the intrinsic sparsity of the recommender datasets, and b) the operations performed within the Lanczos algorithm. The sparsity of the datasets shifts the MV product to be more memory-bound (limited by bandwidth), in the sense that the CPU is not fully exploited each time we read data from the memory (a reality inherent in all methods based on sparse matrix computations). Moreover, the rest of the computations performed in parallel, like the inner products and the orthogonalization phase, are generally dominated by latency and low granularity, which in turn also limits the scalability of the method. Note that to alleviate this, one can use sophisticated parallel schemes that try to overlap communication with computations; however, their analysis goes deep into high-performance computing and lies outside the scope of this paper.

3.4 Computational Tests in a Distributed Environment

In this section, we report computational experiments with our parallel implementation of the EIGENREC scheme. Our main purpose is to measure the actual wall-clock timings to compute a predefined number of eigenvectors of \( A = W^T W \) using the Lanczos algorithm, and which by far represents the main computational bottleneck of our approach. We report results for a varying number of MPI processes, where each MPI process is a single computational core. For each MPI process, we explicitly set the number of threads equal to one (distributed-memory model only). The local BLAS-1 operations to compute the associated inner products and vector updates, as well as the two sparse Matrix-Vector products with \( W^T \) and \( W \) at each Lanczos iteration were performed using routines implemented in the Intel Math Kernel Library (Release 11.3) [34], while the tridiagonal eigenvalue problem was solved using the LAPACK routine \texttt{DSYEV} [35]. The experiments were performed at the Itasca Linux cluster at Minnesota Supercomputing Institute. Itasca is an HP Linux cluster with...
TABLE 1
Computational Experiments

|          | MovieLens10M | MovieLens20M |
|----------|--------------|--------------|
|          | f=50 100 150 200 300 | f=50 100 150 200 300 |
| 8 cores  |               |               |
| 16 cores |               |               |
| 32 cores |               |               |
| 64 cores |               |               |
| 89 | 2.1 3.8 5.5 7.0 9.6 | 11.0 18.0 24.2 28.7 36.8 |
| 16 cores | 1.2 2.3 3.4 4.3 6.0 | 6.9 11.6 15.6 18.9 24.3 |
| 32 cores | 0.7 1.3 1.9 2.4 3.4 | 4.8 8.1 11.1 13.5 17.6 |
| 64 cores | 0.5 0.8 1.1 1.4 2.1 | 3.5 6.0 8.2 9.9 12.5 |

1,091 HP ProLiant BL280c G6 blade servers, each with two-socket, quad-core 2.8 GHz Intel Xeon X5560 “Nehalem EP” processors sharing 24 GB of system memory, with a 40-gigabit QDR InfiniBand (IB) interconnect. In total, Itasca consists of 8,728 compute cores and 24 TB of main memory.

Table 1 reports results obtained in distributed computing environments for the two largest matrices in the MovieLens collection, datasets MovieLens10M and MovieLens20M. Details for these two datasets can be found in [32]. For each dataset, we consider the computation of a varying number of eigenvectors in the proposed latent model. Naturally, as more and more eigenvectors are sought, the computational time to build the latent model grows larger, since more Lanczos steps are necessary. We provide the elapsed time (in seconds) for computing \( f = 50, 100, 150, 200 \) and 300 top eigenvectors, using 8, 16, 32 and 64 computational cores. All timings amount to computing the \( f \) largest eigenvectors in the maximum possible accuracy. From Table 1, we can see that for the largest of the two datasets, MovieLens20M, we can build a sufficiently good latent model in roughly half a minute (the time to compute the 300 largest eigenvectors using a modestly sized system with eight cores). Using more cores to perform the inner computations in Lanczos algorithm reduces the computational time and helps build the latent model in as less as a few seconds. Finally, the scaling of the method was found to be satisfactory and steady up to sixty-four cores.

4 QUALITATIVE EVALUATION

4.1 Datasets and Metrics

The recommendation quality of our method was tested utilizing data originated from two recommendation domains, namely Movie Recommendation – where we exploit the standard MovieLens1M and MovieLens100K datasets [32] that have been used widely for the qualitative evaluation of recommender systems; and Song Recommendation – where we used the Yahoo!R2Music dataset [36] which represents a snapshot of the Yahoo!Music community’s preferences for different songs. More details about the datasets used can be found in [32], [36].

4.1.1 Metrics

For our qualitative experiments, except for the standard Recall and Precision metrics [13], [37], we also use a number of other well known utility-based ranking indices, that assume that the utility of a recommended item is discounted by a factor related to its position in the final recommendation list [38]. Depending on the decay of the positional discount down the list we have the:

**Normalized Discounted Cumulative Gain** which assumes that the ranking positions are discounted logarithmically fast [12], [38] and is defined by:

\[
\text{NDCG}_{@k} = \frac{\text{DCG}_{@k}(y, \pi)}{\text{DCG}_{@k}(y, \pi^*)},
\]

with

\[
\text{DCG}_{@k}(y, \pi) = \sum_{q=1}^{k} \frac{2^{y_{\pi_q}} - 1}{\log_2(2 + q)},
\]

where \( y \) is a vector of the relevance values for a sequence of items, \( \pi_q \) is the index of the \( q \)th item in the recommendation list \( \pi \), and \( \pi^* \) is the optimal ranking of the items with respect to the relevant scores (see [12] for details).

**R-Score**, which assumes that the value of recommendations declines exponentially fast to yield the following score:

\[
R(\alpha) = \sum_{q} \frac{\max(y_{\pi_q} - d, 0)}{2^{\alpha q}},
\]

where \( \alpha \), controls the exponential decline and is referred to as the half-life parameter (see [38] for details).
Mean Reciprocal Rank, which assumes a slower decay than R-Score but faster than NDCG. MRR is the average of the reciprocal rank scores of the users, defined as follows:

$$\text{RR} = \frac{1}{\min_q \{ q : y_{iq} > 0 \}}. \quad (29)$$

4.1.2 Competing Recommendation Methods

We compare EigenRec against a number of methods of the graph-based top-$N$ recommendation family, that are considered to be highly promising in dealing with sparsity [9]. Generally speaking, graph-based recommendation methods represent the recommender database as a bipartite user-item graph, $G = \{ (V, U), E \}$ where $E = \{ e_{ij} | i \in V, j \in U \}$ such that $t_{ij} \in R$, and try to estimate similarity or distance measures between the nodes which can be used for the computation of ranked lists of the items with respect to each user.

The five competing methods used in our experiments are: the Pseudo-Inverse of the user-item graph Laplacian (Lt), the Matrix Forest Algorithm (MFA), the Regularized Commute Time (RCT), the Markov Diffusion Kernel (MD) and the Relative Entropy Diffusion (RED). Below we give their formal definitions.

**The pseudoinverse of the Laplacian.** This matrix contains the inner products of the node vectors in a Euclidean space where the nodes are exactly separated by the commute time distance [21]. For the computation of the $G_{Lt}$ matrix we used the formula:

$$G_{Lt} \triangleq (L - \frac{1}{n+m}ee^T)^{-1} + \frac{1}{n+m}ee^T, \quad (30)$$

where $L$ is the Laplacian of the graph model of the recommender system, $n$, the number of users, and $m$, the number of items (see [23] for details).

**The MFA matrix.** MFA matrix contains elements that also provide similarity measures between nodes of the graph by integrating indirect paths, based on the matrix-forest theorem [40]. Matrix $G_{MFA}$ was computed by

$$G_{MFA} \triangleq (I + L)^{-1}, \quad (31)$$

where $I$, the identity matrix and $L$, defined above.

**Markov Diffusion Kernel.** As discussed in [23] the underlying hypothesis behind this kernel is that similar nodes diffuse in a similar way through the graph. Concretely, if we define a stochastic matrix $P \triangleq D^{-1}A$, where $A$ is the adjacency matrix of the graph and $D$, a diagonal matrix containing the outdegrees of the graph nodes, the Markov diffusion kernel with parameter $t$ is defined by

$$G_{MD} \triangleq Z(t)Z^T(t), \quad \text{with} \quad Z(t) \triangleq \frac{1}{t} \sum_{\tau=1}^{t} P^\tau. \quad (32)$$

Extensive experiments done by the authors in [23] suggest that the Markov diffusion kernel does particularly well in collaborative recommendation tasks.

**Relative Entropy Diffusion Matrix.** This similarity matrix is based on the Kullback-Leibler divergence between distributions and it is defined by

$$G_{RED} \triangleq Z(t) \log(Z^T(t)) + \log(Z(t))Z^T(t), \quad (33)$$

where $Z(t)$ is defined as previous. As with the Markov diffusion kernel, $t$ is a parameter of the model.

**Regularized Commute Time Kernel.** Finally, the Regularized Commute Time is defined by

$$G_{RCT} \triangleq (D - \alpha A)^{-1}, \quad (34)$$

and its $ij$th element denotes the discounted cumulated probability of visiting node $j$ when starting from node $i$ [23], [41].

For our experiments we tested each method for many different values of the parameters for every dataset and we report the best results achieved for each experiment. Table 2 shows the parametric range tested for each method. For further details about the competing methods the reader should see [21], [23], [40] and the references therein.

| Method                        | Parameter | Range Tested |
|-------------------------------|-----------|--------------|
| PseudoInverse of the Laplacian| -         | -            |
| Matrix Forest Algorithm       | -         | -            |
| Markov Diffusion Kernel       | $t$       | $1, 2, \ldots, 10, 50, 100$ |
| Relative Entropy Diffusion Matrix | $t$       | $1, 2, \ldots, 10, 50, 100$ |
| Regularized Commute Time Kernel | $\alpha$ | $10^{-6}, 10^{-5}, \ldots, 0.99$ |
4.2 Quality of Top-N Recommendations

For our recommendation quality comparison tests we used the complete MovieLens1M dataset (denoted ML1M) and – following the dataset preparation approach used by Karypis et al. in [14] – a computationally manageable subset of the Yahoo! Research Alliance Webscope dataset (denoted Yahoo) with 3312 items and 7307 users. The latter was necessary, because of the computational profile of many of the methods we compete against. Note here, all competing methods require handling a graph of \( m + n \) nodes (where \( m \) the number of items and \( n \) the number of users), with the extraction of the recommendation scores many times involving inversions of \((m + n)\)-dimensional square matrices etc.; problems that easily become intractable as the population of users in the system increases.

In the subsections to follow, except for the Standard Recommendation, we also test the performance of our method in dealing with two very challenging and realistic scenarios that are linked to the inherent sparsity of typical recommender systems datasets. Namely, the Long-Tail Recommendation, where we evaluate the ability of our method in making useful recommendations of unpopular items, and the Cold-Start Recommendation, were we evaluate how well it does in recommending items for New Users in an existing recommender system (localized sparsity) as well as making recommendation for a New Community of users in the starting stages of the system.

4.2.1 Standard Recommendation

To evaluate the quality of EIGENREC in suggesting top-\( N \) items, we have adopted the methodology proposed by Cremonesi et al. in [13]. In particular, we form a probe set \( P \) by randomly sampling 1.4% of the ratings of the dataset, and we use each item \( v_j \), rated with 5-star by user \( u_i \) in \( P \) to create the test set \( T \). For each item in \( T \), we select randomly another 1000 unrated items of the same user, we rank the complete lists (containing 1001 items) using each of the competing methods, and we measure the respective recommendation quality.

First we test the recommendation performance of EIGENREC in the MRR metric for scaling factors in the range \([-2, 2]\) using all three similarity matrices. We choose the MRR metric for this test simply because it can summarize the recommendation performance in a single number which allows direct comparisons between different similarity matrices as well as different scaling parameters for each given matrix. Figure 2 reports the MRR scores as a function of the parameter \( d \) for every case, using the number of latent factors that produces the best possible performance for each matrix.

We see that the best performance is achieved for small positive values of parameter \( d \). This was true for every similarity matrix tested, and for both datasets. Notice that this parameter was included in our model as a means to control the sensitivity of the inter-item proximity scores to the prior popularity of the items under consideration. Our results suggest, that while this popularity is important (i.e. every time the best performing scaling factor was strictly positive), its contribution to the final matrix \( A \) should be weighted carefully so as not to overshadow the pure similarity component.

We see that our models outperform PureSVD every time, with the performance gap being significantly larger for the Yahoo dataset, which had a steeper performance decay as the scaling factors moved towards 1 (see Figure 2). Remember here, that the “black box” approach of the traditional PureSVD assumes cosine similarity (which is usually great) with scaling parameter \( d \) equal to 1 (which is usually not). As can be seen in Table 3, simply controlling parameter \( d \) alone results to 51.59% recommendation performance gain with respect to PureSVD. We find this particularly interesting, as it uncovers a fundamental limitation of the traditional PureSVD approach, that can be trivially alleviated through our approach.

Using the Jaccard similarity matrix, we test EIGENREC against the 5 state of the art graph-based methods described earlier. Figure 3 reports the Recall as a function of \( N \) (i.e. the number of items recommended), the Precision as a function

| TABLE 3 |
| --- |
| **Ranking Performance for Different Proximity Matrices** |
| | PureSVD | Cosine | Pearson | Jaccard |
| Yahoo | 22.22% | 33.66% | 33.46% | 34.11% |
| ML1M | 30.48% | 33.12% | 32.84% | 32.40% |

Fig. 2. Testing the recommendation performance of EIGENREC on the MRR metric for scaling factors in the range \([-2, 2]\) using all three similarity matrices.
of the Recall, the Normalized Discounted Cumulative Gain as a function of $N$ and the RScore as a function of the halflife parameter $\alpha$, for the Yahoo (first row) and the MovieLens1M (second row) datasets. As for Recall($N$) and NDCG@N, we consider values of $N$ in the range $[1, \ldots, 20]$; larger values can be safely ignored for a typical top-$N$ recommendation task [13]. As we can see, EIGENREC outperforms every other method considered, for all datasets and in all metrics, reaching for example, at $N = 10$ a recall around 60%. This means that 60% of the 5-starred items were presented in the top-10 out of the 1001 places in the recommendation lists of the respective users.

Finally, for completeness, we also run EIGENREC on the standard MovieLens100K dataset using the five predefined splittings into training and test sets. In this test, and in order to allow direct comparisons with many different results to be found in the literature [19], [22], [42], [43], [44], [45], we use the Degree of Agreement metric. Degree of Agreement (DOA) is a variant of Somer’s D statistic, that have been used by many authors for the performance evaluation of top-$N$ recommendations on the MovieLens100K dataset (see [42] for details). EIGENREC obtained a macroaveraged DOA score of 92.81 and a microaveraged DOA of 91.18. To the best of our knowledge these scores are the highest achieved thus far on this benchmark dataset.

### 4.2.2 Long-Tail Recommendation

The distribution of rated items in recommender systems is long-tailed, i.e. most of the ratings is concentrated in a few very popular items, leaving the rest of the itemspace unevenly sparse. Of course, the recommendation of popular items is an easy task, that brings little utility in recommender systems; on the other hand, the task of recommending long-tail items adds novelty and serendipity to the users [13], and it is also known to increase substantially the profits of e-commerce companies [18], [46]. The innate sparsity of the problem however – which is aggravated even more for long-tail items – presents a major challenge for the majority of state-of-the-art collaborative filtering methods.

To evaluate EIGENREC in recommending long-tail items, we adopt the methodology described in [13]. In particular, we order the items according to their popularity which was measured in terms of number of ratings, and we partition the test set $T$ into two subsets, $T_{\text{tail}}$ and $T_{\text{head}}$, that involve items originated from the long-tail, and the short-head of the distribution respectively. We discard the items in $T_{\text{head}}$ and we evaluate EIGENREC and the other algorithms on the $T_{\text{tail}}$ test set, using the procedure explained in Section 4.2.1.

Having evaluated the performance of EIGENREC in the MRR metric for all three similarity matrices, we obtained very good results for every case, with marginally better recommendation quality achieved for the Jaccard similarity component with 241 and 270 latent factors and scaling factor 0.2 and 0.4 for the Yahoo and the MovieLens datasets respectively. Proceeding with these parameter settings we run EIGENREC against the other graph-based algorithms and we report the results in Figure 4. It is interesting to notice that MFA and $L_{\dagger}$ do particularly well in the long-tail recommendation task, especially in the sparser Yahoo dataset. They even manage to surpass RED, who had reached the second place when the popular items were included (Figure 3). Once again, we see that EIGENREC achieves the best results, in all metrics and for both datasets.

We have seen that both in standard and in long-tail recommendation scenarios, our approach gives very good results, consistently outperforming – besides PureSVD – a number of elaborate graph-based methods, known to work very well in uncovering nontrivial similarities through the exploitation of transitive relations that the graph representation of the data

**Fig. 3.** Recommendation quality on MovieLens1M and Yahoo/R2Music datasets using Recall@N, Precision and NDCG@N and RScore metrics.
brings to light [9]. In our final set of experiments, presented in the following section, we test the merits of EIGENRec in dealing with sparsity in its most extreme manifestations; the Cold-Start Problems.

### 4.2.3 Cold-Start Recommendation

The cold-start problem refers to the challenging task of making reliable recommendations for a system experiencing an initial lack of ratings [17]. This is a very common problem faced by real recommender systems in their beginning stages, when there is no adequate number of ratings for the collaborative filtering algorithms to uncover similarities between items or users (New Community Problem). The problem however can arise also through the introduction of new users to an existing system (New Users Problem); naturally newly emerging users have not rated many items yet, making it difficult for the collaborative filtering algorithm to produce qualitative personalized recommendations. This can be seen as a type of localized sparsity problem and it represents one of the continuing challenges faced by recommender systems in use [47].

**New Community Problem:** To test the performance of EIGENRec in dealing with the new community problem, we conduct the following experiment: We simulate the phenomenon by randomly selecting to include 33%, and 66% of the Yahoo dataset on two new artificially sparsified versions in such a way that the first dataset is a subset of the second. The idea is that these new datasets represent snapshots of the initial stages of the recommender system, when the community of users was new and the system was lacking ratings [45]. Then, we take the new community datasets and we create test sets following the methodology described in Section 4.2.1; we run all the algorithms and we evaluate their performance using the MRR, which makes it easier to compare the top-\(N\) quality for the different stages in the system’s evolution. We test for both standard and long-tail recommendation and we report the results in Figure 5. We clearly see that EIGENRec outperforms every other algorithm, even in the extremely sparse initial stage where the system is lacking 2/3 of its ratings.

In the figure, we report the qualitative results using the Cosine similarity this time, however, the performance of all three similarity components we propose was found to be equally good.

![Fig. 5. New Community recommendation quality on Yahoo datasets using MRR metric.](image)

![Fig. 4. Long-Tail recommendation quality on MovieLens1M and Yahoo!R2Music datasets using Recall@N, Precision and NDCG@N and RScore metrics.](image)
**New Users Problem**: In order to evaluate the performance of our algorithm in coping with new users problem, we again use the Yahoo dataset and we run the following experiment. We randomly select 50 users having rated 100 items or more, and we randomly delete 95% of their ratings. The idea is that the modified data represent an “earlier version” of the dataset, when these users were new to the system, and as such, had fewer ratings. Then, we take the subset of the dataset corresponding to these new users and we create the test set as before, using 10% as a cut-off for the Probe Set this time, in order to have enough 5-rated movies in the Test Set to estimate reliably the performance quality. The results are presented in Figure 6. We see that EIGENREC manages to outperform all competing algorithms in all metrics as before.

### 4.3 Discussion

The qualitative results presented above indicate that our method is able to produce high-quality recommendations, alleviating significant problems related to sparsity. Let us mention here that the competing algorithms are considered among the most promising methods in the literature to address sparsity problems [9], [22]. This was verified in our experiments as well. Indeed, our results clearly show that the graph-based methods perform very well with their comparative performance increasing with the sparsity of the underlying dataset, and reaching its maximum in the cold-start scenarios. EIGENREC nonetheless managed to perform even better, in every recommendation setting considered; a fact, that together with its significantly friendlier computational profile, makes it a qualitative and feasible option for realistic top-$N$ recommendation settings.

The choice of the scaling factor was found to be particularly significant for each and every pure similarity component. For the cosine similarity, in particular, we observed that the best results were always achieved for scaling parameters away from 1, making the traditional PureSVD algorithm, “qualitatively dominated” in every case considered. Regarding the best choice for the pure similarity component, the differences in recommendation quality observed in our experiments were relatively small. Therefore, our observations suggest that, at least for the recommendation scenarios considered in this work, all three simple inter-item proximity matrices present good candidates for high-quality recommendations, with the $K_{cos}$ being slightly more convenient to handle computationally.

### 5 Additional Remarks on Related Work

Factorization of a sparse similarity matrix was used to predict ratings in the EigenTaste system [27]. The authors first calculate the Pearson’s correlation scores between the jokes and then form a denser latent space in which they cluster the users following a scheme related to the well-known spectral clustering method [48]. The predicted rating of a user about a particular item is then calculated as the mean rating of this item, made by the rest of the users in the same cluster. The approach followed here differs significantly. The fact that we pursue ranking-based recommendations grants us the flexibility of not caring about the exact recommendation scores and allows us to introduce our novel proximity matrix, which except its pure similarity core also includes an important scaling component which was found to greatly influence the overall quality in every recommendation scenario.

In the literature there have been proposed many “content aware” methods (both learning-based [49], [50], [51] and graph-based [16], [19]) that deal with cold-start problems exploiting meta-information about the items and/or the users outside the ratings matrix (e.g. the genre or the director of a movie, the composer of a piece of music etc.). EIGENREC is a pure collaborative filtering method, i.e. it neither assumes nor exploits any information about the users of the items other than the rating matrix. In our experiments, we have tested its innate ability to alleviate sparsity related problem and found that it managed to do very well even in their most extreme manifestations.

At the computational core of our method lays the classic Lanczos algorithm that has been extensively used in the context of numerical linear algebra for the computation of the eigenvectors and/or singular triplets of large sparse matrices [29]. From a qualitative perspective, Blom and Ruhe [52] suggested the use of an algorithm closely related to Latent Semantic Indexing, which employs the Lanczos bidiagonalization technique to generate two sets of vectors that essentially replace
the left and right singular vectors, lowering the computational cost. Chen and Saad [53] have recently examined the use of Lanczos vectors in applications where the major task can be reduced to computing a matrix-vector product in the principal singular directions of the data matrix; they demonstrated the effectiveness of this approach on two different problems originated from information retrieval and face recognition. Finally, in an earlier version of this work presented in [54], we examined the use of Lanczos vectors for a very fast “crude” construction of a latent space that avoids overfitting extremely sparse datasets.

6 Conclusions

In this work, we propose EigenRec; an expressive and computationally efficient latent factor framework for top-\(N\) recommendation that generalizes and refines the standard PureSVD method. EigenRec “uncovers” PureSVD’s modeling core and treats it as a special case of a general inter-element proximity matrix consisting of a similarity and a scaling component. We propose three simple inter-item proximity matrices that are capable of increasing significantly the recommendation quality with respect to PureSVD, and we show how to handle them efficiently in a way that allows building the latent model without assembling or explicitly forming them in the process.

Based on the formulation of the problem developed in this paper and using our computational approach, one can achieve a more economical computation of PureSVD, with respect to the application of the standard truncated singular value decomposition on the rating matrix. In particular, the reduced dimension subspace built by EigenRec is based on the eigendecomposition of a significantly lower-dimensional matrix, which can be performed very efficiently by exploiting the Lanczos method. Having implemented the suggested approach in distributed computing environments and after performing several experiments on different datasets of varying size and complexity, we find that the computational profile of our method makes it readily applicable in real-world large-scale systems.

Finally, from a qualitative perspective, a comprehensive set of experiments on real datasets shows that EigenRec achieves very good results in widely used metrics against several state-of-the-art collaborative filtering techniques. Our method was also shown to behave particularly well even when the sparsity of the dataset is severe – as in the New Community and the New Users versions of the Cold-Start problem – where it was found to outperform all other methods considered, including the very promising for their anti-sparsity properties graph-based techniques. In conclusion, our findings suggest that both the computational profile of EigenRec and its qualitative performance make it a promising candidate for the Standard, Long-Tail and Cold-Start recommendation tasks even in big data scenarios.

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ACKNOWLEDGMENTS

We are grateful to the Minnesota Supercomputing Institute, for providing us with computational resources for our large-scale experiments. We would also like to thank Efratios Gallopoulos, for insightful discussions.