Some parametric congruences involving generalized central trinomial coefficients

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Received: 30 January 2023 / Accepted: 3 September 2023
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Abstract
For \( n = 0, 1, 2, \ldots \) and \( b, c \in \mathbb{Z} \), the \( n \)th generalized central trinomial coefficient \( T_n(b, c) \) is the coefficient of \( x^n \) in the expansion of \( (x^2 + bx + c)^n \). In particular, \( T_n = T_n(1, 1) \) (\( n = 0, 1, 2, \ldots \)) are central trinomial coefficients. Let \( p \) be an odd prime. For any \( b, c \in \mathbb{Z} \) with \( p \nmid bc(b + 2c) \), we determine

\[
\sum_{k=0}^{p-1} \frac{(2k) T_k(b, c^2)}{4^k (b + 2c)^k} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{(2k) T_k(b, c)}{(4b)^k}
\]

modulo \( p^2 \). As consequences,

\[
\sum_{k=0}^{p-1} \frac{1}{12^k} T_k \equiv \left( \frac{P}{3} \right) \frac{3^{p-1} + 3}{4} \pmod{p^2}
\]

provided \( p > 3 \) (where \((-\)) denotes the Legendre symbol), and

\[
\sum_{k=0}^{p-1} \frac{(2k) T_k(2, -1)}{8^k} \equiv \begin{cases} 
2x - p/(2x) & \pmod{p^2} \\
0 & \pmod{p^2}
\end{cases} \text{ if } p = x^2 + 4y^2 \text{ for } (x, y) \in \mathbb{Z} \text{ and } 4 \mid x - 1,
\]

\[
\text{if } p \equiv 3 \pmod{4}.
\]

Keywords Congruences · Generalized central trinomial coefficients · Binomial coefficients · Harmonic numbers

Mathematics Subject Classification Primary 11A07 · 11B75; Secondary 05A10 · 11B65

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1 Introduction

For any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $b, c \in \mathbb{Z}$, the generalized central trinomial coefficient $T_n(b, c)$ (cf. [12]) is defined as the coefficient of $x^n$ in the expansion of $(x^2 + bx + c)^n$ (or the constant term in the expansion of $(x + b + c/x)^n$). By the multinomial theorem, it is clear that

$$T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{n}{k} b^{n-2k} c^k,$$

where $\lfloor x \rfloor$ is the floor function. The generalized central trinomial coefficients have many interesting combinatorial interpretations; for example, from (1.1), it is easy to see that $T_n(b, c)$ with $b, c \in \mathbb{N}$ counts the colored lattice paths from $(0, 0)$ to $(n, 0)$ using only steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, where $H$ and $D$ may have $b$ and $c$ colors, respectively. Note that $T_n := T_n(1, 1)$ is the $n$th central trinomial coefficient and $T_n(2, 1)$ is exactly the central binomial coefficient $\binom{2n}{n}$. The generalized central trinomial coefficients are also related to the well-known Legendre polynomials

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\frac{x-1}{2}\right)^k = \sum_{k=0}^{n} \binom{n}{2k} (\frac{\sqrt{x^2-1}}{2})^k \left(x - \sqrt{x^2-1}\right)^{n-k}$$

(cf. [4, p. 38]) via the following identity (see [12, 15, 16]):

$$T_n(b, c) = (\sqrt{d})^n P_n \left(\frac{b}{\sqrt{d}}\right),$$

where $d = b^2 - 4c \neq 0$.

It is known that sums involving products of some binomial coefficients (e.g., $\binom{2k}{k}, \binom{2k}{k}^2, \binom{2k}{3k}, \binom{2k}{k}^3$) usually have some interesting congruence properties. Since $T_n(b, c)$ is a natural extension of $\binom{2n}{n}$, Sun [15, 16] investigated congruences for sums involving generalized central trinomial coefficients systematically. In particular, Sun [16, Theorem 2.1] determined

$$\sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c) \equiv \left(\frac{b}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}}{64^k} \equiv \left(\frac{P}{5}\right) \pmod{p},$$

for any $b, c, m \in \mathbb{Z}$ and odd prime $p$ with $p \nmid m$. As a corollary, he obtained that

$$\sum_{k=0}^{p-1} \frac{(2k)}{12k} T_k(b, c) \equiv \left(\frac{6}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}}{64^k} \equiv \left(\frac{P}{5}\right) \pmod{p},$$

where $(-)$ denotes the Legendre symbol. For more congruence properties of the generalized central trinomial coefficients, one may consult [2, 5, 6, 10, 15, 16].

As in [7], for any $n \in \mathbb{N}$ and $x \in \mathbb{C}$, we define

$$w_n(x) := \begin{cases} \frac{(x+1)x^n-(x^{-1}+1)x^{-n}}{x-x^{-1}}, & \text{if } x \neq \pm 1, \\ 2n + 1, & \text{if } x = 1, \\ (-1)^n, & \text{if } x = -1, \end{cases}$$

where $\alpha = x + \sqrt{x^2-1}$.

The first purpose of this paper is to establish the following parametric congruence as a generalization of (1.4).
Some parametric congruences

Theorem 1.1  Let $p$ be an odd prime and let $b, c \in \mathbb{Z}$ with $p \nmid c(b + 2c)$. Then

$$
\sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c^2) \equiv w(p-1)/2 \left( \frac{b - 6c}{b + 2c} \right) \pmod{p^2}.
$$

(1.5)

Applying Theorem 1.1 with $b = c = 1$, we obtain the following result conjectured by Sun [16, Conjecture 2.1].

Corollary 1.1  For any prime $p > 3$, we have

$$
\sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c^2) \equiv \left( \frac{p}{3} \right) \frac{3p-1}{4} \pmod{p^2}.
$$

(1.6)

Now we state our second theorem.

Theorem 1.2  Let $p$ be an odd prime. For any $b, c \in \mathbb{Z}$ with $p \nmid b$, we have

$$
\sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c) \equiv p \left[ \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \left( \frac{c}{b^2} \right)^k \right] \pmod{p^2}.
$$

(1.7)

Consequently,

$$
\sum_{k=0}^{p-1} \binom{2k}{k} T_k(2, -1) \equiv \begin{cases} 
2x - p/(2x) & \text{if } p = x^2 + 4y^2 (x, y \in \mathbb{Z}) \text{ and } 4 \mid x - 1,
0 & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
$$

(1.8)

To deduce (1.8) from (1.7), we need the following auxiliary result.

Theorem 1.3  Let $p$ be an odd prime. Then

$$
p \sum_{k=0}^{(p-1)/2} \binom{2k}{k} (-4)^k(4k + 1) \equiv \begin{cases} 
2x - p/(2x) & \text{if } p = x^2 + 4y^2 (x, y \in \mathbb{Z}) \text{ and } 4 \mid x - 1,
0 & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
$$

(1.9)

We are going to prove Theorem 1.1 and Corollary 1.1 in the next section, and show Theorems 1.2 and 1.3 in Sect. 3.

2 Proofs of Theorem 1.1 and Corollary 1.1

In order to show Theorem 1.1, we need the following transformation of $T_n(b, c^2)$ which follows from (1.3) and [4, (3.136)].

Lemma 2.1  For $n \in \mathbb{N}$ and $b, c \in \mathbb{Z}$ we have

$$
T_n(b, c^2) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} (b + 2c)^n (-c)^k.
$$

(2.1)
**Proof** Denote the right-hand side of (2.1) by $a_n$. Via the Zeilberger algorithm (cf. [13]), we find the following recurrence:
\[-(b - 2c)(b + 2c)(n + 1)a_n + b(2n + 3)a_{n+1} - (n + 2)a_{n+2} = 0 \quad (n \in \mathbb{N}).\]

Note that the same recurrence holds for $T_n(b, c^2)$. Moreover, it is easy to see that $T_0(b, c^2) = a_0$ and $T_1(b, c^2) = a_1$. Thus (2.1) follows by induction on $n$.\hfill \Box

**Lemma 2.2** Let $n, j \in \mathbb{N}$ with $n \geq j$. Then we have
\[
\sum_{k=j}^{n} \frac{\binom{2k}{k}}{4^k} = \frac{n+1}{2^{2n+1}(2j+1)} \binom{n}{j} \binom{2n+2}{n+1}.
\] (2.2)

**Proof** This can be easily proved by induction on $n$.\hfill \Box

Kh. Hessami Pilehrood, T. Hessami Pilehrood and R. Tauraso [7, Theorem 2] completely determined
\[
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} t^k}{2k+1} \equiv \frac{w(p-1)/2(1 - 8t) - (-16t)(p-1)/2}{p} \quad \text{(mod $p^3$)},
\]
where $p$ is an odd prime and $t$ is a $p$-adic unit. We need their result in the modulus $p$ case.

**Lemma 2.3** (cf. [7, Theorem 2]) For any odd prime $p$ and $t \in \mathbb{Z}_{p}^*$, we have
\[
\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} t^k}{2k+1} \equiv \frac{w(p-1)/2(1 - 8t) - (-16t)(p-1)/2}{p} \quad \text{(mod $p^3$)}.
\] (2.3)

**Lemma 2.4** For any odd prime $p$, we have
\[
\binom{2p}{p} \equiv 2 \quad \text{(mod $p^2$)} \quad \text{and} \quad \binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \quad \text{(mod $p^2$)}.
\]

**Proof** These can be verified directly for $p = 3$. For $p > 3$, we even have $\binom{2p}{p} \equiv 2 \quad \text{(mod $p^3$)}$ by J. Wolstenholme [18] and $\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \quad \text{(mod $p^3$)}$ by F. Morley [11].\hfill \Box

**Proof of Theorem 1.1** By Lemmas 2.1 and 2.2, we have
\[
\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(b, c^2)}{4^k(b + 2c)^k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k} }{4^k} \sum_{l=0}^{k} \binom{k}{l} \binom{2l}{l} \left( \frac{-c}{b + 2c} \right)^l
\]
\[
= \sum_{l=0}^{p-1} \binom{2l}{l} \left( \frac{-c}{b + 2c} \right)^l \sum_{k=l}^{p-1} \frac{\binom{2k}{k} }{4^k} \binom{k}{l}
\]
\[
= \frac{p^{(2p)}_{2}}{2^2 p - 1} \sum_{l=0}^{p-1} \binom{2l}{l} \binom{(p-1)/2}{l} \left( \frac{-c}{b + 2c} \right)^l / 2l + 1.
\]

Note that $\binom{2l}{l} / (2l + 1) \equiv 0 \quad \text{(mod $p$)}$ for $l \in \{(p + 1)/2, \ldots, p - 1\}$ and $\binom{(p-1)/2}{l} \equiv (-1)^l \quad \text{(mod $p$)}$ for $0 \leq l < p$. Thus we have
\[
\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(b, c^2)}{4^k(b + 2c)^k} \equiv \frac{2p^{(p-1)/2}_{p}}{2^2 p - 1} \left( \frac{-c}{b + 2c} \right)^{(p-1)/2} + \frac{p^{(2p)}_{p}}{2^2 p - 1} \sum_{l=0}^{p-1} \binom{2l}{l} \binom{(p-3)/2}{l} \left( \frac{-c}{b + 2c} \right)^l \quad \text{(mod $p^2$)}.
\]
In view of Lemma 2.4, we arrive at
\[
\sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c) \equiv \left( \frac{-16c}{b+2c} \right)^{(p-1)/2} + \frac{p}{4p-1} \sum_{l=0}^{(p-3)/2} \frac{(2l)(\frac{c}{b+2c})}{2l+1} \pmod{p^2}.
\]
Then we complete the proof by Lemma 2.3 and Fermat’s little theorem. \( \Box \)

**Proof of Corollary 1.1** To show (1.6), it remains to prove
\[
w_{(p-1)/2} \left( -\frac{5}{3} \right) \equiv \left( \frac{p}{3} \right) \frac{3^{p-1} + 3}{4} \pmod{p^2}. \tag{2.4}
\]
It is easy to see that
\[
w_{(p-1)/2} \left( -\frac{5}{3} \right) = \frac{(-1)^{(p-1)/2}}{4} \left( \frac{1}{3} \right) (1 + ph).
\]
From [8, p. 51], we know that \( a^{(p-1)/2} \equiv \left( \frac{a}{p} \right) \pmod{p} \) for any integer \( a \neq 0 \pmod{p} \). Thus we may write \( 3^{(p-1)/2} \) as \( (\frac{3}{p})(1 + ph) \), where \( h \) is a \( p \)-adic integer. In view of this,
\[
3^{p-1} = (3^{(p-1)/2})^2 \equiv 1 + 4ph \pmod{p^2}.
\]
By the above and with the help of the law of quadratic reciprocity (cf. [8]), we get
\[
w_{(p-1)/2} \left( -\frac{5}{3} \right) = \frac{(-1)^{(p-1)/2}}{4} \left( \left( \frac{3}{p} \right)(1 + ph) \right)
\equiv \frac{(-1)^{(p-1)/2}}{4} \left( \frac{3}{p} \right) (4 + 2ph)
\equiv \frac{3^{p-1} + 3}{4} \left( \frac{3}{p} \right) \left( -\frac{1}{p} \right)
\equiv \frac{p}{3} \frac{3^{p-1} + 3}{4} \pmod{p^2}.
\]
This proves (2.4). \( \Box \)

**3 Proofs of Theorems 1.2 and 1.3**

To show Theorem 1.3, we need the following identity due to Kummer (cf. [1, p. 126]).

**Lemma 3.1** For any \( a, b \in \mathbb{C} \) with \( a, a - b, a/2 - b \not\in \{-1, -2, -3, \ldots\} \), we have
\[
\sum_{k=0}^{\infty} \frac{(-1)^k(a)_k(b)_k}{(1)_k(a - b + 1)_k} = \frac{\Gamma(a - b + 1) \Gamma(\frac{a}{2} + 1)}{\Gamma(a + 1) \Gamma(\frac{a}{2} - b + 1)},
\]
where \((x)_k = \prod_{0 \leq j < k} (x + j)\) is the Pochhammer symbol and \( \Gamma(\cdot) \) stands for the Gamma function.

We also need Morita’s \( p \)-adic Gamma function \( \Gamma_p \) (cf. [14]), where \( p \) is an odd prime. Recall that \( \Gamma_p(0) := 1 \) and
\[
\Gamma_p(n) := (-1)^n \prod_{1 \leq k < n, 1 \text{ for } n = 1, 2, 3, \ldots \}
\]
\[\sum_{k=0}^{\infty} \frac{(-1)^k(a)_k(b)_k}{(1)_k(a - b + 1)_k} = \frac{\Gamma(a - b + 1) \Gamma(\frac{a}{2} + 1)}{\Gamma(a + 1) \Gamma(\frac{a}{2} - b + 1)},
\]
Let \( \mathbb{Z}_p \) denote the ring of all \( p \)-adic integers. The definition of \( \Gamma_p \) can be extended to \( \mathbb{Z}_p \) since \( \mathbb{N} \) is a dense subset of \( \mathbb{Z}_p \) with respect to the \( p \)-adic norm. It follows that

\[
\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} 
-x, & \text{if } x \neq 0 \pmod{p}, \\
-1, & \text{if } x \equiv 0 \pmod{p}.
\end{cases}
\]

It is known (cf. [14, p. 369]) that for any \( x \in \mathbb{Z}_p \) we have

\[
\Gamma_p(x)\Gamma_p(1-x) = (-1)^{(x)_p^{-1}},
\]

where \( (x)_p \) is the least nonnegative residue of \( -x \) modulo \( p \). It is also known (cf. [17]) that for \( \alpha, \tau \in \mathbb{Z}_p \) we have

\[
\Gamma_p(\alpha + \tau p) \equiv \Gamma_p(\alpha) \left( 1 + \tau p(\Gamma_p'(0) + H_{p-1-(\tau \alpha)_p}) \right) \pmod{p^2},
\]

where \( H_n = \sum_{k=1}^n 1/k \) denotes the \( n \)th harmonic number.

**Proof of Theorem 1.3** It is easy to verify that

\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)_k}{(-4)^k(4k+1)} \equiv \sum_{k=0}^{(p-1)/2} \frac{(1/2)_k(1/4)_k(-1)_k}{(1)_k(5/4)_k} \pmod{p^2}.
\]

We first assume that \( p \equiv 1 \pmod{4} \). In this case, \( \text{ord}_p(p/(5/4)_k) \geq 0 \) for all \( k \) among \( 0, 1, \ldots, (p-1)/2 \), where \( \text{ord}_p(\cdot) \) stands for the \( p \)-adic order. It is easy to verify that

\[
p \sum_{k=0}^{(p-1)/2} \frac{(1-p/2)_k(1-2p/4)_k(-1)_k}{(1/2)_k(5/4)_k} \equiv \left( 1 - \frac{p}{2} \sum_{j=0}^{k-1} \frac{1}{1/2 + j} - \frac{p}{2} \sum_{j=0}^{k-1} \frac{1}{1/4 + j} \right) \pmod{p^2}
\]

and

\[
p \sum_{k=0}^{(p-1)/2} \frac{(2-p/4)_k(1-p/4)_k(-1)_k}{(1)_k(5/4)_k} \equiv \left( 1 - \frac{p}{4} \sum_{j=0}^{k-1} \frac{1}{1/2 + j} - \frac{p}{4} \sum_{j=0}^{k-1} \frac{1}{1/4 + j} \right) \pmod{p^2}.
\]

On the other hand, by Lemma 3.1 we have

\[
\sum_{k=0}^{(p-1)/2} \frac{(1-p/2)_k(1-2p/4)_k(-1)_k}{(1)_k(5/4)_k} = \lim_{t \to 1} \sum_{k=0}^{\infty} \frac{(1-tp/2)_k(1-2tp/4)_k(-1)_k}{(1)_k(5/4)_k}
\]

\[
= \lim_{t \to 1} \frac{\Gamma(5/4)\Gamma(3-tp/2)\Gamma(4+tp/4)}{\Gamma(3-tp/2)\Gamma(4+tp/4)} = \frac{\Gamma(5/4)\Gamma(p-1/4)}{\Gamma(3-tp/2)\Gamma(4+tp/4)} \lim_{t \to 1} \frac{\sin(3-tp/2\pi)}{\sin(5-tp/2\pi)}
\]

\[
= (-1)^{(p-1)/4} \frac{2\Gamma(5/4)\Gamma(p-1/4)}{\Gamma(3-tp/2)\Gamma(4+tp/4)}.
\]
where we have used the well-known formula \( \Gamma(x)\Gamma(1 - x) = \pi / \sin(\pi x) \) (cf. [14, p. 371]). Also,
\[
(p^{-1}/2) \sum_{k=0}^{p-1/2} \frac{(2 - p)k(1 - p)k(-1)^k}{(1)_k(\frac{5}{4})_k} = \frac{\Gamma(\frac{5}{4})\Gamma(10 - p)}{\Gamma(\frac{6 - p}{4})\Gamma(\frac{8 + p}{8})}.
\]
Combining the above we obtain
\[
p\sum_{k=0}^{(p^{-1}/2)} \frac{(1/2)k(1/2)k(-1)^k}{(1)_k(\frac{5}{4})_k} = \sigma_1 - \sigma_2 \pmod{p^2},
\]
where
\[
\sigma_1 := \frac{2p\Gamma(\frac{5}{4})\Gamma(10 - p)}{\Gamma(\frac{6 - p}{4})\Gamma(\frac{8 + p}{8})} \quad \text{and} \quad \sigma_2 := (-1)^{(p-1)/4} \frac{2p\Gamma(\frac{5}{4})\Gamma(p^{-1})}{\Gamma(\frac{p-1}{4})\Gamma(\frac{4 + p}{4})}.
\]
By [9], we have
\[
H_{[p/2]} \equiv -2q_p(2) \pmod{p} \quad \text{and} \quad H_{[p/4]} \equiv -3q_p(2) \pmod{p},
\]
where \( q_p(2) \) denotes the Fermat quotient \((2^{p-1} - 1)/p\). It is easy to see that
\[
\frac{\Gamma(10 - p)}{\Gamma(\frac{8 + p}{8})} = (-1)^{(p-1)/4} \frac{8\Gamma_p(10 - p)}{p\Gamma_p(\frac{8 + p}{8})}.
\]
Thus, by (3.1) and (3.2) we have
\[
\sigma_1 = \frac{16\Gamma_p(\frac{5}{4})\Gamma_p(\frac{5}{4} - \frac{p}{8})}{\Gamma_p(\frac{3}{2} - \frac{p}{4})\Gamma_p(1 + \frac{p}{8})}
\]
\[
\equiv - \frac{16\Gamma_p(\frac{5}{4})^2}{\Gamma_p(\frac{3}{2})^2} \left( 1 - \frac{p}{8}H_{[p/4]} + \frac{p}{4}H_{[p/2]} \right)
\]
\[
\equiv -2\Gamma_p \left( \frac{1}{4} \right)^2 \Gamma_p \left( \frac{1}{2} \right) \left( 1 - \frac{p}{8}q_p(2) \right) \pmod{p^2}.
\]
Similarly, it is not hard to find that
\[
\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{4 + p}{4})} = (-1)^{(p-1)/4} \frac{4\Gamma_p(\frac{5}{4})}{p\Gamma_p(\frac{4 + p}{4})}.
\]
Thus, by (3.1) and (3.2) we arrive at
\[
\sigma_2 = (-1)^{(p-1)/4} \frac{8\Gamma_p(\frac{5}{4})\Gamma_p(\frac{p-1}{2})}{\Gamma_p(\frac{p-1}{4})\Gamma_p(\frac{4 + p}{4})}
\]
\[
\equiv (-1)^{(p+3)/4} \frac{8\Gamma_p(\frac{5}{4})\Gamma_p(-\frac{1}{2})}{\Gamma_p(-\frac{1}{4})} \left( 1 + \frac{p}{2}H_{[p/2]} - \frac{p}{4}H_{[p/4]} \right)
\]
\[
\equiv - \Gamma_p \left( \frac{1}{4} \right)^2 \Gamma_p \left( \frac{1}{2} \right) \left( 1 - \frac{p}{4}q_p(2) \right) \pmod{p^2}.
\]
In view of the above, we have

\[
p \sum_{k=0}^{(p-1)/2} \left( \frac{1}{2} \right)_k \left( \frac{1}{4} \right)_k (-1)^k \equiv -\Gamma_p \left( \frac{1}{4} \right)^2 \Gamma_p \left( \frac{1}{2} \right) \pmod{p^2}.
\]

By [3], if \( p = x^2 + 4y^2 \) (\( x, y \in \mathbb{Z} \)) with \( x \equiv 1 \pmod{4} \) then

\[
-\Gamma_p \left( \frac{1}{4} \right)^2 \Gamma_p \left( \frac{1}{2} \right) \equiv 2x - \frac{p}{2x} \pmod{p^2}.
\]

Thus, with aid of (3.3), we have (1.9) in the case \( p \equiv 1 \pmod{4} \).

Now we consider the remaining case \( p \equiv 3 \pmod{4} \). Note that \( \text{ord}_p(4k + 1) = 0 \) for \( k = 0, 1, \ldots, (p - 1)/2 \). Therefore, by Lemma 3.1 we have

\[
p \sum_{k=0}^{(p-1)/2} (-1)^k \left( \frac{1}{2} \right)_k \left( \frac{1}{4} \right)_k \equiv 0 \pmod{p^2},
\]

where in the last step we have used the fact that

\[
\lim_{u \to (1-p)/2} \frac{\Gamma(u)^2}{\Gamma(u+1)} = 0 \pmod{p^2}.
\]

Combining this with (3.3), we find that (1.9) also holds in the case \( p \equiv 3 \pmod{4} \).

By the above, we have completed the proof of Theorem 1.3. \( \square \)

**Proof of Theorem 1.2** In view of (1.1) and Lemma 2.2, we have

\[
\sum_{k=0}^{p-1} \left( \frac{2k}{k} \right) T_k(b, c) = \sum_{k=0}^{(p-1)/2} \frac{(2k)}{4^k} \sum_{j=0}^{[k/2]} \binom{k}{2j} \left( \frac{2j}{b^2} \right)^j \left( \frac{c}{b^2} \right)^j = \sum_{j=0}^{(p-1)/2} \binom{2j}{j} \left( \frac{c}{b^2} \right)^j \sum_{k=2j}^{p-1} \frac{(2k)(2j)}{4^k}.
\]

If \( p \equiv 3 \pmod{4} \), then \( p \nmid (4j + 1) \) for all \( j = 0, 1, \ldots, (p - 1)/2 \). In this case, by Lemma 2.4 and Fermat’s little theorem, we obtain from the above

\[
\sum_{k=0}^{p-1} \left( \frac{2k}{k} \right) T_k(b, c) = p \sum_{j=0}^{(p-1)/2} \frac{(2j)}{4j + 1} \left( \frac{c}{b^2} \right)^j \pmod{p^2}.
\]
Now suppose $p \equiv 1 \pmod{4}$. Then, by Lemma 2.4 and the first paragraph of this proof, we have
\[
\sum_{k=0}^{p-1} \frac{(2^k)}{k} T_k(b, c) \equiv \frac{p(2p)}{2^{2p-1}} \sum_{0 \leq j \leq (p-1)/2} \sum_{j \neq (p-1)/4} \frac{\binom{2j}{j}(p-1)}{4j+1} \left( \frac{c}{b^2} \right)^j + \frac{2p}{2^{2p-1}} \left( \frac{p-1}{2} \right) \left( \frac{p-1}{4} \right) \left( \frac{c}{b^2} \right)^{(p-1)/4} \\
= p \sum_{0 \leq j \leq (p-1)/2} \sum_{j \neq (p-1)/4} \frac{\binom{2j}{j}}{4j+1} \left( \frac{c}{b^2} \right)^j + \left( \frac{p-1}{2} \right) \left( \frac{p-1}{4} \right) \left( \frac{c}{b^2} \right)^{(p-1)/4} \\
= \sum_{j=0}^{(p-1)/2} \frac{\binom{2j}{j}}{4j+1} \left( \frac{c}{b^2} \right)^j \pmod{p^2}.
\]
Combining the above, we have proved (1.7).

In light of Theorem 1.3,
\[
\sum_{k=0}^{p-1} \frac{(2^k)}{k} T_k(2, -1) \equiv \frac{p}{8} \sum_{k=0}^{(p-1)/2} \frac{(2^k)}{k} \pmod{p^2}.
\]
Combining this with (1.7), we immediately obtain (1.8). This ends our proof. 

Acknowledgements We are grateful to the anonymous referee for valuable suggestions. This work is supported by the National Natural Science Foundation of China (Grants 12201301 and 12371004 respectively)

Data availability Data sharing not applicable since this paper contains no data.

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