Anomalous diffusion and Tsallis statistics in an optical lattice

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We point out a connection between anomalous quantum transport in an optical lattice and Tsallis’
generalized thermostatistics. Specifically, we show that the momentum equation for the semiclassical
Wigner function that describes atomic motion in the optical potential, belongs to a class of transport
equations recently studied by Borland [PLA 245, 67 (1998)]. The important property of these
ordinary linear Fokker–Planck equations is that their stationary solutions are exactly given by Tsallis
distributions. Dissipative optical lattices are therefore new systems in which Tsallis statistics can
be experimentally studied.

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Non-Gaussian distributions occur frequently in systems that do not follow the prescriptions of standard sta-
tistical mechanics. Prominent examples of non–Gaussian statistics are Lévy stable distributions [1]. The proba-
bility density of a one–dimensional symmetric Lévy stable distribution is defined by its Fourier transform as
\( \mathcal{L}_x(x) = 1/(2\pi) \int dk \exp[ikx - C|k|^\alpha] \), where \( 0 < \alpha \leq 2 \).
A key feature of such a stable distribution is the presence of an asymptotic, non–Gaussian, power–law tail,
\( \mathcal{L}_x(x) \sim 1/|x|^{\alpha+1} \), when \( \alpha < 2 \). This leads to the im-
portant consequence that, except in the Gaussian case \( \alpha = 2 \), a Lévy probability density has a divergent sec-
ond moment [2]. Signatures of Lévy statistics have been experimentally observed in a variety of physical systems
[3] ranging from micelle systems [4] to porous glasses [5] and subrecoil laser cooling [6]. Another non–Gaussian
distribution, which naturally arises within the framework of nonextensive statistical mechanics [7,8], is the Tsallis
distribution, \( P_x(x) = Z_q^{-1}[1 - \beta(1 - q)x^2]^{1/(1-q)} \) with \( 1 \leq q < 3 \) [9]. Like a Lévy stable distribution, the function
\( P_x(x) \) exhibits an asymptotic, non–Gaussian, algebraic tail, \( P_x(x) \sim 1/x^{2/(\alpha-1)} \), for a Tsallis index \( q \neq 1 \). Typ-
ical systems where Tsallis’ generalized thermostatistics has been applied are those involving long–range correlations,
such as self–gravitating systems [8] or long–range magnetic systems [10], and systems with fluctuating tem-
perature [11]. In the last decade the theory of nonextensive statistical physics witnessed a tremendous develop-
ment [12] and there is now also growing experimental evidence of the relevance of Tsallis statistics in describing
physical processes [8]. As an example we mention fully developed turbulence, where it has recently been shown
that velocity fluctuations are well described by a Tsallis–like distribution [13]. However, there is a need for more experimental support.

Our aim in this paper is to show that there is a connec-
tion between Tsallis statistics and anomalous quantum transport in optical lattices. An optical lattice is a
standing–wave potential that can be obtained by superposition of counter–propagating laser beams with linear orthogonal polarizations (for a recent review see [14]).

The optical potential so produced is spatially periodic and, as a consequence, shares many common properties
with crystalline lattices in solid–state physics, such as Bragg scattering and Bloch oscillations. The main ad-
vantage of an optical crystal compared to its condensed–matter counterpart is that the optical periodic potential
is exactly known and, furthermore, easily modified in a precise and controlled way. Originally designed for laser
cooling (Sisyphus cooling) (for an introduction see [15]), optical lattices rapidly evolved to an active field of investi-
gation of its own [14].

An important issue in this context is the understanding of atomic transport in the optical lattice. Depending on
the depth of the optical potential, three different regimes can be identified [14,16–20]: (i) diffusive motion in deep
potentials, (ii) ballistic motion in shallow potentials and (iii) an intermediate regime in between (of main interest
here) where anomalous (non–Gaussian) diffusion takes place. The existence of Lévy–like diffusion with long
jumps below a given potential threshold has been predicted by Marksteiner et al. [19] and later experimentally
verified by a group at the MPQ in Garching by studying the dynamics of a single ion in a one–dimensional optical
lattice [20]. In the following, we show that the equation governing the evolution of the semiclassical momentum
distribution of the atom in the optical potential belongs to a family of ordinary linear Fokker–Planck equations re-
cently defined by Borland [21]. The interesting property of these equations is that their stationary solutions are
exactly given by Tsallis statistics. This allows us not only to express the indices \( q \) and \( \beta \) of the Tsallis distribution in
terms of the microscopic parameters of the quantum optical problem, but also to give a physical explanation for
the non-normalizability of the distribution, as well as for the divergence of its variance in some range of parameters
to be specified. We also introduce the Rayleigh equation for a quantum Lévy process and compare the properties
of its stationary solution with those of the Tsallis distribution. We find that the two processes lead to different,
experimentally observable, predictions.

Starting from the microscopic Hamiltonian that de-
scribes the atom–laser interaction in the optical lattice, an atomic quantum master equation can be derived [22]. After spatial averaging, the Rayleigh equation for the corresponding semiclassical Wigner function \(W(p,t)\) can be written as [16,17,19],

\[
\frac{\partial W}{\partial t} = -\frac{\partial}{\partial p} [K(p)W] + \frac{\partial}{\partial p} \left[ D(p) \frac{\partial W}{\partial p} \right].
\]

(1)

Equation (1) has the form of an ordinary linear Fokker–Planck equation with momentum–dependent drift and diffusion coefficients,

\[
K(p) = -\frac{\alpha p}{1 + (p/p_c)^2}, \quad D(p) = D_0 + \frac{D_1}{1 + (p/p_c)^2}.
\]

(2)

These two quantities have a simple physical interpretation: The drift \(K(p)\) represents a cooling force (due to the Sisyphus effect) with damping coefficient \(\alpha\). This force acts only on slow particles with a momentum smaller than the capture momentum \(p_c\). This is an important point as we shall discuss below. The diffusion factor \(D(p)\), on the other hand, describes stochastic momentum fluctuations and accounts for heating processes. We note that \(D(p)\) has two contributions [16]: A constant part \(D_0\) which corresponds to fluctuations due to spontaneous photon emissions and fluctuations in the difference of photons absorbed in the two laser beams, plus a term proportional to \(D_1\) which stems from fluctuations in the dipolar forces. This last term has the same limited momentum range \(p_c\) as the drift force. Interestingly, we remark that for vanishing \(D_1\), Eq. (1) exactly reduces to the Fokker–Planck equation considered by Starilo and shown to give rise to nonextensive statistics [23].

It is easily seen from Eq. (2) that \(K(p)\) and \(D(p)\) satisfy the following condition,

\[
\frac{K(p)}{D(p)} = -\frac{\beta}{1 - (1 - q)U(p)} \frac{\partial U(p)}{\partial p},
\]

with

\[
\beta = \frac{\alpha}{2(D_0 + D_1)}, \quad q = 1 + \frac{2D_0}{\alpha p_c^2} \quad \text{and} \quad U(p) = p^2.
\]

(3)

(4)

Equation (3) has been first obtained by Borland [21]. We mention that in her original work, Borland considered the Ito–form of the Fokker–Planck equation, whereas here, Eq. (3) applies to the Stratonovich–form (1). The condition (3) implies that the stationary solution \(W_q(p)\) of the Rayleigh equation (1) is given by the Tsallis distribution,

\[
W_q(p) = Z_q^{-1}[1 - (1 - q)U(p)]^{1/(1-q)}.
\]

(5)

Equation (5) is the exact general stationary solution of Eq. (1) with the requirement \(W_q(p) \to 0\) when \(p \to \pm \infty\), the constant \(Z_q\) being a normalizing factor. The fact that the steady–state solution of Eq. (1) is non–Gaussian is of course well-known [16,17,19]. Surprisingly, however, it has not been realized that this precisely corresponds to a Tsallis distribution. Among infinitely many non–Gaussian distributions, Eq. (3) singles out the nonextensive Tsallis distribution (5) (see Fig. 1). It is worth noting that the Tsallis indices \(q\) and \(\beta\) can be simply expressed in terms of the microscopic parameters of the problem [see Eqs. (4)]. In particular, we see that \(q\) depends on the ratio of the diffusion constant \(D_0\) to the product of the friction coefficient \(\alpha\) with the square of the capture momentum \(p_c\), and does not depend on \(D_1\). Equations (4) thus provide a link between the macroscopic Tsal- lis distribution (5) and the underlying microscopic dynamics in the optical potential. This allows us to give a physical interpretation of the characteristics of the nonextensive distribution (5).

Let us first remind that the distribution (5) is not normalizable for a Tsallis index \(3 \leq q\) or, equivalently, for \(\alpha p_c^2 < D_0\). Physically, this means that the cooling force is too weak, compared to the random momentum fluctuations, to maintain the particle in a steady state around \(p = 0\) (this is often referred to as décrochage [16,17]). On the other hand, in the limit where \(q \to 1\) \((D_0 \ll \alpha p_c^2)\), the stationary solution (5) reduces to the standard Maxwell–Boltzmann distribution, \(W_1(p) = Z_1^{-1}\exp[-\beta U(p)]\), with an inverse temperature \(\beta\). In this case the cooling force is much stronger than the random momentum fluctuations. It thus appears that the Tsallis index \(q\) is intimately related to the interplay between stochastic heating processes (momentum fluctuations, as measured by \(D_0\)) and the cooling force with capture momentum \(p_c\). It is important to remark that the finiteness of the latter is directly responsible for the occurrence of the non–Gaussian Tsallis distribution in this problem. This is confirmed by the observation that for infinite \(p_c\), Eq. (1) reduces to the Ornstein–Uhlenbeck equation with well–known Gaussian dynamics. Using the parametrization of Ref. [16], the index \(q\) can be further written as \(q = 1 + 4AE_R/U_0\), where \(U_0\) is the potential depth and \(E_R\) the recoil energy. We thus see that the Tsallis index can be related to the ratio of the recoil energy to the potential depth. This means that the nature of the atomic dynamics can be simply tuned by varying the depth of the optical lattice. We also notice that the inverse temperature \(\beta\) is written as the ratio of the friction coefficient to the sum of the diffusion coefficients, in analogy with the fluctuation–dissipation relation. We hasten to add that Eq. (5) corresponds to a steady state and not to an equilibrium state, and as such temperature is not well–defined in this problem.

We now turn to the intermediate regime with a Tsallis index \(5/3 < q < 3\) \((D_0 < \alpha p_c^2 < 3D_0)\). Here the second moment, \(\langle p^2 \rangle = \int p^2W_q(p)dp\), of the Tsallis distribution is infinite. As a consequence, the mean kinetic energy of the particle, \(E_K = \langle p^2 \rangle / 2m\), diverges. In this regime, rare but large momentum fluctuations occur that shove
the particle outside the range of the cooling force before it is recaptured again. This leads to anomalous momentum diffusion. The transition from Gaussian to anomalous diffusion as the depth of the optical lattice is decreased has recently been investigated experimentally and the divergence of the mean kinetic energy has been observed [20]. This is a clear signature of the underlying non–Gaussian statistics. A dissipative optical lattice hence appears as a unique system that allows experimental investigation of the Tsallis distribution in a whole range of \( q \) by simply varying a single parameter, the depth of the optical potential.

We emphasize that the non–Gaussian Tsallis statistics is here generated by an ordinary linear Fokker–Planck equation (3), which is often associated with the usual Boltzmann–Gibbs statistics. To our knowledge, atomic transport in an optical lattice constitutes the only physical system known so far where this occurs. Again, this results from the subtle interplay between the deterministic (drift) and stochastic (diffusion) forces (2) that act on the particle [21]. This is for instance at variance with the fully developed turbulence problem discussed in Ref. [13]. In the latter case, nonextensive statistics is obtained from a generalized Langevin equation with fluctuating friction and diffusion coefficients. For comparison, the Langevin equation that corresponds to the Rayleigh equation (1) reads

\[
\dot{p} = K(p) + \frac{\partial D(p)}{\partial p} + \sqrt{2D(p)} \eta(t),
\]

where \( \eta(t) \) is a centered Gaussian random force with variance \( \langle \eta(t)\eta(t') \rangle = \delta(t-t') \). Equation (6) is a Langevin equation with multiplicative white noise and deterministic coefficients.

![Diagram](image)

**FIG. 1.** Tsallis distribution \( W(q)(p) \) (5) for three values of the optical potential: (a) \( U_0 \gg 44E_R \), (b) \( U_0 = 44E_R \) and (c) \( U_0 = 24E_R \).

Anomalous diffusion in optical lattices shares common properties with a Lévy process and is often associated with it [19,20]. So let us next compare in some detail the two quantum processes based on Tsallis and Lévy statistics. The Klein–Kramers equation for a quantum Lévy process has been recently derived in [24] using a path–integral approach. The corresponding Rayleigh equation for a free quantum Lévy process can be obtained by integrating over the position,

\[
\frac{\partial W}{\partial t} = \frac{\gamma}{m} \frac{\partial}{\partial p}[pW] + D \frac{\partial^\alpha W}{\partial |p|^\alpha},
\]

where \( \gamma \) and \( D \) respectively denote the friction and diffusion coefficients. In Eq. (7), we have introduced the Riesz fractional derivative which is defined through its Fourier transform as [25,26],

\[
-\frac{\partial^\alpha}{\partial |p|^\alpha} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \exp[-ipy]|y|^\alpha.
\]

We point out that contrary to the ordinary Fokker–Planck equation (1), Eq. (7) is a generalized, fractional transport equation. It reduces to the Ornstein–Uhlenbeck equation when \( \alpha = 2 \). The stationary solution of Eq. (7) can be easily found by Fourier transformation [27]. It is given by the Lévy distribution, \( W_\alpha(p) = \mathcal{L}_\alpha(p) \) with \( C = mD/\alpha \gamma \). As already indicated in the introduction, the function \( W_\alpha(p) \) asymptotically decays according to a power–law, \( W_\alpha(p) \sim 1/|p|^{\alpha+1} \). This is reminiscent of a Tsallis distribution with index \( q = (3 + \alpha)/(1 + \alpha) \). However, there is an important difference between a Lévy and a Tsallis distribution: In the Lévy case, the second moment is finite for only one single value of the parameter \( \alpha = 2 \). By direct contrast, in the case of a Tsallis distribution, the second moment is finite in a parameter interval \( 1 < q < 5/3 \). This implies that the mean kinetic energy of the Tsallis particle is finite over a complete range of potential depths and not only for a particular value of \( U_0 \), as this would be the case for a particle obeying Lévy statistics. This is in agreement with what is measured experimentally [see Figure (4) of Ref. [20]].

It is further instructive to write down the Langevin equation that corresponds to the Rayleigh equation (7). It reads

\[
\dot{p} = -\frac{2}{m}p + \xi(t).
\]

Here \( \xi(t) \) is a centered Lévy distributed stochastic force. Its characteristic function is given by \( \hat{P}(k) = \int d\xi \exp[ik\xi]P(\xi) = \exp[-D|k|^\alpha] \). We see that for a Lévy process, the anomalous behavior of the particle finds its origin in the non–Gaussian properties of the noise. This is in contrast to the Langevin equation (6) where the noise is Gaussian. From all this we conclude that the strange kinetics that occurs in an optical lattice, in the semiclassical limit considered here, does not correspond to a quantum Lévy process, strictly speaking, but rather a “quantum Tsallis process”.

Interestingly, broad momentum distributions and Lévy statistics play an important role in another branch of atomic physics, namely subrecoil laser cooling — Lévy statistics being even used as a tool to optimize the cooling process [6]. The physical mechanism that leads here to
power–law momentum distributions of the cold atoms is based on the presence of a trap, \(|p| < p_{\text{trap}}\), around \(p = 0\) where particles remain for a long time during their random walk in momentum space, before leaving again. The broad momentum distribution can then be seen as resulting from the competition between the rates of entry and of departure in the trap [6].

In conclusion, we have shown that Tsallis statistics naturally appears in anomalous quantum transport in an optical lattice. Remarkably the Tsallis distribution is here generated by an ordinary linear Fokker–Planck equation and not by some generalized (non–linear) diffusion equations. Furthermore, the Tsallis index \(q\) can be simply expressed in terms of the microscopic parameters of the quantum–optical problem, in particular, the potential depth \(U_0\). This shows that the shape of the distribution can be straightforwardly modified — from a Gaussian to a uniform distribution — by solely varying \(U_0\). This opens the possibility of experimentally studying Tsallis distributions and non–Gaussian dynamics for any index \(q\).

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