Bounds for avalanche critical values of the Bak-Sneppen model

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Abstract
We study the Bak-Sneppen model on locally finite transitive graphs \( G \), in particular on \( \mathbb{Z}^d \) and on \( T_\Delta \), the regular tree with common degree \( \Delta \). We show that the avalanches of the Bak-Sneppen model dominate independent site percolation, in a sense to be made precise. Since avalanches of the Bak-Sneppen model are dominated by a simple branching process, this yields upper and lower bounds for the so-called avalanche critical value \( p^{BS}_c(G) \). Our main results imply that \( \frac{1}{\Delta+1} \leq p^{BS}_c(T_\Delta) \leq \frac{1}{\Delta-1} \), and that \( \frac{1}{2d+1} \leq p^{BS}_c(\mathbb{Z}^d) \leq \frac{1}{2d} + \frac{1}{(2d)^2} + O(d^{-3}) \), as \( d \to \infty \).

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1 Introduction and main results

The Bak-Sneppen model was originally introduced as a simple model of evolution by Per Bak and Kim Sneppen [1]. Their original model can be defined as follows. There are \( N \) species (vertices) arranged on a circle, each of which has been assigned a random fitness. The fitnesses are independent and uniformly distributed on \((0, 1)\). At each discrete time step the system evolves by locating the lowest fitness and replacing this fitness, and those of its
two neighbours, by independent and uniform \((0, 1)\) random variables. We say that a vertex whose fitness is changed by this procedure has been updated.

It is not particularly significant that the underlying graph of the model is the circle, or \(\mathbb{Z}\) in the thermodynamic limit. Bak-Sneppen models can be defined on a wide range of graphs using the same update rule as above: the vertex with minimal fitness and its neighbours are updated. Unlike particle systems such as percolation or the contact process, the Bak-Sneppen model has no tuning parameter. Therefore, it has been described as exhibiting self-organised critical behaviour, see [7] for a discussion.

One of the ways to analyse Bak-Sneppen models is to break them down into a series of avalanches. An avalanche from a threshold \(p\), referred to as a \(p\)-avalanche, is said to occur between times \(s\) and \(s + t\) if at time \(s\) all the fitnesses are equal to or greater than \(p\) with at most one vertex where equality holds, and time \(s + t\) is the first time after \(s\) at which all fitnesses are larger than \(p\). The vertex with minimal fitness at time \(s\) is called the origin of the avalanche. A \(p\)-avalanche can be considered as a stochastic process in its own right. The key feature of the origin is that it has the minimal fitness (as it will be updated immediately). Hence, we can consider its fitness to be any value, as long as this value is minimal. Vertices with fitness below the threshold are called active, others are called inactive. Note that the exact fitness value of an inactive vertex is irrelevant for the avalanche, since this value can never be minimal during the avalanche. This motivates the following formal definition of an avalanche.

**Definition 1.1** A \(p\)-avalanche with origin \(v\) on a graph \(G\) (with vertex set \(V(G)\)) is a stochastic process with state space \([0, p]^\mathbb{A}, A \subset V(G)\) and initial state \(p^{(v)}\). The process follows the update rules of the Bak-Sneppen model. Any vertex with a fitness smaller than or equal to \(p\) is included. Any vertex with a fitness larger than \(p\) is not included. The process terminates when it is the empty set.

Studying avalanches has considerable advantages. A Bak-Sneppen model on an infinite graph is not well-defined: when there are infinitely many vertices, there may not be a vertex with minimal fitness. However, Bak-Sneppen avalanches can be defined on any locally finite graph as follows: at time 0 all vertices have fitness 1, apart from one vertex, the origin of the avalanche, which has fitness \(p\). We then apply the update rules of the Bak-Sneppen
model, until all fitnesses are above $p$. This is consistent with our previous notion, as it is only the fitnesses updated during the avalanche that determine the avalanche’s behaviour.

The ability to look directly at infinite graphs is very desirable, because the most interesting behaviour of the Bak-Sneppen model is observed in the limit as the number of vertices in the graph tends to infinity.

In the literature alternative types of avalanches have been proposed, see [5, 6]. The definition given here corresponds to the most commonly used notion of an avalanche and was introduced by Bak and Sneppen [1]. For a more thorough coverage readers are directed to Meester and Znamenski [8, 9]. Note that unlike the Bak-Sneppen model itself, the avalanches do have a tuning parameter, namely the threshold $p$.

In this paper, we look mainly at transitive graphs. The behaviour of an avalanche on a transitive graph is independent of its origin: an avalanche with origin at vertex $v$ behaves the same as an avalanche with origin 0. When analysing avalanches on transitive graphs, it is therefore natural to talk about a typical $p$-avalanche without specifying its origin. To analyse avalanches, some definitions are needed. The set of vertices updated by an avalanche is referred to as its range set, with the range being the cardinality of this range set. Letting $r_{BS}^G(p)$ denote the range of a $p$-avalanche on a transitive graph $G$, we define the (avalanche) critical value of the Bak-Sneppen model as

$$p_{c}^{BS}(G) = \inf\{p : \mathbb{P}(r_{BS}^G(p) = \infty) > 0\}. \quad (1)$$

Numerical simulations [1] suggest that the stationary marginal fitness distributions for the Bak-Sneppen model on $N$ sites tend to a uniform distribution on $(p_{c}^{BS}(\mathbb{Z}), 1)$, as $N \to \infty$. It has been proved in [2] that this is indeed the case if $p_{c}^{BS} = \overline{p}_{c}^{BS}(\mathbb{Z})$, where $\overline{p}_{c}^{BS}(\mathbb{Z})$ is another critical value, based on the expected range, and is defined as

$$\overline{p}_{c}^{BS}(G) = \inf\{p : \mathbb{E}[r_{BS}^G(p)] = \infty\}. \quad (2)$$

It is widely believed, but unproven, that these two critical values are equal.

It should now be clear that knowledge about the value of $p_{c}^{BS}(G)$ is vital in determining the self-organised limiting behaviour of the Bak-Sneppen model, even though there is no tuning parameter in the model. Although in this paper we focus on the critical value (1), our bounds for the critical value (1) also hold for the critical value (2), see Section 6.
The approach of this paper is to compare Bak-Sneppen avalanches with two well-studied processes, namely branching processes and independent site percolation. A simple comparison with branching processes gives a lower bound on the critical value, whereas a more complex comparison with site percolation gives an upper bound. To warm up, we first give the (easy) lower bound.

**Proposition 1.2** On any locally finite transitive graph $G$ with common vertex degree $\Delta$, we have

$$p_{c}^{BS}(G) \geq \frac{1}{\Delta + 1}.$$  

**Proof:** At every discrete time step of the system, we draw $\Delta + 1$ independent uniform $(0, 1)$ random variables to get the new fitnesses of the vertex with minimal fitness, and of its $\Delta$ neighbours. Each of these $\Delta + 1$ new fitnesses is below the threshold $p$ with probability $p$, independent of each other. This induces a coupling with a simple branching process with binomial $(\Delta + 1, p)$ offspring distribution, where every active vertex in the Bak-Sneppen avalanche is represented by at least one particle in the branching process. Hence, if the branching process dies out, then so does the Bak-Sneppen avalanche. Therefore the critical value of the Bak-Sneppen avalanche can be no smaller than the critical value of the branching process. \hfill $\square$

The main result of this paper is the following upper bound for the critical value $p_{c}^{BS}(G)$ of the Bak-Sneppen model on a locally finite transitive graph $G$. The critical value for independent site percolation on $G$ is denoted by $p_{c}^{site}(G)$. We recall that for site percolation on $G$ with parameter $p$, the probability of an infinite cluster at the origin is positive for all $p > p_{c}^{site}(G)$, and 0 for all $p < p_{c}^{site}(G)$.

**Theorem 1.3** On any locally finite transitive graph $G$, we have

$$p_{c}^{BS}(G) \leq p_{c}^{site}(G).$$

This result implies that on many locally finite transitive graphs, $p_{c}^{BS}$ is non-trivial. For the Bak-Sneppen avalanche on $\mathbb{Z}$, Theorem gives a trivial upper bound, but in this case we know from that $p_{c}^{BS}(\mathbb{Z}) \leq 1 - \exp(-68)$.  

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Since the critical value of site percolation on $T_\Delta$, the regular tree with common degree $\Delta$, equals $1/(\Delta - 1)$, the following corollary holds.

**Corollary 1.4** The critical value of the Bak-Sneppen model on a regular tree $T_\Delta$, with common degree $\Delta$, satisfies

$$\frac{1}{\Delta + 1} \leq p_{BS}(T_\Delta) \leq \frac{1}{\Delta - 1}.$$

Applying the expansion for the critical value of site percolation on $\mathbb{Z}^d$ given by Hara and Slade [4], we also have the following corollary.

**Corollary 1.5** The critical value of the Bak-Sneppen model on $\mathbb{Z}^d$ satisfies

$$\frac{1}{2d + 1} \leq p_{BS}(\mathbb{Z}^d) \leq \frac{1}{2d} + \frac{1}{(2d)^2} + O(d^{-3}), \quad d \to \infty.$$

The paper is organised as follows. In Section 2 we take some preliminary steps by describing an alternative way of constructing a Bak-Sneppen avalanche. Section 3 uses this construction to couple the Bak-Sneppen avalanche and another stochastic process. The proof that the critical value of the Bak-Sneppen avalanche is larger than that of the coupled stochastic process is given in Section 4. The proof of Theorem 1.3 is completed in Section 5 where we show that the coupled process in fact constructs the cluster at the origin of site percolation with the origin always open. In Section 6 we discuss some implications and generalisations of our methods and results.

### 2 An alternative construction of the Bak-Sneppen model

In the introduction the Bak-Sneppen model was defined in its original format and then generalised to locally finite graphs. However, for our purposes it is more convenient to work with an alternative construction of the Bak-Sneppen model. We call this new construction the *forgetful* Bak-Sneppen model, as the exact fitness values will be no longer fixed (or remembered). This idea borrows heavily from the ‘locking thresholds representation’ in [8], and was used in a much simpler form in [2]. The forgetful Bak-Sneppen model is defined below and then argued to be equivalent to the normal Bak-Sneppen model, in the sense that at all times, the fitnesses have the same distributions.
Consider a Bak-Sneppen model on a finite transitive graph $G$ with $N$ vertices. To start with, all $N$ vertices have independent uniformly $(0, 1)$ distributed fitnesses. In the forgetful Bak-Sneppen model, all $N$ vertices have fitness distributions, instead of fitness values. At time 0, all vertices have uniform $(0,1)$ fitness distributions. The system at time $n$ is generated from the system at time $n-1$ by the following procedure.

1. We draw $N$ new independent random variables according to the appropriate fitness distributions at time $n-1$.

2. The minimum of the fitnesses is found and fixed.

3. All the other fitness values are discarded, and replaced by the conditional distribution of these fitnesses, given that they are larger than the observed minimal fitness.

4. The vertex with minimal fitness and its neighbours have their value or fitness distributions replaced by uniform $(0,1)$ distributions. (So now all vertices have some distribution associated with them.) This is the state of the system at time $n$.

It is easy to see that the fitness distributions at time $n$ generated by this procedure are the same as the fitness distributions in the normal Bak-Sneppen model at time $n$.

Furthermore, all fitness distributions have the convenient property that they are uniform distributions. Indeed, suppose that a random variable $Y$ has a uniform $(y, 1)$ distribution, denoted by $F_y$. If we condition on $Y > z$, then $Y$ has distribution $F_{y\vee z}$, where $y \vee z = \max\{y, z\}$. All our fitnesses initially have uniform $(0,1)$ distributions. Two things can change these distributions. They can be reset to $F_0$ by being updated, or they can be conditioned to be bigger than some given value; in both cases they remain uniform.

The above construction gives a forgetful Bak-Sneppen model on a finite graph, but this is easily extended to a forgetful Bak-Sneppen model on a locally finite graph. The only difference is that initially we assign the fitness distribution $F_0$ to the origin. The remaining vertices have a fitness distribution with all mass in the point 1, denoted by $F_1$. The avalanche ends when all the fitnesses within the avalanche are above the threshold, which is equivalent to saying that the minimal fitness is above the threshold. It is possible to see when the avalanche has finished by checking the value of the minimum fitness (phase 2 above). Thus we can use the forgetful method to generate avalanches.
3 The construction of the coupling

This section is divided into three parts. To begin with, some intuition behind the main result (Theorem 1.3) is given. This is followed by a precise description of the coupling, and then we give an example for added clarity.

3.1 Intuition

We are interested in comparing the Bak-Sneppen avalanche with the open cluster at the origin of independent site percolation, with the proviso that the origin is open with probability 1 rather than with probability $p$. This clearly has no effect on the critical value.

Typically, site percolation is studied as a static random structure, but it is also possible to build up the open cluster at the origin dynamically. This is standard (we refer to [3] for details) but the idea can be described as follows. Starting with just the origin, we can evaluate one of the neighbours and decide whether this neighbour is open or not. If it is, we add it to the cluster, if it isn’t, we declare it closed. One can continue in this fashion, each time step evaluating neighbours of the current cluster one by one. If the probability that a vertex is open, given the full history of this process, is always equal to $p$, then in fact we do create the site-percolation open cluster of the origin. When there are no more unevaluated neighbours, the process stops, and the cluster is finite in that case.

The growth of both a Bak-Sneppen avalanche and the open cluster at the origin is driven by the extremal vertices. In a Bak-Sneppen avalanche, the extremal vertices are those vertices that are contained within the avalanche and have neighbours outside the avalanche. It is only through one of the extremal vertices having the minimal fitness that the range of the avalanche can increase. For site percolation, the extremal vertices are those having a neighbour in the open cluster at the origin, but that are themselves unknown as to be open or closed. These are exactly the vertices at the edge of the cluster, and they will increase the size of the cluster by being open. Since it is the extremal vertices that drive the spread of both processes, the task is to relate the two sets of extremal vertices to each other.

The major difficulty to overcome is that in the Bak-Sneppen model an extremal vertex may be updated by neighbouring activity before having minimal fitness itself, whereas in site
percolation a vertex is either open or closed. So in the Bak-Sneppen model it is possible that a previously active extremal vertex never has minimal fitness, having been made inactive by a subsequent neighbouring update. Conversely, an originally inactive vertex can be made active. Hence, in the Bak-Sneppen model the neighbour of an active vertex will not necessarily be updated, while in our construction of the open cluster at the origin in site percolation, the neighbour of an open site is always considered. This means that it is not useful to couple the two models in the natural manner by realising the fitness and determining if the vertex is open and closed immediately with the same random variable.

The following heuristics make Theorem 1.3 plausible. If a vertex’s fitness is not minimal, then its conditional distribution based on this information is stochastically larger than its original uniform $(0, 1)$ distribution. So if a vertex is updated by a neighbour having minimal fitness, this makes its fitness stochastically smaller, making the vertex more likely to be active and therefore, intuitively at least, the avalanche is more likely to continue. This means that on average the interference from the non-extremal vertices of the Bak-Sneppen model on the extremal vertices should be beneficial to the spread of the avalanche.

### 3.2 The coupling

We now describe the construction of a process that we will refer to as the *coupled process*. As we shall see later, this process is constructed in such a way that it is stochastically dominated by the Bak-Sneppen avalanche, which is crucial for our argument. In Section 5 we show that this coupled process in fact constructs the cluster at the origin of site percolation.

Let $V(G)$ be the vertex set of the graph $G$. The coupled process is a stochastic process with values in $\{([0, 1] \times \{f, d\})^A, A \subset V(G)\}$. An entry $(a, f)$ means that the value of that vertex is fixed at $a$ forever, while an entry $(a, d)$ means that the value of that vertex is distributed uniformly on $(a, 1)$. The coupled process is coupled to a forgetful Bak-Sneppen avalanche, and is constructed as follows.

Fix an avalanche threshold $p$. We start with two copies of the graph $G$, denoted by $G_B$ (for the Bak-Sneppen avalanche) and $G_C$ (for the coupled process). Initially we assign the value $0$ to the origin of $G_B$ and $(0, f)$ to the origin of $G_C$, and we call the origin in $G_C$ open (as anticipated before). Then all the $\Delta$ neighbours of the origin of both graphs get
distribution $F_0$. On $G_C$, we define the extremal set $\mathcal{E}$ as the set of all points that have been assigned a distribution, but not (yet) an exact value.

The Bak-Sneppen avalanche on $G_B$ is generated according to the aforementioned (forgetful) construction, i.e., we sample new fitnesses, fix the minimal value and then calculate the fitness distributions accordingly. In the coupled process, only the vertices contained in $\mathcal{E}$ are considered. We apply the following procedure to all vertices in $\mathcal{E}$.

Consider a vertex $v_C \in \mathcal{E}$ with $G_B$-counterpart $v_B$. Let $F_z$ and $F_y$ be their respective fitness distributions. We realise the fitnesses of the vertices in $G_B$, and in particular realise the fitness of $v_B$ with an independent uniformly $(0,1)$ distributed random variable $U$ via $y + (1 - y)U$. Let $M$ be the minimal fitness in $G_B$. As long as the vertex with minimal fitness in the avalanche is active, i.e., $M \leq p$, we have the following two options, with corresponding rules for the coupled process. One should bear in mind that the main goal of the coupling is the stochastic domination. In Section 3.3 below, these are illustrated by an explicit example.

1. The fitness of $v_B$ is not minimal.
   
   We alter the distribution of $v_C$ by conditioning on the extra information that the fitness of $v_B$ must be bigger than $M$. Since the fitness of $v_B$ is not minimal, we have $y + (1 - y)U > M$, and hence $U > (M - y)^{+}/(1 - y)$. The new distribution of $v_C$ is $\hat{F}_z$, where
   
   $$\hat{z} = z + (1 - z)\frac{(M - y)^{+}}{(1 - y)}.$$  

2. The fitness of $v_B$ is minimal, so it has value $M$.
   
   It follows that $y + (1 - y)U = M$. The fitness of $v_C$ is now fixed at
   
   $$z + (1 - z)U = z + (1 - z)\frac{(M - y)}{(1 - y)}.$$  

   If this value is less than $p$, we say that $v$ is open, remove $v$ from $\mathcal{E}$, add the neighbours of $v$ that have an undetermined state to $\mathcal{E}$, and give them distribution $F_0$. If the value of $v$ is larger than $p$, then $v$ is closed and removed from $\mathcal{E}$.

The final step of the construction is as follows. The first time that the vertex with minimal fitness in $G_B$ is inactive (that is, $M > p$), the Bak-Sneppen avalanche has finished.
As soon as this happens, we fix *all* the values of the vertices in $\mathcal{E}$ in the following way, similar to rule 2 above. Let $v_C \in \mathcal{E}$ and $v_B$ have fitness distributions $F_z$ and $F_y$ respectively, and let $U$ be the associated uniform $(0,1)$ random variable. Then $U$ satisfies $y + (1-y)U \geq M$, i.e., $U \geq (M-y)/(1-y)$. The new distribution of $v_C$ is $F_\hat{z}$ with $\hat{z} = z + (1-z)(M-y)/(1-y)$.

As final step of the coupling, we realise the fitness of $v_C$ as $\hat{z} + (1-\hat{z})X$, where $X$ is an independent uniformly $(0,1)$ distributed random variable. In Section 4 we show that as soon as the Bak-Sneppen avalanche ends, this fitness value is at least $p$, and hence all the vertices in $\mathcal{E}$ will be closed. Before that, we give an example to illustrate the coupling procedure described above.

### 3.3 An example

The behaviour of the processes is illustrated by the following example, displayed in Figure 1. In this example the graph $G$ is a tree. For illustration purposes, we show only the part of the graph where the activity takes place.

Consider the forgetful avalanche at time $n$ say, in the following situation, see Figure 1, graph a: all fitnesses shown have distribution $F_0$. Then in graph b, uniform $(0,1)$ random variables are drawn. The random variables $U_1, U_2, \text{and } U_3$ are associated with the vertices visible in the picture. Other random variables are of course drawn for the other vertices. We then use the full set of random variables to determine the location and the magnitude of the new minimal fitness. This happens to be the vertex corresponding to the random variable $U_2$ (graph c). Finally, the new fitness distributions for time $n+1$ are determined (graph d). Note that the forgetful Bak-Sneppen model never actually assumes the values given in graph b.

During the same time step, the coupled process evolves as follows. We only consider the vertices in the extremal set $\mathcal{E}$, see Figure 1, graph e. Before the time step, the vertices have fitness distributions $F_x$, for some $x \in (0,1)$. Given the location of the minimal fitness in the Bak-Sneppen avalanche, the vertices in $\mathcal{E}$ are classified according to the rules 1 and 2 above (graph f). From the location and magnitude of the minimal fitness of the avalanche, it follows that $U_3 \geq M$, so $\hat{x} = x + (1-x)M$. Finally, the value of the vertex that corresponds with the vertex with minimal fitness in the avalanche is fixed, according to rule 2. Its value
Figure 1: Graphs a) – d) show a time step in the forgetful Bak-Sneppen process, and graphs e) – h) show a time step in the coupled process. The encircled vertex has the minimal fitness. In the coupled process, black points are open, white points are undetermined, and closed points are omitted.
$f$ is given by $f = x + (1 - x)M$. Now there are two possible cases: either $f > p$, and the vertex is closed (graph $g$), or $f \leq p$, and the vertex is open and its undetermined neighbours are added to $\mathcal{E}$ with distribution $F_0$ (graph $h$).

4 A domination principle

To show that the critical value of the coupled process can be no smaller than the critical value of the Bak-Sneppen avalanche, we use a domination argument. The propositions below show that the coupled process can finish no later than the Bak-Sneppen avalanche (so that the avalanche can be said to dominate the coupled process).

**Proposition 4.1** For every $v_C \in G_C$ and corresponding $v_B \in G_B$, at all times, the (conditional) fitness distribution of $v_C$ is stochastically larger than the (conditional) fitness distribution of $v_B$.

**Proof:** It should be noted that this proposition only makes sense for vertices in $\mathcal{E}$. Furthermore, it is safe to assume that the $p$-avalanche is still in progress, so the minimal fitness is less than $p$. The proof proceeds by induction. When new vertices are added to the coupled process, they (and their equivalents in $G_B$) have uniform $(0,1)$ distributed fitnesses. This is by definition for the coupled process, but also holds for $G_B$, since vertices in $G_B$ corresponding to new vertices added to $\mathcal{E}$ are always neighbours of the vertex with minimal fitness. This means that the statement of the proposition holds for new vertices added to $\mathcal{E}$.

To make the induction step, consider $v_C \in \mathcal{E}$ with corresponding vertex $v_B$ in $G_B$, and let $F_z$ and $F_y$ be the fitness distributions of $v_C$ and $v_B$ at time $n$, where $y \leq z < 1$. Let $u$ be the realisation of the uniform $(0,1)$ random variable associated with $v_C$ and $v_B$ at the intermediate step, and let $m$ be the minimal fitness.

Assume first that $v_B$ does not have the minimal fitness. This provides information on the value of $u$, namely that $y + (1 - y)u > m$, and hence $u > (m - y)/(1 - y)$. If $y \geq m$, this information is useless: we already knew that $u > 0$, and the fitness distributions of $v_C$ and $v_B$ are not changed. If $y < m$, then the inequality for $u$ does contain information, and we can calculate the corresponding inequality for the fitness of $v_C$:

$$z + (1 - z)u > z + \frac{(1 - z)(m - y)}{1 - y} = m + \frac{(1 - m)(z - y)}{1 - y} := \hat{z}.$$
So at time $n+1$, $v_B$ has distribution $F_{y\lor m}$ and $v_C$ has distribution $F_{\hat{z}}$. Since $m, y < 1$ and $y \leq z$, we have $\hat{z} \geq m$. Hence $(y \lor m) \leq \hat{z}$, and the desired property holds.

Second, we consider the case that a neighbour of $v_B$ had minimal fitness. In that case the fitness distribution of $v_B$ is reset to $F_0$, and there is nothing left to prove. □

**Proposition 4.2** At the moment that the $p$-avalanche ends, all vertices in $E$ are closed. As a consequence, if the probability of an infinite $p$-avalanche is zero, then there cannot be an infinite cluster of open sites in the coupled process, almost surely.

**Proof:** By Proposition 4.1, at all times every point in $E$ has a fitness that is stochastically larger than the fitness of the corresponding vertex in the avalanche. Hence, if the $p$-avalanche ends, then in the coupled process all neighbours in the set $E$ will be closed, as their fixed values can not be smaller than those in the avalanche, which are already greater than $p$ as the avalanche has ended. This removes all vertices from $E$ and ensures that no more are added, implying that in the coupled process no more vertices will be added to the open cluster around the origin. □

We conclude this section by giving an example where the coupled process is finite, but the Bak-Sneppen avalanche is infinite. This shows that the stochastic domination described in this section is not a stochastic equality.

Let $G = \mathbb{Z}$ and $p = 0.7$. Suppose that both in the first step and the second step in the Bak-Sneppen model, the origin is minimal with fitness 0.5. In the coupled process, the neighbours of the origin have fitness distribution $F_{0.5}$ after the first step, and $F_{0.5 + (1-0.5)0.5} = F_{0.75}$ after the second step. Since $0.75 > p$, this implies that the neighbours of the origin will eventually be closed, and the cluster in the coupled process is finite. However, the Bak-Sneppen avalanche may very well be infinite.

5 The cluster at the origin of site percolation

To complete the proof of Theorem 1.3, it remains to show that the coupled process in fact constructs the open cluster at the origin of independent site percolation, with the proviso that the origin is open with probability 1. To get into the right frame of mind for the proof,
we first give an example. At the same time, the example illustrates the construction of the coupled process in action.

5.1 An example

Consider the Bak-Sneppen avalanche and the coupled process defined on $\mathbb{Z}$ with parameter $p$. We wish to calculate the probability that in the coupled process both neighbours of the origin are closed. Note that for the site percolation cluster this probability is $(1 - p)^2$, so our aim is to show that this probability is also $(1 - p)^2$ for the coupled process. To calculate this probability, we introduce the following, more general probability: for all $0 \leq x \leq p$, let $g_p(x)$ be the probability that both neighbours of the origin will be declared closed, given that their current fitness distributions both are $F_x$. In this notation, the desired probability is equal to $g_p(0)$.

Starting with the distributions $F_x$ for both neighbours, we call the first subsequent step, the first time step. For the coupled process, both neighbours of the origin are declared closed if their realised values are above $p$. Noting that both neighbours have distribution $F_x$, this will happen at the first time step if in the Bak-Sneppen model all three values are above $(p - x)/(1 - x)$. If the minimum, which has density $3(1 - b)^2$, is below $(p - x)/(1 - x)$, and located at the origin (which happens with probability $1/3$), then we have to look at subsequent updates in the Bak-Sneppen model.

In this second case, the three fitness distributions in the Bak-Sneppen model are reset to $F_0$. However, in the coupled process, the fitnesses of $-1$ and $1$ are now $F_{x+(1-x)b}$, where $b$ is the avalanche minimum at the first time step. For the second time step, we are now in a similar situation as for the first, except that the fitness distribution has a different parameter: $x + (1 - x)b$ instead of $x$. This similarity holds for any starting level $x$, and leads to the following expression for $g_p(x)$:

$$g(x) := g_p(x) = \left(\frac{1 - p}{1 - x}\right)^3 + \frac{1}{3} \int_0^{p-x} 3(1 - b)^2 g(x + (1 - x)b) db.$$ \hfill (3)

Substituting $y = x + (1 - x)b$, equation (3) becomes

$$g(x) = \frac{1}{(1 - x)^2} \left((1 - p)^3 + \int_x^p (1 - y)^2 g(y) dy\right).$$ \hfill (4)
Using (4), a little algebra yields that for small $h$,
\begin{align*}
g(x + h) - g(x) &= \frac{(1 - x)^3 - (1 - x - h)^3}{(1 - x - h)^3} g(x) - \frac{1}{(1 - x - h)^3} \int_x^{x+h} (1 - y)^2 g(y) dy.
\end{align*}
Since $0 \leq g \leq 1$, it follows from (5) that $g(x + h) - g(x) \to 0$ for $h \to 0$, so $g$ is continuous. Hence, we can calculate the differential quotient:
\begin{align*}
\lim_{d \to 0} \frac{g(x + h) - g(x)}{h} &= 3\frac{(1 - x)^2 g(x)}{(1 - x)^3} - \frac{(1 - x)^2 g(x)}{(1 - x)^3} = \frac{2g(x)}{1 - x}.
\end{align*}
The same holds for the left-hand limit, so $g(x)$ is differentiable, and $g'(x) = 2g(x)/(1 - x)$.
This differential equation has a unique solution for each $p$, given by $g(x) = c(p)/(1 - x)^2$.
Using the boundary condition $g_p(p) = 1$, we find $c(p) = (1 - p)^2$, so that
\begin{align*}
g_p(x) &= \frac{(1 - p)^2}{(1 - x)^2}.
\end{align*}
In particular, the desired probability that in the coupled process both neighbors are closed is given by $g_p(0) = (1 - p)^2$, as required.

Although this example gave us what we wanted, clearly this type of calculation does not generalise to more complicated events. Therefore, the proof that the coupled process constructs the site percolation open cluster, which we turn to now, necessarily has a different flavour.

### 5.2 The proof

Our first goal is to determine the distribution of the information we use to generate the coupled process. More precisely, consider an arbitrary step of the forgetful Bak-Sneppen model, when there are $n$ vertices in the avalanche range so far. We enumerate these vertices $1, \ldots, n$, and suppose that all $n$ vertices in the avalanche have just been assigned a (conditional) distribution $F_{y_1}, \ldots, F_{y_n}$. (Recall that these are just uniform distributions above the respective $y_i$'s.) We sample from this random vector, using independent uniform $(0,1)$ distributed random variables $U_1, \ldots, U_n$: a sample from $F_{y_i}$ is realised via $y_i + (1 - y_i)U_i$.
We locate the minimum $M$, at vertex $K$ say; note that both $M$ and $K$ are random. Hence,
\begin{align*}
U_K &= \frac{M - y_K}{1 - y_K}.
\end{align*}
Conditional on $K$ and $M$, the remaining values $U_i$, $i \neq K$, are uniformly distributed above $\max\{y_i, M\}$ respectively, that is, we know that

$$U_i > \frac{(M - y_i)}{1 - y_i}, \quad i \neq K.$$

When we now also sample from all the other entries $i \neq K$, (which are uniformly distributed above $(M - y_i)^+/\{1 - y_i\}$ respectively) we have described a somewhat complicated way of sampling from the original vector $(U_1, \ldots, U_n)$, that is, such a sample yields independent uniform $(0, 1)$ distributed entries, see also Figure 2 and its caption. Note that we do not claim that $U_K$ is uniformly distributed on $(0, 1)$: it is not. However, since the index $K$ is random, this does not contradict the fact that the vector $(U_1, \ldots, U_n)$ consists of independent uniform $(0, 1)$ random variables.

Looking back to Section 3.2, it should be clear that in the coupled process independent uniform $(0, 1)$ random variables generated in the above way are used to alter the values of the vertices contained in $E$. Note that using $|E|$ entries rather than $n$ does not affect their marginal distributions or dependence structure, as the values of $U_i$s do not depend on whether the associated vertices are in $E$ or not.

It is now possible to give a direct description of the construction of the coupled process. We start with the origin being open and look at the neighbours of the origin, which initially have distribution $F_0$. These distributions are realised as follows, using the independent uniform $(0, 1)$ random variables described above. At each time step at most one value becomes fixed and the rest are given distributions. The fixed value corresponds to the case that $K \in E$. To calculate the new values of vertices in $E\setminus\{K\}$, we use the information that the $U_i$’s are independently and uniformly distributed above $(M - y_i)^+/\{1 - y_i\}$. This means that we do not fix their actual values at that time step, but instead change their distributions conditioned on this information. Once a vertex has a fixed value, it is declared open if and only if this value falls below $p$. Whenever a vertex is declared open, the neighbours that neither have a fixed value nor belong to $E$ are added to $E$ with distribution $F_0$.

Since fitnesses are initially independent uniform $(0, 1)$ when added to $E$ and the information we use to update the distributions is also independent uniform $(0, 1)$, the following holds: if at any time point the procedure is stopped and all the distributions are realised, one will recover an independent uniform $(0, 1)$ sample. Hence, all considered vertices (except
the origin) are open independently and with probability \( p \). It should now be obvious that our procedure is no different to building a site percolation cluster at the origin by the iterative method of assigning independent uniform \((0, 1)\) random variables to all undetermined neighbours of the cluster and calling a vertex open if its random variable takes a value less than \( p \). This completes the proof of Theorem 1.3. \( \square \)

Figure 2: In the forgetful BS-avalanche, before the update, vertices \( a \), \( b \) and \( c \) have fitness distributions \( F_{0.2} \), \( F_{0.3} \), and \( F_{0.6} \), respectively. After realising these distributions, vertex \( a \) is minimal with value \( M = 0.5 \). This means that \( U_a = \frac{0.5-0.2}{1-0.2} = \frac{3}{8} \), \( U_b \geq \frac{0.5-0.3}{1-0.3} = \frac{2}{7} \), and \( U_c \geq 0 \). This sample, namely \( U_a = 3/8 \) combined with a sample from a uniform \((2/7, 1)\) and a uniform \((0, 1)\) distribution, is a sample of three i.i.d. uniform \((0, 1)\) random variables.

Note that in case of an infinite Bak-Sneppen avalanche, some vertices in the coupled process may never get a fixed value. This is not a problem, because this is just what happens if an infinite open cluster around the origin is built up dynamically: not all vertices will be tested in the process of constructing this cluster.

6 Final remarks and extensions

Throughout this paper we have only considered locally finite transitive graphs. We assumed transitivity to avoid technicalities that would have obscured the main lines of reasoning. However, our results also hold in a more general setting, namely for any locally finite graph. The following observations explain this generalisation. The lower bound (Proposition 1.2)
can easily be adapted by considering a branching process with binomial($\Delta^* + 1, p$) offspring, where $\Delta^*$ is the maximal degree of the graph. Note that the lower bound is trivial if $\Delta^* = \infty$.

The coupling argument used to prove that the Bak-Sneppen avalanche dominates site percolation, at no point used the transitivity of the underlying graph, and hence also holds for non-transitive graphs. However, for non-transitive graphs, the choice of the origin affects the behaviour of the avalanche. The upper bound (Theorem 1.3) is generalised by the following observation: although the distribution of the size of the open cluster around the origin in site percolation does depend on the choice of the origin, standard arguments yield that the critical value does not.

Another consequence of our methods is the following. The careful reader may have noticed that the proofs actually yield a stronger result than stated in Theorem 1.3, namely stochastic domination. Define the range of site percolation to be the cardinality of the open cluster around the origin plus all its closed neighbours (these closed neighbours correspond to updated vertices in the Bak-Sneppen avalanche that were never minimal). The proof of Theorem 1.3 then demonstrates that the range of the $p$-avalanche is stochastically larger than the range of site percolation with parameter $p$.

Although not explicitly stated in the proof of Proposition 1.2, a similar extension also applies there. The set of offspring of a branching process with a binomial $(n - 1, p)$ offspring distribution is equivalent to the open cluster around the origin (root) of site percolation with parameter $p$ on $T_{n}^*$, where $T_{n}^*$ is a rooted tree where the root has degree $n - 1$, and all other vertices have degree $n$. In this case we get that the range of a $p$-avalanche on a transitive graph with common vertex degree $\Delta$ is stochastically smaller than the range of site percolation on $T_{\Delta + 2}^*$.

Finally, we argue that Theorem 1.3 holds as well for the critical value $p_c$. It is well-known that for site percolation on $\mathbb{Z}^d$ or on a tree, $p_c^{\text{site}}(G)$ is equal to the critical value associated with the expected size of the open cluster at the origin, see Grimmett [3]. Since each vertex in the open cluster contributes at most $\Delta$ closed neighbours to the range of site percolation, the range is always less than $\Delta$ times the size of the cluster. Hence, the critical values associated with the expectation of these two objects are the same. As a consequence, the stochastic bounds given above imply that the bounds in Proposition 1.2 and Theorem
1.3 also hold for the critical value (2).

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