Spectral mapping theorems for differentiable $C_0$ semigroups

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Abstract

Let $(T(t))_{t \geq 0}$ be a $C_0$ semigroup on a Banach space $X$ with infinitesimal generator $A$. In this work, we give conditions for which the spectral mapping theorem $\sigma_e(T(t)) \setminus \{0\} = \{e^{it}, \lambda \in \sigma_e(A)\}$ holds, where $\sigma_e$ can be equal to the essential, Browder and Kato spectrum. Also, we will be interested in the relations between the spectrum of $A$ and the spectrum of the $n^{th}$ derivative $T(t)^{(n)}$ of a differentiable $C_0$ semigroup $(T(t))_{t \geq 0}$.

Keywords Semigroup · Differentiable · Essential spectrum

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1 Introduction and preliminaries

Let $X$ be a complex Banach space, $B(X)$ denote the algebra of all bounded linear operators on $X$ and $C(X)$ the set of all linear closed operators from $X$ to $X$. We write $D(A)$, $A^*$, $R(A)$, $N(A)$, $\rho(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$ and $\sigma_r(A)$ respectively for the domain, the adjoint, the range, the kernel, the resolvent, the spectrum, the point spectrum, the approximate point spectrum and residual spectrum of an operator $A \in C(X)$. The function resolvent of $A \in C(X)$ is defined for all $\lambda \in \mathbb{C}$ by $R(\lambda, A) = (\lambda - A)^{-1}$. The ascent $a(A)$ and the descent $d(A)$ of an operator $A \in C(X)$ are defined respectively by $a(A) = \inf\{k \in \mathbb{N} : N(A^k) = N(A^{k+1})\}$ and $d(A) = \inf\{k \in \mathbb{N} : R(A^k) = R(A^{k+1})\}$, with the convention $\inf(\emptyset) = \infty$. The ascent and descent spectra are defined by $\sigma_{asc}(A) = \{\lambda \in \mathbb{C} : a(\lambda - A) = \infty\}$ and $\sigma_{desc}(A) = \{\lambda \in \mathbb{C} : d(\lambda - A) = \infty\}$. If the range $R(A)$ is closed and $\dim N(A) < \infty$ (codim $R(A) < \infty$), then $A$ is called an upper
(a lower) semi-Fredholm operator. If both \( \dim N(A) < \infty \) and \( \codim R(A) < \infty \) are finite, then \( A \) is called a Fredholm operator, see ([10]). The upper semi-Fredholm spectrum \( \sigma_{uf}(A) \), the lower semi-Fredholm spectrum \( \sigma_{lf}(A) \) and the essential spectrum \( \sigma_e(A) \) of \( A \) are defined by \( \sigma_{uf}(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not upper semi-Fredholm} \} \), \( \sigma_{lf}(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not lower semi-Fredholm} \} \), and \( \sigma_e(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not Fredholm} \} \). We say that an operator \( A \in \mathcal{C}(X) \) is upper semi-Browder if it is upper semi-Fredholm and has finite ascent. Similarly, \( A \) is lower semi-Browder if it is lower semi-Fredholm and has finite descent. An operator \( A \) is Browder if it is both lower and upper semi-Browder, see ([11]).

The upper semi-Browder spectrum \( \sigma_{ub}(A) \), the lower semi-Browder spectrum \( \sigma_{lb}(A) \) and the Browder spectrum \( \sigma_b(A) \) of \( A \) are defined by \( \sigma_{ub}(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not upper semi-Browder} \} \), \( \sigma_{lb}(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not lower semi-Browder} \} \), and \( \sigma_b(A) = \{ \lambda \in \mathbb{C} : \lambda - A \text{ is not Browder} \} \). Recall that a closed operator \( A \) is said to be semi-regular if \( R(A) \) is closed and \( N(A) \subseteq \mathcal{R}^\infty(A) \), where \( \mathcal{R}^\infty(A) = \bigcap_{n \geq 0} R(A^n) \). The semi-regular spectrum of \( A \) is defined by \( \sigma_s(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not semi-regular} \} \), see [4]. Next, \( A \) is said to have the single valued extension property at \( \lambda_0 \in \mathbb{C} \) (SVEP) if for every open neighborhood \( U \subseteq \mathbb{C} \) of \( \lambda_0 \), the only analytic function \( f : U \to D(A) \) which satisfies the equation \( (A - zI)f(z) = 0 \) for all \( z \in U \) is the function \( f \equiv 0 \). \( A \) is said to have the SVEP if \( A \) has the SVEP for every \( \lambda \in \mathbb{C} \), see [1]. Many equations of mathematical physics can be cast in the abstract form:

\[
u'(t) = Au(t), \quad t \geq 0, \quad u(0) = u_0
\]

on a Banach space \( X \). Here \( A \) is a given linear operator with domain \( D(A) \) and the initial value \( u_0 \). The semigroups can be used to solve a large class of problems commonly known as the Cauchy problem. The solution of (1) will be given by \( u(t) = T(t)u_0 \) for an operator semigroup \( (T(t))_{t \geq 0} \) on \( X \). In order to understand the behavior of the solutions in terms of the data concerning \( A \), one seeks information about the spectrum of \( T(t) \) in terms of the spectrum of \( A \). Unfortunately the spectral mapping theorem \( e^{\sigma_x(A)} = \sigma_x(T(t)) \setminus \{0\} \) often fails, sometimes in dramatic ways. However, the inclusion

\[
e^{\sigma_x(A)} \subseteq \sigma_x(T(t)) \setminus \{0\}
\]

always holds, where \( \sigma_s \in \{ \sigma, \sigma_e, \sigma_b \} \).

A one-parameter family \( (T(t))_{t \geq 0} \) of bounded operators on a Banach space \( X \) is called a \( C_0 \) semigroup of operators or a strongly continuous semigroup of operators if:

1. \( T(0) = I \).
2. \( T(t + s) = T(t)T(s) \), for all \( t, s \geq 0 \),
3. \( \lim_{t \to 0} T(t)x = x \), for all \( x \in X \).

\( (T(t))_{t \geq 0} \) has a unique infinitesimal generator \( A \) defined on the domain \( D(A) \) by:

\[
D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\},
\]

\[
Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}, \text{ for all } x \in D(A).
\]

Also, \( A \) is a densely defined closed operator, see [3, 5]. We introduce the following operator acting on \( X \) and depending on the parameters \( \lambda \in \mathbb{C} \) and \( t \geq 0 \),
\[
B_\lambda(t)x = \int_0^1 e^{\lambda(t-s)} T(s) x ds, \quad \text{for all } x \in X. \]
It is well known that \(B_\lambda(t)\) is a bounded linear operator from \(X\) to \(D(A)\) and we have ([3, 5]):

1. \((e^{\lambda t} - T(t))^n x = (\lambda - A)^n B_\lambda^n(t) x\), \quad \text{for all } x \in X, n \in \mathbb{N};
2. \((e^{\lambda t} - T(t))^n x = B_\lambda^n(t)(\lambda - A)^n x\), \quad \text{for all } x \in D(A^n), n \in \mathbb{N};
3. \(R((e^{\lambda t} - T(t))^n) \subseteq R((\lambda - A)^n);\)
4. \(N((\lambda - A)^n) \subseteq N(e^{\lambda t} - T(t))^n).\)

A strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\) is called eventually differentiable if there exists \(t_0 \geq 0\) such that \(t \mapsto T(t)x\) is differentiable on \((t_0, \infty)\) for every \(x \in X\). The semigroup is called differentiable if \(t_0\) can be chosen as \(t_0 = 0\). If \((T(t))_{t \geq 0}\) is differentiable, then \(B_\lambda(t)x\) is differentiable and \(B'_\lambda(t)x = T(t)x + \lambda B_\lambda(t)x\) for every \(t > 0\) and \(x \in X\). Furthermore, (see [5])

1. \((\lambda e^{\lambda t} - AT(t))x = (\lambda - A)B'_\lambda(t)x\), \quad \text{for all } x \in X;
2. \((\lambda e^{\lambda t} - AT(t))x = B'_\lambda(t)(\lambda - A)x\), \quad \text{for all } x \in D(A).

One of the classical approaches to finding information about the solution \(T(t)\) is to study the spectrum of the semigroup directly. In several applications, we have the explicit expression of the generator \(A\). Thus we need information on the spectrum of the semigroup \(T(t)\) in terms of that of the generator \(A\). In addition, the study of the spectral mapping theorem for the different parts of the spectrum proves to be essential.

The spectral inclusions for various reduced spectra of a differentiable \(C_0\) semigroup were studied by Tajmouati et al. [6–8]. But, the equality (spectral mapping theorem) is not examined. So it is natural to ask the question: can we establish equality?

In this work, we will continue in this direction, we will prove, under some conditions, the spectral equality for \(C_0\) semigroup for essential spectrum, Browder spectrum, upper(lower)semi-Fredholm spectrum. Also, we give an affirmative answer to the question we just asked.

## 2 Main results

We start with the following results, they will be needed later.

**Lemma 1** [5] Let \((T(t))_{t \geq 0}\) be a \(C_0\) semigroup which is eventually differentiable and let \(A\) be its infinitesimal generator, then there is an \(t_0 > 0\), such that

1. \(\text{For } t > nt_0\text{ and } n \in \mathbb{N}^+, T(t) : X \to D(A^n)\) and \(A^n T(t)\) is a bounded linear operator.
2. \(\text{For } t > nt_0\text{ and } n \in \mathbb{N}^+, T^{(n-1)}(t)\) is continuous in the uniform operator topology.

**Lemma 2** Let \((T(t))_{t \geq 0}\) be an eventually differentiable \(C_0\) semigroup on \(X\) with infinitesimal generator \(A\). Then \((T(t)F)_{t \geq 0}\) is an uniformly continuous semigroup, where \(F = N(e^{\lambda t} - T(s)), \lambda \in \mathbb{C}\) and \(s > 0\).

**Proof** It is clear that the closed subspace \(F\) is both \(T(t)\) and \(A\) invariant, furthermore \(D(A_{|F}) = D(A) \cap F \subseteq F\). Let \(x \in F\), then \(x_n = n \int_0^{\frac{1}{n}} T(s) x ds \in D(A) \cap F\) for all \(n \in \mathbb{N}\) and
\[ x_n \xrightarrow{n \to \infty} x, \text{ so } D(A_{|F}) = F. \] Let \( \mu \in \rho(A) \), we want to show that \( R(\mu, A_{|F}) \) is bounded below. By way of contradiction, assume that there exists \( (x_n)_n \subseteq F \) such that \( \| x_n \| = 1 \) and \( R(\mu, A_{|F})x_n \xrightarrow{n \to \infty} 0 \). From Lemma 1, then there is \( t_0 > 0 \), such that \( AT(t_0) \) is bounded, then \((\mu - A_{|F})T(t_0)R(\mu, A_{|F})x_n \xrightarrow{n \to \infty} 0 \). This implies that \( T(t_0)x_n \xrightarrow{n \to \infty} 0 \). If \( t > t_0 \), then \( T(t)x_n = T(t_0)T(t-t_0)x_n = T(t-t_0)T(t_0)x_n \xrightarrow{n \to \infty} 0 \), then \( e^{\mu t}x_n = T(t)x_n \xrightarrow{n \to \infty} 0 \), which is impossible. If \( 0 < t \leq t_0 \), there exists \( p \in \mathbb{N}^* \) such that \( pt > t_0 \), then \( T(pt)x_n = T(pt-t_0)T(t_0)x_n \xrightarrow{n \to \infty} 0 \), so \( e^{\mu t}x_n = T(t_0)x_n \xrightarrow{n \to \infty} 0 \), which contradicts the fact that \( \| x_n \| = 1 \). Hence, there exists \( M \in \mathbb{R}^+ \) such that for all \( x \in F \) \( \| R(\mu, A)x \| \geq M \| x \| \). Consequently, for all \( x \in D(A_{|F}) \), \( \| A_{|F}x \| \leq (|\mu| + M^{-1}) \| x \| \). By [5, Theorem I. 1.2], we conclude that \( (T(t)|_{|F})_{t \geq 0} \) is a uniformly continuous semigroup.

**Lemma 3** Let \((T(t))_{t \geq 0}\) be an eventually differentiable \( C_0 \) semigroup with infinitesimal generator \( A \) on a reflexive Banach space \( X \). Then \((T^*(t))_{t \geq 0}\) is eventually differentiable \( C_0 \) semigroup with generator \( A^* \).

**Proof** Since \( X \) is reflexive, by [9, Corollary 1.3.2], \((T^*(t))_{t \geq 0}\) is \( C_0 \) semigroup with generator \( A^* \). Assume that \((T(t))_{t \geq 0}\) is differentiable for \( t > t_0 \). Let \( x^* \in X^* \) and \( t > t_0 \). For \( x \in X \) arbitrary we have,

\[
\lim_{h \to 0} \frac{1}{h} \langle T^*(h)T^*(t)x^* - T^*(t)x^*, x \rangle = \lim_{h \to 0} \frac{1}{h} \langle x^*, T(h)T(t)x - T(t)x \rangle = \langle x^*, T'(t)x \rangle = \langle (T'(t))^*x^*, x \rangle
\]

This shows that \( T^*(t)x^* \in D(A^*) \) and \((T^*(t))_{t \geq 0}\) is eventually differentiable \( C_0 \) semigroup.

**Theorem 1** Let \((T(t))_{t \geq 0}\) be an eventually differentiable \( C_0 \) semigroup on \( X \) with infinitesimal generator \( A \). Then

\[
\sigma_{\mathfrak{a}}(T(t))\{0\} = \{ e^{\lambda t}, \lambda \in \sigma_{\mathfrak{a}}(A) \}
\]

If \( X \) is reflexive, we have

\[
\sigma_{\mathfrak{b}}(T(t))\{0\} = \{ e^{\lambda t}, \lambda \in \sigma_{\mathfrak{b}}(A) \}
\]
\[
\sigma_{\mathfrak{c}}(T(t))\{0\} = \{ e^{\lambda t}, \lambda \in \sigma_{\mathfrak{c}}(A) \}
\]

**Proof** If \( e^{\mu t} - T(t) \) is upper semi-Fredholm, then \( \dim N(e^{\mu t} - T(t)) < \infty \) and \( R(e^{\mu t} - T(t)) \) is closed. By [7, Lemma 2.3], we have \( R(\lambda - A) \) is closed. Since \( N(\lambda - A) \subseteq N(e^{\mu t} - T(t)) \), so \( \dim N(\lambda - A) < \infty \). This implies that \( \lambda - A \) is upper semi-Fredholm. Conversely, let \( \mu = e^{\mu t} \in \sigma_{\mathfrak{a}}(T(t))\{0\} \) and \( F = N(e^{\mu t} - T(t)) \). From Lemma 2, \( A_{|F} \) is the generator of the uniformly continuous semigroup \((T(t)|_{|F})_{t \geq 0}\). Then \( \{ e^{\lambda t}, \lambda \in \sigma_{\mathfrak{a}}(A_{|F}) \} \subseteq \sigma_{\mathfrak{a}}(T(t)|_{|F})\{0\} \). Since \( A_{|F} \) is bounded, then \( \sigma_{\mathfrak{a}}(A_{|F}) \neq \emptyset \). Let \( \lambda_0 \in \sigma_{\mathfrak{a}}(A_{|F}) \), then \( e^{\lambda_0 t} \in \sigma_{\mathfrak{a}}(T(t)|_{|F}) \subseteq \sigma(T(t)|_{|F}) = \{ \mu \} \). Then \( \mu = e^{\lambda_0 t} \). We must show that \( \lambda_0 \in \sigma_{\mathfrak{a}}(A) \). Suppose on the contrary that \( \lambda_0 \not\in \sigma_{\mathfrak{a}}(A) \), then \( A - \lambda_0 \) is upper semi Fredholm, so \( \dim N(\lambda_0 - A) < \infty \) and \( R(\lambda_0 - A) \) is closed. Since \( N(\lambda_0 - A_{|F}) \subseteq N(\lambda_0 - A) \), then \( \dim N(\lambda_0 - A_{|F}) < \infty \). Let \( (\lambda_0 - A_{|F})x_n \xrightarrow{n \to \infty} y \) with \( (x_n)_n \subseteq F \). Since \( R(\lambda_0 - A) \) is closed,
then there exists $x \in D(A)$ such that $y = (\lambda_0 - A)x$. $B_{\lambda_0}(t)$ is bounded implies that $B_{\lambda_0}(t)(\lambda_0 - A)x_n \xrightarrow{\text{n} \to \infty} B_{\lambda_0}(t)(\lambda_0 - A)x$, then $(\mu - T(t))x_n \xrightarrow{\text{n} \to \infty} (\mu - T(t))x$. Hence $(\mu - T(t))x = 0$, consequently $x \in F$ and $y \in R(\lambda_0 - A_{\mu}^*)$, which is absurd. Thus $\sigma_{uf}(T(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_{uf}(A)\}$. We now turn to the second statement. Since $A$ is the generator of the eventually differentiable $C_0$ semigroup $(T(t))_{t \geq 0}$, then from Lemma 3, $A^*$ is the generator of the eventually differentiable $C_0$ semigroup $(T^*(t))_{t \geq 0}$. Then $\sigma_{uf}(T^*(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_{uf}(A^*)\}$. By duality, we have $\sigma_{lf}(T(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_{lf}(A)\}$ and consequently $\sigma_s(T(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_s(A)\}$. 

The following corollary is an immediate consequence of the previous theorem.

**Corollary 1** Let $(T(t))_{t \geq 0}$ be an eventually differentiable $C_0$ semigroup on $X$ with infinitesimal generator $A$. Then

$$\sigma_{ub}(T(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_{ub}(A)\}$$

If $X$ is reflexive, we have

$$\sigma_{lb}(T(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_{lb}(A)\}$$

$$\sigma_b(T(t))\{\emptyset\} = \{e^{\lambda t}, \lambda \in \sigma_b(A)\}$$

**Remark 1** Note that the inclusion $\{e^{\lambda t}, \lambda \in \sigma_s(A)\} \subseteq \sigma_s(T(t))\{\emptyset\}$, where $\sigma_s \in \{\sigma_s, \sigma_{uf}, \sigma_{lf}, \sigma_{ub}, \sigma_{lb}, \sigma_b\}$ holds in the general setting where $(T(t))_{t \geq 0}$ is a $C_0$ semigroup. This inclusion is strict as shown in the following example.

**Example 1** Consider the translation group on the space $C_{2\pi}(\mathbb{R})$ of all $2\pi$ periodic continuous functions on $\mathbb{R}$ and denote its generator by $A$ (see [3, Paragraph I.4.15]). From [3, Examples 2.6.iv] we have, $\sigma(A) = i\mathbb{Z}$, then $e^{t\sigma(A)}$ is at most countable, therefore $e^{t\sigma(A)}$ are also. The spectra of the operators $T(t)$ are always contained in $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and contain the eigenvalues $e^{\lambda k}$ for $k \in \mathbb{Z}$. Since $\sigma(T(t))$ is closed, it follows from [3, Theorem IV.3.16] that $\sigma(T(t)) = \Gamma$ whenever $t/2\pi \notin \mathbb{Q}$, then $\sigma(T(t))$ is not countable, so $\sigma_s(T(t))$ are also. Therefore the inclusions of the Remark 1 are strict. From Theorem 1 and Corollary 1, the semigroup $(T(t))_{t \geq 0}$ is not differentiable.

**Example 2** On $C([0, 1])$ we consider the operator $B$ given by

$$Bu := mu'' + qu'$$

where $m(0) = m(1) = 0$, $\sqrt{m} \in C[0, 1]$, $q \in C[0, 1]$, and $q/\sqrt{m}$ is bounded in $]0, 1[$. The operator $(B, D(B))$ generates a analytic semigroup $(T(t))_{t \geq 0}$ of angle $\frac{\pi}{2}$ in $C[0, 1]$, see [5, Theorem 4.20], where $D(B) := \{u \in C[0, 1] \cap C^2(0, 1), Bu \in C[0, 1]\}$. Since $(T(t))_{t \geq 0}$ is an analytic semigroup, $(T(t))_{t \geq 0}$ matches the condition of the Theorem 1, then

$$\sigma_s(T(t))\{\emptyset\} = \{e^{\lambda s}, \lambda \in \sigma_s(B)\}.$$

We have the following result.

**Proposition 1** Let $(T(t))_{t \geq 0}$ be a $C_0$ semigroup on $X$ with infinitesimal generator $A$. Then

$$\{e^{\lambda s}, \lambda \in \sigma_k(A)\} \subseteq \sigma_k(T(t))\{\emptyset\} \subseteq \{e^{\lambda s}, \lambda \in \sigma_{ap}(A)\}$$
Furthermore, if A has the SVEP, we have $\{e^{\lambda t}, \lambda \in \sigma_A(t)\} = \sigma_A(T(t)) \setminus \{0\}$.

**Proof** According to [2, Theorem 2.2], we have the inclusion $\{e^{\lambda t}, \lambda \in \sigma_A(t)\} \subseteq \sigma_A(T(t)) \setminus \{0\}$.

Now, let $\mu \in \sigma_A(T(t)) \setminus \{0\}$ and $F = N(\mu - T(t))$, then $F$ is both $T(t)$ and $A$ invariant closed subspace of $X$. From Lemma 2, $A_{1F}$ is a bounded operator that generates the $C_0$ semigroup $(T(t)_{1F})_{t \geq 0}$. We have $\{e^{\lambda t}, \lambda \in \sigma_A(A_{1F})\} \subseteq \sigma_A(T(t)_{1F}) \setminus \{0\}$. Since $A_{1F}$ is bounded, then $\sigma_A(A_{1F}) \neq \emptyset$. Let $\lambda_0 \in \sigma_A(A_{1F})$, then $e^{\lambda_0 t} \in \sigma_A(T(t)_{1F}) = \sigma(T(t)_{1F}) = \{\mu\}$. Then $\mu = e^{\lambda_0 t} \in \{e^{\lambda t}, \lambda \in \sigma_A(A_{1F})\} \cup \{0\}$. Now, let us show that $\lambda_0 \in \sigma_A(A)$. If $\lambda_0 \notin \sigma_A(A)$, so $\lambda_0 \notin \sigma_A(A_{1F})$, then $\lambda_0 \notin \sigma_A(A_{1F})$, which is absurd.

**Lemma 4** Let $(T(t))_{t \geq 0}$ be an eventually differentiable $C_0$ semigroup on $X$ with infinitesimal generator $A$. Let $\mu \in \sigma(AT(t)) \setminus \{0\}$ and $F = N(\mu - AT(t))$. Then $(T(t)_{1F})_{t \geq 0}$ is a uniformly continuous semigroup on $X$ with infinitesimal generator $A_{1F}$.

**Proof** Clearly, the closed subspace $F$ is both $T(t)$ and $A$ invariant. We show that $D(A_{1F}) = F$. Let $x \in F$ and $x_n = n \int_0^1 T(t)x \, dt$ for $n \geq 1$, then $x_n \in D(A) \cap F = D(A_{1F})$ and $x_n \xrightarrow{n \to \infty} x$. Therefore $x \in D(A_{1F})$ and consequently $D(A_{1F}) = F$. We show that $A_{1F}$ is bounded. Let $\lambda_0 \in \rho(A_{1F})$, then $R(\lambda_0, A_{1F})$ is bounded below. Suppose the contrary, then there exists $(y_n) \in F$ such that $\|y_n\| = 1$ and $R(\lambda_0, A_{1F})y_n \xrightarrow{n \to \infty} 0$. Since $(T(t))_{t \geq 0}$ is differentiable, then $AT(t_x)$ is bounded, so $(\lambda_0 - A)T(t)R(\lambda_0, A_{1F})y_n \xrightarrow{n \to \infty} 0$, this implies that $y_n \xrightarrow{n \to \infty} 0$ which contradicts the fact that $\|y_n\| = 1$. Hence, there exists $C > 0$ such that for all $x \in F$, $\|R(\lambda_0, A_{1F})x\| \geq C\|x\|$. Then, for all $x \in D(A_{1F})$, $\|A_{1F}x\| \leq (\|\lambda_0\| + C^{-1})\|x\|$.

**Theorem 2** Let $(T(t))_{t \geq 0}$ be an eventually differentiable $C_0$ semigroup on $X$ with infinitesimal generator $A$. Then for $t > 0$, we have

$$
\sigma(AT(t)) \cup \{0\} = \{\lambda e^{\lambda t}, \lambda \in \sigma_A(A) \cup \{0\}\}
$$

$$
\sigma_{ap}(AT(t)) \cup \{0\} = \{\lambda e^{\lambda t}, \lambda \in \sigma_{ap}(A) \cup \{0\}\}
$$

**Proof** From [5, Lemma 4.6], we have $\{\lambda e^{\lambda t}, \lambda \in \sigma_A(A)\} \subseteq \sigma(AT(t))$. We prove the other inclusion, let $\mu \in \sigma(AT(t)) \setminus \{0\}$, by Lemma 4, $(T(t)_{1F})_{t \geq 0}$ is a uniformly continuous $C_0$-semigroup on $X$ with infinitesimal generator $A_{1F}$, where $F = N(\mu - AT(t))$. Then $(T(t)_{1F}) = e^{\lambda \mu}$. Since the function $f : z \mapsto ze^{\frac{z}{C}}$ is analytic on $C$, from the spectral mapping theorem we have $\sigma((AT(t))_{1F}) = \sigma(A_{1F}T(t)_{1F}) = \sigma(A_{1F}e^{\lambda \mu}) = \sigma(f(A_{1F})) = f(\sigma(A_{1F}))$. Since $A_{1F}$ is bounded, then $\sigma(A_{1F}) \neq \emptyset$. Let $\lambda_0 \in \sigma(A_{1F})$, then $\lambda_0 e^{\lambda_0 t} = \lambda_0 \mu \in \sigma((AT(t))_{1F}) = \{\mu\}$. Therefore, $\mu = \lambda_0 e^{\lambda_0 t}$. We show that $\lambda_0 \in \sigma(A)$. Suppose on the contrary that $\lambda_0 - A$ is bijective, then $\lambda_0 - A_{1F}$ is injective. Let $y \in F$, since $\lambda_0 - A$ is surjective, then there exists $x \in D(A)$ such that $y = (\lambda_0 - A)x$, therefore $(\lambda_0 e^{\lambda_0 t} - AT(t))y = (\lambda_0 - A)B_{\lambda_0}^t(t)y = 0$, so $B_{\lambda_0}^t(t)y = 0$. Let $\lambda_0 \in \sigma(A_{1F})$, then $\lambda_0 e^{\lambda_0 t} - \lambda_0 \mu = \lambda_0 \mu \in \sigma_{ap}(A_{1F})$. Conversely, let $\mu \in \sigma_{ap}(AT(t)) \setminus \{0\}$, then $\sigma_{ap}(AT(t))_{1F} = \sigma(AT(t))_{1F} = f(\sigma_{ap}(A_{1F})) = \{\mu\}$. Since $A_{1F}$ is bounded, then $\sigma_{ap}(A_{1F}) \neq \emptyset$. Let $\lambda_0 \in \sigma_{ap}(A_{1F})$, then $\mu = \lambda_0 e^{\lambda_0 t}$. We prove that $\lambda_0 \in \sigma_{ap}(A)$.
Suppose that \( \lambda_0 \notin \sigma_{ap}(A) \), then \( \lambda_0 - A \) is bounded below, so \( \lambda_0 - A_{|F} \) is injective. Let \( (\lambda_0 - A_{|F})x_n \to y \). Since \( R(\lambda_0 - A) \) is closed then there exists \( x \in D(A) \) such that \( y = (\lambda_0 - A_{|F})x \). Since \( B'_{\lambda_0}(t) \) is bounded, then
\[
0 = B'_{\lambda_0}(t)(\lambda_0 - A_{|F})x_n \to B'_{\lambda_0}(t)(\lambda_0 - A_{|F})x = (\lambda_0 e^{\lambda t} - AT(t))x.
\]
we conclude that \( x \in F \). Therefore \( \lambda_0 - A_{|F} \) is bounded below, a contradiction.

**Corollary 2** Let \( (T(t))_{t \geq 0} \) be an eventually differentiable \( C_0 \) semigroup on \( X \) with infinitesimal generator \( A \). Then for all \( t > 0 \) and \( n \in \mathbb{N} \), we have
\[
\sigma_+(T(t)^{(a)}) \cup \{0\} = \{ \lambda^n e^{\lambda t}, \lambda \in \sigma_+(A) \cup \{0\} \},
\]
where \( \sigma_+ = \sigma, \sigma_{ap} \).

**Proof** Let \( n \in \mathbb{N} \) and \( t > 0 \). Since \( (T(t))_{t \geq 0} \) is a differentiable \( C_0 \) semigroup, then \( T(t)^{(a)} \) is a bounded operator, furthermore \( T(t)^{(n)} = A^n T(t) = \left( AT\left( \frac{t}{n} \right) \right)^n \). From the spectral mapping theorem, we have
\[
\sigma_+(T(t)^{(a)}) \cup \{0\} = \sigma_+(AT\left( \frac{t}{n} \right))^n \cup \{0\}
= \left\{ \mu^n : \mu \in \sigma_+(AT\left( \frac{t}{n} \right)) \right\} \{0\}
= \left\{ \left( \lambda e^{\frac{\lambda t}{n}} \right)^n, \lambda \in \sigma_+(A) \cup \{0\} \right\}
= \{ \lambda^n e^{\lambda t}, \lambda \in \sigma_+(A) \cup \{0\} \}
\]

**Example 3** Let \( X := \ell^p \) and \( M_q \) be the multiplication operator defined by
\[
M_q(x_n)_{n \in \mathbb{N}} = (q_n x_n)_{n \in \mathbb{N}} \text{ with } q = (q_n)_{n \in \mathbb{N}} \text{ and } q_n = -n + in^2 .
\]
Then \( \sigma(M_q) = \{ q_n : n \in \mathbb{N} \} \). From [3, Counterexamples II.14.16] the semigroup \( (T_q(t))_{t \geq 0} \) generated by \( M_q \) is differentiable. By Corollary 2, for \( t > 0 \), we have \( \sigma(T_q(t)^{(ap)}) \cup \{0\} = \{(-n + in^2)^p e^{-(n^2 + in^2)} : n \in \mathbb{N} \} \).

**Example 4** Consider the heat equation in \( L^p(0, \pi) \).
\[
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) &= \frac{\partial^2 u}{\partial x^2}(t, x), (t, x) \in \mathbb{R}^+ \times (0, \pi) \\
\frac{\partial^2 u}{\partial x^2}(t, x), (t, x) \in \mathbb{R}^+ \times (0, \pi) \\
u(0, x) &= 0 = u(t, \pi), t \geq 0 \\
u(0, x) &= f(x) \quad x \in (0, \pi)
\end{aligned}
\]

Let \( p > 2 \). On \( X = L^p(0, \pi) \) consider the operator defined by \( Af(x) = f''(x) \) with domain \( D(A) = W^{2,p}(0, \pi) \cap W^1_0(0, \pi), x \in (0, \pi) \) where \( W^p_0 = \{ f \in W^p(0, \pi) : f(0) = f(\pi) = 0 \} \). The operator \( A \) is self-adjoint. The corresponding semigroup \( (T(t))_{t \geq 0} \) is analytic, so is differentiable. For each \( f \in W^{2,p}(0, \pi) \cap W^1_0(0, \pi) \) the unique solution of the equation is given by \( u(t, x) = (T(t)f)(x) \). The spectrum of \( A \) is \( \sigma(A) = \{-n^2 : n \geq 1\} \). Since \( \text{int}(\sigma(A)) = \emptyset \), then \( A \) and \( A^* \) have SVEP. So, \( \sigma(A) = \sigma_k(A) = \sigma_{ap}(A) = \sigma_{su}(A) \). From Proposition 1, we have \( \sigma_+(T(t)) \cup \{0\} = \{e^{-n^2}, n \geq 1\} \cup \{0\} \). Also, by Corollary 2, we have \( \sigma(T(t)^{(ap)}) \cup \{0\} = \sigma_{ap}(T(t)^{(ap)}) \cup \{0\} = \{(-1)^p n^2 e^{-n^2}, n \geq 0\} \).
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