Simple physical applications of a groupoid structure

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Abstract

Motivated by Quantum Mechanics considerations, we expose some cross product constructions on a groupoid structure. Furthermore, critical remarks are made on some basic formal aspects of the Hopf algebra structure.

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1. Introduction

In [11] was recovered the notion of E-groupoid [EBB-groupoid], whose groupoid algebra led to the so-called E-groupoid algebra [EBB-groupoid algebra]. Following a suggestion of A. Connes (see [6, § I.1]), some elementary algebraic structures of Matrix Quantum Mechanics can arise from certain representations of such groupoids and related group algebras. For instance, the map \((i,j) \rightarrow \nu_{ij}\) of [11, § 5], provides the kinematical time evolution of the observables \(q\) and \(p\), given by the Hermitian matrices [11, § 5, (✠)], that, by means of the Kuhn-Thomas relation, satisfies the celebrated Heisenberg canonical commutation relations (see [23, § 14.4] and [3, § 2.3]), whereas the HBJ EBB-groupoid algebra constructed in [11, § 6], is the algebra of physical observables according to W. Heisenberg.

The structure of groupoid has many considerable applications both in pure and applied mathematics (see [4, 25]): here, let us consider the following, particular example, drew from Quantum Field Theory (QFT).

• A toy model of QFT. In Renormalization of Quantum Field Theory (see [26] and references therein), a Feynman graph \(\Gamma\) may be analytically represented as sum of iterated
divergent integrals defined as follows

\[ \Gamma^1(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\epsilon}}, \quad \Gamma^n(t) = \int_t^\infty \frac{dp_1}{p_1^{1+\epsilon}} \int_{p_1}^\infty \frac{dp_2}{p_2^{1+\epsilon}} \ldots \int_{p_{n-1}}^\infty \frac{dp_n}{p_n^{1+\epsilon}} \quad n \geq 2 \]

for \( \epsilon \in \mathbb{R}^+ \); such integrals diverges logarithmically for \( \epsilon \to 0^+ \).

It can be proved that these iterated integrals form a Hopf algebra of rooted trees (see [7, 26]), say \( H_R \), and called the Connes-Kreimer Hopf algebra (see [7]). The renormalization of these integrals requires a regularization, for instance through a linear multiplicative functional \( \phi_a \) (bare Green function, defining a Feynman rule) on them, which represents a certain way of evaluation of the Feynman graphs, at the energy scale \( a \), of the type

\[ \phi_a \left( \prod_{i \in I} \Gamma^i(t) \right) = \prod_{i \in I} \Gamma^i(a), \]

where \( I \) denotes an arbitrary finite ordered subset of \( \mathbb{N} \), and \( \Gamma_a = \Gamma^0(a) \) are the normalized coupling constants at the energy scale (or renormalization point) \( a \); if \( \Gamma \) is a Feynman graph, then \( \phi_a(\Gamma) \) is the corresponding regularized Feynman amplitude, according to the renormalization scheme parametrized by \( a \).

So, every Feynman rule is a character \( \phi_a : H_R \to \mathbb{C} \) of the Hopf algebra \( H_R \), and their set is a (renormalization) group \( G_R \) under the group law given by the usual convolution law \( \ast \) of \( H_R \). Thus, the coalgebra structure of \( H_R \) endows \( G_R \) with a well-defined group structure.

If \( S \) denotes the antipode of \( H_R \), then let us consider the following deformed antipode \( S_a = \phi_a \circ S \) : in [26], it is considered a particular modification of the usual antipode axiom \( S \ast id = id \ast S = \eta \circ \varepsilon \) (see [11, § 8]), precisely

\[ \varepsilon_{a, b} = S_a \ast id_b = (\phi_a \circ S) \ast \phi_b = m \circ (S_a \otimes \phi_b) \circ \Delta \quad \text{(renormalized Green functions)} \]

Hence, in [26, § 5], it is proved to be true the following pair groupoid law \( \varepsilon_{a, b} \ast \varepsilon_{b, c} = \varepsilon_{a, c} \), deduced from the Hopf algebra properties of \( H_R \). Moreover, if we consider the renormalized quantities \( \varepsilon_{a, b}(\Gamma^n(t)) = \Gamma^n_{a, b} \), then we have

\[ \Gamma^1_{a, b} = \int_b^a \frac{dp}{p^{1+\epsilon}}, \quad \Gamma^2_{a, b} = \int_b^a \frac{dp_1}{p_1^{1+\epsilon}} \int_{p_1}^a \frac{dp_2}{p_2^{1+\epsilon}}, \ldots, \]

with every \( \Gamma^n_{a, b} \) finite for \( \epsilon \to 0^+ \), and zero for \( a = b \). \( \varepsilon_{a, b} \) is said to be a renormalized character of \( H_R \) at the energy scales \( a, b \). The correspondence (renormalization schemes) \( \phi_b, \phi_b \to \varepsilon_{a, b} \) is what renormalization typically achieves.

Finally, from the relation \( \varepsilon_{a, b} \ast \varepsilon_{b, c} = \varepsilon_{a, c} \) and the coproduct rule of \( H_R \), it is possible to obtain the following relation

\[ \Gamma^n_{a, c} = \Gamma^n_{a, b} + \Gamma^n_{b, c} + \sum_{j=1}^{i-1} \Gamma^i_{a, b} \Gamma^{i-j}_{b, c} \quad i \geq 2, \]

that is a generalization of the so-called Chen’s Lemma (see [5, 13]); this relation describes what happens if we change the renormalization point.

If \( \Gamma \in H_R \) is a Feynman graph, then we have the following asymptotic expansion \( \Gamma = \Gamma^0 + \Gamma^1 + \Gamma^2 + \Gamma^3 + \ldots \); in general, such a series may be divergent, and, in this case, it can be renormalized to an finite, but undetermined, value. We have that \( \varepsilon_{a, b}(\Gamma) = \Gamma_{a, b} \)
is the result of the regularization of $\Gamma$ at the energy scales $a, b$, and, respect to the scale change $\phi_a \rightarrow \phi_b$ (which allows us to renormalizes in a non-trivial manner), we have the following rule for the shift of the normalized coupling constants $\Gamma_b = \Gamma_a + \sum_{i \in \mathbb{N}} \Gamma^i_{a,b}$, in dependence of the running coupling constants $\Gamma^i_{a,b}$, $i \geq 1$.

In short, the comparison among different renormalization schemes (via the variation of the renormalization point) is regulated by the fundamental pair groupoid law $\varepsilon_{a,b} \hat{\ast} \varepsilon_{b,c} = \varepsilon_{a,c}$.

Furthermore, we point out as this groupoid combination law, connected with a variation of the renormalization points, leads us to further formal properties of renormalization as, for instance, the cohomological ones or the Callan-Symanzik type equations.

In [11, § 5], we have defined a specific EBB-groupoid, called the Heisenberg-Born-Jordan EBB-groupoid (or HBJ EBB-groupoid), whose group algebra, said HBJ EBB-groupoid algebra, may be endowed with a (albeit trivial) Hopf algebra structure, obtaining the so-called HBJ EBB-Hopf algebra (or HBJ EBBH-algebra), that is a first, possible example of generalization of the structure of Hopf algebra: indeed, it is a particular weak Hopf algebra, or quantum groupoid (see [19, § 2.5], [20, § 2.1.4] and [24, § 2.2]), in the finite-dimensional case.

We remember that group algebras were basic examples of Hopf algebras, so that groupoid algebras may be considered as basic examples of a class of structures generalizing the ordinary Hopf algebra structure; this class contains the so-called weak Hopf algebras, the Lu’s and Xu’s Hopf algebroids, and so on (see [1, 2, 14, 27]).

In this paper, starting from the E-groupoid structure exposed in [11], we want to introduce another, possible generalization of the ordinary Hopf algebra structure, following the notions of commutative Hopf algebroid (see [22]) and of quantum semigroup (see [9, § 1, p. 800]).

Moreover, at the end of [11, § 8], it has been mentioned both the triviality of the Hopf algebra structure there introduced (on the HBJ EBB-algebra), and some non-trivial duality questions related to the (possible) not finite generation of the HBJ EBB-algebra.

At the § 5. of the present paper we’ll try to settle these questions by means of some fundamental works of S. Majid.

Moreover, it should be interesting to go into the question related to possible, further roles that the groupoid structures may play in Renormalization. See, also, the conclusions of § 7 of the present paper.

See, for instance, the formalization of the quantum mechanics motivations adduced by V. G. Drinfeld in [9, § 1].

That must be considered as a necessary continuation of [1].
Indeed, in [16] and [17], Majid has constructed non-trivial examples of non-commutative and non-cocommutative Hopf algebras (hence, non-trivial examples of quantum groups), via his notion of bicrossproduct. This type of structures involves group algebras and their duals; furthermore, these structures have an interesting physical meaning, since they are an algebraic representation of some quantum mechanics problems (see also [18, Chap. 6]).

Finally, we’ll recall some other cross product constructions, among which the group Weyl algebra and the (Drinfeld) quantum double, that provides further, non-trivial examples of a quantum group having a really physical meaning.

Moreover, if we consider such structures applied to the EBJ EBBH-algebra (of [11, § 8]), then it is possible to get structures that represents an algebraic formalization of some possible quantum mechanics problems on a groupoid, in such a way to obtain new examples of elementary structures of a Quantum Mechanics on groupoids.

2. The Notion of E-semigroupoid

For the notions of E-groupoid and EBB-groupoid, with relative notations, we refer to [11, § 1].

An E-semigroupoid is an algebraic system of the type \((G, G^{(0)}, G^{(1)}, r, s, i, \star)\), where \(G, G^{(0)}, G^{(1)}\) are non-void sets such that \(G^{(0)}, G^{(1)} \subseteq G, r, s : G \to G^{(0)}, i : G^{(1)} \to G^{(1)}\) and \(G^{(2)} = \{(g_1, g_2) \in G \times G; s(g_1) = r(g_2)\}\), satisfying the following conditions:

1. \(s(g_1 \star g_2) = s(g_2), r(g_1 \star g_2) = r(g_1), \forall (g_1, g_2) \in G^{(2)};\)
2. \(s(g) = r(g) = g, \forall g \in G^{(0)};\)
3. \(g \star \alpha(s(g)) = \alpha(r(g)) \star g = g, \forall g \in G;\)
4. \((g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3), \forall g_1, g_2, g_3 \in G;\)
5. \(\forall g \in G^{(1)}, \exists g^{-1} \in G^{(1)} : g \star g^{-1} = \alpha(r(g)), g^{-1} \star g = \alpha(s(g)),\)

being \(\alpha : G^{(0)} \to G\) the immersion of \(G^{(0)}\) into \(G\), and \(i : g \to g^{-1}\). The maps \(r, s\) are called, respectively, range and source, \(G\) is the support, \(G^{(0)}\) is the set of units, and \(G^{(1)}\) is the set of inverses, of the given E-semigroupoid.

\(\text{4Whenever the relative } \star\text{-products are well-defined.}\)
For simplicity, we write \( r(g), s(g) \) instead of \( \alpha(r(g)), \alpha(s(g)) \).

We obtain an E-groupoid when \( G^{(1)} = G \) (see [11, § 1]), whereas we obtain a monoid when \( G^{(0)} = \{e\} \). Moreover, if an E-semigroupoid also verify the condition of [11, § 1, •6], then we have an EBB-semigroupoid.

3. The Notion of Linear \( \mathbb{K} \)-algebroid

A linear algebra (over a commutative scalar field \( \mathbb{K} \)) is an algebraic system of the type \((V_\mathbb{K}, +, \cdot, m, \eta)\), where \((V_\mathbb{K}, +, \cdot)\) is a \( \mathbb{K} \)-linear space and \((V, +, m)\) is a unital ring, satisfying certain compatibility conditions; in particular, \((V, m)\) is a unital semigroup (that is, a monoid).

Following, in part, [21] (where it is introduced the notion of vector groupoid), if it is given an E-semigroupoid \((G, G^{(0)}, G^{(1)}, r, s, i, \ast)\) such that

i) \( G_\mathbb{K} = (G, +, \cdot) \) is a \( \mathbb{K} \)-linear space, and \( G^{(0)}, G^{(1)} \) are its linear subspaces;

ii) \( r, s \) and \( i, \ast \), are linear maps;

iii) \( g_1 \ast (\lambda g_2 + \mu g_3 - s(g_1)) = \lambda (g_1 \ast g_2) + \mu (g_1 \ast g_3) - g_1, \) \( (\lambda g_1 + \mu g_2 - r(g_3)) \ast g_3 = \lambda (g_1 \ast g_3) + \mu (g_2 \ast g_3) - g_3 \), for every \( g_1, g_2, g_3 \in G \) and \( \lambda, \mu \in \mathbb{K} \) for which there exists the relative \( \ast \)-products,

then we say that \((G_\mathbb{K}, G^{(0)}, G^{(1)}, r, s, i, \ast)\) is a linear \( \mathbb{K} \)-algebroid.

4. The Notion of E-Hopf Algebroid

Let \( \mathfrak{G}_\mathbb{K} = (G_\mathbb{K}, G^{(0)}, G^{(1)}, r, s, i, \ast) \) be a linear \( \mathbb{K} \)-algebroid. If we set

\[
G^{(2)} = \{g_1 \otimes_\ast g_2 \in G \times G; s(g_1) = r(g_2)\} \cong G \otimes_\ast G,
\]

\[
m_\ast(g_1 \otimes_\ast g_2) = g_1 \ast g_2,
\]

then, more specifically, with \((\mathfrak{G}_\mathbb{K}, m_\ast, \{\eta^{(e)}_r\}_{e \in G^{(0)}}, \{\eta^{(e)}_s\}_{e \in G^{(0)}})\), we’ll denote such a linear \( \mathbb{K} \)-algebroid where, for each \( e \in G^{(0)} \), we put \( \eta^{(e)}_r, \eta^{(e)}_s : \mathbb{K} \to G \) in such a way that \( \eta^{(e)}_r(k) = \{e\} \) and \( \eta^{(e)}_s(k) = \{e\} \), \( \forall k \in \mathbb{K} \). Hence, the unitary and associativity properties \( \bullet_3 \) and \( \bullet_4 \), of the given linear \( \mathbb{K} \)-algebroid, are as follow:

1. \( m_\ast \circ (\eta^{(e)}_r \otimes_\ast \text{id}) = m_\ast \circ (\text{id} \otimes_\ast \eta^{(e')}_s), \quad \forall e, e' \in G^{(0)}, \)

\(\text{Henceforth, every partial } \ast \text{-operation (as } \otimes_\ast \text{, and so on) that we consider, is assumed to be defined.}\)
We define a partial comultiplication by a map

\[
m_\ast \circ (\eta_{r}^{(e)} \otimes \ast, m_{\ast}) = m_{\ast} \circ (m_{\ast} \otimes \ast, \eta_{s}^{(e')}) , \quad \forall e, e' \in G^{(0)},
\]

where id is the identity of \( G \).

Let us introduce, now, a cosemigroupoid structure as follows. We define a partial comultiplication by a map \( \Delta_{\ast} : G \rightarrow G \otimes_{\ast} G \), in such a way that, when the following condition holds:

\[
3. \quad (\text{id} \otimes_{\ast} \Delta_{\ast}) \circ \Delta_{\ast} = (\Delta_{\ast} \otimes_{\ast} \text{id}) \circ \Delta_{\ast},
\]

then we say that \( (\mathfrak{G}_{K}, \Delta_{\ast}) \) is a cosemigroupoid.

If we require to subsist suitable homomorphism conditions for the maps \( m_{\ast}, \Delta_{\ast}, \{ \eta_{r}^{(e)} \}_{e \in G^{(0)}}, \{ \eta_{s}^{(e)} \}_{e \in G^{(0)}} \), then we may to establish a certain \textit{quantum semigroupoid} structure (in analogy to the quantum semigroup structure - see \[9, \S 1, p. 800\]) on \( (\mathfrak{G}_{K}, m_{\ast}, \{ \eta_{r}^{(e)} \}_{e \in G^{(0)}}, \{ \eta_{s}^{(e)} \}_{e \in G^{(0)}}) \).

Following, in part, the notion of commutative Hopf algebroid given in \[22, \text{Appendix 1}\], if we define certain counits by maps \( \varepsilon_{r}^{(e)}, \varepsilon_{s}^{(e)} : G \rightarrow K \) for each \( e \in G^{(0)} \), then we may to require that further counit properties holds, chosen among the following

\[
\begin{align*}
4.1. \quad (\text{id} \otimes_{\ast} \varepsilon_{r}^{(e)}) \circ \Delta_{\ast} &= (\varepsilon_{r}^{(e')} \otimes_{\ast} \text{id}) \circ \Delta_{\ast} = \text{id}, \quad \forall e, e' \in G^{(0)}, \\
4.2. \quad (\text{id} \otimes_{\ast} \varepsilon_{s}^{(e)}) \circ \Delta_{\ast} &= (\varepsilon_{s}^{(e')} \otimes_{\ast} \text{id}) \circ \Delta_{\ast} = \text{id}, \quad \forall e, e' \in G^{(0)}, \\
4.3. \quad (\text{id} \otimes_{\ast} \varepsilon_{r}^{(e)}) \circ \Delta_{\ast} &= (\varepsilon_{r}^{(e')} \otimes_{\ast} \text{id}) \circ \Delta_{\ast} = \text{id}, \quad \forall e, e' \in G^{(0)}, \\
4.4. \quad (\text{id} \otimes_{\ast} \varepsilon_{s}^{(e)}) \circ \Delta_{\ast} &= (\varepsilon_{s}^{(e')} \otimes_{\ast} \text{id}) \circ \Delta_{\ast} = \text{id}, \quad \forall e, e' \in G^{(0)},
\end{align*}
\]

with a set of compatibility conditions chosen among the following (or a suitable combination of them)

\[
\begin{align*}
5.1. \quad \varepsilon_{r}^{(e)} \circ \eta_{r}^{(e')} &= \varepsilon_{r}^{(e)} \circ \eta_{r}^{(e')} = \text{id}, \quad \eta_{r}^{(e)} \circ \varepsilon_{r}^{(e')} = \eta_{r}^{(e)} \circ \varepsilon_{r}^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)}, \\
5.2. \quad \varepsilon_{s}^{(e)} \circ \eta_{s}^{(e')} &= \varepsilon_{s}^{(e)} \circ \eta_{s}^{(e')} = \text{id}, \quad \eta_{s}^{(e)} \circ \varepsilon_{s}^{(e')} = \eta_{s}^{(e)} \circ \varepsilon_{s}^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)}, \\
5.3. \quad \varepsilon_{r}^{(e)} \circ \eta_{s}^{(e')} &= \varepsilon_{r}^{(e)} \circ \eta_{s}^{(e')} = \text{id}, \quad \eta_{r}^{(e)} \circ \varepsilon_{s}^{(e')} = \eta_{r}^{(e)} \circ \varepsilon_{s}^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)}, \\
5.4. \quad \varepsilon_{s}^{(e)} \circ \eta_{r}^{(e')} &= \varepsilon_{s}^{(e)} \circ \eta_{r}^{(e')} = \text{id}, \quad \eta_{s}^{(e)} \circ \varepsilon_{r}^{(e')} = \eta_{s}^{(e)} \circ \varepsilon_{r}^{(e')} = \text{id}_{G^{(0)}}, \quad \forall e, e' \in G^{(0)},
\end{align*}
\]

in such a case, we may define a suitable \textit{linear \( K \)-coalgebroid} structure of the type \( (\mathfrak{G}_{K}, \Delta_{\ast}, \{ \varepsilon_{r}^{(e)} \}_{e \in G^{(0)}}, \{ \varepsilon_{s}^{(e)} \}_{e \in G^{(0)}}) \), whence a \textit{linear \( K \)-coalgebroid} structure of the type \( (\mathfrak{G}_{K}, m_{\ast}, \Delta_{\ast}, \{ \eta_{r}^{(e)} \}_{e \in G^{(0)}}, \{ \eta_{s}^{(e)} \}_{e \in G^{(0)}}, \{ \varepsilon_{r}^{(e)} \}_{e \in G^{(0)}}, \{ \varepsilon_{s}^{(e)} \}_{e \in G^{(0)}}) \).
If, when it is possible, we impose certain $K$-algebroid homomorphism conditions for the maps $\Delta_\ast, \{\varepsilon^{(e)}\}_e \in G^{(0)}, \{\varepsilon^{(e)}\}_e \in G^{(0)}$, and/or certain $K$-coalgebroid homomorphism conditions for the maps $m_\ast, \{\eta^{(e)}\}_e \in G^{(0)}, \{\eta^{(e)}\}_e \in G^{(0)}$, then we may to establish a certain linear $K$-bialgebroid structure on $B\mathfrak{g}_K$, having posed $B\mathfrak{g}_K = (G_K, m_\ast, \Delta_\ast, \{\eta^{(e)}\}_e \in G^{(0)}, \{\eta^{(e)}\}_e \in G^{(0)}, \{\varepsilon^{(e)}\}_e \in G^{(0)}, \{\varepsilon^{(e)}\}_e \in G^{(0)})$.

Finally, if, for each $f, g \in \text{End} (G_K)$ such that $f \otimes \ast g$ there exists, we put

$$G \xrightarrow{\Delta_\ast} G \otimes \ast G \xrightarrow{f \otimes \ast g} G \otimes \ast G \xrightarrow{m_\ast} G,$$

then it is possible to consider the following (partial) convolution product

$$f \ast_\ast g = m_\ast \circ (f \otimes \ast g) \circ \Delta_\ast \in \text{End} (G_K).$$

Thus, an element $a \in \text{End} (G_K)$ may to be said an antipode of a $K$-bialgebroid structure $B\mathfrak{g}_K$, when there exists $a \ast_\ast \text{id}, \text{id} \ast_\ast a$, and

$$a \ast_\ast \text{id} = \text{id} \ast_\ast a = 2^{th} \text{ condition of } 5_i,$$

if $B\mathfrak{g}_K$ has the property $5_i, \ i = 1, 2, 3, 4$.

A $K$-bialgebroid structure with, at least, one antipode, is said to have an $E$-Hopf algebroid structure. If $G^{(0)} = \{e\}$, then we obtain an ordinary Hopf algebra structure.

5. The Majid’s quantum gravity model

S. Majid, in [16] and [17], have introduced a particular noncommutative and noncocommutative bicrossproduct Hopf algebra that should be viewed as a toy model of a physical system in which both quantum effects (the noncommutativity) and gravitational curvature effects (the noncocommutativity) are unified. The Majid construction, being a noncommutative noncocommutative Hopf algebra, may be viewed as a non-trivial example of quantum group having an important physical meaning. We’ll apply this model to the EBJ EBB-groupoid algebra (eventually equipped with a riemannian structure).

Following [15, Chap. III, § 1], an $E$-groupoid $(G, G^{(0)}, r, s, \ast)$ is said to be a differentiable $E$-groupoid (or a $E$-groupoid manifold) if $G, G^{(0)}$ are differentiable manifolds, the maps $r, s : G \to G^{(0)}$ are surjective submersions, the inclusion $\alpha : G^{(0)} \hookrightarrow G$ is smooth, and the partial multiplication $\ast : G^{(2)} \to G$ is smooth (if we understand $G^{(2)}$ as submanifold of $G \times G$).
A locally trivial differentiable E-groupoid is said to be a Lie E-groupoid.

Following [10], a differentiable E-groupoid \((G,G^0,r,s,⋆)\) is said to be a Riemannian E-groupoid if there exists a metric \(g\) over \(G\) and a metric \(g_0\) over \(G^0\) in such a way that the inversion map \(i:G \to G\) is an isometry, and \(r,s\) are Riemannian submersions of \((G,g)\) onto \((G^0,g_0)\).

Let \(A_{HBJ} = A_K(G_{HBJ}(\mathcal{F})) \cong A_K(G_{Br}(I))\) be the HBJ EBB-algebra of [11, § 6]: as seen there, it represents the algebra of physical observables according to the Matrix Quantum Mechanics.

The question of which metric to choose, for instance over \(A_{HBJ}\), or over \(G_{HBJ} = G_{HBJ}(\mathcal{F}) \cong G_{Br}(I)\) (for this last groupoid isomorphism, see [11, § 5]), is not trivial and not a priori dictated (see, [12, II] for a discussion of a similar question related to a tentative of metrization of the symplectic phase-space manifold of a dynamical system, in order that be possible to define a classical Brownian motion on it).

Let us introduce a Majid’s toy model of quantum mechanics combined with gravity, following [16, 17, 18].
S. Majid ([16]) follows the abstract quantization formulation of I. Segal, whereby any abstract \(C^*\)-algebra can be considered as the algebra of observables of a quantum system, and the positive linear functionals on it as the states.
He, first, consider a pure algebraic formulation of the classical mechanics of geodesic motion on a Riemannian spacetime manifold, precisely on a homogeneous spacetime, following the well-known Mackey’s quantization procedure on homogeneous spacetimes (see [18, Chap. 6], and references therein).

The basis of the Majid’s physical picture lies in a new interpretation of the semidirect product algebra as quantum mechanics on homogeneous spacetime (according to [8]). Namely, he consider the semidirect product \(\mathbb{K}[G_1] \ltimes_\alpha \mathbb{K}(G_2)\) where \(G_1\) is a finite group that acts, through \(\alpha\), on a set \(G_2\); here, \(\mathbb{K}[G_1]\) denotes the group algebra over \(G_1\), whereas \(\mathbb{K}(G_2)\) denotes the algebra of \(\mathbb{K}\)-valued functions on \(G_2\).

[17, section 1.1] motivates the search for self-dual algebraic structures in general, and Hopf algebras in particular, so that it is natural to search the self-dual structure of \(\mathbb{K}[G_1] \ltimes_\alpha \mathbb{K}(G_2)\), as follows.

\(^6\)For the notion of local triviality of a topological groupoid, see [15, Chap. II, § 2].
To this end, we assume that $G_2$ is also a group that acts back by an action $\beta$ on $G_1$ as a set; so, one can equivalently view that $\beta$ induces a coaction of $K[G_2]$ on $K(G_1)$, and defines the corresponding semidirect coproduct coalgebra which we denote $K[G_1]^{\beta} \rtimes K(G_2)$. Such a bicrossproduct structure will be denoted $K[G_1]^{\beta} \bowtie K(G_2)$.

Majid’s model fit together the semidirect product by $\alpha$ with the semidirect coproduct by $\beta$, to form a Hopf algebra; in such a way, we’ll have certain (compatibility) constraints on $(\alpha, \beta)$ that gives a bicrossproduct Hopf algebra structure to $K[G_1] \otimes K(G_2)$, that it is of self-dual type; this structure is non-commutative [non-cocommutative] when $\alpha$ [$\beta$] is non-trivial.

Majid’s model of quantum gravity starts from the physical meaning of a bicrossproduct structure relative to the case $K = \mathbb{C}$, $G_1 = G_2 = \mathbb{R}$ and $\alpha_{\text{left}}(u(s)) = \hbar u + s$, with $\hbar$ a dimensionful parameter (Planck’s constant), achieved as a particular case of the classical self-dual $*$-Hopf algebra of observables (according to I. Segal) $C^{\ast}(G_1) \otimes C(G_2)$, where $G_1, G_2$ has a some group structure and $C^{\ast}(G_1)$ is the convolution $C^{\ast}$-algebra on $G_1$. Further, with a suitable compatibility conditions (see [16, (9)] or [18, (6.15)]) for $(\alpha, \beta)$ – that can be viewed as certain (Einstein) “second-order gravitational field equations” for $\alpha$ (that induces metric properties on $G_2$), with back-reaction $\beta$ playing the role of an auxiliary physical field – we obtain a bicrossproduct Hopf algebra $C^{\ast}(G_1)^{\beta} \bowtie^{\alpha} C(G_2)$, with the following self-duality $(C^{\ast}(G_1)^{\beta} \bowtie^{\alpha} C(G_2))^* \cong C^{\ast}(G_2)^{\alpha} \bowtie^{\beta} C(G_1)$. Moreover (see [16]), in the Lie group setting, the non-commutativity of $G_2$ (whence, the non-cocommutativity of the coalgebra structure) means that the intrinsic torsion-free connection on $G_2$, has curvature (cogravity), that is to say, the non-cocommutativity plays the role of a Riemannian curvature on $G_2$ (in the sense of non-commutative geometry).

We, now, consider a simple quantization problem (see [17, § 1.1.2]). Let $G_1 = G_2$ be a group and $\alpha$ the left action; hence, the algebra $W(G) \triangleleft K^*[G] \kappa_{\text{left}} K(G)$ will be called the group Weyl algebra of $G$, and it represents the algebraic quantization of a particle moving on $G$ by translations.

Finally, if we want to apply these considerations to the case $G_1 = G_2 = G_{HBJ}$, then we must consider both the finite-dimensional and the infinite-dimensional case, in such a way that be possible to determine the dual (or the restricted dual) of $A_K(G_{HBJ})$. 

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Taking into account the physical meaning of $G_{HBJ}$, it follows that it is possible to consider the above mentioned algebraic structures (with their physical meaning) in relation to the case study $G_1 = G_2 = G_{HBJ}$, with consequent physical interpretation (where possible), in such a way to get non-trivial examples of quantum groupoids having a possible quantic meaning.

6. A particular Weyl Algebra (and other structures)

The cross and bicross (or double cross) constructions provides the basic algebraic structures on which to build up non-trivial examples of quantum groups, even in the infinite-dimensional case. In this paragraph, we expose some examples of such constructions.

Let $\mathcal{F}_{HBJ} = \mathcal{F}_K(G_{HBJ}(\mathcal{F}))$ be the linear $K$-algebra of $K$-valued functions defined on $G_{HBJ}$.

Let $(G, \cdot)$ be a finite group. If $\mathcal{F}_K(G)$ is the linear $K$-algebra of $K$-valued functions on $G$, then, since $\mathcal{F}_K(G) \otimes \mathcal{F}_K(G) \cong \mathcal{F}_K(G \times G)$, it follows that such an algebra can be endowed with a natural structure of Hopf algebra by the following data

1. coproduct $\Delta : \mathcal{F}_K(G) \to \mathcal{F}_K(G \times G)$ given by $\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2)$ for all $g_1, g_2 \in G$;

2. counit $\varepsilon : \mathcal{F}_K(G) \to K$, with $\varepsilon(f) = 1$ for each $f \in G$;

3. antipode $S : \mathcal{F}_K(G) \to \mathcal{F}_K(G)$, defined by $S(f)(g) = f(g^{-1})$ for all $g \in G$.

If we consider a finite groupoid instead of a finite group, then 3. even subsists, but 1. and 2. are no longer valid because of the groupoid structure.

If $G = (G, G^{(0)}, r, s, \ast)$ is a finite groupoid, then, following [24, § 2.2], the most natural coalgebra structure on $\mathcal{F}_K(G)$, is given by the following data

1’. coproduct $\Delta(g_1, g_2) = f(g_1 \ast g_2)$ if $(g_1, g_2) \in G^{(2)}$, and $= 0$ otherwise;

2’. counit $\varepsilon(f) = \sum_{e \in G^{(0)}} f(e)$.

7Such a groupoid may be, eventually, endowed with a further topological and/or metric (as the Riemannian one) structure.

8If we consider a non-commutative non-cocommutative Hopf algebra as a model of quantum group.
In such a way, via 1’, 2’ and 3, it is possible to consider a Hopf algebra structure on $\mathcal{F}_K(\mathcal{G})$.

In the finite-dimensional case, we have $A^* \cong A^* \mathcal{K}(\mathcal{G}) \cong \mathcal{F}_K(\mathcal{G})$ (see [18, Example 1.5.4]) as regard the dual Hopf algebra $A^* \mathcal{K}(\mathcal{G})$ of $A \mathcal{K}(\mathcal{G})$, whereas, in the infinite-dimensional case, we have the restricted dual $A^* \mathcal{K}(\mathcal{G}) \cong \mathcal{F}_K(\mathcal{G})(\mathcal{HBJ})$ (in the finite-dimensional case, it is $\mathcal{F}_K(\mathcal{G})(\mathcal{HBJ})$). Hence, if we define the action

$$\alpha : (b, a) \rightarrow b \triangleright a = \langle b, a_{(1)}\rangle a_{(2)} \quad \forall a \in A \mathcal{K}(\mathcal{HBJ}), \quad \forall b \in \mathcal{F}_K(\mathcal{HBJ}),$$

then it is possible to define the left cross product algebra

$$\mathcal{H}(A \mathcal{K}(\mathcal{HBJ}) \cong A \mathcal{K}(\mathcal{HBJ}) \ltimes \alpha \mathcal{F}_K(\mathcal{HBJ}),$$

called the Heisenberg double of $A \mathcal{K}(\mathcal{HBJ})$.

If $V$ is an $A$-module algebra, with $A$ a Hopf algebra, then let $V \ltimes A$ be the corresponding left cross product, and $(v \otimes a) \triangleright w = v(a \triangleright w)$ the corresponding Schrödinger representation (see [18, § 1.6]) of $V$ on itself.

If $V$ has a Hopf algebra structure and $V^{(o)}$ is its restricted dual via the pairing $\langle \cdot, \cdot \rangle$, then (see [18, Chap. 6]) the action $\sigma$ given by $\phi \triangleright v = v_{(1)}\langle \phi, v_{(2)}\rangle$ for all $v \in V, \phi \in V^{(o)}$, make $V$ into a $V^{(o)}$-module algebra and $V \otimes V^{(o)}$ into an algebra with product given by

$$(v \otimes \phi)(w \otimes \psi) = vw_{(1)} \otimes \langle w_{(2)}, \phi_{(1)}\rangle \phi_{(2)}\psi,$$

so that let $W(V) \doteq V \ltimes \sigma V^{(o)}$ be the corresponding left cross product algebra. Hence, it is possible to prove (see [18, Chap. 6]) that the related Schrödinger representation (see [18, § 6.1. p. 222]) give rise to an algebra isomorphism $\chi : V \ltimes \sigma V^{(o)} \rightarrow \text{Lin}(V)$ (= algebra of $K$-endomorphisms of $V$), given by $\chi(v \otimes \psi)w = vw_{(1)}\langle \phi, w_{(2)}\rangle$; $W(V) \doteq V \ltimes \sigma V^{(o)}$ is said to be the (restricted) group Weyl algebra of the Hopf algebra $V$.

This last construction is an algebraic generalization of the usual Weyl algebra of Quantum Mechanics on a group (see [17, § 1.1.2]), whose finite-dimensional prototype is as follows.

Let $G$ be a finite group, and let’s consider the strict dual pair given by the $K$-valued functions on $G$, say $\mathbb{K}(G)$, and the free algebra on $G$, say $K^G$. Hence, the right action of $G$ on itself given by $\psi_u(s) = su$, establishes a left
cross product algebra structure, say $\mathbb{K}(G) \ltimes \mathbb{K}G$, on $\mathbb{K}(G) \otimes \mathbb{K}G$; such an action induces (see [18, Chap. 6]), also, a left regular representation of $G$ into $\mathbb{K}G$, so that we can consider the related Schrödinger representation generated by it and by the action of $\mathbb{K}G$ on itself by pointwise product. Thus, if $V = \mathbb{K}(G)$, with $\mathbb{K}(G)$ endowed with the usual Hopf algebra structure, then we have that the Weyl algebra $\mathbb{K}(G) \ltimes \mathbb{K}G$ (with $V^{(o)} = \mathbb{K}(G)^* = \mathbb{K}G$ since $G$ is finite) is isomorphic to $\text{Lin}(\mathbb{K}(G))$ via the Schrödinger representation.

As already said in the previous paragraph, such a Weyl algebra formalizes the algebraic quantization of a particle moving on $G$ by translations.

If we apply what has been said above to $\mathcal{G}_{HBJ}$ in the finite-dimensional case (that is, when $\text{card } I < \infty$ that correspond to a finite number of energy levels — see [11, § 6]), since $V^{(o)} = \mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ}) = \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$, then we have that

$$\mathcal{W}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})) = \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$$

represents the algebraic quantization of a particle moving on the groupoid $\mathcal{G}_{HBJ}$ by translations, remembering the quantic meaning (according to I. Segal) of $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ (as set of states) and $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ (as set of observables).

Instead, in the infinite-dimensional case, we have that $\mathcal{F}_{\mathbb{K}}^{(o)}(\mathcal{G}_{HBJ})$ is isomorphic to a sub-Hopf algebra of $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ (this is the HBJ EBBH-algebra — see [11, § 8]), so that

$$\mathcal{W}(\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})) = \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ});$$

Finally, the right adjoint action (see [18, § 1.6]) of $\mathcal{G}_{HBJ}$ on itself given by $\psi_g(h) = g^{-1} \star h \star g$ if there exists, and $= 0$ otherwise, make $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ into an $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$-module algebra. In the finite-dimensional case, we have $\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$, so that

$$\mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ}) = \mathcal{A}_{\mathbb{K}}^{**}(\mathcal{G}_{HBJ}) \cong \mathcal{F}_{\mathbb{K}}^{*}(\mathcal{G}_{HBJ}),$$

whence $\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ})$ is a $\mathcal{F}_{\mathbb{K}}^{*}(\mathcal{G}_{HBJ})$-module algebra too. Therefore, we may consider the following left cross product algebra

$$\mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{F}_{\mathbb{K}}^{*}(\mathcal{G}_{HBJ}) \cong \mathcal{F}_{\mathbb{K}}(\mathcal{G}_{HBJ}) \ltimes \mathcal{A}_{\mathbb{K}}(\mathcal{G}_{HBJ})$$

that the tensor product coalgebra makes into a Hopf algebra, called the (Drinfeld) quantum double of $\mathcal{G}_{HBJ}$, and denoted $\mathcal{D}(\mathcal{G}_{HBJ})$; even in the finite-dimensional case, it represent the algebraic quantization of a particle constrained to move on conjugacy classes of $\mathcal{G}_{HBJ}$ (quantization on homogeneous

\footnote{From here, we may speak of a Quantum Mechanics on a groupoid.}
Besides, it has been proved, for a finite group $G$, that this (Drinfeld) quantum double $D(G_{HBJ})$, has a quasitriangular structure (see [18, Chap. 6]) given by

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{u^{-1}su,t} \delta_{t} \otimes uv,$$

$$\Delta(\delta_s \otimes u) = \sum_{ab=s} \delta_a \otimes w \delta_b \otimes u,$$

$$\varepsilon(\delta_s \otimes u) = \delta_{s,e},$$

$$S(\delta_s \otimes u) = \delta_{u^{-1}a^{-1}u} \otimes u^{-1},$$

$$R = \sum_{u \in G} \delta_u \otimes e \otimes 1 \otimes u,$$

where we have identifies the dual of $K^G$ with $K(G)$ via the idempotents $p_g, g \in G$ such that $p_gp_h = \delta_{g,h}p_g$ (see [19, § 2.5] and [18, § 1.5.4]).

Such a quantum double represents the algebra of quantum observables of a certain physical system with symmetry group $G$.

Hence, even in the finite-dimensional case, we may consider, with suitable modifications, an analogous quasitriangular structure on $G_{HBJ}$, obtaining the (Drinfeld) quantum double $D(G_{HBJ})$ on $G_{HBJ}$; thus, if we consider a quasitriangular Hopf algebra as a model of quantum group, the (Drinfeld) quantum double $D(G_{HBJ})$ provides a non-trivial example of quantum group having a (possible) quantic meaning (related to a quantum mechanics on a groupoid).

7. Conclusions

From what has been said above, and in [11], it rises a possible role played by the groupoid structures in Quantum Mechanics and Quantum Field Theory.

For instance, such a groupoid structure might takes place a prominent rule in Renormalization, as well as in the Majid’s model of quantum gravity. Indeed, a central problem in quantum gravity concerns its nonrenormalizability due to the existence of UV divergences; in turn, the UV divergences arise from the assumption that the classical configurations being summed over are defined on a continuum.

So, the discreteness given by groupoid structures may turn out to be of some usefulness in such a (renormalization) QFT problems.

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10 The infinite-dimensional case is not so immediate.
11 Eventually equipped with further, more specific structures, as the topological or metric ones.
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