TOPOLOGICAL CONSTRUCTIONS FOR TIGHT SURFACE GRAPHS

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Abstract. We investigate properties of sparse and tight surface graphs. In particular we derive topological inductive constructions for \((2, 2)\)-tight surface graphs in the case of the sphere, the plane, the twice punctured sphere and the torus. In the case of the torus we identify all 116 irreducible base graphs and provide a geometric application to configurations of circular arcs in the spirit of the Koebe-Andreev-Thurston circle packing theorem.

1. Introduction

Given a graph (loop edges and parallel edges are allowed) \(\Gamma\) and a non-negative integer \(k\), define \(\gamma_k(\Gamma) = kn - m\) where \(\Gamma\) has \(n\) vertices and \(m\) edges. Intuitively we think of \(\gamma_k\) as a functional that measures the \(k\)-dimensional ‘degrees of freedom’ of the graph. If \(l \leq k\), we say that \(\Gamma\) is \((k, l)\)-sparse if \(\gamma_k(\Gamma') \geq l\) for every nonempty subgraph \(\Gamma'\) of \(\Gamma\). If, in addition, \(\gamma_k(\Gamma) = l\), then we say that \(\Gamma\) is \((k, l)\)-tight.

The definition must be modified a little to obtain a useful definition of sparsity in the case \(k < l\). Although we will not consider this case in our work, for the purposes of the introductory discussion we mention that for \((2, 3)\)-sparsity we only require the sparsity inequality to be satisfied for subgraphs with at least one edge. For \((2, 4)\)-sparsity we require that the graph have no loop edges and that the sparsity inequality need only be satisfied for subgraphs with at least two edges.

Sparse and tight graphs for various values of \((k, l)\) arise naturally in several contexts. The well known tree packing theorem of Nash-Williams and Tutte can be viewed as a characterisation of \((k, k)\)-tight graphs (see [17] and [22]). More recently much of the interest in sparsity has been inspired by geometric rigidity theory, starting with the foundational result of Pollaczek-Geiringer, independently rediscovered by Laman, that characterises \((2, 3)\)-tight graphs as precisely those that have a generic realisation as rigid plane bar-joint frameworks (see [18], [19] and [13]). Many other similar results relating various classes of sparse and tight graphs to rigidity properties have followed.

Sparse and tight graphs arise naturally in other geometric settings. We will see below that \((2, l)\)-sparse graphs arise naturally in the combinatorics of certain arrangements of curves in the plane and other surfaces.

In the context of topological graph theory sparsity counts again appear naturally. For example, plane triangulations are \((3, 6)\)-tight. More pertinent to the present work, it is an

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easy consequence of Euler’s polyhedral formula that a plane graph is \((2, 4)\)-tight if and only if it is a quadrangulation.

In all of the settings mentioned above, inductive arguments based on vertex splitting moves play an important role. We mention in particular the following: Nakamoto and Ota \([16]\) have shown that for any compact closed surface there are finitely many irreducible quadrangulations such that all other quadrangulations can be constructed from one of these by a sequence quadrilateral splitting moves. This result follows earlier work of Barnette and other proving results of a similar nature for triangulations of surfaces \([2]\). More recently Fekete, Jordán and Whiteley have given a topological inductive characterisation of plane Laman graphs using vertex splitting and, in a similar vein, Alam et al. have characterised plane \((2, 2)\)-tight graphs (see \([8]\) and \([1]\)). We also mention that, motivated by applications in geometric rigidity, the first three authors have considered inductive characterisations of block-and-hole graphs (which are modified plane triangulations) and partial triangulations of a torus (see \([3]\) and \([4]\)).

In the present work we consider \((2, l)\)-sparsity for graphs embedded in various surfaces and in particular we investigate the properties of vertex splitting type operations on such embedded graphs. We give some general results that apply to arbitrary surfaces and sparsity counts before specialising to the consideration of \((2, 2)\)-tight graphs in orientable surfaces of genus at most one. Here we are able to give explicit inductive characterisations for various classes of graphs and, inspired by the results of Alam et al. mentioned above, we give a geometric application of our main inductive characterisation in the context of non overlapping collections of circular arcs in the flat torus.

1.1. **Outline of the paper and summary of main results.** Section 2 summarises the necessary background material from graph theory and topology in particular.

Section 3 contains a review of the basic facts about \((2, l)\)-sparse graphs. We also prove a key result, Theorem 3.5, which provides an important connection between the topological and combinatorial properties of a tight surface graph. Special cases of this have appeared elsewhere (notably in \([8]\), \([3]\) and \([4]\)).

In Section 4 we give a careful analysis of certain operations which later form the basis of our main inductive construction results.

Having defined our class of inductive moves we turn, in Section 5, to the investigation of those surface graphs that are irreducible with respect to the specified set of operations. We conjecture that for any surface of finite type there are finitely many such irreducibles and we are easily able to establish this in the case of the sphere, plane and twice punctured sphere.

In Section 6 we show that a tight subgraph of an irreducible surface graph is also irreducible (Theorem 6.6). This is an important result for our later work and we note that we can prove this for arbitrary orientable surfaces (not just for small genus) which we hope may prove useful in future work on higher genus.

In Section 7 we consider the class of irreducibles on the torus which, in contrast to the three other surfaces considered earlier, requires considerable effort. We are able to establish an upper bound for the size of an irreducible, thus reducing the problem to a finite search. In Section 8 we outline the data structures and algorithms that we used in a computer assisted search to find all examples.

Finally in Section 9 we give a geometric application of our inductive construction for \((2, 2)\)-tight torus graphs.
2. Graphs, surfaces, embeddings and rotation systems

In this section we review the basic concepts that we need from topological graph theory.

2.1. Graphs and surface graphs. We will use the word graph as a synonym of one-dimensional combinatorial cell complex. Thus, a graph is a quadruple \( \Gamma = (\Sigma, E, s, t) \) where \( V, E \) are finite sets (vertices and edges respectively) and \( s, t : E \to V \) are functions that encode the incidence relation between the edges and vertices. So graphs can have multiple parallel edges and/or loop edges. If \( e \in \{ s(e), t(e) \} \) then we say that the vertex \( v \) is incident to the edge \( e \) or vice versa. We say that edges \( e, f \) are adjacent if they are incident to a common vertex and we say that vertices \( u, v \) are adjacent if they are incident to a common edge. Given \( E' \subset E, V(E) \) is the set of vertices spanned by \( E' \). Given \( V' \subset V, E(V) \) is the set of edges spanned by \( V' \). A walk of length \( k \) in \( \Gamma \) is a sequence \( w = v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1} \) where for \( i = 1, \ldots, k \), \( e_i \) is incident to both \( v_i \) and \( v_{i+1} \). Note that in the case that \( e_i \) is a loop edge, some extra care is required to specify which direction of \( e_i \) is intended. However since loop edges cannot arise in the context of our later results we ignore this ambiguity in the notation. We say that \( w \) is a cycle if \( v_{k+1} = v_1 \) and we say that such a cycle is simple if that is the only repeated vertex.

The geometric realisation of \( \Gamma \) is the compact topological space \( |\Gamma| = V \cup (E \times [0, 1]) / \sim \) where \((e, 0) \sim s(e) \) and \((e, 1) \sim t(e) \). Suppose that \( \Sigma \) is a surface (a real two dimensional smooth manifold). A \( \Sigma \)-graph is a pair \( G = (\Gamma, \varphi) \) where \( \Gamma \) is a graph and \( \varphi : |\Gamma| \to \Sigma \) is an embedding (that is to say a continuous injective map). Suppose that for \( i = 1, 2 \), \( G_i = (\Gamma_i, \varphi_i) \) is a \( \Sigma_i \)-graph. We say that \( G_1 \) and \( G_2 \) are isomorphic if there is a homeomorphism \( h : \Sigma_1 \to \Sigma_2 \) and a graph isomorphism \( l : \Gamma_1 \to \Gamma_2 \) such that \( h \circ \varphi_1 = \varphi_2 \circ l \). Up to isomorphism, surface graphs can be described by a combinatorial data structure called a rotation system and it is common in the topological graph theory literature to work exclusively with rotation systems: see Section 3 for the basic definition and [15] for a more thorough treatment of rotation systems and combinatorial maps. Generally we will use topological descriptions of the objects and trust that the reader can, if desired, make the appropriate translation to the language of rotation systems and combinatorial maps. In Section 3 we do make use of rotation systems in order to describe a suitable data structure for making computations with surface graphs.

Note that we will apply some of the standard terminology of graph theory to surface graphs with the understanding that we are referring to the underlying graph where appropriate. For example if \( G = (\Gamma, \varphi) \) we say that \( G \) is connected if \( \Gamma \) is connected in the standard sense of undirected graphs. Similarly we will understand vertices and edges of \( G \) to mean vertices and edges of \( \Gamma \), or possibly their images under \( \varphi \). We will also apply standard set theoretic operations such as subset, union or intersection, understanding that we refer to the underlying graph. For example if \( G_i = (\Gamma_i, \varphi_i), i = 1, 2 \) are both subgraphs of a \( \Sigma \)-graph \( G = (\Gamma, \varphi) \), then \( G_1 \cup G_2 \) is the \( \Sigma \)-graph \( (\Gamma_1 \cup \Gamma_2, \varphi|_{\Gamma_1 \cup \Gamma_2}) \), where \( \Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2, s, t) \).

2.2. Constructions and deletions. Let \( e \) be an edge of \( \Gamma \) that is not a loop edge. The Jordan-Schoenhflies Theorem implies that there is a homeomorphism \( \Sigma/e \to \Sigma \) that is the identity outside a open ball around \(|e|\). Moreover this homeomorphism is unique up to isotopy. Clearly \( \varphi \) induces an embedding \( |\Gamma/e| \to \Sigma/\varphi(e) \) and we compose this with the homeomorphism \( \Sigma/\varphi(e) \to \Sigma \) to obtain an embedding \( \varphi/e : |\Gamma/e| \to \Sigma \). Let \( G/e = (\Sigma, \varphi/e) \).

Also, if \( e \) is any edge (loop or not) of \( G \) we define \( G - e \) to be the \( \Sigma \)-graph \( (\Gamma - e, \varphi|_{\Gamma - e}) \).

In later sections we adopt the following notational convention: if \( H \) is a subgraph of \( G \) and \( e \) is an edge of \( G \) then \( H/e = H - e = H \) in the case where \( e \) is not an edge of \( H \).
2.3. Faces and subgraphs. A face of $G$ is a component of $\Sigma - \varphi([\Gamma])$. The boundary of $F$, denoted $\partial F$ is the $\Sigma$-subgraph of $G$ consisting of all those edges and vertices in $F$, the topological closure of $F$ in $\Sigma$. There is a well defined collection of closed walks, called the boundary walks of $F$ that cover the underlying graph of $\partial F$ (see [2] for details). We say that $F$ has a non degenerate boundary if no vertex is repeated among all the boundary walks of $F$.

We say that a face is cellular if it is homeomorphic to $\mathbb{R}^2$ and we say that $G$ is cellular if every face of $G$ is cellular. For a cellular face $F$, the degree, denoted $|F|$ is the length of the unique boundary walk of $F$. We write $f_i$ for the number of cellular faces of degree $i$. Note that if $\Sigma$ is connected then $f_0 \geq 1$ if and only if $\Sigma$ is a sphere and $\Gamma$ is a single vertex.

Suppose that $G = (\Gamma, \varphi)$ is a $\Sigma$-graph as above. A subgraph $\Omega$ of $\Gamma$ induces a $\Sigma$-graph $H = (\Omega, \varphi|_{\Omega})$ and we refer to $H$ as a subgraph of $G$. Let $F$ be a face of $H$. Let $\Lambda$ be the subgraph of $\Gamma$ consisting of all those vertices and edges of $\Gamma$ whose image under $\varphi$ is contained in $\overline{F}$. Define

$$\text{int}_G(F) = (\Lambda, \varphi|_{\Lambda}).$$

Observe that any face of $\text{int}_G(F)$ that is contained in $F$ is also a face of $G$. On the other hand, there are one or more faces of $\text{int}_G(F)$ which are contained in $\Sigma - \overline{F}$. We call such a face an external face of $\text{int}_G(F)$. Such an external face need not be a face of $G$. Note that if $F$ has a unique boundary walk that is a simple cycle, then $\text{int}_G(F)$ has just one external face. In general it may have more than one external face.

Similarly, let $\Phi$ be the subgraph of $\Gamma$ consisting of those vertices and edges of $\Gamma$ whose image under $\varphi$ is contained in $\Sigma - F$. Define

$$\text{ext}_G(F) = (\Phi, \varphi|_{\Phi})$$

and observe that $\text{ext}_G(F)$ has one exceptional face, namely $F$, such that all other faces of $\text{ext}_G(F)$ are also faces of $G$.

Finally we observe that $\partial F = \text{int}_G(F) \cap \text{ext}_G(F) = \text{int}_G(F) \cap H$.

2.4. Simple loops in surfaces. A loop in a surface $\Sigma$ is a continuous map $\alpha : S^1 \to \Sigma$. We say that $\alpha$ is simple if it is injective. We say that $\alpha$ is non separating if $\Sigma - \alpha(S^1)$ has the same number of connected components as $\Sigma$. Given a simple loop $\alpha$ in a surface $\Sigma$ we say that $\Sigma - \alpha(S^1)$ is the surface obtained by cutting along $\alpha$. Given a surface $\Sigma$ with boundary we can cap a boundary component by gluing a copy of a closed disc to the surface along the given boundary component.

If $\Sigma$ is a surface without boundary and of genus $g$ and $\alpha$ is a non separating simple loop in $\Sigma$ then we form $\Sigma^g$ by cutting along $\alpha$ and then capping the two resulting boundary components. Clearly $\Sigma^g$ is a surface without boundary of genus $g - 1$.

Given simple loops $\alpha, \beta$ in $\Sigma$, recall that the geometric intersection number is defined by

$$i(\alpha, \beta) = \min |\alpha'(S^1) \cap \beta'(S^1)|$$

where $\alpha'$, respectively $\beta'$, varies over all simple loops that are homotopic to $\alpha$, respectively $\beta$. We review some basic facts about this invariant that we will need later. Proofs of all of the assertions below can found in (or at least easily deduced from) many sources (for example [2]). If $i(\alpha, \beta) \neq 0$ then both $\alpha$ and $\beta$ are essential: that is to say they are not null homotopic. If $i(\alpha, \beta) = 1$ then both $\alpha$ and $\beta$ are non separating in $\Sigma$. Finally in the special case that $\Sigma$ is the torus, if $i(\alpha, \beta) = 0$ and $i(\beta, \delta) = 0$ then $i(\alpha, \delta) = 0$.

Suppose that $G = (\Gamma, \varphi)$ is a $\Sigma$-graph and let $F$ be a face of $G$. Further suppose that $\alpha$ is a non separating loop in $\Sigma$ such that $\alpha(S^1) \subset F$. By cutting and capping $\Sigma$ along $\alpha$ we
can form a $\Sigma^\alpha$-graph, denoted $G^\alpha$ which has the same underlying graph as $G$. Observe that all faces of $G^\alpha$ except the one(s) corresponding to $F$ are also faces of $G$. In this way we can, given any $\Sigma$-graph $G$ whose underlying graph is connected, construct a cellular graph $\tilde{G}$, by cutting and capping along a collection of non separating curves contained in the non cellular faces of $G$.

Finally, some terminology. If $G = (\Gamma, \varphi)$ is a $\Sigma$-graph and $\alpha$ is a loop in $\Sigma$, we say that $\alpha$ is contained in $G$ if $\alpha(S^1) \subset \varphi(|\Gamma|)$.

3. Sparsity

For a graph $\Gamma = (V, E, s, t)$ as above, define $\gamma(\Gamma) = 2|V| - |E|$. For $l \leq 2$ we say that $\Gamma$ is $(2, l)$-sparse (or just sparse if $l$ is clear from the context) if, $\gamma(\Gamma') \geq l$ for every nonempty subgraph $\Gamma'$ of $\Gamma$. We say that $\Gamma$ is $(2, l)$-tight if it is $(2, l)$-sparse and $\gamma(\Gamma) = l$. We will be particularly interested in $(2, 2)$-sparse graphs. Note that $(2, 2)$-tight graphs cannot have loop edges but can have parallel edges (but not triples of parallel edges).

We record some elementary lemmas for later use. The proofs are straightforward and we omit them.

**Lemma 3.1.** Suppose that $\Gamma_1, \Gamma_2$ are subgraphs of $\Gamma$. Then

\[
\gamma(\Gamma_1 \cup \Gamma_2) = \gamma(\Gamma_1) + \gamma(\Gamma_2) - \gamma(\Gamma_1 \cap \Gamma_2)
\]

□

**Lemma 3.2.** Suppose that $\Gamma$ is $(2, 2)$-sparse and that $\gamma(\Gamma') \leq 3$ for some subgraph $\Gamma'$ of $\Gamma$. Then $\Gamma'$ is connected. □

**Lemma 3.3.** Suppose that $\Gamma_1, \Gamma_2$ are $(2, 2)$-tight subgraphs of a $(2, 2)$-sparse graph $\Gamma$. If $\Gamma_1 \cap \Gamma_2$ is not empty then both $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ are $(2, 2)$-tight. □

Now we consider sparsity in the context of surface graphs.

**Theorem 3.4.** If $\Sigma$ is a connected boundaryless compact orientable surface of genus $g$ and $G$ is a cellular $\Sigma$-graph then

\[
\sum_{i \geq 0} (4 - i)f_i = 8 - 8g - 2\gamma(G)
\]

**Proof.** Use the Euler polyhedral formula and the fact that $\sum if_i = 2|E|$. □

Now our first significant new result. We note that related results and special cases of this have appeared elsewhere (notably [8] and [3]).

**Theorem 3.5.** Suppose that $l \leq 2$ and that $G$ is a $(2, l)$-tight $\Sigma$-graph. If $H$ is a subgraph of $G$ and $F$ is a face of $H$, then $\gamma(H \cup \text{int}_G(F)) \leq \gamma(H)$.

**Proof.** By Lemma [1] we have

\[
\gamma(H \cup \text{int}_G(F)) = \gamma(H) + \gamma(\text{int}_G(F)) - \gamma(H \cap \text{int}_G(F)).
\]
Now, $H \cap \text{int}_G(F) = \text{ext}_G(F) \cap \text{int}_G(F)$ and using Lemma 1 again, we see that
\[
\gamma(H \cap \text{int}_G(F)) = \gamma(\text{ext}_G(F) \cap \text{int}_G(F)) \\
= \gamma(\text{int}_G(F)) + \gamma(\text{ext}_G(F)) - \gamma(\text{ext}_G(F) \cup \text{int}_G(F)) \\
= \gamma(\text{int}_G(F)) + \gamma(\text{ext}_G(F)) - \gamma(G) \\
= \gamma(\text{int}_G(F)) + \gamma(\text{ext}_G(F)) - l \\
\geq \gamma(\text{int}_G(F)).
\]

The last inequality above follows from applying the sparsity of $G$ to the nonempty subgraph $\text{ext}_G(F)$.

\[\square\]

**Corollary 3.6.** Suppose that $l \leq 2$ and that $G$ is a $(2,l)$-tight $\Sigma$-graph. If $H$ is a subgraph of $G$ and $F$ is a face of $H$, then $\gamma(\text{ext}_G(F)) \leq \gamma(H)$.

**Proof.** Let $J_1, \cdots, J_k$ be all the faces of $H$ that are different from $F$. Then $\text{ext}_G(F) = H \cup \bigcup_{i=1}^k \text{int}_G(J_i)$. Now the conclusion follows from repeated applications of Theorem 3.5. \[\square\]

We remark that all of the results of this section admit straightforward adaptations to the function $\gamma_k$ for any positive integer $k$. We have specialised to the case $\gamma = \gamma_2$ since this will be our main interest later and we wish to avoid excessive notational clutter.

### 4. Inductive operations on surface graphs

In this section we will focus on topological inductive operations on graphs that are natural in the context of $(2,l)$-tight graphs. Let $G$ be a $\Sigma$-graph. A digon, respectively triangle, respectively quadrilateral is a cellular face of degree two, respectively three, respectively four.

#### 4.1. Digon contractions.

Suppose that $D$ is a digon of $G$ with boundary walk $v_1, e_1, v_2, e_2, v_1$ such that $v_1 \neq v_2$ and $e_1 \neq e_2$. Let $G_D = (G/e_1) - e_2$. Observe that $(G/e_1) - e_2$ is canonically isomorphic to $(G/e_2) - e_1$, so $G_D$ depends only on the digon and not the particular choice of labelling the edges.

We remark that, for a connected surface $\Sigma$, while a digon in a $(2,2)$-sparse $\Sigma$-graph necessarily has distinct vertices, it may have degenerate boundary, but only in the case that the graph is a single edge and $\Sigma$ is a sphere.

**Lemma 4.1.** $G$ is $(2,l)$-sparse if and only if $G_D$ is $(2,l)$-sparse

**Proof.** Let $z$ be the vertex of $G_D$ that corresponds to the contracted edge $e_1$. It is clear that any subgraph of $G_D$ that does not contain $z$ is isomorphic to a subgraph of $G$. Also if $H$ is a subgraph of $G_D$ that does contain $z$ then there is a subgraph $K$ of $G$ such that $e_1, e_2 \in K$ and $(K/e_1) - e_2 = H$. Thus $\gamma(K) = \gamma(H)$. So we have shown that if $G$ is sparse then so is $G_D$.

For the converse, suppose that $H$ is a subgraph of $G$ such that $\gamma(H) < l$. If $\{v_1, v_2\} \not\subset H$, then $H$ is isomorphic to a subgraph of $G_D$. If $\{v_1, v_2\} \subset H$, then $\gamma((H/e_1) - e_2) \leq \gamma(H) < l$. So in either case $G_D$ is not sparse. \[\square\]
4.2. Triangle contractions. Now suppose that $T$ is a triangle in $G$ with boundary walk $v_1,e_1,v_2,e_2,v_3,e_3,v_1$ such that $v_1 \neq v_3$ and $e_1 \neq e_2$. Let $G_{T,e_1} = (G/e_1) - e_2$. We note that a triangle in a $(2,2)$-sparse surface graph necessarily has a non degenerate boundary walk, since any degeneracy would entail a (forbidden) loop edge.

**Lemma 4.2.** Suppose that $G$ is $(2,l)$-sparse and that $G_{T,e_1}$ is not $(2,l)$-sparse. Then there is a subgraph $H$ of $G$ that contains $e_1$ but not $v_3$ such that $\gamma(H) = l$

*Proof.* Let $K$ be a subgraph of $G_{T,e_1}$ satisfying $\gamma(K) \leq l - 1$ and let $z$ be the vertex of $G_{T,e_1}$ corresponding to the contracted edge $e_1$. Clearly $z \in K$, otherwise $K$ is also a subgraph of $G$. Also, we can clearly assume that $K$ is an induced subgraph of $G$.

Now let $H$ be the maximal subgraph of $G$ such that $H/e_1 - e_2 = K$. Now $\gamma(K) \geq \gamma(H) - 1$ with equality if and only if $e_2 \notin H$. But $\gamma(K) \leq l - 1$ and $\gamma(K) \geq l$, so we do indeed have equality. Since $K$ and therefore $H$ are induced subgraphs, it follows that $v_3 \notin H$. \qed

We refer to the graph $H$ whose existence is asserted in Lemma 4.2 as a blocker for the contraction $G_{T,e_1}$.

As noted above, in a $(2,2)$-sparse surface graph a triangle necessarily has a non degenerate boundary walk. Thus there are three possible contractions (one for each of the edges) associated to any such face.

**Lemma 4.3.** Suppose that $G$ is a $(2,2)$-sparse $\Sigma$-graph and that $T$ is a triangle with edges $e_1,e_2,e_3$. Then at least two of the $\Sigma$-graphs $G_{T,e_1},G_{T,e_2},G_{T,e_3}$ are $(2,2)$-sparse.

*Proof.* Suppose that there are blockers $H_1$, respectively $H_2$, for $G_{T,e_1}$ respectively $G_{T,e_2}$. Then $v_1,v_3 \in H_1 \cup H_2$. However $v_3 \notin H_1$ and $v_1 \notin H_2$ so $e_3 \notin H_1 \cup H_2$. However $v_2 \in H_1 \cap H_2$ so by Lemma 3.3 $H_1 \cup H_2$ is $(2,2)$-tight. This contradicts the sparsity of $G$. \qed

4.3. Quadrilateral contractions. Finally, suppose that $Q$ is a quadrilateral of $G$ with boundary walk $v_1,e_1,v_2,e_2,v_3,e_3,v_4,e_4,v_1$. Suppose that $v_1 \neq v_3$ and $e_1 \neq e_3$. Let $d$ be a new edge that joins $v_1$ and $v_3$ and is embedded as a diagonal of the quadrilateral $Q$. Define $G_{Q,v_1,v_3}$ to be $(G \cup \{d\})/d - \{e_1,e_3\}$. Clearly the underlying graph of $G_{Q,v_1,v_3}$ is obtained from $\Gamma$ by identifying the vertices $v_1$ and $v_3$ and then deleting $e_1$ and $e_3$. Thus $\gamma(G) = \gamma(G_{Q,v_1,v_3})$. However this quadrilateral contraction move does not necessarily preserve $(2,l)$-sparsity.

**Lemma 4.4.** Suppose that $G$ is $(2,l)$-sparse but $G_{Q,v_1,v_3}$ is not $(2,l)$-sparse. Then at least one of the following statements is true.

1. There is some subgraph $H$ of $G$ such that $v_1,v_3 \in H$, exactly one of $v_2,v_4$ is in $H$ and $\gamma(H) = l$. (H is called a type 1 blocker.)
2. There is some subgraph $K$ of $G$ such that $v_1,v_3 \in K$, $v_2,v_4 \not\in K$ and $\gamma(K) = l + 1$. (K is called a type 2 blocker.)

*Proof.* Let $K$ be a maximal subgraph of $G_{Q,v_1,v_3}$ satisfying $\gamma(K) \leq l - 1$. Let $z$ be the vertex of $G_{Q,v_1,v_3}$ corresponding to $v_1$ and $v_3$. Clearly $z \in K$, otherwise $K$ would also be a subgraph of $G$. Let $H$ be the maximal subgraph of $G$ satisfying $(H \cup \{d\})/d - \{e_1,e_3\} = K$. It is clear that $H$ is an induced subgraph, since $K$ is an induced subgraph. If $\{v_2,v_4\} \subset H$, then $\gamma(H) = \gamma(K) \leq l - 1$ which contradicts the sparsity of $G$. So at most one of $v_2,v_4$ belongs to $H$. Also, it is clear that $l \leq \gamma(H) \leq \gamma(K) + 2 \leq l - 1$. So $\gamma(H) = l$ or $l + 1$. If $\gamma(H) = l$ and one of $v_2,v_4 \in H$ then (1) is true. If $\gamma(H) = l$ and neither of $v_2,v_4$ is in $H$, then let $H' = H \cup \{v_2\} \cup \{e_1,e_2\}$. Now observe that $e_1 \neq e_2$ since $v_1 \neq v_3$. Thus $\gamma(H') = \gamma(H) = l$.
Figure 1. A $(2, 2)$-tight projective plane graph. Here we are using the representation of the projective plane as a disc with antipodal boundary points identified. This surface graph has a single quadrilateral face, with a degenerate boundary walk.

and, again, (1) is true. Finally if $\gamma(H) = l + 1$. Then $\gamma(H) = \gamma(K) + 2$ and since $H$ is a induced graph, it follows that neither of $v_2, v_4$ belongs to $H$. Thus (2) is true in this case. □

In the special case that $l = 2$, various degeneracies are forbidden. Now suppose that $G$ is a $(2, 2)$-sparse $\Sigma$-graph and that $Q$ is a quadrilateral face of $G$ with boundary walk $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. We adopt the notational convenience that $v_5 = v_1$ and $e_5 = e_1$.

Lemma 4.5. For $i = 1, 2, 3, 4$, $v_i \neq v_{i+1}$.

Proof. Loop edges are forbidden in a $(2, 2)$-sparse graph. □

Lemma 4.6. Suppose that $\Sigma$ is orientable and that $G$ is $(2, 2)$-tight. Then $e_i \neq e_j$ for $1 \leq i < j \leq 4$.

Proof. We observe that since $\Sigma$ is orientable, a repeated edge in $\partial Q$ entails the existence either of a vertex of degree one or a loop edge. Both of these are forbidden in a $(2, 2)$-tight graph. □

Lemma 4.7. Suppose that $\Sigma$ is orientable, $G$ is $(2, 2)$-tight and that $v_1 = v_3$. Then $v_2 \neq v_4$. Furthermore $G_{Q,v_2,v_4}$ is also $(2, 2)$-tight.

Proof. Suppose that $v_2 = v_4$. By Lemma 4.5 and the sparsity of $G$, $\partial Q$ has exactly two vertices and two edges. This contradicts Lemma 4.6. Thus $v_2 \neq v_4$.

Now suppose that $G_{Q,v_2,v_4}$ is not $(2, 2)$-tight. Then by Lemma 4.4 there is a blocker for this contraction. Since $v_1 = v_3$ by assumption, the blocker must be a type 2 blocker. Thus we have a subgraph $K$ such that $\gamma(K) = 3$, $v_2, v_4 \in K$ and $v_1 \notin K$. However, by Lemma 4.6 there are at least four edges joining $v_1$ to $K$, contradicting the sparsity of $G$. □

See Figure 1 for an example of $(2, 2)$-tight projective plane graph whose only face is a quadrilateral with repeated edges in the boundary walk. This example shows that orientability is not a redundant hypothesis in the statements of Lemmas 4.6 or 4.7.

Lemma 4.8. Suppose that $\Sigma$ is orientable, $G$ is $(2, 2)$-tight and $Q$ is a quadrilateral face of $G$ such neither $G_{Q,v_1,v_3}$ nor $G_{Q,v_2,v_4}$ is $(2, 2)$-sparse. Then $Q$ has a non degenerate boundary. Furthermore, if $H_1$ and $H_2$ are blockers for $G_{Q,v_1,v_3}$ respectively $G_{Q,v_2,v_4}$, then both $H_1$ and $H_2$ are type 2 blockers and $H_1 \cap H_2 = \emptyset$.

Proof. The non degeneracy of the boundary walk of $Q$ follows immediately from Lemmas 4.5, 4.6 and 4.7.
Now suppose that one of the blockers, say $H_1$, is of type 1 and suppose that $v_2 \notin H_1$. Then $v_4 \in H_1 \cap H_2$. So $\gamma(H_1 \cup H_2) = \gamma(H_1) + \gamma(H_2) - \gamma(H_1 \cap H_2) \leq 2 + \gamma(H_2) - 2 = \gamma(H_2)$. Now if $H_2$ is also type 1 then $\gamma(H_1 \cup H_2) = 2$. However $v_1, v_2, v_3 \in H_1 \cup H_2$ but $H_1 \cup H_2$ does not contain one of $e_1, e_2$ which contradicts the sparsity of $G$. Similarly if $H_2$ is type 2, then $\gamma(H_1 \cup H_2) \leq 3$, but $H_1 \cup H_2$ does not contain either of $e_1, e_2$, again contradicting the sparsity of $G$.

So both $H_1$ and $H_2$ are type 2 blockers. Moreover $v_1, v_2, v_3, v_4 \in H_1 \cup H_2$ but $e_1, e_2, e_3, e_4 \notin H_1 \cup H_2$. Now

$$2 \leq \gamma(H_1 \cup H_2 \cup \{e_1, e_2, e_3, e_4\})$$
$$= \gamma(H_1) + \gamma(H_2) - \gamma(H_1 \cap H_2) - 4$$
$$= 2 - \gamma(H_1 \cap H_2)$$

So $\gamma(H_1 \cap H_2) \leq 0$ which implies that $H_1 \cap H_2 = \emptyset$. \qed

In the situation described in the statement of Lemma 4.8 we say that the quadrilateral $Q$ is blocked. Observe that the blocker $H_1$ is connected so it is possible to find a simple walk from $v_1$ to $v_3$ in $H_1$. By concatenating the geometric realisation of this walk with the diagonal of $Q$ joining $v_3$ and $v_1$ we obtain a simple loop in $\Sigma$, which we denote by $\alpha_1$. Similarly we construct another simple loop, denoted $\alpha_2$, by concatenating a walk in $H_2$ with the diagonal of $Q$ that joins $v_4$ and $v_2$. Now since $H_1 \cap H_2$ is empty, we can choose these loops so that they intersect transversely at exactly one point (where the diagonals meet). Thus these loops have geometric intersection number equal to one. In particular, we note that both $\alpha_1$ and $\alpha_2$ must be non separating loops in $\Sigma$. These loops will play an important role in the following sections.

5. Irreducible surface graphs

Let $G$ be a $(2, 2)$-tight $\Sigma$-graph. In light of Lemmas 4.1 and 4.3 we say that $G$ is irreducible if it has no digons, no triangles and if, for every quadrilateral face of $G$, both of the possible contractions result in graphs that are not $(2, 2)$-sparse.

For each of the contractions described in Section 4 there are the corresponding vertex splitting moves. More precisely, if $G' = G_D$, respectively $G' = G_T, e$, respectively $G' = G_{Q, u, v}$ for some digon $D$, respectively triangle $T$ and edge $e \in \partial T$, respectively quadrilateral $Q$ and vertices $u, v \in \partial Q$, then we say that $G$ is obtained from $G'$ by a digon, respectively triangle, respectively quadrilateral split. Thus every $(2, 2)$-tight $\Sigma$-graph can be constructed from some irreducible by applying a sequence of digon/triangle/quadrilateral splits. Our goal is to identify, for various surfaces, the set of irreducibles.

Conjecture 5.1. If $\Sigma$ is a surface with finite genus and finitely many boundary components and punctures, then there are finitely many distinct isomorphism classes of irreducible $(2, 2)$-tight $\Sigma$-graphs.

We will address some special cases of Conjecture 5.1 in this and later sections. Let $S$ be the 2-sphere.

Theorem 5.2. If $G$ is a $(2, 2)$-tight $S$-graph with at least two vertices then $G$ has at least two faces of degree at most 3. In particular, any $(2, 2)$-tight $S$-graph can be constructed from a single vertex by a sequence of digon and/or triangle splits.
Figure 2. The two non cellular irreducible torus graphs. Here and in subsequent diagrams we use the standard representation of the torus as a square with opposite edges identified appropriately. Note that by cutting the torus along a non separating loop these graphs can also be viewed as graphs in the twice punctured sphere.

Proof. By Lemma 3.2, $G$ is connected and therefore cellular. Since $G$ has at least two vertices, $f_0 = 0$. Also $f_1 = 0$ by sparsity, so by Theorem 3.4, we see that $2f_2 + f_3 \geq 4$. □

The case of plane graphs is similarly straightforward.

Corollary 5.3. If $G$ is a $(2,2)$-tight $\mathbb{R}^2$-graph with at least two vertices then $G$ has at least one cellular face of degree at most 3. In particular, any $(2,2)$-tight $\mathbb{R}^2$-graph can be constructed from a single vertex by a sequence of digon and/or triangle splits.

Proof. Cap (i.e fill in the puncture) the non cellular face of $G$ and then apply Theorem 5.2. □

Let $A$ be the twice punctured sphere $\mathbb{R}^2 - \{(0,0)\}$. There are two obvious examples of irreducible $(2,2)$-tight $A$-graphs, with one vertex and two vertices respectively: see Figure 2.

Theorem 5.4. If $G$ is an irreducible $(2,2)$-tight $A$-graph, then $G$ is isomorphic to one of the $A$-graphs shown in Figure 2.

Proof. There are two cases to consider. First suppose that $G$ does not separate the two punctures of $A$. Then there is a unique non cellular face of $G$. By capping this face (i.e. filling in the two punctures) we create a cellular $(2,2)$-tight $S$-graph $\tilde{G}$. As in the proof of Theorem 5.2, we see that either this graph has a single vertex or it has at least two faces that are digons or triangles. In the latter case, one of these faces must also be a face of $G$ and so, in this case, if $G$ has at least two vertices then it is not irreducible.

Now suppose that $G$ does separate the punctures of $A$. Clearly $G$ has exactly two non cellular faces. By capping these two faces, we create a $(2,2)$-tight $S$-graph $\tilde{G}$. This graph satisfies $2f_2 + f_3 = 4 + f_5 + 2f_6 + \cdots$ and since all but two of the faces of $\tilde{G}$ are also faces of the irreducible $G$, it follows that the two exceptional faces of $\tilde{G}$ are digons and all other faces are quadrilateral faces of $G$. Thus it suffices to show that there cannot be any quadrilateral faces in $G$.

For a contradiction, suppose that $Q$ is a quadrilateral. Since $G$ is irreducible, both possible contractions of $Q$ are blocked and we infer the existence of simple loops $\alpha_1$ and $\alpha_2$ as described at the end of Section 4. Recall that these loops intersect transversely at exactly one point and thus $\alpha_1$ is non separating in $A$. However the Jordan Curve Theorem tells us that that any simple loop in $A$ must be separating. □

We note that, for any positive integer $n$, it is straightforward to construct an $A$-graph that has no digons or triangles, but has $n$ quadrilateral faces. So, in contrast to the cases of the
sphere or plane, we do require the quadrilateral contraction move in order to have finitely many irreducible \((2,2)\)-tight \(\Sigma\)-graphs.

6. Subgraphs of irreducibles

Throughout this section, let \(\Sigma\) be an orientable surface and let \(G = (\Gamma, \varphi)\) be an irreducible \((2,2)\)-tight \(\Sigma\)-graph. Let \(H = (\Lambda, \varphi|_{\Lambda})\) be a subgraph of \(G\). We say that \(H\) is inessential if there is some embedded open disc \(U \subset \Sigma\) such that \(\varphi(\Lambda) \subset U\). If there is no such disc \(U\) then \(H\) essential.

We observe that if \(F\) is a cellular face of \(G\) that has a non degenerate boundary walk, then \(\partial F\) is inessential: let \(U\) be an open disc neighbourhood of the embedded closed disc \(\overline{F}\). We also note that if \(H\) is inessential and is connected then it has at most one non cellular face \(F\). Moreover if we cut and cap along a maximal non separating set of loops in \(F\) we obtain an \(S\)-graph which, in this section, we will denote by \(\hat{H}\).

Let \(K_1\) be the graph with one vertex and no edges. Let \(K_2\) be the complete graph on two vertices. For \(n \geq 2\) let \(\mathcal{C}_n\) be the \(n\)-cycle graph (in particular \(\mathcal{C}_2\) has exactly two parallel edges).

Lemma 6.1. Suppose that \(H\) is a subgraph of \(G\) whose underlying graph is isomorphic to either \(\mathcal{C}_2\) or \(\mathcal{C}_3\). Then \(H\) is essential.

Proof. Suppose that the underlying graph of \(H\) is isomorphic to \(\mathcal{C}_2\). The other case is similar. Suppose that \(H\) is inessential. Let \(U\) be an open disc that contains \(\varphi(\Lambda)\). Clearly there is a digon face \(D\) of \(H\) that is contained in \(U\). Now let \(K\) be the \(S\)-graph obtained by cutting and capping the external face of \(\text{int}_G(D)\). By Theorem 3.5 \(\gamma(K) = 2\) and by Theorem 3.4 \(K\) has at least two faces of degree at most 3. One of these faces is also a face of \(G\) contradicting the irreducibility of \(G\).

Lemma 6.2. Suppose that \(H\) is an inessential subgraph of \(G\) and that \(\gamma(H) = 2\). Then the underlying graph of \(H\) is \(K_1\).

Proof. Suppose that \(H\) has at least two vertices. Then by Theorem 3.4 \(\hat{H}\) has at least two faces of degree at most 3. If one of these is a triangle or a digon with non degenerate boundary then the underlying graph of \(H\) contains a copy of \(\mathcal{C}_2\) or \(\mathcal{C}_3\) which contradicts Lemma 6.1. Therefore \(\hat{H}\) must have two digon faces both of which have degenerate boundaries. However, as pointed out in Section 4.1 no \(S\)-graph can have more than one degenerate digon. □

Lemma 6.3. Suppose that \(H\) is an inessential subgraph of \(G\) and that \(\gamma(H) = 3\). Then the underlying graph of \(H\) is \(K_2\).

Proof. By Theorem 3.4 \(\hat{H}\) satisfies \(2f_2 + f_3 = 2 + f_5 + 2f_6 + \cdots\). As in the proof of Lemma 6.2 we see that \(\hat{H}\) cannot have a triangle or a digon with non degenerate boundary. So the only possibility is that \(\hat{H}\) has a digon face with degenerate boundary. As pointed out in Section 4.1 there is only one \(S\)-graph with a degenerate digon face and its underlying graph is indeed \(K_2\). □

The case of a subgraph isomorphic to \(\mathcal{C}_4\) is a little more involved.

Lemma 6.4. Suppose that \(H\) is an inessential subgraph of \(G\) whose underlying graph is isomorphic to \(\mathcal{C}_4\). Then \(H\) is the boundary of some quadrilateral face of \(G\).
**Proof.** Suppose that $U$ is an embedded disc containing $\varphi(|\Lambda|)$ and let $R$ be the face of $H$ that is contained in $U$. First observe that $\gamma(H) = 4$, so by Theorem 3.5 $\gamma(\text{int}_G(R)) \leq 4$. Now, by Lemma 6.1, $\text{int}_G(R)$ has no digons or triangles and it follows easily from Theorem 3.4 that $\gamma(\text{int}_G(R)) = 4$ and that all the cellular faces of $\text{int}_G(R)$ are quadrilaterals: that is to say that $\text{int}_G(R)$ is in fact a quadrangulation of $\overline{R}$.

Now, let $Q$ (with boundary vertices $v_1, v_2, v_3, v_4$) be a quadrilateral face of $\text{int}_G(R)$ that is contained in $R$. Since $G$ is irreducible, we have blockers $H_1$ and $H_2$ for the two possible contractions of $Q$, as described in Lemma 4.8. Also we have simple loops $\alpha_1$ and $\alpha_2$ as described in Section 4. These loops intersect transversely at one point in $Q$. If $w_1, w_2, w_3, w_4$ are the vertices of $\partial R$ in cyclic order, it follows that one of the loops, say $\alpha_1$, contains $w_1$ and $w_2$ and that $\alpha_2$ contains $w_2$ and $w_4$. Thus $\alpha_2$ divides $R$ into disjoint open subsets $R_1$ and $R_3$ (see Figure 3) where $w_1, v_1 \in \overline{R_1}$ and $w_3, v_3 \in \overline{R_3}$. Now we can decompose the blocker $H_1$ as $K_e \cup K_1 \cup K_3$, where $K_e = \text{ext}_G(R) \cap H_1$, $K_1$ is the part of $H_1$ contained in $\overline{R_1}$ and $K_3$ is the part of $H_1$ contained in $\overline{R_3}$. It is clear that $K_e \cap K_1 = \{w_1\}$ and $K_e \cap K_3 = \{w_3\}$. Therefore, by Lemma 11

$$3 = \gamma(H_1) = \gamma(K_e) + \gamma(K_1) + \gamma(K_3) - 4.$$  

Using the sparsity of $G$ it follows that at least one of $\gamma(K_1)$ or $\gamma(K_3)$ is equal to 2. Now $K_1$ and $K_3$ are both inessential subgraphs of $G$ since $\overline{R_1}$ and $\overline{R_3}$ are both embedded closed discs in $\Sigma$. It follows from Lemma 6.2 that at least one of $K_1$ or $K_3$ is a single vertex. So either $v_1 = v_2$ or $v_3 = v_4$. We have shown that at least one of $v_1$ or $v_3$ actually lies in the boundary of $R$. Similarly at least one of $v_2$ or $v_4$ lies in the boundary of $R$.

Thus we have shown that if $Q$ is any quadrilateral face of $G$ contained in $R$ then $\partial Q$ and $\partial R$ share at least one edge. Now it is an elementary exercise to show that in any quadrangulation of $R$ that has this property, either there are no quadrilaterals properly contained in $R$, or some quadrilateral has a boundary vertex with degree 2. Clearly, by Lemma 4.8 no quadrilateral face of the irreducible graph $G$ can have a boundary vertex of degree 2. It follows that there are no quadrilateral faces of $G$ that are properly contained in $R$ and so $R$ is itself a face of $G$.  

We say that a subgraph $H = (\Lambda, \varphi|\Lambda|)$ of $G$ is annular if it is essential and $\varphi(|\Lambda|)$ is contained in some embedded open annulus of $\Sigma$. Let $\mathfrak{B}$ be the graph $\{u, v, w\}, \{e, f, g, h\}, s, t$, where $s(e) = s(f) = s(g) = s(h) = u, t(e) = t(f) = v$ and $t(g) = t(h) = w$.

**Lemma 6.5.** Suppose that $H$ is a subgraph of $G$ whose underlying graph is isomorphic to $\mathfrak{B}$. Then $H$ is not annular.

**Proof.** Suppose, seeking a contradiction, that $H$ is annular. Let $U$ be an open annulus containing $\varphi(|\Lambda|)$ and let $R$ be the face of $H$ that is contained in $U$. Observe that $\gamma(H) = 2$, so by Theorem 3.5 $\gamma(\text{int}_G(R)) = 2$. Let $K$ be the $S$-graph obtained by cutting and capping the external faces of $\text{int}_G(R)$ (there could be more than one in this case). Now $K$ is a $(2,2)$-tight $S$-graph with two digon faces. Since all other faces of $K$ are also faces of the irreducible $G$, it follows easily from Theorem 3.4 that all other faces of $K$ are quadrilaterals. Thus, all faces of $G$ that are contained in $R$ are in fact quadrilaterals.

Now we can argue, using a straightforward modification of the argument from the proof of Lemma 6.4, that any quadrilateral face of $G$ that is contained in $R$ must in fact share a boundary edge with $R$. Again, following the proof of Lemma 6.4 it follows that $R$ itself must be a face of $G$. However this contradicts Lemma 4.8, where we showed that any quadrilateral face of an irreducible has a non degenerate boundary.  

$\square$
Figure 3. From the proof of Lemma 6.4 the shaded region represents the blocker for the contraction $G_{Q,v_1,v_3}$.

Now the main result of this section: a tight subgraph of an irreducible is also irreducible.

**Theorem 6.6.** Suppose that $G = (\Gamma, \varphi)$ is an irreducible $(2,2)$-tight $\Sigma$-graph and $\Lambda$ is a $(2,2)$-tight subgraph of $\Gamma$. Then $H = (\Lambda, \varphi|_{\Lambda})$ is an irreducible $\Sigma$-graph.

**Proof.** We see that $H$ cannot have any triangle or digon, since the boundary of such a face would contradict Lemma 6.1. Now suppose that $Q$ is a quadrilateral face of $H$. It is not clear, a priori, that the boundary of $Q$ is non degenerate, so we must prove that before proceeding.

Applying Lemma 4.6 to $H$, we see that there are no repeated edges in the boundary of $Q$. If $\partial Q$ is inessential then, since $B$ contains a copy of $C_2$, this contradicts Lemma 6.1. On the other hand, if $\partial Q$ is inessential then it must be annular and this contradicts Lemma 6.5. Thus we see that in fact $Q$ must have a non degenerate boundary.

By Lemma 6.4 this means that $Q$ is also a face of $G$ and so there are blockers $H_1, H_2$ as described by Lemma 4.8. Now consider the $\Sigma$-graph $K = H_1 \cup H_2 \cup \partial Q$. This is $(2,2)$-tight, so, by Lemma 3.3, $K \cap H$ is also $(2,2)$-tight. Now, $K \cap H = (H_1 \cap H) \cup (H_2 \cap H) \cup \partial Q$. Using Lemma 1, $H_1 \cap H_2 = \emptyset$, $H_1 \cap H \cap \partial Q = \{v_1, v_3\}$ and $H_2 \cap H \cap \partial Q = \{v_2, v_4\}$, we have

\[
2 = \gamma(K \cap H) \\
= \gamma(\partial Q) + \gamma(H_1 \cap H) + \gamma(H_2 \cap H) - \gamma(H_1 \cap H \cap \partial Q) - \gamma(H_2 \cap H \cap \partial Q) \\
= 4 + \gamma(H_1 \cap H) + \gamma(H_2 \cap H) - 4 - 4.
\]
Thus $\gamma(H_1 \cap H) + \gamma(H_2 \cap H) = 6$. If $\gamma(H_1 \cap H) = 2$ then $(H_1 \cap H) \cup \{v_2\} \cup \{e_1, e_2\}$ would be a type 1 blocker for the contraction $G_{Q,v_1,v_3}$, contradicting Lemma 4.8. So $\gamma(H_1 \cap H) \geq 3$ and similarly $\gamma(H_2 \cap H) \geq 3$. It follows that $\gamma(H_1 \cap H) = \gamma(H_2 \cap H) = 3$ and that $H_1 \cap H$ and $H_2 \cap H$ are blockers for the contractions $H_{Q,v_1,v_3}$ and $H_{Q,v_1,v_3}$ respectively. Thus both possible contractions of $Q$ are blocked in $H$ as required.

For example, suppose that $\Gamma$ is the simple $(2,2)$-tight graph obtained by adding a vertex of degree two to $K_4$. Is it possible to embed $\Gamma$ into the torus to create an irreducible torus graph? There are several possible embeddings to consider, however we can significantly narrow the search space by observing that since $K_4$ is tight, by Theorem 6.6 any irreducible embedding of $\Gamma$ must extend an irreducible embedding of $K_4$. Now, it is not hard to see that there is only one irreducible embedding of $K_4$ in the torus, up to isomorphism. Moreover, one readily checks that there is no way to add the remaining vertex of $\Gamma$ to this embedding without creating a face of degree at most 4. Thus there is no irreducible embedding of $\Gamma$ in the torus.

7. IRREDUCIBLE TORUS GRAPHS

Let $\mathbb{T} = S^1 \times S^1$ be the torus. Throughout this section let $G = (\Gamma, \varphi)$ be an irreducible $(2,2)$-tight $\mathbb{T}$-graph. Our goal in this section is to show that there are only finitely many isomorphism classes of such graphs by establishing an upper bound for the number of vertices of $G$.

In the case that $G$ is not cellular we will see that we can essentially reduce the problem to the sphere or the annulus. If $G$ is cellular then using Theorem 3.4 and $f_2 = f_3 = 0$ we see that $G$ satisfies $f_5 + 2f_6 + 3f_7 + 4f_8 = 4$ and $f_i = 0$ for $i \geq 9$. Since $|V| = 2 + \sum_{i \geq 2} f_i$, the problem reduces to establishing a bound for the number of quadrilateral faces that an irreducible $\mathbb{T}$-graph can have.

First we deal with the non cellular case.

Lemma 7.1. Suppose that $G$ is not cellular. Then $\Gamma$ is either isomorphic to $K_1$ or to $C_2$. Furthermore, in the latter case, $G$ is annular.

Proof. Since $\Gamma$ is connected it is clear $G$ has a single non cellular face. By cutting along a non separating loop in this face we obtain an $A$-graph $\tilde{G}$. Observe that any face of $G$ that is not also a face of $G$ is non cellular. It follows that $\tilde{G}$ is an irreducible $A$-graph. Now the conclusion follows from Theorem 5.4.

For the remainder of the section, assume that $G$ is cellular. Let $Q$ be a quadrilateral face of $G$ with boundary walk $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. As described in Section 4 we have blockers $H_1$ and $H_2$ and simple loops $\alpha_1$ and $\alpha_2$ that intersect transversely at one point.

Lemma 7.2. At least one of $H_1$ or $H_2$ is an inessential subgraph of $G$.

Proof. Suppose that both are essential. Then there are non separating simple loops $\beta_1$ contained in $H_1$ and $\beta_2$ contained in $H_2$. Now $H_1 \cap H_2 = \emptyset$, so $i(\beta_1, \beta_2) = 0$. However, it is also clear that $i(\alpha_1, \beta_2) = i(\alpha_2, \beta_1) = 0$. As pointed out in Section 2.4 this implies that $i(\alpha_1, \alpha_2) = 0$, contradicting the fact that these curves intersect transversely at one point.

For the remainder of the section, suppose that $H_1$ is an inessential blocker. By Lemma 6.3 the graph of $H_1$ is $K_2$. Furthermore we will assume that $H_2$ is a maximal blocker with respect to inclusion and let $J$ be the face of $H_2$ that contains $v_1, v_3$. See Figure 4 for an illustration of these assumptions in the case where $H_2$ is an essential blocker.
Figure 4. A quadrilateral face with an essential blocker. The shaded region represents the essential blocker $H_2$.

Lemma 7.3. Any face of $H_2$ that is not $J$ is also a face of $G$.

Proof. Suppose that $F \neq J$ is a face of $H_2$. Then $\gamma(H_2 \cup \text{int}_G(F)) \leq \gamma(H_2) = 3$, by Theorem 3.5. Also $v_1, v_3 \notin \text{int}_G(F)$, since $F \neq J$. If follows that $H_2 \cup \text{int}_G(F)$ is a blocker for $G_{Q,v_2,v_4}$ and so by the maximality of $H_2$, $\text{int}_G(F) \subset H_2$ as required. □

Next we want to examine the structure of $H_2$. It turns out that there are exactly ten distinct possibilities. If $H_2$ is inessential then, by Lemma 6.3, it has graph $K_2$ (Figure 5). On the other hand, if $H_2$ is essential we have the following.

Lemma 7.4. Suppose that $H_2$ is essential. Then it is isomorphic to one of the nine torus graphs shown in Figures 6 and 7.

Proof. Since $\gamma(H_2) = 3$, it is connected by Lemma 3.2. Let $K$ be the $S$-graph obtained by cutting and capping $H_2$ along a non-separating loop in $J$. Clearly $K$ has two exceptional faces $J^+$ and $J^-$ such that all other faces of $K$ are faces of $G$ (using Lemma 7.3). Now, since $J^+$ and $J^-$ are the only faces of $K$ that could have degree less than 4, Theorem 3.4 implies that $K$ satisfies

\[
2f_2 + f_3 = 2 + f_5 + 2f_6
\]

and $f_i = 0$ for $i \geq 7$. There are two cases to consider.

(a) There is no quadrilateral face of $G$ in $H_2$. There are various subcases:

1. $|J^+| = |J^-| = 2$. Then, from Equation 3 we get $f_5 + 2f_6 = 2$. So either $f_5 = 0$ and $f_6 = 1$ and we have the example shown in Figure 3(a), or, $f_5 = 2$ and $f_6 = 0$ and we have one of the examples shown in Figure 3(b) or (c).

2. $|J^+| = 2$ and $|J^-| = 3$. Then we have $f_5 = 1$. There is one possibility: Figure 3(d).

3. $|J^+| = |J^-| = 3$. In this case, Equation 3 implies that $J^+$ and $J^-$ are the only faces of $K$. So we have the example shown in Figure 3(e).

4. $|J^+| = 2$ and $|J^-| = 4$. In this case, Equation 3 implies that $J^+$ and $J^-$ are the only faces of $K$ and we have the example shown in Figure 3(f).
(b) There is some quadrilateral face of $G$ in $H_2$. This case requires a little more effort as we must first establish that there is no more than one such face. Let $G' = ∂Q ∪ H_1 ∪ H_2$. Clearly $G'$ is $(2,2)$-tight and so by Theorem 6.6, it is also irreducible.

Suppose that $R$ is a quadrilateral face of $G$, with boundary vertices $w_1, w_2, w_3, w_4$, that is contained in $H_2$ (and so is also a face of $G'$). By Lemma 7.2 we know that there is a blocker for one of the contractions of $R$ in $G'$ whose graph is $K_2$. Without loss of generality assume that a blocker $L_1$ for the contraction $G'_{R,w_1,w_3}$ has graph $K_2$. Now we claim that $L_1 ∩ H_2$. If not then it is clear that $L_1$ must intersect $H_1$. Since the vertices of $L_1$ are both in $H_2$ this contradicts $H_1 ∩ H_2 = ∅$, thus establishing our claim.

Now consider a maximal blocker, $L_2$, for the contraction $G'_{R,w_2,w_4}$. We have

$$3 = γ(L_2) = γ(L_2 ∩ (∂Q ∪ H_1)) + γ(L_2 ∩ H_2) - γ(L_2 ∩ (∂Q ∪ H_1) ∩ H_2) = γ(L_2 ∩ (∂Q ∪ H_1)) + γ(L_2 ∩ H_2) - γ(L_2 ∩ \{v_2, v_4\})$$

Now it is clear that $\{v_2, v_4\} ⊂ L_2$ since $L_2$ is connected, so we have

$$γ(L_2 ∩ H_2) = 7 - γ(L_2 ∩ (∂Q ∪ H_1))$$

Furthermore, it is also clear that $L_1$ separates $v_2$ from $v_4$ in $H_2$, so $L_2 ∩ H_2$ has at least two components. Also $L_2 ∩ (∂Q ∪ H_1)$ is a subgraph of $∂Q ∪ H_1$ that contains the vertices $v_2, v_4$. It follows easily that $γ(L_2 ∩ (∂Q ∪ H_1)) ≥ 3$ with equality only if $L_2 ∩ (∂Q ∪ H_1) = ∂Q ∪ H_1$. Therefore the only way that 4 can be satisfied is that $∂Q ∪ H_1 ⊂ L_2$ and $L_2 ∩ H_2$ has exactly two components $X_2 \ni v_2$ and $X_4 \ni v_4$ such that $γ(X_2) = γ(X_4) = 2$. In particular it follows from Theorem 6.6 and Lemma 7.1 that the underlying graph of $X_2$, and also of $X_4$, is isomorphic to $K_1$ or $C_2$. Now since $L_1$ also separates $w_2$ and $w_4$ in $H_1$ we can, without loss of generality, assume that $v_2, w_2 \in X_2$ and $v_4, w_4 \in X_4$.

Let $Z_2$, respectively $Z_4$, be the maximal $(2,2)$-tight subgraph of $H_2$ that contains $v_2$, respectively $v_4$. By Lemma 3.3 we see that $X_2 ⊂ Z_2$ and $X_4 ⊂ Z_4$. Furthermore we see that since $Z_2$ and $Z_4$ are both disjoint from $α_1$, they are either annular or inessential. By Lemma 7.1, $Z_2$ has graph $K_1$ (inessential case) or $C_2$ (annular case). Similar comments apply to $Z_4$. Now the argument in the paragraph above shows that every quadrilateral face of $H_2$ has a boundary vertex in $Z_2$ and a diagonally opposite vertex in $Z_4$. It follows easily that there is at most one such quadrilateral face in $H_2$.

Now we can argue as in case (a) but with the proviso that there is exactly one quadrilateral face, $R$, of $H_2$ that is also a face of $G$. We observe that there is a cycle of length 3 in $H_2$ (formed by two edges of $∂R$ and the inessential blocker for $R$) and so also in $K$. It is not hard to see that it follows that $K$ must have at least two faces of odd degree: at least one on either ‘side’ of the cycle of length 3. We find the following subcases.

1. $|J^+| = |J^-| = 2$. From Equation (3) we have $f_5 + 2f_6 = 2$. Since $K$ has some face of odd degree we can rule out the possibility $f_5 = 0, f_6 = 1$. Therefore $f_5 = 2$ and $f_6 = 0$. There is only one possibility for $H_2$: Figure 7(a).

2. $|J^+| = 2$ and $|J^-| = 3$. Then, as in case (a) we have $f_5 = 1$ and there is one possibility: Figure 7(b).

3. $|J^+| = |J^-| = 3$. In this case, Equation 3 implies that $R, J^+$ and $J^-$ are the only faces of $K$: Figure 7(c).
\begin{figure}
\centering
\includegraphics[width=0.2\textwidth]{unique_inessential_blocker}
\caption{The unique inessential blocker}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{essential_blockers_no_quad_face}
\caption{Essential blockers with no quadrilateral face}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.7\textwidth]{essential_blockers_quad_face}
\caption{Essential blockers with a quadrilateral face}
\end{figure}

(4) \(|J^+| = 4\) and \(|J^-| = 2\). Since \(K\) must have at least two faces of odd degree, Theorem 3.4 would imply that there is a triangle or digon in \(K\) that is also a face of \(G\), contradicting its irreducibility. Thus this subcase cannot arise.

It remains to rule out the possibility of a quadrilateral face that is neither \(Q\) nor a face of \(H_2\). In fact we can prove something a little more general than that.

\textbf{Lemma 7.5.} \(K\) be a \((2, 2)\)-tight subgraph of \(G\) and suppose that \(F\) is a cellular face of \(K\). There is no quadrilateral face of \(G\) properly contained within \(F\).
Proof. Suppose that \( R \), with vertices \( w_1, w_2, w_3, w_4 \), is a quadrilateral face of \( G \) properly contained within \( F \) and let \( B_1 \) and \( B_2 \) be blockers for contractions of \( R \) in \( G \). If \( B_1 \subset F \), then since \( F \) is cellular, \( B_1 \) would separate \( w_3 \) from \( w_4 \) which contradicts \( B_1 \cap B_2 = \emptyset \). Therefore \( B_1 \) is not contained in \( F \) or equivalently, since \( B_1 \) is connected, \( B_1 \cap K \neq \emptyset \). Similarly \( B_2 \cap K \neq \emptyset \).

Now, let \( M = \partial R \cup B_1 \cup B_2 \) and observe that \( M \) is \((2,2)\)-tight and therefore, by Lemma 3.3 \( M \cap K \) is also \((2,2)\)-tight. Now it is clear that \( M \cap K = (B_1 \cap K) \cup (B_2 \cap K) \cup \{\text{a type 1 blocker for } 2 = \partial R \cap K\} \). Therefore

\[
2 = \gamma(M \cap K) = \gamma(B_1 \cap K) + \gamma(B_2 \cap K) - |E(\partial R \cap K)| \tag{5}
\]

Now, we observe that \( |E(\partial R \cap K)| \in \{0,1,2,4\} \) since \( K \) is an induced subgraph of \( G \). If \( |E(\partial R \cap K)| = 4 \) then clearly \( R \) must be a face of \( K \) which contradicts our assumption that \( R \) is properly contained within \( F \). On the other hand if \( |E(\partial R \cap K)| \leq 1 \), then (5) yields \( \gamma(B_1 \cap K) + \gamma(B_2 \cap K) \leq 3 \) which contradicts the fact that both \( B_1 \cap K \) and \( B_2 \cap K \) are nonempty. Finally if \( |E(\partial R \cap K)| = 2 \) then it is clear that \( K \) contains exactly three of the vertices \( w_1, w_2, w_3, w_4 \). However in this case (5) implies that \( \gamma(B_1 \cap K) = \gamma(B_2 \cap K) = 2 \). It follows that \( K \) contains at most one of the vertices \( w_1, w_3 \), otherwise \( (B_1 \cap K) \cup \{w_2\} \) would span a type 1 blocker for \( G_{R,w_1,w_3} \), contradicting Lemma 4.8. Similarly \( K \) contains at most one of the vertices \( w_2, w_4 \). Thus \( K \) contains at most two of the vertices \( w_1, w_2, w_3, w_4 \) yielding the required contradiction.

Finally we have our main theorem about torus graphs.

**Theorem 7.6.** Suppose that \( G \) is an irreducible \((2,2)\)-tight \( \mathcal{T} \)-graph. Then \( G \) has at most two quadrilateral faces.

**Proof.** Suppose, as above, that \( Q \) is a quadrilateral face of \( G \), with maximal blockers \( H_1 \) and \( H_2 \). Also assume that \( H_1 \) is inessential. We have seen that there is at most one other quadrilateral face of \( G \) contained among faces of \( H_2 \). Now let \( K = \partial Q \cup H_1 \cup H_2 \). Clearly \( K \) is a \((2,2)\)-tight subgraph of \( G \). Now we consider the faces of \( K \) that are not also faces of \( G \). If \( H_2 \) is also inessential then there is at most one such face and this face is cellular of degree 8. If \( H_2 \) is essential then there at most two such faces and each such face is cellular and has degree at least 5. So by Lemma 7.5 there is no quadrilateral face of \( G \) that is not also a face of \( K \).

**Corollary 7.7.** There are finitely many distinct isomorphism classes of irreducible \((2,2)\)-tight torus graphs. In particular any such irreducible torus graph has at most eight vertices.

**Proof.** We may as well assume that \( G \) is cellular, since in the non cellular case we know that \( G \) has at most two vertices. Since \( \gamma(G) = 2 \) we have \( |V| = 1 + \frac{1}{2} \sum if_i \), so we must maximise \( \sum if_i \). Since \( G \) is irreducible, \( f_i = 0 \) for \( i = 0, 1, 2, 3 \) and \( f_4 \leq 2 \). From Theorem 3.4 we have \( f_5 + 2f_6 + 3f_7 + 4f_8 = 4 \) and \( f_i = 0 \) for \( i \geq 9 \). Clearly the maximum value for \( \sum if_i \) is attained by having \( f_4 = 2, f_5 = 4, f_6 = 0 \) for \( i \neq 4, 5 \); see Figure 20 for examples for which these bounds are achieved. In that case \( |V| = 8 \). Now there are finitely many isomorphism classes of \((2,2)\)-tight graphs with at most eight vertices. Moreover, for each such graph, there are finitely many isomorphism classes of torus graphs with that underlying graph.

8. Searching for irreducibles

Given Corollary 7.7 a naive algorithm to find all the irreducibles mentioned therein would be
(1) Find all $(2,2)$-tight graphs with at most 8 vertices.
(2) For each such graph, find all isomorphism classes of torus embeddings.
(3) Eliminate all embeddings that are not irreducible.

It is impractical to carry out this procedure without the assistance of a computer as step (1) will already yield many thousands of distinct graphs, each of which could have many different torus embeddings.

However, since we have a lot of structural information about irreducibles, we can narrow the search space significantly. For example, it is clear from the proof of Corollary 7.7 that any irreducible with 8 vertices must have 2 quadrilateral faces, 4 faces of degree 5 and no other faces. Moreover, we know that each quadrilateral face has one essential blocker and one other blocker which must be one of the 10 graphs described in Section 7. It is not too difficult to deduce that any 8 vertex irreducible must be isomorphic to one of the examples shown in Figure 20.

Similarly for torus graphs with at most 4 vertices there are relatively few possibilities for the underlying graph: 13 in total. Now, using Lemmas 6.1, 6.4 and 6.5 we can easily deduce that an irreducible with at most 4 vertices is isomorphic to one of the examples shown in Figures 15 or 16. For the cases of 5, 6 and 7 vertices the naive approach becomes excessively laborious. We have used the computer algebra system SageMath [20] to automate much of the search process in these cases. We briefly outline the relevant data structures and algorithms here.

8.1. Data structures. In order to carry out our computer assisted search we needed to implement two key data structures, one to model graphs (the native SageMath Graph class is not particularly well adapted to our purposes), and one to model surface graphs.

8.1.1. Graphs. A dart (or half-edge) of a graph $\Gamma = (V, E, s, t)$ is a pair $(e, r)$ where $e \in E$ and $r \in \{s, t\}$. Let $D$ be the set of darts of $\Gamma$ and observe that there is a partition $\mathcal{V}$ of $D$ defined by $P_v = \{(e, r) \in D : r(e) = v\}$. There is another partition $\mathcal{E}$ of $D$ defined by $Q_e = \{(e, s), (e, t)\}$. Using this construction one readily sees that there is a correspondence between graphs and triples $(X, \mathcal{P}, \mathcal{Q})$ where $X$ is a set, $\mathcal{P}$ is a partition of $X$ and $\mathcal{Q}$ is a partition of $X$ each of whose parts has two elements. We use this observation to implement a class in SageMath that accurately models our notion of graph. We have subclassed the native SageMath Graph class in order to take advantage of the built-in graph theoretic functionality in SageMath. We have also implemented methods modelling various standard graph theoretic operations including vertex splitting and edge contractions. We adapted the built-in SageMath graph isomorphism checker to work with our subclass.

We also need to check $(2,2)$-sparsity for our graphs and for this purpose we created a very basic implementation of the pebble game algorithm of Lee and Streinu [14].

8.1.2. Surface graphs. Let $S_k$ be the group of permutations of the set $\{1, \ldots, k\}$. An oriented rotation system is a pair $(\sigma, \tau)$ where $\sigma$ is some element of $S_{2n}$ and $\tau$ is a fixed point free involution in $S_{2n}$. By a theorem of Edmonds [6] there is a correspondence between isomorphism classes of oriented rotation systems and isomorphism classes of cellular surface graphs whose underlying surface is orientable. For a contemporary exposition of this theory see [15].

Oriented rotation systems provide a convenient data structure for carrying out computations with surface graphs. In particular it is straightforward to compute boundary walks of faces, the genus and components of the underlying surface etc. Furthermore it is straightforward to implement the topological edge contraction and deletion operations discussed in Section 4.
as well as other standard operations such as adding a new vertex within a specified face and adding a new edge that subdivides a face in a specified way. We have implemented this data structure in SageMath along with methods corresponding to the invariants and operations mentioned above.

8.2. The **search algorithm**. In order to search for irreducibles, we make use of Theorem 6.6 in a relatively straightforward way. Observe that if a \((2,2)\)-tight graph has a vertex of degree 2 then deleting this vertex yields a smaller \((2,2)\)-tight graph. If we know all possible irreducible torus embeddings of this smaller graph then we need only work out all possible ways to add back in the deleted vertex `topologically`. That is to say we must add the vertex within a face together with edges to the required neighbours that must lie in the boundary of the face. This is substantially more efficient than searching among all possible embeddings of the original graph. On the other hand if the graph has minimum degree 3 then we carry out a brute force search among all possible rotation systems whose underlying graph is the given one. A slightly more formal description of this idea is given in Algorithm 1.

Algorithm 1 An inductive algorithm for finding all irreducibles with \(n\) vertices

1: **Input:** lists \(\mathcal{G}_{n-1}\), respectively \(\mathcal{G}_n\), of all \((2,2)\)-tight graphs with \(n - 1\), respectively \(n\) vertices, a list \(\mathcal{I}_{n-1}\) of all the irreducible torus graphs with \(n - 1\) vertices and a mapping \(f_{n-1} : \mathcal{I}_{n-1} \rightarrow \mathcal{G}_{n-1}\) that maps each irreducible to its underlying graph.
2: **Output:** A list \(\mathcal{I}_n\) of all the irreducible torus graphs with \(n\) vertices together with a mapping \(f_n : \mathcal{I}_n \rightarrow \mathcal{G}_n\).
3: for \(\Gamma \in \mathcal{G}_n\) do
4: if \(\Gamma\) has a vertex of degree 2 then
5: Let \(\Theta\) be the graph obtained by deleting the vertex of degree 2
6: for \(G \in f_{n-1}^{-1}(\Theta)\) do
7: Identify any face whose boundary contains both neighbours of the deleted vertex. See if a new vertex can be added within the face and adjacent to the two neighbours without creating any face of degree 2, 3 or 4. Add all resulting rotation systems to the list \(\mathcal{I}_n\), check to see if any new entry is isomorphic to any existing one and remove the new one if it is. Update the mapping \(f_n\) mapping all the new entries in \(\mathcal{I}_n\) to the appropriate Henneberg extension of \(\Theta\) in \(\mathcal{G}_n\).
8: end for
9: else
10: Label the darts of \(\Gamma, 1, \cdots, 2n\) and identify the partitions \(V\), respectively \(E\) corresponding to the vertices, respectively edges. Let \(\tau\) be the involution whose cycle partition is \(E\).
11: for each \(\sigma \in S_{2n}\) whose cycle partition is \(V\) do
12: Check that the rotation system \((\sigma, \tau)\) corresponds to an irreducible torus graph. Check to see if it is isomorphic to any existing entry in \(\mathcal{I}_n\). If not then add to \(\mathcal{I}_n\) and update \(f_n\) appropriately.
13: end for
14: end if
15: end for
8.3. Computational results. Using a SageMath implementation of Algorithm 1 we have found the complete list of irreducible torus graphs: there are 116 in total. See [5] for the SageMath code together with data files describing the rotation systems corresponding to each of cellular irreducible torus graphs and corresponding diagrams.

9. Application: contacts of circular arcs

In this section we describe an application to the study of contact graphs. The foundational result in this area is the well known Koebe-Andreev-Thurston Circle Packing Theorem ([12]) which says that every plane simple graph can be realised as the contact graph of some arrangement of circles with non overlapping interiors in the Euclidean plane. Following this theorem, contact graphs arising in many other geometric contexts have been investigated. We consider contact graphs arising from certain families of curves in surfaces of constant curvature. We begin by giving a model for a general class of contact problems and then specialise to a case of particular interest.

Let \( \alpha : [0, 1] \to \Sigma \) be a curve. We say that \( \alpha \) is non selfoverlapping if it is injective on the open interval \((0, 1)\). Now suppose that \( \alpha, \beta : [0, 1] \to \Sigma \) are distinct curves in \( \Sigma \). We say that \( \alpha \) and \( \beta \) are non overlapping if \( \alpha((0, 1)) \cap \beta((0, 1)) = \emptyset \). Let \( C \) be a collection of curves in \( \Sigma \) having the following properties

- Every \( \alpha \in C \) is non selfoverlapping.
- For every distinct \( \alpha, \beta \in C \), \( \alpha \) and \( \beta \) are non overlapping.

We want to construct a combinatorial object that describes the contact properties of such a collection. In order to do this we impose some further non degeneracy conditions on \( C \) as follows.

- \( \alpha(0) \neq \alpha(1) \) for every \( \alpha \in C \)
- For every distinct \( \alpha, \beta \in C \), \( \{\alpha(0), \alpha(1)\} \cap \{\beta(0), \beta(1)\} \) is empty.

In other words, we allow the end of one curve to touch another curve (or to touch itself), but the point that it touches cannot be an endpoint of that curve. We say that \( C \) is a non degenerate collection of non overlapping curves. Note that if a collection fails the non degeneracy conditions, it can typically be made degenerate by an arbitrarily small perturbation. A contact of \( C \) is a quadruple \((\alpha, \beta, x, y)\) where \( \alpha, \beta \in C \), \( x \in \{0, 1\} \) and \( \alpha(x) = \beta(y) \).

Now we can define a graph \( \Gamma_C \) as follows. The vertex set is \( C \) and the edge set is \( T \), the set of contacts of \( C \). We define the required incidence functions by \( s(\alpha, \beta, x, y) = \alpha \) and \( t(\alpha, \beta, x, y) = \beta \). Finally we can construct an embedding \(|\Gamma_C| \to \Sigma\). For \( \beta \in C \), suppose that \( t^{-1}(\beta) = \{(\alpha_1, \beta, x_1, y_1), \cdots, (\alpha_k, \beta, x_k, y_k)\} \). Let \( J_\beta \) be a nonempty closed subinterval of \([0, 1]\) with the following properties.

1. \( \{y_1, \cdots, y_k\} \subset J_\alpha \)
2. \( 0 \in J_\beta \) if and only there is no contact \((\beta, \gamma, 0, y)\) in \( T \).
3. \( 1 \in J_\beta \) if and only there is no contact \((\beta, \gamma, 1, y)\) in \( T \).

In other words \( J_\beta \) is a subinterval that covers all the ‘points of contact’ in \( \beta \) together with any endpoints of \( \beta \) that do not touch a curve. Now we observe that \( \beta : J_\beta \to \Sigma \) is a homeomorphism onto its image. So it follows from the Jordan-Schoenflies Theorem that \( \Sigma/\beta(J_\beta) \) is homeomorphic to \( \Sigma \). Furthermore, since \( \beta(J_\beta) \cap \delta(J_\delta) = \emptyset \) for \( \beta \neq \delta \) it follows that \( \Sigma \) is homeomorphic to \( \Sigma/\sim \) where \( \sim \) is the equivalence relation that collapses each \( \beta(J_\beta) \) to a point, for all \( \beta \in C \). Using this homeomorphism we construct an embedding by mapping each vertex of \( \Gamma_C \) (i.e. element of \( C \)) to the corresponding point of \( \Sigma/\sim \). Since an edge of
Figure 8. The construction of the contact graph associated to a collection of curves. On the left we have a collection of curves. The bold section of $\alpha$ represents $\alpha(J_\alpha)$. On the right is the corresponding graph with edge orientations as indicated.

$\Gamma_C$ is a contact $(\alpha, \beta, x, y)$, we can construct the corresponding edge embedding by using the restriction of $\alpha$ to the component of $[0, 1] - J_\alpha$ that contains $x$: see Figure 8.

We are interested in the recognition problem for contact graphs: can we find necessary and/or sufficient conditions for a surface graph to be the contact graph of a collection of curves? Typically we are looking for conditions for which there are efficient algorithmic checks. As noted in the introduction, there are efficient algorithms for checking whether or not a given graph is $(2, l)$-sparse. See [14] and [10] for details.

Hliněný ([11]) has shown that a plane graph admits a representation by contacts of curves if and only if it is $(2, 0)$-sparse. This result easily generalises to other surfaces. We include the statement to provide some context for our later result.

Lemma 9.1. Let $G$ be a $\Sigma$-graph. Then $G \cong G_C$ for some $C$ as above if and only if $G$ is $(2, 0)$-sparse. $\square$

It is worth noting here that the definition of the contact graph used in [11] and elsewhere is different to the one we have given above. In the literature the contact graph is typically defined as the intersection graph of the collection of curves. This definition works well in the plane. However for non simply connected surfaces we propose that it is more natural to define the contact graph as above.

Now we suppose that $\Sigma$ is also equipped with a metric of constant curvature. In this context we can distinguish many interesting subclasses of non selfoverlapping curves. For example, a circular arc is a curve of constant curvature and a line segment is a locally geodesic curve. For collections of such curves the representability question can depend on the embedding of the graph and not just the graph itself (in contrast to Lemma 9.1). For example, if $\Pi$ is the graph consisting of two vertices joined by two parallel edges, then $\Pi$ cannot be represented by a collection of line segments in the flat plane. However, if $\Pi$ is embedded as a non separating cycle in the torus, then it is easy to construct a representation of the resulting surface graph as a collection of line segments in the flat torus.

Given a $\Sigma$-graph $G$ and a non degenerate non overlapping collection of circular arcs $C$ such that $G \cong G_C$ we say that $C$ is a CCA representation of $C$ (abbreviating Contacts of Circular Arcs). See Figure 9 for an example in the torus. Alam et al. ([11]) have shown that any $(2, 2)$-sparse plane graph has a CCA representation in the flat plane. We prove an analogous result for the flat torus.
First we need a lemma to show that every sparse surface graph can be obtained by deleting only edges from a tight surface graph.

**Lemma 9.2.** Suppose that $\Sigma$ is a connected surface, $l \leq 2$ and $G$ is a $(2, l)$-sparse $\Sigma$-graph. There is some $(2, 2)$-tight $\Sigma$-graph $H$ such that $V(H) = V(G)$ and $G$ is a subgraph of $H$.

**Proof.** Clearly it suffices to show that if $\gamma(G) \geq l + 1$ then we can add an edge $e$ within some face of $G$ so that $G \cup \{e\}$ is $(2, l)$-sparse.

Now if $G$ has no tight subgraph then we can add any edge without violating the sparsity count. So we assume that $G$ has some nonempty tight subgraph. Let $L$ be a maximal tight subgraph of $G$. If $L$ spans all vertices of $G$ then $L = G$ and $G$ is already tight, so we assume that $L$ is not spanning in $G$. Since $\Sigma$ is connected there is some face $F$ of $G$ whose boundary contains vertices $u \in L$ and $v \notin L$. Let $e$ be a new edge that joins $u$ and $v$ through a path in $F$. We claim that $G \cup \{e\}$ is $(2, l)$-sparse. If not then there must be some tight subgraph $K$ of $G$ such that $u, v \in K$. But $K \cap L$ is nonempty, so by Lemma 3.3, $K \cup L$ is $(2, l)$-tight. This contradicts the maximality of $L$. $\square$

**Theorem 9.3.** Every $(2, 2)$-sparse torus graph admits a CCA representation in the flat torus.

**Proof.** First observe that edge deletion is CCA representable: just shorten one of the arcs slightly. So by Lemma 9.2 it suffices to prove the theorem for $(2, 2)$-tight torus graphs. To that end we must show that

(a) each irreducible $(2, 2)$-tight torus graph has a CCA representation.

(b) if $G \to G'$ is a digon, triangle or quadrilateral contraction move and $G'$ has a CCA representation, then $G$ also has a CCA representation. In other words the relevant vertex splitting moves are CCA representable.

For (a) it is possible to give an explicit CCA representation for each of the 116 irreducibles listed in Appendix A. We will not describe those here but below we shall explain a simple method to make these constructions easily. Full details are given in [21].

For (b), see Figure 10 for an illustration of the CCA representation of the quadrilateral split. It is easily seen that any quadrilateral split is similarly CCA representable. The digon and triangle vertex splits are also representable and indeed have already been dealt with in the plane context in [1]. We observe that the constructions described there work equally well for torus graphs. $\square$
Figure 10. A CCA representation of a quadrilateral splitting move. The contact graph of the configuration on the left is a quadrilateral contraction of the contact graph of the configuration on the right.

In order to construct the CCA representations of the irreducibles mentioned in the proof we can make use of topological Henneberg moves. We remind the reader that a Henneberg vertex addition move is the operation of adding a new vertex to a graph and two edges from that vertex to the existing graph. Note that we allow the two new edges to be parallel. Moreover in the context of surface graphs we insist that the new vertex is placed in some face of the existing graph and the two edges are incident with vertices in the boundary of that face. We refer to such an operation as a topological Henneberg move. Clearly a Henneberg move is the inverse operation to divalent vertex deletion. It is well known (and elementary) that divalent vertex deletions preserve \((2, l)-\)sparseness for all \(l\). On the other hand Henneberg moves preserve \((2, l)-\)sparseness for \(l \leq 2\), and for \(l = 3\) if we also insist that the new edges are not parallel.

It turns out that there are just 12 irreducibles that have no vertices of degree 2. In Figure 12 we given diagrams of each of these torus graphs and in Figure 13 we give sample CCA representations of each of these in the flat torus. We observe that each of the 116 irreducible graphs can be constructed by a sequence topological Henneberg moves from one of the torus graphs in Figure 12; indeed one easily sees that at most five Henneberg moves are required.

It remains to show that the required topological Henneberg moves are CCA representable. A CCA representation of a topological Henneberg move is illustrated in Figure 11. In general of course, topological Henneberg moves can fail to be CCA representable given a fixed representation of the initial graph (Figure 14). However, it is readily verified that given the CCA representations in Figure 13 it is possible to represent all the necessary Henneberg moves that are required to construct CCA representations of the full set of 116 irreducible graphs. See [21] for complete details of this.

Finally we observe that allowing divalent vertex additions we have the following inductive construction for \((2, 2)\)-tight torus graphs.

**Theorem 9.4.** If \(G\) is a \((2, 2)\)-tight torus graph then \(G\) can be constructed from one of the torus graphs in Figure 12 by a sequence of moves each of which is is either a digon split, triangle split, quadrilateral split or a divalent vertex addition. \(\Box\)
Figure 11. A CCA representation of a topological Henneberg move. The bold arc on the right represents the new vertex, that touches two of the initial arcs.

Figure 12. The irreducibles that have no vertex of degree 2.

Appendix A. Irreducible Torus Graphs

Up to isomorphism there are 116 distinct irreducible $(2,2)$-tight torus graphs. We describe them all in this section, grouped according to the number of vertices. Our descriptions will consist of a diagram of a representative of each class. Each diagram is a standard representation of a torus as a rectangle with the usual side identifications.

For graphs with at most three vertices (Figure 15) the situation is straightforward. There are four $(2,2)$-tight graphs and each has a unique irreducible embedding in the torus.
Figure 13. CCA representations of the 12 irreducible torus graphs with no vertices of degree two: $C^i_j$ is a CCA representation of $G^i_j$ from Figure 12.

Among the graphs with four vertices (Figure 16) we see the first instance of an irreducible torus graph with a quadrilateral face. Also we have a pair of non isomorphic irreducibles that have the same underlying graph.

Given that there are 23 irreducibles with five vertices and 47 with six, one might expect even larger numbers for the cases of seven or eight vertices. However irreducibles with seven, respectively eight, vertices must contain at least one, respectively two, quadrilateral faces. This follows easily from Theorem 3.4. The presence of these quadrilateral faces enforces a lot of additional structure, hence the relatively small number of examples in these cases.
Figure 14. It is impossible to insert a circular arc that touches both $u$ and $v$. This illustrates a topological Henneberg move that cannot be represented by contacts of circular arcs given this representation of the initial graph.

Figure 15. Irreducible torus graphs with at most three vertices

Figure 16. Irreducible torus graphs with four vertices
Figure 17. Irreducible torus graphs with five vertices
Figure 18. Irreducible torus graphs with six vertices
Figure 19. Irreducible torus graphs with seven vertices

Figure 20. Irreducible torus graphs with eight vertices
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