We consider holographic superconductors in a rotating black string spacetime. In view of the mandatory introduction of the $A_c$ component of the vector potential we are left with three equations to be solved. Their solutions show that the effect of the rotating parameter $a$ influences the critical temperature $T_c$ and the conductivity $\sigma$ in a simple but not trivial way.

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I. INTRODUCTION

The use of the AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence has been recently proposed by Hartnoll, Herzog and Horowitz for the investigation of the strongly correlated condensed matter physics from the gravitational dual \cite{1}. It generally shows that the instability of black strings corresponds to a second-order phase transition from the normal state to a superconductor state with the spontaneous $U(1)$ symmetry breaking and leads to interesting physics in the related lower dimensional theories.

Such a breakthrough result has been widely used more recently to model conductivity and other condensed matter physics properties, in particular, holographic superconductor in various spacetimes \cite{2}. Most authors considered, up to now, the holographic superconductor on a static background. However, we still have room for the other hair, namely angular momentum, on the top of mass and charge. Or, equivalently, we can say that a real black string (or black hole) could have angular momentum in the background. It is thus our goal to study the holographic superconductor related to the 4 dimensional rotating black hole. We consider the simplest uncharged background case and we will use both analytical and numerical methods to find the critical temperature and conductivity of the holographic superconductor related to the rotating black hole background.

The paper is planned as follows. In section II, we introduce the metric of the uncharged rotating black hole and calculate the Hawking radiation and Hawking temperature. Then, the analytical and numerical methods are applied to find the conductivity of such a black hole in sections III and IV respectively. Section V includes a summary and a conclusion.
angular Hamilton-Jacobi equations as follows,
\[ f^2 R^2 - \frac{(\omega_0 - m_0 \omega)^2}{(a^2 - \Xi^2)^2} + f \left[ \frac{(\omega_0 - m_0 \Xi)^2}{r^2 (a^2 - \Xi^2)^2} + \frac{\lambda}{r^2} + \mu_0^2 \right] = 0 , \]
and get
\[ \left( \frac{dY}{d\theta} \right)^2 = \lambda , \] (2.5)
where \( \lambda \) is a constant. At the event horizon \( r_0 \), we can expand the function \( f \) as
\[ f(r) = f'(r_h)(r - r_h) + \frac{f''(r_h)}{2}(r - r_h)^2 + \cdots , \] (2.6)
and get
\[ R_{\pm} = \pm \frac{i \pi}{f'(r_h)} \frac{\omega_0 - m_0 a}{\Xi - \frac{\Xi}{a^2}} , \] (2.7)
where \( R_+ \) and \( R_- \) are the radial outgoing and incoming modes respectively. Therefore, the tunneling rate at the event horizon is
\[ \Gamma = e^{-2(\Xi R_+ - \Xi R_-)} = \exp \left( - \frac{4 \pi}{f'(r_h)} \frac{\omega_0 - m_0 a}{\Xi - \frac{\Xi}{a^2}} \right) , \] (2.8)
while the Hawking temperature is given by
\[ T_h = \frac{\Omega_h}{4 \rho \pi} = \frac{3r_h}{4 \rho \pi} \] (2.9)
where \( \rho = \frac{\Xi}{a^2 - \Xi^2} = \frac{1 + a^2}{1 + a^2} \).

We thus find that the form of the Hawking temperature is very similar to the temperature of static black hole, while the \( \rho \) in Eq. (2.9) depends on the rotation parameter \( a \). We use the results to analyse the holographic superconductor in the next section.

### III. HOLOGRAPHIC SUPERCONDUCTOR MODES AND ANALYTICAL INVESTIGATION

In the rotating black hole spacetime background, the Lagrangian density of the simplest holographic superconductor model with a Maxwell field and a charged complex scalar field is
\[ L = \frac{1}{4} F^{\mu \nu} F_{\mu \nu} - |\partial \Psi - i A \Psi|^2 - \frac{m_0^2}{\Xi^2} \Psi^2 \] , (3.1)
where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \Psi = \Psi(r) \) since \( g_{\mu \nu} \) only depends on \( r \). As in the static background spacetime case, \( A_\mu = \delta_\mu^r \Phi(r) \), but in the rotating black string background, we must set \( A_\mu = \delta_\mu^r \Phi(r) + \delta_\mu^\omega \Omega(r) \) in view of the presence of the \( g_{\mu \omega} \) term.

From the variation of Eq. (3.1), we get three equations. They are
\[ \Psi'' + \left( \frac{f'}{f} + \frac{2}{r} \right) \Psi' + \frac{(1 + a^2) \Phi + a \Omega}{(1 + a^2 + 4a^2 f)} \Psi \]
\[ + \frac{(1 + a^2) \Omega + a \Phi}{(1 + a^2 + 4a^2 f)} \] (3.2)
\[ \Phi'' + \frac{2(1 + a^2)^2}{(1 + a^2 + 4a^2 f)} - \frac{a^2 f'}{r(1 + a^2 + 4a^2 f)} \Psi' = 0 \] (3.3)
\[ \Omega'' - \frac{2a^2}{(1 + a^2 + 4a^2 f)} - \frac{(1 + a^2)^2 f'}{(1 + a^2 + 4a^2 f)} \Omega' = 0 \] (3.4)
The boundary condition at the horizon requests \( A_\mu \) to be finite. The solutions of above equations at infinity are given by
\[ \Psi = \frac{\sqrt{\Omega} \Phi}{r \Delta^3} + \frac{\sqrt{\Omega} \Phi}{r^3 \Delta^3} + \cdots , \]
\[ \Phi = \mu - \frac{\rho}{r} \cdots , \]
\[ \Omega = \nu - \frac{\rho}{r} \cdots , \] (3.5)
where \( \Delta = \frac{3}{2} \pm \sqrt{\frac{3}{2} - \frac{a^2}{4}} \) implying that \( \Delta \) satisfies \( \frac{1}{2} < \Delta < 3 \). In the following, we set \( L = 1 \).

Siopsis and Therrien proposed an effective analytic method to calculate the critical temperature of holographic superconductor in static spacetime. We generalize such a method for the rotating black hole case in this section.

According to that procedure, we change the coordinates as \( z = r_0/r \) and the field equations can be rewritten as
\[ z \Psi_{zz} - \frac{2 + z^3}{1 - z^3} \Psi_z + \frac{z \left[ (a^2 + 1) \Phi + a \Omega \right]^2}{(a^4 + a^2 + 1)^2 \frac{m_0^2}{(1 - z^3)^2}} \Psi \]
\[ + \frac{z \left( (a^2 + 1) \Omega + a \Phi \right)^2}{(a^4 + a^2 + 1)^2 \frac{m_0^2}{(1 - z^3)^2}} \Psi - \frac{m_0^2}{z (1 - z^3)} \Psi = 0 , \] (3.6)
\[ \Phi_{zz} - \frac{3a^2 z^2 \Phi_z}{(a^4 + a^2 + 1)(z^3 - 1)} - \frac{2\Phi^2 z}{z^2 (1 - z^3)} \Phi \]
\[ - \frac{3a (a^2 + 1)^2 z^2 \Omega_z}{(a^4 + a^2 + 1)(z^3 - 1)} = 0 , \] (3.7)
\[ \Omega_{zz} + \frac{3 (a^2 + 1)^2 z^2 \Phi_z}{(a^4 + a^2 + 1)(z^3 - 1)} - \frac{2 \Omega^2 z}{z^2 (1 - z^3)} \]
\[ + \frac{3a (a^2 + 1)^2 z^2 \Phi_z}{(a^4 + a^2 + 1)(z^3 - 1)} = 0 . \] (3.8)
At the critical temperature $T_c$, we have $\Psi = 0$, thus Eqs. (3.7) and (3.8) become
\[
\Phi_{zz} - \frac{3\alpha^2z^2\Phi_z}{(a^4 + a^2z^2 + 1)(z^3 - 1)} - \frac{3\alpha (a^2 + 1)z^2\Omega_z}{(a^4 + a^2z^2 + 1)(z^3 - 1)} = 0
\]
\[
\Omega_{zz} + \frac{3 (a^2 + 1)^2 z^2\Omega_z}{(a^4 + a^2z^2 + 1)(z^3 - 1)} + \frac{3\alpha (a^2 + 1)z^2\Phi_z}{(a^4 + a^2z^2 + 1)(z^3 - 1)} = 0. \quad (3.9)
\]

Therefore, we can rewrite above equations as $\Omega_z = \Omega_z(\Phi_{zz}, \Phi_z)$ and $\Phi_z = \Phi_z(\Omega_{zz}, \Omega_z)$, and then substitute into Eq. (3.9) again, so that the decoupling equations are given by
\[
\Phi_{zz} + \frac{2 \Phi_{zz} + 1}{z} \Phi_{zz} = 0
\]
\[
\Omega_{zz} + \frac{2 \Omega_{zz} + 1}{z} \Omega_{zz} = 0 \quad (3.10)
\]
and the solutions are
\[
\Phi = \mu - \rho z + C_1 \left[ \sqrt{12} \arctan \left( \frac{1 + 2z}{\sqrt{3}} \right) + \ln \left( \frac{1 + z + z^2}{1 - z^2} \right) \right],
\]
\[
\Omega = \nu - \zeta z + C_2 \left[ \sqrt{12} \arctan \left( \frac{1 + 2z}{\sqrt{3}} \right) + \ln \left( \frac{1 + z + z^2}{1 - z^2} \right) \right], \quad (3.11)
\]
Considering the boundary condition at the horizon, we require $\Phi|_{z=1} = \Omega|_{z=1} = 0$, so that we may set
\[
\Phi = \lambda r_{hc} (1 - z),
\]
\[
\Omega = \lambda r_{hc} (1 - z), \quad (3.12)
\]
where $\lambda = \frac{\rho}{r_{hc}}$, and $r_{hc}$ is the radius of the horizon at critical temperature. Next, substituting Eq. (3.12) into Eq. (3.6), we find $\lambda = -\frac{\alpha}{1 + \alpha^2}\lambda$.

According to the idea of Siopsis and Terrien, we introduce $\Psi(z) = \sqrt{2\alpha}z^2 F(z)$ (where $F|_{z=0} = 1$, and $F_z|_{z=0} = 0$) to match the boundary condition, and substitute Eq. (3.12) into Eq. (3.6), so, finally, Eq. (3.6) becomes
\[
-F_{zz} + \frac{1}{z} \left( \frac{2 + z^3}{1 - z^3} - 2\Delta \right) F_z + \frac{\Delta^2 z^2}{1 - z^3} F = \frac{\lambda^2}{(1 + z + z^2)^2} F, \quad (3.13)
\]
where $\lambda = \frac{\lambda}{1 + \alpha^2}$. We observe that the form of Eq. (3.13) is the same as the results of [6], so we can directly use the mathematical conclusions about the eigenvalue $\tilde{\lambda}$
\[
\tilde{\lambda}^2 = \int_0^1 dz \left\{ z^{2\Delta - 2} \left[ (1 - z^3)(F_z)^2 + \Delta^2 z^2 F^2 \right] \right\} / \int_0^1 dz \left[ z^{2\Delta - 2} - \frac{1 - z z^2}{1 + z z^2} F^2 \right], \quad (3.14)
\]
and assume a very simple trial function $F = F_\alpha(z) \equiv 1 - \alpha z^2$. We can thus compute the minimizing value of $\lambda^2$. Then, using the temperature’s formula, the critical temperature is given by the expression
\[
T_c = \frac{3}{4\rho hc} \frac{\rho}{\sqrt{\lambda}} = \frac{3}{4\rho hc} \frac{\rho}{\sqrt{\lambda}} \quad (3.15)
\]
where
\[
\eta = \frac{1}{\sqrt{1 + \alpha^2}} = \frac{1 + a^2 + a^4}{(1 + a^2)^{3/2}}. \quad (3.16)
\]
From above equations, we can get the relation between critical temperature $T_c$ and $\lambda$ which depends on $\Delta$. In table I we write the results of the analytical method in the rotating black strings spacetime.

| $\Delta$   | $T_c$   |
|------------|---------|
| 0.6        | 0.45504 $\eta\sqrt{\rho}$ |
| 0.8        | 0.29124 $\eta\sqrt{\rho}$ |
| 1          | 0.22496 $\eta\sqrt{\rho}$ |
| 1.2        | 0.18638 $\eta\sqrt{\rho}$ |
| 1.4        | 0.16066 $\eta\sqrt{\rho}$ |
| 1.6        | 0.14214 $\eta\sqrt{\rho}$ |
| 1.8        | 0.12810 $\eta\sqrt{\rho}$ |
| 2          | 0.11704 $\eta\sqrt{\rho}$ |
| 2.2        | 0.10809 $\eta\sqrt{\rho}$ |
| 2.4        | 0.10067 $\eta\sqrt{\rho}$ |
| 2.6        | 0.09440 $\eta\sqrt{\rho}$ |
| 2.8        | 0.08903 $\eta\sqrt{\rho}$ |

TABLE I: Critical Temperature $T_c$ with the $\Delta$.
In Fig. 1 we plot the relationship between \( \eta \) and \( a \). We find that \( \eta < 1 \) as \( |a| < a_1 \) where

\[
a_1 = \sqrt{\frac{21}{18}} \left[ (25 + 3 \sqrt{69})^{1/3} + (25 - 3 \sqrt{69})^{1/3} \right] - \frac{1}{3} \approx 0.868837.
\]

In this region, the correction from \( a \) leads the critical temperature \( T_c \) to decrease. The minimum value of \( \eta \) is \( \eta_{\text{min}} = \sqrt{\frac{20}{21}} - 1 \approx 0.937744 \) at \( a_{\text{min}} = \sqrt{\frac{20}{21}} \approx 0.550251 \). However, the effect of \( \eta \) leads to an increase of the critical temperature \( T_c \) as \( |a| > a_1 \).

Further details of holographic superconductor shall be investigated by a numerical method in the next section.

**IV. NUMERICAL INVESTIGATION AND CONDUCTIVITY**

In this section, we set \( m^2 = -2 \), \( r_h = 1 \), and then use the the shooting method [1], combining with the boundary condition (3.5) and the main equations (3.2)-(3.4), to calculate the condensate as a function of temperature. The results are shown in FIG. 2.

In Fig. 2, it is easy to find that, as the rotating parameter \( a \) increases, the curved line gets lower.

Finally, we study the conductivity. Considering the perturbed Maxwell field \( dA = A_x(r)e^{-i\omega t} dr \), we obtain the equation

\[
A''_x + \frac{f'}{f} A'_x - \alpha^2 \left[ 1 - \frac{f a^2}{(1 + a^2)^2 r^2} \right] \omega^2 \frac{f^2}{r^2} A_x - 2 \frac{\omega^2}{r} A_x = 0 \quad (4.1)
\]

The boundary condition at event horizon requires

\[
A_x(r) \sim f(r)^{-i \omega / 3r_h}, \quad (4.2)
\]

while the behavior of \( A_x \) in the asymptotic AdS region is

\[
A_x = A^{(0)} + \frac{A^{(1)}}{r}. \quad (4.3)
\]

Therefore, we can get the conductivity of the superconductor by using the AdS/CFT dictionary [1]

\[
\sigma = \frac{-i A^{(1)}}{\omega A^{(0)}}. \quad (4.4)
\]

We use the above equations to calculate the conductivity in Figs. 3 4. It can be seen that the effect of \( a \) drives both the real and imaginary parts smaller and smaller.

We also plot \( -\omega \text{Im} \sigma \) with small \( \omega \), and the results show that \( -\omega \text{Im} \sigma \) with different \( a \) go to the same constant as \( \omega \) goes to 0. This implies superconductivity for zero frequency.

**V. CONCLUSIONS**

We have considered holographic superconductors in 3+1 dimensional rotating black strings. The investigation shows that the \( A_x = \Omega \) term can be ignored in 3+1 dimensional static spacetime. The effect from rotating parameter \( a \) leads both the real and imaginary parts of the conductivity lower. Superconductivity, however, remains.

The spacetime we considered in this paper is the simplest black string spacetime. It might be more meaningful to investigate some more general case, such as Kerr-Newmann anti-de Sitter as well as higher rotating black strings. On the other hand, the holographic superconductor model with a Maxwell field and a charged complex scalar field (5.1) is the simplest form. Also, we have never consider the backreactions of spacetime. Further study is under way.

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FIG. 2: The condensate as a function of the temperature for the two operators $O_1$ and $O_2$ in rotating spacetime.

FIG. 3: The real part of conductivity for the two operators $O_1$ and $O_2$ in rotating spacetime.

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FIG. 4: The imaginary part of conductivity for the two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ in rotating spacetime.

FIG. 5: $-\omega \text{Im}\sigma$ for the two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ in rotating spacetime.