Lyapunov Exponent and Charged Myers Perry Spacetimes

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Abstract We compute the proper time Lyapunov exponent for charged Myers Perry black hole spacetime and investigate the instability of the equatorial circular geodesics (both timelike and null) via this exponent. We also show that for more than four spacetime dimensions ($N \geq 3$), there are no Innermost Stable Circular Orbits (ISCOs) in charged Myers Perry black hole spacetime. We further show that among all possible circular orbits, timelike circular orbits have longer orbital periods than null circular orbits (photon spheres) as measured by asymptotic observers. Thus, timelike circular orbits provide the slowest way to orbit around the charged Myers Perry black hole.

Keywords ISCO, Lyapunov exponent, Charged Myers Perry black hole.

1 Introduction

Geodesics, especially equatorial circular geodesics in four dimensional ($3 + 1$) spacetimes \cite{23}, have been extensively discussed in the literature. In ($N + 1$) spacetime dimensions, such studies are confined to specific geometries inspired to some extent by string theoretic solutions. It has been shown that \cite{2} higher dimensional black hole spacetimes have certain distinct features beyond their four dimensional counterparts. For example, higher dimensional spacetimes admit black-ring \cite{6}, black-string \cite{4} and black-Saturn \cite{13} solutions, four dimensional analogues of which do not exist. Again higher dimensional spacetimes permit a variety of horizon topologies, whereas in ($3 + 1$) spacetime the horizon topologies are usually $R \times S^2$. 

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On the other hand, four dimensional black holes have a number of remarkable features, such as their obeisance of uniqueness theorems, dynamical stability, and the laws of black hole mechanics. It is thus important to investigate what happens to these issues in higher dimensional spacetimes. Although the laws of black hole mechanics are known to hold for Lorentzian spacetimes of arbitrary dimension, the uniqueness theorem is violated in higher dimensional spacetimes. In [8], the authors argue that there may be a possibility of creation of higher dimensional black holes in a future Large Hadron Collider, thereby bringing such black holes into the realm of reality.

Tangerlini [11] was the first to derive static black hole solutions of the Einstein equation in $> 4$ dimensions. These solutions generalize spherically symmetric Schwarzschild and Reissner-Nordstrøm black holes of 3 + 1 dimension in Einstein’s general relativity. For higher dimensional static spherically symmetric black holes, a uniqueness theorem still exists [9]. However, as mentioned above, Myers-Perry black holes present a different situation.

In this paper, we study properties of causal geodesics in charged Myers Perry blackhole spacetime. We further compute the proper time Lyapunov exponent for such geodesics in this spacetime. Using this exponent we shall prove that for ($N \geq 3$), there are no ISCOs in charged Myers Perry black-hole space-times. Note that the principal Lyapunov exponents($\lambda$) have been computed in [2,10,15] using a coordinate time $t$, where $t$ is measured by the asymptotic observers. Thus, these exponents are explicitly coordinate dependent and therefore have a degree of unphysicality. Here we compute the principal Lyapunov exponent ($\lambda$) analytically by using the proper time which is coordinate invariant.

Thus the proper time Lyapunov exponent can be derived as in section 2 as well as in our previous work for Reissner-Nordstrøm black holes [21] and Kerr-Newman spacetimes [22]:

$$\lambda_{\text{proper}} = \pm \sqrt{\frac{c^2}{2}}. \quad (1)$$

The paper is organized as follows: in section 2 we give the basic definition of Lyapunov exponent and also show that it may be expressed in terms of the radial effective potential. In section 3 we demonstrate that the reciprocal of the critical exponent can be expressed in terms of the effective radial potential. In section 4 we fully describe the equatorial circular geodesics, both time-like and null, for charged Myers Perry space-times. In section 5 we show that the Lyapunov exponent can be used to study the instability of time-like circular geodesics. In section 6 we compute the reciprocal of the Critical exponent explicitly; we conclude with discussions in section 7.

2 Proper time Lyapunov exponent and Radial Potential:

In any classical phase space the Lyapunov exponent [24] gives a measure of the average rate of expansion and contraction of a trajectories(geodesics) sur-
rounding it. A positive Lyapunov exponent indicate a divergence between two nearby geodesics, i.e. the paths of such a system are extremely sensitive to changes of the initial conditions. A negative Lyapunov exponent implies a convergence between two nearby geodesics and the vanishing Lyapunov exponent indicate the existence of marginal stability.

The rate of exponential expansion or contraction in the direction of $y(0)$ on the trajectory passing through $X_0$ (trajectory at $t = 0$) is given by

$$
\lambda_i = \lim_{t \to \infty} \left( \frac{1}{t} \right) \ln \left( \frac{\| y(t) \|}{\| y(0) \|} \right) .
$$

(2)

where $\| \|$ denotes a vector norm. The asymptotic quantity $\lambda_i$ is called the Lyapunov exponent.

If there exists a set of $n$ Lyapunov exponents associated with an $n$-dimensional autonomous system and they can be ordered by size that is

$$
\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n .
$$

(3)

The set of $n$-numbers $\lambda_i$ is called the Lyapunov Spectrum [25].

For $p$-dimensions, a $p$-dimensional Lyapunov exponent $\lambda$ is defined as

$$
\lambda^p = \lim_{t \to \infty} \left( \frac{1}{t} \right) \ln \left( \frac{\| y_1(t) \wedge y_2(t) \wedge \ldots \wedge y_p(t) \|}{\| y_1(0) \wedge y_2(0) \wedge \ldots \wedge y_p(0) \|} \right) .
$$

(4)

where $\wedge$ is an exterior or vector cross product.

To derive the Lyapunov exponent in terms of the radial equation we shall first derive the 2nd derivative of the square of the radial component of the four velocity in terms of Lyapunov exponent. Now the Lagrangian for a test particle in the equatorial plane for any stationary axi-symmetric space-time can be written as

$$
\mathcal{L} = \frac{1}{2} \left[ g_{tt} \dot{t}^2 + 2g_{t\phi} \dot{t} \dot{\phi} + g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2 \right] .
$$

(5)

Now we defining the canonical momenta as

$$
p_q = \frac{\delta \mathcal{L}}{\delta \dot{q}} .
$$

(6)

Now from the Euler-Lagrange equations of motion

$$
\frac{dp_q}{d\tau} = \frac{\delta \mathcal{L}}{\delta \dot{q}} .
$$

(7)

Using it we get the non-linear differential equation in 2-dimensional phase space with phase space variables $X_i(t) = (p_r, r)$.

$$
\frac{dp_r}{d\tau} = \frac{\delta \mathcal{L}}{\delta r} \text{ and } \frac{dr}{d\tau} = \frac{p_r}{g_{rr}} .
$$

(8)
Now linearizing the equations of motion about circular orbits of constant \( r \), we get the infinitesimal evolution matrix as

\[
M_{ij} = \left( \frac{0}{\frac{d}{dr} \left( \frac{dL}{dr} \right)} \right) \bigg|_{r=r_0} .
\]  

(9)

Now for circular orbits of constant \( r = r_0 \) have the characteristic values of the matrix gives the information about stability of the orbits. The eigen values of this matrix are the principal Lyapunov exponent. Therefore the eigen values of the evolution matrix along the circular orbit can be written as

\[
\lambda^2 = \frac{1}{2g_{rr}} \frac{d}{dr} \left( \frac{\delta L}{\delta r} \right) \bigg|_{r=r_0} .
\]  

(10)

Again from Lagrange’s equation of motion

\[
\frac{d}{d\lambda} \left( \frac{\delta L}{\delta \dot{r}} \right) - \frac{\delta L}{\delta r} = 0 .
\]  

(11)

Thus the Lyapunov exponent(which is the inverse of the instability time scale associated with the geodesic motions) in terms of the square of the radial velocity \((\dot{r}^2)\) can be written as

\[
\frac{\delta L}{\delta r} = \frac{1}{2g_{rr}} \frac{d}{dr} \left( \dot{r}g_{rr} \right)^2 .
\]  

(12)

Finally the principal Lyapunov exponent can be rewritten as

\[
\lambda^2 = \frac{1}{2g_{rr}} \frac{d}{dr} \left[ \frac{1}{2g_{rr}} \left( \dot{r}g_{rr} \right)^2 \right] .
\]  

(13)

Again for circular geodesics \[23\]

\[
\dot{r}^2 = (\dot{r}^2)' = 0 .
\]  

(14)

where prime denotes for a derivative with respect to \( r \). Therefore the equation \[24\] for proper time Lyapunov exponent \[24\] must be reduces to

\[
\lambda = \pm \sqrt{\frac{(\dot{r}^2)''}{2}} .
\]  

(15)

where we may defined \( \dot{r}^2 \) as radial potential or effective radial potential. In general the Lyapunov exponent come in \( \pm \) pairs to conserve the volume of phase space. From now we shall take only positive Lyapunov exponent. The circular orbit is unstable when the \( \lambda \) is real, the circular orbit is stable when the \( \lambda \) is imaginary and the circular orbit is marginally stable or saddle point when \( \lambda = 0 \).

(Note that in reference \[15\], the authors use a different definition of the Lyapunov exponent(Coordinate time), \( \lambda = \sqrt{\frac{V''}{2}} \) with \( V_r = \dot{r}^2 \).)
3 Critical exponent and Radial potential:

Following Pretorius and Khurana\[14\], we can define Critical exponent which is the ratio of Lyapunov time scale $T_\lambda$ and Orbital time scale $T_\Omega$ may be written as

$$\gamma = \frac{T_\lambda}{2\pi T_\Omega} = \frac{\text{Lyapunov Timescale}}{\text{Orbital Timescale}}.$$  \hfill (16)

where we have introduced $T_\lambda = \frac{1}{\lambda}$ and $T_\Omega = \frac{2\pi}{\Omega}$, which is important for blackhole merger in the ring down radiation. In terms of the square of the proper radial velocity $(\dot{r}^2)$, Critical exponent can be written as

$$\gamma = \frac{T_\lambda}{T_\Omega} = \frac{1}{2\pi} \sqrt{\frac{2\Omega^2}{(\dot{r}^2)''}}.$$  \hfill (17)

Alternatively the reciprocal of critical exponent is proportional to the effective radial potential which is given by

$$\frac{1}{\gamma} = \frac{T_\Omega}{T_\lambda} = 2\pi \sqrt{\frac{1}{2\Omega^2} (\dot{r}^2)''}.$$  \hfill (18)

4 Instability of Equatorial Circular Geodesics of the Charged Myers-Perry space-time:

We will start our journey with a charged Myers-Perry black hole of $N + 1$ dimension which rotates in a single plane with only one non-zero angular momentum parameter $a$ and is a solution of the vacuum Einstein equation. Therefore the space-time metric in terms of Boyer-Lindquist coordinates is given by\[3,12,16\]

$$ds^2 = -\left(1 - \frac{mr^{4-N}}{\Sigma} + \frac{q^2 r^{2(3-N)}}{\Sigma}\right)dt^2 - \frac{2a}{\Sigma} \left(m r^{4-N} - q^2 r^{2(3-N)}\right) \sin^2 \theta dt d\phi$$

$$+ \left( r^2 + a^2 + \frac{a^2 (m r^{4-N} - q^2 r^{2(3-N)}) \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2 + \frac{r^{N-2} \Sigma}{\Delta} dr^2$$

$$+ \Sigma d\theta^2 + r^2 \cos^2 \theta d\Omega_{N-3}^2,$$  \hfill (19)

where,

$$\Delta = r^{N-2}(r^2 + a^2) - mr^2 + q^2 r^{4-N},$$  \hfill (20)

$$\Sigma = r^2 + a^2 \cos^2 \theta,$$  \hfill (21)

$$d\Omega_{N-3}^2 = d\chi_1^2 + \sin^2 \chi_1 [d\chi_2^2 + \sin^2 \chi_2 (\cdots d\chi_{N-3}^2)].$$  \hfill (22)
The electromagnetic potential one form for the space-time \( (19) \) is

\[
A = A_\mu dx^\mu = -\frac{Q}{(N-2)r^{N-4}}(dt - a \sin^2 \theta d\phi),
\]

(23)

The determinant \((g)\) of the metric \((19)\) gives

\[
\sqrt{-g} = \sqrt{\gamma \Sigma r^{N-3} \sin \theta \cos^{N-3} \theta},
\]

(24)

where \(\gamma\) is the determinant of the metric \((22)\).

The parameters \(m, a, q\) are related with the physical mass \((M)\), angular momentum \((J)\) and charge \((Q)\) through the relations given by

\[
M = A_{N-1}(N-1) \frac{1}{16\pi G} m.
\]

(25)

\[
J = A_{N-1} \frac{8\pi G}{ma}.
\]

(26)

\[
Q^2 = \frac{(N-2)(N-1)A_{N-1}q^2}{8\pi G}.
\]

(27)

Here \(A_{N-1}\) is the area of the unit sphere in \(N-1\) dimensions

\[
A_{N-1} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \cos^{N-3} \theta d\theta \prod_{i=1}^{N-3} \int_0^\pi \sin^{(N-3)-i} \chi_i d\chi_i = \frac{2\pi^{N/2}}{\Gamma(N/2)}
\]

(28)

The position of the event horizon is represented by the largest root of the polynomial \(\Delta_{r=r_+} = 0\). The angular velocity at the event horizon is given by

\[
\Omega_H = \frac{a}{r_+^2 + a^2}.
\]

(29)

The semiclassical Bekenstein-Hawking entropy reads\([16]\)

\[
S = \frac{A_H}{4} = \pi r_+^{N-3}(r_+^2 + a^2)\bar{A}_{N-3}.
\]

(30)

where

\[
\bar{A}_{N-3} = \prod_{i=1}^{N-3} \int_0^\pi \sin^{(N-3)-i} \chi_i d\chi_i = \frac{\pi^{N-1}}{\Gamma(N/2 - 1)}.
\]

(31)

The surface gravity of this space-times \((19)\) is given by

\[
\kappa = \frac{(N-4)(r_+^2 + a^2) + 2r_+^2 - (N-2)q^2r_+^{2(3-N)}}{2r_+(r_+^2 + a^2)}.
\]

(32)

and the Hawking temperature reads

\[
T_H = \frac{\kappa}{2\pi} = \frac{(N-4)(r_+^2 + a^2) + 2r_+^2 - (N-2)q^2r_+^{2(3-N)}}{4\pi r_+(r_+^2 + a^2)}.
\]

(33)
In appropriate limits the metric \( m_{19} \), the BH entropy \( m_{30} \), surface gravity \( m_{32} \) and Hawking temperature \( m_{33} \) reproduces the \( N + 1 \) dimensional spherically symmetric, static Schwarzschild, Reissner-Nordstrom \( m_{11} \) and axially symmetric, Myers-Perry spacetime\[2].

To compute the geodesics in the equatorial plane for the Charged Myers Perry space-time we follow\[23\]. To determine the geodesic motions of a test particle in this plane we set \( \dot{\theta} = 0 \) and \( \theta = \text{constant} = \frac{\pi}{2} \).

Therefore the necessary Lagrangian for this motion is given by

\[
2\mathcal{L} = - \left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2N-4}} \right) \dot{t}^2 - \left( \frac{2ma}{r^{N-2}} - \frac{2aq^2}{r^{2N-4}} \right) \dot{\phi} + \frac{r^N}{\Delta} \dot{r}^2 + \left( r^2 + a^2 \right) \frac{ma^2}{r^{2N-4}} - \frac{a^2q^2}{r^{2N-4}} \dot{\phi}^2. \tag{34}
\]

The generalized momenta can be derived from it are

\[
p_t = - \left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2N-4}} \right) \dot{t} - \left( \frac{ma}{r^{N-2}} - \frac{aq^2}{r^{2N-4}} \right) \dot{\phi} = -E = \text{Const}. \tag{35}
\]

\[
p_\phi = - \left( \frac{ma}{r^{N-2}} - \frac{aq^2}{r^{2N-4}} \right) \dot{t} + \left( r^2 + a^2 \right) \frac{ma^2}{r^{2N-4}} - \frac{a^2q^2}{r^{2N-4}} \dot{\phi} = L = \text{Const}. \tag{36}
\]

\[
p_r = \frac{r^N}{\Delta} \dot{r}. \tag{37}
\]

Here \((\dot{t}, \dot{r}, \dot{\phi})\) denotes differentiation with respect to proper time\( (\tau) \). Since the Lagrangian does not depends on ‘\( t \)’ and ‘\( \phi \)’, so \( p_t \) and \( p_\phi \) are conserved quantities. The independence of the Lagrangian on ‘\( t \)’ and ‘\( \phi \)’ manifested, the stationarity and the axisymmetric character of the Charged Myers Perry space-time. The Hamiltonian is given by

\[
\mathcal{H} = p_t \dot{t} + p_\phi \dot{\phi} + p_r \dot{r} - \mathcal{L}. \tag{38}
\]

In terms of the metric the Hamiltonian is

\[
2\mathcal{H} = - \left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2N-4}} \right) \dot{t}^2 - \left( \frac{2ma}{r^{N-2}} - \frac{2aq^2}{r^{2N-4}} \right) \dot{\phi} + \frac{r^N}{\Delta} \dot{r}^2 + \left[ r^2 + a^2 + \frac{ma^2}{r^{2N-4}} - \frac{a^2q^2}{r^{2N-4}} \right] \dot{\phi}^2. \tag{39}
\]

Since the Hamiltonian is independent of ‘\( t \)’, therefore we can write it as

\[
2\mathcal{H} = - \left[ \left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2N-4}} \right) i + \left( \frac{ma}{r^{N-2}} - \frac{aq^2}{r^{2N-4}} \right) \phi \right] \dot{t} + \frac{r^N}{\Delta} \dot{r}^2 + \left[ \left( \frac{ma}{r^{N-2}} - \frac{aq^2}{r^{2N-4}} \right) i + \left( r^2 + a^2 + \frac{ma^2}{r^{N-2}} - \frac{a^2q^2}{r^{2N-4}} \right) \phi \right] \dot{\phi}. \tag{40}
\]

\[
= -E \dot{t} + L \dot{\phi} + \frac{r^N}{\Delta} \dot{r}^2 = \epsilon = \text{const}. \tag{41}
\]
Here \( \epsilon = -1 \) for time-like geodesics, \( \epsilon = 0 \) for light-like geodesics and \( \epsilon = +1 \) for space-like geodesics. Solving equations (35) and (36) for \( \dot{t} \), we find

\[
\dot{t} = r^{N-2} \left[ \left( 1 - \frac{m}{r^{N-2}} + \frac{q^2}{r^{2N-4}} \right) L + \left( \frac{ma}{r^{N-2}} - \frac{aq^2}{r^{2N-4}} \right) E \right].
\]

Inserting these solutions in equations (41) we obtain the radial equation

\[
r^2 \dot{r}^2 = r^2 E^2 + \left( \frac{m}{r^{N-2}} - \frac{q^2}{r^{2N-4}} \right) (aE - L)^2 + (a^2 E^2 - L^2) + \epsilon \frac{\Delta}{r^{N-2}}.
\]

### 4.1 Circular Null Geodesics:

For null geodesics \( \epsilon = 0 \), introducing new quantities \( m = 2M \) and \( q = Q \) for simplicity the radial equation (44) becomes

\[
r^2 \dot{r}^2 = r^2 E^2 + \left( \frac{2M}{r^{N-2}} - \frac{Q^2}{r^{2N-4}} \right) (aE - L)^2 + (a^2 E^2 - L^2).
\]

The equations finding the radius of \( r_c \) of the unstable circular ‘photon orbit’ at \( E = E_c \) and \( L = L_c \) are

\[
E_c^2 r_c^2 + \left( \frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{2N-4}} \right) (aE_c - L_c)^2 + (a^2 E_c^2 - L_c^2) = 0.
\]

\[
2r_c E_c^2 + \left( - (N - 2) \frac{2M}{r_c^{N-1}} + (2N - 4) \frac{Q^2}{r_c^{2N-3}} \right) (aE_c - L_c)^2 = 0.
\]

Now introducing the impact parameter \( D_c = \frac{L_c}{E_c} \), the above equations may be written as

\[
r_c^2 + \left( \frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{2N-4}} \right) (a - D_c)^2 + (a^2 - D_c^2) = 0.
\]

\[
r_c = (N - 2) M \frac{Q^2}{r_c^{N-3}} - (N - 2) \frac{Q^2}{r_c^{2N-3}} (a - D_c)^2 = 0.
\]

From equation (45) we have

\[
D_c = a \mp \sqrt{\frac{r_c^{N-1}}{M(N - 2)r_c^{N-2} - (N - 2)Q^2}}.
\]

The equation (45) is valid if and only if \( |D_c - a| > a \). For counter rotating orbit, we have \( |D_c - a| = -(D_c - a) \), which correspond to upper sign in the above equation and co-rotating \( |D_c - a| = + (D_c - a) \), which correspond to lower sign.
in the above equation. Inserting equation (50) in (48) we find an equation for the radius of null circular orbits

\[ r_c^{2N-4} - NMr_c^{N-2} + 2ar_c^{N-3} \sqrt{(N-2)Mr_c^{N-2} - (N-1)Q^2} + (N-1)Q^2 = 0. \]

(51)

When \( N = 3 \), we recover the well known photon sphere [7] equation for the Kerr Newman space-times [22]. Another important relation can be derived using equations (48) and (50) for null circular orbits are

\[ D_c^2 = a^2 + r_c^2 \left( \frac{NMr_c^{N-2} - (N-1)Q^2}{(N-2)Mr_c^{N-2} - (N-2)Q^2} \right). \]

(52)

Now we will derive an important quantity associated with the circular null geodesics is the angular frequency which is denoted by \( \Omega_c \)

\[ \Omega_c = \frac{\left[ \left( 1 - \frac{2M}{r_c^{N-2}} + \frac{Q^2}{r_c^{2N-4}} \right) D_c + \left( \frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{N-2}} \right) a \right]}{\left( r_c^2 + a^2 + \frac{2Ma^2}{r_c^{N-2}} - \frac{a^2Q^2}{r_c^{2N-4}} \right) - a \left( \frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{N-2}} \right) D_c} = \frac{1}{D_c}. \]

(53)

Using equations (50) and (48) we show that the angular frequency \( \Omega_c \) of the circular null geodesics is inverse of the impact parameter \( D_c \), which generalizes the result of Kerr Newmann case [22] to the charged Myers Perry black-hole space-time. It proves that this is a general feature of any higher dimensional stationary space-time.

4.2 Circular Timelike Geodesics:

The time-like geodesics equation (44) can be written as by setting \( \epsilon = -1 \)

\[ r^2 \dot{r}^2 = r^2 E^2 + a^2 \left( \frac{2M}{r_c^{N-2}} - \frac{Q^2}{r_c^{2N-4}} \right) (aE - L)^2 + (a^2E^2 - L^2) - \frac{\Delta}{r_c^{2N-4}}. \]

(54)

where \( E \) is the energy per unit mass of the particle describes the trajectory.

Now we shall find the radial equation of the timelike circular geodesics in terms of reciprocal radius \( u = 1/r \) as the independent variable, can be expressed as

\[ V(u) = u^{-4}a^2 = E^2 + 2Mu^N (aE - L)^2 - u^{2N-2}Q^2 (aE - L)^2 + (a^2E^2 - L^2) a^2 - a^2a^2 + 2Mu^{N-2} - Q^2a^{2N-4}. \]

(55)

The conditions for the occurrence of circular orbits at \( r = r_0 \) or reciprocal radius \( u = u_0 \) are

\[ V(u) = 0. \]

(56)
Now setting \( x = L_0 - aE_0 \), where \( L_0 \) and \( E_0 \) are the values of energy and angular momentum for circular orbits at the radius \( r_0 = \frac{1}{u_0} \). Therefore using (55, 57) we get the following equations

\[
-x^2Q^2u_0^{2N-2} + 2Mx^2u_0^N - (x^2 + 2axE_0)u_0^2 - a^2u_0^2 - Q^2u_0^{2N-4} + 2Mx_0^{N-2} - 1 + E_0^2 = 0 .
\]

and

\[
-(N-1)x^2Q^2u_0^{2N-3} + NMx^2u_0^{N-1} - (x^2 + 2axE_0)u_0 - a^2u_0
\]

\[-(N-2)Q^2u_0^{2N-5} + (N-2)Mu_0^{N-3} = 0 .
\]

Using (58, 59) we find an equation for \( E_0^2 \) as

\[
E_0^2 = 1 + (N-4)Mu_0^{N-2} + (N-2)Mx_0^N - (N-2)Q^2u_0^{2N-2} - (N-3)Q^2u_0^{2N-4} .
\]

with the aid of equation (60), equation (59) gives us

\[
2axE_0u_0 = x^2[NMu_0^{N-1} - (N-1)Q^2u_0^{2N-3} - u_0] - a^2u_0 - (N-2)Q^2u_0^{2N-5} + (N-2)Mu_0^{N-3} .
\]

Eliminating \( E_0 \) between these equations, we have obtained the following quadratic equation for \( x^2 \) i.e

\[
Ax^4 + Bx^2 + C = 0 .
\]

where

\[
A = u_0^2 [NMu_0^{N-1} - (N-1)Q^2u_0^{2N-3} - u_0]^2 - 4a^2u_0^2 [(N-2)Mu_0^{N-2} - (N-2)Q^2u_0^{2N-4}]
\]

\[
B = -2 [a^2u_0 + (N-2)Q^2u_0^{2N-5} - (N-2)Mu_0^{N-3}] [NMu_0^{N-1} - (N-1)Q^2u_0^{2N-3} - u_0]
\]

\[-4a^2u_0^2 [1 + (N-4)Mu_0^{N-2} - (N-3)Q^2u_0^{2N-4}]
\]

\[
C = [a^2u_0 + (N-2)Q^2u_0^{2N-5} - (N-2)Mu_0^{N-3}]^2
\]

The solution of this equations (62) are

\[
x^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} .
\]
where the discriminant of this equation is
\begin{equation}
\mathcal{D} = 4a \Delta u_0 \sqrt{(N - 2)M u_0^N - (N - 2)Q^2 u_0^{2N-2}}.
\end{equation}

and
\begin{equation}
\Delta u_0 = 1 + a^2 u_0^2 - 2M u_0^N - 2Q^2 u_0^{2N-4}.
\end{equation}

The solutions becomes simpler form by writing
\begin{equation}
[1 - NM u_0^{N-2} + (N - 1)Q^2 u_0^{2N-4}]^2
- 4a^2 [(N - 2)M u_0^N - (N - 2)Q^2 u_0^{2N-2}] = Z_+ Z_-.
\end{equation}

Thus we get the solution as
\begin{equation}
x^2 u_0^2 = \frac{-B \pm \mathcal{D}}{Z_+ Z_-}.
\end{equation}

Thus we find
\begin{equation}
x^2 u_0^2 = \frac{\Delta u_0 - Z_\mp}{Z_\mp}.
\end{equation}

Again we can write
\begin{equation}
\Delta u_0 - Z_\mp = \left[a u_0 \pm \sqrt{(N - 2)M u_0^{N-2} - (N - 2)Q^2 u_0^{2N-4}}\right]^2.
\end{equation}

Therefore the solutions for \( x \) thus may be written as
\begin{equation}
x = -\frac{a \sqrt{u_0}}{\sqrt{u_0 Z_\mp}} \pm \frac{(N - 2)M u_0^{N-3} - (N - 2)Q^2 u_0^{2N-5}}{u_0 Z_\mp}.
\end{equation}

Here the upper sign in the foregoing equations applies to counter-rotating orbits, while the lower sign applies to co-rotating orbits. Replacing the solution (71) for \( x \) in equation (60), we obtain the energy
\begin{equation}
E_0 = \frac{1}{\sqrt{Z_+}} \left[1 - 2M u_0^{N-2} + a \sqrt{(N - 2)M u_0^N - (N - 2)Q^2 u_0^{2N-2} + Q^2 u_0^{2N-4}}\right].
\end{equation}
and the value of angular momentum associated with the circular orbit is given by

\[ L_0 = \pm \frac{1}{u_0 Z} \left[ (1 + a^2 u_0^2) \sqrt{(N-2)M u_0^{N-3} - (N-2)Q^2 u_0^{2N-5}} \pm 2aM \sqrt{u_0^{2N-3} + aQ^2 u_0^{4N-7}} \right] \]  

(73)

As we previously defined \( E_0 \) and \( L_0 \) followed by equations (72) and (73) are the energy and the angular momentum per unit mass of a particle describing a circular orbit of radius \( u \).

To compute the stability of timelike circular orbit we must calculate the 2nd order derivative of effective potential with respect to \( u \) for the values of \( E_0 \) and \( L_0 \) specific to circular orbits.

Now the 2nd order derivative of effective potential becomes

\[ \frac{d^2V}{du^2} = (N-2)u^{N-4} \left[ (NM - 2(N-1)Q^2 u_0^{N-2}) - 2(N-3)Q^2 u_0^{N-2} + (N-4)M \right] \]  

(74)

Using (69) we find

\[ \frac{d^2V}{du^2} \bigg|_{u=u_0} = \frac{2(N-2)u_0^{N-4}}{Z} \left[ (NM - 2(N-1)Q^2 u_0^{N-2}) \Delta_{u_0} - (4M - 4Q^2 u_0^{N-2}) Z \right] \]  

(75)

The 2nd order derivative of effective potential shows that it explicitly depends on space-time dimensionality \( N \). So, to determine the stability of equatorial circular geodesics we must check the sign of 2nd order derivative of the function \( V(u) \) which will be helpful to distinguish between different values of \( N \). Since \( E_0 \), \( L_0 \) and \( x = L_0 - aE_0 \) must be real, the function \( \Delta_{u_0} \) and \( Z \) are such that

\[ \Delta_{u_0} \geq Z \geq 0 \].  

(76)

Case I: For \( N \geq 4 \) i.e five dimensional case, the above equation leads to

\[ [NM - 2(N-1)Q^2 u_0^{N-2}] \Delta_{u_0} \geq [4M - 4Q^2 u_0^{N-2}] Z \].  

(77)

which immediately suggests that

\[ \frac{d^2V}{du^2} \geq 0 \].  

(78)

Thus we conclude that there are no ISCO and stable timelike circular orbit around the rotating five-dimensional charged Myers-Perry blackhole spacetime, at least in the “equatorial” planes. Which generalizes the previous work by Frolov and Stojkovic[11] on five dimensional rotating black hole.

Case II: Now in four dimension \( N = 3 \), the above equation (75) reduces to

\[ \frac{d^2V}{du^2} \bigg|_{u=u_0} = \frac{2}{u_0 Z} \left[ (3M - 4Q^2 u_0) \Delta_{u_0} - (4M - 4Q^2 u_0) Z \right] \]  

(79)
For $\Delta_{u_0} \geq Z_\pm \geq 0$, the equation (79) leads to
\[ (3M - 4Q^2 u_0) \Delta_{u_0} < (4M - 4Q^2 u_0) Z_\pm. \]
which implies
\[ \frac{d^2V}{du^2} < 0. \tag{81} \]
This suggests that there exist ISCO and stable timelike circular orbit
around the rotating four dimensional Kerr-Newman space-time\cite{22}.

Case III. For $N \geq 3$ i.e arbitrary dimension, the above equation (75) leads
to
\[ [NM - 2(N - 1)Q^2 u_0^{N-2}] \Delta_{u_0} \geq [4M - 4Q^2 u_0^{N-2}] Z_\pm. \tag{82} \]
which immediately suggests that
\[ \frac{d^2V(u)}{du^2} \geq 0. \tag{83} \]
Thus we conclude that in space-time dimensions greater than four i.e $N \geq 3$, there are no ISCOs and stable timelike circular orbits around the rotating higher dimensional charged Myers-Perry blackhole space-time, at least in the “equatorial” planes. Which generalizes the previous work by Cardoso\cite{15} on higher dimensional Myers-Perry blackhole space-time.

This may suggested that the absence of bounded stable circular orbit in
the black hole exterior is a generic property of higher dimensional charged black holes. Which generalizes the previous work by Tangerlini\cite{1} on non-rotating higher dimensional black hole and by Frolov and Stojkovic\cite{11} on five dimensional rotating black hole.

4.2.1 Angular velocity of Timelike Circular Orbit

Now we compute the orbital angular velocity for timelike circular geodesics at
$r = r_0$ is given by
\[ \Omega_0 = \frac{\dot{\phi}}{\dot{t}} = \left[ \frac{L_0 - 2M u_0^{N-2} x + Q^2 u_0^{2N-4} x}{(1 + a^2 u_0^2) E_0 - 2axu_0^2 (2M u_0^{N-2} - Q^2 u_0^{2N-4})} \right]. \tag{84} \]
Again this can be rewritten as
\[ \Omega_0 = \frac{\left[ L_0 - 2M u_0^{N-2} x + Q^2 u_0^{2N-4} x \right] u_0^2}{(1 + a^2 u_0^2) E_0 - 2axu_0^2 (2M u_0^{N-2} - Q^2 u_0^{2N-4})}. \tag{85} \]
Now the previously mentioned expression can be simplified as

\[ L_0 - 2M xu_0^{N-2} + Q^2 xu_0^{2N-4} = \mp \frac{\sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-5}}}{u_0 \Delta u_0} . \quad (86) \]

\[ (1 + a^2u_0^2)E_0 - 2aM xu_0^N + axQ^2u_0^{2N-2} = \frac{\Delta u_0}{Z_0} \left[ 1 \mp a \sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}} \right] . \quad (87) \]

Substituting (86) and (87) into (85) we get the angular velocity for timelike circular geodesics is given by

\[ \Omega_0 = \mp \frac{\sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}}}{1 \mp a \sqrt{(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-2}}} . \quad (88) \]

4.2.2 Ratio of Angular velocity of time like circular orbit to Null Circular Orbit

Since we have already proved that for timelike circular geodesics the angular velocity is given by the equation (88)

Again we obtained for circular null geodesics \( \Omega_c = \frac{1}{d} \), so we can deduce similar expression for it is given by

\[ \Omega_c = \mp \frac{\sqrt{(N-2)Mu_c^N - (N-2)Q^2u_c^{2N-2}}}{1 \mp a \sqrt{(N-2)Mu_c^N - (N-2)Q^2u_c^{2N-2}}} . \quad (89) \]

Resultantly we obtain the ratio of angular frequency for time-like circular geodesics to the angular frequency for null circular geodesics is

\[ \frac{\Omega_0}{\Omega_c} = \left( \frac{\sqrt{Mr_0^{N-2} - Q^2}}{\sqrt{Mr_c^{N-2} - Q^2}} \right) \left( \frac{r_c^{N-1} \mp a \sqrt{(N-2)Mr_c^{N-2} - (N-2)Q^2}}{r_0^{N-1} \mp a \sqrt{(N-2)Mr_0^{N-2} - (N-2)Q^2}} \right) \quad (90) \]

which is proportional to the radial coordinates \( r_0 \). For \( r_0 = r_c \), \( \Omega_0 = \Omega_c \), i.e., when the radius of time-like circular geodesics is equal to the radius of null circular geodesics, the angular frequency corresponds to that geodesic are equal, which demands that the intriguing physical phenomena could occur in the curved four dimensional space-time, for example, possibility of exciting Quasi Normal Modes (QNM) by orbiting particles, possibly leading to instabilities of the curved space-time [15]. It would be very interesting to investigate such phenomenon occur in higher dimensional space-time.

For \( r_0 > r_c \), we proved that for Schwarzschild black-hole, Reissner Nordstrom black-hole [21] and Kerr Black-hole the null circular geodesics have the largest angular frequency as measured by asymptotic observers than the time-like circular geodesics. We therefore conclude that null circular geodesics
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provide the fastest way to circle black holes\cite{20}. This generalizes the case of axi-symmetric symmetry Kerr Newmann space-time\cite{22} to the more general case of stationary, axisymmetry charged Myers Perry blackhole space-times.

Now the ratio of time period for time-like circular geodesics to the time period for null circular geodesics is given by

$$
\frac{T_0}{T_c} = \left( \frac{\sqrt{Mr_c^{N-2} - Q^2}}{\sqrt{Mr_0^{N-2} - Q^2}} \right) \left( \frac{r_0^{N-1} \mp a \sqrt{(N-2)Mr_0 - (N-2)Q^2}}{r_c^{N-1} \pm a \sqrt{(N-2)Mr_c - (N-2)Q^2}} \right) \quad (91)
$$

This ratio is valid for $r_0 \neq r_c$. For $r_0 = r_c$, $T_0 = T_c$, i.e. time period of both geodesics are equal, which possibly leading to the excitations of Quasi Normal Modes. For $r_0 > r_c$, $T_0 > T_c$, which implies that the orbital period for time-like circular geodesics is greater than the orbital period for null circular geodesics. For $r_0 = r_{\text{particle}}$ and $r_c = r_{\text{photon}}$, therefore the ratio of time period for the orbit of massive particles ($r_0 = r_{\text{timelike}}$) to the time period for photon-sphere ($r_c = r_{\text{photon}}$) for charged Myers Perry black-hole is given by

$$
\frac{T_{\text{particle}}}{T_{\text{photon}}} = \left( \frac{\sqrt{Mr_c^{N-2} - Q^2}}{\sqrt{Mr_0^{N-2} - Q^2}} \right) \left( \frac{r_0^{N-1} \pm a \sqrt{(N-2)Mr_0 - (N-2)Q^2}}{r_c^{N-1} \mp a \sqrt{(N-2)Mr_c - (N-2)Q^2}} \right) \quad (92)
$$

This implies that $T_{\text{particle}} > T_{\text{photon}}$, therefore we conclude that timelike circular geodesics provide the slowest way to circle the charged Myers Perry black-hole space-time.

Here we may note that we recover from \cite{51} the condition for the occurrence of the well known unstable circular null geodesics by taking the limit $E_0 \rightarrow \infty$, when

$$
Z_{\pm} = [1 - NMu_0^{N-2} + (N-1)Q^2u_0^{2N-4}] \pm 2a \sqrt{[(N-2)Mu_0^N - (N-2)Q^2u_0^{2N-4}] = 0} \quad (93)
$$

or alternatively for $r_0 = r_c$

$$
r_c^{2N-4} - NMr_c^{N-2} \pm 2a r_c^{N-3} \sqrt{(N-2)Mr_c^{N-2} - (N-2)Q^2 + (N-1)Q^2 = 0} \quad (94)
$$

Here $(-)$ sign indicates for direct orbit and $(+)$ sign indicates for retrograde orbit. The real positive root of the equation is the closest circular photon orbit of the blackhole.
5 Lyapunov exponent and Timelike Circular Geodesics:

Now we evaluated the Lyapunov exponent for timelike circular geodesics as follows, using equations (15) we get

\[ \lambda_{\text{time}} = \sqrt{\frac{(N-2)}{r_0^{N-2}Z_{\mp 0}}[r_0^{N-4}(NMr_0^{N-2} - 2(N - 1)Q^2)\Delta r_0 - (4Mr_0^{N-2} - Q^2)Z_{\mp 0}]} \]

(95)

where

\[ \Delta r_0 = r_0^N + a^2r_0^{N-2} - 2Mr_0^2 + Q^2 r_0^{4-N} \]

\[ Z_{\mp 0} = r_0^{2N-4} - NMr_0^{N-2} \mp 2a^3r_0^{-3}\sqrt{(N - 2)Mr_0^{N-2} - (N - 2)Q^2 + (N - 1)Q^2}. \]

Since \( \Delta r_0 \geq Z_{\mp 0} \geq 0 \) and for \( r_0^{N-4}(NMr_0^{N-2} - 2(N - 1)Q^2)\Delta r_0 \geq (4Mr_0^{N-2} - 4Q^2)Z_{\mp 0} \), \( \lambda \) is real, so we conclude that there are no ISCOs around the charged Myers-Perry blackhole space-time.

6 Critical exponent and Timelike Circular Geodesics:

Now we determine the Critical exponent of charged Myers Perry black hole space-time for equatorial timelike circular geodesics. From that we shall prove the instability of equatorial timelike circular geodesics via Critical exponent. Thus the reciprocal of Critical exponent is given by

\[ \frac{1}{\gamma} = 2\pi (r_0^{N-1} \pm a\sqrt{(N - 2)M r_0 - (N - 2)Q^2}) \times \]

\[ \sqrt{\left[ r_0^{N-4}(NM r_0^{N-2} - 2(N - 1)Q^2)\Delta r_0 - (4Mr_0^{N-2} - 4Q^2)Z_{\mp 0}\right]} \]

(96)

Since \( Z_{\mp 0} \geq 0 \), \( \Delta r_0 \geq 0 \) and \( (NM r_0^{N-2} - 2(N - 1)Q^2)\Delta r_0 \geq (4Mr_0^{N-2} - 4Q^2)Z_{\mp 0} \), so \( \frac{1}{\gamma} \) is real, which also implies that equatorial timelike circular geodesics is unstable.

7 Lyapunov exponent and Null Circular Geodesics:

For null circular geodesics the Lyapunov exponent and K-S entropy of charged Myers Perry blackhole are given by

\[ \lambda_{\text{null}} = \sqrt{\frac{(N-2)(L_c - aE_c)^2[NMr_c^{N-2} - 2(N - 1)Q^2]}{r_c^{2N}}} \]

(97)
Since $NMr_c^{N-2} > 2(N-1)Q^2$ therefore $\lambda$ is real so the null circular geodesics are unstable. In the appropriate limits, we can obtain the Lyapunov exponent for Myers Perry space-times, Tangherlini RN space-times and Tangherlini Schwarzschild space-times.

8 Critical Exponent and Null Circular Geodesics:

Therefore the reciprocal of Critical exponent is given by

$$\frac{1}{\gamma} = 2\pi \left( r_c^{N-1} \mp a \sqrt{(N-2)Mr_c - (N-2)Q^2} \right) \times$$

$$\sqrt{\frac{(L_c - aE_c)^2[NMr_c^{N-2} - 2(N-1)Q^2]}{r_c^{2N}(Mr_c^{N-2} - Q^2)}}. \quad (98)$$

9 Discussion

The study demonstrates that Lyapunov exponent may be used to give a full description of time-like circular geodesics and null circular geodesics in charged Myers Perry black hole space-time. We showed that the Lyapunov exponent can be used to determine the instability of equatorial circular geodesics, both time-like and null case for charged Myers Perry black hole space-times. By computing Lyapunov exponent, we proved that for more than four space time dimensions ($N \geq 3$), there is no ISCO in charged Myers Perry black hole space-times. The other point we have studied that for circular geodesics around the central black-hole, time-like circular geodesics is characterized by the smallest angular frequency as measured by the asymptotic observers-no other circular geodesics can have a smallest angular frequency. Thus such types of space-times always have $\Omega_{\text{particle}} < \Omega_{\text{photon}}$ for all time-like circular geodesics. Alternatively it was shown that the orbital period for time-like circular geodesics is characterized by the longest orbital period than the null circular geodesics. Hence, we conclude that time-like geodesics provide the slowest way to circle the black hole.

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