ON THE SCALING CRITICAL REGULARITY OF THE YANG-MILLS SYSTEM IN THE LORENZ GAUGE

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Abstract. In this paper, we prove the local well-posedness of the Yang-Mills system in the Lorenz gauge for initial data in the Besov space $B^{\frac{1}{2}}_{2,1} \times B^{-\frac{1}{2}}_{2,1}$ with additional angular regularity. To the best of our knowledge, our study is the first result on $(1 + 3)$-dimensional Yang-Mills system with initial data in the scaling critical regularity.

1. Introduction

Yang-Mills field theory suggested by C. N. Yang and R. Mills [30] in early 1954 has worthy in that it is the first generalisation of the abelian gauge group to non-abelian gauge theory, which gives an extension of Maxwell field theory and unify the strong and weak interaction with electromagnetic field and then provide base for the Standard Model. However, the Yang-Mills field equations are also interesting to the field of mathematics, especially in dispersive partial differential equations since it is written as nonlinear wave equations containing bilinear forms, which have an interesting structure in Minkowski space $\mathbb{R}^{1+d}$.

1.1. Yang-Mills system in the covariant form. Let $G$ be a compact Lie group and $\mathfrak{g}$ its Lie algebra. For the sake of simplicity, we may assume $G = SU(n, C)$, $n \geq 2$, the group of unitary matrices of determinant one. Then $\mathfrak{g} = su(n, C)$ is the algebra of trace-free skew-Hermitian matrices whose infinitesimal generators are denoted by $T^a$, $a = 1, 2, \ldots, n^2 - 1$, and are traceless Hermitian matrices. For example, if $\mathfrak{g} = su(2, C)$, then $T^a$ is chosen to be Pauli matrices; $i\sigma^a$, $a = 1, 2, 3$.

For a $\mathfrak{g}$-valued 1-form $A$ on the Minkowski space $\mathbb{R}^{1+3}$ with signature $(-, +, +, +)$, we denote by $F = F[A]$ the associated curvature $F = dA + [A, A]$. More explicitly, given 1-form $A_\mu : \mathbb{R}^{1+3} \to \mathfrak{g}$, we define $F_{\mu\nu}$ by

\begin{equation}
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
\end{equation}

where $[\cdot, \cdot]$ denotes the matrix commutator. Then Yang-Mills system is written by

\begin{equation}
D^\mu F_{\mu\nu} = 0,
\end{equation}

where $D^\mu = \partial^\mu + [A^\mu, \cdot]$ is the covariant derivative associated to the gauge potential $A^\mu$. Here, the physically interesting quantity is the curvature $F$; $A$ is not necessarily unique. Indeed, we consider the following gauge transformation:

\begin{equation}
A_\mu \to A'_\mu = UA_\mu U^{-1} - (\partial_\mu U)U^{-1},
\end{equation}

2010 Mathematics Subject Classification. M35Q55, 35Q40.

Key words and phrases. Yang-Mills system, wave equation, bilinear estimates, null structure, angular regularity, Besov space.
for sufficiently smooth function $U: \mathbb{R}^{1+3} \to G$. Let us denote $F' = F[A']$ and $D'_\mu = \partial_\mu + [A'_\mu, \cdot]$. Then we see that
\begin{equation}
F' = UFU^{-1}, \quad D'_\mu F' = U(D_\mu F)U^{-1}.
\end{equation}
Now this implies that
\begin{equation}
D'^\mu F'_{\mu\nu} = \partial'^\mu F'_{\mu\nu} + [A'^\mu, F'_{\mu\nu}] = 0,
\end{equation}
which concludes that (1.2) is invariant under the gauge transformation (1.3).

1.2. **Yang-Mills system in the Lorenz gauge.** Now we derive the Yang-Mills equations as nonlinear wave equations. We expand (1.2) in terms of the gauge potentials $A_\mu$ to get the following nonlinear PDE:
\begin{equation}
\Box A_\nu = \partial_\nu \partial^\mu A_\mu - [\partial^\mu A_\mu, A_\nu] - [A_\mu, \partial^\nu A_\nu] - [A^\mu, \partial_\nu A_\mu] + [A_\mu, \partial_\nu A_\nu] - [A^\mu, [A_\mu, A_\nu]].
\end{equation}
We impose the Lorenz gauge condition: $\partial^\mu A_\mu = 0$ and then our equation (1.6) becomes the nonlinear wave equation:
\begin{equation}
\Box A_\nu = -2[A^\mu, \partial_\mu A_\nu] + [A^\mu, \partial_\nu A_\mu] - [A^\mu, [A_\mu, A_\nu]].
\end{equation}
Also, for any choice of gauge condition, $F$ satisfies the following nonlinear wave equation:
\begin{equation}
\Box F_{\mu\nu} = -[A^\lambda, \partial_\lambda F_{\mu\nu}] - \partial^\lambda [A_\lambda, F_{\mu\nu}] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] - 2[F^\lambda_{\mu}, F_{\nu\lambda}].
\end{equation}
To derive (1.8), we first recall the Jacobi identity:
\[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \]
which implies
\[ D_\mu D_\nu X - D_\nu D_\mu X = [F_{\mu\nu}, X]. \]
In particular, $D^\mu D_\nu F_{\lambda\mu} = D_\nu D^\mu F_{\lambda\mu} + [F^\mu_{\nu}, F_{\lambda\mu}]$ and $D^\mu D_\lambda F_{\mu\nu} = D_\lambda D^\mu F_{\mu\nu} + [F^\mu_{\lambda}, F_{\mu\nu}]$. Then applying $D^\mu$ to both sides of the Bianchi identity
\[ D^\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0 \]
gives
\[ D^\mu D_\mu F_{\nu\lambda} + D_\nu D^\mu F_{\lambda\mu} + [F^\mu_{\nu}, F_{\lambda\mu}] + D_\lambda D^\mu F_{\mu\nu} + [F^\mu_{\lambda}, F_{\mu\nu}] = 0. \]
Here, by Yang-Mills equations (1.2), we have $D^\mu F_{\lambda\mu} = 0$ and $D^\mu F_{\mu\nu} = 0$ Also by antisymmetry of $F_{\mu\nu}$, we get $[F^\mu_{\nu}, F_{\lambda\mu}] = [F^\mu_{\nu}, F_{\lambda\mu}] = -2[F^\mu_{\nu}, F_{\lambda\mu}] = -[F^\mu_{\nu}, F_{\lambda\mu}]$. Hence,
\[ D^\mu D_\mu F_{\nu\lambda} + 2[F^\mu_{\nu}, F_{\lambda\mu}] = 0. \]
Finally, in view of the identity
\[ D^\mu D_\mu X = \Box X + [A^\mu, \partial_\mu X] + \partial^\mu [A_\mu, X] + [A^\mu, [A_\mu, X]], \]
we obtain (1.8). Now we expand the second and fourth terms of (1.8) and impose the Lorenz gauge condition, which yields
\begin{equation}
\Box F_{\mu\nu} = -2[A^\lambda, \partial_\lambda F_{\mu\nu}] + 2[\partial_\nu A^\lambda, \partial_\lambda A_\mu] - 2[\partial_\mu A^\lambda, \partial_\lambda A_\nu] + 2[\partial^\lambda A_\mu, \partial_\nu A_\lambda] + 2[\partial^\lambda A_\nu, \partial_\mu A_\lambda] - [A^\lambda, [A_\lambda, F_{\mu\nu}]] + 2[F_{\lambda\nu}, [A^\lambda, A_\mu]] - 2[[A^\lambda, A_\mu], [A_\lambda, A_\nu]].
\end{equation}
Our aim is to study the Cauchy problem of the nonlinear wave equations \((1.7)\) and \((1.9)\). To do this, we first consider initial data for \(A\) and \(F\) at \(t = 0\), namely,

\[
A(0) = a, \quad \partial_t A(0) = \dot{a}, \quad F(0) = f, \quad \partial_t F(0) = \dot{f}.
\]

We note that the initial data \(f\) for \(F\) is determined by \((a, \dot{a})\). Indeed, we have

\[
\begin{align*}
\Omega_{ij} &= \partial_i a_j - \partial_j a_i + [a_i, a_j], \\
\Omega_{0i} &= \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\
\dot{\Omega}_{ij} &= \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\
\dot{\Omega}_{0i} &= \partial^i \Omega_{ji} + [a^i, f_{\alpha i}].
\end{align*}
\]

We also note that the Lorenz gauge condition and \((1.2)\) with \(\nu = 0\) require the constraints

\[
\begin{align*}
\dot{a}_0 &= \partial^i a_i, \\
\partial^i \Omega_{0i} &= [a^i, \dot{a}_i].
\end{align*}
\]

We are interested in the well-posedness of given equations, which means that given initial data in adapted function spaces, a solution exists, and it is unique, and its flow moves continuously in its existence time in the adapted function spaces. Consequently, whether a given solution is well-posed or ill-posed is decided by the roughness of the adapted function spaces. Typically, it is known that the scaling critical index decides well-posedness \([25]\). In other words, if the adapted function space is more regular than scaling critical Sobolev space, then the given solution is well-posed, whereas the solution is ill-posed if function space is rougher. For this reason, it is an important study to lower the regularity and to attain the scaling critical regularity of several dispersive equations, e.g., \([27]\), \([2]\), \([5]\). We refer the readers to \([25]\ Chapter 3) for more information on scaling critical regularity and well-posedness.

However, to attain the critical regularity is not a simple problem that can be treated by Strichartz estimates \([23]\), \([7]\) and multilinear estimates \([29]\), \([24]\) since, for example, given a product of two functions, if two input frequency are collinear, which results in resonance interaction, then oscillation does not play a role. Thus it is heavily required to reveal further cancellation. Indeed, it is the cancellation property by an angle between input frequency called null structure \([8]\). Especially in Yang-Mills equations, it is Q-type null form given by

\[
\begin{align*}
Q_{ij}(u, v) &= \partial_i u \partial_j v - \partial_j u \partial_i v, \\
Q_0(u, v) &= \partial^i u \partial_i v - \nabla u \cdot \nabla v.
\end{align*}
\]

Motivated by this structure, null form estimates are studied \([6]\), \([9]\), \([12]\), and also many researchers have exploited null form estimates to study the low regularity problem of Yang-Mills equations \([10]\), \([16]\).

In addition, since the sharpness of Strichartz estimates is proved by Knapp counterexample, which is not radial, it is natural to expect that the \(p, q\) range of Strichartz estimates can be extended if we impose radial assumption \([21]\), \([3]\). In fact, it is possible to attain critical regularity by spherical harmonic expansion and imposing additional angular regularity on initial data, \([2]\), \([22]\), \([27]\).

Inspired by the previous work, we study and prove the scaling critical regularity of \((1 + 3)\) dimensional Yang-Mills equations in the Lorenz gauge. We introduce the infinitesimal generators of the rotations on Euclidean space

\[
\Omega_{i,j} = x_i \partial_{x_j} - x_j \partial_{x_i}
\]

and define the spherical Laplacian \(\Delta_{S^2} = \sum_{i<j} \Omega_{i,j}^2\). Let \(\langle \Omega \rangle^p = (1 - \Delta_{S^2})^{\frac{p}{2}}\). We state our main result.
Theorem 1.1. Let $\sigma \geq 1$. Suppose that given initial data $(a, \dot{a}, f, \dot{f})$ which satisfy (1.10) and (1.12) in the function space:
\[
\langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}} \times \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}} \times \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}} \times \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}}.
\]
Then there exists local existence time $T = T(\|\langle \Omega \rangle^{\sigma} a\|_{B_{2,1}^{\frac{1}{4}}}, \|\langle \Omega \rangle^{\sigma} \dot{a}\|_{B_{2,1}^{\frac{1}{4}}}, \|\langle \Omega \rangle^{\sigma} f\|_{B_{2,1}^{\frac{1}{4}}}, \|\langle \Omega \rangle^{\sigma} \dot{f}\|_{B_{2,1}^{\frac{1}{4}}}) > 0$ such that there exist solutions
\[
\begin{align*}
A &\in C([-T, T]; \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}}) \cap C^1((-T, T); \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}}), \\
F &\in C([-T, T]; \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}}) \cap C^1((-T, T); \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{4}}),
\end{align*}
\]
of (1.7) and (1.9), respectively. Also, the solutions $A$ and $F$ have the regularity
\[
\begin{align*}
(A \pm \frac{1}{i|\nabla|} \partial_t A) &\in \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}}(S_T), \\
(F \pm \frac{1}{i|\nabla|} \partial_t F) &\in \langle \Omega \rangle^{-\sigma} B_{2,1}^{\frac{1}{2}}(S_T),
\end{align*}
\]
where $S_T = (-T, T) \times \mathbb{R}^3$ is the restricted domain, and $B_{2,1}^{s,b}$ is the Besov type $X_s^{\alpha,b}$ space.

Here $B_{2,1}^s$ is the usual inhomogeneous Besov space such that $\|f\|_{B_{2,1}^s} = \sum_{N \geq 1} N^s \|P_{[\xi] \approx N} f\|_{L^2}$, where $P_{[\xi] \approx N}$ is the Littlewood-Paley projection on $\{\xi \in \mathbb{R}^3 : |\xi| \approx N\}$.

1.3. Previous results. In advance to discussion on the main strategy of the proof of Theorem 1.1 we summarise the previous work on Yang-Mills system. So far, the most interesting Yang-Mills system has been $(1 + 4)$ dimensional system whose scaling critical Sobolev space is $\dot{H}^1(\mathbb{R}^4)$, which means energy critical, where Yang-Mills system has its conserved quantity called energy:
\[
\mathcal{E}(t) := \sum_{\mu, \nu=0}^d \int_{\mathbb{R}^d} |F_{\mu\nu}(t, x)|^2 \, dx.
\]

Recently, Krieger and Tataru [11] showed the global well-posedness for data with small energy in $(1 + 4)$ dimensional Yang-Mills system in the Coulomb gauge. Sterbenz [22] also proved the global result in the Lorenz gauge for small initial data with angular regularity. Oh [13, 14] developed a new approach via Yang-Mills heat flow to recover the well-posedness result for finite energy initial data.

On the $(1 + 3)$ dimension, which we now focus on, Selberg and Tesfahun [18] first discovered null structure in the bilinear forms in Yang-Mills system and studied local well-posedness with finite energy initial data. Then Tesfahun [20] revealed a partial null structure hidden in Yang-Mills equations and then proved local well-posedness for initial data below energy space. Later, Pecher [15] improved this result and showed the local well-posedness for initial data $(a, f) \in H^s \times H^r$, $(s, r) = (\frac{7}{8}+, -\frac{1}{7}+)$). However, the scaling critical regularity exponents in $(1 + 3)$ dimension are $(s_c, r_c) = (\frac{5}{2}, -\frac{1}{2})$. Thus there is still a gap between the critical regularity and the previous results.

The novelty of Theorem 1.1 is to attain the scaling critical regularity exponents. To the best of our knowledge, Theorem 1.1 is the first result on the well-posedness of $(1 + 3)$ Yang-Mills equations concerning the scaling critical regularity.

1.4. Strategy of the proof. We prove the local well-posedness for initial data in Besov space with angular regularity. To do this, instead of using global estimates [11], which the authors of [18, 20, 15] exploited, we make the use of fully dyadic decomposition of space-time Fourier sides and utilize the null structure, which is revealed by [18, 20]. Then we handle the nonlinear terms by using sharp bilinear estimates of wave type localised in a thickened null cone [17].
One fortune in dealing with bilinear estimates is that we don’t need to be worried about high-high-low frequency. Indeed, since there is no complex conjugation in the product, the spatial Fourier transform of product of two functions is written as

\[\hat{u}_1\hat{u}_2(\xi_0) = \int_{\xi_0 = \xi_1 + \xi_2} \hat{u}_1(\xi_1)\hat{u}_2(\xi_2) \, d\xi_1 d\xi_2.\]

In the high-high-low interaction, \(|\xi_0| \ll |\xi_1| \approx |\xi_2|\), the angle between two input frequency cannot be small; \(\theta_{12} \approx 1\). Hence the null structure plays no role, and we have high modulation - low frequency, in which case we gain better estimates. Thus in the presence of bilinear estimates, we are only concerned with high-low-high interaction; \(\min(|\xi_1|, |\xi_2|) \ll |\xi_0| \approx \max(|\xi_1|, |\xi_2|)\) with \(\theta_{12} \ll 1\).

However, this approach is not sufficient to attain the critical regularity. The problem is that we must encounter a bilinear estimate of the form:

\[\|(D^{-1}F_{\pm 1})F_{\pm 2}\|_{\mathcal{B}^{s_0}_{2,1}B^{s_1}_{2,1}B^{s_2}_{2,1}},\]

which gives us regularity at most \(r = -\frac{1}{4}\). (See the proof of (6.1) and Remark 6.1.) To overcome this, we apply spherical harmonic expansion and then impose angular regularity. We are willing to gain some regularity via spherical harmonic projections. Indeed, by the mercy of Lemma 2.3 we can attack the above problem as follows:

\[\|e^{\mp itD}P_{K_{\mathcal{N},L}} H_t f\|_{L^5_{t,x}(\mathbb{R}^{1+3})} \lesssim N^{-\frac{1}{4}}L^{\frac{1}{4}}l\|P_{K_{\mathcal{N},L}} H_t f\|_{L^5_{t,x}(\mathbb{R}^{1+3})},\]

where \(N, L, l\) are dyadic numbers, \(N\) is a spatial frequency, \(L\) is a modulation and \(H_t\) is a spherical harmonic projection. Note that we get additional modulation \(L^{\frac{1}{4}}\), which forces us to impose the condition \(b = \frac{1}{4}\). One expect that it may be possible to attack the critical Sobolev space \(H^{\frac{1}{2}} \times H^{-\frac{1}{2}}\) by using Lemma 2.3 to both terms; however, one should note that we need to deal with summation by angular regularity:

\[\sum_{l_0, l_1, l_2} l_0^4 l_1 l_2 \|H_{l_1}u_1\| \|H_{l_2}u_2\|,\]

which cannot give us the required estimates. Consequently, when we prove bilinear estimates, we should use Lemma 2.3 only once and obtain \(l_{12}^{12}\) to avoid the summation problem on angular regularity. We also observe that in dealing with such a bilinear estimate, in view of Lemma 2.3 the most serious interaction appears when \(l_2 \ll l_1\) in the low-low-high modulation: \(L_1 \lesssim L_2 \ll L_0\). (See also Remark 5.2.)

When we treat higher-order terms, we can obtain even better estimates. In fact, we shall use Bernstein’s inequality and apply Lemma 2.3 twice. It makes no trouble for the summation of angular regularity. We gain much more regularity than the loss of using Bernstein’s inequality. (See also Remark 7.2) Even if we don’t lose any regularity by duality and then study quadrilinear estimates, its treatment is quite formidable.

We shall show the proof of (6.4) by duality and concerning quadrilinear estimate in Appendix to explain the advantage of applying angular regularity. (See also Remark 7.2)

We also note that to attain the critical Sobolev space \(H^s \times H^{-s}\), the further study must be the low-low-high modulation in bilinear estimates. Since for all other cases, we already prove better estimates.

This is the main scheme of the proof of our result. We end this section with organisation of this paper and notation used throughout this paper.

**Organisation.** We give some preliminaries in the next section, which introduces the decomposition of d’Alembertian and the Besov type \(X^{s,b}\) space and basic facts about analysis on the sphere. In Section 3, we present bilinear estimates of wave type, which plays a key role in the proof of well-posedness and also discuss
the null structure. We construct Picard’s iterates in Section 4. Section 5 and 6 are the proof of nonlinear estimates to show the well-posedness. Finally, Appendix is devoted to the proof of trilinear estimate, which is significantly different from our mainstream of the proof of Theorem 1.1 and the proof of energy estimate lemma and the notation from quantum field theory. 

**Notations.** Since we only use $L_{t,x}^2$ norm, by $\|F\|$ we abbreviate $\|F\|_{L_{t,x}^2} := \sum_a \|F_a\|_{L_{t,x}^2}$ for $F = F_a T_a$, where $\{T_a\}_{a=1}^{n^2-1}$ is the set of infinitesimal generators of given Lie algebra $g = su(n, \mathbb{C})$. As usual different positive constants independent on dyadic numbers such as $N$ and $L$ are denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

The spatial Fourier transform and space-time Fourier transform on $\mathbb{R}^3$ and $\mathbb{R}^{1+3}$ are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx, \quad \hat{u}(X) = \int_{\mathbb{R}^{1+3}} e^{-i(t\tau + x \cdot \xi)} u(t, x) \, dt \, dx,$$

where $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^3$, and $X = (\tau, \xi) \in \mathbb{R}^{1+3}$. Also we denote $\mathcal{F}(u) = \hat{u}$. Then we define space-time Fourier projection operator $P_E$ by $P_E u(\tau, \xi) = \chi_{E} \hat{u}(\tau, \xi)$, for $E \subset \mathbb{R}^{1+3}$. We define spatial Fourier projection operator, similarly. For example, $P_{|\xi| \approx N}$ is the Littlewood-Paley projection on $\{|\xi| \approx N\}$.

Since we prefer to use the differential operator $|\nabla|$ rather than $-i\nabla$, for the sake of simplicity, we put $D := |\nabla|$ whose symbol is $|\xi|$.

For brevity, we denote the maximum, median, and minimum of $N_0$, $N_1$, $N_2$ by

$$N_{\text{max}}^{012} = \max(N_0, N_1, N_2), \quad N_{\text{med}}^{012} = \text{med}(N_0, N_1, N_2), \quad N_{\text{min}}^{012} = \min(N_0, N_1, N_2).$$

2. Preliminaries

2.1. Decomposition of d’Alembertian. In this section, we introduce the standard decomposition of d’Alembertian to rewrite (1.7) and (1.9) as first order system. We use the following transform:

$$(A, \partial_t A, F, \partial_t F) \rightarrow (A_+, A_-, F_+, F_-),$$

where

$$A_{\pm} = \frac{1}{2} \left( A \pm \frac{1}{iD} \partial_t A \right), \quad F_{\pm} = \frac{1}{2} \left( F \pm \frac{1}{iD} \partial_t F \right).$$

Recall the system

$$\begin{cases}
\Box A_\nu = \mathcal{M}_\nu(A, \partial_t A, F, \partial_t F), \\
\Box F_{\mu\nu} = \mathcal{N}_{\mu\nu}(A, \partial_t A, F, \partial_t F),
\end{cases}$$

(2.1)

where

$$\mathcal{M}_\nu(A, \partial_t A, F, \partial_t F) = -2[A^\lambda, \partial_\nu A_\mu] + [A^\mu, \partial_\nu A_\mu] - [A^\mu, [A_\mu, A_\nu]],$$

$$\mathcal{N}_{\mu\nu}(A, \partial_t A, F, \partial_t F) = -2[A^\lambda, \partial_\mu F_{\nu\lambda}] + 2(\partial_\nu A^\lambda, \partial_\lambda A_\mu) - 2(\partial_\mu A^\lambda, \partial_\lambda A_\nu) + 2(\partial^\lambda A_\mu, \partial_\nu A_\lambda) + 2(\partial_\mu A^\lambda, \partial_\nu A_\lambda) - [A^\lambda, [A_\lambda, F_{\nu\lambda}]] + 2[F_{\mu\nu}, [A^\lambda, A_\mu]] - 2[F_{\lambda\nu}, [A^\lambda, A_\mu]]$$

Then our system (2.1) is rewritten as

$$\begin{cases}
(i\partial_t \pm D) A_{\pm} = \mp \frac{1}{2D} \widetilde{\mathcal{M}}_{\nu}(A_+, A_-, F_+, F_-), \\
(i\partial_t \pm D) F_{\pm} = \mp \frac{1}{2D} \widetilde{\mathcal{N}}_{\mu\nu}(A_+, A_-, F_+, F_-),
\end{cases}$$

(2.2)
Lemma 2.1. Let us consider the integral equation:
\[ v(t) = e^{\mp itD} f + \int_0^t e^{\mp i(t-t')D} F(t') dt' \]
with sufficiently smooth \( f \) and \( F \). If \( T \leq 1 \), then for any \( s \in \mathbb{R} \), we have
\[ \|v\|_{B^{s+\frac{1}{2}}_{\pm}(S_T)} \lesssim T^{\frac{s}{2}} \|f\|_{B^{s}_{\pm}} + \|F\|_{B^{s-\frac{1}{2}}_{\pm}(S_T)}. \]

The proof is very straightforward. We refer the readers to [5] for the proof, however, for the convenience to readers, we leave its brief proof in Appendix.

Also the initial data transforms to
\[ A_{\pm}(0) = a_{\pm} := \frac{1}{2} \left( a \pm \frac{1}{iD} \right) \in (\Omega)^{-\sigma}B^{\pm}_{2,1}, \]
\[ F_{\pm}(0) = f_{\pm} := \frac{1}{2} \left( f \pm \frac{1}{iD} \right) \in (\Omega)^{-\sigma}B^{-\frac{3}{2}}_{2,1}. \]

2.2. Besov type \( X^{s,b} \) spaces. We adapt the Besov type \( X^{s,b} \) space. Instead of \((\ell^2)\) summation in the Sobolev space, by using \((\ell^1)\) summation in the Besov space, we can focus on the critical regularity and uniqueness of solutions to (1.7) and (1.9) without difficulty. In this section, we also introduce linear estimate of the adapted function space.

We first define the function space as follows. For dyadic number \( N \geq 1 \) and \( L \), we define the set
\[ K^{\pm}_{N,L} = \{ (\tau, \xi) \in \mathbb{R}^{1+3} : \|\xi\| \approx N, \quad \|\tau \pm |\xi|| \approx L \}. \]
Then the adopted function space is given by
\[ B^{s,b}_{\pm} = \left\{ u \in L^2 : \|u\|_{B^{s,b}_{\pm}} = \sum_{N,L} N^s L^b \|P_{K_{N,L}} u\| < \infty \right\}. \]

Note that we have assumed \( N \geq 1 \) to avoid the singularity at the origin in the spatial Fourier side because of \( D^{-1} \) term in the right-hand-side of (2.2).
Since we are only concerned with local time existence \( T \leq 1 \) throughout this paper, it is convenient to utilize our function space in the local time setting. Hence we introduce the restriction space. The time-slab which is the subset of \( \mathbb{R}^{1+3} \) is given by
\[ S_T = (-T,T) \times \mathbb{R}^3. \]
Then we define the restriction norm \( B^{s,b}_{\pm}(S_T) \) for a function \( u \) on a time slab \( S_T \) by
\[ \|u\|_{B^{s,b}_{\pm}(S_T)} = \inf_{v=u \text{ on } S_T} \|v\|_{B^{s,b}_{\pm}}. \]
This becomes a semi-norm on \( B^{s,b}_{\pm} \), however, it is a norm if we identify elements which agree on \( S_T \) and the resulting space is denoted by \( B^{s,b}_{\pm}(S_T) \). In other words, \( B^{s,b}_{\pm}(S_T) \) is the quotient space \( B^{s,b}_{\pm}/X \), where \( X = \{ v \in B^{s,b}_{\pm} : v = 0 \text{ on } S_T \} \). Since \( X \) is a closed subspace in \( B^{s,b}_{\pm} \), we conclude that the quotient space \( B^{s,b}_{\pm}(S_T) \) is a Banach space.

Now we state the energy estimate lemma:

Lemma 2.1. Let us consider the energy estimate lemma:
\[ \sum_{\pm} (\mathcal{N}_\nu(A_+, A_-, F_+, F_-) \mathcal{M}_\nu(A, \partial_\tau A, F, \partial_\tau F), \]
\[ \sum_{\pm} (\tilde{\mathcal{N}}_{\mu\nu}(A_+, A_-, F_+, F_-) \tilde{\mathcal{M}}_{\mu\nu}(A, \partial_\tau A, F, \partial_\tau F)). \]
2.3. **Analysis on the sphere.** We recall some basic facts from Fourier analysis in spherical coordinates. We refer the readers to [2], [21] for the following discussion. One can also find more systematic introduction on spherical harmonics in [21] Chapter 4. We start with the homogeneous polynomials $P_l$ on $\mathbb{R}^3$ of degree $l$ which satisfy

\begin{equation}
\left( \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \Delta_{S^2} \right) P_l = 0.
\end{equation}

Given nonnegative integer $l \geq 0$, there exists a finite dimensional set $Y_l$ of homogeneous harmonic polynomials of degree $l$ which satisfy

\begin{equation}
- \Delta_{S^2} y_{l,n} = l(l+1) y_{l,n}, \quad y_{l,n} \in Y_l.
\end{equation}

Put $\omega_2 = |S^2|$. The the natural inner product on $S^2$ is given by

\begin{equation}
\langle F, G \rangle_{L^2_2(S^2)} = \frac{1}{\omega_2} \int_{S^2} F(\omega) \overline{G(\omega)} d\sigma(\omega).
\end{equation}

We state the basic properties of the set $Y_l$. 

**Lemma 2.2** (Lemma 3.1 of [21]).

1. The dimension of the space $Y_l$ is $|Y_l| = 2l + 1$.
2. The spaces $Y_l$ are mutually orthogonal.
3. Let \{y_{l,n}\}_{n=0}^{2l} be any orthonormal basis of $Y_l$. Then the identity $\sum_{n=0}^{2l} |y_{l,n}(\omega)|^2 = |Y_l|$ holds for all $\omega \in S^2$.
4. For each $y_{l,n} \in Y_l$, the following identity holds:

$$\sum_{l \leq j} \|\Omega_{l,j} Y_l\|_{L^2_2(S^2)}^2 = l(l+1) \|y_{l,n}\|_{L^2_2(S^2)}^2.$$ 

Now given $f \in L^2(\mathbb{R}^3)$, we have the orthogonal decomposition:

$$f(x) = \sum_{m \geq 0} \sum_{n=0}^{2m} (f(|x|\omega), y_{m,n}(\omega))_{L^2_2(S^2)} y_{m,n} \left( \frac{x}{|x|} \right).$$

We let $l \geq 1$ be a dyadic number and define the spherical Littlewood-Paley projections

\begin{align*}
(H_l f)(x) &= \sum_{m \geq 0} \sum_{n=0}^{2m} \rho \left( \frac{m}{l} \right) (f(|x|\cdot), y_{m,n}(\omega))_{L^2_2(S^2)} y_{m,n} \left( \frac{x}{|x|} \right), \\
(H_1 f)(x) &= \sum_{m \geq 0} \sum_{n=0}^{2m} \rho_{\leq l}(m) (f(|x|\cdot), y_{m,n}(\omega))_{L^2_2(S^2)} y_{m,n} \left( \frac{x}{|x|} \right),
\end{align*}

where $\rho \in C_0^\infty(\mathbb{R})$ is a fixed smooth function supported in \{t : \frac{1}{2} < t < 2\} and $\sum_{N:dyadic} \rho_1(\frac{N}{2}) = 1$. Here, $\rho_1 = \sum_{N \leq 1} \rho_1(\frac{N}{2}) = 1$ with $\rho_1(0) = 1$. We also define the angular derivatives

\begin{equation}
\langle \Omega \rangle^\sigma f = \sum_{l:dyadic} l'' H_l f.
\end{equation}

The following lemma will play crucial role in the proof of our main result.

**Lemma 2.3.** Let $N, L,$ and $l$ be dyadic numbers. We have the following linear estimates:

$$\|P_{K_{N,L}} H_l u\|_{L^2_{l,\nu}} \lesssim N^{-\frac{1}{2}} L^\frac{1}{2} l \|P_{K_{N,L}} H_l u\|_{L^2_{l,\nu}}.$$
Proof. This is followed by Lemma 8.2 of [2], especially, by the following estimate:

\[ \|e^{itD}P_{|\xi| \approx N}H_tf\|_{L^2_t \dot{B}^{-\frac{3}{2}}_x} \lesssim N^{-\frac{1}{4}}l\|P_{|\xi| \approx N}H_tf\|_{L^2_x}. \]

We replace \( f \) by \( P_{|\tau| \pm |\xi| \approx L}f \). Then the proof is readily followed as [25, Lemma 2.9]. Indeed, we have

\[ \|e^{itD}P_{N,L}^+H_tf\|_{L^2_t \dot{B}^{-\frac{3}{2}}_x} \lesssim N^{-\frac{1}{4}}l\|P_{N,L}^+H_tf\|_{L^2_x}. \]

By Fourier inversion we have

\[ u(t, x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^{4+1}} \tilde{u}(\tau, \xi)e^{it\tau}e^{ix\cdot\xi} \, d\tau \, d\xi. \]

Set

\[ P_{N,L}^+H_tf(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \tilde{u}(\tau \pm |\xi|, \xi)e^{ix\cdot\xi} \, d\xi \]

and then we get

\[ e^{itD}P_{N,L}^+H_tf(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \tilde{u}(\tau \pm |\xi|, \xi)e^{\pm it|\xi|}e^{ix\cdot\xi} \, d\xi. \]

Thus we have the following representation:

\[ u(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\tau}e^{itD}P_{N,L}^+H_tf \, d\tau. \]

The required estimate is then followed by Minkowski’s inequality and Cauchy-Schwarz inequality:

\[ \|P_{N,L}^+H_tf\|_{L^2_t \dot{B}^{-\frac{3}{2}}_x} \lesssim \int_{\mathbb{R}} \|e^{it\tau}e^{itD}P_{N,L}^+H_tf\|_{L^2_t \dot{B}^{-\frac{3}{2}}_x} \, d\tau \]

\[ = N^{-\frac{1}{4}}l \int_{\mathbb{R}} \|P_{N,L}^+H_tf\|_{L^2_t} \, d\tau \]

\[ = N^{-\frac{1}{4}}lL^\frac{1}{2}\|P_{N,L}^+H_tf\|_{L^2_t \dot{B}^{-\frac{3}{2}}_x}. \]

\[ \square \]

From now on, since we often use the projection operators \( P_{N,L}^\pm \) throughout this paper, we abbreviate it by \( P_{N,L}^I \) for simplicity. Now given \( s \in \mathbb{R}, \ b \geq \frac{3}{4} \) and \( \sigma \geq 1 \), we let

\[ \|u\|_{\mathcal{B}^{s,b,\sigma}_{K,L}} = \sum_{N \geq 1} \sum_{L} \sum_{l} N^s L^{b_1}l^\sigma \|P_{N,L}^I u\|_{L^2_{t,x}}. \]

Then we define the Banach space

\[ \mathcal{B}^{s,b,\sigma}_{K,L} = \{ u \in C([-T, T]; \Omega^{-\sigma}\mathcal{B}^{s}_{2,1}) : \|u\|_{\mathcal{B}^{s,b,\sigma}_{K,L}} < \infty \}. \]

3. Bilinear estimates and Null structures

3.1. Bilinear estimates. Given dyadic numbers \( N, L \), we invoke that

\[ K_{N,L}^\pm = \{(\tau, \xi) \in \mathbb{R}^{4+1} : |\xi| \approx N, \ |\tau \pm |\xi| \approx L \}. \]

We introduce the key ingredient to handle multilinear estimates.

Theorem 3.1 (Theorem 1.1. of [17]). For all \( u_1, u_2 \in L^2(\mathbb{R}^{4+1}) \) such that \( \tilde{u}_j \) is supported in \( K_{N_j,L_j}^\pm \), then the estimates

\[ \|P_{N_0,L_0}^\pm (u_1 u_2)\| \leq C \|u_1\| \|u_2\|. \]
holds with

\begin{align}
(3.1) & \quad C \sim (N^{12}_{\min}N^{12}_{\min}L_1L_2)^{\frac{1}{2}}, \\
(3.2) & \quad C \sim (N^{12}_{\min}N^{02}_{\min}L_0L_j)^{\frac{1}{2}}, \quad j = 1, 2, \\
(3.3) & \quad C \sim ((N^{012}_{\min}L_{012}^{\min}))^{\frac{1}{2}},
\end{align}

for any choice of signs \((\pm_0, \pm_1, \pm_2)\).

**Remark 3.2.** Here, \((3.3)\) is followed by a trivial volume estimate. In the proof of Theorem 1.1, we use \((3.1)\) and \((3.2)\) for the low modulation - high frequency; \(L \ll N\), as otherwise \((3.3)\) is better estimate.

### 3.2. Modulation analysis

In the proof of Theorem 1.1, we typically treat bilinear estimates and via duality we are concerned with the following integral:

\begin{equation}
(3.4) \quad \int_{\mathbb{R}^{1+3}} u_1u_2 \cdot \varphi \, dt \, dx,
\end{equation}

and on the space-time Fourier side it is written by

\begin{equation}
(3.5) \quad \int_{\mathbb{R}^{1+3}} \hat{\varphi}(X_1 + X_2)\hat{u_1}(X_1)\hat{u_2}(X_2) \, dX_1 \, dX_2.
\end{equation}

Now suppose that

\begin{equation}
(3.6) \quad |\tau_1 + \tau_2 \pm_0 |\xi_1 + \xi_2| | \approx L_0, \quad |\tau_1 \pm_1 |\xi_1| | \approx L_1, \quad |\tau_2 \pm_2 |\xi_2| | \approx L_2,
\end{equation}

where \(L_0, L_1, L_2\) are dyadic numbers. We see that

\[ |\pm_0 |\xi_1 + \xi_2| \mp_1 |\xi_1| \mp_2 |\xi_2| | \leq |\tau_1 + \tau_2 \pm_0 |\xi_1 + \xi_2| | + |\tau_1 \pm_1 |\xi_1| | + |\tau_2 \pm_2 |\xi_2| |. \]

Motivated by this inequality, we fix \(\pm_0 = +\) and define the resonance function

\begin{equation}
(3.7) \quad \mathcal{R}^{\pm_1, \pm_2}(\xi_1, \xi_2) = |\xi_1 + \xi_2| \mp_1 |\xi_1| \mp_2 |\xi_2| |.
\end{equation}

We note that the above integral \((3.4)\) vanishes unless

\begin{equation}
(3.8) \quad |\mathcal{R}^{\pm_1, \pm_2}(\xi_1, \xi_2)| \lesssim L^{012}_{\max}.
\end{equation}

**Lemma 3.3** (Lemma 2.2 of [1]). Given the resonance function

\[ \mathcal{R}^{\pm_1, \pm_2}(\xi_1, \xi_2) = |\xi_1 + \xi_2| \mp_1 |\xi_1| \mp_2 |\xi_2| |,
\]

we have the following bounds:

If \((\pm_1, \pm_2) = (-, -)\) or \((\pm_1, \pm_2) = (+, +)\) and \(|\xi_1 + \xi_2| \ll \min(|\xi_1|, |\xi_2|)\), then

\begin{equation}
(3.9) \quad |\mathcal{R}^{\pm_1, \pm_2}(\xi_1, \xi_2)| \gtrsim \max(|\xi_1 + \xi_2|, |\xi_1|, |\xi_2|).
\end{equation}

If \(\pm_1 \neq \pm_2\) or \((\pm_1, \pm_2) = (+, +)\) and \(|\xi_1 + \xi_2| \gtrsim \min(|\xi_1|, |\xi_2|)\), then

\begin{equation}
(3.10) \quad |\mathcal{R}^{\pm_1, \pm_2}(\xi_1, \xi_2)| \gtrsim \frac{|\xi_1||\xi_2|}{|\xi_1 + \xi_2|} \angle(\pm_1\xi_1, \pm_2\xi_2)^2.
\end{equation}

Furthermore, with any choice of signs, we have

\begin{equation}
(3.11) \quad |\mathcal{R}^{\pm_1, \pm_2}(\xi_1, \xi_2)| \gtrsim \min(|\xi_1|, |\xi_2|) \angle(\pm_1\xi_1, \pm_2\xi_2)^2.
\end{equation}
3.3. **Null structure.** Before discussion on the null structure, we note that the spatial part of vector potential \( A = (A_1, A_2, A_3) \) can be split into divergence-free and curl-free parts:

\[
A = A^d + A^c,
\]

where

\[
A^d = (-\Delta)^{-1} \nabla \times \nabla \times A, \\
A^c = (-\Delta)^{-1} \nabla (\nabla \cdot A).
\]

Also we define the Riesz transform given by

\[
R_j = D^{-1} \partial_j = \frac{\partial_j}{D}.
\]

Now we introduce the standard null forms:

\[
Q_0(u, v) = \partial_\alpha u \partial^\alpha v \\
Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v.
\]

Then we define a commutator version of null forms by

\[
Q_0[u, v] = [\partial_\alpha u, \partial^\alpha v] \\
Q_{\alpha\beta}[u, v] = [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v].
\]

Here we give some remark on commutator version of null forms. For \( su(n) \)-valued functions \( u \) and \( v \), we write \( u = u_a T^a \) and \( v = v_b T^b \), where \( a, b = 1, 2, \cdots, n^2 - 1 \) and \( u_a, v_b \) are smooth scalar functions. Then there holds

\[
Q_{\alpha\beta}[u, v] = [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] \\
= [\partial_\alpha u_a T^a, \partial_\beta v_b T^b] - [\partial_\beta u_a T^a, \partial_\alpha v_b T^b] \\
= \partial_\alpha u_a \partial_\beta v_b [T^a, T^b] - \partial_\beta u_a \partial_\alpha v_b [T^a, T^b] \\
= (\partial_\alpha u_a \partial_\beta v_b - \partial_\beta u_a \partial_\alpha v_b) [T^a, T^b] \\
= Q_{\alpha\beta}(u_a, v_b) [T^a, T^b] \\
= Q_{\alpha\beta}(u_a, v_b) i f^{ab}_{\ c} T^c.
\]

For the last equality see Appendix below. Then for a function space \( X'(\mathfrak{su}(n)) \) defined by the functions with value in \( \mathfrak{su}(n) \), we observe that

\[
\|Q_{\alpha\beta}[u, v]\|_{X'(\mathfrak{su}(n))} = \sum_c \|Q_{\alpha\beta}(u_a, v_b)\|_{X'(C)} |f^{ab}_{\ c}|.
\]

Hence we conclude that the \( X'(\mathfrak{su}(n)) \) norm of commutator version of null forms is reduced to the \( X'(\mathbb{C}) \) norm of null forms of scalar functions.

The following lemma is on null structure hidden in several bilinear terms. We adapt the notation by [26].

We define

\[
\Omega[u, v] = -\frac{1}{2} \epsilon^{ijk} \epsilon_{klm} Q_{ij}[R^i u^m, v] - Q_{0i}[R^i u_0, v],
\]

where \( R^i = D^{-1} \partial^i \) is the Riesz transform.
Lemma 3.4 (Lemma 1, Section 2.1. of [26]). In the Lorenz gauge, we have the identities

\begin{align}
[A^\mu, \partial_\mu \phi] &= \Omega [D^{-1} A, \phi] \\
[\partial_\mu A^\mu, \partial_\mu \phi] &= Q_0 [A^\mu, \phi].
\end{align}

Lemma 3.5 (Lemma 2, Section 2.2. of [26]). In the Lorenz gauge, we have the identity

\[ [A^\lambda, \partial_\mu A_\lambda] = \sum_{j=1}^{3} \Gamma^j_\mu (A, \partial A, F, \partial F), \]

where

\begin{align*}
\Gamma^1_\mu (A, \partial A, F, \partial F) &= -[A_0, \partial_\mu A_0] + [A^\mu, \partial_\mu A^\mu] \\
\Gamma^2_\mu (A, \partial A, F, \partial F) &= \frac{1}{2} \epsilon^{ijk} \epsilon_{klm} (Q_{ij} [D^{-1} R^n A_m, D^{-1} R^l \partial_\mu A_m] \\
&\quad + Q_{ij} [D^{-1} R^n \partial_\mu A^n, D^{-1} R^l A_m]), \\
\Gamma^3_\mu (A, \partial A, F, \partial F) &= [D^{-2} \nabla \times F, D^{-2} \nabla \times \partial_\mu F] \\
&\quad - [D^{-2} \nabla \times F, D^{-2} \partial_\mu \nabla \times (A \times A)] \\
&\quad - [D^{-2} \nabla \times (A \times A), D^{-2} \partial_\mu \nabla \times (A \times A)] \\
&\quad + [D^{-2} \nabla \times (A \times A), D^{-2} \partial_\mu \nabla \times (A \times A)].
\end{align*}

Here \( F = (F_{23}, F_{31}, F_{12}) \).

Proof. We write

\[ [A^\lambda, \partial_\mu A_\lambda] = -[A_0, \partial_\mu A_0] + [A^\mu, \partial_\mu A^\mu] = \sum_{j=1}^{3} \Gamma^j_\mu (A, \partial A, F, \partial F), \]

where

\begin{align*}
\Gamma^1_\mu (A, \partial A, F, \partial F) &= -[A_0, \partial_\mu A_0] + [A^\mu, \partial_\mu A^\mu], \\
\Gamma^2_\mu (A, \partial A, F, \partial F) &= [A^\mu, \partial_\mu A^\mu] + [A^\mu, \partial_\mu A^\mu], \\
\Gamma^3_\mu (A, \partial A, F, \partial F) &= [A^\mu, \partial_\mu A^\mu].
\end{align*}

We first consider \( \Gamma^1_\mu \):

\[ \Gamma^1_\mu = (-A_0 \partial_\mu A_0 + A^\mu \partial_\mu A^\mu) + (\partial_\mu A_0 A_0 - \partial_\mu A^\mu \partial_\mu A^\mu). \]

Using the Lorenz gauge: \( \partial_\mu A_0 = \nabla \cdot A \), we see that

\begin{align*}
-A_0 \partial_\mu A_0 + A^\mu \partial_\mu A^\mu &= -A_0 \partial_\mu A_0 + D^{-2} \nabla (\nabla \cdot A) \cdot \partial_\mu D^{-2} \nabla (\nabla \cdot A) \\
&= -A_0 \partial_\mu A_0 + D^{-2} \nabla (\partial_\mu A_0) \cdot D^{-2} \nabla (\partial_\mu A_0) \\
&= -A_0 \partial_\mu A_0 + D^{-1} R_j (\partial_\mu A_0) D^{-1} R^j \partial_\mu (\partial_\mu A_0).
\end{align*}

Similarly, we have

\[ \partial_\mu A_0 A_0 - \partial_\mu A^\mu \partial_\mu A^\mu = (\partial_\mu A_0) A_0 - D^{-1} R^j \partial_\mu (\partial_\mu A_0) D^{-1} R_j (\partial_\mu A_0). \]
Combining these, we get the desired identity for $\Gamma_1^\mu$. Now consider $\Gamma_2^\mu$. We let

$$B = -D^{-2}(\nabla \cdot A), \quad C = D^{-2} \nabla \times A.$$ 

Then $A^{cf} = \nabla B$ and $A^{df} = \nabla \times C$. Using this, we see that

$$A^{cf} \cdot \partial_\mu A^{df} = \nabla B \cdot (\nabla \times \partial_\mu C)$$

$$= (\nabla B)_i (\nabla \times \partial_\mu C)_i$$

$$= \partial_i B \epsilon^{ijk} \partial_j \partial_k C_k$$

$$= -\partial_i D^{-2}(\nabla \cdot A) \epsilon^{ijk} \partial_j \partial_k \epsilon_{klm} D^{-2} \partial^m A^m$$

$$= -\epsilon^{ijk} \epsilon_{klm} \partial_i D^{-1} R^m A^m \partial_j D^{-1} \partial_\mu R^l A_m$$

$$= -\frac{1}{2} \epsilon^{ijk} \epsilon_{klm} Q_{ij}(D^{-1} R^l A_m, D^{-1} R^m A_n).$$

Similarly, we get

$$\partial_\mu A^{df} \cdot A^{cf} = -\frac{1}{2} \epsilon^{ijk} \epsilon_{klm} Q_{ij}(D^{-1} R^l \partial_\mu A_m, D^{-1} R^m A_n).$$

The second term $A^{df} \cdot \partial_\mu A^{cf} - \partial_\mu A^{cf} \cdot A^{df}$ is treated similarly and we omit the details. Summing these identities gives the desired identity for $\Gamma_2^\mu$.

Finally, we recall (1.1) and write

$$\nabla \times A = F - A \times A,$$

where $F = (F_{23}, F_{31}, F_{12})$. Then we have

$$A^{df} = D^{-2}(\nabla \times F - \nabla \times (A \times A)).$$

Inserting this in place of $A^{df}$ gives the desired identity for $\Gamma_2^\mu$. \hfill \Box

Since $\Gamma^1$ is non-\(Q\)-type, we shall reveal its explicit cancellation. We write $\Gamma^1 = (\Gamma_0^1, \Gamma_1^1)$, $i = 1, 2, 3$. Then we see that

$$\Gamma_0^1 = -[A_0, \partial_i A_0] + [D^{-1} R_j (\partial_i A_0), D^{-1} R^i \partial_l (\partial_\mu A_0)]$$

$$= -[A_0, \partial^i A_i] + [D^{-1} R_j (\partial_\mu A_0), D^{-1} R^i \partial^j (\partial_\mu A_1)].$$

Here we only treat the first terms of each commutator:

$$-A_0 \partial^i A_i + D^{-1} R_j (\partial_i A_0) D^{-1} R^i \partial^j (\partial_\mu A_i) = -\sum_{\pm_1, \pm_2} (A_{0, \pm_1} \partial^i A_{i, \pm_2} + R_j (\pm_1 A_{0, \pm_1}) R^j (\pm_2 \partial^i A_{i, \pm_2}))$$

$$= -\sum_{\pm_1, \pm_2} \mathbf{P}_{\pm_1, \pm_2} (A_{0, \pm_1}, \partial^i A_{i, \pm_2}),$$

where

$$\mathbf{P}_{\pm_1, \pm_2} (u, v) = uv + R_j (\pm_1 u) R^j (\pm_2 v)$$

and its symbol is given by

$$p_{\pm_1, \pm_2} (\xi_1, \xi_2) = 1 - \frac{(\pm_1 \xi_1) \cdot (\pm_2 \xi_2)}{||\xi_1|| ||\xi_2||}.$$
Hence we have

\begin{align}
\Gamma_0 &= -\sum_{\pm,\pm_2} (P_{\pm,\pm_2}(A_{0,\pm_1}, \partial^i A_{i,\pm_2}) - P_{\pm,\pm_2}(\partial^i A_{i,\pm_1}, A_{0,\pm_2})) \\
\Gamma_i &= -\sum_{\pm,\pm_2} (P_{\pm,\pm_2}(A_{0,\pm_1}, \partial_i A_{0,\pm_2}) - P_{\pm,\pm_2}(\partial_i A_{0,\pm_1}, A_{0,\pm_2}))
\end{align}

and the symbol \(p_{\pm,\pm_2}\) satisfies the following estimate:

\[|p_{\pm,\pm_2}(\xi_1, \xi_2)| \lesssim |\zeta(\pm_1 \xi_1, \pm_2 \xi_2)|^2.\]

4. Local well-posedness: Proof of Theorem 1.1

To prove Theorem 1.1, we shall construct Picard iteration in the space of \(B^{s,b,\sigma}_{+,-}\). First we recall our nonlinear wave equations (2.2):

\[i(\partial_t \pm D)A_{\pm} = \mp \frac{1}{2D} M_e(A_+, A_-, F_+, F_-),\]

\[i(\partial_t \pm D)F_{\pm} = \mp \frac{1}{2D} \tilde{N}_{\mu \nu}(A_+, A_-, F_+, F_-).\]

Then by Duhamel’s principle the above system is equivalent to the following integral equations:

\begin{align}
A_{\nu,\pm}(t) &= A^{\text{hom}}_{\nu,\pm}(t) \mp \int_0^t e^{-i(t-t')} D \frac{1}{2iD} \tilde{M}_{\nu}(A_+, A_-, F_+, F_-)(t') dt', \\
F_{\mu \nu,\pm}(t) &= F^{\text{hom}}_{\mu \nu,\pm}(t) \mp \int_0^t e^{-i(t-t')} D \frac{1}{2iD} \tilde{N}_e(A_+, A_-, F_+, F_-)(t') dt',
\end{align}

where the homogeneous part is given by

\[\phi^{\text{hom}}_{\alpha,\pm} = \frac{1}{2} e^\pm iD \left( \phi\alpha(0, x) \mp \frac{1}{iD} \partial_\alpha(0, x) \right).\]

To construct Picard’s iterates \(A^{(n)}_{\nu,\pm}\) and \(F^{(n)}_{\mu \nu,\pm}\), we set \(A_{\nu,\pm}^{(0)}\) and \(F_{\mu \nu,\pm}^{(0)}\) to be identically zero, and \(A_{\nu,\pm}^{(0)}\) and \(F_{\mu \nu,\pm}^{(0)}\) are defined by \(A_{\nu,\pm}^{(0)}\) and \(F_{\mu \nu,\pm}^{(0)}\), respectively. Then the general inductive step \(A_{\nu,\pm}^{(n)}\) and \(F_{\mu \nu,\pm}^{(n)}\) are given by solving (4.1) and (4.2) on \(S_T\) with the previous step \(A_{\nu,\pm}^{(n-1)}\) and \(F_{\mu \nu,\pm}^{(n-1)}\) inserted on the right-hand side, respectively.

To prove the local well-posedness of our system, it suffices to show that the iterates \((A_{\nu,\pm}^{(n)}, F_{\mu \nu,\pm}^{(n)}, F_{\mu \nu,\pm}^{(n)})\) is a Cauchy sequence in the space \(\mathcal{B}^{s,b,\sigma}_{+,-} \times \mathcal{B}^{s,b,\sigma}_{+,-} \times \mathcal{B}^{s,b,\sigma}_{+,-} \times \mathcal{B}^{s,b,\sigma}_{+,-}.\) Thus using the standard contraction argument, the local well-posedness is followed by the following estintes:

\begin{align}
\|\tilde{M}_{\nu}(A_+, A_-, F_+, F_-)\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} &\lesssim \mathcal{S}(1 + \mathcal{S}^3), \\
\|\tilde{N}_{\mu \nu}(A_+, A_-, F_+, F_-)\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} &\lesssim \mathcal{S}(1 + \mathcal{S}^3).
\end{align}

Here \(\mathcal{S} = \sum_{\pm} (\|A_\pm\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} + \|F_\pm\|_{\mathcal{B}^{s,b,\sigma}_{+,-}})\) and \(\|A_\pm\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} = \sum_{\mu=0}^3 \|A_{\mu,\pm}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} + \|F_{\pm,\mu}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} = \sum_{\mu=0}^3 \|F_{\mu \nu,\pm}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}}.\)

We also need the estimates of homogeneous parts:

\begin{align}
\|\Omega^\sigma A^{\text{hom}}_{\nu,\pm}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} &\lesssim \|\langle \Omega \rangle^\sigma a\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} + \|\langle \Omega \rangle^\sigma \hat{a}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}}, \\
\|\langle \Omega \rangle^\sigma F^{\text{hom}}_{\mu \nu,\pm}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} &\lesssim \|\langle \Omega \rangle^\sigma f\|_{\mathcal{B}^{s,b,\sigma}_{+,-}} + \|\langle \Omega \rangle^\sigma \hat{f}\|_{\mathcal{B}^{s,b,\sigma}_{+,-}},
\end{align}

which can be easily shown. Thus we will focus on the proof of the estimates (4.3) and (4.4).
5. Bilinear forms: Terms of the form $[A^\lambda, \partial_\lambda \phi]$, $[\partial_\mu A^\lambda, \partial_\lambda \phi]$, and $[\partial^\lambda A_\mu, \partial_\lambda A_\nu]$.

5.1. Reduction to nonlinear estimates. Now we reduce the above estimates to nonlinear estimates. We have already observed in Section 3 that the matrix structure in null forms plays no crucial role in the estimates. Also, we claim that estimates of higher-order terms of $A_\mu F_{\mu\nu}$ are reduced to the nonlinear estimates of scalar functions. For example, we consider the cubic terms $[A^\mu_{\pm 1}, [A_{\mu, \pm 2}, A_{\nu, \pm 3}]]$. We write $A^\mu_{\pm 1} = A^\mu_{\pm 1,a} T^a$, $A_{\mu, \pm 2} = A_{\mu, \pm 2,b} T^b$, and $A_{\nu, \pm 3} = A_{\nu, \pm 3,c} T^c$. Then for a function space $\mathcal{X}(\mathfrak{su}(n))$, the norm of $[A^\mu_{\pm 1}, [A_{\mu, \pm 2}, A_{\nu, \pm 3}]]$ is given by

$$\| [A^\mu_{\pm 1}, [A_{\mu, \pm 2}, A_{\nu, \pm 3}]] \|_{\mathcal{X}(\mathfrak{su}(n))} = \sum_c \| A^\mu_{\pm 1,a} A_{\mu, \pm 2,b} A_{\nu, \pm 3,c} \|_{\mathcal{X}(\mathbb{C})} \| f^b_c \| f^a_d \|.$$

and thus the norm of cubic terms of $\mathfrak{su}(n)$-valued fields is reduced to the norm of cubic terms of scalar fields. Therefore, from now on, we consider the $\mathfrak{su}(n)$-valued functions $A_\mu F_{\mu\nu}$ as $\mathbb{C}$-valued functions.

In this section we give the proof of the required estimates for the bilinear forms which has the form $[A^\lambda, \partial_\lambda \phi]$, $[\partial_\mu A^\lambda, \partial_\lambda \phi]$, and $[\partial^\lambda A_\mu, \partial_\lambda A_\nu]$:

\begin{align}
(5.1) \quad & \| [A^\mu_{\pm 1}, \partial_\mu A_\nu] \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A^\mu_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A_\nu \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.2) \quad & \| [A^\mu_{\pm 1}, \partial_\mu F_{\mu\nu}] \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A^\mu_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| F_{\mu\nu} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.3) \quad & \| \partial_\mu A^\mu_{\pm 1}, \partial_\mu A_\nu \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A^\mu_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A_\nu \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.4) \quad & \| [\partial_\mu A^\mu_{\pm 1}, \partial_\mu A_\nu] \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A^\mu_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A_\nu \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.5) \quad & \| [\partial^\lambda A_\mu, \partial_\lambda A_\nu] \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A^\mu_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A^\nu_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}.
\end{align}

In view of Lemma 3.3 it suffices to prove the following:

\begin{align}
(5.6) \quad & \| Q_{ij}(D^{-1} A_{\pm 1}, A_{\pm 2}) \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A_{\pm 2} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.7) \quad & \| Q_{ij}(D^{-1} A_{\pm 1}, F_{\pm 2}) \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| F_{\pm 2} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.8) \quad & \| Q_{ij}(A_{\pm 1}, A_{\pm 2}) \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A_{\pm 2} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}, \\
(5.9) \quad & \| Q_0(A_{\pm 1}, A_{\pm 2}) \|_{B^{\pm \frac{1}{2}, \frac{3}{2}}_{2, T}} \lesssim \| A_{\pm 1} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}} \| A_{\pm 2} \|_{B^{\frac{1}{2}, \frac{3}{2}}_{2, T}}.
\end{align}

where we omit the Riesz transform $R^i$, since it is bounded in our function spaces.

To prove the above bilinear estimates via null forms for functions $u$ and $v$, we recall the substitution:

\begin{align}
&\begin{align*}
&u = u_+ + u_-, \quad \partial_t u = iD(u_+ - u_-), \\
&v = v_+ + v_-, \quad \partial_t v = iD(v_+ - v_-).
\end{align*}
\end{align}

Then we have

\begin{align}
Q_{j0}(u, v) &= \sum_{\pm 1, \pm 2} (\pm 1)(\pm 2) ((\pm 1 \partial_3 u_{\pm 1})(\pm 2 i D v_{\pm 2}) - (\pm 1 i D u_{\pm 1})(\pm 2 \partial_3 v_{\pm 2})) , \\
Q_{jk}(u, v) &= \sum_{\pm 1, \pm 2} (\pm 1)(\pm 2) ((\pm 1 \partial_3 u_{\pm 1})(\pm 2 \partial_4 v_{\pm 2}) - (\pm 1 \partial_3 u_{\pm 1})(\pm 2 \partial_4 v_{\pm 2})) , \\
Q_0(u, v) &= \sum_{\pm 1, \pm 2} (\pm 1)(\pm 2)(|D| u_{\pm 1}|D| v_{\pm 2} - (\pm 1 \partial^3) u_{\pm 1}(\pm 2 \partial_3) v_{\pm 2}) ,
\end{align}
and their symbols are given by
\[
q_{0}(\xi_{1}, \xi_{2}) = -\xi_{1,j}|\xi_{2}| + |\xi_{1}|\xi_{2,j},
\]
\[
q_{j,k}(\xi_{1}, \xi_{2}) = -\xi_{1,j}\xi_{2,k} + \xi_{1,k}\xi_{2,j},
\]
\[
q_{0}(\xi_{1}, \xi_{2}) = |\xi_{1}||\xi_{2}| - \xi_{1} \cdot \xi_{2}.
\]

For these symbols, we have the following estimates.

**Lemma 5.1** (Lemma 5, Section 5.2. of [18]). For \(\xi_{1}, \xi_{2} \in \mathbb{R}^{3}\) with \(\xi_{1}, \xi_{2} \neq 0\),
\[
|q_{j0}(\xi_{1}, \xi_{2})|, |q_{jk}(\xi_{1}, \xi_{2})| \lesssim |\xi_{1}||\xi_{2}| |\langle \xi_{1}, \xi_{2} \rangle|,
\]
\[
|q_{0}(\xi_{1}, \xi_{2})| \lesssim |\xi_{1}||\xi_{2}| |\langle \xi_{1}, \xi_{2} \rangle|^{2}.
\]

In view of Lemma 5.1 it suffices to prove the estimates (5.6), (5.7), and (5.8).

### 5.2. Proof of (5.6).
We first consider the following \(L_{t,x}^{2}\) norm:
\[
\|P_{K_{N},L_{0}}^{\phi_{0}} Q_{ij}(D^{-1}A_{1,2}, A_{2,2})\| \lesssim \sum_{N,L,l} \|P_{K_{N},L_{0}}^{\phi_{0}} Q_{ij}(D^{-1}A_{N1,L1}, A_{N2,L2})\|
\]
\[
\lesssim \sum_{N,L,l} \theta_{ij} N_{2}C_{N,L}^{0,12} \|A_{N1,L1,l1}\| \|\|A_{N2,L2,l2}\|
\]
\[
\lesssim \sum_{N,L,l} \theta_{ij} N_{2}C_{N,L}^{0,12} \|A_{N1,L1,l1}\| \|\|A_{N2,L2,l2}\|,
\]
where we write \(A_{N1,L1,l1} = P_{K_{N},L_{0}}^{\phi_{0}} A_{\pm}\) for brevity and we used Lemma 5.1. Here, \(C_{N,L}^{0,12}\) is (3.1) or (3.2) or (5.8). Thus we see that by definition of \(B_{\pm,0,\sigma}\) and Lemma 2.1 to prove the estimate (5.6), we need to show that
\[
J_{A}^{1}(N,L) := \sum_{N_{0},L_{0} \geq 1} N_{0}^{-\frac{1}{2}}L_{0}^{-\frac{1}{2}} \theta_{ij} N_{2}C_{N,L}^{0,12} \|A_{N1,L1,l1}\| \|\|A_{N2,L2,l2}\|
\]
\[
\lesssim (N_{1}N_{2})^{\frac{1}{2}}(L_{1}L_{2})^{\frac{1}{2}} \|\|A_{N1,L1,l1}\| \|\|A_{N2,L2,l2}\|.
\]
If we had the estimate (5.10), then the \((l_{1})\) summation on \(N_{1}, N_{2}\) and \(L_{1}, L_{2}\) and then \(l_{0}, l_{1}, l_{2}\) would give the required estimate (5.6). We also assume by symmetry \(L_{1} \leq L_{2}\). The standard Littlewood-Paley trichotomy shows that we have the following two important relations:
\[
N_{0} \ll N_{1} \approx N_{2},
\]
\[
N_{12} \lesssim N_{0} \approx N_{12}^{\min}.
\]
By Lemma 3.3 we note that in the high-high-low frequency; \(N_{0} \ll N_{1} \approx N_{2}\) we obtain better estimates than we required. Indeed, if \(|\xi_{0}| \ll |\xi_{1}| \approx |\xi_{2}|\), we must have \(\pm_{1} \neq \pm_{2}\) and hence \(\theta_{12} \approx 1\). Thus for this interaction, the null structure plays no crucial role and it gives us high modulation - low frequency; \(N \lesssim L\).

We use a volume estimate (5.8) instead of (3.1) and (3.2) and then
\[
J_{A}^{1}(N,L) \lesssim \sum_{N_{0},L_{0}} N_{0}^{-\frac{1}{2}}L_{0}^{-\frac{1}{2}} N_{2}(N_{0}^{3}R_{\min}^{0})^{\frac{1}{2}} \|A_{N_{1},L_{1},l_{1}}\| \|\|A_{N_{2},L_{2},l_{2}}\|
\]
\[
\lesssim \sum_{N_{0}} N_{0}N_{2}(N_{\min}^{12})^{-\frac{1}{2}}(L_{1}L_{2})^{\frac{1}{2}} \|A_{N_{1},L_{1},l_{1}}\| \|\|A_{N_{2},L_{2},l_{2}}\|.
\]
Hence in the following three cases we are only concerned with \(N_{\min}^{12} \ll N_{\max}^{12} \approx N_{0}\) and low modulation - high frequency; \(L \ll N\), since \(N \lesssim L\) gives us better estimate.
5.2.1. Case 1: $L_0 \ll L_2$.

$$J_1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{2}{5}} L_0^{-\frac{2}{5}} \left( \frac{L_0}{N_{\min}^{12}} \right)^{\frac{2}{5}} N_2 (N_{\min}^{012} N_0^{01} L_0 L_1)^{\frac{2}{5}} N_1^{\frac{1}{5}} L_1^{\frac{2}{5}} l_1^2 \| A_{N_1, L_1, l_1}^{\pm, 1} \| A_{N_2, L_2, l_2}^{\pm, 2} \| .$$

$$\lesssim \sum_{N_0, L_0} N_2 N_0^{-\frac{2}{5}} (N_{\min}^{01})^{\frac{1}{5}} l_0 L_0^{12} \| A_{N_1, L_1, l_1}^{\pm, 1} \| A_{N_2, L_2, l_2}^{\pm, 2} \| .$$

$$\lesssim (N_1 N_2)^{\frac{2}{5}} (L_1 L_2)^{\frac{2}{5}} l_1^2 \| A_{N_2, L_2, l_2}^{\pm, 2} \|. $$

5.2.2. Case 2: $L_2 \ll L_0$. We consider two cases: $l_1 \ll l_2$ and $l_2 \ll l_1$. For $l_1 \ll l_2$, apply Lemma 2.3 to $A_{\pm, 2}$ to obtain

$$J_1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{2}{5}} L_0^{-\frac{2}{5}} \left( \frac{L_0}{N_{\min}^{12}} \right)^{\frac{2}{5}} N_2 (N_{\min}^{012} N_0^{12} L_1 L_2)^{\frac{2}{5}} N_1^{-\frac{2}{5}} L_1^{\frac{2}{5}} l_1^2 \| A_{N_1, L_1, l_1}^{\pm, 1} \| A_{N_2, L_2, l_2}^{\pm, 2} \| .$$

$$\lesssim \sum_{N_0} \sum_{L_0 \leq N_1} N_2 N_0^{-\frac{2}{5}} (N_{\min}^{01})^{\frac{1}{5}} N_1^{\frac{1}{5}} l_0 L_0^{12} L_1^{\frac{2}{5}} l_1^2 \| A_{N_1, L_1, l_1}^{\pm, 1} \| A_{N_2, L_2, l_2}^{\pm, 2} \| .$$

$$\lesssim (N_1 N_2)^{\frac{2}{5}} (L_1 L_2)^{\frac{2}{5}} l_1 \| A_{N_1, L_1, l_1}^{\pm, 1} \| A_{N_2, L_2, l_2}^{\pm, 2} \|. $$

This completes the proof of (5.9).

**Remark 5.2.** We observe that in the high-high-low modulation or high-low-high modulation: $L_0 \lesssim L_2$, we don’t need to exploit angular regularity, Lemma 2.3. On the other hand, for the low-low-high modulation: $L_2 \ll L_0$, the existence of $L_0^{\frac{2}{5}}$ forces us to use angular regularity. One should note that $l_2 \ll l_1$ is the most serious interaction: for $l_1 \ll l_2$ we obtain even better estimate $(N_1 N_2)^{\frac{2}{5}}$.

5.3. Proof of (5.7). We write

$$\| Q_{ij}(D^{-1} A_{\pm, 1}, F_{\pm, 2}) \|_{L^{p, \frac{2}{5}}_{x_0, \theta_0}} = \sum_{N_0, L_0, l_0} N_0^{-\frac{2}{5}} L_0^{-\frac{2}{5}} l_0^\sigma \| P_{L_0}^{l_0} Q_{ij}(D^{-1} A_{\pm, 1}, F_{\pm, 2}) \| .$$

$$\lesssim \sum_{N, L, l} N_0^{-\frac{2}{5}} L_0^{-\frac{2}{5}} l_0^\sigma \| P_{L_0}^{l_0} Q_{ij}(D^{-1} A_{N_1, L_1, l_1}^{\pm, 1}, F_{N_2, L_2, l_2}^{\pm, 2}) \| .$$

$$\lesssim \sum_{N, L, l} N_0^{-\frac{2}{5}} L_0^{-\frac{2}{5}} l_0^\sigma N_2 \theta_{12} C_{N, L}^{012} \| A_{N_1, L_1, l_1}^{\pm, 1} \| \| F_{N_2, L_2, l_2}^{\pm, 2} \|. $$

To prove (5.7), it suffices to show the following estimate:

$$J_1(N, L) := \sum_{N_0, L_0} N_0^{-\frac{2}{5}} L_0^{-\frac{2}{5}} N_2 \theta_{12} C_{N, L}^{012} \| A_{N_1, L_1, l_1}^{\pm, 1} \| \| F_{N_2, L_2, l_2}^{\pm, 2} \| .$$

$$\lesssim N_1^{\frac{2}{5}} N_2^{\frac{2}{5}} (L_1 L_2)^{\frac{2}{5}} l_0 \| A_{N_1, L_1, l_1}^{\pm, 1} \| \| F_{N_2, L_2, l_2}^{\pm, 2} \|. $$
If $N_0 \ll N_1 \approx N_2$, then by (5.3.2) we have

\[
\mathcal{J}_F^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{2}{L_0} - \frac{1}{2}} \left( L_2^2 \frac{L_0}{N_{12} \min} \right)^{\frac{1}{2}} \left( N_0^{12} N_{10} L_0 \right)^{\frac{1}{2}} A_{N_1, L_1, l_1} \| F_{N_2, L_2, l_2}^{\pm 2} \|
\]

\[
\lesssim \sum_{N_0} \left( N_0^{12} N_{10} L_0 \right)^{\frac{1}{2}} A_{N_1, L_1, l_1} \| F_{N_2, L_2, l_2}^{\pm 2} \|
\]

\[
\lesssim N_1^3 N_2^{-\frac{2}{2}}, (L_1 L_2)^{\frac{7}{2}} L_2^2 || A_{N_1, L_1, l_1} || F_{N_2, L_2, l_2}^{\pm 2} ||.
\]

Now we consider $N_{12} \min \ll N_0 \approx N_{12} \max$.

5.3.2. Case 1: $L_0 \ll L_2$.

\[
\mathcal{J}_F^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{2}{L_0} - \frac{1}{2}} N_2 \left( L_2^2 \frac{L_0}{N_{12} \min} \right)^{\frac{1}{2}} \left( N_0^{12} N_{10} L_1 \right)^{\frac{1}{2}} A_{N_1, L_1, l_1} \| F_{N_2, L_2, l_2}^{\pm 2} \|
\]

\[
\lesssim \sum_{N_0, L_0} \left( N_0^{12} N_{10} L_1 \right)^{\frac{1}{2}} N_2^{-\frac{2}{L_0} - \frac{1}{2}} (L_1 L_2)^{\frac{7}{2}} L_1^2 A_{N_1, L_1, l_1} \| F_{N_2, L_2, l_2}^{\pm 2} \|
\]

\[
\lesssim N_1^3 N_2^{-\frac{2}{2}}, (L_1 L_2)^{\frac{7}{2}} L_2^2 || A_{N_1, L_1, l_1} || F_{N_2, L_2, l_2}^{\pm 2} ||.
\]

5.3.3. Case 3: $L_2 \ll L_0$, $l_2 \ll l_1$.

\[
\mathcal{J}_F^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{2}{L_0} - \frac{1}{2}} N_2 \left( L_0 \frac{L_0}{N_{12} \min} \right)^{\frac{1}{2}} \left( N_0^{12} N_{10} L_1 \right)^{\frac{1}{2}} N_2^{-\frac{2}{L_0} - \frac{1}{2}} (L_1 L_2)^{\frac{7}{2}} L_2^2 A_{N_1, L_1, l_1} \| F_{N_2, L_2, l_2}^{\pm 2} \|
\]

\[
\lesssim \sum_{N_0, L_0} \left( N_0^{12} N_{10} L_1 \right)^{\frac{1}{2}} N_2^{-\frac{2}{L_0} - \frac{1}{2}} (L_1 L_2)^{\frac{7}{2}} L_2^2 N_2^{-\frac{2}{L_0} - \frac{1}{2}} l_2 A_{N_1, L_1, l_1} \| F_{N_2, L_2, l_2}^{\pm 2} \|
\]

\[
\lesssim N_1^3 N_2^{-\frac{2}{2}}, (L_1 L_2)^{\frac{7}{2}} L_2^2 || A_{N_1, L_1, l_1} || F_{N_2, L_2, l_2}^{\pm 2} ||,
\]

where we used $N_0^{-\frac{2}{L_0}} \ll N_2^{-\frac{2}{L_0}}$, since for $N_2 \ll N_0$ or $N_2 \approx N_0$, we have $N_0^{-\delta} \ll N_2^{-\delta}$ for any $\delta > 0$.

This completes the proof of (5.7).

5.4. Proof of (5.8).

\[
\| Q_{ij}(A_{\pm 1}, A_{\pm 2}) \|_{L^p_{L_0}, L^p_{L_0}, L^p_{L_0}} = \sum_{N_0, L_0, l_0} N_0^{-\frac{2}{L_0} - \frac{1}{2}} l_0 \| P_{K_{N_0, L_0}}^L Q_{ij}(A_{\pm 1}, A_{\pm 2}) \|
\]

\[
\lesssim \sum_{N_L, L} N_0^{-\frac{2}{L_0} - \frac{1}{2}} l_0 \| P_{K_{N_0, L_0}}^L Q_{ij}(A^{\pm 1}_{N_1, L_1, l_1}, A^{\pm 2}_{N_2, L_2, l_2}) \|
\]

\[
\lesssim \sum_{N_L, L} N_0^{-\frac{2}{L_0} - \frac{1}{2}} l_0 N_1 N_2 \theta_{l_2} C_{N_L}^{12} \| A^{\pm 1}_{N_1, L_1, l_1} \| A^{\pm 2}_{N_2, L_2, l_2} \|.
\]
We need to show that
\[
\mathcal{J}_F^2(N, L) := \sum_{N_0, L_0} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} N_1 N_2 \theta_{12} C_{N,L}^{012} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim (N_1 N_2) \frac{2}{7} (L_1 L_2)^{\frac{1}{2}} l_{\min}^{12} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|.
\]

We first consider \( N_0 \ll N_1 \approx N_2 \).

\[
\mathcal{J}_F^2(N, L) \lesssim \sum_{L_0} N_1 N_2 L_0^{-\frac{1}{2}} (L_0^{012})^\frac{1}{2} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim N_1 N_2 (N_{\max}^{12})^{-\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim (N_1 N_2) \frac{2}{7} (L_1 L_2)^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|.
\]

We are left to treat \( N_{\min}^{12} \ll N_0 \approx N_{\max}^{12} \).

5.4.1. Case 1: \( L_0 \ll L_2 \). We write \( N_1 N_2 = N_{\max}^{12} N_{\min}^{12} \). Since \( N_0 \approx N_{\max}^{12} \), we see that

\[
\mathcal{J}_F^2(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} N_{\max}^{12} N_{\min}^{12} \left( \frac{L_2}{N_{\max}^{12}} \right)^{\frac{1}{2}} (N_{\max}^{12} N_{\min}^{12} L_0 L_1)^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim \sum_{N_0, L_0} N_0^{-\frac{3}{2}} N_{\max}^{12} N_{\min}^{12} L_0^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim (N_1 N_2) \frac{2}{7} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|.
\]

5.4.2. Case 2: \( L_2 \ll L_0, l_1 \ll l_2 \).

\[
\mathcal{J}_F^2(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} N_{\max}^{12} N_{\min}^{12} \left( \frac{L_2}{N_{\max}^{12}} \right)^{\frac{1}{2}} (N_{\max}^{12} N_{\min}^{12} L_0 L_1)^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim \sum_{N_0, L_0} \sum_{L_0 \ll N_{\max}^{12}} N_0^{-\frac{3}{2}} N_{\max}^{12} N_{\min}^{12} N_0^{-\frac{1}{2}} L_0^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} l_1 \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim \frac{N_{\min}^{12}}{2} (N_{\max}^{12})^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} l_1 \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|.
\]

5.4.3. Case 2: \( L_2 \ll L_0, l_2 \ll l_1 \).

\[
\mathcal{J}_F^2(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} N_{\max}^{12} N_{\min}^{12} \left( \frac{L_2}{N_{\max}^{12}} \right)^{\frac{1}{2}} (N_{\max}^{12} N_{\min}^{12} L_0 L_1)^{\frac{1}{2}} \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim \sum_{N_0, L_0} \sum_{L_0 \ll N_{\max}^{12}} N_0^{-\frac{3}{2}} N_{\max}^{12} N_{\min}^{12} N_0^{-\frac{1}{2}} L_0^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} l_2 \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|
\lesssim \frac{N_{\min}^{12}}{2} (N_{\max}^{12})^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} l_2 \| A_{N_1,L_1,l_1}^{\pm 1} \| \| A_{N_2,L_2,l_2}^{\pm 2} \|.
\]
6. Bilinear forms: Terms of the form \([A^\lambda, \partial_\mu A_\lambda]\) and \([\partial_\nu A^\lambda, \partial_\mu A_\lambda]\)

To the end of the proof of local well-posedness, we need to consider \(\Gamma^j, j = 1, 2, 3\) in Lemma 3.5. However, the treatment of \(\Gamma^1\) and \(\Gamma^2\) is exactly same as the previous section. Hence we focus on \(\Gamma^3\):

\[
\| [D^{-2} \nabla \times F, D^{-2} \nabla \times \partial_\nu F] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \| F_{\pm 1} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \| F_{\pm 2} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

\[
\| [D^{-2} \nabla \times F, D^{-2} \nabla \times \partial_\nu F] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \| F_{\pm 1} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \| F_{\pm 2} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

\[
\| [D^{-2} \nabla \times F, D^{-2} \nabla \times \partial_\nu \nabla \times (A \times A)] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \| F_{\pm 1} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \prod_{j=1}^{2} \| A_{\pm j} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

\[
\| [D^{-2} \nabla \times \partial_\mu F, D^{-2} \partial_\nu \nabla \times (A \times A)] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \| F_{\pm 1} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \prod_{j=1}^{2} \| A_{\pm j} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

\[
\| [D^{-2} \nabla \times \partial_\mu (A \times A), D^{-2} \nabla \times \partial_\nu F] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \| F_{\pm 1} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \prod_{j=1}^{2} \| A_{\pm j} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

\[
\| [D^{-2} \nabla \times (A \times A), D^{-2} \partial_\nu \nabla \times (A \times A)] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \prod_{j=1}^{4} \| A_{\pm j} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

\[
\| [D^{-2} \nabla \times \partial_\mu (A \times A), D^{-2} \partial_\nu \nabla \times (A \times A)] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \prod_{j=1}^{4} \| A_{\pm j} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

We also have higher-order terms. However, all of them are absorbed in the above estimates except for one:

\[
\| [A_{\pm 1}^\mu, [A_{\mu, \pm 2}, A_{\nu, \pm 3}]] \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \lesssim \| A_{\pm 1} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \| A_{\pm 2} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}} \| A_{\pm 3} \|_{B^{\frac{1}{2}, 2}_{\frac{1}{2}, \frac{3}{2}}},
\]

Even though the proof of (6.9) is similar to others, we shall treat it quite differently; instead of Bernstein’s inequality, we make the use of duality and study quadrilinear estimates. By duality, we don’t need to be worried about the loss of regularity. However, the proof is much more cumbersome, and this example shows the advantage of applying angular regularity. We leave its proof to Appendix.

6.1. Proof of (6.1). We write

\[
\| P_{K_{N_0, L_0}} f_{\pm 1} f_{\pm 2} \| \lesssim \sum_{N_1, L_1} N_1^{-1} C_{N_1, L_1}^{0, 12} \| f_{\pm 1}^{N_1, L_1, 1} \| \| f_{\pm 2}^{N_2, L_2, 1} \|.
\]

Thus to prove (6.1), we need to show that

\[
\mathcal{I}^{1}(N, L) := \sum_{N_0, L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} N_1^{-1} C_{N_1, L_1}^{0, 12} \| f_{\pm 1}^{N_1, L_1, 1} \| \| f_{\pm 2}^{N_2, L_2, 1} \| \lesssim (N_1 N_2)^{-\frac{1}{2}} (L_1 L_2)^{-\frac{1}{2}} \| f_{\pm 1}^{N_1, L_1, 1} \| \| f_{\pm 2}^{N_2, L_2, 1} \|.
\]
We first consider high-low-low interaction: \( N_0 \ll N_1 \approx N_2 \). Then by Proposition 3, we have

\[
\mathcal{T}^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} L_0^{-\frac{1}{2}} N_1^{-\frac{1}{2}} N_0^{\frac{1}{3} L_0^{(012)}} \left( L_{(012)}^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( N_1^{(012)} \right)^{\frac{1}{2}} F_{N_1, L_1, l_1}^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim \sum_{N_0} N_1^{-1} N_0 (L_{(012)}^{\frac{1}{2}})^{\frac{1}{2}} F_{N_1, L_1, l_1}^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim \sum_{N_0} N_1^{-1} N_0 (N_{(012)}^{\frac{1}{2}})^{\frac{1}{2}} (L_1 L_2)^{\frac{1}{2}} F_{N_1, L_1, l_1}^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim (N_1 N_2)^{-\frac{1}{3}} (L_1 L_2)^{\frac{1}{2}} F_{N_1, L_1, l_1}^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}.
\]

Hence we focus on high-low-high interaction: \( N_{(012)}^{\frac{1}{2}} \ll N_0 \approx N_{(012)}^{\frac{1}{2}} \max \).

6.1.1. Case 1: \( L_0 \ll L_2, \ l_1 \ll l_2 \).

\[
\mathcal{T}^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} L_0^{-\frac{1}{2}} N_1^{-1} (N_{(012)}^{\frac{1}{2}} N_{(012)}^{\frac{1}{2}} L_0 L_1)^{\frac{1}{2}} N_1^{-\frac{1}{2}} L_1^{\frac{1}{2}} l_1 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} N_1^{-\frac{1}{2}} N_0^{-1} (N_{(012)}^{\frac{2}{5}} N_{(012)}^{\frac{2}{5}} L_0 L_1)^{\frac{1}{2}} L_0^{\frac{1}{2}} L_1^{\frac{1}{2}} l_1 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim (N_1 N_2)^{-\frac{1}{3}} L_1^{\frac{2}{2}} L_2^{\frac{2}{2}} l_1 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}.
\]

6.1.2. Case 2: \( L_0 \ll L_2, \ l_2 \ll l_1 \).

\[
\mathcal{T}^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} L_0^{-\frac{1}{2}} N_1^{-1} (N_{(012)}^{\frac{1}{2}} N_{(012)}^{\frac{1}{2}} L_0 L_1)^{\frac{1}{2}} N_2^{-\frac{1}{2}} L_2^{\frac{1}{2}} l_2 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} N_2^{-\frac{1}{2}} N_1^{-\frac{1}{2}} (N_{(012)}^{\frac{2}{5}} N_{(012)}^{\frac{2}{5}} L_0 L_1)^{\frac{1}{2}} L_0^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} l_2 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim (N_1 N_2)^{-\frac{1}{3}} L_1^{\frac{2}{2}} L_2^{\frac{2}{2}} l_2 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}.
\]

6.1.3. Case 3: \( L_2 \ll L_0, \ l_1 \ll l_2 \).

\[
\mathcal{T}^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} L_0^{-\frac{1}{2}} N_1^{-1} (N_{(012)}^{\frac{1}{2}} N_{(012)}^{\frac{1}{2}} L_0 L_1)^{\frac{1}{2}} N_1^{-\frac{1}{2}} L_1^{\frac{1}{2}} l_1 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} N_1^{-\frac{1}{2}} N_1^{-1} (N_{(012)}^{\frac{2}{5}} N_{(012)}^{\frac{2}{5}} L_0 L_1)^{\frac{1}{2}} L_0^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} l_1 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim (N_1 N_2)^{-\frac{1}{3}} L_1^{\frac{2}{2}} L_2^{\frac{2}{2}} l_1 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}.
\]

6.1.4. Case 4: \( L_2 \ll L_0, \ l_2 \ll l_1 \).

\[
\mathcal{T}^1(N, L) \lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} L_0^{-\frac{1}{2}} N_1^{-1} (N_{(012)}^{\frac{1}{2}} N_{(012)}^{\frac{1}{2}} L_0 L_1)^{\frac{1}{2}} N_2^{-\frac{1}{2}} L_2^{\frac{1}{2}} l_2 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim \sum_{N_0, L_0} N_0^{-\frac{1}{3}} N_1^{-\frac{1}{2}} N_1^{-1} (N_{(012)}^{\frac{2}{5}} N_{(012)}^{\frac{2}{5}} L_0 L_1)^{\frac{1}{2}} L_0^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} l_2 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}
\]

\[
\lesssim (N_1 N_2)^{-\frac{1}{3}} L_1^{\frac{2}{2}} L_2^{\frac{2}{2}} l_2 \left( F_{N_1, L_1, l_1}^{\frac{1}{2}} \right)^{\frac{1}{2}} F_{N_2, L_2, l_2}^{\frac{1}{2}}.
\]

This completes the proof of (6.1).

Remark 6.1. We observe that in the proof of (6.1), without using angular regularity, the regularity we can attain is at most \(-\frac{1}{4}\). Indeed, suppose that the initial data \((a, f)\) is given by \(a \in \mathcal{H}^*\) and \(f \in \mathcal{H}'\). Since bilinear estimates give us essential term \(N^1\), we see that \(N^{s-1} N^{-1} N^1 \leq N^{2r}\) and hence we have the relation between \(s\) and \(r\): \(s - 1 \leq 2r\).
6.2. Proof of (6.2). As we have seen the proof of (6.1), to prove the estimate, we need to show that

\[ T^2(N, L) := \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N,L}^{-1/2} \|F_{N,L,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\| \]

\[ \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\|. \]

We can obtain better estimate for the high-high-low interaction: \( N_0 \ll N_1 \approx N_2 \). On the other hand, we have \( N_1 \ll N_0 \) or \( N_1 \approx N_0 \), and hence \( N_1 \lesssim N_0 \). This implies \( N_0^{-1} \lesssim N_1^{-1} \) and then we see that

\[ T^2(N, L) = \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N,L}^{-1/2} \|F_{N,L,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\| \]

\[ \lesssim \sum_{N,L} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} N_1^{-1} C_{N,L}^{-1/2} \|F_{N,L,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\| \]

\[ \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\|. \]

where the last inequality is one that we have already proven. This completes the proof of (6.2).

6.3. Proof of (6.3). We write

\[ \|A_{\pm_1} A_{\pm_2}^{-1} F_{x_0} \|_{B^{\frac{1}{2},-\frac{1}{2},p}_{\infty,0}} = \sum_{N_0,L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N,L}^{-1/2} \|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2}^{-1} F_{x_0})\| \]

\[ \lesssim \sum_{N_0,L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N,L}^{-1/2} \|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2})\|\|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2})\| \]

\[ \lesssim \sum_{N_0,L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N,L}^{-1/2} \|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2})\|\|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2})\| \]

\[ \lesssim \sum_{N_0,L_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{4}} C_{N,L}^{-1/2} \|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2})\|\|P_{K_{N_0},L_0} (A_{\pm_1} A_{\pm_2})\| \]

where to obtain the second inequality we used Bernstein’s inequality:

\[ (6.10) \quad \|P_{\xi} \|_{L^p(X^2)} \lesssim N^{d(\frac{1}{p} - \frac{1}{2})} \|P_{\xi} \|_{L^2(X^2)} \]

for \( q < p \leq \infty \). Thus we see that to prove (6.3), we need to show that

\[ T^3(N, L) := L_0^{-\frac{1}{2}} C_{N,L} \|A_{N_1,L_1,i,j}^{1/2}\|\|A_{N_2,L_2,i,j}^{1/2}\| \]

\[ \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\|. \]

6.3.1. Case 1: \( L_0 \ll L_2 \).

\[ T^3(N, L) \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\| \]

\[ \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\|. \]

6.3.2. Case 2: \( L_2 \ll L_0 \).

\[ T^3(N, L) \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\| \]

\[ \lesssim (N_1N_2)^{-1/2} (L_1L_2)^{1/2} l_{\min}^{-1} \|F_{N_1,L_1,i,j}^{1/2}\|\|F_{N_2,L_2,i,j}^{1/2}\|. \]
Thus we need to prove that

\[ \| A_{\pm,1} A_{\pm,2} F_{\pm,0} \|_{B_{\pm,0}^{2,4}} = \sum_{N_0, L_0, l_0} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} l_0^{-\frac{1}{2}} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \| \]

\[ \lesssim \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} l_0^{-\frac{1}{2}} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \|_{L^4} \| P^l_{K^{\pm,0}_{N_0,L_0}} F_{\pm,0} \|_{L^4} \]

\[ \lesssim \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} l_0^{-\frac{1}{2}} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \| \| P^l_{K^{\pm,0}_{N_0,L_0}} F_{\pm,0} \|_{L^4} \]

\[ \lesssim \sum_{N_0, L_0, l_0} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \| \| P^l_{K^{\pm,0}_{N_0,L_0}} F_{\pm,0} \|_{L^4} \]

\[ \lesssim \sum_{N_0, L_0, l_0} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \| \| P^l_{K^{\pm,0}_{N_0,L_0}} F_{\pm,0} \|_{L^4} \]

We note that the proof of (6.5), (6.6) are same as the proof of (6.4).

6.5. Proof of (6.7). We also use Bernstein’s inequality. Indeed,

\[ \| D^{-1} (A_{\pm,1} A_{\pm,2}) (A_{\pm,3} A_{\pm,4}) \|_{B_{\pm,0}^{2,4}} = \sum_{N_0, L_0, l_0} N_0^{-\frac{3}{2}} L_0^{-\frac{1}{2}} l_0^{-\frac{1}{2}} \| D^{-1} (A_{\pm,1} A_{\pm,2}) (A_{\pm,3} A_{\pm,4}) \| \]

\[ \lesssim \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} l_0^{-\frac{1}{2}} \| D^{-1} (A_{\pm,1} A_{\pm,2}) \|_{L^4} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,3} A_{\pm,4}) \|_{L^4} \]

\[ \lesssim \sum_{N_0, L_0, l_0} N_0^{-\frac{1}{2}} L_0^{-\frac{1}{2}} l_0^{-\frac{1}{2}} \| D^{-1} (A_{\pm,1} A_{\pm,2}) \|_{L^4} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,3} A_{\pm,4}) \|_{L^4} \]

\[ \lesssim \sum_{N_0, L_0, l_0} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \| \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,3} A_{\pm,4}) \|_{L^4} \]

\[ \lesssim \sum_{N_0, L_0, l_0} \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,1} A_{\pm,2}) \| \| P^l_{K^{\pm,0}_{N_0,L_0}} (A_{\pm,3} A_{\pm,4}) \|_{L^4} \]

Thus we need to prove that

\[ I^4(N, L) := \sum_{N_0, L_0} C_{N_0}^{0,12} C_{N_0}^{0,34} \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \]

\[ \lesssim (N_1 N_2 N_3 N_4)^{\frac{1}{2}} (L_1 L_2 L_3 L_4)^{\frac{1}{2}} \min_{min} \min_{l_0} \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \]

We also assume \( L_1 \leq L_2, L_3 \leq L_4 \).

6.5.1. Case 1: \( L_0 \ll L_2, L_4 \).

\[ I^4(N, L) \lesssim \sum_{N_0, L_0} (N_0^{12} N_0^{10})^{\frac{1}{2}} (N_0^{34} N_0^{3})^{\frac{1}{2}} L_0^{\frac{1}{2}} \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \|

\[ \lesssim (N_1 N_2 N_3 N_4)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{2}} \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \|

6.5.2. Case 2: \( L_2 \ll L_0 \ll L_4 \).

\[ I^4(N, L) \lesssim \sum_{N_0, L_0} (N_0^{12} N_0^{10})^{\frac{1}{2}} (N_0^{34} N_0^{3})^{\frac{1}{2}} L_0^{\frac{1}{2}} \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \|

\[ \lesssim (N_1 N_2 N_3 N_4)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{2}} \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \| \| A_{N_0, L_0, l_0} \|.
6.5.3. Case 3: \( L_4 \ll L_0 \ll L_2 \).

\[
\mathcal{I}^4(N, L) \lesssim \sum_{N_0, L_0} (N_0^2 L_0^4)^{\frac{1}{2}} (N_0^4 L_0^3)^{\frac{1}{2}} L_0^2 \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2} \right\| \left\| A_{N_3, L_3, l_3} \right\| \left\| A_{N_4, L_4, l_4} \right\|
\]

\[
\lesssim (N_1 N_2 N_3 N_4)^{\frac{1}{2}} (L_1 L_3 L_4)^{\frac{1}{2}} L_2^2 \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2} \right\| \left\| A_{N_3, L_3, l_3} \right\| \left\| A_{N_4, L_4, l_4} \right\|
\]

6.5.4. Case 4: \( L_2, L_4 \ll L_0, l_1 \ll l_2 \).

\[
\mathcal{I}^4(N, L) \lesssim \sum_{N_0, L_0} (N_0^2 L_0^4)^{\frac{1}{2}} (N_0^4 L_0^3)^{\frac{1}{2}} L_0^2\left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2} \right\| \left\| A_{N_3, L_3, l_3} \right\| \left\| A_{N_4, L_4, l_4} \right\|
\]

\[
\lesssim (N_1 N_2 N_3 N_4)^{\frac{1}{2}} (L_1 L_3 L_4)^{\frac{1}{2}} L_1^2 \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2} \right\| \left\| A_{N_3, L_3, l_3} \right\| \left\| A_{N_4, L_4, l_4} \right\|
\]

6.5.5. Case 5: \( L_2, L_4 \ll L_0, l_2 \ll l_1 \).

\[
\mathcal{I}^4(N, L) \lesssim \sum_{N_0, L_0} (N_0^2 L_0^4)^{\frac{1}{2}} (N_0^4 L_0^3)^{\frac{1}{2}} \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2} \right\| \left\| A_{N_3, L_3, l_3} \right\| \left\| A_{N_4, L_4, l_4} \right\|
\]

\[
\lesssim (N_1 N_2 N_3 N_4)^{\frac{1}{2}} (L_1 L_3 L_4)^{\frac{1}{2}} L_2^2 \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2} \right\| \left\| A_{N_3, L_3, l_3} \right\| \left\| A_{N_4, L_4, l_4} \right\|
\]

This completes the proof of (6.7). We also note that the proof of (6.8) is same.

Remark 6.3. Although we used Lemma 2.3 only once to \( A_{\pm 2} \) in Case 5, it is possible to apply Lemma 2.3 to \( A_{\pm 3} \), or \( A_{\pm 4} \), to obtain sharp estimate. Hence we obtain sharp estimates of the higher-order terms of (1.7) and (1.9).

This is the end of the proof of Theorem 1.1.

7. Appendix

7.1. Proof of (6.9). In this section, to show (6.9), instead of Bernstein’s inequality, which we enjoy in the treatment of higher-order terms, we shall make the use of duality to get quadrilinear estimate. Indeed, we consider the following \( L^2 \) norm:

\[
\left\| P_{K_{N_4, L_4}}^{2} A_{\pm 1} A_{\pm 2} A_{\pm 3} \right\| = \sup_{\| \varphi \| = 1} \left| \int A_{\pm 1} A_{\pm 2} A_{\pm 3} \varphi_{N_4, L_4} dx \right|
\]

\[
\lesssim \sum_{N_0, L_0} \sup_{\| \varphi \| = 1} \left| \int A_{N_1, L_1, l_1}^{\pm} A_{N_2, L_2, l_2}^{\pm} A_{N_3, L_3, l_3}^{\pm} \varphi_{N_4, L_4} dx \right|
\]

\[
\lesssim \sum_{N_0, L_0} \sup_{\| \varphi \| = 1} \left| \int P_{K_{N_0, L_0}}^{2} (A_{N_1, L_1, l_1}^{\pm} A_{N_2, L_2, l_2}^{\pm}) || P_{K_{N_0, L_0}}^{2} (A_{N_3, L_3, l_3}^{\pm} \varphi_{N_4, L_4}) \right|
\]

\[
\lesssim \sum_{N_0, L_0} C_{K_{N_0, L_0}}^{012} C_{N_0, L_0}^{014} \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2}^{\pm} \right\| \left\| A_{N_3, L_3, l_3}^{\pm} \right\|
\]

To show (6.9), it suffices to prove that

\[
\mathcal{K}_{A}^{4}(N, L) := \sum_{N_0, L_0} \sum_{L_0} (N_0^4)^{\frac{1}{2}} L_0^{\frac{1}{2}} C_{N, L}^{012} C_{N, L}^{014} \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2}^{\pm} \right\| \left\| A_{N_3, L_3, l_3}^{\pm} \right\|
\]

\[
\lesssim (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{2}} L_1^2 \left\| A_{N_1, L_1, l_1}^{\pm} \right\| \left\| A_{N_2, L_2, l_2}^{\pm} \right\| \left\| A_{N_3, L_3, l_3}^{\pm} \right\|
\]

We give a remark on the angular regularity.
Remark 7.1. In dealing with quadrilinear estimate, we essentially consider the following spherical integral:

$$\sum_{l} \int_{S^2} y_1(\omega)y_2(\omega)y_3(\omega)y_4(\omega)dS(\omega).$$

Then by Littlewood-Paley trichotomy, in general we are concerned with

$$l_{012}^{001} \sim l_{012}^{001} \geq l_{012}^{001}, \quad l_{034}^{001} \sim l_{034}^{001} \geq l_{034}^{001}.$$ 

However, there are several cases which must be excluded. For example, if $l_0 \ll l_1 \ll l_0 \ll l_0 \ll l_1 \ll l_4$, the orthogonality of spherical harmonics forces it should be $l_{12}^{001} \sim l_{34}^{001}$. We also note that the cases $l_1 \ll l_0 \ll l_2 \ll l_3 \ll l_4$ and $l_2 \ll l_0 \ll l_1 \ll l_3 \ll l_4$ cannot appear in the summation of angular regularity.

By symmetry, we assume $L_1 \leq L_2$ and $L_3 \leq L_4$.

7.1.1. Case 1: $L_0 \ll L_2$, $L_0 \ll L_4$. This case is treated straightforwardly.

$$\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_0^{012} N_4^{012} L_0 L_1) \frac{1}{L_1} \frac{1}{L_4} (N_0^{034} N_4^{034} L_0 L_3) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|,$$

$$\lesssim \sum_{N_0, N_4} (N_0^{012} N_4^{012} N_0^{034} N_4^{034}) \frac{1}{L_1} \frac{1}{L_4} (L_1 L_2) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\| \lesssim (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{3}} A_{N_1, L_1, L_1} \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|.$$

7.1.2. Case 2: $L_4 \ll L_0 \ll L_2$. This is also straightforward. Indeed,

$$\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_0^{012} N_4^{012} L_0 L_1) \frac{1}{L_1} \frac{1}{L_4} (N_0^{034} N_4^{034} L_3 L_4) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|,$$

$$\lesssim \sum_{N_0, N_4} (N_0^{012} N_4^{012} N_0^{034} N_4^{034}) \frac{1}{L_1} \frac{1}{L_4} (L_1 L_2) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\| \lesssim (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{3}} A_{N_1, L_1, L_1} \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|.$$

7.1.3. Case 3: $L_2 \ll L_0$, $L_4 \ll L_0$. By (3.1), we get

$$\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} N_4^{-\frac{3}{4}} L_4^{-\frac{1}{4}} (N_0^{012} N_4^{012} L_0 L_2) \frac{1}{L_1} \frac{1}{L_4} (N_0^{034} N_4^{034} L_3 L_4) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|,$$

$$\lesssim \sum_{N_0, N_4} (N_0^{012} N_4^{012} N_0^{034} N_4^{034}) \frac{1}{L_1} \frac{1}{L_4} (L_1 L_2) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\| \lesssim (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{3}} A_{N_1, L_1, L_1} \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|.$$

If $l_1 \ll l_2$, we use Lemma 2.3 to $A_{\pm 1}$ and we get

$$\mathcal{K}_A^1(N, L) \lesssim \sum_{L_0, L_4} (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{3}} L_4^{-\frac{1}{3}} L_1 \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|,$$

$$\lesssim (N_0^{12}) \frac{1}{L_1} \frac{1}{L_4} (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2) \frac{1}{L_3} \frac{1}{L_4} \|A_{N_1, L_1, L_1}\| \|A_{N_2, L_2, L_2}\| \|A_{N_3, L_3, L_3}\|.$$
7.1.4. **Case 4**: $L_2 \ll L_0 \ll L_4$

\[
\mathcal{K}_A(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( N_{\min}^{12} N_{\min}^{03} L_1 L_2 \right)^{\frac{1}{2}} \left( N_{\min}^{03} N_{\min}^{03} L_0 L_3 \right)^{\frac{1}{2}} \left\| A_{N_1, L_1, l_1}^{1, 1} \right\| \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| \left\| A_{N_3, L_3, l_3}^{1, 3} \right\|.
\]

If $L_0 \ll L_3$, then we are done. Now consider $L_3 \ll L_0$. We further divide this into $L_2 \ll L_3$ and $L_3 \ll L_2$. We first consider $L_3 \ll L_2 \ll L_0 \ll L_4$. If $l_3 \ll l_4$, then

\[
\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} \left( N_{\min}^{12} N_{\min}^{03} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( L_1 L_2 L_3 \right)^{\frac{1}{2}}
\]

\[
\times L_d^+ L_0^+ N_3^{-\frac{1}{2}} L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}
\]

\[
\lesssim \sum_{N_0, N_4} \left( N_{\min}^{12} N_{\min}^{03} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( L_1 L_2 L_3 \right)^{\frac{1}{2}}
\]

\[
\times \sum_{L_4 \ll L_3} L_d^+ N_3^{-\frac{1}{2}} L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}
\]

\[
\lesssim (N_1 N_2) L_3^2 N_3^{-\frac{1}{2}} \left( L_1 L_2 \right)^{\frac{1}{2}} L_2^+ L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}.
\]

For $l_3 \ll l_2$, we further consider $l_1 \ll l_2$ and $l_2 \ll l_1$. Let us consider first $l_1 \ll l_2$ and $l_4 \ll l_3$. We apply Lemma 2.8 on $\varphi$ to get

\[
\left\| \mathcal{D}_{K_0, l_0}^1 \right\| \ll C_{N, N_1} L^{-\frac{1}{2}}_4 L_4^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| \varphi^{\frac{1}{4}}_{N_4, L_4, l_4} \right\|
\]

and also on $A_{l_1}$. We see that

\[
\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} \left( N_{\min}^{12} N_{\min}^{03} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( L_1 L_2 L_3 \right)^{\frac{1}{2}}
\]

\[
\times L_d^+ L_0^+ N_3^{-\frac{1}{2}} L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}
\]

\[
\lesssim \sum_{N_0, N_4} \left( N_{\min}^{12} N_{\min}^{03} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( L_1 L_2 L_3 \right)^{\frac{1}{2}}
\]

\[
\times \sum_{L_4 \ll L_3} \sum_{L_4 \ll L_3} \left( N_1 N_2 \right)^{-\frac{1}{2}} L_0^+ L_4^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}
\]

\[
\lesssim (N_1 N_2 N_3) L_3^2 L_3^+ L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}.
\]

Now we are left to treat $l_2 \ll l_1$ and $l_4 \ll l_3$. Note that in this case our assumption $L_3 \ll L_2$ plays no role and hence we postpone this case to $L_2 \ll L_3$. We assume $L_2 \ll L_3 \ll L_0 \ll L_4$. If $l_3 \ll l_4$, then

\[
\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} \left( N_{\min}^{12} N_{\min}^{03} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( L_1 L_2 L_3 \right)^{\frac{1}{2}}
\]

\[
\times L_d^+ L_0^+ N_3^{-\frac{1}{2}} L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}
\]

\[
\lesssim \sum_{N_0, N_4} \left( N_{\min}^{12} N_{\min}^{03} \right)^{\frac{1}{2}} N^{-\frac{1}{2}}_4 L^{-\frac{1}{2}}_4 \left( L_1 L_2 L_3 \right)^{\frac{1}{2}}
\]

\[
\times \sum_{L_4 \ll L_3} \left( N_1 N_2 \right)^{-\frac{1}{2}} L_0^+ L_4^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}
\]

\[
\lesssim (N_1 N_2) L_3^2 \left( L_1 L_3 \right)^{\frac{1}{2}} L_2^+ L_3^+ \left[ A_{N_1, L_1, l_1}^{1, 1} \right] \left\| A_{N_2, L_2, l_2}^{1, 2} \right\| A_{N_3, L_3, l_3}^{1, 3}.
\]
For \( l_4 \ll l_3 \) with \( l_1 \ll l_2 \),
\[
\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} (N_{12} N_{03} N_{01} N_{12}) \frac{1}{2} N_4^{-\frac{1}{2}} (N_{03} N_{01} N_{12}) \frac{1}{2} (L_1 L_2 L_3) \frac{1}{2} \\
\times L_4^{-\frac{1}{2}} L_0^\frac{1}{2} (N_1 N_4)^{-\frac{1}{2}} (L_1 L_4) \frac{1}{2} l_1 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \| \\
\lesssim \sum_{N_0, N_4} (N_{12} N_{03} N_{01} N_{12}) \frac{1}{2} N_4^{-\frac{1}{2}} (N_{03} N_{01} N_{12}) \frac{1}{2} (L_1 L_2) \frac{1}{2} L_3 \frac{1}{2} \\
\times \sum_{L_0, L_4} \sum_{N_4} (N_1 N_4)^{-\frac{1}{2}} L_0^\frac{1}{2} L_4^\frac{1}{2} l_1 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \| \\
\lesssim (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_2) \frac{1}{2} L_3 \frac{1}{2} l_1 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \|.
\]

Finally, we deal with \( l_2 \ll l_1 \) and \( l_4 \ll l_3 \) which gives the most serious interaction.
\[
\mathcal{K}_A^1(N, L) \lesssim \sum_{N_0, N_4} \sum_{L_0, L_4} (N_{12} N_{03} N_{01} N_{12}) \frac{1}{2} N_4^{-\frac{1}{2}} (N_{03} N_{01} N_{12}) \frac{1}{2} (L_1 L_2 L_3) \frac{1}{2} \\
\times L_4^{-\frac{1}{2}} L_0^\frac{1}{2} (N_2 N_4)^{-\frac{1}{2}} (L_2 L_4) \frac{1}{2} l_2 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \| \\
\lesssim \sum_{N_0, N_4} (N_{12} N_{03} N_{01} N_{12}) \frac{1}{2} N_4^{-\frac{1}{2}} (N_{03} N_{01} N_{12}) \frac{1}{2} (L_1 L_3) \frac{1}{2} L_2 \frac{1}{2} \\
\times \sum_{N_2, N_4} (N_2 N_4)^{-\frac{1}{2}} (L_2 L_4) \frac{1}{2} l_2 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \| \\
\lesssim \sum_{N_0, N_4} (N_{12} N_{03} N_{01} N_{12}) \frac{1}{2} N_4^{-\frac{1}{2}} (N_{03} N_{01} N_{12}) \frac{1}{2} (L_1 L_3) \frac{1}{2} L_2 \frac{1}{2} \\
\times \sum_{L_0, L_4} \sum_{N_4} (N_2 N_4)^{-\frac{1}{2}} (L_2 L_4) \frac{1}{2} l_2 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \| \\
\lesssim (N_1 N_2 N_3)^{\frac{1}{2}} (L_1 L_3) \frac{1}{2} L_2 \frac{1}{2} l_2 l_4 \| A_{N_1, L_1, l_1} \| \| A_{N_2, L_2, l_2} \| \| A_{N_3, L_3, l_3} \|.
\]

This completes the proof of (6.9).

Remark 7.2. One should note that although we don't apply Bernstein's inequality to avoid the loss of regularity, the assumption \( b = \frac{3}{4} \) forces us to use angular regularity. By the problem of summation of angular regularity, we must obtain \( l_4 \ll l_3 \). Hence in the case \( l_4 \ll l_1, l_4 \ll l_3, \) and \( L_2 \ll L_0 \ll L_4 \), which is the most serious interaction, we can obtain at most critical Besov norm.

7.2. Proof of Lemma 2.1. Let \( G \in B^{s, -\frac{1}{4}} \) be an arbitrarily chosen representative of \( F \in B^{s, -\frac{1}{4}} (S_T) \). Then we need to show that \( \| v \|_{B_{\infty}^{s, -\frac{1}{4}} (S_T)} \lesssim T^\frac{1}{4} \| f \|_{B_{\infty}^{s, -\frac{1}{4}}} + \| G \|_{B^{s, -\frac{1}{4}} (S_T)} \). By density, we may assume \( G \in S(\mathbb{R}^{1+3}) \). Since \( \mathcal{F}[e^{itD} f] = \delta(t \pm |\xi|) \hat{f}(\xi) \), by taking a compactly supported smooth function \( \rho \) with value 1 on \([-1, 1]\) and \( \rho(t) = \rho(\frac{1}{t}) \), we have
\[
\| e^{itD} f \|_{B_{\infty}^{s, -\frac{1}{4}} (S_T)} = \| \rho T e^{itD} f \|_{B_{\infty}^{s, -\frac{1}{4}}} = \sum_{N, L} N^s L^\frac{1}{2} |\chi| |\xi| N^s |\partial\tau| (\tau \pm |\xi|) \hat{f}(\xi) \| L^2_{\tau}, \xi \\
\lesssim \sum_{N, L} N^s L^\frac{1}{2} \| P_{|\tau| N^s} \| |\partial\tau| \| f \|_{B_{\infty}^{s, -\frac{1}{4}}} \lesssim \| f \|_{B_{\infty}^{s, -\frac{1}{4}}}.
\]

We write \( w(t) = \int_0^t e^{it(t-t')} |\xi| G(t', \xi) dt' \) and by taking the spatial Fourier transform we get
\[
\hat{w}(t, \xi) = \int_0^t e^{it(t-t')} \hat{G}(t', \xi) \frac{1}{2\pi} \int_0^t \frac{e^{it\lambda} - e^{-it|\xi|}}{i(\lambda \pm |\xi|)} \hat{G}(\lambda, \xi) d\lambda.
\]
Thus we have
\[ \tilde{w}(\tau, \xi) = \frac{\tilde{G}(\tau, \xi)}{i(\tau \pm |\xi|)} - \delta(\tau \pm |\xi|) \int \frac{\tilde{G}(\lambda, \xi)}{i(\lambda \pm |\xi|)} d\lambda. \]

Now we split \( G = G_1 + G_2 \) corresponding to the Fourier domains: \( |\tau + |\xi|| \gg 1 \) and \( |\tau + |\xi|| \lesssim 1 \), respectively. We write \( w = w_1 + w_2 \) accordingly. By Taylor’s expansion we get
\[
\tilde{w}_1(t) = e^{\mp i|\xi|} \sum_{n \geq 1} \frac{(it(\lambda \pm |\xi|))^n}{n!} \chi_{|\tau + |\xi|| \lesssim 1} \tilde{G}(\lambda, \xi) d\lambda.
\]

Hence we write \( w_1(t) = \sum_{n \geq 1} \frac{t^n}{n!} e^{\mp itD} f_n \), where
\[
\tilde{f}_n(\xi) = \int (i(\lambda \pm |\xi|))^{n-1} \chi_{|\lambda \pm |\xi|| \leq 1} \tilde{G}(\lambda, \xi) d\lambda.
\]

Now from the support condition \( \{ |\lambda \pm |\xi|| \leq 1 \} \) we see that \( \| f_n \|_{B^s_{2,1}} \lesssim \| G \|_{B^s_{2,-1}} \). Thus we get
\[
\| w_1 \|_{B^s_{2,1}(S_T)} \lesssim \| \rho_T w_1 \|_{B^s_{2,0}} \lesssim \sum_{n \geq 1} \frac{1}{n!} \| t^n \rho_T(t) e^D f_n \|_{B^s_{2,0}}
\]
\[
\leq \sum_{n \geq 1} \frac{1}{n!} \| t^n \rho_T(t) \|_{B^s_{2,1}} \| f_n \|_{B^s_{2,1}} \lesssim \left( \sum_{n \geq 1} \frac{n^{2n}}{n!} \right) \| G \|_{B^s_{2,-1}} \lesssim \| G \|_{B^s_{2,-1}},
\]
where we used \( T \leq 1 \) and \( \| t^n \rho_T \|_{B^s_{2,1}} \lesssim \| t^n \rho(t) \|_{H^1} \lesssim n^{2n} \). Next, we deal with the estimate of \( w_2 \). To do this, we first split \( w_2 = a - b \) where
\[
\tilde{a}(\tau, \xi) = \frac{\chi_{|\tau \pm |\xi|| \gg 1} \tilde{G}(\tau, \xi)}{i(\tau \pm |\xi|)},
\]
\[
\tilde{b}(\tau, \xi) = \delta(\tau \pm |\xi|) \int \frac{\chi_{|\lambda \pm |\xi|| \gg 1} \tilde{G}(\lambda, \xi)}{i(\lambda \pm |\xi|)} d\lambda.
\]

Thus
\[
\| a \|_{B^s_{2,0}(S_T)} \lesssim \| \rho_T a \|_{B^s_{2,0}} \lesssim \sum_{N \geq 1} \sum_{L \geq 1} L^s L^{-1} \| P_{K_{N,L}}^+ G \| \lesssim \| G \|_{B^s_{2,-1}},
\]
\[
\| b \|_{B^s_{2,0}(S_T)} \lesssim \| \rho_T b \|_{B^s_{2,0}} \lesssim \sum_{N \geq 1} \sum_{L \geq 1} L^s L^{-1} \| P_{K_{N,L}}^+ G \| \lesssim \| G \|_{B^s_{2,-1}}.
\]

This completes the proof of Lemma 2.1.

7.3. Notation from Quantum Field Theory. Let \( G \) be a compact Lie group and \( \mathfrak{g} \) its Lie algebra. In this paper we have assumed \( G = SU(n, \mathbb{C}) \), \( n \geq 2 \) and then \( \mathfrak{g} = \mathfrak{su}(n, \mathbb{C}) \) for the sake of simplicity. We introduce the set of infinitesimal generators which are traceless Hermitian matrices denoted by \( \{ T^a \}_{a=1}^{n^2-1} \). Then given \( \mathfrak{g} \)-valued function \( u \) we shall denote it by \( u = u_a T^a \). We also introduce so-called structure coefficients \( f^a_{bc} \) of given Lie algebra \( \mathfrak{g} \) such that \( [T^a, T^b] = i f^a_{bc} T^c \). More generally, for \( \mathfrak{g} \)-valued functions \( A \), we have
\[
[A_a T^a, A_b T^b] = i A_a A_b f^a_{bc} T^d.
\]

The readers may find more discussion on non-abelian gauge symmetry in [19] and [28].
Acknowledgments

This research is supported in part by NRF-2018R1D1A3B07047782, NRF-2018R1A2B2006298, and NRF-2016K2A9A2A13003815. I would like to thank Professor Yonggeun Cho, who encouraged me to study the Cauchy problem of the Yang-Mills system. I am truly appreciative of Professor Sebastian Herr, mentor during my stay at Bielefeld, for fruitful discussion and significant contribution to this work. I am also grateful to all IRTG 2235 members in Bielefeld and staff of Zentrum für interdisziplinäre Forschung der Universität Bielefeld for kind hospitality, where this work was completed.

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