\textbf{$l_0$ Sparse Inverse Covariance Estimation}

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\textbf{Abstract—}Recently, there has been focus on penalized log-likelihood covariance estimation for sparse inverse covariance (precision) matrices. The penalty is responsible for inducing sparsity, and a very common choice is the convex $l_1$ norm. However, the best estimator performance is not always achieved with this penalty. The most natural sparsity promoting “norm” is the non-convex $l_0$ penalty but its lack of convexity has deterred its use in sparse maximum likelihood estimation. In this paper we consider non-convex $l_0$ penalized log-likelihood inverse covariance estimation and present a novel cyclic descent algorithm for its optimization. Convergence to a local minimizer is proved, which is highly non-trivial, and we demonstrate via simulations the reduced bias and superior quality of the $l_0$ penalty as compared to the $l_1$ penalty.

\textbf{Index Terms}—sparsity, inverse covariance, log-likelihood, $l_0$ penalty, $l_1$ penalty, non-convex optimization

I. INTRODUCTION

Graphical models have a long history [1]–[3] and provide a systematic way of analyzing dependencies in high dimensional data. The structure of the graph identifies meaningful interactions among the data variables. When the data is Gaussian with mean $0_{p \times 1}$ and covariance $\Sigma_{p \times p}$, the graphical model is an undirected graph specified by the non-zeros in the precision (inverse covariance) matrix $\Omega = \Sigma^{-1}$. In this Gaussian case the graph captures conditional dependency (Markovian) properties of the variables: the absence of an edge between nodes $i$ and $j$, $i \neq j$, in the graph reflects conditional independence of variables $i$ and $j$ given the other variables. Letting $\omega_{ij}$ denote the $ij$-th component of $\Omega$, this in turn corresponds to having $\omega_{ij} = 0$, [1]–[3].

Following the parsimony principle, the estimation objective is to choose the simplest model, i.e., the sparsest graph that adequately explains the data. The sparsity requirement improves the interpretability of the model and reduces over-fitting. In order to estimate a sparse $\Omega$, much attention has been given to minimizing a sparsity Penalized Log-Likelihood (PLL) objective function. The log-likelihood promotes goodness-of-fit of the estimator while the penalty promotes many of its entries to become zero.

Even though the $l_0$ “norm” is the natural sparsity promoting penalty, the $l_1$ norm has become its dominant replacement. The primary justification is the convexity of the $l_1$ penalty and this has resulted in its widespread use in sparse linear regression [4]. As the $l_1$-PLL objective function is convex, convex optimization approaches can be applied to obtain sparse penalized Maximum-Likelihood (ML) estimators. As a result, there has been extensive research in the development of efficient methods for solving the $l_1$-PLL problem. Examples include [5]–[16], and an overview is given [17], [18]. These methods range from cyclic descent type algorithms [5], [7], [9], [14], to alternating linearization algorithms [8], [10], [11], and projected sub-gradient methods [15]. Newton-type methods that incorporate cyclic descent, conjugate gradient as well as iterative shrinkage methods [19], are considered in [12], [13].

Despite the high popularity of the $l_1$ norm in sparsity penalized ML estimation problems, it has certain drawbacks. One drawback is that $l_1$ penalization induces shrinkage of the parameter estimates, which introduces negative biases [20]–[23]. Another drawback is that for very sparse problems $l_1$-PLL does not produce sufficiently sparse estimates [20], [22], [24], [25], resulting in the recovery of less parsimonious models. Hence, it is natural to ask the question: can the $l_0$ penalized estimator of inverse covariance provide improvement over the $l_1$ penalized estimator? The $l_0$ penalty has been considered in other sparsity penalized problem formulations, for example, in sparse linear regression [26]–[32], sparse signal recovery [29], PCA and low rank matrix completion [22], [33], [34]. The $l_0$ penalty induces maximum sparsity and would be expected to have superior prediction accuracy relative to $l_1$ penalized PLL, especially for very sparse $\Omega$.

In this paper we develop an algorithm for solving the non-convex $l_0$-PLL problem for inverse covariance estimation. We propose a novel Cyclic Descent (CD) algorithm to implement the optimization. We prove convergence of the algorithm to a local minimizer of the $l_0$-PLL objective function.

CD algorithms developed for optimizing the $l_1$-PLL objective function are proposed in [5]–[9], [12]. The GLASSO method in [5] and its variant in [6] are block-type CD procedures, which are derived using duality arguments and convergence analysis is performed using convexity arguments. The method in [7] applies the CD procedure to the elements of the Cholesky decomposition of each iterate. The SINCO method in [9] is a greedy-type algorithm derived using an equivalent reformulation of the $l_1$-PLL problem by exploiting the piecewise linearity of the $l_1$ penalty. The ALM algorithm in [8] uses linearization to find solutions of the objective function surrogates, which are updated in an alternating fashion. These iterates eventually converge to a single solution. The QUIC algorithm in [12] is a quasi-Newton type method, which applies an efficient CD procedure on a second order approximation of the $l_1$-PLL objective function. Inexact line search is then used to achieve descent. QUIC is a special case of the Newton-type methods proposed in [13]. To minimize the second order approximation, [13] also considers the nonlinear conjugate gradient method and the FISTA algorithm from [19]. The latter

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\textsuperscript{1}The $l_q$ function is not a norm for $q < 1$. 

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is a Majorization-Minimization or a proximal-type method. A monotone version of FISTA, called M-FISTA, from [35] can also be considered to improve stability.

Due to non-linearity and non-convexity of the $l_0$-PLL objective function, we cannot exploit any of the above ideas to derive $l_0$-PLL algorithms and analyze their convergence. Alternating linearization procedures are extremely hard to analyse in the non-convex setting, and could result in unstable algorithms if applied blindly. Furthermore, we cannot exploit second order approximations because the inexact line-search algorithms if applied blindly. Additionally, we cannot exploit any of the above ideas for the trace of $X$ for the determinant of $X$, and could result in unstable minimizers of (3).

Lastly, the objective function is obtained by replacing the $l_0$ penalty in (3) by the $l_1$ norm of the matrix entries, i.e., by:

$$\|X\|_1 = \sum_{i=1}^{p} \sum_{j=1}^{p} |x_{ij}|$$

see [5]–[16].

An important question is whether the solution of the $l_1$-PLL problem, with some tuning parameter $\mu > 0$, is also a minimizer of (3). The answer is no, as given in the following theorem:

**Theorem 1.** Suppose $\hat{\Omega}_1(\mu)$ is a global minimizer of the $l_1$-PLL objective function with tuning parameter $\mu > 0$. Denote the set of all local minimizers of (3) by $S_{l_0}(\lambda)$. Then $\hat{\Omega}_1(\mu) \notin S_{l_0}(\lambda)$ for any $\mu > 0$.

**Proof.** See Appendix B.

Since all global minimizers are also local minimizers, Theorem 1 implies that any solution of the $l_1$-PLL problem will not be a global minimizer of (3). As a result, this theorem motivates a different approach to minimizing (3).

### IV. Algorithm Development

In this section we derive a Coordinate Descent (CD) algorithm for finding local minima of (3).

The basic concept of the algorithm is to fix all entries except for one selected entry of the current (symmetric) iterate $X_0 > 0$. $\mathcal{L}(\cdot)$ is then minimized with respect to (w.r.t.) the selected entry. Once the new value of this entry is calculated, $X_0$ is updated and $\mathcal{L}(\cdot)$ is minimized w.r.t. the next selected entry. The update equation is:

$$Z_{ij}(x_{ij}) = X_{0} + \begin{cases} (x_{ii} - x_{0,ii})e_i e_i^T & \text{if } i = j \\ (x_{ij} - x_{0,ij})U_{ij}U_{ji}^T & \text{otherwise} \end{cases}$$

where $U_{ij}$ is defined in (1). For what follows we define:

$$Y_0 = X_0^{-1}$$

as well as:

$$\phi_{x_{0,ij}}(x_{ij}) = -\log \det(Z_{0,ij}(x_{ij})) + s_{ij}x_{ij}$$

for any $i, j$. We will also rely on the standard determinant and matrix inverse identities given in Appendix A.
A. Element-wise Minimizers of $\mathcal{L}(\cdot)$ when $i = j$

The minimizers of $\mathcal{L}(X_{0,ii}(x))$ are given by:

$$
\begin{align*}
\arg\min_x \phi_{X_{0,ii}}(x) &= \arg\min_x \phi_{X_{0,ii}}(x) \\
&= \arg\min_x \phi_{X_{0,ii}}(x) \\
&= \arg\min_x \mathcal{L}(Z_{0,ii}(x)),
\end{align*}
$$

where $\phi_{X_{0,ii}}(\cdot)$ is defined in (7). Noting that $\phi_{X_{0,ii}}(\cdot)$ is differentiable, the minimizers are given by solving the equation:

$$
\frac{\partial \phi_{X_{0,ii}}}{\partial x}(x) = -[Z_{0,ii}(x)^{-1}][x] + s_{ii} = 0.
$$

We substitute $\delta = \delta(x)$ and $X = Z_{0,ii}(x)$ in the matrix inverse identity (34) to obtain:

$$
[Z_{0,ii}(x)^{-1}][x] = \frac{y_{0,ii}}{1 + \delta(x) y_{0,ii}},
$$

(9)

So, substituting (9) in (8) and solving for $x_{ii}$, the (unique) minimizer is given by:

$$
m_{ii} = x_{0,ii} + \frac{y_{0,ii} - s_{ii}}{y_{0,ii} s_{ii}}.
$$

(10)

We lastly need to check that $Z_{0,ii}(m_{ii}) > 0$, i.e., is invertible. By observing (31) or (34), this requires that $1 + \delta(m_{ii})y_{0,ii} > 0$, which can easily be confirmed.

B. Element-wise Minimizers of $\mathcal{L}(\cdot)$ when $i \neq j$

The minimizers of $\mathcal{L}(X_{0,ij}(x))$ are given by:

$$
\begin{align*}
\arg\min \phi_{X_{0,ij}}(x) &= \arg\min \phi_{X_{0,ij}}(x) \\
&= \arg\min \phi_{X_{0,ij}}(x) \\
&= \arg\min \mathcal{L}(Z_{0,ij}(x)),
\end{align*}
$$

where $\phi_{X_{0,ij}}(\cdot)$ is again defined in (7). In this case, $\phi_{X_{0,ij}}(\cdot)$ has a single discontinuity at $x = 0$ but only if $0$ is in the domain of $\phi_{X_{0,ij}}(\cdot)$, i.e., if $\det(Z_{0,ij}(0)) > 0$. Otherwise, $\phi_{X_{0,ij}}(\cdot)$ would be continuous everywhere. The continuous (and differentiable) part of $\phi_{X_{0,ij}}(\cdot)$ is given by:

$$
c_{X_{0,ij}}(x_{ij}) = -\log \det(Z_{0,ij}(x_{ij})) + 2s_{ij}x_{ij} + 2\lambda.
$$

(11)

First consider the case that $\det(Z_{0,ij}(0)) > 0$, in which case we can equivalently express $\phi_{X_{0,ij}}(\cdot)$ as:

$$
\phi_{X_{0,ij}}(x_{ij}) = c_{X_{0,ij}}(x_{ij})[x_{ij} \neq 0] + (c_{X_{0,ij}}(x_{ij}) - 2\lambda)[x_{ij} = 0].
$$

(12)

Now we see that the minimizers of $\phi_{X_{0,ij}}(\cdot)$ are the minimizers of $c_{X_{0,ij}}(\cdot)$ or $x_{ij} = 0$. Since $c_{X_{0,ij}}(\cdot)$ is strictly convex, it has a unique minimizer obtained as the solution to:

$$
c'_{X_{0,ij}}(x) = -2[Z_{0,ij}(x)^{-1}][x] + 2s_{ij} = 0.
$$

(13)

Substituting $\delta = \delta(x)$ and $X = Z_{0,ij}(x)$ into the matrix inverse identity (35), we obtain:

$$
[Z_{0,ij}(x)^{-1}][x] = \frac{-\Delta_{0,ij}\delta(x) + y_{0,ij}}{-\Delta_{0,ij}\delta(x)^2 + 2y_{0,ij}\delta(x) + 1},
$$

where

$$
\Delta_{0,ij} = \Delta_{ij}(Y_0) > 0,
$$

(15)

and $\Delta_{ij}(\cdot)$ is given by (33). Substituting (14) into (13) and solving for $x$, the (unique) minimizer is:

$$
m_{ij} = y_{0,ij} + \frac{y_{0,ij}}{-\Delta_{0,ij}\delta(x)^2 + 2y_{0,ij}\delta(x) + 1}.
$$

(16)

when $s_{ij} = 0$, $Z_{0,ij}(m_{ij}) > 0$ since by (32) $-\Delta_{0,ij}\delta(m_{ij})^2 + 2y_{0,ij}\delta(m_{ij}) + 1 > 0$.

When $s_{ij} \neq 0$, by substituting (14) into (13), (16) is equivalent to:

$$
\Delta_{0,ij}s_{ij}\delta(x_{ij})^2 - (\Delta_{0,ij} + 2y_{0,ij}s_{ij})\delta(x_{ij}) + (y_{0,ij} - s_{ij}) = 0.
$$

The discriminant of the above quadratic equation is $\Delta_{0,ij}^2 + 4s_{ij}y_{0,ij}y_{0,ij} > 0$, and so, there are two solutions. However, only one of these, given by:

$$
m_{ij} = x_{0,ij} + \frac{y_{0,ij} + \Delta_{0,ij} - \sqrt{\Delta_{0,ij}^2 + 4s_{ij}y_{0,ij}y_{0,ij}}}{2a_{0,ij}s_{ij}},
$$

(17)

yields $-\Delta_{0,ij}\delta(m_{ij})^2 + 2y_{0,ij}\delta(m_{ij}) + 1 > 0$, i.e., $Z_{0,ij}(m_{ij}) > 0$. Note that, from L'Hôpital’s rule, (17) approaches (16) as $s_{ij} \to 0$.

Lastly, $\det(Z_{0,ij}(0)) \leq 0$ implies that the (unique) minimizer of $\phi_{X_{0,ij}}(\cdot)$ in (7) is equal to $m_{ij}$.

The above results are summarized in the following theorem:

**Theorem 2.** When $i \neq j$, the minimizers $\hat{x}_{ij}$ of $\phi_{X_{0,ij}}(\cdot)$ in (7) satisfy:

- when, $\det(Z_{0,ij}(0)) \leq 0$:

$$
\hat{x}_{ij} = m_{ij},
$$

(18)

- when, $\det(Z_{0,ij}(0)) > 0$:

$$
\hat{x}_{ij} = \begin{cases} 
0 & \text{if } \phi_{X_{0,ij}}(0) < \phi_{X_{0,ij}}(m_{ij}) \\
\{0, m_{ij}\} & \text{if } \phi_{X_{0,ij}}(0) = \phi_{X_{0,ij}}(m_{ij}) \\
\{m_{ij}\} & \text{if } \phi_{X_{0,ij}}(0) > \phi_{X_{0,ij}}(m_{ij}),
\end{cases}
$$

(19)

where $m_{ij} = m_{ij}(x_{ij})$ is given by (16) when $s_{ij} = 0$, and is given by (17) otherwise, and $\phi_{X_{0,ij}}(\cdot)$ is given by (12).

C. Dealing with $\det(X(0))$, $\phi_{X_{0,ij}}(0)$ and $\phi_{X_{0,ij}}(m_{ij})$

Computing (18) and (19) requires two operations:

(a) comparing 0 to $\det(Z_{0,ij}(0))$

(b) comparing $\phi_{X_{0,ij}}(0)$ to $\phi_{X_{0,ij}}(m_{ij})$

Even though all the mentioned quantities contain $\det(Z_{0,ij}(\cdot))$, (a) and (b) must be done efficiently without explicitly calculating the determinant.

For (a), we substitute $\delta = -x_{0,ij}$ and $X = X_0$ into the determinant identity (32) and, since $\det(X_0) > 0$,

$$
-\Delta_{0,ij}x_{0,ij}^2 - 2y_{0,ij}x_{0,ij} + 1 > 0.
$$

(20)

For (b), we again substitute $\delta = -x_{0,ij}$ and $X = X_0$ into (32) to obtain an expression for $\phi_{X_{0,ij}}(0)$, i.e.,

$$
\phi_{X_{0,ij}}(0) = -\log \det(Z_{0,ij}(0)) = -\log \det(X_0) - \log \{\Delta_{0,ij}x_{0,ij}^2 - 2y_{0,ij}x_{0,ij} + 1\}.
$$

(21)
Then, substituting \( \delta = m_{ij} - x_{0,ij} \) and \( X = X_0 \) in (32) we obtain an expression for \( \phi_{X_0,ij}(m_{ij}) \)

\[
\phi_{X_0,ij}(m_{ij}) = c_{X_0,ij}(m_{ij}) = -\log \det(X_0) - \log ( -\Delta_{0,ij} \delta(m_{ij})^2 + 2y_{0,ij} \delta(m_{ij}) + 1) + 2s_{ij}m_{ij} + 2\lambda. \tag{22}
\]

When comparing \( \phi_{X_0,ij}(0) \) to \( \phi_{X_0,ij}(m_{ij}) \), expressions (21) and (22) lead to an expression that is minimized without the need for explicit calculation of any matrix determinants.

### D. Updating \( Y_0 = X_0^{-1} \)

Since \( Y_0 \) is needed to compute the entry update (10), (16) and (17), \( Y_0 \) needs to be updated as well. An efficient way to do this is to use the matrix inverse identities (34) and (35) with substitutions \( \delta = \delta(m_{ij}) \) and \( X = X_0 \).

After every off-diagonal entry update the proposed CD algorithm needs to compute a new matrix inverse \( Y_0 \), which requires \( O(p^2) \) multiplications. As a result, there are order \( \frac{1}{2}p^2 \times O(p^2) \) multiplications for each matrix sweep. Now, note that if:

\[
x_{0,ij} = 0 \quad \text{and} \quad \phi_{x_0,ij}(0) \leq \phi_{x_0,ij}(m_{ij}), \tag{23}
\]

then there is no change in \( X_0 \), and hence \( Y_0 \) would not need to be updated. In practice, the sparser the problem we are dealing with the larger the set of entries that satisfy (23) becomes, resulting in a smaller ("active") set of entries for which \( Y_0 \) is updated. Thus, the \( \frac{1}{2}p^2 \) factor in the inverse updating can in practice be reduced to something close to just half the number of off-diagonal non-zeros in \( X_0 \): a much smaller number.

**Remark 1.** To make sure that the size of the "active" set is small the CD algorithm should be initialized with a very sparse matrix, e.g., a diagonal matrix.

### E. Coordinate Descent (CD) Algorithm for the \( l_0 \) Penalized Log-Likelihood (l₀-PLL) Problem

Here we state the CD algorithm for minimizing (3).

1) **Initialization:** From [36, Theorem 3] we know that a necessary condition for existence of a solution to (3) is \( 1/s_{ii} > 0 \). In order to guarantee a small active set, following Remark 1, we initialize the CD algorithm with \( 1/s_{ii} \) for every \( i = 1, \ldots, p \).

2) **Updating the Entries:** Note that only the diagonal entries and only half of the off-diagonal entries need to be updated. Denote the set of indices of all these entries by \( S_A \), which is easily computed off-line. For very large and very sparse problems the CD algorithm can be sped-up by only updating the non-zero components after a sufficiently large number of matrix sweeps. Updating only a subset of entries per matrix sweep is used for CD algorithm speed-ups for minimizing the convex \( l_1 \)-PLL objective function [12, 13].

### The Coordinate Descent (CD) Algorithm

1) Suppose \( X^k = [x^k_{ij}] \) and \( Y^k = (X^k)^{-1} = [y^k_{ij}] \) are the current iterates (symmetric).

2) Let \( X_0 = X^k \) and \( Y_0 = Y^k \), and for each \( (i, j) \in S_A \), repeat (i) to (vi):

   (i) \( m_{ij}^k = m^k_{ij}(x^k_{ij}) \) is set according to:
   - (10) if \( i = j \).
   - (16) if \( i \neq j \) and \( s_{ij} = 0 \).
   - (17) if \( i \neq j \) and \( s_{ij} \neq 0 \).

   (ii) If \( -\Delta_{ij}(x^k_{ij})^2 - 2y^k_{ij}x^k_{ij} + 1 > 0 \) and \( i \neq j \), compute:

   \[
   A(x^k_{ij}) = \begin{cases} 
   0 & \text{if } \phi_{X^k_{ij}}(0) < \phi_{X^k_{ij}}(m_{ij}^k) \\
   m_{ij}^k \mathbb{I}(x^k_{ij} \neq 0) & \text{if } \phi_{X^k_{ij}}(0) = \phi_{X^k_{ij}}(m_{ij}^k) \\
   m_{ij}^k & \text{if } \phi_{X^k_{ij}}(0) > \phi_{X^k_{ij}}(m_{ij}^k)
   \end{cases}
   \tag{24}
   \]

   (iii) If \( -\Delta_{ij}(x^k_{ij})^2 - 2y^k_{ij}x^k_{ij} + 1 \leq 0 \) or \( i = j \), compute:

   \[
   A(x^k_{ij}) = m_{ij}^k
   \tag{25}
   \]

   (iv) Update \( x^k_{ij} \) (and \( x_{ij}^k \) if \( i \neq j \)) with:

   \[
   x^k_{ij} = A(x^k_{ij})
   \tag{26}
   \]

   (v) Denote the matrix with the updated \( x^k_{ij} \) by \( X^{k+1} \). Then, calculate \( Y^{k+1} = (X^{k+1})^{-1} \) using the Sherman Morrison Woodbury formula:

   \[
   \delta = x^{k+1}_{ij} - x^k_{ij}
   \]

   If \( \delta \neq 0 \), then:
   - for \( i = j \):
     \[
     Y^{k+1}_{ij} = Y^k_{ij} - \frac{y^k_{ij} X^k_{ij}}{1 + \delta y^k_{ij}}
     \]
   - for \( i \neq j \):
     \[
     [y^k_{[i]} y^k_{[j]}] \begin{bmatrix} 1 + \delta y^k_{ij} & -\delta y^k_{ij} \\ -\delta y^k_{ij} & 1 + \delta y^k_{ij} \end{bmatrix} [y^k_{[i]} y^k_{[j]}]
     \]

   (vi) Increment the counter \( k \) by 1.

(3) Go to (1).

**Remark 2.** The map \( A(x_{0,ij}) \) depends on \( X_0 \) in step (2) of the algorithm as well as indices \( ij \). It is given by the element-wise minimizer \( \tilde{x}_{ij} \) in (18) and (19). Since in (19) we see that there are two minimizers 0 and \( m_{ij} \), we have set \( A(x_{0,ij}) \) to 0 when the current value is 0, and to \( m_{ij} \) otherwise. The motivation for this choice is Theorem 3 in the next section.

### V. Convergence Analysis

Convergence of CD methods for sparse and general problems have been previously analysed [5], [7], [9], [14], [39–43]. The analysis in [41]–[43] holds only for convex functions, and is not applicable. Convergence has been proved in [40] under weaker convexity assumptions. However, these assumptions do not hold for the \( l_0 \)-PLL problem. Lastly, the global convergence theorem in [41], [44] fails because \( \mathcal{L}(\cdot) \) is not continuous, furthermore the lack of differentiability prevents us from using any analysis in [39].
In the following convergence analysis we firstly use the algorithm map \( A(\cdot) \) to show that the fixed points of the algorithm are strict local minimizers. Then, under two necessary conditions it is subsequently shown that the whole sequence converges to a single local minimizer.

**Remark 3.** The statement \( x^k_{ij} \to x^*_{ij} \) as \( k \to \infty \) applies to the fixed \( ij \)-th entry of \( X^k \). Due to the cyclic nature of the CD algorithm, this means that \( k \) is a function of \( (i,j) \), i.e., \( k = k(i,j) = i + (j-1)p + p^2r \) and \( r = 0, 1, 2, \ldots \). For example, if the size of \( X^k \) is \( p = 4 \) and we focus on entry \((3,2)\), then \( k = 7, 23, 39, \ldots, \infty \) corresponds to the iterations where this entry is updated. In order to simplify notation the iteration counter in \( x^k_{ij} \) will simply be denoted by \( k \), noting that we actually mean \( k(i,j) \). Since the fixed \( ij \)-th entry in statement \( x^k_{ij} \to x^*_{ij} \) is arbitrary, the statement is therefore equivalent to the statement \( X^k \to X^* \).

The set of fixed points of the algorithm is defined as:

\[
F = \bigcap_{ij} F_{ij}, \quad \text{where } F_{ij} = \{ X > 0 : x_{ij} = A(x_{ij}) \}. \tag{27}
\]

where \( F_{ij} \) is the set of positive definite matrices that satisfy the fixed point equation \( x_{ij} = A(x_{ij}) \). The definition of \( A(\cdot) \) in (24) asserts that \( X^k \) converges to a fixed point \( X^* \) of \( A(\cdot) \):

**Theorem 3.** If \( x^k_{ij} \to x^*_{ij} \) as \( k \to \infty \), then \( x^k_{ij} = A(x^*_{ij}) \), i.e., \( X^* \in F \).

**Proof.** See Appendix B.

The following theorem establishes that the fixed points are isolated points and hence strict local minimizers of (3):

**Theorem 4.** \( X \in F \) is a strict local minimizer of \( \mathcal{L}(\cdot) \). Specifically, there exists \( \epsilon > 0 \) such that for any symmetric \( \Delta = [\delta_{ij}] \) satisfying \( 0 < \| \Delta \|_F < \epsilon \):

\[
\mathcal{L}(X) < \mathcal{L}(X + \Delta). \tag{28}
\]

**Proof.** See Appendix B.

Theorems 3 and 4 imply that a convergent algorithm must converge to a local minimizer.

Next, consider the following two assumptions:

(A1). Assume there exists a \( K > 0 \) and \( \alpha \in (0, \infty) \) such that \( X^k \preceq \alpha I \) for all \( k > K \).

(A2). For any subsequence \{\( x^{k_n}_{ij} \)\} such that \( \lim_{n \to \infty} x^{k_n}_{ij} \in \mathcal{X}_{\alpha} = \{ x_{0,ij} : i \neq j, \phi_{x_{0,ij}}(0) = \phi_{x_{0,ij}}(m_{ij}) \} \), assume:

(a) \( x^{k_n}_{ij} = 0 \ \forall n > N \) implies \( x^{k_n+1}_{ij} = 0 \ \forall n > N \),

(b) \( x^{k_n}_{ij} \neq 0 \ \forall n > N \) implies \( x^{k_n+1}_{ij} \neq 0 \ \forall n > N \),

where \( N > 0 \).

**Remark 4.** (A1) implies that \{\( X^k \)\} has limit points. Observe that the set \( \mathcal{X}_{\alpha} \) defined in (A2) is of measure zero. Condition (A2) is obviously much weaker than the statement: \( x^{k_n+1}_{ij} - x^{k_n}_{ij} \to 0 \) as \( n \to \infty \), which is a necessary condition for algorithm convergence and is proved in Proposition 3. (A2) will hold if we have that \( x^{k+1}_{ij} - x^{k}_{ij} \to 0 \) as \( k \to \infty \), which is much easier to check in practice, but is an overly strong assumption.

We have the following convergence theorem:

**Theorem 5.** If (A1) and (A2) hold then \( X^k \to X^* \) as \( k \to \infty \), where \( X^* \) is a local minimizer of \( \mathcal{L}(\cdot) \).

**Proof.** See Appendix B.

The proof of Theorem 5 requires several propositions and lemmas given in Appendix B. We note some of those propositions here and provide a short summary of how they are used. In Proposition 3 we show that the difference of the successive iterates converges to zero, a necessary convergence condition. Then, in Proposition 4 we show that the limit points of the algorithm sequence are fixed points. Ostrowski’s result from [45] with Propositions 3 and 4 can subsequently be used to establish that the algorithm sequence converges to a closed and connected subset of fixed points, which is Proposition 5. By Theorem 4, the set of fixed points is a discrete set of local minimizers, and hence the connected subset to which the algorithm sequence converges must be comprised of a single point only, establishing Theorem 5.

VI. Simulations

Here the performance of the \( l_0 \) and \( l_1 \) penalized estimators \( \Omega \) of the true precision matrix \( \Omega_{p \times p} \) are compared. For the \( l_0 \) penalized estimator we use the proposed CD algorithm, while the \( l_1 \) penalized estimator is obtained using the \( l_1 \) COV algorithm from [36], [37] with \( q = 1 \), which converges to a unique solution by convexity of the \( l_1 \) -PLL objective function [12]. Both algorithms are initialized at the same point, as indicated in Section IV-E1. If \( X_0 \) denotes the current iterate and \( X_0^+ \) denotes the update of \( X_0 \) after a single sweep, then these algorithms are terminated when:

\[
|\mathcal{L}(X_0) - \mathcal{L}(X_0^+)|/|\mathcal{L}(X_0)| < 10^{-8}.
\]

A. The Considered Configurations of \( \Omega \)

We let \( p = 100 \), and consider reconstructing small-world (s.w.) and non small-world (n.s.w.) sparse inverse covariances \( \Omega \). Non-small-world \( \Omega \)'s are constructed using the Matlab function sprandsym, see [46]. Small-world \( \Omega \)'s are based on the model in [47], and the Matlab code used for construction is from [48]. In these constructions the locations of the zeros and non-zeros in \( \Omega \) are specified by the adjacency matrix of a sparse random graph. Both n.s.w. and s.w. \( \Omega \)'s have normally distributed off-diagonal non-zeros but the vertex degree distributions of the associated random graphs are very different, see Figure 1.

B. Varying the Sparsity in \( \Omega \)

The true sparse inverse covariances \( \Omega = \Omega(\alpha) \) are varied as a function of the sparsity level \( \alpha \in [0, 1] \), where \( \Omega(1) \) is the most sparse and \( \Omega(0) \) is the least sparse matrix. Specifically, we generate \( \Omega(1) \) and \( \Omega(0) \) with \( \|\Omega(1)\|_0 = 0.015 \times p^2 \) and \( \|\Omega(0)\|_0 = 0.22 \times p^2 \) using n.s.w. and s.w. models. To generate \( \Omega(\alpha) \) for any \( \alpha \in (0, 1) \), we stochastically combine \( \Omega(1) \) and
for $i,j$ iteration to minimize the $l_1$ oracle estimator, denoted by $\hat{M}$. The average of these penalized $l_1$ oracle estimators will be referred to as the average penalized $l_1$ oracle estimator. This is especially true for off-diagonal non-zeros.

When $\lambda$ is tuned to give minimum KL divergence, the solution to $\min_{\lambda} \text{KL}$ will be referred to as the penalized ML oracle estimator. The average of these penalized ML oracle estimators (over $M$ trials) will be referred to as the average penalized ML oracle estimator, denoted by $\hat{\Omega}_{av}$. Lastly, $\hat{\Omega}_{av}$ and KL superscripted by $l_0$ and $l_1$ correspond to these quantities by minimizing the $l_0$ and $l_1$ penalized PLL objective function, respectively.

In practice we do not have access to $\Omega$ so to demonstrate that the proposed method is practically useful we also show results for which $\lambda$ has been selected using the computationally efficient Extended Bayesian Information Criterion (EBIC) [49]–[51]. Unlike the classical methods such as the BIC and Cross Validation, the EBIC is known to work well for sparse graphs when $n$ and $p$ are of similar size [50]. To measure the average practical performance we compute:

$$\hat{\text{KL}}_{l_0} = \frac{1}{M} \sum_{i=1}^{M} \text{KL}(\hat{\Omega}(\lambda), \Omega), \quad \hat{\lambda} = \arg\min_{\lambda} \text{EBIC}(\hat{\Omega}(\lambda))$$

where $\text{EBIC}(\hat{\Omega}(\lambda))$ is stated in [49].

D. Results for Non Small-World (n.s.w.) $\Omega$

Figure 2 shows that as the sparsity in $\Omega$ increases, the $l_0$ penalized ML estimator outperforms the $l_1$ penalized ML estimator (error bars are 95% confidence intervals). The performance advantage holds both for under-determined $n = 0.7 \times p$ and over-determined $n = 6 \times p$ scenarios.

Figures 3 and 4 illustrate that the $l_1$ penalized ML oracle estimator has over-estimated the number of non-zero components, and that the $l_0$ penalized ML oracle estimator produces relatively sparser solutions. For $n = 0.7 \times p$, Figures 5, 6 and 7 confirm the significant shrinkage biases in the larger components of the $l_1$ penalized ML oracle estimator due to the effect of linear penalization in the $l_1$ penalty. We see that no such biases are present in the $l_0$ penalized ML oracle estimator. For $n = 6 \times p$, Figures 8, 9 and 10 also confirm the biases of the $l_1$ penalty.

Lastly, for $\alpha = 1$ and $n = 0.7 \times p$ we computed averages of ensemble goodness of fit according to KL divergence (29). Figure 11 shows the results. On the left the ratio between the $\text{KL}_{l_0}$ and $\text{KL}_{l_1}$ is 2.97, which is close to that in Figure 2. On the right, the ROC curves quantitatively establish the superior performance of the $l_0$ penalty.

E. Results for Small-World (s.w.) $\Omega$

Figure 12 demonstrates that for a very sparse $\Omega = \Omega(\alpha)$ the $l_0$ penalized ML estimator has better performance than the $l_1$ penalized ML estimator. This is especially true for $n = 6 \times p$ case. However, for less sparse scenarios, i.e., for $\alpha < 0.5$, we see that the opposite is true, and using the $l_1$ penalty seems to be a better choice in terms of oracle fit in KL divergence. This could be because the proposed $l_0$ approach might be
Fig. 3. The true non-small-world (n.s.w.) $\Omega = \Omega(\alpha)$ and the corresponding average penalized ML oracle estimators $\hat{\Omega}_{av}(\alpha)$ with $l_0$ and $l_1$ sparsity penalties when $n = 0.7 \times p$. The true inverse covariance is a single realization of the n.s.w. configuration. (a) $\alpha = 1$, (b) $\alpha = 0.9$, and (c) $\alpha = 0.8$. For ease of visualization the inverse covariance estimators have their off-diagonal values magnified 300 times. Notice that the estimates $\hat{\Omega}_{av,l_1}$ contain many spurious small valued non-zeros unlike the proposed $l_0$ penalized ML estimates $\hat{\Omega}_{av,l_0}$.

Fig. 4. The true non-small-world (n.s.w.) $\Omega = \Omega(\alpha)$ and the corresponding average penalized ML oracle estimators $\hat{\Omega}_{av}(\alpha)$ with $l_0$ and $l_1$ sparsity penalties when $n = 6 \times p$. The true inverse covariance is a single realization of the n.s.w. configuration. (a) $\alpha = 1$, (b) $\alpha = 0.9$, and (c) $\alpha = 0.8$. For ease of visualization the inverse covariance estimators have their off-diagonal values magnified 300 times. Notice that the estimates $\hat{\Omega}_{av,l_1}$ contain many spurious small valued non-zeros unlike the proposed $l_0$ penalized ML estimates $\hat{\Omega}_{av,l_0}$.

Fig. 5. Comparison of amplitudes of the off-diagonal entries in non-small-world (n.s.w.) $\Omega = \Omega(\alpha)$, $\hat{\Omega}_{av,l_0}$ and $\hat{\Omega}_{av,l_1}$, where $n = 0.7 \times p$. As it can be seen, the $l_1$ exhibits significant shrinkage bias. More prone to converge to a local minimizer for lower sparsity levels.

Figures 13 and 14 show a similar trend as Figure 4, i.e., the average $l_0$ penalized ML oracle estimator is sparser than the average $l_1$ penalized ML oracle estimator, where the latter again contains many more small valued non-zero values.

Figures 15, 16 and 17 again confirm the biases in the non-zero entries of the average $l_1$ penalized ML oracle estimator unlike the average $l_0$ penalized ML oracle estimator.

For $n = 6 \times p$, Figures 18, 19 and 20 indicate that the biases of the $l_1$ penalty are less evident.

Lastly, similarly to the case of n.s.w. $\Omega$, we computed the ensemble average performance by repeating the entire simulation procedure in Section VI-C 15 times and averaging out the different random draws of s.w. $\Omega = \Omega(1)$. Figure 21 shows the results. On the left the ratio between the average $\hat{KL}_{l_0}$ and $\hat{KL}_{l_1}$ is given by 1.29, which is close to that in Figure 12. On the right, the ROC curves show better performance of the $l_0$ penalty.

VII. CONCLUSION

We have proposed using the non-convex $l_0$ penalized log-likelihood for estimation of the inverse covariance matrix in Gaussian graphical models as an alternative to the convex $l_1$ penalized log-likelihood approach. We proved that the solutions to the $l_0$ and $l_1$ penalized likelihood maximizations are...
not generally the same. We developed a novel cyclic descent algorithm for the non-convex optimization and established convergence to a strict local minimizer.

Comparisons between the penalized Maximum-Likelihood (ML) estimators corresponding to the $l_0$ and the $l_1$ penalty demonstrated two advantages of the proposed $l_0$ penalty for both non small-world and small-world configurations of $\Omega$. First, for very sparse inverse covariance we have shown that on average the $l_1$ penalized ML estimators are insufficiently sparse as compared to the $l_0$ penalized ML estimators. Second, we have shown that on average the $l_1$ penalty produces non-zero components that have significantly higher bias due to the shrinkage effect induced by the $l_1$ penalty, which is not induced by the $l_0$ penalty.

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APPENDIX A

For the proofs of results in the paper some standard determinant and matrix inverse identities will be needed.

In what follows, matrix $X$ is symmetric and invertible and $Y = X^{-1}$. The first result is on the determinant of a perturbed matrix $X$:

$$\det(X + \delta e_i e_i^T) = \det(X)(1 + \delta y_{ii}),$$ 

(31)

where $e_i$ is a unit vector with a 1 in the $i$th entry and 0 in all other entries. Furthermore:

$$\det(X + \delta U_{ij} U_{ji}^T) = \det(X) \det(I + \delta U_{ij}^T Y U_{ij})$$

$$= \det(X)(-\Delta_{ij}\delta^2 + 2y_{ij}\delta + 1),$$

(32)

where, as defined in (1), $U_{ij} = [e_i \ e_j]$ and we define:

$$\Delta_{ij} = \Delta_{ij}(Y) = y_{ii}y_{jj} - y_{ij}^2 > 0.$$ 

(33)

The standard Sherman-Morrison-Woodbury identity gives:

$$(X + \delta U_{ij} U_{ji}^T)^{-1} = Y - \frac{\delta Y e_i e_i^T Y}{1 + \delta y_{ii}} = Y - \frac{\delta y_{[i]} Y_{[i]}}{1 + \delta y_{ii}}$$

(34)

assuming $1 + \delta y_{ii} \neq 0$, and:

$$(X + \delta U_{ij} U_{ji}^T)^{-1}$$

$$= Y - \delta Y U_{ij}(I + \delta U_{ji}^T Y U_{ij})^{-1} U_{ji}^T Y$$

$$= Y - \frac{\delta [y_{[i]} \ y_{[j]}]}{-\Delta_{ij}\delta^2 + 2y_{ij}\delta + 1}$$

(35)

assuming $-\Delta_{ij}\delta^2 + 2y_{ij}\delta + 1 \neq 0$. 


Fig. 6. Comparison of amplitudes of the off-diagonal entries in non-small-world (n.s.w.) $\Omega = \Omega(\alpha), \hat{\Omega}_{av,l_0}$ and $\hat{\Omega}_{av,l_1}$, where $n = 0.7 \times p$. As it can be seen, the $l_1$ exhibits significant shrinkage bias.

Fig. 7. Comparison of amplitudes of the off-diagonal entries in non-small-world (n.s.w.) $\Omega = \Omega(\alpha), \hat{\Omega}_{av,l_0}$ and $\hat{\Omega}_{av,l_1}$, where $n = 0.7 \times p$. As it can be seen, the $l_1$ exhibits significant shrinkage bias.
APPENDIX B

Proof of Theorem 1: There are two scenarios to consider: (1) the set of local minimizers $S_{l_0}(\lambda)$ contains diagonal matrices only, vs. (2) $S_{l_0}(\lambda)$ contains at least one matrix with off-diagonal non-zero entries. We cover both simultaneously.

We first derive the necessary optimality condition for a non-zero off-diagonal entry of a local minimizer of (3): Let $X \in S_{l_0}(\lambda)$ and define $Y = X^{-1}$. Denote the set of non-zero entries in $X$ by:

$$Z_C(X) = \{(i,j) : x_{ij} \neq 0\},$$

which is non-empty by assumption. Now, since $X$ is a local minimizer, by definition there exists an $\epsilon > 0$ such that:

$$L(X + \Delta) \geq L(X) \text{ for any } \Delta \text{ with } \|\Delta\|_F < \epsilon,$$  \hspace{1cm} (36)

where $\Delta = [\delta_{ij}]$ is a symmetric matrix perturbation. Letting $(i,j) \in Z_C(X)$ and consider:

$$\Delta = \begin{cases} 
\delta_{ii} e_i e_i^T & \text{if } i = j \\
\delta_{ij} U_{ij} U_{ji}^T & \text{otherwise.}
\end{cases}$$

By substituting $\delta = \delta_{ij}$ into (32) and (33), we have:

(i) If $i = j$, then:

$$L(X + \Delta) - L(X) = -\log \det(X + \Delta) + \text{tr}(S(X + \Delta))$$
$$+ \lambda \|X + \Delta\|_0 + \log \det(X) - \text{tr}(SX)$$
$$- \lambda \|X\|_0$$
$$= -\log \{\det(X)(1 + \delta_{ii}y_{ii})\} + \text{tr}(S\Delta)$$
$$+ \log \det(X) + 2\lambda \sum_{i=1} x_{ii} + \delta_{ii} \neq 0$$
$$- 2\lambda \sum_{i=1} x_{ii} \neq 0 = -\log(1 + \delta_{ii}y_{ii}) + s_{ii}\delta_{ii}.$$

(ii) If $i \neq j$, then:

$$L(X + \Delta) - L(X) = -\log \det(X + \Delta) + \text{tr}(S(X + \Delta))$$
$$+ \lambda \|X + \Delta\|_0 + \log \det(X) - \text{tr}(SX)$$
$$- \lambda \|X\|_0$$
$$= -\log \{\det(X)(-\Delta_{ij}\delta_{ij}^2 + 2y_{ij}\delta_{ij} + 1)\}$$
$$+ \text{tr}(S\Delta) + \log \det(X)$$
$$+ 2\lambda \sum_{i=1} x_{ij} + \delta_{ij} \neq 0$$
$$- 2\lambda \sum_{i=1} x_{ij} \neq 0$$
$$- 2\lambda \delta_{ij} + 2\lambda \sum_{i=1} (x_{ij} + \delta_{ij} \neq 0) - 2\lambda,$$
Fig. 10. Comparison of amplitudes of the off-diagonal entries in non-small-world (n.s.w.) Ω = Ω(α), Ω_{av,l_0} and Ω_{av,l_1}, where n = 6 × p. As it can be seen, the l_1 exhibits significant shrinkage bias.

Fig. 11. (Left) Box and Whisker plot of \( \hat{KL}_{l_0} \) and \( \hat{KL}_{l_1} \) for 15 independent draws from the non-small-world (n.s.w.) ground truth model Ω = Ω(1). The mean \( \bar{KL}_{l_0} \) and mean \( \bar{KL}_{l_1} \) are denoted by the red horizontal line and are given by 1.81 and 5.38, respectively. The box represents the standard deviation of \( \hat{KL} \), while the whiskers denote the lowest and highest \( \hat{KL} \) value. (Right) ROC curves plotted using the average true and false positive rates (TPR and FPR). Each TPR and FPR instance is obtained by calculating the TPR and FPR for each \( \hat{Ω} \) (note that there are \( M \) of these \( \hat{Ω} \)) and then taking the average.

where \( \Delta_{ij} = \Delta_{ij}(Y) \) and \( \Delta_{ij}(\cdot) \) is defined in (33).

Suppose that:

\[
|\delta_{ij}| < \min\{|x_{ij}|, \epsilon/2\}.
\]

Since \( |\delta_{ij}| < |x_{ij}| \), we have:

\[
I(x_{ij} + \delta_{ij} \neq 0) = I(x_{ij} \neq 0) = 1 \quad \text{for} \quad i \neq j,
\]

and (36) is equivalent to:

\[
0 \leq f(\delta_{ij}) = \begin{cases} 
-\log(1 + \delta_{ij}y_{ij}) + s_{ij}\delta_{ij} & \text{if} \quad i = j \\
-\log(-\Delta_{ij}\delta_{ij}^2 + 2y_{ij}\delta_{ij} + 1) + 2\delta_{ij}s_{ij} & \text{otherwise}
\end{cases}
\]

for any \( |\delta_{ij}| < \min\{|x_{ij}|, \epsilon/2\} \). Noting that \( f(0) = 0 \), and \( f(\delta_{ij}) \geq 0 \) in a small region around \( \delta_{ij} = 0 \), we must have \( f'(0) = 0 \). Thus, by differentiating \( f(\delta_{ij}) \) and letting \( \delta_{ij} \to 0 \):

\[
y_{ij} + s_{ij} = 0, \quad \text{for any} \quad (i,j) \in \mathcal{Z}(X).
\]

This is the necessary condition for \( X \) to be in \( S_{l_0}(\lambda) \). To finish
have:

Namely, we might have \( q = 6 \) and suppose \( q > 0 \). The estimators have their off-diagonal values magnified 300 times. As in Figure 4, we relate (37) to (38) need to hold simultaneously for some \( \mu > 0 \). But, this is not possible, which completes the proof.

The following simple lemma will be useful for the subsequent proofs:

**Lemma 1.** Suppose \( x_{ij}^{k_n} \to x_{ij}^\star \) as \( n \to \infty \). Define:

\[
q(x_{ij}^{k_n}) = -\Delta_{ij}^{k_n}(x_{ij}^{k_n})^2 - 2g_{ij}^{k_n}x_{ij}^{k_n} + 1
\]

and suppose \( q(x_{ij}^\star) = 0 \). Then for a large enough \( N > 0 \) we have:

\[
A(x_{ij}^{k_n}) = m_{ij}^{k_n} \text{ for all } n > N
\]

**Proof.** \( q(x_{ij}^{k_n}) \) is continuous w.r.t. \( x_{ij}^{k_n} \), and \( q(x_{ij}^\star) = 0 \) can in general be reached in an oscillating fashion as \( n \to \infty \). Namely, we might have \( q(x_{ij}^{k_n}) \leq 0 \) for some \( n \), and \( q(x_{ij}^{k_n}) > 0 \) for some other \( n \). Since \( A(x_{ij}^{k_n}) = m_{ij}^{k_n} \) for those

\[
\text{for some } \tilde{\phi}(\cdot)
\]

The biases of the penalized ML oracle estimators \( \tilde{\phi}(\cdot) \) contain many spurious small valued non-zeros unlike the proposed \( l_1 \) penalized ML estimates \( \tilde{\phi}(\cdot) \).

Therefore, \( \tilde{\phi}(\cdot) \) has non-zero bias. Hence, we have that:

\[
\text{for some } \tilde{\phi}(\cdot)
\]

and this implies:

\[
\text{for some } \tilde{\phi}(\cdot)
\]

which is finite. However, having \( q(x_{ij}^{k_n}) \to 0 \) implies:

\[
\text{for some } \tilde{\phi}(\cdot)
\]

Therefore, \( \tilde{\phi}(\cdot) \) has non-zero bias. Hence, we have that:

\[
\text{for some } \tilde{\phi}(\cdot)
\]

Next, recalling (22) we have that:

\[
\text{for some } \tilde{\phi}(\cdot)
\]
where:
\[
\bar{q}(x_{ij}^k) = -\Delta_{ij}^k \delta(m_{ij}^k)^2 + 2y_{ij}^k \delta(m_{ij}^k) + 1
\]
\[
= \begin{cases} 
    1 + \left(\frac{y_{ij}^k}{\Delta_{ij}^k}\right)^2 & \text{if } s_{ij} = 0 \\
    \frac{\sqrt{(\Delta_{ij}^k)^2 + 4\gamma_{ij}^k y_{ij}^k \Delta_{ij}^k}}{2s_{ij}} & \text{otherwise.}
\end{cases}
\]

The second equality for \(\bar{q}(x_{ij}^k)\) can easily be shown using the results in Section IV-B. Now, notice that \(\bar{q}(x_{ij}^k) > 0\) for all \(n\) and \(\Delta_{ij}^k, y_{ij}^k, y_{jj}^k\), which are themselves strictly positive for all \(n\). Since:
\[
\Delta_{ij}^k \to \Delta_{ij}^* > 0, \quad y_{ii}^k \to y_{ii}^* > 0, \quad \text{and} \quad y_{jj}^k \to y_{jj}^* > 0
\]
we must have that:
\[
\bar{q}(m_{ij}^k) \to \bar{q}^* > 0.
\]

Also note that \(m_{ij}^k \to m_{ij}^*\), which is finite by the same reason that \(q^*\) is finite (see the definition of \(m_{ij}\)). This means:
\[
c_{X_{k,i,j}}(m_{ij}^k) \to c^*.
\]

which is finite. Thus, there has to exist a large enough \(N > 0\) such that:
\[
\phi_{X_{k,i,j}}(0) > c_{X_{k,i,j}}(m_{ij}^k) \quad \text{for all } n > N,
\]

implying \(\mathcal{A}(x_{ij}^k) = m_{ij}^k\) for all \(n > N\).

**Proof of Theorem 3:** Firstly, note that if \(x_{ij}^k \to x_{ij}^*\), then we must have \(x_{ij}^{k+1} \to x_{ij}^*\) as well. As a result:
\[
\mathcal{A}(x_{ij}^k) \to x_{ij}^*.
\]

and so, all that needs to be shown is that \(\mathcal{A}(x_{ij}^k) \to \mathcal{A}(x_{ij}^*)\).

When \(i = j\), we have \(\mathcal{A}(x_{ij}^k) = m_{ij}^k\), which is continuous w.r.t. \(x_{ij}^k\). Thus,
\[
x_{ij}^k \to x_{ij}^* \quad \text{implies} \quad \mathcal{A}(x_{ij}^k) \to \mathcal{A}(x_{ij}^*)
\]

Now suppose \(i \neq j\) and consider the fact that:
\[
q(x_{ij}^k) \to q(x_{ij}^*),
\]

where \(q(\cdot)\) is defined in (39). There are two cases:

- **[C1]**: Suppose \(q(x_{ij}^*) \leq 0\). By the definition of \(\mathcal{A}(\cdot)\) and Lemma 1 (with \(k_n = k\), we have \(\mathcal{A}(x_{ij}^k) = m_{ij}^k\) for all \(k > K\). Since \(m_{ij}^k\) is a continuous function of \(x_{ij}^k\) we must have \(\mathcal{A}(x_{ij}^k) \to \mathcal{A}(x_{ij}^*)\).

- **[C2]**: Suppose \(q(x_{ij}^*) > 0\), and noting that \(\phi_{X_{k,i,j}}(m_{ij}^k) = c_{X_{k,i,j}}(m_{ij}^k)\) define:
\[
\Phi_{X_{k,i,j}}(x_{ij}^k) = \phi_{X_{k,i,j}}(0) - c_{X_{k,i,j}}(m_{ij}^k),
\]

which is continuous w.r.t. \(x_{ij}^k\), in which case:
\[
\Phi_{X_{k,i,j}}(x_{ij}^k) \to \Phi_{X_{k,i,j}}(x_{ij}^*).
\]
There are now two scenarios:

(i) \( \Phi_{X^*,ij}(x^*_{ij}) \neq 0 \): This implies that for a sufficiently large \( K > 0 \), we have:

\[ \Phi_{X^*,ij}(x^k_{ij}) < 0 \text{ or } \Phi_{X^*,ij}(x^k_{ij}) > 0 \text{ for all } k > K. \]

In the former case, \( \mathcal{A}(x^k_{ij}) = 0 \), and in the latter case \( \mathcal{A}(x^k_{ij}) = m^k_{ij} \) for all \( k > K \). These are both continuous w.r.t. \( x^k_{ij} \) implying \( \mathcal{A}(x^k_{ij}) \to \mathcal{A}(x^*_{ij}) \).

(ii) \( \Phi_{X^*,ij}(x^*_{ij}) = 0 \): Since \( \mathcal{A}(x^*_{ij}) \to x^*_{ij} \), for a large enough \( K \) we have to have \( \Phi_{x^*,ij}(x^k_{ij}) \) approach 0 either from below or from above for all \( k > K \). So, suppose:

\[ \Phi_{X^*,ij}(x^k_{ij}) < 0 \text{ for all } k > K. \]

Then \( \mathcal{A}(x^k_{ij}) = 0 \) for all \( k > K \), which implies:

\[ \mathcal{A}(x^k_{ij}) \to 0, \]

and thus, \( x^*_{ij} = 0 \). So, using the definition of \( \mathcal{A}(\cdot) \), at \( x^*_{ij} \) we have:

\[ \mathcal{A}(x^*_{ij}) = m^0_{ij} \cdot 1(x^*_{ij} \neq 0) = m^0_{ij} \cdot 0 = 0, \]

implying \( \mathcal{A}(x^k_{ij}) \to \mathcal{A}(x^*_{ij}) \).

Alternatively, suppose:

\[ \Phi_{X^*,ij}(x^k_{ij}) > 0 \text{ for all } k > K. \]

Then \( \mathcal{A}(x^k_{ij}) = m^k_{ij} \) for all \( k > K \), which implies:

\[ \mathcal{A}(x^k_{ij}) \to m^*_{ij}, \]

and thus, \( x^*_{ij} = m^*_{ij} \). Therefore, using the definition of \( \mathcal{A}(\cdot) \), at \( x^*_{ij} \) we have:

\[ \mathcal{A}(x^*_{ij}) = m^*_{ij} \cdot 1(x^*_{ij} \neq 0) = m^*_{ij} \cdot 1 = m^*_{ij}, \]

implying \( \mathcal{A}(x^k_{ij}) \to \mathcal{A}(x^*_{ij}) \). This completes the proof. \( \square \)

The proof of Theorem 4 requires Lemmas 2 and 3:

**Lemma 2.** Suppose \( X_0 \in \mathcal{F} \) and \( x_{0,ij} \neq 0 \). Define:

\[ Y_0 = X_0^{-1}. \]

Then, \( y_{0,ij} = s_{ij} \).

**Proof.** Having \( X_0 \in \mathcal{F} \) implies \( x_{0,ij} = \mathcal{A}(x_{0,ij}) \).

When \( i = j \) we have \( \mathcal{A}(x_{0,ii}) = m_{ii} \), where \( m_{ij} \) is from (10). This implies:

\[ x_{0,ii} = x_{0,ii} + \frac{y_{0,ii} - s_{ii}}{y_{0,ii}s_{ii}}, \]

which reduces to \( y_{0,ii} = s_{ii} \) after simplification.

When \( i \neq j \), having \( x_{0,ij} \neq 0 \) implies \( \mathcal{A}(x_{0,ij}) = m_{ij} \), where \( m_{ij} \) is defined in (16) and (17). As a result, we have
where $\Delta_{0,ij} = \Delta_{ij}(Y_0)$ and $\Delta_{ij}(\cdot)$ is from (33). After simplification we can easily obtain that $y_{0,ij} = s_{ij}$.

Lemma 3. Suppose $X_0 \in \mathcal{F}$ and $x_{0,ij} = 0$, where $i \neq j$. Letting $Y_0 = X_0^{-1}$, there exists $\delta > 0$ that depends on $\lambda$, $X_0$ and $S$ such that:

$$|y_{0,ij} - s_{ij}| \leq \delta.$$  

Proof. As in Lemma 3, $X_0 \in \mathcal{F}$ implies $x_{0,ij} = \mathcal{A}(x_{0,ij})$. Having $x_{0,ij} = 0$ means $\mathcal{A}(\cdot)$ is given by (24) and:

$$\phi(x_{0,ij}(0)|_{x_{0,ij}=0} \leq \phi(x_{0,ij}(m_{ij}))(x_{0,ij}=0).$$  

(41)

Recall the following standard inequalities:

$$\log(a) \geq \frac{(a-1)}{a}, \text{ for any } a > 0$$  

(42)

$$\sqrt{a^2 + b} - a \geq \sqrt{b} - a, \text{ for any } a, b \geq 0$$  

(43)

Dealing with (41) requires two cases:

Case 1: $s_{ij} = 0$. It can easily be shown that (41) reduces to:

$$\log \left(1 + \frac{y_{0,ij}^2}{\Delta_{0,ij}}\right) \leq 2\lambda.$$  

So, using (42) with:

$$a = 1 + \frac{y_{0,ij}^2}{\Delta_{0,ij}},$$

we obtain that:

$$|y_{0,ij}| \leq \sqrt{2\lambda y_{0,ij}y_{0,jj}}.$$  

Since $|y_{0,ij} - s_{ij}| = |y_{0,ij}|$, the proof is complete for $[C_1]$.

Case 2: $s_{ij} \neq 0$. It can easily be shown that (41) reduces to:

$$\log \left(\frac{\sqrt{\Delta_{0,ij} - 4s_{ij}^2y_{0,ij}y_{0,jj}}}{{2s_{ij}^2}}\right) + \frac{\sqrt{\Delta_{0,ij} - 2s_{ij}y_{0,ij} - \Delta_{0,ij}}}{\Delta_{0,ij}} \leq 2\lambda,$$

where

$$\Delta_{0,ij} = \Delta_{ij}^2 + 4s_{ij}^2y_{0,ij}y_{0,jj} > 0,$$

noting that $\Delta_{0,ij} > 0$ as well. So, using (42) with:

$$a = \frac{\sqrt{\Delta_{0,ij} - 4s_{ij}^2y_{0,ij}y_{0,jj}}}{2s_{ij}^2},$$

we have:

$$\log \left(\frac{\sqrt{\Delta_{0,ij} - 4s_{ij}^2y_{0,ij}y_{0,jj}}}{{2s_{ij}^2}}\right) + \frac{\sqrt{\Delta_{0,ij} - 2s_{ij}y_{0,ij} - \Delta_{0,ij}}}{\Delta_{0,ij}} \leq 2\lambda.$$  

(44)

The last inequality in (44) comes from the fact that:

$$\sqrt{\Delta_{0,ij} - \Delta_{0,ij}} > 0.$$  

Next, substituting:

$$a = \Delta_{0,ij} \text{ and } b = 4s_{ij}^2y_{0,ij}y_{0,jj}$$

in (43), we obtain:

$$\sqrt{\Delta_{0,ij} - \Delta_{0,ij}} \geq 2|s_{ij}|\sqrt{y_{0,ij}y_{0,jj}}.$$  

□
Thus:
\[
(\star\star) \geq \frac{2|s_{ij}|\sqrt{|y_{0,ij}y_{0,ij}} - \Delta_{0,ij} - 2s_{ij}y_{0,ij}}{\Delta_{0,ij}} = \frac{2|s_{ij}|\sqrt{|y_{0,ij}y_{0,ij}} - (y_{0,ij}y_{0,ij} - y_{0,ij}^2) - 2s_{ij}y_{0,ij}}{\Delta_{0,ij}}
\]
\[
= \frac{(y_{0,ij} - s_{ij})^2 - |s_{ij}| - \sqrt{|y_{0,ij}y_{0,ij}}|^2}{\Delta_{0,ij}}.
\]
(45)

As a result, (44), (45) and the fact that \((\star) + (\star\star) \leq 2\lambda\) imply (after re-arrangement) that: \(|y_{0,ij} - s_{ij}| < \delta\) for some \(\delta > 0\). This completes the proof.

\(\square\)

**Proof of Theorem 4:** Let \(Y = X^{-1}\) and introduce:
\[
l_\lambda(X) = -\log \det(X) + \text{tr}(SX).
\]
The Hessian is equal to:
\[
\nabla^2 l_\lambda(X) = Y \otimes Y > 0.
\]
Since any eigenvalue of \(\nabla^2 l_\lambda(X)\) is a continuous function of \(X\), there exists a small neighbourhood of \(X\), denoted by:
\[
\mathcal{U}_\epsilon(X) = \{X' = X + \Delta : 0 \leq \|\Delta\|_F < \epsilon_0\},
\]
such that \(\nabla^2 l_\lambda(X') > 0\) for all \(X' \in \mathcal{U}_\epsilon(X)\). In other words, there exists a constant \(\mu > 0\) such that:
\[
\nabla^2 l_\lambda(X') \geq \mu I \text{ for all } X' \in \mathcal{U}_\epsilon(X),
\]
which in turn implies that \(l_\lambda(\cdot)\) is strongly convex in \(\mathcal{U}_\epsilon(X)\). Recalling the standard inequality for a strongly convex function:
\[
l_\lambda(X + \Delta) \geq l_\lambda(X) + \text{tr}(\nabla l_\lambda(X) \Delta) + \frac{1}{2\mu}\|\Delta\|_F^2,
\]
(46)
where \(\|\Delta\|_F < \epsilon_0\). The equality in (46) comes from using:
\[
\nabla l_\lambda(X) = -Y + S.
\]
Now, using the fact that \(X \in \mathcal{F}\) implies \(x_{ij} \in \mathcal{F}_{ij}\), we introduce the following sets:
\[
\mathcal{Z}_X = \{(i,j) : i \neq j, x_{ij} = 0\},
\]
\[
\mathcal{Z}_X^c = \{(i,j) : (i,j) \notin \mathcal{Z}_X\}
\]
Using (46) we obtain:
\[
\mathcal{L}(X + \Delta) \geq \mathcal{L}(X) + R_\lambda(\Delta),
\]
where it can be easily shown that:
\[
R_\lambda(\Delta) = \sum_{ij} \frac{1}{2} \mu \delta_{ij}^2 + (-y_{ij} + s_{ij})\delta_{ij} + \lambda \|x_{ij} + \delta_{ij} \neq 0\| - \lambda \|x_{ij} \neq 0\|
\]
\[
= \sum_{(i,j) \in \mathcal{Z}_X} \frac{1}{2} \mu \delta_{ij}^2 + (-y_{ij} + s_{ij})\delta_{ij} + \lambda \|\delta_{ij} \neq 0\|
\]
\[
+ \sum_{(i,j) \in \mathcal{Z}_X^c} \frac{1}{2} \mu \delta_{ij}^2 + (-y_{ij} + s_{ij})\delta_{ij} + \lambda \|x_{ij} + \delta_{ij} \neq 0\| - \lambda.
\]
In the above define \(S_Z(\delta_{ij})\) and \(S_Z(\delta_{ij})\) to be the summands corresponding to \((i,j) \in \mathcal{Z}_X\) and \((i,j) \in \mathcal{Z}_X^c\) respectively.

Now, \(R_\lambda(0) = 0\), and so, the idea is to show that there exists \(\epsilon > 0\) such that \(R_\lambda(\Delta) > 0\) for any \(\Delta\) satisfying \(0 < \|\Delta\|_F < \epsilon\). This, with \(\epsilon = \min\{\epsilon_0, \epsilon'\}\), will then imply the result (28). We proceed by dealing with each summand in \(R_\lambda(\cdot)\). There are two cases:

\[\text{[C}_1\text{]}\] Regarding \(S_Z(\cdot)\). We have \(S_Z(0) = 0\), so suppose \(\delta_{ij} \neq 0\). Then:
\[
S_Z(\delta_{ij}) > 0 \quad \text{when} \quad 0 < \|\delta_{ij}\| < \epsilon_1.
\]
where the last \(\geq\) comes from using Lemma 3 with \(\delta = c_{ij} > 0\).

\[\text{[C}_2\text{]}\] Regarding \(S_Z(\cdot)\). We have \(S_Z(0) = 0\), so suppose \(\delta_{ij} \neq 0\). Then, defining:
\[
\epsilon_2 = \min_{(i,j) \in \mathcal{Z}_X} |x_{ij}|,
\]
which is clearly strictly positive, it follows from (47) that:
\[
S_Z(\delta_{ij}) > 0 \quad \text{when} \quad 0 < \|\delta_{ij}\| < \epsilon_1.
\]

Therefore:
\[
S_Z(\delta_{ij}) = \left(\frac{1}{2} \mu \delta_{ij}^2 + (-y_{ij} + s_{ij})\delta_{ij} + \frac{1}{2} \mu \delta_{ij}^2 > 0,
\]
where the last \(\text{equality is due to Lemma 2, i.e., -y_{ij} = s_{ij}}\).

Letting \(\epsilon' = \min\{\epsilon_1, \epsilon_2\}\), completes the proof.

\(\square\)

**Proposition 1.** Let \(x_{0,ij}^+ = A(x_{0,ij})\), and define:
\[
\mathcal{D}(x_{0,ij}, x_{0,ij}^+) = \mathcal{L}(Z_{0,ij}(x_{0,ij})) - \mathcal{L}(Z_{0,ij}(x_{0,ij}^+))
\]
Then, \(x_{0,ij}^+ = x_{0,ij}\) if and only if \(\mathcal{D}(x_{0,ij}, x_{0,ij}^+) = 0\).

**Proof.** Clearly, \(x_{0,ij}^+ = x_{0,ij}\) implies \(\mathcal{D} = 0\). Now, suppose \(\mathcal{D} = 0\), in which case:
\[
\mathcal{D}(x_{0,ij}, x_{0,ij}^+) = \phi x_{0,ij}(x_{0,ij}) - \phi x_{0,ij}(x_{0,ij}^+) = 0
\]
\[
\iff \phi x_{0,ij}(x_{0,ij}) = \phi x_{0,ij}(x_{0,ij}^+)
\]
\[
\iff \phi x_{0,ij}(x_{0,ij}) = \min_{Z} \phi x_{0,ij}(z).
\]
Letting:
\[
g(x_{0,ij}) = -\Delta_{0,ij}(x_{0,ij})^2 - 2y_{0,ij}x_{0,ij} + 1,
\]
there are two cases:
[C1]: $q(x_{0,i,j}) \leq 0$, $\phi_{X_{0,i,j}}(\cdot)$ has a unique minimizer given by $x_{0,i,j}^+$, see (18) in Theorem 2. Thus, (49) implies $x_{0,i,j} = x_{0,i,j}^+$.  

[C2]: $q(x_{0,i,j}) > 0$, $\phi_{X_{0,i,j}}(\cdot)$ has a minimizer given by 0 and/or by $m_{ij}$, where the latter is the unique minimizer of $c_{X_{0,i,j}}$. Note that $m_{ij} \neq 0$ by Theorem 2. There are now two subcases:

(i) $\phi_{X_{0,i,j}}(0) \neq \phi_{X_{0,i,j}}(m_{ij})$: The minimizer of $\phi_{X_{0,i,j}}(\cdot)$ is unique, and is either 0 or $m_{ij}$, see expression (19) and Figure 22 (Top). Therefore, (49) implies $x_{0,i,j} = x_{0,i,j}^+$.  

(ii) $\phi_{X_{0,i,j}}(0) = \phi_{X_{0,i,j}}(m_{ij})$: $\phi_{X_{0,i,j}}(\cdot)$ has two minimizers, 0 and $m_{ij}$, see expression (19). By the definition of $A(\cdot)$, we have:

$$x_{0,i,j}^+ = m_{ij} \cdot \mathbb{I}(x_{0,i,j} \neq 0).$$

Using (50), $x_{0,i,j}^+ = 0$ implies $x_{0,i,j} = 0$, and thus, $x_{0,i,j} = x_{0,i,j}^+$. If $x_{0,i,j}^+ \neq 0$, then $x_{0,i,j} \neq 0$ as well. This indicates that $\phi_{X_{0,i,j}}(\cdot)$ can only have $m_{ij}$ as its minimizer, see Figure 22 (Bottom). Thus, (49) implies $x_{0,i,j} = x_{0,i,j}^+$.  

![Figure 22](image)

We have the following sub-sequential result:  

**Proposition 2.** Assume that (A2) is satisfied. Suppose:

$$\left(x_{ij}^{k_n}, A(x_{ij}^{k_n}) \right) \rightarrow (x_{ij}^*, x_{ij}^{**})$$

as $n \rightarrow \infty$. Then, $x_{ij}^{**} = A(x_{ij}^*)$.  

**Proof.** Recalling from Theorem 3, when $i \neq j$, the result easily follows by the continuity of $A(\cdot)$. Using the same notation as in Theorem 3, when $i \neq j$, the result also follows if $q(x_{ij}^*) \leq 0$, or $q(x_{ij}^*) > 0$ with $\Phi_{X_{ij}}(x_{ij}^*) \neq 0$. Next, assume $q(x_{ij}^*) > 0$ and $\Phi_{X_{ij}}(x_{ij}^*) = 0$.  

Note that $x_{ij}^{k_n}$ is an iterate that is thresholded, i.e., it can only be 0 or $m_{ij}^{k_{n-1}} \neq 0$ for every $n$, where the latter can only converge to a nonzero number, say, $m_{ij}^\star$. Also note that $x_{ij}^{k_n} \rightarrow x_{ij}^*$ implies:

$$|x_{ij}^{k_{n+1}} - x_{ij}^{k_n}| \rightarrow 0.$$  

(i) Suppose $x_{ij}^* = 0$. Then for a large enough $N > 0$ we must have $x_{ij}^{k_n} = 0$ for all $n > N$, otherwise (51) would be violated. Then,

$$A(x_{ij}^{k_n}) \rightarrow 0$$

by (A2), and again by the definition of $A(\cdot)$, at $x_{ij}^*$ we have:

$$A(x_{ij}^*) = m_{ij}(x_{ij}^*) \cdot I(x_{ij}^* \neq 0) = m_{ij} \cdot 0 = x_{ij}^*.$$  

(ii) Suppose $x_{ij}^* \neq 0$. Then for a large enough $N > 0$ we must have $x_{ij}^{k_n} = m_{ij}^{k_{n-1}}$ for all $n > N$, otherwise (51) would be violated. Then, (A2) implies $A(x_{ij}^{k_n}) = m_{ij}^\star$, which in turn implies:

$$A(x_{ij}^{k_n}) \rightarrow m_{ij}^\star,$$

where $m_{ij}^\star = m_{ij}(x_{ij}^*) \neq 0$. So, by the definition of $A(\cdot)$, at $x_{ij}^*$ we have:

$$A(x_{ij}^*) = m_{ij} \cdot I(x_{ij}^* \neq 0) = m_{ij} \cdot 1 = x_{ij}^*,$$

completing the proof.  

The remaining results follow from Proposition 2 and require (A1) and (A2). Before proceeding to Proposition 3, two lemmas are needed:  

**Lemma 4.** Supposing the statement in Proposition 2,

$$D(x_{ij}^{k_n}, A(x_{ij}^{k_n})) \rightarrow D^\infty$$

implies $D^\infty \geq D(x_{ij}^*, x_{ij}^{**})$.  

**Proof.** First, due to the update of two equal matrix entries at a time, it is obvious that $D(\cdot, \cdot)$ is given by:

$$D(x_{ij}^{k_n}, A(x_{ij}^{k_n})) = \phi_{X_{ij}^{k_n},ij}(x_{ij}^{k_n}) - \phi_{X_{ij}^{k_n},ij}(A(x_{ij}^{k_n})).$$

When $i = j$, $\phi_{X_{ij}^{k_n},ij}(\cdot)$ is continuous w.r.t. its argument, and so, by (53) we have $D^\infty = D(x_{ij}^*, x_{ij}^{**})$.  

When $i \neq j$, define:

$$\Pi_{k_n} = \Pi(x_{ij}^{k_n} \neq 0) \text{ and } \Pi_{k_{n+1}} = \Pi(A(x_{ij}^{k_n}) \neq 0),$$

and:

$$\Pi^* = \Pi(x_{ij}^* \neq 0) \text{ and } \Pi^{**} = \Pi(x_{ij}^{**} \neq 0).$$

With these definitions:

$$D(x_{ij}^{k_n}, A(x_{ij}^{k_n})) = \left(\phi_{X_{ij}^{k_n},ij}(x_{ij}^{k_n}) - \phi_{X_{ij}^{k_n},ij}(A(x_{ij}^{k_n}))\right) + 2\lambda(\Pi_{k_n} - \Pi_{k_{n+1}})$$

$$\rightarrow D(x_{ij}^*, x_{ij}^{**}) - 2\lambda(\Pi^* - \Pi^{**}) + 2\lambda D^\infty,$$
as $n \to \infty$, where:
\[
\Delta l^\infty = \lim_{n \to \infty} l_k^n - l_k^{n+1}
\]
The first two terms in (54) result from the continuity of $c_{X_k, n, i, j}(\cdot)$ w.r.t. its argument. As a result, in order to show (52), by observation of (54), all we have to show is that:
\[
\Delta l^\infty \geq l^* - l^{**}.
\]
There are four cases:

\[\text{[C1]}: \text{Suppose } x_{ij} = 0 \text{ and } x_{ij}^* = 0. \text{ If } x_{ij} = 0, \text{ then } x_{ij} = x_{ij}^*, \text{ and so, } A(x_{ij}^*) = A(x_{ij}^*). \text{ But, by Proposition 2 we also have } A(x_{ij}^*) = x_{ij}^*, \text{ and thus,}
\]
\[
\Delta l^\infty = l^* - l^{**}.
\]
If $x_{ij}^* \neq 0$, by the definition of the $l_0$ function we have $l_k^n \to 1$. As a result,
\[
\Delta l^\infty \geq 0 = l^* - l^{**}.
\]

\[\text{[C2]}: \text{Suppose } x_{ij}^* \neq 0 \text{ and } x_{ij}^* \neq 0. \text{ Then, } l_k^n \to 1, \text{ and so,}
\]
\[
\Delta l^\infty \geq 0 = l^* - l^{**}.
\]
For the remaining cases, we consider the continuous functions $q(x_{ij}^*)$ and $\Phi_{X_k, n, i, j}(x_{ij}^*)$ from (39) and (40), respectively.

\[\text{[C3]}: \text{Suppose } x_{ij}^* = 0 \text{ and } x_{ij}^* \neq 0. \text{ If } x_{ij}^* = 0 \text{ then } x_{ij} = x_{ij}^*, \text{ and so,}
\]
\[
A(x_{ij}^*) = A(x_{ij}^*) = x_{ij}^*,
\]
where the last equality is due to Proposition 2. Then we have:
\[
\Delta l^\infty = 0 - 1 = l^* - l^{**}.
\]
Next, supposing $x_{ij}^* \neq 0$ implies $l_k^n \to 1$. Since $q(x_{ij}^*) = q(0) = 1 > 0$, $A(x_{ij}^*)$ is given by (24) and by Proposition 2 we also have $x_{ij}^* = A(x_{ij}^*)$. So, by the fact that $x_{ij}^* \neq 0$ and the definition of $A(\cdot)$, either:

(i) $\Phi_{X, n, i, j}(x_{ij}^*) > 0$, or

(ii) $\Phi_{X, n, i, j}(x_{ij}^*) = 0$ and $x_{ij}^* \neq 0$ holds. Clearly, only (i) can be valid in this case, and so, for a large enough $N > 0$ we must also have $\Phi_{X_k, n, i, j}(x_{ij}^*) > 0$ for all $n > N$. Therefore,
\[
A(x_{ij}^*) = m_{ij}^k \neq 0 \text{ for all } n > N.
\]
This implies $l_k^n \to 1$, and so,
\[
\Delta l^\infty = 1 - 1 = 0 > -1 = l^* - l^{**}.
\]

\[\text{[C4]}: x_{ij}^* \neq 0 \text{ and } x_{ij}^* = 0. \text{ We firstly have that } l_k^n \to 1. \text{ We cannot have } x_{ij}^* = m_{ij}^k, \text{ where } m_{ij}^k \to m_{ij}^*, \text{ because } m_{ij}^* \neq 0. \text{ Then, by Proposition 2, } A(\cdot) \text{ can only be given by (24), where from the two resulting possibilities:}
\]

(i) $\Phi_{X, n, i, j}(x_{ij}^*) < 0$, or

(ii) $\Phi_{X, n, i, j}(x_{ij}^*) = 0$ and $x_{ij}^* = 0$ only (i) can be valid. So, for a large enough $N > 0$ we must have $\Phi_{X_k, n, i, j}(x_{ij}^*) < 0$ for all $n > N$, which implies $A(x_{ij}^*) = 0$ for all $n > N$. Thus, $A(x_{ij}^*) \to 0$, which in turn implies $l_k^n \to 1$. So,
\[
\Delta l^\infty = 1 - 0 = l^* - l^{**}.
\]

\[\text{Lemma 5. The sequence } \{L(X^k)\}_k \text{ is bounded from below.}
\]
\[\text{Proof. } \text{Firstly, } ||X^k||_0 > 0. \text{ Also, having } X^k > 0 \text{ and } S \geq 0 \text{ implies tr}(SX^k) \geq 0. \text{ As a result,}
\]
\[
L(X^k) > -\log \det(X^k),
\]
and by (A1),
\[
-\log \det(X^k) \geq -p \log \alpha,
\]
Thus, we have that:
\[
L(X^k) > -p \log \alpha
\]
which completes the proof.

\[\text{Proposition 3. } x_{ij}^k - x_{ij}^{k+1} \to 0 \text{ as } k \to \infty.
\]
\[\text{Proof. } \text{We show the result by establishing a contradiction. So, suppose } x_{ij}^k - x_{ij}^{k+1} \neq 0, \text{ which means there exists a subsequence:}
\]
\[
\{x_{ij}^k - x_{ij}^{k+1}, x_{ij}^k - x_{ij}^{k+2}, \ldots \} \to \delta \neq 0.
\]
We note that any subsequence of the sequence in (56) must converge to $\delta$ in order for (56) to hold. Since the sequence $\{x_{ij}^k, x_{ij}^k, \ldots \}$ is bounded by (A1), it has at least one limit point. Denote one of these limit points by $x_{ij}^*$ and suppose:
\[
\{x_{ij}^*, x_{ij}^*, \ldots \} \to x_{ij}^*,
\]
where
\[
\{l_1, l_2, \ldots \} \subseteq \{k_1, k_2, \ldots \}.
\]
Now, consider the sequence $\{x_{ij}^* + 1, x_{ij}^* + 2, \ldots \}$, which must have at least one limit point since it is also bounded by (A1). Denote one of these limit points by $x_{ij}^{**}$, and suppose:
\[
\{x_{ij}^{**} + 1, x_{ij}^{**} + 2, \ldots \} \to x_{ij}^{**},
\]
where
\[
\{r_1, r_2, \ldots \} \subseteq \{l_1, l_2, \ldots \}.
\]
But now:
\[
\{x_{ij}^*, x_{ij}^{**, \ldots} \} \to x_{ij}^*,
\]
since this sequence is a subsequence of the sequence in (57). As a result:
\[
\{x_{ij}^* - x_{ij}^{*, 1}, x_{ij}^* - x_{ij}^{*, 2}, \ldots \} \to x_{ij}^* - x_{ij}^{**}.
\]
Next, let $L_k = L(X^k)$, and we obviously have $L_k \geq L_{k+1}$. So, the sequence $\{L_k\}_k$ is non-increasing and by Lemma 5 it must have a finite limit, say, $L^*$. Since:
\[
L_k - L_{k+1} \to L^* - L^* = 0,
\]
by the definition of $D(\cdot, \cdot)$ in (48), this means:

$$D(x^k_{ij}, x^{k+1}_{ij}) \to 0,$$

(61)

and so, $D(x^n_{ij}, x^{n+1}_{ij}) \to 0$. Then, using Lemma 4 we have:

$$0 \geq D(x^\bullet_{ij}, x^{\bullet\bullet}_{ij}).$$

(62)

Since we also have $x^{n+1}_{ij} = A(x^n_{ij})$, we can use (58), (59) and Proposition 2 to obtain that: $x^\bullet_{ij} = A(x^\bullet_{ij})$. Thus:

$$D(x^\bullet_{ij}, x^{\bullet\bullet}_{ij}) = D(x^\bullet_{ij}, A(x^\bullet_{ij})) \geq 0.$$

(63)

The $\geq$ in (63) comes from the definition of $D(\cdot, \cdot)$ and the fact that:

$$L(Z_{0,ij}(x^\bullet_{ij})) \geq L(Z_{0,ij}(A(x^\bullet_{ij}))).$$

As a result, (62) and (63) imply $D(x^\bullet_{ij}, A(x^\bullet_{ij})) = 0$, which by Proposition 1 means $x^\bullet_{ij} = x^{\bullet\bullet}_{ij}$. Consequently, the limit in (60) is 0. Because that sequence is a subsequence of the sequence in (56) we obtain a contradiction, implying (56) cannot hold, which completes the proof.

Proposition 4. $\{x^i_{ij}\}_k$ has limit points, which are all fixed points.

Proof. By (A1), the sequence $\{(x^i_{ij}, x^{i+1}_{ij})\}_k$ is bounded, and so, has at least one limit point. Denote one of the limit points by $x^\bullet_{ij}$. Then, we can find a subsequence $\{(x^n_{ij}, x^{n+1}_{ij})\}_n$ such that:

$$(x^n_{ij}, x^{n+1}_{ij}) \to (x^\bullet_{ij}, x^{\bullet\bullet}_{ij})$$

as $n \to \infty$.

By Proposition 3 we have:

$$x^n_{ij} - x^{n+1}_{ij} \to 0,$$

and so, $x^\bullet_{ij} = x^{\bullet\bullet}_{ij}$. Lastly, by Proposition 2 we have $x^{\bullet\bullet}_{ij} = A(x^\bullet_{ij})$, which implies that $x^\bullet_{ij} = A(x^\bullet_{ij})$. □

Proposition 5. $X^k \to F'$ as $k \to \infty$, where $F' \subseteq F$ is a closed and connected set.

Proof. By Proposition 4, $F' \subseteq F$ is the set of limit points of $\{X^k\}_k$. Then, from Proposition 3 we have:

$$X^k - X^{k+1} \to 0$$

and $\{X^k\}_k$ is bounded by (A1). Due to these two facts, we can apply Ostrowski’s Theorem 26.1 in [45, p.173], which states that the set of limit points of $\{X^k\}_k$ is closed and connected. □

Proof of Theorem 5: Define the set of strict local minimizers of $L(\cdot)$:

$$\mathcal{M} = \{X^\bullet: \text{there exists } \epsilon > 0 \text{ such that}$$

$$L(X^\bullet) < L(X^\bullet + \Delta), \text{ for all } 0 < \|\Delta\|_F < \epsilon\}.$$  

This set is derived by considering Theorem 4, by which for $X \in F$ we have $X \in \mathcal{M}$. This implies $\mathcal{M} \neq \emptyset$ and $F \subseteq \mathcal{M}$. Since $\mathcal{M}$ is the set of distinct local minimizers it must be discrete i.e. consists only of isolated points. If not, there exists a connected subset which is a continuum and this violates the strict inequality. Therefore, the subset $F$ is a discrete set as well. However, by Proposition 5 the limit point set of $\{X^k\}_k$ is a connected subset of $F$. Hence, the limit point set must contain only a single point, say $X^\bullet$, and the result follows. □

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