Photon spheres in Einstein and Einstein-Gauss-Bonnet theories and circular null geodesics in axially-symmetric spacetimes

Emanuel Gallo\(^1,2,\) and J. R. Villanueva\(^3,4,\)

\(^1\) FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina
\(^2\) Instituto de Física Enrique Gaviola (IFEG) CONICET Ciudad Universitaria, 5000 Córdoba, Argentina.
\(^3\) Instituto de Física y Astronomía, Universidad de Valparaíso, Gran Bretaña 1111, Valparaíso, Chile,
\(^4\) Centro de Astrofísica de Valparaíso, Gran Bretaña 1111, Playa Ancha, Valparaíso, Chile.

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In this article we extend a recent theorem proven by Hod (Phys. Lett. B, 727, 345–348, 2013) to \(n\)-dimensional Einstein and Einstein-Gauss-Bonnet theories, which gives an upper bound for the photon sphere radii of spherically symmetric black holes. As applications of these results we give a universal upper bound for the real part of quasinormal modes in the WKB limit and a universal lower bound for the position of the first relativistic image in the strong lensing regime produced by these type of black holes. For the axially-symmetric case, we also make some general comments (independent of the underlying gravitational theory) on the relation between circular null geodesics and the fastest way to circle a black hole.

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I. INTRODUCTION

As is well known, the study of null geodesics at a given spacetime is not only relevant from a theoretical point of view, but also from a practical perspective. Their analysis, in addition to information about the causal structure of the spacetime, can also establish observable consequences of various astrophysical phenomena \([1–7]\).

The role of so-called circular null geodesics is not trivial. In some situations, these geodesics can allow the existence of what is known as a photon surface, which, in the context of four-dimensional spacetimes, is a three-dimensional non-spacelike manifold \(\mathcal{P}\), such that every null geodesic whose tangent vector \(l^\mu\) at a given point \(p \in \mathcal{P}\) is contained in the tangent space \(T_p \mathcal{P}\) of \(\mathcal{P}\) at \(p\), always remains in \(\mathcal{P}\). In particular, in the case of spherically symmetric spacetimes, a photon sphere can be defined as a \(SO(3) \times \mathbb{R}\)-invariant photon surface. For rigorous definitions see \([8]\).

The study of photon spheres in the case of spherical symmetry and circular null geodesics in axial symmetric spacetimes is important in astrophysics for several reasons:

a) They appear explicitly in the study of relativistic images produced by black holes in the strong lensing regime \([9, 10]\).

b) They play an important role in the analysis of quasinormal modes in black hole perturbations \([11, 12]\).

c) They determine the shadow of black holes, or (which is the same), how they look to outside observers \([13]\).

d) In the case of hairy black holes, they determine a lower bound on the size of the hair \([14]\).

e) They allow a link between points a) and b), or more precisely between gravitational waves and lensing, as recently shown in \([13, 10]\).

They have also been analyzed in situations that do not contain black holes, as for example, in Boson stars \([17]\), where the authors showed that in certain configurations photons spheres can occur, or in certain classes of regular metrics with non-negative trace of the energy-momentum tensor, where the existence of at least two photon spheres was shown \([18]\). Recently, a study of the thermodynamics of a quantum version of photon spheres was presented \([19]\).

Although in axially-symmetric spacetimes there is no notion yet of the analog of photon surfaces, there are circular null geodesics on the equatorial plane, and they are also useful for the discussion of the previous points. For all these considerations, the characterization and localization of these particular surfaces or circular null geodesics is relevant.

Recently \([20]\), Hod made some interesting observations about this type of null geodesics. In particular, in the framework of four-dimensional general relativity (GR), he analyzed a general family of spherically symmetric black holes, which satisfy some natural asymptotic and energy conditions, finding an upper bound for the radius of photon spheres in terms of their ADM mass. In the case of axial symmetry, he studied circular null geodesics on the equatorial plane of a Kerr spacetime, finding that they are also the fastest way to circle a Kerr black hole \([21]\) and conjecturing an expression for the minimum orbital time to circle any compact object in GR in terms of its mass. As shown by Pradhan \([22]\), the conclusion that circular null geodesics are the fastest way to circle black...
holographic phenomena. In this context, the effect of those Gauss-Bonnet terms is nontrivial for higher dimensions, so the theory of gravity, which includes Gauss-Bonnet terms, is called Einstein-Gauss-Bonnet (EGB) gravity.

Even without these Gauss-Bonnet corrections, the n-dimensional version of GR is usually studied in astrophysical and theoretical contexts, and in general it is also the case using other alternative gravitational theories. For some works on these topics in EGB or pure GR see [11, 26, 28, 31].

Because of the evidence garnered of the importance of circular null geodesics in the characterization of spherically and axially-symmetric black holes, we extend and generalize some of the Hod results to higher-dimensional gravitational theories. In particular we study photon spheres in n-dimensional GR and EGB theory and circular null geodesics on the equatorial plane for generic axially-symmetric spacetimes.

The article is organized as follows. In section II we review EGB theory, and prepare the setting for the discussion of photon spheres. In section III, we establish and prove two theorems which state upper bounds on the real part of quasinormal frequencies in the WKB limit and a lower bound for the first relativistic per bound for the real part of quasinormal frequencies in higher dimensions. The area of the unit (n - 2)-sphere

$$\Omega_{n-2} = 2\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right),$$

with $$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$, the gamma function.

The Einstein-Gauss-Bonnet equations in terms of the components of the energy momentum tensor, $$T^{\mu}_{\nu} = -ho$$, $$T^{\tau}_{r} = p_r$$ and $$T^\theta_\phi = p_\phi$$ reads

$$\kappa T_{ab} = G_{ab} = G^{(0)}_{ab} + G^{(1)}_{ab} + G^{(2)}_{ab},$$

where $$\kappa = 8\pi G/c^4$$ is the gravitational constant, $$T_{ab}$$ is the energy-momentum tensor, representing the matter-field distribution resulting from the variation $$\delta S_{\text{matt}}/\delta g^{ab}$$, and

$$G^{(0)}_{ab} = \Lambda g_{ab},$$

$$G^{(1)}_{ab} = R_{ab} - \frac{1}{2} R g_{ab},$$

$$G^{(2)}_{ab} = -\alpha \left[ \frac{1}{2} g_{ab} (R_{cjke} R^{jke} - 4R_{cj} R^{cj} + R^2) - 2R R_{ab} + 4 R_{ac} R^{c}_{\ b} + 4 R_{cj} R^{cj}_{\ ab} - 2 R_{acj} R^{c_{j}e}_{\ ab} \right].$$

From now on, we will use units where $$\hbar = 1$$ and we adopt $$\alpha$$ positive since this condition arises from the string theory. We also assume asymptotically flat spacetimes, and therefore $$\Lambda = 0$$.

Let us consider a spherically symmetric metric given by

$$ds^2 = -e^{-2b(r)}(\rho(r)dt^2 + \mu(r)^{-1}dr^2 + r^2 d\Omega_{n-2}^2),$$

with $$d\Omega_{n-2}^2$$ being the metric of the (n - 2)-sphere

$$d\Omega_{n-2}^2 = d\theta_i^2 + \sum_{i=2}^{n-2} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2,$$

solution to the Einstein-Gauss-Bonnet equations in higher dimensions. The area of the unit (n - 2)-sphere is given by

$$S_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)},$$

where $$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$, the gamma function.

The action that describes Einstein-Gauss-Bonnet gravity coupled with matter fields reads:

$$S = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left[ R - 2\Lambda + \alpha (R_{abcd} R^{abcd} + R^2 - 4 R_{ab} R^{ab}) + S_{\text{matt}} \right],$$

where $$S_{\text{matt}}$$ is the action associated with the matter fields, and $$\alpha$$ is the Gauss-Bonnet coupling constant associated in the string models, with the tension of these strings. This constant introduces a length scale. In fact, the corrections that this theory produces to GR, are noted at short distances, given by the scale $$l = \sqrt{4\alpha}$$.

The equations of motion resulting from $$\delta S = 0$$ are

$$\kappa T_{ab} = G_{ab} = G^{(0)}_{ab} + G^{(1)}_{ab} + G^{(2)}_{ab},$$

where $$\kappa = 8\pi G/c^4$$ is the gravitational constant, $$T_{ab}$$ is the energy-momentum tensor, representing the matter-field distribution resulting from the variation $$\delta S_{\text{matt}}/\delta g^{ab}$$, and

$$G^{(0)}_{ab} = \Lambda g_{ab},$$

$$G^{(1)}_{ab} = R_{ab} - \frac{1}{2} R g_{ab},$$

$$G^{(2)}_{ab} = -\alpha \left[ \frac{1}{2} g_{ab} (R_{cjke} R^{jke} - 4R_{cj} R^{cj} + R^2) - 2R R_{ab} + 4 R_{ac} R^{c}_{\ b} + 4 R_{cj} R^{cj}_{\ ab} - 2 R_{acj} R^{c_{j}e}_{\ ab} \right].$$

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solution to the Einstein-Gauss-Bonnet equations in higher dimensions. The area of the unit (n - 2)-sphere is given by

$$S_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)},$$

with $$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$, the gamma function.

The Einstein-Gauss-Bonnet equations in terms of the components of the energy momentum tensor, $$T^{\mu}_{\nu} = -\rho$$, $$T^{\tau}_{r} = p_r$$ and $$T^\theta_\phi = p_\phi$$ reads

$$\kappa \rho = \frac{1}{2 r^2} (n - 2) [\rho_\mu'(r) + (n - 3)(\mu - 1)]$$

$$+ \frac{\alpha}{2 r^4} (n - 2)(\mu - 1)[2\rho_\mu'(r) + (n - 5)(\mu - 1)],$$

$$\kappa p_r = -(n - 2) \left\{ \frac{1}{2 r^2} [2\rho_\mu'(r) - \rho_\mu'(r) - (n - 3)(\mu - 1)]$$

$$- \frac{\alpha}{2 r^4} (\mu - 1)[-2\rho_\mu'(r) + 4\rho_\mu'(r) - (\mu - 1)(n - 5)] \right\},$$

$$\kappa p_\phi = -(n - 2) \left\{ \frac{1}{2 r^2} [2\rho_\mu'(r) - \rho_\mu'(r) - (n - 3)(\mu - 1)]$$

$$- \frac{\alpha}{2 r^4} (\mu - 1)[-2\rho_\mu'(r) + 4\rho_\mu'(r) - (\mu - 1)(n - 5)] \right\},$$

$$\kappa p_\phi = -(n - 2) \left\{ \frac{1}{2 r^2} [2\rho_\mu'(r) - \rho_\mu'(r) - (n - 3)(\mu - 1)]$$

$$- \frac{\alpha}{2 r^4} (\mu - 1)[-2\rho_\mu'(r) + 4\rho_\mu'(r) - (\mu - 1)(n - 5)] \right\}.$$
where $\tilde{\alpha} = \alpha(n - 3)(n - 4)$. We have not written the angular-angular equations because we do not need them. Instead, we will use the only nontrivial component of the energy-momentum conservation equation $\nabla_a T^a_\gamma = 0$, which reads

$$p_r' = -\frac{(e^{-2\tilde{\alpha}} \mu)'(\rho + p_r) + n - 2}{r}(\rho - p_r). \quad (6)$$

From eqs.(4) and (5) it follows that

$$\mu(2\tilde{\alpha}(\mu - 1) - r^2)\delta = \kappa \frac{r^3}{(n-2)}(\rho + p_r). \quad (7)$$

We are interested in regular black holes. In particular, we assume that at the horizon $r_H$, $\mu(r)$ and $\delta(r)$ satisfy

$$\mu(r_H) = 0, \quad \delta'(r_H) \leq \infty. \quad (8)$$

These conditions, together with (7) imply

$$\rho(r_H) + p_r(r_H) = 0. \quad (10)$$

Because of the asymptotic flatness requirement, we also assume

$$\mu(r \to \infty) = 1, \quad \delta(r \to \infty) \to 0. \quad (11)$$

Note that, by requiring a GR limit as $\tilde{\alpha} \to 0$, the general solution of (4) can be written as

$$\mu = 1 + \frac{r^2}{2\tilde{\alpha}} \left(1 - \sqrt{1 + \frac{8\kappa \tilde{\alpha} M(r)}{(n-2)S_{n-2} r^{n-1}}} \right), \quad (13)$$

with $M(r)$ given by

$$M(r) = M_H + S_{n-2} \int_{r_H}^r \rho r^{n-2} dr; \quad (14)$$

which can be shown to be the generalized Misner-Sharp mass. In particular, $M_H$ is the horizon mass, and when $r$ goes to infinity, $M(r)$ goes to the ADM mass $M$. In order to have a finite ADM mass, $\rho$ must satisfy

$$\lim_{r \to \infty} r^{n-1} \rho = 0. \quad (15)$$

In the vacuum case ($T^a_\gamma = 0$), we have

$$\mu = 1 + \frac{r^2}{2\tilde{\alpha}} \left(1 - \sqrt{1 + \frac{8\kappa \tilde{\alpha} M}{(n-2)S_{n-2} r^{n-1}}} \right); \quad (16)$$

and the associated vacuum metric is known as the Boulware-Deser-Wheeler (BDW) black hole. If $\tilde{\alpha} \to 0$, Eq.(13) reduces to its GR limit,

$$\mu = 1 - \frac{2\kappa M(r)}{r^{n-2}}. \quad (17)$$

Finally,

$$T = -\rho + p_r + (n-2)p_\perp; \quad (18)$$

denotes the trace of the energy momentum tensor and we assume that matter satisfies the dominant energy condition (DEC) $\rho \geq 0, \rho \geq [p_r, |p_\perp|]$. From the DEC, we see then that $M_H \leq M(r) \leq M$. 

### III. Upper bound on the photon sphere radii

We are now prepared to establish the following theorems:

**Theorem 1.** Hod’s theorem (n-dimensional GR version): Let $(M, g_{ab})$ be a spherically symmetric spacetime, such that: i) it has a regular event horizon, ii) it is asymptotically flat, iii) the dominant energy condition is satisfied and the energy-momentum trace is nonpositive, iv) it satisfies the n-dimensional Einstein equations. Then this metric admits at least one photon sphere, which in the coordinates given by (4) is characterized by a radius $r_\gamma$, which is bounded by the following expression in terms of its total ADM mass $M$:

$$r_\gamma \leq \left(\frac{\kappa(n-1)}{(n-2)S_{n-2}} M \right)^{\frac{1}{n-3}}. \quad (19)$$

This theorem can be extended to Einstein-Gauss-Bonnet theory in n-dimensions. In particular, in five dimensions we can obtain an upper bound for the photon sphere radius in terms of the ADM mass and the constant $\tilde{\alpha}$. For $n > 5$, although we cannot give a general expression for the photon sphere radius bound, in terms of the ADM mass (with the exception of $n = 9$), we can state a more general result. These properties are summarized in the following theorem.

**Theorem 2.** n-dimensional EGB version: Let $(M, g_{ab})$ be a spherically symmetric spacetime such that: a) it satisfies conditions i), ii) and iii) of Theorem 1, b) it satisfies the n-dimensional EGB equations. Then this metric admits at least one photon sphere, whose radius $r_\gamma$ in the coordinates given by (4) is always bounded by the photon sphere radius $r_\gamma^*$ of a vacuum spherically symmetric BDW black hole with the same total ADM mass $M$, i.e. $r_\gamma \leq r_\gamma^*$. In particular in five dimensions the photon sphere radius is bounded by the following expression in terms of its total ADM mass $M$:

$$r_\gamma \leq \frac{\sqrt{6}[\kappa M(\kappa M - 3\tilde{\alpha} \pi^2)]^{1/4}}{3\pi}. \quad (20)$$

Note that the inequality (20) is saturated for a five-dimensional BDW black hole with mass $M$.

**Proof:** Common part. Since the Einstein equations can be recovered from (4) and (5) with $\tilde{\alpha} = 0$, in the proof there is a common part for both theorems. Basically, we follow the same steps as Hod’s proof [20], although some changes are needed to incorporate the n-dimensional case. Because of the spherical symmetry, null geodesics may without loss of generality be taken to be on the equatorial plane. From the metric (4) it is easy to show that the equation for circular null geodesics
\[ \dot{\gamma} = (\dot{r}_\gamma)' = 0 \] (where a dot means derivative with respect to an affine parameter and \( ' \) means derivative with respect to \( r \)) reads

\[ \tilde{N}(r_\gamma) = 0; \quad (21) \]

with

\[ \tilde{N}(r) = 2e^{-2\delta(r)} \mu(r) - r[e^{-2\delta(r)} \mu(r)]'. \quad (22) \]

By defining the function

\[ N(r) = e^{2\delta(r)} \tilde{N}(r), \quad (23) \]

we similarly obtain

\[ N(r_\gamma) = 2[1 + r_\gamma \delta'(r_\gamma)]\mu(r_\gamma) - r_\gamma \mu'(r_\gamma) = 0. \quad (24) \]

From the trace of the energy-momentum tensor and \( (23) \), we can write \( (6) \) as

\[ p'_r = \frac{1}{2\mu}(N(\rho + p_r) + 2\mu T - 2n\mu p_r). \quad (25) \]

Now if we define:

\[ P = r^n p_r, \quad (26) \]

we arrive at

\[ P'(r) = \frac{r^{n-1}}{2\mu} (N(\rho + p_r) + 2\mu T). \quad (27) \]

In terms of the energy-momentum components, \( N \) reads

\[ N = \frac{1}{2\alpha(\mu - 1) - r^2} \left\{ (n - 1)\alpha \mu^2 - [(n - 1)r^2 + 2(n - 3)\alpha]\mu + (n - 5)\alpha + (n - 3)r^2 + \frac{2}{n - 2}\alpha r p_r \right\}; \quad (28) \]

which in the \( n \)-dimensional GR case reduces to

\[ N_{GR} = (n - 1)\mu - (n - 3) - \frac{2\alpha}{(n - 2)} r^2 p_r. \quad (29) \]

Let us observe that \( (24) \) admits at least one solution as follows from the fact that the conditions \( (8) \) and \( (23) \) imply \( N(r_H) \leq 0 \), and taking into account that

\[ \lim_{r \to \infty} r^2 p_r = 0 \] (which follows from the dominant energy condition and \( (15) \)), we see that \( N(r \to \infty) \to 2 \); therefore, there is a \( r_\gamma \) where \( N(r_\gamma) = 0 \). What is more, because we are interested in the innermost null circular orbit, \( N(r) \) must satisfy

\[ N(r_H) \leq r < r_\gamma < 0. \quad (30) \]

Then, by using the dominant energy condition we obtain

\[ p_r(r_H) = -\rho(r_H) \leq 0, \quad \Rightarrow p_r(r_H) \leq 0 \Leftrightarrow P(r_H) \leq 0. \quad (31) \]

Similarly from \( (27), (30) \) and the condition of nonpositive trace \( T \leq 0 \), we see that \( P(r_H \leq r < r_\gamma) \leq 0 \). From this last condition and \( (31) \) we have

\[ p_r(r_\gamma) \leq 0. \quad (32) \]

Let us analyze the implications of this relation in the Einstein and Einstein-Gauss-Bonnet cases separately.

**Completion of the proof of Theorem 1.** In the Einstein case, from \( (32) \) and \( (29) \) we conclude that

\[ \mu(r_\gamma) \leq \frac{n - 3}{n - 1}, \quad (33) \]

which by using \( (17) \) implies

\[ r_\gamma \leq \left( \frac{\kappa(n - 1)}{(n - 2)S_{n-2}} \right) \frac{n - 3}{n - 1}. \quad (34) \]

or in terms of the total ADM mass \( M \),

\[ r_\gamma \leq \left( \frac{\kappa(n - 1)}{(n - 2)S_{n-2}} \right) \frac{n - 3}{n - 1}. \quad (35) \]

**Completion of the proof of Theorem 2.** Let us start by noting that in \( (28) \) the following expression appears in the denominator

\[ 2\alpha(\mu - 1) - r^2 = \frac{r^2}{2\alpha} \sqrt{1 + \frac{8\alpha\alpha M(r)}{(n - 2)S_{n-2}}} < 0. \quad (36) \]

Therefore from \( (28) \) and \( (32) \) we deduce that at \( r_\gamma \),

\[ N^* = \left\{ (n - 1)\alpha \mu^2 - [(n - 1)r^2 + 2(n - 3)\alpha]\mu + (n - 5)\alpha + (n - 3)r^2 \right\} = -\frac{2}{n - 2}\alpha r p_r \geq 0. \quad (37) \]

The implications of this equation for the allowed \( r_\gamma \) will be discussed first in five dimensions because this case is exactly solvable, and after that, we will extend the analysis to higher dimensions.

By writing \( N^* \) as

\[ N^* = (n - 1)\alpha \mu - B_{(n)-} |(\mu - B_{(n)+}), \quad (38) \]

we see that in five dimensions, \( (37) \) implies that at \( r_\gamma \), some of these conditions hold:

\[ \mu \leq -\frac{1}{2\alpha}(\alpha + r^2 + \sqrt{\alpha^2 + r^4}) \equiv B_{(5)-}, \quad (39) \]

or

\[ \mu \geq \frac{1}{2\alpha}(\alpha + r^2 + \sqrt{\alpha^2 + r^4}) \equiv B_{(5)+}; \quad (40) \]

where in order to write these inequalities we use the fact that \( B_{(5)-} \) and \( B_{(5)+} \) are both positive real functions and that \( B_{(5)-} < B_{(5)+} \).
We also note [using $M_H \leq M(r) \leq M$] that independently of the dimension, $\mu$ is bounded from below by

$$\mu \geq 1 + \frac{v^2}{2\alpha} \left(1 - \sqrt{1 + \frac{8\alpha \delta M}{(n - 2)S_{n-2}p^{n-1}}} \right) \equiv \mu_{(n)M};$$

and by

$$\mu \leq 1 + \frac{v^2}{2\alpha} \left(1 - \sqrt{1 + \frac{8\alpha \delta M H}{(n - 2)S_{n-2}p^{n-1}}} \right) \equiv \mu_{(n)MH};$$

from above. Consequently, the validity of the conditions (43) or (44) implies

$$\mu_{(5)M}(r_\gamma) \leq B_{(5)-}(r_\gamma);$$

or

$$\mu_{(5)MH}(r_\gamma) \geq B_{(5)+}(r_\gamma);$$

respectively. Moreover, the four functions $\mu_M(r)$, $\mu_{(5)M}(r)$, $B_{(5)+}(r)$ and $B_{(5)-}(r)$ are all monotonically increasing functions. From this, and observing that $B_{(5)+}(r) \geq 1$, and $\mu_{(5)MH}(r \to \infty) = 1$, it follows that

$$\mu(r) \leq \mu_{(5)MH}(r) < B_{(5)+}(r) \forall r,$$

thus, it is not possible for any real $r_\gamma$ to satisfy the inequality (40).

Let us study now the inequality (41). Observing that for $r \geq r_H$,

$$0 < B_{(5)-}(r) < B_{(5)-}(r \to \infty) = \frac{1}{2},$$

and

$$\mu_{(5)M}(r = r_H) \leq 0, \quad \mu_{(5)M}(r \to \infty) = 1,$$  

we see that (43) will be valid for all $r$ such that $r_H \leq r \leq r_\gamma^*$, with $r_\gamma^*$ solution of

$$\mu_{(5)M}(r_\gamma^*) = B_{(5)-}(r_\gamma^*).$$

The only positive real solution of this equation is

$$r_\gamma^* = \frac{\sqrt{6}(\kappa M - 3\alpha \pi^2)}{3\pi}. $$

On the other hand, as $\mu(r_H) = 0$, $B_{(5)-}(r_H) > 0$, and $\mu(r) \geq \mu_{(5)M}(r)$, we conclude that $\mu(r) > B_{(5)-}(r)$ for all $r > r_\gamma^*$. Consequently, the radius $r_\gamma$ which satisfies (39) must necessarily be bounded from above by $r_\gamma^*$, that is

$$r_\gamma \leq \frac{\sqrt{6}(\kappa M - 3\alpha \pi^2)}{3\pi}.$$  

In the $n$-dimensional case, as was mentioned above, we cannot find an explicit upper bound in terms of the ADM mass; however, we can show that even in these situations, the photon sphere radius will always be bounded by the photon sphere radius of a BDW black hole with the same mass. It can be shown as follows. First, in the general case, following a similar analysis as in Eqs. (38), (39) and (40), we see that (37) is satisfied at $r_\gamma$ if

$$\mu \leq \frac{1}{2\alpha} \left( \frac{2(n-3)}{n-1} \alpha + r^2 - \sqrt{16 \left( \frac{n-1}{2} \alpha^2 + r^4 \right)} \right) \equiv B_{(n)-},$$

or if

$$\mu \leq \frac{1}{2\alpha} \left( \frac{2(n-3)}{n-1} \alpha + r^2 + \sqrt{16 \left( \frac{n-1}{2} \alpha^2 + r^4 \right)} \right) \equiv B_{(n)+},$$

which using (41) and (42) implies

$$\mu_{(n)M}(r_\gamma) \leq B_{(n)-}(r_\gamma);$$

or

$$\mu_{(n)MH}(r_\gamma) \geq B_{(n)+}(r_\gamma).$$

For the same reasons as in the five-dimensional case $B_{(n)+}(r) \geq 1$, $\mu_{(n)MH}(r \to \infty) = 1$, and both being monotonically increasing functions, we conclude that there is no real $r_\gamma$ so that (53) and hence (52) can be satisfied. On the other hand, from the fact that $B_{(n)-}$ and $\mu_{(n)M}$ are monotonically increasing functions, and taking into account that

$$0 \leq B_{(n)-}(r) < B_{(n)-}(r \to \infty) = \frac{n-3}{n-1} < 1 \forall r \geq r_H,$$

and $\mu_{(n)M}(r_H) \leq 0$, $\mu_{(n)M}(r \to \infty) = 1$, we see that (53) holds for all $r_H \leq r < r_\gamma^*$ with $r_\gamma^*$ satisfying

$$\mu_{(n)M}(r_\gamma^*) = B_{(n)-}(r_\gamma^*).$$

As $\mu(r_H) = 0$, $B_{(n)-}(r_H) > 0$ and $\mu(r) \geq \mu_{(n)M}(r)$; and following the same arguments as in the five-dimensional situation, we conclude that the radius $r_\gamma$ where (39) holds, will also satisfy $r_\gamma \leq r_\gamma^*$. Furthermore, from (28), (39) and (38), and observing that

$$N^* = [\hat{\alpha}(\mu - 1) - r^2]N - \frac{2}{n-2}\kappa r^4 p, $$

we see that (56) is equivalent to requiring $N(r_\gamma^*) = 0$ for a BDW black hole with ADM-mass $M$ ($\rho = p_r = 0$). In other words, $r_\gamma^*$ can be interpreted as the radius of the photon sphere for a BDW black hole with mass $M$. As a result, the bound is saturated in these cases.
eight dimensions this quotient reaches its minimum value. However, as for \( n > 4 \), the dependence in the mass is not linear, the dimension which minimizes the value of the upper bound for the radius \( r_\gamma \) at a given fixed \( M \) will depend on the value of this mass.

\[
\frac{r_\gamma}{(GM)^{1/3}}
\]

Let us also observe that if we take \( \alpha \to 0 \) in (50), we recover the upper bound that we found in the GR case.

As a final comment, we mention that (58) can also be solved in nine dimensions, obtaining the result

\[
r_\gamma \leq 98(14)^{1/3} \left[ \frac{\hat{h}^{2/3} - 686(14)^{2/3} \pi^3 \hat{\kappa} \hat{M}}{\pi^3 \hat{h}^{1/3}} \right]^{1/4},
\]

with

\[
h = \pi \kappa \hat{M} \left[ 18 \kappa \hat{M} + \sqrt{2 \kappa \hat{M} (162 \kappa \hat{M} + 7 \pi^4 \hat{\kappa}^3) \} \right].
\]

Let us note that, in our study cases, \( \hat{\alpha} \geq 0 \) and \( r_\gamma \geq r_H \); therefore,

\[
\Omega_\infty \leq \left[ \frac{\hat{n} \hat{\alpha} + \frac{3}{n+1}}{\hat{n} \hat{\alpha} + \frac{3}{n+1} + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + \frac{4}{\gamma} \hat{r}_H^4}} \right]^{1/2}.
\]

In order to compare this with the Hod bound, we will give a weaker inequality, but one which is simpler and more universal than (65) in the sense that it does not depend on the particular properties of the black hole. Let us note that, in our study cases, \( \hat{\alpha} \geq 0 \) and \( r_\gamma \geq r_H \); therefore,

\[
\omega_l \leq \left[ \frac{\hat{n} \hat{\alpha} + \frac{3}{n+1}}{\hat{n} \hat{\alpha} + \frac{3}{n+1} + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + \frac{4}{\gamma} \hat{r}_H^4}} \right]^{1/2}.
\]

Combining this inequality with (62), we finally obtain

\[
\omega_l \leq \left[ \frac{\hat{n} \hat{\alpha} + \frac{3}{n+1}}{\hat{n} \hat{\alpha} + \frac{3}{n+1} + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + \frac{4}{\gamma} \hat{r}_H^4}} \right]^{1/2};
\]

which implies

\[
\omega_l \hat{r}_H \leq \left[ \frac{\hat{n} \hat{\alpha} + \frac{3}{n+1}}{\hat{n} \hat{\alpha} + \frac{3}{n+1} + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + \frac{4}{\gamma} \hat{r}_H^4}} \right]^{1/2}.
\]
This expression generalizes to $n$-dimensional EGB theory, the universal bound found by Hod in the framework of four-dimensional GR\cite{20}. In particular, in the GR case it reduces to
\begin{equation}
\omega_l r_H \leq l \left[ \frac{n - 3}{n - 1} \right]^{1/2}, \tag{69}
\end{equation}
which, when $n = 4$ agrees with the expression given in \cite{20}. Note also that in five dimensions the first factor of \cite{58} does not survive, and we get the simple bound
\begin{equation}
\omega_l r_H \leq \frac{1}{\sqrt{2}} l. \tag{70}
\end{equation}

At this point, we can ask if there is a similar universal bound for the Lyapunov exponent; however, the answer is negative. The Lyapunov exponent for circular null geodesics is defined by \cite{11}
\begin{equation}
\lambda = \sqrt{\frac{r_0^2 [e^{-2\delta(r_0)} \mu(r_0)]}{2L^2} V''(r_0)}, \tag{71}
\end{equation}
with
\begin{equation}
V''(r_0) = \frac{L^2}{r_0^2} \left\{ e^{-2\delta(r_0)} \mu(r_0) - r_0^2 [e^{-2\delta(r_0)} \mu(r_0)]'' \right\}, \tag{72}
\end{equation}
and $L$ the orbital angular momentum of the circular null geodesics. After some simple computations it can be shown that in $n$-dimensional GR,
\begin{equation}
V''(r_0) = \frac{L^2 N'(r_0)}{r_0^3}. \tag{73}
\end{equation}
This makes it possible to write \cite{71} as
\begin{equation}
\lambda = \sqrt{\frac{[e^{-2\delta(r_0)} \mu(r_0)] N'(r_0)}{2r_0}}. \tag{74}
\end{equation}
From \cite{29} we have
\begin{align}
N'(r_0) &= (n - 1) \mu'(r_0) - \frac{2\kappa}{(n - 2)} \left[ 2r_0 p_r(r_0) + r_0^2 p_r'(r_0) \right] \\
&\geq (n - 1) \mu'(r_0) + 2\kappa r_0 p_r(r_0), \tag{75}
\end{align}
where in order to establish the last inequality, we have taken into account that at $r_0$, $p_r$ and $P''$ are both non-positive numbers. However, from the first factor in \cite{74}, which is the factor $e^{-2\delta(r_0)} \mu(r_0)$, we can only ensure that $e^{-2\delta(r_0)} \mu(r_0) \leq \mu(r_0)$, and therefore this inequality and \cite{75} impose bounds in opposite directions, thereby preventing a universal bound (independent of the matter content) from being written for the Lyapunov exponent.

Coming back to the inequality \cite{68}, it can also be used to impose a universal bound over the location of the first relativistic image in the strong lensing regime. In general, if we assume that the observer is at a distance $D_{ol}$ from the lens, then the first relativistic image subtends an angle $\theta_{\infty} [10,15]$, which can be expressed in terms of the circular orbital frequency $\Omega_{\infty}$ as
\begin{equation}
\theta_{\infty} = \frac{1}{D_{ol} \Omega_{\infty}}. \tag{76}
\end{equation}
From this relation and using \cite{68}, we obtain the universal and simple lower bound
\begin{equation}
\theta_{\infty} \geq \frac{r_H}{D_{ol}} \left[ \frac{n - 3}{(n - 1)r_H^2 + \frac{n - 3}{n - 1}} \right]^{1/2}. \tag{77}
\end{equation}
In the GR limit it reduces to
\begin{equation}
\theta_{\infty} \geq \frac{r_H}{D_{ol}} \left[ \frac{n - 1}{n - 3} \right]^{1/2}, \tag{78}
\end{equation}
which, for the 4-dimensional case gives
\begin{equation}
\theta_{\infty} \geq \sqrt{3} \frac{r_H}{D_{ol}}. \tag{79}
\end{equation}

V. GENERAL COMMENT ON THE FASTEST WAY TO CIRCLE AXIALLY-SYMMETRIC SPACETIMES

It would be very interesting to find a generalization of some of the previous results for the case of axial symmetry. Even in this case, the circular null geodesics on the equatorial plane share some of the properties found in the spherical symmetric case. For example, Wei and Liu \cite{10} found universal relations between the first relativistic image and the quasinormal frequencies by analyzing equatorial circular null geodesics in arbitrary axially-symmetric black holes, thus extending some of the results of spherical symmetry. Recently, Hod\cite{21} also showed that the fastest way to circle a black hole in general relativity is through circular null geodesics. In particular, he demonstrated that for any spherically symmetric spacetime in four dimensions, the minimum orbital time as measured by an asymptotic observer is realized by circular null geodesics, independently of the underlying gravitational theory. Moreover, he showed that a similar result can be obtained in GR for the case of a Kerr black hole by noting that the equatorial circular orbit with minimum traveling time coincides with the solution obtained by solving the equation governing circular null geodesics. More recently, in \cite{22}, Pradhan, in his study of charged Myers-Perry black holes in higher dimensions, made the observation that Hod’s conclusion regarding the fastest way to circle a black hole remains valid in this more general family of metrics.

As mentioned, the fact that circular null geodesics minimize the orbital period in the case of spherically symmetric spacetimes is always valid. However, in \cite{21}, when Hod analyzed the Kerr metric, he obtained two different equations (one for the fastest circular orbit, and another
for the circular null geodesics, see Eqs. (21) and (31) in his paper), and he showed that they admit the same kind of solutions; therefore, it was not clear from his discussion how general these results were. In other words, one can ask whether the agreement between equatorial circular null geodesics and the fastest way to circle black holes is a property unique to GR or it is a geometrical property general to any axially-symmetric metric, independently of any gravity theory.

Additionally, Hod made the interesting conjecture that there should be a lower bound for the traveling time, given in terms of the ADM mass of the black hole [21]. More precisely, he conjectured a lower bound on the orbital periods $T_{\infty}$ of circular null geodesics on the equatorial plane. In mathematical terms he states that (in units with $G = c = 1$)

$$T_{\infty} \geq 4\pi M. \quad (80)$$

In particular, the equality is satisfied by an extreme Kerr black hole. This conjecture is physically motivated by taking into account the rotational dragging which is maximal in the case of an extremal black hole. If this conjecture were correct, it could also be used to establish another universal bound for the possible location of the first relativistic image of an object lensed by a black hole. In fact, if (80) were valid we should obtain $\Omega_{\infty} = 2\pi/T_{\infty} \leq 1/M$, which would imply

$$\theta_{\infty} \geq \frac{2M}{D_{od}}. \quad (81)$$

In other words, the expression (81) would also be valid for a generic rotating black hole, at least in the GR regime.

Even given the reasonableness of conjecture (80), it has not yet been proven. However, it leads one to wonder whether this conjecture can be kept as potentially valid in alternative gravitational theories. It is our purpose in this section to answer these questions. In particular, we make the observation that the orbital period for circular orbits always coincides with the circular null geodesics for any axially-symmetric spacetime, independently of the gravitational theory. We also answer the second question negatively on the validity of (81) for alternative black hole candidates by giving an explicit counterexample where the bound assumed by Hod is not satisfied. What is more, we will show that the conjecture can be violated even in the case of a nonrotating black hole. In order to do that, we will discuss a special family of Kaluza-Klein black holes.

Let us start with the first question. We begin, with the more general conformally stationary and axially-symmetric metric in $n$-dimensions, without making reference to any gravitational theory. These kinds of metrics are given by

$$ds^2 = e^{2\phi(r,\theta, t, \cdot)} (g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2 + r^2 \cos(\theta)^2 d\Omega^2_{n-3}), \quad (82)$$

with

$$d\Omega^2_{n-3} = d\phi_3^2 + \sin^2 \phi_2 \left[ d\phi_3^2 + \sin^2 \phi_3 (\cdots d\phi_3^2) \right], \quad (83)$$

and with the components of the metric not depending on $\phi$. As this metric is axially-symmetric, there are equatorial orbits, and in particular we can compute from this metric the orbital time for circular null curves, i.e., curves such that $ds^2 = 0, \theta = \pi/2$ and $r = r_\gamma$. The orbital period for these curves is:

$$T_{\infty} = 2\pi \frac{g_{t\phi} \pm \sqrt{\Delta}}{g_{tt}}, \quad (84)$$

where

$$\Delta = g_{\phi\phi}^2 - g_{tt} g_{\phi\phi}, \quad (85)$$

and the $+/-$ signs correspond to counter-rotating/co-rotating orbits, respectively. If we compute their extremes we obtain

$$\frac{dT}{dr} = \frac{\pi}{\sqrt{\Delta}} G = 0, \quad (86)$$

with

$$G_{\pm} = \frac{1}{g_{tt}} \left[ (\mp 2g_{t\phi} \sqrt{\Delta} + \Delta + g_{\phi\phi}^2) \frac{dg_{tt}}{dr} ight. \left. + (-2g_{t\phi} g_{\phi\phi} \pm 2g_{tt} \sqrt{\Delta}) \frac{dg_{t\phi}}{dr} + g_{\phi\phi}^2 \frac{dg_{\phi\phi}}{dr} \right]. \quad (87)$$

So, if we assume that $\Delta$ is a regular function and we only consider the region exterior to all horizons (where $\Delta = 0$), the minimum orbital period is obtained for those curves which satisfy $G_{\pm} = 0$.

On the other hand, we can study the condition for the existence of circular null geodesics on the equatorial plane. We start from the conservation equations

$$E = - \left( g_{tt} \dot{t} + g_{t\phi} \dot{\phi} \right), \quad (88)$$

$$L = g_{t\phi} \dot{t} + g_{\phi\phi} \dot{\phi}, \quad (89)$$

$$p_r = g_{rr} \dot{r}, \quad (90)$$

which represent the energy of the null particle as measured by an asymptotic observer, the orbital angular momentum and the radial component of the linear momentum, respectively. These equations can be solved for $\dot{t}$, and $\dot{\phi}$ and replaced in the Hamiltonian $H$,

$$H = -E \dot{t} + L \dot{\phi} + p_r \dot{r} = 0; \quad (91)$$

obtaining the following equation for $\dot{r}$:

$$\dot{r}^2 = V_r; \quad (92)$$

with

$$V_r = - \frac{E^2}{g_{rr} \Delta} \left( g_{\phi\phi} + 2g_{t\phi} b + g_{tt} b^2 \right), \quad (93)$$
and $b = L/E$, the impact parameter. Circular null geodesics must satisfy $\dot{r}_\gamma = \dot{r}_\gamma' = 0$, or equivalently $V_r = V_r' = 0$. From $V_r = 0$, we obtain

$$b = \pm \frac{\sqrt{\Delta}}{g_{tt}}, \quad (94)$$

and from $V_r' = 0$ it follows that

$$\frac{d}{dr} g_{\phi\phi} + 2b \frac{d}{dr} g_{t\phi} + b^2 \frac{d}{dr} g_{tt} = 0, \quad (95)$$

Finally, by replacing the expression for $b$ given by (94) in the last equation, we conclude that the circular null geodesics, if they exist, must satisfy

$$\frac{1}{g_{tt}} \left[ (\pm 2g_{t\phi}\sqrt{\Delta} + \Delta + g_{t\phi}^2) \frac{dg_{tt}}{dr} \right] = 0,$$

which agrees with the expression for $G_{\pm}$ given previously. As a result, its critical points always coincide with the values of the circular geodesics, generalizing in this way the results given by Hod for spherically symmetric spaces, and the particular cases of Kerr and Myers Perry. Let us remark that there is not inconsistency between this result and the fact that in [21] Hod obtained two different algebraic equations [Eqs. (21) and (31) in [21] for the minimum orbital period and geodesic motion, respectively]. In fact, the circular null geodesics are obtained by solving $\dot{r}_\gamma = 0$ and $\dot{r}_\gamma' = 0$, both being functions of $b$. If we first solve $\dot{r}_\gamma' = 0$ for $b$ and replace its value in $\dot{r}_\gamma = 0$, (as the path followed by Hod) we obtain Eq. (31) of [21]. Alternatively, if we first solve for $b$ from $\dot{r}_\gamma = 0$ and replace it in $\dot{r}_\gamma' = 0$, we arrive to his Eq. (21).

As mentioned above, in [21], by comparing the total orbital period for rotating Kerr black holes with those of Schwarzschild ones, a lower bound for the orbital periods $T_\infty$ (as measured from far away observers) of circular null geodesics around compact objects with mass $M$, Eq. (30) was conjectured. Now, by giving an explicit counterexample, we will show that, in contrast to the geometrical status of fastest way to circle black holes, this conjecture cannot be generally valid in other black holes candidates coming from alternative gravitational theories.

The counterexample is based on the Kaluza-Klein black hole. Shadows of these kinds of black holes were recently analyzed in [33]. Let us consider the following metric, which satisfies the vacuum Einstein equations in five dimensions.

$$ds^2 = -(1 - \frac{2m}{r}) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) + dy^2.$$  

(97)

Then, by doing a compactification of the extra dimension and a boost transformation with velocity $v$ in the $y$-direction, and by projecting in the 4-manifold, a new metric is obtained representing a charged spherically symmetric black hole together with a dilaton field [32]. The four-dimensional metric reads:

$$ds^2 = - \frac{1 - 2m}{B} dt^2 + \frac{B}{1 - \frac{2m}{r}} dr^2 + B r^2 (d\theta^2 + \sin(\theta)^2 d\phi^2), \quad (98)$$

with

$$B = \sqrt{1 + \frac{2mv^2}{r(1 - v^2)}}. \quad (99)$$

This metric together with the scalar field

$$\Phi = - \frac{\sqrt{3}}{2} \ln B, \quad (100)$$

and the electromagnetic potential $A$

$$A = \frac{v}{2(1 - v^2)} \frac{1 - 2m}{B^2} dt, \quad (101)$$

satisfies the set of Einstein-Maxwell-dilaton field equations that follow from the action:

$$S = \int d^4x \sqrt{-g} \left[ R + 2(\nabla \Phi)^2 - e^{2\sqrt{3}\Phi} F^2 \right]. \quad (102)$$

This metric represents a black hole with ADM mass $M$ and charge $Q$ given in terms of the parameter $m$ and the velocity $v$ by:

$$M = m \left[ 1 + \frac{v^2}{2(1 - v^2)} \right], \quad (103)$$

$$Q = \frac{mv}{1 - v^2}. \quad (104)$$

One of the characteristics of this type of metric is that the ADM mass, the dilaton field and the charge depend on the boost parameter. Note that the event horizon is located at $r_H = 2m$, which for a fixed mass $M$, shrinks to zero when $v$ goes to 1. Let us compute now the minimum orbital time that a circular null geodesic takes to orbit this black hole. In order to do this, we must solve for the co-rotating orbits, that is $G_- = 0$, which, using the expression (87) in this case gives

$$(v^2 - 2)r^2 - 2M(v^2 - 2)(4v^2 - 3)r - 16M^2 v^2 (1 - v^2) = 0. \quad (105)$$

The physical solution of this equation is

$$r_\gamma = \frac{3 - 4v^2 + \sqrt{9 - 8v^2}}{2 - v^2} M. \quad (106)$$

By replacing this radius in (87), we obtain

$$T_\infty = 2\pi \frac{3 - 4v^2 + \sqrt{9 - 8v^2} + 9}{2 - v^2} M.$$  

(107)
and it goes to zero when (Schwarzschild case) it takes the value $T_v$. A plot of this function is shown in Fig. 2. At a radius of photon spheres in $\pi$ are smaller than $4\pi M$, we found that for $v > 0$, $T_v$ there is a value of $v$ with mass $M$ can be made arbitrarily small. Therefore, there is a value of $v$ from which the Hod conjecture cannot be satisfied for this type of black holes. Numerically we found that for $v > 0.96591$ all the orbital periods $T_\gamma$ are smaller than $4\pi M$.

VI. SUMMARY

In this work, we have found upper bounds for the radius of photon spheres in $n$-dimensional GR and EGB theories, thereby generalizing the previous work of Hod in [20]. In the general situation of a black hole dressed with matter fields, we have seen that the photon sphere radius is always smaller than that corresponding to a vacuum black hole with the same mass. It would be interesting to know how generic these results are, in the sense of how much they depend on gravitational field equations. In particular, it would be natural to study whether these results hold for the more general Lovelock gravitational theory or if they can be extended to other theories like $f(R)$ ones.

With respect to the study of circular null geodesics in axial symmetry, we have observed that equatorial circular null geodesics always minimize the orbital time around a black hole. It is a geometrical result. Given the astrophysical importance of this type of geodesics, it would also be interesting to attempt a proof of the Hod conjecture [50]. We wish to deal with these and other associated problems in the near future.

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[1] Z. Stuchlík and M. Calvani, Null geodesics in black hole metrics with nonzero cosmological constant. Gen. Relat. Gravit. 23, 507–519 (1991).
[2] N. Cruz, M. Olivares, J.R. Villanueva, The geodesic structure of the Schwarzschild anti-de sitter black hole. Class. Quantum Gravity 22, 1167–1190 (2005).
[3] J.R. Villanueva, J. Saavedra, M. Olivares and N. Cruz, Photons motion in charged anti-de Sitter black holes. Astrophys. Space Sci 344, 437–446 (2013).
[4] S. Fernando, Null geodesics of charged black holes in string theory. Phys. Rev. D 85, 024033 (2012).
[5] J.R. Villanueva and M. Olivares, On the null trajectories in conformal Weyl gravity. J. Cosmol. Astropart. Phys. 06, 040 (2013).
[6] J.R. Villanueva and Y. Vásquez, About the coordinate time for photons in Lifshitz space-times. Eur. Phys. J. C 73, 2587 (2013).
[7] A. García, E. Hackmann, J. Kunz, C. Lämmerzahl and A. Macias, Motion of test particles in a regular black hole spacetime. J. Math. Phys. 56, 032501 (2015).
[8] Clarissa-Marie Claudel, K.S. Virbhadra and G.F.R. Ellis. The geometry of photon surfaces. J.Math.Phys, 42:818-838,(2001).
[9] V. Bozza. Gravitational Lensing by Black Holes. Gen.Rel.Grav., 42:2269-2300, (2010).
[10] V. Bozza. Gravitational lensing in the strong field limit. Phys. Rev. D, 66: 103001 (2002).
[11] V.Cardoso,A.S.Miranda,E.Berti,H.Witek,V.T.Zanchin. Geodesic stability, Lyapunov exponents, and quasinormal modes. Phys.Rev.D 79:064016 (2009).
[12] Yves Décanini, Antoine Folacci and Bernard Raffaelli. Unstable circular null geodesics of static spherically symmetric black holes, Regge poles and quasinormal frequencies. Phys.Rev.D 81:104039,(2010).
[13] F. Atamurotov, A. Abduljabbarov and B. Ahmedov. Shadow of rotating non-Kerr black hole. Phys. Rev. D, 88:064004 (2013).
[14] Shahrar Hod. Hairy black holes and null circular geodesics. Phys.Rev.D, 84:124030 (2011).
[15] I. Zh. Stefanov , S. S. Yazadjiev and G. G. Gyulchev. Connection between Black-Hole Quasinormal Modes and Lensing in the Strong Deflection Limit. Phys. Rev. Lett. 104:251103 (2010).
[16] Shao-Wen Wei and Yu-Xiao Liu. Establishing a
universal relation between gravitational waves and black hole lensing. *Phys. Rev. D*, 89:047502 (2014).

[17] Dubravko Horvat, Sasa Iljic, Anamaria Kirin and Zoran Narancic. Formation of photon spheres in boson stars with a nonminimally coupled field. *Class. Quantum Grav.*, 30:095014, (2013).

[18] Shahar Hod. Self-gravitating field configurations: The role of the energy-momentum trace. *Physics Letters B*, 739, 383 (2014).

[19] M. C. Baldiotti, Walace S. Elias, C. Molina and Thiago S. Pereira. Thermodynamics of quantum photon spheres. *Phys. Rev. D*, 90:104025 (2014).

[20] Shahar Hod. Upper bound on the radii of black-hole photon spheres. *Phys. Lett. B*, 727, 345-348 (2013).

[21] Shahar Hod. The fastest way to circle a black hole. *Phys.Rev.D*, 84, 104024 (2011).

[22] Partha Pratim Pradhan. Lyapunov Exponent and Charged Myers-Perry Spacetimes. *Eur. Phys. J. C* 73:2477 (2013).

[23] D. J. Gross and J. H. Sloan. The Quartic Effective Action For The Heterotic String. *Nucl. Phys. B* 291,41 (1987).

[24] R. R. Metsaev and A. A. Tseytlin. Order $\alpha'$ (two-loop) equivalence of the string equations of motion and the $\sigma$-model Weyl invariance conditions: Dependence on the dilaton and the antisymmetric tensor. *Nucl. Phys. B* 293, 385 (1987).

[25] H. Maeda and M. Nozawa. Generalized Misner-Sharp quasi-local mass in Einstein-Gauss-Bonnet gravity. *Phys.Rev.D* 77:064031, (2008).

[26] Naoki Tsukamoto, T. Kitamura, K. Nakajima, H. Asada. Gravitational lensing in Tangherlini spacetime in the weak gravitational field. and the strong gravitational field *Phys. Rev. D*, 90:064043 (2014).

[27] S. Chandrasekhar The Mathematical Theory of Black Holes. *Oxford University Press, New York* (1983).

[30] J. Man and H. Cheng. The time delay in strong gravitational lensing with Gauss-Bonnet correction. *JCAP* 1406 (2014) 028.

[31] S. Iyer and C. M. Will. Black-hole normal modes: A WKB approach. I. Foundations and application of a higher-order WKB analysis of potential-barrier scattering. *Phys.Rev.D* 35:3621 (1987).

[32] A. N. Aliev and G. D. Esmer. Hidden symmetries and geodesics of Kerr spacetime in Kaluza-Klein theory. *Phys.Rev.D* 87:084022,(2013).

[33] L. Amarilla and E.F. Eiroa. Shadow of a Kaluza-Klein rotating dilaton black hole. *Phys.Rev.D* 87:044057, (2013).