Time fractional CGMY model for the numerical pricing of European call options

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Abstract. Evolutionary equations containing fractional derivatives have been widely used in financial models and can describe anomalous diffusion and transmission dynamics in some complex systems. We assumes that option pricing obeys the infinite jump CGMY process and treats stock price fluctuations as a fractal transmission system, A time fractional CGMY option pricing (TFCGMY) model is obtained. The L1 approximation of the Caputo fractional derivative and the modified GL approximation are used to discretize the time and space fractional order operators; the first order space derivative is discretized using the central difference quotient. The TFCGMY model and the above numerical techniques are used for the pricing analysis of European call options. By comparing and analyzing the option pricing model with the Black-Scholes model and the CGMY model shows that the introduction of time fractional differentiation can better capture the jump characteristics of stock prices of options.

1. Introduction

The most exciting issue in financial markets is the law of movement of prices in financial markets, and which distribution function obeys the rate of return of assets is an important research area in financial econometrics. The normal distribution [1] has excellent mathematical properties, and plays an important role in finance. Black-Scholes [2] proved the differential equation that the price of the underlying asset that does not pay dividends must be satisfied, and established a classic Black-Scholes option pricing model. However, the assumption that the underlying asset price in the Black-Scholes model obeys the log-normal distribution and constant volatility is not consistent with the empirical characteristic facts. A lot of empirical analysis shows that the empirical distribution of return on assets is biased, showing the characteristics of "spikes and fat tails", which is far from the normal distribution.

At present, the discontinuity of financial asset prices and the non-normality of the distribution of returns have been recognized. In terms of price movements in financial markets, the jump model has been developed in two ways: a jump-diffusion model and an infinite activity jump model. Merton [3] first proposed a compound Poisson process with finite jumps to characterize a small number of discontinuous jumps in asset prices. The double exponential model established by Kou [4] is also a finite jump model. Intuitively, the purely jumping Lévy model of infinite activity can better describe the process of financial asset prices, especially in reality, only discrete samples can be obtained. Therefore, in the current research on derivative product pricing theory, a large number of researches on pricing theory based on asset prices obeying the Lévy process appear. Madan & Milne [5] used the Lévy process that incrementally obeys the Variance Gamma (VG) distribution to describe stock returns. In addition, the Lévy process can be used to continuously check and capture the dynamic
jumps in the stock price in the financial market. The models introduced also include Normal Inverse Gaussian (NIG) \[^6\], Log Stable \[^7\], and CGMY \[^8\], and Generalized Hyperbolic process \[^9\] and so on. A better alternative to the above two types of models is the CGMY method introduced by Carr et al. This process is a special type of infinite activity Lévy process with four key parameters and controlling its basic characteristics. By using appropriate values, these four parameters further allow diffusion and jumping to have limited and infinite mobility and variability.

Fama proved that stock prices move randomly \[^10\], and proposed the Efficient Market Hypothesis (EMH). In fact, it can be known from theory and practice that the financial system is neither completely random nor completely deterministic. It is a system between random and deterministic. In recent years, the latest international research results show that there is a fractal structure in the financial system, which gives people the confidence to describe and study the fractal characteristics of the financial system. Based on the theoretical basis of the heterogeneous market hypothesis, Peters E E proposed the Fractal Market Hypothesis (FMH), which makes up for the deficiency of the efficient market hypothesis.

2. The time fractional CGMY Model

The Lévy process \[\left\{L_t, t \in [0, T]\right\}\] is an independent fixed incremental process with a logarithmic characteristic function. According to the different distribution assumptions of the incremental \[L_t - L_s, (t > s)\], the Lévy process has different manifestations. The Lévy process can be represented by triples \((\mu, \sigma, \omega)\), that is, the Lévy distribution can be decomposed into a linear combination of time variables, Brownian motion, and purely jump Lévy processes. And its characteristic index function satisfies the following Lévy-Khintchine formula

\[
\psi(u) = -iu\mu + 1/2\sigma^2u^2 - \int_{-\infty}^{\infty} \left( e^{iu\omega} - 1 - iu\omega\right) v(dx),
\]

Here, \(\mu \in \mathbb{R}\), \(\nu\) is called the Lévy measure and \(\nu(R) = \lambda \leq \infty\), which is used to measure the arrival rate of jumps and it may occur numerous times at any time; \(\sigma\) is the volatility, \(\sigma \geq 0\); \(\psi(-\xi)\) is the characteristic index of the Lévy process, \(q(x)\) is a truncation function, and \(i = \sqrt{-1}\) is Virtual unit.

Under the risk neutral measure (EMM), the CGMY model assumes that the option price follows the geometric Lévy process

\[
dx_t = (r - \nu)dt + dL_t.
\]

Here, \(x_t = \ln S_t\), \(\nu\) is the convexity adjustment and \(\nu = CT(1)\left(M - 1\right) - M^y + (G + 1) - G^y\), \(r\) is the risk-free interest rate, \(dL_t\) is the CGMY process controlled by four parameters \(C, G, M, Y\). Parameter \(C\) measures the overall activity level; parameters \(G, M\) respectively control the exponential decay rate of the left and right sides of the Lévy measure. When \(G = M\), the distribution of the CGMY model is symmetrical; parameter \(Y\) determines whether the CGMY process has a completely monotonic Lévy measure and whether the Lévy process is a finite or infinite jump.

This article \[^11\] pointed out that the following FPDE equations are satisfied in the CGMY model,

\[
\frac{\partial V(x,t)}{\partial t} + (r - \nu)\frac{\partial V(x,t)}{\partial x} + C\Gamma\left(-Y\right)e^{\mu x}D^\mu_x\left(e^{-\mu x}V(x,t)\right) + C\Gamma\left(-Y\right)e^{\sigma x}D^\sigma_x\left(e^{\sigma x}V(x,t)\right) = (r + C\Gamma\left(-Y\right)\left(M^y + G^y\right))V(x,t).
\]

Among them, the left and right Riemann-Liouville(R-L) fractional order differentials are given by the following formulas, respectively,
\[
_{a}D^{\alpha}_{x}f(x) = \Gamma^{-1}(n-\alpha)\frac{\partial^{n}}{\partial x^{n}} \int_{0}^{x} (x-y)^{-\alpha-n} f(y) \, dy,
\]
for \(0 < n-1 \leq \alpha < n\).

\[
_{a}D^{\alpha}_{x}f(x) = (-1)^{n} \Gamma^{-1}(n-\alpha)\frac{\partial^{n}}{\partial x^{n}} \int_{x}^{0} (y-x)^{-\alpha-n} f(y) \, dy,
\]
for \(0 < n-1 \leq \alpha < n\).

The reference [12] treats the fluctuation of option price as a fractal transmission system. When \(0 < \gamma < 1\), equation (3) becomes

\[
A_{t}e^{\gamma t}, D^{\gamma}_{t}V(x,t) + (r - \nu) \frac{\partial V(x,t)}{\partial x} + CT(-Y)e^{\nu t}, D^{\gamma}_{t}(e^{\nu t}V(x,t))
\]

\[
+ CT(-Y)e^{\nu t}D^{\gamma}_{t}(e^{\nu t}V(x,t)) - \left(r + CT(-Y)(M^{\gamma} + G^{\gamma})\right)V(x,t) = 0.
\]

Here, \(1 < Y < 2\), and \(T\) is the term of the option contract, \(d_{i}\) is the Hausdorff dimension of the transmission system, and \(A_{t}\) is a constant and \(\gamma\) is a transmission exponent. The \(D^{\gamma}_{t}\) is same as the right-R fractional order differentials.

This article discusses and analyzes European call options, and the TFCGMY model has the following boundary conditions and terminal value conditions:

\[
V(B, t) = 0, V(B, t) = e^{bt} - K \cdot e^{nt}, t \in (0, T]; V(x, 0) = \max \{e^{x} - K, 0\}, x \in (B_{a}, B_{b}).
\]

We call equation (4-5) as the TFCGMY model of option pricing. When \(\gamma = 1\), the model degenerates into the CGMY model. And when \(\gamma = 1\), \(Y \rightarrow 2\), if choose \(C = \sigma^{2}/(4\Gamma(-Y))\) and \(G = M^{\gamma}\) in advance, can be obtained from Theorem 1 in [13], the TFCGMY model (16) degenerates into the B-S model. The CGMY process in this model combines the features of many continuous-time models and captures their essential differences in the case of parameters. At the same time, the fractal transmission system based on the nonlinear dynamical system is used to introduce the time fractional differential, and the liquidity and investment starting point are used to explain various market phenomena that the efficient market hypothesis cannot explain.

3. Discrete format of TFCGMY model

In this section, we will build a higher-order convergence numerical format for the TFCGMY model. For convenience, let's assume \(A_{t} = d_{i} = 1\). By using the expiration time \(\tau = T - t\), we convert the model (4-5) into an initial value problem,

\[
D^{\gamma}_{t}V(x, \tau) + a \frac{\partial V(x, \tau)}{\partial x} + b(x)_{\gamma} D^{\gamma}_{x}(f(x)V(x, \tau)) + c(x)_{\gamma} D^{\gamma}_{x}(h(x)V(x, \tau)) + DV(x, \tau) = 0,
\]

\[
V(B_{a}, \tau) = 0, V(B_{b}, \tau) = p(\tau), \tau \in [0, T]; V(x, 0) = u(x), x \in (B_{a}, B_{b}).
\]

Here, \(f(x) = e^{\alpha x}\), \(h(x) = e^{\beta x}\), \(D^{\gamma}_{\tau}(r + CT(-Y)(M^{\gamma} + G^{\gamma}))\), \(a = r - \nu\), \(b(x) = CT(-Y)e^{\alpha x}\), \(c(x) = CT(-Y)e^{\beta x}\), \(v = CT(Y)(M^{\gamma} - M^{\gamma} + G^{\gamma})\).

In fact, when \(0 < \gamma < 1\), the modified R-L fractional derivative \(D^{\gamma}_{\tau}V(x, \tau)\) is consistent with the Caputo fractional derivative \(\frac{\partial^{n}}{\partial x^{n}}V(x, \tau)\), that is,

\[
D^{\gamma}_{\tau}V(x, \tau) = \Gamma^{-1}(1 - \gamma) \frac{d}{d\tau} \int_{0}^{\tau} \frac{V(x, \eta) - V(x, 0)}{(\tau - \eta)^{\gamma}} d\eta = \Gamma^{-1}(1 - \gamma) \frac{\partial V(x, \eta)}{\partial \eta} \frac{(\tau - \eta)^{\gamma}}{(\tau - \eta)^{\gamma}} d\eta = \frac{\partial}{\partial \eta}V(x, \tau).
\]

Note \(\Delta \tau = k \Delta x\); \(x_{i} = B_{a} + i \Delta x\); \(k = 0, 1, 2, ..., n\); \(i = 0, 1, 2, ..., m\); where \(\Delta x = (B_{a} - B_{b})/m\) and \(\Delta \tau = T / n\) are the space step and time step, respectively. Using the L1 approximation of the Caputo fractional derivative, \(\frac{\partial}{\partial \eta}V(x, \tau)\) is discretized at point \((x_{i}, \tau_{k+1})\),

\[
\frac{\partial^{n}}{\partial x^{n}}V(x, \tau) \approx \frac{V(x_{i+1}, \tau_{k+1}) - V(x_{i-1}, \tau_{k+1})}{2 \Delta x}, \quad \frac{\partial^{n}}{\partial \eta^{n}}V(x, \tau) \approx \frac{V(x_{i}, \tau_{k+1}) - V(x_{i}, \tau_{k})}{\Delta \tau}.
\]
\[
\alpha D^\gamma_t V(x_i, \tau_{k+1}) = 0 D^\gamma_t V(x_i, \tau_{k+1}) + O((\Delta \tau)^{-2-\gamma})
\]
\[
= \Gamma^\gamma(1-\gamma) \sum_{j=0}^k \left( V(x_i, \tau_{k+1+j}) - V(x_i, \tau_{k+1}) \right)(\Delta \tau)^{-\gamma} \left[ (j+1)^{-\gamma} - j^{-\gamma} \right] + O((\Delta \tau)^{-2-\gamma}).
\]

For the first derivative in space, the center difference quotient is used to discretize,
\[
\frac{\partial V(x_i, \tau_{k+1})}{\partial x} = \frac{V(x_{i+1}, \tau_{k+1}) - V(x_{i-1}, \tau_{k+1})}{2\Delta x} + O((\Delta x)^2).
\]

For the space fractional derivative operator, the weighted shift generalized difference (WSGD) operator with second-order derivative approximation in [14] is used for approximation. We selected \((p,q) = (1,0)\),
\[
b_{0i} D^\gamma_t u(x_i, \tau_{k+1}) = (\Delta x)^{-\gamma} \sum_{j=0}^{i+1} \omega_j V(x_{i-j+1}, \tau_{k+1}) + O((\Delta x)^2),
\]
\[
. D^\gamma_t u(x_i, \tau_{k+1}) = (\Delta x)^{-\gamma} \sum_{j=0}^{M-i} \omega_j V(x_{i+j-1}, \tau_{k+1}) + O((\Delta x)^2).
\]

Here, \(\omega_0 = Y/2 g_0, \omega_j = Y/2 g_j + (2-Y) / 2g_{j-1}, g_0 = 1, g_j = (1+(Y+1)/j)g_{j-1}, j = 1, 2, \ldots\)

Let \(i = 0, 1, 2, \ldots, M\), \(k = 0, 1, 2, \ldots, N\), \(V^k_i = V(x_i, \tau_k)\), \(b_i = b(x_i)\), \(c_i = c(x_i)\), \(f_i = f(x_i)\), \(h_i = h(x_i)\), and substitute the equation (7-9) into (6), get the discrete form of the equation,
\[
(\Delta \tau)^{-\gamma} \left( \Gamma(2-\gamma) \right) \sum_{j=0}^k (V^k_{i+j} - V^{i+j}_{i-j}) d_j = \frac{d}{2} (V^k_{i+1} - V^{i+1}_{i-1})(\Delta x)^{-1} + b_i (\Delta x)^{-\gamma} \sum_{j=0}^{M-i} \omega_j f_{i+j-1} V^{i+j}_{i+j-1}
\]
\[
+ c_i (\Delta x)^{-\gamma} \sum_{j=0}^{M-i} \omega_j h_{i+j-1} V^{i+j}_{i+j-1} + D V^k_{i+1} + O((\Delta \tau)^{-2-\gamma} + (\Delta x)^2).
\]

Where \(d_j = (j+1)^{-\gamma} - j^{-\gamma}\).

The initial boundary conditions are discrete as follows,
\[
V^0_0 = 0, V^0_i = p(\tau_i), k = 1, 2, \ldots, N - 1; V^0_i = u(x_i), i = 1, 2, \ldots, M - 1.
\]

Let \(V^i_k\) be the approximate solution of \(V^i_k\), and omit the truncation error \(O((\Delta \tau)^{-2+\gamma} + (\Delta x)^2)\), then you can get the discrete format of the TFGCMY model (6),
\[
V^k_i - \frac{\phi a}{2} (\Delta x)^{-\gamma} (V^k_{i+1} - V^{k}_{i-1}) - \phi b_i (\Delta x)^{-\gamma} \sum_{j=0}^{i+1} \omega_j f_{i+j-1} V^{k}_{i+j-1} - \phi c_i (\Delta x)^{-\gamma} \sum_{j=0}^{M-i} \omega_j h_{i+j-1} V^{k}_{i+j-1} - \phi D V^k_i = V^0_i,
\]
\[
V^{k+1}_i - \frac{\phi a}{2} (\Delta x)^{-\gamma} (V^{k+1}_{i+1} - V^{k+1}_{i-1}) - \phi b_i (\Delta x)^{-\gamma} \sum_{j=0}^{i+1} \omega_j f_{i+j-1} V^{k+1}_{i+j-1} - \phi c_i (\Delta x)^{-\gamma} \sum_{j=0}^{M-i} \omega_j h_{i+j-1} V^{k+1}_{i+j-1}
\]
\[
- \phi D V^{k+1}_i = \sum_{j=0}^k (d_j - d_{j+1}) V^{k-j}_i + d_k V^0_i.
\]

Here, \(\phi = (\Delta x)^\gamma \Gamma(2-\gamma) > 0\).

Let \(\tilde{V}^i = (\tilde{V}^i_0, \ldots, \tilde{V}^i_M)^T\), the matrix form of this discrete format is as follows,
\[
\left[(1 - \phi D) I - L - P - Q\right] \tilde{V}^i = \tilde{V}^0 + F^i, k = 0,
\]
\[
\left[(1 - \phi D) I - L - P - Q\right] \tilde{V}^{k+1} = \sum_{j=0}^k (d_j - d_{j+1}) \tilde{V}^{k-j} + d_k \tilde{V}^0 + F^{k+1}, 1 \leq k \leq N - 1.
\]
Here, $I$ is the identity matrix of order $(M-1)\times(M-1)$, $\eta_i = \phi b_i (\Delta x)^{\gamma}$, $\theta_i = \phi c_i (\Delta x)^{\gamma}$.

$$L = \begin{bmatrix}
0 & \frac{\phi a}{2\Delta x} & \cdots \\
-\frac{\phi a}{2\Delta x} & 0 & \frac{\phi a}{2\Delta x} \\
& -\frac{\phi a}{2\Delta x} & 0 & \ddots \\
& & & \ddots & \ddots \\
& & & & -\frac{\phi a}{2\Delta x} & 0
\end{bmatrix}_{M-1},$$

$$P = \begin{bmatrix}
a_0 f_1 \eta_1 & a_0 f_2 \eta_1 & \cdots & a_0 f_{M-1} \eta_{M-1} \\
a_0 f_1 \eta_2 & a_0 f_2 \eta_2 & \cdots & a_0 f_{M-2} \eta_{M-2} \\
& \ddots & \ddots & \ddots \\
& & a_0 f_1 \eta_{M-1} & a_0 f_2 \eta_{M-1} & \cdots & a_0 f_{M-2} \eta_{M-2} & a_0 f_{M-1} \eta_{M-1}
\end{bmatrix},$$

$$Q = \begin{bmatrix}
a_0 h_1 \theta_1 & a_0 h_2 \theta_1 & \cdots & a_0 h_{M-1} \theta_{M-1} \\
a_0 h_1 \theta_2 & a_0 h_2 \theta_2 & \cdots & a_0 h_{M-2} \theta_{M-2} \\
& \ddots & \ddots & \ddots \\
& & a_0 h_1 \theta_{M-1} & a_0 h_2 \theta_{M-1} & \cdots & a_0 h_{M-2} \theta_{M-2} & a_0 h_{M-1} \theta_{M-1}
\end{bmatrix}_{M-1},$$

$$F^k = V^k_M \left\{ a_0 h_1 \theta_1, \cdots, a_0 h_M \theta_{M-1}, \frac{\phi A}{2\Delta x} + a_0 f_M \eta_{M-1} + a_0 h_M \theta_{M-1} \right\}^T.$$

The discrete format (13) of the TFCGMY model for the stability and convergence analysis of initial values can be found in [12] and [14], and the numerical examples can be found in [15].

4. Pricing analysis for European call options

The initial value problem of the B-S model has a closed solution derived from the normal distribution function. In this section, we consider setting different the order of fractional derivatives, comparing the numerical solution results of the option price obtained by the TFCGMY model with B-S and CGMY models, and analyzing the effects of the $\gamma$ and $Y$ values on the option price.

In the calculations reported here to price European call options, it is assumed that the options are struck at $K=80$, the risk-free interest rate $r=0.1$, the term of the option contract $T=1/2$, the volatility $\sigma=0.35$, and $M=150$, $N=100$. The down barrier is located at $B_d=0$, and the up barrier is located at $B_u=\log(140)$. Analyzing European call options, when $\gamma=1.0$, $Y$ take 1.8 and 1.5 respectively, the numerical solution of the TFCGMY model is consistent with the CGMY model, as shown in Figure 1; When $\gamma=1.0$ and $Y \rightarrow 2$, take $C=\sigma^2 / (4\Gamma(-Y))$, $G=M$, the numerical solution of the TFCGMY model is consistent with the classical CGMY model and the B-S model, as shown in Figure 2. The above conclusions show that the algorithm proposed in this paper is effective.
Figure 1. Comparison of numerical solution between the proposed model and the CGMY model.

Figure 2. Comparison of numerical solution between the proposed model, the CGMY model and B-S model.

Figure 3 shows the difference between the numerical solution of the TFCGMY model and the numerical solution of the CGMY model when $Y = 1.8$ and $\gamma$ take different values. It can be seen that the smaller the fractional order $\gamma$, the greater the price of the option. It is shown that the introduction of the time-fractional order can capture the stock price jump characteristics of in the money options and out-of-the-money options.
5. Conclusions
In the study of the nonlinearity of financial systems, the nonlinear dynamic characteristics cannot usually be obtained directly from the data of financial variables. For option pricing, when the option price is considered as the basic fractal transmission system and the stock price follows the infinite jump Lévy distribution, this article a new TFCGMY model is derived, and a higher-order numerical method is obtained. Finally, the model established in this paper is applied to the pricing of European call options, and the option prices obtained by the proposed model are compared with the results obtained by the B-S and CGMY models. The results show that, with the B-S and CGMY models, the proposed TFCGMY model can better capture the jumping characteristics of stock prices for real or imaginary options. In recent years, the latest international research results show that there is a fractal structure in the financial system, which gives people the confidence to describe and study the fractal characteristics of the financial system. The numerical techniques proposed in this paper can also be used for other similar scores; this method can also provide a basis for the further application of fractional calculus in financial market models.

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