1. Introduction

In this paper we are going to discuss the constrained approximation, that is, find a best approximation from a convex subset to a given compact subset subject to certain constraints. Essentially, the problem is based on the monotone best approximation given by Roulier [6] and Lorentz and Zeller [4,5], as well as Taylor’s papers [7,8,9] on best approximation by algebraic polynomials with restricted ranges. In [6] Roulier has discussed the monotone best approximation to a given function with same property. Later on Lorentz and Zeller generalized this problem to a general form by assuming that $\epsilon_i D^k i p(x) \geq 0$, $i = 1, 2, \ldots, r$, where $\epsilon_i = \pm 1$, $1 \leq k_1 \leq k_2 \leq \ldots \leq k_r < n$ and $p(x)$ is polynomial of degree $\leq n$. Meanwhile, Taylor has considered a problem concerning best approximation by algebraic polynomials with restricted ranges, i.e. to find a polynomial $P_0(x)$ of degree $\leq n$ which is a best approximation to a given function $f(x)$ such that $f(x) - P_0(x) \leq u(x)$ where $f, u$ are given functions in $C[a, b]$ such that $f(x) \leq u(x)$ for all $x \in [a, b]$. This leads us to question that whether it is possible to develop a more general theory with regard to these problems. To answer the question, let us consider the following problem:

(P): Given a compact subset $F \subset X$, a real Banach Space, to find $y' \in Y$, a subset of a subspace $G$ of $X$, such that

$$d_F(y') = \inf_{y \in Y} d_F(y) = \inf_{y \in Y} \max_{f \in F} \max_{k \in K \subset X^*} <k, f - y>$$

where $K$ is a symmetric $\sigma(X^*, X)$ - compact subset of $X^*$, the real dual space of $X$, and $Y = Y_1 \cap Y_2$ with

$$Y_1 = \{ y \in G : <h, y - \psi_i> \geq 0 \ \forall \ h \in B_i, \ i = 1, 2, \ldots, m \}$$

$$Y_2 = \{ y \in G : <h, y - \rho_i> \leq 0 \ \forall \ h \in C_i, \ i = 1, 2, \ldots, m \}$$

for some elements $\psi_i$ and $\rho_i$ in $X$.  

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We introduce, for $y \in X$,
\[ p(y) = \max_{k \in K} < k, y > \]
and
\[ d_F(y) = \max_{f \in F} p(f - y) \]

By a proper choice of $B_i, C_i, \psi_i$ and $\rho_i$, we may assume that $Y \neq \emptyset$. A solution for (P) will be called a best approximation to $F$ from $Y$, or briefly, a "best approximation".

We begin with discussing some general questions on the existence and characterization of the solution. Indeed, the theory in which we shall develop will give a definite answer to the previous question.

2. General results

**Lemma 2.1.** Let $G$ be $n$-dimensional subspace of $X$ and $Y$ be defined as above. Assume that the restriction of $p(\cdot)$ to $G$ is a norm and each $B_i, C_i \subset X^*(i = 1, 2, \ldots, m)$. Then, there exists $y' \in Y$ such that

\[ d_F(y') = \inf_{y \in Y} d_F(y). \]

**Proof.** Let $y_j$ be a sequence in $Y$ such that $\lim_{j \to \infty} d_F(y_j) = \inf_{y \in Y} d_F(y)$. Moreover,

\[ p(y_j) \leq \max_{f \in F} p(f - y_j) + \max_{f \in F} p(f) \leq M, \]
for some real $M$, since $\max_{f \in F} p(f)$ is fixed and $d_F(y_j)$ are terms of a convergent sequence. As the restriction is a norm, $\{y_j\}$ is a bounded sequence in $G$. Hence, there exists $y' \in G$ such that $\lim_{j \to \infty} y_j = y'$. We will show that, in fact, $y' \in Y$. For each $i$, we have

\[ < h, y_j - \psi_i > \geq 0 \forall h \in B_i \text{ and } < h, y_j - \rho_i > \leq 0 \forall h \in C_i \text{ and all } j. \]
Since $B_i, C_i \subset X^*$,

\[ < h, y' - \psi_i > = \lim_{j \to \infty} < h, y_j - \psi_i > \geq 0 \forall h \in B_i \text{ and } < h, y' - \rho_i > = \lim_{j \to \infty} < h, y_j - \rho_i > \leq 0 \forall h \in C_i. \]
Therefore, $y' \in Y$. Further, for each $j$,

\[ 0 \leq d_F(y_j) - \inf_{y \in Y} d_F(y) \leq \left[ \max_{f \in F} p(f - y_j) + p(y_j - y') \right] - \inf_{y \in Y} d_F(y) \]

\[ = \left[ d_F(y_j) - \inf_{y \in Y} d_F(y) \right] + p(y_j - y'). \]
Since the term in brecket tends to zero as $j \to \infty$ and also $\lim_{j \to \infty} p(y_j - y') = 0$, this shows that $d_F(y') = \inf_{y \in Y} d_F(y)$, which proves the lemma.

For the sake of simplicity, we will assume that $B_i \cap B_j = \emptyset, C_i \cap C_j = \emptyset$ for $i \neq j$ and $B_i \cap (-B_j) = \emptyset, C_i \cap (-C_j) = \emptyset$ for $i \neq j$, where $-B_j = \{-h : h \in B_j\}$, $-C_j = \{-h : h \in C_j\}$. Define the sets

\[ H(y) = \{ \epsilon k \in K : \exists f \in F, < \epsilon k, f - y > = d_F(y) \} \quad \text{where } \epsilon = \pm 1; \]
CONSTRAINED APPROXIMATION OF A COMPACT SET

\[ N_{1,i}(y) = \{ h \in B_i : < h, y - \psi_i >= 0 \}, \]
\[ N_{2,i}(y) = \{ h \in C_i : < h, y - \rho_i >= 0 \}, \]
\[ J = \{1, \ldots, m\}, \]
\[ I_1(y) = \{ i \in J : N_{1,i}(y) \neq \emptyset \}, \]
\[ I_2(y) = \{ i \in J : N_{2,i}(y) \neq \emptyset \}, \]

and
\[ N(y) = \left[ \bigcup_{i \in I_1(y)} N_{1,i}(y) \right] \cup \left[ \bigcup_{i \in I_2(y)} N_{2,i}(y) \right]. \]

Let us consider two particular cases which are not of general interest. First, suppose that, for some \( y_0 \in Y \) and \( k \in K \), there exist \( f_1, f_2 \in F \) such that
\[ < k, f_1 - y_0 > = d_F(y_0) \quad \text{and} \quad < k, f_2 - y_0 > = -d_F(y_0). \]

Then \( y_0 \) is obviously a best approximation, as no approximation can make the error smaller at \( k \). We will therefore call \( k \) a straddle point of \( F \). In case, we have \( \{H^+(y') \cap N_{2,i}(y')\} \cup \{H^-(y') \cap N_{1,i}(y')\} \neq \emptyset \) for some \( i, j \in J \), where \( H^+(y') = \{+k \in H(y')\} \) and \( H^-(y') = \{-k \in H(y')\} \), then any attempt to decrease the error \( d_F(y') \) would remove \( y' \) from \( Y \) through failure of the necessary inequalities for \( < h, y' - \psi_j > \) and \( < h, y' - \rho_i > \) at all \( h \in \{H^+(y') \cap N_{2,i}(y')\} \cup \{H^-(y') \cap N_{1,j}(y')\} \). Thus, \( y' \) is clearly a best approximation. In the following discussion, unless otherwise stated, we will rule out these two cases.

Theorem 2.1. Let \( G \) be a subspace of \( X \) and \( Y \) a subset of \( G \) defined as before. Then \( y' \in Y \) is a best approximation to \( F \) if and only if there does not exist \( y \in G \) such that
\[ < k, y > < 0 \quad \forall \ k \in H(y') \]
\[ < h, y > \leq < h, y' - \psi_i > \quad \forall \ h \in B_i, \quad i = 1, 2, \ldots, m \] (2.1)

and
\[ < -h, y > \leq < h, \rho_i - y' > \quad \forall \ h \in C_i, \quad i = 1, 2, \ldots, m. \]

Proof. Suppose that \( y' \) is not a best approximation to \( F \), then there is a \( y_1 \) in \( Y \) such that
\[ d_F(y_1) < d_F(y'). \]

Observe that for each \( k \in H(y') \) there is a \( f \) in \( F \) such that
\[ < k, f - y_1 > < < k, f - y' >. \]

Therefore, we have
\[ < k, y' - y_1 > < 0 \quad \text{for all} \quad k \in H(y'). \]
Moreover,
\[
< h, y' - y_1 > - < h, y' - \psi_i > = < h, -y_1 + \psi_i > = < h, y_1 - \psi_i > \leq 0
\]
for all \( h \in B_i, i = 1, \ldots, m \),

and
\[
< h, y' - y_1 > - < h, y' - \rho_i > = < h, -y + \rho_i > = < h, \rho_i - y_1 > \geq 0
\]
\( \forall h \in C_i, i = 1, 2, \ldots, m \).

Consequently, if we put \( y = y' - y_1 \), then we have the required result. This proves the sufficiency.

On the other hand, if there is a \( y \in G \) such that
\[
< k, y > < 0 \text{ for all } k \in H(y'),
\]
\[
< h, y > \leq < h, y' - \psi_i > \text{ for all } h \in B_i, i = 1, \ldots, m,
\]

and
\[
- < h, y > \leq < h, \rho_i - y' > \text{ for all } h \in C_i, i = 1, \ldots, m.
\]

Then, by compactness of \( H(y') \), there exists a real \( b > 0 \) and an open subset \( V \) of \( K \) such that
\[
\inf_{k \in H(y')} | d_F(y') - k, y | = b,
\]

and
\[
\max_{f \in F} < k, f - y' > < k, y > < -b/2 \text{ for all } k \in V.
\]

Obviously, \( H(y') \subseteq V \). Hence, there is a real \( c > 0 \) such that
\[
d_F(y') - \max_{f \in F} \max_{k \in K \backslash V} < k, f - y' > \geq c.
\]

Take \( r = \min \left[ b/2p(y)^2, c/2p(y) \right] \), and observe that, for \( k \in V \),
\[
\max_{f \in F} < k, f y_t > = \max_{f \in F} < k, f - y' > + t < k, y >
\]
\[
< \max_{f \in F} < k, f - y' > \leq d_F(y') \text{ for sufficiently small } t \in (0, r],
\]

where \( y_t = y' - ty \), while for \( k \in K \backslash V, t \in (0, r] \),
\[
\max_{f \in F} | < k, f - y_t > | = \max_{f \in F} \max_{k \in K \backslash V} | < k, f - y' > | + t | < k, y > |
\]
\[
\leq d_F(y') - c + c/2 = d_F(y') - c/2.
\]

It follows that \( d_F(y_t) < d_F(y') \) for sufficiently small \( t \in (0, r] \). Finally, we will show that there exists \( t \in (0, r] \) such that \( y_t \) is in \( Y \). Since \( < h, y > \leq < h, y' - \psi_i > \) and
<h, y' - \psi_i \geq 0 \text{ for all } h \in B_i, \ i = 1, \cdots, m, \text{ we have } <h, ty > \leq <h, y' - \psi_i > \text{ for all } h \in B_i, \ i = 1, \cdots, m \text{ and } t \in (0, 1]. \text{ Therefore, } <h, y' - ty - \psi_i > \geq 0 \text{ for all } h \in B_i, \ i = 1, \cdots, m \text{ and } t \in (0, 1]. \text{ Similarly, we have } <h, \rho_i - y' + ty > \geq 0 \text{ for all } h \in C_i, \ i = 1, 2, \cdots, m \text{ and } t \in (0, 1]. \text{ Consequently, this shows that there exists } t \in (0, r] \text{ such that } y_t \text{ is in } Y \text{ and } d_F(y_t) < d_F(y'). \text{ Hence, } y' \text{ is not a best approximation to } F, \text{ proving the theorem.}

The condition (2.1) of Theorem 2.1 is too restrictive. Therefore, for the practical purpose we would like to replace condition (2.1) by a less restrictive condition. First, we assume that \( B_i, C_i \subseteq X^* \ (i = 1, 2, \cdots, m) \) are \( \sigma(X^*, X) \)-compact.

With the aid of the additional assumptions, we have

Theorem 2.2. Suppose that \( B_i, C_i \subseteq X^* \ (i = 1, 2, \cdots, m) \) are \( \sigma(X^*, X) \)-compact and there exists \( y \in G \) such that \( <r, y > \geq a > 0 \) for \( r \in N_1, (y') \cup [-N_2, (y')] \), \( i = 1, 2, \cdots, m \), for some given \( y' \in Y \). Then \( y' \) is a best approximation to \( F \) if and only if

\[
\overline{\co}(H(y') \cup N(y')) \cap G^1 \neq \emptyset
\]
where \( G^1 = \{ u \in X^* : <u, y > = 0 \ \forall \ y \in G \} \).

Proof. For the sake of convenience, let \( M = H(y') \cup N(y') \). As in Theorem 2.1, we know that if \( y' \) is not a best approximation, there is an \( y_1 \in Y \) such that

\[
<k, y' - y_1 > < 0 \quad k \in H(y')
\]

\[
<h, y' - y_1 > \leq <h, y' - \psi_i > \quad \forall \ h \in B_i, \ i = 1, 2, \cdots, m
\]

and

\[
<-h, y' - y_1 > \leq <h, \rho_i - y' > \quad \forall \ h \in C_i, \ i = 1, 2, \cdots, m.
\]

Hence, \( <h, y' - y_1 > \geq 0 \ \forall h \in N_1, (y'), \ i \in I_1(y') \) and \( <-h, y' - y_1 > \geq 0 \ \forall h \in N_2, (y'), \ i \in I_2(y') \). If

\[
<r, y' - y_1 > < 0 \quad \forall \ r \in N(y')
\]

then, putting \( y = y' - y_1 \), we get

\[
<r, y > < 0 \quad \forall \ r \in M.
\]

Hence \( \overline{\co}(M) \cap G^1 = \emptyset \). Otherwise, if exists \( r \in N(y') \) such that \( <r, y' - y_1 >= 0 \). By assumption, we have a \( y \in G \) such that \( <r, y > \geq a > 0 \ \forall \ r \in N(y'), \) and, by the compactness of \( H(y') \), there exist real \( b_1 > 0 \) and \( b_2 > 0 \) such that \( \inf_{k \in H(y')} |<k, y' - y_1 >| = b_1 \) and \( \max_{k \in H(y')} |<k, y >| = b_2 \). Take \( c = b_1/2b_2 \), then, for \( y_t = y' - y_1 - ty \), where \( t \in (0, c] \), we have

\[
<k, y_t > = <k, y' - y_1 > - t <k, y > < 0 \ \forall \ k \in H(y');
\]

and

\[
<r, y_t > = <r, y' - y_1 > - t <r, y > < 0 \ \forall \ r \in N(y').
\]

This again shows that \( \overline{\co}(M) \cap G^1 = \emptyset \), proving the sufficiency.
Conversely, suppose $\overline{\partial}(M) \cap G^\perp = \emptyset$, by a separation theorem [2] and the fact that the dual space of $X^*$ under $\sigma(X^*, X)$-topology is $X$, there is an $y \in G$ such that

$$< k, y > < 0 \quad \forall \ k \in H(y')$$

and

$$< r, y > < 0 \quad \forall r \in N(y').$$

By a similar argument to that in the proof of Theorem 2.1, there exist real $b, c, r > 0$ and an open subset $V$ of $K$ such that

$$\left[ \max_{f \in F} < k, f - y_t > \right]^2 \leq \left[ d_F(y') \right]^2 - tb/2 \forall t \in (0, r], \ k \in V,$$

$$\max_{f \in F} < k, f - y_t > \leq d_F(y') - c/2 \quad \forall k \in K \mid V, \ t \in (0, r]$$

where $y_t = y' - ty$.

Since $B_i, C_i$ are compact and $N_{1, i}(y'), N_{2, i}(y')$ are closed subsets of $B_i, C_i$, there exist $\epsilon_{j, i} > 0$ such that

$$\min_{h \in N_{1, i}(y')} |< h, y >| = e_{1, i} \quad \forall \ i \in I_1(y')$$

and

$$\min_{h \in N_{2, i}(y')} |< -h, y >| = e_{2, i} \quad \forall \ i \in I_2(y').$$

Define the open set $U_{j, i}$ as

$$U_{1, i} = \{ h \in B_i : < h, y > < -\frac{\epsilon_{1, i}}{2}, \ \forall \ i \in I_1(y') \}$$

$$U_{2, i} = \{ h \in C_i : < -h, y > < -\frac{\epsilon_{2, i}}{2}, \ \forall \ i \in I_2(y') \}$$

$$U_{1, i} = \emptyset \quad \forall \ i \in J \mid I_1(y')$$

and

$$U_{2, i} = \emptyset \quad \forall \ i \in J \mid I_2(y').$$

Clearly, $N_{j, i}(y') \subset U_{j, i} \forall \ i \in I_j(y')$. Hence there exists $c_{j, i} > 0$ such that

$$< h, y' - \psi_i > \geq c_{1, i} \quad \forall \ h \in B_i \mid U_{1, i}, \ i \in J$$

$$< -h, y' - \rho_i > \geq c_{2, i} \quad \forall \ h \in C_i \mid U_{2, i}, \ i \in J$$

Put $\mu_i = \max_{h \in B_i} |< h, y >|$, $\nu_i = \max_{h \in C_i} |< h, y >|$, $a_{j, i} = \min \{ \frac{c_{j, i}}{2\mu_i}, \frac{\epsilon_{j, i}}{2\nu_i} \}$ and

$$r_0 = \left\{ \min_{\epsilon \in J} \{ a_{1, i} \}, \ \min_{\epsilon \in J} \{ a_{2, i} \}, \ r \right\}.$$  Observe that, for $h \in U_{1, i}$ and $t \in (0, r_0)$,

$$< h, y' - ty - \psi_i > = < h, y' - \psi_i > - t < h, y > > 0$$
and for $h \in B_i \cup U_{1,i}$, $t \in (0, r_0)$
\[ < h, y' - ty - \psi_i > = < h, y' - \psi_i > - t < h, y > \geq c_{1,i} - t < h, y > \geq 0. \]

Similary, for $h \in U_{2,i}$ and $t \in (0, r_0]$,
\[ < -h, y' - ty - \rho_i > = < -h, y' - \rho_i > - t < -h, y > \geq 0. \]

and for $h \in C_i \cup U_{2,i}, t \in (0, r_0]$
\[ < -h, y' - ty - \rho_i > = < -h, y' - \rho_i > - t < -h, y > \geq c_{2,i} - t < -h, y > \geq 0. \]

This shows $y' - ty$ is in $Y$ for $t \in (0, r_0]$. Moreover, we have $d_F(y' - ty) < d_F(y')$. Hence $y'$ is not a best approximation to $F$ which proves the theorem.

In the case where $G$ is of dimension $n$, we have the following useful result without the additional assumption of continuity of linear functional in $B_i, C_i, i = 1, 2, \ldots, m$.

**Theorem 2.3.** Suppose $G$ is an $n$-dimensional subspace of $X$ and $B_i, C_i \subset X^*(i = 1, 2, \ldots, m)$. Furthermore, assume that the restriction $B_i \mid G, C_i \mid G$ are closed and bounded. If there exists $y \in G$ such that
\[ < \tau, y > \geq a > 0, \forall \tau \in N_{1,i}(y') \cup [-N_{2,i}(y')], i = 1, 2, \ldots, m, \]
for some given $y' \in Y$, then $y'$ is a best approximation to $F$ if and only if there exist $s$ functionals, $k_1, \ldots, k_s \in H(y')$, $\ell_i$ functionals $h_{i,1}, \ldots, h_{i,\ell_i} \in N_{1,i}(y')$ for $i \in I_1(y')$ and $t_i$ functionals $q_{i,1}, \ldots, q_{i,t_i} \in N_{2,i}(y')$ for $i \in I_2(y')$ and $s + \sum_{i \in I_1(y')} \ell_i + \sum_{i \in I_2(y')} t_i$ scalars $a_1, \ldots, a_s, b_{i,1}, \ldots, b_{i,\ell_i}, c_{i,1}, \ldots, c_{i,t_i} > 0$, such that
\[
\begin{align*}
\sum_{i \in I_1(y')} \ell_i + \sum_{i \in I_2(y')} t_i & \leq n + 1, \\
\sum_{i=1}^s a_i & + \sum_{i \in I_1(y')} \ell_i \sum_{j=1}^{\ell_i} b_{i,j} + \sum_{i \in I_2(y')} t_i \sum_{j=1}^{t_i} c_{i,j} = 1 \\
\end{align*}
\]

and
\[
\sum_{i=1}^s a_i < \kappa_i, y > + \sum_{i \in I_1(y')} \ell_i \sum_{j=1}^{\ell_i} b_{i,j} < h_{i,j}, y > - \sum_{i \in I_2(y')} t_i \sum_{j=1}^{t_i} c_{i,j} < q_{i,j}, y > = 0 \forall y \in G.
\]

**Proof.** Define the set $W$ of $n$-tuples as follows:
\[
W = \left\{ (< k, y_1 >, \ldots, < k, y_n >) : k \in H(y') \right\} \\
\cup \left\{ (< h_{i,j}, y_1 >, \ldots, < h_{i,j}, y_n >) : h_{i,j} \in N_{1,i}(y'), i \in I_1(y') \right\} \\
\cup \left\{ (< -q_{i,j}, y_1 >, \ldots, < -q_{i,j}, y_n >) : q_{i,j} \in N_{2,i}(y'), i \in I_2(y') \right\}
\]

where $y_1, \ldots, y_n$ is a basis for $G$. Obviously, $W$ is a compact subset of $R^n$, since $B_i \mid G, C_i \mid G$ are closed and bounded.
First of all, we shall show that $y'$ is a best approximation if and only if $O \in \text{co}(W)$.

Suppose that $y'$ is not a best approximation, then, as in the proof of sufficiency of Theorem 2.2, there exists a $y \in G$ such that

$$< k, y > < 0 \quad \forall k \in H(y')$$

and

$$< \tau, y > < 0 \quad \forall \tau \in N(y').$$

Therefore, by a known result in [1, p.19], $O \notin \text{co}(W)$.

On the other hand, suppose $O \notin \text{co}(W)$, then, by a known result in [1, p.19], there exists $y \in G$ such that

$$< k, y > < 0 \quad \forall k \in H(y')$$

and

$$< \tau, y > < 0 \quad \forall \tau \in N(y').$$

By a similar argument to that in the proof of necessity of Theorem 2.2, we can find $y \in Y$ such that $d_F(y) < d_F(y')$. Therefore, $y'$ is not a best approximation.

Thus, we have shown that $y'$ is a best approximation if and only if $O \in \text{co}(W)$. By Caratheodory's Theorem, $0 \in \text{co}(W)$ if and only if there exist $k_1, \ldots, k_s \in H(y')$, $h_{i,1}, \ldots, h_{i,t_i} \in N_1(i'(y'))$ for $i \in I_1(y')$ and $q_{i,1}, \ldots, q_{i,t_i} \in N_2(i'(y'))$ for $i \in I_2(y')$ and $s + \sum_{i \in I_1(y')} t_i + \sum_{i \in I_2(y')} t_i$ scalars $a_1, \ldots, a_s, b_{i,1}, \ldots, b_{i,t_i}$ and $c_{i,1}, \ldots, c_{i,t_i} > 0$ such that

$$s + \sum_{i \in I_1(y')} t_i + \sum_{i \in I_2(y')} t_i \leq n + 1,$$

$$\sum_{i=1}^{s} a_i + \sum_{i \in I_1(y')} \sum_{j=1}^{t_i} b_{i,j} + \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{i,j} = 1$$

and

$$\sum_{i=1}^{s} a_i (< k_{i_1}, y_{1_1}, \ldots, k_{i_n}, y_{n_1} > \ldots, < k_{i_s}, y_{n_s} >) + \sum_{i \in I_1(y')} \sum_{j=1}^{t_i} b_{i,j}(< h_{i,j_1}, y_{1_1}, \ldots, < h_{i_j}, y_{n_1} > \ldots, < q_{i,j}, y_{n_s} >)$$

$$- \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{i,j}(< q_{i,j}, y_{1_1}, \ldots, < q_{i,j}, y_{n_s} >) = 0.$$

On multiplying by any vector $\overline{d} = (d_1, d_2, \ldots, d_n)$, it follows that

$$\sum_{i=1}^{s} a_i < k_{i, y} > + \sum_{i \in I_1(y')} \sum_{j=1}^{t_i} b_{i,j} < h_{i,j}, y >$$

$$- \sum_{i \in I_2(y')} \sum_{j=1}^{t_i} c_{i,j} < q_{i,j}, y > = 0 \quad \forall y \in G.$$
which proves the theorem.

3. Application to space $C[a,b]$

We now turn to a concrete application of the results of Section 2. Let $G$ be an $n$-dimensional subspace of $C^r[a,b]$, the set of all $r$ times continuous differentiable functions in $C[a,b]$. We denote the point evaluation functional of $k$-th derivative at $x$ by $\hat{\varphi}^k(f) = D^k f(x)$ for all $f \in C^r[a,b]$ ($k \leq r$). The semi-norm $p(\cdot)$ is defined as

$$p(f) = \max_{x \in T} |f(x)|$$

where $T$ is a closed subset of $[a,b]$. Obviously, the corresponding set $K = \{ \varepsilon \hat{\varphi}^k : x \in T \}$ where $\varepsilon = \pm 1$. First, we consider $B_i = C_i = \{ \hat{\varphi}^k_i : x \in [a,b] \}$ and $\psi_i, \varphi_i \in C^r[a,b]$, $i = 1,2,\ldots,m$, such that $\hat{\varphi}^k_i(\psi_i) < \hat{\varphi}^k_i(\varphi_i)$ $\forall \hat{\varphi}^k_i \in B_i$, $i = 1,2,\ldots,m$, $1 \leq k_1 \leq k_2 \leq \cdots \leq k_m < r$. Hence the set

$$Y = \{ g \in G : \hat{\varphi}^k_i(\psi_i) \leq \hat{\varphi}^k_i(g) \leq \hat{\varphi}^k_i(\varphi_i), \forall \hat{\varphi}^k_i \in B_i, \ i = 1,2,\ldots,m \}.$$

Obviously the restriction $B_i |G$ is closed and bounded. By virtue of Theorem 2.3, we have,

**Theorem 3.1.** Suppose there exists a $g \in G$ such that $\tau(g) \geq \beta > 0 \ \forall \ \tau \in N_{l_1,i}(g_0)$ $\cup$ $[-N_{2,i}(g_0)]$ $i = 1,2,\ldots,m$, for some given $g_0 \in Y$. Then $g_0$ is best approximation to a compact subset $F \subset C[a,b]$ if and only if there exist $x_1,\ldots,x_s \in T$, $y_1,\ldots,y_s \in [a,b]$ for $i \in I_1(g_0)$, $z_1,\ldots,z_s \in [a,b]$, for $i \in I_2(g_0)$, $s$ functions $f_1,\ldots,f_s \in F$ (not necessarily distinct) and $s + \sum_{i \in I_1(g_0)} \ell_i + \sum_{i \in I_2(g_0)} t_i$ scalars $a_1,\ldots,a_s$, $b_{i,1},\ldots,b_{i,s}$ and $c_{i,1},\ldots,c_{i,s},i > 0$ such that

$$s + \sum_{i \in I_1(g_0)} \ell_i + \sum_{i \in I_2(g_0)} t_i \leq n + 1,$$

$$\sum_{i=1}^s a_i + \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} + \sum_{i \in I_2(g_0)} \sum_{j=1}^{t_i} c_{i,j} = 1,$$

$$\sum_{i=1}^s a_i \sigma(x_i)g(x_i) + \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} D^k g(y_{i,j})$$

$$- \sum_{i \in I_2(g_0)} \sum_{j=1}^{t_i} c_{i,j} D^k g(z_{i,j}) = 0 \ \forall \ g \in G.$$

$$| f_i(x_i) - g_0(x_i) | = d_F(g_0), \ i = 1,2,\ldots,s.$$
\[ D^{k_i} \left[ y_{i,j} - \psi_i(y_{i,j}) \right] = 0, \quad j = 1, \ldots, \ell_i, \quad i \in I_1(g_0) \]

and

\[ D^{k_i} \left[ z_{i,j} - \rho_i(z_{i,j}) \right] = 0, \quad j = 1, \ldots, \ell_i, \quad i \in I_2(g_0), \]

where \( \sigma(x_i) = \text{sign}(f_i(x_i) - g_0(x_i)) \), \( i = 1, 2, \ldots, s \).

If we put \( B_i = \{ \varepsilon_i \hat{x}^{k_i} : x \in [a, b] \} \), \( C_i = \emptyset \), and \( \psi_i = 0 \), \( i = 1, 2, \ldots, m \), where \( \varepsilon_i = \pm 1 \), \( 1 \leq k_1 \leq k_2 \leq \cdots \leq k_m < r \), then \( Y = \{ g \in G : \varepsilon_i k_i D^{k_i} g(x) \geq 0, \forall x \in [a, b], \ i = 1, 2, \ldots, m \} \). Suppose \( G \) is the set of all polynomials of degree \( \leq n \). Then, since \( G \) is the set of all polynomials of degree \( \leq n \), there always exist \( g \in G \) such that \( \varepsilon_i \hat{x}^{k_i}(g) > 0 \), \( \forall \varepsilon_i \hat{x}^{k_i} \in N_{1,i}(g_0), \ i \in I_1(g_0) \), for some \( g_0 \in Y \), and by virtue of Theorem 2.3, we have

**Theorem 3.2.** An element \( g_0 \in Y \) is a best approximation to a compact subset \( F \subset C[a, b] \), if and only if there exist \( x_1, \ldots, x_s \in T, \ y_{i,1}, \ldots, y_{i,t_i} \in [a, b] \) for \( i \in I_1(g_0) \), \( s \) functions \( f_1, \ldots, f_s \in F \) (not necessarily distinct) and \( s + \sum_{i \in I_1(g_0)} \ell_i \) scalars \( a_1, \ldots, a_s \) and \( b_1, \ldots, b_{\ell_i} > 0 \) such that \( s + \sum_{i \in I_1(g_0)} \ell_i \leq n + 2 \),

\[
\sum_{i=1}^{s} a_i + \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} = 1.
\]

\[
\sum_{i=1}^{s} a_i \sigma(x_i) g(x_i) + \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} \varepsilon_i b_{i,j} k_i D^{k_i} g(y_{i,j}) = 0 \quad \forall g \in G \quad \text{(3.2)}
\]

\[
| f_i(x_i) - g_0(x_i) | = d_F(g_0), \quad i = 1, 2, \ldots, s
\]

and

\[ D^{k_i} g_0(y_{i,j}) = 0, \quad j = 1, 2, \ldots, \ell_i, \quad i \in I_1(g_0), \]

where \( \sigma(x_i) = \text{sign}(f_i(x_i) - g_0(x_i)) \), \( i = 1, 2, \ldots, s \).

In the case \( F \) consists of a single function and \( p(\cdot) \) is the usual supremum-norm, Theorem 3.2 is, in fact, a known result given in [5].

Now, consider the case. \( m = 1 \), \( B_1 = C_1 = \{ \hat{x} : x \in [a, b] \} \) and \( G \) is an \( n \)-dimensional subspace of \( C[a, b] \). Let \( \psi_1, \rho_1 \) be two given functions in \( C[a, b] \) such that \( \psi_1(x) < \rho_1(x) \) \( \forall x \in [a, b] \). Then, by Theorem 2.3, we have

**Theorem 3.3.** Suppose there exists a \( g \in G \) such that \( \tau(g) \geq \beta > 0 \) \( \forall \tau \in N_{1,2}(g_0) \cup \left[ -N_{2,1}(g_0) \right] \), for some given \( g_0 \in Y \). Then \( g_0 \) is a best approximation to a compact subset \( F \subset C[a, b] \) if and only if there exist \( s \) points, \( x_1, \ldots, x_s \in T, \) \( \ell \) points \( y_1, \ldots, y_t \in [a, b], \) \( t \) points \( z_1, \ldots, z_t \in [a, b] \), \( s \) functions \( f_1, \ldots, f_s \in F \) and \( s + \ell + t \) scalars \( a_1, \ldots, a_s, b_1, \ldots, b_{\ell}, \) \( c_1, \ldots, c_t > 0 \) such that \( s + \ell + t \leq n + 1 \),

\[
\sum_{i=1}^{s} a_i + \sum_{i=1}^{\ell} b_i + \sum_{i=1}^{t} c_i = 1
\]
CONSTRANED APPROXIMATION OF A COMPACT SET

\[ \sum_{i=1}^{s} a_i \sigma(x_i) g(x_i) + \sum_{i=1}^{t} b_i g(y_i) - \sum_{i=1}^{t} c_i g(z_i) = 0 \quad \forall g \in G \] (3.3)

\[ | f_i(x_i) - g_0(x_i) | = d_F(g_0), \quad i = 1, 2, \ldots, s; \]
\[ g_0(y_i) - \psi_1(y_i) = 0, \quad i = 1, 2, \ldots, \ell; \]

and

\[ g_0(z_i) - \rho_1(z_i) = 0, \quad i = 1, 2, \ldots, t, \]

where

\[ \sigma(x_i) = \text{sgn}(f_i(x_i) - g_0(x_i)), \quad i = 1, 2, \ldots, s. \]

In this case, if \( G \) is the set of all polynomials of degree \( \leq n \), \( F \) consists of a single function \( f \) say, \( \psi_1 = f + \gamma_1, \rho_1 = f + \gamma_2 \), for some \( \gamma_1, \gamma_2 \in C[a, b] \) such that \( \gamma_1(x) < \gamma_2(x) \), and \( p(\cdot) \) is the usual supremum norm, then Theorem 3.3 is, in fact, a known result in [7].

4. Application to space \( C[a, b] \) endowed with \( L_\mu \)-norm \((\mu \geq 1)\)

Let \( G \) be an \( n \)-dimensional subspace of \( C^*[a, b] \) and \( B_i, C_i, \psi_i, \rho_i, Y \) be defined as in the beginning of Section 3. Suppose \( p(\cdot) \) is defined to be a \( L_\mu \)-norm. Then, by virtue of Theorem 2.3, we have

**Theorem 4.1.** Suppose there exists a \( g \in G \) such that \( \tau(g) \geq \beta > 0 \) \( \forall \tau \in N_{1,i}(g_0) \) \( \cup [-N_2,i(g_0)], \quad i \in J \) for some given \( g_0 \in Y \). Then \( g_0 \) is a best approximation to a compact subset \( F \subset C[a, b] \) if and only if there exist \( s \) functions \( u_1(x), \ldots, u_s(x) \in L_\nu[a, b] \), \( s \) functions \( f_1, \ldots, f_s \in F \) (not necessarily distinct), \( \Sigma_{i \in I_1(g_0)} \ell_i \) points \( y_{i,1}, \ldots, y_{i,\ell_i} \in [a, b] \), \( \Sigma_{i \in I_2(g_0)} t_i \) points \( z_{i,1}, \ldots, z_{i,t_i} \in [a, b] \) and \( s + \Sigma_{i \in I_1(g_0)} \ell_i + \Sigma_{i \in I_2(g_0)} t_i \) scalars \( a_1, \ldots, a_s, b_{1,1}, \ldots, b_{1,\ell_1} \) and \( c_{1,1}, \ldots, c_{1,\ell_1} > 0 \) such that \( s + \Sigma_{i \in I_1(g_0)} \ell_i + \Sigma_{i \in I_2(g_0)} t_i \leq n + 1 \) and

\[
\begin{align*}
\sum_{i=1}^{s} a_i \int_{a}^{b} g(x) u_i(x) dx & + \sum_{i \in I_1(g_0)} \sum_{j=1}^{\ell_i} b_{i,j} D^{k_i}g(y_{i,j}) \\
& + \sum_{i \in I_2(g_0)} \sum_{j=1}^{t_i} c_{i,j} D^{k_i}g(z_{i,j}) = 0 \quad \forall g \in G
\end{align*}
\] (4.1)

where \( \frac{1}{\mu} + \frac{1}{\nu} = 1 \), \( \int_{a}^{b} (f_i - g_0) u_i = d_F(g_0), \quad i = 1, 2, \ldots, s \),

\[ D^{k_i} \left[ g_0(y_{i,j}) - \psi_i(y_{i,j}) \right] = 0, \quad j = 1, 2, \ldots, \ell_i, \quad i \in I_1(g_0) \]

and

\[ D^{k_i} \left[ g_0(z_{i,j}) - \rho_i(z_{i,j}) \right] = 0, \quad j = 1, 2, \ldots, t_i, \quad i \in I_2(g_0). \]
Finally, we consider the case $m = 1$, $e_1 = -1$, $B_1 = \{ e_1 \hat{x} : x \in [a, b] \}$, $C_1 = \emptyset$ and $F = \{ f \}$. Suppose $G$ is an $n$-dimensional subspace of $C[a, b]$ containing constant functions. Define the set $Y$ as before by taking $\psi_1 = f$. This again leads to one-sided approximation in $C[a, b]$ with $L_\mu$-norm. However, it is clear that $f(x) - g(x)$ does not change sign for each $g \in Y$, hence, if $u(x) \in L_\nu[a, b]$ such that $\int (f - g)u = \| f - g \|_\mu$, then $u(x) = (f - g)^{\mu-1}(x)/\| f - g \|_\mu^{\mu-1}$. Consequently, we have

**Theorem 4.3.** A function $g_0 \in Y$ is a best approximation to $f \in C[a, b]$ if and only if there exist $y_1, \ldots, y_\ell \in N_{1,1}(g_0)$ and $\ell$ scalars $b_1, \ldots, b_\ell > 0$ such that $\ell \leq n$ and

$$\int_a^b g(x)(f(x) - g_0(x))^{\mu-1}dx - \sum_{i=1}^\ell b_i g(y_i) = 0 \quad \text{for all } g \in G. \quad (4.2)$$

If $\mu = 1$, then the equality $(4.2)$ can be written as

$$\int_a^b g(x)dx = \sum_{i=1}^\ell b_i g(y_i) \quad \text{for all } g \in G_n.$$

**Remark** In the case $G$ is the set of all polynomials of degree $\leq n - 1$, then $2\ell - e_1 \geq n$ where $\ell$ and $e_1$ are the number of points in $N_{1,1}(g_0)$ and $N_{1,1}(g_0) \cap \{ a, b \}$, respectively, for otherwise, we would find a $g \in G$ such that $g$ has double zeros on $N_{1,1}(g_0) \cap \{ a, b \}$ and simple zeros on $N_{1,1}(g_0) \cap \{ a, b \}$, which would contradict the relation (4.2). Moreover, if $Dg_0 \neq 0$, then $2\ell - e_1$ is, in fact, actually equal to $n$.

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Department of Mathematics, University of Malaya, Kuala Lumpur, Malaysia.