The Stokes phenomenon for the $q$-difference equation satisfied by the basic hypergeometric series $3\varphi_1(a_1, a_2, a_3; b_1; q, x)$

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Abstract

We show the connection formula for the basic hypergeometric series $3\varphi_1(a_1, a_2, a_3; b_1; q, x)$ between around the origin and infinity by the using of the $q$-Borel-Laplace transformations. We also show the limit $q \to 1 - 0$ of the new connection formula.

1 Introduction

In this paper, we show the connection formula for the divergent basic hypergeometric series

$$3\varphi_1(a_1, a_2, a_3; b_1; q, x) = \sum_{n \geq 0} \frac{(a_1, a_2, a_3; q)_n}{(b_1; q)_n(q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n$$

(1)

between around the origin and around infinity by the using of the $q$-Borel-Laplace resummation methods. Here, the function $(a; q)_n$ is the $q$-shifted factorial (see section 2 and [1] for more details of the $q$-shifted factorials and the basic hypergeometric series $r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x)$):

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \ldots (1 - aq^{n-1}), & n \geq 1. \end{cases}$$

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The series (1) satisfy the third order linear $q$-difference equation
\[
(a_1a_2a_3x - \frac{b_j}{q^2})u(q^3x) - \left\{ (a_1a_2 + a_2a_3 + a_3a_1)x - \left( \frac{b_1}{q^2} + \frac{1}{q} \right) \right\} u(q^2x) + \left\{ (a_1 + a_2 + a_3) - \frac{1}{q} \right\} u(qx) - xu(x) = 0. \quad (2)
\]

Equation (2) also has a fundamental system of solutions around infinity:

\[
v_1(x) := x^{-a_3} \phi_2 \left( a_1, \frac{a_1q}{b_1}, 0; \frac{a_1q}{a_2}, \frac{a_1q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x} \right) \quad (3)
\]

\[
v_2(x) := x^{-a_2} \phi_2 \left( a_2, \frac{a_2q}{b_1}, 0; \frac{a_2q}{a_1}, \frac{a_2q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x} \right) \quad (4)
\]

\[
v_3(x) := x^{-a_3} \phi_2 \left( a_3, \frac{a_3q}{b_1}, 0; \frac{a_3q}{a_2}, \frac{a_3q}{a_3}; q, \frac{qb_1}{a_1a_2a_3x} \right) \quad (5)
\]

where $a_j = q^{a_j}, j = 1, 2$ and 3. In section 3, we show the connection formula between (3), (4) and (1).

We review the connection problems on the linear $q$-difference equations. Connection problems on the linear $q$-difference equations with regular singular points were studied by G. D. Birkhoff [1]. Connection formulae for the second order linear $q$-difference equations are given by the matrix form
\[
\begin{pmatrix}
u_1(x) \\ u_2(x)
\end{pmatrix} =
\begin{pmatrix}
C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x)
\end{pmatrix}
\begin{pmatrix}
u_1(x) \\ u_2(x)
\end{pmatrix}.
\]

The pair $(u_1(x), u_2(x))$ is a fundamental system of solutions around the origin and the pair $(v_1(x), v_2(x))$ is a fundamental system of solutions around infinity. The connection coefficients $C_{jk}(x) (1 \leq j, k \leq 2)$ are given by $q$-periodic and unique valued functions
\[
\sigma_q C_{jk}(x) = C_{jk}(x), \quad C_{jk}(e^{2\pi i}x) = C_{jk}(x),
\]

namely, the elliptic functions.

The first example of the connection formula was given by G. N. Watson [11] in 1910. Watson gave the connection formula for Heine’s basic hypergeometric series
\[
\phi_1(a, b; c; q, x) := \sum_{n \geq 0} \frac{(a, b; q)_n}{(c; q)_n(q; q)_n} x^n
\]
around the origin and around the infinity \([4, \text{page 117}]. \) Heine’s \(2\varphi_1(a, b; c; q, x)\) satisfies the \(q\)-difference equation

\[
\big( (c - abq x) \sigma_q^2 - \{ (c + q) - (a + b) qx \} \sigma_q + q(1 - x) \big) u(x) = 0. \tag{6}
\]
The equation (6) also has a fundamental system of solutions around the infinity:

\[
y^{(a,b)}_\infty(x) = x^{-\alpha} 2\varphi_1 \left( a, \frac{aq}{c}, \frac{aq}{b}, q, \frac{cq}{abx} \right)
\]
and

\[
y^{(b,a)}_\infty(x) = x^{-\beta} 2\varphi_1 \left( b, \frac{bq}{c}, \frac{bq}{a}, q, \frac{cq}{abx} \right),
\]
provided that \(a = q^\alpha\) and \(b = q^\beta\). Watson’s connection formula for \(2\varphi_1(a, b; c; q, x)\) is given by

\[
2\varphi_1(a, b; c; q, x) = \frac{(b, c/a; q)_\infty \theta(-ax)_\infty \theta(x)}{(c, b/a; q)_\infty \theta(-bx)_\infty \theta(x)} y^{(a,b)}_\infty(x) + \frac{(a, c/b; q)_\infty \theta(-bx)_\infty \theta(x)}{(c, a/b; q)_\infty \theta(-x)_\infty \theta(bx)} y^{(b,a)}_\infty(x).
\]
Here, the notation \(\theta(x)\) is the theta function of Jacobi (see section two for more details). We remark that the connection coefficients are given by the \(q\)-elliptic functions.

But connection formulae for \(q\)-difference equations with irregular singular points had not known for a long time. We remark that A. Duval and C. Mitschi gave connection matrices for degenerated differential equations [3]. The irregularity of \(q\)-difference equations are studied by the using of the Newton polygons by J.-P. Ramis, J. Sauloy and C. Zhang [9]. C. Zhang gave connection formulae for some confluent type basic hypergeometric series [12, 13, 14] where he uses the \(q\)-Borel-Laplace transformations. In [6, 7], the author gave the connection formula for the Hahn-Exton \(q\)-Bessel function and the \(q\)-confluent type function by the \(q\)-Borel-Laplace transformations. These resummation methods are powerful tools for connection problems on linear \(q\)-difference equations with irregular singular points.

**Definition 1.** We assume that \(f(x)\) is a formal power series \(f(x) = \sum_{n\in\mathbb{Z}} a_n x^n, \quad a_0 = 1.\)
1. The $q$-Borel transformation is

$$(B_q^+ f) (\xi) := \sum_{n \in \mathbb{Z}} a_n q^{n(n-1)/2} \xi^n (=: \psi(\xi)).$$

2. For any analytic function $\psi(\xi)$ around $\xi = 0$, the $q$-Laplace transformation is

$$(L_{q,\lambda}^+ \psi) (x) := \frac{1}{1 - q} \int_0^\infty \frac{\varphi(\xi)}{\theta_q(\xi/x)} \frac{d\xi}{\xi} = \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q(\lambda q^n/x)}.$$

Here, this transformation is given by Jackson’s $q$-integral [4, page 23].

The definition is a special case of one of the $q$-Laplace transformations in [2, 12]. The $q$-Borel transformation is the formal inverse of the $q$-Laplace transformation as follows:

**Lemma 1** (Zhang, [12]). For any entire function $f(x)$, we have

$$L_{q,\lambda}^+ \circ B_q^+ f = f.$$

Thanks to these methods, some connection formulae for the second order $q$-difference equations were found. However, the connection formulae for more higher order linear $q$-difference equations have not been known. In this paper, especially we apply the $q$-Borel-Laplace transformations to the divergent series (1) to study the connection problem on the third order $q$-difference equation. In the section 3, we show the following theorem:

**Theorem.** For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have

$$3f_1(a_1, a_2, a_3; b_1; q; \lambda, x) := (L_{q,\lambda}^+ \circ B_q^+ 3\varphi_1(a_1, a_2, a_3; b_1; q, x)) (x)$$

$$= \frac{(a_2, a_3, b_1/a_1; q)_{\infty}}{(b_1, a_2/a_1, a_3/a_1; q)_{\infty}} \frac{\theta(a_1 \lambda) \theta(a_1 qx/\lambda)}{\theta(a_1) \theta(a_1 x)} v_1(x)$$

$$+ \frac{(a_1, a_3, b_1/a_2; q)_{\infty}}{(b_1, a_1/a_2, a_3/a_2; q)_{\infty}} \frac{\theta(a_2 \lambda) \theta(a_2 qx/\lambda)}{\theta(a_2) \theta(a_2 x)} v_2(x)$$

$$+ \frac{(a_2, a_1, b_1/a_3; q)_{\infty}}{(b_1, a_2/a_3, a_1/a_3; q)_{\infty}} \frac{\theta(a_3 \lambda) \theta(a_3 qx/\lambda)}{\theta(a_3) \theta(a_3 x)} v_3(x).$$

Here, $(L_{q,\lambda}^+ \circ B_q^+ 3\varphi_1(a_1, a_2, a_3; b_1; q, x)) (x)$ is the $q$-Borel-Laplace transform of the divergent series $3\varphi_1(a_1, a_2, a_3; b_1; q, x)$.
We remark that the connection coefficients (with the new parameter $\lambda$) are given by the $q$-elliptic functions. These coefficients are also the new example of the Stokes phenomenon \cite{2} for the $q$-difference equation \eqref{2}.

In the last section, we also give the limit $q \to 1 - 0$ of the new connection formula.

2 Basic notations

In this section, we review our notations. The $q$-shifted operator $\sigma_q$ is given by $\sigma_q f(x) = f(qx)$. For any fixed $\lambda \in \mathbb{C}^* \setminus q\mathbb{Z}$, the set $[\lambda; q]$-spiral is $[\lambda; q] := \lambda q^k \{ k \in \mathbb{Z} \}$. The function $(a; q)_n$ is the $q$-shifted factorial such that

\[ (a; q)_0 := 1, \quad (a; q)_n := (1 - a)(1 - aq) \dots (1 - aq^{n-1}), \quad n \geq 1. \]

Moreover, $(a; q)_\infty := \lim_{n \to \infty} (a; q)_n$ and

\[ (a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty. \]

The basic hypergeometric series with the base $q$ \cite[page 4]{4} is

\[ \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x)}{(b_1, \ldots, b_s; q, q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n. \]

The radius of convergence is $\infty$, $1$ or $0$ according to whether $r-s < 1$, $r-s = 1$ or $r-s > 1$. The theta function of Jacobi is important in connection problems on linear $q$-difference equations. The theta function with the base $q$ is

\[ \theta_q(x) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \quad \forall x \in \mathbb{C}^*. \]

The theta function has the triple product identity

\[ \theta_q(x) = (q, -x, -\frac{q}{x}; q)_\infty. \]  

\[ \theta_q(x) = (q, -x, -\frac{q}{x}; q)_\infty. \]  

\[ \theta_q(q^k x) = q^{\frac{n(n-1)}{2}} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}. \]

The theta function also has the inversion formula $\theta_q(1/x) = \theta_q(x)/x$. 

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We remark that \( \theta(\lambda q^k / x) = 0 \) if and only if \( x \in [-\lambda; q] \). The function \( \theta(x) / \theta(q^\alpha x), \forall \alpha \notin \mathbb{Z} \) satisfies a \( q \)-difference equation
\[
\theta(qx) = q^\alpha \theta(x),
\]
which is also satisfied by the function \( \theta(x) = x^\alpha \).

3 The connection formula

In this section, we give the new connection formula for the basic hypergeometric series \( 3\varphi_1(a_1, a_2, a_3; b_1; q, x) \). In section 3.1, we review the connection formula of non-degenerated series \( 3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x) \).

3.1 The non-degenerated case

The non-degenerated convergent series
\[
3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x) := \sum_{n \geq 0} \frac{(a_1, a_2, a_3, q)_{n}}{(b_1, b_2, q)_{n}} q^{rac{n(n-1)}{2}} x^n
\]
satisfies the third order \( q \)-difference equation
\[
\left\{ \left( a_1 a_2 a_3 x - \frac{b_1 b_2}{q^2} \right) \sigma_q^3 - \left\{ (a_1 a_2 + a_2 a_3 + a_3 a_1)x - \left( \frac{b_1 b_2}{q^2} + \frac{b_2}{q} + \frac{b_1}{q} \right) \right\} \sigma_q^2 \\
\left\{ (a_1 + a_2 + a_3)x - \left( \frac{b_1}{q} + \frac{b_2}{q} + 1 \right) \right\} \sigma_q - (x - 1) \right\} u(x) = 0.
\]
Equation (9) also has a fundamental system of solutions around infinity:
\[
\tilde{v}_1(x) = \frac{\theta(a_1 x)}{\theta(x)} 3\varphi_2 \left( \frac{a_1}{b_1}, \frac{a_1 q}{b_1}, \frac{a_1 q}{b_2}, \frac{a_1 q}{b_3}, \frac{a_1 q}{a_3}, q, \frac{q b_1 b_2}{a_1 a_2 a_3 x} \right),
\]
\[
\tilde{v}_2(x) = \frac{\theta(a_2 x)}{\theta(x)} 3\varphi_2 \left( \frac{a_2}{b_1}, \frac{a_2 q}{b_1}, \frac{a_2 q}{b_2}, \frac{a_2 q}{b_3}, \frac{a_2 q}{a_3}, q, \frac{q b_1 b_2}{a_1 a_2 a_3 x} \right),
\]
\[
\tilde{v}_3(x) = \frac{\theta(a_3 x)}{\theta(x)} 3\varphi_2 \left( \frac{a_3}{b_1}, \frac{a_3 q}{b_1}, \frac{a_3 q}{b_2}, \frac{a_3 q}{b_3}, \frac{a_3 q}{a_3}, q, \frac{q b_1 b_2}{a_1 a_2 a_3 x} \right).
\]
The connection formula between the solutions (10), (11), (12) and (8) can be found in [4, page 121]. We remark that the following formula was essentially given by L. J. Slater.
Theorem 1 (Slater, [10]). For any \( x \in \mathbb{C}^* \), we have

\[
3\varphi_2(a_1, a_2, a_3; b_1, b_2; q, x) = \frac{(a_2, a_3, b_1/a_1, b_2/a_1; q)_\infty \theta(-a_1 x) \theta(x)}{(b_1, b_2/a_1, a_3/a_1; q)_\infty \theta(-x) \theta(a_1 x)} \tilde{v}_1 + \text{idem}(a_1; a_2, a_3).
\]

Provided that the notation \( \text{idem}(a_1; a_2, a_3) \) after an expression stands for the sum expressions obtained from the preceding expression by interchanging \( a_1 \) with each \( a_2 \) and \( a_3 \).

This Theorem can be considered as the higher order extension of Watson’s formula. By Theorem 1 we obtain the following key Lemma.

Lemma 2. For any \( x \in \mathbb{C}^* \), we have

\[
3\varphi_2(a_1, a_2, a_3; b_1, 0; q, x) = \frac{(a_2, a_3, b_1/a_1; q)_\infty \theta(-a_1 x)}{(b_1, a_2/a_1, a_3/a_1; q)_\infty \theta(-x)} 2\varphi_2 \left( a_1, a_1 q, a_1 q, a_1 q, q^2 b_1 \right) + \text{idem}(a_1; a_2, a_3).
\]

Proof. We take the limit \( b_2 \to 0 \) in Theorem 1 we obtain the conclusion. \( \square \)

In the next section, we prove our new connection formula by Lemma 2 and the \( q \)-Borel-Laplace transformations.

### 3.2 Proof of main Theorem

In this section, we prove the following Theorem.

Theorem 2. For any \( x \in \mathbb{C}^* \setminus [-\lambda; q] \), we have

\[
3f_1(a_1, a_2, a_3; b_1; q, \lambda, x) := \left( \mathcal{L}_q^+ \circ \mathcal{B}_q^+ \right) 3\varphi_1(a_1, a_2, a_3; b_1; q, x) (x)
\]

\[
= \frac{(a_2, a_3, b_1/a_1; q)_\infty \theta(a_1 \lambda) \theta(a_1 q x / \lambda) \theta(x)}{(b_1, a_2/a_1, a_3/a_1; q)_\infty \theta(\lambda) \theta(q x / \lambda) \theta(a_1 x)} v_1(x)
+ \frac{(a_1, a_3, b_1/a_2; q)_\infty \theta(a_2 \lambda) \theta(a_2 q x / \lambda) \theta(x)}{(b_1, a_1/a_2, a_3/a_2; q)_\infty \theta(\lambda) \theta(q x / \lambda) \theta(a_2 x)} v_2(x)
+ \frac{(a_2, a_1, b_1/a_3; q)_\infty \theta(a_3 \lambda) \theta(a_3 q x / \lambda) \theta(x)}{(b_1, a_2/a_3, a_1/a_3; q)_\infty \theta(\lambda) \theta(q x / \lambda) \theta(a_3 x)} v_3(x).
\]
Proof. We apply the $q$-Borel transformation to the series $3\varphi_1(a_1, a_2, a_3; b_1; q, x)$.

$$(B_q^+ 3\varphi_1(a_1, a_2, a_3; b_1; q, x))(\xi) = 3\varphi_2(a_1, a_2, a_3; b_1, 0, -\xi) =: \varphi(\xi).$$

By Lemma 2, we have another expression of the function $\varphi(\xi)$. We also apply the $q$-Laplace transformation $L_{q, \lambda}$ to the function $\varphi(\xi)$, we obtain the conclusion. 

Remark 1. We remark that the fundamental system of solutions for equation (2) are given by

$\begin{align*}
v_1(x) &:= \frac{\theta(a_1 x)}{\theta(x)} 3\varphi_2(a_1, a_1 q \frac{b_1}{a_2}, a_1 q \frac{a_1 q}{a_2}, a_1 q \frac{q b_1}{a_1 a_2 a_3 x}), \\
v_2(x) &:= \frac{\theta(a_2 x)}{\theta(x)} 3\varphi_2(a_2, a_2 q \frac{b_1}{a_1}, a_2 q \frac{a_2 q}{a_1}, a_2 q \frac{q b_1}{a_1 a_2 a_3 x}), \\
v_3(x) &:= \frac{\theta(a_3 x)}{\theta(x)} 3\varphi_2(a_3, a_3 q \frac{b_1}{a_2}, a_3 q \frac{a_3 q}{a_2}, a_3 q \frac{q b_1}{a_1 a_2 a_3 x})
\end{align*}$

in the Theorem 2.

Remark 2. By the $q$-difference equation of the theta function, we can check out that the connection coefficients (with the new parameter $\lambda$)

$\begin{align*}
C_1(x) &:= \frac{(a_2, a_3, b_1/a_1; q)_{\infty} \theta(a_1 x)}{(b_1, a_2/a_1, a_3/a_1; q)_{\infty} \theta(\lambda) \theta(q x/\lambda) \theta(a_1 x)} \\
C_2(x) &:= \frac{(a_1, a_3, b_1/a_2, a_3/a_2; q)_{\infty} \theta(a_2 x)}{(b_1, a_1/a_2, a_3/a_2; q)_{\infty} \theta(\lambda) \theta(q x/\lambda) \theta(a_2 x)} \\
C_3(x) &:= \frac{(a_2, a_1, b_1/a_3, a_1/a_3; q)_{\infty} \theta(a_3 x)}{(b_1, a_2/a_3, a_1/a_3; q)_{\infty} \theta(\lambda) \theta(q x/\lambda) \theta(a_3 x)}
\end{align*}$

are the $q$-elliptic functions.

4 The limit $q \to 1 - 0$ of the connection formula

The aim of this section is to give the limit $q \to 1 - 0$ of the new connection formula as follows:
Theorem 3. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have the following limit $q \to 1 - 0$ of the connection formula

$$\lim_{q \to 1 - 0} {}_3F_1(q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_3}; q^{\beta_1}; q; \lambda, x) = \frac{\Gamma(\beta_1)\Gamma(\alpha_2 - \alpha_1)\Gamma(\alpha_3 - \alpha_1)}{\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\beta_1 - \alpha_1)} x^{-\alpha_1} \, _2F_2\left(\alpha_1, \alpha_1 + 1 - \beta_1; \alpha_1 + 1 - \alpha_2, \alpha_1 + 1 - \alpha_3; \frac{1}{x}\right)$$

$$+ \frac{\Gamma(\beta_1)\Gamma(\alpha_1 - \alpha_2)\Gamma(\alpha_3 - \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_3)\Gamma(\beta_1 - \alpha_2)} x^{-\alpha_2} \, _2F_2\left(\alpha_2, \alpha_2 + 1 - \beta_1; \alpha_2 + 1 - \alpha_1, \alpha_2 + 1 - \alpha_3; \frac{1}{x}\right)$$

$$+ \frac{\Gamma(\beta_1)\Gamma(\alpha_2 - \alpha_3)\Gamma(\alpha_1 - \alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_1)\Gamma(\beta_1 - \alpha_3)} x^{-\alpha_3} \, _2F_2\left(\alpha_3, \alpha_3 + 1 - \beta_1; \alpha_3 + 1 - \alpha_2, \alpha_3 + 1 - \alpha_1; \frac{1}{x}\right),$$

provided that $-\pi < \arg x < \pi$.

The following proposition [13] is important to consider the limit $q \to 1 - 0$ of our connection formula.

Proposition 1. For any $x \in \mathbb{C}^* (-\pi < \arg x < \pi)$, we have

$$\lim_{q \to 1 - 0} \frac{\theta(q^\beta x)}{\theta(q^\alpha x)} = x^{\alpha - \beta} \tag{16}$$

and

$$\lim_{q \to 1 - 0} \frac{\theta \left( \frac{q^\alpha x}{(1 - q)} \right)}{\theta \left( \frac{q^\beta x}{(1 - q)} \right)} (1 - q)^{\beta - \alpha} = x^{\beta - \alpha}. \tag{17}$$

We also review the $q$-gamma function. The $q$-gamma function $\Gamma_q(x)$ is

$$\Gamma_q(x) := \frac{(q; q)_x}{(q^x; q)_x} (1 - q)^{1 - x}, \quad 0 < q < 1.$$

The limit $q \to 1 - 0$ of $\Gamma_q(x)$ gives the gamma function [4, page 20]

$$\lim_{q \to 1 - 0} \Gamma_q(x) = \Gamma(x). \tag{18}$$

We give the proof of the Theorem [3].
Proof. At first, we put \( a_j := q^{\alpha_j} \) (\( j = 1, 2, 3 \)), \( b_1 := q^{\beta_1} \) and \( x \mapsto x/(1 - q) \). We remark that the limit \( q \to 1 - 0 \) of the left hand-side of Theorem 3 formally converges the hypergeometric series

\[
_3F_1(\alpha_1, \alpha_2, \alpha_3; \beta_1; x) = \sum_{n \geq 0} \frac{(\alpha_1, \alpha_2, \alpha_3)_n}{(\beta_1)_n n!} x^n.
\]

We consider the right hand-side. The connection formula can be rewritten as follows:

\[
_3f_1(q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_3}; q^{\beta_1}; q, \lambda, x)
= \frac{(q^{\alpha_2}, q^{\alpha_3}, q^{\beta_1-\alpha_1}; q)_\infty}{(q^{\beta_1}, q^{\alpha_2-\alpha_1}, q^{\alpha_3-\alpha_1}; q)_\infty} \frac{\theta(q^{\alpha_1} \lambda)}{\theta(q^{\beta_1} \lambda)} \frac{\theta(q^{\alpha_1+1} x)}{\theta(q^{\beta_1} \lambda)}
\times 3\phi_2\left( q^{\alpha_1}, q^{\alpha_1+1-\beta_1}, 0; q^{\alpha_1+1-\alpha_2}, q^{\alpha_1+1-\alpha_3}, q; \frac{q^{1+\beta_1}(1 - q)}{q^{\alpha_1+\alpha_2+\alpha_3} x} \right)
+ \text{idem}(q^{\alpha_1}; q^{\alpha_2}, q^{\alpha_3})
\]

\[
= \frac{\Gamma_q(\beta_1)\Gamma_q(\alpha_2 - \alpha_1)\Gamma_q(\alpha_3 - \alpha_1)}{\Gamma_q(\alpha_2)\Gamma_q(\alpha_3)\Gamma_q(\beta_1 - \alpha_1)} \frac{\theta(q^{\alpha_1} \lambda)}{\theta(q^{\beta_1} \lambda)} \left\{ \frac{\theta(q^{\alpha_1+1} x)}{\theta(q^{\beta_1} \lambda)} \right\} (1 - q)^{-\alpha_1}
\times 3\phi_2\left( q^{\alpha_1}, q^{\alpha_1+1-\beta_1}, 0; q^{\alpha_1+1-\alpha_2}, q^{\alpha_1+1-\alpha_3}, q; \frac{q^{1+\beta_1}(1 - q)}{q^{\alpha_1+\alpha_2+\alpha_3} x} \right)
+ \text{idem}(q^{\alpha_1}; q^{\alpha_2}, q^{\alpha_3}).
\]

By (16), (17) and (18), we obtain the conclusion. □

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