δ₁² Without Sharps

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δ₁² denotes the supremum of the lengths of Δ₁² prewellorderings of the reals. A result of Kunen and Martin (see Martin[77]) states that δ₁² is at most ω₂ and it is known that in the presence of sharps the assumption δ₁² = ω₂ is strong: it implies the consistency of a strong cardinal (see Steel-Welch[?]).

In this paper we show how to obtain the consistency of δ₁² = ω₂ in the absence of sharps, without strong assumptions.

Theorem. Assume the consistency of an inaccessible. Then it is consistent that δ₁² = ω₂ and ω₁ is inaccessible to reals (i.e., ω₁[L[x]] is countable for each real x).

The proof is obtained by combining the Δ₁-coding technique of Friedman-Velickovic [95] with the use of a product of Jensen codings of Friedman [94].

We begin with a description of the Δ₁-coding technique.

Definitions. Suppose x is a set, ⟨x, ∈⟩ satisfies the axiom of extensionality and A ⊆ ORD. x preserves A if ⟨x, A ∩ x⟩ ≅ ⟨x, A ∩ x⟩ where x = transitive collapse of x. For any ordinal δ, x[δ] = {f(γ)|γ < δ, f ∈ x, f a function, γ ∈ Dom(f)}. x strongly preserves A if x[δ] preserves A for every cardinal δ. A sequence x₀ ⊆ x₁ ⊆ ... is tight if it is continuous and for each i,

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\[ \langle \bar{x}_j | j < i \rangle \] belongs to the least ZF\(^-\) model which contains \( \bar{x}_i \) as an element and correctly computes \( \text{card}(\bar{x}_i) \).

**Condensation Condition for A.** Suppose \( t \) is transitive, \( \delta \) is regular, \( \delta \in t \) and \( x \in t \). Then:

(a) There exists a continuous, tight \( \delta \)-sequence \( x_0 < x_1 < \cdots < t \) such that \( \text{card}(x_i) = \delta, x \in x_0 \) and \( x_i \) strongly preserves \( A \), for each \( i \).

(b) If \( \delta \) is inaccessible then there exist \( x_i \)'s as above but where \( \text{card}(x_i) = \aleph_i \).

The following is proved in Friedman-Velickovic [95].

**\( \Delta_1 \)-Coding.** Suppose \( V = L \) and the Condensation Condition holds for \( A \). Then \( A \) is \( \Delta_1 \) in a class-generic real \( R \), preserving cardinals.

Now we are ready to begin the proof of the Theorem. Suppose \( \kappa \) is the least inaccessible and \( V = L \). Let \( \langle \alpha_i | i < \kappa^+ \rangle \) be the increasing list of all \( \alpha \in (\kappa, \kappa^+) \) such that \( L_\alpha = \text{Skolem hull}(\kappa) \) in \( L_\alpha \). For each \( i < \kappa^+ \) define \( f_i : \kappa \rightarrow \kappa \) by \( f_i(\gamma) = \text{ordertype}(\text{ORD} \cap \text{Skolem hull}(\gamma) \text{ in } L_{\alpha_i}) \). By identifying \( f_i \) with its graph and using a pairing function we can think of \( f_i \) as a subset of \( \kappa \).

The following is straightforward.

**Lemma 1.** Each \( f_i \) obeys the Condensation Condition. Indeed \( \langle f_i | i < \kappa^+ \rangle \) jointly obeys the Condensation Condition in the following sense: Suppose \( t \) is transitive, \( \delta \) is regular, \( \delta \in t, x \in t \). Then there exists a tight \( \delta \)-sequence \( x_0 < x_1 < \cdots < t \) such that \( \text{card}(x_i) = \delta, x \in x_0 \) and each \( x_i \) strongly preserves all \( f_j \) for \( j \in x_i \) (and if \( \delta = \kappa \) then we can alternatively require \( \text{card}(x_i) = \aleph_i \)).

Now, following Friedman [94] we use a “diagonally-supported” product of Jensen-style codings. For each \( i < \kappa^+ \) let \( \mathcal{P}(i) \) be the forcing from Friedman-Velickovic [95] to make \( f_i \) \( \Delta_1 \)-definable in a class-generic real. Then \( \mathcal{P} \) consists of all \( p \in \prod_{i < \kappa^+} \mathcal{P}(i) \) such that for infinite ordinals \( \gamma \), \( \{ i | p(i)(\gamma) \neq (\phi, \phi) \} \) has cardinality at most \( \alpha \) and in addition \( \{ i | p(i)(0) \neq (\phi, \phi) \} \) is finite.

Now note that for successor cardinals \( \gamma < \kappa \) the forcing \( \mathcal{P} \) factors as \( \mathcal{P}_\gamma \ast \mathcal{P}^G_\gamma \) where \( \mathcal{P}_\gamma \) forces that \( \mathcal{P}^{G_\gamma} \) has the \( \gamma^+-\text{CC} \). Also the joint Condensation Condition of Lemma 1 implies that the argument of Theorem 3 of Friedman-
Velickovic [95] can be applied here to show that $P_\gamma$ is $\leq \gamma$-distributive, and also that $P$ is $\Delta$-distributive (if $\langle D_i | i < \kappa \rangle$ is a sequence of predense sets then it is dense to reduce each $D_i$ below $\aleph_{i+1}$). So $P$ preserves cofinalities.

Thus in a cardinal-preserving forcing extension of $L$ we have produced $\kappa^+$ reals $\langle R_i | i < \kappa^+ \rangle$ where $R_i$ $\Delta_1$-codes $f_i$ and hence there are well-orderings of $\kappa$ of any length $< \kappa^+$ which are $\Delta_1$ in a real. Finally Lévy collapse to make $\kappa = \omega_1$ and we have $\delta^1_2 = \omega_2$, $\omega_1$ inaccessible to reals.

The above proof also shows the following, which may be of independent interest.

**Theorem 2.** Let $\delta^1_1(\kappa)$ be the sup of the lengths of wellorderings of $\kappa$ which are $\Delta_1$ over $L_\kappa[x]$ for some $x$, a bounded subset of $\kappa$. Then (relative to the consistency of an inaccessible) it is consistent that $\kappa$ be weakly inaccessible and $\delta^1_1(\kappa) = \kappa^+$.

**Remark.** The conclusion of Theorem 2 cannot hold in the context of sharps: if $\kappa$ is weakly inaccessible and every bounded subset of $\kappa$ has a sharp then $\delta^1_1(\kappa) < \kappa^+$. This is because $\delta^1_1(\kappa)$ is then the second uniform indiscernible for bounded subsets of $\kappa$, which can be written as the direct limit of the second uniform indiscernible for subsets of $\delta$, as $\delta$ ranges over cardinals less than $\kappa$; so $\delta^1_1(\kappa)$ has cardinality $\kappa$.

Using the least inner model closed under sharp, we can also obtain the following.

**Theorem 3.** Assuming it is consistent for every set to have a sharp, then this is also consistent with $\delta^1_3 = \omega_2$.

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