Wigner separability entropy and complexity of quantum dynamics

Giuliano Benenti, †, Gabriel G. Carlo, ‡, and Tomáš Prosen †

1 CNISM, CNR-INFM & Center for Nonlinear and Complex Systems, Università degli Studi dell’Insubria, Via Valleggio 11, 21100 Como, Italy
2 Istituto Nazionale di Fisica Nucleare, Sezione di Milano, via Celoria 16, 20133 Milano, Italy
3 Departamento de Física, Comisión Nacional de Energía Atómica, Avenida del Libertador 8250, (1429) Buenos Aires, Argentina
4 Department of Physics, Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia.

(Dated: December 21, 2013)

We propose the Wigner separability entropy as a measure of complexity of a quantum state. This quantity measures the number of terms that effectively contribute to the Schmidt decomposition of the Wigner function with respect to a chosen phase space decomposition. We prove that the Wigner separability entropy is equal to the operator space entanglement entropy, measuring entanglement in the space of operators, and, for pure states, to twice the entropy of entanglement. The quantum to classical correspondence between the Wigner separability entropy and the separability entropy of the classical phase space Liouville density is illustrated by means of numerical simulations of chaotic maps. In this way, the separability entropy emerges as an extremely broad complexity quantifier in both the classical and quantum realms.

PACS numbers: 05.45.Mt, 03.65.Sq, 05.45.Pq

I. INTRODUCTION

Measuring complexity in a simple and unified way has been a major and long quest in both quantum and classical dynamics. While there exists a direct connection between chaos and algorithmic complexity of trajectories in classical physics [1], the problem is particularly elusive for quantum mechanics [2], where the notion of trajectory is forbidden by the Heisenberg uncertainty principle and complexity can be attributed not only to the lack of integrability but also to the tensor-product structure of the Hilbert space, that is, to entanglement.

The phase space representation of quantum mechanics is a very convenient framework to investigate quantum complexity, in that one can compare classical and quantum dynamical evolutions of distributions in phase space. In this context, the number of Fourier harmonics of the Wigner function has already been used to characterize the complexity of a quantum state, both for single-particle [3] and many-body [4] quantum dynamics, in particular to detect quantum phase transitions. This complexity measure can be equally applied to classical and quantum mechanics, with the Liouville density used instead of the Wigner function in the classical case. However, this quantity has the disadvantage of being basis-dependent. Moreover, knowledge of the whole Fourier harmonics spectrum of the Wigner function seems in general not necessary to compute expectation values of physically relevant observables. For instance, even though the number of harmonics grows exponentially in time for both integrable and non-integrable quantum chaotic Ising chains [4], the resources required to simulate both local and extensive observables grow exponentially in the chaotic case but only polynomially for the integrable model [3].

Recently, a new complexity indicator has been introduced for classical dynamics, the separability entropy [6], measuring the logarithm of the effective number of terms in the Schmidt (or singular value) decomposition of the Liouville density, with respect to an arbitrary phase space decomposition. This quantity estimates the minimal amount of computational resources required to simulate the classical Liouvillian evolution and grows linearly in time for dynamics that cannot be efficiently simulated. In this paper, we extend this notion of complexity to the quantum realm, by defining the Wigner separability entropy as the number of terms that effectively contribute to the Schmidt decomposition of the Wigner function. We prove that such quantity is equal to the operator space entanglement entropy [6], constructed from the Schmidt decomposition of the density operator in the space of Hilbert-Schmidt operators and quantifying the complexity of time-dependent density-matrix renormalization group simulations. Furthermore, for pure states the Wigner separability entropy is twice the entanglement entropy. We illustrate the quantum to classical correspondence for the separability entropy and its link to the entanglement entropy by means of numerical simulations of quantum chaotic maps.

The paper is organized as follows. In Sec. II we define the Wigner separability entropy $h[W]$ and prove its relation to the operator space entanglement entropy and, for pure states, to the entanglement entropy. The dynamical evolution of $h[W]$ is studied for quantum chaotic map models in Sec. III. We finish with concluding remarks in Sec. IV.

---

[1] Electronic address: giuliano.benenti@uninsubria.it
[2] Electronic address: carlo@tandar.cnea.gov.ar
[3] Electronic address: tomas.prosen@fmf.uni-lj.si
II. WIGNER SEPARABILITY ENTROPY

Given a system described in a 2d-dimensional compact phase space $\Omega$ by the Wigner function $W(z)$ (with the normalization constraint $\int dW(z) = 1$) and an arbitrary phase space decomposition, $\Omega = \Omega_1 \oplus \Omega_2$, into two set of coordinates, $z \equiv (x, y)$, we can write the Schmidt (singular value) decomposition of the Wigner function:

$$W(x, y) = \sum_n \mu_n v_n(x) w_n(y),$$

with $n \in \mathbb{N}$, $\{v_n\}$ and $\{w_n\}$ orthonormal bases for $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively, and the Schmidt coefficients (singular values) $\mu_1 \geq \mu_2 \geq \ldots \geq 0$ satisfying $\sum_n \mu_n^2 = \int dW^2(z)$. We then define the Wigner separability entropy as

$$h[W] = -\sum_n \mu_n^2 \ln \mu_n^2,$$

where

$$\mu_n = \sqrt{\int dW^2(z)}.$$

The coefficients $\{\mu_n\}$ satisfy $\sum_n \mu_n^2 = 1$ and are the Schmidt coefficients of the singular value decomposition of $\hat{W} = W/\sqrt{\int dW^2(z)}$. That is, $W$ is normalized in $L^2(\Omega)$: $\int dW^2(z) = 1$. Note that for pure states

$$W(z) = (2\pi\hbar)^{-d/2} W(z).$$

The Wigner separability entropy $h[W]$ quantifies the logarithm of the number of terms that effectively contribute to decomposition [1] and therefore provides a measure of separability of the Wigner function with respect to the chosen phase space decomposition.

The main advantage of defining the separability entropy in phase space by means of the Wigner function is that such quantity can be directly translated to classical mechanics. The classical analog of the Wigner separability entropy is the s-entropy $h[\rho]$ defined in Ref. [2], where the classical phase space distribution $\rho_c(z)$ is used instead of the Wigner function $W(z)$.

It is interesting to establish a connection between the Wigner separability entropy and the operator space entanglement entropy [7], constructed from the Schmidt decomposition of the density operator $\hat{\rho}$ acting on the Hilbert space $\mathcal{H}$. Since $\text{Tr}(\hat{\rho}^2) \leq 1$, the density operator is a Hilbert-Schmidt operator, that is, an operator with finite Hilbert-Schmidt norm $\|\hat{\rho}\|_{\text{HS}} = \sqrt{\text{Tr}(\hat{\rho}^2)}$. Therefore, $\hat{\rho}$ can be interpreted as a vector belonging to the Hilbert space $\mathcal{B}(\mathcal{H})$ of Hilbert-Schmidt operators, with the Hilbert-Schmidt inner product $(\hat{A}, \hat{B})_{\text{HS}} = \text{Tr}(\hat{A}^\dagger \hat{B})$. Therefore, given $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the density operator has a Schmidt decomposition,

$$\hat{\rho} = \sum_n \mu_n \hat{\sigma}_n \otimes \hat{\tau}_n,$$

where $\{\hat{\sigma}_n\}$ and $\{\hat{\tau}_n\}$ are orthonormal ($\text{Tr}(\hat{\sigma}_n^\dagger \hat{\tau}_n) = \delta_{mn}$, $\text{Tr}(\hat{\sigma}_m^\dagger \hat{\tau}_n) = \delta_{mn}$) bases for $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$, respectively, and the Schmidt coefficients $\mu_1 \geq \mu_2 \geq \ldots \geq 0$ satisfying $\sum_n \mu_n^2 = \text{Tr}(\hat{\rho}^2) = \|\hat{\rho}\|_{\text{HS}}^2$. The operator space entanglement entropy [7] is then given by

$$h[\hat{\rho}] = -\sum_n \mu_n^2 \ln \mu_n^2, \quad \mu_n = \frac{\|\mu_n\|_{\text{HS}}}{\|\hat{\rho}\|_{\text{HS}}}.$$

In what follows, we prove that $h[\hat{\rho}] = h[W]$, that is, the operator space entanglement entropy equals the Wigner separability entropy. This result follows from the fact that the Weyl correspondence establishes an isomorphism between Hilbert-Schmidt operators and $L^2(\Omega)$ functions on classical phase space $\mathbb{R}^d$. Since the density operator $\hat{\rho}$ is the integral kernel of a unique linear one-to-one transformation mapping the operators $\hat{\sigma}_n \leftrightarrow \hat{\tau}_n$, $\forall n$, i.e. $\text{Tr}_1((\hat{\sigma}_n^\dagger)\hat{\rho}) = \mu_n \tau_n$, and vice versa for the inverse transformation, Tr$_2((\hat{\tau}_n^\dagger)\hat{\rho}) = \mu_n \sigma_n$, and since the Weyl transform of the density operator is the Wigner function, it follows that $\hat{\rho}$ and $W$ have the same Schmidt coefficients, and therefore

$$h[\hat{\rho}] = h[W].$$

In order to explicitly illustrate the above result, we consider a density operator $\hat{\rho}(\hat{a}_1, ..., \hat{a}_d, \hat{a}_1^\dagger, ..., \hat{a}_d^\dagger)$ written in terms of a set of bosonic creation-annihilation operators $\{\hat{a}_i, \hat{a}_j^\dagger\} = 0$, $\{\hat{a}_i^\dagger, \hat{a}_j\} = \delta_{ij}$, and define the Wigner function as

$$W(\alpha, \alpha^*) = \frac{1}{(2\pi\hbar)^d} \int d\eta d\eta^* \exp \left( \frac{\eta \cdot \alpha}{\sqrt{\hbar}} - \frac{\eta^* \cdot \alpha^*}{\sqrt{\hbar}} \right) \cdot \text{Tr}[\hat{\rho} \hat{D}(\eta)],$$

where $\eta = (\eta_1, ..., \eta_d) \in \mathbb{C}^d$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{C}^d$ are $d$-dimensional complex variables, the integration runs over the complex $\eta_i$-planes for $i = 1, ..., d$, $\hat{D}$ is the displacement operator

$$\hat{D}(\eta) = \hat{D}(\eta_1, ..., \eta_d) = \exp \left[ \sum_{i=1}^d \left( \eta_i \hat{a}_i^\dagger - \eta_i^* \hat{a}_i \right) \right],$$

and $|\alpha\rangle$ are the coherent states

$$|\alpha\rangle = |\alpha_1 \alpha_2 ... \alpha_d\rangle = \hat{D} \left( \frac{\alpha}{\sqrt{\hbar}} \right) |0\rangle,$$

with $|\alpha_i\rangle$ being an eigenstate of the annihilation operator $\hat{a}_i$, i.e., $\hat{a}_i |\alpha_i\rangle = \frac{\alpha_i}{\sqrt{\hbar}} |\alpha_i\rangle$, and $|0\rangle$ being the vacuum state $\hat{a}_j |0\rangle = 0$. Using the singular value decomposition

$$\hat{\rho} = \sum_n \mu_n \hat{\sigma}_n \otimes \hat{\tau}_n \otimes \hat{\tau}_n^\dagger (\hat{a}_d^\dagger/\sqrt{2}, ..., \hat{a}_d, \hat{a}_d^\dagger, ..., \hat{a}_d^\dagger)$$

we obtain

$$\text{Tr}[\hat{\rho} \hat{D}(\eta)] = \sum_n \mu_n \hat{\sigma}_n (\eta_1, ..., \eta_d, \eta_1^*, ..., \eta_d^*) \hat{\tau}_n \otimes \hat{\tau}_n^\dagger (\eta_d/\sqrt{2}, ..., \eta_d, \eta_d^*, ..., \eta_d^*),$$

where

$$\hat{\sigma}_n = (\hat{a}_d^\dagger/\sqrt{2}, ..., \hat{a}_d, \hat{a}_d^\dagger, ..., \hat{a}_d^\dagger), \quad \hat{\tau}_n = (\eta_d/\sqrt{2}, ..., \eta_d, \eta_d^*, ..., \eta_d^*).$$
where
\[ \tilde{\sigma}_n = \text{Tr}_{1,\ldots,d/2} [\tilde{\sigma}_n \tilde{D}(\eta_1, \ldots, \eta_{d/2})], \]
\[ \tilde{\tau}_n = \text{Tr}_{d/2+1,\ldots,d} [\tilde{\tau}_n \tilde{D}(\eta_{d/2+1}, \ldots, \eta_d)]. \]
Finally, we derive
\[ W(\alpha, \alpha^*) = \sum_n \mu_n v_n (\alpha_1, \ldots, \alpha_{d/2}, \alpha^*_1, \ldots, \alpha^*_{d/2}) \]
\[ \times w_n (\alpha_{d/2+1}, \ldots, \alpha_d, \alpha^*_{d/2+1}, \ldots, \alpha^*_d), \]
where
\[ v_n = \frac{1}{\pi^d h^{d/2}} \int d\eta_1 d\eta_1^* \cdots d\eta_{d/2} d\eta_{d/2}^* \]
\[ \times \exp \left[ \sum_{i=1}^{d/2} \left( \frac{\eta_i^* \alpha_i}{\sqrt{\hbar}} - \frac{\eta_i \alpha_i^*}{\sqrt{\hbar}} \right) \right] \tilde{\sigma}_n, \]
\[ w_n = \frac{1}{\pi^d h^{d/2}} \int d\eta_{d/2+1} d\eta_{d/2+1}^* \cdots d\eta_d d\eta_d^* \]
\[ \times \exp \left[ \sum_{i=d/2+1}^d \left( \frac{\eta_i^* \alpha_i}{\sqrt{\hbar}} - \frac{\eta_i \alpha_i^*}{\sqrt{\hbar}} \right) \right] \tilde{\tau}_n. \]

Eq. 18 is precisely the singular value decomposition of the Wigner function (the orthogonality of the Schmidt vectors \( \{ \mu_n \} \) can be checked straightforwardly) and has the same Schmidt coefficients \( \{ \mu_n \} \) as the singular value decomposition \( \{ \lambda_n \} \) of the density operator, so we conclude that \( h[W] = h[\rho] \) \( \square \).

When the density operator \( \hat{\rho} \) describes a pure state, \( \hat{\rho} = |\psi\rangle \langle \psi| \), there exists a simple relation between the Wigner separability entropy and the entanglement content of the state \( |\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). The Schmidt decomposition of \( |\psi\rangle \) reads \( |\psi\rangle = \sum_j \lambda_j |\phi_j\rangle \otimes |\xi_j\rangle \),

\[ |\psi\rangle = \sum_j \lambda_j |\phi_j\rangle \otimes |\xi_j\rangle, \]

with \( \{ \phi_j \} \) \( \{ |\xi_j\rangle \} \) orthonormal basis for \( \mathcal{H}_1 \) \( \mathcal{H}_2 \), \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \), and \( \sum_j \lambda_j^2 = 1 \). On the other hand, we can also write the Schmidt decomposition of the operator \( \hat{\rho} = |\psi\rangle \langle \psi| \):

\[ \hat{\rho} = \sum_{j,k} \lambda_j \lambda_k |\phi_j\rangle \langle \phi_k| \otimes |\xi_j\rangle \langle \xi_k|. \]

The comparison between 19 and 15 implies \( \{ \mu_n \}_{n \in \mathbb{N}} = \{ \lambda_j \lambda_k \}_{j,k \in \mathbb{N}} \). Therefore,

\[ h[W] = -\sum_n \mu_n^2 \ln \mu_n^2 = -\sum_{j,k} \lambda_j^2 \lambda_k^2 \ln(\lambda_j^2 \lambda_k^2), \]

\[ = -2 \sum_j \lambda_j^2 \ln \lambda_j^2 = -2S(\hat{\rho}_1) = -2S(\hat{\rho}_2), \]

where \( \hat{\rho}_1 = \text{Tr}_1(\hat{\rho}) \) and \( \hat{\rho}_2 = \text{Tr}_2(\hat{\rho}) \) are the reduced density operators for subsystems 1 and 2 and \( S \) is the von Neumann entropy. Since for a pure state \( |\psi\rangle \) von Neumann entropy of the reduced density matrix quantifies the entanglement \( E \) of \( |\psi\rangle \) \( [10, 11] \),

\[ E(|\psi\rangle) = S(\hat{\rho}_1) = S(\hat{\rho}_2), \]

we can conclude that the Wigner separability entropy is twice the entanglement entropy \( E(|\psi\rangle) \): \( h[W] = 2E(|\psi\rangle). \)

For pure states \( S(\hat{\rho}) = 0 \), and therefore the Wigner separability entropy is equal to the quantum mutual information

\[ I(1:2) = S(\hat{\rho}_1) + S(\hat{\rho}_2) - S(\hat{\rho}). \]

Quantum mutual information measures total correlations, both of classical and quantum nature, between subsystems 1 and 2 and for pure states classical correlations \( C(|\psi\rangle) \) are equal to quantum correlations, measured by \( E(|\psi\rangle) \) \( [12] \).

### III. WIGNER SEPARABILITY ENTROPY FOR CHAOTIC MAPS

In order to illustrate the quantum to classical correspondence for the separability entropy, we study the evolution in time of \( h[W] \) and of its classical counterpart \( h[\rho_c] \) for three quantum maps: (i) the quantum baker’s map, (ii) the perturbed quantum cat map, and (iii) a model consisting of two coupled perturbed cat maps. The first model (i) exhibits an atypical failure of the quantum to classical correspondence due to discontinuity in the mapping producing drastic quantum diffusion effects. The second model (ii) is typical in that the Wigner separability entropy \( h[W] \) follows the classical separability entropy \( h[\rho_c] \) up to the Ehrenfest time scale \( t_E \), until Wigner function evolution (whose time derivative is given by the Moyal bracket) is well approximated by the evolution of the classical Liouville density (whose time derivative is given by the Poisson bracket, the first order in \( \hbar \) expansion of the Moyal bracket), i.e.

\[ W(t = 0) = \rho_c(t = 0) \Rightarrow h[W(t)] = h[\rho_c(t)] \text{ for } t < t_E. \]

Note that for chaotic dynamics with Lyapunov exponent \( \lambda \), the Ehrenfest time scales as \( t_E \sim (\lambda^{-1}) \). The third model (iii) illustrates for pure states the connection between the separability entropy and the entanglement entropy.

We study the quantum versions of classical chaotic maps on the 2-torus \( [0, 1] \times [0, 1] \). Quantization imposes the Planck constant to coincide with the inverse of the dimension \( N \) of the Hilbert space, i.e., \( \hbar = 1/(2\pi N) \). The baker map has been studied in detail in many works (see for example \( [14, 15] \)). The classical baker map is defined
by

\[
(q_{t+1}, p_{t+1}) = \begin{cases} 
(2q_t, \frac{1}{2}p_t), & \text{if } 0 \leq q_t \leq \frac{1}{2}, \\
(2q_t - 1, \frac{1}{2}p_t + \frac{1}{2}), & \text{if } \frac{1}{2} < q_t < 1,
\end{cases}
\]

(25)

where the discrete time \( t \in \mathbb{Z} \) measures the number of map iterations. The quantum analogue of the classical baker map is a unitary operator acting on an \( N \)-dimensional Hilbert space (assuming \( N \) to be even). In the position-\((q)\)-representation its matrix reads:

\[
B = G_N^{-1} \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix},
\]

(26)

where \( G_N \) is the \( N \)-dimensional antiperiodic Fourier matrix:

\[
(G_N)_{lj} = \frac{1}{\sqrt{N}} e^{-2\pi i(j(l+1/2)/(l+1/2))/N},
\]

(27)

with \( 0 \leq l, j \leq N - 1 \).

The classical perturbed cat map reads \[16\]

\[
\begin{pmatrix} q_{t+1} \\ p_{t+1} \end{pmatrix} = \mathcal{M} \begin{pmatrix} q_t \\ p_t + \epsilon(q_t) \end{pmatrix},
\]

(28)

where \( q \) and \( p \) are taken mod (1), and where \( \epsilon(q_t) = -[K/(2\pi)]\sin(2\pi q_t) \). Throughout this paper we have used

\[
\mathcal{M} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},
\]

(29)

satisfying the symplectic condition \( \det \mathcal{M} = 1 \), and \( K = 0.5 \). The perturbed quantum cat map in the \( q \)-representation is given by the \( N \times N \) unitary matrix \( \hat{M} \) whose elements read

\[
M_{lj} = A \exp \left[ \frac{\pi i}{N(M_{11}l^2 - 2lj + M_{22}j^2) + F} \right],
\]

(30)

where

\[
A = [1/(iN M_{12})]^{1/2},
\]

\[
F = [iKN/(2\pi)] \cos (2\pi l/N).
\]

Finally, we have considered a 2D model (in 4D phase space \((q^1, q^2, p^1, p^2) \in [0, 1)^4\) consisting of two coupled perturbed cat maps. In order to do this in a symplectic way we have used a symmetric coupling in the position coordinates \[17\]. In this case the classical map is given by (superscripts indicate which ‘cat’ the coordinates refer to)

\[
\begin{pmatrix} q_{t+1}^1 \\ p_{t+1}^1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} q_t^1 \\ p_t^1 + \epsilon(q_t^1) + \epsilon'(q_t^1, q_t^2) \end{pmatrix},
\]

and,

\[
\begin{pmatrix} q_{t+1}^2 \\ p_{t+1}^2 \end{pmatrix} = \mathcal{M} \begin{pmatrix} q_t^2 \\ p_t^2 + \epsilon(q_t^2) + \epsilon'(q_t^1, q_t^2) \end{pmatrix},
\]

(31)

where \( q^i \) and \( p^i \) \((i = 1, 2)\) are again taken mod (1), and where we have used the same perturbation \( \epsilon(q) \) as before (for both maps). Here, \( \epsilon'(q_1^1, q_2^2) = -[K/(2\pi)]\sin(2\pi q_1^1 + 2\pi q_2^2) \) is the coupling term.

The quantized version of the 2D perturbed cat map is obtained by multiplying a separable (tensor) product of quantized 1D perturbed cat maps with a simple exponentiated coupling matrix (which is diagonal in the \( q \)-representation). More explicitly, \( N^2 \times N^2 \) unitary matrix \( \hat{M}^{2D} \) is given by

\[
\hat{M}^{2D}_{(i_1, j_1, i_2, j_2)} = M_{i_1, j_1} M_{i_2, j_2} C_{j_1, j_2}
\]

(32)

where

\[
C_{j_1, j_2} = \exp\{[iK\pi/(2\pi)] \cos[(2\pi/N)(j_1 + j_2)]\}
\]

(33)

and \( j_1, j_2, l_1, l_2 \in \{0, \ldots, N - 1\} \). In the following calculations we take \( K = 1.0 \).

We have studied the classical and quantum separability entropies. In all cases we have used the initial states given in terms of a Gaussian phase space distribution with dispersion equal to \( \sqrt{\pi} \), and its quantum analogue, a coherent state on the torus. Both distributions are centered in the middle of the phase space, i.e., at \((q, p) = (0.5, 0.5)\). We have evolved 10 time steps (iterations) for all maps.

In Fig. 1 we show the separability entropy [8] as a function of time (in units of map steps) for the baker map [25,26]. We have used the same discretization number \( N = 2^9 \) for both, the classical and the quantum simulations and removed the effects of the torus periodicity on the Wigner distributions [18]. We have decomposed the 2-torus in coordinates \( q \) and \( p \). While the classical separability entropy saturates to a value of order \( \sim 1 \) bit, after an initial short transient growth, due to the exact solvability of its Liouville evolution [19], and drops back towards zero for \( t \sim \log N \) due to coarse-graining of classical Liouvillean evolution, the Wigner separability entropy grows with time due to quantum interference and diffraction patterns produced by discontinuity of the map. The baker’s map is therefore non generic, in that it exhibits very different classical and quantum results even within the Ehrenfest time range due to diffraction effects.

In order to investigate a case with generic behavior we have taken the smooth (continuous) perturbed cat map [29,30]. In Fig. 2 we show the same quantities as in Fig. 1. Now there is quantum to classical correspondence, up to the Ehrenfest time \( t_E \propto \log N \). Note that, similarly to the case of baker map, the classical result also exhibits discretization time scale after which the classical distribution experiences a complexity reduction which is a numerical artifact due to finiteness of \( N \). Indeed, due to coarse graining the Liouville evolution becomes homogeneous in phase space, and this implies asymptotically vanishing separability entropy. Finally, and for comparison purposes only, we show the quantum results without removing the effects of the ghost images that appear on
FIG. 1: (Color online) Classical and quantum separability entropies behavior for the baker map. The (red) gray line with crosses corresponds to the classical result and the black line with squares to the quantum one. The quantum curve has been calculated using the Wigner function without ghost images [18]. In both cases $N = 2^9$.

account of the torus periodicity (this is shown just for the lower value of $N$). Wigner separability entropy without ghost images removed exhibits some deviations for shorter times (then, the fact that we have ghost peaks in Wigner distribution is more important than at later times), the removal of ghost images is therefore needed in order to obtain quantum to classical correspondence.

Finally, we have tested the Wigner separability entropy and its relation to pure-state entanglement for two coupled perturbed cat maps [31,32] (in this case we have evaluated the Wigner distributions according to Ref. [20]). In practice we have decomposed the four-dimensional phase space in the coordinates $(q^1, p^1)$ and $(q^2, p^2)$ corresponding to each of the two ‘cats’, respectively. We have also compared the results with the classical separability entropy [6]. It is worth mentioning that in order to obtain the classical complexity measure we have used the same phase space decomposition for the four dimensional Liouville distribution that we have previously applied to the four-dimensional Wigner function. All these results are shown in Fig. 3. The agreement between the quantum measures is accurate within the machine precision, i.e. $h[W] = 2E(\langle |\psi\rangle)$ (in the figure we have rescaled von Neumann entropy by a factor of 2), reflecting the match between singular value decompositions for the density matrix and its Wigner distribution. In this case we did not remove ghost images in the Wigner distributions since all the coherences measured by the separability entropy in these coordinate pairs are relevant to measure entanglement. Regarding the classical separability entropy we find a reasonable agreement up to time $\sim \log N$ where due to coarse graining $h[\rho_c]$ starts to drop. It is important to underline that the classical saturation is entirely due to the phase space coarse graining chosen for the numerical simulations, while the quantum one is an unavoidable phenomenon fixed by the finite size of $\hbar$. Note that, as recently pointed out [21], the initial growth of entanglement can be reproduced in the semiclassical regime by purely classical computations.

FIG. 2: (Color online) Same as in Fig. 1 but now for the perturbed cat map. The black lines with squares correspond to quantum results and the gray (red) ones with crosses to classical ones. The light gray (green) dotted line corresponds to the quantum values using the Wigner distribution with ghost images which exhibits small deviations at shorter times. The lower (three) curves correspond to $N = 2^{11}$ while the upper (two) curves correspond to $N = 2^{13}$.

FIG. 3: (Color online) Classical and quantum separability entropies behavior for two coupled cat maps (the gray (red) line with crosses and the black line with squares, respectively). Two times von Neumann entropy is also shown by means of a light gray (green) dotted line with triangles. We have used discretization number $N = 2^9$ in both the classical and quantum cases, for each map.
IV. CONCLUSIONS

In this paper we have proposed a new measure of complexity of quantum states, the Wigner separability entropy. This quantity turns out to be equal to the operator space entanglement entropy, it quantifies the minimal amount of computational resources required to simulate the quantum dynamical evolution of a system by means of time-dependent density-matrix renormalization group [5]. Moreover, due to its relation with the entropy of entanglement for pure states and to the existence of the analogous s-entropy of Liouville densities in classical dynamics, the separability entropy emerges as an extremely broad complexity quantifier in both the classical and quantum realms.

With regard to a previously proposed phase-space quantum complexity indicator, that is, the number of Fourier harmonics of the Wigner function [3, 4], the Wigner separability entropy has the advantage of being basis-independent. More importantly, numerical indications [5] suggest that the Wigner separability entropy should be able to distinguish between quantum chaotic and quantum regular motion also for many-body systems. Finally, we point out that the Wigner separability entropy is well defined also for mixed states and could therefore be used to quantify the complexity of decoherent quantum dynamics.

Acknowledgments

We thank Italo Guarneri for useful discussions. Partial financial support from Conicet, Argentina (GC), and the grants P1-0044 and J1-2208 of Slovenian Research Agency (TP) is gratefully acknowledged.

[1] J. Ford, Phys. Today 36, 40 (1983); V.M. Alekseev and M.V. Jacobson, Phys. Rep. 75, 287 (1981).
[2] R. Alicki and M. Fannes, Quantum dynamical systems (Oxford University Press, Oxford, 2001); T. Prosen, J. Phys. A 40, 7881 (2007), and references therein.
[3] V.V. Sokolov, O.V. Zhirov, G. Benenti, and G. Casati, Phys. Rev. E 78, 046212 (2008); G. Benenti and G. Casati, Phys. Rep. E 79, 025201(R) (2009).
[4] V. Balachandran, G. Benenti, G. Casati, and J. Gong, Phys. Rev. E 82, 046216 (2010).
[5] T. Prosen and M. Žnidarič Phys. Rev. E 75, 015202(R) (2007).
[6] T. Prosen, Phys. Rev. E 83, 031124 (2011).
[7] T. Prosen and I. Pizorn, Phys. Rev. A 76, 032316 (2007).
[8] J. T. C. Pool, J. Math. Phys. 7, 66 (1966).
[9] G. B. Folland, Harmonic analysis in phase space (Princeton University Press, Princeton, New Jersey, 1989).
[10] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge University Press, Cambridge, 2000).
[11] G. Casati, I. Guarneri, and J. Reslen, preprint arXiv:1109.0907v2 [quant-ph].