On astigmatic solutions of the wave and the Klein-Gordon-Fock equations with exponential fall–off

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Abstract

Highly localized explicit solutions to multidimensional wave and Klein–Gordon–Fock equations are presented. Their Fourier transform is also found explicitly. Solutions depend on a set of parameters, and demonstrate astigmatic properties. Asymptotic analysis for large and moderate time shows that constructed solutions have Gaussian localisation near a point moving with the group speed.

1 Introduction

Seeking localized solutions to (non-) linear differential equations has a very long history, starting probably with the famous observation by John Scott Russell of a solitary wave in the Union Canal [1]. Since then, many a research were made. An interest to the theoretical study of localized solutions of linear equations was renewed after the discovery of lasers and further progress in technologies of emitting ultra-short pulses. Nowadays, there are also numerous potential applications of such solutions, for example, for the localized low-loss energy transmission, communication, medical imaging or remote sensing. The state of the art in this field, along with its history, is presented in recent books [2, 3].

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In free space (or non-dispersive media) the problems of wave propagation are usually described using the wave equation (WE)

\[ \square \phi \equiv \partial^2_t \phi - \Delta \phi = 0, \]  

(1)

where \( \partial_t(\cdot) \equiv \frac{\partial}{\partial t}(\cdot) \), etc., and

\[ \Delta \phi \equiv \partial^2_x \phi + \partial^2_y \phi + \partial^2_z \phi \]

is three dimensional Laplace operator. Here and throughout the paper we put the constant of speed of light equal to unity, \( c = 1 \).

On the other hand, the waves in dispersive medium are often described by the Klein–Gordon–Fock equation (KGFE)

\[ (\square + m^2)u = 0, \]  

(2)

where \( m \) is a mass parameter having different physical meaning in different systems. The KGFE is important in studying of electromagnetic waves in the isotropic cold collisionless plasma [4], waves of charge density in Drude metals, or the high-frequency acoustic waves in the gas of charged particles treated in the hydrodynamic approach [5].

A relatively recent spike in activity in investigating localized solutions was promoted by the paper by Brittingham [6], who outlined a new one–parametric family of beam-like localized explicit solutions to the WE which he named the focus wave mode (FWM, which we also call the Gaussian beam). This solution is localized in the Gaussian way along a straight line in space and has infinite energy. This solution cannot be obtained by the separation of variables in coordinates \( \mathbf{r} = (x, y, z) \) and time. However introduction of new variables

\[ \alpha = z - t, \quad \beta = z + t, \]  

(3)

where \( z \) is coordinate along the propagation axis, enables one to do this. The KGF analogue of the Brittingham FWM for the WE was given by Ziolkowski in [7] by the separation of variables (3). In [8] he also suggested to seek other solutions as weighted superpositions over a free parameter of FWM and in doing so he obtained a solution of finite energy and a power–law localization. Another way of construction of highly localized solutions, the so called bidirectional representation, is based on the Fourier integral in new variables (3). Taking the Fourier weight in the proper way Besieres, Shaarawi and Ziolkowski [9], Donelly and Ziolkowski [10] found new solutions with finite energy of WE and KGFE respectively.

All highly localized solutions both for the WE and for KGFE spread propagating. There were found however the so called undispersive solutions, which propagate without spreading, but have a power-law localization from their moving amplitude maximum. For the WE these are the Bessel beams, solutions found by Durnin [11] by separation of variables in initial space–time coordinates, and the X–waves found by weighted superposition of Bessel beams [12]. For the KGFE solutions the same property is possessed by MacKinnon’s solution [13] which can be found by combination of the separation of variables and the Lorentz transformations.
The particle-like exact solution to the WE which decreases exponentially in all directions away from a point moving along a straight line was first presented by Kiselev and Perel in [14]. Perel and Sidorenko [15] considered this solution from the point of view of wavelet analysis, investigated numerically the uncertainty relation for this solution and found explicitly its Fourier transform. In [15] the solution was also analyzed from the point of view of complex sources. It was shown that it can be generated by a pulse source moving with the speed of wave propagation. An integral representation of this solution in terms of Gaussian beams due to ideas of [8] was given by Perel and Fialkovsky in the paper [16] which was however mainly devoted to the KGFE. In [16] it was also suggested a class of explicit exponentially localized packet-like solutions for KGFE and investigated some of their properties. One of solutions from the class obtained in [16] coincides with one from [17]. Kiselev, Plachenov and Chamorro-Posada [18] created astigmatic beam-like and packet-like solutions of the WE.

Finally, we shall also mention the work by Overfelt [19] who got a class of solutions which generalizes Gaussian beams of [6] and the Bessel beams of [11] and which have better localization near the propagation axis than Gaussian beams. In Besieres, Shaarawi and Ziolkowksi [17] there was suggested a new method of design of solutions of three dimensional KGFE reducing them to a solution of one dimensional KGFE equation with new ‘time’ and ‘coordinate’ containing an arbitrary function. On this way they found the counterparts of the Gauss–Bessel pulses [19] and some other solutions. Interested reader can find more detailed references and discussion of various constructions of the localized solutions both for the WE and the KGFE in books [2, 3] and reviews [21, 20].

The present work is a continuation of works [16] and [18]. We present nonseparable solutions with exponential localization, both beam-like and particle-like ones. The former solutions have Gaussian localization near a straight line and traveling wave-fronts. The latter ones in addition to localization near a line are localized in a Gaussian manner near a point moving along this line.

The structure of the paper is following. In the next section we revisit known results on exact exponentially localized solutions for the wave equation. By doing so, we also construct the astigmatic generalizations of the known beam–like and particle–like solutions for the WE. In Section 3 we apply the developed methods to the construction of the multidimensional astigmatic solutions to the Klein–Gordon–Fock equation. We proceed by asymptotic investigation of the KGF solutions in Section 4 where we consider both small/moderate and large times regimes and discuss the choice of parameters which enable us to govern the localization properties of solutions. We conclude our research by obtaining the Fourier transformation of all constructed families of solutions in Section 5, and by providing some final remarks and numerical studies of the solutions in Section 6. In Appendix A we present the asymptotic investigation of obtained solutions in the Fourier domain. All our results are valid for the space–time with any number of spatial dimensions.
2 Wave Equation Revisited

In this section we construct a generalizations of the known localized exact solutions to the WE

\[ \partial_t^2 \phi - \Delta_n \phi = 0, \]

where \( \Delta_n \) stands for \( n \) dimensional Laplacian operator. By doing so we also revive the necessary techniques to be used also for the Klein–Gordon–Fock equation. In constructing particle–like solutions of the wave equation, we follow the idea by Ziolkowski [8] and seek such solutions in the form of a superposition of Gaussian beams.

The exponentially localized solutions we are focusing on may be considered as “relatively undistorted progressive waves” by Courant and Hilbert [22], or “nondispersive waves” by Hillion [23, 24]. They are of the form of a ray series which comprises one term only

\[ \phi = g(r, t)f(\theta), \]

where \( \theta \) is a solution of the eikonal equation for WE

\[ (\partial_t \theta)^2 - (\nabla \theta)^2 = 0, \]

function \( g(r, t) \) depends on the form of \( \theta \) and satisfies two equations

\[ (\partial_t \theta)(\partial_t g_0) - \langle \nabla \theta, \nabla g_0 \rangle + g_0 \Box \theta = 0, \]

\[ (\partial_t g)^2 - (\nabla g)^2 = 0, \]

and \( f(\theta) \) is an arbitrary function. Here \( \langle ab \rangle = \sum_{i=1}^{n} a_i b_i \).

The focus wave modes by Brittingham [6] belonging to the so called Bateman–Hillion class are based on the following eikonal [23, 25]

\[ \theta = \alpha + \frac{r_1^2}{\beta - i \varepsilon}, \quad r = (r_\perp, z), \]

where \( z \equiv x_n, \alpha \) and \( \beta \) are defined in (3), and \( \varepsilon \) is an arbitrary positive constant. The function \( g(r, t) \) in this case is of the form \( g(r, t) = (\beta - i \varepsilon)^{-d/2}, \) \( d = n - 1. \) Choosing the arbitrary function \( f(\theta) \) as a pure exponent [6, 26] \( f(\theta) = \exp\{i\eta \theta\} \), where \( \eta \) is a positive parameter we obtain FWM [6], or Gaussian beam, which reads

\[ \phi \equiv \frac{e^{i\eta \theta}}{(\beta - i \varepsilon)^{d/2}} = \frac{1}{(\beta - i \varepsilon)^{d/2}} \exp\left\{ i \left( \eta \alpha + \frac{\beta}{\varepsilon} \frac{r_1^2}{\Delta_\perp^2} \right) - \frac{r_1^2}{\Delta^2_\perp} \right\}. \]

We separated imaginary and real parts in the exponent to stress exponential localization of the solution near \( z \)-axis, with \( \Delta_\perp = \sqrt{\frac{\beta^2 + \varepsilon^2}{\eta \varepsilon}} \) we denote the width of the beam in all transverse directions. Such axisymmetric solutions are called the stigmatic ones.
One of the obvious but far reaching generalizations of the solution (9) is achieved by formal linear transformation of the transverse coordinates in the eikonal function $\theta$ (8) (see, e.g., [18])

$$\theta = \alpha + (r_\perp, \Gamma(\beta)r_\perp),$$

where $\Gamma$ is a $d \times d$ complex matrix depending on $\beta$, whose properties are to be discussed later. Astigmatic generalization of (9) can be obtained by putting $g(r, t) \equiv g(z + t) = \sqrt{\det \Gamma(\beta)}$ and is given by

$$\phi_b(\alpha, \beta, r_\perp) = c_b \sqrt{\det \Gamma(\beta)} \exp \{i \eta \theta\},$$

where $c_b$ and $\eta > 0$ are arbitrary constants. The WE (4) is indeed resolved by (11) if $\Gamma(\beta)$ satisfies the Bernoulli equation $\partial_\beta \Gamma = -\Gamma^2$ and therefore reads

$$\Gamma(\beta) = \Gamma_0(E + \beta \Gamma_0)^{-1},$$

where $E$ is the unity $d \times d$ matrix and $\Gamma_0 = \Gamma(0)$ — constant non-degenerate one. Function $\phi_b$ has no singularities if $\Gamma_0$ does not have nonzero real eigenvalues, and it has the Gaussian localization around the axis $\phi$ if the matrix $\text{Im} \Gamma(\beta)$ is positively defined. Both conditions are fulfilled if $\Gamma_0$ has a positive definite imaginary part. Indeed, the regularity condition is fulfilled since $\text{Im}(r_\perp, \Gamma_0r_\perp) > 0$ for all nonzero vectors $r_\perp$ (real or complex) including eigenvectors of $\Gamma_0$ and hence all its eigenvalues have positive imaginary parts. To show the localization of solution for all values of $\beta$ we note that once $\Gamma_0^{-1}$ has negative definite imaginary part, $\text{Im} \Gamma^{-1}(\beta) \equiv \text{Im} \Gamma_0^{-1}$ is also negative definite matrix. Then $\text{Im} \Gamma(\beta)$ stays positively defined for all $\beta$. We remind here that positive definiteness of $\text{Im} \Gamma$ and negative definiteness of $\text{Im} \Gamma^{-1}$ are equivalent: $\text{Im}(r_\perp, \Gamma r_\perp) = \text{Im}(\Gamma^{-1} p, p) = -\text{Im}(p, \Gamma^{-1} p)$, where $p = \Gamma r_\perp$.

The axisymmetrical stigmatic beam (9) may be obtained from (11) if $\Gamma(\beta) = E(\beta - z_0 - i\varepsilon)^{-1}$ ($z_0$ and $b > 0$ are real constants). The width of the Gaussian curve in the transverse direction depends in this case on the propagation direction $z$ and time. For fixed time the width has a minimum which is called a waist.

The simplest non-axisymmetrical solutions, the **aligned simple astigmatic** ones, correspond to a diagonal $\Gamma$-matrix: $\Gamma_{jk} = \delta_{jk}(\beta - z_j - i\varepsilon_j)^{-1}$ with $z_j$ and $\varepsilon_j > 0$ being real constants, $j, k = 1, 2, \ldots n$. When $\Gamma(\beta)$ can be diagonalized by a orthogonal rotation of axes the solutions are called **rotated simple astigmatic**. In two–dimensional case it is given by

$$\Gamma = U \Lambda U^{-1}, \quad U = \begin{pmatrix} \cos \Phi & -\sin \Phi \\ \sin \Phi & \cos \Phi \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ \frac{1}{\beta - z_1 - i\varepsilon_1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{\beta - z_2 - i\varepsilon_2} & 0 \end{pmatrix}.$$
The case of general astigmatism is characterized by the dependence on time and $z$ of the direction of the main axes of the localization ellipsoid. It happens when the eigenvectors of $\Gamma$ (which are constant due to (12)) are complex and do not coincide with those of $\text{Re} \Gamma$ and $\text{Im} \Gamma$. The real eigenvectors of the latter matrices do depend on $\beta$. Then the main axes of the ellipsoids (or hyperboloids) of constant phase and of modulus levels rotate with time and/or coordinate $z$. The absolute value of the total angle of rotation is equal to $\pi$ for both of them (see [27] or [28]). There is no definition of the waist in the case of general astigmatism.

Arnaud and Kogelnik realised [27] that in two dimensions a general astigmatic solution can be obtained by assigning a complex value to $\Phi$ in (13). Indeed, (11) is still a solution of the wave equation (4) in this case but the eigenvectors of $\text{Re} \Gamma$ and $\text{Im} \Gamma$ will be different. The solution with such $\Gamma$ is localized in the neighborhood of the $z$–axis if [27] $\varepsilon_1, 2 > 0$ and $\cosh^2 (2 \text{Im } \Phi) ((z_2 - z_1)^2 + (\varepsilon_2 - \varepsilon_1)^2) < ((z_2 - z_1)^2 + (\varepsilon_2 + \varepsilon_1)^2)$. We note, that the smaller is $\text{Im } \Phi$, the closer is the solution to a simple astigmatic one.

All the above discussion of the astigmatic properties is equally applicable to the particle–like solutions of the WE of the next section, and to all solutions of the KGFE considered in Section 3.

2.1 Particle–like solutions for the wave equation

We seek the particle-like solutions of Eq. (4) in the form of a superposition of Gaussian beams $\phi_b$ obtained in (11)

$$\phi^{(\nu)}_p(\alpha, \beta, r_\perp) = \int_0^\infty d\eta F^{(\nu)}(\eta) \phi_b(\alpha, \beta, r_\perp, \eta),$$

where $F^{(\nu)}(\eta)$ is a particular function depending on the parameter $\nu$. We put

$$F^{(\nu)}(\eta) = \eta^{-\nu-1}e^{-\eta(\eta + \omega^2/\eta)},$$

where $\nu$, $\omega$, and $\gamma$ are arbitrary constants, $\omega > 0$, $\gamma > 0$. Such particular choice of the spectral function $F$ is motivated by consideration of the Fourier transformation of one of the previously known solution to the WE, given below in (18). For the first time this spectral weight appears in [9] in regard of developing the so called ‘bidirectional’ representation for the solutions of the WE.

It can easily be shown that (14) is reduced to an integral representation of the MacDonald function $K_\nu$ [29] (also called the modified Bessel function of the second kind)

$$\int_0^\infty x^{l-1}\exp\left\{-\frac{a}{x} - bx\right\} \, dx = 2 \left(\frac{a}{b}\right)^{1/2} K_l\left(2\sqrt{ab}\right),$$

which is valid if $\text{Re } a > 0$, $\text{Re } b > 0$. We put $l = -\nu$, $a = \omega^2 \gamma$, $b = \gamma - i\theta$ and obtain from (14)

$$\phi^{(\nu)}_p(x, y, z, t) = c_p \sqrt{\det \Gamma(\beta)} s^\nu K_\nu(s), \quad s = 2\omega \gamma \left(1 - i\theta/\gamma\right)^{1/2},$$
where \( \theta \) is given by (10), and \( c_p \) can be expressed through numerical parameters of (11) as \( c_p = 2(2x^2\gamma)^{-\nu}c_b \), but can also be treated as an arbitrary numerical constant. It is worth noting that \( s \) satisfies the eikonal equation (6) for wave equation (4).

For particular case of \( d = 2 \) and diagonal \( \mathbf{\Gamma}_0 = i\varepsilon^{-1}\text{diag}(1, 1) \), the formula (17) yields a family of axisymmetric solutions first presented in [16]. For \( \nu = 1/2 \) it gives

\[
\phi_p^{(1/2)}(x, y, z, t) = \exp\left\{-2\varpi\gamma\sqrt{1 - i\theta/\gamma}\right\}(\beta - i\varepsilon)^{-1},
\]

which was first obtained in [14]. It corresponds to a following choice of \( f(\theta) \) in (5): \( f(\theta) = e^{-p\sqrt{1 - i\theta/\gamma}} \), were \( p \) is a positive real constant.

All of the solutions (17) have a Gaussian localization in vicinity of a point running with wave velocity along a straight line provided the free parameters satisfy for certain relations, similar to those obtained in [16]. However their detailed investigation is out of scope of the current paper.

Finally we note, that all the solutions of the form of (17) can be interpreted as part of ‘arbitrary waveform solutions’ (5) with \( f(\theta) = s^\nu K_\nu(s) \) and \( g = \sqrt{\det \mathbf{\Gamma}(\beta)} \). Apart from being more constructive the integral representation (14) used here with the weight function (15) will be important in construction of the solution for the KGFE, where no waveform freedom is available.

### 3 Klein–Gordon-Fock equation

Operating only in the space–time domain we construct now both beam–like and particle–like solutions of the multidimensional Klein–Gordon–Fock equation

\[
(\partial_t^2 - \Delta_n + m^2)u = 0
\]

where \( \Delta_n \) is the Laplace operator in \( n \) spatial dimensions.

#### 3.1 Gaussian beams

To elaborate a particle–like solution \( u_p \) of the Klein–Gordon–Fock equation as a superposition of the beam–like solutions \( u_b \), we shall first construct the latter ones. In doing so, we shall consider the solution (11) of the WE in spacial dimension increased by one, \( r_\perp = (x, y, \ldots) \rightarrow r_\perp^\xi = (\zeta, x, y, \ldots) \), and calculate its Fourier transform with respect to \( \zeta \)

\[
u_b(t, z, r_\perp) \equiv \int_{-\infty}^{\infty} d\zeta \phi_b(t, z, r_\perp, \zeta) e^{-im\zeta},
\]

where \( \phi_b \) is given by (11) in \( d + 1 \) dimensions. We shall further assume that the enlarged \( (d + 1) \times (d + 1) \) matrix \( \tilde{\mathbf{\Gamma}} \) is such that

\[
(r_\perp^\xi, \tilde{\mathbf{\Gamma}}(\beta)r_\perp^\xi) = (r_\perp, \mathbf{\Gamma}(\beta)r_\perp) + \zeta^2/(\beta - i\varepsilon_m)
\]
with a positive constant \( \varepsilon_m \), and \( \Gamma \) is a \( d \times d \)–matrix as before. Then
\[
 u_b = c_b \sqrt{\det \Gamma(\beta)} e^{im\theta} (\beta - i\varepsilon_m)^{-1/2} \int_{-\infty}^{\infty} d\zeta e^{i\zeta^2/(\beta - i\varepsilon_m) - i\zeta}.
\]

Taking the integral and introducing a new numerical constant, \( C_b = c_b e^{-\varepsilon_m \eta^2/4\eta + \pi/4} \sqrt{\pi/\eta} \), we obtain a non-axisymmetric generalization of solution found in [7]
\[
 u_b = C_b \sqrt{\det \Gamma(\beta)} \exp\{imS_b\}, \quad S_b = \frac{\theta \eta}{m} - \frac{\beta m}{4\eta},
\]
and \( \theta \) is defined in (10).

The absolute value of this solution does not depend on \( \alpha \)
\[
 |u_b| = \left| C_b \sqrt{\det \Gamma(\beta)} \right| \exp\{-\eta(r_\perp, \text{Im} \Gamma(\beta) r_\perp)\}.
\]

The level surfaces of (23) are moving with a unit velocity along the \( z \)–axis in the negative direction, and \( |u_b| \) is exponentially localized in transversal directions provided the matrix \( \Gamma \) satisfies the conditions discussed in the Section 2.

As it follows from (12), \( \Gamma(\beta) \sim \beta^{-1} E - \beta^{-2} \Gamma_0^{-1} \) for \( |\beta| \to \infty \), and thus localization degree around \( z \)–axis is decreasing with time and the solution becomes more axisymmetric. Apart from this, the absolute value of the pre-exponential factor is also decreasing as \( |\beta|^{-d/2} \) with \( |\beta| \to \infty \). Thus, at every given moment of time the solution (22) has a Gaussian localization along the transversal coordinates and power-law localization along the longitudinal ones. The total energy of the beam is infinite. Thus, the considered solution is indeed a Gaussian beam, one can also call it a Focus Wave Mode for the Klein–Gordon-Fock equation.

### 3.2 Gaussian packets for the Klein-Gordon-Fock equation

Acting along the lines of Subsection 2.2.1 we seek the particle–like solutions of the KGFE in the form of a superposition of the beam–like solutions \( u_b \)

\[
 u_p^{(\nu)}(\alpha, \beta, r_\perp) = \int_0^\infty d\eta F^{(\nu)}(\eta) u_b(\alpha, \beta, r_\perp; \eta).
\]

By choosing \( F^{(\nu)}(\eta) \) as (15) and plugging it into (24), we immediately recognize the same integral representation for the modified Bessel function (16) with \( a = (4\gamma x^2 + \varepsilon_m m^2 + i\beta m^2)/4 \) and \( b = \gamma - i\theta \), and arrive at
\[
 u_p^{(\mu-1/2)} = C_p \sqrt{\det \Gamma(\beta)} \left( \frac{S_p}{\tau + i\beta} \right)^\mu K_\mu (mS_p),
\]
where we use notation for the complex phase function
\[
 S_p = [(\gamma - i\theta)(\tau + i\beta)]^{1/2},
\]
here $\tau = 4\gamma^2/m^2 + \varepsilon_m$, $C_p = 2^{\mu+1} \sqrt{\pi} e^{\pi/4} c_b/m^\mu$ and $\mu = \nu + 1/2$ can be treated as new independent parameters. The square root with the positive real part is assumed in (26). The constructed family of solutions $u_p$ is nonaxisymmetric multidimensional generalizations of the solutions obtained for $d = 2$ and $d = 3$ in [16].

We stress here that (16) is indeed applicable for (24). First, Re $a$ is positive since $\tau > 0$ and $\beta$ is real. Secondly, Re $b = \gamma + \text{Im} \theta$ is positive as well, because $\gamma > 0$ and $\text{Im} \theta > 0$, as it follows from its definition (10) and the fact that the imaginary part of the quadratic form $(r_\perp, \Gamma(\beta) r_\perp)$ is positively defined by assumption.

We note that for $m \to 0$ this solution transforms to the localized solution of the wave equation $\phi_p^{(\mu)} (17)$ in the account that $\tau$ changes with $m$ in such a way that $m^2(\tau(m) + i\beta) \to m \to 0 4\gamma^2$.

Let us consider now some particular examples of the constructed solutions. The modified Bessel function reduces to elementary functions if $\mu$ is half-integer. For the values $\mu = 1/2$ and $\mu = -1/2$, we have $K^{(1/2)}_\pm (m S_p) = \sqrt{\pi/(2m S_p)} \exp\{-m S_p\}$ and the formula (25) yields

$$u_p^{(0)} = C_2 \sqrt{\text{det} \Gamma(\beta)} \frac{e^{-m S_p}}{(\tau + i\beta)^{1/2}}, \quad u_p^{(-1)} = C_3 \sqrt{\text{det} \Gamma(\beta)} \frac{e^{-m S_p}}{(\gamma - i\theta)^{1/2}}, \quad (27)$$

where $C_2$ and $C_3$ are numerical constants. We further note, that in two-dimensional space (i.e. with $\Gamma \equiv \Gamma_{xx} = (\beta - i\tau)^{-1}$) we can reduce the second solution of (27) to a function depending on one variable $S_p$ only

$$u_p^{(-1)}(S_p) = C_3 \frac{e^{-m S_p}}{S_p}. \quad (28)$$

In this case, the phase $S_p$ (26) is given by

$$S_p = \sqrt{(\gamma - i\alpha)(\tau + i\beta) + x^2} = \sqrt{x^2 + (z - i e)^2 - (t - i f)^2}, \quad (29)$$

with $e = (\tau - \gamma)/2$, $f = (\tau + \gamma)/2$. We note that such $S_p$ can be interpreted as a distance in the (euclidian) space–time with imaginary time

$$S_p = |R|, \quad R = (x, z - z_0, it - t_0),$$

where $z_0 = i e$, $t_0 = -f$. From this point of view, (28) can be thought of as a point source solution $G(R) \equiv u_p^{(-1)}(|R|)$ of the equation

$$\Delta_3 G - m^2 G = \delta(R), \quad (30)$$

where $\delta(R)$ is the three–dimensional Dirac delta–function, $\Delta_3$ is the Laplacian in three dimensions. So, our solution of the KGFE in two spatial dimensions is the point source solution of the elliptic equation (30) in three dimensions which is analytically continued to the complex plane.
Finally, the solutions (22) and (25) may be treated from the point of view of ray method approach discussed for KGFE by Maslov [30]. Phase functions $S_b$ and $S_p$ satisfy the eikonal equation
\[(\partial_t S)^2 - (\nabla S)^2 - 1 = 0\] and solutions $u_b$ and $u_p$ may be regarded as ray expansions $u = e^{imS(r,t)} \sum_{k \geq 0} (im)^{-k} g_k(r,t)$ comprising a single term.

4 Asymptotic investigation in space–time domain

We prove here that formula (25) gives a family of particle-like solutions of the Klein-Gordon-Fock equation. To do so, we first note that in all cases when $|S_p| \to \infty$, $-3\pi/2 < \arg S_p < \pi/2$ we can use the following asymptotic expression of the modified Bessel function
\[K_{\mu}(mS_p) \sim e^{-mS_p} \sqrt{\frac{\pi}{2mS_p}} (1 + O(1/S_p)).\]
Then all the solutions of our family behave as
\[u_p^{(\mu-1/2)} \sim C_p \frac{\pi^{1/2}}{(2m)^{1/2}} \sqrt{\det \Gamma(\beta)} S_p^{\mu-1/2} \frac{S_p}{(\tau + i\beta)^\mu} \exp\{-mS_p\}.\]
Basing on this expression we will develop in what follows asymptotical expansions of (25).

4.1 Behaviour at spatial infinity

We show now that the solution (25) has exponential decay at spatial infinity, i.e. for $z, r_\perp \to \infty$ and finite times, and therefore has a finite energy.

To prove the applicability of expression (33) for $z, r_\perp \to \infty$ and fixed time, first we give the estimate from below for the absolute value of $S_p$

\[|S_p| = |(\tau + i\beta) (\gamma - i(\alpha + (r_\perp, \Gamma r_\perp)))|^{1/2} \geq (\tau^2 + \beta^2)^{1/4}(\gamma - |r_\perp|^2 h(\beta))^{1/2} = (\tau^2 + \beta^2)^{1/4}(\gamma^2 + 2|\gamma - i\theta| h(\beta) + |r_\perp|^4 h^2(\beta))^{1/4} \geq (\beta^2 \gamma^2 + Q|r_\perp|^2)^{1/4}.\]

At the first line of (34) we have used that $|\gamma - i\theta| \geq \Re(\gamma - i\theta) = \gamma + (r_\perp, \Im \Gamma r_\perp)$ and introduced the notation
\[h(\beta) = \left(\frac{r_\perp}{|r_\perp|}, \Im \Gamma \frac{r_\perp}{|r_\perp|}\right),\]
while to proceed to the second line we further used that
\[\Gamma = \frac{\beta^{-1}}{E + \beta^{-1} \Gamma^{-1}_0} = \beta^{-1}E - \beta^{-2} \Gamma^{-1}_0 E^{-1} + \frac{\beta^{-2} \Gamma^{-1}_0}{E + \beta^{-1} \Gamma^{-1}_0}.\]
and the fact that the imaginary part of $\Gamma$ is positively defined. Together with the continuity and boundedness of $h$ as a function of $\beta$ and $r_\perp$, it allows us to conclude that there exists a constant $Q > 0$ such that for any $\beta$ and $r_\perp$

$$2\gamma(\tau^2 + \beta^2)h(\beta) \geq Q.$$        

This justifies the last inequality of (34) and thus proves that $|S_p|$ does indeed grow with $z \to \infty$ and/or $r_\perp \to \infty$.

On the other hand, since $\text{Re} (\tau + i\beta) > 0$ and $\text{Re} (\gamma - i\theta) > 0$ for any finite $\beta$ we have $|\arg(\tau + i\beta)| < \pi/2 - \delta$, $|\arg(\gamma - i\theta)| < \pi/2$, where $\delta \equiv \delta(\beta) > 0$ and thus

$$|\arg S_p| = |\arg ((\tau + i\beta)(\gamma - i\theta))^{1/2}| < \pi/2 - \delta.$$  \hspace{1cm} (36)  

This already proves applicability of the asymptotical expression (33). Furthermore, assuming that $|z|$ is big enough we deduce that $\text{sgn} (\arg(\tau + i\beta)) = \text{sgn}(z)$, while

$$\text{sgn} (\arg(\gamma - i\theta)) = -\text{sgn} ((\alpha + \text{Re}(r_\perp, \Gamma r_\perp))) \simeq -\text{sgn} ((z + |r_\perp|^2/z)) = -\text{sgn}z,$$

where we again used (35). This tells us that for $z \to \infty$, the arguments of the factors $\tau + i\beta$ and $\gamma - i\theta$ in $S_p$ have opposite signs and (at least partially) cancel each other. Then the estimate (36) can be further strengthen

$$|\arg S_p| = |\arg ((\tau + i\beta)(\gamma - i\theta))^{1/2}| < \pi/4.$$ \hspace{1cm} (37)  

The estimates (34), (37) and the formula (33) show that the absolute value of any solution $|u_p^{(\mu-1/2)}|$ decreases exponentially with growing coordinates, and therefore all these solutions have finite energy.

### 4.2 Asymptotics for small and moderate $z$ and time $t$

We intend to show now that for some relation (to be discussed below) between the mass $m$ and the solution parameters $\gamma$, $\tau$ the solution (25) is a wave packet with the Gaussian envelop moving with group speed.

First we assume that the coordinates and time are small enough (in what follows we clarify the formal meaning of the smallness) for the square root in (26) to be expanded in Taylor series up to the terms of the second order in time and coordinates

$$S_p = m\sqrt{\gamma\tau} \left(1 + \frac{i\beta^2}{2\tau} + \ldots \right) \left(1 - \frac{i\theta}{2\gamma} + \frac{\alpha^2}{8\gamma^2} + \ldots \right).$$ \hspace{1cm} (38)  

We use $\alpha$ instead of $\theta$ in the last term because we are interested only in the terms quadratic in coordinates and time. However, we postpone expanding $\Gamma(\beta)$ till we work out the applicability conditions. Simplifying (38) and collecting the terms we obtain

$$S_p = \sqrt{\gamma\tau} \left(1 - i\frac{(\tau - \gamma)z - (\tau + \gamma)t}{2\gamma\tau} + (z - v_\tau t)^2 \frac{(\gamma + \tau)^2}{8(\gamma\tau)^2} - \frac{i}{2\gamma} (r_\perp, \Gamma(\beta) r_\perp) + \ldots \right),$$ \hspace{1cm} (39)
\[ v_{gr} = \frac{\tau - \gamma}{\tau + \gamma}. \]  

(40) \[ v_{gr} \]

We choose for definiteness that \( \gamma < \tau \). It ensures that the solution propagates forward along the \( z \)-axis, i.e. that \( v_{gr} > 0 \).

We notice now that the expansion (38) requires the conditions

\[ |\beta| \ll \tau, \quad |\theta| < |\alpha| + |(r_{1\perp}, \Gamma(\beta) r_{1\perp})| \ll \gamma \]

which are reduced under assumption that \( \gamma < \tau \) to the following (actually, stronger) ones

\[ |t| \ll \tau, \quad |z - t| \ll \gamma, \quad |(r_{1\perp}, \Gamma(\beta) r_{1\perp})| \leq r_{1\perp} \| \Gamma(\beta) \| \ll \gamma \]  

(41) \[ \text{cond-t-mod-3} \]

where \( \| \cdot \| \) is an appropriate matrix norm, e.g. euclidian.

The possible expansion of the \( \Gamma(\beta) \) depends on the range of values of \( t \). First we note that if

\[ \| \Gamma_0 \| \ll \tau^{-1} \]  

(42) \[ \text{G0_co} \]

then \( |z + t| \| \Gamma_0 \| \ll 1 \), and we can expand \( \Gamma(\beta) \) in the following way

\[ \Gamma(\beta) \equiv \Gamma_0 (E + \beta \Gamma_0)^{-1} = \Gamma_0 (1 - (z + t) \Gamma_0 + \ldots). \]  

(43) \[ \text{hel-Lam} \]

The same is true if (42) is not satisfied but \( t \) is small enough for the condition \( |z + t| \| \Gamma_0 \| \ll 1 \), to be fulfilled. If both conditions are not meet, we can still expand \( (r_{1\perp}, \Gamma(\beta) r_{1\perp}) \) by using

\[ \Gamma(\beta) = \frac{\Gamma_0}{(E + 2t \Gamma_0)} + \alpha \Gamma_0 = \frac{\Gamma_0}{(E + 2t \Gamma_0)} \left( 1 - \frac{\alpha \Gamma_0}{E + 2t \Gamma_0} + \ldots \right) \]  

(44) \[ \text{hel-Lam} \]

the latter expansion is valid if

\[ \| (z - t) \Gamma_0 (E + 2t \Gamma_0)^{-1} \| \ll \gamma \| \Gamma(2t) \| \ll 1. \]  

(45) \[ \text{cond-t-mod-1} \]

Thus, the dependence on \( t \) can be important even for small times, \( t \ll \tau \), if the \( \| \Gamma_0 \| \) is large enough.

Thus we conclude, that if the conditions (41) and either one of the (42) and (45) are valid, then we may use the expansion (39) of \( S_p \) and the asymptotics (32) of the modified Bessel function to obtain the asymptotics of the packet (25)

\[ u_p^{(\mu-1/2)} \approx A(\zeta) \exp \left\{ i(K \tilde{z} - \Omega \tilde{t} + g|\tilde{r}_{1\perp}|^2) \right\} \exp \left\{ -\frac{(\tilde{z} - v_{gr} \tilde{t})^2}{2\Delta^2} \right\} \exp \left\{ -\frac{|\tilde{r}_{1\perp}|^2}{2\Delta^2} \right\}, \]  

(46) \[ \text{pack-mod-t} \]

where we used the dimensionless coordinates, time and mass

\[ \tilde{t} = t/\sqrt{\tau \gamma}, \quad \tilde{z} = z/\sqrt{\tau \gamma}, \quad \tilde{r} = r/\sqrt{\tau \gamma}, \quad p = m/\sqrt{\tau \gamma}. \]  

(47) \[ \text{non-dim} \]
All the characteristics of the asymptotic — $K$, $\Omega$, $v_{gr}$, $\Delta_\parallel$ and $\Delta_\perp$ in (46) are expressed in terms of non-dimensional parameters $p$, $\tau/\gamma$ and $\tau \text{ Im} \Gamma$ as follows

$$\Omega = p \left( \sqrt{\tau/\gamma} + \sqrt{\gamma/\tau} \right)/2, \quad K = p \left( \sqrt{\tau/\gamma} - \sqrt{\gamma/\tau} \right)/2, \quad v_{gr} = K/\Omega, \quad (48)$$

$$\Delta_\parallel^2 = \frac{p}{\Omega^2}, \quad \Delta_\perp^2 = \frac{1}{p^2} (e_\perp, \text{Im} \Gamma(\zeta)e_\perp), \quad \zeta = \begin{cases} 0, & |t| ||\Gamma_0|| \ll 1 \\ 2t, & \text{otherwise} \end{cases}, \quad (49)$$

here $e_\perp = r_\perp/|r_\perp|$. The amplitude factor $A$ and correction term in the phase $g$ read

$$A = C_p \pi^{1/2} \sqrt{\text{det} \Gamma(\zeta)} \left( \frac{\gamma^{1/4}}{\tau^{1/4}} \right) \exp\{-m\sqrt{\gamma/\tau}\}, \quad g = \frac{p}{2} (e_\perp, \text{Re} \Gamma(\zeta)e_\perp). \quad (50)$$

The solution $u_b$ describes a wave with frequency $\Omega$ and wave number $K$, which propagates along the $z$ axis and has the Gaussian envelop moving with group velocity $v_{gr}$. The localization near the $z$ axis is determined by the $\Delta_\perp$ which depends on the orientation of $r_\perp$ (the result of astigmatic nature of the considered solution) and time. Note that for $t > ||\Gamma_0||^{-1}$ (while still being much less then $\tau$) the width $\Delta_\perp$ starts growing linearly with time. We call this regime as moderate times' one. It can only show up when $||\Gamma_0|| \ll \tau$, otherwise only two asymptotic regimes can be identified for our solution: small times or large ones.

Now we will check, that the formulae (46)-(50) describe correctly the field up to the distances where the packet becomes exponentially small. To this end we first compare (by order of magnitude) the longitudinal width of the packet, $\Delta_\parallel \sim |\tilde{z} - v_{gr}\tilde{t}|$, with the distance from the point $\tilde{z}_0 = v_{gr}\tilde{t}$, where (46) is still applicable as defined by the second condition (41). This distance is $|\tilde{z} - v_{gr}\tilde{t}| \leq |\tilde{z} - \tilde{t}| + |\tilde{t}(1 - v_{gr})| \ll \sqrt{\gamma/\tau}$ with account of (40). Secondly, the transverse width of the packet $|\tilde{r}_\perp| \sim \Delta_\perp$ should be inside the zone determined by the third condition of (41). Thus, it must hold that

$$\Delta_\parallel \ll \sqrt{\gamma/\tau}, \quad \Delta_\perp \ll (\tau ||\Gamma(\zeta)||)^{-1/2}. \quad (51)$$

Thirdly, we demand that our solution must travel according to (46) on distances which are much larger then its longitudinal width $\Delta_\parallel$, i.e.,

$$\Delta_\parallel \ll \tilde{z}_{\text{max}} \sim v_{gr}\sqrt{\gamma/\tau}. \quad (52)$$

The two widths $\Delta_\parallel$ and $\Delta_\perp$ contain $p$ in the denominator. Therefore all of the conditions (51) and (52) are satisfied if $\tau/\gamma$ is fixed and $p \to \infty$. The last condition (52) is the more restrictive. In terms of the parameters $\gamma/\tau$ and $p$ it reads

$$\frac{1}{p} \ll \frac{\tau}{4\gamma} \left( 1 - \frac{\gamma}{\tau} \right)^2, \quad \text{i.e.} \quad p \ll \frac{\gamma^2}{\Omega - K}. \quad (53)$$

Finally, we may compare our results in the limit $m \to 0$ with the formulas for the packet–like (stigmatic) solution for the wave equation obtained in (37) of [15]. In doing so we must put
\( m^2 \tau \to 4\gamma \kappa^2 \), where \( \kappa \) is a constant used in [15] (compare with the note after (25)) and also \( \Gamma_0 = i\varepsilon^{-1} E \) and thus obtain

\[
\Delta_{\Pi} = \frac{4(\gamma \tau)^{3/2}}{m(\gamma + \tau)^2} \to \frac{2\gamma}{\kappa}, \quad \Delta_\perp = i \frac{\sqrt{\gamma}}{m\sqrt{\tau}} (-i\varepsilon) \to \frac{\varepsilon}{2\kappa}
\]

in complete agreement with [15].

### 4.3 Large-time behavior

Let us find the asymptotics of (25) for large times and large distances \( r = (r_\perp, z) \). We assume that \( r = vt \) for \( t \to \infty \), but \( v = (v_\perp, v_z) \) is fixed. The asymptotics of \( \Gamma \) (12) and \( \theta \) (10) are as follows

\[
\Gamma \sim \frac{E}{(v_z + 1)t} - \frac{\Gamma_0^{-1}}{(v_z + 1)^2 t^2} + O(t^{-3}), \quad \theta \sim -i t - \frac{1 - v^2}{2(1 + v_z)\sqrt{1 - v^2}} + O(t^{-1})
\]

and

\[
\Gamma \approx \frac{t\sqrt{1 - v^2}}{2(1 + v_z)\sqrt{1 - v^2}} + \frac{\gamma(1 + v_z)}{2\sqrt{1 - v^2}} + \frac{\tau \sqrt{1 - v^2}}{2(1 + v_z)} + O(t^{-1})
\]

In the first line here we took the square root with the positive real part. Thus, \( |S_p| \to \infty \) for \( t \to \infty \) and we can use the asymptotics (32) of \( K_\mu(mS_p) \), and (33) for the whole solution.

Now, we suppose that \( p \equiv m\sqrt{\gamma} \tau \gg 1 \). Introducing for the sake of brevity new variables

\[
\varpi = 1/\sqrt{1 - v^2}, \quad \chi = v/\sqrt{1 - v^2},
\]

we rewrite the solution (33) in the form that allows for its further simplification

\[
u\upmu_{p(\mu-1/2)} \approx \mathcal{A}(t, v) \exp \left\{-imt\sqrt{1 - v^2}\right\} \exp\left\{-p\Phi(v)\right\},
\]

\[
\Phi = i \frac{\varpi_\perp^2 (e_\perp, \Gamma_0^{-1} e_\perp)}{\sqrt{\gamma} \tau (\varpi + \chi_z)} + \frac{\sqrt{\gamma/\tau} (\varpi + \chi_z)}{2(\varpi + \chi_z)}, \quad e_\perp = \frac{\chi_\perp}{|\chi_\perp|}
\]

\[
\mathcal{A}(t, v) \approx C_P \sqrt{\frac{\pi}{2m}} e^{iS_p - i(1/d + \text{sgn}(t)\pi/4)} \left[ t/(d+1)^{2/2}(1 + v_z)\mu + \frac{1 - v^2\mu^2}{2} t^{d+1/2} \right],
\]

which we obtained by using that \( S_p \approx -i(\mu - 1/2)\text{sgn}(t)\pi/2 \left[ t/(d+1)^{2/2}(1 + v_z)\mu + \frac{1 - v^2\mu^2}{2} t^{d+1/2} \right] \) for large \( |t| \) due to \( S_p \approx -i(\mu - 1/2)\text{sgn}(t)\pi/2 \left[ t/(d+1)^{2/2}(1 + v_z)\mu + \frac{1 - v^2\mu^2}{2} t^{d+1/2} \right] \). We used also that \( \text{Re}(\tau + i\beta) > 0 \), and the fact that \( \sqrt{\det(-i\Gamma(\beta))}/(\tau + i\beta)\mu \approx (i\beta)^{-d/2-\mu} = |\beta|^{-\mu-d/2}e^{-i(\mu+d/2)\text{sgn}(t)\pi/2} \). To get the latter equality we note that for
\[ \beta \to \infty \] we get \((-i \Gamma) \approx E/(i\beta + O(1))\), where \(O(1)\) is positive. The branch of the square root \(\sqrt{\det(-i \Gamma(\beta))}\) is fixed by the asymptotics for \(|t| \to \infty\): \(\arg \sqrt{\det(-i \Gamma)} \to -\text{sgn}(t)d\pi/4\). We do not specify the branch of the square root in \(\sqrt{\det \Gamma}\) and introduce the argument

\[ \delta \Gamma = \arg \sqrt{\det \Gamma} - \arg \sqrt{\det (-i \Gamma)}. \]  

(61)

When \(p \gg 1\), the modulus of the second exponent in (58) has a sharp maximum and we will use quadratic approximation of \(\Phi\) in its vicinity. We seek its position in the spherical coordinate system, i.e., \(\chi_z = \chi \cos \vartheta, \chi_\perp = \chi \mathbf{e}_\perp \sin \vartheta\). We note that the first term in \(\Phi\) (59) has nonnegative real part, thus its least value is equal to zero when \(\vartheta = 0\) or \(\vartheta = \pi\). The two other terms of \(\Phi\) are mutually inverse (up to a factor of 1/2), thus their sum reaches its least value equal to unity if

\[ \varpi + \chi_z = \sqrt{\tau/\gamma}. \]  

(62)

Together with the condition \(\vartheta = 0\) this gives us

\[ \mathbf{v}_\perp = 0, \quad v_z = v_{gr}, \quad \varpi = \Omega/m, \quad \chi = \mathcal{K}/m, \]  

(63)

where \(\mathcal{K}\) and \(\Omega\) are defined in (48). It is easy to check that \(\vartheta = \pi\) is incompatible with (62) for \(v < 1, \tau/\gamma > 1\). Finally, we obtain

\[ u_p^{(\mu-1/2)} \approx \mathcal{A}(t, v_{gr}) \exp \left\{ -m\sqrt{\gamma t} - im \sqrt{1-v^2} - ip\vartheta^2 \frac{\Im \Phi''_{\vartheta\vartheta}}{2} \right\} \exp \left\{ -\frac{(r - v_{gr}t)^2}{2p^2\Delta_v^2} - \frac{\vartheta^2}{2\Delta_\vartheta^2} \right\}. \]  

(64)

We have used here that \(v_z = z/t, \mathbf{v}_\perp = \mathbf{r}_\perp/t\) and that the derivative \(\Phi''_{\vartheta\chi}\) is zero. The notation for \(\mathcal{A}(t, \mathbf{v})\) was introduced in (60). The widths of the packet \(\Delta_v\) and \(\Delta_\vartheta\) are expressed through the second derivatives of \(\Phi(\mathbf{v})\) as follows

\[ \Delta_v^2(t) = \left[ p\Phi''_{\chi\chi}(\chi_v')^2 \right]^{-1} = \frac{(1 - v_{gr}^2)^2}{p} = \frac{p^2}{\Omega^2}, \]  

(65)

\[ \Delta_\vartheta^2(t) = (p \Im \Phi''_{\vartheta\vartheta})^{-1} = \frac{\tau(1 - v_{gr}^2)}{p v_{gr}^2(\mathbf{e}_\perp, -\Im \Gamma_0^{-1}\mathbf{e}_\perp)} = \frac{\tau}{\mathcal{K}_2^2(\mathbf{e}_\perp, -\Im \Gamma_0^{-1}\mathbf{e}_\perp)}, \]  

(66)

where \(\Phi''_{\chi\chi} \equiv \frac{\partial^2 \Phi}{\partial \vartheta^2} |_{\vartheta=0,v=v_{gr}}, \Phi''_{\vartheta\vartheta} \equiv \frac{\partial^2 \Phi}{\partial v^2} |_{\vartheta=0,v=v_{gr}}, \chi_v' = \frac{d\chi}{dv} |_{v=v_{gr}}\). It is worth mentioning here, that due to the fact that \(\Im \Gamma_0\) is positively defined, so is \(-\Im \Gamma_0^{-1}\), and thus \(\Delta_\vartheta^2\) is positive.

According to (64) the field is concentrated in the intersection of a cone and a spherical annulus. The width of the annulus increases with time and may be estimated as \(2t\Delta_v\). We will require that the speed of the packet center exceeds the speed of the packet enlarging. The angle of the cone does not depend on time and we can assume that it is small. Finally we have conditions

\[ \Delta_v \ll v_{gr}, \quad \Delta_\vartheta \ll 1, \]  

(67)
which can be written in the simplest stigmatic case, $\Gamma = E/(\beta - i\varepsilon)$, as follows

$$p \gg 16(\tau/\gamma - \gamma/\tau)^{-2}, \quad p \gg 4 \left(\sqrt{\tau/\gamma} - \sqrt{\gamma/\tau}\right)^{-2}/\tau/\varepsilon.$$  \hfill (68)

These conditions are the more restrictive the closer are $\tau$ and $\gamma$ to each other. If $p \to \infty$ for fixed other parameters the localization is more pronounced. If $\gamma/\tau \to 0$ the localization both in angle and along the propagation direction is better. In the case of general astigmatism the term $(e_\perp, -\mathrm{Im}\Gamma_0^{-1}e_\perp)$ should stand in the last inequality instead of $\varepsilon$.

Now we turn to the applicability conditions of the obtained formulas. Time $t$ will be considered large if expansions of $\Gamma$ and $\theta$ (54) could be limited to their first terms. For that we will require

$$|t| \gg \|\Gamma_0^{-1}\|/|1 + v_z|, \quad |t|(1 - v^2) \gg v^2_\perp\|\Gamma_0^{-1}\|/|1 + v_z|. \hfill (69)$$

For the expansion of $S_p$ (56) being valid we additionally need

$$\frac{\theta}{\gamma} \approx \frac{|t|(1 - v^2)}{|1 + v_z|} \gg 1, \quad \frac{\beta}{\tau} = \frac{|t(1 + v_z)|}{\tau} \gg 1. \hfill (70)$$

Both conditions (69) and (70) contain $v$, but we can substitute it with $v_{gr}$ by recalling that $|v - v_{gr}| \approx \Delta_v \ll v_{gr}$ according to (67). The group velocity itself can be expressed in terms of $\tau$ and $\gamma$ by using (40). Two conditions (70) are reduced to just one then, $|t| \gg \tau$. Taking into account that $v^2_\perp \approx v^2_{gr}, \vartheta^2 \ll 1$ we replace (69) by a stronger inequality. Combining the two, we obtain

$$|t| \gg 4\|\Gamma_0^{-1}\|\tau/\gamma, \quad |t| \gg \tau. \hfill (71)$$

These conditions specify large times.

### 5 Fourier analysis

Let us introduce a Fourier transformation relevant to the problem in hand

$$\mathcal{F}[f](\omega, k_\perp, k_z) = \int_{\mathbb{R}^{d+2}} dt \, d^d r_\perp \, dz \, f(t, r_\perp, z) \, e^{i(\omega t - k_\perp r_\perp - k_z z)}, \hfill (72)$$

where $k_\perp$ denotes the Cartesian components of the wave vector, $k_\perp = (k_x, k_y, \ldots)$, perpendicular to $k_z$, $d$ is the number of transversal dimensions. The inverse transformation is

$$f(t, r_\perp, z) = \frac{1}{(2\pi)^{d+2}} \int_{\mathbb{R}^{d+2}} d\omega \, d^d k_\perp \, dk_z \, \mathcal{F}[f](\omega, k_\perp, k_z) \, e^{-i(\omega t - k_\perp r_\perp - k_z z)}. \hfill (73)$$

Performing the Fourier transformation of multidimensional WE (4) or KGFE (19) one obtains the following equation in terms of generalized functions

$$(\omega^2 - \varpi^2(k_\perp) - k_z^2)\mathcal{F}[f](\omega, k_\perp, k_z) = 0, \hfill (74)$$
here \( \kappa^2 = k_1^2 \) for WE and \( \kappa^2 = k_1^2 + m^2 \) for KGFE.

Any solution of (73) must be representable as
\[
\mathcal{F}[f](\omega, k_{\perp}, k_z) = \delta(\omega^2 - \kappa^2(k_{\perp}) - k_z^2) \hat{f}(k_{\perp}, k_z)
\]
with a suitable well–behaved function \( \hat{f}(k_{\perp}, k_z) \). It is this function \( \hat{f} \) which we call the Fourier image in what follows.

Constructing particular solutions to the WE or KGFE, we are free to choose any particular subspace of the surface \( \omega^2 = \kappa^2(k_{\perp}) + k_z^2 \) in the phase space. For instance, in [10] it was considered a solution of the from
\[
F[f](\omega, k_{\perp}, k_z) = \Xi(\tilde{\eta}) \delta(\tilde{\eta} - \kappa^2/4\tilde{\eta}) \delta(\omega + (\tilde{\eta} + \kappa^2/4\tilde{\eta}))
\]
where \( \tilde{\eta} \) is an arbitrary (real) parameter (to avoid conflict of notation we changed the original notation of [10]), and \( \Xi(\tilde{\eta}) \) is an arbitrary weight function.

Now we obtain the Fourier image both for the beam–like solutions \( \phi_b, u_b \) and particle–like ones \( \phi_p, u_p \). Apart from revealing the connection of our solutions with aforementioned ones, it will also be used in constructing asymptotic expansions in Appendix A.

5.1 Wave Equation

The Fourier transform (72) of the solution (11) of the WE is given by
\[
\mathcal{F}[\phi](\omega, k) = c_b \int dt \, dz \, d^4r_{\perp} \, e^{i(\omega t - k_z z - k_{\perp} \cdot r_{\perp})} \sqrt{\det \Gamma} \, e^{i\eta(\alpha + (r_{\perp}, \Gamma r_{\perp}))} = c_b \int dt \, dz \, e^{i\delta - i k_{\perp} z} e^{i\eta \alpha} I(\eta, r_{\perp}),
\]
the definition of \( \delta \Gamma \) see in (61). We recall that all the eigenvalues of the matrix \( (-i\Gamma) \) have positive real part, and thus the last integral is convergent.

Substituting the last formula into (76) and taking into account that \( \Gamma^{-1} = \Gamma_0^{-1} + \beta E \) we have
\[
\mathcal{F}[\phi_b](k_{\perp}, k_z) = c_b e^{i\delta} e^{i\eta \alpha} \int dt \, e^{i\omega t - i(\eta + \frac{1}{4\eta} k_{\perp}^2)} \int dz \, e^{-iz(k_z - \eta + \frac{1}{4\eta} k_{\perp}^2)}
\]
\[
= c_b e^{i\delta} e^{-\frac{i}{4\eta}(k_{\perp}, \Gamma_0^{-1} k_{\perp})} 16 \pi^{2+\frac{d}{2}} \eta^{-d/2} \delta \left( \eta - \frac{k_z + \sqrt{k_z^2 + k_{\perp}^2}}{2k_{\perp}^2 + 4\eta^2} \right) \delta \left( \omega - \left( \eta + \frac{1}{4\eta} k_{\perp}^2 \right) \right).
\]
Using the properties of the delta functions we can rewrite it finally as

\[
\mathcal{F}[\phi_b](k_\perp, k_z) = \hat{c}_b \frac{\mathrm{e}^{i(k_\perp \cdot \Gamma_0^{-1} k_\perp)}}{\omega(k_z + \omega)^{d/2-1}} \delta \left( \eta - \frac{k_z + \omega}{2} \right) \delta \left( \omega - \sqrt{k_\perp^2 + k_z^2} \right),
\]

(79) $F_{b,k_fn}$

\[
\hat{c}_b \equiv c_b \pi (2\pi)^{1+d/2} \mathrm{e}^{i\delta r}.
\]

We remind that $\eta > 0$ is a free parameter of our solution, along with $\Gamma_0$. The Fourier image is defined now as (assuming $\omega = \sqrt{k_\perp^2 + k_z^2}$)

\[
\hat{\phi}_b(k_\perp, k_z) = \hat{c}_b \mathrm{e}^{-i(k_\perp \cdot \Gamma - 10 k_\perp)} \delta \left( \eta - \frac{k_z + \omega}{2} \right).
\]

(80) $\phi_b(k)$

If compared with the considerations of [10] (see eq. (75)) we can see that in our case $\tilde{\eta} = \eta$ and

\[
\Xi \equiv \Xi(k_\perp, k_z) = \hat{c}_b \mathrm{e}^{-i(k_\perp \cdot \Gamma - 10 k_\perp)} \mathrm{e}^{-\gamma(k_\perp \cdot \Gamma - 10 k_\perp)} \delta \left( \eta - \frac{k_z + \omega}{2} \right).
\]

This shows that our spectral function has more variables and less symmetries depending on $k_\perp$ and $k_z$ separately.

For obtaining the Fourier image of a particle like solution $\phi_p(k)$ we employ (14)

\[
\hat{\phi}_p^{(\nu)}(k_\perp, k_z) = \int_0^\infty d\eta \hat{\phi}_b(k_\perp, k_z) F^{(\nu)}(\eta)
\]

\[
= \hat{c}_b \frac{\mathrm{e}^{-i(k_\perp \cdot \Gamma_0^{-1} k_\perp)}}{\omega(k_z + \omega)^{d/2-1}} \int_0^\infty d\eta \eta^{-\nu-1} \mathrm{e}^{-\gamma(\eta + \kappa^2/\eta)} \delta \left( \eta - \frac{k_z + \omega}{2} \right).
\]

Then, we can write

\[
\hat{\phi}_p^{(\nu)}(k_\perp, k_z) = 2^{\nu+1} \hat{c}_b \frac{\mathrm{e}^{-\gamma(k_z + \omega)} - 4\gamma^{\nu+1} \frac{\mathrm{e}^{i(k_\perp \cdot \Gamma_0^{-1} k_\perp)}}{\omega(k_z + \omega)^{\nu+d/2}}}{\omega(k_z + \omega)^{\nu+d/2}}.
\]

(81) $\phi_p(k)$

5.2 KGF equation

Now we shall construct the Fourier transform of the KGF solution (22). We accomplish it by acting similar to the Section 33.1. First we increase the number of dimensions by one, $d \to d+1$, and put the additional momenta component equal to mass, $k_{d+1} \equiv m$.

Thus, we have to substitute everywhere in (80) $d$ by $d+1$ and $k_\perp$ by $k_\perp^m = (k_\perp, m)$

\[
\hat{u}_b(k_\perp, k_z) \equiv \hat{\phi}_b(k_\perp^m, k_z) = \hat{\phi}_b((k_\perp, m), k_z).
\]
At the same time we must assume similar to (21) that
\[(k^m_{\perp}, \tilde{\Gamma}_0^{-1}k^m_{\perp}) = (k_{\perp}, \Gamma_0^{-1}k_{\perp}) - i\varepsilon m k_{\perp},\]
where \(\tilde{\Gamma}_0\) is a \((d + 1) \times (d + 1)\) matrix, and \(\Gamma_0\) is \(d \times d\) one as before.

Thus, the Fourier image of the beam–like solution is
\[
\hat{u}_b(k^m_{\perp}, k_z) = \hat{C}_b e^{-\varepsilon m k_{\perp}^2 + i(k_{\perp}, \Gamma_0^{-1}k_{\perp})} \frac{\varepsilon m k_{\perp} + \sqrt{(k_{\perp} + \omega)^2 + \varepsilon m k_{\perp}^2}}{\omega(k_z + \omega)^{d/2-1/2}} \delta \left( \eta - \frac{k_z + \omega}{2} \right),
\]
(82) \[\hat{u}_b(k_{\perp}, k_z)\]
\[\hat{C}_b = c_b \pi (2\pi)^{3/2+d/2} e^{i\delta r}.\]
Here it is assumed that \(\omega = \sqrt{m^2 + k_{\perp}^2 + k_z^2}\).

For obtaining the Fourier image of the particle–like solution \(u_p(k_{\perp}, k_z)\) we perform the integral transformation (24) of the (82)
\[
\hat{u}_p^{(\nu)}(k_{\perp}, k_z) = \int_0^\infty d\eta \hat{u}_b((k_{\perp}, m), k_z) F^{(\nu)}(\eta)
\]
\[\hat{u}_p(k_{\perp}, k_z)\]
\[= \hat{C}_b e^{\frac{\varepsilon m k_{\perp} + (k_{\perp}, \Gamma_0^{-1}k_{\perp})}{2(k_z + \omega)}} \int_0^\infty d\eta \eta^{-\nu-1} e^{-\gamma(\eta + \varepsilon^2/\eta)} \delta \left( \eta - \frac{k_z + \omega}{2} \right).\]

Then we arrive at
\[
\hat{u}_p^{(\nu)}(k_{\perp}, k_z) = \hat{C}_p e^{-\frac{\gamma}{2}(k_z + \omega)} \frac{\tau m^2 + (k_{\perp}, \Gamma_0^{-1}k_{\perp})}{2(k_z + \omega)^{\nu+d/2+1/2}} \delta \left( \eta - \frac{k_z + \omega}{2} \right),
\]
(83) \[\hat{u}_p(k_{\perp}, k_z)\]
where we used the notation of the previous section, \(\tau = 4\gamma/\varepsilon^2/m^2 + \varepsilon_m\), and put \(\hat{C}_p = 2^{\nu+1}\hat{C}_b\).

We notice a remarkable difference in localization properties of the Fourier images for KGFE and WE: in \(\hat{u}_b(k_{\perp}, k_z)\) as compared with \(\hat{\phi}_b(k_{\perp}, k_z)\) it is the absence the exponential suppression of small \((k_z + \omega)\) via terms of the type of \(e^{-c/(k_z + \omega)}\).

## 6 Discussion of the results

In the present paper we have elaborated four families of explicit exact exponentially localized solutions to the wave equation (11), (17), and to the Klein–Gordon–Fock one (22), (25). The families (11), (22) represent beam–like solutions localized exponentially near a ray, while (17) and (25) are particle–like ones localized exponentially near a point moving with group velocity along one of the axis. All of the presented solutions are astigmatic multi-dimensional generalizations of those obtained before by the authors [18, 16, 14], as well as by other researches [31, 17], etc.

Unlike most of the others works, we performed all the analysis in space–time domain, which proved to be both convenient and efficient. Focusing on the particle–like solutions of
the KGF equation which are somewhat less studied in the literature, we investigated in detail the asymptotic properties of the central result of our work — the particle–like solutions to the KGF equation distinguishing several regimes: small times, moderate times and large times. We also presented explicit Fourier transformation of all constructed solutions and confirmed our asymptotic consideration obtained in space–time domain by investigating the Fourier integral.

Now we summarize briefly the contents of the Section 4. The constructed solutions contain several parameters: $\tau$, $\gamma$, $\Gamma_0$ (or $\varepsilon$ in the stigmatic case). If the non-dimensional mass

$$p = m \sqrt{\gamma \tau} \quad (84)$$

is large the solution behaves as a packet with the Gaussian envelop filled with oscillations which on the axis of the packet has the wave number $K$ and the frequency $\Omega = \sqrt{K^2 + p^2}$, and moves with the group speed $v_{gr} = K/\Omega$. Packet–like behaviour takes place for all times. Below we discuss properties of the field in different regimes.

For small times, $t \ll \|\Gamma_0\|^{-1}$ (assuming that $\|\Gamma_0\|^{-1} < \tau$) the solution behaves according to (46) with $\zeta = 0$. This regime is characterized by complete absence of any distortion during the propagation. The longitudinal width as well as transversal one, is time independent. In the transversal direction the astigmatic properties are practically frozen and do not depend neither on time, nor on propagated distance. The localization ellipse is defined by $\Gamma_0$ itself (compare with large times). The maximum of the propagated distance is of the order of $2v_{gr}\|\Gamma_0\|^{-1}$. This
regime is exemplified at the Fig.1 for the following values of parameters (in units of mass $m$)
$\gamma = 800$, $\tau = 8 \cdot 10^5$, $q_1 = 1 + i\varepsilon$, $q_2 = 14 + 3i\varepsilon$, $\varepsilon = 3 \cdot 10^3$, $\Phi = -0.31i$.

Moderate times are characterized by condition $\|\Gamma_0\|^{-1} \leq t \ll \tau$. The solution in this case can be described rather good by the asymptotic formula (46) with $\zeta = 2t$. The distortion of the solution in this regime is twofold. First of all, the absolute value of the solution is decreasing linearly with time due to the dependence on $t$ of the prefactor $A$ (50) via $\sqrt{\det \Gamma}$. Secondly, the transverse width of the solution $\Delta_\perp$ also grows linearly with time. Both these features are clearly visible on the Fig.2, where the absolute value of $u_p$ is plotted at the same values of the parameters as before. We also note that at this stage the astigmatic properties can already be seen — the localization ellipse is slowly rotating.

For the large time regime $t$ satisfies conditions (71). The amplitude of the solution also decreases with time in this case, as it follows from (60). The packet is concentrated in the intersection of a cone and a spherical annulus as it is seen from (64). The width of the annulus increases linearly with time as $\Delta_\perp t$, while the angular width of the cone is given by $\Delta_\vartheta$, see (65), (66). Their connection with the widths at small times is

$$\Delta_\perp^2 = p\Delta_\parallel^4, \quad \Delta_\vartheta^2 = \frac{1}{(K\Delta_\perp)^2}.$$  \hfill (85)

We see that the larger is the transverse width for small time the narrower is a cone. This property reflects the uncertainty principle. Large-time behaviour is presented on the Fig.3,
where the absolute value of $u_p$ is plotted over quite large period of time at the same values of the parameters as before. The astigmatic properties are already frozen at this regime, the axis of the localisation ellipse are rotated to the full angle $\pi$ as compared with its position at $t \rightarrow -\infty$. The latter fact can be understood if one notes that the localization ellipse for large times is defined by $\Gamma_0^{-1}$, see (66), contrary to the case of small times where it is defined by $\Gamma_0$. Under inversion, the smaller eigenvalue (the ellipse axis) becomes the bigger one, so, the localization ellipse effectively rotates by an angle of $\pi/2$ as compared to small times.

From a practical point of view, it can be more convenient to characterize the solutions not by choosing the initial parameters $\tau, \gamma$ and $\varepsilon$, but by specifying their asymptotic properties — the wave number, group velocity and localization widths either at small times, or at large ones. The former are unambiguously expressed through the latter as we see from (85).

A legitimate practical problem is to find for a given KGFE (i.e. for a given value of $m$) a solution with particular values of, e.g., $\Omega$ and the width of the packet for small times $\Delta_\parallel$. From (49) we can deduce then the value of $p$, which must satisfy $p \gg 1$, if we wish the asymptotics be applicable Next, we calculate the wave number, $K^2 = \Omega^2 - p^2$ and the group speed, $v_{gr} = K/\Omega$. Knowing $p$ and $v_{gr}$ we can derive both the product $\tau \gamma$ by (84) and the ratio $\gamma/\tau = (1 - v_{gr})/(1 + v_{gr})$, which together gives us all the parameters of the desired solution but its astigmatic properties. We can deduce the latter by choosing, for instance, the transversal width at small times, $\Delta_\perp$. Now all the parameters for the solution are known. Parameter $\mu$ cannot be derived by considering the asymptotic properties in the highest order.

We have however some restrictions to be satisfied, if we wish our solution possesses good localization properties

$$p \ll K^2 \Omega + K \Omega - K, \quad p^3 \ll (\Omega K)^2, \quad p \ll K^2 \frac{\varepsilon}{\tau}. \tag{86}$$

The first condition makes a longitudinal width of the packet for small and moderate times smaller than the distance where asymptotics works, see (53). The second and the third ones concern the large-time asymptotics, they originate from (67). The second condition ensures that the speed of increasing of the longitudinal width of the packet is smaller then the group speed. The third condition means that the angle of the cone is small. All of these conditions can be satisfied, for example, if we assume that $\Omega \sim p$, $\Delta_\parallel \sim 1/\sqrt{p}$. Then $K$ is of order of $O(p)$. If we take parameters in such a way that $K \gg \sqrt{p}$ as well, the restriction conditions (86) will be satisfied.

We expect that our results may be useful for prediction of waves propagation in media with dispersion. The obtained solution have non-zero angular momentum [33] which is studied intensively for the wave equation in context of manipulating of nanoparticles. The investigation of this momentum for the waves in dispersive media is a very appealing for the future research. Another possible application concerns two-dimensional solutions of KGFE. Such solution may be a base for further design of localized solutions of the Dirac equations which may find application in prediction and modelling of waves in epitaxial graphen.
Figure 3: Large times behaviour of absolute value of $u_p$, for consequent times $t = \tau, 2\tau, 3\tau$ as a function of $y$ and $z$ for $x = 0$ (upper pictures) and as function of $x$ and $y$ for $z = v_{gr} t$ (lower pictures). Convenient normalization for the $|u_p|$ is chosen, the coordinates and time are in the units of mass $m$. See text for the values of all parameters.

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A Asymptotic behaviour of KGF solutions

A.1 General properties of solutions of the KGF equation

We show here how the properties of solutions of the KGFE can be found within a general approach based on Fourier representation.

Any solution of the KGF equation can be written as a Fourier integral

$$ u(r, t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} d^n k d\omega \hat{u}(\omega, k) \exp\{i k r - i \omega t\} \delta(\omega - \omega(k)), $$

where $\omega(k) = \sqrt{k^2 + m^2}$, and $n = d + 1$. It is convenient to introduce new dimensionless variables $\chi = k/m$, $\varpi = \omega/m$. We assume that the modulus of the Fourier transform $\hat{u}(k)$ has a sharp maximum and $\hat{u}(k)$ can be written in the form

$$ \hat{u}(k) = a(\chi) \exp\{-p\Phi(\chi)\}, \quad p \gg 1, $$

where $\Phi(\chi) = \sqrt{\chi^2 + 1}$.
i.e. we assume that $\Phi$ has stationary point in the minimum of its real part. The formula (87) may be rewritten in the form suitable for the analysis by the method of the steepest descent now

$$u(r, t) = \frac{m^n}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} d^n x \; a(x) \exp\{-p\Psi(x)\},$$

(89)

where

$$\Psi(x) = \Phi(x) + i\tilde{\omega}(x - \chi v), \quad v = r/t, \quad \tilde{\omega} = mt/p, \quad \omega = \sqrt{\chi^2 + 1}. \quad (90)$$

The main term of asymptotics of the integral (89) for $p \to \infty$ reads

$$u(r, t) \approx \exp\{-p\Psi(x_*)\} a(x_*) \frac{m^n}{(2\pi)^n} \sqrt{\frac{(2\pi)^n}{\det \Psi_*}} \exp(1 + O(p^{-1})), \quad (91)$$

where $\det \Psi''_*$ is the determinant of the Jakobi matrix $\Psi''$ calculated in the saddle point $x_*$, i.e., the matrix of the second derivatives of $\Psi$ with respect to $\chi$. The saddle point $x_*$ should be found from the equation

$$\nabla \Psi(x_*) = \nabla \Phi(x_*) + i\tilde{\omega}(\nabla \chi_*(x_*) - v) = 0. \quad (92)$$

For small times (in comparison with $p/m$) we seek the saddle point as an expansion $x_* = x_*^{(0)} + i\tilde{t}x_*^{(1)} + \ldots$ and get

$$\nabla \Phi(x_*^{(0)}) = 0, \quad x_*^{(1)} = -\Phi''_0^{-1}(v_{gr} - v), \quad v_{gr} \equiv \nabla \chi_*(x_*^{(0)}), \quad (93)$$

where $\Phi''_0$ is the matrix of the second derivatives of $\Phi$ with respect to $\chi$ calculated in the point $x_*^{(0)}$. Corrections are to be taken into account in the formula (91) only in the exponential term containing the large parameter $p$

$$\Psi(x_*) \approx \Phi(x_*^{(0)}) + i\tilde{\omega}(x_*^{(0)} - x_*^{(0)} v) - \frac{\tilde{t}^2}{2} (\Phi''_0 x_*^{(1)}, x_*^{(1)}) - i\tilde{t}^2 ((v_{gr} - v), x_*^{(1)}). \quad (94)$$

Substituting (94) in the (91), neglecting the correction terms in the amplitude and recalling that $vt = r$ we obtain

$$u(r, t) \approx \mathcal{A}_1 \exp\{-i(\omega_0 t - k_0 \cdot r)\} \exp\left\{ \frac{m^2}{2p} ((r - v_{gr} t), (\Phi''_0)_{xt}^{-1}(r - v_{gr} t)) \right\} \mathcal{A}_1 = \frac{m^n}{2\pi (2\pi p)^{n/2}} \sqrt{\det \Phi''_0}, \quad k_0 = m\chi_*(0), \quad \omega_0 = m\tilde{\omega}(\chi_*(0)). \quad (95)$$

This formula can be applied to the exact solution $u_p$ presented in Section 3.2. Its Fourier image (83) can be given in the form (88) as follows

$$a(\chi) = \frac{\hat{C}_p}{m^{n+1+n/2}} \frac{1}{\omega(\omega + \chi_z)^{n+n/2}}, \quad (97)$$

$$\Phi(\chi) = i \frac{(X_1 \Gamma_{\gamma}^{-1} X_1)}{2\sqrt{\pi(\omega + \chi_z)}} + \frac{\sqrt{\gamma/\tau}}{2} (\omega + \chi_z) + \frac{\sqrt{\tau/\gamma}}{2(\omega + \chi_z)}, \quad p = m\sqrt{\tau\gamma}. \quad (98)$$

It is easy to check that formula (95) with account of (97) and (98) gives (46).
A.2 Large-time behaviour of the particle-like solution

Now we turn to the large time behaviour and assume that $\hat{u}(k)$ in (87) changes slowly as compared with the oscillatory term. Thus we are able to proceed with stationary phase method and obtain [32]

$$u(r, t) \sim \frac{\hat{u}(k_*)}{(2\pi)^{n/2}} \frac{1}{|\text{det}(\omega''_*)|^{1/2}} \exp \left\{-it(\omega(k_*)) - k_* \cdot \frac{m\pi}{4} \text{sgn}(t) \right\}. \quad (99)$$

Here $k_* = k_*(v)$ is the solution of the equation

$$\nabla \omega(k_*) = v \equiv r/t, \quad (100)$$

where $\omega(k_* = \sqrt{k_*^2 + m^2}$. It is easy to check that $k_* = m\chi$, $\omega(k_* = m\varpi$ where $\chi(v)$ and $\varpi(v)$ are given by (57). By $\omega''_* $ in (99) we denote the $n \times n$ matrix of second derivatives of $\omega$ with respect to components of $k$ calculated in the point $k_*(v)$. It is easy to check that

$$\det \omega''_* = \frac{1}{\omega_*^n} - \frac{k_*^2}{\omega_*^{n+2}} = \frac{m^2}{\omega_*^{n+2}} = \frac{(1 - v^2)^{n/2+1}}{m^n}. \quad (101)$$

Formula (99) demonstrates complicated dependence of $r$ and $t$ through $v$. It reads

$$u(r, t) \sim \frac{m^{n/2}}{(2\pi)^{n/2}} \frac{\exp \left\{-im\sqrt{t^2 - r^2} \right\}}{|t|^{n/2} (1 - v^2)^{(n+2)/4}} e^{-i \frac{2\pi}{4} \text{sgn}(t)} \hat{u} \left( \frac{mv}{\sqrt{1 - v^2}} \right). \quad (102)$$

After substitution of $\hat{u}$ from (83) with account of (57) we obtain the formula which is in agreement with previously found formula (64). It is important to note that the obtained formula cannot be applied when $m \to 0$. It is due to the fact that the second derivative of the phase function from (99) in this case tends to zero, thus the region of validity of the asymptotic (102) is approaching spacial infinity. The stationary phase method which we used is not applicable in this case.

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