Worldvolume Dynamics of D-branes in a D-brane Background

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ABSTRACT

We study the embedding of D(8-p)-branes in the background geometry of parallel Dp-branes for \( p \leq 6 \). The D(8-p)-brane is extended along the directions orthogonal to the Dp-branes of the background. The D(8-p)-brane configuration is determined by its Dirac-Born-Infeld plus Wess-Zumino-Witten action. We find a BPS condition which solves the equation of motion. The analytical solution of the BPS equation for the near-horizon and asymptotically flat geometries is given. The embeddings we obtain represent branes joined by tubes. By analyzing the energy of these tubes we conclude that they can be regarded as bundles of fundamental strings. Our solution provides an explicit realization of the Hanany-Witten effect. When \( p = 6 \) the solution of the BPS equation must be considered separately and, in general, the embeddings of the D2-branes do not admit the same interpretation as in the \( p < 6 \) case.

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1 Introduction

D-branes have been shown to play a fundamental role in the non-perturbative structure of string theory. Originally [1], they were introduced as hypersurfaces in space-time on which open strings with Dirichlet boundary conditions are allowed to end. The extended objects so defined are charged under the Ramond-Ramond (RR) sector of the theory and are essential to understand the duality structure of string theories. Moreover, the low energy physics of D-branes can be described by supersymmetric gauge theory and, actually, super Yang-Mills theory arises as a low energy description of parallel D-branes.

D-branes also appear as solutions of the supergravity effective action of string theory [2]. These solutions are extended generalizations of the Schwarzschild geometry which describe the gravitational field created by an object carrying Ramond-Ramond charge. This gravitational field acts on the dimensions orthogonal to the D-brane world volume which, from the point of view of the gauge theory, can be regarded as “extra dimensions”.

The interplay between gauge theory and gravity is on the basis of the Maldacena conjecture [3]. Indeed, Maldacena has argued that there is a remarkable duality between classical supergravity in the near-horizon region and the large $N$ 't Hooft limit of the $SU(N)$ Yang-Mills theory. In particular, the near-horizon geometry of parallel D3-branes is equivalent to the space $AdS_5 \times S^5$, whose boundary can be identified with four-dimensional Minkowsky space-time. In this case one gets the so-called $AdS/CFT$ correspondence between type IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ non-abelian super Yang-Mills theory in four dimensions [4, 5, 6]. Within this context, one can compute Wilson loop expectation values by considering a fundamental open string placed on the interior of the $AdS_5$ space and having its ends on the boundary [7]. This formalism has been used to extract quark potentials both in the supersymmetric and non-supersymmetric theory. Moreover, Witten [8] has proposed a way to incorporate baryons by means of a D5-brane wrapped on the $S^5$, from which fundamental strings that end on the D3-branes emanate [9, 10].

In ref. [11], the problem of a D5-brane moving under the influence of a D3-brane background was considered. The D5-brane dynamics is governed by an action which is the sum of a Dirac-Born-Infeld and a Wess-Zumino term. The equations of motion of the static D5-brane are solved if one assumes that the fields satisfy a certain first-order BPS differential equation that was found previously by Imamura [12]. This BPS condition can be integrated exactly in the near-horizon region of the background D3-brane geometry and the solutions are spikes of the D5-brane worldvolume which can be interpreted as bundles of fundamental strings ending on the D3-branes. Actually, these kinds of spikes are a general characteristic of the Dirac-Born-Infeld non-linear gauge theory and can be used to describe strings attached to branes [13, 14].

The BPS condition for the full asymptotically flat D3-brane metric was analyzed numerically in ref. [11]. As a result of this study one gets a precise description of the string creation process which takes place when two D-branes cross each other, i.e. of the so-called Hanany-Witten effect [15, 16]. These results were generalized in ref. [17] to the case of baryons in confining gauge theories. Moreover, in ref. [18] the BPS differential equation was obtained as a condition which must be fulfilled in order to saturate a lower bound on the energy.
In this paper we shall study the worldvolume dynamics of a D(8-p)-brane moving under the action of the gravitational and Ramond-Ramond fields of a stack of background Dp-branes for \( p \leq 6 \). The D(8-p)-brane is extended along the directions orthogonal to the worldvolume of the background Dp-branes. For \( p = 3 \) this system is the one analyzed in ref. [11]. By using the method of ref. [18], we will find a BPS first-order differential equation whose solutions also verify the equation of motion. Remarkably, we will be able to integrate analytically this BPS equation both in the near-horizon and asymptotically flat metrics. For \( p \leq 5 \) the solutions we will find are similar to the ones described in ref. [11], i.e. the D(8-p)-brane worldvolume has spikes which can be interpreted as flux tubes made up of fundamental strings. Our analytical solution will allow us to give a detailed description of the shape of these tubes, their energy and of the process of string creation from D-brane worldvolumes.

The organization of this paper is the following. In section 2 we formulate our problem. The BPS differential equation is obtained in section 3. In section 4 this BPS condition is integrated for \( p \leq 5 \) in the near-horizon region. The \( p = 6 \) system requires a special treatment, which is discussed in section 5. The solution of the BPS equation for the full Dp-brane geometry is obtained in section 6. The detailed analysis of this solution is performed in appendix A. Finally, in section 7, we summarize our results and point out some directions for future work.

## 2 Worldvolume Dynamics

The ten-dimensional metric (in the string frame) corresponding to a stack of \( N \) coincident extremal Dp-branes in type II superstring theory is [4]:

\[
ds^2 = \left[ H_p(r) \right]^{-\frac{2}{7}} (-dt^2 + dx_i^2) + \left[ H_p(r) \right]^{\frac{1}{4}} (dr^2 + r^2 d\Omega_{8-p}^2) .
\] (2.1)

In eq. (2.1) \( x_i \) are \( p \)-dimensional cartesian coordinates along the branes, \( r \) is a radial coordinate parametrizing the distance to the branes and \( d\Omega_{8-p}^2 \) is the line element of an unit \( 8-p \) sphere. The harmonic function \( H_p(r) \) appearing in eq. (2.1) is given by:

\[
H_p(r) = a + \left( \frac{R}{r} \right)^{7-p},
\] (2.2)

where \( a = 0, 1 \) and the ‘radius’ \( R \) can be written in terms of the number \( N \) of branes, the Regge slope \( \alpha' \) and the string coupling constant \( g_s \) as follows:

\[
R^{7-p} = N g_s 2^{5-p} \pi \frac{\Gamma (\frac{7-p}{2})}{\Gamma (\frac{7-p}{2})} \frac{\pi^{\frac{7-p}{2}}}{\Gamma (\frac{7-p}{2})}.
\] (2.3)

We have chosen our conventions in such a way that the tension of the fundamental string \( T_f \) is:

\[
T_f = \frac{1}{2\pi \alpha'},
\] (2.4)

The value \( a = 1 \) in eq. (2.2) corresponds to the full geometry of the stack of Dp-branes. Actually, in order to have an asymptotically flat space-time, we shall restrict ourselves to
the case \( p < 7 \). Taking \( a = 0 \) in the harmonic function \( H_p(r) \) is equivalent to approximate the Dp-brane geometry by the near-horizon metric of the ‘throat’ region of (2.1).

The Dp-brane solution we are considering is also characterized by a dilaton field \( \phi(r) \) and a Ramond-Ramond (RR) \( p + 2 \)-form field \( G \). The corresponding values are:

\[
\begin{align*}
e^{-\tilde{\phi}(r)} &= \left[ H_p(r) \right]^{\frac{a-3}{4}}, \\
G_{t, x_1^\parallel, \ldots, x_p^\parallel, r} &= \frac{d}{dr} \left[ H_p(r) \right]^{-1},
\end{align*}
\]  

(2.5)

where \( \tilde{\phi}(r) = \phi(r) - \phi_\infty \) and \( \phi_\infty \) is the value of the dilaton field at infinite distance of the Dp-branes (i.e. \( \phi_\infty = \lim_{r \to \infty} \phi(r) \)). As was pointed out in ref. [3], in order to trust this type II supergravity solution, both the curvature in string units and the dilaton must be small. This fact introduces restrictions in the values of the radial coordinates for which the correspondence between the supergravity and gauge theory descriptions is valid (see also ref. [19]).

Let us now consider a D(8-p)-brane embedded in the transverse directions of the stack of the background Dp-branes. The dynamics of this D(8-p)-brane is determined by its action, which is a sum of a Dirac-Born-Infeld and a Wess-Zumino term:

\[
S = -T_{8-p} \int d^{9-p}\xi e^{-\tilde{\phi}} \sqrt{-\det (g + F)} + T_{8-p} \int d^{9-p}\xi A \wedge *G,
\]

(2.6)

where \( g \) is the induced metric on the worldvolume of the D(8-p)-brane, \( A \) is a worldvolume abelian gauge field and \( F = dA \) its field strength. In eq. (2.6) \( *G \) is a (8-p)-form, which is the pullback of the Hodge dual of the background RR (p+2)-form \( G \). Notice that this RR field acts as a source for the worldvolume gauge field \( A \). The coefficient \( T_{8-p} \) appearing in the action (2.6) is the tension of a D(8-p)-brane, which is given by [1]:

\[
T_{8-p} = (2\pi)^{p-8} (\alpha')^{\frac{p-2}{2}} (g_s)^{-1}.
\]

(2.7)

Let \( \theta^1, \theta^2, \ldots, \theta^{8-p} \) be coordinates which parametrize the \( S^{8-p} \) transverse sphere. The worldvolume coordinates \( \xi^\alpha (\alpha = 0, \ldots, 8-p) \) will be taken as:

\[
\xi^\alpha = (t, \theta^1, \ldots, \theta^{8-p}).
\]

(2.8)

We shall assume that the \( \theta 's \) are spherical angles on \( S^{8-p} \) and that \( \theta \equiv \theta^{8-p} \) is the polar angle \( (0 \leq \theta \leq \pi) \). Therefore, the \( S^{8-p} \) line element \( d\Omega^2_{8-p} \) can be decomposed as:

\[
d\Omega^2_{8-p} = d\theta^2 + (\sin \theta)^2 d\Omega^2_{7-p},
\]

(2.9)

where \( d\Omega^2_{7-p} \), which only depends on \( \theta^1, \ldots, \theta^{7-p} \), is the line element of a \( S^{7-p} \) sphere. We are going to consider static configurations of the D(8-p)-brane which only depend on the polar angle \( \theta \). Actually, we shall restrict ourselves to the case in which the only non-vanishing fields are \( r(\theta), x(\theta) \) and \( A_0(\theta), x \) being a direction parallel to the background Dp-branes. It
is straightforward to verify that the action for such configurations can be written as:

\[
S = TT_{8-p} \Omega_{7-p} \int d\theta \ (\sin \theta)^{7-p} \times \\
\times \left[ -r^{7-p} H_p(r) \sqrt{r^2 + r'^2 + \frac{x'^2}{H_p(r)}} - F_{0,\theta}^2 + (-1)^{p+1} (7 - p) A_0 R^{7-p} \right],
\]

(2.10)

where the prime denotes derivative with respect to \( \theta \) and \( T = \int dt \). In eq. (2.10) \( \Omega_{7-p} \) is the volume of the unit \((7-p)\)-sphere, given by:

\[
\Omega_{7-p} = \frac{2\pi^{\frac{8-p}{2}}}{\Gamma\left(\frac{8-p}{2}\right)}.
\]

(2.11)

It is interesting at this point to remember that we are considering static solutions of the gauge field that only depend on \( \theta \). The electric field for these configurations is:

\[
E = F_{0,\theta} = -\partial_\theta A_0.
\]

(2.12)

Let us now define the displacement field \( D_p \) as:

\[
D_p(\theta) \equiv \frac{(-1)^{p+1}}{TT_{8-p} \Omega_{7-p}} \frac{\partial S}{\partial E} = (-1)^{p+1} \frac{(\sin \theta)^{7-p} r^{7-p} H_p(r) E}{\sqrt{r^2 + r'^2 + \frac{x'^2}{H_p(r)}} - E^2}.
\]

(2.13)

The extra factors appearing in the definition (2.13) have been included for convenience. The Euler-Lagrange equation for \( A_0 \) implies an equation for \( D_p \) which is easy to find, namely:

\[
\frac{d}{d\theta} D_p(\theta) = -(7 - p) R^{7-p} (\sin \theta)^{7-p}.
\]

(2.14)

Notice that the right-hand side of eq. (2.14) only depends on \( \theta \). Therefore (2.14) can be integrated and, as a result, one can obtain \( D_p \) as a function of \( \theta \). We shall work out the explicit form of \( D_p(\theta) \) at the end of this section. At present we only need to use the fact that \( D_p(\theta) \) satisfies eq. (2.14). Actually, substituting \((7 - p) R^{7-p} (\sin \theta)^{7-p}\) by \(-\partial_\theta D_p\) in the Wess-Zumino term of the action and integrating by parts, one can recast \( S \) as:

\[
S = -TU,
\]

(2.15)

with \( U \) given by:

\[
U = T_{8-p} \Omega_{7-p} \int d\theta \left[ (-1)^{p+1} E D_p(\theta) + \\
+ (\sin \theta)^{7-p} r^{7-p} H_p(r) \sqrt{r^2 + r'^2 + \frac{x'^2}{H_p(r)}} - E^2 \right].
\]

(2.16)
Notice that, as is evident from their relation (2.13), $S$ and $U$ give rise to the same Euler-Lagrange equation. Actually, since we have eliminated $A_0$ in favor of $D_p$, we can regard $U$ as the Legendre transform of $S$ and, therefore, $U$ can be considered as an energy functional for the embedding of the D(8-p)-brane in the Dp-brane background. The fields $E$ and $D_p$ are related, as is obvious from eq. (2.13). It is not difficult to invert eq. (2.13) and get $E$ in terms of $D_p$. The result is:

$$E = (-1)^{p+1} \sqrt{\frac{r^2 + r' r^2 + \frac{x' r^2}{H_p(r)}}{(D_p(\theta))^2 + [(\sin \theta)^{7-p} r^{7-p} H_p(r)]^2}} D_p(\theta).$$

(2.17)

Using this relation we can eliminate $E$ from the expression of $U$:

$$U = T_{8-p} \Omega_{7-p} \int d\theta \sqrt{r^2 + r' r^2 + \frac{x' r^2}{H_p(r)}} \sqrt{(D_p(\theta))^2 + [(\sin \theta)^{7-p} r^{7-p} H_p(r)]^2}.$$

(2.18)

Recall that $D_p(\theta)$ is a known function of $\theta$ (see below). Therefore eq. (2.18) gives the energy functional $U$ in terms of $x(\theta)$ and $r(\theta)$. These functions must be solutions of the Euler-Lagrange equations obtained from $U$. From the study of the functional $U$ in several situations we will determine the shape of the D(8-p)-brane embedding in the background geometry. Let us consider, first of all, the near-horizon approximation, which is equivalent to taking $a = 0$ in the harmonic function (2.2). In this case $r^{7-p} H_p(r) = R^{7-p}$ and $U$ is given by:

$$U = T_{8-p} \Omega_{7-p} \int d\theta \sqrt{r^2 + r' r^2 + \frac{r^{7-p}}{R^{7-p}}} \sqrt{(D_p(\theta))^2 + [(\sin \theta)^{2(7-p)} R^{2(7-p)}]}.$$

(2.19)

By inspecting eq. (2.19) one easily concludes that $U$ transforms homogeneously under a simultaneous rescaling of the radial and parallel coordinates of the form:

$$(r, x) \rightarrow (\alpha r, \alpha \frac{p-5}{2} x),$$

(2.20)

where $\alpha$ is a constant. It follows that the equations of motion derived from $U$ are invariant under the transformation (2.20). Moreover, the homogeneous character of $U$ under the transformation (2.20) implies the following scaling law:

$$x \sim \frac{R^{7-p}}{r^{\frac{5-p}{2}}},$$

(2.21)

where the power of $R$ has been determined by imposing dimensional homogeneity of both sides of (2.21). Eq. (2.21) is precisely the holographic UV/IR relation found in ref. [20]. According to eq. (2.21), for $p < 5$, large radial distances ($r \rightarrow \infty$) correspond to small values of the parallel coordinate $x$. For $p = 5$, the distance $x$ is insensitive to changes in $r$, whereas, for $p = 6$, $x$ increases when $r$ grows. The consequences of this relation for the correspondence between field theories and near-horizon supergravities have been discussed in ref. [20] (see also ref. [21]). In our approach we shall see that, indeed, the $p = 6$ case is
special and the embedding of the D2-brane in the D6-brane background geometry has new characteristics, which must be studied separately.

Let us finish this section by giving the expressions of the displacement fields $D_p$. The $\theta$-dependence of $D_p$ can be obtained by integrating the right-hand side of eq. (2.14). The results one gets for the different cases are:

\begin{align*}
D_0(\theta) &= R^7 \left[ \cos \theta \left( \sin^6 \theta + \frac{6}{5} \sin^4 \theta + \frac{8}{5} \sin^2 \theta + \frac{16}{5} \right) + \frac{16}{5} (2\nu - 1) \right], \\
D_1(\theta) &= R^6 \left[ \cos \theta \left( \sin^5 \theta + \frac{5}{4} \sin^3 \theta + \frac{15}{8} \sin \theta \right) + \frac{15}{8} (\pi \nu - \theta) \right], \\
D_2(\theta) &= R^5 \left[ \cos \theta \left( \sin^4 \theta + \frac{4}{3} \sin^2 \theta + \frac{8}{3} \right) + \frac{8}{3} (2\nu - 1) \right], \\
D_3(\theta) &= R^4 \left[ \cos \theta \left( \sin^3 \theta + \frac{3}{2} \sin \theta \right) + \frac{3}{2} (\pi \nu - \theta) \right], \\
D_4(\theta) &= R^3 \left[ \cos \theta \left( \sin^2 \theta + 2 \right) + 2 (2\nu - 1) \right], \\
D_5(\theta) &= R^2 \left[ \cos \theta \sin \theta + \pi \nu - \theta \right], \\
D_6(\theta) &= R \left[ \cos \theta + 2\nu - 1 \right],
\end{align*}

\[ (2.22) \]

where we have parametrized the additive constant of integration by means of a parameter $\nu$. We have chosen the $\theta$-independent term in $D_p(\theta)$ in such a way that the value of the displacement field at $\theta = \pi$ is given by:

\[ D_p(\pi) = -2\sqrt{\pi} \frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R^{7-p} (1 - \nu) . \]

\[ (2.23) \]

Eq. (2.23) allows to give a precise meaning to the parameter $\nu$ [11]. In fact, as we shall verify below, there exist solutions for which $r \to \infty$ and $x' \to 0$ when $\theta \to \pi$ in a way that simulates a ‘flux tube’ attached to the D(8-p)-brane. The ‘tension’ (i.e. the energy per unit radial distance) for one of these spikes can be obtained from the expression of $U$ in eq. (2.18). Actually, as $\theta \to \pi$ one can check that the terms with $r'$ dominate over the other terms in the first square root in eq. (2.18), whereas the second square root can be approximated by $|D_p(\pi)|$. As a result [11], the tension of the spike is $T_{8-p} \Omega_{7-p} |D_p(\pi)|$. Using the value of $D_p(\pi)$ given in eq. (2.23) and the fact that (see eqs. (2.3), (2.7) and (2.11)):

\[ T_{8-p} \Omega_{7-p} R^{7-p} = \frac{NT_f}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{7-p}{2}\right)}{\Gamma\left(\frac{8-p}{2}\right)} , \]

\[ (2.24) \]

we conclude that the tension of the $\theta = \pi$ spike is $(1 - \nu) N T_f$, where $T_f$ is the tension of the fundamental string written in eq. (2.4). This result implies that we can interpret the
\[ \theta = \pi \] tube as a bundle of \((1 - \nu)N\) fundamental strings which, from the gauge theory point of view, corresponds to a baryon formed by \((1 - \nu)N\) quarks. It is clear that, although in the classical theory \(\nu\) is a continuous parameter, upon quantization \(\nu\) should be a multiple of \(1/N\) taking values in the range \(0 \leq \nu \leq 1\).

Let us finally point out that there exist other solutions for which \(r \to \infty\) and \(x' \to 0\) when \(\theta \to 0\). The asymptotic tension for these solutions is \(T_{8-p} \Omega_{7-p} \mid D_p(0) \mid\). The value of \(D_p(\theta)\) at \(\theta = 0\) can be obtained from the values given in eq. (2.22). The result is:

\[ D_p(0) = 2\sqrt{\pi} \frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} R^{7-p} \nu. \tag{2.25} \]

Using again eq. (2.24), one can verify that \(T_{8-p} \Omega_{7-p} \mid D_p(0) \mid = \nu N T_f\), which corresponds to a bundle of \(\nu N\) fundamental strings. This fact provides an interpretation of \(\nu\) for this second class of solutions.

### 3 BPS conditions

In the remaining of this paper we are going to study solutions of the equations of motion of the brane probes which are not extended in the directions parallel to the background branes. This amounts to take \(x' = 0\) (i.e. \(x(\theta) = \text{constant}\)) in our previous equations. From the expression (2.18) of \(U\) we can get the differential equation which determines \(r\) as a function of \(\theta\). Indeed, the Euler-Lagrange equation derived from \(U\) is:

\[
\frac{d}{d\theta} \left[ \frac{r'}{\sqrt{r'^2 + r^{7-p}}} \sqrt{(D_p(\theta))^2 + \left[ (\sin \theta)^{7-p} r^{7-p} H_p(r) \right]^2} \right] = \\
= \frac{r}{\sqrt{r'^2 + r^{7-p}}} \sqrt{(D_p(\theta))^2 + \left[ (\sin \theta)^{7-p} r^{7-p} H_p(r) \right]^2} + \\
+ (7-p) a \frac{\sqrt{r'^2 + r^{7-p}}}{r} \frac{\left( r \sin \theta \right)^{2(7-p)} H_p(r)}{\sqrt{(D_p(\theta))^2 + \left[ (\sin \theta)^{7-p} r^{7-p} H_p(r) \right]^2}}.
\tag{3.1}
\]

Trying to obtain a solution for such a complicate second-order differential equation seems, a priori, hopeless. A possible strategy to solve eq. (3.1) consists of finding a first integral for this system. Notice that the integrand of \(U\) depends explicitly on \(\theta\) (see eq. (2.18)) and, therefore, there is no first integral associated to the invariance under shifts of \(\theta\) by a constant. Nevertheless, we will be able to find a first-order equation such that any function \(r(\theta)\) satisfying it is a solution of the equation (3.1). This first-order condition is much simpler than the Euler-Lagrange equation (3.1) and, indeed, we will be able to solve it analytically,
both in the near-horizon \((a = 0)\) and asymptotically flat \((a = 1)\) geometries. For \(p = 3\) and \(a = 0\), the first order equation was found by Imamura \[12\] as a BPS \[22\] condition for the implementation of supersymmetry in the worldvolume theory of a D5-brane propagating in a D3 background. This condition was subsequently extended to \((p = 3, \ a = 1)\) and \((p = 4, a = 0)\) in refs. \[11\] and \[17\] respectively. In order to generalize these results, we shall follow here the approach of ref. \[18\] (see also ref. \[23\]), where the BPS equation was found by requiring the saturation of a certain bound on the energy functional.

Following ref. \[18\], let us define the quantity:

\[
\Delta_p(r, \theta) \equiv r^{7-p} H_p(r) (\sin \theta)^{7-p}.
\]  

(3.2)

In terms of \(\Delta_p(r, \theta)\), we define the function \(f_p(r, \theta)\) as follows:

\[
f_p(r, \theta) \equiv \frac{\Delta_p(r, \theta) \sin \theta + D_p(\theta) \cos \theta}{\Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta}.
\]  

(3.3)

Making use of the function \(f_p\) it is easy to verify that the energy can be put in terms of a square root of a sum of squares. Indeed, it is a simple exercise to demonstrate that \(U\) can be written as:

\[
U = T_{8-p} \Omega_{7-p} \int d\theta \times \sqrt{Z^2 + r^2 \left( \Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta \right)^2 \left( \frac{r'}{r} - f_p(r, \theta) \right)^2},
\]  

(3.4)

where \(Z\) is given by:

\[
Z = r \left( \Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta \right) \left[ 1 + \frac{r'}{r} f_p(r, \theta) \right].
\]  

(3.5)

In view of eq. (3.4), it is clear that the energy of the D(8-p)-brane is bounded as:

\[
U \geq T_{8-p} \Omega_{7-p} \int d\theta \left| Z \right|.
\]  

(3.6)

This bound is saturated when \(r(\theta)\) satisfies the following first-order differential equation:

\[
\frac{r'}{r} = f_p(r, \theta).
\]  

(3.7)

It is straightforward to prove that any function \(r(\theta)\) that satisfies eq. (3.7) is also a solution of the Euler-Lagrange equation (3.1). Notice that the condition (3.7) involves the displacement field \(D_p(\theta)\) (see eq. (3.3)). The expressions of the \(D_p\)'s have been given at the end of section 2 (eq. (2.22)). However, in order to demonstrate that the solutions of eq. (3.7) verify the equation of motion (3.1), we do not need to use the explicit form of \(D_p(\theta)\). The only property required for this proof is the value of the derivative of \(D_p(\theta)\), given in eq. (2.14). Moreover, as pointed out in ref. \[18\] for the \(p = 3\) case, \(Z\) can be written as a total derivative:
\[ Z = \frac{d}{d\theta} \left[ D_p(\theta) r \cos \theta + \left( \frac{a}{8-p} + \frac{R_7^{7-p}}{r^{7-p}} \right) (r \sin \theta)^{8-p} \right]. \] (3.8)

It is important to stress the fact that eq. (3.8) can be proved without using the condition (3.7) or the Euler-Lagrange equation (3.1) (only eq. (2.14) has to be used). Eq. (3.8) implies that the bound (3.6) does not depend on the detailed form of the function \( r(\theta) \). Actually, as a consequence of eq. (3.8), only the boundary values of \( r(\theta) \) matter when one evaluates the right-hand side of eq. (3.6). The functions \( r(\theta) \) which solve eq. (3.7) correspond to those D(8-p)-brane embeddings which, for given boundary conditions, have minimal energy. Due to this fact we shall refer to eq. (3.7) as the BPS condition. The search for the solutions of the BPS equation and their interpretation will be the subject of the next three sections.

4 The near-horizon solution for \( p \leq 5 \)

In this section we are going to solve the BPS first-order differential equation in the near-horizon geometry for \( p \leq 5 \). It will become clear in the process of finding the solution that the \( p = 6 \) case is singled out (as expected from the holographic relation (2.21)). This \( p = 6 \) case will be discussed separately in section 5.

Let us start by defining the function \( \Lambda_p(\theta) \) by means of the equation:

\[ \Lambda_p(\theta) \equiv \frac{1}{R^{7-p}} \left[ R_7^{7-p} (\sin \theta)^{6-p} \cos \theta - D_p(\theta) \right]. \] (4.1)

The right-hand side of eq. (4.1) is known and, thus, \( \Lambda_p(\theta) \) is a known function of \( \theta \) whose explicit expression can be obtained by substituting the values of \( D_p(\theta) \), given in eq. (2.22), in (4.1). Moreover, the BPS condition (3.7) for \( a = 0 \) can be written in terms of \( \Lambda_p(\theta) \) as:

\[ \frac{r'}{r} = \frac{(\sin \theta)^{6-p} - \Lambda_p(\theta) \cos \theta}{\Lambda_p(\theta) \sin \theta}. \] (4.2)

A key point in what follows is that \( \Lambda_p(\theta) \) has a simple derivative, which can be easily obtained from eq. (2.14), namely:

\[ \frac{d}{d\theta} \Lambda_p(\theta) = (6 - p) (\sin \theta)^{5-p}. \] (4.3)

By inspecting the right-hand side of this equation we observe that \( p = 6 \) is a special case. Indeed, when \( p \neq 6 \), we can represent the \((\sin \theta)^{6-p}\) term appearing in the BPS condition as:

\[ (\sin \theta)^{6-p} = \frac{\sin \theta}{6-p} \frac{d}{d\theta} \Lambda_p(\theta), \quad (p \neq 6). \] (4.4)

After doing this, the right-hand side of eq. (4.2) is immediately recognized as a logarithmic derivative and, as a consequence, the BPS condition can be readily integrated. The result one arrives at is:
Figure 1: The functions $\Lambda_p(\theta)$ and $D_p(\theta)$ for $0 \leq \theta \leq \pi$. For illustrative purposes we have plotted these functions for $p = 4$ and $\nu = 1/4$.

$$r(\theta) = C \left[ \frac{\Lambda_p(\theta)}{\sin \theta} \right]^{\frac{1}{p-1}} ,$$

(4.5)

where $C$ is a positive constant. For $p < 5$ there is a fractional power of $\Lambda_p(\theta)$ in the right-hand side of eq. (1.5) and, thus, the solution we have found only makes sense for those values of $\theta$ such that $\Lambda_p(\theta) \geq 0$. Moreover, it is clear from eq. (4.3) that:

$$\frac{d\Lambda_p}{d\theta} \geq 0 , \quad (p < 6, \ 0 \leq \theta \leq \pi) ,$$

(4.6)

and, therefore, $\Lambda_p(\theta)$ is a monotonically increasing function in the interval $0 \leq \theta \leq \pi$. The values of $\Lambda_p(\theta)$ at $\theta = 0, \pi$ are:

$$\Lambda_p(0) = -\frac{D_p(0)}{R^{2-p}} = -2\sqrt{\pi} \frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} \nu \leq 0 ,$$

$$\Lambda_p(\pi) = -\frac{D_p(\pi)}{R^{2-p}} = 2\sqrt{\pi} \frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} (1 - \nu) \geq 0 ,$$

(4.7)

where, in order to establish the last inequalities, we have used the fact that $0 \leq \nu \leq 1$. From this discussion it follows that there exists a unique value $\theta_0$ of the polar angle such that:

$$\Lambda_p(\theta_0) = 0 .$$

(4.8)

The form of the function $\Lambda_p(\theta)$ has been displayed in figure 1. It is clear that the solution (1.5) is valid for $\theta_0 \leq \theta \leq \pi$. Notice that $\theta_0$ depends on $\nu$ and, actually, is a monotonically increasing function of $\nu$. In fact, from eq. (4.7) one concludes that $\theta_0 = 0$ for $\nu = 0$, whereas $\theta_0 = \pi$ for $\nu = 1$.

For $p = 5$ the function appearing in the right-hand side of eq. (4.3) is:

$$\Lambda_5(\theta) = \theta - \pi \nu .$$

(4.9)
Thus, our solution in this case is:

\[ r(\theta) = C \frac{\theta - \pi \nu}{\sin \theta}, \quad (p = 5) \]  \hspace{1cm} (4.10)

As the radial coordinate \( r \) must be non-negative, from eq. (4.10) one immediately concludes that, also in this \( p = 5 \) case, \( \theta_0 \leq \theta \leq \pi \), where now \( \theta_0 \) depends linearly on \( \nu \), namely:

\[ \theta_0 = \pi \nu, \quad (p = 5) \]  \hspace{1cm} (4.11)

The solution we have found coincides with the one obtained in refs. \[11, 17\] for \( p = 3, 4 \). Our result (4.5) generalizes these solutions for any \( p < 6 \) (the solution for \( p = 6 \) will be given in the next section). By inspecting eq. (4.5) it is easy to conclude that, for \( \nu \neq 1 \), \( r(\theta) \) diverges when \( \theta \to \pi \). Actually, when \( \theta \approx \pi \) and \( \nu \neq 1 \), \( r(\theta) \) behaves as:

\[ r(\theta) \approx C \left[ \Lambda_p(\pi) \right]^{6-p}, \quad (\nu \neq 1), \]  \hspace{1cm} (4.12)

which corresponds to a “tube” of radius \( C \left[ \Lambda_p(\pi) \right]^{6-p} \). We will check below that the energy of these tubes corresponds to a bundle of \( (1 - \nu)N \) fundamental strings emanating from the D(8-p)-brane. When \( \nu \neq 0 \) the critical angle \( \theta_0 \) is greater than zero and, as a consequence, eq. (4.3) gives \( r(\theta_0) = 0 \). Actually, when \( \nu \neq 0 \), the solution has the shape represented in figure 2a. On the contrary, when \( \nu = 0 \), the function \( r(\theta) \) takes a non-vanishing value at the critical value \( \theta_0 = 0 \). Notice that, in this last case, both numerator and denominator on the right-hand side of eq. (4.5) vanish. The behaviour of \( \Lambda_p(\theta) \) near \( \theta = 0 \) can be obtained by a Taylor expansion. The successive derivatives of \( \Lambda_p(\theta) \) can be easily computed from the value of its first derivative, given in eq. (4.3). It is easy to prove that the first non-vanishing derivative of \( \Lambda_p(\theta) \) at \( \theta = 0 \) is:

\[ \left. \frac{d^{6-p}}{d\theta^{6-p}} \Lambda_p(\theta) \right|_{\theta=0} = (6-p)! \]  \hspace{1cm} (4.13)

Moreover, the \( (7-p) \)th derivative of \( \Lambda_p(\theta) \) at \( \theta = 0 \) vanishes, i.e.:

\[ \left. \frac{d^{7-p}}{d\theta^{7-p}} \Lambda_p(\theta) \right|_{\theta=0} = 0 \]  \hspace{1cm} (4.14)

Therefore, the expression of \( \Lambda_p(\theta) \) for values of \( \theta \) close to zero takes the form:

\[ \Lambda_p(\theta) \approx \Lambda_p(0) + \theta^{6-p} + c_p \theta^{8-p} + \cdots, \]  \hspace{1cm} (4.15)

where \( c_p \) is a non-vanishing coefficient. Substituting this expansion in the right-hand side of eq. (4.5) for \( \nu = 0 \) leads to the conclusion that:

\[ r(0) = C, \quad (\nu = 0), \]  \hspace{1cm} (4.16)

i.e. \( r(0) \neq 0 \) for \( \nu = 0 \), as claimed above. Moreover, \( r'(0) \) vanishes in this case:

\[ r'(0) = 0, \quad (\nu = 0). \]  \hspace{1cm} (4.17)
Figure 2: Plot of the near-horizon solution (4.5) for $(p = 3, \nu = 1/4)$ (a) and $(p = 3, \nu = 0)$ (b). The discontinuous line represents the polar axis ($\theta = \pi$ at the top). The dot corresponds to the origin $r = 0$. The background branes are located at the origin and extend in the directions orthogonal to the plane of the figure.

The shape of the solution for $\nu = 0$ has been plotted in figure 2b. Notice that the $\nu = 0$ tube corresponds to the ordinary baryon with $N$ quarks.

The solution of the near-horizon BPS condition written in eq. (4.5) is not the only one. Following ref. [11], we can construct a new solution, valid for values of $\theta$ in the range $0 \leq \theta \leq \theta_0$, in which $\Lambda_p(\theta) \leq 0$, as follows:

$$\tilde{r}(\theta) = C \left[ -\Lambda_p(\theta) \right]^{\frac{1}{6-p}} \frac{\sin \theta}{\sin \theta_0}.$$  \hspace{1cm} (4.18)

Eq. (4.18) describes a spike at $\theta = 0$. We shall refer to the solution (4.18) as a “lower tube” solution, in contrast to the “upper tube” solution of eq. (4.5). Actually, these two types of solutions are related. In order to verify this fact, let us point out that the functions $D_p(\theta)$ and $\Lambda_p(\theta)$ change their sign under the transformation $\theta \rightarrow \pi - \theta$, $\nu \rightarrow 1 - \nu$:

$$D_p(\theta; \nu) = -D_p(\pi - \theta; 1 - \nu),$$

$$\Lambda_p(\theta; \nu) = -\Lambda_p(\pi - \theta; 1 - \nu).$$  \hspace{1cm} (4.19)

and, therefore, the solutions (4.5) and (4.18) are simply related, namely:

$$\tilde{r}(\theta; \nu) = r(\pi - \theta; 1 - \nu).$$  \hspace{1cm} (4.20)

It follows from this relation that the lower flux tubes correspond to $\nu N$ quarks.
Let us now evaluate, following ref. [18], the energy for the two types of near-horizon solutions we have found. These solutions saturate the bound (3.6) and, thus, the energy is precisely the right-hand side of this equation. Moreover, when \( r \) is a solution of eq. \((3.7)\), \( Z \) can be written as:

\[
Z = r \left[ \Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta \right] \left[ 1 + \left( \frac{r'}{r} \right)^2 \right].
\]  

(4.21)

It is obvious from this equation that, for our solutions, the sign of \( Z \) is just the sign of \( \Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta \). Moreover, using the definitions of \( \Delta_p \) (eq. (3.2)) and \( \Lambda_p \) (eq. (4.1)), one can prove that:

\[
\Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta = a r^{7-p} (\sin \theta)^{7-p} \cos \theta + R^{7-p} \Lambda_p(\theta) \sin \theta,
\]  

(4.22)

where we have used the general harmonic function \((2.2)\). In the near-horizon case \((a = 0)\) the first term on the right-hand side of eq. (4.22) is absent and, therefore, the sign of \( \Delta_p(r, \theta) \cos \theta - D_p(\theta) \sin \theta \) is just the sign of \( \Lambda_p(\theta) \). In the solutions (4.5) and (4.18), the angle \( \theta \) can take values in a range such that \( \Lambda_p(\theta) \) has a well defined sign. Therefore, we can write:

\[
\int d\theta \left| \frac{\partial Z}{\partial \theta} \right| = \int d\theta \left| Z \right|.
\]  

(4.23)

If we now define \( Z \) as:

\[
Z \equiv T_{8-p} \Omega_{7-p} \int d\theta \left| Z \right|
\]  

(4.24)

it is immediate that the energy of our near-horizon BPS solutions is:

\[
U_{BPS} = \left| Z \right|.
\]  

(4.25)

In order to evaluate \( Z \) for the near-horizon solutions, we make use of the representation \((3.8)\) of \( Z \) as a total derivative. Recall that \( \theta \) varies in the range \( \theta_i \leq \theta \leq \theta_f \) where \( \theta_i = \theta_0 \) \((\theta_i = 0)\) and \( \theta_f = \pi \) \((\theta_f = \theta_0)\) for the upper (lower) tube solution. It follows that \( Z \) can be written as the sum \([1]\):

\[
Z = Z_s + Z_{gs},
\]  

(4.26)

where \( Z_s \) and \( Z_{gs} \) are given by:

\[
Z_s = T_{8-p} \Omega_{7-p} D_p(\theta) r(\theta) \cos \theta \bigg|_{\theta_i}^{\theta_f},
\]

\[
Z_{gs} = T_{8-p} \Omega_{7-p} R^{7-p} r(\theta) (\sin \theta)^{8-p} \bigg|_{\theta_i}^{\theta_f}.
\]  

(4.27)

It is clear that \( Z \) only depends on the values of \( r(\theta) \) at the boundaries \( \theta = \theta_i \) and \( \theta = \theta_f \). In this sense we can regard \( Z \) as a topological quantity which, for fixed boundary conditions, is invariant under local variations of the fields \([18]\).
Let us now compute $Z_{gs}$. After substituting the solutions (4.5) and (4.18) of the BPS equations, we get:

$$Z_{gs} = T_{8-p} \Omega_{7-p} R^{7-p} C (\sin \theta)^{7-p} \left[ \pm \Lambda_p(\theta) \right]^{1/p}_{\theta_i} \cos \theta f_{\theta_i}, \quad (4.28)$$

where the $+$ ($-$) sign corresponds to the upper (lower) tube solution. The angles $\theta_i$ and $\theta_f$ can take the values 0, $\pi$, and $\theta_0$. For $\theta = 0, \pi$ the right-hand side of eq. (1.28) vanishes due to the $(\sin \theta)^{7-p}$ factor, whereas $\theta = \theta_0$ gives also a vanishing contribution because $\Lambda_p(\theta_0) = 0$ (see eq. (4.8)). In conclusion, we can write:

$$Z_{gs} = 0. \quad (4.29)$$

In the same way one can obtain the values of $Z_s$. One can check in this case that the contribution of $\theta = \theta_0$ to the right-hand side of the first equation in (4.21) vanishes and, as a consequence, only $\theta = 0, \pi$ contribute. After using eqs. (2.23)-(2.25), one gets the result:

$$Z_s = \begin{cases} (1 - \nu) NT_f L, & \text{(upper tube)} \\ -\nu NT_f L, & \text{(lower tube)} \end{cases} \quad (4.30)$$

where $L = r(\pi)$ ($L = r(0)$) for the upper (lower) tube solution. It is now evident that $U_{BP S} = |Z_s|$ is equal to the energy of a bundle of $(1 - \nu)N$ or $\nu N$ fundamental strings. Notice that this result is the same as the one found at the end of section 2 for the energy of the $\theta = 0$ and $\theta = \pi$ spikes. Let us finally mention that in ref. [18] it has been argued in favor of interpreting $Z_s$ as a central charge in the worldvolume supersymmetry algebra.

### 5 The near-horizon D6-D2 system

In this section we are going to integrate the BPS condition for $p = 6$. Notice that, according to the definition (4.1) and the expression of $D_6(\theta)$ given in eq. (2.23), $\Lambda_6(\theta)$ is constant, i.e.:

$$\Lambda_6(\theta) = 1 - 2\nu. \quad (5.1)$$

For $p = 6$ we cannot use eq. (4.4) and, hence, we have to deal directly with eq. (4.2). Actually, eq. (4.2) makes sense only for $\nu \neq 1/2$ since $\Lambda_6(\theta)$, and therefore the denominator of eq. (4.2) vanishes identically for $\nu = \frac{1}{2}$. With this restriction, the $p = 6$ BPS condition is:

$$\frac{r'}{r} = \frac{1}{1 - 2\nu} \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta}. \quad (5.2)$$

It is not difficult to realize that the right-hand side of eq. (5.2) can be written as a total derivative, namely:

---

2 The fact that the $p = 6, \nu = 1/2$ BPS condition is ill-defined is an artifact of the near-horizon approximation.
\[
\frac{r'}{r} = \frac{d}{d\theta} \left[ \log \left( \frac{\tan \frac{\theta}{2}}{\sin \theta} \right)^{\frac{1}{1-2\nu}} \right].
\] (5.3)

Therefore, the BPS equation can be immediately integrated. It is convenient to put the result in the form:

\[
r(\theta) = A \left[ \frac{\sin \frac{\theta}{2}^{\frac{2\nu}{1-2\nu}}}{\cos \frac{\theta}{2}^{\frac{2(1-\nu)}{1-2\nu}}} \right],
\] (5.4)

where \( A \) is a positive constant. The nature of the solution (5.4) depends critically on the value of \( \nu \). First of all, the range of values of \( \theta \) is not restricted, i.e. \( 0 \leq \theta \leq \pi \). If \( 0 \leq \nu < 1/2 \), the \( \cos (\theta/2) \) term present in the denominator of (5.4) makes \( r \to \infty \) at \( \theta \approx \pi \), whereas for \( 1/2 < \nu \leq 1 \) the solution diverges at \( \theta \approx 0 \). Actually, the solution for \( 0 \leq \nu < 1/2 \) is related to the one for \( 1/2 < \nu \leq 1 \) by means of the equation:

\[
r(\theta; \nu) = r(\pi - \theta; 1 - \nu).
\] (5.5)

Eq. (5.5) can be checked easily from the expression (5.4). Notice that for \( 0 < \nu < 1/2 \) (or \( 1/2 < \nu < 1 \)) the D2 brane passes through the point \( r = 0 \). This behaviour is similar to the \( p < 6 \) case. However, the present solution differs substantially from those studied in section 4. In order to make this difference manifest, let us introduce the cylindrical coordinates \( (z, \rho) \) as follows:

\[
\begin{align*}
  z &= -r \cos \theta, \\
  \rho &= r \sin \theta.
\end{align*}
\] (5.6)

The behaviour of the coordinate \( \rho \) in the region in which \( r \to \infty \) is of special interest to interpret the asymptotic behaviour of the brane. It is not difficult to find how \( \rho \) depends on \( \theta \) for the solution (5.4):

\[
\rho = A \left( \tan \frac{\theta}{2} \right)^{\frac{1}{1-2\nu}}.
\] (5.7)

Clearly, when \( \nu < 1/2 \), \( \rho \to \infty \) for \( \theta \to \pi \), whereas if \( \nu > 1/2 \) the coordinate \( \rho \) diverges for \( \theta \approx 0 \). This implies that our solution does not behave as a tube \(^3\). This fact is more explicit if we rewrite our solution (5.4) as a function \( z = z(\rho) \). After a short calculation, we get:

\[
\begin{align*}
  z &= \frac{\rho^{2\nu}}{2A^{1-2\nu}} \left( \rho^{2(1-2\nu)} - A^{2(1-2\nu)} \right).
\end{align*}
\] (5.8)

When \( \rho \to \infty \), the function \( z(\rho) \) behaves as:

\[
\lim_{\rho \to \infty} z(\rho) \sim \begin{cases} 
  \rho^{2(1-\nu)}, & \text{if } \nu < \frac{1}{2}, \\
  -\rho^{2\nu}, & \text{if } \nu > \frac{1}{2},
\end{cases}
\] (5.9)

\(^3\)A similar calculation for \( p < 6 \) gives \( \rho \to \text{constant at the spikes.} \)
Figure 3: Representation of the D2 embedding for $\nu = 1/4$ (a) and $\nu = 0$ (b). The conventions are the same as in figure 2.

and thus, as expected, $z \to +\infty$ ($z \to -\infty$) for $\nu < 1/2$ ($\nu > 1/2$). Moreover, eq. (5.8) implies that the asymptotic shape of the solution is that of a paraboloid \(\nabla\) rather than a tube. We have thus an “upper paraboloid” for $\nu < 1/2$ and a “lower paraboloid” for $\nu > 1/2$. In figure 3 we have plotted the $p = 6$ solution for different values of $\nu$. Notice that for $\nu \neq 0,1$ $z(\rho)$ has an extremum for some value of $\rho \neq 0$, while for $\nu = 0,1$ the extremum is located at $\rho = 0$. This extremum is a minimum (maximum) for $\nu < 1/2$ ($\nu > 1/2$). It is interesting to point out that, although the solution for $p = 6$ differs from the ones found for $p \leq 5$, the energy $U$ is still given by $|Z_s|$, where $Z_s$ is the same as in eq. (4.30).

It follows from our previous analysis that $\nu = 1/2$ is a critical point in the behaviour of the D6-D2 system. The BPS condition (5.2) is not defined in this case and we have to come back to the equation of motion (3.1). Fortunately, we will be able to find the general solution of the field equation in this case. The key point in this respect is the observation that, for $p = 6$ and $\nu = 1/2$, the near-horizon energy density does not depend on $\theta$ explicitly. Indeed, a simple calculation shows that:

$$ U = T_2 \Omega_1 R \int d\theta \sqrt{r^2 + r'^2} \, , \quad \quad (\nu = \frac{1}{2}) \, . \quad \quad (5.10) $$

This $\theta$-independence implies the “conservation law”:

$$ r' \frac{\partial U}{\partial r'} - U = \text{constant} \, , \quad \quad (5.11) $$

or, using the explicit form of $U$ given in eq. (5.10):

\[ ^4 \text{Strictly speaking, only for } \nu = 0,1 \text{ eq. (5.8) represents a paraboloid.} \]
Figure 4: Solution of the equation of motion for \( p = 6 \) and \( \nu = 1/2 \) for the near-horizon metric (eq. (5.13)).

\[
\frac{r^2}{\sqrt{r^2 + r'^2}} = C ,
\]
where \( C \geq 0 \) is constant. It is not difficult to integrate the first-order equation (5.12). The result is:

\[
r(\theta) = \frac{C}{\sin (\theta + \alpha)} ,
\]
\( \alpha \) being a new integration constant. The solution (5.13) represents a cone with opening half-angle \( \alpha \) and with its vertex located at a distance \( C/\sin \alpha \) from the origin (see figure 4). This fact is neatly shown if we rewrite (5.13) in terms of the cylindrical coordinates \( (\rho, z) \) defined in eq. (5.6). In these coordinates, eq. (5.13) becomes:

\[
\rho \cos \alpha - z \sin \alpha = C .
\]

It is clear from figure 4 that \( \alpha \) can take values in the range \(-\pi/2 \leq \alpha \leq \pi/2\). When \( \alpha = 0 \) the cone degenerates in the cylinder \( \rho = C \), while for \( \alpha = \pm \pi/2 \) the solution (5.14) is the plane \( z = \mp C \). Notice that eq. (5.14) is invariant under the transformation \( \alpha \rightarrow -\alpha \), \( z \rightarrow -z \). Thus, for \( \alpha > 0 \) (\( \alpha < 0 \)) \( z \) goes to \(+\infty\) (\(-\infty\)) when the angle \( \theta \) takes the value \( \pi - \alpha \) (\(-\alpha\)). It is interesting to point out that, when \( \rho \rightarrow \infty \), \( z \) diverges linearly with \( \rho \), which is, certainly, the limiting case of the \( \nu \neq 1/2 \) situation (see eq. (5.3)). This softer behaviour for \( \nu = 1/2 \) will be also reflected in the solution for the full Dp-brane metric, as we shall check in the next section, where we will find the solutions of the BPS equation for the full Dp-brane brane metric (2.1). We will check that for \( p = 6 \) and \( \nu = 1/2 \) this solution is equivalent, in the near-horizon region, to our general solution (5.14) with \( C = 0 \), i.e. only those cones passing through the origin correspond to BPS solutions of the asymptotically flat metric.
In this section we are going to study the solutions of the BPS condition beyond the near-horizon approximation. Recall that the full metric of the Dp-brane background is obtained by taking $a = 1$ in the harmonic function (2.2). The full Dp-brane metric is asymptotically flat and, therefore, one expects that a brane placed far away from the location of the background branes is not bent by the gravitational field. We shall concentrate first in the study of this asymptotic behaviour and, afterwards, we will consider the complete solution.

It is convenient for our purposes to rewrite the BPS condition (3.7) in the coordinates $(z, \rho)$ of eq. (5.6). In these coordinates, the D(8-p)-brane embedding is characterized by a function $z(\rho)$. The BPS equation gives $dz/d\rho$ in terms of $(z, \rho)$. Actually, it is not difficult to relate $z(\rho)$ and its derivative to $r(\theta)$ and $r'(\theta)$. By using only the relation between both coordinate systems one can verify that:

$$\frac{dz}{d\rho} = \sin \theta - \cos \theta \frac{r'}{r}. \quad (6.1)$$

By substituting the value of the ratio $r'/r$ given by eq. (3.7) for $a = 1$ on the right-hand side of this equation, one arrives at the following first-order differential equation for the function $z(\rho)$:

$$\frac{dz}{d\rho} = -\frac{[\rho^2 + z^2]^{\frac{7-p}{2}}}{R^{7-p} + [\rho^2 + z^2]^{\frac{7-p}{2}}} D_p \left( \arctan \left( -\frac{\rho}{z} \right) \right) \frac{D_p(\pi/2)}{\rho^{7-p}}. \quad (6.2)$$

It is now elementary to evaluate the right-hand side of eq. (6.2) in the asymptotic region in which $\rho \to \infty$, $z/\rho \to 0$ (and $\theta \to \pi/2$). In this limit, eq. (6.2) takes the form:

$$\frac{dz}{d\rho} \sim -\frac{D_p(\pi/2)}{\rho^{7-p}} + \cdots, \quad (6.3)$$

where we have only kept the first term in the expansion in powers of $R/\rho$. Notice that $z'(\rho) \to 0$ as $\rho \to \infty$, according to our expectations. However, also in this approach, the behaviour of the $p = 6$ case differs from that corresponding to $p < 6$. Indeed, for $p = 6$, $z'(\rho) \sim \rho^{-1}$ and $z(\rho)$ diverges logarithmically in the asymptotic region while, for $p < 6$, $z(\rho)$ approaches a constant value as $\rho \to \infty$. Let us specify further these two kinds of behaviours. The coefficient multiplying the power of $\rho$ on the right-hand side of eq. (6.3) is $D_p(\pi/2)$, which can be computed from eq. (2.22):

$$D_p(\pi/2) = -2\sqrt{\pi} \frac{\Gamma\left( \frac{8-p}{2} \right)}{\Gamma\left( \frac{7-p}{2} \right)} R^{7-p} \left( \frac{1}{2} - \nu \right). \quad (6.4)$$

Using this result we can integrate eq. (6.3). For $p < 6$, we get:

$$z(\rho) \sim z_\infty - \left( \frac{1}{2} - \nu \right) \sqrt{\pi} \frac{\Gamma\left( \frac{6-p}{2} \right)}{\Gamma\left( \frac{7-p}{2} \right)} R^{7-p} \frac{1}{\rho^{\theta-p}} + \cdots, \quad (p \neq 6), \quad (6.5)$$
where $z_\infty$ is a constant representing the asymptotic value of $z$. From eq. (6.5) one concludes that the sign of $z_\infty - z$ depends on the sign of $\frac{1}{2} - \nu$. i.e. if $\nu < 1/2$ ($\nu > 1/2$) the brane reaches its asymptotic value of $z$ from below (above). If $p = 6$ we get the expected logarithmic dependence:

$$z (\rho) \sim (1 - 2\nu) R \log \rho + \cdots .$$

(6.6)

Notice that when $\rho \to \infty$ in eq. (6.6), $z \to +\infty$ if $\nu < 1/2$ while, on the contrary, $z \to -\infty$ for $\nu > 1/2$, in agreement with our near-horizon analysis of section 5. It is interesting to point out that, although in this case $z$ diverges as $\rho \to \infty$, $z/\rho$ still vanishes in this limit, as assumed in the derivation of eq. (6.3).

For $\nu = 1/2$, the leading term in the asymptotic expansion vanishes and we have to compute the next-to-leading term. This does not change significantly the analysis of the $p < 6$ case. However, for $p = 6$, things change drastically when $\nu = 1/2$. In fact, one can prove from eq. (6.6) that, in this case, $z$ approaches a constant value $z_\infty$ as $\rho \to \infty$ (and, actually, $z - z_\infty$ decreases as $\rho^{-1}$ for $\rho >> R$).

Let us now see how one can integrate exactly the BPS condition for the full metric. The resulting solution must reproduce the asymptotic behaviour we have just described. For $p = 3$, eq. (6.2) was integrated numerically in ref. [11]. As we shall show below, our analytical solution agrees with these numerical results. It is more convenient to come back to our original $(r, \theta)$ coordinates. The basic strategy we will adopt to integrate the BPS equation is to write it in terms of the same function $\Lambda_p(\theta)$ that we have already used to find the near-horizon solutions. For illustrative purposes, we will do it for a general value of the parameter $a$ of the harmonic function. After using the definition (4.1), eq. (3.7) can be put in the form:

$$r' = a r^{7-p} (\sin \theta)^{8-p} + R^{7-p} (\sin \theta)^{6-p} - R^{7-p} \Lambda_p(\theta) \cos \theta$$

(6.7)

$$a r^{7-p} (\sin \theta)^{7-p} \frac{d}{d\theta} (r \cos \theta) = R^{7-p} \left[ r (\sin \theta)^{6-p} - \Lambda_p(\theta) \frac{d}{d\theta} (r \sin \theta) \right].$$

(6.8)

On the right-hand side of this expression we recognize a term containing $(\sin \theta)^{6-p}$, whose representation as a derivative of the function $\Lambda_p(\theta)$ was crucial to integrate the near-horizon BPS condition for $p \neq 6$. Actually, assuming that $p \neq 6$ and using eq. (4.4), our differential equation becomes:

$$a r^{7-p} (\sin \theta)^{7-p} \frac{d}{d\theta} (r \cos \theta) =$$

$$= \frac{R^{7-p}}{6-p} \left[ r \sin \theta \frac{d}{d\theta} \Lambda_p(\theta) - (6-p) \Lambda_p(\theta) \frac{d}{d\theta} (r \sin \theta) \right], \quad (p \neq 6).$$

(6.9)
Taking $a = 0$ we recover the solutions (4.5) and (4.18). From now on we shall put $a = 1$ in all our expressions. In this case we can pass the $r^{7-p} (\sin \theta)^{7-p}$ factor in the left-hand side of eq. (5.9) to the right-hand side and, remarkably, this equation can be written as:

$$\frac{d}{d\theta} (r \cos \theta) = \frac{R^{7-p}}{6-p} \frac{d}{d\theta} \left[ \frac{\Lambda_p(\theta)}{(r \sin \theta)^{6-p}} \right], \quad (p \neq 6).$$  \hspace{1cm} (6.10)

The integration of eq. (6.10) is now immediate:

$$r \cos \theta = \frac{R^{7-p}}{6-p} \frac{\Lambda_p(\theta)}{(r \sin \theta)^{6-p}} - z_\infty, \quad (p \neq 6),$$  \hspace{1cm} (6.11)

where $z_\infty$ is a constant of integration which can be identified with the asymptotic value of $z$ introduced in eq. (5.5). Actually, in terms of the coordinates $(z, \rho)$, our solution (6.11) can be written as:

$$z = z_\infty - \frac{R^{7-p}}{6-p} \frac{\Lambda_p(\arctan(-\rho/z))}{\rho^{6-p}}, \quad (p \neq 6).$$  \hspace{1cm} (6.12)

Eqs. (6.11) and (6.12) give the analytical solution of the BPS differential equation for $p \neq 6$. It is easy to verify that eq. (6.12) for $\rho \to \infty$ is reduced to eq. (6.5). Moreover, by derivating both sides of eq. (6.12) with respect to $\rho$, one can extract the value of $dz/d\rho$. The result in terms of $(z, \rho)$ is precisely the one written in (6.2), i.e. the function $z(\rho)$ defined by (6.12) is a solution of the BPS condition in cylindrical coordinates.

Let us now study the analytical solution we have found and, specially, how it interpolates between the near-horizon and asymptotic regions. Notice, first of all, that eqs. (6.11) and
(6.12) give \( r(\theta) \) and \( z(\rho) \) in implicit form. In particular, eq. (6.11) determines \( r(\theta) \) as the root of a polynomial of degree \( 7 - p \) which, for \( p \geq 3 \), can be solved algebraically. However, the implicit equations (6.11) and (6.12) are much easier to handle and, actually, can be used to plot the embeddings. In figure 5 we have represented these embeddings for different values of \( \nu \) and \( z_\infty \). Moreover, by studying eqs. (6.11) and (6.12) we can characterize the general features of our BPS solutions. The detailed analysis of these solutions is performed in appendix A. It follows from this analysis that our solution (6.12) coincides with the near-horizon embeddings in a region close to the origin. Actually, as can be seen from the plots of figure 5, the integral of the BPS differential equation for the full metric contains cylindrical regions which resemble closely the tubes found in the near-horizon analysis. It is not difficult to relate the parameter \( z_\infty \) of eq. (6.12) to the constant \( C \) appearing in the lower and upper tube solutions (eqs. (4.5) and (4.18)). For large \( |z_\infty| \) it can be seen that both types of solutions are approximately equal if we are sufficiently near the origin and if the following relation:

\[
C^{6-p} = \frac{R^{7-p}}{(6-p) |z_\infty|}, \tag{6.13}
\]

is satisfied.

For \( 0 < \nu < 1 \) the solution reaches the origin \( r = 0 \) by means of a tube, which is an upper or lower tube depending on the sign of \( z_\infty \) (see figure 5). For \( |z_\infty| \to \infty \) we will verify that this tubes can be regarded as bundles of fundamental strings, exactly in the same way as in the near-horizon case.

The behaviour of the \( \nu = 0 \) solution is different from that we have just described for \( 0 < \nu < 1 \). When \( z_\infty \to -\infty \), the D(8-p)-brane is nearly flat. As we increase \( z_\infty \), an upper tube develops and for \( z_\infty \to +\infty \) a bundle of \( N \) fundamental strings, connecting the Dp and D(8-p) branes, is created. As pointed out in ref. [11], this solution provides a concrete realization of the so-called Hanany-Witten effect [15].

Let us now study the energy of our solutions. It is clear from the plots of figure 5 that \( r \) is not a single-valued function of \( \theta \). It is thus more convenient to use the \((z, \rho)\) coordinates in order to have a global description of the energy functional. From eq. (2.18) it is straightforward to obtain the expression of \( U \) as an integral over \( \rho \). The result can be put in the form:

\[
U = T_{8-p} \Omega_{7-p} \int d\rho \sqrt{\left( \Delta_p - D_p \frac{dz}{d\rho} \right)^2 + \left( D_p + \Delta_p \frac{dz}{d\rho} \right)^2}. \tag{6.14}
\]

In eq. (6.14) \( D_\rho \) depends on \( \rho \) and \( z(\rho) \) as in eq. (5.2) and \( \Delta_\rho \) is the function defined in eq. (3.2). It is important to stress that eq. (6.14) gives the energy for any embedding \( z(\rho) \). Notice that on the right-hand side of eq. (6.14) we have the sum of two squares. The second of these two terms vanishes when the BPS condition \( dz/d\rho = -D_\rho/\Delta_\rho \) (see eq. (6.2)) holds. It is clear from these considerations that, if we define:

\[
X \equiv \Delta_p - D_p \frac{dz}{d\rho}, \tag{6.15}
\]

the solution for \( \nu = 1 \) can be reduced to that for \( \nu = 0 \) (see eq. (A.1)).
we obtain the following lower bound for the energy of any embedding:

\[ U \geq T_{8-p} \Omega_{7-p} \int d\rho \left| X \right|. \]  

(6.16)

The bound (6.16) is saturated precisely when the embedding satisfies the first-order BPS equation (6.2). A remarkable aspect of the function \( X \) appearing in the right-hand side of eq. (6.16) is that it can be put as a total derivative. Indeed, one can verify that:

\[ X = -\frac{d}{d\rho} \left[ -z D_p + \left( \frac{1}{8-p} + \frac{R^{7-p}}{(\rho^2 + z^2)^{\frac{1}{2}}} \right) \rho^{8-p} \right]. \]  

(6.17)

Eq. (6.17) is the analog in these coordinates of eq. (3.8). As it happened with eq. (3.8), in order to prove eq. (6.17) one only has to use eq. (2.14) and, therefore, (6.17) is valid for any function \( z(\rho) \). An important consequence of eq. (6.17) is that the bound (6.16) only depends on the boundary conditions of the embedding at infinity. Thus, one can say that the BPS embeddings are those that minimize the energy for a given value of \( z(\rho) \) at \( \rho \to \infty \).

From eq. (6.14) it is very easy to obtain the energy \( U_{BPS} \) of one of such BPS solutions. The result one arrives at is:

\[ U_{BPS} = -T_{8-p} \Omega_{7-p} \int d\rho \left[ \frac{dz}{d\rho} + \left( \frac{dz}{d\rho} \right)^{-1} \right] D_p. \]  

(6.18)

Apart from being simple, eq. (6.18) is specially suited for our purposes. Notice, first of all, that the integrand on the right-hand side of (6.18) is always non-negative due to eq. (6.2). Secondly, we can use (6.18) to evaluate the energy of a tubular portion of the brane. Indeed, for large \( |z_\infty| \), \( \left| \frac{dz}{d\rho} \right| \) is large in one of these tubes and the argument of \( D_p \) is almost constant and equal to \( \pi \) (0) for an upper (lower) tube. Neglecting the term containing \( \left( \frac{dz}{d\rho} \right)^{-1} \) on the right-hand side of eq. (6.18), and defining the length of the tube as:

\[ L_{tube} = \int_{tube} d\rho |\frac{dz}{d\rho}|, \]  

(6.19)

we get the following values for the energy of the tubes:

\[ U_{tube} = \begin{cases} -T_{8-p} \Omega_{7-p} D_p(\pi) L_{tube}, & \text{(upper tube)}, \\ T_{8-p} \Omega_{7-p} D_p(0) L_{tube}, & \text{(lower tube)}. \end{cases} \]  

(6.20)

Taking into account eqs. (2.23) and (2.25), and using eq. (2.24), one can easily show from the result in eq. (6.20) that \( U_{tube} \) coincides with \( |Z_s| \), where the values of \( Z_s \) are given in eq. (4.30). This confirms our interpretation of the tubes as bundles of \((1-\nu)N\) and \(\nu N\) fundamental strings.

It is not difficult to calculate the energy of the whole brane. Actually, by means of eq. (6.17) we can perform the integral appearing in the right-hand side of eq. (6.18). As we are calculating the energy of an infinite brane, this integral is divergent. In order to regulate this divergence, let us introduce a cutoff \( \rho_c \), in such a way that the integral in eq. (6.18) is performed between \( \rho = 0 \) and \( \rho = \rho_c \). It is not difficult to check from the properties of our
solutions that the contribution of the lower limit \( \rho = 0 \) is zero and, thus, only the value at the cutoff contributes. After this process, \( U_{BPS} \) can be put as:

\[
U_{BPS} = -T_{8-p} \Omega_{7-p} \left[ z \left( \frac{1}{8 - p} + \frac{R_{7-p}}{(\rho^2 + z^2)^{\frac{7-p}{2}}} \right) \rho^{8-p} \right]_{\rho = \rho_c} .
\]  

(6.21)

We have now to take \( \rho_c \to \infty \) and, therefore, \( z(\rho_c) \to z_{\infty} \). Notice that, in this limit, the argument of the function \( D_p \) is \( \pi/2 \). Using eq. (6.4), \( U_{BPS} \) can be put as a sum of two terms, namely:

\[
U_{BPS} = \left( \frac{1}{2} - \nu \right) NT_f z_{\infty} + T_{8-p} \Omega_{7-p} \left[ \frac{\rho_c^{8-p}}{8 - p} + R_{7-p} \rho_c \right] .
\]  

(6.22)

Notice that the first term on the right-hand side of eq. (6.22) is finite and depends linearly on \( z_{\infty} \), while the second term diverges when \( \rho_c \to \infty \) and is independent of \( z_{\infty} \). According to ref. [18], one can give the following interpretation to this divergence. Let us consider a D(8-p)-brane embedded in the metric (2.1) along the plane \( z = 0 \) or, equivalently, such that its worldvolume is determined by the equation \( \theta = \pi/2 \). Notice that in this configuration the brane is not bent at all and, for this reason, it will be referred to as the “ground state” of the brane. Let \( g_{gs} \) be the induced metric on the worldvolume of the D(8-p)-brane for this ground state configuration. Putting the worldvolume gauge fields to zero and substituting \( g \) by \( g_{gs} \) in eq. (2.6), we get the action of the ground state:

\[
S_{gs} = -T_{8-p} \int d^{9-p} \xi e^{-\tilde{\phi}} \sqrt{-\det (g_{gs})} .
\]  

(6.23)

The energy \( E_{gs} \) of the ground state is obtained from \( S_{gs} \) as:

\[
E_{gs} = -\frac{S_{gs}}{T} ,
\]  

(6.24)

where \( T = \int dt \). Using the metric given in eq. (2.1) and the value of the dilaton field displayed in eq. (2.5), the calculation of \( E_{gs} \) is a simple exercise. By comparing the result of this computation with the right-hand side of eq. (6.22), one discovers that \( E_{gs} \) is equal to the divergent contribution to \( U_{BPS} \). Therefore, one can subtract \( E_{gs} \) from \( U_{BPS} \) and define a renormalized energy \( U_{ren} \) as:

\[
U_{ren} \equiv U_{BPS} - E_{gs} .
\]  

(6.25)

It follows from eq. (6.22) that \( U_{ren} \) is given by:

\[
U_{ren} = \left( \frac{1}{2} - \nu \right) NT_f z_{\infty} .
\]  

(6.26)

By derivating \( U_{ren} \) with respect to \( z_{\infty} \), we learn that there is a net constant force acting on the D(8-p)-brane, which is equal to \( \left( \frac{1}{2} - \nu \right) NT_f \), i.e. equivalent to the tension of \( \left( \frac{1}{2} - \nu \right) N \) fundamental strings. In ref. [11] this force was interpreted as a consequence of the fact that, due to the \( p + 2 \)-form flux captured by the D(8-p)-brane, the latter is endowed with an effective charge equal to \( \left( \frac{1}{2} - \nu \right) N \) units.

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Let us now integrate the BPS differential equation for $p = 6$. Using the value of $\Lambda_6(\theta)$ in eq. (6.8), we get (for $a = 1$):

$$r \sin \theta \frac{d}{d\theta} (r \cos \theta) = R \left[r + (2\nu - 1) \frac{d}{d\theta} (r \sin \theta)\right], \quad (p = 6). \quad (6.27)$$

This equation can be recast in the form:

$$\frac{d}{d\theta} (r \cos \theta) = R \frac{d}{d\theta} \left[\log \left[\tan \left(\frac{\theta}{2}\right) (r \sin \theta)^{2\nu - 1}\right]\right], \quad (p = 6), \quad (6.28)$$

and, therefore, its integration is immediate. It is more interesting to write the result in cylindrical coordinates. After a short calculation one gets:

$$\left(z + \sqrt{\rho^2 + z^2}\right) \rho^{2(\nu - 1)} = e^{\frac{k-z}{\pi}} , \quad (p = 6), \quad (6.29)$$

where $k$ is a constant of integration. It is easy to check that the function $z(\rho)$ parametrized by eq. (6.29) has the asymptotic behaviour displayed in eq. (6.6). It follows that, as expected, $z(\rho)$ does not reach a constant value for $p = 6$ and $\nu \neq 1/2$. The profiles of the $\nu \neq 1/2$ curves are similar to the near-horizon ones, plotted in figure 3. The only difference is that, instead of the behaviour written in eq. (5.9), $|z(\rho)|$ grows logarithmically as $\rho \to \infty$. Moreover, similarly to what happens to the $p \leq 5$ solution, eq. (6.29) is invariant under the transformation:

$$\rho \to \rho, \quad z \to -z, \quad k \to -k, \quad \nu \to 1 - \nu. \quad (6.30)$$

It is also easy to check that the function $z(\rho)$ defined by eq. (6.29) coincides, in the near-horizon region, with the one written down in eq. (5.8) if the constants $A$ and $k$ are identified as follows:

$$A^{2\nu - 1} = e^{\frac{k}{\pi}}, \quad (\nu \neq \frac{1}{2}). \quad (6.31)$$

For $\nu = 1/2$ the behaviour of the solution changes drastically. Indeed, in this case, one can solve eq. (5.29) for $\rho$ and get $\rho = \rho(z)$. The result is:
\[ \rho = \frac{z}{\sinh \frac{k - z}{R}}, \quad (\nu = \frac{1}{2}, p = 6). \]  

The function (6.32) has been represented graphically in figure 6. From this plot one notices that \( \lim_{\rho \to \infty} z(\rho) = k \). Actually, one can prove using eq. (6.32) that for \( k \geq 0 \) \( (k \leq 0) \) the variable \( z \) takes values in the range \( 0 \leq z \leq k \) \( (k \leq z \leq 0) \). Thus, in this \( (p = 6, \nu = 1/2) \) case, the constant \( k \) plays the same role as \( z_\infty \) did for \( p \leq 5 \). In particular, for \( k = 0 \) the solution (6.32) is equivalent to the equation \( z = 0 \), which is precisely the same solution we found for \( p \leq 5, \nu = 1/2 \) and \( z_\infty = 0 \), i.e. one obtains the “ground state” solution in both cases.

Let us finally point out that, by linearizing eq. (6.32) in \( \rho \) and \( z \), one gets the equation of a cone passing through the origin and with half-angle equal to:

\[ \tan \alpha = \frac{1}{\sinh \left( \frac{\pi}{R} \right)}. \]  

Therefore, with the identification (6.33), we get a perfect agreement with the near-horizon solutions (5.14) with \( C = 0 \).

7 Summary and Conclusions

In this paper we have studied the embedding of a D(8-p)-brane in the background geometry of a stack of coincident Dp-branes. This embedding is governed by the worldvolume action of the D(8-p)-brane (eq. (2.1)), which determines the equation of motion (3.1). By using a BPS argument we have found a bound for the energy of the system such that those embeddings which saturate it are also a solution of the equation of motion. This equation of motion (eq. (3.7)) is a first-order differential equation which, amazingly, can be solved analytically both in the near-horizon and asymptotically flat geometries.

The solutions of the BPS equations give the deformation of the D(8-p)-brane under the influence of the gravitational and RR fields created by the background branes. Generically, these solutions contain tubes connecting the D(8-p)-brane to the background branes which, after analyzing its energy, can be interpreted as bundles of fundamental strings. From the point of view of the gauge theory defined on the worldvolume of the background branes, these tubes represent baryonic multiquark states.

There are several topics which were not covered by our analysis which, in our opinion, it would be interesting to consider in a future work. Let us mention some of them. First of all, it would be interesting to understand the relation between the BPS condition and supersymmetry. One expects, following the line of ref. [12], that the BPS differential equation is precisely the requirement one must impose to the brane embedding in order to preserve some fraction of the space-time supersymmetry. An analysis of the supersymmetry algebra, by applying the methods of ref. [24], could shed light on this aspect of the problem.

Another topic which would be worth to explore is the use that our exact results could have in the study of baryons in gauge theories both in the supersymmetric and non-supersymmetric
models. In the latter case we have to extend our results to the situation in which the background brane configuration is not extremal.

Finally, it would be interesting to find out if there exist BPS conditions, similar to the ones found here, for brane embeddings in more general brane geometries, such as the ones corresponding to the intersection of several branes of different types.

The final goal of these studies is twofold. On the one hand, we would like to know what string theory can teach us about the non-perturbative structure of gauge theories while, at the same time, we would like to uncover aspects of a more complete formulation of string theory.

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APPENDIX A

In this appendix we are going to carry out a study of the solution found in section 6 of the BPS differential equation for the asymptotically flat metric for \( p \leq 5 \) (eqs. (6.11) and (6.12)).

The first thing we want to point out in this respect is that, due to the property of \( \Lambda_p \) displayed in eq. (4.19), our solution (6.12) is invariant under the transformation:

\[
\rho \rightarrow \rho , \quad z \rightarrow -z , \quad z_\infty \rightarrow -z_\infty , \quad \nu \rightarrow 1 - \nu .
\]  

(A.1)

Due to this invariance we can restrict ourselves to the case \( 0 \leq \nu \leq 1/2 \). The solutions outside this range of \( \nu \) can be obtained by performing a reflection with respect to the \( z = 0 \) axis. Therefore, unless otherwise stated, we will assume in what follows that \( \nu \leq 1/2 \).

Moreover, it is more convenient for our purposes to rewrite eq. (6.12) in the equivalent form:

\[
(z_\infty - z ) \rho^{6-p} = \frac{R^{7-p}}{6-p} \Lambda_p \left( \arctan \left( -\rho/z \right) \right) .
\]  

(A.2)

We will start our analysis of the solution by studying its cut with the \( \rho = 0 \) axis. By looking at eq. (A.2) it follows immediately that \( \Lambda_p \) must vanish for \( \rho = 0 \). Thus, the angle \( \theta \) at which the solution reaches the \( \rho = 0 \) axis must be \( \theta = \theta_0 \), \( \theta_0 \) being the same angle appearing in the near-horizon solution (see eq. (4.8)). Following eq. (5.5), \( \rho = r(\theta) \sin \theta \) and, therefore, if \( \rho = 0 \) for \( \theta = \theta_0 \) one must have \( r(\theta_0) \sin \theta_0 = 0 \). There are two possibilities to fulfill this equation. If, first of all, \( \nu \neq 0 \), the angle \( \theta_0 \) is non-vanishing, which means that, necessarily, \( r(\theta_0) = 0 \), i.e. the solution reaches the origin \( r = 0 \) at an angle \( \theta_0 \). Notice that, as it should occur near \( r = 0 \), this is precisely what happens in the near-horizon solution. The second possibility is \( \nu = 0 \). In this case \( \theta_0 = 0 \) and the vanishing of \( \rho \) does not require \( r = 0 \). Again,
this is in agreement with the near-horizon solution. Actually, the distance \( r(0) \) at which the \( \nu = 0 \) solution cuts the \( \rho = 0 \) axis can be determined, as in section 4, by expanding \( \Lambda_p(\theta) \) around \( \theta_0 = 0 \). Using eq. (4.15) one gets that \( r(0) \) must satisfy:

\[
[r(0)]^{6-p} [r(0) + z_\infty] = \frac{R^{7-p}}{6-p}, \quad (\nu = 0) .
\] (A.3)

As the right-hand side of this equation is positive, it follows that \( r(0) > -z_\infty \). When this condition is satisfied, the left-hand side of eq. (A.3) is a monotonically increasing function of \( r(0) \) and, therefore, there exists a unique solution for \( r(0) \).

Let us now consider the cut with the \( z = 0 \) axis. We shall denote the corresponding value of the \( \rho \) coordinate by \( \rho_0 \), i.e. \( \rho_0 = \rho( z = 0 ) \). If \( \rho_0 \neq 0 \), the value of the angle \( \theta \) for \( z = 0 \) is \( \theta = \pi/2 \) and, after evaluating the function \( \Lambda_p \) for this value of \( \theta \), eq. (A.2) gives:

\[
\rho_0^{6-p} = 2\sqrt{\pi} \frac{\Gamma(\frac{8-p}{2})}{\Gamma(\frac{2-p}{2})} \left( \frac{1}{2} - \nu \right) \frac{R^{7-p}}{z_\infty} .
\] (A.4)

Notice that for \( \nu \neq 1/2 \) the right-hand side of eq. (A.4) only makes sense for \( z_\infty > 0 \). Therefore, only for \( z_\infty > 0 \) the solution for \( \nu \neq 1/2 \) cuts the \( z = 0 \) axis at a finite non-vanishing value of the coordinate \( \rho \). For \( \nu = 1/2 \) and \( z_\infty \neq 0 \) eq. (A.4) has no solution for \( \rho_0 \neq 0 \). All these features appear in the plots of figure 5.

From eq. (A.2) one can extract the range of allowed values of the coordinate \( z \). Indeed, it follows from eq. (A.2) that the signs of \( z_\infty - z \) and \( \Lambda_p \) are the same. On the other hand, we know that \( \Lambda_p(\theta) \) is positive for \( \theta > \theta_0 \) and negative for \( \theta < \theta_0 \). Thus, when \( \theta > \theta_0 \) (\( \theta < \theta_0 \)) one must have \( z < z_\infty \) (\( z > z_\infty \)), while for \( \theta = \theta_0 \) either \( \rho = 0 \) or else \( z = z_\infty \). As \( \theta_0 \leq \pi/2 \) for \( \nu \leq 1/2 \), one can easily see that these results imply that \( z \leq 0 \) for \( z_\infty \leq 0 \), whereas \( z \) can be positive or negative if \( z_\infty > 0 \).

It is easy to prove that \( z \), as a function of \( \rho \), must have a unique extremum. For \( \nu < 1/2 \) this extremum is actually a minimum, as we are going to verify soon. From eq. (6.2) it is clear that \( dz/d\rho \) vanishes if and only if \( D_p \) is zero. Recall that the Gauss’ law (eq. (2.14)) implies that \( D_p \) is a monotonically decreasing function of \( \theta \) (see figure 1). Moreover, it follows from eqs. (2.25) and (2.26) that \( D_p(0) \geq 0 \) and \( D_p(\pi) \leq 0 \). Thus, it must necessarily exist a unique value \( \theta_m \) of \( \theta \) such that:

\[
D_p(\theta_m) = 0 .
\] (A.5)

Clearly, at the point of the curve \( z(\rho) \) at which \( \theta = \theta_m \) the derivative \( dz/d\rho \) is zero. It is interesting to compare \( \theta_m \) with the angle \( \theta_0 \) for which \( \Lambda_p \) is zero. By substituting \( \theta = \theta_0 \) in the equation which relates \( D_p(\theta) \) and \( \Lambda_p(\theta) \) (eq. (I.1)), one gets:

\[
D_p(\theta_0) = R^{7-p}(\sin \theta_0)^{6-p} \cos \theta_0 .
\] (A.6)

For \( \nu \leq 1/2 \) one has \( \theta_0 \leq \pi/2 \) and, thus, eq. (A.6) gives \( D_p(\theta_0) \geq 0 \). The monotonic character of \( D_p(\theta) \) implies that \( 0 \leq \theta_0 \leq \theta_m \leq \pi/2 \). Notice that \( \theta_0 = \theta_m \) if \( \nu = 0 \) \( (\theta_0 = \theta_m = 0) \) or \( \nu = 1/2 \) \( (\theta_0 = \theta_m = \pi/2) \). The coordinate \( z \) of the extremum is \( z_m = -r(\theta_m)\cos \theta_m \). As \( \theta_m \leq \pi/2 \), one must have \( z_m \leq 0 \). Moreover, since \( \theta_m \geq \theta_0 \), then \( \Lambda_p(\theta_m) \geq 0 \) and eq. (A.2) gives \( z_m \leq z_\infty \). By using the value of \( \Lambda_p \) at \( \theta = \theta_m \), which is:
\[ \Lambda_p(\theta_m) = (\sin \theta_m)^{6-p} \cos \theta_m, \quad (A.7) \]

(see eq. (4.1)) one can obtain an expression which determines \( z_m \), namely:

\[ |z_m|^{6-p} (z_\infty - z_m) = \frac{R^{7-p}}{6-p} \left( \cos \theta_m \right)^{7-p}. \quad (A.8) \]

It is not difficult to verify now that, when \( \nu < 1/2 \), the extremum at \( \theta = \theta_m \) is a minimum. In order to prove it, one must evaluate \( d^2z/d\rho^2 \). This can be done by derivating eq. (6.2). After putting \( \rho = \rho_m = r(\theta_m) \sin \theta_m \), and using the fact that \( dz/d\rho \) vanishes for \( \rho = \rho_m \), one arrives at:

\[ \frac{d^2z}{d\rho^2} \bigg|_{\rho=\rho_m} = -\left( \rho_m^2 + z_m^2 \right) \left[ R^{7-p} + (\rho_m^2 + z_m^2)^{7-p} \right] z_m. \quad (A.9) \]

When \( \nu < 1/2 \), one has \( z_m < 0 \) and, as a consequence, the right-hand side of eq. (A.9) is strictly positive. Therefore \( z(\rho) \) has a minimum for \( \rho = \rho_m \) as claimed. Notice that, when \( \nu = 1/2 \), i.e. \( \theta_m = \pi/2 \), the right-hand side of eq. (A.8) vanishes and, thus, \( z_m = 0 \) (the value \( z = z_\infty \) is reached asymptotically).

Eq. (A.8) can be solved immediately if \( z_\infty = 0 \). Indeed, in this case, one has:

\[ |z_m|^{7-p} = \frac{R^{7-p}}{6-p} \left( \cos \theta_m \right)^{7-p}, \quad (z_\infty = 0). \quad (A.10) \]

For general values of \( z_\infty \) we cannot give the general solution of eq. (A.8). However, one can extract valuable information from this equation. An important point concerning eq. (A.8) is the fact that its right-hand side does not depend on \( z_\infty \) (it only depends on \( p \) and \( \nu \)) and, therefore, it remains constant when we change the asymptotic value of the \( z \) coordinate. This allows us to study the behaviour of the minimum in different limiting situations. Let us suppose, first of all, that \( z_\infty \to +\infty \) for \( \nu < 1/2 \). As \( z_m < 0 \), the difference \( z_\infty - z_m \to +\infty \). In order to keep the right-hand side of eq. (A.8) finite, one must have \( z_m \to 0^- \). Actually \( z_m \) should behave as:

\[ z_m \sim -\frac{1}{(z_\infty)^{6-p}}, \quad (z_\infty \to +\infty, \nu < \frac{1}{2}). \quad (A.11) \]

On the other hand if \( z_\infty \to -\infty \), again for \( \nu < 1/2 \), the only possibility to maintain \( |z_m|^{6-p} (z_\infty - z_m) \) constant is that \( z_\infty - z_m \to 0^+ \) (recall that \( z_m \leq z_\infty \) and, thus, \( z_m \to 0 \) is impossible if \( z_\infty \to -\infty \)). The actual behaviour of \( z_\infty - z_m \) in this limit is:

\[ z_\infty - z_m \sim \frac{1}{(z_\infty)^{6-p}}, \quad (z_\infty \to -\infty, \nu < \frac{1}{2}). \quad (A.12) \]

The \( \nu = 0 \) embedding presents some characteristics which make it different from the \( \nu \neq 0 \) ones. We have already shown that the \( \nu = 0 \) solution cuts the \( \rho = 0 \) axis at a coordinate \( z = -r(0) \), with \( r(0) \) determined by eq. (A.3). Moreover, for \( \nu = 0 \) the function \( \Lambda_p \) on the right-hand side of eq. (A.2) is non-negative for all \( \rho \neq 0 \), which implies that \( z \leq z_\infty \) for these solutions. Recall that, in this case, \( z(\rho) \) has a minimum at \( \rho = 0 \).
$z_\infty \to -\infty$, due to eq. (A.12), $z(\rho)$ is approximately a constant function, i.e. $z(\rho) \approx z_\infty$. If, on the contrary, $z_\infty \to +\infty$, an “upper tube” connecting the point $z = -r(0) \to 0$ (see eq. (A.11)) and the asymptotic region is developed (see figure 5).

$\nu = 1/2$ is also a special case. Indeed, for $\nu = 1/2$ the angle $\theta_0$ is equal to $\pi/2$ and, therefore, the function $\Lambda_p$ in eq. (A.2) takes positive (negative) values for $z > 0$ ($z < 0$). By studying the signs of both sides of eq. (A.2) one easily concludes that for $z_\infty > 0$ one must have $0 \leq z < z_\infty$, whereas for $z_\infty < 0$ the coordinate $z$ takes values on the range $z_\infty < z \leq 0$ (see figure 5). When $z_\infty = 0$, the analysis of eq. (A.2) leads to the conclusion that the solution is the hyperplane $z = 0$.

It is also possible to determine the range of possible values that the coordinate $\theta$ can take. Actually $\theta$ has generically an extremal value, which we shall denote by $\theta_*$. The existence of $\theta_*$ can be proved by computing the derivative $d\theta/d\rho$ for our solution. A short calculation proves that this derivative vanishes if the coordinate $z$ takes the value:

$$z_* = \frac{6 - p}{7 - p} z_\infty . \tag{A.13}$$

Alternatively, one could determine $z_*$ as the point at which the denominator of the BPS condition (3.7) vanishes. By computing the second derivative $d^2\theta/d\rho^2$ at $\theta = \theta_*$, one can check that $\theta_*$ is a minimum (maximum) if $z_\infty < 0$ ($z_\infty > 0$). It is also interesting to point out that for $z = z_*$ the function $Z$ of eq. (4.21) changes its sign. This is consistent with the extremal nature of $\theta_*$. 

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*For $\nu = 0$ and $z_\infty < 0$, the value $z_\infty = 0$ is not reached (recall that $z \leq z_\infty$ when $\nu = 0$). In this case the coordinate $\theta$ varies monotonically along the embedding.*

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