MIRROR SYMMETRY FOR CONCAVEX VECTOR BUNDLES ON PROJECTIVE SPACES

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Abstract. Let $X \subset Y$ be smooth, projective manifolds. Assume that $\iota : X \hookrightarrow \mathbb{P}^s$ is the zero locus of a generic section of $V^+ = \oplus_{i \in I} O(k_i)$ where all the $k_i$’s are positive. Assume furthermore that $N_{X/Y} = \iota^*(V^-)$ where $V^- = \oplus_{j \in J} O(-l_j)$ and all the $l_j$’s are negative. We show that under appropriate restrictions, the generalized Gromov-Witten invariants of $X$ inherited from $Y$ can be calculated via a modified Gromov-Witten theory on $\mathbb{P}^s$. This leads to local mirror symmetry on the A-side.

1. Introduction

Let $V^+ = \oplus_{i \in I} O(k_i)$ and $V^- = \oplus_{j \in J} O(-l_j)$ be vector bundles on $\mathbb{P}^s$ with $k_i$ and $l_j$ positive integers. Suppose $X \hookrightarrow \mathbb{P}^s$ is the zero locus of a generic section of $V^+$ and $Y$ is a projective manifold such that $X \rightarrow Y$ with normal bundle $N_{X/Y} = \iota^*(V^-)$. The relations between Gromov-Witten theories of $X$ and $Y$ are studied here by means of a suitably defined equivariant Gromov-Witten theory in $\mathbb{P}^s$. We apply mirror symmetry to the latter to evaluate the gravitational descendants of $Y$ supported in $X$.

Section 2 is a collection of definitions and techniques that will be used throughout this paper. In section 3, using an idea from Kontsevich, we introduce a modified equivariant Gromov-Witten theory in $\mathbb{P}^s$ corresponding to $V = V^+ \oplus V^-$. The corresponding $D$-module structure ([6, [15, [29]) is computed in section 4. It is generated by a single function $\tilde{J}_V$. In general, the equivariant quantum product does not have a nonequivariant limit. It is shown in Lemma 4.1.1 that the generator $\tilde{J}_V$ does have a limit $J_V$ which takes values in $H^* \mathbb{P}^m[[q,t]]$. It is this limit that plays a crucial role in this work.

Let $Y$ be a smooth, projective manifold. The generator $J_Y$ of the pure $D$-module structure of $Y$ encodes one-pointed gravitational descendents of $Y$. It takes values in the completion of $H^*Y$ along the semigroup (Mori cone) of the rational curves of $Y$. The pullback map $j^* : H^*Y \rightarrow H^*X$ extends to a map between the respective completions. In Theorem 4.2.2 we describe one aspect of the relation between pure Gromov-Witten theory of $X \rightarrow Y$ and the modified Gromov-Witten theory of $\mathbb{P}^s$. Under natural restrictions, the pull back $j^*(J_Y)$ pushes forward to $J_V$. It follows that although defined

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on $\mathbb{P}^n$, $J_V$ encodes the gravitational descendants of $Y$ supported in $X$, hence the contribution of $X$ to the Gromov-Witten invariants of $Y$.

The only way that $X$ remembers the ambient variety $Y$ in this context is by the normal bundle. $Y$ can therefore be substituted by a local manifold. This suggests that there should be a local version of mirror symmetry (see the Remark at the end of section 4). This was first realized by Katz, Klemm, and Vafa [20]. The principle of local mirror symmetry in general has yet to be understood. Some interesting calculations that contribute toward this goal can be found in [8].

In section 5 we give a proof of the Mirror Theorem which allows us to compute $J_V$. A hypergeometric series $I_V$ that corresponds to the total space of $V$ is defined. The Mirror Theorem 5.0.3 states that $I_V = J_V$ up to a change of variables. Hence, the gravitational descendants of $Y$ supported on $X$ can be computed in $\mathbb{P}^n$.

Two examples of local Calabi-Yau threefolds are considered in section 6. For $X = \mathbb{P}^1$ and $V = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, we obtain the Aspinwall-Morrison formula for multiple covers. If $X = \mathbb{P}^2$ and $V = \mathcal{O}(-3)$, the quantum product of $Y$ pulls back to the modified quantum product in $\mathbb{P}^2$. The mirror theorem in this case yields the virtual number of plane curves on a Calabi-Yau threefold.

The rich history of mirror symmetry started in 1990 with a surprising conjecture by Candelas, de la Ossa, Green and Parkes [7] which predicts the number $n_d$ of degree $d$ rational curves on a quintic threefold. In [15], Givental presented a clever argument which, as shown later by Bini et al. in [6] and Pandharipande in [29], yields a proof of the mirror conjecture for Fano and Calabi-Yau (convex) complete intersections in projective spaces. Meanwhile, in a very well written paper [26], Lian, Liu and Yau used a different approach to obtain a complete proof of mirror theorem for convex complete intersections on projective spaces. An alternative proof of the convex Mirror Theorem has been given by Bertram [5]. In this paper we use Givental’s approach to study the local nature of mirror symmetry and to present a proof of the concavex Mirror Theorem.

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2. Stable maps and localization

2.1. Genus zero stable maps. Let $\overline{M}_{0,n}(X, \beta)$ be the Deligne-Mumford moduli stack of pointed stable maps to $X$. For an excellent reference on
the construction and its properties we refer the reader to [13]. We recall some of the features on \( \overline{M}_{0,n}(X, \beta) \) and establish some notation. For each marking point \( x_i \) let \( e_i : \overline{M}_{0,n}(X, \beta) \to X \) be the evaluation map at \( x_i \) and \( \mathcal{L}_i \) the cotangent line bundle at \( x_i \). The fiber of this line bundle over a moduli point \( (C, x_1, ..., x_n, f) \) is the cotangent space of the curve \( C \) at \( x_i \).

Let \( \pi_k : \overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,n-1}(X, \beta) \) be the morphism that forgets the \( k \)-th marked point. The obstruction theory of the moduli stack \( \overline{M}_{0,n}(X, \beta) \) is described locally by the following exact sequence

\[
0 \to \text{Ext}^0(\Omega_C(\sum_{i=1}^n x_i), \mathcal{O}_C) \to H^0(C, f^*TX) \to T_M \to \text{Ext}^1(\Omega_C(\sum_{i=1}^n x_i), \mathcal{O}_C) \to H^1(C, f^*TX) \to \Upsilon \to 0.
\]

(1)

(Here and thereafter we are naming sheaves after their fibres). To understand the geometry behind this exact sequence we note that \( T_M = \text{Ext}^1(f^*\Omega_X \to \Omega_C, \mathcal{O}_C) \) and \( \Upsilon = \text{Ext}^2(f^*\Omega_X \to \Omega_C, \mathcal{O}_C) \) are respectively the tangent space and the obstruction space at the moduli point \( (C, x_1, ..., x_n, f) \).

The spaces \( \text{Ext}^0(\Omega_C(\sum_{i=1}^n x_i), \mathcal{O}_C) \) and \( \text{Ext}^1(\Omega_C(\sum_{i=1}^n x_i), \mathcal{O}_C) \) describe respectively the infinitesimal automorphisms and infinitesimal deformations of the marked source curve. It follows that the expected dimension of \( \overline{M}_{0,n}(X, \beta) \) is \(-K_X \cdot \beta + \text{dim} X + n - 3\).

A smooth projective manifold \( X \) is called convex if \( H^1(\mathbb{P}^1, f^*TX) = 0 \) for any morphism \( f : \mathbb{P}^1 \to X \). For a convex \( X \), the obstruction bundle \( \Upsilon \) vanishes and the moduli stack is unobstructed and of the expected dimension. Examples of convex varieties are homogeneous spaces \( G/P \).

In general this moduli stack may behave badly and have components of larger dimensions. In this case, a Chow homology class of the expected dimension has been constructed [3] [27]. It is called the virtual fundamental class and denoted by \( \overline{M}_{0,n}(X, \beta)^{\text{virt}} \). Although its construction is quite involved, we will be using mainly two relatively easy properties. The virtual fundamental class is preserved when pulled back by the forgetful map \( \pi_n \). A proof of this fact can be found in section 7.1.5 of [3]. If the obstruction sheaf \( \Upsilon \) is free, the virtual fundamental class refines the top Chern class of \( \Upsilon \). This fact is proven in Proposition 5.6 of [3].

2.2. Equivariant cohomology and localization theorem. The notion of equivariant cohomology and the localization theorem is valid for any compact connected Lie group. For a detailed exposition on this subject we suggest Chapter 9 of [9]. Below we state without proof the results that will be used in this work.

The complex torus \( T = (\mathbb{C}^*)^{s+1} \) is classified by the principal \( T \)-bundle

\[
ET = (\mathbb{C}^{s+1} - \{0\})^{s+1} \to BT = (\mathbb{C}P^{s+1})^{s+1}.
\]

(2)
Let \( \lambda_i = c_1(\tau_i^*(\mathcal{O}(1))) \) and \( \lambda := (\lambda_0, ..., \lambda_s) \). We will use \( \mathcal{O}(\lambda_i) \) for the line bundle \( \tau_i^*(\mathcal{O}(1)) \). Clearly \( H^*(BT) = \mathbb{C}[\lambda] \). If \( T \) acts on a variety \( X \), we let \( X_T := X \times_T ET \).

**Definition 2.2.1.** The equivariant cohomology of \( X \) is

\[
H^*_T(X) := H^*(X_T).
\]

If \( X = x \) is a point then \( X_T = BT \) and \( H^*_T(x) = \mathbb{C}[\lambda] \). For an arbitrary \( X \), the equivariant cohomology \( H^*_T(X) \) is a \( \mathbb{C}[\lambda] \)-module via the equivariant morphism \( X \to x \).

Let \( \mathcal{U} \) be a vector bundle over \( X \). If the action of \( T \) on \( X \) can be lifted to an action on \( \mathcal{U} \) which is linear on the fibers, \( \mathcal{U} \) is an equivariant vector bundle and \( \mathcal{U}_T \) is a vector bundle over \( X_T \). The equivariant chern classes of \( E \) are \( c^*_T(\mathcal{U}) := c_k(\mathcal{U}_T) \). We will use \( \text{E}(\mathcal{U}) \) (\( \text{E}_T(\mathcal{U}) \)) to denote the nonequivariant (equivariant) top chern class of \( \mathcal{U} \).

Let \( X^T = \bigcup_{j \in J} X_j \) be the decomposition of the fixed point locus into its connected components. \( X_j \) is smooth for all \( j \) and the normal bundle \( N_j \) of \( X_j \) in \( X \) is equivariant. Let \( i_j : X_j \to X \) be the inclusion. The following corollary of the localization theorem will be used extensively here:

**Theorem 2.2.1.** Let \( \alpha \in H^*_T(X) \otimes \mathbb{C}(\lambda) \). Then

\[
\int_{X_T} \alpha = \sum_{j \in J} \int_{(X_j)_T} \frac{i_j^*(\alpha)}{E_T(N_j)}.
\]

A basis for the characters of the torus is given by \( \varepsilon_i(t_0, ..., t_s) = t_i \). There is an isomorphism between the character group of the torus and \( H^2(BT) \) sending \( \varepsilon_i \) to \( \lambda_i \). We will say that the weight of the character \( \varepsilon_i \) is \( \lambda_i \).

For an equivariant vector bundle \( \mathcal{U} \) over \( X \) it may happen that the restriction of \( \mathcal{U} \) on a fixed point component \( X_j \) is trivial (for example if \( X_j \) is an isolated point). In that case \( \mathcal{U} \) decomposes as a direct sum \( \bigoplus_{i=1}^{m_i} \mu_i \otimes \mathcal{O}(\lambda_i) \) of characters of the torus. If the weight of \( \mu_i \) is \( \rho_i \), then the restriction of \( c^*_T(\mathcal{U}) \) on \( X_j \) is the symmetric polynomial \( \sigma_k(\rho_1, ..., \rho_m) \).

Our interest here is for \( X = \mathbb{P}^s \). For any action of \( T \) on \( \mathbb{P}^s \) we will denote

\[
\mathcal{P} := H^*_T(\mathbb{P}^s).
\]

\[
\mathcal{R} := \mathcal{P} \otimes \mathbb{C}(\lambda)
\]

Consider the diagonal action of \( T = (\mathbb{C}^*)^{s+1} \) on \( \mathbb{P}^s \) with weights \( (-\lambda_0, ..., -\lambda_s) \) i.e.

\[
(t_0, t_1, ..., t_s) \cdot (z_0, z_1, ..., z_s) = (t_0^{-1}z_0, ..., t_s^{-1}z_s).
\]

Then \( \mathbb{P}^s_T = \mathbb{P}(\bigoplus_{i \in I} \mathcal{O}(-\lambda_i)) \). There is an obvious lifting of the action of \( T \) on the tautological line bundle \( \mathcal{O}(-1) \). It follows that \( \mathcal{O}(k) \) is equivariant for all \( k \). Let \( p = c^*_T(\mathcal{O}_{\mathbb{P}^s}(1)) \) be the equivariant hyperplane class. We obtain \( \mathcal{P} = \mathbb{C}[\lambda, p]/\prod_j (p - \lambda_j) \) and \( \mathcal{R} = \mathbb{C}(\lambda)[p]/\prod_j (p - \lambda_j) \). The locus of the fixed points consists of points \( p_j \) for \( j = 0, 1, ..., s \) where \( p_j \) is the point whose \( j \)-th coordinate is 1 and all the other ones are 0. On the
level of the cohomology the map $i^*_j$ sends $p$ to $\lambda_j$. A basis for $R$ as a $\mathbb{C}(\lambda)$-vector space is given by $\phi_j = \prod_{k \neq j} (p - \lambda_k)$ for $j = 0, 1, ..., s$. Also $i^*_j(\phi_j) = \prod_{k \neq j} (\lambda_j - \lambda_k) = \text{Euler}_T(N_j)$. The localization theorem 2.2.1 says that for any polynomial $F(p) \in \mathbb{C}(\lambda)[p]/\prod_{i=1}^s (p - \lambda_i)$

$$\int_{\mathbb{P}_s^T} F(p) = \sum_j \frac{F(\lambda_j)}{\prod_{k \neq j} (\lambda_j - \lambda_k)}.$$  

By translating the target of a stable map we get an action of $T$ on $\overline{M}_{0,n}(\mathbb{P}^s, d)$. In [24] Kontsevich identified the fixed point components of this action in terms of decorated graphs. If $f : (C, x_1, ..., x_n) \to \mathbb{P}^s$ is a fixed stable map then $f(C)$ is a fixed curve in $\mathbb{P}^s$. The marked points, collapsed components and nodes are mapped to the fixed points $p_i$ of the $T$-action on $\mathbb{P}^s$. A noncontracted component must be mapped to a fixed line $\overline{p_ip_j}$ on $\mathbb{P}^s$. The only branch points are the two fixed points $p_i$ and $p_j$ and the restriction of the map $f$ to this component is determined by its degree. The graph $\Gamma$ corresponding to the fixed point component containing such a map is constructed as follows. The vertices correspond to the connected components of $f^{-1}\{p_0, p_1, ..., p_s\}$. The edges correspond to the noncontracted components of the map. The graph is decorated as follows. Edges are marked by the degree of the map on the corresponding component, and vertices are marked by the fixed point of $\mathbb{P}^s$ where the corresponding component is mapped to. To each vertex we associate a leg for each marked point that belongs to the corresponding component. For a vertex $v$, let $n(v)$ be the number of legs or edges incident to that vertex. Also for an edge $e$ let $d_e$ be the degree of the stable map on the corresponding component. Let

$$\overline{M}_\Gamma := \prod_v \overline{M}_{0,n(v)}.$$  

There is a finite group of automorphisms $G_\Gamma$ acting on $\overline{M}_\Gamma$. The order of the automorphism group $G_\Gamma$ is

$$a_\Gamma = \prod_e d_e \cdot |\text{Aut}(\Gamma)|.$$  

The fixed point component corresponding to the decorated graph $\Gamma$ is

$$\overline{M}_{\Gamma} = \overline{M}_\Gamma / G.$$  

Let $i_\Gamma : \overline{M}_{\Gamma} \to \overline{M}_{0,n}(\mathbb{P}^s, d)$ be the inclusion of the fixed point component corresponding to $\Gamma$ and $N_\Gamma$ its normal bundle. This bundle is $T$-equivariant. Let $\alpha$ be an equivariant cohomology class in $H^*_T(\overline{M}_{0,n}(\mathbb{P}^s, d))$ and $\alpha_\Gamma := i^*_\Gamma(\alpha)$. Theorem 2.2.1 says:

$$\int_{\overline{M}_{0,n}(\mathbb{P}^s, d)} \alpha = \sum_\Gamma \int_{(\overline{M}_\Gamma)_T} \frac{\alpha_\Gamma}{a_\Gamma \text{Euler}_T(N_\Gamma)}.$$  

Explicit formulas for Euler$_T(N_\Gamma)$ in terms of chern classes of cotangent line bundles in $H^*_T(\overline{M}_\Gamma)$ have been found by Kontsevich in [24].
23. Linear and nonlinear sigma models for a projective space. Two
compactifications of the space of degree $d$ maps $\mathbb{P}^1 \to \mathbb{P}^s$ will be very
important in this paper. $M_d := \overline{M}_{0,0}(\mathbb{P}^s \times \mathbb{P}^1, (d, 1))$ is called the degree $d$ nonlinear sigma model of $\mathbb{P}^s$ and $N_d := \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{s+1})$ is called
the degree $d$ linear sigma model of the projective space $\mathbb{P}^s$. An element in
$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{s+1}$ is an $s + 1$-tuple of degree $d$ homogeneous polynomials
in two variables $w_0$ and $w_1$. As a vector space, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))^{s+1}$ is
generated by the vectors $v_{ir} = (0, ..., 0, w_0^{d-r}w_1^r, 0, ..., 0)$ for $i = 0, 1, ..., s$ and
$r = 0, 1, ..., d$. The only nonzero component of $v_{ir}$ is the $i$-th one.

The action of $T' := T \times \mathbb{C}^*$ in $\mathbb{P}^s \times \mathbb{P}^1$ with weights $(-\lambda_0, ..., -\lambda_s)$ in the
$\mathbb{P}^s$ factor and $(-h, 0)$ in the $\mathbb{P}^1$ factor gives rise to an action of $T'$ in $M_d$
by translation of maps. $T'$ also acts in $N_d$ as follows: for $\ell = (t_0, ..., t_s) \in T$
and $t \in \mathbb{C}^*$

\begin{equation}
(\ell, t) \cdot [P_0(w_0, w_1), ..., P_s(w_0, w_1)] = [t_0P_0(tw_0, w_1), ..., t_sP_s(tw_0, w_1)]
\end{equation}

There is a $T'$-equivariant morphism $\psi : M_d \to N_d$. Here is a set-theoretical
description of this map (for a proof that it is a morphism see [15] or [25]). Let $q_i$ for $i = 1, 2$
be the projection maps on $\mathbb{P}^s \times \mathbb{P}^1$. For a stable map
$(C, f) \in M_d$ let $C_0$ be the unique component of $C$ such that $q_2 \circ f : C_0 \to \mathbb{P}^1$
is an isomorphism. Let $C_1, ..., C_s$ be the irreducible components of $C - C_0$
and $d_i$ the degree of the restriction of $q_1 \circ f$ on $C_i$. Choose coordinates
on $C_0 \cong \mathbb{P}^1$ such that $q_2 \circ f(y_0, y_1) = (y_1, y_0)$. Let $C_0 \cap C_i = (a_i, b_i)$
and $q_1 \circ f = [f_0 : f_1 : ... : f_s] : C_0 \to \mathbb{P}^s$. Then

\begin{equation}
\psi(C, f) := \prod_{i=1}^{n}(b_iw_0 - a_iw_1)^{d_i}[f_0 : f_1 : ... : f_s].
\end{equation}

Let $p_{ir}$ be the points of $N_d$ corresponding to the vectors $v_{ir}$. The fixed point
loci of the $T'$-action on $N_d$ consists of the points $p_{ir}$. We write $\kappa$ for the
equivariant hyperplane class of $N_d$. The restriction of $\kappa$ at the fixed point
$p_{ir}$ is $\lambda_i + rh$. The restriction of the equivariant Euler class of the tangent
space $TN_d$ at $p_{ir}$ is $[25]$

\begin{equation}
E_{ir} = \prod_{(j,t) \neq (i,r)} (\lambda_i - \lambda_j + rh - th).
\end{equation}

Fixed point components of $M_d$ are obtained as follows. Let $\Gamma_{d_j}^i$ be the
d graph of a $T$-fixed point component in $\overline{M}_{0,1}(\mathbb{P}^s, d_j)$ where the marking
is mapped to $p_i$ and $d_1 + d_2 = d$. Let $(d_1, d_2)$ be a partition of $d$. We identify
$\overline{M}_{\Gamma_{d_1}^i} \times \overline{M}_{\Gamma_{d_2}^j}$ with a $T'$-fixed point component $M_{d_1d_2}^i$ in $M_d$ in the following
manner. Let $(C_1, x_1, f_1) \in \overline{M}_{\Gamma_{d_1}^i}$ and $(C_2, x_2, f_2) \in \overline{M}_{\Gamma_{d_2}^j}$. Let $C$
be the nodal curve obtained by gluing $C_1$ with $\mathbb{P}^1$ at $x_1$ and $0 \in \mathbb{P}^1$ and $C_2$
with $\mathbb{P}^1$ at $x_2$ and $\infty \in \mathbb{P}^1$. Let $f : C \to \mathbb{P}^s \times \mathbb{P}^1$ map $C_1$ to the slice $\mathbb{P}^s \times \infty$ by
means of $f_1$ and $C_2$ to $\mathbb{P}^s \times 0$ by means of $f_2$. Finally $f$ maps $\mathbb{P}^1$ to $p_1 \times \mathbb{P}^1$
by permuting coordinates. $\psi$ maps $M_{d_1d_2}^i$ to $p_{id_2} \in N_d$, hence the equivariant
restriction of $\psi^*(\kappa)$ in $M^i_{d_1d_2}$ is $\lambda_i + d_2h$. The normal bundle $N_{\Gamma_{d_1d_2}}$ of this component in the above identification can be found by splitting it in five pieces: smoothing the nodes $x_1$ and $x_2$ and deforming the restriction of the map to $C_1, C_2, \mathbb{P}^1$. Using Kontsevich’s calculations, Givental obtained [15]

$$\frac{1}{E_T(N_{\Gamma_{d_1d_2}})} = \frac{1}{\prod_{k \neq i}(\lambda_i - \lambda_k)} E_T(N_{\Gamma_{d_1}}) E_T(N_{\Gamma_{d_2}}) - h(-h - c_1) h(h - c_2)$$

where $c_j, j = 1, 2$ is the first Chern class of the cotangent line bundle on $\overline{M}_{\Gamma_i}^j$.

3. A Gromov-Witten theory induced by a vector bundle

3.1. The obstruction class of a concavex vector bundle. The notion of concavex vector bundle is due to Lian, Liu and Yau [25] and is central to this work.

**Definition 3.1.1.**

1. A line bundle $\mathcal{L}$ on $X$ is called **convex** if $H^1(C, f^*(\mathcal{L})) = 0$ for any genus zero stable map $(C, x_1, ... x_n, f)$.

2. A line bundle $\mathcal{L}$ on $X$ is called **concave** if $H^0(C, f^*(\mathcal{L})) = 0$ for any nonconstant genus zero stable map $(C, x_1, ... x_n, f)$.

3. A direct sum of convex and concave line bundles on $X$ is called a **concavex** vector bundle.

A concavex vector bundle $V$ in a projective space $\mathbb{P}^s$ has the form

$$V = V^+ \oplus V^- = (\oplus_{i \in I} \mathcal{O}(k_i)) \oplus (\oplus_{j \in J} \mathcal{O}(-l_j))$$

where $k_i$ and $l_j$ are positive numbers. Denote $E^+ := E(V^+)$ and $E^- := E(V^-)$.

Let $d > 0$. Consider the following diagram

$$\begin{array}{ccc}
\overline{M}_{0,n+1}(\mathbb{P}^s, d) & \xrightarrow{e_{n+1}} & \mathbb{P}^s \\
\downarrow{\pi_{n+1}} & & \\
\overline{M}_{0,n}(\mathbb{P}^s, d) & & \\
\end{array}$$

Since $V$ is concavex, the sheaf

$$V_d := V^+_d \oplus V^-_d = \pi_{n+1}^* e_{n+1}(V^+) \oplus R^1 \pi_{n+1}^* e_{n+1}(V^-)$$

is locally free.

**Definition 3.1.2.** The obstruction class corresponding to $V$ is defined to be:

$$E_d := E(V_d) = E(V^+_d)E(V^-_d) := E^+_d E^-_d$$

For a $T$-action on $\mathbb{P}^s$ that lifts to a linear action on the fibers of $V = V^+ \oplus V^-$, let $E^+ := E_T(V^+)$ and $E^- := E_T(V^-)$. Assume that $E^-$ is invertible.
Definition 3.1.3. The modified equivariant integral $\omega_V : \mathcal{R} \to \mathbb{C}(\lambda)$ corresponding to $V$ is defined as follows

$$\omega_V(\alpha) := \int_{\mathbb{P}^n} \alpha \cup \frac{E^+}{E^-}. \quad (19)$$

Consider the trivial action of $T = (\oplus_{i \in I} C^*) \oplus (\oplus_{j \in J} C^*)$ on $\mathbb{P}^s$. In this case $\mathbb{P}^s_T = \mathbb{P}^s \times (\oplus_{i \in I} \mathbb{P}^\infty) \times (\oplus_{j \in J} \mathbb{P}^\infty)$ and $\overline{M}_{0,n}(\mathbb{P}^s, d)_T = \overline{M}_{0,n}(\mathbb{P}^s, d) \times (\oplus_{i \in I} \mathbb{P}^\infty) \times (\oplus_{j \in J} \mathbb{P}^\infty)$. It follows that $\mathcal{P} = H^*(\mathbb{P}^s, \mathbb{C}[\lambda])$ and $\mathcal{R} = H^*(\mathbb{P}^s, \mathbb{C}(\lambda))$. Let $p$ denote the equivariant hyperplane class. The $T$-action lifts to a linear action on the fibers of $V$ with weights $((-\lambda_i)_{i \in I}, (-\lambda_j)_{j \in J})$. Let $q_i$ and $q_j$ denote the projection maps on $\overline{M}_{0,n}(\mathbb{P}^s, d)_T$. Both $V^+_d$ and $V^-_d$ are $T$-equivariant bundles and

$$(V^+_d)_T = V^+_d \otimes (\oplus_{i \in I} q^* i_{\mathbb{P}^\infty}(\lambda_i)) \quad (20)$$

The equivariant obstruction class is

$$E_d := E_T(V_d) = E_T(V^+_d)E_T(V^-_d) = E^+_dE^-_d. \quad (21)$$

The modified equivariant integral for the trivial action of $T$ on $\mathbb{P}^s$ gives rise to a modified perfect pairing in $\mathcal{R}$

$$\langle a, b \rangle_V := \omega_V(a \cup b).$$

Let $T_0 = 1, T_1 = p, ..., T^s = p^s$ be a basis of $\mathcal{R}$ as a $\mathbb{C}(\lambda)$-vector space. The intersection matrix $g_{rt} := \langle (T_r, T_t)_V \rangle$ has an inverse $(g^{rt})$. Let $T^s = \sum_{j=0}^s g^{ij} T_j$ be the dual basis with respect to this pairing. Clearly

$$T^i = T_{m-i} \cdot \left(\frac{E^-}{E^+}\right). \quad (22)$$

This implies that in $H^*(\mathbb{P}^s \times \mathbb{P}^s) \otimes \mathbb{C}(\lambda)$ we have

$$\sum_{i=1}^s T_i \otimes T^i = \Delta \cdot \left(1 \otimes \frac{E^-}{E^+}\right) \quad (23)$$

where $\Delta = \sum_{i=0}^s T_i \otimes T_{s-i}$ is the class of the diagonal in $\mathbb{P}^s \times \mathbb{P}^s$.

Recall that the morphism $\pi_k : \overline{M}_{0,n}(\mathbb{P}^s, d) \to \overline{M}_{0,n-1}(\mathbb{P}^s, d)$ forgets the $k$-th marked point.

Lemma 3.1.1. $\pi_k^*(E_d) = E_d$ and $\pi_k^*(E_d) = E_d$.

**Proof.** For simplicity we will consider the case $V = \mathcal{O}(k) \oplus \mathcal{O}(-l)$ and $k = n$. The general case is similar. Let $M_k = \overline{M}_{0,k}(\mathbb{P}^s, d)$ and $M_{n,n} =$
We now use the spectral sequence
\[ \pi_{n+1}^* \mathcal{O}(-l) = R^1 \pi_{n+1}^* \mathcal{O}(-l) = R^1 \pi_{n+1}^* \mathcal{O}(-l). \]
By the projection formula
\[ \mu_* \mu^* \alpha^* e_n^* k = \alpha^* e_n^* k \otimes \mu_* (\mathcal{O}_{M_{n+1}}). \]
Since the map \( \mu \) is birational and \( M_{n+1} \) is normal \( \mu_* (\mathcal{O}_{M_{n+1}}) = \mathcal{O}_{M_{n,n}} \) hence
\[ \mu_* \mu^* \alpha^* e_n^* k = \alpha^* e_n^* k. \]
Substituting in (24) and applying base extension properties (\( \pi_n \) is flat) yields
\[ \pi_{n+1}^* \mathcal{O}(-l) = \beta_* \alpha^* e_n^* k = \pi_n^* (\pi_n^* e_n^* k). \]
For the case of a negative line bundle we have
\[ R^1 \pi_{n+1}^* \mathcal{O}(-l) = R^1 (\pi_{n+1}^* \pi^* e_n^* k) = R^1 \pi_{n+1}^* \mu^* \alpha^* e_n^* k = R^1 \pi_{n+1}^* \mu^* \alpha^* e_n^* k = R^1 \pi_n^* e_n^* k. \]
We now use the spectral sequence
\[ R^0 \beta_* (R^q \mu_* \mathcal{F}) \implies R^{p+q} \pi_{n+1}^* \mathcal{F} \]
where \( \mathcal{F} \) is a sheaf of \( \mathcal{O}_{M_{n+1}} \)-modules. The map \( \mu \) is birational. If we think of \( M_n \) as the universal map of \( M_{n-1} \), then the map \( \mu \) has nontrivial fibers only over pairs of stable maps in \( M_n \) that represent the same special point (i.e. node or marked point) of a stable map in \( M_{n-1} \). These nontrivial fibers are isomorphic to \( \mathbb{P}^1 \). Since \( \mathcal{F} = e_{n+1}^* \mathcal{O}(-l) \) we obtain \( R^q \mu_* \mathcal{F} = 0 \) for \( q > 0 \). It follows that this spectral sequence degenerates, giving
\[ R^1 \pi_{n+1}^* \mathcal{O}(-l) = R^1 \beta_* \mu_* \mu^* \alpha^* e_n^* k = R^1 \pi_n^* e_n^* k. \]
Now we proceed as in (26) to conclude
\[ R^1 \pi_{n+1}^* \mathcal{O}(-l) = \pi_n^* (R^1 \pi_n^* e_n^* k - l). \]
The lemma is proven.†

Remark 3.1.1. The previous lemma justifies the omission of \( n \) from the notation of the obstruction class.
3.2. Modified equivariant correlators and quantum cohomology.

Let $\gamma_i \in \mathcal{R}$ for $i = 1, \ldots, n$ and $d > 0$. Introduce the following modified equivariant Gromov-Witten invariants:

\[ \tilde{I}_d(\gamma_1, \ldots, \gamma_n) := \int_{\overline{M}_{0,n}(\mathbb{P}^s, d)} e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_n) \cup E_d \in \mathbb{C}(\lambda) \]

Now $\overline{M}_{0,n}(\mathbb{P}^s, 0) = \overline{M}_{0,n} \times \mathbb{P}^s$ and all the evaluation maps equal the projection $q_2$ to the second factor. The integrals in this case are defined as follows

\[ \tilde{I}_0(\gamma_1, \ldots, \gamma_n) := \int_{\overline{M}_{0,n}(\mathbb{P}^s, 0)} e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_n) \cup q_2^*E(V) \in \mathbb{C}(\lambda) \]

The modified equivariant gravitational descendants are defined similarly to Gromov-Witten invariants:

\[ \tilde{I}_d(\tau_1, \gamma_1, \ldots, \tau_n \gamma_n) := \int_{\overline{M}_{0,n}(\mathbb{P}^s, d)} e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_n) \cup E_d. \]

Lemma 3.1.1 is essential in proving that the modified correlators satisfy the same properties, such as fundamental class property, divisor property, point mapping axiom etc., that the usual Gromov-Witten invariants do. The proofs are similar to the ones in pure Gromov-Witten theory. As an illustration, we prove one of these properties.

**Fundamental class property.** Let $\gamma_n = 1$ and $d \neq 0$. The forgetful morphism $\pi_n : \overline{M}_{0,n}(\mathbb{P}^s, d) \to \overline{M}_{0,n-1}(\mathbb{P}^s, d)$ is equivariant. Using Lemma 3.1.1 we obtain:

\[ e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_{n-1}) \cup e_n^*(1) \cup E_d = \pi^*(e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_{n-1}) \cup E_d). \]

Therefore:

\[ \tilde{I}_d(\gamma_1, \ldots, \gamma_{n-1}, 1) = \int_{\overline{M}_{0,n}(\mathbb{P}^s, d)} \pi^*(e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_{n-1}) \cup E_d) = \int_{\pi_n^*\overline{M}_{0,n}(\mathbb{P}^s, d)} e_1^*(\gamma_1) \cup \ldots \cup e_n^*(\gamma_{n-1}) \cup E_d = 0. \]

The last equality is because the fibers of $\pi_n$ are positive dimensional. If $d = 0$, by the point mapping property we know that the integral is zero unless $n = 3$. In that case: $\tilde{I}_0(\gamma_1, \gamma_2, 1) = \langle \gamma_1, \gamma_2 \rangle.\dagger$

We will now prove a technical lemma which will be very useful later. Let $A \cup B$ be a partition of the set of markings and $d = d_1 + d_2$. Let $D = D(A, B, d_1, d_2)$ be the closure in $\overline{M}_{0,n}(\mathbb{P}^s, d)$ of stable maps of the following type. The source curve is a union $C = C_1 \cup C_2$ of two lines meeting at a node $x$. The marked points corresponding to $A$ are on $C_1$ and those corresponding to $B$ are on $C_2$. The restriction of the map $f$ on $C_i$ has degree $d_i$ for $i = 1, 2$. $D$ is a boundary divisor in $\overline{M}_{0,n}(\mathbb{P}^s, d)$. Let $M_1 := \overline{M}_{0,|A|+1}(\mathbb{P}^s, d_1)$ and $M_2 := \overline{M}_{0,|B|+1}(\mathbb{P}^s, d_2)$. Let $e_x$ and $\tilde{e}_x$ be the
evaluation maps at the additional marking in $M_1$ and $M_2$ and $\mu := (e_x, \tilde{e}_x)$. The boundary divisor $D$ is obtained from the following fibre diagram
\[
\begin{array}{c}
M_1 \times M_2 \\
\downarrow \nu \\
\mathbb{P}^s
\end{array}
\begin{array}{c}
\overrightarrow{\iota} \\
\downarrow \mu
\end{array}
\begin{array}{c}
\mathbb{P}^s \times \mathbb{P}^s
\end{array}
\]
where $\nu$ is the “evaluation map at the node $x$” and $\delta$ is the diagonal map.

**Lemma 3.2.1.** For any classes $\gamma_1, \ldots, \gamma_n$ in $\mathcal{R}$:
\[
\int_D \prod_{i=1}^n e_i^*(\gamma_i)E_d = \sum_{a=0}^s \left( \int_{M_1} \prod_{i \in A} e_i^*(\gamma_i)e_x(T_a)E_{d_1} \right) \times \left( \int_{M_2} \prod_{j \in B} e_j^*(\gamma_j)\tilde{e}_x(T^a)E_{d_2} \right).
\]

**Proof.** This lemma is the analogue of the Lemma 16 in [13]. The proof needs a minor modification. Let $\alpha : D \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^s, d)$. Consider the normalization sequence at $x$:
\[
(35) \quad 0 \to \mathcal{O}_C \to \mathcal{O}_C' \oplus \mathcal{O}_C'' \to \mathcal{O}_x \to 0.
\]
Twisting it by $f^*(V^+)$ and $f^*(V^-)$ and taking the cohomology sequence yields the following identities on $D$:
\[
(36) \quad \alpha^*(E_d^+)\nu^*(E^+) = \iota^*(E_{d_1}^+ \times E_{d_2}^+),
\]
and
\[
(37) \quad \alpha^*(E_d^-) = \iota^*(E_{d_1}^- \times E_{d_2}^-)\nu^*(E^-).
\]
By combining equations (36) and (37) we obtain the restriction of $E_d$ in the divisor $D$:
\[
(38) \quad \alpha^*(E_d) = \iota^*(E_{d_1} \times E_{d_2})\nu^* \left( \frac{E^-}{E^+} \right).
\]
Using formula (23) we obtain
\[
(39) \quad \iota_* \nu^* \left( \frac{E^-}{E^+} \right) = \mu^* \left( 1 \otimes \frac{E^-}{E^+} \right) \mu^*(\Delta).
\]
Therefore
\[
\int_D \prod_{i=1}^n e_i^*(\gamma_i) \cup E_d = \int_{M_1 \times M_2} \prod_{i=1}^n e_i^*(\gamma_i) \cup E_{d_1} \cup E_{d_2} \cup \mu^* \left( 1 \otimes \frac{E^-}{E^+} \right) \cup
\]
\[
\cup \mu^*(\Delta) = \int_{M_1 \times M_2} \prod_{i=1}^n e_i^*(\gamma_i) \cup E_{d_1} \cup E_{d_2} \cup \mu^* \left( \sum_a T_a \otimes T^a \right) =
\]
\[
\left( 40 \sum_{a=0}^m \left( \int_{M_1} \prod_{i=1}^{n_1} e_i^*(\gamma_i) \cup e_x(T_a) \cup E_{d_1} \right) \times \left( \int_{M_2} \prod_{j=1}^{n_2} e_j^*(\gamma_j) \cup \tilde{e}_x(T^a) \cup E_{d_2} \right) \right).
\]
The lemma is proven.†
The same proof can be used to show that the previous splitting lemma is true for gravitational descendants as well.

**Corollary 3.2.1.** The following modified topological recursion relations hold:

\[ \tilde{I}_d(\tau_{k_1+1}; \gamma_1, \tau_{k_2}; \gamma_2, \tau_{k_3}; \gamma_3, \prod_{i=4}^{n} \tau_s; \omega_i) = \]

\[ \sum_{i \in I_1} \tilde{I}_{d_1}(\tau_{k_1}; \gamma_1, \prod_{i \in I_1} \tau_s; \omega_i, T_a) \tilde{I}_{d_2}(T^a; \tau_{k_2}; \gamma_2, \tau_{k_3}; \gamma_3, \prod_{i \in I_2} \tau_s; \omega_i) \]

where the sum is over all splittings \( d_1 + d_2 = d \) and partitions \( I_1 \cup I_2 = \{4, \ldots, n\} \) and over all indices \( a \).

**Proof.** Let \( A \) and \( B \) be two disjoint subsets of \( \{1, 2, \ldots, n\} \). We will denote by \( D(A, B) \) the sum of boundary divisors \( D(E, F, d_1, d_2) \) such that \( E, F \) is a partition of \( \{1, 2, \ldots, n\} \) and \( A \subset E \) and \( B \subset F \) and \( d_1 + d_2 = d \). The notation \( D(A, B) \) does not reflect neither the number \( n \) of marked points nor the degree \( d \) of the maps but they will be clear from the context. Consider the morphism \( \pi : \overline{M}_{0,3}(\mathbb{P}^s, d) \to \overline{M}_{0,3} \) that forgets the map and all but the first 3 markings. Since \( \overline{M}_{0,3} \) is a point, the cotangent line bundle at the first marking is trivial. But \( \pi^*(\mathcal{L}_1) = \mathcal{L}_1 - D(\{1\}, \{2, 3\}) \) therefore \( \mathcal{L}_1 = D(\{1\}, \{2, 3\}) \) in \( \overline{M}_{0,n}(\mathbb{P}^s, d) \). Multiply both sides of the previous equation by \( \prod_{j=1}^{3} c_1(\mathcal{L}_j)^k_j \cup e_j^*(\gamma_j) \cup \prod_{i=4}^{n} c_1(\mathcal{L}_i)^{s_i} \cup e_i^*(\omega_i) \cup E_d \) and integrate. The corollary follows from the splitting lemma for gravitational descendents.†

In the process of finding solutions to the WDVV equations, Kontsevich suggested the following modified equivariant Gromov-Witten potential

\[ \tilde{\Phi}(t_0, t_1, \ldots, t_m) := \sum_{n \geq 3} \sum_{d \geq 0} \frac{1}{n!} \tilde{I}_d(\gamma^{\otimes n}) \]

where \( \gamma = t_0 + t_1 p + \ldots + t_s p^s \) and \( t_i \in \mathbb{C}(\lambda) \). Let \( \tilde{\Phi}_{ijk} = \frac{\partial^3 \tilde{\Phi}}{\partial t_i \partial t_j \partial t_k} \).

**Definition 3.2.1.** The modified, equivariant quantum product on \( \mathcal{R} \) is defined to be the linear extension of

\[ T_i \ast_V T_j := \sum_{k=0}^{m} \tilde{\Phi}_{ijk} T^k. \]

**Theorem 3.2.1.** \( \mathcal{Q}H^n_{/\mathbb{P}^s_T} := (\mathcal{R}, \ast_V) \) is a commutative, associative algebra with unit \( T_0 \).

**Proof.** A simple calculation shows that:

\[ \tilde{\Phi}_{ijk} = \sum_{n \geq 0} \sum_{d \geq 0} \frac{1}{n!} \tilde{I}_d(T_i, T_j, T_k, \gamma^{\otimes n}). \]

The commutativity of the modified, equivariant quantum product follows from the symmetry of the new integrals. \( T_0 \) is the unit due to the fundamental class property for the modified Gromov-Witten invariants. To proving
the associativity we proceed as in Theorem 4 in \[12\]. Let \( \check{\Phi}_{ijk} = \frac{\partial^3 \check{\Phi}}{\partial t_i \partial t_j \partial t_k} \).

We compute

\[
(T_i \ast_V T_j) \ast_V T_k = \sum \sum \check{\Phi}_{ijeg} \check{\Phi}_{fkl} g^{ld} T_d,
\]

\[
T_i \ast_V (T_j \ast_V T_k) = \sum \sum \check{\Phi}_{jkeg} \check{\Phi}_{fil} g^{ld} T_d.
\]

Since the matrix \((g^{ld})\) is nonsingular, \((T_i \ast_V T_j) \ast_V T_k = T_i \ast_V (T_j \ast_V T_k)\) is equivalent to

\[
\sum_{e,f} \check{\Phi}_{ijeg} \check{\Phi}_{fkl} = \sum_{e,f} \check{\Phi}_{jkeg} \check{\Phi}_{fil}.
\]

Equation (44) is the WDVV equation for the modified potential \(\check{\Phi}\). To prove this equation let \(q, r, s, t\) be four different integers in \(\{1, 2, \ldots, n\}\). There exists an equivariant morphism:

\[
\pi : \overline{M}_{0,n}(\mathbb{P}^s, d) \to \overline{M}_{0,4} = \mathbb{P}^1
\]

that forgets the map and all the marked points but \(q, r, s, t\). Obviously the divisors \(D(\{q, r\}, \{s, t\})\) and \(D(\{q, s\}, \{r, t\})\) are linearly equivalent in \(\overline{M}_{0,4}\) hence, via the pullback \(\pi^*\), they are linearly equivalent in \(\overline{M}_{0,n}(\mathbb{P}^s, d)\). Now integrate the class

\[
\prod_{i=1}^{n-4} (e_i^*(\gamma)) \cup e_{n-3}^*(T_i) \cup e_{n-2}^*(T_j) \cup e_{n-1}^*(T_k) \cup e_n^*(T_l) \cup E_d
\]

over \(D(\{q, r\}, \{s, t\})\) and use Lemma 3.2.1 to obtain WDVV equation hence the associativity.\[†\]

If we restrict \(\check{\Phi}_{ijk}\) to the divisor classes \(\gamma = tp\), and use the divisor property for the modified Gromov-Witten invariants, we obtain the small product:

\[
T_i \ast_V T_j := T_i \cup T_j + \sum_{d>0} q^d \sum_{k=0}^m \check{I}_d(T_i, T_j, T_k) T_k^k.
\]

Here \(q = e^t\). We extend this product to \(R \otimes_{\mathbb{C}} \mathbb{C}[[q]]\) to obtain the small equivariant quantum cohomology ring \(SQH^*_V\mathbb{P}^s\). We will use \(*_V\) to denote both the small and the big quantum product. The difference will be clear from the context.

**Remark 3.2.1.**

- Equation (38) and Lemma 3.1.1 are the basis for building a modified equivariant Gromov-Witten theory similar to pure Gromov-Witten theory.
- One can see from (22) that the only potential problem with the existence of the nonequivariant limit of (45) is the presence of \(E^+\) in the denominator of \(T^k\). Hence if \(V = V^-\) is a pure negative line bundle
the nonequivariant limit of this product exists. An example of this situation is treated in the last section.

4. A $\mathcal{D}$-module structure induced by $V$

4.1. Equivariant quantum differential equations. Recall from section 2.3 the generator $\hbar$ of $H^2(BC^*)$. Consider the system of first order differential equations on the modified, big quantum cohomology ring $QH^*_V(\mathbb{P}_T)$

\begin{equation}
\frac{\partial}{\partial t_i} = T_i * \hbar \cdot : i = 1, \ldots, m.
\end{equation}

**Theorem 4.1.1.** The space of solutions of these equations has the following basis:

\begin{equation}
s_a = T_a + \sum_{j=0}^m \sum_{n=0}^\infty \sum_{d=0}^\infty \sum_{k=0}^\infty \hbar^{- (k+1)} \frac{1}{n!} \tilde{I}_d(\tau_k T_a, T_j, \gamma \otimes n) T_j =
\end{equation}

\begin{equation}
T_a + \sum_{j=0}^m \sum_{n=0}^\infty \sum_{d=0}^\infty \frac{1}{n!} \tilde{I}_d \left( \frac{T_a}{\hbar - c}, T_j, \gamma \otimes n \right) T_j
\end{equation}

where $c$ is a formal symbol that stands for $c_1^T(\mathcal{L}_1)$ and $\frac{T_a}{\hbar - c}$ should be expanded in powers of $\frac{c}{\hbar}$.

**Proof.** On one hand

\begin{equation}
\frac{\partial s_a}{\partial t_i} = \sum_{j=0}^m \sum_{n=0}^\infty \sum_{d=0}^\infty \sum_{k=0}^\infty \hbar^{- (k+1)} \frac{1}{n!} \tilde{I}_d(\tau_k T_a, T_j, \gamma \otimes n) T_j.
\end{equation}

On the other hand

\begin{equation}
T_i * s_a = T_i * T_a + \sum_{j=0}^m \sum_{n=0}^\infty \sum_{d=0}^\infty \sum_{k=0}^\infty \hbar^{- (k+1)} \frac{1}{n!} \tilde{I}_d(\tau_k T_a, T_j, \gamma \otimes n)(T_i * T_j) =
\end{equation}

\begin{equation}
\sum_{n,d,e} \frac{1}{n!} \tilde{I}_d(T_i, T_a, T_e, \gamma \otimes n) T_e + \sum_{j=0}^m \sum_{n=0}^\infty \sum_{d=0}^\infty \sum_{k=0}^\infty \hbar^{- (k+1)} \frac{1}{n!} \tilde{I}_d(\tau_k T_a, T_j, \gamma \otimes n)
\end{equation}

\begin{equation}
\times \sum_{m,d2,e} \frac{1}{m!} \tilde{I}_{d2}(T_i, T_j, T_e, \gamma \otimes m) T_e.
\end{equation}

The theorem follows from the topological recursion relations $\mathbb{H}$.

Restrictions $\tilde{s}_a$ of the sections $s_a$ to $\gamma \in H^0(\mathbb{P}^m) \oplus H^2(\mathbb{P}^m)$ are solutions of

\begin{equation}
\frac{\partial}{\partial t_i} = T_i * \hbar \cdot : i = 0, 1.
\end{equation}

Repeated use of the divisor axiom yields

\begin{equation}
\tilde{s}_a = e^{\frac{t_0 + pt_1}{\hbar}} \cup T_a + \sum_{d=1}^m \sum_{j=0}^\infty q^d \tilde{I}_d \left( \frac{e^{\frac{t_0 + pt_1}{\hbar}} \cup T_a}{\hbar - c}, T_j \right) T_j,
\end{equation}

\begin{equation}
\text{where } c_1^T(\mathcal{L}_1).
\end{equation}
where \( q := e^{t_1} \)

**Definition 4.1.1.** The module of differential operators that annihilate \( \langle \tilde{s}_a, 1 \rangle_V \) for all \( a \) is called the modified equivariant \( \mathcal{D} \)-module of \( \mathbb{P}^s \) induced by \( V \).

This module is generated by the following \( R[[t_0, t_1, q]] \)-valued function

\[
\tilde{J}_V = \sum_{a=0}^q \langle \tilde{s}_a, 1 \rangle_V T^a.
\]

Recall that \( e_1 : \overline{M}_{0,2}(\mathbb{P}^s, d) \to \mathbb{P}^s \) is the evaluation map at the first marked point and \( c \) is the chern class of the cotangent line bundle at the first marked point. Substituting (50) in (51) and using the projection formula we obtain:

\[
\tilde{J}_V = \exp \left( \frac{t_0 + pt_1}{\hbar} \right) \cdot 
\left( 1 + \sum_{d>0} q^d PD^{-1} \left( e_1 \left( \frac{E}{\hbar - c} \cap \overline{M}_{0,2}(\mathbb{P}^s, d) \right) \right) \cup \left( \frac{E^-}{E^+} \right) \right).
\]

In the above expression \( PD : H^*(\overline{M}_{0,2}(\mathbb{P}^s, d)) \to H_{s+d+d-1-s} \overline{M}_{0,2}(\mathbb{P}^s, d) \) is the Poincaré duality isomorphism.

It will be convenient for us to work with the moduli space of one pointed stable maps. To that end we note that

\[
e_1 \left( \frac{E}{\hbar - c} \cap \overline{M}_{0,2}(\mathbb{P}^s, d) \right) = e_1 \left( \frac{E}{h(h-c)} \cap \overline{M}_{0,1}(\mathbb{P}^s, d) \right).
\]

This identity follows easily from the fact that if \( \pi_2 : \overline{M}_{0,2}(\mathbb{P}^s, d) \to \overline{M}_{0,1}(\mathbb{P}^s, d) \) forgets the second marked point and \( D \) is the image of the universal section of \( \pi \) induced by the marked point, then \( c = \pi_2^*(c) + D \) and \( E_d = \pi_2^*(E_d) \).

The final expression for \( \tilde{J}_V \) is

\[
\tilde{J}_V = \exp \left( \frac{t_0 + pt_1}{\hbar} \right) \cdot 
\left( 1 + \sum_{d>0} q^d PD^{-1} \left( e_1 \left( \frac{E}{h(h-c)} \cap \overline{M}_{0,1}(\mathbb{P}^s, d) \right) \right) \cup \left( \frac{E^-}{E^+} \right) \right).
\]

From this presentation we see that the presence of the equivariant class \( E^+ \) in the denominator of \( \tilde{J}_V \) is a potential problem for the existence of the nonequivariant limit.

**Lemma 4.1.1.** \( \tilde{J}_V \in \mathcal{P}[[q]] \) therefore it has a nonequivariant limit.

**Proof.** Let \( V'_d \) be the subbundle of \( V^+_d \) whose fiber consists of those sections of \( H^0(C, f^*(V^+)) \) that vanish at the marked point. Let \( E'_d := c^T_{top}(V'_d) \). There is an exact sequence of equivariant bundles on \( \overline{M}_{0,1}(\mathbb{P}^s, d) \):

\[
0 \to V'_d \to V^+_d \to e_1^*(V^+) \to 0.
\]
Taking the top Chern classes we obtain

\[ E_d^+ = E'_d \cdot e_1^*(E^+). \]  

(55)

We compute

\[
PD^{-1} e_1^* \left( \frac{E_d}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) = \\
PD^{-1} \left( e_1^* \left( \frac{E'_d E_d^-}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \right) = \\
PD^{-1} \left( E^+ \cup e_1^* \left( \frac{E'_d E_d^-}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \right) = \\
E^+ \cup PD^{-1} \left( e_1^* \left( \frac{E'_d E_d^-}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \right).
\]

(56)

Therefore

\[
\tilde{J}_V = \exp \left( \frac{t_0 + pt_1}{h} \right) \cdot \\
\left( 1 + \sum_{d > 0} q^d PD^{-1} e_1^* \left( \frac{E'_d E_d^-}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \cup E^- \right).
\]

(57)

It is now visible from this presentation that \( \tilde{J}_V \in \mathcal{P}[[q]] \) and

\[
J_V := \lim_{\lambda \to 0} \tilde{J}_V = \\
\exp \left( \frac{t_0 + Ht_1}{h} \right) \left( 1 + \sum_{d > 0} q^d PD^{-1} e_1^* \left( \frac{E'_d E_d^-}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \cup E^- \right).
\]

(58)

The lemma is proven.†

4.2. A local property of the J-function. Let \( Y \) be a smooth projective variety and \( j : \mathbb{P}^s \hookrightarrow Y \) an embedding. Suppose that \( \mathcal{N}_{\mathbb{P}^s/Y} = V^- = \mathcal{O}(-l) \) for some \( l > 0 \). Let \( C \) be a curve in \( \mathbb{P}^s \). The map \( j \) gives rise to an embedding

\[
\overline{M}_{0,n}(\mathbb{P}^s, [C]) \hookrightarrow \overline{M}_{0,n}(Y, j_*([C]))
\]

(59)

Lemma 4.2.1. Let \( C \) be a degree \( d \) rational curve in \( \mathbb{P}^s \). Then \( \overline{M}_{0,n}(\mathbb{P}^s, d) = \overline{M}_{0,n}(Y, j_*([C])) \).

Proof. Let \( (C', x_1, \ldots, x_n, f) \in \overline{M}_{0,n}(Y, j_*([C])) \) and \( f(C') = C_1 \cup C_2 \cup \ldots \cup C_p \) be the irreducible decomposition. Then \( d[\text{line}] = [C_1] + \ldots + [C_p] \). Let \( I_1 = \{ i : C_i \subset \mathbb{P}^s \} \) and \( I_2 = \{ 1, 2, \ldots, n \} - I_1 \). Assume that \( I_2 \) is nonempty. If

\[
d[\text{line}] - \sum_{i \in I_1} [C_i]
\]

has nonpositive degree in \( \mathbb{P}^s \), we intersect with an ample divisor in \( Y \) to see that

\[
d[\text{line}] - \sum_{i \in I_1} [C_i] = \sum_{i \in I_2} [C_i]
\]
is impossible. Otherwise, we intersect with \([\mathbb{P}^s]\) to get the same contradiction. Hence \(I_2\) is empty and all the curves \(C_i\) lie in \(\mathbb{P}^s\). It follows that \(f\) factors through \(\mathbb{P}^s\) and therefore \((C', x_1, \ldots, x_n, f) \in \mathcal{M}_{0,n}((\mathbb{P}^s, d))\). On the other hand \(\mathcal{M}_{0,n}((\mathbb{P}^s, d))\) is a component of \(\mathcal{M}_{0,n}(Y, j_*(\{C\}))\) (see for example section 7.4.4 in [3]). These two arguments imply the lemma.†

Denote \(\mathcal{M}_{0,n}(Y, d) := \mathcal{M}_{0,n}(Y, j_*(\{C\}))\), where \(C\) is any rational curve of degree \(d\) in \(\mathbb{P}^s\). The following Lemma is a special case of a conjecture by Cox, Katz and Lee in [10] which was proved in [23].

**Lemma 4.2.2.** \(\mathcal{M}_{0,n}(Y, d)\)^{virt} = \(E_d \cap \mathcal{M}_{0,n}(\mathbb{P}^s, d)\).

At this point we introduce a new object. For any smooth projective variety \(Y\) and any ring \(\mathcal{A}\), we define the formal completion of \(\mathcal{A}\) along the semigroup of the Mori cone of \(Y\) to be

\[
\mathcal{A}([q^\beta]) := \{ \sum_{\beta} a_{\beta} q^\beta, \ a_{\beta} \in \mathcal{A}, \ \beta - \text{effective} \).
\]

where \(\beta \in H_2(Y, \mathbb{Z})\) is effective if it is a positive linear combination of algebraic curves. This new ring behaves like a power series since for each \(\beta\), the set of \(\alpha\) such that \(\alpha + \beta\) is effective is finite. For example, in the case of \(\mathbb{P}^s\) we obtain the power series \(\mathcal{A}[[q]]\).

Choose generators \(D_1, \ldots, D_r\) of \(H^2(Y, \mathbb{Q})\) such that \(j^*(D_1) = H\) and \(j^*(D_i) = 0\) for \(i \geq 2\). Elements of \(H^0(Y, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q})\) are of the form \(t_0 + tD := t_0 + t_1 D_1 + \ldots + t_r D_r\). It is shown in [15] that the generator of the quantum \(D\)-module for the pure Gromov-Witten theory of \(Y\) is

\[
J_Y = \exp\left(\frac{t_0 + tD}{\hbar}\right) \sum_{\beta \in H_2(Y, \mathbb{Q})} q^\beta P D^{-1} \left( e_{1*} \left( \mathcal{M}_{0,1}(Y, \beta) \right)^{virt} \right)
\]

The moduli spaces \(\mathcal{M}_{0,1}(Y, \beta)\) are empty unless \(\beta\) is effective. Hence we will consider \(J_Y\) as an element of the ring \(H^*Y[[[t_0, t_1, \ldots, t_r]][[q^\beta]]\).

We extend the map \(j^* : H^*Y \to H^*\mathbb{P}^s\) to a homomorphism

\[
j^* : H^*Y[[[t_0, t_1, \ldots, t_r]][[q^\beta]]] \to H^*\mathbb{P}^s[[t_0, t_1]][[q]]
\]

by defining \(j^*(t_i) = 0\) for \(i > 1\) and \(j^*(q^\beta) = q^\beta\) for \(\beta \in j_*(H_2(\mathbb{P}^s, \mathbb{Z}))\) and \(j^*(q^\beta) = 0\) for \(\beta \in H_2(Y, \mathbb{Z}) - j_*(H_2(\mathbb{P}^s, \mathbb{Z}))\). The following results show that \(J\)-function is local.

**Theorem 4.2.1.** \(j^*(J_Y) = J_Y\).

**Proof.** Notice that

\[
j^*J_Y = \exp\left(\frac{t_0 + t_1H}{\hbar}\right) \sum_{d_1 = 0}^{\infty} q^{d_1} J^*P D^{-1} \left( e_{1*} \left( \mathcal{M}_{0,1}(Y, d_1) \right)^{virt} \right)
\]
Consider the following fiber diagram

\[ \overline{M}_{0,1}(\mathbb{P}^s, d_1) \xrightarrow{e_1} \overline{M}_{0,1}(Y, d_1) \]

By excess intersection theory and the previous lemma

\[ j^* \left( e_1^* \left( \frac{\overline{M}_{0,1}(Y, d_1)^{\text{virt}}}{h(h - c)} \right) \right) = E^\bot \cap e_1^* \left( \frac{E_{d_1}}{h(h - c)} \cap \overline{M}_{0,1}(\mathbb{P}^s, d_1) \right). \]

The theorem follows easily:

**Theorem 4.2.2.** Let \( V = V^+ \oplus V^- = \mathcal{O}(k) \oplus \mathcal{O}(-l) \) on \( \mathbb{P}^s \). Let \( \iota : X \hookrightarrow \mathbb{P}^s \) be the zero locus of a generic section of \( V^+ \). Assume that \( X \) is smooth and \( \dim X > 2 \). Let \( Y \) be a smooth projective variety such that \( j : X \hookrightarrow Y \) with \( N = N_{X/Y} = \iota^*(V^-) \). Assume that if \( C \subset Y \) is a curve with \( [C] \in M_X \) then all the irreducible components \( C_i \) of \( C \) satisfy \( C_i \subset X \). Let \( j^* \) be the map constructed as in \([12]\). Let \( J_Y \) be the generator of the pure \( D \)-module of \( Y \) \([14]\). Then

\[ \iota^*(j^*(J_Y)) = E(V^+)j_Y. \]

where \( \iota \) is the Gysin map on cohomology.

**Proof.** Since \( \dim X > 2 \) it follows that \( H^2X \) is generated by \( \iota^*(H) \). Let \( \beta_1 \) be the Poincaré dual to \( \iota^*(H) \) and let \( D_1, D_2, \ldots, D_r \) be a set of generators of \( H^2(Y, \mathbb{Q}) \). We may assume that \( j^*(D_1) = \iota^*(H) \) and \( j^*(D_i) = 0 \) for \( i > 1 \). Let \( tD := t_1D_1 + \ldots + t_rD_r \). Now

\[ J_Y = \exp \left( \frac{t_0 + tD}{h} \right) \sum_{\beta \in H^2(Y, \mathbb{Q})} q^\beta PD^{-1} e_1^* \left( \frac{\overline{M}_{0,1}(Y, \beta)^{\text{virt}}}{h(h - c)} \right) \]

Consider the following diagram

\[ \overline{M}_{0,1}(Y, d_1 \beta_1) \xrightarrow{j_1} \overline{M}_{0,1}(X, d_1 \beta_1) \xrightarrow{\iota_1} \overline{M}_{0,1}(\mathbb{P}^s, d_1) \]

\[ Y \xleftarrow{j} X \xrightarrow{\iota} \mathbb{P}^s \]

The square on the left is a fibre diagram. We repeatedly use the projection formula:

\[ \iota^*(j^*(J_Y)) = \iota^! \left( \exp \left( \frac{t_0 + t_1 \iota^*(H)}{h} \right) \sum_{d_1=0}^{\infty} q^{d_1} PD^{-1} j^* e_1^* \left( \frac{\overline{M}_{0,1}(Y, d_1 \iota_1^*(\beta_1))^{\text{virt}}}{h(h - c)} \right) \right) \]

\[ = \iota^! \left( \exp \left( \frac{t_0 + t_1 \iota^*(H)}{h} \right) \sum_{d_1=0}^{\infty} q^{d_1} \iota^*(E^-) \cup PD^{-1} e_1^* j_1^* \left( \frac{\overline{M}_{0,1}(Y, d_1 \iota_1^*(\beta_1))^{\text{virt}}}{h(h - c)} \right) \right) \]

\[ = \exp \left( \frac{t_0 + t_1 H}{h} \right) \left( E^+ + \sum_{d_1=1}^{\infty} q^{d_1} (E^-) \cup PD^{-1} e_1^* j_1^* \left( \frac{\overline{M}_{0,1}(Y, d_1 \iota_1^*(\beta_1))^{\text{virt}}}{h(h - c)} \right) \right) \]
We apply Proposition 9

There is the following fibre square:

Therefore:

We compute:

\[
\exp \left( \frac{t_0 + t_1 H}{\hbar} \right) \left( \mathcal{E}^+ + \sum_{d_1 = 1}^{\infty} q_1^{d_1} (E^-) \cup PD^{-1} e_{1^*} j_1^* \left( \frac{[\overline{M}_{0,1}(Y, d_1 j_*(\beta_1))]^{\text{virt}}}{h(h - c)} \right) \right).
\]

The equality in the second row follows from excess intersection theory in the left square. An argument similar to Lemma 4.2.1 implies that

\[
\overline{M}_{0,1}(X, d_1 \beta_1) = \overline{M}_{0,1}(Y, d_1 j_*(\beta_1)).
\]

There are two obstruction theories in this moduli stack corresponding to the moduli problems of maps to \( X \) and \( Y \) respectively. They differ exactly by the bundle \( R^1 \pi_2 e_2^*(\mathcal{N}) \) where

\[
\pi_2 : \overline{M}_{0,2}(X, d_1 \beta_1) \to \overline{M}_{0,1}(X, d_1 \beta_1)
\]
is the map that forgets the second marked point and \( \mathcal{N} = \mathcal{N}_{X/Y} \). It follows that:

\[
j_1^*([\overline{M}_{0,1}(Y, d_1 j_*(\beta_1))]^{\text{virt}}) = E(R^1 \pi_2 e_2^*(\mathcal{N}) \cap [\overline{M}_{0,1}(X, d_1 \beta_1)]^{\text{virt}}.
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\overline{M}_{0,2}(X, d_1 \beta_1) & \xrightarrow{e_2} & X \\
\downarrow {\iota_2} & & \downarrow {\iota} \\
\overline{M}_{0,2}(\mathbb{P}, d_1) & \xrightarrow{e_2} & \mathbb{P}
\end{array}
\]

We compute:

\[
e_2^*(\mathcal{N}) = e_2^*(\iota^*(\mathcal{O}(-l))) = \iota_2^* e_2^*(\mathcal{O}(-l)).
\]

There is the following fibre square:

\[
\begin{array}{ccc}
\overline{M}_{0,2}(X, d_1 \beta_1) & \xrightarrow{\iota_2} & \overline{M}_{0,2}(\mathbb{P}, d_1) \\
\downarrow {\pi_2} & & \downarrow {\pi_2} \\
\overline{M}_{0,1}(X, d_1 \beta_1) & \xrightarrow{\iota_1} & \overline{M}_{0,1}(\mathbb{P}, d_1)
\end{array}
\]

We apply Proposition 9.3 in [18] to obtain:

\[
R^1 \pi_2 e_2^*(\mathcal{N}) = R^1 \pi_2 {\iota_2}^* e_2^*(\mathcal{O}(-l)) = \iota_1^* (R^1 \pi_2 e_2^*(\mathcal{O}(-l))) = \iota_1^* (V_{d_1^-}).
\]

Therefore:

\[
j_1^*([\overline{M}_{0,1}(Y, d_1 j_*(\beta_1))]^{\text{virt}}) = E(R^1 \pi_2 e_2^*(\mathcal{N}) \cap [\overline{M}_{0,1}(X, d_1 \beta_1)]^{\text{virt}} = (67) \ i_1^*(E_{d^-}) \cap [\overline{M}_{0,1}(X, d_1 \beta_1)]^{\text{virt}}.
\]

On the other hand, Proposition 11.2.3 of [9] says that:

\[
\iota_1^* [\overline{M}_{0,1}(X, d_1 \beta_1)]^{\text{virt}} = E_d^+ \cap [\overline{M}_{0,1}(\mathbb{P}, d_1)].
\]
Substituting (67) and (68) in (65) we obtain
\[
\iota_2(j^*(J_Y)) = \exp \left( \frac{t_0 + t_1 H}{h} \right) \cdot 
\left( E^+ + \sum_{d=1}^{\infty} q_1^d PD^{-1} e_1\ast \left( \frac{E_d}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \cup (E^-) \right). 
\]
(69)

Recall that on \( H^*(\overline{M}_{0,1}(\mathbb{P}^s, d)) \) we have \( E_d = E_d' E_d^+ e_1^+(E^+) \). Substituting this in (69) and using the projection formula we obtain
\[
\iota_1(j^*(J_Y)) = \exp \left( \frac{t_0 + t_1 H}{h} \right) \cup E^+ \cup 
\left( 1 + \sum_{d=1}^{\infty} q_1^d PD^{-1} e_1\ast \left( \frac{E_d'}{h(h - c)} \cap [\overline{M}_{0,1}(\mathbb{P}^s, d)] \right) \cup (E^-) \right). 
\]
(70)

The theorem is proven.†

**Remark 4.2.1.** This naturally leads to local mirror symmetry. For example, let \( Y \) be a Calabi-Yau threefold that contains \( X = \mathbb{P}^2 \). By adjunction formula, the normal bundle of \( \mathbb{P}^2 \) in \( X \) is \( K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3) \). The last theorem asserts that the restriction of \( J_Y \) in \( X \) depends only on \( V = \mathcal{O}_{\mathbb{P}^2}(-3) \) i.e. in a neighborhood of \( X \) in \( Y \). Hence \( J_Y \) encodes Gromov-Witten correlators of the total space of \( \mathcal{O}_{\mathbb{P}^2}(-3) \) which is a local Calabi-Yau. In the next section we will see that mirror symmetry can be applied to \( J_Y \) establishing that mirror symmetry is local at least on the A-side. Interesting calculations in this direction can be found in [3].

5. Mirror Theorem

In this section we will formulate and prove The Mirror Theorem which computes the generator \( J_Y \). Recall that \( V = (\oplus_{i \in I} \mathcal{O}(k_i)) \oplus (\oplus_{j \in J} \mathcal{O}(-l_j)) = V^+ \oplus V^- \) with \( k_i, l_j > 0 \) for all \( i \in I \) and \( j \in J \). Consider the \( H^* \mathbb{P}^s \)-valued hypergeometric series
\[
I_V(t_0, t_1) := \exp \left( \frac{t_0 + t_1 H}{h} \right) \sum_{d=0}^{\infty} q_1^d \prod_{i \in I} k_i^d \prod_{j \in J} \prod_{m=1}^{l_j d - 1} (-l_j H - mh) \prod_{m=1}^{d} (H + mh)^{s+1}. 
\]
(71)

**Theorem 5.0.3.** (Mirror theorem). Assume that \( \sum_{i \in I} k_i + \sum_{j \in J} l_j \leq s + 1 \) and that \( J \) is nonempty. If \( |J| > 1 \) or \( \sum_{i \in I} k_i + \sum_{j \in J} l_j < s + 1 \) then \( J_V = I_V \). Otherwise, there exists a power series \( I_1 \) of \( q \) such that \( J_V(t_0, t_1 + I_1) = I_V(t_0, t_1) \) as power series of \( q \).

**Remark 5.0.2.** The case in which \( J \) is empty has been treated in [5], [6], [15], [20]. It was suggested by Givental that his techniques should apply in the case in which \( J \) is nonempty.
5.1. The Equivariant Mirror Theorem. We use Givental’s approach for complete intersections in projective spaces [15] to prove an equivariant version of the theorem. For the remainder of this paper we will use the standard diagonal action of $T = (\mathbb{C}^*)^{s+1}$ on $\mathbb{P}^s$ with weights $(-\lambda_0, ..., -\lambda_s)$. Recall from section 2.2 that $\mathcal{P} = H^*_T(\mathbb{P}^s) = \mathbb{C}[\lambda]/\prod_{i=0}^{s}(p - \lambda_i)$ and $\mathcal{R} = \mathbb{C}(\lambda)[p]/\prod_{i=0}^{s}(p - \lambda_i)$. Denote

$$J_{eq}^V := \exp \left( \frac{t_0 + pt_1}{\hbar} \right) \sum_{d=0}^{\infty} q^d e_{1*} \left( \frac{E_d E_d^{-1}}{\hbar(h - c)} \right) \cup \left( \prod_{j \in J} -l_j p \right) = \exp \left( \frac{t_0 + t_1 p}{\hbar} \right) S(q, h).$$

(72)

and

$$I_{eq}^V := \exp \left( \frac{t_0 + t_1 p}{\hbar} \right) \sum_{d=0}^{\infty} q^d \prod_{i \in I} \prod_{m=1}^{k_{id}} (k_i p + m\hbar) \prod_{j \in J} \prod_{m=0}^{l_{jd} - 1} (-l_j p - m\hbar) \prod_{m=0}^{d} \prod_{i=0}^{s}(p - \lambda_i + m\hbar) \prod_{j \in J} \prod_{m=0}^{l_{jd} - 1} (-l_j p - m\hbar) = \exp \left( \frac{t_0 + t_1 p}{\hbar} \right) S'(q, h).$$

(73)

Obviously the nonequivariant limits of $J_{eq}^V$ and $I_{eq}^V$ are respectively $J_V$ and $I_V$. The mirror theorem will follow as a nonequivariant limit of the following

**Theorem 5.1.1. (The Equivariant Mirror Theorem).** The same change of variables from theorem 5.0.3 transforms $I_{eq}^V$ into $J_{eq}^V$.

**Remark 5.1.1.** As the reader will see, the central part of the proof of the Mirror Theorem (up to section 5.5) involves lengthy formulae and algebraic manipulations. To simplify the presentation, we will assume during this part that $V = \mathcal{O}(k) \oplus \mathcal{O}(-l)$. The general case is similar. We will return to the general case $V = V^+ \oplus V^-$ in section 5.5.

Recall that the equivariant Thom classes $\phi_i$ of the fixed points $p_i$ form a basis of $\mathcal{R}$ as a $\mathbb{C}(\lambda)$-vector space. Let $S_i$ and $S_i'$ be the restrictions of $S$ and $S'$ at the fixed point $p_i$. By the localization theorem in $\mathbb{P}^s$ they determine $S$ and $S'$. By the projection formula

$$S_i = \int_{\mathbb{P}^s} S \cup \phi_i = 1 + \sum_{d=1}^{\infty} q^d \int_{\mathbb{P}^s/\mathbb{P}^0} e^\Phi_{q}(-l_{1p}\phi_i) \frac{E_d E_d^{-1}}{\hbar(h - c)}. $$

(74)

The proof of the equivariant mirror theorem is based on exhibiting similar properties of the correlators $S_i$ and $S_i'$. The extra property $S_i = 1 + o(h^{-2})$ determines $S_i'$ uniquely. After the change of variables that property is satisfied by $S_i'$ as well which implies $S_i = S_i'$.

We now proceed with displaying properties of the correlators $S_i$ and $S_i'$.
5.2. Linear recursion relations. The first property is given by this

**Lemma 5.2.1.** The correlators $S_i$ satisfy the following linear recursion relations:

\[
S_i = 1 + \sum_{d=1}^{\infty} q^d R_{id} + \sum_{d=1}^{\infty} \sum_{j \neq i} q^d C_{ijd} S_j \left( q, \frac{\lambda_j - \lambda_i}{d} \right)
\]

where $R_{id} \in \mathbb{C}(\lambda)[h^{-1}]$ are polynomials in $h^{-1}$ and

\[
C_{ijd} = \frac{(\lambda_j - \lambda_i) \prod_{m=1}^{d}(k\lambda_i + m\frac{\lambda_i - \lambda_j}{d}) \prod_{m=0}^{d-1}(-l\lambda_i + m\frac{\lambda_i - \lambda_j}{d})}{dh(dh + \lambda_i - \lambda_j) \prod_{m=1}^{d} \prod_{k=0,(k,m) \neq (j,d)}(\lambda_i - \lambda_k + m\frac{\lambda_i - \lambda_j}{d})}.
\]

**Proof.** We will see during the proof that $S_i$ is regular at $h = \frac{\lambda_j - \lambda_i}{d}$. The integrals that appear in the formula for $S_i$ can be evaluated using localization theorem

\[
\int_{\overline{M}_{0,1}(\mathbb{P}^s, d)} e^\star_i(-lp^\star \phi_i) \frac{E_d^\prime}{h(h-c)} E_d = \sum_{\Gamma} \int_{(\overline{M}_\Gamma)_{\mathbb{R}}} \frac{1}{a_{\Gamma EN}(N_{\Gamma})} \left( e^\star_i(-lp^\star \phi_i) \frac{E_d^\prime}{h(h-c)} E_d \right) \Gamma
\]

There are three types of fixed point components $M_{\Gamma}$ of $\overline{M}_{0,1}(\mathbb{P}^s, d)$. The first one consists of those $M_{\Gamma}$ where the component of the curve that contains the marked point is collapsed to $p_i$. We denote the set of these components by $\mathcal{F}_{1,d}$. Let $\mathcal{F}_{2,d}$ be the set of those $M_{\Gamma}$ in which the marked point is mapped at $p_i$ and its incident component is a multiple cover of the line $p_i, \overline{p}_j$ for some $j \neq i$. Finally let $\mathcal{F}_{0,d}$ be the rest of the fixed point components. Notice first that:

\[
\sum_{\Gamma \in \mathcal{F}_{0,d}} \int_{(\overline{M}_\Gamma)_{\mathbb{R}}} \frac{1}{a_{\Gamma EN}(N_{\Gamma})} \left( e^\star_i(-lp^\star \phi_i) \frac{E_d^\prime}{h(h-c)} E_d \right) \Gamma = 0.
\]

Indeed, let $\Gamma_j \in \mathcal{F}_{0,d}$ represent a fixed point component with the marked point mapped to the fixed point $p_j$ for some $j \neq i$. Since $(e^\star_i(\phi_i))_{\Gamma_j} = 0$ we are done. Next, in each fixed point component that belongs to $\mathcal{F}_{1,d}$ the class $c$ is nilpotent. Indeed, if $\Gamma$ is the decorated graph that represents such a fixed point component, let $\overline{M}_{0,k}$ correspond to the vertex of $\Gamma$ that contains the marked point. Then $k \leq d + 1$. There is a morphism:

\[
\varphi : M_{\Gamma} \rightarrow \overline{M}_{0,k}
\]

such that $\varphi^\star(c) = c_{\Gamma}$. For dimension reasons $c^{d-1} = 0$ on $\overline{M}_{0,k}$ therefore $\frac{1}{h(h-c)}$ is a polynomial of $c$ in $\overline{M}_{\Gamma}$. Hence

\[
\sum_{\Gamma \in \mathcal{F}_{1,d}} \int_{(\overline{M}_\Gamma)_{\mathbb{R}}} \frac{1}{a_{\Gamma EN}(N_{\Gamma})} \left( e^\star_i(-lp^\star \phi_i) \frac{E_d^\prime}{h(h-c)} E_d \right) \Gamma = R_{id}
\]

is a polynomial in $h^{-1}$. 


We now consider the fixed point components in $\mathcal{F}_{2,d}$. Again let $\Gamma$ represent such a component. For a stable map $(C, x_1, f)$ in $\Gamma$ let $C'$ be the component of $C$ containing $x_1$, $C''$ the rest of the curve, $x = C' \cap C''$ and $f(x) = p_j$ for some $j \neq i$. Let $d'$ be the degree of the map $f$ on the component $C'$ and $d'' = d - d'$. Then $(C'', x, f|_{C''})$ is a fixed point in $M_{0,1}(\mathbb{P}^s, d'')$. Denote its decorated graph by $\Gamma''$. Choose the coordinates on $C'$ such that the restriction of $f$ on $C'$ is given by $f(y_0, y_1) = (0, ..., z_i = y_0^{d''}, ..., z_j = y_1^{d''}, ..., 0)$. As $\Gamma$ moves in $\mathcal{F}_{2,d}$, the set of all such $\Gamma''$ exhausts all the fixed points in $M_{0,1}(\mathbb{P}^s, d'')$ where the first marked point is not mapped to $p_i$. Since $\text{Aut}(\Gamma) = \text{Aut}(\Gamma'')$ it follows from (10) that:

$$a_\Gamma = d'a''_{\Gamma''}.$$  

The local coordinate at $p_i$ on the component $C'$ is $z = \frac{y_1}{y_0}$. The weight of the $T$-action on $y_l$ is $\frac{\lambda_l}{d'}$ for $l = 0, 1$. It follows that the weight of the action on the coordinate $z$ and hence on $T^*_{p_i} C'$ is $\frac{\lambda_j - \lambda_i}{d'}$ therefore $c_T = \frac{\lambda_j - \lambda_i}{d'}$. Now $E_T(N_\Gamma)$ can be split in three pieces: smoothing the node $x$ and deforming the maps $f|_{C''}$ and $f|_{C'}$. It follows [15]

$$E_T(N_\Gamma) = \left(\frac{\lambda_j - \lambda_i}{d'} - c_T\right)E_T(N_{\Gamma''}) \cdot \prod_{m=0}^{d'-1} \prod_{k=0, (m,k)\neq (0,i)}^{s} \left(\lambda_i - \lambda_k + m \frac{\lambda_j - \lambda_i}{d'}\right).$$

Next, we find the localization of $E_d^+$ and $E_d^-$ on the fixed point component $M_{d_1,d_2}$. Consider the normalization sequence at the node $x$

$$0 \rightarrow \mathcal{O}_{C'} \rightarrow \mathcal{O}_{C''} \oplus \mathcal{O}_{C''} \rightarrow \mathcal{O}_{x} \rightarrow 0.$$  

Twisting it by $f^*(V^+)$ and $f^*(V^-)$ respectively and taking the cohomology sequence yields

$$(E_d^-)_\Gamma = (-l\lambda_j)(E_d^-)_{\Gamma''}(E_d^-)_\Gamma$$

and

$$(E_d^+)_\Gamma = \frac{(E_d^+)_{\Gamma''}(E_d^+)_{\Gamma''}}{k\lambda_j}.$$  

An explicit basis for $H^1(C', f^*(V^-)) = H^1(\mathcal{O}_{\mathbb{P}^s}(-ld'))$ consists of

$$\frac{y_0^{s}y_1^{ld'-2-s}}{(y_0y_1)^{ld'-1}} \cdot \frac{1}{y_0^{ld'-s-1}y_1^{1+s}} : s = 0, 1, ..., ld' - 2.$$  

It allows us to compute:

$$(E_d^-)_{\Gamma''} = \prod_{s=0}^{ld'-2} \left(1 + s - \frac{ld'}{d'}\lambda_i - \frac{1 + s}{d'}\lambda_j\right) = \prod_{s=1}^{ld'-1} \left(-l\lambda_i + s \frac{\lambda_j - \lambda_i}{d'}\right).$$
Therefore we have:

\[ (E_d')^\Gamma = (-l \lambda_j) \prod_{s=1}^{ld'-1} \left( -l \lambda_i + s \frac{\lambda_j - \lambda_i}{d'} \right) (E_{d''}')^\Gamma. \]

A basis for \( H^0(C', f^*(V^+)) = H^0(C'_d(kd')) \) consists of monomials \( y_0^* y_1^{kd'-s} \) for \( s = 0, \ldots, kd' \). It can be used to calculate \((E_{d'}')^\Gamma\) similarly to \((E_d')^\Gamma\) above. Recall from (53) that \( E_{d'}^+ = e_1^*(E(V^+))E_{d'}' \). The line bundle \( e_1^*(V^+) \) is trivial on \( M_\Gamma \) but the torus acts on it with weight \( k \lambda_i \). Hence \((E_{d'}')^\Gamma = k \lambda_i (E_d')^\Gamma\).

Substituting in (83) yields

\[ (E_d')^\Gamma = \prod_{r=1}^{kd'} \left( k \lambda_i + r \frac{\lambda_j - \lambda_i}{d'} \right) (E_{d''}')^\Gamma. \]

We pause here to show that \( S_i \) is regular at \( h = \frac{\lambda_j - \lambda_i}{d} \) for any \( j \neq i \) and any \( d > 0 \). It follows from (83) and (79) that:

\[ S_i = 1 + \sum_{d=1}^{\infty} q^d R_{id} + \sum_{\Gamma \in \mathcal{F}_{d},d} q^d \int_{(M_{\Gamma})^T} \frac{(-l \lambda_j) \prod_{k \neq i} (\lambda_i - \lambda_k)(E_d' \cdot E_d')^\Gamma}{h(h - c_{\Gamma})a_{\Gamma} E_T(N_{\Gamma})}. \]

From this representation of \( S_i \) it is clear that the coefficients of the power series \( S_i = \sum_{d=0}^{\infty} S_{id} q^d \) belong to \( \mathbb{Q}(\lambda, h) \). But \( c_{\Gamma} = \frac{\lambda_i - \lambda_j}{d} \) for some \( d' \leq d \) and \( R_{id} \) has poles only at \( h = 0 \) therefore \( S_i \) is regular at \( h = \frac{\lambda_i - \lambda_j}{d} \). We use the equations (81), (86) and (85) to compute

\[ \sum_{\Gamma \in \mathcal{F}_{d},d} q^d \int_{(M_{\Gamma})^T} \frac{(-l \lambda_j) \prod_{k \neq i} (\lambda_i - \lambda_k)(E_d' \cdot E_d')^\Gamma}{h(h - c_{\Gamma})a_{\Gamma} E_T(N_{\Gamma})} = \]

\[ \sum_{d'=1}^{\infty} \sum_{j \neq i} q^{d'} C_{ijd'} \sum_{d''} q^{d''} \int_{(M_{\Gamma'})^T} \frac{-l \lambda_j \prod_{k \neq j} (\lambda_j - \lambda_k)(E_{d''} \cdot E_{d''})^\Gamma}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_i) - c_{\Gamma''}^d a_{\Gamma''} E_T(N_{\Gamma''})} = \]

\[ \sum_{d'=1}^{\infty} \sum_{j \neq i} q^{d'} C_{ijd'} S_j \left( q, \frac{\lambda_j - \lambda_i}{d'} \right). \]

The lemma follows by substituting the above identity into (87)\( \dagger \).

**Lemma 5.2.2.** The correlators \( S_i' \) satisfy the same linear recursion relations as \( S_i \).

**Proof.** We know that \( S_i' = \sum_{d=0}^{\infty} q^d S_{id}' \) with

\[ S_{id}' = \frac{\prod_{m=1}^{kd}(k \lambda_i + mh) \prod_{m=0}^{ld-1}(-l \lambda_i - mh)}{dh \prod_{m=1}^{d} \prod_{j=0,(j,m) \neq (i,d)}^{d} (\lambda_i - \lambda_j + mh)}. \]
Note that $S'_{id} \in \mathbb{C}(\lambda, h)$ is a proper rational expression of $h$. It has multiple poles at $h = 0$ and simple poles at $h = \frac{\lambda_r - \lambda_i}{m}$ for any $r \neq i$ and any $1 \leq m \leq d$. Applying calculus of residues in the $h$-variable yields:

$$S'_{id} = R_{id} + \sum_{m=1}^{d} \sum_{r \neq i} \frac{1}{dh(\lambda_i - \lambda_r + mh)}$$

(90)

$$\times \frac{\prod_{n=1}^{kd} (k\lambda_i + n\frac{\lambda_r - \lambda_i}{m}) \prod_{n=0}^{ld-1} (-l\lambda_i - n\frac{\lambda_r - \lambda_i}{m})}{\prod_{n=1, (j,n) \neq (r,m)}^{d} \prod_{j=0, (j,n) \neq (i,d)}^{s} (\lambda_i - \lambda_j + n\frac{\lambda_r - \lambda_i}{m})}$$

for some polynomials $R_{id} \in \mathbb{C}(\lambda)[h^{-1}]$ such that $R_{id}(0) = 0$. Substitute equation (90) in (89) to obtain:

$$S'_{i} = 1 + \sum_{d=1}^{\infty} q^d R_{id} + \sum_{d=1}^{\infty} q^d \sum_{r \neq i} \frac{1}{d^m (\lambda_i - \lambda_r + mh)}$$

(91)

$$\times \frac{\prod_{n=1}^{kd} (k\lambda_i + n\frac{\lambda_r - \lambda_i}{m}) \prod_{n=0}^{ld-1} (-l\lambda_i - n\frac{\lambda_r - \lambda_i}{m})}{\prod_{n=1, (j,n) \neq (r,m)}^{d} \prod_{j=0, (j,n) \neq (i,d)}^{s} (\lambda_i - \lambda_j + n\frac{\lambda_r - \lambda_i}{m})}.$$
integral in the proof of the following Lemma so that it is easier to see how to proceed in the general case.

**Lemma 5.3.1.** If \( z \) is a variable, the expression:

\[
P(z, h) = \omega_V \left( e^{pz} S(qe^{zq}, h) S(q, -h) \right)
\]

belongs to \( \mathbb{Q}(\lambda)[h[[q, z]] \).

**Proof.** In Section 2.3 we introduced the action of \( T' = T \times \mathbb{C}^* \) on \( \mathbb{P}^s \times \mathbb{P}^1 \) with weights \((-\lambda_0, \ldots, -\lambda_s)\) on the first factor and \((-h, 0)\) in the second factor. Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}_{0,1}(\mathbb{P}^s \times \mathbb{P}^1, (d, 1)) & \xrightarrow{\epsilon_1} & \mathbb{P}^s \times \mathbb{P}^1 \\
\pi
\end{array}
\]

Define:

\( W_d = W^+_d \oplus W^-_d \) := \( \pi_*(\pi'_*(\mathcal{O}_{\mathbb{P}^s}(k) \otimes \mathcal{O}_{\mathbb{P}^1})) \oplus R^1 \pi'_*(\mathcal{O}_{\mathbb{P}^s}(-l) \otimes \mathcal{O}_{\mathbb{P}^1}) \)

The lemma will follow from the identity:

\[
P(z, h) = \sum_{d=0}^{\infty} q^d \int_{(\mathcal{M}_d)_{T'}} e^{z\psi^*k} E_{T'}(W_d)
\]

where \( \psi \) and \( \kappa \) were defined in Section 2.3. The localization formula for the diagonal action of \( T \) on \( \mathbb{P}^s \) applied to the left side gives

\[
P(z, h) = \sum_{i=0}^{s} S_i(qe^{zq}, h) e^{z\lambda_i} S_i(q, -h) \left( \frac{k\lambda_i}{-l\lambda_i} \right).
\]

We recall from identity \( \text{[74]} \)

\[
S_i = 1 + \sum_{d=1}^{\infty} q^d \int_{\mathcal{M}_{0,1}(\mathbb{P}^s, d)_{T'}} \frac{e^*(-l \phi_i)}{h(h - c)} E_d^* E_d^-. \]

To compute the integrals on the right side of \( \text{[96]} \) we will use localization for the action of \( T' \) on \( \mathcal{M}_d \). In Section 2.3 we found that the components of the fixed point loci have the form \( \mathcal{M}_d^i = \overline{\mathcal{M}_d^i} \times \overline{\mathcal{M}_d^j} \) for some \( i = 0, 1, \ldots, s \) and a splitting \( d = d_1 + d_2 \). We first compute the restriction of \( E_{T'}(W_d) \) in such a component. Consider the following normalization sequence

\[
0 \to \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}_0} \oplus \mathcal{O}_{\mathbb{C}_1} \oplus \mathcal{O}_{\mathbb{C}_2} \to \mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2} \to 0
\]

Twist \( \text{[99]} \) by \( f^*(\mathcal{O}(-l) \otimes \mathcal{O}_{\mathbb{P}^1}) \) and take the corresponding long exact cohomology sequence. We obtain

\[
0 \to \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}_1}(-l) \oplus \mathcal{O}_{\mathbb{C}_2}(-l) \to W^-_d \to W^-_{d_1} \oplus W^-_{d_2} \to 0.
\]

The first piece is trivial since it comes from the isomorphism

\[
\mathcal{O}_{\mathbb{P}^s}(-l) \otimes \mathcal{O}_{\mathbb{P}^1})|_{\mathbb{C}} \cong \mathcal{O}_{\mathbb{C}} \cong \mathbb{C}.
\]
The left hand side is generated by $\frac{1}{z_i}$ therefore the weight of that piece is $-l\lambda_i$. It follows that

$$E_{T'}(W_d^-) = (l\lambda_i)E_{d_1}^-E_{d_2}^-.$$  

Similarly, twisting the normalization sequence (99) by $f^*(\mathcal{O}(k) \otimes \mathcal{O}_{\mathbb{P}^1})$ and taking the corresponding cohomology sequence we obtain:

$$E_{T'}(W_d^+) = (k\lambda_i)E_{d_1}E_{d_2}'.$$

We now use the localization theorem to calculate the integrals on the right side of (98). The equivariant Euler class of the fixed point component $M^{i_1}_{d_1d_2}$ has been calculated at the end of section 2.3.

$$\int_{(M_{d_1})_{d_1}} e^{z\psi_{\nu}} E_{T'}(W_d) = \sum_{\Gamma'_{d_1},\Gamma'_{d_2}} (k\lambda_i)(l\lambda_i) \prod_{k \neq i} (\lambda_i - \lambda_k)$$

If we use localization to compute $S_i$ in (98) and then substitute in (97) we obtain the right side of the last equation.\

**Lemma 5.3.2.** If $z$ is a variable, the expression:

$$P'(z, h) = \omega_V \left( S'(q e^{z\hbar}, h) e^{pz} S'(q, -\hbar) \right)$$

belongs to $\mathbb{Q}(\lambda)[h][[q, z]]$.

**Proof.** The lemma will follow from the identity

$$P'(z, h) = \sum_{d=0}^{\infty} q^d \int_{(N_d)_{d'}} e^{z\kappa} \prod_{m=0}^{kd} (k\kappa - mh) \prod_{m=1}^{ld-1} (-l\kappa + mh).$$

For $d = 0$ the convention

$$\int_{(N_0)_{d'}} e^{z\kappa} \prod_{m=0}^{kd} (k\kappa + mh) \prod_{m=1}^{ld-1} (-l\kappa + mh) = \int_{\mathbb{P}_T} e^{pz} \left( \frac{\prod_{i \in I} k_i \rho_i}{\prod_{j \in J} -l_j \rho_j} \right)$$
is taken. Apply the localization formula to the integrals \([104]\).

\[
P'(z, h) = \sum_{i=0}^{s} k\lambda_i e^{\lambda_i z} \frac{(-l\lambda_i) \prod_{j \neq i}(\lambda_i - \lambda_j)}{\prod_{m=1}^{d} \prod_{j=0}^{s}(\lambda_i - \lambda_j + mh)}
\]

\[
\times \sum_{d_1=0}^{\infty} (qe^{zh})^{d_1} \frac{\prod_{m=1}^{d_1}(k\lambda_i + mh) \prod_{m=0}^{d_1-1}(-l\lambda_i - mh)}{\prod_{m=1}^{d_2} \prod_{j=0}^{s}(\lambda_i - \lambda_j + mh)}
\]

\[
\times \sum_{d_2=0}^{\infty} q^{d_2} e^{(\lambda_i + d_1 h)z} \frac{\prod_{m=1}^{d_2}(k\lambda_i - mh) \prod_{m=0}^{d_2-1}(-l\lambda_i + mh)}{\prod_{m=1}^{d_2} \prod_{j=0}^{s}(\lambda_i - \lambda_j - mh)}
\]

\[
\sum_{i=0}^{s} \frac{1}{\prod_{j \neq i}(\lambda_i - \lambda_j)} \sum_{d_1=0}^{\infty} q^{d_1} e^{(\lambda_i + d_1 h)z} \frac{\prod_{m=1}^{d_1}(k\lambda_i + mh) \prod_{m=0}^{d_1-1}(-l\lambda_i - mh)}{\prod_{m=1}^{d_2} \prod_{j=0}^{s}(\lambda_i - \lambda_j + mh)}
\]

\[
= \sum_{d_1=0}^{\infty} \prod_{m=1}^{d_1}(k\lambda_i + mh) \prod_{m=0}^{d_1-1}(-l\lambda_i - mh) \prod_{m=1}^{d_2} \prod_{j=0}^{s}(\lambda_i - \lambda_j + mh)
\]

But for \(d_1, d_2 > 0\)

\[
\prod_{m=1}^{d_1}(k\lambda_i + mh) \prod_{m=0}^{d_1-1}(-l\lambda_i - mh) \prod_{m=1}^{d_2} \prod_{j=0}^{s}(\lambda_i - \lambda_j + mh)
\]

\[
= \prod_{j \neq i}(\lambda_i - \lambda_j) \prod_{m=1}^{d_1} \prod_{j=0}^{s}(\lambda_i - \lambda_j + mh) \prod_{m=1}^{d_2} \prod_{j=0}^{s}(\lambda_i - \lambda_j - mh)
\]

\[
\prod_{m=1}^{d_1+d_2}(k(\lambda_i + d_1 h) - mh) \prod_{m=0}^{d_1+d_2-1}(-l(\lambda_i + d_1 h) + mh)
\]

\[
\prod_{j=0}^{s} \prod_{m=0}^{d_1+d_2}(\lambda_i - \lambda_j + mh)
\]

Therefore

\[
P'(z, h) = \sum_{d=0}^{\infty} q^{d} \sum_{d_1=0}^{d} \cdots \sum_{i=0}^{s} e^{(\lambda_i + d_1 h)z}
\]

\[
\times \frac{\prod_{m=0}^{d}(k(\lambda_i + d_1 h) - mh) \prod_{m=0}^{d-1}(-l(\lambda_i + d_1 h) + mh)}{\prod_{j=0}^{s} \prod_{m=0}^{d}(\lambda_i - \lambda_j + mh)}
\]

(104)

By the localization formula in \(N_d\) we can see that

\[
P'(z, h) = \sum_{d=0}^{\infty} q^{d} \int_{(N_d)_{\mathbb{R}^r}} e^{z\kappa} \frac{\prod_{m=0}^{kd}(k\kappa - mh) \prod_{m=1}^{ld-1}(-l\kappa + mh)}{\prod_{m=0}^{d}(k\kappa - mh) \prod_{m=1}^{ld-1}(-l\kappa + mh)}
\]

The lemma is proven.†

5.4. **Mirror transformation and uniqueness.** The following two theorems carry over from \([15]\). The first lemma deals with uniqueness.

**Lemma 5.4.1.** Let \(S = \sum_{d=0}^{\infty} S_d q^d\) and \(S' = \sum_{d=0}^{\infty} S'_d q^d\) be two power series with coefficients in \(\mathbb{R}[[h^{-1}]]\) that satisfy the following conditions:

1. \(S_0 = S'_0 = 1\)
2. They both satisfy the recursion relations of Lemma \(5.2.1\)
3. They both have the double polynomiality property of Lemma \(5.3.2\)
4. For any \(d\), \(S_d \equiv S'_d\) mod \((h^{-2})\).
Then $S = S'$.

The second lemma describes a transformation which preserves the properties of lemma 5.4.1.

**Lemma 5.4.2.** Let $I_1$ be a power series in $q$ whose first term is zero. Then $\exp\left(\frac{I_1 p}{\hbar}\right) S(qe^{I_1}, \hbar)$ satisfies conditions 1, 2, 3 of Lemma 5.4.1.

### 5.5. The conclusion of the proof of Mirror Theorem.

Recall that $I_{eq} V = \exp\left( t_0 + pt \frac{\hbar}{h} \right) \left( 1 + \sum_{d=1}^{\infty} q^d \frac{\prod_{i \in I} \prod_{m=1}^{k_i d} (k_i p + m \hbar) \prod_{j \in J} \prod_{m=0}^{l_j d-1} (-l_j p - m \hbar)}{\prod_{m=1}^{d} \prod_{i=0}^{s}(p - \lambda_i + m \hbar)} \right)$.

We are assuming that there is at least one negative line bundle. We expand the second factor of $I_{eq} V$ as a polynomial of $\hbar^{-1}$. Each negative line bundle produces a factor of $\frac{p}{\hbar}$. For example, in the case $V = O(k) \oplus O(-l)$ the expansion yields:

$$I_{eq}^V = \exp\left( t_0 + pt \frac{\hbar}{h} \right) \left( 1 + \frac{p}{\hbar} \sum_{d=1}^{\infty} q^d \frac{(-1)^d (ld - 1)! (kd)!}{(d!)^{s+1}} \frac{1}{\hbar^{d(s+1-k-l)}} + o\left(\frac{1}{\hbar^2}\right) \right)$$

If $V$ contains two or more negative line bundles it follows that $I_{eq}^V = \exp\left( t_0 + pt \frac{\hbar}{h} \right) \left( 1 + o\left(\frac{1}{\hbar^2}\right) \right)$.

Lemma (5.4.1) and (5.4.2) imply that $J_{eq}^V = I_{eq}^V$. If $\sum_{i \in I} k_i + \sum_{j \in J} l_j < s + 1$ the presence of $\frac{1}{\hbar^{d(s+1-k-l)}}$ in the above expansion shows that again

$$I_{eq}^V = \exp\left( t_0 + pt \frac{\hbar}{h} \right) \left( 1 + o\left(\frac{1}{\hbar^2}\right) \right)$$

hence $J_{eq} = I_{eq}$. We may assume that $\sum_{i \in I} k_i + \sum_{j \in J} l_j = s + 1$ and $|J| = 1$. In this case

$$I_{eq}^V = \exp\left( t_0 + pt \frac{\hbar}{h} \right) \left( 1 + I_1 \frac{p}{\hbar} + o\left(\frac{1}{\hbar^2}\right) \right)$$

where $I_1$ is a power series of $q$ whose first term is zero. For example if $V = O(k) \oplus O(-l)$ the power series $I_1$ is

$$I_1 = \sum_{d=1}^{\infty} q^d \frac{(-1)^d (ld - 1)! (kd)!}{(d!)^{s+1}}$$

Recall that $S = 1 + o(\hbar^{-2})$. Therefore

$$\exp\left(\frac{I_1 p}{\hbar}\right) S(qe^{I_1}, \hbar) = 1 + I_1 \frac{p}{\hbar} + o(\hbar^{-2}).$$
Lemma 5.4.2 implies that both \( \exp \left( \frac{I_1 P}{h} \right) S(qe^{t_1}, h) \) and \( S'(q, h) \) satisfy the conditions of the Lemma 5.4.1. It follows that

\[
\exp \left( \frac{I_1 P}{h} \right) S(qe^{t_1}, h) = S'(q, h)
\]

Multiplying both sides of this identity by \( \exp \left( \frac{t_0 + pt}{h} \right) \) yields

\[
(107) \quad J_{V'}^e(t_0, t + I_1) = I_{V'}^e(t_0, t).
\]

This completes the proof.†

**Corollary 5.5.1.** Let \( V = (\oplus_{i \in I} \mathcal{O}(k_i)) \oplus (\oplus_{j \in J} \mathcal{O}(-l_j)). \) For \(|J| > 1 \) or \( k + l < s + 1 \)

\[
e_{1*}\left( \frac{E_d^eE_d^-}{h(h - c)} \right) = \prod_{i \in I} \prod_{m=1}^{k_i} (k_i H + mh) \prod_{j \in J} \prod_{m=1}^{l_j} (-l_j H - mh) \prod_{m=1}^{d} (H + mh)^{s+1}.
\]

**Proof.** As mentioned above in this case we have \( J_{V'}^e = I_{V'}^e. \) Recall that

\[
J_{V'}^e = \exp \left( \frac{t_0 + pt}{h} \right) \left( 1 + \sum_{d > 0} q^d e_{1*} \left( \frac{E_d^eE_d^-}{h(h - c)} \right) \cup \prod_{j \in J} (-l_j p) \right).
\]

We obtain the equivariant identity:

\[
e_{1*}\left( \frac{E_d^eE_d^-}{h(h - c)} \right) \cup \prod_{j \in J} (-l_j p) = \prod_{i \in I} \prod_{m=1}^{k_i} (k_i p + mh) \prod_{j \in J} \prod_{m=0}^{l_j} (-l_j p - mh) \prod_{m=1}^{d} \prod_{k=0}^{p - \lambda_k + mh}.
\]

The restriction of \( p \) to any fixed point \( p_i \) is nonzero. This implies that \( p \) is invertible. Therefore we obtain

\[
e_{1*}\left( \frac{E_d^eE_d^-}{h(h - c)} \right) = \prod_{i \in I} \prod_{m=1}^{k_i} (k_i p + mh) \prod_{j \in J} \prod_{m=1}^{l_j} (-l_j p - mh) \prod_{m=1}^{d} \prod_{k=1}^{p + 1} (p - \lambda_k + mh).
\]

The nonequivariant limit of this identity reads

\[
e_{1*}\left( \frac{E_d^eE_d^-}{h(h - c)} \right) = \prod_{i \in I} \prod_{m=1}^{k_i} (k_i H + mh) \prod_{j \in J} \prod_{m=1}^{l_j} (-l_j H - mh) \prod_{m=1}^{d} (H + mh)^{s+1}.
\]

The theorem is proven.†

This corollary is particularly useful when Euler(\( V^- \)) = 0 in \( \mathbb{P}^s. \) In this case \( J_V = I_V = \exp \left( \frac{t_0 + Ht_1}{h} \right) \) hence the mirror theorem is true trivially.

An example of such a situation is \( V = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \) which is treated in the next section.
6. Examples

6.1. Multiple covers. Let \( C \) be a smooth rational curve in a Calabi-Yau threefold \( X \) with normal bundle \( N = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) and \( \beta = [C] \in H_2(X, \mathbb{Z}) \). Since \( K_X = \mathcal{O}_X \) the expected dimension of the moduli space \( \overline{M}_{0,0}(X, d\beta) \) is zero. However, this moduli space contains a component of positive dimension, namely \( \overline{M}_{0,0}(\mathbb{P}^1, d) \). Indeed, let \( f : \mathbb{P}^1 \to C \) be an isomorphism, and \( g : \mathbb{P}^1 \to \mathbb{P}^1 \) a degree \( d \) multiple cover. Then \( f \circ g \) is a stable map that belongs to \( \overline{M}_{0,0}(X, d\beta) \). For a proof of the fact that \( \overline{M}_{0,0}(\mathbb{P}^1, d) \) is a component of \( \overline{M}_{0,0}(X, d\beta) \) see section 7.4.4 in [9]. Let \( N_d \) be the degree of \( \overline{M}_{0,0}(X, d\beta) \)^{\text{virt}}. We want to compute the contribution \( n_d \) of \( \overline{M}_{0,0}(\mathbb{P}^1, d) \) to \( N_d \). Kontsevich asserted in [24] and Behrend proved in [2] that the restriction of \( \overline{M}_{0,0}(X, d\beta) \)^{\text{virt}} to \( \overline{M}_{0,0}(\mathbb{P}^1, d) \) is precisely \( E_d \) for \( V = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). Therefore:

\[
 n_d = \int_{\overline{M}_{0,0}(\mathbb{P}^1, d)} E_d.
\]

Note that \( \dim \overline{M}_{0,0}(\mathbb{P}^1, d) = 2d - 2 \) and the rank of the bundle \( V_d \) is also \( 2d - 2 \). We use the mirror theorem to compute numbers \( n_d \). Since \( V \) contains two negative line bundles we can apply Corollary 5.5.1

\[
e_1^*(\frac{E_d}{h(h-c)}) = \frac{\prod_{m=1}^{d-1}(-H-mh)^2}{\prod_{m=1}^{d}(H+mh)^2} = \frac{1}{(H+dh)^2}.
\]

An expansion of the left hand side using the divisor property for the modified gravitational descendants yields

\[
e_1^*(\frac{E_d}{h(h-c)}) = \frac{dn_d}{h^2} + \frac{H}{h^3} \int_{\overline{M}_{0,1}(\mathbb{P}^1, d)} c \cup E_d.
\]

where \( c \) is the chern class of the cotangent line bundle at the marked point. On the other hand:

\[
\frac{1}{(H+dh)^2} = \frac{1}{d^2h^2} - \frac{2H}{d^3h^3}.
\]

We obtain the Aspinwall-Morrison formula

\[
n_d = \frac{1}{d^3},
\]

which has been proved by several different methods [24],[28],[32]. We also obtain

\[
\int_{\overline{M}_{0,1}(\mathbb{P}^1, d)} c \cup E_d = -\frac{2}{d^3}.
\]

6.2. Virtual numbers of plane curves. Let \( X \) be a Calabi-Yau threefold containing a \( \mathbb{P}^2 \). As we saw in Remark 4.2.1, the normal bundle of \( \mathbb{P}^2 \) in \( X \) is \( K_{\mathbb{P}^2} = \mathcal{O}(-3) \). Let \( C \) be a rational curve of degree \( d \) in \( \mathbb{P}^2 \). Since \( K_X = \mathcal{O}_X \), the expected dimension of the moduli space \( \overline{M}_{0,0}(X, [C]) \) is zero.
Lemma 4.2.1 says that $\overline{M}_{0,0}(\mathbb{P}^2,d) = \overline{M}_{0,0}(X,[C])$, hence the dimension of this moduli stack is $3d - 1$. Recall the diagram

$$
\begin{array}{ccc}
\overline{M}_{0,1}(\mathbb{P}^2,d) & \xrightarrow{e_1} & \mathbb{P}^2 \\
\downarrow \pi_1 & & \\
\overline{M}_{0,0}(\mathbb{P}^2,d)
\end{array}
$$

From Lemma 4.2.2, the virtual fundamental class of $\overline{M}_{0,0}(X,[C])$ is the refined top Chern class of the bundle $V_d = R^1\pi_* (e^*_1(K_{\mathbb{P}^2}))$ over $\overline{M}_{0,0}(\mathbb{P}^2,d)$. The zero pointed Gromov-Witten invariant:

$$
N_d := \deg[\overline{M}_{0,0}(X,[C])]^\text{virt} = \int_{\overline{M}_{0,0}(\mathbb{P}^2,d)} \mathbb{E}_d
$$

is called the virtual number of degree $d$ rational curves in $X$. As promised in Remark 3.2.1, we will show that the modified equivariant quantum product in this case has a nonequivariant limit. We will also use The Mirror Theorem to calculate these numbers $N_d$.

The modified pairing on $\mathbb{P}^2\mathbb{C}^*$ corresponding to $V = \mathcal{O}_{\mathbb{P}^2}(-3)$ is

$$
\langle a, b \rangle := \int_{(\mathbb{P}^2)\mathbb{C}^*} a \cup b \cup \left( \frac{1}{-3p - \lambda} \right).
$$

Recall that $p$ denotes the equivariant hyperplane class in $\mathbb{P}^2$. Then $1, p, p^2$ is a basis for $R$ as a $\mathbb{Q}(\lambda)$-module. A simple calculation shows that $-\lambda p^2, -3p^2 - \lambda H, -3p - \lambda$ is its dual basis with respect to the above pairing. Since both bases and $E_d$ are polynomials in $\lambda$, we can restrict $I_d$ and $\ast_V$ in $P = H^*(\mathbb{P}^2, \mathbb{Q}[\lambda])$ and take the nonequivariant limit. Denote by $H$ the nonequivariant limit of $p$. We obtain the following nonequivariant quantum product on $H^*\mathbb{P}^2 \otimes \mathbb{Q}[\![q]\!]$

$$
a \ast_V b := a \cup b + \sum_{d=1}^{\infty} q^d T^k I_d(a,b,-3HT_k)
$$

where $T^0 = 1, T^1 = H, T^2 = H^2$ and for $\gamma_1, \gamma_2, \ldots, \gamma_n \in H^*\mathbb{P}^2$

$$
I_d(\gamma_1, \gamma_2, \ldots, \gamma_n) = \int_{\overline{M}_{0,n}(\mathbb{P}^2,d)} e^*_1 \gamma_1 \cup e^*_2 \gamma_2 \cup \ldots \cup e^*_n \gamma_n \cup \mathbb{E}_d.
$$

For example, using the divisor axiom we obtain

$$
H \ast_V H = H^2(1 - 3 \sum_{d>0} q^d d^2 N_d).
$$

Theorem 3.2.1 implies the following:

**Theorem 6.2.1.** $(H^*\mathbb{P}^2 \otimes \mathbb{Q}[\![q]\!], \ast_V)$ is an associative, commutative and unital ring with unity $1 = [\mathbb{P}^2]$. 

Denote by \( i \) the embedding \( i : \mathbb{P}^2 \hookrightarrow X \). Since the normal bundle of \( \mathbb{P}^2 \) in \( X \) is \( \mathcal{O}_{\mathbb{P}^2}(-3) \), it follows that \( i^* \left( -\frac{1}{3}[\mathbb{P}^2] \right) = T^1 \) and \( i^* \left( -\frac{1}{3}[l] \right) = T^2 \). Therefore the map \( i^* : (H^* X, \mathbb{Q}) \to (H^* \mathbb{P}^2, \mathbb{Q}) \) is surjective. Consider the small quantum cohomology rings \( SQH^* X = (H^* X \otimes \mathbb{Q}[[\beta]], \ast) \) and \( SQH^*_V \mathbb{P}^2 := (H^* \mathbb{P}^2 \otimes \mathbb{Q}[[q]], \ast_V) \) where the products are given by three point correlators. Recall from section 4.2 the extension of \( i^* : H^* (X, \mathbb{Q}) \to H^* (\mathbb{P}^2, \mathbb{Q}) \) to \( \tilde{i}^* : SQH^* X \to SQH^*_V \mathbb{P}^2 \). There is a natural relation between the modified quantum product in \( \mathbb{P}^2 \) and the pure product in \( X \).

**Theorem 6.2.2.** The map \( \tilde{i}^* \) is a ring homomorphism.

**Proof.** Complete \( \tau^0 = [X], \tau^1 = -\frac{1}{3}[\mathbb{P}^2], \tau^2 = -\frac{1}{3}H \) into a basis of \( (H^* X, \mathbb{Q}) \) by adding elements from \( \text{Ker}(i^*) \). Let \( \tau_0 = [pt], \tau_1 = H, \tau_2 = [\mathbb{P}^2], \ldots \) be the dual basis. Let \( a, b \in H^* X \). We want to show

\[
\tilde{i}^*(a \ast b) = i^*(a) \ast_V i^*(b).
\]

But

\[
a \ast b = \sum_{\beta \in H_2(X, \mathbb{Q})} \sum_r q^r \tau^r \int_{[M_{0,3}(X, \beta)]_{\text{virt}}} e_1^* a \cup e_2^* b \cup e_3^* \tau_r.
\]

Note that this formula is true for a \( \mathbb{Z} \)-basis, but due to the uniqueness of the quantum product, it is true for any \( \mathbb{Q} \)-basis as well. Now,

\[
\tilde{i}^*(a \ast b) = \sum_{d \geq 0} \sum_r q^d i^* \tau^r \int_{[M_{0,3}(\mathbb{P}^2, d)]_{\text{virt}}} e_1^* (i^* a) \cup e_2^* (i^* b) \cup e_3^* (i^* \tau_r) \mathcal{E}_d.
\]

But \( i^*(\tau^r) = T^k \) for \( r = 0, 1, 2 \) and \( i^*(\tau^r) = 0 \) for \( r \geq 2 \). The theorem follows from the readily checked fact: \( i^*(\tau_k) = -3HT_k \) for \( k = 0, 1, 2 \).

Using the divisor and fundamental class properties of the modified gravitational descendants it is easy to show that:

\[
J_V = \exp \left( \frac{t_0 + t_1 H}{h} \right) \left( 1 - \frac{H^2}{2h^2} \sum_{d=1}^{\infty} q^d d N_d \right).
\]

The hypergeometric series corresponding to the total space of \( V = \mathcal{O}_{\mathbb{P}^2}(-3) \) is:

\[
I_V := \exp \left( \frac{t_0 + t_1 H}{h} \right) \sum_{d=0}^{\infty} q^d \prod_{m=0}^{3d-1} (3H - mh) \prod_{m=1}^{d} (H + mh)^3.
\]

We expand this function

\[
I_V = \exp \left( \frac{t_0 + t_1 H}{h} \right) \left( 1 + I_1 \frac{H}{h} + o\left( \frac{1}{h} \right) \right)
\]

where

\[
I_1 = 3 \sum_{d=1}^{\infty} q^d (-1)^d \frac{(3d-1)!}{(d!)^3}.
\]
The mirror theorem for this case says that \( J(t_0, t_1 + I_1) = I_V(t_0, t_1) \). This theorem allows us to compute the virtual number of rational plane curves in the Calabi-Yau \( X \). The first few numbers are \( 3, -45, 244, 9 \).

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