Toric moment mappings and Riemannian structures

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Abstract

We give an interpretation of the symplectic fibrations of coadjoint orbits for the group $SU(4)$ in terms of the compatibility of Riemannian $G$-reductions in six dimensions. An analysis of moment polytopes associated to a standard Hamiltonian toric action on the coadjoint orbits highlights these relations. This theory leads to a geometrical application to intrinsic torsion classes of 6-dimensional Riemannian structures.

Introduction

In an article of Abbena, Garbiero and Salamon [2] (itself based on [1]), the authors exploit a solid tetrahedron to describe the set of orthogonal almost complex structures on the Iwasawa and other 6-dimensional nilmanifolds. Their employment of the tetrahedron was justified on purely combinatorial grounds, enabling the 16 Gray-Hervella classes of almost Hermitian structures to be represented in terms of unions of vertices, faces, edges and other segments. Our aim is to insert this theory into a more universal setting.

In the present article, we consider more general reductions of a Riemannian structure specified by a subgroup $G$ of $SO(N)$ stabilizing a 2-form. The relevance of 2-forms to the intrinsic torsion of Riemannian structures arises from the isomorphism $\Lambda^2 T^*_p M \cong \mathfrak{so}(N)$ of the space of 2-forms with the orthogonal Lie algebra. Moreover, the orthogonal complement $g^\perp$ of the Lie algebra of $G$ in $\mathfrak{so}(N)$ is a model of the vertical space of the bundle $P/G$ parametrizing the $G$-structures.

Our point of view emphasizes the natural role that symplectic geometry plays in the classification of Riemannian structures. The flag varieties parametrizing reductions of the type we are considering are merely adjoint (equivalently, coadjoint) orbits of $SO(6)$. The coadjoint orbits of any compact Lie group $G$ are precisely the manifolds on which $G$ acts transitively as a compact group of symplectic automorphisms (see [9] [8]). This fact enables us to prove that every $G$-structure defined by a 2-form on a 6-manifold is associated to a moment mapping from a flag variety to $\mathfrak{so}(6)^\ast$. Restriction to a maximum torus gives rise to a moment polytope in $\mathbb{R}^3$, and the shape of the polytope is sensitive to the exact
2-form chosen. This construction then provides a precise geometrical interpretation of the tetrahedron introduced in [2].

Suitable Riemannian $G$-structures can be considered as reductions of spin structures. Since $Spin(6)$ is isomorphic to $SU(4)$, the approach of studying coadjoint orbits of $SU(4)$ is advantageous. In fact, the spinor language implicit in this paper gives one the possibility of enlarging the discussion to include other structures not necessarily defined by a 2-form. An important example in six dimensions is the case of $SU(3)$-structures, whose intrinsic torsion measures the extent to which a manifold fails to be Calabi–Yau.

In our set-up, phenomena involving symplectic fibrations of coadjoint orbits, such as symplectic quotients and other operations analysed in [9], have a direct and detailed interpretation in terms of compatibility conditions for specific Riemannian structures in six real dimensions. A discussion of certain invariant subsets in Section 3 enables us to identify the image of toric moment mappings, and characterize the resulting faces using almost complex structures. Then, in Section 4, we interpret the well-known Klein correspondence of projective geometry between elements in $\mathbb{CP}^3$ and $G_2(\mathbb{C}^4)$ from this viewpoint. In the majority of cases, this and similar correspondences can be clearly visualized in terms of the moment polytopes characterizing the structures involved.

The aim of the last section is to illustrate some implications of the theory that go beyond mere algebraic and combinatorial aspects. Since nilmanifolds are parallelizable in a natural way, they provide a rich source of examples of structures defined globally in terms of invariant tensors. We describe an application of the theory involving the classes of various types of Riemannian structures on the Iwasawa manifold characterized by specific constraints on their intrinsic torsion.

This paper is based on parts of the author’s doctoral thesis [11], and a summary of related results was presented in a conference proceeding [12].

1 SU(4) coadjoint orbits

A fundamental result guarantees that any orbit of the adjoint action of a compact Lie group $G$ on its Lie algebra $\mathfrak{g}$ intersects the closure of each Weyl chamber in a single point [6, 4]. This property implies that the set of adjoint orbits can be parametrized by the closed fundamental Weyl chamber. The standard identification $\mathfrak{g} \cong \mathfrak{g}^*$ realized by the Killing form, also allows us to identify the orbits of the adjoint and the coadjoint actions.

It is always possible to define a symplectic structure on a coadjoint orbit $\mathcal{O}$ such that the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ plays the role of a moment map associated to the Hamiltonian action of $G$ (see [10]). This is carried out by means of Konstant–Kirillov–Souriau (KKS) symplectic structure, defined by

$$\omega_\lambda(\mathcal{X}, \mathcal{Y}) = (\lambda, [X, Y]), \quad X, Y \in T_\lambda \mathcal{O},$$
where \( \mathcal{X}, \mathcal{Y} \) are the vector fields generated by \( X, Y \in \mathfrak{g} \) i.e. \( \text{ad}_X X = \mathcal{X} \) and \( \text{ad}_Y Y = \mathcal{Y} \).

Restricting the group action to the maximal torus \( T \subset G \), we obtain a Hamiltonian toric action on the orbit. In any case, the moment map \( \mu_T \) associated to this action consists in orthogonal projection to the subalgebra \( t \subset \mathfrak{g} \). Any coadjoint orbit \( \mathcal{O} \) intersects \( t \) in a single orbit of the Weyl group:

\[
\mathcal{O} \cap t = W \cdot \lambda,
\]

for some \( \lambda \in t \). The points in the intersection of the orbit and \( t \) are exactly the points fixed by the action of \( T \), and none of these are found in the interior of the convex polytope determined by the Weyl orbit of \( \lambda \). The celebrated Atiyah and Guillemin–Sternberg (AGS) Convexity Theorem then states that the image by \( \mu_T \) of an orbit passing trough \( \lambda \in t \) is the convex hull of the Weyl group orbit of \( \lambda \):

\[
\mu_T(\mathcal{O} \cdot \lambda) = \text{conv}(W \cdot \lambda).
\]

See \[3, 9\] for more details.

We first apply this general theory to provide a complete description of the set of \( SU(4) \) coadjoint orbits, diffeomorphic to the homogeneous spaces listed below. Consider the maximum torus \( T \subset SU(4) \) consisting of matrices \( \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}) \) with \( \sum \theta_i = 0 \). Using the basis

\[
\{ \text{diag}(-1, -1, 1, 1), \, \text{diag}(-1, 1, -1, 1), \, \text{diag}(-1, 1, 1, -1) \},
\]

we can identify \( t \in \mathfrak{su}(4) \) isometrically with \( \mathbb{R}^3 \). Relative to this basis, the fundamental weights are:

\[
\lambda_1 = (1, 1, 1), \quad \lambda_2 = (0, 0, 1), \quad \lambda_3 = (1, -1, 1).
\]

The fundamental Weyl chamber \( B \) is determined by the system (see Figure 2):

\[
x_3 > x_1, \quad x_1 > x_2, \quad x_1 > -x_2
\]

A generic coadjoint orbit, passing through an interior point of \( B \), is the real 12-dimensional manifold

\[
\mathcal{O}^{SU(4)} = \frac{SU(4)}{S(U(1) \times U(1) \times U(1) \times U(1))} \cong \frac{SU(4)}{T}
\]

of “full” complex flags in \( \mathbb{C}^4 \). The image \( \mu_T(\mathcal{O}^{SU(4)}) \) of such a generic orbit is the dodecahedron in Figure 1. A point belonging to the faces or the edges of the closed fundamental Weyl chamber \( \bar{B} \) admits a non-trivial stabilizer and thus gives rise to a “singular” orbit. Points in \( \bar{B} \) with the corresponding stabilizer and coadjoint orbit are listed in the following table.


| point in $\mathcal{B}$ | stabilizer | orbit | image by $\mu_T$ (Figure 3) |
|------------------------|------------|-------|-----------------------------|
| $\lambda_1$            | $U(3) \times U(1)$ | $\mathcal{P}^+ = \frac{SU(4)}{SU(3) \times U(1)} \cong \mathbb{C}P^3$ | tetrahedron $\Delta_{\mathcal{P}^+}$ (a) |
| $\lambda_2$            | $U(1) \times U(3)$ | $\mathcal{P}^- = \frac{SU(4)}{SU(1) \times U(3)} \cong \mathbb{C}P^3$ | tetrahedron $\Delta_{\mathcal{P}^-}$ (c) |
| $\lambda_3$            | $U(2) \times U(2)$ | $\mathcal{G} = \frac{SU(4)}{SU(2) \times U(2)} \cong \text{Gr}_2(\mathbb{C}^4)$ | octahedron $\Delta_{\mathcal{G}}$ (e) |
| $a\lambda_1 + b\lambda_2$ | $U(2) \times U(1) \times U(1)$ | $\mathcal{F}_{211} = \frac{SU(4)}{SU(2) \times U(1) \times U(1)}$ | truncated tetrahedron $\Delta_{\mathcal{F}_{211}}$ (b) |
| $a\lambda_1 + b\lambda_3$ | $U(1) \times U(2) \times U(1)$ | $\mathcal{F}_{121} = \frac{SU(4)}{SU(1) \times U(2) \times U(1)}$ | skew-cuboctahedron $\Delta_{\mathcal{F}_{121}}$ (f, g) |
| $a\lambda_2 + b\lambda_3$ | $U(1) \times U(1) \times U(2)$ | $\mathcal{F}_{112} = \frac{SU(4)}{SU(1) \times U(1) \times U(2)}$ | truncated tetrahedron $\Delta_{\mathcal{F}_{112}}$ (d) |

The following result shows how the orbits are interrelated:

**Proposition 1.** Suppose that $G_\alpha$ is not a maximal subgroup of $SU(N)$. Then there exists a subgroup $H$ such that $G_\alpha \subset H \subset SU(N)$ and $\mathcal{O}_\alpha$ is a holomorphic bundle over $H \setminus SU(N)$ with fibre $G_\alpha \setminus H$.

We express this in symbols by $\mathcal{O}_\alpha \cong G_\alpha \setminus H \ltimes H \setminus SU(N)$.

This proposition holds for any compact semisimple Lie group. Here are some examples:

The only orbit of $SU(2)$ is $\mathcal{O}^{SU(2)} = \frac{SU(2)}{U(1)} \cong \mathbb{C}P^1$.

The orbits of $SU(3)$ are $\mathcal{O}^{SU(3)} = \frac{SU(3)}{U(1) \times U(1)}$ and $\mathcal{O}_s^{SU(3)} = \frac{SU(3)}{SU(2) \times U(1)} \cong \mathbb{C}P^2$.

The generic $SU(3)$ orbit fibres over the degenerate one:

$$\mathcal{O}^{SU(3)} \cong \mathcal{O}_s^{SU(3)} \times \mathcal{T}^{SU(2)} \cong \mathbb{C}P^2 \times \mathbb{C}P^1$$  (3)

The three 10-dimensional orbits shown in the table are obviously diffeomorphic as homogeneous manifolds, but play distinct roles in the theory that we shall explain below.

Since the diagonal of the bounding cube is simultaneously in the closure of six Weyl chambers, we conclude that the Weyl orbit of $\lambda_1$ consists of itself and the points $(1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ in $\mathbb{R}^3 \cong t^*$. The resulting tetrahedron is denoted by $\Delta_{\mathcal{P}^+}$. The specific position of a point in $\mathcal{B}$ in Figure 2 determines the analogous polytopes illustrated in Figure 3. Opposite faces are parallel, and the rectangular faces and the hexagonal faces in each of $\Delta_{\mathcal{F}_{211}}, \Delta_{\mathcal{F}_{112}}$ and $\Delta_{\mathcal{G}}$ are mutually congruent.
Figure 3: Continuous family of moment polytopes for singular $SU(4)$ orbits

The generic $SU(4)$ orbit fibres over the degenerate one in the following way:

$$\mathcal{O}^{SU(4)} \cong \mathcal{P} \rtimes \mathcal{O}^{SU(3)} \cong \mathbb{C}P^3 \rtimes \mathbb{C}P^2 \rtimes \mathbb{C}P^1$$

(4)

More generally, $\mathcal{O}^{SU(N)} \cong \mathbb{C}P^{N-1} \rtimes \mathcal{O}^{SU(N-1)}$.

In the case of $SU(4)$, inclusion relations between the stabilizer groups determine the following fibrations with fibres $\mathbb{C}P^1$ and $\mathbb{C}P^2$:
For more details see [11, 12]. Actually these fibrations are symplectic in the sense that their fibre $\pi^{-1}(p) = F$ is a symplectic manifold for which the transition mappings induce symplectomorphisms of $F$. This is equivalent to a certain connection being flat [9]. More generally, for any compact Lie group $G$, we have:

**Proposition 2.** Let $x, \lambda$ be two points in the same Weyl chamber. If the isotropy Lie algebras satisfy $g_x \subset g_\lambda$, then the map of orbits $G \cdot x \to G \cdot \lambda$ given by $g \cdot x \mapsto g \cdot \lambda$ is a symplectic fibration with fibre a coadjoint orbit of $G_\lambda$.

Properties of symplectic fibrations allow one to generalize a fundamental fact regarding toric varieties. Namely, the image by the toric moment map of a toric variety is a Delzant polytope and toric varieties associated to the same moment polytope are mutually symplectomorphic. Of course, $\mathbb{CP}^1$ and $\mathbb{CP}^2$ are toric manifolds with image a line segment and filled triangle respectively.

**Remark 3.** The AGS Convexity Theorem combined with (i) the simple observation that fixed points of a higher orbit belong to fibres over fixed points in the base one, (ii) the fact that the base variety of the symplectic fibration is a symplectic manifold, allows us to “recognize” some coadjoint orbits from their images by $\mu_T$. For example, the image of a generic $SU(3)$ orbit is a hexagon, and in Figure 4 we see how the symplectic fibration (3) is detected by the moment map. In fact, the $\mathbb{CP}^1$-fibres over the fixed points in $\mathbb{CP}^2$ (vertices of the triangle) are mapped to segments anchored at the fixed points of the generic orbit.

Similar phenomena occur in the case of $SU(4)$. A more detailed explanation of Figure 5 will be given in the next section.

## 2 Riemannian geometry in six dimensions

Let $G, H$ be Lie groups, with $H$ a subgroup of $G$. It is a well known fact that a reduction of a $G$-structure to an $H$-structure can be realized by selecting a tensor $\xi$, stabilized by $H$
in a suitable $G$-module $V$. The parameter space of such reductions is given by the $G$-orbit of $\xi$ in $V$.

Consider a Riemannian manifold $(M,g)$ of dimension $N$, and the $SO(N)$-module associated to the space $\Lambda^2 T^* M$ of 2-forms at each point. Any smooth 2-form $\omega \in \Lambda^2 T^* M$ determines a skew-symmetric endomorphism $\mathfrak{F}$ of each tangent space $T_p M$ via

$$\omega(X,Y) = g(\mathfrak{F}X,Y).$$

(6)

The endomorphism $\mathfrak{F}$ can be regarded as a smooth section of the tensor bundle $T^* M \otimes TM$. The action of $\mathfrak{F}$ on $T_p M$ can be analysed in terms of its kernel and other eigenspaces, which (as $p$ varies) give rise to distributions. Integrability properties of such distributions help to characterize the structure on $M$, and we shall see some practical examples in Section 5.

If an endomorphism $\mathfrak{F}$ is preserved by the action of a non-trivial subgroup $H \subset SO(N)$, it determines a reduction of the structure group $SO(N)$ to $H$. There are many relevant examples of such a procedure. When $\omega$ is a simple 2-form, $(\ker \mathfrak{F})^\perp$ has dimension 2, and $-\mathfrak{F}^2$ is the projection to this subspace. At the other extreme, when $N = 2n$ is even and $\mathfrak{F} = J$ is an almost complex structure, the eigenspaces of $\mathfrak{F}$ are maximal complex isotropic subspaces of $(T_p M)_c$. In both of these examples, $\mathfrak{F}$ satisfies

$$g(X,Y) = g(\mathfrak{F}X,\mathfrak{F}Y), \quad X,Y \in (\ker \mathfrak{F})^\perp,$$

and thus induces an orthogonal transformation on $(\ker \mathfrak{F})^\perp$.

For the orthogonal group, the coadjoint orbits are exactly the sets of isospectral skew-symmetric matrices $A$, or equivalently endomorphisms $\mathfrak{F}$ characterized by the same set of eigenvalues. This is because each orbit contains a complex diagonal matrix of the form $\text{Ad}_P(A) = P^{-1}AP$ with $P$ unitary, and the dimensions of its eigenspaces are invariant.

**Proposition 4.** A given coadjoint orbit of $SO(N)$ determines an orthogonal splitting of $\mathbb{R}^N$ as a direct sum of orthogonal subspaces (the eigenspaces of $\mathfrak{F}$).

The use of 2-forms to define a geometrical structure leads one naturally to consider coadjoint orbits for $SO(N)$. However, the fact that $\text{Spin}(6)$ is isomorphic to $SU(4)$ allows one to identify the coadjoint orbits of $SO(6)$ with those of $SU(4)$. This is the approach
that we adopt in the present paper. In order to render this correspondence explicit, it is first necessary to relate representations of $SO(6)$ and $SU(4)$.

We start from a complex 4-dimensional vector space $V$ endowed with a Hermitian inner product $h$ and a fixed volume form $\phi \in \Lambda^4 V$. We define an anti-linear operator $\ast$ in the standard way by

$$\xi \wedge (\ast \psi) = h(\xi, \psi) \phi, \quad \xi, \psi \in \Lambda^2 V.$$ 

This operator induces a real structure on the 6-dimensional complex vector space $\Lambda^2 V$. The elements fixed by $\ast$ form a real 6-dimensional vector space $W$ endowed with a positive definite inner product.

Fix a unitary basis $\{v_0, v_1, v_2, v_3\}$ of $V$ such that $v_1 \wedge v_2 \wedge v_3 \wedge v_4 = \phi$. We use the abbreviation $v_{ij} = v_i \wedge v_j$, so that $(1 + \ast)v_01 = v_01 + v_23$. With a suitable extension of $h$ to $\Lambda^2 V$, we obtain the following basis of $W$:

$$e^1 = v_01 + v_23, \quad e^2 = -i(v_01 - v_23), \quad e^3 = v_02 + v_31, \quad e^4 = -i(v_02 - v_31), \quad e^5 = v_03 + v_12, \quad e^6 = -i(v_03 - v_12).$$

(7)

The elements $e^{ij} = e^i \wedge e^j$ form a basis for the space $\Lambda^2 W$. In a formal way, we compute

$$i e^{12} = (v_01 + v_23) \wedge (v_01 - v_23) - (v_01 - v_23) \wedge (v_01 + v_23).$$

Using the map $\Lambda^2 V \otimes \Lambda^2 V \rightarrow V \otimes \Lambda^3 V$, this can be contracted into

$$v_0 \otimes v_{123} - v_1 \otimes v_{023} + v_2 \otimes v_{013} - v_3 \otimes v_{012},$$

(8)

up to an overall constant.

Given $\phi$, we have $\Lambda^3 V \cong \overline{V}$, so we can convert (8) into an element of $V \otimes \overline{V}$. Repeating this procedure, we obtain:

$$i e^{12} = -v_0 \otimes \overline{v}_0 - v_1 \otimes \overline{v}_1 + v_2 \otimes \overline{v}_2 + v_3 \otimes \overline{v}_3,$$

$$i e^{34} = -v_0 \otimes \overline{v}_0 + v_1 \otimes \overline{v}_1 - v_2 \otimes \overline{v}_2 + v_3 \otimes \overline{v}_3,$$

$$i e^{56} = -v_0 \otimes \overline{v}_0 + v_1 \otimes \overline{v}_1 + v_2 \otimes \overline{v}_2 - v_3 \otimes \overline{v}_3.$$ 

(9)

This is exactly the basis (1) of $t^3 \subset \mathfrak{su}(4)$.

Let us denote by $\sigma$ the map $W \rightarrow \Lambda^2 V$ defined by (7). The natural action of a circle $T^1 \subset T^3 \subset SO(6)$ on the space $\langle e^1, e^2 \rangle$ is given by:

$$\Gamma(\theta)e^1 = e^1 \cos \theta + e^2 \sin \theta = (v_01 + v_23) \cos \theta - i(v_01 - v_23) \sin \theta = e^{-i\theta} v_01 + e^{i\theta} v_{23}. $$

8
The last equality can be re-expressed as:
\[
\Gamma(\theta)e^1 = (e^{-i\frac{\theta}{2}}v_0) \wedge (e^{-i\frac{\theta}{2}}v_1) + (e^{i\frac{\theta}{2}}v_2) \wedge (e^{i\frac{\theta}{2}}v_3) = e^{i\frac{\theta}{2}}\lambda_1 \cdot \sigma^{-1}(e^1).
\] (10)
This simple observation shows how a circle \(U(1) \subset SU(4) \cong Spin(6)\) gives a double covering of the corresponding circle \(SO(2) \subset SO(6)\); as expected, one loop in \(Spin(6)\) produces two loops in \(SO(6)\). This phenomenon is a special case of

**Proposition 5.** There is an \(SU(4)\)-equivariant bijective correspondence between the adjoint representations \(su(4)\) and those \(so(6)\).

In the above example, we restricted the adjoint action to \(U(1)\), and considered invariant subsets. We retain the notation \(\mathcal{P}^+, \mathcal{P}^-, \mathcal{Q}\) for the \(SO(6)\) orbits corresponding to those of Section 1.

The reason for the specific choice of basis (9) arising from (7) becomes obvious. It is a realization of the isomorphism

\[
\mathbb{R}^{15} \cong su(4) \cong so(6) \cong \Lambda^2 \mathbb{R}^6,
\]
in such a way that the images of the root spaces arising from (11) are the subspaces \(\langle e^1, e^2 \rangle, \langle e^3, e^4 \rangle, \langle e^5, e^6 \rangle\). Thus, the Lie algebra \(t\) is mapped to the 3-dimensional subspace of \(\Lambda^2 T_p^*M\) spanned by the elements \(\{e^{12}, e^{34}, e^{56}\}\), and \(T\) acts on the three real 2-dimensional subspaces in the standard block-diagonal way.

The inequalities defining the fundamental Weyl chamber of \(SO(6)\) are exactly the same as (2) (see [4]). This reflects the fact that the Dynkin diagrams of \(SU(4)\) and \(SO(6)\) coincide. Following this construction the image \(\Lambda^2 T_p^*M\) of the fundamental Weyl chamber is generated by the elements

\[
\mu_1 = e^{12} + e^{34} + e^{56}, \quad \mu_2 = e^{56}, \quad \mu_3 = e^{12} - e^{34} + e^{56}.
\] (11)

From the general theory, we know that every coadjoint orbit has a unique representative element in the closure of the fundamental Weyl chamber \(\bar{B}\).

**Proposition 6.** Any 2-form at a point of a 6-dimensional oriented Riemannian manifold is equivalent under the action of \(SO(6)\) to some linear combination \(\sum_{i=1}^{3} a_i \mu_i\) with \(a_i \geq 0\).

We can now introduce the Riemannian structure defined by a fixed 2-form. Such a structure is determined by a smooth section of the fibre bundle \(M \times_{SO(6)} \mathcal{O}\), where \(\mathcal{O}\) is a coadjoint orbit. As a result of the previous discussion, the position of the representative of the orbit inside the image of \(\bar{B}\) determines the structure group of the reduction. The following table relates positions inside \(\bar{B}\) to \(SO(6)\) orbits:
We comment briefly case by case:

Case 1 parametrizes the set of possible $T^3$-reductions of the Riemannian structure, consisting of a choice of three orthogonal complementary 2-dimensional spaces in each tangent space.

Case 2 parametrizes the orthogonal almost complex structures on $\mathbb{R}^6$ compatible with a fixed orientation. The parameter space at each point is $\mathcal{P}^+$.

Case 3 parametrizes the orthogonal almost complex structures inducing the opposite orientation on $\mathbb{R}^6$; this is the meaning of the “tilde” over $\mathcal{P}$. The orbit is $\mathcal{P}^-$.

Case 4 features the Grassmannian $\mathbb{G}_2(\mathbb{R}^6)$ of oriented 2-planes in $\mathbb{R}^6$, isomorphic (by Proposition 5) to the Grassmannian $\mathbb{G}_2(\mathbb{C}^4)$ of 2-planes in $\mathbb{C}^4$. A simple 2-form defines via (6) a splitting $T_pM = \mathcal{V} \oplus \mathcal{H}$ with $\mathcal{V}$ an oriented 2-plane, and $\mathcal{H} = \ker \tilde{\mathcal{F}}_3$ a 4-plane whose orientation is not specified. The orbit $\mathcal{G}$ parametrizes a set of orthogonal almost product structures, essentially in the sense of Naveira [13].

Cases 5-9 are related to the 10-dimensional “intermediate” complex flag manifold; in this case the Ad action is characterized by two distinct pairs of imaginary eigenvalues. The corresponding isotropy subgroup of $SO(6)$ is isomorphic to $U(1) \times U(2)$, which leads to:

**Definition 7.** Let $M$ be a Riemannian manifold of dimension $N$. A mixed structure on $M$ is a reduction of its structure group $O(N)$ to $U(p) \times U(q)$, where $2(p+q) = N$.

Such a structure is equivalent to the simultaneous assignment of an orthogonal almost complex structure $J$ (so $J^2 = -I$) and an orthogonal almost product structure $P$ (so $P^2 = I$) such that $JP = PJ$. Working pointwise, without implying integrability, we shall refer to the former as an OCS, and the latter as an OPS. In our case in which $p = 1$ and $q = 2$ we set $\mathcal{V} = \ker(P - I)$ so that $P$ is the identity on the 2-plane.
Case 5. The 2-form belongs in the plane generated by $\mu_1$ and $\mu_2$. The position of this point inside the Weyl chamber $B$ reflects the fact that $\omega$ is a linear combination of a 2-form arising from an almost complex structure $J$ and a simple 2-form arising from a positively-oriented $J$-invariant 2-plane.

**Proposition 8.** The orbit $\mathcal{F}_1$ parametrizes mixed structures determined by an OCS $J$ in $\mathcal{P}^+$ and an OPS in $\mathcal{G}$ whose 2-plane is $J$-invariant and oriented consistently with $J$.

Case 6 is completely analogous; $\mathcal{F}_2$ parametrizes mixed structures with $J \in \mathcal{P}^-$ together with an OPS whose 2-plane is $J$-invariant and oriented consistently with $J$.

Cases 7 and 9. This time, the position of the 2-form in $\bar{B}$ exhibits the 2-form as a weighted linear combination of two compatible OCS’s $J_+ \in \mathcal{P}^+$ and $J_- \in \mathcal{P}^-$. By “compatible”, we mean that both OCS’s admit a common invariant 2-plane on which $J_+$ and $J_-$ differ by sign, whilst on the orthogonal complement the two OCS’s induce the same orientation. In the specific case displayed, the offending 2-plane is $\langle e^3, e^4 \rangle$. Thus,

**Proposition 9.** The orbit $\mathcal{F}_3^+$ parametrizes mixed structures determined by an OCS $J$ in $\mathcal{P}^+$, and an OPS in $\mathcal{G}$ whose 2-plane is oriented consistently with $-J$.

Case 8 is the special case in which the contributions of the two OCS’s have the same weight. The 2-dimensional subspace is determined by the kernel of $\omega$ and thus the orientation is not specified by the tensor. This corresponds exactly to Yano’s definition of Riemannian $f$-structure [14, 15], further developed by Blair [5]. The $(1, 1)$-tensor satisfies the condition $\bar{\mathfrak{f}}^3 + \bar{\mathfrak{f}} = 0$, and in the terminology introduced in [5] this case represents an “$\mathcal{F}$-structure”.

In conclusion, $SO(6)$-inequivalent (but $O(6)$-equivalent) mixed structures are parametrized at each point by the various $\mathcal{F}$’s. The fact that $f$-structures in six dimensions provide a special case of mixed structures is due to the isomorphism $SO(2) \cong U(1)$.

In the previous section, we analysed a coadjoint orbit $\mathcal{O}$ as a symplectic manifold. The Riemannian structures under consideration are now realized as smooth sections of fibre bundles with fibre $\mathcal{O}$. Recall Proposition 5. The mapping

$$\frac{SO(6)}{G} \rightarrow \Lambda^2 T^* M$$

which associates the 2-form to a specific $G$-reduction is proportional (at each point) to the moment map

$$\frac{SO(6)}{G} \rightarrow \mathfrak{g} \mathfrak{o}(6)^* \cong \mathfrak{su}(4)$$

associated to the KKS symplectic structure. Combining this mapping with the orthogonal projection $\mathfrak{su}(4) \rightarrow \mathfrak{t}$ (where $\mathfrak{t}$ denotes the Lie algebra of $T$) gives us the moment mapping

$$\mu_T : \frac{SO(6)}{G} \rightarrow \mathfrak{t} \cong \mathbb{R}^3$$

11
for the action of $T$ itself. To sum up,

**Proposition 10.** *The Hamiltonian action of the maximum torus $T$ of $SO(6)$ on $\mathcal{O}$ associates a characteristic “moment polytope” to each of the Riemannian structures defined above by a 2-form.*

**Remark 11.** Our construction refines the description of geometrical structures, in the sense that we put more emphasis on the defining tensor rather than merely the isotropy subgroup. A real projective class in $\overline{\mathcal{B}}$ gives rise to the same isotropy subgroup. In the special case of orthogonal almost complex, almost product and $f$-structures, there is a unique projective class and so these structures are parametrized by projective classes of forms in the walls of $\overline{B}$.

Observe that the singular polytope corresponding to $F_3$ (Figure 3g) is a special case of $\Delta_{\mathcal{F}_{121}}$, with rectangular faces being exactly squares.

### 3 Toric invariant subsets

The analysis of subsets of a coadjoint $SU(4)$ orbit $\mathcal{O}$ invariant by a circle $T^1$ leads to a precise description of the planes defining the moment polytopes in terms of 2-forms and compatibility conditions. The subsets in $\mathcal{O}$ invariant by a non-trivial subgroup of the maximal torus $T = T^3$ are those on which $\mu_T$ is singular.

From another point of view, invariant subsets are detected by particular endomorphisms associated to 2-forms. For example, an OCS $J$ acts as a simultaneous rotation by $\pi/2$ on each of a triple of invariant 2-planes and is itself an element of a suitable maximal torus. In particular, the structure $J_0$ associated to the 2-form $\omega_0 = e^{12} + e^{34} + e^{56}$ acts as $\exp(i\pi/2 \cdot \lambda_1)$. Similarly, the endomorphism associated to the simple 2-form $e^{56} = e^5 \wedge e^6$ corresponds to $\exp(i\pi/2 \cdot \lambda_2)$.

Let $G$ be a compact Lie group. Having chosen a maximal torus, we denote by $\lambda_1, \ldots, \lambda_N$ a set of fundamental weights in $\mathfrak{g}^*$, by $F_1, \ldots, F_N$ be the stabilizer group of each weight and by $W_i$ the Weyl group of the structure $(F_i, T)$. Observe that $W_i$ is generated by the reflections induced by the roots orthogonal to $\lambda_i$. As $F_i$ leaves invariant $\lambda_i$, we have

$$\text{ad}_X(\lambda_i) = [X, \lambda_i] = 0, \quad X \in \mathfrak{f}_i,$$

where $\mathfrak{f}_i$ is the Lie algebra of $F_i$. The previous equation can be read “backwards” to give $\text{ad}_{\lambda_i}(X) = 0$, so the elements of subalgebra $\mathfrak{f}_i \subset \mathfrak{g}$ are preserved by the action of the circle subgroup $C_i = \{\exp(t\lambda_i) : t \in \mathbb{R}\}$.

**Theorem 12.** *The singular points of the moment map $\mu_T : \mathcal{O}_\lambda \to \mathfrak{t}$ are the symplectic manifolds*

$$F_i \cdot w\lambda, \quad w \in W, \quad i = 1, \ldots, N.$$
The singular values of the toric moment map $\mu_T$ are the corresponding convex polytopes $\text{conv}(W_i \cdot w\lambda)$.

For the proof of this result we refer the reader to [9].

An application of this theorem to the subsets of $SU(4)$ coadjoint orbits invariant by the action of a fundamental circle $C_i$ leads to:

**Proposition 13.** Given a vertex $\alpha$ of the singular polytope of an $SU(4)$-orbit, the intersection of the Weyl group orbit $W_i\alpha \cap W\lambda_i$ is given by the set of all vertices staying on the plane passing through $\alpha$ and orthogonal to $\lambda$.

Figure 6 shows the directions of some roots orthogonal to $\lambda_1$ and $\lambda_2$ and some invariant sets. The case of the tetrahedron is obvious. The roots generating $W_1$ and $W_2$ can be viewed as inward pointing normal vectors of the polytopes $\text{conv}(W_i \cdot w\lambda) = \mu_T(F_i \cdot w\lambda)$.

![Figure 6: Roots orthogonal to $\lambda_1$ and $\lambda_2$. Projections of some $C_1$ and $C_2$ invariant subsets](image)

The sets simultaneously invariant under the action of 1-tori are projected to the lines determined by the intersections of the planes which are projections of sets invariant by a 1-torus. For example the edges of any polytope but also internal segments and segments contained in faces. Observe that certain segments appear as projections of sets invariant under $C_2$. This fact is quite easy to justify in terms of sums of roots.

The results of our analysis can be read directly from Figure 6 in the light of:

**Remark 14.** Sets invariant under the action of some subtorus carry an effective action of the complementary subtorus. The symplectic nature of the fibrations, combined with the general properties ([8] and well as [3] and [4]) guarantee that the invariant subsets of the orbits anchored at fixed points are symplectomorphic to
$S^2 \cong \mathbb{CP}^1$ when they map to line segments,
$\mathbb{CP}^2$ when they map to (filled) triangles,
$S^2 \times S^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ when they map to rectangles or quadrangles,
generic $SU(3)$ orbits (see 3) when they map to hexagons,
and so on.

We get the tetrahedra associated to both cases of almost complex structures as follows:

**Proposition 15.** The images of $\mathcal{P}^+$ and $\mathcal{P}^-$ by the toric moment map $\mu_T$ are specified by the inequalities:

\[
\mu_T(\mathcal{P}^+) : \begin{cases} 
  x + y + z \geq -1 \\
  x - y - z \geq -1 \\
- x + y - z \geq -1 \\
- x - y + z \geq -1
\end{cases} \quad \mu_T(\mathcal{P}^-) : \begin{cases} 
  x + y + z \leq 1 \\
  x - y - z \leq 1 \\
- x + y - z \leq 1 \\
- x - y + z \leq 1
\end{cases}
\]

**Proof.** To save time, we shall only describe one case of equality, as this technique will be useful later. Consider the form $\omega_0 = e^{12} + e^{34} + e^{56}$ associated to $J_0$. Observe that if we change the sign of $\omega$ on a 4-dimensional $J_0$-invariant subspace, the form obtained in this way still belongs to the orbit $SO(6)\omega_0$. The prototypes of such forms are the remaining three vertices $\omega_1, \omega_2, \omega_3$ of $\Delta_{\mathcal{P}^+}$ (see (13) and (14) below). A generic 2-form with this property is given by:

\[
\omega = -\omega_0 + 2e \wedge J_0 e
\]

where $e = \sum_{i=1}^6 x_ie^i$ and $\sum_{i=1}^6 x_i^2 = 1$. Recall that $\mu_T$ is the projection to the space $\langle e^{12}, e^{34}, e^{56} \rangle$. Since

\[
J_0 e = x_1 e^2 - x_2 e^1 + x_3 e^4 - x_4 e^3 + x_5 e^6 - x_6 e^5,
\]

we have

\[
\mu_T(\omega) = (-1 + 2(x_1^2 + x_2^2), -1 + 2(x_3^2 + x_4^2), -1 + 2(x_5^2 + x_6^2)) = (x, y, z).
\]

Thus the projections of this form satisfy $x + y + z = -1$, and lie in the plane passing through $\omega_i$ with $i = 1, 2, 3$. The planes perpendicular to each $\omega_i$ and passing trough the complementary vertices of the tetrahedron are obtained in the same way. If the same consideration is applied to a generic 2-plane generated keeping the form inside $\mathcal{P}^+$, and applying the Cauchy-Schwarz inequality we obtain the inequalities in the claim.

The same calculation with $\omega = -e^{12} - e^{34} - e^{56} \in \mathcal{P}^-$, leads to a plane which coincides with a face of $\Delta_{\mathcal{P}^-}$, namely $x + y + z = 1$. \qed
The above proof is based on a correspondence between subsets in $\mathcal{P}^+, \mathcal{P}^-$ projecting to the faces of $\Delta_{\mathcal{P}^+}, \Delta_{\mathcal{P}^-}$, and classes of suitably-oriented $J$-invariant 2-planes. The same observation leads to the octahedron as the image of $\mathcal{G}$, consisting of simple unit 2-forms $\alpha \wedge \beta$. Let $J$ be a vertex of $\Delta_{\mathcal{P}^+}$ or $\Delta_{\mathcal{P}^-}$; we can ask “what is the image in $\Delta_{\mathcal{G}}$ of the $J$-invariant planes”. This is substantially the argument applied in [11, 12]:

**Proposition 16.** The image of $\mu_T(\mathcal{G})$ satisfies

$$|x| + |y| + |z| \leq 1,$$

with equality if and only if $\beta = J\alpha$ where $J$ is one of the almost complex structures $\omega = \pm e^{12} \pm e^{34} \pm e^{56}$.

The octahedron combines the inequalities defining $\Delta_{\mathcal{P}^+}$ and $\Delta_{\mathcal{P}^-}$; thus $\Delta_{\mathcal{G}}$ can be obtained as an intersection of the tetrahedras as in Figure 7. The same technique, exploiting contributions of invariant 2-planes, can be applied to $\mathcal{F}_i$ and $\mathcal{O}_{SO(6)}$ (the faces of the polytopes are always parallel to the faces of $\Delta_{\mathcal{P}^\pm}$ or to the coordinate planes).

### 4 A Klein correspondence

In Section 2, we introduced the symplectic fibrations of $SU(4)$ coadjoint orbits. Consider the lower part of the first (left-hand) diagram in (5). The projections $\pi_1$ and $\pi_2$ can be understood in terms of the classical Klein correspondence in which $\mathcal{G} \cong \mathcal{G}_{r2}(\mathbb{C}^4)$ is identified with a non-degenerate quadric in $\mathbb{P}(\Lambda^2 \mathbb{C}^4)$. Here is a summary:

1. $\mathcal{G}$ parametrizes the projective lines $\mathbb{CP}^1$ in $\mathbb{CP}^3$.

2. A point $x \in \mathbb{CP}^3$ determines an $\alpha$-plane in $\mathcal{G}$, consisting of all the lines passing through that point.

3. A point $y \in (\mathbb{CP}^3)^*$ determines a $\beta$-plane in $\mathcal{G}$, consisting of all the lines lying in the plane $y$.

In the light of our realization of coadjoint orbits as parameter spaces of Riemannian structures, this correspondence assumes a completely new interpretation. In it, $\mathcal{G}$ is identified with the *real* Grassmannian $\mathcal{G}_{r2}(\mathbb{R}^4)$ that parametrizes compatible almost product structures. Namely,

1’. Given a decomposition $T_pM = \mathcal{V} \oplus \mathcal{H}$ arising from an orthogonal almost product structure $P$, there is a $\mathbb{CP}^1$ worth of compatible almost complex structures parametrized by $\omega \in S^2 \subset \Lambda^2_+ \mathcal{H}^*$. This is our projective line in $\mathcal{P}^+$.
2'. Given an orthogonal almost complex structure $J$ we have the $J$-invariant 2-planes generated by $\{v, Jv\}$ and each one determines an almost product structure.

3'. Likewise, given an orthogonal almost complex structure $J$ we have the $J$-invariant oppositely-oriented 2-planes generated by $\{v, -Jv\}$.

To better understand 1', we recall that $\Lambda^2_+ \mathcal{H}^*$ denotes the 3-dimensional space of self-dual 2-forms, and that the 2-sphere $S^2$ of unit self-dual forms parametrizes those almost complex structures on $\mathcal{H} = \mathbb{R}^4$ compatible with both metric and orientation. When combined with a standard almost complex structure on $\mathcal{V}$, we obtain a positively-oriented almost complex structure on $\mathbb{R}^6$.

The results of Section 3 can be exploited to establish a mapping between singular polytopes induced by the Klein correspondence. We explain this next.

A mixed structure in $\mathcal{F}_1 = \mathcal{F}_{211}$ determines an OCS $J$, namely its projection via $\pi_1$. This $J$ identifies the tangent space $T_p M$ with $\mathbb{C}^3$, and $J$-invariant splittings of $\mathbb{R}^6$ are parametrized by complex 1-dimensional (or complementary 2-dimensional) subspaces in $\mathbb{C}^3$, i.e. by the projective space $\mathbb{C}P^2$. Thus, the set of mixed structures fibres over the set of compatible OCS’s with fibre $\mathbb{C}P^2$. The inverse image by $\pi_1$ of an OCS $J \in \mathcal{P}^+$ inside $\mathcal{F}_1$ is determined by answering the question: “which are the OPS’s whose 2-plane is both $J$-invariant and oriented consistently with $J$?” Working in terms of 2-forms, we take the non-degenerate 2-form $\omega$ associated to $J$ add to it a simple 2-form $f \wedge Jf$ representing the $J$-Invariant plane in question. For instance, $\mu_T(\pi_1^{-1}(J_0))$ is the face generated by the vertices

$$e^{12} + e^{34} + e^{56} + \alpha e^{12}, \quad e^{12} + e^{34} + e^{56} + \alpha e^{34}, \quad e^{12} + e^{34} + e^{56} + \alpha e^{56}$$

in the truncated tetrahedron. In view of the results of Section 2 and the symplectic nature of the fibrations (recall Proposition 10), we conclude that the set of mixed structures compatible with $J$ is a subset of $\mathcal{F}_1$ invariant by $\exp(\frac{i}{2} \pi \cdot \lambda_1)$. Figure 8 represents the map between polytopes induced by the projections $\pi_1$ and $\pi_2$.

![Figure 8: A polytope map induced by the Klein correspondence.](image)

Let $J_i$ be the OCS’s that correspond to the 2-forms $\omega_i$ in the proof of Proposition 15. These are fixed by $T = T^3$ and therefore determine vertices in the tetrahedron $\mu_T(\mathcal{P}^+)$. Adopting notation from [2], we denote by $E_{12}$ the edge joining these two vertices as in
Figure 9. The inverse image of this edge is the projective line

\[ L = \mu_T^{-1}(E_{12}) = \{ [0, z_2, z_3, 0] : z_i \in \mathbb{C} \} = \Lambda_2^-(e^1, e^2, e^3, e^4), \]

using the notation of (7).

Let us determine explicitly the subset

\[ \mu_T(\pi_1^{-1}(L)) \subset \Delta_{F_{211}}. \]  \hspace{1cm} (12)

A generic 2-form belonging to \( L \) has the expression

\[ \omega = a(e^{12} - e^{34}) + b(e^{13} - e^{42}) + c(e^{14} - e^{23}) - e^{56}. \]

Without loss of generality, fix a unit 1-form

\[ e = \alpha e^1 + \beta e^3 + \gamma e^5, \quad \alpha^2 + \beta^2 + \gamma^2 = 1. \]

Let \( J \) be the OCS corresponding to \( \omega \). Then

\[ Je = \alpha(\alpha e^2 + \beta e^3 + \gamma e^5 + \beta(-ae^4 - be^1 + ce^2) - \gamma e^6 \]

and

\[ \mu_T(e \wedge Je) = (\alpha^2 \alpha + \alpha \beta c)e^{12} + (\alpha \beta c - \beta^2 \alpha)e^{34} - \gamma^2 e^{56} \]

\[ = (\alpha \beta c + \alpha^2 \beta, \alpha \beta c - \beta^2 \alpha, -\gamma^2). \]

With this parametrization, \( E_{12} = \{(a, -a, -1) : a \in [-1, 1] \} \), so at this point consider

\[ v = (x, y, z) = \mu_T(\omega + t e \wedge Je) \]

\[ = (a + t(\alpha^2 \alpha + \alpha \beta c), -a + t(\alpha \beta c - \beta^2 \alpha), -1 - t\gamma^2). \]

We see that \( x - y - az = a(3 + t) \), so for each point on the edge, \( v \) belongs to a plane with a varying normal vector \((1, -1, a)\). In particular observe that for \( a = \pm 1 \), we obtain exactly the “vectors” \( \omega_2 \) and \( \omega_3 \) normal to the triangular faces of \( \Delta_{F_{211}} \) and for \( a = 0 \) we get a vector parallel to the edge. The next step is to fix the value of the parameter \( t \), i.e. the specific \( \mathcal{F}_3^+ \)-orbit realizing the mixed structure. By definition \( t \) is positive, so the set \( \mu_T(\omega + t e \wedge Je) \) is characterized by \( z \leq -1 \); furthermore being a subset of \( \Delta_{F_{211}} \) it is bounded by its faces. Thus:

**Proposition 17.** The moment map \( \mu_T \) projects \( \pi_1^{-1}(L) \) onto the closed solid region of \( \mathbb{R}^3 \) bounded by the bold lines in Figure 9.
Remark 18. We emphasize that the condition \( z = -1 \) implies that \( \gamma = 0 \), and corresponds to a 2-form with fixed \( e^{56} \)-component. The set mapped to the internal rectangle defined by that condition is isomorphic to \( \mathbb{C}P^2 \), which as expected is a subset of \( \mathcal{F}_1 \) invariant by an \( S^1 \)-action in the plane \( \langle e^5, e^6 \rangle \) anchored at four fixed points.

This observation generalizes Remark 3 to subsets of the lower orbit and confirms that the mapping between polytopes arises from symplectic phenomena. The moment map captures the essence of the symplectic fibration. The inverse image of each point of \( \mathcal{P}^+ \) in \( \mathcal{F}_1 \) is isomorphic to a \( \mathbb{C}P^2 \), and the edge of the tetrahedron is the projection of a \( \mathbb{C}P^1 \). The inverse image of the entire set is thus \( \mathbb{C}P^1 \times \mathbb{C}P^2 \). The subset (12) is “a triangle times a line”!

Analogously, the inverse image by \( \pi_2^{-1} \) of an element in \( \mathcal{G} \) is parametrized by suitably-oriented OCS’s on \( \mathcal{H} \). Given an orthonormal basis \( \{f_1, f_2\} \) of \( \mathcal{V} \), we extend the 2-form \( f^{12} \) by adding a unit element of \( \Lambda^2 \mathcal{H} \) or inside \( \Lambda^2 \mathcal{H} \) so as to obtain an OCS on \( T_p M \). It follows that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) admit \( SO(4)/U(2) \cong \mathbb{C}P^1 \) as fibres in the first and third diagrams in (5). For example, \( e^{56} \) can be completed to

\[
\omega_0 = e^{12} + e^{34} + e^{56}, \quad \omega_3 = -e^{12} - e^{34} + e^{56} \in \mathcal{F}_1, \quad (13)
\]

and a whole 2-sphere of similar non-degenerate 2-forms. The same simple 2-form can be also completed to

\[
-\omega_1 = -e^{12} + e^{34} + e^{56}, \quad -\omega_2 = e^{12} - e^{34} + e^{56} \in \mathcal{F}_2, \quad (14)
\]

and an \( S^2 \) worth of OCS’s with the opposite orientation.

If we apply the same criterion, involving the convex hull of fixed points, we can claim without further ado that:

**Proposition 19.** Let \( K \) denote the subset of \( \mathcal{G} \) mapped by \( \mu_T \) onto the intersection of \( \Delta_\mathcal{G} \) with the plane \( \langle e^{12}, e^{34} \rangle \). Then \( \mu_T(\pi_2^{-1}(K)) \) is the polytope represented in bold on the right of Figure 10.

The set \( K \) occurs again in Theorem 22 below.
Remark 20. Since $K \cong \mathcal{G}r_2(\mathbb{R}^4)$, we expect the subset $\pi_2^{-1}(K)$ of $\mathcal{F}_1$ to be isomorphic to $K \times \mathbb{CP}^1 \cong S^2 \times S^2 \times \mathbb{CP}^1$. The intersection of this polytope with any plane orthogonal to $\langle e^{56} \rangle$ is a rectangle, so $\mu_T(\pi_2^{-1}(K))$ is “a rectangle times a line”. The rectangular face anchored at the fixed points again corresponds to forms with fixed $e^{56}$ component, and represents exactly the set invariant under the rotation as in Remark 18. In the present context it should be interpreted as a projection of $S^2 \times S^2 \times \mathbb{CP}^1$ having 2-spheres in respectively $\Lambda_2^2 \langle e^1, e^2, e^3, e^4 \rangle$ and $\Lambda_2^2 \langle e^1, e^2, e^3, e^4 \rangle$.

The inverse image in $\mathcal{F}_3^+$ of a point $\omega \in \mathcal{P}^\pm$ is given by all the forms obtained by adding to $\omega$ a “small” contribution of a negatively-oriented $J$-invariant 2-plane. The inverse image of a $J$ in $\mathcal{F}_3$ can be recovered replacing the positively-oriented 2-plane of the case $\mathcal{F}_1$ with a negatively-oriented one. All the triangular faces of $\Delta_{\mathcal{F}_121}$ are projections by $\mu_T$ of inverse images of OCS’s corresponding to each vertex of the auxiliary cube. This concludes our “2-form interpretation” of the middle symplectic fibration in (5).

We may likewise consider the inverse image in $\mathcal{F}_3$ of a point in $\mathcal{G}$. Consider the simple 2-form $e^{34}$. The natural candidates for elements in the inverse images with $b < 1$ are:

$$
-e^{12} + e^{56} - be^{34}, \quad e^{12} - e^{56} - be^{34} \in \mathcal{F}_3^+,
$$

$$
e^{12} + e^{56} - be^{34}, \quad -e^{12} - e^{56} - be^{34} \in \mathcal{F}_3^-.
$$

The case of $\mathcal{F}_3^+$ is displayed in Figure [11]. The internal segment $AB$ is the projection of the inverse image of $e^{34}$ which remarkably is obtained as the intersection of subsets in $\mathcal{F}_3^+$ invariant by the rotations generated by the roots $\lambda_1$ (corresponding to $J_0$) and $\lambda_3$ (corresponding to an OCS in $\mathcal{P}^-$). The case of $\mathcal{F}_3$ requires a 2-plane that is invariant simultaneously by $J^+$ and $J^-$. In Figure [11] the sets invariant by the respective $T^1$-actions project onto the filled hexagons.

The inverse image of $e^{34}$ in $\mathcal{P}^+$ consists of all the OCS’s $J$ for which this form represents the contribution of a negatively-oriented $J$-invariant 2-plane (for example, full-rank forms involving $-e^{34}$). It is easy to see that this set projects onto the edge determined by $J_1$ and $J_3$. As another example, consider the form $e^{15} - e^{26} - ae^{34}$. Its image by $\mu_T$ is
the midpoint of the segment $AB$. It can be interpreted as the sum of $e^{15} - e^{26} - e^{34} \in \mathcal{P}^+$ and $(1-a)e^{34}$. The form $e^{15} - e^{26} - e^{34}$ maps exactly to the midpoint of the edge $J_1J_3$.

In the case of $f$-structures parametrized by $\mathcal{F}_3^0$, there is no way to define a map between singular polytopes corresponding to the symplectic fibrations in the middle diagram of (5). The lack of such a mapping is due to the degeneracy of the characteristic 2-form. In fact, in the case of the remaining $\mathcal{F}_3$ orbits, there are two distinguished internal segments corresponding to the projections of the same 2-plane taken with two different orientations. In this case, the 2-form is not capable of determining the compatible orientation on its kernel. Graphically, this is expressed by the fact that the two segments (corresponding to the different orientations of the 2-dimensional kernel) intersect in the origin (see Figure 11).

Here is an example. The form $e^{14} + e^{23} \in \mathcal{F}_3^0$ determines a splitting

$$\langle \mathbb{R}^6 \rangle^* = \langle e^1, e^2, e^3, e^4 \rangle \oplus \langle e^5, e^6 \rangle,$$

and the form can be extended to $\omega = e^{14} + e^{23} + e^{56}$ to yield a compatible OCS in $\mathcal{P}^+$. The image of $e^{14} + e^{23}$ by $\mu_T$ is the origin of $\mathbb{R}^3$, whereas the image of $\omega$ is the midpoint $(0, 0, 1)$ of an edge of the tetrahedron.

5 An application

A standard way to classify a Riemannian $G$-structure is by means of the $G$-irreducible components of the associated space of intrinsic torsion. The prototype of such a classification gave rise to the sixteen classes of almost Hermitian manifolds à la Gray–Hervella [7]. More recently, there has been work on the problem of embedding classes of $G$-structures inside the corresponding parameter spaces or orbits.

Our approach enables one to compare the intrinsic torsion of interrelated structures, and it was observed in [12] that the intrinsic torsion of a mixed structure is completely determined by knowledge of the intrinsic torsion of the underlying OCS and OPS. An analysis of “null-torsion classes” of structures (obtained by setting various components of the intrin-
sic torsion equal to zero) on the Iwasawa manifold and other nilmanifolds was carried out in [1, 2]. We now combine these techniques.

The Iwasawa manifold \( N \) is defined as the set of right cosets \( \mathbb{Z}^6 \backslash G_H \), where \( G_H \) is the complex Heisenberg group and \( \mathbb{Z}^6 \) the natural lattice:

\[
G_H = \left\{ \begin{pmatrix} 1 & z^1 & z^2 \\ 0 & 1 & z^3 \\ 0 & 0 & 1 \end{pmatrix} : z^k \in \mathbb{C} \right\}, \quad \mathbb{Z}^6 = \left\{ \begin{pmatrix} 1 & a^1 & a^2 \\ 0 & 1 & a^3 \\ 0 & 0 & 1 \end{pmatrix} : a^k \in \mathbb{Z}[i] \right\}
\]

It has a standard Riemannian metric induced from a left-invariant tensor on \( G_H \).

The following result was effectively proved in [1], though we can now give it a moment map interpretation.

**Theorem 21.** The set of complex structures on \( N \) is given by the disjoint union of the point \( \omega_0 \) and a \( \mathbb{C}P^1 \). This is a \( T \)-invariant subset of \( \mathcal{Q}^+ \) and its image by \( \mu_T \) is the union of a vertex and the opposite edge \( E_{12} \) of \( \Delta_{\mathcal{Q}^+} \).

A justification of the invariance of this set by a maximal torus based upon the fact that \( G_H \) is a complex Lie group was given in [12].

The following results were established in [12]:

**Theorem 22.** The class of OPS’s \( P \in \mathcal{G} \) on \( N \) characterized by \( [\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H} \) is the complex submanifold

\[
K \cong \text{Gr}_2(\mathbb{R}^4) \cong \mathbb{C}P^1 \times \mathbb{C}P^1
\]

of \( \mathcal{G} \) whose image \( \mu_T(K) \) is the intersection \( \Delta_{\mathcal{Q}} \cap \langle e^{12}, e^{34} \rangle \).

Analogously, we consider the class of OPS’s for which \( [\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V} \). Let us denote the corresponding subset of \( \mathcal{G} \) by \( K' \).

**Theorem 23.** The subset \( K \cap K' \) of \( \mathcal{G} \) is the disjoint union of two 2-spheres. Its image by the moment map is

\[
\mu_T(K \cap K') = \{ xe^{12} + ye^{34} : x + y = \pm 1, \ |x|, |y| \leq 1 \},
\]

and consists of two line segments.

The properties of the fibrations of coadjoint orbits, described in the present article, enable us to detect directly some null-torsion classes of mixed structures as subsets of the 10-dimensional orbit that parametrizes them. Consider an OCS \( J \in \mathcal{Q}^+ \) and an OPS \( P \in \mathcal{G} \) with known intrinsic torsion. From what we have said, the points of \( \mathcal{F}_i \) lying in \( \pi_1^{-1}(J) \cap \pi_2^{-1}(P) \) have pre-determined intrinsic torsion. If we consider the inverse image of entire classes of OCS and OPS, their intersection determines the inclusion of the corresponding null-torsion class of mixed structures inside \( \mathcal{F} \).
Corollary 24. The class of mixed structures on $N$ of type $\mathcal{F}_1$ obtained from a complex structure and an almost product structure with integrable 4-dimensional distribution is isomorphic to a disjoint union $(\mathbb{CP}^1 \times \mathbb{CP}^1) \sqcup \mathbb{CP}^1$. Its projection to $\Delta_{\mathcal{F}_1}$ is shown in Figure 12.

Figure 12: Image of integrable structures in $\mathcal{F}_1$ lying over $K$

This result is proved by Remarks 18 and 20.

The same considerations applied to Proposition 23 yield

Corollary 25. The class of mixed structures on $N$ of type $\mathcal{F}_1$ obtained from a complex structure and an almost product structure for which both the 4-dimensional and 2-dimensional distributions are integrable is isomorphic to $\mathbb{CP}^1 \sqcup \mathbb{CP}^1 \sqcup \mathbb{CP}^1$. Its projection to $\Delta_{\mathcal{F}_1}$ is shown in Figure 13.

Figure 13: Image of integrable structures in $\mathcal{F}_1$ lying over $K \cap K'$

A classification of null-torsion classes on the Iwasawa manifold $N$ relative to its standard metric (and on other nilmanifolds, relative to various metrics) was given in [2]. Our last two corollaries are indicative of a class of results that should lead to a similar classification of null-torsion classes for other structures on $N$. We hope that the techniques of this paper will help establish the extent to which these subsets of the various coadjoint orbits for $SO(6)$ or $SU(4)$ are invariant by tori.
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