Asymptotic Solutions of the Kinetic Boltzmann Equation, Multicomponent Non-equilibrium Gas Dynamics and Turbulence

S. A. Serov

Institute of Theoretical and Mathematical Physics, Sarov, Russia

S. S. Serova

St. Petersburg State University, St. Petersburg, Russia

(Dated: February 7, 2014)

Abstract

In the article correct method for the kinetic Boltzmann equation asymptotic solution is formulated, the Hilbert’s and Enskog’s methods are discussed. The equations system of multicomponent non-equilibrium gas dynamics is derived, that corresponds to the first order in the approximate (asymptotic) method for solution of the system of kinetic Boltzmann equations. It is shown, that the velocity distribution functions of particles, obtained by the proposed method and by Enskog’s method, within Enskog’s approach, are equivalent up to the infinitesimal first order terms of the asymptotic expansion, but, generally speaking, differ in the next order. Interpretation of turbulent gas flow is proposed, as stratified on components gas flow, which is described by the derived equations system of multicomponent non-equilibrium gas dynamics.

PACS numbers: 02.30.Mv, 05.20.Dd, 47.27.Ak

Keywords: kinetic Boltzmann equation, multicomponent non-equilibrium gas dynamics, turbulence

serov@vniief.ru
I. INTRODUCTION

In 1912 Hilbert considered the kinetic Boltzmann equation for one-component gas as an example of integral equation and proposed a "recipe" for its approximate (asymptotic) solution (see [1], Chapter XXII). Hilbert's "recipe" was inconvenient for practical use, because the five arbitrary functional parameters of the first and the following approximations of the velocity distribution function had to be found by solving the differential equations in partial derivatives (equations of gas dynamics of the first and higher orders). Five years later Enskog proposed to use zero conditions, conditions (57)-(59) below with zero right-hand sides, to determine the five arbitrary functional parameters of the first and following approximations of the velocity distribution function. As we will see in the section III (in case of one-component gas), the imposition of the zero conditions leads, in fact, to using different comparison scales in the asymptotic expansion of the velocity distribution function and in the asymptotic expansion of the particle number density, the mean (mass) velocity and the temperature, that are derived from the asymptotic expansion of the velocity distribution function by integration over velocities with different weighting functions. As a result of paralogism of the method of successive approximations (one has to set variable coefficients of the same terms of the unified comparison scale equal to each other) partial time derivatives vanish in the necessary conditions of solutions existence of integral equations of higher orders (see below) and with them terms of gas-dynamic equations, corresponding to viscosity, heat conduction, ... vanish. Enskog "improved" the situation by the introducing (see, for example, [2], Chapter 7, § 1, Section 5) of the unsubstantiated expansion of partial time derivative.

The approach of Struminskii, who had proposed in 1974 in [3] his approximate (asymptotic) method of solution of the system of kinetic Boltzmann equations for multicomponent gas, differs from the approach of Enskog to asymptotic solution of the Boltzmann equations system for gas mixture in that, how the infinitesimal parameter is introduced in the Boltzmann equations system for gas mixture, i.e. the solution is constructing in another asymptotic limit. It could be noted, that approaches, that are similar to Struminskii's approach to asymptotic solution of the kinetic Boltzmann equations system, were regarded earlier in the kinetic theory of plasma (for instance, see [4], § 7.5). In substance, Struminskii’s method of solution of kinetic equations system is the same as Enskog’s method (Struminskii used the
partial time derivative expansion, as Enskog did).

After some preliminary remarks in section II, in section III there will be proposed the correct method of asymptotic solution of the kinetic Boltzmann equations system for multicomponent gas mixture for the approach, that combines Enskog’s and Struminskii’s approaches; in particular, it will be shown, how one has to modify Enskog’s method – in addition to asymptotic expansion of the velocity distribution function of \( i \)-component particles of gas mixture it is necessary to determine and to use the expansion of the particle number density \( n_i \) of \( i \)-component, mean mass velocity \( u \) and temperature \( T \) of the gas mixture.

Further, in the section IV the system of infinitesimal first order equations of multicomponent non-equilibrium gas dynamics, appearing during the process of the solution of the system of Boltzmann equations by successive approximations method in the section III as necessary condition of the existence of approximate (asymptotic) solution of the integral equations system, is considered in more detail.

In section V the expressions for the velocity distribution function of particles of gas mixture in the infinitesimal first order are obtained for the approach, that Enskog’s approach and Struminskii’s approach combines, in particular it is shown, that the velocity distribution functions, that are derived within Enskog’s approach by Enskog’s method and the proposed method, are equivalent up to the infinitesimal first order terms of the asymptotic expansion (inclusive), but, generally speaking, differ in the next order of the asymptotic expansion; the difference is a possible reason, why the transition to a higher order approximation in Enskog’s method does not lead to any essential improvements of the result.

In section VI an interpretation of turbulent flows is proposed within the multicomponent gas dynamics.

Struminskii’s article, and, as a result, the followings articles, that referred to the paper [3] (for example, [5]), contained errors in calculation of collision integrals. In appendix A general analytic expressions of these collision integrals (that depend on the interaction cross-section) are derived in general case, when separate components (with Maxwell velocity distribution function of particles) have different mean velocities and temperatures.

For the elementary model of an interaction of gas mixture particles – the interaction of particles as rigid spheres – collision integrals can be simply calculated completely, the expressions for the interaction potential of rigid spheres are written in appendix B.

Notations, used below, are close to notations in [2]; it is assumed, that all regarded
functions are continuous and continuously differentiable so many times as it is necessary, if their derivatives are considered, and all regarded integrals converge.

II. SOME REMARKS

In kinetic theory of gases and gas dynamics, closely connected with it, there are no small dimensionless constants, similar, for example, to the fine structure constant in quantum electrodynamics, in terms of which it would be possible to construct perturbation theory expansions, paying special attention to convergence of derivable expansions. In particular applications of kinetic theory of gases and gas dynamics it is possible often to make up from parameters of the problem a variable dimensionless value (in gas dynamics such variable dimensionless values are named numbers: the Mach number, the Reynolds number . . . ), we will denote this value by \( \theta \), and to consider asymptotic expansions with variable coefficients (6, Chapter V, § 2, Section 5) of physical values in terms of integral powers of \( \theta - \{\theta^r\}_{r \geq 0} \), when \( \theta \) tends to 0 (most important asymptotic expansions, further just those asymptotic expansions are considered; generally speaking, it is possible to use also integer negative powers \( r \), but we shall not do it), or more rarely \( \{\theta^{-r}\}_{r \geq 0} \), when \( \theta \) tends to \(+\infty\) or to \(-\infty\); changing \( \theta \) to \( 1/\theta \), it is possible to switch easily from one of these cases to another. In the theory of asymptotic expansions variable coefficients are considered to be comparable to the unit (and may depend on \( \theta \) in general case, but not in this article) and functions of a comparison scale \( \{\theta^r\}_{r \geq 0} \) – to be infinitesimal. If an asymptotic solution to specified precision \( \{\theta^r\} \) relative to a comparison scale \( \{\theta^r\}_{r \geq 0} \) exists, it is unique (see, for example, 6, Chapter V, § 2, Section 2) and, if asymptotic expansions are considered at \( \theta \to 0 \), is exact, at least, at \( \theta = 0 \), therefore it is convenient to consider the asymptotic expansions at \( \theta \to 0 \) (and integer \( r \geq 0 \)).

In applications of the theory of asymptotic expansions, solving physical equations by the successive approximations method, it is convenient not to extract the dimensionless parameter \( \theta^r \) from a term of asymptotic expansion \( \theta^r K^{(r)}(x) \) as a really small multiplier, but to use \( \theta^r \) as an indicator of infinitesimal order of corresponding (infinitely) small variable coefficient \( K^{(r)}(x) \), setting in the resultant expressions \( \theta = 1 \); with such agreement the formal "small" parameter \( \theta \) can be introduced in an equation arbitrarily, but the meaning of obtained asymptotic solutions of such equation with small parameter is determined by
physical validity of the introduction of the small parameter in the original equation.

By method of successive approximations we obtain, generally speaking, an asymptotic solution of the problem. If some additional conditions exist (for example, initial conditions or boundary conditions for a differential equation), the conditions must be expanded in asymptotic series, and the obtained equations system have to be solved, setting variable coefficient at the same terms of unified comparison scale equal to each other. If the system is solved successfully, we obtain an asymptotic solution of the initial equation, that meets (asymptotically) the additional conditions. In some cases the found asymptotic solution turns out to be regular solution ([7], Chapter 1), i.e. it depends analytically on $\theta$, or even exact solution of the problem. For example, the equation (see [8], Chapter V, § 2 or [9], Chapter IV, § 7.1)

$$\theta \frac{df}{dt} + f = 0, \quad f(0) = 0$$

(1)

with zero initial condition has asymptotic solution $f^{(r)}(t) \equiv 0$, ($r = 0, 1, 2 \ldots$) relative to the comparison scale, formed of functions $\theta^r$, that is also the exact solution of the problem. But the problem

$$\theta \frac{df}{dt} + f = 0, \quad f(0) = 1$$

(2)

has no analogous asymptotic solution, because the zero order asymptotic solution of the differential equation $f^{(0)}(t) \equiv 0$ contradicts to the initial condition $f^{(0)}(0) = 1$. It is not a serious drawback of the method of successive approximations. In the problem (2) one can introduce new function $g(t)$:

$$f(t) = \exp(-t/\theta) g(t)$$

(3)

(cf. with [7]; though not all considered by Lomov expansions are the asymptotic expansions with variable coefficients, and, accordingly, Lomov’s approach in toto seems not quite correct, in the monograph [7] it is actually shown, that similar replacements of functions allow to obtain asymptotic solutions for a wide class of problems) and obtain asymptotic solution of the problem by the successive approximations method $g^{(0)}(t) \equiv \text{const} = 1$, $g^{(r)}(t) \equiv \text{const} = 0$ ($r = 1, 2 \ldots$), that is again the exact solution.

Criticism of the successive approximations method in [8], Chapter V, § 2 and [9], Chapter IV, § 7.1 reflects, possibly, dissatisfaction of the authors with the introduction of the
unsubstantiated expansion of partial time derivative in Enskog’s method for the kinetic Boltzmann equation asymptotic solution, see below.

Hilbert noted in [1], Chapter XXII, that the expansion of the velocity distribution function

\[ F = \frac{\Phi}{\lambda} + \Psi + X\lambda + \cdots \]  

(4)

[similar to (34) below; Hilbert considered one-component gas only; we use Hilbert’s notations here, however, the meaning of the notations is sufficiently understandable from context] is the power series in (a small parameter) \( \lambda \), satisfying to the Boltzmann equation and such, that expressions [cf. with (50)-(53) and (54)-(56) below]

\[
\int \psi^{(i)} F \, d\omega = \frac{1}{\lambda} \int \psi^{(i)} \Phi \, d\omega + \int \psi^{(i)} \Psi \, d\omega + \lambda \int \psi^{(i)} X \, d\omega + \cdots \quad (i = 1, 2, 3, 4, 5)
\]

(5)

for \( t = t_0 \) change to power series

\[
\Lambda^{(i)} = \frac{f^{(i)}}{\lambda} + g^{(i)} + \lambda h^{(i)} + \cdots \quad (i = 1, 2, 3, 4, 5),
\]

(6)

and in the theorem, completing the work, he has formulated a ”recipe” to get an asymptotic solution for the Boltzmann equation, in which he proposed to determine five arbitrary functional parameters of functions \( \Phi, \Psi, X \ldots ” \) from five partial differential equations” [analogous (82), (99) below], ”setting at \( t = t_0 \)”

\[
\int \psi^{(i)} \Phi \, d\omega = \lambda \Lambda^{(i)} \quad (i = 1, 2, 3, 4, 5),
\]

(7)

\[
\int \psi^{(i)} \Psi \, d\omega = 0 \quad (i = 1, 2, 3, 4, 5),
\]

(8)

\[
\int \psi^{(i)} X \, d\omega = 0 \quad (i = 1, 2, 3, 4, 5).
\]

(9)

In notations from (42), (50)-(59) below, Hilbert proposed simply to set special initial values

\[
n(r, t_0) = n^{(0)}(r, t_0),
\]

(10)

\[
u(r, t_0) = \nu^{(0)}(r, t_0),
\]

(11)

\[
T(r, t_0) = T^{(0)}(r, t_0)
\]

(12)

or

\[
\int \psi^{(i)} f^{(r)} \, dc \bigg|_{t=t_0} \equiv 0 \quad (l = 1, 2, 3)
\]

(13)
"For the further substantiation of the gas theory" it might be as well to supplement Hilbert’s theorem with explicit expressions of five arbitrary functional parameters of the functions $f_i^{(r)}$, found on the $r$-step ($r = 0, 1, 2 \ldots$) of the successive approximations method, through the physical parameters of the gas [see (88)-(90), (105)-(107) below]; Hilbert had not done it.

Enskog supplemented Hilbert’s ”recipe”. However, at that Enskog made a logical mistake. He used zero conditions (13) identically at any $t$, not at $t = t_0$ only (see [2], Chapter 7, § 1, Section 1):

$$
\int \psi^{(l)} f^{(r)} \, dc \equiv 0 \quad (l = 1, 2, 3) \tag{14}
$$

for $r = 1, 2 \ldots$. From the viewpoint of the theory of asymptotic expansions, Enskog [instead of (54)-(56) below] supposed

$$
n (r, t, \theta) = \theta^n n (r, t, \theta) + \theta^1 0 + \theta^2 0 + \cdots, \tag{15}
$$

$$
u (r, t, \theta) = \theta^n u (r, t, \theta) + \theta^1 0 + \theta^2 0 + \cdots, \tag{16}
$$

$$
T (r, t, \theta) = \theta^n T (r, t, \theta) + \theta^1 0 + \theta^2 0 + \cdots. \tag{17}
$$

If $n$, $u$ and $T$ did not depend on $r$ and $t$, that would mean, that Enskog used different scales of comparison in the method of successive approximations at the same time \{$(r, t, \theta)$, $(r, t, \theta)$, \}(13), $(r, t, \theta)$, \}(13), that is in itself wrong. In general case, when $n$, $u$ and $T$ depend on $r$ and $t$, sums (15)-(17) can not even be considered as asymptotic expansions with variable coefficients.

Violation of logic of the successive approximations method shows itself immediately in vanishing of partial time derivatives in gas-dynamics equations systems of the $(r + 1)$-order [analogous (82) ($r = 1, 2 \ldots$) below, the necessary conditions of solutions existence of the integral higher orders equations in Enskog’s approach], according to (14),

$$
\int \psi^{(l)} \frac{\partial f^{(r)}}{\partial t} \, dc = \frac{\partial}{\partial t} \int \psi^{(l)} f^{(r)} \, dc = 0 \quad (l = 1, 2, 3), \tag{18}
$$

and with them terms of gas-dynamic equations, corresponding to viscosity, heat conduction, \ldots, vanish.

Hence Enskog should have concluded, that the use of (14) is incorrect, as in proof of theorem by contradiction, but instead of that Enskog proposed to use the unsubstantiated
expansion of partial time derivative
\[ \frac{\partial}{\partial t} = \sum_{r=0}^{\infty} \theta^r \frac{\partial}{\partial t}. \] (19)

III. CORRECT METHOD OF SOLUTION OF THE KINETIC BOLTZMANN EQUATIONS SYSTEM

The Boltzmann equations system, that describes change of dependent on \( t \) and spatial coordinates, prescribed by radius-vector \( \mathbf{r} \), the velocity distribution functions \( f_i (t, \mathbf{r}, \mathbf{c}_i) \) due to collision with particles of other components of mixture of rarefied monatomic gases, where \( \mathbf{c}_i \) are the velocities of particles of \( i \)-component of the mixture \{see [2], Chapter 8, (1.1); discussion of the derivation of the Boltzmann equations system and its applicability range see, for example, in [2], Chapters 3 and 18, [10], Chapter 7, § 1, [11], Chapter 3 and the Bogolyubov paper [12]; below the central interaction of molecules are considered only, when the force, with which each molecule acts on the other, is directed along the line, connecting the centers of the molecules\}, could be written as:

\[ \frac{\partial f_i}{\partial t} + \mathbf{c}_i \cdot \frac{\partial f_i}{\partial \mathbf{r}} + \frac{X_i}{m_i} \cdot \frac{\partial f_i}{\partial \mathbf{c}_i} = \sum_{j \in N} \int \int \int (f'_i f'_j - f_i f_j) g_{ij} b \, db \, d\epsilon \, dc_j 
= \sum_{j \in N} \int \int (f'_i f'_j - f_i f_j) k_{ij} d\mathbf{k} \, dc_j \quad (i \in N); \] (20)

in (20) \( N \) is a set of indexes, that are numbering components of the mixture; \( X_i \) is an external force, which acts on the molecule of the \( i \)-component; \( m_i \) is the mass of the molecule of the \( i \)-component; \( g_{ij} \) is the modulus of the relative velocity of colliding particles \( \mathbf{g}_{ij} = \mathbf{c}_i - \mathbf{c}_j \); \( b \) is the impact distance, \( \epsilon \) is the azimuth angle, \( \mathbf{k} \) is the unit vector, directed to the center of mass of the colliding particles from the point of closest approach – see [2], Chapter 3, Fig. 3; the scalar function \( k_{ij} (\mathbf{g}_{ij}, \mathbf{k}) \) is determined by equality

\[ g_{ij} b \, db \, d\epsilon \overset{\text{def}}{=} k_{ij} d\mathbf{k}; \] (21)

by prime in (20) and below the velocities and the functions of velocities after the collision are denoted.

Let us introduce following notations:
\[ J_i (f_i, f) = \int \int (f_i f - f'_i f') k_i d\mathbf{k} \, dc, \] (22)
\[ J_{ij} (f_i, f_j) = \int \int (f_i f_j - f'_i f'_j) k_{ij} d\mathbf{k} \, dc_j; \] (23)
to differ velocities of colliding molecules of the same kind in (22) the one velocity is denoted by \( c_j \) and the other is denoted by \( c \) (without any index) and the index of the corresponding velocity distribution function \( f \) is omitted.

In Enskog’s approach the differential parts of the Boltzmann equations (20), that are denoted by \( D_i f_i \) below, are considered to be small as compared with the right-hand sides of equations (20) – see [2], Chapter 7, § 1, Section 5 – therefore the indicator of infinite smallness \( \theta \) is formally introduced in the Boltzmann equations system in the following way:

\[
\theta D_i f_i = - \sum_j J_{ij} (f_i, f_j) \quad (i \in N). \tag{24}
\]

In Struminskii’s approach to the asymptotic solution of the Boltzmann equations system the differential parts of the Boltzmann equations (20) and the collision integrals of the particles of \( i \)-component with the particles of the other components are considered to be small as compared with the collision integral of the particles of \( i \)-component between each other, therefore the indicator of infinitesimal smallness \( \theta \) is introduced in the Boltzmann equations system in another way:

\[
\theta D_i f_i = - J_i (f_i, f) - \theta \sum_{j \neq i} J_{ij} (f_i, f_j) \quad (i \in N). \tag{25}
\]

It is possible to combine Enskog’s approach with Struminskii’s approach. For this purpose we divide the set of mixture components \( N \) into two subsets: the subset of components, that we call formally inner components (we could consider the case, when there are some subsets of inner components, but this case does not fundamentally differ from the one, considered below, the only difference is that the notation become more complicated) and the subset of components, that we call external components. To differ the two groups of mixture components we denote the subset of indexes of inner components \( \hat{N} \) as well as the indexes of inner components \( \hat{i} \in \hat{N} \) by symbol “” and the subset of indexes of external components \( \check{N} \) as well as the indexes of external components \( \check{i} \in \check{N} \) by symbol “”; the intersection of the sets \( \hat{N} \) and \( \check{N} \) is the empty set – \( \hat{N} \cap \check{N} = \emptyset \) and the union of these sets is the set of indexes of all mixture components \( \hat{N} \cup \check{N} = N \); if an assertion concerns both kinds of components the symbols “” and “” will be omitted. In new notations the Boltzmann equations system
can be rewritten as:

\[ \theta D_i f_i = -\sum_{j \in \hat{N}} J_{ij} (f_i, f_j) - \theta \sum_{j \in \hat{N}} J_{ij} (f_i, f_j) \quad (\hat{i} \in \hat{N}), \]  
\[ \theta D_i f_i = -J_i (f_i, f) - \theta \sum_{j \neq i} J_{ij} (f_i, f_j) \quad (\hat{i} \in \hat{N}). \]  

(26a)

(26b)

Let us write the asymptotic expansion of the velocity distribution function \( f_i \) of particles of \( i \)-component as formal series of successive approximations in powers of \( \theta \): 

\[ f_i = f_i^{(0)} + \theta f_i^{(1)} + \theta^2 f_i^{(2)} + \cdots. \]  

(27)

The differential parts of the equations (26) are written as:

\[ D_i f_i = \left( \frac{\partial}{\partial t} + c_i \cdot \frac{\partial}{\partial r} + \frac{X_i}{m_i} \cdot \frac{\partial}{\partial c_i} \right) \left( f_i^{(0)} + \theta f_i^{(1)} + \cdots \right) \]

\[ = D_i^{(0)} + \theta D_i^{(1)} + \theta^2 D_i^{(2)} + \cdots, \]  

(28)

where

\[ D_i^{(r)} = \frac{\partial f_i^{(r)}}{\partial t} + c_i \cdot \frac{\partial f_i^{(r)}}{\partial r} + \frac{X_i}{m_i} \cdot \frac{\partial f_i^{(r)}}{\partial c_i} \quad (r = 0, 1, 2 \ldots), \]  

(29)

– cf. with [2], Chapter 7, § 1, Sections 4, 5 and [3]. In (28)-(29) the partial time derivative expansion (19) is not used in contrast to that, how it was made by Enskog and further by Struminskii. As result, described below method for solution of the system of kinetic Boltzmann equations differ fundamentally from Enskog’s method and Struminskii’s method.

Substituting (27) and (28) in (26a) and equating coefficients at the same powers of \( \theta \) to each other, we obtain the equations system of the method of successive approximations for finding the velocity distribution functions of inner components particles of gas mixture \( f_i^{(r)} \); taking introduced notations (22), (23) and (29) into account, the system can be rewritten as:

\[ \sum_{j \in \hat{N}} J_{ij} \left( f_i^{(0)}, f_j^{(0)} \right) = 0 \quad (\hat{i} \in \hat{N}), \]  

(30)

\[ D_i^{(r-1)} + \sum_{j \in \hat{N}} J_{ij} \left( f_i^{(r)}, f_j^{(0)} \right) + \sum_{j \in \hat{N}} \sum_{s=1}^{r-1} J_{ij} \left( f_i^{(r-s)}, f_j^{(s)} \right) + \sum_{j \in \hat{N}} J_{ij} \left( f_i^{(0)}, f_j^{(r)} \right) \]

\[ + \sum_{j \in \hat{N}} \sum_{s=0}^{r-1} J_{ij} \left( f_i^{(r-1-s)}, f_j^{(s)} \right) = 0 \quad (\hat{i} \in \hat{N}, r = 1, 2 \ldots). \]  

(31)
Similarly substituting (27) and (28) in (26b) and equating coefficients at the same powers of $\theta$ to each other, we obtain the equations system of the method of successive approximations for finding the velocity distribution functions of particles of external components of gas mixture $f^{(r)}_i$:

$$J_i\left(f^{(0)}_i, f^{(0)}\right) = 0 \quad (\hat{i} \in \hat{N}),$$

$$D^{(r-1)}_i + J_i\left(f^{(r)}_i, f^{(0)}\right) + \sum_{s=1}^{r-1} J_i\left(f^{(r-s)}_i, f^{(s)}\right) + J_i\left(f^{(0)}_i, f^{(r)}\right)$$

$$+ \sum_{j \neq i} \sum_{s=0}^{r-1} J_{ij}\left(f^{(r-1-s)}_i, f^{(s)}_j\right) = 0 \quad (\hat{i} \in \hat{N}, \ r = 1, 2 \ldots).$$

(33)

It could be noted also, that in original Hilbert’s paper [1] and Enskog’s paper (see. [2], Chapter 7, § 1, Sections 4) the formal parameter $\theta$ was not directly introduced in the Boltzmann equation, but in the series of successive approximations for the velocity distribution function the parameter $\theta$ was introduced in another way (not as in (27)):

$$f^{(0)}_i = 1 \theta f_i^{(0)} + f^{(1)}_i + \theta f^{(2)}_i + \cdots ;$$

(34)

Obviously, with the same result it is possible to use the expansion (27) for the velocity distribution function, but to enter the multiplier $\theta$ in the Boltzmann equation, as it was made in (24). Successive approximations $f^{(0)}_i, f^{(1)}_i, f^{(2)}_i \ldots$, that are calculated within Enskog’s approach, turn out to be (inversely) ordered by the number density of molecules of the mixture $n$: $f^{(0)}_i$ is proportional to $n$, $f^{(1)}_i$ does not depend directly on $n$ etc. Thus, there is a physical justification to use the method of successive approximations to find an asymptotic solution of the Boltzmann equation. For Struminskii’s approach it is difficult to determine explicitly a small physical variable to construct asymptotic expansions in powers of the variable.

Speaking about an order of approximation below, we assume the order to be equal to the value of index $r$ in (31), (33). According to (23), (30), in zero order approximation we have the following system of integral equations to find the velocity distribution functions of particles of inner components of gas mixture $f^{(0)}_i$:

$$\sum_{j \in \hat{N}} J_{ij}\left(f^{(0)}_i, f^{(0)}_j\right) = \sum_{j \in \hat{N}} \int \int \left(f^{(0)}_i f^{(0)}_j - f^{(0)}_i f^{(0)}_j\right) k_{ij} dk dc_j = 0 \quad (\hat{i} \in \hat{N}).$$

(35)
In the kinetic theory of gases the following equality is often used:

\[ \int \int \int \int \phi_i f_i' f_j' g_{ij} b \, db \, dc_i \, dc_j = \int \int \int \phi_i f_j g_{ij} b \, db \, dc_i \, dc_j \]  \tag{36}

in (36), as well as in (20) velocities and velocity functions of particles after collision are denoted by prime. To each collision of two gas molecules, that transforms the velocities of the molecules before the collision \( c_i, c_j \) into \( c_i', c_j' \), the collision of two gas molecules corresponds bijectively, that transforms the velocities of the molecules \( c_i', c_j' \rightarrow -c_i', -c_j' \rightarrow -c_i, -c_j \rightarrow c_i, c_j \), and for considered central interactions it follows from the energy conservation law, that magnitudes of relative velocities of molecules before and after the collision are equal

\[ g_{ij} = g_{ij}' \]  \tag{37}

from the angular momentum conservation law it follows, that the impact parameters are also equal

\[ b = b' \]  \tag{38}

for the collisions; thus, the equality (36) follows directly from that the Jacobian determinant of the transformation of not primed velocities into primed velocities equals to unity; the equality does not depend on form of the functions \( \phi, f \), the only requirement is that integrals are determined and converge – cf. with \cite{2}, Chapter 3, § 5, Section 3.

Following Hecke’s idea (see \cite{1}, Chapter XXII), we multiply both sides of the equation (35) on \( \ln f_i \), integrate over \( c_i \), sum over \( \hat{i} \) and transform integrals, taking (21), (36) into account; as a result we obtain:

\[ \frac{1}{4} \sum_{i,j \in \mathbb{N}} \int \int \ln \left( \frac{f_i^{(0)} f_j^{(0)}'}{f_i^{(0)'} f_j^{(0)}} \right) \left( f_i^{(0)} f_j^{(0)} - f_i^{(0)'} f_j^{(0)'} \right) k_{ij} \, dk \, dc_i \, dc_j = 0. \tag{39} \]

Integrands in (39) can not be (strictly) less than zero, therefore the sum in (39) can be equal to zero under the condition only, that all integrands for all values of integration variables are vanished (all considered functions are assumed to be continuous in each point of their definition domain), i.e.

\[ f_i^{(0)'} f_j^{(0)^{'}} \equiv f_i^{(0)} f_j^{(0)} \]  \tag{40}

or

\[ \ln f_i^{(0)'} + \ln f_j^{(0)'} - \ln f_i^{(0)} - \ln f_j^{(0)} \equiv 0. \]  \tag{41}
Hence, \( \ln f_i^{(0)} \) should be expressed linearly in terms of *summational invariants* of collision

\[
\psi_i^{(1)} = m_i, \quad (42a)
\]

\[
\psi_i^{(2)} = m_i c_i, \quad (42b)
\]

\[
\psi_i^{(3)} = \frac{1}{2} m_i c_i^2; \quad (42c)
\]

for a collision of the \( \hat{i} \)-molecule with the \( \hat{j} \)-molecule conservation of the collision invariant \( \psi_i^{(l)} \) is expressed by equality:

\[
\psi_i^{(l)\prime} + \psi_j^{(l)\prime} - \psi_i^{(l)} - \psi_j^{(l)} = 0 \quad (l = 1, 2, 3). \quad (43)
\]

Linear combination of the summational invariants \( \psi_i^{(l)} \) is an additive invariant. There are no other summational invariants, that depend on velocities of molecules and are linear independent of \( \psi_i^{(l)} \): six scalar unknown quantities (six components of velocities of the molecules after the collision \( c_i', c_j' \)) are completely determined by six known components of velocities of the molecules before the collision \( c_i, c_j \) by two free geometric parameters, that define a collision, for example, such as impact parameter \( b \) and azimuth angle \( \epsilon \) (see above), that gives us two constraint equations, and from four scalar equations (43), corresponding to the conservation of energy and three components of momentum; Existence of summational invariant of collision, depending on velocities of molecules and linearly independent of \( \psi_i^{(l)} \), would give similar to (43) redundant constraint equation for velocities of molecules after collision and velocities of molecules before collision.

Therefore,

\[
\ln f_i^{(0)} = \alpha_i^{(1,0)} + \alpha_i^{(2,0)} \cdot m_i c_i + \alpha_i^{(3,0)} \frac{1}{2} m_i c_i^2, \quad (44)
\]

where \( \alpha_i^{(1,0)} \) and \( \alpha_i^{(3,0)} \) are some, independent of \( c_i \), scalar functions of spatial coordinates (that are defined by the radius vector \( r \)) and time \( t \), and \( \alpha_i^{(2,0)} \) is a vector function of \( r \) and \( t \) [as it follows from the equations (43) for \( l = 2, 3 \), the functions \( \alpha_i^{(2,0)} \) and \( \alpha_i^{(3,0)} \) are the same for all inner components of mixture]. Or

\[
\ln f_i^{(0)} = \ln \alpha_i^{(0,0)} + \alpha_i^{(3,0)} \frac{1}{2} m_i \left( c_i + \alpha_i^{(2,0)}/\alpha_i^{(3,0)} \right)^2, \quad (45)
\]

where \( \alpha_i^{(0,0)} \) is a new scalar function of \( r \) and \( t \). I.e. the general solution of the equations system (35) can be written as a set of the Maxwell functions:

\[
f_i^{(0)} = \beta_i^{(1,0)} \left( \frac{m_i}{2 \pi k \beta_i^{(3,0)}} \right)^{3/2} e^{- \frac{m_i (c_i - \beta_i^{(2,0)})^2}{2 k \beta_i^{(3,0)}}} \quad (\hat{i} \in \hat{N}), \quad (46)
\]
where $k$ is the Boltzmann constant,

$$
\beta_i^{(1,0)} = \alpha_i^{(0,0)} \left( -\frac{2\pi}{m_i \alpha_i^{(3,0)}} \right)^{3/2},
$$

(47)

$$
\beta_i^{(2,0)} = -\frac{\alpha_i^{(2,0)}}{\alpha_i^{(3,0)}},
$$

(48)

$$
\beta_i^{(3,0)} = -\frac{1}{k \alpha_i^{(3,0)}}.
$$

(49)

Particle number density $n_i$ of the $i$-component, mean mass velocity $u \cdot$ and temperature $T \cdot$ of inner components of mixture are introduced by definitions:

$$
n_i \overset{\text{def}}{=} \int f_i \, dc_i,
$$

(50)

$$
u \sum_{i \in \hat{N}} n_i \cdot m \overset{\text{def}}{=} \sum_{i \in \hat{N}} m_i \cdot c_i f_i \, dc_i,
$$

(51)

$$
\frac{3}{2} kT \cdot \sum_{i \in \hat{N}} n_i \overset{\text{def}}{=} \sum_{i \in \hat{N}} \int \frac{1}{2} m_i (c_i - u \cdot)^2 f_i \, dc_i,
$$

(52)

in (52) $k$ is the Boltzmann constant. From (50)-(52) the equality is obtained:

$$
\frac{3}{2} kT \cdot \sum_{i \in \hat{N}} n_i + \frac{1}{2} u^2 \sum_{i \in \hat{N}} n_i \cdot m_i = \sum_{i \in \hat{N}} \int \frac{1}{2} m_i c_i^2 f_i \, dc_i,
$$

(53)

that is convenient to use below instead of definition (52).

According to definitions (50), (51), (53), in addition to the asymptotic expansion (27) it is necessary to determine asymptotic expansions for particle number density $n_i$ of the $i$-component

$$
n_i = n_i^{(0)} + \theta n_i^{(1)} + \theta^2 n_i^{(2)} + \cdots,
$$

(54)

mean mass velocity $u \cdot$

$$
u \cdot = u \cdot^{(0)} + \theta u \cdot^{(1)} + \theta^2 u \cdot^{(2)} + \cdots
$$

(55)

and temperature $T \cdot$ of inner components of mixture

$$
T \cdot = T \cdot^{(0)} + \theta T \cdot^{(1)} + \theta^2 T \cdot^{(2)} + \cdots.
$$

(56)

Substituting (27) and (54)-(56) in (50), (51), (53) and equating terms of the same infinitesimal order we obtain Card $\hat{N} + 4$ scalar relations, that connect asymptotic expansions (27) and (54)- (56):

$$
\int f_i^{(r)} \, dc_i = n_i^{(r)} \quad (i \in \hat{N}),
$$

(57)
\[
\sum_{i \in \hat{N}} \int \frac{1}{2} m_i c_i^2 f_i^{(r)}(r_i) \, d\mathbf{c}_i = \frac{3}{2} k \sum_{i \in \hat{N}} \left( n_i T_i^{(r)}(r_i) + \frac{1}{2} \sum_{i \in \hat{N}} m_i (n_i u_i^2)^{(r)} \right)
\]
\[
= \frac{3}{2} k \sum_{i \in \hat{N}} \sum_{s=0}^{r} n_i^{(r-s)} T_i^{(s)} + \frac{1}{2} \sum_{i \in \hat{N}} m_i \sum_{s=0}^{r} \sum_{q=0}^{s} n_i^{(r-s)} u_i^{(s-q)} \cdot \mathbf{u}^{(q)}
\]
\[
= \frac{3}{2} k \sum_{s=0}^{r} \hat{n}^{(r-s)} T_i^{(s)} + \frac{1}{2} \sum_{s=0}^{r} \sum_{q=0}^{s} \hat{\rho}^{(r-s)} u_i^{(s-q)} \cdot \mathbf{u}^{(q)}. \tag{59}
\]

In \(58\), \(59\) the notations are introduced

\[
\hat{\rho}^{(r-s)} = \sum_{i \in \hat{N}} m_i n_i^{(r-s)}, \tag{60}
\]
\[
\hat{n}^{(r-s)} = \sum_{i \in \hat{N}} n_i^{(r-s)}. \tag{61}
\]

In particular, for \(r = 0\) from \(57\) through \(59\) we obtain expressions for arbitrary functions \(\beta_i^{(1,0)}(r, t)\), \(\beta_i^{(2,0)}(r, t)\) and \(\beta_i^{(3,0)}(r, t)\) in \(46\) through the zero order approximations to local values of the \(i\)-component number density, the mean mass velocity and the temperature of inner components of the mixture:

\[
\beta_i^{(1,0)}(r, t) = n_i^{(0)}(r, t), \tag{62a}
\]
\[
\beta_i^{(2,0)}(r, t) = u_i^{(0)}(r, t), \tag{62b}
\]
\[
\beta_i^{(3,0)}(r, t) = T_i^{(0)}(r, t). \tag{62c}
\]

According to \(22\), \(32\), zero order integral equations, from which the velocity distribution functions \(f_i^{(0)}(r_i)\) of particles of outer components of the mixture are found:

\[
J_i \left( f_i^{(0)}, f_i^{(0)} \right) = \int \int \left( f_i^{(0)} f_i^{(0)'} - f_i^{(0)} f_i^{(0)'} \right) k_i d\mathbf{k} d\mathbf{c} = 0 \quad (i \in \hat{N}), \tag{63}
\]

are simpler than equations \(35\) and differ actually from \(35\) only by lack of summation over components. Therefore, similarly \(46\), the general solution of the equations system \(63\) can be written as a set of the Maxwell functions:

\[
f_i^{(0)} = \beta_i^{(1,0)} \left( \frac{m_i}{2 \pi k_i \beta_i^{(3,0)}} \right)^{3/2} e^{-\frac{m_i (\epsilon_i - \beta_i^{(2,0)})^2}{2 k_i \beta_i^{(3,0)}}} \quad (i \in \hat{N}), \tag{64}
\]
where $\beta_i^{(1,0)}$ and $\beta_i^{(3,0)}$ are some, independent of $c_i$, scalar functions of spatial coordinates, defined by the radius vector $r$, and time $t$, and $\beta_i^{(2,0)}$ is a vector function of $r$ and $t$.

Let’s add to the definition of the number density of particles of $i$-component definitions of mean velocity $u_i$ and temperature $T_i$ of outer component of mixture:

$$u_i n_i m_i \text{def} = \int m_i c_i f_i d c_i,$$

$$\frac{3}{2} k T_i n_i \text{def} = \int \frac{1}{2} m_i (c_i - u_i)^2 f_i d c_i;$$

from (50), (65), (66) the equality is obtained:

$$\frac{3}{2} k T_i n_i + \frac{1}{2} u_i^2 n_i m_i = \int \frac{1}{2} m_i c_i^2 f_i d c_i,$$

that is convenient to use below instead of definition (66).

Let’s enter similar (55)-(56) asymptotic expansions of outer $i$-component mean velocity $u_i$

$$u_i = u_i^{(0)} + \theta u_i^{(1)} + \theta^2 u_i^{(2)} + \ldots$$

and outer $i$-component temperature $T_i$

$$T_i = T_i^{(0)} + \theta T_i^{(1)} + \theta^2 T_i^{(2)} + \ldots.$$

Substituting (27), (54), (68), (69) in (50), (65), (67) and equating terms of the same infinitesimal order we obtain for each $i$ 5 (scalar) relations, that connect asymptotic expansions (27), (54), (68), (69):

$$\int f_i^{(r)} d c_i = n_i^{(r)},$$

$$\int m_i c_i f_i^{(r)} d c_i = m_i (n_i u_i)^{(r)} = m_i \sum_{s=0}^{r} n_i^{(r-s)} u_i^{(s)} = \sum_{s=0}^{r} \rho_i^{(r-s)} u_i^{(s)},$$

$$\int \frac{1}{2} m_i c_i^2 f_i^{(r)} d c_i = \frac{3}{2} k (n_i T_i)^{(r)} + \frac{1}{2} m_i (n_i u_i^2)^{(r)}$$

$$= \frac{3}{2} k \sum_{s=0}^{r} n_i^{(r-s)} T_i^{(s)} + \frac{1}{2} m_i \sum_{s=0}^{r} \sum_{q=0}^{s} n_i^{(r-s)} u_i^{(s-q)} \cdot u_i^{(q)}$$

$$= \frac{3}{2} k \sum_{s=0}^{r} n_i^{(r-s)} T_i^{(s)} + \frac{1}{2} \sum_{q=0}^{r} \sum_{s=0}^{r} \rho_i^{(r-s)} u_i^{(s-q)} \cdot u_i^{(q)}.$$
cf. with \((57)-(59)\). In \((71)\), \((72)\) the notation is used
\[
\rho^{(r-s)}_i = m_i n^{(r-s)}_i. \tag{73}
\]

For \(r = 0\) from \((70)-(72)\) we obtain expressions for arbitrary functions \(\beta^{(1,0)}_i(r,t), \beta^{(2,0)}_i(r,t)\) and \(\beta^{(3,0)}_i(r,t)\) in \((64)\) through the zero order approximations to local values of the number density, the mean velocity and the temperature of outer \(\hat{i}\)-component of the mixture:
\[
\begin{align*}
\beta^{(1,0)}_i(r,t) &= n^{(0)}_i(r,t), \tag{74a} \\
\beta^{(2,0)}_i(r,t) &= u^{(0)}_i(r,t), \tag{74b} \\
\beta^{(3,0)}_i(r,t) &= T^{(0)}_i(r,t). \tag{74c}
\end{align*}
\]

For \(r \geq 1\) the velocity distribution functions of inner components of gas mixture \(f^{(r)}_i\) are found from the integral equations system \((31)\), which, taking \((23)\) and \((40)\) into account, can be rewritten in the form
\[
\begin{align*}
\mathcal{D}^{(r-1)}_i + \sum_{j \in N} \sum_{s=1}^{r-1} J_{ij} \left( f^{(r-s)}_i, f^{(s)}_j \right) + \sum_{j \in N} \sum_{s=0}^{r-1} J_{ij} \left( f^{(r-1-s)}_i, f^{(s)}_j \right) \\
= - \sum_{j \in N} J_{ij} \left( f^{(0)}_i \chi^{(r)}_i, f^{(0)}_j \right) - \sum_{j \in N} J_{ij} \left( f^{(0)}_i, f^{(0)}_j \chi^{(r)}_j \right) \\
= - \sum_{j \in N} \int \int f^{(0)}_i f^{(0)}_j \left( \chi^{(r)}_i + \chi^{(r)}_j - \chi^{(r)}_i - \chi^{(r)}_j \right) k_{ij} \, dk \, dc_j \quad \left( \hat{i} \in \hat{N}, \right) \tag{75}
\end{align*}
\]
in \((75)\) functions \(f^{(r)}_i\) are written as \(f^{(r)}_i = f^{(0)}_i \chi^{(r)}_i\), where \(\chi^{(r)}_i\) are new unknown functions.

The left-hand sides of equations \((75)\) involves functions, that are known from the previous step of the successive approximations method. Unknown functions \(\chi^{(r)}_i\) enter, linearly, only into the right-hand sides of equations \((75)\). Therefore the general solution of the system of equations \((31)\) is a family of functions of a form \(\{f^{(r)}_i = \Xi^{(r)}_i + \chi^{(r)}_i\} \in \hat{N}\), where \(\{\Xi^{(r)}_i = f^{(0)}_i \phi^{(r)}_i\} \in \hat{N}\), \(\{\chi^{(r)}_i = f^{(0)}_i \phi^{(r)}_i\} \in \hat{N}\), a family of functions \(\{\phi^{(r)}_i\} \in \hat{N}\) is a particular solution of the system of inhomogeneous equations \((75)\) and a family of functions \(\{\phi^{(r)}_i\} \in \hat{N}\) is the general solution of the system of homogeneous equations
\[
0 = \sum_{j \in N} \int \int f^{(0)}_i f^{(0)}_j \left( \phi^{(r)}_i + \phi^{(r)}_j - \phi^{(r)}_i - \phi^{(r)}_j \right) k_{ij} \, dk \, dc_j \quad \left( \hat{i} \in \hat{N}, \right) \tag{76}
\]

Multiplying equations \((70)\) by \(\phi^{(r)}_i\), integrating over all values of \(c_i\), summing over \(\hat{i}\) and transforming integrals, as it has been made in deriving \((39)\), we obtain
\[
\frac{1}{4} \sum_{i,j \in N} \int \int f^{(0)}_i f^{(0)}_j \left( \phi^{(r)}_i + \phi^{(r)}_j - \phi^{(r)}_i - \phi^{(r)}_j \right)^2 k_{ij} \, dk \, dc_i \, dc_j = 0. \tag{77}
\]
From (77) we conclude, cf. with (39) and (44), that \( \phi_i^{(r)} \) are linear combinations of the summational invariants of the collision \( \psi_i^{(l)} \) \( (l = 1, 2, 3) \):
\[
\phi_i^{(r)} = \alpha_i^{(1,r)} + \alpha_i^{(2,r)} \cdot m_i c_i + \alpha_i^{(3,r)} \frac{1}{2} m_i c_i^2,
\]
where \( \alpha_i^{(1,r)} \) and \( \alpha_i^{(3,r)} \) are some, independent of \( c_i \), scalar functions of spatial coordinates, defined by the radius vector \( r \), and time \( t \), and \( \alpha_i^{(2,r)} \) is a vector function of \( r \) and \( t \) (as well as above, arbitrary functions \( \alpha_i^{(2,r)} \) and \( \alpha_i^{(3,r)} \) are identical for all inner components of the mixture), and, hence,
\[
\xi_i^{(r)} = f_i^{(0)} \left( \alpha_i^{(1,r)} + \alpha_i^{(2,r)} \cdot m_i c_i + \alpha_i^{(3,r)} \frac{1}{2} m_i c_i^2 \right) \quad \left( \hat{i} \in \hat{N} \right).
\]
To simplify further evaluations according to the expression for \( f_i^{(0)} \), see (46) and (62), let us rewrite (79) as
\[
\xi_i^{(r)} = f_i^{(0)} \left[ \beta_i^{(1,r)} + \beta_i^{(2,r)} \cdot m_i \left( c_i - u^{(0)} \right) + \beta_i^{(3,r)} \frac{1}{2} m_i \left( c_i - u^{(0)} \right)^2 \right] \quad \left( \hat{i} \in \hat{N} \right),
\]
where \( \beta_i^{(1,r)} \), \( \beta_i^{(2,r)} \) and \( \beta_i^{(3,r)} \) are new functions of \( r \) and \( t \). Family of functions \( \{ \chi_i^{(r)} \}_{i \in \hat{N}} \) is a solution of the system of inhomogeneous equations
\[
F_i^{(r)} = \sum_{j \in \hat{N}} \int_{\hat{N}} f_i^{(0)} f_j^{(0)} \left( \chi_i^{(r)} + \chi_j^{(r)} - \chi_i^{(r')} - \chi_j^{(r')} \right) k_{ij} dk dc_j \quad \left( \hat{i} \in \hat{N} \right),
\]
where \( F_i^{(r)} \) denote left-hand sides of the equations (75), taken with opposite sign.

Multiplying equations (81) by \( \psi_i^{(l)} \) \( (l = 1, 2, 3) \), integrating over all values of \( c_i \) and transforming integrals as above, we obtain, taking (43) into account, as necessary condition for the existence of solutions of the system of integral equations (81), the necessity of the fulfillment of equalities:
\[
\int \psi_i^{(1)} F_i^{(r)} dc_i = 0 \quad \left( \hat{i} \in \hat{N} \right),
\]
\[
\sum_{i \in \hat{N}} \int \psi_i^{(l)} F_i^{(r)} dc_i = 0 \quad (l = 2, 3).
\]

Among (infinitesimal) set of particular solutions of the system of equations (81), different from each other on some solution of the system of homogeneous equations (76), unique solution \( \{ \Phi_i^{(r)} \}_{i \in \hat{N}} \) may be chosen such that
\[
\int \psi_i^{(1)} f_i^{(0)} \Phi_i^{(r)} dc_i = 0 \quad \left( \hat{i} \in \hat{N} \right),
\]
\[
\sum_{i \in \hat{N}} \int \psi_i^{(l)} f_i^{(0)} \Phi_i^{(r)} dc_i = 0 \quad (l = 2, 3).
\]
Having substituted expression for $f_i^{(r)} (\hat{i} \in \hat{N})$

$$f_i^{(r)} = \Xi_i^{(r)} + \zeta_i^{(r)}$$

$$= f_i^{(0)} \Phi_i^{(r)} + f_i^{(0)} \left[ \beta_i^{(1,r)} + \beta_i^{(2,r)} \cdot m_i (c_i - u^{(0)}) + \beta_i^{(3,r)} \frac{1}{2} m_i (c_i - u^{(0)})^2 \right]$$  \hspace{1cm} (84)

in (57)-(59), taking (46), (60)-(62) and (83) into account, we obtain a system of Card (\hat{N}) + 4 algebraic equations [constraint equations for asymptotic expansions (27) and (54)-(56)]:

$$n_i^{(0)} \beta_i^{(1,r)} + \frac{3}{2} n_i^{(0)} kT^{(0)} \beta_i^{(3,r)} = n_i^{(r)} \quad (\hat{i} \in \hat{N}),$$  \hspace{1cm} (85)

$$u^{(0)} \sum_{i \in \hat{N}} m_i n_i^{(0)} \beta_i^{(1,r)} + \hat{\rho}^{(0)} kT^{(0)} \beta_i^{(2,r)} + \frac{3}{2} \hat{\rho}^{(0)} kT^{(0)} u^{(0)} \beta_i^{(3,r)} = \sum_{s=0}^{r} \hat{\rho}^{(r-s)} u^{(s)},$$  \hspace{1cm} (86)

$$= \frac{3}{2} k \sum_{s=0}^{r} \hat{n}^{(r-s)} T^{(s)} + \frac{1}{2} \sum_{s=0}^{r} \sum_{q=0}^{s} \hat{\rho}^{(r-s)} u^{(s-q)} \cdot u^{(q)},$$  \hspace{1cm} (87)

from which we find expressions for functions $\beta_i^{(1,r)} (r, t), \beta_i^{(2,r)} (r, t)$ and $\beta_i^{(3,r)} (r, t)$ through (variable) coefficients of asymptotic expansions of the particle number density of $\hat{i}$-component, of the mean mass velocity and of the temperature of inner components of the mixture

$$\beta_i^{(1,r)} = \frac{n_i^{(r)}}{n_i^{(0)}} - \frac{3}{2} \frac{1}{\hat{n}^{(0)} T^{(0)}} \left[ \sum_{s=0}^{r} \left( \hat{n}^{(r-s)} T^{(s)} - \hat{n}^{(r)} T^{(0)} \right) \right]$$

$$- \frac{1}{2} \frac{1}{\hat{n}^{(0)} kT^{(0)}} \left[ \sum_{s=0}^{r} \sum_{q=0}^{s} \hat{\rho}^{(r-s)} u^{(s-q)} \cdot u^{(q)} - \hat{\rho}^{(r)} \left( u^{(0)} \right)^2 \right]$$

$$+ \frac{1}{\hat{n}^{(0)} kT^{(0)}} u^{(0)} \left[ \sum_{s=0}^{r} \hat{\rho}^{(r-s)} u^{(s)} - \hat{\rho}^{(r)} u^{(0)} \right],$$  \hspace{1cm} (88)

$$\beta_i^{(2,r)} = \frac{1}{\hat{\rho}^{(0)} kT^{(0)}} \left[ \sum_{s=0}^{r} \left( \hat{\rho}^{(r-s)} u^{(s)} - \hat{\rho}^{(r)} u^{(0)} \right) \right],$$  \hspace{1cm} (89)
\[
\beta_{(3,r)} = \frac{k}{\hat{n}^{(0)} (kT^{(0)})^2} \left[ \sum_{s=0}^{r} \left( \hat{n}^{(r-s)} T^{(s)} - \hat{n}^{(r)} T^{(0)} \right) \right]
\]
\[
+ \frac{1}{3} \frac{1}{\hat{n}^{(0)} (kT^{(0)})^2} \left[ \sum_{s=0}^{r} \sum_{q=0}^{s} \hat{\rho}^{(r-s)} u^{(s-q)} \cdot u^{(q)} - \hat{\rho}^{(r)} \left( u^{(0)} \right)^2 \right]
\]
\[
- \frac{2}{3} \frac{1}{\hat{n}^{(0)} (kT^{(0)})^2} u^{(0)} \cdot \left[ \sum_{s=0}^{r} \left( \hat{\rho}^{(r-s)} u^{(s)} \right) - \hat{\rho}^{(r)} u^{(0)} \right].
\] (90)

Then the fulfillment of equalities (82) can be considered as the differential equations, the \(r\)-order equations of gas dynamics, for finding \(n_i^{(r-1)}\), \(u_i^{(r-1)}\), \(T_i^{(r-1)}\) (\(r = 1, 2 \ldots\)).

For \(r = 1\) from (88)-(90) we have
\[
\beta_{(1,1)}^{(1,1)} = \frac{n_i^{(1)}}{n_i^{(0)}} - 3 \frac{T_i^{(1)}}{2 T_i^{(0)}},
\] (91)
\[
\beta_{(2,1)}^{(2,1)} = \frac{u_i^{(1)}}{k T_i^{(0)}},
\] (92)
\[
\beta_{(3,1)}^{(3,1)} = \frac{1}{k T_i^{(0)}} T_i^{(1)}. \] (93)

The partial solution of the system of inhomogeneous equations (81) \(\{ \Phi_i^{(r)} \}_{i \in \hat{N}}\), satisfying (83), may be constructed, for example, using expansion of \(\Phi_i^{(r)} (c_i)\) in series in terms of Sonine polynomials with expansion coefficients, depending on \(r\) and \(t\) (see [2] or [10]); such construction proves existence of solutions of the system of integral equations (75) (one could simply use here and above Fredholm’s theorems [13], [14]).

For \(r \geq 1\) the velocity distribution functions of outer components of gas mixture \(f_i^{(r)}\) are found from the integral equations system (33), which, taking account of (22) and analogous (40) equality
\[
f_i^{(0)} f^{(0)'} = \hat{f}_i^{(0)} f^{(0)},
\] (94)
can be rewritten in the form
\[
D_i^{(r-1)} + \sum_{s=1}^{r-1} J_i \left( f_i^{(r-s)} f^{(s)} \right) + \sum_{j \neq i} \sum_{s=0}^{r-1} J_{ij} \left( f_i^{(r-1-s)} f_j^{(s)} \right)
= -J_i \left( f_i^{(0)} \chi^{(r)} f^{(0)} \right) - J_i \left( f_i^{(0)} f^{(0)} \chi^{(r)} \right)
= - \int \int f_i^{(0)} f^{(0)} \left( \chi_i^{(r)} + \chi^{(r)} - \chi_i^{(r)} - \chi^{(r)} \right) \hat{k}_i d\hat{k} d\hat{c} \quad (i \in \hat{N}),
\] (95)
in (95) functions \( f_i^{(r)} = f_i^{(0)} \chi_i^{(r)} \), where \( \chi_i^{(r)} \) are new unknown functions.

Equations (95) differ from equations (75) only in the left-hand sides, that are known from the previous step of the successive approximations method, and in the absence of summation over components in the right-hand sides of equations (95). Therefore similarly to, how it has been done above for inner components, we obtain expression for the general solution of the system of homogeneous integral equations, corresponding to (95),

\[
\phi_i^{(r)} = \alpha_i^{(1,r)} + \alpha_i^{(2,r)} \cdot m_i c_i + \alpha_i^{(3,r)} \frac{1}{2} m_i c_i^2,
\]

(96)

Hence,

\[
\xi_i^{(r)} = f_i^{(0)} \left( \alpha_i^{(1,r)} + \alpha_i^{(2,r)} \cdot m_i c_i + \alpha_i^{(3,r)} \frac{1}{2} m_i c_i^2 \right) \quad (i \in \tilde{N})
\]

(97)

or

\[
\xi_i^{(r)} = f_i^{(0)} \left[ \beta_i^{(1,r)} + \beta_i^{(2,r)} \cdot m_i \left( c_i - u_i^{(0)} \right) + \beta_i^{(3,r)} \frac{1}{2} m_i \left( c_i - u_i^{(0)} \right)^2 \right] \quad (i \in \tilde{N}).
\]

(98)

Necessary condition for the existence of solutions of the system of integral equations (95) can be written as

\[
\int \psi_i^{(l)} F_i^{(r)} d c_i = 0 \quad (i \in \tilde{N}, \ l = 1, 2, 3),
\]

(99)

where \( F_i^{(r)} \) denote left-hand sides of the equations (95), taken with opposite sign.

Among (infinitesimal) set of particular solutions of the system of equations (95) by the condition

\[
\int \psi_i^{(l)} f_i^{(0)} F_i^{(r)} d c_i = 0 \quad (i \in \tilde{N}, \ l = 1, 2, 3)
\]

(100)

unique solution \( \{ \Phi_i^{(r)} \}_{i \in \tilde{N}} \) may be chosen. Having substituted expression for \( f_i^{(r)} \) \((i \in \tilde{N})\)

\[
f_i^{(r)} = \Xi_i^{(r)} + \xi_i^{(r)}
\]

\[
= f_i^{(0)} \Phi_i^{(r)} + f_i^{(0)} \left[ \beta_i^{(1,r)} + \beta_i^{(2,r)} \cdot m_i \left( c_i - u_i^{(0)} \right) + \beta_i^{(3,r)} \frac{1}{2} m_i \left( c_i - u_i^{(0)} \right)^2 \right]
\]

(101)

in (70)-(72), taking (64), (73)-(74) and (100) into account, we obtain for each index \( \tilde{i} \) a system of 5 algebraic equations [constraint equations for asymptotic expansions (27) and (54), (68)-(69)]:

\[
n_i^{(0)} \beta_i^{(1,r)} + \frac{3}{2} n_i^{(0)} k T_i^{(0)} \beta_i^{(3,r)} = n_i^{(r)},
\]

(102)
\[ \mathbf{u}_i^{(0)} v_i n_i^{(0)} \beta_i^{(1r)} + \rho_i^{(0)} k T_i^{(0)} \beta_i^{(2r)} + \frac{3}{2} \rho_i^{(0)} k T_i^{(0)} \mathbf{u}_i^{(0)} \beta_i^{(3r)} = \sum_{s=0}^{r} \rho_i^{(r-s)} \mathbf{u}_i^{(s)}, \quad (103) \]

\[ \frac{1}{2} n_i^{(0)} \left[ 3k T_i^{(0)} + m_i \left( \mathbf{u}_i^{(0)} \right)^2 \right] \beta_i^{(1r)} \]
\[ + \rho_i^{(0)} k T_i^{(0)} \mathbf{u}_i^{(0)} \beta_i^{(2r)} \]
\[ + \frac{3}{4} k T_i^{(0)} \left[ 5n_i^{(0)} k T_i^{(0)} + \rho_i^{(0)} \left( \mathbf{u}_i^{(0)} \right)^2 \right] \beta_i^{(3r)} \]
\[ = \frac{3}{2} k \sum_{s=0}^{r} n_i^{(r-s)} T_i^{(s)} + \frac{1}{2} \sum_{s=0}^{r} \sum_{q=0}^{s} \rho_i^{(r-s)} \mathbf{u}_i^{(s-q)} \cdot \mathbf{u}_i^{(q)}, \quad (104) \]

from which we find expressions for functions \( \beta_i^{(1r)} (r, t) \), \( \beta_i^{(2r)} (r, t) \) and \( \beta_i^{(3r)} (r, t) \) through (variable) coefficients of asymptotic expansions of the particle number density of \( \tilde{i} \)-component, of the mean velocity and of the temperature of the \( \tilde{i} \)-component

\[ \beta_i^{(1r)} = \frac{n_i^{(r)}}{n_i^{(0)}} - \frac{3}{2} \frac{1}{n_i^{(0)} T_i^{(0)}} \left[ \sum_{s=0}^{r} \left( n_i^{(r-s)} T_i^{(s)} \right) - n_i^{(r)} T_i^{(0)} \right] \]
\[ - \frac{1}{2} \frac{1}{n_i^{(0)} k T_i^{(0)}} \left[ \sum_{s=0}^{r} \sum_{q=0}^{s} \rho_i^{(r-s)} \mathbf{u}_i^{(s-q)} \cdot \mathbf{u}_i^{(q)} - \rho_i^{(r)} \left( \mathbf{u}_i^{(0)} \right)^2 \right] \]
\[ + \frac{1}{n_i^{(0)} k T_i^{(0)}} \mathbf{u}_i^{(0)} \cdot \left[ \sum_{s=0}^{r} \left( \rho_i^{(r-s)} \mathbf{u}_i^{(s)} \right) - \rho_i^{(r)} \mathbf{u}_i^{(0)} \right], \quad (105) \]

\[ \beta_i^{(2r)} = \frac{1}{\rho_i^{(0)} k T_i^{(0)}} \left[ \sum_{s=0}^{r} \left( \rho_i^{(r-s)} \mathbf{u}_i^{(s)} \right) - \rho_i^{(r)} \mathbf{u}_i^{(0)} \right], \quad (106) \]

\[ \beta_i^{(3r)} = \frac{k}{n_i^{(0)} \left( k T_i^{(0)} \right)^2} \left[ \sum_{s=0}^{r} \left( n_i^{(r-s)} T_i^{(s)} \right) - n_i^{(r)} T_i^{(0)} \right] \]
\[ + \frac{1}{3} \frac{1}{n_i^{(0)} \left( k T_i^{(0)} \right)^2} \left[ \sum_{s=0}^{r} \sum_{q=0}^{s} \rho_i^{(r-s)} \mathbf{u}_i^{(s-q)} \cdot \mathbf{u}_i^{(q)} - \rho_i^{(r)} \left( \mathbf{u}_i^{(0)} \right)^2 \right] \]
\[ - \frac{2}{3} \frac{1}{n_i^{(0)} \left( k T_i^{(0)} \right)^2} \mathbf{u}_i^{(0)} \cdot \left[ \sum_{s=0}^{r} \left( \rho_i^{(r-s)} \mathbf{u}_i^{(s)} \right) - \rho_i^{(r)} \mathbf{u}_i^{(0)} \right]. \quad (107) \]

The fulfillment of equalities (39) can be considered as the differential equations, the \( r \)-order equations of gas dynamics, for finding \( n_i^{(r-1)} , \mathbf{u}_i^{(r-1)} , T_i^{(r-1)} \) \( (r = 1, 2, \ldots) \).
For \( r = 1 \) from (105)-(107) we have, in particular:

\[
\begin{align*}
\beta_i^{(1,1)} &= \frac{n_i^{(1)}}{n_i^{(0)}} - \frac{3 T_i^{(1)}}{2 T_i^{(0)}}, \\
\beta_i^{(2,1)} &= \frac{u_i^{(1)}}{k T_i^{(0)}}, \\
\beta_i^{(3,1)} &= \frac{1}{k T_i^{(0)}} \frac{T_i^{(1)}}{T_i^{(0)}},
\end{align*}
\]

(108)

(109)

(110)

cf. with (91)-(93).

The partial solution \( \{ \Phi_i^{(r)} \}_{i \in \hat{N}} \) of the system of inhomogeneous integral equations (95), satisfying (100), may be constructed similarly to, how satisfying (83) partial solution \( \{ \phi_i^{(r)} \}_{i \in \hat{N}} \) of the system of inhomogeneous integral equations (81) is constructed, see above.

IV. THE SYSTEM OF FIRST ORDER EQUATIONS OF MULTICOMPONENT NON-EQUILIBRIUM GAS DYNAMICS

Let us consider in more detail the system of infinitesimal first order equations (82), (99) \((r = 1)\), derived above as the necessary (and sufficient) condition of the solution existence of the first order integral equations system (75), (95) \((r = 1)\).

To simplify transformations, according to the expressions for velocity distribution functions of particles of infinitesimal zero order (46), (64), functions \( \Psi_i^{(l)} \), \( \Psi_i^{(l)} \) may be used in (82) and (99) \((r = 1)\) rather than functions \( \psi_i^{(l)} \), \( \psi_i^{(l)} \), respectively:

\[
\begin{align*}
\Psi_i^{(1)} &= m_i, \\
\Psi_i^{(2)} &= m_i C_i, \\
\Psi_i^{(3)} &= \frac{1}{2} m_i C_i^2,
\end{align*}
\]

(111a)

(111b)

(111c)

for inner components \( C_i = c_i - u_i^{(0)} \), for outer components \( C_i = c_i - u_i^{(0)} \).

At transformation of differential parts of the equations (82) and (99) we use equalities:

\[
\begin{align*}
\int \Psi_i^{(l)} \frac{\partial f_i^{(0)}}{\partial t} \, dc_i &= \frac{\partial}{\partial t} \int \Psi_i^{(l)} f_i^{(0)} \, dc_i - \int \frac{\partial \Psi_i^{(l)}}{\partial t} f_i^{(0)} \, dc_i \\
&= \frac{\partial}{\partial t} \left( n_i \Psi_i^{(l)} \right) = n_i \frac{\partial \Psi_i^{(l)}}{\partial t},
\end{align*}
\]

(112)
\[ \int \Psi_i^{(r)} c_i \cdot \frac{\partial f_i^{(0)}}{\partial r} \, dc_i = \frac{\partial}{\partial r} \cdot \int \Psi_i^{(r)} c_i f_i^{(0)} \, dc_i - \int c_i \cdot \frac{\partial \Psi_i^{(r)}}{\partial r} f_i^{(0)} \, dc_i \]
\[ = \frac{\partial}{\partial r} \cdot n_i \Psi_i^{(0)}\overline{c_i} - n_i c_i \cdot \frac{\partial \Psi_i^{(r)}}{\partial r}, \quad (113) \]
\[ \int \Psi_i^{(r)} \frac{X_i}{m_i} \cdot \frac{\partial f_i^{(0)}}{\partial c_i} \, dc_i = - \int \left( \frac{\partial}{\partial c_i} \cdot \frac{\Psi_i^{(r)} X_i}{m_i} \right) f_i^{(0)} \, dc_i \]
\[ = -n_i \frac{\partial}{\partial c_i} \cdot \frac{\Psi_i^{(r)} X_i}{m_i}. \quad (114) \]

In (112)-(114) the bar above symbol with index \((0)\) denotes the average of the value:
\[ \overline{V}^{(0)} = \frac{1}{n_i} \int V f_i^{(0)} \, dc_i; \quad (115) \]

\(r\) and \(c_i\) are considered as independent variables; At averaging in (114) it is assumed, that external force \(X_i\), acting on the particle of species \(i\), is independent of the particle velocity, it is assumed also, that integrals, depending on external forces \(X_i\), are convergent, and product \(\Psi_i^{(r)} X_i f_i^{(0)}\) tends to zero, when \(c_i\) tends to infinity.

After simple transformations from (82) and (99) \((r = 1)\) we obtain following system of infinitesimal first order equations of multicomponent non-equilibrium gas dynamics:
\[ \frac{\partial n_i^{(0)}}{\partial t} = - \frac{\partial}{\partial r} \cdot n_i^{(0)} u_i^{(0)} \quad (i \in \hat{N}), \quad (116) \]
\[ \dot{\rho}^{(0)} \frac{\partial u_i^{(0)}}{\partial t} + \frac{\partial}{\partial r} \cdot \dot{\rho}^{(0)} + \sum_{i \in \hat{N}, j \in \hat{N}} J_{p,ij}^{(0)} = \sum_{i \in \hat{N}} n_i^{(0)} X_i - \rho^{(0)} u_i^{(0)} \cdot \frac{\partial}{\partial r} u_i^{(0)}, \quad (117) \]
\[ \frac{\partial \dot{E}_i^{(0)}}{\partial t} + \frac{\partial}{\partial r} \cdot \dot{q}_i^{(0)} + \dot{p}_i^{(0)} : \frac{\partial u_i^{(0)}}{\partial r} + \sum_{i \in \hat{N}, j \in \hat{N}} J_{E,ij}^{(0)} = - \frac{\partial}{\partial r} \cdot \dot{E}_i^{(0)} u_i^{(0)}, \quad (118) \]
\[ \frac{\partial n_i^{(0)}}{\partial t} = - \frac{\partial}{\partial r} \cdot n_i^{(0)} u_i^{(0)} \quad (i \in \hat{N}), \quad (119) \]
\[ n_i^{(0)} m_i \frac{\partial u_i^{(0)}}{\partial t} + \frac{\partial}{\partial r} \cdot p_i^{(0)} + \sum_{j \neq i} J_{j,ij}^{(0)} = n_i^{(0)} X_i - n_i^{(0)} m_i u_i^{(0)} \cdot \frac{\partial}{\partial r} u_i^{(0)} \quad (i \in \hat{N}), \quad (120) \]
\[ \frac{\partial E_i^{(0)}}{\partial t} + \frac{\partial}{\partial r} \cdot q_i^{(0)} + p_i^{(0)} : \frac{\partial u_i^{(0)}}{\partial r} + \sum_{j \neq i} J_{E,ij}^{(0)} = - \frac{\partial}{\partial r} \cdot E_i^{(0)} u_i^{(0)} \quad (i \in \hat{N}). \quad (121) \]
In accordance with the general definition of pressure tensor of $i$-component of gas mixture

$$p_i \overset{\text{def}}{=} \int m_i (c_i - u_i) (c_i - u_i) f_i \, dc_i$$  \hspace{1cm} (122)$$

and with the general definition of $i$-component heat flux vector

$$q_i \overset{\text{def}}{=} \int \frac{1}{2} m_i (c_i - u_i)^2 (c_i - u_i) f_i \, dc_i$$  \hspace{1cm} (123)$$

(cf. with [2], Chapter 2, §§ 3, 4) in (116)-(121)

$$\hat{p}^{(0)} = \sum_{i \in \hat{N}} n_i^{(0)} m_i \left( c_i - \hat{u}_i \right) \left( c_i - \hat{u}_i \right)^{(0)} = \hat{n}^{(0)} kT^{(0)} U = \hat{p}^{(0)} U$$  \hspace{1cm} (124)$$

is inner components pressure tensor of zero order, $\hat{p}^{(0)}$ is inner components hydrostatic pressure of zero order, $U$ is the unit tensor, double product of two second rank tensors $w$ and $w'$ ([2], Chapter 1, § 3) is the scalar $w : w' = \sum_\alpha \sum_\beta w_\alpha \beta w'_\beta \alpha = w' : w$,

$$\hat{q}^{(0)} = \frac{1}{2} \sum_{i \in \hat{N}} n_i^{(0)} m_i \left( c_i - \hat{u}_i \right)^2 \left( c_i - \hat{u}_i \right)^{(0)} = 0$$  \hspace{1cm} (125)$$

is inner components heat flux vector of zero order,

$$\hat{E}^{(0)} = \frac{1}{2} \sum_{i \in \hat{N}} n_i^{(0)} m_i \left( c_i - \hat{u}_i \right)^2 = \frac{3}{2} \hat{n}^{(0)} kT^{(0)}$$  \hspace{1cm} (126)$$

is zero order internal energy of particles of inner components per unit volume, which is equal, in this case, to energy of their translational chaotic motion, however, the energy transfer equations, written in form (118) and (121) can be used in more general cases as well (cf. with [10], Chapter 7, § 6), in (124)-(126) averaging (115) is performed with Maxwell function $f_i^{(0)}$ from (46);

$$p_i^{(0)} = n_i^{(0)} m_i \left( c_i - u_i \right) \left( c_i - u_i \right)^{(0)} = n_i^{(0)} kT_i^{(0)} U = p_i^{(0)} U$$  \hspace{1cm} (127)$$

is $\check{i}$-component pressure tensor of zero order, $p_i^{(0)}$ is $\check{i}$-component hydrostatic pressure of zero order,

$$q_i^{(0)} = \frac{1}{2} n_i^{(0)} m_i \left( c_i - u_i \right)^2 \left( c_i - u_i \right)^{(0)} = 0$$  \hspace{1cm} (128)$$

is $\check{i}$-component heat flux vector of zero order,

$$E_i^{(0)} = \frac{1}{2} n_i^{(0)} m_i \left( c_i - u_i \right)^2 = \frac{3}{2} n_i^{(0)} kT_i^{(0)}$$  \hspace{1cm} (129)$$
is zero order internal energy of particles of $i$-component per unit volume, in (127)-(129) averaging (113) is performed with Maxwell function $f^{(0)}_i$ from (63).

Analytic expressions for integrals $J^{(0)}_{p,ij}$, $J^{(0)}_{E,ij}$ from (117), (118) and (120), (121) are given below – see (A24), (A28) and (B2), (B3).

In particular, if mean velocities and temperatures coincide for all component of mixture (the set of outer components is empty), then from (116)-(118) we obtain the system of infinitesimal first order gas-dynamics equations of the Enskog-Chapman theory [it may be noted in passing, that the sums over all indexes $i$ of collision integrals in (99) are equal to zero in all orders of the method of successive approximations, it is easy to check, using equality (36), from the physical point of view this statement is reduced to, that total impulse and (kinetic) energy of particles of gas do not change in their collisions between themselves]:

\[
\frac{\partial n_i^{(0)}}{\partial t} = - \frac{\partial}{\partial r} \cdot n_i^{(0)} u_i^{(0)} \quad \left( i \in \hat{N} \right),
\]

(130)

\[
\hat{\rho}^{(0)} \frac{\partial u_i^{(0)}}{\partial t} + \frac{\partial}{\partial r} \cdot \hat{p}^{(0)} = \sum_{i \in \hat{N}} n_i^{(0)} X_i - \hat{\rho}^{(0)} u_i^{(0)} \cdot \frac{\partial}{\partial r} u_i^{(0)},
\]

(131)

\[
\frac{\partial \hat{E}^{(0)}}{\partial t} + \frac{\partial}{\partial r} \cdot \hat{q}^{(0)} + \hat{p}^{(0)} : \frac{\partial u_i^{(0)}}{\partial r} = - \frac{\partial}{\partial r} \cdot \hat{E}^{(0)} u_i^{(0)}.
\]

(132)

It follows from (130), (132), taking (124)-(126) into account, that the flow of gas, described by the system of first order gas-dynamics equations of the Enskog-Chapman theory, is adiabatic:

\[
\frac{D}{Dt} \left[ \hat{n}^{(0)} \left( T^{(0)} \right)^{-3/2} \right] = 0,
\]

(133)

in the adiabatic equation (133)

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i^{(0)} \cdot \frac{\partial}{\partial r},
\]

(134)

V. VELOCITY DISTRIBUTION FUNCTIONS OF INFINITESIMAL FIRST ORDER

Using the system of infinitesimal first order gas-dynamics equations it is possible to specify the form of velocity distribution functions of the first order. In particular, we shall see,
that the velocity distribution functions, obtained by the proposed method within Enskog’s approach, i.e. when outer components are absent, and by the Enskog’s method are equivalent up to the infinitesimal first order terms of the asymptotic expansion (inclusive).

As

$$\ln f_i^{(0)} = \text{const.} + \ln n_i^{(0)} - \frac{3}{2} \ln T^{(0)} - \frac{m_i (c_i - u_i^{(0)})^2}{2kT_i^{(0)}},$$

(135)

see (46) and (62), the system of integral equations (75) \((r = 1)\) from which \(\{\chi_i^{(1)}\}_{i \in \hat{N}}\) are found, excluding time derivatives by equations (116)-(118), taking (124)-(126) into account, can be written in the form:

$$\frac{\partial f_i^{(0)}}{\partial t} + c_i \cdot \frac{\partial f_i^{(0)}}{\partial r} + \frac{X_i}{m_i} \cdot \frac{\partial f_i^{(0)}}{\partial c_i} + \sum_{j \in \hat{N}} J_{ij} \left( f_i^{(0)}, f_j^{(0)} \right)$$

$$= f_i^{(0)} \left( \frac{\partial \ln f_i^{(0)}}{\partial t} + c_i \cdot \frac{\partial \ln f_i^{(0)}}{\partial r} + \frac{X_i}{m_i} \cdot \frac{\partial \ln f_i^{(0)}}{\partial c_i} \right) + \sum_{j \in \hat{N}} J_{ij} \left( f_i^{(0)}, f_j^{(0)} \right)$$

$$= f_i^{(0)} \left[ \left( c_i^2 - \frac{5}{2} \right) C_i \cdot \frac{\partial T_i^{(0)}}{\partial r} + 2 \left( C_i \cdot d_i \right) : \frac{\partial u_i^{(0)}}{\partial r} + \tilde{\rho}_i^{(0)} C_i \cdot d_i + s_i \right]$$

$$= -\sum_{j \in \hat{N}} \int \int_{\hat{N}} f_i^{(0)} f_j^{(0)} \left( \chi_j^{(1)} + \chi_j^{(1)} - \chi_i^{(1)} - \chi_j^{(1)} \right) k_{ij} dk d\ell \quad (i \in \hat{N}).$$

(136)

In (136), as well as in the previous section, \(C_i = c_i - u_i^{(0)}\),

$$C_i = \left( \frac{m_i}{2kT_i^{(0)}} \right)^{1/2} C_i,$$

(137)

\(C_i\) is the modulus of vector \(C_i\); for arbitrary second rank tensor \(w\)

$$\tilde{w} = w - \frac{1}{3} U (U : w)$$

(138)

is tensor with zero trace;

$$d_i = \frac{\partial}{\partial r} \left( \frac{n_i^{(0)}}{\tilde{n}^{(0)}} \right) + \left( \frac{n_i^{(0)}}{\tilde{n}^{(0)}} - \frac{n_i^{(0)} m_i}{\tilde{\rho}^{(0)}} \right) \frac{\partial \ln \tilde{\rho}^{(0)}}{\partial r} - \left( \frac{n_i^{(0)} m_i}{\tilde{\rho}^{(0)}} \right) \left[ \frac{\tilde{\rho}^{(0)}}{m_i} \chi_i^{(1)} - \sum_{j \in \hat{N}} n_j^{(0)} X_j \right];$$

(139)

$$s_i = -\frac{\tilde{\rho}_i^{(0)} m_i}{\tilde{\rho}^{(0)}} C_i \cdot \sum_{i \in \hat{N}, j \in \hat{N}} J_{ij}^{(0)} \left( \frac{2}{3} C_i^2 - 1 \right) \sum_{i \in \hat{N}, j \in \hat{N}} J_{ij}^{(0)} \left( f_i^{(0)}, f_j^{(0)} \right) \right] + \frac{1}{f_i^{(0)}} \sum_{j \in \hat{N}} J_{ij}^{(0)} \left( f_i^{(0)}, f_j^{(0)} \right).$$

(140)
The general solution of the system of homogeneous integral equations, corresponding to the system of equations (136), was constructed above in the section III. A partial solution \{\Phi_{i}^{(1)}(c_{i})\}_{i \in \hat{N}} of the system of linear integral equations (136) can be sought in the form:

$$\Phi_{i}^{(1)} = -A_{i} \cdot \frac{\partial \ln T^{(0)}}{\partial r} - B_{i} : \frac{\partial u^{(0)}}{\partial r} - \hat{n}^{(0)} \sum_{j \in \hat{N}} D_{i}^{j} \cdot d_{j} - S_{i}. \quad (141)$$

Substituting (141) in (136), we find, that family of vector functions \{A_{i}(c_{i})\}_{i \in \hat{N}} must satisfy the system of integral equations

$$f_{i}^{(0)} \left( C_{i}^{2} - \frac{5}{2} \right) C_{i} = \sum_{j \in \hat{N}} \int \int f_{i}^{(0)} f_{j}^{(0)} \left( A_{i} + A_{j} - A_{i}' - A_{j}' \right) k_{ij} d\mathbf{k} d\mathbf{c}_{j} \quad (i \in \hat{N}), \quad (142)$$

family of tensor functions \{B_{i}(c_{i})\}_{i \in \hat{N}} must satisfy the system of integral equations

$$2 \int \int f_{i}^{(0)} (C_{i} \circ \mathbf{c}_{i}) = \sum_{j \in \hat{N}} \int \int f_{i}^{(0)} f_{j}^{(0)} \left( B_{i} + B_{j} - B_{i}' - B_{j}' \right) k_{ij} d\mathbf{k} d\mathbf{c}_{j} \quad (i \in \hat{N}). \quad (143)$$

Double family of vector functions \{D_{i}^{j}(c_{i})\}_{i, j \in \hat{N}} must satisfy the system of integral equations

$$\frac{1}{n_{i}^{(0)}} f_{i}^{(0)} C_{i} \cdot d_{i} = \sum_{j, h \in \hat{N}} \int \int f_{i}^{(0)} f_{j}^{(0)} d_{h} \cdot \left( D_{i}^{h} + D_{j}^{h} - D_{i}^{h} - D_{j}^{h} \right) k_{ij} d\mathbf{k} d\mathbf{c}_{j} \quad (i \in \hat{N}). \quad (144)$$

It follows from the definition \(d_{i}\), see (139), that

$$\sum_{i \in \hat{N}} d_{i} = 0, \quad (145)$$

therefore for every \(\hat{i}\) one of coefficients \(D_{\hat{i}}^{\hat{j}}\) may be considered as equal to zero; for symmetry of expressions it is usually assumed, that \(D_{\hat{i}}^{\hat{j}} = 0\). Using (145), the system of equations (144) may be rewritten in the form:

$$\frac{1}{n_{i}^{(0)}} f_{i}^{(0)} C_{i} \left( \delta_{\hat{i} \hat{h}} - \delta_{\hat{i} \hat{k}} \right) = \sum_{j \in \hat{N}} \int \int f_{i}^{(0)} f_{j}^{(0)} \left( D_{\hat{i}}^{\hat{h}} + \hat{D}_{i}^{\hat{h}} - D_{\hat{i}}^{\hat{h}} - \hat{D}_{i}^{\hat{h}} + \hat{D}_{\hat{i}}^{\hat{h}} + \hat{D}_{i}^{\hat{h}} \right) k_{ij} d\mathbf{k} d\mathbf{c}_{j} \quad (\hat{i}, \hat{h}, \hat{k} \in \hat{N}). \quad (146)$$

cf. with [10], Chapter 7, § 3, (3.32). In (146) \(\delta_{ik}\) is Kronecker’s delta symbol:

$$\delta_{ik} = \begin{cases} 0 & (i \neq k), \\ 1 & (i = k). \end{cases} \quad (147)$$
Family of scalar functions \( \{S_i(c_i)\}_{i \in \hat{N}} \) must satisfy the system of integral equations

\[
f^{(0)}_i s_i = \sum_{j \in \hat{N}} \iint f^{(0)}_i f^{(0)}_j \left( S_i + S_j - S'_{i} - S'_{j} \right) k_{ij} dk dc_j \quad \left( i \in \hat{N} \right).
\]

Solving of the system of integral equations (148) (this question will be considered, possibly, in our next article) becomes a little simpler, if functions \( S_i \) are written in the form:

\[
S_i = S_i^0 \cdot \sum_{i \in \hat{N}, j \in \hat{N}} J^{(0)}_{p,i,j} + S_i^s. \tag{149}
\]

Conditions of solubility of the system of equations (142), (143), (144), (148), following from (82) [in (82) \( F^{(r)}_i \) must be replaced, respectively, by left-hand sides of equations (142), (143), (144) and (148)], are fulfilled, because conditions (82) in the form (116)-(118) were used in the derivation of the integral equations system (136).

Because \( \mathbf{r} \) and \( t \) do not enter explicitly in equations (142), (143) and (144), \( u^{(0)} \) enters only in combination \( (c_i - u^{(0)}) = C_i \) (after replacement of integration over \( c_j \) by integration over \( C_j \)) and the left-hand sides of equations (143) are symmetric tensors with zero trace, functions \( \{A_i(c_i)\}_{i \in \hat{N}}, \{B_i(c_i)\}_{i \in \hat{N}} \) and \( \{D^j_i(c_i)\}_{i \in \hat{N}} \) can be sought in the form:

\[
A_i = A_i \left( n_i^{(0)}, c_i, T^{(0)} \right) C_i, \tag{150}
\]

\[
B_i = B_i \left( n_i^{(0)}, c_i, T^{(0)} \right) (C_i C_i), \tag{151}
\]

\[
D^j_i = D^j_i \left( n_i^{(0)}, c_i, T^{(0)} \right) C_i, \tag{152}
\]

where \( A_i \left( n_i^{(0)}, c_i, T^{(0)} \right), B_i \left( n_i^{(0)}, c_i, T^{(0)} \right) \) and \( D^j_i \left( n_i^{(0)}, c_i, T^{(0)} \right) \) are scalar functions of \( n_i^{(0)}, c_i \) and \( T^{(0)} \).

Imposing on the functions \( \{\Phi^{(1)}_i\}_{i \in \hat{N}} \) the condition (83), from (84) and (91)-(93) we obtain the following expression for the infinitesimal first order velocity distribution functions of inner components particles of gas mixture \( f^{(1)}_i \left( i \in \hat{N} \right): \)

\[
f^{(1)}_i = f^{(0)}_i \left( -A_i \frac{\partial \ln T^{(0)}}{\partial r} - B_i \frac{\partial}{\partial r} u^{(0)} - n_i^{(0)} \sum_{j \in \hat{N}} D^j_i \cdot d_j - S_i \right) + f^{(0)}_i \left( \frac{3 T^{(1)}}{2 T^{(0)}} + \frac{u^{(1)}}{kT^{(0)}} \cdot m_i C_i + \frac{1}{kT_i^{(0)}} \cdot \frac{T^{(1)}}{T_i^{(0)}} \cdot \frac{1}{2} m_i^2 C_i^2 \right). \tag{153}
\]

Similarly, the system of integral equations (95) \( (r = 1) \), from which \( \{\chi_i^{(1)}\}_{i \in \hat{N}} \) are found, excluding time derivatives by equations (119)-(121) taking (127)-(129) into account, can be
written in the form:

\[
\frac{\partial f^{(0)}_i}{\partial t} + c_i \cdot \frac{\partial f^{(0)}_i}{\partial r} + \frac{X_i}{m_i} \cdot \frac{\partial f^{(0)}_i}{\partial c_i} + \sum_{j \neq i} J_{ij} \left( f^{(0)}_i, f^{(0)}_j \right) = f^{(0)}_i \left( \frac{\partial \ln f^{(0)}_i}{\partial t} + c_i \cdot \frac{\partial \ln f^{(0)}_i}{\partial r} + \frac{X_i}{m_i} \cdot \frac{\partial \ln f^{(0)}_i}{\partial c_i} \right) + \sum_{j \neq i} J_{ij} \left( f^{(0)}_i, f^{(0)}_j \right)
\]

\[
= f^{(0)}_i \left[ \left( C^2 - \frac{5}{2} \right) C_i \cdot \frac{\partial \ln T^{(0)}_i}{\partial r} + 2 \left( C \circ C_i : \frac{\partial}{\partial r} u^{(0)}_i + s_i \right) \right]
\]

\[
= - \int \int f^{(0)}_i f^{(0)}_j \left( \chi^{(1)}_i + \chi^{(1)}_j - \chi^{(1)'}_i + \chi^{(1)'}_j \right) k_i \, dk \, dc \quad (i \in \mathbb{N}), \quad (154)
\]

where \( C_i = c_i - u^{(0)}_i \),

\[
C_i = \left( \frac{m_i}{2kT^{(0)}_i} \right)^{1/2} C_i,
\]

\( C_i \) is the modulus of vector \( C_i \);

\[
s_i = -\frac{1}{p^{(0)}_i} C_i \cdot \sum_{j \neq i} J^{(0)}_{p,ij} - \frac{1}{p^{(0)}_i} \left( \frac{2}{3} C^2_i - 1 \right) \sum_{j \neq i} J^{(0)}_{E,ij}
\]

\[
+ \frac{1}{f^{(0)}_i} \sum_{j \neq i} J_{ij} \left( f^{(0)}_i, f^{(0)}_j \right).
\]

A partial solution \( \{ \Phi^{(1)}_i \}_{i \in \mathbb{N}} \) of the system of equations \( (154) \) can be sought in the form:

\[
\Phi^{(1)}_i = -A_i \cdot \frac{\partial \ln T^{(0)}_i}{\partial r} - B_i : \frac{\partial}{\partial r} u^{(0)}_i - S_i.
\]

Substituting \( (157) \) in the system of linear integral equations \( (154) \), we find, that family of vector functions \( \{ A_i (C_i) \}_{i \in \mathbb{N}} \) must satisfy the system of integral equations

\[
f^{(0)}_i \left( C^2 - \frac{5}{2} \right) C_i = \int \int f^{(0)}_i f^{(0)}_j \left( A_i + A - A_i' - A' \right) k_i \, dk \, dc \quad (i \in \mathbb{N}), \quad (158)
\]

family of tensor functions \( \{ B_i (C_i) \}_{i \in \mathbb{N}} \) must satisfy the system of integral equations

\[
2f^{(0)}_i (\circ C_i C_i) = \int \int f^{(0)}_i f^{(0)}_j \left( B_i + B - B_i' - B' \right) k_i \, dk \, dc \quad (i \in \mathbb{N}); \quad (159)
\]

family of scalar functions \( \{ S_i (C_i) \}_{i \in \mathbb{N}} \) must satisfy the system of integral equations

\[
f^{(0)}_i s_i = \int \int f^{(0)}_i f^{(0)}_j \left( S_i + S - S_i' - S' \right) k_i \, dk \, dc \quad (i \in \mathbb{N}), \quad (160)
\]

30
solving of the system of integral equations \((160)\) becomes a little simpler, if functions \(S_i\) are written in the form:

\[
S_i = S_i^v \cdot \sum_{j \neq i} J_{p, ij}^{(0)} + S_i^s. \tag{161}
\]

Conditions of solubility of the systems of equations \((158), (159), (160)\), following from \((99)\) [in \((99)\) \(F_i^{(r)}\) must be replaced, by left-hand sides of equations \((158), (159)\) and \((160)\), are fulfilled, because conditions \((99)\) in the form \((119)-(121)\) were used in the derivation of the integral equations system \((154)\).

As \(r\) and \(t\) do not enter explicitly in equations \((158)\) and \((159)\), \(u_i^{(0)}\) enters only in combination \((c_i - u_i^{(0)}) = C_i^0\) (after replacement of integration over \(c\) by integration over \(C\)) and the left-hand sides of equations \((159)\) are symmetric tensors with zero trace, functions \(\{A_i(c_i)\}_{i \in \bar{N}}\) and \(\{B_i(c_i)\}_{i \in \bar{N}}\) can be sought in the form:

\[
A_i = A_i\left(n_i^{(0)}, C_i, T_i^{(0)}\right) C_i, \tag{162}
\]

\[
B_i = B_i\left(n_i^{(0)}, C_i, T_i^{(0)}\right) (C_i^0 C_i), \tag{163}
\]

where \(A_i\left(n_i^{(0)}, C_i, T_i^{(0)}\right)\) and \(B_i\left(n_i^{(0)}, C_i, T_i^{(0)}\right)\) are scalar functions of \(n_i^{(0)}, C_i\) and \(T_i^{(0)}\).

Imposing on the functions \(\{\Phi_i^{(1)}\}_{i \in \bar{N}}\) the condition \((100)\), from \((101)\) and \((108)-(110)\) we obtain the expression for the infinitesimal first order velocity distribution functions of outer components particles of gas mixture \(f_i^{(1)}\) \((i \in \bar{N})\):

\[
f_i^{(1)} = f_i^{(0)} - A_i \cdot \frac{\partial \ln T_i^{(0)}}{\partial r} - B_i \cdot \frac{\partial}{\partial r} u_i^{(0)} - S_i + f_i^{(0)}\left(n_i^{(1)} - \frac{3}{2} T_i^{(1)} + \frac{u_i^{(1)}}{k T_i^{(0)}} \cdot m_i C_i + \frac{1}{k T_i^{(0)}} \frac{T_i^{(1)}}{T_i^{(0)}} \frac{1}{2} m_i C_i^2\right). \tag{164}
\]

Let us consider a case when outer components in a mixture are absent. Up to the infinitesimal first order terms (inclusive – see \([6]\), Chapter V, § 2, Section 2, definition 2) the expression, see \((153)\),

\[
f_i^{(0)} + f_i^{(0)}\left(n_i^{(1)} - \frac{3}{2} T_i^{(1)} + \frac{u_i^{(1)}}{k T_i^{(0)}} \cdot m_i C_i + \frac{1}{k T_i^{(0)}} \frac{T_i^{(1)}}{T_i^{(0)}} \frac{1}{2} m_i C_i^2\right) \tag{165}
\]

coincides with asymptotic expansion of the solution \(\tilde{f}_i^{(0)}\) in the Enskog-Chapman theory

\[
\tilde{f}_i^{(0)} = n_i^{[1]} \left(\frac{m_i}{2 \pi k T_i^{[1]}\cdot}\right)^{3/2} e^{-\frac{m_i (c_i - u_i^{[1]} - 1\cdot)^2}{2 k T_i^{[1]}\cdot}}, \tag{166}
\]
where \( n_i^{[1]} = n_i^{(0)} + n_i^{(1)} \), \( u_i^{[1]} = u_i^{(0)} + u_i^{(1)} \) and \( T^{[1]} = T^{(0)} + T^{(1)} \), cf. with the Taylor expansion of \( \tilde{f}_i^{(0)} \) at the point \( (n_i^{(0)}, u_i^{(0)}, T^{(0)}) \). This statement can be written in the form:

\[
\tilde{f}_i^{(0)} \sim f_i^{(0)} + f_i^{(1)} = f_i^{(0)} \left( \frac{n_i^{(1)}}{n_i^{(0)}} - \frac{3 T_i^{(1)}}{2 T_i^{(0)}} + \frac{u_i^{(1)}}{k T_i^{(0)}} \cdot m_i C_i + \frac{1}{k T_i^{(0)}} T_i^{(1)} \right).
\] (167)

Equations (142), (143) and (146) differ from analogous equations [10], Chapter 7, (3.34), (3.33) and (3.32) in the Enskog-Chapman theory essentially only in use of \( n_i^{(0)}, u_i^{(0)} \) and \( T^{(0)} \) instead of \( n_i, u \) and \( T \) (i.e. \( n_i^{[1]}, u_i^{[1]} \) and \( T^{[1]} \)). Because functions \( \{ A_i(c_i) \}_{i \in \hat{N}}, \{ B_i(c_i) \}_{i \in \hat{N}} \) and \( \{ D_i^j(c_i) \}_{i, j \in \hat{N}} \) are of the infinitesimal first order, in the first, second and third terms in the right-hand side of (141) and also in (150)-(152) functions \( n_i^{(0)}, u_i^{(0)} \) and \( T_i^{(0)} \) can be, respectively, replaced by functions \( n_i^{[1]}, u_i^{[1]} \) and \( T_i^{[1]} \). Therefore up to first infinitesimal order terms the expression, cf. with (153),

\[
\tilde{f}_i^{(0)} \left( -A_i \cdot \frac{\partial \ln T_i^{(0)}}{\partial r} - B_i : \frac{\partial u_i^{(0)}}{\partial r} - \hat{n}^{(0)} \sum_{j \in \hat{N}} D_i^j \cdot d_j \right).
\] (168)

coincides with the expression for the function \( \tilde{f}_i^{(1)} \) in the Enskog-Chapman theory

\[
\tilde{f}_i^{(1)} = f_i^{(0)} \left( -\tilde{A}_i \cdot \frac{\partial \ln T_i^{(0)}}{\partial r} - \tilde{B}_i : \frac{\partial u_i^{(0)}}{\partial r} - \hat{n}^{(0)} \sum_{j \in \hat{N}} \tilde{D}_i^j \cdot d_j \right).
\] (169)

Consequently, up to first infinitesimal order terms the functions \( f_i^{[1]} = f_i^{(0)} + f_i^{(1)} \) coincide with the functions \( \tilde{f}_i^{[1]} = \tilde{f}_i^{(0)} + \tilde{f}_i^{(1)} \), obtained in the Enskog-Chapman theory:

\[
f_i^{[1]} \sim \tilde{f}_i^{[1]}.
\] (170)

As a result, with the same precision (cf. with [2], Chapter 8, § 4 and [10], Chapter 7, § 4) the expressions for the diffusion velocities of components of gas mixture, for the \( \hat{i} \)-component heat flux vector, \( \ldots \) coincide, in particular, the expressions for \( \hat{i} \)-component pressure tensor
coincide:

\[
p_i^{[1]} = \int m_i (c_i - u^{[1]}_i) (c_i - u^{[1]}_i) f_i^{[1]} dc_i
\]

\[
\overset{1}{\sim} \left( n_i^{[0]} kT^{[0]} + n_i^{[1]} kT^{[0]} + n_i^{[0]} kT^{[1]} \right) U
\]

\[
- \int m_i (c_i - u^{[0]}_i) (c_i - u^{[0]}_i) f_i^{[0]} B_i \left[ \left( C_i \right) : \frac{\partial}{\partial r} u^{[0]}_i \right] dc_i
\]

\[
\overset{1}{\sim} n_i^{[1]} kT^{[1]} U - 2 \left[ \frac{1}{15} \frac{m_i^2}{2kT^{[1]}} \int (c_i - u^{[1]}_i)^4 f_i^{[0]} B_i \right] dc_i
\]

\[
= n_i^{[1]} kT^{[1]} U - 2\tilde{\mu}_i \frac{\partial}{\partial r} u^{[1]}_i = \tilde{p}_i^{[1]}, \quad (171)
\]

– cf. with incorrect opposite assertions (about Hilbert’s method for the kinetic Boltzmann equation asymptotic solution, which, in principle, should lead to the same results, as the proposed in this article method, see section II), for example, in [15], [8], [9]; in (171) the notations are used: \( \mu_i \) is viscosity coefficient for \( \hat{i} \)-component, for arbitrary second rank tensor \( w \)

\[
(\overline{w})_{\alpha\beta} = \frac{1}{2} (w_{\alpha\beta} + w_{\beta\alpha}) \quad (172)
\]

is symmetric tensor, corresponding to it.

VI. TURBULENCE AS MULTICOMPONENT GAS DYNAMICS

As is well known, laminar flow becomes turbulent flow, when some parameter characterizing the flow, namely, Reynolds number

\[
Re = \frac{\rho uL}{\mu} > 1. \quad (173)
\]

In (173), \( \rho \) is the density of gas, \( u \) and \( L \) are some characteristic macroscopic velocity and linear size of the flow, \( \mu \) is the coefficient of viscosity. Having rewritten (173) as

\[
Re = \frac{\rho u^2}{\mu \frac{u}{L}}, \quad (174)
\]

cf. with the expression for viscosity tensor in (171), the Reynolds number can be treated as the ratio of the macroscopic momentum flux, proportional to \( f^{[0]} \), to the viscosity-induced microscopic momentum flux, proportional to \( \tilde{f}^{[1]} \) (from the point of view of the kinetic
theory of gases instead of Reynolds number it would be more natural to consider the ratio $Re = f^{(0)}/f^{(1)}$. Roughly speaking, viscosity, "aligning" the gas molecules according to a Maxwellian distribution with the same mean velocities and temperatures for different components of mixture, can "process" only the microscopic momentum flux. But if the macroscopic flux exceeds the microscopic, the gas flow, necessarily, has begun to stratify on components [and other mechanism, leading to an equilibrium state of gas, "turns on" – see integral terms in (116)-(121), however, from the point of view of the kinetic theory of gases it is the same mechanism, collisions of gas particles with each other]. The flow stratification on components can be also caused by external factors.

The gas-dynamic equations system of the second approximation order (i.e. with diffusion, viscosity and heat conduction) of the Enskog-Chapman theory, in principle, can not describe gas flow with developed turbulence, as

$$Re \sim \frac{f^{(0)}}{\tilde{f}^{(1)}} \to \infty,$$

when $\tilde{f}^{(1)} \to 0$; with increasing Reynolds number, according to this system of equations, the entropy production decreases until the complete cessation of growth of the entropy in the gas, see the adiabatic equation (133) (terms of the gas-dynamic equations system of the second approximation order of the Enskog-Chapman theory, additional to the gas-dynamic equations system of the first approximation order of the Enskog-Chapman theory, corresponding to the transition of gas to an equilibrium state and, hence, to the increasing of its entropy, are proportional to $\tilde{f}^{(1)}$), whereas the entropy of gas grows with growth of Reynolds number in experiments, i.e. theoretical (Enskog-Chapman) and experimental dependencies of entropy of gas on a Reynolds number are different.

Mixing is absent in the gas-dynamic equations system of the first approximation order of the Enskog-Chapman theory (130)-(132), because all components of gas mixture, according to the mass transfer equations (130), move with the same mean mass velocity of mixture. The substance from some physical region (even with moving boundary) cannot cross region boundary in any way. Mixing in the Enskog-Chapman system of gas-dynamic equations, i.e. a possibility to gas particles to cross boundary of physical region, as well as other mechanisms (viscosity and heat conduction), leading gas to an equilibrium, and, hence, increasing entropy of gas, appears only in the following infinitesimal order of asymptotic expansion of velocity distribution functions of gas particles and is related to diffusion velocities ($\sim \tilde{f}^{(1)}_i$) in mass
transfer equations. The ratio of diffusion velocity of some component of gas mixture to mean velocity of this component \( \sim 1/Re \sim \frac{\dot{f}_i^{(1)}}{\dot{f}_i^{(0)}} \) tends to zero, when \( \dot{f}_i^{(1)} \to 0 \). Therefore it is also not possible within Enskog-Chapman theory to describe observed intensive turbulent mixing in gas-dynamic flows with great Reynolds numbers.

If gas-dynamic equations do not describe turbulent gas flows, then either something has been missed during the transition from the exact solution of the system of kinetic Boltzmann equations to its approximate solution (by the Enskog method) and next to the gas-dynamic equations, or the system of kinetic Boltzmann equations does not describe turbulent gas flows and requires replacement. However the last, i.e. necessity of replacement of the system of kinetic Boltzmann equations on another system of kinetic equations in transition from gas laminar flow to turbulent gas flow, seems ill-founded.

The gas dynamics of the components with the velocity distribution functions, close to the Maxwell functions, with different mean velocities and temperatures, should be described by equations (116)-(121). From this point of view, the observed chaotic character of the turbulent flow is similar to the chaotic character of the Brownian motion. They differ in scale: in the Brownian motion that particle moves stochastically, whose mass is comparable to the mass of other gas molecules, whereas in the turbulent flow that body moves stochastically, whose mass is comparable to the mass of separate gas components. In (117)-(118), (120)-(121) the integral terms (proportional to \( n_i, n_j \)) can be huge, it explains unexpected (for those, who tries to describe turbulent flow by the gas-dynamic equations system of the second approximation order of the Enskog-Chapman theory) power of turbulent effects.

**Appendix A: Calculation of collision integrals**

In this section we are dealing with calculation of definite multidimensional integrals

\[
\int \int \int \Psi^{(l)}_i \left( f_i^{(0)} f_j^{(0)} - f_i^{(0)} f_j^{(0)} \right) g_{ij} b db d\epsilon d\mathbf{c}_i d\mathbf{c}_j. \tag{A1}
\]

In (A1) \( \Psi^{(1)}_i = m_i, \Psi^{(2)}_i = m_i C_i, \Psi^{(3)}_i = \frac{1}{2} m_i C_i^2, C_i = c_i - u_i; \)

\[
f_i^{(0)} = n_i \left( \frac{m_i}{2\pi k T_i} \right)^{3/2} e^{-\frac{m_i (c_i - u_i)^2}{2k T_i}} \tag{A2}
\]

is the Maxwell velocity distribution function of \( i \)-component particles, the prime in the distribution function implies, that the distribution of particles velocities \( c'_i \) after the collision
is considered; to diminish a little inconvenience of notation, the upper index "(0)" at \( n_i, u_i, T_i \) is omitted. The other notation is specified above.

According to (36), the integral (A1) can be transformed as follows:

\[
\int \int \int \int \Psi_i (l) f_i (0\prime) f_j (0\prime) g_{ij} b db d\epsilon d\mathbf{c}_i d\mathbf{c}_j = \int \int \int \int \Psi_i (l) f_i (0\prime) f_j (0\prime) g'_{ij} b db d\epsilon d\mathbf{c}_i d\mathbf{c}_j
\]

\[
- \int \int \int \int \Psi_i (l) f_i (0) f_j (0) g_{ij} b db d\epsilon d\mathbf{c}_i d\mathbf{c}_j = \int \int \int \int \left( \Psi_i (l) - \Psi_i (l) \right) f_i (0) f_j (0) g_{ij} b db d\epsilon d\mathbf{c}_i d\mathbf{c}_j.
\] (A3)

As the particle mass is conserved in the collision, integral (A3) vanishes for \( \Psi_i (1) = m_i \). In two other cases, generally speaking, this is not true.

Statements of the two following simple propositions are used below several times. The proposition 2 is taken from [6], Chapter II, §1, Section 5. Regulated functions (see [6], Chapter II, §1, Section 3) in formulations of propositions can be replaced with more known continuous functions (any continuous function on \( R \) is regulated). As complete normed vector spaces over the field of real numbers \( R \) further in the article vector spaces \( R \) or \( R^3 \) over \( R \) are considered, with the usual modulus of a real number or a three-dimensional vector as norm.

**Proposition 1.** \( f \) is assumed to be a regulated function on \( R \) with values in \( R \), \( w \in R^3 \) be a fixed nonzero vector, \( n \in R^3 \) be a unit vector. In this case

\[
\int_{\Omega_n} f (w \cdot n) n d\Omega_n = \frac{2\pi w}{w} \int_0^{\pi} f (w \cos (\theta)) \cos (\theta) \sin (\theta) d\theta.
\] (A4)

In the left-hand side of (A4) the integral is taken over all directions of vector \( n \), \( w \cdot n \) is the scalar product of vectors \( w \) and \( n \).

**Remark.** If \( w \) is the zero vector, then the right-hand side of (A4) may be set equal to 0.

**Proof.** Select the system of spherical coordinates, such that the polar axis direction be the same as the direction of the vector \( w \). Resolve the vector \( n \) into two components: parallel \( n_\parallel \) and perpendicular \( n_\perp \) to the vector \( w \),

\[
n = n_\parallel + n_\perp = \frac{(w \cdot n) w}{w^2} + n_\perp.
\] (A5)
Having substituted expression \((A3)\) for the vector \(n\) into the left-hand side of \((A4)\) and integrating over the azimuth angle, we obtain the required equality, as in the integration over the azimuth angle the term, containing \(n_\perp\), vanishes. 

\[\]

**Proposition 2.** \(E\) and \(F\) is assumed to be two complete normed spaces over field \(R\), \(u\) be a continuous linear map of \(E\) into \(F\). In this case, if \(f\) is a regulated function on interval \(I \subset R\) with values in \(E\), then \(u \circ f\) is the regulated function on \(I\) with values in \(F\) and

\[
\int_a^b u(f(t)) \, dt = u \left( \int_a^b f(t) \, dt \right). 
\]

(A6)

**Proof.** Equality \((A6)\) follows immediately from the expression for the derivative of composite function \(u \circ f\); details of the proof can be found in [6], Chapter II, § 1, Section 5.

Major difficulties of evaluation of integral \((A3)\) are caused by that parameters of Maxwell functions for the \(i\)-th and the \(j\)-th components are not equal:

\[
u_i \neq u_j, \quad T_i \neq T_j.
\]

(A7)

As result, it is not easy get rid of scalar products of vectors in the exponent (it is desirable to have the expression for the exponent as simple as possible).

As the scattering angle depends on the modulus of relative velocity of colliding particles \{see, for example, [2], Chapter 3, § 4, Section 2 or [10], Chapter 1, (5.26)\}, it is natural to proceed in \((A3)\) to new variables – center-of-mass velocity \(G_{ij}\) and relative velocity of colliding particles \(g_{ij}\), which are related to particles velocities \(c_i\) and \(c_j\) by

\[
c_i = G_{ij} + \frac{m_j}{m_i + m_j} g_{ij},
\]

(A8)

\[
c_j = G_{ij} - \frac{m_i}{m_i + m_j} g_{ij},
\]

(A9)

– cf. with [2], Chapter 9, § 2. For further simplification of the exponent vector \(G_{ij}\) can be replaced by vector \(\tilde{G}_{ij}\) resulting from \(G_{ij}\) in an arbitrary affine transformation, which is a composition of shift, homothety (multiplication by a scalar) and rotation. The rotation arbitrariness is reduced to the freedom in choosing of direction of the polar axis in the transition to the spherical coordinate system. Similarly, the vector \(g_{ij}\) can be replaced by the vector \(\tilde{g}_{ij}\), resulting from \(g_{ij}\) in composition of arbitrary homothety and arbitrary
rotation. The shift of the origin of the vector $g_{ij}$ would lead to parametric dependence of the final integral on vectors $u_i$ and $u_j$ (cf. with [16], Chapter 3), that is undesirable, as integral (A3) is supposed to be reduced to Chapman-Cowling integral $\Omega_{ij}^{(l,s)}$ [see [2], Chapter 9, § 3, (3.29) and [10], Chapter 7, (4.34)], depending on the modulus of the relative velocity of colliding particles $g_{ij}$ only.

In view of the aforesaid, make the following substitution of variables $G_{ij}$ and $g_{ij}$:

$$g_{ij} = z_1 \tilde{g}_{ij}, \quad G_{ij} = \frac{\sqrt{2}}{m_i m_j} \left( m_i T_j + m_j T_i \right) $$

In (A10)-(A11) the scalar factors $z_1$, $z_2$, and $z_3$ are selected from the condition that the coefficients of $\tilde{g}_{ij}^2$ and $\tilde{G}_{ij}^2$ in the exponent be equal to 1 and the coefficient of the scalar product $\tilde{g}_{ij} \cdot \tilde{G}_{ij}$ be equal to 0 (cf. with the method of separation of variables):

$$z_1 = \sqrt{\frac{2 (m_i T_j + m_j T_i)}{m_i m_j}} ,$$  \hspace{1cm} (A12)

$$z_2 = \sqrt{\frac{2 T_i T_j}{m_i T_j + m_j T_i}} ,$$  \hspace{1cm} (A13)

$$z_3 = \frac{2 (T_i - T_j)}{m_i + m_j} \sqrt{\frac{m_i m_j}{2 (m_i T_j + m_j T_i)}} .$$  \hspace{1cm} (A14)

Analogous substitutions of variables can be used in more complicated situations, for example, discussed in [16], Chapter 3.

With new variables the exponent can be written in the following form:

$$- \left[ \tilde{g}_{ij}^2 + \tilde{G}_{ij}^2 + a_0 w^2 + a_1 \tilde{g}_{ij} \cdot w + a_2 \tilde{G}_{ij} \cdot w \right] ,$$  \hspace{1cm} (A15)

where

$$w = \frac{u_i - u_j}{2} ,$$  \hspace{1cm} (A16)

$$a_0 = \frac{m_i}{2T_i} + \frac{m_j}{2T_j} ,$$  \hspace{1cm} (A17)

$$a_1 = -2 \sqrt{\frac{2 m_i m_j}{m_i T_j + m_j T_i}} ,$$  \hspace{1cm} (A18)

$$a_2 = \left( \frac{m_j}{T_j} - \frac{m_i}{T_i} \right) \sqrt{\frac{2 T_i T_j}{m_i T_j + m_j T_i}} .$$  \hspace{1cm} (A19)

It is easy to see, that by the above-specified transformations of variables only, without using the shift of the origin of vector $g_{ij}$, it is impossible to get rid of the constant term in
exponent (A15) and, hence, of the constant exponential factor, which will appear hereafter in all expressions containing integrals of form (A1), (A3). Such factors are missing in [3], (8).

Determine Jacobian of transformation of variables \((c_i, c_j) \rightarrow (\tilde{g}_{ij}, \tilde{G}_{ij})\) [see (A10)-(A11)]:

\[
\frac{\partial (c_i, c_j)}{\partial (\tilde{g}_{ij}, \tilde{G}_{ij})} = \frac{\partial (c_i, c_j)}{\partial (g_{ij}, G_{ij})} \frac{\partial (g_{ij}, G_{ij})}{\partial (\tilde{g}_{ij}, \tilde{G}_{ij})} = z_1^3 z_2^3 \frac{\partial (c_i, c_j)}{\partial (g_{ij}, G_{ij})} = z_1^3 z_2^3 \frac{\partial (c_i, c_j)}{\partial (g_{ij}, G_{ij})} = z_1^3 z_2^3.
\]

(A20)

Now consider the case, when \(\Psi_i^{(t)} = \Psi_i^{(2)} = m_i (c_i - u_i)\). In view of (A15), (A20), (A10)-(A11) and the equality, following from the definition of \(k\) above,

\[
m_i (c'_i - c_i) = \frac{m_i m_j}{m_i + m_j} (g'_{ij} - g_{ij}) = -2 \frac{m_i m_j}{m_i + m_j} (g_{ij} \cdot k) k
\]

(A21)

integral (A3) can be rewritten as:

\[
\int \int \int m_i (c'_i - c_i) \int f_i^{(0)} f_j^{(0)} g_{ij} b db dc_i dc_j
\]

\[
= -2 \frac{m_i m_j}{m_i + m_j} z_1^5 z_2^3 n_i \left( \frac{m_i}{2 \pi k T_i} \right)^{3/2} n_j \left( \frac{m_j}{2 \pi k T_j} \right)^{3/2} \int \int \int (\tilde{g}_{ij} \cdot k) k
\]

\[
\times \exp \left( - \left\{ \tilde{g}_{ij}^2 + \tilde{G}_{ij}^2 + a_0 w^2 + a_1 \tilde{g}_{ij} \cdot w + a_2 \tilde{G}_{ij} \cdot w \right\} \right)
\]

\[
\times \tilde{g}_{ij} b db dc_i dc_j \tilde{G}_{ij} d\tilde{g}_{ij}.
\]

(A22)

Integrating over \(\varepsilon\) in (A22) (with fixed \(\tilde{g}_{ij}\) and \(\tilde{G}_{ij}\)), we resolve vector \(k\) into two components: the ones parallel and perpendicular to vector \(\tilde{g}_{ij}\) — cf. with the proof of Proposition \(\text{II}\):

\[
\int (\tilde{g}_{ij} \cdot k) k d\varepsilon = 2 \pi \cos^2 \left( \frac{\pi - \chi}{2} \right) \tilde{g}_{ij} = \pi (1 - \cos \chi) \tilde{g}_{ij}.
\]

(A23)

When integrating over \(\tilde{G}_{ij}\) and directions of vector \(\tilde{g}_{ij}\), we use Proposition \(\text{II}\). As a result we obtain

\[
J_{p, ij}^{(0)} = - \int \int \int \int m_i C_i \left( f_{i}^{(0)} f_{j}^{(0)} - f_{i}^{(0)} f_{j}^{(0)} \right) g_{ij} b db dc_i dc_j
\]

\[
= - \int \int \int \int m_i (c'_i - c_i) f_{i}^{(0)} f_{j}^{(0)} g_{ij} b db dc_i dc_j
\]

\[
= - 16 n_i n_j \frac{m_i T_j + m_j T_i}{m_i + m_j} \frac{w \sqrt{\pi}}{\xi^2} e^{-2 m_i m_j w^2}
\]

\[
\times \int \int e^{-\tilde{g}_{ij}^2} [\tilde{g}_{ij} \xi \cosh (\tilde{g}_{ij} \xi) - \sinh (\tilde{g}_{ij} \xi)] \tilde{g}_{ij}^2 (1 - \cos \chi) b db d\tilde{g}_{ij}.
\]

(A24)
factor \(a_1\) is determined by formula (A18). It is easy to check, that the singularity at \(\xi = 0\), which is possible when \(w = 0\), is actually absent in the right-hand side. Expression (A24) differs substantially from Struminskii’s expression [3], (8).

The case, when \(\Psi(l) = \Psi(3) = \frac{1}{2} m_i (c_i - u_i)^2\), differs from the just considered one in the factor of the exponent in the right-hand side of (A22). Transform difference \(\Psi(l') - \Psi(l)\) according to (A10), (A11) and [2], Chapter 3, (4.9) and taking account of that only the relative particles velocity direction changes during the collision \((g_{ij} = g'_{ij})\):

\[
\Psi^{(3)r} - \Psi^{(3)} = \frac{m_i}{2} \left[(c'_i - u_i)^2 - (c_i - u_i)^2\right] = \frac{m_i}{2} (c'_i - c_i) \cdot (c'_i + c_i - 2u_i)
\]

\[
= \frac{m_im_j}{m_i + m_j} (g'_{ij} - g_{ij}) \cdot (G_{ij} - u_i)
\]

\[
= -2 z_1 \frac{m_im_j}{m_i + m_j} (g_{ij} \cdot k) \left[k \cdot \left\{z_2 \bar{G}_{ij} + z_3 \bar{g}_{ij} - \frac{u_i - u_j}{2}\right\}\right].
\]

With respect to its arguments the scalar product is a bilinear continuous function, therefore Proposition 2 can be applied. After integration over \(\epsilon\), similarly to (A23), we arrive at:

\[
-2 z_1 \frac{m_im_j}{m_i + m_j} \int (\bar{g}_{ij} \cdot k) \left[k \cdot \left\{z_2 \bar{G}_{ij} + z_3 \bar{g}_{ij} - \frac{u_i - u_j}{2}\right\}\right] d\epsilon
\]

\[
= -2\pi z_1 \frac{m_im_j}{m_i + m_j} (1 - \cos \chi) \left(\bar{g}_{ij} \cdot \left\{z_2 \bar{G}_{ij} + z_3 \bar{g}_{ij} - \frac{u_i - u_j}{2}\right\}\right).
\]

Perform the integration over \(\bar{G}_{ij}\) and directions of vector \(\bar{g}_{ij}\), using proposition 1:

\[
J_{E,ij}^{(0)} = - \int \int \int \frac{1}{2} m_i C_i^2 \left[f_i^{(0)r} f_j^{(0)} - f_i^{(0)} f_j^{(0)}\right] g_{ij} b db de dc_i dc_j
\]

\[
= - \int \int \int \frac{m_i}{2} (c'_i - u_i)^2 - (c_i - u_i)^2 \left[f_i^{(0)} f_j^{(0)} g_{ij}\right] b db de dc_i dc_j
\]

\[
= J_{e,ij}^{(0)} - u_i \cdot J_{p,ij}^{(0)}
\]

\[
= 16 n_i n_j \sqrt{\pi} e^{-\frac{2m_im_j w^2}{m_i m_j}}
\]

\[
\times \int \int e^{-\bar{g}_{ij}^2} \left\{D_{1,ij} \frac{w}{z} \left[\bar{g}_{ij} \xi \cosh (\bar{g}_{ij} \xi) - \sinh (\bar{g}_{ij} \xi)\right] + 2 D_{2,ij} \bar{g}_{ij}^2 \sinh (\bar{g}_{ij} \xi)\right\}
\]

\[
\times \bar{g}_{ij}^2 (1 - \cos \chi) b db d\bar{g}_{ij}.
\]
In (A28):

\[ D_{1,ij} = \frac{2 m_j T_i}{m_i + m_j}, \]  
(A29)

\[ D_{2,ij} = \frac{m_i m_j (T_i - T_j)}{2 (m_i + m_j)^2} \sqrt{\frac{2 T_i}{m_i} + \frac{2 T_j}{m_j}}. \]  
(A30)

The other notations are the same as in (A24).

It is interesting to note, that for \( u_i = u_j \) integral (A24) and the first term in (A28) vanish, the second term in (A28) is proportional \((T_i - T_j)\), that corresponds to energy transfer from the "hot" components to the "cold", see the gas-dynamic equations system (110)-(121). In view of the sign of \( a_1 \), (A18) and definition of \( \xi \) (A25) the first term leads to \textit{temperature increase} when \( u_i \neq u_j \).

**Appendix B: Expressions of collision integrals for interaction potential of rigid spheres**

Collision integrals (A24) and (A28) are complicated functions of mean velocities and temperatures of separate components, mainly, because of the complicated dependence of deflection angle \( \chi \) on (the modulus of) relative velocity of colliding particles \( g_{ij} \) – cf. with [10], Chapter 1, (5.26):

\[ \chi (b, g_{ij}) = \pi - 2b \int_{r_{min}}^{\infty} \frac{dr/r^2}{\sqrt{1 - \frac{2\varphi(r)}{m_i g_{ij}^2} - \frac{b^2}{r^2}}}. \]  
(B1)

In (B1) \( m_{ij} = m_i m_j / (m_i + m_j) \) is reduced mass of colliding particles, \( \varphi (r) \) – central interaction potential of particles, depending on distance \( r \) between them.

In the simplest case of particles, interacting as rigid spheres with diameters \( \sigma_i \) and \( \sigma_j \), the following analytical expressions for collision integrals have been derived from (A24) and (A28):

\[ J^{(0)}_{p,ij} = -n_i n_j \frac{m_j T_j + m_i T_i}{m_i + m_j} \frac{w}{w} \frac{\sqrt{\pi}}{2\xi^2} \sigma_{ij}^2 \]
\[ \times \left[ e^{-\xi^2/4} 2\xi (\xi^2 + 2) + \sqrt{\pi} (\xi^4 + 4\xi^2 - 4) \operatorname{erf} \left( \frac{\xi}{2} \right) \right]. \]  
(B2)
\[ J_{E,ij}^{(0)} = n_i n_j \frac{\sqrt{\pi}}{2\xi^2} \sigma_{ij}^2 e^{-\xi^2/4} \left[ 2D_{1,ij} w\xi (\xi^2 + 2) + 2D_{2,ij} \xi^2 (\xi^2 + 10) \right] \]
\[ + n_i n_j \frac{\pi}{2\xi^2} \sigma_{ij}^2 \left[ D_{1,ij} w (\xi^4 + 4\xi^2 - 4) + D_{2,ij} \xi (\xi^4 + 12\xi^2 + 12) \right] \text{erf} \left( \frac{\xi}{2} \right). \]  

(B3)

In (B2)-(B3) \( \sigma_{ij} = (\sigma_i + \sigma_j)/2 \) and notations from (A16), (A18), (A25), (A29)-(A30) are used.

[1] David Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen* (Teubner, Leipzig and Berlin, 1912) [in German].

[2] Sydney Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press, Cambridge, 1952).

[3] V. V. Struminskii, Prikladnaya Mathematica i Mechanica 38, 203 (1974) [Applied Mathematics and Mechanics 38, 203 (1974)].

[4] K. P. Gurov, *Foundations of Kinetic Theory* (Nauka, Moscow, 1966) [in Russian].

[5] V. V. Struminskii and V. Ju. Velikodniy, Soviet Physics-Doklady 266, 28 (1982) [Sov. Phys. Dokl. 266, 28 (1982)].

[6] N. Bourbaki, *Functions of a Real Variable*, Elements of Mathematics, Book IV (Springer, Berlin, 2004).

[7] S. A. Lomov, *Introduction to the General Theory of Singular Perturbations* (Nauka, Moscow, 1981) [in Russian]; English transl. in: *Mathematical Monographs*, Vol. 112, (American Mathematical Society, 1992).

[8] Carlo Cercignani, *Theory and Application of the Boltzmann Equation* (Scottish Academic Press, Edinburgh and London, 1975).

[9] P. Réridois and M. De Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).

[10] J. O. Hirschfelder, Ch. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1954).

[11] J. H. Ferziger and H. G. Kaper, *Mathematical theory of transport processes in gases* (North-Holland Publishing Company, Amsterdam, 1972).

[12] N. N. Bogolyubov, *Problems of the Dynamic Theory in Statistical Physics* (Gostechizdat,
Moscow-Leningrad, 1946) [in Russian]; English transl. in: *Studies in statistical mechanics*, Vol. 1, edited by J. De Boer and G. E. Uhlenbeck (North-Holland Publishing Company, Amsterdam, 1962).

[13] I. Fredholm, Acta Mathematica, 27, 365-390 (1903).

[14] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1 (Wiley, New York, 1989).

[15] V. V. Struminskii, Soviet Physics-Doklady 158, 70 (1964) [Sov. Phys. Dokl. 158, 70 (1964)].

[16] V. N. Oraevskiy, J. V. Konikov, and G. V. Khazanov, *Transport Processes in Anisotropic Near-Earth Plasma* (Nauka, Moscow, 1985) [in Russian].