Kelvin-Helmholtz instability in a weakly ionized layer

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Abstract We study the linear theory of Kelvin-Helmholtz instability in a layer of ions and neutrals with finite thickness. In the short wavelength limit the thickness of the layer has a negligible effect on the growing modes. However, perturbations with wavelength comparable to layer’s thickness are significantly affected by the thickness of the layer. We show that the thickness of the layer has a stabilizing effect on the two dominant growing modes. Transition between the modes not only depends on the magnetic strength, but also on the thickness of the layer.

Keywords instabilities · ISM: general · ISM: jets and outflows

1 Introduction

When two fluids are in relative motion on either side of a common boundary, the Kelvin-Helmholtz (KH) instability can occur. This instability, the simplest example of shear flow instability, is a well-known phenomenon in fluid mechanics and astrophysics (e.g. Chandrasekhar 1961). For example, the interactions between the solar wind and planetary magnetospheres are studied based on the KH instability (e.g. Nagano 1979). The KH instability, because of the velocity shear in the mass flow between coronal plume structure and the interplume region of the sun, could be a potential source for some of the Alfvénic fluctuations observed in the solar wind (Andries, Tirry & Goossens 2000). The structure and the helical wave motion in the ionized cometary tails
as studied by Ershkovich & Mendis (1986), Ershkovich, Prialnik & Eviatar (1986) and Ershkovich, Flammer & Mendis (1986) using the KH theory. The KH instability theory has been successfully used to interpret the structure of the pc-scale jet in the radio source 3C273 (Lobanov & Zensus 2001). Birk et al. (2000) studied the role of KH instabilities in superwinds of primeval galaxies. The effect of radiative losses on the evolution of the KH instability in jets and outflows has been studied by many authors, either in linear regime (e.g., Massaglia et al. 1992; Hardee & Stone 1997) or by direct numerical simulations (e.g., Rossi et al. 1997; Downes & Ray 1998). Recently, Michikoshi & Inutsuka (2006) studied the KH instability of the dust layer in protoplanetary discs to understand the effect of relative motion between gas and dust.

However, in many astrophysical flows the single fluid approximation is inappropriate. For example, the fact that the winds and the intergalactic medium may be only partially ionized should be considered in treatment of role of the KH instability in superwinds of primeval galaxies. We may have a similar situation at the interface between a jet and its cocoon if the KH instability is to be viewed as a possible mechanism for entrainment. The problem becomes more complicated if we note that the development of the KH instability is strongly influenced by the magnetic field (e.g. Malagoli et al. 1996; Jones et al. 1997). Therefore it seems a two-fluid treatment of KH instability, taking account of the different motions of ions and neutrals and of the magnetic field, rather than a pure one fluid MHD description is a more appropriate approach.

By neglecting the thermal pressure forces of the neutrals or perturbations in the neutral gas bulk velocities, the KH instability of a system consisting of ions and neutrals with applications to partially ionized cometary ionopause, has been studied for both compressible (Ershkovich & Mendis 1986) and incompressible cases (Ershkovich, Prialnik & Eviatar 1986; Ershkovich, Flammer & Mendis 1986). Chhajlani & Vyas (1991) incorporated the effects of finite resistivity and rotation on the KH instability of two superposed fluids slipping past each other. They showed that small rotation and the presence of neutral particles destabilize the system. Birk et al. (2000) studied the KH instability by taking into account the full incompressible dynamics of both the neutral and the ionized gas components with applications to multi-phase galactic outflow winds. In another similar study, Watson et al. (2004) (hereafter WZHC) investigated the KH instability in the linear, partially ionized regime to determine its possible effect on entrainment in massive bipolar outflows. They showed that for much of the relevant parameter space, neutral and ions are sufficiently decoupled that the neutrals are unstable while ions are held in place by the magnetic field. Birk & Wiechen (2002) focused on unstable shear flows in partially ionized dense dusty plasma. They considered the dust and neutral gas components so that dust and neutral collisions is the dominant momentum transfer mechanism and dust component can interact with magnetic field lines, although dust charge fluctuations are negligible. They showed long wavelength modes can be stabilized by dust and neutral gas collisional momentum transfer. By doing multifluid numerical simulations, Wiechen (2006)
studied KH modes in partially ionized, dusty plasmas for different masses and charges of dust.

Analytical studies of KH instability in weakly ionized medium have so far considered one interface between two flows. For example, WHZC studied incompressible KH instability in a two-fluid system (charged and neutral) in order to understand entrainment in outflows from massive stars. They did the calculations in planar geometry with an interface at \( z = 0 \). However, it is not unreasonable to suppose that we may have a flowing layer with finite thickness and two interfaces. In this paper, we generalize the WHZC analysis by considering a finite thickness for a weakly ionised flow. We will show that the growth time-scale of the instability increases due to the thickness of the layer in particular in the long wavelength limit. In the next section, we will present the basic equations and assumptions of the model. In the section 3, we will discuss growing modes and their dependence on the thickness of the layer.

2 General Formulation

We follow the analysis of Chandrasekhar (1961) but take account of the different bulk velocities and densities of the neutral and ions, both inside and outside the layer. These two components of our model are coupled by collisional interactions. We assume incompressibility for the analytical calculations and also that the convective term in the induction equation dominates the resistive one. For doing linear analysis, the unperturbed properties of the system are important. We suppose that the streaming takes place in the \( x \)-direction with velocity \( U(z) \),

\[
U(z) = \begin{cases} 
+U & \text{for } |z| \leq d \\
-U & \text{for } |z| > d 
\end{cases}
\]

(\( U \) is constant) and the neutral and ion components have the same velocity in the unperturbed state. The magnetic field is assumed parallel to the interface; that is, \( B = B_x e_x \). Finally, all unperturbed physical quantities are assumed constant in each medium. The system under consideration consists of charged and neutral fluid components coupled by collisional interactions.

Equations for the neutrals become

\[
-k_x \rho_n u'_n + i \rho_n \frac{du'_n}{dz} = 0, \tag{1}
\]

\[
\phi \rho_n u'_n = -k_x p'_n + i \gamma_n \rho_n \rho_i (u'_n - u'_i), \tag{2}
\]

\[
\phi \rho_n w'_n = i \frac{dp'_n}{dz} + i \gamma_n \rho_n \rho_i (w'_n - w'_i), \tag{3}
\]

where \( \phi = \omega + k_x U \). Also, the equations of the ions are

\[
-k_x \rho_i u'_i + i \rho_i \frac{dw'_i}{dz} = 0, \tag{4}
\]
\[
\phi \rho_i u_i' = -k_x p_i' + i \gamma_{ni} \rho_n \rho_i (u_i' - u_n'), \quad (5)
\]
\[
\phi \rho_i w_i' = i \frac{dp_i'}{dz} - \frac{B_i^2}{4\pi \phi} \frac{d^2 w_i'}{dz^2} + \frac{k_i^2 B_i^2}{4\pi} \frac{w_i'}{\phi} + i \gamma_{ni} \rho_n \rho_i (w_i' - w_n'). \quad (6)
\]

We can reduce the above differential equations to a set of two differential equations for \(w'_i\) and \(w'_n\) as follows (WZHC)

\[
(r_i \phi - \frac{B_i^2 k_i^2}{4\pi \phi}) D w_i' = i \gamma_{ni} \rho_n \rho_i D (w_i' - w_n'), \quad (7)
\]
\[
\rho_n \phi D w_n' = i \gamma_{ni} \rho_n \rho_i D (w_n' - w_i'), \quad (8)
\]

where \(D \equiv \frac{d^2}{dz^2} - k_i^2\). The behavior of the flow both inside and outside the layer is determined by the general solutions of the linear differential equations \(S\) and \(T\). One can easily show that the general solutions of these equations are linear combinations of \(\exp(\pm k_x z)\) and \(\exp(-k_x z)\). We impose the condition that the perturbed quantities do not diverge at \(z = \pm \infty\). Thus, the general solutions are

\[
w_i'(z) = \begin{cases} C_1 e^{+k_x z} + C_2 e^{-k_x z} & \text{for } 0 \leq z \leq d \\ C_3 e^{-k_x z} & \text{for } d \leq z \end{cases} \quad (9)
\]
\[
w_n'(z) = \begin{cases} C_4 e^{+k_x z} + C_5 e^{-k_x z} & \text{for } 0 \leq z \leq d \\ C_6 e^{-k_x z} & \text{for } d \leq z \end{cases} \quad (10)
\]

where \(C_1, C_2, C_3, C_4, C_5\) and \(C_6\) are constants which are determined by the boundary conditions. We note that, because of symmetry of the unperturbed state, there are solutions where \(w_i(z) = -w_i(-z)\) and \(w_n(z) = -w_n(-z)\) (odd solutions) and solutions where \(w_i(z) = w_i(-z)\) and \(w_n(z) = w_n(z)\) (even solutions). Thus, we can write the boundary condition at \(z = 0\) as

\[
C_1 + SC_2 = 0, \quad (11)
\]
\[
C_4 + SC_5 = 0, \quad (12)
\]

where \(S = +1\) and \(S = -1\) are corresponding to even and odd solutions, respectively.

Another boundary condition is the continuity of the ion and neutral displacements across the interfaces. So, the boundary conditions at \(z = d\) are

\[
\frac{C_1 e^{k_x d} + C_2 e^{-k_x d}}{\omega + k_x U} = \frac{C_3 e^{-k_x d}}{\omega - k_x U}, \quad (13)
\]
\[
\frac{C_4 e^{k_x d} + C_5 e^{-k_x d}}{\omega + k_x U} = \frac{C_6 e^{-k_x d}}{\omega - k_x U}. \quad (14)
\]

Having equations (11), (12), (13) and (14), we can rewrite \(w_i\) and \(w_n\) as

\[
w_i'(z) = \begin{cases} C_2 (e^{-k_x z} - Se^{k_x z}) & \text{for } 0 \leq z \leq d \\ C_2 \frac{k_x}{1 + k_x^2} (1 - Se^{2k_x z}) & \text{for } d \leq z \end{cases} \quad (15)
\]
\( w_n'(z) = \begin{cases} C_5(e^{-kx}z - Se^{kx}z) & \text{for } 0 \leq z \leq d \\ C_5\frac{-kU}{1-S}{(1 - Se^{2kx}z)} & \text{for } d \leq z \end{cases} \) \hspace{1cm} (16)

Clearly, the differential equations (8) and (7) are not valid at the interfaces \( z = d \) (or \( z = -d \)), where velocities of the species are not analytic. However, we can integrate each equation over an infinitesimal region around the interface \( z = d \) and use standard Gaussian pillbox arguments. Then we have

\[
\triangle \{ \rho_n \phi \} \frac{dw_n'}{dz} = i \triangle \{ \gamma_\Omega \rho_n \rho_i \frac{d}{dz} (w_n' - w_i') \}, \hspace{1cm} (17)
\]

\[
\triangle \{ \rho_i \phi \} \frac{dw_i'}{dz} = i \triangle \{ \gamma_\Omega \rho_n \rho_i \frac{d}{dz} (w_i' - w_n') \} + \frac{B_i^2 k_\parallel^2}{4\pi} \triangle \{ \frac{1}{\phi} \frac{dw_n'}{dz} \}, \hspace{1cm} (18)
\]

By substituting equations (15) and (16) into the above equations and after lengthy (but straightforward) mathematical manipulations, we obtain

\[
[\xi^2 + m\xi - (1 - \lambda^2)h^2 + a^2]C_2 - m\xi C_5 = 0, \hspace{1cm} (19)
\]

\[
- \xi C_2 + [\xi^2 + \xi - (1 - \lambda^2)h^2]C_5 = 0, \hspace{1cm} (20)
\]

where

\[
m = \frac{\rho_n}{\rho_i}, h = \frac{U k_x}{\nu}, a = \frac{k \nu A_i}{\nu} - \lambda h = -i\xi, \hspace{1cm} (21)
\]

and \( \lambda = S^{-1} \exp(-2kd) \) and the ion Alfvén velocity \( \nu A_i \) is defined as \( B/\sqrt{4\pi\rho_i} \).

By the condition that the algebraic equations (19) and (20) have a nontrivial solution, we obtain the dispersion relation

\[
\xi^4 + (m + 1)\xi^3 - [2(1 - \lambda^2)h^2 - a^2]\xi^2
\]

\[
- [(m + 1)(1 - \lambda^2)h^2 - a^2]\xi + (1 - \lambda^2)((1 - \lambda^2)h^2 - a^2)h^2 = 0. \hspace{1cm} (22)
\]

This algebraic equation (22) is the basis of our stability analysis. As we see, the effect of the thickness of the layer appears by term \( \lambda^2 \) which depends on the thickness of the layer.

3 Analysis

We note that equation (22) reduces to equation (29) of WZHC if we set \( \lambda = 0 \). Roots of equation (22) with positive real parts correspond to growing unstable modes. This equation is applicable for various weakly ionized systems. However, in this work the input parameters are chosen to be appropriate for molecular outflows. This makes for easier comparison with the analysis of WZHC. In our illustrative examples, the relative velocity \( 2U \) is 20 Km s\(^{-1}\), and the ratio, \( m \), of neutral to ionized density is assumed to be around \( 10^6 \). The collision frequency is \( \nu = 1.5 \times 10^{-13} \text{ s}^{-1} \) (Draine et al. 1983). The magnetic field strength is taken to be either 1 \( \mu \text{G} \) (weak-field case) or 1 mG (strong-field case). Having
Fig. 1 Growth rate of the instability vs. wavelength of the perturbations for two cases $B = 1 \, \mu G$ (top) and $B = 1 \, mG$ (bottom). In both cases, it is assumed $U = 10 \, \text{Km s}^{-1}$. The dashed curves correspond to the WZHC analysis and the other curves are labeled by the thickness of layer. Black circles in the bottom plot denote the wavelength at which the growing mode changes from $(1 - \lambda^2)^{1/2}h$ to $(1 - \lambda^2)h^2$. 
these input parameters, we can solve the algebraic equation (22) numerically to find the dispersion relation and the fastest growing modes.

Figure 1 shows plots of the dispersion relation for the above input parameters. When $\lambda \to 0$, the dispersion relation tends to WZHC analysis (dashed curves in Figures 1), as expected. Each curve is also labeled by the thickness of the layer, i.e. $2d$. It can be seen from Figure 1 that the effect of the thickness of the layer is not significant for perturbations with short wavelength. However, as the wavelength of the perturbations increases, the thickness of the layer has a stabilizing effect on the growing modes and the dispersion relation deviates from the analysis of WZHC both in the weakly and strongly magnetized cases. Considering the thickness of the layer implies longer growth time-scale for the perturbations, in particular in the long wavelength limit. Moreover, as one would expect, as the thickness of the layer decreases the deviation from the WZHC analysis occurs at shorter wavelength, and also the growth time-scale becomes longer.

Interestingly, the roots of equation (22) which correspond to growing modes can be described by approximate analytical solutions. There are three approximate positive roots for this equation depending on its coefficients:

(i) $(1 - \lambda^2)^{1/2}h$, when $(1 - \lambda^2)^{1/2}h > 1$;

(ii) $(1 - \lambda^2)h^2$, when $(1 - \lambda^2)^{1/2}h < 1$ and $(1 - \lambda^2)^{1/2} < (a/h)m^{-1/2}$;

(iii) $(1 - \lambda^2)^{1/2}h(1 - (a/h)^2(1 - \lambda^2)^{-1}m^{-1})^{1/2}$ and $(1 - \lambda^2)^{1/2} > (a/h)m^{-1/2}$.

The agreement between these approximate roots and numerical roots of the equation (22) is excellent. Thus, as long as $(1 - \lambda^2)^{1/2}h > 1$ the only growing mode is the first growing mode, irrespective of the strength of the magnetic field (see top plot of Figure 1). Once this inequality is violated, we may have the second growing mode, depending on whether $(1 - \lambda^2)^{1/2} < (a/h)m^{-1/2}$. For example, for $B = 1$ mG we have $(a/h)m^{-1/2} = 4.5 > 1 \geq (1 - \lambda^2)^{1/2}$ and so the transition from the first to the second growing modes occurs when $(1 - \lambda^2)^{1/2}h = 1$. We can denote this transition wavelength by $\lambda_t$ and is shown by black circles in Figure 1 (bottom plot). But in the weakly magnetic case where $B = 1$ $\mu$G, we have $(a/h)m^{-1/2} = 4.5 \times 10^{-3} < (1 - \lambda^2)^{1/2}$. So, for wavelengths greater than $\lambda_t$, the third growing mode appears, although the second term inside the parenthesis of this root is much smaller than unity and the third mode actually tends toward first mode (top plot). Considering our ranges of the input parameters, the third mode is very close to the first mode.

The first mode corresponds to the usual hydrodynamic instability, but the second mode appears in our weakly ionized case as a new mode (WZHC). The transition between these two modes depends on the thickness of the layer. Figure 2 shows the transition wavelength $\lambda_t$ versus the the half thickness of the layer. For thin layers, the transition wavelength is smaller than the thick layers. In fact, as the thickness of the layer increases, this transition wavelength increase and tends to an asymptotic value for the very thick layers. WZHC analysis corresponds to this asymptotic regime.
Fig. 2 The transition wavelength $\lambda_t$ vs. the half thickness $d$ of the layer.

4 Conclusions

In this study we have extended the analysis of WZHC by considering a layer of ions and neutrals with finite thickness interacting with an ambient medium in the incompressible limit. However, this model can not be applied directly for systems, such as outflows or jets because of our simplifying assumptions. We
can, however, gain insight into the possible effects of shear between a moving layer of ions and neutrals and the surrounding medium. We have shown that the growth time-scale of KH instability in a layer of ions and neutrals increases when compared to a system with one interface, this effect being more evident in the long wavelength limit. There are two dominant growing modes of KH instability in such a system, depending on the magnetic strength and the thickness of the layer. As the thickness of the layer decreases, the transition between these two unstable modes occurs at shorter wavelengths. Given these results, it remains to future work to extend this analysis to compressible case and make direct comparisons with jets or outflows with finite thickness.

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