Bifurcation in the angular velocity of a circular disk propelled by symmetrically distributed camphor pills

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(Dated: 20 September 2018)

We studied rotation of a disk propelled by a number of camphor pills symmetrically distributed at its edge. The disk was placed on a water surface and could rotate around a vertical axis located at the disk center. In such systems, the driving momentum comes from nonuniform surface tension resulting from inhomogeneous surface concentration of camphor molecules released from the pills. We investigated the stationary angular velocity as a function of the disk radius and of the number of pills. It was observed that for a small disk radius the angular velocity dropped to zero after a critical number of pills was exceeded. Such behavior was confirmed by a numerical model of rotor evolution. We also present bifurcation analysis of the conditions at which the transition between a still and a rotating disk appears.

I. INTRODUCTION

Studies on self-propelled objects have become popular in the recent years because the behaviour of many such systems shows similar character of motion to that expressed by living organisms. Self-propelled physicochemical objects can be divided into two classes. One of them includes these with embedded asymmetry. For such objects, the direction of motion is determined by their structure and geometry. For example, Janus particles, characterized by different rates of surface reactions at different parts of their surface, can move in the direction determined by their chemical activity. There are also objects in which the boundaries direct a jet of reaction products and force the motion. Self-motion can also appear in systems without chemical reactions if there are other processes generating nonequilibrium conditions. For example, if a piece of camphor is placed on the water surface then molecules of camphor hardly dissolve in water, but majority of them form a layer on the water surface. In a typical experimental conditions this layer is unstable, because camphor molecules continuously evaporate from the surface. The presence of camphor layer modifies the surface tension. It is known that water surface tension is a decreasing function of camphor surface concentration. The averaged force acting on a camphor piece is directed towards the neighboring region with the lowest camphor surface concentration. These properties of camphor can be used to make self-propelled objects. The simplest and most known is a camphor boat i.e., a boat-shaped piece of plastic with a piece of camphor glued at its stern. Such location of camphor breaks system symmetry. The surface concentration of released camphor molecules around the stern is higher than that around the bow, that decreases the surface tension in the stern area. As the result the boat moves forward.

The second class of self-propelled system includes the symmetric objects in which the symmetry is broken by asymmetry of processes that generate the motion. Such systems include droplets where Belousov-Zhabotinsky (BZ) reaction proceeds. The interfacial tension between a droplet and the surrounding oil phase is related to the level of catalyst oxidation. If a droplet is sufficiently large then homogeneous oscillations and, for yet larger droplets a propagating excitation pulse, can appear. The related changes in interfacial tension can generate a jump of the droplet in the direction of pulse propagation. However, the direction of an excitation pulse propagation is random, thus a symmetric BZ droplet can be shifted in random directions and there are no factors that can stabilize the direction of motion.

In the case of a symmetric object propelled by camphor particles, we observe a positive feedback between the generated force (or momentum) and the direction of object motion. Let us consider a symmetric camphor disk reclining on the water surface. It releases camphor molecules around, but both formation of a camphor layer around the disk and the evaporation of camphor molecules from the water surface are subject to fluctuations. If an area characterized by a low surface concentration of camphor appears close to the disk then the disk is shifted towards it because the disk is attracted by the region with higher interfacial tension. When the disk is shifted from the original position the surface camphor concentration in front of the disk is lower than in the region behind the disk, because the area in front of disk has been more distant from the camphor source than the region behind.
the disk. Therefore, the disk motion continues up to the moment the disk hits the boundary or it is repelled by water meniscus near the boundary.

Studies on self-propelled rotational motion are interesting since such motion occurs in a confined space, thus we can neglect the effect of boundaries. There have been several reports on systems that show spontaneous rotation. They can be also divided into two classes. One includes systems with broken chiral symmetry and the rotational motion is determined by the asymmetry embedded into the system. The systems in which rotation occurs through the spontaneous breaking of chiral symmetry belong to the other class. An important work on a system belonging to the latter class was done by Pimienta et al., who studied spontaneous rotation of a dichloromethane droplet on aqueous phase. Another interesting system was studied by Takabatake et al., where a droplet with a small soap fragment performed rotational motion.

Alternatively, fluctuations of surface camphor concentrations can induce initial rotation that is supported by coupling between direction of motion and the concentration gradient, like in the motion of a camphor disk mentioned above. The recent paper of Nakata et al. reported rotation of a regular hexagon propelled by camphor pills located at the corners. The system remained symmetric during the time its evolution was observed, however the state in which the hexagon did not rotate was unstable. The stable states were symmetry breaking clockwise and anti-clockwise rotations.

There are not too many studies in which mathematical modeling of self-propelled rotational motion has been compared with experimental results. In this respect systems that are propelled by camphor particles are worth considering because a simplified model of their evolution based on reaction-diffusion equation for camphor surface concentration coupled with the Newtonian equation of motion for the camphor particles can be formulated. One of the analyzed problems was rotational motion of an elliptic camphor object. The elliptic shape has no chirality and thus the rotational motion of an elliptic camphor particle appears as the result of spontaneous symmetry breaking between clockwise and anticlockwise rotational motion. Though the system is simple, there is some difficulty in the analytical description of its evolution. In our previous papers we considered yet simpler system: a camphor rotor made of two camphor pills attached at the ends of a plastic stripe. The pills were floating on a water surface whereas the stripe was elevated above the surface. The system was allowed to rotate around a vertical axis located at the center of the stripe. The driving momentum came from the nonuniformity of surface tension resulting from inhomogeneous surface concentration of camphor molecules around the pills. We observed that such rotor starts to move only after a distance between the camphor pills was larger than the critical one. For such system the rotor radius can be considered as a bifurcation parameter.

The mathematical model of such spontaneous symmetry breaking can be formulated in terms of pitchfork bifurcation in dynamical systems.

In this paper we continue our study on symmetrical rotors that move after symmetry of the medium is broken as the result of internal fluctuations. We consider a disk that can rotate on a water surface because it is powered by a number of camphor pills. The pills are symmetrically distributed at the disk edge. There are two parameters that describe the system: the disk radius and the number of pills. We have performed experiments and analyzed the stationary angular velocity of the disk as the function of both parameters. The results are reported in Section II. It comes out that for small disk radius the angular velocity drops to zero when the number of pills is large. If the disk is large then in the same range of pill number the angular velocity weakly depends on the number of pills attached. In Section III we present a mathematical model of disk rotation. Results of numerical simulations presented in Section IV allowed us to determine the values of model parameters for which the qualitative agreement with experimental results is obtained. Section V is concerned with the analytical methods used to study disk evolution and with the analysis of bifurcation between the rotating and the still disk.

II. EXPERIMENTS

We study the angular velocity of a disk propelled by a number of camphor pills located on water surface. The system is illustrated in Fig. 1. The disk could rotate around a vertical axis located at its center. The pills were symmetrically distributed close to the disk edge. They were glued to columns located below the disks such that the pills are in contact with water, whereas the disk was elevated over the water surface to avoid generation of an extensive hydrodynamic flows by the moving disk. The profile of camphor surface concentration on water results from the balance between the inflow of camphor molecules from the pills and camphor evaporation and dissolution in the water. The driving momentum comes from nonuniform surface tension resulting from inhomogeneous surface concentration of camphor molecules released from the pills.

Like in the camphor disk motion discussed in the Introduction there is a positive feedback between the disk angular shift and the force generating the torque. We investigate the stationary angular velocity as a function of the number of pills $N$ and of the distance between the axis and the pill center $\ell$. The pills of mass $m$ and radius $\rho$ were tangent to the disk, so the disk radius $R = \ell + \rho$. The experiments were performed on water surface in a square tank (tank side 120 mm) and the water level was 10 mm. Water was purified using a Millipore system (Elix 5) and its temperature was $22 \pm 1^\circ$C. In experiments we used commercially available camphor (99%
purity, Sigma-Aldrich) without further purification. The pills were made by pressing camphor in a pill maker. The radius of each camphor pill was $\rho = 1.5$ mm and it was 1 mm high.

Figure 2 illustrates the analysis of experimental data. A dot was glued on the disk surface. Its position was recorded on a movie and analyzed using ImageJ program. We applied two methods to obtain the angular velocity. In one of them we calculated the angular velocity as a function of time using the difference in dot location observed on consecutive frames of the movie. In the second method we analyzed time dependent color intensity in the region marked as a red line in Fig. 2(b). We observed a minimum in color intensity every time the dot passed this region. The angular velocity averaged over a single rotation was calculated from the time difference corresponding to the successive minima. Both methods gave similar results. In a typical experiment the angular velocity was quite stable (cf. Fig. 2(c)) and weakly depended on time at the time scale of a few minutes, which in the presented case corresponded to over 100 rotations. A small decrease in time can be related to the increase of concentration of camphor dissolved in water.

Figure 3 summarizes the experimental results obtained for 3 different disk radii. It shows the average angular velocity $\omega(\ell,N)$ measured within first 6 minutes of rotation. For a small number of pills ($N \leq 10$) the angular velocity for fixed $N$ is a decreasing function of $\ell$. If the dependence of $\omega(\ell,N)$ on $N$ is concerned we observed that for small disk radii the angular velocity rapidly decreases with the increasing number of pills. The disk of $\ell = 8.5$ mm powered by $N = 14$ pills randomly rotated in both directions, but it did not show unidirectional rotation lasting more than a second. For a larger disk radii ($\ell = 13.5$ mm and $\ell = 18.5$ mm) the angular velocity is slowly decreasing with $N$. The fits suggest that for all considered values of $\ell$ the angular velocity is a decreasing function of $N$ if $N > 5$. However in all experimental results $\omega(N = 3) < \omega(N = 4)$, so we cannot exclude a maximum of angular velocity at small $N \in \{4, 5\}$.
where $\rho$ corresponds to the camphor pill radius and $\Theta(\xi)$ is the Heaviside’s step function, i.e., $\Theta(\xi) = 1$ for $\xi \geq 0$ and $\Theta(\xi) = 0$ for $\xi < 0$.

The camphor molecules reduce the surface tension of the water surface, and the camphor pills are driven by the surface tension around it, which induces the spinning motion of a circular disk. The dynamics on this motion is described as

$$I \frac{d^2 \phi}{dt^2} = -\eta r \frac{d\phi}{dt} + T. \quad (4)$$

Here, $I$ is the momentum of inertia of the rotor, and it is described using the mass of one camphor pill $m$ as

$$I = Nm\ell^2 = \pi N\sigma \rho^2 \ell^2, \quad (5)$$

where $\sigma$ is the area density of the camphor pills. $-\eta r \frac{d\phi}{dt}$ is the torque originating from the friction force working on the camphor pills and $\eta r$ is described using the friction coefficient per unit area, $\kappa$,

$$\eta r = \pi N\kappa \rho^2 \ell^2, \quad (6)$$

as is derived in the previous paper\textsuperscript{37,43}. $T$ is the torque exerting on the disk, which is represented as

$$T = \sum_{j=0}^{N-1} \ell_j \times F_j, \quad (7)$$

where $F_j$ is the driving force of the $j$-th camphor pill induced by the surface tension gradient. It should be noted that constraint force should work on each pill to maintain the composition of the rotor, but the direction of the constraint force working on the $j$-th pill is the same as $\ell_j$ and therefore it does not affect the torque. $F_j$ is described as

$$F_j = \int_0^{2\pi} \gamma(c(\ell_j + \rho e(\theta))) e(\theta) \rho d\theta, \quad (8)$$

where $\gamma(c)$ is the surface tension depending on the camphor surface concentration, and $e(\theta)$ is a unit vector in the direction of $\theta$, i.e., $e(\theta) = \cos \theta e_x + \sin \theta e_y$. For simplicity, we set

$$\gamma(c) = \gamma_0 - kc, \quad (9)$$

where $\gamma_0$ is water surface tension, and $k$ is a positive constant. Hereafter, we set $k = 1$.

Taken in all, we obtain

$$\pi N\sigma \rho^2 \ell^2 \frac{d^2 \phi}{dt^2} = -\pi N\kappa \rho^2 \ell^2 \frac{d\phi}{dt} + T, \quad (10)$$

or

$$\sigma \frac{d^2 \phi}{dt^2} = -\kappa \frac{d\phi}{dt} + \frac{1}{\pi N\rho^2 \ell^2} T, \quad (11)$$

It is noted that we used the equations with dimensionless variables. The length, time, and concentration are nondimensionalized with the diffusion length.

\[ \text{FIG. 3: Experimental results on the angular velocity } \omega(\ell, N) \text{ as a function of the number of camphor pills } N \text{ for } 3 \text{ selected values of } \ell \text{ – the distance between disk axis and the dot center (8.5, 13.5 and 18.5 mm). Dots represent experimental data and curves show the fit using a quadratic polynom. Numbers next to the dots give the corresponding value of } \ell. \]

In this section, we introduce a mathematical model for a disk propelled with $N$ camphor pills attached. We set the coordinates so that the center of a circular disk, i.e., the rotation axis, is located at the origin. The motion of disk is described by the characteristic angle $\phi(t)$. All $N$ camphor pills are located at the distance of $\ell$ from the origin and have equivalent spacing. Thus the position of $j$-th camphor pill, $\ell_j(t)$, can be described as:

$$\ell_j(t) = \ell \left[ \cos \left( \phi(t) + \frac{2\pi j}{N} \right) e_x + \sin \left( \phi(t) + \frac{2\pi j}{N} \right) e_y \right], \quad (1)$$

for $j = 0, \cdots, N-1$, where $e_x$ and $e_y$ are the unit vectors in $x$- and $y$-directions, respectively.

The camphor pills release camphor molecules to the water surface, and the camphor molecules diffuse at the water surface. Some camphor molecules sublime to the air and some dissolve to the water bulk phase. All these processes are taken into account in the equation for the time evolution of the surface concentration of camphor molecules:

$$\frac{\partial c}{\partial t} = \nabla^2 c - c + \sum_{j=0}^{N-1} f(r; \ell_j), \quad (2)$$

where $c(r,t)$ is the surface concentration of camphor molecules at the position $r$ and time $t$. The first term corresponds to the diffusion, and the second one to sublimation to the air and dissolution to the water bulk. The last term $f(r; \ell_j(t))$ denotes the release of camphor molecules from the $j$-th pill, which is represented as

$$f(r, \ell_j(t)) = \frac{1}{\pi \rho^2} \Theta(\rho - |r - \ell_j(t)|), \quad (3)$$

for $j = 0, \cdots, N-1$; $r = \sum_{j=0}^{N-1} \ell_j(t)$; $\Theta(\xi)$ is the Heaviside’s step function, i.e., $\Theta(\xi) = 1$ for $\xi \geq 0$ and $\Theta(\xi) = 0$ for $\xi < 0$.
$\sqrt{D/a}$, the characteristic time of sublimation/dissolution $1/a$, and the ratio between the release rate and sublimation/dissolution rate, $f/a$, where $D$ is the diffusion constant of camphor molecules, $a$ is the sublimation/dissolution rate, and $f$ is the release rate of camphor molecules from one camphor pill per unit time. It should also be noted that we neglect the hydrodynamic effect. Due to the surface tension gradient, the Marangoni flow should occur. Therefore, we regard $D$ as an “effective” diffusion constant including the Marangoni effect.

IV. NUMERICAL SIMULATIONS

Based on the model introduced in the previous section, we performed numerical calculation. The release of camphor molecules, described by Eq. (3), was approximated using the expression:

$$f(r, \ell_j(t)) = \frac{1}{2\pi\rho^2} \left[ 1 + \tanh \left( \frac{(r - \ell_j - \rho)}{\delta} \right) \right], \quad (12)$$

in order to reduce the effect of discretization, where $\delta$ is a positive constant for smoothing.

The parameters were set to be $\sigma = 0.001$, $\rho = 0.15$, and $\delta = 0.025$. The distance between the disk center to the camphor pill center $\ell$, the number of camphor pills $N$, and the coefficient of the friction $\kappa$ were varied as parameters. The time evolution was calculated with the Euler algorithm and the diffusion was calculated with the explicit method. Time step was $10^{-4}$ and the spatial mesh was 0.025. The force working on the camphor disk was calculated by summing the surface tension at 32 discrete points along the periphery. To avoid the effect of the boundary, we calculated the surface concentration of camphor up to 5 distance units from the axis. The Neumann conditions were applied at the boundaries. The calculation started from the initial condition that $\phi = 1$ and $d\phi/dt = 0.1$. The terminal angular velocity $\omega$ is set to be $d\phi/dt$ at $t = 100$, since we confirmed that the system had already reached close to the stationary stable state at $t = 100$.

The numerical results are shown in Fig. 4 in which we simultaneously plotted $\omega$ against $N$ for $\ell = 0.85, 1.35$, and 1.85. It should be noted that the ratios between the camphor pill radius and the circular disk radius are the same in experiments and numerical simulation. In Fig. 4 (a), (b), and (c), we plotted the results on $\omega$ against $N$ for different values of $\kappa$. The crosspoints of the plots with different $\ell$ changed depending on $\kappa$, and therefore we hope we can estimate $\kappa$ from the experimental results. In this case, the plot in (b) is most close to the experimental results, and thus we can estimate $\kappa \simeq 0.01$.

V. THEORETICAL ANALYSIS

In this section, we derive the reduced evolution equation for the spinning of a disk with $N$ camphor pills. We adopted Eq. (2) for the time evolution of concentration field, assuming that the support of function describing the release of camphor molecules is infinitesimally small:

$$f(r, \ell_j(t)) = \delta (r - \ell_j(t)), \quad (13)$$

instead of a finite size (cf. Eq. (3)). Here $\delta(\cdot)$ is Dirac’s delta function in a two-dimensional space. The source term given by Eq. (13) delivers the same amount of cam-
against radius of camphor disks to be 0 as those described by Eqs. (3) and (12).

In our previous study, we derived the explicit form of the concentration field expanded with respect to the velocity, acceleration, jerk (i.e. the rate of change of acceleration), and so on of a camphor pill; the concentration field at the position \( \mathbf{r} \) originating from a camphor pill whose position is \( \ell(t) \) is described as

\[
\begin{align*}
    c_s(r; \ell) &= \frac{1}{2\pi} K_0(d) - \frac{1}{4\pi} K_0(d) \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] \\
    &+ \frac{1}{16\pi} d K_1(d) \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] - \frac{1}{16\pi} d K_1(d) \left| \dot{\mathbf{e}} \right|^2 \\
    &+ \frac{1}{16\pi} d K_0(d) \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] - \frac{1}{32\pi} d K_1(d) \left| \dot{\mathbf{e}} \right|^2 \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] \\
    &- \frac{1}{96\pi} d K_0(d) \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] + \frac{1}{32\pi} d K_2(d) \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] \\
    &- \frac{1}{96\pi} d K_1(d) \left[ \mathbf{d} \cdot \dot{\mathbf{e}} \right] + \left( \text{higher order terms} \right),
\end{align*}
\]

where \( \mathbf{d} = \mathbf{r} - \ell \) and \( d = |\mathbf{d}| \). \( K_n \) is the second-kind modified Bessel function of \( n \)-th order, and a dot (\( \cdot \)) means the time derivative. In this expansion, we assume that the velocity of the camphor pill is sufficiently small.

In the case of a disk with \( N \) camphor pills, the concentration field is expressed by summing up the concentration field originating from each camphor pill since the evolution equation for concentration field is linear. Thus, the concentration field made by a rotor whose center is located at the origin is given by

\[
c(\mathbf{r}) = \sum_{j=0}^{N-1} c_s(\mathbf{r}; \ell_j). 
\]

Considering that the driving force originates from the imbalance of surface tension, the driving force working on the \( j \)-th pill \( F_j \) can be calculated as follows:

\[
\begin{align*}
    \frac{1}{\pi \rho^2} F_j &= -\sum_{k=0}^{N-1} \lim_{\rho \to +0} \frac{1}{\pi \rho^2} \int_0^{2\pi} c_s(\ell_j + \rho e(\theta); \ell_k)e(\theta) \rho d\theta \\
    &= \frac{1}{4\pi} \left( -\gamma + \log \frac{2}{\rho} \right) \dot{\ell}_j - \frac{1}{16\pi} \dot{\ell}_j - \frac{1}{32\pi} \left| \dot{\ell}_j \right|^2 \dot{\ell}_j \\
    &+ \frac{1}{48\pi} \ddot{\ell}_j - \sum_{k \neq j} \nabla c_s(\ell_j; \ell_k) + O(\rho^0),
\end{align*}
\]

where \( \rho \) is considered to be an infinitesimally small parameter corresponding to the radius of a camphor pill. It should be noted that \( \nabla c_s(\ell_j; \ell_k) = \nabla c_s(\mathbf{r}; \ell_k)|_{\mathbf{r} = \ell_j} \).

By explicitly calculating \( \nabla c_s(\ell_j; \ell_k) \) and substituting the results into Eq. (16), the torque working on the \( j \)-th pill, \( \tau_j \), is obtained by taking the vector product of the radial vector and the force,

\[
\tau_j = \ell_j \times F_j. 
\]

Since the torques acting on pills are identical due to the geometric symmetry, we finally obtain the total torque \( \tau \) working on the rotor

\[
\tau = \sum_{j=0}^{N-1} \tau_j = N \tau_0. 
\]
Therefore the reduced equation for the time evolution of the angle describing disk position $\phi(t)$ reads:

\[
(\sigma + B(\ell, N)) \frac{d^2 \phi}{dt^2} = (A(\ell, N) - \kappa) \frac{d\phi}{dt} + C(\ell, N) \left( \frac{d\phi}{dt} \right)^3,
\]

where $A(\ell, N)$, $B(\ell, N)$, and $C(\ell, N)$ are given as

\[
A(\ell, N) = \frac{1}{4\pi} \left( -\gamma_{\text{Euler}} + \log \frac{2}{\rho} \right) + \sum_{j=1}^{N-1} \left[ -\ell K_1 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \frac{\sin^2 \left( \frac{2\pi j}{N} \right)}{2 \sin \left( \frac{\pi j}{N} \right)} \right] + K_0 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \cos \left( \frac{2\pi j}{N} \right),
\]

\[
B(\ell, N) = \frac{1}{16\pi} \left( 1 - \sum_{j=1}^{N-1} \left[ K_0 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \ell^2 \sin^2 \left( \frac{2\pi j}{N} \right) \right] - 2K_1 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \ell \sin \left( \frac{\pi j}{N} \right) \cos \left( \frac{2\pi j}{N} \right) \right),
\]

\[
C(\ell, N) = \frac{1}{192\pi} \left( -6\ell^2 - 4 \right) + \sum_{j=1}^{N-1} \left[ 12K_0 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \ell^4 \sin^2 \left( \frac{2\pi j}{N} \right) \cos \left( \frac{2\pi j}{N} \right) \right] - K_1 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \ell^2 \sin^4 \left( \frac{2\pi j}{N} \right) \sin \left( \frac{\pi j}{N} \right) + 4K_1 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \ell^3 \sin \left( \frac{\pi j}{N} \right) \times \left\{ -3 \cos^2 \left( \frac{2\pi j}{N} \right) + 4 \sin^2 \left( \frac{2\pi j}{N} \right) \right\} + 8K_2 \left( 2\ell \sin \left( \frac{\pi j}{N} \right) \right) \ell^2 \sin^2 \left( \frac{2\pi j}{N} \right) \cos \left( \frac{2\pi j}{N} \right) \right).
\]

where $\gamma_{\text{Euler}}$ is Euler-Mascheroni constant ($\approx 0.577$). In the process of calculation, we have neglected the term proportional to $d^3\phi/dt^3$. The detailed process of calculation is shown in Appendix A. It should be noted that $B(\ell, N)$ is always positive, while $A(\ell, N)$ and $C(\ell, N)$ change their sign depending on the parameters. The positive value of $B(\ell, N)$ means that the stationary solution and its stability does not change in the viscosity limit when $\sigma$ in Eq. (19) goes to zero.

At the bifurcation point, the coefficient of $d\phi/dt$ in the right side of Eq. (19) should be 0, i.e., $\kappa = A(\ell, N)$. The friction coefficient $\kappa$ depends on the water level, but in our experiments the water level was fixed so it can be regarded as a constant. The coefficient $A(\ell, N)$ is plotted against $N$ in Fig. 5(a) with $\ell = 2, 3, 4, \text{and} 5$. It is noted that the length scale is normalized with the characteristic length of the concentration field, which is related to the diffusion and sublimation of camphor molecules on water surface. The coefficient $A(\ell, N)$ monotonically decreases with an increase in $N$ for every $\ell$.

If $A(\ell, N)$ is smaller than $\kappa$, then the disk stops. Thus, monotonical decrease of $A(\ell, N)$ with the increase in $N$ means that there is a critical $N_0$ such that the disk stops for $N > N_0$.

The stable angular velocity $\omega(\ell, N)$ is given by $\omega(\ell, N) = \pm \sqrt{(A(\ell, N) - \kappa)/(-C(\ell, N))}$, if $A(\ell, N) - \kappa$ is positive and $C(\ell, N)$ is negative. Figure 5(c) shows the plot of $\omega$ against $N$ with $\ell = 2, 3, 4, \text{and} 5$. When the coefficient $C(\ell, N)$ is positive, then the bifurcation is subcritical one, and a higher order term (e.g. fifth-order term) is supposed to suppress the divergence of the angular velocity, which is out of scope of the present analysis.

VI. DISCUSSION AND CONCLUSION

In the paper we analyzed bifurcation between still and rotating states of a disk propelled by a number of camphor pills.

Experiments on the stationary velocity of a disk located on the water surface were motivation for presented study. The experiments revealed highly nonlinear behavior of the angular velocity $\omega(\ell, N)$ as the function of the disk size $\ell$ and of the number of pills propelling it $N$. We observed that for a moderate number of pills ($5 \leq N \leq 12$) the angular velocity was a decreasing function of the disk size: larger disks rotated slower than smaller ones. For a fixed disk radius the angular velocity $\omega(\ell, N)$ was a decreasing function of $N$. If the disk radius $R \geq 15$ mm then in the range $5 \leq N \leq 16$ the decrease in $\omega(\ell, N)$ was slow. However, if the disk radius was $R = 10$ mm ($\ell = 8.5$ mm) then $\omega(\ell, N)$ rapidly dropped to zero after the number of propelling pills exceeded a critical value $N_\tau = 13$. For $N > N_\tau$ the disk did not rotate.

We have investigated if the nonlinear behavior of angular velocity and the bifurcation between rotating and still disk can be reproduced by a mathematical model describing the dynamics of camphor combustion, its influence on the surface tension and interaction of floating objects through concentration dependent surface tension. The model we have used describes hydrodynamic effects and associated camphor transport via effective diffusion constant, that should be optimized for a particular system. It uses scaled variables selected such that the effective diffusion constant $D$ as well as the combined rates of camphor evaporation and dissolution in water are equal to 1. One of the adjustable model parameters is the friction coefficient $\kappa$. We have found that for $\kappa = 0.01$ numerical simulations give correct $N_\tau$ for the
ACKNOWLEDGMENTS

The authors acknowledge Professor S. Nakata for his helpful discussion. This work was supported by JSPS-PAN Bilateral Joint Research Program “Spatio-temporal patterns of elements driven by self-generated, geometrically constrained flows.” between Japan and the Polish Academy of Sciences. One of the authors (N.A.) has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734276. Another author (Y.K.) is grateful for the support within JSPS KAKENHI Grant Number JP17J05270 and the Cooperative Research Program of “Network Joint Research Center for Materials and Devices” No. 20181023.

Appendix A: Detailed calculation

In this section, we show the detailed derivation of Eq. (16) with Eqs. (20) to (22).

In Eq. (16), the last listed term represents the force originating from the camphor concentration released from the other pills, while the first four terms describe the force originating from the camphor concentration released from the considered pill itself. Here we set

$$F_j = F_{j}^{(\text{self})} + \sum_{k \neq j} F_{j,k}^{(\text{other})}, \quad (A1)$$

where

$$\frac{1}{\pi \rho^2} F_{j}^{(\text{self})} = \frac{1}{4\pi} \left( -\gamma + \log \frac{2}{\rho} \right) \dot{e}_j - \frac{1}{10\pi} \ddot{e}_j - \frac{1}{32\pi} \dot{e}_j^2 \dddot{e}_j + \frac{1}{48\pi} \dddot{e}_j + O(\rho^3), \quad (A2)$$

and

$$\frac{1}{\pi \rho^2} F_{j,k}^{(\text{other})} = -\nabla c_s(r_j; \ell_k). \quad (A3)$$

By explicitly calculating the gradient of $c_s(r; \ell)$ as

$$\nabla c_s(r; \ell) = \frac{\partial}{\partial r} c_s(r; \ell) = \frac{1}{2\pi} K'_0 \left( |r - \ell| \right) \frac{r - \ell}{|r - \ell|} - \frac{1}{4\pi} K_0 \left( |r - \ell| \right) \frac{\left( r - \ell \right) \cdot \dot{\ell}}{|r - \ell|} - \frac{1}{4\pi} K_0 \left( |r - \ell| \right) \left( \frac{r - \ell}{|r - \ell|} \right), \quad (A4)$$

we can obtain the explicit form of $F_{j,k}^{(\text{other})}$. Here we used the equality on modified Bessel function as

$$z K'_{\nu}(z) + \nu K_\nu(z) = -z K_{\nu+1}(z).$$

The torque $T_{j,\dot{e}}$ working on the $j$-th camphor pill orig-
\[
\frac{1}{\pi \rho^2} T_{j,j} = \lim_{r \to \ell_0} \left[ \frac{1}{4\pi} K_0(|r - \ell_0|) \left( \ell_0 \times \dot{\ell}_0 \right) - \frac{1}{16\pi} |r - \ell| K_1(|r - \ell_0|) \left( \ell_0 \times \dot{\ell}_0 \right) \right.
\]
\[
- \frac{1}{32\pi} |r - \ell_0| K_1(|r - \ell_0|) |\dot{\ell}_0|^2 \left( \ell_0 \times \dot{\ell}_0 \right) + \frac{1}{96\pi} |r - \ell_0|^2 K_2(|r - \ell_0|) (\ell_0 \times \ddot{\ell}_0) \left. \right]
\]
\[
= \frac{1}{4\pi} \left( -\gamma_{\text{Euler}} + \log \frac{2}{e} \right) \ell_0^2 \ddot{\phi} - \frac{1}{16\pi} \ell_0^2 \dddot{\phi} - \frac{1}{32\pi} \ell_0^4 \dddot{\phi}^3 + \frac{1}{48\pi} \ell_0^2 \ddot{\phi} \dot{\phi} \dot{\phi}. \quad (A6)
\]

Here we used (A5) \( \lim_{x \to +0} K_0(x) = -\gamma_{\text{Euler}} + \log(2/e) \), \( \lim_{x \to +0} x K_1(x) = 1 \), \( \lim_{x \to +0} x^2 K_2(x) = 2 \).

From Eq. (15), we only have to obtain \( T_0 \) considering the system symmetry, and it is calculated as:
\[
T_0 = \ell_0 \times F_0
\]
\[
= \ell_0 \times F_0^{(\text{self})} + \ell_0 \times \sum_{k=1}^{N-1} F^{(\text{other})}_{0,k}
\]
\[
= T_{0,0} + \sum_{k=1}^{N-1} T_{0,k}. \quad (A7)
\]

Therefore, \( T_{0,k} \) is obtained from Eq. (A5) as...
\[
\frac{1}{\pi \rho^2} T_{0,k} = - [\ell_0 \times \nabla c_s (\ell_0 ; \ell_k)] = \frac{1}{2 \pi} \mathcal{K}_0' (|\ell_0 - \ell_k|) \frac{\ell_0 \times \ell_k}{|\ell_0 - \ell_k|} - \frac{1}{4 \pi} \mathcal{K}_0' (|\ell_0 - \ell_k|) \left( \ell_0 \cdot \dot{\ell}_k \right) \frac{\ell_0 \times \ell_k}{|\ell_0 - \ell_k|} + \frac{1}{4 \pi} \mathcal{K}_0 (|\ell_0 - \ell_k|) \left( \ell_0 \times \dot{\ell}_k \right)
\]

\[
- \frac{1}{16 \pi} \mathcal{K}_0 (|\ell_0 - \ell_k|) \left[ \ell_0 - \ell_k \cdot \dot{\ell}_k \right] \left( \ell_0 \times \dot{\ell}_k \right) - \frac{1}{16 \pi} |\ell_0 - \ell_k| \mathcal{K}_1 (|\ell_0 - \ell_k|) \left( \ell_0 \times \dot{\ell}_k \right)
\]

\[
+ \frac{1}{16 \pi} |\ell_0 - \ell_k| \left( \ell_0 \cdot \dot{\ell}_k \right)^2 \left( \ell_0 \times \dot{\ell}_k \right) + \frac{1}{16 \pi} \mathcal{K}_0' (|\ell_0 - \ell_k|) \left( \ell_0 \cdot \dot{\ell}_k \right)^2 \frac{\ell_0 \times \ell_k}{|\ell_0 - \ell_k|}
\]

\[
- \frac{1}{8 \pi} \mathcal{K}_0 (|\ell_0 - \ell_k|) \left( \ell_0 \cdot \dot{\ell}_k \right) \left( \ell_0 \times \dot{\ell}_k \right) - \frac{1}{32 \pi} \mathcal{K}_0 (|\ell_0 - \ell_k|) |\ell_0 - \ell_k| \mathcal{K}_1 (|\ell_0 - \ell_k|) \left( \ell_0 \cdot \dot{\ell}_k \right) \left( \ell_0 \times \dot{\ell}_k \right)
\]

\[
- \frac{1}{32 \pi} |\ell_0 - \ell_k| \mathcal{K}_1 (|\ell_0 - \ell_k|) \left( \ell_0 \cdot \dot{\ell}_k \right) \left( \ell_0 \times \dot{\ell}_k \right) - \frac{1}{96 \pi} |\ell_0 - \ell_k| \mathcal{K}_2 (|\ell_0 - \ell_k|) \left( \ell_0 \times \dot{\ell}_k \right)
\]

\[
= \frac{1}{4 \pi} \mathcal{K}_0' \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \frac{\ell_0 \sin \left( \frac{\pi k}{N} \right)}{\left| \sin \left( \frac{\pi k}{N} \right) \right|} + \frac{1}{8 \pi} \mathcal{K}_0' \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell_0 \sin \left( \frac{2 \pi k}{N} \right) \sin \left( \phi \right)
\]

\[
+ \frac{1}{4 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^2 \cos \left( \frac{2 \pi k}{N} \right) \phi
\]

\[
+ \frac{1}{16 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^4 \sin \left( \frac{2 \pi k}{N} \right) \left( \phi^2 \left( \cos \left( \frac{2 \pi k}{N} \right) - 1 \right) + \phi \sin \left( \frac{2 \pi k}{N} \right) \right)
\]

\[
- \frac{1}{8 \pi} \mathcal{K}_1 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^3 \sin \left( \frac{\pi k}{N} \right) \left( \phi \cos \left( \frac{2 \pi k}{N} \right) - \phi^2 \sin \left( \frac{2 \pi k}{N} \right) \right)
\]

\[
+ \frac{1}{16 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^4 \sin \left( \frac{2 \pi k}{N} \right) \phi^2
\]

\[
+ \frac{1}{32 \pi} \mathcal{K}_2 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^5 \sin \left( \frac{2 \pi k}{N} \right) \phi^3 + \frac{1}{8 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^4 \sin \left( \frac{2 \pi k}{N} \right) \cos \left( \frac{2 \pi k}{N} \right) \phi^2
\]

\[
+ \frac{1}{32 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^6 \sin^2 \left( \frac{2 \pi k}{N} \right) \phi^3 - \frac{1}{16 \pi} \mathcal{K}_1 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^5 \sin \left( \frac{\pi k}{N} \right) \cos \left( \frac{2 \pi k}{N} \right) \phi^3
\]

\[
+ \frac{1}{192 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^7 \sin^4 \left( \frac{2 \pi k}{N} \right) \phi^3 + \frac{1}{32 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^6 \sin^2 \left( \frac{2 \pi k}{N} \right) \cos \left( \frac{2 \pi k}{N} \right) \phi^3
\]

\[
- \frac{1}{16 \pi} \mathcal{K}_1 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^5 \sin \left( \frac{\pi k}{N} \right) \sin \left( \frac{2 \pi k}{N} \right) \phi \phi
\]

\[
+ \frac{1}{32 \pi} \mathcal{K}_0 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^6 \sin^2 \left( \frac{2 \pi k}{N} \right) \left[ \phi^2 \left( \cos \left( \frac{2 \pi k}{N} \right) - 1 \right) + \phi \sin \left( \frac{2 \pi k}{N} \right) \right] \phi
\]

\[
- \frac{1}{16 \pi} \mathcal{K}_1 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^5 \sin \left( \frac{\pi k}{N} \right) \cos \left( \frac{2 \pi k}{N} \right) \phi^2 \left( \cos \left( \frac{2 \pi k}{N} \right) - 1 \right) + \phi \sin \left( \frac{2 \pi k}{N} \right) \phi
\]

\[
+ \frac{1}{16 \pi} \mathcal{K}_1 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^5 \sin \left( \frac{\pi k}{N} \right) \sin \left( \frac{2 \pi k}{N} \right) \phi^2 \sin \left( \frac{2 \pi k}{N} \right) - \phi \cos \left( \frac{2 \pi k}{N} \right) \phi
\]

\[
+ \frac{1}{48 \pi} \mathcal{K}_2 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^5 \sin \left( \frac{\pi k}{N} \right) \sin \left( \frac{2 \pi k}{N} \right) \left( \phi - \phi^3 \right) \sin \left( \frac{2 \pi k}{N} \right) + 3 \phi \phi \left( \cos \left( \frac{2 \pi k}{N} \right) - 1 \right)
\]

\[
+ \frac{1}{24 \pi} \mathcal{K}_2 \left( 2 \ell \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \ell^4 \sin^2 \left( \frac{\pi k}{N} \right) \left( \phi^2 \cos \left( \frac{2 \pi k}{N} \right) - 3 \phi \phi \sin \left( \frac{2 \pi k}{N} \right) \right)
\]
Reducing the number of terms we used:

\[
\begin{align*}
\sum_{k=1}^{N-1} f \left( \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \sin \left( \frac{2\pi k}{N} \right) &= 0, \quad (A10) \\
\sum_{k=1}^{N-1} f \left( \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \sin^3 \left( \frac{2\pi k}{N} \right) &= 0, \quad (A11) \\
\sum_{k=1}^{N-1} f \left( \left| \sin \left( \frac{\pi k}{N} \right) \right| \right) \sin \left( \frac{4\pi k}{N} \right) &= 0. \quad (A12)
\end{align*}
\]

Thus, the equation for the position of the disk reads:
\[
N \sigma \ell^2 \phi \\
= - \kappa N \ell^2 \phi + \frac{N \ell^2 \phi}{4\pi} \left( -\gamma + \log \frac{2}{\rho} + \sum_{k=1}^{N-1} \left[ -\ell K_1 \left( 2\ell \sin \left( \frac{\pi k}{N} \right) \right) \sin^2 \left( \frac{2\pi k}{N} \right) + K_0 \left( 2\ell \sin \left( \frac{\pi k}{N} \right) \right) \cos \left( \frac{2\pi k}{N} \right) \right] \right) \\
+ \frac{N \ell^2}{16\pi} \left( -1 + \sum_{k=1}^{N-1} \left[ \frac{\ell}{K_0} \left( \sin \left( \frac{\pi k}{N} \right) \right) \right] \ell^2 \sin^2 \left( \frac{2\pi k}{N} \right) - 2K_1 \left( 2\ell \sin \left( \frac{\pi k}{N} \right) \right) \right) \left[ \sin \left( \frac{\pi k}{N} \right) \cos \left( \frac{2\pi k}{N} \right) \right] \right) \\
+ \frac{N \ell^2}{192\pi} \left( -6 \ell^2 - 4 + \sum_{k=1}^{N-1} \left[ 12K_0 \left( 2\ell \sin \left( \frac{\pi k}{N} \right) \right) \right] \ell^4 \sin^2 \left( \frac{2\pi k}{N} \right) + \cos \left( \frac{2\pi k}{N} \right) - \ell K_1 \left( 2\ell \sin \left( \frac{\pi k}{N} \right) \right) \right) \left[ \sin \left( \frac{\pi k}{N} \right) \cos \left( \frac{2\pi k}{N} \right) \right] \right) \\
+ \left[ 4K_1 \left( 2\ell \sin \left( \frac{\pi k}{N} \right) \right) \right] \sin \left( \frac{\pi k}{N} \right) \left[ -3 \cos \left( \frac{2\pi k}{N} \right) + 4\sin \left( \frac{2\pi k}{N} \right) \right] \right) \phi_3. \quad (A13)
\]
The frictional force working on the $j$-th camphor pill is represented as $-\pi \rho^2 \kappa \ell_j$. Therefore the torque originating from this force is $-\pi \rho^2 \kappa \ell_j \times (d\ell_j/dt) = -\pi \rho^2 \kappa \ell_j^2 (d\phi/dt)$. By summing up the torque on $N$ particles, we obtain Eq. (6).