Class field theory for open curves over local fields

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1 Introduction

In this note, we investigate the class field theory for an open (=non proper) curve over a local field with arbitrary characteristic. Although a large number of studies have been made on such a theory for higher dimensional varieties over a $p$-adic field (e.g., [17], [8], [19] and [20]), little is known over a local field with positive characteristic. Precisely, let $k$ be a local field of characteristic $p \geq 0$. Here, a local field means a complete discrete valuation field with finite residue field. Let $U$ be a regular and geometrically connected curve over a local field $k$. Because one can find the smooth compactification $X$ of $U$ uniquely which contains $U$ as a dense open subvariety, we often say that $U$ is an open curve. A topological group $C(U)$ which is called the idèle class group, and the reciprocity homomorphism $\rho_U : C(U) \to \pi_1^{ab}(U)$ are introduced as in [7] (see Sect. 2). In this note, we show the following theorem.

**Theorem 1.1** (Thm. 4.6). The kernel $\text{Ker}(\rho_U)$ is the maximal divisible subgroup of $C(U)$.

On the Pontrjagin dual group $H^1(U, \mathbb{Q}/\mathbb{Z}) \cong \pi_1^{ab}(U)^\vee$, we also prove the following theorem.

**Theorem 1.2** (Thm. 4.1). The reciprocity homomorphism $\rho_U$ induces a surjective homomorphism $\rho_U^\vee : H^1(U, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow C(U)^\vee$ with kernel $\text{Ker}(\rho_U^\vee) \cong (\mathbb{Q}/\mathbb{Z})^r$ for some $r \geq 0$, where $C(U)^\vee$ is the group of all continuous homomorphisms $C(U) \to \mathbb{Q}/\mathbb{Z}$ of finite order.

Here, the invariant $r$ is the rank of the smooth compactification $X$ of $U$ ([17], Def. 2.5) which depends on the type of the reduction of $X$. For example, $r = 0$ if it has good reduction. In the terms of the fundamental groups we have

$$\pi_1^{ab}(U)/\text{Im}(\rho_U) \cong \hat{\mathbb{Z}}^r,$$

where $\text{Im}(\rho_U)$ is the topological closure of the image of $\rho_U$. This quotient $\pi_1^{ab}(U)/\text{Im}(\rho_U)$ classifies completely split coverings of $U$, that is, a finite abelian covering of $U$ in which any closed point $P \in U$ splits completely (cf. [17], II, Def. 2.1). As in the classical class field theory, the theorem above implies the following:

**Corollary 1.3.** (i) We have a one to one correspondence between the set of abelian étale coverings of $U$ which are not completely split and the one of finite index open subgroups of $C(U)$.

(ii) For an abelian étale covering $f : V \rightarrow U$ which is not completely split, the reciprocity homomorphism $\rho_U$ and the norm homomorphism $N_{V/U} : C(V) \rightarrow C(U)$ (will be introduced in Sect. 5) gives an isomorphism

$$C(U)/N_{V/U}C(V) \xrightarrow{\cong} \pi_1^{ab}(U)/f_\ast \pi_1^{ab}(V)$$

of finite abelian groups.
The theorems above are known when the curve $U$ itself is proper (17), (21) or the case of char($k$) = 0 (7). So our main interest is in the case of char($k$) = $p > 0$. The proof is essentially same as in the proof of the class field theory for curves over global fields due to K. Kato and S. Saito (13, Thm. 3, 14, Thm. 9.1). One of the reasons that their proof works here is the cohomological dimensions of the base fields. Since we divided the proof into two: the prime to $p$-part and the $p$-part, the proof is much simpler than the one for curves over global fields.

In Section 5, we also introduce the ray class field theory

$$\rho_{X,D} : C(X, D) \to \pi_1^{ab}(X, D)$$

which describes bounded ramification along a given Weil divisor $D$ on $X$ with support $|D| = X \setminus U$. Here, the fundamental group $\pi_1^{ab}(X, D)$ is introduced by using Kato’s ramification subgroup of the abelian Galois group of 2-dimensional local fields (see Sect. 5).

**Notations**

Throughout this paper, $k$ is a local field, namely a complete discrete valuation field with finite residue field. A curve over a field is an integral separated scheme of dimension 1 over the field. For a field $F$, we denote by $F^s$ the separable closure of $F$ and $F^{ab}$ denotes the maximal abelian extension of $F$.

For an abelian group $A$ and a positive integer $n$, we denote by $A/n$ the cokernel of the map $n : A \to A$ defined by $x \mapsto nx$.

For any topological abelian group $G$, we denote by $G^\vee$ the set of all continuous homomorphisms $G \to \mathbb{Q}/\mathbb{Z}$ of finite order, where $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology.

For a profinite group $G$ and a prime number $l$, we define the $l$-part of $G$ denoted by $G_{l}$ to be $\operatorname{lim}_{\leftarrow} N G/N$, where $N$ runs over all open normal subgroups of $G$ with $G/N$ is a $l$-group.

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**2 Local class field theory**

A 2-dimensional local field is a complete discrete valuation field whose residue field is a local field. In this section, we review some definitions and results of class field theory for 2-dimensional local fields following [9], [10]. Here we emphasize our attention to the case of positive characteristic. Let $K$ be a 2-dimensional local field of char($K$) = $p > 0$ with residue field $k$.

**2-dimensional local class field theory**

The class field theory of $K$ describes the abelian Galois group $G_K^{ab} = \operatorname{Gal}(K^{ab}/K)$ of $K$ by constructing a canonical continuous homomorphism called the reciprocity homomorphism

$$\rho_K : K_2(K) \to G_K^{ab},$$

where $K_2(K)$ is the Milnor $K$-group of $K$. The multiplicative group $K^\times$ and the Milnor $K$-group $K_2(K)$ have good topologies (introduced in [9], Sect. 7) and this makes $\rho_K$ continuous. We omit the detailed exposition on the definitions of these topologies. But under the topologies, the unit group $U_K = \theta_K^\circ$ is open in $K^\times$. The symbol map

$$K^\times \times K^\times \to K_2(K); \ (x, y) \mapsto \{x, y\}$$

and the norm map

$$N_L/K : K_2(L) \to K_2(K)$$

for a finite extension $L/K$ are continuous. Note also that any continuous homomorphism $K_2(K) \to \mathbb{Q}/\mathbb{Z}$ is automatically of finite order with respect to this topology (10, Sect. 3.5, Rem. 4).
For a field $F$ of characteristic $p > 0$ and $n \geq 0$, define
$$H^n(F, \mathbb{Z}/p^n \mathbb{Z}(n)) := H^{n-1}(F, W_i \mathbb{Q}_p),$$
where $W_i \mathbb{Q}_p$ is the Galois module defined by the étale sheaf of the logarithmic part of the de Rham-Witt complex. This Galois module plays the role of $\mathbb{Z}/m \mathbb{Z}(n) := \mu_m^\infty(\overline{F})$ for $m$ prime to $p$, where $\mu_m(\overline{F})$ is the Galois module of $m$-th roots of unity. We define (following [10], Sect. 3.2, Def. 1)
$$H^0(F) := \lim_{(m,p)=1} H^0(F, \mathbb{Z}/m \mathbb{Z}(-1)), \quad \text{and}
H^n(F) := \lim_{(m,p)=1} H^n(F, \mathbb{Z}/m \mathbb{Z}(n-1)) \oplus \lim_{r \to 1} H^n(F, \mathbb{Z}/p^n \mathbb{Z}(n-1)),
$$
where $m$ ranges over all integers $m$ which are prime to $p$ and $\mathbb{Z}/m \mathbb{Z}(-1) := \text{Hom}(\mu_m(\overline{F}), \mathbb{Q}/\mathbb{Z})$.

**Theorem 2.1** ([11], Thm. 3). For any prime number $l \neq p$, we have
$$H^1(K)[l] \cong H^1(k)[l] \oplus H^0(k)[l],$$
where $A[l]$ is the $l$-part of a torsion abelian group $A$.

**Theorem 2.2** ([10], Sect. 3.1, 3.5, [17], Chap. I, Th. 3.1). (i) The reciprocity homomorphism $\rho_K$ induces an isomorphism
$$\rho_K^\vee : H^1(K) = (G^\text{ab}_K)^\vee \overset{\sim}{\to} K_2(K)^\vee.$$

(ii) We have the following commutative diagram
$$\begin{array}{ccc}
K_2(K) & \overset{\rho_K}{\longrightarrow} & \text{Gal}(K^\text{ab}/K) \\
\downarrow{\partial} & & \downarrow{\text{restriction}} \\
K^\times & \overset{\rho_K}{\longrightarrow} & \text{Gal}(K^\text{ab}/k),
\end{array}$$
where $\partial : K_2(K) \to K^\times$ is the boundary homomorphism, and $\rho_K$ is the reciprocity homomorphism of $k$. In particular, an element $\chi \in H^1(K)$ is unramified, that is, the corresponding cyclic extension of $K$ is unramified if and only if $\rho_K^\vee(\chi)$ annihilates $U^n K_2(K)$.

From the above theorem, the correspondence $L \mapsto N_{L/K} K_2(L)$ gives a one to one correspondence between the set of finite abelian field extensions of $K$ and the one of finite index open subgroups of $K_2(K)$.

**Theorem 2.3** ([4], Thm. 4.5). $\ker(\rho_K) = \bigcap_{m \geq 1} m K_2(K)$ and is a divisible group.

**Ramification theory**

For $n \geq 1$, let $U^n = 1 + m^n$ be the higher unit groups of $K$, where $m$ is the maximal ideal of the valuation ring of $K$. Denote by $U^n K_2(K)$ the subgroup of $K_2(K)$ generated by the image of the symbol map
$$U^n \otimes K^\times \to K_2(K); u \otimes x \mapsto \{u, x\}.
$$
We also have an increasing filtration $\{ \text{fil}_n H^0(K) \}_{n \geq 0}$ on $H^0(K)$ ([12], Def. 2.1) with $H^0(K) = \bigcup_{n \geq 0} \text{fil}_n H^0(K)$.

**Proposition 2.4** ([12], Prop. 6.5, see also Rem. 6.6). Let $K$ be a 2-dimensional local field. For $\chi \in H^1(K)$, $\chi \in \text{fil}_n H^1(K)$ if and only if $\rho_K^\vee(\chi) \in K_2(K)^\vee$ annihilates $U^{n+1} K_2(K)$.

On the abelian Galois group $G^\text{ab}_K := \text{Gal}(K^\text{ab}/K)$, this filtration on $H^1(K)$ induces the ramification filtration $\{ G^\text{ab,n}_K \}_{n \geq 1}$ which is defined by
$$G^\text{ab,n}_K := (H^1(K) / \text{fil}_{n-1} H^1(K))^\vee = \ker(G^\text{ab}_K \to \text{fil}_{n-1} H^1(K)^\vee).$$

The above proposition implies that the reciprocity homomorphism $\rho_K$ induces a bijection
$$\rho_K^\vee : (G^\text{ab,n}_K)^\vee \overset{\sim}{\to} U^n K_2(K)^\vee.
$$
Since we are only considering abelian extensions, one adopt the Kato’s ramification subgroups. However, in our case of $\text{char}(K) = p > 0$, it is known that the filtration coincides with Abbes-Saito’s logarithmic version of ramification subgroups ([11]) which is defined on the absolute Galois group $G_K = \text{Gal}(\overline{K}/K)$ more generally ([2], Cor. 9.12).
3 Curves over local fields

Let $k$ be a local field of positive characteristic $p$. Let $U$ be a regular and geometrically connected curve over $k$ with function field $K = k(U)$ and $U_0$ the set of closed points in $U$. For a point $P \in U_0$, we denote by $k(P)$ the residue field at $P$ that is a finite extension of $k$, and $K_P$ the completion of $K$ at $P$. The field $K_P$ is known to be a 2-dimensional local field with residue field $k(P)$. By the desingularization at the point outside $U$ there exists uniquely a proper regular curve $X$ over $k$ which contains $U$ as a dense open subvariety. We put $U_\infty := X \setminus U$.

Idèle Class Groups

First we define a homomorphism

$$\partial : K_2(K) \to \bigoplus_{P \in U_0} k(P)^* \oplus \bigoplus_{P \in U_\infty} K_2(K_P)$$

as follows:

- For $P \in U_0$, the inclusion $\iota_P : K \hookrightarrow K_P$ induces $K_2(K) \to K_2(K_P)$.
- For $P \in U_\infty$, the boundary map $\partial_P : K_2(K_P) \to k(P)^*$.

The direct sum of these homomorphisms gives the required $\partial$.

**Definition 3.1.** The cokernel $C(U) : = \text{Coker}(\partial)$ of $\partial$ is called the idèle class group of $U$.

The idèle class group $C(U)$ is a quotient of the restricted product $\prod_{P \in X_0} K_2(K_P)$ with respect to the closed subgroup $\prod_{P \in X_0} K_2(K_P) = \text{Ker}(\partial_P : K_2(K_P) \to k(P)^*)$ which is a topological group induced from the topology on $K_2(K_P)$ (cf. Sect. 2). The idèle class group $C(U)$ is endowed with the quotient topology.

Next, as in Section 2 of [7], the 2-dimensional local class field theory $\rho_{K_P} : K_2(K_P) \to G_{K_P}^{ab} = \pi_1^{ab}(\text{Spec}(K_P))$ induces a homomorphism

$$\prod_{P \in X_0} K_2(K_P) \to G_{K_P}^{ab} \to \pi_1^{ab}(U).$$

It factors through $C(U)$ by the reciprocity law of $K$ ([7], Chap. II, Prop. 1.2). The induced homomorphism

$$\rho_U : C(U) \to \pi_1^{ab}(U)$$

is the reciprocity homomorphism of $U$. Furthermore, the norm map $N_{k[P]/k} : k(P)^* \to k^*$ for $P \in U_0$ and the composition $N_{k[P]/k} \circ \partial_P : K_2(K_P) \to k^*$ for $P \in U_\infty$ induce a homomorphism $N : C(U) \to k^*$. They make the following diagram commutative:

$$\begin{array}{ccc}
0 & \longrightarrow & C(U)^0 \\
\downarrow & & \downarrow \rho_U \\
0 & \longrightarrow & \pi_1^{ab}(U)^0
\end{array}$$

$$\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow \rho_U \\
\prod_{P \in X_0} K_2(K_P) & \longrightarrow & \pi_1^{ab}(U)
\end{array}$$

$$\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow \rho_U \\
G_{K_P}^{ab} & \longrightarrow & 0,
\end{array}$$

where $f : U \to \text{Spec}k$ is the structure morphism and the groups $C(U)^0$ and $\pi_1^{ab}(U)^0$ are defined by the exactness of the horizontal rows.

Norm homomorphisms

Here, we introduce the norm homomorphisms on the idèle class groups. Let $f : V \to U$ be a (finite) étale covering. This extends uniquely to the smooth compactifications $Y \to X$ which is also denoted by $f$. We also define $V_\infty : = Y \setminus V$.

**Definition 3.2.** Define the norm homomorphism

$$N_{V/U} : C(V) \to C(U)$$

for $f$ as follows: Let $L/K$ be the extension of function fields corresponding to $f : V \to U$. 

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• For $Q \in V_0$, $P = f(Q)$, we have the norm homomorphism $K_2(L_Q) \to K_2(K_P)$ of the Milnor $K$-groups.
• For $Q \in V_0$, $P = f(Q)$, we have the norm homomorphism $k(Q)^\times \to k(P)^\times$.

These give a homomorphism

$$
\bigoplus_{Q \in V_0} k(Q) \oplus \bigoplus_{Q \in V_0} K_2(L_Q) \to \bigoplus_{P \in U_0} k(P) \oplus \bigoplus_{P \in U_0} K_2(K_P).
$$

Since the norm $K_2(L) \to K_2(K)$ is compatible with above norms, we obtain $N_{V/U} : C(V) \to C(U)$.

Since the reciprocity homomorphisms of local class field theory commutes with norm homomorphisms, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
C(V) & \xrightarrow{N_{V/U}} & C(U) \\
\rho_U \downarrow & & \downarrow \rho_U \\
\pi_1^{ab}(V) & \xrightarrow{f_*} & \pi_1^{ab}(U).
\end{array}
$$

**Unramified class field theory**

The following theorem for the compactification $X$ is known due to Saito [17] and Yoshida [21] in which $C(X)$ is denoted by $SK_1(X)$.

**Theorem 3.3.** (i) $\text{Ker}(\rho_X)$ and $\text{Ker}(\rho_X^0)$ are the maximal divisible subgroups of $C(X)$ and $C(X)^0$ respectively.
(ii) $\pi_1^{ab}(X)/\text{Im}(\rho_X) \cong \hat{\mathbb{Z}}^\prime$ for some $r \geq 0$, where $\text{Im}(\rho_X)$ is the topological closure of the image $\text{Im}(\rho_X)$.
(iii) The image $\text{Im}(\rho_X^0)$ is finite.

In particular, the fundamental group $\pi_1^{ab}(X)^0$ is written as

$$
0 \to \pi_1^{ab}(X)^0_{\text{tor}} \to \pi_1^{ab}(X)^0 \to \hat{\mathbb{Z}}^\prime \to 0,
$$

where $\pi_1^{ab}(X)^0_{\text{tor}}$ is the torsion part of $\pi_1^{ab}(X)^0$ which is finite. If $X$ has a $k$-rational point, this extension splits. For any positive integer $m$, the reciprocity homomorphism $\rho_U$ of $U$ in general induces

$$
\rho_{U/m} : C(U)/m \to \pi_1^{ab}(U)/m.
$$

On the dual groups of $\rho_{X,m}$, Theorem 3.3 implies the following.

**Proposition 3.4.** For any prime $l$ (which may be $p$) and a positive integer $n$, $\rho_X^\vee : H^1(X, \mathbb{Z}/l^n\mathbb{Z}) \to (C(X)/l^n)^\vee$ is surjective.

**Proof.** By Theorem 3.3(i), $\rho_X^0$ induces an injection

$$
\rho_X^0 : C(X)^0/l^n \hookrightarrow \pi_1^{ab}(X)^0/l^n
$$

and the quotient $C(X)^0/l^n$ is a finite group (Thm. 3.3 (iii)). From the Pontrjagin duality theorem for profinite groups, we obtain the surjection

$$
(\rho_X^0)^\vee : (\pi_1^{ab}(X)^0/l^n)^\vee \twoheadrightarrow (C(X)^0/l^n)^\vee
$$

on the dual groups. Put $\overline{X} := X \otimes_k \overline{k}$. The spectral sequence $H^i(G_k, H^j(\overline{X}, \mathbb{Z}/l^n\mathbb{Z})) \Rightarrow H^{i+j}(X, \mathbb{Z}/l^n\mathbb{Z})$ gives an exact sequence

$$
0 \to \pi_1^{ab}(\overline{X})_{G_k} \to \pi_1^{ab}(X) \to G_k \to 0,
$$

where $\pi_1^{ab}(\overline{X})_{G_k}$ is the $G_k$-coinvariant module (cf. [17], Chap. II, Lem. 3.2). In particular, $\pi_1^{ab}(X)^0 \cong \pi_1^{ab}(\overline{X})_{G_k}$. The commutative diagram 2 induces the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \to & H^1(k, \mathbb{Z}/l^n\mathbb{Z}) & \to H^1(X, \mathbb{Z}/l^n\mathbb{Z}) & \to H^0(k, H^1(\overline{X}, \mathbb{Z}/l^n\mathbb{Z})) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & \\
(k^\times/l^n)^\vee & \to (C(X)/l^n)^\vee & \to (C(X)^0/l^n)^\vee & & & \\
\end{array}
$$

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By local class field theory, the left vertical map $\rho_{X,p}^\vee$ is bijective. Because of
$$H^0(k, H^1(\bar{X}, \mathbb{Z}/l^n\mathbb{Z})) \cong ((\pi^{ab}_1(\bar{X})/l^n)_{\text{et}})^\vee \cong (\pi^{ab}_1(X)/l^n)^\vee,$$
the right vertical map in the diagram above is surjective. Therefore, $\rho_{X,p}^\vee$ is surjective.

**Corollary 3.5.** The induced homomorphism
$$\rho_X^\vee : H^1(X, \mathbb{Q}/\mathbb{Z}) \to C(X)^\vee$$
from the reciprocity homomorphism $\rho_X$ satisfies the following:
(i) $\rho_X^\vee$ is surjective.
(ii) $\ker(\rho_X^\vee) \cong (\mathbb{Q}/\mathbb{Z})^r$ for some $r \geq 0$.

**Proof.** The assertion (ii) follows from Theorem 3.3 (ii). From Proposition 3.4, the reciprocity homomorphism $\rho_X$ induces the surjective homomorphism
$$\rho_{X,m}^\vee : (\pi^{ab}_1(X)/m)^\vee \to H^1(X, \mathbb{Z}/m\mathbb{Z}) \to (C(X)/m)^\vee$$
for any positive integer $m$. By taking the inductive limit, this gives
$$H^1(X, \mathbb{Q}/\mathbb{Z}) = \lim_m H^1(X, \mathbb{Z}/m\mathbb{Z}) \to \lim_m (C(X)/m)^\vee.$$

Take a character $\chi \in C(X)^\vee$. By the very definition of $C(X)^\vee$, the character $\chi$ has finite order (cf. Notations in Section 1). Hence there exists $m$ such that $\chi \in (C(X)/m)^\vee$ and thus $\rho_X^\vee$ is surjective.

## 4 Proof of the main theorems

We keep the notation of Section 3. In this section, we show the following theorem.

**Theorem 4.1.** Let $U$ be a regular and geometrically connected curve over a local field $k$ of characteristic $p > 0$. The induced homomorphism
$$\rho_U^\vee : H^1(U, \mathbb{Q}/\mathbb{Z}) \to C(U)^\vee$$
from the reciprocity homomorphism $\rho_U$ satisfies the following:
(i) $\rho_U^\vee$ is surjective.
(ii) $\ker(\rho_U^\vee) \cong (\mathbb{Q}/\mathbb{Z})^r$ for some $r \geq 0$.

First we determine $\ker(\rho_U^\vee)$ by comparing with $\ker(\rho_X^\vee)$ for the compactification $X$ of $U$.

**Proof of Thm. 4.1 (ii).** For any $P \in U_{\text{et}}$, put $Y := \text{Spec}(\Theta_{X,p}^\vee)$, where $\Theta_{X,p}^\vee$ is the completion of $\Theta_X^\vee$. The localization sequence of the étale cohomology groups on $i : P \hookrightarrow Y$ gives an exact sequence
$$0 \to H^1(Y, \mathbb{Q}/\mathbb{Z}) \to H^1(\text{Spec} K_p, \mathbb{Q}/\mathbb{Z}) \to H^2_p(Y, \mathbb{Q}/\mathbb{Z}) \to H^2(Y, \mathbb{Q}/\mathbb{Z}) \to \cdots.$$  

In terms of the Galois cohomology groups, we have
$$H^n(Y, \mathbb{Q}/\mathbb{Z}) \stackrel{\sim}{\longrightarrow} H^n(P, \mathbb{Q}/\mathbb{Z}) \cong H^n(k(P), \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad H^n(\text{Spec} K_p, \mathbb{Q}/\mathbb{Z}) \cong H^n(K_p, \mathbb{Q}/\mathbb{Z}).$$

The Tate duality theorem for local fields implies $H^2(k(P), \mathbb{Q}/\mathbb{Z}) = 0$. The excision theorem induces $H^2_p(Y, \mathbb{Q}/\mathbb{Z}) \cong H^2_p(X, \mathbb{Q}/\mathbb{Z})$ (cf. [10], Chap. III, Cor. 1.28) and we obtain the commutative diagram below:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1(k(P)) & \longrightarrow & H^1(K_p) & \longrightarrow & H^2_p(X, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow \rho_{\text{et}} & & \downarrow \rho_{\text{et}}^\vee & & \downarrow \rho_{\text{et}}^\vee & & \downarrow \rho_{\text{et}} & & \downarrow \rho_{\text{et}} \\
0 & \longrightarrow & (k(P)^\vee)^\vee & \longrightarrow & K_2(K_p)^\vee & \longrightarrow & \eta^0 K_2(K_p)^\vee & ,
\end{array}
$$
where \( \rho_{U} \) and \( \rho_{K_p} \) are the reciprocity homomorphisms of \( k(P) \) and \( K_P \) respectively (Thm. 2.2). Next we consider the following commutative diagram:

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & H^1(X, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^1(U, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \bigoplus_{p \in \mathcal{U}_o} H^0_p(X, \mathbb{Q}/\mathbb{Z}) \\
& & \downarrow \rho_X^U & & \downarrow \rho_U & & \downarrow \phi_{\mathcal{U}_o} \\
0 & \longrightarrow & C(X)^0 & \longrightarrow & C(U)^0 & \longrightarrow & \bigoplus_{p \in \mathcal{U}_o} U^0 K_2(K_P)^0. \\
\end{array}
\]

Here, the upper horizontal sequence is the localization sequence, and is exact. The diagram \((3)\) gives

\[ \text{Ker}(\rho_X^U) = \text{Ker}(\rho_U^U). \]

The former group is known to be isomorphic to \((\mathbb{Q}/\mathbb{Z})^r\) for some \( r \geq 0 \) (Cor. 3.5(ii)). So the assertion (ii) follows from this.

To determine the prime to \( p \)-part of \( \text{Ker}(\rho_U^U) \), we introduce some notations. For an abelian profinite group \( G \), we denote by \( G[p^r] \) the prime to \( p \)-part of \( G \). We denote by

\[
\begin{align*}
\rho_U^U : C(U)^0 & \twoheadrightarrow \pi_1^{ab}(U) \rightarrow \pi_1^{ab}(U)[p^r], \\
\rho_U^U : C(U)^0 & \twoheadrightarrow \pi_1^{ab}(U) \rightarrow \pi_1^{ab}(U)[p^r].
\end{align*}
\]

**Proposition 4.2.**

(i) \( \text{Ker}(\rho_U^U) \) and \( \text{Ker}(\rho_U^U) \) are l-divisible for any prime \( l \neq p \).

(ii) The image \( \text{Im}(\rho_U^U) \) is finite.

**Proof.** The proof is basically as the one in the case of curves over \( p \)-adic field (18). Here, we give a sketch of the proof.

(ii) We have the following diagram with exact rows:

\[
\begin{array}{cccccccccccc}
\bigoplus_{p \in \mathcal{U}_o} U^0 K_2(K_P) & \longrightarrow & C(U)^0 & \longrightarrow & C(X)^0 & \longrightarrow & 0 \\
& & \downarrow \rho_U^U & & \downarrow \rho_U^U & & \downarrow \phi_{\mathcal{U}_o} \\
0 & \longrightarrow & \text{Ker}(j_\ast) & \longrightarrow & \pi_1^{ab}(U)[p^r] & \longrightarrow & \pi_1^{ab}(X)^0[p^r] & \longrightarrow & 0,
\end{array}
\]

where \( j : U \hookrightarrow X \) is the open immersion. It is known that \( \text{Im}(\rho_U^U) \) is finite (Thm. 3.2 (iii)). Since we are considering the prime to \( p \)-part, the left vertical map factors through \( \oplus_p U^0 K_2(K_P) \rightarrow U^0 K_2(K_P) \). By Merkurjev's theorem (15, IX, Thm. 4.3), \( K_2(k(P)) \) is a sum of a finite group and a divisible subgroup. Since a profinite group contains no non-trivial divisible elements, the image of \( \rho_{K_p} : U^0 K_2(K_P) \twoheadrightarrow \text{Ker}(j_\ast) \) is finite. Hence the image of \( \rho_U^U \) is finite in \( \pi_1^{ab}(U)^0[p^r] \).

(i) For any positive integer \( m \) which is prime to \( p \), we have \( H^3(U, \mathbb{Z}/m\mathbb{Z}(2)) = H^3(X, j^! \mathbb{Z}/m\mathbb{Z}(2)) \), where \( \mathbb{Z}/m\mathbb{Z}(n) = \mu_{mn} \) and \( j : U \hookrightarrow X \) is the open immersion. We define a commutative diagram:

\[
\begin{array}{cccccccccccc}
K_2(K)/m \bigoplus_{p \in \mathcal{U}_o} k(P)^0/m & \longrightarrow & \bigoplus_{p \in \mathcal{U}_o} K_2(K_P)/m & \longrightarrow & C(U)/m & \longrightarrow & 0 \\
& & \downarrow \phi \downarrow k & & \downarrow h & & \\
H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) & \longrightarrow & \bigoplus_{p \in \mathcal{U}_o} H^3_p(X, j^! \mathbb{Z}/m\mathbb{Z}(2)) & \longrightarrow & H^3(X, j^! \mathbb{Z}/m\mathbb{Z}(2)).
\end{array}
\]

Here, the horizontal sequences are exact, and the left vertical map \( h^2 \) is bijective by the Merkurjev-Suslin theorem (15). The middle vertical map \( h \) is also bijective from the Kummer theory

\[ K_1(k(P))/m \xrightarrow{\sim} H^1(k(P), \mathbb{Z}/m\mathbb{Z}(1)) \cong H^3_\mathbb{Q}(X, j^! \mathbb{Z}/m\mathbb{Z}(2)) \]
for \( P \in U_0 \) and the Merkurjev-Suslin theorem again

\[
K_2(K_P/m) \overset{\sim}{\longrightarrow} H^2(K_P, \mathbb{Z}/m\mathbb{Z}(2)) \cong H^3(X, j_!(\mathbb{Z}/m\mathbb{Z}(2))
\]

for \( P \in U_\infty \). Thus, the induced homomorphism \( C(U)/m \to H^3(U, \mathbb{Z}/m\mathbb{Z}(2)) \) is injective. By the duality theorem \([13]\), we have \( \pi_1^{ab}(U)/m \cong H^2(U, \mathbb{Z}/m\mathbb{Z}(2)) \) so that the reciprocity homomorphism \( \rho_{U,m} : C(U)/m \to \pi_1^{ab}(U)/m \) is injective. Therefore,

\[
\text{Ker}(\rho^0_{U,l}) = \bigcup_{m > 0, (m,p) = 1} mC(U) \quad \text{and} \quad \text{Ker}(\rho^0_{U,l} \gamma) = \bigcup_{m > 0, (m,p) = 1} mC(U) \cap C(U)^0.
\]

The finiteness of the image of \( (\rho^0_{U,l}) \gamma \) implies that there exists a positive integer \( M \) which is prime to \( p \) such that \( \text{Ker}(\rho_{U,l}^0 \gamma) = MC(U) \cap C(U)^0 \) (see the proof of Thm. 5.1 in [17], for this argument). From this, \( \text{Ker}(\rho_{U,l}^0 \gamma) \) is \( l \)-divisible for any \( l \neq p \).

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker}(\rho^0_{U,l} \gamma) & \longrightarrow & \text{Ker}(\rho_{U,l}^0 \gamma) & \longrightarrow & k^	imes \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(U)^0 & \longrightarrow & \text{Ker}(\rho_{U,l}^0 \gamma) & \longrightarrow & C(U) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1^{ab}(U)^0[p'] & \longrightarrow & \pi_1^{ab}(U)[p'] & \longrightarrow & G_k^{ab}[p'] & \longrightarrow & 0.
\end{array}
\]

From local class field theory, \( \text{Ker}(\rho_{U,l}^0 \gamma) \) is \( l \)-divisible for any prime \( l \neq p \). The image of the norm \( N : C(U) \to k^\times \) is open and has finite index in \( k^\times \). Therefore, \( \text{Im}(N) \cap \text{Ker}(\rho_{U,l}^0 \gamma) = \text{Ker}(\rho_{U,l}^0 \gamma \vert_{\text{Im}(N)}) \) is \( l \)-divisible and we obtain the assertion.

**Proposition 4.3.** Let \( l \) be a prime with \( l \neq p \) and \( n \) a positive integer. The reciprocity homomorphism \( \rho_{U,lp} : C(U)/lp \to \pi_1^{ab}(U)/lp \) induces a surjection \( \rho_{U,lp}^0 : H^2(U, \mathbb{Z}/lp\mathbb{Z}) \to (C(U)/lp)^\gamma \).

**Proof.** From Proposition 4.2(i), \( \rho_{U,lp}^0 \) induces an injection

\[
\rho_{U,lp}^0 : C(U)^0/lp \hookrightarrow \pi_1^{ab}(U)^0/lp
\]

and the quotient \( C(U)^0/lp \) is a finite group (Prop. 4.2(ii)). Hence, we obtain a surjection

\[
(\rho_{U,lp}^0 \gamma)^0 : (\pi_1^{ab}(U)^0/lp)^\gamma \to (C(U)^0/lp)^\gamma
\]

on the dual groups.

If we assume \( U(k) \neq \emptyset \), then the short exact sequence

\[
0 \to \pi_1^{ab}(U)^0 \to \pi_1^{ab}(U) \to G_k^{ab} \to 0
\]

splits. Hence the upper sequence in the following diagram is exact:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(k, \mathbb{Z}/lp\mathbb{Z}) & \longrightarrow & H^1(U, \mathbb{Z}/lp\mathbb{Z}) & \longrightarrow & (\pi_1^{ab}(U)^0/lp)^\gamma & \longrightarrow & 0 \\
\text{ker} & \downarrow & \rho_{U,l}^0 & \downarrow & \rho_{U,l}^0 \gamma & \downarrow & \text{ker} \gamma & \downarrow & \rho_{U,l}^0 \gamma \\
(k^\times/lp)^\gamma & \overset{N^\gamma}{\longrightarrow} & (C(U)/lp)^\gamma & \overset{N^\gamma}{\longrightarrow} & (C(U)^0/lp)^\gamma & \overset{N^\gamma}{\longrightarrow} & (C(U)/lp)^\gamma & \overset{N^\gamma}{\longrightarrow} & (C(U)^0/lp)^\gamma
\end{array}
\]

where the left vertical homomorphism is induced by local class field theory and is bijective. Therefore, \( \rho_{U,lp}^0 \) is surjective.

In general, one can find a finite extension \( k'/k \) such that \( U(k') \neq \emptyset \). Put \( U' := U \otimes_k k' \) and \( f : U' \to U \) be the induced morphism. Since the norm homomorphism \( N := N_{U'/U} : C(U') \to C(U) \) is compatible with the base change, we have the following commutative diagram:

\[
\begin{array}{cccccc}
H^1(U, \mathbb{Z}/lp\mathbb{Z}) & \overset{f}{\longrightarrow} & H^1(U', \mathbb{Z}/lp\mathbb{Z}) \\
\rho_{U,l}^0 & \downarrow & \rho_{U',l}^0 & \downarrow & \rho_{U',lp}^0 \\
(C(U)/lp)^\gamma & \overset{N^\gamma}{\longrightarrow} & (C(U')/lp)^\gamma & \overset{N^\gamma}{\longrightarrow} & (C(U')/lp)^\gamma.
\end{array}
\]
To show the assertion we may assume that $G := \text{Gal}(k'/k)$ is a $l$-group. In fact, we denote by $g : V \to U$ corresponding to the $l$-Sylow subgroup of $G$. This induces $g_* : \pi_{1}^{ab}(V)/l^n \rightarrow \pi_{1}^{ab}(U)/l^n$. Furthermore, by induction on the extension degree of $k'/k$ we may assume that the Galois group $G$ is a cyclic group of the order $l$. From the spectral sequence $H^i(G, H^j(U', Z'/l^mZ')) \Rightarrow H^{i+j}(U, Z'/l^mZ)$, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & G^p & \to & H^1(U, Z/p^nZ) & \to & H^1(U', Z/p^nZ)^G & \to & 0 \\
0 & \to & \text{Ker}(N^p) & \to & (C(U)/l^n)^N & \to & (C(U')/l^n)^G & \to & 0
\end{array}
\]

(4)

Here, for a $G$-module $M$, let $M^G$ be the $G$-invariant submodule of $M$. From Lemma 4.4 below, the left vertical map is surjective. In fact, $\text{Ker}(N^p) \cong \{0\}$ or $G^p$. In either case, $\rho^p_{U'}$ is surjective by applying the 5-lemma to the above diagram. □

**Lemma 4.4.** For any prime $l$ (which may be $p$), let $k'/k$ be a Galois extension of degree $l$ with Galois group $G$ and put $U' := U \otimes_k k'$. Then, the left vertical map in the commutative diagram below is surjective:

\[
\begin{array}{ccccccccc}
0 & \to & G^p & \to & H^1(U, Q/Z) & \to & H^1(U', Q/Z) & \to & 0 \\
0 & \to & \text{Ker}(N^p) & \to & C(U)^N & \to & C(U')^N & \to & 0
\end{array}
\]

where $N := N_{U'/U} : C(U') \to C(U)$ is the norm homomorphism.

**Proof.** Let $\psi$ be an element of the kernel of $N^p$. It induces an element $\psi_p$ of $K_2(K_p)^p$ for each $P \in U_{\infty}$. Since $\psi_p$ is annihilated by the unramified extension $K_pk'$ of $K_p$, so that $\psi_p$ annihilates $U^nK_2(K_p)$ (Thm. 2.1 (iii)). Thus, the assertion is reduced to the case of $U = X$. From Corollary 3.5 (i), $\rho_X^p$ and $\rho_X^p$ are surjective.

Recall that the quotient $\pi_{1}^{ab}(X)/\text{Im}(\rho_X)$ classifies complete split abelian coverings of $X$. If the induced covering $f : X' := X \otimes_k k' \to X$ is completely split, then

\[\pi_{1}^{ab}(X')/\text{Im}(\rho_X) \to \pi_{1}^{ab}(X)/\text{Im}(\rho_X) \to G \to 0\]

is exact so that $N^p : C(X)^p \to C(X')^p$ is injective. On the other hand, if $f : X' \to X$ is non-completely split covering, then $f_* : \pi_{1}^{ab}(X')/\text{Im}(\rho_X) \to \pi_{1}^{ab}(X)/\text{Im}(\rho_X)$. The kernel of $N^p : C(X)^p \to C(X')^p$ is isomorphic to $G^p$. □

Next, we investigate the $p$-part of the reciprocity homomorphism $\rho_U$.

**Proposition 4.5.** For any positive integer $n$, $\rho^p_{U^n} : H^1(U, Z/p^nZ) \to (C(U)/p^n)^p$ is surjective.

**Proof.** Instead of using $Z/p^nZ$ with $Q/Z$ in (3), the localization sequence gives

\[
\begin{array}{ccccccccc}
0 & \to & H^1(X, Z/p^nZ) & \to & H^1(U, Z/p^nZ) & \to & \bigoplus_{P \in U_{\infty}} H^2_F(X, Z/p^nZ) & \to & H^2(X, Z/p^nZ) \\
0 & \to & (C(X)/p^n)^p & \to & (C(U)/p^n)^p & \to & \bigoplus_{P \in U_{\infty}} (U^nK_2(K_p)/p^n)^p & \to & (U^nK_2(K_p)/p^n)^p
\end{array}
\]

(5)
From Proposition 3.4, \( \rho_{U,p}^{\vee} \) is surjective. To show that \( \rho_{\psi \phi}^{\vee} \) is surjective, take \( \varphi \in (C(U)/p^n)^\vee \). Since the homomorphism \( i \) above comes from the inclusion \( U^0 K_2(K_p) \rightarrow K_2(K_p) \) for \( P \in U_{\infty} \), the 2-dimensional local class field theory implies \( \text{Im}(i) \subset \text{Im}(\iota) \). There exists \( \gamma \in \bigoplus_p H_2^1(X, \mathbb{Z}/p^n \mathbb{Z}) \) such that \( i(\gamma) = (\iota \psi \phi)(\gamma) \). From \( H_2^1(X \otimes \overline{K}, \mathbb{Z}/p^n \mathbb{Z}) = 0 \), the image of \( j(\gamma) \) by the homomorphism \( H_2^1(X, \mathbb{Z}/p^n \mathbb{Z}) \rightarrow H_2^1(X \otimes \overline{K}, \mathbb{Z}/p^n \mathbb{Z}) \) becomes zero for some finite Galois extension \( K \) of \( k \). Put \( U' := U \otimes K \) and let \( f : U' \rightarrow U \) be the induced morphism. Since the norm homomorphism \( \mathcal{N} := \mathcal{N}_U : C(U') \rightarrow C(U) \) is compatible with the base change, we have the following commutative diagram:

\[
\begin{array}{ccc}
H^1(U, \mathbb{Z}/p^n \mathbb{Z}) & \xrightarrow{\rho_{U,p}^{\vee}} & H^1(U', \mathbb{Z}/p^n \mathbb{Z}) \\
\downarrow \rho_{U,p} & & \downarrow \rho_{U',p}^{\vee} \\
(C(U)/p^n)^\vee & \xrightarrow{\mathcal{N}^\vee} & (C(U')/p^n)^\vee
\end{array}
\]

Thus, there exists \( \chi' \in H^1(U', \mathbb{Z}/p^n \mathbb{Z}) \) such that \( \varphi' := \mathcal{N}^\vee(\varphi) = \rho_{U',p}^{\vee}(\chi') \) in \( (C(U')/p^n)^\vee \). We prove that \( \varphi \) comes from \( H^1(U, \mathbb{Z}/p^n \mathbb{Z}) \). To this we may assume that \( G := \text{Gal}(k'/k) \) is a \( p \)-group by considering the \( p \)-Sylow subgroup of \( G \). Furthermore, by induction on the extension degree of \( k'/k \) we may assume that the Galois group \( G \) is a cyclic group of the order \( p \). We have the following commutative diagram:

\[
\begin{array}{cccc}
0 & \xrightarrow{} & G^\vee & \xrightarrow{} & H^1(U, \mathbb{Z}/p^n \mathbb{Z}) & \xrightarrow{\rho_{U,p}^{\vee}} & H^1(U', \mathbb{Z}/p^n \mathbb{Z})^G & \xrightarrow{} & 0 \\
& & \downarrow \rho_{U,p} & & \downarrow \rho_{U',p}^{\vee} & & \downarrow \rho_{U',p}^{\vee} & & \\
0 & \xrightarrow{} & \text{Ker}(\mathcal{N})^\vee & \xrightarrow{} & (C(U)/p^n)^\vee & \xrightarrow{\mathcal{N}^\vee} & (C(U')/p^n)^\vee & \xrightarrow{G} & 0
\end{array}
\]

The upper sequence comes from the spectral sequence \( H^p(G, H^q(U', \mathbb{Z}/p^n \mathbb{Z})) \Rightarrow H^{p+q}(U, \mathbb{Z}/p^n \mathbb{Z}) \) and is exact. Since \( \varphi' \) is fixed by \( G \), \( \chi' \) is also fixed by \( G \). From Lemma 4.3, we have \( \text{Ker}(\mathcal{N}^\vee) = \{ 0 \} \) or \( G^\vee \). In either case, there exists \( \chi \in H^1(U, \mathbb{Z}/p^n \mathbb{Z}) \) such that \( \rho_{U,p}^{\vee}(\chi) = \varphi \) from the diagram above. Therefore, \( \rho_{U,p}^{\vee} \) is surjective.

Now, we complete the proof of Theorem 4.4.

**Proof of Thm. 4.4 (i).** From Proposition 4.3 and 4.5, the reciprocity homomorphism \( \rho_U \) induces the surjective homomorphism

\[
\rho_{U,m}^{\vee} : (\pi_1^{ab}(U)/m)^\vee = H^1(U, \mathbb{Z}/m \mathbb{Z}) \rightarrow (C(U)/m)^\vee
\]

for any positive integer \( m \). This gives

\[
H^1(U, \mathbb{Q}/\mathbb{Z}) = \lim_{m \rightarrow \infty} H^1(U, \mathbb{Z}/m \mathbb{Z}) \rightarrow \lim_{m \rightarrow \infty} (C(U)/m)^\vee.
\]

For any \( \chi \in (C(U)/m)^\vee \), there exists \( m \) such that \( \chi \in (C(U)/m)^\vee \) since \( \chi \) has finite order. Hence, \( \rho_U^{\vee} \) is surjective.

Finally, we show the following theorem on the kernel of \( \rho_U \).

**Theorem 4.6.** The kernel \( \text{Ker}(\rho_U) \) is the maximal divisible subgroup of \( C(U) \).

**Proof.** From Proposition 4.2 it is enough to show that \( \text{ker}(\rho_U) \) is \( p \)-divisible. The short exact sequence

\[
\bigoplus_{P \in U_{\infty}} U^0 K_2(K_P) \rightarrow C(U) \rightarrow C(X) \rightarrow 0
\]

induces a short exact sequence

\[
\bigoplus_{P \in U_{\infty}} \text{Ker}(\rho_{K_P}^0) \rightarrow \text{Ker}(\rho_U) \rightarrow \text{Ker}(\rho_X) \rightarrow 0,
\]

10
where \( \rho^0_{K_P} := \rho_{K_P}|_{U^0K_2(K_P)} \) is the restriction of \( \rho_{K_P} \). Since \( \text{Ker}(\rho_X) \) is \( p \)-divisible (Thm. 5.3(i)), it is enough to show that \( \text{Ker}(\rho^0_{K_P}) \) is also \( p \)-divisible for each \( P \in U_{\infty} \).

For \( P \in U_{\infty} \), it is also known that \( \text{Ker}(\rho_{K_P}) \) is \( p \)-divisible (Thm. 2.3). For any \( \xi \in \text{Ker}(\rho^0_{K_P}) \), there exists \( \eta \in \text{Ker}(\rho_{K_P}) \) such that \( \xi = p\eta \). We have to show \( \eta \in \text{Ker}(\rho^0_{K_P}) = U^0K_2(K_P) \cap \text{Ker}(\rho_{K_P}) \). Recall that \( U^0K_2(K_P) := \text{Ker}(\partial_P) \) and \( \partial_P : K_2(K_P) \to k(P)^{\times} \) is the boundary map. By \( \eta \in U^0K_2(K_P) = \text{Ker}(\partial_P) \), we have \( 1 = \partial_P(\xi) = \partial_P(p\eta) = \partial_P(\eta)^p \) in \( k(P)^{\times} \). Since \( k(P)^{\times} \) contains no non-trivial \( p \)-torsion elements, we have \( \partial_P(\eta) = 1 \). In particular, \( \eta \in U^0K_2(K_P) \) and thus \( \text{Ker}(\rho^0_{K_P}) \) is \( p \)-divisible. \( \square \)

### 5 Restricted Ramification

In this section, we study the abelian coverings of \( U \) restricting the ramification along a given divisor as a modulus. Let \( D = \sum_{P \in U_{\infty}} m_P P > 0 \) be an effective Weil divisor on \( X \) with support \( |D| = U_{\infty} = X \setminus U \). Considering \( D \) as a modulus, we define the fundamental group \( \pi_1(X, D)^{ab} \) with bounded ramification by:

\[
\pi_1^{ab}(X, D) := \text{Coker} \left( \bigoplus_{P \in U_{\infty}} G^{ab, m_P}_{K_P} \rightarrow \pi_1^{ab}(U) \right),
\]

where \( G^{ab, m_P}_{K_P} \) is the ramification subgroup of \( G^{ab}_{K_P} := \text{Gal}(K^{ab}_{K_P}/K_P) \) in the upper numbering (Sect. 2). The fundamental group \( \pi_1(X, D) \) can be defined by constructing some Galois category of coverings of \( X \) with restricted ramification along \( D \) (cf. 6). The information on the ramification of 2-dimensional local field \( K_P \) at \( P \in U_{\infty} \) is related to the natural filtration \( U^mK_2(K_P) \) (Prop. 2.4). Corresponding to this fundamental group, we define the idele class group of \( U \) with modulus \( D \) by:

\[
C(X, D) := \text{Coker} \left( \bigoplus_{P \in U_{\infty}} U^mK_2(K_P) \rightarrow C(U) \right).
\]

The 2-dimensional class field theory (Prop. 2.4) gives the reciprocity homomorphism

\[
\rho_{XD} : C(X, D) \rightarrow \pi_1(X, D)^{ab}.
\]

Furthermore, the norm maps \( N_{k(P)/k} : k(P)^{\times} \rightarrow k^{\times} \) induce \( N : C(X, D) \rightarrow k^{\times} \) and the following diagram commutative:

\[
\begin{array}{cccccc}
0 & \rightarrow & C(X, D)^0 & \rightarrow & C(X, D) & \rightarrow & N \rightarrow & k^{\times} \\
& & \uparrow{\rho^0_{XD}} & & \uparrow{\rho_{XD}} & & \uparrow{\rho} & \\
0 & \rightarrow & \pi_1^{ab}(X, D)^0 & \rightarrow & \pi_1^{ab}(X, D) & \rightarrow & \pi_1^{ab}(\text{Spec}(k)) & \rightarrow & 0.
\end{array}
\]

Here, the groups \( C(X, D)^0 \) and \( \pi_1^{ab}(X, D)^0 \) are defined by the exactness of the horizontal rows.

**Theorem 5.1.** (i) \( \rho_{XD}^\vee : \pi_1^{ab}(X, D)^\vee \rightarrow C(X, D)^\vee \) is surjective.

(ii) \( \text{Ker}(\rho_{XD}^\vee) = (\mathbb{Q}/\mathbb{Z})^\vee \).

(iii) \( \text{Ker}(\rho_{XD}) \) is the maximal divisible subgroup of \( C(X, D) \).

**Proof.** (i) By the very definitions, we have

\[
\begin{array}{cccccc}
0 & \rightarrow & \pi_1^{ab}(X, D)^\vee & \rightarrow & H^1(U, \mathbb{Q}/\mathbb{Z}) & \rightarrow & \bigoplus_{P \in U_{\infty}} (C^{ab, m_P}_{K_P})^\vee \\
& & & & & & = \oplus \mathcal{P}_{K_P} \\
0 & \rightarrow & C(X, D)^\vee & \rightarrow & C(U)^\vee & \rightarrow & \bigoplus_{P \in U_{\infty}} U^mK_2(K_P)^\vee.
\end{array}
\]

The right vertical map is bijective (Prop. 2.4) and \( \rho_U^\vee \) is surjective (Thm. 4.1), so that \( \rho_{XD}^\vee \) is surjective.
(ii) From the surjections
\[ \pi_1^{ab}(U)/\text{Im}(\rho_U) \to \pi_1^{ab}(X, D)/\text{Im}(\rho_{X,D}) \to \pi_1^{ab}(X)/\text{Im}(\rho_X), \]
we obtain \( \pi_1^{ab}(X, D)/\text{Im}(\rho_{X,D}) \cong \hat{\ZZ} \).

(iii) The surjection \( C(U) \to C(X, D) \) gives a surjection \( \ker(\rho_U) \to \ker(\rho_{X,D}) \). Theorem 4.6 implies the assertion. \( \square \)

As in the proof of Theorem 4.2 we obtain the following lemma:

**Lemma 5.2.** If \( m_P = 1 \) for all \( P \in |D| \), then the image of \( \rho^{0}_{X,D} \) is finite.

For a general modulus \( D = \sum_P m_P P > 0 \), we have the following commutative diagram:

\[
\begin{array}{cccc}
\bigoplus_{P \in |U|} U^{m_P} K_2(K_P)/U^{m_P+1} K_2(K_P) & \to & C(X, D + P)^0 & \to & C(X, D)^0 & \to & 0 \\
\downarrow \rho^{m_P}_{X,D} & & \downarrow \rho^{0}_{X,D} & & \downarrow \rho^{0}_{X,D} & & \downarrow \rho^{0}_{X,D} \\
\bigoplus_{P \in |U|} \pi_1^{ab}(X, D + P)^0 & \to & \pi_1^{ab}(X, D)^0 & \to & 0 \\
\end{array}
\]

The graded quotients of the filtration \( \{U^{m} K_2(K_P)\} \) are written in terms of the absolute Kähler differentials of the residue field \( k(P) \) (cf. [3]).

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