FUNCTIONS CONCERNED WITH DIVISORS OF ORDER $r$

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Abstract. N. Minculete has introduced a concept of divisors of order $r$: integer $d = p_1^{b_1} \cdots p_k^{b_k}$ is called a divisor of order $r$ of number $n = p_1^{a_1} \cdots p_k^{a_k}$ if $d \mid n$ and $b_j \in \{r, a_j\}$ for $j = 1, \ldots, k$. One can consider respective divisor function $\tau^{(r)}$ and sum-of-divisors function $\sigma^{(r)}$.

In the present paper we investigate the asymptotic behaviour of

$$\sum_{n \leq x} \tau^{(r)}(n) \quad \text{and} \quad \sum_{n \leq x} \sigma^{(r)}(n)$$

and improve several results of [10] and [11]. We also provide conditional estimates under Riemann hypothesis.

1. Introduction

Recently N. Minculete in his PhD Thesis [10], devoted to the functions using exponential divisors, and in further paper [11] introduced a concept of divisors of order $r$: integer $d = p_1^{b_1} \cdots p_k^{b_k}$ is called a divisor of order $r$ of number $n = p_1^{a_1} \cdots p_k^{a_k}$ if $d \mid n$ in the usual sense and $b_j \in \{r, a_j\}$ for $j = 1, \ldots, k$. We also suppose that $1$ is a divisor of any order of itself (but not of any other number). Let us denote respective divisor and sum-of-divisor functions as $\tau^{(r)}$ and $\sigma^{(r)}$. These functions are multiplicative and

$$\tau^{(r)}(p^a) = \begin{cases} 1, & a \leq r, \\ 2, & a > r. \end{cases}$$

$$\sigma^{(r)}(p^a) = \begin{cases} p^a, & a \leq r, \\ p^a + p^r, & a > r. \end{cases}$$

In a special case of $r = 0$ we get well-studied unitary divisors. For example, it was proved in [3] that

$$\sum_{n \leq x} \tau^{(0)}(n) = \frac{x}{\zeta(2)} \left( \log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + O(x^{1/2}),$$

(under Riemann hypothesis error term is $O(x^{22/608+\varepsilon})$ due to [7]) and in [14] it was proved that

$$\sum_{n \leq x} \sigma^{(0)}(n) = \frac{x^2}{12 \zeta(3)} + O(x \log^{5/3} x).$$

In another special case of $r = 1$ we get so-called by Minculete exponential semiproper divisors and denote $\tau^{(1)} := \tau^{(1)}$, $\sigma^{(1)} := \sigma^{(1)}$. An integer $d$ is an exponential semiproper divisor of $n$ if $\ker d = \ker n$ and $(d/\ker n, n/d) = 1$, where $\ker n = \prod_{p \mid n} p$.

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Minculete proved in [10, (3.1.17–19)] that
\[
\limsup_{n \to \infty} \frac{\log \tau(r)(n) \log \log n}{\log n} = \frac{\log 2}{r + 1},
\]
\[
\sum_{n \leq x} \tau(r)(n) = \frac{\zeta(r + 1)}{\zeta(2r + 2)}x + Ax^{1/(r + 1)} + O(x^{1/(r + 2) + \varepsilon}),
\]
\[
\limsup_{n \to \infty} \frac{\sigma(r)(n)}{n \log \log n} = \frac{6e\gamma}{\pi^2}.
\]

In the present paper we improve the error term in (6) and establish asymptotic formulas for \(\sum_{n \leq x} \sigma(r)(n)\) with \(O\)- and \(\Omega\)-estimates of the error term.

2. Notation

In asymptotic relations we use \(\sim\), \(\asymp\), Landau symbols \(O\) and \(o\), big omegas \(\Omega\) and \(\Omega_{\pm}\), Vinogradov symbols \(\ll\) and \(\gg\) in their usual meanings. All asymptotic relations are given as an argument tends to the infinity.

Letter \(p\) with or without indexes denote rational prime.

As usual \(\zeta(s)\) is Riemann zeta-function. For complex \(s\) we denote \(\sigma := \Re s\) and \(t := \Im s\).

We use abbreviations \(\log x := \log \log x\), \(\ll\log x := \log \log \log x\).

Letter \(\gamma\) denotes Euler–Mascheroni constant, \(\gamma \approx 0.577\).

Everywhere \(\varepsilon > 0\) is an arbitrarily small number (not always the same even in one equation).

We write \(f \ast g\) for Dirichlet convolution: 
\[
(f \ast g)(n) = \sum_{d \mid n} f(d)g(n/d).
\]

Function \(\ker\): \(\mathbb{N} \to \mathbb{N}\) stands for \(\ker n = \prod_{p \mid n} p\).

For a set \(A\) notation \(#A\) means the cardinality of \(A\).

3. Preliminary estimates

Consider
\[
\tau(a, b; n) = \sum_{k+l=n, \gcd(k, l) = 1} 1, \quad T(a, b; x) = \sum_{n \leq x} \tau(a, b; n), \quad 1 \leq a < b.
\]

One can directly check that
\[
\sum_{n=1}^{\infty} \frac{\tau(a, b; n)}{n^s} = \zeta(as)\zeta(bs), \quad \sigma > 1
\]

Lemma 1.

\[
T(a, b; x) = H(a, b; x) + \Delta(a, b; x)
\]

where
\[
H(a, b; x) = \begin{cases} 
\frac{\chi(b/a)x^{1/a} + \chi(a/b)x^{1/b}}{x^{1/a} \log x + (2\gamma - 1)x^{1/a}}, & 1 \leq a < b, \\
\frac{\chi(a/b)x^{1/b}}{x^{1/a}}, & a = b,
\end{cases}
\]

and
\[
x^{1/2(a+b)} \ll \Delta(a, b; x) \ll \begin{cases} 
x^{1/(2a+b)} & 1 \leq a < b, \\
x^{1/3a}\log x & a = b.
\end{cases}
\]

Proof. See [3] Th. 5.1, Th. 5.3, Th. 5.8.

In fact \(\Delta(a, b; x)\) can be estimated more precisely. For our goals we are primarily interested in the behaviour of \(\Delta(1, b; x)\). Let us suppose that
\[
\Delta(1, b; x) \ll x^\theta \log^\varphi x,
\]
then due to [8, Th. 5.11] we can choose
\[ \theta_b = \frac{1}{b + 7/2}, \quad \theta'_b = 1, \quad b \geq 7. \]

Estimates for \( b \leq 16 \) are given in Table 1. Estimate for \( b = 1 \) belongs to Huxley [5], and estimate for \( b = 2 \) belongs to Graham and Kolesnik [4]. We have found no references on the best known results for \( b \geq 3 \), so we calculated them with the use of [8, Th. 5.11, Th. 5.12] selecting appropriate exponent pairs carefully. It seems that some of these estimates may be new.

**Lemma 2.** Let \( \alpha \) and \( \beta \) be positive real numbers with \( \beta + 1 \leq \alpha \). Then
\[
\sum_{mn^\beta} \frac{\zeta(2\alpha - \beta)}{2} x^2 + D(\alpha, \beta; x), \quad D(\alpha, \beta; x) \ll \begin{cases} 
\frac{x \log^{2/3} x}{x}, & \beta + 1 = \alpha, \\
\frac{\varepsilon}{x}, & \beta + 1 < \alpha.
\end{cases}
\]

**Proof.** See [13, Th. 1].

For \( k > 0 \) one can define a multiplicative function \( \mu_k \) implicitly by
\[
\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)}, \quad \sigma > 1.
\]

| \( b \) | \( \theta_b \) | \( \theta'_b \) | Exponent pair or reference |
|------|----------|-------------|---------------------------|
| 1    | 131/416 + \varepsilon \approx 0.314904 | 0 | [9] |
| 2    | 1057/4789 + \varepsilon \approx 0.220899 | 0 | [4] |
| 3    | 1486/8647 + \varepsilon \approx 0.171852 | 0 | AB(AB)^3H |
| 4    | 1448/10331 + \varepsilon \approx 0.140161 | 0 | AH |
| 5    | 71318556275/587475333596 + \varepsilon \approx 0.121398 | 0 | (A^2B)^2(AB)^2A^3(AB)^2BA^3BH |
| 6    | 669/6305 + 106106 | 0 | (A^2B)^4(AB)^3A^4BI |
| 7    | 338866613/3586241504 + \varepsilon \approx 0.094491 | 0 | A^2B^3(AB)^2A^3B^6AB(A^2B^2)^2ABI |
| 8    | 2000836147/23452762172 + \varepsilon \approx 0.085314 | 0 | (AB)^3(A^3B)^6A^3BI |
| 9    | 372854099/4786779707 + \varepsilon \approx 0.077892 | 0 | (AB)^3(A^2B)^3BA^2B^6ABI |
| 10   | 150509/2096993 + \varepsilon \approx 0.071774 | 0 | (A^2B)^3ABH |
| 11   | 1048/15811 + \varepsilon \approx 0.066283 | 0 | A^3H |
| 12   | 64/1037 + \varepsilon \approx 0.061716 | 0 | A^3H |
| 13   | 2516635/43324033 + \varepsilon \approx 0.058089 | 0 | A^3BA^2BA^2B(AB)^2H |
| 14   | 75/1373 + \varepsilon \approx 0.054625 | 0 | A^2(AB)^2A^2BI |
| 15   | 13514730527/262064292044 + \varepsilon \approx 0.051570 | 0 | A(A^2B)^3A^4B^7A^3BB^4A^4BH |
| 16   | 15/307 + \varepsilon \approx 0.048860 | 0 | A^3BA^2BA^2BI |

Table 1. Values of \( \theta_b \) and \( \theta'_b \) in [8] for \( b \leq 16 \). Exponent pairs are written in terms of \( A \)- and \( B \)-processes [8, Th. 2.12, 2.13]. We abbreviate \( B := BA \). Here \( I = (0, 1) \) and \( H = (32/205 + \varepsilon, 269/410 + \varepsilon) \) is Huxley exponent pair from [5].
Lemma 4. Let \( \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x^{1/k}} \mu(n) \ll x^{1/k} \exp(-CN(x)), \)

where \( C > 0, N(x) = \log^{3/5} x \log^{-1/5} x. \) See [6, Th. 12.7] for the proof of the last estimate. Assuming Riemann hypothesis (RH) we get much better result

\[
M_k(x) \ll x^{1/2k+\varepsilon} \quad \text{[15, Th. 14.25 (C)]}.
\]

Lemma 3. Let \( K \in \mathbb{N}, J \in \mathbb{N} \cup \{0\}, m_1 \leq \cdots \leq m_K, n_1 \leq \cdots \leq n_J, \) where all \( m_k, n_j \in \mathbb{N}, \) and suppose that

\[
\sum_{n=1}^{\infty} a(n) n^{\sigma} = \frac{\zeta(m_1 s) \cdots \zeta(m_K s)}{\zeta(n_1 s) \cdots \zeta(n_J s)}.
\]

Let

\[
\alpha = \frac{K-1}{2 \sum_{k=1}^{K} m_k}.
\]

If \( 1/\alpha < 2n_j \) for all \( j = 1, \ldots, J \) then for arbitrary \( H(x) \) of the form

\[
H(x) = \sum_{i=1}^{t} x^{\beta_i} P_i(\log x), \quad \beta_i \in \mathbb{C}, \quad \alpha < \Re \beta_i \leq 1, \quad P_i \text{ are polynomials},
\]

we have

\[
\sum_{n \leq x} a_n = H(x) + \Omega(x^\alpha).
\]

Proof. This is a simplified version of [9, Th. 2].

4. ASYMPTOTIC PROPERTIES OF \( \sum \tau^{(r)}(n) \)

Lemma 4. Let \( F_r(s) \) be Dirichlet series for \( \tau^{(r)}; \)

\[
F_r(s) := \sum_{n=1}^{\infty} \frac{\tau^{(r)}(n)}{n^s}.
\]

Then

\[
F_r(s) = \frac{\zeta(s) \zeta((r+1)s)}{\zeta((2r+2)s)}, \quad \sigma > 1.
\]

Proof. Let us transform Bell series for \( \tau^{(r)}; \)

\[
\tau^{(r)}_p(x) = \sum_{k=0}^{\infty} \tau^{(r)}(p^k) x^k = \sum_{k=0}^{r} x^k + 2 \sum_{k>r} x^k = \sum_{k=0}^{\infty} x^k + \sum_{k>r} x^k =
\]

\[
= (1 + x^{r+1}) \sum_{k=0}^{\infty} x^k = \frac{1 + x^{r+1}}{1 - x} = \frac{1 - x^{2r+2}}{(1 - x)(1 - x^{r+1})}.
\]

The representation of \( F_r \) in the form of an infinite product by \( p \) completes the proof:

\[
F_r(s) = \prod_p \tau^{(r)}_p(p^{-s}) = \prod_p \frac{1 - p^{-(2r+2)s}}{(1 - p^{-s})(1 - p^{-(r+1)s})} = \frac{\zeta(s) \zeta((r+1)s)}{\zeta((2r+2)s)}.
\]

It follows from [9] that

\[
\tau^{(r)} = \tau(1, r+1; \cdot) \ast \mu_{2r+2}
\]
Theorem 1. If $\Delta$ is estimated as in (8) then for $r > 0$
\[
\sum_{n \leq x} \tau^{(r)}(n) = Ax + Bx^{1/(r+1)} + \mathcal{E}_{r+1}(x), \quad \mathcal{E}_r(x) = O \left( x^{\max(\theta_r, 1/2r)} \log^{\theta_r} x \right),
\]
where constants $A$ and $B$ are specified below in (11).

Proof. Taking into account (10) we have for $r > 0$
\[
\sum_{n \leq x} \tau^{(r)}(n) = \sum_{n \leq x} \mu_{2r+2}(n)T(1, r; x/n) = \zeta(r+1)x \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n} +
\]
\[+ \zeta(1/(r+1))x^{1/(r+1)} \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n^{1/(r+1)}} + \sum_{n \leq x} \mu_{2r+2}(n)\Delta(1, r+1, x/n).
\]

But for $s \geq 1/k$
\[
\sum_{n \leq x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} - \sum_{n > x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} + O(x^{1/k-s})
\]

and
\[
\sum_{n \leq x} \mu_{2k}(n)\Delta(1, k, x/n) = \sum_{n \leq x^{1/2k}} \mu(n)\Delta(1, k, x/n^{2k}) \ll
\]
\[\ll \sum_{n \leq x^{1/2k}} \left( \frac{x}{n^{2k}} \right)^\theta n^{\theta_k} \log^\theta x \ll x^{\theta_k} \log^\theta x \left( 1 + x^{1/2k - \theta_k} \right) \ll x^{\max(\theta_k, 1/2k)} \log^\theta x.
\]

So
\[
\sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r+1)}{\zeta(2r+2)}x + \frac{\zeta(r+1)}{\zeta(2)}x^{1/r} + O \left( x^{\max(\theta_{r+1}, 1/(2r+2))} \log^{\theta_r+1} x \right).
\]

For the case $r = 0$ see (3) above.

Lemma 5. Let $r > 0$, $x^\varepsilon \leq y \leq x^{1/2r}$. Then under RH we have
\[
\mathcal{E}_r(x) = \sum_{n \leq y} \mu(n)\Delta(1, r; x/n^2) + O(x^{1/2+r}y^{1/2-r} + x^\varepsilon).
\]

Proof. We follow the approach of Montgomery and Vaughan (see (12) or (11)).

First of all consider
\[
g_y(s) = \frac{1}{\zeta(s)} - \sum_{d \leq y} \frac{\mu(d)}{d^s}.
\]

Then for $\sigma > 1$
\[
g_y(s) = \sum_{d > y} \frac{\mu(d)}{d^s}.
\]

Assuming RH we have by (15, Th. 14.25]
\[
\sum_{d \leq y} \frac{\mu(d)}{d^s} = \zeta^{-1}(s) + O(y^{1/2-\sigma+\varepsilon}(|t|^\varepsilon + 1)) \quad \text{for} \quad \sigma > 1/2 + \varepsilon,
\]
so
\[
g_y(s) \ll y^{1/2-\sigma+\varepsilon}(|t|^\varepsilon + 1) \quad \text{for} \quad \sigma > 1/2 + \varepsilon.
\]

Now let us split $\sum_{n \leq x} \tau^{(r-1)}(n)$ into two parts:
\[
\sum_{n \leq x} \tau^{(r-1)}(n) = \sum_{d^{2r} \leq x} \mu(d)T(1, r; x/d^{2r}) = S_1 + S_2,
\]

where constants $A$ and $B$ are specified below in (11).
where

\[ S_1 := \sum_{d \leq y} \mu(d)T(1, r; x/d^{2r}) = \zeta(r)x \sum_{d \leq y} \frac{\mu(d)}{d^{2r}} + \zeta(1/r)x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^r} + \sum_{d \leq y} \mu(d)\Delta(1, r; x/d^{2r}) \]

and \( S_2 \) is the rest of \( \sum_{n \leq x} \tau^{(r-1)}(n) \). We note that under RH by taking into account \( y \leq x^{1/2r} \) we have

\[ x^{1/r} \sum_{d > y} \frac{\mu(d)}{d^2} \ll x^{1/r} y^{-3/2 + \varepsilon} \ll x^{1/2} y^{1/2-r+\varepsilon} \]

and so

\[ x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} = \frac{x^{1/r}}{\zeta(2)} + O(x^{1/2} y^{1/2-r+\varepsilon}). \]

Next, let

\[ h_y(s) := \zeta(s)\zeta(rs)g_y(2rs)x^s s^{-1}. \]

Then by Perron formula with \( c = 1 + \varepsilon, T = x^2 \) one can estimate

\[ S_2 = \frac{1}{2\pi i} \int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} h_y(s)ds + O(x^\varepsilon). \]

By moving line of integration to \([1/2 + \varepsilon - ix^2, 1/2 + \varepsilon + ix^2]\) we obtain

\[ S_2 = \text{res}_{s=1} h(s) + O(I_1 + I_2 + I_3), \]

where

\[ I_1 = \int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} h(s)ds, \quad I_2 = \int_{1/2+\varepsilon-ix^2}^{1/2+\varepsilon+ix^2} h(s)ds, \quad I_3 = \int_{1/2+\varepsilon-ix^2}^{1/2+\varepsilon+ix^2} h(s)ds. \]

Due to (13) and estimates of \( \zeta \) under RH we have

\[ g_y(2rs) \ll y^{1/2-r}(|t|^\varepsilon + 1) \quad \text{for } \sigma > 1/2 + \varepsilon, \]

\[ h(s) \ll y^{1/2-r}(|t|^\varepsilon + 1)x^s s^{-1} \quad \text{for } \sigma > 1/2 + \varepsilon, \]

and

\[ I_{1,3} \ll y^{1/2-r+\varepsilon} \max_{\sigma \in [1/2+\varepsilon, 1+\varepsilon]} x^{\sigma-2} \ll y^{1/2-r+\varepsilon}, \]

\[ I_2 \ll y^{1/2-r+\varepsilon} \int_1^{x^{2}} t^{-1/2} dt \ll y^{1/2-r+\varepsilon} x^{1/2+\varepsilon}. \]

Identity

\[ \text{res}_{s=1} h(s) = \zeta(r)x \sum_{d > y} \frac{\mu(d)}{d^{2r}} \]

completes the proof.

**Theorem 2.** If \( \Delta \) is estimated as in (3) and \( \theta_r < 1/2r \) then under RH

\[ E_r(x) = O(x^\alpha), \quad \alpha = \frac{1 - \theta_r}{2r + 1 - 4r\theta_r}. \]
FUNCTIONS CONCERNED WITH DIVISORS OF ORDER \( r \)

**Proof.** Let us start with (12):

\[
E_r(x) = \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2 + \varepsilon} y^{1/2 - r} + x^{\varepsilon}) \ll \sum_{n \leq y} \left( \frac{x}{n^{2r}} \right)^{\theta_r + \varepsilon} + O(x^{1/2 + \varepsilon} y^{1/2 - r} + x^{\varepsilon})
\]

If \( \theta_r < 1/2r \) then

\[
E_r(x) \ll x^{\varepsilon} \left( x^{\theta_r} y^{1-2r\theta_r} + x^{1/2} y^{1/2-r} \right).
\]

Choice \( y = x^{\beta} \), where

\[
\beta = \frac{1 - 2\theta_r}{2r + 1 - 4r\theta_r},
\]

accomplishes the proof.

For the values of \( \theta_b \) from Table 1 we have

\[
\max(\theta_r, 1/2r) = \begin{cases} 
1/2r, & r \leq 2, \\
\theta_r, & r > 2.
\end{cases}
\]

So currently the only non-trivial case of the previous theorem is an estimation for \( \tau^{(1)} \equiv \tau^{(r)} \). We get under assumption of RH that

\[
\sum_{n \leq x} \tau^{(1)}(n) = \frac{\zeta(2)}{\zeta(4)} x + \frac{\zeta(1/2)}{\zeta(2)} x^{1/2} + O(x^{\alpha + \varepsilon}),
\]

where

\[
\alpha = \frac{1 - \theta_2}{5 - 8\theta_2} = \frac{3728}{15249} \approx 0.241 < 1/4.
\]

**Theorem 3.**

\[
E_r(x) = \Omega \left( x^{1/(2r+2)} \right).
\]

**Proof.** Equation (14) is implied by the substitution \( m_1 = 1, m_2 = r, n_1 = 2r \) into Lemma 3. The choice of parameters plainly follows from (9). We obtain

\[
\alpha = \frac{1}{2r+2},
\]

which is an exponent in the required \( \Omega \)-term.

5. ASYMPTOTIC PROPERTIES OF \( \sum \sigma^{(r)} \)

**Lemma 6.** Let \( G_r(s) \) be Dirichlet series for \( \sigma^{(r)} \):

\[
G_r(s) := \sum_{n=1}^{\infty} \frac{\sigma^{(r)}(n)}{n^s}.
\]

Then

\[
G_r(s) = \frac{\zeta(s-1)\zeta((r+1)s-r)}{\zeta((r+2)s-r-1)} H_r(s), \quad \sigma > 2,
\]

where Dirichlet series \( H_r(s) \) converges absolutely for \( \sigma > (2r+2)/(2r+3) \).

**Proof.** Consider Bell series for \( \sigma^{(r)} \):

\[
\sigma^{(r)}_p(x) := \sum_{k=0}^{\infty} \sigma^{(r)}(p^k)x^k = \sum_{k=0}^{r} p^k x^k + \sum_{k>r} (p^r + p^k)x^k = \sum_{k=0}^{r} p^k x^k + \sum_{k>r} p^r x^k = \frac{1}{1 - px} + \frac{p^r x^{r+1}}{1 - x}.
\]
Then
\[(1 - px)\sigma_p^{(r)}(x) = 1 + \frac{p^r x^{r+1}(1 - px)}{1 - x} = 1 + \sum_{k=0}^{\infty} (p^r x^{r+1+k} - p^r x^{r+2+k})\]
and
\[\frac{(1 - px)(1 - p^r x^{r+1})}{1 - p^{r+1} x^{r+2}} \sigma_p^{(c)}(x) = 1 + p^r x^{r+2}(1 - px)(1 - p^r x^r) \frac{(1-x)(1-p^{r+1} x^{r+2})}{(1-x)} := h_p(x).\]
For \(\sigma > 1\) we have
\[h_p(p^{-s}) \ll p^{-2}.\]
For \(1 \geq \sigma \geq (2r+2)/(2r+3) + \varepsilon\) we have
\[h_p(p^{-s}) \ll p^{2r+1-(2r+3)s} \ll p^{-1-\varepsilon}.\]
Now (15) follows from the representation of \(G_r\) in the form of infinite product by \(p\):
\[G_r(s) = \prod_{p} \sigma_p^{(r)}(p^{-s}).\]

Following theorem generalizes (4).

**Theorem 4.**
\[\sum_{n \leq x} \sigma^{(r)}(n) = Dx^2 + O(x \log^{5/3} x), \quad D = \frac{\zeta(r+2)H_r(2)}{2\zeta(r+3)}.\]

**Proof.** For a fixed \(r\) let \(z(n)\) be the coefficient at \(n^{-s}\) of the Dirichlet series
\[\frac{\zeta(s-1)\zeta((r+1)s-r)}{\zeta((r+2)s-r-1)}\]
and let \(h(n)\) be the coefficient of the Dirichlet series \(H_r(s)\). It follows from (15) that \(\sigma^{(r)} = z \ast h\). One can verify that
\[z(n) = \sum_{ab^{r+1}c^{r+2} = n} ab^r c^{r+1} \mu(c).\]
Taking into account Lemma (2) with \((\alpha, \beta) = (r+1, r)\) we obtain
\[\sum_{n \leq x} z(n) = \sum_{c \leq x^{1/(r+2)}} c^{r+1} \mu(c) \left( \frac{\zeta(r+2)}{2} \frac{x^2}{c^{r+4}} + O(x^{c-r-2} \log^{2/3} x) \right)\]
\[= \frac{\zeta(r+2)}{2\zeta(r+3)} x^2 + O(x \log^{5/3} x)\]
because
\[\sum_{c \leq x^{1/(r+2)}} \frac{\mu(c)}{c^{r+3}} = \frac{1}{\zeta(r+3)} - \sum_{c > x^{1/(r+2)}} \frac{\mu(c)}{c^{r+3}} = \frac{1}{\zeta(r+3)} + O(x^{-1})\]
and
\[\sum_{c \leq x^{1/(r+2)}} \frac{\mu(c)}{c} \ll \sum_{c \leq x} \frac{1}{c} \ll \log x.\]
Now
\[\sum_{n \leq x} \sigma^{(r)}(n) = \sum_{n \leq x} h(n) \left( \frac{\zeta(r+2)}{2\zeta(r+3)} \frac{x^2}{n^2} + O\left( \frac{x}{n} \log^{5/3} x \right) \right)\]
\[= \frac{\zeta(r+2)}{2\zeta(r+3)} x^2 \sum_{n \leq x} \frac{h(n)}{n^2} + O\left( x \log^{5/3} x \sum_{n \leq x} \frac{h(n)}{n} \right).\]
But \( H_r(s) \) converges absolutely at \( \sigma \geq (2r+2)/(2r+3) + \varepsilon \), so

\[
\sum_{n \leq x} \frac{h(n)}{n} \ll O(1)
\]

and

\[
\sum_{n \leq x} \frac{h(n)}{n^2} = H_r(2) - \sum_{n > x} \frac{h(n)}{n^2} = H_r(2) + O(x^{-(2r+4)/(2r+3)+\varepsilon}).
\]

\[\blacksquare\]

**Theorem 5.** For a fixed \( r > 0 \)

\[
\sum_{n \leq x} \sigma^{(r)}(n) = Dx^2 + \Omega_x(x \log x).
\]

**Proof.** The proof almost replicates the proof of [13, Th. 3] with following changes (in notations of [13]):

\[
\kappa(n) := \frac{\sigma^{(r)}(n)}{n},
\]

\[
\sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} = \frac{\zeta(s)\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1),
\]

\[
u := \mu \ast \kappa,
\]

\[
\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \frac{\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1),
\]

\[
v(p^a) = \frac{\sigma^{(r)}(p^a)}{p^{2a}} - \frac{\sigma^{(r)}(p^{a-1})}{p^{2a-1}} = \begin{cases} 0, & a \leq r + 1, \\ 1/p, & a = r + 1, \\ p^{r-a} - p^{r-a+1}, & a > r + 1. \end{cases}
\]

We take \( m := \log^{1/(4r+4)} x \) and

\[
A := \prod_{p \leq m} p^{r+1} \sim e^{(r+1)m} \sim \exp(\log^{1/4} x),
\]

then

\[
G = \sum_{k \leq u(x)} \frac{v(k)}{k} \gcd(A, k) = \sum_{n^{r+1} | A} \sum_{k \leq u(x)/n^{r+1}} \frac{v(n^{r+1}k)}{v(n^{r+1})k}.
\]

Here \( \sum_k \) means summation over \( k \) such that for every \( p \mid k \) we have \( p \mid n \) or \( p \nmid A \).

Taking into account \( v(p^{r+1}) = 1/p \) we get

\[
G = \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \geq 1} \frac{v(n^{r+1}k)}{v(n^{r+1})k} + o(1) =
\]

\[
= \sum_{n^{r+1} | A} v(n^{r+1}) \prod_{p \mid n} \left( 1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-2}} \right) \prod_{p > m} \left( 1 + \sum_{\nu \geq r+1} \frac{v(p^\nu)}{p^\nu} \right) + o(1).
\]

Since \( |v(p^\nu)| \leq 1/p \) we obtain

\[
\sum_{\nu \geq r+1} \frac{v(p^\nu)}{p^\nu} \ll p^{-r-2}.
\]
Since \( v(n^{r+1}) = 1/n \) for \( n^{r+1} | A \) and \( \log m \asymp \log x \) we have

\[
G = (1 + o(1)) \sum_{n^{r+1} | A} \frac{1}{n} \prod_{p | n} \left( 1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu - r - 2}} \right) =
\]

\[
= (1 + o(1)) \prod_{p \leq m} \left( 1 + \frac{1}{p} + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu - r - 1}} \right).
\]

But \( v(p^\nu) \geq 1/2p^{\nu - r - 1} \) for \( a \geq r + 2 \). So

\[
\sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu - r - 1}} \geq \sum_{\nu \geq r+2} \frac{1}{2(p^{\nu - r - 1})^2} \geq \frac{1}{2p^2}.
\]

Hence

\[
G \geq (1 + o(1)) \prod_{p \leq m} (1 + p^{-1} + p^{-2}/2) \gg \prod_{p \leq m} (1 + p^{-1}) \gg \log m \gg \log x.
\]

\[\square\]

6. Some remarks

The estimate (5) implies that \( \tau^{(r)}(n)/n \to 0 \) as \( n \to \infty \). Thus it is natural to ask what is the maximum value of this ratio.

**Lemma 7.** For \( n \geq 1 \) we have

\[\tau^{(r)}(n) \leq n,\]

where the equality has place only if \( n = 1 \) or if \( n = 2 \) and \( r = 0 \).

**Proof.** Recalling the definition (1) we obtain that the least value of \( a \) for which \( \tau^{(r)}(p^a) \) is different from 1 is \( a = r + 1 \). So

\[\tau^{(r)}(n) = 2^{#(p^{r+1}|n)} \leq 2^{(\log_2 n)/(r+1)} = n^{1/(r+1)}\]

and the statement of the lemma easily follows. \[\square\]

One can see that (7) implies

\[\frac{\sigma^{(r)}(n)}{n} \to +\infty, \quad n \to \infty.\]

**Theorem 6.** Consider the distribution function

\[S_N(q,r;\lambda) := \frac{1}{N} \#\{n \leq N \mid \sigma^{(r)}(n^q) \leq \lambda n^q\}, \quad q, r \in \mathbb{N}.\]

Then \( S_N(q,r;\lambda) \) weakly converges to a function \( S(q,r;\lambda) \) which is continuous if and only if \( q > r \).

**Proof.** Let us fix arbitrary \( q \) and \( r \) and let

\[f(n) := \ln \frac{\sigma^{(r)}(n^q)}{n^q},\]

here \( f \) is an additive function. It is enough to prove that

\[F_N(\lambda) := \frac{1}{N} \#\{n \leq N \mid f(n) \leq \lambda}\]

converges weakly to some \( F(\lambda) \) as \( N \to \infty \) and \( F \) is continuous if and only if \( q > r \).

By definition (2)

\[\sigma^{(r)}(p^q) = \begin{cases} p^q, & r \geq q, \\ p^q + p^r, & r < q. \end{cases}\]
So
\[ f(p) = \begin{cases} 0, & r \geq q, \\ \ln(1 + p^{r-q}) \ll p^{r-q}, & r < q, \end{cases} \]
and \( f(p) = |f(p)| \leq 1 \). Also
\[ \sum_{p} f(p) \ll \begin{cases} 0, & r \geq q, \\ \sum_{p} p^{r-q-1} \ll \sum_{p} p^{-2}, & r < q, \end{cases} < +\infty. \]
and the same is valid for \( \sum_{p} f^2(p)/p \). Now by Erdős—Wintner theorem [2, Th. i] we get that \( F_N(\lambda) \) converges weakly to \( F(\lambda) \) as \( N \to \infty \). Taking into account
\[ \sum_{f(p) \neq 0} 1/p \ll \begin{cases} 0, & r \geq q, \\ \infty, & r < q, \end{cases} \]
we see that due to [2, Th. v] the distribution \( F \) is continuous if and only if \( r < q \). ■

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