SMOOTH AND SINGULAR TRAVELING WAVE SOLUTIONS FOR THE SERRE-GREEN-NAGHDI EQUATIONS

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Abstract. In this paper, we consider the traveling wave solutions of the one-dimensional Serre-Green-Naghdi (SGN) equations which are proposed to model dispersive nonlinear long water waves in a one-layer flow over flat bottom. We decouple the traveling wave system of SGN equations into two ordinary differential equations. By studying the bifurcations and phase portraits of each bifurcation set of one equation, we obtain the exact traveling wave solutions of SGN equations for the variable \(u(x, t)\) which represents average horizontal velocity of water wave. For the compacted orbits intersecting with the singular line in phase plane, we obtained two families of solutions: a family of smooth traveling wave solutions including periodic wave solutions and solitary wave solutions, and a family of compacted singular solutions which have continuous first-order derivative but discontinuous second-order derivative.

1. Introduction. The Serre-Green-Naghdi (SGN) equations, also known as the fully-nonlinear Boussinesq wave equations, can be applied to describe the behaviour of dispersive shoaling water waves\([10, 11]\). Even though various methods have been proposed to study the fully nonlinear shallow water wave SGN equations\([2, 6, 7, 9, 15, 14, 16, 24]\), most of them studied the numerical solutions. By applying qualitative theory and the method of dynamical systems, Deng \([7]\) considered the equation for a linearized version of the SGN equations (Boussinesq equations) and Li \([16]\) investigated the bifurcations and exact travelling wave solutions of the celebrated Green-Naghdi equations. In \([9]\), Favrie et al investigated numerically the solitary wave solution of the following one-dimensional Serre-Green-Naghdi equations

\[h_t + (hu)_x = 0, \quad (hu)_t + (hu^2 + p)_x = 0\] \hspace{1cm} (1)
which is proposed to describe dispersive non-linear long water waves in a one-layer flow over a flat bottom without considering the dissipative effects, where

\[ p = \frac{1}{2} gh^2 + \frac{1}{3} h^2 t^\frac{\partial}{\partial x} + u \frac{\partial}{\partial x} (h_t + uh_x) \]

and \( h > 0 \) is the water depth, \( g > 0 \) is the acceleration of gravity and \( u \) is the average horizontal velocity.

In recent decades, there have been amounts of literature on nonlinear partial differential equations (PDEs), such as the symmetries and conservation laws\cite{22, 23}, the blow-up solutions \cite{18, 19}, the well-posedness \cite{17, 28, 30} and even some special kinds of invariant solutions (like traveling wave solutions) and so on \cite{1, 13, 20, 21, 25, 29}. Traveling wave solutions are a kind of special solutions of PDEs which usually admit the two Lie point symmetries \( X_1 = \frac{\partial}{\partial x} \) and \( X_2 = \frac{\partial}{\partial t} \). Then \( \xi = x - ct \) is an invariant which can be applied to reduce nonlinear PDEs to ordinary differential equations (ODEs), where \( c \) is usually named as wave speed.

It is not difficult to check that system (1) is a special case that \( \tau = 0 \) of the Serre-Green-Naghdi (SGN) system with surface tension \cite{5, 8, 10, 11, 26}

\[ h_t + (hu)_x = 0, \quad u_t + uu_x + gh_x = \frac{1}{3h} [h^3 (u_x + uu_x - u_x^2)]_x + \tau h_{xxx}, \]

where \( \tau \) is a constant surface tension coefficient divided by the density. Steady wave solutions of system (2) was investigated in \cite{5} and the solitary wave solutions and their interactions were examined numerically in \cite{8}. The localized traveling waves are analyzed in \cite{26}. In this paper, we study the travelling wave solutions of system (1) by the qualitative analysis methods of planar dynamical systems and obtain the bifurcations and some exact bounded travelling wave solutions.

This paper is organized as follows. At first, we reduce the SGN equations into ordinary differential equations by introducing the traveling wave invariant variable, and then decoupled this systems into two ordinary differential equations in Section 2. In Section 3, we discuss the dynamical behaviour of system (8), and study the bifurcations and the phase portraits in each bifurcation set. In Section 4, we derive the explicit smooth bounded travelling wave solutions including solitary and periodic wave solutions. We study the singular bounded travelling wave solutions in \( C^1(\mathbb{R}) \) and obtained a family of compacted solutions in Section 5.

2. Traveling wave system of the SGN equations. It is easy to see that the SGN equations (1) admit the two Lie point symmetries \( X_1 = \frac{\partial}{\partial x} \) and \( X_2 = \frac{\partial}{\partial t} \). Let \( \xi = x - ct \) and

\[ u(t, x) = \varphi(\xi), \quad h(t, x) = \eta(\xi). \]

Then (1) reduces to the following ODEs:

\[ -c\eta + (\varphi \eta) = 0, \quad -c(\varphi \eta) + \eta^2 + \left( \frac{1}{2} g \eta^2 \right) + \left( \frac{1}{3} \eta^2 \left( (\varphi - c)^2 \eta \xi + (\varphi - c) \varphi \eta \xi \right) \right) = 0. \]

Integrating (4) once with respect to \( \xi \) gives

\[ -c\eta + \varphi \eta = A, \quad -c\varphi \eta + \eta^2 + \left( \frac{1}{2} g \eta^2 \right) + \left( \frac{1}{3} \eta^2 \left( (\varphi - c)^2 \eta'' + (\varphi - c) \varphi' \eta' \right) = B, \]
where \(A\) and \(B\) are constants of integration. Here we take \(A \neq 0\), from the first equation in (5). For \(\varphi \neq c\), we have

\[
\eta = \frac{A}{\varphi - c}, \quad \eta' = \frac{-A\varphi'}{(\varphi - c)^2}, \quad \eta'' = \frac{-A\varphi''(\varphi - c) + 2A\varphi'^2}{(\varphi - c)^3},
\]

where \(\cdot\) denotes the derivative with respect to \(\xi\). Substituting (6) into the second equation in (5) leads to

\[
\frac{1}{3}A^3(\varphi - c)\varphi'' = (\varphi - c)((A\varphi - B)(\varphi - c)^2 + \frac{1}{2}gA^2) + \frac{1}{3}A^3\varphi'^2.
\]

Let \(v = \varphi\xi\), then equation (7) is equivalent to the following dynamical system

\[
\varphi' = v, \quad v' = \frac{f(\varphi) + A^3v^2}{A^3(\varphi - c)}
\]

provided that \(\varphi - c \neq 0\), where \(f(\varphi) = 3(\varphi - c)((A\varphi - B)(\varphi - c)^2 + \frac{1}{2}gA^2)\). Clearly, system (8) has a singular line \(\varphi = c\). The transformation \(d\xi = A^3(\varphi - c)d\tau\) transforms system (8) into an analytic system

\[
\dot{\varphi} = A^3(\varphi - c)v, \quad \dot{v} = f(\varphi) + A^3v^2,
\]

where \(\cdot\) denotes the derivative with respect to \(\tau\).

As is well known that the bounded orbits of the associated traveling wave system of a nonlinear wave equation correspond to bounded traveling wave solutions of nonlinear wave equations. The periodic orbit usually corresponds to a periodic wave solution, a homoclinic orbit determines a solitary wave solution and heteroclinic orbit corresponds to a kink or anti-kink solutions. However, we note that the water equation (3) in this paper have a singular line \(\varphi = c\). The dynamical system (8) is derived under the assumption \(\varphi \neq c\). Thus for the orbits which are disjoint with the singular line \(\varphi = c\), the corresponding relation mentioned above is valid. However, for the orbits intersecting with the singular line \(\varphi = c\), there might be some singular phenomena.

In this paper, we will examine the effects of singular line \(\varphi = c\) on the traveling wave solution of equation (1), that is, we will consider the traveling wave solutions \((\varphi(\xi), \eta(\xi))\) of (4) for which there exits \(\xi_0\) such that \(\varphi(\xi_0) = c\). To include this special case, at first we show that \(\varphi(\xi)\) of equation (4) is determined by

\[
A^3(\varphi - c)\varphi'' = f(\varphi) + A^3\varphi'^2.
\]

Actually from the first equation of (5), we obtain

\[
\eta(\varphi - c) = A, \quad \eta'(\varphi - c)^2 = -A\varphi', \quad \eta''(\varphi - c)^3 = -A\varphi''(\varphi - c) + 2A\varphi'^2.
\]

Multiplying the second equation of (5) on the both sides by \((\varphi - c)^3\) and substituting (11) into the result, one has (10). Note that \(\varphi = c\) is a trivial solution of system (10) and if there exists \(\xi_0\) such that \(\varphi(\xi_0) = c\), then \(\eta(\xi_0) = c\). Let \(\varphi' = v\), equation (10) is equivalent to the dynamical system

\[
\dot{\varphi}' = v, \quad A^3(\varphi - c)v' = f(\varphi) + A^3v^2.
\]

Clearly, for \(\varphi \neq c\), equation (12) is equivalent to system (8) which has the same phase portraits as (9) except for the singular straight line \(\varphi = c\). Consequently, the smooth traveling wave solutions \(\varphi \neq c\) can be derived from the bounded orbits of
the analytic system (9) which are disjoint with the singular line. The orbits with \( \varphi \neq c \) are determined by the first integral

\[
H(\varphi, v) = \frac{1}{2(\varphi - c)^2}(-3A\varphi^2(\varphi-c)^2 + 6B\varphi(\varphi-c)^2 + 3gA^2(\varphi-c) + A^3v^2) = h. \tag{13}
\]

3. Qualitative analysis and phase portraits of the traveling wave system of (9). In this section, we study the equilibrium points of system (9) firstly to obtain their phase portraits and bifurcations. Suppose \((\varphi_0, v_0)\) is an equilibrium of system (9). To get the local qualitative properties of the equilibrium \((\varphi_0, v_0)\), we now study the eigenvalues of the coefficient matrix of the linearized system of (9) at this point, i.e.,

\[
M(\varphi_0, v_0) = \begin{bmatrix} A^3v_0 & A^3(\varphi_0 - c) \\ f'(\varphi_0) & 2A^3v_0 \end{bmatrix}_{(\varphi_0, v_0)}
\]

By the theory of planar dynamical system [4], we know if \(M(\varphi_0, v_0)\) has two real eigenvalues with opposite signs then the equilibrium \((\varphi_0, v_0)\) is a saddle point; if \(M(\varphi_0, v_0)\) has two complex eigenvalues, then the equilibrium \((\varphi_0, v_0)\) is a center or a focus. In our case, if \(M(\varphi_0, v_0)\) has two complex eigenvalues, then \(M(\varphi_0, v_0)\) must be a center since we know that (9) has a first integral. For study the critical points of system (9), we need to solve equations

\[
A^3(\varphi - c)v = 0, \quad f(\varphi) + A^3v^2 = 0. \tag{14}
\]

If \( \varphi = c \), clearly \( v = 0 \), then \((c, 0)\) is a critical point. If \( v = 0 \), we will give the analysis for the number of the zero of \( f(\varphi) \). Let

\[
(A\varphi - B)(\varphi - c)^2 + \frac{1}{2}gA^2 = w(\varphi), \tag{15}
\]

then \( f(\varphi) = 3(\varphi - c)w(\varphi) \). And the equation

\[
w'(\varphi) = (\varphi - c)(3A\varphi - 2B - Ac) = 0 \tag{16}
\]

has two roots \( \varphi_1 = c \) and \( \varphi_2 = \frac{Ac + 2B}{3A} \). Then we have the following results.

**Lemma 3.1.** While \( A > 0 \), the following conclusions hold for system (9).

1. If \( w(\varphi_2) > 0 \), i.e. \( c > \frac{B}{A} + \frac{2}{3}\sqrt{Ag} \), system (9) has two equilibrium points, denoted as \( (\varphi_{e1}, 0), (c, 0) \). For \( f'(\varphi_{e1}) < 0 \), we have \( D = -A^3(\varphi_{e1} - c)f'(\varphi_{e1}) < 0 \). Then \( (\varphi_{e1}, 0) \) is a saddle point and \((c, 0)\) is degenerate.

2. If \( \varphi_2 > c \) and \( w(\varphi_2) = 0 \), i.e. \( c = \frac{B}{A} + \frac{2}{3}\sqrt{Ag} \), system (9) has three equilibrium points denoted as \( (\varphi_{e1}, 0), (c, 0), (\varphi_{e2}, 0) \) and \( \varphi_{e1} < c < \varphi_{e2} = \varphi_2 \). \((\varphi_{e1}, 0)\) is a saddle point, \((c, 0)\) is degenerate and \((\varphi_{e2}, 0)\) is a cusp point.

3. If \( \varphi_2 > c \) and \( w(\varphi_2) < 0 \), i.e. \( c < \frac{B}{A} - \frac{2}{3}\sqrt{Ag} \), system (9) has four equilibrium points, denoted as \( (\varphi_{e1}, 0), (c, 0), (\varphi_{e2}, 0), (\varphi_{e3}, 0) \), and \( \varphi_{e1} < c < \varphi_{e2} < \varphi_{e3} \). For \( f'(\varphi_{e2}) < 0 \), we have \( D = -A^3(\varphi_{e2} - c)f'(\varphi_{e1}) > 0 \). For \( f'(\varphi_{e3}) > 0 \), we have \( D = -A^3(\varphi_{e2} - c)f'(\varphi_{e1}) < 0 \). Then \( (\varphi_{e1}, 0) \) is a saddle point, \((c, 0)\) is degenerate, \((\varphi_{e2}, 0)\) is a center point, \((\varphi_{e3}, 0)\) is a saddle point.

The phase portraits of system (9) which are shown in Figure 1 are obtained by using Lemma 3.1 and the first integral (13). Here we ignore to study the case when \( A < 0 \) in consideration of the symmetry. The following we study the orbits of
system (9) which are determined by \( h = H(\varphi, v) \) for different values of \( h \) for \( A > 0 \). Let \( h_0 = H(\varphi_0, 0) \).

**Case (1).** \( c \geq \frac{B}{A} - \frac{3}{2} \sqrt{Ag} \). For \( \varphi_{e1} < \varphi < c \) and \( \varphi_0 = \varphi_{e1} \), \( H(\varphi, v) = h_0 \) determines two heteroclinic orbits of the system (9) connecting \((\varphi_{e1}, 0)\) and \((c, 0)\); For \( \varphi_{e1} < \varphi_0 < c \) and \( \varphi_0 < \varphi < c \), \( H(\varphi, v) = h_0 \) determines a family of homoclinic orbits of the system (9) at \((c, 0)\).

**Case (2).** \( c < \frac{B}{A} - \frac{3}{2} \sqrt{Ag} \). Setting \( \varphi_0 = \varphi_{e1} \) and for \( \varphi_{e1} < \varphi < c \), \( H(\varphi, v) = h_0 \) determines two heteroclinic orbits of the system (9) connecting \((\varphi_{e1}, 0)\) and \((c, 0)\); For \( \varphi_{e1} < \varphi_0 < c \) and \( \varphi_0 < \varphi < c \), \( H(\varphi, v) = h_0 \) determines a family of homoclinic orbits of the system (9) at \((c, 0)\); For \( \varphi_{e2} < \varphi_0 < \varphi_{e3} \) and \( \varphi_0 < \varphi < \varphi_{e3} \), \( H(\varphi_2, 0) < H(\varphi, v) < H(\varphi_3, 0) \), \( H(\varphi, v) \) determines a periodic orbit of the system (9); For \( \varphi_{e3} < \varphi < c \) and \( \varphi_0 = \varphi_{e3} \), \( H(\varphi, v) = h_0 \) determines a homoclinic orbit of the system (9) at \((\varphi_{e3}, 0)\).

![Figure 1](image)

**Figure 1.** The phase portraits of system (9) in each bifurcation set for \( A > 0 \). (1) \( c > \frac{B}{A} - \frac{3}{2} \sqrt{Ag} \); (2) \( c = \frac{B}{A} - \frac{3}{2} \sqrt{Ag} \); (3) \( c < \frac{B}{A} - \frac{3}{2} \sqrt{Ag} \).

4. **Bounded smooth traveling wave solutions of (1).** We know that the orbits of systems (8), (9) and (12), with \( \varphi \neq c \), are all determined by the first integral (13) which can be rewritten as

\[
\varphi'^2 = (3A\varphi^2 - 6B\varphi + 2h)(\varphi - c)^2 - 3gA^2. \tag{17}
\]

Obviously, (17) is valid for \( \varphi = c \). We can show that solutions of (10) can be derived by (17) and the following lemma.

**Lemma 4.1.** For arbitrary real number \( h \) and an open interval \( I \in R \), if \( \varphi(\xi) \) satisfies (17) and \( \varphi(\xi) \) is not identically equal to a constant on any subinterval of \( I \), then \( \varphi(\xi) \) solves (10).

For the proof one can refer to [12, 27]. Note that there are four parameters \( A, B, c \) and \( h \) involving in equation (17). So we have to study equation (17) in different
cases determined by the values of the parameters $A, B, c$ and for different energy levels $h$. Next we are going to study the solutions of (17) for different cases and hence derive the traveling wave solutions of (3).

**Lemma 4.2.** [12] Let $F(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$ with $x_1 < x_2 < x_3 < x_4$. For arbitrary $Y \in (x_2, x_3)$,

1. if $x_1 < x_2 < x_3 < x_4$, then $\int_{x_1}^{Y} \frac{1}{\sqrt{F(x)}} \, dx$ converges, where $i = 2$ or $i = 3$;
2. if $x_1 = x_2 = x_3 = x_4$, then $\int_{x_1}^{Y} \frac{1}{\sqrt{F(x)}} \, dx$ diverges, where $i = 2$ ($i = 3$).

In terms of the bifurcation and phase portraits in each bifurcation set, we integrate along the bounded orbits and then get the bounded traveling wave solutions of (1).

### 4.1. Bounded smooth traveling wave solutions of (1)

For the smooth traveling wave solutions of (1), we have the following conclusion.

**Case (I).** $c \geq \frac{B}{2} - \frac{1}{2} \sqrt{A^2b}$. 

(a). For $\varphi_0 \in (\varphi_{e_1}, c)$, the compact orbits of system (12) passing by $(\varphi_0, 0)$ is determined by

$$v^2 = \frac{3(\varphi - \varphi_0)(\varphi - \varphi_-)(c - \varphi)(\varphi_+ - \varphi)}{A^2},$$

where $\varphi \in (\varphi_0, c)$, $\varphi_\pm = \frac{1}{2}(\frac{2B}{A} + c - \varphi_0) \pm \frac{1}{2} \sqrt{(\frac{2B}{A} + c - \varphi_0)^2 - 4(\frac{2Bc}{A} + \frac{2A}{\varphi_0 - c} - c\varphi_0)}$, and $\varphi_- < \varphi_0 < \varphi < c < \varphi_+$. Let

$$T = \int_{\varphi_0}^{c} \frac{A\, dw}{\sqrt{3(w - \varphi_0)(w - \varphi_-)(w - c)(w - \varphi_+)}}.$$  

By Lemma 4.2, we know that $T < \infty$, which means that along this compact orbit of system (12) it takes finite time in terms of $\xi$ to reach the singular line $\varphi = c$. Integrating (18) [3] yields a exact periodic solution to (10) (See Fig 2(2)) given by:

$$\varphi = \varphi_1(\xi) = \varphi_- + \frac{(\varphi_0 - \varphi_-)(c - \varphi_-)}{(c - \varphi_-) - (c - \varphi_0)sn^2(p\xi, k)},$$

where $p = \frac{1}{2A} \sqrt{3(\varphi_+ - \varphi_0)(c - \varphi_-)}$, $k = \sqrt{\frac{(c - \varphi_0)(\varphi_+ - \varphi_-)}{(c - \varphi_0)(c - \varphi_-)}}$ and $\xi = x - ct$. Obviously, (20) is a periodic traveling wave solution with period $2T$ which satisfies $\varphi(T) = c$, which means that this homoclinic orbit of (9) correspond to a smooth periodic solution of (10) which is a periodic traveling wave solution of (1).

(b). For $\varphi_{e_1} < \varphi < c$, the compact orbits $H(\varphi, v) = H(\varphi_{e_1}, 0)$ of system (12) are determined by

$$v^2 = \frac{3}{A^2}(\varphi - \varphi_{e_1})^2(\varphi - c)(\varphi - \varphi_1),$$

where $\varphi_1 = \frac{2B}{A} + c - 2\varphi_{e_1}$ and $\varphi_{e_1} < \varphi < c < \varphi_1$, which correspond to the two heteroclinic orbits of analytic system (9) connecting $(\varphi_{e_1}, 0)$ and $(c, 0)$. Integrating along the orbit above $x-$axis from $c$ to $\varphi$ for $\varphi \in (\varphi_{e_1}, c)$, that is

$$\phi(\varphi; c, A, B) = \int_{c}^{\varphi} \frac{A\, dw}{(w - \varphi_{e_1}) \sqrt{3(w - c)(w - \varphi_1)}}.$$  

From Lemma 4.2, we know that $\phi(\varphi_{e_1}, c, A, B) = \infty$ and $\phi(\varphi, c, A, B) < \infty$ for any $\varphi \in (\varphi_{e_1}, c)$. Then solving $\phi(\varphi, c, A, B) = \xi$ for $\varphi$ yields the smooth solitary
wave solution to (10) (See Fig 3(2)):
\[ \varphi = \varphi_2(\xi) = \varphi_{e_1} + \frac{4ae^{-\frac{x^2}{A}}}{e^{2\frac{x^2}{A}} + 4(2\varphi_{e_1} - c - \frac{B}{A})e^{-\frac{x^2}{A}} + (2\frac{B}{A} - 2\varphi_{e_1})^2}, \tag{23} \]
where \( a = (\varphi_{e_1} - c)(3\varphi_{e_1} - 2\frac{B}{A} - c). \)

**Case (II).** \( c < \frac{B}{A} - \frac{2}{3}\sqrt{\frac{A}{B}}. \)

(a). For \( \varphi_0 \in (\varphi_{e_1}, c) \), the compact orbit \( H(\varphi, v) = h_0 \) is determined by (18) for \( \varphi_0 < \varphi < c < \varphi_+ \). Similar analysis as what we perform in case (I), we know the homoclinic loops at \((c, 0)\) of (9) correspond to a family of smooth periodic solutions of (10) and hence the periodic traveling wave solutions of (1) which can be expressed as (4.4). For the compact orbits \( H(\varphi, v) = h_0 \) with \( h_0 = H(\varphi_{e_1}, 0) \) are determined by (21) for \( \varphi_{e_1} < \varphi < c < \varphi_1 \), which correspond to the two heteroclinic orbits of analytic system (9) connecting \((\varphi_{e_1}, 0)\) and \((c, 0)\). By similar analysis we know that the heteroclinic loop correspond to a smooth solitary solution given by (4.7).

(b). For \( \varphi_0 \in (\varphi_{e_2}, \varphi_{e_3}) \), then the compact orbits of system (7) passing by \((\varphi_0, 0)\) is determined by (18) with \( c < \varphi_0 < \varphi < \varphi_0 < \varphi_+ \). Integrating (18) and then solving for \( \varphi \) yields an exact periodic solution:
\[ \varphi = \varphi_3(\xi) = c + \frac{(\varphi_0-c)(\varphi_0-c)}{(\varphi_0-c)(\varphi_0-c) - (\varphi_0-c)sn^2(p\xi, k)}, \tag{24} \]
where \( p = \frac{1}{2A}\sqrt{3(\varphi_{e_3} - \varphi_0)(\varphi_0 - c)}, k^2 = \frac{(\varphi_{e_3} - \varphi_0)(\varphi_{e_3} - c)}{(\varphi_{e_3} - \varphi_0)(\varphi_{e_3} - c)} \) and \( \xi = x - ct. \)

The homoclinic orbit passing by the saddle \((\varphi_{e_3}, 0)\) of system (12) is determined by
\[ v^2 = \frac{3}{A^2}(\varphi - \varphi_{e_3})^2(\varphi - c)(\varphi - \varphi_3), \tag{25} \]
where \( \varphi_3 = \frac{2B}{A} + c - 2\varphi_{e_3} \). Integrating along the orbit yields
\[ \varphi = \varphi_2(\xi) = \varphi_{e_3} + \frac{-4ae^{\frac{x^2}{A}}}{e^{2\frac{x^2}{A}} + 4(2\varphi_{e_3} - c - \frac{B}{A})e^{\frac{x^2}{A}} + (2\frac{B}{A} - 2\varphi_{e_3})^2}, \tag{26} \]
which is a solitary wave solution of (1). Here \( a = (\varphi_{e_3} - c)(3\varphi_{e_3} - 2\frac{B}{A} - c). \)

**Figure 2.** Phase orbit of (12) and the corresponding bounded traveling wave solutions. (1) a compact orbit of (12) intersecting with the singular line; (2) a smooth periodic traveling wave solution; (3) compacton.
4.2. **Singular traveling wave solutions of (1).** We now focus on the solutions of (1) corresponding to orbits intersecting with the singular line \( \phi = c \). We see from section 3 that for arbitrary \( \phi_{e_1} < \phi_0 < c \), there is a unique orbit on \((\phi, v)\)-phase plane passing through \((\phi_0, 0)\) and intersecting with the singular line \( \phi = c \), which is determined by the algebraic equation (4.2). Integrating along this closed orbit, we derived a periodic traveling wave solution in last section. Remind that this class of orbits intersecting with the singular line \( \phi = c \) which may effect the solutions of equation and arise some singular traveling wave solutions [12, 16, 27].

According to the analysis in the previous two sections, we see that for arbitrary \( c \) there exists a family of compacted orbits intersecting with the singular line \( \phi = c \) (see Fig 2(1)). Suppose that \( \phi(0) = \phi_0 \), then along the orbit \( \phi \) approaches \( c \) in finite time \( T \). Therefore we defined a function with compact support (see Fig 2(3)) as follows:

\[
\varphi_{c_1}(\xi) = \begin{cases} 
\varphi_+ - \frac{(\varphi_0 - \varphi_-)(c - \varphi_-)}{(c - \varphi_0) \text{sn}^2(p\xi, k)}, & |\xi| < T, \\
\varphi_0 - k^2 & |\xi| \geq T,
\end{cases}
\]

where \( p = \frac{1}{2\mathcal{A}} \sqrt{3(\varphi_+ - \varphi_0)(\varphi_- - c)} \), \( k^2 = \frac{(\varphi_+ - \varphi_0)(\varphi_- - c)}{(\varphi_+ - \varphi_0)(\varphi_- - c)} \) and \( \xi = x - ct \).

It is easy to know the following properties of the function defined by (27): it has continuous first-order derivative but the second-order derivative fails to exist at \(|\xi| = T\); it satisfies equation (8) except \(|\xi| = T\). This class of equations are usually named as compactons of nonlinear wave equations.

![Figure 3. Phase orbit of (12) and the corresponding bounded traveling wave solutions. (1) a compact orbit of (12) intersecting with the singular line; (2) a smooth periodic traveling wave solution; (3) compacton.](image1)

For the orbit passing by the saddle point \((\varphi_{e_1}, 0)\) and intersecting with the singular line \( \phi = c \) (see Fig 3(1)), we can construct a corresponding solution in \( C^1(\mathbb{R}) \).
Let

\[ \varphi_{c_2}(\xi) = \begin{cases} 
\varphi_{c_1} + \frac{4ae^{-\frac{\sqrt{3}a\xi}{2}}}{e^{-2\frac{\sqrt{3}a\xi}{4}} - 2be^{-\frac{\sqrt{3}a\xi}{4}} + (2B - 2\varphi_{c_1})^2}, & \xi < 0, \\
c & \xi \geq 0,
\end{cases} \] (28)

Then \( \varphi = \varphi_{c_2}(\pm \xi) \) define two singular solutions which approach \( c \) in finite time (See Fig 3(3) and 3(4)).

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