Ricci almost solitons on semi-Riemannian warped products

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Abstract
It is shown that a gradient Ricci almost soliton on a warped product, $(B^n \times_h F^m, g, f, \lambda)$ whose potential function $f$ depends on the fiber, is either a Ricci soliton or $\lambda$ is not constant and the warped product, the base and the fiber are Einstein manifolds, which admit conformal vector fields. Assuming completeness, a classification is provided for the gradient Ricci almost solitons on warped products, whose potential functions depend on the fiber. An important decomposition property of the potential function in terms of functions which depend either on the base or on the fiber is proven. In the case of a complete gradient Ricci soliton, the potential function depends only on the base.

KEYWORDS
conformal fields, Einstein manifolds, Ricci almost solitons, Ricci solitons, warped products

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1 | INTRODUCTION

A Ricci almost soliton $(M, g, X, \lambda)$ is a Riemannian or semi-Riemannian manifold $(M, g)$ with a vector field $X$ and a smooth function $\lambda : M \to \mathbb{R}$ satisfying the following fundamental equation

$$\text{Ric} + \frac{1}{2} \nabla^2 \lambda g = \lambda g.$$  

(1.1)

where $f$ is called the potential function and $\nabla f$ is the Hessian of $f$ with respect to the metric $g$. We say that a Ricci almost soliton is shrinking, steady, expanding or undefined if the function $\lambda$ is positive, null, negative or changes sign, respectively. The concept of Ricci almost soliton was introduced in [38] generalizing the notion of Ricci soliton, which is the case when the function $\lambda$ is constant. The importance of the Ricci solitons is due to their relation with the Ricci flow. In fact, as one can see for instance in [17] they are stationary solutions of the Ricci flow, that was introduced by Hamilton [24]. If the function $\lambda$ is not constant, then Ricci almost solitons evolve under the Ricci flow changing only by conformal diffeomorphisms (see [40] and [23] page 4). Another relation with geometric flows is obtained by choosing specific functions for $\lambda$, for which the corresponding Ricci almost solitons are self similar solutions of the so called Ricci–Bourguignon flow [9]. This flow
is an interpolation between the Ricci flow and the Yamabe flow [16]. On the other hand gradient Ricci almost solitons can be viewed as a generalization of Einstein manifolds, since by considering $f$ to be a constant function (1.1) reduces to $Ric = \lambda g$, which defines an Einstein manifold.

Motivated by these relations with geometric flows, Ricci solitons and Einstein manifolds we are interested in investigating the geometry of such manifolds and we consider problems such as when a Ricci almost soliton becomes a Ricci soliton or even an Einstein manifold. In [1, 3, 4, 6, 22, 25, 38] and [41] the authors proved that under certain geometric constraints a Ricci almost soliton becomes a Ricci soliton or an Einstein manifold carrying a conformal field.

Examples of Ricci almost solitons can be found in [5, 20, 34] and [38]. We observe that in [38] there are examples of Riemannian manifolds that do not admit Ricci almost soliton structures. For some results on Ricci almost solitons under certain geometric obstructions, we refer the reader to the papers [2–5, 9, 13, 21] and references therein.

The warped products played an important role in the construction of non trivial Ricci solitons, as well as of Einstein manifolds. The cigar soliton, and more generally, the Bryant soliton, were obtained by using local warped product construction. More examples can be found in [17, 19, 39] and in references therein. It is important to note that warped products naturally arose in the studies of related structures on semi-Riemannian manifolds, as one can see for instance in [1, 25] and [41].

Ricci almost solitons on warped products were studied firstly in [20]. The authors gave a systematic approach using the hypothesis that the function $\lambda$ depends only on the base. Under this condition, they proved that the potential function $f$ depends only on the base.

In this paper, we start by characterizing gradient Ricci almost solitons on non trivial warped products in both situations, i.e., when the potential function depends on the fiber (Theorem 2.1) and when it does not depend on the fiber (Theorem 2.3), completing the previous study. We show that in both cases, the fiber is necessarily an Einstein manifold.

We then concentrate our main results in the case when the potential function depends on the fiber. In this case, we show a rigity result in the sense that a gradient Ricci almost soliton $(B^n \times_{h} F^m, g, f, \lambda)$ reduces either to a gradient Ricci soliton or $\lambda$ is not constant and $(B^n \times_{h} F^m, g)$ is an Einstein manifold (see Corollary 2.6). Moreover, assuming completeness, Theorem 2.7 provides the classification of the gradient Ricci almost solitons on warped products, whose potential functions depend on the fiber.

These main results are obtained as a consequence of the following steps: Proposition 4.1 shows that for any gradient Ricci almost soliton the potential function $f$ decomposes in terms of functions that depend either on the base or on the fiber. Then, using the characterization given in Theorem 2.1, we prove in Theorem 2.4, that if the gradient of the warping function $h$ is an improper vector field, then the gradient Ricci almost soliton is actually a gradient Ricci soliton, the base is a Brinkman space and the fiber is Ricci flat, moreover, we classify the fiber when it is complete. When the gradient of the warping function is a proper vector field, then we prove in Theorem 2.5 that it is either a gradient Ricci soliton, i.e., $\lambda$ is constant or $\lambda$ is not constant and $(B^n \times_{h} F^m, g)$, the base and the fiber are Einstein manifolds. Moreover, Theorems 2.1 and 2.5 show that the base, the fiber and the warped product admit conformal vector fields. The existence of such vector fields plays an important role in the classification of complete Riemannian and semi-Riemannian Einstein manifolds and it is used in the proof of Theorem 2.7.

We observe that the assumption that the potential function depends only on the base is made in all constructions of gradient Ricci solitons using warped products, as we can see in [11] and [18], to name a few. As a consequence of our approach, we conclude in Corollary 2.2 that the potential function of a gradient Ricci soliton on a complete warped product depends only on the base, showing that this hypothesis can be eliminated. This result was also considered in [39] with a different approach.

This paper is organized as follows. In Section 2, we state our main results. Section 3, contains basic definitions and classical results needed for the proofs. In Section 4, we prove our main results. In the Appendix we collect all the results on conformal vector fields that we use throughout the paper.

2 MAIN RESULTS

In this section, we will state our main results. The proofs will be given in Section 4. In this paper, we are considering warped products $(B^n \times_{h} F^m, g)$, where $(B^n, g_B)$ and $(F^m, g_F)$ are either Riemannian or semi-Riemannian manifolds and $g = g_B + h^2 g_F$. We are assuming that $h$ is not constant. In this case we say that the warped product is not trivial. In what follows we will denote the connection, the Ricci curvature and other tensors defined using the metric $g_B$ with a subscript $B$, as $\nabla_B, Ric_B$. Similar notation will be considered for the metric $g_F$. 
Our first result says that for a gradient Ricci almost soliton on a warped product, when the potential function depends on the fiber then the fundamental Equation (1.1) on a warped product reduces to a system of equations on the base and on the fiber, in the following way:

**Theorem 2.1.** Let \( B^n \times_h F^m \) be a non trivial warped product where the base \( (B^n, g_B) \) or the fiber \( (F^m, g_F) \) can be either a Riemannian or a semi-Riemannian manifold. Then \( (B^n \times_h F^m, g, f, \lambda) \) is a gradient Ricci almost soliton, with \( f \) non constant on \( F \) if and only if \( f = \beta + h\varphi \), where \( \varphi : F \to \mathbb{R} \) is not constant and \( \beta : B \to \mathbb{R} \) are differentiable functions such that

\[
\begin{align*}
\nabla_B \nabla_B h + ah g_B &= 0, \\
\text{Ric}_B + \nabla_B \nabla_B \beta &= \left[ h^{-1} (\nabla_B h) \beta - bh^{-1} + (n-1)a \right] g_B, \\
\nabla_F \nabla_F \varphi + (c\varphi + b) g_F &= 0, \\
\text{Ric}_F &= (m-1)cg_F, \\
\end{align*}
\]

(2.1)

for some constants \( a, b, c \in \mathbb{R} \), the function \( \lambda \) is given by

\[
\lambda = h^{-1} \left( \nabla_B h \right) \beta - bh^{-1} + (m+n-1)a - ah\varphi, 
\]

(2.2)

and the constants \( a \) and \( c \) are related to \( h \) by the equation

\[
|\nabla_B h|^2 + ah^2 = c. 
\]

(2.3)

Equations such as the first or third equations of (2.1) have appeared in many contexts. They appeared for example in concircular transformations [42], in conformal transformations between Einstein spaces [30] and in conformal vector fields on Einstein manifolds [31].

A function satisfying Equation (2.3) is said to have constant energy, following [15], where the author investigated properties of such functions. Equation (2.3) also appeared in the Critical Point Equation (CPE) conjecture (see [33]), where it is shown that the potential function of a CPE metric satisfies an equation similar to (2.3) if and only if it is an Einstein manifold (i.e., the CPE Conjecture holds).

As an application of Theorem 2.1 we can prove that for a complete warped product gradient Ricci solitons (that is, when \( \lambda \) is a constant) the potential function does not depend on the fiber.

**Corollary 2.2.** Let \( (B \times_h F, g, f, \lambda) \) be a gradient Ricci soliton on a complete non trivial Riemannian or semi-Riemannian warped product. Then \( f \) does not depend on the fiber.

Corollary 2.2 shows that, for complete gradient Ricci solitons on semi-Riemannian warped product, the potential function depends only on the base. It was first considered in [39] with a different approach. Roughly speaking, the authors decompose the gradient of the potential function \( f \in C^\infty(B \times F) \) into two vector fields and use Proposition 35 and Corollary 43 of [36] (on pages 208 and 211, respectively). This procedure seems to require an extra condition on \( f \).

Our next result characterizes gradient Ricci almost solitons i.e., Equation (1.1), on warped products, when the potential function depends only on the base.

**Theorem 2.3.** Let \( B^n \times_h F^m \) be a non trivial warped product where \( (B^n, g_B) \) or \( (F^m, g_F) \) can be either a Riemannian or a semi-Riemannian manifold. Then \( (B^n \times_h F^m, g, f, \lambda) \) is a gradient Ricci almost soliton, with \( f \) constant on \( F \) if and only if

\[
\begin{align*}
\text{Ric}_B + \nabla_B \nabla_B f - mh^{-1} \nabla_B \nabla_B h &= \lambda g_B, \\
\lambda h^2 &= h(\nabla_B h) f - (m-1)|\nabla_B h|^2 - h \Delta_B h + c(m-1), \\
\text{Ric}_F &= c(m-1)g_F, \\
\end{align*}
\]

(2.4)

for some constant \( c \in \mathbb{R} \).
The Riemannian version of Theorem 2.3 was also considered in [20], where the authors gave some explicit solutions to the system. The essence of both Theorems 2.1 and 2.3 is to express the condition for a warped product to be a gradient Ricci almost soliton in terms of conditions on the base and on the fiber. Note that the first and third equations in (2.1) say that the corresponding gradient fields are conformal vector fields (see Appendix for definitions). In addition, the fourth equation of (2.1) and the third equation of (2.4) show that the fiber is an Einstein manifold in both cases.

We say that a semi-Riemannian manifold \((M, g)\) is a Brinkmann space if it admits a parallel light like vector field \(X\), called a Brinkmann field. These spaces play an important role in General Relativity [10] and they were introduced by Brinkmann [10] when the author studied conformal transformations between Einstein manifolds.

We say that a vector field \(X\) is improper if there is an open set where \(X\) is lightlike. If there is no such an open set then \(X\) is called a proper vector field. Our next two results show the rigidity of a gradient Ricci almost soliton on a warped product \((B^n \times_h F^m, g, f, \lambda)\), when the potential function depends on the fiber. Namely, we show that such a gradient Ricci almost soliton is either a gradient Ricci soliton (i.e. \(\lambda\) is constant) or \(\lambda\) is not constant but \((B \times h F, g)\) is an Einstein manifold.

Theorems 2.4 and 2.5 consider respectively the case when \(\nabla B h\) is an improper vector field and a proper one and the potential function of the warped product depends on the fiber.

**Theorem 2.4.** Let \(B^n \times_h F^m, n \geq 2\), be a non trivial warped product where the base \((B^n, g_B)\) is a semi-Riemannian manifold and the fiber \((F^m, g_F)\) can be either a Riemannian or a semi-Riemannian manifold. Then \((B^n \times_h F^m, g, f, \lambda)\) is a gradient Ricci almost soliton, with \(f\) non constant on \(F\) and \(\nabla B h\) an improper vector field on \(B\) if and only if \(\lambda\) is constant, i.e. it is a Ricci soliton and \(f = \beta + h \varphi\), where \(\varphi : F \to \mathbb{R}\) non constant and \(\beta : B \to \mathbb{R}\) are smooth functions satisfying

\[
g(\nabla_B h, \nabla_B \beta) = \lambda h + b, \quad \text{Ric}_B + \nabla_B \nabla_B \beta = \lambda g_B, \quad \nabla_F \nabla_F \varphi + b g_F = 0
\]

for a constant \(b \in \mathbb{R}\). \(B\) is a Brinkmann space with \(\nabla_B h\) as a Brinkmann field and \(F\) is Ricci flat.

If in addition \(F\) is complete, then it is isometric to

1. \(\pm \mathbb{R} \times F^{m-1}\), where \(F\) is Ricci flat, if \(b = 0\);
2. \(\mathbb{R}^m\), if \(b \neq 0\).

Our next result which considers \(\nabla_B h\) to be a proper vector field is divided into two cases according to \(\nabla_B h\) being homothetic or not. A vector field is homothetic if its local flow acts by translations. Otherwise it is called non-homothetic.

**Theorem 2.5.** Let \(B^n \times_h F^m\) be a non trivial warped product where the base \((B^n, g_B)\) or the fiber \((F^m, g_F)\) can be either a Riemannian or a semi-Riemannian manifold and suppose that \((B^n \times_h F^m, g, f, \lambda)\) is a gradient Ricci almost soliton with \(f\) non constant on \(F\) and \(\nabla_B h\) a proper vector field. Then

i) If \(\nabla_B h\) is homothetic, then \(\lambda\) is constant, i.e, it is a gradient Ricci soliton.

ii) If \(\nabla_B h\) is non-homothetic, then \(\lambda\) is not constant, \(B^n \times_h F^m, B\) and \(F\) are Einstein manifolds such that

\[
\text{Ric}_{B \times_h F} = a(n + m - 1)g, \quad \text{Ric}_B = a(n - 1)g_B, \quad \text{Ric}_F = c(m - 1)g_F,
\]

where the constants \(a \neq 0\) and \(c\) are related to \(h\) by \(|\nabla_B h|^2 + ah^2 = c\). Moreover, \(\nabla f\) and \(\nabla_B h\) are conformal gradient fields on \(B^n \times_h F^m\) and on \(B^n\), respectively, satisfying

\[
\nabla f + (af + a_0)g = 0, \quad \nabla_B \nabla_B h + ah g_B = 0.
\]

and

\[
\lambda = -af + a(m + n - 1) - a_0,
\]

for some constant \(a_0 \in \mathbb{R}\).

A direct corollary of both Theorem 2.4 and Theorem 2.5 is the following rigidity result. Other rigidity results can be found in [3, 4, 22] or [38].
Corollary 2.6. If \((B^n \times_h F^m, g, f, \lambda)\) is a warped product gradient Ricci almost soliton, with \(f\) non constant on \(F\), then one of the following holds

i) \(\lambda\) is constant, i.e., it is a gradient Ricci soliton;

ii) \(\lambda\) is not constant, \((B^n \times_h F^m, g), (B, g_B)\) and \((F, g_F)\) are Einstein manifolds, \(\nabla_B h\) is a proper and non-homothetic conformal vector field, and \(\nabla f, \nabla_F \varphi\) are conformal.

Observe that Corollaries 2.6 and 2.2 imply that a complete gradient Ricci almost soliton \((B^n \times_h F^m, g, f, \lambda)\) on a warped product, with \(f\) non constant on the fiber \(F\), will have necessarily non constant \(\lambda\).

Hence it follows from Corollary 2.6, item ii), that the warped product, the base and the fiber are Einstein manifolds which admit conformal vector fields. This fact has a fundamental role in the classification result of such gradient Ricci almost solitons, given below in Theorem 2.7, since Einstein manifolds carrying such vector fields are completely classified.

In order to better organize the paper we provide the classification of such Einstein manifolds in the Appendix. We conclude this section with Theorem 2.7. In order to do so, we consider the following classes of \(n\)-dimensional complete Einstein manifolds (see Theorems A.6–A.7 in the Appendix). The division of such manifolds into the classes \(I\) and \(II\) below is introduced in order to simplify the statement of Theorem 2.7.

Class I

1. \(\mathbb{R} \times N^{n-1}\) where \((N, g_N)\) is a complete Riemannian or semi-Riemannian Einstein manifold.

2. A Brinkman space of dimension \(n \geq 3\), i.e. a semi-Riemannian manifold \((M^n, g)\) admitting a parallel light like vector field.

Class II

1. \(S^n_\varepsilon(1/\sqrt{|c|})\), when \(0 \leq \varepsilon \leq n - 2\); the covering of \(S^n_{n-1}(1/\sqrt{|c|})\) when \(\varepsilon = n - 1\) and the upper part of \(S^n_\varepsilon(1/\sqrt{|c|})\) when \(\varepsilon = n\) with \(c > 0\).

2. \(H^n_\varepsilon(1/\sqrt{|c|})\), when \(2 \leq \varepsilon \leq n - 1\); the covering of \(H^n_\varepsilon(1/\sqrt{|c|})\) when \(\varepsilon = 1\) and the upper part of \(H^n_\varepsilon(1/\sqrt{|c|})\) when \(\varepsilon = 0\), with \(c < 0\).

3. \((\mathbb{R} \times N^{n-1}, \pm dt^2 + \cosh^2(\sqrt{|c|}t)g_N)\), where \((N^{n-1}, g_N)\) is a Riemannian or semi-Riemannian Einstein manifold.

4. \((\mathbb{R} \times N^{n-1}, \pm dt^2 \pm e^{2\sqrt{|c|}t}g_N)\), where \((N^{n-1}, g_N)\) is a Riemannian Einstein manifold.

The following result classifies the complete gradient Ricci almost solitons on warped products, whose potential functions depend on the fiber. It also shows that \(\lambda\) depends on the fibre.

Theorem 2.7. Let \(M^{n+m} = B^n \times_h F^m\) be a non trivial warped product where \((B^n, g_B)\) or \((F^m, g_F)\) can be either a Riemannian or a semi-Riemannian manifold. Then \((B^n \times_h F^m, g, f, \lambda)\) is a complete gradient Ricci almost soliton with \(f\) non constant on \(F\) if and only if there exist constants \(a \neq 0, a_0, c \in \mathbb{R}\) such that \(f = a^{-1}(−\lambda + a(m + n − 1) − a_0)\) and

1. if \(n = 1\) then \(B^1\) is isometric to \((\mathbb{R}, \text{sgn} a \, dt^2)\)

\[
h = \begin{cases} 
 Ae \sqrt{|a|}t & \text{if } c = 0, \\
 \sqrt{|c/a|} \cosh (\sqrt{|a|}t + B) & \text{if } c \neq 0, 
\end{cases}
\]  

(2.8)

where \(A \neq 0\) and \(B \in \mathbb{R}\). Moreover, \(M\) is an Einstein manifold satisfying \(\text{Ric}_M = (m + n - 1)ag\) and if \(m \geq 2\), \(F\) is an Einstein manifold satisfying \(\text{Ric}_F = (m - 1)c g_F\).

2. If \(n \geq 2\) and \(m \geq 2\) then

- \(M^{n+m}\) is an Einstein manifold isometric either to a manifold of Class II.1 (resp. II.2) when \(a > 0\) (resp. \(a < 0\)) and \(f\) has some critical point or it is isometric to a manifold of Class II.3 or II.4 if \(f\) has no critical points.

- \(B\) is a complete Einstein manifold isometric either to a manifold of Class II.1 (resp. Class II.2) and index \(\varepsilon_B = n\) (resp. \(\varepsilon_B = 1\)) if \(a > 0\) (resp. \(a < 0\)) and \(h\) has critical points or to a manifold of Class II.3 or II.4 if \(h\) has no critical points.
\[ F \text{ is a complete Einstein manifold isometric to either } \mathbb{R}^n, \text{ or to a manifolds of Class I when } c = 0 \text{ and it is isometric to a manifold of Class II when } c \neq 0. \]

3. Moreover, \( F^m \) for \( m \geq 1 \) is positive definite, i.e., has index zero (resp. negative definite, i.e., has index \( n \)), if \( B^n \) for \( n \geq 1 \) is positive definite (resp. negative definite).

Remarks 2.8.

1. As we will see in the next sections, the proofs of our main results rely strongly on an important decomposition property of the potential function, namely \( f = \beta + h\varphi \), where \( \beta \) and \( h \) are defined on the base and \( \varphi \) is defined on the fiber \( F \) (see Proposition 4.1). By considering this decomposition, in Theorem 2.7 item 2, when \( c \neq 0 \), the fiber \( F \) is isometric to a manifold of Class II 1 (resp. Class II 2) when \( c > 0 \) (resp. \( c < 0 \)) and \( \varphi \) has some critical point, while \( F \) is isometric to a manifold of Class II 3 or II 4 when \( \varphi \) has no critical points (see proof of Theorem 2.7).

2. We observe that, when we are in the Riemannian setting, Theorem 2.4 does not occur. Moreover, Class I only contains the product of \( \mathbb{R} \times N^{n-1} \), where \( (N, g_N) \) is a complete Riemannian Einstein manifold and Class II is restricted to the Riemannian manifolds. The classification in the Riemannian case was first obtained in [38].

3 | PRELIMINARIES

In this section we recall some definitions and results that will be used in Section 4, for the proofs of the main results.

3.1 | Warped products

Along this subsection we follow the notation and results of [8] and [36]. See also [7].

Consider two semi-Riemannian manifolds \((B^n, g_B)\) and \((F^m, g_F)\). Given a smooth function \( h : B \to (0, +\infty) \), we can consider the warped product \( B \times_h F \), with warping function \( h \), as the product manifold \( B \times F \) endowed with the metric \( g = g_B + h^2g_F \), defined by

\[
g = \pi^*g_B + (h \circ \pi)^2\sigma^*g_F, \tag{3.1}\]

where \( \pi : B \times F \to B \) and \( \sigma : B \times F \to F \) are the canonical projections. So \( B \times_h F \) is a semi-Riemannian manifold of dimension \( n + m \).

In what follows, we will consider on the product lifted vector fields from the base and from the fiber identifying these vector fields with the corresponding vector fields on the base and on the fiber, respectively. The set of all such liftings from the base will be denoted by \( \mathfrak{X}(B) \subset \mathfrak{X}(B \times F) \) and the set of all liftings from the fiber will be denoted by \( \mathfrak{X}(F) \subset \mathfrak{X}(B \times F) \). Vector fields lifted from the base will be denoted by \( X, Y, Z \in \mathfrak{X}(B) \) and vector fields lifted from the fiber will be denoted by \( U, V, W \in \mathfrak{X}(F) \). For more information about lifting vector fields see for example [36].

The propositions below express the geometry of the warped product in terms of the base and fiber geometries and the properties of the warping function. They can be used to produce examples of metrics satisfying some prescribed properties, as one can see for example in [8].

**Proposition 3.1** ([8], page 211). Let \( M = B^n \times_h F^m \) be a Riemannian or semi-Riemannian warped product. Then the Ricci tensor of \( M \) is given by

\[
\begin{cases}
\text{Ric}(X, Y) = \text{Ric}_B(X, Y) - m h^{-1} \nabla_B h(X, Y), \\
\text{Ric}(X, U) = 0, \\
\text{Ric}(U, V) = \text{Ric}_F(U, V) - \left[ h \Delta_B h + (m - 1) |\nabla_B h|^2 \right] g_F(U, V).
\end{cases}
\tag{3.2}
\]

The next result is a direct consequence of Proposition 3.1 and it will be useful for the proofs of our main results.
Proposition 3.2 ([28], Corollary 3). A semi-Riemannian warped product, $B^n \times_h F^m$, is an Einstein space with Einstein constant $a \in \mathbb{R}$ if and only if there is a constant $c \in \mathbb{R}$ so that

$$
\begin{align*}
\text{Ric}_B - m h^{-1} \nabla_B h & = a(m + n - 1)g_B, \\
h \Delta_B h + (m - 1) |\nabla_B h|^2 + a(m + n - 1) h^2 & = c(m - 1), \\
\text{Ric}_F & = c(m - 1)g_F.
\end{align*}
$$

(3.3)

If the base $B$ is a connected interval $I \subset \mathbb{R}$, then Proposition 3.2 takes a simpler form, that we state below, for future references.

Corollary 3.3. A semi-Riemannian warped product of the form $I \times_h F^m$, where $I \subset \mathbb{R}$, is an Einstein space if and only if $(F^m, g_F)$ is an Einstein space and the function $h$ satisfies

$$
h'' + ah = 0 \quad \text{and} \quad \pm (h')^2 + ah^2 = c,
$$

where $a$ is the Einstein constant of $I \times_h F^m$ and $c$ is the Einstein constant of $F$.

As an immediate consequence of the properties of the connection in a warped product, proved in [8], page 206, one obtains

Proposition 3.4. Let $M = B^n \times_h F^m$ be a semi-Riemannian warped product. Then the Hessian of a function $f : M \to \mathbb{R}$ is given by

$$
\begin{align*}
\nabla \nabla f(X, Y) & = \nabla_B \nabla_B f(X, Y), \\
\nabla \nabla f(X, U) & = X(U(f)) - h^{-1} X(h) U(f), \\
\nabla \nabla f(U, V) & = \nabla_F \nabla_F f(U, V) + h(\nabla_B h) f g_F(U, V).
\end{align*}
$$

(3.5)

By a complete semi-Riemannian manifold we mean a semi-Riemannian manifold where each geodesic can be extended to $\mathbb{R}$. In the Riemannian case one shows the following.

Proposition 3.5 ([8], page 209). A Riemannian warped product $B \times_h F$ is complete if and only if $B$ and $F$ are complete.

Been and Busemann showed that $(\mathbb{R} \times \mathbb{R}, dx^2 - e^{2x} dy^2)$ is not a complete semi-Riemannian manifold. In fact, they showed that there are light like geodesics that cannot be extended to $\mathbb{R}$ [12] (see also [36], page 209). Their example shows that there is no result similar to Proposition 3.5 for indefinite metrics i.e., with nonzero index.

For our purposes we have the following result that guarantees the non completeness of the semi-Riemannian warped product, whenever the gradient of the warping function is a parallel vector field on the base. For the sake of completeness we include its proof. For more results on completeness of semi-Riemannian manifolds see [14].

Proposition 3.6. Let $B \times_h F$ be a non trivial warped product, where $(B^n, g_B)$ or $(F^m, g_F)$ can be either a Riemannian or a semi-Riemannian manifold. If $\nabla_B h$ is a parallel vector field on $B$, then $B \times_h F$ is not complete.

Proof. Suppose by contradiction that $B \times_h F$ is complete. Consider $p_0 \in B$ and $v_0 \in T_{p_0}B$ such that $dh_{p_0}v_0 \neq 0$. Let $\gamma$ be the geodesic such that $\gamma(0) = p_0$ and $\gamma'(0) = v_0$. Since $\nabla_B h$ is parallel, it follows that

$$
(h \circ \gamma)''(t) = \gamma''(\gamma(h)) = \gamma''(\gamma(h)) - \nabla_{\gamma'} \gamma'(h) = \nabla_B \nabla_B h(h', h') = 0.
$$

Therefore, there exist constants $a_0, b_0 \in \mathbb{R}$, so that

$$
(h \circ \gamma)(t) = a_0 t + b_0.
$$
Observe that

\[ a_0 = (h \circ \gamma)'(0) = dh_{p_0}v_0 \neq 0. \]

By assumption $\gamma$ is defined on $\mathbb{R}$, hence we may consider $t_0 = -b_0/a_0 \in \mathbb{R}$. However, $h(\gamma(-b_0/a_0)) = 0$, which contradicts the fact that $h \neq 0$. \hfill $\Box$

### 3.2 Bochner formula

In this section we will state a version of the Bochner formula that will be used in the next section. For a proof in the Riemannian case, see Lemma 2.1 of [37]. We observe that the same proof is valid for any signature.

**Theorem 3.7 ([37]).** Let $(M, g)$ be a Riemannian or semi-Riemannian manifold and let $\varphi : M \to \mathbb{R}$ be a smooth function. Then

\[ \text{div}(\nabla \nabla \varphi)(X) = \text{Ric}(\nabla \varphi, X) + X(\Delta \varphi), \quad (3.6) \]

for all $X \in \mathfrak{X}(M)$.

With this version of Bochner formula, we can provide a simple proof of the proposition below when $n \geq 2$. For another proof when $n \geq 3$ in a more general setting see ([31]).

**Proposition 3.8.** Let $(M^n, g)$ be an Einstein manifold with dimension $n \geq 2$ and Einstein constant $a$. If $\varphi : M \to \mathbb{R}$ is a smooth function such that $\nabla \varphi$ is a conformal vector field satisfying

\[ \nabla \nabla \varphi + \phi g = 0 \]

for some smooth function $\phi : M \to \mathbb{R}$, then there is a constant $b \in \mathbb{R}$ such that $\phi = -a \varphi - b$.

**Proof.** It is easy to see that $\Delta \varphi = n \phi$ and that $\text{div}(\nabla \nabla \varphi)(X) = X(\phi)$, for all $X \in \mathfrak{X}(M)$. Using Bochner formula, we have

\[ (n - 1)X(\phi + a \varphi) = 0. \]

Since $X$ is an arbitrary field and $n \geq 2$, it follows that there is a constant $b$ satisfying the assertion. \hfill $\Box$

### 4 PROOF OF THE MAIN RESULTS

We start with an important decomposition property of the potential function of a gradient Ricci almost soliton on a warped product. We prove that the potential function decomposes in terms of functions which depend either on the base or on the fiber.

**Proposition 4.1.** Let $(B^n \times_B F^m, g, f, \lambda)$ be a gradient Ricci almost soliton defined on a warped product manifold, where the base $(B^n, g_B)$ or the fiber $(F^m, g_F)$ are either Riemannian or semi-Riemannian manifolds, $h : B \to \mathbb{R}$ is a positive smooth function and $g = g_B + h^2 g_F$. Then the potential function $f$ can be decomposed as

\[ f = \beta + h \varphi, \quad (4.1) \]
where $\beta : B \to \mathbb{R}$ and $\phi : F \to \mathbb{R}$ are smooth functions. Furthermore, the fundamental equation (1.1) is equivalent to the system

$$
\begin{align*}
Ric_B + \nabla_B \nabla_B \beta + (\phi - mh^{-1}) \nabla_B \nabla_B h &= \lambda g_B, \\
Ric_F + h \nabla_F \nabla_F \phi &= \left[ h \Delta h + (m - 1) |\nabla_B h|^2 - h(\nabla_B h) \beta - \phi h(\nabla_B h) h + \lambda h^2 \right] g_F.
\end{align*}
$$

(4.2)

**Proof.** In view of Proposition 3.1 and Proposition 3.4 we can rewrite the fundamental Equation (1.1) as follows

$$
\begin{align*}
Ric_B(X, Y) - mh^{-1} \nabla_B h(X, Y) + \nabla_B \nabla_B f(X, Y) &= \lambda g_B(X, Y), \\
Ric_F(U, V) + \nabla_F \nabla_F f(U, V) &= \left[ \lambda h^2 + (m - 1) h^{-2} |\nabla_B h|^2 + h^{-1} \Delta_B h - h(\nabla_B h) f \right] g_F(U, V), \\
X(U(f)) &= h^{-1} X(h(U(f)).
\end{align*}
$$

(4.3)

Observe that $X(U(f)) - h^{-1} U(f) X(h) = 0$ implies

$$
X(U(fh^{-1})) = X(U(f))h^{-1} = X(U(f))h^{-1} - U(f)h^{-2} X(h) = 0,
$$

for all $X \in \mathfrak{L}(B)$ and all $U \in \mathfrak{L}(F)$. Hence $fh^{-1} = \tilde{\beta} + \phi$, where $\tilde{\beta} \in C^\infty(B)$ and $\phi \in C^\infty(F)$.

Therefore, there are smooth functions $\beta : B \to \mathbb{R}$ and $\phi : F \to \mathbb{R}$ such that the potential function $f$ decomposes as in (4.1), where $\beta = h\tilde{\beta}$. Substituting (4.1) in the first two equations of (4.3), a straightforward computation implies that (4.2) holds.

In order to analyse the system (4.2), we will consider separately the cases where the potential function $f$ depends or not on the fiber. We observe that when the warping function $h$ is constant, the warped product reduces to the Riemannian or semi-Riemannian product. In this case, the base and the fiber must be gradient Ricci solitons, as we can easily see from (4.2). So, from now on, we will assume that $h$ is not constant.

For the proof of Theorem 2.1, we will need the following lemma.

**Lemma 4.2.** Let $B^n \times F^m$ be a product manifold and let $h : B^n \to \mathbb{R}$, $\phi : F^m \to \mathbb{R}$ be non constant differentiable functions. Let $\mu_1, \rho_1 : D \subset B \to \mathbb{R}$ and $\mu_2, \rho_2 : G \subset F \to \mathbb{R}$ be differentiable functions, such that $D \times G$ is connected. Then

$$
h(p)\mu_2(q) + \phi(q)\mu_1(p) = \rho_1(p) + \rho_2(q), \quad \text{for all } (p, q) \in D \times G.
$$

(4.4)

if and only if there are constants $b, \tilde{b}, c, \tilde{c} \in \mathbb{R}$ such that

$$
\begin{align*}
\mu_1 &= ch + \tilde{c}, \\
\rho_1 &= bh + \tilde{b}, \\
\mu_2 &= -c\phi - b, \\
\rho_2 &= \tilde{c}\phi - \tilde{b},
\end{align*}
$$

(4.5)

for all $p \in D$ and $q \in G$.

**Proof.** Assume that the relation (4.4) holds. Since $h$ and $\phi$ are not constant functions, we consider $(p_0, q_0) \in D \times G$ such that $p_0$ and $q_0$ are regular points of the functions $h$ and $\phi$, respectively. Then there exists a vector field $X_1$ on a connected neighborhood $D_1 \subset D$ of $p_0$ and a vector field $U_1$ on a connected neighborhood $G_1 \subset G$ of $q_0$ such that

$$
X_1(h(p)) \neq 0, \quad U_1(\phi)(q) \neq 0, \quad \text{for all } p \in D_1, \quad \text{for all } q \in G_1.
$$
Consider \(X_1, X_2, \ldots, X_n\) and \(U_1, U_2, \ldots, U_m\) orthogonal frames locally defined in (neighborhoods that we still denote by) \(D_1\) and \(G_1\) respectively. Applying the vector fields \(X_k, k = 1, \ldots, n\) and \(U_\alpha, \alpha = 1, \ldots, m\) to the relation (4.4) we get that

\[
X_k(h)U_\alpha(\mu_2) = -X_k(\mu_1)U_\alpha(\varphi), \quad \text{for all } k, \alpha.
\]  

(4.6)

In particular, we have

\[
\frac{X_1(\mu_1)}{X_1(h)} = \frac{U_1(\mu_2)}{U_1(\varphi)} = c, \quad \text{in } D_1 \text{ and } G_1,
\]

for some constant \(c \in \mathbb{R}\). Hence

\[
X_1(\mu_1) = cX_1(h) \quad \text{in } D_1 \quad \text{and} \quad U_1(\mu_2) = -cU_1(\varphi) \quad \text{in } G_1.
\]  

(4.7)

We want to show that this expression holds for all \(X_i\) and \(U_\alpha\). Fix \(p_1 \in D_1\) and consider \(X_i(h)(p_1)\) for \(i \geq 2\). If \(X_i(h)(p_1) \neq 0\), shrinking \(D_1\) if necessary, we can assume that \(X_i(h) \neq 0\) in \(D_1\). Then it follows from (4.6) and (4.7) that in \(D_1\)

\[
\frac{X_i(\mu_1)}{X_i(h)} = -\frac{U_1(\mu_2)}{U_1(\varphi)} = c.
\]

Therefore,

\[
X_i(\mu_1) = cX_i(h) \quad \text{in } D_1.
\]

If \(X_i(h)(p_1) = 0\), then it follows from (4.6) that \(U_1(\varphi)X_i(\mu_1)(p_1) = 0\) therefore \(X_i(h)(p_1) = cX_i(\mu_1)(p_1) = 0\). We conclude that for all \(i\) we have

\[
X_i(\mu_1 - ch) = 0 \quad \text{in } D_1.
\]

Similarly, for all \(\alpha\) we get that

\[
U_\alpha(\mu_2 + c\varphi) = 0 \quad \text{in } G_1.
\]

From the last two expressions we conclude that there exist constants \(\tilde{c}, b \in \mathbb{R}\) such that

\[
\mu_1 - ch = \tilde{c}, \quad \text{in } D_1 \quad \mu_2 + c\varphi = -b, \quad \text{in } G_1.
\]

It follows from (4.4) that

\[
\rho_1 + bh = \tilde{c}\varphi - \rho_2 = \tilde{b}.
\]

Therefore, we obtained (4.5) in \(D_1 \times G_1\).

If there is \(p_1 \in D \setminus D_1\), using (4.4) in \(p_1\) and (4.5) in \(q \in G_1\) we have

\[
\varphi(q)(\mu_1(p_1) - \tilde{c}) = \rho_1(p_1) + bh(p_1) - \tilde{b},
\]

for all \(q \in G_1\). Applying \(X_1\) on the above identity and how \(\varphi\) is not constant on \(G_1\) it follows that (4.5) holds on \(D \times G_1\).

Analogously if there is \(q_1 \in G \setminus G_1\), we can use (4.4) in \(q_1\), (4.5) in \(p \in D_1\) and the non constancy of \(h\) on \(D_1\) to prove (4.5) on whole \(D \times G\).

Conversely, if (4.5) holds, straightforward computations show that (4.4) is satisfied. \(\square\)
Proof of Theorem 2.1. If \((B^n \times_h F^m, g, f, \lambda)\) is a gradient Ricci almost soliton then it follows from Proposition 4.1 that \(f = \beta + h\varphi\) and the system (4.2) is satisfied. We are assuming that \(h\) is not constant and \(f\) depends on the fibers. Hence \(\varphi\) is not constant.

Considering the system (4.2) evaluated at pairs of orthogonal vector fields \((X, Y), X, Y \in \mathfrak{X}(B)\) and \((U, V), U, V \in \mathfrak{X}(F)\) locally defined on a neighborhood of any point \((p, q) \in B \times F\), we have

\[
\begin{aligned}
Ric_B(X, Y) + \nabla_B \nabla_B \beta(X, Y) + (\varphi - mh^{-1}) \nabla_B \nabla_B h(X, Y) &= 0, \\
Ric_F(U, V) + h \nabla_F \nabla_F \varphi(U, V) &= 0.
\end{aligned}
\] (4.8)

Fix \(p_1 \in B\) and consider an open neighborhood \(G_1 \subset F\) of regular points \(q\) of \(\varphi\) and \(W\) a vector field on \(F\) such that \(W(\varphi) \neq 0\) in \(G_1\). Considering the first equation of (4.8) at the points \((p_1, q)\) and applying \(W\) to this equation, we get that

\[
\begin{aligned}
\nabla_B \nabla_B h(X, Y)(p_1) &= 0, \\
Ric_B(X, Y)(p_1) + \nabla_B \nabla_B \beta(X, Y)(p_1) &= 0, \quad \text{for all} \quad p_1 \in B.
\end{aligned}
\]

Similarly, by fixing \(q_1 \in F\) and considering an open neighborhood \(D_1 \subset B\) of regular points \(p\) of \(h\), we obtain from the second equation of (4.8) that

\[
\begin{aligned}
\nabla_F \nabla_F \varphi(U, V)(q_1) &= 0, \\
Ric_F(U, V)(q_1) &= 0, \quad \text{for all} \quad q_1 \in F.
\end{aligned}
\]

Therefore, for any pairs of orthogonal vector fields \((X, Y)\) and \((U, V)\), locally defined in \(B \times F\), we have

\[
\begin{aligned}
\nabla_B \nabla_B h(X, Y) &= 0, \\
Ric_B(X, Y) + \nabla_B \nabla_B \beta(X, Y) &= 0, \\
\nabla_F \nabla_F \varphi(U, V) &= 0, \\
Ric_F(U, V) &= 0.
\end{aligned}
\] (4.9)

Let \((p_0, q_0) \in B \times F\) such that \(p_0\) and \(q_0\) are regular points of the functions \(h\) and \(\varphi\) respectively. Then there exist vector fields \(X_1\) and \(U_1\) defined on open connected sets \(D \subset B\) and \(G \subset F\) with \(p_0 \in D\) and \(q_0 \in G\), such that

\[
X_1(h)(p) \neq 0, \quad \text{for all} \quad p \in D, \\
U_1(\varphi)(q) \neq 0, \quad \text{for all} \quad q \in G.
\] (4.10)

Let \(\{X_j\}_{j=2}^n\) and \(\{U_\alpha\}_{\alpha=2}^m\) be orthogonal vector fields on \(D\) and \(G\) respectively. Without loss of generality we may consider

\[
\begin{aligned}
g_B(X_j, X_k) &= \varepsilon_j \delta_{jk} h^2, \quad \text{for all} \quad j, k \in \{1, ..., n\}, \\
g_F(U_\alpha, U_\gamma) &= \varepsilon_\alpha \delta_{\alpha\gamma}, \quad \text{for all} \quad \alpha, \gamma \in \{1, ..., m\},
\end{aligned}
\] (4.11)

where \(\varepsilon_j\) and \(\varepsilon_\alpha\) denote the signatures of the vector fields.

Now we consider the system (4.2) evaluated at the pairs \((X_j, X_j)\) and \((U_\alpha, U_\alpha)\). Subtracting the first equation multiplied by \(\varepsilon_j\) from the second one multiplied by \(\varepsilon_\alpha\), we get the following expression

\[
\varphi(q)\mu_{1j}(p) + h(p)\mu_{2j}(q) = \rho_{1j}(p) + \rho_{2j}(q), \quad \text{for all} \quad (p, q) \in D \times G.
\] (4.12)
where \(1 \leq j \leq n\), \(1 \leq \alpha \leq m\) and
\[
\begin{align*}
\mu_{1j} &= -\varepsilon_j \nabla_B \nabla_B h(X_j, X_j) + h|\nabla_B h|^2, \\
\rho_{1j} &= h \Delta_B h + (m - 1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \varepsilon_j [\text{Ric}_B + \nabla_B \nabla_B \beta - mh^{-1} \nabla_B \nabla_B h](X_j, X_j), \\
\mu_{2\alpha} &= \varepsilon_\alpha \nabla_F \nabla_F \varphi(U_\alpha, U_\alpha), \\
\rho_{2\alpha} &= -\varepsilon_\alpha \text{Ric}_F(U_\alpha, U_\alpha).
\end{align*}
\] (4.13)

In view of Lemma 4.2, it follows from (4.12) that, for each pair \((j, \alpha)\), there exist constants \(a_{j\alpha}, b_{j\alpha}, c_{j\alpha}, d_{j\alpha}\), such that
\[
\begin{align*}
\mu_{1j} &= c_{j\alpha} h + \bar{c}_{j\alpha}, \\
\rho_{1j} &= -b_{j\alpha} h + \bar{b}_{j\alpha}, \\
\mu_{2\alpha} &= -c_{j\alpha} \varphi - b_{j\alpha}, \\
\rho_{2\alpha} &= \bar{c}_{\alpha} \varphi - \bar{b}_{j\alpha}.
\end{align*}
\] (4.14)

Therefore,
\[
\begin{align*}
\frac{X_1(\mu_{1j})}{X_1(h)} &= c_{j\alpha}, \\
\frac{X_1(\rho_{1j})}{X_1(h)} &= -b_{j\alpha}.
\end{align*}
\]
\[
\begin{align*}
\frac{U_1(\mu_{2\alpha})}{U_1(\varphi)} &= -c_{j\alpha}, \\
\frac{U_1(\rho_{2\alpha})}{U_1(\varphi)} &= \bar{c}_{\alpha},
\end{align*}
\]
i.e., \(c_{j\alpha}, b_{j\alpha}\) do not depend on \(\alpha\), and \(c_{j\alpha}\) and \(\bar{c}_{j\alpha}\) do not depend on \(j\). Hence we denote \(c_{j\alpha} = c, b_{j\alpha} = b\) and \(\bar{c}_{j\alpha} = \bar{c}_j\). Moreover, it follows from (4.14) that
\[
\mu_{1j} - ch = \bar{c}_j \quad \text{and} \quad \mu_{2\alpha} + c\varphi = -b_j.
\]

Therefore, \(\bar{c}_j\) does not depend on \(\alpha\) and \(b_j\) does not depend on \(j\). Hence we may denote \(\bar{c}_j = \bar{c}, b_j = b\) and
\[
\rho_{1j} + bh = \bar{b}_{j\alpha}, \quad \rho_{2\alpha} - \bar{c}\varphi = -\bar{b}_{j\alpha}.
\]

We conclude that \(\bar{b}_{j\alpha}\) does not depend on \(j\) and \(\alpha\) and we can denote \(\bar{b}_{j\alpha} = \bar{b}\). Therefore, it follows from (4.13) and (4.14) that in \(D \times G\) we have
\[
\begin{align*}
-\varepsilon_j \nabla_B \nabla_B h(X_j, X_j) + h|\nabla_B h|^2 &= ch + \bar{c}, \\
h \Delta_B h + (m - 1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \varepsilon_j [\text{Ric}_B + \nabla_B \nabla_B \beta - mh^{-1} \nabla_B \nabla_B h](X_j, X_j) &= -bh + \bar{b}, \\
\varepsilon_\alpha \nabla_F \nabla_F \varphi(U_\alpha, U_\alpha) &= -c\varphi - b, \\
-\varepsilon_\alpha \text{Ric}_F(U_\alpha, U_\alpha) &= \bar{c}\varphi - \bar{b}.
\end{align*}
\] (4.15)

Considering (4.9) for the orthogonal vector fields \(\{X_j\}_{j=1}^n\), \(\{U_\alpha\}_{\alpha=1}^m\), it follows from (4.15) that in \(D \times G\) we have
\[
\begin{align*}
\nabla_B \nabla_B h + h^{-1} \bar{c} h^{-2} - h^{-1} |\nabla_B h|^2 \text{g}_B &= 0, \\
\text{Ric}_B + \nabla_B \nabla_B \beta + \{h \Delta_B h - |\nabla_B h|^2 - h(\nabla_B h)\beta - \bar{b} + bh + m\bar{c}h^{-1} + mc\} h^{-2} \text{g}_B &= 0, \\
\nabla_F \nabla_F \varphi + (c\varphi + b) \text{g}_F &= 0, \\
\text{Ric}_F + (\bar{c}\varphi - \bar{b}) \text{g}_F &= 0.
\end{align*}
\] (4.16)

We will now prove that (4.16) holds in \(B \times F\). Let \(p_1 \in B\) and \(X \in T_{p_1}B\) such that \(\text{g}_B(X, X) = \varepsilon_X h^2(p_1)\), where \(\varepsilon_X = \pm 1\). Consider \(q \in G\) and the system (4.2) at the pair of vectors \((X, X)\) and the pair of vectors fields \((U_1, U_1)\) at \((p_1, q)\), \(q \in G\).
Multiplying the first equation by $-\varepsilon_X$ and adding to the second equation multiplied by $\varepsilon_1$, we get

$$\varphi(q)\mu_1X(p_1) + h(p_1)\mu_2X(q) = \rho_1X(p_1) + \rho_2(q), \quad \text{for all } q \in G,$$

where

$$\begin{cases}
\mu_1X &= -\varepsilon_X \nabla_B \nabla_B h(X,X) + h[\nabla_B h]^2, \\
\rho_1X &= h\Delta_B h + (m-1)[\nabla_B h]^2 - h(\nabla_B h)\beta + \varepsilon_1[Ric_B + \nabla_B \nabla_B \beta - mh^{-1} \nabla_B \nabla_B h](X,X), \\
\mu_2 &= -c\varphi - b, \\
\rho_2 &= \varepsilon \varphi - \bar{b},
\end{cases}$$

(4.18)

where the last two equalities follow from (4.15) and the fact that $q \in G$. Therefore, (4.17) reduces to

$$[-c h(p_1) + \mu_1X(p_1) - \bar{c}]\varphi(q) = bh(p_1) + \rho_1X(p_1) - \bar{b}, \quad \text{for all } q \in G.$$

Applying the vector field $U_1$ to this equation, we conclude that

$$\mu_1X(p_1) = c h(p_1) + \bar{c}, \quad \rho_1X(p_1) = -b h(p_1) + \bar{b}. \quad (4.19)$$

Similarly, considering $q_1 \in F$ and $U \in T_{q_1}F$ such that $g_F(U, U) = \varepsilon_U = \pm 1$, for all $p \in D$ the equations of (4.2) evaluated at the pairs $(X_1, X_1)$ and $(U, U)$ will imply that

$$\varphi(q_1)\mu_{11}(p) + h(p)\mu_{2U}(q_1) = \rho_{11}(p) + \rho_{2U}(q_1),$$

where

$$\mu_{2U} = \varepsilon_U \nabla_F \nabla_F \varphi(U, U), \quad \rho_{2U} = -\varepsilon_U Ric_F(U, U).$$

Analogue arguments as before will imply that

$$\mu_{2U}(q_1) = -c\varphi(q_1) - b, \quad \rho_{2U}(q_1) = \varepsilon \varphi(q_1) - \bar{b}. \quad (4.20)$$

Since $p_1 \in B$ and $q_1 \in F$ are arbitrary, we conclude that for any locally defined vector fields $X \in \mathfrak{X}(B)$ and $U \in \mathfrak{X}(F)$, such that $g_B(X, X) = \varepsilon_X h^2$ and $g_F(U, U) = \varepsilon_U$ we have that (4.19) and (4.20) hold. We now consider any point $(p_1, q_1) \in B \times F$ and orthogonal fields locally defined $Y_1, ..., Y_n$ in $\mathfrak{X}(B)$, $V_1, ..., V_m$ in $\mathfrak{X}(F)$ such that $g_B(Y_j, Y_j) = \varepsilon_j h^2$ and $g_F(V_\alpha, V_\alpha) = \varepsilon_\alpha$. Then

$$\begin{cases}
-\varepsilon_j \nabla_B \nabla_B h(Y_j, Y_j) + h[\nabla_B h]^2 = c h + \bar{c}, \\
h\Delta_B h + (m-1)[\nabla_B h]^2 - h(\nabla_B h)\beta + \varepsilon_1[Ric_B + \nabla_B \nabla_B \beta - mh^{-1} \nabla_B \nabla_B h](Y_j, Y_j) = -b h + \bar{b}, \\
\varepsilon_\alpha \nabla_F \nabla_F \varphi(V_\alpha, V_\alpha) = -c\varphi - b \\
-\varepsilon_\alpha Ric_F(V_\alpha, V_\alpha) = \varepsilon \varphi - \bar{b}.
\end{cases}$$

Considering (4.9) for the orthogonal vector fields $\{Y_j\}_{j=1}^n$ and $\{V_\alpha\}_{\alpha=1}^m$, it follows that (4.16) holds in $B \times F$.

We will now use Bochner formula (3.6) to prove that

$$\bar{c} = 0, \quad \bar{b} = (m-1)c. \quad (4.21)$$

In fact, it follows from the third equation of (4.16) that

$$U_1(\Delta_F \varphi) = -cmU_1(\varphi).$$
From the fourth equation, we have

\[ Ric_F(∇_Fφ, U_1) = (−cφ + b)U_1(φ). \]

Moreover,

\[
div(∇_F∇_Fφ)(U_1) = \sum_{α=1}^{m} (∇_F U_α)(U_1, U_α)
\]

\[ = \sum_{α=1}^{m} (∇_F U_α (−(cφ + b)g_F))(U_1, U_α) \]

\[ = -cg_F(U_1, \sum_{α=1}^{m} U_α(φ)U_α) \]

\[ = -cU_1(φ). \]

Now Bochner formula implies that

\[ [cφ − b + c(m − 1)]U_1(φ) = 0. \]

Since \( U_1(φ) \neq 0 \), we conclude that (4.21) holds.

Therefore, on \( B × F \) the system (4.16) reduces to

\[
\begin{align*}
\nabla_B \nabla_B h + (c - |\nabla_B h|^2)h^{-1}g_B &= 0, \\
Ric_B + ∇_B ∇_B β + (h^{-1} [Δ_B h - (∇_B h)β + b] + h^{-2}(c - |∇_B h|^2))g_B &= 0, \\
∇_F ∇_F φ + (cφ + b)g_F &= 0, \\
Ric_F - (m - 1)c g_F &= 0.
\end{align*}
\]

(4.22)

Observe that for any \( X ∈ \mathfrak{X}(B) \), we have the following expressions

\[
\begin{align*}
ν_B ν_B h(X, ν_B X) &= g_B(ν_X ν_B h, ν_B h) = \frac{1}{2}X(ν_B h^2), \\
ν_B ν_B h(X, ν_B X) &= (ν_B h^2 - c)h^{-1}X(h),
\end{align*}
\]

where the second equality follows from (4.22). Therefore,

\[
\frac{X(|ν_B h|^2)}{2} - (|ν_B h|^2 - c)h^{-1}X(h) = 0,
\]

which implies that

\[ X[(c - |∇_B h|^2)h^{-2}] = 0. \]

Hence there exists a constant \( α \) such that

\[ (c - |∇_B h|^2)h^{-2} = α, \]

i.e., (2.3) holds. Moreover, the first equation of (4.22) reduces to

\[ ∇_B ∇_B h + a h g_B = 0. \]
and $\Delta_B h = -anh$. Hence the second equation of (4.22) reduces to

$$\text{Ric}_B + \nabla_B \nabla_B \beta = [(n-1)a + h^{-1}(\nabla_B h)\beta - bh^{-1}]g_B.$$  

Finally, it follows from these two last equations that the first equation of (4.2) provides

$$\lambda = h^{-1}(\nabla_B h)\beta + (m + n - 1)a - bh^{-1} - ah\varphi.$$  

Therefore, the functions $f$, $h$ and $\lambda$ satisfy the system (2.1). The converse is a straightforward computation. This concludes the proof of Theorem 2.1.  

Proof of Corollary 2.2. Suppose by contradiction that $f$ depends on the fiber, then it follows from Theorem 2.1 that $f = \beta + h\varphi$ where $\varphi$ is not constant. Moreover, $\beta$, $h$, $\varphi$ and $\lambda$ satisfy (2.1)–(2.3). Hence there exists a vector field $U \in \mathfrak{X}(F)$ such that $U(\varphi) \neq 0$ on an open subset of $F$. Since $\lambda$ is constant, taking the derivative of (2.2) with respect to $U$, we obtain $0 = U(\lambda) = -ahU(\varphi)$. Hence $a = 0$ and the first equation of (2.2) reduces to $\nabla_B \nabla_B h = 0$. However, it follows from Proposition 3.6 that if $B \times_h F$ is complete then $\nabla_B h$ is not parallel, which is a contradiction.  

Proof of Theorem 2.3. It follows from Proposition 4.1 that if $(B^n \times_h F^m, g, f, \lambda)$ is a gradient Ricci almost soliton and $f$ is constant on $F$, then in the decomposition of $f$ given by (4.1) we may consider $\varphi = 0$. Therefore, from the first equation of (4.2) we get that the first equation of (2.4) holds and that $\lambda$ is a function constant on $F$, hence it depends only on $B$. In order to obtain the other equations of (2.4), we observe that if $U \in \mathfrak{X}(F)$ is a unitary vector field satisfying $g_F(U, U) = \varepsilon \in \{-1, 1\}$ we obtain from the second equation of (4.2):

$$\varepsilon \text{Ric}_F(U, U) = h\Delta_B h + (m - 1)|\nabla_B h|^2 - h(\nabla_B h)\beta + \lambda h^2.$$  

Since the left hand side is a function defined only on $F$ and the right hand side is a function defined only on $B$, there is a constant $\tilde{c} \in \mathbb{R}$ independent of the fixed field $U$, (as we can see using the right hand side of the above equality), such that

$$\lambda h^2 = h(\nabla_B h)\beta - (m - 1)|\nabla_B h|^2 - h\Delta_B h + \tilde{c},$$  

and

$$\text{Ric}_F = \tilde{c}g_F.$$  

In order to normalize the Einstein constant, we consider $\tilde{c} = (m - 1)c$. This proves that (2.4) holds. The converse is a simple calculation.  

Proof of Theorem 2.4. From Theorem 2.1, we have that $f = \beta + h\varphi$ and Equations (2.1)–(2.3) are satisfied. If $\nabla_B h$ is an improper vector field on $B$, it follows from Equation (2.3) that $a = c = 0$. Hence, (2.1) and (2.2) imply that $\nabla_B h$ is a parallel light like vector field, $(F, g_F)$ is Ricci flat and

$$\begin{cases} 
\text{Ric}_B + \nabla_B \nabla_B \beta = \lambda g_B, \\
\lambda = h^{-1}[(\nabla_B h)\beta - b], \\
\nabla_F \nabla_F \varphi + bg_F = 0. 
\end{cases}$$  

(4.23)

Now we will prove that $\lambda$ is constant. If $\lambda = 0$ there is nothing to prove. Otherwise there is an open set $U \subset M$ where $\lambda$ does not vanish. Then it follows from the second equation of (4.23) that

$$\frac{1}{2}X(\ln(\lambda^2)) = \frac{1}{2}X(\ln(h^{-2}[g(\nabla_B h, \nabla_B \beta) - b]^2))$$  

$$= -h^{-1}X(h) + [g(\nabla_B h, \nabla_B \beta) - b]^{-1}X(g(\nabla_B h, \nabla_B \beta))$$  

$$= -h^{-1}X(h) + [g(\nabla_B h, \nabla_B \beta) - b]^{-1}\nabla_B \nabla_B \beta(X, \nabla_B h).$$  

(4.24)
Since \( \nabla_B h \) is a parallel vector field, Bochner’s Formula implies that \( \text{Ric}(X, \nabla_B h) = 0 \), hence from the first equation of (4.23), we get that \( \nabla_B \nabla_B \beta(X, \nabla_B h) = \lambda g_B(X, \nabla_B h) \). We conclude, using the second equation of (4.24) that (4.24) reduces to
\[
\frac{1}{2} X(\ln(\lambda^2)) = -h^{-1} X(h) + h^{-1} X(h) = 0,
\]
which proves that \( \lambda \) is constant. The converse is immediate.

Now suppose that \( (F, g_F) \) is complete. Since \( \nabla_F \nabla_F \varphi + b g_F = 0 \), the result follows from Theorem A.5 in the Appendix, if \( b \neq 0 \) and from Theorem A.6 if \( b = 0 \).

**Proof of Theorem 2.5.** If \( (B^n \times_h F^m, g, f, \lambda) \) is a gradient Ricci almost soliton with \( h \) non constant and \( f \) depending on the fiber then, it follows from Theorem 2.1 that there are functions \( \beta : B \to \mathbb{R} \) and \( \varphi : F \to \mathbb{R} \) and constants \( a, b, c \in \mathbb{R} \), such that \( f = \beta + h \varphi \) where \( \beta, h, \varphi \) and \( \lambda \) satisfy (2.1)-(2.3).

If \( \nabla_B h \) is a homothetic vector field, then \( a = 0 \). It means that this vector field is parallel, and by the same argument as in the proof of Theorem 2.4, we see that \( \lambda \) is constant, which proves that \( (B^n \times_h F^m, g, f, \lambda) \) is a Ricci soliton.

From now on we will suppose that \( \nabla_B h \) is a non homothetic vector field, that is, that \( a \neq 0 \).

If \( n = 1 \), from (2.1)-(2.3) we get
\[
\begin{align*}
\begin{cases}
h''' + ah = 0, \\
\pm(h'')^2 + ah^2 = c, \\
\text{Ric}_F = (m - 1)c g_F,
\end{cases}
\end{align*}
\]
where \( g_B = \pm dt^2 \). Therefore, \( B^1 \times_h F^m \) is an Einstein manifold with normalized Einstein constant \( a \), as a consequence of Corollary 3.3.

If \( n \geq 2 \), it follows from the second equation of (2.1) that \( (B, g_B, \beta, \tilde{\lambda}) \) is a gradient Ricci almost soliton, i.e.,
\[
\text{Ric}_B + \nabla_B \nabla_B \beta = \tilde{\lambda} g_B
\]
where
\[
\tilde{\lambda} = h^{-1}(\nabla_B h)\beta - bh^{-1} + (n - 1)a. \tag{4.26}
\]
From the first equation of (2.1), we get that \( \nabla_B h \) is a gradient conformal field satisfying
\[
\nabla_B \nabla_B h + a h g_B = 0, \tag{4.27}
\]
i.e., \( (B, g_B, \beta, \tilde{\lambda}) \) is a gradient Ricci almost soliton. Moreover, \( \nabla_B \nabla_B h - \Delta_B h / n g = 0 \). By hypothesis, \( \nabla_B h \) is a non homothetic vector field hence \( \nabla_B h \) is a proper vector field, and therefore \( a \neq 0 \) and \( h \) admits regular points. Fixing a regular point of \( h, p \in B \), it follows from Proposition A.10 that there exists a connected open set \( D \subset B \), containing \( p \), such that \( D \) is diffeomorphic to \( (-\varepsilon, \varepsilon) \times N^{n-1} \) for \( \varepsilon > 0 \) and a regular level \( N^{n-1} \) of \( h \), in such a way that \( h \) does not depend on \( N^{n-1} \) and \( (D, g_D) \) is isometric to \( ((-\varepsilon, \varepsilon) \times N^{n-1}, \pm dt^2 + h'(t)^2 g_N) \), where \( g_D = g_B|_D \) and \( g_N = g_B|_N \). By restricting \( \beta \) and \( \tilde{\lambda} \) to \( D \), we have that \( (D, g_D, \beta, \tilde{\lambda}) \) is a gradient Ricci almost soliton, therefore
\[
((-\varepsilon, \varepsilon) \times_h N^{n-1}, \pm dt^2 + h'(t)^2 g_N, \beta, \tilde{\lambda}), \tag{4.28}
\]
is also a gradient Ricci almost soliton. We are going to use this coordinate system to conclude that \( (D, g_D) \) is an Einstein manifold with normalized Einstein constant \( a \). This is equivalent to proving that following equations hold
\[
\begin{align*}
\begin{cases}
h'''' + ah' = 0, \\
\pm(h'')^2 + ah'^2 = c, \\
\text{Ric}_N = (n - 2)c g_N,
\end{cases}
\end{align*}
\]
as one can see from Corollary 3.3. In order to do so, we must consider two cases whether \( \beta \) depends on \( N^{n-1} \) or not.
Suppose that $\beta$ depends on $N^{n-1}$, then we can apply Theorem 2.1 to (4.25), when restricted to $D$, given as in (4.28). From the first and fourth equations of (2.1) we get that the following equations hold

\[
\begin{aligned}
& h''' \pm \hat{a}h' = 0, \\
& \operatorname{Ric}_N = (n - 2)\hat{c}g_N,
\end{aligned}
\]  

(4.30)

for some constants $\hat{a}, \hat{c} \in \mathbb{R}$. Moreover, from (2.3) the constants $\hat{a}$ and $\hat{c}$ are related to $h'$ by the equation $\pm(h'')^2 + \hat{a}(h')^2 = \hat{c}$.

It follows from the first equation of (2.1) and (4.30) that $a = \hat{a}$. This proves (4.29) for this case.

Suppose that $\beta$ does not depend on $N^{n-1}$, then since (4.25) holds, we can apply Theorem 2.3 to $D$ given as in (4.28). Then (2.4) reduces to

\[
\begin{aligned}
& \beta'' - (n - 1)(h')^{-1}h''' = \pm \hat{\lambda}, \\
& (h')^2 \hat{\lambda} = \pm h' h'' \beta' \mp (n - 2)(h'')^2 \mp h''' + \hat{c}(n - 2), \\
& \operatorname{Ric}_N = \hat{c}(n - 2)g_N,
\end{aligned}
\]  

(4.31)

for some constant $\hat{c} \in \mathbb{R}$. Moreover, the first equation of (2.1) restricted to $D$ gives $h'' \pm ah = 0$ and hence $h''' = ah' = 0$. These two equations substituted into the first two equations of (4.31) implies that

\[
\begin{aligned}
& \beta'' \pm (n - 1)a = \pm \hat{\lambda}, \\
& (h')^2 \hat{\lambda} = -ahh' \beta' \mp (n - 2)a^2 h^2 + a(h')^2 + \hat{c}(n - 2).
\end{aligned}
\]  

(4.32)

Substituting (4.26) into both equations of (4.32), and using (2.3) we conclude that the following equations hold

\[
\begin{aligned}
& ((\beta^2 h^{-1})')' = \mp b h^{-2}, \\
& \beta^2 h^{-1} = bh^{-1}h' + (n - 2)(\hat{c} \mp ac)(h')^{-1}, \\
& h'' \pm ah' = 0, \\
& \pm (h'')^2 + a(h')^2 = \pm ac, \\
& \operatorname{Ric}_N = \hat{c}(n - 2)g_N.
\end{aligned}
\]  

(4.33)

Therefore, in order to prove that (4.29) holds, we need to show the equality $\hat{c} = \pm ac$. If $c = 0$ it follows from the second equation of (4.33) that $\hat{c} = 0$. If $c \neq 0$, then we substitute the second equation of (4.33) into the first one to obtain

\[
a(\hat{c} \mp ac) = 0,
\]

which implies $\hat{c} = \pm ac$, since $a \neq 0$. Therefore, we have proved that (4.29) also holds when $\beta$ does not depend on $N^{n-1}$.

Now from Proposition A.9, we know that the set of regular points of $h$ is a dense subset of $B$, and the argument above implies that $(B, g_B)$ is an Einstein manifold with normalized Einstein constant $a$. As a consequence we have

\[
\begin{aligned}
& \operatorname{Ric}_B = a(n - 1)g_B, \\
& \nabla_B \nabla_B h + ahg_B = 0, \\
& \operatorname{Ric}_F = c(m - 1)g_F,
\end{aligned}
\]  

(4.34)

which implies from Proposition 3.2 that $B \times_h F$ is itself an Einstein with normalized Einstein constant $a$.

From the fundamental equation (1.1), we obtain that

\[
\nabla \nabla f + ((m + n - 1)a - \lambda)g = 0,
\]

and $\nabla f$ is a gradient conformal field on an Einstein manifold. Proposition 3.8 says that there is a constant $a_0$ such that

\[
\lambda = -af + a(m + n - 1) + a_0,
\]  

(4.35)
in view of $n + m \geq 2$. Hence $\nabla \nabla f + af - a_0 = 0$. Moreover, since $f$ is non constant on $F$, (4.35) implies that $\lambda$ is not constant. This concludes the proof of Theorem 2.5.

In order to prove Theorem 2.7 that provides the classification of complete gradient Ricci almost solitons whose potential function depends on the fiber, we will use the classification of Einstein manifolds carrying conformal vector fields, available in the Appendix, Theorem A.8, Theorem A.7 and Theorem A.5

**Proof of Theorem 2.7.** Suppose that $(B^n \times_h F^m, g, f, \lambda)$ is a complete gradient Ricci almost soliton, with $h$ non constant and $f$ depending on $F$. Then it follows from Theorem 2.1 that there are functions $\beta : B \to \mathbb{R}$ and $\varphi : F \to \mathbb{R}$ and constants $a, b, c \in \mathbb{R}$ such that $f = \beta + h \varphi$, where $\beta$, $h$, $\varphi$ and $\lambda$ satisfy (2.1)–(2.3). From Proposition 3.6 the completeness of $B^n \times_h F^m$ implies that $\nabla_B h$ is not a parallel vector field on $B$ and hence it follows from the first equation of (2.1) that $a \neq 0$, therefore $\nabla_B h$ is not homothetic. Applying Theorem 2.5 we have that $B^n \times_h F^m$, $B$ and $F$ are Einstein manifolds satisfying (2.5) for constants $a \neq 0$ and $c, a_0 \in \mathbb{R}$, $\nabla_B h, \nabla_F \varphi$ and $\nabla F$ are conformal vector fields satisfying (2.6) and $\lambda$ is given by (2.7).

If $n = 1$ then $g_B = \pm dt^2$ and from the first equation of (2.1) and (2.3) we have that $h'' \pm ah = 0$ and $\pm (h')^2 + a h^2 = c$. Since $B$ is not compact it follows that $B^1 = \mathbb{R}$ and the non vanishing of $h$ implies that $\pm a < 0$. Therefore $h$ satisfies

$$h'' - |a|h = 0,$$

$$\langle h' \rangle^2 - |a|h^2 = \pm c,$$

and hence (2.8) holds, i.e.

$$h = \begin{cases} A e^{\sqrt{|a|} t} & \text{if } c = 0, \\ \sqrt{|a|} |\cosh (\sqrt{|a|} t + \theta) & \text{if } c \neq 0, \end{cases}$$

where $A \neq 0$ and $\theta \in \mathbb{R}$.

If $n \geq 2$ and $m \geq 2$, it follows that $B^n$ and $F^m$ are complete Einstein manifolds satisfying (2.5).

Since $f$ satisfies the first equation of (2.6) it follows that $\hat{f} = f - a_0/a$ is a solution of $\nabla \nabla \hat{f} + a \hat{f} g = 0$, therefore from Theorem A.7 we conclude that when $f$ has some critical point then $B \times_h F$ is isometric to a manifold of Class II 1 (resp Class II 2) when $a > 0$ (resp. $a < 0$) and $f$ is a height function on $S^n_\infty (1/\sqrt{|a|})$ (resp. $H^n_\infty (1/\sqrt{|a|})$) (see Examples A.2 and A.3); when $f$ has no critical points then $B \times_h F$ is isometric to a manifold of Class II 3 or II 4.

Since $h$ satisfies the second equation of (2.6) then it follows from Theorem A.7 that if $h$ has no critical points then $B$ is isometric to one of the manifolds of Class II 3 or II 4 and if $h$ has some critical point then $B$ is isometric to a manifold of Class II 1 or II 2 according to the sign of $a$, moreover, $h$ is a height function. However, since $h$ does not vanish it induces a restriction on the index of $B$, in fact, it follows from Proposition A.8 that when $a > 0$ (resp. $a < 0$) $B$ is isometric to $S^n_\infty (1/\sqrt{|a|})$ (resp. $H^n_\infty (1/\sqrt{|a|})$).

Since $\varphi$ satisfies the third equation of (2.1), i.e. $\nabla_F \nabla_F \varphi + (c \varphi + b) g_F = 0$, it follows from Theorem A.5 that if $c = 0$ and $b \neq 0$, then $F$ is isometric to a semi Euclidean space $\mathbb{R}^m_\epsilon$. If $c = b = 0$ then Theorem A.6 implies that $F$ is isometric to a manifold of Class I. Finally, if $c \neq 0$ then Theorem A.7 implies that $F$ is isometric to a manifold of Class II 1 (resp. Class II 2) when $c > 0$ (resp. $c < 0$) and $\varphi$ has some critical point while $F$ is isometric to a manifold of Class II 3 or 4 when $\varphi$ has no critical points.

We conclude by observing that, since $B \times F$ is complete, in order to avoid the phenomena of Been–Buseman example (see [12] or [36], page 209) one must have $F^m$, $m \geq 1$ positive definite (resp. negative definite) if $B$ is positive definite (resp. negative definite).

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APPENDIX A: CONFORMAL FIELDS

Let $(M^n, g)$ be a semi-Riemannian manifold of dimension $n \geq 2$. For a pair of constants $b, c \in \mathbb{R}$, consider a function $\varphi : M \to \mathbb{R}$ satisfying

$$\nabla \nabla \varphi + (c \varphi + b)g = 0. \quad (A.1)$$

The vector field $\nabla \varphi$ of a smooth function $\varphi$ satisfying the above equation is said to be conformal. If $c \neq 0$ we can assume that $b = 0$ replacing $\varphi$ by $\varphi - b/c$. Equation (A.1) has been largely studied since 1920. It started with Brinkman’s work [10] on conformal transformations between semi-Riemannian Einstein manifolds.

Before stating some classification results for complete manifolds that admit non-constant solutions to Equation (A.1), we will present examples of spaces carrying such solutions. In this section, we are following the notation used in [36].

Example A.1. Let $\mathbb{R}^n_\varepsilon$ be the linear space $\mathbb{R}^n$ with the semi-Riemannian metric of index $\varepsilon$

$$\langle v, w \rangle_\varepsilon = \sum_{j=1}^n \varepsilon_j v_j w_j.$$

If $\varphi$ is a non-constant solution of (A.1), then a straightforward calculation shows that $c$ must be zero and that, for all $b \in \mathbb{R}$, a generic solution to (A.1) in $\mathbb{R}^n_\varepsilon$ is given by

$$\varphi(x_1, \ldots, x_n) = -(b/2) \sum_{j=1}^n \varepsilon_j x_j^2 + \langle A_\varepsilon, x \rangle_\varepsilon + A_{n+1} \quad (A.2)$$

where $A_\varepsilon = (\varepsilon_1 A_1, \ldots, \varepsilon_n A_n) \in \mathbb{R}^n_\varepsilon$ and $A_{n+1} \in \mathbb{R}$.

Example A.2. The pseudosphere, with dimension $n$ and index $\varepsilon$, is defined as

$$\mathbb{S}^n_\varepsilon \left(1/\sqrt{c}\right) = \{ x \in \mathbb{R}^n_{\varepsilon+1}; \langle x, x \rangle_\varepsilon = 1/c \}, \quad \text{ where } c > 0.$$  

It is connected if and only if $0 \leq \varepsilon \leq n - 1$ and simply connected if and only if $0 \leq \varepsilon \leq n - 2$. Furthermore, each connected component of $\mathbb{S}^n_\varepsilon \left(1/\sqrt{c}\right)$ is a complete semi-Riemannian manifold of dimension $n$, index $\varepsilon$ and constant curvature $c$. It is not difficult to see that the functions

$$\varphi_A(x) = \langle A_\varepsilon, x \rangle_\varepsilon$$

provide all functions defined in $\mathbb{S}^n_\varepsilon \left(1/\sqrt{c}\right)$ that satisfy (A.1). Note that $\varphi_{A_\varepsilon}(x)$ is the height function with respect to $A_\varepsilon$ on the pseudosphere.

Example A.3. Similarly to the example above, the pseudo-hyperbolic space [36], with dimension $n$ and index $\varepsilon$, is defined as

$$\mathbb{H}^n_\varepsilon \left(1/\sqrt{-c}\right) = \{ x \in \mathbb{R}^n_{\varepsilon+1}; \langle x, x \rangle_{\varepsilon+1} = 1/c \}, \quad \text{ where } c < 0.$$  

It is connected if and only if $2 \leq \varepsilon \leq n$ and simply connected if and only if $1 \leq \varepsilon \leq n - 2$. Furthermore each connected component of $\mathbb{H}^n_\varepsilon \left(1/\sqrt{-c}\right)$ is a complete semi-Riemannian manifold of dimension $n$, index $\varepsilon$ and constant curvature $c$. As in the previous example $\varphi_{A_{\varepsilon+1}}(x) = \langle A_{\varepsilon+1}, x \rangle_{\varepsilon+1}$ provide all functions defined in $\mathbb{H}^n_\varepsilon \left(1/\sqrt{-c}\right)$ that satisfy (A.1). Note that $\varphi_{A_{\varepsilon+1}}(x)$ is the height function with respect to $A_{\varepsilon+1}$ on the pseudo-hyperbolic space.

Example A.4. Let $\pm I \times h : N^{n-1}$ be a warped product manifold, where $I \subset \mathbb{R}$ is a connected interval and $N^{n-1}$ is an arbitrary Riemannian or semi-Riemannian manifold. Then a simple calculation shows that the function

$$\varphi(s, p) = \int_{s_0}^s h(t) \, dt$$
solves Equation (A.1), when \( h \) satisfies

\[
h'' \pm ch = 0. \tag{A.3}
\]

The following Theorems A.5–A.7 are of fundamental importance in the proofs of Section 4. They provide the classification results of complete semi-Riemannian Einstein manifolds that admit a non constant solution of Equation (A.1). These theorems assert the uniqueness of the examples given above, when \( V\varphi \) is proper. The improper case was analyzed by Brinkman [10] showing, among other things, that \( V\varphi \) must be parallel. Since then, spaces carrying parallel improper vector fields are called Brinkman spaces.

**Theorem A.5** ([27]). A complete semi-Riemannian manifold, \((M^n, g)\), with \( n \geq 2 \), admits a non constant solution of the equation \( \nabla \nabla \varphi + bg = 0 \), \( b \neq 0 \), if and only if it is isometric to the semi-Euclidean space \( \mathbb{R}^n_\epsilon \).

This result is a particular case of a theorem proved by Kerbrat [27], where the author classifies spaces carrying vector fields satisfying more general equations.

**Theorem A.6.** A complete semi-Riemannian Einstein manifold, \((M^n, g)\), with \( n \geq 2 \), admits a non constant solution \( \varphi \) of the equation \( \nabla \nabla \varphi = 0 \) if and only if it is isometric to

1. [27] \( \mathbb{R} \times N^{n-1} \), where \((N, g_N)\) is a complete semi-Riemannian Einstein manifold, if \( V\varphi \) is a proper vector field
2. [10] a Brinkman space, if \( V\varphi \) is an improper vector field and \( n \geq 3 \).

The theorem below is a compilation of the classification of Einstein manifolds carrying non-homothetic conformal fields. The Riemannian case was settled essentially by Obata [35] and [26], while the case with positive signature was handled by Kerbrat [27].

**Theorem A.7.** A complete semi-Riemannian Einstein manifold, \((M^n, g)\), with \( n \geq 2 \) and index \( \epsilon \), admits a non constant solution of the equation \( \nabla \nabla \varphi + c\varphi g = 0 \) with \( c \neq 0 \) if and only if it is isometric to

1. \( \mathbb{S}^n_\epsilon(1/\sqrt{\epsilon}) \), when \( 0 \leq \epsilon \leq n - 2 \); the covering of \( \mathbb{S}^n_{n-1}(1/\sqrt{\epsilon}) \) when \( \epsilon = n - 1 \) and the upper part of \( \mathbb{S}^n_\epsilon(1/\sqrt{\epsilon}) \) when \( \epsilon = n \) if \( c > 0 \) and \( \varphi \) has some critical point
2. \( \mathbb{H}^n_\epsilon(1/\sqrt{|\epsilon|}) \), when \( 2 \leq \epsilon \leq n - 1 \); the covering of \( \mathbb{H}^n_\epsilon(1/\sqrt{|\epsilon|}) \) when \( \epsilon = 1 \) and the upper part of \( \mathbb{H}^n_\epsilon(1/\sqrt{|\epsilon|}) \) when \( \epsilon = 0, \) if \( c < 0 \) and \( \varphi \) has some critical point;
3. \((\mathbb{R} \times N^{n-1}, \pm dt^2 + \cosh^2(\sqrt{|\epsilon|} t)g_N)\), where \((N^{n-1}, g_N)\) is a semi-Riemannian Einstein manifold, if \( \varphi \) has no critical points
4. \((\mathbb{R} \times N^{n-1}, \pm dt^2 + e^{2\sqrt{|\epsilon|} t}g_N)\), where \((N^{n-1}, g_N)\) is a Riemannian Einstein manifold, if \( \varphi \) has no critical points.

For our purposes it is important to know if a height function has zeros or not. This is because height functions can occur as warping functions and warping functions do not admit zeros. The next proposition, for which we did not find any reference, reveals which hyperquadrics admit such functions.

**Proposition A.8.** Let \( \varphi_A : \mathbb{R}^{n+1}_\epsilon \to \mathbb{R} \) be the height function with respect to \( A \in \mathbb{R}^{n+1}_\epsilon \), \( A \neq 0 \) and \( n \geq 2 \). Then \( \varphi_A \) has no zeros on \( \mathbb{S}^n_\epsilon(1/\sqrt{\epsilon}) \) (resp. \( \mathbb{H}^n_\epsilon(1/\sqrt{-\epsilon}) \)) if and only if \( \epsilon = n \) (resp. \( \epsilon = 1 \)) and \( A \) is a space like (resp. time like) or light like vector.

**Proof.** We first prove the proposition in the case of the sphere. Since we are considering \( \mathbb{S}^n_\epsilon(1/\sqrt{\epsilon}) \neq \emptyset \), we can assume \( 0 \leq \epsilon \leq n \), i.e., \( \epsilon \neq n + 1 \). Moreover, \( \varphi_A \) is a linear function, hence \( \mathbb{R}^{n+1}_\epsilon = \text{Ker}(\varphi_A) \oplus \text{Im}(\varphi_A) \), where \( \text{Ker}(\varphi_A) = (A)^\perp \subset \mathbb{R}^{n+1}_\epsilon \). Since \( A \neq 0 \), it follows that \( \dim \{ \text{Ker}(\varphi_A) \} = n \geq 2 \) and \( \dim \{ \text{Im}(\varphi_A) \} = 1 \). In what follows, we will analyze each case according to \( A \) being a time like, space like or light like vector. We will consider an appropriate orthonormal basis in each case, \( \{ e_1, ..., e_\epsilon, e_{\epsilon+1}, ..., e_{n+1} \} \) for \( \mathbb{R}^{n+1}_\epsilon \) such that \( e_1, ..., e_\epsilon \) are time like and \( e_{\epsilon+1}, ..., e_{n+1} \) are space like.

Suppose that \( A \) is time like. In this case, \( 1 \leq \epsilon \leq n \) and we choose the basis such that \( e_\epsilon = A/\sqrt{|\epsilon(A, A)_\epsilon|} \). Therefore, \( e_{n+1} \) and \( e_\epsilon \) are orthogonal hence, \( (1/\sqrt{\epsilon})e_{n+1} \in (A^\perp \cap \mathbb{S}^n_\epsilon(1/\sqrt{\epsilon})) \), i.e., \( \varphi_A \) has zeros on the sphere.
Suppose that $A$ is space like. We consider the basis on $\mathbb{R}^{n+1}$, such that $e_{\varepsilon+1} = A_e / |A_e|$. If $0 \leq \varepsilon \leq n - 1$, then $e_{\varepsilon+1}$ and $e_{n+1}$ are orthogonal and hence $(1/\sqrt{c})e_{n+1} \in A \cap S^\varepsilon (1/\sqrt{c})$. If $\varepsilon = n$ then $A \cap S^\varepsilon (1/\sqrt{c}) = 0$, i.e., $\varphi_A$ has no zeros on the sphere.

Suppose that $A$ is light like, then $1 \leq \varepsilon \leq n$ and it is not so difficult to see that there exist orthogonal vectors $V_1, V_2 \in \mathbb{R}^{n+1}$ such that $V_1 \neq 0$ is time like, $V_2 \neq 0$ is space like and $A = V_1 + V_2$. We consider the basis so that $e_\varepsilon = V_1 / \sqrt{\langle V_1, V_1 \rangle}$ and $e_{\varepsilon+1} = V_2 / |V_2|$. If $\varepsilon \leq n - 1$, then $(1/\sqrt{c})e_{\varepsilon+2} \in A \cap S^\varepsilon (1/\sqrt{c})$. Therefore, $\varphi_A$ has no zeros on the sphere if and only if $\varepsilon = n$.

This completes the proof for the case of the sphere. Considering adequate changes, the proof for the hyperbolic space is similar. □

The next result is due to Kerbrat [27] and it can be found in Kuhnel’s paper [32].

**Proposition A.9 ([27]).** Let $\varphi : M \to \mathbb{R}$ be a solution of Equation (A.1). Then the critical points of $\varphi$ are isolated.

The local classification below can be found in [10] or [29] and it is of fundamental importance for the classification of complete manifolds admitting solutions to Equation (A.1).

**Proposition A.10 ([10]).** Let $(M, g)$ be a Riemannian or semi-Riemannian-manifold. The following are equivalent:

1. There is a non constant solution $\varphi$ of

   $$\nabla \nabla \varphi - (\Delta \varphi / n) g = 0,$$

   in a neighborhood of a point $p \in M$ such that $g(\nabla \varphi, \nabla \varphi) \neq 0$.

2. There is a neighborhood $U$ of $p \in M$, a smooth function $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $\varphi'(t) \neq 0$, for all $t \in (-\varepsilon, \varepsilon)$ and a pseudo-Riemannian manifold $(N, g_N)$ such that $(U, g)$ is isometric to the warped product

   $$((-\varepsilon, \varepsilon) \times_{\varphi'} N, \pm dt^2 + (\varphi')^2 g_N),$$

   where $\text{sgn}(g(\varphi', \varphi')) = \pm 1$. 

