Purity Results on $F$-crystals

Jinghao Li

Let $n \in \mathbb{N}$. Let $k$ be a perfect field of characteristic $p > 0$. Let $W(k)$ be the ring of Witt vectors with coefficient in $k$ and let $B(k)$ be its fractional field.

Let $\sigma$ be the absolute Frobenius automorphism:

$\sigma : W(k) \rightarrow W(k)$

$x = (x_0, x_1, \cdots) \mapsto \sigma(x) = (x_0^p, x_1^p, \cdots)$.

We extend $\sigma$ naturally to an automorphism of $B(k)$.

A $\sigma^n$-$F$-crystal (or $F^n$-crystal or $F$-crystal if $n = 1$) over $k$ (or Spec $k$) is a pair $(M, F)$ consisting of a free $W(k)$-module $M$ of finite rank, together with a $\sigma^n$-linear endomorphism $F : M \rightarrow M$, i.e. $F$ is additive and $F(ax) = \sigma^n(a)F(x)$ for all $a \in W(k)$ and $x \in M$, which induces an automorphism of $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The exterior powers of an $F^n$-crystal $(M, F)$ are the $F^n$-crystals $(\wedge^i M, \wedge^i F)$ ($i = 0, 1, 2, \cdots$) with underlying modules $\wedge^i M$ and $\sigma^n$-endomorphisms $\wedge^i F$ defined by

$\wedge^i F(m_1 \wedge \cdots \wedge m_i) = F(m_1) \wedge \cdots \wedge F(m_i)$.

For $i = 0$ and $(M, F) \neq 0$, we define $(\wedge^0 M, \wedge^0 F)$ to be $(W(k), \sigma)$.

A morphism of $F^n$-crystals $f : (M, F) \rightarrow (M', F')$ is a $W(k)$-linear map $f : M \rightarrow M'$ such that $fF = F'f$.

The category of $F^n$-crystals up to isogeny is obtained from the category of $F^n$-crystals by keeping the same objects, but tensoring $\text{Hom}$ groups, which are $\mathbb{Z}_p$ modules, over $\mathbb{Z}_p$ with $\mathbb{Q}_p$.

An isogeny between $F^n$-crystals is a morphism of $F^n$-crystals which becomes an isomorphism in the category of $F^n$-crystals up to isogeny.

An $F^n$-crystal is said to be divisible by $\lambda > 0$ if for all $m \in \mathbb{N}$, we have $F^m = 0 \mod p^{\lfloor m/\lambda \rfloor}$, where $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ is the floor function.

The Hodge slopes of an $F^n$-crystal $(M, F)$ are the integers defined as follows. The image $F(M)$ is a $W(k)$-submodule of $M$ of rank $r = \text{rank}(M)$, so by the theory of elementary divisors, since $W(k)$ is a discrete valuation ring, in particular a principal ideal domain, there exist $W(k)$-bases $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ of $M$ such that for all $1 \leq i \leq r$ we have

$F(v_i) = p^{a_i}w_i$

for some integers $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$. These integers are called the Hodge slopes of $(M, F)$.

The Hodge polygon of $(M, F)$ is the graph of the Hodge function on $[0, r]$ defined on integers $0 \leq i \leq r$ by
Hodge_F(i) = least Hodge slope of ($\Lambda^i M, \lambda^i F$) = \begin{cases} 
0, & i = 0, \\
\lambda_1 + \cdots + \lambda_i, & 1 \leq i \leq r 
\end{cases}
and then extended linearly between successive integers.

The Newton slopes of an $F^n$-crystal $(M, F)$ are the sequence of $r = \text{rank}(M)$ rational numbers $0 \leq \lambda_1 \leq \cdots \leq \lambda_r$ defined in the following way.

Pick an algebraically closed field extension $k'$ of $k$, and consider the $F^n$-crystal over $k'$: $(M \otimes W(k), F \otimes \sigma^n)$ obtained from $(M, F)$ by extension of scalars. For each non-negative rational number $\lambda = \frac{a}{b}$ with $a \in \mathbb{Z}, b \in \mathbb{N}, (a, b) = 1$, we denote by $E(\lambda)$ the $F^n$-crystal over $k'$ defined by:

$E(\lambda) = ((\mathbb{Z}_p[T]/(T^b - p^a)) \otimes \mathbb{Z}_p) W(k')$, (multiplication by $T^n \otimes \sigma^n$).

**Theorem 0 (Dieudonné–Manin, [De72, (Theorem, Page 85)])** If $k'$ is an algebraically closed extension of $k$, then for any $F^n$-crystal $(M, F)$ over $k$, $(M \otimes W(k)) B(k')$ is isomorphic to a finite direct sum of $E(\lambda_i) \otimes W(k') B(k')$'s with $\lambda_i \in \mathbb{Q}_{\geq 0}$.

By the above theorem, we can write $(M \otimes W(k)) B(k', F \otimes \sigma^n) \cong \bigoplus_{i=1}^s E(a_i/b_i) \otimes B(k')$ for an increasing sequence $a_1/b_1 \leq a_2/b_2 \leq \cdots \leq a_s/b_s$ with $\sum b_i = r$. The Newton slopes of $(M, F)$ are defined to be the sequence of $r$ rational numbers $(\lambda_1, \cdots, \lambda_r) := (a_1/b_1 \text{ repeated } b_1 \text{ times}, a_2/b_2 \text{ repeated } b_2 \text{ times}, \cdots, a_s/b_s \text{ repeated } b_s \text{ times}).$

The Newton polygon of $(M, F)$ is the graph of the Newton function on $[0, r]$, defined on integers $0 \leq i \leq r$ by

$\text{Newton}_F(i) = \text{least Newton slope of } (\Lambda^i M, \lambda^i F) = \begin{cases} 
0, & i = 0, \\
\lambda_1 + \cdots + \lambda_i, & 1 \leq i \leq r 
\end{cases}$
and then extended linearly between successive integers. Let $\{\mu_1, \mu_2, \cdots, \mu_t\} = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ with $\mu_1 < \mu_2 < \cdots < \mu_t$ for $1 \leq t \leq r$ and let $r_i$ be the multiplicity of $\mu_i$ for $1 \leq i \leq t$.

Here is the graph of the Newton polygon of $(M, F)$:
For another characterization of the Newton slopes, we choose an auxiliary integer $N \geq 1$ which is divisible by $r!$, where $r = \text{rank}(M)$, and consider the discrete valuation ring $R = W(k')[X]/(X^N - p) = W(k'[p^{1/N}])$. We can extend $\sigma$ to an automorphism of $R$ by requiring that $\sigma(X + (X^N - p)) = X + (X^N - p)$. For each $\lambda \in \frac{1}{N} \mathbb{Z}$, we can speak about $p^\lambda = X^N + (X^N - p)$ in $R$.

Let $K = \text{Frac}(R)$. By an analogue of Dieudonné–Manin’s theorem over $K$, we know that $M \otimes_{W(k)} K$ admits a $K$-basis $e_1, \ldots, e_r$ which transforms under the $\sigma$–linear endomorphism $F \otimes \sigma$ by the formula $(F \otimes \sigma)(e_i) = p^\lambda_i e_i$ for $1 \leq i \leq r$. An equivalent characterization of the Newton slopes is by the existence of an $R$-basis $u_1, \ldots, u_r$ of $M \otimes_{W(k)} R$ with respect to which the matrix of $F \otimes \sigma$ is upper-triangular, with $p^\lambda_i$'s along the diagonal, i.e. $F(u_i) \equiv p^\lambda_i u_i \mod \sum_{1 \leq j \leq i} Ru_j$ for all $1 \leq i \leq r$.

The second characterization of Newton slopes shows:

1. The Newton slopes of the $m^{th}$ iterate $(M, F^m)$ of $(M, F)$ are $(m\lambda_1, \ldots, m\lambda_r)$.
2. All Newton slopes $\lambda_i$ of $(M, F)$ are equal to 0 if and only if $F$ is a $\sigma^n$-linear automorphism of $M$.
3. All Newton slopes $\lambda_i$ of $(M, F)$ are $> 0$ if and only if $F$ is topologically nilpotent on $M$, i.e. if and only if we have $F^r(M) \subset pM$ where $r = \text{rank}(M)$.

The break points of an $F^n$-crystal $(M, F)$ are defined to be the points where the Newton polygon changes slopes.
Remark: From the definition of the Newton slopes, we have that all break points have integer coordinates.

Let \( m \in \mathbb{N} \). Let \( R \) be an \( \mathbb{F}_p \)-algebra, let \( W_m(R) \) be the ring of Witt vectors of length \( m \) with coefficients in \( R \) and let \( \sigma \) be the Frobenius endomorphisms of \( W_m(R) \) for any \( m \in \mathbb{N} \). Let \( \mathcal{M}^\sigma(W_m(R)) \) be the abelian category whose objects are \( W_m(R) \)-modules endowed with \( \sigma^m \)-linear endomorphisms and whose morphisms are \( W_m(R) \)-linear maps that respect the \( \sigma^m \)-linear endomorphisms. We identify \( \mathcal{M}^\sigma(W_m(R)) \) with a full subcategory of \( \mathcal{M}^\sigma(W_{m+1}(R)) \). With a little abuse of terminology, we call the following morphism modulo \( p^m \):

\[
\mathcal{M}^\sigma(W_{m+1}(R)) \to \mathcal{M}^\sigma(W_m(R))
\]

\[
O \mapsto O \otimes_{W_{m+1}(R)} W_m(R),
\]

for any \( s \in \mathbb{N}_{>0} \). If \( S \) is an \( \mathbb{F}_p \)-scheme, in a similar way we define \( \mathcal{M}^\sigma(W_m(S)) \). If \( S = \text{Spec} \, R \), we identify \( \mathcal{M}^\sigma(W_m(R)) = \mathcal{M}^\sigma(W_m(S)) \).

In general, we can define an \( F^n \)-crystal \( \mathcal{C} \) over any \( \mathbb{F}_p \)-algebra \( R \), cf. [Ka79, (2.1)]. The evaluation of the \( F^n \)-crystal \( \mathcal{C} \) at the thickening \((R \hookrightarrow W_m(R))\) is a triple \((m\mathcal{C}, F, m\nabla)\), where \( m\mathcal{C} \) is a locally free \( W_m(R) \)-module of finite rank, \( F : m\mathcal{C} \to m\mathcal{C} \) is a \( \sigma^m \)-linear endomorphism, and \( m\nabla \) is an integrable and topologically nilpotent connection on \( m\mathcal{C} \) that satisfies certain axioms.

In this paper, connections as \( m\nabla \) will play no role. A morphism \( \phi : \mathcal{C} \to \mathcal{C}_1 \) of \( F^n \)-crystals over \( R \) defines naturally a morphism in the category of \( \mathcal{M}^\sigma(W_m(R)) \)

\[
m\phi : m\mathcal{C} \to m\mathcal{C}_1.
\]

The association \( \phi \to m\phi \) defines a \( \mathbb{Z}_p \)-linear (evaluation) functor from the category of \( F^n \)-crystals over \( R \) into the category \( \mathcal{M}^\sigma(W_m(R)) \).

Let \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be two objects of \( \mathcal{M}^\sigma(W_m(R)) \) such that their underlying \( W_m(R) \)-modules are locally free of finite ranks. Let \( S = \text{Spec} \, R \). We consider the functor

\[
\text{Hom}(\mathcal{O}_1, \mathcal{O}_2) : \text{Sch}^S \to \text{SET}
\]
from the category $\text{Sch}^S$ of $S$-schemes to the category $\text{SET}$ of sets, with the property that $\text{Hom}(\mathcal{O}_1, \mathcal{O}_2)(S_1)$ is the set underlying the $\mathbb{Z}/p^n\mathbb{Z}$-module of morphisms of $\mathcal{M}(W_m(S_1))$ that are between $f'(\mathcal{O}_1)$ and $f'(\mathcal{O}_2)$; Here $f : S_1 \to S$ is the structural morphism of $S_1$ and $f^*$ is the pullback to $S_1$

\[
\begin{array}{ccc}
  f'^*(\mathcal{O}_1) & \to & \mathcal{O}_1 \\
  \downarrow & & \downarrow \\
  S_1 & \to & S \\
\end{array}
\quad
\begin{array}{ccc}
  f'^*(\mathcal{O}_2) & \to & \mathcal{O}_2 \\
  \downarrow & & \downarrow \\
  S_1 & \to & S. \\
\end{array}
\]

**Lemma 0** The functor $\text{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ is representable by an affine $S$-scheme which locally is of finite presentation.

**Proof of Lemma 0:** In [Va06, (Lemma 2.8.4.1)], Vasiu proved the lemma in the case when $n = 1$. We note that the proof for a general natural number $n$ goes the same way.

Now let $R$ be a reduced $\mathbb{F}_p$-algebra. Let $R^\text{perf}$ be its perfect closure.

Example: If $R = k[x_1, \cdots, x_n]$, then $R^\text{perf} = \bigcup_{u \geq 1} k[x_1^u, \cdots, x_n^u]$.

Talking about an $F^n$-crystal over $S = \text{Spec } R$, we can look at its pullback to its perfect closure $R^\text{perf}$. The pullback of such an $F$-crystal to $R^\text{perf}$ is a finite rank locally free $W(R^\text{perf})$-module $M$ equipped with a Frobenius linear map $F : M \to M$ such that $F(M) \supset p'M$ for some $t \geq 0$.

Let $S = \text{Spec } R$ be a reduced affine $\mathbb{F}_p$-scheme. Let $\mathcal{C}$ be an $F^n$-crystal over $S$. For any $\mathbb{F}_p$-homomorphism $\phi : R \to k$, the pullback $\mathcal{C}^{(\phi)}$ is an $F^n$-crystal over $k$. Its Newton polygon depends only on the underlying point $\text{Ker}(\phi) \in S$. This allows us to speak about the Newton slopes and Newton polygons of $\mathcal{C}$ at various points of $S$.

We define a function

\[ f^{\mathcal{C}} : S \to \text{Set of Newton Polygons} \]

\[ s \mapsto \text{NP}(\mathcal{C}_s), \text{ where } \mathcal{C}_s \text{ is the pullback of } \mathcal{C} \text{ to the algebraic closure } \bar{k(s)} \text{ of the residue field } k(s) \text{ of the point } s \in S, \text{ i.e.,} \]

\[
\begin{array}{ccc}
  \mathcal{C}_s & \to & \mathcal{C} \\
  \downarrow & & \downarrow \\
  \text{Spec } \bar{k(s)} & \to & S. \\
\end{array}
\]

By the Newton polygon stratification of an $\mathbb{F}_p$-scheme $S$ defined by an $F^n$-crystal $\mathcal{C}$ over $S$ we mean the stratification of $S$ with the property that each stratum of it is of the form $f^{\mathcal{C}}_u^{-1}(\text{a fixed Newton Polygon})_{\text{red}}$. For a given Newton polygon $\nu$, let $S_\nu = f^{\mathcal{C}}_u^{-1}(\nu)_{\text{red}}$.

Clearly we have a set partition: $S = \bigsqcup_{\nu \in I} S_\nu$, where $I$ is the set of all possible Newton Polygons.

Example: Let $R = k[t]$ and $R^\text{perf} = k[t]_{\text{perf}}$. Let $S = \text{Spec } R$ be as above. Let $I \in W(R^\text{perf})$ be the image of $(t, 0, 0, \cdots) \in W(R)$. Suppose $M$ is an $F$-crystal over $R$ such that its pullback to $R^\text{perf}$ is an $F$-crystal $(M^\text{perf}, F)$ where $M^\text{perf} = W(R^\text{perf}) \oplus W(R^\text{perf})$ with $[e_1, e_2]$ as a basis.
and \( F(e_1) = te_1 + pe_2, \) \( F(e_2) = pe_1 \). We observe that when \( t = 0 \) the corresponding Newton polygon \( \nu_1 \) has no break point in the middle (i.e., it is not the starting or ending point) and a unique slope 1. When \( t \neq 0 \), the corresponding Newton polygon \( \nu_2 \) has the break point \((1, 0)\) in the middle and two Newton slopes 0 and 2. Thus in this case as sets we have \( S_{\nu_1} = \{(t)\}, \) \( S_{\nu_2} = S \setminus \{(t)\} \) and \( S_{\nu} = \emptyset \) for \( \nu \neq \nu_1 \) or \( \nu_2 \).

**Theorem 1** (Grothendieck–Katz, [Ka79, Theorem 2.3.1]) Let \( S \) be an \( \mathbb{F}_p \)-scheme, \( \mathcal{C} \) be an F-crystal over \( S \) and \( \nu \) be a Newton polygon. Then the set
\[
S_{\geq \nu} = \{s \in S | NP(\mathcal{C}_s) \text{ lies above } \nu\}
\]
is Zariski closed in \( S \).

**Corollary 1** Let \( S \) be an \( \mathbb{F}_p \)-scheme, \( \mathcal{C} \) be an F-crystal over \( S \) and \( \nu \) be a Newton polygon. All the strata \( S_{\nu} \) of the Newton polygon stratification of \( S \) defined by \( \mathcal{C} \) are locally closed subschemes of \( S \).

**Types of purity notions:**

Let \( S \) be an \( \mathbb{F}_p \)-scheme. Let \( T \) be a reduced locally closed subscheme of \( S \). Let \( \overline{T} \) be the schematic closure of \( T \), i.e., topologically \( \overline{T} \) is the Zariski closure of \( T \) in \( S \) and endowed with the reduced ringed structure.

(à la Nagata–Zariski) Suppose \( S \) is locally Noetherian. We say \( T \) is weakly pure in \( S \), if each non-empty irreducible component of \( \overline{T} \setminus T \) has pure codimension 1 in \( \overline{T} \).

Suppose \( S \) is locally Noetherian. We say \( T \) is universally weakly pure in \( S \), if for every locally Noetherian scheme \( S_1 \) equipped with a morphism \( S_1 \rightarrow S \), the locally closed subscheme \( (T_\times S_1)_{\text{red}} \) is weakly pure in \( S_1 \).

(Vasiu) We say \( T \) is pure in \( S \), if \( T \) is an affine \( S \)-scheme.

(folklore) We say \( T \) is strongly pure in \( S \), if locally in the Zariski topology of \( \overline{T} \) there exists a global function \( f \) on \( \overline{T} \) such that \( T = \overline{T}_f \) is the largest open subscheme of \( \overline{T} \) over which \( f \) is invertible.

**Lemma 1** Let \( S \) be a reduced locally Noetherian \( \mathbb{F}_p \)-scheme. If \( T \subset S \) is locally closed and \( S \)-affine, then \( \overline{T} \setminus T \) is either \( \emptyset \) or of pure codimension 1 in \( S \). In particular, purity implies weak purity.

**Proof of Lemma 1:** Since this is a local statement, we can assume both \( S = \text{Spec } R \) and \( T = \text{Spec } A \) are reduced affine schemes. By replacing \( S \) by \( \overline{T} \), we can assume \( T \) is open dense in \( S \). If \( T = S \), the statement is proved. Otherwise, let \( x \) be a generic point of an irreducible component of \( S \setminus T \) and let \( d = \dim \mathcal{O}_{S,x} \). We need to show \( d = 1 \). Since \( T \) is dense in \( S \) and \( S \setminus T \neq \emptyset \), we know \( d \geq 1 \). Consider the pullback \( \hat{T} \) of \( T \) in the following commutative diagram:

\[
\begin{array}{ccc}
\hat{T}_\text{red} & \xrightarrow{\epsilon} & \text{Spec } \mathcal{O}_{S,x}_\text{red} \\
\downarrow & & \downarrow \\
T' & \xrightarrow{\phi} & S = \text{Spec } R.
\end{array}
\]
By replacing \( S \) by \( \text{Spec} \, \tilde{\mathcal{O}}_{S,x_{\text{red}}} \) and \( T \) with \( \hat{T}_{\text{red}} \), we can assume \( R \) is a local, reduced, complete noetherian \( \mathbb{F}_p \)-algebra and all the points in \( S \setminus T \) are of codimension \( \geq d \) in \( S \). As \( R \) is local, complete ring, it is also excellent (cf. [Hi80, (34.B)]). Thus the normalization \( S^n \) of \( S \) is a finite \( S \)-scheme. Consider the following commutative diagram:

\[
\begin{array}{ccc}
T^n & \longrightarrow & S^n = \text{Spec} \, R^n \\
\downarrow & & \downarrow \\
T^c & \longrightarrow & S = \text{Spec} \, R.
\end{array}
\]

Since the morphism \( S^n \to S \) is both finite and surjective, for any preimage of \( x \) in \( S^n \), say \( \tilde{x} \in S^n \), we have \( \dim \mathcal{O}_{S^n,\tilde{x}} = \dim \mathcal{O}_{S,x} \). By replacing \( x \) by \( \tilde{x} \), \( T \) by \( T^n \) and \( S \) by \( S^n \), we can assume both \( T = \text{Spec} \, A \) and \( S = \text{Spec} \, R \) are affine, normal, reduced, Noetherian \( \mathbb{F}_p \)-schemes and all the points in \( S \setminus T \) are of codimension \( \geq d \). For \( 1 \leq i \leq n \), let \( T_i = \text{Spec} \, A_i \subset S_i = \text{Spec} \, R_i \) be the irreducible components of \( T \) and \( S \) respectively and we have \( A_i \) and \( R_i \) are all normal, integral, Noetherian \( \mathbb{F}_p \)-algebras. Now suppose \( d \geq 2 \). At the level of rings, the codimension 1 points are height 1 prime ideals. Therefore for any \( 1 \leq i \leq n \) we have,

\[
R_i \hookrightarrow A_i = \bigcap_{p \text{ a prime of height 1}} A_{i,p} = \bigcap_{p \text{ a prime of height 1}} R_{i,p} = R_i
\]

(cf. [Hi80, (17.H)] for the first and third equalities). Therefore

\[
S = \text{Spec} \prod_{i=1}^n R_i = \text{Spec} \prod_{i=1}^n A_i = T,
\]

which is a contradiction. Hence \( d = 1 \) and Lemma 1 is proved. □

We have the following obvious implications and identifications:

Strong purity \( \Rightarrow \) purity \( \Rightarrow \) univeral weak purity \( \Rightarrow \) weak purity
Purity = universal purity
Strong purity = universal strong purity

**Theorem 2** (A. J. de Jong and F. Oort, [JO00, (Theorem 4.1)]) Let \( S \) be a reduced locally Noetherian \( \mathbb{F}_p \)-scheme and let \( \mathcal{C} \) be an \( F \)-crystal over \( S \). Then the Newton polygon stratification of \( S \) defined by \( \mathcal{C} \) is universally weakly pure in \( S \).

**Theorem 3** (A. Vasiu, [Va06, (Theorem 6.1)]) Let \( \mathcal{C} \) be an \( F \)-crystal over a reduced locally Noetherian \( \mathbb{F}_p \)-scheme \( S \). Then the Newton polygon stratification of \( S \) defined by \( \mathcal{C} \) is pure in \( S \).

Let \( P_0 \) be a point in the \( xy \)-coordinate plane and let

\[
S_{P_0} = \{ s \in S \mid NP(\mathcal{C}, s) \text{ has } P_0 \text{ as a break point} \}.
\]

It can be shown that topologically \( S_{P_0} \) is locally closed in \( S \) (We will prove this in the proof of Theorem 5), and we endow it with the reduced ringed structure.

**Theorem 4** (Y. Yang, [Ya10, (Theorem 1.1)]) Let \( S \) be a reduced locally Noetherian \( \mathbb{F}_p \)-scheme and let \( \mathcal{C} \) be an \( F \)-crystal over \( S \). Fix a point \( P_0 \) in the \( xy \)-coordinate plane. Then \( S_{P_0} \)
is universally weakly pure in $S$.

Our main result is the following theorem, which will imply Theorems 2 to 4:

**Theorem 5 (J. Li)** Let $S$ be a reduced locally Noetherian $\mathbb{F}_p$-scheme. Let $\mathcal{C}$ be an $F^n$-crystal over $S$, $n \geq 1$. Fix a point $P_0$ in the xy-coordinate plane. Then $S_{P_0}$ is pure in $S$.

**Proof of Theorem 5:** It will be done in 5 steps.

**Step 1. Reduction step.**

Since purity is a local statement, we first assume $S = \text{Spec } R$ is affine and we need to show that $S_{P_0}$ is an affine scheme. Let $P_0 = (a, b)$. If $(a, b) \notin \mathbb{N}^2$ or $a = 0$ and $b \neq 0$, then $S_{P_0} = \emptyset$ and the theorem holds trivially. If $(a, b) = (0, 0)$, then $S_{P_0} = S$. Again the theorem holds trivially. Now we suppose $a, b \in \mathbb{N}_{>0}$. By replacing $\mathcal{C}$ by $\wedge^a \mathcal{C}$, we see that $S_{(a, b)} = \{ s \in S | \text{NP}(\wedge^a \mathcal{C}_s) \text{ has } (1, b) \text{ as a break point} \}$. Therefore, we can assume $a = 1$. By replacing $\mathcal{C}$ by $\mathcal{C}^\infty$ with $c$ a large integer (For example $c = r!$, where $r = \text{rank}(\mathcal{C})$), we can assume that for each point $s \in S$, all Newton slopes of $\mathcal{C}_s$ are integers. Now let $S_{\geq v_1} = \{ s \in S | \text{NP}(\mathcal{C}_s) \geq v_1 \}$, where $v_1$ is the following Newton polygon:

From Theorem 1, $S_{\geq v_1}$ is closed in $S$, thus affine and since $(1, b)$ is a break point of $v_1$, we have $S_{P_0} \subset S_{\geq v_1}$. By replacing $S$ by $S_{\geq v_1}$, we can assume $\text{NP}(\mathcal{C}_s) \geq v_1$ for any $s \in S$. Let $S_{\geq v_2} = \{ s \in S | \text{NP}(\mathcal{C}_s) \geq v_2 \}$, where $v_2$ is the following Newton polygon:
We have $S_{P_0} = S\setminus S_{\geq 2}$ and this shows that $S_{P_0}$ is locally closed in $S$. If $S_{\geq 2} = \emptyset$, then $S_{P_0} = S$ is affine and the theorem is proved. Now we suppose $S_{\geq 2} \neq \emptyset$ and $S_{P_0} = S\setminus S_{\geq 2}$ is an open subscheme of $S = \text{Spec } R$ and we need to show $S_{P_0}$ is affine. The statement is local in the faithfully flat topology of $S$ and thus we can assume that $S$ is local. Let $\hat{R}$ be the completion of $R$ and let $\hat{S} := \text{Spec } \hat{R}$. As $\hat{S}$ is a faithfully flat $S$-scheme, to show that $S_{P_0}$ is affine it suffices to show that $S_{P_0} \times_S \hat{S}$ is affine. Let $\hat{S}_1 = \text{Spec } \hat{R}_1, \cdots, \hat{S}_j = \text{Spec } \hat{R}_j$ be the irreducible components of the reduced scheme of $\hat{S}$ (Here $j \in \mathbb{N}$); They are spectra of local, complete, integral, Noetherian $\mathbb{F}_p$-algebras. The scheme $S_{P_0} \times_S \hat{S}$ is affine if and only if the irreducible components $S_{P_0} \times_S \hat{S}_1, \cdots, S_{P_0} \times_S \hat{S}_j$ of the reduced scheme of $S_{P_0} \times_S \hat{S}$ are all affine, cf. Chevalley’s theorem in [Gr61, Ch. II, Cor. (6.7.3)]. So to prove the theorem we can assume $R = \hat{R} = \hat{R}_1$. As $R$ is a local, complete ring, it is also excellent, cf. [Hi80, (34.B)]. Thus the normalization $S''$ of $S$ is a finite $S$-scheme. So $S''$ is a semilocal, complete, integral, normal scheme. This implies $S''$ is local. But $S_{P_0}$ is affine if and only if $S_{P_0} \times_S S''$ is affine, cf. [Va06, (Lemma 2.9.2)]. Thus to prove the theorem, we can also assume $S$ is normal, i.e., $S = S''$. We emphasize that for the rest of the proof we will use the fact that $R$ is a complete, integral, local, normal $\mathbb{F}_p$-algebra and $U := S_{P_0}$ is an open subscheme of $S = \text{Spec } R$. We can assume $U \neq \emptyset$ and hence it is dense in $S$. Let $k_S$ be the field of fractions of $R$ and $k$ be its algebraic closure.

**Step 2. The affine $S$-scheme $H_m$.**

Now let $\mathcal{C}_0$ be an $F^n$-crystal of rank 1 and slope $b$ over $\mathbb{F}_p$, that is $\mathcal{C}_0 = (\mathbb{Z}_p, p^b \sigma^n)$. Let $\mathcal{C}_{0,S}$ be its pullback to $S$:

\[
\begin{array}{ccc}
\mathcal{C}_{0,S} & \xrightarrow{\sigma} & \mathcal{C}_0 \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & \text{Spec } \mathbb{F}_p.
\end{array}
\]

Let $m \in \mathbb{N}$ and $m >> 0$. Let $m \mathcal{C}_{0,S}$ be the evaluation of $\mathcal{C}_{0,S}$ at $W_m(S) = W_m(R)$ and $m\mathcal{C}$ be the evaluation of $\mathcal{C}$ at $W_m(S)$. We view these evaluations as $W_m(R)$-modules equipped with $\sigma^n$-linear endomorphisms. By Lemma 0, the functor $\text{HOM}(m\mathcal{C}_{0,S}, m\mathcal{C})$ is representable by an affine $S$-scheme $H_m$ which is of finite presentation and it is Noetherian since $S$ is Noetherian. In other words, we have an affine morphism of finite type: $H_m \rightarrow S$. Let $x$ be a point of $U$ in $S$. Consider the stalk at $x$: $V_x := \mathcal{O}_{U,x} = \mathcal{O}_{S,x}$. It is an integrally closed noetherian
local domain of dimension one. Therefore, $V_s$ is a discrete valuation ring. Let $W_s$ be a discrete valuation ring with an algebraically closed residue field such that $V_s \hookrightarrow W_s$ and to simplify our notation, let $W := W_s^{\text{perf}}$ be its perfect closure. Consider the pullback of $\mathcal{C}$ to $W$:

\[
\begin{array}{ccc}
\mathcal{C}_W & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\text{Spec } W & \longrightarrow & S.
\end{array}
\]

As $S = S_{\mu}$, all Newton slopes of $\mathcal{C}_W$ are greater or equal to $b$ at every point of $\text{Spec } W$. By [Ka79, (Theorem 2.6.1)], we have an isogeny $\phi: \mathcal{C}_W \rightarrow \mathcal{C}'_W$, where $\mathcal{C}'_W$ is an $F^n$–crystal over $W$ which is divisible by $b$ and the cokernel of $\phi$ is annihilated by $p^t$ for $t = (r - 1)b$. Let $\varphi$ be the isogeny from $\mathcal{C}'_W$ to $\mathcal{C}_W$. Next we prove the following lemma:

**Lemma 2** Let $W$ be a perfect discrete valuation ring of characteristic $p$ with an algebraically closed residue field. Let $(M, F)$ be an $F^n$-crystal over $W$ such that at each point of $\text{Spec } W$, all its Newton slopes are greater than or equal to $b$ and it is divisible by $b$. Assume the multiplicity of its Newton slope $b$ is 1 at each point of $\text{Spec } W$, then there exists a unique sub-$F^n$-crystal $(M_1, F_1)$ of $(M, F)$ which is also a direct summand, where $(M_1, F_1)$ is an $F^n$-crystal over $W$ of rank $1$, Newton slope $b$.

**Proof of Lemma 2:** Since $(M, F)$ is divisible by $b$, we have $F(M) \subset p^b M$. Let $w$ be an arbitrary point in $\text{Spec } W$ and let $(M_w, F_w)$ be the pullback of $(M, F)$ to $k(w)$, where $k(w)$ is the algebraic closure of the residue field $k(w)$ at $w$.

\[
\begin{array}{ccc}
(M_w, F_w) & \longrightarrow & (M, F) \\
\downarrow & & \downarrow \\
\text{Spec } k(w) & \longrightarrow & \text{Spec } W.
\end{array}
\]

We have $F_w(M_w) \subset p^b M_w$ and $(1, b)$ is a break point of the Newton polygon of $(M_w, F_w)$ for every $w \in \text{Spec } W$. If $F_w(M_w) \subset p^{b+1} M_w$, then all the Newton slopes of $(M_w, F_w)$ will be greater than $b$, which is a contradiction. Thus $F_w(M_w) \not\subset p^{b+1} M_w$ and $F_w(M_w) \subset p^b M_w$. Since for every point $w \in \text{Spec } W$ the multiplicity of the Newton slope $b$ of $(M_w, F_w)$ is 1, therefore the point $(1, b)$ lies on the Hodge polygon of $(M_w, F_w)$ at every point $w \in \text{Spec } W$ and by [Ka79, (Theorem 2.4.2)], lemma holds. ■

Applying Lemma 2, we have a monomorphism: $\tilde{\mathcal{C}}_{0, W} \hookrightarrow \mathcal{C}'_W$, where $\tilde{\mathcal{C}}_{0, W}$ is an $F^n$–crystal over $W$ of rank $1$ and Newton slope $b$ and $j_m$ admits a unique splitting. Modulo $p^m$, at the level of evaluation we get a monomorphism: $\tilde{\mathcal{C}}_{0, W} \hookrightarrow m\mathcal{C}'_W$. Let $\mathcal{C}_{0, W}$ be the pullback of $\mathcal{C}_{0, S}$ to $\text{Spec } W$:

\[
\begin{array}{ccc}
\mathcal{C}_{0, W} & \longrightarrow & \mathcal{C}_{0, S} \\
\downarrow & & \downarrow \\
\text{Spec } W & \longrightarrow & S.
\end{array}
\]

Since $\mathcal{C}_{0, W}$ is also an $F^n$-crystal over $W$ of rank $1$ and Newton slope $b$, modulo $p^m$ we have $m\mathcal{C}_{0, W}$ isomorphic to $m\mathcal{C}_{0, W}$. Now we have a morphism $i_W(m)$ from $m\mathcal{C}_{0, W} \rightarrow m\mathcal{C}_W$ by composing the following morphisms: $m\mathcal{C}_{0, W} \cong m\mathcal{C}_{0, W} \hookrightarrow m\mathcal{C}'_W \rightarrow m\mathcal{C}_W$, where $\varphi_m$ is the isogeny $\varphi$.
modulo $p^n$, and thus its cokernel is annihilated by $p'$. Let $f_m$ be the composition of the morphisms: $m\mathcal{C}_{0,W} \cong m\tilde{\mathcal{C}}_{0,W} \hookrightarrow m\mathcal{C}'_W$ and we know that $f_m$ is a monomorphism that splits.

**Step 3. Gluing morphisms.**

Before we glue the morphisms, let us first discuss three useful cases of inductive limits.

Let $V \hookrightarrow V_1$ be a monomorphism of commutative $\mathbb{F}_{p'}$-algebras. Suppose we have an inductive limit $V_1 = \text{ind lim} \, V_\alpha$ of commutative $V$-subalgebras of $V_1$ indexed by the set of objects $\Lambda$ of a filtered, small category. For $\alpha \in \Lambda$, let $f^\alpha : \text{Spec } V_\alpha \to \text{Spec } V$ be the natural morphism. Let $(O, \phi_O)$ and $(O', \phi_{O'})$ be objects of $M'(W_m(V))$ such that $O$ and $O'$ are free $W_m(V)$-modules of finite rank. Let $(O_1, \phi_{O_1})$ and $(O'_1, \phi_{O'_1})$ be the pullbacks of $(O, \phi_O)$ and $(O', \phi_{O'})$ (respectively) to objects of $M'(W_m(V_1))$. We consider a morphism

$$u_1 : (O_1, \phi_{O_1}) \to (O'_1, \phi_{O'_1})$$

of $M'(W_m(V_1))$. We fix ordered $W_m(V)$-bases $B_O$ and $B'_O$ of $O$ and $O'$ (respectively). Let $B_1$ be the matrix representation of $u_1$ with respect to the ordered $W_m(V_1)$-basis of $O_1$ and $O'_1$ defined naturally by $B_O$ and $B'_O$ (respectively). Let $V_{u_1}$ be the $V$-subalgebra of $V_1$ generated by the components of the Witt vectors of length $m$ with coefficients in $V_1$ that are entries of $B_1$. As $V_{u_1}$ is a finitely generated $V$-algebra, there exists $\alpha_0 \in \Lambda$ such that $V_{u_1} \hookrightarrow V_{\alpha_0}$. This implies that $u_1$ is the pullback of a morphism

$$u_{\alpha_0} : f^{\alpha_0}_m(O, \phi_O) \to f^{\alpha_0}_m(O', \phi_{O'})$$

of $M'(W_m(V_{\alpha_0}))$. Here are three special cases of interest.

(a) If $V$ is a field and $V_1$ is an algebraic closure of $V$, then as $V_\alpha$’s we can take the finite field extensions of $V$ that are contained in $V_1$.

(b) If $V_1$ is a local ring of an integral domain $V$, then as $V_\alpha$’s we can take the $V$-algebras of global functions of open, affine subschemes of Spec $V$ that contain Spec $V_1$.

(c) We consider the case when $V$ is a discrete valuation ring that is an $N - 2$ ring in the sense of [Hi80, (31.A)], when $V_1$ is a faithfully flat $V$-algebra that is also a discrete valuation ring, and when each $V_{\alpha}$ is a $V$-algebra of finite type. The flat morphism $f^{\alpha_0}_m : \text{Spec } V_{\alpha_0} \to \text{Spec } V$ has quasi-sections, cf. [Gr64, Ch. IV, Cor. (17.16.2)]. In other words, there exists a finite field extension $\tilde{k}$ of $k$ and a $V$-subalgebra $\tilde{V}$ of $\tilde{k}$ such that: (i) $\tilde{V}$ is a local, faithfully flat $V$-algebra of finite type of $\tilde{k}$, and (ii) we have a morphism $f^{\alpha_0}_m : \text{Spec } \tilde{V} \to \text{Spec } V_{\alpha_0}$ such that $\tilde{f} := f^{\alpha_0}_m \circ \phi_{\alpha_0}$ is the natural morphism Spec $\tilde{V} \to$ Spec $V$. As $V$ is an $N - 2$ ring, its normalization in $\tilde{k}$ is a finite $V$-algebra and so a Dedekind domain. This implies that we can assume $\tilde{V}$ is a discrete valuation ring. For future use, we recall that any excellent ring is a Nagata ring (cf. [Hi80, (34.A)]) and so also an $N - 2$ ring (cf. [Hi80, (31.A)]). Let

$$\tilde{u} : \tilde{f}^*_m(O, \phi_O) = f^{\alpha_0}_m(f^{\alpha_0}_m(O, \phi_O)) \to \tilde{f}^*_m(O', \phi_{O'}) = f^{\alpha_0}_m(f^{\alpha_0}_m(O', \phi_{O'}))$$

be the pullback of $u_{\alpha_0}$ to a morphism of $M'(W_m(\tilde{V}))$. If $V$ is the local ring of an integral $\mathbb{F}_{p'}$-scheme $\tilde{U}$, then $\tilde{V}$ is a local ring of the normalization of $\tilde{U}$ in $\tilde{k}$. So from (b) we get that there exists an open subscheme $\tilde{U}$ of this last normalization that has $\tilde{V}$ as a local ring and that has the property that $\tilde{u}$ extends to a morphism of $M'(W_m(\tilde{U}))$. 

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Facts:

1. If $u_1$ is a monomorphism and $(O_1, \phi_{O_1})$ is a direct summand of $(O'_1, \phi_{O'_1})$, then $u_{a_0}$ is a monomorphism and $f_m^{a_n}(O, \phi_O)$ is a direct summand of $f_m^{a_n}(O', \phi_{O'})$.

2. If $u_1$ is a morphism such that its cokernel is annihilated by $p'$, by enlarging $V_{u_1}$, we can assume $\text{Coker}(u_{a_0})$ is also annihilated by $p'$, cf. [Va06, 2.8.3].

Now let

$$v := \max \{ v(1, 1, b, c) \mid c = 0, 1, 2, \ldots \}, \text{Maximum hodge slope of } \mathbb{C}_k,$$

cf. [Va06, 5.1.1(b)] for the function $v(\cdot, \cdot, \cdot, \cdot)$ with $M_1 = \mathbb{C}_0 \cdot k$ and $M_2 = \mathbb{C}_k$ ($v$ does not depend on $m$). Replacing $m$ by $m + v$, from the above discussion (case (c)), we get that there exists a finite field extension $k_{\bar{S}, \tilde{\varphi}_v}$ of $k_{\bar{S}}$ and an open, affine subscheme $U_{\tilde{\varphi}_v}$ of the normalization of $U$ in $k_{\bar{S}, \tilde{\varphi}_v}$, such that $U_{\tilde{\varphi}_v}$ has a local ring $\tilde{V}_x$ which is a discrete valuation ring that dominates $V_x$ and moreover we have a morphism

$$i_{U_{\tilde{\varphi}_v}}(m + v) : m + v \mathbb{C}_{0, U_{\tilde{\varphi}_v}} \to m + v \mathbb{C}_{U_{\tilde{\varphi}_v}},$$

where $m + v \mathbb{C}_{0, U_{\tilde{\varphi}_v}}$ and $m + v \mathbb{C}_{U_{\tilde{\varphi}_v}}$ are the pullbacks of $m + v \mathbb{C}_{0, S}$ and $m + v \mathbb{C}$ to $\text{Spec } U_{\tilde{\varphi}_v}$ respectively. Modulo $p^m$, we have a morphism

$$i_{U_{\tilde{\varphi}_v}}(m) : m \mathbb{C}_{0, U_{\tilde{\varphi}_v}} \to m \mathbb{C}_{U_{\tilde{\varphi}_v}},$$

where $m \mathbb{C}_{0, U_{\tilde{\varphi}_v}}$ and $m \mathbb{C}_{U_{\tilde{\varphi}_v}}$ are the pullbacks of $m \mathbb{C}_{0, S}$ and $m \mathbb{C}$ to $\text{Spec } U_{\tilde{\varphi}_v}$ respectively. Let $I_m$ be the set of morphisms $m \mathbb{C}_{0, k} \to m \mathbb{C}_k$ that are reductions modulo $p^m$ of morphisms $m + v \mathbb{C}_{0, k} \to m + v \mathbb{C}_k$. From [Va06, (Theorem 5.1.1(b)+ Remark 5.1.2)], we get that each morphism in $I_m$ lifts to a morphism $m \mathbb{C}_{0, k} \to \mathbb{C}_k$. Thus $I_m$ is a finite set. Based on the above discussion (case (a)), by replacing $(S, U)$ by its normalizations $(\bar{S}, \bar{U})$ in a finite field extension of $k_{\bar{S}}$, we can assume that $I_m$ is the set of pullbacks of a set of morphisms $L_{m}$ of $\mathcal{M}^0(W_m(k_{\bar{S}}))$. Since $i_{U_{\tilde{\varphi}_v}}(m)$ is the reduction modulo $p^m$ of the morphism $i_{U_{\tilde{\varphi}_v}}(m + v)$, thus $i_{U_{\tilde{\varphi}_v}}(m) \in I_m$ and the pullback of $i_{U_{\tilde{\varphi}_v}}(m)$ to a morphism of $\mathcal{M}^0(W_m(k_{\bar{S}, \tilde{\varphi}_v}))$ is also the pullback of a morphism in $L_m$. As $V_x = \tilde{V}_x \cap k_{\bar{S}}$, inside $W_m(k_{\bar{S}, \tilde{\varphi}_v})$ we have $W_m(V_x) = W_m(\tilde{V}_x) \cap W_m(k_{\bar{S}})$. This implies that the pullback of $i_{U_{\tilde{\varphi}_v}}(m)$ to a morphism of $\mathcal{M}^0(W_m(\tilde{V}_x))$ is in fact the pullback of a morphism of $\mathcal{M}^0(W_m(V_x))$. From the above discussion (case (b)) (applied with $(V_1, V)$ replaced by $(V_x, R)$), we get the existence of an open subscheme $U_{V_x}$ of $U$ that has $V_x$ as a local ring and such that we have a morphism

$$i_{U_{V_x}}(m) : m \mathbb{C}_{0, U_{V_x}} \to m \mathbb{C}_{U_{V_x}},$$

where $m \mathbb{C}_{0, U_{V_x}}$ and $m \mathbb{C}_{U_{V_x}}$ are the pullbacks of $m \mathbb{C}_{0, S}$ and $m \mathbb{C}$ to $\text{Spec } U_{V_x}$ respectively. Using the above Facts 1 and 2, if we apply similar arguments on $j_m : m \mathbb{C}_{0, W} \cong m \mathbb{C}_W$ and $\varphi_m : m \mathbb{C}_W \to m \mathbb{C}_{U_{V_x}}$, we can assume that $i_{U_{V_x}}(m)$ is the composition of two morphisms $j_{U_{V_x}}(m) : m \mathbb{C}_{0, U_{V_x}} \to m \mathbb{C}_{U_{V_x}}$ and $\varphi_{U_{V_x}}(m) : m \mathbb{C}_{U_{V_x}} \to m \mathbb{C}_{U_{V_x}}$, where $j_{U_{V_x}}(m)$ is a monomorphism that splits and the cokernel of $\varphi_{U_{V_x}}(m)$ is annihilated by $p'$. Now at the level of modules consider the following diagram:

$$\begin{array}{ccc}
  m \mathbb{C}_{0, U_{V_x}} & \xrightarrow{i_{U_{V_x}}(m)} & m \mathbb{C}_{U_{V_x}} \\
  m \mathbb{C}_{0, k} \downarrow & & \downarrow m \mathbb{C}_{k} \\
  m \mathbb{C}_{U_{V_x}} \xrightarrow{i_{U_{V_x}}(m)} & & m \mathbb{C}_{U_{V_x}} \\
\end{array}$$

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Let $m\mathcal{C}_{0,k} = (W_m(k), p^b\sigma^m)$. Since the Newton polygon of $\mathcal{C}_k$ has $(1,b)$ as a break point, at the level of evaluation there is a unique rank 1 free sub-$W_m(k)$-module $m\mathcal{E}_k = (W_m(k), p^b\sigma^m)$ of $m\mathcal{C}_k$. The morphism $i_{U_{V_x}}(m)$ is nonzero mod $p^t$ (assuming $m > t$) since $i_{U_{V_x}}(m) = \varphi_{U_{V_x}}(m) \circ j_{U_{V_x}}(m)$, where $\text{Coker}(\varphi_{U_{V_x}}(m))$ is annihilated by $p^t$ and $j_{U_{V_x}}(m)$ is a monomorphism that splits. Therefore the morphism $i_{k,a}(m)$ is not a zero morphism if $m > t$, thus it must factor through $m\mathcal{E}_k$

$$i_{k,a}(m) : m\mathcal{C}_{0,k} \xrightarrow{f_{i}(m)} m\mathcal{E}_k \hookrightarrow m\mathcal{C}_k.$$ 

Let $\epsilon = f_{i}\epsilon(1) \in W_m(k)$ and consider the following commutative diagram

$$m\mathcal{C}_{0,k} = W_m(k) \xrightarrow{f_{i}(m)} m\mathcal{E}_k = W_m(k)$$

We have $p^b\sigma^m(\epsilon) = p^b\sigma^m(f_{i}(m)\epsilon(1)) = f_{i}(m)(p^b\sigma^m(\epsilon)) = f_{i}(m)(p^b) = p^b f_{i}(m)(1) = p^b \epsilon$. Therefore, $\sigma^m(\epsilon) = \epsilon \mod p^{m-b}$. Let $\epsilon = p^t \epsilon'$, where $\epsilon' \in W_m(k)$ is invertible. Since the cokernel of $f_{i}(m)$ is annihilated by $p^t$ (as the pullback to $\text{Spec} k$ of $\varphi_{U_{V_x}}(m)$ is annihilated by $p^t$ and the pullback to $\text{Spec} k$ of $j_{U_{V_x}}(m)$ splits), we have $1 \leq t_k \leq t$. We now have $\sigma^m(\epsilon') = \epsilon' \mod p^{m-b-t_k}$. Therefore $\sigma^m((\epsilon')^{-1}) = (\epsilon')^{-1} \mod p^{m-b-t_k}$. This shows that multiplication by $(\epsilon')^{-1}$ is an automorphism of $W_{m-b-t_k}(k)$ and of $W_{m-b-t_k}(k)$ since $t_k \leq t$. Replacing $m$ by $m-b-t_k$ and composing $f_{i}(m)$ with this automorphism $W_{m-b-t_k}(k)$

$$W_m(k) \xrightarrow{\text{multiplication by } (\epsilon')^{-1}} W_m(k)$$

we can assume $f_{i}(m)(1) = 1 \in W_m(k)$.

Now let $y \in U$ be another point of codimension 1. Similarly we can construct a morphism $f_{i}(m)$ and by the above construction $f_{i}(m)$ and $f_{i}(m)$ coincide. This tells us that the pullbacks of $i_{U_{V_x}}(m)$ and $i_{U_{V_x}}(m)$ to morphisms of $\mathcal{M}'(W_m(U_{V_x} \cap U_{V_y}))$ coincide and therefore they glue together. Now we glue the morphisms $i_{U_{V_x}}(m)$ for all $x \in U$ with codimension 1 and we obtain a morphism $i_{U_{V_x}}(m) : m\mathcal{C}_{0,U_0} \rightarrow m\mathcal{C}_{U_0}$, where $U_0$ is an open subscheme of $U$ and $S$ and codim $(U \setminus U_0) \geq 2$. By the definition of $H_m$, we have constructed an $S$-section $j : U_0 \rightarrow H_m$.

**Step 4. Sections of $H_m$.**

By the gluing argument of Step 3, we have an open subscheme $U_0$ of $S$ with codim$(U \setminus U_0) \geq 2$ and a section $j : U_0 \rightarrow H_m$ such that the following diagram commutes:

$$\begin{array}{ccc}
U_0 & \xrightarrow{j} & H_m \\
\downarrow & & \downarrow \\
U & \xrightarrow{\text{section } j} & S.
\end{array}$$

Let $J_m$ be the schematic closure of $j(U_0)$ in $H_m$. As $H_m$ is affine and Noetherian, $J_m$ is also an affine, Noetherian $S$-scheme. Since $S$ is Noetherian, normal and integral, we get that $U_0$ is also Noetherian, normal and integral and therefore $J_m$ is integral. Now we have the following commutative diagram:
Now consider the pullback $\tilde{J}_m$ of $J_m$ to $U$:

\[
\begin{array}{c}
U_0 \xrightarrow{\text{open}} U \xrightarrow{\text{affine}} S.
\end{array}
\]

Claim: The morphism $g$ is an isomorphism, i.e., $\tilde{J}_m \cong U$.

Proof of Claim: To prove that $g$ is an isomorphism, we can assume $U = \text{Spec } A$ is affine. As $g$ is an affine morphism, we can also assume $\tilde{J}_m = \text{Spec } B$ is also affine. Since $U_0$ is open dense in both $U$ and $\tilde{J}_m$, therefore $U$ and $\tilde{J}_m$ are birationally equivalent. Thus their fractional fields Frac$(A)$ and Frac$(B)$ are equal. Since $U_0$ is in both $U$ and $\tilde{J}_m$, we have $A_p = B_p$ for any prime $p \in \text{Spec } A$ of height $1$. As $A$ is a Noetherian normal domain, we have

\[
A \hookrightarrow B \subset \bigcap_{q \in \text{Spec } b \text{ of height } 1} B_q \subset \bigcap_{p \in \text{Spec } A \text{ of height } 1} A_p = A
\]

(cf. [Hi80, (17.H)] for the equality part; The first monomorphism is given by $g^\#$). Therefore $A = B$ and the claim is proved.

Now we have a section from $U \cong \tilde{J}_m \hookrightarrow H_m$.

**Step 5. Final output:** $U = J_m$ for $m >> 0$.

Claim: If $m >> 0$, then $U = J_m$.

Proof of Claim: Suppose not. Let $\eta$ be the generic point of an irreducible component of $J_m \setminus U$. Consider the stalk at $\eta$: $\mathcal{O}_{J_m, \eta} := R_p$, which is a normal, local Noetherian ring but not a field since $U$ is dense in $J_m$. Now we have dim$(R_p) \geq 1$. Therefore, we can find a prime ideal $q \in U \subset S_p$, such that dim$(R_p/q) = 1$. Now the normalization $(R_p/q)^\alpha$ of $R_p/q$ is an integrally closed Noetherian local ring of dimension one, thus a discrete valuation ring. The morphism Spec$(R_p/q) \to J_m$ obtained from composing the natural morphisms Spec$(R_p/q)^\alpha \to \text{Spec } R_p/q \to \text{Spec } R_p \to J_m$ sends the generic point of Spec$(R_p/q)^\alpha$ to $U$ and the closed point to $J_m \setminus U$. With the help of a little commutative algebra, we can further assume the generic point of Spec$(R_p/q)^\alpha$ is mapped into $U_0 \subset U$. Let $D$ be the completion of the discrete valuation ring $(R_p/q)^\alpha$, it is isomorphic to a power series ring $l[[t]]$ for some field $l$ of characteristic $p$ by Cohen structure theorem, cf. [Ei07, (Theorem 7.7)]. By injecting $l[[t]]$ to $\tilde{l}[[t]]$, we can further assume $l$ is algebraically closed. Now we look at the pullback of $\mathcal{C}$ to Spec $D$:

\[
\begin{array}{c}
\mathcal{C}_D \xrightarrow{\text{open}} \mathcal{C} \xrightarrow{\text{affine}} S.
\end{array}
\]
It is an $F^\alpha$-crystal such that at the generic point $\text{Spec}(\text{Frac}(D))$ of $D$ its Newton polygon has $(1, b)$ as a break point and at the closed point $\text{Spec} l$ of $D$ all Newton slopes are at least $b + 1$. Suppose the generic point $\text{Spec}(\text{Frac}(D))$ is mapped to $z \in U_0$, we pull back the following morphisms to $\text{Spec}(D)$:

$$
\begin{array}{ccc}
\text{Spec} E & \xrightarrow{\psi_{U_0}(m)} & \text{Spec} D \\
\text{mC}_{0, D} & \xrightarrow{j_{U_0}(m)} & \text{mC}'_{U_0} \\
\text{mC}_0 & \xrightarrow{m \mathcal{C}_{U_0}} & \text{mC}'_{U_0} \\
\end{array}
$$

Let $E = D_{\text{Perf}}$ and $m_{0, E}, m_{E}$ and $m_{E}'$ be the pullback of $m_{0, D}, m_{D}$ and $m_{D}'$ to $\text{Spec} E$ respectively:

$$
\begin{align*}
\text{Spec} E & \xrightarrow{\psi_{U_0}(m)} \text{Spec} D \\
\text{mC}_{0, E} & \xrightarrow{j_{U_0}(m)} \text{mC}'_{E} \\
\text{mC}_0 & \xrightarrow{m \mathcal{C}_{U_0}} \text{mC}'_{E} \\
\end{align*}
$$

Recall $r = \text{rank}(E) = \text{rank}(m_{E}) = \text{rank}(m_{E}') = \text{rank}(m_{E})$. Let $m_{0, E} = (W_m(E), F^t)$, let $m_{E} = (W_m(E), F^t)$ and let $m_{0, E}' = (W_m(E), p^b \sigma^m)$. Since $C'$ is divisible by $b$, we can assume $F' = p^b G$. Let $x$ be a basis element of $M_{0}/p^m M_{0}$ (We can assume $\sigma^m x = x$) and consider the following morphism at the level of $W_m(E)$-modules:

$$
m_{E}' \xrightarrow{\gamma} m_{E}' \xrightarrow{\gamma} m_{E}
$$

where $y = (y_1, y_2, \cdots, y_r)$ and $y_i \in W_m(E)$ for $i = 1, 2, \cdots, r$. We have $p^h G(y) = p^h G(y(x)) - p^h \gamma(x) = \bar{F}(y(x)) - \gamma(p^h x) = \gamma(p^h \sigma^m x) = \gamma(p^h x) = 0$. Therefore $G(y) = y \mod p^m b$. Since the cokernel of $\gamma : m_{E}'_{E} \to m_{E}$ is annihilated by $p^t$ and $y : m_{0, E}' \to m_{E}'$ is a monomorphism that splits, we can write $y = p^h z$ for $0 \leq t_0 \leq t$ and $z = (z_1, z_2, \cdots, z_r)$ is not divisible by $p$, where $z_i = (z_{i, 0}, z_{i, 1}, z_{i, 2}, \cdots, z_{i, m-1})$ with $z_{i, j} \in E$ for $1 \leq i \leq r, 0 \leq j \leq m - 1$.

**Subclaim 1**: $y \neq 0 \mod p^h$ at the closed point $\text{Spec} l$ of $\text{Spec} E$.

**Proof of Subclaim 1**: Recall $E = l[[T]]^{\text{Perf}}$. As $G(y) = y \mod p^{m-b} x$ with $0 \leq t_0 \leq t$, we have $G(z) = z \mod p^{m-b-t}$. Since $z = (z_1, z_2, \cdots, z_r)$ is not divisible by $p$, modulo $p$ in $E'/l[[T]]^{\text{Perf}}$ we have $(z_{1, 0}, z_{2, 0}, \cdots, z_{r, 0}) \neq 0$. Suppose $z = 0$ at the closed point $\text{Spec} l$, then for some $v \in \mathbb{N}$, we have $z_{i, 0} \in l[[T]]^{\text{Perf}}$ and $z_{i, 0} = 0 \mod T^v$ for all $i = 1, 2, \cdots, r$. Now $\tilde{T} = T^v$ and let $z_{i, 0} = \tilde{T} f_{i, 0}^v$ for some $f \in \mathbb{N}$, for all $i = 1, 2, \cdots, r$ and $z_{i, 0}^v \neq 0 \mod \tilde{T}$ for some $1 \leq i_0 \leq r$. Let $\overline{\mathcal{C}}$ be $\mathcal{C}$ mod $p$. We have $\overline{\mathcal{C}}((z_{1, 0}^v, z_{2, 0}^v, \cdots, z_{r, 0}^v)) = \overline{G((z_{1, 0}, z_{2, 0}, \cdots, z_{r, 0}))} = \overline{G(\tilde{T} f_{1, 0}^v, \cdots, z_{r, 0}^v))} = \overline{T^v \frac{G((z_{1, 0}, z_{2, 0}, \cdots, z_{r, 0}))}{\mathcal{C}}}$. This contradicts the fact that $z_{i, 0}^v \neq 0 \mod \tilde{T}$ and thus $z \neq 0$ at the closed point. As $y = p^h z$, we have $y \neq 0 \mod p^h$ at the closed point and Subclaim 1 is proved.

Evaluating the morphism $\gamma$ at the closed point $\text{Spec} l$ of $\text{Spec} E$ we have a morphism $$\beta : (W_m(l), p^h \sigma^m) \to (N, \hat{F})$$
where \((N, \hat{F}) = (\mathcal{C}_l \mod p^m)\) and \(\xi_l\) is the pullback of \(\mathcal{C}_D\) to the closed point \(\text{Spec} \, l\). Since \(y \neq 0 \mod p^h\) at \(\text{Spec} \, l\), the morphism \(\beta\) is non-zero when reduced modulo \(p^h\).

**Subclaim 2:** \(\mathcal{C}_l\) has Newton slope \(b\).

**Proof of Subclaim 2:** Let \(\mathcal{C}_l = (M_l, F_l)\) and suppose all Newton slopes of \(\mathcal{C}_l\) are at least \(b + 1\). By [Ka79, (Sharp Slope Estimate 1.5.1)] we have \(F_l^i(M_l) \subset p^{(b+1)u-h} M_l\) for all \(u \in \mathbb{N}\), where \(h > 0\) is a fixed number. If \(u > h + t_0 + 1\), we have \(F_l^u(M_l) \subset p^{t_0 + h + 1} M_l\). Let \((N, \hat{F}^u) = (M_l, F_l^u) \mod p^m\) and we also have \(\hat{F}^u(N) \subset p^{t_0 + h + 1} N\). Now consider the following morphism

\[
\alpha : (W_m(l), (p^b \sigma^m)^y) \rightarrow (N, \hat{F}^u)
\]

where as a function \(\alpha = \beta\). Since the morphism \(\beta\) is non-zero when reduced modulo \(p^h\), so is the morphism \(\alpha\). However, \(p^b u e = p^b u \alpha(1) = \alpha(p^b u) = \alpha((p^b \sigma^m)^y(1)) = \hat{F}^u(\alpha(1)) = \hat{F}^u(e) \in p^{m + t_0 + 1} N\) and thus \(p^{m - t_0} \alpha(1) = p^{m - t_0} e = p^{m - t_0} p^b u e \in p^{m + t_0 + 1} N = p^{m + 1} N = 0\) if \(m \geq b u + t + 1 \geq b u + t_0 + 1\), which is a contradiction since \(\alpha \mod p^h\) is non-zero. Subclaim 2 is therefore proved.

By Subclaim 2, the \(F^n\)-crystal \(\mathcal{C}_D\) has Newton slope \(b\) at the closed point \(\text{Spec} \, l\), a contradiction by our assumption. Therefore the claim is proved, i.e., \(U = J_m\).

Now we have \(S \rho_0 = U = J_m\) and \(J_m\) is affine, therefore \(S \rho_0\) is affine and Theorem 5 is thus proved. ■

First application of Theorem 5:

**Proposition 1** Theorem 5 implies Theorem 3, i.e., if \(S \rho_0\) is pure in \(S\) for each point \(P_0\) in the xy-coordinate plane, then the Newton polygon stratification of \(S\) defined by \(\mathcal{C}\) is pure in \(S\).

**Proof of Proposition 1:** Let \(S\) and \(\mathcal{C}\) be as above and let \(\nu_0\) be a Newton polygon. We need to show that the set

\[
S_{\nu_0} = \{ s \in S | NP(\mathcal{C}_s) = \nu_0 \}
\]

is an affine \(S\)-scheme. To prove that \(S_{\nu_0}\) in \(S\) is affine, we can further assume \(S\) is affine. Let the break points of \(\nu_0\) be \(Q_0, Q_1, \cdots, Q_t\) and let

\[
S_{\geq \nu_0} = \{ s \in S | NP(\mathcal{C}_s) \geq \nu_0 \}.
\]

Notice for any \(s \in S\), the Newton polygon \(NP(\mathcal{C}_s)\) is \(\nu_0\) if and only if \(NP(\mathcal{C}_s) \geq \nu_0\) and moreover \(Q_0, Q_1, \cdots, Q_t\) are all break points of \(NP(\mathcal{C}_s)\). Therefore \(S_{\nu_0} = S_{\geq \nu_0} \cap S_{Q_0} \cap S_{Q_1} \cap \cdots \cap S_{Q_t}\). By Theorem 1, \(S_{\geq \nu_0}\) is closed in \(S\), thus affine. By Theorem 5, \(S_{Q_i}\) is affine for \(i = 0, 1, \cdots, t\). Since \(S\) is affine thus separated, we conclude that \(S_{\nu_0}\) is affine. ■

Second application of Theorem 5:

**Proposition 2** Theorem 5 implies Theorem 4.

**Proof of Proposition 2:** From Theorem 5, we know that \(S \rho_0\) is pure. As purity implies universal weak purity, we have \(S \rho_0\) being universally weakly pure. ■
Now we have the following implications:

Theorem 5 ⇒ Theorem 4 ⇒ Theorem 2.

Theorem 5 ⇒ Theorem 3 ⇒ Theorem 2.

Third application of Theorem 5:

Let $C = (M, F)$ be an $F^n$-crystal over an algebraically closed field $k$ of char $p > 0$. Let $\overline{M}$ be the reduction modulo $p$ of $M$ and let $\overline{F} : \overline{M} \to \overline{M}$ be the reduction modulo $p$ of $F$. Then the $p$-rank $t$ of $C$ can be defined equivalently as follows:

(i) It is $\dim_{\mathbb{F}_p}(|x \in \overline{M}| \overline{F}(x) = x)$.

(ii) It is the multiplicity $t$ of the Newton slope 0 of $C$.

(iii) It is the unique non-negative integer such that $(t, 0)$ is a break point of the Newton polygon of $C$.

Let $S$ be a locally Noetherian $\mathbb{F}_p$-scheme and let $\mathfrak{C}$ be an $F^n$-crystal over $S$. For each $t \in \mathbb{N}$, let $Y_t$ be the reduced, locally closed subscheme of $S$ formed by those points $s \in S$ with the property that the $p$-rank of the pullback of $\mathfrak{C}$ to $k(s)$, where $k(s)$ is the algebraic closure of the residue field of $s$, is exactly $t$. We call $Y_t$ the stratum of $p$-rank $t$ of the $p$-rank stratification of $S$ defined by $\mathfrak{C}$.

Based on (iii), one gets the following corollary:

**Corollary 2** Let $S$ be a locally Noetherian $\mathbb{F}_p$-scheme and $\mathfrak{C}$ be an $F$-crystal over $S$, then the $p$-rank strata of $S$ defined by $\mathfrak{C}$ are pure in $S$.

**Proof of Corollary 2:** Let $Y_t$ be a stratum of the $p$-rank stratification of $S$ as above. A point $s \in S$ belongs to $Y_t$ if and only if $(t, 0)$ is a break point of the Newton polygon of $\mathfrak{C}$, $s$. Using the same notation as in the statement of Theorem 5, this shows that $Y_t = S_{(t,0)}$. By Theorem 5, $Y_t$ is affine in $S$. ■

Since purity implies weak purity, Corollary 2 implies the following theorem:

**Theorem 6 (Th. Zink, [Zi01, (Proposition 5)])** Let $S$ be a locally Noetherian $\mathbb{F}_p$-scheme and $\mathfrak{C}$ be an $F$-crystal over $S$, then the $p$-rank strata of $S$ defined by $\mathfrak{C}$ are weakly pure in $S$.

Fourth application of Theorem 5:

Let $R$ be an $\mathbb{F}_p$-algebra and let $A$ be an $n \times n$ matrix with coefficients in $R$, i.e., $A \in M_{n \times n}(R)$.

Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be an $n$-dimensional vector whose entries are variables $x_1, x_2, \ldots, x_n$ and let $\underline{b}$
be a constant $n$–dimensional vector with coefficients in $R$, i.e., $\underline{b} \in R^n$. Let $\underline{x}^{[p]} = \begin{bmatrix} x_1^p \\ x_2^p \\ \vdots \\ x_n^p \end{bmatrix}$.

Consider the following Artin-Schreier equation:

$$\underline{x} = A\underline{x}^{[p]} + \underline{b}$$

Let $S$ be a $R$-scheme. Define

$$\phi : S \to \mathbb{N}$$

$$s \mapsto d = \dim_{\mathbb{F}_p}(k(s) \mid x = A_s\underline{x}^{[p]} + \underline{b}_s),$$

where $k(s)$ is the algebraic closure of the residue field $k(s)$ of $s \in S$, $A_s$ is the canonical image of $A$ in $M_{n \times n}(k(s))$ and $\underline{b}_s$ is the canonical image of $\underline{b}$ in $k(s)^n$.

Let $S$ be an $R$-scheme. For any $d \in \mathbb{N}$, let $Y_d$ be the reduced, locally closed subscheme of $S$ formed by those points $s \in S$ with the property $\phi(s) = d$. We call $Y_d$ the stratum of the Artin-Schreier stratification of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]} + \underline{b}$.

By the first proof of Theorem 2.4.1 (b) in [Va13], the Artin-Schreier stratification of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]} + \underline{b}$ is equivalent to the Artin-Schreier stratification of $S$ defined by the equation $\underline{x} = \tilde{A}\underline{x}^{[p]}$ for some $\tilde{A} \in M_{n \times n}(R)$. Therefore from now on, we will always assume $\underline{b} = \underline{0}$ in the Artin-Schreier stratification.

Corollary 3 Let $S = \text{Spec } R$ be a locally Noetherian affine $\mathbb{F}_p$-scheme. Let $\underline{x}$, $\underline{x}^{[p]}$ and $A$ be defined as above. Then each Artin-Schreier stratum of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]}$ is pure in $S$.

Proof of Corollary 3: Let $S_0$ be a stratum of the Artin-Schreier stratification of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]}$. By replacing $S$ by the schematic closure $\overline{S_0}$ of $S_0$ in $S$, we can assume that $S_0$ is open dense in $S$. We can further assume that the $\mathbb{F}_p$-scheme $S$ is reduced. Consider the pullback of $S_0$ in the following commutative diagram:

$$\begin{array}{ccc}
S^{\text{perf}}_0 & \xrightarrow{\text{integral}} & S_0 \\
\downarrow \text{open} & & \downarrow \text{open} \\
S^{\text{perf}} & \xrightarrow{\text{integral}} & S = \text{Spec } R
\end{array}$$

Since the morphism $S^{\text{perf}}_0 \to S_0$ is integral and $S_0$ is an open subscheme of $S$, to prove that $S_0$ is affine it suffices to prove that $S^{\text{perf}}_0$ is affine, cf. [Va06, (Lemma 2.9.2)]. Therefore, we can assume $R = R^{\text{perf}}$, i.e., $R$ is a reduced, Noetherian perfect $\mathbb{F}_p$–algebra.
Now consider the $F$-crystal $\mathcal{C} = (W(R)^{2n}, F)$ over $R$, where $F = g \begin{bmatrix} I_n & 0_n \\ 0_n & pI_n \end{bmatrix} \sigma_{W(R)}$. Here $g$ is a fixed invertible matrix in $GL_{2n}(W(R))$ lifting $\overline{g} = \begin{bmatrix} A & I_n \\ I_n & 0_n \end{bmatrix} \in GL_{2n}(R)$, $I_n$ is the $n \times n$ identity matrix, $0_n$ is the $n \times n$ zero matrix and $\sigma_{W(R)}$ is the Frobenius endomorphism of $W(R)$.

Claim: For any $d \in \mathbb{N}$, the stratum of $p$-rank $d$ of the $p$-rank stratification of $S$ defined by $\mathcal{C}$ is the same as the stratum $Y_d$ (using the same notation as in the definition of Artin-Schreier stratification) of the Artin-Schreier stratification of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]}$.

Proof of Claim: Let $s \in S = \text{Spec } R, \overline{k(s)}$ be the algebraic closure of its residue field, $\sigma_{W(\overline{k(s)})}$ be the Frobenius endomorphism of $W(\overline{k(s)})$ and $F_s = F \otimes_{W(\overline{k(s)})} g_s \begin{bmatrix} I_n & 0_n \\ 0_n & pI_n \end{bmatrix} \sigma_{W(\overline{k(s)})}$, where $g_s$ is the canonical image of $g$ in $GL_{2n}(W(\overline{k(s)}))$. Consider the pullback $\mathcal{C}_s = (W(\overline{k(s)})^{2n}, F_s)$ of $\mathcal{C}$ to Spec $k(s)$:

$$\xymatrix{ \mathcal{C}_s & \mathcal{C} \\
\text{Spec } k(s) \ar[r] \ar[d] & S = \text{Spec } R \ar[d] 
}$$

We investigate the $\mathbb{F}_p$-Vector space $V = \{\underline{z} \in \mathcal{C}_s|F_s(\underline{z}) = \underline{z}\}$, where $F_s = \overline{g}_s \begin{bmatrix} I_n & 0_n \\ 0_n & 0_n \end{bmatrix} \sigma_{W(\overline{k(s)})} = \begin{bmatrix} A_s & 0_n \\ I_n & 0_n \end{bmatrix}$ is the reduction modulo $p$ of $F_s$, where $A_s$ is the canonical image of $A$ in $M_{n \times n}(\overline{k(s)})$ and $g_s$ is $g_s$ mod $p$, and $\overline{\mathcal{C}_s} = (\overline{k(s)}^{2n}, \overline{F_s})$ is the reduction modulo $p$ of $\mathcal{C}_s$. Suppose

$$\underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \end{bmatrix} \in \overline{k(s)}^{2n}.$$ We have $\underline{z} \in V$ if and only if $\begin{bmatrix} A_s & 0_n \\ I_n & 0_n \end{bmatrix} \begin{bmatrix} z_1^p \\ \vdots \\ z_{2n}^p \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_{2n} \end{bmatrix}$ if and only if

$$A_s \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_1^p \\ z_2^p \\ \vdots \\ z_n^p \end{bmatrix}$$ and $z_{n+1} = z_1^p, z_{n+2} = z_2^p, \ldots, z_{2n} = z_n^p$. Therefore

$$\dim_{\mathbb{F}_p} V = \dim_{\mathbb{F}_p}(\{\underline{z} \in \overline{k(s)}^{2n}|\underline{z} = A\underline{z}^{[p]}\}) := d,$$

which means the stratum of $p$-rank $d$ in the $p$-rank stratification of $S$ defined by $\mathcal{C}$ is the same as the stratum $Y_d$ in the Artin-Schreier stratification of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]}$. Since $s \in S$ is arbitrary chosen and thus $d$ is arbitrary, Claim is proved.

By Corollary 2, each Artin-Schreier stratum of $S$ defined by the equation $\underline{x} = A\underline{x}^{[p]}$ is pure in $S$. This ends the proof of Corollary 3. ■

Remark: Deligne and Vasiu also obtained the results of Corollary 2 and 3 using different methods, cf. [De11] and [Va14].
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