Dynamical extensions for shell-crossing singularities.

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Abstract. We derive global weak solutions of Einstein’s equations for spherically symmetric dust-filled space-times which admit shell-crossing singularities. In the marginally bound case, the solutions are weak solutions of a conservation law. In the non-marginally bound case, the equations are solved in a generalized sense involving metric functions of bounded variation. The solutions are not unique to the future of the shell-crossing singularity, which is replaced by a shock wave in the present treatment; the metric is bounded but not continuous.

Submitted to: Class. Quantum Grav.

PACS numbers: 04.20.Dw, 04.20.Ex, 02.30.Jr
1. Introduction

The first examples of naked singularities occurring in gravitational collapse which were constructed with a view to testing and refining the cosmic censorship hypothesis were given in [1]. The singularities in question arise in the collapse of an inhomogeneous dust sphere. One can consider the dust sphere to be foliated by infinitely thin shells of matter; the singularities occur when outer shells overtake inner ones, and are known as shell-crossing singularities (SCS). These singularities are not considered to be serious counter-examples to the cosmic censorship hypothesis for reasons which we review now and will return to below.

The first and most convincing reason is that the fluid matter model is not appropriate for the study of gravitational phenomenon on the smallest scales. The fluid model is a macroscopic approximation which works well when it can be interpreted as representing the smoothed out behaviour of matter fields in a region of space-time, as for instance in cosmology. Applying this model all the way down to a scale where two infinitely thin shells of fluid intersect is not appropriate. This view of the matter model is now part of the cosmic censorship hypothesis, see e.g. [2]; the matter model being used must be such that it does not develop singularities in special relativity. (It should be pointed out that this view is not universally supported and that some authors consider the shell-focussing singularity in fluid models to be a valid counter-example to the cosmic censorship hypothesis; see e.g. [3].) In the case of the dust matter model, which is a macroscopic model of collisionless matter, this point of view has received independent support in [4]. It was shown here that the spherically symmetric Einstein-Vlasov system, which involves a kinetic theoretic description of collisionless matter and in which dust arises as a singular limit, does not admit shell-crossing singularities. Note that the occurrence of shell-crossing is not entirely due to the vanishing pressure; an SCS can also occur when the pressure is non-zero [1]. It has also been shown that in spherically symmetric perfect fluids, the global visibility of a shell-crossing singularity is related to an unphysical situation, namely the vanishing of the sound speed at the singularity [5].

The second reason relates to the view that certain geometric properties of shell-crossing singularities indicate the possibility of extending the space-time beyond the singularity, which would then not be considered genuine. While it is generally accepted that geodesic incompleteness means that space-time is singular, there does not exist a universally accepted definition of what constitutes a genuine space-time singularity in the sense of boundary constructions and levels of differentiability. However an extension of even very low differentiability would allow one to consider an SCS as an interior point of the space-time rather than a boundary point (singularity). In particular, one could then test Clarke’s notion of generalized hyperbolicity for the SCS [6]. For example, it has been claimed that the metric can be written in a form which is \( C^0 \) and non-degenerate at the singularity [7] (but see below). Also in [7], it was shown that the SCS is gravitationally weak in the sense of Tipler: Jacobi fields carried along radial
null geodesics running into the singularity have finite limits at the singularity. It has been claimed that this is evidence that an extension through the singularity may be constructed. However not enough is currently known about the connection between this definition of the gravitational strength of a singularity and the question of extendibility of the space-time to make this characterization useful.

The third reason relates to actual attempts to construct an extension. In [8], a dynamical extension was constructed for a (very) special case of an SCS. In [9], the general case was treated. Motivated by behaviour observed in the Newtonian case, the authors analysed the possibility of constructing an extension through the singularity into a region filled with three superimposed dust flows. The analysis is incomplete, due to the awkward nature of the system of evolution equations obtained, but strongly indicates the existence of a solution to the extension problem. An extension along these lines for the Newtonian case has recently been given in [10]. A model of shell-crossing involving the collision of actual (distributional) shells of dust was studied in [11]. The authors find that in order to single out an extension to the future of the collision, one must put in extra information by hand.

So the question remains open: does there exist a natural extension through a shell-crossing singularity, and if so, is the extension unique? We address these questions here and find affirmative and negative answers respectively. The key is to introduce co-ordinates which cast the field equations in a form in which the shell-crossing singularity is replaced by a shock wave. We use $8\pi G = c = 1$.

2. Existence of weak solutions

In co-moving co-ordinates, the line element for spherical dust collapse is [12]

$$ds^2 = -dt^2 + \frac{r^2}{1 + E}dR^2 + r^2(R, t)d\Omega^2,$$

where $d\Omega^2$ is the line element for the unit 2-sphere. The field equations yield

$$r_{,t} = -\sqrt{E + \frac{m}{r}},$$

$$\rho = \frac{m'(R)}{r^2 r_{,R}},$$

the latter equation defining the density $\rho(R, t)$ of the dust; $m = m(R)$ and $E = E(R)$ are preserved by the fluid flow. These functions are arbitrary (subject to certain energy and differentiability conditions) and constitute the initial data for the problem, set at $t = 0$. $m$ is twice the Misner-Sharp mass. The evolution equation (2) is readily solved; in the general case ($E \neq 0$) one obtains an implicit solution. From this, $r_{,R}$ may be calculated. Regular initial data can lead to two types of singularity: that occurring when $\rho$ diverges at $r = 0$ (called the shell-focussing singularity), and that occurring when $r_{,R} = 0$ but $r \neq 0$. This latter is the shell-crossing singularity. According to the implicit function theorem, the SCS can be represented locally by a function $t = t_{SC}(R) > 0$; this can only
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be calculated explicitly in the marginally bound case $E = 0$. The time $t = t_{SF}(R) > 0$ of the shell-focussing singularity can always be calculated explicitly. See e.g. [7]. We say that a shell-crossing singularity occurs if $t_{SC}(R) < t_{SF}(R)$ for some $R > 0$; this indicates that the shell-crossing singularity precedes the shell-focussing singularity. Note that if $t_{SC}$ and $t_{SF}$ are sufficiently smooth (in fact only continuous), then the SCS occurs at values of $R$ forming disjoint open subsets. Furthermore, it is possible to identify these open subsets in terms of the initial data $E(R), m(R)$ [13]. The co-ordinate freedom can be used to set $r(R,0) = R$, and so the initial density is $\mu := \rho(R,0) = m'/R^2$. For positive density in the matter filled region, we require $m'(R) > 0$ for $R > 0$, and so we see that $\rho$ must diverge at the SCS. Note that if there exists a region $t_{SC} < t < t_{SF}$, then the density given by (3) is negative in this region. One needs a new form of the solution which can be used beyond the SCS.

On one level, the SCS can be viewed simply as a breakdown of the co-moving co-ordinate system: for fixed $t$, $R \mapsto r(R,t)$ is no longer one-to-one. So we make the transformation $(t,R) \rightarrow (t,r(R,t))$. This transformation was used in [7]. The resulting line element is

$$ds^2 = (1 + E)^{-1}[-(1 - \frac{m}{r})dt^2 + 2udt dr + dr^2] + r^2 d\Omega^2,$$

where $u = \sqrt{E + m/r}$. This appears to yield a $C^0$ and non-degenerate metric at the SCS. However this does not take account of the fact that $m, E$ are functions of $t$ and $r$. In particular, a space-time event on the shell crossing corresponds to (at least) two values of the co-moving co-ordinate $R$, and so one cannot uniquely determine the value of $m(R)$ or $E(R)$ at the shell-crossing. Thus the line element above does not provide an extension through the shell crossing unless the functional dependence of $m$ and $E$ on the proper time and proper radius is given; this principal aim of this paper is to do this. The dependence in question is worked out by solving the field equations in the new co-ordinate system. These are simply the conditions that $m, E$ are preserved by the fluid flow, and read

$$m, t - um, r = 0, \quad (4)$$

$$m, t - E, r = 0. \quad (5)$$

We define the characteristics of this system to be solutions of

$$\frac{dr}{dt} = -u. \quad (6)$$

Then we see from (4) and (5) that $m, E$ are constant along characteristics and we can write down an implicit solution obtained by integrating (6). This solution holds until such time as characteristics cross; the characteristics are the fluid flow lines and so this is equivalent to shell-crossing. To extend beyond the SCS, we let $f(u) = -u^2/2$ and rewrite the system as

$$u, t + f, r = \frac{m}{2r^2}, \quad (7)$$

$$m, t - um, r = 0. \quad (8)$$
This system is fundamentally non-conservative, although \( \text{(7)} \) is an inhomogeneous conservation law which admits the weak solution \( u, m \in L^\infty(\mathbb{R}_+^2) \) if
\[
\int_{\mathbb{R}_+^2} (u\psi_t - \frac{u^2}{2}\psi_r + \frac{m}{2r^2}\psi) \, dr \, dt = 0
\] (9)
for all test functions \( \psi \in C^\infty_0(\mathbb{R}_+^2) \). \( \text{(9)} \) holds for differentiable solutions of \( \text{(7)} \) and extends the space of functions in which we may seek solutions to \( L^\infty(\mathbb{R}_+^2) \), where \( \mathbb{R}_+^2 = (0, \infty) \times (0, \infty) \). In particular, a weak solution need not be differentiable. Noting that \( \text{(7)}, \text{(8)} \) share the characteristic speed \( -u \), we must expect that discontinuities in \( u, m \) will occur at the same points. Then for \( \text{(8)} \) we will need an alternative solution concept to that used for the conservation law \( \text{(7)} \). We use an approach pioneered by Volpert [14] and developed by Le Floch [15]. In this approach, we seek functions \( u, m \in BV(\Omega) \), where \( \Omega = \mathbb{R}_+^2 \) and \( BV(\Omega) \) is the space of real-valued functions of bounded variation on \( \Omega \), i.e. functions locally integrable on \( \Omega \) whose first order partial derivatives are locally finite Borel measures [14]. Such functions have at worst jump discontinuities almost everywhere (a.e.) on \( \Omega \). If \( u, v \in BV(\Omega) \) and (nonlinear) \( g \in C^1(\mathbb{R}, \mathbb{R}) \) then using a certain regularization \( \hat{g}(u) \) of \( g(u) \), the expression \( \hat{g}(u)\partial v/\partial x^i \) is a finite Borel measure which coincides with \( g(u)\partial v/\partial x^i \) a.e. on \( \Omega \). See [14, 15] for details of these last two statements. This allows us to make sense of the left hand side of \( \text{(8)} \) when \( u, m \) are both discontinuous. We define a weak solution of \( \text{(7)}, \text{(8)} \) to be a pair \( u, m \in BV(\Omega) \) which solve \( \text{(7)} \) in the usual weak sense (i.e. which satisfy \( \text{(9)} \) for all \( \psi \in C^\infty_0(\Omega) \)) and which satisfy \( m_t - \hat{u} m_r = 0 \) in the sense of Borel measures. The global existence of such solutions may be demonstrated as follows. This extends the standard derivation of weak solutions for conservation laws [16].

As noted above, \( E, m \) are constant along characteristics and so a classical solution is defined until such time as characteristics intersect. Recall that shell-crossing (i.e. intersection of characteristics) occurs in disjoint open subsets of the parameter space labelled by \( R \), which is the characteristic label. So we may restrict attention to the case of one such open subset and treat others in the same way. Let us first consider the nature of the characteristics.

**Lemma 1** Let \( u_0, m_0 \in C^1(\mathbb{R}_+) \) satisfy \( u'_0 > 0, m'_0 \geq 0 \), where \( u_0(r) = u(r, 0) > 0 \) and \( m_0(r) = m(r, 0) \geq 0 \).

(i) The characteristic curves are \( C^\infty \) on \( \Omega \) and through each point of \( \Omega \), there passes at least one characteristic. Each characteristic reaches \( r = 0 \) in a finite amount of time.

(ii) If the characteristics \( R_1, R_2 \) (with \( R_1 < R_2 \)) intersect in the region \( r > 0 \), then their slopes at the point of intersection satisfy
\[
0 > \left. \frac{dt}{dr} \right|_{R_1} > \left. \frac{dt}{dr} \right|_{R_2}.
\]

(iii) A given pair of characteristics intersects at most once and does so at some time \( t > 0 \).
Proof: Part (i) is proven by noting that integrating (6) is equivalent to integrating the evolution equation (2) in the co-moving co-ordinate system. The characteristic $R$ (i.e. the solution of (6) which satisfies $r|_{t=0} = R$) reaches $r = 0$ at the shell focussing time $t_{SF}(R)$. To prove part (ii), let $u_1, u_2$ denote the values of $u$ at the point of intersection on the characteristics $R_1, R_2$ respectively. It suffices to show that $u_2^2 > u_1^2$. Let $r_3$ be the value of $r$ at which the intersection occurs. Then $r_3 < R_1 < R_2$, since $R_1$ is the initial value of $r$ along the characteristic $R_1$ and the characteristics have negative slope. Then

$$u_2^2 - u_1^2 = \frac{m_0(R_2) - m_0(R_1)}{r_3} + E_0(R_2) - E_0(R_1)$$

$$\geq \frac{m_0(R_2) - m_0(R_1)}{R_2} + E_0(R_2) - E_0(R_1)$$

$$> \frac{m_0(R_2) - m_0(R_1)}{R_1} + E_0(R_2) - E_0(R_1)$$

$$= u_0^2(R_2) - u_0^2(R_1) > 0$$

as required. The proof of part (iii) follows from the conclusion of part (ii); the characteristic $R_1$ can only intersect $R_2$ from below. $\square$

We note also that the regions in which characteristics intersect form connected subsets of $\Omega$ which may be written explicitly in terms of the characteristic label $R$ as

$$Z = \{(r, t) \in \Omega : r = r(t; R), t_{SC}(R) \leq t \leq t_{SF}(R), R_1 \leq R \leq R_2\}$$

where $r = r(t; R)$ represents the characteristic with label $R$, and $R_1, R_2$ are least and greatest values of $R$ for which the corresponding characteristics meet others in $\Omega$.

Let $p$ be any point in $Z^o$, the interior of $Z$. We can identify the left-most and the right-most characteristics which intersect at $p$. Let these correspond to values $R_L < R_R$ of the characteristic label respectively. There exists an open neighbourhood $N_L$ of $R_L$ and a time interval $I$ such that the change of coordinates $\gamma : (R, t) \mapsto (r(t; R), t)$ is a diffeomorphism from $N_L \times I$ onto an open neighbourhood $N_p$ of $p$. (The Jacobean of this transformation is singular only at the shell-crossing singularity, which corresponds to the boundary of $Z$.) We can then define a function $u_L : N_p \to \mathbb{R}$ by $u_L(r, t) = u_0(R)$, where $R$ is defined by $(R, t) = \gamma^{-1}(r, t)$. From the assumptions and part (i) of Lemma 1, we see that $u_L$ is $C^1$ on $N_p$. This extends to a $C^1$ function on the interior of $Z$ and a corresponding $C^1$ map $u_R$ is defined in a similar fashion. Part (ii) of Lemma 1 implies that $u_L, u_R$ satisfy what we will refer to as a gap property: $u_L(r, t) < u_R(r, t)$ for $(r, t) \in Z^o$.

Let $(r_0, t_0)$ be the co-ordinates of that point $p_0$ on the shell-crossing singularity (i.e. on the boundary of $Z$) for which $t_0$ is minimal. This is the globally earliest occurrence of characteristic crossing, and we assume for convenience that this defines a unique point. (This will involve an assumption about the initial data $u_0, m_0$.) The existence, smoothness and gap property of $u_L, u_R$ imply that there exist $C^0$ piecewise $C^1$ curves $r = \phi(t)$ extending through $Z^o$ from $p_0$ to $r = 0$ with the property that

$$-u_L(\phi(t), t) > \phi'(t) > -u_R(\phi(t), t),$$
for all values of $t$ for which $\phi'(t)$ exists. The relevance of such curves is that they provide a means of constructing a unique foliation of $\Omega$ by characteristics. $\Omega - Z$ already admits such a foliation. In $Z$ and to the left of $\Phi$ (the graph of $\phi$), we use the characteristics which approach $\Phi$ from the left (those used to define $u_L$ along $\Phi$). In $Z$ and to the right of $\Phi$, we use the characteristics which approach $\Phi$ from the right. We note that all uses of the words ‘right’ and ‘left’ are well-defined by the gap property and the definition of $\phi$.

We are now in a position to write down weak solutions of \ref{eq:1} and \ref{eq:2}. For each path $\phi$ described above, we define $m(\phi,0) = m_0(\phi)$ and then define $m_\phi$ throughout $\Omega - \Phi$ by taking $m_\phi$ to be constant along the characteristics of the foliation constructed around $\phi$. Taking $m_0 \in C^1(\mathbb{R}_\pm)$, Lemma 1 shows that $m_\phi \in C^1(\Omega - \Phi)$, with a jump discontinuity across $\Phi - p_0$. This is sufficient to guarantee that $m_\phi \in BV(\Omega)$. We do likewise to obtain $E_\phi \in BV(\Omega)$. This procedure yields BV functions $m_\phi, E_\phi$ which solve the field equations \ref{eq:4} and \ref{eq:5} along individual characteristics. However more is required in order to have a weak solution. A standard argument (see for example [16]) shows that in order for $u, m$ to satisfy \ref{eq:3} for all $\psi \in C_0^\infty(\Omega)$, the Rankine-Hugoniot condition must be satisfied:

$$[u]\dot{\phi} = [-\frac{u^2}{2}]. \tag{10}$$

For any function $T$ on $\Omega$, $[T] = T_+ - T_-$ where $T_\pm$ denotes the right and left-hand limits of $T(r,t)$ as the discontinuity is approached along $t =$constant. Thus when $u_L \neq u_R$, i.e. anywhere in the interior of $Z$, we must have

$$\dot{\phi}(t) = -\frac{1}{2}(u_L(\phi(t),t)) + u_R(\phi(t),t)). \tag{11}$$

The smoothness of $u_L, u_R$ implies the existence and uniqueness of a $C^2$ solution $r = \phi_*(t)$ of the initial value problem consisting of \ref{eq:6} and the initial condition $r(t_0) = r_0$.

It remains to verify that \ref{eq:3} is satisfied in the sense of measures. This is trivially the case on $\Omega - \Phi_*$, where the equation is solved in smooth functions. So consider the measure $\sigma = m_\alpha - \dot{u}m_{\alpha, r}$. For any point $(r,t)$ on the shock, this measure evaluates to

$$\sigma(\{(r,t)\}) = -[m]\dot{\phi} - [m]\int_0^1 (u_L + \alpha(u_R - u_L)) d\alpha = 0.$$ 

We have used here Volpert’s definition of the regularization $\hat{u}$ of a discontinuous function, and applied the Rankine-Hugoniot condition after evaluating the integral. This concludes the proof of the existence of weak solutions of the system \ref{eq:1}, \ref{eq:2}. We summarise as follows.

**Proposition 1** Let $u_0, m_0 \in C^1(0,\infty)$ satisfy $u_0' > 0$, $m_0' \geq 0$. Then there exists $u, m \in BV(\Omega)$ giving a weak solution of \ref{eq:1}, \ref{eq:2} as defined above with $u(r,0) = u_0(r), m(r,0) = m_0(r)$ and which satisfies $m(x+h,t) \geq m(x,t)$ for all $h > 0$ and all $(x,t) \in \Omega$. $\Box$

The last statement here is immediate from the construction and indicates that the solutions have positive distributional density. Note that the solutions are global in $t$,
but that this does not imply the absence of a future singularity. Figure One illustrates
the result of this Proposition for the initial data $E_0 = 0$ and

$$m_0(\xi) = \begin{cases} M(\xi), & 0 \leq x \leq 2.25; \\ M(2.25), & x > 2.25, \end{cases}$$

(12)

where

$$M(\xi) = \frac{\xi^2}{(1 + 12\xi - 9\xi^2 + 2\xi^3)^2}.$$  

Thus the boundary of the dust sphere is initially at $x = 2.25$.

3. Uniqueness considerations

In both the marginally and non-marginally bound cases, uniqueness fails on a
fundamental level. To see this we focus on the marginally bound case. In this case,
$E = 0$ and there is only field equation (4) which can be written in conservative form by
taking $x = -\frac{2}{3} r^{3/2}$;

$$m_t - \sqrt{mm} x = m_t + f_x = 0$$

where $f = -\frac{2}{3} m^{3/2}$. For this form of the equation, the Rankine-Hugoniot condition is
$[m] \phi = [f]$. For any smooth function $M(m)$, the solution of the pde’s for $M$ and $m$
agree in the smooth domain. However the conservative form of the equation for $M$
will not be the same as that for $m$. This leads to a different shock speed $\Box$, and hence a different
location for the shock. For example, $M = \ln m$ leads to $[\ln m] \phi = [-2\sqrt{m}]$. Thus we do
not have a unique weak solution for given initial data; we could only expect to obtain
such if the field equations were fundamentally integral conservation laws which is not
the case.

However it is possible to obtain a uniqueness result on a different level for the
marginally bound case. (There are no global uniqueness results for weak solutions
of non-conservative first order systems.) We note that as with entropy solutions, the
condition which yields uniqueness has a strong physical motivation, being equivalent
in the present case to positivity of the energy density considered as a measure. The
proof involves a variation of Oleinik’s proof of the uniqueness of entropy solutions of
conservation laws with convex flux functions (see chapter 16 of [16]).

**Proposition 2** Let $m_0 \in L^\infty(\mathbb{R}_+)$ satisfy $m_0(x + h) \geq m_0(x)$ for all $x \geq 0$ and $h > 0$. Let $f(m) = -\frac{2}{3} m^{3/2}$. Then there exists a unique weak solution on $\mathbb{R}_+^2$ of the conservation law

$$m_t + f(m)_x = 0$$

(13)

which satisfies

$$m(x + h, t) \geq m(x, t)$$

for all $h > 0$ and for all $(x, t) \in \mathbb{R}_+^2$, and $m(x, 0) = m_0(x)$.  

Proof: We refer to a solution described in this proposition as a positive density solution.

(i) As regards existence, the proof of Proposition 1 is easily adapted to the present case. The regularity of the data can be lowered by virtue of the fact that the characteristics in this case are straight lines satisfying

\[ \frac{dx}{dt} = -\sqrt{m}. \]

(ii) For uniqueness, our aim is to prove that there is at most one positive density solution in \( L^\infty(\Omega) \). So let \( m, \tilde{m} \) be two such solutions with the same initial data. Then each satisfies the weak solution condition

\[ \int_{\Omega} (m \psi_t + f(m) \psi_x) \, dx dt = 0 \]

for all \( \psi \in C^\infty_0(\Omega) \), and so we can write

\[ 0 = \int_{\Omega} [(m - \tilde{m}) \psi_t + (f(m) - f(\tilde{m})) \psi_x] \, dx dt \]

\[ = \int_{\Omega} w[\psi_t + G(m(x,t), \tilde{m}(x,t)) \psi_x] \, dx dt, \quad (14) \]

where \( w = m - \tilde{m} \) and

\[ G = \int_0^1 f'(\lambda m(x,t) + (1 - \lambda)\tilde{m}(x,t)) \, d\lambda. \]

The proof proceeds by attempting to replace the term multiplying \( w \) in the integrand of (14) by an arbitrary test function \( \phi \in C^\infty_0(\Omega) \). The result would then be immediate, but the non-smoothness of \( G \) must be dealt with carefully. As mentioned above, this proof follows almost identically Oleinik’s proof of uniqueness of entropy solutions as given in [16].

(iii) For any \( \epsilon \in (0,1) \), we define the smooth functions \( m^\epsilon = \eta_\epsilon \ast m \) and \( \tilde{m}^\epsilon = \eta_\epsilon \ast \tilde{m} \) where \( \eta_\epsilon \) is the standard mollifier and \( \ast \) represents convolution. Then \( \|m^\epsilon\|_\infty \leq \|m\|_\infty \) and \( m^\epsilon \to m \) pointwise a.e. in \( \Omega \), with similar results holding for \( \tilde{m}^\epsilon \). The positive density condition \( (m_0 \text{ non-decreasing}) \) implies that

\[ m^\epsilon_{,x}(x,t) \geq 0 \quad (15) \]

in \( \Omega \).

Next, we define \( G^\epsilon = G(m^\epsilon, \tilde{m}^\epsilon) \) and write (14) as

\[ 0 = \int_{\Omega} w[\psi_t + G^\epsilon \psi_x] \, dx dt + \int_{\Omega} w(G - G^\epsilon) \psi_x \, dx dt. \quad (16) \]

Now let \( \phi \in C^\infty_0(\Omega) \) be arbitrary, and take \( T > 0 \) such that \( \text{supp}(\phi) \subseteq (0, \infty) \times (0,T) \). Let \( \psi^\epsilon \) be the solution of the terminal value problem

\[ \psi^\epsilon_t + G^\epsilon \psi^\epsilon_x = \phi \quad \text{in} \quad (0, \infty) \times (0,T) \]

\[ \psi^\epsilon = 0 \quad \text{on} \quad (0, \infty) \times \{t = T\}. \quad (17) \]
This problem may be solved by the method of characteristics and admits a unique smooth solution. Furthermore, since $|G^e|$ is bounded in the compact set $\text{supp}(\phi) \subset \Omega$, the solution $\psi^e$ also has compact support in $(0, \infty) \times [0, T)$.

(iv) We next establish the following upper bound:

$$|\psi^e_{,x}(x, t)| \leq e^T \|\phi_{,x}\|_\infty \, \text{in} \, \text{supp}(\psi^e). \quad (18)$$

Note that

$$G^e_{,x} = \int_0^1 f''(\lambda m^e + (1 - \lambda) \bar{m}^e)(\lambda m^e_{,x} + (1 - \lambda) \bar{m}^e_{,x}) \, d\lambda.$$  

Since $m > 0$ in $\Omega$, $f'' = -\frac{1}{2}m^{-1/2}$ is negative and bounded on any compact subset of $\Omega$. The positive density condition $\text{supp}(\psi^e) \subset \Omega$, then yields $G^e_{,x} \leq 0$ in $\text{supp}(\psi^e)$.

Let $a = e^t \psi^e_{,x}$. From (17), we obtain

$$a_{,t} + G^e a_{,x} = (1 - G^e_{,x})a + e^t \phi_{,x}, \quad (19)$$

and we note that $a = 0$ on $t = T$. Let $s > 0$. In the compact set $K_s = \text{supp}(\psi^e) \cap \{s \leq t \leq T\}$, $a$ attains its maximum and minimum values. Let its maximum occur at $(x_0, t_0)$; we must have $a(x_0, t_0) \geq 0$. If $(x_0, t_0) \in \partial K_s$, with $t_0 > s$, then $a(x_0, t_0) = 0$. If $s \leq t_0 < T$ and $(x_0, t_0)$ is in the interior of $K_s$, then we must have

$$a_{,t} \leq 0, \quad a_{,x} = 0$$

at the maximum. Then from (19), at $(x_0, t_0)$ we have

$$0 \geq (1 - G^e_{,x})a + e^{t_0} \phi_{,x}$$

$$\geq a + e^{t_0} \phi_{,x}.$$  

From this we obtain

$$a_{\text{max}} \leq -e^{t_0} \phi_{,x}$$

$$\leq e^{t_0} \|\phi_{,x}\|_\infty$$

$$\leq e^T \|\phi_{,x}\|_\infty.$$  

Similarly, at the non-positive minimum we must have either $a = 0$ or

$$a_{,t} \geq 0, \quad a_{,x} = 0.$$  

In this case, (19) yields

$$0 \leq (1 - G^e_{,x})a + e^{t_0} \phi_{,x}$$

$$\leq a + e^{t_0} \phi_{,x}.$$  

From this we obtain

$$a_{\text{min}} \geq -e^{t_0} \phi_{,x}$$

$$\geq -e^T \phi_{,x}$$

$$\geq -e^T \|\phi_{,x}\|_\infty.$$  

Hence

$$|a| \leq e^T \|\phi_{,x}\|_\infty,$$
and so we obtain the \( s \)-independent bound

\[
|\psi^{\epsilon}_{x}| \leq e^{T} \|\phi_{x}\|_{\infty}
\]
on \text{supp}(\psi^{\epsilon}) \cap \{s \leq t \leq T\} for any \( s > 0 \), and so we may let \( s \to 0 \) to complete the derivation of (18).

(v) It remains to show that for any \( \phi \in C_{0}^{\infty}(\Omega) \) and \( \delta > 0 \),

\[
\left| \int_{\Omega} w\phi \, dx \, dt \right| < \delta.
\]

In (16), take \( \psi \) to be \( \psi^{\epsilon} \), the solution of (17). Then

\[
\int_{\Omega} w\phi \, dx \, dt = -\int_{\Omega} w(G - G^{\epsilon})\psi^{\epsilon}_{x} \, dx \, dt.
\]
The domain of integration on the right hand side may be replaced by \( \text{supp}(\psi^{\epsilon}) \), whereon (18) applies to bound \( |\psi^{\epsilon}_{x}| \) by a constant. Then the pointwise limit \( G - G^{\epsilon} \to 0 \) yields the result by taking \( \epsilon \) to be sufficiently small. \( \square \)

We note that this proof carries through for different formulations of the field equation (13). That is, for a diffeomorphism \( G : \mathbb{R}^{+} \to \mathbb{R}^{+} \) with \( G(0) = 0 \), the corresponding equation for \( M = G(m) \) with data corresponding to positive density has unique weak positive density solutions. These solutions will however disagree with that of Proposition 2.

4. Discussion

The solutions presented here show that one can extend space-time beyond a shell-crossing singularity in a natural way; the extension arises almost immediately when one steps out of the co-moving co-ordinate system. The drawback is that the extension is not unique. Our interpretation of this fact is that, leaving aside the question of the matter model, shell-crossing singularities should be included in a discussion of cosmic censorship. This is based on the following argument originally put forward in [17].

Strong cosmic censorship asserts that generically, initial data evolve to give a maximal globally hyperbolic space-time. It has been shown that such an evolution must be unique [18]. Thus the lack of a unique evolution indicates a violation of strong cosmic censorship. Genericity aside, this is the situation we have found here: there are multiple \( L^{\infty} \) weak solutions of Einstein’s equation for spherical dust for certain open subsets (cf. [13]) of the space of initial data.

Having extended beyond the shell-crossing singularity, one can now investigate the question of the global structure of the space-time and generalized hyperbolicity and study geodesics impinging on the singularity. This work is in progress, utilizing Colombeau theory [19] as has recently been done for space-times admitting conical singularities [20], [21] and for hypersurface singularities [22].

The problem of uniqueness seems hard to resolve, as it would seem to involve deciding on which formulation of the conservation law (7) is ‘correct’. One possible way to obtain uniquely the shock surface is to consider the conservation law \( \nabla_{a}T^{ab} = 0 \).
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This was the route taken in [23], in which the authors generalize the classic results of Israel [24] to Lipschitz continuous metrics and so deal with a different variety of gravitational shocks. In our the present case we do not have Lipschitz continuity. Indeed $T^{ab}$ is measure-valued and it is far from clear how to extract jump conditions from the conservation law. An approach based on the recently developed geometric theory of nonlinear generalized functions [25] may yield the required results.

There is an interesting analogy between shell-crossing singularities and shock waves in gas dynamics. In each case, the singularity disappears when a more appropriate matter model is used (dust replaced by the Vlasov model, inviscid fluid replaced by viscous). In the latter case, the shocks observed in the inviscid system reflect large derivatives which can arise in the more realistic model. It would be of interest to see if this pattern is repeated in the dust-Vlasov pair, where the existence of large derivatives in the latter model may be to all intents and purposes physical singularities.

Acknowledgments

I am grateful to G. Hörmann, M. Kunzinger, M. Oberguggenberger and R. Steinbauer for enlightening discussions and to the Royal Irish Academy and the Austrian Academy of Sciences for financial support. I also thank C. Barrabès and P. Hogan for their support and the referees for their comments.

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Figure 1. The diagram shows the characteristics (solid and dashed lines) corresponding to the mass function \( m(x, t) \). The dashed lines are the characteristics that emanate from the vacuum region of the initial data surface \( t = 0 \). The shell-crossing singularity corresponds to the boundary of the region \( Z \) in which characteristics cross, the earliest point of which is roughly at \( (x, t) = (0.496, 3.695) \). The shock, shown bold, evolves from this point. (In the figure, this has been constructed numerically using a second order Euler scheme.) The solution in co-moving co-ordinates holds outside \( Z \); the solution constructed in Proposition One holds throughout \( \Omega \) and so extends through the shell-cross.