QUASI-POTENTIALS OF THE ENTROPY FUNCTIONALS
FOR SCALAR CONSERVATION LAWS

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Abstract. We investigate the quasi-potential problem for the entropy
cost functionals of non-entropic solutions to scalar conservation laws
with smooth fluxes, as defined in [10, 15, 2, 13]. We prove that the
quasi-potentials coincide with the integral of a suitable Einstein entropy.

1. Introduction

For a real function $f$, consider the scalar conservation law in the unknown
$u \equiv u(t, x)$

$$u_t + f(u)x = 0$$  (1.1)

where $t \in [0, T]$ for some $T > 0$, $x \in \mathbb{T}$ (the one-dimensional torus), and sub-
scripts denote partial derivatives. Equation (1.1) does not admit in general
classical solutions for the associated Cauchy problem, even if the initial da-
tum is smooth. On the other hand, if $f$ is non-linear, there exist in general
infinitely many weak solutions. An admissibility condition, the so-called
entropic condition, is then required to recover uniqueness for the Cauchy
problem in the weak sense [6]. The unique solution satisfying such a condi-
tion is called the Kruzkhov solution.

A classical result [6, Chap. 6.3] states that the Kruzkhov solution can
be obtained as limit for $\varepsilon \downarrow 0$ of the solution $u^\varepsilon$ to the Cauchy problem
associated with the equation

$$u_t + f(u)_x = \frac{\varepsilon}{2} (D(u)u_x)_x$$  (1.2)

provided that the initial data also converge. Here the diffusion coefficient
$D$ is a uniformly positive smooth function, and we remark that convergence
takes place in the strong $L_p([0, T] \times \mathbb{T})$ topology. The Kruzkhov solution to
(1.1) has also been proved to be the hydrodynamical limit of the empirical
density of stochastic particles systems under hyperbolic scaling, when the
number of particles diverges to infinity [11, Chap. 8]. These results legitimize
the Kruzkhov solution as the physically relevant solution to (1.1), and the
entropic condition as the appropriate selection rule between the infinitely
many weak solutions to (1.1).

1991 Mathematics Subject Classification. Primary 35L65; Secondary 49S05.
Key words and phrases. Quasi-potential, conservation laws, entropy.
Provided the flux $f$ and the diffusion coefficient $D$ are chosen appropriately (depending on the particles system considered), one may say that (1.2) is a continuous version for the evolution of the empirical density of particles system, in which the small stochastic effects are neglected (or averaged) and $\varepsilon$ is the inverse number of particles. The convergence of both (1.2) and the empirical measure of the density of particles to the same solution of (1.1) confirms somehow that this approximation is reliable.

In [10, 15], the long standing problem of providing a large deviations principle for the empirical measure of the density of stochastic particles systems under hyperbolic scaling is addressed. In particular, the totally asymmetric simple exclusion process is investigated (which in particular corresponds to $f(u) = u(1 - u)$ in the hydrodynamical limit equation (1.1)), and the large deviations result partially established. Roughly speaking, when the number of particles $N$ diverges to infinity, the asymptotic probability of finding the density of particles in a small neighborhood of a path $u : [0, T] \times \mathbb{T} \to \mathbb{R}$ is $e^{-N H^J_V(v)}$, where $H^J_V$ is a suitable large deviations rate functional (see Section 2).

A continuous mesoscopic mean field counterpart of this large deviations result is provided in [2, 13]. In [13] a large deviations principle for a stochastic perturbation to (1.2) (driven by a fluctuation coefficient $\sigma$) is investigated in the limit of jointly vanishing stochastic noise and (deterministic) diffusion. In [2] a purely variational problem is addressed, namely the investigation of the $\Gamma$-limit of a family of functionals $H_\varepsilon$ associated with (1.2) (see Section 2). The candidate large deviations functional $H$ introduced in [13] and the candidate $\Gamma$-limit introduced in [2] coincide, and in the case $f(u) = u(1 - u)$ they are expected to coincide with the functional $H^J_V$ introduced in [10, 15] (the equality can be proved on functions of bounded variations, but it is missing in the general case). The functional $H$ thus provides a generalization of the functional $H^J_V$, for arbitrary fluxes $f$ (in particular, not necessarily convex or concave), diffusion coefficients $D$ and fluctuation coefficients $\sigma$. The functionals $H_\varepsilon$, $H$ and $H^J_V$ are nonnegative; $H_\varepsilon$ vanishes only on solutions to (1.2), so that $H_\varepsilon$ can be interpreted as the cost of violating the flow (1.2). On the other hand, $H$ and $H^J_V$ are $+\infty$ off the set of weak solutions to (1.1), they vanish only on Kruzkov solution to (1.1), and thus they can be interpreted as the cost of violating the entropic condition for the flux (1.1). Section 2 of the paper is devoted to the precise definition of the functionals $H_\varepsilon$, $H$ and $H^J_V$.

Redirecting the reader to Section 3 for a more detailed discussion, here we briefly recall a general definition of the quasi-potential associated with a family of functionals. Suppose we are given a topological space $U$, and for each $T > 0$ a set $X_T \subset C([0, T]; U)$ and a functional $I_T : X_T \to [0, +\infty]$. For the sake of simplicity, let us also fix a point $m \in U$. The quasi-potential $V : U \to [0, +\infty]$ associated with $\{I_T\}$ is then defined as $V(u) := \inf_{T > 0} \inf_w I_T(w)$, where the infimum is carried over all the $w \in X_T$.
such that \( w(0) \equiv m \) and \( w(T) = u \). A natural choice for the reference point \( m \) should be an attractive point for the minima of the functionals \( I_T \) (see e.g. Theorem 4.4 and Theorem 5.5 for the case of \( H_\varepsilon, H \) and \( H^{JV} \)). Indeed, in such a case, the investigation of the quasi-potential is a classical subject both in dynamical optimal control theory and in large deviations theory, as it quantifies “the cheapest cost” to move from the stable point \( m \) to a general one \( u \). Moreover, from the optimal control theory point of view, the quasi-potential describes the long time limit of the functionals \( I_T \), see e.g. [3]. Furthermore, there is a broad family of stochastic processes for which the quasi-potential is expected to be the large deviations rate functional of their invariant measures, provided \( I_T \) is the large deviations rate functional of the laws of such processes up to time \( T > 0 \) (see e.g. [9, Theorem 4.4.1] for the classical finite-dimensional case, and [4] for a more general discussion and applications to particles systems). Moreover, see [9, Chap. 4], the quasi-potential of the large deviations rate functionals provides a valuable tool to investigate long time behavior of the processes (e.g. average time to be waited for the process to leave an attractive point, and the path to follow when the process performs such a deviation). Finally, in the context of non-equilibrium statistical mechanics in which the functionals \( H_\varepsilon, H \) and \( H^{JV} \) have been introduced, the quasi-potential has been proposed as a dynamical definition of the free energy functional for systems out of equilibrium [5].

Since \( H_\varepsilon \) is a functional associated with a control problem (see [2]) and it can be also retrieved as large deviations rate functional of some particles systems (e.g. weakly asymmetric particle systems, see [11]) and stochastic PDEs (see [13]), the quasi-potential problem is relevant for such a functional. Similarly, \( H \) and \( H^{JV} \) are the (candidate) large deviations rate functionals for both particles systems processes and stochastic PDEs, see [10, 15, 13].

Given a bounded measurable map, \( u_i : \mathbb{T} \to \mathbb{R} \), it is well known that the (entropic) solutions to the Cauchy problems for (1.1) and (1.2) with initial datum \( u_i \) will converge to the constant \( m = \int_T dx u_i(x) \) namely constant profiles are attractive points for the zeros of the functionals \( H_\varepsilon, H \) and \( H^{JV} \). Given \( m \in \mathbb{R} \), and two positive, smooth maps on \( \mathbb{T} \), interpreted as the diffusion coefficient \( D \) and fluctuation coefficient \( \sigma \), the Einstein entropy is defined as the unique nonnegative function \( h_m \) on \( \mathbb{R} \) such that
\[
h_m''(m) = D.
\]
In this paper, we establish an explicit formula for the quasi-potential problem associated with the functionals \( H_\varepsilon, H \) and \( H^{JV} \) (which of course will depend on a time parameter \( T \)) with reference point the constant maps on the torus, proving that these three quasi-potential functionals coincide and are equal to the integral of the Einstein entropy. More precisely, given \( u_f \in L_\infty(\mathbb{T}) \), the quasi-potential \( V(m, u_f) \) of \( H_\varepsilon, H \) and \( H^{JV} \) with reference constant \( m \in \mathbb{R} \) is equal to \( \int_\mathbb{T} dx h_m(u_f(x)) \) if \( \int_T dx u_f(x) = m \) and it is \(+\infty\) otherwise (see Theorem 3.1).

As remarked above, both the large deviations results in [10, 15, 13] and the \( \Gamma \)-limit results in [2] are incomplete, due to little knowledge of structure.
Theorems and regularity results for weak solutions to conservation laws [11]. These results on the quasi-potential give therefore an additional heuristic argument supporting the actual identification of $H$ as the $\Gamma$-limit of $H_\varepsilon$. Similarly, since it is easily seen that the large deviations rate functional (in the hydrodynamical limit) of the invariant measures of the totally asymmetric simple exclusion process is also given by the integral of the Einstein entropy, these results also support the conjecture that $H$ and $H^{JV}$ may coincide at least in the case $f(u) = u(1 - u)$ and that they are in fact the large deviations rate functional of the totally asymmetric simple exclusion process. Finally, we remark that the integral of the Einstein entropy is expected to rule the long time behavior of well-behaving physical systems, and the result provided in this paper thus also supports the universality of the Jensen and Varadhan functional $H^{JV}$ (or in general of $H$) as a relevant universal entropy functional for asymmetric conservative, closed physical systems.

2. Preliminaries

Our analysis will be restricted to equibounded “densities” $u$, and for the sake of simplicity we let $u$ take values in $[-1, 1]$. Let $U$ denote the compact Polish space of measurable functions $u : \mathbb{T} \to [-1, 1]$, equipped with the $H^{-1}(\mathbb{T})$ metric

$$d_U := \sup \left\{ \langle u, \varphi \rangle, \varphi \in C^\infty_c(\mathbb{T}), \langle \varphi_x, \varphi_x \rangle + \langle \varphi, \varphi \rangle = 1 \right\}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_2(\mathbb{T})$. Given $T > 0$, let $X_T$ be the Polish space $C([0, T]; \mathbb{R})$ endowed with the metric

$$d_{X_T}(u, v) := \sup_{t \in [0, T]} d_U(u(t), v(t)) + \|u - v\|_{L_1([0, T] \times \mathbb{T})}$$

Hereafter we assume $f$ a Lipschitz function on $[-1, 1]$. Moreover we let $D, \sigma \in C([-1, 1])$ with $D$ uniformly positive, and $\sigma$ strictly positive in $(-1, 1)$.

2.1. The functional $H_\varepsilon$. For $\varepsilon > 0$, $T > 0$, we define $H_{\varepsilon; T} : X_T \to [0, +\infty]$ as (hereafter we may drop the explicit dependence on integration variables inside integrals when no misunderstanding is possible)

$$H_{\varepsilon; T}(u) := \begin{cases} 
\sup_{\varphi \in C_c^{\infty}([0, T] \times \mathbb{T})} \varepsilon^{-1} \left[ - \int_0^T dt \langle u, \varphi_t \rangle + \langle f(u), \varphi_x \rangle - \frac{\varepsilon}{2} \langle D(u)u_x, \varphi_x \rangle \right] & \text{if } u_x \in L^2([0, T] \times \mathbb{T}) \\
- \frac{1}{2} \int_0^T dt \langle \sigma(u)\varphi_x, \varphi_x \rangle & \text{otherwise} \\
+\infty & \text{otherwise}
\end{cases}$$

Note that $H_{\varepsilon; T}(u) = 0$ iff $u \in X_T$ is a weak solution to (1.2). $H_{\varepsilon; T}$ is a lower-semicontinuous and coercive functional on $X_T$ (see [2, Proposition 3.3, Theorem 2.5]). Moreover if $H_{\varepsilon; T}(u) < +\infty$ then $u \in C([0, T]; L^1(\mathbb{T}))$ (see [2, Lemma 3.2]).
2.2. Entropy-measure solutions. We say that \( u \in \mathcal{X}_T \) is a weak solution to (1.1) iff for each \( \varphi \in C^\infty_c((0, T) \times \mathbb{R}) \)
\[
\int_0^T dt \langle u, \varphi_t \rangle + \langle f(u), \varphi_x \rangle = 0
\]
A function \( \eta \in C^2([-1, 1]) \) is called an entropy, and its conjugated entropy flux \( q \in C([-1, 1]) \) is defined up to an additive constant by \( q(w) := \int w dv \eta'(v)f'(v) \). For \( u \in \mathcal{X}_T \) a weak solution to (1.1), for \( (\eta, q) \) an entropy–entropy flux pair, the \( \eta \)-entropy production is the distribution \( \varphi_{\eta,u} \) acting on \( C^\infty_c((0, T) \times \mathbb{R}) \) as
\[
\varphi_{\eta,u}(\varphi) := -\int_0^T dt \langle \eta(u), \varphi_t \rangle + \langle q(u), \varphi_x \rangle
\]
Let \( C^{2,\infty}_c([-1, 1] \times (0, T) \times \mathbb{T}) \) be the set of compactly supported maps \( \vartheta : [-1, 1] \times (0, T) \times \mathbb{R} \ni (v, t, x) \mapsto \vartheta(v, t, x) \in \mathbb{R} \), that are \( C^2 \) in the \( v \) variable, with derivatives continuous up to the boundary of \([-1, 1] \times (0, T) \times \mathbb{T} \), and \( C^\infty \) in the \((t, x)\) variables. For \( \vartheta \in C^{2,\infty}_c([-1, 1] \times (0, T) \times \mathbb{T}) \) let \( \vartheta' \) and \( \vartheta'' \) denote its partial derivatives with respect to the \( v \) variable. We say that a function \( \vartheta \in C^{2,\infty}_c([-1, 1] \times (0, T) \times \mathbb{T}) \) is an entropy sampler, and its conjugated entropy flux sampler \( Q : [-1, 1] \times (0, T) \times \mathbb{T} \) is defined up to an additive function of \( (t, x) \) by \( Q(w, t, x) := \int w \vartheta'(v, t, x)f'(v)dv \). Finally, given a weak solution \( u \) to (1.1), the \( \vartheta \)-sampled entropy production \( P_{\vartheta,u} \) is the real number
\[
P_{\vartheta,u} := -\int_{(0,T)\times T} dt dx \left[ (\partial_t \vartheta)(u(t,x),t,x) + (\partial_x Q)(u(t,x),t,x) \right] \tag{2.2}
\]
If \( \vartheta(v, t, x) = \eta(v)\varphi(t, x) \) for some entropy \( \eta \) and some \( \varphi \in C^\infty_c((0, T) \times \mathbb{T}) \), then \( P_{\vartheta,u} = \varphi_{\eta,u}(\varphi) \).

The next proposition introduces a suitable class of solutions to (1.1) which will be needed in the sequel. We denote by \( M_T \) the set of finite measures on \((0, T) \times \mathbb{T}\) that we consider equipped with the weak* topology. In the following, for \( \varrho \in M_T \) we denote by \( \varrho^\pm \) the positive and negative part of \( \varrho \).

**Proposition 2.1.** [3 Proposition 2.3], [7]. Let \( u \in \mathcal{X}_T \) be a weak solution to (1.1). The following statements are equivalent:

(i) for each entropy \( \eta \), the \( \eta \)-entropy production \( \varphi_{\eta,u} \) can be extended to a Radon measure on \((0, T) \times \mathbb{T}\), namely \( \|\varphi_{\eta,u}\|_{TV} := \sup \{ \varphi_{\eta,u}(\varphi), \varphi \in C^\infty_c((0, T) \times \mathbb{T}), \|\varphi\| \leq 1 \} < +\infty \).

(ii) there exists a bounded measurable map \( \varrho_u : [-1, 1] \ni \varrho_u(v; dt, dx) \in M_T \) such that for any entropy sampler \( \vartheta \)
\[
P_{\vartheta,u} = \int_{[-1,1] \times (0,T) \times \mathbb{T}} dv \varrho_u(v; dt, dx) \vartheta''(v, t, x) \tag{2.3}
\]
We say that a weak solution \( u \in \mathcal{X}_T \) is an **entropy-measure solution** to (1.1) if it satisfies the equivalent conditions of Proposition 2.1. The set of entropy-measure solutions to (1.1) is denoted by \( \mathcal{E}_T \subset \mathcal{X}_T \). In general \( \mathcal{E}_T \not\subseteq BV([0, T] \times \mathbb{T}) \), the main regularity result for \( \mathcal{E}_T \) being \( \mathcal{E}_T \subset C([0, T]; L^1(\mathbb{T})) \), provided \( f \in C^2([-1, 1]) \) is such that there is no interval in which \( f \) is affine (see [2, Lemma 5.1]). A Kruzhkov solution to (1.1) is a weak solution \( u \in C([0, T]; L^1(\mathbb{T})) \) such that \( \varphi_{\eta,u} \leq 0 \) in distributional sense, for each convex entropy \( \eta \). Since a weak solution \( u \) such that \( \varphi_{\eta,u} \leq 0 \) can be shown to be an entropy-measure solution, the entropic condition is equivalent to \( \varphi_u(v; dt, dx) \leq 0 \) for a.e. \( v \in [-1, 1] \).

2.3. **\( \Gamma \)-entropy cost of non-entropic solution.** For \( T > 0 \), we introduce the functional \( H_T : \mathcal{X}_T \to [0, +\infty] \) as

\[
H_T(u) := \begin{cases} 
\int dv \frac{D(v)}{\sigma(v)} \varphi^+_u(v; dt, dx) & \text{if } u \in \mathcal{E}_T \\
+\infty & \text{otherwise}
\end{cases}
\]

(2.4)

In [2, Proposition 2.6] it is proved that \( H_T \) is coercive and lower semicontinuous, and that it vanishes only on Kruzhkov solutions to (1.1).

As noted in [2, Remark 2.7], if \( u \in \mathcal{X}_T \cap BV([0, T] \times \mathbb{T}) \) is a weak solution to (1.1), then \( u \in \mathcal{E}_T \). Let \( J_u \) be the jump set of \( u \in \mathcal{E}_T \cap BV([0, T] \times \mathbb{T}) \), \( \mathcal{H}^1 \mathbb{1}_{J_u} \) the one-dimensional Hausdorff measure restricted to \( J_u \) and, at a point \((s, y) \in J_u \), let \( n = (n^t, n^x) \equiv n(s, y) \) be the normal to \( J_u \) and \( u^- \equiv u^-(s, y) \) (respectively \( u^+ \equiv u^+(s, y) \)) be the left (respectively right) trace of \( u \) (these are well defined \( \mathcal{H}^1 \mathbb{1}_{J_u} \) a.e., since \( n^x \) can be chosen uniformly positive, see [2, Remark 2.7]). Then

\[
H_T(u) = \int_{J_u} d\mathcal{H}^1 |n^x| \int dv \frac{D(v)}{\sigma(v)} \frac{\rho^+(v, u^+, u^-)}{|u^+ - u^-|}
\]

(2.5)

where

\[
\rho(v, u^+, u^-) := [f(u^-)(u^+ - v) + f(u^+)(v - u^-) - f(v)(u^+ - u^-)] \mathbb{1}_{[u^- \wedge u^+, u^- \vee u^+]}(v)
\]

(2.6)

and \( \rho^+ \) denotes the positive part of \( \rho \).

In [2], a suitable set \( \mathcal{S}_T \subset \mathcal{E}_T \) of **entropy-splittable** solutions to (1.1) is also introduced, and the next result is proved.

**Theorem 2.2.** [2, Theorem 2.5]. For each \( T > 0 \), the following statements hold.

(i) The sequence of functionals \( \{H_{\varepsilon,T}\} \) satisfies the \( \Gamma \)-liminf inequality

\[
\Gamma \lim_{\varepsilon \to 0} H_{\varepsilon,T} \geq H_T \text{ on } \mathcal{X}_T.
\]

(ii) Assume that there is no interval where \( f \) is affine. Then the sequence of functionals \( \{H_{\varepsilon,T}\} \) is equicoercive on \( \mathcal{X}_T \).
(iii) Assume furthermore that \( f \in C^2([-1, 1]) \), and \( D, \sigma \in C^\alpha([-1, 1]) \) for some \( \alpha > 1/2 \). Define

\[
\overline{H}_T(u) := \inf \left\{ \lim_{n} H_T(u_n), \{u_n\} \subset S_T : u_n \to u \text{ in } X_T \right\}
\]

Then the sequence of functionals \( \{H_{\varepsilon,T}\} \) satisfies the \( \Gamma \)-limsup inequality \( \Gamma \lim_{\varepsilon \to 0} H_{\varepsilon,T} \leq \overline{H}_T \) on \( X_T \).

Note that \( \Gamma \)-limsup inequality is not complete, as it is not known that \( \overline{H}_T = H_T \).

2.4. The Jensen and Varadhan functional. Suppose that \( \sigma \) is such that there exists \( h \in C^2([-1, 1]) \) such that \( \sigma h'' = D \), and let \( g \) be such that \( g' = h' f' \). For \( T > 0 \), we also introduce the Jensen and Varadhan functional \( H^{J\text{V}}_T : X_T \to [0, +\infty] \) as

\[
H^{\text{J\text{V}}}_T(u) := \sup_{\varphi \in C^\infty([0,T] \times [0,1])} \left\{ \int_T dx \left[ h(u(T, x)) \varphi(T, x) - h(u(0, x)) \varphi(0, x) \right] - \int_0^T dt \langle h(u), \varphi_t \rangle + \langle g(u), \varphi_x \rangle \right\}
\]

\[
+ \infty \quad \text{if } u \text{ is a weak solution to } (1.1)
\]

\[
\text{otherwise}
\]

(2.7)

Note that the definition of \( H^{\text{J\text{V}}}_T \) does not depend on the choice of \( h \), provided it satisfies \( \sigma h'' = D \). This functional has been introduced in \([10]\) (in the case \( D \equiv 1 \) and \( f(u) = \sigma(u) = u(1-u) \)). In \([2]\) it is proved that \( H^{\text{J\text{V}}}_T \leq H_T \), that \( H^{\text{J\text{V}}}_T(u) = H_T(u) \) if \( f \) is convex or concave and \( u \) has bounded variation, and that \( H^{\text{J\text{V}}}_T < H_T \) if \( f \) is neither convex or concave.

3. QUASI-POTENTIALS

We want to study the quasi-potentials \( V_\varepsilon, V, V^{\text{J\text{V}}} : [-1, 1] \times U \to [0, +\infty] \) associated respectively with \( H_{\varepsilon,T}, H_T \) and \( H^{\text{J\text{V}}}_T \), and defined as

\[
V_\varepsilon(m, u_f) := \inf \left\{ H_{\varepsilon,T}(u), T > 0, u \in X_T : u(0) \equiv m, u(T) = u_f \right\} \quad (3.1)
\]

\[
V(m, u_f) := \inf \left\{ H_T(u), T > 0, u \in X_T : u(0) \equiv m, u(T) = u_f \right\} \quad (3.2)
\]

\[
V^{\text{J\text{V}}}(m, u_f) := \inf \left\{ H^{\text{J\text{V}}}_T(u), T > 0, u \in X_T : u(0) \equiv m, u(T) = u_f \right\} \quad (3.3)
\]

Note that, if \( u_f \equiv m \), then \( V_\varepsilon(m, u_f) = V(m, u_f) = V^{\text{J\text{V}}}(m, u_f) = 0 \). On the other hand, whenever \( m = +1 \) or \( m = -1 \), then if \( u_f \neq m \), necessarily \( \int dx u_f(x) \neq m \) and thus \( V_\varepsilon(m, u_f) = V(m, u_f) = V^{\text{J\text{V}}}(m, u_f) = +\infty \), since \( H_{\varepsilon,T}(u) = H_T(u) = H^{\text{J\text{V}}}_T(u) = +\infty \) whenever \( u \in X_T \) is such that \( \int_0^T dx u(s, x) \neq \int_0^T dx u(t, x) \) for some \( s, t \in [0, T] \). Therefore, in the following we focus on the case \( m \in (-1, 1) \).
Our main result is the following. For \( m \in (-1, 1) \) define the Einstein entropy \( h_m \in C([-1, 1]; [0, +\infty]) \cap C^2((-1, 1)) \) as the unique function such that \( \sigma(v)h_m''(v) = D(v) \) for \( v \in (-1, 1) \), \( h_m(m) = 0 \), \( h_m'(m) = 0 \), and let

\[
W_m(u_f) := \int_T dx h_m(u_f(x)) \in [0, +\infty]
\]

Note that, if \( \int_T dx u_f(x) = m \), \( W_m(u_f) \) can also be written by the more explicit but less evocative formula

\[
\int_T dx \int_m^{u_f(x)} dw [u_f(x) - w] \frac{D(w)}{\sigma(w)}
\]

**Theorem 3.1.**

(i) Assume

\[
\lim_{\alpha \downarrow 0} \alpha^2 \left[ \frac{1}{\sigma(-1 + \alpha)} + \frac{1}{\sigma(1 - \alpha)} \right] = 0 \tag{3.4}
\]

Then

\[
V^\varepsilon(m, u_f) = \begin{cases} W_m(u_f) & \text{if } \int_T dx u_f = m \\ +\infty & \text{otherwise} \end{cases}
\]

for any \( \varepsilon > 0 \), for any \( m \in (-1, 1) \) and \( u_f \in U \).

(ii) Assume \( f \in C^2([-1, 1]) \) is such that there is no interval in which \( f \) is affine. Assume also that for some \( \delta_0 > 0 \), the set \( \{v \in [-1, 1] : f''(v) = 0\} \cap [m - \delta_0, m + \delta_0] \) is finite. Then

\[
V(m, u_f) = \begin{cases} W_m(u_f) & \text{if } \int_T dx u_f = m \\ +\infty & \text{otherwise} \end{cases}
\]

for any \( m \in (-1, 1) \) and \( u_f \in U \).

(iii) Assume the same hypotheses of (ii) and furthermore that there exists \( h \in C^2([-1, 1]) \) such that \( \sigma h'' = D \). Then

\[
V^{JV}(m, u_f) = \begin{cases} W_m(u_f) & \text{if } \int_T dx u_f = m \\ +\infty & \text{otherwise} \end{cases}
\]

for any \( m \in (-1, 1) \) and \( u_f \in U \).

Note that (3.4) is verified if \( \sigma \) does not vanish, or vanishes slower than quadratically in \(-1\) and \(+1\).

Observe that \( H_{\varepsilon:T} \) has a quadratic structure (see (4.1)), so that the proof of Theorem 3.1 (i) is an infinite-dimensional version of Freidlin-Wentzell theorem [9, Theorem 4.3.1]. However this is not the case for \( H_T \). In particular, since \( H_T(u) = +\infty \) if \( u \) is not a (entropy-measure) solution to (1.1), the main technical difficulty in the proof of Theorem 3.1 (ii) is to show that one can find a solution \( u \) to (1.1) such that \( u \) connects in finite time a profile \( v \in U \) close in \( L^\infty(T) \) to the constant profile \( m \), to \( m \) itself. We remark that the quasi-potential problem for \( H_T \) is at this time being addressed in \[1\] in the case of Dirichlet boundary conditions. While this setting is quite similar to ours, the difficulties are completely different. In the boundary
driven case, the entropic evolution connects a non-constant profile to a constant in finite time, so for $T$ large it is not difficult to solve the minimization problem (3.2) far from the boundaries. On the other hand, new challenging difficulties appear in (3.2) when dealing with weak solutions to (1.1) featuring discontinuities at the boundary (boundary layers). Of course, this problem does not appear at all on a torus.

**Remark 3.2.** Let $T_1, T_2 > 0$, and let $u_1 \in \mathcal{X}_{T_1}$, $u_2 \in \mathcal{X}_{T_2}$. Define the measurable function $u : [0, T_1 + T_2] \times \mathbb{T} \rightarrow [-1, 1]$ by $u(t, x) = u_1(t, x)$ if $t \in [0, T_1]$, and $u(t, x) = u_2(t - T_1, x)$ if $t \in (T_1, T_1 + T_2]$. Then $u \in \mathcal{X}_{T_1 + T_2}$ iff $u_1(T_1) = u_2(0)$ and in such a case

$$H_{\varepsilon; T_1 + T_2}(u) = H_{\varepsilon; T_1}(u_1) + H_{\varepsilon; T_2}(u_2)$$

Furthermore if the hypotheses of Theorem 3.1-(ii) hold, then

$$H_{T_1 + T_2}(u) = H_{T_1}(u_1) + H_{T_2}(u_2)$$

**Proof.** A change of variables in the definition (2.1) shows that $H_{\varepsilon; T_1}(u_1) + H_{\varepsilon; T_2}(u_2)$ can be still written in the form (2.1) with $T = T_1 + T_2$, where however the supremum is carried over all the test functions $\varphi \in C_c^\infty((0, T_1) \cup (T_1, T_1 + T_2) \times \mathbb{T})$. However, if $u_1(T_1) = u_2(0)$, the supremum carried over such test functions coincides with the supremum carried over $C_c^\infty((0, T_1 + T_2) \times \mathbb{T})$. Namely, $H_{\varepsilon; T_1}(u_1) + H_{\varepsilon; T_2}(u_2)$ equals the definition of $H_{\varepsilon; T_1 + T_2}(u)$.

By (2.4) it follows that $H_{T_1 + T_2}(u) = +\infty$ whenever $H_{T_1}(u_1) = +\infty$ or $H_{T_2}(u_2) = +\infty$. Assume instead $H_{T_1}(u_1), H_{T_2}(u_2) < \infty$. Under the assumptions of Theorem 3.1-(ii), the boundedness of $H_T$ implies strong continuity in $L_1(\mathbb{T})$ as remarked below Proposition 2.1. Therefore if $u_1(T_1) = u_2(0)$ then $u \in C([0, T_1 + T_2]; L_1(\mathbb{T}))$ and $u \in \mathcal{E}_{T_1 + T_2}$. By (2.2), (2.3) and the $L_1(\mathbb{T})$ continuity of $u_1, u_2$ and $u$, it follows that $\varphi_u(v; \{T_1\} \times \mathbb{T}) = \varphi_{u_1}(v; \{T_1\} \times \mathbb{T}) = \varphi_{u_2}(v; \{0\} \times \mathbb{T}) = 0$ for a.e. $v \in [-1, 1]$. Thus $\varphi_u(v; dt, dx) = \varphi_{u_1}(v; dt, dx)$ in $[0, T_1] \times \mathbb{T}$ and $\varphi_u(v; dt, dx) = \varphi_{u_2}(v; dt, dx)$ in $[T_1, T_1 + T_2] \times \mathbb{T}$, and the equality follows from (2.4).

Since $H_{\varepsilon; T}(m) = H_T(m) = 0$, by Remark 3.2 the infima in (3.1), (3.2) are attained in the limit $T \rightarrow +\infty$.

4. **Proof of Theorem 3.1 for $V_\varepsilon$**

Given a bounded measurable function $a \geq 0$ on $[0, T] \times \mathbb{T}$ let $\mathcal{D}^1_{a; T}$ be the Hilbert space obtained by identifying and completing the functions $\varphi \in C^\infty([0, T] \times \mathbb{T})$ with respect to the seminorm $\left[\int_0^T dt \langle \varphi_x, a \varphi_x \rangle\right]^{1/2}$. Let $\mathcal{D}^{-1}_{a; T}$ be its dual space. The corresponding norms are denoted respectively by $\| \cdot \|_{\mathcal{D}^1_{a; T}}$ and $\| \cdot \|_{\mathcal{D}^{-1}_{a; T}}$.

**Remark 4.1.** Let $a \geq 0$ be a bounded measurable function on $[0, T] \times \mathbb{T}$. Let $F, G \in L_2([0, T] \times \mathbb{T})$ be such that $F_x, (aG)_x \in \mathcal{D}^{-1}_{a; T}$. Assume that
\[ \int_T dx G(t, x) = 0 \text{ for a.e. } t \in [0, T]. \] Then
\[ \left( F_x, (aG)_x \right)_{D_{\alpha T}^{-1}} = \int_0^T dt \langle F, G \rangle \]
where \( \langle \cdot, \cdot \rangle_{D_{\alpha T}^{-1}} \) denotes the scalar product in \( D_{\alpha T}^{-1}. \)

By a standard application of the Riesz representation theorem (see [2, Lemma 3.1]), we have that if \( H_{\varepsilon; T}(u) < +\infty \) then
\[ H_{\varepsilon; T}(u) = \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x - \frac{\varepsilon}{2} (D(u)u_x)_x \right\|^2_{D_{\sigma(u); T}^{-1}} \tag{4.1} \]

If \( \int_T dx u_f(x) \neq m \), then Theorem 3.1(i) follows from the conservation of the total mass of functions \( u \in X_T \) with \( H_{\varepsilon; T}(u) < +\infty \). On the other hand, if \( \int_T dx u_f(x) = m \), the proof of the theorem is a consequence of the following Lemmata. In fact from Lemma 4.2 we get \( V_{\varepsilon}(m, u_f) \geq W_m(u_f) \), and from Lemma 4.3 and Lemma 4.4 we have \( V_{\varepsilon}(m, u_f) \leq W_m(u_f) + \gamma \) for each \( \gamma > 0 \).

**Lemma 4.2.** Assume (3.1), let \( T > 0 \) and \( u \in X_T \) be such that \( H_{\varepsilon; T}(u) < +\infty \), \( u(0, x) \equiv m \), \( u(T, x) = u_f(x) \). Then \( \int_T dx h_m(u_f(x)) < +\infty \), \( u_t + f(u)_x, (D(u)u_x)_x \in D_{\sigma(u); T}^{-1} \) and
\[ H_{\varepsilon; T}(u) = \int_T dx h_m(u_f(x)) + \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x \right\|^2_{D_{\sigma(u); T}^{-1}} \]

**Lemma 4.3.** For each \( \gamma > 0 \), there exists \( T > 0 \) and \( u \in X_T \) such that \( H_{\varepsilon; T}(u) < +\infty \), \( u(0, x) = m \), \( u(T, x) = u_f(x) \) and
\[ \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x \right\|^2_{D_{\sigma(u); T}^{-1}} \leq \gamma \]

**Proof of Lemma 4.2.** We first assume that there exists \( \delta > 0 \) such that for a.e. \( (t, x) \in [0, T] \times \mathbb{T}, -1 + \delta \leq u(t, x) \leq 1 - \delta \), so that \( \sigma(u) \) is uniformly positive. It follows that \( (D(u)u_x)_x, f(u)_x \in D_{\sigma(u); T}^{-1} \) so that, since \( H_{\varepsilon; T}(u) < +\infty \), by (4.1) we also have \( u_t \in D_{\sigma(u); T}^{-1} \). In particular there exists \( \theta \in L^2([0, T] \times \mathbb{T}) \) such that \( u_t = \theta_x \) weakly. Therefore
\[ H_{\varepsilon; T}(u) = \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x - \frac{\varepsilon}{2} (D(u)u_x)_x \right\|^2_{D_{\sigma(u); T}^{-1}} \]
\[ = \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x \right\|^2_{D_{\sigma(u); T}^{-1}} - \left( \theta_x + f(u)_x, (D(u)u_x)_x \right)_{D_{\sigma(u); T}^{-1}} \]
\[ = \frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x \right\|^2_{D_{\sigma(u); T}^{-1}} - \int_0^T dt \langle \theta, \frac{D(u)}{\sigma(u)} u_x \rangle + \langle f(u), \frac{D(u)}{\sigma(u)} u_x \rangle \]
where in the last line we used Remark 4.1 as for each $t \in [0, T]$

$$\int_T dx \frac{D(u(t,x))}{\sigma(u(t,x))} u_x(t,x) = \int_T dx h_m'(u(t,x))_x = 0$$

Similarly we have $\langle f(u(t)), \frac{D(u(t))}{\sigma(u(t))} u_x(t) \rangle = 0$ and integrating by parts:

$$- \int_0^T dt \langle \theta, \frac{D(u)}{\sigma(u)} u_x \rangle = \int_0^T dt \langle \theta_x, h_m'(u) \rangle = \int_0^T dt \langle u_t, h_m'(u) \rangle$$

$$= \int_T dx h_m(u(T,x)) - h_m(u(0,x))$$

Lemma 4.2 is therefore established for each $u \in \mathcal{X}_T$ bounded away from $-1$ and $+1$. For a general $u \in \mathcal{X}_T$ such that $u(0, \cdot) \equiv m \in (-1, 1)$, and $\delta > 0$, let us define

$$u^\delta(t,x) = (1 - \delta)u(t,x) + \delta m$$

Provided \( (3.4) \) holds, the sequence $\{u^\delta\} \subset \mathcal{X}_T$ converges to $u$ as $\delta \to 0$, and is such that: for $\delta > 0$, $u^\delta$ is bounded away from $-1$ and $+1$; $u^\delta(0, \cdot) \equiv m$, $\int_T dx h(u^\delta(T,x)) \to \int_T dx h(u(T,x))$; $H_{\varepsilon,T}(u^\delta) \to H_{\varepsilon,T}(u)$;

$$\left\| u_t^\delta + f(u^\delta)_x + \frac{\varepsilon}{2} (D(u^\delta)u_x)_x \right\|_{\mathcal{D}_{\sigma(u^\delta),T}^{-1}}^2 \to \left\| u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x \right\|_{\mathcal{D}_{\sigma(u),T}^{-1}}^2$$

Therefore, since Lemma 4.2 holds for $u^\delta$ for each $\delta > 0$, it also holds for $u$. \( \square \)

The following result is well known [8].

**Theorem 4.4.** Let $u_f \in U$ and let $v : [0, \infty) \times T \to \mathbb{R}$ be the solution to (1.2) with initial datum $u_f$. Then $\lim_{t \to -\infty} \|v(t) - m\|_{L_\infty([0,T] \times T)} = 0$ where $m = \int_T dx u_f(x)$.

**Proof of Lemma 4.4.** Let $v : [0, \infty) \times T \to \mathbb{R}$ be the solution to (1.2) with initial datum $v(0,x) = u_f(-x)$, and for $T_1, T_2 > 0$ let $u \in \mathcal{X}_{T_1+T_2}$ be defined as

$$u(t,x) = \begin{cases} (1 - \frac{t}{T_1})m + \frac{t}{T_1}v(T_2, -x) & \text{for } t \in [0, T_1] \\ v(T_1 + T_2 - t, -x) & \text{for } t \in [T_1, T_1 + T_2] \end{cases}$$

Since $u$ satisfies $u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x = 0$ for $t \in [T_1, T_1 + T_2]$, we have by Remark 3.2

$$\frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)_x + \frac{\varepsilon}{2} (D(u)u_x)_x \right\|_{\mathcal{D}_{\sigma(u), T_1+T_2}^{-1}}^2$$

$$\leq \frac{3\varepsilon^{-1}}{2} \left[ \left\| u_t \right\|_{\mathcal{D}_{\sigma(u), T_1}^{-1}}^2 + \left\| f(u)_x \right\|_{\mathcal{D}_{\sigma(u), T_1}^{-1}}^2 + \left\| \frac{\varepsilon}{2} (D(u)u_x)_x \right\|_{\mathcal{D}_{\sigma(u), T_1}^{-1}}^2 \right]$$

(4.2) \( \square \)
Let now $\delta > 0$ (to be chosen below) be small enough to have $-1 < m - \delta < m + \delta < 1$, and define

$$C_{\sigma, \delta} := \max_{v \in [m-\delta, m+\delta]} \frac{1}{\sigma(v)} < +\infty$$

$$C_f(t) := \int_T dx \sigma(u(t, x)) \int_T dx \frac{f(u(t, x))}{\sigma(u(t, x))}$$

$$C_{f, \delta} := \max_{v \in [m-\delta, m+\delta]} f(v) - \min_{v \in [m-\delta, m+\delta]} f(v)$$

$$C_D := \max_{v \in [0, 1]} \frac{D(v)^2}{2}$$

Let also $\theta \in L_2([0, T_1] \times T)$ be defined by

$$\theta(x, t) = \frac{v(T_2, -x) - m}{T_1}$$

$$\int_T \frac{\theta(t, x)}{\sigma(u(t, x))} = 0$$

By Theorem 4.4 there exists $\tau_\delta > 0$ such that $\|v(t) - m\|_{L_\infty(T)} \leq \delta$ for each $t \geq \tau_\delta$. By Remark 4.1 and (4.2), since $u_t = \theta$ weakly, we have for each $T_2 \geq \tau_\delta$

$$\frac{\varepsilon^{-1}}{2} \left\| u_t + f(u)x + \varepsilon \left\langle D(u)u_x \right\rangle_x \right\|_{D_{\sigma(u); T_1 + T_2}^{-1}}^2$$

$$\leq \frac{3\varepsilon^{-1}}{2} \int_0^{T_1} dt \left\langle \theta, \frac{\theta}{\sigma(u)} \right\rangle + \left\langle f(u) - cf_f, \frac{f(u) - cf_f}{\sigma(u)} \right\rangle + \left\langle D(u)u_x, \frac{D(u)u_x}{\sigma(u)} \right\rangle$$

$$\leq \frac{3\varepsilon^{-1}}{2} C_{f, \delta} \left[ \int_0^{T_1} dt \left\langle \theta, \theta \right\rangle + T_1 C_{f, \delta}^2 + T_1 C_D \left\langle v(T_2)_x, v(T_2)_x \right\rangle \right] \quad (4.3)$$

By standard parabolic estimates we have

$$\int_0^{+\infty} dt \left\langle v_x, v_x \right\rangle < +\infty$$

In particular there exists a $T_{2, \delta} > \tau_\delta$ such that $\left\langle v(T_{2, \delta})_x, v(T_{2, \delta})_x \right\rangle < \delta$. Note that, as $\delta \to 0$, $C_{f, \delta}$ stays bounded, while $C_{f, \delta}, \left\langle \theta, \theta \right\rangle$ and $\left\langle v(T_{2, \delta})_x, v(T_{2, \delta})_x \right\rangle$ vanish. Therefore the right hand side of (4.3) can be made arbitrarily small provided $\delta$ is small enough. \hfill \Box

5. PROOF OF THEOREM 3.1 FOR $V$ AND $V^{JV}$

Define the parity operator $P : U \to U$ by $Pu(x) = u(-x)$ and for $T > 0$ the time-space parity operator $P^T : \mathcal{X}_T \to \mathcal{X}_T$ by $P^T u(t, x) = u(T - t, -x)$. Define the time reversed quasi-potential $V : U \times [-1, 1] \to [0, \infty]$ as

$$V(u_i, m) := \inf \left\{ H_T(u), T > 0, u \in \mathcal{X}_T : u(0) = u_i, u(T) \equiv m \right\}$$
Lemma 5.1. Assume $f \in C^2([-1,1])$ is such that there is no interval in which $f$ is affine. Let $T > 0$, $u_f \in U$ and $m = \int_T dx u_f(x)$. Then

$$V(m,u_f) = V(Pu_f,m) + W_m(u_f)$$

Proof. By the assumptions on $f$, as remarked below Proposition 2.1, $\mathcal{E}_T \subset C([0,T];L^1(\mathbb{T}))$. In particular, equations (2.2)-(2.3) extend to any $\vartheta \in C^{2,\infty}([-1,1] \times [0,T] \times \mathbb{T})$ (now $\vartheta(0)$ and $\vartheta(T)$ need not to vanish) as

$$\int_T dx \vartheta(u(T,x),T,x) - \vartheta(u(0,x),0,x)$$

$$- \int_{[0,T] \times T} dt dx \left[ (\partial_t \vartheta)(u(t,x),t,x) + (\partial_x Q)(u(t,x),t,x) \right]$$

$$= \int_{[-1,1] \times [0,T] \times T} dv \varrho_u(v;dt,dx) \vartheta''(v,t,x) \tag{5.1}$$

Note that for $u \in \mathcal{E}_T$ and $v \in [-1,1]$,

$$\varrho_{P\tau_u}(v;dt,dx) = -\varrho_u(v;d(T-t),d(-x))$$

Therefore assuming also $u(0) \equiv m$, $u(T) = u_f$, we have for each $\eta \in C^2([-1,1])$ with $\eta(m) = 0$

$$\int dv \eta''(v) \varrho_{u_+}(v;dt,dx) - \int dv \eta''(v) \varrho_{P\tau_u}(v;dt,dx)$$

$$= \int dv \eta''(v) \varrho_{u_+}(v;dt,dx) - \int dv \eta''(v) \varrho_u(v;d(T-t),-dx)$$

$$= \int dv \eta''(v) \varrho_u(v;dt,dx) - \int dv \eta''(v) \varrho_u(v;dt,dx)$$

$$= \int dv \eta''(v) \varrho_u(v;dt,dx) = \int dx \eta(u(T,x)) - \eta(u(0,x))$$

$$= \int dx \eta(u_f(x)) \tag{5.2}$$

where we used (5.1) with $\theta(v,t,x) = \eta(v)$. If $\sigma$ is bounded away from 0, then (5.2) evaluated for $\eta = h_m$ immediately yields

$$H_T(u) = H_T(P^T u) + W_m(u_f) \tag{5.3}$$

If $\sigma$ vanishes at $-1$ or $+1$, then (5.3) is obtained by monotone convergence, when considering in (5.2) a sequence $\{\eta^n\} \subset C^2([-1,1])$ such that: $\eta^n(m) = 0$; $0 \leq (\eta^n)'' \leq h_m''$; and for all $v \in [-1,1]$, $\eta^n(v) \uparrow h_m(v)$ and $(\eta^n)''(v) \uparrow h''_m(v)$.

Optimizing in (5.3) over $T$ and $u$ we get $V(m,u_f) \geq V(Pu_f,m) + W_m(u_f)$. Replacing $u_f$ by $Pu_f$ and thus $Pu_f$ by $P(Pu_f) = u_f$, we get the reverse inequality.

Definition 5.2. We say that $u_i \in U$ is piecewise constant iff there is a finite partition of $\mathbb{T}$ in intervals such that $u_i$ is constant on each interval. For $T > 0$, we say that $u \in X_T$ is piecewise constant iff $u \in C([0,T];L^1(\mathbb{T}))$.
and there exists a finite partition of $[0, T] \times \mathbb{T}$ in connected sets with Lipschitz boundary such that $u$ is constant on each set of these.

The following lemma is the main technical difficulty of this paper, and its proof is postponed at the end of this section.

**Lemma 5.3.** Assume the same hypotheses of Theorem 3.1-(ii). For each $\gamma > 0$, there exist $T^{\gamma}$, $\delta^{\gamma} > 0$ such that the following holds. For each piecewise constant $u_i \in U$ satisfying $\int_T dx \, u_i(x) = m$ and $\|u_i - m\|_{L_{\infty}(\mathbb{T})} \leq \delta^{\gamma}$, there exists $u^\gamma \in \mathcal{X}_{T^\gamma}$ such that $u^\gamma(0) = u_i$, $u^\gamma(T^\gamma) \equiv m$ and $H_{T^\gamma}(u^\gamma) \leq \gamma$.

The next corollary relaxes the condition in Lemma 5.3 requiring $u_i$ to be piecewise constant.

**Corollary 5.4.** Assume the same hypotheses of Theorem 3.1-(ii). For each $\gamma > 0$, there exist $T^{\gamma}$, $\delta^{\gamma} > 0$ such that the following holds. For each $u_i \in U$ satisfying $\int_T dx \, u_i(x) = m$ and $\|u_i - m\|_{L_{\infty}(\mathbb{T})} \leq \delta^{\gamma}$, there exists $u^\gamma \in \mathcal{X}_{T^\gamma}$ such that $u^\gamma(0) = u_i$, $u^\gamma(T^\gamma) \equiv m$ and $H_{T^\gamma}(u^\gamma) \leq \gamma$.

**Proof.** For a fixed $\gamma > 0$, let $T^{\gamma}$ and $\delta^{\gamma} > 0$ be as in Lemma 5.3. For $u_i \in U$ such that $|u_i - m| \leq \delta^{\gamma}$, where $m = \int_T dx \, u_i(x)$, let $\{u_i^n\} \subset U$ be a sequence of piecewise smooth functions converging to $u_i$ in $U$ and satisfying $\int_T dx \, u^n_i(x) = m$ and $\|u_i - m\|_{L_{\infty}(\mathbb{T})} \leq \delta^{\gamma}$. For each $n$, by Lemma 5.3 there exist $u^{n,\gamma}$ such that $u^{n,\gamma}(0) = u^n_i$, $u^{n,\gamma}(T^\gamma) \equiv m$ and $H_{T^\gamma}(u^{n,\gamma}) \leq \gamma$. Therefore, since $H_{T^\gamma}$ has compact sublevel sets (see [2] Proposition 2.6), there is a (not relabeled) subsequence $\{u^{n,\gamma}\}$ converging to a $u^\gamma \in \mathcal{X}_{T^\gamma}$, and $H_{T^\gamma}(u^\gamma) \leq \gamma$. By the definition of convergence in $\mathcal{X}_{T^\gamma}$, $u^{n,\gamma}(0)$ and $u^{n,\gamma}(T^\gamma)$ converge in $U$ to $u^\gamma(0)$ and $u^\gamma(T^\gamma)$ respectively, and thus $u^\gamma(0) = u_i$ and $u^\gamma(T^\gamma) \equiv m$. \qed

We recall a result in [6, Chap. 11.5].

**Theorem 5.5.** Assume $f \in C^2([-1, 1])$, and that there is no interval in which $f$ is affine. Let $u_i \in U$ and let $\bar{u} : [0, \infty) \times \mathbb{T} \to \mathbb{R}$ be the Kruzhkov solution to (1.1) with initial datum $u_i \in U$. Then

$$
\lim_{t \to \infty} \|\bar{u}(t) - m\|_{L_{\infty}([0,T] \times \mathbb{T})} = 0
$$

where $m = \int_T dx \, u_i(x)$.

**Proof of Theorem 5.5-(ii).** Fix $u_i \in U$ and $\gamma > 0$. Let $T^{\gamma}$ and $\delta^{\gamma}$ be as in Corollary 5.4 let $\bar{u} : [0, \infty) \to U$ be the Kruzhkov solution to (1.1) with initial datum $u_i$, and let $m = \int_T dx \, u_i(x)$. By Theorem 5.3 there exists $\tau^{\gamma}$ such that $|\bar{u}(\tau) - m| \leq \delta^{\gamma}$. By Corollary 5.4 there exists $\bar{u} \in \mathcal{X}_{T^\gamma}$ such that $\bar{u}(0) = \bar{u}(\tau)$, $\bar{u}(T^\gamma) \equiv m$ and $H_{T^\gamma}(\bar{u}) \leq \gamma$. Define $u \in \mathcal{X}_{T^{\gamma} + T^\gamma}$ by

$$
u(t, x) := 
\begin{cases}
\bar{u}(t, x) & \text{if } t \leq \tau^{\gamma} \\
\bar{u}(t - \tau, x) & \text{if } \tau^{\gamma} \leq t \leq \tau^{\gamma} + T^\gamma
\end{cases}
$$

Then, by Remark 3.2 $H_{T^{\gamma} + T^\gamma}(u) = H_{T^{\gamma}}(\bar{u}) + H_{T^\gamma}(\bar{u}) \leq \gamma$. Therefore $V(u_i, m) = 0$ and the proof is thus complete since Lemma 5.1 holds. \qed
The remaining of this section is devoted to the proof of Lemma 5.3.

**Remark 5.6.** Let \( m \in (-1,1) \), assume the same hypotheses of Theorem 3.1, (ii), and let \( \delta_0 \) be defined accordingly. Then, taking perhaps a smaller \( \delta_0 \), one can assume \([m - \delta_0, m + \delta_0] \subset (-1,1)\) and that one (and only one) of the following holds:

(A) in the interval \([m - \delta_0, m + \delta_0] \), \( f \) is either strictly convex or strictly concave.

(B) \( f \) is either strictly convex in \([m - \delta_0, m] \) and strictly concave in \([m, m + \delta_0] \), or strictly convex in \([m, m + \delta_0] \) and strictly convex in \([m, m + \delta_0] \).

With no loss of generality, we will assume \( f \) convex in \([m - \delta_0, m + \delta_0] \) if case (A) holds, and \( f \) concave in \([m - \delta_0, m] \) and convex in \([m, m + \delta_0] \) if (B) holds.

**Remark 5.7.** Let \( T > 0 \) and assume \( u \in \mathcal{E}_T \) to be piecewise constant according to Definition 5.2. Then the jump set of \( u \) consists of a finite number of segments in \([0, T] \times \mathbb{T}\). In particular there exist a finite sequence

\[ 0 = T^0 < T^1 < \ldots < T^n = T, \]

and, for \( k = 1, \ldots, n \), finite sequences \( \{w_j^k\}_{j=1}^{N_k} \subset (-1,1) \) such that for \( t \in (T^{k-1}, T^k) \), \( u(t) \) is piecewise constant with jump set consisting of a finite set of points \( \{x_j^k(t)\}_{j=1}^{N_k} \in \mathbb{T} \), and the traces of \( u(t) \) at \( x_j^k(t) \) are \( w_j^k \) (from the right) and \( w_{j-1}^k \) (from the left, where we understand \( w_0^k \equiv w_{N_k}^k \)).

In particular, by (2.5) we have that

\[
H_T(u) = \sum_{k=1}^n (T^k - T^{k-1}) \sum_{j=1}^{N_k} \int_{[0,T]} D(v) \frac{\rho^+(v, w_j^k, w_{j-1}^k)}{\sigma(v)} |w_j^k - w_{j-1}^k| \quad (5.4)
\]

If \( u \in \mathcal{E}_T \) is piecewise constant, and \( u^- \), \( u^+ \) are the left and right traces of \( u \) at a given point in the jump set of \( u \), we say that the shock between \( u^- \) and \( u^+ \) is entropic iff \( \rho(v, u^-, u^+) \leq 0 \) for almost every \( v \), while it is anti-entropic iff \( \rho(v, u^-, u^+) \geq 0 \) for almost every \( v \). If \( f \) is convex or concave, each shock is either entropic or anti-entropic, but in the general case the sign of \( \rho(v, u^-, u^+) \) may depend on \( v \).

**Lemma 5.8.** Let \( m \in (-1,1) \), and \( \delta_0 \equiv \delta_0(m) > 0 \) be as in Remark 5.6. Let \( u_i \in U \) be piecewise constant and such that \( \int_T dx u_i(x) = m \) and \( \|u_i - m\|_{L^\infty(T)} \leq \delta_0 \). Then for each \( T, \gamma > 0 \) there exists \( w \in \mathcal{X}_T \) piecewise constant such that \( \|w - m\|_{L^\infty([0,T] \times \mathbb{T})} \leq \|u_i - m\|_{L^\infty(T)} \), \( w(0) = u_i \), and \( H_T(w) \leq \gamma \).

The proof of Lemma 5.8 will be divided in three steps. The main idea is to construct a piecewise smooth weak solution \( w \), by splitting each shock appearing in the initial datum in an entropic part and an anti-entropic part, the anti-entropic part being split itself in \( M \) small anti-entropic shocks, with \( M \) a large integer, see Figure 1. For such a solution to exist, the points at
which the shocks are split have to be carefully chosen. We are then able to
define \( w \) up to the first time at which two (or more) shocks collide. Defining
then \( w \) recursively, we prove that there can be only a finite number of times
at which the shocks collide, and thus \( w \) is well-defined globally in time.
Finally, we show that \( H_T(w) \) can be made arbitrarily small by choosing \( M \)
large.

**Proof of Lemma 5.8.** As noted in Remark 5.6, we assume \( f \) to be strictly
convex in \([m, m + \delta_0] \). Hereafter we let \( \delta := \| u_i - m \|_{L_\infty(T)} \leq \delta_0 \).

**Step 1: Evolution of shocks.** Let \( x_1, \ldots, x_N \in \mathbb{T} \) be the points at which
the discontinuities of \( u_i \) are located. With a little abuse of notation, we also
denote by \( u_i : \mathbb{R} \to [-1, 1] \) and \( x_j \in [0, 1] \subset \mathbb{R} \) the lift of \( u_i \) and \( x_j \) to \( \mathbb{R} \), and
we assume \( x_j < x_{j+1} \) for \( j = 1, \ldots, N - 1 \). For \( j = 1, \ldots, N \) and \( n \in \mathbb{Z} \), let \( x_{j+nN} = x_j + n \in \mathbb{R} \), and for \( j \in \mathbb{Z} \) let \( u_j^- \) and \( u_j^+ \) be respectively the right
and left traces of \( u_i \) at \( x_j \). Define

\[
U_j := \begin{cases}
\max\{ w \in [u_j^-, u_j^+] : \rho(v, w, u_j^-) \leq 0, \forall v \in [-1, 1]\} & \text{if } u_j^- < u_j^+ \\
\min\{ w \in [u_j^+, u_j^-] : \rho(v, w, u_j^+) \leq 0, \forall v \in [-1, 1]\} & \text{if } u_j^+ < u_j^-
\end{cases}
\]

Since \( f \) is convex in \([m, m + \delta_0] \), if \( u_j^- < u_j^+ \) and \( U_j \leq v \leq v' \leq u_j^+ \), or if
\( u_j^+ < u_j^- \) and \( u_j^- \leq v' \leq v \leq U_j \) then

\[
\frac{f(u_j^+) - f(U_j)}{u_j^- - U_j} \leq \frac{f(U_j) - f(v)}{U_j - v} \leq \frac{f(v) - f(v')}{v - v'} \leq \frac{f(u_j^-) - f(U_j)}{U_j - u_j^-}
\]

where we understand \( \frac{f(w) - f(w)}{w - w} = f'(w) \) for \( w \in [-1, 1] \).

**Figure 1.** In the figure, we have \( f(u) = u^3 - u, \ m = 0 \)
and \( M = 2 \). Consider a discontinuity between the values \( u_1^- = -\delta \) and \( u_1^+ = \delta \). Then \( U_1 \) is chosen as the abscissa of
the point at which a line passing in \((-\delta, f(-\delta))\) is tangent to
the graph of \( f \). The values \( U_1 + \frac{k}{M}(u_k^+ - U_1) \) for \( k = 1, 2 \) are
the abscissas of the intersections of the dashed lines with the
graph of \( f \).
Therefore, fixed an integer $M \geq 2$, it is possible to define the map $w^u_j : [0, +\infty) \times \mathbb{R} \to [m - \delta, m + \delta]$ as

$$w^u_j(t, x) := \begin{cases} 
    u_j^- & \text{if } x - x_j \leq \frac{f(u_j^-) - f(U_j)}{u_j^- - U_j} t \\
    U_j & \text{if } x - x_j \in \left[\frac{f(u_j^-) - f(U_j)}{u_j^- - U_j} t, M\frac{f(U_j) - f(U_j + \frac{u_j^+ - U_j}{u_j^+ - u_j})}{U_j - u_j^+} t\right] \\
    U_j + \frac{k}{M}(u_j^+ - U_j) & \text{if } x - x_j \in \left[M\frac{f(U_j + \frac{k}{M}(u_j^+ - U_j)) - f(U_j + \frac{k}{M}(u_j^+ - U_j))}{U_j - u_j^+} t, M\frac{f(U_j + \frac{k}{M}(u_j^+ - U_j)) - f(U_j + \frac{k}{M}(u_j^+ - U_j))}{U_j - u_j^+} t\right] \\
    U_j & \text{for } k \in \{1, \ldots, M - 1\} \\
    U_j + \frac{k}{M}(u_j^+ - U_j) & \text{if } x - x_j \geq M\frac{f(U_j + \frac{k}{M}(u_j^+ - U_j)) - f(u_j^+)}{U_j - u_j^+} t
    \end{cases} \quad (5.5)$$

Note that this definition makes sense in the case $U_j = u_j^-$ or $U_j = u_j^+$. We also let $X_j^-(t) := x_j + \frac{f(u_j^-) - f(U_j)}{u_j^- - U_j} t$, $X_j^+(t) := x_j + \frac{f(U_j + \frac{k}{M}(u_j^+ - U_j)) - f(u_j^+)}{U_j - u_j^+} t$ and

$$T(u_i) := \inf\{t \geq 0 : \min_j[X_j^-(t) - X_{j-1}^+(t)] = 0\}$$

We next define $w^{u_i} : [0, T(u_i)] \times \mathbb{R} \to [m - \delta, m + \delta]$ as

$$w^{u_i}(t, x) = w^u_j(t, x) \quad \text{if } x \in [X_j^+(t), X_{j+1}^-(t)]$$

$w^{u_i}$ is a weak solution to (1) in $[0, T(u_i)] \times \mathbb{R}$, since it is piecewise constant and satisfies the Rankine-Hugoniot condition along the shocks. Since it is also 1-periodic on $\mathbb{R}$, it defines a map $w^{u_i} : [0, T(u_i)] \times \mathbb{T} \to [m - \delta, m + \delta]$ such that $w^{u_i}(0) = u_i$ and $w^{u_i} \in \mathcal{E}_{\mathcal{T}(u_i)}$.

**Step 2:** There is a finite number of shocks merging. We next define recursively, for $k \geq 1$, $T^k \in [0, \mathbb{T}]$ and $w^k : [T^{k-1}, T^k] \times \mathbb{T} \to [m - \delta, m + \delta]$ (where $T^0 = 0$) by

$$T^1 := T(u_i) \wedge \mathbb{T}$$

$$w^1 := w^{u_i}$$

$$qT^k := [T^{k-1} + T(w^{k-1}(T^{k-1}))] \wedge \mathbb{T} \quad \text{for } k \geq 2$$

$$w^k(t, x) := w^{w^{k-1}(T^{k-1})}(t - T^{k-1}, x) \quad \text{for } k \geq 2$$

We want to show that there exists a $K \in \mathbb{N}$ such that $T^K = \mathbb{T}$.

By definition, for each $k \geq 1$ and $t \in (T^{k-1}, T^k)$, the discontinuities of $w^k(t)$ are either entropic or non-entropic. On the other hand, at the times $T^k$ at which two or more shocks collide, one and only one of the following may happen.

- At a point $y \in \mathbb{T}$, two or more entropic shocks of $w^k$ merge at time $T^k$. Then $w^{k+1}$ has one entropic shock in $[T^k, T^{k+1}]$ starting at $y$.
- At a point $y \in \mathbb{T}$ one or more entropic shocks of $w^k$ merge with one or more anti-entropic shocks. Then either $w^{k+1}$ has one entropic
shock starting at \( y \), or \( w^{k+1} \) has \( a \) anti-entropic shocks starting at \( y \), for some integer \( a, 0 \leq a \leq M \).

Note that, at a time \( T^k \), one or more of the merging here described may happen at different points \( y \in T \), but at no point there can be a shock merging involving only anti-entropic shocks. Therefore, at a given shocks merging:

- either the number of entropic shocks stays constant and the number of anti-entropic shocks decreases by at least one; or the number of entropic shocks decreases by at least one, and the number of anti-entropic shocks may either decrease, or increase (by at most \( M \)). It follows that there can be at most a finite number of shocks merging, and in particular a finite number of times at which shocks merge. Recalling that \( N \) was the total number of discontinuity points of \( u_i \), and that by definition \( w^1 \) has at most \( N \) entropic shocks and \( N M \) anti-entropic shocks, it follows that for each \( k \), \( w^k \) has at most \( (2N - 1)M \) anti-entropic shocks, the remaining shocks being entropic.

Therefore the sequence \( \{T^k\} \) has no accumulation points before \( T \), and it will hit \( T \) for \( k \) large enough.

**Figure 2.** In the figure, we have \( f(u) = u^3 - u, m = 0 \), and the initial datum \( u_i \), having two jumps between the values \(-\delta\) and \( \delta \). The figure shows \( w \) at different times \( t \in [0, T(u_i)] \).

**Step 3: Computing \( H_T \).** We can thus define \( w : [0, T] \times T \to [m - \delta, m + \delta] \) by requiring \( w(t, x) = w^k(t, x) \) for \( t \in [T^{k-1}, T^k] \). \( w \) is piecewise constant and it satisfies the Rankine-Hugoniot condition along the shocks, therefore, since \( w^{k-1}(T^k) = w^k(T^k) \), \( w \in E_T \). As noted above, in each time interval \([T^{k-1}, T^k]\), \( w \) has at most \((2N - 1)M\) shocks. Moreover, by the definition
Lemma 5.9. Assume the same hypotheses of Theorem 3.1-(ii) and let $\gamma > 0$. Let $m \in (-1, 1)$ and let $\delta_0 = \delta_0(m)$ be defined as in Remark 5.6. Then for each $\delta_1 \in (0, \delta_0)$ there exists $\delta_2 \equiv \delta_2(\delta_1) \in (0, \delta_0)$ such that for each $\delta_2 \in (0, \delta_2)$ the following holds. For a fixed arbitrary $x_0 \in \mathbb{T}$ let $u_d \equiv u^\delta_1, \delta_2 \in U$.
be defined as
\[
u_d(x) := \begin{cases} 
m + \delta_1 & \text{if } |x - x_0| \leq \frac{\delta_2}{2(\delta_1 + \delta_2)} \\
m - \delta_2 & \text{otherwise}
\end{cases}
\]
and let
\[
\tau \equiv \tau^{\delta_1, \delta_2} := \frac{1}{|R(\delta_1, 0) - R(0, -\delta_2)|}
\]
Then \(\tau < \infty\), and there exists \(u \in \mathcal{X}_\tau\) such that \(u(0) = u_d\), \(u(\tau) \equiv m\) and
\[
H_\tau(u) \leq \left[ \max_{v \in [m - \delta_0, m + \delta_0]} \frac{D(v)}{\sigma(v)} \right] \frac{C(\delta_1, 0)^+ + C(0, -\delta_2)^+}{|R(\delta_1, 0) - R(0, -\delta_2)|} \tag{5.6} \]

Proof. Fix \(\delta_1 \in (0, \delta_0)\). By the definition of \(\delta_0\), \(R(\delta_1, 0) \neq f'(m)\) and assuming \(f\) strictly convex in \([m, m + \delta_0]\) (see Remark \ref{5.6}), we have \(R(\delta_1, 0) > f'(m)\). Recalling the definition of \(\rho\) in \((2.6)\), still by the convexity of \(f\) in \([m, m + \delta_0]\), we have \(\rho(v, m, m + \delta_1) < 0\) for \(v \in (m, m + \delta_1)\) and \(C(\delta_1, 0) > 0\). In particular there exists \(\delta_2\) small enough such that for each \(\delta_2 \in (0, \delta_2)\) and each \(v \in (m - \delta_2, m + \delta_1)\)
\[
R(\delta_1, 0) - R(0, -\delta_2) > 0 \tag{5.7} \]
\[
\rho(v, m - \delta_2, m + \delta_1) < 0 \tag{5.8} \]

Let us now fix \(\delta_2 \in (0, \delta_2)\). By \((5.7)\), \(\tau^{\delta_1, \delta_2}\) is finite. With no loss of generality we may assume \(x_0 = \frac{\delta_2}{2(\delta_1 + \delta_2)}\), as the general case is obtained by a space translation of the solution \(u\) given below by the quantity \(x_0 - \frac{\delta_2}{2(\delta_1 + \delta_2)}\).
Define
\[
u(t, x) := \begin{cases} 
m & \text{if } |x - [R(\delta_1, 0) + R(0, -\delta_2)]| \leq \frac{\delta_2}{2(\delta_1 + \delta_2)} \\
m + \delta_1 & \text{if } |x - \frac{\delta_2}{2(\delta_1 + \delta_2)} - [R(\delta_1, 0) + R(\delta_1, -\delta_2)]| \leq \frac{\delta_2}{2(\delta_1 + \delta_2)} \\
m - \delta_2 & \text{otherwise}
\end{cases}
\]
It follows that \(u(0) = u_d\) and \(u(\tau) \equiv m\). Moreover \(u\) is a piecewise constant weak solution to \((1.1)\). For a fixed \(t \in (0, T)\), \(u(t)\) has three discontinuity jumps, where its value jumps from \(m\) to \(m + \delta_1\), from \(m + \delta_1\) to \(m - \delta_2\) and from \(m - \delta_2\) to \(m\). In particular \(H_\tau(u)\) can be calculated by \((5.4)\). The shock between the values \(m + \delta_1\) and \(m - \delta_2\) is entropic by \((5.3)\), and thus it gives no contributions to the sum \((5.4)\). By the convexity assumption on \(f\) in \([m, m + \delta_0]\), the shock between \(m\) and \(m + \delta_1\) is anti-entropic, namely \(\rho(v, m + \delta_1, m) \geq 0\). Moreover the shock between \(m - \delta_2\) and \(m\) is either entropic (if case (A) in Remark \ref{5.6} holds) or anti-entropic (if case (B) in
Remark 5.6 holds). Therefore (5.4) yields

\[ H_T(u) = \tau \left[ \int dv \frac{D(v) \rho^+(v,m,m-\delta)}{\sigma(v)} + \int dv \frac{D(v) \rho^+(v,m+\delta_1,m)}{\delta_1} \right] \]

\[ = \left[ \int dv \frac{D(v) \rho(v,m,m-\delta_2)}{\delta_2} + \int dv \frac{D(v) \rho(v,m+\delta_1,m)}{\delta_1} \right] \frac{R(\delta_1,0) - R(0,-\delta_2)}{R(\delta_1,0) - R(0,-\delta_2)} \]

\[ \leq \max_{v \in [m-\delta_0,m+\delta_0]} \frac{D(v)}{\sigma(v)} \left[ \int dv \frac{\rho(v,m,m-\delta_2)}{\delta_2} + \int dv \frac{\rho(v,m+\delta_1,m)}{\delta_1} \right] \frac{R(\delta_1,0) - R(0,-\delta_2)}{R(\delta_1,0) - R(0,-\delta_2)} \]

namely (5.6).

Proof of Lemma 5.3. Fix \( \gamma > 0 \). Recall the definition of \( \delta_0 \equiv \delta_0(m) \) in Remark 5.6 as noted in Remark 5.6; we may assume \( f \) to be strictly convex in \( [m, m+\delta_0] \). We thus have \( R(\delta_1,0) > f'(m), \rho(v,m+\delta_1,m) \geq 0 \) for each \( \delta_1 \in (0, \delta_0) \). Then by explicit computation

\[ \lim_{\delta_1 \downarrow 0} \frac{\sup_{\delta_2 \in [0, \delta_0]} \left| C(\delta_1, \delta') + |C(\delta', -\delta_2)| + |C(\delta_1, -\delta_2)| \right|}{R(\delta_1,0) - f'(m)} = 0 \]

In particular, defining \( \delta_2(\cdot) \) as in Lemma 5.9, there exist \( \delta_1 \equiv \delta_1^* \in (0, \delta_0) \), \( \delta_2 \equiv \delta_2^* \in (0, \delta_2(\delta_1)) \) and \( \delta \equiv \delta^* \in (0, \delta_1 \wedge \delta_2) \) such that

\[ \left[ \max_{v \in [m-\delta_0,m+\delta_0]} \frac{D(v)}{\sigma(v)} \right] \sup_{\delta' \in [-\delta, \delta]} \frac{C(\delta_1,0) + |C(\delta', -\delta_2)| + |C(\delta_1, -\delta_2)|}{R(\delta_1,0) - f'(m)} \leq \frac{\gamma}{8} \]

\[ \frac{R(\delta_1,0) - f'(m)}{4} \leq \frac{\sup_{\delta', \delta'' \in [-\delta, \delta]} |R(\delta', -\delta_2) - R(\delta', \delta'')|}{2} \]

\[ \leq \inf_{\delta', \delta'' \in [-\delta, \delta]} R(\delta_1, \delta') - R(\delta'', -\delta_2) \]

\[ \leq \frac{1}{4} \int dv \rho(v,m,\delta,m+\delta_1) > 0 \quad \text{for } v \in (m-\delta, m+\delta_1) \]

\[ \frac{1}{4} \int dv \rho(v,m,\delta,m-\delta_2) > 0 \quad \text{for } v \in (m-\delta_2, m+\delta) \]

Let now \( u_i \in U \) be an arbitrary piecewise constant profile such that \( \|u_i - m\|_{L^\infty(T)} \leq \delta \). Fix

\[ \mathcal{T} := \frac{4}{R(\delta_1,0) - f'(m)} \]

By Lemma 5.8 there exists a piecewise constant map \( w \equiv w_{\mathcal{T},\gamma/4} : [0, \mathcal{T}] \times \mathcal{T} \to [m-\delta, m+\delta] \) such that \( w(0) = u_i \) and \( H_T(w) \leq \gamma/4 \).
Let the Lipschitz map \( s_1, s_2 : [0, +\infty) \to \mathbb{T} \) be defined as the solutions to the Cauchy problems
\[
\begin{align*}
\dot{s}_1(t) &= \frac{f(m+\delta_1)-f(w(t,s_1(t)))}{m+\delta_1-w(t,s_1(t))} \equiv R(\delta_1, w(t,s_1(t))-m) \\
\dot{s}_1(0) &= 0 \\
\dot{s}_2(t) &= \frac{f(m-\delta_2)-f(w(t,s_2(t)))}{m-\delta_2-w(t,s_2(t))} \equiv R(w(t,s_2(t))-m, -\delta_2) \\
\dot{s}_2(0) &= 0
\end{align*}
\]

Despite the right hand sides are discontinuous, these equations are well-posed since \( w \) is piecewise constant and conditions (5.11)-(5.12) hold.

With a little abuse of notation, we also denote by \( s_1 \) and \( s_2 \) the lift of \( s_1 \) and \( s_2 \) on \( \mathbb{R} \). Note that, by (5.10), \( s_1(t) - s_2(t) \) is increasing in \( t \) and letting \( T > 0 \) be the first time \( t \) at which \( s_1(t) - s_2(t) = 1 \), we have still by (5.10)
\[
T \leq T
\]

We also set \( x_0 := s_1(T) = s_2(T) \in \mathbb{T} \), and let \( u_d \equiv u_{\delta_1,\delta_2} \), \( \tau \equiv \tau_{\delta_1,\delta_2} \) be defined as in Lemma 5.9 (with \( \delta_1, \delta_2 \) and \( x_0 \) defined as above in this proof), and let \( v \in \mathcal{X}_\tau \) be the solution to (1.1) whose existence is proved in Lemma 5.9. We finally let
\[
u^\tau(t,x) := \begin{cases} 
m + \delta_1 & \text{if } t \in [0,T] \text{ and } x \in A_1(t) \\
-m - \delta_2 & \text{if } t \in [0,T] \text{ and } x \in A_2(t) \\
\text{or}(t,x) & \text{if } t \in [0,T] \text{ and } x \notin A_1(t) \cup A_2(t) \\
v(t-T,x) & \text{if } t \in [T,T+\tau] \end{cases}
\]

where for \( t \geq 0 \)
\[
A_1(t) := \{ x \in \mathcal{T} : |x - \frac{1}{2} [s_1(t) + R(\delta_1,-\delta_2) t] | \leq \frac{1}{2} |s_1(t) - R(\delta_1,-\delta_2) t| \} \\
A_2(t) := \{ x \in \mathcal{T} : |x - \frac{1}{2} [R(\delta_1,-\delta_2) t + s_2(t)] | \leq \frac{1}{2} |R(\delta_1,-\delta_2) t - s_2(t)| \}
\]

Note that \( u^\tau_{[0,T]} \in \mathcal{X}_T \) is piecewise constant, and it is the gluing of solutions to (1.1) satisfying the Rankine-Hugoniot condition at the borders of \( \{ (t,x) \in [0,T] \times \mathbb{T} : x \in A_i(t) \} \) (for \( i = 1, 2 \)). We thus have \( u^\tau_{[0,T]} \in \mathcal{E}_T \) and \( u^\tau \in \mathcal{E}_{T+\tau} \).

In order to calculate \( H_T(u^\tau_{[0,T]}) \), we will use Remark 5.7. Note that for each \( t \in [0,T] \) the set of discontinuity points of \( u^\tau(t) \) consists of the discontinuity points of \( w(t) \), and the discontinuities at \( s_1(t) \), at \( s_2(t) \) and at \( R(\delta_1,-\delta_2) t \). Because of assumptions (5.11)-(5.12), there is at most a finite number of times \( t \in [0,T] \) at which \( s_1(t) \) and \( s_2(t) \) may overlap with a discontinuity of \( w(t,\cdot) \). Note that assumption (5.13) implies \( \rho(v,w,m+\delta_1) \leq 0 \), for each \( v \in [-1,1] \) and \( w \in [m-\delta,m+\delta] \), so that the shock of \( u^\tau \) at \( s_1 \) is entropic
Figure 3. In the figure, we have $f(u) = u^3 - u$, $m = 0$, and the initial datum $u_i$ having two jumps between the values $-\delta$ and $\delta$. The figure shows $u^\gamma$ at a small time $0 < t < T$ and at time $T$.

and it does not appear in the sum (5.4). Conversely $\rho(v, m - \delta_2, m + \delta_1) \geq 0$, so that the shock along the curve $t \mapsto R(\delta_1, -\delta_2)t$ appears in the sum (5.4). Finally, by (5.14), $\rho(v, w(t, s_2(t), m - \delta_2)$ is either negative or positive for
Figure 4. In the figure, we have $f(u) = u^3 - u$, $m = 0$, and the initial datum $u_i$ having two jumps between the values $-\delta$ and $\delta$. The figure shows $u^\gamma$ at different times $t \in (T, T + \tau]$.

Each $t \in [0, T]$ and $v \in [-1, 1]$, depending whether case (A) or (B) of Remark 5.6 holds for $f$. By Remark 3.2 and recalling that $v$ satisfies (5.6)

\[
H_{T+t}(u^\gamma) = H_r(v) + H_T(v_{|0,T}^\gamma) \leq H_r(v) + H_T(v)
\]

\[
+ \int_0^T dt \left[ \int dv \frac{D(v) \rho^+(v, m - \delta_2, w(t, s_2(t)))}{\sigma(v)} \right]
\]

\[
= \frac{\gamma}{4} + \left[ \max_{v \in [m - \delta_0, m + \delta_0]} \frac{D(v)}{\sigma(v)} \right] \frac{C(\delta_1, 0)^+ + C(0, -\delta_2)^+}{|R(\delta_1, 0) - R(0, -\delta_2)|}
\]

\[
+ \int_0^T dt \left[ \int dv \frac{D(v) \rho(v, m - \delta_2, w(t, s_2(t)))}{\delta_2} \right]
\]

\[
+ \int dv \frac{D(v) \rho(v, m + \delta_1, m - \delta_2)}{\delta_1 + \delta_2}
\]

\[
\leq \left\{ \begin{array}{ll}
\frac{\gamma}{4} + \left[ \max_{v \in [m - \delta_0, m + \delta_0]} \frac{D(v)}{\sigma(v)} \right] C(\delta_1, 0)^+ + C(0, -\delta_2)^+ \\
+ T C(\delta_1, -\delta_2) + T \sup_{\delta' \in [-\delta, \delta]} C(\delta', -\delta_2)^+
\end{array} \right. 
\]
By (5.15) and (5.10) we thus obtain
\[
H_{T+\tau}(u^\gamma) \leq \frac{\gamma}{4} + \left[ \max_{v \in [m-\delta_0,m+\delta_0]} \frac{D(v)}{\sigma(v)} \right]
2C(\delta_1,0)^+ + 2C(0,-\delta_2)^+ + 4C(\delta_1,-\delta_2) + 4 \sup_{\delta' \in [-\delta,\delta]} C(\delta',-\delta_2)^+
\]
Therefore \( H_{T+\tau}(u^\gamma) \leq \gamma \) by (5.9). \( \square \)

**Proof of Theorem 3.1-(iii).** We assume \( \int_T dx \, u_f(x) = m \), the proof being trivial otherwise. Since \( H_{J^V}^{J^V} \leq H_T \) we have \( V^{J^V} \leq W_m(u_f) \) by Theorem 3.1-(ii). The converse inequality is obtained by taking \( \varphi \equiv 1 \) in the very definition of \( H_{J^V}^{J^V} \).

**Acknowledgment:** We are indebted to Lorenzo Bertini and Matteo Novaga for enlightening discussions.

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