INTEGRAL THEORY FOR QUASI-HOPF ALGEBRAS

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Abstract. We generalize the fundamental structure Theorem on Hopf (bi)-modules by Larson and Sweedler to quasi-Hopf algebras \(H\). For \(\dim H < \infty\) this proves the existence and uniqueness (up to scalar multiples) of integrals in \(H\). Among other applications we prove a Maschke-type Theorem for diagonal crossed products as constructed by the authors in [HN, HN99].

Contents

1. Introduction 1
2. Quasi-Hopf Algebras 3
3. Quasi-Hopf Bimodules 6
4. Integral Theory 13
5. Fourier Transformations 15
6. The Comodulus and Radford’s Formula 19
7. Cocentral Bilinear Forms 20
8. Semisimplicity of Diagonal Crossed Products 24
References 27

1. Introduction

A (left) integral \(l\) in a (quasi-) Hopf algebra \(H\) is an element of \(H\) satisfying for all \(a \in H\) (\(\epsilon\) denoting the counit)

\[ al = \epsilon(a)l. \]

For finite dimensional Hopf algebras \(H\) with dual \(\hat{H}\), one may identify \(H \cong \hat{\hat{H}}\) and a left integral may equivalently be characterized as a functional on \(\hat{H}\) being invariant under the canonical right action of \(H\) on \(\hat{H}\), i.e. satisfying \(l(\psi \leftarrow a) = \epsilon(a)l(\psi), \quad \forall a \in H, \psi \in \hat{H}\). Choosing \(H\) to be the group algebra of a finite group \(G\), \(\hat{H}\) becomes the algebra of functions on \(G\) (under pointwise multiplication) and a (left) integral in \(H\) yields a (right) translation invariant measure on \(G\) (in this case \(l\) is given by the sum over all elements of \(G\) and is also a right integral in \(H\)). Thus the notion of integrals in Hopf algebras is a natural generalization of the notion of translation invariant group measures.

It is wellknown [Abe80, Swe69], that a (left) integral in a finite dimensional Hopf algebra always exists and is unique up to normalization. The proof in [Abe80, Swe69, LS69] uses the fundamental structure Theorem on Hopf modules by Larson.

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and Sweedler [LS69]. A (right) Hopf $H$-module $M$ of a Hopf algebra $H$ is a (right) $H$-module which is also a (right) $H$-comodule such that the corresponding $H$-action ($\cdot$) and $H$-coaction ($\rho$) are compatible in a natural way (i.e. $\rho(m \cdot a) = \rho(m) \cdot \Delta(a)$, $m \in M, a \in H$). Denoting the subspace of coinvariants of $M$ (i.e. of elements satisfying $\rho(m) = m \otimes 1$) by $M^{coH}$, the structure Theorem states that

$$M \cong M^{coH} \otimes H \quad \text{as Hopf } H\text{-modules}, \quad (1.1)$$

where the Hopf $H$-module structure on $M^{coH} \otimes H$ is the trivially extended Hopf $H$-module structure on $H$ (given by right multiplication and coproduct) and the isomorphism $M^{coH} \otimes H \to M$ is given by $m \otimes x \mapsto m \cdot x$.

For finite dimensional Hopf algebras the isomorphism (1.1) immediately implies the existence and uniqueness of (left) integrals, since in this case also the dual $\hat{H}$ becomes a Hopf $H$-module by dualizing the Hopf module structure of $H$. Then the left integrals in $\hat{H}$ are precisely the coinvariants and due to finite dimensionality the subspace of coinvariants has to be one-dimensional. This proves existence and uniqueness of integrals in $\hat{H}$, and therefore also in $H$, by interchanging the role of $H$ and $\hat{H}$.

In this paper we generalize the structure Theorem to quasi-Hopf algebras as introduced by Drinfeld [Dri90], and show that many results of the integral theory of Hopf algebras carry over to the quasi-Hopf case. We will recall the precise definition of quasi-Hopf algebras in Section 2. The main difference compared to ordinary Hopf algebras is the weakening of the coassociativity axiom, i.e. the coproduct $\Delta$ satisfy

$$(\text{id} \otimes \Delta)(\Delta(a)) \phi = \phi (\Delta \otimes \text{id})(\Delta(a)), \quad a \in H \quad (1.2)$$

with invertible $\phi \in H \otimes H \otimes H$, whereas in the Hopf case $\phi$ would be trivial. In particular this implies that the dual space $\hat{H}$ in general is not an algebra. Equation (1.2) already suggests that there is an appropriate notion of quasi-Hopf $H$-bimodules (the guiding example being $H$ itself), whereas a meaningful notion of quasi-Hopf $H$-modules does not exist. However the structure Theorem for Hopf modules may as well be formulated for Hopf $H$-bimodules and we will generalize this version in Section 3 to quasi-Hopf algebras. We then provide for finite dimensional $H$ a quasi-Hopf $H$-bimodule structure on the dual $\hat{H}$. As in [LS69] this implies

$$\hat{H} \cong \mathcal{L} \otimes H \quad \text{as quasi-Hopf } H\text{-bimodules},$$

where $\mathcal{L}$ is a one-dimensional subspace of $\hat{H}$, whose elements will be called (left) cointegrals on $H$ (recall that $\hat{H}$ is no algebra, therefore neither does it make sense to speak of integrals in $\hat{H}$ nor is it possible to simply interchange the role of $H$ and $\hat{H}$ to arrive at the existence and uniqueness of integrals in $H$). We then show that every nonzero cointegral is nondegenerate (as a functional on $H$), i.e. every finite dimensional quasi-Hopf algebra is Frobenius, which implies existence and uniqueness of (left) integrals. The existence part has already been proved before by F. Panaite [PV] using different methods, see below.

In Section 5 we develop a notion of Fourier transformations and show that these are given in terms of cointegrals by the same formula as for ordinary Hopf algebras. We then give a characterization of semisimple quasi-Hopf algebras which essentially is due to F. Panaite [Pan98]. Moreover we show that $H$ is unimodular iff the modular automorphism of any (left) cointegral is given by the square of the antipode $S$, or equivalently, if every left integral is $S$-invariant. In Section 6 we generalize
Radford’s formula for the fourth power of the antipode to quasi-Hopf algebras. In Section 7 we provide a characterization of cointegrals in terms of cosemisimplicity needed to prove in Section 8 a Machke type Theorem for diagonal crossed products $A \bowtie H$ as constructed in [HN] - an important example being the quantum double $D(H) = H \bowtie H$, see [HN99]. In fact this has been our original motivation to develop an integral theory for quasi-Hopf algebras.

We remark that there is an alternative - and in fact very short - approach to prove the existence and uniqueness of integrals in finite dimensional Hopf algebras, by providing a direct formula for a nonzero integral, this way avoiding the use of the structure Theorem, see [Dae97]. This approach has been successfully generalized to quasi-Hopf algebras by [PV], proving the existence of integrals. Yet the generalization of the uniqueness result along the lines of [Dae97] seems to be out of reach.

2. Quasi-Hopf Algebras

We recall some definitions and properties of quasi-Hopf algebras as introduced by Drinfeld [Dri90] and fix our notations. Throughout $k$ will be a field and all algebras and linear spaces will be over $k$. All algebra morphisms are supposed to be unital. The dual of a linear space $V$ is denoted $\hat{V} := \text{Hom}_k(V, k)$.

A quasi-bialgebra $(H, \Delta, \epsilon, \phi)$ is a unital $k$-algebra $H$ with algebra morphisms $\Delta : H \to H \otimes H$ (the coproduct) and $\epsilon : H \to k$ (the counit), and an invertible element $\phi \in H \otimes H \otimes H$ (the reassociator), such that

\begin{align*}
(\text{id} \otimes \Delta)(\Delta(a)) &= \phi(\Delta \otimes \text{id})(\Delta(a)), \quad a \in H, \\
(\text{id} \otimes \text{id} \otimes \Delta)(\phi)(\Delta \otimes \text{id} \otimes \text{id})(\phi) &= (1 \otimes \phi)(\text{id} \otimes \Delta \otimes \text{id})(\phi)(\phi \otimes 1), \\
(\epsilon \otimes \text{id} \otimes \Delta) &= \text{id} = (\text{id} \otimes \epsilon) \circ \Delta, \\
(\text{id} \otimes \epsilon \otimes \text{id})(\phi) &= 1 \otimes 1.
\end{align*}

The identities (2.1)-(2.4) also imply $(\epsilon \otimes \text{id} \otimes \text{id})(\phi) = (\text{id} \otimes \text{id} \otimes \epsilon)(\phi) = 1 \otimes 1$. As for Hopf algebras we denote $\Delta(a) = \sum_i a_{(1)}^i \otimes a_{(2)}^i = a_{(1)} \otimes a_{(2)}$, but since $\Delta$ is only quasi-coassociative we adopt the further convention

\begin{align*}
(\Delta \otimes \text{id})(\Delta(a)) &= a_{(1,1)} \otimes a_{(1,2)} \otimes a_{(2)}, \\
(\text{id} \otimes \Delta)(\Delta(a)) &= a_{(1)} \otimes a_{(2,1)} \otimes a_{(2,2)}.
\end{align*}

Furthermore we denote $\Delta^{op}(a) := a_{(2)} \otimes a_{(1)}$ and

\begin{align*}
\phi &= X^j \otimes Y^j \otimes Z^j, \\
\phi^{-1} &= \bar{X}^j \otimes \bar{Y}^j \otimes \bar{Z}^j,
\end{align*}

suppressing the summation symbol $\sum_j$.

A quasi-bialgebra $H$ is called quasi-Hopf algebra, if there is a linear antimorphism $S : H \to H$ (the antipode) and elements $\alpha, \beta \in H$ satisfying for all $a \in H$

\begin{align*}
S(a_{(1)})a_{(2)} &= \alpha(a), \\
S(a_{(1)})\beta S(a_{(2)}) &= \beta S(\alpha(a)), \\
X^j S(Y^j) \alpha Z^j &= 1 = S(\bar{X}^j)\alpha \bar{Y}^j \beta S(\bar{Z}^j).
\end{align*}

We will always assume $S$ to be invertible. Equations (2.5),(2.6) imply $\epsilon \circ S = \epsilon$ and $\epsilon(\alpha \beta) = 1$. Thus, by rescaling $\alpha$ and $\beta$ we may without loss assume $\epsilon(\alpha) = \epsilon(\beta) = 1$.

We will also frequently use the following notation: If $\psi = \sum \psi_1^i \otimes \psi_2^i \otimes \cdots \psi_m^i \in H^{\otimes n}$, then, for $m \leq n$, $\psi_{n_1,n_2,\ldots,n_m}^m \in H^{\otimes n}$ denotes the element in $H^{\otimes n}$ having $\psi_k^i$ in the $n_k$th slot and $1$ in the remaining ones.
Next we recall that an invertible element \( F \in H \otimes H \) satisfying \((e \otimes \text{id})(F) = (\text{id} \otimes e)(F) = 1\), induces a so-called "twist transformation"
\[
\Delta_F(a) := F\Delta(a)F^{-1}, \quad \phi_F := (1 \otimes F)(\text{id} \otimes \Delta)(F)\phi(\Delta \otimes \text{id})(F^{-1} \otimes \text{id})(F^{-1} \otimes 1),
\]
and \((H, \Delta_F, \epsilon, \phi_F)\) is again a quasi-bialgebra.

It is well known that the antipode of a Hopf algebra is also an anti coalgebra morphism, i.e. \( \Delta(a) = (S \otimes S)(\Delta^{op}(S^{-1}(a))) \). For quasi-Hopf algebras this is true only up to a twist. Following Drinfeld we define the elements \( \gamma, \delta \in H \otimes H \) by
\[
\gamma := (S(U^i) \otimes S(T^i)) (\alpha \otimes \alpha) (V^i \otimes W^i), \quad (2.8)
\]
\[
\delta := (K^j \otimes L^j) (\beta \otimes \beta) (S(N^j) \otimes S(M^j)), \quad (2.9)
\]
where
\[
T^i \otimes U^i \otimes V^i \otimes W^i = (1 \otimes \phi^{-1}) (\text{id} \otimes \text{id} \otimes \Delta)(\phi),
\]
\[
K^j \otimes L^j \otimes M^j \otimes N^j = (\phi \otimes 1) (\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1}).
\]

With these definitions Drinfeld has shown in [Dri90], that \( f \in H \otimes H \) given by
\[
f := (S \otimes S)(\Delta^{op}(X^i)) \gamma \Delta(Y^i \beta S(Z^i)) \quad (2.10)
\]
is invertible and satisfies for all \( a \in H \)
\[
f\Delta(a)f^{-1} = (S \otimes S)(\Delta^{op}(S^{-1}(a))). \quad (2.11)
\]
The elements \( \gamma, \delta \) and the twist \( f \) fulfill the relations
\[
f \Delta(a) = \gamma, \quad \Delta(\beta)f^{-1} = \delta. \quad (2.12)
\]
Furthermore, the corresponding twisted reassociator \((2.7)\) is given by
\[
\phi_f = (S \otimes S \otimes S)(\phi^{321}). \quad (2.13)
\]
Setting
\[
h := (S^{-1} \otimes S^{-1})(f^{21}), \quad (2.14)
\]
the above relations imply
\[
h\Delta(a)h^{-1} = (S^{-1} \otimes S^{-1})(\Delta^{op}(S(a))), \quad (2.15)
\]
\[
\phi_h = (S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321}). \quad (2.16)
\]
Let \( H_{op} \) denote the algebra with opposite multiplication, then \( H_{op} \) becomes a quasi-Hopf algebra by setting \( \phi_{op} := \phi^{-1}, S_{op} := S^{-1}, \alpha_{op} := S^{-1}(\beta), \beta_{op} := S^{-1}(\alpha) \), implying \( h_{op} = f^{-1} \).

We will also need a generalization of Hopf algebra formulae of the type \( a_{(1)} \otimes a_{(2)} S(a_{(3)}) = a \otimes 1 \) to the quasi-coassociative setting. For this one uses the following elements \( q_R, p_R, q_L, p_L \in H \otimes H \), see e.g. [HN, HN99],
\[
q_R := X^i \otimes S^{-1}(\alpha Z^i) Y^i, \quad p_R := \bar{X}^i \otimes Y^i \beta S(\bar{Z}^i), \quad (2.17)
\]
\[
q_L := S(\bar{X}^i) \alpha \bar{Y}^i \otimes \bar{Z}^i, \quad p_L := Y^i S^{-1}(X^i \beta) \otimes Z^i. \quad (2.18)
\]

\[\text{suppressing summation symbols}\]
They obey the relations (for all $a \in H$)
\[
[1 \otimes S^{-1}(a_{(2)})] q_R \Delta(a_{(1)}) = [a \otimes 1] q_R, \tag{2.19}
\]
\[
\Delta(a_{(1)}) p_R [1 \otimes S(a_{(2)})] = p_R [a \otimes 1], \tag{2.20}
\]
\[
[S(a_{(1)})] \otimes 1] q_L \Delta(a_{(2)}) = [1 \otimes a] q_L, \tag{2.21}
\]
\[
\Delta(a_{(2)}) p_L [S^{-1}(a_{(1)})] \otimes 1] = p_L [1 \otimes a], \tag{2.22}
\]
and (writing $q_R = q^1_R \otimes q^2_R$, etc. suppressing the summation symbol and indices)
\[
\Delta(q^1_R) p_R [1 \otimes S(q^2_R)] = 1 \otimes 1, \quad [1 \otimes S^{-1}(p^2_R)] q_R \Delta(p^1_R) = 1 \otimes 1, \tag{2.23}
\]
\[
\Delta(q^2_L) p_L [S^{-1}(q^1_L)] \otimes 1] = 1 \otimes 1, \quad [S(p^1_L) \otimes 1] q_L \Delta(p^2_L) = 1 \otimes 1. \tag{2.24}
\]

Moreover, the following formula has been proved in [HN]
\[
(q_R \otimes 1) (\Delta \otimes \text{id})(q_R) \phi^{-1} = [1 \otimes S^{-1}(Z^i) \otimes S^{-1}(Y^i)] [1 \otimes h] (\text{id} \otimes \Delta)(q_R \Delta(X^i)). \tag{2.25}
\]

We remark that the algebraic properties of a quasi-Hopf algebra $H$ may be translated into corresponding properties of its representation category $\text{Rep} H$. More precisely $\text{Rep} H$ is a rigid monoidal category [Dri90]. This implies that also the elements $f, h, q_R, q_L, p_R, p_L$ and their properties as stated above may be nicely described and understood in categorical terms, i.e. they define natural transformations, see [HN]. One may also use a graphical description to obtain and describe their properties, see [HN99].

We denote $\Gamma(H) := \text{Alg}(H, k)$ the group of 1-dimensional representations of $H$, where for $\mu, \nu \in \Gamma(H)$ we put $\mu \nu := (\mu \otimes \nu) \circ \Delta$. Note that for 1-dimensional representations this product is indeed strictly associative with unit $\tilde{1} \equiv \epsilon$. Clearly, for any $\mu \in \Gamma(H)$ Eq. (2.6) implies $\mu(\alpha) \neq 0 \neq \mu(\beta)$. Putting $\bar{\mu} := \mu \circ S$ we conclude $\bar{\mu} \mu = \mu \bar{\mu} = \epsilon$ by (2.5) and therefore $\bar{\mu} = \mu^{-1}$. Thus $\Gamma(H)$ is indeed a group.

Finally we introduce $\hat{H}$ as the dual space of $H$ with its natural ‘multiplication’ $\hat{H} \otimes \hat{H} \to \hat{H}$, which however is no longer associative
\[
\langle \varphi \psi | a \rangle := \langle \varphi \otimes \psi | \Delta(a) \rangle, \quad \langle 1 | a \rangle := \epsilon(a),
\]
where $\varphi, \psi \in \hat{H}, a \in H$ and where $\langle \cdot | \cdot \rangle : \hat{H} \otimes H \to k$ denotes the dual pairing. We have $\hat{1} \varphi = \varphi \hat{1} = \varphi$. Transposing the right and left multiplication on $H$ one obtains on $\hat{H}$ the left and right $H$-actions
\[
(a \rightarrow \varphi | b) := \langle \varphi | ba \rangle, \quad (\varphi \leftarrow a | b) := \langle \varphi | ab \rangle, \quad a, b \in H, \varphi \in \hat{H} \tag{2.26}
\]
satisfying $a \rightarrow (\varphi \psi) = (a_{(1)} \rightarrow \varphi)(a_{(2)} \rightarrow \psi)$ and $\varphi \leftarrow a = (\varphi \leftarrow a_{(1)})(\varphi \leftarrow a_{(2)})$.

For $\varphi \in \hat{H}$ and $a \in H$ we also use the dual notation
\[
\varphi \rightarrow a := a_{(1)} \langle \varphi | a_{(2)} \rangle, \quad a \leftarrow \varphi := \langle \varphi | a_{(1)} \rangle a_{(2)}.
\]
If $H$ is finite dimensional, then $\hat{H}$ is also equipped with a coassociative coalgebra structure $(\hat{\Delta}, \hat{\epsilon})$ given by $(\hat{\Delta}(\varphi) | a \otimes b) := \langle \varphi | ab \rangle$ and $\hat{\epsilon}(\varphi) := \langle \varphi | 1 \rangle$. In this case we have
\[
a \rightarrow \varphi = \varphi_{(1)} \langle \varphi_{(2)} | a \rangle, \quad \varphi \leftarrow a = \varphi_{(2)} \langle \varphi_{(1)} | a \rangle, \quad a \in H, \varphi \in \hat{H}
\]
and \( \hat{\Delta}(\varphi \psi) = \hat{\Delta}(\varphi)\hat{\Delta}(\psi) \).

3. Quasi-Hopf Bimodules

Throughout let \( H \) denote a quasi-Hopf algebra over the field \( k \). As a natural generalization of the notion of Hopf bimodules we define

**Definition 3.1.** Let \( M \) be an \( H \)-bimodule and let \( \rho : M \to M \otimes H \) be an \( H \)-bimodule map. Then \((M, \rho)\) is called a quasi-Hopf \( H \)-bimodule, if

\[
\begin{align*}
(\text{id}_M \otimes \epsilon) \circ \rho &= \text{id}_M, \\
\phi \cdot (\rho \otimes \text{id}_M)(\rho(m)) &= (\text{id}_M \otimes \Delta)(\rho(m)) \cdot \phi,
\end{align*}
\]

and \( \hat{\Delta}(\varphi \psi) = \hat{\Delta}(\varphi)\hat{\Delta}(\psi) \).

More specifically, one may also call \((M, \rho)\) a right quasi-Hopf \( H \)-bimodule, whereas a left quasi-Hopf \( H \)-bimodule \((M, \Lambda)\) would be defined analogously, except that \( \Lambda : M \to H \otimes M \) would now be a quasi-coassociative left coaction. If not mentioned explicitly, we will always work with right coactions.

A trivial example is of course given by \( M = H \) and \( \rho = \Delta \). More generally, one straightforwardly checks the following

**Lemma 3.2.** Let \((V, \triangleright)\) be a left \( H \)-module. Then \( V \otimes H \) becomes a quasi-Hopf \( H \)-bimodule by putting for \( a, b, x \in H \) and \( v \in V \)

\[
\begin{align*}
a \cdot (v \otimes x) \cdot b &:= (a_{(1)} \triangleright v) \otimes a_{(2)} x b, \\
\rho_{V \otimes H}(v \otimes x) &:= \tilde{X}^i \triangleright v \otimes \tilde{Y}^i x_{(1)} \otimes \tilde{Z}^i x_{(2)}.
\end{align*}
\]

In the special case \( V = k \) with \( H \)-action given by some \( \gamma \in \Gamma(H) \) we may identify \( V \otimes H \cong H \) to obtain a quasi-Hopf \( H \)-bimodule structure on \( H \), denoted by \((H, \rho_\gamma)\). This leads to

**Corollary 3.3.** Let \( \gamma \in \Gamma(H) \) and put \( T_\gamma := \gamma(\tilde{X}^i)\tilde{Y}^i \otimes \tilde{Z}^i \in H \otimes H \). Denote \( H_\gamma := H \) considered as an \( H \)-bimodule with actions

\[
\begin{align*}
a \cdot x \cdot b &:= (a \leftarrow \gamma) x b, \\
a, b \in H, x \in H_\gamma.
\end{align*}
\]

Put \( \rho_\gamma : H_\gamma \to H_\gamma \otimes H, \rho_\gamma(x) := T_\gamma \Delta(x) \). Then \((H_\gamma, \rho_\gamma)\) provides a quasi-Hopf \( H \)-bimodule.

Our aim is to generalize the fundamental structure Theorem on Hopf (bi-)modules by Larson and Sweedler [LS69] to quasi-Hopf algebras \( H \). Thus, any quasi-Hopf \( H \)-bimodule \( M \) will be shown to be isomorphic to some \( V \otimes H \) as in Lemma 3.2, where \( V \equiv M^{coH} \subset M \) will be a suitably defined subspace of coinvariants.

To this end we first provide what will be a projection \( E : M \to M^{coH} \), which for ordinary Hopf algebras \( H \) would be given by \( E(m) = m_{(0)} \cdot S(m_{(1)}) \). In this case \( E(M) \equiv M^{coH} \) would be invariant under the adjoint \( H \)-action \( a \triangleright m := a_{(1)} \cdot m \cdot S(a_{(2)}) \) and we would have \( E(a \cdot m) = a \triangleright E(m), \forall a \in H, m \in M \).

In the quasi-Hopf case we first appropriately generalize this last property. Here the basic idea is that the "would-be-adjoint" action \( \triangleright \) should satisfy \( a \cdot m =
(a(1) • m) • a(2). In fact, we will see that this only holds for \( m \in M^{coH} \). For \( m \in M \) and \( a \in H \) we now define

\[
E(m) := q^1_R \cdot m_{(0)} \cdot \beta S(q^2_R m_{(1)}) \tag{3.3}
\]

\[
a • m := E(a \cdot m), \tag{3.4}
\]

where \( q^1_R \otimes q^2_R \equiv q_R \in H \otimes H \) is defined in (2.17).

**Proposition 3.4.** Let \( M \) be a quasi-Hopf \( H \)-bimodule and let \( E \) and \( \triangleright \) be given as above. Then for all \( a, b \in H \) and \( m \in M \)

(i) \( E(m \cdot a) = E(m) \epsilon(a) \)

(ii) \( E^2 = E \)

(iii) \( a \triangleright E(m) = E(a \cdot m) \equiv a \triangleright m \)

(iv) \( (ab) \triangleright m = a \triangleright (b \triangleright m) \)

(v) \( a \cdot E(m) = [a(1) \triangleright E(m)] \cdot a(2) \)

(vi) \( E(m_{(0)}) \cdot m_{(1)} = m. \)

(vii) \( E(E(m)_{(0)}) \otimes E(m)_{(1)} = E(m) \otimes 1. \)

**Proof.** All properties (i) - (vii) follow easily from the properties of quasi-Hopf algebras as stated in Section 2. We will give a rather detailed proof such that the unexperienced reader may get used to the techniques used when handling formulae involving iterated non coassociative coproducts. We will in the following denote \( q_R = q = q^1 \otimes q^2 \). Equality (i) follows directly from the antipode property \( a(1)\beta S(a(2)) = \epsilon(a)\beta \). To show (ii) one uses (i) to compute

\[
E^2(m) = E(q^1 \cdot m_{(0)} \cdot \beta S(q^2 m_{(1)}))
\]

\[
= E(q^1 \cdot m_{(0)}) \epsilon(\beta S(q^2 m_{(1)}))
\]

\[
= E(m) \epsilon(\alpha \beta) = E(m)
\]

by (2.4) and (3.1). Equality (iii) is obtained similarly:

\[
a \triangleright E(m) = E(a \cdot E(m)) = E(aq^1 \cdot m_{(0)} \cdot \beta S(q^2 m_{(1)})) = E(a \cdot m).
\]

Property (iv) follows immediately from (iii). To show (v) we use (2.19):

\[
a \cdot E(m) = aq^1 \cdot m_{(0)} \cdot \beta S(q^2 m_{(1)})
\]

\[
= q^1 a(1,1) \cdot m_{(0)} \cdot \beta S(q^2 a(1,2)m_{(1)}) a(2)
\]

\[
= E(a(1) \cdot m) \cdot a(2)
\]

\[
= [a(1) \triangleright E(m)] \cdot a(2).
\]

Part (vi) follows by using formula (2.17) for \( q = q_R \), (3.2) and then (2.3),(2.4)

\[
E(m_{(0)}) \cdot m_{(1)} = X^i \cdot m_{(0,0)} \cdot \beta S(Y^i m_{(0,1)}) \alpha Z^i m_{(1)}
\]

\[
= m \cdot X^i \beta S(Y^i) \alpha Z^i = m.
\]
Finally, we prove (vii) by using part (i), Eq. (3.2) and the left identity in (2.23) to compute
\[
E(E(m)_{(0)}) \otimes E(m)_{(1)} = E(q_{(1)}^1 \cdot m_{(0,0)}) \otimes q_{(2)}^1 m_{(0,1)} \beta S(q^2 m_{(1)}) \\
= E(q_{(1)}^1 \bar{X}^i \cdot m_{(0)}) \otimes q_{(2)}^1 \bar{Y}^i m_{(1,1)} \beta S(q^2 \bar{Z}^i m_{(1,2)}) \\
= E(q_{(1)}^1 \bar{X}^i \cdot m) \otimes q_{(2)}^1 \bar{Y}^i \beta S(\bar{Z}^i) S(q^2) \\
= E(m) \otimes 1. 
\]

Due to part (ii), (vi) and (vii) of Proposition 3.4 the following notions of coinvariants all coincide

**Definition 3.5.** The space of coinvariants of a quasi-Hopf \( H \)-bimodule \( M \) is defined to be
\[
M^{coH} := E(M) \equiv \{ n \in M \mid E(n) = n \} \equiv \{ n \in M \mid E(n_{(0)}) \otimes n_{(1)} = E(n) \otimes 1 \}
\]

Let us first check this definition for the type of quasi-Hopf \( H \)-bimodules described in Lemma 3.2.

**Lemma 3.6.** The coinvariants of the quasi-Hopf \( H \)-bimodule \( V \otimes H \) in Lemma 3.3 are given by \((V \otimes H)^{coH} = V \otimes 1\), and for \( v \in V \), \( x \in H \), we have \( E(v \otimes x) = v \otimes \epsilon(x) 1 \).

**Proof.** The definitions in Lemma 3.3 imply \((v \otimes x) = (v \otimes 1) \cdot x\), which by Proposition 3.4 (i) further implies
\[
E(v \otimes x) = E(v \otimes 1) \epsilon(x).
\]
Thus we are left to show the identity \( E(v \otimes 1) = v \otimes 1 \) (denoting \( q = q_R\)):
\[
E(v \otimes 1) = q^1 \cdot [\bar{X}^i \triangleright v \otimes \bar{Y}^i] \cdot \beta S(q^2 \bar{Z}^i) \\
= q_{(1)}^1 \bar{X}^i \triangleright v \otimes q_{(2)}^1 \bar{Y}^i \beta S(\bar{Z}^i) S(q^2) \\
= v \otimes 1,
\]
where the last equality follows from (2.23). \( \square \)

For later purposes let us also consider the \( k \)-dual \( \hat{M} \) as a left \( H \)-module by transposing the right \( H \)-action on \( M \). Denote \( \hat{M}^H \) the invariants under this action, i.e.
\[
\hat{M}^H := \{ \psi \in \hat{M} \mid \psi(m \cdot a) = \psi(m) \epsilon(a), \forall m \in M, a \in H \}.
\]

**Lemma 3.7.** Let \( E^T : \hat{M} \rightarrow \hat{M} \) denote the transpose projection of \( E \). Then
\[
E^T(\hat{M}) = \hat{M}^H.
\]

**Proof.** Clearly, part (i) of Proposition 3.4 implies \( E^T(\hat{M}) \subseteq \hat{M}^H \). Conversely, if \( \psi \in \hat{M}^H \) then (3.3) implies \( \psi \circ E = \psi \), hence \( \psi \in E^T(\hat{M}) \).

We are now in the position to provide the fundamental Theorem for quasi-Hopf \( H \)-bimodules. By a morphism \( (M, \rho) \rightarrow (M', \rho') \) of quasi-Hopf \( H \)-bimodules we mean an \( H \)-bimodule map \( f : M \rightarrow M' \) satisfying \( \rho' \circ f = (f \otimes \id) \circ \rho \).
Theorem 3.8. Let $M$ be a quasi-Hopf $H$-bimodule. Consider $N \equiv M^{coH}$ as a left $H$-module with $H$-action $\triangleright$ as in (3.3), and $N \otimes H$ as a quasi-Hopf $H$-bimodule as in Lemma 3.3. Then

$$\nu : N \otimes H \ni n \otimes a \mapsto n \cdot a \in M$$

provides an isomorphism of quasi-Hopf $H$-bimodules with inverse given by

$$\nu^{-1}(m) = E(m_{(0)}) \otimes m_{(1)}.$$  

Proof. Using Proposition 3.4 (i), (vii) and $n = E(n)$ we compute

$$\nu^{-1} \circ \nu(n \otimes a) = E(n_{(0)} \cdot a_{(1)}) \otimes n_{(1)} a_{(2)} = E(n_{(0)}) \otimes n_{(1)} a = E(n) \otimes a = n \otimes a.$$  

Conversely $\nu \circ \nu^{-1}(m) = E(m_{(0)}) \cdot m_{(1)} = m$ by Proposition 3.4 (vi). Thus $\nu$ is indeed an isomorphism of vector spaces.

We are left to show that $\nu$ also respects the quasi-Hopf $H$-bimodule structures. By definition we have $a \cdot (n \otimes x) \cdot b = E(a_{(1)} \cdot n) \otimes a_{(2)} x b$ and therefore

$$\nu(a \cdot (n \otimes x) \cdot b) = E(a_{(1)} \cdot n) \cdot a_{(2)} x b = [a_{(1)} \triangleright E(n)] \cdot a_{(2)} x b = a \cdot E(n) \cdot x b = a \cdot n \cdot x b = a \cdot \nu(n \otimes x) \cdot b.$$  

Here we have used Proposition 3.4 (iii) and (v) in the second and third line, respectively. Thus $\nu$ is an $H$-bimodule map. Finally, we show that $\nu^{-1}$ (and therefore $\nu$) are also $H$-comodule maps.

$$\rho_{N \otimes H}(\nu^{-1}(m)) = E(\bar{X}^i \cdot m_{(0)}) \otimes \bar{Y}^i m_{(1,1)} \otimes \bar{Z}^i m_{(1,2)} = E(m_{(0,0)}) \otimes m_{(0,1)} \otimes m_{(1)} = (\nu^{-1} \otimes \text{id})(\rho_M(m)).$$  

Here we have used part (iii) of Proposition 3.4 in the first line and part (i) together with the quasi-coassociativity (3.3) in the second line.

Corollary 3.9. For any quasi-Hopf $H$-bimodule $M$ we have

$$M^{coH} = \{ n \in M \mid \rho(n) = (\bar{X}^i \triangleright n) \cdot \bar{Y}^i \otimes \bar{Z}^i \}.$$  

Proof. If $\rho(n) = (\bar{X}^i \triangleright n) \cdot \bar{Y}^i \otimes \bar{Z}^i$ then by Proposition 3.4(i) and (vi) $E(n) = E(n_{(0)}) \cdot n_{(1)} = n$, whence $n \in N := M^{coH}$. The inverse implication follows from $\rho_M = (\nu \otimes \text{id}) \circ \rho_{N \otimes H} \circ \nu^{-1}$ and $\nu^{-1}(N) = N \otimes 1$.  

Theorem 3.8 shows that there is a one-to-one correspondence (up to equivalence) between left $H$-modules and quasi-Hopf $H$-bimodules. As for ordinary Hopf algebras, this is actually an equivalence of monoidal categories. To see this we note that if $M$ and $N$ are quasi-Hopf $H$-bimodules, then so is $M \otimes H N$ with its natural $H$-bimodule structure, the coaction being given by

$$\rho_{H \otimes M} : (m \otimes n) := (m_{(0)} \otimes n_{(0)}) \otimes m_{(1)} n_{(1)}.$$  

In this way the category $H \mathcal{M}_H^H$ of quasi-Hopf $H$-bimodules becomes a strict monoidal category with unit object given by $H$. Moreover, we have
Lemma 3.10. Let $M$ and $N$ be quasi-Hopf $H$-bimodules. Then the map

$$i_{MN} : M^{coH} \otimes N^{coH} \ni m \otimes n \mapsto (X^i \triangleright m) \otimes_H (Y^i \triangleright n) \cdot Z^i \in (M \otimes_H N)^{coH}$$

provides an isomorphism of left $H$-modules.

Proof. Denote $E_M$, $E_N$ and $E_{M \otimes_H N}$ the projections onto the corresponding coinvariants. To prove that $i_{MN} : M^{coH} \otimes N^{coH} \to (M \otimes_H N)^{coH}$ is bijective we claim

$$\tilde{i}_{MN} = id$$

$$i_{MN} \circ \tilde{i}_{MN} = E_{M \otimes_H N},$$

where $\tilde{i}_{MN} : M \otimes_H N \to M^{coH} \otimes N^{coH}$ is given by

$$\tilde{i}_{MN}(m \otimes_H n) := E_M(m_0) \otimes E_N(m_1 \cdot n).$$

By Proposition 3.11(i) $\tilde{i}_{MN}$ is well defined and we have

$$\tilde{i}_{MN}(m \otimes_H n) = m \otimes E_N(n), \quad \forall m \in M^{coH}$$

thus proving (3.7). To prove (3.6) let $m \in M^{coH}$ and $n \in N$. Then $(q := q_R)$

$$E_{M \otimes_H N}(m \otimes_H n) = q^1 \cdot m_0 \otimes_H n_0 \cdot \beta S(q^2 m_1 n_1)$$

$$= q^1 \cdot (X^i \triangleright m) \cdot Y^i \otimes_H n_0 \cdot \beta S(q^2 Z^i n_1)$$

$$= q^1 \cdot X^i \triangleright m \otimes_H q_1 \cdot Y^i \cdot n_0 \cdot \beta S(q^2 Z^i n_1)$$

$$= X^k \triangleright m \otimes_H q^1 \cdot Y^k \cdot n_0 \cdot \beta S(q^2 Z^i (1) n_1) Z^k$$

$$= X^k \triangleright m \otimes_H (Y^k \triangleright n) \cdot Z^k,$$

where in the second line we have used Corollary 3.9, in the third line Proposition 3.4(v) and in the fourth line the identity

$$(\Delta \otimes id)(q_R)^{-1} = [X^k \otimes 1 \otimes S^{-1}(Z^k)] [1 \otimes q_R \Delta(Y^k)]$$

which follows easily from (2.2) and (2.3). Thus, by Proposition 3.4(vi) and (iii) we conclude for general $m \in M, n \in N$

$$E_{M \otimes_H N}(m \otimes_H n) = E_{M \otimes_H N}(E(m_0) \otimes_H m_1 \cdot n)$$

$$= X^k \triangleright E(m_0) \otimes_H [Y^k \triangleright (m_1 \cdot n)] \cdot Z^k$$

$$= (i_{MN} \circ \tilde{i}_{MN})(m \otimes_H n).$$

This proves that $i_{MN}$ is bijective. To prove that it is $H$-linear we compute for $a \in H, m \in M^{coH}$ and $n \in N^{coH}$

$$a \cdot i_{MN}(m \otimes n) = a \cdot (X^i \triangleright m) \otimes_H (Y^i \triangleright n) \cdot Z^i$$

$$= a_{(1)} X^i \triangleright m \otimes_H a_{(2,1)} Y^i \triangleright n \cdot a_{(2,2)} Z^i$$

$$= i_{MN}(a_{(1,1)} \triangleright m \otimes a_{(1,2)} \triangleright n) \cdot a_{(2)} .$$

$H$-linearity of $i_{MN}$ follows by applying $E_{M \otimes_H N}$ to both sides and using Proposition 3.4(i). \qed

Denoting the category of left $H$-modules by $HM$, Lemma 3.11 leads to

Proposition 3.11. There is an equivalence of monoidal categories $HM^{coH} \cong HM$ given on the objects by $M \mapsto M^{coH}$ and on the morphisms by $f \mapsto f \mid M^{coH}$. 
Proof. If \( f : M \rightarrow N \) is a morphism of quasi-Hopf \( H \)-bimodules, then \( f(\mathcal{M}^{coH}) \subset \mathcal{N}^{coH} \); \( f \upharpoonright \mathcal{M}^{coH} \) is \( H \)-linear and

\[
f(m) = f(E(m_{(0)}) \cdot m_{(1)}) = f(E(m_{(0)})) \cdot m_{(1)}.
\]

Thus, \( f \) is uniquely determined by its restriction \( f \upharpoonright \mathcal{M}^{coH} \), and by Theorem 3.5 \((M \mapsto \mathcal{M}^{coH}, f \mapsto f \upharpoonright \mathcal{M}^{coH})\) provides an equivalence of categories with reverse functor given by \( V \mapsto V \otimes H \) and \((f : V \rightarrow W) \mapsto (f \otimes \text{id} : V \otimes H \rightarrow W \otimes H)\).

By Lemma 3.10, these functors preserve the monoidal structures provided we show that for any three objects \( M, N, K \) in \( _H\mathcal{M}^{coH}_H \) the following diagram commutes

\[
\begin{array}{ccc}
(M^{coH} \otimes N^{coH}) \otimes K^{coH} & \overset{\phi}{\longrightarrow} & M^{coH} \otimes (N^{coH} \otimes K^{coH}) \\
\phi \downarrow & & \downarrow \text{id} \otimes i_N \\
(M \otimes H N)^{coH} \otimes K^{coH} & \overset{i_{MN} \otimes \text{id}}{\longrightarrow} & (M \otimes H N \otimes H K)^{coH} \\
\end{array}
\]

(3.9)

To this end let \( m \in M^{coH}, n \in N^{coH}, k \in K^{coH} \), then

\[
(i_{(M \otimes H N)K} \circ (i_{MN} \otimes \text{id}))(m \otimes n \otimes k) = i_{MN}(X_{(1)} \triangleright m \otimes X_{(2)} \triangleright n) \otimes_H (Y_{(1)} \triangleright k) \cdot Z^i
\]

\[
= X^j X_{(1)} \triangleright m \otimes_H X^j X_{(2)} \triangleright n \otimes_H (Z^j_{(1)} Y_{(1)} \triangleright k) \cdot Z^j_{(2)} Z^j
\]

\[
= X^j X_{(1)} \triangleright m \otimes_H X^k Y_{(1)} Y_{(2)} \triangleright n \otimes_H (Y^j_{(1)} Y^j_{(2)} \triangleright k) \cdot Z^j
\]

\[
= (i_{MN} \otimes \text{id}) \circ (i_{NK})(X_{(1)} \triangleright m \otimes Y^j_{(1)} Y^j_{(2)} \triangleright n \otimes Z^j \triangleright k),
\]

where we have used Lemma 3.10 in the first line, \( H \)-linearity of \( i_{MN} \) in the second line, (2.2) in the third line and again \( H \)-linearity of \( i_{NK} \) in the last line.

Note that \( _H\mathcal{M}^{coH}_H \) is strictly monoidal whereas \( _H\mathcal{M} \) is not. It will be shown elsewhere that \( _H\mathcal{M}^{coH}_H \) naturally coincides with the category of representations of the two-sided crossed product \( A := H \triangleright H \bowtie H \) constructed in [HN]. Thus, by methods of P. Schauenburg [Sch98], \( A \) becomes a \( H \)-bialgebra (also called quantum groupoid) in the sense of Takeuchi [Tak77]. If \( H \) is finite dimensional and Frobenius-separable (i.e. admits a nondegenerate functional of index one), then by [NS], the Takeuchi quantum groupoids are the same as the weak Hopf algebras of [BS06, XL, BNS]. This will approach a proof of an announcement of [Nil], saying that \( H \) to any f.d. semi-simple quasi-Hopf algebra \( H \) there is a (strictly coassociative!) weak Hopf algebra structure on \( A := H \triangleright H \bowtie H \), whose representation category obeys the same fusion rules.

With the application to integral theory in mind we finally show, that for finite dimensions any left quasi-Hopf \( H \)-bimodule \((M, \Lambda)\) naturally gives rise to a dual right

\(^2\) over an algebraically closed field \( k \) of characteristic zero
quasi-Hopf $H$-bimodule $(M^*, \rho)$. As a linear space we put $M^* = \tilde{M} \equiv \text{Hom}_k(M, k)$ with $H$-bimodule structure given for $a, b \in H$, $m \in M$ and $\psi \in \tilde{M}$ by

$$\langle a \cdot \psi \cdot b \mid m \rangle := \langle \psi \mid S^{-1}(a) \cdot m \cdot S(b) \rangle. \quad (3.10)$$

To define the right $H$-coaction on $M^*$ we first deform the left coaction $\Lambda$ on $M$ by putting $\bar{\Lambda} : M \to H \otimes M$ according to

$$\bar{\Lambda}(m) := V \cdot \Lambda(m) \cdot U,$$

where $U, V \in H \otimes H$ are given by

$$U := f^{-1}(S \otimes S)(q_R^2), \quad (3.12)$$

$$V := (S^{-1} \otimes S^{-1})(p_R^2) h, \quad (3.13)$$

the elements $f, h, q_R, p_R \in H \otimes H$ being defined in (2.11)-(2.18). With these definitions the linear map $\bar{\Lambda}$ satisfies $(e \otimes \text{id}_M) \circ \bar{\Lambda} = \text{id}_M$ and

$$[1 \otimes S^{-1}(a)] \cdot \bar{\Lambda}(m) \cdot [1 \otimes S(b)] = [a_{(2)} \otimes 1] \cdot \bar{\Lambda}(S^{-1}(a_{(1)}) \cdot m \cdot S(b_{(1)})) \cdot [b_{(2)} \otimes 1], \quad (3.14)$$

$$[Y^i \otimes Z^j \otimes 1] \cdot (\text{id} \otimes \bar{\Lambda})(S^{-1}(X^i) \cdot m) = (\Delta \otimes \text{id})\bar{\Lambda}(m \cdot S(X^i)) \cdot [Y^i \otimes Z^j \otimes 1], \quad (3.15)$$

see Lemma 3.13 below. These identities imply the following

**Proposition 3.12.** Let $(M, \Lambda)$ be a left quasi-Hopf $H$-bimodule and assume $H$ or $M$ finite dimensional. Consider $M^*$ as an $H$-module as in (3.10) and define $\rho : M^* \to M^* \otimes H$ by identifying $M^* \otimes H \cong \text{Hom}_k(M, H)$ and putting for $\psi \in M^*$ and $m \in M$

$$[\rho(\psi)](m) := (\text{id}_H \otimes \psi)(\bar{\Lambda}(m)) \in H. \quad (3.16)$$

Then $(M^*, \rho)$ provides a right quasi-Hopf $H$-bimodule.

**Proof.** $H$-linearity of $\rho$ follows straightforwardly from (3.14). Identifying $\text{Hom}_k(M, H \otimes H) \cong M^* \otimes H \otimes H$, quasi-coassociativity of $\rho$ is equivalent to the identity

$$(Y^i \otimes Z^j) \cdot [(\rho \otimes \text{id})\rho(\psi)](S^{-1}(X^i) \cdot m) = [(\text{id} \otimes \Delta)\rho(\psi)](m \cdot S(X^i)) \cdot (Y^i \otimes Z^j),$$

which follows from (3.13). \qed

We conclude this section with the proof of the identities (3.14) and (3.15).

**Lemma 3.13.** For all $a, b \in H$ and $m \in M$ we have

$$U [1 \otimes S(a)] = \Delta(S(a_{(1)})) U [a_{(2)} \otimes 1] \quad (3.17)$$

$$[1 \otimes S^{-1}(a)] V = [a_{(2)} \otimes 1] V \Delta(S^{-1}(a_{(1)})) \quad (3.18)$$

$$\phi^{-1}(\text{id} \otimes \Delta)(U) (1 \otimes U) = (\Delta \otimes \text{id})(\Delta(S(X^i))U) (Y^i \otimes Z^j \otimes 1) \quad (3.19)$$

$$(\Delta \otimes \text{id})(V) \phi^{-1} = (Y^i \otimes Z^j \otimes 1)(1 \otimes V) (\text{id} \otimes \Delta)(V \Delta(S^{-1}(X^i))), \quad (3.20)$$

implying (3.14) and (3.15).
Proof. Equation (3.20) is equivalent to
\[(1 \otimes V)(id \otimes \Delta)(V)\phi = (\tilde{Y}^i \otimes \tilde{Z}^i \otimes 1)(\Delta \otimes id)(V \Delta(S^{-1}(\tilde{X}^i))).\]
Thus, noting that in $H_{op}$ the roles of $U$ and $V$ interchanged, (3.18) and (3.20) reduce to (3.17) and (3.19), respectively, in $H_{op}$. To prove (3.17) we compute, using (2.11) and (2.19) and denoting $q := q_R$
\[\Delta(S(a_{(1)}))U[a_{(2)} \otimes 1] = f^{-1}(S \otimes S)([S^{-1}(a_{(2)}) \otimes 1]q^{21} \Delta^{op}(a_{(1)}))\]
\[= f^{-1}(S \otimes S)([1 \otimes a]q^{21})\]
\[= U[1 \otimes S(a)].\]
To prove (3.19) we compute
\[\phi^{-1}(id \otimes \Delta)(U)(1 \otimes U)\]
\[\overset{(2.11)}{=} \phi^{-1}(id \otimes \Delta)(f^{-1})(1 \otimes f^{-1})(S \otimes S \otimes S)((1 \otimes q^{21})(id \otimes \Delta^{op})(q^{21}))\]
\[\overset{(2.13)}{=} (\Delta \otimes id)(f^{-1})(1 \otimes f^{-1})(S \otimes S \otimes S)(([q \otimes 1](\Delta \otimes id)(q)\phi^{-1}]^{321})\]
\[\overset{(2.23)}{=} (\Delta \otimes id)(f^{-1})(1 \otimes f^{-1})(S \otimes S \otimes S)((\Delta^{op} \otimes id)(q^{21} \Delta^{op}(X^i)))\]
\[(f \otimes 1)(Y^i \otimes Z^i \otimes 1)\]
\[\overset{(2.11)}{=} (\Delta \otimes id)(\Delta(S(X^i)))(\Delta \otimes id)(f^{-1})(\Delta \otimes id)((S \otimes S)(q^{21}))\]
\[(f^{-1} \otimes 1)(f \otimes 1)(Y^i \otimes Z^i \otimes 1)\]
\[= (\Delta \otimes id)(\Delta(S(X^i)))(\Delta \otimes id)(U)(Y^i \otimes Z^i \otimes 1).\]
Now (3.14) follows immediately from (3.17), (3.18) and (3.15) from (3.13), (3.20).

4. Integral Theory

As in the original work of Larson and Sweedler [LS69], the first application of our previous results provides a theory of integrals and Fourier transformations for finite dimensional quasi-Hopf algebras.

Definition 4.1. An element $l \in H$ ($r \in H$) is called a left (right) integral, if $al = \epsilon(a)l$ ($ra = \epsilon(r)a$), $\forall a \in H$. If $l$ is a left and a right integral, then it is called two-sided. A left (right) integral $l$ is called normalized, if $\epsilon(l) = 1$. A Haar integral $e$ is a normalized two-sided integral.

Note that being the unit in the ideal of two-sided integrals a Haar integral $e \in H$ is unique, provided it exists. In particular $S(e) = e$. More generally, for any $\gamma \in \Gamma(H)$ denote
\[L_\gamma := \{l \in H \mid al = \gamma(a)l, \forall a \in H\},\]
then $L \equiv L_e$ is the space of left integrals.

From now on we assume dim $H < \infty$. To prove dim $L = 1$ we will show below that $H$ is a Frobenius algebra, implying dim $L_\gamma = 1$ for all $\gamma \in \Gamma(H)$. Indeed, let $\omega : H \rightarrow k$ be non degenerate and denote $\omega_R : H \rightarrow \tilde{H}$, $\langle \omega_R(a) \mid b \rangle := \omega(ba)$, then
\[L_\gamma = \omega_R^{-1}(k\gamma).\]
Moreover, in this case we have for all \( l \in L_\gamma \) and all \( a \in H \)
\[
la = \hat{\gamma}(a)l,
\]
where \( \hat{\gamma} = \gamma \circ \theta_\omega^{-1} \), and where \( \theta_\omega = \omega_R^{-1} \circ \omega_L \in \text{Aut } H \) denotes the modular (or Nakayama) automorphism of \( \omega \) (i.e. solving \( \omega(ab) = \omega(b)\omega(a), \forall a, b \in H \)). Note that since \( \theta_\omega \) is unique up to inner automorphisms, \( \hat{\gamma} \) only depends on \( \gamma \). As in ordinary f.d. Hopf algebras, we call
\[
\mu := \tilde{\epsilon} \equiv \epsilon \circ \theta_\omega^{-1} \in \Gamma(H) \tag{4.1}
\]
the \textit{modulus} of \( H \) (\( \mu \) is also called the \textit{distinguished grouplike element} of \( \hat{H} \) [Rad93]). Thus \( H \) is unimodular, i.e. \( \mu = \epsilon \), iff one (and therefore all) nonzero integrals are two-sided. In particular if \( H \) is symmetric (i.e. admits a non degenerate trace), then it is unimodular.

We now generalize the methods of [LS69] to prove that indeed all f.d. quasi-Hopf algebras are Frobenius, such that the above arguments apply. First, we consider \((H, \Delta)\) as a left quasi-Hopf \( H \)-bimodule and choose the dual (right) quasi-Hopf \( H \)-bimodule structure \((H^*, \rho)\) as in Proposition 3.12. Thus, as a linear space \( H^* = \hat{H} \) with \( H \)-bimodule structure given for \( a, b \in H \) and \( \psi \in H^* \) by (see (2.26) for the notation)
\[
a \cdot \psi \cdot b = S(b) \rightarrow \psi \leftarrow S^{-1}(a). \tag{4.2}
\]
Following Proposition 3.12 the \( H \)-coaction \( \rho : H^* \rightarrow H^* \otimes H \) is given by
\[
\rho(\psi) := \sum_i b_i^* \psi \otimes b_i, \tag{4.3}
\]
where \( \{b_i\} \subset H \) is a \( k \)-basis with dual basis \( \{b_i^*\} \subset \hat{H} \), and where according to (3.11) the (non-associative) "multiplication" \( \ast : H^* \otimes H^* \rightarrow H^* \) is given by
\[
\langle \varphi \ast \psi \mid a \rangle := \langle \varphi \otimes \psi \mid \hat{\Delta}(a) \rangle = \langle \varphi \otimes \psi \mid V \Delta(a) U \rangle, \quad a \in H, \varphi, \psi \in H^*. \tag{4.4}
\]
Now in ordinary Hopf algebras the coinvariants of \( H^* \) would precisely be the left integrals in \( \hat{H} \) [LS69]. Thus we propose

**Definition 4.2.** The coinvariants \( \lambda \in \hat{H}^{coH} \) are called \textit{left cointegrals} on \( H \) and we denote the space of left cointegrals as \( L := \hat{H}^{coH} \).

We will give some equivalent characterizations of left cointegrals in Section 5 and also in Section 7. Theorem 3.8 and Proposition 3.12 now immediately imply

**Theorem 4.3.** Let \( H \) be a f.d. quasi-Hopf algebra. Then \( \dim L = 1 \) and all nonzero left cointegrals on \( H \) are nondegenerate. In particular \( H \) is a Frobenius algebra and therefore the space of left (right) integrals in \( H \) is one dimensional.

**Proof.** The first statement follows from \( \dim H = \dim H^* \) and the fact that by Theorem 3.8 \( \mathcal{L} \otimes H \ni \lambda \otimes a \mapsto \lambda \cdot a := (S(a) \rightarrow \lambda) \in H^* \) provides an isomorphism of quasi-Hopf \( H \)-bimodules. For the second statement see the remarks above. \( \square \)

Note that according to (3.3), (3.10) and (4.3) the projection \( E : H^* \rightarrow \mathcal{L} \) is given for \( \varphi \in H^* \) and \( a \in H \) by
\[
\langle E(\varphi) \mid a \rangle = \sum_i \langle b_i^* \otimes \varphi \mid \hat{\Delta}(S^{-1}(q_R^1)aS^2(q_R^2b_i)S(\beta)) \rangle. \tag{4.5}
\]
Lemma 4.4. The transpose $E^T : H \to H$ is given by

$$E^T(a) = \sum_i (b^i \otimes \text{id})(\bar{\Delta}(S^{-1}(q_R^i)aS^2(q_R^i)b_i)S(\beta))$$

and provides a projection onto the space of right integrals $R \subset H$. Moreover, the dual pairing $\mathcal{L} \otimes R \ni \lambda \otimes r \mapsto \langle \lambda \mid r \rangle \in k$ is nondegenerate.

Proof. By Lemma 3.3 and the definition (3.10) $E^T(H) = \{r \in H \mid rS(a) = r\epsilon(a), \forall a \in H\}$, which is the space of right integrals $R \subset H$. Thus $E$ and $E^T$ induce the splittings $H^* = \mathcal{L} \oplus R^\perp$ and $H = R \oplus L^\perp$, respectively. 

Whether the projection onto the space of left integrals given by $[\mathcal{P}]$ is functorially related to our formula remains unclear at the moment.

5. Fourier Transformations

In this Section we first determine the modular automorphism of a nonzero left cointegral in terms of the modulus $\mu$ of $H$, just as for ordinary Hopf algebras. In particular, $H$ will be unimodular if the modular automorphism of any left cointegral on $H$ is given by the square of the antipode. We then develop a notion of Fourier transformation for quasi-Hopf algebras and show that these are given in terms of cointegrals by the same formula as for ordinary Hopf algebras. This will finally lead to a characterization of symmetric or semi-simple f.d. quasi-Hopf algebras just like in the coassociative case.

First note that by (3.4) and Proposition 3.4 $\mathcal{L}$ carries a unital representation of $H$ given by

$$a \triangleright \lambda := E(\lambda \leftarrow S^{-1}(a)), \quad a \in H, \lambda \in \mathcal{L}.$$  

Since $\dim \mathcal{L} = 1$ there must exist a unique $\gamma \in \Gamma(H)$ such that

$$a \triangleright \lambda = \gamma(a)\lambda, \quad \forall a \in H, \lambda \in \mathcal{L}.$$  (5.1)

Proposition 3.4 (v) then implies for all $a \in H, \lambda \in \mathcal{L}$

$$\lambda \leftarrow S^{-1}(a) = \gamma(a(1))(S(a(2)) \leftarrow \lambda) = S(a \leftarrow \gamma) \rightarrow \lambda.$$  (5.2)

Lemma 5.1. Let $H$ be a f.d. quasi-Hopf algebra with modulus $\mu$. Then $\gamma = \mu$ and the modular automorphism of any nonzero $\lambda \in \mathcal{L}$ is given by

$$\theta_\lambda(a) = S(S(a) \leftarrow \mu), \quad a \in H.$$  

In particular, $H$ is unimodular iff $\lambda(ab) = \lambda(bS^2(a))$ for all $\lambda \in \mathcal{L}$ and all $a, b \in H$.

Proof. By the defining relation of $\theta_\lambda$,

$$\lambda \leftarrow a = \theta_\lambda(a) \rightarrow \lambda, \quad \forall a \in H,$$

we conclude from (5.2) $\theta_\lambda(a) = S(S(a) \leftarrow \gamma)$. This implies $\gamma^{-1} \circ S^{-1} \circ \theta_\lambda = \epsilon \circ S = \epsilon$ and therefore $\mu \equiv \epsilon \circ \theta_\lambda^{-1} = \gamma^{-1} \circ S^{-1} = \gamma$. 

Corollary 5.2. Let $r \in H$ be a right integral. Then $ar = \mu^{-1}(a)r$ for all $a \in H$.

Proof. Clearly $ar$ is a right integral and for any $0 \neq \lambda \in \mathcal{L}$ we have $\langle \lambda \mid ar \rangle = \langle S(a) \triangleright \lambda \mid r \rangle = \mu^{-1}(a)\langle \lambda \mid r \rangle$, from which the statement follows by the nondegeneracy of the pairing $\mathcal{L} \otimes R \to k$. 

□
We now generalize the notion of a Fourier transformation. Since \( \dim \mathcal{L} = 1 \), any nonzero \( \lambda \in \mathcal{L} \) induces an identification \( \mathcal{L} \otimes H \cong H_\mu \) as quasi-Hopf bimodules, see Corollary 3.3. Thus, by Theorem 3.3, \( H_\mu \cong H^* \) as quasi-Hopf bimodules.

**Definition 5.3.** A Fourier transformation is a nonzero morphism of quasi-Hopf \( H \)-bimodules \( \mathcal{F} : H_\mu \to H^* \), i.e., by (4.2)-(4.4) a linear map satisfying for \( a, b \in H \), \( \psi \in H^* \) and \( T_\mu \in H \otimes H \) defined in Corollary 3.3:

\[
\mathcal{F}(ab) = S(b) \mathcal{F}(a) = \mathcal{F}(b) S^{-1}(a \leftarrow \mu^{-1}) \\
\psi \ast \mathcal{F}(a) = (\mathcal{F} \otimes \psi)(T_\mu \Delta(a)).
\]

We will see that given (5.4) the two conditions in (5.3) are actually equivalent. To this end, for \( \lambda \in H^* \) define \( \mathcal{F}_\lambda : H \to H^* \) and \( \mathcal{F}_\lambda' : H \to H^* \) by

\[
\mathcal{F}_\lambda(a) := S(a) \mathcal{F}(\lambda), \quad \mathcal{F}_\lambda'(a) := \lambda \leftarrow S^{-1}(a \leftarrow \mu^{-1}).
\]

Then by (5.3) any Fourier transformation \( \mathcal{F} \) satisfies \( \mathcal{F} = \mathcal{F}_\lambda = \mathcal{F}_\lambda' \), where \( \lambda := \mathcal{F}(1) \). Moreover, as for ordinary Hopf algebras, \( \lambda = \mathcal{F}(1) \) is a left cointegral. More precisely, generalizing results of [Nil94], we have

**Proposition 5.4.** Let \( H \) be a f.d. quasi-Hopf algebra with modulus \( \mu \) and let \( 0 \neq \lambda \in \hat{H} \). Then the following are equivalent

\[(i) \ \lambda \text{ is a left cointegral} \]
\[(ii) \ \mathcal{F}_\lambda \text{ is a Fourier transformation} \]
\[(iii) \ \rho(\mathcal{F}_\lambda(a)) = (\mathcal{F}_\lambda \otimes \text{id})(T_\mu \Delta(a)), \quad \forall a \in H \]
\[(iv) \ \rho(\mathcal{F}_\lambda'(a)) = (\mathcal{F}_\lambda' \otimes \text{id})(T_\mu \Delta(a)), \quad \forall a \in H \]

**Proof.** (i)\(\Rightarrow\)(ii): By Theorem 3.3 \( \mathcal{L} \otimes H \ni \lambda \otimes a \mapsto \mathcal{F}_\lambda(a) \in H^* \) is an isomorphism of quasi-Hopf \( H \)-bimodules, implying \( \mathcal{F}_\lambda : H_\mu \to H^* \) to be a Fourier transformation.

(ii)\(\Rightarrow\)(iii+iv): Holds by condition (5.4) in Definition 5.3.

(iii)\(\Rightarrow\)(i): Pick \( 0 \neq \chi \in \mathcal{L} \) and put \( f := \mathcal{F}_\chi^{-1} \circ \mathcal{F}_\lambda \). Since \( \mathcal{F}_\chi \) and \( \mathcal{F}_\chi' \) are right \( H \)-module maps, we get \( f(a) = T_\mu \Delta(f(a)), \forall a \in H \). Moreover, (iii) implies \( (f \otimes \text{id})(T_\mu \Delta(a)) = T_\mu \Delta(f(a)), \forall a \in H \), and in particular \( (f(1) \otimes 1)T_\mu = T_\mu \Delta(f(1)) \). Applying \( \epsilon \otimes \text{id} \) gives \( f(1) = \epsilon(f(1))1 \) and therefore \( \lambda = \epsilon(f(1))\chi \).

The implication (iv)\(\Rightarrow\)(i) follows similarly by considering \( f' := (\mathcal{F}_\chi')^{-1} \circ \mathcal{F}_\chi \) and noting \( f'(a) = a f'(1) \).

Next, we determine the Frobenius basis associated with a non-zero left cointegral \( \lambda \) on \( H \). By this we mean the unique solution \( \sum_i u_i \otimes v_i \equiv Q_\lambda \in \mathcal{H} \otimes H \) of

\[
\sum_i \lambda(a u_i) v_i = a = \sum_i u_i \lambda(v_i a), \quad \forall a \in H.
\]

Note that for \( \dim H < \infty \) and \( \lambda \in \hat{H}, Q_\lambda \in \mathcal{H} \otimes H \) as above exists if and only if \( \lambda \) is nondegenerate. (Choose a basis \( \{u_i\} \) of \( H \) and let \( \{v_i\} \) be the unique basis of \( H \) such that \( \lambda(v_i u_j) = \delta_{i,j} \). In this case the two conditions above are equivalent and we have \( \sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a \) for all \( a \in H \). Also note that

\[
\sum_i u_i \otimes v_i = \sum_i \theta_\lambda(v_i) \otimes u_i.
\]

If \( H \) is a f.d. Hopf algebra and \( \lambda \in \hat{H} \) is a nonzero left integral in \( \hat{H} \), then by results of [LS69], \( Q_\lambda = (S \otimes \text{id})(\Delta(r)) \), where \( r \in H \) is the unique right integral satisfying \( \langle \lambda \mid r \rangle = 1 \). For f.d. quasi-Hopf algebras, this result generalizes as follows:
Proposition 5.5. Let $H$ be a f.d. quasi-Hopf algebra and let $\lambda \in \mathcal{L}$ be a nonzero left cointegral on $H$. Then

$$\mathcal{F}_\lambda^{-1}(\psi) = (\text{id} \otimes \psi)(\bar{\Delta}(r)) \quad \text{and} \quad Q_\lambda = (S \otimes \text{id})(\bar{\Delta}(r)),$$

where $r \in R \subset H$ is the unique right integral satisfying $\langle \lambda \mid r \rangle = 1$.

Proof. Since $\mathcal{F}_\lambda(a) = \nu(\lambda \circ a)$, according to Theorem 3.8 and Lemma 1.4 the inverse $\mathcal{F}_\lambda^{-1}$ is given by

$$\mathcal{F}_\lambda^{-1}(\psi) = \langle E(\psi(0)) \mid r \rangle \psi(1) = \langle \psi(0) \mid r \rangle \psi(1) = (\text{id} \otimes \psi)(\bar{\Delta}(r)).$$

Here we have used $E^T(r) = r$ and the definition (3.16) applied to $(M, \Lambda) := (H, \Delta)$. Hence we conclude $(\text{id} \otimes S(a) \rightarrow \lambda)(\bar{\Delta}(r)) = a, \forall a \in H$, and therefore $\sum_i u_i \otimes v_i := (S \otimes \text{id})(\bar{\Delta}(r))$ satisfies $\sum_i u_i \lambda(v_i a) = a, \forall a \in H$. \hfill \Box

We conclude this section by characterizing symmetric or semi-simple f.d. quasi-Hopf algebras quite analogously as in the Hopf case.

Proposition 5.6. A f.d. quasi-Hopf algebra $H$ is

(i) unimodular if and only if one (and hence all) nonzero left (right) integrals are $S$-invariant;

(ii) symmetric if and only if it is unimodular and $S^2$ is an inner automorphism.

Proof. (i): If there is a nonzero $S$-invariant integral in $H$, then it is two-sided, hence all integrals are two-sided and $S$-invariant (by the one-dimensionality of the space of left/right integrals), implying $\mu = \epsilon$. Conversely, if $H$ is unimodular then all integrals are two-sided and by Lemma 5.1 the modular automorphism of any nonzero $\lambda \in \mathcal{L}$ is given by $S^2$. Thus by (3.10) the Frobenius basis $\sum_i u_i \otimes v_i \equiv Q_\lambda \in H \otimes H$ satisfies $\sum_i u_i \otimes v_i = \sum_j S^2(v_j) \otimes u_j$ and therefore $\sum_i \epsilon(u_i)v_i = \sum_i u_j \epsilon(v_j)$. But with $(\epsilon \otimes \text{id})(\bar{\Delta}(r)) = r$ and $(\text{id} \otimes \epsilon)(\bar{\Delta}(r)) = S^{-1}(\beta)r\alpha = r$ (since $r$ is a two-sided integral and $\epsilon(S^{-1}(\beta)\alpha) = \epsilon(\beta\alpha) = 1$) Proposition 5.7 implies $r = \epsilon(u_i)v_i = u_i \epsilon(v_i) = S(r)$.

(ii): If $H$ is symmetric then it is unimodular and all modular automorphisms are inner. Hence $S^2$ is inner. Conversely, if $H$ is unimodular and $S^2$ is inner pick $0 \neq \lambda \in \mathcal{L}$ and $g \in H$ invertible such that $g x g^{-1} = S^2(x), \forall x \in H$. Then $\tau := g^{-1} \lambda$ is a nondegenerate trace on $H$. \hfill \Box

In [Pan98] F. Panaite has shown recently, that a quasi-Hopf algebra is semi-simple Artinian if and only if it contains a normalized left (or right) integral. For finite dimensional quasi-Hopf algebras we have in addition (the equivalence (v) in Theorem 5.7 is essentially also due to [Pan98])

Theorem 5.7. For a f.d. quasi-Hopf algebra $H$ the following are equivalent:

(i) $H$ is semi-simple.

(ii) $H$ has a normalized left (or right) integral.

(iii) The left cointegral $\lambda_e := E(\epsilon) \equiv \sum_i S^2(b_i)S(\beta)\alpha \rightarrow b^i \in \mathcal{L}$ is nonzero.

(iv) $H$ has a Haar integral $e$.

(v) $H$ is a separable $k$-algebra.

Moreover, in this case the Haar integral $e$ satisfies $\langle \lambda_e \mid e \rangle = 1$.

\footnote{where as above $b_i$ denotes a basis of $H$ with dual basis $b^i \in \hat{H}$}
Proof. (i)⇒(ii): see Theorem 2.3 in [Pan98].

(ii)⇒(iii): Let \( r \in R \) be a normalized right integral, then \( \langle E(\epsilon) \mid r \rangle = \epsilon(r) = 1 \).
Moreover, to verify the above formula for \( \lambda_e \), we use (4.5) and \((\epsilon \otimes \text{id})(\Delta(x)) = S^{-1}(\beta)x\alpha\) to obtain
\[
\lambda_e(a) = \sum_i (b^i | S^{-1}(q^1_R(\beta)aS^2(q^2_Rb_i)S(\beta)\alpha)
= \sum_i (q^1_RS^{-1}(q^1_R(\beta)aS^2(b_i)S(\beta)\alpha)
= \sum_i (b^i | aS^2(b_i)S(\beta)\alpha)
\]
and therefore indeed \( \lambda_e = \sum S^2(b_i)S(\beta)\alpha \mapsto b^i \). Here we have used (2.6) and the identity \( \sum b^i \otimes yb_i = \sum b^i \leftarrow y \otimes b_i \).

(iii)⇒(iv): If \( \lambda_e \neq 0 \) there exists a unique right integral \( e \in H \) such that \( \lambda_e(e) = 1 \) yielding \( \epsilon(e) = \epsilon(E^T(e)) = 1 \). Thus \( e \) is normalized. Moreover, using again the identity \( \sum b^i \otimes yb_i = \sum b^i \leftarrow y \otimes b_i \) one easily verifies that \( \lambda_e(aS^2(b)) = \lambda(ab) \), i.e. the modular automorphism of \( \lambda_e \) is given by \( S^2 \). Hence, by Lemma 5.4, \( H \) is unimodular and \( e \) is the Haar integral in \( H \).

(iv)⇒(v): Following [Pan98], if \( e \) is a normalized left integral then
\[
P := (\text{id} \otimes S)(gR\Delta(e)(\beta \otimes 1))
\]
provides a separating idempotent in \( H \langle H \rangle_\text{op} \), i.e. \( P_1^\dagger P_2^\dagger = 1 \) and \((a \otimes 1)P = P(1 \otimes a)\) for all \( a \in H \).

(v)⇒(i): This is a standard textbook exercise, see e.g. [Pie82]. \( \square \)

Let us conclude with an explicit formula for \( \lambda_e \) in the case of \( H \) being a multimatrix algebra, i.e. \( H \cong \otimes_i \text{Mat}_k(n_I) \). In this case \( H \) is symmetric, i.e. it admits a non degenerate trace. Hence Proposition 5.6 applies and we may pick \( g \in H \) invertible such that \( g(x)g^{-1} = S^2(x) \), \( \forall x \in H \). Put \( c_I := tr_I(g^{-1}S(\beta)\alpha) \), where \( tr_I \) denotes the standard trace on \( \text{Mat}_k(n_I) \), and denote \( L(a) \), \( R(b) \) the operators of left multiplication with \( a \) and right multiplication with \( b \), respectively, on \( H \). Then
\[
\lambda_e(a) = \sum_i (b^i | agb_i g^{-1}S(\beta)\alpha)
= Tr(L(ag) \circ R(g^{-1}S(\beta)\alpha))
= \sum_i c_I tr_I(ag),
\]
where \( Tr \) denotes the standard trace on \( \text{End}_kH \) and where we have used that \( Tr(L(x)R(y)) = \sum_I tr_I(x)tr_I(y) \). By the nondegeneracy of \( \lambda_e \) we conclude \( c_I \neq 0 \), \( \forall I \). Proposition 5.6 then implies for the Haar integral \( e \in H \)
\[
(S \otimes \text{id})(\Delta(e)) = \sum c_I^\mu e_I^\mu g_I^{-1} \otimes e_I^\nu,
\]
where the \( e_I^\mu \)'s denote the matrix units in \( \text{Mat}_k(n_I) \) and \( g_I \) the component of \( g \) in the matrix block \( \text{Mat}_k(n_I) \).
6. The Comodulus and Radford’s Formula

Recall that if $H$ is a finite dimensional Hopf algebra we may identify the modulus of $\hat{H}$ with an element $u \in H$, i.e.

$$\lambda \psi = \psi(u) \lambda$$

for all left cointegrals $\lambda \in \mathcal{L}$ and all $\psi \in \hat{H}$. If $r \in H$ satisfies $\lambda(r) = 1$ we get

$$u = (\lambda \otimes \text{id})(\Delta(r)).$$

Choosing $r$ to be a right integral, $r \in R$, this is the definition which appropriately generalizes to quasi-Hopf algebras:

**Definition 6.1.** Let $H$ be a f.d. quasi-Hopf algebra and let $\lambda \in \mathcal{L}$ and $r \in R$ satisfy $\langle \lambda | r \rangle = 1$. Then we call $u := (\lambda \otimes \text{id})(\bar{\Delta}(r)) \equiv (\lambda \otimes \text{id})(V\Delta(r)U) \in H$ the comodulus of $H$.

The good use of this definition stems from the fact that it gives rise to a generalization of Radford’s Formula [Rad76] expressing $S^4$ as a composition of the inner and coinner automorphisms, respectively, induced by $u^{-1}$ and $\mu$.

**Proposition 6.2.** (Radford’s Formula) Let $H$ be a f.d. quasi-Hopf algebra with modulus $\mu \in \hat{H}$ and comodulus $u \in H$. For $a \in H$ put $S_\mu(a) := S(a) - \mu$ and $v := (\text{id} \otimes \lambda \circ S)(\bar{\Delta}(r))$. Then

$$u^{-1} = S^2(v) = S^{-2}_\mu(v) \quad (6.1)$$

$$u^{-1}a u = S^2(S^{-2}_\mu(a)), \quad \forall a \in H. \quad (6.2)$$

**Proof.** Using [3.14] and Corollary 5.2 we have for all $a, b \in H$

$$[1 \otimes S^{-1}(a)] \bar{\Delta}(r) [1 \otimes S(b)] = [a - \mu \otimes 1] \bar{\Delta}(r) [b \otimes 1]$$

and therefore

$$au = \langle \lambda \otimes \text{id} | [S_\mu(a) \otimes 1] \bar{\Delta}(r) \rangle$$

$$= \langle \lambda \otimes \text{id} | \bar{\Delta}(r) [S(S^{-2}_\mu(a)) \otimes 1] \rangle$$

$$= uS^2(S^{-2}_\mu(a)),$$

where we have used that according to Lemma 5.1 the modular automorphism of $\lambda$ is given by $\theta_\lambda = S \circ S_\mu$. We are left to show that $S^{-2}_\mu(v)$ is a left inverse of $u$, implying by the above calculation $S^2(v)$ to be a right inverse and therefore $u^{-1} = S^2(v) = S^{-2}_\mu(v)$. First note that the second line of the above calculation also gives

$$au = (\mathcal{F}_\lambda(S^2_\mu(a)) \otimes \text{id} | \bar{\Delta}(r))$$

for all $a \in H$. Hence

$$S^{-2}_\mu(v)u = (\mathcal{F}_\lambda(v) \otimes \text{id} | \bar{\Delta}(r))$$

$$= (\lambda \otimes \text{id})(Q_\lambda) = 1$$

where we have used that by Proposition 5.5 $\nu = \mathcal{F}^{-1}_\lambda(\lambda \circ S)$ and that $Q_\lambda := (S \otimes \text{id})(\bar{\Delta}(r)) \in H \otimes H$ provides the Frobenius basis of $\lambda$. □
To get more similarity with Radford’s original formulation in \[Rad76\] we put \( f_\mu := (\mu \otimes \text{id})(f) \) and use \( \ref{eq:mu} \) to get \( S_\mu^b(a) = f_\mu^{-1} S(\mu^{-1} \rightarrow (S(a) \leftarrow \mu)) f_\mu \). Writing \( b = \mu^{-1} \rightarrow (S(a) \leftarrow \mu) \) we conclude

**Corollary 6.3.** Under the conditions of Proposition \ref{prop:main} we have for all \( b \in H \)

\[
S^4(b) = S^3(f^{-1}_{\mu} S(u)[(\mu \rightarrow b) \leftarrow \mu^{-1}] S(u^{-1}) S^3(f_\mu)
\]

7. Cocentral Bilinear Forms

Recall \[Abe80\] that a Hopf algebra \( H \) is cosemisimple iff it admits a bilinear form \( \Sigma : H \otimes H \rightarrow k \) satisfying for all \( a, b \in H \)

\[
(id \otimes \Sigma)(\Delta(a) \otimes b) = (\Sigma \otimes \text{id})(a \otimes \Delta(b)), \tag{7.1}
\]

\[
\Sigma \circ \Delta = \epsilon. \tag{7.2}
\]

In particular, if \( \lambda \in \hat{H} \) is a normalized left cointegral on \( H \) (i.e. \( \lambda(1) = 1 \) and \( \lambda \rightarrow a = \lambda(a)1, \forall a \in H \)), then such a \( \Sigma \) is obtained by

\[
\Sigma(a \otimes b) := \lambda(aS(b)).
\]

In this case \( \Sigma \) is right invariant, i.e. for all \( a, b, c \in H \)

\[
\Sigma((a \otimes b)\Delta(c)) = \Sigma(a \otimes b)\epsilon(c).
\]

Conversely, any right invariant bilinear form \( \Sigma \) is obtained this way, where \( \lambda(a) := \Sigma(a \otimes 1) \), and where \( \Sigma \) satisfies \( \ref{eq:7.1} \) iff \( \lambda \in \hat{H} \) is a left cointegral on \( H \). Moreover, in this case the normalization condition \( \ref{eq:7.2} \) is equivalent to \( \Sigma(1 \otimes 1) \equiv \lambda(1) = 1 \).

In this Section we generalize these relations to f.d. quasi-Hopf algebras. Our motivation is two-fold. On the one hand, this will provide a characterization of left cointegrals which is less implicit than Definition \ref{def:ci} and more reminiscent to ordinary Hopf algebra theory. On the other hand, although we have no sensible notion of cosemisimplicity for quasi-Hopf algebras, we precisely need the analogues of \( \ref{eq:7.1} \) and \( \ref{eq:7.2} \) to prove in Theorem \ref{thm:CI} semisimplicity of the diagonal crossed product \( A \bowtie \hat{H} \) constructed in \[HN\], for any semisimple two-sided \( H \)-comodule algebra \( A \). In particular this will imply the quantum double \( D(H) \) of a f.d. quasi-Hopf algebra \( H \) to be semisimple, iff \( H \) is semisimple and admits a left cointegral \( \lambda_0 \in \mathcal{L} \) satisfying the normalization condition \( \lambda_0(\beta S(a)) = 1 \).

We start with first preparing a more general formalism

**Definition 7.1.** Let \((K, \Lambda_K)\) and \((M, \rho_M)\) be a left and a right quasi-Hopf \( H \)-bimodule, respectively. We call a bilinear form \( \Sigma : K \otimes M \rightarrow k \) \( H \)-biinvariant, if for all \( a \in H, \ k \in K, \ m \in M \)

\[
\Sigma ((k \otimes m) \cdot \Delta(a)) = \epsilon(a)\Sigma(k \otimes m) = \Sigma(\Delta(a) \cdot (k \otimes m)).
\]

We call \( \Sigma \) \( H \)-cocentral, if

\[
(id \otimes \Sigma)(\phi \cdot [\Lambda_K(k) \otimes m] \cdot \phi^{-1}) = (\Sigma \otimes \text{id})(\phi^{-1} \cdot [k \otimes \rho_M(m)] \cdot \phi)
\]

If the \( H \)-(co)actions are understood we also just use the words biinvariant and cocentral. To motivate our terminology note that if \( H \) is a Hopf algebra, then \((K \otimes M)^\wedge \) naturally becomes an \( H \)-bimodule by putting

\[
(\psi \cdot \Sigma \cdot \varphi)(k \otimes m) := (\varphi \otimes \Sigma \otimes \psi)(\Lambda_K(k) \otimes \rho_M(m)), \ \varphi, \psi \in \hat{H}, \ \Sigma \in (K \otimes M)^\wedge.
\]

In this case \( \Sigma \) is \( H \)-cocentral iff \( \varphi \cdot \Sigma = \Sigma \cdot \varphi, \forall \varphi \in \hat{H} \).
Let now $K^* = \text{Hom}_k(K,k)$ be the $H$-bimodule with $H$-actions defined in (3.10). Recall from Proposition 3.12 that if $H$ or $K$ is finite dimensional, then there exists a right $H$-coaction $\rho_{K^*} : K^* \to K^* \otimes H$ making $(K^*, \rho_{K^*})$ a right quasi-Hopf $H$-bimodule. We are aiming to characterize cocentral biinvariant elements $\Sigma \in (K \otimes M)^\wedge$ in terms of quasi-Hopf $H$-bimodule morphisms $f : M \to K^*$. To this end we need

**Lemma 7.2.** Let $(K, \Lambda)$ be a left quasi-Hopf $H$-bimodule and (assuming $K$ or $H$ finite dimensional) let $(K^*, \rho)$ denote its dual right quasi-Hopf $H$-bimodule, see Proposition 7.1. Then for all $k \in K$ and $\psi \in K^*$

$$(\text{id} \otimes \psi)(q_R \cdot \Lambda(k) \cdot p_R) = (k \otimes \text{id})(q_L \cdot \rho(\psi) \cdot p_L).$$

**Proof.** We first show the identities

$$q_R = [q_L^2 \otimes 1] V \Delta(S^{-1}(q_L^1)), \quad \text{(7.3)}$$

$$p_R = \Delta(S(p_L^1)) U [p_L^2 \otimes 1]. \quad \text{(7.4)}$$

Noting that in $H_{op}$ the roles of $U$ and $V$ as well as of $q_R$ and $p_R$ and of $q_L$ and $p_L$ interchange reduces (7.4) to (7.3). For the left hand side of (7.3) we obtain, using (3.13) and (2.14)

$$[q_L^2 \otimes 1] V \Delta(S^{-1}(q_L^1)) = (S^{-1} \otimes S^{-1})(f^{21} \Delta^{op}(q_L^1) p_R^{21} [S(q_L^2) \otimes 1])$$

$$= (S^{-1} \otimes S^{-1})(S \otimes S)(\Delta(\bar{X}^i)) \gamma^{21} \Delta^{op}(\bar{Y}^i) p_R^{21} [S(\bar{Z}^k) \otimes 1])$$

$$= (S^{-1} \otimes S^{-1}) \left((S \otimes S)(X^j \otimes Y^j \Delta(\bar{X}^k \bar{X}^i)) [a \otimes a] \right.$$

$$\left.[\bar{Z}^k \otimes Z^j \bar{Y}^k] \Delta^{op}(\bar{Y}^i) [\bar{Y}^m \otimes \bar{X}^m] [\beta S(\bar{Z}^j \bar{Z}^m) \otimes 1] \right)$$

$$= (S^{-1} \otimes S^{-1}) \sigma(\Psi), \quad \text{(7.5)}$$

where $\sigma : H^{\otimes 5} \to H^{\otimes 2}$ denotes the map

$$a \otimes b \otimes c \otimes d \otimes e \mapsto S(a) \alpha d \beta S(e) \otimes S(b) \alpha c,$$

and $\Psi \in H^{\otimes 5}$ is given by

$$\Psi := [\phi \otimes 1 \otimes 1] ([\Delta \otimes \text{id} \otimes \text{id}] (\phi^{-1}) \otimes 1) (\Delta \otimes \Delta \otimes \text{id})(\phi^{-1}) [1 \otimes 1 \otimes \phi^{-1}].$$

Now we use the pentagon equation for the last three factors of $\Psi$ to obtain

$$\Psi = [\phi \otimes 1 \otimes 1] ([\Delta \otimes \text{id} \Delta \otimes \text{id} \otimes \text{id}] (\phi^{-1}) (\Delta \otimes \text{id} \otimes \Delta)(\phi^{-1})$$

$$= (\text{id} \otimes \Delta \Delta \otimes \text{id} \otimes \text{id}) (\phi^{-1}) [\phi \otimes 1 \otimes 1] (\Delta \otimes \text{id} \otimes \Delta)(\phi^{-1}).$$

Using the antipode properties (2.3), under the evaluation of $\sigma$ the third factor may be dropped and the first factor may be replaced by $(\phi^{-1})^{145}$. Hence we get

$$\sigma(\Psi) = \sigma([\bar{X}^i \otimes 1 \otimes 1 \otimes \bar{Y}^i \otimes \bar{Z}^i] [\phi \otimes 1 \otimes 1])$$

$$= S(X^k) S(\bar{X}^i) \alpha \bar{Y}^i \beta S(\bar{Z}^i) \otimes S(Y^k) \alpha Z^k$$

$$= S(X^k) \otimes S(Y^k) \alpha Z^k,$$

where we have used (2.6). Thus we finally arrive at

$$(S^{-1} \otimes S^{-1})(\sigma(\Psi)) = X^k \otimes S^{-1}(\alpha Z^k) Y^k = q^R.$$

By (7.3) we have proved (7.3).
Using (2.3), (2.4) and the definition of $\rho$ given in (3.10), (3.11) and (3.16), one obtains

$$(k \otimes \text{id})(q_L \cdot \rho(1) \cdot p_L) = (S^{-1}(q_L) \cdot k \cdot S(p_L) \otimes \text{id}) \left( \left[ 1 \otimes q_L^2 \right] \cdot \rho(1) \cdot [1 \otimes p_L^2] \right)$$

$$= (\text{id} \otimes \rho) \left( \left[ q_L^2 \otimes 1 \right] \cdot S^{-1}(q_L) \cdot k \cdot S(p_L) \right) \cdot [p_L^2 \otimes 1]$$

$$= (\text{id} \otimes \rho)(q_R \cdot \Lambda(k) \cdot p_R).$$

This finishes the proof of Lemma 7.2. \(\square\)

**Theorem 7.3.** Let $(K, \Lambda_K)$ and $(M, \rho_M)$ be a left and a right quasi-Hopf $H$-bimodule, respectively.

(i) Then the assignment $f \mapsto \Sigma_f$, where

$$\Sigma_f(k \otimes m) := \langle f(m) | S^{-1}(a) \cdot k \cdot b \rangle,$$

provides a one-to-one correspondence between $H$-bimodule maps $f : M \to K^*$ and biinvariant elements $\Sigma_f \in (K \otimes M)^\wedge$. The inverse assignment is given by $\Sigma \mapsto f_\Sigma$, where

$$\langle f_\Sigma(m) | k \rangle := \langle p \cdot (k \otimes m) \cdot q \rangle$$

and where $p = p_L$ or $p = p_R$ and $q = q_L$ or $q = q_R$, $f_\Sigma$ being independent of these choices.

(ii) Assume in addition $H$ or $K$ finite dimensional. Then $\Sigma_f$ is cocentral if and only if $f$ is a morphism of quasi-Hopf $H$-bimodules, i.e. $(f \otimes \text{id}) \circ \rho_M = \rho_K \circ f$.

**Proof.** For $a, b \in H$ we have by (2.5)

$$\Sigma_f(\Delta(a) \cdot (k \otimes m) \cdot \Delta(b)) = \langle a_{(2)} \cdot f(m) \cdot b_{(2)} | S^{-1}(a_{(1)}a_{(1)} \cdot k \cdot b_{(1)}) \rangle$$

$$= \langle f(m) | S^{-1}(S(a_{(1)})a_{(2)}) \cdot k \cdot b_{(1)} \cdot b_{(2)} \rangle$$

$$= \epsilon(a)\epsilon(b)\Sigma_f(k \otimes m)$$

and from (2.6) we conclude for $p = p_{L/R}$ and $q = q_{L/R}$

$$\Sigma_f(p \cdot (k \otimes m) \cdot q) = \langle p^2 \cdot f(m) \cdot q^2 | S^{-1}(a)p^1 \cdot k \cdot q^1 \rangle$$

$$= \langle f(m) | S^{-1}(S(p^1) \cdot a \cdot p^2) \cdot k \cdot q^1 \cdot b \rangle$$

$$= \langle f(m) | k \rangle.$$

Thus, $\Sigma_f$ is biinvariant and $f_\Sigma = f$ independently of the choice of $q = q_{L/R}$ and $p = p_{L/R}$. Conversely, let $\Sigma \in (M \otimes K)^\wedge$ be biinvariant, then for all choices $q = q_{L/R}$ and $p = p_{L/R}$

$$\Sigma(p \cdot (k \otimes a \cdot m)) = \Sigma(p \cdot (S^{-1}(a) \cdot k \otimes m))$$

$$\Sigma((k \otimes m \cdot b) \cdot q) = \Sigma((k \otimes S(b) \otimes m) \cdot q)$$

by the identities (2.19) - (2.22). Hence, $f_\Sigma$ is an $H$-bimodule map for all choices of $q = q_{L/R}$ and $p = p_{L/R}$:

$$\langle f_\Sigma(a \cdot m \cdot b) | k \rangle = \langle f \cdot (k \otimes a \cdot m \cdot b) \cdot q \rangle$$

$$= \Sigma(p \cdot (S^{-1}(a) \cdot k \cdot S(b) \otimes m) \cdot q)$$

$$= \langle a \cdot f_\Sigma(m) \cdot b | k \rangle.$$
Finally, if $\Sigma$ is biinvariant we may use the identities (2.23) and (2.24) to conclude
\[
\Sigma((k \cdot \beta \otimes m) \cdot q) = \Sigma((k \otimes m \cdot S^{-1}(\beta)) \cdot q) = \Sigma(k \otimes m)
\]
\[
\Sigma(p \cdot (S^{-1}(\alpha) \cdot k \otimes m)) = \Sigma(p \cdot (k \otimes \alpha \cdot m)) = \Sigma(k \otimes m)
\]
for all choices of $q = q_{L/R}$ and $p = p_{L/R}$. Hence, $\Sigma_{f_{\Sigma}} = \Sigma$ for all these choices and therefore $f_{\Sigma}$ is independent of these choices. This proves the first statement of Theorem 7.3. To prove the second statement we compute
\[
(id \otimes \Sigma_f)(\phi \cdot [\Lambda(k) \otimes m] \cdot \phi^{-1}) = \langle id \otimes Z^i \cdot f(m) \cdot Z^j | (X^i \otimes S^{-1}(\alpha))Y^i \cdot \Lambda(k) \cdot (\bar{X}^j \otimes \bar{Y}^j \beta) \rangle = \langle id \otimes f(m) | q_R \cdot \Lambda(k) \cdot p_R \rangle = \langle k \otimes id | q_L \cdot \rho_K \cdot (f(m)) \cdot p_L \rangle
\]
where in the last line we have used Lemma 7.2. On the other hand
\[
(id \otimes \Sigma_f)(\phi^{-1} \cdot [k \otimes \rho_M(m)] \cdot \phi) = \langle f(\bar{Y}^i \cdot m_{(0)} \cdot Y^j) | S^{-1}(\alpha)X^i \cdot k \cdot X^j \beta \rangle Z^i m_{(1)} Z^j
\]
\[
= \langle f(q_L \cdot m_{(0)} \cdot p_L) | k \rangle q_L^2 m_{(1)} p_L^2.
\]
Thus, $\Sigma_f$ is cocentral if and only if for all $m \in M$
\[
q_L \cdot \rho_K \cdot (f(m)) \cdot p_L = (f \otimes id)(q_L \cdot \rho_M(m) \cdot p_L).
\]
Clearly, (7.6) holds, if $f$ is an $H$-bimodule and an $H$-comodule map. Conversely, let $f$ be an $H$-bimodule map satisfying (7.6). Then by (2.24)
\[
\rho_K \cdot (f(m)) = [(S(p_L^1) \otimes 1)q_L \Delta(p_L^2)] \cdot \rho_K \cdot (f(m)) \cdot [\Delta(q_L^1)p_L(S^{-1}(q_L^1) \otimes 1)]
\]
\[
= [(S(p_L^1) \otimes 1)q_L] \cdot \rho_K \cdot (f(p_L^2 \cdot m \cdot q_L^2)) \cdot [p_L(S^{-1}(q_L^1) \otimes 1)]
\]
\[
= [S(p_L^1) \otimes 1] \cdot (f \otimes id)(q_L \cdot \rho_M(p_L^2 \cdot m \cdot q_L^2) \cdot p_L) \cdot [S^{-1}(q_L^1) \otimes 1]
\]
\[
= (f \otimes id)([S(p_L^1) \otimes 1]q_L \Delta(p_L^2)] \cdot \rho_M(m) \cdot [\Lambda(q_L^1)p_L(S^{-1}(q_L^1) \otimes 1)]
\]
\[
= (f \otimes id)(\rho_M(m))
\]
Hence $f$ is an $H$-comodule map. This proves the second statement of Theorem 7.3.

We now apply Theorem 7.3 to our theory of integrals by putting $(K, \Lambda_K) = (H, \Delta)$ and $(M, \rho_M) = (H_{\mu}, \rho_{\mu})$, see Corollary 7.3, where $\mu \in \Gamma(H)$ is the modulus of $H$. In this case, by Definition 5.3 and Proposition 4.3 the quasi-Hopf bimodule morphisms $H_{\mu} \rightarrow H^*$ are precisely our Fourier transformations $F_{\lambda}$, $\lambda \in \mathcal{L}$, given in (5.5). Thus, we conclude

**Corollary 7.4.** Let $H$ be a finite dimensional quasi-Hopf algebra with modulus $\mu$. Then the assignment $\lambda \mapsto \Sigma_\lambda := \Sigma_{f_{\lambda}}$, i.e.
\[
\Sigma_\lambda(a \otimes b) := \lambda(S^{-1}(\alpha)a \beta S(b)),
\]
provides a one-to-one correspondence between left cointegrals $\lambda \in \mathcal{L}$ and cocentral biinvariant forms $\Sigma : H \otimes H_{\mu} \rightarrow k$. The inverse assignment is given by $\Sigma \mapsto \lambda_\Sigma := f_{\Sigma}(1)$, i.e. (putting $p_{\mu} := p^1 \otimes p^2 \leftarrow \mu$)
\[
\lambda_\Sigma(\alpha) := \Sigma(p_{\mu}(a \otimes 1)q)
\]
which is independent of the choices $q = q_{L/R}$ and $p = p_{L/R}$. 
Corollary 7.4 shows that on a finite dimensional quasi-Hopf algebra the space of biinvariant cocentral forms $\Sigma : H \otimes H_\mu \to k$ is one dimensional. Moreover, we also have

**Corollary 7.5.** Let $H$ be a finite dimensional quasi-Hopf algebra with modulus $\mu$ and let $\gamma \in \Gamma(H)$. If there exists a nonzero cocentral biinvariant form $\Sigma : H \otimes H_\gamma \to k$, then $\gamma = \mu$.

**Proof.** Pick $0 \neq \lambda \in \mathcal{L}$ and put $f' := F_\lambda^{-1} \circ f_\Sigma$. Then $f' : H_\gamma \to H_\mu$ is a nonzero morphism of quasi-Hopf $H$-bimodules, whence $f'(a) = ba = (a \mapsto \gamma^{-1}\mu)b$ for all $a \in H$, where $b := f'(1) \neq 0$. Moreover, $T_\mu \Delta(b) \equiv \rho_\mu(f'(1)) = (f' \otimes \text{id})(T_\gamma) = (b \otimes 1)T_\gamma$. Applying $\epsilon \otimes \text{id}$ gives $b = \epsilon(b)1$ and therefore $\gamma = \mu$.

Putting $\gamma = \epsilon$ Corollary 7.4 in particular implies that nonzero cocentral biinvariant forms $\Sigma : H \otimes H \to k$ exist if and only if $H$ is unimodular.

Let us conclude with considering the normalization condition (7.2). We call a cocentral biinvariant $\Sigma : H \otimes H_\mu \to k$ normalized, if $\Sigma(1 \otimes 1) = 1$, which by the right $H$-invariance is equivalent to (7.2). If such a normalized $\Sigma$ exists, it is unique and we denote it by $\Sigma_0$. Correspondingly, we also call the associated left cointegral $\lambda_0$ normalized, which means $\lambda_0(S^{-1}(\alpha)\beta) = 1$.

It is instructive to look at the condition for the existence of $\lambda_0$ (and therefore $\Sigma_0$) in the case of $H$ being a multi-matrix algebra over an algebraically closed field $k$. In this case $\mu = \epsilon$ and we have to check, wether the nonzero left cointegral $\lambda_c$ of Theorem 5.7 (iii) satisfies $\lambda_c(S^{-1}(\alpha)\beta) = \lambda_c(\beta S(\alpha)) \neq 0$, which would give $\lambda_0 = \lambda_c(\beta S(\alpha))$ and therefore $\gamma = \epsilon$.

If $k$ is algebraically closed we may choose $g$ in (5.7) obeying $S(g) = g^{-1}$ to conclude

$$\lambda_c(\beta S(\alpha)) = \sum_I d_I d_I^* =: d_H,$$

where

$$d_I = \text{tr}_I(\beta S(\alpha)g), \quad d_I^* = \text{tr}_I(S(\beta S(\alpha)g)).$$

Note that in the ribbon quasi-Hopf scenario of [AC92] the $d_I$’s are precisely the quantum dimensions of the irreducible $H$-modules labelled by $I$. Thus one might call $d_H$ the quantum dimension of $H$, considered as an $H$-module with respect to the adjoint action. In summary we conclude

**Corollary 7.6.** A f.d. semisimple quasi-Hopf algebra $H$ over an algebraically closed field $k$ admits a normalized left cointegral $\lambda_0$ (and therefore a normalized cocentral biinvariant form $\Sigma_0 : H \otimes H \to k$), if and only if the quantum dimension of $H$ with respect to the adjoint action on itself is nonzero.

8. Semisimplicity of Diagonal Crossed Products

As an application we will now prove a Maschke type Theorem for diagonal crossed products $A \rhd_\delta \hat{H}$ as defined in [HN, HN99]. We recall that by an twosided $H$-comodule algebra $(A,\delta,\Psi)$ associated with a quasi-Hopf algebra $H$ we mean an

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4 $S(g)g$ being a central $S$-invariant invertible element in $H$ it has a central $S$-invariant invertible square root $z \in Z(H)$. Thus, $g' := z^{-1}g$ satisfies $S(g') = g'^{-1}$.

5 In our description the $d_I$’s are not yet uniquely fixed, however the products $d_I d_I^*$ are so.
algebra $A$ together with an algebra map $\delta : A \to H \otimes A \otimes H$ and an invertible element $\Psi \in H \otimes H \otimes A \otimes H \otimes H$ satisfying (denoting $\text{id} \equiv \text{id}_H$, $1 \equiv 1_H$)

$$
(\text{id} \otimes \delta \otimes \text{id})(\delta(a)) \Psi = \Psi (\Delta \otimes \text{id}_A \otimes \Delta)(\delta(a)), \quad \forall a \in A, \quad (8.1a)
$$

$$
(1 \otimes \Psi \otimes 1)(\text{id} \otimes \Delta \otimes \text{id}_A \otimes \Delta \otimes \text{id})(\Psi)(\phi \otimes 1_A \otimes \phi^{-1})
= (\text{id} \otimes \text{id} \otimes \delta \otimes \text{id} \otimes \text{id})(\Psi)(\Delta \otimes \text{id} \otimes \text{id}_A \otimes \text{id} \otimes \Delta)(\Psi), \quad (8.1b)
$$

$$
(\epsilon \otimes \text{id}_A \otimes \epsilon) \circ \delta = \text{id}_A, \quad (8.1c)
$$

$$
(\text{id} \otimes \epsilon \otimes \text{id}_A \otimes \epsilon \otimes \text{id})(\Psi) = (\epsilon \otimes \text{id} \otimes \text{id}_A \otimes \text{id} \otimes \epsilon)(\Psi) = 1 \otimes 1_A \otimes 1, \quad (8.1d)
$$

We call $(\delta, \Psi)$ a two-sided $H$-coaction on $A$ and denote

$$
\delta(a) = a_{(-1)} \otimes a_{(0)} \otimes a_{(1)}
$$

$$
\varphi \triangleright a \triangleleft \psi := (\psi \otimes \text{id}_A \otimes \varphi)(\delta(a)), \quad a \in A, \varphi, \psi \in \hat{H},
$$

$$
\Omega := (h^{-1})^{21}(S^{-1} \otimes S^{-1} \otimes \text{id}_A \otimes \text{id} \otimes \text{id})(\Psi), \quad (8.2)
$$

with $h$ defined in (2.14).

**Definition 8.1.** [HN] Given a two-sided coaction $(\delta, \Psi)$ of a quasi-Hopf algebra $H$ on a unital algebra $A$ we define the right diagonal crossed product $A \bowtie \hat{H}$ to be the vector space $A \otimes \hat{H}$ with multiplication rule

$$(a \otimes \varphi)(b \otimes \psi) := ab_{(0)}\Omega^3 \otimes [(\Omega^2 S^{-1}(b_{(1)}) \triangleright \varphi \triangleleft b_{(1)} \Omega^4)(\Omega^1 \triangleright \psi \triangleleft \Omega^5)]
$$

$$
= \big[a (\varphi_{(1)} \triangleright b \triangleleft S^{-1}(\varphi(3))) \Omega^3\big] \otimes \big[(\Omega^2 \triangleright \varphi(2) \triangleleft \Omega^4)(\Omega^1 \triangleright \psi \triangleleft \Omega^5).\big]
$$

where the second line holds if $\hat{H}$ has a coalgebra structure dual to the algebra $H$, in particular if $H$ is finite dimensional.

In [HN] it has been shown that $A \bowtie \hat{H}$ is a unital algebra containing $A \equiv A \otimes \mathbf{1}$ as a unital subalgebra.\footnote{The proof in \[HN\] straightforwardly generalizes to infinite dimensions, in which case the generating matrix $R$ above has to be viewed as a map $R : H \to A \bowtie \hat{H}$.} Defining the generating matrix $R \in H \otimes (A \bowtie \hat{H})$ by $R := \sum_i b_i \otimes (1 \otimes b')$, where $\{b_i\}$ is a basis in $H$ with dual basis $\{b'_i\} \subset \hat{H}$, then $R$ obeys $(\epsilon \otimes \text{id})(R) = 1_A \otimes 1$ and

$$
R[1 \otimes a] = [a_{(1)} \otimes a_{(0)}] R [S^{-1}(a_{(-1)}) \otimes 1_A], \quad (8.3)
$$

$$
R^{13} R^{33} = [\Omega^2 \otimes \Omega^3 \otimes \Omega^3] (\Delta \otimes \text{id})(R) [\Omega^2 \otimes \Omega^3 \otimes 1_A]. \quad (8.4)
$$

Moreover, $A \bowtie \hat{H}$ is the unique smallest unital algebra extension $B \subset A \equiv A \otimes \mathbf{1}$ such that there exists $R \in H \otimes B$ satisfying $(\epsilon \otimes \text{id})(R) = 1$ together with (8.3) and (8.4), see [HN].

To prove the following Theorem we rely on the existence of a normalized cocentral biinvariant form $\Sigma : H \otimes H \to k$, which is why by Corollary [7.7] we require $H$ unimodular. However, we don’t know whether these conditions are also necessary.

**Theorem 8.2.** Let $H$ be a f.d. unimodular quasi-Hopf algebra and let $(A, \delta, \Psi)$ be a twosided $H$-comodule algebra. If $A$ is semisimple Artinian and if $H$ admits a normalized left cointegral $\lambda_0 \in L$, then the diagonal crossed product $A \bowtie H$ is semisimple Artinian.
Proof. We show that for any two \((A \bowtie \hat{H})\)-modules \(V,W\), where \(W \subset V\) is a submodule, there exists a \((A \bowtie \hat{H})\)-linear surjection \(\tilde{p} : V \to W\). Denoting the canonical embedding \(i : W \hookrightarrow V\) and \(R_W = (id \otimes \pi_W)(R) \in H \otimes \text{End}(W)\) and \(R_V = (id \otimes \pi_V)(R)\) it therefore suffices to find a surjection \(\tilde{p} : V \to W\) satisfying

\[
\tilde{p} \circ i = id_W, \quad (8.5)
\]

\[
\tilde{p} \circ a = a \circ \tilde{p}, \quad \forall a \in A, \quad (8.6)
\]

\[
(id \otimes \tilde{p}) \circ R_V = R_W \circ (id \otimes \tilde{p}), \quad (8.7)
\]

where at the l.h.s. and r.h.s. of (8.6) we have used the shortcut notation \(a \equiv \pi_V(a) \in \text{End}(V)\) and \(a \equiv \pi_W(a) \in \text{End}(W)\), respectively. This notation will also be used frequently below. We proceed as follows. Viewing \(V\) and \(W\) as \(A\)-modules, the semisimplicity of \(A\) implies the existence of an \(A\)-linear surjection \(p : V \to W\), satisfying (8.3) and (8.4). Denoting \(\Omega := \Omega^{-1}\), we now define the map \(\tilde{p} : V \to W\) in terms of \(p\) by

\[
\tilde{p} := (\Sigma_0 \otimes id_W) \circ (\Omega^4 \otimes \Omega^5 \otimes \Omega^3) \circ R_W^{13} \circ (id \otimes id \otimes p) \circ R_V^{23} \circ (\Omega^2 \otimes \Omega^1 \otimes id_V),
\]

where \(\Sigma_0\) is the normalized biinvariant cocentral form associated with \(\lambda_0\). To show (8.7) note that \(R_V \circ (1 \otimes i) = (1 \otimes i) \circ R_W\), since by assumption the embedding \(i\) is \(A \bowtie \hat{H}\)-linear. Using \(p \circ i = id_W\) this implies

\[
R_W^{13} \circ (id \otimes id \otimes p) \circ R_V^{23} \circ (id \otimes id \otimes i) = R_W^{13} \circ R_V^{23}
\]

and therefore

\[
\tilde{p} \circ i = (\Sigma_0 \otimes id_W) \circ (\Omega^4 \otimes \Omega^5 \otimes \Omega^3) \circ R_W^{13} \circ (\Omega^2 \otimes \Omega^1 \otimes id_W) = (\Sigma_0 \otimes id_W) \circ (\Delta \otimes id)(R_W) \equiv (\epsilon \otimes id_W)(R_W) = id_W,
\]

where we have used (8.4). Then the normalization condition \(\Sigma_0 \circ \Delta = \epsilon\) and finally the identity \((\epsilon \otimes id)(R) = 1 \otimes 1\). Thus we have proven (8.3).

\(A\)-linearity (8.6) follows from (8.3) and biinvariance of \(\Sigma_0\), since one computes

\[
\tilde{p} \circ a = (\Sigma_0 \otimes id_W) \circ (\cdot \cdot \cdot) \circ (1 \otimes 1 \otimes a) = (\Delta(S^{-1}(a_{(-1)})) \cdot \Sigma_0 \cdot \Delta(a_{(1)})) \circ (\cdot \cdot \cdot) = a \circ \tilde{p}.
\]

We are left to show (8.7). For the l.h.s. we get

\[
(id \otimes \tilde{p}) \circ R_V = (\Sigma_0 \otimes id \otimes id_W) \circ (\Omega^4 \otimes \Omega^5 \otimes id \otimes \Omega^3) \circ R_W^{14} \circ (id^3 \otimes p) \circ R_V^{24} \circ R_W^{14} \circ (\Omega^2 \otimes \Omega^1 \otimes id \otimes id_V) = (\Sigma_0 \otimes id \otimes id_W) \circ (\Omega^4 \Omega^3 \otimes \Omega^5 \Omega^4 \otimes \Omega^3 \Omega^1(0_{(0)})) \circ R_W^{14} \circ (id^3 \otimes p) \circ (\Delta \otimes id)(R_V) \equiv (S^{-1}(\Omega^3 \Omega^1(0_{(0)})) \Omega^2 \otimes \Omega^1 \otimes \Omega^1 \otimes id_V) \circ (id^3 \otimes p) \circ (\Delta \otimes id)(R_V) \equiv (\phi^{-1} \otimes id_W) \circ (\Omega^4 \Delta(\Omega^5) \otimes \Omega^3) \circ R_W^{14} \circ (id^3 \otimes p) \circ (\Delta \otimes id)(R_V) \equiv (\phi \otimes id_V).
\]

(8.8)
Here we have used \([8.3, 8.4]\) and the identity
\[
\Omega^1 \otimes \Omega^2 \bar{\Omega}^1 \otimes S^{-1}(\Omega^3_{(-1)}) \bar{\Omega}^2 \otimes \bar{\Omega}^3 \Omega^3_{(0)} \otimes \bar{\Omega}^4 \Omega^3_{(1)} \otimes \bar{\Omega}^5 \Omega^4 \otimes \bar{\Omega}^5
\]
\[
= \left[ (\Delta^{op}(\Omega^1) \otimes \bar{\Omega}^2) \phi^{21} (\Omega^1 \otimes \Delta^{op}(\Omega^2)) \right] \otimes \bar{\Omega}^3 \Omega^3
\]
\[
\otimes \left[ (\Delta(\Omega^4) \otimes \Omega^5) \phi^{-1} (\bar{\Omega}^4 \otimes \Delta(\Omega^5)) \right]
\]
- following from \([8.2], (8.1b)\) and \((8.16)\) - and the biuniqueness of \(\Sigma_0\). A similar calculation yields for the r.h.s. of \((8.7)\)
\[
R_W \circ (id \otimes \phi) = (id \otimes \Sigma_0 \otimes id_W) \circ (\phi \otimes id_W) \circ (\Delta(\bar{\Omega}^1) \otimes \bar{\Omega}^5 \otimes \bar{\Omega}^3)
\]
\[
\circ \left[ (\Delta \otimes id)(R_W) \right]^{124} \circ (id^3 \otimes \phi) \circ R^{34}_{V} \circ (\Delta(\Omega^2) \otimes \bar{\Omega}^1 \otimes id_V) \circ (\phi^{-1} \otimes id_V)
\]
Comparing \((8.8)\) and \((8.9)\), both expressions coincide since \(\Sigma_0\) is cocentral. This proves \((8.7)\) and therefore concludes the proof of Theorem 8.2.

\[\square\]

Theorem 8.2 and Theorem 7.3 imply

**Corollary 8.3.** Let \(H\) be a f.d. semisimple unimodular quasi-Hopf algebra, admitting a normalized cointegral \(\lambda_0\), i.e. \(\lambda_0(\beta S(\alpha)) = 1\). Then the quantum double \(D(H) = H \bowtie \bar{H}\) of \(H\) is simple.

**Conjecture:** More generally, for \(\lambda \in \mathcal{L}\) and \(r \in R\), we expect \((\beta \mapsto \lambda \bowtie r) \in D(H)\) to be a left integral, at least if \(H\) is unimodular.\(^7\) Moreover, we expect \(D(H)\) to be always unimodular as in the Hopf algebra case. Note that the counit on \(D(H)\) is given by \(\epsilon_{\beta}(\psi \bowtie a) = \psi(S^{-1}(\alpha)) \epsilon(a)\). Hence, under the conditions of Corollary 8.3 the Haar integral in \(D(H)\) should be given by \((\beta \mapsto \lambda_0 \bowtie e)\).

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\[\text{\footnote{Here we have used the convention } D(H) = \bar{H} \bowtie H, \text{ see } [45].}\]
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