THE DISK COMPLEX AND TOPOLOGICALLY MINIMAL SURFACES IN THE 3-SPHERE

MARION CAMPISI AND LUIS TORRES

ABSTRACT. We show that the disk complex of a genus $g > 1$ Heegaard surface for the 3-sphere is homotopy equivalent to a wedge of $(2g - 2)$-dimensional spheres. This implies that genus $g > 1$ Heegaard surfaces of the 3-sphere are topologically minimal with index $2g - 1$.

1. INTRODUCTION

Let $\Sigma$ be a compact, connected, orientable surface of genus $g > 1$, properly embedded in a compact, orientable 3-manifold $M$. The disk complex $\Gamma(\Sigma)$ is defined to be the simplicial complex such that:

- vertices correspond to isotopy classes of curves bounding non-trivial compressing disks for $\Sigma$,
- a set of $k$ vertices $\{v_1, \ldots, v_k\}$ with mutually disjoint representative curves in each isotopy class defines a $(k - 1)$-simplex in $\Gamma(\Sigma)$ with $v_1, \ldots, v_k$ as its vertices. We denote the $(k - 1)$-simplex by $\langle \alpha_1, \ldots, \alpha_k \rangle$, where $\alpha_i$ is a representative curve for the isotopy class corresponding to $v_i$.

In [2], Bachman introduced the topological index of $\Sigma$ as the smallest $n$ such that $\pi_{n-1}(\Gamma(\Sigma))$ is nontrivial when $\Gamma(\Sigma)$ is non-empty or 0 if $\Gamma(\Sigma)$ is empty. If $\Sigma$ has a well-defined topological index, i.e. either $\Gamma(\Sigma) = \emptyset$ or it has a nontrivial homotopy group, then we say $\Sigma$ is topologically minimal.

Topologically minimal surfaces generalize two well-known classes of surfaces. Surfaces of index 0 are precisely those which are incompressible, and surfaces of index 1 are those which are strongly irreducible. Moreover, a number of well-known results for incompressible and strongly irreducible surfaces generalize to topologically minimal surfaces. For example, given an incompressible surface and a topologically minimal surface in an irreducible 3-manifold, Bachman [2] showed that they may be isotoped so that their intersection consists of loops which are essential on both surfaces.

After Bachman’s initial work on topologically minimal surfaces, Bachman and Johnson [3] exhibited the first examples of topologically minimal surfaces with arbitrarily high index. This was done by a construction involving gluing copies of two-bridge link complements in chains along their boundary tori.
This established that higher index surfaces exist, as previously only surfaces of indices 0, 1, and 2 were known. Lee generalized their result to show that every 3-manifold containing an incompressible surface contains arbitrarily high topological index surfaces [8]. Campisi and Rathbun then showed that there exist hyperbolic 3-manifolds containing arbitrarily high index surfaces [4]. Although these examples establish the existence of topologically minimal surfaces in a variety of 3-manifolds, they are obtained through highly specific and artificial constructions. The exception to this is a result of Lee, which shows that bridge spheres of the unknot are topologically minimal [9], though their precise index is unknown.

The topology of the disk complex is not well-understood, which makes it difficult to study its homotopy groups. This in turn makes it difficult to identify topologically minimal surfaces and their indices. In this paper, we adapt machinery of Harer [6] and Ivanov [7] used in the study of the curve complex to the study of the disk complex. This leads to the two following results.

Let $\Sigma$ be a compact, connected surface with $-\chi(\Sigma) \geq 2$, embedded in $S^3$ as a Heegaard surface possibly with open disks removed.

**Theorem 1.** The disk complex $\Gamma(\Sigma)$ is

- $(-\chi(\Sigma) - 1)$-connected if $\Sigma$ is closed,
- $(-\chi(\Sigma) - 2)$-connected if $\Sigma$ has 1 boundary component,
- $(-\chi(\Sigma) - 3)$-connected if $\Sigma$ has 2 or more boundary components.

**Theorem 2.** Let $g$ and $b$ denote the genus and number of boundary components of $\Sigma$, respectively. The disk complex $\Gamma(\Sigma)$ is homotopy equivalent to a space of dimension

- $2g - 2$ for $g > 0, b = 0$,
- $2g - 3 + b$ for $g > 0, b > 0$,
- $b - 4$ for $g = 0, b \geq 4$,

and is empty otherwise.

Bachman conjectured that there were no topologically minimal surfaces in $S^3$ [2]. This was shown to be false by Appel and Gabai in Appel’s senior thesis [1], although this result was not published or publicly available. Our present work independently confirms that the conjecture is false. Indeed, combining Theorems 1 and 2 gives the following result.

**Theorem 3.** If $\Sigma$ is a genus $g > 1$ Heegaard surface for $S^3$, then it is topologically minimal with index $2g - 1$.

In Section 2 of this paper, we introduce a correspondence between simplicial maps into the disk complex and families of smooth, nondegenerate, real-valued
maps, which is based on the work of Ivanov [7] for the curve complex. This correspondence allows us to prove Theorem 1. In Section 3, we investigate the homotopy type of the disk complex using combinatorial methods and prove Theorems 2 and 3.

2. Smooth functions and the disk complex

2.1. Preliminaries. Throughout this paper, we adopt the convention that whenever we consider a surface $\Sigma$ to be embedded in $S^3$, we consider it embedded as a Heegaard surface possibly with open disks removed. We will emphasize such an embedding by calling it a standard embedding. Moreover, we will always consider $\Sigma$ to be standardly embedded in $S^3$ when considering its disk complex $\Gamma(\Sigma)$.

Let $\Sigma$ denote a compact, connected, orientable surface with $\chi(\Sigma) \leq -2$ throughout Section 2.

**Definition 2.1.1.** Let $f : \Sigma \to \mathbb{R}$ be a smooth function. If $\Sigma$ has boundary, we will assume that $f(\partial \Sigma) = 0$, where 0 is a regular value of $f$ and $f \geq 0$. A component $\alpha$ of $f^{-1}(A)$ is called non-singular if $\alpha$ does not contain a critical point of $f$.

Note that the non-singular components of such smooth functions are simple closed curves.

**Definition 2.1.2.** A smooth function $f : \Sigma \to \mathbb{R}$ is called non-degenerate if there is a non-trivial simple closed curve among the non-singular components of its level sets.

The following sections will be devoted to establishing a correspondence between simplicial maps into $\Gamma(\Sigma)$ and families of smooth functions. This correspondence will allow us to study simplicial maps into $\Gamma(\Sigma)$ from the perspective of smooth functions. In particular, the convexity of the space of smooth functions and the simplicial approximation theorem will allow us to prove that any continuous mapping of a sphere of small enough dimension into $\Gamma(\Sigma)$ is contractible. This in turn proves that the homotopy groups of $\Gamma(\Sigma)$ up to a certain dimension (which depends on $\chi(\Sigma)$) are trivial.

2.2. Constructing a simplicial map from a family of smooth non-degenerate functions. Given a family $\{f_t : \Sigma \to \mathbb{R}\}_{t \in P}$ of smooth non-degenerate functions, where $P$ is a parameter space, we now show how to construct a non-unique abstract simplicial complex $C$ and a simplicial map $|C| \to \Gamma(\Sigma)$, where $|C|$ is the geometric realization of $C$.

Since the functions $f_t$ are non-degenerate, then for each $t \in P$ there is an $A_t \in \mathbb{R}$ and a non-singular component $\alpha_t$ of $f_t^{-1}(A_t)$ which is a non-trivial
simple closed curve. Perturb \( f_t \) to a Morse function \( g_t \) with \( g_t \) injective on its critical points and \( f_t^{-1}(A_t) = g_t^{-1}(A_t) \).

**Definition 2.2.1.** A handlebody filling of \( \Sigma \) is a pair \((H, \Sigma)\) such that \( H \) is a 3-dimensional handlebody with \( \partial H = \Sigma \).

**Lemma 2.2.2** ([5], Lemma 8). Given a surface \( \Sigma \) and a Morse function \( f : \Sigma \to \mathbb{R} \) which is injective on critical points, there is a handlebody filling \((H, \Sigma)\) of \( \Sigma \) with the property that every regular level component of \( f \) bounds a disk in \( H \).

Let \( \hat{\Sigma} \) be the closed surface that results from capping off any boundary components of \( \Sigma \) with disks. Extending \( g_t \) to a Morse function for \( \hat{\Sigma} \) which is injective on critical points, by Lemma 2.2.2 we obtain a handlebody filling \((H, \hat{\Sigma})\) of \( \hat{\Sigma} \). Now consider \( H \) standardly embedded in \( S^3 \) with \( \hat{\Sigma} \) identified with its image under this embedding. Then \( \alpha_t \subset \hat{\Sigma} \) bounds a nontrivial compressing disk for \( \hat{\Sigma} \subset S^3 \). Deleting the capping disks that produce \( \hat{\Sigma} \) shows that \( \alpha_t \subset \Sigma \) bounds a nontrivial compressing disk for \( \Sigma \subset S^3 \).

Observe now that if \( u \) lies in a sufficiently small neighborhood \( U_t \) of \( t \in P \), then one of the components of \( f^{-1}_u(A_t) \) must be non-singular and isotopic to \( \alpha_t \). Denote this component by \( \alpha_{tu} \). The neighborhoods \( \{U_t : t \in P\} \) form a covering of the space \( P \). Let \( \{U_t : t \in V\} \) be a subcover (possibly finite in the case of \( P \) compact) of this covering, and let \( C \) be the nerve of this subcover. That is, \( C \) is an abstract simplicial complex whose vertices are identified with the elements of \( V \), and \( t_0, \ldots, t_n \) form an \( n \)-simplex in \( C \) if and only if \( U_{t_0} \cap \cdots \cap U_{t_n} \neq \emptyset \). We now show that the correspondence \( t \mapsto \langle \alpha_t \rangle \) gives a well-defined simplicial map \( |C| \to \Gamma(\Sigma) \). Let \( \sigma \) be a simplex of \( C \), and let \( W \) be the set of elements of \( V \) corresponding to the vertices of \( \sigma \). By definition, we have \( \bigcap_{t \in W} U_t \neq \emptyset \), so let \( u \in \bigcap_{t \in W} U_t \). Then

(i) for all \( t \in W \), \( \alpha_{tu} \) is isotopic to \( \alpha_t \), i.e. \( \langle \alpha_{tu} \rangle = \langle \alpha_t \rangle \);

(ii) the simple closed curves in \( \{\alpha_{tu} : t \in W\} \) are non-singular components of level sets of \( f_u \), so any two of these simple closed curves are either disjoint or isotopic.

By (ii), \( \{\langle \alpha_{tu} \rangle : t \in W\} \) determines a simplex in \( \Gamma(\Sigma) \), and by (i), this simplex coincides with the image \( \{\langle \alpha_t \rangle : t \in W\} \) of \( W \). Identifying \( C \) with its geometric realization \( |C| \), we see that the correspondence \( t \mapsto \langle \alpha_t \rangle \) indeed determines a well-defined simplicial map \( |C| \to \Gamma(\Sigma) \).

A simplicial map obtained by the above process is said to realize the family of smooth maps \( \{f_t : \Sigma \to \mathbb{R}\}_{t \in P} \).
Note that we do not require all smooth maps in the family to be non-degenerate. Indeed, in the construction above, we only needed the functions $f_t$ for $t \in V$ to be non-degenerate. In Section 2.3, we will show that every simplicial map realizes a family of smooth non-degenerate maps.

**Remark 2.2.3.** Let $\{f_t : \Sigma \to \mathbb{R}\}_{t \in P}$ be a family of smooth non-degenerate functions, $Q$ be a subset of $P$, and $\{f_t\}_{t \in Q}$ be a subfamily of $\{f_t\}_{t \in P}$. Given a simplicial map $F : |C_Q| \to \Gamma(S)$ that realizes $\{f_t\}_{t \in Q}$, $F$ can be extended to a simplicial map $F' : |C_P| \to \Gamma(S)$ such that $F'$ realizes $\{f_t\}_{t \in P}$ and $F'$ realizes $\{f_t\}_{t \in Q}$ when restricted to $|C_Q|$. By the construction above, note that $C_Q$ is the nerve of a covering $\{U_t\}_{t \in V}$ of $Q$, and $|C_Q| \to \Gamma(\Sigma)$ is determined by a fixed choice of non-singular components $\alpha_t$ of $f_t^{-1}(A_t)$, $t \in V$. Every $U_t$ has the form $U_t' \cap Q$, where $U_t'$ is an open set in $P$. Moreover, $U_t'$ may be chosen so that $f_u^{-1}(A_t)$ has a non-singular component isotopic to $\alpha_t$ for both $u \in U_t$ and $u \in U_t'$. Let $C_P$ be the nerve of any covering containing $\{U_t' : t \in V\}$. Now take any choice of simple closed curves determining $F' : |C_P| \to \Gamma(\Sigma)$ that includes $\alpha_t$, $t \in V$. Then $F'$ realizes $\{f_t\}_{t \in P}$ and, by construction, realizes $\{f_t\}_{t \in Q}$ when restricted to $|C_Q|$.

2.3. Constructing a family of smooth non-degenerate functions from a simplicial map. To establish a correspondence between simplicial maps into $\Gamma(\Sigma)$ and families of smooth non-degenerate functions, it remains to show that we may construct a family of smooth non-degenerate functions from a simplicial map. Given such a simplicial map, we first show it realizes a family of smooth maps which are not necessarily non-degenerate.

**Lemma 2.3.1.** Every simplicial map $h : X \to \Gamma(\Sigma)$ realizes a family of smooth functions $\{f_t : \Sigma \to \mathbb{R}\}_{t \in X}$.

Ivanov (§1.3, [7]) showed this in the case of simplicial maps into the curve complex $C(\Sigma)$. Since $\Gamma(\Sigma) \subset C(S)$, it is clear that it also holds for simplicial maps into the disk complex. We present Ivanov’s proof below.

**Proof.** Let $h : X \to \Gamma(\Sigma)$ be a simplicial map and let $V$ be the vertex set of $X$. For any vertex $v \in V$, let $St_v$ denote the closed star of $v$ in the first barycentric subdivision of $X$. The set of barycentric starts $\{St_v : v \in V\}$ form a closed covering of $X$, and the geometric realization of its nerve coincides with $X$ (see Lemma 2.11 of [10] for a proof).

If $U_v$ is a sufficiently small neighborhood of $St_v$ for each $v \in V$, then the geometric realization of the nerve of the open covering $\{U_v : v \in V\}$ also coincides with $X$.

We now construct the family $\{f_t : \Sigma \to \mathbb{R}\}_{t \in X}$. Equipping $\Sigma$ with a (complete) Riemannian metric of constant curvature with geodesic boundary, for
every vertex of $\Gamma(\Sigma)$ there is a unique geodesic representative in the corresponding isotopy class. Since $h$ is simplicial, vertices of $X$ are mapped to vertices of $\Gamma(\Sigma)$, so each $v \in V$ corresponds to a geodesic $\alpha_v$ under $h$. For each $v \in V$, choose a closed neighborhood $N_v$ of $\alpha_v$ such that $N_v \cap \partial \Sigma = \emptyset$ and $N_v \cap N_w = \emptyset$ when $\alpha_v \cap \alpha_w = \emptyset$. For each $v \in V$ choose a smooth function $g_v : N_v \to \mathbb{R}$ and a regular value $A_v$ of $g_v$ such that $g_v^{-1}(A_v) = \alpha_v$ and $g_v > 0$.

Lastly, choose a closed neighborhood $N_0$ of $\partial \Sigma$ such that $N_0 \cap N_v = \emptyset$ for all $v \in V$, and choose a smooth function $g_0 : N_0 \to \mathbb{R}$ such that $0$ is a regular value of $g_0$, $g_0^{-1}(0) = \partial \Sigma$, and $g_0 \geq 0$. Letting $K = (X \times N_0) \cup \left( \bigcup_{v \in V} \text{St}_v \times N_v \right) \subset X \times \Sigma$, define $G : K \to \mathbb{R}$ by

$$G(t, x) = \begin{cases} g_0(x) & \text{if } (t, x) \in X \times N_0, \\ g_v(x) & \text{if } (t, x) \in \text{St}_v \times N_v. \end{cases}$$

Observe that if $\text{St}_v \cap \text{St}_w \neq \emptyset$, then $\{v, w\}$ is a simplex in $X$ and so $\alpha_v \cap \alpha_w = \emptyset$ and $N_v \cap N_w = \emptyset$. Therefore the sets $\text{St}_v \times N_v$ are pairwise disjoint and do not intersect $X \times N_0$, so $G$ is well-defined. Now extend $G$ to a smooth function $F : T \times \Sigma \to \mathbb{R}$. For each $t \in X$, define $f_t : \Sigma \to \mathbb{R}$ by $f_t(x) = F(t, x)$. Then $\{f_t\}_{t \in X}$ is a family of smooth functions, and $h$ realizes $\{f_t\}_{t \in X}$ by construction.

If the simplicial complex $X$ is of sufficiently small dimension (which will be made precise below), we will show that every simplicial map $h : X \to \Gamma(\Sigma)$ also realizes a family of smooth non-degenerate functions $\{f_t\}_{t \in X}$. The following theorem of Ivanov will be instrumental.

**Theorem 2.3.2** ([7], Theorem 2.5). Let

$$d = \begin{cases} -\chi(\Sigma) & \text{if } \Sigma \text{ is closed,} \\ -\chi(\Sigma) - 1 & \text{if } \Sigma \text{ has exactly 1 boundary component,} \\ -\chi(\Sigma) - 2 & \text{otherwise.} \end{cases}$$

Then any family of smooth functions $\{f_t : \Sigma \to \mathbb{R}\}_{t \in P}$ with $\dim P \leq d$ can be approximated arbitrarily well by a family of smooth non-degenerate functions.

**Lemma 2.3.3.** Let

$$d = \begin{cases} -\chi(\Sigma) & \text{if } \Sigma \text{ is closed,} \\ -\chi(\Sigma) - 1 & \text{if } \Sigma \text{ has exactly 1 boundary component,} \\ -\chi(\Sigma) - 2 & \text{otherwise.} \end{cases}$$
Every simplicial map $h : X \to \Gamma(\Sigma)$ with $\dim X \leq d$ realizes a family of smooth non-degenerate functions \{\{f_t : \Sigma \to \mathbb{R}\}_{t \in X}\}.

**Proof.** By Lemma 2.3.1, any simplicial map $h : X \to \Gamma(\Sigma)$ realizes a family of smooth functions \{\{f_t : \Sigma \to \mathbb{R}\}_{t \in X}\}. Let \{\alpha_v : v \in V\} be the simple closed curves described in the proof of Lemma 2.3.1. By Theorem 2.3.2, \{\{f_t : \Sigma \to \mathbb{R}\}_{t \in X}\} can be approximated arbitrarily well by a family of smooth non-degenerate functions \{\{f'_t\}_{t \in X}\}. Recalling the construction described in Section 2.2, we will repeat the arguments and show how to obtain $h$ from such an approximation \{\{f'_t\}_{t \in X}\} via this construction. This will prove that $h$ realizes a family of smooth non-degenerate functions.

Take sufficiently small neighborhoods $U'_v \subset U_v$ of $\text{St}_v$ for each $v \in V$ together with a good enough approximation \{\{f'_t\}_{t \in X}\} such that, for each $v \in V$, there is a regular value $A'_t \in [A_t - \varepsilon, A_t + \varepsilon]$ for some $\varepsilon > 0$ such that $(f'_t)^{-1}(A'_t)$ contains a non-singular component $\alpha'_t$ which is isotopic to $\alpha_t$. Following the rest of the construction, the correspondence $t \mapsto \langle \alpha'_t \rangle$ obtained is exactly the same as the correspondence $t \mapsto \langle \alpha_t \rangle$ since $\langle \alpha_t \rangle = \langle \alpha'_t \rangle$, so $t \mapsto \langle \alpha'_t \rangle$ determines a mapping $X \to \Gamma(\Sigma)$ that coincides with $h$. Therefore $h$ realizes \{\{f'_t\}_{t \in X}\}. □

### 2.4. Connectivity of the disk complex

We are now ready to prove the main result of Section 2.

**Theorem 2.4.1.** Let

$$d = \begin{cases} 
-\chi(\Sigma) & \text{if } \Sigma \text{ is closed}, \\
-\chi(\Sigma) - 1 & \text{if } \Sigma \text{ has exactly 1 boundary component}, \\
-\chi(\Sigma) - 2 & \text{otherwise.}
\end{cases}$$

Then $\Gamma(\Sigma)$ is $(d - 1)$-connected.

**Proof.** Let $N \leq d - 1$. It suffices to show that any continuous map $S^N \to \Gamma(\Sigma)$ is homotopic to a constant map. In this direction, let $g : S^N \to \Gamma(\Sigma)$ be a continuous map. By the simplicial approximation theorem, there exists a triangulation $T$ of $S^N$ with homeomorphism $\varphi : S^N \to T$ and a simplicial map $h : T \to \Gamma(\Sigma)$ such that $g$ is homotopic to $|h| = h \circ \varphi$. By Lemma 2.3.1, $h$ realizes a family of smooth functions \{\{f_t\}_{t \in X}\}. Since the space of smooth functions is contractible, \{\{f_t\}_{t \in X}\} can be extended to a family of smooth functions \{\{f'_t\}_{t \in B}\}, where $B$ is a simplicial $(N + 1)$-ball with boundary $T$. We have $\dim B \leq d$, so Theorem 2.3.2 shows that \{\{f'_t\}_{t \in B}\} can be approximated arbitrarily well by a family of smooth non-degenerate functions \{\{f''_t\}_{t \in B}\}. Note that the proof of Lemma 2.3.3 shows that $h$ realizes \{\{f''_t\}_{t \in T}\} if \{\{f'_t\}_{t \in B}\}$ is a good enough approximation to \{\{f_t\}_{t \in B}\}. Taking such an approximation \{\{f''_t\}_{t \in B}\}, Remark 2.2.4 shows that $h$ can be extended to a simplicial map $h' : B \to \Gamma(\Sigma)$. Therefore $h$ and thus also $g$ is homotopic to a constant map. □
3. THE HOMOTOPY TYPE OF THE DISK COMPLEX AND HEegaard SURFACES OF THE 3-sPHERE

For the remainder of this paper, let $\Sigma$ be a compact, connected surface of genus $g > 1$ with $b$ boundary components, standardly embedded in $S^3$. A subcomplex of either the curve complex or disk complex has dimension of at most $3g - 4 + b$ since a pants decomposition has at most $3g - 3 + b$ curves. In Section 3.1, we show that we can do better than this in terms of homotopy dimension. Explicitly, we show that $\Gamma(\Sigma)$ is homotopy equivalent to a space of dimension

- $2g - 2$ for $g > 0$, $b = 0$,
- $2g - 3 + b$ for $g > 0$, $b > 0$, or
- $b - 4$ for $g = 0$, $b \geq 4$.

In Section 3.2, we combine this result with Theorem 2.4.1 to show, in the case where $\Sigma$ is closed, $\Gamma(\Sigma)$ is homotopy equivalent to a wedge of spheres of dimension $-\chi(\Sigma)$. This immediately proves that genus $g > 1$ Heegaard surfaces for $S^3$ are topologically minimal with index $2g - 1$, which is the main result of this paper.

3.1. Homotopy dimension of the disk complex. In Theorem 3.5 of [6], Harer shows that the curve complex is homotopy equivalent to a wedge of spheres of the same dimension as in each of the cases of Theorem 3.1.1 below. Employing the techniques used in his proof, we obtain the following result. For convenience, let $\Sigma^b_g$ denote a compact, connected surface of genus $g$ with $b$ boundary components, standardly embedded in $S^3$.

**Theorem 3.1.1.** The disk complex $\Gamma(\Sigma^b_g)$ is homotopy equivalent to a space of dimension

- $2g - 2$ for $g > 0$, $b = 0$,
- $2g - 3 + b$ for $g > 0$, $b > 0$,
- $b - 4$ for $g = 0$, $b \geq 4$,

and is empty otherwise.

**Proof.** Assume first that $g = 0$. Note that a sphere with one, two, or three boundary components does not admit any essential simple closed curves, so $\Gamma(\Sigma^b_0)$ is empty for $b < 4$. For $b \geq 4$, a pants decomposition of $\Sigma^b_0$ consists of $b - 3$ curves, and such a set of curves gives a $(b - 4)$-simplex in $\Gamma(S^b_0)$ which is of maximal dimension.

Assume now that $g > 0$. We will proceed by double induction over $g$ (outer induction) and $b$ (inner induction), so assume that the theorem has been proven for all surfaces of genus smaller than $g$ with arbitrarily many boundary components.
Let $\partial \Sigma^b_g = \{P_1, \ldots, P_b\}$ be the boundary components of $\Sigma^b_g$. Define $\hat{\Gamma}(\Sigma^b_g)$ to be the subcomplex of $\Gamma(\Sigma^b_g)$ consisting of simplices $\langle \alpha_0, \ldots, \alpha_k \rangle$ for which no curve $\alpha_i$ bounds a region on $\Sigma^b_g$ which contains $P_1$ and exactly one other boundary component $P_j$. Note that $\hat{\Gamma}(\Sigma^b_g) = \Gamma(\Sigma^b_g)$ for $b = 1$. Attaching a disk to $P_1$ defines a map $\Phi : \hat{\Gamma}(\Sigma^b_g) \to \Gamma(\Sigma^b_{g-1})$.

Lemma 3.1.2. If $b > 0$, the map $\Phi$ is a homotopy equivalence.

Proof. Let $F_0$ be the surface obtained from $\Sigma^b_g$ by attaching a disk to $P_1$. Choose a hyperbolic metric for $F_0$ so that every simple closed curve in $F_0$ is represented by a unique geodesic. There is at least one point in $F_0$ and a neighborhood of $F_0$ through which no simple geodesic passes. Selecting this point to lie on the disk attached to $P_1$ and the neighborhood to contain the disk defines a map $\Psi : \Gamma(\Sigma^b_{g-1}) \to \hat{\Gamma}(\Sigma^b_g)$. Note that $\Phi \circ \Psi$ is the identity on $\Gamma(\Sigma^b_{g-1})$. The proof will be complete once we show that the homomorphisms induced by $\Phi$ on homotopy groups are injective.

Let $T$ be a simplicial $n$-sphere and let $f : T \to \hat{\Gamma}(\Sigma^b_g)$ be a simplicial map. We will show that $f$ is homotopic to $f' = \Psi \circ \Phi \circ f$. Let $v_1, \ldots, v_k$ be the vertices of $T$, and write $\langle \alpha_i \rangle = f(v_i)$ and $\langle \alpha'_i \rangle = f'(v_i)$, where $\alpha_i$ and $\alpha'_i$ are the geodesic representatives of each isotopy class. Each $\alpha_i$ is isotopic to $\alpha'_i$ in $F_0$, and if $\alpha'_i$ and $\alpha'_j$ are isotopic in $F_0$ then they are already equal in $\Sigma^b_g$ by construction.

For all $i$, the isotopy of $\alpha_i$ to $\alpha'_i$ describes a properly embedded annulus $A_i$ in $F_0 \times [0,1]$ with $A_i \cap (F_0 \times 0) = \alpha_i$ and $A_i \cap (F_0 \times 1) = \alpha'_i$. We may arrange the annuli so that they are pairwise transverse and that each one is transverse to $P_1 \times [0,1]$. However, it is necessary to make the $A_i$ so that if $\alpha_i$ and $\alpha_j$ are disjoint, then either $A_i$ and $A_j$ are disjoint or $A_i \cap A_j = \alpha'_i = \alpha'_j$.

To do this, consider $A_i$ and $A_j$ with $\alpha_i$ and $\alpha_j$ disjoint. The intersection $A_i \cap A_j$ is a properly embedded submanifold in both $A_i$ and $A_j$. At one end, $A_i$ and $A_j$ are disjoint and at the other end they are either equal or disjoint, hence any unwanted intersections between them are circles. Circles of intersection which are inessential can be removed by an innermost circle argument. Without loss of generality, suppose now that $\alpha$ is a circle of intersection that is nontrivial in $\pi_1(A_i)$. Since $\pi_1(F_0 \times [0,1]) = \pi_1(F_0)$, then $\alpha$ must also be nontrivial in $\pi_1(A_j)$, otherwise it would be an inessential circle in $F_0$. Hence $\alpha'_i = \alpha'_j$. Now interchange the portion of $A_i$ containing $\alpha'_i$ with the portion of $A_j$ containing $\alpha'_j$ so that the intersection at $\alpha$ is eliminated. Continue in this way until all essential circles of intersection are removed.

After we have removed these intersection circles, the annuli are in the desired positions. Move the $A_i$ slightly (if necessary) so that $P_1 \times [0,1]$ intersects only one annulus at a time, and let $0 < t_1 < \cdots < t_m < 1$ be the times that
these intersections occur. Suppose $A_i$ meets $P_1$ at time $t_1$. Let $\beta_i$ be the curve $A_i \cap (F_0 \times (t_1 + \epsilon))$, where $\epsilon > 0$ is chosen so that $t_1 + \epsilon < t_2$. By construction, $\beta_i$ can be isotoped to be disjoint from $\alpha_i$, and the positioning of the annuli guarantee that if $\alpha_i$ is disjoint from $\alpha_j$ then so must $\beta_i$. If $\Lambda = \langle \nu_i, \nu_j, \ldots, \nu_n \rangle$ is a face of $T$ in the star of $v_i$, then add the simplex $\langle v_i, \beta_i, \nu_j, \ldots, \nu_n \rangle$ to $T$ along $\Lambda$. The map $f$ is then extended by taking $\beta_i$ to $\langle \alpha_i \rangle$. Continuing in this way, we add simplices and extend $f$ for each of the intersections at each time $t_i$ as we did for $t_1$. Completing this process gives the desired homotopy of $f$ to $f'$.

Returning to the proof of Theorem 3.1.1, consider the first barycentric subdivision, $\Gamma^0(\Sigma^b_g)$, of $\Gamma(\Sigma^b_g)$, which is defined to be the simplicial complex such that

(i) $\Gamma^0(\Sigma^b_g)$ has a vertex $v$ for every simplex in $\Gamma(\Sigma^b_g)$, where the weight of $v$ is defined as the dimension of the simplex it represents,

(ii) a chain of $t+1$ proper inclusions of simplices in $\Gamma(\Sigma^b_g)$ defines a $t$-simplex in $\Gamma^0(\Sigma^b_g)$ with its vertices being those that represent the simplices in the chain.

Define $X_k$ to be the subcomplex of $\Gamma^0(\Sigma^b_g)$ whose vertices have weight greater than or equal to $k$. We will construct $\Gamma^0(\Sigma^b_g)$ starting from $X_{3g-4}$ by adding vertices of decreasing weight together with the simplices they span in each step.

The subcomplex $X_{3g-4}$ is just a discrete set of points, and so its dimension is less than $2g - 2$. Suppose that $X_{k+1}$ has been shown to be homotopy equivalent to a space of dimension no more than $2g - 2$. Consider a vertex $v = \langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle$ of $X_k - X_{k+1}$. If $F_1, \ldots, F_t$ are the components of $\Sigma^b_g$ obtained by splitting along $\alpha_0, \ldots, \alpha_k$, then the link of $v$ in $X_{k+1}$ is readily identified with the join of the complexes $\Gamma^0(F_1), \ldots, \Gamma^0(F_t)$. Supposing that, for $i = 1, \ldots, t$, each component $F_i$ has genus $g_i$ and $b_i$ boundary components, observe that $\chi(F_i) = 2 - 2g_i - b_i$. Note that $g_i < g$ for each $i = 1, \ldots, t$ and that $\sum_{i=1}^t b_i = 2k + 2$.

Assume now that $b = 0$. Then we have

$$\chi(\Sigma^b_g) = \sum_{i=1}^t \chi(F_i)$$

$$= \sum_{i=1}^t (2 - b_i - 2g_i)$$

$$= 2t - 2k - 2 - \sum_{i=1}^t 2g_i$$
and hence
\[ g = k - t + 2 + \sum_{i=1}^{t} g_i. \]

Inductively, for all \( i \), \( \Gamma_0(F_i) \) is homotopy equivalent to a space of dimension \( 2g_i + b_i - 3 \), so the link of \( v \) in \( X_{k+1} \) is homotopy equivalent to a space of dimension
\[ (t - 1) + \sum_{i=1}^{t} (2g_i + b_i - 3) = 2g - 3. \]

The join of \( v \) with its link in \( X_{k+1} \) is thus homotopy equivalent to a space of dimension \( 2g - 2 \). To finish the proof for \( b = 0 \), observe that we obtain \( X_k \) by adding the join of every vertex in \( X_k - X_{k+1} \) with its link in \( X_{k+1} \), so \( X_k \) is indeed homotopy equivalent to a space of dimension \( 2g - 2 \). By this inductive construction of \( \Gamma(\Sigma_{g}^b) \), it follows that \( \Gamma(\Sigma_{g}^b) \) is homotopy equivalent to a space of dimension \( 2g - 2 \), so the Theorem holds for \( b = 0 \). By Lemma 3.1.2, it also holds for \( b = 1 \).

Now assume \( b > 2 \), and suppose that \( \Gamma(\Sigma_{g}^{b-1}) \) is homotopy equivalent to a space of dimension \( 2g - 3 + (b - 1) = 2g - 4 + b \). Lemma 3.1.2 shows that \( \hat{\Gamma}(\Sigma_{g}^b) \) is homotopy equivalent to the same space. Now \( \Gamma(\Sigma_{g}^b) \) is obtained from \( \hat{\Gamma}(\Sigma_{g}^b) \) by adding the stars of the omitted vertices along their links: the curves corresponding to these vertices each bound regions in \( \Sigma_{g}^b \) containing \( P_1 \) and exactly one other boundary component, and if two such curves are disjoint then they are isotopic. The map \( \Phi \) sends the link of such a vertex homomorphically onto \( \Gamma(\Sigma_{g}^{b-1}) \). Therefore \( \Gamma(\Sigma_{g}^b) \) has the homotopy type of the join of \( \Gamma(\Sigma_{g}^{b-1}) \) with a set of points, and hence is homotopy equivalent a space of dimension \( 2g - 3 + b \). \( \square \)

3.2. **Topological index of Heegaard surfaces for the 3-sphere.** We are now ready to state the main results of this paper. Theorem 3.1.1 and Theorem 2.4.1 together prove the following result.

**Theorem 3.2.1.** If \( \Sigma \) is a genus \( g > 1 \) Heegaard surface for \( S^3 \), then it is topologically minimal with index \( 2g - 1 \).

This result confirms that the conjecture that the 3-sphere does not contain topologically minimal surfaces is false.

**References**

[1] Daniel Appel, *Topological indices of surfaces in the 3-sphere*, Bachelor’s thesis, Princeton University, 2010.

[2] David Bachman, *Topological index theory for surfaces in 3-manifolds*, Geometry & Topology **14** (2010), no. 1, 585–609.
[3] David Bachman and Jesse Johnson, *On the existence of high index topologically minimal surfaces*, Mathematical Research Letters **17** (2009), no. 3, 389–394.

[4] Marion Campisi and Matt Rathbun, *Hyperbolic manifolds containing high topological index surfaces*, Pacific Journal of Mathematics **296** (2018), no. 2, 305–319.

[5] David Gay, *Functions on surfaces and constructions of manifolds in dimensions three, four and five*, arXiv:1701.01711.

[6] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Inventiones mathematicae **84** (1986), no. 1, 157–176.

[7] N. V. Ivanov, *Complexes of curves and Teichmüller spaces*, Mathematical notes of the Academy of Sciences of the USSR **49** (2016), no. 5, 479–484.

[8] Jung Hoon Lee, *On topologically minimal surfaces of high genus*, Proceedings of the American Mathematical Society **143** (2015), 2725–2730.

[9] Jung Hoon Lee, *Bridge spheres for the unknot are topologically minimal*, Pacific Journal of Mathematics **282** (2016), no. 2, 437–443.

[10] Andrzej Nagórko, *Characterization and topological rigidity of Nöbeling manifolds*, Memoirs of the American Mathematical Society **223** (2013), no. 1048.

San José State University, San Jose, CA 95192
*E-mail address:* marion.campisi@sjsu.edu

San José State University, San Jose, CA 95192
*E-mail address:* luis.torres@sjsu.edu