Fractional charge and statistics in the fractional quantum spin Hall effect

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Abstract
In this paper, we consider there exist two types of fundamental quasihole excitation in the fractional quantum spin Hall state and investigate their topological properties by both Chern–Simons field theory and the Berry phase technique. By the two different techniques, we obtain the identical charge and statistical angle for each type of quasihole, as well as the identical mutual statistics between two different types of quasihole excitation.

1. Introduction
In condensed matter systems, most states of matter can be characterized by Landau’s symmetry breaking theory. However, topological states of quantum matter cannot be described by this theory. The first examples of topological quantum states discovered in nature are the quantum Hall (QH) states [1, 2], which opened up a new chapter in condensed matter physics. In the noninteracting limit, the integer quantum Hall (IQH) state is characterized by the TKNN invariant [3] or first Chern number. For the fractional quantum Hall (FQH) state, interaction between electrons turns out to be crucial, and an Abelian FQH state can be characterized by an integer symmetric \( K \) matrix and an integer charge vector \( \mathbf{t} \), up to a \( SL(n, \mathbb{Z}) \) equivalence [4]. However, both the IQH and FQH states appear in a strong magnetic field which breaks time reversal symmetry.

In a seminal paper, Haldane [5] has shown that the IQH state can be realized in a tight-binding graphene lattice model without net magnetic field. This state, however, breaks time reversal symmetry due to a local magnetic flux density of a zero net flux through the unit cell. Recently, Kane and Mele [6, 7] proposed a novel class of topological state, i.e. the integer quantum spin Hall (IQSH) state, which can be viewed as two spin-dependent copies of Haldane’s model that preserve time reversal symmetry. Bernevig and Zhang proposed the QSH state for semiconductors [8], where the Landau levels arise from the strain gradient, rather than the external magnetic field. In the Landau level picture, the QSH state can be understood as the state in which the spin-\( \uparrow \) and spin-\( \downarrow \) electrons are in two opposing effective orbital magnetic fields \( \mp B \) realized by spin–orbit (SO) coupling, respectively. For the QSH state, the bulk is gapped and insulating while there are gapless edge states in a time reversal invariant system with SO coupling. Bernevig, Hughes and Zhang predict a QSH state in the HgTe/CdTe heterostructure [9], which has been confirmed by experiment [10].

By analogy with the relation between the IQHE and FQHE, it is natural to ask the question whether there can exist a fractional QSH (FQSH) state. An explicit wavefunction for the FQSH state was first proposed by Bernevig and Zhang [8], and the edge theory was discussed by Levin [11]. Recently, a set of exactly soluble lattice electron models for the FQSH state were constructed [12], and the generic wavefunction for the FQSH state was proposed through the Wannier function approach [13].

As is well known, quasihole (or quasiparticle) excitations above the FQH ground states have fractional charge and fractional statistics [14, 15], such as the \( \nu = 1/k \) Laughlin states, which have quasihole excitations with charge \( e/k \) and statistical angle \( \theta = \pi/k \). So we have to ask what the properties of the excitations will be in the FQSH state. In order to answer this question, we investigate the charge and statistics of the quasihole excitations in the FQSH state. All five quantities for the fractional charge and statistics are identically given in two different ways. This is the main task in this paper.

The paper is organized as follows. In section 2, we introduce the theoretical model of the QSHE and the...
wavefunction for the FQSH state proposed by Bernevig and Zhang. In section 3, we first analyse the quasihole excitations in the FQSH state and write down the wavefunction for each type of fundamental quasihole excitation. Next, the identical charge and statistics of each type of excitation as well as the mutual statistics between different types of excitations are obtained by two different methods in sections 3.1 and 3.2, respectively. Finally, section 4 is devoted to conclusions.

2. Wavefunction of fractional quantum spin Hall state

Now, we will briefly review the theoretical model proposed by Bernevig and Zhang [8]. The simplest case of the QSHE can be viewed as superposing two QH systems with opposite spins. The spin-↑ QH state has positive charge Hall conductance ($\sigma_{xy} = +e^2/h$) while the spin-↓ QH state has negative charge Hall conductance ($\sigma_{xy} = -e^2/h$). As such, the charge Hall conductance of the whole system vanishes. However, the spin Hall conductance remains finite and quantized in units of $e^2/2\pi$ since the spin-↑ and the spin-↓ QH states have opposite chirality.

To realize this QSH state we need a spin-dependent effective orbital magnetic field, which can be created by the spin–orbit coupling in conventional semiconductors in the presence of a strain gradient. When the off-diagonal (shear) components of the strain symmetric tensor are $\epsilon_{xy}(\leftrightarrow E_x) = 0$, $\epsilon_{xz}(\leftrightarrow E_z) = g y$, $\epsilon_{yz}(\leftrightarrow E_z) = g x$, respectively, and $g$ denotes the magnitude of the strain gradient, a single electron in a symmetric quantum well in the $xy$ plane can be described by

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} C_3 \frac{g(y p_x - x p_y)}{h} + D(x^2 + y^2),$$

where the second term corresponds to the spin–orbit coupling ($\vec{p} \times \vec{E} \cdot \vec{\sigma}$), and the third term is the confining potential. The constant $C_3$ is $8 \times 10^5$ m$^{-1}$ for GaAs. We introduce $l = (8mD)^{-1/4}$ and $\omega = (8D/m)^{1/4}$, which have length dimension and frequency dimension separately. When we introduce the dimensionless quantities $x \rightarrow x/l$, $y \rightarrow y/l$, $p_x \rightarrow l p_x/h$, $p_y \rightarrow l p_y/h$ and $H \rightarrow H/h\omega$, the above Hamiltonian can be expressed as

$$H = \frac{1}{4}[p_x^2 + p_y^2 + \frac{1}{2}(x^2 + y^2) + R(y p_x - x p_y)]$$

with $R = C_3 g/\sqrt{mD}$. Since the eigenvalue of $\sigma_z$ is a good quantum number, we can use the spin along the $z$ direction to characterize the state and we have

$$H = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix},$$

$$H_{\uparrow,\downarrow} = \frac{1}{4}[p_x^2 + p_y^2 + (x^2 + y^2) \pm R(x p_x - y p_y)].$$

We focus on the special case of $R = 1$, i.e. $D = D_0 = mg^2 C_3^2/8$, where the Hamiltonian (1) becomes

$$H = \frac{1}{2m} \left( \vec{p} + \vec{\tau} \vec{\sigma} \right)^2,$$

$$\vec{\tau} = \frac{mg C_3}{2e} (y, -x) = \frac{R}{2} (y, -x),$$

where $B = mgC_3/e$, $\vec{B} = \nabla \times \vec{A} = -\vec{B}_\perp$ plays the role of effective magnetic field, which essentially arises from the gradient of the strain. We notice, moreover, that when $R = 1$, the two quantities $l$ and $\omega$ become $l_0 = (8mD)^{-1/4} = \sqrt{l/eB}$ and $\omega_0 = (8D/m)^{1/2} = eB/m$, which are exactly the definitions of magnetic length $l_B$ and cyclotron frequency $\omega_c$ in QH. Thus we can imagine the electrons of the two-dimensional system experience a spin-dependent effective magnetic field $-\vec{B}_\perp \vec{\tau} \vec{\sigma}$. We introduce the complex coordinate $z = x + i y = re^{i\theta}$, $\bar{z} = x - i y = re^{-i\theta}$, and define the harmonic-oscillator ladder operators as

$$a^\dagger_\uparrow = \frac{1}{\sqrt{2}} \left( \frac{z^*}{2} + 2\theta \right), \quad a^\dagger_\downarrow = \frac{1}{\sqrt{2}} \left( \frac{z^*}{2} - 2\theta \right),$$

$$a^\dagger_\downarrow = \frac{1}{\sqrt{2}} \left( \frac{z^*}{2} + 2\theta \right), \quad a^\dagger_\uparrow = \frac{1}{\sqrt{2}} \left( \frac{z^*}{2} - 2\theta \right).$$

They satisfy $[a_\sigma, a^\dagger_\sigma'] = \delta_{\sigma,\sigma'}$. $[a_\sigma, a_\sigma] = 0$, $(\sigma, \sigma' = \{\uparrow, \downarrow\})$. In the vicinity of $R = 1$, the Hamiltonian can be written as

$$H_\uparrow = a^\dagger_\uparrow a_\uparrow + \frac{1}{2}, \quad H_\downarrow = a^\dagger_\downarrow a_\downarrow + \frac{1}{2}.$$

We denote $m_\uparrow$ and $m_\downarrow$ as the eigenvalues of $a^\dagger_\uparrow a_\uparrow$ and $a^\dagger_\downarrow a_\downarrow$, respectively. The $z$ component of the angular momentum operator is given as $L_z = -i\hbar \partial_\theta = z^* \partial_\theta - z^* \partial_\theta = a^\dagger_\uparrow a_\downarrow - a^\dagger_\downarrow a_\uparrow$, which commutes with both $H_\uparrow$ and $H_\downarrow$, and has eigenvalues $m_\uparrow - m_\downarrow$. For spin-$\uparrow$ electrons, the lowest Landau level (LLL) corresponds to $m_\uparrow = 0$ and the single particle wavefunction is

$$\psi_{m_\uparrow} = \frac{z^{m_\uparrow}}{\sqrt{2\pi \times \sigma^\uparrow}} \exp \left( -\frac{1}{4} |z|^2 \right)$$

with the angular momentum $m_\uparrow$. While for spin-$\downarrow$ electrons, the LLL corresponds to $m_\downarrow = 0$ and the single particle wavefunction is

$$\psi_{m_\downarrow} = \frac{(-1)^{m_\downarrow} z^{m_\downarrow}}{\sqrt{2\pi \times \sigma^\downarrow}} \exp \left( -\frac{1}{4} |z|^2 \right)$$

with the angular momentum $-m_\downarrow$. It is easy to see that the QSH system is equivalent to a bilayer system: in one layer we have spin-$\uparrow$ electrons in the presence of a down-magnetic field ($-\vec{B}_\perp$), the spin-$\uparrow$ electrons are chiral and have positive charge Hall conductance, while in other layer we have spin-$\downarrow$ electrons in the presence of an up-magnetic field ($+\vec{B}_\perp$), the spin-$\downarrow$ electrons are antichiral and have negative charge Hall conductance.

In addition, the polynomial part of the wavefunctions of spin-$\uparrow$ and spin-$\downarrow$ are holomorphic and anti-holomorphic functions, respectively. If we consider the intra-layer correlations and ignore inter-layer correlations, the many-body wavefunction would be

$$\psi = \prod_{i<j} (z_{ij} - z_{ij}^*) \psi_{m_\uparrow} \prod_{i<j} (\bar{z}_{ij}^* - \bar{z}_{ij}) \psi_{m_\downarrow} \exp \left( -\frac{1}{4} \sum_{i<j} |z_{ij}|^2 + \sum_{i<j} |\bar{z}_{ij}|^2 \right),$$

where $z_{ij}$ and $\bar{z}_{ij}$ denote the spin-$\uparrow$ and spin-$\downarrow$ coordinates, respectively, and the filling factors of the spin-$\uparrow$ and spin-$\downarrow$ layers are $1/m_\uparrow$ and $1/m_\downarrow$, respectively, the odd integer $m_\sigma$ plays the role of the relative angular momentum between electrons with spin orientation $\sigma$. Here, $m_\uparrow = m_\downarrow$ because of the chiral–antichiral symmetry.

However, the electrons in the QSH state reside in the same quantum well and may possibly experience the additional interaction between the spin-$\uparrow$ and spin-$\downarrow$ layers. Hence, the many-body wavefunction of the whole FQSH
system should take the form [8]
\[
\Psi_{m_1m_2\ldots m_n}(z_{11}, \ldots z_{1j}, z_{j1}, \ldots z_{jj}) = \prod_{i<j}(z_{1i} - z_{1j})^{m_1} \prod_{i<j}(z_{ij} - z_{ji})^{m_j} \times \prod_{i,j}(z_{1i} - z_{ij}^*)^n e^{-i\frac{1}{2\hbar} \sum_{i} |z_{1i}|^2 + \sum_{i} |z_{ji}|^2}. \tag{7}
\]

The above wavefunction is similar to the Halperin wavefunction [16], which is a generalized Laughlin wavefunction to a two-component system. The Halperin wavefunctions are holomorphic for both \(z_{1j}\) and \(z_{ji}\), since for the bilayer QH system, the electrons of both layers experience the same external magnetic field. Whereas for the FQSH system, the spin-\(\uparrow\) electrons experience a down effective magnetic field and spin-\(\downarrow\) electrons experience an up effective magnetic field, this fact leads to the wavefunction of FQSH incorporating both holomorphic and anti-holomorphic coordinates. We should point out here that the picture of the QSH state we mentioned above can be only realized in the case of \(R = 1\) [8], hence the following discussions and results are limited to the case of \(R = 1\).

3. Fractional charge and fractional statistics in the FQSH system

As we mentioned in section 2, the FQSH system is analogous to a bilayer FQH system, but the electrons in the FQSH system experience a spin-dependent effective magnetic field. For bilayer FQH system, there are two kinds of fundamental quasihole excitation, which are in each layer, respectively [17]. Hence it is natural to speculate that there exist two types of fundamental quasihole excitation in the FQSH state. A quasihole in the spin-\(\uparrow\) layer is described by the wavefunction

\[
\Psi^{(\uparrow)}(\xi^{\uparrow}) = \prod_i (\xi^{\uparrow}_i - z_{1i}) \Psi_{m_1m_2\ldots m_n}(z_{11}, \ldots z_{1j}, z_{j1}, \ldots z_{jj}), \tag{8}
\]

and in the spin-\(\downarrow\) layer is described by

\[
\Psi^{(\downarrow)}(\xi^{\downarrow}) = \prod_i (\xi^{\downarrow}_i - z_{1i}^*) \Psi_{m_1m_2\ldots m_n}(z_{11}, \ldots z_{1j}, z_{j1}, \ldots z_{jj}), \tag{9}
\]

where \(\xi^{\uparrow}\) and \(\xi^{\downarrow}\) denote the coordinates of quasiholes of the spin-\(\uparrow\) and spin-\(\downarrow\) layers respectively. In following, we calculate the charge and statistics of the two types of the quasihole excitation as well as the mutual statistics between a quasihole in the spin-\(\uparrow\) layer and a quasihole in the spin-\(\downarrow\) layer, and give all five quantities for the fractional charge and statistics by two different methods in sections 3.1 and 3.2, separately.

3.1. Chern–Simons field theory approach

The topological structure of the FQH states can be understood by several approaches, and the most general is the low-energy effective Chern–Simons theory [4, 18]. Wen and Zee [4] pointed out that the Abelian FQH liquids can be characterized by the integer valued \(K\)-matrices and the integer valued charged vectors, up to \(SL(n, \mathbb{Z})\) equivalences, and the quasiparticle quantum numbers, such as fractional charge and fractional statistics, can be calculated from them. So we will search for the corresponding \(K\)-matrix and the charged vector of the FQSH state. Now we first construct the Chern–Simons theory for the FQSH state described by wavefunction (7).

When \(n = 0\), the spin-\(\uparrow\) electrons and spin-\(\downarrow\) electrons decouple, the FQSH state is just two independent FQH states with opposite magnetic fields. We introduce two \(U(1)\) gauge fields \(a_{1\mu}\), \(a_{1\mu}\) to describe the conserved electromagnetic current \(J_{1\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_{1\lambda}\) and \(J^m_{1\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a^{m}_{1\lambda}\), separately. Coupling the system to an external electromagnetic gauge potential \(A_{\mu}\) (here the notation \(A_{\mu}\) is not the ‘gauge potential’ of the effective orbital magnetic field in the FQSH system), then the total effective Lagrangian is

\[
\mathcal{L} = \mathcal{L}_{\uparrow} + \mathcal{L}_{\downarrow},
\]

with

\[
\mathcal{L}_{\uparrow} = \frac{m_1}{4\pi} e^{\mu\nu\lambda} a_{1\mu} \partial_\nu a_{1\lambda} + \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} A_{\mu} \partial_\nu a_{1\lambda}, \tag{11}
\]

\[
\mathcal{L}_{\downarrow} = \frac{m_1}{4\pi} e^{\mu\nu\lambda} a_{1\mu} \partial_\nu a_{1\lambda} + \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} A_{\mu} \partial_\nu a_{1\lambda}. \tag{12}
\]

When we integrate out \(a_{1\uparrow}\) and \(a_{1\downarrow}\) from equations (11) and (12), one can obtain the linear response of the both layers to the external electromagnetic fields. In order to correctly reflect the fact that the spin-\(\uparrow\) FQH layer has positive charge Hall conductance while the spin-\(\downarrow\) FQH layer has negative charge Hall conductance, we must use the negative sign in front of \(a_{1\uparrow} \wedge da_{1\uparrow}\) and the positive sign in front of \(a_{1\downarrow} \wedge da_{1\downarrow}\).

We now turn to the case for \(n \neq 0\), i.e. there exist inter-layer correlations between the spin-\(\uparrow\) electrons and spin-\(\downarrow\) electrons. We start with the spin-\(\downarrow\) FQH layer, namely \(\prod_{i,j}(z_{1j} - z_{1j}^*)^{m_i} \exp(-\frac{1}{4\hbar} \sum_{i} |z_{1i}|^2)\), which is a \(1/m_1\) Laughlin state and can be described by the effective Lagrangian (12). Since the factor \(\prod_{i,j}(z_{1j} - z_{1j})^{m_i}\) in (8) corresponds to a fundamental quasihole excitation in the spin-\(\uparrow\) layer, from the factor \(\prod_{i,j}(z_{1j} - z_{1j}^*)^{m_i}\) in (7) we can conclude that an electron in the spin-\(\uparrow\) layer is bound to a ‘large’ quasihole excitation in the spin-\(\downarrow\) layer. Such a quasihole is an \(n\)-fold fundamental quasihole and carries an \(a_{1\mu}\) charge of \(-n\). While in the mean field theory (MFT), we view the \(a_{1\mu}\) field as a fixed background and ignore the response of the \(a_{1\mu}\) field, then the quasihole gas behaves like bosons in the ‘magnetic’ field \(nb_{1\downarrow} = -ne\gamma_0 a_{1\downarrow}\). We can introduce a bosonic field \(\phi\) to describe the quasihole excitation by

\[
\phi^\dagger(b_0 + ina_{1\downarrow})\phi + \frac{1}{2m^2} \phi^\dagger(b_0 + ina_{1\downarrow})^2 \phi. \tag{13}
\]

When we attach a spin-\(\uparrow\) electron to each quasihole (i.e. bosonic field \(\phi\) in the MFT), the bound state behaves like a fermion. We denote \(\psi\) as the fermion field. These fermions experience an effective magnetic field \(-eB + nb_{1\downarrow}\), this process can be described by

\[
\psi^\dagger(b_0 - ieA_0 + ina_{1\downarrow})\psi + \frac{1}{2m^2} \psi^\dagger(b_0 - ieA_1 + ina_{1\downarrow})^2 \psi. \tag{14}
\]
When the fermion density satisfies
\[ \psi^\dagger \psi = \frac{1}{m} - e B + n b_\downarrow \frac{1}{2\pi}, \]
(the effective orbital magnetic field is negative for spin-\(\uparrow\) electrons as mentioned in section 2), i.e. the fermions have filling factor \(1/m\), the ground state of the electrons in the spin-\(\uparrow\) layer condense to a \(1/m\) Laughlin state, which corresponds to the factor \(\prod_{\ell \neq j}(z_{1\ell} - z_{j\ell})^{m\ell}\) in equation (7). Next, we introduce another gauge field \(a_{l\mu}\) (from now on, we use \(a_{l\mu}\) instead of \(a_{l\mu}^\dagger\) for notational convenience) to describe the conserved fermion current \(j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_{l\lambda}\) in the spin-\(\uparrow\) layer, the effective theory of the \(1/m\) Laughlin state in the spin-\(\uparrow\) layer is given by
\[ \mathcal{L}'_{\uparrow} = - \frac{m\ell}{4\pi} e^{\mu\nu\lambda} a_{l\mu} \partial_\nu a_{l\lambda}. \] (15)

The effective theory of the bound states has a form
\[ \mathcal{L}_b = (e A_{l\mu} - na_{l\mu}) j^\mu \]
\[ = \frac{e}{2\pi} e^{\mu\nu\lambda} A_{l\mu} \partial_\nu a_{l\lambda} = \frac{n}{2\pi} e^{\mu\nu\lambda} a_{l\mu} \partial_\nu a_{l\lambda}. \] (16)

Putting (12), (15) and (16) together, we eventually obtain the Chern–Simons theory of the FQSH state, i.e.
\[ \mathcal{L} = - \frac{1}{4\pi} \sum_{IJ} a_{l\mu} K_{IJ} e^{\mu\nu\lambda} \partial_\nu a_{l\lambda} + \frac{e}{2\pi} \sum_I t_I A_{l\mu} e^{\mu\nu\lambda} \partial_\nu a_{l\lambda}, \]
(17)
where \(I, J = \{\uparrow, \downarrow\}\) and
\[ \mathbf{K} = \begin{pmatrix} m & n \\ n & -m \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \] (18)

A generic quasiparticle (or quasihole) in the FQSH state is a bound state which carries \(l_\uparrow\) units of \(a_{l\mu}\) charge and \(l_\downarrow\) units of \(a_{l\mu}^\dagger\) charge, we can denote it by a vector \(\mathbf{l} = (l_\uparrow, l_\downarrow)^T\). If there are \(k\) quasiparticles, the system is labelled as \((L) = (l_{1\uparrow}, l_{1\downarrow})^T, (L) = (l_{1\uparrow}, l_{1\downarrow})^T, \ldots, (l_{k\uparrow}, l_{k\downarrow})^T, \ldots, (l_{k\uparrow}, l_{k\downarrow})^T\), moreover, we use \(j\) to denote the current of the \(l\)th quasiparticle, then the effective Lagrangian of these quasiparticles has the form
\[ \Delta \mathcal{L} = \sum_I a_{l\mu} j^\mu_I, \]
(19)
where
\[ j^\mu_I = j_I^{(1)} + j_I^{(2)} + \cdots + j_I^{(k)} = \sum_{L=1}^{k} j_I^{(L)} f_I^\mu. \] (20)

So the total low-energy effective theory of the FQSH state with quasiparticle excitations is the sum of (17) and (19):
\[ \mathcal{L}_\downarrow = - \frac{1}{4\pi} \sum_{IJ} a_{l\mu} K_{IJ} e^{\mu\nu\lambda} \partial_\nu a_{l\lambda} \]
\[ + \frac{e}{2\pi} \sum_I t_I A_{l\mu} e^{\mu\nu\lambda} \partial_\nu a_{l\lambda} + \sum_I a_{l\mu} j^\mu_I. \] (21)

Integrating out the gauge field \(a_I\) entirely,
\[ e^I f^I d^3 x \mathcal{L}_{\text{Nord}} = \int D\alpha e^I f^I d^3 x \mathcal{L}_I, \]
we obtain a matrix version of the nonlocal Hopf-Lagrangian [19]
\[ \mathcal{L}_{\text{Hopf}} = \pi j_I J_{IJ} \left( \epsilon^{\mu\nu\lambda} \partial_\nu \frac{\partial_\lambda}{\partial^2} \right) j_J, \] (22)
with the modified currents
\[ j_I^{\mu} = j_I^{\mu} + e \left( \frac{1}{2\pi} \int \epsilon^{\mu\nu\lambda} \partial_\nu a_{l\lambda} \right). \] (23)

The Lagrangian (22) contains three types of terms: \(AA\), \(AJ\) and \(jj\). The \(AJ\) term determine the electronic charge of the \(l\)th quasiparticle:
\[ Q^{(L)}/e = \sum_{IJ} j_I J_{IJ}^{-1} j_J = \mathbf{t}^T \mathbf{K}^{-1} \mathbf{l}. \] (24)

The quasiparticles interact with each other via the \(jj\) term, and the value of \(S = \pi \int \mathcal{K}_{-1} \mathcal{K}^{-1} \mathcal{K}_1 / \pi \) can be determined by relating the Hopf invariant to the Gauss linking number between the trajectories of two quasiparticles [20]. So, when we move the \(L\)th quasiparticle all the way around the \(L\)th quasiparticle the wavefunction acquires a phase angle \(\theta^{(L)}\):
\[ \theta^{(L')} = \sum_{IJ} j_I J_{IJ}^{-1} j_J = \mathbf{t}^T \mathbf{K}^{-1} \mathbf{l}. \] (25)

For \(L = L'\), the statistical angle \(\theta^{(L)} = \theta^{(LL)} / 2\) associated with exchanging two identical particles is given by:
\[ \theta^{(L)} = \sum_{IJ} j_I J_{IJ}^{-1} j_J = \mathbf{t}^T \mathbf{K}^{-1} \mathbf{l}. \] (26)

The two types of fundamental quasihole excitations in the FQSH state are labelled by
\[ I_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad I_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (27)

Then, according to equations (24) and (27), the physical electronic charge of the fundamental quasihole excitations of the spin-\(\uparrow\) and spin-\(\downarrow\) layers are given as
\[ Q^{(\uparrow)} = e \frac{m\uparrow + n}{m\uparrow m\downarrow + n^2}, \quad Q^{(\downarrow)} = e \frac{n}{m\uparrow m\downarrow + n^2}. \] (28)

According to equations (26) and (27), the statistical angles for the two types of the fundamental quasihole excitations are given as
\[ \theta^{(\uparrow)} = \frac{\pi}{m\uparrow m\downarrow + n^2}, \quad \theta^{(\downarrow)} = -\frac{\pi}{m\uparrow m\downarrow + n^2}. \] (29)

Applying equations (25) and (27), we can get the mutual statistics between a quasihole in the spin-\(\uparrow\) layer and a quasihole in the spin-\(\downarrow\) layer:
\[ \theta^{(\uparrow\downarrow)} = 2\pi \frac{n}{m\uparrow m\downarrow + n^2}. \] (30)

### 3.2. Berry phase technique

The fractional charge and fractional statistics of the excitations of the \(\nu = 1/k\) Laughlin FQH state can be
calculated directly by Berry phase, which is a simple but profound concept relating to the adiabatic theorem in quantum mechanics. The basic idea is as follows [21]: we compute the Berry phase under the assumption that the quasi-hole is slowly transported around a loop. The calculation will enable us to infer the charge of the quasi-hole, while the statistical angle can be obtained by computing the Berry phase when one quasi-hole winds around another. Now we will use the similar idea to calculate the fractional statistics in the FQSH state.

As we mentioned in section 2, the wavefunction of spin-↑ (−↓) electrons are holomorphic (anti-holomorphic), which is caused by the opposite direction of the effective magnetic field for two layers. In other words, if we change the direction of the effective magnetic field in the spin-↓ layer, then the wavefunction of spin-↓ electrons becomes holomorphic, meanwhile, the ‘effective’ filling factor of spin-↓ electrons becomes a negative number. In this sense, we consider there exists a one-to-one correspondence between the wavefunction (7) and the following wavefunction

\[ \Psi_{m_1-m_n}(\{z_{1i}\}, \{z_{2i}\}) = \prod_{i<j}(z_{1i} - z_{1j})^{m_1} \prod_{i<j}(z_{2i} - z_{2j})^{-m_n} \]
\[ \times \prod_{i,j} (z_{1i} - z_{2i})^b \exp\left( -\frac{1}{\sqrt{2} B} \sum_i |z_{1i}|^2 + \sum_i |z_{2i}|^2 \right). \tag{31} \]

The corresponding wavefunctions of the quasi-hole excitations in the spin-↑ and the spin-↓ layers are

\[ \Psi^{(1)}(\xi_1) = \prod_i (\xi_1 - z_{1i}) \Psi_{m_1-m_N}(\{z_{1i}\}, \{z_{2i}\}), \tag{32} \]
\[ \Psi^{(1)}(\xi_1) = \prod_i (\xi_2 - z_{2i}) \Psi_{m_1-m_N}(\{z_{1i}\}, \{z_{2i}\}). \tag{33} \]

The plasma analogy [22] results from writing the quantum distribution function, the square of the modulus of the many-body wavefunction, as a classical statistical mechanics distribution function for interacting particles in an external potential, i.e.

\[ |\Psi|^2 = e^{-U}. \tag{34} \]

The classical system described by equation (34) is a two-dimensional generalized Coulomb plasma [23], in which there exist two species of particle and interactions between them are logarithmic potentials with three independent coupling constants. According to equations (32) and (34), the classical potential energy corresponding to \( \Psi^{(1)} \) is

\[ U^{(1)} = m_1 \sum_{i<j} (-2 \ln |z_{1i} - z_{1j}|) + m_2 \]
\[ \times \sum_{i<j} (-2 \ln |z_{2i} - z_{2j}|) + n \sum_{i<j} (-2 \ln |z_{1i} - z_{2j}|) \]
\[ + \sum_i \frac{|z_{1i}|^2}{2\rho_B^2} + \sum_i \frac{|z_{2i}|^2}{2\rho_B^2} + \sum_i (-2 \ln |\xi_1 - z_{1i}|). \tag{35} \]

This potential energy function (35) describes a system consisting of pseudospin-↑ and pseudospin-↓ particles. (Here we only apply ↑ and ↓ to denote the two species of particles in two-dimensional plasma, the particle index is called ‘pseudospin’.) All particles have mutual two-dimensional Coulomb interactions with coupling constant \( m_1 \) between two pseudospin-↑ particles, coupling constant \(-m_1\) between two pseudospin-↓ particles, and coupling constant \( n \) between a pseudospin-↑ particle and pseudospin-↓ particle. Meanwhile, both the pseudospin-up and pseudospin-down particles interact with unit coupling constant with the uniform background, and the uniform background has charge density \( (2\pi \rho_B^{-1}) \). From equation (35) we know that, for \( U^{(1)} \), only pseudospin-↑ particles interact with a unit coupling constant with an pseudospin-↑ impurity particle located at the \( \xi_1 \), while pseudospin-↓ particles do not interact with an pseudospin-↑ impurity particle at all. The notion of ‘impurity particle’ corresponds to the quasi-hole excitation in the FQSH state. The charge densities induced in each species of particle by the impurity can be calculated using the perfect screening properties that result from the long range interactions of the plasma. Far enough from the impurity the net interaction must vanish for each species of particle. In other words, the sum of the impurity charge times its coupling constant plus the induced charge in each plasma component times the coupling strength for that plasma component must vanish, i.e.

\[ \left( \begin{array}{cc} m_1 & n \\ n & -m_1 \end{array} \right) \left( \begin{array}{c} q^{(1)}_\uparrow \\ q^{(1)}_\downarrow \end{array} \right) = \left( \begin{array}{c} -1 \\ 0 \end{array} \right), \tag{36} \]

where \( q^{(1)}_\uparrow \) and \( q^{(1)}_\downarrow \) represent the pseudospin-↑ component and the pseudospin-↓ component of the induced charge by the pseudospin-↑ impurity. From equation (36), we get

\[ q^{(1)}_\uparrow = \frac{-m_1}{m_1 m_\downarrow + n^2}, \quad q^{(1)}_\downarrow = \frac{-n}{m_1 m_\downarrow + n^2}. \tag{37} \]

The total physical electronic charge of the quasihole in the spin-↑ layer is

\[ Q^{(1)} = -e(q^{(1)}_\uparrow + q^{(1)}_\downarrow) = e\frac{m_\downarrow + n}{m_1 m_\downarrow + n^2}. \tag{38} \]

Similarly, for wavefunction \( \Psi^{(1)} \), we get

\[ \left( \begin{array}{cc} m_1 & n \\ n & -m_1 \end{array} \right) \left( \begin{array}{c} q^{(1)}_\uparrow \\ q^{(1)}_\downarrow \end{array} \right) = \left( \begin{array}{c} 0 \\ -1 \end{array} \right), \tag{39} \]

where \( q^{(1)}_\uparrow \) and \( q^{(1)}_\downarrow \) represent the pseudospin-↑ component and the pseudospin-↓ component of the induced charge by the pseudospin-↓ impurity. From equation (39), we get

\[ q^{(1)}_\uparrow = \frac{-n}{m_1 m_\downarrow + n^2}, \quad q^{(1)}_\downarrow = \frac{m_1}{m_1 m_\downarrow + n^2}. \tag{40} \]

The total physical electronic charge of the quasihole in the spin-↓ layer is

\[ Q^{(1)} = -e(q^{(1)}_\uparrow + q^{(1)}_\downarrow) = e\frac{m_\downarrow + n}{m_1 m_\downarrow + n^2}. \tag{41} \]

We find that equations (38) and (41) are in perfect agreement with the results (28) in section 3.1.

The normalized wavefunction corresponding to (32) is given by

\[ \Psi^{(1)}(\xi_1, \xi_1^*) = \frac{1}{\sqrt{N^{(1)}(\xi_1, \xi_1^*)}} \Psi^{(1)}(\xi_1), \tag{42} \]
where $N^{(1)}(\xi_\uparrow, \xi_\downarrow) = (\Psi^{(1)}(\xi_\uparrow)|\Psi^{(1)}(\xi_\downarrow))$ is the normalization factor. Assuming that a quasihole is slowly transported around a loop $\Gamma$, the phase change of $\Psi^{(1)}(\xi_\uparrow(t), \xi_\downarrow(t))$ gives the Berry phase $e^{i\gamma}$,

$$
\gamma = i\int_0^t \frac{d}{dt} (\Psi^{(1)}(\xi_\uparrow(t), \xi_\downarrow(t))) dt
$$

$$
= a_{\xi_\uparrow} \frac{d}{dt} a_{\xi_\uparrow} + a_{\xi_\downarrow} \frac{d}{dt} a_{\xi_\downarrow},
$$

(43)

with

$$
a_{\xi_\uparrow} = i(\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) \frac{\partial}{\partial \xi_\uparrow} |\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow)),
$$

$$
a_{\xi_\downarrow} = i(\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) \frac{\partial}{\partial \xi_\downarrow} |\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow)),
$$

(44)

moreover $a_{\xi_\uparrow}$ and $a_{\xi_\downarrow}$ can be expressed in terms of the normalization factor:

$$a_{\xi_\uparrow} = \frac{i}{2} \frac{d}{d\xi_\uparrow} \ln N^{(1)}, \quad a_{\xi_\downarrow} = -\frac{i}{2} \frac{d}{d\xi_\downarrow} \ln N^{(1)}.\quad (45)$$

From equations (43) and (45), we know that in order to get the Berry phase, we should calculate the normalization factor, which can be obtained from the generalized plasma analogue. Because of the complete screening of the plasma, the net force acting on the volume element of the background vanishes, i.e. $q_i^{(1)} \cdot \mathbf{l} = 0$, where $g^{(1)}$ denotes the coupling constant between the pseudospin-$\uparrow$ impurity and the background charge, then we have

$$g^{(1b)} = -(q_i^{(1)} + q_i^{(1)}) = -\frac{m_i + n}{m_i + n^2}.\quad (46)$$

Let us consider a state characterized by the presence of two quasihole excitations in the spin-$\uparrow$ layer. Assume that the quasihole in $\xi_\uparrow$ is adiabatically moved on a full loop $\Gamma$, while the quasihole in $\xi_\downarrow$ is fixed. Again, due to the complete screening of the plasma, the net force on the quasihole in $\xi_\downarrow$ vanishes, i.e. $q_i^{(1)} \cdot \mathbf{l} = 0$, where $g^{(1)}$ denotes the coupling constant between the two pseudospin-$\uparrow$ impurities, then we have

$$g^{(1)} = -q_i^{(1)} = -\frac{m_i + n}{m_i + n^2}.\quad (47)$$

We note that, after including a term $\lim_{\xi_\uparrow \to \xi_\downarrow} \exp\left(-\frac{m_i + n}{m_i + n^2} |\xi_\uparrow|^2 \right)$ corresponding to the interaction between the pseudospin-$\uparrow$ impurity and the background $g^{(1b)} |\xi_\uparrow|^2$, and a term $\lim_{\xi_\uparrow \to \xi_\downarrow} \frac{m_i}{m_i + n^2}$ corresponding to the interaction between the two pseudospin-$\uparrow$ impurities $g^{(1)}(-2\ln |\xi_\uparrow - \xi_\downarrow|)$, the total energy $U(\xi_\uparrow, \xi_\downarrow, \xi_\uparrow', \xi_\downarrow')$ of the plasma with two pseudospin-$\uparrow$ impurities is given by:

$$e^{-U(\xi_\uparrow, \xi_\downarrow, \xi_\uparrow', \xi_\downarrow')} = \left| e^{-\frac{m_i + n}{m_i + n^2} \frac{1}{4\phi} |(\xi_\uparrow|^2 + |\xi_\downarrow|^2)} \right|

\times |\xi_\uparrow - \xi_\downarrow|^{m_i + n^2} \Psi^{(1)}(\xi_\uparrow, \xi_\downarrow)^2.\quad (48)$$

where

$$\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) = \prod_i (\xi_i - z_{\xi_i}) \prod_i (\xi_i' - z_{\xi_i'}), \quad (49)$$

and the corresponding normalization factor is

$$N^{(1)}(\xi_\uparrow, \xi_\downarrow, \xi_\uparrow', \xi_\downarrow') = (\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow)|\Psi^{(1)}(\xi_\uparrow, \xi_\downarrow')).\quad (50)$$

Due to the complete screening of the plasma, the total energy $U(\xi_\uparrow, \xi_\downarrow, \xi_\uparrow', \xi_\downarrow')$ of the plasma is independent of $\xi_\uparrow(\xi_\downarrow')$ and $\xi_\downarrow(\xi_\uparrow')$, i.e.

$$U(\xi_\uparrow, \xi_\downarrow, \xi_\uparrow', \xi_\downarrow') = \text{const.}\quad (51)$$

From equations (48)–(51), we find that

$$N^{(1)}(\xi_\uparrow, \xi_\downarrow, \xi_\uparrow', \xi_\downarrow') \propto \frac{m_i + n}{m_i + n^2} \frac{1}{4\phi} (|\xi_\uparrow|^2 + |\xi_\downarrow|^2)$$

$$\times |\xi_\uparrow - \xi_\downarrow|^{m_i + n^2}.\quad (52)$$

Without loss of generality, setting $\xi_\downarrow = 0$, combining equations (45) and (52), we have

$$a_{\xi_\uparrow} = \frac{m_i + n}{m_i + n^2} \frac{1}{4\phi} \int \xi_\uparrow \frac{d}{d\xi_\uparrow} \ln N^{(1)} = -\frac{m_i + n}{2m_i + n^2} \xi_\uparrow.\quad (53)$$

From equations (43) and (53), we have

$$\gamma = \frac{m_i + n}{m_i + n^2} \frac{1}{4\phi} \left( \int \xi_\uparrow \frac{d}{d\xi_\uparrow} \Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) - \int \xi_\downarrow \frac{d}{d\xi_\downarrow} \Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) \right)$$

$$- \frac{m_i}{2m_i + n^2} \left( \int \xi_\uparrow \frac{d}{d\xi_\uparrow} \Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) - \int \xi_\downarrow \frac{d}{d\xi_\downarrow} \Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) \right).\quad (54)$$

As we move the quasihole $\xi_\uparrow$ around $\xi_\downarrow$, a circle of radius $r = (\text{const.})$, i.e. $\xi_\downarrow = \rho e^{i\theta}$, then $\xi_\uparrow = \rho e^{i\theta}(\xi_\downarrow)^{-1}$, $\int \Psi^{(1)}(\xi_\uparrow, \xi_\downarrow) - \int \Psi^{(1)}(\xi_\downarrow, \xi_\uparrow) = 2\pi i$, we have

$$\gamma = -\pi r^2 \frac{m_i + n}{m_i + n^2} \frac{1}{4\phi} + 2\pi \frac{m_i}{m_i + n^2}.\quad (55)$$

As we mentioned in section 2, $\phi_B$ is just the effective magnetic field in the FQSH state, and $\Phi_B = (e\phi_B)^{-1}$, where $B$ is the effective magnetic field arising from the strain gradient. So the first term of equation (55) is proportional to the effective magnetic field $B$ and the area. Only the second term corresponds to the fractional statistics of the quasihole in the spin-$\uparrow$ layer. Noting that the statistical angle associated with moving one quasihole halfway around the other, we finally obtain that

$$\theta^{(1)} = \pi \frac{m_i}{m_i + n^2}.\quad (56)$$

which is consistent with the result (29) in section 3.1.

In order to reflect the interrelationships between the two methods we used above more clearly, we introduce

$$a_{\xi} = a_{\xi_\uparrow} + a_{\xi_\downarrow}, \quad a_{\psi} = i(a_{\xi_\uparrow} - a_{\xi_\downarrow}).\quad (57)$$
from equations (43), (53) and (57), and we have

$$\gamma = \oint (a_x \, dx + a_y \, dy)$$

(58)

$$\left(a_x, a_y\right) = e^{-\frac{m_\downarrow + n}{m_\uparrow m_\downarrow + n^2}} \left(A_x, A_y\right) + \frac{m_\uparrow}{m_\uparrow m_\downarrow + n^2} \frac{1}{r^2} (-y, x).$$

(59)

One can easily see that the first term of the right-hand side of equation (59) is due to the effective magnetic field, it corresponds to the A–B effect of the quasihole in the effective magnetic field. While the second term, which is proportional to \(\vec{a}' = (a'_x, a'_y) = \frac{1}{\pi} (y, x) = \nabla \theta, (\theta = \text{arg} \xi_x^\uparrow)\), is nothing but the ‘statistical’ vector potential (i.e. Chern–Simons field) arising from the flux tube [24, 15, 18]. We find that

$$\theta^{(\uparrow)} = \frac{1}{2} \frac{m_\downarrow}{m_\uparrow m_\downarrow + n^2} \oint d\mathbf{r} \cdot \nabla \theta = \pi \frac{m_\downarrow}{m_\uparrow m_\downarrow + n^2}.$$

This is just the result given in equations (29) and (56). Similarly, we can obtain the statistical angle \(\theta^{(\downarrow)}\) and the mutual statistics \(\theta^{(\uparrow \downarrow)}\) by generalized plasma analogy and Berry phase technique, and find that they are in perfect agreement with the results given in (29) and (30) in section 3.1.

### 4. Conclusions

In this paper, we study the topological properties of the two types of fundamental quasihole excitation in the FQSH state by Chern–Simons field theory and the Berry phase technique. We use the two different approaches to calculate the fractional charge and statistical angle of each type of quasihole excitation, as well as the mutual statistics between the two different types of excitation, respectively. All results obtained from the two methods are identical.

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