Three-Dimensional Mixed Convection Flow of Viscoelastic Fluid with Thermal Radiation and Convective Conditions

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Abstract
The objective of present research is to examine the thermal radiation effect in three-dimensional mixed convection flow of viscoelastic fluid. The boundary layer analysis has been discussed for flow by an exponentially stretching surface with convective conditions. The resulting partial differential equations are reduced into a system of nonlinear ordinary differential equations using appropriate transformations. The series solutions are developed through a modern technique known as the homotopy analysis method. The convergent expressions of velocity components and temperature are derived. The solutions obtained are dependent on seven sundry parameters including the viscoelastic parameter, mixed convection parameter, ratio parameter, temperature exponent, Prandtl number, Biot number and radiation parameter. A systematic study is performed to analyze the impacts of these influential parameters on the velocity and temperature, the skin friction coefficients and the local Nusselt number. It is observed that mixed convection parameter in momentum and thermal boundary layers has opposite role. Thermal boundary layer is found to decrease when ratio parameter, Prandtl number and temperature exponent are increased. Local Nusselt number is increasing function of viscoelastic parameter and Biot number. Radiation parameter on the Nusselt number has opposite effects when compared with viscoelastic parameter.

Introduction
Analysis of non-Newtonian fluids is an active area of research for the last few years. Such fluids represent many industrially important fluids including certain oils, shampoos, paints, blood at low shear rate, cosmetic products, polymers, body fluids, colloidal fluids, suspension fluids, pasta, ice cream, ice, mud, dough floor etc. In many fields such as food industry, drilling operations and bioengineering, the fluids, either synthetic or natural, are mixtures of different stuffs such as water, particle, oils, red cells and other long chain molecules. Such combination imparts strong rheological properties to the resulting liquids. The dynamic viscosity in non-Newtonian materials varies non-linearly with the shear rate; elasticity is felt through elongational effects and time-dependent effects. The fluids in these situations have been treated as viscoelastic fluids. Further, all the non-Newtonian fluids in nature cannot be predicted by single constitutive equation. Hence all the contributors in the field are using different models of non-Newtonian fluids in their theoretical and experimental studies (see [1-11] and several refs. therein). The boundary layer flows of non-Newtonian fluids in the presence of heat transfer have special importance because of practical engineering applications such as food processing and oil recovery. Especially the stretching flows in this direction are prominent in polymer extrusion, glass fiber and paper production, plastic films, metal extrusion and many others.

After the pioneering works of Sakiadis [12] and Crane [13], numerous works have been presented for two-dimensional boundary layer flow of viscous and non-Newtonian fluids over a surface subject to linear and power law stretching velocities (see some recent studies [14-21]). It has been noted by Gupta and Gupta [22] that stretching mechanism in all realistic situations is not linear. For instance the stretching is not linear in plastic and paper production industries. Besides these the flow and heat transfer by an exponentially stretching surface has been studied by Magyari and Keller [23]. In this attempt the two-dimensional flow of an incompressible viscous fluid is considered. The solutions of laminar boundary layer equations describing heat and flow in a quiescent fluid driven by an exponentially permeable stretching surface are numerically analyzed by Elbashbashy [24]. Al-Odat et al. [25] numerically discussed the thermal boundary layer on an exponentially stretching surface with an exponential temperature distribution. Here magnetohydrodynamic flow is addressed. Nadeem and Lee [26] presented the steady boundary layer flow of nanofluid over an exponential stretching surface. Sajid and Hayat [27] examined the thermal radiation effect in the boundary layer flow and heat transfer of a viscous fluid. The flow is caused by an exponentially stretching sheet. The thermal radiation effect in steady hydromagnetic mixed convection flow of viscous incompressible fluid past an exponentially stretching sheet is examined by El-Aziz and Nabil [28]. Pal [29] carried out an...
Mathematical Modelling

We consider three dimensional mixed convection boundary layer flow of second grade fluid passing an exponentially stretching surface. The surface coincides with the plane $z=0$ and the flow is confined in the region $z>0$. The surface also possess the convective boundary condition. Influence of thermal radiation through Rosseland’s approximation is taken into account. Flow configuration is given below in Fig. 1.

Three-Dimensional Mixed Convection Flow

The governing boundary layer equations for steady three-dimensional flow of viscoelastic fluid can be put into the forms (see Nazar and Latip [11]):

\begin{equation}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\end{equation}

\begin{equation}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{k_u}{\rho} \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial u}{\partial z} \right),
\end{equation}

\begin{equation}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{k_v}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right),
\end{equation}

where $u$, $v$, and $w$ are the velocity components in the $x$, $y$, and $z$ directions respectively, $k_u$ is the material fluid parameter, $\mu$ is the dynamic viscosity, $\nu=\mu/\rho$ is the kinematic viscosity, $T$ is the fluid temperature, $\rho$ is the fluid density, $g$ is the gravitational acceleration, $\beta_p$ is thermal expansion coefficient of temperature, $c_p$ is the specific heat, $k$ is the thermal conductivity and $q_0$ the radiative heat flux. Note that $w$-momentum equation vanishes by applying boundary layer assumptions (see Schlichting [46]).
By using the Rosseland approximation, the radiative heat flux $q_r$ is given by

$$q_r \sim -\frac{4\sigma_s}{3k_e} \frac{\partial T^4}{\partial z}$$

(5)

Where $\sigma_s$ is the Stefan-Boltzmann constant and $k_e$ the mean absorption coefficient. By using the Rosseland approximation, the present analysis is limited to optically thick fluids. If the temperature differences are sufficiently small then Eq. (5) can be linearized by expanding $T^4$ into the Taylor series about $T_w$, which after neglecting higher order terms takes the form:

$$T^4 = 4T_w^4 - 3T_w^4$$

(6)

By using Eqs. (5) and (6), Eq. (4) reduces to

$$\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial z^2} - \frac{16\sigma_s T_w^3}{3k_e \rho c_p} \frac{\partial^2 T}{\partial z^2}$$

(7)

The boundary conditions can be expressed as

$$u = U_w, \quad v = V_w, \quad w = 0, \quad -k \frac{\partial T}{\partial z} = h(T_f - T), \text{at} \ z = 0,$$

$$u \to 0, \quad v \to 0, \quad T \to T_\infty \text{as} \ z \to \infty,$$

(8)

where subscript $w$ corresponds to the wall condition, $k$ the thermal conductivity, $T_f$ is the hot fluid temperature, $h$ the heat transfer coefficient and $T_\infty$ is the free stream temperature.

The velocities and temperature are taken in the following forms:

$$U_w = U_0 e^{\frac{x+y}{L}}, \quad V_w = V_0 e^{\frac{x+y}{L}}, \quad T_w = T_f = T_\infty + T_0 e^{\frac{4(x+y)}{L}}$$

(9)

in which $U_0, V_0$ are the constants, $L$ is the reference length and $A$ is the temperature exponent.

The mathematical analysis of the problem is simplified by using the transformations (Liu et al. [34]):

![Figure 2. $h$-curves for the functions $f'(\eta), g'(\eta)$ and $\theta'(\eta)$.](doi:10.1371/journal.pone.0090038.g002)

| Table 1. Convergence of series solutions for different order of approximations when $\kappa = 0.1, A = 0.2, Pr = 1.2, z = 0.2, n_f = -0.5, n_g = -0.6$ and $n_0 = -0.7$. |
|---|
| Order of approximations | 1 | 5 | 10 | 15 | 20 | 25 |
| $f''(0)$ | 1.06111 | 1.02482 | 1.02609 | 1.02623 | 1.02618 | 1.02618 |
| $g'(0)$ | 0.544444 | 0.548057 | 0.548092 | 0.548043 | 0.548053 | 0.548053 |
| $\theta'(0)$ | 0.317778 | 0.305581 | 0.305729 | 0.305744 | 0.305738 | 0.305738 |

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Incompressibility condition is now clearly satisfied whereas Eqs. (2)–(7) give

\[
\begin{align*}
  u &= U_0 e^{\frac{x+y}{L}} f'(\eta), \quad v = U_0 e^{\frac{x+y}{L}} g'(\eta), \\
  w &= -\left(\frac{v U_0}{2L}\right)^{1/2} (f + \eta f' + \eta g' + \eta^2 g''), \\
  T &= T_\infty + T_0 e^{\frac{A(x+y)}{2L} - \theta(\eta)}, \quad \theta = \left(\frac{U_0}{2L}\right)^{1/2} e^{\frac{x+y}{L}} z. 
\end{align*}
\]

(10)

In which \(K\) is the viscoelastic parameter, \( \alpha \) is the ratio parameter, \( Pr \) is the Prandtl number, \( Gr \) is the local Grashof number, \( R \) is the radiation parameter, \( A \) is the temperature exponent, \( c \) is the Biot number, \( Re \) is the local Reynold number, \( l \) is the mixed convection parameter and prime denotes the differentiation with respect to \( \eta \). These can be defined as

\[
\begin{align*}
  f''' + (f + g) f'' - 2(f' + g') f' + 6 f'' f' + (3 f'' - 3 f'' + \eta g'') f'' + & (4 g' + 2 \eta g'') f'' - (f' + \eta g') f''' \\
  + (f' + \eta g') f''' = 0, \\
  g''' + (f + g) g'' - 2(f' + g') g' + 6 g'' g' + (3 g'' - 3 g'' + \eta g''') g'' + & (4 f' + 2 \eta f'') g'' - (f' + \eta g') g''' = 0, \\
\end{align*}
\]

(11)

(12)

in which \(K\) is the viscoelastic parameter, \( \alpha \) is the ratio parameter, \( Pr \) is the Prandtl number, \( Gr \) is the local Grashof number, \( R \) is the radiation parameter, \( A \) is the temperature exponent, \( \gamma \) is the Biot number, \( Re \) is the local Reynold number, \( \lambda \) is the mixed convection parameter and prime denotes the differentiation with respect to \( \eta \). These can be defined as

\[
\begin{align*}
  f' &\to 0, g' \to 0, \theta \to 0 \text{ as } \eta \to \infty 
\end{align*}
\]

(13)

(14)

(15)
The skin-friction coefficients in the x and y directions are given by

\[ C_{fx} = \frac{\tau_{wx}}{1/2 \rho U_w^2}, \]
\[ C_{fy} = \frac{\tau_{wy}}{1/2 \rho U_w^2}, \] (17)

where

\[ \tau_{wx}|_{z=0} = \left( \mu \frac{\partial u}{\partial z} + k_0 \right) \left[ \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} \right] \]
\[ + \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial u}{\partial z} \]
\[ \left. + \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial u}{\partial z} \right|_{z=0}, \] (18)

\[ \tau_{wy}|_{z=0} = \left( \mu \frac{\partial v}{\partial z} + k_0 \right) \left[ \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2} \right] \]
\[ + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial v}{\partial z} \]
\[ \left. + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial v}{\partial z} \right|_{z=0}. \]

By using Eq. (18) in Eq. (17) the non-dimensional forms of skin friction coefficients are as follows:

\[ C_{fx} = \left( \frac{Re}{T} \right)^{-1/2} [f^\prime + K(-f^\prime + g^\prime) + 5(f^\prime + g^\prime)g^\prime + 2f^\prime f^\prime + 2g^\prime g^\prime]|_{z=0}. \] (19)

Further the local Nusselt number has the form

\[ Nu = -\frac{\left( \frac{16 \sigma T^3_0}{3 k_{\infty}} + k \right) \frac{\partial T}{\partial z}}{k(T_w - T_{\infty})/x} = -\frac{x}{L} \left( \frac{Re}{T} \right)^{1/2} (1 + \frac{4}{3} \frac{R}{f}(0)). \] (21)

Series Solutions

The initial guesses and auxiliary linear operators in the desired HAM solutions are

\[ f_0(\eta) = (1 - e^{-\alpha}), g_0(\eta) = \alpha (1 - e^{-\alpha}), \theta_0(\eta) = \exp(-\eta) \] (22)

\[ L_{f} = f^\prime - f^\prime', L_{g} = g^\prime - g^\prime, L_{0} = 0 - 0 \] (23)
subject to the properties

\[ L_f(C_1 + C_2 e^\phi + C_3 e^{-\phi}) = 0, \quad L_g(C_4 + C_5 e^\phi + C_6 e^{-\phi}) = 0, \]

\[ L_\theta(C_7 e^\phi + C_8 e^{-\phi}) = 0 \] (24)

in which \( C_i \) (i = 1–8) are the arbitrary constants, \( L_f, L_g \) and \( L_\theta \) are the linear operators and \( f_0(\eta), g_0(\eta) \) and \( \theta_0(\eta) \) are the initial guesses.

Following the idea in ref. [38] the zeroth order deformation problems are

\[ (1-p)L_f\left[ \dot{f}(\eta; p) - f_0(\eta) \right] = ph_fN_f\left[ \dot{f}(\eta; p), \dot{g}(\eta; p) \right], \] (25)

\[ (1-p)L_g\left[ \dot{g}(\eta; p) - g_0(\eta) \right] = ph_gN_g\left[ \dot{f}(\eta; p), \dot{g}(\eta; p) \right], \] (26)

\[ (1-p)L_\theta\left[ \dot{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\thetaN_\theta\left[ \dot{f}(\eta; p), \dot{g}(\eta; p), \dot{\theta}(\eta; p) \right], \] (27)

\[ \mu_0(p) = 0, \dot{\mu}_0(p) = 1, \dot{f}(\infty; p) = 0, \dot{g}(0; p) = 0, \]

\[ \dot{g}(0; p) = \alpha, \dot{g}(\infty; p) = 0, \dot{\theta}(0; p) = -\gamma[1 - \theta(0; p)], \]

\[ \dot{\theta}(\infty; p) = 0, \]

For \( p = 0 \) and \( p = 1 \) one has

\[ \dot{f}(\eta; 0) = f_0(\eta), \dot{g}(\eta; 0) = g_0(\eta), \]

\[ \dot{\theta}(\eta; 0) = \theta_0(\eta), \text{ and } \dot{f}(\eta; 1) = f(\eta), \]

\[ \dot{g}(\eta; 1) = g(\eta), \dot{\theta}(\eta; 1) = \theta(\eta). \] (29)

Note that when \( p \) increases from 0 to 1 then \( f(\eta, p), g(\eta, p) \) and \( \theta(\eta, p) \) vary from \( f_0(\eta), g_0(\eta) \) and \( \theta_0(\eta) \) to \( f(\eta), g(\eta), \text{and } \theta(\eta) \). So as the embedding parameter \( p \in [0, 1] \) increases from 0 to 1, the solutions \( f(\eta; p), g(\eta; p) \) and \( \theta(\eta; p) \) of the zeroth order deforma-

mation equations deform from the initial guesses \( f_0(\eta), g_0(\eta) \) and \( \theta_0(\eta) \) to the exact solutions \( f(\eta), g(\eta), \text{and } \theta(\eta) \) of the original nonlinear differential equations. Such kind of continuous variation is called deformation in topology and that is why the Eqs. (26-28) are called the zeroth order deformation equations. The values of the nonlinear operators are given below:

\[ N_f[f(\eta, p), g(\eta, p), \theta(\eta, p)] = \frac{\partial^2 f(\eta, p)}{\partial \eta^2} - \gamma \frac{\partial f(\eta, p)}{\partial \eta} \frac{\partial g(\eta, p)}{\partial \eta} \frac{\partial \theta(\eta, p)}{\partial \eta} + \frac{f(\eta, p) + g(\eta, p)}{\partial \eta} \frac{\partial f(\eta, p)}{\partial \eta} \frac{\partial g(\eta, p)}{\partial \eta} \frac{\partial \theta(\eta, p)}{\partial \eta} 
\]

\[ + K \left[ \frac{\partial f(\eta, p)}{\partial \eta} + \frac{\partial g(\eta, p)}{\partial \eta} \frac{\partial f(\eta, p)}{\partial \eta} \right] + 2\alpha k(\eta, p). \] (30)

For \( p = 0 \) and \( p = 1 \) one has

\[ \dot{f}(\eta; 0) = f_0(\eta), \dot{g}(\eta; 0) = g_0(\eta), \]

\[ \dot{\theta}(\eta; 0) = \theta_0(\eta), \text{ and } \dot{f}(\eta; 1) = f(\eta), \]

\[ \dot{g}(\eta; 1) = g(\eta), \dot{\theta}(\eta; 1) = \theta(\eta). \] (29)
Taylor series expansion of the nonlinear operators. The convergence of above series strongly depends upon $h_f, h_g,$ and $h_0$. Considering that $h_f, h_g,$ and $h_0$ are chosen in such a manner that Eqs. (33)-(35) converge at $p = 1$ then

$$
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \theta_0(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial p^m} \right|_{p=0},
$$

where the convergence of above series strongly depends upon $h_f, h_g,$ and $h_0$. Considering that $h_f, h_g,$ and $h_0$ are chosen in such a manner that Eqs. (33)-(35) converge at $p = 1$ then

$$
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta),
$$

$$
g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta),
$$

and

$$
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m.
$$

Here $h_f, h_g,$ and $h_0$ are the non-zero auxiliary parameters and $N_f, N_g,$ and $N_0$ the nonlinear operators. Taylor series expansion gives

$$
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, f_0(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial p^m} \right|_{p=0},
$$

$$
g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta) p^m, g_0(\eta) = \frac{1}{m!} \left. \frac{\partial^m g(\eta; p)}{\partial p^m} \right|_{p=0},
$$

$$
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m.
$$
The corresponding problems at mth order deformations satisfy

\[ L_f[f_m(\eta) - \chi_m f_m-1(\eta)] = h_f R^m_f(\eta), \]  
\[ L_g[g_m(\eta) - \chi_m g_m-1(\eta)] = h_g R^m_g(\eta), \]  
\[ L_\theta[\theta_m(\eta) - \chi_m \theta_m-1(\eta)] = h_\theta R^m_\theta(\eta). \]

Equation (39) indicates that the range of admissible values of \( \eta \) to determine the suitable ranges for \( h_f \), \( h_g \), and \( h_\theta \) are \(-0.7 \leq h_f \leq -0.2\), \(-0.7 \leq h_g \leq -0.1\), and \(-0.8 \leq h_\theta \leq -0.2\). Table 1 shows that the series solutions converge in the whole region of \( \eta \) when \( h_f = -0.5, h_g = -0.6 \), and \( h_\theta = -0.7 \).

**Convergence Analysis**

We recall that the series [36-38] contain the auxiliary parameters \( h_f, h_g \) and \( h_\theta \). These parameters are useful to adjust and control the convergence of homotopic solutions. Hence the \( h \) curves are sketched at 15th order of approximations in order to determine the suitable ranges for \( h_f, h_g \) and \( h_\theta \). Fig. 2 denotes that the range of admissible values of \( h_f, h_g \) and \( h_\theta \) are \(-0.7 \leq h_f \leq -0.2\), \(-0.7 \leq h_g \leq -0.1\) and \(-0.8 \leq h_\theta \leq -0.2\). Table 1 shows that the series solutions converge in the whole region of \( \eta \) when \( h_f = -0.5, h_g = -0.6 \) and \( h_\theta = -0.7 \).

### Table 2. Comparative values of \(-f'''(0), -g'''(0)\) and \(f(\infty) + g(\infty)\) for different values \( \alpha \) when \( K = \lambda = \gamma = R = 0 \)

| \( \alpha \) | \(-f'''(0)\) | \(-g'''(0)\) | \(f(\infty) + g(\infty)\) | \(-f'''(0)\) | \(-g'''(0)\) | \(f(\infty) + g(\infty)\) |
|----------|-----------|-----------|-------------------|-----------|-----------|-------------------|
| 0.00     | 1.22180856| 0.090564383 | 1.28181 | 0.905644 | 1.28181 | 0.905644 |
| 0.50     | 1.56988846| 0.78494423 | 1.10918263 | 1.56989 | 0.784944 | 1.10918263 |
| 1.00     | 1.81275105| 1.81275105 | 1.28077378 | 1.81275 | 1.81275 | 1.28077378 |

![Figure 16. Influence of Pr on the temperature \( \theta(\eta) \)](https://doi.org/10.1371/journal.pone.0090038.g016)

Figure 16. Influence of Pr on the temperature \( \theta(\eta) \)  

The corresponding problems at mth order deformations satisfy

\[ R^m_g(\eta) = \frac{\theta_m(\eta) - \chi_m \theta_m-1(\eta)}{g_m(\eta) - \chi_m g_m-1(\eta)} = h_g R^m_g(\eta), \]

\[ R^m_\theta(\eta) = \frac{\theta_m(\eta) - \chi_m \theta_m-1(\eta)}{\theta_m(\eta) - \chi_m \theta_m-1(\eta)} = h_\theta R^m_\theta(\eta). \]

Equation (39) indicates that the range of admissible values of \( \eta \) to determine the suitable ranges for \( h_f, h_g \), and \( h_\theta \) are \(-0.7 \leq h_f \leq -0.2\), \(-0.7 \leq h_g \leq -0.1\), and \(-0.8 \leq h_\theta \leq -0.2\).
Discussion of Results

The effects of ratio parameter $a$, viscoelastic parameter $K$, mixed convection parameter $\lambda$, Biot number $\gamma$, and radiation parameter $R$ on the velocity component $f'(\eta)$ are shown in the Figs. 3-7. It is observed from Fig. 3 that velocity component $f'(\eta)$ and thermal boundary layer thickness are decreasing functions of ratio parameter $a$. This is due to the fact that with the increase of ratio parameter $a$, the $x$-component of velocity coefficient decreases which leads to a decrease in both the momentum boundary layer and velocity component $f'(\eta)$. Fig. 4 illustrates the influence of viscoelastic parameter $K$ on the viscoelastic parameter $f'(\eta)$. It is clear that both the boundary layer and velocity component $f'(\eta)$ increase when the viscoelastic parameter increases. Influence of mixed convection parameter $\lambda$ on the velocity component $f'(\eta)$ is analyzed in Fig. 5. Increase in mixed convection parameter $\lambda$ shows an increase in velocity component $f'(\eta)$. This is due to the fact that the buoyancy forces are much more effective rather than the viscous forces. Effects of Biot number $\gamma$ and the radiation parameter $R$ on the velocity component $f'(\eta)$ can be predicted from Figs. 6 and 7. These Figs. depict that the influences of $\gamma$ and $R$ on both the velocity component $f'(\eta)$ and thermal boundary layer thickness are similar i.e., there is increase in these quantities. Figs. 8 and 9 illustrate the variations of ratio parameter $a$ and viscoelastic parameter $K$ on the velocity component $g'(\eta)$. Variation of ratio parameter $a$ is analyzed in Fig. 8. Through comparative study with Fig. 3 it is noted that $f'(\eta)$ decreases while $g'(\eta)$ increases when $a$ increases. Physically, when $a$ increases from zero, the lateral surface starts moving in y-direction and thus the velocity component $g'(\eta)$ increases and the velocity component $f'(\eta)$ decreases. Fig. 9 is plotted to see the variation of viscoelastic parameter $K$ on the velocity component $g'(\eta)$. It is found that both the velocity component $g'(\eta)$ and momentum boundary layer thicknesses are increasing functions of $K$. It is revealed from Figs. 4 and 9 that the effect of $K$ on both the velocities are qualitatively similar. Figs. 10-16 are sketched to see the effects of ratio parameter $a$, viscoelastic parameter $K$, the temperature exponent $\lambda$, Biot number $\gamma$, mixed convection parameter $\lambda$, Radiation parameter $\alpha$, Prandtl number $Pr$ on the temperature $\theta(\eta)$. Fig. 10 is drawn to see the impact of ratio parameter $a$ on the temperature $\theta(\eta)$. It is noted that the temperature $\theta(\eta)$ and also the thermal boundary layer thickness decreases with increasing $a$. Variation of viscoelastic parameter $K$ on the temperature $\theta(\eta)$ is shown in Fig. 11. Here both the temperature and thermal boundary layer thickness are decreasing functions of $K$. Variation of mixed convection parameter $\lambda$ is analyzed in Fig. 12. It is seen that both the temperature $\theta(\eta)$ and thermal boundary layer thickness are decreasing functions of mixed convection parameter $\lambda$. Fig. 13 presents the plots for the variation of Biot number $\gamma$. Note that $\theta(\eta)$ increases when $\gamma$ increases. The thermal boundary layer thickness is also increasing function of $\gamma$. It is also noted that the fluid temperature is zero when the Biot number vanishes. Influence of temperature exponent $\alpha$ is displayed in Fig. 14. It is found that both the temperature $\theta(\eta)$ and thermal boundary layer thickness decrease when $\alpha$ is increased. Also both the temperature $\theta(\eta)$ and thermal boundary layer thickness are increasing functions of thermal radiation parameter $R$ (see Fig. 15). It is observed that an increase in $R$ has the ability to increase thermal boundary layer. It is due to the fact when the thermal radiation parameter increases, the mean absorption coefficient $k_{e}$ will be decreased which in turn increases the divergence of the radiative heat flux. Hence the rate of radiative heat transfer to the fluid is increased and consequently the fluid temperature increases. Fig. 16 is plotted to see the effects of $Pr$ on $\theta(\eta)$. It is noticed that both the temperature profile and thermal boundary layer thickness are decreasing functions of $Pr$. In fact when $Pr$ increases then thermal diffusivity decreases. This indicates reduction in energy transfer ability and ultimate it results in the decrease of thermal boundary layer.

Table 3. Values of skin friction coefficients for different values of $K$ and $a$ when $\lambda = \gamma = 0.5$, $R = 0.3$, $Pr = 1.2$, and $A = 0.2$.

| $K$   | $a$ | $-(g)^{1/2}/C_{f}$ | $-(f)^{1/2}/C_{f}$ |
|------|----|-----------------|-------------------|
| 0.0  | 0.5| 4.95289         | 4.37363            |
| 0.2  | 0.5| 5.15686         | 3.97055            |
| 0.3  | 0.5| 5.42622         | 3.96130            |
| 0.2  | 0.0| 3.72170         | 1.65409            |
| 0.2  | 0.2| 4.30247         | 2.34617            |
| 0.5  | 0.2| 5.42622         | 3.96130            |

Table 4. Values of local Nusselt number $\theta'(0)$ for different values of the parameters $K$, $a$, $\lambda$, $R$, $Pr$, $A$ and $\gamma$.

| $K$ | $\lambda$ | $a$ | $\gamma$ | $R$ | $Pr$ | $A$ | $\theta'(0)$ |
|-----|-----|----|-----|----|----|----|-------------|
| 0.0 | 0.5 | 0.5 | 0.5 | 0.3 | 1.2 | 0.2 | 0.297492    |
| 0.5 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.282007    |
| 0.2 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.329701    |
| 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0885730   |
| 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.216850    |
| 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.305738    |
| 0.2 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.305738    |
| 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.292750    |
| 0.2 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.292752    |
| 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.288530    |
| 0.2 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.305738    |
| 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.305738    |

Three-Dimensional Mixed Convection Flow
Influence of viscoelastic parameter on the velocities $f'(\eta)$ and $g'(\eta)$ is quite opposite. However the effect of viscoelastic parameter $K$ on the velocities $f'(\eta)$ and $g'(\eta)$ is qualitatively similar.

- Momentum boundary layer thickness increases for larger viscoelastic parameter $K$, mixed convection parameter $\lambda$, ratio parameter $\alpha$, Biot number $\gamma$, Prandtl number $Pr$, and temperature exponent $A$ while it decreases through an increase in radiation parameter $R$.

Table 2. Comparative values of $-f''(0)$, $-g''(0)$ and $f(\infty)+g(\infty)$ for different values $x$ when $K_1 = \lambda = \gamma = R = 0$.

Conclusions

Three-dimensional mixed convection flow of viscoelastic fluid over an exponentially stretching surface is analyzed in this study. The analysis is carried out in the presence of thermal radiation subject to convective boundary conditions. The main observations can be summarized as follows:

- Influence of ratio parameter $\alpha$ on the velocities $f'(\eta)$ and $g'(\eta)$ is quite opposite. However the effect of viscoelastic parameter $K$ on the velocities $f'(\eta)$ and $g'(\eta)$ is qualitatively similar.

- Momentum boundary layer thickness increases for $g'(\eta)$ when ratio parameter $\alpha$ is large. Effect of $\alpha$ on $f'(\eta)$ is opposite to that of $g'(\eta)$.

- Velocity component $f'(\eta)$ is increasing function of mixed convection parameter $\lambda$. However $\theta'(\eta)$ decreases with an increase of mixed convection parameter $\lambda$. The impact of Biot number $\gamma$ and radiation parameter $R$ on $f'(\eta)$ and $\theta(\eta)$ are qualitatively similar.

- Momentum boundary layer is an increasing function of mixed convection parameter $\lambda$ while thermal boundary layer is decreasing function of mixed convection parameter $\lambda$.

- Increase in Prandtl number decreases the temperature $\theta(\eta)$.

- Thermal boundary layer thickness decreases when ratio parameter $\alpha$, viscoelastic parameter $K$, mixed convection parameter $\lambda$, Biot number $\gamma$, Prandtl number $Pr$ and temperature exponent $A$ are increased.

- Influence of viscoelastic parameter $K$ on the $x$ and $y$ direction of skin friction coefficients is opposite.

- Both components of skin friction coefficient increase through an increase in ratio parameter $\alpha$.

- Local Nusselt number is an increasing function of Prandtl number $Pr$, ratio parameter $\alpha$, viscoelastic parameter $K$, mixed convection parameter $\lambda$, Biot number $\gamma$ and temperature exponent $A$ while it decreases for radiation parameter $R$.

Supporting Information

File S1 Appendix.

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Author Contributions

Conceived and designed the experiments: TH MBA HHA MSA. Performed the experiments: TH MBA HHA MSA. Analyzed the data: TH MBA HHA MSA. Contributed reagents/materials/analysis tools: TH MBA HHA MSA. Wrote the paper: TH MBA HHA MSA.
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