Invertible Linear Relations Generated by Integral Equations with Operator Measures

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Abstract. We define a minimal relation $L_0$ generated by an integral equation with operators measures and give a description of the relations $L_0 - \lambda E$, $L_0^* - \lambda E$, where $L_0^*$ is adjoint for $L_0$, $\lambda \in \mathbb{C}$. The obtained results are applied to a description of relations $T(\lambda)$ such that $L_0 - \lambda E \subset T(\lambda) \subset L_0^* - \lambda E$ and $T^{-1}(\lambda)$ are bounded everywhere defined operators.

1. Introduction

In this paper, we consider the integral equation

$$y(t) = x_0 - i \int_a^t dp(s)y(s) - i \int_a^t dm(s)f(s),$$

where $y$ is an unknown function, $a \leq t \leq b$; $f$ is an operator in a separable Hilbert space $H$, $J = J^*$, $J^2 = E$ ($E$ is the identical operator); $p$, $m$ are operator-valued measures defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in $H$; $x_0 \in H$, $f \in L_2(H, dm; a, b)$. We assume that the measures $p$, $m$ have bounded variations and $p$ is self-adjoint, $m$ is non-negative.

We define a minimal relation $L_0$ generated by equation (1) and give a description of the relations $L_0 - \lambda E$, $L_0^* - \lambda E$, where $L_0^*$ is adjoint for $L_0$, $\lambda \in \mathbb{C}$. We apply these results to a description of relations $T(\lambda)$ such that $L_0 - \lambda E \subset T(\lambda) \subset L_0^* - \lambda E$ and $T^{-1}(\lambda)$ are bounded everywhere defined operators and give an explicit form of the operators $T^{-1}(\lambda)$.

If the measures $p$, $m$ are absolutely continuous (i.e., $p(\Delta) = \int_\Delta p(t)dt$, $m(\Delta) = \int_\Delta m(t)dt$ for all Borel sets $\Delta \subset [a, b]$, where the functions $\|p(t)\|$, $\|m(t)\|$ belong to $L_1(a, b)$), then integral equation (1) is transformed to a differential equation with a non-negative weight operator function. Linear relations and operators generated by such differential equations were considered in many works (see [14], [4], [5], further detailed bibliography can be found, for example, in [13], [3]).

The study of integral equation (1) differs essentially from the study of differential equations by the presence of the following features: i) a representation of a solution of equation (1) using an evolutional family of operators is possible if the measures $p$, $m$ have not common single-point atoms (see [6]); ii) the

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Lagrange formula contains summands relating to single-point atoms of the measures \( p, m \) (see [7]). Note that this work partially corrects the errors made in the article [8]. Also note that equation (1) was considered in [9, 10] under the assumption that \( m \) is the usual Lebesque measure on \([a, b] \). In [9], an explicit form of operators \( T^{-1}(\lambda) \) is given in the case when the set of single-point atoms of the measure \( p \) can be arranged as an increasing sequence converging to \( b \). In [9], \( L_0, L^*_0 \) are operators. In [10], a description of \( T^{-1}(\lambda) \) is given in terms of boundary values, i.e., necessary and sufficient conditions are obtained under which a boundary value problem determines relations \( T(\lambda) \) such that \( T^{-1}(\lambda) \) are bounded everywhere defined operators.

2. Preliminary assertions

Let \( H \) be a separable Hilbert space with a scalar product \((\cdot, \cdot)\) and a norm \( \| \cdot \| \). We consider a function \( \Delta \rightarrow P(\Delta) \) defined on Borel sets \( \Delta \subset [a, b] \) and taking values in the set of linear bounded operators acting in \( H \). The function \( P \) is called an operator measure on \([a, b] \) (see, for example, [2, ch. 5]) if it is zero on the empty set and the equality \( P(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} P(\Delta_n) \) holds for disjoint Borel sets \( \Delta_n \) where the series converges weakly. Further, we extend any measure \( P \) on \([a, b] \) to a segment \([a, b_0] \) \((b_0 > b)\) letting \( P(\Delta) = 0 \) for each Borel set \( \Delta \subset (b, b_0) \).

By \( V(\Delta) \) we denote \( V(\Delta) = \rho(\Delta) = \sup \sum_i |P(\Delta_i)|\), where the supremum is taken over all finite sums of disjoint Borel sets \( \Delta_i \subset \Delta \). The number \( V(\Delta) \) is called the variation of the measure \( P \) on the Borel set \( \Delta \). Suppose that the measure \( P \) has the bounded variation on \([a, b] \). Then for \( \rho \)-almost all \( \xi \in [a, b] \) there exists an operator function \( \xi \rightarrow \Psi(\xi) \) such that \( \Psi \) possesses the values in the set of linear bounded operators acting in \( H_x, ||\Psi(\xi)|| = 1 \), and the equality

\[
P(\Delta) = \int_\Delta \Psi(\xi) d\rho(\xi) \tag{2}
\]

holds for each Borel set \( \Delta \subset [a, b] \). The function \( \Psi \) is uniquely determined up to values on a set of zero \( \rho \)-measure. Integral (2) converges with respect to the usual operator norm ([2, ch. 5]).

Further, \( \int_0^t \) stands for \( \int_{[0,t]} \) if \( t_0 < t \), for \( -\int_{[t_0,0]} \) if \( t_0 > t \), and for \( 0 \) if \( t_0 = t \). This implies that \( y(\alpha) = x_0 \) in equation (1). A function \( h \) is integrable with respect to the measure \( P \) on a set \( \Delta \) if there exists the Bochner integral \( \int_{[t_0,t]} \Psi(t) h(t) dP = \int_{[t_0,t]} (dP) h(t) \). Then the function \( y(t) = \int_{[t_0,t]} (dP) h(t) \) is continuous from the left.

By \( S_P \) denote a set of single-point atoms of the measure \( P \) (i.e., a set \( t \in [a, b] \) such that \( P(\{t\}) \neq 0 \)). The set \( S_P \) is at most countable. The measure \( P \) is continuous if \( S_P = \emptyset \), it is self-adjoint if \( (P(\Delta))^* = P(\Delta) \) for each Borel set \( \Delta \subset [a, b] \), it is non-negative if \( P(\Delta_x, x) \geq 0 \) for all Borel sets \( \Delta \subset [a, b] \) and for all elements \( x \in H \).

In following Lemma 2.1, \( p_1, p_2, q \) are operator measures having bounded variations on \([a, b] \) and taking values in the set of linear bounded operators acting in \( H \). Suppose that the measure \( q \) is self-adjoint. We assume that these measures are extended on the segment \([a, b_0] \). \([a, b_0] \) \( \supset [a, b] \) \([a, b] \) in the manner described above.

**Lemma 2.1.** [7] Let \( f, g \) be functions integrable on \([a, b_0] \) with respect to the measure \( q \) and \( y_0, z_0 \in H \). Then any functions

\[
y(t) = y_0 - i \int_0^t d(p_1(s) y(s)) - i \int_0^t d(q_1(s)) f(s), \quad z(t) = z_0 - i \int_0^t d(p_2(s) z(s)) - i \int_0^t d(q_1(s)) g(s) \quad (a \leq t_0 < b_0, \ t_0 \leq t \leq b_0)
\]

satisfy the following formula (analogous to the Lagrange one):

\[
\int_{c_1}^{c_2} (d(q)(f)(t), z(t)) - \int_{c_1}^{c_2} (y(t), d(q)(g)(t)) = (i |f(c_2), z(c_2)) - (i |f(c_1), z(c_1)) + \int_{c_1}^{c_2} \left[ (y(t_0), d(p_2(t) z(t)) \right] - \left[ (y(t_0), z(t)) \right] - \left[ (d(p_1)(t), z(t)) \right] - \sum_{t \in S_P, \{c_1, c_2\}} \left[ (i |p_1(t), y(t), p_2(t) z(t)) \right] - \sum_{t \in S_P, \{c_1, c_2\}} \left[ (i |q(t), f(t), p_2(t) z(t)) \right] - \sum_{t \in S_P, \{c_1, c_2\}} \left[ (i |p_1(t), y(t), q(t), g(t)) \right] - \sum_{t \in S_P, \{c_1, c_2\}} \left[ (i |q(t), f(t), q(t), g(t)) \right], \quad t_0 \leq c_1 < c_2 \leq b_0. \tag{3}
\]
Further we assume that measures \( p, m \) have bounded variations and \( p \) is self-adjoint, \( m \) is non-negative. We consider the equation

\[
y(t) = x_0 - i \int_a^t d\bar{p}(s)y(s) - i \int_a^t d\bar{m}(s)f(s),
\]

where \( x_0 \in H, f \) is integrable with respect to the measure \( m \) on \([a, b], a \leq t \leq b_0\).

We construct a continuous measure \( p_0 \) from the measure \( p \) in the following way. We set \( p_0((t_k)) = 0 \) for \( t_k \not\in S_p \) and we set \( p_0(\Lambda) = p(\Lambda) \) for all Borel sets such that \( \Lambda \cap S_p = \emptyset \). Similarly, we construct a continuous measure \( m_0 \) from the measure \( m \). We denote \( \bar{p} = p - p_0, \bar{m} = m - m_0 \). Then \( \bar{p}(I(t_k)) = p(\{t_k\}) \) for all \( t_k \in S_p \) and \( \bar{m}(\Lambda) = 0 \) for all Borel sets \( \Lambda \) such that \( \Lambda \cap S_p = \emptyset \). The similar equalities hold for the measure \( \bar{m} \). The measures \( p_0, \bar{p}, m_0, \bar{m} \) are self-adjoint and the measures \( m_0, \bar{m} \) are non-negative.

We replace \( p \) by \( p_0 \) and \( m \) by \( m_0 \) in (4). Then we obtain the equation

\[
y(t) = x_0 - i \int_a^t d\bar{p}_0(s)y(s) - i \int_a^t d\bar{m}_0(s)f(s).
\]

Equations (4), (5) have unique solutions (see [6]).

By \( W(t, \lambda) \) denote an operator solution of the equation

\[
W(t, \lambda)x_0 = x_0 - i \int_a^t d\bar{p}_0(s)W(s, \lambda)x_0 - i\lambda \int_a^t d\bar{m}_0(s)W(s, \lambda)x_0,
\]

where \( x_0 \in H, \lambda \in C \) (\( C \) is the set of complex numbers). Using Lemma 2.1, we get

\[
W^*(t, \bar{\lambda})W(t, \lambda) = I
\]

by the standard method (see [9]). The functions \( t \to W(t, \lambda) \) and \( t \to W^{-1}(t, \lambda) = [W^*(t, \bar{\lambda})]^{-1} \) are continuous with respect to the uniform operator topology. Consequently there exist constants \( \varepsilon_1 > 0, \varepsilon_2 > 0 \) such that the inequality

\[
\varepsilon_1 \|x\|^2 \leq \|W(t, \lambda)x\|^2 \leq \varepsilon_2 \|x\|^2
\]

holds for all \( x \in H, t \in [a, b_0], \lambda \in C \subset C \) (\( C \) is a compact set).

**Lemma 2.2.** Suppose that a function \( f \) is integrable with respect to the measure \( m \). A function \( y \) is a solution of the equation

\[
y(t) = x_0 - i \int_a^t d\bar{p}_0(s)y(s)x - i\lambda \int_a^t d\bar{m}_0(s)y(s) - i \int_a^t d\bar{m}(s)f(s), \quad x_0 \in H, \quad a \leq t \leq b_0,
\]

if and only if \( y \) has the form

\[
y(t) = W(t, \lambda)x_0 - W(t, \lambda)i \int_a^t W^*(\xi, \bar{\lambda})d\bar{m}(\xi)f(\xi).
\]

**Proof.** We denote \( \bar{p}_0 = p_0 - \lambda m_0 \). The measure \( \bar{p}_0 \) is continuous. Equation (9) has a unique solution (see [6]). It is enough to prove that if we substitute the function from the right side (10) instead \( y \) in the equation (9), then we get the identity. With this substitution, the right side (10) takes the form

\[
x_0 - i \int_a^t d\bar{p}_0(s) \left( W(s, \lambda)x_0 - W(s, \lambda)i \int_a^t W^*(\xi, \bar{\lambda})d\bar{m}(\xi)f(\xi) \right) - \\
- i\lambda \int_a^t d\bar{m}_0(s) \left( W(s, \lambda)x_0 - W(s, \lambda)i \int_a^t W^*(\xi, \bar{\lambda})d\bar{m}(\xi)f(\xi) \right) - i \int_a^t d\bar{m}(s)f(s) = \\
x_0 - i \int_a^t d\bar{p}_0(s) \left( W(s, \lambda)x_0 - W(s, \lambda)i \int_a^t W^*(\xi, \bar{\lambda})d\bar{m}(\xi)f(\xi) \right) - i \int_a^t d\bar{m}(s)f(s) = \\
x_0 - i \int_a^t d\bar{p}_0(s)W(s, \lambda)x_0 - \int_a^t d\bar{p}_0(s)W(s, \lambda)i \int_a^t W^*(\xi, \bar{\lambda})d\bar{m}(\xi)f(\xi) - i \int_a^t d\bar{m}(s)f(s).
\]
We change the limits of integration in the third term of the right-hand side (11). Then the third term takes the form

$$\int_a^t d\vec{p}_0(s)W(s, \lambda)\int_a^\xi W^*(\xi, \lambda)dm(\xi)f(\xi) = \int_a^t \left(\int_a^\xi d\vec{p}_0(s)W(s, \lambda)\right)\int W^*(\xi, \lambda)dm(\xi)f(\xi) =$$

$$= \int_a^t \left(\int_a^\xi d\vec{p}_0(s)W(s, \lambda)\right)\int W^*(\xi, \lambda)dm(\xi)f(\xi) - \int_a^t \left(\int_a^\xi d\vec{p}_0(s)W(s, \lambda)\right)\int W^*(\xi, \lambda)dm(\xi)f(\xi).$$

The last term in (12) is equal to zero since the measure \(\vec{p}_0\) is continuous. Using (6), we continue equality (11)

$$W(t, \lambda)x_0 - \int_a^t \left(\int_a^\xi d\vec{p}_0(s)W(s, \lambda)\right)\int W^*(\xi, \lambda)dm(\xi)f(\xi) - if\int_a^t dm(s)f(s).$$

(13)

It follows from (6) that (13) is equal to

$$W(t, \lambda)x_0 - \int_a^t i((W(t, \lambda) - E) - (W(\xi, \lambda) - E))\int W^*(\xi, \lambda)dm(\xi)f(\xi) - if\int_a^t dm(s)f(s) =$$

$$= W(t, \lambda)x_0 - i\int_a^t W(t, \lambda)W^*(\xi, \lambda)dm(\xi)f(\xi) + i\int_a^t W(\xi, \lambda)W^*(\xi, \lambda)dm(\xi)f(\xi) - if\int_a^t dm(s)f(s).$$

Taking into account (7), we continue the last equality

$$W(t, \lambda)x_0 - iW(t, \lambda)\int_a^t W^*(\xi, \lambda)dm(\xi)f(\xi) + if\int_a^t dm(\xi)f(\xi) - if\int_a^t dm(s)f(s) = y(t).$$

The Lemma is proved.

3. Linear relations generated by the integral equation

Let \(B\) be a Hilbert space. A linear relation \(T\) is understood as any linear manifold \(T \subset B \times B\). The terminology on the linear relations can be found, for example, in [11], [1]. In what follows we make use of the following notations: \([\cdot, \cdot]\) is an ordered pair; \(D(T)\) is the domain of \(T\); \(\mathcal{R}(T)\) is the range of \(T\); \(\text{ker} \ T\) is a set of elements \(x \in B\) such that \([x, 0] \in T\); \(T^{-1}\) is the relation inverse for \(T\), i.e., the relation formed by the pairs \([x', x]\), where \([x, x'] \in T\). A relation \(T\) is called surjective if \(\mathcal{R}(T) = B\). A relation \(T\) is called invertible or injective if \(\text{ker} \ T = [0]\), i.e., the relation \(T^{-1}\) is an operator; it is called continuously invertible if it is closed, invertible, and surjective (i.e., \(T^{-1}\) is a bounded everywhere defined operator). A relation \(T^*\) is called adjoint for \(T\) if \(T^*\) consists of all pairs \([y_1, y_2]\) such that equality \((x_2, y_1) = (x_1, y_2)\) holds for all pairs \([x_1, x_2] \in T\). A relation \(T\) is called symmetric if \(T \subset T^*\).

It is known (see, for example, [12, ch.3], [11, ch.1]) that the graph of an operator \(T: D(T) \to B\) is the set of pairs \([x, Tx]\) \(\in B \times B\), where \(x \in D(T) \subset B\). Consequently, the linear operators can be treated as linear relations; this is why he notation \([x_1, x_2] \in T\) is used also for the operator \(T\). Since all considered relations are linear, we shall often omit the word “linear”.

Let \(m\) be a non-negative operator measure defined on Borel sets \(\Delta \subset [a, b]\) and taking values in the set of linear bounded operators acting in the space \(H\). The measure \(m\) is assumed to have a bounded variation on \([a, b]\). We introduce the quasi-pseudo product \((x, y)_m = \int_a^b ((dm)x(t), y(t))\) on a set of step-like functions with values in \(H\) defined on the segment \([a, b]\). Identifying with zero functions \(y\) obeying \((y, y)_m = 0\) and making the completion, we arrive at the Hilbert space denoted by \(L_2(H, dm; a, b) = \mathcal{H}\). The elements of \(\mathcal{H}\) are the classes of functions identified with respect to the norm \(\|y\|_m = \langle y, y \rangle^{1/2}_m\). In order not to complicate the terminology, the class of functions with a representative \(y\) is indicated by the same symbol and we write \(y \in \mathcal{H}\). The equality of the functions in \(\mathcal{H}\) is understood as the equality for associated equivalence classes.
Let us define a minimal relation $L_0$ in the following way. The relation $L_0$ consists of pairs $[(y, f)] \in \mathcal{S} \times \mathcal{S}$ satisfying the condition: for each pair $[(y_0, f_0)]$ there exists a pair $[(y, f)]$ such that the pairs $[(y, f_0), (y_0, f_0)]$ are identical in $\mathcal{S} \times \mathcal{S}$ and $[(y, f)]$ satisfies equation (4) and the equalities

$$y(\alpha) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_p; \quad \mathbf{m}(\beta) f_0(\beta) = 0, \quad \beta \in \mathcal{S}_m. \quad (14)$$

Further, without loss of generality it can be assumed that if $[(y, f_0)] \in L_0$, then equalities (4), (14) hold for this pair. In general, the relation $L_0$ is not an operator since a function $y$ can happen to be identified with zero in $\mathcal{S}$, while $f$ is non-zero. It follows from Lemma 2.1 that the relation $L_0$ is symmetric.

**Lemma 3.1.** If a pair $[(y, f)] \in L_0 - \lambda E$, then

$$y(t) = -i \int_a^t dp(s) y(s) - i/\lambda \int_a^t d\mathbf{m}(s)y(s) - i \int_a^t d\mathbf{m}_0(s)f(s). \quad (15)$$

**Proof.** Let $[(y, f)] \in L_0 - \lambda E$. It follows from the definition of the relation $L_0$ that the pair $[(y, f)]$ satisfies the equation

$$y(t) = -i \int_a^t dp(s) y(s) - i/\lambda \int_a^t d\mathbf{m}(s)y(s) - i \int_a^t d\mathbf{m}(s)f(s). \quad (16)$$

Consequently,

$$y(t) = -i \int_a^t dp(s) + \bar{p}(s)) y(s) - i/\lambda \int_a^t d(\mathbf{m}(s) + \bar{\mathbf{m}}(s)) y(s) - i \int_a^t d(\mathbf{m}_0(s) + \bar{\mathbf{m}}(s)) f(s). \quad (17)$$

The pair $[(y, f + \lambda y)]$ belongs to $L_0$. Equalities (14) imply $\mathbf{m}(\beta)(\lambda y(\beta) + f(\beta)) = 0$, $y(\alpha) = 0$, where $\alpha \in \mathcal{S}_p$, $\beta \in \mathcal{S}_m$. Using (17), we obtain (15). The Lemma is proved. \qed

**Corollary 3.2.** Equalities (15), (16) hold together for any pairs $[(y, f)] \in L_0 - \lambda E$.

**Lemma 3.3.** A pair $[(\tilde{y}, \tilde{f})] \in \mathcal{S} \times \mathcal{S}$ belongs to the relation $L_0 - \lambda E$ if and only if there exists a pair $[(y, f)]$ such that the pairs $[(\tilde{y}, \tilde{f})]$, $[(y, f)]$ are identical in $\mathcal{S} \times \mathcal{S}$ and the equalities

$$y(t) = -W(t, \lambda) i \int_a^t W(s, \bar{\lambda}) d\mathbf{m}_0(s)f(s), \quad (18)$$

$$y(\alpha) = W(\alpha, \lambda) i \int_a^\alpha W(s, \bar{\lambda}) d\mathbf{m}_0(s)f(s) = 0, \quad \alpha \in \mathcal{S}_p \cup \{b_0\}, \quad (19)$$

$$\mathbf{m}(\beta)(\lambda y(\beta) + f(\beta)) = 0, \quad \beta \in \mathcal{S}_m. \quad (20)$$

hold.

**Proof.** The desired assertion follows from (14) and Lemmas 2.2, 3.1 and Corollary 3.2. \qed

**Corollary 3.4.** If $y \in D(L_0)$, then $y$ is continuous and $y(b_0) = 0$.

**Corollary 3.5.** Suppose a pair $[(y, f)]$ satisfies equality (18). The function $f \in \mathcal{S}$ belongs to the range $R(L_0 - \lambda E)$ if and only if $f$ satisfies the conditions

$$\int_a^\alpha W(s, \bar{\lambda}) d\mathbf{m}_0(s)f(s) = 0, \quad \mathbf{m}(\beta)(\lambda y(\beta) + f(\beta)) = 0, \quad (21)$$

where $\alpha \in \mathcal{S}_p \cup \{b_0\}$, $\beta \in \mathcal{S}_m$. 

Remark 3.6. The first equality in (21) is equivalent to the following
\[ \int_{\alpha_1}^{\alpha_2} W^*(s, \overline{\lambda}) d\mathbf{m}_0(s) f(s) = 0, \quad \alpha_1, \alpha_2 \in S_p \cup [a] \cup [b_0]. \] (22)

Remark 3.7. It follows from Lemma 3.3, Corollary 3.4 that we can replace \( b_0 \) by \( b \) in (19), (21), (22).

Lemma 3.8. The relation \( L_0 \) is closed.

Proof. Suppose \( \{y_n, f_n\} \in L_0 \). Using (18) – (20) for \( \lambda = 0 \), we obtain
\[ y_n(t) = -W(t, 0) \int_0^t W^*(s, 0) d\mathbf{m}_0(s) f_n(s), \] (23)
\[ y_n(\alpha) = W(\alpha, 0) \int_0^\alpha W^*(s, 0) d\mathbf{m}_0(s) f_n(s) = 0, \quad m(\beta) f_\beta = 0, \] (24)
where \( \alpha \in S_p \cup [b_0], \beta \in S_m \). Suppose that the sequences \( \{y_n, f_n\} \) converge in \( \mathcal{H} \) to \( y, f \), respectively. We note that if a sequence converges in \( \mathcal{H} = L_2(H, d\mathbf{m}; a, b) \), then this sequence converges in \( L_2(H, d\mathbf{m}_0; a, b) \). Moreover,
\[ \|f_n - f\|_2^2 = (m(\beta)(f_n(\beta) - f(\beta)), f_n(\beta) - f(\beta)) = (m(\beta) f_\beta, f_\beta), \]
where \( \beta \in S_m \). Passing to the limit as \( n \to \infty \) in (23), (24), we obtain equalities (18) – (20) for \( \lambda = 0 \). It follows from Lemma 3.3 that the pair \( (y, f) \in L_0 \). The Lemma is proved. \( \square \)

By \( \mathcal{X}_A = \mathcal{X}_A(t) \) denote an operator characteristic function of a set \( A \), i.e., \( \mathcal{X}_A(t) = E \) if \( t \in A \) and \( \mathcal{X}_A(t) = 0 \) if \( t \notin A \). We shall often omit the argument \( t \) in the notation \( \mathcal{X}_A \).

Remark 3.9. Equality (20) means that the function \( \mathcal{X}_{\mathcal{I}_0}(\lambda y(\beta) + f(\beta)) \) is identified with zero in the space \( \mathcal{H} \).

By \( \overline{S}_p \) denote the closure of the set \( S_p \). Let \( S_0 \) be the set \( t \in [a, b] \) such that \( y(t) = 0 \) for all \( y \in \mathcal{D}(L_0) \). It follows from (14) and Corollary 3.4 that \( a, b \in S_0 \) and \( S_p \subset S_0 \). Corollary 3.4 implies that the set \( S_0 \) is closed. Therefore, \( \overline{S}_p \cup [a] \cup [b] \subset S_0 \).

Lemma 3.10. Suppose \( \{y, f\} \in L_0 \). Then \( f(t) = 0 \) for \( m \)-almost all \( t \in S_0 \).

Proof. Using Corollary 3.5 (for \( \lambda = 0 \)) and Remark 3.7, we get
\[ \int_a^t (d\mathbf{m}_0(s)f(s), W(s, 0)x) = 0, \quad m(\beta) f_\beta = 0 \]
for all \( x \in H \) and for all \( \alpha \in S_0, \beta \in S_m \). Hence equality (2) implies
\[ \int_a^t (\Psi_m(s)f(s), W(s, 0)x)d\mathbf{m}_0(s) = 0, \quad m(\beta) f_\beta = 0. \] (25)
We denote
\[ \varphi_x(t) = (\Psi_m(t)f(t), W(t, 0)x), \quad \Phi_x(t) = \int_a^t \varphi_x(s)d\mathbf{m}_0(s). \]

The function \( \Phi_x \) is continuous. Hence it follows from (25) that \( \Phi_x(t) = 0 \) for all \( t \in S_0 \). Therefore, \( \varphi_x(t) = 0 \) for \( \varphi_m \)-almost all \( t \in S_0 \).

Let \( \mathcal{I}_n \) be a countable everywhere dense set in \( H \) and let \( X_n \) be a set \( t \in S_0 \) such that \( \varphi_{\mathcal{I}_n}(t) = 0 \). Then
\[ \varphi_m(X_n) = \varphi_m(S_0). \]
We denote \( X = \cap_n X_n \). Then \( \varphi_m(X) = \varphi_m(S_0) \) and \( \varphi_x(t) = 0 \) for all \( n \). If a sequence
It follows from Lemma 2.2 that the function \( G \) is a solution of equation (9) on the segment \([\alpha_k, \gamma]\), \( \gamma < \beta_k \) (for \( a = \alpha_k, y = G, f = g, x_0 = 0 \)).
Suppose a pair \((y, f) \in L_0 - \overline{\lambda E}\). The pair \((y, f)\) satisfies equation (16) in which \(\lambda\) is replaced by \(\overline{\lambda}\). Therefore we can apply formula (3) to the functions \(y, f, G, g\) for \(c_1 = a_k, c_2 = y, q = m, p_1 = p_0 + \overline{\lambda}m, p_2 = p_0 + \lambda m_0\). Since the measures \(p_0, m_0\) is continuous, self-adjoint, \(m = m_0 + \tilde{m}\), and (20) holds, we obtain

\[
\int_{\mathcal{G}} (\tilde{d} m(s)) f(s), G(s) - \int_{\mathcal{G}} (y, \tilde{d} m(s)) g(s) = (i y(y), G(y)) - \int_{\mathcal{G}} \overline{\lambda} (\tilde{m}(s)) y(s), G(s). \]

Using the equality \(G_\alpha(t) = G(t) - x_{S_m} G(t)\) and (20), we get

\[
\int_{\mathcal{G}} (\tilde{d} m(s)) f(s), G_\alpha(s) - \int_{\mathcal{G}} (y, \tilde{d} m(s)) g(s) = (i y(y), G(y)) - \sum_{s \in S_m \cap [a, b]} \overline{\lambda} (\tilde{m}(s)) y(s), G(s) - \sum_{s \in S_m \cap [a, b]} (\tilde{m}(s)) f(s), G(s) = (i y(y), G(y)). \tag{29}
\]

The function \(y\) is continuous from the left and \(y(\beta_k) = 0\) (also see Corollary 3.4). Hence passing to the limit as \(y \rightarrow \beta_k - 0\) in (29), we obtain

\[
\int_{\mathcal{G}} (\tilde{d} m(s)) f(s), G_\alpha(s) = \int_{\mathcal{G}} (y(s), \tilde{d} m(s)) g(s). \tag{30}
\]

This implies the desired statement. The Lemma is proved. \(\square\)

By \(\mathcal{S}_{10}\) (by \(\mathcal{S}_{11}\)) denote a subspace of functions that belong to \(\mathcal{S}_1\) and vanish on \(S_m\) (on \([a, b] \setminus S_m\), respectively) with respect to the norm in \(\mathcal{S}_1\). So, \(\mathcal{S}_{10}\) (\(\mathcal{S}_{11}\)) consists of functions of the form \(x_{[a,b]\cup\{a,b\}} h\) (of the form \(x_{\mathcal{S}_m} h\), respectively), where \(h \in \mathcal{S}_1\) is an arbitrary function. Therefore,

\[
\mathcal{S}_1 = \mathcal{S}_{10} \oplus \mathcal{S}_{11}, \quad \mathcal{S}_0 = \mathcal{S}_{10} \oplus \mathcal{S}_{11}. \tag{31}
\]

Obviously, the space \(\mathcal{S}_{11}\) is the closure in \(\mathcal{S}_1\) of the linear span of functions that have the form \(x_{[a,b]} \cdot x\), where \(x \in H, \tau \in S_m \cap S_0\). By (14), it follows that \(\mathcal{S}_{11} \subset \ker L_0^\ast\).

**Remark 3.13.** Suppose \(\tau \in S_m \cap S_0\). Then \(x_{[a,b]}(\cdot) x \in \mathcal{S}_0\) for \(x \in H\). Hence (26) implies that the pair \([0, x_{[a,b]}(\cdot) x] \in L_0^\ast\). In particular, Remark 3.11 implies that this is true for \(\tau \in S_m \cap (\bigcup_{k=1}^\infty [a_k, b_k] \cup \{a, b\})\), where \(a_k, b_k\) are boundaries of intervals \((a_k, b_k) = \mathcal{S}_k \subset \mathcal{J}\).

We define an operator \(\mathcal{U}_0(\lambda) : \mathcal{S}_1 \rightarrow \mathcal{S}_1\) by the equation

\[
(\mathcal{U}_0(\lambda) f)(t) = -x_{[a,b],\mathcal{S}_m} w(t, \lambda) \int_a^t w^\tau(s, \overline{\lambda}) d m(s) f(s), \quad f \in \mathcal{S}_1. \tag{32}
\]

The operator \(\mathcal{U}_0(\lambda)\) is bounded. Obviously, \(\mathcal{U}_0(0) = 0\). Taking into account (27) and Lemma 3.12, we obtain that the pair \([\mathcal{U}_0(\lambda) f, x_{[a,b],\mathcal{S}_m} (\cdot) \lambda f] \in L_0^\ast - \lambda E\).

Let \(u_0(t, \lambda, \tau) : H \rightarrow \mathcal{S}_1\) be an operator acting by the formula

\[
u_0(t, \lambda, \tau) x = (\mathcal{U}_0(\lambda) x_{[a,b]}(\cdot) x)(t) = -x_{[a,b],\mathcal{S}_m} w(t, \lambda) \int_a^t w^\tau(s, \overline{\lambda}) d m(s) \lambda x_{[a,b]}(s) x,
\]

where \(x \in H, \tau \in (a_k, b_k) \cap S_m, (a_k, b_k) = \mathcal{J}_k \subset \mathcal{J}\). Then the pair \([u_0(t, \lambda, \tau) x, \lambda x_{[a,b]}(\cdot) x] \in L_0^\ast - \lambda E\). The definition of \(L_0\) implies that the function \(x_{[a,b]}(\cdot) x \in \ker L_0^\ast\). Consequently, \([x_{[a,b]}(\cdot) x, -\lambda x_{[a,b]}(\cdot) x] \in L_0^\ast - \lambda E\). Thus, for any \(x \in H\) the function

\[
u_0(t, \lambda, \tau) x + x_{[a,b]}(\cdot) x \in \ker (L_0^\ast - \lambda E). \tag{33}
\]
Using (31), we get
\[\|u_k(\cdot, \lambda) x\|_0 \leq |\lambda| \gamma \|x_{k1}(\cdot) x\|_0 = |\lambda| \gamma m^{1/2}(\tau) x,\] (33)
where \(\gamma > 0\), \(x \in H\), \(\tau \in (\alpha_k, \beta_k) \cap S_m\).

The linear span of functions of the form \(x_{k1}(\cdot)x\) \((x \in H, \tau \in S_m \setminus S_0)\) is dense in the space \(S_{11}\). It follows from (31), (32) that for any the function \(z_1 \in S_{11}\)
\[\mathcal{U}_k(\lambda) z_1 + z_1 \in \ker(L_{10}^* - \lambda E).\] (34)

**Lemma 3.14.** The linear span of functions of the form \(x_{(\alpha, \beta), S_m} w_k(\cdot, \lambda) x\) is dense in \(S_{10} \cap \ker(L_{10}^* - \lambda E)\). Here \(x \in H; k = 1, \ldots, k_1\) if \(k_1\) is finite and \(k\) is any natural number if \(k_1\) is infinite.

**Proof.** Suppose that \(h_0 \in S_{10} \cap \ker(L_{10}^* - \lambda E)\) and
\[(h_0, x_{(\alpha, \beta), S_m} w_k(\cdot, \lambda) x)_0 = \int_a^b (d\mu(s)) h_0(s), x_{(\alpha, \beta), S_m} w_k(s, \lambda) x = 0\] (35)
for all \(x \in H\) and for all \(k\). Let us prove that \(h_0(t) = 0\) \(m\)-almost everywhere. We denote
\[y(t) = -\bar{W}(t, \lambda) t \int_a^t W(s, \lambda)d\mu_0(s)h_0(s).\] (36)

We define the function \(h\) as follows. We put \(h(t) = h_0(t)\) for \(t \in [a, b] \setminus S_m\), and \(h(t) = -\bar{\bar{\lambda}}^{-1} y(t)\) for \(t \in S_m\), \(\lambda \neq 0\), and \(h(t) = 0\) for \(t \in S_m\), \(\lambda = 0\). The function \(y\) will not change if \(h_0\) is replaced by \(h\) in (36). Moreover, equality (35) will remain with this replacement. Then it follows from Lemma 3.3 and Corollary 3.5 that the pair \((y, h) \in L_{10} - \overline{\lambda E}\). Hence, \((h_0, h)_0 = 0\) since \(h_0 \in \ker(L_{10}^* - \lambda E)\). On the other hand, \((h_0, h)_0 = (h_0, h)_0\). This implies \(h_0 = 0\). The Lemma is proved. \(\square\)

**Lemma 3.15.** The linear span of functions of the form \(x_{(\alpha, \beta), S_m} w_k(\cdot, \lambda) x_0 + u_k(\cdot, \lambda) x_0 + x_{k1}(\cdot) x_0\) is dense in \(\ker(L_{10}^* - \lambda E)\). Here \(x_k, x_0 \in H; \tau \in (\alpha_k, \beta_k) \cap S_m; k = 1, \ldots, k_1\) if \(k_1\) is finite and \(k\) is any natural number if \(k_1\) is infinite.

**Proof.** Let \(z \in \ker(L_{10}^* - \lambda E)\). Then \(z = z_0 + z_1\), where \(z_0 \in S_{10}, z_1 \in S_{11}\). Suppose that the function \(z\) is orthogonal to the functions listed in the condition of the Lemma. We claim that \(z = 0\). The pair \([z_1, -\lambda z_1] \in L_{10}^* - \lambda E\) since \(z_1 \in \ker L_{10}^*\). Therefore, \([z_0, \lambda z_1] \in L_{10}^* - \lambda E\). We denote \(z_k = x_{(\alpha, \beta), S_m} z_0, z_{1k} = x_{(\alpha, \beta), S_m} z_1\). Using Lemma 3.12, we get
\[z_{0k}(t) = -x_{(\alpha, \beta), S_m} w_k(t, \lambda) \int_a^t W(s, \lambda)d\mu_0(s)z_{1k}(s) + h_0(t),\] (37)
where \(h_0 \in \ker(L_{10}^* - \lambda E)\). Moreover, \(h_0 \in S_{10}\) since \(z_{0k} \in S_{10}\) and the first term in (37) belongs to \(S_{10}\). According to Lemma 3.14, \(h_0\) belongs to the closure of linear span of functions that have the form \(x_{(\alpha', \beta'), S_m} w_k(t, \lambda)x'\), \(x' \in H\). Using (30), (37), we obtain \(z_k = \mathcal{U}_k(\lambda) z_{1k} + z_{1k} + h_0\). By assumption, \((z_k, \mathcal{U}_k(\lambda) z_{1k} + z_{1k})_0 = 0\) and \((z_k, h_0)_0 = 0\). Hence, \((z_k, z_{1k})_0 = 0\) for all \(k\). Therefore, \((z, z)_0 = 0\). The Lemma is proved. \(\square\)

**Remark 3.16.** The Lemma 3.15 remains true if functions of the form \(u_k(\cdot, \lambda) x_k + x_{k1}(\cdot) x_k\) are replaced by functions \(u_k(\cdot, \tau) x_k + x_{k1}(\cdot) w_k(\tau, x_k) x_k\). Indeed, by (8), (27), it follows that the operator \(w_k(\tau, x_k)\) is continuously invertible for \(\tau \in \Phi = \{(\alpha_k, \beta_k)\}\). Hence the linear spans of the noted above functions coincide.

Let \(M\) be a set consisting of intervals \(J \in J\) and single-point sets \(\{\tau\}\), where \(\tau \in \mathcal{S}_m \setminus S_0\). The set \(M\) is at most countable. Let \(k\) be the number of elements in \(M\). We arrange the elements of \(M\) in the form of a finite or infinite sequence and denote these elements by \(E_k\), where \(k\) is any natural number if the number of elements in \(M\) is infinite. If \(k \leq k\) if the number of elements in \(M\) is finite.
We shall assign an operator function \( v_k \) to each element \( E_k \in M \) in the following way. If \( E_k \) is the interval, \( E_k = \{ \tau_k \} \in J \), then
\[
v_k(t, \lambda) = x_{[\alpha_k, \beta_k]}'S_n w_k(t, \lambda) . \tag{38}
\]
If \( E_k \) is a single-point set, \( E_k = \{ \tau_k \} \in S_m \setminus S_0 \), and \( \tau_k \in J_n = (\alpha_n, \beta_n) \in J \), then
\[
v_k(t, \lambda) = u_n(t, \lambda, \tau_k) w_n(\tau_k, \lambda) + x_{[\tau_k]}(t) w_n(\tau_k, \lambda) . \tag{39}
\]

**Remark 3.17.** It follows from (27) that equality (38) is equivalent to the following: \( v_k(t, \lambda) = x_{[\alpha_k, \beta_k]}S_n w_k(t, \lambda) \).

**Lemma 3.18.** The linear span of functions \( t \to v_k(t, \lambda)\xi_k (\xi_k \in H) \) is dense in \( \ker(L^*_{10} - \lambda E) \). (Here \( k \in \mathbb{N} \) if \( k = \infty \), and \( 1 \leq k < k \) if \( k \) is finite.)

**Proof.** The required statement follows from Remark 3.16 and Lemma 3.15 immediately. \( \square \)

**Corollary 3.19.** A function \( f \in S_1 \) belongs to the range \( R(L_{10} - \lambda E) \) if and only if the equality \( (f, v_k(\cdot, \lambda))_S = 0 \) holds for all \( k \). (Here \( k \in \mathbb{N} \) if \( k = \infty \), and \( 1 \leq k < k \) if \( k \) is finite.)

**Proof.** The proof follows from the equality \( R(L_{10} - \lambda E) \oplus \ker(L_{10}^* - \lambda E) = S_1 \) and Lemma 3.18. \( \square \)

Further, we denote \( v_k(t, 0) = v_k(t) \). We note that \( u_k(t, 0, \tau) = 0 \) (see (31)).

Let \( Q_{k,0} \) be a set \( x \in H \) such that the functions \( t \to v_k(t)x \) are identical with zero in \( S \). We put \( Q_k = H \cap Q_{k,0} \).

On the linear space \( Q_k \) we introduce a norm \( \| \cdot \|_k \) by the equality
\[
\| \xi_k \|_k = \| v_k(\cdot)\xi_k \|_S , \quad \xi_k \in Q_k . \tag{40}
\]

We note that if \( v_k \) has form (38) with \( \lambda = 0 \), then
\[
\| \xi_k \|_{k,0} = \left( \int_{[\alpha_k, \beta_k]} (d\mathbf{m}(s) w_k(s, 0)(\xi_k, w_k(s, 0)\xi_k))^{1/2} = \left( \int_{[\alpha_k, \beta_k]} (d\mathbf{m}(s) w_k(s, 0)\xi_k, w_k(s, 0)\xi_k) \right)^{1/2} , \quad \xi_k \in Q_k .
\]

If \( v_k \) has form (39) with \( \lambda = 0 \), then
\[
\| \xi_k \|_{k,0} = (d\mathbf{m}(\tau_k)) w_n(\tau_k, 0)\xi_k, w_n(\tau_k, 0)\xi_k)^{1/2} = \left[ \int (d\mathbf{m}(\tau_k)) w_n(\tau_k, 0)\xi_k \right]_S , \quad \xi_k \in Q_k .
\]

By \( Q_k^* \) denote the completion of \( Q_k \) with respect to norm (40). This norm (40) is generated by the scalar product
\[
(\xi_k, \eta_k)_S = (v_k(\cdot)\xi_k, v_k(\cdot)\xi_k)_S . \tag{41}
\]

where \( \xi_k, \eta_k \in Q_k \). From formula (2) in which the measure \( P \) is replaced by \( \mathbf{m} \), it follows that
\[
\| \xi_k \|_k \leq \gamma \| \xi_k \|_S , \quad \xi_k \in Q_k . \tag{42}
\]

where \( \gamma > 0 \) is independent of \( \xi_k \in Q_k \).

It follows from (42) that the space \( Q_k^* \) can be treated as a space with a negative norm with respect to \( Q_k^* \) \([2, \text{ch.} 1], [11, \text{ch.} 2] \). By \( Q_k^+ \) denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that \( Q_k^+ \subset Q_k \subset Q_k^* \). By \( (, )_S \) and \( \| \cdot \|_S \) we denote the scalar product and the norm in \( Q_k^* \), respectively.

**Lemma 3.20.** There exist constants \( \gamma_{1k}, \gamma_{2k} > 0 \) such that the inequality
\[
\gamma_{1k} V(t, x) \leq \| v_k(\cdot, \lambda) x \|_S \leq \gamma_{2k} V(t, x) \| x \|_S \tag{43}
\]
holds for all \( x \in H \).
Proof. Using Lemma 2.2 and (6), we get

\begin{equation}
W(t, \lambda)x_0 = W(t, 0)x_0 - W(t, 0)i\int_0^t W'(s, 0)dm(s)\lambda W(s, \lambda)x_0, \quad x_0 \in H, \tag{44}
\end{equation}

\begin{equation}
W(0, \lambda)x_0 = W(\lambda)\Pi_0 + W(\lambda)i\int_0^t W'(s, \lambda)dm(s)\lambda W(s, \lambda)x_0, \quad x_0 \in H. \tag{45}
\end{equation}

Suppose that \(v_k\) has form (38). Using (27), (44), (45), we obtain

\begin{equation}
v_k(t, \lambda)x_0 = v_k(t, 0)x_0 - v_k(t, 0)i\int_0^t v_k'(s, 0)dm(s)\lambda v_k(s, \lambda)x_0, \quad x_0 \in H, \tag{46}
\end{equation}

\begin{equation}
v_k(t, 0)x_0 = v_k(t, \lambda)x_0 + v_k(t, \lambda)i\int_0^t v_k'(s, \lambda)dm(s)\lambda v_k(s, \lambda)x_0, \quad x_0 \in H. \tag{47}
\end{equation}

Equalities (8), (46), (47) imply (43) in the case when \(v_k\) has form (38). Suppose that \(v_k\) has form (39). Using (39), (31), we get

\[
\|v_k(\cdot, \lambda)x\|_b^2 = \|u_n(\cdot, \tau_k\\w_n(\tau_k, \lambda)x)^2 + \|X_{iJ}(\cdot)w_n(\tau_k, \lambda)x\|^2_b \geq \|X_{iJ}(\cdot)w_n(\tau_k, \lambda)x\|^2_b = \|v_k(\cdot)x\|^2_b .
\]

On the other hand, using (31), (33), we obtain

\[
\|v_k(\cdot, \lambda)x\|_b \leq \|u_n(\cdot, \tau_k\\w_n(\tau_k, \lambda)x\|_b + \|X_{iJ}(\cdot)w_n(\tau_k, \lambda)x\|_b \leq \gamma_3 \|v_k(\cdot)x\|_b ,
\]

where \(\gamma_3 > 0\). The Lemma is proved.

Remark 3.21. By (43), it follows that the set \(Q_{\lambda, 0}\) will not change if the function \(v_k(\cdot, \lambda)\) is replaced by \(v_k(\cdot, 0)\) in the definition of \(Q_{\lambda, 0}\). Moreover, with such a replacement, the space \(Q_+\) will not change in the following sense: the set \(Q_+\) will not change, and the norm in it will be replaced by the equivalent one. The similar statement holds for the space \(Q_+^*\).

Suppose that a sequence \(\{x_{n_l}\}, x_{n_l} \in Q_\lambda\), converges in the space \(Q_\lambda\) to \(x_0 \in Q_\lambda\) as \(n \to \infty\). It follows from Lemma 3.20 that the sequence \(\{v_k(\cdot, \lambda)x_{n_l}\}\) is fundamental in \(\bar{\mathcal{D}}\). Therefore this sequence converges to some element in \(\bar{\mathcal{D}}\). By \(v_k(\cdot, \lambda)x_0\) we denote this element.

Let \(Q_{\lambda}^{-1} = Q_1^{-1} \times \ldots \times Q_N^{-1} \subset Q_\lambda^{-1} \subset Q_\lambda^{-1}\), respectively, and let \(V_N(t, \lambda) = (v_1(t, \lambda), \ldots, v_N(t, \lambda))\) be the operator one-row matrix. It is convenient to treat elements from \(Q_{\lambda}^{-1}\) as one-column matrices, and to assume that \(V_N(t, \lambda)\xi_N = \sum_{k=1}^N v_k(t, \lambda)\xi_k\), where we denote \(\xi_N = \text{col}(\xi_1, \ldots, \xi_N) \in Q_{\lambda}^{-1}\).\(\xi_k \in Q_{\lambda}^{-1}\). Let \(\ker_1(\lambda)\) be a linear space of functions \(t \to v_k(t, \lambda)\xi_k\), \(\xi_k \in Q_{\lambda}^{-1}\). By (40) and Lemma 3.20, it follows that \(\ker_1(\lambda)\) is closed in \(\bar{\mathcal{D}}\). The spaces \(\ker_1(0)\) and \(\ker_1(0)\) are orthogonal for \(k \neq j\). We denote \(\mathcal{K}_N(\lambda) = \ker_1(\lambda) + \ldots + \ker_N(\lambda)\). Obviously, \(\mathcal{K}_N(\lambda) \subset \mathcal{K}_N(\lambda)\) for \(N_1 < N_2\).

Lemma 3.22. The set \(\cup \mathcal{K}_N(\lambda)\) is dense in \(\text{ker}(L_1^* - A_E)\).

Proof. The required statement follows from Lemma 3.18 immediately.
Since $\tilde{Q}_N$ is dense in $\tilde{Q}_N^*$, we obtain

$$V_N(\lambda)f = \int_a^b V_N(s, \lambda)dm(s)f(s).$$ (48)

Thus, we have proved the following statement.

**Lemma 3.23.** The operator $V_N(\lambda)$ maps continuously and one-to-one $\tilde{Q}_N$ onto $\mathcal{K}_N(\lambda)$. The adjoint operator $V_N^*(\lambda)$ maps continuously $\mathcal{K}_N(\lambda)$ onto $\tilde{Q}_N^*$ and acts by formula (48). Moreover, $V_N^*(\lambda)$ maps one-to-one $\mathcal{K}_N(\lambda)$ onto $\tilde{Q}_N^*$.

Let $Q_-, Q_+, Q$ be linear spaces of sequences, respectively, $\tilde{\eta} = \{\eta_k\}$, $\tilde{\varphi} = \{\varphi_k\}$, $\tilde{\zeta} = \{\xi_k\}$, where $\eta_k \in Q_-$, $\varphi_k \in Q_+$, $\xi_k \in Q; k \in \mathbb{N}$ if $k = \infty$, and $1 \leq k \leq k$ if $k$ is finite; $k$ is the number of elements in $\mathcal{M}$. We assume that the series $\sum_{k=1}^{\infty} \|\eta_k\|_2$, $\sum_{k=1}^{\infty} \|\varphi_k\|_2$, $\sum_{k=1}^{\infty} \|\xi_k\|_2$ converge if $k = \infty$. These spaces become Hilbert spaces if we introduce scalar products by the formulas

$$(\tilde{\eta}, \tilde{\zeta}) = \sum_{k=1}^{k} (\eta_k, \zeta_k), \quad (\tilde{\eta}, \tilde{\varphi})_+ = \sum_{k=1}^{k} (\varphi_k, \varphi^*_k), \quad (\tilde{\varphi}, \tilde{\varphi})_+ = \sum_{k=1}^{k} (\varphi_k, \varphi^*_k).$$

In these spaces, the norms are defined by the equalities

$$\|\tilde{\eta}\|^2 = \sum_{k=1}^{k} \|\eta_k\|^2, \quad \|\tilde{\varphi}\|^2 = \sum_{k=1}^{k} \|\varphi_k\|^2, \quad \|\tilde{\zeta}\|^2 = \sum_{k=1}^{k} \|\xi_k\|^2.$$

The spaces $Q_-, Q_+$ can be treated as spaces with positive and negative norms with respect to $Q_\lambda$. Thus, we have proved the following statement.

**Lemma 3.24.** The operator $V(\lambda)$ maps $Q_\lambda$ onto $\ker(L_{10}^\lambda - \lambda E)$ continuously and one to one. A function $z$ belongs to $\ker(L_{10}^\lambda - \lambda E)$ if and only if there exists an element $\tilde{\eta} = [\eta_k] \in Q_\lambda$ such that $z(t) = (V(\lambda)\tilde{\eta})(t) = V(t, \lambda)\tilde{\eta}$. The operator $V(\lambda)$ maps $\mathcal{K}_\lambda$ onto $Q_\lambda$ continuously, and acts by formula (49), and $\ker V(\lambda) = \mathcal{K}_\lambda \oplus \mathcal{R}(L_{10}^\lambda - \lambda E)$. Moreover, $V(\lambda)$ maps $\ker(L_{10}^\lambda - \lambda E)$ onto $Q_\lambda$ one to one.
Theorem 3.25. A pair \( \{\tilde{y}, \tilde{f}\} \in \mathcal{S} \times \mathcal{S} \) belongs to \( L^*_0 - \Lambda E \) if and only if there exist a pair \( \{y, f\} \in \mathcal{S} \times \mathcal{S} \), functions \( y_0, y'_0 \in \mathcal{S}_0 \), \( \tilde{y}, \tilde{f} \in \mathcal{S}_1 \), and an element \( \tilde{\eta} \in \mathcal{Q} \) such that the pairs \( \{\tilde{y}, \tilde{f}\}, \{y, f\} \) are identical in \( \mathcal{S} \times \mathcal{S} \) and the equalities

\[
y = y_0 + \tilde{y}, \quad f = y'_0 + \tilde{f},
\]

\[
\tilde{y}(t) = \tilde{V}(t, \lambda)\tilde{f}(s) = -\sum_{k=1}^{n_{\lambda, \rho}} x_{[a_k, b_k], S_n} w_k(t, \lambda) i_{[a_k, b_k], S_n} \int_{a_k}^{b_k} w_k(s, \lambda) \text{d}m(s) \tilde{f}(s)
\]

hold, where the series in (51) converges in \( \mathcal{S}_1 \), \( k_1 \) is the number of intervals \( J_k \in \mathcal{J} \).

Proof. Equalities (50) follow from (26). Let us prove that equality (51) holds. It follows from Lemma 3.24 that \( \mathcal{V}(\lambda)\tilde{y} \in \ker(L^*_{10} - \Lambda E) \). We prove that if the functions \( \tilde{y}, \tilde{f} \) satisfy equality (51), then the pair \( \{\tilde{y}, \tilde{f}\} \in L^*_0 - \Lambda E \).

If \( k_1 \) is finite, then this statement follows from Lemmas 3.12, 3.24. We assume that \( k_1 = \infty \) and first prove that the series in (51) converges in \( \mathcal{S}_1 \) for each function \( \tilde{f} \in \mathcal{S}_1 \).

The function

\[
\tilde{y}_k(t) = -\sum_{k=1}^{n_{\lambda, \rho}} x_{[a_k, b_k], S_n} w_k(t, \lambda) \int_{a_k}^{b_k} w_k(s, \lambda) \text{d}m(s) \tilde{f}(s) \]

vanishes outside the interval \([a_k, b_k]\). (Here \( \Psi_m, \rho_m \) are functions from formula (2) in which the measure \( P \) is replaced by \( m \).) We denote \( \tilde{f}_k(t) = \chi_{[a_k, b_k]} \tilde{f}(t) \).

Using (52), (8), (2), we get

\[
\|\tilde{y}_k(t)\| \leq \epsilon_1 \|w_k(t, \lambda)\| \left( \int_{a_k}^{b_k} \|w(s, \lambda)\| \|\psi_{m/12}(s)\| \|\tilde{f}(s)\| \|\text{d}m(s)\| \right)^{1/2} \leq \epsilon \left( \int_{a_k}^{b_k} \|\psi_{m/12}(s)\| \|\tilde{f}(s)\| \|\text{d}m(s)\| \right)^{1/2} = \epsilon \|\tilde{f}_k\|_B, \quad \epsilon_1, \epsilon > 0.
\]

This implies

\[
\|\tilde{y}_k\|_B = \int_{a_k}^{b_k} (\Psi_m(t)\tilde{y}_k(t), \tilde{y}_k(t)) \text{d}\rho_m(t) \leq \epsilon^2 \rho_m([a_k, b_k]) \|\tilde{f}_k\|_B^2.
\]

We denote \( S_n(t) = \sum_{k=1}^n \tilde{y}_k(t) \) and prove that the sequence \( \{S_n\} \) converges in \( \mathcal{S} \). From (53), we get

\[
\|S_n\|_B^2 = \sum_{k=1}^n \|\tilde{y}_k\|_B^2 \leq \epsilon^2 \sum_{k=1}^n \rho_m([a_k, b_k]) \|\tilde{f}_k\|_B^2 \leq \epsilon^2 \rho_m([b, b]) \|\tilde{f}\|_B^2.
\]

Hence the sequence \( \{S_n\} \) converges to some function \( S \in \mathcal{S} \) and

\[
S(t) = -\sum_{k=1}^{\infty} x_{[a_k, b_k], S_n} w_k(t, \lambda) \int_{a_k}^{b_k} w_k(s, \lambda) \text{d}m(s) \tilde{f}(s), \quad \|S\|_B \leq \epsilon_2 \|\tilde{f}\|_B, \quad \epsilon_2 > 0.
\]

It follows from Lemma 3.12 that the pair \( \{S_n, \sum_{k=1}^n \tilde{f}_k\} \in L^*_0 - \Lambda E \). The relation \( L^*_0 \) is closed. Therefore, \( \{S, \tilde{f}\} \in L^*_0 - \Lambda E \) and \( \{\tilde{y}, \tilde{f}\} \in L^*_0 - \Lambda E \).

Now we assume that a pair \( \{\tilde{y}, \tilde{f}\} \in L^*_0 - \Lambda E \). For the function \( \tilde{f} \), we find a function \( S \) by formula (54).

Then \( \{S, \tilde{f}\} \in L^*_0 - \Lambda E \). Hence \( \tilde{y} - S \in \ker(L^*_0 - \Lambda E) \). By Lemma 3.24, it follows that there exists an element \( \tilde{\eta} \in \mathcal{Q} \) such that \( \tilde{y} - S = \mathcal{V}(\lambda)\tilde{\eta} \). Therefore \( \tilde{y} \) has form (51). Now (26) implies the desired assertion. The theorem is proved. \( \square \)
4. Continuously invertible extensions of the relation $L_0 - \lambda E$

We denote

$$u_0(t, \lambda) = -x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu(s) \frac{f(s)}{f(s)} =$$

$$= -x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu_0(s) \frac{f(s)}{f(s)},$$

$$\bar{u}_k(t, \lambda) = x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu(s) \frac{f(s)}{f(s)} =$$

$$= x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu_0(s) \frac{f(s)}{f(s)}.$$

It follows from Remark 3.11 that $x_{[a_b]}(S_m \cap S_0) = x_{[a_b]}$ if $a_k \not\in S_m$ and $x_{[a_b]}(S_m \cap S_0) = x_{[a_b]}$ if $a_k \in S_m$ (see also Remark 3.13).

**Lemma 4.1.** Let $\lambda \neq 0$. Equality (51) hold if and only if

$$\tilde{y}(t) = \bar{V}(t, \lambda) \tilde{\rho} + 2^{-1} \sum_{k=1}^{k_1} \left[ y_k(t, \lambda) - x_{S_m \cap S_0} \eta_k(t, \lambda) - x_{S_m \cap (a_b)} \lambda^{-1} \tilde{f}(t) \right] +$$

$$+ 2^{-1} \sum_{k=1}^{k_1} \left[ \bar{u}_k(t, \lambda) - x_{S_m \cap (a_b)} \tilde{u}_k(t, \lambda) - x_{S_m \cap (a_b)} \lambda^{-1} \tilde{f}(t) \right],$$

(55)

where $\tilde{\rho} \in Q_-$.

**Proof.** By standard transformations, equality (51) is reduced to the form

$$\tilde{y}(t) = \bar{V}(t, \lambda) \tilde{\delta} - 2^{-1} \sum_{k=1}^{k_1} x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu(s) \frac{f(s)}{f(s)} +$$

$$+ 2^{-1} \sum_{k=1}^{k_1} x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu_0(s) \frac{f(s)}{f(s)},$$

(56)

where $\tilde{\delta} = \{\delta_k\} \in Q_-$, and $\delta_k = \eta_k$ if $v_k$ has form (39), and $\delta_k = \eta_k - 2^{-1} \int_a^\theta w_k^*(s, \lambda) d\mu(s) \frac{f(s)}{f(s)}$ if $v_k$ has form (38).

Let us write the function

$$w_k(t, \lambda) = -x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu(s) \frac{f(s)}{f(s)}$$

in a different form. Using (57), (30), we get

$$w_k(t, \lambda) = x_{[a_b]}(S_m \cap S_0) \eta_k(t, \lambda) - x_{[a_b]}(S_m \cap S_0) w_k(t, \lambda) \int_a^\theta w_k^*(s, \lambda) d\mu(s) \frac{f(s)}{f(s)} =$$

$$= \eta_k(t, \lambda) - x_{S_m \cap (a_b)} \eta_k(t, \lambda) + x_{S_m \cap (a_b)} \lambda^{-1} \tilde{f}(t) + [x_{S_m \cap (a_b)} \lambda^{-1} \tilde{f}(t) + (\mathcal{U}_k(\lambda) \lambda^{-1} x_{S_m \cap (a_b)}) \tilde{f}(t)].$$

Using (34), we get

$$v_k = x_{S_m \cap (a_b)} \lambda^{-1} f + \mathcal{U}_k(\lambda) \lambda^{-1} x_{S_m \cap (a_b)} \tilde{f} \in \ker(L_{10} - \lambda E).$$
Therefore,
\[ w_k(t, \lambda) = \eta_k(t, \lambda) - [x_{S_m \cap (a, \beta_k)} \eta_k(t, \lambda) + x_{S_m \cap (a, \beta_k)} \lambda^{-1} \hat{f}(t)] + \psi_k(t). \] (58)

Similarly, we transform the function
\[ \tilde{w}_k(t, \lambda) = x_{[a, b] \setminus S_m} \tilde{w}_k(t, \lambda) \int_0^b \tilde{w}_k'(s, \lambda) d\mathcal{m}(s) \tilde{f}(s) \]
to the form
\[ \tilde{w}_k(t, \lambda) = \tilde{\eta}_k(t, \lambda) - [x_{S_m \cap (a, \beta_k)} \tilde{\eta}_k(t, \lambda) + x_{S_m \cap (a, \beta_k)} \lambda^{-1} \tilde{f}(t)] + \tilde{\psi}_k(t), \]
by Lemma 3.15 and (34), it follows that here the last two terms belong to \( \ker(L^*_1 - \lambda E) \). Consequently,
\[ \tilde{w}_k(t, \lambda) = \tilde{\eta}_k(t, \lambda) - [x_{S_m \cap (a, \beta_k)} \tilde{\eta}_k(t, \lambda) + x_{S_m \cap (a, \beta_k)} \lambda^{-1} \tilde{f}(t)] + \tilde{\psi}_k(t), \] (59)
where \( \tilde{\psi}_k \in \ker(L^*_1 - \lambda E) \). Now the desired statement follows from (56), (58), (59) and Lemma 3.24. The Lemma is proved.

**Lemma 4.2.** Let \( \lambda = 0 \). Equality (51) hold if and only if
\[ \bar{y}(t) = \bar{V}(t, 0) \bar{\zeta} + 2^{-1} \sum_{k=1}^{k_0} \int_a^b \bar{w}_k(x, 0) \mathcal{m}(x) x_{S_m \setminus \tilde{f}(s))} \] +
\[ + 2^{-1} \sum_{k=1}^{k_0} \eta_k(t, 0) + x_{[a, b] \setminus S_m} \tilde{w}_k(t, 0) \int_0^b \tilde{w}_k'(s, 0) d\mathcal{m}(s) x_{S_m \setminus \tilde{f}(s))}. \] (60)

**Proof.** Equality (56) holds for \( \lambda = 0 \). We transform the function \( w_k(t, 0) \) (see (57)) in the following way:
\[ w_k(t, 0) = -x_{S_m \cap (a, \beta_k)} \tilde{w}_k(t, 0) \int_0^b \tilde{w}_k'(s, 0) d\mathcal{m}(s) \tilde{f}(s) = \eta_k(t, 0) - x_{S_m \cap (a, \beta_k)} \eta_k(t, 0) -
\[ -x_{[a, b] \setminus S_m} \tilde{w}_k(t, 0) \int_0^b \tilde{w}_k'(s, 0) d\mathcal{m}(s) x_{S_m \setminus \tilde{f}(s))}. \]
Similarly, we transform the function \( \tilde{w}_k(t, 0) \). Since \( x_{S_m \cap (a, \beta_k)} \eta_k(t, 0) \in \ker L^*_1 \), \( x_{S_m \cap (a, \beta_k)} \tilde{\eta}_k(t, 0) \in \ker L^*_1 \), \( x_{[a, b] \setminus S_m} \tilde{w}_k(t, 0) \int_0^b \tilde{w}_k'(s, 0) d\mathcal{m}(s) x_{S_m \setminus \tilde{f}(s))} \in \ker L^*_1 \), we obtain the required statement. The Lemma is proved.

**Theorem 4.3.** Let \( T(\lambda) \) be a linear relation such that \( L^*_1 - \lambda E \subset T(\lambda) \subset L^*_1 - \lambda E \). The relation \( T(\lambda) \) is continuously invertible in the space \( \mathcal{S}_1 \) if and only if there exists a bounded operator \( M(\lambda) : \mathcal{Q} \rightarrow \mathcal{Q} \) such that equalities (61) (for \( \lambda \neq 0 \)) and (62) (for \( \lambda = 0 \)) (see equalities below) hold for any pair \( \{y, f\} \in T(\lambda) \)
\[ \bar{y}(t) = \int_a^b \bar{V}(t, \lambda) M(\lambda) \bar{V}(s, \lambda) d\mathcal{m}(s) \bar{f}(s) +
\[ + 2^{-1} \sum_{k=1}^{k_0} \int_a^b \bar{x}_{[a, b] \setminus S_m} \tilde{x}_{S_m \setminus \tilde{f}(s))} \]
\[ - 2^{-1} \sum_{k=1}^{k_0} \int_a^b \bar{x}_{S_m \cap (a, \beta_k)} \tilde{x}_{S_m \setminus \tilde{f}(s))} \]
\[ = -x_{S_m \cap (a, \beta_k)} \tilde{w}_k(t, \lambda) \sgn(s - t) \int_0^b \tilde{w}_k'(s, \lambda) d\mathcal{m}(s) x_{S_m \setminus \tilde{f}(s))} -
\[ - 2^{-1} \sum_{k=1}^{k_0} \int_a^b \bar{x}_{S_m \cap (a, \beta_k)} \tilde{x}_{S_m \setminus \tilde{f}(s))} - \lambda^{-1} \sum_{k=1}^{k_0} \bar{x}_{S_m \cap (a, \beta_k)} \tilde{f}(t). \] (61)
\[\overline{y}(t) = \int_{a}^{b} \overline{V}(t, 0)M(0)\overline{V}'(s, 0)dm(s)\overline{f}(s) + \]
\[+ 2^{-1} \sum_{k=1}^{k_{0}} \int_{a}^{b} x_{(a, \beta)}|S_{\lambda_{n}}\cap S_{\lambda_{0}}(t)w_{k}(t, 0)sgn(s - t)if_{\lambda_{n}}'(s, 0)dm(s)x_{(a, \beta)}|S_{\lambda_{0}}(s)\overline{f}(s) + \]
\[+ 2^{-1} \sum_{k=1}^{k_{0}} \int_{a}^{b} x_{(a, \beta)}|S_{\lambda}(t)w_{k}(t, 0)sgn(s - t)if_{\lambda_{n}}'(s, 0)dm(s)x_{(a, \beta)}|S_{\lambda_{0}}(s)\overline{f}(s). \quad (62)\]

**Proof.** First note that the range \(\mathcal{R}(L_{10} - \lambda E)\) is closed and \(\ker(L_{10} - \lambda E) = \{0\}\). This follows from the Lemma 3.3. Suppose that the relation \(T^{-1}(\lambda)\) is a boundary everywhere defined operator and \(\overline{y} = T^{-1}(\lambda)\overline{f}\). Then \(\overline{y}\) has form (55) for \(\lambda \neq 0\) and (60) for \(\lambda = 0\). In this equalities, \(\overline{c} \in \mathcal{Q}_{-}\) is uniquely determined by \(\overline{f}\) and \(\lambda\), i.e., \(\overline{c} = \overline{c}(\overline{f}, \lambda)\). Indeed, if \(\overline{f} = 0\), then \(\overline{V}(t, \lambda)\overline{c} = T^{-1}(\lambda)0 = 0\). It follows from Lemma 3.24 that \(\overline{c} = 0\). Moreover, \(\overline{c}\) depends on \(\overline{f}\) linearly. Consequently, \(\overline{c} = S(\lambda)\overline{f}\), where \(S(\lambda) : \mathcal{S}_{1} \rightarrow \mathcal{Q}_{-}\) is a linear operator for fixed \(\lambda\). We claim that the operator \(S(\lambda)\) is bounded. Indeed, if a sequence \(\{\overline{f}_{n}\}\) converges to zero in the space \(\mathcal{S}_{1}\) as \(n \rightarrow \infty\), then the sequence \(\{\overline{y}_{n}\}\subset T^{-1}(\lambda)\overline{f}_{n}\) converges to zero in \(\mathcal{S}_{1}\). Hence the sequence \(\{\mathcal{V}(\lambda)\overline{c}_{n}\}\) (where \(\overline{c}_{n} = S(\lambda)\overline{f}_{n}\)) converges to zero in \(\mathcal{S}_{1}\). By Lemma 3.24, it follows that the sequence \(\{S(\lambda)\overline{f}_{n}\}\) converges to zero in the space \(\mathcal{Q}_{-}\). Therefore \(S(\lambda)\) is the bounded operator.

Now we prove that \(\overline{c}(\overline{f}, \lambda)\) is uniquely determined by the element \(\mathcal{V}(\lambda)\overline{f}\in \mathcal{Q}_{-}\). Suppose \(\mathcal{V}(\lambda)\overline{f} = 0\).

The application of Lemma 3.24 yields \(\overline{f} \in \mathcal{R}(L_{10} - \lambda E)\).

Suppose \(\lambda \neq 0\). Taking into account Lemma 3.3, we determine a function \(y\) by equality (55) in which

\[y_{ \mathcal{S}_{\lambda_{n}}(a, \beta)}(t, \lambda) + y_{ \mathcal{S}_{\lambda_{0}}(a, \beta)}(t, \lambda) + y_{ \mathcal{S}_{\lambda_{n}}(a, \beta)}(t, \lambda) + y_{ \mathcal{S}_{\lambda_{0}}(a, \beta)}(t, \lambda) = 0. \]

By Lemma 3.3 and Remark 3.9, it follows that the pairs \(\{y_{k}, x_{(a, \beta)}f\}, \{y_{k}, x_{(a, \beta)}\overline{f}\} \subset L_{10} - \lambda E\). This and the invertibility of \(T(\lambda)\) imply that \(\overline{c}(\overline{f}, \lambda) = 0\) for all \(\lambda \neq 0\).

Let \(\lambda = 0\). Using Lemma 3.3 (for \(\lambda = 0\)) and Remark 3.9, we determine a function \(y\) by equality (60) in which \(x_{(a, \beta)}f(\tau) = 0\) for \(\tau \in \mathcal{S}_{\lambda}\). Then equality (60) will take the form

\[\overline{y}(t) = \mathcal{V}(t, 0)\overline{c}(\overline{f}, 0) + 2^{-1} \sum_{k=1}^{k_{0}} y_{k}(t, 0) + 2^{-1} \sum_{k=1}^{k_{0}} \overline{y}_{k}(t, 0). \]

It follows from Lemma 3.3 and Remark 3.9 that \(\{y_{k}, x_{(a, \beta)}f\}, \{y_{k}, x_{(a, \beta)}\overline{f}\} \subset L_{10}\). This and the invertibility of \(T(0)\) imply that \(\overline{c}(\overline{f}, 0) = 0\).

Thus \(S(\lambda)\overline{f} = M(\lambda)\mathcal{V}(\lambda)\overline{f}\), where \(M(\lambda) : \mathcal{Q}_{-} \rightarrow \mathcal{Q}_{-}\) is an everywhere defined operator. Let \(\mathcal{V}(\lambda)\overline{f}\) be a restriction of \(\mathcal{V}(\lambda)\overline{f}\) to \(\ker(L_{10}^* - \lambda E)\). By Lemma 3.24, it follows that \(M(\lambda) = S(\lambda)\mathcal{V}_{0}(\lambda)\mathcal{V}(\lambda)^{-1}\). Hence \(M(\lambda)\) is the bounded operator and equalities (61) (for \(\lambda \neq 0\)) and (62) (for \(\lambda = 0\)) hold.

Conversely, suppose that equalities (61) (for \(\lambda \neq 0\)) and (62) (for \(\lambda = 0\)) hold. Then \(\overline{y} = 0 \text{ if } \overline{f} \text{ is in (61), (62)}. \) Therefore, \(T^{-1}(\lambda)\) is an operator. We claim that the operator \(T^{-1}(\lambda)\) is bounded. Indeed, suppose that pairs \(\{\overline{y}_{n}, f_{n}\}\) satisfy the equality (61) or (62) and the sequence \(\{f_{n}\}\) converges to zero in \(\mathcal{S}_{1}\). It follows from Lemma 3.24 and equalities (61), (62) that the sequence \(\{\overline{y}_{n}\}\) converges to zero. So, \(T^{-1}(\lambda)\) is the boundary everywhere defined operator. The Theorem is proved.

**Corollary 4.4.** Let \(\overline{T}(\lambda) \subset \mathcal{S}_{1} \times \mathcal{S}_{1}\) be a linear relation and \(L_{10} - \lambda E \subset \overline{T}(\lambda) \subset L_{10}^* - \lambda E\). Then \(\overline{T}(\lambda)\) is continuously invertible in the space \(\mathcal{S}_{1}\) if and only if \(\overline{T}(\lambda)\) has the form \(\overline{T}(\lambda) = T_{0} \oplus T(\lambda)\), where \(T_{0} \subset \mathcal{S}_{1} \times \mathcal{S}_{0}\), \(T(\lambda) \subset \mathcal{S}_{1} \times \mathcal{S}_{1}\) are linear relations, \(L_{10} - \lambda E \subset T(\lambda) \subset L_{10}^* - \lambda E\), \(T(\lambda)\) is continuously invertible in \(\mathcal{S}_{1}\) (i.e., \(T(\lambda)\) satisfies Theorem 4.3), \(T_{0}\) is any continuously invertible relation in \(\mathcal{S}_{0}\).

**Proof.** The desired statement follows from (26).
Remark 4.5. It follows from Lemma 3.24 that the operator $M(\lambda)$ is uniquely determined by the relation $T(\lambda)$ and by the choice of functions $v_k$.

We shall write equalities (61), (62) in a short form. We denote $\overline{W}(t, \lambda) = \sum_{k=1}^{b} x_{[\alpha_k, \beta_k]}(s) \delta_k(t, \lambda)$, i.e., $\overline{W}(t, \lambda) = w_0(t, \lambda)$ for $t \in (\alpha_k, \beta_k)$, and $\overline{W}(\alpha_k, \lambda) = w_0(\alpha_k, \lambda)$ if $\alpha_k \notin S_m$, and $\overline{W}(\alpha_k, \lambda) = 0$ if $\alpha_k \in S_m$. In (61), (62), the series converge in $G_1$ for any function $f \in G_1$. We denote

$$K(t, s, \lambda) = \overline{V}(t, \lambda)M(\lambda)\overline{V}^*(s, \lambda) + 2^{-1}\overline{W}(t, \lambda)sgn(s - t)j\overline{W}^*(s, \overline{\lambda})\hat{x}_{[\alpha, \beta]}(s) - 2^{-1}\hat{x}_{\overline{S}_m}(t)\overline{W}(t, \lambda)sgn(s - t)j\overline{W}^*(s, \overline{\lambda})\hat{x}_{[\alpha, \beta]}(s), \ \lambda \neq 0;$$

$$K(t, s, 0) = \overline{V}(t, 0)M(0)\overline{V}^*(s, 0) + 2^{-1}\overline{W}(t, 0)sgn(s - t)j\overline{W}^*(s, 0)\hat{x}_{[\alpha, \beta]}(s) + 2^{-1}\hat{x}_{\overline{S}_m}(t)\overline{W}(t, 0)sgn(s - t)j\overline{W}^*(s, 0)\hat{x}_{\overline{S}_m}(s).$$

Then the equalities (61), (62) can be written as

$$\gamma(t) = (T^{-1}(\lambda)\overline{f})(t) = \int_a^b K(t, s, \lambda)dm(s)\hat{f}(s) - \lambda^{-1}\hat{x}_{\overline{S}_m}(s)\overline{f}(t), \ \lambda \neq 0, \ \overline{f} \in G_1;$$

$$y(t) = (T^{-1}(0)\overline{f})(t) = \int_a^b K(t, s, 0)dm(s)\hat{f}(s), \ \overline{f} \in G_1. \quad (63)$$

Let us consider some examples.

Example 4.6. Suppose $p = p_0$ is a continuous measure, $m = \mu$ is the usual Lebesgue measure on $[a, b]$ (i.e., $\mu([\alpha, \beta]) = \beta - \alpha$, where $a \leq \alpha < \beta \leq b$ (we write ds instead of $d\mu(s)$)). In this case, $L_0, L_0^*$ are operators, $k_1 = k = 1$, $S_0 = [0, \infty)$, $Q_{1, 0} = [0, \infty)$, $Q_1 = H = Q_+$. $\overline{V}(t, \lambda) = W(t, \lambda)$. Equality (51) has the form

$$y(t) = W(t, \lambda)\eta - W(t, \lambda)j\int_a^b W^*(s, \overline{\lambda})f(s)ds, \ \eta = (L_0^* - \lambda E)y, \ \eta \in H.$$

For any $\lambda$, equalities (63), (64) take the form

$$y(t) = (T^{-1}(\lambda)f)(t) = \int_a^b K(t, s, \lambda)f(s)ds,$$

where $K(t, s, \lambda) = W(t, \lambda)(M(\lambda) + 2^{-1}sgn(s - t)j)W^*(s, \overline{\lambda})$.

Example 4.7. We assume that measures $p, m$ are continuous. Then $L_0, L_0^*$ are not operators, generally. In this case, $k_1 = k_2 = 1$, $S_0 = [0, \infty)$, $Q_{1, 0} = [0, \infty)$, $Q_1 = H = Q_+$. $\overline{V}(t, \lambda)$ is an extension of the operator $\xi \to W_0(\lambda)\xi$ ($\xi \in Q_1 \subset H$) to the set $Q_+$. $\overline{V}(t, \lambda)\eta = W(t, \lambda)\eta = (W(\lambda)\eta)(t)$ ($\eta \in Q_+$). Equality (51) has the form

$$y(t) = \overline{W}(t, \lambda)\eta - \overline{W}(t, \lambda)j\int_a^b \overline{W}^*(s, \overline{\lambda})dm(s)f(s), \ \{y, f\} \in L_0^* - \lambda E, \ \eta \in Q_+.$$

For any $\lambda$, equalities (63), (64) take the form

$$y(t) = (T^{-1}(\lambda)f)(t) = \int_a^b K(t, s, \lambda)dm(s)f(s),$$

where $K(t, s, \lambda) = \overline{W}(t, \lambda)(M(\lambda) + 2^{-1}sgn(s - t)j)\overline{W}^*(s, \overline{\lambda})$.

Example 4.8. Suppose that $m = \mu$ is the usual Lebesgue measure and the set $S_p$ of single-point atoms of the measure $p$ can be arranged as an increasing sequence converging to $b$. In this case, the description of $T^{-1}(\lambda)$ is obtained in [9].
Example 4.9. Suppose that $\mathcal{S}_m \neq \emptyset$ and $m = \mu + \hat{m}$, where $\mu = m_0$ is the usual Lebesgue measure on $[a, b]$ and $\mu(\Delta) = m(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_m = \emptyset$. So, $\mathcal{S}_m = \mathcal{S}_m$ and $m(\beta) = \hat{m}(\beta)$ for all $\beta \in \mathcal{S}_m$. We arrange the elements of $\mathcal{S}_m$ in the form of a finite or infinite sequence $(\xi_k)$. Let $k_0$ be the number of elements in $\mathcal{S}_m$. We denote $Q_{k_0} = \text{ker}(m|_{(\xi_k)})$, $\tilde{Q}_k = H \oplus Q_{k_0}$, where $\xi_k \in \mathcal{S}_m$. Let $m_k$ be the restriction of the operator $m(\xi_k)$ to $\tilde{Q}_k$. The operator $m_k$ is self-adjoint and $\mathcal{R}(m_k) \subset \tilde{Q}_k$. By $\tilde{Q}_k$ denote the completion of $Q_k$ with respect to norm $\|\xi\| = (m_k \xi, \xi)^{1/2}$, where $\xi \in \tilde{Q}_k$. Let $\tilde{Q}_- = \text{linear space of sequences } \eta_0 = \{\eta_k\}$ such that $\eta_k \in \tilde{Q}_k^\perp$ ($k \in \mathbb{N}$ if $k_2 = \infty$, and $1 \leq k \leq k_2$ if $k_2$ is finite) and the series $\sum_{k=1}^{\infty} \|\eta_k\|^2$ converges if $k_2 = \infty$. Then $\mathcal{S}_- = L_2(H; a, b) \oplus \tilde{Q}_-$. Suppose $p = 0$ and $q \notin \mathcal{S}_m$, $m \notin \mathcal{S}_m$. (The case of an arbitrary continuous measure $p$ can be considered similarly.) Then $\mathcal{S}_0 = \{0\}$, $k_1 = 1$, $W(t, 0) = E$, and $\mathcal{Q}_- = H \oplus \tilde{Q}_-$. It follows from Lemma 3.3 and (14) that a pair $(y, f) \in L_0$ if and only if

$$y(t) = -i \int_a^t f(s) ds, \quad y(b) = 0, \quad m(\beta) f(\beta) = 0 \quad (\beta \in \mathcal{S}_m).$$

Using Theorem 3.25 for $\lambda = 0$, we obtain that a pair $(y, f) \in L_0^\prime$ if and only if

$$y(t) = \eta_0 + \sum_{\tau_k \leq t} X_{[\tau_k]}(t) \eta_k - i \int_a^t d\mathbf{m}(s) f(s), \quad (66)$$

where $\eta_0 \in H$, $\tau_k \in \mathcal{S}_m$, $\eta_k \in \tilde{Q}_k$, and the sequence $\tilde{\eta} = \{\eta_0, \eta_k\}$ belongs to $\mathcal{Q}_-$. (Here $k \in \mathbb{N}$ if $k_2 = \infty$, and $1 \leq k \leq k_2$ if $k_2$ is finite.) It follows from Lemma 3.15 (for $\lambda = 0$) that the function $X_{\mathcal{S}_m}(t) \int_a^t d\mathbf{m}(s) f(s) \in \ker L_0^\prime$. Therefore, equality (66) can be written as

$$y(t) = \xi_0 + \sum_{\tau_k \leq t} X_{[\tau_k]}(t) \xi_k - \hat{X}_{[a,b], \mathcal{S}_m}(t) i \int_a^t d\mathbf{m}(s) f(s), \quad \xi_0 \in H, \quad \xi_k \in \tilde{Q}_k, \quad \tilde{\xi} = [\xi_0, \xi_k] \in \mathcal{Q}_-. \quad (67)$$

By (6), it follows that $W(t, \lambda) = \exp(-i/\lambda t)$. Using (31), we get

$$u_1(t, \lambda, \tau) x = -X_{[a,b], \mathcal{S}_m}(t) W(t, \lambda) i \int_a^t W^*(s, \lambda) d\mathbf{m}(s) X_{[a,b]}(s) x, \quad x \in H, \quad \tau \in \mathcal{S}_m. \quad (68)$$

Hence, $u_1(t, \lambda, \tau) x + -X_{[a,b], \mathcal{S}_m}(t) W(t, \lambda) i \int_a^t W^*(s, \lambda) d\mathbf{m}(s) X_{[a,b]}(s) x$ is equal to zero if $t < \tau$, and $X_{[\tau_k]}(t)x$ if $t = \tau$, and $-\lambda X_{[a,b], \mathcal{S}_m}(t) W(t, \lambda) i \int_a^t W^*(s, \lambda) d\mathbf{m}(s) X_{[a,b]}(s) x$ if $t > \tau$. We denote $u_2(t, \lambda) = X_{[a,b], \mathcal{S}_m}(t) W(t, \lambda) u_2(t, \lambda) = u_1(t, \lambda, \tau) W(t, \lambda) x + X_{[\tau_k]}(t) W(t, \lambda) x (k \in \mathbb{N}$ if $k_2 = \infty$, and $1 \leq k \leq k_2$ if $k_2$ is finite). By Lemma 3.18, it follows that the linear span of functions $v_0(t, \lambda) \xi_0, v_k(t, \lambda) \xi_k$ ($\xi_0, \xi_k \in H$) is dense in $\ker(L_0^\prime - \lambda E)$. The operator $V_N(t, \lambda)$ has the form $V_N(t, \lambda) = (v_0(t, \lambda), ..., v_{N-1}(t, \lambda))$. As above, by $V_\lambda(t)$ we denote the operator $V_\lambda(t) : \mathcal{Q}_- \to \mathcal{S}_-$. The operator $V_\lambda$ has the form $V_\lambda(t) = V_N(t, \lambda) \eta_N$ for all $N \in \mathbb{N}$, where $V_N(t, \lambda)$ is the operator $\xi_0 \mapsto V_N(t, \lambda) \xi_0, \xi_0 \in \tilde{Q}_N$. Thus, in this example, equalities (61), (62) will take form (67), (68), respectively, (see equalities below)
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