A CLASS OF MATRIX-VALUED SCHRÖDINGER OPERATORS WITH PRESCRIBED FINITE-BAND SPECTRA

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Abstract. We construct a class of matrix-valued Schrödinger operators with prescribed finite-band spectra of maximum spectral multiplicity. The corresponding matrix potentials are shown to be stationary solutions of the KdV hierarchy. The methods employed in this paper rely on matrix-valued Herglotz functions, Weyl–Titchmarsh theory, pencils of matrices, and basic inverse spectral theory for matrix-valued Schrödinger operators.

1. Introduction

While basic aspects of inverse spectral theory for matrix-valued Schrödinger operators were established some time ago, finer properties such as isospectral sets (manifolds) of potentials, for instance, in the periodic or algebro-geometric finite-band cases, are still in their infancy. This paper intends to make a modest contribution to this circle of ideas. More precisely, given a closed set \( \Sigma \subset \mathbb{R} \) of the type

\[
\Sigma = \left\{ \bigcup_{j=0}^{n-1} [E_{2j}, E_{2j+1}] \right\} \cup [E_{2n}, \infty),
\]

(1.1)

we construct \( m \times m \) matrix-valued Schrödinger operators \( H_{\Sigma} = -d^2/dx^2 I_m + Q_{\Sigma} \) in \( L^2(\mathbb{R})^{m \times m} \) (\( I_m \) the identity matrix in \( \mathbb{C}^{m \times m} \), \( m \in \mathbb{N} \)) with prescribed spectrum \( \Sigma \),

\[
\text{spec}(H_{\Sigma}) = \Sigma,
\]

(1.3)

of uniform spectral multiplicity \( 2m \). The constructed matrix potentials \( Q_{\Sigma} \) will turn out to be reflectionless in the sense discussed in [23], [40], and [56], that is, the half-line Weyl–Titchmarsh matrices \( \mathcal{M}_{\pm, \Sigma}(z, x) \) associated with \( H_{\Sigma} \), the half-lines \( [x, \pm \infty) \), and a Dirichlet boundary condition at \( x \in \mathbb{R} \), satisfy

\[
\lim_{\varepsilon \downarrow 0} \mathcal{M}_{+, \Sigma}(\lambda + i\varepsilon, x) = \lim_{\varepsilon \downarrow 0} \mathcal{M}_{-, \Sigma}(\lambda - i\varepsilon, x),
\]

(1.4)

\[
\lambda \in \bigcup_{j=0}^{n-1} (E_{2j}, E_{2j+1}) \cup (E_{2n}, \infty), \quad x \in \mathbb{R}.
\]
Especially, \( M_+; \Sigma \cdot, x \) is the analytic continuation of \( M_-; \Sigma \cdot, x \) through the set \( \Sigma \), and vice versa. In other words, \( M_+; \Sigma \cdot, x \) and \( M_-; \Sigma \cdot, x \) are the two branches of an analytic matrix-valued function \( M_\Sigma \cdot, x \) on the two-sheeted Riemann surface of \( \left( \prod_{\ell=0}^{2n} (z - E_\ell) \right)^{1/2} \). These facts imply a purely absolutely continuous spectrum \( \Sigma \) of the associated Schrödinger operator \( H_\Sigma \) of uniform (maximal) multiplicity \( 2m \).

Before we turn to a brief description of the contents of each section, it seems appropriate to mention some of the pertinent results and especially, the most recent activities in connection with (inverse) spectral theory of matrix-valued Schrödinger operators. The basic Weyl–Titchmarsh theory of singular Hamiltonian systems and their basic spectral theory were developed by Hinton and Shaw, Kogan and Rofe-Beketov, Orlov, and others (see, e.g., [43], [45]–[49], [54], [55], [57], [58], [81], [91], [92], [94], [97], [101, Ch. 9], and the references therein). Various aspects of direct spectral theory, including investigations of the nature of the spectrum involved, (regularized) trace formulas, uniqueness theorems, etc., appeared in [10], [12]–[15], [20], [23], [24], [35], [37], [39], [60], [82], [89], [90]. General asymptotic expansions of Weyl–Titchmarsh matrices as the (complex) spectral parameter tends to infinity under optimal regularity assumptions on the coefficients are of relatively recent origin and can be found in [20], [21] (see also [95], [107]). The inverse scattering formalism for Schrödinger operators has been studied by a variety of authors and we refer, for instance, to [1], [2], [3], [7], [75], [77], [78], [108]. General inverse spectral theory, the existence of transformation operators, etc., are discussed in [68], [69], [85], [94], [97], [98], [100], [101], and the references therein. Inverse monodromy problems have recently been discussed in [4], [5], [6], [15], [68], [69], [96], [101], and the literature cited therein. More specific inverse spectral problems, such as compactness of the isospectral set of periodic Schrödinger operators [13], special isospectral matrix-valued Schrödinger operators, and Borg-type uniqueness theorems (for periodic coefficients as well as eigenvalue problems on compact intervals) were recently studied in [17], [18], [19], [21], [23], [26], [51], [52], [68], [69], [101], [102], [103], [104]. Moreover, direct spectral theory in the particular case of periodic Schrödinger operators (i.e., Floquet theory and the like) has been studied in [12], [14], [23], [25], [26], [34], [53], [59], [60], [86], [102], [103], [110]–[112], with many more pertinent references to be found therein. Apart from Floquet theoretic applications in connection with Schrödinger operators already briefly touched upon, we also need to mention applications to random Schrödinger operators associated with strips as discussed, for instance, in [16], [54], [56], and especially to nonabelian completely integrable systems. Since the literature associated with the latter topic is of enormous proportions, we can only refer to a few pertinent publications, such as, [7], [8], [11], [27], [30], [70], [71], [75], [79], [80], [83], [93], [95]–[97], [99]. The interested reader will find a wealth of additional material in these references.

Section 2 summarizes basic results in Weyl–Titchmarsh theory and some elements of inverse spectral theory for matrix-valued Schrödinger operators. Polynomial pencils of matrices are briefly reviewed in Section 3. In Section 4 we present our principal new result, the construction of \( m \times m \) matrix-valued Schrödinger operators \( H_\Sigma \) with spectrum \( \Sigma \) (cf. (1.1)) and uniform maximal spectral multiplicity \( 2m \). In our final Section 5 we prove that \( Q_\Sigma \) satisfies a stationary KdV equation (in fact, we explicitly identify the first equation in the stationary KdV hierarchy satisfied by \( Q_\Sigma \)) and derive matrix-valued trace formulas for \( Q_\Sigma \) and higher-order KdV invariants.
2. Basic Facts on Weyl–Titchmarsh Theory

In this section we briefly recall basic elements of the Weyl–Titchmarsh theory for matrix-valued Schrödinger operators. Throughout this paper all matrices will be considered over the field of complex numbers \( \mathbb{C} \), and the corresponding linear space of \( k \times \ell \) matrices will be denoted by \( \mathbb{C}^{k \times \ell} \), \( k, \ell \in \mathbb{N} \). Moreover, \( I_k \) denotes the identity matrix in \( \mathbb{C}^{k \times k} \), \( M^* \) the adjoint (i.e., complex conjugate transpose), \( M^t \) the transpose of a matrix \( M \), \( \text{diag}(m_1, \ldots, m_k) \in \mathbb{C}^{k \times k} \) a diagonal \( k \times k \) matrix, and \( \text{AC}_{\text{loc}}(\mathbb{R}) \) denotes the set of locally absolutely continuous functions on \( \mathbb{R} \). The spectrum, point spectrum (the set of eigenvalues), essential spectrum, absolutely continuous spectrum, and singularly continuous spectrum of a self-adjoint linear operator \( T \) in a separable complex Hilbert space are denoted by \( \text{spec}(T) \), \( \text{spec}_p(T) \), \( \text{spec}_{\text{ess}}(T) \), \( \text{spec}_{\text{ac}}(T) \), \( \text{spec}_{\text{sc}}(T) \), respectively.

The basic assumption for this section will be the following.

**Hypothesis 2.1.**

(i) Fix \( m \in \mathbb{N} \), suppose \( Q = Q^* \in L^1_{\text{loc}}(\mathbb{R})^{m \times m} \) and introduce the differential expression

\[
\mathcal{L} = -I_m \frac{d^2}{dx^2} + Q, \quad x \in \mathbb{R}.
\]

(ii) Suppose \( \mathcal{L} \) is in the limit point case at \( \pm \infty \).

Given Hypothesis 2.1 (i) we consider the matrix-valued Schrödinger equation

\[
-\psi''(z,x) + Q(x)\psi(z,x) = z\psi(z,x) \quad \text{for a.e. } x \in \mathbb{R},
\]

where \( z \in \mathbb{C} \) plays the role of a spectral parameter and \( \psi \) is assumed to satisfy

\[
\psi(z,\cdot), \psi'(z,\cdot) \in \text{AC}_{\text{loc}}(\mathbb{R})^{m \times m}.
\]

Throughout this paper, \( x \)-derivatives are abbreviated by a prime \( \prime \).

Let \( \Psi(z,x,x_0) \) be a \( 2m \times 2m \) normalized fundamental system of solutions of (2.2) at some \( x_0 \in \mathbb{R} \) which we partition as

\[
\Psi(z,x,x_0) = \begin{pmatrix} \theta(z,x,x_0) & \phi(z,x,x_0) \\ \theta'(z,x,x_0) & \phi'(z,x,x_0) \end{pmatrix}.
\]

Here \( \prime \) denotes \( d/dx \), \( \theta(z,x,x_0) \) and \( \phi(z,x,x_0) \) are \( m \times m \) matrices, entire with respect to \( z \in \mathbb{C} \), and normalized according to

\[
\Psi(z,x_0,x_0) = I_{2m},
\]

that is,

\[
\theta(z,x_0,x_0) = \phi'(z,x_0,x_0) = I_m, \quad \phi(z,x_0,x_0) = \theta'(z,x_0,x_0) = 0.
\]

In this context, we briefly recall a set of formulas needed later in Section 4. Introducing

\[
\mathcal{J} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix},
\]

one infers

\[
\Psi(\bar{z},x,x_0)^* \mathcal{J} \Psi(z,x,x_0) = \mathcal{J},
\]

which implies \( \mathcal{J} \Psi(z,x,x_0) (\Psi(\bar{z},x,x_0)^*)^* = I_{2m} \) and hence

\[
\Psi(z,x,x_0) \mathcal{J} \Psi(\bar{z},x,x_0)^* = \mathcal{J}.
\]
Writing out (2.8) and (2.9) explicitly yields
\[ \theta'(\bar{z}, x, x_0)^* \theta(z, x, x_0) - \theta(\bar{z}, x, x_0)^* \theta'(z, x, x_0) = 0, \]
\[ \phi'(\bar{z}, x, x_0)^* \phi(z, x, x_0) - \phi(\bar{z}, x, x_0)^* \phi'(z, x, x_0) = 0, \]
\[ \phi'(\bar{z}, x, x_0)^* \theta(z, x, x_0) - \phi(\bar{z}, x, x_0)^* \theta'(z, x, x_0) = I_m, \]
\[ \theta(z, x, x_0)^* \phi'(z, x, x_0) - \theta'(z, x, x_0)^* \phi(z, x, x_0) = I_m, \]
and
\[ \phi(z, x, x_0) \theta(\bar{z}, x, x_0)^* - \theta(z, x, x_0) \phi(\bar{z}, x, x_0)^* = 0, \]
\[ \phi'(z, x, x_0)^* \theta(z, x, x_0)^* - \theta'(z, x, x_0)^* \phi(z, x, x_0)^* = 0, \]
\[ \phi'(z, x, x_0) \theta(z, x, x_0)^* - \theta'(z, x, x_0) \phi(z, x, x_0)^* = I_m, \]
\[ \theta(z, x, x_0, \alpha) \phi'(z, x, x_0)^* - \phi(z, x, x_0, \alpha) \theta'(z, x, x_0)^* = I_m. \]

Next, assuming \(-\infty < a < b \leq \infty\), we consider the spaces
\[ N(z, \pm \infty) = \{ \phi \in L^2((c, \pm \infty))^m \mid -\phi'' + Q \phi = z \phi \text{ a.e. on } (c, \pm \infty) \}, \]
for some \(c \in \mathbb{R}\) and \(z \in \mathbb{C}\). (Here \((\phi, \psi)_{C^n} = \sum_{j=1}^{n} \bar{\phi}_j \psi_j\) denotes the standard scalar product in \(C^n\), abbreviating \(\chi \in \mathbb{C}^n\) by \(\chi = (\chi_1, \cdots, \chi_n)^t\).) Both dimensions of the spaces in (2.18), \(\dim_C(N(z, \pm \infty))\) and \(\dim_C(N(z, -\infty))\), are constant for \(z \in \mathbb{C}_\pm = \{ \zeta \in \mathbb{C} \mid \pm \text{Im}(\zeta) > 0 \}\) (see, e.g., [55]). One then recalls that \(L\) in (2.1) is in the limit point case at \(\pm \infty\) whenever
\[ \dim_C(N(z, \pm \infty)) = m \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}. \]

Since the potential \(Q\) to be constructed in Section 4 will automatically lead to the limit point case at \(\pm \infty\), we decided to limit our considerations mainly to this situation. In this context we note the well-known fact that if \(L\) in (2.1) is in the limit point case at \(\pm \infty\), then the \(m \times m\) Weyl–Titchmarsh matrices associated with \(L\), the half-lines \([a, \pm \infty]\), and a Dirichlet boundary condition at \(a\), are given by
\[ \mathcal{M}_\pm(z, x_0) = \Psi_\pm(z, x_0) \Phi_\pm(z, x_0)^{-1}|_{x=x_0}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \]
where \(\Psi_\pm\) satisfy \((L - zI_m)\Psi_\pm(z, \cdot, x_0) = 0\) and
\[ \Psi_\pm(z, \cdot, x_0) \in L^2([x_0, \pm \infty))^m \times m. \]
The actual normalization of \(\Psi_\pm(z, \cdot, x_0)\) is clearly irrelevant and hence \(\Psi_\pm(z, \cdot, x_0)\) can be replaced by \(\Psi_\pm(z, \cdot, x_0)C\), where \(C\) is any nonsingular \(m \times m\) matrix.

For later reference we summarize the principal results on \(\mathcal{M}_\pm(z, x_0)\) in the next theorem. First we recall the following definition.

**Definition 2.2.** A map \(\mathcal{M}: \mathbb{C}_+ \to \mathbb{C}^{n \times n}, \ n \in \mathbb{N}, \) extended to \(\mathbb{C}_-\) by \(\mathcal{M}(\bar{z}) = \mathcal{M}(z)^*\) for all \(z \in \mathbb{C}_+,\) is called an \(n \times n\) Herglotz matrix\(^1\) if it is analytic on \(\mathbb{C}_+\) and \(\text{Im}(\mathcal{M}(z)) \geq 0\) for all \(z \in \mathbb{C}_+\).

Here we denote \(\text{Im}(\mathcal{M}) = (\mathcal{M} - \mathcal{M}^*)/2i\) and \(\text{Re}(\mathcal{M}) = (\mathcal{M} + \mathcal{M}^*)/2\).

In the scalar context \(n = 1\), the condition \(\text{Im}(\mathcal{M}(z)) \geq 0\) in Definition 2.2 can be replaced by \(\text{Im}(m(z)) > 0\) for the corresponding scalar counterpart \(m(z)\).

\(^1\)There appears to be considerable confusion in the literature since Nevanlinna, Pick, Nevanlinna–Pick matrix, in addition to Herglotz matrix, are also in use. In part these discrepancies can be traced back to the use of the upper half-plane \(\mathbb{C}_+\), versus the open unit disk \(D\), and in some cases the geographical location of the author in question determines the preferred notation. Following a tradition in mathematical physics, we adopt the notion of Herglotz functions in this paper.
Theorem 2.3 ([21], [40], [45], [46], [49], [56]). Assume Hypothesis 2.1 and suppose that \( z \in \mathbb{C} \setminus \mathbb{R}, \ x_0 \in \mathbb{R} \). Then

(i) \( \pm M_\pm(z,x_0) \) is an \( m \times m \) matrix-valued Herglotz function of maximal rank. In particular,

\[
\text{Im}(\pm M_\pm(z,x_0)) > 0, \quad z \in \mathbb{C}_+, \tag{2.22}
\]

\[
M_\pm(z,x_0) = M_\pm(z,x_0)^*, \tag{2.23}
\]

\[
\text{rank}(M_\pm(z,x_0)) = m, \tag{2.24}
\]

\[
\lim_{\varepsilon \downarrow 0} M_\pm(\lambda + i\varepsilon, x_0) \text{ exists for a.e. } \lambda \in \mathbb{R}. \tag{2.25}
\]

Local singularities of \( \pm M_\pm(z,x_0) \) and \( \mp M_\pm(z,x_0)^{-1} \) are necessarily real and at most of first order in the sense that

\[
\pm \lim_{\varepsilon \downarrow 0} (i\varepsilon M_\pm(\lambda + i\varepsilon, x_0)) \geq 0, \quad \lambda \in \mathbb{R}, \tag{2.26}
\]

\[
\pm \lim_{\varepsilon \downarrow 0} i\varepsilon M_\pm(\lambda + i\varepsilon, x_0)^{-1} \geq 0, \quad \lambda \in \mathbb{R}. \tag{2.27}
\]

(ii) \( \pm M_\pm(z,x_0) \) admit the representations

\[
\pm M_\pm(z,x_0) = \text{Re}(\pm M_\pm(\pm i, x_0)) + \int_\mathbb{R} d\Omega_\pm(\lambda, x_0) ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \tag{2.28}
\]

where

\[
\int_\mathbb{R} \| d\Omega_\pm(\lambda, x_0) \|_{C^{m \times m}} (1 + \lambda^2)^{-1} < \infty \tag{2.29}
\]

and

\[
\Omega_\pm((\lambda, \mu), x_0) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda + \delta}^{\mu + \delta} d\nu \text{Im}(\pm M_\pm(\nu + i\varepsilon, x_0)). \tag{2.30}
\]

(iii) Define the \( 2m \times m \) matrices

\[
\Psi_\pm(z, x, x_0) = \begin{pmatrix}
\psi_\pm(z, x, x_0) \\
\psi_\pm^*(z, x, x_0)
\end{pmatrix} = \begin{pmatrix}
\theta(z, x, x_0) & \phi(z, x, x_0) \\
\theta'(z, x, x_0) & \phi'(z, x, x_0)
\end{pmatrix} \begin{pmatrix}
I_m \\
M_\pm(z, x_0)
\end{pmatrix}, \tag{2.31}
\]

then

\[
\text{Im}(M_\pm(z,x_0)) = \text{Im}(z) \int_{x_0}^{\pm \infty} dx \psi_\pm(z, x, x_0)^* \psi_\pm(z, x, x_0). \tag{2.32}
\]

(iv) Denote by \( C_\varepsilon \subset \mathbb{C}_+ \) the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle \( \varepsilon \), with \( 0 < \varepsilon < \pi/2 \). Then

\[
M_\pm(z,x_0) \mid_{z \to \pm \infty} \in C_\varepsilon. \tag{2.33}
\]

Necessary and sufficient conditions for \( M_\pm(z,x_0) \) to be the half-line \( m \times m \) Weyl–Titchmarsh matrix associated with a Schrödinger operator on \( [x_0, \pm \infty) \) in terms of the corresponding measures \( \Omega_\pm(\cdot, x_0) \) in the Herglotz representation (2.28) of \( M_\pm(z,x_0) \) can be derived using the matrix-valued extension of the classical inverse spectral theory approach due to Gelfand and Levitan [33], as worked out by Rofe-Beketov [85]. The following result describes sufficient conditions for a monotonically
nondecreasing matrix function to be the matrix spectral function of a half-line Schrödinger operator. It extends well-known results in the scalar case $m = 1$ (cf. [65, Sects. 2.5, 2.9], [66], [76, Sect. 26.5], [106]).

Theorem 2.4 ([85]). Let $\Omega_+$ be a monotonically nondecreasing $m \times m$ matrix-valued function on $\mathbb{R}$ satisfying the following two conditions.

(i) Whenever $f \in C([x_0, \infty))^{m \times 1}$ with compact support contained in $[x_0, \infty)$ and 
\[ \int_{\mathbb{R}} F(\lambda)^* d\Omega_+(\lambda) F(\lambda) = 0, \text{ then } f = 0 \text{ a.e.}, \] (2.34)

where 
\[ F(\lambda) = \lim_{R^+ \to \infty} \int_{x_0}^{R} dx \frac{\sin(\lambda^{1/2}(x-x_0))}{\lambda^{1/2}} f(x), \quad \lambda \in \mathbb{R}. \] (2.35)

(ii) Define 
\[ \tilde{\Omega}_+(\lambda) = \begin{cases} \Omega_+(\lambda) - \frac{2}{3\pi} \lambda^{3/2}, & \lambda \geq 0 \\ \Omega_+(\lambda), & \lambda < 0 \end{cases} \] (2.36)

and assume the limit 
\[ \lim_{R^+ \to \infty} \int_{-\infty}^{R} d\tilde{\Omega}_+(\lambda) \frac{\sin(\lambda^{1/2}(x-x_0))}{\lambda^{1/2}} = \Phi(x) \] (2.37)

exists and $\Phi \in L^\infty([x_0, R])^{m \times m}$ for all $R > x_0$. Moreover, suppose that for some 
$r \in \mathbb{N}_0$, $\Phi^{(r+1)} \in L^1([x_0, R])^{m \times m}$ for all $R > x_0$, and $\Phi(x_0) = 0$.

Then $\Omega_+$ is the matrix spectral function of a self-adjoint Schrödinger operator $H_+$ in $L^2([x_0, \infty))^{m \times m}$ associated with the $m \times m$ matrix-valued differential expression 
$L_+ = -d^2/dx^2 + Q$, $x > x_0$, with a Dirichlet boundary condition at $x_0$, a self-adjoint boundary condition at $\infty$ (if necessary), and a self-adjoint potential matrix $Q$ with $Q^{(r)} \in L^1([x_0, R])^{m \times m}$ for all $R > x_0$.

Next, assuming Hypothesis 2.1, we introduce the self-adjoint Schrödinger operator $H$ in $L^2(\mathbb{R})^m$ by 
\[ H = -\mathcal{I}_m \frac{d^2}{dx^2} + Q, \] (2.38)

\[ \text{dom}(H) = \{ g \in L^2(\mathbb{R})^m \mid g, g' \in AC_{\text{loc}}(\mathbb{R})^m; (-g'' + Qg) \in L^2(\mathbb{R})^m \}. \]

The resolvent of $H$ then reads 
\[ ((H - z)^{-1} f)(x) = \int_{\mathbb{R}} dx' \mathcal{G}(z, x, x') f(x'), \quad z \in \mathbb{C} \setminus \mathbb{R}, \ f \in L^2(\mathbb{R})^m, \] (2.39)

with the Green’s matrix $\mathcal{G}(z, x, x')$ of $H$ given by 
\[ \mathcal{G}(z, x, x') = \psi_+(z, x, x_0)[\mathcal{M}_-(z, x_0) - \mathcal{M}_+(z, x_0)]^{-1}\psi_+(z, x, x_0)^*, \]
\[ x \leq x', \ z \in \mathbb{C} \setminus \mathbb{R}. \] (2.40)

Introducing 
\[ \mathcal{N}_\pm(z, x_0) = \mathcal{M}_-(z, x_0) \pm \mathcal{M}_+(z, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \] (2.41)
the $2m \times 2m$ Weyl–Titchmarsh function $\mathcal{M}(z,x_0)$ associated with $H$ on $\mathbb{R}$ is then given by

$$\mathcal{M}(z,x_0) = (\mathcal{M}_{j,j'}(z,x_0))_{j,j'=1,2} \quad (2.42)$$

$$= \begin{pmatrix}
\mathcal{M}_+(z,x_0)\mathcal{N}_-(z,x_0)^{-1}\mathcal{M}_-(z,x_0) - \mathcal{N}_-(z,x_0)^{-1}\mathcal{N}_+(z,x_0)/2 & \mathcal{N}_-(z,x_0)^{-1} \\
\mathcal{N}_+(z,x_0)\mathcal{N}_-(z,x_0)^{-1}/2 & -\mathcal{N}_-(z,x_0)^{-1}
\end{pmatrix},$$

where $z \in \mathbb{C}\setminus\mathbb{R}$.

The basic results on $\mathcal{M}(z,x_0)$ then read as follows.

**Theorem 2.5** ([40], [45], [46], [49], [56]). Assume Hypothesis 2.1 and suppose that $z \in \mathbb{C}\setminus\mathbb{R}$, $x_0 \in \mathbb{R}$. Then,

(i) $\mathcal{M}(z,x_0)$ is a matrix-valued Herglotz function of rank $2m$ with representation

$$\mathcal{M}(z,x_0) = \text{Re}(\mathcal{M}(i,x_0)) + \int_{\mathbb{R}} d\Omega(\lambda,x_0) \left((\lambda-z)^{-1} - \lambda(1+\lambda^2)^{-1}\right),$$

where

$$\int_{\mathbb{R}} \|d\Omega(\lambda,x_0)\|_{c^{2m\times2m}} (1+\lambda^2)^{-1} < \infty \quad (2.44)$$

and

$$\Omega((\lambda,\mu], x_0) = \lim_{\frac{\delta \downarrow 0}{\varepsilon \downarrow 0}} \frac{1}{\pi} \int_{\lambda+\delta}^{\mu+\delta} d\nu \text{Im}(\mathcal{M}(\nu + i\varepsilon,x_0)). \quad (2.45)$$

(ii) $z \in \mathbb{C}\setminus\text{spec}(H)$ if and only if $\mathcal{M}(z,x_0)$ is holomorphic near $z$.

Here spec$(T)$ denotes the spectrum of $T$. Later on we will denote by spec$_{ac}(T)$ the absolutely continuous spectrum of $T$.

Finally, we state the following characterization of $\mathcal{M}(z,x_0)$ to be used later on. In the scalar context $m = 1$ this has been used by Rofe-Beketov [87], [88] (see also [65, Sect. 7.3]).

**Theorem 2.6** ([87], [88]). Assume Hypothesis 2.1, suppose that $z \in \mathbb{C}\setminus\mathbb{R}$, $x_0 \in \mathbb{R}$, and let $\ell, r \in \mathbb{N}_0$. Then the following assertions are equivalent.

(i) $\mathcal{M}(z,x_0)$ is the $2m \times 2m$ Weyl–Titchmarsh matrix associated with a Schrödinger operator $H$ in $L^2(\mathbb{R})^m$ of the type (2.38) with an $m \times m$ matrix-valued potential $\mathcal{Q} \in L^1_{\text{loc}}(\mathbb{R})$ and $\mathcal{Q} \in \mathcal{C}^4(\langle -\infty, x_0 \rangle)$ and $\mathcal{Q} \in \mathcal{C}^r(\langle x_0, \infty \rangle)$.

(ii) $\mathcal{M}(z,x_0)$ is of the type (2.42) with $\mathcal{M}_\pm(z,x_0)$ being half-line $m \times m$ Weyl–Titchmarsh matrices on $[x_0, \pm\infty)$ corresponding to a Dirichlet boundary condition at $x_0$ and a self-adjoint boundary condition at $-\infty$ and/or $\infty$ (if any) which are associated with an $m \times m$ matrix-valued potential $\mathcal{Q}$ satisfying $\mathcal{Q} \in \mathcal{C}^4(\langle -\infty, x_0 \rangle)$ and $\mathcal{Q} \in \mathcal{C}^r(\langle x_0, \infty \rangle)$, respectively.

If (i) or (ii) holds, then the $2m \times 2m$ matrix-valued spectral measure $\Omega(\cdot,x_0)$ associated with $\mathcal{M}(z,x_0)$ is determined by (2.42) and (2.45).

Next, we consider variations of the reference point $x_0 \in \mathbb{R}$. In analogy to (2.20), we note that in the case where the Schrödinger differential expression $\mathcal{L}$ is in the limit point case at $\pm\infty$,

$$\mathcal{M}_\pm(z,x) = \Psi'_\pm(z,x,x_0)\Psi_\pm(z,x,x_0)^{-1}, \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad (2.46)$$

represents the corresponding half-line Weyl–Titchmarsh matrix on $[x, \pm\infty)$, $x \in \mathbb{R}$, with $\Psi_\pm(z,\cdot,x_0)$ defined in (2.31). Again the actual normalization of $\Psi_\pm$ is, of course, irrelevant. Since $\Psi_\pm$ satisfies the second-order linear $m \times m$ matrix-valued
differential equation (2.2), $M_\pm$ in (2.46) satisfies the matrix-valued Riccati-type equation (independently of any limit point assumptions at $\pm\infty$)

$$M'_\pm(z,x) + M_\pm(z,x)^2 = Q(x) - zI_m, \quad x \in \mathbb{R}, \; z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.47)$$

The asymptotic high-energy behavior of $M_\pm(z,x)$ as $|z| \to \infty$ has recently been determined in [21] under minimal smoothness conditions on $Q$ and without assuming that $L$ is in the limit point case at $\pm\infty$. Here we recall just a special case of the asymptotic expansion proved in [21] which is most suited for our discussion in Section 4. We denote by $C_\varepsilon \subset C_+$ the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle $\varepsilon$, with $0 < \varepsilon < \pi/2$.

**Theorem 2.7** ([21]). Fix $x_0 \in \mathbb{R}$ and let $x \geq x_0$. In addition to Hypothesis 2.1 suppose that $Q \in C^\infty([x_0, \pm\infty)^{m \times m}$ and that $L$ is in the limit point case at $\pm\infty$. Let $M_\pm(z,x), \; x \geq x_0$, be defined as in (2.46). Then, as $|z| \to \infty$ in $C_\varepsilon$, $M_\pm(z,x)$ has an asymptotic expansion of the form

$$M_\pm(z,x) \underset{|z| \to \infty}{\simeq} z^{1/2} + \sum_{k=1}^N M_{\pm,k}(x)z^{-k/2} + o(|z|^{-N/2}), \; N \in \mathbb{N}. \quad (2.48)$$

The expansion (2.48) is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in $C_\varepsilon$ and uniform in $x$ as long as $x$ varies in compact subsets of $[x_0, \infty)$. The expansion coefficients $M_{\pm,k}(x)$ can be recursively computed from

$$M_{\pm,1}(x) = \frac{1}{2}iQ(x), \quad M_{\pm,2}(x) = \frac{1}{4}Q'(x), \quad M_{\pm,k+1}(x) = \frac{1}{2}\left( M_{\pm,k}'(x) + \sum_{\ell=1}^{k-1} M_{\pm,\ell}(x)M_{\pm,k-\ell}(x) \right), \; k \geq 2. \quad (2.49)$$

The asymptotic expansion (2.48) can be differentiated to any order with respect to $x$.

**Remark 2.8.**

(i) Due to the recursion relation (2.49), the coefficients $M_{\pm,k}$ are universal polynomials in $Q$ and its $x$-derivatives (i.e., differential polynomials in $Q$). That the asymptotic expansion (2.48) can be differentiated to arbitrary order in $x$ follows from repeated use of the Riccati-type equation (2.47).

(ii) In the case where $Q$ and its $x$-derivatives are in $L^1(\mathbb{R})^{m \times m}$, or in the case where $Q$ is periodic and hence Floquet theory applies, the proof of the existence of an asymptotic expansion of the type (2.48) follows in a routine manner by iterating appropriate Volterra-type integral equations. The general case, however, is intricate as is evident from the treatment in [21].

Finally, in addition to (2.40), one infers for the $2m \times 2m$ Weyl–Titchmarsh function $M(z,x)$ associated with $H$ on $\mathbb{R}$ in connection with arbitrary half-lines
Moreover, the Riccati-type equations (2.47) imply the following results needed in Section 4.

\[ M(z, x) = (M_{j,j'}(z, x))_{j,j'=1,2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \]  
\[ M_{1,1}(z, x) = M_{\pm}(z, x) [M_{-}(z, x) - M_{+}(z, x)]^{-1} M_{\mp}(z, x) \]
\[ = \psi^*_+(z, x, x_0) [M_{-}(z, x_0) - M_{+}(z, x_0)]^{-1} \psi^*_-(z, x_0), \]
\[ M_{1,2}(z, x) = 2^{-1} [M_{-}(z, x) - M_{+}(z, x)]^{-1} [M_{-}(z, x) + M_{+}(z, x)] \]
\[ = \psi^*_+(z, x, x_0) [M_{-}(z, x_0) - M_{+}(z, x_0)]^{-1} \psi^*_-(z, x_0), \]
\[ M_{2,1}(z, x) = 2^{-1} [M_{-}(z, x) + M_{+}(z, x)] [M_{-}(z, x) - M_{+}(z, x)]^{-1} \]
\[ = \psi^*_+(z, x, x_0) [M_{-}(z, x_0) - M_{+}(z, x_0)]^{-1} \psi^*_-(z, x, x_0), \]
\[ M_{2,2}(z, x) = [M_{-}(z, x) - M_{+}(z, x)]^{-1} \]
\[ = \psi^*_+(z, x, x_0) [M_{-}(z, x_0) - M_{+}(z, x_0)]^{-1} \psi^*_-(z, x, x_0). \]

Introducing the convenient abbreviation,
\[ M(z, x) = \begin{pmatrix} h(z, x) & -g_2(z, x) \\ -g_1(z, x) & g(z, x) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in \mathbb{R}, \]
one then verifies from (2.50)–(2.55) and from \( M(\mp, x)^* = M(z, x), \) \( M_{\pm}(\mp, x)^* = M_{\pm}(z, x) \) that
\[ g(z, x)^* = g(z, x), \quad g_2(z, x)^* = g_1(z, x), \quad h(z, x)^* = h(z, x), \]
\[ g(z, x)g_1(z, x) = g_2(z, x)g(z, x), \]
\[ h(z, x)g_2(z, x) = g_1(z, x)h(z, x), \]
\[ g(z, x) = [M_{-}(z, x) - M_{+}(z, x)]^{-1}, \]
\[ g(z, x)h(z, x) - g_2(z, x)^2 = - (1/4) I_m, \]
\[ h(z, x)g(z, x) - g_1(z, x)^2 = - (1/4) I_m, \]
\[ M_{\pm}(z, x) = \mp (1/2) g(z, x)^{-1} - g(z, x)^{-1} g_2(z, x) \]
\[ + (1/2) g(z, x)^{-1} - g_1(z, x) g(z, x)^{-1}. \]

Moreover, the Riccati-type equations (2.47) imply the following results needed in Section 4.

**Lemma 2.9.** Let \( z \in \mathbb{C} \setminus \mathbb{R} \) and define \( M_{\pm} \) by (2.46) so that \( M_{\pm} \) satisfy the Riccati-type equation (2.47). Then, for a.e. \( x \in \mathbb{R}, \)
\[ g' = -(g_1 + g_2), \]
\[ g'_1 = -(Q - z I_m) g - h \]
\[ = -(g'' + g Q - Q g)/2, \]
\[ g'_2 = -g(Q - z I_m) - h \]
\[ = -(g'' + Q g - Q g)/2, \]
\[ h' = -g_1(Q - z I_m) - (Q - z I_m) g_2, \]
\[ h = [g'' - g(Q - z I_m) - (Q - z I_m) g]/2 \]
if $Q \in L^1_{\text{loc}}(\mathbb{R})^{m \times m}$, and
\begin{align*}
&g''_1 = -2(Q - zI_m)g' - Q'g + g_1Q - Qg_1, \\
&g''_2 = -2g'(Q - zI_m) - gQ' + Qg_2 - g_2Q
\end{align*}
if in addition $Q' \in L^1_{\text{loc}}(\mathbb{R})^{m \times m}$.

Proof. (2.47) rewritten in terms of $g$, $g_1$, $g_2$ yields
\begin{equation}
\pm (1/2)g^{-1}g'g^{-1} + g^{-1}g'g^{-1}g_2 - g^{-2}g' + (1/4)g^{-2} + g^{-1}g_2g^{-1}g_2
\end{equation}
Taking the difference of the two equations in (2.73) yields (2.64). Adding the two equations in (2.73) and using (2.57), (2.60), (2.61), and (2.64) yields (2.65) and (2.67). Combining (2.65), (2.67), and (2.64) implies (2.70). Inserting (2.70) into (2.65) and (2.67) yields (2.66) and (2.68). (2.69) follows from differentiating $h = g_2^2g^{-1} - (1/4)g^{-1}$, inserting $g'$ and $g''_1$ from (2.64) and (2.65), and making repeated use of the identities (2.57), (2.61). Finally, (2.71) (resp. (2.72)) follows from differentiating (2.65) (resp. (2.67)) inserting (2.69) for $h'$. Alternatively, (2.67)–(2.72) follow directly from (2.65)–(2.71) using (2.56).

3. Polynomial Pencils of Matrices in a Nutshell

Since self-adjoint polynomial pencils of matrices play a role in our principal section 4, we briefly review some of the corresponding definitions and basic results, mainly following the monograph of Markus [72] and papers by Markus and Matsaev [73], [74]. While all results below are discussed for operator pencils by Markus and Matsaev, we will only quote them in the matrix context, for simplicity.

Given $m \in \mathbb{N}$, we denote by
\begin{equation}
\mathcal{A}(z) = \sum_{k=0}^{n} A_k z^k, \quad A_k \in \mathbb{C}^{m \times m}, \quad 1 \leq k \leq n, \quad z \in \mathbb{C},
\end{equation}
a polynomial pencil of $m \times m$ matrices (in short, a pencil) in the following. $\mathcal{A}$ is called of degree $n \in \mathbb{N}_0$ if $A_n \neq 0$ and monic if $A_n = I_m$.

**Definition 3.1.** Let $\mathcal{A}$ be a pencil of the type (3.1).

(i) The spectrum of $\mathcal{A}$, denoted by $\text{spec}(\mathcal{A})$, is defined by
\begin{equation}
\text{spec}(\mathcal{A}) = \{ z \in \mathbb{C} \mid \mathcal{A}(z) \text{ is not invertible} \}.
\end{equation}
z$_0$ $\in \mathbb{C}$ is called an eigenvalue of $\mathcal{A}$ if $\mathcal{A}(z_0) f_0 = 0$ has a solution $f_0 \in \mathbb{C}^m \setminus \{0\}$.

(ii) A monic pencil $\mathcal{C}$ is called a (right) divisor of $\mathcal{A}$ if
\begin{equation}
\mathcal{A}(z) = \mathcal{B}(z)\mathcal{C}(z), \quad z \in \mathbb{C}
\end{equation}
for some pencil $\mathcal{B}$. If in addition $\text{spec}(\mathcal{B}) \cap \text{spec}(\mathcal{C}) = \emptyset$, then $\mathcal{C}$ is called a (right) spectral divisor of $\mathcal{A}$.

(iii) $Z \in \mathbb{C}^{m \times m}$ is called a (matrix) root of the pencil $\mathcal{A}$ if
\begin{equation}
\mathcal{A}(Z) = 0,
\end{equation}
where $\mathcal{A}(Z)$ is defined as
\begin{equation}
\mathcal{A}(Z) = \sum_{k=0}^{n} A_k Z^k.
\end{equation}
Theorem 3.2 (Markus [72], Sect. 29). Let $\mathcal{A}$ be a pencil of the type (3.1).

(i) \((zI_m - \mathcal{Z})\) is a divisor of $\mathcal{A}$ if and only if $\mathcal{A}(\mathcal{Z}) = 0$. (This justifies the notation introduced in the last part of Definition 3.1(ii).)

(ii) Let $\mathcal{A}$ be a monic pencil of degree $n$ and $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ spectral roots of $\mathcal{A}$. Then the following assertions are equivalent:

- (a) $\mathcal{V}(\mathcal{Z}_1, \ldots, \mathcal{Z}_n)$ is invertible.
- (b) $\text{spec}(\mathcal{Z}_j) \cap \text{spec}(\mathcal{Z}_k) = \emptyset$, $j \neq k$, $1 \leq j, k \leq n$.
- (c) $\text{spec}(\mathcal{A}) \subseteq \bigcup_{j=1}^{n} \text{spec}(\mathcal{Z}_j)$.

(iii) If $\mathcal{A}$ is a self-adjoint pencil, then $\text{spec}(\mathcal{A})$ is symmetric with respect to $\mathbb{R}$.

Theorem 3.3 (Markus [72], Sect. 31). Let $\mathcal{A}$ be a pencil of the type (3.1).

(i) If $\mathcal{A}$ is a weakly hyperbolic pencil, then $\text{spec}(\mathcal{A}) \subseteq \mathbb{R}$.

(ii) The root zones $\Delta_j(\mathcal{A})$, $1 \leq j \leq n$, of a weakly hyperbolic pencil $\mathcal{A}$ are intervals (possibly degenerating to a point).

(iii) If $\mathcal{A}$ is a weakly hyperbolic pencil of order $n$ with root zones $\{\Delta_j(\mathcal{A})\}_{1 \leq j \leq n}$ and $\Delta_k(\mathcal{A}) = \{\lambda_k\}$ for some $k \in \{1, \ldots, n\}$, then $\mathcal{A}(z) = (z - \lambda_k)\mathcal{B}(z)$, where $\mathcal{B}$ is a weakly hyperbolic pencil of order $n-1$, with root zones

$$\Delta_1(\mathcal{A}), \ldots, \Delta_{k-1}(\mathcal{A}), \Delta_k(\mathcal{A}), \ldots, \Delta_n(\mathcal{A}).$$

(iv) If $\mathcal{A}$ is a weakly hyperbolic pencil then $\Delta_j(\mathcal{A}) \cap \Delta_{j+1}(\mathcal{A})$, $1 \leq j \leq n-1$, consists of at most one point.

(v) If $\mathcal{A}$ is a hyperbolic pencil, then $\Delta_j(\mathcal{A}) \cap \Delta_k(\mathcal{A}) = \emptyset$, $j \neq k$, $1 \leq j, k \leq n$. Thus, a hyperbolic pencil is strongly hyperbolic if and only if

$$\Delta_j(\mathcal{A}) \cap \Delta_{j+1}(\mathcal{A}) = \emptyset, \quad 1 \leq j \leq n-1.$$ (3.9)

(vi) Suppose $\mathcal{A}$ is a self-adjoint pencil of degree $n$, $\mathcal{A}_n > 0$, and $\mathcal{A}(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$. Then $\mathcal{A}$ is a weakly hyperbolic pencil if and only if there exist numbers $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-1}$ such that $(-1)^j\mathcal{A}(\gamma_j) \geq 0$, $1 \leq j \leq n-1$. 

$Z \in \mathbb{C}^{m \times m}$ is called a (matrix) spectral root of the pencil $\mathcal{A}$ if $(zI_m - Z)$ is a spectral divisor of $\mathcal{A}$.

(iv) The pencil $\mathcal{A}$ is called self-adjoint if $\mathcal{A}_k = \mathcal{A}^*_k$ for all $1 \leq k \leq n$ (i.e., $\mathcal{A}(\overline{z})^* = \mathcal{A}(z)$ for all $z \in \mathbb{C}$).

(v) A self-adjoint pencil $\mathcal{A}$ is called weakly hyperbolic if $\mathcal{A}_m > 0$ and for all $f \in \mathbb{C}^{m \times m}$, the roots of the polynomial $(f, \mathcal{A}(\cdot)f)_{\mathbb{C}^m}$ are real. If in addition all these zeros are distinct, the pencil $\mathcal{A}$ is called hyperbolic.

Moreover, the Vandermonde matrix corresponding to a collection $\{\mathcal{Z}_1, \ldots, \mathcal{Z}_n\} \subseteq \mathbb{C}^{m \times m}$ is defined by

$$\mathcal{V}(\mathcal{Z}_1, \ldots, \mathcal{Z}_n) = \begin{pmatrix}
I_m & I_m & \cdots & I_m \\
\mathcal{Z}_1 & \mathcal{Z}_2 & \cdots & \mathcal{Z}_n \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{Z}_1^{n-1} & \mathcal{Z}_2^{n-1} & \cdots & \mathcal{Z}_n^{n-1}
\end{pmatrix}. \quad (3.7)$$
Suppose \( A \) is a self-adjoint pencil of degree \( n \) with \( \lambda_n > 0 \). Then \( A \) is a strongly hyperbolic pencil if and only if there exist numbers \( \gamma_1 < \gamma_2 < \cdots < \gamma_{n-1} \) such that \((-1)^jA(\gamma_j) > 0, 1 \leq j \leq n-1\).

**Theorem 3.4** (Markus [72], Sect. 31). Let \( A \) be a pencil of the type (3.1).

1. Suppose \( A \) is a weakly hyperbolic pencil and \( \Delta_{j_0} \cap \Delta_{j_0+1}(A) = \emptyset \) for some \( j_0 \in \{1, \ldots, n\} \). Then \( A \) has a spectral root \( Z_{j_0} \) such that \( \text{spec}(A) \cap \Delta_{j_0}(A) = \text{spec}(Z_{j_0}) \) and \( Z_{j_0} \) is similar to a self-adjoint matrix.

2. A strongly hyperbolic pencil \( A \) has \( n \) spectral roots \( \{Z_j\}_{1 \leq j \leq n} \) such that \( \text{spec}(Z_j) = \text{spec}(A) \cap \Delta_j(A) \) and each \( Z_j, 1 \leq j \leq n \), is similar to a self-adjoint matrix.

**Theorem 3.5** (Markus [72], Sect. 31, Markus and Matsaev [73], [74]). Let \( A \) be a pencil of the type (3.1).

1. A weakly hyperbolic monic pencil \( A \) of degree \( n \) is decomposable as
   \[
   A(z) = (zI_m - Y_n)(zI_m - Y_{n-1}) \cdots (zI_m - Y_1),
   \]
   with \( \text{spec}(Y_j) \subset \Delta_j(A) \), \( 1 \leq j \leq n \).

2. Let \( A \) be a strongly hyperbolic monic pencil of degree \( n \). Then \( A \) is decomposable as
   \[
   A(z) = (zI_m - Y_n)(zI_m - Y_{n-1}) \cdots (zI_m - Y_1),
   \]
   with \( \text{spec}(Y_j) \subset \Delta_j(A) \), \( 1 \leq j \leq n \). Moreover, each \( Y_j \) is similar to a spectral root \( Z_j \) of \( A \) and hence,
   \[
   \text{spec}(Y_j) = \text{spec}(Z_j) = \text{spec}(A) \cap \Delta_j(A), \quad 1 \leq j \leq n.
   \]

4. A Class of Matrix-Valued Schrödinger Operators with Prescribed Finite-Band Spectra

This section is devoted to the construction of a class of matrix-valued Schrödinger operators with a prescribed finite-band spectrum of uniform maximum multiplicity, the principal result of this paper.

To begin our analysis we start with a useful result on (scalar) Herglotz functions. Even though the result is probably well-known to experts, we provide an elementary proof for completeness.

Let\[
\{E_\ell\}_{0 \leq \ell \leq 2n} \subseteq \mathbb{R}, \quad n \in \mathbb{N}, \quad \text{with } E_\ell < E_{\ell+1}, 0 \leq \ell \leq 2n-1,
\]
and introduce the polynomial
\[
R_{2n+1}(z) = \prod_{\ell=0}^{2n} (z - E_\ell), \quad z \in \mathbb{C}.
\]
Moreover, we define the square root of \( R_{2n+1} \) by
\[
R_{2n+1}^{1/2}(\lambda)^{1/2} = \lim_{\varepsilon \downarrow 0} R_{2n+1}(\lambda + i\varepsilon)^{1/2}, \quad \lambda \in \mathbb{R},
\]
and
\[ R_{2n+1}(\lambda)^{1/2} = |R_{2n+1}(\lambda)|^{1/2} \begin{cases} (-1)^n i & \text{for } \lambda \in (-\infty, E_0), \\ (-1)^{n+j} i & \text{for } \lambda \in (E_{2j-1}, E_{2j}), j = 1, \ldots, n, \\ (-1)^{n+j} & \text{for } \lambda \in (E_{2j}, E_{2j+1}), j = 0, \ldots, n - 1, \\ 1 & \text{for } \lambda \in (E_{2n}, \infty), \end{cases} \lambda \in \mathbb{R} \quad (4.4) \]

and analytically continue \( R_{2n+1}^{1/2} \) from \( \mathbb{R} \) to all of \( \mathbb{C} \setminus \Sigma \), where \( \Sigma \) is defined by
\[ \Sigma = \left\{ \bigcup_{j=0}^{n-1} [E_{2j}, E_{2j+1}] \right\} \cup [E_{2n}, \infty). \quad (4.5) \]

In this context we also mention the useful formula
\[ \frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} = -R_{2n+1}(z)^{1/2}, \quad z \in \mathbb{C}_+. \quad (4.6) \]

**Theorem 4.1.** Let \( z \in \mathbb{C} \setminus \Sigma \) and \( n \in \mathbb{N} \). Define \( R_{2n+1}^{1/2} \) as in (4.1)–(4.4) followed by an analytic continuation to \( \mathbb{C} \setminus \Sigma \). Moreover let \( F_n \) and \( H_{n+1} \) be two monic polynomials of degree \( n \) and \( n+1 \), respectively. Then
\[ \frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} \]
is a Herglotz function if and only if all zeros of \( F_n \) are real and there is precisely one zero in each of the intervals \([E_{2j-1}, E_{2j}], 1 \leq j \leq n\). Moreover, if \( iR_{2n+1}^{1/2}F_n \) is a Herglotz function, then it can be represented in the form
\[ \frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} = \frac{1}{\pi} \int_{\Sigma} \frac{F_n(\lambda)d\lambda}{R_{2n+1}(\lambda)^{1/2}} \frac{1}{\lambda - z}, \quad z \in \mathbb{C} \setminus \Sigma. \quad (4.8) \]

Similarly,
\[ \frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}} \]
is a Herglotz function if and only if all zeros of \( H_{n+1} \) are real and there is precisely one zero in each of the intervals \([-\infty, E_0], [E_{2j-1}, E_{2j}], 1 \leq j \leq n\). Moreover, if \( iR_{2n+1}^{1/2}H_{n+1} \) is a Herglotz function, then it can be represented in the form
\[ \frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}} = \text{Re} \left( \frac{iH_{n+1}(i)}{R_{2n+1}(i)^{1/2}} \right) \]
\[ + \frac{1}{\pi} \int_{\Sigma} \frac{H_{n+1}(\lambda)d\lambda}{R_{2n+1}(\lambda)^{1/2}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right), \quad z \in \mathbb{C} \setminus \Sigma. \quad (4.10) \]

**Proof.** We start with the case of \( F_n(z)/R_{2n+1}(z)^{1/2} \) in (4.7). Consider a closed counterclockwise oriented contour \( \Gamma_{R,\varepsilon} \) which consists of the semicircle \( C_\varepsilon = \{ z \in \mathbb{C} \mid z = E_0 + \varepsilon \exp(i\alpha), -\pi/2 \leq \alpha \leq \pi/2 \} \) centered at \( E_0 \), the straight line \( L_+ = \{ z \in \mathbb{C}_+ \mid z = x + i\varepsilon, E_0 \leq x \leq R \} \), the following part of the circle of radius \((R^2 + \varepsilon^2)^{1/2}\) centered at \( E_0 \), \( C_R = \{ z \in \mathbb{C} \mid z = E_0 + (R^2 + \varepsilon^2)^{1/2} \exp(i\beta), \arctan(\varepsilon/R) \leq \beta \leq 2\pi - \arctan(\varepsilon/R) \} \), and the straight line \( L_- = \{ z \in \mathbb{C}_- \mid z = x - i\varepsilon, E_0 \leq x \leq R \} \).
Then, for \( \varepsilon > 0 \) small enough and \( R > 0 \) sufficiently large, one infers

\[
\frac{i F_n(z)}{R_{2n+1}(z)^{1/2}} = \frac{1}{2\pi i} \int_{\Gamma_{R,\varepsilon}} \frac{1}{\zeta - z} \frac{i F_n(\zeta)}{R_{2n+1}(\zeta)^{1/2}} d\zeta
\]

\[
= \frac{1}{\varepsilon} \int_{\Sigma} \frac{1}{\lambda - z} \frac{F_n(\lambda)}{R_{2n+1}(\lambda)^{1/2}}. \tag{4.11}
\]

Here we used (4.4) to compute the contributions of the contour integral along \([E_0, R]\) in the limit \( \varepsilon \downarrow 0 \) and note that the integral over \( C_R \) tends to zero as \( R \uparrow 0 \) since

\[
\frac{F_n(\zeta)}{R_{2n+1}(\zeta)^{1/2}} \zeta \sim O(|\zeta|^{-1/2}). \tag{4.12}
\]

Next, utilizing the fact that \( F_n \) is monic and using (4.4) again, one infers that \( F_n(\lambda) d\lambda / R_{2n+1}(\lambda)^{1/2} \) represents a positive measure supported on \( \Sigma \) if and only if \( F_n \) has precisely one zero in each of the intervals \([E_{2j-1}, E_{2j}]\), \( 1 \leq j \leq n \). In other words,

\[
\frac{F_n(\lambda)}{R_{2n+1}(\lambda)^{1/2}} \geq 0 \quad \text{on} \quad \Sigma \tag{4.13}
\]

if and only if \( F_n \) has precisely one zero in each of the intervals \([E_{2j-1}, E_{2j}]\), \( 1 \leq j \leq n \). The Herglotz representation theorem, Theorem 2.3, then finishes the proof of (4.8). The proof of (4.10) follows along similar lines taking into account the additional residues at \( \pm i \) inside \( \Gamma_{R,\varepsilon} \) which are responsible for the real part on the right-hand side of (4.10). \( \square \)

Theorem 4.1 can be improved by invoking ideas developed in the Appendix of [61] (cf. also [105]). We will pursue this further in [9].

**Corollary 4.2.** Let \( z \in \mathbb{C} \setminus \Sigma \) and \( m, n \in \mathbb{N} \). Define \( R_{2n+1}^{1/2} \) as in (4.1)–(4.4) followed by an analytic continuation to \( \mathbb{C} \setminus \Sigma \). Moreover let \( F_n \) and \( \mathcal{H}_{n+1} \) be two monic \( m \times m \) matrix pencils of degree \( n \) and \( n + 1 \), respectively. Then \( (i/2)R_{2n+1}^{-1/2}F_n \) is a Herglotz matrix if and only if the root zones \( \Delta_j(\mathcal{F}_n) \) of \( \mathcal{F}_n \) satisfy

\[
\Delta_j(\mathcal{F}_n) \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{4.14}
\]

Analogously, \( (i/2)R_{2n+1}^{-1/2}\mathcal{H}_{n+1} \) is a Herglotz matrix if and only if the root zones \( \Delta_j(\mathcal{H}_{n+1}) \) of \( \mathcal{H}_{n+1} \) satisfy

\[
\Delta_0(\mathcal{H}_{n+1}) \subset (\infty, E_0], \quad \Delta_j(\mathcal{H}_{n+1}) \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{4.15}
\]

If (4.14) (resp., (4.15)) holds, then \( \mathcal{F}_n \) (resp., \( \mathcal{H}_{n+1} \)) is a strongly hyperbolic pencil.

**Proof.** We recall that \( (i/2)R_{2n+1}^{-1/2}F_n \) is an \( m \times m \) Herglotz matrix if and only if \( (f, (i/2)R_{2n+1}^{-1/2}F_n f)_{L^2} \) is a Herglotz function for all \( f \in \mathbb{C}^m \setminus \{0\} \). Thus, it suffices to apply Theorem 4.1, identifying \( F_n \) and \( (f, \mathcal{F}_n f)_{L^2} \), to arrive at (4.14). The same argument applied to \( \mathcal{H}_{n+1} \) yields (4.15). \( \square \)

Next, we define the following \( 2m \times 2m \) matrix \( \mathcal{M}_z(z, x_0) \) which will turn out to be the underlying Weyl–Titchmarsh matrix associated with a class of \( m \times m \) matrix-valued Schrödinger operators with prescribed finite-band spectra. We introduce,
for fixed $x_0 \in \mathbb{R}$,
\[
\mathcal{M}_\Sigma(z, x_0) = (\mathcal{M}_{\Sigma, p,q}(z, x_0))_{1 \leq p,q \leq 2} \quad (4.16)
\]
\[
= \frac{i}{2R_{2n+1}(z)^{1/2}} \begin{pmatrix}
\mathcal{H}_{n+1,\Sigma}(z, x_0) & -\mathcal{G}_{2, n-1, \Sigma}(z, x_0) \\
-\mathcal{G}_{1, n-1, \Sigma}(z, x_0) & \mathcal{F}_{n, \Sigma}(z, x_0)
\end{pmatrix}, \quad z \in \mathbb{C}\setminus\Sigma.
\]

Here $R_{2n+1}(z)^{1/2}$ is defined as in (4.1)–(4.4) followed by analytic continuation into $\mathbb{C}\setminus\Sigma$ and the polynomial matrix pencils $\mathcal{F}_{n, \Sigma}$, $\mathcal{G}_{1, n-1, \Sigma}$, $\mathcal{G}_{2, n-1, \Sigma}$, and $\mathcal{H}_{n+1, \Sigma}$ are introduced as follows:

(i) $\mathcal{F}_{n, \Sigma}(\cdot, x_0)$ is an $m \times m$ monic matrix pencil of degree $n$, that is, $\mathcal{F}_{n, \Sigma}(\cdot, x_0)$ is of the type
\[
\mathcal{F}_{n, \Sigma}(z, x_0) = \sum_{\ell=0}^{n} \mathcal{F}_{n-\ell, \Sigma}(x_0) z^{\ell}, \quad \mathcal{F}_{0, \Sigma}(x_0) = I_m, \quad z \in \mathbb{C} \quad (4.17)
\]
and
\[
\frac{i}{2R_{2n+1}^{1/2}} \mathcal{F}_{n, \Sigma}(\cdot, x_0) \text{ is assumed to be an } m \times m \text{ Herglotz matrix.} \quad (4.18)
\]

Hence $\mathcal{F}_{n, \Sigma}(\cdot, x_0)$ is a self-adjoint (in fact, strongly hyperbolic) pencil,
\[
\mathcal{F}_{n, \Sigma}(z, x_0)^* = \mathcal{F}_{n, \Sigma}(z, x_0), \quad z \in \mathbb{C} \quad (4.19)
\]
and $(i/2)R_{2n+1}^{-1/2}\mathcal{F}_{n, \Sigma}$ and $2iR_{2n+1}^{1/2}\mathcal{F}_{n, \Sigma}^{-1}$ admit the Herglotz representations
\[
\frac{i}{2R_{2n+1}(z)^{1/2}} \mathcal{F}_{n, \Sigma}(z, x_0) = \frac{1}{2\pi} \int_{\Sigma} \frac{d\lambda}{R_{2n+1}(\lambda)^{1/2}} \mathcal{F}_{n, \Sigma}(\lambda, x_0) \frac{1}{\lambda - z}, \quad z \in \mathbb{C}\setminus\Sigma, \quad (4.20)
\]
\[
\frac{iR_{2n+1}(z)^{1/2}}{2} \mathcal{F}_{n, \Sigma}(z, x_0)^{-1} = \frac{1}{\pi} \int_{\Sigma} d\lambda R_{2n+1}(\lambda)^{1/2} \mathcal{F}_{n, \Sigma}(\lambda, x_0)^{-1} \left( \frac{1}{\lambda - z} - \frac{1}{1 + \lambda^2} \right) \\
+ \sum_{k=1}^{N} (z - \mu_k(x_0))^{-1} \Gamma_{\Sigma, k}(x_0), \quad (4.21)
\]
where
\[
\Gamma_{\Sigma, 0}(x_0) = \Gamma_{\Sigma, 0}(x_0)^* \in \mathbb{C}^{m \times m}, \quad 0 \leq \Gamma_{\Sigma, k}(x_0) \in \mathbb{C}^{m \times m}, \quad 1 \leq k \leq N,
\]
\[
\sum_{k=1}^{N} \text{rank}(\Gamma_{\Sigma, k}(x_0)) \leq mn, \quad \mu_k(x_0) \in \bigcup_{j=1}^{n} [E_{2j-1}, E_{2j}], \quad 1 \leq k \leq N. \quad (4.22)
\]

In fact, there are precisely $m$ numbers $\mu_k(x_0)$ in $[E_{2j-1}, E_{2j}]$ for each $1 \leq j \leq n$, counting multiplicity (they are the points $z$ where $\mathcal{F}_{n, \Sigma}(z, x_0)$ is not invertible).
(ii) Given these facts we now define

\[ G_{1,n-1,\Sigma}(z,x_0) = \left( \sum_{k=1}^{N} \frac{\varepsilon_k(x_0)}{z - \mu_k(x_0)} \Gamma_{\Sigma,k}(x_0) \right) F_{n,\Sigma}(z,x_0), \]  

(4.23)

\[ G_{2,n-1,\Sigma}(z,x_0) = F_{n,\Sigma}(z,x_0) \left( \sum_{k=1}^{N} \frac{\varepsilon_k(x_0)}{z - \mu_k(x_0)} \Gamma_{\Sigma,k}(x_0) \right), \]  

(4.24)

\[ \varepsilon_k(x_0) \in \{1,-1\}, \quad 1 \leq k \leq N, \quad z \in \mathbb{C} \setminus \{\mu_k(x_0)\}_{1 \leq k \leq N}, \]  

(4.25)

and

\[ H_{n+1,\Sigma}(z,x_0) = R_{2n+1}(z) F_{n,\Sigma}(z,x_0)^{-1} \]  

(4.26)

\[ + \left( \sum_{k=1}^{N} \frac{\varepsilon_k(x_0)}{z - \mu_k(x_0)} \Gamma_{\Sigma,k}(x_0) \right) F_{n,\Sigma}(z,x_0) \left( \sum_{\ell=1}^{N} \frac{\varepsilon_{\ell}(x_0)}{z - \mu_{\ell}(x_0)} \Gamma_{\Sigma,\ell}(x_0) \right), \]  

\[ z \in \mathbb{C} \setminus \{\mu_k(x_0)\}_{1 \leq k \leq N}. \]

**Lemma 4.3.** Let \( z \in \mathbb{C} \setminus \{\mu_k(x_0)\}_{1 \leq k \leq N}. \) \( G_{p,n-1,\Sigma}(\cdot, x_0), \) \( p = 1, 2, \) are \( m \times m \) polynomial matrix pencils of equal degree at most \( n - 1 \) and \( H_{n+1,\Sigma}(\cdot, x_0) \) is a self-adjoint \( m \times m \) monic matrix pencil of degree \( n + 1. \) Moreover, the following identities hold.

\[ G_{2,n-1,\Sigma}(\overline{z},x_0)^* = G_{1,n-1,\Sigma}(z,x_0), \]  

(4.27)

\[ F_{n,\Sigma}(z,x_0) G_{1,n-1,\Sigma}(z,x_0) = G_{2,n-1,\Sigma}(z,x_0) F_{n,\Sigma}(z,x_0), \]  

(4.28)

\[ H_{n+1,\Sigma}(z,x_0) G_{2,n-1,\Sigma}(z,x_0) = G_{1,n-1,\Sigma}(z,x_0) H_{n+1,\Sigma}(z,x_0), \]  

(4.29)

\[ F_{n,\Sigma}(z,x_0) H_{n+1,\Sigma}(z,x_0) - G_{2,n-1,\Sigma}(z,x_0)^2 = R_{2n+1}(z) I_m, \]  

(4.30)

\[ H_{n+1,\Sigma}(z,x_0) F_{n,\Sigma}(z,x_0) - G_{1,n-1,\Sigma}(z,x_0)^2 = R_{2n+1}(z) I_m. \]  

(4.31)

**Proof.** The identities (4.28)–(4.31) are obvious from (4.23)–(4.25). Similarly, (4.27) is clear from (4.23)–(4.25) and (4.19). By (4.21), one infers

\[ \lim_{z \to \mu_k(x_0)} F_{n,\Sigma}(z,x_0) \Gamma_{\Sigma,k}(x_0) = 0 = \lim_{z \to \mu_k(x_0)} F_{n,\Sigma}(z,x_0) \Gamma_{\Sigma,k}(x_0) \]  

(4.32)

and hence \( G_{p,n-1,\Sigma}(\cdot, x_0), \) \( p = 1, 2, \) are polynomial matrix pencils of degree at most \( n - 1. \) By (4.27),

\[ \deg(G_{1,n-1,\Sigma}(\cdot, x_0)) = \deg(G_{2,n-1,\Sigma}(\cdot, x_0)) = n - 1. \]  

(4.33)

Next, using (4.21) again, one notes that

\[ i R_{2n+1}(\mu_k(x_0))^{1/2} \Gamma_{\Sigma,k}(x_0) = -\Gamma_{\Sigma,k}(x_0) [(d/dz) F_{n,\Sigma}(\mu_k(x_0)) \Gamma_{\Sigma,k}(x_0)], \]  

(4.34)

\[ 1 \leq k \leq N \]

and thus, combining (4.26) and (4.34),

\[ \text{res}_{z=\mu_k(x_0)} H_{n+1,\Sigma}(z,x_0) \]  

(4.35)

\[ = i R_{2n+1}(\mu_k(x_0))^{1/2} \Gamma_{\Sigma,k}(x_0) + \Gamma_{\Sigma,k}(x_0) [(d/dz) F_{n,\Sigma}(\mu_k(x_0)) \Gamma_{\Sigma,k}(x_0)] = 0, \]  

\[ 1 \leq k \leq N. \]

Hence, \( H_{n+1,\Sigma} \) is indeed a polynomial matrix pencil of degree \( n + 1. \)

In fact, \( H_{n+1,\Sigma} \) is a strongly hyperbolic pencil as shown in Theorem 4.8.

In the following it will be convenient to use the following set of assumptions.

\[ \square \]
Hypothesis 4.4. Let $m, n \in \mathbb{N}$. Define $R_{2n+1}$ as in (4.1), (4.2) and $R_{2n+1}^{1/2}$ as in (4.3), (4.4) followed by an analytic continuation to $\mathbb{C}\setminus \Sigma$, with $\Sigma$ introduced in (4.5). Moreover, let the polynomial $m \times m$ matrix pencil $F_{n, \Sigma}(\cdot, x_0)$, $G_{1,n-1, \Sigma}(\cdot, x_0)$, $G_{2,n-1, \Sigma}(\cdot, x_0)$, and $H_{n+1, \Sigma}(\cdot, x_0)$ be defined as in (4.17), (4.18), (4.23)–(4.26).

Next, we introduce
\[
M_{\pm, \Sigma}(z, x_0) = \pm i R_{2n+1}(z)^{1/2} F_{n, \Sigma}(z, x_0)^{-1} - G_{1,n-1, \Sigma}(z, x_0) F_{n, \Sigma}(z, x_0)^{-1}
\]
and
\[
N_{\pm, \Sigma}(z, x_0) = M_{-\Sigma}(z, x_0) \pm M_{+\Sigma}(z, x_0), \quad z \in \mathbb{C}\setminus \{\Sigma \cup \{\mu_k(x_0)\}_{1 \leq k \leq N}\}.
\]

We also introduce the open interior $\Sigma^o$ of $\Sigma$ defined by $\Sigma^o = \bigcup_{j=0}^{n-1} (E_{2j}, E_{2j+1}) \cup (E_{2n}, \infty)$. Then one verifies the following fundamental facts.

Theorem 4.5. Assume Hypothesis 4.4 and let $z \in \mathbb{C}\setminus \{\Sigma \cup \{\mu_k(x_0)\}_{1 \leq k \leq N}\}$. Moreover, introduce the $2m \times 2m$ matrix $M_{\Sigma}(\cdot, x_0)$ as in (4.16), (4.37) and the $m \times m$ matrices $M_{\pm, \Sigma}(\cdot, x_0)$ as in (4.36). Then,

(i) $\pm M_{\pm, \Sigma}(\cdot, x_0)$ are $m \times m$ Herglotz matrices with representations
\[
\pm M_{\pm, \Sigma}(z, x_0) = \frac{1}{\pi} \int_{\Sigma} d\lambda R_{2n+1}(\lambda)^{1/2} F_{n, \Sigma}(\lambda, x_0)^{-1}
\]
\[
\times \left(1 - \frac{\lambda - z}{\lambda - 1 + \lambda^2}\right) + \Gamma_{\Sigma, 0}(x_0) - \sum_{k=1}^{N} \frac{1 \pm \varepsilon_k(x_0)}{z - \mu_k(x_0)} \Gamma_{\Sigma, k}(x_0),
\]
\[
z \in \mathbb{C}\setminus \{\Sigma \cup \{\mu_k(x_0)\}_{1 \leq k \leq N}\}.
\]

Moreover, $M_{\pm, \Sigma}(\cdot, x_0)$ are the half-line $M$-matrices associated with self-adjoint Schrödinger operators $H^D_{\pm, x_0, \Sigma}$ in $L^2((x_0, \pm \infty))^m$, with a Dirichlet boundary condition at the point $x_0$ and an $m \times m$ matrix-valued potential $Q_{\Sigma}$ satisfying
\[
Q_{\Sigma} = Q^*_{\Sigma} \in L^1_{loc}(\mathbb{R})^{m \times m} \cap C^\infty(\mathbb{R}\setminus \{x_0\})^{m \times m},
\]
given by
\[
H^D_{\pm, x_0, \Sigma} = -I_m \frac{d^2}{dx^2} + Q_{\Sigma},
\]
\[
\text{dom}(H^D_{\pm, x_0, \Sigma}) = \{g \in L^2((x_0, \pm \infty))^m \mid g, g' \in AC([x_0, c])^m \text{ for all } c \geq x_0; \lim_{\varepsilon \downarrow 0} g(x_0 \pm \varepsilon) = 0; \ (-g'' + Q_{\Sigma} g) \in L^2((x_0, \pm \infty))^m\}.
\]

(ii) The differential expression $L_{\Sigma} = -I_m \frac{d^2}{dx^2} + Q_{\Sigma}$ is in the limit point case at $\pm \infty$.

(iii) The matrix $M_{\Sigma}(\cdot, x_0)$, defined by
\[
M_{\Sigma}(z, x_0) = \left(M_{\Sigma, p, q}(z, x_0)\right)_{1 \leq p, q \leq 2}
\]
\[
= \begin{pmatrix}
M_{\Sigma}(z, x_0) N_{\Sigma}(z, x_0)^{-1} M_{\Sigma}(z, x_0) & N_{\Sigma}(z, x_0)^{-1} N_{\Sigma}(z, x_0)/2 \\
N_{\Sigma}(z, x_0) N_{\Sigma}(z, x_0)^{-1}/2 & N_{\Sigma}(z, x_0)^{-1}
\end{pmatrix},
\]
is a $2m \times 2m$ Herglotz matrix admitting a representation of the type (2.43), with measure $\Omega_{\Sigma}(\cdot, x)$ given by

$$d\Omega_{\Sigma}(\lambda, x_0) = \begin{cases} \frac{1}{2\pi i \mu_{n+1}(\lambda)} \left( \begin{array}{cc} \mathcal{H}_{n+1, \Sigma}(\lambda, x_0) & -\mathcal{G}_{2, n-1, \Sigma}(\lambda, x_0) \\ -\mathcal{G}_{1, n-1, \Sigma}(\lambda, x_0) & \mathcal{F}_{n, \Sigma}(\lambda, x_0) \end{array} \right) d\lambda, & \lambda \in \Sigma^o, \\ 0, & \lambda \in \mathbb{R} \setminus \Sigma. \end{cases} \tag{4.42}$$

In addition, $\mathcal{M}_{\Sigma}(\cdot, x_0)$ is the Weyl–Titchmarsh $\mathcal{M}$-matrix associated with the self-adjoint Schrödinger operator $H_{\Sigma}$ in $L^2(\mathbb{R})^m$ defined by

$$H_{\Sigma} = -\mathcal{L}_m \frac{d^2}{dx^2} + Q_{\Sigma}, \tag{4.43}$$

$$\text{dom}(H_{\Sigma}) = \{ g \in L^2(\mathbb{R})^m \mid g, g' \in AC_{\text{loc}}(\mathbb{R})^m; (-g'' + Q_{\Sigma} g) \in L^2(\mathbb{R})^m \}. \tag{4.44}$$

(iv) $H_{\Sigma}$ has purely absolutely continuous spectrum $\Sigma$,

$$\text{spec}(H_{\Sigma}) = \text{spec}_{\text{ac}}(H_{\Sigma}) = \Sigma, \quad \text{spec}_p(H_{\Sigma}) = \text{spec}_{\text{sc}}(H_{\Sigma}) = \emptyset, \tag{4.44}$$

with $\text{spec}(H_{\Sigma_n})$ of uniform spectral multiplicity $2m$.

Proof. The representations (4.38) for $\mathcal{M}_{\pm, \Sigma}$ immediately follow from combining (4.21), (4.23), and (4.36a). These representations also prove that $\pm \mathcal{M}_{\pm, \Sigma}(\cdot, x_0)$ are $m \times m$ Herglotz matrices (cf. Theorem 2.3). Combining (4.38) with Theorem 2.4, taking into account Lemma 8.3.2 in [65], then yields the properties stated for $Q_{\Sigma}$. That $\mathcal{L}_\Sigma = -\mathcal{L}_m \frac{d^2}{dx^2} + Q_{\Sigma}$ is in the limit point case at $\pm \infty$ can be proved in analogy to Wienholtz’s proof [109] of a result originally due to Povzner [84], reproduced as Theorem 35 in [41, p. 58]. The corresponding details will be presented in [22]. Equation (4.41) follows from (4.16), (4.23)–(4.26), and (4.36). Relation (4.44) follows from the explicit formula (4.42) of the spectral measure. In particular, the support property $\text{supp}(\Omega_{\Sigma}) = \Sigma$ of the measure $\Omega_{\Sigma}$ in (4.42) proves $\text{spec}(H_{\Sigma}) = \Sigma$, etc. The uniform maximum spectral multiplicity $2m$ then follows from the fact that $\text{rank}(d\Omega_{\Sigma}/d\lambda) = 2m$ on the interior $\Sigma^o$ of $\Sigma$. \qed

At this point we cannot yet infer continuity of $Q_{\Sigma}$ at the boundary point $x_0$. We will subsequently return to this issue in Theorem 4.8.

In the following we will apply these facts to our concrete class of matrix-valued Schrödinger operators discussed in Theorem 4.5. In order to find the corresponding Weyl–Titchmarsh matrices $\mathcal{M}_{\pm, \Sigma}(z, x)$, we need some preparations. We denote by $\psi_{\pm, \Sigma}(z, x, x_0)$ the Weyl solutions (2.31) associated with $Q_{\Sigma}$, that is,

$$\psi_{\pm, \Sigma}(z, x, x_0) = \theta_{\Sigma}(z, x, x_0) + \phi_{\Sigma}(z, x, x_0) \mathcal{M}_{\pm, \Sigma}(z, x_0), \quad z \in \mathbb{C} \setminus \Sigma, \tag{4.45}$$

where, in obvious notation, $\theta_{\Sigma}(z, x, x_0), \phi_{\Sigma}(z, x, x_0)$ denote the fundamental system (2.4) corresponding to $Q_{\Sigma}$. Then straightforward computations of the right-hand sides of (2.46)–(2.49) (taking into account (2.16), (2.17), (4.28), and (4.31)) yield

$$\mathcal{M}_{\Sigma}(z, x) = (\mathcal{M}_{\Sigma, p, q}(z, x))_{1 \leq p, q \leq 2} \tag{4.46}$$

$$= \frac{i}{2 R_{2n+1}(z)^{1/2}} \begin{pmatrix} \mathcal{H}_{n+1, \Sigma}(z, x) & -\mathcal{G}_{2, n-1, \Sigma}(z, x) \\ -\mathcal{G}_{1, n-1, \Sigma}(z, x) & \mathcal{F}_{n, \Sigma}(z, x) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \Sigma,$$
where we abbreviated

\begin{align}
F_{n,\Sigma}(z, x) &= \theta_{\Sigma}(z, x, x_0)F_{n,\Sigma}(z, x_0)\theta_{\Sigma}(\tau, x, x_0)^* \\
&+ \phi_{\Sigma}(z, x, x_0)H_{n+1,\Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&- \phi_{\Sigma}(z, x, x_0)\mathcal{G}_{1, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&- \theta_{\Sigma}(z, x, x_0)\mathcal{G}_{2, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^*,
\end{align}

(4.47)

\begin{align}
\mathcal{G}_{1, n-1, \Sigma}(z, x) &= -\theta_{\Sigma}(z, x, x_0)F_{n,\Sigma}(z, x_0)\theta_{\Sigma}(\tau, x, x_0)^* \\
&- \phi_{\Sigma}(z, x, x_0)H_{n+1,\Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&+ \phi_{\Sigma}(z, x, x_0)\mathcal{G}_{1, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&+ \theta_{\Sigma}(z, x, x_0)\mathcal{G}_{2, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^*,
\end{align}

(4.48)

\begin{align}
\mathcal{G}_{2, n-1, \Sigma}(z, x) &= -\theta_{\Sigma}(z, x, x_0)F_{n,\Sigma}(z, x_0)\theta_{\Sigma}(\tau, x, x_0)^* \\
&- \phi_{\Sigma}(z, x, x_0)H_{n+1,\Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&+ \phi_{\Sigma}(z, x, x_0)\mathcal{G}_{1, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&+ \theta_{\Sigma}(z, x, x_0)\mathcal{G}_{2, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^*,
\end{align}

(4.49)

\begin{align}
\mathcal{H}_{n+1, \Sigma}(z, x) &= \theta_{\Sigma}(z, x, x_0)F_{n,\Sigma}(z, x_0)\theta_{\Sigma}(\tau, x, x_0)^* \\
&+ \phi_{\Sigma}(z, x, x_0)H_{n+1,\Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&- \phi_{\Sigma}(z, x, x_0)\mathcal{G}_{1, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^* \\
&- \theta_{\Sigma}(z, x, x_0)\mathcal{G}_{2, n-1, \Sigma}(z, x_0)\phi_{\Sigma}(\tau, x, x_0)^*.
\end{align}

(4.50)

Considerations of this type can be found in [65, Sect. 8.2] in the special scalar case

\( m = 1 \) and in the matrix context \( m \in \mathbb{N} \) in [101, Sect. 9.4].

Differentiating (4.47)–(4.50) with respect to \( x \) (taking into account that \( \theta'' = (Q_{\Sigma} - zI_m)\theta, \phi'' = (Q_{\Sigma} - z)\phi \) then yields (4.51)–(4.58) below. Alternatively, these results directly follow from Lemma 2.9 identifying \( \mathfrak{g} \) and \((i/2)R_{2n+1}^{-1/2}F_n, \mathfrak{g}_p \) and \((i/2)R_{2n+1}^{-1/2}\mathcal{G}_{p,n-1}, p = 1, 2, \) and \( \mathfrak{h} \) and \((i/2)R_{2n+1}^{-1/2}\mathcal{H}_{n+1} \), respectively.

**Lemma 4.6.** Assume Hypothesis 4.4 and let \((z, x) \in \mathbb{C} \times \mathbb{R}\). Then

\begin{align}
F'_{n,\Sigma} &= -(\mathcal{G}_{1, n-1, \Sigma} + \mathcal{G}_{2, n-1, \Sigma}),
\end{align}

(4.51)

\begin{align}
\mathcal{G}'_{n-1, \Sigma} &= -(Q_{\Sigma} - zI_m)F_{n,\Sigma} - \mathcal{H}_{n+1, \Sigma} \\
&= -(F''_{n,\Sigma} + F_{n,\Sigma}Q_{\Sigma} - Q_{\Sigma}F_{n,\Sigma})/2,
\end{align}

(4.52)

(4.53)

\begin{align}
\mathcal{G}''_{n-1, \Sigma} &= -2(Q_{\Sigma} - zI_m)F'_{n,\Sigma} - Q_{\Sigma}F_{n,\Sigma} + \mathcal{G}_{1, n-1, \Sigma}Q_{\Sigma} - Q_{\Sigma}\mathcal{G}_{1, n-1, \Sigma},
\end{align}

(4.54)

\begin{align}
\mathcal{G}'_{2, n-1, \Sigma} &= -F_{n,\Sigma}(Q_{\Sigma} - zI_m) - \mathcal{H}_{n+1, \Sigma} \\
&= -(F''_{n,\Sigma} + Q_{\Sigma}F_{n,\Sigma} - F_{n,\Sigma}Q_{\Sigma})/2,
\end{align}

(4.55)

(4.56)

\begin{align}
\mathcal{G}''_{2, n-1, \Sigma} &= -2F'_{n,\Sigma}(Q_{\Sigma} - zI_m) - F_{n,\Sigma}Q_{\Sigma} + Q_{\Sigma}\mathcal{G}_{2, n-1, \Sigma} - \mathcal{G}_{2, n-1, \Sigma}Q_{\Sigma},
\end{align}

(4.57)

\begin{align}
\mathcal{H}'_{n+1, \Sigma} &= \mathcal{G}_{1, n-1, \Sigma}(Q_{\Sigma} - zI_m) - (Q_{\Sigma} - zI_m)\mathcal{G}_{2, n-1, \Sigma},
\end{align}

(4.58)

\begin{align}
\mathcal{H}''_{n+1, \Sigma} &= [F''_{n,\Sigma} - F_{n,\Sigma}(Q_{\Sigma} - zI_m) - (Q_{\Sigma} - zI_m)F_{n,\Sigma}]/2.
\end{align}

(4.59)

In particular, one also verifies the following facts from (2.56)–(2.63).
Lemma 4.7. Assume Hypothesis 4.4 and let \((z, x) \in \mathbb{C} \times \mathbb{R}\). Then
\[
\mathcal{F}_{n,\Sigma}(z, x)^* = \mathcal{H}_{n+1,\Sigma}(z, x), \quad \mathcal{H}_{n+1,\Sigma}(z, x)^* = \mathcal{H}_{n+1,\Sigma}(z, x),
\]
\[
\mathcal{G}_{2n-1,\Sigma}(z, x)^* = \mathcal{G}_{1, n-1,\Sigma}(z, x),
\]
\[
\mathcal{F}_{n,\Sigma}(z, x) \mathcal{G}_{1, n-1,\Sigma}(z, x) = \mathcal{G}_{2, n-1,\Sigma}(z, x) \mathcal{F}_{n,\Sigma}(z, x),
\]
\[
\mathcal{H}_{n+1,\Sigma}(z, x) \mathcal{G}_{2, n-1,\Sigma}(z, x) = \mathcal{G}_{1, n-1,\Sigma}(z, x) \mathcal{H}_{n+1,\Sigma}(z, x),
\]
\[
\mathcal{H}_{n+1,\Sigma}(z, x) \mathcal{F}_{n,\Sigma}(z, x) - \mathcal{G}_{1, n-1,\Sigma}(z, x)^2 = R_{2n+1}(z)\mathcal{I}_m,
\]
\[
\mathcal{F}_{n,\Sigma}(z, x) \mathcal{H}_{n+1,\Sigma}(z, x) - \mathcal{G}_{2, n-1,\Sigma}(z, x)^2 = R_{2n+1}(z)\mathcal{I}_m.
\]

Proof. (4.60) is clear from (4.27), the fact that \(\mathcal{F}_{n,\Sigma}(\cdot, x)_0\) and \(\mathcal{H}_{n+1,\Sigma}(\cdot, x)_0\) are self-adjoint \(m \times m\) matrix pencils, (4.48), and (4.49). Similarly, (4.61)–(4.64) follow from elementary (but somewhat tedious) calculations directly from (4.47)–(4.50), invoking (2.10)–(2.17) and (4.28)–(4.31) repeatedly. □

Combining (2.50)–(2.54) and (4.46) then yields
\[
\mathcal{M}_{\pm, \Sigma}(z, x) = \pm iR_{2n+1}(z)^{1/2} \mathcal{F}_{n,\Sigma}(z, x)^{-1} - \mathcal{G}_{1, n-1,\Sigma}(z, x) \mathcal{F}_{n,\Sigma}(z, x)^{-1}
\]
\[
= \pm iR_{2n+1}(z)^{1/2} \mathcal{F}_{n,\Sigma}(z, x)^{-1} - \mathcal{F}_{n,\Sigma}(z, x)^{-1} \mathcal{G}_{2, n-1,\Sigma}(z, x),
\]
\[
z \in \mathbb{C} \setminus \mathbb{R}.
\]

One observes that for each \(x \in \mathbb{R}\), \(\mathcal{M}_{+, \Sigma}(\cdot, x)\) is the analytic continuation of \(\mathcal{M}_{-\Sigma}(\cdot, x)\) through the set \(\Sigma\), and vice versa,
\[
\lim_{\varepsilon \downarrow 0} \mathcal{M}_{+, \Sigma}(\lambda + i\varepsilon, x) = \lim_{\varepsilon \downarrow 0} \mathcal{M}_{-\Sigma}(\lambda - i\varepsilon, x),
\]
\[
\lambda \in \bigcup_{j=0}^{n-1} (E_{2j}, E_{2j+1}) \cup (E_{2n}, \infty), \quad x \in \mathbb{R}.
\]

In other words, for each \(x \in \mathbb{R}\), \(\mathcal{M}_{+, \Sigma}(\cdot, x)\) and \(\mathcal{M}_{-\Sigma}(\cdot, x)\) are the two branches of an analytic matrix-valued function \(\mathcal{M}_{\Sigma}(\cdot, x)\) on the two-sheeted Riemann surface of \(R_{2n+1}^{1/2}\). Thus, the corresponding potential \(Q_{\Sigma}\) is reflectionless in the sense discussed in [23], [40], and [56].

Thus, one obtains the following results.

Theorem 4.8. Assume Hypothesis 4.4 and let \((z, x) \in \mathbb{C} \setminus \mathbb{R}\) and \(x \in \mathbb{R}\). Then
(i) \(\mathcal{M}_{\pm, \Sigma}(z, \cdot)\) in (4.65) satisfy the matrix-valued Riccati-type equation
\[
\mathcal{M}_{\pm, \Sigma}(z, x) + \mathcal{M}_{\pm, \Sigma}(z, x)^2 = Q_{\Sigma}(x) - \epsilon \mathcal{I}_m, \quad x \in \mathbb{R}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

Moreover, \(\mathcal{M}_{\pm, \Sigma}(z, x)\) in (4.65) are the \(m \times m\) Weyl–Titchmarsh matrices associated with \(H_{\Sigma}(\cdot, \cdot)\) on the half-lines \([x, \pm \infty]\) and thus for each \(x \in \mathbb{R}, \mathcal{M}_{\Sigma}(z, x)\) in (4.46) is a \(2m \times 2m\) Weyl–Titchmarsh matrix associated with \(H_{\Sigma}\) on \(\mathbb{R}\). In particular, \(\mathcal{M}_{\Sigma}(\cdot, x)\) is a \(2m \times 2m\) Hermitz matrix of \(H_{\Sigma}\) admitting a representation of the type (2.43), with measure \(\Omega_{\Sigma}(\cdot, \cdot)\) given by
\[
d\Omega_{\Sigma}(\lambda, x) = \frac{1}{2\pi R_{2n+1}(\lambda)^{1/2}} \left( \begin{array}{cc} \mathcal{H}_{n+1, \Sigma}(\lambda, x) & -\mathcal{G}_{2, n-1, \Sigma}(\lambda, x) \\ -\mathcal{G}_{1, n-1, \Sigma}(\lambda, x) & \mathcal{F}_{n, \Sigma}(\lambda, x) \end{array} \right) d\lambda, \quad \lambda \in \Sigma^n, \lambda \in \mathbb{R} \setminus \Sigma.
\]

(ii) \(\mathcal{F}_{n,\Sigma}(\cdot, x)\) and \(\mathcal{H}_{n+1,\Sigma}(\cdot, x)\) are strongly hyperbolic (and hence self-adjoint) \(m \times m\) monic matrix pencils of degree \(n \) and \(n + 1\), respectively, and \(\mathcal{G}_{p,n-1,\Sigma}(\cdot, x)\),
p = 1, 2, are \( m \times m \) matrix pencils of degree \( n - 1 \).

(iii) \( Q_\Sigma \in C^\infty(\mathbb{R})^{m \times m} \).

Proof. (4.67) is clear from Lemma 4.6 and (4.65). Since \( Q_\Sigma \in C^\infty(\mathbb{R})^{m \times m} \), the initial value problems

\[
M_{\pm}(z, x) + M_{\pm}(z, x)^2 = Q_\Sigma(x) - zI_m, \quad x \in \mathbb{R}, \quad z \in \mathbb{C}\setminus\mathbb{R},
\]

(4.69a)

\[
M_{\pm}(z, x_0) = Q_\Sigma(z, x_0), \quad (4.69b)
\]

with \( M_{\pm, \Sigma}(z, x_0) \) given by (4.36), has a unique solution. Since at \( x = x_0 \) this solution coincides with the Weyl–Titchmarsh \( M \)-matrix \( M_{\pm, \Sigma}(z, x) \) at \( x = x_0 \) in (4.65) (using the initial condition (2.5) in (4.47)–(4.50)), \( M_{\pm, \Sigma}(z, x) \) represents the Weyl–Titchmarsh matrices associated with \( H_\Sigma^{2 \times 2} \) on the half-lines \( |x, \pm\infty| \). This proves part (i). By the known leading asymptotic behavior (2.33) of \( M_{\pm, \Sigma}(\cdot, x) \) (valid for each \( x_0 \in \mathbb{R} \), see [21]) and that of the diagonal Green’s matrix \( G_\Sigma(\cdot, x) = M_{\Sigma, 2, 2}(\cdot, x) \) of \( H_\Sigma \) as \( |z| \to \infty \), \( F_{n, \Sigma}(\cdot, x) \) and \( H_{n+1, \Sigma}(\cdot, x) \) are monic matrix pencils of degree \( n \) and \( n + 1 \), respectively, and \( G_{p, n-1, \Sigma}(\cdot, x), p = 1, 2 \), are \( m \times m \) matrix pencils of degree \( n - 1 \). Since the diagonal blocks of each Herglotz matrix are also Herglotz matrices, one concludes that \( (i/2)R_{2n+1}^{-1/2}F_{n, \Sigma} \) and \( (i/2)R_{2n+1}^{-1/2}H_{n+1, \Sigma} \) are Herglotz matrices. By Corollary 4.2 this then proves that \( F_{n, \Sigma} \) and \( H_{n+1, \Sigma} \) are strongly hyperbolic pencils and hence item (ii) holds. As in Theorem 4.5, \( Q_\Sigma \in C^\infty((-\infty, x) \cup (x, \infty))^{m \times m} \). Since \( x \in \mathbb{R} \) is arbitrary, this proves (iii).

It should be emphasized that the construction of \( Q_\Sigma \) along the lines of Section 4 in the scalar case \( m = 1 \) is due to Levitan [62] (see also [63], [64], [65, Ch. 8], [67]).

5. Trace Formulas and Connections with the Stationary Matrix KdV Hierarchy

In this section we introduce the stationary matrix Korteweg–de Vries (KdV) hierarchy and show that the class of finite-band potentials \( Q_\Sigma \) constructed in Section 4 satisfies some (and hence infinitely many) equations of the stationary KdV equations. We also introduce trace formulas for KdV invariants.

In order to extend the recursive approach constructing KdV Lax pairs in the scalar (Abelian) context to the present matrix-valued (non-Abelian) setting, we focus on an efficient approach introduced by Dubrovin [28] (in the scalar case \( m = 1 \)). Recalling (2.50)–(2.63) and Lemma 2.9, we state the following matrix-version of Dubrovin’s generating function approach to higher-order Lax pairs.

We start by introducing the following hypothesis.

**Hypothesis 5.1.** Fix \( m \in \mathbb{N} \), suppose \( Q = Q^* \in C^\infty(\mathbb{R})^{m \times m} \) and introduce the differential expression

\[
\mathcal{L} = -\mathcal{I}_m \frac{d^2}{dx^2} + Q, \quad x \in \mathbb{R}.
\]

(5.1)

Suppose \( \mathcal{L} \) is in the limit point case at \( \pm \infty \) and introduce the corresponding self-adjoint operator \( H \) in \( L^2(\mathbb{R})^m \) by

\[
H = -\mathcal{I}_m \frac{d^2}{dx^2} + Q,
\]

(5.2)

\[
\text{dom}(H) = \{ g \in L^2(\mathbb{R})^m \mid g, g' \in AC_{\text{loc}}(\mathbb{R})^m; (-g'' + Qg) \in L^2(\mathbb{R})^m \}.
\]
Given Hypothesis 5.1, we introduce the generating operator $P_z$ by
\[
P_z = \left( g(z, \cdot) \frac{d}{dx} + g_2(z, \cdot) \right)(L - zI_m)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
(5.3)
\[
dom(P_z) = \bigcup_{E \in \mathbb{C} \setminus \{ \mathbb{R} \cup \{z\} \}} \ker(L - EI_m),
\]
where $\ker(L - EI_m)$ denotes the algebraic nullspace of $L - EI_m$ (as opposed to the functional analytic nullspace in $L^2(\mathbb{R})^m$) and $(L - zI_m)^{-1}$ acts in the obvious manner by
\[
(L - zI_m)^{-1} \psi = (E - z)^{-1} \psi, \quad \psi \in \ker(L - EI_m).
\]
(5.4)

The precise operator theoretic properties of $P_z$ will be irrelevant in the following.

**Lemma 5.2.** Assume Hypothesis 5.1 and let $z \in \mathbb{C} \setminus \mathbb{R}, x \in \mathbb{R}$. Then,
\[
\left( \left( g(z, \cdot) \frac{d}{dx} + g_2(z, \cdot) \right)(L - zI_m)^{-1}, L \right) \bigg|_{\bigcup_{E \in \mathbb{C} \setminus \{ \mathbb{R} \cup \{z\} \}} \ker(L - EI_m)} = -2g'(z, \cdot) \bigg|_{\bigcup_{E \in \mathbb{C} \setminus \{ \mathbb{R} \cup \{z\} \}} \ker(L - EI_m)}.
\]
(5.5)

**Proof.** Fix $\psi \in \ker(L - EI_m)$ for some $E \in \mathbb{C} \setminus \{ \mathbb{R} \cup \{z\} \}$. Then one computes
\[
\left( \left( g(z, \cdot) \frac{d}{dx} + g_2(z, \cdot) \right)(L - zI_m)^{-1}, L \right) \psi
= (E - z)^{-1} \{ [2g' + g'' + gQ - Qg] \psi
\]
\[
+ [g'' + 2g'(Q - zI_m) + gQ' + g_2Q - Qg_2 + 2(z - E)g' \psi]
\]
\[
= -2g'(z, \cdot) \bigg|_{\bigcup_{E \in \mathbb{C} \setminus \{ \mathbb{R} \cup \{z\} \}} \ker(L - EI_m)},
\]
(5.6)
using (2.68) and (2.72).

A variant of Dubrovin’s idea of a generating operator for KdV Lax pairs was also used by Olmedilla, Martínez Alonso, and Guil [79]. Their approach, however, focuses on formal operator expansions and formal pseudo-differential operators. For a different approach we refer to [27, Ch. 15], [32].

Next, we recall that
\[
g(z, x) = [M_-(z, x) - M_+(z, x)]^{-1},
\]
(5.7)
\[
g_1(z, x) = \frac{1}{2} [M_-(z, x) + M_+(z, x)][M_-(z, x) - M_+(z, x)]^{-1},
\]
(5.8)
\[
g_2(z, x) = \frac{1}{2} [M_-(z, x) - M_+(z, x)]^{-1}[M_-(z, x) + M_+(z, x)],
\]
(5.9)
\[
h(z, x) = M_\pm(z, x)[M_-(z, x) - M_+(z, x)]^{-1}M_\mp(z, x),
\]
(5.10)
and note that by Theorem 2.7 the right-hand sides of (5.7) and (5.9) admit asymptotic expansions in cones avoiding the spectrum of $H$. In particular, one thus
obtains the asymptotic expansions

\[ g(z, x) = \frac{i}{2z^{1/2}} \sum_{k=0}^{\infty} \widehat{R}_k(x) z^{-k}, \quad \widehat{R}_0(x) = I_m, \quad (5.11) \]

\[ g_p(z, x) = \frac{i}{2z^{1/2}} \sum_{k=0}^{\infty} \widehat{G}_p,k(x) z^{-k}, \quad \widehat{G}_{p,0}(x) = 0, \quad p = 1, 2, \quad (5.12) \]

\[ h(z, x) = \frac{iz^{1/2}}{2} \sum_{k=0}^{\infty} \widehat{H}_k(x) z^{-k}, \quad \widehat{H}_0(x) = I_m, \quad (5.13) \]

for some coefficients \( \widehat{R}_k, \widehat{G}_{p,k}, p = 1, 2, \) and \( \widehat{H}_k \), which are universal differential polynomials in \( \mathcal{Q} \) by Remark 2.8 (i). Explicitly, one obtains

\[ \widehat{R}_0 = I_m, \quad \widehat{R}_1 = \frac{1}{2} \mathcal{Q}, \quad \widehat{R}_2 = -\frac{i}{4} \mathcal{Q}' + \frac{i}{4} \mathcal{Q}'', \quad (5.14) \]

\[ \widehat{G}_{1,0} = -\frac{i}{2} \mathcal{Q}', \quad \widehat{G}_{1,1} = \frac{i}{16} \mathcal{Q}'' - \frac{1}{8} (\mathcal{Q}'')' - \frac{1}{8} \mathcal{Q}' \mathcal{Q}, \quad (5.15) \]

\[ \widehat{G}_{2,0} = -\frac{i}{2} \mathcal{Q}', \quad \widehat{G}_{2,1} = \frac{i}{16} \mathcal{Q}'' - \frac{1}{8} (\mathcal{Q}'')' - \frac{1}{8} \mathcal{Q}' \mathcal{Q}, \quad (5.16) \]

\[ \widehat{H}_0 = I_m, \quad \widehat{H}_1 = -\frac{i}{2} \mathcal{Q}, \quad \widehat{H}_2 = \frac{i}{8} \mathcal{Q}'' - \frac{1}{8} \mathcal{Q}' \mathcal{Q}, \quad (5.17) \]

e etc.

Motivated by Lemma 5.2, we now introduce the \( m \times m \) matrix-valued differential expressions \( \widehat{P}_{2k+1} \) by

\[ \widehat{P}_{2k+1} = \sum_{\ell=0}^{k} \left( \widehat{R}_\ell \frac{d}{dx} + \widehat{G}_{2,\ell} \right) \mathcal{L}^{k-\ell}, \quad k \in \mathbb{N}_0. \quad (5.18) \]

In analogy to Lemma 5.2 one then obtains the following result.

**Lemma 5.3.** Assume Hypothesis 5.1 and let \( z \in \mathbb{C} \setminus \mathbb{R}, \ x \in \mathbb{R} \). Then,

\[ [\widehat{P}_{2k+1}, \mathcal{L}] = 2\widehat{R}_{2k+1}, \quad k \in \mathbb{N}_0. \quad (5.19) \]

**Proof.** Assuming \( \psi \in \ker(\mathcal{L} - zI_m) \) one computes

\[ [\widehat{P}_{2k+1}, \mathcal{L}] \psi = \sum_{\ell=0}^{k} z^{k-\ell} \left( 2\widehat{G}_{2,\ell}' + \widehat{R}_\ell \mathcal{Q} - \mathcal{Q} \widehat{R}_\ell \right) \psi' \]

\[ + \sum_{\ell=0}^{k} z^{k-\ell} \left( \widehat{G}_{2,\ell}'' + 2\widehat{R}_\ell \mathcal{Q} - \mathcal{Q} \widehat{R}_\ell \right) \psi \]

\[ = \sum_{\ell=0}^{k} z^{k-\ell} \left( 2\widehat{G}_{2,\ell}' + \widehat{R}_\ell \mathcal{Q} - \mathcal{Q} \widehat{R}_\ell \right) \psi' - 2\widehat{R}_0 \psi \]

\[ + \sum_{\ell=0}^{k} z^{k-\ell} \left( \widehat{G}_{2,\ell}'' - 2\widehat{R}_{\ell+1}' + 2\widehat{R}_\ell \mathcal{Q} - \mathcal{Q} \widehat{R}_\ell \right) \psi \]

\[ = 2\widehat{R}_{2k+1} \psi. \quad (5.20) \]

Here we used \( \widehat{R}_0 = 0 \) and

\[ 2\widehat{G}_{2,\ell}' + \widehat{R}_\ell \mathcal{Q} - \mathcal{Q} \widehat{R}_\ell = 0, \quad \ell \in \mathbb{N}_0, \quad (5.21) \]

\[ \widehat{G}_{2,\ell}'' - 2\widehat{R}_{\ell+1}' + 2\widehat{R}_\ell \mathcal{Q} - \mathcal{Q} \widehat{R}_\ell = 0, \quad \ell \in \mathbb{N}_0, \quad (5.22) \]
which follow from inserting the asymptotic expansion (5.11) and (5.12) into (2.68) and (2.72) (which is permitted by Theorem 2.7). Relation (5.20) implies (5.19) since $\hat{P}_2^{k+1}$ and $L$ are $m \times m$ matrix-valued differential expressions of finite-order while $\bigcup_{z \in \mathbb{C}} \ker(L-zI_m)$ is an infinite-dimensional space of $C^\infty(\mathbb{R})^m$-functions.  

Introducing

$$P_{2k+1} = \sum_{\ell=0}^k c_{k-\ell} \hat{P}_{2\ell+1}, \quad k \in \mathbb{N}_0,$$  

(5.23)

where

$$\{c\ell\}_{\ell=1,...,k} \subset \mathbb{C}, \quad c_0 = 1$$  

(5.24)

denotes a set of constants, the pairs $(P_{2k+1}, L)$, $k \in \mathbb{N}_0$, by definition, represent the Lax pairs of the (matrix-valued) KdV hierarchy. More precisely, varying $\{c\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{C}$, the set of evolution equations,

$$\frac{d}{dt}L - [P_{2k+1}, L] = 0, \quad k \in \mathbb{N}_0,$$  

(5.25)

or equivalently, the set of equations,

$$\text{KdV}_k(Q) = Q_t - 2 \sum_{\ell=0}^k c_{k-\ell} \hat{R}_{\ell+1}'(Q,...) = 0, \quad k \in \mathbb{N}_0,$$  

(5.26)

represents the (matrix-valued) KdV hierarchy of evolution equations. The corresponding stationary KdV hierarchy, characterized by

$$Q_t = 0, \text{ or equivalently, by } [P_{2k+1}, L] = 0, \quad k \in \mathbb{N}_0,$$  

(5.27)

is then given by

$$\text{s-KdV}_k(Q) = -2 \sum_{\ell=0}^k c_{k-\ell} \hat{R}_{\ell+1}'(Q,...) = 0, \quad k \in \mathbb{N}_0.$$  

(5.28)

Remark 5.4. By Remark 2.8, each $\hat{R}_\ell$ is a universal polynomial in $Q$ and its $x$-derivatives and occasionally we slightly abuse notation and indicate this by writing $\hat{R}_\ell(Q,...)$ for $\hat{R}_\ell(x)$, $\hat{R}_{\ell+1}'(Q,...)$ for $\hat{R}_{\ell+1}'(x)$, etc. Explicit formulas for the differential polynomials $\hat{R}_\ell(Q,...)$ were derived for instance, in [8] and [83].
In order to make the connection with the finite-band formalism of Section 4 we now recall

\[
\frac{i}{2R_{2n+1}(z)^{1/2}}F_{n,\Sigma}(z, x) = [\mathcal{M}_{-\Sigma}(z, x) - \mathcal{M}_{+\Sigma}(z, x)]^{-1}, \tag{5.29}
\]

\[
\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{G}_{1,n-1,\Sigma}(z, x)
\]

\[
= \frac{1}{2} [\mathcal{M}_{-\Sigma}(z, x) + \mathcal{M}_{+\Sigma}(z, x)] [\mathcal{M}_{-\Sigma}(z, x) - \mathcal{M}_{+\Sigma}(z, x)]^{-1}, \tag{5.30}
\]

\[
\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{G}_{2,n-1,\Sigma}(z, x)
\]

\[
= \frac{1}{2} [\mathcal{M}_{-\Sigma}(z, x) - \mathcal{M}_{+\Sigma}(z, x)]^{-1} [\mathcal{M}_{-\Sigma}(z, x) + \mathcal{M}_{+\Sigma}(z, x)], \tag{5.31}
\]

\[
\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{H}_{n,\Sigma}(z, x)
\]

\[
= \mathcal{M}_{+\Sigma}(z, x) [\mathcal{M}_{-\Sigma}(z, x) - \mathcal{M}_{+\Sigma}(z, x)]^{-1} \mathcal{M}_{+\Sigma}(z, x), \tag{5.32}
\]

and note that (5.29)–(5.32) admit expansions convergent in a neighborhood of infinity. In particular,

\[
\frac{1}{R_{2n+1}(z)^{1/2}}F_{n,\Sigma}(z, x) = \frac{1}{z^{1/2}} \sum_{k=0}^{\infty} \hat{\mathcal{R}}_{k,\Sigma}(x) z^{-k}, \quad \hat{\mathcal{R}}_{0,\Sigma}(x) = \mathcal{I}_{m}, \tag{5.33}
\]

\[
\frac{1}{R_{2n+1}(z)^{1/2}}\mathcal{G}_{p,n-1,\Sigma}(z, x) = \frac{1}{z^{1/2}} \sum_{k=0}^{\infty} \hat{\mathcal{G}}_{p,k,\Sigma}(x) z^{-k}, \quad \hat{\mathcal{G}}_{p,0,\Sigma}(x) = 0, \quad p = 1, 2, \tag{5.34}
\]

\[
\frac{1}{R_{2n+1}(z)^{1/2}}\mathcal{H}_{n,\Sigma}(z, x) = \frac{1}{z^{1/2}} \sum_{k=0}^{\infty} \hat{\mathcal{H}}_{k,\Sigma}(x) z^{-k}, \quad \hat{\mathcal{H}}_{0,\Sigma}(x) = \mathcal{I}_{m}, \tag{5.35}
\]

for \(|z|\) sufficiently large. Here the coefficients \(\hat{\mathcal{R}}_{k,\Sigma}\) and \(\hat{\mathcal{G}}_{p,k,\Sigma}\) are the universal differential polynomials \(\hat{\mathcal{R}}_{k} = \hat{\mathcal{R}}_{k}(\mathcal{Q}_{\Sigma}, \ldots)\) and \(\hat{\mathcal{G}}_{p,k} = \hat{\mathcal{G}}_{p,k}(\mathcal{Q}_{\Sigma}, \ldots)\) in (5.11) and (5.12) (with \(\mathcal{Q}\) replaced by \(\mathcal{Q}_{\Sigma}\)). We also recall

\[
F_{n,\Sigma}(z, x) = \sum_{\ell=0}^{n} F_{n-\ell,\Sigma}(x) z^{\ell}, \quad F_{0,\Sigma}(x) = \mathcal{I}_{m}, \tag{5.36}
\]

\[
\mathcal{G}_{p,n-1,\Sigma}(z, x) = \sum_{\ell=0}^{n-1} \mathcal{G}_{p,n-1-\ell,\Sigma}(x) z^{\ell}, \quad p = 1, 2, \tag{5.37}
\]

\[
\mathcal{H}_{n+1,\Sigma}(z, x) = \sum_{\ell=0}^{n} \mathcal{H}_{n+1-\ell,\Sigma}(x) z^{\ell}, \quad \mathcal{H}_{0,\Sigma}(x) = \mathcal{I}_{m}. \tag{5.38}
\]

Since we seek the connection between the set of coefficients \(\hat{\mathcal{R}}_{k,\Sigma}, \hat{\mathcal{G}}_{p,k,\Sigma}, \hat{\mathcal{H}}_{k,\Sigma}\) and \(\mathcal{F}_{k,\Sigma}, \mathcal{G}_{p,k,\Sigma}, \mathcal{H}_{k,\Sigma}\), we next consider the following elementary expansions. Let

\[
\eta \in \mathbb{C} \text{ such that } |\eta| < \min\{|E_0|^{-1}, \ldots, |E_{2n}|^{-1}\}. \tag{5.39}
\]
Then
\[
\left( \prod_{\ell=0}^{2n} (1 - E\ell \eta) \right)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(E)\eta^k,
\]
where
\[
\hat{c}_0(E) = 1, \quad \hat{c}_1(E) = \frac{1}{2} \sum_{\ell=0}^{2n} E\ell,
\]
\[
\hat{c}_2(E) = \frac{1}{4} \sum_{\ell_1, \ell_2=0}^{2n} E\ell_1 E\ell_2 + \frac{3}{8} \sum_{\ell=0}^{2n} E^2\ell, \quad \text{etc.}
\]

Similarly, one has
\[
\left( \prod_{\ell=0}^{2n} (1 - E\ell \eta) \right)^{1/2} = \sum_{k=0}^{\infty} c_k(E)\eta^k,
\]
where
\[
c_0(E) = 1, \quad c_1(E) = -\frac{1}{2} \sum_{\ell=0}^{2n} E\ell,
\]
\[
c_2(E) = \frac{1}{4} \sum_{\ell_1, \ell_2=0}^{2n} E\ell_1 E\ell_2 - \frac{1}{8} \sum_{\ell=0}^{2n} E^2\ell, \quad \text{etc.}
\]
Lemma 5.5. Assume Hypothesis 5.1 and let \( x \in \mathbb{R} \). Then,

\[
F_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} c_{\ell-k}(E) \hat{R}_{k,\Sigma}(x), \quad \ell = 0, \ldots, n,
\]

(5.46)

\[
\hat{R}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(E) F_{k,\Sigma}(x), \quad \ell = 0, \ldots, n,
\]

(5.47)

\[
\mathcal{G}_{p,\ell,\Sigma}(x) = \sum_{k=0}^{\ell} c_{\ell-k}(E) \mathcal{G}_{p,k,\Sigma}(x), \quad \ell = 0, \ldots, n-1, \quad p = 1, 2,
\]

(5.48)

\[
\hat{\mathcal{G}}_{p,\ell+1,\Sigma}(x) = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(E) \mathcal{G}_{p,k,\Sigma}(x), \quad \ell = 0, \ldots, n-1, \quad p = 1, 2,
\]

(5.49)

\[
\mathcal{H}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} c_{\ell-k}(E) \hat{H}_{k,\Sigma}(x), \quad \ell = 0, \ldots, n+1,
\]

(5.50)

\[
\hat{\mathcal{H}}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(E) H_{k,\Sigma}(x), \quad \ell = 0, \ldots, n+1.
\]

(5.51)

Proof. It suffices to prove (5.46) and (5.47) and so we omit the analogous proofs of (5.48)–(5.51). Since for \( |z| \) sufficiently large,

\[
z^{-n} F_{n,\Sigma}(z, x) = \sum_{\ell=0}^{n} F_{\ell,\Sigma}(x) z^{-\ell} = z^{-n-(1/2)} R_{2n+1}(z)^{1/2} \sum_{\ell=0}^{\infty} \hat{R}_{\ell,\Sigma}(x) z^{-\ell}
\]

\[
= \sum_{k=0}^{\infty} c_k(E) z^{-k} \sum_{\ell=0}^{\infty} \hat{R}_{\ell,\Sigma}(x) z^{-\ell}
\]

\[
= \sum_{\ell=0}^{\infty} \left( \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(E) \hat{R}_{k,\Sigma}(x) \right) z^{-\ell}
\]

(5.52)

and hence (5.46). Equation (5.47) is then clear from (5.46) and

\[
\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(E) c_{\ell}(E) = \delta_{k,0}, \quad k \in \mathbb{N}_0.
\]

(5.53)

The latter follows from multiplying (5.40) and (5.43), comparing coefficients of \( \eta^k \).

Given these preliminaries we can now state the principal result of this section.

Theorem 5.6. The self-adjoint finite-band potential \( Q_{\Sigma} \in C^\infty(\mathbb{R})^{m \times m} \), discussed in Theorems 4.5 and 4.8, is a stationary KdV solution satisfying

\[
\text{s-KdV}_n(Q_{\Sigma}) = -2 \sum_{\ell=0}^{n} c_{n-\ell}(E) \hat{R}_{\ell+1}(Q_{\Sigma}, \ldots) = 0,
\]

(5.54)

with \( c_{\ell}(E) \) defined in (5.44) and \( \hat{R}_{\ell+1} \) the universal differential polynomials (with respect to \( Q \)) in (5.11).
Moreover, one obtains the sequence of trace formulas
\[ \mathcal{P}_{2n+1, \Sigma} = \sum_{\ell=0}^{n} c_{n-\ell}(E) \mathcal{P}_{2\ell+1} = \sum_{\ell=0}^{n} \left( \mathcal{F}_{n-\ell, \Sigma}(\cdot) \frac{d}{dx} + \mathcal{G}_{2,n-1-\ell, \Sigma}(\cdot) \right) \mathcal{L}^{\ell} \] (5.55)
(cf. (5.46), (5.48), and (5.23)), one computes for \( \psi \in \mathrm{ker}(\mathcal{L} - z \mathcal{I}_m), \ z \in \mathbb{C} \),
\[ \mathcal{P}_{2n+1, \Sigma} \psi = \mathcal{F}_{n, \Sigma}(z, x) \psi' + \mathcal{G}_{2,n-1, \Sigma}(z, x) \psi, \] (5.56)
and hence,
\[ [\mathcal{P}_{2n+1, \Sigma}, \mathcal{L}] \psi = [2 \mathcal{G}_{2,n-1, \Sigma} + \mathcal{F}'_{n, \Sigma} \mathcal{F}_{n, \Sigma} Q_{\Sigma} - Q_{\Sigma} \mathcal{F}_{n, \Sigma}] \psi' \] (5.57)
\[ + [\mathcal{G}'_{2,n-1, \Sigma} + 2 \mathcal{F}'_{n, \Sigma}(Q_{\Sigma} - z \mathcal{I}_m) + \mathcal{F}_{n, \Sigma} Q'_{\Sigma} + \mathcal{G}_{2,n-1, \Sigma} Q_{\Sigma} - Q_{\Sigma} \mathcal{G}_{2,n-1, \Sigma}] \psi = 0 \]
by (4.56) and (4.57). Since \( z \in \mathbb{C} \) is arbitrary, this implies
\[ [\mathcal{P}_{2n+1, \Sigma}, \mathcal{L}] = 0, \] (5.58)
completing the proof by (5.27), (5.28).

Next, we turn to a discussion of trace formulas for the finite-band potential \( Q_{\Sigma} \) in terms of (matrix) roots of \( \mathcal{F}_{n, \Sigma} \) and \( \mathcal{H}_{n+1, \Sigma} \).

**Theorem 5.7.** Let \((z, x) \in \mathbb{C} \times \mathbb{R}\) and assume \( Q_{\Sigma} \) to be the self-adjoint finite-band potential discussed in Theorems 4.5 and 4.8. In addition, let the monic self-adjoint matrix pencils \( \mathcal{F}_{n, \Sigma} \) and \( \mathcal{H}_{n+1, \Sigma} \) be given by (4.47) and (4.50). Then \( \mathcal{F}_{n, \Sigma}(\cdot, x) \) and \( \mathcal{H}_{n+1, \Sigma}(\cdot, x) \) are strongly hyperbolic pencils and hence admit the factorizations
\[ \mathcal{F}_{n, \Sigma}(z, x) = (z \mathcal{I}_m - \mathcal{U}_n(x))(z \mathcal{I}_m - \mathcal{U}_{n-1}(x)) \cdots (z \mathcal{I}_m - \mathcal{U}_1(x)), \] (5.59)
\[ \mathcal{H}_{n+1, \Sigma}(z, x) = (z \mathcal{I}_m - \mathcal{V}_n(x))(z \mathcal{I}_m - \mathcal{V}_{n-1}(x)) \cdots (z \mathcal{I}_m - \mathcal{V}_1(x)), \] (5.60)
with
\[ \mathrm{spec}(\mathcal{V}_0(x)) = \mathrm{spec}(\mathcal{H}_{n+1, \Sigma}) \cap \Delta_0(\mathcal{H}_{n+1, \Sigma}) \subset (-\infty, E_0], \] (5.61)
\[ \mathrm{spec}(\mathcal{U}_j(x)) = \mathrm{spec}(\mathcal{F}_{n, \Sigma}) \cap \Delta_j(\mathcal{F}_{n, \Sigma}) \subseteq [E_{2j-1}, E_{2j}], \ 1 \leq j \leq n, \] (5.62)
\[ \mathrm{spec}(\mathcal{V}_j(x)) = \mathrm{spec}(\mathcal{H}_{n+1, \Sigma}) \cap \Delta_0(\mathcal{H}_{n+1, \Sigma}) \subseteq [E_{2j-1}, E_{2j}], \ 1 \leq j \leq n. \] (5.63)

Moreover, one obtains the sequence of trace formulas
\[ \mathcal{U}_{j_1}(x) \cdots \mathcal{U}_{j_k}(x) \mathcal{U}_{j_1}(x) = \sum_{\ell=0}^{k} c_{k-\ell}(E) \tilde{\mathcal{R}}_{\ell, \Sigma}(x), \ 1 \leq k \leq n, \] (5.64)
\[ 0 = \sum_{\ell=0}^{k} c_{k-\ell}(E) \tilde{\mathcal{R}}_{\ell, \Sigma}(x), \ k \geq n + 1, \] (5.65)
\[ \mathcal{V}_{j_1}(x) \cdots \mathcal{V}_{j_k}(x) \mathcal{V}_{j_1}(x) = \sum_{\ell=0}^{k} c_{k-\ell}(E) \tilde{\mathcal{S}}_{\ell, \Sigma}(x), \ 1 \leq k \leq n, \] (5.66)
\[ 0 = \sum_{\ell=0}^{k} c_{k-\ell}(E) \tilde{\mathcal{S}}_{\ell, \Sigma}(x), \ k \geq n + 1. \] (5.67)
In particular, in the special case $k = 1$, $Q$ satisfies the trace formulas
\begin{equation}
Q(x) = \left( \sum_{\ell=0}^{2n} E_{\ell} \right) I_m - 2 \sum_{j=1}^{n} U_j(x), \quad (5.68)
\end{equation}
\begin{equation}
= - \left( \sum_{\ell=0}^{2n} E_{\ell} \right) I_m + 2 \sum_{k=0}^{n} V_k(x). \quad (5.69)
\end{equation}
In addition, one obtains
\begin{equation}
Q^{(r)} \in C^\infty(\mathbb{R})^{m \times m} \cap L^\infty(\mathbb{R})^{m \times m} \text{ for all } r \in \mathbb{N}_0. \quad (5.70)
\end{equation}

**Proof.** By (5.29) and (5.32), $i/R_{2n+1} F_{\Sigma,\Sigma}(\cdot, x)$ and $i/R_{2n+1} H_{n+1,\Sigma}(\cdot, x)$ are Her- glotz matrices and hence (4.14) and (4.15) apply. In particular, $F_{\Sigma,\Sigma}(\cdot, x)$ and $H_{n+1,\Sigma}(\cdot, x)$ are strongly hyperbolic pencils. Since both are monic, the factorizations (5.59) and (5.60), as well as (5.61)–(5.63), hold by Theorem 3.5 (ii). By (5.33) one infers
\begin{equation}
z^{-n} F_{\Sigma,\Sigma}(z, x) = \sum_{k=0}^{n} \left( \sum_{j_1, j_2, \ldots, j_k} U_{j_1}(x) \cdots U_{j_k}(x) \hat{R}_k(x) \right) z^{-k}
\end{equation}
\begin{equation}
= \sum_{k=0}^{\infty} \left( \prod_{\ell=0}^{2n} \left( 1 - \frac{E_{\ell}}{z} \right) \right)^{1/2} \sum_{k=0}^{\infty} \hat{R}_k(x) z^{-k}
\end{equation}
\begin{equation}
= \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{k} c_{\ell} \hat{R}_{k-\ell}(x) \right) z^{-k}. \quad (5.71)
\end{equation}
Comparing coefficients $z^{-k}$, $k \in \mathbb{N}$, then yields (5.64) and (5.65). In the special case $k = 1$ one infers
\begin{equation}
- \sum_{j=1}^{n} U_j(x) = - \frac{1}{2} \left( \sum_{\ell=0}^{2n} E_{\ell} \right) I_m + \hat{R}_{1,\Sigma}(x) \quad (5.72)
\end{equation}
and since $\hat{R}_{1,\Sigma} = Q/2$ by (5.14), the trace formula (5.68) for $Q$ results. (5.66), (5.67), and (5.69) are proved analogously. By (5.62), $U_j \in L^\infty(\mathbb{R})^{m \times m}$, $1 \leq j \leq n$ and hence
\begin{equation}
\hat{R}_k \in C^\infty(\mathbb{R})^{m \times m} \cap L^\infty(\mathbb{R})^{m \times m}, \quad k \in \mathbb{N}_0 \quad (5.73)
\end{equation}
(since $Q \in C^\infty(\mathbb{R})^{m \times m}$ by Theorem 4.8 (iii)). An analysis of the recursion relation (2.49) for $M_{\pm,k}$ combined with (5.7), (5.11) then proves that $\hat{R}_k$ is of the form
\begin{equation}
\hat{R}_k = d_k Q^{(2k-2)} + R_k(Q^{(2k-4)}, \ldots), \quad k \geq 2, \quad (5.74)
\end{equation}
with $d_k \in \mathbb{R}$ appropriate constants and $R_k$ abbreviating a differential polynomial in $Q$ which contains $Q^{(2k-4)}$ as the highest derivative of $Q$. Hence one infers (5.70).
The factorizations (5.59), (5.60), eigenvalue distributions (5.61)–(5.63), and trace formulas (5.64)–(5.69) are extensions of well-known formulas in the scalar case $m = 1$ (see, e.g., [28], [31], [36], [38], [50], [67]).

Finally, the property $Q_{\Sigma} \in C^\infty(\mathbb{R})^{m \times m}$ in Theorem 4.8 (iii) can be improved upon by using the system (4.51), (4.52), (4.55), and (4.58). In fact, writing

\[
F_{n,\Sigma}(z, x) = \sum_{\ell=0}^{n} F_{n-\ell,\Sigma}(x) z^\ell, \quad F_{0,\Sigma}(x) = I_m, \tag{5.75}
\]

\[
G_{p,n-1,\Sigma}(z, x) = \sum_{\ell=0}^{n-1} G_{p,n-1-\ell,\Sigma}(x) z^\ell, \quad p = 1, 2, \tag{5.76}
\]

\[
H_{n+1,\Sigma}(z, x) = \sum_{\ell=0}^{n} H_{n+1-\ell,\Sigma}(x) z^\ell, \quad H_{0,\Sigma}(x) = I_m, \tag{5.77}
\]

one obtains the following result.

**Lemma 5.8.** Assume Hypothesis 4.4 and let $(z, x) \in \mathbb{C} \times \mathbb{R}$. Then the coefficients in (5.75)–(5.77) satisfy the autonomous nonlinear first-order system

\[
\begin{align*}
F_{\ell,\Sigma} &= -(G_{1,\ell-1,\Sigma} + G_{2,\ell-1,\Sigma}), \quad 1 \leq \ell \leq n, \tag{5.78} \\
G'_{1,\ell,\Sigma} &= -(F_{1,\Sigma} - H_{1,\Sigma})F_{\ell+1,\Sigma} + F_{\ell+2,\Sigma} - H_{\ell+2,\Sigma}, \quad 0 \leq \ell \leq n - 1, \tag{5.79} \\
G'_{2,\ell,\Sigma} &= -F_{\ell+1,\Sigma}(F_{1,\Sigma} - H_{1,\Sigma}) + F_{\ell+2,\Sigma} - H_{\ell+2,\Sigma}, \quad 0 \leq \ell \leq n - 1, \tag{5.80} \\
H'_{\ell,\Sigma} &= G_{1,\ell-1,\Sigma} + G_{2,\ell-1,\Sigma} - (F_{1,\Sigma} - H_{1,\Sigma})G_{2,\ell-2,\Sigma}, \quad 1 \leq \ell \leq n + 1, \tag{5.81}
\end{align*}
\]

\[
F_{n+1,\Sigma} = 0, \quad G_{p,n-1,\Sigma} = G_{p,-1,\Sigma} = 0, \quad p = 1, 2. \tag{5.82}
\]

Moreover, $F_{\ell,\Sigma}, 0 \leq \ell \leq n, G_{p,\ell,\Sigma}, 0 \leq \ell \leq n - 1, p = 1, 2, H_{\ell,\Sigma}, 0 \leq \ell \leq n + 1$, and hence $Q_{\Sigma}$, are all analytic in an open neighborhood containing the real axis.

**Proof.** Inserting (5.75)–(5.77) into (4.51), (4.52), (4.55), and (4.58) yields

\[
\begin{align*}
F_{\ell,\Sigma} &= -(G_{1,\ell-1,\Sigma} + G_{2,\ell-1,\Sigma}), \quad 1 \leq \ell \leq n, \tag{5.83} \\
G'_{1,\ell,\Sigma} &= -Q_{\Sigma} F_{\ell+1,\Sigma} + F_{\ell+2,\Sigma} - H_{\ell+2,\Sigma}, \quad 0 \leq \ell \leq n - 1, \tag{5.84} \\
G'_{2,\ell,\Sigma} &= -F_{\ell+1,\Sigma} Q_{\Sigma} + F_{\ell+2,\Sigma} - H_{\ell+2,\Sigma}, \quad 0 \leq \ell \leq n - 1, \tag{5.85} \\
H'_{\ell,\Sigma} &= G_{1,\ell-1,\Sigma} + G_{2,\ell-1,\Sigma} - (G_{1,\ell-2,\Sigma} Q_{\Sigma} + Q_{\Sigma} G_{2,\ell-2,\Sigma}), \quad 1 \leq \ell \leq n + 1, \tag{5.86}
\end{align*}
\]

\[
F_{n+1,\Sigma} = 0, \quad G_{p,n-1,\Sigma} = G_{p,-1,\Sigma} = 0, \quad p = 1, 2, \tag{5.87}
\]

\[
Q_{\Sigma} = F_{1,\Sigma} - H_{1,\Sigma}. \tag{5.88}
\]

Insertion of (5.88) into (5.84)–(5.86) yields the autonomous nonlinear first-order system (5.78)–(5.82). Given the initial conditions

\[
\begin{align*}
F_{\ell,\Sigma}(x_0), \quad 1 \leq \ell \leq n, \\
G_{p,\ell,\Sigma}(x_0), \quad p = 1, 2, 0 \leq \ell \leq n - 1, \tag{5.89} \\
H_{\ell,\Sigma}(x_0), \quad 1 \leq \ell \leq n + 1,
\end{align*}
\]

determined by (5.75)–(5.77) and the half-line Weyl–Titchmarsh matrices $M_\pm(z, x_0)$ in (4.36), the maximal interval of existence of the solution of the autonomous system (5.78)–(5.82), (5.89) is all of $[0, \infty)$ and $(-\infty, 0)$, and hence all of $\mathbb{R}$ (cf. [42, p. 18]), applying (5.70). Thus, one recovers the $C^\infty(\mathbb{R})^{m \times m}$-property $F_{\ell, \Sigma}, G_{p, \ell, \Sigma}, p = 1, 2,$
H \epsilon, and hence that of $Q_\Sigma$. Moreover, since the Picard iterations are convergent in sufficiently small circles in $\mathbb{C}$ centered around each $x \in \mathbb{R}$ (cf. [44, Sect. 2.3]), the unique solution obtained by these Picard iterations is analytic in each of these circles.

We conclude with a remark that puts the construction of $Q_\Sigma$ in Section 4 into proper perspective.

**Remark 5.9.** The simplest examples of potentials $Q_\Sigma$ described in Theorem 4.8 are of the type

$$Q_\Sigma(x) = \text{diag}(q_{1,\Sigma}(x), \ldots, q_{m,\Sigma}(x)), \quad (5.90)$$

where $q_{j,\Sigma}$, $1 \leq j \leq m$, are isospectral algebra-geometric finite-band potentials associated with scalar Schrödinger operators in $L^2(\mathbb{R})$ with spectrum $\Sigma$. The next simplest and closely related set of such examples for $Q_\Sigma$ then will be of the type

$$Q_\Sigma(x) = U \text{diag}(q_{1,\Sigma}(x), \ldots, q_{m,\Sigma}(x))U^{-1}, \quad (5.91)$$

where $U \in \mathbb{C}^{m \times m}$ is a unitary $m \times m$ matrix independent of $x \in \mathbb{R}$. At first sight one might think that perhaps all matrix potentials $Q_\Sigma$ are of the form (5.91). That this is certainly not the case will be argued next. Indeed, assuming that (5.91) holds for some unitary $m \times m$ matrix $U$ independent of $x$, one infers that

$$Q_\Sigma^{(r)}(x) = U \text{diag}(q_{1,\Sigma}^{(r)}(x), \ldots, q_{m,\Sigma}^{(r)}(x))U^{-1} \quad \text{for all } r \in \mathbb{N}_0. \quad (5.92)$$

Since by Remark 2.8 (i) and Lemma 5.5 all coefficients of $F_{n,\Sigma}, G_{p,n-1,\Sigma}, p = 1, 2,$ and $H_{n+1,\Sigma}$ are differential polynomials with respect to $Q_\Sigma$, (5.92) implies

$$UF_{n,\Sigma}(z, x)U^{-1} = \text{diag}(F_{n,\Sigma,1}(z, x), \ldots, F_{n,\Sigma,m}(z, x)), \quad (5.93)$$

$$UG_{p,n-1,\Sigma}(z, x)U^{-1} = \text{diag}(G_{p,n-1,\Sigma,1}(z, x), \ldots, G_{p,n-1,\Sigma,m}(z, x)), \quad p = 1, 2, \quad (5.94)$$

$$UH_{n+1,\Sigma}(z, x)U^{-1} = \text{diag}(H_{n+1,\Sigma,1}(z, x), \ldots, H_{n+1,\Sigma,m}(z, x)), \quad (5.95)$$

$$UM_{\pm,\Sigma}(z, x)U^{-1} = \text{diag}(m_{\pm,\Sigma,1}(z, x), \ldots, m_{\pm,\Sigma,m}(z, x)). \quad (5.96)$$

Consequently, one obtains for all $z, z' \in \mathbb{C}_+, x, x' \in \mathbb{R},$

$$[Q_\Sigma^{(r)}(x), Q_\Sigma^{(s)}(x')] = 0, \quad r \in \mathbb{N}_0, \quad (5.97)$$

$$[F_{n,\Sigma}(z, x), F_{n,\Sigma}(z', x')] = 0, \quad (5.98)$$

$$[G_{p,n-1,\Sigma}(z, x), G_{p,n-1,\Sigma}(z', x')] = 0, \quad p = 1, 2, \quad (5.99)$$

$$[H_{n+1,\Sigma}(z, x), H_{n+1,\Sigma}(z', x')] = 0, \quad (5.100)$$

$$[M_{\pm,\Sigma}(z, x), M_{\pm,\Sigma}(z', x')] = 0. \quad (5.101)$$

In particular,

$$[F_{n,\Sigma}(z, x_0), F_{n,\Sigma}(z', x_0)] = 0 \quad \text{for all } z, z' \in \mathbb{C}_+. \quad (5.102)$$

Thus, whenever

$$[F_{n,\Sigma}(z, x_0), F_{n,\Sigma}(z', x_0)] \neq 0 \quad \text{for some } z, z' \in \mathbb{C}_+ \quad (5.103)$$

(which can easily be arranged for $n \geq 2$), (5.91) cannot hold for a unitary $m \times m$ matrix $U$ independent of $x$. 
Additional results, including extensions of Borg’s and Hochstadt’s theorems in the special cases $n = 0, 1$, respectively, will appear in [9].

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