THREE NEUTRINO OSCILLATIONS IN MATTER

Ara Ioannisian\textsuperscript{1,2} and Stefan Pokorski\textsuperscript{3}

\textsuperscript{1} Yerevan Physics Institute, Alikhanian Br. 2, 375036 Yerevan, Armenia
\textsuperscript{2} Institute for Theoretical Physics and Modeling, 375036 Yerevan, Armenia
\textsuperscript{3} Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, ul. Pasteura 5, PL-02-093 Warsaw, Poland

Following similar approaches in the past, the Schrödinger equation for three neutrino propagation in matter of constant density is solved analytically by two successive diagonalizations of 2x2 matrices. The final result for the oscillation probabilities is obtained directly in the conventional parametric form as in the vacuum but with explicit simple modification of two mixing angles ($\theta_{12}$ and $\theta_{13}$) and mass eigenvalues.

PACS numbers:

The MSW effect \cite{1} for the neutrino propagation in matter attracts a lot of experimental and theoretical attention. Most recently, the discussion is focused on the DUNE experiment \cite{2}. On the theoretical side, a large number of numerical simulations of the MSW effect in matter with a constant or varying density has been performed. Although, in principle, sufficient for comparing the theory predictions with experimental data, they do not provide a transparent physical interpretation of the experimental results. Therefore, several authors have also published analytical or semi-analytical solutions to the Schrödinger equation for three neutrino propagation in matter of constant density, in various perturbative expansions \cite{3–5}. The complexity of the calculation, the transparency of the final result and the range of its applicability depend on the chosen expansion parameter.

In this short note we solve the Schrödinger equation in matter with constant density, using the approximate seesaw structure of the full Hamiltonian in the electroweak basis. This way one can diagonalize the 3x3 matrix by two successive diagonalizations of 2x2 matrices (similar approaches have been used in the past, in particular in ref. \cite{4} and \cite{5}). We specifically have in mind the parameters of the DUNE experiment but our method is applicable for their much wider range. The final result for the oscillation probabilities is obtained directly in the conventional parametric form as in the vacuum but with modified two mixing angles and mass eigenvalues\cite{11}, similarly to the well known results for the two-neutrino propagation in matter. The three neutrino oscillation probabilities in matter have been presented in the same form as here in the recent ref. \cite{6}, where the earlier results obtained in ref. \cite{5} are rewritten in this form. The form of our final results can also be obtained after some simplifications from ref.\cite{4}. Our approach can be easily generalized to non-constant matter density.

The starting point is the Schrödinger equation

\begin{equation}
\frac{d}{dx} \nu = \mathcal{H}_\nu
\end{equation}

where $\mathcal{H}$ is the Hamiltonian in matter. In the electroweak basis it reads

\begin{equation}
\mathcal{H} = U \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\Delta m^2}{2E} & 0 \\
0 & 0 & \frac{\Delta m^2}{2E}
\end{pmatrix} U^\dagger + \begin{pmatrix}
V(x) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\end{equation}

The matrix $U$ is the neutrino mixing matrix in the vacuum. The mass squared differences are defined as $\Delta m^2_\alpha \equiv m^2_\alpha - m^2_\beta (\approx 7.5 \times 10^{-5} eV^2)$ and $\Delta m^2_\alpha \equiv m^2_3 - m^2_1 (\approx \pm 2.5 \times 10^{-3} eV^2$, positive sign is for normal mass ordering and negative sign for inverted one). Here $V(x)$ is the neutrino weak interaction potential energy $V = \sqrt{2}G_F N_e$ ($N_e$ is electron number density) and we take it in this section to be $x$-independent. The neutrino oscillation probabilities are determined by the $S$-matrix elements

\begin{equation}
S_{\alpha\beta} = T e^{-i \int_{x_0}^{x_f} \mathcal{H}(x) dx}
\end{equation}

For a constant $V$ and in order to obtain our results in the same form as for the oscillation probabilities in the vacuum, it is convenient to rewrite the $S$-matrix elements as follows:

\begin{equation}
S_{\alpha\beta} = e^{-i U_m \mathcal{H}_m U_m^\dagger (x_f - x_0)} = U_m e^{-i \mathcal{H}_m L \mathcal{U}_m^\dagger}
\end{equation}
The matrix $\mathcal{H}_m$ is the Hamiltonian in matter in the mass eigenstate basis:

$$
\mathcal{H} = \begin{pmatrix}
\mathcal{H}_1 & 0 & 0 \\
0 & \mathcal{H}_2 & 0 \\
0 & 0 & \mathcal{H}_3
\end{pmatrix}
$$  \hspace{1cm} (5)

and the $U_m$ is the neutrino mixing matrix in matter. Defining $\phi_{21} = (\mathcal{H}_2 - \mathcal{H}_1)L$ and $\phi_{31} = (\mathcal{H}_3 - \mathcal{H}_1)L$, we can write

$$
S_{\alpha\beta} = \left[ U_m \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\phi_{21}} & 0 \\ 0 & 0 & e^{-i\phi_{31}} \end{pmatrix} U^\dagger_m \right]  \hspace{1cm} (6)
$$

The remaining task is to find the eigenvalues of $\mathcal{H}$ and the mixing matrix $U_m$:

$$
\mathcal{H} = U_m \mathcal{H}_m U_m^\dagger
$$  \hspace{1cm} (7)

It is convenient to do it in two steps, first calculating the hamiltonian in a certain auxiliary basis. This way, to an excellent approximation, we can diagonalize the 3x3 matrix by two successive diagonalizations of the 2x2 matrices.

The auxiliary basis $[7, 8]$ is defined by the following equation

$$
\mathcal{H}' = U^\text{aux}\dagger \mathcal{H} U^\text{aux} \quad \text{and} \quad S = U^\text{aux} e^{(-i\mathcal{H} L)} U^\text{aux}\dagger
$$  \hspace{1cm} (8)

where

$$
U^\text{aux} = O_{23} U^\delta O_{13}
$$  \hspace{1cm} (9)

and the rotations $O_{ij}$ are defined by the decomposition of the mixing matrix $U$ in the vacuum (see eq. 2) as follows:

$$
U = O_{23} U^\delta O_{13}
$$

where

$$
U^\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}
$$  \hspace{1cm} (11)

($c_{12} \equiv \cos \theta_{12}, \ s_{12} \equiv \sin \theta_{12}$ etc).

The matrices $O_{ij}$ are orthogonal matrices. It is more convenient to rewrite the matrix $U$ in another form

$$
U \to U \cdot U^\delta = O_{23} U^\delta O_{13}
$$

where

$$
U^\delta = \begin{pmatrix} c_{12} c_{13} + s_{12} s_{13} e^{i\delta} & c_{12} s_{13} - s_{12} c_{13} e^{i\delta} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & c_{23} s_{13} e^{i\delta} \\ c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{12} c_{23} + c_{12} s_{23} s_{13} e^{i\delta} & c_{23} c_{13} e^{i\delta} \end{pmatrix}
$$  \hspace{1cm} (10)

Using eqs. 2, 8 we obtain

$$
\mathcal{H}' = O_{13}^T U^\delta O_{23}^T \mathcal{H} O_{23} U^\delta O_{13}
$$

$$
= \begin{pmatrix}
V c_{13}^2 & s_{12} c_{12} \Delta m^2_{2e} & s_{13} c_{13} V \\
\frac{s_{12} c_{12} \Delta m^2_{2e}}{2E} & (c_{12}^2 - s_{12}^2) \Delta m^2_{2e} & 0 \\
\frac{s_{13} c_{13} V}{2E} & \frac{\Delta m^2_{2e}}{2E} & V s_{13}^2
\end{pmatrix} ,
$$  \hspace{1cm} (13)

$$
\Delta m^2_{ee} = c_{12}^2 \Delta m^2_{2e} + s_{12}^2 (\Delta m^2_{at} - \Delta m^2_{2e})
$$  \hspace{1cm} (14)

(The term $s_{12}^2 \Delta m^2_{2e}$ has been subtracted from the diagonal elements; it gives an overall phase in the $S$-matrix elements.)
This matrix has a see-saw structure, with the (13), (31) elements much smaller than the (33) element and can be put in an almost diagonal form by two rotations

\[
O'^T_{13} T O'_{13} T H' O'_{13} T O_{12} = \begin{pmatrix}
\mathcal{H}_1 & 0 & 0 \\
0 & \mathcal{H}_2 & 0 \\
0 & 0 & \mathcal{H}_3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\Delta m^2}{2E} & 0 \\
0 & 0 & \frac{\Delta m^2}{2E}
\end{pmatrix} + \mathcal{H}_1 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

After the first rotation we have

\[
O'^T_{13} H' O'_{13} = \begin{pmatrix}
\sin^2 \theta'_{13} \frac{\Delta m^2}{2E} + \cos^2(\theta_{13} + \theta'_3) V & \cos \theta'_{13} \psi_{12} \frac{\Delta m^2}{2E} & 0 \\
\cos \theta'_{13} \psi_{12} \frac{\Delta m^2}{2E} & \sin \theta'_{13} \psi_{12} \frac{\Delta m^2}{2E} & \sin^2(\theta_{13} + \theta'_3) V \\
0 & \sin^2 \theta'_{13} \psi_{12} \frac{\Delta m^2}{2E} + \cos^2(\theta_{13} + \theta'_3) V & 0
\end{pmatrix}
\]

where

\[
\sin 2\theta'_{13} = \sqrt{\frac{\epsilon_a \sin 2\theta_{13}}{(\cos 2\theta_{12} - \epsilon_a)^2 + \sin^2 2\theta_{13}}},
\]

and

\[
\epsilon_a = \frac{2EV}{\Delta m^2_{ee}}
\]

We can safely neglect the (23), (32) elements which are generated after the first rotation (see Appendix A) and diagonalize the remaining 2x2 sub-matrix with the second rotation

\[
\sin 2\theta^m_{12} = \sqrt{\cos^2(\theta_{13} + \theta'_3) V + \sin^2(\frac{2EV}{\Delta m^2_{ee}})}, \quad \text{where} \quad \epsilon_\odot = \frac{2EV}{\Delta m^2_{ee}}(\cos^2(\theta_{13} + \theta'_3) + \sin^2(\frac{2EV}{\epsilon_a})).
\]

The eigenvalues of \(\mathcal{H}\) are

\[
\mathcal{H}_2 - \mathcal{H}_1 = \frac{\Delta m^2_{12}}{2E} = \frac{\Delta m^2_{12}}{2E} \sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \sin^2(\theta_{13} + \theta'_3) V},
\]

\[
\mathcal{H}_3 - \mathcal{H}_1 = \frac{\Delta m^2_{23}}{2E} = \cos^2 \theta_{13} \frac{\Delta m^2_{ee}}{2E} + \sin^2(\theta_{13} + \theta'_3) V \\
- \frac{1}{2}[(\epsilon_{12}^2 - \epsilon_{12}^2) \frac{\Delta m^2_{12}}{2E} + \sin^2(\theta_{13} + \theta'_3) V + \cos^2(\theta_{13} + \theta'_3) V] \\
+ \frac{1}{2} \frac{\Delta m^2_{23}}{2E} \\
= \frac{\Delta m^2_{ee}}{2E} \sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}} \\
- \frac{1}{4} \frac{\Delta m^2_{23}}{2E} \sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}} + \frac{1}{4} \frac{\Delta m^2_{23}}{2E} + V + \frac{1}{4} \frac{\Delta m^2_{ee}}{2E} \cos 2\theta_{12} (23)
\]

Finally, for the mixing matrix \(U_m\) in matter we obtain

\[
U_m = U^\text{aux} O'_{13} O_{12} = O_{23} U^\text{dH} O_{13} O_{12} = O_{23} U^\text{dH} O_{13} O_{12},
\]

so that

\[
U_m = U_{23} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\delta}
\end{pmatrix} \begin{pmatrix}
\cos \theta^m_{13} & \sin \theta^m_{13} & 0 \\
- \sin \theta^m_{13} & \cos \theta^m_{13} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\delta}
\end{pmatrix} \begin{pmatrix}
\cos \theta^m_{12} & \sin \theta^m_{12} & 0 \\
- \sin \theta^m_{12} & \cos \theta^m_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
with $\theta_{13}^m = \theta_{13} + \theta_{13}'$ and

$$\sin 2\theta_{13}^m = \frac{\sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}, \quad \cos 2\theta_{13}^m = \frac{\cos 2\theta_{13} - \epsilon_a}{\sqrt{(\cos 2\theta_{13} - \epsilon_a)^2 + \sin^2 2\theta_{13}}}$$

and

$$\begin{align*}
\sin 2\theta_{12}^m &= \frac{\cos \theta_{12}' \sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2 \theta_{12}' \sin^2 2\theta_{12}}}, \\
\cos 2\theta_{12}^m &= \frac{\cos 2\theta_{12} - \epsilon_\odot}{\sqrt{(\cos 2\theta_{12} - \epsilon_\odot)^2 + \cos^2 \theta_{12}' \sin^2 2\theta_{12}}}
\end{align*}$$

In summary the mixing matrix in matter, $U_m$, is given by the following change of the parameters from the vacuum solution:

$$\begin{align*}
\theta_{12} &\rightarrow \theta_{12}^m \quad \text{(eq. 27)} \\
\theta_{13} &\rightarrow \theta_{13}^m \quad \text{(eq. 26)} \\
\theta_{13}^m &\equiv \theta_{23} \\
\delta^m &\equiv \delta.
\end{align*}$$

The mass eigenvalues are given by eqs. 21, 22, 23.

The oscillation probabilities $P_{\nu_\alpha \rightarrow \nu_\beta}$ ($\alpha, \beta = \mu, \tau$) have the same forms as for the vacuum oscillations with mass eigenstates as above and with replacements $\theta_{12} \rightarrow \theta_{12}^m$ and $\theta_{13} \rightarrow \theta_{13}^m$. This approximate solution is valid for all energies. Numerically our result is identical to the approximation of two angles rotation in [21] and the 0th order result of [21]. For anti-neutrino oscillations $P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta}$, $V \rightarrow -V$ and $\delta \rightarrow -\delta$. For normal mass hierarchy $\Delta m^2_{32}$ is positive and for inverted mass hierarchy it is negative.

**Appendix A**

The (23), (32) elements at energies below the second resonance $(23) \approx s_{23}s_{13}\frac{\Delta m^2_{32}}{3m_{\text{pl}}}V \approx 1.1 \cdot 10^{-13}eV$ and for DUNE(T2K) distance, $L=1285 \ (295)\text{km}$, $(23) \cdot L \approx 1.2 \cdot 10^{-3}(3 \cdot 10^{-4})$. $VL \approx 0.7$ for DUNE and it is 0.17 for T2K.

$$\begin{align*}
S &= S_0 + S_1 + \ldots \\
S_0 &= U_m \begin{pmatrix} 1 & 0 & 0 \\
0 & e^{-i\phi_{21}} & 0 \\
0 & 0 & e^{-i\phi_{31}} \end{pmatrix} U_m\dag \\
S_1 &= \sin \theta_{13} \frac{2\Delta m^2_{21}L}{2E} U_m \begin{pmatrix} 0 & 0 & -\sin \theta_{12}^m e^{-i\phi_{31}} \\
0 & 0 & \cos \theta_{12}^m e^{-i\phi_{31}} \\
-\sin \theta_{12}^m e^{-i\phi_{21}} & \cos \theta_{12}^m e^{-i\phi_{21}} & 0 \end{pmatrix} U_m\dag
\end{align*}$$

For normal hierarchy and for neutrinos $\sin \theta_{13}^m \sqrt{\frac{2\Delta m^2_{21}L}{2E}}$, $\sin \theta_{13}^m$ reached its maximal value 0.4% at second resonance. At $E=6.5\text{GeV}$ it is about 0.1%. Other term proportional to $\cos \theta_{12}^m$ is much smaller for all energies.

**Appendix B**

For easy reference we collect here the formulae for oscillation probabilities in vacuum

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \sum_{i,j} U_{\beta i} U^*_{\alpha i} U_{\beta j} U^*_{\alpha j} e^{-i\Delta m^2_{ij}L}$$

$$= \sum_{i,j} \Re[U_{\beta i} U^*_{\alpha i} U_{\beta j} U^*_{\alpha j}] \cos \frac{\Delta m^2_{ij}L}{2E} + 2 \sum_{i>j} \Im[U_{\beta i} U^*_{\alpha i} U_{\beta j} U^*_{\alpha j}] \sin \frac{\Delta m^2_{ij}L}{2E} L$$

$$\phi_{ij} = \frac{\Delta m^2_{ij}L}{2E} \quad i, j = 1, 2, 3$$

$$\sum_{i,j} \Re[U_{\beta i} U^*_{\alpha i} U_{\beta j} U^*_{\alpha j}] \cos \phi_{ij} = \delta_{\alpha \beta} - 4 \sum_{i>j} \Re[U_{\beta i} U^*_{\alpha i} U_{\beta j} U^*_{\alpha j}] \sin^2 \frac{\phi_{ij}}{2}$$

Jerskog invariant, $\mathcal{J}$,
\[\Im [U_{\beta i}U_{\alpha j}^* U_{\beta j}^* U_{\alpha j}] = \pm \mathcal{J}\]

\[\mathcal{J} = c_{13}^2 s_{13} c_{12} s_{23} c_{23} \sin \delta = \frac{1}{8} c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin \delta\]

\[\sin \phi_{32} - \sin \phi_{31} + \sin \phi_{12} = 4 \sin \left( \frac{\phi_{21}}{2} \right) \sin \left( \frac{\phi_{31}}{2} \right) \sin \left( \frac{\phi_{12}}{2} \right) \quad (\phi_{32} = \phi_{31} - \phi_{21})\]

\[2 \sum_{i > j} \Im [U_{\beta i}U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}] \sin \frac{\Delta m_{ij}^2}{2E} = -8 \mathcal{J} \sin \frac{\phi_{21}}{2} \sin \frac{\phi_{31}}{2} \sin \frac{\phi_{32}}{2} \quad (\alpha = \mu, \beta = \tau)\]

\[P_{\nu_x \to \nu_x} = 1 - c_{13}^2 \sin^2 2\theta_{12} \sin^2 \left( \frac{\phi_{21}}{2} \right) - \sin^2 2\theta_{13} (c_{12}^2 \sin^2 \left( \frac{\phi_{31}}{2} \right) + s_{12}^2 \sin^2 \left( \frac{\phi_{32}}{2} \right))\]

\[P_{\nu_x \to \nu_x} = 1 - 4|U_{\mu 1}|^2|U_{\mu 2}|^2 \sin^2 \left( \frac{\phi_{21}}{2} \right) - 4|U_{\mu 1}|^2|U_{\mu 3}|^2 \sin^2 \left( \frac{\phi_{31}}{2} \right) - 4|U_{\mu 2}|^2|U_{\mu 3}|^2 \sin^2 \left( \frac{\phi_{32}}{2} \right)\]

\[P_{\nu_x \to \nu_x} = -4 \mathcal{J} \sin \frac{\phi_{21}}{2} \sin \frac{\phi_{31}}{2} \sin \frac{\phi_{32}}{2}\]

\[\alpha U_{\mu 1} = s_{12}^2 c_{13} + s_{12}^2 c_{23} s_{13}\cos \delta + 2 s_{12} c_{12} c_{23} c_{33} s_{13}\cos \delta\]

\[\beta U_{\mu 2} = c_{12}^2 + s_{12}^2 s_{23}^2 s_{13}\cos \delta - 2 s_{12} c_{12} c_{23} c_{33} s_{13}\cos \delta\]

\[\gamma U_{\mu 3} = c_{13}^2 s_{23}\]

\[\alpha U_{\mu 1} = -c_{13}^2 \sin^2 2\theta_{12} (c_{23}^2 - s_{23}^2 s_{13}) - c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos \delta\]

\[\beta U_{\mu 2} = \sin^2 2\theta_{13} s_{23}^2 c_{12}^2 + \frac{1}{2} c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos \delta\]

\[\gamma U_{\mu 3} = 2 \sin \theta_{13} s_{23}^2 - c_{13} \sin 2\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos \delta\]

Acknowledgements. One of us (A.I.) is grateful to the CERN Theory group for its hospitality and to the DUNE members for useful discussions at Fermilab in summer 2017. A.I. is especially grateful to Pilar Coloma and Maury Goodman for underlining the importance of presenting the oscillation parameters in a simple form. S.P. is partially supported by the National Science Centre, Poland, under research grants DEC-2015/19/B/ST2/02848, DEC-2015/18/M/ST2/00054 and DEC-2014/15/B/ST2/02157.

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[11] The results of this paper have been presented as private communication by one of us (A.I) to the members of the T2HKK collaboration in December 2017.

[12] The definition of $\Delta m^2_{ee}$ coincides with one of the definitions of the effective mass squared differences measured at reactor experiments [9, 10].