MULTIAGENT MODELS IN TIME-VARYING AND RANDOM ENVIRONMENT

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Abstract

In this paper we study multiagent models with time-varying type change. Assume that there exist a closed system of $N$ agents classified into $r$ types according to their states of an internal system; each agent changes its type by an internal dynamics of the internal states or by the relative frequency of different internal states among the others, e.g., multinomial sampling. We investigate the asymptotic behavior of the empirical distributions of the agents’ types as $N$ goes to infinity, by the weak convergence criteria for time-inhomogeneous Markov processes and the theory of Volterra integral equations of the second kind. We also prove convergence theorems of these models evolving in random environment.

Keywords: Multiagent models, Type change, Wright-Fisher model, Time-inhomogeneity, Random environment

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1. INTRODUCTION

Agent-based models (ABMs), or multiagent systems (MAS), arise from many areas of science and social sciences such as ecology, artificial intelligence, communication networks, sociology, economics; see e.g. Ferber (1999, 1995), Ouelhadj (1996), and Wooldridge (1995). Goldstone and Janssen (2006) studied the collective behavior of the agent-based computational models, which build social structures from the ‘bottom-up’. We give some of the attractive features of ABMs presented in their paper. First, ABMs can describe precise mathematical formulation, which make clear, quantitative and objective predictions possible. Second, ABMs can bridge the explanations that

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link the analysis of the individual agent level and the analysis of the emergent group level.

In this paper we will focus on the agent based modeling in social or economic discipline. Föllmer and Schweizer (1993) considered an interacting agent financial model in which they used Black’s (1986) classification of traders: information traders and noise traders. Lux (1995, 1997, 1998) studied a model of three types of traders which can probabilistically change their types. Horst (2000, 2001, 2002, 2005) kept some aspects of Föllmer and Schweizer’s model and considered interacting agent models with local and global interactions. Horst assumed that the set of active agents \( \mathcal{A} \) is countable and there is a sequence of finite sets \( \mathcal{A}_n \) satisfying \( \lim_{n \to \infty} \mathcal{A}_n = \mathcal{A} \). In Horst’s example, the traders are divided into fundamental traders and noise traders, and the fundamental traders are divided into optimistic and pessimistic fundamental traders. Horst introduced the concept of individual mood into his models. At each period \( t \geq 0 \), each fundamental trader has its own mood, e.g., \( x_{t}^{a} = +1 \) or \( x_{t}^{a} = -1 \), that is to say, the fundamental trader is an optimist or a pessimist. Let \( C \) be a fixed set of individual states, i.e., \( x_{t}^{a} \in C \) for each \( a \in \mathcal{A} \) and \( t \geq 0 \). Let \( x_{t} = \{ x_{t}^{a} \}_{a \in \mathcal{A}} \).

Horst defined the empirical distribution, which is called *mood of the market*, as follows:

\[
\rho_{t} = \rho(x_{t}) = \lim_{n \to \infty} \frac{1}{|\mathcal{A}_n|} \sum_{a \in \mathcal{A}_n} \delta_{x_{t}^{a}}(\cdot).
\]

The market mood is one of the main driving forces of Horst’s interacting agent models.

So far, we have seen the importance of the empirical distribution of the individual states which links the behavior of individual agent level and the emergent laws of collective level. The multiagent models of this paper arise from social or economic background; some features of them show similarities with the Wright-Fisher model in population biology, and the Voter model, see e.g. Either and Kurtz (1986), and Holley (1975). Instead of giving the precise definition of the agents, we describe the properties and behaviors of the agents rather intuitively. The multiagent models here share some similarities with ABMs in other disciplines. The most general assumptions about the mechanism of the multiagent models are as follows:

1. The time is in nonnegative integer units, denoted by \( k \geq 0 \).

2. There are fixed \( N \geq 2 \) agents in the multiagent system at all times. There are no
entries of new agents into the system or exits of current agents from the system.

3. There is an internal system with which all agents are concerned. The internal system has \( r \geq 2 \) states which we simply denote by 1, \( \cdots \), \( r \). The internal system will not change with time \( k \). That is to say, at any time \( k \geq 0 \), there is no new state added to the internal system and there is no existing state removed from it. The behavior of internal states are observed by all agents. Each agent has one and only one internal state at each time \( k \geq 0 \). Thus, the agents are classified into \( r \) types, according to their internal states.

4. Assume that \( n^N_1(k) \) (1 \( \leq \) \( i \) \( \leq \) \( r \)) is the number of agents of type \( i \) at time \( k \) and \( n^N(k) = (n^N_1(k), n^N_2(k), \cdots , n^N_r(k)) \) is the distribution of all agents among the \( r \) types. By the second assumption, \( n^N_1(k) + \cdots + n^N_r(k) = N \) and \( \frac{n^N(k)}{N} \) is the empirical distribution of the types at \( k \geq 0 \).

5. Assume that \( \{(p_{N,i,j}(k))_{r \times r}, k \geq 0\} \) is a sequence of deterministic stochastic-matrix valued functions which represent the external environment of the multiagent system.

6. Based on all the information of the agents’ types and the environment up to time \( k \), each agent has an independent strategy of probabilistically choosing its type for the next time unit \( k + 1 \). The strategy of an agent is realized by keeping or changing its type. The agents of the same type have a common strategy. That is to say, from time \( k \) to \( k + 1 \), the agents of type \( i \) switch to type \( j \) with probability \( p_{N,i,j}(k) \). This process of changing types occurs locally among agents of the same type, and it is called internal dynamics.

7. From time \( k \) to \( k + 1 \), there also occurs another process of global type change. When we make this assumption, we would make a minor change on the fifth assumption, to say that the internal dynamics occurs from time \( k \) to \( k + \frac{1}{2} \), rather than from time \( k \) to \( k + 1 \). Based on all the information up to time \( k + \frac{1}{2} \), each agent independently determines its type by \( \frac{n^N(k+\frac{1}{2})}{N} \). That is to say, for any agent, regardless of its type at time \( k + \frac{1}{2} \), the probability of its new type being \( i \) at time \( k + 1 \) is \( \frac{n^N(k+\frac{1}{2})_i}{N} \). Therefore, \( n^N(k+1) \sim \text{multinomial} \left( N, \frac{n^N(k+\frac{1}{2})}{N} \right) \).
Note that we can change the number $\frac{1}{2}$ in the seventh assumption by any number $c$ ($0 < c < 1$). The external environment in the fifth assumption can be external economic fundamentals. An example of this is given by Example 1. Based on the assumptions 1-6, we can construct the multiagent model with internal dynamics (MAMWID). If we have all the seven assumptions, we can construct multiagent model with internal dynamics and multinomial sampling (MAMWIDAMS). The multinomial sampling is a kind of interaction among the agents. When we assume that $\{(p_{N,i,j}(k))_{r \times r}, k \geq 0\}$ is a random sequence, we can construct multiagent model with internal dynamics and random environment (MAMWIDARE); and multiagent model with internal dynamics, multinomial sampling, and random environment (MAMWIDAMSARE) respectively.

In this paper, we will mainly study the asymptotic behaviors of the empirical distribution, $\{\frac{n_N((N(t))}{N}, t \geq 0\}$, of the types, as $N \to \infty$. Lux assumed the number of the agents to be finite, but he didn’t consider the asymptotics of the empirical distribution of the types as $N$ becomes large. Föllmer, Schweizer, and Horst assumed that the number of the agents is countable; they do not have the question we discuss here. Another feature of our multiagent models is that the transition structure of the internal dynamics is time-inhomogeneous.

This paper is the first attempt of a systematic study of the interacting agent financial systems. Another working paper of the author, which goes one step further than this one, focuses on the interacting agent feedback finance models, see Wu (2006).

This paper is organized as follows. In Subsection 2.1 we formulate MAMWID and MAMWIDAMS; and state their convergence in Theorem 2.1. In Subsection 2.2 we formulate MAMWIDARE and MAMWIDAMSARE; and state their joint and annealed convergence in Theorem 2.2. In Section 3 we prove Theorem 2.1 and in Section 4, we prove Theorem 2.2. In Appendix A we state weak convergence criteria for time inhomogeneous Markov processes, which are the main tools of this paper.

2. Formulation of the Multiagent Models and Main Results

2.1. Multiagent models in deterministic time-varying environment

Now we formulate the multiagent model with internal dynamics (MAMWID) based on the assumptions 1-6 in Section 1. Let $A(t) = (a_{i,j}(t))_{r \times r}$ be a $r \times r$ matrix-valued
A càdlàg function on $[0, \infty)$, which satisfies the conditions

1) For each $t \geq 0$, $A(t)e = 0_{r \times 1}$, where $e = [1, \ldots, 1]'$ and "$'$" denotes transpose;

2) For each $t \geq 0$, $1 \leq i, j \leq r, i \neq j$, $a_{i,j}(t) \geq 0$.

Fix $N \geq 1$, let $A_N(t) = (a_{N,i,j}(t))_{r \times r}$ be a $r \times r$ matrix-valued function on $[0, \infty)$, which satisfies the conditions

1) For each $t \geq 0$, $A_N(t)e = 0_{r \times 1}$;

2) For each $t \geq 0$, $1 \leq i, j \leq r, i \neq j$, $a_{N,i,j}(t) \geq 0$;

3) $A_N$ is a stepwise function, i.e., $A_N(t)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$.

Therefore $A(t)$ and $\{A_N(t)\}$ are $Q$-matrix valued functions.

Let $R^r$ be the Euclidean space corresponding to the $r \times r$ square matrix. For each $N \geq 1$, and $k \geq 0$, let

$$P_{N,k} = (p_{N,i,j}(k))_{r \times r} = I + \frac{1}{N}A_N(k),$$

(2.1)

where $I$ is the identity matrix of order $r$, and $p_{N,i,j}(k)$ is the probability of each agent of type $i$ switching to type $j$ at time $k + 1$. The definition of $P_{N,k}$ is valid since for large enough $N$, $P_{N,k}$ is a stochastic matrix, which we call internal transition matrix of MAMWID.

We are ready to formulate MAMWID. For $k \geq 0$, the transition between $n^N_i(k)$ and $n^N_i(k+1)$ is determined by the sixth assumption in Section 1 as follows. For $1 \leq i \leq r$, each agent of type $i$ can change its type to $j$, with probability $p_{N,i,j}(k) (1 \leq j \leq r)$. Since the $n^N_i(k)$ agents independently make their transitions, the distribution of these $n^N_i(k)$ agents among the $r$ types at time $k + 1$ is a random vector denoted by $\Xi_{N,k,i} = (\xi_{N,k,i,1}, \ldots, \xi_{N,k,i,r})$, which satisfies

$$\Xi_{N,k,i} \sim \text{multinomial}(n^N_i(k), P_{N,k,i}),$$

(2.2)

where $P_{N,k,i}$ is the $i$-th row of the matrix $P_{N,k}$. Since agents in different type change their types independently, $\Xi_{N,k,1}, \ldots, \Xi_{N,k,r}$ are independent. The distribution of all the agents at time $k + 1$ is

$$n^N(k+1) \equiv \Xi_{N,k,1} + \cdots + \Xi_{N,k,r}.$$  

(2.3)
The sequence \( \{n^N(k), k \geq 0\} \) defined this way is a time inhomogeneous Markov chain. At last, we define
\[
X^N(t) = \frac{n^N([Nt])}{N}.
\] (2.4)

We introduce some notations. We put \( Z_+ = \{0, 1, \cdots\} \) and \( R_+ = [0, \infty) \),
\[
K_N = \{N^{-1}\alpha : \alpha \in (Z_+)^r, \sum_{i=1}^r \alpha_i = N\},
\]
and
\[
K = \{\alpha : \alpha \in (R_+)^r, \sum_{i=1}^r \alpha_i = 1\}.
\]

We define the time-dependent generator \( \{G_A(t), 0 \leq t < \infty\} \) on \( C^1(K) \): for each \( f \in C^1(K) \) and \( t \geq 0 \),
\[
G_A(t)f(p) = pA(t) \frac{\partial f}{\partial x}.
\] (2.5)
It is clear that, for each \( t \geq 0 \), \( \mathcal{D}(G_A(t)) = C^1(K) \) and \( \mathcal{D}(G_A) = C^1(K) \) is the common domain of the generator \( \{G_A(t), 0 \leq t < \infty\} \), and \( D = C^2(K) \) is a subalgebra contained in \( \mathcal{D}(G_A) \).

Next, we illustrate the transition structure of the multiagent model with internal dynamics and multinomial sampling (MAMWIDAMS) by the following diagram:

![Diagram](image)

**Figure 1:** The transition structure of MAMWIDAMS

Once \( \{n^N(k), k \geq 0\} \) is defined by the Figure 1, we define
\[
Y^N(t) = \frac{n^N([Nt])}{N}.
\] (2.6)

Next, we define the differential operator \( G_B \) on \( C^2(K) \) by
\[
G_B = \frac{1}{2} \sum_{i,j=1}^r b_{ij}(p) \frac{\partial^2}{\partial x_i \partial x_j},
\] (2.7)
where \( b_{ij}(p) = p_i(\delta_{ij} - p_j) \). We also define the generator \( \{G_{AB}(t), 0 \leq t < \infty\} \) on \( C^2(K) \) by
\[
G_{AB}(t) = G_A(t) + G_B, \text{ for any } t \geq 0,
\] (2.8)
whose common domain is \(\mathcal{D}(G_{AB}) = \mathcal{D}(G_A) \cap \mathcal{D}(G_B) = C^2(K)\).

Define the metric \(d_U\) on \(D_{R^+ \times [0, \infty)}\) as follows:

\[
d_U(x, y) = \int_0^\infty e^{-u} \sup_{0 \leq t \leq u} ||x(t) - y(t)||_r \wedge 1du, \ x, y \in D_{R^+ \times [0, \infty)}.
\]

(2.9)

**Theorem 2.1.** Define \(P_{N,k}\) by (2.4). Let \(\mu \in \mathcal{P}(K)\). Assume that \(\lim_{N \to \infty} d_U(A_N, A) = 0\), where \(d_U\) is defined by (2.4).

1) MAMWID. Define \(\{n^N(k), k \geq 0\}\) by the internal dynamics. Define \(X^N\) by (2.4). If \(P(X^N(0))^{-1} \Rightarrow \mu\), then there exists a unique solution \(X^\infty\) of the \(D_K[0, \infty)\) martingale problem for \((G_A, \mu)\) on \(C^2(K)\), and \(X^N \Rightarrow X^\infty\).

2) MAMWIDS. Define \(\{n^N(k), k \geq 0\}\) by the Figure 1. Define \(Y^N\) by (2.4), and \(G_{AB}(t)\) by (2.3). If \(P(Y^N(0))^{-1} \Rightarrow \mu\), then there exists a unique solution \(Y^\infty\) of the \(D_K[0, \infty)\) martingale problem for \((G_{AB}, \mu)\) on \(C^3(K)\), and \(Y^N \Rightarrow Y^\infty\).

### 2.2. Multiagent models in random environment

In this subsection, we assume that \(\{A_N\}\) and \(A\) are random elements which represent an external random environment. We also assume that \(A\) is \(C_{R^+:[0, \infty)}\)-valued. Then, we need the condition \(\lim_{N \to \infty} d(A_N, A) = 0\) instead of the condition \(\lim_{N \to \infty} d_U(A_N, A) = 0\), where \(d\) is the complete separable Skorohod metric on \(D_{R^+:[0, \infty)}\).

Let \((\mathcal{L}, d_U)\) be the subspace of \((D_{R^+:[0, \infty)}, d_U)\) such that each element of \(\mathcal{L}\) satisfies the conditions at the beginning of subsection 2.1. Let \((\mathcal{L}_c, d_U)\) be the subspace of \((C_{R^+:[0, \infty)}, d_U)\) such that \(\mathcal{L}_c = \mathcal{L} \cap C_{R^+:[0, \infty)}\). Let \((\mathcal{P}(D_K[0, \infty)), \rho)\) be the space of probability measures on \(D_K[0, \infty)\) where \(\rho\) is the Prohorov metric on \(\mathcal{P}(D_K[0, \infty))\). Then Theorem 2.1 shows that for any \(A \in \mathcal{L}_c\), there exist unique \(P_A \in \mathcal{P}(D_K[0, \infty))\) such that under \(P_A\), the coordinate process \(Z\) on \(D_K[0, \infty)\) is the unique solution of the martingale problem for the generator \(\{G_A(t), 0 \leq t < \infty\}\) on \(C^2(K)\), and \(P_{AB} \in \mathcal{P}(D_K[0, \infty))\) such that under \(P_{AB}\), \(Z\) is the unique solution of the martingale problem for the generator \(\{G_{AB}(t), 0 \leq t < \infty\}\) on \(C^3(K)\). We define \(\Phi : \mathcal{L}_c \mapsto \mathcal{P}(D_K[0, \infty))\) by \(\Phi(A) = P_A\) and \(\Psi : \mathcal{L}_c \mapsto \mathcal{P}(D_K[0, \infty))\) by \(\Phi(A) = P_{AB}\).

For each \(N \geq 1\), let \((\mathcal{L}_N, d_U)\) be the subspace of \((\mathcal{L}, d_U)\) such that for each \(A_N \in \mathcal{L}_N\), \(A_N(t)\) is a constant on \(t \in \left[\frac{k}{N}, \frac{k+1}{N}\right)\) for each \(k \geq 0\). Then for given \(A_N \in \mathcal{L}_N\), there are unique probability measure \(P_{A_N} \in \mathcal{P}(D_{K_N}[0, \infty))\) which is related to
MAMWID, and unique probability measure \( P_{A_N} \in \mathcal{P}(D_{K_N}[0, \infty)) \) which is related to MAMWIDAMS. Under \( P_{A_N} \) or \( P_{A_NB} \), the coordinate process \( Z_N \) on \( D_{K_N}[0, \infty) \) has the same distributions as those defined for \( X^N \) of MAMWID or \( Y^N \) of MAMWIDAMS. Then we can define \( \Phi_N : \mathcal{L}_N \rightarrow \mathcal{P}(D_{K_N}[0, \infty)) \) and \( \Psi_N : \mathcal{L}_N \rightarrow \mathcal{P}(D_{K_N}[0, \infty)) \) correspondingly.

**Example 1.** Assume that \((H, d_H)\) is a polish space. \( \{h^N(k_N), k \geq 0\} \) is a sequence of external economic fundamentals taking values in \((H, d_H)\). We assume also that \( F \) is a continuous mapping from \((D_H[0, \infty), d_{S_H})\) onto \((\mathcal{L}, d_U)\), where \( d_{S_H} \) represents the complete separable Skorohod metric on \( D_H[0, \infty) \). Let \( A_N(t) = F(h^N(k_N)) \), for any \( N \geq 1 \) and \( t \geq 0 \). Then \( \{h^N(k_N), k \geq 0\} \) constitutes the environment of the multiagent system. Thus we can define our multiagent system which are driven by internal dynamics, interaction among agents, and external economic fundamentals.

Now we have multiagent models evolving in random environment if we assume that \( \{h^N(k_N), k \geq 0\} \) is a random sequence. The idea of external economic fundamentals in deterministic or random environment was used by Horst (2000).

We assume that for each \( N \geq 1 \), \( A_N \) is an \( \mathcal{L}_N \)-valued process defined on some probability space \((\Omega_N, \mathcal{F}_N, Q_N)\), and \( A \) is an \( \mathcal{L}_c \)-valued process defined on \((\Omega, \mathcal{F}, Q)\). Then we can define multiagent model with internal dynamics and random environment (MAMWIDARE), and multiagent model with internal dynamics, multinomial sampling, and random environment (MAMWIDMSARE).

Now, we state the joint and annealed weak convergence theorem for the multiagent models evolving in random environment.

**Theorem 2.2.** (Joint and annealed convergence) Let \( \{A_N\} \) and \( A \) be defined above, \( \mu_N \in \mathcal{P}(K_N) \) for each \( N \geq 1 \) and \( \mu \in \mathcal{P}(K) \). Assume that \( A_N \Rightarrow A \), and \( \mu_N \Rightarrow \mu \).

1) MAMWIDARE. For \( N \geq 1 \) and \( \omega_N \in \Omega_N \), define \( \Phi_N(A_N(\omega_N)) \in \mathcal{P}(D_{K_N}[0, \infty)) \) such that \( \Phi_N(A_N(\omega_N))(Z_N(0))^{-1} = \mu_N \); for each \( \omega \in \Omega \), define \( \Phi(A(\omega)) \in \mathcal{P}(D_{K}[0, \infty)) \).
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\( D_K[0, \infty) \) such that \( \Phi(A(\omega))(Z(0))^{-1} = \mu \). Then

\[
(A_N, \Phi_N(A_N)) \Rightarrow (A, \Phi(A)) \tag{2.10}
\]

\[
\lim_{N \to \infty} \rho \left( \int \Phi(A_N(\omega_N))Q_N(d\omega_N), \int \Phi(A(\omega))Q(d\omega) \right) = 0. \tag{2.11}
\]

2) MAMWIDMSARE. For \( N \geq 1 \) and \( \omega_N \in \Omega_N \), define \( \Psi_N(A_N(\omega_N)) \in D_K[0, \infty) \) such that \( \Psi_N(A_N(\omega_N))(Z_N(0))^{-1} = \mu_N \); for each \( \omega \in \Omega \), define \( \Psi(A(\omega)) \in D_K[0, \infty) \) such that \( \Psi(A(\omega))(Z(0))^{-1} = \mu \). Then

\[
(A_N, \Psi_N(A_N)) \Rightarrow (A, \Psi(A)) \tag{2.12}
\]

\[
\lim_{N \to \infty} \rho \left( \int \Psi(A_N(\omega_N))Q_N(d\omega_N), \int \Psi(A(\omega))Q(d\omega) \right) = 0. \tag{2.13}
\]

3. Proof of Theorem 2.1

At first, we consider some properties of \( \{n_N(k), k \geq 0\} \) defined just by internal dynamics. Let \( V = (v_1, \ldots, v_r) \) be a positive vector. For any \( k \geq 1 \) and \( m \geq 0 \), by (2.2) and (2.3), the independence of \( \Xi_{N,k,i}'s \), and the Markov property of \( \{n_N(k), k \geq 0\} \), we have

\[
E \left[ \prod_{i=1}^{r} n_i^{N}(m+k) \mid n_N(m) \right] = \prod_{i=1}^{r} \left( P_{N,m,i} \prod_{l=1}^{k-1} P_{N,m+l,\cdot} V \right) n_i^{N}(m), \tag{3.1}
\]

where \( \prod_{l=1}^{k-1} P_{N,m+l} = P_{N,m+1} \times \cdots \times P_{N,m+k-1} \). We make the convention that when we denote the product of a sequence of matrices by prod, we actually make the multiplication from the left to the right as the index increases its order.

Then it follows by (3.1) that

\[
E[n_N^N(m+k) \mid n_N^N(m)] = n_N^N(m) \prod_{l=m}^{m+k-1} P_{N,l}, \tag{3.2}
\]

and for \( 1 \leq j \leq r \),

\[
E[n_J^N(m+k)(n_J^N(m+k) - 1) \mid n_N^N(m)]
\]

\[
= \left( n_N^N(m) \prod_{l=m}^{m+k-2} P_{N,l} \right)^2 \left( P_{N,m+k-1,j} \prod_{l=m+1}^{m+k-2} P_{N,l} \right)^2 n_J^N(m),
\]

\[
- \sum_{i=1}^{r} \left( P_{N,m,i} \prod_{l=m+1}^{m+k-2} P_{N,l} \right)^2 n_i^N(m),
\]
where $P_{N,m+k−1,j}$ is $j$-th column of matrix $P_{N,m+k−1}$. Then we can get

$$E[(n_j^N(m+k) - n_j^N(m)]^2 | n^N(m)]$$

$$= \left( n^N(m) \prod_{l=m}^{m+k-2} P_{N,l} \right) [P_{N,m+k−1,j} P_{N,m+k−1,j}]^{2}$$

$$- \sum_{i=1}^{r} \left( P_{N,m+i} \prod_{l=m+1}^{m+k-2} P_{N,l} \right) [P_{N,m+k−1,j}]^{2} n_i^N(m)$$

$$- 2n^N(m) \prod_{l=m}^{m+k-2} P_{N,l} [P_{N,m+k−1,j}]^{2} n_j^N(m) + (n_j^N(m))^2. \quad (3.3)$$

For each $N ≥ 1$, we define the transition operators on $\{n^N(k) : k ≥ 0\}$ as follows:

$$S^N_{N,k} f(p) = E[f(\frac{n^N(k+1)}{N})] = n^N(k) = p, \quad (3.4)$$

for each $k ≥ 0$ and $f ∈ C(K_N), \; p ∈ K_N$.

Lemma 3.1. Define $P_{N,k}$, and $X^N$ in subsection 2.1, and $S^N_{N,k}$ by (3.4). If $\lim_{N → ∞} d_U(A_N, A) = 0$, where $d_U$ is defined by (2.1), then

$$\lim_{N → ∞} \sup_{0 ≤ t ≤ T} \sup_{p ∈ K_N} |N[S^N_{N,[Nt]} - I]|f(p) - G_A(t) f(p)| = 0 \quad (3.5)$$

for any $f ∈ D = C^2(K)$.

Proof. Let $f ∈ D = C^2(K)$, fix $T > 0$. For $0 ≤ t ≤ T$, and $p ∈ K_N$, note that by Taylor’s expansion and (3.3)

$$[S^N_{N,[Nt]} - I] f(p)$$

$$= E[(X^N(t + \frac{1}{N}) - p) \frac{∂f}{∂x}(p)' | X^N(t) = p]$$

$$+ E[(X^N(t + \frac{1}{N}) - p) \frac{∂^2f}{∂x^2}(X^N(t))(X^N(t + \frac{1}{N}) - p)' | X^N(t) = p]$$

$$+ \frac{1}{N} p A_N(\frac{[Nt]}{N}) \frac{∂f}{∂x}(p)'$$

$$+ E[(X^N(t + \frac{1}{N}) - p) \frac{∂^2f}{∂x^2}(X^N(t))(X^N(t + \frac{1}{N}) - p)' | X^N(t) = p],$$

where $X^N(t) = p + θ^N t (X^N(t + \frac{1}{N}) - p)$, for some $θ^N ∈ (0, 1)$. Then

$$|N[S^N_{N,[Nt]} - I]|f(p) - G_A(t) f(p)|$$

$$≤ |p [A_N(\frac{[Nt]}{N}) - A(t)] \frac{∂f}{∂x}(p)' | \quad (3.6)$$

$$+ N |E[(X^N(t + \frac{1}{N}) - p) \frac{∂^2f}{∂x^2}(X^N(t))(X^N(t + \frac{1}{N}) - p)' | X^N(t) = p]|.$$
Denote by $I_1(N, p, t)$ and $I_2(N, p, t)$, the first and second term on the right hand side of (3.7). Let $\|\frac{\partial^2 f}{\partial x^2}\| = \max_{1 \leq r, j \leq r} \|\frac{\partial^2 f}{\partial x_{i_1} \partial x_{j}}\|$. By Hölder inequality,

$$I_2(N, p, t) \leq \|\frac{\partial^2 f}{\partial x^2}\| N \sum_{i=1}^{r} \sum_{j=1}^{r} E[|X_i^N(t + \frac{1}{N}) - p_i| |X_j^N(t + \frac{1}{N}) - p_j||X^N(t) = p|]$$

$$\leq \|\frac{\partial^2 f}{\partial x^2}\| N \sum_{i=1}^{r} \sum_{j=1}^{r} \left(E[(X_i^N(t + \frac{1}{N}) - p_i)^2 |X^N(t) = p| \right. \left. \times E[(X_j^N(t + \frac{1}{N}) - p_j)^2 |X^N(t) = p| \right) \cdot \frac{1}{2}.$$

Since $A_N(t)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$, for fixed $1 \leq j \leq r$, by (3.8), we get

$$N^2 E[(X_j^N(t + \frac{1}{N}) - p_j)^2 |X^N(t) = p|] = p A_{N,j} \left(\frac{[N]}{N}\right) + (p A_{N,j} \left(\frac{[N]}{N}\right))^2 - 2p_j a_{N,j,j} \left(\frac{[N]}{N}\right) - \sum_{k=1}^{r} p_k a_{N, k, j} \left(\frac{[N]}{N}\right)$$

$$\leq \sum_{l=1}^{r} p_l |A_{N,l,j}(t)| + \sum_{l=1}^{r} p_l |A_{N,l,j}(t)|^2 + 2p_j |a_{N,j,j}(t)|$$

$$\leq \sum_{l=1}^{r} |A_{N,l,j}(t)| + \sum_{l=1}^{r} |A_{N,l,j}(t)|^2 + 2|a_{N,j,j}(t)|.$$

Since $\lim_{N \to \infty} d_U(A_N, A) = 0$ and $A$ is bounded on $[0, T]$, there exists $C_T > 0$ such that

$$N^2 E[(X_j^N(t + \frac{1}{N}) - p_j)^2 |X^N(t) = p|] \leq C_T$$

for any $N \geq 1$, $0 \leq t \leq T$, $p \in K_N$, and $1 \leq j \leq r$. Then

$$\sup_{0 \leq t \leq T} \sup_{p \in K_N} I_2(N, p, t) \leq \frac{1}{N} \|\frac{\partial^2 f}{\partial x^2}\| C_T.$$ (3.7)

Thus $\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{p \in K_N} I_2(N, p, t) = 0$.

For matrix $B = (b_{i,j})_{r \times r}$ and vector $y = (y_1, \ldots, y_r)'$, let $\|B\| = \sum_{i,j=1}^{r} |b_{i,j}|$ and $||y||$ be the Euclidean norm of $y$. Then $\|By\| \leq \|B\| \cdot ||y||$. It follows that

$$I_1(N, p, t) \leq \|p\| \cdot \|A_N \left(\frac{[N]}{N}\right) - A(t)\| \cdot \|\frac{\partial f}{\partial x}(p)'\|$$

$$\leq \|A_N \left(\frac{[N]}{N}\right) - A(t)\| \cdot \|\frac{\partial f}{\partial x}(p)'\|. (3.8)$$

Notice that $A_N \left(\frac{[N]}{N}\right) = A_N(t)$, $t \geq 0$, thus

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{p \in K_N} I_1(N, p, t) = 0 \quad (3.9)$$
follows by (3.8) and \( \lim_{N \to \infty} d_U(A_N, A) = 0 \). Therefore (3.9) is proved.

**Corollary 3.1.** With the same conditions as those in Lemma 3.1, we have

\[
\sup_N \sup_{0 \leq t \leq T} \sup_{p \in K_N} |N[S_{N,[N]} - I]f(p)| < \infty
\]

(3.10)

and

\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{p \in K_N} ||S_{N,[N]} - I||f(p)| = 0
\]

(3.11)

for any \( T > 0 \) and \( f \in C^2(K) \).

**Proof.** This follows from Lemma 3.1 immediately.

**Remark 3.1.** We can use Taylor’s expansion of order 1 to prove directly that (3.10) and (3.11) hold for any \( T > 0 \) and \( f \in C^1(K) \).

**Lemma 3.2.** Define the time-dependent generator \( \{G_A(t), 0 \leq t < \infty\} \) on \( C^1(K) \) by (2.5). Let \( \mu \in \mathcal{P}(K) \). Then the \( D_K[0, \infty) \) martingale problem for \( (G_A, \mu) \) has at most one solution.

**Proof.** The limit process \( X^\infty \) of Theorem 2.1 is deterministic such that \((X^\infty)'\) satisfies the linear differential equation

\[
\frac{dx(t)}{dt} = A'(t)x(t).
\]

Then we have the uniqueness of the \( D_K[0, \infty) \) martingale problem for \((G_A, \mu)\).

**Proof of Theorem 2.1, Part 1.** This part follows by Lemma 3.1, 3.2, and Remark A.1

Next we make preparations for proving Theorem 2.1, Part 2.

At first, we consider the multiagent model with only multinomial sampling (MAMWMS). This is illustrated by Figure 2.
This model is similar to the neutral Wright-Fisher model of population genetics. However in contrast to the standard genetics model here the internal state change (which would correspond to mutation) can be influenced by an external random environment. If we don’t include in the Wright-Fisher model selection and gene mutation, we get a model which is very close to MAMWMS. We define a transition operator $T_N$ related to the homogeneous Markov chain \( \{ n_N^{(k+1)}(t), k = 0, 1, \cdots \} \) on $C(K_N)$ as follows:

\[
T_N f(p) = E[f\left(\frac{n_N^{(1 + \frac{1}{N})}}{N}\right)\bigg| n_N^{(1)} = p], \quad f \in C(K_N), \quad p \in K_N.
\]

Let

\[
Z^N(t) = \frac{n_N([Nt] + \frac{1}{2})}{N}
\]

and let $Z$ be a diffusion process in $K$ with the generator $G_B$ defined by (2.7). Then we have the following analog of the classical Wright-Fisher diffusion limit.

**Proposition 3.1.** With the conditions above,

\[
\lim_{N \to \infty} \sup_{p \in K_N} |N(T_N - I)f(p) - G_B f(p)| = 0 \quad (3.12)
\]

for every $f \in C^2(K)$. If $Z^N(0) \Rightarrow Z(0)$ in $K$, then $Z^N \Rightarrow Z$ in $D_K[0, \infty)$.

Define \( \{U_{N,k}, k \geq 0\} \) as follows:

\[
U_{N,k} f(p) = E[f\left(\frac{n_N^{(k+1)}}{N}\right)\bigg| n_N^{(k)} = p], \quad f \in C(K_N), \quad p \in K_N \text{ for any } k \geq 0.
\]

Then by the Figure 1, since $S_{N,k}$ and $T_N$ are one step transition operators related to the internal dynamics and the global multinomial sampling respectively, we have

\[
U_{N,k} = S_{N,k} T_N. \quad (3.13)
\]

**Lemma 3.3.** Assume that $\lim_{N \to \infty} d_U(A_N, A) = 0$. With the definitions above, we have

\[
\lim_{N \to \infty} \sup_{p \in K_N} |N[U_{N,[Nt]} - I]f(p) - G_{AB}(t)f(p)| = 0 \quad (3.14)
\]

for every $f \in C^3(K)$ and $t \geq 0.$
Lemma 3.4. With the same conditions as those in Lemma 3.3, we have

\[
\sup_{N} \sup_{0 \leq t \leq T} \sup_{p \in K_N} |N[U_{N,[N]} - I]f(p) - G_{AB}(t)f(p)| < \infty \tag{3.15}
\]

for any \( T > 0 \) and \( f \in C^3(K) \).

Proof. Let \( f \in C^3(K) \), fix \( T > 0 \). For \( 0 \leq t \leq T \), and \( p \in K_N \), notice that

\[
\sup_{p \in K_N} |N[U_{N,[N]} - I]f(p)| = \sup_{p \in K_N} |N[S_{N,[N]}(T_{N} - I)]f(p)| + |N[S_{N,[N]} - I]f(p)|
\]

\[
\leq \sup_{p \in K_N} |N[T_{N} - I]f(p)| + \sup_{p \in K_N} |N[S_{N,[N]} - I]f(p)|.
\]

It follows by \( S_{N,[N]} \) being a contraction that

\[
\sup_{p \in K_N} |N[U_{N,[N]} - I]f(p) - G_{AB}(t)f(p)|
\]

\[
\leq \sup_{p \in K_N} |N[T_{N} - I]f(p) - G_{B}(p)| + \sup_{p \in K_N} |S_{N,[N]}G_{B}f(p) - G_{B}f(p)|
\]

\[
+ \sup_{p \in K_N} |N[S_{N,[N]} - I]f(p) - G_{A}(t)f(p)|.
\]

Lemma 3.4 is then proved by 3.12, 3.11, Remark 3.1, and 3.5.
Multiagent models in time-varying and random environment

Let \( A^{(n)} = (a_{\alpha,\alpha'}^{(n)})_{\alpha,\alpha'\in\mathbb{N}_n} \in D_{R^n \times c_n} [0, \infty) \) as follows: for each \( \alpha, \alpha' \in \mathbb{N}_n \) and \( t \geq 0 \),

\[
a_{\alpha,\alpha'}^{(n)}(t) = \begin{cases}
\sum_{j=1}^{r} \alpha_j a_j(t), & \text{if } \alpha = \alpha'; \\
\alpha_j a_{kj}(t), & \text{if there exist } 1 \leq j, k \leq r, j \neq k, \text{ such that } \alpha_l = \alpha'_l, \\
0, & \text{otherwise.}
\end{cases}
\]

We arrange the elements of \( A^{(n)} \) along the rows and columns decreasingly by the order \(' >'\). For each \( n \geq 1 \), we define a \( c_n \times c_n \) matrix \( B^{(n)} = (b_{\alpha,\alpha'}^{(n)})_{\alpha,\alpha'\in\mathbb{N}_n} \) as follows: for each \( \alpha, \alpha' \in \mathbb{N}_n \),

\[
b_{\alpha,\alpha'}^{(n)} = \begin{cases}
\frac{1}{2}(n - n^2), & \text{if } \alpha = \alpha'; \\
0, & \text{otherwise.}
\end{cases}
\]

For each \( n \geq 2 \), using \( \mathbb{N}_n \) and \( \mathbb{N}_{n-1} \) as the index sets, we define a \( c_n \times c_{n-1} \) matrix \( D^{(n)} = (d_{\alpha,\alpha'}^{(n)})_{\alpha\in\mathbb{N}_n, \alpha'\in\mathbb{N}_{n-1}} \) as follows: for each \( \alpha \in \mathbb{N}_n \), \( \alpha' \in \mathbb{N}_{n-1} \),

\[
d_{\alpha,\alpha'}^{(n)} = \begin{cases}
\frac{1}{4}\alpha_i (\alpha_i - 1), & \text{if } \alpha' = \alpha_i - 1 \text{ and } \alpha'_j = \alpha_l \text{ for } l \neq i, 1 \leq l \leq r; \\
0, & \text{otherwise.}
\end{cases}
\]

We also arrange the elements of \( B^{(n)} \), \( D^{(n)} \) along the rows and columns decreasingly by the order \(' >'\).

**Lemma 3.5.** Define the time-dependent generator \( \{G_{AB}(t), 0 \leq t < \infty\} \) on \( C^2(K) \) by \( B \). Let \( \mu \in \mathcal{P}(K) \). Then the \( D_K[0, \infty) \) martingale problem for \( (G_{AB}, \mu) \) on \( C^3(K) \) has at most one solution.

**Proof.** Assume that \( \{Y(t) = (Y_1(t), \cdots, Y_r(t)) : 0 \leq t < \infty\} \) is one solution of the martingale problem for \( (G_{AB}, \mu) \) on \( C^3(K) \). Let \( n \geq 1 \), for arbitrary \( \alpha \in \mathbb{N}_n \), and \( t \geq 0 \), define \( f_\alpha(x_1, \cdots, x_r) = x_1^{\alpha_1} \cdots x_r^{\alpha_r} \) and \( y^n_\alpha(t) = E[f_\alpha(Y(t))] \). Define the column vector \( y_n(t) = (y^n_\alpha(t))_{\alpha\in\mathbb{N}_n} \), where the elements of \( y_n(t) \) is arranged decreasingly by the order \(' >'\).

Then, for \( n \geq 1 \) and given \( \alpha \in \mathbb{N}_n \) and \( t > 0 \), we have

\[
E[f_\alpha(Y(t))] = E[f_\alpha(Y(0))] + \int_0^t E[G_{AB}(s)f_\alpha(Y(s))]ds.
\]  

At first, for \( n = 1 \), since \( G_Bf_\alpha \equiv 0 \) for any \( \alpha \in \mathbb{N}_1 \), we have

\[
E[f_\alpha(Y(t))] = E[f_\alpha(Y(0))] + \int_0^t E[G_A(s)f_\alpha(Y(s))]ds.
\]
which implies

\[ y_1(t) = y_1(0) + \int_0^t A^{(1)}(s)y_1(s)ds. \] (3.20)

Next, we calculate \( E[f_a(Y(t)) \] for \( n \geq 2 \). By (3.17), (3.19), and (3.21), we can get

\[
E[G_B f_a(Y(s))] = \frac{1}{2} E[\sum_{i=1}^r b_i(Y(s)) \frac{\partial^2 f_a}{\partial x_i^2}(Y(s))] + \frac{1}{2} E[\sum_{i=1}^r \sum_{j=1, j\neq i}^r b_{ij}(Y(s)) \frac{\partial^2 f_a}{\partial x_i \partial x_j}(Y(s))]
\]

\[
= \frac{1}{2} E[\sum_{i=1}^r Y_i(s)(1 - Y_i(s)) \alpha_i(\alpha_i - 1) Y_1(s)^{\alpha_1} \cdots Y_r(s)^{\alpha_r}]
\]

\[
- \frac{1}{2} \sum_{i=1}^r \sum_{j=1, j\neq i}^r E[\alpha_i \alpha_j Y_1(s)^{\alpha_1} \cdots Y_r(s)^{\alpha_r}]
\]

\[
= \frac{1}{2} \sum_{i=1}^r \alpha_i(\alpha_i - 1) E[Y_1(s)^{\alpha_1} \cdots Y_i(s)^{\alpha_i - 1} \cdots Y_r(s)^{\alpha_r}]
\]

\[
- \frac{1}{2} \sum_{i=1}^r \sum_{j=1}^r \sum_{j\neq i}^r \alpha_i \alpha_j E[Y_1(s)^{\alpha_1} \cdots Y_r(s)^{\alpha_r}]
\]

\[
= D^{(n)}\alpha_{n-1}(s) - \frac{1}{2} \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j - \sum_{i=1}^r \alpha_i E[Y_1(s)^{\alpha_1} \cdots Y_r(s)^{\alpha_r}]
\]

\[
= D^{(n)}\alpha_{n-1}(s) - \frac{1}{2} (n^2 - n) E[Y_1(s)^{\alpha_1} \cdots Y_r(s)^{\alpha_r}]
\]

\[
= D^{(n)}\alpha_{n-1}(s) + B^{(n)}\alpha_n(s),
\]

where \( D^{(n)}\alpha_i \) is the \( i \)-row of the matrix \( D^{(n)} \), \( B^{(n)}\alpha_i \) the \( i \)-row of the matrix \( B^{(n)} \). By (3.19) we can get

\[
\int_0^t E[G_A(s)f_a(Y(s))]ds = \int_0^t E[Y(s)A(s) \frac{\partial f_a}{\partial x}(Y(s))']ds
\]

\[
= \int_0^t \left[ \sum_{j=1}^r \sum_{k=1}^{k-1} \alpha_j a_{j,k}(s)y_n(s) + \sum_{k=1}^r \sum_{j=1}^r \alpha_j a_{k,j}(s)y(\alpha_1, \cdots, \alpha_{j-1}, \cdots, \alpha_{k+1}, \cdots, \alpha_r)(s) \right. \]

\[
+ \left. \sum_{k=1}^r \sum_{j=k+1}^r \alpha_j a_{k,j}(s)y(\alpha_1, \cdots, \alpha_k+1, \cdots, \alpha_{j-1}, \cdots, \alpha_r)(s) \right] ds
\]

\[
= \int_0^t A^{(n)}_{\alpha_i}(s)y_n(s)ds,
\] (3.22)

where \( A^{(n)}_{\alpha_i} \) is the \( i \)-row of the matrix \( A^{(n)} \). Then by (3.19), (3.21), and (3.22), for \( n \geq 2 \), we obtain

\[
y_n(t) = y_n(0) + \int_0^t D^{(n)}y_{n-1}(s)ds + \int_0^t [A^{(n)}(s) + B^{(n)}]y_n(s)ds.
\] (3.23)
Define for $n = 1$,
\[
V_1(s, t) = \begin{cases} 
A^{(1)}(t) & \text{if } 0 \leq t \leq s < \infty, \\
0 & \text{otherwise};
\end{cases}
\]
and for $n \geq 2$,
\[
V_n(s, t) = \begin{cases} 
A^{(n)}(t) + B^{(n)} & \text{if } 0 \leq t \leq s < \infty, \\
0 & \text{otherwise};
\end{cases}
\]
Let $f_1(t) = y_1(0)$, $0 \leq t < \infty$. Then, by (3.20) and the theory of Volterra equations of the second kind (see e.g. Smithies (1958) or Tricomi (1957)), we know that $x_1 = y_1$ is the unique solution of the Volterra equation of the second kind
\[
x(s) = f_1(s) + \int_0^s V_1(s, t)x(t)dt, \quad (0 \leq s \leq T)
\] (3.24)
on the space $L^2([0, T], \mathbb{R}^c)$ for any $T > 0$. We can do this procedure recursively. Assume that we know that for $n \geq 1$ the unique solution $x_n = y_n$ is determined, then we define $f_{n+1}(t) = y_{n+1}(0) + \int_0^t D^{(n+1)}x_n(s)ds$. Then by (3.22), we know that $x_{n+1} = y_{n+1}$ is the unique solution of the Volterra equation of the second kind
\[
x(s) = f_{n+1}(s) + \int_0^s V_{n+1}(s, t)x(t)dt, \quad (0 \leq s \leq T)
\] on the space $L^2([0, T], \mathbb{R}^{c_{n+1}})$ for any $T > 0$. Note that the construction of the Volterra equations does depend just on $y_n(0)$ for $n \geq 1$, and does not depend on $y_n(t)$ for $t > 0$, $n \geq 1$. Then we conclude that all moments of the one-dimensional marginal distribution of any two solutions of the martingale problem for $(G_{AB}, \mu)$ are the same. The uniqueness of the martingale problem for $(G_{AB}, \mu)$ on $C^3(K)$ then follows by Theorem 4.2, Chapter 4 of Either and Kurtz (1986).

**Proof of Theorem 2.1, Part 2.** This part follows by Lemma 3.3, 3.4, 3.5 and Corollary A.2.

4. Multiagent models in random environment

4.1. Measurability with respect to random environment

At first, for $\Phi$ and $\Psi$ define in Subsection 2.2 we have the following Lemma.

**Lemma 4.1.** $\Phi$ and $\Psi$ are continuous mappings from $(\mathcal{L}_c, d_U)$ to $(\mathcal{P}(D_K[0, \infty)), \rho)$.
Proof. This is immediate by (2.5), (2.8) and Corollary A.1.

Now we consider the measurability related to \( A \) if \( A \) is an \( \mathcal{L}_c \)-valued process.

**Lemma 4.2.** Assume that \( A \) is an \( \mathcal{L}_c \)-valued process defined on some probability space \( (\Omega, \mathcal{F}, Q) \). Then \( (A, \Phi(A)), (A, \Psi(A)) \) are \( \mathcal{F}/[\mathcal{B}(\mathcal{L}_c) \otimes \mathcal{B}(\mathcal{P}(D_K[0,\infty)))) \)-measurable.

**Proof.** Since \( K \) is separable, so are \( D_K[0,\infty) \) and \( \mathcal{P}(D_K[0,\infty)) \). Since \( \mathcal{L}_c \) is also separable, to prove that \( (A, \Phi(A)) \) is measurable, it suffices to prove that if \( C \in \mathcal{B}(\mathcal{L}_c) \), \( D \in \mathcal{B}(\mathcal{P}(D_K[0,\infty))) \), \( (A, \Phi(A))^{-1}(C \times D) \in \mathcal{F} \). This is clear by Lemma 4.1 and \( (A, \Phi(A))^{-1}(C \times D) = A^{-1}(C \cap \Phi^{-1}(D)) \). Similarly, we can prove that \( (A, \Psi(A)) \) is measurable.

Next, we consider the measurability related to \( \{A_N\} \). Since \( \{A_N(\frac{k}{N}), k \geq 0\} \) determines the transition structure of a Markov chain by \( \{S_{N,k}, k \geq 0\} \) in MAMWID or by \( \{U_{N,k}, k \geq 0\} \) in MAMWIDAMS, it is clear that \( \Phi_N \) and \( \Psi_N \) are continuous mappings from \( \mathcal{L}_N \) to \( \mathcal{P}(D_K[0,\infty)) \). Then we have the following lemma.

**Lemma 4.3.** Assume that \( A_N \) is an \( \mathcal{L}_N \)-valued process defined on some probability space \( (\Omega_N, \mathcal{F}_N, Q_N) \). Then \( (A_N, \Phi_N(A_N)), (A_N, \Psi_N(A_N)) \) are \( \mathcal{F}/[\mathcal{B}(\mathcal{L}_N) \otimes \mathcal{B}(\mathcal{P}(D_K[0,\infty)))) \)-measurable.

**Remark 4.1.** Using the notations \( \Phi_N, \Psi_N, \Phi, \) and \( \Psi \), we can restate Theorem 2.1 as follows. Let \( \hat{A}_N \in \mathcal{L}_N \), and \( \hat{A} \in \mathcal{L}_c \). Let \( \mu_N \in \mathcal{P}(K_N) \), and \( \mu \in \mathcal{P}(K) \). Assume that \( \lim_{N \rightarrow \infty} d(\hat{A}_N, \hat{A}) = 0 \), and \( \mu_N \Rightarrow \mu \).

1) MAMWID. Define \( \Phi_N(\hat{A}_N) \in \mathcal{P}(D_K[0,\infty)) \) and \( \Phi(\hat{A}) \in \mathcal{P}(D_K[0,\infty)) \), such that \( \Phi_N(\hat{A}_N)Z_N(0)^{-1} = \mu_N \) and \( \Phi(\hat{A})Z(0)^{-1} = \mu \). Then
\[
\lim_{N \rightarrow \infty} \rho(\Phi_N(\hat{A}_N), \Phi(\hat{A})) = 0.
\]

2) MAMWIDAMS. Define \( \Psi_N(\hat{A}_N) \in \mathcal{P}(D_K[0,\infty)) \) and \( \Psi(\hat{A}) \in \mathcal{P}(D_K[0,\infty)) \), such that \( \Psi_N(\hat{A}_N)Z_N(0)^{-1} = \mu_N \) and \( \Psi(\hat{A})Z(0)^{-1} = \mu \). Then
\[
\lim_{N \rightarrow \infty} \rho(\Psi_N(\hat{A}_N), \Psi(\hat{A})) = 0.
\]

4.2. Proof of Theorem 2.2

The next two lemmas will be used in the proof of Theorem 2.2.
Lemma 4.4. Let \( \hat{A} = (\hat{a}_{i,j})_{r \times r} \) be an \( \mathcal{L}_c \)-valued process a.s., defined on some probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \), and \( \hat{A} = (\hat{a}_{i,j})_{r \times r} \) be a \( C_{R^r \times [0, \infty]} \)-valued process defined on some probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \). If \( \hat{P} \hat{A}^{-1} = \hat{P} \hat{A}^{-1} \), then \( \hat{A} \) is also an \( \mathcal{L}_c \)-valued process a.s.

Proof. The proof of this lemma is straightforward.

Lemma 4.5. Let \( A_N \) be an \( \mathcal{L}_N \)-valued process defined on some probability space \( (\Omega_N, \mathcal{F}_N, Q_N) \) a.s., and let \( \hat{A}_N \) be a \( D_{R^r \times [0, \infty]} \)-valued process defined on \( (\hat{\Omega}_N, \hat{\mathcal{F}}_N, \hat{Q}_N) \). If \( Q_N A_N^{-1} = \hat{Q}_N \hat{A}_N^{-1} \), then \( \hat{A}_N \) is also an \( \mathcal{L}_N \)-valued process a.s.

Proof. We omit this proof.

Lemma 4.4 and 4.5 indicate that the conditions specified at the beginning of subsection 2.1 for the processes \( \{A_N\} \) and \( A \) just depend on the distributions of \( \{A_N\} \) and \( A \).

Proof of Theorem 2.2. We just prove part 1). Part 2) can be proved similarly.

At first, we prove 2.2.1. Since \( \{A_N\} \) are \( D_{R^r \times [0, \infty]} \)-valued processes, \( A \) is a \( C_{R^r \times [0, \infty]} \)-valued process, and \( A_N \Rightarrow A \), by Skorohod Representation theorem, there exist some probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{Q}) \) and a sequence of \( D_{R^r \times [0, \infty]} \)-valued processes \( \{\hat{A}_N\} \) and a \( C_{R^r \times [0, \infty]} \)-valued process \( \hat{A} \) satisfying \( \lim_{N \to \infty} d(\hat{A}_N, \hat{A}) = 0 \) \( \hat{Q} \)-a.s, \( \hat{Q} \hat{A}^{-1} = Q A^{-1} \), and \( \hat{Q} \hat{A}_N^{-1} = Q_N A_N^{-1} \) on \( \mathcal{B}(D_{R^r \times [0, \infty]}) \) for each \( N \geq 1 \). By Lemma 4.4 and 4.5, \( \hat{A} \) is an \( \mathcal{L}_c \)-valued process, and \( \hat{A}_N \) is an \( \mathcal{L}_N \)-valued process for each \( N \geq 1 \). By Lemma 4.4 and 4.5, \( (\hat{A}, \Phi(\hat{A})) \) and \( (\hat{A}_N, \Phi_N(\hat{A}_N)) \) are measurable. By Remark 4.1 we have

\[
\lim_{N \to \infty} \rho(\Phi_N(\hat{A}_N(\omega)), \Phi(\hat{A}(\omega))) = 0, \ \hat{Q} \text{-a.s.} \quad (4.1)
\]

For each \( f \in \hat{C}(\mathcal{L}_c) \), and \( g \in \hat{C}(\mathcal{B}(D_{K}[0, \infty])) \), by bounded convergence theorem

\[
\lim_{N \to \infty} \int f(A_N(\omega_N))g(\Phi_N(A_N(\omega_N)))Q(d\omega_N) = \int f(\hat{A}(\omega))g(\Phi(\hat{A}(\omega)))\hat{Q}(d\hat{\omega}) = \int f(\hat{A}(\omega))g(\Phi(\hat{A}(\omega)))\hat{Q}(d\hat{\omega}) = \int f(A(\omega))g(\Phi(A(\omega)))Q(d\omega).
\]
Since $\mathcal{L}_c$ and $\mathcal{P}(D_K[0,\infty))$ are separable, $\overline{C}(\mathcal{L}_c)$ and $\overline{C}(\mathcal{P}(D_K[0,\infty)))$ are convergence determining on $(\mathcal{L}_c, d_U)$ and $(\mathcal{P}(D_K[0,\infty)), \rho)$ respectively, (2.10) is proved.

Secondly, we prove that (2.11). By (4.1), for any open set $G \subset D_K[0,\infty)$, we have

$$\liminf_{N \to \infty} \Phi_N(\hat{A}_N(\hat{\omega})) (G) \geq \Phi(\hat{A}(\hat{\omega}))(G), \hat{Q}\text{-a.s.} \quad (4.2)$$

Then by Fatou Lemma, we get

$$\liminf_{N \to \infty} \int \Phi_N(\hat{A}_N(\hat{\omega}))(G) \hat{Q}(d\hat{\omega}) \geq \int \Phi(\hat{A}(\hat{\omega}))(G) \hat{Q}(d\hat{\omega}), \quad (4.3)$$

which implies that

$$\liminf_{N \to \infty} \int \Phi_N(A_N(\omega_N))(G) Q_N(d\omega_N) \geq \int \Phi(A(\omega))(G) Q(d\omega). \quad (4.4)$$

Since $D_K[0,\infty)$ is separable, (2.11) is proved.

**Appendix A. Weak Convergence Criteria for Time Inhomogeneous Markov Processes**

In this section we state the weak convergence criteria for time inhomogeneous Markov processes. These criteria are concerned with martingale problems with time-dependent generators. We introduce some notations from Either and Kurtz (1986). For $n = 1, 2, \ldots$, let $\{\mathcal{G}^n\}$ be a complete filtration, and let $\mathcal{L}_n$ be the space of real-valued $\mathcal{G}^n$-progressive processes $\xi$ satisfying

$$\sup_{0 \leq t \leq T} E[|\xi(t)|] < \infty$$

for each $T > 0$. Let $\mathcal{A}_n$ be the collection of pairs $(\xi, \varphi) \in \mathcal{L}_n \times \mathcal{L}_n$ such that

$$\xi(t) - \int_0^t \varphi(s)ds$$

is a $\mathcal{G}^n$-martingale.

**Proposition A.1.** Let $(E, \bar{\tau})$ be a Polish space. Let $\{G(s), 0 \leq s < \infty\}$ be a family of operators on $\overline{C}(E)$. Suppose that there exists a countable set $\Gamma_1 \subset [0,\infty)$, such that for each $s \notin \Gamma_1$, $\{G(s), 0 \leq s < \infty\}$ has a common domain denoted by $\mathcal{D}(G)$ and for each $f \in \mathcal{D}(G)$, $G(s)f \in \overline{C}(E)$ for $s \notin \Gamma_1$, and $\|G(s)f\|$ is bounded for $s \in [0,T] \setminus \Gamma_1$ for any $T > 0$. Suppose that there is an algebra $\mathcal{C}_n$ contained in the closure of $\mathcal{D}(G)$ (in
the sup norm) which separates points. Suppose that the $D_E[0, \infty)$ martingale problem for $(G, \nu)$ has at most one solution, where $\nu \in \mathcal{P}(E)$. Suppose for each $n \geq 1$, $X_n$ is a $\{\mathcal{F}_t^n\}$-adapted process with sample paths in $D_E[0, \infty)$. Suppose $P(X_n(0))^{-1} \Rightarrow \nu$ and the compact containment condition holds. Suppose $M \subset \mathcal{C}(E)$ is separating. Then condition (a) implies that there exists a solution $X$ of the $D_E[0, \infty)$ martingale problem for $(G, \nu)$, and $X_n \Rightarrow X$:

(a) There exists a countable set $\Gamma_2 \subset [0, \infty)$ such that for each $f \in \mathcal{D}(G)$, and $T > 0$, there exists $(\xi_n, \varphi_n) \in \mathcal{A}_n$, such that

\[
\sup_n \sup_{0 \leq s \leq T, s \notin \Gamma_1} E[|\xi_n(s)|] < \infty, \quad (A.1)
\]

\[
\sup_n \sup_{0 \leq s \leq T, s \notin \Gamma_1} E[|\varphi_n(s)|] < \infty, \quad (A.2)
\]

\[
\lim_{n \to \infty} E[(\xi_n(t) - f(X_n(t))) \prod_{i=1}^{k} h_i(X_n(t_i))] = 0, \quad (A.3)
\]

\[
\lim_{n \to \infty} E[(\varphi_n(t) - G(t)f(X_n(t))) \prod_{i=1}^{k} h_i(X_n(t_i))] = 0, \quad (A.4)
\]

for all $k \geq 0$, $0 \leq t_1 < t_2 < \cdots < t_k \leq T$ with $t_i, t \notin \Gamma_1 \cup \Gamma_2$, and $h_1, \cdots, h_k \in M$, and

\[
\lim_{n \to \infty} E[\sup_{t \in \bar{Q} \cap [0, T]} |\xi_n(t) - f(X_n(t))|] = 0, \quad (A.5)
\]

\[
\sup_n E[||\varphi_n||_{p,T}] < \infty, \text{ for some } p \in (1, \infty], \quad (A.6)
\]

where $\bar{Q}$ is a countable and dense subset of $R$, $||h||_{p,T} = [\int_0^T |h(t)|^p dt]^{1/p}$ if $p < \infty$;

$||h||_{\infty,T} = \text{esssup}_{0 \leq t \leq T} |h(t)|$.

The conditions of the following two corollaries are more convenient to be verified for our multiagent models.

**Corollary A.1.** Suppose in Proposition A.1 that for each $n \geq 1$, $X_n$ is a $\{\mathcal{F}_t^n\}$-adapted process with sample paths in $D_E[0, \infty)$ and generator $\{G_n(s), 0 \leq s < \infty\}$ on $\mathcal{C}(E)$. Suppose also for each $n \geq 1$ there exists a countable set $\Gamma^n \subset [0, \infty)$, such that for each $s \notin \Gamma^n$, $\{G_n(s), 0 \leq s < \infty\}$ has a common domain denoted by $\mathcal{D}(G_n)$. Then condition (b) implies that there exists a solution $X$ of the $D_E[0, \infty)$ martingale problem for $(G, \nu)$, and $X_n \Rightarrow X$:
(b) For each \( n \geq 1 \), \( \mathcal{F}(G_n) = \mathcal{F}(G) \) and there exists a countable set \( \Gamma_2 \subset [0, \infty) \) such that for each \( f \in \mathcal{F}(G) \), and \( T > 0 \),

\[
\sup_n \sup_{0 \leq s \leq T} \sup_{s \notin \Gamma_1 \cup \Gamma_2} \|G_n(s)f\| < \infty,
\]

where \( \| \cdot \| \) is the sup norm on \( B(E) \), and

\[
\lim_{n \to \infty} \sup_{q \in E} |G_n(t)f(q) - G(t)f(q)| = 0,
\]

for any \( t \) satisfying \( t \notin \Gamma_1 \cup \Gamma_2 \cup \cup_{n=1}^{\infty} \Gamma^n \) and \( 0 \leq t \leq T \).

**Corollary A.2.** Suppose in Proposition A.1 that \( X_n = \eta_n(Y_n((\alpha_n^i))) \) and \( \{\mathcal{F}_n\} = \{\mathcal{F}_{[\alpha_n^i]}\} \), where \( \{Y_n(k), k = 0, 1, 2, \cdots\} \) is a time inhomogeneous Markov chain in a metric space \( E_n \) with transition functions \( \mu_{n,k}(x, \Gamma) \), \( \eta_n : E_n \to E \) is Borel measurable, and \( \alpha_n \to \infty \) as \( n \to \infty \). Define \( T_{n,k} : B(E_n) \to B(E) \) by

\[
T_{n,k}f(x) = \int f(y)\mu_{n,k}(x, dy),
\]

and let \( G_{n,k} = \alpha_n(T_{n,k} - I) \). Then condition (e) implies that there exists a solution \( X \) of the \( D_E[0, \infty) \) martingale problem for \( (G, \nu) \), and \( X_n \Rightarrow X \):

(c) There exists a countable set \( \Gamma_2 \subset [0, \infty) \) such that for each \( f \in \mathcal{F}(G) \), and \( T > 0 \),

\[
\sup_n \sup_{0 \leq t \leq T} \sup_{s \notin \Gamma_1 \cup \Gamma_2} |G_n([\alpha_n^i])f \circ \eta_n(q)| < \infty, \tag{A.7}
\]

and

\[
\lim_{n \to \infty} \sup_{q \in E_n \setminus \Gamma_2} |G_n([\alpha_n^i])f \circ \eta_n(q) - G(t)f \circ \eta_n(q)| = 0, \tag{A.8}
\]

for any \( t \) satisfying \( t \notin \Gamma_1 \cup \Gamma_2 \) and \( 0 \leq t \leq T \).

**Remark A.1.** 1) Assume that \( D \subset \mathcal{F}(G) \), and \( C_a \subset D \) where \( C_a \) is a subalgebra of \( \mathcal{C}(E) \) which separates points. If we can prove that the uniqueness of \( D_E[0, \infty) \) martingale problem for \( (G, \nu) \) holds for functions in \( D \) instead of the common domain \( \mathcal{F}(G) \), then we can replace \( \mathcal{F}(G) \) by \( D \) in the conditions (a), (b) and (c). 2) We can simplify the condition (c) of Corollary A.2 into the following version: there exists a countable set \( \Gamma_2 \subset [0, \infty) \) such that for each \( f \in \mathcal{F}(G) \), and \( T > 0 \),

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \sup_{s \notin \Gamma_1 \cup \Gamma_2} |G_n([\alpha_n^i])f \circ \eta_n(q) - G(t)f \circ \eta_n(q)| = 0. \tag{A.9}
\]

This condition is stronger than the condition (c), since (A.8) implies (A.7) and (A.9).
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