Generalized Statistics and Solar Neutrinos

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Abstract: The generalized Tsallis statistics produces a distribution function appropriate to describe the interior solar plasma, thought as a stellar polytrope, showing a tail depleted respect to the Maxwell-Boltzmann distribution and reduces to zero at energies greater than about 20k_B T. The Tsallis statistics can theoretically support the distribution suggested in the past by Clayton and collaborators, which shows also a depleted tail, to explain the solar neutrino counting rate.

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By imposing a variational principle one can conveniently generalize both Boltzmann-Gibbs statistics and standard thermodynamics [3,4]. The lack of adequacy of the Boltzmann entropy is related to the breakdown of Boltzmann-Gibbs statistics for systems with long-range interactions [3,4] such as, among others, the interior solar plasma, where the particles are exposed to many-body collisions and the energy available to a particular pair of particles is not defined [3].

Even for a non perfect plasma, the ideal gas approximation is usually considered a good one. The Maxwell-Boltzmann (MB) distribution is believed to be highly correct and the many-body physics involved does not determine sensible deviations from the standard statistics used [3]. However, non-Maxwellian (flattened) distributions in plasmas heated by inverse bremsstrahlung (collisional) absorption of sufficiently strong electromagnetic fields have been predicted and recently measured [6] in D - 3He fusion plasma experiments, a nonthermal component of the ion distribution and the possible consequences on the nuclear rates have been investigated [3,11].

Among the distributions that deviate from the MB one, we pose our attention on distributions coming from generalized statistics. We give a very brief outline of the Tsallis generalization of thermodynamics and statistical physics [4,5], suitable for describing systems with long-range interactions [3,4]. If we have a system with W microscopic states, each with probability f_i ≥ 0 normalized as

\[ \sum_{i=1}^{W} f_i = 1 \]

then the entropy is given by

\[ S_q = \frac{k}{q-1} \sum_{i=1}^{W} f_i (1 - f_i^{q-1}) \]

where k and q are constants. In the limit as q approaches unity, the well known expression \( S_1 = -k \sum_i f_i \log f_i \) is recovered and we can fix \( k = k_B \) (the Boltzmann constant), i.e. the Tsallis statistics reduces to the standard one as \( q \to 1 \).

We have shown a generalization of the Tsallis statistics by using a kinetic approach, based on the Fokker-Planck equation, and we have given to the Tsallis parameter q the meaning of a measure of the deviation from constant behavior of the diffusion coefficient \( D(v) \), a quadratic function of the velocity \( v \) [3,4]. The Tsallis generalized statistics is actually widely used in many different physical problems (we send the reader to Refs. [5,6] where many references on the different applications are quoted).

One of the first problems where Tsallis statistics has been applied is that of stellar polytropes [3]. Nobre and Tsallis [4] and other authors have recently discussed the physical needs for departure from Boltzmann-Gibbs statistical mechanics and thermodynamics for gravitational like systems [17] and anomalous diffusion [6,18] (this last subject could be of great interest in the study of solar core, due to the well known problem of the diffusion of light elements like Li and Be [14]).

Plastino and Plastino [5] have sought help from Tsallis entropy to find sensible distribution functions for stellar polytropes while that of MB gives unphysical distribution functions. They found a range of variability of q from a relation between the polytropic index n and q deduced comparing two different but equivalent expressions of the distribution. We obtain a different relation between n and q because we impose a special constraint on the solar gravitational potential.

The internal structure of the sun can be considered polytropic with index n ranging between the value 3/2 (\( \gamma=5/3 \)) and 5 (\( \gamma=6/5 \)) [4] (\( \gamma \) is the adiabatic parameter). In the interior regions where hydrogen ionization is changing rapidly, \( \gamma \) is taken to be very close to unity (\( n = \infty, q = 1 \)).
We want to apply generalized statistics to derive a distribution function for the interior solar plasma, relevant in calculating the nuclear fusion reaction rates responsible for the neutrino flux emitted by the sun \[21,22\]. The following arguments support the program of this work.

It is well known that one of the main problems with solar physics is related to the detected neutrinos arriving from the sun. All the different experiments have confirmed a deficit in the flux relative to the predictions of standard theory of nuclear physics \[19,21\]. In particular, the neutrino flux from the reactions involving \(^7Be\) and \(^8\)B, mainly due to collisions at energies higher than the effective energy of \(pp\) reactions \((4.58\ k_B T)\), is much lower than that predicted by the standard models, which uses MB distributions \[9\].

In the recent past we have tried to contribute to the solution of the solar neutrino problem deriving, as steady state solutions of the Fokker-Planck equation, statistical distributions that differ from the MB distribution for a depleted tail at high energies \[12,13\]. The distribution \(f\) is also the solution of a Boltzmann equation without collisions (at equilibrium the distributions obtained with and without the collision term coincide).

Non-Maxwellian distribution can give thermonuclear reaction rates \(r =< \sigma v >\) smaller than the standard ones \[11\], allowing a reduced flux of neutrinos from the sun interior, particularly at energies above few \(k_B T\).

Very recently we have shown that the Tsallis distribution is a particular case of the family of distributions we derived by means of the Fokker-Planck equation kinetic approach \[13,14\]. A well defined statistics can be derived for any particular pair of values \((M,N)\) indicating the degree of the polynomials, in the velocity variable, used to describe the drift \(J\) and diffusion \(D\) coefficients. The Tsallis classical and quantum statistics are related to the pair \((0,1)\) (in this paper we do not derive the drift and diffusion coefficients of the solar core, this will be investigated elsewhere, rather we use straight way the generalized Tsallis statistics).

In addition to this, let us recall that Clayton and collaborators \[22\] suggested, on phenomenological grounds, that neutrino counting rate will be much reduced if the high energy tail of the MB distribution of relative energies is depleted, the depletion being described by the factor \(\exp\{-\delta (E/k_B T)^2\}\) with a suggested value \(\delta \approx 0.01\).

The distribution that we have derived in Ref. \[12,13\] of the Tsallis type, can take into account this behavior, certainly due to the long-range gravitational interaction. The Tsallis parameter \(q\) can be related to the Clayton parameter \(\delta = (1-q)/2\); our constraints impose a value of \(\delta\) slightly different from the value argued by Clayton. We obtain that the value of \(\delta\) must satisfy the range of variability \(0.02 \leq \delta \leq 0.05\).

To show the analogies of our approach with the distribution suggested by Clayton, we introduce, by means of an expansion, an approximated expression of the Tsallis distribution which is correct in the range of energies of interest here.

The Tsallis distribution function reduces to zero at \(E \approx (10/25) k_B T\) depending on the value of \(\delta\); therefore, high energy collisions are greatly reduced or absent and the neutrino flux from \(^7Be\) and \(^8\)B reactions is smaller than foreseen by standard theories, the importance of the neutrino flux from \(pp\) reactions is consequently increased.

In this work we show that the suggestion of Clayton and collaborators is well founded on theoretical grounds (long-range gravitational interaction), it is equivalent to using the distribution we have derived elsewhere \[12,14\] and that this distribution is well motivated within the Tsallis statistics. The neutrino counting rate, compared to the standard predictions, can be explained using the above prescriptions. A complete treatment of this subject can be carried on only within a complete solar model code \[23,24\], it will be done and reported elsewhere.

Following Tsallis \[1,2\] Plastino and Plastino \[8\] and Boghosian \[17\], the distribution function \(f\) of stellar polytropes is

\[
f \propto [1 + (q - 1)(\alpha + \beta \epsilon)]^{1/(1-q)} ,
\]

where \(q\) is the Tsallis parameter and \(\epsilon = E + \Phi(r)\), where \(E\) is the c.m. kinetic energy, \(\Phi(r)\) is the gravitational potential, \(\alpha = -\beta \mu\), where \(\mu\) is the chemical potential, \(\beta = 1/(k_B T)\). The distribution (3) becomes the MB distribution \(\exp[-(\alpha + \beta \epsilon)]\) as \(q\) goes to one. Let us define the relative gravitational potential \(\Psi\)

\[
\Psi = -\Phi + \Phi_0 ,
\]

where \(\Phi_0\) is a constant to be chosen such that \(\Psi\) vanishes at the edge of the system \[8\]. The following relation between the relative potential \(\Psi\) and the density \(\rho\) holds

\[
\rho^{\gamma-1} = \frac{\gamma-1}{K \gamma} \Psi ,
\]

where \(K = \mathcal{P}/\rho^{\gamma}\), \(\mathcal{P}\) is the pressure and \(\gamma\) is the adiabatic parameter. The polytropic index \(n\) is defined by

\[
\gamma = 1 + \frac{1}{n} .
\]

We introduce the relative energy \(\mathcal{E} = \Psi - E\), the distribution \(f\) can be written as

\[
f \propto [1 + (q - 1)(\alpha + \beta \Phi_0) - (q - 1)\beta \mathcal{E}]^{1/(1-q)} .
\]

The quantity \(\alpha\) can be chosen, without loosing generality, in such a way that \[17\]

\[
1 + (q - 1)(\alpha + \beta \Phi_0) = 0 .
\]
We want to compare the Tsallis statistics $f$ to the distribution introduced by Clayton et al.\textsuperscript{[22]} which is equal to the MB distribution $C \exp(-\beta E)$ times a correction factor

$$f = C e^{-\beta E} e^{\varphi(E)} = e^{\beta \delta_0 - \beta_1 E - \beta_2 E^2},$$  \hspace{1cm} (9)$$

where we set $\beta_1 = \beta + B$ ($B$ is a constant).

We observe that the Tsallis distribution can be written in a classical form $f \propto e^{-\tau}$ where

$$-\dot{\tau} = \frac{1}{1 - q} \{ \log[1 + (q - 1)(\alpha + \beta \varepsilon)] \}. \hspace{1cm} (10)$$

The function $\dot{\tau}$ may be interpreted as a generalized energy that takes into account many body collective interactions\textsuperscript{[20]}. The logarithmic term in Eq.(10) can be expanded in powers of $(q - 1)(\alpha + \beta \varepsilon)$ with the conditions $| (q - 1)(\alpha + \beta \varepsilon) | \leq 1$ and $(q - 1)(\alpha + \beta \varepsilon) \neq -1$.

We impose that

$$\alpha + \beta \Phi = 0,$$  \hspace{1cm} (11)

to allow that $\beta_1 = \beta$, as required for physical reasons in the approximated distribution by Clayton\textsuperscript{[3,22]}. Infact, $B$ does not have any effect and can be ignored because we maintain the solar luminosity at its known value and the increase of the central temperature $T$ to counteract the effect of $B$ on the power of the sun raises the $^8B$ neutrino flux back to the value it had at $B = 0$. The special condition (11) or $\alpha = \beta (\Psi - \Phi_0)$, which is a constraint characteristic of the solar core, shows that $\alpha$ is a function of $\Psi$ or $(E + E)$, therefore the comparison between two different but equivalent expressions of $f$ done to derive a relation between $n$ and $q$ in Ref.s\textsuperscript{[8,17]} is not allowed in this case. Finally, we find that the parameters $\beta_0$, $\beta_1$, $\beta_2$ are: $\beta_0 = \log C$, $\beta_1 = \beta$ and $\beta_2 = (1 - q)\beta_0^2/2 = \delta \beta^2$.

The above constraints (8) and (11) imply

$$1 + (q - 1)\beta \Psi = 0,$$  \hspace{1cm} (12)

and from Eqs.(5), (6) and (12) we obtain

$$q = 1 - \frac{\tau}{n + 1} \hspace{0.5cm} \text{or} \hspace{0.5cm} \delta (n + 1) = \frac{\tau}{2},$$  \hspace{1cm} (13)

with $\tau = k_B T \rho / \mathcal{P}$. This relation, which links $n$ and $q$, differs from the relation given by Plastino and Plastino\textsuperscript{[8]} or from the one given by Boghosian\textsuperscript{[13]} because of the constraint (11) valid for the solar core.

In Ref.\textsuperscript{[3]} it is shown that, if we select $n = 3$, the following values of $\rho$, $\mathcal{P}$ and $T$ of the solar core can be derived: the proton density $\rho = 53.47$ gr/cm$^3 = 0.32 \times 10^{-13}$ protons/fm$^3$, the pressure $\mathcal{P} = 0.77 \times 10^{-16}$ MeV/fm$^3$ and the temperature $k_B T = 1.034 \times 10^{-3}$ MeV which is slightly lower than the temperature fixed by other models $(1.29 \times 10^{-3}$ MeV). These last two figures are determined supposing a particular central composition; changing the composition by increasing the $He$ concentration a greater temperature can be reached. With this selection of values we obtain $\tau = 0.43$ and $\delta = 0.05$.

In the solar core the value of the different physical quantities of interest in this work have been reported by Ichimaru\textsuperscript{[23]} to be: $k_B T = 1.29 \times 10^{-3}$ MeV, $\beta = 0.77 \times 10^3$ MeV$^{-1}$, $\rho = 56.2$ gr/cm$^3$, which is the 36% of 156 gr/cm$^3$, for the free protons and $\mathcal{P} = 2.12 \times 10^{-16}$ MeV/fm$^3$.

By inserting these values into Eq.(13) we obtain (with $n = 3$) $\tau = 0.2$ and $\delta (n + 1) = 0.1$; then we can fix the central composition to maintain the magnitude of $\tau$ to be 0.2 and select different couples of values of $n$ and $\delta$, e.g.: $n = 3/2$ and $\delta = 0.04$, $n = 3$ and $\delta = 0.025$, $n = 5$ and $\delta = 0.017$. In conclusion, we may expect that the appropriate value of $\delta$ be in the range between 0.02 and 0.05. The value proposed by Clayton ($\delta = 0.01$) gives (when $\tau = 0.2$) $n = 9$ which is unphysical, because greater than 5.

The approximated distribution function with $\delta = 0.02$ is

$$f_{approx} = C' e^{-0.774 E_{\nu He} - 0.012 E_{\nu He}^2},$$  \hspace{1cm} (14)$$

where $C'$ is the normalization constant. It is easy to verify that the correct Tsallis distribution reduces to zero at $E = k_B T/(2B)$ forbidding the existence of ions with energies greater than 10$^5$ times the sun temperature. We wish to recall that the Eq.(14) represents a Druyvenstein distribution\textsuperscript{[22]}.

All quantities relevant in solar physics and solar neutrino emission can be expressed and evaluated as functions of the parameter $\delta$. We report here, for instance, the thermonuclear reaction rate corrected respect to the MB rate $r_{MB}$ by the depletion factor and calculated in Ref.\textsuperscript{[2]}

$$r = r_{MB} \left( 1 + \frac{15}{4} \delta - \frac{7}{3} \delta E_0 \frac{E_0}{k_B T} + \cdots \right) e^{-\Delta},$$  \hspace{1cm} (15)$$

where $E_0$ is the most effective energy ($E_0 = 4.5 k_B T$ for $pp$ reactions) and $\Delta$ is a function of $E_0$, $k_B T$ and $\delta$.

The neutrino counting can be reduced sensibly above few $k_B T$ with some changes in the solar model parameters. These can be compensated by small changes in the initial $He$ concentration. Most of this reduction has come at the expenses of the $^7Be$ and $^8B$ neutrino fluxes. Counting rates within the solar model used by Clayton and collaborators can be extrapolated from the curves reported in Fig.2 of their work\textsuperscript{[22]}.

A value of $\delta$ different from zero makes a star more luminous and reduces the rate of energy production at a given temperature. The solar core contracts to higher temperature. As shown by Clayton et al. the increase of temperature at given solar luminosity does not increase the neutrino fluxes that decrease with $\delta$. 

3
We do not discuss further, in this work, the measured results and the predictions of the neutrino fluxes; we leave complete and more definitive discussion after the proposed distribution will be tested within the available solar models [20,24,25,28]. Of course, the results reported in this work depend on the values of the parameters of the solar core one takes as input. We can expect that the trend of definitive results will be on the line of the present description. We hope that the content of this work could be useful to the operating and proposed solar neutrino and underground nuclear astrophysics experiments [29,30].

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