HIGHER ORDER STRONG APPROXIMATIONS OF SEMILINEAR STOCHASTIC WAVE EQUATION WITH ADDITIVE SPACE-TIME WHITE NOISE

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Abstract.
New fully discrete schemes are developed to numerically approximate a semilinear stochastic wave equation (SWE) driven by additive space-time white noise. Based on the spatial discretization done via a spectral Galerkin method, exponential time integrators involving linear functionals of the noise are introduced for the temporal approximation. The resulting fully discrete schemes are very easy to implement and allow for higher strong convergence rate than existing numerical schemes such as the Crank-Nicolson-Maruyama scheme and the stochastic trigonometric method. In particular, the new schemes achieve in time an order of $1-\epsilon$ for arbitrarily small $\epsilon > 0$, which exceeds the barrier order $\frac{1}{2}$ established by Walsh [27]. Finally, numerical results are reported to confirm higher convergence rates and computational efficiency of the new schemes.

Key words. semilinear stochastic wave equation, additive noise, strong approximations, higher order, spectral Galerkin method, exponential Euler scheme

AMS subject classifications. 60H35, 60H15, 65C30

1. Introduction. Wave motion and mechanical vibration are two common physical phenomena that are usually mathematically modelled by hyperbolic partial differential equations. Spurred by the demand of modern applications, random perturbation is often taken into account and a noisy force term is hence included, which leads us to stochastic partial differential equations (SPDEs) of hyperbolic type [4, 26]. Particularly, the stochastic wave equation is one of the fundamental SPDEs of hyperbolic type and is widely used to describe physical processes, including, for example, the motion of a vibrating string [1] and the motion of a strand of DNA [6].

In this article we are interested in the strong approximations [16] of the semilinear stochastic wave equation (SWE) driven by additive space-time white noise,

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + f(x, u) + \dot{W}, \quad 0 \leq t \leq T, \ x \in (0, 1) \\
u(0, x) &= u_0(x), \ \frac{\partial u}{\partial t}(0, x) = v_0(x), \ x \in (0, 1), \\
u(t, 0) &= u(t, 1) = 0, \ t > 0,
\end{align*}
\]

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a smooth nonlinear function satisfying

\[
\begin{align}
|f(x, u)| &\leq L(|u| + 1), \\
\left| \frac{\partial f}{\partial u}(x, u) \right| &\leq L, \quad \left| \frac{\partial^2 f}{\partial x \partial u}(x, u) \right| \leq L, \quad \text{and} \quad \left| \frac{\partial^2 f}{\partial u^2}(x, u) \right| \leq L
\end{align}
\]

for all $x \in [0, 1], u \in \mathbb{R}$ and some constant $L > 0$. Concerning the noise $\dot{W}$, we consider a case of special interest when the forcing term is a space-time white noise,
which best models the fluctuations generated by microscopic effects in a homogeneous physical system [9]. In other words, throughout this paper we restrict ourselves to the cylindrical $I$-Wiener process (i.e., the standard $Q$-Wiener process in [8, 21] with the covariance operator $Q = I$). For more details on general SPDEs, one can, for example, refer to [6, 8, 21, 26].

In the last two decades, much progress has been made in both strong and weak approximations of parabolic SPDEs, especially of stochastic heat equations. Here we do not intend to mention the existing numerous works, but resort to a review article [13] and a relevant monograph [14] for an extensive list of references on numerics of parabolic SPDEs. In contrast to the parabolic case, just a few works have been devoted to numerical study of the stochastic wave equation [2, 5, 10, 17, 18, 19, 20, 22, 23, 27].

For the sake of comparison, we shall review strong convergence rates of numerical methods for SWE in the literature. For space semi-discretisations of SWE with multiplicative space-time white noise on the one-dimensional real line, finite difference method was considered in [22], and its strong convergence rate of $\frac{1}{3}$ was obtained there. Adopting a spectral Galerkin method in spatial approximation of SWE with additive noise, the authors of [2] improved the rate from $\frac{1}{3}$ to $\frac{1}{2} - \epsilon$ for arbitrarily small $\epsilon > 0$. Using an adaptation of "leapfrog" discretisation of the same equation as in [22], Walsh [27] constructed a fully difference scheme that attains the rate of $\frac{1}{2}$ in both time and space. In a series of works on numerical study of linear SWE driven by additive noise [17, 18, 19], spatial approximations are done by finite element method and the time discretisations by rational approximation to the exponential function. For the space-time white noise case (Q = I), the strong convergence results in [17, 19] (Theorem 5.1 in [17] and Theorem 4.6 in [19]) imply convergence rate of $\frac{1}{r+1} \beta$ in space and $\frac{p}{p+1} \beta$ in time, for any $\beta < \frac{1}{2}$ and $p, r \in \mathbb{N}^+$ being method parameters. Recently in [5], a stochastic trigonometric scheme was introduced for the temporal approximation of the linear SWE as in [17, 18, 19], and its strong convergence order was proved to be $\frac{1}{2} - \epsilon$ for arbitrary small $\epsilon > 0$ in the case $Q = I$ (see Theorem 4.1 in [5]).

Obviously, none of the above strong convergence rates (in time and in space) exceeds $\frac{1}{2}$, which seems to be an order barrier for strong approximations of space-time white noise driven SWE. Actually, the limit on the convergence rate of numerical schemes for the SWE [11] has been established in [27] in the sense that no scheme based on the basic increments of white noise strongly converges at a rate faster than $\frac{1}{2}$. An interesting question thus arises as to whether it is possible to overcome the order barrier. In this work, we provide a positive answer to this question and design two fully discrete schemes for [11], which enjoy higher strong convergence order than $\frac{1}{2}$. More precisely, we spatially discretize [11] by a spectral Galerkin method, and then introduce two exponential time integrators involving two linear functionals of the noise. As shown in the main convergence result (Theorem 4.1), the fully discrete schemes achieve convergence rate of $\frac{1}{2} - \epsilon$ in space and rate of $1 - \epsilon$ in time for arbitrarily small $\epsilon > 0$. Comparing with existing schemes mentioned above, our schemes give remarkable improvement in computational efficiency, although the schemes are very easy to implement (see the simulation code presented in Section 5).

Of course, it is necessary to mention that the idea of using linear functionals of the noise in the time-stepping scheme has been exploited in [12], where the authors consider fast strong approximations of semilinear stochastic heat equation with space-time white noise. In [7], such approximations are called accelerated exponential Euler method, in contrast to the usual exponential Euler method involving only basic Wiener increments. To recover the optimal strong convergence rate $1 - \epsilon$ in time for the
accelerated exponential Euler method applied to stochastic heat equation, however, the Fréchet derivative operator of the Nemytskij operator \( F \) defined by (2.7) and the Laplacian \( \Lambda \) are required to commute in some sense (see Assumption 2.4 in [12]). The commutativity conditions are quite restrictive since they are fulfilled for linear function \( f(u) = cu \), but excludes most nonlinear functions such as \( f(u) = \frac{1}{1 + u^2} \). When the driven noise is a smoother \( Q\)-Wiener process with \( Q \) being trace class (i.e., \( \text{Tr}(Q) < \infty \)), Jentzen et al. [15] get rid of such commutativity conditions for the accelerated exponential Euler scheme. In this article we devise two accelerated exponential Euler methods for the stochastic wave equation (1.1). Most importantly, for the tough case of noise being white in time as well as space \((Q = I)\), we obtain the rate \( 1 - \epsilon \) in time of the schemes on the conditions \((1.2)\) and \((1.3)\), but without imposing any restrictive commutativity conditions as required in [12].

Finally, we would like to remark that, throughout this work, \( C \) appearing in the following estimates is a generic constant that may vary from one place to another and depends only on \( T, \epsilon, L \) and initial data, but is independent of \( \Delta t, N \).

The rest of this paper is organized as follows. In the next section, some preliminaries are presented and an abstract framework is formulated, in which some regularity properties of the considered SWE are derived. In Section 2 we analyze strong approximation error arising from spatial discretization done by a spectral Galerkin method. Then two exponential time integrators are introduced and strong convergence of fully discrete approximations are studied in Section 3. Numerical experiments are included in Section 4 to confirm our findings. At the end of this article, a conclusion is drawn and some remarks are made briefly.

2. Preliminaries and framework. Let \((U, \langle \cdot, \cdot \rangle, ||\cdot||)\) and \((H, \langle \cdot, \cdot \rangle, ||\cdot||)\) be two separable Hilbert spaces. By \( \mathcal{L}(U, H) \) we denote the space of bounded linear operators from \( U \) to \( H \) and for simplicity, we denote \( \mathcal{L}(U) = \mathcal{L}(U, U) \). Additionally, we need spaces of nuclear and Hilbert-Schmidt operators [8; 21]. The space of nuclear operators from \( U \) to \( H \) is denoted by \( \mathcal{L}_1(U, H) \), and its norm is given by

\[
||\Gamma||_{\mathcal{L}_1(U,H)} := \inf \left\{ \sum_{i=1}^{\infty} ||a_i|| \cdot ||b_i|| \mid \Gamma x = \sum_{i=1}^{\infty} a_i \langle b_i, x \rangle, \ x \in U \right\}. \quad (2.1)
\]

Let \( \{e_i\}_{i \in \mathbb{N}} \) be an orthonormal basis of \( U \) and also denote \( \mathcal{L}_1(U) := \mathcal{L}_1(U,U) \). If \( \Gamma \in \mathcal{L}_1(U) \) is additionally nonnegative and symmetric, then

\[
||\Gamma||_{\mathcal{L}_1(U)} = \text{Tr}(\Gamma) := \sum_{i=1}^{\infty} \langle Ge_i, e_i \rangle. \quad (2.2)
\]

Let here that the trace of a nuclear operator, namely, \( \text{Tr}(\Gamma) \) for \( \Gamma \in \mathcal{L}_1(U) \), is independent of the particular choice of the basis \( \{e_i\}_{i \in \mathbb{N}} \). By \( \mathcal{L}_2(U, H) \) we denote the space of Hilbert-Schmidt operators, equipped with the norm

\[
||\Gamma||_{\mathcal{L}_2(U,H)} = \left( \sum_{i=1}^{\infty} ||Ge_i||^2 \right)^{1/2}, \quad (2.3)
\]

also not depending on the particular choice of the basis. Analogously, we write \( \mathcal{L}_2(U) = \mathcal{L}_2(U, U) \) for short. If \( \Gamma_1 \in \mathcal{L}(U, H) \) and \( \Gamma_2 \in \mathcal{L}_j(U), \ j = 1, 2 \), then \( \Gamma_1 \Gamma_2 : U \rightarrow H \in \mathcal{L}_j(U,H) \) for \( j = 1, 2 \), and

\[
||\Gamma_1 \Gamma_2||_{\mathcal{L}_j(U,H)} \leq ||\Gamma_1||_{\mathcal{L}(U,H)} \cdot ||\Gamma_2||_{\mathcal{L}_j(U)}, \quad j = 1, 2. \quad (2.4)
\]
Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with a normal filtration \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) and by \( L_p(\Omega, U) \) we denote the space of \( U \)-valued integrable random variables with the norm defined by \( \|u\|_{L_p(\Omega, U)} = (\mathbb{E}\|u\|^p)^{\frac{1}{p}} \) for \( p \geq 2 \) being an integer.

Setting \( U := L^2((0,1), \mathbb{R}) \), the space of real-valued square integrable functions and denoting the Laplacian with Dirichlet boundary condition by \( -\Lambda : \mathcal{D}(\Lambda) \subset U \to U \), one can then rewrite (1.1) in an abstract Itô form

\[
\begin{aligned}
\{ & \dot{u} = -\Lambda u dt + F(u) dt + dW(t), \quad 0 \leq t \leq T, \\
& u(0) = u_0, \quad \dot{u} = v_0,
\end{aligned}
\]  
(2.5)

where the initial data \( u_0, v_0 \) are \( \mathcal{F}_0 \)-measurable random variables and \( \dot{u} \) stands for the time derivative of \( u \). Also, \( u \) is regarded as a \( U \)-valued stochastic process and \( F : U \to U \) is the Nemytskij operator associated to \( f \) as in (1.1) given by

\[
F(u)(x) = f(x, u(x)), \quad x \in (0,1).
\]  
(2.6)

Moreover, the Fréchet derivative operators of \( F \) are given by

\[
\begin{aligned}
F'(u)(\varphi)(x) &= \frac{\partial f}{\partial u}(x, u(x)) \cdot \varphi(x), \\
F''(u)(\varphi, \psi)(x) &= \frac{\partial^2 f}{\partial u^2}(x, u(x)) \cdot \varphi(x) \cdot \psi(x)
\end{aligned}
\]  
(2.7)

for all \( u, \varphi, \psi \in U \). Thanks to (1.2) and (1.3), the Nemytskij operator \( F \) also satisfies

\[
\begin{aligned}
\|F(u)\| &\leq L(\|u\| + 1), \\
\|F(u_1) - F(u_2)\| &\leq L\|u_1 - u_2\|
\end{aligned}
\]  
(2.9)

for all \( u, u_1, u_2 \in U \). Furthermore, the driven stochastic process \( W(t) \) in (2.5) is a cylindrical \( I \)-Wiener process with respect to \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \), which can be represented as follows [8, 21]:

\[
W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i,
\]  
(2.11)

where \( \{e_i = \sqrt{2}\sin(i\pi x), x \in (0,1)\}_{i \in \mathbb{N}} \) form an orthonormal basis of \( U \) consisting of eigenfunctions of \( \Lambda \) with \( \Lambda e_i = \lambda_i e_i \), \( \lambda_i = \pi^2 i^2 \), \( i \in \mathbb{N} \). Additionally, \( \{\beta_i(t)\}_{i \in \mathbb{N}} \) are a family of real Brownian motions mutually independent in the probability space. It is then obvious that

\[
\|\Lambda^{\beta-1}\|^2_{L^2(U)} = \text{Tr}(\Lambda^{\beta-1}) = \pi^{2(\beta-1)} \sum_{i=1}^{\infty} \frac{1}{i^{2(\beta-1)}} \leq C < \infty, \quad \text{for any } \beta < \frac{1}{2},
\]  
(2.12)

To facilitate the convergence analysis below, we shall rewrite (2.5) as a stochastic evolution equation in a new Hilbert space \( H \) to fall into the framework in [8] and apply corresponding results as obtained there. Precisely, introducing the velocity of the solution \( u \) and denoting \( v = \dot{u} \) lead (2.5) to the following Cauchy problem

\[
\begin{aligned}
\{ & dX(t) = AX(t) dt + F(X) dt + BdW(t), \quad 0 \leq t \leq T, \\
& X(0) = X_0,
\end{aligned}
\]  
(2.13)
where \( X_0 = (u_0, v_0)^T \) is an \( \mathcal{F}_0 \)-measurable random variable and
\[
X = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}, \quad F(X) = \begin{bmatrix} 0 \\ F(u) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.
\] (2.14)

Before proceeding further, we introduce some spaces and notations. Let \( \dot{H}^\alpha = \mathcal{D}(\Lambda^\frac{\alpha}{2}) \) with the corresponding norm defined by
\[
\|u\|_\alpha := \|\Lambda^\frac{\alpha}{2} u\| = \left( \sum_{i=1}^{\infty} \lambda_i^\alpha \langle u, e_i \rangle^2 \right)^{\frac{1}{2}}, \quad \alpha \in \mathbb{R}, \ u \in \dot{H}^\alpha.
\] (2.15)

Here and below \( \|\cdot\| \) and \( \langle \cdot, \cdot \rangle \) denote the norm and corresponding inner product in the Hilbert space \( U = L^2(0, 1) \). Then \( \dot{H}^0 = U \) and \( \dot{H}^{-\gamma} \) can be identified with the dual space \( (\dot{H}^\gamma)^* \) for \( \gamma > 0 \), see, e.g., [24]. Furthermore we introduce the product space
\[
\dot{H}^\alpha := \dot{H}^\alpha \times \dot{H}^\alpha^{-1},
\] (2.16)
equipped with the norm
\[
\|X\|_\alpha^2 := \|u\|_\alpha^2 + \|v\|_{\alpha^{-1}}^2, \quad \alpha \in \mathbb{R}, \ X = (u, v)^T.
\] (2.17)

For a special case \( \alpha = 0 \), we denote \( H = \dot{H}^0 := \dot{H}^0 \times \dot{H}^{-1} \). Regarding \( \Lambda \) as an operator \( \dot{H}^1 \to \dot{H}^{-1} \) and defining
\[
\mathcal{D}(A) = \left\{ X \in H : AX = \begin{bmatrix} v \\ -\Lambda u \end{bmatrix} \in H = \dot{H}^0 \times \dot{H}^{-1} \right\} = \dot{H}^1 = \dot{H}^1 \times \dot{H}^0,
\] (2.18)
the operator \( A \) is then the generator of a \( C_0 \)-semigroup \( E(t), t \geq 0 \) on \( H \), given by
\[
E(t) = e^{tA} = \begin{bmatrix} C(t) & \Lambda^{-\frac{\alpha}{2}} S(t) \\ -\Lambda^{\frac{\alpha}{2}} S(t) & C(t) \end{bmatrix}.
\] (2.19)

Here \( C(t) = \cos(t\Lambda^{\frac{\alpha}{2}}) \) and \( S(t) = \sin(t\Lambda^{\frac{\alpha}{2}}) \) are the cosine and sine operators defined by
\[
C(t)u = \sum_{i=1}^{\infty} \cos(t\lambda_i^{1/2}) \langle u, e_i \rangle e_i, \quad S(t)u = \sum_{i=1}^{\infty} \sin(t\lambda_i^{1/2}) \langle u, e_i \rangle e_i
\]
for any \( t \geq 0, u \in \dot{H}^{-1} \). As defined above, \( B \in \mathcal{L}(U,H) \) and \( X_0 \) is an \( \mathcal{F}_0 \)-measurable \( H \)-valued random variable. Now, we look at the existence and uniqueness of the mild solution of \( (2.13) \), which has been discussed in [3, 22] using different frameworks.

**Theorem 2.1.** Suppose conditions [12] and [13] are fulfilled, \( W(t), t \geq 0 \) is a cylindrical \( I \)-Wiener process represented by (2.11), and \( X_0 \) is an \( \mathcal{F}_0 \)-measurable random variable satisfying
\[
\|X_0\|_{L_p(\Omega, \dot{H}^{\frac{\alpha}{2}})} = \left( \mathbb{E} \|X_0\|_{\dot{H}^{\frac{\alpha}{2}}}^p \right)^{\frac{1}{p}} = \left( \mathbb{E} \left[ \left( \|u_0\|_{\dot{H}^{\frac{\alpha}{2}}}^2 + \|v_0\|_{\dot{H}^{-\frac{\alpha}{2}}}^2 \right)^{p/2} \right] \right)^{\frac{1}{p}} < \infty \quad (2.20)
\]
for any \( p \geq 2 \). Then stochastic wave equation (2.13) has a unique mild solution given by
\[
X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))ds + \int_0^t E(t-s)BdW(s), \ t \geq 0, \ a.s.
\] (2.21)
Moreover, it holds for any $0 \leq \beta < \frac{1}{2}, p \geq 2$ and $t \in [0, T]$ that
\[
\|X(t)\|_{L^p(\Omega, H^\beta)} \leq C \left( \|X_0\|_{L^p(\Omega, H^\beta)} + t + \frac{t^2}{2} \|\Lambda^{\frac{p-1}{2}}\|_{L^2(U)} \right). \tag{2.22}
\]

**Proof.** We first claim that the nonlinear operator $F: H \to H$ satisfies the globally Lipschitz condition and linear growth condition. Actually, for arbitrary $X = (u, v)^T, X_1 = (u_1, v_1)^T, X_2 = (u_2, v_2)^T \in H$ we infer that
\[
\|F(X_1) - F(X_2)\| = \|\Lambda^{-\frac{\beta}{2}}(F(u_1) - F(u_2))\| \leq L\|u_1 - u_2\| \leq L\|X_1 - X_2\|
\]
and
\[
\|F(X)\| = \|\Lambda^{-\frac{\beta}{2}}F(u)\| \leq L\|u\| + 1 \leq L(\|X\| + 1)
\]
by combining (2.10), (2.11) and the definitions of $F$ and $\|\cdot\|$. Also, it has been shown in [7] Theorem 3.1 that, with $Q = I$,
\[
\int_0^t \|E(s)B\|_{L^2(U, H^\beta)}^2 ds \leq 2t \|\Lambda^{\frac{p-1}{2}}\|_{L^2(U)}^2 < \infty, \quad \text{for } 0 \leq \beta < \frac{1}{2}, \tag{2.23}
\]
holds for all $t \in [0, T]$ due to (2.12). In view of Theorem 7.6 from [8], one can use standard arguments to readily derive the existence and uniqueness of the mild solution (2.21) so that
\[
\mathbb{E}(\|X(t)\|^p) \leq C_{p,T} (1 + \mathbb{E}(\|X_0\|^p), \quad t \in [0, T]. \tag{2.24}
\]
With this and using (2.10), (2.11), (2.12), (2.23), the Burkholder-Davis-Gundy type inequality ([8] Lemma 7.2]) and elementary inequalities we deduce from (2.21) that
\[
\|X(t)\|_{L^p(\Omega, H^\beta)} \leq \|F(X_0)\|_{L^p(\Omega, H^\beta)} + \int_0^t \|E(t-s)F(X(s))\|_{L^p(\Omega, H^\beta)} ds
\]
\[
+ \left\| \int_0^t E(t-s)BdW(s) \right\|_{L^p(\Omega, H^\beta)}
\]
\[
= \left\| \int_0^t E(t-s)X_0 ds + \right\| \int_0^t E(t-s)BdW(s) \right\|_{L^p(\Omega, H^\beta)}
\]
\[
+ \int_0^t \left( \|\Lambda^{-\frac{\beta}{2}}S(t-s)F(u(s))\|^2 + \|C(t-s)F(u(s))\|^2 \right)^{1/2} ds
\]
\[
\leq 2\|X_0\|_{L^p(\Omega, H^\beta)} + C \left( \int_0^t \|E(s)B\|_{L^2(U, H^\beta)}^2 ds \right)^{\frac{1}{2}} + \sqrt{2} \int_0^t \|F(u(s))\|_{L^p(\Omega, H^\beta)} ds
\]
\[
\leq 2\|X_0\|_{L^p(\Omega, H^\beta)} + \sqrt{2}C\sqrt{t} \|\Lambda^{\frac{p-1}{2}}\|_{L^2(U)} + \sqrt{2}L \int_0^t \|u(s)\| + 1 ds
\]
\[
\leq 2\|X_0\|_{L^p(\Omega, H^\beta)} + \sqrt{2}C\sqrt{t} \|\Lambda^{\frac{p-1}{2}}\|_{L^2(U)} + \sqrt{2}Lt + \sqrt{2}L \int_0^t \|u(s)\|_{L^p(\Omega, U)} ds
\]
\[
< \infty.
\]
for any \( t \in [0, T] \) and \( 0 \leq \beta < \frac{1}{2} \). Here the stability properties of \( S(t), C(t), \Lambda^{-\gamma} \) for \( t, \gamma \in [0, \infty) \) were considered. Further, the above estimate implies
\[
\|X(t)\|_{L^p(\Omega, H^s)} \leq 2\|X_0\|_{L^p(\Omega, H^s)} + \sqrt{2C\sqrt{T}} \|\Lambda^{\beta+\frac{1}{2}}\|_{L^2(U)} + \sqrt{2Lt} + \sqrt{2L} \int_0^t \|X(s)\|_{L^p(\Omega, H^s)} ds
\]
for all \( t \in [0, T] \) and thus the Gronwall inequality yields the desired estimate \( \text{(2.22)} \). □

Writing \( \text{(2.21)} \) out and replacing \( E(t) \) by \( \text{(2.19)} \) yield two components:
\[
\begin{align*}
\{ u(t) &= C(t)u_0 + \Lambda^{-\frac{1}{2}}S(t)v_0 + \int_0^t \Lambda^{-\frac{1}{2}}S(t-s)F(u(s))ds + O_t, \\
v(t) &= -\Lambda^{\frac{1}{2}}S(t)u_0 + \Lambda^{\frac{1}{2}}C(t)v_0 + \int_0^t \Lambda^{\frac{1}{2}}C(t-s)F(u(s))ds + \hat{O}_t,
\end{align*}
\]
where \( t \in [0, T] \) and we used the notations
\[
O_t = \int_0^t \Lambda^{-\frac{1}{2}}S(t-s)dW(s), \quad \hat{O}_t = \int_0^t \Lambda^{\frac{1}{2}}C(t-s)dW(s).
\] (3.26)

Next we are to design approximations of the mild solution \( u(t) \) and measure the discrepancy between the approximations and \( u(t) \). To this end, one has to discretize both the time interval \([0, T]\) and the infinite dimensional space \( U \). For temporal discretizations of SWE, the finite difference method is a common choice \([10,19,20,27]\), while spatial discretizations can be achieved with finite difference\([10,22,27]\), finite element\([17,18,19]\) and spectral Galerkin\([21,23]\) methods. In this work we consider finite difference time discretization and spectral Galerkin discretization in space.

3. The spectral Galerkin approximation of SWE. We first consider the spatial discretizations of \( \text{(2.13)} \), which is done by the Galerkin spectral method. For \( N \in \mathbb{N} \), we define a finite dimensional subspace \( U_N \) of \( U \) by
\[
U_N := \text{span}\{e_1, e_2, \ldots, e_N\},
\] (3.1)
and the projection operator \( P_N : \hat{H}^{\alpha} \to U_N \) by
\[
P_N \xi = \sum_{i=1}^N (\xi, e_i)e_i, \quad \forall \xi \in \hat{H}^{\alpha}, \alpha \geq -1.
\]
Moreover, we define \( \Lambda_N : U_N \to U_N \) by
\[
\Lambda_N \xi = \Lambda P_N \xi = P_N \Lambda \xi = \sum_{i=1}^N \lambda_i (\xi, e_i)e_i, \quad \forall \xi \in U_N.
\] (3.2)

Then applying the Galerkin projection to \( \text{(2.13)} \) gives finite dimensional stochastic differential equations (SDEs) in \( H_N := U_N \times U_N \)
\[
\begin{align*}
dX^N(t) &= A_N X^N(t)dt + F_N(X^N)dt + B_N dW(t), \quad 0 \leq t \leq T, \\
X^N(0) &= X_0^N,
\end{align*}
\]
(3.3)
where \( X_0^N = (P_N u_0, P_N v_0)^T \) and
\[
X^N = \begin{bmatrix} u^N \\ v^N \end{bmatrix}, \quad A_N = \begin{bmatrix} 0 & I \\
-A_N & 0 \end{bmatrix}, \quad F_N(X^N) = \begin{bmatrix} 0 \\
P_N F(u^N) \end{bmatrix}, \quad B_N = \begin{bmatrix} 0 \\ P_N \end{bmatrix}.
\]
Similarly as above, the operator $A$ is the generator of a $C_0$-semigroup $E_N(t)$, $t \geq 0$ on $U_N \times U_N$ and

$$E_N(t) = e^{tA} = \begin{bmatrix} C_N(t) & \frac{1}{2}\Lambda_N^{-\frac{1}{2}}S_N(t) \\ -\frac{1}{2}\Lambda_N^{\frac{1}{2}}S_N(t) & C_N(t) \end{bmatrix}, \quad (3.4)$$

where $C(t) = \cos(t\Lambda_N^\frac{1}{2})$ and $S_N(t) = \sin(t\Lambda_N^\frac{1}{2})$ for $t \geq 0$ are the cosine and sine operators defined in $U_N$. Moreover it can be verified straightforwardly that

$$\cos(t\sqrt{\Lambda_N})P_N u = \cos(t\sqrt{\Lambda})P_N u = P_N \cos(t\sqrt{\Lambda})u$$ \quad (3.5)

and

$$\sin(t\sqrt{\Lambda_N})P_N u = \sin(t\sqrt{\Lambda})P_N u = P_N \sin(t\sqrt{\Lambda})u$$ \quad (3.6)

for any $u \in \dot{H}^{-1}$. The following result guarantees a unique global solution of (3.3).

**Theorem 3.1.** Assume that all conditions in Theorem 2.1 are all fulfilled. Then it holds for all $t \geq 0$.

The proof of (3.8) goes along exactly the same lines as that of (2.22) and is thus omitted here. Similarly to (2.25), (3.7) can be rewritten as

$$X(t) = E_N(t)X_0 + \int_0^t E_N(t-s)F_N(X_N(s))ds + \int_0^t E_N(t-s)B_N dW(s), \quad (3.7)$$

$\mathbb{P}$-a.s. for any $t \geq 0$. Additionally, for any $0 \leq \beta < \frac{1}{2}$, $p \geq 2$ and $t \in [0, T]$

$$\|X(t)\|_{L_p(\Omega,H^\beta)} \leq C \left(\|X_0\|_{L_p(\Omega,H^\beta)} + t + t^\frac{1}{2}\|\Lambda^{\frac{1}{2}}N\|_{L_2(\Omega)}\right). \quad (3.8)$$

The proof of (3.8) goes along exactly the same lines as that of (2.22) and is thus omitted here. Similarly to (2.22), (3.7) can be rewritten as

$$\begin{cases}
  u(t) = C_N(t)u_0^N + \Lambda_N^{-\frac{1}{2}}S_N(t)v_0^N + \int_0^t \Lambda_N^{-\frac{1}{2}}S_N(t-s)P_N F(u(s))ds + C^N, \\
v(t) = -\Lambda_N^{\frac{1}{2}}S_N(t)v_0^N + C_N(t)v_0^N + \int_0^t C_N(t-s)P_N F(u(s))ds + \tilde{C}^N,
\end{cases} \quad (3.9)$$

where for simplicity of presentation we denote $u_0^N = P_N u_0$, $v_0^N = P_N v_0$ and for $t \geq 0$

$$C^N = \int_0^t \Lambda_N^{-\frac{1}{2}}S_N(t-s)P_N dW(s), \quad \tilde{C}^N = \int_0^t C_N(t-s)P_N dW(s). \quad (3.10)$$

Before going to the spatial error analysis, we present the following lemma, which is an immediate consequence of (2.22) and (3.8).

**Lemma 3.2.** Assume that (2.9) is fulfilled, and let $u$ and $u^N$ be given by (2.21) and (3.9), respectively. Then it holds for some constant $C$ that

$$\sup_{0 \leq t \leq T} \|F(u(t))\|_{L_2(\Omega,U)} \leq C, \quad \text{and} \quad \sup_{0 \leq t \leq T} \|F(u^N(t))\|_{L_2(\Omega,U)} \leq C. \quad (3.11)$$

Armed with the above preparations, we are now able to analyze the spatial error.

**Theorem 3.3 (Spatial discretization error).** Suppose that all conditions in Theorem 2.1 are satisfied. Then it holds for all $t \in [0, T]$ that

$$\|u(t) - u^N(t)\|_{L_2(\Omega,U)} \leq C \left(\|u_0\|_{L_2(\Omega,\dot{H}^{-\frac{1}{2}})} + \|v_0\|_{L_2(\Omega,\dot{H}^{-\frac{1}{2}})} + 1\right) N^{-\frac{1}{2}+\epsilon}. \quad (3.12)$$
where \( u(t) \) and \( u^N(t) \) are given by (2.25) and (3.9), respectively.

Proof. Combining (2.25) and (3.9) yields

\[
\begin{align*}
u^N(t) - u(t) &= (C_N P_N - C(t)) u_0 + \left( \Lambda_N^{-\frac{1}{2}} S_N(t) P_N - \Lambda^{-\frac{1}{2}} S(t) \right) v_0 \\
&+ \int_0^t \left( \Lambda_N^{-\frac{1}{2}} S_N(t-s) P_N F(u^N(s)) - \Lambda^{-\frac{1}{2}} S(t-s) F(u(s)) \right) \, ds \\
&+ \int_0^t \left( \Lambda_N^{-\frac{1}{2}} S_N(t-s) P_N - \Lambda^{-\frac{1}{2}} S(t-s) \right) dW(s) \tag{3.13}
\end{align*}
\]

for any \( t \geq 0 \) and therefore for any \( t \geq 0 \)

\[
\begin{align*}
\| u^N(t) - u(t) \|_{L_2(\Omega, U)} &\leq \| (C_N P_N - C(t)) u_0 \|_{L_2(\Omega, U)} + \left\| \left( \Lambda_N^{-\frac{1}{2}} S_N(t) P_N - \Lambda^{-\frac{1}{2}} S(t) \right) v_0 \right\|_{L_2(\Omega, U)} \\
&+ \int_0^t \left\| \Lambda_N^{-\frac{1}{2}} S_N(t-s) P_N F(u^N(s)) - \Lambda^{-\frac{1}{2}} S(t-s) F(u(s)) \right\|_{L_2(\Omega, U)} \, ds \\
&+ \int_0^t \left\| \left( \Lambda_N^{-\frac{1}{2}} S_N(t-s) P_N - \Lambda^{-\frac{1}{2}} S(t-s) \right) dW(s) \right\|_{L_2(\Omega, U)} \\
&:= I_1 + I_2 + I_3 + I_4.
\end{align*}
\]

Note that it holds for all \( t \geq 0 \) that

\[
\left\| \Lambda_N^{-\gamma} S_N(t) P_N - \Lambda^{-\gamma} S(t) \right\|_{\mathcal{L}(U)} = \left\| (S(t) P_N - S(t)) \Lambda^{-\gamma} \right\|_{\mathcal{L}(U)}
= \sup_{i \geq N+1} \left| \sin(\sqrt{\lambda_i} t) \lambda_i^{-\gamma} \right| \leq \lambda_i^{-\gamma} \leq \frac{C}{N^{2\gamma}},
\tag{3.15}
\]

and similarly for any \( t \geq 0 \)

\[
\left\| \Lambda_N^{-\gamma} C_N(t) P_N - \Lambda^{-\gamma} C(t) \right\|_{\mathcal{L}(U)} = \sup_{i \geq N+1} \left| \cos(\sqrt{\lambda_i} t) \lambda_i^{-\gamma} \right| \leq \lambda_i^{-\gamma} \leq \frac{C}{N^{2\gamma}}.
\tag{3.16}
\]

Hence using (3.15) and (3.16) with \( \gamma = \frac{\beta}{2} < \frac{1}{4} \) shows

\[
I_1 + I_2 \leq C \left( \| u_0 \|_{L_2(\Omega, \dot{V}^\beta)} + \| v_0 \|_{L_2(\Omega, \dot{V}^{\beta-1})} \right) N^{-\beta}.
\tag{3.17}
\]

Concerning \( I_3 \), using (2.10), (3.11), (3.15) and also taking into account of \( \Lambda^{-\frac{1}{2}}, S(t) \) and \( P_N \) into account lead to

\[
\begin{align*}
I_3 &\leq \int_0^t \left\| \Lambda_N^{-\frac{1}{2}} S_N(t-s) P_N \left( F(u^N(s)) - F(u(s)) \right) \right\|_{L_2(\Omega, U)} \, ds \\
&+ \int_0^t \left\| \left( \Lambda_N^{-\frac{1}{2}} S_N(t-s) P_N - \Lambda^{-\frac{1}{2}} S(t-s) \right) F(u(s)) \right\|_{L_2(\Omega, U)} \, ds \\
&\leq L \int_0^t \| u^N(s) - u(s) \|_{L_2(\Omega, U)} \, ds + \frac{C}{N} \int_0^t \| F(u(s)) \|_{L_2(\Omega, U)} \, ds \\
&\leq L \int_0^t \| u^N(s) - u(s) \|_{L_2(\Omega, U)} \, ds + \frac{C \gamma}{N}.
\tag{3.18}
\end{align*}
\]
With the aid of the isometry property of stochastic integral, (2.4), (2.12) and (3.15), one can estimate $I_4$ for $t \in [0, T]$ as follows

\[
|I_4|^2 = \int_0^T \left\| \left( \Lambda_N^{-\frac{7}{2}} S_N(t-s) P_N - \Lambda_N^{-\frac{7}{2}} S(t-s) \right) \right\|^2_{L_2(U)} ds \\
\leq \int_0^T \left\| (S(t-s) P_N - S(t-s)) \Lambda_N^{-\frac{7}{2}} \right\|^2_{L_2(U)} \cdot \left\| \Lambda_N^{-\frac{7}{2}} \right\|^2_{L_2(U)} ds \leq \frac{C_T}{N^{2\beta}}
\]

for any $\beta < \frac{1}{2}$. Therefore, inserting (3.17), (3.18) and (3.19) into (3.14) yields

\[
\|u_N^N(t) - u(t)\|_{L_2(\Omega, U)} \leq C \left( \|u_0\|_{L_2(\Omega, H^\beta)} + \|v_0\|_{L_2(\Omega, H^{\beta-1})} + 1 \right) N^{-\beta} \\
+ L \int_0^t \|u_N^N(s) - u(s)\|_{L_2(\Omega, U)} ds
\]

for any $\beta < \frac{1}{2}$. Note that $\|u_N^N(t) - u(t)\|_{L_2(\Omega, U)} \leq \|u_N^N(t)\|_{L_2(\Omega, U)} + \|u(t)\|_{L_2(\Omega, U)} < \infty$ by Theorem 2.1 and Theorem 3.1. Applying the Gronwall inequality to the preceding estimate with setting $\beta = \frac{1}{2} - \epsilon$ gives (3.12). The proof is completed. □

4. Fully discrete approximations and strong convergence. Until now, only semi-discrete approximations in space have been investigated. In this section, we continue to consider temporal discretization of (3.9). Two exponential time integrators (see also [12], [15]) shall be constructed for (3.9) and hence result in full discrete approximations of SWE (1.1).

4.1. The fully discrete approximations and main result. For spatial approximations (3.9), we propose two time-stepping schemes as follows:

\[
\begin{align*}
\frac{u_m^{N+1}}{u_m^{N+1}} &= C_N(\tau)u_m^N + \Lambda_N^{-\frac{7}{2}} S_N(\tau)u_m^N + \Lambda_N^{-1} \left( I - C_N(\tau) \right) P_N F(u_m^N) \\
&\quad + \int_{t_m}^{t_{m+1}} \Lambda_N^{-\frac{7}{2}} S_N(t_{m+1} - s) P_N dW(s), \\
\frac{u_m^{N}}{u_m^{N}} &= -\Lambda_N^{-\frac{7}{2}} S_N(\tau)u_m^N + C_N(\tau)u_m^N + \Lambda_N^{-\frac{7}{2}} S_N(\tau) P_N F(u_m^N) \\
&\quad + \int_{t_m}^{t_{m+1}} C_N(t_{m+1} - s) P_N dW(s),
\end{align*}
\]

and

\[
\begin{align*}
\frac{v_m^{N+1}}{v_m^{N+1}} &= C_N(\tau)v_m^N + \Lambda_N^{-\frac{7}{2}} S_N(\tau)v_m^N + \tau \Lambda_N^{-\frac{7}{2}} S_N(\tau) P_N F(v_m^N) \\
&\quad + \int_{t_m}^{t_{m+1}} \Lambda_N^{-\frac{7}{2}} S_N(t_{m+1} - s) P_N dW(s), \\
v_m^{N+1} &= -\Lambda_N^{-\frac{7}{2}} S_N(\tau)v_m^N + C_N(\tau)v_m^N + \tau C_N(\tau) P_N F(v_m^N) \\
&\quad + \int_{t_m}^{t_{m+1}} C_N(t_{m+1} - s) P_N dW(s)
\end{align*}
\]

for $m = 0, 1, 2, \ldots, M-1$. Here $u_m^N$ and $v_m^N$ are, respectively, temporal approximations of $u^N(t)$ and $v^N(t)$ at the grid point $t_m$, with initial values $u_0^N = P_N u_0, v_0^N = P_N v_0$, and $\tau = T/M$ being the stepsize. It is worthwhile to point out that both schemes are much easier to simulate than it appears at first sight. To show this fact, we take scheme (4.1) for example and make some remarks on the implementations. Note first for $i = 1, 2, \ldots, N, m = 0, 1, \ldots, M-1$ that

\[
\zeta_i := \left( \int_{t_m}^{t_{m+1}} \Lambda_N^{-\frac{7}{2}} S_N(t_{m+1} - s) P_N dW(s), e_i \right) = \lambda_i^{-\frac{7}{2}} \int_{t_m}^{t_{m+1}} \sin \left( (t_{m+1} - s) \lambda_i^{\frac{7}{2}} \right) d\beta_i(s)
\]

\]

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are mutually independent normally distributed random variables satisfying
\[
\mathbb{E} \zeta_i = 0, \quad \mathbb{E} |\zeta_i|^2 = \frac{1}{2\lambda_i} \left( \tau - \frac{\sin(2\tau \sqrt{\lambda_i})}{2\sqrt{\lambda_i}} \right), \quad i = 1, 2, \ldots, N.
\]
Similarly
\[
\tilde{\zeta}_i := \left\langle \int_{t_m}^{t_{m+1}} C_N(t_{m+1} - s) P_N dW(s), e_i \right\rangle = \int_{t_m}^{t_{m+1}} \cos \left( (t_{m+1} - s) \lambda_i^{\frac{1}{2}} \right) d\beta_i(s)
\]
for \( i = 1, 2, \ldots, N, \ m = 0, 1, \ldots, M - 1 \) are mutually independent normally distributed random variables with
\[
\mathbb{E} \tilde{\zeta}_i = 0, \quad \mathbb{E} |\tilde{\zeta}_i|^2 = \frac{1}{2} \left( \tau + \frac{\sin(2\tau \sqrt{\lambda_i})}{2\sqrt{\lambda_i}} \right), \quad i = 1, 2, \ldots, N.
\]
Accordingly, the components of \( u_m^N \) and \( v_m^N \) in (4.1), i.e., \( \langle u_m^N, e_i \rangle \) and \( \langle v_m^N, e_i \rangle \) for \( i = 1, 2, \ldots, N \) and \( m = 0, 1, \ldots, M \), can be calculated by the following recurrence equations:
\[
\langle u_{m+1}^N, e_i \rangle = \cos(\tau \sqrt{\lambda_i}) \langle u_m^N, e_i \rangle + \lambda_i^{-\frac{1}{2}} \sin(\tau \sqrt{\lambda_i}) \langle v_m^N, e_i \rangle + \lambda_i^{-1} (1 - \cos(\tau \sqrt{\lambda_i})) \langle F(u_m^N), e_i \rangle + \frac{1}{\sqrt{2\lambda_i}} \left( \tau - \frac{\sin(2\tau \sqrt{\lambda_i})}{2\sqrt{\lambda_i}} \right) \frac{1}{2} R_m^i, \tag{4.3}
\]
\[
\langle v_{m+1}^N, e_i \rangle = -\lambda_i^{\frac{1}{2}} \sin(\tau \sqrt{\lambda_i}) \langle u_m^N, e_i \rangle + \cos(\tau \sqrt{\lambda_i}) \langle u_m^N, e_i \rangle + \lambda_i^{-\frac{1}{2}} \sin(\tau \sqrt{\lambda_i}) \langle F(u_m^N), e_i \rangle + \frac{1}{\sqrt{2\lambda_i}} \left( \tau + \frac{\sin(2\tau \sqrt{\lambda_i})}{2\sqrt{\lambda_i}} \right) \frac{1}{2} R_m^i. \tag{4.4}
\]
where \( R_m^i \) for \( i = 1, 2, \ldots, N \) and \( m = 0, 1, \ldots, M \) are independent, standard normally distributed random variables. Hence, scheme (4.1) can be implemented easily (see FIG. 5.1 for the implementation code). Now we formulate our main result as follows.

**Theorem 4.1.** Suppose that the nonlinear function in (1.1) satisfies all conditions in (1.2) and (1.3), and that \( W \) is a cylindrical I-Wiener process represented by (2.25). Moreover, assume the initial data satisfy
\[
\|u_0\|_{L_p(\Omega, \mathcal{H})} + \|v_0\|_{L_p(\Omega, \mathcal{H}^o)} < \infty \tag{4.5}
\]
for all \( p \in [2, 4] \). Let \( u(t) \) be a mild solution of (1.1) represented by (2.25) and let \( u_m^N \) be numerical approximations produced by (4.1) or (4.2). Then it holds for all \( m = 0, 1, 2, \ldots, M \) and for arbitrarily small \( \epsilon > 0 \) that
\[
\|u_m^N - u(t_m)\|_{L_2(\Omega, \mathcal{U})} \leq C \left( N^{-\frac{1}{2} + \epsilon} + \tau^{1-\epsilon} \right). \tag{4.6}
\]

This main result indicates that the mean-square approximation error in (4.6) is composed of two parts. The first term arises due to spatial discretization and the second term corresponds to the temporal discretization error. The detailed proof of Theorem 4.1 is postponed to the subsection 4.3.
4.2. Some preparatory results. Before starting proof of the main result, we need some preparatory results, which are crucial in the following convergence analysis.

**Lemma 4.2.** Assume that $S(t)$ and $C(t)$ are the sine and cosine operators as defined above. Then for any $\gamma \in [0,1]$ there exists some constant $c_{\gamma}$ such that

$$\|(S(t) - S(s))A^{-\frac{1}{2}}\|_{L(U)} \leq c_{\gamma}(t-s)^{\gamma}, \quad \|(C(t) - C(s))A^{-\frac{1}{2}}\|_{L(U)} \leq c_{\gamma}(t-s)^{\gamma} \quad (4.7)$$

for all $t \geq s \geq 0$.

**Proof.** To begin with, we recall two obvious facts that, for arbitrary $\gamma \in [0,1]$ there exists $c_{\gamma}$ such that

$$|\sin(x) - \sin(y)| \leq c_{\gamma}|x-y|^{\gamma}, \quad |\cos(x) - \cos(y)| \leq c_{\gamma}|x-y|^{\gamma} \quad (4.8)$$

for all $x \geq 0, y \geq 0$. Accordingly it holds for all $t \geq s \geq 0$ that

$$\|(S(t) - S(s))A^{-\frac{1}{2}}\|_{L(U)} = \sup_{i \in \mathbb{N}} \left| \lambda_i^{-\frac{1}{2}} \left( \sin(t\sqrt{\lambda_i}) - \sin(s\sqrt{\lambda_i}) \right) \right| \leq c_{\gamma}(t-s)^{\gamma}.$$

The first estimate in (4.7) is thus proved. The second one can be derived in exactly the same way. □

Subsequently we present some regularity results on the stochastic processes arising from (3.9), which are crucial in analyzing strong convergence orders of (4.1) and (4.2).

**Lemma 4.3.** Suppose that conditions (2.9), (2.12) and (2.13) are all fulfilled. Then the stochastic convolution $\mathcal{O}_t^N$ (3.10) and the stochastic process $\bar{u}^N(t) := u^N(t) - \mathcal{O}_t^N$ satisfy

$$\|\mathcal{O}_t^N - \mathcal{O}_s^N\|_{L_4(\Omega,V)} \leq C(t-s)\beta; \quad \|\bar{u}^N(t) - \bar{u}^N(s)\|_{L_2(\Omega,H^\beta)} \leq C(t-s) \quad (4.9)$$

for any $\beta \in [0,\frac{1}{2})$, $t-s \in [0,1]$ and a universal constant $C > 0$. Here by $V := L^4((0,1),\mathbb{R})$ we denote the Banach space consisting of integrable functions, equipped with the norm $||\xi||_V = \left( \int_0^1 |\xi(x)|^4 dx \right)^{\frac{1}{4}}$.

**Proof.** According to the definitions above, we split $\mathcal{O}_t^N - \mathcal{O}_s^N$ into two terms:

$$\begin{align*}
\mathcal{O}_t^N - \mathcal{O}_s^N &= \int_0^t \Lambda_N^{-\frac{1}{2}} S_N(t-r)P_N dW(r) - \int_0^s \Lambda_N^{-\frac{1}{2}} S_N(s-r)P_N dW(r) \\
&= \int_0^s \Lambda_N^{-\frac{1}{2}} \left( S_N(t-r) - S_N(s-r) \right)P_N dW(r) + \int_s^t \Lambda_N^{-\frac{1}{2}} S_N(t-r)P_N dW(r) \\
&:= K_1(t,s,x) + K_2(t,s,x).
\end{align*}$$

Exploiting the mutual independence of $\{\beta_i\}_{i \in \mathbb{N}}$, Itô’s isometry, (4.8) and (2.12) yields

$$E|K_1(t,s,x)|^2 = E \left| \sum_{i=1}^N \int_0^s \lambda_i^{-\frac{1}{2}} \left( \sin\left((t-r)\sqrt{\lambda_i}\right) - \sin\left((s-r)\sqrt{\lambda_i}\right) \right) d\beta_i(r) e_i(x) \right|^2$$

$$\leq 2 \sum_{i=1}^N \int_0^s \lambda_i^{-1} \left| \sin\left((t-r)\sqrt{\lambda_i}\right) - \sin\left((s-r)\sqrt{\lambda_i}\right) \right|^2 dr$$

$$\leq 2 \sum_{i=1}^N \int_0^s \lambda_i^{-1} \left| \sin\left((t-r)\sqrt{\lambda_i}\right) - \sin\left((s-r)\sqrt{\lambda_i}\right) \right|^2 dr$$
\[
\leq 2c_\beta \sum_{i=1}^{N} \lambda_i^{\beta-1} (t-s)^{2\beta} \\
\leq C (t-s)^{2\beta}
\]
for \( t-s \in [0,1] \). Similarly, one can obtain for \( t-s \in [0,1] \) that
\[
\mathbb{E}|K_2(t,s,x)|^2 \leq C(t-s)^{2\beta+1}.
\]
Collecting the above two estimates together shows for \( t-s \in [0,1] \) that
\[
\mathbb{E} |O^N_t(x) - \bar{O}^N_t(x)|^2 \leq 2\mathbb{E}|K_1(t,s,x)|^2 + 2\mathbb{E}|K_2(t,s,x)|^2 \leq C (t-s)^{2\beta}.
\] (4.11)
Since \( O^N_t(x) - \bar{O}^N_t(x) \) is a Gaussian real-valued random variable, there exists a new constant \( C \) such that
\[
\mathbb{E}|O^N_t(x) - \bar{O}^N_t(x)|^4 \leq C(t-s)^{4\beta}, \quad t-s \in [0,1],
\]
which obviously shows for \( t-s \in [0,1] \) that
\[
\mathbb{E}\|O^N_t - \bar{O}^N_t\|^4_V = \mathbb{E} \int_0^1 |O^N_t(x) - \bar{O}^N_t(x)|^4dx \leq C(t-s)^{4\beta}.
\] (4.12)
Hence, (4.9) is validated. To get (4.10), we first write
\[
\bar{u}^N(t) - \bar{u}^N(s) = C_N(t)P_N u_0 + \Lambda_N^{-\frac{1}{2}} S_N(t)P_N v_0 + \int_0^t \Lambda_N^{-\frac{1}{2}} S_N(t-r)P_N F(u^N(r))dr \\
- C_N(s)P_N u_0 - \Lambda_N^{-\frac{1}{2}} S_N(s)P_N v_0 - \int_0^s \Lambda_N^{-\frac{1}{2}} S_N(s-r)P_N F(u^N(r))dr
\]
\[
= (C_N(t) - C_N(s))P_N u_0 + \Lambda_N^{-\frac{1}{2}} (S_N(t) - S_N(s))P_N v_0 \\
+ \int_0^s \Lambda_N^{-\frac{1}{2}} (S_N(t-r) - S_N(s-r))P_N F(u^N(r))dr \\
+ \int_s^t \Lambda_N^{-\frac{1}{2}} S_N(t-r)P_N F(u^N(r))dr.
\]
Therefore, using stability of \( P_N \), (3.11) and (4.7) results in
\[
\|\bar{u}^N(t) - \bar{u}^N(s)\|_{L_2(\Omega,\dot{H}_0)} \leq \|\left((C(t) - C(s))P_N u_0\right)\|_{L_2(\Omega,\dot{H}_0)} \\
+ \|\Lambda^{-\frac{1}{2}} (S(t) - S(s))P_N v_0\|_{L_2(\Omega,\dot{H}_0)} \\
+ \int_0^s \|\Lambda^{-\frac{1}{2}} (S(t-r) - S(s-r))P_N F(u^N(r))\|_{L_2(\Omega,\dot{H}_0)} dr \\
+ \int_s^t \|\Lambda^{-\frac{1}{2}} S(t-r)P_N F(u^N(r))\|_{L_2(\Omega,\dot{H}_0)} dr
\]
\[
\leq C \left(\|u_0\|_{L_2(\Omega,\dot{H}^1)} + \|v_0\|_{L_2(\Omega,\dot{H}^0)}\right) (t-s) \\
+ \int_0^s C(t-s) \|F(u^N(r))\|_{L_2(\Omega,\dot{H}_0)} dr \\
+ \int_s^t C(t-r) \|F(u^N(r))\|_{L_2(\Omega,\dot{H}_0)} dr
\]
\[
\leq C(t-s)
\]
for \( t-s \in [0,1] \) and finishes the proof. \( \square \)
4.3. Proof of Theorem 4.1. To measure the overall mean-square error, one can decompose it into temporal and spatial errors:

\[
\|u_m^N - u(t_m)\|_{L^2(\Omega, U)} \leq \|u_m^N - u^N(t_m)\|_{L^2(\Omega, U)} + \|u^N(t_m) - u(t_m)\|_{L^2(\Omega, U)} \\
\leq \|u_m^N - u^N(t_m)\|_{L^2(\Omega, U)} + C \varepsilon^{-\frac{1}{2} + \eta}.
\]

(4.13)

Here \(m = 0, 1, \cdots, M\) and the last inequality holds due to the spatial discretization error obtained in (3.12) and assumption (4.5). Consequently, it only remains to estimate the temporal discretization error \(\|u_m^N - u^N(t_m)\|_{L^2(\Omega, U)}\), which is different as \(u_m^N\) is produced by different schemes, i.e., (4.1) and (4.2). First we deal with (4.1). Noticing that

\[
\int_{t_m}^{t_{m+1}} \Lambda_N^{-\frac{1}{2}} S_N(t_m + 1 - s) ds = \Lambda_N^{-1} (I - C_N(\tau)) , \int_{t_m}^{t_{m+1}} C_N(t_m + 1 - s) ds = \Lambda_N^{\frac{1}{2}} S_N(\tau),
\]

one can rewrite (4.1) in a compact form

\[
X_{m+1}^N = E_N(\tau)X_m^N + \int_{t_m}^{t_{m+1}} E_N(t_m + 1 - s) F_N(X_m^N) ds \\
+ \int_{t_m}^{t_{m+1}} E_N(t_m + 1 - s) B_N dW(s)
\]

(4.14)

for \(m = 0, 1, 2, \cdots, M - 1\), where we denote \(X_m^N = (u_m^N, v_m^N)^T\). Therefore (4.14) implies for all \(m = 0, 1, 2, \cdots, M - 1\) that

\[
X_{m+1}^N = E_N(t_m + 1)X_0^N + \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} E_N(t_m + 1 - s) F_N(X_l^N) ds \\
+ \int_{0}^{t_{m+1}} E_N(t_m + 1 - s) B_N dW(s).
\]

(4.15)

Subtracting (3.7) from (4.15) yields

\[
X_{m+1}^N - X_N(t_m + 1) = \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} E_N(t_m + 1 - s) \left(F_N(X_l^N) - F_N(X_N(s))\right) ds,
\]

(4.16)

which in turn suggests for all \(m = 0, 1, \cdots, M - 1\) that

\[
u_{m+1}^N - u^N(t_m + 1) = \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \Lambda_N^{-\frac{1}{2}} S_N(t_m + 1 - s) P_N (F(u_l^N) - F(u^N(s))) ds.
\]

(4.17)

In what follows we will estimate (4.17) in detail. First of all, using (3.2), (3.5), (3.6), (2.10), stability of \(\Lambda_N^{-\frac{1}{2}}, P_N\) and \(S(t)\) shows for all \(m = 0, 1, \cdots, M - 1\) that

\[
\|u_m^N - u^N(t_m + 1)\|_{L^2(\Omega, U)} \\
\leq \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \left\|\Lambda_N^{-\frac{1}{2}} S(t_m + 1 - s) P_N (F(u_l^N) - F(u^N(t_l)))\right\|_{L^2(\Omega, U)} ds \\
+ \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \Lambda_N^{-\frac{1}{2}} S(t_m + 1 - s) P_N (F(u^N(s) - F(u^N(t_l)))) ds \|_{L^2(\Omega, U)}
\]

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\[\leq \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \| F(u_{t}^{N}) - F(u^{N}(t_{l})) \|_{L_{2}(\Omega, U)} \, ds + J \]
\[\leq L \tau \sum_{l=0}^{m} \| u_{t_{l}}^{N} - u^{N}(t_{l}) \|_{L_{2}(\Omega, U)} + J, \] (4.18)

where for simplicity we denote
\[ J = \left\| \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s) P_{N} (F(u^{N}(s)) - F(u^{N}(t_{l})) \right\|_{L_{2}(\Omega, U)}. \] (4.19)

Using the notations as defined in Lemma 4.3, one can further split \( J \) into two terms as follows:
\[ J = \left\| \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s) P_{N} (F(\bar{u}^{N}(s) + O_{s}^{N}) - F(\bar{u}^{N}(t_{l}) + O_{t_{l}}^{N})) \right\|_{L_{2}(\Omega, U)} \]
\[ \leq \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \left\| \Lambda^{-\frac{1}{2}} S(t_{m+1} - s) P_{N} (F(\bar{u}^{N}(s) + O_{s}^{N}) - F(\bar{u}^{N}(t_{l}) + O_{t_{l}}^{N})) \right\|_{L_{2}(\Omega, U)} \, ds \]
\[ + \left\| \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s) P_{N} (F(\bar{u}^{N}(t_{l}) + O_{t_{l}}^{N}) - F(\bar{u}^{N}(t_{l}) + O_{t_{l}}^{N})) \right\|_{L_{2}(\Omega, U)} \]
\[ := J_{1} + J_{2}. \] (4.20)

Thus the estimate of \( J \) reduces to estimates of \( J_{1} \) and \( J_{2} \). Since \( J_{1} \) is an easy term, we treat it first. Using (2.10), (4.10) and stability of \( \Lambda^{-\frac{1}{2}}, P_{N} \) and \( S(t) \) gives
\[ J_{1} \leq \sum_{l=0}^{m} \int_{0}^{t_{l+1}} \| F(\bar{u}^{N}(s) + O_{s}^{N}) - F(\bar{u}^{N}(t_{l}) + O_{t_{l}}^{N}) \|_{L_{2}(\Omega, U)} \, ds \]
\[ \leq L \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \| \bar{u}^{N}(s) - \bar{u}^{N}(t_{l}) \|_{L_{2}(\Omega, U)} \, ds \]
\[ \leq C \tau. \] (4.21)

Before dealing with \( J_{2} \), one can straightforwardly derive that the Fréchet derivative operator of the Nemyskij operator \( F \) defined as above satisfy
\[ \| F'(u) \varphi \| \leq L \| \varphi \|, \quad \| F''(u)(\psi_{1}, \psi_{2}) \| \leq L \| \psi_{1} \|_{V} \cdot \| \psi_{2} \|_{V} \] (4.22)
for any \( u, \varphi \in U \) and \( \psi_{1}, \psi_{2} \in V = L^{2}((0, 1), \mathbb{R}) \), provided all conditions in (1.3) are fulfilled. Exploiting now Taylor’s formula, Hölder’s inequality, (4.9) and (4.22) leads us to
\[ J_{2} \leq \left\| \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s) P_{N} F'(u^{N}(t_{l})) (O_{s}^{N} - O_{t_{l}}^{N}) \right\|_{L_{2}(\Omega, U)} \]
\[ + \sum_{l=0}^{m} \int_{t_{l}}^{t_{l+1}} \left\| \Lambda^{-\frac{1}{2}} S(t_{m+1} - s) P_{N} \int_{0}^{1} F''(u^{N}(t_{l}) + r(O_{s}^{N} - O_{t_{l}}^{N})) \right. \]
\[ \cdot (O_{s}^{N} - O_{t_{l}}^{N}, O_{s}^{N} - O_{t_{l}}^{N}) dr \right\|_{L_{2}(\Omega, U)} \, ds \]
At this moment it remains to properly estimate the first term in the last step of (4.23). To this end, we split $O^N_s - O^N_{t_i}$ into two terms:

$$O^N_s - O^N_{t_i} = \int_0^s \Lambda^{-\frac{1}{2}} S_N(s-r)P_N dW(r) - S_N(s-t_i)O^N_{t_i} + (S_N(s-t_i) - I)O^N_{t_i} = \int_0^s \Lambda^{-\frac{1}{2}} S_N(s-r)P_N dW(r) + (S_N(s-t_i) - I)O^N_{t_i}.$$  

This implies that

$$\left\| \sum_{l=0}^m \int_{t_l}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s)P_N F'(u^N(t_l)) \left( O^N_s - O^N_{t_l} \right) ds \right\|_{L_2(\Omega,U)} \leq \left\| \sum_{l=0}^m \int_{t_l}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s)P_N F'(u^N(t_l)) \left( \int_{t_l}^{s} \Lambda^{-\frac{1}{2}} S_N(s-r)P_N dW(r) \right) ds \right\|_{L_2(\Omega,U)} + \left\| \sum_{l=0}^m \int_{t_l}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s)P_N F'(u^N(t_l)) \left( S_N(s-t_i) - I \right)O^N_{t_i} ds \right\|_{L_2(\Omega,U)} = J_{21} + J_{22}. \tag{4.24}$$

Thanks to properties of Itô stochastic integrals and (2.4), we arrive at

$$J_{21}^2 = \sum_{l=0}^m \left\| \int_{t_l}^{t_{l+1}} \Lambda^{-\frac{1}{2}} S(t_{m+1} - s)P_N F'(u^N(t_l)) \left( \int_{t_l}^{s} \Lambda^{-\frac{1}{2}} S_N(s-r)P_N dW(r) \right) ds \right\|^2_{L_2(\Omega,U)} \leq \tau \sum_{l=0}^m \int_{t_l}^{t_{l+1}} \left\| \Lambda^{-\frac{1}{2}} S(t_{m+1} - s)P_N F'(u^N(t_l)) \left( \int_{t_l}^{s} \Lambda^{-\frac{1}{2}} S_N(s-r)P_N dW(r) \right) \right\|^2_{L_2(\Omega,U)} \text{ d}rds \leq \tau \sum_{l=0}^m \int_{t_l}^{t_{l+1}} \E \left\| \Lambda^{-\frac{1}{2}} S(t_{m+1} - s)P_N F'(u^N(t_l)) \Lambda^{-\frac{1}{2}} S(s-r)P_N \right\|^2_{L_2(U)} \text{ d}rds \times \left\| \Lambda^{-\frac{1}{2}} P_N \right\|^2_{L_2(U)} \text{ d}rds. \tag{4.25}$$
Furthermore, (2.4) and (2.12) yield that
\[ \int_0^\infty \left( \sum_{l} \sum_{m} \sum_{m'} \sum_{l} \sum_{l} E \left| \psi \right|^\frac{2}{\alpha + 1} \right)^2 \leq \left( \sum_{l} \sum_{m} \sum_{m'} \sum_{l} \sum_{l} \left| \psi \right|^2 \right)^2 \leq C(s - r)^{2\beta}. \]

Furthermore, (2.11) and (2.12) yield that
\[ \left\| \Lambda^{\frac{2}{\alpha - 1}} P_N \right\|_{L^2(U)}^2 \leq \left\| \Lambda^{\frac{2}{\alpha - 1}} \right\|_{L^2(U)}^2 \cdot \left\| P_N \right\|_{L^2(U)}^2 \leq C < \infty, \quad \beta \in (0, \frac{1}{2}). \]  

Inserting the preceding two estimates into (4.25) thus shows for all \( \beta \in (0, \frac{1}{2}) \) that
\[ |J_{21}|^2 \leq C\epsilon^{2\beta + 2}. \]

This completes the estimate of \( J_{21} \). In the next step we start to estimate \( J_{22} \):
\[ J_{22} \leq \int_0^{t_{i+1}} \left( \sum_{l} \sum_{m} \sum_{m'} \sum_{l} \sum_{l} \left| \lambda^{\frac{2}{\alpha - 1}} \right|^2 \right)^2 \leq \left( \sum_{l} \sum_{m} \sum_{m'} \sum_{l} \sum_{l} \left| \lambda \right|^2 \right)^2 \leq C(s - r)^{2\beta}. \]

where \( \epsilon \in (0, 1) \) is arbitrarily small and where stability of \( \Lambda^{\frac{2}{\alpha}} S(t), P_N, \) self-adjointness of \( F'(u) \) and \( \Lambda \), and Hölder’s inequality were invoked. Recall that
\[ (S_N(s - t_i) - I) C_{l_i} = \left( S_N(s - t_i) - I \right) \int_0^{t_i} \Lambda^{\frac{2}{\alpha}} S_N(t_i - r) P_N dW(r) \]
\[ = \int_0^{t_i} \left( S(s - r) - S(t_i - r) \right) \Lambda^{\frac{2}{\alpha}} P_N dW(r) \]
for $s \in [t_i, t_{i+1}]$ and that $\|\Lambda^{-\frac{1+\epsilon}{2}}\|_{L^2(U)}^2 = \sum_{i=1}^{\infty} \lambda_i^{-1+\epsilon} = \pi^{-1-\epsilon} \sum_{i=1}^{\infty} i^{-1-\epsilon} \leq C < \infty$ for arbitrarily small $\epsilon \in (0, 1)$. Therefore it follows for arbitrarily small $\epsilon \in (0, 1)$ and $s \in [t_i, t_{i+1}]$ that

$$
\left\|\Lambda^{-\frac{1+\epsilon}{2}} (S_N(s-t_i) - I) \mathcal{O}_{c}^{N} \right\|_{L^2(U)}^2
\leq C \left( \int_{t_i}^{t_{i+1}} \left\|\left(S(s-r) - S(t_i-r)\right) \Lambda^{-\frac{1+\epsilon}{2}} \cdot \Lambda^{-\frac{1+\epsilon}{2}} P_N \left\|_{L^2(U)}^2 \right. \right. \right.
\left. \left. \left\|_{L^2(U)}^2 \cdot \left\|P_N \right\|_{L^2(U)}^2 \right\|_{L^2(U)}^2 \right) \right)^{\frac{1}{2}}
\leq C \left( \int_{t_i}^{t_{i+1}} \left\|\left(S(s-r) - S(t_i-r)\right) \Lambda^{-\frac{1+\epsilon}{2}} \left\|_{L^2(U)}^2 \cdot \left\|P_N \right\|_{L^2(U)}^2 \right\|_{L^2(U)}^2 \right) \right)^{\frac{1}{2}}
\leq C(s-t_i)^{1-\epsilon},
$$

(4.29)

where the Burkholder-Davis-Gundy type inequality (cf. Lemma 7.2), (4.1) and (4.7) were used. In addition, we denote $C((0, 1), \mathbb{R})$ by the Banach space consisting of continuous functions from $(0, 1)$ to $\mathbb{R}$ equipped with the norm $\|\varphi\|_{C((0, 1), \mathbb{R})} = \sup_{x \in (0, 1)} |\varphi(x)|$. Then combining the equivalent Sobolev-Slobodeckij norm (see, e.g., [25]) and (A.46) in [8] shows for all $u \in \mathcal{D}(\Lambda^{-\frac{1+\epsilon}{2}})$, $\varphi \in \mathcal{D}(\Lambda^{-\frac{1+\epsilon}{2}})$ with $\epsilon \in (0, 1)$ that

$$
\|\Lambda^{-\frac{1+\epsilon}{2}} F'(u)\varphi\| = \|F'(u)\varphi\|_{L^2(U)}
= \|F'(u)\varphi\| + \left( \int_{0}^{1} \int_{0}^{1} \frac{1}{|x-y|^{2+\epsilon}} \left| \frac{\partial f}{\partial y}(x, u(x)) \varphi(x) - \frac{\partial f}{\partial y}(y, u(y)) \varphi(y) \right|^2 dy dx \right)^{\frac{1}{2}}
\leq L \|\varphi\| + \sqrt{3} \left( \int_{0}^{1} \int_{0}^{1} \frac{1}{|x-y|^{2+\epsilon}} \left| \frac{\partial f}{\partial y}(x, u(x)) \varphi(x) - \frac{\partial f}{\partial y}(y, u(y)) \varphi(y) \right|^2 dy dx \right)^{\frac{1}{2}}
+ \sqrt{3} \left( \int_{0}^{1} \int_{0}^{1} \frac{1}{|x-y|^{2+\epsilon}} \left| \frac{\partial f}{\partial u}(y, u(y)) (\varphi(x) - \varphi(y)) \right|^2 dy dx \right)^{\frac{1}{2}}
\leq L \|\varphi\| + \sqrt{3} L \left( \int_{0}^{1} \int_{0}^{1} \left| x-y \right|^\epsilon dy dx \right)^{\frac{1}{2}} \|\varphi\|_{C((0, 1), \mathbb{R})}
+ \sqrt{3} L \left( \int_{0}^{1} \int_{0}^{1} \left| u(x) - u(y) \right|^2 dy dx \right)^{\frac{1}{2}} \|\varphi\|_{C((0, 1), \mathbb{R})}
+ \sqrt{3} L \left( \int_{0}^{1} \int_{0}^{1} \left| \varphi(x) - \varphi(y) \right|^2 dy dx \right)^{\frac{1}{2}} \|\varphi\|_{C((0, 1), \mathbb{R})}
\leq L \|\varphi\| + \sqrt{3} L \|\varphi\|_{C((0, 1), \mathbb{R})} + \sqrt{3} L \|\varphi\|_{C((0, 1), \mathbb{R})} + \sqrt{3} L \|\varphi\|_{C((0, 1), \mathbb{R})}
\leq C \left( \|u\|_{L^\infty} + 1 \right) \|\varphi\|_{L^\infty},
$$

(4.30)
where the fact that $\mathcal{D}(\Lambda_{\frac{1}{2}}^{1+}) \subset C((0,1), \mathbb{R})$ continuously for $\epsilon \in (0,1)$ by Sobolev embedding theorem and (1.3) were used. Further, in view of (3.8) we get

$$\left\| \sup_{\|\psi\| \leq 1} \left\| \Lambda_{\frac{1}{2}}^{1+} F'(u^N(t)) \Lambda_{-\frac{1}{2}}^{1-} \psi \right\|_{L_2(\Omega, \mathbb{R})} \leq C \left\| u^N(t) \right\|_{L_2(\Omega, \mathbb{R})} + 1 \left\| \frac{u^N(t)}{\epsilon} \right\|_{L_2(\Omega, \mathbb{R})} \leq C \left\| u^N(t) \right\|_{L_2(\Omega, \mathbb{H}^{1/2})} + C < +\infty. \quad (4.31)$$

Then inserting (4.29) and (4.31) into (4.28) gives for arbitrarily small $\epsilon \in (0,1)$ that

$$J_{22} \leq \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} C(s - t_l)^{1-\epsilon} ds \leq C \tau^{1-\epsilon}. \quad (4.32)$$

Thus plugging (4.27), (4.32) into (4.24) and setting $\beta = \frac{1-\epsilon}{2}$ in (4.23) show for arbitrarily small $\epsilon \in (0,1)$ that

$$J_2 \leq C \tau^{1-\epsilon}, \quad (4.33)$$

which together with (4.21) implies from (4.20) that

$$J \leq C \tau^{1-\epsilon}. \quad (4.34)$$

for arbitrarily small $\epsilon \in (0,1)$. Hence one can deduce from (4.18) for arbitrarily small $\epsilon \in (0,1)$ and all $m = 0, 1, \cdots, M - 1$ that

$$\| u_m^N - u^N(t_m) \|_{L_2(\Omega, U)} \leq L \tau \sum_{l=0}^{m} \left\| u_l^N - u^N(t_l) \right\|_{L_2(\Omega, U)} + C \tau^{1-\epsilon}. \quad (4.35)$$

This recurrence equation obviously suggests $\| u_m^N - u^N(t_m) \|_{L_2(\Omega, U)} < \infty$ for all $m = 1, 2, \cdots, M$ and thus the discrete Gronwall inequality applied to (4.35) gives for all $m = 1, 2, \cdots, M$ and arbitrarily small $\epsilon \in (0,1)$ that

$$\| u_m^N - u^N(t_m) \|_{L_2(\Omega, U)} \leq C \tau^{1-\epsilon}. \quad (4.36)$$

Finally, inserting (4.36) into (1.13) gives overall discretization error (4.6) for the first numerical scheme (4.1).

For the other scheme (4.2), direct calculation similar to (4.10) shows

$$X_m^N - X^N(t_{m+1}) = \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \left( E_N(\tau) F_N(X_l^N) - E_N(t_{m+1} - s) F_N(X_N(s)) \right) ds,$$

which implies for all $m = 0, 1, 2, \cdots, M - 1$ that

$$\begin{align*}
    u_m^N &- u^N(t_m+1) \\
    &= \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \left( \Lambda_{\frac{1}{2}}^{1+} S_N(\tau) P_N F(u_l^N) - \Lambda_{-\frac{1}{2}}^{1-} S_N(t_{m+1} - s) P_N F(u^N(s)) \right) ds \\
    &= \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \left( \Lambda_{-\frac{1}{2}}^{1-} S(\tau) P_N F(u_l^N) - \Lambda_{-\frac{1}{2}}^{1-} S(t_{m+1} - s) P_N F(u^N(s)) \right) ds \\
    &= \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \Lambda_{-\frac{1}{2}}^{1-} S(\tau) P_N \left( F(u_l^N) - F(u^N(s)) \right) ds \\
    &+ \sum_{l=0}^{m} \int_{t_l}^{t_{l+1}} \left( \Lambda_{-\frac{1}{2}}^{1-} S(\tau) - \Lambda_{-\frac{1}{2}}^{1-} S(t_{m+1} - s) \right) P_N F(u^N(s)) ds.
\end{align*} \quad (4.37)$$
Therefore it follows for all \( m = 0, 1, 2, \cdots, M - 1 \) that

\[
\left\| u_{m+1}^N(t_{m+1}) - u^N(t_{m+1}) \right\|_{L^2(\Omega,U)} \\
\leq \left\| \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \Lambda^{-\frac{s}{2}} S(\tau) P_N \left( F(u_i^N) - F(u^N(s)) \right) ds \right\|_{L^2(\Omega,U)} \\
+ \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \left\| \left( \Lambda^{-\frac{s}{2}} S(\tau) - \Lambda^{-\frac{s}{2}} S(t_{m+1} - s) \right) P_N F(u^N(s)) \right\|_{L^2(\Omega,U)} ds \\
\leq \left\| \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \Lambda^{-\frac{s}{2}} S(\tau) P_N \left( F(u_i^N) - F(u^N(s)) \right) ds \right\|_{L^2(\Omega,U)} + C\tau \\
\leq \left\| \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \Lambda^{-\frac{s}{2}} S(\tau) P_N \left( F(u_i^N) - F(u^N(t_i)) \right) ds \right\|_{L^2(\Omega,U)} + J' + C\tau \\
\leq \Delta t \sum_{i=0}^{m} \left\| u_i^N - u^N(t_i) \right\|_{L^2(\Omega,U)} + J' + C\tau,
\]

where we denote

\[
J' = \left\| \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \Lambda^{-\frac{s}{2}} S(\tau) P_N \left( F(u^N(s)) - F(u^N(t_i)) \right) ds \right\|_{L^2(\Omega,U)},
\]

and additionally (2.10), (3.11) and (4.7) were used in the estimates. Note that the term \( J' \) here only depends on \( u^N(r), r \in [0T], \) not depending on temporal discretizations. Moreover, note that \( J' \) is almost a copy of the term \( J \) in (4.1), with only \( S_N(t_{m+1} - s) \) replaced by \( S_N(\tau) \). Thus, repeating exactly the same way as before one can easily obtain for arbitrarily small \( \epsilon \in (0, 1) \) that

\[
J' \leq C\tau^{1-\epsilon}.
\]

Inserting (4.40) into (4.38) and an application of the discrete Gronwall inequality give the required estimate (4.6) for scheme (4.2). This finishes the proof of Theorem 4.1.

5. Numerical tests. As the first numerical example, we consider a nonlinear SWE, the Sine-Gordon equation driven by space-time white noise:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \sin(u) + \dot{W}, \quad 0 < t \leq 1, \quad x \in (0, 1) \\
u(0, x) &= \frac{\partial u}{\partial t}(0, x) = 0, \quad x \in (0, 1), \\
u(t, 0) &= u(t, 1) = 0, \quad t > 0.
\end{align*}
\]

The corresponding deterministic equation is used to describe the dynamics of coupled Josephson junctions driven by a fluctuating current source[4]. In what follows, we use various numerical schemes to numerically solve (5.1) and compare the corresponding computational errors. To this end, "exact" solutions such as \( u(t) \) and \( u^N(t) \) are always needed, which in the following numerical results are identified with the numerical solutions produced by (4.1) using very small stepsize. Moreover, the expectations are approximated by computing averages over samples.

The MATLAB code of one-path simulation of (5.1) are presented in Fig. 5.1. Here we evoke built-in functions “dst” and “idst” in MATLAB, which are based on fast Fourier transform, to numerically approximate inner products in (4.3)-(4.4) at
\begin{verbatim}
M = 64; N = M^2; T = 4; tau = T/M;
A = pi^2*(1:N).^2; sqrtA = sqrt(A);
CosA = cos(sqrtA*tau); SinA = sin(sqrtA*tau);
SW1 = sqrt((tau-sin(2*sqrtA*tau)./(2*sqrtA))./ (2*A));
SW2 = sqrt((tau+sin(2*sqrtA*tau)./(2*sqrtA))./ 2);
f = @(x) sin(x);
Y1 = zeros(1,N); Y2 = zeros(1,N);
for m = 1:M
    Rd=randn(1,N);
dW1 = SW1.*Rd; dW2 = SW2.*Rd;
y1 = dst( Y1 ) * sqrt(2);
Fy1 = idst( f(y1) )/sqrt(2);
Y10 = Y1;
Y1 = CosA.*Y1 + 1./sqrtA.*SinA.*Y2 + 1./A.*(1-CosA).*Fy1 +dW1;
Y2 = -sqrtA.*SinA.*Y10 + CosA.*Y2 + 1./sqrtA.*SinA.*Fy1 + dW2;
end
plot((0:N+1)/(N+1),[0,dst(Y1)*sqrt(2),0]);
\end{verbatim}

Fig. 5.1. MATLAB code for one-path simulation of (5.1) using (4.1)

cheap costs. The aliasing errors are then neglected. More details and remarks on similar implementation can be found in [28, Section 5.1]

Now let us start with tests on the convergence rates. At first, we consider the spatial convergence rate of the spectral Galerkin method. Fig. 5.2 depicts the spatial approximation errors $\|u(T) - u^N(T)\|_{L^2(\Omega,U)}$ against $\frac{1}{N}$ on a log-log scale, with $T = 1$ and $N = 2^k, k = 4,5,...,9$. One can detect that the errors decrease at a slope of $\frac{1}{2}$ as $\frac{1}{N}$ goes down, which is consistent with the previous assertion on the spatial convergence rate $\frac{1}{2}$— (here and below we write $b$— for the convergence order if the convergence order is higher than $b - \epsilon$ for arbitrarily small $0 < \epsilon < b$).

![Graph showing spatial errors for the spectral Galerkin method applied to SWE](image)

Fig. 5.2. Spatial errors for the spectral Galerkin method applied to SWE (5.1).

With $N = 100$ fixed, below we shall compare several time integrators applied to approximate $u^N(T), T = 1$ in (5.3). In Fig. 5.3 we present approximation er-
rors caused by different temporal discretizations, including the linear implicit Euler scheme, the Crank-Nicolson-Maruyama scheme, the stochastic trigonometric method and our new scheme (4.1). From the left picture of Fig. 5.3, one can easily observe that the new scheme (4.1) performs much better than the other ones. On the one hand, it produces significantly smaller errors than the other three schemes. On the other hand, computational errors of scheme (4.1) decrease faster, i.e., at a rate of 1, comparing with those of others. To clearly show convergence rates of the three existing schemes, we change the scales of coordinate axes and discard computational errors of scheme (4.1). In the right picture of Fig. 5.3, one can find that the approximation errors of the linear implicit Euler (LIE) scheme [19] decrease at a rate of $1^{1/4}$, errors of the Crank-Nicolson-Maruyama (CNM) scheme [11, 19] at a rate of $1^{1/3}$, and errors of the stochastic trigonometric method (STM) [5] at a rate of $1^{1/2}$. These numerical performances coincide with the theoretical findings, see Theorem 4.1 of this work, [19, Theorem 4.12] and [5, Theorem 4.1].

For the second example, we look at another nonlinear SWE as follows

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{1+u}{1+u^2} + \dot{W}, \quad 0 < t \leq 1, \; x \in (0,1) \\
u(0, x) &= 0, \; \frac{\partial u}{\partial x}(0, x) = 1, \; x \in (0,1), \\
u(t, 0) &= \nu(t, 1) = 0, \; t > 0.
\end{align*}
$$

(5.2)

Subsequently, we focus on the overall computational efforts of various fully discrete schemes, with spectral Galerkin discretizations in space. We take the number of realizations of independent random variables needed for approximations as a measure for computational efforts. To get the approximations $u^N_M$ via the numerical schemes mentioned above, one needs to generate $M \times N$ random variables. Recall that the Crank-Nicolson-Maruyama (CNM) scheme and the stochastic trigonometric method (STM) achieve convergence rates of $1^{1/3}$ and $1^{1/2}$ in time, respectively. In space, the spectral Galerkin method always promises convergence order of $1^{1/2}$. In order to balance the errors in space and in time, we set $M = N^{1/3}$ for CNM scheme and $M = N$ for STM. Similarly, one should set $M = N^{1/2}$ for our schemes (4.1) and (4.2). With these settings, the four schemes all result in the overall approximation error $O((1/M)^{1/2})$. The overall approximation errors produced by different schemes are listed in Table 5.1-5.3.
Also, in Fig. 5.4 we plot these approximation errors against numbers of used random variables. Overall approximation errors of the four schemes all decrease at expected rates, as $N$ increases. For example, overall computational errors of schemes (4.1) and (4.2) both enjoy decrease at a slope of $-\frac{1}{3}$, which is an immediate consequence of our previous setting $M = N^{1/2}$. Similarly, overall errors of the other two methods exhibit decay rates of $-\frac{1}{5}$ and $-\frac{1}{4}$ as expected. Given a precision $\varepsilon = 0.02$, we are to compare the required computational costs for the above four schemes. One can first detect that the Crank-Nicolson-Maruyama scheme achieves the given precision $\varepsilon = 0.02$ in the case $N = 2^8, M = 2^{12}$ and requires to generate $2^{20} = 1048576$ random variables. For the stochastic trigonometric method, $2^{18} = 262144$ ($N = M = 2^9$) random variables are needed to promise the error 0.01879 satisfying the precision. Our proposed schemes (4.1) and (4.2), however, achieve the given precision as $N = 2^8, M = 2^4$, which results in generation of only $2^{12} = 4096$ random variables. It turns out that, with the same precision, the proposed schemes (4.1) and (4.2) reduce the number of used random variables greatly and thus improve computational efficiency significantly.

| $N = 2^4$ | $N = 2^6$ | $N = 2^8$ | $N = 2^{10}$ |
|------------|-----------|-----------|-------------|
| 0.14046    | 0.06640   | 0.03244   | 0.01501     |

| $N = 2^6$ | $N = 2^7$ | $N = 2^8$ | $N = 2^9$ |
|-----------|-----------|-----------|-----------|
| 0.05434   | 0.03841   | 0.02690   | 0.01879   |

| $N = 2^4$ | $N = 2^6$ | $N = 2^8$ | $N = 2^{10}$ |
|-----------|-----------|-----------|-------------|
| Scheme (4.1) | 0.05651   | 0.02831   | 0.01396     | 0.00659     |
| Scheme (4.2) | 0.05756   | 0.02838   | 0.01393     | 0.00657     |

6. Conclusion remarks. Finally, we would like to make some conclusion remarks. In this work, two higher order fully discrete schemes have been devised for stochastic wave equation with additive space-time white noise. Both theoretical convergence results and numerical experiments show that the proposed schemes can significantly reduce computational costs, compared with existing methods for SWE. This is due to including more information on stochastic convolutions by exploiting linear functionals of noises in the presented schemes, which leads to a breakthrough of convergence rates in time. We remark that the new schemes can be also successfully applied to SWE driven by a general additive noise, i.e., $Q$-Wiener process, on the condition that $\Lambda$ and $Q$ have a common eigenbasis. In the present setting, we always assume that the drift term $f$ satisfies globally Lipschitz conditions (cf. (2.10)), which excludes many important model equations in applications. In the future, we plan to address this issue and to investigate strong convergence of numerical schemes for nonlinear SWE with non-globally Lipschitz coefficients.
Fig. 5.4. The overall approximation errors against number of used random variables.

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