Abstract

In this paper we obtain quantitative, non-asymptotic, and data dependent, Bernstein-von Mises type bounds on the normal approximation of the posterior distribution in exponential family models with arbitrary centring and scaling. Our bounds, which are stated in the total variation and Wasserstein distances, are valid for univariate and multivariate posteriors alike, and do not require a conjugate prior setting. They are obtained through a refined version of Stein’s method of comparison of operators that allows for improved dimensional dependence in high-dimensional settings, and that may also be of interest in other problems. Our approach is rather flexible and in certain settings allows for the derivation of bounds with rates of convergence faster than the usual $O(n^{-1/2})$ rate (when $n$ is the sample size). We illustrate our findings on a variety of exponential family distributions, including the Weibull, multinomial and linear regression with unknown variance. The resulting bounds have an explicit dependence on the prior distribution and on sufficient statistics of the data from the sample, and thus provide insight into how these factors may affect the quality of the normal approximation. Insights that can be gleaned from our examples include identification of conditions under which faster $O(n^{-1})$ convergence rates occur for Bernoulli data, illustrations of how the quality of the normal approximation is influenced by the choice of standardisation, and dimensional dependence in high-dimensional settings.

Keywords: Exponential family; posterior distribution; prior distribution; Bayesian inference; normal approximation; Stein’s method

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1 Introduction

The Bernstein-von Mises (BvM) Theorem is a cornerstone in Bayesian statistics. Loosely put, this theorem reconciles Bayesian and frequentist large sample theory by guaranteeing that suitable scalings of posterior distributions are asymptotically normal. In particular, this implies that the contribution of the prior vanishes in the asymptotic posterior. Beyond its philosophical implications, this result is central to Bayesian inference, as it provides theoretical justification for Laplace approximation which consists in using Gaussian approximations for (typically intractable) posterior integrals. The BvM Theorem is named after the works [4, 50], but can be traced back to work by Laplace [35]; we refer to [39, Section 8] for an overview, including a historical tour d’horizon.

There exist many different versions of the BvM Theorem; one of the most basic forms goes as follows. Let $\theta \in \mathbb{R}^d \sim \pi_0$ follow the prior distribution, and conditional on $\theta$, let $X_1, X_2, \ldots$ be independent random variables in $\mathbb{R}^r$ from a probability measure $P_\theta$ belonging to a model ($P_\theta : \theta \in \Theta$). For $n \geq 1$ let $x = (x_1, \ldots, x_n)$ be a realisation of $X = (X_1, \ldots, X_n)$ and let $P_\theta(x|\theta)$ denote the posterior distribution of $\theta$ given $X = x$. Let $\hat{\theta}_n$ denote the posterior mode (we call it the MAP, for Maximum A Posteriori estimator) and $I(\hat{\theta}_n)$ the model Fisher information matrix of $P_\theta$ evaluated at
The BvM Theorem then assures us that, under regularity conditions (including invertibility of the Fisher information $I(\hat{\theta}_n)$, quadratic mean differentiability, identifiability),
\begin{equation}
d_{\text{TV}}\left(P_0(\theta | X), N(\hat{\theta}_n, I^{-1}(\hat{\theta}_n)/n)\right) = o_{P_{\theta_0}}(1)
\end{equation}
where $d_{\text{TV}}(P, Q)$ denotes the total variation distance between distributions $P$ and $Q$, and the convergence holds in probability with respect to the law $P_{\theta_0} = P_{\theta_0} \otimes \cdots \otimes P_{\theta_0}$ of the data generating process when the true parameter is $\theta_0$. A similar conclusion holds for a centring around the maximum likelihood estimate (MLE) and, more generally, around any efficient estimator of the true parameter; see [49, Section 10.2] for details and a proof.

Approximate normality of posterior distributions still holds, with a different limiting covariance matrix, in the misspecified model setting that the true underlying distribution $P_0$ is no longer of the form $P_{\theta_0}$. An early reference on misspecified BvM Theorems is [33] where the corresponding version of (1) is obtained under a Local Asymptotic Normality (LAN) condition; see also [34] for a BvM theorem in the context of misspecified, non-i.i.d., hierarchical models, and [7] for an overview of BvMs under misspecification. Extensions to the semiparametric setting have been obtained e.g. in [6] under a LAN condition, or in [15] for smooth functionals, both for i.i.d. and non-i.i.d. samples. Versions of the BvM result also hold true in nonparametric settings, see e.g. [36, 13, 14]. We refer to [42] for an overview of semi- and nonparametric BvMs. The high-dimensional setting is also well studied, see e.g. [23, 10, 28, 9, 27, 43]. Almost sure versions are obtained in [39] for various commonly encountered situations. Finally, extensions of the BvM statement beyond normal approximation have also been considered recently, see e.g. [19] wherein skew normal approximations are studied.

In practice, there is a need for non-asymptotic statements which, arguably, are of a different nature than (1). Here, the objective is to study, for a fixed sample, the proximity between the standardised posterior distribution (conditioned on the observation) and a standard normal distribution. An important reference on this topic is [41] where a finite sample semiparametric version of the BvM theorem is obtained for non-informative and flat Gaussian priors; it is seen furthermore that, under certain conditions, the normal approximation holds as soon as $d^2 \log d \ll n$ (which is an improvement from [23] where the condition $d^4 \log d \ll n$ was identified). Similar spirited fixed sample bounds are also obtained in [40] for Gaussian priors in high-dimensional or nonparametric setups, while [45] considers Gaussian priors and log-concave likelihoods; in [17] the proximity is expressed in terms of the Kullback-Leibler divergence. Recently, [29] proposed fixed sample bounds of the form
\begin{equation}
\mathcal{D}(P^*(\theta | X), N(0, I)) \leq \mathcal{U}_n(x)
\end{equation}
with $\mathcal{D}(P, Q)$ standing for either the total variation distance $d_{\text{TV}}(P, Q)$, the Wasserstein distance $d_{\text{Wass}}(P, Q)$, or the difference of covariances between laws $P$ and $Q$, with $P^*(\theta | X)$ the law of the standardised posterior centred around the MAP or the MLE, and with $\mathcal{U}_n(x)$ a statistic. The quantity $\mathcal{U}_n(x)$ is, in principle, computable explicitly, although exact computation turns out to be challenging in any specific example. Concurrently, the preprints [30, 32] prove that the condition $d^2 \ll n$ is necessary for total variation distance and obtain bounds with explicit dependence on the dimension for specific examples (multinomial, logistic regression). The papers [29, 30, 32] also provide excellent, detailed, and comparative literature reviews on BvM Theorems and Laplace approximation.

In this paper, we obtain bounds of the same spirit as (2), allowing to assess, at fixed sample size and sample, the quality of normal approximation of the posterior under the assumption that the observations are independent (but not necessarily identically distributed) from an exponential family. We make no assumptions (other than a mild differentiability assumption) on the prior $\pi_0$, and consider general centring and scaling. Specifically, the question we shall address can be stated as follows.

**Question:** Given $\theta_0 \in \mathbb{R}^d$ and $K$ a symmetric $d \times d$ matrix, let $\theta \sim P_2(\cdot | X)$ and consider
\begin{equation}
\theta^* = K(\theta - \theta_0).
\end{equation}
Under what conditions on the prior, on the model, on the observation $x$, and on the standardisation $(\theta_0, K)$ is the law of $\theta^*$ close to the standard normal distribution?

We will answer this question both in total variation distance and in Wasserstein (a.k.a. Kantorovitch) distance. In dimension 1, our bounds are sharp (sometimes even equalities); the following example, detailed in Section 3.4, is illustrative of the conclusions we obtain.
Example 1 (Weibull data). Consider an observation sampled independently from Weibull data with density \( p_1(x|\ell, m) = (m/\ell^m)x^{m-1}\exp(-x/\ell^m) \) on the positive real line. We suppose \( m \) is known, and \( \ell \) is the parameter of interest. Fix, for \( \tau_1 > 0 \) and \( \tau_2 > -1 \), a (conjugate) gamma prior on \( \theta = \ell^{-m} \) of the form \( \pi_0(\theta) \propto e^{-\eta(\theta)} \), with \( \eta(\theta) = \tau_1 \theta - \tau_2 \log \theta \) (for \( \tau_1 = 0 \) and \( \tau_2 = -1 \) one obtains the improper Jeffreys prior for which our bounds also hold when \( n \geq 2 \)). We consider the so-called MAP standardisation; in the notation from (3), we set \( \theta_{\text{MAP}}^* = K(\theta - \theta_0) \) with \( \theta_0 = \theta \) the posterior mode and \( K = (\lambda''(\theta))^{1/2} \) where \( \lambda(\theta) = \tau_1 \theta - (\tau_2 + n) \log \theta \). Then our bounds from Section 3.4 are

\[
d_{\text{TV}}(\theta_{\text{MAP}}^*, N) \leq \sqrt{\frac{\pi}{8}} \frac{1}{\sqrt{n + \tau_2}}, \quad d_{\text{Wass}}(\theta_{\text{MAP}}^*, N) = \frac{1}{\sqrt{n + \tau_2}}.
\]

This setting is also considered in [29, Example 5.2] where a Wasserstein upper bound is obtained; in contrast, we achieve equality. Our above bounds do not depend on the sample nor on the parameter \( \tau_1 \); however, letting \( \tau_2 \) be large artificially improves the quality of the Gaussian approximation. In case we standardise around the MLE with a uniform prior, our bounds lead to a similar conclusion as above (in fact, the same bounds with \( \tau_2 \) set to 0), whereas taking a conjugate prior leads to (rather inelegant) bounds on the total variation and Wasserstein distances which now depend on all the parameters and the data \( x \) in an intricate way.

In dimension \( d \geq 2 \) our bounds remain explicitly computable. The different expressions can also be readily gauged in terms of the “typical” behavior of samples, hereby shedding new light for instance on the dimensional dependence in the high-dimensional setting. The next example is illustrative.

Example 2 (Categorical data). In Section 3.5, we consider i.i.d. multinomial data with parameter \( p = (p_1, \ldots, p_k)^T \) and individual likelihood \( p_1(x|p) = \prod_{j=1}^k p_j^{x_j} \) where \( x = (x_1, \ldots, x_k) \in \{0, 1\}^k \) with \( \sum_{j=1}^k x_j = 1 \), and \( 0 < p_1, \ldots, p_k < 1 \) with \( \sum_{j=1}^k p_j = 1 \). We fix a conjugate Dirichlet prior on \( p \) of the form \( \pi_0(p) \propto \prod_{j=1}^k (1 - \sum_{j=1}^{k-1} p_j) \tau_j \prod_{j=1}^{k-1} p_j^{r_j} \) for some parameters \( \tau_1, \ldots, \tau_k > -1 \). The natural parameter is \( \theta = (\theta_1, \ldots, \theta_k)^T \), where \( \theta_j = \log(p_j/p_k) \), \( j = 1, \ldots, k-1 \), and we centre around the MAP as in Example 1. Scaling with the (unique symmetric) square root of the Hessian evaluated at the MAP, we prove that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
d_{\text{TV}}(\theta_{\text{MAP}}^*, N) \leq \frac{C_1 k}{\sqrt{\min \pi}}, \quad d_{\text{Wass}}(\theta_{\text{MAP}}^*, N) \leq \frac{C_2 k^{3/2}}{\sqrt{\min \pi}},
\]

where \( \min \pi = \min_{1 \leq u \leq k} (n \bar{x}_u + \tau_u) \), and the Wasserstein distance bound is derived under the mild assumption that \( \min \pi > \max\{8, 8k\} \) for some constant \( c > 0 \). Moreover, we obtain the estimate \( C_1 \leq 8.46 \), and we also show that \( C_2 \leq 7.40 \) under the mild assumption that \( \sqrt{\min \pi} \geq 7.40k/\sqrt{2} \). Many bounds in total variation distance are available for BdMs with multinomial data, see e.g. [16, 23, 33, 37, 71]; as discussed in [16, 71], bounds on different standardisations are not directly comparable. We are not aware of a Wasserstein bound (though [31] bounds the difference of means).

In dimension 1 (i.e. when \( k = 2 \), with Bernoulli data) our approach leads to refined upper and lower bounds which identify the role of the various sufficient statistics in the convergence, and also display the effect of the value of the ground truth success probability \( p^* \) on the quality of the approximations. For instance, we show the curious fact that when \( p^* = (3 \pm \sqrt{3})/6 \) then the upper and lower bounds on the Wasserstein distance coincide, leading to an explicit expression in this specific case. Moreover, we also observe a phenomenon which, to the best of our knowledge, has not been reported previously: in case of a ground truth success probability \( p^* = 0.5 \), the choice of prior has an influence on the asymptotic behavior of the posterior, in the sense that the rate of convergence of the posterior towards the normal is no longer \( n^{-1/2} \) but rather \( n^{-1} \), at least in case of a MAP centring with conjugate prior or an MLE centring with flat prior. When considering an MLE centring with conjugate prior, the rate of convergence only changes when \( p^* = 0.5 \) for specific choices of the prior parameters. All our claims are supported by simulations, see Figures 1 and 2; Section 3.7 has more details.

A complete statement of our bounds requires notations that would be cumbersome to introduce already, and we refer the reader to the forthcoming Theorem 4. In a nutshell, our bounds are exactly of the form (2) with \( U_n(x) \) depending on several very natural quantities: (i) a first order term on the log-likelihood measuring how close \( \theta_0 \) is to a critical point (this term disappears when \( \theta_0 \) is the MAP); (ii) a second order term reflecting how close \( K \) is to the Hessian (this term also disappears...
when \( \theta_0 \) is the MAP); (iii) a third order term measuring the flatness of the log-likelihood around \( \theta_0 \); (iv) remainder terms (which are negligible in all our examples). In all these quantities the dependence on the data is through sufficient statistics only. We could in principle work from here, as in most literature on the topic, by identifying abstract conditions on the behavior of the log-likelihood which guarantee correct scaling with sample size and dimension; this would come at the cost of transparency and interpretability. We therefore choose rather to focus on specific concrete settings for illustrating the power of our approach.

The method of proof rests on Stein’s method of comparison of operators, see e.g. [38]. This method provides bounds on integral probability metrics (see, e.g., [52]), including the total variation and Wasserstein distances) between arbitrary probability measures under regularity conditions on their unnormalised densities. Using results on the regularity of solutions to the standard normal Stein equation from [22], we will show via an application of Stein’s method of comparison of operators that if \( X \) has differentiable density \( p^X \) and \( N \) is standard normal then (see Lemma [4])

\[
d_{\text{Wass}}(X, N) \leq \sup_{f \in \mathcal{C}_1} |\mathbb{E} [(\nabla \log p^X(X) + X, \nabla f(X))]| \tag{5}
\]

with \( \mathcal{C}_1 \) the collection of functions \( f : \mathbb{R}^d \to \mathbb{R} \) which are twice differentiable with bounded second derivative and such that \( \nabla f(x) \) lies in the unit cube \( \mathcal{C}_1 \) in \( \mathbb{R}^d \), and (see Lemma [6])

\[
d_{\text{TV}}(X, N) \leq C \sup_{f \in \mathcal{C}_1} |\mathbb{E} [(\nabla \log p^X(X) + X, \nabla f(X))]| \tag{6}
\]

with \( C \) an explicit constant and \( \mathcal{G}_1 \) the collection of functions \( f : \mathbb{R}^d \to \mathbb{R} \) which are twice differentiable with bounded first and second derivatives and such that \( \nabla f(x) \) lies in the unit ball \( \mathcal{S}_1 \) in \( \mathbb{R}^d \).

The Wasserstein distance bound \([5]\); while not explicitly stated in the literature, follows directly from existing results. In contrast, the total variation distance bound \([4]\) is genuinely novel. It depends on new estimates for the solution of the standard normal Stein equation, which provide tighter control over \( \nabla f \) in the total variation framework (see Lemma [3]). These estimates allow the supremum in \([4]\) to be taken over a smaller set than what current results permit. This shift from a unit-cube constraint for \( \mathcal{C}_1 \) in \([5]\) to a unit-ball constraint for \( \mathcal{G}_1 \) in \([6]\) ensures an improved dimensional dependence for our bound on the total variation distance. For instance, the Wasserstein distance bound in \([4]\) (which follows from \([5]\)) scales at a higher rate in the dimension than the total variation distance bound in \([4]\) (which follows from \([6]\)). To our knowledge, standard treatments of Stein’s method skipped or simplified this aspect, thus missing the improved dependence on the dimension. This stronger control on \( \nabla f \) may also prove valuable for applications of Stein’s method in other contexts, such as high-dimensional limit theorems or concentration-of-measure inequalities.

The paper is structured as follows. Section 2 contains the core of the paper: the detailed setting and assumptions in Section 2.2, the Stein’s method arguments and related general bounds in Section 2.3 and abstract bounds for BvM-type settings in Section 2.4. Section 3 contains various applications, namely in the univariate case: Bernoulli data (Section 3.1), Poisson data (Section 3.2), normal with known mean data (Section 3.3), Weibull data (Section 3.4), and in the multivariate case: multinomial data (Section 3.5), normal data (Section 3.6) and a linear regression setting (Section 3.7). The Appendix contains further results and the less illuminating technical proofs.

2 Main results

2.1 Notations

Vectors \( t \in \mathbb{R}^d \) are column vectors; the transpose \( t^\top \) is a row vector. Given two vectors \( r, s \in \mathbb{R}^d \) we write \( r \otimes s \) as the matrix with components \( r_u s_v \), for \( 1 \leq u, v \leq d \); higher order tensor products are defined similarly. For two tensors \( R, S \) of the same dimensions we write \( \langle R, S \rangle = \sum_{i_1, \ldots, i_k} R_{i_1 \ldots i_k} S_{i_1 \ldots i_k} \), the sum being taken over all possible indices. If \( t \in \mathbb{R}^d \) we write \( \| t \|_2 = \sqrt{t^\top t} \) for its Euclidean norm. If \( R \) is a symmetric \( d \times d \) matrix we write

\[
\| R \|_{\text{diag}} = \sqrt{\max_{1 \leq u \leq d} |R_{uu}|} \quad \text{and} \quad \| R \|_{\text{sp}} = \sup \{ x^\top R y | x, y \in \mathbb{R}^d \text{ and } \| x \|_2 = \| y \|_2 = 1 \}
\]
The Laplacian operator is denoted $\Delta$, and the Hessian of a twice differentiable function $D$ is the symmetric matrix $H_g(t) = D^2 g(t) = (\partial^2_{u,v} g(t))_{1 \leq u,v \leq d}$. If $H_g(t)$ is invertible, we write $F_g(t)$ for its inverse Hessian at $t$. If in addition $H_g(t)$ is positive definite, it has a unique invertible symmetric square root which we denote $K_g(t)$; we denote the inverse of this last matrix $J_g(t)$. In other words,

$$H_g(t) = D^2 g(t), \quad K_g(t) = (H_g(t))^{1/2}, \quad F_g(t) = (H_g(t))^{-1} \quad \text{and} \quad J_g(t) = (H_g(t))^{-1/2}. \quad (7)$$

We denote the entries of these matrices $H^q_{u,v}(t) = \partial^2_{u,v} g(t), K^q_{u,v}(t), F^q_{u,v}(t), \text{and } J^q_{u,v}(t)$ for $1 \leq u, v \leq d$, respectively.

We shall bound so-called Integral Probability Metrics (IPMs) which are of the form $d_{\mathcal{F}}(X,Y) = \sup_{\phi \in \mathcal{F}} |E[\phi(X)] - E[\phi(Y)]|$, where $\mathcal{F}$ is a class of test functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\phi \in L^1(X) \cap L^2(Y)$, see e.g. [2]. Particular instances are $\mathcal{F}_\text{Wass} = \{\phi : \mathbb{R}^d \to \mathbb{R} : \phi \text{ is Lipschitz, } \|\phi\|_{\text{Lip}} \leq 1\}$, the class of Lipschitz functions on $\mathbb{R}^d$ with Lipschitz constant $\|\phi\|_{\text{Lip}} := \sup_{s \neq t} |\phi(s) - \phi(t)|/\|s - t\|_2 \leq 1$, and $\mathcal{F}_{\text{TV}} = \{\|\cdot\|_1 : B \in B(\mathbb{R}^d)\}$. Then

$$d_{\text{Wass}}(X,Y) = \sup_{\phi \in \mathcal{F}_{\text{Wass}}} |E[\phi(X)] - E[\phi(Y)]| \quad \text{and} \quad d_{\text{TV}}(X,Y) = \sup_{\phi \in \mathcal{F}_{\text{TV}}} |E[\phi(X)] - E[\phi(Y)]|, \quad (8)$$

denote the (1-)Wasserstein and total variation distances between the laws of $X$ and $Y$.

In this paper, $N \sim N(0, I_d)$ stands for a standard normal random vector in $\mathbb{R}^d$, with density $\gamma_d(t)$ on $\mathbb{R}^d$; $Y_1, Y_2, \ldots$ denote independent and identically distributed (i.i.d.) univariate uniform $U(0,1)$ random variables which are independent of all other random elements. For a random vector $X$ in $\mathbb{R}^d$, $L^p(X)$ is the collection of functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $|\phi^{p}(X)|$ has finite mean.

### 2.2 Setup

Let $n, r$ and $d$ be positive integers. Let $X = (X_1, \ldots, X_n)$ be a vector of independent random vectors, each in $\mathbb{R}^r$, from regular $d$-parameter exponential families with respective probability density functions $p_{1,i}(\cdot | \theta), i = 1, \ldots, n$ (each with respect to the same positive, $\sigma$-finite dominating measure $\mu$ defined on the Borel sets of $\mathbb{R}^r$) given in the canonical form

$$p_{1,i}(x | \theta) = \exp\left(\langle h_i(x), \theta \rangle - \alpha_i(x) - \beta_i(\theta)\right), \quad x \in \mathbb{R}^r, \quad i = 1, \ldots, n, \quad (9)$$

where $\theta = (\theta_1, \ldots, \theta_d)^T$ is the common parameter of interest which we suppose to belong to some open set $\Theta \subseteq \mathbb{R}^d$, and the functions $h_i : \mathbb{R}^r \to \mathbb{R}$ are chosen so that the normalising constants $\exp(-\beta_i(\theta))$ are finite for $i = 1, \ldots, n$. For i.i.d. data, we shall write $\beta(\theta) = \beta_i(\theta)$ for all $i = 1, \ldots, n$. Exponential family distributions in canonical form are analytically convenient and come without loss of generality as any exponential family distribution can be reduced to the canonical form via a re-parametrisation, see, e.g., [12]; a review of the results for this paper is in Appendix B.

Now let $x = (x_1, \ldots, x_n) \in \mathbb{R}^{r \times n}$ be a fixed observation from model (9) and choose a (possibly improper) prior $\pi_0$ on $B(\mathbb{R}^d)$, which, for notational convenience, we write as $\pi_0(\theta) = \exp(-\eta(\theta))$ for $\eta : \mathbb{R}^d \to \mathbb{R}$ a scalar function on which appropriate assumptions will be imposed below. The posterior distribution given $x$ is then

$$p_2(\theta | x) = \kappa(x) \exp\left(n(\overline{h}(x), \theta) - n\overline{\beta}(\theta) - \eta(\theta)\right), \quad (10)$$

with $\overline{h}(x) = n^{-1} \sum_{i=1}^n h_i(x_i), \quad \overline{\beta}(t) = n^{-1} \sum_{i=1}^n \beta_i(t) \in \mathbb{R}$ and $\kappa(x)$ the corresponding normalising constant. A key function in this paper is

$$\lambda(\theta) = n\overline{\beta}(\theta) + \eta(\theta). \quad (11)$$

Finally, let $\theta_0 = (\theta_0^1, \ldots, \theta_0^d) \in \Theta$ and $K$ be a symmetric, invertible and positive definite $d \times d$ matrix. In line with the conventions from equation (7) in Section 2.1, we set $J = K^{-1}, F = J^2$ and $H = K^2$. Given $\theta \sim p_2(\cdot | x)$ we define

$$\theta^* = K(\theta - \theta_0). \quad (12)$$
Then \( \theta^* (= \theta^*(x, K, \theta_0)) \) has density on \( \Theta^* = \{ K(\theta - \theta_0) \mid \theta \in \Theta \} \) given by

\[
p^*(t \mid x) = \det(J)p_2(\theta_0 + J t \mid x) = \kappa(x)\det(J) \exp(n(h(x), \theta_0 + J t) - \lambda(\theta_0 + J t)).
\] (13)

We can now reformulate the question from the introduction in a more precise manner.

**Question:** Given \( \theta \sim p_2(\cdot \mid x) \) from model [10] and a standardisation as [12], under what conditions on the prior \( \eta \), on the model \( \beta_1, \ldots, \beta_n \), on \( x \), and on the standardisation \( (\theta_0, K) \) is the law of \( \theta^* \) close to \( N(0, I_\theta) \)?

Our standing assumptions on the model \( (\theta) \) and the corresponding likelihood \( (10) \) are as follows.

**Assumption A.**

0. The maps \( x \mapsto p_{1,i}(x \mid \theta), 1 \leq i \leq n, \) have the same support \( S \subseteq \mathbb{R}^r \) which does not depend on \( \theta \).

1. For all \( 1 \leq i \leq n \) the parameter \( \theta \) is identifiable, in the sense that the maps \( \theta \mapsto p_{1,i}(x \mid \theta) \) are one-to-one for all \( x \in S \). The parameter space \( \Theta \) is open and convex.

2. The posterior density \( \theta \mapsto p_2(\theta \mid x) \) is positive and differentiable throughout \( \Theta \), and for all \( f: \mathbb{R}^d \rightarrow \mathbb{R} \) with uniformly bounded first and second derivatives,

\[
\int_{\Theta} \text{div}(\nabla f(\theta)p_2(\theta \mid x)) \, d\theta = 0.
\] (14)

3. \( \lambda \) is twice differentiable on \( \Theta \) and the Hessian \( H_\lambda(\theta) \) is symmetric positive definite for all \( \theta \in \Theta \).

**Remark 1.** Assumptions A0 and A1 are generic. Assumption A2, which depends on \( x \), ensures that all forthcoming integration by parts can be carried out without boundary terms. We could work under a weaker formulation, at the cost of additional boundary terms; see the forthcoming proof of Lemma 3. Finally, Assumption A3 is very natural but could be weakened, see Remark 6.

Although we will keep the statements of the results as general as possible in terms of \( \theta_0 \) and \( K \), in all the examples worked out in Section 3 we will consider one of the following two situations.

**Example 3** (Standardisation around the MAP). In [12] we fix \( \theta_0 = \tilde{\theta} = \tilde{\theta}_\text{MAP}(x) \) the posterior mode which we suppose to be uniquely defined through the equation \( \nabla \lambda(\tilde{\theta}) = n\bar{h}(x) \). We also take \( \bar{K} = K_\lambda(\tilde{\theta}) \) as the unique symmetric square root of \( H_\lambda(\tilde{\theta}) \), and consider the normal approximation of

\[
\theta^*_{\text{MAP}} = \bar{K}(\theta - \tilde{\theta}),
\] (15)

which is distributed according to the standardised posterior distribution centred at the posterior mode.

**Example 4** (Standardisation around the MLE). In [12] we fix \( \theta_0 = \hat{\theta} = \hat{\theta}_\text{MLE}(x) \) the maximum likelihood estimate which is assumed to be uniquely defined through the equation \( \nabla \bar{\lambda}(\hat{\theta}) = \bar{h}(x) \). We take \( \bar{K} = \sqrt{n}K_{\bar{\lambda}}(\hat{\theta}) \) where \( K_{\bar{\lambda}}(\hat{\theta}) \) is the unique symmetric square root of \( H_{\bar{\lambda}}(\hat{\theta}) \), the Hessian of \( \bar{\lambda} \) evaluated at \( \hat{\theta} \). Thus we consider the normal approximation of

\[
\theta^*_{\text{MLE}} = \bar{K}(\theta - \hat{\theta}),
\] (16)

which is distributed according to the standardised posterior distribution centred at the MLE.

Our approach to tackling our Question rests on two basic ingredients, namely (i) general bounds on the total variation and Wasserstein distances that are obtained from Stein’s method of comparison of operators (see Section 2.3) and (ii) Taylor expansions of the log-likelihood ratio \( \log(p^*(\cdot \mid x)/\gamma_d(\cdot)) \) (see Lemma 9 and Appendix B.2).
2.3 Stein’s method of comparison of operators

To sketch how we use Stein’s method for Gaussian approximation via comparison of operators, first for a random vector $X$ on $\mathbb{R}^d$ with differentiable density $p^X$ with support $S_X$, we introduce the Stein density operator (or Langevin-Stein operator, see [1]), for $x \in S_X$ as

$$T_X f(x) = \frac{\text{div}(p^X(x) \nabla f(x))}{p^X(x)} = \Delta f(x) + \langle \nabla \log p^X(x), \nabla f(x) \rangle. \quad (17)$$

Then $E[T_X f(X)] = \int_{S_X} \text{div}(p^X(x) \nabla f(x)) dx = 0$ for a large class of functions $f : \mathbb{R}^d \to \mathbb{R}$. We extend $T_X$ to act on $\mathbb{R}^d$ by setting it to 0 outside of $S_X$. The class of functions such that $E[T_X f(X)] = 0$ is called the Stein class for $p^X$ (or, equivalently, for $X$); we denote it $\mathcal{F}(X)$. For $N \sim N(0, I_d)$ its density $\gamma_d$ satisfies $\nabla \log \gamma_d(x) = -x$ for all $x \in \mathbb{R}^d$, and $N$ hence has density Stein operator $\mathcal{F}(X)$

$$T_N f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle \text{ on } \mathbb{R}^d; \quad (18)$$

the corresponding Stein class $\mathcal{F}(N)$ contains for instance all twice differentiable functions with bounded gradient. This operator was first used in [2, 25] and has been well studied in the literature. Of particular importance are the PDEs of the form

$$T_N f_\phi(x) = \Delta f_\phi(x) - \langle x, \nabla f_\phi(x) \rangle = \phi(x) - E[\phi(N)]; \quad (19)$$

with $\phi : \mathbb{R}^d \to \mathbb{R}$ belonging to $L^1(N)$. These equations are called the ($\phi$-)standard normal Stein equations and allow to derive the following (essentially well-known) lemma.

**Lemma 1** (Bound on differences of expectations). Let $N \sim \gamma_d$ and $X \sim p^X$ be as above. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be such that there exists a twice differentiable solution $f_\phi$ to the Stein equation (19) which, moreover, satisfies $f_\phi \in \mathcal{F}(X)$. Then

$$|E[\phi(X)] - E[\phi(N)]| \leq |E[\langle \nabla \log p^X(X) + X, \nabla f_\phi(X) \rangle]|. \quad (20)$$

**Proof.** By construction, $(T_N - T_X) f(x) = -\langle \nabla \log p^X(x) + x, \nabla f(x) \rangle$ for all $x$ at which $f$ is differentiable. The stated assumptions are tailored to ensure (20) follows from (17), (18) and (19) along with the fact that $E[T_X f_\phi(X)] = 0$ because $f_\phi \in \mathcal{F}(X)$. \hfill $\square$

Recalling the integral probability metrics introduced at the end of Section 2.1, one sees how the right-hand-side of (20) provides bounds on IPMs, at least formally. To apply Lemma 1 in practice it is crucial to have good information on the behavior of the gradients of solutions to (19). This is a well-studied problem. In dimension $d = 1$ we refer e.g. to [20, Example 2.31] where we can read the following (here we correct a typo in the last line of that example).

**Lemma 2** (Bounds in dimension $d = 1$). For $\phi \in L^1(\gamma_1)$ let $f_\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable function with first derivative $f'_\phi(x) = e^{x^2/2} \int_{-\infty}^{\phi(x)} e^{-u^2/2} du$. Then $f_\phi$ is a twice differentiable solution to the Stein equation $f''(x) - xf'(x) = \phi(x) - E[\phi(N)]$. Moreover,

- if $\phi \in \mathcal{F}_{\text{Wass}}$ (Lipschitz with Lipschitz constant 1) then $\|f_\phi\|_\infty \leq 1$ and $\|f''_\phi\|_\infty \leq \sqrt{2/\pi}$;

- if $\phi \in \mathcal{F}_{\text{TV}}$ (indicators of Borel sets) then $\|f_\phi\|_\infty \leq \sqrt{\pi/8}$ and $\|f''_\phi\|_\infty \leq 2$.

The case $d \geq 2$ is more complicated. We start with a regularity result from [22, Proposition 2.1] which will lead to Wasserstein bounds.

**Lemma 3** (Lipschitz test functions). For $0 < t < 1$ and $x \in \mathbb{R}^d$ we set $N_{t,x} = \sqrt{t}x + \sqrt{1-t}N$. If $\phi$ is differentiable on $\mathbb{R}^d$ then the function $f_\phi$ defined by

$$f_\phi(x) = -\int_0^1 \frac{1}{2t} (E[\phi(N_{t,x})] - E[\phi(N)]) dt \quad (21)$$

is a twice differentiable solution to (19). Let $C_1$ be the unit cube in $\mathbb{R}^d$. If furthermore $\phi \in \mathcal{F}_{\text{Wass}}$ then

$$\nabla f_\phi(x) \in C_1 \text{ for all } x \in \mathbb{R}^d \text{ and } \sup_{x \in \mathbb{R}^d} \|\nabla^2 f_\phi(x)\|_{\text{H.S.}} \leq \sqrt{d}. \quad (22)$$
Proof. The first statement comes from [22, Proposition 2.1] where it is shown to hold under the weaker condition that \( \phi \) is \( \alpha \)-Hölder. If \( \phi \) is Lipschitz then the first claim in [22] follows from [24] and [22, Proposition 2.1]. The second statement in [22] follows from [10, Lemma 3].

Using Lemma 3 (resp., the first statement in Lemma 3) in the bound (20) from Lemma 1 we immediately reap a Wasserstein-1 bound on multivariate (resp., univariate) normal approximation.

**Lemma 4** (Wasserstein Stein bound). Let \( \mathcal{E}_1 \) be as in [5]. Let \( X \) be a random variable on \( \mathbb{R}^d \) with density \( p^X \) such that \( \mathcal{E}_1 \subseteq \mathcal{F}(X) \). Then
\[
\text{d}_{\text{Wass}}(X, N) \leq \sup_{f \in \mathcal{E}_1} \left| \mathbb{E} \left[ (\nabla \log p^X(X) + X, \nabla f(X)) \right] \right|.
\] (23)

Our next two results concern total variation distance. The one-dimensional case is covered by Lemma 2. In dimension \( d \geq 2 \), we note that solutions to (19) being only valid for points \( x \in \mathbb{R}^d \) where \( \phi(x) \) is locally Hölder continuous (see [25, p. 726]), we cannot directly work with the Stein equation (19) with indicator test functions \( \phi \in \mathcal{F}_{TV} \) of the form \( \phi(\cdot) = \mathbb{1}_B(\cdot) \) with \( B \subset \mathcal{B}(\mathbb{R}^d) \). Instead we apply the smoothing approach used in [25, Lemma 2.1].

**Lemma 5.** For \( 0 < t < 1 \) and \( \phi \in \mathcal{F}_{TV} \), let \( \phi_t(x) = \mathbb{E}[\phi(\sqrt{t}N + \sqrt{1-t}x)] \). Then \( \phi_t \) is differentiable on \( \mathbb{R}^d \) and \( \| \partial_j \phi_t \|_\infty < \infty \) for all \( j = 1, \ldots, d \). The functions
\[
f_{\phi,t}(x) := -\int_t^1 \frac{1}{2(1-s)} (\phi_s(x) - \mathbb{E}[\phi(N)]) \, ds
\] (24)
are well-defined twice differentiable solutions of the \( \phi_t \)-Stein equation (19) with bounded first and second derivatives. Moreover, letting \( \mathcal{B}(\delta) \) be the centred ball in \( \mathbb{R}^d \) with radius \( \delta \), we have
\[
\nabla f_{\phi,t}(x) \in \mathcal{B}(\sqrt{\pi}/2) \text{ for all } x \in \mathbb{R}^d \text{ and all } t \in (0,1).
\] (25)

Proof. A change of variables gives
\[
\phi_t(x) = \int_{\mathbb{R}^d} \phi(t^{\frac{1}{2}}y + (1-t)^{\frac{1}{2}}x) \gamma_d(y) \, dy = t^{-d/2} \int_{\mathbb{R}^d} \phi(u) \gamma_d((u - (1-t)^{\frac{1}{2}}x)/t^{\frac{1}{2}}) \, du.
\] (26)
Since \( \phi \) is bounded and \( \gamma_d \) is integrable, we can exchange derivatives with respect to \( x_j \) and integrals, recall that \( \partial_j \gamma_d(x) = -x_j \gamma_d(x) \), and change variables back to get
\[
\partial_j f_{\phi,t}(x) = \left( \frac{1-t}{t} \right)^{\frac{1}{2}} \mathbb{E} \left[ N_j \phi(t^{\frac{1}{2}}N + (1-t)^{\frac{1}{2}}x) \right].
\]
Since \( \phi \) is bounded it follows that \( |\partial_j \phi_t(x)| < \infty \) for all \( x \in \mathbb{R}^d \). This proves the first part of the claim.

The fact that \( f_{\phi,t} \) is a well-defined twice differentiable solution of the \( \phi_t \)-Stein equation (19) follows immediately from [25, Lemma 2.1]. For (25), a change of variables as in (26) yields
\[
f_{\phi,t}(w) = -\frac{1}{2} \int_t^1 \int_{\mathbb{R}^d} \frac{1}{s^{d/2}(1-s)} (\phi(u) - \mathbb{E}[\phi(N)]) \gamma_d \left( \frac{u - (1-s)^{1/2}w}{s^{1/2}} \right) \, du \, ds.
\]
Therefore, by dominated convergence, since \( \phi \) is bounded, we obtain, for \( x \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^d \):
\[
|\langle \xi, \nabla f_{\phi,t}(x) \rangle| = \left| \sum_{j=1}^d \xi_j \partial_j f_{\phi,t}(x) \right|
= \frac{1}{2} \left| \sum_{j=1}^d \xi_j \int_t^1 \int_{\mathbb{R}^d} s^{-(d+1)/2} (u - (1-s)^{1/2}x)_j \phi(u) \, du \, ds \right|
\leq \frac{1}{2} \left| \sum_{j=1}^d \xi_j \gamma_d y_j \left( \phi(s^{1/2}y + (1-s)^{1/2}x) - \mathbb{E}[\phi(N)] \right) \, dy \, ds \right|
\leq \frac{1}{2} ||\phi - \mathbb{E}[\phi(N)]||_\infty \int_0^1 \int_{\mathbb{R}^d} \frac{1}{\sqrt{s(1-s)}} \left| \sum_{j=1}^d \xi_j y_j \right| \gamma_d(y) \, dy \, ds.
\]
If $\phi \in \mathcal{F}_{TV}$ then $\|\phi - \mathbb{E}[\phi(N)]\|_{\infty} \leq 1$. Direct evaluation gives $\int_0^1 (s(1-s))^{1/2} ds = \pi$. Also, letting $N_1, \ldots, N_d$ denote i.i.d. standard normal random variables, we have
\[
\int_{\mathbb{R}^d} \left| \sum_{j=1}^{d} \xi_j y_j \right| \gamma_d(y) dy = \mathbb{E}\left[ \left| \sum_{j=1}^{d} \xi_j N_j \right| \right] = \sqrt{\frac{2}{\pi}} \|\xi\|_2,
\]
because $\sum_{j=1}^{d} \xi_j N_j \sim \mathcal{N}(0, \sum_{j=1}^{d} \xi_j^2)$. Hence $|\langle \xi, \nabla f(x) \rangle| \leq \sqrt{\pi/2} \|\xi\|_2$ for all $t$, $x$ and $\xi$. \hfill $\Box$

**Remark 2.** The statement (25) is independent of $t$; this property is not enjoyed by higher order derivatives of $f_{p,t}$ (see [24, Lemma 2.1]). While the regularity of the solution of multivariate normal Stein equation is well studied, the fact that the gradient of the solution of the $\phi_t$-Stein equation belongs to a centred ball in $\mathbb{R}^d$, rather than just a centred cube in $\mathbb{R}^d$, appears to have been overlooked. This subtle difference will be crucial for obtaining bounds with an improved dimensional dependence in the total variation distance.

Similarly as in the Wasserstein case, Lemma 5 (resp., the second statement in Lemma 2) in [20] leads to the following total variation distance bound on normal approximation.

**Lemma 6** (Global total variation bound). Let $\mathcal{G}_1$ be as in (6). Let $X \in \mathbb{R}^d$ be a random variable with density $p^X$ such that $\mathcal{G}_1 \subseteq \mathcal{F}(X)$. Then
\[
d_{TV}(X, N) \leq C_d \sup_{f \in \mathcal{G}_1} \mathbb{E} \left[ \left| \nabla \log p^X(X) + X, \nabla f(X) \right| \right] \tag{27}
\]
where $C_1 = \sqrt{\pi/8}$ and $C_d = \sqrt{\pi/2}$ for all $d \geq 2$.

**Proof.** The claim in dimension $d = 1$ is known (see e.g. [20]). For $d \geq 2$, using [6] (4.6) and (4.9) we get that for all $0 < t < 1$ there exists absolute constants $A_d \geq 1$ and $B_d \geq 0$ such that $d_{TV}(X, N) \leq A_d \sup_{\phi \in \mathcal{F}_d} \mathbb{E}[\phi(X)] - \mathbb{E}[\phi(N)] + B_d \sqrt{1 - t}$ with $\phi$ as in Lemma 5. We then apply the same approach as in Lemma 4 on the supremum, with the added fact that since the bound on the difference of expectations does not ultimately depend on $t$ we can take $t = 0$ (which on examining the argument of [5] Section 4] allows us to also let $A_d = 1$) to get the conclusion. \hfill $\Box$

Sharper bounds than in Lemma 6 can often be obtained via the following refinement.

**Lemma 7** (Local total variation bound). Let $\mathcal{G}_1$ and $C_d$, $d \geq 1$, be as in Lemma 6. Let $X \in \mathcal{B}(\mathbb{R}^d)$ be compact with piecewise smooth boundary $\delta(X)$. Let $X$ have density $p^X$ with respect to the Lebesgue measure on $\mathbb{R}^d$ which is positive and differentiable on $X$. Finally, suppose that $\mathcal{G}_1 \subseteq \mathcal{F}(X)$. Then
\[
d_{TV}(X, N) \leq C_d \sup_{f \in \mathcal{G}_1} \mathbb{E} \left[ \left| \nabla \log p^X(X) + X, \nabla f(X) \right| \right] + \mathbb{P}[X \notin X] + r(X), \tag{28}
\]
with
\[
r(X) = C_d A_d(\delta(X)) \sup\{p^X(s) \mid s \in \delta(X)\}, \tag{29}
\]
where $A_1(\delta(X)) = 1$ and $A_d(\delta(X))$ is the surface area of $\delta(X) \subseteq \mathbb{R}^d$ for all $d \geq 2$.

**Proof.** First let $B \in \mathcal{B}(\mathbb{R}^d)$. By using the law of total probability, we have
\[
\mathbb{P}[X \in B] - \mathbb{P}[N \in B] = \mathbb{P}[X \in B \mid X \in X] \mathbb{P}[X \in X] + \mathbb{P}[X \in B \mid X \notin X] \mathbb{P}[X \notin X]
- \mathbb{P}[N \in B \mid X \in X] \mathbb{P}[X \in X] - \mathbb{P}[N \in B \mid X \notin X] \mathbb{P}[X \notin X]
\leq \|\mathbb{P}[X \in B] - \mathbb{P}[N \in B]\| \mathbb{P}[X \in X] + \mathbb{P}[X \notin X]
\]
with $X_X$ the random variable with distribution $\mathcal{L}(X \mid X \in X)$ and $N$ independent of $X$. Hence,$\quad d_{TV}(X, N) \leq d_{TV}(X_X, N) \mathbb{P}[X \in X] + \mathbb{P}[X \notin X]$. Again we use the inequality $d_{TV}(X_X, N) \leq \sup_{\phi \in \mathcal{F}_TV} \left| \mathbb{E}[\phi(X)] - \mathbb{E}[\phi(N)] \right|$. Some easy algebraic manipulations give, with $T_N$ as in (18),
\[
\mathbb{E}[\phi_t(X_X)] - \mathbb{E}[\phi_t(N)] \mathbb{P}[X \in X] = \int_X (T_N \tilde{f}_{\phi,t}(x))p^X(x) dx
= -\mathbb{E}\left[ \left| \nabla \log p^X(X) + X, \nabla \tilde{f}_{\phi,t}(X) \right| I_X(X) \right] + \int_X \text{div}(p^X(x)\nabla \tilde{f}_{\phi,t}(x)) dx.
\]

The first term above gives the first term in (28). By the divergence theorem we deduce
\[
\int_X \text{div}(p^X(u)\nabla \tilde{f}_{\phi,t}(u)) \, du = \int_{\delta(X)} p^X(s) \left( \nabla \tilde{f}_{\phi,t}(s), n(s) \right) \, d\sigma(s)
\]
with \(\sigma\) the surface measure and \(n\) the outward pointing unit normal vector. Using the Cauchy-Schwarz inequality with the inequality \(\|
abla \tilde{f}_{\phi,t}\|_2 \leq \sqrt{\pi}/2\) in (25) (and in dimension \(d = 1\) we have \(\|f^\phi_0\|_{\infty} \leq \sqrt{\pi}/8\), from Lemma 2 we get
\[
\int_X \text{div}(p^X(u)\nabla \tilde{f}_{\phi,t}(u)) \, du \leq C_d \int_{\delta(X)} p^X(s) \, d\sigma(s) \leq C_d \sup \{p^X(s) \mid s \in \delta(X)\} A_d(\delta(X)).
\]
Adding the various pieces gives the claim. \(\Box\)

Remark 3. At least formally, (27) follows from (28) by taking \(X = \Theta\) and employing the convention \(r(\Theta) = 0\). This will allow us to simultaneously state both the local and global total variation bounds.

2.4 Back to the Bayes

Recall the exponential family setup and all the corresponding notations from Section 2.2. We now address our Question by applying the Stein’s method developed in Section 2.3 to the specific choice \(X = \theta^* = \mathbf{K}(\theta - \theta_0)\) for some \(\theta\) distributed according to the posterior distribution in an exponential family as in (10), \(\theta_0 \in \Theta\) some arbitrary point, and \(\mathbf{K}\) some symmetric, invertible and positive definite \(d \times d\) matrix. We begin by noting the following.

**Lemma 8.** Let \(p^*\) be the density of \(\theta^*\) on \(\Theta^*\) as given in (13). Under Assumption A, the Stein operator \(T_{\theta^*}\) for \(\theta^*\), given for \(t^* \in \Theta^*\) by
\[
T_{\theta^*} f(t^*) = \Delta f(t^*) + \langle \nabla \log p^*(t^* | x), \nabla f(t^*) \rangle
\]
and set to 0 outside of \(\Theta^*\), satisfies \(\mathbb{E}[T_{\theta^*} f(\theta^*)] = 0\) for all twice differentiable functions \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) with bounded first and second derivatives.

**Proof.** Let \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) be a differentiable function. We note (see e.g. [38]) that for all \(t^* \in \Theta^*\) the Stein operator satisfies \(p^*(t^* | x) T_{\theta^*} f(t^*) = \text{div}(\nabla f(t)p^*(t^* | x))\). Hence
\[
\mathbb{E}[T_{\theta^*} f(\theta^*)] = \int_{\Theta^*} \text{div}(\nabla f(t^*)p^*(t^* | x)) \, dt^*.
\]
The conclusion then follows directly after a change of variables in Assumption (A2). \(\Box\)

Moreover, the following can be seen to hold (see Appendix B.2).

**Lemma 9** (Third order Taylor expansion). Let \(\Theta_0 \subseteq \Theta\) be a measurable neighborhood of \(\theta_0\) and set \(\Theta_0^* = \mathbf{K}(\Theta_0 - \theta_0)\). Let \(\mathbf{J} = \mathbf{K}^{-1}\), set \(\bar{h}\) as in (10) and \(\lambda\) as in (11). If \(\lambda\) is three times differentiable on \(\Theta_0\), then for all \(t^* \in \Theta_0^*\) and all \(\bar{\xi} \in \mathbb{R}^d\) we have
\[
\langle \nabla \log p^*(t^* | x), t^* \rangle, \xi \rangle = \langle \nabla \mathbf{A}(\theta_0) - n\bar{h}(x), \mathbf{J}\xi \rangle + \langle \mathbf{H}(\theta_0) - \mathbf{K}^2, (\mathbf{J}\xi) \otimes (\mathbf{J}t^*) \rangle + \langle \mathbf{Y}^3(\lambda, \theta_0; t^*), (\mathbf{J}\xi) \otimes (\mathbf{J}t^*) \otimes (\mathbf{J}t^*) \rangle
\] (30)
where for \(g : \mathbb{R}^d \rightarrow \mathbb{R}\) three times differentiable on \(\Theta_0\) and \(t^* \in \Theta_0^*\) we set
\[
\mathbf{Y}^3(g, \theta_0; t^*) = \mathbb{E}[Y_1 D^3 g(\theta_0 + Y_1 Y_2 (\mathbf{J}t^*))]
\]
with \(Y_1, Y_2\) i.i.d. uniform random variables on \((0, 1)\) and \(\mathbf{J} = \mathbf{K}^{-1}\).

In order to efficiently state our bounds we will need some final notations.
Definition 1 (Bespoke derivatives). With all notations as above we set

\[ \mathcal{D}_1(g, \theta_0, K | \Theta_0^0) = \sum_{u=1}^{d} \left| \partial_u g(\theta_0) - n \bar{h}_u(x) \right| \mathbb{P}[\theta^* \in \Theta_0^0], \]

\[ \mathcal{D}_2(g, \theta_0, K | \Theta_0^0) = \sum_{u=1}^{d} \sum_{v=1}^{d} \left| \partial_{u,v}^2 g(\theta_0) - (K^2)_{u,v} \right| \mathbb{E} \left[ \left| (J \theta^*)_{u,v} \right| \mathbb{P}[\theta^* \in \Theta_0^0] \right], \]

\[ \mathcal{D}_3(g, \theta_0, K | \Theta_0^0) = \sum_{u=1}^{d} \sum_{v=1}^{d} \sum_{w=1}^{d} \mathbb{E} \left[ \left| Y^3_{u,v,w}(g, \theta_0; \theta^*)(J \theta^*)_{u,v}(J \theta^*)_{w} \right| \mathbb{P}[\theta^* \in \Theta_0^0] \right]. \]

Finally, we set

\[ \Delta_3(g, \theta_0, K | \Theta_0^0) = \sum_{j=1}^{3} \mathcal{D}_j(g, \theta_0, K | \Theta_0^0). \]

Remark 4. In line with the comment from Remark 3, we will drop the notation “\( | \Theta_0^0 \)” in any of the above when \( \Theta_0 = \Theta \) (or, equivalently, when \( \Theta_0^0 = \Theta^* \)).

With these quantities defined, we are ready to state our bounds.

Theorem 1 (Third order bounds). Let \( \theta \) be distributed according to \( (10) \) and suppose Assumption A is satisfied. Let \( \theta^* \) be defined as in \( (12) \) for \( K \) a square symmetric definite positive matrix and \( \theta_0 \in \Theta \); define \( \Theta^* \) accordingly. Let \( J = K^{-1} \) and \( F = J^2 \). Set \( C_1 = \sqrt{\pi/8} \) and \( C_d = \sqrt{\pi/2} \) for \( d \geq 2 \) (as in Lemma 8). Then the following statements hold true:

- If \( \lambda \) is three times differentiable on \( \Theta^* \) then
  \[ d_{\text{Wass}}(\theta^*, N) \leq \sqrt{d} ||F||_{\text{diag}} \Delta_3(\lambda, \theta_0, K), \]
  \[ d_{\text{TV}}(\theta^*, N) \leq C_d ||F||_{\text{diag}} \Delta_3(\lambda, \theta_0, K). \]
- Let \( \Theta_0^0 \subset \Theta^* \subseteq \mathbb{R}^d \) be a compact set with piecewise smooth boundary. If \( \lambda \) is three times differentiable on \( \Theta_0^0 \) then
  \[ d_{\text{TV}}(\theta^*, N) \leq C_d ||F||_{\text{diag}} \Delta_3(\lambda, \theta_0, K | \Theta_0^0) + \mathbb{P}[\theta^* \notin \Theta_0^0] + r_4^*(\Theta_0^0) \]
  where \( r_4^*(\Theta_0^0) = C_d A_4^* p_{\infty}^* \) with \( p_{\infty}^* = \sup \{ p^*(\theta^* | x) | \theta^* \in \delta(\Theta_0^0) \} \), \( A_4^* = 1 \) and \( A_4^* \) is the surface area of \( \delta(\Theta_0^0) \) if \( d \geq 2 \).

Proof. Reading directly from Lemma 8 we see that it suffices to tackle the quantities

\[ \left| \langle \nabla \lambda(\theta_0) - n \bar{h}(x), J \mathbb{E} [\nabla f(\theta^*)] \rangle \right| + \left| \langle H_\lambda(\theta_0) - K^2, J \mathbb{E} [(J \nabla f(\theta^*)) \otimes (J \theta^*)] \rangle \right| \]

\[ + \left| \mathbb{E} \left[ \left| Y^3(\lambda, \theta_0; \theta^*), (J \nabla f(\theta^*)) \otimes (J \theta^*) \right|^2 \right] \right| \]

for \( f \) belonging either to \( \mathcal{C}_1 \) in order to apply Lemma 3 and obtain the Wasserstein bound \( (35) \), or to \( \mathcal{G}_1 \) in order to apply Lemmas 6 and 7 on \( X = B(\delta) \) and obtain the total variation bound \( (37) \).

For \( (35) \), note that for all \( x \in C_1 \) (the unit cube in \( \mathbb{R}^d \)) and all \( 1 \leq u \leq d \) we have

\[ |(J \xi)_u| \leq \sum_{v=1}^{d} \sum_{w=1}^{d} \xi_{uv} \leq \sqrt{d} \sqrt{F_{uu}} \leq \sqrt{d} ||F||_{\text{diag}}. \]

Bound \( (35) \) follows from Lemma 8. For the bound \( (36) \) we apply Lemma 8 along with the fact that for all \( x \in \mathcal{B}_1 \) (the unit disk in \( \mathbb{R}^d \)) and all \( u = 1, \ldots, d \) we have

\[ |(J \xi)_u| \leq \sum_{v=1}^{d} \sum_{w=1}^{d} \xi_{uw} \leq \sqrt{d} \sqrt{F_{uu}} \leq ||F||_{\text{diag}}. \]

Finally, the bound \( (37) \) follows from the same arguments, through Lemma 7 on \( \Theta_0^0 \).
Remark 5. A simple choice of $\Theta_0^*$ is the ball $\mathcal{B}(\delta)$ with radius $\delta$ whose surface area is given by $A_d^* = \delta^{d-1} \pi^{d/2}/\Gamma(d/2 + 1) \sim \pi^{d-1}/(2\pi^{d/2}\delta^{d-1}/d^{d-1}/2)$, see e.g. [43, Section 3]. This quantity is negligible in $d$ as long as $d$ does not grow too fast with the dimension.

Remark 6 (About the assumptions). Lemma 9 is a particular case of a general order Taylor expansion provided in Lemma 14 in Appendix B.2. Although in Theorem 1 we focus on third order expansions, we could work out higher order bounds (under higher order differentiability assumptions) from [31] in Appendix B.2. We could also dispense with the third order differentiability requirement and only take derivatives up to order 2, by using [30]. Also, we could easily dissociate the assumptions on $\beta$ and $\eta$ (for instance imposing higher order regularity on $\beta$ but only very weak differentiability on the prior $\eta$) and expand to different orders in $\beta$ and $\eta$. We do not detail all the possible variants, for conciseness, but provide the following univariate fourth order expansion, which we apply in Example 13 to obtain a faster convergence rate with respect to the sample size $n$ when the model takes a particular form. Let

$$
\Delta_4(g, \theta_0, K | \Theta_0^*) = \sum_{j=1}^2 \mathcal{D}_j(g, \theta_0, K | \Theta_0^*) + \frac{1}{2} |g(\theta_0)| \mathbb{E} [ (J\theta_0)^2 \mathbb{I}_{\Theta_0^*}(\theta^*) ]
+ \mathbb{E} \left[ \mathbb{E} \left[ Y_1 Y_2^2 (\theta_0 + Y_1 Y_2 J \theta^*) \right] (J \theta^*)^3 \mathbb{I}_{\Theta_0^*}(\theta^*) \right],
$$

(38)

where $Y_1, Y_2, Y_3$ are i.i.d. $U(0, 1)$. Then, if $\lambda : \mathbb{R} \to \mathbb{R}$ is four times differentiable on $\Theta^*$,

$$
d_{\text{Wass}}(\theta^*, N) \leq \| \mathbf{F} \|_{\text{diag}} \Delta_4(\lambda, \theta_0, K), \quad d_{\text{TV}}(\theta^*, N) \leq \sqrt{n} \| \mathbf{F} \|_{\text{diag}} \Delta_4(\lambda, \theta_0, K).
$$

(39)

The bounds depend on $\| \mathbf{F} \|_{\text{diag}}$ and $\Delta_3(\lambda, \theta_0, K | \Theta_0^*) = \sum_{j=1}^3 \mathcal{D}_j(\lambda, \theta_0, K | \Theta_0^*)$. The first two summands in this last term simply reflect the influence of the centring and scaling parameters from Appendix B.2 on the Gaussian proximity: $\mathcal{D}_1(\lambda, \theta_0, K | \Theta_0^*)$ indicates how close $\theta_0$ is to a critical point of $p_2(\cdot | \mathbf{x})$ and $\mathcal{D}_2(\lambda, \theta_0, K | \Theta_0^*)$ indicates how close the scaling $K$ is to the Fisher information. We thus have the following simplification under MAP standardisation.

Example 5 (MAP standardisation, continued). Using the notations from Example 3, it follows that

$$
\mathcal{D}_1(\lambda, \hat{\theta}, \bar{K} | \Theta_0^*) = \mathcal{D}_2(\lambda, \hat{\theta}, \bar{K} | \Theta_0^*) = 0
$$

(40)

for all $\Theta_0^*$ (hence also for the unconditional version).

For general standardisation, as $\lambda = n\beta + \eta$, linearity and the triangle inequality yield that

$$
\mathcal{D}_1(\lambda, \theta_0, K | \Theta_0^*) \leq n \mathcal{D}_1(\beta, \theta_0, K | \Theta_0^*) + \sum_{u=1}^d |\partial_u \eta(\theta_0)| \mathbb{P}[\theta^* \in \mathcal{B}(\delta)],
$$

$$
\mathcal{D}_2(\lambda, \theta_0, K | \Theta_0^*) \leq n \mathcal{D}_2(\beta, \theta_0, K | \Theta_0^*) + \sum_{u=1}^d \sum_{v=1}^d |H_{u,v}^\theta(\theta_0)| \mathbb{E} [ (J \theta^*)_u \mathbb{I}_{\Theta_0^*}(\theta^*) ]
$$

and

$$
\mathcal{D}_3(\lambda, \theta_0, K | \Theta_0^*) \leq n \mathcal{D}_3(\beta, \theta_0, K | \Theta_0^*) + \mathcal{D}_3(\eta, \theta_0, K | \Theta_0^*).
$$

(41)

The prior dependent terms vanish in all of the above under a flat (uninformative) prior; under a Gaussian prior we have

$$
\sum_{u=1}^d |\partial_u \eta(\theta_0)| = \sum_{u=1}^d |\theta_0^u| \quad \text{and} \quad H_{u,v}(\theta_0) = I_d.
$$

In the case of MLE standardisation some further simplifications occur, as follows.

Example 6 (MLE standardisation, continued). Using the notations from Example 4, it follows that $\| \mathbf{F} \|_{\text{diag}} = n^{-1/2} \| F_{\beta}(\hat{\theta}) \|_{\text{diag}}$; moreover, $\mathcal{D}_1(\hat{\beta}, \hat{\theta}, \bar{K} | \Theta_0^*) = \mathcal{D}_2(\hat{\beta}, \hat{\theta}, \bar{K} | \Theta_0^*) = 0$ and

$$
\mathcal{D}_1(\lambda, \hat{\theta}, \bar{K} | \Theta_0^*) \leq \sum_{u=1}^d |\partial_u \eta(\hat{\theta})| \mathbb{P}[\theta^* \in \Theta_0^*],
$$

$$
\mathcal{D}_2(\lambda, \hat{\theta}, \bar{K} | \Theta_0^*) \leq \frac{1}{\sqrt{n}} \sum_{u=1}^d \sum_{v=1}^d |H_{u,v}^\theta(\hat{\theta})| \mathbb{E} [ (J_{\hat{\beta}}(\hat{\theta}) \theta^*)_u \mathbb{I}_{\Theta_0^*}(\theta^*) ]
$$

12
Example 7 (MAP standardisation, continued). As \( \lambda = n\overline{\beta} + \eta \), if the various quantities in inequality (4.1) exist, we have that \( \mathcal{D}_3(\lambda, \theta_0, K \mid \Theta_0^*) \leq n\mathcal{D}_3(\beta, \theta_0, K \mid \Theta_0^*) \{ 1 + r_n^2 \} \), where \( r_n^2 \) is a remainder term of order \( O(n^{-1}) \). As the posterior mode \( \tilde{\theta} \) solves \( \nabla \beta(\theta) + n^{-1}\nabla \eta(\theta) = \overline{h}(x) \) whilst the MLE \( \hat{\theta} \) solves \( \nabla \beta(\theta) = \overline{h}(x) \), provided the functions \( \beta \) and \( \eta \) possess suitability regularity, we have that \( \theta = \tilde{\theta} \{ 1 + r_n^2 \} \), where \( r_n^2 \) is a remainder term of order \( o(1) \) as \( n \to \infty \) (in all our examples of Section 3, we have that \( r_n^2 = O(n^{-1}) \)). We therefore obtain from (35) and (36) that, if \( \lambda \) is three times differentiable on \( \Theta^* \),

\[
d_{\text{Wass}}(\theta^*_\text{MAP}, N) \leq \frac{\sqrt{d}}{\sqrt{n}}U_n(x) \{ 1 + r_n(x, \eta) \}, \quad d_{\text{TV}}(\theta^*_\text{MAP}, N) \leq \frac{C_d}{\sqrt{n}}U_n(x) \{ 1 + r_n(x, \eta) \},
\]

where \( U_n(x) = \| F(\hat{\theta}) \|_{\text{diag}} \mathcal{D}_3(n\overline{\beta}, \hat{\theta}, K) \) depends solely on the sample size \( n \) and sample data \( x \) and is independent of the prior distribution, and \( r_n(x, \eta) \) is a remainder term (which is in general dependent on the prior distribution \( \eta \)) of order \( o(1) \) as \( n \to \infty \).

Although we will tend to not spell this out for sake of conciseness, in all our examples in Section 3, our upper bounds are of the type the decomposition \( \mathcal{U}_1(x)n^{-1/2} + \mathcal{U}_2(x, \tau)n^{-3/2} \), where \( \mathcal{U}_1(x) \) is independent of the prior hyperparameter \( \tau \), whilst \( \mathcal{U}_2(x, \tau) \) does in general depend on the prior hyperparameter. In contrast, our upper bounds on \( d_{\text{Wass}}(\theta^*_\text{MLE}, N) \) and \( d_{\text{TV}}(\theta^*_\text{MLE}, N) \) do not enjoy such a decomposition, being of the form \( \mathcal{U}_3(x, \tau)n^{-1/2} \); we also obtain lower bounds of the form \( \mathcal{L}(x, \tau)n^{-1/2} \) which suggest that in general under MLE standardisation we cannot expect to obtain upper bounds of the form \( \mathcal{U}_4(x)n^{-1/2} + \mathcal{U}_3(x, \tau)n^{-3/2} \).

It remains to deal with the third order term in our bounds.

Lemma 10. Suppose that \( g \) is three times differentiable on \( \Theta_0^* \), and let \( S(t) \) be a function such that \( \sum_{u=1}^{d} |\partial^3_{u,v,w}g(t)| \leq S(t) \) for all \( 1 \leq v, w \leq d \) with \( t \in \Theta_0^* \). Then

\[
\mathcal{D}_3(g, \theta_0, K \mid \Theta_0^*) \leq dE \left[ Y_1S(R_2) \| \theta^* \|_F^2 \mathcal{I}_{\Theta_0^*}(\theta^*) \right]
\]

with \( R_2 = \theta_0 + Y_1Y_2J\theta^* \) and \( \| \theta^* \|_F^2 = (\theta^*)^\top F\theta^* \).

Proof. Let \( \Theta_0 = J\Theta_0^* + \theta_0 \). Then, by triangle inequality,

\[
\mathcal{D}_3(g, \theta_0, K \mid \Theta_0^*) \leq \sum_{u=1}^{d} \sum_{v=1}^{d} \sum_{w=1}^{d} E \left[ Y_1|\partial^3_{u,v,w}g(R_2)| |(J\theta^*)_v| |(J\theta^*)_w| \mathcal{I}_{\Theta_0^*}(\theta^*) \right]
\]

\[
\leq \sum_{v=1}^{d} \sum_{w=1}^{d} E \left[ Y_1S(R_2) |\theta_v - \theta^*_v| |\theta_w - \theta^*_w| \mathcal{I}_{\Theta_0^*}(\theta) \right]
\]

\[
= E \left[ Y_1S(R_2) \left( \sum_{v=1}^{d} |\theta_v - \theta^*_v| \right)^2 \mathcal{I}_{\Theta_0^*}(\theta) \right] \leq dE \left[ Y_1S(R_2) \left( \sum_{v=1}^{d} (\theta_v - \theta^*_v)^2 \right) \mathcal{I}_{\Theta_0^*}(\theta) \right],
\]

applying the Cauchy-Schwarz inequality on the sum. Finally, \( \sum_{v=1}^{d} (\theta_v - \theta^*_v)^2 = (J\theta^*)^\top (J\theta^*) \) which gives the claim. \( \square \)

Remark 7. Sharper bounds can be obtained if certain third order partial derivatives of \( g \) vanish. For example, suppose that \( \partial^3_{u,v,w}g(t) = 0 \) if \( u, v, w \) all belong to the set \( \{ 1, \ldots, d-1 \} \) (as in Example 3.7). Then, from the proof of Lemma 10 it is easily seen that

\[
\mathcal{D}_3(g, \theta_0, K \mid \Theta_0^*) \leq \sum_{v=1}^{d} E \left[ Y_1S(R_2)((\theta_v - \theta^*_v)(\theta_d - \theta^*_d) | \mathcal{I}_{\Theta_0^*}(\theta^*) \right].
\]

Lemma 10 is simple to apply when \( S \) is a bounded function. This assumption is natural but the coarseness of the approach leads to a non-optimal dependence on the dimension. We can improve the dependence considerably under the next assumption.

Assumption P1. There exist functions \( p_u(t), 1 \leq u \leq d \), and constants \( C_1, C_2, C_3 \) such that, for all \( t \in \mathcal{X}, |\partial^3_{u,u,u}g(t)| \leq C_1p_u(t), |\partial^3_{u,u,w}g(t)| \leq C_2p_u(t)p_v(t) \) and \( |\partial^3_{u,v,w}g(t)| \leq C_3p_u(t)p_v(t)p_w(t) \) for all \( 1 \leq u, v, w \leq d \).
Lemma 11. Suppose that $g$ satisfies Assumption $[P_4]$ on $\Theta_0^*$. Let $C_\infty(t) := C_1 + 2C_2 \sum_{u=1}^d p_u(t) + C_3 \left( \sum_{u=1}^d p_u(t) \right)^2$. Then

$$
\mathcal{D}_3(g, \theta_0, K | \Theta_0^*) \leq \mathbb{E} \left[ Y_1 C_\infty(R_2) \left\| \theta^* \right\|_{\mathcal{P}(R_2)}^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

with $R_2 = \theta_0 + Y_1 Y_2 \mathbf{J} \theta^*$ and $\mathbf{P}(t) = \text{diag}(p(t))$ is the diagonal matrix with entries $p_u(t), 1 \leq u \leq d$.

Proof. Let $\Theta_0 = \mathbf{J} \Theta_0^* + \theta_0$. Under Assumption $[P_1]$ we can write

$$
\mathcal{D}_3(g, \theta_0, K | \Theta_0^*) \leq \sum_{i=1}^4 I_i(g)
$$

with

$$
I_1(g) = C_1 \sum_{u=1}^d \mathbb{E} \left[ Y_1 p_u(R_2)(\theta_u - \theta_0^u)^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

and

$$
I_2(g) = C_2 \sum_{u=1}^d \sum_{v \neq u} \mathbb{E} \left[ Y_1 p_u(R_2)p_v(R_2)(\theta_v - \theta_0^v)^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

(we switch notation in the summation indices in the last step for clarity),

$$
I_3(g) = C_2 \sum_{u=1}^d \sum_{v \neq u} \mathbb{E} \left[ Y_1 p_u(R_2)p_v(R_2)p_w(R_2) | \theta_u - \theta_0^u | | \theta_v - \theta_0^v | \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

$$
\leq C_2 \mathbb{E} \left[ Y_1 \left( \sum_{u=1}^d p_u(R_2) | \theta_u - \theta_0^u | \right)^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

$$
\leq C_2 \mathbb{E} \left[ Y_1 \left( \sum_{u=1}^d p_u(R_2) \right)^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

where the last inequality follows from applying the Cauchy-Schwarz inequality on the sum, and finally

$$
I_4(g) = C_3 \sum_{u=1}^d \sum_{v \neq u} \sum_{w \neq u, v} \mathbb{E} \left[ Y_1 p_u(R_2)p_v(R_2)p_w(R_2) | \theta_v - \theta_0^v | | \theta_w - \theta_0^w | \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

$$
\leq C_3 \mathbb{E} \left[ Y_1 \left( \sum_{u=1}^d p_u(R_2) \right)^2 \left( \sum_{v=1}^d p_u(R_2) | \theta_v - \theta_0^v | \right)^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

$$
\leq C_3 \mathbb{E} \left[ Y_1 \left( \sum_{u=1}^d p_u(R_2) \right)^2 \left( \sum_{u=1}^d p_u(R_2) | \theta_u - \theta_0^u | \right)^2 \mathbb{I}_{\Theta_0^*}(\theta) \right]
$$

with the same justification.

Remark 8. Assumption $[P_1]$ may seem ad hoc, but it is actually quite natural in our context. Indeed, taking $g = \beta$ as in model $[P_0]$ then standard properties of exponential families in canonical form give (see Appendix $[P_0]$), for all $1 \leq u, v, w \leq d$,

$$
\partial_{u,v,w}^3 \beta(t) = \mathbb{E} \left[ (h_u(X(t)) - \mathbb{E}[h_u(X(t)]))(h_v(X(t)) - \mathbb{E}[h_v(X(t)]))(h_w(X(t)) - \mathbb{E}[h_w(X(t)]) \right]
$$

with $X(t) \sim p_1(\cdot | t)$. If the $h_u(X(t))$ are negatively correlated (as is the case for multinomial data generating process, see Example $[3.5]$) then Assumption $[P_1]$ is satisfied.
Remark 9. Note that for \( x \in \mathbb{R}^d \) and \( X \) a \( d \times d \) symmetric matrix we have \( \|x\|_X = x^\top X x \leq \|X\|_{sp} \|x\|_2^2 \) with \( \|X\|_{sp} \) the spectral norm defined in Section 2.7.

The univariate case follows the following observation in Wasserstein distance. As the identity function \( \text{Id} : \mathbb{R} \to \mathbb{R} : x \mapsto x \) is Lipschitz-1, we have the following lower bound:

\[
d_{\text{Wass}}(\theta^*, N) \geq |E[\theta^*]| = K|E[\theta] - \theta_0|.
\]

(44)

It is of interest to know when the choice \( \text{Id} \) achieves the supremum in the Wasserstein distance. In this context, the notion of stochastic ordering comes into the picture. We recall that two real-valued random variables \( X, Y \) are stochastically ordered if the function \( x \mapsto \mathbb{P}[X \leq x] - \mathbb{P}[Y \leq x] \) does not change sign over \( x \in \mathbb{R} \). The following result can be found e.g. in [44, Theorem 1.A.11].

Proposition 1. Let \( X, Y \) have finite mean. Then \( X, Y \) are stochastically ordered if and only if \( d_{\text{Wass}}(X, Y) = |E[X] - E[Y]| \).

The next proposition ensues (see also [51] for related material).

Proposition 2. If the function \( w \mapsto \lambda'(w) - n\bar{h}(x) - K^2(w - \theta_0) \) does not change sign on the support of \( p^*(\cdot | x) \) then \( \theta^* \) and \( N \) are stochastically ordered, and \( d_{\text{Wass}}(\theta^*, N) = K|E[\theta] - \theta_0| \).

Proof. A well-known sufficient condition for stochastic ordering of two absolutely continuous distributions is when their log-likelihood is monotone (see e.g. [37]). Straightforward computations show that this happens as soon as the stated condition holds.

\[ \square \]

3 Applications

3.1 Bernoulli data

Consider an observation \( x = (x_1, \ldots, x_n) \) sampled independently from a Bernoulli distribution with parameter \( p \in (0, 1) \), a member of an exponential family with natural parameter \( \theta \) with

\[
\theta = \log(p/(1-p)), \quad \Theta = (-\infty, \infty), \quad h_i(x) = x, \quad \beta(\theta) = \log(1 + e^\theta)
\]

(recall (9)); \( \beta \) is infinitely differentiable on \( \Theta \). With prior \( \pi_0(\theta) = \exp(-\eta(\theta)) \) the function from (11) is

\[
\lambda(\theta) = n\bar{\beta}(\theta) + \eta(\theta) = n\beta(\theta) + \eta(\theta) = n\log(1 + e^\theta) + \eta(\theta).
\]

We first consider normal approximation of the posterior distribution centred at its mode as in Example 3 i.e.

\[
\hat{\theta} = (\lambda')^{-1}(n\bar{x}), \quad \hat{K} = \sqrt{\lambda''(\hat{\theta})} \quad \text{and} \quad \theta^*_{\text{MAP}} = \sqrt{\lambda''(\theta - \hat{\theta})}.
\]

(45)

We fix a conjugate prior on \( \theta \), i.e. \( \eta(\theta) = -\tau_1 \theta + \tau_2 \log(1 + e^\theta) \) for some \( \tau_1 > 0, \tau_2 - \tau_1 > 0 \). Then

\[
\lambda(\theta) = -\tau_1 \theta + (n + \tau_2) \log(1 + e^\theta), \quad \lambda'(\theta) = -\tau_1 + (n + \tau_2) \frac{e^\theta}{1 + e^\theta},
\]

\[
\lambda''(\theta) = (n + \tau_2) \frac{e^\theta}{(1 + e^\theta)^2}, \quad \text{and} \quad \lambda'''(\theta) = (n + \tau_2) \frac{e^\theta - e^{2\theta}}{(1 + e^\theta)^3}.
\]

For ease of notation (and also to connect with the forthcoming Example 3.5) we introduce the quantities \( \pi_1 = n\bar{x} + \tau_1 \) and \( \pi_2 = n(1 - \bar{x}) + \tau_2 - \tau_1 \); note that \( \pi_1 + \pi_2 = n + \tau_2 \) does not depend on the sample. Simple computations yield

\[
\hat{\theta} = \log\left(\frac{\pi_1}{\pi_2}\right), \quad \text{and} \quad \lambda''(\hat{\theta}) = \frac{\pi_1 \pi_2}{\pi_1 + \pi_2}.
\]

We have \( \hat{K} = \sqrt{\pi_1 \pi_2 / (\pi_1 + \pi_2)} = \sqrt{\pi_1 \pi_2 / (n + \tau_2)} \) and thus \( \|\hat{F}\|_{\text{diag}} = \tilde{J} = \sqrt{1/\pi_1 + 1/\pi_2} \). Since \( 0 \leq (e^x - e^{2x})/(1 + e^x)^3 \leq 1/(6\sqrt{3}) \) for all \( x \in \mathbb{R} \), we can apply Lemma 10 with \( S(t) = 1/(6\sqrt{3}) \) to get

\[
\Delta_3(\lambda, \hat{\theta}, \hat{K}) \leq \frac{1}{12\sqrt{3}} \frac{(n + \tau_2)^2}{\pi_1 \pi_2} E \left[ (\theta^*_{\text{MAP}})^2 \right].
\]
It follows that the main term in our upper bounds in Theorem 1 satisfies

$$\tilde{J}\Delta_3(\lambda, \hat{\theta}, \bar{K}) \leq \frac{1}{12\sqrt{3}} \frac{1}{\sqrt{n}} \left( \frac{1 + \tau_2/n}{(\bar{x} + \tau_2/n)(1 - \bar{x} + (\tau_2 - \tau_1)/n)} \right)^{3/2} E[(\theta^*_{MAP})^2].$$

The conjugate prior setting allows us to compute the posterior moments explicitly. With $\Gamma$ the gamma function, $\psi(t) = \Gamma'(t)/\Gamma(t)$ the digamma function and $\rho(t) = \log(t) - \psi(t)$ we have

$$E[\theta] - \bar{\theta} = \rho(\pi_1) - \rho(\pi_2),$$

and, since $1 - x \leq 1/(6\sqrt{3}x(1 - x))$ for $0 < x < 1$, we see that (50) can be improved to

$$\lim_{n \to \infty} \sqrt{n} U_n(x, \tau) = \frac{1}{2} \left( |1 - 2p^*| \right) \left( \frac{n + \tau_2}{2} \right)^{1/2} \left( \frac{1}{\pi_1} + \frac{1}{\pi_2} \right)^{1/2} |n(1 - 2\bar{x}) + \tau_2 - 2\tau_1| E[(\theta^*_{MAP})^2]$$

$$+ \frac{n + \tau_2}{24} \left( \frac{1}{\pi_1} + \frac{1}{\pi_2} \right)^{1/2} E[(\theta^*_{MAP})^3]$$

$$= \frac{|1 - 2\bar{x}|}{2\sqrt{n \bar{x}(1 - \bar{x})}} + \frac{1}{24} \frac{1}{\pi (\bar{x}(1 - \bar{x}))^{2n}} + r_n,$$

where $r_n$ is a remainder term of order $O(n^{-3/2})$. In obtaining the above equality, we used that $E[(\theta^*_{MAP})^2] = 1 + O(n^{-1/2})$ and $E[(\theta^*_{MAP})^3] = E[N] + O(n^{-1/2}) = 2\sqrt{2/\pi} + O(n^{-1/2})$. From the inequalities in (39) we conclude that

$$d_{TV}(\theta^*, N) \leq \sqrt{\frac{\pi}{8}} U_{2,n}(x, \tau), \quad d_{Wass}(\theta^*, N) \leq U_{2,n}(x, \tau),$$

and, since $1 - 2x \leq 1/(6\sqrt{3}x(1 - x))$ for $0 < x < 1$, we see that (50) can be improved to

$$\lim_{n \to \infty} \sqrt{n} U_{2,n}(x, \tau) \leq \frac{1}{2} \frac{|1 - 2p^*|}{\sqrt{p^*(1 - p^*)}}.$$
Figure 1: This figure illustrates the results from Example 3.1 (Bernoulli data) using MAP centring and scaling with a conjugate prior. **Orange curves:** True values of the Wasserstein distance, obtained numerically. **Blue curves:** Lower bound of the Wasserstein distance, as given by equation (47). **Red curves:** Upper bound of the Wasserstein distance, as given by equation (51). **Green curves:** Additional bound, computed using equation (48). In panels (a) to (e), the values are scaled by multiplying them by the square root of the sample sizes for the specified values of $p^\star$. In panel (f), the same data as in panel (e) is presented, but the values are instead multiplied by the sample size. Each curve represents the average of 10 simulations performed for sample sizes $n \in \{100, 120, \ldots, 500\}$, all derived from the same Bernoulli dataset. The prior parameters are arbitrarily fixed at $\tau_1 = 0.84$ and $\tau_2 = \sqrt{2}$. In many panels, the blue curves overlap with the orange curves, and the green curves overlap with the red curves, making the blue and green curves largely invisible.
For the sake of illustration we computed the Wasserstein and total variation distance bounds in [46, 53] for Bernoulli samples with success parameters $p^* = i/10$, $i = 1, \ldots, 5$, for sample sizes ranging from $n = 100$ to $n = 500$; we also compute a numerical approximation of the true Wasserstein and total variation distances since, in this simple setting, it is easy to evaluate the corresponding integrals numerically at least for moderate sample sizes up to $n = 500$. We illustrate the results in Figure 1 only in the case of the Wasserstein distance since the conclusions in the total variation distance are similar. Several lessons can be learned from here. First, it appears that (47) is in fact the exact value of the Wasserstein distance in this case (in view of Proposition 1 we conjecture the two distributions are stochastically ordered) and also, up to a multiplicative constant, the exact value of the total variation distance. Second, when $p^* = 0.5$ the true rate of convergence is of order $n^{-1}$; the third order upper bound from (46) misses this information while the fourth order upper bound in (53) captures it. We illustrate this in subfigures (e) and (f), by changing the scaling factor on the same data. We believe that the fact that the blue curve is sometimes slightly below the orange curve is explained by numerical instability in the integrals. Finally, we note that the proximity between the lower and upper bounds visible in panel (b) is the result of an intriguing situation: the lower bound and the two upper bounds yield the same value when $p^* = (3 \pm \sqrt{3})/6$ (i.e. $p^* = 0.211\ldots$ and $p^* = 0.788\ldots$); hence whenever $\bar{x}$ is close to this value, the bounds are nearly identical.

Next, we study normal approximation of the posterior standardised around the MLE. Here $\theta$ is distributed according to $p_2(\cdot | x)$ as in [10] with some prior $\eta : \mathbb{R} \to \mathbb{R}$, which we assume to have bounded third and fourth order derivatives on $\Theta$ and

$$\theta^*_\text{MLE} = \sqrt{n\beta''(\hat{\theta})}(\theta - \hat{\theta})$$

with $\hat{\theta} = (\beta')^{-1}(\bar{x})$.

Since $\beta(\theta) = \log(1 + e^\theta)$, solving $\beta'(\theta) = \bar{x}$, we obtain $\hat{\theta} = \log(\bar{x}/(1 - \bar{x}))$. We also have $\hat{K} = (n\beta''(\hat{\theta}))^{1/2} = \sqrt{n\bar{x}(1 - \bar{x})}$. Then $||\hat{F}||_\text{diag} = \hat{J} = 1/\sqrt{n\bar{x}(1 - \bar{x})}$. Also, again on using Lemma 10 to bound $D_3(\lambda, \hat{\theta}, \hat{K})$, we obtain

$$D_1(\lambda, \hat{\theta}, \hat{K}) \leq |\eta'(\hat{\theta})|, \quad D_2(\lambda, \hat{\theta}, \hat{K}) = |\eta''(\hat{\theta})| \mathbb{E}[|\theta - \hat{\theta}|],$$

$$D_3(\lambda, \hat{\theta}, \hat{K}) \leq \frac{1}{\bar{x}(1 - \bar{x})} \left( \frac{1}{12\sqrt{3}} + \frac{||\eta^{(3)}||_\infty}{n} \right) \mathbb{E}[(\theta^*_\text{MLE})^2],$$

$$\lambda^{(3)}(\hat{\theta}) = \bar{x}(1 - \bar{x})(1 - 2\bar{x}) + \eta^{(3)}(\hat{\theta}), \quad ||\lambda^{(3)}||_\infty \leq n/8 + ||\eta^{(4)}||_\infty.$$

It follows that the main term in the bounds from Theorem 1 satisfies

$$\hat{J}\Delta_3(\lambda, \hat{\theta}, \hat{K}) \leq \frac{|\eta'(\hat{\theta})|}{\sqrt{n\bar{x}(1 - \bar{x})}} + \frac{|\eta''(\hat{\theta})|}{n\bar{x}(1 - \bar{x})} \mathbb{E}[||\theta^*_\text{MLE}||] + \frac{1}{12\sqrt{3} + \frac{2n}{2n\bar{x}(1 - \bar{x})}} \mathbb{E}[||\theta^*_\text{MLE}||^2]$$

(55)

which gives bounds on the total variation and Wasserstein distances in this case as well. Also, bounding the quantity $\Delta_4(\lambda, \hat{\theta}, \hat{K})$, defined in equation (38) gives that

$$\hat{J}\Delta_4(\lambda, \hat{\theta}, \hat{K}) \leq \frac{|\eta'(\hat{\theta})|}{\sqrt{n\bar{x}(1 - \bar{x})}} + \frac{|\eta''(\hat{\theta})|}{n\bar{x}(1 - \bar{x})} \mathbb{E}[||\theta^*_\text{MLE}||] + \frac{1 - 2\bar{x}}{2\sqrt{n\bar{x}(1 - \bar{x})}} (1 + \frac{1}{2\sqrt{n\bar{x}(1 - \bar{x})}}) \mathbb{E}[||\theta^*_\text{MLE}||^2]$$

$$+ \frac{1}{48n(x(1 - \bar{x}))^2} \left( 1 + \frac{8||\eta^{(4)}||_\infty}{n} \right) \mathbb{E}[||\theta^*_\text{MLE}||^3].$$

In the case of a conjugate prior with exponent $\eta(\theta) = -\tau_1\theta + \tau_2 \log(1 + e^\theta)$ we have that

$$\hat{J}\Delta_4(\lambda, \hat{\theta}, \hat{K}) \leq \frac{|\tau_1 - \tau_2\bar{x}|}{\sqrt{n\bar{x}(1 - \bar{x})}} + \frac{\tau_2}{n} \mathbb{E}[||\theta^*_\text{MLE}||] + \frac{\tau_2(1 - 2\bar{x})}{2n\bar{x}(1 - \bar{x})} \mathbb{E}[||\theta^*_\text{MLE}||^2]$$

$$+ \frac{n + \tau_2}{48(n\bar{x}(1 - \bar{x}))^2} \mathbb{E}[||\theta^*_\text{MLE}||^3]$$

(56)

$$= \tau_2\frac{1 - 2\bar{x} + 2|\tau_1 - \tau_2|\bar{x}}{2\sqrt{n\bar{x}(1 - \bar{x})}} + \sqrt{\frac{\pi}{2}} \tau_2 + \frac{1}{24} \sqrt{\frac{2}{\pi}} \left( \frac{1}{(x(1 - \bar{x}))^2} + r_n, \right)$$

(57)
where $r_n$ is a remainder term of order $O(n^{-3/2})$. Observe that the bound (57) is of order $n^{-1}$ if $\bar{x} = 1/2 + O(n^{-1})$ and $\tau_2 = 2\tau_1$. In case of a flat (non-informative) prior we also get that

$$\tilde{\Delta}_4(\lambda, \hat{\theta}, \hat{K}) \leq \frac{|1 - 2\bar{x}|}{2\sqrt{n}\bar{x}(1 - \bar{x})} \mathbb{E}[(\theta_{\text{MLE}})^2] + \frac{1}{48n(\bar{x}(1 - \bar{x})^2} \mathbb{E}[|\theta_{\text{MLE}}|^3].$$

As in the MAP centric case, we performed some numerical explorations over the same configurations. These lead to similar conclusions as those reported in Figure 1 with no further surprise in the case of a flat prior. In the case of a conjugate prior, however, we read from (57) that the rate of convergence will no longer drop to $n^{-1}$ when $p^* = 0.5$ unless $\tau_1 = 2\tau_2$. This is also indicated in our simulations, see Figure 2.

Figure 2: Exactly the same configuration as in Figure 1 but with MLE centring and scaling using a conjugate prior. The values are multiplied by the sample size. **Blue curves:** True values of the Wasserstein distance. **Orange curves:** Bound as defined in equation (55). **Green curves:** Bound as given in equation (57). All curves are multiplied by the sample size. In the left panel (panel (a)), the parameters are set to $\tau_1 = 0.84$ and $\tau_2 = \sqrt{2}$. In the right panel (panel (b)), the parameters are set to $\tau_1 = 1$ and $\tau_2 = 2$. In both panels, $p^* = 0.5$.

### 3.2 Poisson data

Consider i.i.d. Poisson data with parameter $\mu > 0$. This distribution is an exponential family with natural parameter $\theta$ with

$$\theta = \log \mu, \quad \Theta = (-\infty, \infty), \quad h_i(x) = x, \quad \beta(\theta) = e^\theta.$$  

We focus on normal approximation of the posterior distribution centred at its mode so that (15) is the variable of interest. We consider a conjugate gamma prior on $\mu$ of the form $\pi_0(\theta) \propto \exp(-\eta(\theta))$ with $\eta(\theta) = -\tau_1 \theta + \tau_2 e^\theta$ for $\tau_1, \tau_2 > 0$. Then $\lambda(\theta) = -\tau_1 \theta + (n + \tau_2) e^\theta$, so that $\lambda'(\theta) = -\tau_1 + (n + \tau_2) e^\theta$ and $\lambda^{(k)}(\theta) = (n + \tau_2) e^\theta$ for all $k \geq 2$. Solving $\lambda'(\theta) = n\bar{x}$ we obtain

$$\tilde{\theta} = \log \left( \frac{\tau_1 + n\bar{x}}{n + \tau_2} \right) \quad \text{and} \quad \lambda^{(k)}(\tilde{\theta}) = \tau_1 + n\bar{x}, \quad k \geq 2.$$  

In particular, $\tilde{K} = \sqrt{\tau_1 + n\bar{x}}$ and $\tilde{J} = 1/\sqrt{\tau_1 + n\bar{x}}$.

A Wasserstein distance bound is easy to obtain. Indeed, since $\lambda''(\theta)$ is always positive, Proposition 2 applies and we immediately obtain

$$d_{\text{Wass}}(\theta_{\text{MAP}}, N) = \sqrt{n\bar{x} + \tau_1} \left( \log(n\bar{x} + \tau_1) - \psi(n\bar{x} + \tau_1) \right),$$  

and thus, if the ground truth parameter is $\lambda^*$, we get

$$\lim_{n \to \infty} \sqrt{n}d_{\text{Wass}}(\theta_{\text{MAP}}, N) = 1/(2\sqrt{\lambda^*})$$

for all admissible values $\tau_1, \tau_2$. 

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For a total variation distance bound, Lemma 10 (or Lemma 11) can be applied but $\lambda''$ is unbounded so this does not bring a true simplification. We use \eqref{eq:tvbound}. We first compute

$$
E \left[ Y_1 \lambda''(\tilde{\theta} + Y_1 Y_2 w) \right] = (\tau_1 + n\bar{x}) E [ Y_1 \exp (Y_1 Y_2 w)] = \frac{\tau_1 + n\bar{x}}{w^2} (e^w - 1 - w).
$$

Next we note that for $|w| \leq \epsilon$ it holds that $|e^w - 1 - w| \leq (w^2/2)e^\epsilon$ so that with $\Theta^*_\epsilon = (-\tilde{K}\epsilon, \tilde{K}\epsilon)$ we get that the various terms in \eqref{eq:tvbound} satisfy

\[ \mathcal{D}_2(\lambda, \hat{\theta}, \tilde{K} \mid (-\tilde{K}\epsilon, \tilde{K}\epsilon)) \leq \frac{e^\epsilon}{2} E [(\theta^*_{\text{MAP}})^2], \]

\[ P \left[ \theta^*_{\text{MAP}} \notin (-\tilde{K}\epsilon, \tilde{K}\epsilon) \right] \leq \frac{E [(\theta^*_{\text{MAP}})^2]}{\tilde{K}^2 \epsilon^2}, \quad r^*_1(\tilde{K}\epsilon) = \sqrt{\frac{1}{\pi}} \max(p^*(\tilde{K}\epsilon), p^*(-\tilde{K}\epsilon)) \]

(for the first inequality we bound the indicator function by 1, for the second we use Markov’s inequality). Recalling \eqref{eq:tvbound} we obtain after some simplifications

\[ p^*(\pm \tilde{K}\epsilon) = \frac{(n\bar{x} + \tau_1)n\bar{x} + \tau_1 - \frac{1}{2}}{\Gamma(n\bar{x} + \tau_1)} e^{-(n\bar{x} + \tau_1)(e^{\pm \epsilon} - 1)} \leq \frac{1}{\sqrt{2\pi}} \exp \left( - (n\bar{x} + \tau_1)(e^{-\epsilon} + \epsilon - 1) \right) \]

where we used Stirling’s inequality to obtain the first inequality, and the basic inequality $e^y - y \geq e^{-y} + y$, for $y > 0$, to obtain the second inequality. Since $e^{-\epsilon} + \epsilon - 1 > 0$ for all $\epsilon > 0$, the term $r^*_1(\tilde{K}\epsilon) \leq r_n(\epsilon)$ is exponentially negligible in $n$ for $\epsilon$ not too small. We now have for all $\epsilon > 0$,

\[ d_{TV}(\theta^*_{\text{MAP}}, N) \leq \sqrt{n\bar{x} + \tau_1} \left( \sqrt{\frac{\pi}{2}} \frac{e^\epsilon}{\sqrt{2\pi}} + \frac{1}{e^{\tau_1 + n\bar{x}}} \right) E [(\theta^*_{\text{MAP}})^2] + \sqrt{\frac{\pi}{8}} r_n(\epsilon). \tag{59} \]

Since for all $n, \bar{x}$ such that $n\bar{x} + \tau_1 \geq 27$ there exists $\epsilon$ such that $\sqrt{\frac{\pi}{2}} \frac{e^\epsilon}{\sqrt{2\pi}} + \frac{1}{e^{\tau_1 + n\bar{x}}} \leq 1$ (here we can take $\epsilon = 0.85489$), we can conclude that

\[ d_{TV}(\theta^*_{\text{MAP}}, N) \leq \sqrt{n\bar{x} + \tau_1} E [(\theta^*_{\text{MAP}})^2] \tag{60} \]

which gives a bound on the rate of convergence of order $n^{-1/2}$ since $E [(\theta^*_{\text{MAP}})^2] = 1 + o(1)$ as $n \to \infty$. We can, as in Example 3.1, use the conjugate prior setting to compute directly

\[ E[(\theta - \hat{\theta})^2] = \psi'(n\bar{x} + \tau_1) + (\psi(n\bar{x} + \tau_1) - \log(n\bar{x} + \tau_1))^2. \]

This yields

\[ d_{TV}(\theta^*_{\text{MAP}}, N) \leq \sqrt{n\bar{x} + \tau_1} (\psi'(n\bar{x} + \tau_1) + (\psi(n\bar{x} + \tau_1) - \log(n\bar{x} + \tau_1))^2) \lesssim \frac{1}{\sqrt{n\bar{x} + \tau_1}}. \tag{61} \]

Here $\lesssim$ indicates an inequality up to an unspecified absolute constant. Numerical explorations along the lines of Figure 1 indicate that the bound \eqref{eq:tvbound} should be an equality up to a (nearly) constant factor of roughly 8, see Figure 3. Finally, as in the previous example, the total variation and Wasserstein distances appear to be (nearly) proportional to each other.

### 3.3 Normal with known mean data

Consider i.i.d. normal data with known mean $m$ and unknown variance $\sigma^2$. This distribution is written as an exponential family with natural parameter $\theta$ by taking

\[ \theta = \sigma^{-2}, \quad \Theta = (0, \infty), \quad h(x) = -(x - m)^2/2 \text{ and } \beta(\theta) = -(1/2) \log \theta. \]

We denote the prior $\pi_0(\theta) \propto \exp(-\eta(\theta))$ so that $\lambda(\theta) = -(n/2) \log \theta + \eta(\theta)$. Note that the function $\beta$ is infinitely differentiable.
We get, after some straightforward simplifications, \( n \) sample sizes. All curves are computed for the same sample of Poisson data with parameter \( \lambda \) the square root of the sample size. The orange and blue curves are, obviously, indistinguishable in the bounds, computed via equation (61) (right column), also multiplied by the square root of the sample size. The orange and blue curves are, obviously, indistinguishable in the left panel. All curves are computed for the same sample of Poisson data with parameter \( \lambda = 10 \), for sample sizes \( n \in \{100, 120, \ldots, 500\} \). The prior parameters are set to \( \tau_1 = 1 \) and \( \tau_2 = 3 \).

We consider normal approximation of the posterior standardised around the posterior mode, with a conjugate gamma prior on \( \theta \) of the form \( \pi_0(\theta) \propto \exp(-\eta(\theta)) \), where \( \eta(\theta) = \tau_1 \theta - \tau_2 \log \theta \) for \( \tau_1 > 0 \) and \( \tau_2 > -1 \). The posterior distribution has a gamma distribution with shape parameter \( n/2 + \tau_2 + 1 \), and so Assumption A2 is satisfied if \( n + 2\tau_2 > 0 \). Then \( \lambda(\theta) = \tau_1 \theta - (\tau_2 + n/2) \log \theta \), so that, for \( n \geq 2 \),

\[
\lambda'(\theta) = \tau_1 - \frac{n/2 + \tau_2}{\theta}, \quad \lambda''(\theta) = \frac{n/2 + \tau_2}{\theta^2} \quad \text{and} \quad \lambda'''(\theta) = -\frac{n + 2\tau_2}{\theta^3}.
\]

Let \( ns_n^2 = \sum_{i=1}^n (x_i - m)^2 \). Solving \( \lambda'(w) = -ns_n^2/2 \) we obtain

\[
\hat{\theta} = \frac{n + 2\tau_2}{ns_n^2 + 2\tau_1} \text{ so that } \lambda'''(\hat{\theta}) = \frac{(ns_n^2 + 2\tau_1)^2}{2(n + 2\tau_2)}.
\]

In particular, \( \tilde{K} = (ns_n^2 + 2\tau_1)/\sqrt{2(n + 2\tau_2)} \) and \( \tilde{J} = \sqrt{2(n + 2\tau_2)}/(ns_n^2 + 2\tau_1) \).

We have

\[
|\mathbb{E} \left[ Y_1 \lambda'''(\hat{\theta}) + Y_1 Y_2 w \right]| = \mathbb{E} \left[ Y_1 \frac{n + 2\tau_2}{(Y_1 Y_2 w + \frac{n + 2\tau_2}{ns_n^2 + 2\tau_1})^3} \right] = \frac{1}{2} \frac{(ns_n^2 + 2\tau_1)^2}{(n + 2\tau_2)w + \frac{(n + 2\tau_2)}{(ns_n^2 + 2\tau_1)}}.
\]

We get, after some straightforward simplifications,

\[
\mathcal{D}_d(\lambda, \hat{\theta}, \tilde{K}) \leq \frac{(ns_n^2 + 2\tau_1)^2}{2(n + 2\tau_2)} \mathbb{E} \left[ (\theta - \hat{\theta})^2 \right] = \frac{ns_n^2 + \tau_1}{n + 2\tau_2}
\]

(the equality follows since \( \theta \) is gamma distributed with known parameters). Hence from \( \text{(36)} \) we get

\[
d_{TV}(\theta_{MAP}^*, N) \leq \sqrt{\frac{\pi}{4}} \frac{1}{\sqrt{n + 2\tau_2}}.
\]

Similarly, the Wasserstein inequality \( \text{(35)} \) yields \( d_{Wass}(\theta_{MAP}^*, N) \leq \sqrt{2}/\sqrt{n + 2\tau_2} \). Also, we can use inequality \( \text{(44)} \) to obtain a lower bound for the Wasserstein distance \( d_{Wass}(\theta_{MAP}^*, N) \geq \sqrt{2}/\sqrt{n + 2\tau_2} \), whence

\[
d_{Wass}(\theta_{MAP}^*, N) = \frac{\sqrt{2}}{\sqrt{n + 2\tau_2}}.
\]

Numerical explorations indicate that the total variation bounds are off from the simulated total variation distance by a constant factor and the total variation distance appears, as for the previous examples.
and for some parameter constellations, to be a direct multiple of the Wasserstein distance. The specific values taken by the sample have no impact on the Wasserstein distance and appear to have little to no impact on the total variation distance. When \( \tau_2 \) is very large the distance is small.

Next, we study normal approximation of the posterior standardised around the MLE, focusing on Wasserstein distance. Here \( \beta(\theta) = - (1/2) \log \theta \) and \( \beta^{(1)}(\theta) = (1/2)(\ell - 1)/(2\theta^\ell) \) for \( \ell \geq 1 \). Solving \( \beta'(\theta) = - s_n^2/2 \) we obtain

\[
\hat{\theta} = 1/s_n^2
\]

so that \( \beta''(\hat{\theta}) = s_n^4/2 \). Thus, \( \hat{K} = \sqrt{n/2s_n^2} \) and \( \hat{\beta} = \sqrt{2}/(\sqrt{n}s_n^2) \). We take note of the identity

\[
\mathbb{E} \left[ Y_1\beta''(\theta + Y_1w) \right] = \mathbb{E} \left[ \frac{Y_1}{(Y_1w + \theta)^3} \right] = \frac{1}{2(\theta^2)^2} \left[ 1 - \frac{1}{\theta} \right]
\]

which, after simplifications, gives

\[
\mathcal{D}_3(\lambda, \hat{\theta}, \hat{K}) \leq \frac{(s_n^2)^2}{2} \mathbb{E} \left[ \frac{(\theta - \hat{\theta})^2}{|\theta|} \right] + \mathbb{E} \left[ Y_1\eta''(R_2)(\theta - \hat{\theta})^2 \right]
\]

with \( R_2 = \hat{\theta} + Y_1\tau(\theta - \hat{\theta}) \) (we express everything in terms of \( \theta \) rather than \( \theta^* \) because this will simplify the computations slightly). From (45) (also recall Example 3) we get

\[
d_{\text{Wass}}(\theta^*_{\text{MLE}}, N) \leq \sqrt{\frac{2 \eta'(s_n^{-2})}{n}} + \sqrt{\frac{2 \eta''(s_n^{-2})}{s_n^2}} \mathbb{E} [|\theta - \hat{\theta}|] + \frac{s_n^2}{\sqrt{2n}} \mathbb{E} \left[ \frac{(\theta - \hat{\theta})^2}{|\theta|} \right] + \sqrt{\frac{2}{n s_n^2}} \mathbb{E} \left[ Y_1\eta''(R_2)(\theta - \hat{\theta})^2 \right].
\]

The moments can be obtained explicitly, and all terms are easy to obtain once the properties of the prior are known. For instance, if the prior is \( w_\tau = \tau_1\theta - \tau_2 \log \theta \) for \( \tau_1 > 0 \) and \( \tau_2 > -1 \) we have \( \mathbb{E} [|\theta_{\text{MLE}}|^2] \leq n (s_n^4(n + 2(\tau_2 + 1)(\tau_2 + 2)) - 4s_n^2 \tau_1(\tau_2 + 1) + 2\tau_2)/(ns_n^2 + 2\tau_1) \) whence

\[
d_{\text{Wass}}(\theta^*_{\text{MLE}}, N) \leq \frac{\tau_1 - \tau_2 s_n^2}{\sqrt{n/2s_n^2}} \frac{2|\tau_2|}{n} \mathbb{E} [\theta^*_{\text{MLE}}] + \frac{\sqrt{2}}{n} \mathbb{E} [\sqrt{2} - 2\tau_2 (ns_n^4 + 2(s_n^2 + (\tau_1 - s_n^2\tau_2))(ns_n^2 + 2\tau_1)/(n + \tau_2)).
\]

The leading terms in the above give the bound

\[
d_{\text{Wass}}(\theta^*_{\text{MLE}}, N) \leq \sqrt{\frac{2}{n} \frac{|\tau_2 s_n^2 - \tau_1 + s_n^2|}{s_n^2}} + e(n, \tau_1, \tau_2, s_n^2)
\]

with \( e(n, \tau_1, \tau_2, s_n^2) \) an (explicit) error term of lower order \( O(n^{-1}) \). A lower bound can be calculated as well from (44), leading to

\[
d_{\text{Wass}}(\theta^*_{\text{MLE}}, N) \geq \sqrt{\frac{2}{n} \frac{|\tau_2 s_n^2 - \tau_1 + s_n^2|}{s_n^2}} \frac{1}{1 + 2\tau_1/(ns_n^2)},
\]

which is very close to the upper bound. Bounds on the total variation distance follow from exactly the same argument, only with different constants. The MLE depends on all the model parameters as well as the sample in this case, contrarily to the posterior mode centring which led to distances which depend only on \( \tau_2 \) and not on the sample. As before, all these findings are confirmed by numerical explorations; in particular these results indicate that there are parameter constellations leading to equality in the Wasserstein distance with the lower bound.

### 3.4 Weibull data

Consider i.i.d. Weibull data with density on the positive real line

\[
p_1(x|\ell, m) = \frac{m}{\ell m} x^{m-1} \exp \left\{ - (x/\ell)^m \right\} = \exp \left\{ - x^m \ell^{-m} + m \log x - \log m + \log (\ell^{-m}) \right\}.
\]

We suppose \( m \) is known, and \( \ell \) is the parameter of interest. Fix, for \( \tau_1 > 0 \) and \( \tau_2 > -1 \), a (conjugate) gamma prior on \( \theta = \ell^{-m} \) of the form \( \pi_0(\theta) \propto e^{-n(\theta)} \), with \( \eta(\theta) = \tau_1 \theta - \tau_2 \log \theta \) (for \( \tau_1 = 0 \) and
\( \tau_2 = -1 \) one obtains the improper Jeffreys prior for which our bounds also hold for \( n \geq 2 \). Therefore \( \lambda(\theta) = \tau_1 \theta - (\tau_2 + n) \log \theta \) and we take \( \Theta = (0, \infty) \). By the same reasoning as for the previous example, Assumption \( A_2 \) holds if \( n + \tau_2 > 0 \). Simple computations yield \( \hat{\theta} = (1 + \tau_2/n)/(\bar{x}^m + \tau_1/n) \) and \( \hat{K} = \sqrt{n/(\bar{x}^m + \tau_1/n)} \) (here \( \bar{x}^m = n^{-1} \sum_{j=1}^n x_j^m \)). The situation is very similar to the previous example and we immediately get

\[
d_{TV}(\theta_{\text{MAP}}^*, N) \leq \sqrt{\frac{\tau_2}{8 \sqrt{n + \tau_2}}}, \quad d_{Wass}(\theta_{\text{MAP}}^*, N) = \frac{1}{\sqrt{n + \tau_2}}
\]

(equality in the Wasserstein distance follows again because the upper and lower bounds coincide; in particular we deduce that the distributions are stochastically ordered).

We can also work in the MLE standardisation, i.e. standardisation around the MLE. That is, we set \( \theta_{\text{MLE}}^* = \bar{K}(\theta - \hat{\theta}) \) with \( \hat{\theta} = 1/\bar{x}^m \) the MLE and \( \bar{K} = (n \beta''(\hat{\theta}))^{1/2} = \sqrt{n \bar{x}^m} \). Then keeping \( \eta \) unspecified we get

\[
\mathcal{D}_1(\lambda, \hat{\lambda}, \bar{K}) \leq |\eta'(\hat{\lambda})|, \quad \mathcal{D}_2(\lambda, \hat{\lambda}, \bar{K}) \leq |\eta''(\hat{\lambda})|E[|\theta - \hat{\lambda}|],
\]

\[
\mathcal{D}_3(\lambda, \hat{\lambda}, \bar{K}) \leq \frac{n}{2 \tau_2} E \left[ \frac{(\theta - \hat{\lambda})^2}{|\hat{\lambda}|} \right] + E[|\Theta^3(\eta, \hat{\lambda}, \theta_{\text{MLE}}^*)|(|\theta - \hat{\lambda}|^2)].
\]

Also, \( \|F\|_{\text{dirac}} = 1/\sqrt{n \bar{x}^m} \). Taking \( \eta \) the uniform (improper) prior, the first two terms disappear and the bounds become

\[
d_{TV}(\theta_{\text{MLE}}^*, N) \leq \sqrt{\frac{\tau_2}{8 \sqrt{n}}}, \quad d_{Wass}(\theta_{\text{MLE}}^*, N) = \frac{1}{\sqrt{n}}
\]

which do not depend on the sample. Finally, in case a conjugate prior is chosen we set

\[
\Delta = \frac{|\tau_1 - \tau_2 \bar{x}^m|}{\sqrt{n \bar{x}^m}} + \frac{|\tau_2| (1 + \tau_2)}{\sqrt{n(n \bar{x}^m + \tau_1)}} + 2 \tau_2 \frac{\Gamma(n + \tau_2 + 1, n + \tau_1/\bar{x}^m)}{\sqrt{n} \Gamma(n + \tau_2 + 1)} \frac{2 \bar{x}^m \tau_2}{n \bar{x}^m + \tau_1} \frac{n + \tau_2 + 1}{\sqrt{n} \Gamma(n + \tau_2 + 1)} + \frac{(n + \tau_2) (\bar{x}^m)^2 + (\tau_2 - \bar{x}^m)^2}{\sqrt{n} \bar{x}^m} \frac{2 \bar{x}^m (n \bar{x}^m + \tau_1)}{n \bar{x}^m + \tau_1}
\]

and get

\[
d_{TV}(\theta_{\text{MAP}}^*, N) \leq \sqrt{\frac{\pi}{8} \Delta}, \quad \sqrt{\frac{\pi}{8} \frac{|\tau_1 - (1 + \tau_2) \bar{x}^m|}{n \bar{x}^m + \tau_1}} \leq d_{Wass}(\theta_{\text{MAP}}^*, N) \leq \Delta,
\]

and thus bounds in total variation and Wasserstein distances, which, contrarily to the MAP standardisation, now depend on all parameters from the model and the data \( x \) in an intricate way.

### 3.5 Multinomial data

Consider i.i.d. multinomial data with parameter \( p = (p_1, \ldots, p_k)^T \) and individual likelihood \( p_1(x|p) = \prod_{j=1}^k p_j^{x_j} \), where \( 0 < p_1, \ldots, p_k < 1 \) with \( \sum_{j=1}^k p_j = 1 \), and \( x_1, \ldots, x_k \in \{0, 1\} \) with \( \sum_{j=1}^k x_j = 1 \). We express the likelihood as an exponential family by writing

\[
p_1(x|p) = \exp \left\{ -\sum_{j=1}^{k-1} x_j \log(p_j) + \left(1 - \sum_{j=1}^{k-1} x_j\right) \log(p_k) \right\} = \exp \left\{ \sum_{j=1}^{k-1} x_j \log \left( \frac{p_j}{p_k} \right) + \log(p_k) \right\},
\]

so \( h(x_j) = x_j, \ g(x) = 0 \) and the natural parameter is \( \theta = (\theta_1, \ldots, \theta_{k-1})^T \), where \( \theta_j = \log(p_j/p_k) \), \( j = 1, \ldots, k-1 \), and \( \Theta = \mathbb{R}^{k-1} \) (thus \( d = k - 1 \) in the notations of the previous section). Expressing the density in terms of \( \theta \), we get \( p_j(\theta) = p_k e^{\theta_j} \). Also, since \( \sum_{j=1}^k p_j = 1 \), we obtain that \( p_k + p_k \sum_{j=1}^{k-1} e^{\theta_j} = 1 \), so that \( p_k = (1 + \sum_{j=1}^{k-1} e^{\theta_j})^{-1} \). Thus, \( \beta(\theta) = -\log(p_k(\theta)) = \log\left(1 + \sum_{j=1}^{k-1} e^{\theta_j}\right) \). We fix a conjugate Dirichlet prior on \( p = (p_1, \ldots, p_{k-1})^T \) of the form \( \pi_0(p) \propto (1 - \sum_{j=1}^{k-1} p_j)^{\tau_1} \prod_{j=1}^{k-1} p_j^{\tau_j} \) for some parameters \( \tau_1, \ldots, \tau_k > -1 \). Let \( \tilde{\tau} = \sum_{j=1}^k \tau_j \). Expressing everything in terms of \( \theta \) we get \( \pi_0(\theta) \propto \exp(-\eta(\theta)) \), where \( \eta(\theta) = -\sum_{j=1}^{k-1} \tau_j \theta_j + \tilde{\tau} \log(1 + \sum_{j=1}^{k-1} e^{\theta_j}) \). Therefore \( \lambda(\theta) = -\sum_{j=1}^{k-1} \tau_j \theta_j + (n + \tilde{\tau}) \log(1 + \sum_{j=1}^{k-1} e^{\theta_j}) \).
\[ \sum_{j=1}^{k-1} e^{\theta_j} \], and, letting \( \pi_u = \tau_u + n \bar{x}_u \) with \( \bar{x}_u = n^{-1} \sum_{i=1}^{n} x_{i,u}, \; u = 1, \ldots, k \), the posterior is easily seen to be of the form

\[ p_2(\theta | \pi) = \frac{\Gamma(n + \hat{r})}{\prod_{j=1}^{k} \Gamma(\pi_j)} \exp(\sum_{j=1}^{k-1} \pi_j \theta_j) \]

(63)

with \( \pi = (\pi_1, \ldots, \pi_{k-1})^T \) (the normalising constant is known because this is, after a change of variables, a Dirichlet distribution). Note how \( \sum_{u=1}^{k} \pi_u = n + \hat{r} \).

We standardise around the posterior mode i.e. \( \hat{\theta} \) that solves the system of equations \( \partial_u \lambda(\theta) - n \bar{x}_u = 0 \), for \( u = 1, \ldots, k - 1 \). A simple calculation reveals that \( \hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_{k-1})^T \) is given by

\[ \hat{\theta}_u = \log \left( \frac{\tau_u + n \bar{x}_u}{\tau_k + n \bar{x}_k} \right) = \log \left( \frac{\pi_u}{\pi_k} \right), \; u = 1, \ldots, k - 1. \]

The second order partial derivatives of \( \lambda \) are given by

\[ \partial_{u,u} \lambda(\theta) = (n + \hat{r}) \frac{e^{\theta_u} (1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}} - e^{\theta_u})}{(1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}})^2}, \quad \partial_{u,v} \lambda(\theta) = - (n + \hat{r}) \frac{e^{\theta_u + \theta_v}}{(1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}})^2}, \quad u \neq v, \]

from which it follows that

\[ \partial_{u,u} \lambda(\hat{\theta}) = \pi_u - \frac{1}{n + \hat{r}} \pi_u^2, \quad \partial_{u,v} \lambda(\hat{\theta}) = - \frac{1}{n + \hat{r}} \pi_u \pi_v, \quad u \neq v. \]

(64)

The entries in (64) correspond to the elements in the \((k - 1) \times (k - 1)\) Hessian matrix \( \tilde{H} := H_\lambda(\hat{\theta}) \), which can therefore be rewritten as

\[ \tilde{H} = \text{diag}(\pi) - (n + \hat{r})^{-1} \pi \pi^T, \]

where \( \pi \) is the \((k - 1) \times 1\) vector with entries \( \pi_u = \tau_u + n \bar{x}_u \), \( u = 1, \ldots, k - 1 \). Standard arguments (the Sherman-Morrisson formula) give

\[ \tilde{F} := F_\lambda(\hat{\theta}) = \text{diag}(\frac{1}{\pi}) + \frac{1}{\pi_k} 11^T, \]

where \( \text{diag}(1/\pi) \) is the diagonal matrix with diagonal entries \( 1/\pi_u \), \( u = 1, \ldots, k - 1 \) and \( 1 \) is the \((k - 1) \times 1\) vector of 1’s. Also, the matrices \( \tilde{H} \) and \( \tilde{F} \) admit unique symmetric square roots \( \tilde{K} \) and \( \tilde{J} \), respectively, which are, moreover, computable explicitly (diagonal-plus-rank-one matrices are well-studied, see e.g. [12] [47]).

With these preliminaries, we are ready to tackle the bound from Theorem 1. From Example 5 we know that all we need is to control two terms. For the first, we have by definition,

\[ \|\tilde{F}\|_{\text{diag}} = \sqrt{\max_{1 \leq u \leq k-1} |\tilde{F}_{uu}|} \leq \max_{1 \leq u \leq k-1} \sqrt{\frac{1}{\pi_u} + \frac{1}{\pi_k}} \leq \sqrt{\frac{2}{\min \pi}}. \]

(65)

Next, for \( \Omega_{\beta}(\lambda, \hat{\theta}, \tilde{K}) \) the term depending on the third derivatives of \( \lambda \), we start by noting that

\[ \partial_{u,u,u} \lambda(t) = (n + \hat{r}) \frac{e^{\theta_u} (1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}} - e^{\theta_u}) (1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}} - 2 e^{\theta_u})}{(1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}})^3}, \]

for all \( 1 \leq u \leq k - 1 \),

\[ \partial_{u,u,v} \lambda(t) = - (n + \hat{r}) \frac{e^{\theta_u + \theta_v} (1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}} - 2 e^{\theta_u})}{(1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}})^3}, \]

for all \( 1 \leq u \neq v \leq k - 1 \),

\[ \partial_{u,v,w} \lambda(t) = (n + \hat{r}) \frac{2 e^{\theta_u + \theta_v + \theta_w}}{(1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}})^3}, \]

for all \( 1 \leq u \neq v \neq w \leq 1 \).

Let

\[ p_u(t) = \frac{e^{\theta_u}}{1 + \sum_{\ell=1}^{k-1} e^{\theta_{\ell}}}. \]

(66)
Then, since $0 \leq p_u(t) \leq 1$ and $0 \leq \sum_{u=1}^{k-1} p_u(t) \leq 1$ for all $t$, clearly
\[
|\partial_{u,v} \lambda(t)| \leq (n + \bar{\tau}) p_u(t), \quad |\partial_{u,v} \lambda(t)| \leq (n + \bar{\tau}) p_u(t) p_v(t)
\]
and $|\partial_{u,v} \lambda(t)| \leq 2(n + \bar{\tau}) p_u(t) p_v(t) p_w(t)$ for all $1 \leq u \neq v \neq w \leq k - 1$. Assumption [P1] is therefore satisfied on the entire parameter space with $C_1 = C_2 = (n + \bar{\tau})$ and $C_3 = 2(n + \bar{\tau})$, as well as $C_\infty(t) \leq 4(n + \bar{\tau})$ (for all $t$) and we can apply Lemma [H] to get
\[
\Delta_3(\lambda, \hat{\theta}, \bar{K}) = |D_3(\lambda, \hat{\theta}, \bar{K})| \leq 4(n + \bar{\tau}) \mathbb{E} \left[ \sum_{u=1}^{k-1} p_u(R_2)(\hat{\theta}_u - \hat{\theta}_u)^2 \right]^{1/2}
\]
(as before $R_2 = \hat{\theta} + Y_1 Y_2(\theta - \hat{\theta})$). A proof of the following lemma is given in Section [C] of the Appendix.

**Lemma 12.** Under the previous notations, and taking without loss of generality that $\pi_k = \max \pi$, we have, for $u = 1, \ldots, k - 1$,
\[
\mathbb{E}[(\theta_u - \hat{\theta}_u)^2] \leq \frac{2}{\pi_u} \left( 1 + \frac{3}{2\pi_u} \right), \quad \mathbb{E}[(\theta_u - \hat{\theta}_u)^4] \leq \frac{12}{\pi_u^2} \left( 1 + \frac{11}{3\pi_u} + \frac{11}{4\pi_u^2} \right).
\]

Explicit bounds on the total variation and Wasserstein distances now follow easily. We apply (35) and (36) with the simplification $p_u(R_2) \in (0, 1)$ to get
\[
\Delta_3(\lambda, \hat{\theta}, \bar{K}) \leq 2(n + \bar{\tau}) \sum_{u=1}^{k-1} \mathbb{E}[(\theta_u - \hat{\theta}_u)^2] \leq 2(n + \bar{\tau}) \frac{k}{\min \pi} \left( 1 + \frac{3}{2 \min \pi} \right).
\]
Hence
\[
d_{\text{Wasser}}(\theta_{\text{MAP}}, N) \leq 4\sqrt{2(n + \bar{\tau})} \left( \frac{k}{\min \pi} \right)^{3/2} \left( 1 + \frac{3}{2 \min \pi} \right),
\]
\[
d_{\text{TV}}(\theta_{\text{MAP}}, N) \leq 4\sqrt{\pi(n + \bar{\tau})} \left( \frac{k}{(\min \pi)^{3/2}} \right) \left( 1 + \frac{3}{2 \min \pi} \right).
\]
The Wasserstein and total variation bounds are of orders $O((k^6/n)^{1/2})$ and $O((k^5/n)^{1/2})$, respectively, if $\min \pi \approx n/k$ (consistency is proved if $k^6 \log k/n \to 0$ in [23], and if $k^4/n \to 0$ in [3]).

The dependence on the dimension can be reduced in the Wasserstein and total variation distances at the cost of some more work. Here, for the sake of simplicity, we work only up to an absolute constant, but in Appendix [C] we derive bounds with the constants as stated in Example [2]. First, we note that, by definition,
\[
p_u(\hat{\theta}) = \pi_u \frac{1}{n + \bar{\tau}}, \quad u = 1, \ldots, k - 1.
\]

Observe that (68) is of order $O(1/k)$ if $\max \pi \approx n/k$, which is exactly what is needed to reduce the dimensional dependence of our bounds. Following [23] Equation (3.4)], we thus expand $p_u(t)$ around the posterior mode $\hat{\theta}$ to get, for all $u = 1, \ldots, k$,
\[
p_u(R_2) \leq p_u(\hat{\theta}) + \left| p_u(R_2) - p_u(\hat{\theta}) \right| \leq \pi_u \frac{1}{n + \bar{\tau}} + \left| e^{R_{2,u} - \hat{\theta}_u} \right| + \pi_u \sum_{v=1}^{k-1} \left| e^{R_{2,v} - \hat{\theta}_v} \right| \frac{(n + \bar{\tau})}{\pi_k} \sum_{u=1}^{k-1} \left| e^{R_{2,u} - \hat{\theta}_u} \right| (1 + \sum_{\ell=1}^{k-1} e^{R_{2,\ell}}) \left( 1 + \sum_{\ell=1}^{k-1} e^{R_{2,\ell}} \right)
\]
\[
\leq \pi_u \frac{1}{n + \bar{\tau}} + \pi_u M(\theta) \frac{|\theta_u - \hat{\theta}_u|}{(n + \bar{\tau})^2} \sum_{v=1}^{k-1} \pi_v |\theta_v - \hat{\theta}_v|,\]
where $M(\theta) = \max_{1 \leq \ell \leq k-1} e^{2(\theta_u - \hat{\theta}_u)}$. Here we used the basic inequality $|e^{x} - e^{y}| \leq |x - y| e^{x+y}$, from which we obtain that $|e^{R_{2,u} - \hat{\theta}_u} \leq |\theta_u - \hat{\theta}_u| e^{|\theta_u - \hat{\theta}_u|}$. We also used that
\( 1/(1 + \sum_{\ell=1}^{k-1} e^{\ell \delta}) \leq (\pi_k/(n + \tilde{\tau})) \max_{1 \leq \ell \leq k} e^{\ell \delta}, \) which follows from similar considerations along with the fact that \( \sum_{\ell=1}^{k-1} \pi_{\ell} = n + \tilde{\tau}. \) From [67] we get

\[
\Delta_3(\lambda, \hat{\theta}, \hat{K}) \leq 4(n + \tilde{\tau}) \sum_{u=1}^{k-1} \mathbb{E} \left[ Y_1 p_u(R_2)(\theta_u - \hat{\theta}_u)^2 \right] \leq I_1 + I_2 + I_3
\]

with

\[
I_1 = 2 \sum_{u=1}^{k-1} \pi_u \mathbb{E} \left[ (\theta_u - \hat{\theta}_u)^2 \right], \quad I_2 = 2 \sum_{u=1}^{k-1} \pi_u \mathbb{E} \left[ |\theta_u - \hat{\theta}_u|^3 M(\theta) \right], \quad I_3 = \frac{2}{n + \tilde{\tau}} \sum_{u=1}^{k-1} \sum_{v=1}^{k-1} \pi_u \pi_v \mathbb{E} \left[ |\theta_u - \hat{\theta}_v||\theta_u - \hat{\theta}_u|^2 M(\theta) \right].
\]

With Hölder’s inequality we obtain

\[
\mathbb{E} [|\theta_u - \hat{\theta}_u|^3 M(\theta)] \leq (\mathbb{E} [(\theta_u - \hat{\theta}_u)^4])^{3/4} (\mathbb{E} [(M(\theta))^4])^{1/4}, \quad (69)
\]

\[
\mathbb{E} [|\theta_u - \hat{\theta}_u|^4 M(\theta)] \leq (\mathbb{E} [(\theta_u - \hat{\theta}_u)^4])^{3/4} (\mathbb{E} [(M(\theta))^4])^{1/4}. \quad (70)
\]

Note that \( \mathbb{E} [(M(\theta))^4] < \infty \) if \( \pi_k > 8 \) (this condition is easily seen from formula (63) for the posterior density). Recalling that \( e^x = 1 + O(x) \) as \( x \to 0 \) and using the crude bound that \( \max_{1 \leq \ell \leq k} y_{\ell} \leq \sum_{\ell=1}^{k-1} |y_{\ell}| \) we also have that \( \mathbb{E} [(M(\theta))^4] = 1 + O(k(\min \pi)^{-1/2}). \) Therefore applying the approximations of Lemma [12] gives that

\[
I_1 \lesssim k, \quad I_2 \lesssim \frac{k^2}{\sqrt{k \min \pi}} \quad \text{and} \quad I_3 \lesssim \frac{k^2}{n + \tilde{\tau}} \sqrt{\max \pi}.
\]

From the basic inequalities \( \max \pi \leq \sum_{u=1}^{k-1} \pi_u = n + \tilde{\tau} \quad \text{and} \quad n + \tilde{\tau} = \sum_{u=1}^{k-1} \pi_u \geq k \min \pi \) we obtain the bound \( k \sqrt{\max \pi} / (n + \tilde{\tau}) \leq \sqrt{k / \min \pi}. \) Thus, if we assume that \( \min \pi \geq c k \) for some constant \( c > 0, \) then we get from (63) and (66) that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
d_{TV}(\theta_{\text{MAP}}^*, N) \leq \frac{C_1 k}{\sqrt{\min \pi}}, \quad d_{Wass}(\theta_{\text{MAP}}^*, N) \leq \frac{C_2 k^{3/2}}{\sqrt{\min \pi}}. \quad (71)
\]

**Remark 10.** In Appendix [C] we derive an estimate for the constant \( C_2 \) under the assumption that \( \sqrt{\min \pi} \geq 7.40k^{3/2} \sqrt{2}. \) (We can derive an estimate for \( C_1 \) with no need for assumptions on \( \min \pi \) thanks to the trivial bound \( d_{TV}(\theta_{\text{MAP}}^*, N) \leq 1. \) This assumption is mild (the bound will not converge to zero if this condition does not hold), but somewhat arbitrary. However, in this remark, we provide some motivation for our choice. If this assumption does not hold, then we have that \( 7.40k^{3/2} \sqrt{\min \pi} \geq 2k, \) in which case we can obtain the following alternative estimate. Recalling that \( d_{Wass}(\theta_{\text{MAP}}^*, N) = \sup_{\phi \in \mathcal{W}_{\text{wass}}} \| \phi(\theta_{\text{MAP}}^*) - \phi(\theta_{\text{wass}}) \|, \) where \( \mathcal{W}_{\text{wass}} \) is the class of Lipschitz functions on \( \mathbb{R}^d \) such that \( \| \phi \|_{\text{Lip}} := \sup_{s \neq t} \| \phi(s) - \phi(t) \| / \| s - t \| \leq 1, \) we obtain that

\[
d_{Wass}(\theta_{\text{MAP}}^*, N) \leq \mathbb{E} [\| \theta_{\text{MAP}}^* - \theta_{\text{wass}}^* \|] \leq \sqrt{\sum_{u=1}^{k-1} \mathbb{E} [(\theta_u^n - N_u)^2]} = \sqrt{\sum_{u=1}^{k-1} (\mathbb{E} [(\theta_u^n)^2] + 1) = 2(k - 1) + r_n},
\]

where \( r_n \) is a remainder term of order \( o(1) \) as \( n \to \infty \) and \( \theta_{\text{wass}}^* = (\theta_{\text{wass}}^*)_u, 1 \leq u \leq k - 1. \) That \( r_n = o(1) \) as \( n \to \infty \) follows since \( \mathbb{E} [(\theta_u^n)^2] = \mathbb{E} [N_u^2] + o(1) = 1 + o(1), \) where \( N_u \sim N(0, 1). \) So also in this case we obtain a bound that is \( O(k^{3/2}). \)

Finally, we note that the entire procedure can be applied to the MLE standardisation, irrespective of the choice of prior. We have done the computations but the bounds take a rather unsavory form and do not lead to any further improved intuition beyond the fact that, under mild conditions on the prior, \( d_{TV}(\theta_{\text{MLE}}^*, N) \) and \( d_{Wass}(\theta_{\text{MLE}}^*, N) \) converge to zero if \( k / \sqrt{n \min \bar{x}} \to 0 \) and \( k^{3/2} / \sqrt{n \min \bar{x}} \to 0, \) respectively, where \( \min \bar{x} = \min_{1 \leq u \leq k} \bar{x}_u. \) We dispense with the details here and leave the derivation of the bounds to the motivated reader.
Remark 11 (About the literature). As mentioned in the introduction, BeM theorems for multinomial data have been extensively studied, under different parameterisations. As far as we are aware, the closest results to ours are to be found in [40] and [31]. The result from [40, Theorem 2] reads, in our notation, as

\[ d_{TV}(X_n, N_n) \leq \frac{k-1}{\sqrt{n}+\tilde{r}+1} \sqrt{\frac{\max \pi}{\min \pi}}, \]

where \( X_n \sim \text{Dir}(\pi) \) and \( N_n \sim N(\mu_n, \Sigma_n) \) with \( \mu = \pi/(n+\tilde{r}) \) and \( \Sigma_n = (n+\tilde{r}+1)^{-1} (n+\tilde{r})^{-1} (\text{diag}(\pi) - \pi \pi^T) \). [31] considers the exact same setting (with a flat prior \( \tau = 1 \)) and obtains under some mild conditions

\[ \frac{1}{18} \sum_{j=1}^{k} \frac{1}{\sqrt{k}} \leq d_{TV}(X_n, N_n) \leq \sqrt{\frac{k}{\min \pi}} + \frac{k^2}{\min \pi} + r_n \]

(\( r_n \) is a negligible remainder term). Our total variation bound in (72) requires the same condition to go to zero as (72) and (73), and the lower bound in (73) indicates that the condition \( k/\sqrt{n} \to 0 \) is necessary in the setting of [31]. It may be worth noting that [32] also upper bound the difference of means in this case, and obtain an extra \( \sqrt{k} \) in their bound exactly as we do in our Wasserstein distance bounds. To the best of our understanding, the conclusions from (72) and (73) do not carry through to our setting (we are considering normal approximation for a non-trivial transformation of a Dirichlet distribution).

3.6 Univariate normal with mean and variance

Consider i.i.d. normal data with mean \( \mu \) and variance \( \sigma^2 \). Here \( \beta(\theta) = 1/2 \log(2\pi/\theta_2) + \theta_1^2/(2\theta_2) \), \( \theta(x) = 0, h_1(x) = x, h_2(x) = -x^2/2 \) and the natural parameter is \( \theta = (\theta_1, \theta_2)^T = (\mu/\sigma^2, 1/\sigma^2)^T \), which lives on \( \mathbb{R} \times (0, \infty) \).

We consider normal approximation of the posterior standardised around the posterior mode. Fix, for \( \tau_1, \tau_2, \tau_3, \tau_4 \in \mathbb{R} \) with \( \tau_4 > 0, \tau_1 < 3/2, 4\tau_4 \tau_2 > \tau_3^2 \), a (conjugate) prior on \( \theta \) of the form \( \pi_0(\theta) \propto \exp(-\eta(\theta)) \), where \( \eta(\theta) = \log(2\pi/\theta_2)\tau_1 + \theta_1^2\tau_2/\theta_2 - \theta_1\tau_3 + \theta_2\tau_4 \). Therefore

\[ \lambda(\theta) = (n/2 + \tau_2)\theta_1^2/\theta_2 - \theta_1\tau_3 + \theta_2\tau_4 + \left( \frac{n}{2} + \tau_1 \right) \log(2\pi/\theta_2). \]

Letting \( \tilde{\tau}_1 = 1 + 2\tau_1/n, \tilde{\tau}_2 = 1 + 2\tau_2/n, \tilde{\tau}_3 = \tau_3/n \) and \( \tilde{\tau}_4 = 2\tau_4/n \) it follows from a simple calculation that the posterior mode \( \tilde{\theta} \) is given by

\[ \tilde{\theta}_1 = \frac{\tilde{\tau}_1 (\tilde{x} + \tilde{\tau}_3)}{\tilde{\tau}_1 (\tilde{x}^2 + \tilde{\tau}_4) - (\tilde{x} + \tilde{\tau}_3)^2} = \frac{\tilde{x}}{\tilde{x}^2} + O(n^{-1}), \quad \tilde{\theta}_2 = \frac{\tilde{\tau}_1 \tilde{\tau}_2}{\tilde{\tau}_2 (\tilde{x}^2 + \tilde{\tau}_4) - (\tilde{x} + \tilde{\tau}_3)^2} \approx \frac{1}{\tilde{x}^2} + O(n^{-1}), \]

with \( \tilde{x}^2 = \tilde{x}^2 - (\tilde{x})^2 \). We also have that

\[ \tilde{H} := H_\lambda(\theta) = n \begin{pmatrix} \tilde{\tau}_2 & -\tilde{\tau}_1 \theta_1 \\ -\tilde{\tau}_2 \theta_1 & \tilde{\tau}_2 \theta_2^2 + \tilde{\tau}_1 \theta_1 \theta_2 / 2 \theta_2 \end{pmatrix} \quad \text{and} \quad \tilde{F} := F_\lambda(\theta) = n^{-1} \begin{pmatrix} 2 \tilde{\theta}_2^2 + \tilde{\theta}_2 + \theta_1 \theta_2 \\ -\tilde{\tau}_1 \theta_1 / \tilde{\tau}_2 & \tilde{\tau}_1 \theta_2 / \tilde{\tau}_2 \end{pmatrix}. \]

The expressions for \( \tilde{K} := K_\lambda(\tilde{\theta}) \) and \( \tilde{J} := J_\lambda(\tilde{\theta}) \) are quite inelegant but perfectly explicit and can easily be computed. In particular

\[ \| \tilde{F} \|_{\text{diag}} = \frac{1}{n} \max \left( 2 \frac{\tilde{\theta}_2^2}{\tilde{\tau}_1} + \frac{\tilde{\theta}_2}{\tilde{\tau}_2}, 2 \frac{\tilde{\theta}_2^2}{\tilde{\tau}_1} \right) \]

which, as we shall soon see, provides the rate of convergence.

As in the previous example, the main work is to bound

\[ \mathcal{D}_3(\lambda, \tilde{\theta}, \tilde{K} | \Theta^0) = \sum_{u=1}^{2} \sum_{v=1}^{2} \sum_{w=1}^{2} \mathbb{E} \left[ Y_1 | \partial^3_{u,v,w} \lambda(R_2) | \theta_v - \tilde{\theta}_v | \theta_w - \tilde{\theta}_w | \mathbb{E}_h(\theta) \right] \]
(as usual \( R_2 := \hat{\theta} + Y_1 Y_2(\theta - \hat{\theta}) \) and \( \Theta_0 = \hat{\theta} + J \Theta_0^- \)). Simple computations yield

\[
\begin{align*}
\partial_{1,1,1} \lambda(w) &= 0, \quad \partial_{2,2,2} \lambda(w) = -\frac{3(n + 2\tau_2)w_1^2}{w_2^4} - \frac{n + 2\tau_2}{w_2^3}, \\
\partial_{1,1,2} \lambda(w) &= \partial_{1,2,1} \lambda(w) = \partial_{2,1,1} \lambda(w) = -\frac{n + 2\tau_2}{w_2^3}, \\
\partial_{1,2,2} \lambda(w) &= \partial_{2,1,2} \lambda(w) = \partial_{2,2,1} \lambda(w) = \frac{2(n + 2\tau_2)w_1}{w_2^4}.
\end{align*}
\]

As we cannot uniformly bound these third order derivatives, and thus cannot use \([35]\) and \([36]\), we use \([37]\). An application of the basic inequalities \(|w_1| \leq (1 + w_1^2)/2\) and \(|w_2|^{-3} \leq (w_2^2 + w_2^{-4})/2\) shows that the absolute values of each of the third order partial derivatives are bounded above by

\[
(n + 2(\tau_1 + \tau_2))(1 + 3w_1^2) \left( \frac{1}{w_2^3} + \frac{1}{w_2^2} \right) =: (n + 2(\tau_1 + \tau_2))T(w).
\]

We apply Lemma [10] (with \( S(w) = 2(n + 2(\tau_1 + \tau_2))T(w) \)) on \( \Theta_0^* \) where \( \Theta_0 = \{(\theta_1 - \hat{\theta}_1)^2 + (\theta_2 - \hat{\theta}_2)^2 \leq 1\} \), leading to

\[
\mathcal{D}_3(\lambda, \hat{\theta}, \hat{K} \mid \Theta_0^*) \leq 4(n + 2(\tau_1 + \tau_2))E \left[ T(R_2) \sum_{u=1}^{2} (\theta_u - \hat{\theta}_u)^2 \mathbb{I}_{\Theta_0}(\theta) \right].
\]

Since \( \hat{\theta}_1 > 0 \) and \( \hat{\theta}_2 > 0 \), it holds that, for \( \theta \in \Theta_0 = \{(\theta_1 - \hat{\theta}_1)^2 + (\theta_2 - \hat{\theta}_2)^2 \leq 1\} \), we have

\[
T(\hat{\theta} + Y_1 Y_2(\theta - \hat{\theta})) \leq \left( 1 + 6 \left( \frac{\hat{\theta}_1^2}{\theta_1^2} + 1 \right) \left( \frac{1}{\theta_2^2} + \frac{1}{\hat{\theta}_2^2} \right) \right) =: T^*(\hat{\theta}),
\]

so that

\[
\mathcal{D}_3(\lambda, \hat{\theta}, \hat{K} \mid \Theta_0^*) \leq 4(n + 2(\tau_1 + \tau_2))T^*(\hat{\theta}) \sum_{u=1}^{2} E[|\theta_u - \hat{\theta}_u|^2].
\]

Also, \( p_2(\theta \mid x) \) is proportional to

\[
\exp \left( (n\bar{x} + \tau_3)\theta_1 - (n\bar{x}^2/2 + \tau_4)\theta_2 - \frac{(n/2 + \tau_2)^2 \theta_1^2}{\theta_2} + \left( \frac{n}{2} + \tau_1 \right) \log(\theta_2) \right).
\]

Hence the term \( r^*(\Theta_0^*) \) is negligible. Finally, \( \mathbb{P}[\theta \notin \Theta_0^*] = \mathbb{P}[\sum_{u=1}^{2} (\theta_u - \hat{\theta}_u)^2 \geq 1] \leq \sum_{u=1}^{2} \mathbb{E}[|\theta_u - \hat{\theta}_u|^2] \)

so that it only remains to compute \( \sum_{u=1}^{2} \mathbb{E}[|\theta_u - \hat{\theta}_u|^2] = \sum_{u=1}^{2} \{ \text{Var}(\theta_u) + (\mathbb{E}[|\theta_u|] - \hat{\theta}_u)^2 \} \). We can either use properties of the exponential family or direct computations from the above to reape explicit expressions for all moments involved:

\[
\begin{align*}
\mathbb{E}[\theta_1] &= -\frac{(n + 2\tau_1 + 3)(n\bar{x} + \tau_3)}{\bar{x}^2 n^2 + 2\bar{x} n \tau_3 - (n + 2\tau_2)(n\bar{x}^2 + 2\tau_4) + \tau_3^2} = \frac{\bar{x}}{s_x^2} + O(n^{-1}), \\
\mathbb{E}[\theta_2] &= -\frac{(n + 2\tau_1 + 3)(n + 2\tau_2)}{\bar{x}^2 n^2 + 2\bar{x} n \tau_3 - (n + 2\tau_2)(n\bar{x}^2 + 2\tau_4) + \tau_3^2} = \frac{1}{s_x^2} + O(n^{-1}),
\end{align*}
\]

and similarly

\[
\begin{align*}
\text{Var}(\theta_1) &= \frac{1}{n} \frac{\bar{x}^2 + s_x^2}{(s_x^2)^2} + O(n^{-2}), \quad \text{Var}(\theta_2) = \frac{2}{n} \frac{2}{(s_x^2)^2} + O(n^{-2}).
\end{align*}
\]

Wrapping up, on ignoring the lower order terms which we collect into a remainder term \( r_n = O(n^{-3/2}) \) we get

\[
d_{TV}(\theta_{MAP}^*, N) \leq \frac{28\sqrt{\pi} \sqrt{1 + (\bar{x}/s_x)^2}}{\sqrt{n}} \left( 1 + \frac{(\bar{x}/s_x^2)^2}{(1 + (s_x^2)^2)(\bar{x}^2 + s_x^2) + 2} + \frac{\bar{x}^2 + s_x^2 + 2}{n(s_x^2)^2} + r_n. \right)
\]

To leading order the bound only depends on the sample size and the data; the prior hyperparameters only appear in the lower order remainder term \( r_n \).
3.7 Linear regression with unknown variance

We conclude the paper with a non-identically distributed example. Consider $X_1, \ldots, X_n$ from a standard linear regression model with unknown variance, i.e.

$$X = M \xi + \epsilon,$$

where $M \in \mathbb{R}^{n \times k}$ is the design matrix, $\xi \in \mathbb{R}^k$ and $\epsilon \sim N(0, \sigma^2 I_n)$. We will make a standard assumption that $M^T M$ is invertible. We have that $X_i \sim N(m_i, \xi \sigma^2)$, where $m_i = (m_{i1}, \ldots, m_{ik})$ denotes the $i$th row of the matrix $M$, $i = 1, \ldots, n$. The natural parameter is $\theta = (\xi_1/\sigma^2, \ldots, \xi_k/\sigma^2, 1/\sigma^2)^T = (\hat{\theta}, \theta_{k+1})^T \in \mathbb{R}^{k+1}$. Moreover, we have for $i = 1, \ldots, n$ the functions $h_{i,j}(x) = m_{ij}x$, $j = 1, \ldots, k$, $h_{i,k+1}(x) = -x^2/2$ and $\beta_i(\theta) = -\frac{1}{2} \log(\theta_{k+1}) + \frac{1}{2n} (m_i \hat{\theta})^2$. We need the Hessian of

$$\bar{\beta}(\theta) = \frac{1}{n} \sum_{i=1}^n \beta_i(\theta) = -\frac{1}{2} \log(\theta_{k+1}) + \frac{1}{2n} \bar{\theta}^T M^T M \bar{\theta},$$

Direct computations give

$$\nabla_{\bar{\theta}} \bar{\beta}(\theta) = \frac{1}{2n} \bar{\theta} M^T M \bar{\theta},$$

and the second order derivatives

$$\nabla_{\bar{\theta}} \bar{\beta}(\theta) = \frac{1}{2n} \bar{\theta}^T M^T M \bar{\theta},$$

and $\nabla_{\theta_{k+1}} \bar{\beta}(\theta) = \frac{1}{2n} \bar{\theta}^T M^T M \bar{\theta}.$

We standardise around the MLE

$$\hat{\theta} = \frac{n}{r_-(M^T M)^{-1} M^T X, 1}^T \in \mathbb{R}^{k+1},$$

for which

$$H_{\bar{\beta}}(\hat{\theta}) = \frac{r_-}{2n^2} \begin{pmatrix} M^T M & -M^T X \\ -X^T M & r_+ \end{pmatrix},$$

$$F_{\bar{\beta}}(\hat{\theta}) = \frac{2n^2}{r_-} \begin{pmatrix} (M^T M)^{-1} X^T X + (M^T M)^{-1} M^T X M (M^T M)^{-1} (M^T M)^{-1} M^T X & 1 \\ X^T M (M^T M)^{-1} & 1 \end{pmatrix},$$

with $r_\pm = X^T X \pm X^T M (M^T M)^{-1} M^T X$. Recall that $\hat{K} = \sqrt{n} K_{\bar{\beta}}(\hat{\theta})$ and define all other matrices accordingly.

We now choose a conjugate prior of the form

$$\eta(\theta) = \tau_0 \log(\theta_{k+1}) + \bar{\tau}^T \bar{\theta} + \tau_{k+1} \theta_{k+1} + \frac{1}{2\theta_{k+1}} \theta^T \sigma \theta,$$

where $\tau_0 > -1$, $\tau_{k+1} > 0$, $\sigma = (\sigma_{i,j})_{1 \leq i, j \leq k}$ is a symmetric positive definite matrix and $\bar{\tau} = (\tau_1, \ldots, \tau_k)^T \in \mathbb{R}^k$ with $\tau_{k+1} - \sigma^{-1} \bar{\xi}^T \bar{\xi}/2 < 0$. We compute

$$\nabla_{\theta_{k+1}} \eta(\theta) = \frac{\tau_0}{\theta_{k+1}} + \tau_{k+1} - \frac{1}{2\theta_{k+1}^2} \bar{\theta}^T \sigma \theta,$$

as well as

$$\nabla_{\bar{\theta}} \eta(\theta) = -\frac{1}{\theta_{k+1}^2} \sigma \bar{\theta}, \quad \nabla_{\bar{\theta}} \bar{\beta}(\theta) = \frac{1}{2n} \sigma \bar{\theta}, \quad \nabla_{\theta_{k+1}} \eta(\theta) = \frac{-\tau_0}{\theta_{k+1}^2} + \frac{1}{2\theta_{k+1}^2} \bar{\theta}^T \sigma \bar{\theta}.$$

The third order partial derivatives of $\lambda = n \bar{\beta} + \eta$ that are nonzero are given by

$$\nabla_{\bar{\theta}} \lambda(\theta) = \frac{-1}{\theta_{k+1}^2} (M^T M + \sigma), \quad \nabla_{\theta_{k+1}} \lambda(\theta) = -\frac{2}{\theta_{k+1}^3} (M^T M + \sigma) \bar{\theta},$$

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\[ \nabla_{\theta_{k+1}, \theta_{k+2}, \theta_{k+1}} \lambda(\theta) = \frac{2 \tau_0 - n}{\theta_{k+1}^T} - \frac{3}{\theta_{k+1}^T} \tilde{\theta}^T (M^T M + \sigma) \tilde{\theta}. \]

Let \( \kappa = ||M + \sigma||_\infty \). With the inequalities \(|x| \leq (1 + x^2)/2 \) and \(|x|^{-3} \leq (x^{-2} + x^{-4})/2 \) we know that the absolute value of each of the partial derivatives is bounded by

\[ T(\theta) = \left( \frac{1}{\theta_{k+1}^T} + \frac{1}{\theta_{k+1}^T} \right) \left( \frac{n + 2 \tau_0}{2} + 3 \kappa^2 ||\hat{\theta}||^2 + \kappa (1 + ||\hat{\theta}||^2_\infty/2) \right). \]

Now let \( \Theta_0 = B(\hat{\theta}, \epsilon) \) be a ball centred around \( \hat{\theta} \) with radius \( \epsilon = \hat{\theta}_{k+1}/2 > 0 \). Integrating out the uniform component with \( \theta \) fixed we get (computations done with Mathematica)

\[
\mathbb{E}[T(\hat{\theta} + Y_1 Y_2 (\theta - \hat{\theta})) | \theta] \\
\leq \frac{1}{3 \hat{\theta}_{k+1}^3} \left( \frac{n + 2 \tau_0}{2} + \frac{7 \kappa^2}{2} + \kappa \right) \left( \frac{3 (\hat{\theta}_{k+1} - \epsilon) + \hat{\theta}_{k+1}}{2 (\hat{\theta}_{k+1} - \epsilon)^2} + (1 + 3 \hat{\theta}_{k+1}^2) \frac{\log((\hat{\theta}_{k+1}/(\hat{\theta}_{k+1} - \epsilon)))}{\epsilon} \right) \\
= \frac{n}{3 \hat{\theta}_{k+1}^3} \left( \frac{1}{2} + \frac{7 \kappa^2}{8 \hat{\theta}_{k+1}^2} + \hat{\theta}_{k+1} + 1 \right) \left( \frac{5 + 2 \log(2)}{\hat{\theta}_{k+1}} + 6 \log(2) \hat{\theta}_{k+1} \right) \\
= n T^*(\hat{\theta}_{k+1}).
\]

Since \( \nabla_{\hat{\theta}, \hat{\theta}, \hat{\theta}} \lambda(\theta) = 0 \), we can apply the bound (12) to improve the dependence on the dimension \( d = k + 1 \) over directly applying the bound of Lemma [10]. Finally, \( \mathbb{P}[\theta \notin \Theta_0^n] = \mathbb{P}[\sum_{r=1}^{k+1} (\theta_r - \bar{\theta}_r)^2 \geq \hat{\theta}_{k+1}^2/4] \leq (4 \hat{\theta}_{k+1}^2)^k \sum_{r=1}^{k+1} \mathbb{E}[(\theta_r - \bar{\theta}_r)^2] \). We conclude that

\[
d_T V(\theta_{M_{MLE}}; N) \leq \sqrt{\frac{n}{2 \pi}} ||F^2(\hat{\theta})||_{\text{diag}} \left( n(k + 1) T^*(\hat{\theta}_{k+1}) \sum_{r=1}^{k+1} \sqrt{\mathbb{E}[(\theta_{r+1} - \hat{\theta}_{r+1})^2] \mathbb{E}[(\theta_r - \bar{\theta}_r)^2]} \right) \\
+ \sum_{r=1}^{k+1} \mathbb{E}[(\theta_r - \bar{\theta}_r)^2] + n T^*(\Theta_0^n), \quad (74)
\]

where, as in the previous examples, the remainder \( r^*(\Theta_0^n) \) is negligible.

In terms of the dependence on the sample size and the dimension, the bound (74) can be seen to be \( O(k^2/\sqrt{n}) \) provided (i) \( X^T X = \Theta(n) \), (ii) \( r_\infty = \Theta(n) \), (iii) \( \|(M^T M)^{-1} 1 n \|_\infty = \Theta(1) \) and (iv) \( \|(M^T M)^{-1} \|_\infty = \Theta(n^{-1}) \); in each of parts (i)–(iv) we understand the order notation to additionally imply that \( X^T X, r_\infty, \|(M^T M)^{-1} 1 n \|_\infty \) and \( \|(M^T M)^{-1} \|_\infty \) do not grow with the dimension \( k \). Under these conditions, \( ||F^2(\hat{\theta})||_{\text{diag}} = \Theta(1) \), and we also have that \( \mathbb{E}[(\theta_r - \bar{\theta}_r)^2] = O(n^{-1}) \) for \( r = 1, \ldots, k+1 \), from which it is readily seen that the bound (74) is \( O(k^2/\sqrt{n}) \).

We conclude by arguing that conditions (i)–(iv) are natural. Condition (i) is clearly natural. Conditions (ii) and (iii) are also natural since the MLE \( \hat{\theta} \) is a consistent estimator. Finally, we consider condition (iv). For \( i, j = 1, \ldots, k \), \( (M^T M)_{i,j} = \sum_{t=1}^n m_{t,i} m_{t,j} \), where \( m_{i,j} \) is the \( (i, j) \)-th element of the matrix \( M \). Thus, for a typical design matrix \( M \), the elements of \( M^T M \) are of order \( n \), and so \( \det(M^T M) = \Theta(n) \), from which it would follow that \( \det((M^T M)^{-1}) = \Theta(n^{-1}) \), which would in turn imply that \( \|(M^T M)^{-1} \|_\infty = \Theta(n^{-1}) \).

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Although the arguments are elementary, we rely on some cumbersome facts about multivariate derivatives and multivariate Taylor expansions with integral remainder. With $Y_1, Y_2, \ldots$ i.i.d. uniform on $(0, 1)$, we set

\[ U_k = \prod_{j=1}^{k-1} Y_j \quad \text{and} \quad V_k = \prod_{j=1}^{k} Y_j. \]

First recall that if $\phi : \mathbb{R}^d \to \mathbb{R}$ is sufficiently differentiable then the first and second order Taylor expansions of $\phi$ around 0 with integral remainder read as

\[ \phi(t) = \phi(0) + \sum_{\ell_1=1}^{d} \mathbb{E} [\partial_{\ell_1} \phi(Y_1 t)] t_{\ell_1} = \phi(0) + \langle \mathbb{E}[\mathcal{D} \phi(Y_1 t)], t \rangle, \]
\[ \phi(t) = \phi(0) + \sum_{\ell_1=1}^{d} \partial_{t_1} \phi(0) t_1 + \sum_{\ell_1=1}^{d} \sum_{\ell_2=1}^{d} \mathbb{E} \left[ Y_1 \partial_{t_1, t_2}^2 \phi(Y_1 Y_2 t) \right] t_1 t_2 \]
\[ = \phi(0) + \langle D \phi(0), t \rangle + \langle \mathbb{E}[Y_1 D^2 \phi(Y_1 Y_2 t)], t \otimes^2 \rangle \]

(compare with [21, p.20]). To generalise at arbitrary order for \( r \in \{1, \ldots, k\} \) and \( \ell^r = (\ell_1, \ldots, \ell_r) \) we set \( t_{\ell^r} = \prod_{i=1}^{r} t_{\ell_i}, \partial_{t^r} = \partial_{t_{\ell_1}, \ldots, t_{\ell_r}}, \sum_{\ell^r=1}^{d} = \sum_{\ell_1=1}^{d} \cdots \sum_{\ell_r=1}^{d} \). Then
\[ \phi(t) = \phi(0) + \sum_{r=1}^{k} \frac{1}{r!} \sum_{\ell^r=1}^{d} t_{\ell^r} \partial_{t^r} \phi(0) + \sum_{\ell^{k+1}=1}^{d} t_{\ell^{k+1}} \mathbb{E} \left[ U_k \partial_{t^{k+1}}^2 \phi(V_k t) \right] \]
\[ = \phi(0) + \sum_{r=1}^{k} \frac{1}{r!} \langle D^r \phi(0), t \otimes^r \rangle + \langle \mathbb{E}[U_k D^{k+1} \phi(V_k t)], t \otimes^{k+1} \rangle \]

(the tensor notation is much more efficient).

Consequently, if \( \lambda : \mathbb{R}^d \to \mathbb{R} \) is sufficiently differentiable and \( J \) is a symmetric \( d \times d \) matrix then the chain rule combined with the above formulas yields, after a change in the order of summation, for all \( u = 1, \ldots, d \),
\[ \partial_u \lambda(t_0 + J t) = \partial_u \lambda(t_0) + \sum_{v=1}^{d} \mathbb{E}[\partial_{u,v}^2 \lambda(t_0 + Y_1 J t)](J t)_v, \quad (75) \]
\[ \partial_u \lambda(t_0 + J t) = \partial_u \lambda(t_0) + \sum_{v=1}^{d} \partial_{u,v}^2 \lambda(t_0)(J t)_v + \sum_{v=1}^{d} \sum_{w=1}^{d} \mathbb{E} \left[ Y_1 \partial_{u,v,w}^3 \lambda(t_0 + Y_1 Y_2 J t) \right] (J t)_v (J t)_w, \quad (76) \]

and, more generally,
\[ \partial_u \lambda(t_0 + J t) = \partial_u \lambda(t_0) + \sum_{r=1}^{k} \frac{1}{r!} \sum_{v^r=1}^{d} \partial_{u,v^r}^2 \lambda(t_0)(J t)_{v^r} + \sum_{v^r=1}^{d} \sum_{v^{r'}}=1^{d} \mathbb{E} \left[ U_k \partial_{u,v^r}^{k+1} \lambda(t_0 + V_k J t) \right] (J t)_{v^r}. \quad (77) \]

### B Properties of exponential families in canonical form

#### B.1 Generalities

Let \( X_\theta \in \mathbb{R}^r \) be a random variable drawn from an exponential family in canonical form with density (with respect to some dominating measure \( \mu \))
\[ p(x \mid \theta) = \exp \left\{ \langle h(x), \theta \rangle - \alpha(x) - \beta(\theta) \right\} \quad (78) \]

with \( \theta = (\theta_1, \ldots, \theta_d)^T, h(x) \in \mathbb{R}^d, \alpha(x), \beta(\theta) \in \mathbb{R}; \) we suppose that this density is well-defined, that the parameter \( \theta \) is identifiable, and \( p(\cdot \mid \theta) \) has support \( S \subset \mathbb{R}^r \) some open set (not depending on \( \theta \)). This is exactly the setting considered in [9]. By definition the log normalising constant (a.k.a. cumulant generating function) is
\[ \beta(\theta) = \log \int_S \exp \left\{ \langle h(x), \theta \rangle - \alpha(x) \right\} \mu(dx). \]

It then holds that we can take derivatives under the integral sign to get (see [11, Theorem 2.2])
\[ D \beta(\theta) = \mathbb{E}[h(X_\theta)], \quad D^2 \beta(\theta) = \text{Var}[h(X_\theta)], \quad D^3 \beta(\theta) = \mathbb{E}[(h(X_\theta) - \mathbb{E}[h(X_\theta)])^3] \]

(as claimed in Remark 8) and, more generally,
\[ D^k \beta(\theta) = \kappa_k[h(X_\theta)], \]

where \( \kappa_k[h(X_\theta)] \) denotes the \( k \)-th cumulant of \( h(X_\theta) \) for all \( k \geq 1 \).
Lemma 13. Let \( \theta^* \sim p^*(\cdot | x) \) as in (12). Then
\[
\nabla \log p^*(t | x) = J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0 + J t) \right).
\] (79)

Proof. From (13) we see that (\( c \) here is an irrelevant constant that will disappear with the derivatives)

\[
\log p^*(t | x) = c + n \langle \tilde{h}(x), \theta_0 + J t \rangle - \lambda(\theta_0 + J t) = c + n \sum_{v=1}^{d} \tilde{h}_v(x)(J t)_v - \lambda(\theta_0 + J t).
\]

From the chain rule we immediately obtain that the Stein operator \( T_{\theta^*} \) for \( \theta^* \sim p^*(\cdot | x) \) is given by

\[
T_{\theta^*} f(t) = \Delta f(t) + \langle L(t), \nabla f(t) \rangle,
\]

where \( L(t) \) has components

\[
L_{uv}(t) = \partial_{u} \log p^*(t | x) = \sum_{v=1}^{d} J_{uv} \left( n \tilde{h}_v(x) - \partial_{v} \lambda(\theta_0 + J t) \right)
\]

(we also use the symmetry of \( J \)) which gives (79). \( \square \)

Lemma 14 (Taylor expansions of the log likelihood ratio). Let \( Y_1, Y_2, \ldots \) be i.i.d. uniform on \((0, 1)\), independent of all other randomness involved. Let \( p^*(\cdot | x) \) be given by (13). Finally, let \( t^* = K(t - \theta_0) \) with \( t \in \mathbb{R}^d \). The following hold, for all \( \xi \in \mathbb{R}^d \).

- If \( \lambda \) is twice differentiable around \( \theta_0 \), then for all \( t \) sufficiently close to \( \theta_0 \),
  \[
  \langle \nabla \log p^*(t^* | x) + t^*, \xi \rangle = \langle \nabla \lambda(\theta_0) - n \tilde{h}(x), J \xi \rangle + \mathbb{E} \left[ (H_\lambda(r_1(t^*))) - H, (J \xi) \otimes (J t^*) \right]
  \]
  with \( r_1(u) = \theta_0 + Y_1 J u \).

- If \( \lambda \) is three times differentiable around \( \theta_0 \), then Lemma 5 holds.

- If \( \lambda \) is \( k+1 \) times differentiable (for \( k \in \mathbb{N}_0 \)) around \( \theta_0 \), then for all \( t \) sufficiently close to \( \theta_0 \),
  \[
  \langle \nabla \log p^*(t^* | x) + t^*, \xi \rangle = \langle \nabla \lambda(\theta_0) - n \tilde{h}(x), J \xi \rangle + \langle H_\lambda(\theta_0) - H, (J \xi) \otimes (J t^*) \rangle
  + \frac{1}{2} \left\langle D^3 \lambda(\theta_0), (J \xi) \otimes (J t^*) \otimes (J t^*) \right\rangle + \cdots + \frac{1}{k!} \left\langle D^k \lambda(\theta_0), (J \xi) \otimes (J t^*) \otimes (J t^*) \otimes (J t^*) \right\rangle
  - \left\langle \mathbb{E} \left[ U_k D^{k+1} \lambda(r_k(t^*)) \right], (J \xi) \otimes (J t^*) \otimes (J t^*) \otimes (J t^*) \right\rangle,
  \]
  where \( U_k = \prod_{j=1}^{k-1} Y_j \) and \( r_k(u) = \theta_0 + V_k J u \) with \( V_k = \prod_{j=1}^{k} Y_j \).

Proof. Let, as in the previous proof, \( L(t) = \nabla \log p^*(t | x) \). Recall the notation \( H = K^2 \) and \( K = J^{-1} \). We control the expression

\[
L(t) + t = J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0 + J t) \right) + t = J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0 + J t) + K t \right)
\]

by expanding \( \nabla \lambda(\theta_0 + J t) \) around \( t = 0 \) using the material from Section A. Expanding to the first order through (75) gives

\[
L(t) + t = J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0) \right) - J \left[ \mathbb{E}[H_\lambda(t_0 + Y_1 J t)](J t) - (K t) \right]
= J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0) \right) - J \left[ \mathbb{E}[H_\lambda(t_0 + Y_1 J t)] - K^2 \right] (J t).
\]

Replacing \( t \) by \( t^* = K(t - \theta_0) \) and introducing the notation \( R_1 = \theta_0 + Y_1(t - \theta_0) \) yields

\[
L(t^*) + t^* = J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0) \right) - J \left[ \mathbb{E}[H_\lambda(R_1)] - H \right] (t - \theta_0).
\]

It follows, thanks to the symmetry of all matrices involved, that

\[
\langle L(t^*) + t^*, \xi \rangle = \langle J \left( n \tilde{h}(x) - \nabla \lambda(\theta_0) \right), \xi \rangle - \langle J \left[ \mathbb{E}[H_\lambda(R_1)] - H \right] (t - \theta_0), \xi \rangle
\]
\[
\log(x) = \langle n \bar{\theta}(x) - \nabla \lambda(\theta_0), J \xi \rangle - \langle E[H_\lambda(R_1)] - H, (J \xi) \otimes (t - \theta_0) \rangle.
\]

This proves (80). Expanding to the next order through (77) we get
\[
L_u(t) + t_u = \langle J(n \bar{\theta}(x) - \nabla \lambda(\bar{\theta})) u \rangle - \langle J(H_\lambda(\theta_0) - H) J t \rangle u
- \sum_{v=1}^{d} \sum_{s_1 = 1}^{d} \sum_{s_2 = 1}^{d} J_u v E[ Y_1 \partial^3_{s_1,s_2,v} \lambda(\theta_0 + Y_1 Y_2 J t)(J t)_{s_1}(J t)_{s_2}].
\]

Replacing \( t \) with \( t^* = K(t - \theta_0) \) and introducing \( R_2 = \theta_0 + Y_1 Y_2(t - \theta_0) \) leads to
\[
L_u(t^*) + t_u^* = \langle J(n \bar{\theta}(x) - \nabla \lambda(\bar{\theta})) u \rangle - \langle J(H_\lambda(\theta_0) - H)(J \xi) \otimes (t - \theta_0) \rangle u
- \sum_{v=1}^{d} \sum_{s_1 = 1}^{d} \sum_{s_2 = 1}^{d} J_u v E[ Y_1 \partial^3_{s_1,s_2,v} \lambda(R_2)(t - \theta_0)_{s_1}(t - \theta_0)_{s_2}],
\]
and thus
\[
(L(t^*) + t^*, \xi) = \langle n \bar{\theta}(x) - \nabla \lambda(\theta_0), J \xi \rangle - \langle H_\lambda(\theta_0) - H, (J \xi) \otimes (t - \theta_0) \rangle
- \langle E[Y_1 D^3 \lambda(R_2)], (J \xi) \otimes (t - \theta_0)^{\otimes 2} \rangle,
\]
which is (30). The arbitrary order expansion (81) follows along the exact same lines from (77). \( \square \)

C Further proofs from Section 3.5

C.1 Proof of Lemma 12

Proof of Lemma 12 The cumulant generating function for (63) is
\[
\kappa(\pi) = \sum_{j=1}^{k-1} \log \Gamma(\pi_j) + \log \left( n + \tilde{\tau} - \sum_{j=1}^{k-1} \pi_j \right) - \log \Gamma(n + \tilde{\tau}).
\]

With \( \psi \) the digamma function, direct and straightforward computations yield
\[
\partial_i \kappa(\pi) = \psi(\pi_i) - \psi(\pi_k), \quad \partial_{ij} \kappa(\pi) = \psi'(\pi_i) + \psi'(\pi_k), \quad \partial_{ij} \kappa(\pi) = \psi''(\pi_i), \quad i \neq j,
\]
\[
\partial^2_{ij} \kappa(\pi) = \psi''(\pi_i) - \psi''(\pi_k), \quad \partial_{ijk} \kappa(\pi) = \psi'''(\pi_i) + \psi'''(\pi_k),
\]
for all \( i, j = 1, \ldots, k - 1 \). It follows that
\[
\mu_i(\pi) := E[\theta_i] = \psi(\pi_i) - \psi(\pi_k), \quad \sigma^2(\pi) := \text{Var}(\theta_i) = \psi'(\pi_i) + \psi'(\pi_k),
\]
\[
\mu_{3,i}(\pi) := E[(\theta_i - \mu_i(\pi))^3] = \psi'''(\pi_i) - \psi'''(\pi_k),
\]
\[
\mu_{4,i}(\pi) := E[(\theta_i - \mu_i(\pi))^4] = \psi'''(\pi_i) + \psi'''(\pi_k) + 3(\psi'(\pi_i) + \psi'(\pi_k))^2.
\]

The moments needed in the statement of the lemma are centred around the posterior mode \( \tilde{\theta}_i = \log(\pi_i) - \log(\pi_k) \). With the notation \( \rho(x) = \psi(x) - \log(x) \), we get
\[
E[(\theta_i - \tilde{\theta}_i)^2] = \psi'(\pi_i) + \psi'(\pi_k) + (\rho(\pi_i) - \rho(\pi_k))^2 = \sigma^2(\pi) + b_i(\pi)^2 \tag{82}
\]
with \( b_i(\pi) = \mu_i(\pi) - \tilde{\theta}_i = \rho(\pi_i) - \rho(\pi_k) \) the bias. Similarly, we have that
\[
E[(\theta_i - \tilde{\theta}_i)^4] = \mu_{4,i}(\pi) + 4b_i(\pi)\mu_{3,i}(\pi) + 6b_i(\pi)^2\sigma^2(\pi) + b_i(\pi)^4. \tag{83}
\]

In order to bound \( E[(\theta_i - \tilde{\theta}_i)^2] \) and \( E[(\theta_i - \tilde{\theta}_i)^4] \) we will make use of the following bounds for the digamma function (see [26 Lemma 1]). Let \( k \in \mathbb{N} \). Then for \( x > 0 \),
\[
-1/x < \psi(x) - \log(x) < -1/(2x)
\]
and
\[
\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}.
\]
Applying these inequalities and using that without loss of generality \(\pi_k = \max \pi\) we obtain the bounds
\[
|b_i(\pi)| \leq \frac{1}{\pi_i}, \quad \sigma_i^2(\pi) \leq \frac{2}{\pi_i} + \frac{2}{\pi_i},
\]
\[
|\mu_{3,i}(\pi)| \leq \frac{1}{\pi_i} + \frac{2}{\pi_i}, \quad \mu_{4,i}(\pi) \leq \frac{12}{\pi_i} + \frac{28}{\pi_i} + \frac{24}{\pi_i}.
\]
Applying these bounds to the formulas (82) and (83) now yields the desired bounds for \(E\) and \(E[(\theta_i - \tilde{\theta}_i)^2]\) and \(E[(\theta_i - \tilde{\theta}_i)^4]\). \(\square\)

C.2 Derivation of explicit constants

Part 1: Initial bounds

To derive Wasserstein and total variation distance bounds with explicit constants, we require an explicit bound for \(E[(M(\tilde{\theta}))^4]\). In the following lemma, we derive a crude bound, which will suffice for our purposes.

Lemma 15. Let \(G(x) = 2\Gamma(x + 8)\Gamma(x - 8)/(\Gamma(x))^2\). Then, \(G\) is a decreasing function of \(x\) on \((8, \infty)\). Moreover, provided \(\min \pi > 8\), we have that
\[
E[(M(\tilde{\theta}))^4] \leq (k - 1)G(\min \pi). \tag{84}
\]

Proof. Using the crude inequalities \(\max_{1 \leq i \leq k-1} x_i \leq \sum_{i=1}^{k-1} |x_i|\) and \(e^{x} \leq e^{x} + e^{-x}\) we have that
\[
E[(M(\tilde{\theta}))^4] \leq \sum_{u=1}^{k-1} \left( E[e^{8(\theta_u - \tilde{\theta}_u)}] + E[e^{-8(\theta_u - \tilde{\theta}_u)}] \right). \tag{85}
\]
Using formula for the the posterior as given in equation (63) we have that, for \(\pi_k > 8\) (which is ensured since we assume that \(\min \pi > 8\)),
\[
E[e^{8\tilde{\theta}_u}] = \int_{\mathbb{R}^{k-1}} e^{8x_u} \frac{\Gamma(n + \tilde{\tau})}{\prod_{j=1}^{k} \Gamma(\pi_j)} \frac{\exp(\sum_{j=1}^{k-1} \tilde{\pi}_j x_j)}{(1 + \sum_{j=1}^{k-1} e^{x_j})^{n+\tilde{\tau}}} \, dx_1 \cdots dx_{k-1}
\]
\[
= \frac{\Gamma(\pi_u + 8)}{\Gamma(\pi_u)} \frac{\Gamma(\pi_k - 8)}{\Gamma(\pi_k)} \int_{\mathbb{R}^{k-1}} \frac{\Gamma(n + \tilde{\tau})}{\prod_{j=1}^{k} \Gamma(\tilde{\pi}_j)} \frac{\exp(\sum_{j=1}^{k-1} \tilde{\pi}_j x_j)}{(1 + \sum_{j=1}^{k-1} e^{x_j})^{n+\tilde{\tau}}} \, dx_1 \cdots dx_{k-1}
\]
\[
= \frac{\Gamma(\pi_u + 8)}{\Gamma(\pi_u)} \frac{\Gamma(\pi_k - 8)}{\Gamma(\pi_k)}
\]
where \(\tilde{\pi}_u = \pi_u + 8\), \(\tilde{\pi}_k = \pi_k - 8\) and \(\tilde{\pi}_\ell = \pi_\ell\) otherwise. Since \(e^{-8\tilde{\theta}_u} = (\pi_k/\pi_u)^8\), we thus obtain that
\[
E[e^{8(\theta_u - \tilde{\theta}_u)}] = \frac{\Gamma(\pi_u + 8)}{\pi_u^8} \frac{\Gamma(\pi_k - 8)}{\Gamma(\pi_k)}
\]
Now, \(\Gamma(x + 8)/(x^8\Gamma(x)) = \prod_{j=1}^{8} (1 + j/x)\) is clearly a decreasing function of \(x\) on \((8, \infty)\), and similarly \(x^8\Gamma(x-8)/\Gamma(x) = \prod_{j=1}^{8} (1 - j/x)^{-1}\) is clearly a decreasing function of \(x\) on \((8, \infty)\). These monotonicity results allow us to infer that \(G(x)\) is decreasing on the interval \((8, \infty)\), and moreover that
\[
E[e^{8(\theta_u - \tilde{\theta}_u)}] \leq \frac{\Gamma(\min \pi + 8)}{(\min \pi)^8} \frac{\Gamma(\min \pi - 8)}{\Gamma(\min \pi)} = \frac{1}{2} G(\min \pi). \tag{86}
\]
By a similar argument, one can show that \(E[e^{-8(\theta_u - \tilde{\theta}_u)}] \leq G(\min \pi)/2\) for \(\min \pi > 8\). Combining this inequality with inequalities (85) and (86) now yields the desired bound (84). \(\square\)

On combining the bounds (69) and (70) together with the bounds of Lemmas 12 and 15 we can now obtain explicit bounds on the quantities \(I_1, I_2\) and \(I_3\). Provided \(\min \pi > 8\),
\[
I_1 \leq 4(k - 1) \left( 1 + \frac{3}{2 \min \pi} \right) =: (k - 1)A(\min \pi),
\]

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\[
I_a \leq 2 \cdot 12^{3/4}(k-1) \left( 1 + \frac{11}{3 \min \pi} + \frac{15}{4(\min \pi)^2} \right)^{3/4} (k-1)^{1/4} H(\min \pi) \sqrt{\min \pi} =: (k-1)B_k(\min \pi),
\]
\[
I_b \leq 2 \cdot 12^{3/4}(k-1) \left( 1 + \frac{11}{3 \min \pi} + \frac{15}{4(\min \pi)^2} \right)^{3/4} (k-1)^{3/4} H(\min \pi) \sqrt{\min \pi} =: (k-1)C_k(\min \pi),
\]
where \( H(x) := (G(x))^{1/4} \). In deriving the bound for \( I_b \), we used the basic inequalities \( \max \pi \leq \sum_{u=1}^k \pi_u = n + \hat{\tau} \) and \( n + \hat{\tau} = \sum_{u=1}^k \pi_u \geq k \min \pi \) to obtain the bound \( \sqrt{\max \pi/(n+\hat{\tau})} \leq 1/\sqrt{k \min \pi} \).

We thus obtain the bounds
\[
d_{\text{Wass}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{\sqrt{2}(k-1)^{3/2}}{\sqrt{\min \pi}} D_k(\min \pi), \quad d_{\text{TV}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{\sqrt{k-1}}{\sqrt{\min \pi}} D_k(\min \pi),
\]
where \( D_k(\min \pi) = A(\min \pi) + B_k(\min \pi) + C_k(\min \pi) \). These bounds are valid for \( \min \pi > 8 \). Finally, we put these rather unpalatable bounds into a compact form.

**Part 2: Compact total variation distance bounds**

We now complete the derivation of the total variation distance bound given in Example 2 with constant \( C_1 = 8.46 \); that is, we will prove that
\[
d_{\text{TV}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{8.46k}{\sqrt{\min \pi}}. \tag{87}
\]

We note that we may assume that \( \sqrt{\min \pi} \geq 8.46k \), as otherwise the bound is trivial. Observe that this assumption also ensures that we satisfy the requirement that \( \min \pi > 8 \). We begin by verifying that inequality \[87] holds for \( k = 2 \). In this case, \( 8.46k = 16.92 \), and we compute \( D_2(16.92^2) = 5.95640 \). We also observe that since, for fixed \( k \), \( D_k(x) \) is a decreasing function of \( x \), it follows that \( D_2(\min \pi) \leq D_2(16.92^2) = 5.95640 \). Therefore, for \( k = 2 \), we get that
\[
d_{\text{TV}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{\sqrt{\pi(2-1)}}{\sqrt{\min \pi}} \times 5.95640 \leq \frac{10.5574}{\sqrt{\min \pi}} = \frac{5.2787 \times 2}{\sqrt{\min \pi}} < \frac{8.46 \times 2}{\sqrt{\min \pi}},
\]
and so inequality \[87] holds for \( k = 2 \). Similarly, we can verify that inequality \[87] holds for \( 3 \leq k \leq 51 \).

We provide here the details for \( k = 16 \); we do so because, for \( 3 \leq k \leq 51 \), numerical computations confirm that \( ((k-1)/k)D_k((8.46k)^2) \leq (15/16)D_{16}((8.46 \times 16)^2) \). In this case, \( D_{16}((8.46 \times 16)^2) = 5.08787 \) so that
\[
d_{\text{TV}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{\sqrt{\pi(16-1)}}{\sqrt{\min \pi}} \times 5.08787 = \frac{8.45438 \times 16}{\sqrt{\min \pi}} < \frac{8.46 \times 16}{\sqrt{\min \pi}},
\]
as required. Finally, we consider the case \( k \geq 52 \). In this case, we may assume that \( \sqrt{\min \pi} \geq 8.46k \geq 8.46 \times 52 = 439.92 \), and we get that
\[
D_k(\min \pi) \leq D_k(439.92^2)
\]
\[
\leq 4 \left( 1 + \frac{3}{2 \times 439.92^2} \right) + 2 \cdot 12^{3/4} \left( 1 + \frac{11}{3 \times 439.92^2} + \frac{15}{4 \times 439.92^4} \right)^{3/4} \times 
\]
\[
\left( \frac{H(439.92^2)}{8.46 \times 52^{3/4}} + \frac{H(439.92^2)}{8.46 \times 52^{1/4}} \right) = 4.76870.
\]
Thus, for \( k \geq 52 \),
\[
d_{\text{TV}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{\sqrt{\pi k}}{\sqrt{\min \pi}} \times 4.76870 = \frac{8.45231k}{\sqrt{\min \pi}} < \frac{8.46k}{\sqrt{\min \pi}},
\]
as required. We have therefore proved that inequality \[87] holds for all \( k \geq 2 \).

**Part 3: Compact Wasserstein distance bounds**

We now derive the compact bound
\[
d_{\text{Wass}}(\theta^{\star}_{\text{MAP}}, N) \leq \frac{7.40k^{3/2}}{\sqrt{\min \pi}} \tag{88}
\]
that was stated in Example 2. We shall derive this bound under the assumption that $\sqrt{\min \pi} \geq 7.40/\sqrt{2}$. We proceed as in part 2 of the proof, but with a more efficient exposition. The verification that inequality (88) holds for $2 \leq k \leq 18$ follows by a similar approach as in part 2 of the proof. For $2 \leq k \leq 18$, numerical computations show that $((k-1)/k)^{3/2}D_k((7.40k/\sqrt{2})^2) \leq (16/17)^{3/2}D_{17}((7.40 \times 17/\sqrt{2})^2)$. In this case, $D_{17}((7.40 \times 17/\sqrt{2})^2) = 5.72873$ so that

$$d_{\text{Wass}}(\theta_{\text{MAP}}^*, N) \leq \frac{\sqrt{2(17 - 1)^{3/2}}}{\sqrt{\min \pi}} \times 5.72873 = \frac{7.39741 \times 17^{3/2}}{\sqrt{\min \pi}} < \frac{7.40 \times 17^{3/2}}{\sqrt{\min \pi}},$$

as required. We verify that inequality (88) holds for $k \geq 19$. In this case, we may assume that $\sqrt{\min \pi} \geq 7.40k/\sqrt{2} \geq 7.40 \times 19/\sqrt{2} = 99.4192$, and we get that

$$D_k(\min \pi) \leq D_k(99.4192^2) \leq 4 \left( 1 + \frac{3}{2 \times 99.4192^2} \right) + 2 \cdot 12^{3/4} \left( 1 + \frac{11}{3 \times 99.4192^2} + \frac{15}{4 \times 99.4192^4} \right)^{3/4} \times \left( \frac{H(99.4192^2)}{7.40 \times 19^{3/4}} + \frac{H(99.4192^2)}{7.40 \times 19^{1/4}} \right) = 5.22329.$$

Thus, for $k \geq 19$,

$$d_{\text{Wass}}(\theta_{\text{MAP}}^*, N) \leq \frac{\sqrt{2k^{3/2}}}{\sqrt{\min \pi}} \times 5.22329 = \frac{7.38684k^{3/2}}{\sqrt{\min \pi}} < \frac{7.40k^{3/2}}{\sqrt{\min \pi}},$$

as required. This completes the proof. □