Geometrically thick tori around compact objects with a quadrupole moment

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Effective potential and thick accretion tori in static spacetimes

From the normalisation condition \( u_\mu u^\mu = -1 \), we can describe the time component by using the specific angular momentum \( l = \frac{g_{tt}}{g_{\varphi \varphi}} \):

\[
  u_t^{-2} = -g_{tt} - l^2 g_{\varphi \varphi}
\]

Euler equation for a perfect fluid in circular motion:

\[
  \partial_\mu \ln |u_t| - \left( \frac{\Omega}{1 - \Omega l} \right) = -\frac{1}{\rho h} \partial_\mu p
\]

For a barotropic fluid, surfaces of constant \( l \) and \( \Omega \) coincide. If \( dl \neq 0 \), then \( \Omega = \Omega(l) \rightarrow \text{relativistic von Zeipel theorem} \). In that case, integrate Euler equation to find

\[
  \mathcal{W} - \mathcal{W}_{in} := -\int_0^p \frac{dp'}{\rho h} = \ln |u_t| - \ln |(u_t)_{in}| - \int_{l_{in}}^l \frac{\Omega}{1 - \Omega l'} dl''
\]

For constant angular momentum and choosing \( \mathcal{W}_{in} = -\ln |(u_t)_{in}| \), this equation reduces to

\[
  \mathcal{W}(l, r, \vartheta) = -\frac{1}{2} \ln \left( -g_{tt}(r, \vartheta) - l^2 g_{\varphi \varphi}(r, \vartheta) \right)
\]

⇒ 'Polish Doughnuts'
Circular lightlike and timelike geodesics I

Fluid element at the centre moves along circular timelike geodesic → study properties of circular geodesics in the considered spacetimes!
Consider geodesics in the equatorial plane → Lagrangian:

$$\mathcal{L} = \frac{1}{2} \left( g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + g_{\varphi\varphi} \dot{\varphi}^2 \right)$$

Constants of motion:

$$E = g_{tt} \dot{t}, \quad L = g_{\varphi\varphi} \dot{\varphi}, \quad \varepsilon = -g_{tt} \dot{t}^2 - g_{rr} \dot{r}^2 - g_{\varphi\varphi} \dot{\varphi}^2,$$

where $\varepsilon = 0$ for lightlike and $\varepsilon = 1$ for timelike geodesics.
Defining the effective potential $\mathcal{V}$:

$$-g_{tt} g_{rr} \dot{r}^2 + \mathcal{V} = E^2 \quad \text{with} \quad \mathcal{V} = -g_{tt}(r) \left( \frac{L^2}{g_{\varphi\varphi}(r)} + \varepsilon \right)$$

Circular motion ($\dot{r} = 0$ and $\ddot{r} = 0$) is equivalent to

$$\mathcal{V}(\varepsilon, L, r) = E^2, \quad \frac{\partial \mathcal{V}(\varepsilon, L, r)}{\partial r} = 0$$
Circular lightlike and timelike geodesics II

For lightlike geodesics $\varepsilon = 0$, the two conditions are equivalent to

$$\frac{L^2}{E^2} = -\frac{g_{\varphi\varphi}}{g_{tt}}, \quad g_{tt}g'_{\varphi\varphi} = g_{\varphi\varphi}g'_{tt}$$

→ position of the photon circle

For timelike geodesics $\varepsilon = 1$, the two conditions lead to the "Keplerian" constants of motion:

$$L^2_K = \frac{g_{\varphi\varphi}g'_{tt}}{g_{tt}g'_{\varphi\varphi} - g_{\varphi\varphi}g'_{tt}}, \quad E^2_K = -\frac{g_{tt}g'_{\varphi\varphi}}{g_{tt}g'_{\varphi\varphi} - g_{\varphi\varphi}g'_{tt}}$$

→ Keplerian specific angular momentum (KSAM) and Keplerian angular velocity:

$$l^2_K = \left(\frac{L_K}{E_K}\right)^2 = -\frac{\partial_r g_{tt}}{\partial_r g_{\varphi\varphi}}, \quad \Omega^2_K = \left(\frac{g_{tt}L_K}{g_{\varphi\varphi}E_K}\right)^2 = -\frac{\partial_r g_{tt}}{\partial_r g_{\varphi\varphi}}$$

Marginally stable circular orbit (last stable circular orbit):

$$l'_K(r_{ms}) \equiv 0$$

Marginally bound circular orbit:

$$\mathcal{V}(1, L_K(r_{mb}), r_{mb}) \equiv 1 \quad \Leftrightarrow \quad E^2_K(r_{mb}) \equiv 1$$
The q-metric

Simplest exact static exterior solution of the vacuum field equations with non-vanishing quadrupole moment, its metric:

$$ds^2 = - \left( 1 - \frac{2M}{r} \right)^{1+q} dt^2 + \left( 1 - \frac{2M}{r} \right)^{-q} \left[ \left( 1 + \frac{M^2 \sin^2 \vartheta}{r^2 - 2Mr} \right)^{-q(2+q)} \left( \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\vartheta^2 \right) + r^2 \sin^2 \vartheta d\varphi^2 \right] ,$$

where $q$: quadrupole parameter, $M$: mass parameter.

Mass multipole moments (Geroch-Hansen):

$$M_0 = (1 + q)M, \quad M_2 = -\frac{M^3}{3} q(q + 1)(q + 2)$$

For later comparison, express the q-metric in terms of $M_0$ and $M_2$ (restrict discussion to $q > -1$ and $M > 0$):

$$M = M_0 \sqrt{3 \frac{M_2}{M_0^3} + 1}, \quad q = \frac{1}{\sqrt{3 \frac{M_2}{M_0^3} + 1} - 1}$$
Marginally bound and marginally stable circular orbits

For lightlike geodesics, there is exactly one solution for photon circles:

\[ r_c = (3 + 2q)M = \left(2 + \sqrt{1 + 3\frac{M_2}{M_0^3}}\right) \]

Marginally bound circular orbit: only numerically, up to one
Marginally stable circular orbit:

\[ r_{\text{ms}}^\pm = M\left(4 + 3q \pm \sqrt{5q^2 + 10q + 4}\right) \]

→ splits the family of q-metrics intro three classes:

**Class I** : \( \infty > q > -1/2 \text{ or } -1/3 < M_2/M_0^3 < 1 \)
- Schwarzschild-like

**Class II** : \( -1/2 > q \gtrsim -0.553 \text{ or } 1 < M_2/M_0^3 < 4/3 \)
- two marginally stable, but no photon circle anymore

**Class III** : \( -0.553 \gtrsim q > -1 \text{ or } 4/3 < M_2/M_0^3 < \infty \)
- all orbits \( > 2M \) are stable
Depiction of orbits

Abbildung: Circular orbits in the q-metric, depending on the quadrupole moment.
Effective potential in the equatorial plane

\[ \mathcal{W}(r, l, \vartheta) = \frac{1}{2} \ln \left[ \frac{r^2 \sin^2 \vartheta}{(1 - \frac{2M}{r})^{(1+q)}} \right] \left[ \frac{r^2 \sin^2 \vartheta}{1 - l^2 (1 - \frac{2M}{r})^q} \right] \]

\[ \mathcal{W}(r, l, \vartheta) = \frac{1}{2} \ln \left[ \frac{r^2 \sin^2 \vartheta}{(1 - \frac{2M}{r})^{(1+q)}} \right] \left[ \frac{r^2 \sin^2 \vartheta}{1 - l^2 (1 - \frac{2M}{r})^q} \right] \]
Effective potential in Class II spacetimes: the connection of double tori and fish

*Abbildung:* Polar dependency of double tori for Class II spacetimes, depending on the quadrupole moment, forming fish-like structures.
The Erez-Rosen spacetime

First found solution of Einstein’s vacuum field equations identified as describing the gravitational field around a central object with a quadrupole moment

\[ ds^2 = -f dt^2 + \frac{\sigma^2}{f} \left[ e^{2\gamma} (x^2 - y^2) \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2)d\varphi^2 \right] \]

with

\[ f = \frac{x - 1}{x + 1} e^{-2qP_2Q_2} \]

\[ \gamma = \frac{1}{2} (1 + q)^2 \ln \frac{x^2 - 1}{x^2 - y^2} + 2q (1 - P_2) Q_1 + q^2 (1 - P_2) \cdot \left[ (1 + P_2) (Q_1^2 - Q_2^2) + \frac{1}{2} (x^2 - 1) \left( 2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2' \right) \right] \]

with \( Q = Q(x) \) and \( P = P(y) \), and \( q \): quadrupole parameter.

Transformation to Schwarzschild-like coordinates via \( x = r/M - 1 \) and \( y = \cos \vartheta \).

Multipole moments (Geroch-Hansen):

\[ M_0 = M, \quad M_2 = \frac{2}{15} q^3 M^3 \]
Marginally bound and marginally stable circular orbits

For lightlike geodesics, there are up to 2 solutions for photon circles, determined by:

\[ r - 3M - qr(r - 2M)\partial_r Q_2(r/M - 1) = 0 \]

Marginally bound circular orbit: only numerically, up to one
Marginally stable circular orbit: also, only numerically, up to two
Orbit properties divide Erez-Rosen spacetimes into 3:

Class I : \(-\infty < q < 1\) or \(-\infty < M_2/M_0^3 \lesssim 0.13\)
  - Schwarzschild-like

Class IIa : \(1 < q \lesssim 2.25\) or \(0.13 \lesssim M_2/M_0^3 \lesssim 1.52\)
  - two photon circles!

Class IIb : \(2.25 \lesssim q < 4.8\) or \(1.52 \lesssim M_2/M_0^3 \lesssim 25.8\)
  - two marginally stable, but no photon circle anymore - see Class II q-metric

Class III : \(4.8 \lesssim q < \infty\) or \(25.8 \lesssim M_2/M_0^3 < \infty\)
  - all orbits \(> 2M\) are stable
Abbildung: Circular orbits in Erez-Rosen spacetime, depending on the quadrupole moment.
Effective potential in the equatorial plane

\[ \mathcal{W}(r, l, \vartheta) = \frac{1}{2} \ln \left[ \frac{r^2 (r - 2M) e^{-2qP_2(\cos \vartheta)} Q_2(r/M - 1)}{r^3 \sin^2 \vartheta - l^2 (r - 2) e^{-4qP_2(\cos \vartheta)} Q_2(r/M - 1)} \right] \]
Abbildung: Polar dependency of the effective potential for class II spacetimes. The black numbers represent the density at the position of the equipotential surfaces.
Conclusion

- both q-metric and Erez-Rosen spacetime can be distinguished into 3 classes
- In class I, tori are qualitatively similar to the tori in Schwarzschild spacetime
- In class II, there are qualitative differences
  - double tori, fish-like structures
  - two centres
  - no accretion
- In class III, tori cannot have a cusp, thus no accretion