Bayesian and non-Bayesian reliability estimation of multicomponent stress–strength model for unit Weibull distribution

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Abstract
Mazucheli et al. introduced a new transformed model referred as the unit-Weibull (UW) distribution with uni- and anti-unimodal, decreasing and increasing shaped density along with bathtub shaped, constant and non-decreasing hazard rates. Here we consider inference upon stress and strength reliability using different statistical procedures. Under the framework that stress–strength components follow UW distributions with a known shape, we first estimate multicomponent system reliability by using four different frequentist methods. Besides, asymptotic confidence intervals (CIs) and two bootstrap CIs are obtained. Inference results for the reliability are further derived from Bayesian context when shape parameter is known or unknown. Unbiased estimation has also been considered when common shape is known. Numerical comparisons are presented using Monte Carlo simulations and in consequence, an illustrative discussion is provided. Two numerical examples are also presented in support of this study.

Significant conclusion: We have investigated inference upon a stress–strength parameter for UW distribution. The four frequentist methods of estimation along with Bayesian procedures have been used to estimate the system reliability with common parameter being known or unknown.

1. Introduction

Many physical phenomena from different disciplines are often modelled using some known probability models which include, among others, Burr family of distributions, lognormal, gamma, exponential, Weibull distributions. These probability distributions can model a variety of data exhibiting significant variability. One particular probability model of specific significance is the well-known Weibull distribution. Over the last five decades or so, several new modifications of this particular model have been proposed and studied by many researchers, such as exponentiated Weibull [1–3], extended Weibull [4,5], modified Weibull [6–8], odd Weibull [9], Weibull–X class [10], Weibull-G model [11], extended Weibull–G family [12] and so on. These distributions are generally derived by adding some additional parameters to the original probability distribution under consideration. Besides, most of these generalizations share one interesting characteristic that they are based on the support over positive part of the real line. Such inferential trend often leads to insufficient number of distributions with finite support. But at the same time, probability distributions with support on finite range are also of importance in many studies. Physical characteristics of many life test experiments quite often lead to data which may lie in some finite range. Data on fractions, percentages, per capita income growth, fuel efficiency of vehicles, height and weight of individuals, survival times from a deadly disease etc. are likely to lie in some bounded positive intervals. Kumaraswamy [13] studied a two-parameter distribution with support on finite interval and investigated many useful applications of this distribution in meteorological inference. Mazucheli et al. [14,15] derived some useful classical estimates for unknown parameters of a unit-gamma distribution and also introduced unit Birnbaum–Saunders distribution, besides, different estimation methods are used to estimate the model parameters. Further, Mazucheli et al. [16] developed and studied properties of a unit Gompertz distribution and derived inferences for its unknown parameters. Mazucheli et al. [17] initially studied various characteristics of a unit Weibull distribution. Through several applications they showed that this particular model can be treated as an useful alternative to the Kumaraswamy and beta distributions in life test studies. We, here, consider multicomponent reliability estimation under unit Weibull probability distribution. This distribution can assume different shapes – monotonic, unimodal, anti-unimodal based on model parameters and such flexibility makes it quite suitable for many applications.

Keywords
Bayesian point and interval procedures; least square estimator; stress–strength reliability; maximum product of spacing estimator
for fitting various data arising in reliability analysis and industrial experiments.

Traditionally, a system consisting more than one component is referred to as multicomponent system, see [18,19]. Under this framework, a system with \( k \) independent strength components \( X_1, X_2, \ldots, X_k \) properly functions if at least \( s = \text{out-of-} k \) \((s \leq k)\) strength variables exceed the stress \( Y \). Such a system is often called as ‘(s-out-of-k)G’ system. Physical aspects of many investigations may lead to multi-component systems and many such examples abound in practice. Such structures are extensively utilized in industrial experiments, military radio networks, bridge construction, building communication networks, etc. In recent past, many scholars have studied multicomponent stress–strength models largely owing to the growing interest on this topic, see for instance, works of [20–23], Dey and Moala [24] and many others. Seadawy et al. [25], Seadawy et al. [26], Ahmad et al. [27], Riaz et al. [28], Abbasi et al. [29] have also studied some complex structures.

The probability distribution of a unit Weibull distribution, with range in the interval \((0,1)\), is given by

\[
f_X(x, \alpha, \beta) = \alpha \beta (-\ln x)^{\beta-1} x^{-1} e^{-\alpha(-\ln x)^\beta}, \quad \alpha > 0, \quad \beta > 0
\]

and

\[
F_X(x) = e^{-\alpha(-\ln x)^\beta}, \quad \alpha > 0, \quad \beta > 0
\]

where \( \alpha, \beta \) govern shape of UW distribution. We write this distribution as UW(\( \alpha, \beta \)).

Various frequentist procedures are discussed to estimate the unknown multicomponent reliability function. We further estimate this unknown parametric function using Bayes procedure against proper and improper prior distributions under a well-known loss function. Next, we have constructed asymptotic, two bootstrap intervals and highest posterior density (HPD) intervals. Further, UMVUE of the multicomponent reliability is also constructed. Subsequently in Section 3, frequentist and Bayesian inference of multicomponent reliability are derived under the assumption that \( \beta \) is known. Simulations results are evaluated in Section 4 to examine numerical performance of proposed procedures. Two real-life examples are studied in Section 5. Finally, paper is concluded in Section 6.

2. Inference for \( R_{s,k} \) with unknown \( \beta \)

Suppose \( X_1, X_2, \ldots, X_k \) denote strength components with cdf as given in (2). Likewise variable \( Y \) denotes associated stress acting on the system with distribution function \( G_Y(y; \alpha_2, \beta) \). Then we have (see Rao et al. [19])

\[
R_{s,k} = \sum_{i=1}^{k} \binom{k}{i} \int_{-\infty}^{\infty} \left(1 - F(\alpha_2, \beta, y)\right)^i 
\]

\[
\times \left(F(\alpha_2, \beta, y)\right)^{k-i} dG(y).
\]

We next derive some useful inference of this parametric function using different procedures when common shape is unknown or known.

2.1. MLE of \( R_{s,k} \)

Suppose that \( X_1, X_2, \ldots, X_k \) are taken from UW(\( \alpha_1, \beta \)) distribution and \( Y \) follows a UW(\( \alpha_2, \beta \)) distribution where parameters \( \alpha_1, \alpha_2 \) and \( \beta \) are assumed as unknown. Under this framework, we get that

\[
R_{s,k} = \alpha_1 \beta \sum_{i=1}^{k} \binom{k}{i} \int_{0}^{\infty} \left(1 - e^{-\alpha_1(-\ln y)^\beta}\right)^i 
\]

\[
\times \left(e^{-\alpha_1(-\ln y)^\beta}\right)^{k-i} 
\]

\[
\times \alpha_2 \beta (-\ln y)^{\beta-1} y^{-1} e^{-\alpha_2(-\ln y)^\beta} dy.
\]

After some simplification, we have

\[
R_{s,k} = \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \alpha_2 \binom{k}{i} \left[\alpha_1(j + k - i) + \alpha_2\right]^{-1}.
\]
We proceed by assuming that \( n \) structures are subjected to a test. The likelihood, based on observed data \( x_{i1}, x_{i2}, \ldots, x_{ik} \) and \( y_i \) of \((\alpha_1, \alpha_2, \beta)\) is

\[
L (\alpha_1, \alpha_2, \beta; x, y) = \prod_{i=1}^{n} \left( \frac{\alpha_1 \beta (\ln x_i)^{\beta - 1} x_i^{-1} e^{-\alpha_1 (\ln x_i)^{\beta}}}{\sum_{j=1}^{k} x_j} \right) \times (\alpha_2 \beta (-\ln y_i)^{\beta - 1} y_i^{-1} e^{-\alpha_2 (-\ln y_i)^{\beta}}).
\]

The log-likelihood turns out to be

\[
I (\alpha_1, \alpha_2, \beta; x, y) = (\log \alpha_1)(nk) + \sum_{i=1}^{n} \sum_{j=1}^{k} \log(1/x_{ij}) + (\beta - 1) \sum_{i=1}^{n} \sum_{j=1}^{k} \log(-\log x_{ij}) - \alpha_1 \sum_{i=1}^{n} \sum_{j=1}^{k} (-\log x_{ij})^\beta + \alpha_2 \sum_{i=1}^{n} \log(-\log y_i)^\beta + (\log \beta)(n(k + 1)) - \alpha_3 \sum_{i=1}^{n} \sum_{j=1}^{k} (-\log y_i)^\beta + (\log \alpha_2)n.
\]

(5)

After some simplification, we get

\[
\hat{\alpha}_1 = \frac{nk}{\sum_{i=1}^{n} \sum_{j=1}^{k} (-\log x_{ij})^\beta}.
\]

Similarly, from

\[
\frac{\partial I}{\partial \alpha_2} = \frac{n}{\alpha_2} - \sum_{i=1}^{n} (-\log y_i)^\beta = 0,
\]

we get the MLE \( \hat{\alpha}_2 \) as

\[
\hat{\alpha}_2 = \frac{n}{\sum_{i=1}^{n} (-\log y_i)^\beta}.
\]

Finally, \( \partial I/\partial \beta = 0 \) is solved for \( \beta \) to obtain its MLE \( \hat{\beta} \) satisfying

\[
\frac{(nk + n)}{\hat{\beta}} + \sum_{i=1}^{n} \sum_{j=1}^{k} \log(-\log x_{ij}) + \sum_{i=1}^{n} \log(-\log y_i) - \hat{\alpha}_1 \sum_{i=1}^{n} \sum_{j=1}^{k} \log(-\log x_{ij})(-\log y_i)^{\beta} = 0. \tag{8}
\]

The above likelihood equation is nonlinear and can be solved numerically using some iterative technique. Plugging MLE of \( \beta \) in Equations (6) and (7), we can obtain \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \), respectively. Now MLE \( \hat{R}_{sk} \) of \( R_{sk} \) turns out to be

\[
\hat{R}_{sk} = \sum_{i=1}^{k} \sum_{j=0}^{i} (-1)^{i-j} \binom{k}{j} \binom{n}{j+k-i} \hat{\alpha}_1^j \hat{\alpha}_2^{k-i}.
\]

2.2. Asymptotic interval

Here a confidence interval of the multicomponent reliability is given using MLE. The Fisher information of \( \theta = (\alpha_1, \alpha_2, \beta) \) is

\[
I(\theta) = \begin{bmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{bmatrix}.
\]

The elements of \( I(\theta) \) are

\[
I_{11} = E \left( \frac{\partial^2 I}{\partial \alpha_1^2} \right) = -\frac{nk}{\alpha_1^2}, \quad I_{22} = E \left( \frac{\partial^2 I}{\partial \alpha_2^2} \right) = -\frac{n}{\alpha_2^2},
\]

\[
I_{12} = I_{21} = E \left( \frac{\partial^2 I}{\partial \alpha_1 \partial \alpha_2} \right) = 0,
\]

\[
I_{13} = I_{31} = E \left( \frac{\partial^2 I}{\partial \alpha_1 \partial \beta} \right) = \int_0^1 \log(-\log x_{ij})(-\log y_i)^\beta \alpha_1 \times \beta (-\ln x_i)^{\beta - 1} x_i^{-1} e^{-\alpha_1 (\ln x_i)^{\beta}} dx_{ij}.
\]

After some simplification, we get

\[
I_{13} = I_{31} = \frac{nk}{\alpha_1 \beta} \int_0^\infty ue^{-u} \log(u/\alpha_1) \, du.
\]

Similarly, we have

\[
I_{23} = I_{32} = \frac{n}{\alpha_2 \beta} \int_0^\infty ue^{-u} \log(u/\alpha_2) \, du
\]

and

\[
I_{33} = \frac{n(k + 1)}{\beta^2} - \frac{nk}{\beta^2} \int_0^\infty ue^{-u} (\log(u/\alpha_1))^2 \, du
\]

\[- \frac{n}{\beta^2 \alpha_1} \int_0^\infty ue^{-u} (\log(u/\alpha_2))^2 \, du.
\]

From the large sample theory, the MLE of reliability is normal with average given by \( R_{sk} \). The associated variance is computed as

\[
\sigma^2_{R_{sk}} = \left( \frac{\partial R_{sk}}{\partial \alpha_1} \right)^2 I_{11}^{-1} + \left( \frac{\partial R_{sk}}{\partial \alpha_2} \right)^2 I_{22}^{-1} + 2 \left( \frac{\partial R_{sk}}{\partial \alpha_1} \right) \left( \frac{\partial R_{sk}}{\partial \alpha_2} \right) I_{12}^{-1},
\]
where we have
\[
\frac{\partial \hat{R}_{sk}}{\partial \alpha_1} = \sum_{i=0}^{k} \sum_{j=0}^{q_i} (j + k - i) \binom{k}{j} \binom{i}{l} (-1)^{y+1} \alpha_2 \times \left[ \alpha_1 (j + k - i) + \alpha_2 \right]^{-2}
\]
and similarly \(\frac{\partial \hat{R}_{sk}}{\partial \alpha_2}\) is evaluated. Thus 100(1 - \(\eta\))% CI of the parametric function is given by \((\hat{R}_{sk} \pm q_{n/2} \hat{\sigma}_{Rs}),\) where \(q_{n/2}\) is associated percentile of normal \(N(0,1)\) variable and \(\hat{\sigma}_{Rs}\) is computed at respective MLEs.

### 2.3. Bootstrapping

We develop boot-\(p\) and boot-\(t\) intervals for \(R_{sk}\) (see Efron [30] and Hall [31] for further details).

#### 2.3.1. Boot-\(p\)

1. Obtain samples \((y_1^*, \ldots , y_n^*)\) of size \(n\) and also simulate \((x_1^*, \ldots , x_{nk}^*)\) of size \(nk\) where \(i\) is a positive integer ranging from 1 to \(n\). Based on it, a sample estimate of reliability is computed as \(\hat{R}_{sk}^p\).
2. Iterate above stage, say \(n_{boot}\) times.
3. Now if \(F^*(x) = P(\hat{R}_{sk}^* \leq x)\) denote cdf of \(\hat{R}_{sk}^*\) and assume that \(\hat{R}_{sk}^*(x) = F^{*-1}(x)\) for a given \(x\). Then 100(1 - \(\eta\))% boot-\(p\) interval is computed as \((\hat{R}_{sk}^{\alpha/2}, \hat{R}_{sk}^{1-(\alpha/2)})\).

#### 2.3.2. Boot-\(t\)

1. Simulate bootstraps samples \((y_1^*, \ldots , y_n^*)\) of size \(n\) and also generate \((x_1^*, \ldots , x_{nk}^*)\) of size \(nk\) where \(i\) is a positive integer ranging from 1 to \(n\). Based on it, a sample estimate of reliability is computed as \(\hat{R}_{sk}^t\).
2. Evaluate \(T^* = (\sqrt{\text{var}(\hat{R}_{sk}^* - \hat{R}_{sk})})^{-1}(\hat{R}_{sk}^* - \hat{R}_{sk})\).
3. Iterate above two stages sufficient number of times.
4. Suppose \(H_{Rs}^*(x)\) is cdf of variable as defined in Step 2 and define \(\hat{R}_{sk}^{bt} = \hat{R}_{sk}(x) + [H_{Rs}^*(x)](\sqrt{\text{var}(\hat{R}_{sk})})\). Then 100(1 - \(\eta\))% boot-\(t\) interval is computed as \((\hat{R}_{sk}^{\alpha/2}, \hat{R}_{sk}^{1-(\alpha/2)})\).

### 2.4. Least square and weighted least square estimates

The LSE method is initially discussed by Swain et al. [32]. Here we obtain this estimator for the multicomponent reliability \(R_{sk}\). Note that LSE \(\hat{\alpha}_1\) of \(\alpha_1\) can be computed by minimizing \(\sum_{i=1}^{n} [\alpha_1(\text{ln}x_i) - (i/(nk + 1))]^2\), that is, \(\sum_{i=1}^{n} [-\hat{\alpha}_1(\text{ln}x_i) - (i/(nk + 1))]^2\). So LSE \(\hat{\alpha}_1\) is the solution of the following equation:

\[
\sum_{i=1}^{n} [e^{-\hat{\alpha}_1(\text{ln}x_i)} - (i/(nk + 1))] \times e^{-\hat{\alpha}_1(\text{ln}x_i)} (\text{ln}x_i)^\beta = 0,
\]

where \(X_1, X_2, \ldots , X_{nk}\) follow \(UW(\alpha_1, \beta)\) distribution. The LSE \(\hat{\alpha}_{2l}\) of \(\alpha_2\) is computed similarly by solving the following equation:

\[
\sum_{i=1}^{n} [e^{-\hat{\alpha}_1(\text{ln}x_i)} - (i/(nk + 1))] \times e^{-\hat{\alpha}_1(\text{ln}x_i)} (\text{ln}x_i)^\beta = 0,
\]

where \(Y_1, Y_2, \ldots , Y_n\) follow \(UW(\alpha_2, \beta)\) distribution. Thus LSE of \(R_{sk}\) turns out to be

\[
\hat{R}_{sk} = \sum_{i=1}^{k} \sum_{j=0}^{q_i} \left( k \binom{i}{l} \left[ \hat{\alpha}_1 (j + k - i) + \hat{\alpha}_2 \right]^{-1} (1)^j \right).
\]

Next, weighted least square estimates (WLSE) \(\hat{\alpha}_1^w\) of \(\alpha_1\) is computed by minimizing the following function:

\[
\sum_{i=1}^{nk} \frac{(nk + 1)^2 (nk + 2)}{i(nk - i + 1)} [e^{-\hat{\alpha}_1^w(\text{ln}x_i)} - (i/(nk + 1))]^2
\]

and thus the required estimate is determined from the following equation:

\[
\sum_{i=1}^{nk} \frac{(nk + 1)^2 (nk + 2)}{i(nk - i + 1)} [e^{-\hat{\alpha}_1^w(\text{ln}x_i)} - (i/(nk + 1))] \times e^{-\hat{\alpha}_1^w(\text{ln}x_i)} (\text{ln}x_i)^\beta = 0.
\]

The WLSE estimate \(\hat{\alpha}_2^w\) of \(\alpha_2\) is similarly computed from the equation presented below:

\[
\sum_{i=1}^{nk} \frac{(nk + 1)^2 (nk + 2)}{i(nk - i + 1)} [e^{-\hat{\alpha}_2^w(\text{ln}x_i)} - (i/(nk + 1))] \times e^{-\hat{\alpha}_2^w(\text{ln}x_i)} (\text{ln}x_i)^\beta = 0.
\]

So WLSE of \(R_{sk}\) is given by

\[
\hat{R}_{sk} = \sum_{i=1}^{k} \sum_{j=0}^{q_i} \left( k \binom{i}{l} \left[ \hat{\alpha}_1^w (j + k - i) + \hat{\alpha}_2^w \right]^{-1} (1)^j \right).
\]

### 2.5. Maximum product of spacing (MPS) estimates

Here we obtain another estimator called MPS estimates of the multicomponent reliability as discussed in [33]. Here spacing of a random sample of size \(nk\) is defined as \(D_1(\alpha_1) = F_X(x_1) - F_X(x_{n-1}), i = 1, 2, \ldots, nk\), also \(F_X(x_0) = 0 \) and \(F_X(x_{nk+1}) = 1\). The desired estimate \(\hat{\alpha}_1^n\) of \(\alpha_1\) is computed by maximizing the geometric mean \((\prod_{i=1}^{nk+1} D_1(\alpha_1))^{1/(nk+1)}\) of spacing. Alternatively we maximize \((1/(nk + 1)) \sum_{i=1}^{nk+1} \ln D_1(\alpha_1)\) and compute the desired estimate of \(\alpha_1\) from the following equation:

\[
(1/(nk + 1)) \sum_{i=1}^{nk+1} \frac{1}{D_1(\hat{\alpha}_1^n)} [e^{-\hat{\alpha}_1^n(\text{ln}x_i)} - (\text{ln}x_i)^\beta + e^{-\hat{\alpha}_1^n(\text{ln}x_i)} (\text{ln}x_i)^\beta] = 0.
\]
Next using a sample of size \( n \) from the \( Y \) variable, define \( D_i^p(\alpha_2) = F_Y(y_i) - F_Y(y_{i-1}), i = 1, 2, \ldots, n \), also \( F_Y(y_0) = 0 \) and \( F_Y(y_{n+1}) = 1 \). The MPS estimate \( \hat{\alpha}_2^m \) of \( \alpha_2 \) is determined from equation presented as

\[
\left(1/(n + 1) \right) \sum_{i=1}^{n+1} D_i^p(\alpha_2^m) = \int_0^\infty e^{-\hat{\alpha}_2^m(-\ln y_i)^{\beta}} (-\ln y_i)^{\beta} \, dy_i
\]

Thus the MPS estimate of \( R_{s,k} \) can be obtained as

\[
\hat{R}_{s,k}^m = \sum_{i=1}^{k} \sum_{j=0}^{i} \binom{i}{j}\frac{(-1)^j \hat{\alpha}_2^m}{\hat{\alpha}_1^m(j + k - i) + \hat{\alpha}_2^m}.
\]

### 2.6. Bayesian inference

We discuss Bayesian inference of the parametric reliability function. Parameters \((\alpha_1, \alpha_2, \beta)\) are a priori assumed to be independently distributed as gamma variables with hyperparameters \((p_i, q_i)\), where \( p_i \) and \( q_i \), respectively, are shape and reverse of scale, index \( i \) takes values as 1, 2 and 3. The corresponding pdf is

\[
g(x) = \frac{\alpha_i^p}{\Gamma(p_i)} x^{p_i-1} e^{-xq_i}, \quad x, p_i, q_i > 0.
\]

After some simplification, the joint posterior distribution is derived to be

\[
\pi(\alpha_1, \alpha_2, \beta | x, y) \propto \alpha_1^{nk+1} \alpha_2^{n+1} \beta^{n(k+1)+3-1} \exp\left(-\alpha_1 \sum_{i=1}^n \sum_{j=1}^k (-\log x_{ij})^\beta\right) \times e^{-\alpha_2 \sum_{i=1}^n (-\log y_i)^\beta} \times e^{-\beta \sum_{i=1}^n \sum_{j=1}^k (-\log x_{ij})}.
\]

The normalizing constant can easily be computed. The required estimator is

\[
\hat{R}_{s,k}^B = \int_0^\infty \int_0^\infty \int_0^\infty \hat{R}_{s,k} \pi(\alpha_1, \alpha_2, \beta | x, y) \, d\alpha_1 \, d\alpha_2 \, d\beta.
\]

The above integral is relatively difficult to be solved analytically. However, the corresponding posterior expectation can be approximated numerically. We now use [34] procedure and MH algorithm [35,36] to compute \( R_{s,k} \).

#### 2.6.1. Lindley’s method

Here Bayes estimator of the multicomponent reliability is obtained by Lindley’s approximation method. This method is based on asymptotic expansion useful for evaluating Bayes procedures. The expectation of \( u(\theta), \theta = (\alpha_1, \alpha_2, \beta) \), with respect to the given posterior distribution is

\[
E(u(\theta) | x, y) = \frac{\int u(\theta)e^{l(\theta) + \rho(\theta)} \, d\theta}{\int e^{l(\theta) + \rho(\theta)} \, d\theta}
\]

where \( l(\theta) \) and \( \rho(\theta) \) denote the log-likelihood function and logarithm of the associated prior, respectively. From this method, the approximate Bayes estimator is given by

\[
\hat{R}_{s,k} = u(\theta) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5)
\]

Furthermore,

\[
\rho_1 = \frac{p_1 - 1}{\alpha_1} - q_1, \quad \rho_2 = \frac{p_2 - 1}{\alpha_2} - q_2,
\]

\[
\rho_3 = \frac{p_3 - 1}{\beta} - q_3.
\]

Further \( \sigma_{s,k} \) is component \([-l_{s,k}]^{-1} \), \( i, k = 1, 2, 3 \). Next we take the quantity \( u(\theta) \) as \( R_{s,k} \) and respective MLEs are utilized for evaluation purposes. Expressions of Equation (12) are computed below:

\[
l_1 = \frac{nk}{\alpha_1} - \sum_{j=1}^n \log x_{ij}^\beta,
\]

\[
l_1 = -\frac{nk}{\alpha_1^2}, \quad l_{11} = 2nk
\]

\[
l_2 = -\frac{n}{\alpha_2} \sum_{j=1}^n (-\log y_{ij})^\beta, \quad l_{22} = -\frac{n}{\alpha_2^2}, \quad l_{222} = \frac{2n}{\alpha_2^3},
\]

\[
l_3 = \frac{n(k+1)}{\beta} + \sum_{i=1}^n \log(-\log x_{ij})
\]

\[
- \sum_{i=1}^n \sum_{j=1}^k \log(-\log x_{ij})(-\log y_{ij})^\beta
\]
In a likewise manner, we also evaluate \(u_1 = \partial R_{s,k}/\partial \alpha_1, u_2 = \partial R_{s,k}/\partial \alpha_2, u_{11} = \partial^2 R_{s,k}/\partial \alpha_1^2, u_{22} = \partial^2 R_{s,k}/\partial \alpha_2^2\) and also notice that \(u_3 = \partial R_{s,k}/\partial \beta = 0, u_{33} = u_{13} = u_{23} = 0, u_{12} = u_{21} = \partial^2 R_{s,k}/\partial \alpha_1 \partial \alpha_2\).

### 2.6.2. MH algorithm

This iterative procedure is widely used for evaluating Bayes estimates of different parametric functions such as \(R_{s,k}\), among others. The posterior of \((\alpha_1, \alpha_2, \beta)\) given observations is derived in previous section. Thus the marginal posterior of \(\alpha_1, \alpha_2\) are seen to be gamma distributed, respectively. While marginal posterior of other parameter \(\beta\) is not evaluated in known form. We simulate samples for \(\beta\) using normal proposal distribution. Accordingly, Bayes estimate \(\hat{R}_{s,k}^{MH}\) of \(R_{s,k}\) can be obtained from the following procedure.

**Step:**

1. Select an adequate guess \((\alpha_{10}, \alpha_{20}, \beta_0)\) of \((\alpha_1, \alpha_2, \beta)\).
2. In \(i\)th iteration obtain sample \(\beta'\) utilizing a normal distribution with mean \(\beta_{i-1}\) and variance \(\sigma^2\).
3. Obtain a sample \(\alpha'_1\) from the gamma \(G(a, b)\) variable where \(a = (nk + p_1)\) and \(b = q_1 + \sum_{j=1}^{n-k} \frac{1}{-\log y_j}\).
4. Obtain a sample \(\alpha'_2\) from the gamma \(G(a', b')\) variable where \(a' = (n + p_2)\) and \(b' = q_2 + \sum_{j=1}^{n-k} \frac{1}{-\log y_j}\).
5. Set \(m = \min\{1, a'\}\) where \(a' = \pi(\alpha_1', \alpha_2', \beta')\) samples \((\alpha_{11}, \alpha_{21}, \beta_{11} | \text{samples})\)^{-1}.
6. Simulate a number \(u\) from a random variable \(U\) which follows a uniform model on \((0, 1)\).
7. We take the sample as \(\alpha_1 = \alpha_1', \alpha_2 = \alpha_2', \beta = \beta'\) provided the inequality \(u \leq m\) holds.
8. Based on the above sample, we compute \(R_{s,k}^{\sigma}\) at \((\alpha_1', \alpha_2', \beta')\).
9. Now iterate stages 2 to 7 to obtain posterior estimates \(R_{s,k}^{\sigma}\), \(l = 1, 2, \ldots, M\).

Now the required Bayes estimate \(\hat{R}_{s,k}^{MH}\) is given by

\[
\hat{R}_{s,k}^{MH} = \frac{1}{M - M_0} \sum_{i=M_0+1}^{M} R_{s,k}^{\sigma}.
\]

where \(M_0\) is burn-in samples. The HPD interval of \(R_{s,k}\) is constructed from the posterior samples (see Chen and Shao [37]).

### 3. Inference for \(R_{s,k}\) with known \(\beta\)

We now proceed to derive some more frequentist and Bayesian inference of unknown parametric function under assumption that \(\beta = \beta_0\) where \(\beta_0\) is a known constant. We mention that the expression of the reliability is identically same as considered in previous sections since it does not depend on \(\beta_0\).

#### 3.1. MLE

Suppose that \((x_1, x_2, \ldots, x_k)\) denotes an observed value generated from the model as listed in Equation (2) where \(\alpha_1\) being unknown shape and \(\beta_0\) being known shape. Similarly sample \(y\) is drawn when \(\alpha_2\) is unknown. The likelihood is then given by

\[
L(\alpha_1, \alpha_2, \beta_0; x, y) = \prod_{i=1}^{n} \left( \frac{1}{k} \sum_{j=1}^{k} \left( \alpha_1 \beta_0 (-\ln x_j)^{\beta_0-1} x_j^{-1} e^{-\alpha_1 (-\ln x_j)^{\beta_0}} \right) \times \left( \alpha_2 \beta_0 (-\ln y_j)^{\beta_0-1} y_j^{-1} e^{-\alpha_2 (-\ln y_j)^{\beta_0}} \right) \right).
\]

The log-likelihood function can be obtained as

\[
l(\alpha_1, \alpha_2, \beta_0; x, y) = nk \log \alpha_1 + n \log \alpha_2 - \alpha_1 \sum_{i=1}^{n} \frac{1}{k} \sum_{j=1}^{k} (-\log x_j)^{\beta_0} - \alpha_2 \sum_{i=1}^{n} (-\log y_j)^{\beta_0}.
\]
We partially differentiate the above function with respect to unknown parameters \( \alpha_1 \) and \( \alpha_2 \), respectively. Then solving these likelihood equations, respective MLEs of \( \alpha_1 \) and \( \alpha_2 \) are obtained as

\[
\hat{\alpha}_1 = \frac{n k}{\sum_{i=1}^{n} \sum_{j=1}^{k} (-\log x_j)^{\beta_0}}
\]

and

\[
\hat{\alpha}_2 = \frac{n}{\sum_{i=1}^{n} (-\log y_i)^{\beta_0}}.
\]

Thus the estimate \( \hat{R}_{sk} \) of \( R_{sk} \) is evaluated using the invariance property. From large sample theory, we know that this MLE is normally distributed. The associated mean is given by \( R_{sk} \). Similarly variance is

\[
\sigma^2_{R_{sk}} = \left( \frac{\partial R_{sk}}{\partial \alpha_1} \right)^2 \frac{\alpha_1^2}{n k} + \left( \frac{\partial R_{sk}}{\partial \alpha_2} \right)^2 \frac{\alpha_2^2}{n}.
\]

The 100(1 - \( p \))% confidence interval of the reliability \( R_{sk} \) is \( (\hat{R}_{sk} \pm q_{p/2} \sigma_{R_{sk}}) \) where \( q_{p/2} \) is the corresponding percentile of the normal \( N(0, 1) \) distribution. It is also noted that \( \hat{\sigma}_{R_{sk}} \) is the estimated value of \( \sigma_{R_{sk}} \).

### 3.2. UMVUE of \( R_{sk} \)

Here UMVUE \( \hat{R}_{sk}^U \) of the reliability \( R_{sk} \) is derived. It is enough to find the UMVUE of \( \xi(\alpha_1, \alpha_2) = \alpha_2/(\alpha_1 + j + k - i + \alpha_2) \) as UMVUEs satisfy linearity property. We see that \( (Y^*, Z^*) \) is a complete sufficient statistic for \( (\alpha_1, \alpha_2) \). Further with \( Y^* \) has a \( G(nk, \lambda_1) \) pdf and \( Z^* \) also has a \( G(n, \lambda_2) \) pdf where \( (Y^*, Z^*) = (\sum_{i=1}^{n} \sum_{j=1}^{k} (-\log x_j)^{\beta_0}, \sum_{i=1}^{n} (-\log y_i)^{\beta_0}) \). Now let \( \phi(Y, Z) = 1, Y > (j + k - i)Z \) and \( \phi(Y, Z) = 0 \), elsewhere, where \( Y = (-\log x_j)^{\beta_0}, Z = (-\log y_i)^{\beta_0} \). We verify that this is unbiased for estimating \( \xi(\alpha_1, \alpha_2) \).

Now UMVUE \( \hat{\xi}_{U}(\alpha_1, \alpha_2) \) of \( \xi(\alpha_1, \alpha_2) \) is derived using the Lehmann–Scheffe theorem as follows:

\[
\hat{\xi}_{U}(\alpha_1, \alpha_2) = E(\phi(Y, Z) \mid Y^* = y^*, Z^* = z^*) = P(Y > (j + k - i)Z \mid Y^* = y^*, Z^* = z^*)
\]

\[
= \int_{0}^{\infty} \int_{y^*}^{\infty} f_{Y^*}(y^*) f_{Z^*}(z^*) \text{dydz}
\]

where \( A_0 = \{y, z : 0 < y < y^*, 0 < z < z^*, y > (j + k - i)z \} \). The conditional distributions involved in the previous equation is derived using [38]. We evaluate following cases: (i) \( z^* < y^*(j + k - i)^{-1} \), (ii) \( z^* > y^*(j + k - i)^{-1} \), and (iii) \( z^* = y^*(j + k - i)^{-1} \).

**Case (i):**

\[
\hat{\xi}_{U}(\alpha_1, \alpha_2) = (1 - n)(1 - nk) \int_{0}^{\infty} \int_{j+k-i}^{\infty} [z^* y^*]^{-1}
\]

\[
\times \left( 1 - \frac{z^*}{z^* - y^*} \right)^{-(2-n)} \left( y^* \right)^{-(2-nk)} \text{d}z \text{d}y
\]

\[
= \sum_{l=0}^{nk-1} (-1)^l \left( \frac{z^* (j + k - i)}{y^*} \right)^l \left( \frac{(nk - 1)^n}{(n + 1)^n} \right)
\]

**Case (ii):**

\[
\hat{\xi}_{U}(\alpha_1, \alpha_2) = \int_{0}^{\infty} \int_{j+k-i}^{\infty} \left( 1 - \frac{z^*}{z^*} \right)^{n-2} \left( 1 - \frac{y^*}{y^*} \right)^{nk-2} \text{d}z \text{d}y
\]

\[
= 1 - \sum_{l=0}^{n-1} (-1)^l \left( \frac{y^*}{z^* (j + k - i)} \right)^l \left( \frac{(nk - 1)^n}{(n + 1)^n} \right)
\]

**Case (iii):**

\[
\hat{\xi}_{U}(\alpha_1, \alpha_2) = (1 - n)(1 - nk) \int_{0}^{\infty} \int_{j+k-i}^{\infty} [z^* y^*]^{-1}
\]

\[
\times \left( 1 - \frac{z^*}{z^*} \right)^{-(2-n)} \left( y^* \right)^{-(2-nk)} \text{d}z \text{d}y
\]

\[
= \frac{n - 1}{nk + n - 2}
\]

The desired UMVUE is now obtained as

\[
\hat{R}_{sk}^U = \sum_{j=1}^{k} \left( \frac{1}{k} \right) \hat{\xi}_{U}(\alpha_1, \alpha_2) \left[ \sum_{j=0}^{i} (-1)^j \right].
\]

### 3.3. Bayesian inference

Bayesian inference for the reliability is discussed here when \( \beta \) is known. It is assumed that \( \alpha_1 \) and \( \alpha_2 \) are independently distributed as gamma \( G(p_1, q_1) \) and \( G(p_2, q_2) \), respectively. The corresponding joint posterior distribution is given by

\[
\pi(\alpha_1, \alpha_2 \mid \beta_0, x, y) = \frac{(q_1 + y^*)^{nk+1} (q_2 + z^*)^{n+2}}{\Gamma(nk + p_1) \Gamma(n + p_2) \alpha_1^{nk + p_1 - 1} \alpha_2^{n + p_2 - 1} e^{-a_1 (q_1 + y^*) - a_2 (q_2 + z^*)}}
\]

The Bayes estimator has the following form:

\[
\hat{R}_{sk}^{B} = \sum_{j=1}^{k} \sum_{i=0}^{l} \left( \frac{1}{k} \right) (-1)^j \left[ \int_{0}^{\infty} \int_{0}^{\infty} \alpha_2 \right]
\]

\[
\times \left( \frac{\alpha_2}{\alpha_1 (j + K - i) + \alpha_2} \right) \pi(\alpha_1, \alpha_2 \mid \beta_0, x, y) \text{d}\alpha_1 \text{d}\alpha_2.
\]

Next we try to evaluate the above expression. We take \( u_1 = \alpha_2/(\alpha_1 (j + k - i) + \alpha_2) \). Also assume that \( u_2 = \alpha_1 (j + k - i) + \alpha_2 \). We observe that \( u_1 \in (0, 1) \), \( u_2 \in (0, \infty) \). The inverse transformations are \( \alpha_1 = \)
\[ u_2(1 - u_1)/(j + k - i) \] and \( \alpha_2 = u_1 u_2 \) with \( J(u_1, u_2) = -u_2/(j + k - i) \). The above double integral is given by

\[
\frac{(q_1 + y^*)^{nk+p_1}(q_2 + z^*)^{n+p_2}}{\Gamma(nk+p_1)\Gamma(n+p_2)(j+k-1)^{nk+p_1}} \times \int_0^1 \int_0^1 \exp \left( \frac{(1-u_1)(q_1+y^*)}{j+k-i} \right) \left( \frac{u_1 q_2 + z^*}{j+k-i} \right) du_1 du_2
\]

where \( \tau_j = 1 - (q_2 + z^*)/(q_1 + y^*) \) and \( \rho_0 = nk + n + p_1 + p_2 \). Using hypergeometric series representation the above integral is expressed as

\[
\frac{1}{B(b,c-b)} \int_0^1 \left( \frac{t}{1-t} \right)^{b-1}(1-t)^{-c-1} d\tau
\]

when \( |\eta| < 1 \), \( \text{Re}(c) > 0 \) and \( \text{Re}(\tau) > 0 \), also \( |\eta| < 1 \) is the convergence region. We now have

\[
\frac{R^8_{i,j}}{\sum_{i=0}^{\min(nk,p_1)}(n+p_2)(j+k-i)} \left( \begin{array}{c} k \vdots j \vdots i \vdots 1 \vdots \eta \vdots \frac{1}{\eta-1} \\ \eta \end{array} \right) \left( \begin{array}{c} k \vdots j \vdots i \vdots 1 \vdots \eta \vdots \frac{1}{\eta-1} \\ \eta \end{array} \right)
\]

3.3.1. Bayes from Lindley technique

Bayesian inference of \( U(\theta_1, \theta_2) \) from this particular technique turns out as

\[
\hat{u}_k(x,y) = u(\theta_1, \theta_2) + 0.5u_0 + \gamma_0 \tilde{\xi}_{12} + \gamma_2 \xi_{12} + \gamma_0 \xi_{21} + \gamma_3 \xi_{21}
\]

where \( b_0 = \sum_{i=1}^{\min(nk,p_1)} \sum_{j=1}^{\min(nk,p_2)} u_i \tau_j \). We also have \( \gamma_j = \partial \alpha^*/\partial \alpha_1 \partial \alpha_2 \) for positive integer indices \( i \) and \( j \) taking values as 1, 2, 3, with their sum \( i + j \) being three. Also with \( i \) and \( j \) taking values as 1 and 2, we have \( \gamma_j = \partial^2 \alpha_2/\partial \alpha_1 \partial \alpha_2 \). Also \( \tilde{\xi}_{ij} = \left( u_i \tau_{ij} + u_j \tau_{ij} \right) \tau_{ij} \), \( \xi_{ij} = \left( u_i \tau_{ij} + u_j \tau_{ij} \right) \tau_{ij} \).

3.3.2. M-H algorithm

Note that marginal posteriors of \( \alpha_1 \) and \( \alpha_2 \) are gamma \( G(nk + p_1, q_1 + y^*) \) and \( G(n + p_2, q_2 + z^*) \), respectively. We generate samples using the following algorithm and then apply it to compute the Bayes procedure.

Step:

1. Select the adequate choice \( (\alpha_1, \alpha_2) \) of \( (\alpha_1, \alpha_2) \).
2. Obtain a sample \( \alpha_1^* \) using \( G(\alpha, b^*) \) variable where \( a^* = nk + p_1 \) and \( b^* = q_1 + y^* \).
3. Obtain a sample \( \alpha_2^* \) using \( G(\alpha_1^*, b_2^*) \) variable where \( \alpha_1^* = nk + p_2 \) and \( b_2^* = q_2 + z^* \).
4. Evaluate \( m = \min(1, \alpha_0^*) \) where \( \alpha_0^* = (\pi(\alpha_1^*, \alpha_2^*) \) samples)\()^{-1} \).
5. Simulate a number \( u \) from a random variable \( U \) which follows a uniform model on \((0, 1)\).
6. We take the sample as \( \alpha_1 \leftarrow \alpha_1^*, \alpha_2 \leftarrow \alpha_2^* \) provided the inequality \( u \leq m \) holds.
7. Compute \( R_{i,j,k}^L \) at \((\alpha_1, \alpha_2, \beta_1)\).
8. Now iterate stages 2 to 7 to obtain posterior estimates \( R_{i,j,k}^L, i = 1, 2, \ldots, M \).

Considering \( M_0 \) as discarded samples, the estimated value of the parameter \( R_{i,j,k} \) turns out to be \( R_{i,j,k}^L = 1/(M - M_0) \sum_{i=1}^{M_0} R_{i,j,k}^L \).

The credible intervals are evaluated similarly as computed for unknown \( \beta \) case.

4. Numerical experiments

1. Here simulation experiments are performed to compare the performance of various methods proposed for estimating the reliability function. This study is performed against various sample sizes by assuming different values of \( \beta, \alpha_1, \alpha_2 \) and prior distribution parameters.
2. Evaluation of estimators is done under mean square error (MSE) and average biases (ABs).
3. The performance of asymptotic, boot-t, boot-p and HPD intervals is evaluated using interval length and coverage probability criteria.
Table 1. Estimation results for $R_{jk}$ with unknown $\beta$.

| n  | $R_{14}(T)$ | $\hat{R}_{14}$ | $\hat{R}_{14}^T$ | $R_{25}(T)$ | $\hat{R}_{25}$ | $\hat{R}_{25}^T$ | $R_{25}(T)$ | $\hat{R}_{25}$ | $\hat{R}_{25}^T$ |
|----|-------------|----------------|------------------|-------------|----------------|----------------|-------------|----------------|----------------|
| 10 | 0.75        | 0.73416        | 0.67756          | 0.68289     | 0.75246        | 0.77835        | 0.74279     | 0.59211       | 0.57789        |
|    |             | -0.01584       | -0.07244         | -0.06711    | 0.00246        | 0.02835        | 0.00721     | -0.01422      | -0.02903       |
| 15 | 0.73536     | 0.72372        | 0.73945          | 0.76535     | 0.74636        | 0.74082        | 0.58483     | 0.60355       | 0.60883        |
|    |             | 0.00358        | 0.0478           | 0.03854     | 0.0024         | 0.0021         | 0.00241     | 0.00619       | 0.03496        |
|    |             | -0.01164       | -0.02628         | -0.01055    | 0.01535        | -0.00364       | 0.00918     | -0.00728      | 0.01145        |
| 20 | 0.74421     | 0.74814        | 0.75533          | 0.77225     | 0.74956        | 0.74693        | 0.57677     | 0.62817       | 0.63327        |
|    |             | 0.00427        | 0.03378          | 0.02968     | 0.00237        | 0.00172        | 0.00185     | 0.00536       | 0.0239         |
|    |             | -0.00579       | -0.00186         | 0.00533     | 0.02225        | -0.00443       | 0.00307     | -0.01534      | 0.03606        |
| 25 | 0.74465     | 0.76912        | 0.77259          | 0.77821     | 0.74745        | 0.7466         | 0.58862     | 0.63824       | 0.64394        |
|    |             | 0.00192        | 0.02219          | 0.02014     | 0.00209        | 0.00153        | 0.00155     | 0.0036        | 0.02           |
|    |             | -0.00535       | 0.01912          | 0.02259     | 0.02821        | -0.00255       | 0.00397     | -0.00348      | 0.00613        |
| 30 | 0.74361     | 0.77987        | 0.78451          | 0.77938     | 0.74581        | 0.74568        | 0.58683     | 0.65038       | 0.65459        |
|    |             | 0.00172        | 0.01529          | 0.01224     | 0.00201        | 0.00142        | 0.0014      | 0.00309       | 0.01641        |
|    |             | -0.00639       | 0.02987          | 0.03451     | 0.02938        | -0.00419       | 0.00432     | -0.00527      | 0.05827        |
| 35 | 0.74554     | 0.78848        | 0.78838          | 0.7820      | 0.74706        | 0.74701        | 0.58595     | 0.64951       | 0.65136        |
|    |             | 0.00134        | 0.00893          | 0.00903     | 0.00199        | 0.00116        | 0.00112     | 0.00245       | 0.01359        |
|    |             | -0.00446       | 0.03848          | 0.03838     | 0.03         | -0.00294       | 0.00299     | -0.00616      | 0.0574         |
| 40 | 0.74485     | 0.7925         | 0.79528          | 0.78265     | 0.74598        | 0.7462         | 0.58883     | 0.65617       | 0.65785        |
|    |             | 0.00118        | 0.00588          | 0.00431     | 0.00192        | 0.00106        | 0.001      | 0.00227       | 0.0111         |
|    |             | -0.00515       | 0.0425           | 0.04528     | 0.03265        | -0.00402       | 0.00398     | -0.00327      | 0.06407        |
| 45 | 0.74536     | 0.79105        | 0.79449          | 0.78611     | 0.74624        | 0.74638        | 0.58669     | 0.65677       | 0.65755        |
|    |             | 0.00115        | 0.00863          | 0.00649     | 0.00201        | 0.00106        | 0.00102     | 0.002        | 0.01109        |
|    |             | -0.00464       | 0.04105          | 0.04449     | 0.03611        | -0.00376       | 0.00362     | -0.00541      | 0.06467        |
| 50 | 0.74775     | 0.79521        | 0.79619          | 0.7865      | 0.74843        | 0.7485         | 0.58809     | 0.65782       | 0.66148        |
|    |             | 0.00094        | 0.0053           | 0.00467     | 0.002         | 0.00087        | 0.00843     | 0.00172       | 0.00922        |
|    |             | -0.00225       | 0.04521          | 0.04619     | 0.0365         | -0.00157       | 0.0015      | -0.00402      | 0.06572        |

**Note:** The table entries represent the estimated values for different parameters under various conditions.
Table 2. Estimation results for $R_{s,k}$ with unknown $\beta$.

| $n$ | $R_{s,k}(T)$ | $\tilde{R}_{s,k}$ | $R_{s,k}^{LS}$ | $R_{s,k}^{RWLS}$ | $R_{s,k}^{RMPS}$ | $R_{s,k}^{RMH}$ | $R_{s,k}^{RLS}$ | $R_{s,k}^{MB}$ |
|-----|---------------|------------------|----------------|----------------|----------------|----------------|--------------|----------|
| 10  | 0.7619        | 0.7727           | 0.7716         | 0.7564         | 0.69381        | 0.57445        | 0.60952      | 0.5844     |
|     | 0.0067        | 0.0088           | 0.0106         | 0.0039         | 0.0058         | 0.0023         | 0.0015      | 0.0014     |
|     | -0.0114       | -0.0124          | -0.0144        | -0.0156        | -0.0167        | -0.0179        | -0.0181     | -0.0183    |
| 15  | 0.7320        | 0.7521           | 0.7619         | 0.7627         | 0.6601         | 0.56763        | 0.57683      | 0.56992    |
|     | 0.0032        | 0.0034           | 0.0037         | 0.0026         | 0.0012         | 0.0017         | 0.0026      | 0.0026     |
|     | -0.0096       | -0.0099          | -0.0111        | -0.0131        | -0.0157        | -0.0176        | -0.0197     | -0.0201    |
| 20  | 0.7253        | 0.7685           | 0.7738         | 0.7782         | 0.6456         | 0.55661        | 0.56619      | 0.56124    |
|     | 0.0023        | 0.0029           | 0.0019         | 0.0019         | 0.0024         | 0.0017         | 0.0020      | 0.0020     |
|     | -0.0008       | 0.0018           | 0.0017         | 0.0013         | 0.0011         | 0.0012         | 0.0010      | 0.0010     |
| 25  | 0.7216        | 0.7804           | 0.7873         | 0.7894         | 0.6172         | 0.57634        | 0.58640      | 0.57921    |
|     | 0.0019        | 0.0018           | 0.0008         | 0.0011         | 0.0011         | 0.0013         | 0.0011      | 0.0011     |
|     | -0.0088       | 0.0017           | 0.0025         | 0.0017         | 0.0027         | 0.0022         | 0.0024      | 0.0024     |
| 30  | 0.7215        | 0.7867           | 0.7897         | 0.7831         | 0.6089         | 0.55255        | 0.56647      | 0.56144    |
|     | 0.0001        | 0.0003           | 0.0003         | 0.0001         | 0.0004         | 0.0003         | 0.0003      | 0.0003     |
|     | -0.0039       | 0.0025           | 0.0027         | 0.0012         | 0.0010         | 0.0006         | 0.0006      | 0.0006     |
| 35  | 0.7216        | 0.7903           | 0.795          | 0.7845         | 0.6025         | 0.56571        | 0.56619      | 0.56142    |
|     | 0.0013        | 0.0032           | 0.0041         | 0.0014         | 0.0009         | 0.0008         | 0.0008      | 0.0008     |
|     | -0.0067       | 0.0031           | 0.0030         | 0.0022         | 0.0008         | 0.0007         | 0.0007      | 0.0007     |
| 40  | 0.7198        | 0.7906           | 0.7926         | 0.7826         | 0.6049         | 0.56227        | 0.56602      | 0.56287    |
|     | 0.0011        | 0.0033           | 0.0033         | 0.0013         | 0.0008         | 0.0008         | 0.0008      | 0.0008     |
|     | -0.0092       | 0.0031           | 0.0036         | 0.0023         | 0.0000         | 0.0000         | 0.0000      | 0.0000     |
| 45  | 0.7196        | 0.7929           | 0.7927         | 0.7826         | 0.6034         | 0.56398        | 0.56449      | 0.56473    |
|     | 0.0001        | 0.0003           | 0.0003         | 0.0001         | 0.0006         | 0.0006         | 0.0006      | 0.0006     |
|     | -0.0004       | 0.0034           | 0.0035         | 0.0025         | 0.0000         | 0.0000         | 0.0000      | 0.0000     |
| 50  | 0.7185        | 0.7956           | 0.7964         | 0.7856         | 0.6032         | 0.56616        | 0.60678      | 0.60792    |
|     | 0.0005        | 0.0026           | 0.0027         | 0.0012         | 0.0006         | 0.0007         | 0.0007      | 0.0007     |
|     | -0.0033       | 0.0049           | 0.0045         | 0.0023         | 0.0002         | 0.0002         | 0.0002      | 0.0002     |

(4) All estimates are computed when $n$ is assigned as 10, 15, 20, ... , 50.

(5) Both the cases, $\beta$ known or unknown, are taken into consideration for estimating the reliability.

(6) First consider unknown $\beta$ case. In this situation, strength–stress components are observed assuming $(\alpha_1,\alpha_2,\beta)$ as (1,2,2) and (2,2,2,2) respectively. The exact values of $R_{s,k}$, when $(s,k)$ being (1,4) and (2,5), are 0.7500 and 0.59211 for the first set of parameters, and 0.7619 and 0.60952 for the second set of parameters, respectively. The sample size $n$ is assigned as 10, 15, 20, ... , 50

(7) In Tables 1, 2, 5 and 6, for a given $n$, three values are listed columnwise for each estimate. Among these, the first value denotes the estimated value of $R_{s,k}$, immediate lower value denotes the MSE and last value is the AB of that estimator.

(8) In Tables 1–2, estimation results for $R_{s,k}$ are presented based on classical methods which are obtained by ML ($\hat{R}_{s,k}$), least square (LS, $R_{s,k}^{LS}$), maximum product spacing (MPS, $R_{s,k}^{MPS}$) and Bayesian methods. Recall that Bayesian inference is evaluated from Lindley ($R_{s,k}^{RLS}$) and M–H ($R_{s,k}^{RMH}$) methods when true value of the reliability is $(R_{s,k})$. In a similar way, interval widths and associate probabilities of 95% asymptotic, boot-t, boot-p and Bayes intervals are
Table 4. Interval estimation results for $R_{a,k}$ with unknown $\beta$.

| $n$ | $R_{a,k}$ | $R_{a,k}^*(\theta_1, \theta_2)$ | $R_{a,k}^*(\theta_1, \theta_2) = (1.5, 2)$ |
|-----|----------|-------------------------------|---------------------------------|
| 10  | 0.7619   | 0.26651                       | 0.29277                         |
|     | 0.74042 | 0.28486                       | 0.60592                         |
|     | 0.74919 | 0.36684                       | 0.29433                         |
|     | 0.74524 | 0.38263                       | 0.42722                         |
| 15  | 0.21886 | 0.968                          | 0.912                           |
|     | 0.22193 | 0.967                          | 0.932                           |
|     | 0.22913 | 0.927                          | 0.969                           |
|     | 0.22971 | 0.947                          | 0.941                           |
| 20  | 0.18574 | 0.957                          | 0.948                           |
|     | 0.19678 | 0.939                          | 0.941                           |
|     | 0.18822 | 0.939                          | 0.935                           |
| 25  | 0.16629 | 0.955                          | 0.946                           |
|     | 0.17291 | 0.934                          | 0.946                           |
|     | 0.16388 | 0.934                          | 0.939                           |
| 30  | 0.15235 | 0.954                          | 0.948                           |
|     | 0.15638 | 0.934                          | 0.948                           |
|     | 0.1503  | 0.934                          | 0.948                           |

Table 5. Estimation results for $R_{a,k}$ with known $\beta (=2)$.

| $n$ | $R_{a,k}(T)$ | $R_{a,k}^\text{asy}$ | $R_{a,k}^\text{cred}$ | $R_{a,k}^\text{bot-p}$ | $R_{a,k}^\text{bot-t}$ |
|-----|--------------|-----------------------|-----------------------|------------------------|------------------------|
| 10  | 0.75         | 0.74042               | 0.74524               | 0.76651                | 0.75993                |
|     |              | 0.75993               | 0.59211               | 0.57751                | 0.60447                |
|     |              | 0.75974               | 0.58946               | 0.5915                 | 0.60653                |
| 15  |              |                       |                       |                        |                        |
|     |              |                       |                       |                        |                        |
| 20  |              |                       |                       |                        |                        |
|     |              |                       |                       |                        |                        |
| 25  |              |                       |                       |                        |                        |
|     |              |                       |                       |                        |                        |
| 30  |              |                       |                       |                        |                        |
|     |              |                       |                       |                        |                        |
| 50  |              |                       |                       |                        |                        |
|     |              |                       |                       |                        |                        |

(9) We mention that in Tables 3, 4, 7 and 8, for a given $n$, two values are listed columnwise for each estimate. Among these, the first value denotes average length (AL) of interval of $R_{a,k}$, immediate lower value denotes the coverage probability (CP) of that estimator.

(10) In Tables 3 and 4, we have tabulated AL and coverage probabilities of various intervals. From these tables, it is observed that the average lengths of the intervals decrease with the increase in sample size, as expected. The average interval length of HPD interval is smaller than their counter parts.

(11) Next $\beta$ known case is presented. Tables 5 and 6 contain estimation results of reliability under frequentist and Bayes techniques when $\beta_0$ is 2 and 3, respectively.

(12) The strength and stress characteristics are observed for $(\theta_1, \theta_2) = (1.5, 2)$ and $(2.5, 2.5)$. The corresponding true values of $R_{a,k}$ with the given combinations $(k, \theta)$, and $R_{a,k}$, also these estimates are 0.68442 and 0.53391 when $(\theta_1, \theta_2) = (1.5, 2)$, also these estimates are 0.68442 and 0.53391 when $(\theta_1, \theta_2) = (2.5, 2.5)$, respectively. In Bayesian case, the following pairs of hyper-parameters $a_1 = 3, b_1 = 2, a_2 = 2, b_2 = 1$ and $a_1 = 2, b_1 = 1, a_2 = 5, b_2 = 2$ are used for $(\theta_1, \theta_2) = (1.5, 2)$ and $(2.5, 2.5)$, respectively.
(13) Moreover, in Tables 7 and 8, we provide widths and associated probabilities of 95% intervals. From Tables 5 and 6, we observe that the MSE and ABs for the estimates of $R_{k,l}$ based on all methods of estimation decreases as the sample size increases in all cases, as expected.

(14) Note that maximum likelihood and unbiased estimators show good performance in comparison with Bayes counterparts, however, Bayes estimates have an advantage over these two estimates in terms of bias and MSE values. Tabulated values also indicate that MLEs show good behaviour than UMVU estimates.

(15) We further notice that Lindley and M–H inferences slightly vary when compared with corresponding exact Bayes inference and for large sample sizes their MSE and ABs tend to become close to each other. The Bayes estimates of $R_{k,l}$ under the SE loss function have smaller MSE and ABs than their counterparts for all cases. It is observed that in general, average length of HPD intervals are smaller than asymptotic intervals. We also observe that the length of both intervals tends to become smaller with large sample sizes. Also coverage probabilities of these intervals exhibit satisfactory behaviour.

5. Data analysis

Data Set I: We present a real-life numerical example in support of estimation procedures considered for evaluating the system reliability based on UW distribution.
Table 8. Interval estimation results for $R_{s,k}$ with known $\beta = 3$.

| $n$ | $R_{1,4}$ asy | cred | bot-p | bot-t $(\theta_1, \theta_2) = (2.25, 1.5)$ | $R_{2.5}$ asy | cred | bot-p | bot-t |
|-----|----------------|-------|--------|-------------------------------------------|----------------|-------|--------|--------|
| 10  | 0.7619         | 0.25499 | 0.20835 | 0.26367                                   | 0.26856         | 0.60952 | 0.35834 | 0.29395 |
| 15  | 0.20641        | 0.17843 | 0.21269 | 0.22327                                   | 0.29375         | 0.25334 | 0.29387 | 0.32469 |
| 20  | 0.17876        | 0.15935 | 0.18123 | 0.18822                                   | 0.25615         | 0.22711 | 0.25462 | 0.27239 |
| 25  | 0.16005        | 0.14506 | 0.16349 | 0.16678                                   | 0.22821         | 0.20641 | 0.22706 | 0.24402 |
| 30  | 0.14485        | 0.13121 | 0.14816 | 0.15498                                   | 0.20943         | 0.19216 | 0.20826 | 0.21757 |
| 35  | 0.13519        | 0.12525 | 0.13505 | 0.1395                                    | 0.1936          | 0.17919 | 0.19244 | 0.20558 |
| 40  | 0.12621        | 0.11761 | 0.12675 | 0.12938                                   | 0.18092         | 0.16849 | 0.18071 | 0.18766 |
| 45  | 0.11884        | 0.11134 | 0.11867 | 0.12173                                   | 0.17083         | 0.16022 | 0.16938 | 0.1784  |
| 50  | 0.11271        | 0.10641 | 0.11335 | 0.11288                                   | 0.16176         | 0.15258 | 0.16129 | 0.16696 |

Table 9. Test of goodness results for Data I.

| Distribution | $\hat{\theta}_1$ | $\hat{\beta}_1$ | kolm prob | $\hat{\theta}_2$ | $\hat{\beta}_2$ | kolm prob |
|--------------|-------------------|-----------------|------------|-------------------|-----------------|------------|
| Unit Weibull | 1.036154          | 0.996209        | 0.10942    | 0.8941            | 2.114108        | 0.7038222  |
| Log-Lindley  | 1.35382           | 1.657287        | 0.12827    | 0.7586            | 2.818937        | 1.870996   |
| Kumaraswamy  | 0.9796485         | 0.9180274       | 0.097555   | 0.9526            | 1303.455        | 267962.3   |
| Unit Gamma   | 1.034798          | 1.531143        | 0.11054    | 0.8874            | 2.431368        | 3596707    |

The data represent breaking strength of Jute fibre [39]. We have considered 15mm gauge length data for analysis. The diameters of jute fibres are measured with an XSP-8CA digital biological microscope. Consider $s = 1$ and $k = 5$, then we observe that $Y_1$ becomes identical with the sixth observation of considered data. Thus we see that observations $X_{1k}$ are from first to fifth observations. Similarly $Y_2$ is the 12th observation and also $X_{2k}$ are from 7 to 11th data points. Here $k$ is a positive integer ranging from 1 to 5. Data processing for total 30 observations gives $n$ as 5. Observation $(X, Y)$ is given below:

$X = \begin{bmatrix} 594.40 & 202.75 & 168.37 & 574.86 & 225.65 \\ 156.67 & 127.81 & 813.87 & 562.39 & 468.47 \\ 72.24 & 497.94 & 355.56 & 569.07 & 640.48 \\ 550.42 & 748.75 & 489.66 & 678.06 & 457.71 \end{bmatrix}$

$Y = \begin{bmatrix} 76.38 \\ 135.09 \\ 200.76 \\ 106.73 \\ 193.42 \end{bmatrix}$

Figure 1. PP–QQ plots of the unit Weibull distribution for Data I.
Figure 2. Plotting of profile likelihood of Unit Weibull distribution for Data I.

We also update observations $X$ and $Y$ by $X_{ij}/(\max(X) + 1)$, $Y_{ij}/(\max(Y) + 1)$, respectively and then checked using goodness test whether the considered data sets can be fitted to the UW model. The MLEs of respective unknown parameters of all the competing distributions are evaluated numerically. The KS (Kolmogorov–Smirnov) test statistic is taken into consideration for model evaluation. Table 9 contains associated probability values for all the competing models. It is seen that UW distribution can be considered a good representative model given data. We have also plotted P–P and Q–Q plot (Figure 1) of the given data sets and observe that data fitted satisfactorily. In Figure 2, we plotted curve of the profile likelihood function. This is concave in nature, i.e. unimodal function and maximum at $\beta = 1.0001$ for strength data as well as $\beta = 0.7001$ for stress data set. In Table 10, estimation results for the unknown reliability are evaluated for the complete data case. In fact all intervals are also computed. Bayesian inference is performed against improper prior distributions.

(1) Further equivalence testing between UW stress and strength parameters $\beta_1$ and $\beta_2$ is performed. The likelihood ratio test is applied to derive the inference.

$$H_0 : \beta_1 = \beta_2 \text{ versus } H_1 : \beta_1 \neq \beta_2.$$ The test statistic is

$$\Delta = -2(\log_0 - \log_1)$$

which is $\chi^2(1)$.

(2) The estimates of corresponding log likelihoods under data I with $H_0 : \beta_1 = \beta_2$ and alternative hypotheses $H_1 : \beta_1 \neq \beta_2$ are $\log_0 = 1.607646$ and $\log_1 = 1.979536$, respectively. Here, we obtain $\Delta = 0.7437795$ with probability value 0.388478.

(3) We see that null hypothesis $\beta_1 = \beta_2$ indicates favourable result under 5% significance. Thus we conclude that $\beta_1$ and $\beta_2$ may be equal for the considered numerical example.

Data Set II:

The second data set defines 12 core samples from petroleum reservoirs that were sampled by 4 cross-sections. There are a total of 48 observations in this study. Core samples are measured for permeability and each cross-section defined using following components: total area of pores, total perimeter of pores and shape. The data set is available in R language [40], especially on data.frame named rock. Table 11 contains associated estimates for all competing models. We see that UW distribution is a good representative model.
Table 10. Estimation results for Data I.

| \( \hat{\theta}_1 \) | \( \hat{\theta}_2 \) | \( \hat{\beta} \) | \( R_s^k \) | \( R_s^k \) | \( R_s^k \) | \( R_s^k \) | \( \hat{R}_{s,j}^{\text{RLM}} \) | Boot-p | Boot-t | ACI | HPD |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.056552 | 2.366079 | 0.937018 | 0.6906617 | 0.7699177 | 0.7597839 | 0.5719439 | 0.8608635 | (0.3446559, 0.4110284, 0.4799626, 0.3660185, 0.8383069, 0.8660306, 0.9013608) | 0.5719439 | 0.8608635 |

Table 11. Goodness of fit for Data II.

| Distribution | \( \hat{\theta}_1 \) | \( \hat{\beta} \) | kolm | prob | \( \hat{\theta}_2 \) | \( \hat{\beta} \) | kolm | prob |
|---|---|---|---|---|---|---|---|---|
| Unit Weibull | 0.0685 | 4.917 | 0.0836 | 0.9306 | 0.0073 | 9.006 | 0.2812 | 0.6361 |
| Log-Lindley | 70.89 | 0.6458 | 0.4220 | 6.36e−07 | 103.7524 | 0.6233 | 0.5462 | 0.0332 |
| Kumaraswamy | 38.98 | 2.647 | 0.152 | 0.2801 | 216.6 | 3.533 | 0.2403 | 0.8074 |
| Unit Gamma | 0.6323 | 1027753 | 0.4221 | 6.3e−07 | 0.6139 | 1114084 | 0.5463 | 0.0332 |

Figure 3. PP–QQ plots of the unit Weibull distribution for Data II.

Figure 4. Plotting of profile likelihood of Unit Weibull distribution for Data II.
for the data. We have also plotted P–P and Q–Q plots (Figure 3) of the data. We observe that data fitted satisfactorily by the UW model. In Figure 4, we plotted curve of the profile likelihood function. This is concave in nature, i.e. unimodal function with maximum at $\beta = 5$ for strength data and $\beta = 9$ for stress data. Following [41], we further explore histogram plot of bootstrap samples for the logit of $R_{s,k}$ based on 3000 replications, see Figure 5. The histogram resembles of a normal distribution with approximate mean 1.727348 and standard deviation 0.5171634. Further associated K–S estimate for strength data and $\beta = 0.029392$ and $\sigma = 0.5171634$. Further associated K–S estimate $\beta = 0.0588$ 0.0671 5.1585 0.8598 0.7792 0.7881 0.7338 0.8012 0.8741; (0.6285, 0.9348) (0.6192, 0.9534) (0.7567, 0.9630) (0.7782, 0.9537)

Figure 5. Plotting of logit $R_{s,k}$ of Unit Weibull distribution for Data II.

6. Conclusion

We have investigated inference upon a stress–strength parameter for UW distribution. The four frequentist methods of estimation have been used to obtain the system reliability with known and unknown $\beta$. Besides, asymptotic and two bootstrap CIs are obtained. Further Bayes inferences of $R_{s,k}$ are studied against different approximation techniques. The Bayes credible interval is discussed utilizing posterior samples generated from an MCMC method. Besides, unbiased estimation of reliability is also evaluated for known $\beta$ case. Our numerical results suggest that Bayes procedure yields better inference compared to MLE and UMVUE. It is also seen that asymptotic, bootstrap and Bayesian intervals exhibit relatively good CP behaviour. However, width of Bayes intervals remains narrower than the corresponding asymptotic and bootstrap CIs. We present two numerical examples to demonstrate and observe the reliability for one system configuration. We found that WLSE provides marginally better results than the corresponding LSE estimates. Further inference obtained from the MPS method shows good behaviour than the WLSE method. We further note that MLE also improves LSE and WLSE. However, MLE and MPS methods are incomparable as in some case MLE performs better than MPS and in other case opposite behaviour is observed. We observe that if proper prior is available then Bayesian procedures have an advantage over the other methods. Overall, better inference of this particular parametric function can be derived under known $\beta$ case. Although inference is studied when stress–strength components have the UW distributions, however, such studies can be conducted under assumption stress–strength components have different distributions. Moreover, estimation for the multicomponent stress–strength reliability for system with dependent components seems also of interest and importance in practice, which will be studied in the

| $\hat{\beta}$ | $\hat{\sigma}$ | $\hat{\alpha}$ | $\hat{\beta}_s$ | $\hat{\beta}_k$ | $\hat{R}_{s,k}$ | $\hat{R}_{s,k}^{WS}$ | $\hat{R}_{s,k}^{LS}$ | $\hat{R}_{s,k}^{MS}$ | $\hat{R}_{s,k}^{WLS}$ | Boot-p | Boot-t | ACI | HPD |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|---------|---------|-----|------|
| 0.0588         | 0.0671         | 5.1585         | 0.8598         | 0.7792         | 0.7881         | 0.7338         | 0.8012         | 0.8741         | (0.6285, 0.9348) | (0.6192, 0.9534) | (0.7567, 0.9630) | (0.7782, 0.9537) |
future. In near future, we also try to study such problems under some censoring schemes.

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Disclosure statement

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