The Wiedemann-Franz law in the SU(N) Wolff model

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We study the electrical and thermal transport through the SU(N) Wolff model with the use of bosonization. The Wilson ratio reaches unity as N grows to infinity. The electric conductance is dominated by the charge channel, and decreases monotonically with increasing interaction. The thermal conductivity enhances in the presence of local Hubbard $U$. The Wiedemann-Franz law is violated, the Lorentz number depends strongly on the interaction parameter, which can be regarded as a manifestation of spin-charge separation.

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The Wiedemann-Franz law is one of the basic properties of a Fermi liquid. Simply stated, it reflect the fact, that the ability of quasiparticles to carry charge is the same as to transport heat. A possible breakdown of this relation is interpreted in terms of spin-charge separation. Spinons can carry heat in much the same way as charges do, and contribute to heat transport. On the other hand, they fail in electric transport, while charge excitations contribute to both. Another explanation requires inelastic scattering.

This phenomenon is demonstrated in the SU(N) Wolff model. It consists of N species of electrons interacting with each other only at a single site. The model is studied with Abelian bosonization for arbitrary N. We consider fermions with SU(N) spin index or alternatively we take the ensemble of spin and orbital degrees of freedom into account by the N indices. These additional degrees of freedom can be realized through orbital degeneracy, for example, as in Mn oxides. As a result, the additional degrees of freedom can be called flavour or color index. The Wolff model is one of the simplest impurity models, where electron correlation is still present. Also it is the basis of studying the effect of Coulomb interaction on resonant tunneling through a single quantum level.

The Wilson ratio approaches unity as N grows, it crosses over to non-interacting (mean-field) behaviour as N grows to infinity, a phenomenon characteristic to SU(N) model. The electric conduction involves only the charge sector, and increasing interaction increases the resistivity. Heat is transported by both spin and charge excitation, and heat conduction is favoured by any finite value of $U$, regardless to its sign. The respective conductivities depend strongly on the local interaction, leading to the violation of the Wiedemann-Franz law even in the large N limit. This suggests that spin-charge separation is the origin of this breakdown.

The Hamiltonian describing N different species of electrons interacting only at the origin is given by:

$$H = \sum_{m=1}^{N} \left[ -iv \int_{-L/2}^{L/2} dx \Psi_m^+(x) \partial_x \Psi_m(x) + E : \Psi_m^+(0) \Psi_m(0) : + \frac{U}{2} \sum_{n=1, n \neq m}^{N} : \Psi_m^+(0) \Psi_m(0) : \Psi_n^+(0) \Psi_n(0) : \right]$$

and only the radial motion of the particles is accounted for by chiral (right moving) fermion fields. The model can be bosonized via:

$$\Psi_m(x) = \frac{1}{\sqrt{2\pi L}} e^{i\sqrt{4\pi} \phi_m(x)}$$

and after introducing charge and spin fields as

$$\Phi_c(x) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \Phi_m(x),$$

$$\Phi_{n,s}(x) = \frac{1}{\sqrt{n(n+1)}} \sum_{m=1}^{n} \Phi_m(x) - n \Phi_{n+1}(x)$$

$n = 1 \ldots N - 1$, the Hamiltonian separates into different sectors: the spin sector is described by $N - 1$ identical decoupled bosonic modes, and the charge sector transforms into a similar massless bosonic mode as

$$H_s = \sum_{m=1}^{N-1} \left[ v \int_{-L/2}^{L/2} dx (\partial_x \Phi_{m,s}(x))^2 - \frac{U}{2\pi} (\partial_x \Phi_{m,s}(0))^2 \right]$$

$$H_c = v \int_{-L/2}^{L/2} dx (\partial_x \Phi_c(x))^2 + E \sqrt{\frac{N}{\pi}} \partial_x \Phi_c(0) + \frac{(N - 1)U}{2\pi} (\partial_x \Phi_c(0))^2$$

These can readily be diagonalized by introducing pure bosonic representation of the $\Phi(x)$ fields as

$$\Phi(x) = \sum_{q>0} \frac{1}{\sqrt{2Lq}} (e^{iqx} b_q + e^{-iqx} b_q^+) e^{-\alpha q^2/2}$$
where $\alpha$ is the ultraviolet cutoff. Usually $x \in [-L/2, L/2]$, but $L \to \infty$ in actual calculations. The charge Hamiltonian is rewritten as

$$H_c = \sum_{q>0} \left\{ v q b_q^+ b_q + i E \sqrt{\frac{qN}{2L\pi}} (b_q - b_q^+) + \sum_{k>0} \frac{-\sqrt{qk}}{2L} (b_q - b_q^+) (b_k - b_k^+) \right\}, \quad (8)$$

which describes spinless bosons scattered by $U$, and a source term $E$. The latter can be transformed out by a linear shift of the bosonic field, while the former can be considered exactly via Dyson equation. Similar equations describe the spin sector. Within the realm of bosonization, $U$ is restricted to $(-U_0/(N-1), U_0)$ with $U_0 = \pi v/n_0$, $n_0$ is the average density per spin in the homogeneous case.

As a first step to understand the response of our model, it is instructive to investigate the Wilson ratio. The Wilson ratio characterizes to what extent the impurity interaction influences the conduction electron properties. Using Ref. $5$, it is calculated for the SU(N) Wolff model as

$$R = \frac{\chi_{\text{imp}}}{\chi_0} \frac{\partial C_0/\partial T}{\partial C_{\text{imp}}/\partial T} = 1 + \frac{U}{U_0 + (N-2)U}, \quad (9)$$

It interpolates smoothly between zero in the attractive case to $N/(N-1)$ in the repulsive case. Interestingly, the latter value was calculated in the $N$-fold degenerate Coqblin-Schrieffer model as well. This might suggest that the $U \to U_0$ limit of the Wolff model found by bosonization may describe the $U \to \infty$ limit of the original model. The shortcoming of bosonization is to underestimate the strength of interaction as it does for simple impurity scattering.

As $N$ grows, $R$ approaches unity (i.e. the non-interacting limit). Indeed, the ground state of the $N \to \infty$ limit of impurity models is the mean-field solution, because fluctuations around the saddle-point are suppressed by $1/N$, and vanish as $N$ reaches infinity. As a result, both quantities determining the Wilson ratio are proportional to the density of states at the Fermi energy, hence their ratio is 1. The general behaviour of the Wilson ratio for various $N$’s is shown in Fig. 1. The SU(2) case was also investigated in Ref. 11.

The calculation of the conductivity can be carried out in several alternative ways. The most common one relies on the physical assumption that the whole voltage drop occurs at the impurity site. The total number of electrons to the right and left of the impurity are $N_R$ and $N_L$, respectively. Then the current satisfying the continuity equation reads as

$$I = \frac{e}{2} h (N_R - N_L) = \frac{e}{2} [H, N_R - N_L]. \quad (10)$$

From this, the linear conductivity can be calculated by the Kubo formula.

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**FIG. 1:** (Color online) The Wilson ratio of the SU(N) Wolff model is plotted for $N = 2, 3, 4, 8$ and $\infty$ (red) from top to bottom.

On the other hand, one might think, that the conductivity at the impurity site is given by the autocorrelator of the current flowing through it. It turns out that these definitions of the current yield identical result, and we are in position to calculate the conductivity. From the continuity equation ($\partial_t n + \partial_x j = 0$), the local conductance at the impurity site is obtained, if we proceed following Ref. 11-19. The local density is determined by

$$n(x) = \sqrt{\frac{N}{\pi}} \partial_x \Phi(x), \quad (11)$$

from which the local current operator is calculated as

$$j(x) = -e \sqrt{\frac{N}{\pi}} \partial_x \Phi(x) = -evn(x). \quad (12)$$

Only charge excitations participate in electric transport, spinons only transport heat, not charge, as will be demonstrated below. The expectation value of $\partial_x \Phi(x)$ is finite as seen in Ref. 13 which is a natural consequence of the same chirality of all $N$ channels. The current-current correlation function can be evaluated from the bosonic representation of the $\Phi(x)$ field through the equation of motion method. After solving the closed set of equations using Eq. 6, the frequency dependent ac conductance is evaluated as

$$g(\omega) = (ev)^2 \text{Re} \frac{N \chi_0(\omega)}{i\omega(1 + (N-1)U \chi_0(\omega))}, \quad (13)$$

where $\chi_0(\omega)$ is the local charge correlator per spin at $U = 0$. Identical result was obtained for the charge correlator using the generating functional. It is shown in Fig. 2 for $N = 4$. Similar curves describe the $N \neq 4$ behaviour as well. Its dc part yields to

$$g(0) = \frac{e^2}{2\pi} \frac{NU_0^2}{(U_0 + (N-1)U)^2}. \quad (14)$$
which decreases smoothly for repulsive interaction, but increases rapidly in the attractive sector. Throughout the calculations we set \( \hbar = 1 \), which explains the \( e^2/2\pi \) factor in front of the right hand side of Eq. 14. By restoring the original units, this gets replaced by the universal channel conductance unit, \( e^2/h \). Qualitatively similar behaviour was identified in the conductance of a one dimensional interacting electron gas (Luttinger liquid) through a weak link.\(^{23}\) Repulsive interaction caused perfect reflection, attractive one resulted in perfect transmission. Although the details are different, our results exhibit the same phenomenon as seen from Eq. 14 and from the inset of Fig. 2. In the \( N \to \infty \) limit, the dc conductance is zero for any finite \( U \). The charge sector involved in electric transport is governed by Eq. 6, where the scattering term \( \sim U \) is zero for any finite \( U \).\(^{22,23}\) and reads as

\[
J_Q = v^2 \sum_{m=1}^{N} (\partial_x \Phi_m(x))^2 = v^2 \left[ (\partial_x \Phi_c(x))^2 + \sum_{n=1}^{N-1} (\partial_x \Phi_{n,s}(x))^2 \right].
\]  

(15)

The Hamiltonians in the different sectors commute with each other, hence their contributions can be determined independently. The thermal conductivity can be calculated from energy current-current correlation function given by

\[
\Pi(i\omega_n) = - \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_\tau J_Q(\tau) J_Q(0) \rangle,
\]  

(16)

and the retarded response function is obtained after analytic continuation as

\[
\frac{\kappa}{T} = - \frac{\text{Im}(\Pi(i\omega_n \to \omega + i\delta))}{T^2\omega}.
\]  

(17)

Its calculation amounts to determine the \( \langle T_\tau \partial_x \Phi(0, \tau) \rangle^2 (\partial_x \Phi(0, \tau))^2 \) correlator. Possible pairings are of \( \langle T_\tau \partial_x \Phi(0, \tau) \partial_x \Phi(0, 0) \rangle \) type in all sectors, which equals to the the charge and spin correlation functions, obtained in Ref. 2. The Hamiltonians describe scattering of free bosons on a localized ”impurity”, hence only self energy corrections can be taken into account (vertex corrections are absent). The calculation can be carried out in a straightforward manner, and by taking all the channels into account, we find

\[
\frac{\kappa}{T} = \frac{1}{4T^3 \pi} \int_{-\infty}^{\infty} \frac{dx}{\sinh^2(\beta x/2)} \times \left[ (\text{Im} G_c(x))^2 + (N-1)(\text{Im} G_s(x))^2 \right],
\]  

(18)

where \( G_c(x) = v^2 \pi \chi_0(x)/(1 + (N-1)U \chi_0(x)) \) and \( G_s(x) = v^2 \pi \chi_0(x)/(1 - U \chi_0(x)) \). As \( T \to 0 \), this can readily be evaluated to yield

\[
\frac{\kappa}{T} = \frac{\pi}{6} \left( \frac{U_0^4}{(U_0 + (N-1)U)^4} + \frac{(N-1)U_0^4}{(U_0 - U)^4} \right).
\]  

(19)

FIG. 2: (Color online) The frequency dependent conductance is shown for \( N = 4 \) for \( U/U_0 = -0.33, -0.3, -0.2 \) below the dashed line indicating \( U = 0 \) with increasing spectral weight. The other curves correspond to \( U/U_0 = 0.2, 0.5 \) and 0.9 with decreasing peak close to \( \omega = W \), where \( W \) is a sharp momentum space cut-off. The inset shows the ac conductance as a function of the Hubbard interaction.

FIG. 3: (Color online) The Lorentz number in the SU(N) Wolff model is plotted for \( N = 2 \) (solid line), 3 (dashed line), 4 (dashed-dotted line), 8 (dotted line) on a semilogarithmic scale.
In the absence of $U$, it gives the pure result $\kappa = (\pi/6)NT$ (or $\pi^2 k_B^2 NT/3h$ upon restoring original units). In the presence of finite $U$, it increases regardless to the sign of the interaction. In other words, local Coulomb repulsion or attraction facilitates heat transport. From these, the Lorentz number reads as

$$L = \frac{\kappa}{T g(0)} = \frac{L_0}{N} \times \left[ \frac{U_0^2}{U_0 + (N-1)U} + (N-1) \frac{U_0^2 (U_0 + (N-1)U)^2}{(U_0 - U)^4} \right], \quad (20)$$

where $L_0 = \pi^2 k_B^2 / 3e^2$ is the non-interacting result. For any finite $U$ and $N$, the Wiedemann-Franz law is broken, except a single value of attractive $U$ for any given $N$. This breakdown is a natural consequence of spin-charge separation, as seen in Eqs. (5) and (6). Had we chosen $N = 1$, the Wiedemann-Franz law would hold. As $U$ increases in the repulsive regime, it enhances strongly. In the attractive case, a minimum value is reached for decreasing $U$ before the divergence at $-U_0/(N-1)$. For finite $T$, the Lorentz number decreases monotonically. Violation of the Wiedemann-Franz law was reported in one-dimensional electron gas (Luttinger liquid)\textsuperscript{23} with repulsive interaction. For any finite interaction, $L$ increased from the Fermi liquid value, similarly to our case. From these similarities in transport and in X-ray response, we conclude that the physical quantities evaluated within our model and its lattice version (Hubbard model) exhibit the same type of behaviour as a function of interaction. Other failures of the Wiedemann-Franz law have been seen in mean-field theories as well\textsuperscript{22,24}. It is plotted in Fig. 4 for various $N$. Although the long time asymptotics of the Green’s function is of Fermi liquid character, still spin-charge separation occurs on microscopic level, as seen in Eqs. (5) and (6), causing the breakdown of the Wiedemann-Franz law. Similar phenomenon is expected to occur in the single- and multichannel realizations of the Kondo effect\textsuperscript{25}.

In conclusion, we have studied the electric and heat transport properties of the SU($N$) Wolff model. As $N$ grows to infinity, the systems crosses over to non-interacting behaviour. At the same time, the Wiedemann-Franz law remains broken due to spin-charge separation.

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1. A. A. Abrikosov, Fundamentals of the Theory of Metals (North-Holland, Amsterdam, 1998).
2. P. A. Wolff, Phys. Rev. 124, 1030 (1961).
3. P. Schlottmann, Phys. Rev. B 17, 2497 (1978).
4. D. C. Mattis, Ann. Phys. (NY) 89, 45 (1975).
5. B. Dóra, Phys. Rev. B (in press).
6. A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge, 1998).
7. E. Szirmai and J. Sólyom, Phys. Rev. B 71, 205108 (2005).
8. R. Assaraf, P. Azaria, M. Caffare, and P. Lecheminant, Phys. Rev. B 60, 2299 (1999).
9. M. Imada, A. Fujimori, and Y. Tokura, Rev. Mod. Phys. 70, 1039 (1998).
10. G.-M. Zhang, Commun. Theor. Phys. 34, 211 (2000).
11. A. Oguri, Phys. Rev. B 52, 16727 (1995).
12. T. K. Ng and P. A. Lee, Phys. Rev. Lett. 61, 1768 (1988).
13. E. Witten, Nucl. Phys. B 145, 110 (1978).
14. P. Coleman, Phys. Rev. B 35, 5072 (1987).
15. I. Affleck and A. W. W. Ludwig, Nucl. Phys. B 360, 641 (1992).
16. J. van Delft and H. Schoeller, Ann. Phys. (Leipzig) 7, 225 (1998).
17. A. C. Hewson, The Kondo Problem to Heavy Fermions (Cambridge University Press, Cambridge, Great Britain, 1993).
18. W. Izumida, O. Sakai, and Y. Shimizu, J. Phys. Soc. Jpn. 66, 717 (1997).
19. D. S. Fisher and P. A. Lee, Phys. Rev. B 23, 6851 (1981).
20. C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. 68, 1220 (1992).
21. M. Sindel, W. Hofstetter, J. von Delft, and M. Kindermann, Phys. Rev. Lett. 94, 196602 (2005).
22. A. Houghton, S. Lee, and J. B. Marston, Phys. Rev. B 65, 220503 (2002).
23. C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. 76, 3192 (1996).
24. S. G. Sharapov, V. P. Gusynin, and H. Beck, Phys. Rev. B 67, 144509 (2003).