Adjoint QCD, Symmetry-Enriched TQFT and Higher Symmetry-Extension

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Recent work explores the candidate phases of the 4d adjoint quantum chromodynamics (QCD\textsubscript{4}) with an SU(2) gauge group and two massless adjoint Weyl fermions. Both Cordova-Dumitrescu and Bi-Senthil propose possible low energy 4d topological quantum field theories (TQFTs) to saturate the higher \textquoteleft t Hooft anomalies of adjoint QCD\textsubscript{4} under a renormalization-group (RG) flow from high energy. In this work, we generalize the symmetry-extension method of Wang-Wen-Witten [arXiv:1705.06728, Phys. Rev. X 8, 031048 (2018)] to higher symmetries, and formulate higher group cohomology and cobordism theory approach to construct higher-symmetric TQFTs. We prove that the symmetry-extension method saturates certain anomalies, but also prove that neither $A_{P_2(B_2)}$ nor $P_2(B_2)$ can be fully trivialized, with the background 1-form field $A$, Pontryagin square $P_2$ and 2-form field $B_2$. Surprisingly, this indicates an obstruction to constructing a fully 1-form center and 0-form chiral symmetry (full discrete axial symmetry) preserving 4d TQFT with confinement, a no-go scenario via symmetry-extension for specific higher anomalies. We comment on the implications and constraints on deconfined quantum criticality and ultraviolet-infrared (UV-IR) duality in 3+1 spacetime dimensions.

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I. INTRODUCTION AND SUMMARY OF MAIN RESULTS

Recent work explores the candidate phases of the adjoint quantum chromodynamics in 4 dimensional spacetime (QCD\textsubscript{4}) with an SU(2) gauge group and two massless adjoint Weyl fermions (equivalently, two massless adjoint Majorana fermions, or one massless adjoint Dirac fermion) \cite{1-4}.\textsuperscript{1} This adjoint QCD\textsubscript{4} has a 1-form electric $Z_2$ center global symmetry, which is a generalized global symmetry of higher differential form \cite{5}. This adjoint QCD\textsubscript{4} has the SU(2) gauge theory coupling to the matter fields in the adjoint representation, thus it gains a 1-form electric $Z_2$ center symmetry; while the usual fundamental QCD\textsubscript{4} has the gauge theory coupling to the matter

\textsuperscript{1} In this case, we denote the adjoint Weyl fermion flavor $N_f = 2$ and the gauge group $N_c = 2$ for SU($N_c$).
fields in the fundamental representation, which lacks the 1-form symmetry. We will soon learn that this 1-form symmetry plays a crucial role to constrain the higher ‘t Hooft anomaly-matching \cite{6} of the quantum phases of the adjoint QCD$_4$. (See Sec. II for more detailed information regarding the global symmetries and ‘t Hooft anomalies of this adjoint QCD$_4$.)

Given the adjoint QCD$_4$ at the high energy scale, it is known that this theory is weakly coupled thus asymptotically free at ultraviolet (UV) free when the number of Weyl fermion flavor $N_f \leq 5$. Viewing the adjoint QCD$_4$ as a UV completion of a quantum field theory (QFT), we should ask what this QFT flows to under a renormalization-group (RG) flow from ultraviolet (UV) to the low energy at infrared (IR). Both Cordova-Dumitrescu \cite{2} and Bi-Senthil \cite{3} propose its low energy candidate phases at IR, saturating the higher ‘t Hooft anomalies involving the 1-form symmetry.

In particular, Bi-Senthil \cite{3} suggests a fully-symmetric 4d TQFT to saturate higher ‘t Hooft anomalies without breaking any UV global symmetries of the adjoint QCD$_4$. Namely, an interesting RG flow from Bi-Senthil \cite{3} speculates that:

\begin{equation}
\text{Adjont QCD}_4 \text{ at UV} \xrightarrow{\text{RG flow (?) to a long distance}} \text{Massless 1 Dirac (2 Weyl) fermion + 4d TQFT at IR (?).}
\end{equation}

The IR theory only involves a massless free 1 Dirac (or 2 Weyl) fermion, and a decoupled 4d TQFT. Since the massless 1 Dirac fermion has only the ordinary 0-form symmetry but no 1-form symmetry, so the massless fermion sector alone cannot saturate the higher anomaly of the adjoint QCD$_4$. Thus the crucial and nontrivial check on Bi-Senthil \cite{3} proposal of this UV-IR duality eq. (1) relies on the explicit construction of the fully-symmetric 4d TQFT to saturate all higher ‘t Hooft anomalies involving 1-form symmetry. One of the motivation of our present work is to rigorously verify the validity of this symmetric anomalous 4d TQFT.

In this work, we have two goals:

1. We generalize the symmetry-extension method of Wang-Wen-Witten \cite{7} to higher symmetries. We formulate a higher group cohomology or a higher cobordism theory approach of Ref. \cite{7} to construct “symmetric anomalous TQFTs” that can live on the boundary of symmetry protected topological states (SPTs). The “symmetric anomalous TQFTs” is an abbreviation of “the TQFTs that saturates the (higher) ‘t Hooft anomalies of a given global symmetry by preserving the global symmetry.” Previous works in condensed matter physics suggest that the long-range entangled anomalous topological order (whose effective low energy theory is a TQFT) can live on the boundary of a short-range entangled SPT state, see \cite{8} and References therein on this exotic phenomenon. The boundary of SPTs protected by symmetry group $G$ (called $G$-SPTs) has the ‘t Hooft anomaly of symmetry $G$. Ref. \cite{7} provides a systematic way to construct the symmetric anomalous TQFTs for a $G$-SPTs of a given symmetry $G$. In particular, among other results, Ref. \cite{7} proves that:

“For any bosonic $G$-SPTs protected by a finite group $G$ (unitary or anti-unitary time-reversal symmetry) in a 2-dimensional spacetime (2d) or above ($\geq 2d$), there always exists a finite group $K$ bosonic gauge theory which is a TQFT, saturating the $G$-'t Hooft anomaly, that can live on the boundary of $G$-SPTs, based on the symmetry-extension method via a short exact sequence $1 \rightarrow K \rightarrow H \rightarrow G \rightarrow 1$, where all $G, K$ and $H$ are finite groups of 0-form symmetry.”

In this article, we will explore the related phenomenon of Ref. \cite{7} but we improve the formulation by replacing the 0-form $G$ symmetry to include generalized higher symmetries of Ref. \cite{5}.

2. We apply the above generalized higher symmetry-extension method from Ref. \cite{7} either to construct the higher-symmetric anomalous TQFTs, for adjoint QCD$_4$; or to show the invalidity of the TQFTs via a symmetry-extension method.

Specifically, we find an obstruction to construct certain symmetric 4d TQFTs via symmetry extension, for the mixed anomaly mixing between the discrete axial symmetry (here the 0-form $\mathbb{Z}_{2N_cN_f} = \mathbb{Z}_8$ symmetry, with $N_c = N_f = 2$) and the 1-form electric center symmetry (denoted as $\mathbb{Z}_{2^{2N_c|N_f}} = \mathbb{Z}_{2^{2|2}}$). This higher anomaly is abbreviated as the Type I higher anomaly in Ref. \cite{3}. The Type I anomaly in 4d has a $\mathbb{Z}_4$ class (below $k \in \mathbb{Z}_4$ class), one can explicitly write down the 5d topological (abbreviated as “topo.”) invariant \cite{2} which is a cobordism invariant (see mathematical details in [9] and Sec. A),

\begin{equation}
\text{Type I anomaly/topo. invariant : } \epsilon^i \frac{1}{i} \int_{A^{i-2} \cap P_{2} (B_2)} F.\ (2)
\end{equation}

Here $A$ is the $\mathbb{Z}_4$-valued background 1-form gauge field coupling to the 0-form $Z_8/Z_2^4 = Z_4$ part of the axial global symmetry. The $Z_2^{4}$ is the fermionic parity symmetry which is $(-1)^{N_F}$, assigning a minus to the state of system when there is an odd number of total number of fermions $N_F$. The $B_2$ is the $\mathbb{Z}_2$-valued background 2-form gauge field coupling to the 1-form $Z_2^{4}\sim$-symmetry. The $\cup$ is the cup product, and the $P_2$ is the Pontryagin square, see more details in Sec. II. In Sec. A, we will prove the non-existence of anomalous symmetric 4d TQFTs (of finite groups or higher groups) for this 4d higher anomaly (or equivalently, 5d higher SPTs) of eq. (2), via the symmetry extension method. However, we clarify that our proof does not necessarily imply a no go theorem for
the anomalous symmetric 4d TQFTs for Bi-Senthil [3] in general, it could be due to the limitation of the symmetry extension [7] we used. Nevertheless, it is known that [7]'s method is general and systematic enough to construct symmetric TQFT for all bosonic anomalies of the ordinary 0-form finite group symmetries; thus the obstruction from [7] is severe and interesting by itself to be presented here. This proof indicates a no-go scenario for anomalous symmetric 4d TQFTs if we only limit the construction under the symmetry-extension construction of TQFTs.

In contrast, we find that the generalized symmetry-extension method can indeed construct another symmetric 4d TQFT saturating a different higher mixed anomaly, mixing between the background gravity (or the curved spacetime geometry) and the 1-form center symmetry (denoted as $Z_2$). This higher anomaly is abbreviated as the Type II higher anomaly in Ref. [3]. We can explicitly write down the 5d topological (abbreviated as “topo.”) invariant of the 4d symmetric anomalous TQFT for this 4d higher anomaly (or equivalently, 5d higher SPTs) of eq. (3) in Sec. III.

**Type II anomaly/topo. invariant:**

$$e^{i\pi f} w_2(TM) Sq^1 B_2 = e^{i\pi f} w_3(TM) B_2.$$  \(3\)

Here $w_j(TM)$ has the $w_j$ as the $j$-th Stiefel-Whitney (SW) class [10], as the probed background spacetime $M$ connection over the spacetime tangent bundle $TM$. The $Sq^1$ is the Steenrod operation. We demonstrate the explicit construction of the 4d symmetric anomalous TQFT for this 4d higher anomaly (or equivalently, 5d higher SPTs) of eq. (3) in Sec. III.

Physically, the above description concerns the physics side of the story, relating to quantum field theory, QCD or the strongly-correlated systems in condensed matter physics.

Mathematically, we ask the following questions (corresponding to the physics story above) and find an obstruction to a positive answer for a Bi-Senthil’s scenario [3] via the symmetry-extension alone, generalizing the method of [7]:

**Question 1.** Can we trivialize the topological term $A \cup \mathcal{P}_2(B_2)$ via extending the global symmetry by 0-form symmetry and 1-form symmetry? To answer this, we deal with the trivialization problem of the cobordism invariant $A \cup \mathcal{P}_2(B_2)$ of the bordism group $\Omega_{d}^{Spin \times Z_2}$ (B$^2Z_2$). We prove that the answer is negative.

**Question 2.** We also solve the trivialization problem of the cobordism invariant $\mathcal{P}_2(B_2)$ of the bordism group $\Omega_{4}^{Spin \times Z_2}$ (B$^2Z_2$): Can we trivialize the topological term $\mathcal{P}_2(B_2)$ via extending the global symmetry by 0-form symmetry and 1-form symmetry? We prove that the answer is also negative.

The plan of the article goes as follows.

In Sec. II, we detail the related global symmetries and higher anomalies relevant for our goal, following a remarkable Ref. [2].

In Sec. III, we discuss the higher symmetry-extension generalization of [7], and successfully apply the method to construct a 4d symmetric anomalous TQFT for Type II anomaly eq. (3). But this method shows an obstruction for the Type I anomaly eq. (2).

We leave rigorous but more formal and mathematical details of the calculation in Appendices.

In Appendix A, we find a potential obstruction: The Type I anomaly eq. (2) cannot be saturate by a symmetric anomalous finite group/higher group TQFT, at least by a symmetry extension method.

In Appendix B, we give a counter example as the proof for the failure of the symmetry extension method applying to trivializing the 5d $A \cup \mathcal{P}_2(B_2)$.

In Appendix C, we show a similar obstruction: The 4d $\mathcal{P}_2(B_2)$ cannot be saturated by a symmetric anomalous finite group/higher group TQFT, at least by a symmetry extension method.

We note that the Appendix A, B, and C are more technical and mathematical demanding. For readers who are not familiar with the mathematical background for these three sections, one can either consult [9] and [11] (e.g. the Appendix of [11]), or simply skip them and proceed to the conclusion Sec. IV which we summarize the physics interpretations of the above three sections.

We conclude in Sec. IV.

The mathematical details of our cobordism calculations can be found in a companion paper [9].

**II. THEORY OF ADJOINT QCD$_4$**

We have an SU(2) gauge theory coupled to $2 \times 3$ ($N_f = 2$ for the $2$, and the $3$ for the triplet) adjoint Weyl fermions in the adjoint representation of SU(2). The path integral (or partition function) of this adjoint QCD$_4$ in the Minkowski signature, viewed as a UV QFT theory can be written as:

\[ i \text{tr} \left[ \int d^4x \mathcal{L} \right] \]

where $\mathcal{L}$ is the Lagrangian density of the theory. The 't Hooft anomaly. We denote the bordism group $\Omega_{d}^{G}$, while we denote the cobordism group $\Omega_{d}^{G}$. 

\[ 2 \] In this work, we will use the term $dd$ “cobordism invariant” to describe the $dd$ topological term or $dd$ (higher) SPTs. On a manifold with boundary, the boundary of such a cobordism
\[ Z_{\text{UV}} = \int [D\psi][D\bar{\psi}][Da] \exp(iS_{\text{UV}}), \]
\[ S_{\text{UV}} = \int d^4x \sum_{j=1,2} \frac{1}{2} \bar{\psi}^j \gamma^\mu (\partial_\mu - ig_a^\mu (T^a)_{b\dot{b}}) \psi^j_{\dot{b}} - \frac{1}{g^2} \int \text{Tr}(F \wedge F) + \ldots. \]

The eq. (5) contains the first term as the Dirac Lagrangian, and the second term as the Yang-Mills Lagrangian. The \([D\ldots]\) is the path integral measure for the quantum fields. The \(\sigma^\mu (T^a)_{b\dot{b}}\) contains the standard Pauli sigma matrices \(\hat{\sigma}\).

Here we recap the results and will write the results suitable for the cobordism analysis later in Appendices A to C.

1. Flavor symmetry \(\frac{SU(2) \times Z_8}{Z_2^0}\): The classical flavor symmetry of 2 triplet Weyl fermions is the flavor \(U(2) = \frac{SU(2) \times U(1)_A}{Z_2^0}\). However, the axial symmetry \(U(1)_A\) is broken down to a discrete axial symmetry \(Z_{2N,N_f,A}\), which is \(Z_{2N,N_f,A} = Z_8\) here, due to the Adler-Bell-Jackiw (ABJ) anomaly.\(^3\) It is a standard calculation of the \(U(1)_A\)-axial symmetry is explicitly broken by the dynamical SU\((N_c)\)-gauge instanton down to \(Z_{2N,N_f,A}\)-axial symmetry.

So the flavor symmetry is simply \(\frac{SU(2) \times Z_8}{Z_2^0}\) for the quantum theory. The SU(2) is also written as the SU(2)\(_R\) as the R-symmetry thanks to the standard convention in \(\mathcal{N} = 2\) supersymmetric Yang-Mills theory (SYM) \([13]\). In the \(\mathcal{N} = 2\) SYM, the adjoint fermions are gauginos.

2. The 1-form center symmetry \(Z_{2[1]}^c\): The adjoint QCD has the matter in adjoint representation, so the SU\((N_c)\) (here SU(2)) fundamental Wilson line is charged under the 1-form electric center symmetry \(Z_{2[1]}^c\) measured by a 2-surface “charge” operator. The “charged” fundamental Wilson line (spin-1/2 representation of SU(2)) has an odd \(Z_2\) charge. The odd half integer spin-\(n/2\) representation of SU(2) has an odd \(Z_2\) charge of 1-form symmetry. Wilson lines of other integer spin-\(n\) representation (e.g., the adjoint) of SU(2) has a trivial (namely even) \(Z_2\) charge of 1-form symmetry.

Importantly the 1-form center symmetry \(Z_{2[1]}^c\) is preserved means that the electric Wilson loop (loop) is unbreakable, or called tension-ful \([3]\). Since the adjoint QCD has the 1-form center symmetry, we can use the 1-form center symmetry charged object to detect:

- Confinement: If 1-form symmetry is preserved, and all the Wilson loops (of all representations) obey the area law.
- Deconfinement: If 1-form symmetry is spontaneously broken, then the Wilson loops of odd half integer spin-\(n/2\) representation (e.g., fundamental representation) obey the perimeter law.

3. Spacetime symmetry: In Lorentz signature, we have the Poincaré group symmetry which contains the Lorentz group. We also have the discrete CPT

\(^3\) For a clarification of different meanings of anomalies, such as the three different types of physics of anomalies: (1) Classical global symmetry is violated at the quantum theory: ABJ anomaly. (2) Quantum global symmetry is well-defined and preserved but with the ’t Hooft anomaly. (3) Dynamical gauge anomaly; the readers can consult, for example, the Section 1 Introduction of \([12]\) and References therein.
symmetries. There is no charge conjugation C for SU(2) gauge theory due to the lack of SU(2) outer automorphism. So there is only T and P symmetry interchangeably thanks to the CPT theorem. If we focus on orientable spacetime for the adjoint QCD in 4d, we can consider the Spin(d) spacetime symmetry, for the purpose of classifying the ’t Hooft anomalies through the cobordism theory [9, 14]. If we consider the non-orientable spacetime for the adjoint QCD in 4d, we should consider the Pin−(d) spacetime symmetry, for the purpose of classifying the ’t Hooft anomalies through a cobordism theory, See Ref. [9, 14]. This adjoint QCD is a fermionic theory, the spacetime symmetry G_{spacetime} and the internal symmetry G_{internal} shares the fermionic parity Z_2^F, so the precise way to write the full global symmetry would be:

\[
\frac{G_{spacetime} \times G_{internal}}{Z_2^F} = G_{spacetime} \times Z_2^F G_{internal}, \tag{7}
\]

where the common Z_2^F is mod out, while the “\times Z_2^F” notation follows [14].

By combining the internal global symmetry (flavor and 1-form center symmetries) and the spacetime global symmetry above, the overall global symmetry can be written as:

\[
\text{Spin} \times Z_2^F \left( \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right) \times Z_2^F, \tag{8}
\]

\[
\text{Pin}^- \times Z_2^F \left( \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right) \times Z_2^F, \tag{9}
\]

Below we follow Ref. [9], which generalizes a theorem in a remarkable work of Freed-Hopkins [14]. Freed-Hopkins [14] formulates a cobordism theory — whose cobordism group, of the ordinary 0-form global symmetries, classifies a class of symmetric invertible TQFTs, which is relevant to the SPT classification. Ref. [9] generalizes [14] to a cobordism theory of the higher global symmetries (e.g. including 0-form global symmetries and 1-form global symmetries) and computes some examples of such cobordism groups.

In terms of bordism group notation, which later will be helpful for identifying all the (higher) ’t Hooft anomalies and the SPT classes via the computations of [9], we write their corresponding bordism groups Ω_d as:

- **Bordism group for eq. (8):**

\[
\Omega_d \text{Spin} \times Z_2^F \left\{ \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right\} \times BZ_2^F, \tag{10}
\]

\[
\equiv \Omega_d \text{Spin} \times Z_2^F \left\{ \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right\} (B^2 Z_2^F). \tag{11}
\]

For adjoint QCD_4 in 4d, the higher ’t Hooft anomalies are classified by the dimension d = 5 for the above bordism groups. See more details in Ref. [12].

**B. Anomalies**

Now consider the d = 5 bordism groups above in eq. (10) and eq. (11), we like to match their selective 5d cobordism invariants to the anomalies captured by the 4d adjoint QCD_4.

Cordova-Dumitrescu [2] have captured several anomalies, which we now overview:

1. The SU(2) Witten anomaly [15] for the flavor SU(2)_f sector, due to that there is an odd number of SU(2)_f flavor doublet. The appearance of SU(2) Witten anomaly also indicates the IR fate of this adjoint QCD_4 is gapless instead of fully gapped.

2. The (Z_{8,A})^3 anomaly captured by a perturbative anomaly (i.e., a triangle 1-loop Feynman diagram in 4d).

3. The (Z_{8,A})-(gravity)^2 anomaly captured by a perturbative anomaly (i.e., a triangle 1-loop Feynman diagram in 4d). The gravity part is due to the diffeomorphism of the background geometry.

4. The (Z_{8,A})-(SU(2)_f)^2 anomaly captured by a perturbative anomaly (i.e., a triangle 1-loop Feynman diagram in 4d).

Ref. [2] explains the two interesting mixed ’t Hooft higher anomalies involving 1-form symmetry, the Type I eq. (2) and Type II eq. (3) anomalies earlier.

5. Type I higher anomaly: mixing between the 1-form electric center symmetry (Z_2^{c}_{2,1][1]} and the 0-form discrete axial symmetry (Z_{2^{c}}_{2,1]1} = Z_8). We can write eq. (2) as:

\[
e^{i \frac{k}{2}} \int A \cup F_2 (B_2) = e^{i \frac{k}{2}} \int A (B_2) \cup B_2 + B_2 \cup \delta B_2.
\]

\[
e^{i \frac{k}{2}} \int A (B_2) \cup B_2 + B_2 (2S q^1 B_2), \tag{12}
\]

4 Here BG means the classifying space of G, and pt means the point.

5 On the other hand, if we aim to know the 4d SPTs compatible with the symmetry of adjoint QCD_4, then we need to consider the dimension d = 4 for the above bordism groups. This research direction is pursued by Ref. [11] for the related SU(N_c) Yang-Mills gauge theories.
follows. Consider the mathematical and algebraic topology (denoted as $\mathbb{Z}_{2,\{1\}}$) and the background geometry (or the curved spacetime geometry) in eq. (3).

The UV theory as an adjoint QCD$_4$ has all of the above ‘t Hooft anomalies, captured also by a particular 5d cobordism invariant, in eq. (10) and eq. (11).

Following our Introduction, in the next Sec. III, we formulate the higher symmetry-extension generalizing [7], and successfully construct a 4d symmetric anomalous TQFT for Type II anomaly eq. (3). But we will soon show an obstruction to construct symmetric TQFT for the Type I anomaly eq. (2).

III. HIGHER SYMMETRY-EXTENSION

A. Summary of Ordinary Symmetry-Extension

Ref. [7] sets up the symmetry-extension problem as follows. Consider the 4d SPTs protected by an internal symmetry group $G$, whose boundary theory has $(d-1)d$ 't Hooft anomaly in $G$. There are three different ways to phrase the question asked by [7], but their underlying meanings are the same:

Q1. Condensed matter statement: Can we find a total group $H$ such that $G$ is its quotient group, and such that the $G$-SPTs becomes a trivial gapped vacua in $H$? More precisely, there is a local unitary transformation preserving the symmetry $H$ (but breaking the symmetry $G$), such that when the $G$-SPTs is viewed as an $H$-SPTs, it can be deformed to a trivial gapped insulator in $H$ via a local unitary transformation, without breaking $H$ and without any phase transition.$^6$

Q2. QFT or high energy particle physics statement: Given a $(d-1)d$ 't Hooft anomaly in $G$, can we find an enlarged group $H$, with a total group $H$ having $G$ as its quotient group, such that the 't Hooft anomaly in $G$ becomes anomaly-free in $H$? (i.e., the $G$-anomaly becomes trivial in $H$.)

Q3. Mathematical and algebraic topology statement: Given a 4d topological term of a group $G$, here the topological term can be:

- the 4d co/bordism invariant for a 4-d th co/bordism group $\Omega^d_{4}(BG, U(1))$ or bordism group $\Omega^d_4(BG)$ or bordism group, in a cobordism theory,$^7$

  can we find an extended group $H$ with $G$ its quotient group, via a short exact sequence

$$1 \to K \to H \to G \to 1,$$

  such that the topological term of a group $G$ can be pulled back to a trivial topological term of a group $H$?

Suppose the above answer is positive, and suppose that $G$, $H$ and $K$ are finite groups, then Ref. [7] shows, valid for both the lattice Hamiltonian and the path integral construction, that the $G$-SPTs in $d$ can allow:

- $H$-symmetry extended gapped boundary in any spacetime dimension $d \geq 2$,
- $G$-symmetry preserving and topological $K$-gauge theory gapped boundary: Topological emergent $K$-gauge theory with preserving global symmetry $G$ on a bulk $d \geq 3$.

Ref. [7] addresses the above questions Q1, Q2 and Q3, by proving that at least for a finite group $G$ (with $G$ a unitary symmetry group or anti-unitary symmetry group involving time-reversal symmetry), by the following positive answers, with the always-existences on the validity of the symmetric gapped boundary construction:

A1. For any bosonic SPT state with a finite onsite symmetry group $G$, including both unitary and anti-unitary symmetry, there always exists an $H$-symmetry-extended (or $G$-symmetry-preserving) gapped boundary via a nontrivial group extension by a finite $K$, given the bulk spacetime dimension $d \geq 2$.

A2. For any $G$-anomaly in $(d-1)d$ given by a cocycle $\nu^G_d \in H^d(BG, U(1))$ of group cohomology of a finite group $G$, there always exists a pull back to a finite group $H$ via a certain group extension $1 \to K \to H \to G \to 1$, extended by a finite $K$, such that $G$-anomaly becomes $H$-anomaly free, given the dimension $d \geq 2$.

A3. For any $G$-cocycle $\nu^G_d \in H^d(BG, U(1))$ of a finite group $G$, there always exists a pull back to a finite group $H$ via a certain short exact sequence of a group extension $1 \to K \to H \to G \to 1$ by a finite $K$, such that

$$r^*\nu^H_d = \delta \mu^H_d \in H^d(H, U(1)).$$

$^6$ This procedure has been demonstrated explicitly in a many body quantum system recently in Ref. [16], which constructs an explicit path in the enlarged $H$-symmetric quantum Hilbert space.

$^7$ Here the $-$ can be chosen as co/bordism with different structures such as special/orthogonal SO/O, spin/pin Spin/Pin$^\pm$ structures.
Here \( r \) is the pullback operation, and \( \delta \) is the coboundary operation. Namely, a \( G \)-cocycle becomes a \( H \)-coboundary, which splits to a one-lower dimensional \( H \)-cochains \( \mu^H_{d-1} \), given the dimension \( d \geq 2 \).

The proof of [7] has also been verified later by [17]. The related constructions similar to [7] are explored also in specific cases or from different perspectives in [18, 19].

\[ \text{B. Higher Symmetry Generalization} \]

Now we generalize the approach in [7]. The short exact sequence of a group extension \( 1 \to K \to H \to G \to 1 \) extended by a finite \( K \) given in [7] also implies an induced fiber sequence from the fibration

\[ BK \to BH \to BG, \tag{14} \]

where all \( G, K \) and \( H \) are finite groups of 0-form symmetry such that the \( G \)-SPTs protected by a finite group \( G \) becomes trivial \( H \)-SPTs by pulling back \( G \) to \( H \), under the above criteria A1, A2 and A3.

We consider the higher symmetry-extension problem. Our goal is to find a fibration

\[ BK[0] \times B^2K[1] \to BH \to BG \tag{15} \]

where \( G \) and \( H \) are 2-groups, \( K[0] \) and \( K[1] \) are finite abelian groups of 0-form symmetry and 1-form symmetry respectively such that the higher \( G \)-SPTs protected by a 2-group \( G \) becomes trivial higher \( H \)-SPTs by pulling back \( G \) to \( H \).

Similar to questions in Q1, Q2 and Q3 of Sec. IIIA, we ask a set of generalized questions:

\[ \text{Q4. Condensed matter statement:} \text{ Can we find a total 2-group } \mathbb{H} \text{ as a total space such that } \mathbb{B} G \text{ is } \mathbb{H} G \text{'s orbit (or base space), and such that the } \mathbb{G} \text{-SPTs becomes a trivial gapped vacua in } \mathbb{H}?! \text{ More precisely, there is a local unitary transformation preserving the symmetry } \mathbb{H} \text{ (but breaking the symmetry } \mathbb{G} \text{), such that when the } \mathbb{G} \text{-SPTs is viewed as an } \mathbb{H} \text{-SPTs, it can be deformed to a trivial gapped insulator in } \mathbb{H} \text{ via a } local \text{ unitary transformation (note that the locality also need to be generalized to higher dimensional extended object such as a line instead of just a point, due to the 2-group structure), without breaking } \mathbb{H} \text{ and without any phase transition in the enlarged } \mathbb{H} \text{-symmetric quantum Hilbert space.} \]

\[ \text{Q5. QFT or high energy particle physics statement:} \text{ Given a } (d-1)d \text{'t Hooft anomaly in a higher group } \mathbb{G} \text{, can we find an enlarged group } \mathbb{H} \text{, with a total group } \mathbb{H} \text{ obeying eq. (15), such that the } \text{'t Hooft anomaly in } \mathbb{G} \text{ becomes } \text{anomaly-free in } \mathbb{H}? \text{ (i.e., the } \mathbb{G}\text{-anomaly becomes trivial in } \mathbb{H}.) \]

\[ \text{Q6. Mathematical and algebraic topology statement:} \text{ Given a } dd \text{ topological term of a higher group } \mathbb{G} \text{, here the topological term can be:} \]

\[ \bullet \text{ the } dd \text{ cocycle for a } d\text{-th cohomology group } \mathbb{H}^d(\mathbb{B}G, U(1)) \text{ in a higher group cohomology theory.} \]

\[ \bullet \text{ the } dd \text{ co/bordism invariant for a } d\text{-th cobordism group } \Omega^d(\mathbb{B}G, U(1)) \text{ or bordism group } \Omega^d_\mathbb{B}(BG) \text{ or bordism group, in a cobordism theory;} \]

\[ \text{can we find an extended group } \mathbb{H} \text{ obeying eq. (15) such that the topological term of a group } \mathbb{G} \text{ can be pulled back to a trivial topological term of a group } \mathbb{H}? \]

In the next two subsections, we implement the strategy eq. (15) by asking the questions in Q4, Q5 and Q6, for the two examples: Type I anomaly/topo. invariant in eq. (2), and Type II anomaly/topo. invariant in eq. (3).

We relegate more formal and mathematical details of the calculation of the above two subsections into Appendices A, B, and C.

\[ \text{C. Saturate Type II anomaly: Symmetric TQFTs} \]

We first try to do higher symmetry extension to trivialize 4d Type II higher anomaly (given by a 5d topological invariant) eq. (3)

\[ e^{i \pi \int w_2(TM) Sq^1 B_2} = e^{i \pi \int w_3(TM) B_2}. \]

We have found that eq. (3) is a topological invariant in \( d = 5 \), for:

\[ \bullet \mathbb{H}^d(B^2Z_2, U(1)) \text{ group cohomology of a higher classifying space finite group, as well as} \]

\[ \bullet \Omega^d_\mathbb{B}(B^2Z_2) \text{ cobordism group of a higher classifying finite group. Below we can either use the group cohomology or the cobordism group viewpoint to understand the trivialization of 4d Type II higher anomaly.} \]

1. The first way to trivialize this 4d Type II higher anomaly is extending the spacetime symmetry from special orthogonal group \( \text{SO}(d) = \text{Spin}(d)/Z^F_2 \) to \( \text{Spin}(d) \):

\[ BZ_2 \to B\text{Spin}(d) \times B^2Z_2 \to B\text{SO}(d) \times B^2Z_2. \tag{16} \]

8 For the related physics topics on higher group symmetries and higher SPTs, the readers can find from the recent developments [20–25] and References therein.

9 Here the “−" follows the earlier footnote 7.
This extension works since \( w_2(TM) = 0 \) vanishes on Spin manifold. Thus, eq. (3) is trivialized once we pull back eq. (3) into \( B\text{Spin}(d) \times B^2Z_2 \). According to the interpretation in Sec. III A and Ref. [7], the fibration \( B\text{Spin}(d) \) contains an emergent 0-form global symmetry which is anomaly-free and can be dynamically gauged. Indeed, the natural way to interpret the eq. (16) as the generalized construction of [7] is that there is an emergent 1-form \( Z_2 \) gauge theory (dynamically gauged from emergent 0-form global symmetry \( B\text{Spin}(d) \)), such that the \( Z_2 \) gauge theory has additional emergent fermionic particle excitations due to the emergent spin structure (the \( \text{Spin}(d) \) in the total space in eq. (16)). In terms of the full 4d symmetric TQFT saturating the higher 't Hooft anomaly (coupling to the 5d higher SPTs), we can write the involved QFT sectors into a partition function, which looks like the following locally:

\[
\begin{align*}
\exp(i2\pi \int_{\partial M} w_2(TM) \text{Sq}^4 B_2) \cdot \sum_{a \in C^1((\partial M)^4, Z_2)} \exp(i2\pi \int_{(\partial M)^4} \frac{1}{2} (ba) + \ldots )
\end{align*}
\]

locally a 4d \( Z_2 \)-TQFT with emergent fermions and spin-structure

Here \( a \) is the \( Z_2 \)-valued 1-form gauge field (the standard notation as the 1-cochain in \( C^1 \)), \( b \) is the \( Z_2 \)-valued 2-form gauge field (the standard notation as the 2-cochain in \( C^2 \)), the \( \delta \) is the coboundary operator here \( \delta = 2\text{Sq}^1 \), and we use the cup product \( \cup \). See also our previous explanations around eq. (3) for notations. The \( \ldots \) are additional coupling terms between dynamical gauge fields and background fields. The \( \ldots \) also include additional sectors from the UV adjoint \( \text{QCD}_4 \) from eq. (4), in order to saturate the other anomalies. Note that the similar emergent dynamical spin structure with \( Z_2 \) gauge field has been studied in Ref. [26]. The important thing is that the 1-form gauge field \( a \) can be regarded as the difference between two spin-structures, while the gauge field \( a \) becomes dynamical.

Moreover, we can write the extension of eq. (16) in terms of the full symmetry eq. (8):

\[
\begin{align*}
\text{BZ}_2 & \rightarrow \text{B(Spin} \times (\frac{\text{SU}(2) \times \tilde{Z}_8, A}{\tilde{Z}_2}) ) \times B^2Z_4 \rightarrow \text{B(Spin} \times \tilde{Z}_8) \times B^2Z_2 \\text{,}
\end{align*}
\]

while the physical interpretation remains the same as eq. (16) and eq. (17).

2. The second way to trivialize this 4d Type II higher anomaly is extending the 1-form symmetry:

\[
\begin{align*}
B^2Z_2 & \rightarrow B\text{SO}(d) \times B^2Z_4 \rightarrow B\text{SO}(d) \times B^2Z_2.
\end{align*}
\]

This way works since \( B \) is pulled back to \( \tilde{B} \in H^2(B^2Z_4, Z_2) \), and \( \text{Sq}^1 \tilde{B} = 0 \) (see Appendix A 2 d).

According to the interpretation in Sec. III A and Ref. [7], the fibration \( B^2Z_2 \) is associated to an emergent 1-form global symmetry \( Z_{2,[1]} \) which is anomaly-free and can be dynamically gauged. Indeed, the natural way to interpret the eq. (19) as the generalized construction of [7] is that there is an emergent 2-form \( Z_2 \) gauge theory (dynamically gauged from emergent 1-form global symmetry \( B\text{Spin}(d) \)) with a 2-form gauge field \( b' \). The original 1-form \( Z_{2,[1]} \)-symmetry acts projectively on the emergent 2-form \( Z_2 \) gauge theory, but the extended 1-form \( Z_{4,[1]} \)-symmetry acts on it faithfully.

We can write the involved QFT sectors into a following partition function, which looks like the following locally:

\[
\begin{align*}
\exp(i\pi \int_{\partial M} w_2(TM) \text{Sq}^1 B_2) \cdot \sum_{a' \in C^1((\partial M)^1, Z_2), \delta a' \in C^1((\partial M)^1, Z_2)} \exp(i\pi \int_{(\partial M)^1} \frac{1}{2} (\delta b') + \ldots )
\end{align*}
\]

locally a 4d \( Z_2 \)-TQFT, on which the 1-form \( Z_{2,[1]} \)-symmetry acts projectively

Here \( b' \) is the \( Z_2 \)-valued 2-form gauge field (the standard notation as the 2-cochain in \( C^2 \)), \( a' \) is the \( Z_2 \)-valued 1-form gauge field (the standard notation as the 1-cochain in \( C^1 \)), while other notations are explained around eq. (3) and eq. (20). The \( \ldots \) are additional coupling terms between dynamical gauge fields and background fields.
The ... also include additional sectors from the UV adjoint QCD$_4$ from eq. (4), in order to saturate the other anomalies. We can also write the extension of eq. (19) in terms of the full symmetry eq. (8):

\[ B^2Z_{2,[1]} \to B(\text{Spin} \times Z^F_2 \left( \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right)) \times B^2Z_{4,[1]} \to B(\text{Spin} \times Z^F_2 \left( \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right)) \times B^2Z_{2,[1]}, \]  

(21)

while the physical interpretation remains the same as eq. (19).

**D. Saturate Type I anomaly: Obstruction**

We now try to do higher symmetry extension to trivialize 4d Type I higher anomaly (given by a 5d topological invariant of higher SPTs) eq. (2)

\[ e^{i \frac{\pi}{2}} f : A \cup B_2 \to \mu \in \mathbb{Z}_2 \]

Below we show that

1. When \( k = 2 \in \mathbb{Z}_4 \), the Type I anomaly eq. (2) can be trivialized, thanks to the fact that we can rewrite eq. (2) as

\[
e^{i \pi} f : A \cup B_2, \]

\[= e^{i \pi} f : A \cup (B_2 \cup B_2 \cup 2 B_2 \cup 2 B_2 \cup 2 B_2 \cup 2 B_2), \]

\[= e^{i \pi} f : A \cup (B_2 \cup B_2), \]

\[= e^{i \pi} f : A \cup (B_2 \cup B_2) \]

\[= e^{i \pi} f : w_2(TM) + w_1(TM)^2, \]

(22)

where we have used the fact that \( \text{Sq}^1A = 0 \) where \( \tilde{A} = A \mod 2 \), and the Wu formula. See also useful information in [9].

So when \( k = 2 \in \mathbb{Z}_4 \), if we extend the global symmetry by

\[ BZ_2 \to B(\text{Spin} \times (\text{SU}(2) \times Z_2 \times Z_8)) \times B^2Z_2 \to B(\text{Spin} \times Z_2 \times (\text{SU}(2) \times Z_2 \times Z_8)) \times B^2Z_2, \]

(23)

then the Type I anomaly eq. (2) vanishes. This extension works since \( w_1(TM) = 0 \) vanishes on Spin manifolds. Thus, eq. (2) is trivialized once we pull back eq. (2) into \( B(\text{Spin} \times (\text{SU}(2) \times Z_2 \times Z_8)) \times B^2Z_2 \).

2. When \( k = 1, 3 \in \mathbb{Z}_4 \), or \( k \) odd, the Type I anomaly eq. (2) cannot be trivialized by extensions.

We have tried three approaches, which we relegate the details in Appendix A 2 while we summarize the physics story and implication here.

- The first approach (Appendix A 2 b) is a breaking case since we set \( B \) to be zero. Physically this means that in order to saturate the ’t Hooft anomaly, we can break 1-form \( Z_2 \)-symmetry to nothing. In comparison, this 1-form \( Z_2 \)-symmetry breaking is a different scenario from [2, 3].

- In the second approach (details and notations explained in Appendix A 2 c), we define \( G \) to be a group which sits in a homotopy pullback square

\[ \begin{array}{ccc}
B \mathbb{G} & \to & B^2 \mathbb{Z}_2 \\
\downarrow & & \downarrow_{z_2} \\
B(\text{Spin} \times \text{SU}(2) \times Z_8) & \to & B \mathbb{Z}_8 & \to & B^2 \mathbb{Z}_2.
\end{array} \]

Hence we have a fiber sequence

\[ B \mathbb{Z}_2 \to B \mathbb{G} \to B(\text{Spin} \times \text{SU}(2) \times Z_8) \times B^2 \mathbb{Z}_2 \to B^2 \mathbb{Z}_2. \]

(25)

In this case, \( B_2 = B \) is identified with \( \hat{A}(B_2,Z_8) \) where \( A \in H^1(BZ_8,Z_8) \), and \( \text{Sq}^1B = 0 \), but \( \hat{A} \cup \frac{\text{B}^2_2}{2} \) is still not trivialized. This case is also a breaking case, since \( B \) is locked with \( A \). In physics, the locking between two probed background fields means that the global symmetry between two sectors are locked together, thus which results in global symmetry breaking.

Physically this means that in order to saturate the ’t Hooft anomaly, we still need to break symmetry in some way.

- In the third approach, we extend both the 0-form symmetry and the 1-form symmetry:

\[ BZ_2 \times B^2Z_2 \to B(\text{Spin} \times (\text{SU}(2) \times Z_2 \times Z_8)) \times B^2Z_4 \to B(\frac{\text{Spin} \times (\text{SU}(2) \times Z_2 \times Z_8)}{Z_2^F}) \times B^2Z_2. \]

(26)

But in this case, \( \hat{A} \cup \frac{\text{B}^2_2}{2} \) is still not, and cannot be, trivialized.

In summary, we finally conclude that when \( k \) is odd, \( k = 1, 3 \in \mathbb{Z}_4 \), the Type I anomaly eq. (2) cannot be trivialized by extensions and give a proof in Appendix B. In comparison, Ref. [3] proposes a full symmetry-preserving TQFT different all of our scenarios above, which contradicts to our proof in Appendix B.
When \( k = 2 \), such that the Type I anomaly survives as only a \( \mathbb{Z}_2 \) subclass (even \( k \)) in the original \( k \in \mathbb{Z}_4 \) class (of \( kA \cup P_2(B_2) \)), however, we can actually trivialize the \( \mathbb{Z}_2 \) subclass of Type I anomaly and the full Type II anomaly together via the fibration:

\[
\text{BZ}_2 \to \text{B}(\text{Spin} \times \left( \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right)) \times \text{B}^2 \mathbb{Z}_{2,[1]} \to \text{B}(\text{Spin} \times \left( \frac{\text{SU}(2) \times Z_{8,A}}{Z_2^F} \right)) \times \text{B}^2 \mathbb{Z}_{2,[1]}.
\]

The above is achieved by combining both eq. (16) and eq. (23) into eq. (18). Since we only care \( k = 2 \), this also means that the \( Z_{8,A} \)-symmetry only needs to be survived as a \( Z_{4,A} \)-symmetry. Physically this means that \( Z_{8,A} \)-symmetry can be spontaneously broken down to a \( Z_{4,A} \)-symmetry. Thus the eq. (27) really implies a fibration of a smaller symmetry (e.g. a smaller classifying space) as:

\[
\text{BZ}_2 \to \text{B}(\text{Spin} \times \left( \frac{\text{SU}(2) \times Z_{4,A}}{Z_2^F} \right)) \times \text{B}^2 \mathbb{Z}_{2,[1]} \to \text{B}(\text{Spin} \times \left( \frac{\text{SU}(2) \times Z_{4,A}}{Z_2^F} \right)) \times \text{B}^2 \mathbb{Z}_{2,[1]}.
\]

For such a 4d TQFT preserving a \( \left( \frac{\text{SU}(2) \times Z_{4,A}}{Z_2^F} \right) \)-chiral symmetry and 1-form \( \mathbb{Z}_{2,[1]} \)-symmetry (from UV adjoint QCD\(_4\)), saturating the higher ’t Hooft anomaly (coupling to the 5d higher SPTs), we can write the involved QFT sectors into a partition function, which looks like the following locally:

\[
\exp(i\pi \int_{\mathcal{A}(\text{B}_2 \cup \text{B}_2)} \cdot \sum_{\mathcal{A} \in C^1(\partial \mathcal{M}) \times \mathbb{Z}_2} \exp \left( i2\pi \int_{\partial \mathcal{M}} \frac{1}{2}(b\delta a + \ldots) \right),
\]

again the 1-form gauge field \( a \) can be regarded as the difference between two spin-structures; the 1-form emergent dynamical \( \mathbb{Z}_2 \) gauge field \( a \) is associated to a dynamical spin structure (similar to a situation in Ref. [26]).

We note that the . . . terms can involve additional ’t Hooft anomaly cancellation for the UV’s adjoint QCD\(_4\), such as the gapless sector proposed in [1–4]. Besides, the . . . terms also involve the coupling terms between dynamical gauge fields and background fields, so that the full partition function can be made gauge-invariant. Although eq. (28) already suggests a formation definition of TQFTs (based on the extension construction of bulk-boundary coupled TQFTs, see [7] and related constructions in [27, 28]), it may be worthwhile to formulate a cochain or continuum TQFT description following [27, 28] — which we leave them for future work. It may also be worthwhile to give a continuum 4d TQFT formulation for the higher-form gauge theory analogous to Dijkgraaf-Witten [29] like gauge theory, similar to the continuum TQFT formulation given in [22, 30].

### IV. CONCLUSION

We conclude by summarizing the implications of the higher-symmetry extension construction of TQFTs on the low energy dynamics of QCD\(_4\). Then we comment about the constraints on the deconfined quantum critical phenomena, or so called the deconfined quantum critical point (dQCP) [31], in 3+1 spacetime dimensions [3].

#### A. The Fate of the Dynamics of QCD\(_4\)

##### 1. Possible Fates of the Dynamics of Fundamental QCD\(_4\) with \( N_f \) Dirac fermions

First, we recall the possible fates of the dynamics of QCD\(_4\) with \( N_f \) Dirac fermions in fundamental representations of SU\(_c\). The conventional wisdom teaches us that the phase structure of dynamics of QCD\(_4\) via tuning \( N_f \) (with a fix \( N_c \)), shown in Fig. 1, is that:
• At lower $N_f$, there should be a confinement (IR confinement) and chiral symmetry breaking (IR ChSB).
• At larger $N_f$, there is a range of $N_f$ such that at IR, the QFT flows to an interacting conformal field theory ("CFT"), this is known as the range of conformal window phenomena studied by Bank-Zaks [32] and others.
• Let $N_f^{\text{asym.free}} = \frac{11}{2} N_c$, when $N_f < N_f^{\text{asym.free}}$, the UV theory is weak coupling known as the asymptotic freedom (or UV free) [33, 34]. When $N_f > N_f^{\text{asym.free}}$, the UV theory becomes strongly coupled while the coupling $g$ flows weak at IR, at least perturbatively.

2. Possible Fates of the Dynamics of Adjoint QCD$_4$ with $N_f$ Weyl fermions

Now we organize the possible fates of the dynamics of QCD$_4$ with $N_f$ Weyl fermions in adjoint representations of SU($N_c$). The possible phase structure of dynamics of QCD$_4$ via tuning $N_f$ (with a fix $N_c$) is shown in Fig. 2. We remark that the candidate adjoint phases are summarized very elegantly in [2], we recap into a concise Fig. 2, while also list down the related Scenario 1, 2, 3, and 4, from Ref. [2, 3], and from the list summarized in Sec. IV A 2.

![FIG. 1. Candidate phases of fundamental QCD$_4$ and their possible dynamical fates. “ChSB” means the “chiral symmetry breaking phase.” “Pure YM" means the pure Yang-Mills gauge theory with a SU($N_c$) gauge group. “CFT” means conformal field theory. “UV free” or “asym. free” means the asymptotic free. The question mark “?” means the detailed structure of the phase boundaries requires further studies.](image1)

![FIG. 2. Candidate phases of adjoint QCD$_4$ with an SU(2) gauge group ($N_c = 2$) and their possible dynamical fates. “ChSB” means the “chiral symmetry breaking phase.” The Scenario 1, 2, 3, and 4 are from the list summarized in Sec. IV A 2. The question marks “?, ??, and ???” means the detailed structure of the phase boundaries requires further studies.](image2)

The conventional wisdom teaches us that the phase structure of dynamics of adjoint QCD$_4$ via tuning $N_f$ (with a fix $N_c$), shown in Fig. 2, is that:

• At $N_f = 0$, it is a pure SU($N_c$) Yang-Mills gauge theory (say SU(2)), potentially with a $\theta$-term eq. (6). At $\theta = 0$, the phase is a trivially gapped confined phase (IR confinement) with no SPT state. However, at $\theta = \pi$, the phase has mixed higher anomalies [35] and potentially newly found higher ’t Hooft anomalies [12].
• At $N_f = 1$, it is a pure $\mathcal{N} = 1$ supersymmetric Yang-Mills gauge theory (SYM) [36]. Moreover, there are $N_c$ supersymmetric breaking vacua due to gaugino condensation [37], which breaks $Z_{2N_c}$ down to $Z_2$ (simply $Z_2^F$). This $\mathcal{N} = 1$ SYM phase is also known to be confined through monopole condensation, by embedding into a $\mathcal{N} = 2$ SYM theory with $N_c = 2$ [13].
• At lower $N_f$, there should be a confinement (IR confinement) and chiral symmetry breaking (IR ChSB).
• At larger $N_f$, one expects again a range of $N_f$ with a range of conformal window phenomena of Bank-Zaks [32].

To proceed further, we recall that the UV internal global symmetry is $(\text{SU}(2) \times Z_4^\chi) \times Z_2$, and now we organize a list of possible fates of the dynamics of adjoint QCD$_4$ with $N_f$ Weyl fermions proposed from [2, 3]. There are four scenarios, summarized in Table 1 and below:
1. The $N_c$ copies of (or more specifically here $N_c = 2$) of 4d $\mathbb{CP}^1$ sigma model at low energy with spontaneous symmetry breaking Goldstone modes, proposed by [2]. Its global symmetry:

$$O(2) \times \mathbb{Z}_{2,[1]}^\mathbb{C}. \quad (30)$$

In summary, the scenario 1 has:

“chiral symmetry breaking, and confinement.” (31)

To digest better about the target space of $\mathbb{CP}^1$ sigma model, here we can consider the breaking of the 0-form symmetry group $G$ as the total space $E$ breaking to a smaller fiber $F$ (a subgroup or a normal subgroup, as the fiber or the stabilizer), where the order parameter parametrizes the base manifold $B$ (the base space or the orbit). In short, we formally and mathematically write:

$$F \leftrightarrow E \quad \text{stabilizer} \leftrightarrow \text{total space} \quad \downarrow \quad \text{orbit}.$$ \quad (32)

Then we obtain a relation for the scenario 1:

$$S^4 = U(1)_R \leftrightarrow S^3 = SU(2)_R \downarrow \quad S^2 = \mathbb{CP}^1,$$ \quad (33)

or more precisely a relation:

$$O(2)_R = U(1) \times \mathbb{Z}_2 \leftrightarrow \left( \frac{SU(2)_R \times Z_{8,A}}{Z_2^F} \right) \downarrow \quad \mathbb{CP}^1 \times \frac{Z_{8,A}}{Z_2 \times Z_2^F}. \quad (34)$$

The $\mathbb{CP}^1 \times \frac{Z_{8,A}}{Z_2 \times Z_2^F}$ has two copies of $\mathbb{CP}^1$ as the target space, parametrizing the order parameter of the base manifold $B$ (the base space or the orbit).

2. A free massless Dirac fermion (equivalently, two massless Weyl fermions, or two massless Majorana fermions) and a $\mathbb{Z}_2$ discrete gauge theory as a 4d TQFT with a $\mathbb{Z}_4$ symmetry (spontaneously broken from the $\mathbb{Z}_8$ symmetry), proposed by [2]. The IR symmetry is

$$\left( \frac{SU(2) \times Z_{4,A}}{Z_2^F} \right) \times \mathbb{Z}_{2,[1]}^\mathbb{C}. \quad (35)$$

In summary, the scenario 2 has:

“chiral symmetry breaking $\mathbb{Z}_8 \to \mathbb{Z}_4$, and confinement.” (36)

However, as explained in [2], there is an additional emergent new deconfined $\mathbb{Z}_2$-TQFT with emergent new $\mathbb{Z}_{2,[1]}$ symmetries spontaneously broken.

3. A free massless Dirac fermion (equivalently, two massless Weyl fermions, or two massless Majorana fermions) and a 4d TQFT preserving the full $\mathbb{Z}_8$ symmetry, proposed by [3]. The two massless Weyl fermions actually have a $U(2)$ continuous global symmetry. The IR symmetry we focus is:

$$\left( \frac{SU(2) \times Z_{8,A}}{Z_2^F} \right) \times \mathbb{Z}_{2,[1]}^\mathbb{C}. \quad (37)$$

In summary, the scenario 3 proposed that:

“chiral symmetry fully preserved, and confinement.” (38)
4. A 4d U(1) gauge theory in Coloumb phase with a $Z_{2N_f} \times Z_2$ symmetry, proposed by [2]. The IR symmetry we focus on is:

$$\left( \frac{SU(2) \times Z_{8,\Lambda}}{Z_2} \right) \times U(1)^{[1]} \times U(1)^{[1]} \quad (39)$$

The $\left( \ldots \right)$ means a spontaneous symmetry breaking of ... for 1-form symmetry breaking here, it leads to a deconfinement of U(1) gauge theory. In summary, the scenario 4 proposed that:

"chiral symmetry preserved, and deconfinement." (40)

5. Note that there is another scenario from Ref. [1] proposing only a free massless Dirac fermion at IR (equivalently, two massless Weyl fermions, or two massless Majorana fermions), and two vacua (two degenerate ground states) due to $Z_{8,\Lambda} \to Z_{4,\Lambda}$, without any 1-form symmetry. This scenario is certainly incomplete due to the lack of matching the higher 't Hooft anomalies of 1-form symmetry. As Ref. [1] also notices later, the more complete scenario is adding a TQFT sector, following the Scenario 2.

| Scenario | Internal global symmetry $G$ | Chiral Symmetry | 1-form $Z_2^{[1]}$ Symmetry; De-/Confinement | Anomaly matched with UV | Plausible Candidates |
|-----------------|-----------------------------|----------------|---------------------------------|------------------------|---------------------|
| 1. Ref. [2]     | $O(2) \times U(1)^{[1]}$   | SSB            | Enhanced and preserved; Confined. | Yes                    | Yes (favored by energetic?) |
| 2. Ref. [2]     | $(SU(2) \times Z_{1,\Lambda}) \times Z_{2}^{[1]} \times \ldots$ | SSB            | Preserved; Confined. But + new deconfined $Z_2$-TQFT with emergent new $Z_{2}^{[1]}$ SSB. | Yes                    | Yes |
| 3. Ref. [3]     | $(SU(2) \times Z_{8,\Lambda}) \times Z_{2}^{[1]}$ | Preserved      | Preserved; Confined.             |                         | Obstruction |
| 4. Ref. [2]     | $(SU(2) \times Z_{8,\Lambda}) \times U(1)^{[1]} \times U(1)^{[1]}$ | Preserved      | Enhanced but SSB; Deconfined.    | Yes                    | Yes |

TABLE I. The Scenario 1, 2, 3, and 4 are from the list summarized in Sec. IV A 2. The “SSB” stands for “spontaneous symmetry breaking.” The $\left( \ldots \right)$ means that symmetry (...) leads to SSB. We find an obstruction for Scenario 3 based on the higher symmetry-extension construction alone. The Scenario 1 is consistent with the supersymmetry (SUSY) breaking of $N = 2$ SYM from the magnetic monopole point and dyon point (as 2 copies of CP$^1$ model). The Scenario 2 is consistent with the SUSY breaking of $N = 2$ SYM from the generic point $u \neq 0$, and $u \neq \pm \Lambda^2$. The Scenario 4 is consistent with the SUSY breaking of $N = 2$ SYM from the $u = 0$ with a self-dual coupling $\tau$.

**B. Deconfined Quantum Criticality in 3+1 Dimensions and More Comments**

In this work, we obtain a higher-symmetry extension generalization of Ref. [7]’s method to construct symmetric anomalous TQFT saturating higher 't Hooft anomalies. We have obtained a symmetric anomalous TQFT, valid for Scenario 2 from Cordova-Dumitrescu (Ref. [2]), see eq. (28) and eq. (29). However, we are unable to obtain a symmetric anomalous TQFT proposed by Scenario 3 motivated by Bi-Senthil (Ref. [3]) based on a symmetry-extension construction.

It is worthwhile to digest the exotic and interesting physics of Scenario 3 better. The Scenario 3 is motivated by the deconfined quantum criticality in 3+1 dimensions. It is proposed that a critical theory can be realized as a phase transition between two conventional Landau-Ginzburg symmetry-breaking orders [31], or a phase transition between two different SPT orders (see [3] and References therein). The adjoint QCD$_4$ is a UV description (UV side of eq. (1)) of the phase transition, while the IR description is currently unclear (IR side of eq. (1)).

The novelty of Scenario 3 is that the gapless sector is a free CFT as two free Weyl fermions (a single free Dirac fermion). So the hope is that the possible UV-IR duality eq. (1) in 3+1D is between a strongly coupled and interacting UV gauge theory and a free non-interacting massless IR theory, up to a gapped fully-symmetric TQFT sector to saturate the higher 't Hooft anomalies.

Our present work shows an obstruction for Scenario 3 from a symmetry-extension construction alone. The implications of our finding are follows:

I. We should remind the readers that the symmetry-extension construction is fairly general enough to saturate a large class of higher 't Hooft anoma-
lies of bosonic systems. Although the adjoint QCD$_4$ is a fermionic system (the UV completion requires fermionic degrees of freedom, where there are gauge-invariant fermionic operators), the Type I and II anomalies, eq. (2) and eq. (3), are bosonic anomalies in nature.

II. Despite the fact that fully-symmetric TQFT under Scenario 3 cannot be obtained via our symmetry-extension construction, we may still be able to use the symmetry-extension construction to derive other symmetric anomalous TQFTs, suitable to propose new candidate phases of other deconfined quantum criticality (dQCP), in 3+1 and other dimensions.

We should also notice that the recent numerical attempts [38, 39] suggest that the adjoint QCD$_4$ with SU(2) gauge group and $N_f$ number of adjoint Weyl fermions may have IR dynamics as follows:

- At $N_f = 2$, (as 1 adjoint Dirac fermion), according to [39], the IR theory may be very close to the onset of the conformal window, instead of the conventional confining behavior. In addition, the anomalous dimension of the fermionic condensate is reported to be close to 1. The numerical data seems to suggest the IR theory can be an interacting CFT (more exotic), instead of a free CFT (all the proposed scenarios so far, discussed in Table 1).

- At $N_f = 4$, (as 2 adjoint Dirac fermion), Ref. [38] discusses the candidate IR theory. Ref. [38] points out the theory is gapless (or massless), while future endeavor is required to distinguish whether it shows the confinement or the conformal behavior.

To unambiguously determine the IR dynamics, apart from the given numerical inputs [38, 39], we note that further lattice studies are still necessary.

Finally, we remark that many anomalies discussed in Sec. II B, following [2], are non-perturbative global anomalies instead of perturbative anomalies. The non-perturbative anomalies have classifications from finite groups (e.g. $\mathbb{Z}_n$ classes), instead of a $\mathbb{Z}$ classification. Examples include the old and the new SU(2) anomalies [15, 26], and also the recent higher 't Hooft anomalies of SU(N) YM gauge theory, see [35] and [12], and References therein. For these non-perturbative global anomalies, we can saturate certain 't Hooft anomalies of ordinary or higher global symmetries by symmetry-preserving TQFTs or so-called the long-ranged entangled topological order sectors, via our higher symmetry-extension approach, see a companion work along this direction [12].

V. ACKNOWLEDGMENTS

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Appendix A: Cobordism Theory and Higher Symmetry-Extension: Construction of Symmetric TQFTs

By the result of the page 251 of Ref. [40], the cohomology ring of the infinite Lens space $B\mathbb{Z}_2^n = S^\infty/\mathbb{Z}_2^n$ with coefficients $\mathbb{Z}_2^n$ is the polynomial ring generated by $a$ and $b$ over $\mathbb{Z}_2^n$ quotient by the relation $a^2 = 2^n-1 b$:

$$H^*(B\mathbb{Z}_2^n, \mathbb{Z}_2^n) = \mathbb{Z}_2^n[a,b]/(a^2 = 2^n-1 b) \text{ for } n \geq 2,$$

where $a \in H^1(B\mathbb{Z}_2^n, \mathbb{Z}_2^n)$, $b \in H^2(B\mathbb{Z}_2^n, \mathbb{Z}_2^n)$.

$$H^*(B\mathbb{Z}_2^n, \mathbb{Z}_2^n) = \Lambda_{\mathbb{Z}_2^n} \tilde{a} \otimes \mathbb{Z}_2^n [\tilde{b}] \text{ for } n \geq 2,$$  \hspace{1cm} (A2)

where $\tilde{a} = a \mod 2$, $\tilde{b} = b \mod 2$, there is a $(2,2^n)$-Bockstein $\beta_{(2,2^n)}$ with $\beta_{(2,2^n)}(a) = \tilde{b}$. Here $H^*$ is the cohomology ring, $\Lambda_{\mathbb{Z}_2^n}$ denotes the exterior algebra over $\mathbb{Z}_2^n$, $\otimes$ is the tensor product. The $(2, 2^n)$-Bockstein homomorphism $\beta_{(2,2^n)} : H^*(\cdot, \mathbb{Z}_2^n) \to H^{*+1}(\cdot, \mathbb{Z}_2^n)$ is associated to the extension $\mathbb{Z}_2 \to \mathbb{Z}_{2^{n+1}} \to \mathbb{Z}_{2^n}$.

Notice that the notation $a \in H^1(B\mathbb{Z}_2^n, \mathbb{Z}_2^n)$ and $b \in H^2(B\mathbb{Z}_2^n, \mathbb{Z}_2^n)$ will be abused later, since we will encounter the cases $n = 2$ and $n = 3$. We will use the uniform notation and explain wherever they appear.

1. Pullback trivialization of $AP_2(B_2)$ in $\Omega_{S^4}^{BM} \times_{S^2} (SU(2) \times S^2 \mathbb{Z}_2) (B^2 \mathbb{Z}_2)$

Follow the mathematical conventional notation, we will also denote the 5d topological term

$$A \cup \mathcal{P}_2(B_2) \text{ as } a \cup \mathcal{P}_2(x_2) \text{ (A3)}$$

in Appendix A and after. The $a$ here is a background probed field, which should not be confused with the SU(2) dynamical gauge field.
\textit{a. Computation}

\[
E_2^{s,t} = \operatorname{Ext}^{s,t}_{A_2}(H^*(MT(\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8))), Z_2) \otimes H^*(B^2 Z_2, Z_2) \Rightarrow \Omega_{t-s}^{\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8)}(B^2 Z_2). \quad (A4)
\]

Here Ext is the Ext functor, \( A_2 \) is the mod 2 Steenrod algebra, more precisely, \( \operatorname{Ext}^{s,t}_{A_2} \) is the internal degree \( t \) part of the \( s \)-th derived functor of \( \operatorname{Hom}^*_{A_2} \). \( MT(\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8)) \) is the Madsen-Tillmann spectrum of the group \( \operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8) \), the bordism group \( \Omega_{t-s}^{\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8)}(B^2 Z_2) = \pi_d(MT(\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8))) \wedge (B^2 Z_2)_+ \) is the stable homotopy group of the spectrum \( MT(\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8))) \wedge (B^2 Z_2)_+ \), here \( \wedge \) is the smash product, \( X_+ \) is the disjoint union of the space \( X \) and a point. “\( \Rightarrow \)” means “convergent to”.

For more detail, see [9].

Similarly as the discussion in [11, 41], we know

\[
MT(\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8)) = MSpin \wedge \Sigma^{-3} \operatorname{MSO}(3) \wedge (BZ_4)^{2\xi} \quad (A5)
\]

where \( 2\xi \) is twice the sign representation, \( (BZ_4)^{2\xi} \) is the Thom space \( \operatorname{Thom}(BZ_4; 2\xi) \), \( MSpin \) is the Thom spectrum of the group \( \operatorname{Spin} \), \( \operatorname{MSO}(3) \) is the Thom spectrum of the group \( \operatorname{SO}(3) \), \( \Sigma \) is the suspension.

Note that \( (BZ_4)^{2\xi} = \Sigma^{-2} MZ_4 \).

We have a homotopy pullback square

\[
\begin{array}{ccc}
\operatorname{B}(\operatorname{Spin} \times \mathbb{Z}_2 (SU(2) \times \mathbb{Z}_2 Z_8)) & \longrightarrow & \operatorname{BSO}(3) \times BZ_4(A6) \\
\downarrow w_2 & & \downarrow w_2 + \tilde{b} \\
\operatorname{BSO} & \longrightarrow & B^2 Z_2
\end{array}
\]

where \( \tilde{b} \) is the generator of \( H^2(BZ_4, Z_2) \), \( w_2 = w_2(TM) \) is the Stiefel-Whitney class of the tangent bundle \( TM \), \( w_2' = w_2'(\operatorname{SO}(3)) \) is the Stiefel-Whitney class of the universal \( \operatorname{SO}(3) \) bundle.

Hence we have the constraint

\[
w_2(TM) = w_2'(\operatorname{SO}(3)) + \tilde{b} \quad (A7)
\]

Since \( H^*(MSpin, Z_2) = A_2 \otimes A_2(1) \{ Z_2 \oplus M \} \) where \( A_2(1) \) is the subalgebra of \( A_2 \) generated by \( \operatorname{Sq}^1 \) and \( \operatorname{Sq}^2 \) and \( M \) is a M as a graded \( A_2(1) \)-module with the degree \( i \) homogeneous part \( M_i = 0 \) for \( i < 8 \).

For \( t-s < 8 \), we can identify the \( E_2 \) page with

\[
\begin{align*}
\operatorname{Ext}^{s,t}_{A_2(1)}(H^{+3}(\operatorname{MSO}(3), Z_2) \otimes H^{+2}(MZ_4, Z_2) \\
\otimes H^*(B^2 Z_2, Z_2), Z_2).
\end{align*} \quad (A8)
\]

\( H^{+3}(\operatorname{MSO}(3), Z_2) = Z_2[w'_2, w'_3]U \) where \( U \) is the Thom class and \( w'_i \) is the Stiefel-Whitney class of the universal \( \operatorname{SO}(3) \) bundle.

The \( A_2(1) \)-module structure of \( H^{+3}(\operatorname{MSO}(3), Z_2) \) is shown in Figure 4.

\[ \begin{array}{c}
\uparrow \\
\tilde{b}U
\end{array} \]

\[ \begin{array}{c}
\uparrow \\
\tilde{a}U
\end{array} \]

FIG. 4. The \( A_2(1) \)-module structure of \( H^{+3}(\operatorname{MSO}(3), Z_2) \). Each dot indicates a \( Z_2 \), the short straight line indicates a \( \operatorname{Sq}^2 \), the curved line indicates a \( \operatorname{Sq}^3 \).

\( H^{+2}(MZ_4, Z_2) = (Z_2[\tilde{b}] \otimes A_{Z_3}(\tilde{a})U) \) where \( U \) is the Thom class, \( \tilde{a} \) is the generator of \( H^1(BZ_4, Z_2) \), \( \tilde{b} \) is the generator of \( H^2(BZ_4, Z_2) \).

The \( A_2(1) \)-module structure of \( H^{+2}(MZ_4, Z_2) \) is shown in Figure 5.

\[ \begin{array}{c}
\uparrow \\
\tilde{b}U
\end{array} \]

\[ \begin{array}{c}
\uparrow \\
\tilde{a}U
\end{array} \]

FIG. 5. The \( A_2(1) \)-module structure of \( H^{+2}(MZ_4, Z_2) \). The dashed lines indicate a \( (2, 4) \)-Bockstein. Each dot indicates a \( Z_2 \), the short straight line indicates a \( \operatorname{Sq}^1 \), the curved line indicates a \( \operatorname{Sq}^2 \).

\( H^2(B^2 Z_2, Z_2) = Z_2[x_2, x_3, x_5, \ldots] \) where \( x_2 \) is the generator of \( H^2(B^2 Z_2, Z_2) \), \( x_3 = \operatorname{Sq}^1 x_2, x_5 = \operatorname{Sq}^2 x_3, \) etc.

The \( A_2(1) \)-module structure of \( H^*(B^2 Z_2, Z_2) \) is shown in Figure 6.

The \( A_2(1) \)-module structure of \( H^{+3}(\operatorname{MSO}(3), Z_2) \otimes H^{+2}(MZ_4, Z_2) \otimes H^*(B^2 Z_2, Z_2) \) is shown in Figure 7.

There is a differential \( d_2 \) corresponds to the \( (2, 4) \)-Bockstein [42] as indicated in Figure 5. There is also a differential \( d_2 \) maps \( x_2 x_3 + x_5 \) to \( x_2^2 h_0^2 \) since \( \beta_{(2,4)}(P_2(x_2)) = x_2 x_3 + x_5 \) [9]. Since \( \beta_{(2,4)}(aP_2(x_2)) = \tilde{b}x_2^2 + \tilde{a}(x_2 x_3 + x_5) \), there is a differential \( d_2 \) maps \( \tilde{b}x_2^2 + \tilde{a}(x_2 x_3 + x_5) \) to \( \tilde{a}x_2^2 h_0^2 \).

Note that the \( A_2(1) \)-module structure of \( H^{+3}(\operatorname{MSO}(3), Z_2) \otimes H^{+2}(MZ_4, Z_2) \) is contained in that
FIG. 6. The $A_2(1)$-module structure of $H^*(B^2Z_2, Z_2)$. Each dot indicates a $Z_2$, the short straight line indicates a $Sq^1$, the curved line indicates a $Sq^2$.

FIG. 7. The $A_2(1)$-module structure of $H^{*+3}(MSO(3), Z_2) \otimes H^{*+2}(MZ_4, Z_2) \otimes H^*(B^2Z_2, Z_2)$. Each dot indicates a $Z_2$, the short straight line indicates a $Sq^1$, the curved line indicates a $Sq^2$. Each label indicates its degree.

we draw the $E_2$ page for it individually in Figure 8. The rest part is shown in Figure 9.

FIG. 8. $\Omega^3_{Spin \times Z_2}(SU(2) \times Z_2Z_8)$. The arrows indicate differentials.

| $i$ | $\Omega_i^{Spin \times Z_2(SU(2) \times Z_2Z_8)}$ | cobordism invariants |
|-----|---------------------------------|----------------------|
| 0   | $Z$                             |                      |
| 1   | $Z_4$                           | $a$                  |
| 2   | $0$                             |                      |
| 3   | $Z_4$                           | $ab$                 |
| 4   | $Z^2$                           |                      |
| 5   | $Z \times Z_2 \times Z_4$       | $w'_2w'_3$, $ab^2$   |

TABLE II. Bordism group $\Omega_i^{Spin \times Z_2(SU(2) \times Z_2Z_8)}$ in dimensions $i$. Here $a$ is the generator of $H^1(BZ_4, Z_4)$, $b$ is the generator of $H^2(BZ_4, Z_4)$. $w'_i$ is the Stiefel-Whitney class of the universal $SO(3)$ bundle.

FIG. 9. $(\Omega^3_{Spin \times Z_2(SU(2) \times Z_2Z_8)}(B^2Z_2))/(\Omega^3_{Spin \times Z_2(SU(2) \times Z_2Z_8)})$. The arrows indicate differentials.
Now we determine the manifold generator of the $Z_4$-valued invariant $a \cup P_2(x_2)$.

$$\Omega^\text{Spin}_5(\text{Spin}(2) \times Z_2)(B^2Z_2)$$

Here bordism is an equivalence relation. $(M, f, g)$ and $(M', f', g')$ are bordant if there exists a 6-manifold $M$ and maps $F : M \to B(\text{Spin} \times Z_2 (\text{SU}(2) \times Z_2 Z_8))$, $G : M \to B^2Z_2$ such that the boundary of $M$ is the disjoint union of $M$ and $M'$ and the induced $\text{Spin} \times Z_2 (\text{SU}(2) \times Z_2 Z_8)$ structures on $M$ and $M'$ from that determined by $F$ on $M$ coincide with those determined by $f$ and $f'$ respectively, and $G|_M = g, G|_{M'} = g'$.

We have the homotopy pullback square (A6).

In order to give a map $f : M \to B(\text{Spin} \times Z_2 (\text{SU}(2) \times Z_2 Z_8))$, we need only give maps $f_1 : M \to BSO$, $f_2 : M \to BSO(3)$ and $f_3 : M \to BZ_4$ with $f_1(w_2) = f_2^*(w_2) + f_3^*(b)$.

The bordism invariant $a \cup P_2(x_2)$ is actually $f_3^*(a) \cup P_2(g^*(x_2)) = f_3 \cup P_2(g).

Now let $M$ be the Lens space $S^5/\mathbb{Z}_4$, $M$ is orientable but not spin.

Take $f_1 = TM$ (since $M$ is orientable, the tangent bundle $TM$ determines a map $M \to BSO$), $f_2 = 0$, $f_3$ be the generator of $H^1(M, Z_4)$.

By the cell structure of the Lens space, $f_3$ induces a chain map between the cellular chain complexes of $M$ and $BZ_4$, we draw the chain map below degree 2:

$$\begin{array}{ccc}
Z & \xrightarrow{4} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{4} & Z
\end{array}$$

\[ \xrightarrow{1} \]

So $f_3^*(b)$ is non-zero, since $f_1^*(w_2)$ is also non-zero, the cohomology group $H^2(M, Z_2)$ is $Z_2$, we have a commutative diagram

\[ M \xrightarrow{f_3} BZ_4 \]

\[ \xrightarrow{f_1} \]

\[ \text{BSO} \xrightarrow{w_2} B^2Z_2. \]

So we get a map $f : M \to B(\text{Spin} \times Z_2 (\text{SU}(2) \times Z_2 Z_8))$.

Take $g = w_2(TM)$.

$$\int_M f_3 \cup P_2(g) = 1 \mod 4. \quad (A12)$$

The partition function

$$Z(M) = i f_M f_3 \cup P_2(g) = 1. \quad (A13)$$

So $(M, f, g)$ is the manifold generator of the $Z_4$-valued invariant $f_3 \cup P_2(g)$.

### 2. Pullback trivialization

Consider the pullback of $B(\text{Spin} \times Z_2 (\text{SU}(2) \times Z_2 Z_8))$ to $B\text{Spin} \times B(\text{SU}(2) \times Z_2 Z_8)$:

$$BZ_2 \to B\text{Spin} \times B(\text{SU}(2) \times Z_2 Z_8) \to$$

$$B(\text{Spin} \times Z_2 (\text{SU}(2) \times Z_2 Z_8)). \quad (A14)$$

Since $w_2 = 0$ in Spin, so $w_2 x_3 = w_3 x_2$ is trivialized. Furthermore, consider the pullback of $B\text{Spin} \times B(\text{SU}(2) \times Z_2 Z_8)$ to $B\text{Spin} \times B\text{SU}(2) \times BZ_2$:

$$BZ_2 \to B\text{Spin} \times B\text{SU}(2) \times BZ_2 \to$$

$$B\text{Spin} \times B(\text{SU}(2) \times Z_2 Z_8). \quad (A15)$$

To simplify the computation, we only compute $\Omega^\text{Spin}_5(BZ_2 \times B^2Z_2)$ which is a subgroup of $\Omega^\text{Spin}_5(BSU(2) \times BZ_2 \times B^2Z_2)$.

Note that $P_2(x_2) = x_3^2 = Sq^2(x_2) = (w_1(TM) + w_2(TM)^2) x_2 = 0 \mod 2$ on Spin manifolds where we have used the Wu formula, so $P_2(x_2)$ can be divided by 2.

#### a. Computation

We have the Adams spectral sequence

$$E_2^{s,t} = Ext_{\mathbb{Z}_2}^{s,t}(H^*(M\text{Spin}, \mathbb{Z}_2) \otimes H^*(BZ_8, \mathbb{Z}_2)), \quad H^*(B^2Z_2, \mathbb{Z}_2) \Rightarrow \Omega^\text{Spin}_8(BZ_8 \times B^2Z_2). \quad (A16)$$

For $t - s < 8$,

$$\Ext_{\mathbb{Z}_2}^{s,t}(H^*(BZ_8, \mathbb{Z}_2) \otimes H^*(B^2Z_2, \mathbb{Z}_2))$$

$$\Rightarrow \Omega^\text{Spin}_{s+t}(BZ_8 \times B^2Z_2). \quad (A17)$$

\[ \text{Table III. Bordism group } \Omega^\text{Spin}_5(\text{Spin}(2) \times Z_2)(B^2Z_2) \text{ in dimensions } i. \] Here $a$ is the generator of $H^1(BZ_4, \mathbb{Z}_2)$, $b$ is the generator of $H^2(BZ_4, \mathbb{Z}_2)$. $\tilde{a} = a \mod 2$, $\tilde{b} = b \mod 2$. $w_2 = w_2(TM)$ is the Stiefel-Whitney class of the universal $SO(3)$ bundle.
The $A_2(1)$-module structure of $H^*(B\mathbb{Z}_8, \mathbb{Z}_2) \otimes H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2)$ is shown in Figure 10.

Note that the $A_2(1)$-module structure of $H^*(B\mathbb{Z}_8, \mathbb{Z}_2)$ is contained in that of $H^*(B\mathbb{Z}_8, \mathbb{Z}_2) \otimes H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, we draw the $E_2$ page for it individually in Figure 11. The rest part is shown in Figure 12.

There is a differential $d_3$ corresponding to the $(2, 8)$-Bockstein [42] as indicated in Figure 10 and a differential $d_2$ corresponding to the $(2, 4)$-Bockstein $\beta_{(2,4)}(P_2(x_2)) = x_2 x_3 + x_5$.

One $\mathbb{Z}_2$-valued bordism invariant of $\Omega Spin(B\mathbb{Z}_8 \times B\mathbb{Z}_8)$ is $\tilde{a} \cup \frac{P_2(x_2)}{2}$. Here $\tilde{a}$ is the generator of $H^1(B\mathbb{Z}_8, \mathbb{Z}_2)$, $x_2$ is the generator of $H^2(B\mathbb{Z}_8, \mathbb{Z}_2)$.

b. Further trivialization: first approach

Define $G$ to be a group which sits in a homotopy pullback square

\[
\begin{array}{ccc}
B\mathbb{G} & \longrightarrow & B^2\mathbb{Z}_2 \\
\downarrow & & \downarrow \\
B(Spin \times_{\mathbb{Z}_2} (SU(2) \times_{\mathbb{Z}_2} \mathbb{Z}_8)) & \longrightarrow & B^2\mathbb{Z}_2 \\
\downarrow j_1 & & \downarrow j_2 \\
B\mathbb{S}O & \longrightarrow & B\mathbb{S}O(3) \times \mathbb{B}\mathbb{Z}_4 \\
\end{array}
\]

where $j_1^*(w_2) = j_2^*(w_2' + \tilde{b})$, $\tilde{b}$ is the generator of $H^2(B\mathbb{Z}_4, \mathbb{Z}_2)$, $w_2 = w_2(TM)$ is the Stiefel-Whitney class of the tangent bundle $TM$, $w_2' = w_2'(SO(3))$ is the Stiefel-Whitney class of the universal $SO(3)$ bundle.

In general, if we have a homotopy pullback square

\[
\begin{array}{ccc}
P & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z,
\end{array}
\]

then there is a fiber sequence

\[
\Omega Z \rightarrow P \rightarrow X \times Y \rightarrow Z
\]
Here we proposed that this 5d cobordism invariant of $Z_{16}$ is the Postnikov square. A caveat is that we have not yet been able to fully verify the Postnikov square is the only possibility, although partial evidences suggest this to be true. The readers need to be cautious using this particular result. The verification is left for future work.

TABLE V. Bordism group $\Omega_i^{Spin}(BZ_8 \times B^2Z_2)$ in dimensions $i$. Here $\tilde{\eta}$ is the 1d eta invariant, $\text{Arf}$ is the Arf invariant, $\Psi$ is the Postnikov square. $a$ is the generator of $H^1(BZ_8, Z_8)$, $b$ is the generator of $H^2(BZ_8, Z_8)$. $\tilde{a} = a \ mod \ 2$, $\tilde{b} = b \ mod \ 2$.

where $\Omega Z$ is the loop space of $Z$.

So there is a fiber sequence

$$BZ_2 \rightarrow B\tilde{G} \rightarrow B(\text{Spin} \times_{Z_2} (SU(2) \times_{Z_2} Z_8)) \times B^2Z_2 \rightarrow B^2Z_2$$

(A21)

where the last map is $(u, v) \mapsto j_1^*(w_2)(u) - x_2(v) = j_2^*(w_2' + \tilde{b})(u) - x_2(v)$.

Then we define $G'$ to be a group which sits in a homotopy pullback square

$$BZ_2 \rightarrow B\tilde{G}' \rightarrow B\text{Spin} \times B(SU(2) \times_{Z_2} Z_8) \times B^2Z_2 \rightarrow B^2Z_2$$

(A22)

Since $w_2$ is identified with $x_2$ in $BG$, it is trivialized in $BG'$ because $x_2 = w_2 = 0$ due to the spin structure, so $a \cup \mathcal{P}_2(x_2)$ is clearly trivialized by being pulled back to $\Omega_5'$.

Although our starting point was the symmetry-extension, this is a symmetry breaking case in disguise.

\[c. \text{Further trivialization: second approach}\]

Define $G$ to be a group which sits in a homotopy pullback square

$$BG \rightarrow B^2Z_2 \quad (A23)$$

$$B(\text{Spin} \times SU(2) \times Z_8) \xrightarrow{j_3} BZ_8 \xrightarrow{\tilde{b}} B^2Z_2.$$
In general, if we have a homotopy pullback square

\[
P \longrightarrow Y \\
\downarrow \quad \downarrow \\
X \longrightarrow Z,
\]

then there is a fiber sequence

\[
\Omega Z \to P \to X \times Y \to Z
\]

where \(\Omega Z\) is the loop space of \(Z\).

So there is a fiber sequence

\[
\mathrm{B}Z_2 \to \mathrm{B}G \to \mathrm{B}(\mathrm{Spin} \times \mathrm{SU}(2) \times \mathbb{Z}_8) \times \mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2\mathbb{Z}_2
\]

where the last map is \((u, v) \mapsto j_2^8(b)(u) - x_2(v)\).

Since \(P_2(x_2) = x_2 \cup x_2 + x_2 \cup \delta x_2\), \(\delta x_2 = 2\mathrm{Sq}^1 x_2\), \(x_2 \cup x_2 = x_2\), so \(\frac{P_2(x_2)}{2} = x_2^2 + x_2 \cup \mathrm{Sq}^1 x_2\).

Since \(x_2\) is identified with \(\tilde{b} = \beta_{(2,8)}a\) in \(\mathrm{B}G\) where \(a \in H^1(\mathrm{B}Z_8, \mathbb{Z}_8)\) and \(\mathrm{Sq}^1 \beta_{(2,8)} = 0\) [9], so \(\tilde{a} \cup (x_2 \cup \mathrm{Sq}^1 x_2)\) is trivialized in \(\Omega Z\).

Note that \(\tilde{a} \cup \frac{x_2^2}{2}\) is still not trivialized.

This is also a symmetry breaking case, since \(x_2\) is locked with \(a\). In physics, the locking between two probed background fields means that the global symmetry between two sectors are locked together, thus which results in global symmetry breaking.

d. Further trivialization: third approach

Consider the pullback of \(\mathrm{B}^2\mathbb{Z}_2\) to \(\mathrm{B}^2\mathbb{Z}_4\):

\[
\mathrm{B}^2\mathbb{Z}_2 \to \mathrm{B}^2\mathbb{Z}_4 \to \mathrm{B}^2\mathbb{Z}_2.
\]

Since \(\frac{P_2(x_2)}{2} = x_2^2 + x_2 \cup \mathrm{Sq}^1 x_2\), \(x_2 \in H^2(\mathrm{B}^2\mathbb{Z}_2, \mathbb{Z}_2)\) is pulled back to \(\tilde{x}_2 \in H^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2)\) and the following diagram

\[
\begin{array}{ccc}
H^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) & \xrightarrow{\mathrm{Sq}^1} & H^3(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) \\
\downarrow{\beta} & & \downarrow{\mathrm{id}} \\
H^2(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2) & \cong & H^3(\mathrm{B}^2\mathbb{Z}_4, \mathbb{Z}_2)
\end{array}
\]

is commutative by the naturality of Bockstein homomorphism, we have \(\mathrm{Sq}^1 \tilde{x}_2 = 0\), so \(\tilde{a} \cup (x_2 \cup \mathrm{Sq}^1 x_2)\) is trivialized in \(\Omega^5_{\mathrm{Spin}}(\mathrm{B}Z_8 \times \mathrm{B}^2\mathbb{Z}_4)\).

Note that \(\tilde{a} \cup \frac{x_2^2}{2}\) is still not trivialized.

d. Summary

The term \(\tilde{a} \cup \frac{x_2^2}{2}\) cannot be trivialized.

Consider \(M = S^1 \times S^2 \times S^2\), the partition function

\[
Z(M) = (-1)^k f_{M} \tilde{a} \cup \frac{x_2^2}{2} = (-1)^k f_{M} \tilde{a} \cup x_2^2.
\]

Since

\[
H^n(S^2 \times S^2, \mathbb{Z}) = \begin{cases}
\mathbb{Z} & n = 2 \text{ or } 3, n \geq 4 \\
\mathbb{Z} & n = 0, 4
\end{cases}
\]

where the two generators of \(H^2(S^2 \times S^2, \mathbb{Z})\) are \(a, b\), the generator of \(H^4(S^2 \times S^2, \mathbb{Z})\) is \(ab\).

No matter how to pullback, when \(x_2 = a + b \mod 2\), \((-1)^k f_{M} \tilde{a} \cup x_2^2 = (-1)^k\) can be nontrivial.

This conclusion will be stated more formally and proved in the next appendix.

In this appendix, we compute the bordism group \(\Omega^5_{\mathrm{Spin}}(\mathrm{B}Z_8 \times \mathrm{B}^2\mathbb{Z}_2)\) and find a bordism invariant \(a \cup P_2(x_2)\) of it, then we can find the manifold generator of \(a \cup P_2(x_2)\), and consider the pullback trivialization problem of \(a \cup P_2(x_2)\), we first compute the bordism group \(\Omega^5_{\mathrm{Spin}}(\mathrm{B}Z_8 \times \mathrm{B}^2\mathbb{Z}_2)\), and find that \(a \cup P_2(x_2)\) becomes \(a \cup \frac{P_2(x_2)}{2}\) in \(\Omega^5_{\mathrm{Spin}}(\mathrm{B}Z_8 \times \mathrm{B}^2\mathbb{Z}_2)\). Moreover, we find that the summand \(\tilde{a} \cup (x_2 \cup \mathrm{Sq}^1 x_2)\) of \(\tilde{a} \cup \frac{x_2^2}{2}\) can be trivialized \((P_2(x_2) = x_2^2 + 2x_2 \cup \mathrm{Sq}^1 x_2)\), but \(\tilde{a} \cup \frac{x_2^2}{2}\) cannot be trivialized. We conclude that \(a \cup P_2(x_2)\) cannot be trivialized via via extending the global symmetry by 0-form symmetry and 1-form symmetry.

Appendix B: Proof: a counterexample

By direct computation, we find that \(a \cup P_2(x_2)\) is a bordism invariant of \(\Omega^5_{\mathrm{Spin} \times \mathrm{B}(\mathrm{Spin} \times \mathrm{SU}(2) \times \mathbb{Z}_8 \times \mathbb{Z}_8)}(\mathrm{B}^2\mathbb{Z}_2)\).

We consider the trivialization problem: Can we trivialize the topological term \(a \cup P_2(x_2)\) via extending the global symmetry by 0-form \(K_{[0]}\) symmetry and 1-form \(K_{[1]}\) symmetry?

We can reframe it mathematically: Can we find finite abelian groups \(K_{[0]}\) and \(K_{[1]}\) such that

\[
\begin{aligned}
\text{BK}_{[0]} \times \text{B}K_{[1]} & \to \text{BG} \xrightarrow{\phi} \text{B}(\mathrm{Spin} \times \mathrm{Z}_2, (\mathrm{SU}(2) \times \mathrm{Z}_8) \times \mathrm{B}^2\mathbb{Z}_2) \\
(M, g) & \to (M, f g)
\end{aligned}
\]

is a fibration and \((fg)^* (a \cup P_2(x_2)) = 0\) for any 5-manifold \(M\) and any map \(g : M \to \text{BG}\).

There is a group homomorphism:

\[
\begin{aligned}
\Omega^5_{\mathrm{Spin} \times \mathrm{B}(\mathrm{Spin} \times \mathrm{SU}(2) \times \mathrm{Z}_8) \times \mathrm{B}^2\mathbb{Z}_2} & \to (M, g) \\
(M, g) & \to (M, f g)
\end{aligned}
\]

So the trivialization problem is asking whether we can find \(f\) and \(s\) such that \(\phi^* (a \cup P_2(x_2)) = 0\) for any \((M, g) \in \Omega^5_{\mathrm{Spin}}\).

By direct computation, we find that \(a \cup P_2(x_2)\) becomes \(a \cup \frac{P_2(x_2)}{2}\) in \(\Omega^5_{\mathrm{Spin}}(\mathrm{B}Z_8 \times \mathrm{B}^2\mathbb{Z}_2)\).
Our main result is

Claim 1: We cannot find finite abelian groups \( K[0] \) and \( K[1] \) such that

\[
BK[0] \times B^2K[1] \to BG \overset{f}{\to} B\text{Spin} \times BSU(2) \times BZ_8 \times B^2Z_2
\]

is a fibration and \((fg)^*(\alpha \cup \mathcal{P}_2(x_2)) = 0\) for any 5-manifold \( M \) and any map \( g : M \to BG \).

Claim 2: We cannot find finite abelian groups \( K[0] \) and \( K[1] \) such that

\[
BK[0] \times B^2K[1] \to BG \overset{f}{\to} B\text{Spin} \times BSU(2) \times BZ_8 \times B^2Z_2
\]

is a fibration and \((fg)^*(\alpha \cup \mathcal{P}_2(x_2)) = 0\) for any 5-manifold \( M \) and any map \( g : M \to BG \).

To prove Claim 2, we need only prove

\[
\text{Claim 3: We cannot find finite abelian groups } K[0] \text{ and } K[1] \text{ such that }
BK[0] \times B^2K[1] \to BG \overset{f}{\to} B\text{Spin} \times BSU(2) \times BZ_8 \times B^2Z_2
\]

is a fibration and \((fg)^*(\alpha \cup \mathcal{P}_2(x_2)) = 0\) for any Spin 5-manifold \( M \) and any map \( g : M \to BG \).

We prove Claim 3 by finding a counterexample.

For \( M = S^1 \times S^2 \times S^2 \), let \( a, b \) be the generators of \( H^2(S^2 \times S^2, \mathbb{Z}_8) \), \( c \) the generator of \( H^1(S^1, \mathbb{Z}_8) \), and \( h : M \to BZ_8 \times B^2Z_2 \) by \( (c, a + b) \). The lifting problem

\[
g \quad \text{cannot exist, but } (c \mod 2) \cup \mathcal{P}_2(a+b) \neq 0.
\]

In general, if \( F \to E \overset{p}{\to} B \overset{q}{\to} X \) is a fiber sequence, then \([M, F] \to [M, E] \overset{p}{\to} [M, B] \overset{q}{\to} [M, \Sigma F] \) is an exact sequence of abelian groups, so the lifting problem has a solution if and only if \( q_s(h) = 0 \) where \( q : BZ_8 \times B^2Z_2 \to B^2K[0] \times B^4K[1] \). We need prove that \( q \circ h = 0 \). We can write \( q = (q_1, q_3, q_2, q_4) \) where

\[
q_1 \in H^2(BZ_8, K[0]), \quad q_2 \in H^2(B^2Z_2, K[0]), \quad q_3 \in H^3(BZ_8, K[1]), \quad q_4 \in H^3(B^2Z_2, K[1]).
\]

\( q \circ h = (q_1 \circ c + q_2 \circ (a+b), q_3 \circ c + q_4 \circ (a+b)) \), we need only prove \( q_2 \circ (a+b) = 0 \) since other terms vanish for \( M = S^1 \times S^2 \times S^2 \). We assume that \( q_2 = 0 \) to ensure that the 1-form symmetry (here the 1-form \( Z_2 \)-symmetry) is not broken.

In this appendix, we give a proof of the conclusion in the previous appendix. This answers the first question (Question 1) in Sec. I.

**Appendix C: Pullback trivialization of \( \mathcal{P}_2(B_2) \) in \( \Omega^4_{SO}(B^2Z_2) \)**

There is a group homomorphism:

\[
\Omega^4_{SO}(X) \overset{\rho}{\to} \Omega^4_{SO}(B^2Z_2) \quad (M, g) \mapsto (M, fg)
\]

We want to extend the 1-form \( Z_2 \) symmetry by 0-form \( K[0] \) symmetry and 1-form \( K[1] \) symmetry such that \( \rho^* \mathcal{P}_2(g) = \mathcal{P}_2(fg) = 0 \) for any \( (M, g) \in \Omega^4_{SO}(X) \) where \( (M, g) \in \Omega^4_{SO}(B^2Z_2) \).

We consider the trivialization problem: Does there exist a fibration \( f : X \to B^2Z_2 \) with fiber \( BK[0] \times B^2K[1] \)?
where $K_{[0]}$ and $K_{[1]}$ are finite abelian groups such that $P_2(fg) = 0$ for any oriented 4-manifold $M$ and any map $g : M \to X$.

The answer to this problem is negative, for $M = S^2 \times S^2$, let $a, b$ be the generators of $H^3(S^2 \times S^2, \mathbb{Z}_2)$. The lifting problem

\[
\begin{array}{ccc}
S^2 \times S^2 & \overset{g}{\longrightarrow} & \mathbb{B}^2 \mathbb{Z}_2 \\
\downarrow & \searrow f & \downarrow \uparrow \mathbb{Z}_2 \\
X & \stackrel{X}{\longrightarrow} & \end{array}
\]  

always has a solution, but $P_2(fg) = P_2(a + b)$ is non-trivial. Similarly as before, we need only prove that the composition map $S^2 \times S^2 \overset{a+b}{\longrightarrow} \mathbb{B}^2 \mathbb{Z}_2 \overset{q}{\longrightarrow} \mathbb{B}^3 K_{[0]} \times \mathbb{B}^3 K_{[1]}$ is zero. Similarly as before, we write $q'$ as $q' = (q_2, q_4)$. We assume that $q_2 = 0$ to ensure that the 1-form symmetry (here the 1-form $Z_2$-symmetry) is not broken. Note that $q_4 \in H^3(\mathbb{B}^2 \mathbb{Z}_2, K_{[1]})$, clearly $q_4 \circ (a + b) = 0$ on $S^2 \times S^2$ since $H^3(S^2 \times S^2, K_{[1]}) = 0$.

So $P_2(x_2)$ cannot be trivialized.

In this appendix, we consider the pullback trivialization problem of $P_2(x_2)$, we give a similar proof that $P_2(x_2)$ also cannot be trivialized via via extending the global symmetry by 0-form symmetry and 1-form symmetry. This answers the second question (Question 2) in Sec. I.

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