On the value of the non-perturbative field renormalization constant $Z$ in gauge theories

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Abstract

We show that the rigorous nonperturbative field strength (or wavefunction) renormalization $Z$, contrary to general belief, is not necessarily equal to zero in case the two-point function of an interacting quantum field theory is, as expected, more singular at the light-cone (with proper definitions) than the corresponding free field two-point function. However, if (massless) photons or composite (unstable) particles are present, the condition $Z = 0$ should hold true due to arguments seldom presented in the literature. In fact, we argue (within a natural framework for gauge theories) that the condition $Z = 0$ is a universal feature of realistic models of elementary particle physics, which include massless or unstable particles.

1 Introduction

In his recent recollections, ’t Hooft ([Hoo], Sect. 5) emphasizes that an asymptotic (divergent) series, such as the power series for the scattering (S) matrix in the coupling constant $\alpha = \frac{1}{137}$ in quantum electrodynamics ($\text{qed}_{1+3}$) does not define a theory rigorously (or mathematically). In the same token, Feynman was worried about whether the qed S matrix would be “unitary at order 137” ([Wig67a], discussion on p. 126), and in Section 5 of [Hoo], ’t Hooft remarks that the uncertainties in the S matrix amplitudes at order 137 for qed

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are comparable to those associated to the “Landau ghost” (or pole) \[\text{Lan55}\].

The situation is considerably worse for strong interactions. Except for certain processes accounted for by renormalization group techniques \[\text{Wei96a}\], the only approach to such fundamental issues such as the proton-neutron mass difference \[\text{ca}\] or the magnetic moment of the neutron \[\text{ea14}\] has remained that of numerical lattice gauge theory (LGT). However, LGT is only an effective theory in the sense of \[\text{Wei96b}\], p. 523. For such theories many of the pleasant features of quantum field theory, such as invariance or symmetry properties, are lost.

The main reason to believe in quantum field theory (see also \[\text{Wig79}\]) is, therefore, strongly tied to non-perturbative approaches. This is, in particular, true for quantum electrodynamics in the Coulomb gauge in three space dimensions - \(\text{qed}_{1+3}\) - for which a recent result \[\text{JW18}\] establishes positivity of the (renormalized) energy, uniformly in the volume \(V\) and ultraviolet \(\Lambda\) cutoffs. This stability result may indication the absence of Landau poles or ghosts in \(\text{qed}_{1+3}\) (see also \[\text{Frö}\]). In order to achieve this, the theory in Fock space (for fixed values of the cutoffs) is exchanged for a formulation in which (in the words of Lieb and Loss \[\text{LL02}\]) “the electron Hilbert space is linked to the photon Hilbert space in an inextricable way”. Thereby, “dressed photons” and “dressed electrons” arise as new entities. For \(\text{qed}_{1+3}\) (as considered in \[\text{JW18}\]), a non-Fock representation should arise when the cutoffs are removed (see, e.g., \[\text{Wig67b}\] or \[\text{Wig79}\] for comprehensive, still very readable, discussions).

## 2 The Field Algebra

In order to formulate the above-mentioned phenomenon, we consider a field algebra \(\mathcal{F}^0 \equiv \{\bar{A}^0_\mu, \psi^0, \bar{\psi}^0\}\) (containing the identity) generated by the free vector potential \(\bar{A}^0_\mu, \mu = 0, 1, 2, 3, \) and the electron-positron fields \(\psi^0, \bar{\psi}^0\) . We may assume that the field algebra is initially defined on the Fock-Krein (in general, indefinite-metric, see \[\text{Bog74}\]) tensor product of photon and fermion Fock spaces. However, of primary concern for us will be an inequivalent representation of the field algebra \(\mathcal{F}^0\) on a (physical) Hilbert space \(\mathcal{H}\), with generator of time-translations - the physical Hamiltonian \(H\) - satisfying positivity, i.e.,

\[
H \geq 0 ,
\]

(1)
and such that
\[ H\Omega = 0 , \tag{2} \]
where \( \Omega \in \mathcal{H} \) is the vacuum vector, and
\[
A_\mu \equiv A_\mu^0(A_\mu^0, \psi^0, \bar{\psi}^0) , \tag{3}
\]
\[
\Psi \equiv \Psi(A_\mu^0, \psi^0, \bar{\psi}^0) , \tag{4}
\]
\[
\bar{\Psi} \equiv \bar{\Psi}(A_\mu^0, \psi^0, \bar{\psi}^0) . \tag{5}
\]

### 2.1 Gauge transformations

We may define, as usual, the (restricted) class of c-number \( U(1) \) local gauge transformations, acting on \( F_0^0 \), by the maps
\[
A_\mu^0(f) \rightarrow A_\mu^0(f) + c \langle f, \partial^\mu u \rangle , \tag{6}
\]
\[
\psi^0(f) \rightarrow \psi^0(e^{iu}f) , \tag{7}
\]
\[
\bar{\psi}^0(f) \rightarrow \bar{\psi}^0(e^{-iu}f) , \quad f \in \mathcal{S}(\mathbb{R}^{1+s}) . \tag{8}
\]

In QED, \( c = \frac{1}{e} \) and \( u \) satisfies certain regularity conditions, which guarantee that
\[
e^{\pm iuf} \in \mathcal{S}(\mathbb{R}^{1+s}) \text{ if } f \in \mathcal{S}(\mathbb{R}^{1+s}) , \quad \text{and} \quad |\langle f, \partial^\mu u \rangle| < \infty .
\]

In the following, \( \mathcal{S} \) denotes Schwartz space (see, e.g., [BB03]), \( \langle \cdot , \cdot \rangle \) denotes the \( L^2(\mathbb{R}^{1+s}) \) scalar product and \( s \) is the space dimension.

The **observable algebra** is assumed to consist of gauge-invariant objects, namely the fields
\[
F_{\mu,\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \tag{9}
\]
with \( A_\mu \) given by (3), describing the dressed photons, and gauge-invariant quantities \( \Psi, \bar{\Psi} \) in [11], [15], which create-destroy electrons-positrons “with their photon clouds”.

While we hope that the results in [JW18] will eventually lead to an (implicit or explicit) expression for \( \Psi, \bar{\Psi} \), it should be emphasized that this is a very difficult, open problem; see the important work of Steinmann in perturbation theory [Ste84]. We also note that we shall only consider the vacuum sector, i.e., only the combinations
\[
A(f,g) \equiv \bar{\Psi}(f)\Psi(g) \text{ with } f, g \in \mathcal{S}(\mathbb{R}^{1+s}) ,
\]
\[
B(f,g) \equiv \Psi(f)\bar{\Psi}(g) \text{ with } f, g \in \mathcal{S}(\mathbb{R}^{1+s}) , \tag{10}
\]
will occur in our two-point functions. The existence of charged sectors is a related open problem, which will not concern us.

The theory could be given by its \textit{n-point Wightman functions} [SW64] or, alternatively, by a Haag-Kastler net [HK64]. In fact, it has been shown in [HK64] that the physical content of a theory can be expressed in terms of its observable algebra. In the case of gauge theories, the latter corresponds to the algebra generated by gauge invariant quantities [SW74]. Hence, as Lowenstein and Swieca [LS71] argued, the representations of the observable algebra should not depend on the field algebra, from which it is constructed. Thus, \textit{n}-point functions constructed over the observable algebra should be the same in all gauges. These remarks seem to justify the usage of non-covariant gauges, which, as we shall see, are of particular importance in a non-perturbative framework. For scattering theory and particle concepts within a theory of local observables, see [AH67], [BPS91], and [BS05] for a lucid review.

There exist various arguments supporting the use of non-covariant gauges in relativistic quantum field theory: they are of both physical and mathematical nature. In part one of his treatise, Weinberg notes ([Wei96b], p. 375, Ref. 2): “the use of Coulomb gauge in electrodynamics was strongly advocated by Schwinger [Sch63b] on pretty much the same grounds as here: that we ought not to introduce photons with helicities other than $\pm 1$”. Indeed, as shown by Strocchi [Str70], a framework excluding “ghosts” necessarily requires the use of non-manifestly covariant gauges, such as the Coulomb gauge in $\text{qed}_{1+3}$, the Weinberg or unitary gauge in the Abelian Higgs model [Wei73], and the Dirac [Dir49] or light-cone gauge in quantum chromodynamics [SB02]. Another instance of the physical-mathematical advantage of a non-covariant gauge is the “$\alpha = \sqrt{\pi}$” gauge in massless $\text{qed}_{1+1}$ - the Schwinger model [Sch62] - see [LS71], [RW81]. As in the Coulomb gauge in $\text{qed}_{1+3}$, there is no need for indefinite metric in this gauge, \textit{i.e.}, the zero-mass longitudinal part of the current is gauged away, and one has a solution of Maxwell’s equations (as an operator-valued, distributional identity)

$$\partial_\nu F^{\mu,\nu}(x) = -e j^\mu(x)$$ (11)

on the whole Hilbert space. This is an important ingredient in Buchholz’s theorem [Buc86], to which we come back in the sequel.

The structure of the observable algebra is quite simple in the Coulomb gauge: the field (2) is just the electric field, which is defined in terms of a
massive scalar field, the quantities (10) are, in this gauge, rigorous versions of the (path-dependent) quantities

$$\psi(x) e^{ie \int_x^y dt A^\mu(t)} \psi^*(y)$$

and their adjoints (in the distributional sense), see [LS71] and [RW81]. In the case of $\text{qed}_{1+3}$, such quantities are plagued by infrared divergences, see the discussion in [Ste84]. As a consequence of the simple structure of the observable algebra, one arrives at physically and mathematically sensible picture for spontaneous symmetry breaking ([LS71], [RW81]); in covariant gauges this picture is masked by the presence of spurious gauge excitations.

In [SW64], pp. 107-110, it is shown that if the (n-point) Wightman functions satisfy

a.) the relativistic transformation law;

b.) the spectral condition;

c.) hermiticity;

d.) local commutativity;

e.) positive-definiteness,

then they are the vacuum expectation values of a field theory satisfying the Wightman axioms, except, eventually, uniqueness of the vacuum state. We refer to [SW64] or [RS75] for an account of Wightman theory, and for the description of these properties. It has been shown in [LS71], [RW81] that $\text{qed}_{1+1}$ in the "$\alpha = \sqrt{\pi}$" gauge satisfies a.) – e.). The crucial positive-definiteness condition e.) has been shown in [LS71] to be a consequence of the positive-definiteness of a subclass of the n-point functions of the Thirring model [Thi58] in the formulation of Klaiber [Kla68]. Positive-definiteness of the Klaiber n-point functions was rigorously proved by Carey, Ruijsenaars and Wright [CJW85]. Unicity of the vacuum holds in each irreducible subspace of the (physical) Hilbert space $\mathcal{H}$ [RW81], as a result of the cluster property; see also [LS71].

For $\text{qed}_{1+3}$ in the Coulomb gauge, we shall assume a.) – e.) for the n-point functions of the observable fields. We conjecture that this framework is also adequate for other relativistic quantum gauge field theories, as previously discussed. Concerning the case of $\text{qed}_{1+3}$ in the Coulomb gauge,
for both the electron and the photon propagators in perturbation theory, dynamics “restores” Lorentz covariance, \( i.e. \), the instantaneous Coulomb interaction and the transverse part combine to yield a Lorentz invariant expression (see \[Sak67\], Sections 4-4 and 4-6). Assuming Lorentz covariance, positivity of the energy \((1)\) in the physical Hilbert space \( \mathcal{H} \) yields the spectral condition \(b.\). The crucial mathematical reason for choosing a non-covariant gauge is, as we shall see, the positive-definiteness condition \(e.\). We shall assume uniqueness of the vacuum state; this may be achieved by restricting to an irreducible component of \( \mathcal{H} \), as in \( \text{qed}_{1+1} \).

2.2 The Källén-Lehmann representation

In every quantum (field) theory, it is useful to distinguish between static properties (which are invariant under the time evolution) and dynamical properties (for instance those, which depend on the outcome of a scattering process). Among the former is the joint spectrum of the energy-momentum operator \( P^\mu, \mu = 0, 1, 2, 3, \) about which Maison’s theorem \[\text{Mai68}\], which will be needed in the sequel, states:

**Theorem 2.1** (Maison \[\text{Mai68}\]). In a Poincaré or Galilei invariant quantum (field) theory, the joint energy-momentum spectrum in the orthogonal complement of the vacuum vector is absolutely continuous.

A major dynamical issue in quantum field theory is the (LSZ or Araki-Haag) asymptotic condition (see, \( e.g. \), \[BS05\] and references given there), which relates the theory, whose objects are the fundamental observable fields, to particles, described by physical parameters (mass and charge in \( \text{qed} \)). This issue is equivalent to the renormalization (or normalization) of perturbative quantum field theory, which itself is related to the construction of continuous linear extensions of certain functionals, such as to yield well-defined Schwartz distributions (see \[Sch01\] and references given there). On the other hand, in a non-perturbative framework, a theory of renormalization of masses and fields also exists, and “has nothing directly to do with the presence of infinities” (\[Wei96\], p. 441, Sect. 10.3). We adopt a related proposal, which we formulate here, for simplicity, for a theory of a self-interacting scalar field \( A \) of mass \( m \) satisfying the Wightman axioms (modifications are mentioned in the sequel). We assume that \( A \) is an operator-valued tempered distribution on the Schwartz space \( \mathcal{S} \) (see \[RS75\], Ch. IX).
We have the following result, concerning the spectral representation of the two-point function $W_2$ ([RS75], p. 70, Theorem IX-34):

**Theorem 2.2** (The Källén-Lehmann representation).

$$W_2^m(x-y) = \langle \Omega, A(x)A(y)\Omega \rangle = \frac{1}{i} \int_0^\infty d\rho(m_\circ^2) \Delta_+^{m_\circ}(x-y),$$  \hspace{1cm} (13)

where $\Omega$ denotes the vacuum vector, $x = (x_0, \vec{x})$, and

$$\Delta_+^{m_\circ}(x) = \frac{i}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k e^{-ix_0 \sqrt{m_\circ^2 + k^2} + i\vec{x} \cdot \vec{k}}$$  \hspace{1cm} (14)

is the two-point function of the free scalar field of mass $m_\circ$, and $\rho$ is a polynomially-bounded measure on $[0, \infty)$, i.e.,

$$\int_0^L d\rho(m_\circ^2) \leq C(1 + L^N)$$  \hspace{1cm} (15)

for some constants $C$ and $N$. It is further assumed that

$$\langle \Omega, A(f)\Omega \rangle = 0 \quad \forall f \in \mathcal{S}.$$  \hspace{1cm} (16)

Note that (13) is symbolic; for its proper meaning, which relies on (15), see [RS75].

**Proposition 2.3.** For a scalar field of mass $m \geq 0$, the measure $d\rho(m_\circ^2)$ appearing in the Källén-Lehmann spectral representation allows a decomposition

$$d\rho(m_\circ^2) = Z\delta(m_\circ^2 - m^2) + d\sigma(m_\circ^2),$$  \hspace{1cm} (17)

where

$$0 < Z < \infty$$  \hspace{1cm} (18)

and

$$\int_0^L d\sigma(a^2) \leq C_1(1 + L^{N_1})$$  \hspace{1cm} (19)

for some constants $C_1$ and $N_1$. 

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Proof. By the Lebesgue decomposition (see, e.g., Theorems I.13, I.14, p. 22 of [RS72])
\[ d\rho = d\rho_{p.p.} + d\rho_{s.c.} + d\rho_{a.c.}, \]  
(20)
where p.p. denotes the pure point, s.c. denotes the singular continuous and a.c. denotes the absolutely continuous parts of \( d\rho \). Since there exists only a finite number of masses in nature, there is no accumulation point of the values of the mass in a realistic theory, and the pure point part of the measure is, in fact, discrete. For a scalar field of mass \( m \), we obtain
\[ d\rho_{p.p.}(m^2) = Z\delta(m^2 - m^2), \]  
(21)
where \( Z \) satisfies (18), and, by (15), (17) and (20), \( d\sigma \) satisfies (19).

Remark 2.1. Of course, \( Z = 0 \) in (21), if there is no discrete component of mass \( m \) in the total mass spectrum of the theory. In general \( Z \) in each discrete component of \( d\rho \) has only to satisfy
\[ 0 \leq Z < \infty, \]  
(22)
the latter is a consequence of the positive-definiteness conditions e.) (or the positive-definite Hilbert space metric).

Remark 2.2. Expression (21) corresponds precisely to ([Wei96b], p. 461, Equ. (10.7.20)). Thus, Proposition 2.3 is just a mathematical statement of the nonperturbative renormalization theory, as formulated by Weinberg. The physical interpretation of \( 0 < Z < \infty \) is that \( Z \) is the field strength renormalization constant, due to the fact that the physical field \( A_{\text{phys}} \) is normalized (or renormalized) by the one-particle condition ([Wei96b], (10.3.6)) which stems from the LSZ (or Haag-Ruelle) asymptotic condition (see also Sections 2 and 5 of [BS05] and references given there for the appropriate assumptions). A general field, as considered in (9), does not have this normalization. By the same token, the quantity \( m^2 \) in (21) is interpreted as the physical (or renormalized) mass associated to the scalar field. No further conditions are expected to be imposable on quantum fields and, indeed, we shall see that the additional requirement that the interacting fields satisfy the equal-time commutation relations (ETCR) leads to an entirely different picture. As we shall see, it is the latter picture which seems to agree with the general belief ([Wei96b], [Bar63], [Leh54], [Kä53], [Haa90], [Wig67b]) that the condition
\[ Z = 0 \]  
(23)
is a general condition in relativistic quantum field theory (rQFT). It is immediate, however, that this proposed universality of (23) contradicts the definition of $Z$ given by (17) of Proposition 2.3, because the latter implies the absence of mass-shells. Their presence is, however, the central building block of scattering theory [BS05].

For the purposes of identification with Lagrangian field theory, one may equate the $A(.)$ of (13) with the “bare” scalar field $\phi_B$ ([Wei96b], p. 439), whereby

$$A = \sqrt{Z} A_{\text{phys}}.$$  \hspace{1cm} (24)

Similarly, for the electron and photon fields, conventionally,

$$\Psi = \sqrt{Z_2} \Psi_{\text{phys}},$$  \hspace{1cm} (25)

$$F_{\mu,\nu} = \sqrt{Z_3} F_{\mu,\nu,\text{phys}}.$$  \hspace{1cm} (26)

Above, we refer to the previously discussed fields in (4), (5) and (9).

For $F_{\mu,\nu}$ and $\Psi$, we have the analogues of (13), namely

$$\langle F_{\mu,\nu}(x)\Omega, F_{\mu,\nu}(y)\Omega \rangle = \int d\rho_{\text{ph}}(m_0^2) \int \frac{d^3p}{2p_0}(-p^2_{\mu\nu} - p^2_{\nu\mu})e^{ip(x-y)},$$  \hspace{1cm} (27)

with $\mu \neq \nu$, no summations involved, and $p_0 = \sqrt{\vec{p}^2 + m^2_0}$. Denoting spinor indices by $\alpha, \beta$, we have

$$S^+_{\alpha,\beta}(x-y) = \langle \Omega, \Psi_\alpha(x)\bar{\Psi}_\beta(y)\Omega \rangle$$  \hspace{1cm} (28)

\begin{align*}
= & \int_0^\infty d\rho_1(m_0^2)S^+_{\alpha,\beta}(x-y;m_0^2) + \delta_{\alpha,\beta} \int_0^\infty d\rho_2(m_0^2)\Delta^+(x-y;m_0^2),
\end{align*}

with $d\rho_{\text{ph}}, d\rho_1, d\rho_2$ positive, polynomially bounded measures, and $\rho_1$ satisfying certain bounds with respect to $\rho_2$ ([Leh54], p. 350). Again, as in (21),

$$d\rho_{\text{ph}}(m_0^2) = Z_3\delta(m_0^2) + d\sigma_{\text{ph}}(m_0^2)$$  \hspace{1cm} (29)

and

$$d\rho_1(m_0^2) = Z_2\delta(m_0^2 - m_e^2) + d\sigma_1(m_0^2),$$  \hspace{1cm} (30)

with $m_e$ the renormalized electron mass, according to conventional notation.
We have, by the general condition (22),

\[ 0 \leq Z_3 < \infty, \quad (31) \]
\[ 0 \leq Z_2 < \infty. \quad (32) \]

When \( Z_3 > 0 \), the renormalized electron charge follows from (Wei96b, (10.4.18)). Assumption (16), which is also expected to be generally true on physical grounds, becomes

\[ \langle \Omega, F_{\mu,\nu}(f)\Omega \rangle = 0 \quad \forall f \in S, \quad (33) \]
\[ \langle \Omega, \Psi_\alpha(f)\Omega \rangle = 0 \quad \forall f \in S \text{ and } \alpha \text{ a spinor index}. \quad (34) \]

In summary, Proposition 2.3 provides a rigorous (non-perturbative) definition of the field-strength (or wave-function) renormalization constant. In the proof of Theorem 2.2 ([RS75], p. 70), the positive-definiteness condition plays a major role. Thus, the definition of \( Z \) and its range (22) (which depends on the positivity of the measure \( d\rho \)) strongly hinge on the fact that the underlying Hilbert space has a positive metric. Parenthetically, the positive-definiteness condition on the Wightman functions is “beyond the powers of perturbation theory”, as Steinmann aptly observes [Ste84].

3 The Singularity Hypothesis

When \( 0 < Z < \infty \) in (18), the assumption of ETCR for the physical fields may be written (in the distributional sense)

\[ \left[ \frac{\partial A_{\text{phys}}(x_0, \vec{x})}{\partial x_0}, A_{\text{phys}}(x_0, \vec{y}) \right] = -i \frac{Z}{2} \delta(\vec{x} - \vec{y}). \quad (35) \]

Together with (13), (34) yields (see also [Leh54], [Kä53], [Bar63], [Wei96b])

\[ \frac{1}{Z} = \int_0^\infty d\rho(m_0^2). \quad (36) \]

It should be noted that (35) would yield

\[ \frac{1}{Z} = Z + \int_0^\infty d\sigma(m_0^2), \quad (37) \]
Since $d\sigma$ is a positive measure, we obtain from (37) the inequality
\[ Z \leq 1 \]  
(Wei96b, p. 361, Bar63, Equ. (9.19)).

Formulas (35) and (36) have been extensively used as a heuristic guide, even by two of the great founders of axiomatic (or general) quantum field theory, Wightman and Haag, to substantiate the previously mentioned conjecture that $Z = 0$ is expected to be a general condition for interacting fields. One reason is that (35) suggests that, if $Z = 0$ holds, the light-cone singularity of the two-point function of interacting fields is expected to be stronger than the one exhibited by the free field - we shall refer to this assumption, shortly, as the “singularity hypothesis”. Indeed, in [Wig67b], p. 201, it is observed that “\( \int_0^\infty \rho(m_2^2) = \infty \) is what is usually meant by the statement that the field-strength renormalization is infinite”. This follows from (36), with “field-strength renormalization” interpreted as $\frac{1}{Z}$. The connection with the singularity hypothesis comes next ([Wig67b], p. 201), with the observation that, by (13), $W_2$ will have the same singularity, as $(x - y)^2 = 0$, as does $\Delta_+(x - y; m^2)$ if $\int_0^\infty \rho(m_2^2) < \infty$.

4 Steinmann Scaling Degree

In order to formulate the singularity hypothesis in rigorous terms, we recall the Steinmann scaling degree $sd$ of a distribution [Ste71]: for a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, let $u_\lambda$ denote the “scaled distribution”, defined by
\[ u_\lambda(f) \equiv u(f(\lambda^{-1} \cdot)) \, . \]

As $\lambda \to 0$, we expect that $u_\lambda \approx \lambda^{-\omega}$ for some $\omega$, the “degree of singularity” of the distribution $u$. Hence, we set
\[ sd(u) \equiv \inf \{ \omega \in \mathbb{R} \mid \lim_{\lambda \to 0} \lambda^\omega u_\lambda = 0 \} \, , \]  
(39)

with the proviso that if there is no $u$ satisfying the limiting condition above, we set $sd(u) = \infty$. For the free scalar field of mass $m \geq 0$, it is straightforward to show from the explicit form of the two-point function in terms of modified Bessel functions that
\[ sd(\Delta_+) = 2 \, . \]  
(40)
In (40), and the forthcoming equations, we omit the mass superscript. From Theorem 2.2, we have that for \( f \in \mathcal{S}(\mathbb{R}^4) \) the interacting two-point function satisfies

\[
W_+(f) = \int_0^\infty d\rho(0) \int_{\mathbb{R}^3} \frac{d\vec{p}}{\sqrt{\vec{p}^2 + m_0^2}} \tilde{f} \left( \sqrt{\vec{p}^2 + m_0^2}, \vec{p} \right). \tag{41}
\]

Here \( \tilde{f} \in \mathcal{S}(\mathbb{R}^4) \) denotes the Fourier transform of \( f \).

**Definition 4.1.** We say that the singularity hypothesis holds for an interacting scalar field if

\[
sd(W_+) > 2. \tag{42}
\]

**Proposition 4.2.** If the total spectral mass is finite, i.e.,

\[
\int_0^\infty d\rho(a^2) < \infty, \tag{43}
\]

then

\[
sd(W_+) \leq 2; \tag{44}
\]

i.e., the scaling degree of \( W_+ \) cannot be strictly greater than that of a free theory, and thus, by Definition 4.1, the singularity hypothesis (42) is not satisfied.

**Proof.** The scaled distribution corresponding to \( W_+ \) is given by

\[
W_{+,\lambda}(f) = \lambda^{-2} \int_0^\infty d\rho(m_0^2) \int_{\mathbb{R}^3} \frac{d\vec{p}}{\sqrt{\vec{p}^2 + \lambda^2 m_0^2}} \tilde{f} \left( \sqrt{\vec{p}^2 + \lambda^2 m_0^2}, \vec{p} \right). \tag{45}
\]

Assume the contrary to (44), i.e., that \( sd(W_+) = \omega_0 > 2 \). Under this assumption, \( \omega < \omega_0 \) implies that

\[
\lim_{\lambda \to 0} \lambda^\omega W_{+,\lambda}(f) \neq 0. \tag{46}
\]

Choosing

\[
\omega = \omega_0 - \delta > 2 \tag{47}
\]
in (46), we obtain from (45) and (46) that

$$\lim_{\lambda \to 0} \lambda^{\omega-2} \int_0^\infty d\rho(m_0^2) \int_{\mathbb{R}^3} \frac{d\tilde{p}}{\sqrt{\tilde{p}^2 + \lambda^2 m_0^2}} \tilde{f} \left( \sqrt{\tilde{p}^2 + \lambda^2 m_0^2}, \tilde{p} \right) \neq 0.$$  \hspace{1cm} (48)

The limit, as $\lambda \to 0$, of the term inside the brackets in (48), is readily seen to be finite by the Lebesgue dominated convergence theorem due to the assumption (43) and the fact that $\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$; but this contradicts (46) because of (47).

**Remark 4.1.** Proposition 4.2 holds in dimension $n = 2$ by the same proof. For dimension $n = 1$, the proof shows that we must replace $\mathcal{S}'(\mathbb{R})$ by $\mathcal{S}_0'(\mathbb{R}) \equiv \{ f \in \mathcal{S}(\mathbb{R}) \mid \tilde{f}(0) = 0 \}$.

Proposition 4.2 makes Wightman’s previously mentioned remark precise, and suggests, together with (36) the following expectation:

**Assumption 1**  The singularity hypothesis (Definition 4.1) implies $Z = 0$; alternatively, $Z = 0$ is a necessary condition for the singularity hypothesis.

Assumption 1 is also stated in slightly different words by Haag ([Haa96], p. 55), who remarks: “In the renormalized perturbation expansion one relates formally the true field $A_{\text{phys}}$ to the canonical field $A$ (our notation) which satisfies (24), where $Z$ is a constant (in fact, zero). This means that the fields in an interacting theory are more singular objects than in the free theory, and we do not have the ETCR.”

If we use the rigorous definition of $Z$ (Proposition 2.3), we find:

**Theorem 4.3.**

a.) If the ETCR is not assumed, Assumption 1 is false;

b.) if the ETCR is assumed, Assumption 1 is true.
Proof.

a.) By (17), $\int_0^\infty d\rho(m_o^2) = \infty$ holds whatever value of $Z$ satisfying (22) is chosen — in particular, $Z = 0$ — whenever $\int_0^\infty d\sigma(m_o^2) = \infty$.

b.) If the ETCR is assumed, (36) holds whenever $0 < Z < \infty$, in which case $\int_0^\infty d\rho(m_o^2) < \infty$. Thus, by (22), $Z = 0$ is the only possibility to render $\int_0^\infty d\rho(m_o^2) = \infty$.

The hypothesis of ETCR has been in serious doubt for a long time, see, e.g., the remarks in [SW64], p. 101. Its validity has been tested [Wre71] in a large class of models in two-dimensional space-time; the Thirring model [Thi58], the Schroer model [Sch63a], the Thirring-Wess model of vector mesons interacting with zero-mass fermions (see [TW64], [DT67]), and the Schwinger model [Sch62], using, throughout, the formulation of Klaiber [Kla68] for the Thirring model, and its extension to the other models by Lowenstein and Swieca [LS71] - for the Schwinger model, the previously mentioned non-covariant gauge $\alpha = \sqrt{\pi}$ was adopted. Except for the Schwinger model, whose special canonical structure is due to its equivalence (in an irreducible sector) to a theory of a free scalar field of positive mass, the quantity

\[ \{\psi(x),\psi(y)\} - \langle\Omega,\{\psi(x),\psi(y)\}\rangle \Omega \cdot 1, \tag{49} \]

where the $\psi$’s are the interacting fermi fields in the models and $\{\ldots\}$ denotes the anti-commutator and $\Omega$ denotes the vacuum, do not exist in the equal time limit as operator-valued distributions, for a certain range of coupling constants. Two different definitions of the equal time limit were used and compared, one of them due to Schroer and Stichel [SS68].

Thus, the ETCR is definitely not true in general. In perturbation theory, $Z_3(\Lambda)$ ([Wei96b], p. 462) satisfies $Z \leq 1$ (see (38)) for all ultraviolet cutoffs $\Lambda$, but it is just this condition which relies on the ETCR assumption and is not expected to be generally valid. In the limit $\Lambda \to \infty$, however, $Z_3(\Lambda)$ tends to $-\infty$ and hence violates (18) maximally. In fact, (18) is violated even for finite, sufficiently large $\Lambda$.

The models also provide examples of the validity of the singularity hypothesis (for the currents, analogous assertions hold if the commutator is used in place of the anti-commutator).
By b.) of Proposition 4.2 it follows that $Z = 0$ (see (23)) need not be valid, even if the singularity hypothesis is valid. The question may now be posed: what is then the physical meaning of $Z = 0$, or, alternatively: when is $Z = 0$ valid?

In order to try to answer this question, we recall that, in the presence of massless particles (photons), Buchholz [Buc86] used Gauss’ law to show, in a beautiful paper, that the discrete spectrum of the mass operator

$$P_\sigma P^\sigma = M^2 = P_0^2 - \vec{P}^2$$

is empty. Above, $P_0$ is the generator of time translations in the physical representation, i.e., the physical hamiltonian $H$, and $\vec{P}$ is the physical momentum. This fact is interpreted as a confirmation of the phenomenon that particles carrying an electric charge are accompanied by clouds of soft photons.

Buchholz begins by formulating adequate assumptions, viz., given that one wishes to determine the electric charge of a physical state $\Phi$ with the help of Gauss’ law, the space-like asymptotic electromagnetic field of this state must i.) be measurable and ii) with sufficient precision. Let

$$F_{\mu,\nu}(\phi_R) \equiv \int d^4x \frac{\phi(x/R)}{R^2} F_{\mu,\nu}(x).$$

Above, $\phi$ is an arbitrary real test function with compact support in the space-like complement of the origin in Minkowski space and $R > 0$ is a scaling parameter. As $R$ increases the electromagnetic field in (51) is averaged over regions whose diameter and space-like distance from the origin grows like $R$; this average is rescaled by the scaling dimension of the field. He expresses i.) for a state $\Phi$ in the form

$$\lim_{R \to \infty} \langle \Phi, F_{\mu,\nu}(\phi_R) \Phi \rangle = f_{\mu,\nu}(\phi);$$

and ii.) as the boundedness of the mean square deviation of (51):

$$\limsup_{R \to \infty} \| (F_{\mu,\nu}(\phi_R) - f_{\mu,\nu}(\phi) \hat{1}) \Phi \|^2 < \infty.$$
holds in the sense of distributions on $\mathcal{S}(\mathbb{R}^n)$, which allows to determine the electric charge of $\Phi$ by

$$
\langle \Phi, Q \Phi \rangle = \lim_{R \to \infty} \int d^4x \frac{\chi(x/R)}{R} \langle \Phi, j_0(x) \Phi \rangle = f_{i0}(\partial^i \chi),
$$

where $f_{\mu,\nu}$ is the functional (51) and $\chi$ is any test function, whose spatial derivatives $\partial^i \chi$ have support in the region $\{ x \mid x^2 < 0 \}$ and which is normalized such that $\int d^4x \chi(x)\delta^3(x) = 1$. There is a further technical assumption in Buchholz’s paper, in order to avoid domain questions, for which we must refer to his paper (see his footnote 1), but omit from the statement of the following result, which we chose to formulate as a theorem due to the central role it plays in our considerations. We leave, however, the discussion of the applicability of the assumptions to concrete models as qed$_{1+3}$ to a latter stage. In doing so, we also wish to emphasize the connection of Buchholz’s result with Theorem 2.1, and thereby with the theory of unstable systems (see the next section).

**Theorem 4.4 (Buchholz [Buc86]).** Let $\Phi$ be a state in the complement of the vacuum satisfying Gauss’s law in the sense of (54) as well as (52) and (53). Suppose, in addition, that

$$
P_{\sigma}P^\sigma \Phi = m^2 \Phi
$$

for some $m^2 \geq 0$, then

$$
f_{\mu,\nu} = 0 ;
$$

i.e., according to (51) the state $\Phi$ is chargeless.

**Proof.** We only emphasize a change at the very final stage of the proof in [Buc86]. Consider the states

$$
\Phi(y) \doteq \exp(iy \cdot P)\Phi
$$

and the function

$$
y \mapsto F(y) \equiv \langle \Phi(y), \Phi \rangle .
$$

The support of the Fourier transform $\tilde{F}$ of $F$, in the sense of distributions is, by (56), the manifold $\mathcal{M}$ defined by

$$
\mathcal{M} = \{ p \in \mathbb{R}^4 \mid p^\sigma p_\sigma = m^2 \} .
$$

By Theorem 2.1 the joint spectral measure of the operators $P^\mu$ is absolutely continuous in the complement of the vacuum, and therefore the dimension
of $\mathcal{M}$ is three. Let, now, $\phi$ be a test function of the form $\frac{1}{R}\phi(x/R)$, where $\phi$ has the properties assumed after (51). The proof in [Buc86] ends up to the statement that, if

$$f_{\mu,\nu}(\partial_\sigma \phi) \neq 0,$$

(61)

then, the Fourier transform $\tilde{F}$ of $F$, defined by (59), has support in the intersection of the manifold $\mathcal{M}$ with the “plane”

$$\mathcal{N} = \{ p \in \mathbb{R}^4 \mid p^\sigma f_{\mu,\nu}(\partial_\sigma \phi) = 0 \}.$$

(62)

Again by Theorem 2.1 we have that under condition (61) the dimension of $\mathcal{N}$ is three. The following general formula is valid in $\mathbb{R}^n$:

$$\dim(\mathcal{M} \cap \mathcal{N}) \leq \max\{\dim(\mathcal{M}) + \dim(\mathcal{N}) - n, 0\}.$$  

(63)

By (63), $\tilde{F}$, under condition (61), has support of dimension at most two, which is strictly smaller than that of $\mathcal{M}$, which we have seen to be the support of $\tilde{F}$. Thus (61) cannot hold, i.e., $f_{\mu,\nu}(\partial_\sigma \phi) = 0$, and, since $f_{\mu,\nu}$ is a homogeneous distribution of degree $-2$, it follows that $f_{\mu,\nu} = 0$.

\begin{proof}

Remark 4.2. Our reason for making (63) explicit is not a trivial one. Buchholz remarks that $\dim(\mathcal{M} \cap \mathcal{N}) \leq 2$ is impossible since the joint spectrum of the spatial momentum operators is Lebesgue-absolutely continuous on the orthogonal complement of the vacuum as a consequence of the locality of the observables, and cites [Bor66] for a proof. In fact, Borchers shows (see also [Bor62]) that the matrix elements of the translation operator $T(a) = \exp(iP.a)$ between states in the complement of the vacuum decrease to zero for any direction of translation, but this does not suffice to show absolute continuity of the joint energy-momentum spectrum because of the existence of finite singular continuous Borel measures whose Fourier-Stieltjes transform decay at infinity (Rajchman measures, see Definition 3.8 of [M W13]). For such measures, (63) may fail, as the next remark shows. In addition, the proof of Maison’s Theorem 2.1 [Mai68] demonstrates that absolute continuity of the spectrum in the orthogonal complement of the vacuum vector follows from the properties of irreducible representations of the Poincaré group, and not from locality.

Remark 4.3. It is interesting to observe that (63) does not hold in general for singular continuous joint spectra of $P^\mu$, with the dimension interpreted as the Hausdorff dimension (see, e.g., [Fal04]). Consider the example in $\mathbb{R}^2$ of the intersection of the symmetric Cantor set of ratio $1/3$ (see,
e.g., [MW13], appendix C) and the Koch curve (see [Kel06] and references given there). Assume that \( N \) is the Koch curve (Hausdorff dimension \( H \) unknown, fractal dimension \( \log(4)/\log(3) \)), and that \( M \) is the symmetric Cantor set of ratio \( 1/3 \), with \( H = \log(2)/\log(3) \). If the intersection is at the top of the curve, the Hausdorff dimension of \( M \cap N \) is zero (smaller than \( \dim(M) = \log(2)/\log(3) \), but if it is along the bottom of the Koch curve, the Hausdorff dimension of the intersection is (by construction of the Koch curve, see [Kel06] and references given there) \( \log(2)/\log(3) = \dim(M) \); thus: (65) does not hold, because, whatever \( H \leq 1 \), \( \dim(M) + \dim(N) - 2 = H + \log(2)/\log(3) - 2 < 0 \), apparently indicating that the curves do not intersect (but they do); moreover, \( \dim(M \cap N)_b = \dim(M) \), for the intersection (.)\(_b\) along the bottom of the Koch curve, which implies that comparison of dimensions fails, in general, to arrive at a contradiction in Theorem 4.4, if the joint spectrum of \( P^\mu \) is just assumed to be continuous.

When endeavouring to apply Theorem 4.4 to concrete models such as \( \text{qed}_{1+3} \), problems similar to those occurring in connection with the charge superselection rule [SW74] arise. The most obvious one is that Gauss’ law (54) is only expected to be valid (as an operator equation in the distributional sense) in non-covariant gauges (see (11)), the Coulomb gauge in the case of \( \text{qed}_{1+3} \), but not in covariant gauges [SW74]. We adopted, however, the option of staying with the Coulomb gauge and defining the theory in terms of the \( n \)-point Wightman functions of observable fields, i.e., gauge-invariant fields, thus maintaining Hilbert-space positivity. Within this framework, the hypotheses of Theorem 4.4 are in consonance with the requirements of Wightman’s theory [SW64], and Buchholz’s theorem should be applicable to \( \text{qed}_{1+3} \). Recalling (27) and (28), we also have the following result:

**Corollary 4.5.** The assertion of Theorem 4.4 is equivalent to the assertion of the following conditions:

\[
Z_3 = 0 \quad (64)
\]

and

\[
Z_2 = 0 \quad (65)
\]

It is interesting to recall, in connection with Corollary 4.5, that in [JW18], both the photon field and the electron-positron field are “dressed”.
5 Unstable Particles

We now turn to a different, but, surprisingly perhaps, related subject, that of composite or unstable particles. Classically speaking, these are particles whose field does not appear in the Lagrangian ([Wei96b], p. 461, [Wei62], for a different approach [Lan88] and references given there). However, the connection of Buchholz’s theorem with Theorem 2.1 points in this direction. Indeed, by Theorem 2.1 excited states of a relativistic atom are necessarily embedded in the absolutely continuous part of the joint energy-momentum spectrum, and are therefore expected to be unstable states or resonances.

We follow [Wei96b], p. 461, but with some modifications: 1) Weinberg formulated his criterion in terms of functional integrals, we prefer to formulate it in terms of classical field theory; 2) some details are somewhat simplified and / or stated in a different form.

Turning to scalar fields for simplicity, we consider the case of a scalar particle \( C \), of mass \( m_C \), which may decay into a set of two (for simplicity) stable particles, each of mass \( m \). We have energy conservation in the rest frame of \( C \), i.e.,

\[
m_C = \sum_{i=1}^{2} \sqrt{\vec{q}_i^2 + m_i^2} \geq \sum_{i=1}^{2} m_i ,
\]

with \( m_i = m, i = 1, 2 \), and \( \vec{q}_i \) the momenta of the two particles in the rest frame of \( C \):

\[
m_C > 2m .
\]

(66)

Let the stable particle of mass \( m \) be described by a scalar field \( A(x) \), and let us assume that the field \( C(x) \) of particle \( C \) is a functional of the field \( A(x) \), i.e.,

\[
C(x) = F(A(x)) ,
\]

(67)

with \( F \) some, for the present purpose, unspecified function (which must satisfy some regularity conditions; see below). (67) expresses the fact that the theory is equivalent to one in which \( C(x) \) does not appear, i.e., it represents a particle composed of two stable \( A \) particles, the latter’s field \( A(x) \) being the only fundamental constituent of the Lagrangian density. In the above argument, the fields in (67) are the physical (asymptotic) fields, which define the particle structure in the classical limit.
The equations (67) are the Euler-Lagrange equations associated to the classical Lagrangian density

$$L_I(x) \equiv \frac{1}{2} \left( C(x) - F(A(x)) \right)^2. \quad (68)$$

The free Lagrangian density $L_0$ for a scalar field is defined in terms of a “bare” or unrenormalized field $C_0$, defined by

$$C_0(x) = Z_C^{1/2} C(x) \quad \text{where} \quad 0 \leq Z_C < \infty. \quad (69)$$

We justify (69) by the quantum version (24): in conformance with the classical limit, we keep the same correspondence in the classical Lagrangian density. We have, therefore,

$$L_0(x) = \frac{1}{2} \partial_\mu C_0(x) \partial^\mu C_0(x) - \frac{1}{2} m_C^2 C_0(x)^2. \quad (70)$$

Equ. (67) motivates the proposal of the following Ansatz for the full classical Lagrangian density associated to particle $C$:

$$L(x) = L_0(x) + L_I(x). \quad (71)$$

By (69), the Euler-Lagrange equations for $L$ are

$$Z_C^{1/2} \left[ \frac{\partial}{\partial x^\mu} C(x) + m_C^2 C(x) \right] - \left( C(x) - F(A(x)) \right) = 0. \quad (72)$$

We have now the following result:

**Proposition 5.1.** A composite field $C(\cdot)$ of the form (67) is either a free field, or the corresponding field strength renormalization constant $Z_C$ is zero, i.e.,

$$Z_C = 0. \quad (73)$$

**Proof.** By (72), if (73) holds, (67) is obtained, i.e., $C(\cdot)$ is a composite field. On the other hand, if (67) holds, and $Z_C \neq 0$, (72) yields

$$\Box C(x) + m_C^2 C(x) = 0;$$

that is, $C(\cdot)$ is a free field of mass $m_C$. \qed
It is simple to formulate proper regularity conditions on the field \( C(\,.) \) and \( F \) in order to turn Proposition 5.1 into a mathematical theorem, they are the same as for the Noether theorem in the classical theory of fields, see, e.g., [BB82], V.6, p. 118. A theorem which applies to the quantum case would require the development of functional integral methods, see [GJS81]. Note, however, that \( Z_C \) in (73) is the quantum field strength renormalization constant, which is not altered in the classical limit, by (24). A pleasant feature of Proposition 5.1 is the dichotomy: either \( C(\,.) \) is a composite interacting field - in this case, (73) is satisfied - or it it is a free field.

There exists a quantum model of a composite (unstable) particle, satisfying (66), where (73) can be tested, that of Araki et al. [AMKG57], which we hope to revisit in a further publication. For atomic resonances, examples exist [HS95], but the situation there is entirely different from the particle case, because, for a bound-state problem, exponential decay of the wave-functions at infinity provides a natural ultraviolet cutoff. For particles, the issue is to perform the limit of no ultraviolet cutoffs, keeping the physical quantities (decay rates) fixed.

In conclusion, (64), (65) and (73) seem to assert, by totally different arguments, that the physical reason for the occurrence of (23) is the presence of photons or composite (unstable) particles in the theory. As Weinberg remarks ([Wei96b], p. 461), (73) signals that the particle is “maximally coupled to its constituents”. In the case of theories of massless photons, the “dressing” of electrons by photon clouds or of photons by electron-positron clouds may also be viewed similarly: each particle loses its identity as a single object. In the case of unstable particles, this identity is not recovered for asymptotic times, in the case of stable particles the “clouds” hopefully disappear asymptotically in the charge sectors, allowing the construction of a scattering theory even under the assumptions (64) and (65) - the parameters of the free particles being determined by recourse to the nonrelativistic limit of the theory. Alternatively, and most interestingly, relaxing condition (53), a scattering theory for QED was constructed by Alazzawi and Dybalski [ADyb] using entirely new ideas and methods, related to the concept of superselection sector introduced in [BuchRo].

It should be remarked, however, that the crucial test of nonperturbative relativistic gauge theory will be to furnish a measurable number. For QED the most famous ones are not related to scattering theory, but to spectral properties of certain Hamiltonians, for the electron g-factor see [HFC], for the Lamb shift see [Sak67],[DaNu]. In the latter reference, the importance of
the natural line shape for the Lamb shift - an unstable bound state problem - is discussed. So far, the only experimental consequence of a non-perturbative quantum field theoretic model concerns the Thirring model [Th58], which, in its lattice version, the Luttinger model (whose first correct solution is due to Lieb and Mattis [LiMa]) yields a well-established picture of conductivity along one-dimensional quantum wires [MaMa]. Hence, it would be very important to find the ground state energy of positronium using the methods of [JW18].

The picture offered in this paper is, of course, radically different from that of free quantum fields, which do satisfy the ETCR. However, it turns out that, since the photon has a hadronic component [Ber14] and all but the lightest particles are unstable, the condition $Z = 0$ is a universal condition in particle physics; but not for the reasons hitherto assumed!

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