CHARATHEODORY AND SMIRNOV TYPE THEOREM FOR HARMONIC MAPPINGS

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Abstract. We prove a version of Smirnov type theorem and Charatheodory type theorem for a harmonic homeomorphism of the unit disk onto a Jordan surface with rectifiable boundary. Further we establish the classical isoperimetric inequality and Riesz-Zygmund inequality for Jordan harmonic surfaces without any smoothness assumptions of the boundary.

1. Introduction

Throughout this paper \( n \geq 1 \) will be an integer. By \( \langle \cdot , \cdot \rangle \) and \( |\cdot| \) are denoted the standard inner product and Euclidean norm in the space \( \mathbb{R}^n \). In particular \( \mathbb{C}^n = \mathbb{R}^{2n} \), where \( \mathbb{C} = \mathbb{R}^2 \) is the complex plane. By \( U = \{ z = x + iy \in \mathbb{C} : |z| < 1 \} \) we denote the unit disk and by \( T = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \) is denoted the unit circle in the complex plane.

Let \( f = (f^1, \ldots, f^n) : U \to \mathbb{R}^n \) be a continuous mapping defined in the unit disc having partial derivatives of first order in \( U \). The formal derivative (Jacobian matrix) of \( f \) is defined by

\[
\nabla f = \begin{pmatrix}
    f_x^1 & f_y^1 \\
    \vdots  & \vdots  \\
    f_x^n & f_y^n
\end{pmatrix}
\]

Jacobian determinant of \( \nabla f \) is defined by

\[
J_f = (\det[\nabla f^T \cdot \nabla f])^{1/2} = \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y \rangle^2}.
\]

A mapping \( f = (f^1, \ldots, f^n) : U \to \mathbb{R}^n \) is called harmonic if \( f^j, j = 1, \ldots, n \) are harmonic functions in \( U \), that is if \( f^j \) is twice differentiable and satisfies the Laplace equation \( \Delta f^j = 0 \), \( j = 1, \ldots, n \).

Let

\[
P(r, t) = \frac{1 - r^2}{2\pi(1 - 2r \cos t + r^2)}, \quad 0 \leq r < 1, \quad 0 \leq t \leq 2\pi
\]

denote the Poisson kernel for the disc \( U \). It is well known that every bounded harmonic mapping \( f : U \to \mathbb{R}^n \) has the representation as Poisson integral

\[
f(z) = P[F](z) = \int_0^{2\pi} P(r, t - \theta) F(e^{i\theta}) \, dt, \quad z = re^{i\theta} \in U,
\]

where \( F : T \to \mathbb{R}^n \) is a measurable and bounded in the unit circle.

A homeomorphic image of the unit circle \( T \) in \( \mathbb{R}^n \) is called a Jordan curve. A Jordan surface \( \Sigma \subseteq \mathbb{R}^n \) is a homeomorphic image of the closed unit disk, i.e. \( \Sigma = \Phi(U) \), where \( \Phi \) is a homeomorphism. We say that \( \Sigma \) is spanned by the Jordan curve \( \Gamma = \partial \Sigma = \Phi(T) \).

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It will be always assumed that $\Gamma$ is at least rectifiable; we denote by $|\Gamma|$ its length. The surface $\Sigma \subseteq \mathbb{R}^n$ is regular if $\Sigma = \tau(U)$, where $\tau = \tau(x,y)$ is a $C^1$ injective mapping, with positive Jacobian $J_\tau$ in $U$. Thus the tangent vectors $\tau_x$, $\tau_y$ are linearly independent for $z = x + iy \in U$ or equivalently the Jacobian matrix $\nabla f$ has full rank 2 in $U$. The mapping $\tau$ is called parametrization of $\Sigma$. Certainly, it is not unique. The area $|\Sigma|$ of the surface $\Sigma$ equals

$$|\Sigma| = \int_U J_\tau \, dA,$$

where $dA(z) = dx dy$ is Lebesgue measure in the complex plane.

We call a Jordan surface $\Sigma \subseteq \mathbb{R}^n$ a simple–connected harmonic surface if there exist a homeomorphic harmonic mapping $\tau : U \to \Sigma$ (it need not have a homeomorphic extension to $\overline{U}$). Let us point out that $\tau$ need not be a regular parametrization of $\Sigma$, i.e. the strict inequality

$$J_\tau = \sqrt{|\tau_x|^2|\tau_y|^2 - (\tau_x, \tau_y)^2} > 0$$

need not hold except in the planar case (in view of Lewy’s theorem, see [9]). In other words, we allow that the harmonic surfaces have branch points, i.e. the points with zero Jacobian.

Together with this introduction, the paper contains two more sections. In the second section it is proved that a harmonic mapping of the unit disk onto a Jordan surface has BV extension onto the boundary. In addition it is proved the Smirnov theorem for harmonic mappings of the unit disk onto a Jordan surface which assert that, the angular derivative of a harmonic homeomorphism $h$ belong to the Hardy class $h^1(U)$ if and only if the boundary of the surface is rectifiable. In the third section it is proved the isoperimetric inequality for harmonic surfaces. More precisely, if $A$ is the area of a Jordan harmonic surface $\Sigma$ and $L$ is its circumference, then there hold the inequality $4\pi A \leq L^2$. This results is not surprising, and it can be find in the literature in various formulations, but we believe that our inequality contains some new information regarding the isoperimetric inequality, especially because it contains the optimal relaxing condition of smoothness of boundary. We finish the paper by establishing the Riesz-Zygmund inequality which implies that the perimeter of a harmonic surface is bigger than two “diameters”.

2. Carathéodory and Smirnov Theorem for Harmonic Mappings

Recall that a real-valued (or more generally complex-valued) function $f$ on the real line is said to be of bounded variation (BV function) on a chosen interval $[a,b] \subseteq \mathbb{R}$ if its total variation $V^a_b(f)$ is finite, i.e. $f \in BV([a,b])$ if and only if $V^a_b(f) < \infty$. The graph of a function having this property is well behaved in a precise sense. A characterization states that the functions of bounded variation on a closed interval are exactly those $f$ which can be written as a difference $g - h$, where both $g$ and $h$ are bounded monotone: this result is known as the Jordan decomposition. Moreover, if $f$ is absolutely continuous on $[a,b] \subseteq \mathbb{R}$, then $V^a_b(f) = \int_a^b |f'(t)| \, dt$.

**Theorem 2.1** (Helly selection theorem, [14]). Let $(\varphi_n)$ be a sequence of uniformly bounded functions of uniformly bounded variation on a segment $[a,b]$. Then there exists a subsequence $(\varphi_{n_k}) \subseteq (\varphi_n)$ such that $\varphi_{n_k}(x) \to \varphi(x)$ for every $x \in [a,b]$ and $\varphi$ is of bounded variation. Moreover if all of $\varphi_n$ are monotone increasing (or decreasing), then so is $\varphi$.

**Lemma 2.2.** Let $f : U \to \Sigma$ be a harmonic homeomorphism of the unit disk onto a Jordan surface $\Sigma$ with rectifiable boundary $\Gamma$. Then there exists a function $\phi : \Gamma \to \Sigma$ with
bounded variation and with at most countable set of points of discontinuity, where it has the left and the right limit such that \( f = P[\phi] \).

**Proof.** Let \( \Phi : U \to \Sigma \) be a homeomorphism which has extension on \( \overline{U} \) and let \( g = \Phi^{-1} \) be a homeomorphism of \( \Sigma \) onto \( U \). The function \( F = g \circ f : U \to U \) is also a homomorphism. Let \( U_n = \{ z : |z| < \frac{n-1}{n} \} \) and \( \Delta_n = F^{-1}(U_n) \) and let \( g_n \) be a Riemann conformal mapping of \( U \) onto the domain \( \Delta_n \) such that \( g_n(0) = 0 \) and \( g_n'(0) > 0 \). Assume w.l.g. that \( 0 \in \Delta_n \). Then the function

\[
F_n = \frac{n}{n-1} (F \circ g_n) = \frac{n}{n-1} (g \circ f \circ g_n) : \overline{U} \to \overline{U}
\]

is a homeomorphism.

Then \( \varphi_n(e^{i\theta}) = e^{i\phi_n(\theta)} \) such that \( \phi_n(\theta) \) is a monotone function. Let \( \varphi_n = F_n |_{\mathbb{T}} \) and assume that \( (\varphi_{n_k}) \) is a convergent subsequence \( (\varphi_n) \) provided to us by Lemma 2.1. Let \( \varphi_0 = \lim \varphi_{n_k} \). Then \( \varphi_0 \) is monotone. Therefore

\[
\frac{n_k}{n_k-1} (g \circ f \circ g_{n_k})|_{\mathbb{T}} \to \varphi_0.
\]

It follows that

\[
\lim_{k \to \infty} (f \circ g_{n_k})(e^{i\theta}) = \Phi(\varphi_0(e^{i\theta})) \text{ for every } \theta
\]

because \( g = \Phi^{-1} \) is a homeomorphism \( \Sigma \) onto \( \overline{U} \). Since \( \Gamma \) is a rectifiable curve by Scheffe’s theorem ([15]), the function \( \Phi \) has bounded variation in \( T \). Since \( \varphi_0 \) is monotone, it follows that the mapping \( \varphi(e^{i\theta}) = \Phi(\varphi_0(e^{i\theta})) \) has bounded variation.

Since \( \phi_k = (f \circ g_{n_k})|_{\mathbb{T}} \) is continuous and \( f \circ g_{n_k} \) is harmonic, according to Lebesgue Dominated Convergence Theorem, because \( g^{-1} \circ \varphi_0 \) is bounded we obtain

\[
\lim_{k \to \infty} f \circ g_{n_k} = \lim_{k \to \infty} P[\phi_k] = P[\Phi \circ \varphi_0].
\]

It follows that the sequence \( g_{n_k} \) converges. Let \( g_0(z) = \lim_{k \to \infty} g_{n_k}(z) \). Since \( g_0 \) is a conformal mapping of the unit disk onto itself such that \( g_0(0) = 0 \) and \( g_0'(0) > 0 \), it follows that \( g_0 = id \). Therefore \( f = P[\phi] \), where \( \phi = \Phi \circ \varphi_0 \). Since \( \Phi \) is continuous and \( \varphi_0 \) is monotone, the mapping \( \phi \) is continuous except in a countable set of points \( X \) where it has the left and the right limit.

The following two propositions are well-known. For the first one see e.g. [5, Section 1.4].

**Proposition 2.3.** Let \( f : U \to \Sigma \) be a harmonic mapping of the unit disk onto the Jordan surface \( \Sigma \). If \( f = P[\phi] \) and if for some \( \theta_0 \in [0, 2\pi) \) holds

\[
\lim_{\theta \to \theta_0} \phi(e^{i\theta}) = A_0
\]

and

\[
\lim_{\theta \to \theta_0} \phi(e^{i\theta}) = B_0,
\]

then for \( \lambda \in [-1, 1] \) and a Jordan arc \( \Gamma_\lambda(s) \subseteq U \), \( 0 \leq s < 1 \) emanating at \( \Gamma_\lambda(1) = e^{i\theta} \) and forming the angle \( -\frac{\lambda \pi}{2} \) with \( e^{i\theta} \) we have:

\[
\lim_{s \to 1^-} f(\Gamma_\lambda(s)) = \frac{1}{2}(1 - \lambda) A_0 + \frac{1}{2}(1 + \lambda) B_0.
\]

**Proposition 2.4.** If \( f = P[\phi] \) and if \( \phi \) is continuous at \( \zeta \in \mathbb{T} \), then \( f \) has continuous extension on \( \zeta \cup \mathbb{U} \).
**Definition 2.5.** At any point $\zeta$ of the unit circle $\mathbb{T}$, the cluster set, $C_U(f, \zeta)$, is defined as follows: $\alpha \in C_U(f, \zeta)$ if there exists a sequence $(z_n) \subseteq \mathbb{U}$ such that $\lim_{n \to \infty} z_n = \zeta$ while $\lim_{n \to \infty} f(z_n) = \alpha$. It is known that for any $\zeta$ the cluster set $C_U(f, \zeta)$ is nonempty and closed.

**Theorem 2.6.** Let $f : \mathbb{U} \to \Sigma$ be a harmonic homeomorphism of the unit disk onto a Jordan surface with rectifiable boundary $\Gamma$. Then

1. Then there exists a function $\phi : \mathbb{T} \to \Gamma$ with bounded variation and with at most countable set of points of discontinuity, where it has the left and the right limit such that $f = P[\phi]$.
2. If the boundary $\Gamma$ of $\Sigma$ does not contain any segment, then $f$ has a continuous extension up to the boundary.
3. If $e^{i\theta}$ is a point of discontinuity of $\phi$, then there exist $A_0 = \lim_{t \to s^-} \phi(t)$, $B_0 = \lim_{t \to s^+} \phi(t)$ and

$$C_U(f, e^{i\theta}) = [A_0, B_0] \subseteq \Gamma.$$  

**Proof.** The item (1) is contained in Lemma 2.2.

Take $\theta_0 \in [0, 2\pi)$. From item (1), there exist the left and the right boundary values of $\psi$ at $\theta_0$. Let $\lim_{\theta \to \theta_0} f(e^{i\theta}) = A_0$ and $\lim_{\theta \uparrow \theta_0} f(e^{i\theta}) = B_0$. For $R > 0$ and for $-1 \leq \lambda \leq 1$ let

$$z_R = e^{i\theta_0} \frac{R e^{i\lambda \pi/2} - 1}{R e^{i\lambda \pi/2} + 1}.$$  

Then $z_R \to e^{i\theta_0}$ as $R \to \infty$ and the angle between tangent of $\Gamma_{\lambda} = \{ z_R : 1 \leq R \leq \infty \}$ at $R = \infty$ and the point $e^{i\theta_0}$ is equal to $-\lambda \pi/2$. In view of Lemma 2.3 we have

$$\lim_{R \to \infty} f(z_R) = \frac{1}{2} (1 - \lambda) A_0 + \frac{1}{2} (1 + \lambda) B_0.$$  

It follows that $[A_0, B_0] \subseteq C_U(f, e^{i\theta_0})$. Since $f : \mathbb{U} \to \Sigma$ is a homeomorphism, it follows that $C_U(f, e^{i\theta_0}) \subseteq \Gamma$. Therefore $[A_0, B_0] \subseteq \Gamma$. If $\Gamma$ does not contain any segment then $A_0 = B_0$, i.e. $\phi$ is continuous at $e^{i\theta}$. This proves the item (2). To finish the proof of (3) we need to show that $C_U(f, e^{i\theta_0}) \subseteq [A_0, B_0]$. Let $z_n \to e^{i\theta_0}$ and assume that $X = \lim_{n \to \infty} f(z_n)$. Since $z_n$ is a bounded sequence, it exists a Jordan arc in $\mathbb{U}$ emanating at $e^{i\theta_0}$, forming the angle $-\lambda_0 \pi/2$, $\lambda_0 \in [-1, 1]$, with $e^{i\theta_0}$ and containing a subsequence $z_{n_k}$ of $z_n$. Thus

$$X = \lim_{n \to \infty} f(z_{n_k}) = \frac{1}{2} (1 - \lambda_0) A_0 + \frac{1}{2} (1 + \lambda_0) B_0 \in [A_0, B_0].$$

The previous theorem may be considered as Carathéodory theorem for harmonic surfaces.

**Example 2.7.** [5] Assume that $m > 2$ is an integer and $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = 2\pi$ and define

$$\varphi = \sum_{n=0}^{m} \theta_n \chi_{[\theta_n, \theta_{n+1}]}.$$  

Then $f = P[e^{i\varphi(0)}]$ is a harmonic diffeomorphism of the unit disk onto a Jordan domain inside the polygonal line with vertices $e^{i\theta_l}$, $l = 0, 1, \cdots, m$.  

Lemma 2.9. Assume $\Sigma \subset \mathbb{R}^n$ is a Jordan surface spanning a rectifiable curve $\Gamma$, parametrized by harmonic coordinates $f$. Let $0 < r < 1$ and $\Gamma_r = f(rT)$. Then $|\Gamma_r|$ is increasing and
\[
|\Gamma_r| \leq |\Gamma|.
\]

Proof. The total variation of a function $f \in BV$ is
\[
\text{Var}_{\Gamma} f = \sup \sum_{i=1}^m |f(re^{i\pi k + \frac{1}{2}}) - f(re^{i\pi k})|,
\]
where $\sup$ is taken for all finite division of the unit circle $T$.

Put $z = re^{it}$. We proved that $f = P[F]$, where $F \in BV$. Then by (1.1), using integration by parts, it follows that $f_r$ equals the Poisson-Stieltjes integral of $dF$:
\[
f_r(re^{it}) = \int_0^{2\pi} \partial_r P(r, \tau - t)F(t)dt
\]
\[
= -\int_0^{2\pi} \partial_t P(r, \tau - t)F(t)dt
\]
\[
= -P(r, \tau - t)F(t)|_{t=0}^{2\pi} + \int_0^{2\pi} P(r, \tau - t)dF(t)
\]
\[
= \int_0^{2\pi} P(r, \tau - t)dF(t).
\]
Thus
\[
\text{Var}_{\Gamma} \partial_r f = \int_{-\pi}^{\pi} |\partial_r f(re^{it})|dt
\]
\[
= \int_0^{2\pi} \left| \int_0^{2\pi} P(r, \tau - t)dF(\tau) \right| dt
\]
\[
\leq \int_0^{2\pi} |dF(\tau)| \int_0^{2\pi} P(r, \tau - t)dt
\]
\[
= \text{Var}_{\Gamma} F(t) = |\Gamma|.
\]

Lemma 2.8. Assume $\Sigma \subset \mathbb{R}^n$ is a Jordan surface spanning a rectifiable curve $\Gamma$, parametrized by harmonic coordinates $f$. Let $0 < r < 1$ and $\Gamma_r = f(rT)$. Then $|\Gamma_r|$ is increasing and
\[
|\Gamma_r| \leq |\Gamma|.
\]

Proof. The total variation of a function $f \in BV$ is
\[
\text{Var}_{\Gamma} f = \sup \sum_{i=1}^m |f(re^{i\pi k + \frac{1}{2}}) - f(re^{i\pi k})|,
\]
where $\sup$ is taken for all finite division of the unit circle $T$.

Put $z = re^{it}$. We proved that $f = P[F]$, where $F \in BV$. Then by (1.1), using integration by parts, it follows that $f_r$ equals the Poisson-Stieltjes integral of $dF$:
\[
f_r(re^{it}) = \int_0^{2\pi} \partial_r P(r, \tau - t)F(t)dt
\]
\[
= -\int_0^{2\pi} \partial_t P(r, \tau - t)F(t)dt
\]
\[
= -P(r, \tau - t)F(t)|_{t=0}^{2\pi} + \int_0^{2\pi} P(r, \tau - t)dF(t)
\]
\[
= \int_0^{2\pi} P(r, \tau - t)dF(t).
\]
Thus
\[
\text{Var}_{\Gamma} \partial_r f = \int_{-\pi}^{\pi} |\partial_r f(re^{it})|dt
\]
\[
= \int_0^{2\pi} \left| \int_0^{2\pi} P(r, \tau - t)dF(\tau) \right| dt
\]
\[
\leq \int_0^{2\pi} |dF(\tau)| \int_0^{2\pi} P(r, \tau - t)dt
\]
\[
= \text{Var}_{\Gamma} F(t) = |\Gamma|.
\]

Lemma 2.9. Let $f : U \to \Sigma \subset \mathbb{R}^n$ be a homeomorphism onto the Jordan surface $\Sigma$ bounded by rectifiable curve $\Gamma$. Suppose that $f$ has continuous extension on the $\Gamma \setminus E$, where $E$ is a countable union of segments of $\Gamma$ (if there is any). Further, let curves $\Gamma_r$, $0 < r < 1$ defined by $f(re^{it})$, $0 \leq t \leq 2\pi$ be rectifiable. Then
\[
\lim_{r \to 1} \sup |\Gamma_r| \geq |\Gamma|.
\]

Proof. Let $d(x, y) = |x - y|$ be the distance between points $x$ and $y$ in $\mathbb{R}^n$. Let $\varepsilon > 0$ be fixed. There exist points $\omega_0$, $\omega_1, \ldots, \omega_n \in \Gamma$ such that
\[
\sum_{j=0}^n d(\omega_j, \omega_{j+1}) > |\Gamma| - \varepsilon/2,
\]
where we set $\omega_{n+1} = \omega_0$. We may suppose w.l.g. that these points do not lie in $\Gamma \setminus E$.

Since $f$ has continuous extension on the boundary of $\Sigma$ without segments, we can find points $\zeta_j \in \mathbb{T}$ such that $f(\zeta_j) = \omega_j$ for all $j = 0, 1, \ldots, n$. Let $\omega'_j = f(r\zeta_j) \in \Gamma_r$. The
distance between $r \zeta_j$ and $\zeta_j$ is $1 - r$ for all $j$. Since $n$ is fixed and since $f$ has continuous extension on $\Gamma \setminus E$, there exist $r$ close enough to 1 such that

$$s_n := \sum_{j=0}^{n} d(\omega_j', \omega_j) < \varepsilon/4$$

Using the triangle inequality

$$d(\omega_j, \omega_{j+1}) \leq d(\omega_j, \omega_j') + d(\omega_j', \omega_{j+1}) + d(\omega_{j+1}', \omega_{j+1}),$$

we get

$$|\Gamma_r| \geq \sum_{j=0}^{n} d(\omega_j', \omega_{j+1}) \geq \sum_{j=0}^{n} d(\omega_j, \omega_{j+1}) - 2s_n > |\Gamma| - \varepsilon.$$

Since $\varepsilon$ is an arbitrary positive number, it follows $\limsup_{r \to 1} |\Gamma_r| \geq |\Gamma|$.

Smirnov theorem for holomorphic functions can be generalized to harmonic quasiconformal mappings ([8]). The following version of Smirnov only request harmonicity of a homeomorphism and somehow is optimal.

**Theorem 2.10.** Let $f : \mathbb{U} \to \Sigma \subset \mathbb{R}^n$ be a harmonic homeomorphism on the unit disk onto the Jordan surface bounded by the curve $\Gamma$ and let $\Gamma_r$, $0 < r < 1$ be curves defined by $f(re^{it})$, $0 \leq t \leq 2\pi$. Then $\partial_r f \in h^1(\mathbb{U})$ if and only if $f$ is rectifiable. In this settings, $|\Gamma_r| \to |\Gamma|$ as $r \to 1$.

**Proof.** If $\Gamma$ is rectifiable, according to Lemma 2.8 we have $|\Gamma_r| \leq |\Gamma|$ what means

$$\int_0^{2\pi} |\partial_r f(re^{it})| \, dt \leq |\Gamma|.$$

Thus $\partial_r f \in h^1(\mathbb{U})$. On the other hand, if $\partial_r f \not\in h^1(\mathbb{U})$, then $|\Gamma_r|$ is bounded and according to previous lemma $|\Gamma|$ is finite. Since we have harmonic parametrization, $|\Gamma_r|$ is an increasing sequence, thus $\lim_{r \to 1} |\Gamma_r| \leq |\Gamma|$, by Lemma 2.8. Using lemma 2.9 we have the reverse inequality. It follows $\lim_{r \to 1} |\Gamma_r| = |\Gamma|$.

In the settings of the previous theorem, in general, parametrization for $\Gamma$ which is induced by $f$ is not always absolutely continuous (or even continuous). In particular, if $n = 2$ and $f$ is holomorphic, then $f$ induce on $\Gamma$ an absolutely continuous parametrization (this is Smirnov theorem). Thus there is difference between harmonic and holomorphic concerning the property of absolute continuity; see Proposition 2.1 in [2].

### 3. Some classical inequalities for harmonic surfaces revisited

Our first aim in this section is to establish the classical isoperimetric inequality for harmonic surfaces with rectifiable boundary and without any smoothness assumption on the boundary. We expect that some of results we prove in this section are well-known, but due to missing quick references, we include their proofs here.

#### 3.1 Gaussian curvature of a smooth surface

The first fundamental form of a surface $\Sigma \subset \mathbb{R}^n$ (not necessary a Jordan surface) parametrized by a smooth mapping $\tau(z) = (\tau_1(z), \ldots, \tau_n(z)) : \Omega \to \Sigma$, $z = x + iy$ is given by

$$ds^2 = E dx^2 + 2G dx dy + F dy^2$$

where $E = g_{11} = |\tau_x|^2$, $F = g_{12} = \langle \tau_x, \tau_y \rangle$ and $G = g_{22} = |\tau_y|^2$ satisfy $EG - F^2 > 0$ on $\Omega$. 

The Gaussian curvature is usually expressed as a function of the first and second fundamental form. However for the surface which are not embedded in $\mathbb{R}^3$ the second fundamental form is not defined because it depends on Gauss normal, which is not defined in a usual way in $\mathbb{R}^n$, $n \geq 4$. The Brioschi formula for the Gaussian curvature gives us an opportunity to express it by

$$K(x, y) = \frac{-\frac{1}{2}E_{yy} + F_{xy} - \frac{1}{2}G_{xx} \quad \frac{1}{2}E_x \quad \frac{1}{2}F_x - \frac{1}{2}G_y \quad 0 \quad \frac{1}{2}E_y \quad \frac{1}{2}G_x}{F_y - \frac{1}{2}G_x \quad E \quad F \quad \frac{1}{2}E_y \quad E \quad F \quad \frac{1}{2}E_y \quad E \quad F \quad \frac{1}{2}G_x \quad F \quad G} \cdot (EG - F^2)^2$$

This is indeed an alternative formulation of the fundamental Gauss’s Theorem Egregium and consequently the Gaussian curvature does not depend whether the surface is embedded on $\mathbb{R}^3$ or in some other Riemann manifold.

For three vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ we define the matrix

$$[a, b, c] := \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}.$$

**Lemma 3.1.** Let $\Sigma$ be a surface in $\mathbb{R}^n$ with parametrization $\tau = \tau(x, y) = (\tau_1, \ldots, \tau_n)$ which is enough smooth. The Gaussian curvature can be expressed as

$$(3.1) \quad K(x, y) = \frac{\det([\tau_{xx}, \tau_{x}, \tau] \times [\tau_{yy}, \tau_{x}, \tau]^T) - \det([\tau_{xy}, \tau_{x}, \tau]^T)}{(\|\tau_x\|^2)^2 - (\|\tau_x\|^2)^2}$$

**Remark 3.2.** In standard expressions for Gaussian curvature, it appears the third derivative of the parametrization. In formula (3.1) we have only the first and the second derivative which is intriguing, but the proof depends on the third derivative of $\tau$ as well and thus we should assume that the regularity of $\tau$ is something more than class $C^2$.

**Proof.** First of all we have the equalities

$$E_y = 2 \langle \tau_{xy}, \tau_x \rangle, \quad E_{yy} = 2 \langle \tau_{xyy}, \tau_x \rangle + 2 \|\tau_x\|^2,$$

$$F_x = \langle \tau_{xx}, \tau_y \rangle + \langle \tau_x, \tau_{xy} \rangle, \quad F_{xy} = \langle \tau_{xx}, \tau_y \rangle + \langle \tau_{xy}, \tau_y \rangle + \|\tau_y\|^2 + \langle \tau_x, \tau_{xyy} \rangle,$$

$$G_x = 2 \langle \tau_{xy}, \tau_y \rangle, \quad G_{xx} = 2 \langle \tau_{xxy}, \tau_y \rangle + 2 \|\tau_{xy}\|^2$$

and

$$-\frac{1}{2}E_{yy} + F_{xy} - \frac{1}{2}G_{xx} = \langle \tau_{xx}, \tau_{yy} \rangle - \|\tau_{xy}\|^2.$$

Then

$$\det([\tau_{xy}, \tau_x, \tau_y] \times [\tau_{xy}, \tau_x, \tau_y]^T) = \begin{vmatrix} \|\tau_{xy}\|^2 & \frac{1}{2}E_y & \frac{1}{2}G_x \\ \frac{1}{2}E_y & E & F \\ \frac{1}{2}G_x & F & G \end{vmatrix} = \begin{vmatrix} \|\tau_{xy}\|^2 & 0 & 0 \\ \frac{1}{2}E_y & E & F \\ \frac{1}{2}G_x & F & G \end{vmatrix} + \begin{vmatrix} \|\tau_{xy}\|^2 & 0 & 0 \\ \frac{1}{2}E_y & E & F \\ \frac{1}{2}G_x & F & G \end{vmatrix}$$
Theorem 3.3. If $\Sigma$ is a simple connected harmonic surface which allows regular harmonic parametrization $\tau$, then the Gaussian curvature of $\Sigma$ is nonpositive.

Proof. Let $\Sigma$ be a simple connected harmonic surface with regular harmonic parametrization $\tau$, that is, let $\Delta \tau = (0, \ldots, 0)$. Since $\tau_{yy} = -\tau_{xx}$ we obtain
\[
\det([\tau_{xx}, \tau_x, \tau_y]) \times [\tau_{yy}, \tau_x, \tau_y]^T) = \left| \begin{array}{ccc} \tau_{xy}^2 - \frac{1}{2} E_{yy} + F_{xy} - \frac{1}{2} G_{xx} & \frac{1}{2} E_x - \frac{1}{2} F_x & \frac{1}{2} F_x - \frac{1}{2} G_x \\ F_y - \frac{1}{2} G_x & E & F \\ \frac{1}{2} G_y & F & G \end{array} \right|
\]
\[
= \left| \begin{array}{ccc} \tau_{xy}^2 & 0 & 0 \\ F_y - \frac{1}{2} G_x & E & F \\ \frac{1}{2} G_y & F & G \end{array} \right| + \left| \begin{array}{ccc} -\frac{1}{2} E_{yy} + F_{xy} - \frac{1}{2} G_{xx} & \frac{1}{2} E_x - \frac{1}{2} F_x & \frac{1}{2} F_x - \frac{1}{2} G_x \\ F_y - \frac{1}{2} G_x & E & F \\ \frac{1}{2} G_y & F & G \end{array} \right|.
\]

The equality of the lemma now follows from Brioullin formula for Gaussian curvature. □

Theorem 3.4. If $\Sigma$ is a Jordan harmonic surface with rectifiable boundary $\Gamma$, then we have the classical isoperimetric inequality
\[
4\pi |\Sigma| \leq |\Gamma|^2.
\]

Proof. Let $\tau : \mathbb{U} \to \Sigma$ be a harmonic parametrization of $\Sigma$. Since $\tau$ is not necessarily regular, as in [16], let us perturb the surface $\Sigma$ in $\mathbb{R}^{n+2}$ by taking for $e > 0$ and $0 < r < 1$ the harmonic homeomorphism $\tau^e_\tau(z) = (r(z), e z) \in \mathbb{R}^{n+2}$, $z \in \mathbb{U}$ we obtain a harmonic parametrization of a regular simple-connected harmonic surface $\Sigma^e_\tau = \tau^e_\tau(\mathbb{U}) \subseteq \mathbb{R}^{n+2}$ with smooth boundary. Since the Gaussian curvature of $\Sigma^e_\tau$ is non-positive, applying the classical result, we obtain
\[
4\pi |\Sigma^e_\tau| \leq |\Gamma^e_\tau|^2.
\]

Letting first $e \to 0$ and then $r \to 1$, by the inequality (2.1) we obtain (3.2).

We offer another proof of Theorem 3.4 by using the result of Beeson [1], but this case we make use of Theorem 2.10. Since $\tau^e_\tau$ converges to $\tau$ and $|\Gamma^e_\tau|$ converges to $|\Gamma|$, it follows that $|\Sigma^e_\tau|$ converges to $|\Sigma|$, and in view of (3.3) the inequality (3.2) follows immediately. □

Remark 3.5. Theorem 3.4 can be considered as an extension of theorems of Shiffman [16]. Namely Shiffman in order to prove the isoperimetric inequality for harmonic surfaces $\Sigma$ used the assumption that the harmonic parametrization $\tau$ is a homeomorphism with $|\tau_\Gamma \tau| \in BV$. Our proof shows that the condition $|\tau_\Gamma \tau| \in BV$ is somehow redundant, but we make a...
topological condition that $\Sigma$ is a Jordan surface. We decide to present this inequality in this paper, because it is not well-known. Additional motivation why we consider this problem comes from the famous Courant book [4] (see the proof of [4, Theorem 3.7]), which has been published some years after the paper of Shiffman. Indeed Courant proved for $n = 3$ the inequality
\[ 4|\Sigma| \leq |\Gamma|^2, \]
under the condition $\Sigma = \tau(U)$, where $\tau$ is a harmonic parametrization with absolutely continuous boundary data.

If we restrict ourselves to regular surfaces with smooth boundaries, our Theorem 3.4 does not bring any new information, because it is well-known the following fact, the Riemann surface enjoys the isoperimetric inequality (in compact smooth Jordan sub-surfaces) if and only if the Gaussian curvature is non-positive (cf. [3, 7, 12]) (This is a theorem of Beckenbach and Radó). However we believe that the Theorem 3.4 bring some new light on this problem. We strongly believe that Theorem 3.4 is well-known for Jordan minimal surfaces and this particular case can be proved without Theorem 2.6. Recall that Enneper-Weierstrass parameterization
\[ \tau(z) = (p_1(z), \ldots, p_n(z)), \quad z \in \mathbb{U}, \]
of a simple-connected minimal surface $\Sigma$ has harmonic coordinates $p_j(z), j = 1, \ldots, n$ such that $p_j(z) = \text{Re}(a_j(z))$, where $a_j, j = 1, \ldots, n$ are analytic functions on the unit disk satisfying the equation $\sum_{j=1}^{n} (a_j'(z))^2 = 0$.

3.2. Riesz-Zygmund inequality. The following is a classical inequality.

**Proposition 3.6** (Riesz-Zygmund inequality). [13, Theorem 6.1.7] If $f \in h^1(\mathbb{U})$ is a harmonic function then
\[ \int_{-1}^{1} |\partial_r f(re^{is})|dr \leq \frac{1}{2} \int_{0}^{2\pi} |\partial_t f(e^{it})|dt. \]
The constant $1/2$ is the best possible.

As a corollary we have the next inequality.

**Corollary 3.7.** Assume that $f$ is a harmonic diffeomorphism from unit disc $\mathbb{U}$ onto a Jordan domain $\Omega$ with the rectifiable boundary $\Gamma$ and let $d$ be an arbitrary diameter of $\mathbb{U}$. Then, if by $| \cdot |$ we denote the corresponding length, we have
\[ |f(d)| \leq |\Gamma|/2. \]

Now we prove the following extension of Proposition 3.6.

**Theorem 3.8** (Riesz-Zygmund inequality for harmonic surfaces). Assume $\Sigma \subseteq \mathbb{R}^n$ is a harmonic surface spanning a rectifiable curve $\Gamma$ parametrized by harmonic coordinates. Then for every $s \in [0, 2\pi]$
\[ \int_{-1}^{1} |\partial_r \tau(re^{is})| dt \leq \frac{1}{2} \int_{0}^{2\pi} |\partial_t \tau(e^{it})|dt. \]
In other words, the length of the image of an arbitrary diameter $d$ of the unit disk under a harmonic parametrization $\tau$ is less than one half of the perimeter of the surface $\Sigma$.

**Proof.** Assume that $\tau$ are harmonic coordinates. Let $\tau = (\text{Re}(a_1), \ldots, \text{Re}(a_n))$, where $a_j, j = 1, \ldots, n$ are analytic function in the unit disk. Then
\[ \partial_t \tau + ir\partial_r \tau = (a_1', \ldots, a_n') \subset \mathbb{C}^n \]
and thus \( r \partial_r \tau \) is the harmonic conjugate of \( \partial_t \tau \). It follows that

\[
(3.5) \quad r \partial_r \tau (re^{is}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\text{Im}F(re^{it})) \partial_t \tau (e^{it(s-t)}) dt,
\]

where \( F(z) = 2z/(1-z) \). As in the proof of [13, Theorem 6.1.7] we find out that

\[
(3.6) \quad \int_{-1}^{1} |r^{-1} \text{Im}F(re^{it})| dr = \pi
\]

for \( 0 < |t| < \pi \). By Fubini’s theorem, (3.5) and (3.6) we obtain

\[
\int_{-1}^{1} |\partial_r \tau (re^{is})| dr \leq \frac{1}{2\pi} \int_{0}^{2\pi} |\partial_t \tau (e^{it})| dt \int_{-1}^{1} |r^{-1} \text{Im}F(re^{it})| dr = \frac{1}{2} \int_{0}^{2\pi} |\partial_t \tau (e^{it})| dt.
\]

\( \square \)

**Remark 3.9.** It is worth to notice the following important fact. For a minimal surface \( \Sigma \) over a domain in the complex plane, every isothermal parametrization is a harmonic parametrization and it coincides with Enneper–Weierstrass parametrization of the minimal surface.

Let \( \Sigma \subseteq \mathbb{R}^n \) be a regular surface. For two points \( P, Q \in \Sigma \) we define the intrinsic distance as follows

\[
d_I(P, Q) = \inf_{c \in \mathcal{C}} |c|,
\]

where \( \mathcal{C} \) is the set of all Jordan arcs \( c \) of \( \Sigma \) with the length \( |c| \) connecting \( P \) and \( Q \). It should be noted the following fact, for close enough points \( P \) and \( Q \) it exists a geodesic line \( \gamma \) connecting \( P \) and \( Q \) such that \( d_I(P, Q) = |\gamma| \). We define the (geodesic) diameter of \( \Sigma \) as

\[
diam(\Sigma^2) = \sup_{P, Q \in \Sigma} d_I(P, Q).
\]

We can now deduce the following geometric application of Theorem 3.8.

**Theorem 3.10.** If \( \Sigma \subseteq \mathbb{R}^n \) is an arbitrary simply connected harmonic surface with rectifiable boundary \( \Gamma \) then:

\[
(3.7) \quad diam(\Sigma) \leq \frac{1}{2} |\Gamma|.
\]

The constant \( 1/2 \) is the best possible even for minimal surfaces lying over the unit disk.

**Proof.** Without lost of generality, let \( \tau : U \to \Sigma \) be regular harmonic parametrization of the surface \( \Sigma \) (if not we can perturb surface in \( \mathbb{R}^{n-2} \) as in the proof of isoperimetric inequality). Let \( P, Q \in \Sigma \). Then there exist a conformal mapping \( a \) of the unit disk \( U \) onto itself such that \( \tau(a(-x)) = P \) and \( \tau(a(x)) = Q \), \( 0 < x \leq 1 \). Take \( \nu_\delta(z) = \tau \circ a(\delta z) \), \( x < \delta < 1 \). Then by Theorem 3.8 and relation (2.1) we have

\[
d_I(P, Q) \leq \int_{-1}^{1} |\partial_r \nu_\delta(r)| dr \leq \frac{1}{2} \int_{0}^{2\pi} |\partial_t \nu_\delta(e^{it})| dt \leq \frac{1}{2} |\gamma|.
\]

By \( d_I(P, Q) < |\gamma|/2 \) we obtain (3.7).

Show that the constant \( 1/2 \) is sharp. Assume, as we may that \( n = 3 \). Let \( d = [-e^{it}, e^{it}] \) be an arbitrary diameter of the unit disk and let

\[
\tau(x, y) = (x, y, m(x+y))
\]
where $m$ is a large constant. We can express the perimeter of the minimal surface $\tau$ by Elliptic integral of the second kind $E$ i.e.

$$|\gamma| = 2(E[\pi/4, -2m^2] + E[3\pi/4, -2m^2]).$$

The length of $\tau(d)$ is $2\sqrt{1 + m^2 + m^2 \sin 2t}$. The maximal diameter is attained for $t = \pi/4$ and is equal $2\sqrt{1 + 2m^2}$. Then

$$\lim_{m \to \infty} \frac{2\sqrt{1 + 2m^2}}{2(E[\pi/4, -2m^2] + E[3\pi/4, -2m^2])} = \frac{1}{2}.$$

\[ \square \]

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