Tetrahedral Family Symmetry
and the Neutrino Mixing Matrix

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Abstract

In a new application of the discrete non-Abelian symmetry $A_4$ using the canonical
seesaw mechanism, a three-parameter form of the neutrino mass matrix is derived. It
predicts the following mixing angles for neutrino oscillations: $\theta_{13} = 0$, $\sin^2 \theta_{23} = 1/2$,
and $\sin^2 \theta_{12}$ close, but not exactly equal to $1/3$, in one natural symmetry limit.
The symmetry group of the tetrahedron is also that of the even permutation of four objects, i.e. $A_4$. It is a non-Abelian finite subgroup of $SO(3)$ as well as $SU(3)$. It has twelve elements and four irreducible representations: $1, 1', 1'', \text{ and } 3$. It has been shown to be useful in describing the family structure of quarks and leptons. In most previous applications, the lepton doublets $(\nu_i, l_i)$ are assigned to the $3$ representation of $A_4$, whereas the charged-lepton singlets $l_i^c$ are assigned to the three inequivalent one-dimensional representations $1, 1', 1''$. Here as in the two papers of Ref. [7], both $(\nu_i, l_i)$ and $l_i^c$ are $3$ instead.

Three heavy neutral fermion singlets $N_i$ are assumed, transforming as $1, 1', 1''$ under $A_4$. [In the first paper of Ref. [7], they transform as $3$; in the second, they are absent.] The multiplication rule $1' \times 1'' = 1$ implies that the Majorana mass matrix of $N_i$ invariant under $A_4$ is given by

$$M_N = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix}. \quad (1)$$

The multiplication rule

$$3 \times 3 = 1 + 1' + 1'' + 3 + 3 \quad (2)$$

allows the charged-lepton mass matrix to be diagonal by having three Higgs doublets transforming as $1, 1', 1''$, resulting in a diagonal $M_l$ with

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 v_1 \\ h_2 v_2 \\ h_3 v_3 \end{pmatrix}, \quad (3)$$

where $\omega = \exp(2\pi i/3)$ and $v_{1,2,3}$ are the vacuum expectation values of these three Higgs doublets.

As for the Dirac mass matrix linking $\nu_i$ to $N_j$, three other Higgs doublets are assumed, transforming as $3$ under $A_4$. [They are distinguished from the previous three Higgs doublets]
by a discrete $Z_2$ symmetry. Thus

$$
\mathcal{M}_D = \begin{pmatrix}
  f_1 u_1 & f_2 u_1 & f_3 u_1 \\
  f_1 u_2 & f_2 \omega u_2 & f_3 \omega^2 u_2 \\
  f_1 u_3 & f_2 \omega^2 u_3 & f_3 \omega u_3
\end{pmatrix} = \begin{pmatrix}
  u_1 & 0 & 0 \\
  0 & u_2 & 0 \\
  0 & 0 & u_3
\end{pmatrix} \begin{pmatrix}
  1 & 1 & 1 \\
  1 & \omega & \omega^2 \\
  1 & \omega^2 & \omega
\end{pmatrix} \begin{pmatrix}
  f_1 & 0 & 0 \\
  0 & f_2 & 0 \\
  0 & 0 & f_3
\end{pmatrix}.
$$

Using the canonical seesaw mechanism \[9\], the Majorana neutrino mass matrix is then given by

$$
\mathcal{M}_\nu = \mathcal{M}_D \mathcal{M}_N^{-1} \mathcal{M}_D^T = \begin{pmatrix}
  u_1 & 0 & 0 \\
  0 & u_2 & 0 \\
  0 & 0 & u_3
\end{pmatrix} \begin{pmatrix}
  a & b & b \\
  b & a & b \\
  b & b & a
\end{pmatrix} \begin{pmatrix}
  u_1 & 0 & 0 \\
  0 & u_2 & 0 \\
  0 & 0 & u_3
\end{pmatrix},
$$

where

$$
a = f_1^2/A + 2f_2f_3/B, \quad b = f_1^2/A - f_2f_3/B,
$$

and $u_{1,2,3}$ are the vacuum expectation values of the second set of Higgs doublets which transform as $\overline{3}$ under $A_4$.

If $u_1 = u_2 = u_3 = u$, then a residual $Z_3$ symmetry exists, and the eigenvalues of $\mathcal{M}_\nu$ are simply $u^2(a + 2b)$, $u^2(a - b)$, and $u^2(a - b)$. However, since the first eigenvalue corresponds to the eigenstate $(\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$, this is not a realistic scenario. Consider now the case

$$
u_2 = \nu_3 = u \neq \nu_1.
$$

This makes $\mathcal{M}_\nu$ of the form advocated in Ref. \[10\] and results in $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. Since $\theta_{13} = 0$ implies that $CP$ is conserved in neutrino oscillations, the condition $u_2 = u_3$ should be considered “natural” in the sense that it is protected by a symmetry. Note that this alone does not imply $\theta_{23} = \pi/4$, which needs also $A_4$ for it to be true. \[It certainly does not come from $\nu_\mu - \nu_\tau$ exchange as often suggested, because that would imply $\mu - \tau$ exchange as well, which cannot be sustained in the complete Lagrangian of the theory as a symmetry because $m_\mu \neq m_\tau.\]

Using the condition of Eq. (7), $\mathcal{M}_\nu$ of Eq. (5) can be rewritten as

$$
\mathcal{M}_\nu = \begin{pmatrix}
  \lambda^2 a & \lambda b & \lambda b \\
  \lambda b & a & b \\
  \lambda b & b & a
\end{pmatrix}.
$$
In the basis $\nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}$, and $(-\nu_\mu + \nu_\tau)/\sqrt{2}$, this becomes

$$\mathcal{M}_\nu = \begin{pmatrix} \lambda^2 a & \sqrt{2} \lambda b & 0 \\ \sqrt{2} \lambda b & a + b & 0 \\ 0 & 0 & a - b \end{pmatrix},$$

yielding one exact eigenvalue and eigenstate:

$$m_3 = a - b, \quad \nu_3 = (-\nu_\mu + \nu_\tau)/\sqrt{2}. \quad (10)$$

In the submatrix spanning $\nu_e$ and $(\nu_\mu + \nu_\tau)/\sqrt{2}$, consider

$$\mathcal{M}_\nu \mathcal{M}_\nu^\dagger = \begin{pmatrix} |\lambda|^4 |a|^2 + 2|\lambda|^2 |b|^2 & \sqrt{2} \lambda (|b|^2 + a^* b + |\lambda|^2 a^* b) \\ \sqrt{2} \lambda^* (|b|^2 + a b^* + |\lambda|^2 a b^*) & |a + b|^2 + 2|\lambda|^2 |b|^2 \end{pmatrix}. \quad (11)$$

The limit $|m_1|^2 = |m_2|^2$ is reached if

$$|a + b|^2 - |\lambda|^4 |a|^2 = 0, \quad |b|^2 + a^* b + |\lambda|^2 a b^* = 0, \quad (12)$$

both of which are satisfied if $b = -a(1 + |\lambda|^2)$. In this limit, $\Delta m^2_{sol} = 0$ and

$$\Delta m^2_{atm} \equiv |m_3|^2 - (|m_1|^2 + |m_2|^2)/2 = 2|a|^2 (1 - |\lambda|^4)(2 + |\lambda|^2). \quad (13)$$

To obtain a nonzero $\Delta m^2_{sol}$ and the value of $\theta_{12}$, consider

$$b = -a(1 + |\lambda|^2 + \epsilon), \quad (14)$$

then

$$\Delta m^2_{sol} \equiv |m_2|^2 - |m_1|^2 = |a|^2[(\epsilon + \epsilon^*)|\lambda|^2 + |\epsilon|^2] + 8|\lambda|^2 |\epsilon|^* + \epsilon|\lambda|^2 + |\epsilon|^2]^{1/2}, \quad (15)$$

and

$$\tan^2 2\theta_{12} = \frac{8|\lambda|^2 |\epsilon|^* + \epsilon|\lambda|^2 + |\epsilon|^2]}{[(\epsilon + \epsilon^*)|\lambda|^2 + |\epsilon|^2]^2}. \quad (16)$$

There are two natural limits of the parameter $\lambda$. (A) $\lambda = 1$ corresponds to $u_1 = u_2 = u_3 = u$, which is protected by a residual $Z_3$ symmetry as discussed already. (B) $\lambda = 0$
corresponds to $m_{\nu_e} = 0$ and the decoupling of $\nu_e$ from $\nu_\mu$ and $\nu_\tau$, which is protected by a chiral U(1) symmetry. Hence values of $\lambda$ near 1 and 0 will be considered from now on.

(A) For $|\lambda| \approx 1$, $\epsilon$ is expected to be small compared to it in Eq. (14). In that case,

$$\Delta m_{sol}^2 \approx 2|a|^2|\lambda||\Re\epsilon|^2(2 + |\lambda|^2)(1 + 2|\lambda|^2) + 2|\Im\epsilon|^2(1 - |\lambda|^2)^2]^{1/2},$$

and

$$\tan^2 2\theta_{12} \approx 8 \left[ \frac{1 + |\lambda|^2}{2|\lambda|} \right]^2 + \frac{|\Im\epsilon|^2}{(\Re\epsilon)^2} \left( \frac{1 - |\lambda|^2}{2|\lambda|} \right)^2. \tag{18}$$

This means that $|\tan 2\theta_{12}| > 2\sqrt{2}$, or equivalently $\sin^2 \theta_{12} > 1/3$, to be compared with the current experimental fit of $\sin^2 \theta_{12} = 0.31 \pm 0.03$.

Using the typical experimental values

$$\Delta m_{atm}^2 = 2.5 \times 10^{-3} \text{ eV}^2, \quad \Delta m_{sol}^2 = 8.0 \times 10^{-5} \text{ eV}^2, \tag{19}$$

and assuming $\epsilon$ to be real, its value and those of $\sin^2 \theta_{12}$ and $|\lambda^2 a|$ are given in Table 1.

It shows that $\sin^2 \theta_{12}$ is very near 1/3 and cannot be distinguished in practice from being exactly 1/3 [11], as in some models. The last column corresponds to the expected value of the effective neutrino mass measured in neutrinoless double beta decay.

Table 1: Values of $\sin^2 \theta_{12}$, $\epsilon$, and $|\lambda^2 a|$ as functions of $|\lambda|$.

| $|\lambda|$ | $\sin^2 \theta_{12}$ | $\epsilon$ | $|\lambda^2 a|$ |
|-----------|------------------|----------|----------------|
| 0.7       | 0.342            | 0.027    | 0.013 eV       |
| 0.8       | 0.337            | 0.020    | 0.018 eV       |
| 0.9       | 0.334            | 0.011    | 0.029 eV       |
| 1.0       | 0.333            | –        | –              |
| 1.1       | 0.334            | 0.014    | 0.035 eV       |
| 1.2       | 0.336            | 0.032    | 0.026 eV       |
| 1.3       | 0.338            | 0.055    | 0.023 eV       |
| 1.4       | 0.341            | 0.082    | 0.021 eV       |
(B) For $|\lambda| \simeq 0$, consider $|\epsilon|$ also to be of order $|\lambda|$, then

\[
\Delta m_{atm}^2 \simeq 4|a|^2, \quad (20)
\]

\[
\Delta m_{sol}^2 \simeq |a|^2|\epsilon|\sqrt{|\epsilon|^2 + 8|\lambda|^2}, \quad (21)
\]

\[
\tan^2 2\theta_{12} \simeq 8|\lambda|^2/|\epsilon|^2. \quad (22)
\]

In this case, $|a| = 0.025$ eV, and $\sin^2 \theta_{13} < 1/3$ can be obtained for $|\lambda| < |\epsilon|$. Suppose it is fixed at 0.31, then $|\lambda| = 0.19$, $|\epsilon| = 0.22$, and $|\lambda^2 a| = 9.0 \times 10^{-4}$ eV.

In conclusion, it has been shown in this paper that a new application of the non-Abelian discrete symmetry $A_4$ in the context of the canonical seesaw mechanism is successful in obtaining a realistic neutrino mixing matrix with $\theta_{13} = 0$, $\theta_{23} = \pi/4$, and a prediction of $\sin^2 \theta_{12}$ very near $1/3$ in a particular symmetry limit. As Eq. (13) shows, the normal (inverted) hierarchy of neutrino masses is obtained for $|\lambda|$ less (greater) than 1. Typical values of the effective neutrino mass measured in neutrinoless double beta decay are given in Table 1.

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References

[1] E. Ma and G. Rajasekaran, Phys. Rev. D64, 113012 (2001); E. Ma, Mod. Phys. Lett. A17, 289 (2002); ibid. A17, 627 (2002).

[2] K. S. Babu, E. Ma, and J. W. F. Valle, Phys. Lett. B552, 207 (2003); E. Ma, Mod. Phys. Lett. A17, 2361 (2002); M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle, and A. Villanova del Moral, Phys. Rev. D69, 093006 (2004).

[3] E. Ma, hep-ph/0208077, hep-ph/0208097, hep-ph/0307016, hep-ph/0311215.
[4] E. Ma, Phys. Rev. D70, 031901(R) (2004); New J. Phys. 6, 104 (2004); hep-ph/0409075.

[5] E. Ma, Mod. Phys. Lett. A20, 1953 (2005).

[6] G. Altarelli and F. Feruglio, hep-ph/0504165; E. Ma, Phys. Rev. D72, 037301 (2005).

[7] S.-L. Chen, M. Frigerio, and E. Ma, hep-ph/0504181; M. Hirsch, A. Villanova del Moral, J. W. F. Valle, and E. Ma, hep-ph/0507148.

[8] K. S. Babu and X.-G. He, hep-ph/0507217.

[9] M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity, edited by P. van Nieuwenhuizen and D. Z. Freedman (North-Holland, Amsterdam, 1979), p. 315; T. Yanagida, in Proceedings of the Workshop on the Unified Theory and the Baryon Number in the Universe, edited by O. Sawada and A. Sugamoto (KEK Report No. 79-18, Tsukuba, Japan, 1979), p. 95; R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).

[10] E. Ma, Phys. Rev. D66, 117301 (2002).

[11] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B530, 167 (2002).