Quantum dynamical semigroups generated by noncommutative unbounded elliptic operators

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Abstract

We study quantum dynamical semigroups generated by noncommutative unbounded elliptic operators which can be written as Lindblad type unbounded generators. Under appropriate conditions, we first construct the minimal quantum dynamical semigroups for the generators and then use Chebotarev and Fagnola’s sufficient conditions for conservativity to show that the semigroups are conservative.

Keywords: Quantum dynamical semigroups; noncommutative unbounded elliptic operators; conservativity.

1. Introduction

The purpose of this work is to study quantum dynamical semigroups(q.d.s.) generated by noncommutative unbounded elliptic operators. The generators can be expressed as Lindblad type (unbounded) generators. Under appropriated conditions on coefficients, we first construct the minimal quantum dynamical semigroups for the generators and then use Chebotarev and Fagnola’s sufficient conditions for conservativity to show that the semigroups are conservative. For the details, see Section 3.

Let us first describe briefly the background of this study. In [BP], using a quantum version of Feynman-Kac formula, the authors constructed the Markovian...
semigroup generated by the following noncommutative elliptic operator $\mathcal{L}$ on a von Neumann algebra $\mathcal{M}$ acting on a separable Hilbert space $\mathfrak{h}$:

\begin{align}
D(\mathcal{L}) &= D(\delta^2), \\
\mathcal{L}(X) &= \frac{1}{2}\delta^2(X) + a\delta(X) + \delta(X)a - \frac{1}{2}[a, [a, X]], \quad X \in D(\mathcal{L}),
\end{align}

where $a$ is a self-adjoint element of $\mathcal{M}$, $\delta$ is the generator of a weak*-continuous group of *-automorphisms $(\alpha_t)_{t \in \mathbb{R}}$ of $\mathcal{M}$ and $[A, B] = AB - BA$.

Let $\mathcal{M} = B(\mathfrak{h})$ and $b$ be a self-adjoint operator on $\mathfrak{h}$. Let $\alpha_t(X) = e^{itb}Xe^{-itb}$, $X \in \mathcal{M}$, be the corresponding one parameter group of automorphisms of $\mathcal{M}$. Then

$$\delta(X) = i[b, X], \quad X \in D(\delta).$$

Put

$$L := a - ib, \quad H := \frac{1}{2}(ab + ba).$$

The generator $\mathcal{L}$ in (1.1) can be represented by the following Lindblad type generator:

\begin{align}
\mathcal{L}(X) &= i[H, X] - \frac{1}{2} L^*LX + L^*XL - \frac{1}{2} XL^*L, \quad X \in D(\mathcal{L}),
\end{align}

where $[A, B] = AB - BA$.

In this paper, we consider the following situation: Let $\mathfrak{h} = L^2(\mathbb{R}^d)$ and $W_l(x_1, \cdots, x_d)$, $l = 1, 2, \cdots, d$, denoted by $W_l(x)$, be real valued twice differentiable functions on $\mathbb{R}^d$. For each $l = 1, 2, \cdots, d$, let $\partial_l$ be the differential operator $\frac{\partial}{\partial x_l}$ with respect to the $l$-th coordinate. For each $l = 1, 2, \cdots, d$, we choose

\begin{align}
a_l = -W_l \quad \text{and} \quad b_l = -i\partial_l.
\end{align}

Then by (1.3)

\begin{align}
L_l := -(W_l + \partial_l) \quad \text{and} \quad H_l := \frac{i}{2}(W_l \partial_l + \partial_l W_l).
\end{align}

We are interested in the following (formal) generator $\mathcal{L}$:

\begin{align}
\mathcal{L}(X) &= \sum_{l=1}^{d} \left( i[H_l, X] - \frac{1}{2} L_l L_l X + L_l X L_l - \frac{1}{2} X L_l L_l \right) \\
&= \sum_{l=1}^{d} \left( \frac{1}{2} [\partial_l, [\partial_l, X]] - W_l [\partial_l, X] - [\partial_l, X] W_l - \frac{1}{2} [W_l, [W_l, X]] \right).
\end{align}
It is worth to mention that if $X$ is a smooth function with a compact support on $\mathbb{R}^d$ (a multiplication operator on $L^2(\mathbb{R}^d)$), then $[W_l, X] = 0$, $l = 1, 2, \cdots, d$ and the generator given in (1.6) can be rewritten as

$$\mathcal{L}(X) = \frac{1}{2} \Delta X - 2W \cdot \nabla X, \quad (1.7)$$

where $W = (W_1, \cdots, W_d)$, $\nabla X = (\partial_1 X, \cdots, \partial_d X)$ and $\Delta X = \sum_{l=1}^d \partial_{ll} X$. Thus the operator $\mathcal{L}$ given in (1.6) is a noncommutative generalization of the elliptic operator given in (1.7).

The aim of this paper is to construct the conservative minimal q.d.s. with generator $\mathcal{L}$ given in (1.6) for an unbounded multiplication operator $W_l$, $l = 1, 2, \cdots, d$. Because of the unboundedness, the method of the quantum Feynman-Kac formula in [BP, LS] can not be applied. In [BK], the authors employed the theory of the minimal quantum dynamical semigroup to construct the Markovian semigroup with generator $\mathcal{L}$ in (1.4) under the condition $[a, b]$ is bounded. This condition means that $[W_l, i\partial_l]$ is bounded for any $l = 1, 2, \cdots, d$ in our case. In this paper, we will improve the condition. Suppose that there exist positive constants $k_1$ and $k_2$ such that the bounds

$$\left| \frac{\partial W^2_l}{\partial x_k} \right| \leq k_1 (W^2_1 + \cdots + W^2_d) + k_2, \quad l, k = 1, 2, \cdots, d, \quad (1.8)$$

hold(see Assumption 3.1). Under additional conditions (see Assumption 3.1 and Assumption 3.2), we construct the minimal q.d.s. with generator $\mathcal{L}$ given by (1.6) and show its conservativity by using the result of Fagnola and Chebotarev [CF1, CF2].

The paper is organized as follows: In section 2, we review the theory of the minimal q.d.s. and give Chebotarev and Fagnola’s sufficient conditions for conservativity [CF2]. In section 3, we give the assumptions and example for $W$ and state main results. First, we introduce a proposition related to the perturbation of generator of a strongly continuous contraction semigroup, and then construct the minimal q.d.s. with (formal) generator $\mathcal{L}$. Under additional condition, we show that the q.d.s. is conservative. Section 4 is devoted to proofs of main results.
2. Review on the minimal quantum dynamical semigroups

Let \( \mathfrak{h} \) be a separable Hilbert space with the scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( \mathcal{B}(\mathfrak{h}) \) denote the Banach space of bounded linear operators on \( \mathfrak{h} \). The uniform norm in \( \mathcal{B}(\mathfrak{h}) \) is denoted by \( \| \cdot \|_\infty \) and the identity in \( \mathfrak{h} \) is denoted by \( I \). We denote by \( D(G) \) the domain of operator \( G \) in \( \mathfrak{h} \).

**Definition 2.1** A quantum dynamical semigroup (q.d.s.) on \( \mathcal{B}(\mathfrak{h}) \) is a family \( T = (T_t)_{t \geq 0} \) of bounded operators in \( \mathcal{B}(\mathfrak{h}) \) with the following properties:

(i) \( T_0(X) = X \), for all \( X \in \mathcal{B}(\mathfrak{h}) \),

(ii) \( T_{t+s}(X) = T_t(T_s(X)) \), for all \( s, t \geq 0 \) and all \( X \in \mathcal{B}(\mathfrak{h}) \),

(iii) \( T_t(I) \leq I \), for all \( t \geq 0 \),

(iv) (completely positivity) for all \( t \geq 0 \), all integers \( n \) and all finite sequences \( (X_j)_{j=1}^n \), \( (Y_l)_{l=1}^n \) of elements of \( \mathcal{B}(\mathfrak{h}) \), we have

\[
\sum_{j,l=1}^n Y_l^* T_t(X_l^* X_j) Y_j \geq 0,
\]

(v) (normality or \( \sigma \)-weak continuity) for every sequence \( (X_n)_{n \geq 1} \) of elements of \( \mathcal{B}(\mathfrak{h}) \) converging weakly to an element \( X \) of \( \mathcal{B}(\mathfrak{h}) \) the sequence \( (T_t(X_n))_{n \geq 1} \) converges weakly to \( T_t(X) \) for all \( t \geq 0 \),

(vi) (ultraweak or weak* continuity) for all trace class operator \( \rho \) on \( \mathfrak{h} \) and all \( X \in \mathcal{B}(\mathfrak{h}) \) we have

\[
\lim_{t \to 0^+} Tr(\rho T_t(X)) = Tr(\rho X).
\]

We recall that as a consequence of properties (iii), (iv), for each \( t \geq 0 \) and \( X \in \mathcal{B}(\mathfrak{h}) \), \( T_t \) is a contraction, i.e.,

\[
\|T_t(X)\|_\infty \leq \|X\|_\infty,
\]

and as a consequence of properties (iv), (vi), for all \( X \in \mathcal{B}(\mathfrak{h}) \), the map \( t \mapsto T_t(X) \) is strongly continuous.

**Definition 2.2** A q.d.s. \( \mathcal{T} = (T_t)_{t \geq 0} \) is called to be conservative or Markovian if \( T_t(I) = I \) for all \( t \geq 0 \).
The natural generator of q.d.s. would be the Lindblad type generator \[ L(X) = i[H, X] - \frac{1}{2} XM + \sum_{l=1}^{\infty} L_l^* XL_l - \frac{1}{2} MX, \quad X \in B(\mathfrak{h}) \]

where \( M = \sum_{i=1}^{\infty} L_l^* L_l \), \( L_l \) is densely defined and \( H \) a symmetric operator on \( \mathfrak{h} \). The generator can be formally written by

\[
L(X) = XG + G^* X + \sum_{l=1}^{\infty} L_l^* XL_l,
\]

where \( G = -iH - \frac{1}{2} M \). A very large class of q.d.s. was constructed by Davies satisfying the following assumption. It is basically corresponding to the condition \( L(I) = 0 \).

**Assumption 2.1** The operator \( G \) is the infinitesimal generator of a strongly continuous contraction semigroup \( P = (P(t))_{t \geq 0} \) in \( \mathfrak{h} \). The domain of the operators \( (L_l)_{l=1}^{\infty} \) contains the domain \( D(G) \) of \( G \). For all \( v, u \in D(G) \), we have

\[
\langle v, Gu \rangle + \langle Gv, u \rangle + \sum_{l=1}^{\infty} \langle L_l v, L_l u \rangle = 0. \quad (2.2)
\]

As a result of Proposition 2.5 of [3], we can assume only that the domain of the operators \( L_l \) contains a subspace \( D \) which is a core for \( G \) and (2.2) holds for all \( v, u \in D \).

For all \( X \in B(\mathfrak{h}) \), consider the sesquilinear form \( L(X) \) on \( \mathfrak{h} \) with domain \( D(G) \times D(G) \) given by

\[
\langle v, L(X)u \rangle = \langle v, XGu \rangle + \langle Gv, Xu \rangle + \sum_{l=1}^{\infty} \langle L_l v, XL_l u \rangle. \quad (2.3)
\]

Under the Assumption 2.1 one can construct a q.d.s. \( T = (T_t)_{t \geq 0} \) satisfying the equation

\[
\langle v, T_t(X)u \rangle = \langle v, Xu \rangle + \int_{0}^{t} \langle v, L(T_s(X))u \rangle ds \quad (2.4)
\]

for all \( v, u \in D(G) \) and all \( X \in B(\mathfrak{h}) \). Indeed, for a strongly continuous family \( (T_t(X))_{t \geq 0} \) of elements of \( B(\mathfrak{h}) \) satisfying (2.1), the followings are equivalent:

(i) equation (2.4) holds for all \( v, u \in D(G) \),
(ii) for all \( v, u \in D(G) \) we have
\[
\langle v, T_t(X)u \rangle = \langle P(t)v, XP(t)u \rangle \quad (2.5)
\]
\[
+ \sum_{l=1}^{\infty} \int_0^t \langle L_lP(t-s)v, T_s(X)L_lP(t-s)u \rangle ds.
\]

We refer to the proof of Proposition 2.3 in [CF2]. A solution of the equation (2.5) is obtained by the iterations
\[
\langle u, T_t^{(0)}(X)u \rangle = \langle P(t)u, XP(t)u \rangle \quad (2.6)
\]
\[
\langle u, T_t^{(n+1)}(X)u \rangle = \langle P(t)u, XP(t)u \rangle
\]
\[
+ \sum_{l=1}^{\infty} \int_0^t \langle L_lP(t-s)u, T_s^{(n)}(X)L_lP(t-s)u \rangle ds
\]
for all \( u \in D(G) \). In fact, for all positive elements \( X \in \mathcal{B}(\mathfrak{h}) \) and all \( t \geq 0 \), the sequence of operators \((T_t^{(n)}(X))_{n \geq 0}\) is non-decreasing. Therefore it is strongly convergent and its limits for \( X \in \mathcal{B}(\mathfrak{h}) \) and \( t \geq 0 \) define the minimal solution \((T_t')_{t \geq 0}\) of (2.5) in the sense that, given another solution \((T_t')_{t \geq 0}\) of (2.4), one can easily check that
\[
T_t(X) \leq T_t'(X) \leq \|X\|_\infty I
\]
for any positive element \( X \) and all \( t \geq 0 \). For details, we refer to [Ch1, Fa]. From now on, the minimal solution \((T_t)_{t \geq 0}\) is called the minimal q.d.s.

Chebotarev and Fagnola gave a criteria to verify the conservativity of minimal q.d.s. \((T_t)_{t \geq 0}\) obtained under Assumption 2.1. Here we give their result.

**Theorem 2.1** [Theorem 4.4 in [CF2]] Suppose that there exists a positive self-adjoint operator \( C \) in \( \mathfrak{h} \) with the following properties:

(a) The domain of the positive square root \( C^{1/2} \) contains the domain \( D(G) \) of \( G \) and \( D(G) \) is a core for \( C^{1/2} \),

(b) the linear manifolds \( L_l(D(G^2)) \), \( l \geq 1 \), are contained in the domain of \( C^{1/2} \),

(c) there exists a positive self-adjoint operator \( \Phi \), with \( D(G) \subset D(\Phi^{1/2}) \) such that, for all \( u \in D(G) \), we have
\[
-2Re\langle u, Gu \rangle = \sum_{l=1}^{\infty} \|L_lu\|^2 = \|\Phi^{1/2}u\|^2,
\]
(d) $D(C) \subset D(\Phi)$, and for all $u \in D(C)$ we have $\|\Phi^{1/2} u\| \leq \|C^{1/2} u\|$, 

(e) there exists a positive constant $k$ such that 

\[ 2\text{Re}(C^{1/2}u, C^{1/2}Gu) + \sum_{l=1}^{\infty} \|C^{1/2}L_l u\|^2 \leq k \|C^{1/2}u\|^2, \]  

(2.7) 

for all $u \in D(G^2)$. 

Then the minimal q.d.s. $(T_t)_{t \geq 0}$ is conservative.

3. Conservative minimal quantum dynamical semi-groups: Main results

Let $\mathfrak{h} = L^2(\mathbb{R}^d)$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$, the space of $C^\infty$-functions with compact support. We denote by $\partial_l = \frac{\partial}{\partial x_l}$ ($l = 1, 2, ..., d$) differential operators with respect to the $l$-th coordinate and $\partial_{lk} = \frac{\partial^2}{\partial x_l \partial x_k}$ ($l, k = 1, 2, ..., d$). For any measurable function $T$, we denote the (distributional) derivative $\frac{\partial T}{\partial x_l}$ by $(T)_l$, $l = 1, 2, ..., d$. The Laplacian and the gradient operators are denoted by $\Delta$ and $\nabla$, respectively.

Let a function (vector field) $W : \mathbb{R}^d \to \mathbb{R}^d$, $W = (W_1, W_2, ..., W_d)$, be given, where each component function $W_l(x)$, $l = 1, 2, ..., d$, is a real valued twice differentiable function on $\mathbb{R}^d$. We will denote 

\[ W^2 = \sum_{l=1}^{d} W_l^2, \quad x^2 = \sum_{l=1}^{d} x_l^2, \quad |x| = \left( \sum_{l=1}^{d} x_l^2 \right)^{1/2}. \]

In the rest of this paper we suppose that $W$ satisfies the following assumption.

**Assumption 3.1** The function $W = (W_1, W_2, ..., W_d)$ satisfies the following properties:

(C-1) $W_l \in C^2(\mathbb{R}^d)$, $l = 1, 2, ..., d$,

(C-2) for any $\varepsilon \in (0, 1)$ there exists a positive constant $c(\varepsilon)$, depending on $\varepsilon$, such that 

\[ |(W_l)_k| \leq \varepsilon W + c(\varepsilon) \]  

(3.1) 

for any $l, k = 1, 2, ..., d$,

(C-3) there exist positive constants $c_1$, $c_2$ such that 

\[ |(W_l)_{jk}| \leq c_1 |W| + c_2, \quad l, j, k = 1, 2, ..., d. \]  

(3.2)
Remark 3.1 (a) By (C-1), $W^2_l \in L^2_{\text{loc}}(\mathbb{R}^d)$, $l = 1, 2, ..., d$. Due to Theorem X.28 of [RS], $-\Delta + W^2$ is essentially self adjoint on $\mathcal{D}$.

(b) The condition (C-2) implies that for any $\varepsilon \in (0, 1)$ there exist positive constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$, depending on $\varepsilon$, such that for any $l, k = 1, 2, ..., d$ and $u \in \mathcal{D}$

\[ \| (W^2_l)_k u \|^2 \leq \varepsilon^2 \| W^2 u \|^2 + c_1(\varepsilon) \| u \|^2, \]  
\[ \| (W_l)_k u \|^2 \leq \varepsilon^2 \| W^2 u \|^2 + c_2(\varepsilon) \| u \|^2. \]  

(c) Using the fact that $|W| \leq \frac{1}{2}(\alpha W^2 + \alpha^{-1}), \alpha > 0$, we get from (3.2) that for any $\varepsilon \in (0, 1)$ there exist a positive constant $c_3(\varepsilon^{-1})$, depending on $\varepsilon$

\[ |(W_l)_jk| \leq \varepsilon W^2 + c_3(\varepsilon^{-1}), \ l, j, k = 1, 2, ..., d. \]  

Example 3.1 Let $V : \mathbb{R}^d \to \mathbb{R}$ be the function (potential) given by

\[ V(x) = \sum_{l=1}^{d} a_l x_l^{2n} + Q(x), \]

where $a_l > 0$, $l = 1, 2, ..., d$, and $Q(x)$ is a polynomial with degree less than or equal to $2n - 1$. Choose $W = (W_1, W_2, ..., W_d)$, $W_l = \frac{1}{4}(V)_l$, $l = 1, 2, ..., d$. That is, $W = \frac{1}{4}\nabla V$. Then it is easy to check that for any $l, k = 1, 2, ..., d$,

\[ |(W^2_l)_k(x)| \leq \alpha_1 |x|^{4n-3} + \beta_1, \]
\[ W^2(x) \geq \alpha_2 |x|^{4n-2} - \beta_2, \]  

for some positive constants $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$. Notice that for any $\varepsilon > 0$

\[ |x|^{4n-3} \leq \varepsilon |x|^{4n-2} \text{ if } |x| \geq \varepsilon^{-1}, \]
\[ |x|^{4n-3} \leq \varepsilon^{-(4n-3)} \text{ if } |x| \leq \varepsilon^{-1}. \]  

Combining (3.6) and (3.7), we get that the inequality (3.1) holds. The inequality (3.2) can be checked similarly. Thus $W$ satisfies Assumption 3.1.
Consider the operators $L_l$, $H$, $G_0$ and $G$ on a domain $\mathcal{D}$

\[
L_l u = -(W_l + \partial_l)u, \quad l = 1, \ldots, d, \quad L_l = 0, \quad l > d, \quad (3.8)
\]

\[
H u = \frac{i}{2} \sum_{l=1}^{d} (W_l \partial_l + \partial_l W_l)u = \frac{i}{2} \sum_{l=1}^{d} (2W_l \partial_l + (W_l)l)u, \quad (3.9)
\]

\[
G_0 u = -\frac{1}{2} \sum_{l=1}^{d} L_l^* L_l u = -\frac{1}{2} \sum_{l=1}^{d} (W_l - \partial_l)(W_l + \partial_l)u \quad (3.10)
\]

\[
G_0 u = -\frac{1}{2} (-\Delta + W^2 - \sum_{l=1}^{d} (W_l)l)u,
\]

Clearly $H$ is a densely defined symmetric operator on $\mathcal{D}$. Recall that $-\Delta + W^2$ is essentially self adjoint on $\mathcal{D}$.

**Lemma 3.1** Suppose that $W = (W_1, W_2, \ldots, W_d)$ satisfies Assumption 3.1. Then the derivative $\sum_{l=1}^{d} (W_l)l$ is relatively $-\Delta + W^2$-bounded with relative bound less than 1 on $\mathcal{D}$. Moreover $-G_0$ is positive, essentially self adjoint on $\mathcal{D}$.

The proof of Lemma 3.1 will be given in Section 4. The operator $G_0$ generates a strongly continuous contraction semigroup on $\mathfrak{h}$. Since the adjoint operator $G^*$ of $G$ is given by $G^* = iH + G_0$ on $\mathcal{D}$, $G$ is closable. Denote by $G$ again its closure.

We consider the elliptic operator $\mathcal{L}$ on $B(\mathfrak{h})$ formally given by

\[
\mathcal{L}(X) = i[H, X] - \frac{1}{2} \sum_{l=1}^{d} L_l^* L_l X + \sum_{l=1}^{d} L_l^* X L_l - \frac{1}{2} \sum_{l=1}^{d} XL_l^* L_l,
\]

\[
= G^* X + XG + \sum_{l=1}^{d} L_l^* X L_l, \quad X \in D(\mathcal{L}). \quad (3.12)
\]

**Remark 3.2** In case that $[W_l, i\partial_l]$ is bounded on $\mathcal{D}$ and $d = 1$, the elliptic operator $\mathcal{L}$ in (3.12) was studied in [BK]. In this paper, we will remove the boundedness (see (3.1)).

As mentioned in Introduction, we will construct the minimal q.d.s. with the formal generator (3.12) under Assumption 3.1 and adding appropriate conditions (Assumption 3.2), show the conservativity of the semigroup.

We state our main results. First let us introduce a proposition to show that $G$ is the generator of a strongly continuous contraction semigroup on $\mathfrak{h}$. 
Proposition 3.1 Let \((A, D(A))\) be the generator of a strongly continuous contraction semigroup on a Hilbert space \(\mathfrak{h}\) and let \((B, D(B))\) be a symmetric operator on \(\mathfrak{h}\). Assume that the following properties hold:

(a) there is a dense set \(D\) such that \(D \subset D(A) \cap D(B)\) and \(D\) is a core for \(A\),

(b) there are positive constants \(a, b\) such that the bound

\[
\|Bu\|^2 \leq a^2 \|Au\|^2 + b^2 \|u\|^2
\]

(3.13)

holds for any \(u \in D\),

(c) for any \(\varepsilon > 0\) there is a constant \(\bar{c}(\varepsilon) > 0\), depending on \(\varepsilon\), such that the bound

\[
\pm i \left( \langle Au, Bu \rangle - \langle Bu, Au \rangle \right) \leq \varepsilon \|Au\|^2 + \bar{c}(\varepsilon) \|u\|^2
\]

(3.14)

holds for any \(u \in D\).

Then for any \(\alpha \in \mathbb{R}\) the operator \((A + i\alpha B, D(A))\) generates a strongly continuous contraction semigroup on \(\mathfrak{h}\). Moreover \(D\) is a core for \(A + i\alpha B\).

Now consider the sesquilinear form \(L(X)\) on \(\mathfrak{h}\) with domain \(D \times D\) given by

\[
\langle v, L(X)u \rangle = \langle v, XGu \rangle + \langle Gv, Xu \rangle + \sum_{l=1}^{d} \langle L_lv, XL_lu \rangle
\]

(3.15)

and the semigroup \(\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}\) satisfying the equation

\[
\langle v, T_t(X)u \rangle = \langle v, Xu \rangle + \int_0^t \langle v, L(T_s(X))u \rangle \, ds
\]

(3.16)

for all \(u, v \in D\) and for all \(X \in B(\mathfrak{h})\).

Theorem 3.1 Suppose that \(W = (W_1, W_2, ..., W_d)\) satisfies Assumption 3.1.

(a) The operator \(G\) defined as in (3.11) generates a strongly continuous contraction semigroup on \(\mathfrak{h}\). Moreover \(D = C^\infty_0(\mathbb{R}^d)\) is a core for \(G\).

(b) There exists the minimal q.d.s. \(\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}\) satisfying (3.16).

Next, to show that the minimal q.d.s. \(\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}\) is conservative, let us introduce another assumption for \(W = (W_1, W_2, ..., W_d)\).
Assumption 3.2 There exists a constants $c_4 \in \mathbb{R}$ such that

(C-4) $((W_l)_k) \geq -c_4$ in the sense that for any complex numbers $\xi_1, \xi_2, \ldots, \xi_d$,

$$\sum_{l,k=1}^d \bar{\xi}_k(W_l)_k \xi_l \geq -c_4 \sum_{k=1}^d |\xi_k|^2.$$ 

Remark 3.3 Let $W = (W_1, W_2, \ldots, W_d)$ be given as in Example 3.1. Then (C-4) means that $\text{Hess} V \geq -c_5$, where $\text{Hess} V$ is the Hessian of $V$. 

Theorem 3.2 Suppose that $W = (W_1, W_2, \ldots, W_d)$ satisfies Assumption 3.1 and Assumption 3.2. Then the minimal q.d.s. $T = (T_t)_{t \geq 0}$ satisfying (3.16) is conservative. 

4. Proofs of main results.

In this section, we produce the proofs of Lemma 3.1, Proposition 3.1, Theorem 3.1 and Theorem 3.2. We first give the proof of Lemma 3.1.

Proof of Lemma 3.1 We compute that for $u \in D$

$$\|(-\Delta + W^2)u\|^2 = \|\Delta u\|^2 + \|W^2 u\|^2 + 2\Re\langle -\Delta u, W^2 u \rangle$$

$$= \|\Delta u\|^2 + \|W^2 u\|^2 + 2 \sum_{l=1}^d \left( \langle \partial_l u, W^2 \partial_l u \rangle + \Re \langle \partial_l u, (W^2)_l u \rangle \right)$$

$$\geq \|\Delta u\|^2 + \|W^2 u\|^2 - 2 \sum_{l=1}^d \|\partial_l u\| \|(W^2)_l u\|$$

$$\geq \|\Delta u\|^2 + \|W^2 u\|^2 - \sum_{l=1}^d (\|\partial_l u\|^2 + \|(W^2)_l u\|^2). \quad (4.1)$$

Notice that for any $\bar{\varepsilon} \in (0, 1)$

$$\sum_{l=1}^d \|\partial_l u\|^2 \leq \langle -\Delta u, u \rangle \leq \|\Delta u\| \|u\|$$

$$\leq \frac{1}{2} (\bar{\varepsilon}^2 \|\Delta u\|^2 + \bar{\varepsilon}^{-2} \|u\|^2). \quad (4.2)$$

Choosing $\bar{\varepsilon}$ sufficiently small, we conclude from (4.1), (4.2) and the bound in (3.3) that there exist constants $b_1 > 1$ and $b_2 > 0$ such that

$$\|\Delta u\|^2 + \|W^2 u\|^2 \leq b_1 \|(-\Delta + W^2)u\|^2 + b_2 \|u\|^2 \quad (4.3)$$
for any \( u \in \mathcal{D} \).

Combining (4.3) and (3.4), and choosing \( \varepsilon \) sufficiently small, we obtain that

\[
\| \sum_{l=1}^{d} (W_l)_{l} u \|^{2} \leq b_3 \|(-\Delta + W^2)u\|^2 + b_4 \|u\|^2
\]

(4.4)

for \( u \in \mathcal{D} \) and some \( 0 < b_3 < 1, 0 < b_4 \). This yields the proof of lemma. \( \square \)

**Proof of Proposition 3.1**: Replacing \( B \) by \( a^{-1}B \), we may assume that \( a = 1 \). It follows from (3.14) that for any \( \gamma_1 > 0, \gamma_2 > 0 \) and \( u \in \mathcal{D} \)

\[
\|(A + i\gamma_1 B)u\|^2 - \gamma_2^2 \|Bu\|^2
\]

\[
= \|Au\|^2 + i\gamma_1 (\langle Au, Bu \rangle - \langle Bu, Au \rangle) + (\gamma_2^2 - \gamma_1^2) \|Bu\|^2
\]

\[
\geq (1 - \gamma_1 \varepsilon) \|Au\|^2 + (\gamma_2^2 - \gamma_1^2) \|Bu\|^2 - \gamma_1 \tilde{c}(\varepsilon) \|u\|^2.
\]

By choosing \( \varepsilon < \gamma_1^{-1} \), we conclude that for any \( 0 < \gamma_2 \leq \gamma_1 \) and \( u \in \mathcal{D} \) the bound

\[
\gamma_2^2 \|Bu\|^2 \leq \|(A + i\gamma_1 B)u\|^2 + \gamma_1 \tilde{c}(\varepsilon) \|u\|^2
\]

(4.5)

holds.

Since \( \mathcal{D} \) is a core for \( A \), the bound (3.13) (with \( a = 1 \)) holds for all \( u \in \mathcal{D}(A) \). Thus for any \( 0 < \beta < 1, \beta B \) is relatively \( A \)-bounded with relative bound less than 1. Since \( (B, \mathcal{D}(B)) \) is symmetric, it is dissipative. Therefore the operator \( (A + i\beta B, \mathcal{D}(A)) \) generates a strongly continuous contraction semigroup on \( \mathcal{H} \)(see Corollary 3.3 of [Paz, Chap. 3].) Moreover \( \mathcal{D} \) is a core for \( A + i\beta B \) by (3.13)

The bound (4.5) with \( \gamma_1 = \gamma_2 = \beta \) implies that for \( 0 < \gamma < 1, \beta \gamma B \) is relatively \( A + i\beta B \)-bounded with relative bound less than 1 and so \( (A + i\beta(1+\gamma)B, \mathcal{D}(A)) \) generates a strongly continuous contraction semigroup and \( \mathcal{D} \) is a core for the operator. Since \( \beta \gamma < \gamma_2 = \gamma_1 = \beta(1+\gamma) \), the bound (4.5) implies that \( (A + i\beta(1+2\gamma)B, \mathcal{D}(A)) \) generates a strongly continuous contraction semigroup.

By using an induction argument, we conclude that for any \( \beta, \gamma \in (0, 1) \) and \( n = 1, 2, 3, ... \), the operator \( (A + i\beta(1+n\gamma)B, \mathcal{D}(A)) \) generates a strongly continuous contraction semigroup and \( \mathcal{D} \) is a core for generator. For given \( \alpha > 0 \), one can choose \( \beta, \gamma \in (0, 1) \) and \( n \) such that \( \alpha = \beta(1+n\gamma) \), and for given \( \alpha < 0 \), \( B \) replaces by \(-B \). This completes the proof of the theorem. \( \square \)

In order to show that the operator \( G \) defined as in (3.11) is a generator of a strongly continuous contraction semigroup on \( \mathcal{H} \), we only need to check the conditions of Proposition 3.1.
**Proof of Theorem 3.1** (a) To prove the part (a) of theorem we apply Proposition 3.1 for $A = G_0$, $B = H$ and $D = D$. Clearly $H$ is a symmetric operator on $D$. By Lemma 3.1, $G_0$ is negative, essential self-adjoint on $D$, and so it generates a strongly continuous contraction semigroup. Thus the condition (a) of Proposition 3.1 holds. Let us show the condition (b) of Proposition 3.1. A direct computation yields that for $u \in D$

$$\|Hu\|^2 = \frac{1}{4} \sum_{l=1}^{d} \|2W_l \partial_l + (W_l)_{il}\|^2$$

$$\leq \frac{d}{4} \sum_{l=1}^{d} \|2W_l \partial_l + (W_l)_{il}\|^2$$

$$\leq \frac{d}{2} \sum_{l=1}^{d} \left(4 \|W_l \partial_l u\|^2 + \|(W_l)_{il} u\|^2\right),$$

and

$$\|W_l \partial_l u\|^2 = \langle W_l^2 \partial_l u, \partial_l u \rangle$$

$$= \langle \partial_l W_l^2 u, \partial_l u \rangle - \langle (W_l^2)_{il} u, \partial_l u \rangle$$

$$\leq \|W_l^2 u\| \|\partial_l^2 u\| + \|(W_l^2)_{il} u\| \|\partial_l u\|$$

$$\leq \frac{1}{2} \left(\|W_l^2 u\|^2 + \|\partial_l^2 u\|^2 + \|(W_l^2)_{il} u\|^2 + \|\partial_l u\|^2\right),$$

which implies

$$\|Hu\|^2 \leq d \sum_{l=1}^{d} \left(\|W_l^2 u\|^2 + \|(W_l^2)_{il} u\|^2 + \frac{1}{2} \|(W_l)_{il} u\|^2\right)$$

$$+ d \sum_{l=1}^{d} \left(\|\partial_l^2 u\|^2 + \|\partial_l u\|^2\right).$$

(4.6)

Note that for $u \in D$

$$\sum_{l=1}^{d} \|W_l^2 u\|^2 \leq \|W^2 u\|^2;$$

(4.7)

$$\sum_{l=1}^{d} \|\partial_l^2 u\|^2 \leq \|\sum_{l=1}^{d} \partial_l^2 u\|^2 = \|\Delta u\|^2,$$

where we have used that for $l, k = 1, 2, ..., d$

$$\langle \partial_l^2 u, \partial_k^2 u \rangle = \langle \partial_{kl} u, \partial_{kl} u \rangle \geq 0.$$
Applying (4.2), (4.7) and (3.3) into (4.6), we get that there exist constants $a_1 > d$ and $a_2 > 0$ such that for any $u \in D$

$$\|Hu\|^2 \leq a_1(\|\Delta u\|^2 + \|W^2 u\|^2) + a_2\|u\|^2. \tag{4.8}$$

On the other hand, for any $\varepsilon \in (0, 1)$ and $u \in D$, we have

$$\|G_0 u\|^2 = \frac{1}{4}\|(-\Delta + W^2 - \sum_{l=1}^d (W_l)_{il}) u\|^2 \geq \frac{1}{4}\|(-\Delta + W^2)u\| - \|\sum_{l=1}^d (W_l)_{il}u\|^2 \geq \frac{1}{4}\left((1 - \varepsilon)\|(-\Delta + W^2)u\|^2 + (1 - \varepsilon^{-1})\|\sum_{l=1}^d (W_l)_{il}u\|^2\right) \geq \frac{1}{4}\left((1 - \varepsilon)\|(-\Delta + W^2)u\|^2 - \varepsilon^{-1}\|\sum_{l=1}^d (W_l)_{il}u\|^2\right). \tag{4.9}$$

Substituting (4.1) into (4.9), we have

$$\|G_0 u\|^2 \geq \frac{1}{4}(1 - \varepsilon)(\|\Delta u\|^2 + \|W^2 u\|^2) - \frac{1}{4}\sum_{l=1}^d (\|\partial_l u\|^2 + \|(W_{il})_{il}\|^2 + \varepsilon^{-1}\|\sum_{l=1}^d (W_l)_{il}u\|^2). \tag{4.10}$$

Choosing $\varepsilon$ sufficiently small, we conclude from (4.10), (4.2) and the bound in (3.3) that there exist constants $a_3 > 4$ and $a_4 > 0$ such that

$$\|\Delta u\|^2 + \|W^2 u\|^2 \leq a_3\|G_0 u\|^2 + a_4\|u\|^2 \tag{4.11}$$

for any $u \in D$. Combining (4.11) and (4.8), we obtain that

$$\|Hu\|^2 \leq a_5\|G_0 u\|^2 + a_6\|u\|^2, \quad u \in D \tag{4.12}$$

for some $a_5 > 4d$ and $a_6 > 0$. This proves the inequality (3.13).

Next we consider the commutator estimate in (3.14). Recall that

$$A = G_0 = -\frac{1}{2}\sum_{l=1}^d L^*_l L_l = -\frac{1}{2}(-\Delta + W^2 - \sum_{l=1}^d (W_l)_{il}),$$

$$B = H = \frac{i}{2}\sum_{l=1}^d (W_l \partial_l + \partial_l W_l). \tag{4.13}$$
We can write that
\[
\pm i \left[ -\frac{1}{2} \sum_{l=1}^{d} L_l^* L_l, H \right] = \pm \frac{1}{4} \left[ -\Delta + W^2 - \sum_{l=1}^{d} (W_l)_{ll}, \sum_{k=1}^{d} (W_k \partial_k + \partial_k W_k) \right]
\]
\[
= \pm \frac{1}{4} \sum_{k=1}^{d} [-\Delta, W_k \partial_k + \partial_k W_k]
\]
\[
\pm \frac{1}{4} \sum_{k=1}^{d} [W^2 - \sum_{l=1}^{d} (W_l)_{ll}, W_k \partial_k + \partial_k W_k]. \tag{4.14}
\]

Notice that
\[
[-\Delta, W_k \partial_k + \partial_k W_k] = [-\Delta, W_k] \partial_k + \partial_k [-\Delta, W_k] \tag{4.15}
\]
\[
= - \sum_{l=1}^{d} ((W_k)_l \partial_k + (W_k)_l \partial_k + \partial_k (W_k)_l + \partial_k (W_k)_l \partial_l)
\]
\[
= - \sum_{l=1}^{d} ((W_k)_l \partial_k + 2(W_k)_l \partial_k + 2 \partial_k (W_k)_l - \partial_k (W_k)_l)\]

and
\[
[W^2 - \sum_{l=1}^{d} (W_l)_{ll}, W_k \partial_k + \partial_k W_k] \tag{4.16}
\]
\[
= [W^2, W_k \partial_k + \partial_k W_k] - \sum_{l=1}^{d} [(W_l)_{ll}, W_k \partial_k + \partial_k W_k]
\]
\[
= -2W_k ((W^2)_k - \sum_{l=1}^{d} (W_l)_{lk})
\]
\[
= -2 \sum_{l=1}^{d} W_k ((W^2)_k - (W_l)_{lk})
\]
as bilinear forms on \( D \). Substituting (4.15) and (4.16) into (4.14), we obtain that for \( u \in D \)
\[
\pm \langle u, i[-\frac{1}{2} \sum_{l=1}^{d} L_l^* L_l, H]u \rangle = \pm \sum_{l,k=1}^{d} \Re \left( \langle (W_k)_l u, \partial_l u \rangle + \frac{1}{2} \langle (W_k)_l u, \partial_k u \rangle \right)
\]
\[
\pm \frac{1}{2} \sum_{l,k=1}^{d} \langle W_k u, ((W_l^2)_k - (W_l)_{lk}) u \rangle. \tag{4.17}
\]
Notice that for $\varepsilon \in (0, 1)$ and $u \in D$,

$$\sum_{l,k=1}^{d} |\langle (W_k)_l u, \partial_{kl} u \rangle| \leq \sum_{l,k=1}^{d} \| (W_k)_l u \| \| \partial_{kl} u \|$$

$$\leq \sum_{l,k=1}^{d} \frac{1}{2} \left( \varepsilon \| \partial_{kl} u \|^{2} + \varepsilon^{-1} \| (W_k)_l u \|^{2} \right)$$

$$\leq \sum_{l,k=1}^{d} \frac{1}{2} \left( \varepsilon \| \partial_{kl}^2 u \| + \varepsilon \| W^2 u \|^{2} + b_1 \| u \|^{2} \right)$$

$$= \frac{1}{2} \left( \varepsilon \| \Delta u \|^{2} + d \varepsilon \| W^2 u \|^{2} + b_1 d \| u \|^{2} \right)$$

(4.18)

for some constant $b_1 > 0$. Here we have used (3.4) in third inequality. Similarly, for $\varepsilon \in (0, 1)$ and $u \in D$ we get from (3.5) and (4.2) that

$$\sum_{l,k=1}^{d} |\langle (W_k)_l u, \partial_k u \rangle| \leq \sum_{l,k=1}^{d} \frac{1}{2} \left( \| (W_k)_l u \|^{2} + \| \partial_k u \|^{2} \right)$$

$$\leq \frac{1}{2} \left( \varepsilon d \| \Delta u \|^{2} + d^2 \varepsilon \| W^2 u \|^{2} + b_2 \| u \|^{2} \right)$$

(4.19)

and also by (3.3) and (3.5),

$$\sum_{l,k=1}^{d} |\langle W_k u, ((W^2)_l)_k - (W_l)_k u \rangle|$$

(4.20)

$$\leq \sum_{l,k=1}^{d} \frac{1}{2} \left( \varepsilon \| W_k u \|^{2} + \varepsilon^{-1} \| (W^2)_l k u \|^{2} + \| (W_l)_k u \|^{2} \right)$$

$$\leq \varepsilon b_3 \| W^2 u \|^{2} + b_4 \| u \|^{2}$$

for some constants $b_2, b_4$ depending on $\varepsilon$ and $b_3 > 0$, where we have used

$$\sum_{k=1}^{d} \| W_k u \|^{2} = \sum_{k=1}^{d} \langle u, W_k^2 u \rangle = \langle u, W^2 u \rangle \leq \frac{1}{2} (\| u \|^{2} + \| W^2 u \|^{2}).$$

Then substituting (4.18), (4.19) and (4.20) into (4.17), one has that $\tilde{\varepsilon} \in (0, 1)$ and some $b_5 > 0$

$$\pm \langle u, i \left[-\frac{1}{2} \sum_{l=1}^{d} L_l^* L_l, -H \right] u \rangle \leq \tilde{\varepsilon} (\| \Delta u \|^{2} + \| W^2 u \|^{2}) + b_5 \| u \|^{2}.$$  

(4.21)
Here we can choose $\varepsilon$ as small as possible. Hence two inequalities (4.11) and (4.21) produces that for any $\varepsilon'>0$ there is a constant $b_6$, depending on $\varepsilon'$, such that the bound

$$\pm \langle u, i \left[ -\frac{1}{2} \sum_{l=1}^{d} L_l^* L_l, -H \right] u \rangle \leq \varepsilon' \|G_0 u\|^2 + b_6 \|u\|^2.$$  

(4.22)

The part (a) of the proof is completed.

(b) By (a), $G$ generates a strongly continuous contraction semigroup on $\mathfrak{h}$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$ is a core for $G$. We get from (3.10) and (3.11) that we have

$$\langle v, Gu \rangle + \langle Gv, u \rangle + \sum_{l=1}^{d} \langle L_l v, L_l u \rangle = 0$$  

(4.23)

for all $u, v \in \mathcal{D}$. Thus $G$ and $L_l$, $l = 1, 2, \ldots, d$, satisfy the condition (2.2) on $\mathcal{D}$, a core for $G$, and so Assumption 2.1 is satisfied. Therefore, as mentioned in Section 2, by the iterations, we can construct a minimal q.d.s. $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ satisfying the equation (3.16). □

**Proof of Theorem 3.2** Applying Theorem 2.1 we show that the minimal q.d.s. is conservative. Let us choose the operator $C$

$$C = -2G_0 = \sum_{l=1}^{d} L_l^* L_l = -\Delta + W^2 - \sum_{l=1}^{d} (W_l)_l,$$  

(4.24)

$$D(C) = \{u \in L^2(\mathbb{R}^d) | \text{the distribution } Cu \in L^2(\mathbb{R}^d) \}.$$  

Recall that $\mathcal{D}$ is a core for $C$. We have that as bilinear forms on $\mathcal{D}$

$$G^* G = (iH + G_0)(-iH + G_0)$$

$$= H^2 + G_0^2 + i[H, G_0]$$

$$\geq G_0^2 + i[H, G_0].$$  

(4.25)

It follows from (4.25) and (4.22) that we have

$$\|G_0 u\|^2 \leq a \|Gu\|^2 + b \|u\|^2, \quad u \in \mathcal{D}$$  

(4.26)

for some $a, b > 0$. Using the relations (4.21), (4.26) and the fact that $-iH$ is relatively bounded perturbation of $G_0$, we obtain that $G$ and $C$ are relatively bounded with respect to each other and so $D(G) = D(C)$.  

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We will check that the operator $C$ satisfies Theorem 2.1. Hypothesis (a) and (b) of Theorem 2.1 are trivially fulfilled. To check the condition (e) of Theorem 2.1, we estimate

$$CG + G^*C + \sum_{l=1}^{d} L_l^* CL_l = i[H, C] + \frac{1}{2} \sum_{l=1}^{d} \left( L_l^*[C, L_l] + (L_l^*[C, L_l])^* \right)$$

(4.27)

as bilinear forms on $D$.

We obtain from (4.17) and $C = \sum_{l=1}^{d} L_l^* L_l$ that

$$i[H, C] = -\sum_{l,k=1}^{d} \left( (W_k)_l \partial_{kl} + \partial_{lk}(W_k)_l + \frac{1}{2}((W_k)_{ll}\partial_k - \partial_k(W_k)_{ll}) \right)$$

$$-\sum_{l,k=1}^{d} W_k \left( (W_l^2)_k - (W_l)_lk \right)$$

$$= -\sum_{l,k=1}^{d} \left( 2\partial_l(W_k)_l \partial_k + \partial_l(W_k)_{lk} - \frac{1}{2}((W_k)_{ll}\partial_k + \partial_k(W_k)_{ll}) \right)$$

(4.28)

$$-\sum_{l,k=1}^{d} \left( 2W_kW_l(W_l)_k - W_k(W_l)_{lk} \right)$$

as bilinear forms on $D$. On the other hand, we have

$$[C, L_l] = [-\Delta + W^2 - \sum_{k=1}^{d} (W_k)_k, W_l + \partial_l]$$

$$= -[\Delta, W_l] + [W^2, \partial_l] - \sum_{k=1}^{d} [(W_k)_k, \partial_l]$$

$$= \sum_{k=1}^{d} \left( -\partial_k(W_l)_k - (W_l)_k \partial_k - 2W_k(W_k)_l + (W_k)_{kl} \right)$$

$$= \sum_{k=1}^{d} \left( -2\partial_k(W_l)_k + (W_l)_{kk} - 2W_k(W_k)_l + (W_k)_{kl} \right),$$
which implies

\[
\frac{1}{2} \sum_{l=1}^{d} (L_l^*[C, L_l] + (L_l^*[C, L_l])^*)
\]

\[
= \frac{1}{2} \sum_{l,k=1}^{d} \left\{ ((W_l - \partial_l)(-2\partial_k(W_l)_k + U(k, l)) + (2(W_l)_k\partial_k + U(k, l))(W_l + \partial_l) \right\}
\]

\[
= \sum_{l,k=1}^{d} \left\{ ((W_l)_k\partial_k W_l - W_l\partial_k(W_l)_k) + W_l U(k, l) \right\}
\]

\[
+ \sum_{l,k=1}^{d} \left\{ (\partial_{kl}(W_l)_k + (W_l)_k\partial_{lk}) - \frac{1}{2} [\partial_l, U(k, l)] \right\}
\]

as bilinear forms on \( \mathcal{D} \), where \( U(k, l) = (W_l)_kk - 2W_l(W_k)_l + (W_k)_{kl} \). Notice that

\[
[\partial_l, U(k, l)] = [\partial_l, (W_l)_kk + (W_k)_{kl}] - 2((W_k)_l)^2 - 2W_k(W_k)_{ll}
\]

as bilinear forms on \( \mathcal{D} \). Thus we have

\[
\frac{1}{2} \sum_{l=1}^{d} (L_l^*[C, L_l] + (L_l^*[C, L_l])^*)
\]

\[
= \sum_{l,k=1}^{d} \left\{ ((W_l)_k^2 - W_l(W_l)_{kk}) + W_l ((W_l)_kk - 2W_k(W_k)_l + (W_k)_{kl}) \right\}
\]

\[
+ \sum_{l,k=1}^{d} (2\partial_k(W_l)_k\partial_l + \partial_k(W_l)_{kl} - (W_l)_{kk}\partial_l)
\]

\[
+ \sum_{l,k=1}^{d} \left( - \frac{1}{2} [\partial_l, (W_l)_{kk} + (W_k)_{kl}] + ((W_k)_l)^2 + W_k(W_k)_{ll} \right)
\]

\[
= \sum_{l,k=1}^{d} \left\{ 2((W_l)_k^2 + \partial_k(W_l)_k\partial_l - W_lW_k(W_k)_{ll}) + (W_l(W_k)_{kl} + W_k(W_k)_{ll}) \right\}
\]

\[
+ \sum_{l,k=1}^{d} \left( (\partial_k(W_l)_{kl} - (W_l)_kk\partial_l) - \frac{1}{2} [\partial_l, (W_l)_kk + (W_k)_{kl}] \right), \quad (4.29)
\]

as bilinear forms on \( \mathcal{D} \). Exchanging \( l \) and \( k \) in (4.28), and substituting (4.28) and
into (4.27), one has

\[
CG + G^*C + \sum_{l=1}^{d} L_l^* C L_l
\]

(4.30)

\[
= \sum_{l,k=1}^{d} \left( -4W_l W_k (W_k)_l - \frac{1}{2} \partial_l (W_k)_{kl} \right)
\]

\[
+ \sum_{l,k=1}^{d} \left( 2((W_k)_l)^2 + 2W_l (W_k)_{kl} + W_k (W_k)_l \right)
\]

as bilinear forms on \( \mathcal{D} \). By (C-4), we get that for \( u \in \mathcal{D} \)

\[
-4 \langle u, \sum_{l,k=1}^{d} W_l (W_k)_l W_k u \rangle = -4 \sum_{l=1}^{d} \langle W_l u, \sum_{k=1}^{d} (W_k)_l W_k u \rangle
\]

\[
\leq 4c_5 \sum_{l=1}^{d} \langle W_l u, W_l u \rangle = 4c_5 \langle u, W^2 u \rangle.
\]

(4.31)

And it follows from (C-2) and (C-3) that

\[
|2((W_k)_l)^2 + 2W_l (W_k)_{kl} + W_k (W_k)_l| \leq b_1 W^2 + b_2,
\]

(4.32)

\[
\sum_{l,k=1}^{d} |\langle u, \partial_l (W_k)_{kl} u \rangle| \leq \sum_{l,k=1}^{d} \frac{1}{2} \left( \|\partial_l u\|^2 + \|(W_k)_{kl} u\|^2 \right)
\]

\[
\leq \frac{1}{2} \left( d \langle u, -\Delta u \rangle + d^2 \langle u, (b_3 W^2 + b_4) u \rangle \right),
\]

for \( u \in \mathcal{D} \) and some positive constants \( b_i \), \( i = 1, 2, 3, 4 \). Applying (4.31) and (4.32) into (4.30), we have

\[
2Re\langle Cu, Gu \rangle + \sum_{l=1}^{d} \langle L_l u, C L_l u \rangle \leq b_5 \langle u, (-\Delta + W^2) u \rangle + b_6 \langle u, u \rangle, \quad u \in \mathcal{D}
\]

for some \( b_5, b_6 > 0 \). By (C-2), for \( u \in \mathcal{D} \) and \( \varepsilon \in (0, 1) \) we have

\[
\langle u, Cu \rangle = \langle u (-\Delta + W^2) u \rangle - \sum_{l=1}^{d} \langle u, (W_l)_l u \rangle
\]

\[
\geq \langle u, (-\Delta + W^2) u \rangle - \varepsilon \langle u, W^2 u \rangle - b_7\|u\|^2
\]

\[
\geq (1 - \varepsilon) \langle u, (-\Delta + W^2) u \rangle - b_7\|u\|^2,
\]

for some constant \( b_7 > 0 \). Thus for \( u \in \mathcal{D} \) and some \( b_8, b_9 > 0 \)

\[
2Re\langle Cu, Gu \rangle + \sum_{l=1}^{d} \langle L_l u, C L_l u \rangle \leq b_8 \langle u, Cu \rangle + b_9 \langle u, u \rangle.
\]

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Redefine \( C = \sum_{i=1}^{d} L_i^* L_i + \frac{b_8}{b_9} \), then by \((4.23)\) we have

\[
2Re\langle Cu, Gu \rangle + \sum_{i=1}^{d} \langle L_i u, CL_i u \rangle \leq b_8 \langle u, Cu \rangle, \quad u \in \mathcal{D}. \tag{4.33}
\]

We want to extend the inequality \((4.33)\) to the domain \( D(G) \). Since \( G \) and \( C \) are relatively bounded with respect to each other, there exists a sequence \( \{u_n\} \) of elements of \( \mathcal{D} \) such that

\[
\lim_{n \to \infty} u_n = u, \quad \lim_{n \to \infty} Cu_n = Cu, \quad \lim_{n \to \infty} Gu_n = Gu, \quad u \in D(G).
\]

Then the relation \((4.33)\) implies that \( \{C^{1/2}L_i u_n\}_{n \geq 1} \) is a Cauchy sequence. Therefore it is convergent and it is easy to deduce that \((4.33)\) holds for \( u \in D(G) \).

Note that \( \Phi = \sum_{i=1}^{d} L_i^* L_i \leq C (= \Phi + \frac{b_8}{b_9}) \) as bilinear forms on \( \mathcal{D} \). Hence the conditions (c), (d) of Theorem 2.1 also hold and the minimal q.d.s. is conservative.\( \square \)

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