STRING QUANTUM SYMMETRIES AND THE SL(2,\(\mathbb{Z}\)) GROUP

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ABSTRACT

We prove, using arguments relying only on the "special Kähler" structure of the moduli space of the Calabi-Yau three-fold, that in the case of one single modulus the quantum modular group of the string effective action corresponding to Calabi-Yau vacua can not be \(\text{SL}(2,\mathbb{Z})\).
A lot of effort has been recently devoted in order to understand the local and global properties of the moduli space of the Calabi-Yau manifolds [1-3]. Some of the low-energy parameters, corresponding to the vacua of the heterotic string described by compactifications on a Calabi-Yau three-fold or equivalently by a N=2, c=9 world-sheet Superconformal Field Theory (SCFT), have been computed by using techniques of algebraic geometry [2]. The relevant information about these couplings is encoded in a 4th-order linear holomorphic differential equation, particular case of the so called "Picard-Fuchs" equations that can be written for any Calabi-Yau d-fold [4]. It was also realised that the existence of such differential equations implies the geometrical structure of the moduli space to be of restricted type, namely "special Kähler" [1,6]. A crucial role in this contest is played by a group of discrete symmetries, the so called modular (duality) group, which acts isometrically on the Kähler manifold. Owing to their "stringy" origin they are believed to be preserved by non perturbative world-sheet effects, consequently they have been used to constrain the form of the low-energy couplings [7,8]. Despite a lot of effort in trying to understand the meaning of these symmetries we are still far away of having full control of them. What seems however clear is that the modular group related to toroidal and orbifolds compactifications, namely SL(2,\mathbb{Z}), has not a big chance to survive as the right "quantum" modular group of the string. In fact for the case of the exactly soluble SCFT given by the pair of Calabi-Yau manifolds of Candelas, the modular group was shown to be not SL(2,\mathbb{Z}) but a different discrete subgroup of SL(2,\mathbb{R}) [2]. Nevertheless nothing in principle prevents that some specific Calabi-Yau three-fold could exist whose modular group is exactly SL(2,\mathbb{Z}). In this letter we will rule out this possibility for the one modulus case
by showing that the only choices for the Yukawa coupling, compatible with the $\text{SL}(2,\mathbb{Z})$ symmetry, are either $W = \text{const} \neq 0$ \(^1\), corresponding to toroidal compactifications i.e the large radius limit of Calabi-Yau compactifications, or $W \sim \eta^8$ ($\eta$ is the Dedekind function). The second possibility is of no direct physical interest, at least for the one modulus case, because gives $W = 0$ in the large radius limit.

The moduli space of Calabi-Yau manifolds is locally a special Kähler manifold defined by the Kähler potential [1,6] (we will only consider the one modulus case)

$$K = -\ln(V(iQ)V\dagger),$$

where $Q$ is the symplectic metric

$$Q = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix},$$

and

$$V = (X^A(z), F_A(z)) := (X^0, X^1, F_0, F_1), \quad F_A = \frac{\partial F}{\partial X^A}. \quad (3)$$

$X^A$ and $F$ are both holomorphic and $F(X^A)$ is homogeneous of degree two in $X^A$. The Kähler potential of eq. (1) is manifestly invariant under $\text{Sp}(4, \mathbb{R})$ global transformations. In the following we will consider ”special coordinates” defined by

$$t = \frac{X^1}{X^0}, \quad X^0 = 1. \quad (4)$$

In these coordinates the Yukawa coupling $W$ is given in terms of $F$ by

$$W = \partial^3 F. \quad (5)$$

\(^1\) We use $W$ for the Yukawa couplings. The reader should avoid any confusion with the standard supergravity notation where $W$ denotes the superpotential.
All special Kähler geometries in one dimension lead to a 4th-order differential equation characterised by some coefficients $\omega_2$ and $\omega_4$ [5]

$$\left[ \partial^4 + \omega_2 \partial^2 + \omega'_2 \partial + \frac{3}{10} \omega''_2 + \frac{9}{100} \omega''_2 + \omega_4 \right] V = 0. \quad (6)$$

Under coordinates change $\tilde{z} = \tilde{z}(z)$, $\omega_2$, $\omega_4$ and $V$ transform as

$$\tilde{\omega}_2 = \xi^{-2} [\omega_2 - 5 \{\tilde{z}, z\}] ,$$

$$\tilde{\omega}_4 = \xi^{-4} \omega_4 ,$$

$$\tilde{V} = \xi^{\frac{3}{2}} V , \quad (7)$$

where $\xi = \partial \tilde{z} / \partial z$ and $\{\tilde{z}, z\}$ is the Schwarzian derivative.

The modular group associated to the special Kähler manifold is the subgroup $\Gamma$ of $\text{Sp}(4, \mathbb{Z})$ which acts isometrically

$$K (\Gamma z) = K(z) - \Lambda(z) - \Lambda(\bar{z}). \quad (8)$$

The key point is that in terms of the 4th-order differential equation imposing the condition (8) amounts to require that under the action of $\Gamma$, $\omega_2$ and $\omega_4$ are form-invariant, i.e

$$\tilde{\omega}_2(\tilde{z}) = \omega_2(\tilde{z}) , \quad \tilde{\omega}_4(\tilde{z}) = \omega_4(\tilde{z}). \quad (9)$$

The previous equations impose severe constraints on the form of $\omega_2$ and $\omega_4$ and eventually enable one to write them explicitly. Let us now consider $\Gamma = \text{SL}(2, \mathbb{Z})$ and assume that in terms of the special coordinate $t$ it is realised as the linear fractional transformation (the general case will be discussed later on this paper)

$$\Gamma : \quad \tilde{t} = \frac{at + b}{ct + d} , \quad ad - bc = 1 , \quad a, b, c, d \in \mathbb{Z} . \quad (10)$$
From the first two eqs. in (7) it follows that for Γ given by (10) eqs. (9) admit only the two solutions

\[ \omega_2 = \omega_4 = 0, \]  

\[ \omega_2(t) = (ct + d)^4 \omega_2(t), \quad \omega_4(t) = (ct + d)^8 \omega_4(t). \]  

In order to discuss the previous solutions we need the expressions relating ω₂ and ω₄ to the Yukawa coupling W. They can be found in Ref. [5]. We have

\[ \omega_2 = \frac{1}{2W^2} (4W\dot{W} - 5W'^2), \]  

\[ \omega_4 = \frac{1}{100W^4} \left( 175W'^4 - 280W\dot{W}^2W'' + 49W^2W'^2 + 70W^2W'^{'} + 10W^3W''\right). \]

To put these equations in a more manageable form we introduce a new variable U related to W by \( W = U^{-4} \), in terms of which we get

\[ \omega_2 = -8 \frac{U''}{U}, \]  

\[ \omega_4 = - \left( \frac{U'}{U} \right)^2 \omega_2 - \frac{1}{2} \left( \frac{U'}{U} \right) \omega_2' - \frac{1}{20} \omega_2'' + \frac{7}{200} \omega_2^2. \]  

Let us now discuss the two solutions (11) and (12). The solution (11) corresponds to \( W = \text{const} \) or equivalently to cubic \( F \) functions which describe toroidal compactifications (the large radius limit of Calabi-Yau compactifications). The differential equation (6) is solved by [5] (Our solution differs from that of Ref. [5] owing to the choice of the symplectic metric in (2))

\[ V = \left( 1, t, -\frac{1}{6}t^3, \frac{1}{2}t^2 \right). \]  

Note that in this case \( \text{SL}(2,\mathbb{Z}) \) appears as a discrete subgroup of \( \text{Sp}(4,\mathbb{R}) \). In fact from the action of \( M \in \text{Sp}(4,\mathbb{R}) \) on V

\[ \tilde{V} = VM, \]  

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and from (10) we get the embedding of $\text{SL}(2,\mathbb{Z})$ in $\text{Sp}(4,\mathbb{R})$ for the case under consideration\(^2\)

\[
M = \begin{pmatrix}
d^3 & bd^2 & -b^3/6 & b^2d/2 \\
3cd^2 & 2bcd + ad^2 & -ab^2/2 & b^2c/2 + abd \\
-6c^3 & -6ac^2 & a^3 & -3a^2c \\
6c^2d & 2bc^2 + 4acd & -a^2b & 2abc + a^2d
\end{pmatrix}.
\]

By changing the symplectic basis one can achieve $M \in \text{Sp}(4,\mathbb{Z})$.

We turn now to the solutions defined by (12). Eq. (13) together with the first equation in (12) enables one to find how $U$ and $W$ transform under $\text{SL}(2,\mathbb{Z})$. We get

\[
U(\tilde{t}) = (ct + d)^{-1} U(t),
\]

\[
W(\tilde{t}) = (ct + d)^4 W(t).
\]

Eqs. (19) and (12) tell us that $\omega_2$, $\omega_4$ and $W$ have to transform under $\text{SL}(2,\mathbb{Z})$ as modular functions of definite weight. They should therefore be expressible in $B$ terms of modular forms of $\text{SL}(2,\mathbb{Z})$. We will concentrate in the following only on the Yukawa coupling $W$, the coefficients $\omega_2$, $\omega_4$ being determined by (13) and (14) once $W$ is known. The most natural way to implement eq. (19) is to set

\[
W(t) = \eta^8(t),
\]

where $\eta$ is the Dedekind eta-function

\[
\eta(t) = \exp \left( \frac{i\pi}{12} t \right) \prod_n [1 - \exp (2i\pi nt)] .
\]

The solution of the differential equation (6) can be now written as

\[
V = (1, t, F_0, F_1),
\]

\(^2\) This matrix was first given in Ref. [11]
with \( F_0 \) and \( F_1 \) satisfying
\[
\partial^2 F_0 = -t \eta^8, \tag{23}
\]
\[
\partial^2 F_1 = \eta^8. \tag{24}
\]

Eqs. (20-24) describe a well defined solution to one-dimensional special Kähler geometry with \( \text{SL}(2,\mathbb{Z}) \) isometries. However, as one can easily see from the asymptotic behaviour of \( \eta \), as \( t \to i \infty \), \( W \to 0 \). The large radius limit does not correspond to the usual one \( W = \text{const} \) for the Calabi-Yau moduli space. Naturally one could search for solutions of (19) different from (20). The only possible generalizations of eq. (20) satisfying the modular constraint (19) are either \( W(t) = \eta^8(t)S(J) \) or \( W(t) = G_4(t)H(J) \), where \( G_4(t) \) is the Einsestein function of weight four and \( S(J), H(J) \) are rational functions of the absolute modular invariant \( J(t) \). A definition of the modular forms \( G_4(t) \) and \( J(t) \) can be found in Ref. [8]; for more details on the modular forms of \( \text{SL}(2,\mathbb{Z}) \) see e.g Ref. [9]. One can easily check, using the asymptotic expansions of \( \eta, G_4 \) and \( J \), that there is no choice for \( S(J) \) or \( H(J) \) which in the large radius limit reproduces \( W = \text{const} \).

To complete our analysis on the solutions defined by (12) we have to find the embedding of \( \text{SL}(2,\mathbb{Z}) \) in \( \text{Sp}(4,\mathbb{Z}) \) which is consistent with (20-24). To this purpose we write down explicitly the general \( \text{Sp}(4,\mathbb{Z}) \) transformation (16) in special coordinates
\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{25}
\]
\[
\frac{\tilde{X}^1}{X^0} = \tilde{t} = \frac{a_{11} + a_{22}t + c_{12}F_0 + c_{22}F_1}{a_{11} + a_{21}t + c_{11}F_0 + c_{21}F_1}, \tag{26}
\]
\[
\tilde{F}_0 = b_{11} + b_{21}t + d_{11}F_0 + d_{21}F_1, \tag{27}
\]
\[
\tilde{F}_1 = b_{12} + b_{22}t + d_{12}F_0 + d_{22}F_1. \tag{28}
\]
where \(a_{ij}, b_{ij}, c_{ij}, d_{ij}, i, j = 1, 2\) are the matrix elements of the matrices \(A, B, C, D\) respectively and we took \(M \in \text{Sp}(4, \mathbb{Z})\), i.e the entries of \(M\) are integers verifying \(M^T Q M = M\).

The embedding of \(\text{SL}(2, \mathbb{Z})\) in \(\text{Sp}(4, \mathbb{Z})\) is now easily found setting in (25)

\[
B = C = 0, \quad A = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad D = (A^T)^{-1}, \quad ad - bc = 1.
\]  

(29)

The transformation (26) reduces now to the \(\text{SL}(2, \mathbb{Z})\)-one given by (10) whereas (27) and (28) become

\[
\tilde{F} = F D,
\]  

(30)

where the matrix \(D\) is given as in (29) and \(F\) is the row vector \(F_A\). On the other hand the transformation of \(F_A\) leaving invariant the solutions of (23-24) is found using (10) and (19):

\[
F(t) = (ct + d)^{-1} F(\tilde{t}) D.
\]  

(31)

The action of the modular group leaves the Kähler potential invariant up to a Kähler transformation. This implies in general

\[
\tilde{F}(t) = u(t) F(\tilde{t}).
\]

With the identification \(u(t) = (ct + d)\) (30) and (31) are equivalent, demonstrating that the embedding (29) is the right one. Our prove is now complete. There is no solution to one dimensional Kähler geometry exhibiting \(\text{SL}(2, \mathbb{Z})\) isometries and having as large radius limit \(W = \text{const}\), except the limiting case itself. Because any Calabi-Yau compactification should result in a particular special Kähler geometry for the moduli space, one can exclude, in the one-dimensional case, \(\text{SL}(2, \mathbb{Z})\) as the ”quantum” modular group of the string, at least if one wants to recover for large radii the usual field theoretical limit.
Our argumentation is based on the form (10) for the $\text{SL}(2,\mathbb{Z})$ transformations. One can wonder if there is some different realization of the $\text{SL}(2,\mathbb{Z})$ symmetry for which our prove does not hold. This possibility can be ruled out by the following arguments. The group $\text{Sp}(4,\mathbb{R})$ has only two independent $\text{SL}(2,\mathbb{R})$ subgroups. We can identify the two realizations of the $\text{SL}(2,\mathbb{Z})$ group defined by (17) and (29) as coming from a discretisation of these two $\text{SL}(2,\mathbb{R})$. There can not be any other independent $\text{SL}(2,\mathbb{Z}) \subset \text{Sp}(4,\mathbb{R})$ because the existence of such a subgroup not coming from a discretisation of an $\text{SL}(2,\mathbb{R})$ would imply that $\text{Sp}(4,\mathbb{R})$ splits in disconnected parts related exactly by this discrete subgroup. We know that this is not the case. As a consequence every realization of $\text{SL}(2,\mathbb{Z}) \subset \text{Sp}(4,\mathbb{R})$ should be related either to (17) or to (29) by a change of the symplectic basis, i.e $M \rightarrow g^{-1}Mg$. Because we should regard these realizations as equivalent, we conclude that (17) and (29) exhaust all the possible physical situations.

Let us end with some final comments. The solutions described by (20-24) even though owing to their asymptotic behaviour do not seem to be interesting in the one modulus case could play a role in the many moduli case. In this situation the behaviour $W \rightarrow 0$ as $t \rightarrow i\infty$ for some of the moduli should not be rejected as unphysical. The derivation of (20-24) was made exclusively on the ground of special Kähler geometry. Hence we do not know if and how this has a correspondence in terms of Calabi-Yau compactifications. In the above discussion we considered only Calabi-Yau compactifications. However special Kähler geometry is a property of general $(2,2)$ compactifications. For this general situation the large radius behaviour of the Yukawa couplings can not be used, even in the one-modulus case, to discard the solution (20); in this context it might be of physical relevance.
The final remark concerns the form (20) for the Yukawa coupling and gives an intuitive explanation of the kind of obstruction one is faced with, when trying to promote $\text{SL}(2,\mathbb{Z})$ to the "quantum" modular group of the string. It has been stressed in the literature that the translation symmetry, part of the $\text{SL}(2,\mathbb{Z})$ group, $t \to t + 1$, has a "stringy" origin and should therefore always appear as part of the modular group. The general form of the Yukawa coupling required by this symmetry is [10]

$$W(t) = \sum_{n=0}^{\infty} d_n \exp(2\pi i nt). \quad (32)$$

The presence of the constant term $d_0$ in the summation is crucial for getting the large radius behaviour $W = \text{const}$. The extension of the translations to the full $\text{SL}(2,\mathbb{Z})$ group brings the inversion $t \to -1/t$ into play. Invariance under the inversion forbids the constant term $d_0$ and restricts (32) to be proportional to the $\eta$ function as in (20), with the consequent ill behaviour in the large radius limit.

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