PURITY RESOLUTIONS, LINEAR CODES, AND BETTI NUMBERS

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Abstract. We consider the minimal free resolution of Stanley-Reisner rings associated to a linear code and give an intrinsic characterization of linear codes having a minimal free resolution. We use this characterization to quickly deduce the minimal free resolution of Stanley-Reisner ring associated to MDS codes as well as constant weight codes. Further, we compute the Betti numbers of the Stanley-Reisner ring associated to first order Reed-Muller codes and prove that resolutions of Stanley-Reisner ring associated to binary Reed-Muller codes are not pure in general. Using these results we determine the nature of the minimal free resolution of Stanley-Reisner rings corresponding to several known classes of two-weight codes.

1. Introduction

One of the interesting developments in algebraic coding theory in the recent past is the association of a fine set of invariants, called Betti numbers, to linear error correcting codes. This is due to Johnsen and Verdure [15] and their idea is as follows. Some of the basic terminology used below is reviewed in the next section.

Let $C$ be a $q$-ary linear code of length $n$ and dimension $k$ and let $H$ be a parity check matrix of $C$. The vector matroid corresponding to $H$ is a pure simplicial complex, say $\Delta$, and its Stanley-Reisner ring $R_\Delta$ over $\mathbb{F}_q$ is a finitely generated standard graded $\mathbb{F}_q$-algebra of dimension $n-k$. As such it has a minimal graded free resolution. Moreover, $\Delta$ is shellable, thanks to a classical result that goes back to Provan [22] (see also Björner [3, §7.3]). Hence $R_\Delta$ is Cohen-Macaulay. (See, for example, [12, Ch. 6, §2]). So by the Auslander-Buchsbaum formula, the length of any minimal free resolution of $R_\Delta$ is $n-(n-k)=k$, and it looks like

$$F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R_\Delta \longrightarrow 0$$

where $F_0 = R := \mathbb{F}_q[X_1, \ldots, X_n]$ and each $F_i$ is a graded free $R$-module of the form

$$F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}} \text{ for } i = 0, 1, \ldots, k.$$

The nonnegative integers $\beta_{i,j}$ thus obtained depend only on $C$ (and not on the choice of $H$ or the minimal free resolution of $R_\Delta$), and are the Betti numbers of $C$. Thus we may refer to (1) as a (graded minimal free) resolution of $C$. Such a resolution is said to be pure of type $(d_0, d_1, \ldots, d_k)$ if for each $i = 0, 1, \ldots, k$, the Betti number
\( \beta_{i,j} \) is nonzero if and only if \( j = d_i \). If, in addition, \( d_0, d_1, \ldots, d_k \) are consecutive, then the resolution is said to be \textit{linear}. Johnsen and Verdure \cite{JohnsenVerdure1993} showed that the Betti numbers of a code \( C \) contain information about all the generalized Hamming weights \( d_i(C) \) of \( C \). In fact, they showed that

\[
    d_i(C) = \min \{ j : \beta_{i,j} \neq 0 \} \quad \text{for} \quad i = 1, \ldots, k.
\]

More recent work of Johnsen, Roksvold and Verdure \cite{JohnsenRoksvoldVerdure2000} shows that the Betti numbers of \( C \) and its elongations determine the so-called generalized weight polynomial of \( C \). Thus, if we combine this with the results of Jurrius and Pellikaan \cite{JurriusPellikaan2009}, then we obtain a direct relation between the generalized weight enumerator of \( C \) and the Betti numbers of \( C \) and of its elongations.

It is clear therefore that explicit determination of Betti numbers of codes would be useful and interesting. On the other hand, it is usually a hard problem, except in some special cases. The simplest class of codes for which Betti numbers are completely determined is that of MDS codes where the minimal free resolution is linear. The next case is that of simplex codes or dual Hamming codes, which are essentially the prototype of constant weight codes (indeed, by a classical result of Bonisoli \cite{Bonisoli1986}, every constant weight code is a concatenation of simplex codes, possibly with added 0-coordinates). For such codes, the Betti numbers were explicitly determined by Johnsen and Verdure in another paper \cite{JohnsenVerdure1992}. In this case, it turns out that the resolution is pure, although not necessarily linear.

In general, Betti numbers of pure resolutions are relatively easy to determine, thanks to a formula of Herzog and K"uhl \cite{HerzogKuhl1997}, which in the case of linear codes provides an expression for the Betti numbers in terms of the generalized Hamming weights. So the result for simplex codes can be deduced from it if one knows that their (minimal free) resolutions are necessarily pure. Partly with this in view, we consider the question of obtaining an intrinsic characterization for a linear code to have a pure resolution. This is then applied to show that the first order Reed-Muller codes have a pure resolution and all their Betti numbers can be described explicitly. On the other hand, we show that Reed-Muller codes of order 2 or more do not, in general, have a pure resolution. As a corollary, it is seen that the property of admitting a pure resolution is not preserved when passing to the dual.

The first order Reed-Muller codes are examples of two-weight codes, and it is natural to ask if a similar result holds for every two-weight code. However, unlike constant weight codes, the structure of two-weight codes is far more complicated and it is a topic of considerable research in coding theory and finite projective geometry. We refer to the survey of Calderbank and Kantor \cite{CalderbankKantor1986} and the references therein for a variety of examples of two-weight codes. We also take up the question of determining the Betti numbers of many of these codes. It is seen that the resolution is not always pure and thus we can not appeal to the Herzog-K"uhl
formula. Nonetheless we succeed in determining the Betti numbers of many two-weight codes, partly by using a set of equations due to Boij and Söderberg [5]. It appears that the technique of Boij-Söderberg equations used here could be fruitful in the determination of Betti numbers of many important classes of linear codes.

We remark that although our results on the Betti numbers of simple $x$ and first order Reed-Muller codes using the Herzog-Kühl formula were obtained independently in early 2015, Trygve Johnsen [14] has informed us that similar formulas are obtained in the Ph.D. thesis of Armenoff [1] and the Masters thesis of Karpova [21]. In any case, our emphasis here is on the general characterization of purity and the determination of Betti numbers of many two-weight codes, besides the first order Reed-Muller code.

2. Preliminaries

Fix, throughout this paper, positive integers $n, k$ with $k \leq n$ and a finite field $\mathbb{F}_q$ with $q$ elements. We denote by $[n]$ the set $\{1, \ldots, n\}$ of first $n$ positive integers. Also, $2^{[n]}$ denotes the set of all subsets of $[n]$. For any finite set $\sigma$, we denote by $|\sigma|$ the cardinality of $\sigma$. By a $[n, k]_q$-code, we shall mean a $q$-ary linear code of length $n$ and dimension $k$, i.e., a $k$-dimensional subspace of $\mathbb{F}_n^q$.

2.1. Codes and Matroids. Let $C$ be a $[n, k]_q$-code and let $H$ be a parity check matrix of $C$. For $i \in [n]$, let $H_i$ denote the $i$-th column of $H$. Define

$$\Delta := \{\sigma \in 2^{[n]} : \{H_i : i \in \sigma\} \text{ is linearly independent over } \mathbb{F}_q\}.$$ 

The ordered pair $([n], \Delta)$ is a matroid and we call it the matroid associated to the code $C$. Elements of $\Delta$ are called independent sets of this matroid. A maximal independent set in $\Delta$ is called a basis of the matroid. It is well known that every basis of a matroid has the same cardinality and this number is called the rank of the matroid. If $\sigma \subseteq [n]$ and if we let $\Delta|\sigma := \{\tau \in \Delta : \tau \subseteq \sigma\}$, then $(\sigma, \Delta|\sigma)$ is a matroid, called the restriction of the matroid $([n], \Delta)$ to $\sigma$; the rank of this restricted matroid is called the rank of $\sigma$ and denoted by $r(\sigma)$; the difference $|\sigma| - r(\sigma)$ is denoted by $\eta(\sigma)$ and called the nullity of $\sigma$. Evidently, the rank of the matroid $([n], \Delta)$ is the rank of $H$, which is $n-k$, and so the nullity of any $\sigma \subseteq [n]$ ranges from 0 to $k$. For $0 \leq i \leq k$, we define

$$N_i := \{\sigma \subseteq [n] : \eta(\sigma) = i\}.$$

2.2. Stanley-Reisner Rings and Betti Numbers. Suppose $([n], \Delta)$ is as in the previous subsection. Then $\Delta$ is a simplicial complex. We denote by $I_\Delta$ the ideal of the polynomial ring $R := \mathbb{F}_q[X_1, \ldots, X_n]$ generated by all monomials of the form $\prod_{i \in \tau} X_i$, where $\tau \in 2^{[n]} \setminus \Delta$. The quotient $R_\Delta = R/I_\Delta$ is called the Stanley-Reisner ring or the face ring associated to $\Delta$. As noted in the Introduction, $R_\Delta$ has a minimal free resolution of the form (1). Furthermore, since $I_\Delta$ is a monomial
ideal generated by squarefree monomials, we can choose the free $R$-modules $F_i$ in (1) to be not only $\mathbb{Z}$-graded as in (2), but also $\mathbb{Z}^n$-graded so as to write

\[ F_i = \bigoplus_{\sigma \in \mathbb{Z}^n} R(-\sigma)^{\beta_{i,\sigma}} \quad \text{for } i = 1, \ldots, k. \tag{4} \]

In fact, the $\mathbb{Z}^n$-graded Betti numbers $\beta_{i,\sigma}$ have the property that $\beta_{i,\sigma} = 0$ unless the $n$-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ has all its coordinates in $\{0, 1\}$. Such $n$-tuples in $\{0, 1\}^n$ can be naturally identified with subsets of $[n]$ where $(\sigma_1, \ldots, \sigma_n)$ corresponds to the subset $\{i \in [n] : \sigma_i = 1\}$ of $[n]$ that we shall also denote by $\sigma$. Thus we may index the direct sum in (4) by $\sigma \in 2^n$. The relation between the $\mathbb{Z}$-graded and $\mathbb{Z}^n$-graded Betti numbers is simply that

\[ \beta_{i,j} = \sum_{|\sigma| = j} \beta_{i,\sigma} \quad \text{for } i = 1, \ldots, k. \tag{5} \]

Johnsen and Verdure [15] proved an important relationship between the $\mathbb{Z}^n$-graded Betti numbers and subsets of a given nullity. Namely, for $1 \leq i \leq k$ and $\sigma \subseteq [n]$,

\[ \beta_{i,\sigma} \neq 0 \iff \sigma \in N_i \text{ and } \sigma \text{ is a minimal element of } N_i. \tag{6} \]

This result will be very useful for us in the sequel.

Finally, we recall the following general result, which was alluded to in the Introduction. A proof can be found in [3]. We note that a graded module $M$ over a polynomial ring $R$ having projective dimension $k$ will have a minimal free resolution such as (1) with $R_\Delta$ replaced by $M$, except in this case $F_0$ may not be equal to $R$.

In general, we let $\beta_i := \mathrm{rank}_R(F_i) = \sum_j \beta_{i,j}$. Note that if $M$ has a pure resolution of type $(d_0, d_1, \ldots, d_k)$, then $\beta_i := \beta_{i,d_i}$ for $i = 0, 1, \ldots, k$.

**Theorem 2.1** (Boij-Söderberg). Let $R$ be the polynomial ring over a field and let $M$ be a graded $R$-module of finite projective dimension $k$. Then $M$ is Cohen-Macaulay if and only if its graded Betti numbers satisfy the equations

\[ \sum_{i=0}^{k} \sum_{j \in \mathbb{Z}} (-1)^{i+j} \beta_{i,j} = 0 \quad \text{for } \ell = 0, \ldots, k-1 \tag{7} \]

In particular, if the minimal free resolution of $M$ is pure of type $(d_0, d_1, \ldots, d_k)$, then (7) implies the Herzog-Kühl formula [13]:

\[ \beta_i = \beta_0 \prod_{j \neq i} \frac{d_j}{(d_j - d_i)} \quad \text{for } i = 1, \ldots, k. \tag{8} \]

As noted in the Introduction, Stanley-Reisner rings associated to linear codes (or more generally, simplicial complexes corresponding to a matroid) are Cohen-Macaulay, and hence the above theorem is applicable; moreover, in this case, $\beta_0 = 1$. If a $[n, k]_q$-code $C$ has a pure resolution of type $(d_0, \ldots, d_k)$, then $d_0 = 0$ and for $1 \leq i \leq k$, $d_i$ is precisely the $i$-th generalized Hamming weight of $C$, thanks to [3]: we will refer to $\beta_i = \beta_{i,d_i}$ as the Betti numbers of $C$ in this case.
3. Pure Resolution of Linear Codes

In this section we will give a characterization of the purity of the resolution of the Stanley-Reisner ring associated to a linear code in terms of the support weight of certain subcodes of the code. We will then outline some simple applications.

Let $C$ be a $[n,k]_q$ code and let $H = [H_1 \ldots H_n]$ be a parity check matrix $C$, where, as before, $H_i$ denotes the $i$th column of $H$. For any subset $\sigma$ of $[n]$, define $S(\sigma)$ to be the subspace $\langle H_i : i \in \sigma \rangle$ of $\mathbb{F}_q^{n-k}$ spanned by the columns of $H$ indexed by $\sigma$. Note that $r(\sigma) = \dim S(\sigma)$. Let us also define a related subspace of $\mathbb{F}_q^n$ by

$$\hat{S}(\sigma) := \{ x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n : x_i = 0 \text{ for } i \notin \sigma \text{ and } \sum_{i \in \sigma} x_i H_i = 0 \}.$$ 

Recall that for any subcode $D$ of $C$, i.e., a subspace $D$ of $C$, the support of $D$ is the set $\text{Supp}(D)$ of all $i \in [n]$ for which there is $x = (x_1, \ldots, x_n) \in D$ with $x_i \neq 0$.

**Lemma 3.1.** Let $\sigma \subseteq [n]$. Then $\hat{S}(\sigma)$ is a subcode of $C$ and $\text{Supp}(\hat{S}(\sigma)) \subseteq \sigma$.

**Proof.** Since $C = \{ x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n : \sum_{i=1}^n x_i H_i = 0 \}$, it is clear that $\hat{S}(\sigma)$ is a subcode of $C$. The inclusion $\text{Supp}(\hat{S}(\sigma)) \subseteq \sigma$ is obvious. \qed

For any $\sigma \subseteq [n]$, let $\mathbb{F}_q^\sigma$ denote the set of all ordered $|\sigma|$-tuples $(x_i)_{i \in \sigma}$ of elements of $\mathbb{F}_q$ indexed by $\sigma$. Consider the map

$$\phi_\sigma : \mathbb{F}_q^\sigma \rightarrow S(\sigma) \text{ defined by } \phi_\sigma(x) = \sum_{i \in \sigma} x_i H_i.$$ 

Clearly $\phi_\sigma$ is a surjective $\mathbb{F}_q$-linear map.

**Lemma 3.2.** Let $\sigma \subseteq [n]$ and let $\phi_\sigma$ be as in (9). Then $\ker \phi_\sigma$ is isomorphic (as a $\mathbb{F}_q$-vector space) to $\hat{S}(\sigma)$. Consequently,

$$\dim S(\sigma) = |\sigma| - \dim \hat{S}(\sigma) \quad (10)$$

**Proof.** Consider the map $\psi : \mathbb{F}_q^\sigma \rightarrow \mathbb{F}_q^n$ given by $\psi(x) = (v_1, v_2, \ldots, v_n)$, where

$$v_i = \begin{cases} x_i & \text{if } i \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that the restriction of $\psi$ to $\ker \phi_\sigma$ gives an isomorphism of $\ker \phi_\sigma$ onto $\hat{S}(\sigma)$. The second assertion follows from the Rank-Nullity theorem. \qed

For $0 \leq i \leq k$, let $\mathbb{G}_i(C)$ denote the Grassmannian of all $i$-dimensional subspaces of $C$. We call $D \in \mathbb{G}_i(C)$ an $i$-minimal subcode of $C$ if $\text{Supp}(D)$ is minimal among the supports of all $i$-dimensional subcodes of $C$, i.e., $\text{Supp}(D') \nsubseteq \text{Supp}(D)$ for any $D' \in \mathbb{G}_i(C)$ with $D' \neq D$. We let

$$\mathcal{D}_i = \text{the set of all } i\text{-minimal subcodes of } C.$$ 

Note that if $i = 0$, then the only element of $\mathbb{G}_i(C)$ is $\{0\}$, and its support is $\emptyset$, which is clearly $i$-minimal. Moreover, $r(\emptyset) = 0 = |\emptyset|$, and thus $\text{Supp}(\{0\}) \in N_0$. In
Proposition 3.4. Suppose $0 \leq i \leq k$ and $D \in \mathcal{D}_i$. Then $\text{Supp}(D) \subseteq N_i$.

Proof. Let $\sigma := \text{Supp}(D)$. Then for any $x \in D$, clearly $x_i = 0$ for all $i \in [n]$ with $i \notin \sigma$. Also, since $D \subseteq C$, we see that $\sum x_iH_i = 0$ for each $x = (x_1, \ldots, x_n) \in D$. It follows that $D \subseteq \hat{S}(\sigma)$. In particular, $\dim \hat{S}(\sigma) \geq i$. Further, by Lemma 3.1

$$\sigma = \text{supp}(D) \subseteq \text{supp}(\hat{S}(\sigma)) \subseteq \sigma.$$ 

Therefore $\text{supp}(\hat{S}(\sigma)) = \sigma$. In case $\dim(\hat{S}(\sigma)) > i$, we can choose some $j \in \sigma$ and observe that $\{x \in \hat{S}(\sigma) : x_j = 0\}$ is a subspace of dimension $\dim \hat{S}(\sigma) - 1$, and its support is contained in $\sigma \setminus \{j\}$. This can be used to construct an $i$-dimensional subcode $D'$ of $\hat{S}(\sigma)$ with support a proper subset of $\sigma$. But then the minimality of the support of $D$ is contradicted. It follows that $\dim \hat{S}(\sigma) = i$, and hence $D = \hat{S}(\sigma)$.

Now equation (10) shows that $r(\sigma) = |\sigma| - i$, that is, $\sigma \in N_i$. □

It turns out that a partial converse of the above proposition is also true.

Proposition 3.5. Suppose $0 \leq i \leq k$ and $\sigma$ is a minimal element of $N_i$ (with respect to inclusion). Then there exists $D \in \mathcal{D}_i$ such that $\sigma = \text{Supp}(D)$.

Proof. Since $\sigma \in N_i$, we see that $\dim S(\sigma) = r(\sigma) = |\sigma| - i$. Hence equation (10) implies that $\dim \hat{S}(\sigma) = i$. Let $D := \hat{S}(\sigma)$ and $\sigma' := \text{Supp}(D)$. Then $D$ is an $i$-dimensional subcode of $C$ and by Lemma 3.1 $\sigma' \subseteq \sigma$. We claim that $D \in \mathcal{D}_i$. To see this, assume the contrary. Then there exists $D' \in \mathcal{G}_i(C)$ with $D' \neq D$ such that $\text{Supp}(D') \subseteq \sigma'$. Replacing $D'$ by an $i$-dimensional subcode with smaller support, if necessary, we may assume that $D'$ is $i$-minimal. But then by Proposition 3.1 $\text{Supp}(D') \in N_i$, which contradicts the minimality of $\sigma$ in $N_i$. Thus $D \in \mathcal{D}_i$. □

Corollary 3.5. Suppose $0 \leq i \leq k$ and $\sigma \subseteq [n]$. Then $\sigma$ is a minimal element of $N_i$ if and only if there exist an $i$-minimal subcode $D$ of $C$ with $\text{Supp}(D) = \sigma$.

Proof. Follows from Propositions 3.3 and 3.4 □

Theorem 3.6. Let $C$ be an $[n, k]_q$ code and $d_1 < \cdots < d_k$ its generalized Hamming weights. Then any $\mathbb{N}$-graded minimal free resolution of $C$ is pure if and only if for each $i = 1, \ldots, k$, all the $i$-minimal subcodes of $C$ have support weight $d_i$.

Proof. From (9) and Corollary 3.5 we see that for $1 \leq i \leq k$ and $\sigma \subseteq [n]$,

$$\beta_{i, \sigma} \neq 0 \iff \sigma = \text{Supp}(D) \text{ for some } D \in \mathcal{D}_i.$$ 

Thus the desired result follows from (9) and (15). □

Remark 3.7. Let $C$ be an $[n, k]_q$ code and $h$ a positive integer $\leq k$. Given a resolution of $C$, say (1), by its left part after $h$ steps, we mean the exact sequence

$$F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_h$$
which is a minimal free resolution of the cokernel of the last map \( F_{h-1} \rightarrow F_h \). Now let \( d_1 < \cdots < d_k \) be the generalized Hamming weights of \( C \). It is clear that the proof of Theorem 3.6 also shows that the left part after \( h \) steps of any \( \mathbb{N} \)-graded minimal free resolution of \( C \) is pure if and only if for each \( i = h, \ldots, k \), all the \( i \)-minimal subcodes of \( C \) have support weight \( d_i \).

We now show how a characterization due to Johnsen and Verdure [15] of MDS codes can be deduced from our characterization of purity, and moreover, how the minimal free resolution of an MDS code can then be readily determined using the Herzog-Kühl formula.

**Corollary 3.8.** Let \( C \) be a nondegenerate \([n, k]_q\)-code and \( h \) a positive integer \( \leq k \). Then \( C \) is h-MDS if and only if the left part after \( h \) steps of its resolution after \( h \) steps is linear. In particular, \( C \) is an MDS code if and only if its resolution is linear. Moreover, if \( C \) is MDS, then its Betti numbers are given by

\[
\beta_i = \binom{n-k+i-1}{i-1} \binom{n}{k-i} \quad \text{for } i = 1, \ldots, k.
\]

**Proof.** Suppose the left part after \( h \) steps of a resolution of \( C \) is linear. Since \( C \) is nondegenerate, \( d_k = n \), and so from the linearity together with equation (8), we obtain \( d_i = n - k + i \) for \( h \leq i \leq k \). Taking \( i = h \), we see that \( C \) is h-MDS.

Conversely, suppose \( C \) is h-MDS. Then from the strict monotonicity of generalized Hamming weights [24, Thm. 1], we see that \( d_i = n - k + i \) for \( h \leq i \leq k \). Now fix \( i \in \{h, \ldots, k\} \) and let \( D \) be an \( i \)-minimal subcode of \( C \). Let \( \sigma := \text{Supp}(D) \). By Proposition 3.3 \( \sigma \in N_i \). Also, \( n - k + i = d_i \leq |\sigma| \). Consequently, \( n - k \leq |\sigma| - i = r(\sigma) \leq n - k \). It follows that \( |\text{Supp}(D)| = d_i \). Thus, in view of Remark 3.7, we conclude that the left part after \( h \) steps of any resolution of \( C \) is linear.

Now assume that \( C \) is MDS. Then, in view of (8), we see that for \( 1 \leq i \leq k \),

\[
\beta_i = \prod_{j \neq i} \frac{d_j}{|d_j - d_i|} = \prod_{j \neq i} \frac{n - k + j}{|j - i|} = \left( \prod_{j=1}^{i-1} \frac{n - k + j}{i - j} \right) . \left( \prod_{j=i+1}^{k} \frac{n - k + j}{j - i} \right) ,
\]

and an easy calculation shows that this is equal to \( \binom{n-k+i-1}{i-1} \binom{n}{k-i} \). \( \square \)

Let us also show how the result of Johnsen and Verdure [16] about the minimal free resolution of constant weight codes can be deduced from Theorem 3.6.

**Corollary 3.9.** Let \( C \) be an \([n, k]_q\)-code in which each nonzero codeword has constant weight \( d \). Then the \( \mathbb{N} \)-graded resolution of \( C \) is pure. Moreover, the generalized Hamming weights (or the shifts) and the Betti numbers of \( C \) are given by

\[
d_i = q^k - 1 \binom{q^i - 1}{q-1} \quad \text{and} \quad \beta_i = \binom{k}{i} q^\binom{q^{-i}-1}{q^{-1}}, \quad \text{for } i = 1, \ldots, k,
\]

where \( \binom{k}{i}_q \) denotes the Gaussian binomial coefficient.
Proof. It is well known (see, e.g., [19, Thm. 1]) that every $j$-dimensional subcode of the constant weight code $C$ has support weight $d_j$, where

$$d_j = \frac{d(q^j - 1)}{q^{j-1}(q - 1)}, \quad \text{for } j = 1, \ldots, k.$$ 

Hence by Theorem 3, $C$ has a pure resolution. Evidently, the numbers $d_i$ defined above are the generalized Hamming weights of $C$. Moreover, for $i, j = 1, \ldots, k$,

$$d_i - d_j = \frac{d(q^{i-j} - 1)}{q^{i-1}(q - 1)}, \quad \text{if } j < i, \quad \text{and } \quad d_j - d_i = \frac{d(q^{i-j} - 1)}{q^{j-1}(q - 1)}, \quad \text{if } j > i.$$ 

Hence the Herzog-Kühl formula (8) implies that for $i = 1, \ldots, k$,

$$\beta_i = \prod_{j \neq i} \frac{d_j}{d_j - d_i} = \left(\prod_{j=1}^{i-1} \frac{q^{i-j} - 1}{q^{j-1} - 1}\right) \left(\prod_{j=i+1}^{k} \frac{q^{i-j} - 1}{q^{j-1} - 1}\right) = q^{\frac{i(i+1)}{2}} \left[\frac{k}{i}\right]_q,$$

where the last equality follows by noting that for $i = 1, \ldots, k$,

$$\left[\frac{k}{i}\right]_q = \left[\frac{k}{k-i}\right]_q = \frac{(q^{k-1} - 1)(q^{k-2} - 1)\cdots(q^{i+1} - 1)}{(q^{k-i-1} - 1)(q^{k-i-2} - 1)\cdots(q - 1)} = \prod_{j=i+1}^{k} \frac{q^{j-1} - 1}{q^{j-i-1} - 1}.$$ 

This proves the desired result. \hfill \square

4. Reed-Muller Codes

In this section we consider generalized Reed-Muller codes and prove that the resolution of the first order Reed-Muller code is pure, whereas for other Reed-Muller codes, it is non-pure. Let us begin by recalling the construction of (generalized) Reed-Muller codes. Fix integers $r, m$ such that $m \geq 1$ and $0 \leq r \leq m(q-1)$. Define

$$V_q(r, m) = \{ f \in \mathbb{F}_q[X_1, \ldots, X_m] : \deg f \leq r \text{ and } \deg_{X_i} f < q \text{ for } i = 1, \ldots, m\}.$$ 

Fix an ordering $P_1, \ldots, P_{q^m}$ of the elements of $\mathbb{F}_q^m$. Consider the evaluation map

$$\text{Ev} : V_q(r, m) \rightarrow \mathbb{F}_q^m \quad \text{defined by} \quad f \mapsto (f(P_1), \ldots, f(P_{q^m})).$$

The image of Ev is called the generalized Reed-Muller code of order $r$ and we denote it by $\mathcal{RM}_q(r, m)$. It is well-known that $\mathcal{RM}_q(r, m)$ is an $[n, k, d]_q$-code, with

$$n = q^m, \quad k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{m+r-iq}{m}, \quad \text{and} \quad d = (q-s)q^{m-t-1},$$

where $t, s$ are unique integers satisfying $r = t(q-1) + s$ and $0 \leq s \leq q-2$. Further, for any $\omega_0, \omega_1, \ldots, \omega_r \in \mathbb{F}_q$ with $\omega_0 \neq 0$ and any distinct $\omega'_1, \ldots, \omega'_s \in \mathbb{F}_q$, the polynomial

$$f(X_1, \ldots, X_m) = \omega_0 \prod_{i=1}^{t} (1 - (X_i - \omega_i)^{q-1}) \prod_{j=1}^{s} (X_{i+1} - \omega'_j)$$

is in $V_q(r, m)$ and Ev($f$) is a minimum weight codeword of $\mathcal{RM}_q(r, m)$. Moreover, up to a (nonhomogeneous) linear substitution in the $X'_i$s, every minimum weight
codeword of $\mathcal{RM}_q(r, m)$ is of this form; see, e.g., Theorems 2.6.2 and 2.6.3 of [11]. It is also well-known (see, e.g., [2] §5.4) that the dual of $\mathcal{RM}_q(r, m)$ is given by
\begin{equation}
\mathcal{RM}_q(r, m)^\perp = \mathcal{RM}_q(r^\perp, m) \quad \text{where} \quad r^\perp + r + 1 = m(q - 1). \tag{12}
\end{equation}

In particular, if $r = m(q - 1) - 1$, then $\mathcal{RM}_q(r, m)$ is a MDS code (being the dual of $\mathcal{RM}_q(0, m)$, which is the 1-dimensional code of length $q^m$ generated by the all-1 vector). Also if $r = m(q - 1)$, then $\mathcal{RM}_q(r, m)$ is a MDS code, being the full space $F_q^m$. Finally, if $m = 1$, then $\mathcal{RM}_q(r, m)$ is a Reed-Solomon code, and in particular, a MDS code. Thus in all these “trivial cases”, $\mathcal{RM}_q(r, m)$ has a pure, and in fact, linear, resolution. The following result deals with the first nontrivial case of $r = 1$.

**Theorem 4.1.** The $\mathbb{N}$-graded minimal free resolution of the first order Reed-Muller code $\mathcal{RM}_q(1, m)$ is pure and is given by

$$R(d_{m+1})^{\beta_{m+1}} \rightarrow R(d_m)^{\beta_m} \rightarrow \cdots \rightarrow R(d_1)^{\beta_1} \rightarrow R$$

where $d_i = q^m - |q^{m-i}|$ for $1 \leq i \leq m + 1$, and

$$\beta_i = \begin{cases} q^\frac{i+1}{2} \prod_{j=1}^{m-i} \frac{q^{m+1-j} - 1}{q^{m+1-i-j} - 1} & \text{if } 1 \leq i \leq m, \\ \prod_{j=1}^{m-i} (q^j - 1) & \text{if } i = m + 1. \end{cases}$$

**Proof.** First, note that $\dim \mathcal{RM}_q(1, m) = m + 1$. Let $i$ be a positive integer $\leq m + 1$. If $i = m + 1$, then the only $i$-dimensional subcode of $\mathcal{RM}_q(1, m)$ is $\mathcal{RM}_q(1, m)$ itself, and this has support weight $q^m$. Now suppose $1 \leq i \leq m$. Let $D$ be a subcode of $\mathcal{RM}_q(1, m)$ of dimension $i$. Then the support weight of $D$ is clearly

$$q^m - |Z(f_1, \ldots, f_i)|,$$

where $f_1, \ldots, f_i \in V_q(1, m)$ are linearly independent polynomials whose images under $Ev$ form a basis of $D$, and where $Z(f_1, \ldots, f_i)$ denotes the set of common zeros in $F_q^m$ of $f_1, \ldots, f_i$. Now $f_1 = \cdots = f_i = 0$ is a system of $i$ linearly independent (not necessarily homogeneous) linear equations in $m$ variables, and thus it has either no solutions (when the system is inconsistent) or exactly $q^{m-i}$ solutions (when the system is consistent). Accordingly, the support weight of $D$ is either $q^m$ or $q^m - q^{m-i}$. Moreover, if the former holds, then $\text{Supp}(D) = \{1, \ldots, q^m\}$, and so $D$ can’t be an $i$-minimal subcode of $\mathcal{RM}_q(1, m)$. It follows that all $i$-minimal subcodes of $\mathcal{RM}_q(1, m)$ have the same support weight $d_i = q^m - |q^{m-i}|$ for $1 \leq i \leq m + 1$.

Thus, by Theorem 3.1, $\mathcal{RM}_q(1, m)$ has a pure resolution. Consequently, the Betti numbers of $\mathcal{RM}_q(1, m)$ are determined by the Herzog-Kühler equation (3) as follows.

\begin{equation}
\beta_{m+1} = m \prod_{j=1}^{m} \frac{d_j}{d_{m+1} - d_j} = m \prod_{j=1}^{m} \frac{q^m - q^{m-j}}{q^m - q^{m-j}} = \prod_{j=1}^{m} (q^j - 1),
\end{equation}

\footnote{Strictly speaking, for the formula (12) to be valid, we should note that the definition of $\mathcal{RM}_q(r, m)$ is meaningful also when $r = -1$ in which case it is the zero code of length $q^m$.}
whereas for $1 \leq i \leq m$,

$$
\beta_i = \frac{d_{m+1}}{d_{m+1} - d_i} \prod_{m+1 > j > i} \frac{d_j}{d_j - d_i} \prod_{j < i} \frac{d_j}{d_j - d_i} 
$$

$$
= q^i \prod_{j=i+1}^{m} \frac{q^{m-j}(q^j - 1)}{q^{m-j}(q^j - 1)} \prod_{j=1}^{i-1} \frac{q^{m-j}(q^j - 1)}{q^{m-j}(q^j - 1)} 
$$

$$
= q^{\frac{i(i+1)}{2}} \prod_{j=i+1}^{m} \frac{(q^j - 1)}{(q^{j-1} - 1)} = q^{\frac{(i+1)}{2}} \prod_{j=1}^{m-i} \frac{(q^{m+1-j} - 1)}{(q^{m+1-i-j} - 1)}.
$$

This proves the theorem. \qed

**Remark 4.2.** Observe that the pure resolution of $\mathcal{RM}_q(1, m)$ in Theorem 4.1 is linear only when either $m = 1$ or $m = 2 = q$. As noted earlier, $\mathcal{RM}_q(1, m)$ is a MDS code in this case.

Next, we shall show that the minimal free $\mathbb{F}$-resolutions of many generalized Reed-Muller codes of order higher than one are not pure. It will be convenient to consider various cases separately. As usual, we shall say that an element $c$ of a linear code $C$ is a minimal codeword if either $c = 0$, or if $c \neq 0$ and the support of the 1-dimensional subspace $\langle c \rangle$ of $C$ spanned by $c$ is minimal among the supports of all 1-dimensional subcodes of $C$. Evidently, a codeword of minimum weight is minimal, but the converse may not be true.

### 4.1. Binary Case

In this subsection we consider the binary case, i.e., when $q = 2$. We will use the following simple, but useful, observation. It is stated, for instance, in Ashikhmin and Barg [3, Lemma 2.1]. The proof is obvious and is omitted.

**Lemma 4.3.** Let $C$ be a binary linear code and let $d = d(C)$ be its minimum distance. If $c \in C$ is not a minimal weight codeword, then $c = c_1 + c_2$ for some nonzero $c_1, c_2 \in C$ such that $\text{Supp}(\langle c_1 \rangle)$ and $\text{Supp}(\langle c_2 \rangle)$ are disjoint and $\text{Supp}(\langle c_i \rangle) \subseteq \text{Supp}(\langle c \rangle)$ for $i = 1, 2$. In particular, if $c \in C$ has $\text{wt}(c) < 2d$, then $c$ is a minimal codeword of $C$.

The following result shows that all “nontrivial” binary Reed-Muller codes of order greater than 1 have a non-pure resolution.

**Proposition 4.4.** Assume that $m \geq 4$ and $1 < r \leq m - 2$. Then any minimal free $\mathbb{F}$-resolution of the binary Reed-Muller code $\mathcal{RM}_2(r, m)$ is not pure.

**Proof.** The minimum distance of $\mathcal{RM}_2(r, m)$ is $d := 2^{m-r}$ and if we let

$$
Q(X_1, \ldots, X_m) = X_1X_2 \cdots X_{r-2}(X_{r-1}X_r + X_{r+1}X_{r+2}),
$$

then clearly, $Q \in V_2(r, m)$. Moreover, the corresponding codeword $c_Q = \text{Ev}(Q)$ has weight $6 \times 2^{m-r-2} = 3d/2$. Indeed, $Q(a_1, \ldots, a_m) \neq 0$ for $(a_1, \ldots, a_m) \in \mathbb{F}_2^n$ precisely when $a_1 = \cdots = a_{r-2} = 1$, $(a_{r-1}, a_r, a_{r+1}, a_{r+2})$ is one among $(0, 1, 1, 1)$,
Proof. Choose distinct elements $N \leq$ Reed-Solomon (and hence MDS) code for $1$. Assume that $\text{Proposition 4.6.}$

Remark 4.5. As Alexander Barg has pointed out to one of us, the last assertion in Lemma 4.3 can be extended to the $q$-ary case to show that codewords of weight less that $dq/(q-1)$ are minimal in $C$, where $C$ is a $q$-ary linear code with minimum distance $d$. However, for $q > 2$, this is often a restrictive hypothesis, and in the next subsections, we will deal with $q$-ary Reed-Muller codes using a different strategy.

4.2. The Case of $t = 0$. Let $t, s$ be as in (11) so that $r = t(q-1) + s$ and $0 \leq s < q-1$. We will consider the case of Reed-Muller codes of order $r > 1$ for which $t = 0$ (so that $r = s$). Note that such codes are necessarily non-binary, and in fact, $q \geq 4$. We shall also exclude the case when $m = 1$, since $\mathcal{RM}_q(r, 1)$ is a Reed-Solomon (and hence MDS) code for $1 \leq r \leq (q-1)$.

Proposition 4.6. Assume that $m \geq 2$ and $1 < r < q-1$. Then any minimal free $\mathbb{N}$-resolution of the Reed-Muller code $\mathcal{RM}_q(r, m)$ is not pure.

Proof. Choose distinct elements $\omega_1, \ldots, \omega_{r-1} \in \mathbb{F}_q$ and an arbitrary $\omega \in \mathbb{F}_q$. Define

$$Q(X_1, \ldots, X_m) = (X_2 - \omega) \prod_{i=1}^{m-1} (X_1 - \omega_i).$$

Clearly, $Q \in V_q(r, m)$ and the corresponding codeword $c_Q = \text{Ev}(Q)$ has weight $(q-r+1)(q-1)q^{m-2}$. On the other hand, by (11), the minimum distance of $\mathcal{RM}_q(r, m)$ is $(q-r)q^{m-1}$. Observe that

$$(q-r+1)(q-1)q^{m-2} - (q-r)q^{m-1} = (r-1)q^{m-2} > 0 \quad \text{since } r > 1.$$

It follows that $c_Q$ is not a minimum weight codeword. If $c_Q$ is a minimal codeword, then Theorem 5.0 implies the desired result. Now suppose $c_Q$ is not a minimal codeword of $\mathcal{RM}_q(r, m)$. Then we can find $F \in V_q(r, m)$ such that $c_F$ is a minimal codeword of $\mathcal{RM}_q(r, m)$ and $\text{Supp}(c_F) \subset \text{Supp}(c_Q)$. Again, if $c_F$ is not a minimal codeword of $\mathcal{RM}_q(r, m)$, then we are done. Otherwise, by the characterization of minimum weight codewords of $\mathcal{RM}_q(r, m)$, we must have

$$F(X_1, \ldots, X_m) = \prod_{j=1}^{r} (L - \omega_j')$$

for some distinct elements $\omega_1', \ldots, \omega_r' \in \mathbb{F}_q$ and some nonzero linear polynomial $L$ in $\mathbb{F}_q[X_1, \ldots, X_m]$ that we can assume to be homogeneous (by adjusting $\omega_j'$, if necessary). Write $L = a_1X_1 + \cdots + a_mX_m$. Since $\text{Supp}(c_F) \subset \text{Supp}(c_Q)$, it follows that $L$ vanishes whenever we substitute $X_1 = \omega_i$ for some $i \in \{1, \ldots, r\}$ or we substitute $X_2 = \omega$. In particular, $a_1\omega_1 + a_2\omega_2 + \cdots + a_mX_m = \omega_j'$ for some $j \in \{1, \ldots, r\}$. Comparing the degree in each of the variables $X_2, \ldots, X_m$, we
obtain \( a_2 = \cdots = a_m = 0 \) so that \( L = a_1 X_1 \). But then \( L \) does not vanish when we substitute \( X_2 = \omega \), and we obtain a contradiction. This proves the proposition. \( \square \)

4.3. **The case of** \( 0 < t < m-1 \) **and** \( 1 < s < q-1 \). The arguments here will be similar to those in the previous subsection, except that we have to deal with an additional factor of degree \( t(q-1) \). Note that \( 1 < s < q-1 \) implies that \( q \geq 4 \).

**Proposition 4.7.** Assume that \( 1 < r < m(q-1) \) and moreover, \( r = t(q-1) + s \) with \( 0 < t < m-1 \) and \( 1 < s < q-1 \). Then any minimal free \( \mathbb{N} \)-resolution of the Reed-Muller code \( \mathcal{R}_q(r, m) \) is not pure.

**Proof.** Choose distinct elements \( \omega_1, \ldots, \omega_{s-1} \in \mathbb{F}_q \) and an arbitrary \( \omega \in \mathbb{F}_q \). Define

\[
Q(X_1, \ldots, X_m) = \left( \prod_{i=1}^{t} (X_i^{q-1} - 1) \right) \left( \prod_{j=1}^{s-1} (X_{t+1} - \omega_j) \right) (X_{t+2} - \omega)
\]

Clearly, \( Q \in V_q(r, m) \) and the corresponding codeword \( c_Q = \text{Ev}(Q) \) has weight \( (q-s+1)(q-1)q^{m-t-2} \). On the other hand, by (11), the minimum distance of \( \mathcal{R}_q(r, m) \) is \( (q-s)q^{m-t-1} \). Observe that

\[
(q-s+1)(q-1)q^{m-t-2} - (q-s)q^{m-t-1} = (s-1)q^{m-t-2} > 0 \quad \text{since} \ s > 1.
\]

Thus, as in the proof of Proposition 4.6 it suffices to show that if there is \( F \in V_q(r, m) \) such that \( c_F \) is a minimum weight codeword with \( \text{Supp}(c_F) \subseteq \text{Supp}(c_Q) \), then we arrive at a contradiction. Again, any such \( F \) has to be of the form

\[
F(X_1, \ldots, X_m) = \left( \prod_{i=1}^{t} (L_i^{q-1} - 1) \right) \left( \prod_{j=1}^{s} (L_{t+1} - \omega_j) \right)
\]

for some distinct \( \omega_1', \ldots, \omega_s' \in \mathbb{F}_q \), and linearly independent linear polynomials \( L_1, \ldots, L_{t+1} \in \mathbb{F}_q[X_1, \ldots, X_m] \) with \( L_{t+1} \) homogeneous. Note that \( \text{Supp}(c_Q) \) is contained in the linear space \( A = \{ (a_1, \ldots, a_m) \in \mathbb{F}_q^m : a_i = 0 \text{ for } i = 1, \ldots, t \} \), which can be identified with \( \mathbb{A}^{m-t} \), while \( \text{Supp}(c_F) \) is contained in the affine space \( A' := \{ a \in \mathbb{F}_q^m : L_i(a) = 0 \text{ for } i = 1, \ldots, t \} \) of dimension \( m-t \). Further, since \( \text{Supp}(c_F) \subseteq \text{Supp}(c_Q) \), we obtain \( \text{Supp}(c_F) \subseteq A \cap A' \). Now if \( A \neq A' \), then \( \dim(A \cap A') \leq m - t - 1 \), and so \( (q-s)q^{m-t-1} \leq q^{m-t-1} \), which is impossible because \( s < q-1 \). This shows that \( A = A' \). Consequently,

\[
F(0, \ldots, 0, X_{t+1}, \ldots, X_m) = \prod_{j=1}^{s} (L_{t+1}(0, \ldots, 0, X_{t+1}, \ldots, X_m) - \omega_j')
\]

gives a minimum weight codeword in \( \mathcal{R}_q(s, m-t) \) whose support contains the support of the codeword of \( \mathcal{R}_q(s, m-t) \) associated to \( Q(0, \ldots, 0, X_{t+1}, \ldots, X_m) \). But then this leads to a contradiction exactly as in the proof of Proposition 4.6. \( \square \)
4.4. The case of \( s = 0 \). Since the binary case and the case \( t = 0 \) have already been dealt with in subsections 4.1 and 4.2, we shall assume that \( q \geq 3 \) and \( 1 \leq t \leq m - 1 \). Then \( s = 0 \) implies that \( r = t(q - 1) > 1 \).

**Proposition 4.8.** Assume that \( q \geq 3 \) and \( r = t(q - 1) \) with \( 1 \leq t \leq m - 1 \). Then any minimal free \( \mathbb{F}_q \)-resolution of the Reed-Muller code \( \mathcal{R}M_q(r, m) \) is not pure.

**Proof.** Write \( \mathbb{F}_q = \{\omega_1, \ldots, \omega_q\} \) and pick any \( \omega \in \mathbb{F}_q \). Consider

\[
Q(X_1, \ldots, X_m) = \left( \prod_{i=1}^{t-1} (X_i^{q-1} - 1) \right) \left( \prod_{j=3}^{q} (X_{t+1} - \omega_j) \right) (X_{t+2} - \omega)
\]

Then \( \deg Q = (t - 1)(q - 1) + (q - 2) + 1 = t(q - 1) = r \) and so \( Q \in V_q(r, m) \). Also, we can write \( \text{Supp}(c_Q) = A_1 \cup A_2 \), where for \( i = 1, 2 \),

\[
A_i := \{a = (a_1, \ldots, a_m) \in \mathbb{F}_q^m : a_1 = \cdots = a_t = 0, a_{t+1} = \omega_i, \text{ and } a_{t+2} \neq \omega \}.
\]

Clearly, \( A_1, A_2 \) are disjoint and so \( \text{wt}(c_Q) = 2(q-1)q^{m-t-1} \). The minimum distance of \( \mathcal{R}M_q(r, m) \) in this case is \( q^{m-t} \), and \( 2(q-1)q^{m-t-1} > q^{m-t} \), since \( q \geq 3 \). Thus \( c_Q \) is not a minimum weight codeword. As in the proof of Proposition 4.10, it suffices to show that the existence of \( F \in V_q(r, m) \) such that \( c_F \) is a minimum weight codeword with \( \text{Supp}(c_F) \subset \text{Supp}(c_Q) \) leads to a contradiction. By the characterization of minimum weight codewords of \( \mathcal{R}M_q(r, m) \), any such \( F \) has to be of the form \( F(X_1, \ldots, X_m) = \prod_{i=1}^{t-1} (L_i^{q-1} - 1) \) for some linearly independent linear polynomials \( L_1, \ldots, L_t \) in \( \mathbb{F}_q[X_1, \ldots, X_m] \). Hence \( \text{Supp}(c_F) \) is the affine space \( A' := \{a \in \mathbb{F}_q^m : L_i(a) = 0 \text{ for } i = 1, \ldots, t\} \). Since \( \text{Supp}(c_F) \subset \text{Supp}(c_Q) \), we can argue as in the proof of Proposition 4.7 to deduce that \( A' \) is in fact, the linear space \( \{a \in \mathbb{F}_q^m : a_1 = \cdots = a_t = 0\} \). We now claim that \( \text{Supp}(c_F) \) is either disjoint from \( A_1 \) or from \( A_2 \). Indeed, if this is not the case then there are \( P_i \in \text{Supp}(c_F) \cap A_i \) for \( i = 1, 2 \). But then \( P_\lambda := P_1 + \lambda(P_2 - P_1) \in \text{Supp}(c_F) \) for any \( \lambda \in \mathbb{F}_q \), since \( \text{Supp}(c_F) = A' \) is linear. Also since \( q \neq 3 \), we can pick \( \lambda \in \mathbb{F}_q \) such that \( \lambda \neq 0 \) and \( \lambda \neq 1 \). Now \( \text{Supp}(c_F) \subset \text{Supp}(c_Q) \cup A_1 \cup A_2 \) leads to a contradiction since the \( t \)-th coordinate of \( P_\lambda \) is neither \( \omega_1 \) nor \( \omega_2 \). This proves the claim. It follows that \( A' = \text{Supp}(c_F) \subset A_i \) for some \( i \in \{1, 2\} \). But then \( q^{m-t} \leq (q-1)q^{m-t-1} \), which is a contradiction. This proves the proposition. \( \square \)

4.5. The case of \( t = m - 1 \) and \( 1 < s < q - 2 \). We will now consider the last case of nontrivial Reed-Muller codes \( \mathcal{R}M_q(r, m) \) of order \( r = t(q - 1) + s \), where \( r > 1 \) and \( s \neq 1 \), namely, when \( t = m - 1 \) and \( s > 1 \). Note that if we allow \( s = q - 2 \), then \( \mathcal{R}M_q(r, m) \) becomes a MDS code and so we shall assume that \( 1 < s < q - 2 \). In particular, this implies that \( q \geq 5 \).

**Proposition 4.9.** Assume that \( r = (m - 1)(q - 1) + s \) with \( 1 < s < q - 2 \). Then any minimal free \( \mathbb{F}_q \)-resolution of the Reed-Muller code \( \mathcal{R}M_q(r, m) \) is not pure.
Proof. As in the proof of Proposition 4.8 write $\mathbb{F}_q = \{\omega_1, \ldots, \omega_q\}$ and pick any $\omega \in \mathbb{F}_q$. Also let $\nu_1, \ldots, \nu_{s+1}$ be any distinct elements of $\mathbb{F}_q$. Consider

$$Q(X_1, \ldots, X_m) = \left( \prod_{i=1}^{m-2} (X^q_i - 1) \right) \left( \prod_{j=3}^{q} (X_{m-1} - \omega_j) \right) \left( \prod_{j=1}^{s+1} (X_m - \nu_j) \right).$$

Then $\deg Q = (m-2)(q-1) + (q-2) + (s+1) = (m-1)(q-1) + s = r$ and so $Q \in V_q(r, m)$. Also, $\text{wt}(c_Q) = 2(q-s-1)$ and $\text{Supp}(c_Q) \subseteq A_1 \cup A_2$, where $A_i$ denotes the affine line $\{a = (a_1, \ldots, a_m) \in \mathbb{F}_q^m : a_1 = \cdots = a_{m-2} = 0, a_{m-1} = \omega_i\}$ for $i = 1, 2$. The minimum distance of $\mathcal{RM}_q(r, m)$ in this case is $q-s$ and it is less than $2(q-s-1)$, since $s < q-2$. As in the proof of Proposition 4.6, it suffices to show that the existence of $F \in V_q(r, m)$ such that $c_F$ is a minimum weight codeword with $\text{Supp}(c_F) \subseteq \text{Supp}(c_Q)$ leads to a contradiction. By the characterization of minimum weight codewords of $\mathcal{RM}_q(r, m)$, any such $F$ has to be of the form

$$F(X_1, \ldots, X_m) = \prod_{i=1}^{m-1} (L_i^{q-1} - 1) \prod_{j=1}^{s-1} (L_j - \omega_j')$$

for some linearly independent linear polynomials $L_1, \ldots, L_m$ in $\mathbb{F}_q[X_1, \ldots, X_m]$ and distinct $\omega_1', \ldots, \omega_s' \in \mathbb{F}_q$. Also, arguing as in the proof of Theorem 4.1 we see that $\text{Supp}(c_F)$ is contained in the affine line $A' := \{a \in \mathbb{F}_q^m : L_i(a) = 0 \text{ for } i = 1, \ldots, m-1\}$. Now if any two points of $\text{Supp}(c_F)$ belong to different affine lines $A_1$ and $A_2$, then $A_i \cap A'$ is nonempty for $i = 1, 2$ and dimension considerations imply that $A_1 = A_2 = A'$, which is a contradiction. Hence the $(q-s)$ points of $\text{Supp}(c_F)$ are contained in $\text{Supp}(c_Q) \cap A_i$ for a unique $i \in \{1, 2\}$. But then $q-s \leq q-s-1$, which is a contradiction. This proves the proposition.

An easy consequence of the above result is that unlike linear resolutions (which correspond to MDS codes), purity of a resolution is not preserved when passing to the dual.

**Corollary 4.10.** There exists linear codes $C$ with a pure resolution such that $C^\perp$ does not have a pure resolution.

**Proof.** By Theorem 4.1 the first order Reed-Muller code $\mathcal{RM}_q(1, m)$ has a pure resolution. But the dual of $\mathcal{RM}_q(1, m)$ is $\mathcal{RM}_q((m-1)(q-1) + (q-3), m)$ and it does not have a pure resolution, thanks to Proposition 4.9.

We can consolidate the results in subsections 4.1, 4.6 to obtain the following.

**Theorem 4.11.** Assume that $m \geq 2$ and $1 < r < m(q-1) - 1$. Write $r = t(q-1) + s$, where $0 \leq t \leq m-1$ and $0 \leq s < q-1$. Suppose $s \neq 1$. Then any minimal free $\mathbb{N}$-resolution of the Reed-Muller code $\mathcal{RM}_q(r, m)$ is not pure.

**Proof.** Follows from Propositions 4.4, 4.6, 4.7, 4.8 and 4.9.
5. More Examples two weight codes and the resolution of the code

This section is devoted to two weight codes. A linear code $C$ is said to be two weight code if there are two distinct positive integers $d_1$ and $d'_1$ such that every nonzero codeword of $C$ has weight either $d_1$ or $d'_1$. We have seen in Corollary 3.9 that the resolution of constant weight codes are pure and that the Betti numbers of these codes are explicitly known. The first order Reed-Muller codes are examples of a two weight codes, and Theorem 4.1 shows that their resolutions are pure and the Betti numbers can be explicitly determined. Thus, it is natural to ask if every two weight code has pure resolution. In this section we will choose several examples of two weight codes given by Calderbank and Kantor [10] and see that some of them have pure resolution and others does not. In [10], these codes are referred to by a nomenclature such as RT1, TF1, TF1$^d$, etc., and this is indicated in parenthesis at the beginning of each of the examples considered here. We also compute the Betti numbers of some of the two weight codes irrespective of whether or not their resolution is not pure. The examples of two weight codes given in [10] are defined geometrically. So before considering them here, we a recall a geometric language for codes and translate our characterization of purity (Theorem 3.6) in this language.

As before, fix positive integers $n, k$ with $k \leq n$ and a prime power $q$. We denote by $\mathbb{P}^{k-1}$ the $(k-1)$-dimensional projective space over the finite field $\mathbb{F}_q$. A (nondegenerate) $[n,k]_q$ projective system is a multiset of $n$ points of $\mathbb{P}^{k-1}$ that do not lie on a hyperplane of $\mathbb{P}^{k-1}$. Let $\mathcal{P}$ be a $[n,k]_q$ projective system. For $r = 1, \ldots, k$, the $r$th generalized Hamming weight, or the $r$th higher weight of $\mathcal{P}$ is defined by

$$d_r(\mathcal{P}) = n - \max\{|\mathcal{P} \cap \Pi_r| : \Pi_r \text{ linear subspace of } \mathbb{P}^{k-1} \text{ with codim } \Pi_r = r\}.$$

Here the “cardinality” $|\mathcal{P} \cap \Pi_r|$ is understood as the sum of multiplicities of points of $\mathcal{P}$ that are in $\Pi_r$. Note that the only linear subspace of codimension $k$ in $\mathbb{P}^{k-1}$ is the empty set, whereas those of codimension $k-1$ consist of a single point. Thus

$$d_k(\mathcal{P}) = n \quad \text{and} \quad d_{k-1}(\mathcal{P}) = n - 1.$$

We can naturally associate a nondegenerate $[n,k]_q$-linear code to $\mathcal{P}$ as follows. Choose representatives $P_1, \ldots, P_n$ in $\mathbb{F}_q^k$ corresponding to the $n$ points of $\mathcal{P}$. Let $(\mathbb{F}_q^k)^*$ be the dual space of the vector space $\mathbb{F}_q^k$. Consider the evaluation map

$$\text{Ev} : (\mathbb{F}_q^k)^* \to \mathbb{F}_q^n \text{ defined by } \text{Ev}(f) = (f(P_1), \ldots, f(P_n)).$$

The image of $\text{Ev}$ is a linear subspace $C$ of $\mathbb{F}_q^n$ such that $\dim C = k$ and $C$ is not contained in a coordinate hyperplane of $\mathbb{F}_q^n$. This, then, is the $[n,k]_q$-linear code associated to $\mathcal{P}$. We refer to Tsfasman, Vladuţ and Nogin [23] for more on projective systems and simply remark that the above association gives rise to a one-to-one correspondence between the equivalence classes of $[n,k]_q$ projective systems and nondegenerate $[n,k]_q$-linear codes; which preserves generalized Hamming weights. Also, subcodes of $C$ of dimension $r$ correspond to linear subspaces of $\mathbb{P}^{k-1}$
of codimension $r$. Thus, we define the support of a linear subspace $\Pi_r$ of $\mathbb{P}^{k-1}$ with $\text{codim} \Pi_r = r$, to be the multiset $\mathcal{P} \setminus \mathcal{P} \cap \Pi_r$. This corresponds precisely to the support of the corresponding subcode of $C$. As a consequence, we obtain the following geometric translation of our characterization of purity.

**Theorem 5.1.** Let $\mathcal{P} \subseteq \mathbb{P}^{k-1}$ be an $[n,k]_q$ projective system and let $C$ be the corresponding $[n,k]_q$-code. The $\mathbb{N}$-graded resolution of $C$ is pure if and only if for every $1 \leq r \leq k-1$ and every linear subspace $\Pi_r \subset \mathbb{P}^{k-1}$ of codimension $r$ there exist a linear subspace $H(\Pi_r) \subset \mathbb{P}^{k-1}$ of codimension $r$ with $\Pi_r \cap \mathcal{P} \subseteq H(\Pi_r) \cap \mathcal{P}$ and $|H(\Pi_r) \cap \mathcal{P}| = n - d_r(\mathcal{P})$.

**Proof.** Follows from Theorem 3.6 $\square$

**Corollary 5.2.** $\mathcal{P} \subseteq \mathbb{P}^{k-1}$ be an $[n,k]_q$ projective system and let $C$ be the corresponding linear code. Then the $\mathbb{N}$-graded resolution of $C$ is always pure at the $(k-1)^{\text{th}}$ and $k^{\text{th}}$ step.

**Proof.** From [13], we see that $C$ is $(k-1)$-MDS. Thus the desired result follows from Corollary 5.8 and Theorem 5.1 $\square$

The following definition from [10] is a geometric counterpart of two weight codes.

**Definition 5.3.** A projective $(n,k,h_1,h_2)$ system is an $[n,k]$ projective system $\mathcal{P}$ with the property that every hyperplane of $\mathbb{P}^{k-1}$ intersects $\mathcal{P}$ either at $h_1$ points or at $h_2$ points.

Note that if $\mathcal{P}$ is a projective $(n,k,h_1,h_2)$ system, then every nonzero codeword of the corresponding $[n,k]_q$ code $C$ is of Hamming weight $n - h_1$ or $n - h_2$.

We are now ready to discuss several examples from [10] of two weight codes, and investigate their purity and minimal free resolutions. We use the following notation.

$$p_j = p_j(q) := |\mathbb{P}^j(\mathbb{F}_q)| = \begin{cases} q^j + q^{j-1} + \cdots + q + 1 & \text{if } j \geq 0, \\ 0 & \text{if } j < 0. \end{cases}$$

**Example 5.4 (RT1).** Take the base field as $\mathbb{F}_{q^2}$ and let $\mathcal{P} = \mathbb{P}^{k-1}(\mathbb{F}_{q^2})$. Consider $\mathcal{P} = \mathbb{P}^{k-1}(\mathbb{F}_{q^2})$ as a projective system in $\mathbb{P}$. If $\Pi$ is a hyperplane in $\mathbb{P}$, then it is given by an equation of the form $\sum_{i=1}^k z_iX_i = 0$, where $z_1, \ldots, z_k \in \mathbb{F}_{q^2}$, not all zero. Fix a $\mathbb{F}_q$-basis $\{1, \theta\}$ of $\mathbb{F}_{q^2}$ and write $z_i = a_i + \theta b_i$, where $a_i, b_i \in \mathbb{F}_q$ for $i = 1, \ldots, k$. Then $\mathcal{P} \cap \Pi$ consists of points $(c_1 : \cdots : c_k) \in \mathbb{P}^{k-1}(\mathbb{F}_q)$ satisfying $\sum a_i c_i = 0$ and $\sum b_i c_i = 0$. Now if $a_i = 0$ for all $i = 1, \ldots, k$, or if $b_i = 0$ for all $i = 1, \ldots, k$, then $\mathcal{P} \cap \Pi$ corresponds to a $\mathbb{F}_q$-rational hyperplane in $\mathbb{P}^{k-1}(\mathbb{F}_q)$. Otherwise it corresponds to a codimension 2 linear subspace of $\mathbb{P}^{k-1}(\mathbb{F}_q)$. Thus $|\mathcal{P} \cap \Pi| = p_{k-2}(q)$ or $p_{k-3}(q)$. It follows that the linear code corresponding to $\mathcal{P}$, say $C$, is a two-weight code of length $p_{k-1}(q)$ and dimension $k$ over $\mathbb{F}_{q^2}$. Also, it is clear that as $\Pi_r$ varies over $\mathbb{F}_{q^2}$-linear subspaces of codimension $r$ in $\mathbb{P}$, the
maximum possible value of $|\mathcal{P} \cap \Pi_r|$ is attained when $\Pi_r$ is $\mathbb{F}_q$-rational, and in that case $|\mathcal{P} \cap \Pi_r| = p_{k-1-r}(q)$ for $r = 1, \ldots, k$. It follows that the higher weights of $\mathcal{P}$ are given by $d_r = p_{k-1} - p_{k-1-r}$.

To determine purity of minimal free resolutions of $C$, fix a $\mathbb{F}_{q^2}$-linear subspaces $\Pi$ of codimension $r$ in $\mathbb{P}$. Let $t := \dim_{\mathbb{F}_q}(\Pi \cap \mathcal{P})$. If $\Pi$ is not $\mathbb{F}_q$-linear, then $t < k - 1 - r$. Let $\{f_1, \ldots, f_{k+1}\}$ be a $\mathbb{F}_q$ basis of $\Pi \cap \mathcal{P}$. Extend this to a linearly independent set $\{f_1, \ldots, f_{k+r}\} \subset \mathcal{P}$. Note that the set $\{f_1, \ldots, f_{k-r}\}$ is linearly independent over $\mathbb{F}_{q^2}$. (This can be seen, as before, by expressing the coefficients in a linear dependence relation in terms of $1, \theta$.) Now if $H = H(\Pi_r)$ is the linear subspace of $\mathcal{P}$ spanned by $\{f_1, \ldots, f_{k-r}\}$, then $\Pi \cap \mathcal{P} \subset H \cap \mathcal{P}$ and $|H \cap \mathcal{P}| = n - d_r(\mathcal{P})$. Thus, Theorem 5.1 shows that a minimal free resolution of $C$ is pure. Moreover, it is of the form

$$0 \rightarrow R(-d_k)^{\beta_k} \rightarrow \cdots \rightarrow R(-d_2)^{\beta_2} \rightarrow R(-d_1)^{\beta_1} \rightarrow R$$

where $d_r = p_{k-1} - p_{k-1-r}$ and $\beta_i$ are given by Herzog-Kühl equation. In fact, this is precisely the resolution for constant weight codes given in Corollary 3.9.

Remark 5.5. One can similarly consider $\mathcal{P} = \mathbb{P}^{k-1}(\mathbb{F}_q) \subseteq \mathbb{P}^{k-1}(\mathbb{F}_{q^m})$ for any $m \geq 2$, and show that the resolution of the linear code corresponding to this projective system is pure and of the form similar to that in Example 5.4 even though this code is not a two weight code when $m > 2$.

Example 5.6 (TF1). Let $q$ be even and $\mathbb{P}^2$ be the projective plane over $\mathbb{F}_q$. Let $\mathcal{P} \subseteq \mathbb{P}^2$ be a hyperoval. Then by definition, $\mathcal{P}$ is a set of $n = q + 2$ points, no three collinear, with the property that if $L$ is a line in $\mathbb{P}^2$ then $|L \cap \mathcal{P}| = 0$ or 2. In this case the corresponding code is an MDS $[q + 2, 3, q]$ code and the resolution of this code is given by Corollary 3.8.

Example 5.7 (TF1'). Let $\hat{\mathbb{P}}^2$ be the dual projective plane and consider $\hat{\mathcal{P}} = \{L \subseteq \mathbb{P}^2 \text{ is a line and } |L \cap \mathcal{P}| = 2\}$. Note that $\hat{\mathcal{P}} \subseteq \hat{\mathbb{P}}^2$ and points of the projective space $\mathbb{P}^2$ are lines in $\hat{\mathbb{P}}^2$. Note that any two point of $\mathcal{P}$ correspond to a unique line $L$ in $\mathbb{P}^2$ such that $L \in \hat{\mathcal{P}}$. And this proves $|\hat{\mathcal{P}}| = \binom{q+2}{2}$. Now consider a line in $\hat{\mathbb{P}}^2$ i.e. a point of $\mathcal{P} \in \hat{\mathbb{P}}^2$. Counting the intersection of this line with $\hat{\mathcal{P}}$ corresponds to counting lines $L \subseteq \mathbb{P}^2$ that passes through this point $P$ and intersect the hyperoval $\mathcal{P}$ in exactly two points. It depends only on whether the chosen point $P$ lies on $\mathcal{P}$ or not. More precisely if the point $P$ lies on $\mathcal{P}$ then any line passing through this point will intersect the hyperoval $\mathcal{P}$ in two points and there are exactly $(q + 1)$ such lines. On the other hand if if the point $P$ does not lie on $\mathcal{P}$ then choosing any point $Q$ on $\mathcal{P}$ will correspond a line passing through $P$ and $Q$ and intersect $\mathcal{P}$ in one more point. Further, since all of these lines intersect at the point $P$, these lines will intersect in mutually distinct points of $\mathcal{P}$ and hence there are $\frac{(q+2)^2}{2}$ many such lines.
This correspond to an \(\left(\frac{q^2+2}{2}\right), 3, (q + 1), \frac{q+2}{2}\) projective system or equivalently an \(\left(\frac{q^2+2}{2}\right), 3, \frac{q(q+1)}{2}, \frac{q(q+2)}{2}\) code. From the discussion it is clear that there are \((q+2)\) lines in \(\tilde{P}^2\) (points of \(P\)) that will intersect \(\tilde{P}\) in \((q+1)\) many points and \((q^2-1)\) lines in \(\tilde{P}^2\) (points not on \(P\)) that will intersect \(\tilde{P}\) in \(\frac{q^2+2}{2}\) many points. In other words, the weight spectrum of the code corresponding to \(\tilde{P}\) is

\[
A_{w_1} = (q + 2)(q - 1) \quad \text{and} \quad A_{w_2} = (q^2 - 1)(q - 1).
\]

Also, from the discussion we get that both kinds of hyperplane sections of \(\tilde{P}\) are maximal or equivalently, every nonzero codeword has minimal support. Consequently, the resolution of \(C\) is not pure at the first level, but have two twists. But from the construction of the resolution, it is clear that \(\beta_{1,j}\) denote the number of 1-dimensional subspaces of \(C\) whose support is minimal and support weight is \(j\). Hence, we obtain

\[
\beta_{1, \frac{q(q+1)}{2}} = (q + 2) \quad \text{and} \quad \beta_{1, \frac{q(q+2)}{2}} = (q^2 - 1).
\]

Hence, in the view of Corollary 5.2 the resolution of the code \(C\) is of the form:

\[
R(-d_3)^{\beta_3,d_3} \to R(-d_2)^{\beta_2,d_2} \to R(-d_1')^{\beta_1,d_1'} \oplus R(-d_1)^{\beta_1,d_1}
\]

where \(d_1 = \frac{q(q+1)}{2}, \ d_1' = \frac{q(q+2)}{2}, \ d_2 = \frac{q(q+3)}{2}\) and \(d_3 = \frac{(q+1)(q+2)}{2}\). If we write \(X = \beta_{1,d_2}, \ X' = \beta_{1,d_1'}, \ Y = \beta_{2,d_2}\) and \(Z = \beta_{3,d_3}\), then Boij-Söderberg equation (7) gives

\[
1 - (X + X') + Y - Z = 0
\]

\[
-d_1 X - d_1' X' + d_2 Y - d_3 Z = 0
\]

\[
-d_2^2 X - d_1'^2 X' + d_2^2 Y - d_3^2 Z = 0
\]

Putting the values of \(d_1, \ d_1', \ d_2, \ d_3, \ X\) and \(X'\), we get the values of \(Y = \frac{q(q+1)(q+2)}{2}\) and \(Z = \frac{q^2(q+1)}{2}\). This determines the resolution of the code corresponding to \(\tilde{P}\).

Example 5.8 (TF2). Let \(q\) be even and \(\mathbb{P}^2\) is the projective space over \(\mathbb{F}_q\). Let TF2 is the projective system \(P \subseteq \mathbb{P}^2\) of \(n = 1 + (q + 1)(h - 1)\), where \(h|q\) and \(1 < h < q\). The set \(P\) has the property that if \(L\) is a line then \(|L \cap P| = 0\) or \(h\). Let \(C\) be the corresponding code. Then \(C\) is a two-weight linear code of length \(n\), dimension 3 and weight \(q(h - 1)\) and \(n\). In this case the resolution of the code is pure, since the second weight of the code is the length of the code. Hence the resolution of this code is of the form:

\[
R(-d_3)^{\beta_3,d_3} \to R(-d_2)^{\beta_2,d_2} \to R(-d_1)^{\beta_1,d_1}
\]

where \(d_1 = q(h - 1), \ d_2 = (q + 1)(h - 1)\) and \(d_3 = 1 + (q + 1)(h - 1)\). Using the Herzog-Kühl formula, one can compute the Betti numbers and they are

\[
\beta_{1,d_1} = (q + 1)^2 - \frac{q}{h}, \quad \beta_{2,d_2} = qn, \quad \text{and} \quad \beta_{3,d_3} = (h - 1)^2(q + 1)\frac{q}{h}.
\]
Example 5.9 (TF2). Let \( \hat{\mathcal{P}} = \{ H \in \hat{\mathbb{P}}^2 : |H \cap \mathcal{P}| = h \} \) where \( \hat{\mathbb{P}}^2 \) denote the dual of projective plane \( \mathbb{P}^2 \). That is \( \hat{\mathcal{P}} \) is the set of lines in \( \mathbb{P}^2 \) that intersect \( \mathcal{P} \) in exactly \( h \) points. From every point of \( \mathcal{P} \) there are exactly \( (q + 1) \) lines passing through that point and each of these lines will intersect \( \mathcal{P} \) in exactly \( h \) points. Hence

\[
|\hat{\mathcal{P}}| = \frac{(q + 1)(1 + (q + 1)(h - 1))}{h} = n \text{(say)}.
\]

Next we want to understand the intersection of \( \hat{\mathcal{P}} \) with a hyperplane of \( \hat{\mathbb{P}}^2 \). Note that a hyperplane of \( \hat{\mathbb{P}}^2 \) is a point in \( \mathbb{P}^2 \) and

\[
P \cap \hat{\mathcal{P}} = \{ L \subset \mathbb{P}^2 : L \text{ is a line passing through } P \text{ and } |L \cap \mathcal{P}| = h \}
\]

Therefore \( P \cap \hat{\mathcal{P}} = (q + 1) \) or \( \frac{1 + (q + 1)(h - 1)}{h} \) depending on whether point \( P \) lies on \( \mathcal{P} \) or not. Therefore \( \hat{\mathcal{P}} \) is an \((n, 3, (q + 1), \frac{1 + (q + 1)(h - 1)}{h})\) projective system and equivalently corresponding linear code is a two weight code with parameters with distinct nonzero weights given by

\[
w_1 = \frac{q(q + 1)(h - 1)}{h} \quad \text{and} \quad w_2 = \frac{q(1 + (q + 1)(h - 1))}{h}.
\]

Further, from the discussion, it is clear that the weight spectrum of this code is

\[
A_{w_1} = (q - 1)(1 + (q + 1)(h - 1)) \quad \text{and} \quad A_{w_2} = (q - 1)(q + 1)(q - h + 1).
\]

As in the previous case, all of the sets \( P \cap \hat{\mathcal{P}} \) is maximal for every \( P \in \mathcal{P} \). In particular, every codeword of the code associated to \( \hat{\mathcal{P}} \) is minimal weight codeword and hence the resolution of this code is of the form

\[
R(-d_3)^{\beta_3,d_3} \rightarrow R(-d_2)^{\beta_2,d_2} \rightarrow R(-d_1)^{\beta_1,d_1} \oplus R(-d_1)^{\beta_1,d_1}
\]

where

\[
\beta_1,d_1 = (1 + (q + 1)(h - 1)) \quad \text{and} \quad \beta_1,d_1 = (q^2 - h + 2)(q + 1) - 1.
\]

As in Example 5.7 using Boij-Söderberg equation (7) and putting all known values, we get three equations in 2 variables and we can solve them to determine the resolution of this code.

Example 5.10 (TF3). Let \( \mathcal{P} \subset \mathbb{P}^3 \) be the ovoid of \( q^2 + 1 \) many points with the property that no three points of \( \mathcal{P} \) are collinear, and if \( H \) is a hyperplane the \( |H \cap \mathcal{P}| = 1 \) or \( q + 1 \). Let \( C \) be the corresponding linear code. Then \( C \) is two-weight code of length \( n = q^2 + 1 \), dimension \( k = 4 \) and weights \( w_1 = q(q - 1) \) and \( w_2 = q^2 \). The resolution of this code \( C \) is pure. To see this, note that if \( \Pi \) is a hyperplane in \( \mathbb{P}^3 \) intersecting \( \mathcal{P} \) at only one point, then there is another hyperplane \( H \) with \( |H \cap \mathcal{P}| = q + 1 \) and \( \Pi \cap \mathcal{P} \subset H \cap \mathcal{P} \). More precisely, let \( \Pi \cap \mathcal{P} = \{ P \} \) and let \( Q \in \mathcal{P} \) be any point other than \( P \). Let \( H \) be a hyperplane passing through \( P \) and \( Q \). Since \( 2 \leq |H \cap \mathcal{P}| \) we get \( |H \cap \mathcal{P}| = q + 1 \). Further, \( \Pi \cap \mathcal{P} \subset H \cap \mathcal{P} \). In
particular, all minimal codewords are minimum weight codewords of the code and hence the resolution is pure at the first step. Now, observe that
\[ d_2(C) = q^2 - 1. \]
The code is 2-MDS and hence by Corollary 3.8, the resolution is linear after the second step. This proves that the resolution of this code is pure and is of the form
\[ R(-((q^2+1))^{\beta_4,q^2+1} \rightarrow R((-q^2))^{\beta_3,q^2} \rightarrow R(-(q^2-1))^{\beta_2,q^2-1} \rightarrow R(-(q(q-1)))^{\beta_1,q(q-1)}). \]
Solving the Herzog-Kühl equation for Betti numbers, we obtain
\[
\begin{align*}
\beta_4,q^2+1 &= \frac{q^3(q-1)^2}{2}, \\
\beta_3,q^2 &= (q-1)(q^2-1)(q^2+1) \\
\beta_2,q^2-1 &= \frac{q^3(q^2+1)}{2} \quad \text{and} \quad \beta_1,q(q-1) = q(q^2+1)
\end{align*}
\]

**Example 5.11 (RT3).** Let \( P \subset \mathbb{P}^k(F_{q^2}) \) is the projective variety defined by the equation
\[ X_1^{q+1} + \cdots + X_k^{q+1} = 0. \]
The variety \( P \) is known as the Hermitian variety. The linear code corresponding to \( P \) is a two weight code. The intersection of this variety with linear subspaces of different dimension is well understood from the work of Bose and Chakravarti [7]. And it is not hard to see that the resolution of this code is not pure if \( k \geq 4 \). We will discuss the nature of the resolution of this code when \( k = 3 \) and \( k = 4 \). In the case of \( k = 3 \), \( P \) is the Hermitian curve consisting of \( q^3 + 1 \) points. In [7] it is proved that if \( L \) is a line then \( |L \cap P| = 1 \) or \( q + 1 \). So let \( L_1 \) be a line \( \mathbb{P}^2(F_{q^2}) \) that intersects \( P \) at exactly one point, say \( \{P\} \). Let \( Q \in P \) be another point on the curve other than \( P \) and \( L_2 \) be a line passing through \( P \) and \( Q \). Then
\[ |L_2 \cap P| = q + 1 \quad \text{and} \quad L_1 \cap P \subset L_2 \cap P. \]
In other words, all minimal 1-subcode of the corresponding code has support weight \( |P| - (q + 1) = q(q^2 - 1) \) and hence the resolution is pure at first level. To see that the resolution is pure at the second level we argue exactly as in example TF1\( ^d \).
Therefore, in this case, the resolution is pure and is of the form
\[ R(-((q^3+1))^{\beta_5,q^3+1} \rightarrow R((-q^3))^{\beta_2,q^3} \rightarrow R(-(q(q^2-1)))^{\beta_1,q(q^2-1)}). \]
Solving Herzog-Kühl equation \( \mathbb{S} \) for Betti numbers, we get
\[
\begin{align*}
\beta_5,q^3+1 &= q^2(q^2-q+1), \\
\beta_2,q^3 &= (q^3+1)(q^2-1) \quad \text{and} \quad \beta_3,q^3+1 = q(q^2-1)(q^2-q+1).
\end{align*}
\]
If \( k = 4 \), and \( P \) is the Hermitian surface in \( \mathbb{P}^4(F_{q^2}) \). Then the corresponding linear code is a two-weight code of length \((q^2 + 1)(q^3 + 1)\), dimension 4, weights \( w_1 = q^3 \) and \( w_2 = q^6 + q^2 \). In [7], it have been proved that the intersection of \( P \) with a hyperplane in \( \mathbb{P}^3(F_{q^2}) \) is another Hermitian variety in \( \mathbb{P}^2(F_{q^2}) \) of rank 2 or 3. Further, neither of them is contained in another hence the resolution of the code is not pure at the first step. To understand the nature of the resolution at the second step, we use that fact from [7] that a line can intersect the Hermitian surface in the
following three ways: (i) at a point; (ii) at \((q+1)\) points, and (iii) at \((q^2+1)\) points, in which case, the line is contained in the Hermitian surface. Case (ii) and (iii) will correspond to minimal 2-subcodes and hence there is two twist at the second step in the resolution. The resolution of the code is pure after the second step. The argument is exactly the same as in Example 5.6 for TF1\(d\). Hence, the resolution of this code is of the form

\[
R(-d_1)^{\beta_1,d_1} \to R(-d_2)^{\beta_2,d_2} \to R(-d_2)^{\beta_2,d_2} \oplus R(-d_2)^{\beta_2,d_2} \to R(-d_4)^{\beta_4,d_4} \oplus R(-d_1)^{\beta_1,d_1}
\]

where 

\[
d_1 = q^5, \quad d_2 = q^5 + q^3, \quad d_2 = q^3(q^2 + 1), \quad d_2 = q(q + 1)(q^3 - q^2 + 2q - 1),
\]

\[
d_3 = (q^2 + 1)(q^3 + 1) - 1 \quad \text{and} \quad d_4 = (q^2 + 1)(q^3 + 1).
\]

Further, from Table 2b of [10] and the fact that \(\beta_{1,j}\) counts number of 1 minimal subcodes of weight \(j\), we see that

\[
\beta_{1,d_1} = \frac{A_{w_1}}{q-1} = (q^3 + 1)(q^3 + q^2 + q + 1) \quad \text{and} \quad \beta_{1,d_1} = \frac{A_{w_2}}{q-1} = q^4(q^4 - 1)
\]

Putting all these values in Boij-Söderberg equation (7), we get 4 distinct equations in 4 variables with a unique solution. This gives the complete resolution of the code corresponding to the projective system of Hermitian surface.

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**References**

[1] N. Armenoff, *Free Resolutions associated to Representable Matroids*, Ph.D. Thesis, Univ. Kentucky, Lexington, KY, 2015.

[2] E. F. Assmus Jr. and J. D. Key, *Designs and their Codes*, Cambridge: Cambridge University Press, 1992.

[3] A. Ashikhmin and A. Barg, Minimal vectors in linear codes, *IEEE Trans. Inform. Theory* 44 (1998), 2010–2017.

[4] A. Björner, The homology and shellability of matroids and geometric lattices, *Matroid Applications, Encyclopedia Math. Appl.*, 40 (1992), 226–283.

[5] M. Boij and J. Söderberg, Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture, *J. Lond. Math. Soc.* 78 (2008), 85–106.

[6] A. Bonisoli, Every equidistant linear code is a sequence of dual Hamming codes, *Ars Combin.* 18 (1984), 181–186.

[7] R. C. Bose and I. M. Chakravarti, Hermitian Varieties in a Finite Projective Space \(PG(N,q^2)\), *Canad. J. Math.* 18 (1966), 1161–1182.
[8] T. Britz, Higher support matroids, *Discrete Math.*, **307** (2007), 2300–2308.
[9] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, 2nd Ed., Cambridge Univ. Press, 2009.
[10] R. Calderbank and W. M. Kantor, The geometry of two-weight codes, *Bull. London Math. Soc.* **18** (2) (1986), 97–122.
[11] P. Delsarte, J. M. Goethals and F. J Mac Williams, On generalized Reed-Muller codes and their relatives, *Information and Control* **16** (1970), 403–442.
[12] S. R. Ghorpade, A. R. Shastri, M. K. Srinivasan and J. K. Verma (Eds.), *Combinatorial Topology and Algebra*, RMS Lect. Notes Ser. **18**, Ramanujan Math. Soc., Mysore, 2013.
[13] J. Herzog and M. Kühn, On The Betti Numbers of Finite Pure and Linear Resolutions, *Comm. Algebra* **12** (1984), 1627–1646.
[14] T. Johnsen, *Private communication*, April 2015.
[15] T. Johnsen and H. Verdure, Hamming weights and Betti numbers of Stanley- Reisner rings associated to matroids, *Appl. Algebra Engrg. Comm. Comput.* **24** (2013), 73–93.
[16] T. Johnsen, H. Verdure, Stanley-Reisner resolution of constant weight codes, *Des. Codes Cryptogr.*, **72** (2014), 471–481.
[17] T. Johnsen, J. Roksvold, and H. Verdure, A generalization of weight polynomials to matroids, *Discrete Math.*, **339** (2016), 632–645.
[18] R. Jurrius and R. Pellikaan, Extended and generalized weight enumerators, in: *Proc. WCC 2009*, Selmer Center, Univ. Bergen, Norway, 2009.
[19] Z. Liu and W. Chen, Notes on the value function, *Des. Codes Cryptogr.*, **54** (2010), 11–19.
[20] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes*, Elsevier, New York, 1977.
[21] A. Karpova, *Homological Methods applied to Theory of Codes and Matroids*, Masters Thesis, The Arctic Univ. Norway, Tromso, Norway, 2015.
[22] J. S. Provan, *Decompositions, Shellings, and Diameters of Simplicial Complexes and Convex Polyhedra*, Ph.D. Thesis, Cornell Univ., Ithaca, NY, 1977.
[23] M. Tsfasman, S. Vladuț and D. Nogin, *Algebraic Geometric Codes: Basic Notions*, Math. Surv. Monogr., vol. 139, Amer. Math. Soc., Providence, 2007.
[24] V. K. Wei, Generalized Hamming Weights for Linear Codes, *IEEE Trans. Inform. Theory* **37** (1991), 1412–1418.

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