Equation of motion for relativistic compact binaries
with the strong field point particle limit:
the second and half post-Newtonian order

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We study the equation of motion appropriate to an inspiralling binary star system whose constituent stars have strong internal gravity. We use the post-Newtonian approximation with the strong field point particle limit by which we can introduce into general relativity a notion of a point-like particle with strong internal gravity without using Dirac delta distribution. Besides this limit, to deal with strong internal gravity we express the equation of motion in surface integral forms and calculate these integrals explicitly. As a result we obtain the equation of motion for a binary of compact bodies accurate through the second and half post-Newtonian (2.5 PN) order. This equation is derived in the harmonic coordinate. Our resulting equation perfectly agrees with Damour and Deruelle 2.5 PN equation of motion. Hence it is found that the 2.5 PN equation of motion is applicable to a relativistic compact binary.

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I. INTRODUCTION

A relativistic compact binary system in an inspiralling phase is one of the most promising sources of gravitational waves for detectors under construction such as LIGO [1], VIRGO [2], GEO600 [3], and TAMA300 [4]. Among those detectors, TAMA300 has made rapid progress and has recently started observation [3].

To detect gravitational wave signals directly, which are expected to be buried in noisy data streams, the matched filtering technique will be (and, in fact, has been) used. To use this technique, we have to construct reliable wave form templates, i.e., theoretical prediction of gravitational wave form accurate enough to match a data stream with templates without missing one phase of the expected signal. The equation of motion for the source system is one of the necessary ingredients for making such templates.

In an inspiralling phase, since the separation of the binary is sufficiently larger than the radius of the stars, the interbody gravity is weak and orbital velocities of stars are small compared to the velocity of light. Hence the post-Newtonian approximation with a monopole-truncated object [5] description is useful. In fact, various authors have used it to derive an equation of motion of the system [24–27,7–14]. To obtain sufficiently accurate templates, we have to derive a highly accurate, say, the 4 PN equation of motion [15–19] (though, the 2 PN templates [20–22] might be enough to construct search templates [23]). What we are aiming at is to derive such an accurate post-Newtonian equation of motion. In this paper we derive the 2.5 PN equation of motion as a step toward this goal.

The 2.5 PN equation of motion for two monopole-truncated bodies has been derived by various authors; Damour and Deruelle [24,27] used the post-Minkowskian formalism [28], Kopejkin and Grishchuk worked on a spherical body [7,8], Schäfer [10] used the Hamiltonian approach, and Blanchet, Faye and Ponsot used the post-Newtonian approach [12]. Their resulting equations of motion agree with each other. Then what can we add? It
seems to us that these derivations have some uncomfortable features. In the derivation in [28,24–27,10,12], though armed with the ”Dominant Schwartshild” condition [27], one needs to regularize divergences caused by their use of Dirac delta distribution. In the derivation free from any divergences [7,8], one applies the post-Newtonian approximation even to the inside of a star (recall that a neutron star has strong internal gravity). Though the derivation in [12] is mathematically systematic and well-devised, one assumes that a point mass follows a regularized geodesic equation. Instead, in our method we can take into account the strong gravity inside stars by use of the strong-field point particle limit [29,30] and surface integrals, without assuming a geodesic equation.

One of our motivations to derive an higher post-Newtonian equation of motion with our method comes from the recent result on the 3 PN equation of motion, which has been derived by two groups in independent manners (see also [31]). Jaranowski and Schäfer have used Hamiltonian approach and succeeded to determine the 3 PN order Hamiltonian [11] except two terms whose coefficients are arbitrary; the kinetic part [11] and the static part [33]. The former has been recently determined by imposing poincaré invariance on their equation of motion [34]. Blanchet and Faye have also succeeded to derive the 3 PN equation of motion [13,14] using their generalized Hadamard partie finie regularization [35,36]. Though their regularization method is declared to be well-defined unlike the method used by Jaranowski and Schäfer [11], they found that there remains one arbitrary constant corresponding to the static part in the work of Jaranowski and Schäfer. Although the method of Blanchet and Faye seems mathematically rigorous and well-devised, what is uncomfortable, besides the above arbitrary constant, comes from the fact that they rely on some ”naturality” : They assume that a mass point follows a regularized geodesic equation, but the assumption should be verified. Hence, it is interesting to derive the 3 PN equation of motion using yet another method. Besides this, It is important to derive the equation of motion with various methods to confirm the equation, which appears through long calculations.

In our previous paper ([37], referred as paper I henceforth), we have derived the 1 PN equation of motion using the post-Newtonian approximation with the strong filed point
particle limit \[29,30\] and the general equation of motion expressed in surface integrals \[37\] as in the Einstein-Infeld-Hoffmann approach \[32\]. The spin-orbit and quadrupole-orbit coupling forces which appear at the 2 PN order in our ordering have been also derived. Following the initial value formulation on the field by Schutz and Futamase \[38–41\], in this paper we shall derive the 2.5 PN equation of motion.

This paper is organized as follows. In Sec. II, we will review the post-Newtonian approximation and the strong field point particle limit. In Sec. III we shall show how to solve the Einstein equation. There we shall define multipole moments of stars, and introduce a general form of the equation of motion. It is entirely expressed in surface integral form. From Sec. IV to Sec. VII, we shall derive the equation of motion for two compact bodies accurate though the 2.5 PN order, the order at which the radiation reaction effect first appears. It will be found that the temporal component of the equation of motion and its (functional) solution play an important role in our derivation. The section VIII is devoted to conclusion and discussion. Useful formulas and explanations on subtleties of our method will be given in appendices.

Throughout we use units where \( c = 1 = G \). \( \vec{x} \) denotes Euclidean three vector. We raise or lower its index with Kronecker delta.

**II. POST-NEWTONIAN APPROXIMATION WITH STRONG FIELD POINT PARTICLE LIMIT**

In this section we shall review the strong field point particle limit associated with the post-Newtonian approximation. Also scalings of initial data for matter and a body zone will be explained. These scalings enable us to incorporate a self-gravitating point-like particle into post-Newtonian approximation and ensure that tidal effect is the lowest order effect which causes an internal motion of the stars except their spinning motion. See \[29,30\] for detail.

In the inspiralling phase, the post-Newtonian approximation with a point particle de-
scription is suitable to derive the equation of motion. In the post-Newtonian approximation we assume the balance between Newtonian gravitational force and centrifugal force, i.e., \( m/L^2 \sim v_{orb}^2/L \), where \( m \), \( L \), and \( v_{orb} \) are typical scales of a mass of a star, the separation of the binary, and an orbital velocity of the star. Then a natural time scale \( \tau \), which we call Newtonian dynamical time, is
\[
\tau = \epsilon t \]  
where \( \epsilon \) is a non-dimensional parameter and represents the smallness of the orbital velocity;
\[
v_{orb}^i = \frac{dx^i}{dt} = \epsilon \frac{dx^i}{d\tau}, \tag{2.1}
\]
and we set \( dx/d\tau = O(\epsilon^0) \). Therefore we take \( \epsilon \) as the post-Newtonian expansion parameter. Then, let \( \epsilon \) go to zero with \( \tau \) remaining constant. The meaning of \( \tau \) is that two events at the same \((\tau, x^i)\) but having different \( \epsilon \) are in the nearly same phase in the orbit \([39]\). Then from the Newtonian force balance, \( m \sim \epsilon^2 \) when we keep the orbital separation constant.

Now we achieve point particle limit by letting the radius of the star, say, \( R \), shrink at the same rate as the mass of the star; \( m/R = O(\epsilon^0) = \text{constant} \) while \( R = O(\epsilon^2) \to 0 \) and \( m = O(\epsilon^2) \to 0 \) as \( \epsilon \to 0 \). Because of finite \( m/R \), we say this point particle has strong internal gravity. This scaling implies that the density changes proportional to \( \epsilon^{-4} \) (in the \((t, x^i)\) coordinate). The scalings of \( R \) and \( m \) motivate us to define the body zone of the star \( A \) \((A=1,2)\), \( B_A = \{x^i||\vec{x} - \vec{z}_A(\tau)|| < \epsilon R_A\} \) and a body zone coordinate of the star \( A \), \( \alpha^i_A = \epsilon^{-2}(x^i - z^i_A(\tau)) \). \( z^i_A(\tau) \) is a representative point of the star \( A \), e.g., the center of the mass of the star \( A \). \( R_A \) is an arbitrary length scale (much smaller than \( L \) and is not the radius of the star) and constant (, i.e., \( dR_A/d\tau = 0 \)). With the body zone coordinate, the star does not shrink, while the boundary of the body zone goes to infinity. Then it is appropriate to define star’s characteristic quantities such as a mass, a spin, and so on with the body zone coordinate. On the other hand the body zone serves us with a surface \( \partial B_A \), through which gravitational energy momentum flux flows and in turn it amounts to gravitational force exerting on the star \( A \). Because the body zone boundary \( \partial B_A \) is far away from the surface of the star \( A \), we can evaluate the gravitational energy momentum flux at \( \partial B_A \) with the post-Newtonian gravitational field. In fact we shall express our equation of motion in
terms of integrals over $\partial B_A$ and be able to evaluate them explicitly.

Let us move on to the scaling of the matter variables. As in the paper I, we consider an initial value problem to solve the Einstein equation. We take two nearly stationary solutions of the exact Einstein equation as the initial data for the matter variables (and gravitational field). Then we assume that these solutions have the following scalings; the density changes proportional to $\epsilon^{-4}$, the velocity of spinning motion is $O(\epsilon)$. The slow spinning motion assumption is not crucial: In fact, it is straightforward to incorporate a rapidly spinning compact body into our formalism. The scaling of the density suggests that the natural dynamical time (free fall time) $\eta$ inside the star may be $\eta = \epsilon^{-2} t$. Then if we can not assume the nearly stationary condition on the stars, it is difficult to use the post-Newtonian approximation [29,30].

From these initial data we have the following scalings of the star A’s stress energy tensor components in the body zone coordinate, $T^{\mu\nu}_A$; $T^{\tau\tau}_A = O(\epsilon^{-2})$, $T^{\tau\dot{i}}_A = O(\epsilon^{-4})$, $T^{\dot{i}\dot{j}}_A = O(\epsilon^{-8})$. Here the underlined indices mean that for any tensor $A^i$, $A^\dot{i} = \epsilon^{-2} A^i$. In the paper I, we have transformed $T^{\mu\nu}_N$, the components of the stress energy tensor of the matter in the near zone coordinate, to $T^{\mu\nu}_A$ using the transformation of Fermi normal coordinate at the 1 PN order. It is difficult, however, to construct the Fermi normal coordinate at an higher post-Newtonian order. Therefore we shall not use it. We simply assume that for $T^{\mu\nu}_N$ (or rather $\Lambda^{\mu\nu}_N$, see Eq. (3.4)),

$$T^{\tau\tau}_N = O(\epsilon^{-2}), \quad (2.2)$$
$$T^{\tau\dot{i}}_N = O(\epsilon^{-4}), \quad (2.3)$$
$$T^{\dot{i}\dot{j}}_N = O(\epsilon^{-8}), \quad (2.4)$$

as their leading scalings. We have found that these scalings are enough to derive the 2.5 PN equation of motion for two compact bodies.

Henceforth we call the coordinate $(\tau, x^i)$ the near zone coordinate.
III. FIELD EQUATION AND THE GENERAL FORM OF THE EQUATION OF MOTION

A. Field equation

As discussed in the previous section, we shall express our equation of motion in terms of surface integrals over the body zone boundary where it is assumed that the metric slightly deviates from the flat metric \( \eta^{\mu\nu} = \text{diag}(\epsilon^2, 1, 1, 1) \) (in the near zone coordinate). Thus we define a deviation field \( h^{\mu\nu} \) as

\[
h^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g} g^{\mu\nu},
\]

where \( g \) is the determinant of the metric. Our \( h^{\mu\nu} \) differs from the corresponding field in [12] in a sign.

Now we choose the harmonic coordinate condition on the metric

\[
h^{\mu\nu,\nu} = 0, \tag{3.2}
\]

where the comma denotes the partial derivative. Then, we recast the Einstein equation into the relaxed form,

\[
\Box h^{\mu\nu} = -16\pi \Lambda^{\mu\nu}, \tag{3.3}
\]

where \( \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu \) is the flat d’Alembertian and

\[
\Lambda^{\mu\nu} = \Theta^{\mu\nu} + \chi^{\mu\nu\alpha\beta}, \tag{3.4}
\]

\[
\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t^{\mu\nu}_{LL}), \tag{3.5}
\]

\[
\chi^{\mu\nu\alpha\beta} = \frac{1}{16\pi} (\eta^{\alpha\beta} h^{\mu\nu} - h^{\alpha\beta} h^{\mu\nu}). \tag{3.6}
\]

Here, \( T^{\mu\nu} \) and \( t^{\mu\nu}_{LL} \) denote the stress-energy tensor of the stars and the Landau-Lifshitz pseudo-tensor [12]. \( \chi^{\mu\nu\alpha\beta} \) originates from our use of the flat d’Alembertian instead of the curved space d’Alembertian. In consistency with the harmonic condition, the conservation law is expressed as

\[
\Box h^{\mu\nu} = -16\pi \Lambda^{\mu\nu}, \tag{3.3}
\]

where \( \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu \) is the flat d’Alembertian and

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\[ \Lambda^{\mu\nu,\nu} = 0. \]  

(3.7)

Now we rewrite the relaxed Einstein equation into an integral form,

\[ h^{\mu\nu} = 4 \int_{C(\tau, x^k; \epsilon)} d^3y \frac{\Lambda^{\mu\nu}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon)}{|\vec{x} - \vec{y}|}, \]  

(3.8)

where \( C(\tau, x^k; \epsilon) \) means the past light cone emanating from the event \((\tau, x^k)\) (about the dependence of \( C \) on \( \epsilon \), see [39]). We ignored the homogeneous solution, which corresponds to the initial data for the field, for simplicity. Even if we take random initial data for the field [38] supposed to be of the 1 PN order [40], they are irrelevant to the dynamics of the binary system up to the radiation reaction order inclusively [40].

We shall solve the relaxed Einstein equation as follows. First, we make retarded expansion of Eq. (3.8) and change the integral region \( C \) to \( N \) (\( N \) denoting near zone), a \( \tau = \) constant spatial hypersurface (note that such retarded expansion is understood to be formal, since it gives divergent integrals. Because such divergent integrals do not appear up to the 2.5 PN order and we are concerned with the 2.5 PN equation of motion in this paper, we use formal retarded expansion. See below).

\[ h^{\mu\nu} = 4 \sum_{n=0} (-\epsilon)^n \left( \frac{\partial}{\partial \tau} \right)^n \int_N d^3y |\vec{x} - \vec{y}|^{n-1} \Lambda^{\mu\nu}_N(\tau, y^k; \epsilon). \]  

(3.9)

Here we added a subscript \( N \) to \( \Lambda^{\mu\nu} \) to clarify that \( \Lambda^{\mu\nu} \) here are components in the near zone coordinate. Simple retarded expansion gives divergent integrals at the 3 PN order. In this paper we take the near zone as a sphere centered at some fixed point and enclosing the binary system. The radius of this sphere is set to be \( \mathcal{R}/\epsilon \), where \( \mathcal{R} \) is arbitrary but much larger than the size of the binary. Then we find that up to the 2.5 PN order, the terms dependent on \( \mathcal{R} \) take forms such as \( \mathcal{R}^{-n}, \partial_i \ln \mathcal{R}, \partial_i \ln \mathcal{R}, \partial_i \mathcal{R}^n, \) or \( \partial_i \mathcal{R}^n \) \((n \geq 1)\), where \( \partial_\mu = \partial/\partial x^\mu \). Hence we simply let \( \mathcal{R} \) go to infinity.

Second we split the integration into two parts; contribution from the body zone \( B_A \), and from elsewhere, \( N/B \). Schematically we evaluate the following type integration (we omit indices of the field),
\[ h = h_B + h_{N/B}, \quad (3.10) \]
\[ h_B = \epsilon^6 \sum_{A=1,2} \int_{B_A} d^3 \alpha_A \frac{f(\tau, \vec{z}_A + \epsilon^2 \vec{\alpha}_A)}{|\vec{r}_A - \epsilon^2 \vec{\alpha}_A|^{1-n}}, \quad (3.11) \]
\[ h_{N/B} = \int_{N/B} d^3 y \frac{f(\tau, \vec{y})}{|\vec{x} - \vec{y}|^{1-n}}, \quad (3.12) \]

where \( \vec{r}_A = \vec{x} - \vec{z}_A \). We shall deal with these two contributions successively.

1. **Body zone contribution**

As for the body zone contribution, we make multipole expansion with care over the scaling of the integrand, i.e., \( \Lambda^{\mu\nu} \) in the body zone. For example, \( n = 0 \) part in Eq. (3.9), \( h^{\mu\nu}_{Bn=0} \), gives
\[ h^{\tau\tau}_{Bn=0} = 4 \epsilon^4 \sum_{A=1,2} \left( \frac{P^\tau_A}{r_A} + \epsilon^2 \frac{D^k_A r^k_A}{r_A^3} + \epsilon^4 \frac{3 I^{kl}_A r^k_A r^l_A}{2 r_A^5} \right) + O(\epsilon^{10}), \quad (3.13) \]
\[ h^{\tau i}_{Bn=0} = 4 \epsilon^4 \sum_{A=1,2} \left( \frac{P^i_A}{r_A} + \epsilon^2 \frac{J^{ki}_A r^k_A}{r_A^3} \right) + O(\epsilon^{8}), \quad (3.14) \]
\[ h^{ij}_{Bn=0} = 4 \epsilon^2 \sum_{A=1,2} \left( \frac{Z^{ij}_A}{r_A} + \epsilon^2 \frac{Z^{kij}_A r^k_A}{r_A^3} + \epsilon^4 \frac{3 Z^{klij}_A r^k_A r^l_A}{2 r_A^5} \right) + O(\epsilon^{8}). \quad (3.15) \]

Here the quantity with a hat is symmetric and tracefree on its dummy indices, and \( r_A = |\vec{r}_A| \). To derive the 2.5 PN equation of motion, \( h^{\tau\tau} \) up to \( O(\epsilon^9) \) and \( h^{\tau i} \) and \( h^{ij} \) up to \( O(\epsilon^7) \) are needed.

In the above equations we defined multipole moments as
\[ I^L_A = \epsilon^2 \int_{B_A} d^3 \alpha_A \Lambda^\tau_A \alpha^L_A, \quad (3.16) \]
\[ J^{Li}_A = \epsilon^4 \int_{B_A} d^3 \alpha_A \Lambda^{\tau i}_A \alpha^L_A, \quad (3.17) \]
\[ Z^{Li j}_A = \epsilon^8 \int_{B_A} d^3 \alpha_A \Lambda^{\tau lj}_A \alpha^L_A, \quad (3.18) \]

where we introduced multi index notations \( L = i_1 i_2 \cdots i_l \) and \( \alpha^L_A = \alpha^{i_1}_A \alpha^{i_2}_A \cdots \alpha^{i_l}_A \). Then \( P^\tau_A = I^0_A, D^i_A = I^i_A, P^i_A = J^i_A \). We simply call \( P^\mu_A \) the four momentum of the star \( A \), \( P^i_A \) the momentum, and \( P^\tau_A \) the energy. Also we call \( D^k_A \) the dipole moment of the star \( A \), and \( I^{kl}_A \) the quadrupole moment of the star \( A \).
Then we transform these moments into more convenient forms. By the conservation law Eq. (3.7), we have

$$\Lambda_{i}^{\tau} = (\Lambda_{N}^{\tau} y_{A}^{i}),_{\tau} + (\Lambda_{N}^{\tau} y_{A}^{i}),_{\phi} + v_{A}^{i} \Lambda_{N}^{\tau}, \quad (3.19)$$

$$\Lambda_{ij}^{\tau} = (\Lambda_{N}^{\tau(i)j}),_{\tau} + (\Lambda_{N}^{\tau(i)j}),_{\phi} + v_{A}^{i} \Lambda_{N}^{\tau}, \quad (3.20)$$

where \( v_{A}^{i} = \dot{z}_{A}^{i} \), an overdot denotes \( \tau \) time derivative, and \( y_{A} = \bar{y} - \bar{z}_{A}. \) Using these equations and noticing that the body zone remains unchanged (in the near zone coordinate), i.e., \( \dot{R}_{A} = 0 \), we have

$$P_{A}^{i} = P_{A}^{\tau} v_{A}^{i} + Q_{A}^{i} + e^{2} \frac{d D_{A}^{i}}{d \tau}, \quad (3.21)$$

$$J_{A}^{ij} = \frac{1}{2} \left( M_{A}^{ij} + e^{2} \frac{d I_{A}^{ij}}{d \tau} \right) + v_{A}^{(i} D_{A}^{j)} + \frac{1}{2} e^{-2} Q_{A}^{ij}, \quad (3.22)$$

$$Z_{A}^{ij} = e^{2} P_{A}^{\tau} v_{A}^{i} v_{A}^{j} + \frac{1}{2} e^{6} \frac{d^{2} I_{A}^{ij}}{d \tau^{2}} + 2 e^{4} v_{A}^{(i} d D_{A}^{j)} + e^{4} \frac{d v_{A}^{(i} d D_{A}^{j)}}{d \tau} + e^{2} Q_{A}^{(i} v_{A}^{j)} + e^{2} R_{A}^{(ij)}, \quad (3.23)$$

$$Z_{A}^{kij} = \frac{3}{2} A_{A}^{kij} - A_{A}^{(ij)k}, \quad (3.24)$$

where

$$M_{A}^{ij} = 2 e^{4} \int_{B_{A}} d^{3} \alpha \Lambda_{A}^{i} \Lambda_{A}^{j}, \quad (3.25)$$

$$Q_{A}^{Li} = e^{-4} \oint_{\partial B_{A}} d S_{k} \left( \Lambda_{A}^{k} - v_{A}^{k} \Lambda_{N}^{\tau} \right) y_{A}^{i} y_{A}^{j}, \quad (3.26)$$

$$R_{A}^{ij} = e^{-4} \oint_{\partial B_{A}} d S_{k} \left( \Lambda_{A}^{k} - v_{A}^{k} \Lambda_{N}^{\tau} \right) y_{A}^{l} y_{A}^{j}, \quad (3.27)$$

and

$$A_{A}^{kij} = e^{2} J_{A}^{k(i} v_{A}^{j)} + e^{2} v_{A}^{k} J_{A}^{(ij)} + R_{A}^{k(ij)} + e^{4} \frac{d J_{A}^{k(ij)}}{d \tau}. \quad (3.28)$$

[ ] and ( ) attached to indices denote anti-symmetrization and symmetrization. \( M_{A}^{ij} \) is the spin of the star A and Eq. (3.21) is the momentum-velocity relation. In general, we have

$$J_{A}^{Lij} = J_{A}^{(Lij)} + \frac{2l}{l+1} J_{A}^{(L-1[i]ij)} \quad (3.29)$$

$$Z_{A}^{Lij} = \frac{1}{2} \left[ Z_{A}^{(Lij)} + \frac{2l}{l+1} Z_{A}^{(L-1[i]ij)} + Z_{A}^{(Lij)i} + \frac{2l}{l+1} Z_{A}^{(L-1[i]ij)i} \right] \quad (3.30)$$
and

\[
J^{(L_l)}_A = \frac{1}{l+1} \varepsilon^2 \frac{dI^{L_l}_A}{d\tau} + v^{(i)}_A J^{(iL_l)}_A + \frac{1}{l+1} \varepsilon^{-2l} Q^{L_l}_A, \tag{3.31}
\]

\[
Z^{(L_l)j}_A + Z^{(L_j)i}_A = \varepsilon^2 v^{(i)}_A J^{(L_j)}_A + \frac{2}{l+1} \varepsilon^4 \frac{dJ^{(L_j)_i}}{d\tau} + \varepsilon^{-2l+2} R^{L_j}_A, \tag{3.32}
\]

where \(l\) is the number of indices in the multi index \(L\).

Now, from the above equations, especially Eq. (3.23), we find that the body zone contribution, \(h^{\mu\nu}_{Bn=0}\), are of order \(O(\varepsilon^4)\). Note that if we can not or do not assume the (nearly) stationarity of the initial data for the stars, then, instead of Eq. (3.23) we have

\[
Z^{ij}_A = \varepsilon^2 v^i_A v^j_A + \frac{1}{2} \frac{d^2 R^{ij}_A}{d\eta^2} + \cdots,
\]

where we used the dynamical time \(\eta\) (see Sec [I]). In this case the lowest metric differs from the Newtonian form. From our (nearly) stationary assumption the remaining motion inside a star, except the spinning motion, is caused only by the tidal effect by the companion star and from Eq. (3.23), it appears at the 3 PN order [30].

To obtain the lowest order \(h^{\mu\nu}_{Bn=0}\), we have to evaluate the surface integrals \(Q^{L_l}_A, R^{L_j}_A\).

Generally, in \(h^{\mu\nu}_{Bn=0}\) the moments \(J^{L_l}_A\) and \(Z^{L_j}_A\) appear formally at the order \(\varepsilon^{2l+4}\) and \(\varepsilon^{2l+2}\). Thus \(Q^{L_l}_A\) and \(R^{L_j}_A\) appear as

\[
h^{ij}_{Bn=0} \sim \cdots + \varepsilon^4 \frac{r^{L_l}_A \hat{Q}^{L_l}_A}{r^{2l+1}_A} + \cdots,
\]

\[
h^{ij}_{Bn=0} \sim \cdots + \varepsilon^4 \frac{r^{L_j}_A \hat{R}^{L_j}_A}{r^{2l+1}_A} + \cdots,
\]

where we omitted irrelevant terms and numerical coefficients. Thus one may expect that \(Q^{L_l}_A\) and \(R^{L_j}_A\) appear at the order \(\varepsilon^4\) for any \(L\) and we have to calculate an infinite number of moments. This is not the case. We address this problem in the appendix A. The important thing here is that \(\varepsilon^4 Q^{L_l}_A\) and \(\varepsilon^4 R^{L_j}_A\) are at most \(O(\varepsilon^4)\) in \(h^{\mu\nu}_{Bn=0}\).

Finally, since the order of \(h^{L_l\nu}_{Bn}(n \geq 1)\) is higher than that of \(h^{\mu\nu}_{Bn=0}\), \(h^{\mu\nu}_B = O(\varepsilon^4)\).
2. N/B contribution

About the N/B contribution, since the integrand $\Lambda^\mu_\nu_N = -g^\mu_\nu_{LL} + \chi^{\mu\nu\alpha\beta}_{,\alpha\beta}$ is at most quadratic in the small deviation field $h^\mu_\nu$, we make the post-Newtonian expansion in the integrand. Then, basically, with the help of a potential $g(\vec{x})$ which satisfies $\Delta g(\vec{x}) = f(\vec{x})$, $\Delta$ denoting Laplacian, we have for each integral (, e.g., $n = 0$ term in Eq. (3.12))

$$\int_{N/B} d^3y \frac{f(\vec{y})}{|\vec{x} - \vec{y}|} = -4\pi g(\vec{x}) + \oint_{\partial(N/B)} dS_k \left[ \frac{1}{|\vec{x} - \vec{y}|} \frac{\partial g(\vec{y})}{\partial y^k} - g(\vec{y}) \frac{\partial}{\partial y^k} \left( \frac{1}{|\vec{x} - \vec{y}|} \right) \right]. \quad (3.33)$$

The last surface integral really contributes to the 2.5 PN gravitational field. For $n \geq 1$ terms in Eq. (3.12), we use potentials many times to convert all the volume integrals into surface integrals and “$-4\pi g(\vec{x})$” terms.

Now the lowest order integrands can be evaluated with the body zone contribution $h^\mu_\nu_B$, and since $h^\mu_\nu_B$ is $O(\epsilon^4)$, we find (see appendix B)

$$\Lambda^\tau_\tau_N = O(\epsilon^6), \quad (3.34)$$

$$\Lambda^\tau_i_N = O(\epsilon^6), \quad (3.35)$$

$$\Lambda^{ij}_N = (-g)t^{ij}_{LL} + O(\epsilon^8) = \epsilon^4 \frac{1}{64\pi} \left( \delta^i_k \delta^j_l - \frac{1}{2} \delta^{ij} \delta_{kl} \right) h^{\tau\tau,k}_B h^{\tau\tau,l}_B + O(\epsilon^5). \quad (3.36)$$

where we expanded $h^\mu_\nu_B$ in an $\epsilon$ series;

$$h^\mu_\nu_B = \sum_{n=0}^\infty \epsilon^{4+n} n h^\mu_\nu_B.$$  

Similarly, in the following we expand $h^\mu_\nu$ in an $\epsilon$ series. From these equations we find that the deviation field in N/B, $h^\mu_\nu$, is $O(\epsilon^4)$ (It should be noted that in the body zone $h^\mu_\nu$ is assumed to be of order unity and within our method we can not calculate $h^\mu_\nu$ there explicitly. To obtain $h^\mu_\nu$ in the body zone, we have to solve the internal problem).

The Landau-Lifshitz pseudo-Tensor expanded with $h^\mu_\nu$ is listed in the appendix B.

B. General Form of the Equation of Motion

From the definition of the four momentum
\[ P_A^\mu(\tau) = \epsilon^2 \int_{\partial B_A} d^3 \alpha A_\tau^\mu, \tag{3.37} \]

and the conservation law, Eq. (3.4), we have the evolution equation for the four momentum;

\[ \frac{dP_A^\mu}{d\tau} = -\epsilon^{-4} \oint_{\partial B_A} dS_k \Lambda_N^{k\mu} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Lambda_N^{\tau\mu}. \tag{3.38} \]

Here we used the fact that the size and the shape of the body zone are defined to be fixed (in the near zone coordinate).

Inserting the momentum-velocity relation, Eq. (3.21), into the spatial component of Eq. (3.38), we obtain the general form of the equation of motion for the star A;

\[ P_A^\mu \frac{dv_A^\mu}{d\tau} = -\epsilon^{-4} \oint_{\partial B_A} dS_k \Lambda_N^{k\mu} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Lambda_N^{\tau\mu} + \epsilon^{-4} v_A^i \left( \oint_{\partial B_A} dS_k \Lambda_N^{k\tau} - v_A^k \oint_{\partial B_A} dS_k \Lambda_N^{\tau\tau} \right) \]

\[ - \frac{dQ_A^i}{d\tau} - \epsilon^2 \frac{d^2 D_A^i}{d\tau^2}. \tag{3.39} \]

All the right hand side terms in Eq. (3.39) except the dipole moment are surface integrals. We can specify the value of \( D_A^i \) freely to determine the representative point \( z_A^i(\tau) \) of the star A. In this paper we take \( D_A^i = 0 \) and simply call \( z_A^i \) the center of the mass of the star A. Note that to obtain the spin-orbit coupling force in the same form as the previous works [26,43,44], we have to make another choice for \( z_A^i \) (paper I). For completeness we show in the appendix C the spin-orbit coupling force on the condition that \( D_A^i = 0 \). Henceforth we shall restrict our attention on the mass monopoles.

In Eq. (3.39), \( P_A^\tau \) rather than the mass of the star A appears. Hence we have to derive a relation between the mass and \( P_A^\tau \). We shall achieve this by solving the temporal component of the evolution equation (3.38) functionally.

Then since all the equations are expressed with surface integrals except \( D_A^i \) to be specified, we can derive the equation of motion for the strongly self-gravitating star using the post-Newtonian approximation at least before the order where we have to be concerned with the internal problem.
C. On the Arbitrary Constant $R_A$

Since we introduce the body zone by hand, the arbitrary defined body zone size $R_A$ seems to appear in the metric, the multipole moments of the stars, and the equation of motion. More specifically $R_A$ appears in them because of 1) the splitting of the deviation field into two parts (i.e., $B$ and $N/B$ contributions), the definition of the moments, and 2) the surface integrals that we evaluate to derive the equation of motion.

1. $R_A$ Dependence Of The Field

$B$ and $N/B$ contributions to the field depend on the body zone boundary, $\epsilon R_A$. But $h^{\mu\nu}$ itself does not depend on $\epsilon R_A$. Thus it is natural to expect that there are renormalized multipole moments which are independent of $R_A$ since we use non-singular matter sources. One possible practical obstacle for this expectation might be $\log(\epsilon R_A)$ dependence of multipole moments. We have checked that at least up to the 2.5 PN order, there is no such log-term.

Though we use the same symbol for the moments henceforth as before for notational simplicity, it should be understood that they are the renormalized ones (we use the symbol $"P_\mu^A"$ for the renormalized $P_\mu^A$).

2. $R_A$ Dependence Of The Equation Of Motion

Since we make integrations over the body zone boundary, in general the resulting equation of motion seems to depend on the size of the body zone boundary, $\epsilon R_A$. But it does not depend on $R_A$.

In the derivation of Eq. (3.39), if we did not use the conservation law (3.7) until the final step, we have

$$P_\mu^A \frac{dv_i^A}{d\tau} + \epsilon^{-4} \int_{\partial BA} dS_k \Lambda_N^{ki} - \epsilon^{-4} v_A^k \int_{\partial BA} dS_k \Lambda_N^{\tau i}$$

$$- \epsilon^{-4} v_A^i \left( \int_{\partial BA} dS_k \Lambda_N^{k\tau} - v_A^k \int_{\partial BA} dS_k \Lambda_N^{\tau \tau} \right)$$
\[
\begin{align*}
+ \frac{dQ_A^i}{d\tau} + \epsilon^2 \frac{d^2 D_A^i}{d\tau^2} \\
= \epsilon^{-4} \int_{B_A} d^3y \Lambda_N^\nu_{\alpha,\nu} - \epsilon^{-4} v_A^i \int_{B_A} d^3y \Lambda_N^\nu_{\alpha,\nu} + \epsilon^{-4} \frac{d}{d\tau} \left( \int_{B_A} d^3y \Lambda^\nu_{\alpha,\nu} y_A^i \right).
\end{align*}
\] (3.40)

Now the conservation law is satisfied for whatever value we take as \( R_A \), then the right hand side of the above equation is zero for any \( R_A \). Hence the equation of motion Eq. (3.39) does not depend on \( R_A \) (Similar argument can be found in [32]).

Along the same line, the momentum-velocity relation (3.21) does not depend on \( R_A \).

Up to the 1 PN order in paper I, we have explicitly shown the irrelevance of the equation of motion to \( R_A \) by checking the cancellation among the \( R_A \) dependent terms.

**IV. NEWTONIAN AND THE 1 PN EQUATION OF MOTION**

In this section we shall list the Newtonian and the 1 PN equation of motion for later convenience. As for the details of the derivation in our method, see the paper I.

Let us review how to derive the field and the equation of motion order by order.

First of all, from Eqs. (3.34), (3.35), and the time component of Eq. (3.38), we have
\[
\frac{dP_A^\tau}{d\tau} = O(\epsilon^2).
\] (4.1)

Then we define the mass of the star A as the integrating constant of this equation;
\[
m_A = \lim_{\epsilon \to 0} P_A^\tau.
\] (4.2)

\( m_A \) is the ADM mass that the star A had if A were isolated (we took \( \epsilon \) zero limit in Eq. (4.2) to ensure that the mass defined above does not include the effect of the companion star and the orbital motion of the star itself. By this limit we ensure that this mass is the integrating constant of Eq. (4.1). Some subtleties about this definition are discussed in the appendix [D]). By definition \( m_A \) is constant. Then we obtain the lowest order \( h^{\tau\tau} \);
\[
4h^{\tau\tau} = 4 \sum_{A=1,2} \frac{m_A}{r_A}.
\] (4.3)
Second, from Eq. (3.26) with Eqs. (3.34) and (3.35) we obtain \( Q_A^i = 0 \) at the lowest order. Thus we have the Newtonian momentum-velocity relation \( P_A^i = m_A v_A^i + O(\epsilon^2) \) from Eq. (3.21) (we set \( D_A^i = 0 \)). Then from Eq. (3.39), we can derive the Newtonian equation of motion. This completes the Newtonian order calculations.

Next, at the 1 PN order, we need \( 6 h^{\tau\tau}, 4 h^{\mu\nu} \). The \( n = 1 \) term in the retardation expansion series of \( h^{\tau\tau} \), Eq. (3.9), gives no contribution at the 1 PN order by the constancy of the mass \( m_A \), i.e., \( 5 h^{\tau\tau} = 0 \).

Now we obtain \( 4 h^{\tau i} \) from the Newtonian momentum-velocity relation.

\[
4 h^{\tau i} = 4 \sum_{A=1,2} \frac{m_A v_A^i}{r_A}.
\]

From \( 4 h^{\tau\nu} \) and Eqs. (B3) and (B4) we can evaluate surface integrals in the evolution equation for \( P_A^\tau \) at the 1 PN order. The result is (for the star 1)

\[
\frac{dP_1^\tau}{d\tau} = \epsilon^2 m_1 \frac{d}{d\tau} \left( \frac{1}{2} v_1^2 + \frac{3m_2}{r_{12}} \right) + O(\epsilon^3), \tag{4.4}
\]

where \( \vec{r}_{12} = \vec{z}_1 - \vec{z}_2 \) and \( r_{12} = |\vec{r}_{12}| \), and we used the Newtonian equation of motion. From this equation, we have the mass-energy relation at the 1 PN order,

\[
P_1^\tau = m_1 \left[ 1 + \epsilon^2 \left( \frac{1}{2} v_1^2 + \frac{3m_2}{r_{12}} \right) \right] + O(\epsilon^3). \tag{4.5}
\]

Then we have to calculate the 1 PN order \( Q_A^i \) from Eqs. (B3), (B4) and (3.26). The result is that \( Q_A^i = \epsilon^2 m_A^2 v_A^i / (6e R_A) \). As \( Q_A^i \) depends on \( R_A \), we ignore it (see Sec. III C). As a result we obtain the momentum-velocity relation at the 1 PN order, \( P_A^i = P_A^\tau v_A^i + O(\epsilon^3) \) from Eq. (3.21).

Now as for \( 4 h^{ij} \), we first calculate the surface integrals \( Q_A^{ij}, R_A^{ij}, R_A^{kij} \) from Eqs. (3.26) and (3.27) using Eqs. (B2), (B3), and (B4). We then find that they depend on \( R_A \), hence we ignore them and obtain

\[
h_B^{ij} = 4 \epsilon^4 \sum_{A=1,2} \frac{m_A v_A^i v_A^j}{r_A} + O(\epsilon^5).
\]

To obtain \( 6 h^{\tau\tau} \) and \( 4 h^{ij} \), we have to evaluate non-compact support integrals for \( 6 h^{\tau\tau}_{N/B} \) and \( 4 h^{ij}_{N/B} \), and \( n = 2 \) term in Eq. (3.9) for \( h^{\tau\tau} \). It can be done with the help of Laplacian
inverse; $\Delta \ln r_1 = 1/r_1^2$ and $\Delta \ln S = 1/r_1 r_2$. Using the Newtonian equation of motion we obtain

$$h^{\tau \tau} = 4\epsilon^4 \sum_{A=1,2} \frac{P^A_\tau}{r_A} + \epsilon^6 \left[ -2 \sum_{A=1,2} \frac{m_A}{r_A} \left( (\vec{n}_A \cdot \vec{v}_A)^2 - v_A^2 \right) + 2 \frac{m_1 m_2}{r_{12}^2} \vec{n}_{12} \cdot (\vec{n}_1 - \vec{n}_2) \\
+ 7 \sum_{A=1,2} \frac{m_A^2}{r_A^2} + 14 \frac{m_1 m_2}{r_1 r_2} - 14 \frac{m_1 m_2}{r_{12}} \sum_{A=1,2} \frac{1}{r_A^2} \right],$$

(4.6)

$$h^{ij} = 4\epsilon^4 \sum_{A=1,2} \frac{m_A v^i_A v^j_A}{r_A} + \epsilon^4 \left[ \sum_{A=1,2} \frac{m_A^2}{r_A^2} n^i_A n^j_A - \frac{8m_1 m_2}{r_{12} S^2} n^i_{12} n^j_{12} \\
- 8 \left( \delta^i_k \delta^j_l - \frac{1}{2} \delta^i_l \delta^j_k \right) \frac{m_1 m_2}{r_{12} S^2} (\vec{n}_{12} - \vec{n}_1)^i (\vec{n}_{12} + \vec{n}_2)^j \right].$$

(4.7)

Here $S = r_1 + r_2 + r_{12}, \vec{n}_{12} = \vec{r}_{12}/r_{12}, \vec{n}_A = \vec{r}_A/r_A$.

Inserting the field $4h^{\mu \nu}, \epsilon h^{\tau \tau}$ into Eqs. (B4) and (B5) and evaluating the surface integrals in Eq. (3.39), we obtain the 1 PN equation of motion;

$$m_1 \frac{dv^i_1}{d\tau} = -\frac{m_1 m_2}{r_{12}^2} n^i_{12} \\
+ \epsilon^2 \frac{m_1 m_2}{r_{12}^2} \left[ n^i_{12} \left( -v_1^2 - 2v^2 + \frac{3}{2} (\vec{n}_{12} \cdot \vec{v}_2)^2 \right) + 4 (\vec{v}_1 \cdot \vec{v}_2) + \frac{5m_1}{r_{12}} + \frac{4m_2}{r_{12}} \right] \\
+ V^i \left( 4(\vec{n}_{12} \cdot \vec{v}_1) - 3(\vec{n}_{12} \cdot \vec{v}_2) \right),$$

(4.8)

where we defined the relative velocity as $\vec{V} = \vec{v}_1 - \vec{v}_2$ and we used the Newtonian equation of motion and Eq. (4.5).

Finally let us give a summary of our procedure. With the $n-1$ PN order equation of motion and $h^{\mu \nu}$ in hand, we first derive the $n$ PN evolution equation for $P^A_\tau$. Then we solve it functionally and obtain the mass-energy relation at the $n$ PN order. Next we calculate $Q^i_A$ at the $n$ PN order and derive the momentum-velocity relation at the $n$ PN order. Then we calculate $Q^i_A$ and $R^{ij}_A$. With the $n$ PN mass-energy relation, the $n$ PN momentum-velocity relation, $Q^i_A$, and $R^{ij}_A$, we next derive the $n$ PN deviation field $h^{\mu \nu}$. Finally we evaluate the surface integrals which appear in the right hand side of Eq. (3.39) and obtain the $n$ PN.
equation of motion. In the above calculations we use n-1 PN order equation of motion to reduce the order, e.g., when we meet $\epsilon^2 d\nu_i/d\tau$ in the right hand side of the equation motion and we have to evaluate this up to $\epsilon^2$, then we use the Newtonian equation of motion and replace it by $-\epsilon^2 m_2 r_{12}^3 / r_{12}^3$. Basically we shall derive the 2.5 PN equation of motion with the procedure as described above.

V. THE 1.5 PN EQUATION OF MOTION

In the appendix $[\text{A}]$, we show that $Q_{Li}^A$ and $R_{Lij}^A$ do not contribute to $h^{\mu\nu}$ up to the 2.5 PN order. There, we also derive the momentum-velocity relation up to the 1.5 PN order

$$P_i^A = P_i^A v_i^A + O(\epsilon^4),$$

and the relations between $Z_{Lij}^A$ and some moments; the results are that $Z_{A}^{ij} = \epsilon^2 P_A^r v_i^A v_j^A + O(\epsilon^4)$ and $Z_{A}^{kij} = \epsilon^2 M_A^{(i} v^{j)}_{A} + O(\epsilon^4)$. Henceforth we shall use Eqs. (A9) and (A10).

Now let us derive the 1.5 PN equation of motion, which is $O(\epsilon^3)$ correction to the Newtonian equation of motion. Thus as the integrands in Eq. (3.38), we shall use $\tau[-g^{\mu\nu}]$ and hence we have to obtain $\tau h^{\tau \tau}$ and $g h^{\nu i}$. The fields that we shall evaluate are

$$h^{\tau \tau} = \leq \nu h^{\tau \tau} - 4 \epsilon^7 \frac{d}{d\tau} \int_{\partial N} dS k_6 [-g^{\tau k}_{LL}] - \frac{2}{3} \epsilon^7 \frac{\partial^3}{\partial \tau^3} \sum_{A=1,2} P_A^r r^2_A + O(\epsilon^8), \quad (5.1)$$

$$h^{\tau i} = \leq \nu h^{\tau i} - 4 \epsilon^5 \int_{\partial N} dS k_4 [-g^{\tau k}_{LL}] + O(\epsilon^6), \quad (5.2)$$

$$h^{ij} = \leq \nu h^{\tau \tau} - 4 \epsilon^5 \frac{d}{d\tau} \left( \sum_{A=1,2} P_A^r v_A^i v_A^j + \epsilon^4 \int_{N/B} d^3 y [-g^{\tau ij}_{LL}] \right) + O(\epsilon^6). \quad (5.3)$$

Here $\leq \nu h^{\mu\nu}$ denotes the field complete up to the order $\epsilon^n$.

Now let us follow the procedure described in the previous section. From Eqs. (B6) and (B7), the mass-energy relation at the 1.5 PN order becomes

$$P_A^r = (P_A^r)_{\leq 1PN} + O(\epsilon^4)$$

Here $(P_A^r)_{\leq 1PN}$ stands for the 1 PN order mass-energy relation, Eq. (4.3) (without $O(\epsilon^3)$ symbol).
A straightforward calculation for the $N/B$ contribution results in

\[ \leq 7h_{\tau\tau} = 4\epsilon^4 \sum_{A=1,2} \frac{P_{A}^\tau}{r_A} \]

\[ + \epsilon^6 \left[ 2 \sum_{A=1,2} \frac{\partial^2}{\partial\tau^2} (P_{A}^\tau r_A) ight. \]

\[ + 7 \sum_{a,b=1,2} \frac{P_{a}^\tau P_{b}^\tau}{r_ar_b} - 14 \frac{P_{1}^\tau P_{2}^\tau}{r_{12}} \left( \sum_{A=1,2} \frac{1}{r_A} \right) \left( \sum_{A=1,2} \frac{1}{r_A} \right) \]

\[ - \epsilon^7 \frac{d}{3 d\tau} \left( \sum_{A=1,2} P_{A}^\tau v_{A}^2 - \frac{P_{1}^\tau P_{2}^\tau}{r_{12}} \right), \] \hspace{1cm} (5.4)

\[ \leq 5h_{\tau i} = 4\epsilon^4 \sum_{A=1,2} \frac{P_{A}^\tau v_{A}^i}{r_A}, \] \hspace{1cm} (5.5)

\[ \leq 5h_{ij} = 4\epsilon^4 \sum_{A=1,2} \frac{P_{A}^\tau v_{A}^i v_{A}^j}{r_A} \]

\[ + \epsilon^4 \left[ \sum_{A=1,2} \frac{P_{A}^\tau}{r_A} n_i^A n_j^A - \frac{8P_{1}^\tau P_{2}^\tau}{r_{12}S} n_{12}^i n_{12}^j \right. \]

\[ - 8 \left( \delta_{k}^i \delta_{l}^j - \frac{1}{2} \delta_{ij} \delta_{kl} \right) \frac{P_{1}^\tau P_{2}^\tau}{S^2} (\vec{n}_{12} - \vec{n}_1)^k (\vec{n}_{12} + \vec{n}_2)^l \]

\[ - 2\epsilon^5 (3) I_{\text{orb}}^{ij}. \] \hspace{1cm} (5.6)

Here $I_{\text{orb}}^{ij} = \sum_{A=1,2} P_{A}^\tau A_{i}^A A_{j}^A$ and $(n)I$ denotes $d^n I/d\tau^n$.

Note that $5h_{ij}$ and $7h_{\tau\tau}$ depend only on time, thus from Eq. (B8), the equation of motion at the 1.5 PN order becomes

\[ \frac{dv_{A}^i}{d\tau} = \left( \frac{dv_{A}^i}{d\tau} \right)_{\leq 1PN} + O(\epsilon^4). \]

**VI. THE 2 PN EQUATION OF MOTION**

In this and next section, we shall derive the 2.5 PN equation of motion using the procedure described in the section [IV]. The main problem that we have to solve is to derive $h_{N/B}^{\mu\nu}$. Thus, from Eq. (5.33), the problem is reduced to solve the Poisson equations $\Delta g(\vec{x}) = f(\vec{x})$. Fortunately, all the solutions of the Poisson equations that we need to derive the 2.5 PN equation of motion are obtained in [1211].
As the first step of the 2 PN order calculation, we have to derive \( _6 h^{\tau i} \) which will be needed to obtain the 2 PN order evolution equation for \( P_1^\tau \).

\[
_s h^{\tau i} = 4e^4 \sum_{A=1,2} \frac{P_1^\tau v_A^i}{r_A} + 4e^6 \int_{N/B} d^3 y \frac{6[-gt^{\tau i}_{LL}]}{|\vec{x} - \vec{y}|} + 2e^6 \frac{\partial^2}{\partial \tau^2} \left( \sum_{A=1,2} P_2^\tau v_A^i r_A \right) \tag{6.1}
\]

Now we shall show calculations for the \( N/B \) contribution in detail. The integrand is

\[
6[-16\pi gt^{\tau i}_{LL}] = 2_4 h^{\tau k} h^{\tau (k,i)} - \frac{3}{4} _4 h^{\tau k} \partial_4 h^{\tau, i}.
\]

Then the first integrand contributes

\[
\frac{2}{16\pi} \int_{N/B} d^3 y \frac{4_4 h^{\tau k} h^{\tau (k,i)}}{|\vec{x} - \vec{y}|} = \frac{2}{\pi} \sum_{a, b=1,2} \frac{P_1^\tau}{r_A} P_1^\tau v_b^i \int_{N/B} d^3 y \frac{\delta^i_{[a} \partial^j_b r^A_k}{|\vec{x} - \vec{y}| r^A_a r^A_b} \]

\[
= 2 \left[ \sum_{A=1,2} \frac{P_1^\tau}{r_A} (n_A^i n_A^k - \delta^i_{[a} \partial^j_b r^A_k - 4P_1^\tau P_2^\tau v_b^i \partial_2 |\vec{x} - \vec{y}| \partial_1 |\vec{x} - \vec{y}|) - 4P_1^\tau P_2^\tau v_b^i \partial_2 |\vec{x} - \vec{y}| \partial_1 |\vec{x} - \vec{y}| \right], \tag{6.2}
\]

where \( \partial_A = \partial/\partial z_A^i \). In the second equality of the above equation, we used the following formulas,

\[
\int_{N/B} d^3 y \frac{r^A_{[a} \partial^j_b r^A_k}}{r_A^6} = \frac{1}{8} \left( \delta^i_{[a} \delta^j_b + \delta^j_{[a} \delta^i_b \right) \int_{N/B} d^3 y \frac{(\Delta \ln r_A)^{kl}}{|\vec{x} - \vec{y}|}, \tag{6.3}
\]

\[
\int_{N/B} d^3 y \frac{r^A_{[a} \partial^j_b r^A_k}}{r_A^3 r^3_{[a} r^3_{b]} = \int_{N/B} d^3 y \frac{1}{|\vec{x} - \vec{y}| r_A^3 r_A^3} \frac{\partial}{\partial z_A^i} \frac{\partial}{\partial z_A^j} (\Delta \ln S), \tag{6.4}
\]

and Eq. (3.33). Applying the same formulas to the other integrand, we find at last,

\[
4 \int_{N/B} d^3 y \frac{6[-gt^{\tau i}_{LL}]}{|\vec{x} - \vec{y}|} = \sum_{A=1,2} \frac{P_2^\tau}{r_A^2} \left\{ (\vec{n}_A \cdot \vec{v}_A) n_A^i + 7v_A^i \right\} + \frac{4P_1^\tau P_2^\tau}{S r_{12}} (v_{1k} + v_{2k}) (\delta^{ki} - n_{12}^i n_{12}^k)
\]

\[
+ 8P_1^\tau P_2^\tau (v_1^i + v_2^i) \left( -\frac{1}{r_{12}} \sum_{A=1,2} \frac{1}{r_A} + \frac{1}{r_1 r_2} \right)
\]

\[
- 16P_1^\tau P_2^\tau \left\{ v_1^i (n_{12}^i - n_1^i) (n_{12}^k + n_2^k) + v_2^i (n_{12}^k - n_1^k) (n_{12}^i + n_2^i) \right\}
\]

\[
+ 12P_1^\tau P_2^\tau \left\{ v_1^k (n_{12}^k - n_1^k) (n_{12}^i + n_2^i) + v_2^k (n_{12}^i - n_1^i) (n_{12}^k + n_2^k) \right\} \tag{6.5}
\]

We show \( _6 h^{\tau i} \) in the appendix E.
A. The 2 PN evolution equation for $P^\tau_A$

In this subsection we shall derive the 2 PN evolution equation for $P^\tau_A$. The surface integrals we have to evaluate are

$$\frac{dP^\tau_A}{d\tau} = -\epsilon^2 \oint_{\partial B_A} dS_k [-gt^k_{LL}] + \epsilon^2 v^k_A \oint_{\partial B_A} dS_k [-gt^\tau_{\tau LL}]$$
$$- \epsilon^4 \oint_{\partial B_A} dS_{k8} \Lambda^k_{\tau N} + \epsilon^4 v^k_A \oint_{\partial B_A} dS_{k8} \Lambda^\tau_{\tau N} + O(\epsilon^5).$$

(6.6)

Here $O(\epsilon^2)$ terms become

$$- \oint_{\partial B_1} dS_k [-gt^k_{LL}] + v^i \oint_{\partial B_1} dS_k [-gt^\tau_{\tau LL}] = -\frac{P^\tau_A P^\tau_2}{r^8_{12}} [4(\vec{n}_{12} \cdot \vec{u}_1) - 3(\vec{n}_{12} \cdot \vec{v}_2)].$$

(6.7)

This is the 1 PN order evolution equation with the mass $m_A$ replaced by the energy $P^\tau_A$.

From the 2 PN order, $\chi^{\mu\nu\alpha\beta}_{N,\alpha\beta}$ comes into play. We can see this by the definition of $P^\tau_A$, Eq. (3.37). Noticing $\chi^{\tau\alpha\beta}_{N,\alpha\beta} = \chi^{\tau\tau ij}_{N,ij}$, we rewrite Eq. (3.37) as

$$P^\tau_A(\tau) = \epsilon^2 \int_{B_A} d^3 \alpha_A \Theta^\tau_{N} + \epsilon^4 \oint_{\partial B_A} dS_i \chi^{\tau\tau ij}_{N,i}. \quad (6.8)$$

Since $\chi^{\tau\alpha\beta}_{N,\alpha\beta}$ is of order $\epsilon^8$, the surface integral of it first appears at the order $\epsilon^4$, i.e., at the 2 PN order.

Now since $\Theta^\mu_{N,\mu}$ and $\chi^{\mu\nu\alpha\beta}_{N,\alpha\beta}$ are conserved separately, i.e., $\Theta^\mu_{N,\mu} = 0 = \chi^{\mu\nu\alpha\beta}_{N,\alpha\beta\mu}$, we shall deal with $\Theta^\mu_{N,\mu}$ and $\chi^{\mu\nu\alpha\beta}_{N,\alpha\beta}$ separately. Thus we split $P^\tau_A$ into two parts; $P^\tau_{A\Theta}$ and $P^\tau_{A\chi}$.

$$P^\mu_{A\Theta}(\tau) = \epsilon^2 \int_{B_A} d^3 \alpha_A \Theta^\mu_{N}, \quad (6.9)$$
$$P^\mu_{A\chi}(\tau) = \epsilon^2 \int_{B_A} d^3 \alpha_A \chi^{\mu\nu\alpha\beta}_{N}, \quad (6.10)$$

We consider the evolution equation of $P^\mu_{A\chi}$ in the appendix [3]. The evolution equation for $P^\tau_{A\Theta}$ becomes

$$\frac{dP^\tau_{A\Theta}}{d\tau} = \left(\frac{dP^\tau_A}{d\tau}\right)_{\leq 1.5PN}$$
$$- \epsilon^4 \oint_{\partial B_A} dS_{k8} [-gt^k_{LL} \tau ] + \epsilon^4 v^k_A \oint_{\partial B_A} dS_{k8} [-gt^{\tau\tau}_{LL}].$$

(6.11)

From Eqs. (39) and (B10), the relevant combinations of $h^{\mu\nu}$ are;
 Evaluating surface integrals of these ten types of integrands, we obtain,

\[ - \oint_{\partial B_1} dS_{k8} [-g^{\tau k}_{LL}] = - \frac{m_1 m_2}{r_{12}^2} \left[ 9 (\bar{n}_{12} \cdot \bar{v}_2)^2 \right. \\
+ 3 (\bar{v}_1 \cdot \bar{v}_2) (\bar{n}_{12} \cdot \bar{v}_2) - \frac{4 (\bar{v}_1 \cdot \bar{v}_2) (\bar{n}_{12} \cdot \bar{v}_1)}{3} \\
- \left. \frac{7 v_2^2 (\bar{n}_{12} \cdot \bar{v}_2)}{2} + \frac{7 v_2^2 (\bar{n}_{12} \cdot \bar{v}_1)}{6} \right] \\
- \frac{m_1 m_2}{r_{12}^2} \left[ -2 (\bar{n}_{12} \cdot \bar{v}_2) - \frac{2 (\bar{n}_{12} \cdot \bar{v}_1)}{3} \right] \\
- \frac{m_1^2 m_2}{r_{12}^3} \left[ \frac{19 (\bar{n}_{12} \cdot \bar{v}_2)}{2} - \frac{31 (\bar{n}_{12} \cdot \bar{v}_1)}{6} - \frac{7 (\bar{n}_{12} \cdot \bar{V})}{2} \right], \quad (6.12) \]

and

\[ \nu_1^k \oint_{\partial B_1} dS_{k8} [-g^{\tau k}_{LL}] = \frac{m_1 m_2}{r_{12}^2} \left[ -\nu_1^2 (\bar{n}_{12} \cdot \bar{v}_2) + \frac{7 (\bar{n}_{12} \cdot \bar{v}_1) (\bar{n}_{12} \cdot \bar{v}_2)^2}{2} + \frac{4 v_2^2 (\bar{n}_{12} \cdot \bar{v}_1)}{15} \right. \\
- (\bar{v}_1 \cdot \bar{v}_2) (\bar{n}_{12} \cdot \bar{v}_2) + \frac{8 (\bar{v}_1 \cdot \bar{v}_2) (\bar{n}_{12} \cdot \bar{v}_1)}{3} - \frac{5 v_2^2 (\bar{n}_{12} \cdot \bar{v}_1)}{6} \left. \right] \\
+ \frac{4 m_1 m_2}{3} (\bar{n}_{12} \cdot \bar{v}_1) \\
+ \frac{28 m_1^2 m_2}{3} (\bar{n}_{12} \cdot \bar{v}_1). \quad (6.13) \]

Combining these results, the evolution equation for \( P^\tau_4 \) becomes

\[ \left( \frac{dP^\tau_4}{d\tau} \right)_{\leq 2PN} = -\epsilon^2 \frac{m_1 m_2}{r_{12}^2} [4 (\bar{n}_{12} \cdot \bar{v}_1) - 3 (\bar{n}_{12} \cdot \bar{v}_2)] \\
+ \epsilon^4 \frac{m_1 m_2}{r_{12}^2} \left[ -9 (\bar{n}_{12} \cdot \bar{v}_2)^2 + \frac{v_1^2 (\bar{n}_{12} \cdot \bar{v}_2)^2}{2} + 6 (\bar{n}_{12} \cdot \bar{v}_1) (\bar{n}_{12} \cdot \bar{v}_2)^2 - 2 v_1^2 (\bar{n}_{12} \cdot \bar{v}_1) \right. \\
+ 4 (\bar{v}_1 \cdot \bar{v}_2) (\bar{n}_{12} \cdot \bar{V}) + 5 v_2^2 (\bar{n}_{12} \cdot \bar{v}_2) - 4 v_2^2 (\bar{n}_{12} \cdot \bar{v}_1) \left. \right] \\
+ \epsilon^4 \frac{m_1 m_2}{r_{12}^3} [-10 (\bar{n}_{12} \cdot \bar{v}_1) + 11 (\bar{n}_{12} \cdot \bar{v}_2)] \\
+ \epsilon^4 \frac{m_1^2 m_2}{r_{12}^3} [-4 (\bar{n}_{12} \cdot \bar{v}_2) + 6 (\bar{n}_{12} \cdot \bar{v}_1)]. \quad (6.14) \]

We can functionally integrate this equation and the result is
Here
\[ 2\Gamma_1 = \frac{1}{2} v^2_1 + \frac{3m_2}{r_{12}}, \] (6.16)
\[ 4\Gamma_1 = -\frac{3m_2}{2r_{12}} (\vec{v}_1 \cdot \vec{v}_2)^2 + \frac{2m_2}{r_{12}} v^2_2 + \frac{7m_2}{2r_{12}} v^4_1 - \frac{4m_2}{r_{12}} (\vec{v}_1 \cdot \vec{v}_2) + \frac{3}{8} v^4_1 + \frac{7m^2}{2r^2_{12}} - \frac{5m_1m_2}{2r^2_{12}}. \] (6.17)

Eqs. (F2) and (6.15) give the 2 PN order mass-energy relation.

**B. The 2 PN gravitational field**

To derive the 2 PN equation of motion, we need $h_{\tau\tau} + 6h^{l}_{l}$ (see Eqs. (B11) and (3.39)).

The field $h^{\mu\nu}$ that we shall evaluate from now on are as follows.

\[
\begin{align*}
    h_{\tau\tau} &= \lesssim h_{\tau\tau} + 4\epsilon_8 \int_{N/B} d^3y \frac{8\Lambda^T_N}{|\vec{x} - \vec{y}|} + 2\epsilon_8 \frac{\partial^2}{\partial\tau^2} \left( \int_{N/B} d^3y |\vec{x} - \vec{y}| [-g_{LL}^{\tau\tau}] \right) \\
    &\quad + \frac{1}{6} \epsilon_8 \frac{\partial^4}{\partial\tau^4} \left( \sum_{A=1,2} P^{r\tau A}_{A} \right) + O(\epsilon^9),
\end{align*}
\] (6.18)

\[
\begin{align*}
    h^{ij} &= \lesssim h^{ij} + 4\epsilon_6 \int_{N/B} \frac{6[-g^{ij}_{LL}]}{|\vec{x} - \vec{y}|} \\
    &\quad + 2\epsilon_6 \frac{\partial^2}{\partial\tau^2} \left( \int_{N/B} d^3y |\vec{x} - \vec{y}| [_{-g}^{ij}_{LL}] + \sum_{A=1,2} P^{r\tau A}_{A} v^j_A r^i_A \right) + O(\epsilon^7).
\end{align*}
\] (6.19)

Then $h_{\tau\tau} + \epsilon^2 h^{l}_{l} - (\lesssim h_{\tau\tau} + \epsilon^2 \lesssim h^{l}_{l})$ gives

\[
\begin{align*}
    4 \int_{N/B} d^3y \frac{8\Lambda^T_N}{|\vec{x} - \vec{y}|} + 6[-g_{LL}^{k}_{k}] \\
    &+ 2 \frac{\partial^2}{\partial\tau^2} \left[ \sum_{A=1,2} P^{r\tau A}_{A} v^3_A r_A + \int_{N/B} d^3y |\vec{x} - \vec{y}| (6[-g^{ij}_{LL}] + 4[-g_{LL}^{k}_{k}]) \right] \\
    &+ \frac{1}{6} \frac{\partial^4}{\partial\tau^4} \sum_{A=1,2} (P^{r\tau A}_{A})^3.
\end{align*}
\] (6.20)

The integrand $8\Lambda^T_N + 6[-g_{LL}^{k}_{k}]$ contains

\[
16\pi 8\Lambda^T_N + 6[-g_{LL}^{k}_{k}] = 2h^{\tau i}_{4} h^{\tau j}_{4,i,j} - 4h^{ij}_{4} h^{\tau r}_{4,i,j} - 4h^{\tau r}_{4} h^{ij}_{4,i,j} + 2h^{\tau r}_{4} h^{r l}_{4,i,k} - \frac{1}{2} h^{\tau r}_{4} h^{r l}_{4,i,k} \\
- 2h^{\tau r}_{4} h^{\tau r}_{4,k} + \frac{9}{8} h^{\tau r}_{4} h^{\tau r}_{4,k} h^{r l}_{4,k}.
\] (6.21)
The evaluation of the first three terms may call for great care. Let us show the results.

\[
\int_{N/B} \frac{-d^3y}{4\pi |\vec{x} - \vec{y}|} (-2\delta^{ij} \partial_i \partial_j) = 8 \sum_{A=1,2} \frac{P_i^j}{r_A^2} \left( 3n_A^j r_A^{ij} - \delta^{ij} \right)
\]

\[
-32 P_i^j P_i^j (\partial_1 \partial_1 + \partial_2 \partial_2) \ln S
\]

\[
+ \frac{32(\vec{P}_1 \cdot \vec{P}_2)}{3r_{12}} \sum_{A=1,2} \frac{1}{r_A},
\]

where the last term comes from the boundary term (surface integral term) in Eq. (3.33).

\[
\int_{N/B} \frac{-d^3y}{4\pi |\vec{x} - \vec{y}|} 4 h^{ij} h^{ij} \epsilon_{ij} = \frac{4 P_i^j P_i^j}{r_1^2} (\delta^{ij} - 3n_1 n_1^j)
\]

\[
+ 16 P_i^j P_i^j v_i^j v_i^j \partial_2 \partial_2 \ln S
\]

\[
+ \frac{4 P_i^{\tau 3}}{3 r_1^3} + P_1^{\tau 2} P_2^{\tau} (-16 H_1 - K_1)
\]

\[
- \frac{16 P_1^{\tau 2} P_2^{\tau}}{3} + 4 P_1^{\tau 2} P_2^{\tau} \left( -\frac{4}{3 r_1^2 r_1^{12}} + \frac{1}{6 r_1^2 r_1^{12}} \right) + (1 \leftrightarrow 2),
\]

where \((1 \leftrightarrow 2)\) means that there are the same terms but with the labels 1 and 2 exchanged.

The last two terms arise from the body zone boundary term in the Eq. (3.33). \(H_1\) and \(K_1\) have been derived by Blanchet, Damour, and Iyer [46]. They satisfy the following poisson equations in the sense of usual function in N/B:

\[
\Delta K_1 = 2 \partial_i \partial_j \left( \frac{1}{r_2^3} \right) \partial_i \partial_j (\ln r_1),
\]

\[
\Delta H_1 = 2 \partial_i \partial_j \left( \frac{1}{r_1^3} \right) \partial_i \partial_j \ln S.
\]

Their explicit expressions are

\[
K_1 = -\frac{1}{r_2^2} + \frac{1}{r_2 r_2^{12}} - \frac{1}{r_1^2 r_1^{12}} + \frac{r_2}{r_1 r_2^{12}} + \frac{r_2^2}{2 r_1^2 r_2^{12}} + \frac{r_2}{r_1^2 r_2^{12}},
\]

\[
H_1 = -\frac{1}{2r_1^3} - \frac{1}{4r_1^3} - \frac{1}{4r_1^3 r_2^{12}} - \frac{r_2}{2 r_1^2 r_2^{12}} + \frac{r_2}{2 r_1 r_2^{12}} + \frac{3r_2}{2 r_1^2 r_2^{12}} + \frac{r_2^2}{2 r_1^2 r_2^{12}} - \frac{r_2^3}{2 r_1^3 r_2^{12}}.
\]

We note that they go to zero when \(r \to \infty\). The third term of Eq. (6.21) can be evaluated as

\[
\int_{N/B} \frac{-d^3y}{4\pi |\vec{x} - \vec{y}|} 4 h^{ij} h^{ij} \epsilon_{ij} = 2 \sum_{A=1,2} \frac{P_i^j a_i^k}{r_A^2} (3 \partial_A a_A^i \partial_A^j - \delta^{ij} \Delta_A) \ln r_A
\]

\[
+ 8 \sum_{A=1,2} \frac{P_i^j a_i^k}{r_A^2} \partial_A \partial_A^k \ln r_A
\]
+ 16P_1^\tau P_2^\tau (v_1^k v_1^l \partial_1^k \partial_1^l + v_2^k v_2^l \partial_2^k \partial_2^l) \ln S
+ 16P_1^\tau P_2^\tau (a_1^k \partial_1^k + a_2^k \partial_2^k) \ln S
- \frac{16P_1^\tau P_2^\tau}{3r_{12}} \sum_{A=1,2} v_A^2, \quad (6.28)

of the star A,

\begin{align*}
a_A^i &= \frac{dv_A^i}{d\tau}.
\end{align*}

The last term again comes from the boundary term in Eq. (3.33).

The complete $h^{\mu\nu}$ are shown in the appendix E.

C. The 2 PN equation of motion

To derive the 2 PN equation of motion, we have to derive the momentum-velocity relation valid up to $O(\epsilon^4)$. Then, when the integrand is $\mathcal{A}_N^{\tau\mu}Q_A^i$, $Q_A^i$ becomes

\begin{align*}
Q_A^i &= Q_{\text{ALL}}^i + Q_{\chi}^i, \quad (6.29) \\
Q_{\text{LLL}}^i &= \frac{m_1^2 m_2 v_1^i}{3\epsilon R_1 r_{12}} - \frac{m_1^2 m_2 v_1^i}{15\epsilon R_1} - \frac{2m_1^2 m_2 v_2^i}{3\epsilon R_1 r_{12}}, \quad (6.30) \\
Q_{\chi}^i &= 0 \quad (6.31)
\end{align*}

where $Q_{\text{ALL}}^i$ and $Q_{\chi}^i$ come from the integrands $-gt_{\text{LL}}^{\mu\nu}$ and $\chi_N^{\mu\nu\alpha\beta}$. Thus the momentum-velocity relation becomes

\begin{align*}
P_{A\Theta}^i &= P_{A\Theta}^\tau v_A^i + O(\epsilon^5), \quad (6.32) \\
P_{A\chi}^i &= P_{A\chi}^\tau v_A^i + O(\epsilon^5). \quad (6.33)
\end{align*}

Now, by the divergencefree nature of $\chi_N^{\mu\nu\alpha\beta}$, $P_{A\chi}^\mu$ is itself conserved (see the appendix F). Hence we shall consider $P_{A\Theta}^\mu$ only. Up to the 2 PN order, we have to evaluate

\begin{align*}
\frac{dP_{A\Theta}^i}{d\tau} &= -\sum_{n=0}^2 \epsilon^{2n} \int_{\partial B_A} dS_{k4+2n}[-gt_{\text{LL}}^{ik}] + \sum_{n=1}^2 \epsilon^{2n} v_A^k \int_{\partial B_A} dS_{k4+2n}[-gt_{\text{LL}}^{ir}] + O(\epsilon^5). \quad (6.34)
\end{align*}

Evaluating the above surface integrals and using the mass-energy relation valid up to the 2 PN order, we obtain the 2 PN equation of motion. We show it and the 2.5 PN order correction in the next section.
VII. THE 2.5 PN EQUATION OF MOTION

In this section we shall derive the 2.5 PN equation of motion.

Now we first determine the mass-energy relation up to the 2.5 PN order. The surface integrals we have to evaluate are

\[
\frac{dP_{A\Theta}}{d\tau} = \left( \frac{dP_{A\Theta}}{d\tau} \right)_{\leq 2PN} - \epsilon^5 \oint_{\partial B_A} dS_k \left[ -gt_{\tau\tau}^{LL} \right] + \epsilon^5 v_A^k \oint_{\partial B_A} dS_k \left[ -gt_{\tau\tau}^{LL} \right].
\]

(7.1)

The evolution equation for \(P_{A\chi}^\tau\) is shown in the appendix [F]. Evaluating these surface integrals, we find that

\[
\left( \frac{dP_{A\Theta}}{d\tau} \right)_{2.5PN} = 0.
\]

Thus we have the mass-energy relation (of \(P_{A\Theta}^\tau\) part)

\[
P_{1\Theta}^\tau = m_1 \left[ 1 + \epsilon^2 \Gamma_1 + \epsilon^4 \Gamma_1 \right] + O(\epsilon^6).
\]

(7.2)

Here the definition of \(n_\Gamma_A\) are given by Eqs. (6.16) and (6.17). We discuss an interesting interpretation of this mass-energy relation in the appendix [F].

Next we have to derive the momentum-velocity relation. By evaluating explicitly, we find that \(Q_{iA}^j = 0\). Hence we have

\[
P_{iA}^\tau = P_{A\Theta}^\tau v_A^i + O(\epsilon^6).
\]

Finally we have to derive the deviation field \(h^{\mu\nu}\) up to the 2.5 PN order; \(g h^{\tau\tau}, \gamma h^{\mu\nu}\). We have

\[
e^7 \gamma h^{\tau\tau} + e^9 g h^{\tau\tau} = -\frac{4}{3} \epsilon^7 \frac{d}{d\tau} \left[ \sum_{A=1,2} P_{A}^{\tau} v_{A}^2 - \frac{P_{1}^{\tau} P_{2}^{\tau}}{r_{12}} \right]
\]

\[
+ 4 \epsilon^9 \left( \sum_{A=1,2} \frac{m_{A}}{r_{A}} \right) \left[ n_{A}^{i} n_{A}^{j} - \frac{1}{3} \delta_{ij} \right]
\]

\[
- \frac{1}{30} \epsilon^9 \frac{\partial^5}{\partial \tau^5} \left[ \sum_{A=1,2} m_{A} r_{A}^4 \right]
\]

+ (irrelevant terms),

(7.3)
\[
\epsilon^7 h^{r_1} = -\epsilon^7 \frac{2}{3} \frac{\partial^3}{\partial r^3} \sum_{A=1,2} m_A v_A^2 r_A^2, \quad (7.4)
\]

\[
epsilon^5 h^{l_1} + \epsilon^7 h^{l_1} = -2\epsilon^{5(3)} f^{l_{orb}}_{12}
- \epsilon^7 \frac{2}{3} \frac{\partial^3}{\partial r^3} \left( \sum_{A=1,2} m_A v_A^2 r_A^2 - \frac{m_1 m_2}{2r_{12}} \sum_{A=1,2} r_A^2 \right) \]

+ (irrelevant terms). \quad (7.5)

Evaluating the surface integrals in Eq. (3.39) with the deviation field above, we at last obtain the 2.5 PN order equation motion.

\[
m_1 \frac{dv^i_{12}}{d\tau} = -\frac{m_1 m_2}{r_{12}^2} n^i_{12}
+ \epsilon^2 \frac{m_1 m_2}{r_{12}^2} n^i_{12} \left[ -v_1^2 - 2v_2^2 + 4(\vec{v}_1 \cdot \vec{v}_2) + \frac{3}{2} (\vec{n}_{12} \cdot \vec{v}_2)^2 \right]
+ \epsilon^2 \frac{m_1 m_2}{r_{12}^2} v^i \left[ 4(\vec{n}_{12} \cdot \vec{v}_1) - 3(\vec{n}_{12} \cdot \vec{v}_2) \right]
+ \epsilon^4 \frac{m_1 m_2}{r_{12}^2} n^i_{12} \left[ -2v_2^4 + 4v_2^2(\vec{v}_1 \cdot \vec{v}_2) - 2(\vec{v}_1 \cdot \vec{v}_2)^2 + \frac{3}{2} v_1^2(\vec{n}_{12} \cdot \vec{v}_2)^2 + \frac{9}{2} v_2^2(\vec{n}_{12} \cdot \vec{v}_2)^2
- 6(\vec{v}_1 \cdot \vec{v}_2)(\vec{n}_{12} \cdot \vec{v}_2)^2 - \frac{15}{8} (\vec{n}_{12} \cdot \vec{v}_2)^4
+ \frac{m_1}{r_{12}} \left( -\frac{15}{4} v_1^2 + \frac{5}{4} v_2^2 - \frac{5}{2} (\vec{v}_1 \cdot \vec{v}_2) + \frac{39}{2} (\vec{n}_{12} \cdot \vec{v}_1)^2 - 39(\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) + \frac{17}{2} (\vec{n}_{12} \cdot \vec{v}_2)^2 \right)
+ \frac{m_2}{r_{12}} \left( 4v_2^2 - 8(\vec{v}_1 \cdot \vec{v}_2) + 2(\vec{n}_{12} \cdot \vec{v}_1)^2 - 4(\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) + 6(\vec{n}_{12} \cdot \vec{v}_2)^2 \right)
- \frac{57 m_1^2}{4 r_{12}^2} - 9 \frac{m_2^2}{r_{12}^2} + \frac{69 m_1 m_2}{2 r_{12}^2} \right] \]

+ \epsilon^4 \frac{m_1 m_2}{r_{12}^2} v^i \left[ v_1^2(\vec{n}_{12} \cdot \vec{v}_2) + 4v_2^2(\vec{n}_{12} \cdot \vec{v}_1) - 5v_2^2(\vec{n}_{12} \cdot \vec{v}_2) - 4(\vec{v}_1 \cdot \vec{v}_2)(\vec{n}_{12} \cdot \vec{V})
- 6(\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2)^2 + \frac{9}{2} (\vec{n}_{12} \cdot \vec{v}_2)^3
+ \frac{m_1}{r_{12}} \left( -\frac{63}{4} (\vec{n}_{12} \cdot \vec{v}_1) + \frac{55}{4} (\vec{n}_{12} \cdot \vec{v}_2) \right)
+ \frac{m_2}{r_{12}} \left( -2(\vec{n}_{12} \cdot \vec{v}_1) - 2(\vec{n}_{12} \cdot \vec{v}_2) \right) \right]
+ \epsilon^4 \frac{m_1^2 m_2}{5 r_{12}^3} \left[ n^i_{12}(\vec{n}_{12} \cdot \vec{V}) \left( -6 \frac{m_1}{r_{12}} + \frac{52 m_2}{3 r_{12}} + 3V^2 \right) + v^i \left( 2 \frac{m_1}{r_{12}} - 8 \frac{m_2}{r_{12}} - V^2 \right) \right]. \quad (7.6)

This is the 2.5 PN equation of motion for the relativistic compact binary. We used the harmonic coordinate to derive this equation. This equation perfectly agrees with the previous
works\cite{24,25,8,12}. Thus it is found that up to the 2.5 PN order the post-Newtonian equation of motion is applicable to the binary system even whose constituent stars have strong internal gravity.

VIII. CONCLUSION AND DISCUSSION

In this paper, we derived the equation of motion for two compact bodies accurate through the 2.5 post-Newtonian order, where the radiation reaction effect first appears. In the paper I, we have derived the spin-orbit and quadrupole-orbit coupling forces which are the only multipole-orbit coupling forces up to the 2.5 PN order in our ordering. Combing these multipole-orbit coupling forces and the result in this paper, we obtain an equation of motion appropriate to an inspiralling binary star system up to the 2.5 PN order. We have derived all these accelerations in the harmonic coordinate.

To deal with a strongly self-gravitating object such as a neutron star, we used the general form of the equation of motion with the strong field point particle limit. We imposed the scalings on matter and field variables on the initial hypersurface. This scalings enabled us to introduce a self-gravitating point-particle and ensured the (quasi-)stationarity of the stars. And in turn, the (quasi-)stationarity ensured that the internal motion of the star is stimulated primary by the tidal effect. In our method, we did not \textit{a priori} assume a geodesic equation, but used only the local conservation law of the total stress energy-momentum tensor (matter’s stress-energy tensor plus Landau-Lifshitz stress-energy pseudo-tensor). Using this conservation law, we introduced the general form of the equation of motion which is expressed entirely in surface integrals. The evaluation of these integrals were made explicitly. The resulting 2.5 PN equation of motion agrees with Damour and Deruelle 2.5 PN equation of motion \cite{24,25,8,12}. Hence we find that the 2.5 PN equation of motion is applicable to a relativistic compact binary, modulo the scalings imposed initially on the matter and field variables.

Throughout our derivation, the evolution equation for the time component of the four
momentum of the star A, $P^A_\tau$, has played an important role. As a by-product, we obtain an interesting and natural relation between $P^A_\tau$ and the binary’s characteristics, Eq. (H3).

We ignored the dependence on the body zone size $R_A$ in the estimation of the field, the multipole moments of the compact bodies, and the equation of motion. Since we introduced $R_A$ by hand, it is reasonable to expect that the field does not depend on $R_A$. We consider multipole moments as renormalized moments. It is proved that whatever $R_A$ we take as the radius of the body zone, the general equation of motion does not depend on $R_A$.

Discarding $R_A$ dependent terms when calculating the field is similar to regularizations such as Hadamard partie finie. Then, there is a possibility that at the 3 PN order our method might give log$(r_A/\epsilon R_A)$ type ambiguity that might correspond to the one reported in [33,13,14]. If such log term should arise we could not discard $\epsilon R_A$ even practically.

Our method, however, still differs from other works [33,13,14] in some points. Thus, besides as a check for the 3 PN equation of motion obtained in [33,13,14], it is interesting to derive it to study the log type ambiguity. We will be tackling the problem and try to derive the 3 PN equation of motion.

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**APPENDIX A: $Q^{Li}_A$ AND $R^{Lij}_A$**

In $h^{\mu\nu}_{Bn=0}$, Eqs. (3.14), and (3.15), the moments $J^{Li}_A$ and $Z^{Lij}_A$ appear formally at the order $\epsilon^{2l+4}$ and $\epsilon^{2l+2}$. Thus $Q^{Li}_A$ and $R^{Lij}_A$ appear as

$$h^{\tau i}_{Bn=0} \sim \cdots + \epsilon^{4l_L^A} Q^{Li}_A t_A^{2l+1} + \cdots,$$
\[ h_{Bn=0}^{ij} \sim \cdots + \epsilon^4 \frac{R_{A}^{Li} \hat{R}_{A}^{Lij}}{r_{A}^{2l+1}} + \cdots , \]

where we omitted irrelevant terms and numerical coefficients. These equations seem to suggest that \( Q_{A}^{Li} \) and \( R_{A}^{Lij} \) for any \( l \) contribute to \( h^{\mu\nu} \) at the order \( \epsilon^4 \).

We solve this problem by declaring that \( h^{\mu\nu} \) does not depend on the size of the body zone boundary, \( \epsilon R_{A} \) (see Sec. III C). By discarding the terms depending on \( \epsilon R_{A} \), we find that a finite number of \( Q_{A}^{Li} \) and \( R_{A}^{Lij} \) come into play.

Now, let us consider the n PN order integrand \( \epsilon^{2n+2}2_{n+2}\Lambda_{N}^{\mu\nu} \) (where the n PN order integrand means the integrand which appears at the n PN order in the evaluation of \( h_{Bn=0}^{\mu\nu} \). For example, \( \Lambda_{N}^{ij} \) is 1 PN integrand and \( \Lambda_{N}^{\mu i} \) is 2 PN integrand). These integrands, say, for the star A, schematically have forms such as

\[ \epsilon^{2n+2}2_{n+2}\Lambda_{N}^{\tau i} \sim \epsilon^{2n+2} \left( \frac{m}{r} \right)^s \frac{m^2}{r^4} \lambda^{\tau i}_{A}(\vec{n}_{A} \cdot \vec{n}_{AC}), \tag{A1} \]

\[ \epsilon^{2n+2}2_{n+2}\Lambda_{N}^{ij} \sim \epsilon^{2n+2} \left( \frac{m}{r} \right)^t \frac{m^2}{r^4} \lambda^{ij}_{A}(\vec{n}_{A} \cdot \vec{n}_{AC}), \tag{A2} \]

where the label C denotes the companion star (if \( A = 1, C = 2 \)). In these integrands, we expanded \( r_{C} \) around the center of the star A, and expressed the angular dependence of the integrands by the function \( \lambda^{\mu\nu}_{A} \) whose forms are irrelevant here. \( s = 0, 1, \cdots [n - 2] \) (\( s = 0 \) if \( n = 0, 1 \)) and \( t = 0, 1, \cdots [n - 1] \) (\( t = 0 \) if \( n = 0 \)), (where \([a]\) (\( a: \) real number) denotes the biggest integer smaller than \( a \)). \( m^k \) and \( r^k \) are \( m_{1}^{l} m_{3}^{n} \) and \( r_{1}^{l} r_{12}^{n} \) respectively where \( k, l, \) and \( n \) are positive integers and \( l + n = k \). Then from the definition of \( Q_{A}^{Li} \) and \( R_{A}^{Lij} \), Eqs. (3.26) and (3.27), Eqs. (A1) and (A2) give

\[ \epsilon^4 Q_{A}^{Li} \sim (\epsilon R_{A})^{l-s-1}\epsilon^{2n+2}, \]

\[ \epsilon^4 R_{A}^{Lij} \sim (\epsilon R_{A})^{l-t-1}\epsilon^{2n+2}, \]

where only the \( \epsilon \) and \( \epsilon R_{A} \) dependence are shown. Now from the argument in Sec. III C, we are concerned with the terms which do not depend on \( \epsilon R_{A} \). Thus when we use the n PN order integrand, we evaluate \( Q_{A}^{S_{a+1}i} \) and \( R_{A}^{T_{i+1}ij} \).
To derive $h^{\mu\nu}_{Bn=0}$ at the N PN order, we have to know $h^{\mu i}_{Bn}=0$ up to $O(\epsilon^{2N+2})$. Then we have to use $2n+2\Lambda^{\mu\nu}_N$ to evaluate $Q^{Li}_A$ and $R^{Lij}_A$ where $n \leq N$. Furthermore, since we can discard the $\epsilon R_A$ dependent terms, we evaluate only $Q^{Si}_A$ and $R^{Tij}_A$ where $s = 1, \cdots [N - 1]$ and $t = 1, \cdots [N]$. Thus at the Newtonian order, $N = 0$, $Q^{Li}_A$ and $R^{Lij}_A$ do not contribute. At the 1 PN order, $N = 1$, we have to evaluate $R^{kij}_A$ using $4[-g^{t\mu\nu}_{LL}]$. Up to the 2.5 PN order, we have to evaluate only $Q^{i}_A$, $Q^{ki}_A$, $R^{ij}_A$, $R^{kij}_A$, and $R^{klij}_A$ using $4[-g^{t\mu\nu}_{LL}]$ and $6[-g^{t\mu\nu}_{LL}]$.

Explicit calculations required up to the 2.5 PN order are as follows.

First we calculate
\[ \frac{1}{16\pi} \oint_{\partial B_1} dS_{44} h^{\tau\tau,i}_4 h^{\tau\tau,k}_4 y^L_1 y^j_1 = 4(\epsilon R_1)^{l+1} R^{2^{ll}j}_k C^{iklj}_m + O((\epsilon R_1)^{l+2}), \] (A3)
for \( l \) even,
\[ = O((\epsilon R_1)^{l+1}), \] (A4)
for \( l \) odd, and where
\[ C^L = \frac{1}{4\pi} \int d\Omega n^L. \]

Then, $Q^{Li}_A$ ($l \leq 3$) becomes
\[ \epsilon^4 Q^{Li}_A = \epsilon^6 \oint_{\partial B_A} dS_{k} y^L_1 y_A \bigl( [ -g^{t\mu\nu}_{LL} ] - v^k_A [-g^{t\mu\nu}_{LL}] \bigr) + O(\epsilon^8) \]
\[ = \frac{1}{2} \epsilon^6 (\epsilon R_A)^{l+1} P^{2k}_A v^k_C C^{lij}_k + O(\epsilon^6 (\epsilon R_A)^{l+2}, \epsilon^8), \] (A5)
for \( l \) even,
\[ = O(\epsilon^6 (\epsilon R_A)^{l+1}, \epsilon^8), \] (A6)
for \( l \) odd. Thus $Q^{Li}_A$ does not contribute to $h^{\mu\nu}$ up to the 2.5 PN order. The momentum-velocity relation becomes
\[ P^i_A = P^{j}_A v^j_A + O(\epsilon^4), \]
where we omitted the $\epsilon R_A$ dependence.

As for $R^{Lij}_A$ ($l \leq 2$),
\[ \epsilon^4 R_{A}^{Lij} = \epsilon^4 \oint_{\partial B} dS_k y_A^i y_A^j [-g_{LL}^{ik}] + \epsilon^6 \oint_{\partial B} dS_k y_A^i y_A^j (\delta [-g_{LL}^{ik}] - \eta^k_{A6} [-g_{LL}^{i\tau}]) \]
\[ + \epsilon^7 \oint_{\partial B} dS_k y_A^i y_A^7 [-g_{LL}^{ik}] + O(\epsilon^8) \]
\[ = \frac{1}{2} \epsilon^4 (\epsilon R_A)^{-1} P^k_{\alpha} C^{Lijk}_k + O(\epsilon^6 (\epsilon R_A)^{-1}, \epsilon^8) \]  
(A7)

for \( l \) even,
\[ = O(\epsilon^4 (\epsilon R_A)^{l+1}, \epsilon^6 (\epsilon R_A)^l, \epsilon^8), \]  
(A8)

for \( l \) odd. Thus \( R_{A}^{Lij} \) also does not contribute to \( h^{\mu \nu} \) up to the 2.5 PN order.

Finally, since \( Z_{A}^{Lij} \) contains \( J_{A}^{Lij} \), while \( J_{A}^{Lij} \) contains \( Q_{A}^{Lij} \), when we evaluate \( Z_{A}^{Lij} \), we have to evaluate not only \( R_{A}^{Lij} \) but also \( Q_{A}^{Lij} \). From this reason, up to the 2.5 PN order, \( Q_{A}^{kij} \) and \( Q_{A}^{klij} \) appear (see Eqs. (3.31) and (3.32)). From Eqs. (A5) and (A6), \( Q_{A}^{kij} = O(\epsilon^4, \epsilon^2 (\epsilon R_A)) \) and \( Q_{A}^{klij} = O(\epsilon^2) \). These results show that they do not contribute to \( h^{\mu \nu} \) up to the 2.5 PN order.

Thus, up to the 2.5 PN order, we have
\[ h_{Bn=0}^{ij} = 4 \epsilon^4 \sum_{A=1,2} \left( \frac{P_A^{\alpha} v_{A}^{j}}{r_A} + \epsilon^2 \frac{M_{\alpha}^{k} r_{A}^{k}}{2 r_A^3} \right) + O(\epsilon^8), \]  
(A9)
\[ h_{Bn=0}^{ij} = 4 \epsilon^4 \sum_{A=1,2} \left( \frac{P_A^{\alpha} v_{A}^{i} v_{A}^{j}}{r_A} + \epsilon^2 \frac{M_{\alpha}^{k(i) v_{A}^{j}} r_{A}^{k}}{r_A^3} \right) + O(\epsilon^8), \]  
(A10)

where we used Eqs. (3.21), (3.22), (3.23), and (3.24) and omitted the \( \epsilon R_A \) dependence.

**APPENDIX B: LANDAU-LIFSHITZ PSEUDO-TENSOR EXPANDED WITH EPSILON**

The Landau-Lifshitz pseudo-tensor [12] in terms of \( h^{\mu \nu} \) which satisfies the harmonic condition is as follows.
\[ (-16 \pi g) g^{\mu \nu}_{LL} = g_{\alpha \beta} g^{\gamma \delta} h^{\mu \alpha} h^{\nu \beta} + \frac{1}{2} g^{\mu \nu} g_{\alpha \beta} h^{\alpha \gamma} h^{\beta \delta} - 2 g_{\alpha \beta} g^{\gamma (\mu} h^{\nu) \alpha} \delta h^{\beta \delta} \gamma \]
\[ + \frac{1}{2} \left( g^{\mu \alpha} g^{\nu \beta} - \frac{1}{2} g^{\mu \nu} g^{\alpha \beta} \right) \left( g_{\gamma \delta} g_{\epsilon \zeta} - \frac{1}{2} g_{\gamma \epsilon} g_{\delta \zeta} \right) h^{\gamma \delta \epsilon \zeta} \]  
(B1)
We expand the deviation field $h^{\mu\nu}$ in a power series of $\epsilon$;

$$h^{\mu\nu} = \sum_{n=0}^{\infty} \epsilon^{4+n} h^{\mu\nu}_{n+4}. $$

Using this equation, we expand $\eta^{\mu\nu}_{LL}$ with $\epsilon$. The point to note is that 1) we raise or lower indices with the flat metric $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, 2) $\eta^{\tau\tau} = -\epsilon^2$ and $\eta_{\tau\tau} = -\epsilon^{-2}$, 3) $5 h^{ij}_{,k} = 0$ and $\gamma h^{\tau\tau}_{,k} = 0$, 4) $5 h^{\tau\mu} = 0$. Below, vertical strokes denote that indices between the strokes are excluded from (anti-)symmetrization.

1. $O(\epsilon^4)$

$$4[-16\pi g t^{\tau\tau}_{LL}] = 0, $$

$$4[-16\pi g t^{\tau i}_{LL}] = 0, $$

$$4[-16\pi g t^{ij}_{LL}] = \frac{1}{4}\left(4 h^{\tau\tau,i} h^{\tau\tau,j} - \frac{1}{2} \delta^{ij} 4 h^{\tau\tau,k} 4 h^{\tau\tau}_{,k}\right). \tag{B2}$$

2. $O(\epsilon^6)$

$$6[-16\pi g t^{\tau\tau}_{LL}] = -\frac{7}{8} 4 h^{\tau\tau,k} 4 h^{\tau\tau}_{,k}, \tag{B3}$$

$$6[-16\pi g t^{\tau i}_{LL}] = 2 4 h^{\tau\tau,k} 4 h^{\tau,i} \left(\frac{3}{4} h^{\tau k} 4 h^{\tau,i} - h^{\tau\tau,k} 4 h^{\tau,i}\right), \tag{B4}$$

$$6[-16\pi g t^{ij}_{LL}] = 4 h^{(i}^{(k} 4 h^{j)}_{[k}^{j]} - 2 4 h^{\tau\tau,(i} 4 h^{j)k} 4 h^{\tau\tau,k}$$

$$+ \frac{1}{2} h^{\tau\tau,(i} 6 h^{[\tau|j]} + \frac{1}{2} h^{\tau\tau,(i} 4 h^{[k|j]} - \frac{1}{2} \delta^{ij} 4 h^{\tau\tau,i} 4 h^{\tau\tau,j}$$

$$+ \delta^{ij} \left[\frac{3}{8} h^{\tau k} 4 h^{\tau l} + 4 h^{\tau\tau,k} 4 h^{k,l} + 4 h^{\tau\tau,k} 4 h^{k,l}\right]$$

$$- \frac{1}{4} 4 h^{\tau\tau,k} 4 h^{\tau\tau,k} - \frac{1}{4} 4 h^{\tau\tau,k} 4 h^{\tau,k} + \frac{1}{4} 4 h^{\tau\tau} 4 h^{\tau\tau} 4 h^{\tau\tau,k}\right]. \tag{B5}$$

3. $O(\epsilon^7)$

$$7[-16\pi g t^{\tau\tau}_{LL}] = 0, \tag{B6}$$

$$7[-16\pi g t^{\tau i}_{LL}] = 0, \tag{B7}$$

$$7[-16\pi g t^{ij}_{LL}] = \frac{1}{2}\left(4 h^{\tau\tau,(i} 5 h^{[k|j]} + 4 h^{\tau\tau,(i} 7 h^{[\tau|j]} \right)$$

$$- \frac{1}{2} \delta^{ij} \left\{4 h^{\tau\tau,l} 5 h^{k,l} + 4 h^{\tau\tau,k} 7 h^{\tau\tau,k}\right\}. \tag{B8}$$
4. \(O(\varepsilon^8)\)

\[8[-16\pi g t_{\tau\tau}^{ij}] = \frac{7}{4} h^{\tau\tau,k} h^{\tau\tau,k} + \frac{3}{8} h^{\tau\tau,k} h^{\tau\tau,l} + \frac{1}{4} h^{\tau\tau,k} h^{\tau\tau,l} + \frac{7}{8} h^{\tau\tau,k} h^{\tau\tau,k}. \quad (B9)\]

\[8[-16\pi g t_{\tau\tau}^{ij}] = 24 h^{\tau\tau,k} h^{\tau\tau,l} + \frac{1}{4} h^{\tau\tau,k} h^{\tau\tau,l} + \frac{3}{4} h^{\tau\tau,k} h^{\tau\tau,l} + \frac{1}{8} h^{\tau\tau,k} h^{\tau\tau,l}. \quad (B10)\]
\[-4h^{\tau \tau}_4 h^{\tau \tau}(i_6 h^{[\tau \tau],j}) - \frac{1}{2}h^{\tau \tau}_4 h^{\tau \tau,i}_4 h^{\tau \tau,j} \]
\[+ \frac{3}{4}(h^{\tau \tau})^2_4 h^{\tau \tau,i}_4 h^{\tau \tau,j} \]
\[+ \delta_{ij} \left( \frac{3}{8}h^{\tau \tau}_4 h^{\tau \tau,k}_4 h^{\tau \tau,l}_l - 4h^{\tau \tau}_4 h^{\tau \tau,k}_4 h^{\tau \tau,l}_l - 4h^{\tau \tau}_4 h^{\tau \tau,k}_l h^{\tau \tau,l}_l \right) \]
\[+ \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,l}_l h^{\tau \tau,i}_i - \frac{1}{2}h^{\tau \tau}_4 h^{\tau \tau,l}_l h^{\tau \tau,i}_i \]
\[+ \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,l}_l h^{\tau \tau,i}_i + \frac{1}{8}h^{\tau \tau}_4 h^{\tau \tau,l}_l h^{\tau \tau,i}_i \]
\[+ 4h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l - \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[+ \frac{1}{2}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,m}_m h^{\tau \tau,l}_l + \frac{1}{8}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l + \frac{1}{2}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[+ 4h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l - \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[+ 6h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l - \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[+ \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[+ \frac{3}{8}(h^{\tau \tau})^2_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[+ \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[- \frac{1}{8}h^{\tau \tau}_4 h^{\tau \tau,k}_k h^{\tau \tau,l}_l \]
\[\text{(B11)} \]

5. \( O(\epsilon^9) \)

\[9[-16\pi g t^{\tau \tau}_{LL}] = 0, \quad \text{(B12)} \]
\[9[-16\pi g t^{\tau \tau}_{ij}] = 2h^{\tau \tau}_4 h^{\tau \tau}(i_6 h^{[\tau \tau],j}) - \frac{1}{4}h^{\tau \tau}_4 h^{\tau \tau,i}_i h^{\tau \tau,j} + \frac{3}{4}h^{\tau \tau}_4 h^{\tau \tau,i}_i h^{\tau \tau,j}, \quad \text{(B13)} \]
\[9[-16\pi g t^{\tau \tau}_{ij}] = \frac{1}{2}h^{\tau \tau}(i_6 h^{[\tau \tau],j}) + \frac{1}{4}h^{\tau \tau}(i_6 h^{[\tau \tau],j}) - \frac{1}{4}h^{\tau \tau}(i_6 h^{[\tau \tau],j}) \]
\[+ \frac{1}{8}h^{\tau \tau,k}_k h^{\tau \tau,k}_k h^{\tau \tau,k}_k h^{\tau \tau,k}_k - \frac{1}{2}h^{\tau \tau,k}_k h^{\tau \tau,k}_k h^{\tau \tau,k}_k h^{\tau \tau,k}_k \]
\[- 2h^{\tau \tau,k}_k h^{\tau \tau,k}_k h^{\tau \tau,k}_k h^{\tau \tau,k}_k \]
\[+ 2h^{\tau \tau}(i_7 h^{[\tau \tau],j}) \]
\[- 2h^{\tau}(i_7 h^{[\tau \tau],j}) + 2h^{\tau}(i_7 h^{[\tau \tau],j}) \]
\[- 2h^{\tau}(i_7 h^{[\tau \tau],j}) + 2h^{\tau}(i_7 h^{[\tau \tau],j}) \]
\[35 \]
\[ + \delta_{ij} \left[ 24h^{\tau k,l} h^\tau_{[k,l]} - 4h^{\tau \tau,k} h^{\tau k,\tau} + 4h^{\tau k,k,l} h^{kl,\tau} \right] \\
+ \frac{1}{4} h^{\tau \tau,\tau 5 h^l_{l,\tau} - \frac{3}{4} h^{\tau \tau,\tau 7 h^{\tau \tau,\tau} + \frac{1}{8} 4h^{\tau \tau,k} h^{\tau \tau,\tau 5 h^{kl,\tau}} \\
+ \frac{1}{4} h^{\tau \tau,k} h^{\tau \tau,\tau 5 h^l_{l,k} - \frac{1}{4} 4h^{\tau \tau,k} h^{\tau \tau,\tau 5 h_{k,k}} \right]. \] (B14)

**APPENDIX C: SPIN-ORBIT COUPLING FORCE**

It is well known that definition of a dipole moment of the star, which we equate to zero to determine the center of the mass of the star, affects form of the spin-orbit coupling force (e.g., [44]). In the paper I we have chosen \( d_A^i = 0 \), where

\[ D_A^i = d_A^i + \epsilon^2 M_A^i v_A^i + O(\epsilon^4). \]

Instead, in this paper we choose \( D_A^i = 0 \). The corresponding spin-orbit coupling force is

\[
\left( m_1 \frac{dv_1^i}{d\tau} \right)_{SO} = -\epsilon^4 \frac{2V^k}{\nu^3_{12}} \left[ \left( m_1 M_2^{il} + m_2 M_1^{il} \right) \Delta^{lk} + \left( m_1 M_2^{lk} + m_2 M_1^{lk} \right) \Delta^{li} \right] \\
+ \epsilon^4 \frac{1}{\nu^3_{12}} \left[ m_2 M_1^{lk} v_1^k - m_1 M_2^{lk} v_2^k \right] \Delta^{li},
\]

where \( \Delta^{ij} = \delta^{ij} - 3n_{12}^i n_{12}^j \).

**APPENDIX D: DEFINITION OF THE MASS**

In Sec. [4], we defined the mass of the star A as

\[ m_A = \lim_{\epsilon \to 0} P_A^\tau. \] (D1)

We shall explain the meaning of this definition. Also we shall explain the reason we define the mass as above.

Let us consider the origin of the \( \epsilon \) dependence of \( P_A^\tau \). The origin can be classified into three categories: 1) The effect of the motion of the star A itself and the gravity of the companion star. 2) the \( \epsilon R_A \) dependence which comes from our splitting of space time, and
3) the deviation of the scaling law which we impose on $\Lambda^\mu_\nu$ on the initial hypersurface. We shall explain them respectively.

1) This type of $\epsilon$ dependence of $P^\tau_A$ can be specified by solving the evolution equation for $P^\tau_A$. In fact, for example we can rewrite Eq. (4.5) as

$$P^\tau_1 = m_1 + \epsilon^2 \frac{1}{2} m_1 v^2_1 + \epsilon^2 \frac{m_1 m_2}{r_{12}} + \epsilon^2 \frac{2 m_1 m_2}{r_{12}} + O(\epsilon^4).$$

This is the Newtonian energy (times $\sqrt{-g}$) of the star 1, if one lowers the upper index $\tau$ of $P^\tau_1$ down with Newtonian metric and multiply it minus sign. At least up to the 2.5 PN order, this interpretation really works (see the appendix [I]). Thus we can specify the type 1 $\epsilon$ dependence of $P^\tau_A$ by solving the evolution equation for $P^\tau_A$ functionally.

In fact, our definition of the mass, Eq. (D1), comes from this observation; if the star A stayed at rest and there were no companion star, $P^\tau_A$ would be the ADM mass of the star A.

2) Since we defined $P^\tau_A$ as the volume integral over the body zone, it must depend on the size of the body zone, $\epsilon R_A$. As we have stated in Sec. [II.C], we ignore this $\epsilon R_A$ dependence. But even not doing so, when we let $\epsilon$ go to zero, this $\epsilon R_A$ dependence disappears. Let us explain how this occurs.

First, because we assume a non-singular source for the star, the gravitational field on the body zone boundary must be smooth. In particular we can use the post-Newtonian expanded gravitational field near and just inside the body zone boundary. Thus we can estimate the $\epsilon R_A$ dependence using the post-Newtonian expanded gravitational field. Then, the most dangerous term (i.e., the term which seems to diverge when $\epsilon$ goes to zero) in the integrand of the volume integral Eq. (3.37) at the n PN order has a form (see the appendices [A] and [B])

$$\Lambda^\mu_\nu \sim \epsilon^{2n+4} h^{n-1} h_i h_j \sim \left( \frac{m_A}{r_A} \right)^{n-1} \frac{m_A}{r_A^2} \frac{m_A}{r_A^2},$$

where we omitted the indices. Then we can estimate the $\epsilon R_A$ dependence at the n PN order as

$$P^\mu_A \sim \epsilon^{-4} \left( \epsilon R_A \right)^3 \epsilon^{2n+4} \left( \frac{m_A}{\epsilon R_A} \right)^{n-1} \frac{m_A^2}{\left( \epsilon R_A \right)^4} \sim \epsilon^n.$$
An explicit calculation (see paper I) shows that up to the 1 PN order (i.e., \( n \leq 1 \) in the above equation) the \( \epsilon R_A \) dependence takes a form like \( \epsilon^2 m_A^2 / \epsilon R_A \). Thus the \( \epsilon R_A \) dependence vanishes when \( \epsilon \) goes to zero.

3) Though we impose scaling law on \( T_{\mu \nu}^N \) as Eqs. (2.2), (2.3), and (2.4), there must be deviation from these scalings in the integrand \( \Lambda_{\mu \nu}^N \).

One reason of this deviation is that since the velocity of the spinning motion of the star is set to be \( O(\epsilon) \) (slow rotation), the scaling law on the initial hypersurface can not be exactly satisfied by \( \Lambda_{\mu \nu}^N \). Besides this, the possibility of obtaining a sequence of solutions which has the scaling imposed in this paper has not been established. We only assume the existence of such a sequence.

The deviation may also come from the evolution effect. For instance, as the separation of the binary gets shorter and shorter, tidal effect stimulates oscillation of the constituent stars. Such an oscillation may let the stars not satisfy the scaling law.

If we can construct a sequence of solutions which has the scaling law (Eqs. (2.2), (2.3), and (2.4)) at least as their leading order, and when we consider a binary system whose separation is long enough, then we can incorporate the above three deviation into our formalism by expanding the density \( \rho_A \) in an \( \epsilon \) series (Let us use the \( \rho_A \) instead of \( \Lambda_{\mu \nu}^N \) for notational simplicity).

\[
\rho_A = \epsilon^{-4} (\rho_A)_{-4} + \epsilon^{-3} (\rho_A)_{-3} + \cdots.
\]

From this expansion we have an expansion of the mass in the \( \epsilon \) series.

\[
m_A = (0)m_A + \epsilon(1)m_A + \cdots,
\]

where

\[
(\epsilon^n)m_A = \lim_{\epsilon \to 0} \int_{B_A} d^3 y (\rho_A)_{n-4}.
\]

Here we need to take \( \epsilon \) zero limit to describe the star A as a point-like particle.

Thus the \( \epsilon \) zero limit in Eq. (D1) is taken to 1) remove the effect of the motion of the star A itself and the effect of the existence of the companion star, 2) remove the dependence
of $m_A$ on the body zone boundary, $\epsilon R_A$, and 3) achieve the point particle limit, while not removing the effect of the deviation of $\Lambda^\mu_\nu_N$ from the scaling law imposed initially.

**APPENDIX E: THE 2.5 PN GRAVITATIONAL FIELD**

We list the deviation filed $h^{\mu\nu}$ up to the 2.5 PN order. We have checked the harmonic condition up to the 2.5 PN order; $\lesssim h^{\mu\nu,\mu} = O(\epsilon^8)$. We can transform $h^{\mu\nu}$ into the metric by the formulas given in the appendix G. The resulting metric perfectly agrees with the result in [12].

Here are some notations.

$$f^{(\ln S)} = \frac{1}{36}(-r_1^2 + 3r_1 r_2 + r_2^2 - 3r_1 r_2 + 3r_1 r_2 - r_2^2)$$

$$+ \frac{1}{12}(r_1^2 - r_2^2 + r_2^2) \ln S,$$

$$f^{(1,-1)} = \frac{1}{18}(-r_1^2 - 3r_1 r_2 - r_2^2 + 3r_1 r_2 - 3r_1 r_2 + r_2^2)$$

$$+ \frac{1}{6}(r_1^2 + r_2^2 - r_2^2) \ln S.$$

They satisfy $\Delta f^{(\ln S)} = \ln S$ and $\Delta f^{(1,-1)} = r_1/r_2$. These two potentials (and other useful inverse Laplacian formulas) can be found in the appendix of [11].

$$h^{\tau\tau} = 4\epsilon^4 \sum_{A=1,2} \frac{P^\tau_A}{r_A}$$

$$+ \epsilon^6 \left[ 7 \sum_{a,b=1,2} \frac{P^\tau_a P^\tau_b}{r_a r_b} - \frac{14P^\tau_1 P^\tau_2}{r_{12}} \sum_{A=1,2} \frac{1}{r_A} + 2 \frac{\partial^2}{\partial \tau^2} \left( \sum_{A=1,2} P^\tau_A r_A \right) \right]$$

$$- \frac{4}{3} \epsilon^7 \frac{d}{d\tau} \left( \sum_{A=1,2} P^\tau_A v^2_A - \frac{P^\tau_1 P^\tau_2}{r_{12}} \right)$$

$$+ \epsilon^8 \left( 8 \sum_{A=1,2} \frac{P^\tau_3}{r^3_A} + \sum_{A=1,2} \frac{P^\tau^2_A}{r^2_A} \left( 8(\vec{n}_A \cdot \vec{v}_A)^2 - 7v^2_A \right) \right)$$

$$+ \epsilon^8 \frac{d}{d\tau} \left[ - \frac{P^\tau_1 P^\tau_2 (r_1 + r_2)}{2r_{12}} \right]$$

$$+ 14 \epsilon^8 \frac{\partial^2}{\partial \tau^2} \left[ - \frac{P^\tau_1 P^\tau_2 (r_1 + r_2)}{2r_{12}} \right]$$

$$+ \frac{1}{2} \sum_{A=1,2} P^\tau_A \ln \left( \frac{r_A}{R/\epsilon} \right) + P^\tau_1 P^\tau_2 \ln \left( \frac{S}{2R/\epsilon} \right)$$
\[ + \frac{1}{6} \epsilon^8 \frac{\partial^4}{\partial r^4} \left( \sum_{A=1,2} P_A^5 r_A^3 \right) \]
\[ + \epsilon^9 \left[ 4^{(3)} I_{\text{orb}}^{ij} \sum_{A=1,2} \frac{m_A n_A n_A^i}{r_A} - \frac{1}{30} \frac{\partial^5}{\partial r^5} \left( \sum_{A=1,2} m_A r_A^4 \right) \right] \]
\[ - \frac{4}{3} \frac{d}{d \tau} \left\{ \frac{m_1 m_2}{2 r_{12}} \sum_{A=1,2} \left( v_{A}^2 - 3 (\vec{n}_{12} \cdot \vec{v}_A)^2 \right) + \frac{5 m_1 m_2 (m_1 + m_2)}{2 r_{12}^2} \right. \]
\[ + \frac{5 m_1 m_2}{2 r_{12}} (\vec{n}_{12} \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2) - \frac{13 m_1 m_2}{2 r_{12}} (\vec{v}_1 \cdot \vec{v}_2) \left\} \right] + O(\epsilon^{10}). \]  

(E3)

Here

\[ s H^{rr} = P_1^r P_2^r \left( \frac{8}{r_{12}^3} + \frac{3}{r_{12}^2} + \frac{173}{3r_{12}^2} - \frac{52}{r_{12}^2} + \frac{1}{r_{12}} - \frac{r_{12}^2}{2r_{12}^2} - \frac{r_{12}}{2r_{12}^2} - \frac{21}{r_{12}^2} \right) \]
\[ + \frac{9r_1}{2r_{12}^2} + \frac{80}{3r_{12}^2} - \frac{2}{r_{12}^2} - \frac{232}{r_{12}^2} + \frac{15}{2r_{12}^2} - \frac{82}{r_{12}^2} - \frac{4r_2^2}{r_{12}^2} - \frac{8r_2^2}{r_{12}^2} + \frac{8r_2^2}{r_{12}^2} \]
\[ + P_1^r P_2^r \left( \frac{16}{S^2} - \frac{2}{r_{12}^2} \right) \left( \vec{n}_1 \cdot \vec{v}_1 \right)^2 - \frac{16 (\vec{n}_1 \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_1)}{S^2} \]
\[ + \left( \frac{7}{r_{12}^2} - \frac{2}{S^2} - \frac{2}{r_{12}^2} \right) \left( \vec{n}_{12} \cdot \vec{v}_1 \right)^2 \]
\[ + \frac{28}{3r_{12}^2} - \frac{4}{r_{12}^2} + \frac{28}{3r_{12}^2} - \frac{16}{r_{12}^2} + \frac{16}{r_{12}^2} - \frac{16}{S^2} \left( \vec{v}_1 \cdot \vec{v}_2 \right) + \frac{16 (\vec{n}_1 \cdot \vec{v}_2)(\vec{n}_{12} \cdot \vec{v}_1)}{S^2} \]
\[ + \left( \frac{24}{S^2} + \frac{32}{r_{12}^2} \right) \left( \vec{n}_1 \cdot \vec{v}_1 \right)(\vec{n}_1 \cdot \vec{v}_2) - \frac{12 (\vec{n}_1 \cdot \vec{v}_1)(\vec{n}_{12} \cdot \vec{v}_2)}{S^2} \]
\[ + \left( \frac{2}{S^2} - \frac{2}{r_{12}^2} \right) \left( \vec{n}_{12} \cdot \vec{v}_1 \right)(\vec{n}_2 \cdot \vec{v}_2) + \frac{10 (\vec{n}_1 \cdot \vec{v}_1)(\vec{n}_2 \cdot \vec{v}_2)}{S^2} \right) + (1 \leftrightarrow 2) \]  

(E4)

\[ h^{ri} = 4 \epsilon^4 \sum_{A=1,2} \frac{P_A^r v_A^i}{r_A} \]
\[ + \epsilon^6 \left\{ \sum_{A=1,2} \frac{P_A^r}{r_A} \left( (\vec{n}_A \cdot \vec{v}_A) n_A^i + 7 v_A^i \right) \right. \]
\[ + \frac{4P_1^r P_2^r}{S r_{12}} (v_1^k + v_2^k) \left( \delta^{ki} - n_{12}^i n_{12}^k \right) \]
\[ + 8P_1^r P_2^r (v_1^i + v_2^i) \left( - \frac{1}{r_{12}} \sum_{A=1,2} \frac{1}{r_A} + \frac{1}{r_{12}} \right) \]
\[ - 16 \frac{P_1^r P_2^r}{S^2} \left\{ v_1^k (n_{12}^i - n_1^i)(n_{12}^i + n_2^i) + v_2^k (n_{12}^i - n_1^i)(n_{12}^i + n_2^i) \right\} \]
\[ + 12 \frac{P_1^r P_2^r}{S^2} \left\{ v_1^k (n_{12}^i - n_1^i)(n_{12}^i + n_2^i) + v_2^k (n_{12}^i - n_1^i)(n_{12}^i + n_2^i) \right\} \]

40
\[ h_{ij} = 4\epsilon^4 \sum_{A=1,2} \frac{P_A^r v^i_A v^j_A}{r_A} \]

\[ + 2 \sum_{A=1,2} \frac{\partial^2}{\partial r^2} \left( P_A^r v^i_A r_A \right) \]

\[- \frac{2}{3} \epsilon^7 \frac{\partial^3}{\partial r^3} \left( \sum_{A=1,2} m_A v^i_A r_A^2 \right) + O(\epsilon^8). \quad \text{(E5)} \]

\[ 6H_{ij} = 4P^2 r^2 \left\{ \left( \frac{23}{24r^2_1S} - \frac{3r_1}{4r^3_1S} - \frac{163}{24r^2_1S} - \frac{11}{12r_1r_2S} - \frac{17}{8r_1r_2S} - \frac{r_1^2}{8r^3_1r_2S} + \frac{17r_1}{8r^2_1r_2S} + \frac{1}{8r_1r_2S} \right)^{ij} \right. \]

\[ + \frac{11r_2}{12r^3_1S} + \frac{17r_2}{8r^2_1r_2S} + \frac{23r_2}{24r^2_1r_2S} + \frac{11r^2_2}{12r_1r^3_2S} - \frac{23r^2_2}{24r^3_1r^2_2S} - \frac{23r^3_2}{24r^3_1r^2_2S} \delta^{ij} \]

\[ + \left. \left( \frac{-1}{24r^3_1S} + \frac{49}{24r^2_1r_2S} + \frac{r_2^2}{24r^2_1r_2S} \right) n^i_A n^j_A + \left( \frac{11}{3r^3_1r_2S} - \frac{26}{3r^2_1r_2S^2} - \frac{2}{3r^2_1S} - \frac{2}{r^2_1S} \right) n^i_A n^j_A \right\} \]

\[ + \left. \left( \frac{1}{3r^3_1S} + \frac{9}{r^2_1r_2S} + \frac{2r_2}{3r^2_1r_2S} + \frac{15}{r^2_1S} \right) n^i_{12} n^j_{12} \right\} \]

\[ \text{(E6)} \]
\[
+ \left( \frac{-r_2}{3r_{12}^2S^2} + \frac{r_1^2}{6r_{12}^3S^2} + \frac{r_1}{3r_{12}^2S^2} - \frac{17}{2r_{12}S^2} - \frac{r_2^2}{6r_{12}^3S^2} \right) n_{12}^{(i,j)} r_{12}^{ni} n_{12}^{(i,j)} \] 
+ 4P_1^r P_2^r \left\{ -2v_1^2 \left( \delta^{ik} \delta^{jl} - \frac{1}{2} \delta^{ij} \delta^{kl} \right) \frac{\partial^2 \ln S}{\partial z_1^k \partial z_2^l} + 8 \left( \delta^{ik} \delta^{jl} - \frac{1}{2} \delta^{ij} \delta^{kl} \right) \frac{\partial^2 \ln S}{\partial z_1^m \partial z_2^n} (v_1^m v_1^n) \right. 
+ 4(\vec{v}_1 \cdot \vec{v}_2) \left( \delta^{ik} \delta^{jl} - \frac{1}{2} \delta^{ij} \delta^{kl} \right) \frac{\partial^2 \ln S}{\partial z_1^k \partial z_2^l} 
- 8\delta^{(i} v_{1j)} v_2^k \frac{\partial^2 \ln S}{\partial z_1^k \partial z_2^l} + 4v_1^i v_2^j \delta^{kl} \frac{\partial^2 \ln S}{\partial z_1^k \partial z_2^l} 
+ \delta^{ij} \left( 2 \frac{\partial^2 \ln S}{\partial z_1^k \partial z_2^l} + 3 \frac{\partial^2 \ln S}{\partial z_1^k \partial z_2^l} \right) v_1^i v_2^j 
- \left( \delta^{ik} \delta^{jl} - \frac{1}{2} \delta^{ij} \delta^{kl} \right) \frac{\partial^4 f^{(1,-1)}}{\partial z_1^k \partial z_2^l \partial z_1^m \partial z_2^n} v_1^m v_1^n \right\} 
+ (1 \leftrightarrow 2). \tag{E7}
\]

Here in the terms (1 \leftrightarrow 2), \( f^{(1,-1)} \) is replaced by \( f^{(-1,1)} \), and this function satisfies \( \Delta f^{(-1,1)} = r_2/r_1 \). Its explicit form is the same as Eq. (E2) but with the labels 1 and 2 exchanged.

**APPENDIX F: \( \chi \) PART**

Since \( \chi_{\mu\nu\alpha\beta}^{\mu\nu\alpha\beta} \) itself is conserved, i.e., \( \chi_{\mu\nu\alpha\beta}^{\mu\nu\alpha\beta} = 0 \), the \( P_{\chi}^\mu \) by itself is conserved, here we define \( P_{\chi}^\mu \) as

\[
P_{\chi}^\mu = \epsilon^{-4} \int_{B_A} d^3 y \chi^\mu_{\tau\alpha\beta}. \tag{F1}
\]

First we derive the functional dependence of \( P_{\chi}^\mu \) on \( m_A, v_A^i, \) and \( r_{12} \). By the definition of \( \chi_{\mu\nu\alpha\beta}^{\mu\nu\alpha\beta} \),

\[
16\pi \chi_{\tau\alpha\beta}^{\tau\alpha\beta} = (h^{\tau k} h^{\tau l} - h^{\tau l} h^{\tau k}),_{kl},
\]

\[
16\pi \chi_{\tau\alpha\beta}^{\tau\alpha\beta} = (h^{\tau k} h^{\tau l} - h^{\tau l} h^{\tau k}),_{\tau k} + (h^{\tau k} h^{\tau l} - h^{\tau l} h^{\tau k}),_{\tau l},
\]

thus, we can obtain the functional expressions of \( P_{\chi}^\mu \) using Gauss’ law. To obtain these expressions up to the 2.5 PN order, we need \( \delta h^{\mu\nu} \) and \( \delta h^{\mu\nu} \). See the Sec. [7]. The results are
\[ P_{1\chi} = \epsilon^4 \frac{m_1 m_2}{3 r_{12}} \left[ 4 V^2 + \frac{m_2}{r_{12}} - \frac{2m_1}{r_{12}} \right] \]
\[ - \epsilon^5 \frac{2}{3} m_1^{(3)} l_{orbl} + O(\epsilon^6), \quad (F2) \]
\[ P_{A\chi}^i = P_{A\chi}^\tau v_A^i + O(\epsilon^6). \quad (F3) \]

Now we move on to the equation of motion for \( P_{A\chi}^\mu \).

\[ \frac{dP_{A\chi}^\mu}{d\tau} = -\epsilon^{-4} \oint_{\partial B_A} dS_k \chi_{\mu \alpha \beta}^{k} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \chi_{\mu \alpha \beta}^{k}, \quad (F4) \]

Evaluation of surface integrals gives

\[ \frac{dP_{1\chi}^\tau}{d\tau} = -\epsilon^{-4} \frac{m_1 m_2}{3 r_{12}} \left[ 4 V^2 + \frac{m_2 + 4m_1}{r_{12}} \right] \]
\[ - \epsilon^5 \frac{2}{3} m_1^{(4)} l_{orbl} + O(\epsilon^6), \quad (F5) \]
\[ \frac{dP_{1\chi}^i}{d\tau} = -\epsilon^{-4} \frac{m_1 m_2}{3 r_{12}} \left[ \vec{n}_{12} \cdot \vec{V} \right] v_1^i \left[ 4 V^2 + \frac{m_2}{r_{12}} - \frac{2m_1}{r_{12}} \right] \]
\[ - \frac{2}{3} \epsilon^5 \left[ m_1 v_1^{(4)} l_{orbl} - \frac{m_1 m_2 v_1^{(3)} l_{orbl}}{r_{12}^2} \right]. \quad (F6) \]

These two equations, Eqs. (F5) and (F6), are obviously consistent with Eqs. (F2) and (F3).

**APPENDIX G: METRIC EXPANDED WITH EPSILON**

Here we give the metric expanded in an \( \epsilon \) series for convenience.

\[ \epsilon^{-1} \sqrt{-g} = 1 + \frac{1}{2} \epsilon^2 4 h^{\tau \tau} \]
\[ + \epsilon^4 \frac{1}{2} \left[ 6 h^{\tau \tau} - 4 h^{k \tau} - \frac{1}{4} (4 h^{\tau \tau})^2 \right] \]
\[ + \epsilon^5 \frac{1}{2} (7 h^{\tau \tau} - 5 h^{k \tau}) + O(\epsilon^6) \]
\[ \epsilon^{-1} g_{\tau \tau} = -\epsilon^{-2} + \frac{1}{2} 4 h^{\tau \tau} + \epsilon^2 \frac{1}{2} \left[ 6 h^{\tau \tau} + 4 h^{k \tau} - \frac{3}{4} (4 h^{\tau \tau})^2 \right] \]
\[ + \epsilon^3 \frac{1}{2} [7 h^{\tau \tau} + 5 h^{k \tau}] \quad (G1) \]
\[ + \epsilon^4 \frac{1}{2} \left[ 8h^{\tau \tau} + 6h^{k}_{k} - \frac{3}{2} h^{\tau \tau} 4h^{\tau \tau} - \frac{1}{2} h^{k}_{k} 4h^{\tau \tau} + 4h^{\tau k} 4h^{\tau k} + \frac{5}{8} (4h^{\tau \tau})^3 \right] \]
\[ + \epsilon^5 \frac{1}{2} \left[ 9h^{\tau \tau} + 3h^{k}_{k} - \frac{3}{2} h^{\tau \tau} 7h^{\tau \tau} - \frac{1}{2} h^{\tau \tau} 5h^{k}_{k} \right] \]
\[ + O(\epsilon^6) \] (G2)
\[ \epsilon^{-1} g_{\tau i} = -\epsilon^2 4h^{\tau i} - \epsilon^4 \left[ 6h^{\tau i} - \frac{1}{2} 4h^{\tau \tau} 4h^{\tau i} \right] - \epsilon^5 7h^{\tau i} + O(\epsilon^6) \] (G3)
\[ \epsilon^{-1} g_{ij} = \delta^{ij} \left[ 1 + \epsilon^2 \frac{1}{2} 4h^{\tau \tau} \right] \]
\[ + \epsilon^4 \left[ 4h^{ij} + \frac{\delta^{ij}}{2} \left( 6h^{\tau \tau} - 4h^{k}_{k} - \frac{1}{4} (4h^{\tau \tau})^2 \right) \right] \]
\[ + \epsilon^5 \left[ 5h^{ij} + \frac{\delta^{ij}}{2} \left( 7h^{\tau \tau} - 5h^{k}_{k} \right) \right] \]
\[ + O(\epsilon^6) \] (G4)

**APPENDIX H: MEANING OF \( P^\tau_4 \)**

In this section we explain a meaning of \( P^\tau_4 \). First of all, we expand in an \( \epsilon \) series the four velocity of the star A normalized as \( g^{\mu \nu} u^\mu_A u^\nu_A = -\epsilon^{-2} \), where \( u^\mu_A = u^\tau_A u^i_A \). By the help of the appendix C, we have
\[ u^\tau_A = 1 + \epsilon^2 \left[ \frac{1}{2} v^2_A + \frac{1}{4} 4h^{\tau \tau} \right] \]
\[ + \epsilon^4 \left[ \frac{1}{4} 4h^{\tau \tau} + \frac{1}{2} 4h^{k}_{k} - \frac{3}{32} (4h^{\tau \tau})^2 + \frac{5}{8} 4h^{\tau \tau} v^2_A - 4h^{k}_{k} v^k_A + \frac{3}{8} v^4_A \right] \]
\[ + \epsilon^5 \frac{1}{4} \left[ 7h^{\tau \tau} + 5h^{k}_{k} \right] + O(\epsilon^6). \] (H1)

This is a formal series since the metric derived using the point particle description diverges at the point A. Now let us regularize this equation by letting \( r_A \) zero while simply discarding the divergent terms. For example, by this procedure \( \lesssim 6h^{\tau \tau} \) becomes (see Eq. (40))
\[ [\lesssim 6h^{\tau \tau}]_{12}^{ext} = 4 \epsilon^4 \frac{D^\tau}{r_{12}} \]
\[ + \epsilon^6 \left[ -2 \frac{m_2}{r_{12}} \left( \vec{m}_{12} \cdot \vec{v}_2 \right)^2 - v_2^2 \right] - 16 \frac{m_1 m_2}{r_{12}^2} + 7 \frac{m_2^2}{r_{12}^2} \] (H2),

for the star 1. In the above equation, \([f]_{A}^{ext} \) means that we regularize the quantity f at the star A by simply discarding the divergent terms.
Evaluating Eq. (H1) and (G1) by this procedure, then comparing with Eq. (7.2) with Eqs. (6.16) and (6.17), we find at least up to the 2.5 PN order

\[ P_{A\Theta}^\tau = m_A[\sqrt{-g}u_A^\tau]_{\text{ext}}. \]  

(H3)

It is important that even after discarding all the divergent terms, there remains non-linear effect. See Eq. (6.17) and Eq. (H2).

Eq. (H3) is natural. And note that we have never assumed this relation in advance. This relation has been derived by solving the evolution equation for \( P_{A\Theta}^\tau \) functionally.

The result of the above procedure, ”discarding all the divergent terms”, coincides with the result of some regularization scheme, such as Hadamard partie finie, at least up to the 2.5 PN order. We stress, however, that this coincidence is true for the calculation of \( P_{A}^\tau \) up to the 2.5 PN order. This happens because only \( 4h^{\mu\nu}, 5h^{ij}, 7h^{\tau\tau}, \) and \( 6h^{\tau\tau} \) are needed for the calculation of the 2.5 PN \( P_{A}^\tau \) (see Eq. (H1)). When one calculates the 3 PN \( P_{A}^\tau \), for example, \( 8h^{\tau\tau} \) is needed. Then ”discarding all the divergent terms” and Hadamard partie finie give different results (to see this, regularize \( H_1 \) by both ways, for example).

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A monopole-truncated object is spherically symmetric object which is not necessarily compact. Though an equation of motion for such objects is useful for describing inspiralling binaries, we stress that we have to take into account the effect of multipole moments of stars to derive accurate templates especially for binaries in late inspiralling phase.

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