High-dimensional nonlinear approximation by parametric manifolds in Hölder-Nikol’skii spaces of mixed smoothness

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Abstract

We study high-dimensional nonlinear approximation of functions in Hölder-Nikol’skii spaces $H_{\infty}^\alpha(I^d)$ on the unit cube $I^d := [0,1]^d$ having mixed smoothness, by parametric manifolds. The approximation error is measured in the $L_\infty$-norm. In this context, we explicitly constructed methods of nonlinear approximation, and give dimension-dependent estimates of the approximation error explicitly in dimension $d$ and number $N$ measuring computation complexity of the parametric manifold of approximants. For $d = 2$, we derived a novel right asymptotic order of noncontinuous manifold $N$-widths of the unit ball of $H_{\infty}^\alpha(I^2)$ in the space $L_\infty(I^2)$. In constructing approximation methods, the function decomposition by the tensor product Faber series and special representations of its truncations on sparse grids play a central role.

Keywords and Phrases: High-dimensional problem; Nonlinear approximation; Parametric manifold; Mixed smoothness; Sparse grids

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1 Introduction

Some problems in approximation theory and numerical analysis driven by a lot of applications in Information Technology, Mathematical Finance, Chemistry, Quantum Mechanics, Meteorology, and, in particular, in Uncertainty Quantification and Deep Machine Learning, are formulated in high dimensions when the number of involved variables are very large. Numerical methods for such problems may require computational cost increasing exponentially in dimension which makes the computation intractable when the dimension of input data is very large. Hyperbolic crosses and sparse grids promise to rid this “curse of dimensionality” in some problems when high-dimensional data belongs to certain classes of functions having mixed smoothness. Function spaces having mixed smoothness appear naturally in many models of real world problem in mathematical physics, finance and other fields, for instance, the regularity properties eigenfunctions of the electronic Schrödinger operator\textsuperscript{33} or...
the existence of solution of Navier-Stokes equations when initial data belonging to spaces with mixed smoothness [29, Chapter 6]. Approximation methods and sampling algorithms for functions having mixed smoothness constructed on hyperbolic crosses and sparse grids give a surprising effect since hyperbolic crosses and sparse grids have the number of elements much less than those of standard domains and grids but give the same approximation error. This essentially reduces the computational cost, and therefore makes the problem tractable. Sparse grids for approximate sampling recovery and integration were first considered by Smolyak [24]. In computational mathematics, the sparse grid approach was initiated by Zenger [34]. There has been a very large number of papers on hyperbolic cross and sparse-gird approximation and numerical applications to count all of them. We refer the reader to [1, 11] for surveys and for recent further developments and results. We also refer to the monographs [20, 21] for concepts and results on high dimensional problems and computation complexity.

Let us mention some recent results on different aspects of the problem of dimension-dependent error estimation in high-dimensional approximation which are directly related to our papers. The papers [2, 3, 4, 6, 7, 10, 16, 18] are on this problem for hyperbolic cross approximation of functions with mixed smoothness in terms of various n-widths and ε-dimensions. The authors of [6, 10] in particular, extended these problems for infinite-dimensional approximation with applications to stochastic and parametric PDEs. Preasymptotic estimation of high-dimensional problems were also treated in [16, 17, 18]. Related high-dimensional problems were studied in [22, 30] based on ANOVA decomposition. The paper [11] has investigated dimension-dependent estimates of the approximation error for linear algorithms of sampling recovery on Smolyak grids of functions from the space with Hölder–Zygmund mixed smoothness. It proved some upper bounds and lower bounds of the error of the optimal sampling recovery on Smolyak grids, explicit in dimension. All of the above mentioned papers considered only linear problems of high-dimensional approximation.

While linear methods utilize approximation from finite-dimensional spaces, nonlinear approximation means that the approximants do not come from linear spaces but rather from sets of nonlinear structure such as nonlinear manifolds, set of finite cardinality,… It is well understood that nonlinear methods of approximation and numerical methods derived from them often produce superior performance when compared with linear methods. Several notions of linear and nonlinear widths have been introduced to quantify optimality of approximation methods. Let us recall some of them.

Let X be a normed space, F and G subsets in X. We consider the problem of approximation of f ∈ F by elements g ∈ G. The approximation error is measured by ∥f − g∥X. The worst case error of the approximation of elements f ∈ F by elements g ∈ G is defined as

\[ E(F,G,X) := \sup_{f \in F} E(f,G,X) := \sup_{f \in F} \inf_{g \in G} \|f - g\|_X. \]

In numerical applications, an (linear and nonlinear) approximation method is usually based on a finite information in the form of the N values \(b_1(f), \ldots, b_N(f)\) of functionals. Such an approximation method can be seen as

\[ Q_N(f) = P_N(a_N(f)) \quad \text{for a pair of mappings} \quad a_N : F \to \mathbb{R}^N \quad \text{and} \quad P_N : \mathbb{R}^N \to X. \tag{1.1} \]

The approximant set \(G_N := P_N(\mathbb{R}^N)\) can be seen as a manifold in X parameterized by \(\mathbb{R}^N\). The parameter N characterizes computation complexity of the approximation method. We specify approximation methods having common properties by a certain set \(Q_N\) of pairs \((a_N, P_N)\), and look for an optimal method \(Q_N \in Q_N\) of approximation of f ∈ F in terms of the quantity

\[ d(F,Q_N,X) := \inf_{(a_N,P_N) \in Q_N} \sup_{f \in F} \|f - P_N(a_N(f))\|_X. \tag{1.2} \]
It is remarkable that the definition (1.2) is fit to notion of some important quantities of best linear and nonlinear approximation. Thus, if linear approximation is understood as approximation by elements from a finite-dimensional linear subspace, the well-known Kolmogorov widths $d_N(F, X)$ and linear $N$-widths $\lambda_N(F, X)$ being different quantities of best linear approximation can be defined as

$$d_N(F, X) := d(F, Q^d_N, X) = \inf_{\text{linear subspaces } X_N} \sup_{\dim X_N \leq N} E(F, X_N),$$

and $\lambda_N(F, X) := d(F, Q^\lambda_N, X)$, where $Q^d_N$ is the set of all pairs of mappings $(a_N, P_N)$ such that $P_N$ maps $\mathbb{R}^N$ to some linear subspace $X_N \subset X$ of dimension at most $N$, and $Q^\lambda_N$ is the set of all pairs of linear mappings $(a_N, P_N)$. Here the right-hand side of (1.3) is the traditional definition of Kolmogorov $N$-widths (see, e.g., [11] for a traditional definition of linear $N$-widths).

We next discuss the definition (1.2) for some quantities of best nonlinear approximation. The first notion given in [14], is (continuous) manifold $N$-width and defined by requiring $(a_N, P_N)$ to be continuous:

$$\delta_N(F, X) := d(F, Q^\delta_N, X) := \inf_{(a_N, P_N) \in Q^\delta_N} \sup_{f \in F} \|f - P_N(a_N(f))\|_X,$$

where $Q^\delta_N$ is the set of all pairs of continuous mappings $(a_N, P_N)$. Here the approximant set $G_N := P_N(\mathbb{R}^N)$ is a continuous manifold in $X$.

The requirement of continuity on $a_N, P_N$ is too minimal and does not give stability used in practice. To have stability in the numerical implementation one can restrict mappings $a_N$ and $P_N$ to be Lipschitz continuous. Based on this idea, in [5] the authors have introduced a notion of stable manifold $N$-widths by the formula $\delta^\gamma_N(F, X) := d(F, Q^\gamma_N, X)$, where $Q^\gamma_N$ is the subset in $Q^\delta_N$ of all Lipschitz mappings $a_N, P_N$ with some fixed constant $\gamma \geq 1$, that is $|a_N(f) - a_N(g)| \leq \gamma \|f - g\|_X$, and $\|P_N(x) - P_N(y)\|_X \leq \gamma|x - y|$, $x, y \in \mathbb{R}^N$ with the Euclid norm $|\cdot|$.

However, in many numerical applications approximation methods do not have continuous properties. The nonlinear $N$-width which is not based on continuity condition was suggested by Kolmogorov (1955) in the form of inverse quantity, $\varepsilon$-entropy. This is entropy $N$-width

$$\varepsilon_N(F, X) := d(F, Q^\varepsilon_N, X) = \inf_{X_N \subset X, |X_N| \leq 2^N} E(F, X_N),$$

where $Q^\varepsilon_N$ is the set of all pairs of mappings $(a_N, P_N)$ such that $a_N$ maps $F$ into $\{0, 1\}^N \subset \mathbb{R}^N$ and $|X_N|$ denotes the cardinality of $|X_N|$.

Another way to avoid the continuous restriction to require the approximant set $G_N := P_N(\mathbb{R}^N)$ which is in general, noncontinuous manifold parameterized by $\mathbb{R}^N$, to be contained in a finite-dimensional linear subspace. This leads to a notion of (noncontinuous) nonlinear manifold $N, M$-width $d_{N,M}(F, X)$ of a subset $F$ in $X$ as

$$d_{N,M}(F, X) := d(F, Q^{d}_{N,M}, X) = \inf_{(a_N, P_N) \in Q^{d}_{N,M}} \sup_{f \in F} \|f - P_N(a_N(f))\|_X,$$

where $Q^{d}_{N,M}$ is the set of all pairs of mappings $(a_N, P_N)$ such that $P_N$ maps $\mathbb{R}^N$ to some linear subspace $X_M \subset X$ of dimension at most $M$. The parameter $M$ in some sense only controls the linear dimension of the parametric manifold $P_N(\mathbb{R}^N)$, but is not related to computation complexity of the approximation method which is as above mentioned, characterized by the parameter $N$. Notice that with $N \leq M$

$$d_M(F, X) \leq d_{N,M}(F, X) \leq d_N(F, X) \quad \text{and} \quad d_{N,N}(F, X) = d_N(F, X).$$ (1.4)
One may assume that $N$ and $M$ are comparable, in particular, take $M = M(N)$ with the restriction $N \leq M(N) \leq CN(\log N)^\kappa$ for some $\kappa \geq 0$ and $C \geq 1$. With this assumption $d_{N,M(N)}(F,X)$ now only depends on $N$. It is surprising that for some cases, $d_{N,M(N)}(F,X)$ may have asymptotic order less than asymptotic order of any known nonlinear $N$-widths. This is confirmed at least by the following example.

Let $\hat{U}_{\infty}^{\alpha,d}$ be the unit ball of the Hölder-Nikol’skii space of functions on the unit cube $\mathbb{I}^d$ of mixed smoothness $0 < \alpha \leq 1$ with zero on the boundary of $\mathbb{I}^d$ (see Section 2 for a definition). Then when $d = 1$ we have that
\[
d_N(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I})) \asymp \delta_N(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I})) \asymp N^{-\alpha},
\]
and
\[
d_{N, \lfloor N \log N \rfloor}(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I})) \asymp (N \log N)^{-\alpha}.
\]

The asymptotic order of $d_N(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I}))$ in (1.5) is well-known, see, e.g., [25] for references. The asymptotic order of $\delta_N(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I}))$ in (1.5) was proven in [14]. The upper bound of (1.6) follows from recent results obtained in [32] for the case $\alpha = 1$ and in [13] for the case $\alpha \in (0,1)$. The lower bound of (1.6) follows from the inequalities
\[
d_{N, \lfloor N \log N \rfloor}(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I})) \geq d_{N, \lfloor N \log N \rfloor}(\hat{U}_{\infty}^{\alpha,1}, L_{\infty}(\mathbb{I})) \asymp (N \log N)^{-\alpha}.
\]

Here, we want to emphasize that the right asymptotic order of Kolmogorov width $d_N(\hat{U}_{\infty}^{\alpha,d}, L_{\infty}(\mathbb{I}^d))$ and linear $N$-widths $\lambda_N(\hat{U}_{\infty}^{\alpha,d}, L_{\infty}(\mathbb{I}^d))$ as well as of nonlinear $N$-widths for $d \geq 2$ is open problems except the case $d = 2$ (see [11, Chapter 4] for detailed comments).

All the above remarks and comments motivate us to consider high-dimensional nonlinear approximation by parametric manifolds for functions from the unit ball $U_{\infty}^{\alpha,d}$ of Hölder-Nikol’skii spaces having mixed smoothness $\alpha$. The approximation error is measured in the $L_{\infty}$-norm. In this context, we investigate the explicit construction of approximation methods of the form (1.1) with $(a_N, P_N) \in \mathcal{Q}_{N,M(N)}^d$ for approximation of $f \in \hat{U}_{\infty}^{\alpha,d}$, and explicit estimates in dimension $d$ and $N$ of the approximation error. We also treat the problem of right asymptotic order of $d_{N,M(N)}(U_{\infty}^{\alpha,d}, L_{\infty})$. (Here and in what follows, we use the abbreviation: $L_{\infty} := L_{\infty}(\mathbb{I}^d)$ and $\| \cdot \|_{L_{\infty}} := \| \cdot \|_{L_{\infty}}$.)

Let us briefly describe our main contribution. Let $N \in \mathbb{N}$ with $N \geq N(d)$ be given and $M(N) := \left[ \frac{12d^d(d-1)}{(d-1)!} N(\log N)(\log \log N)^{-d+1} \right]$ where $N(d)$ is a certain number (see (3.21)). Then we can explicitly construct a $M(N)$-dimensional subspace $F_{M(N)}$ of continuous functions on $\mathbb{I}^d$ spanned by tensor product Faber basis functions, and maps
\[
\lambda_N : \hat{U}_{\infty}^{\alpha,d} \to \mathbb{R}^N \quad \text{and} \quad G_N : \mathbb{R}^N \to F_{M(N)} \subset C(\mathbb{I}^d),
\]
so that
\[
\sup_{f \in \hat{U}_{\infty}^{\alpha,d}} \| f - G_N^* (\lambda_N(f)) \|_{L_{\infty}} \leq C_\alpha \left( \frac{K^d}{(d-1)!} \right)^{2\alpha+1} \frac{\log N(\log \log N)(d-\alpha+1)}{(N \log N)^\alpha} (log \log N)^{(d-1)\alpha},
\]
and
\[
\sup_{f \in \hat{U}_{\infty}^{\alpha,d}} \| f - G_N^* (\lambda_N(f)) \|_{L_{\infty}} \geq C_{\alpha,d} \frac{(\log N)^{(d-1)\alpha+1}}{(N \log N)^\alpha} (log \log N)^{(d-1)\alpha},
\]
where $K := (4^\alpha 6/(2^\alpha - 1))^{1/(2\alpha+1)}$, $C_\alpha := 27\alpha^2/(2^\alpha - 1)$ and the constant $C_{\alpha,d}$ depends on $\alpha, d$ only.

In the case $d = 1$, (1.8) follows from results on approximation by deep ReLU networks which have
been proven in [32] ($\alpha = 1$) and [13] ($\alpha \in (0, 1)$). Notice the term \(\left(\frac{K^{d-1}}{(d-1)!}\right)^{2\alpha+1}\) in the right-hand of (1.8) decays super-exponentially when \(d \to \infty\). To our knowledge (1.8) is the first result on dimension-dependent error estimation of nonlinear approximation of functions having mixed smoothness.

From (1.8) and (1.9) we also derived some upper and lower estimates for the noncontinuous manifold widths \(d_{N,M(N)}(\hat{U}^{\alpha,d}_\infty, L_\infty)\) (see Corollary 3.5). Especially, when \(d = 2\) we obtain the novel right asymptotic order

\[
d_{N,|N(\log N)(\log \log N)-1|}(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2)) \asymp N^{-\alpha} \log N(\log \log N)^{\alpha}.
\]

(The case \(d = 1\) already is given in (1.6).) Let us compare this asymptotic order with the asymptotic order of other well-known \(N\)-widths. We have for \(\alpha \in (0, 1)\) that

\[
\begin{align*}
\varepsilon_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2)) &\asymp \delta_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2)) \\
&\asymp d_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2)) \asymp \lambda_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2)) \asymp N^{-\alpha}(\log \log N)^{\alpha+1}.
\end{align*}
\]

The results in (1.10) were proven in [26] for entropy \(N\)-widths, and in [27] Kolmogorov \(N\)-widths. For the asymptotic order of \(\lambda_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2))\) in (1.10) see [11, p. 67]. The asymptotic order of \(\delta_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2))\) in (1.10) follows from a Carl’s type inequality between \(\varepsilon_N\) and \(\delta_N\) [5] and the inequality \(\delta_N \leq \lambda_N\). To know the authors, the asymptotic order of \(\delta_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2))\) is not known, except the upper bound via the inequality \(\delta_N \leq d_N\) and (1.10). Comparing the asymptotic order of \(d_{N,|N(\log N)(\log \log N)-1|}(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2))\) with the asymptotic order of the “smallest” entropy \(N\)-widths \(\varepsilon_N(\hat{U}^{\alpha,2}_\infty, L_\infty(\mathbb{I}^2))\) and the other \(N\)-widths in (1.10), we find that the first one is smallest in logarithm scale.

In construction of approximation and estimation of the approximation error, a representation of functions in \(U^{\alpha,d}_\infty\) by tensorized Faber series plays a central role. We primarily approximate \(f \in U^{\alpha,d}_\infty\) by the truncations of tensorized Faber series \(R_m(f)\) on sparse grids, and then approximate the function \(f - R_m(f)\) by combining a sparse-grid interpolation approximation and an approximation by sets of finite cardinality.

The outline of this paper is as follows. In Section 2, we present some auxiliary knowledge: a definition of Hölder-Nikol’skii spaces of mixed smoothness \(H^{a}_\infty(\mathbb{R}^d)\) and a representation of functions in \(H^{a}_\infty(\mathbb{R}^d)\) based on tensorized Faber basis. In this section, we also study auxiliary approximation of functions \(f \in U^{\alpha,d}_\infty\) by truncations of tensorized Faber series \(R_m(f)\) on sparse grids, and approximation of \(f\) by sets of finite cardinality. Section 3 is devoted to construction of manifold approximation for functions in Hölder-Nikol’skii spaces and estimation of the approximation error.

Notation. As usual, \(\mathbb{N}\) is the natural numbers, \(\mathbb{Z}\) is the integers, \(\mathbb{R}\) is the real numbers and \(\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}; \mathbb{N}_- = \mathbb{N}_0 \cup \{-1\}\). The letter \(d\) is reserved for the underlying dimension of \(\mathbb{R}^d, \mathbb{N}^d\), etc. We use \(x_i\) to denote the \(i\)th coordinate of \(x \in \mathbb{R}^d\), i.e., \(x := (x_1, \ldots, x_d)\). For \(x, y \in \mathbb{R}^d\), \(xy\) denotes the Euclidean inner product of \(x, y\), and \(2^x := (2^{x_1}, \ldots, 2^{x_d})\). For \(k, s \in \mathbb{N}_0^d\), we denote \(2^{-k}s := (2^{-k_1}s_1, \ldots, 2^{-k_d}s_d)\). For \(x \in \mathbb{R}^d\), we denote \(|x| := |x_1| + \ldots + |x_d|\). We use the abbreviation: \(L_\infty := L_\infty(\mathbb{I}^d)\) and \(\|\cdot\|_\infty := \|\cdot\|_{L_\infty}\). Universal constants or constants depending on parameter \(\alpha, d\) are denoted by \(C\) or \(C_{\alpha, d}\), respectively. Values of constants \(C\) and \(C_{\alpha, d}\) in general, are not specified except the case when they are precisely given, and may be different in various places. For two sequences \(a_n\) and \(b_n\) we will write \(a_n \lesssim b_n\) if there exists a constant \(C > 0\) such that \(a_n \leq C b_n\) for all \(n\), and \(a_n \asymp b_n\) if \(a_n \lesssim b_n\) and \(b_n \lesssim a_n\). \(|A|\) denotes the cardinality of the finite set \(|A|\).
2 Approximation by truncated Faber series

This section presents some preliminaries. We first provide a definition of Hölder-Nikol’skii spaces of mixed smoothness $H^{\alpha}_{\infty}(\mathbb{I}^d)$ and certain properties of these spaces. As a preparation for the manifold approximation in the next section we recall a representation of continuous functions on the unit cube by the tensorized Faber series. We then give an estimate for the representation coefficients of functions from Hölder-Nikol’skii spaces and the error of the approximation of $f \in H^{\alpha}_{\infty}(\mathbb{I}^d)$ by truncations of the tensorized Faber series $R_m(f)$. In the last part of this section, a set of finite cardinality is explicitly constructed to approximate functions in $H^{\alpha}_{\infty}(\mathbb{I}^d)$ and approximation error is given explicitly in $d$.

2.1 Hölder-Nikol’skii spaces of mixed smoothness

This subsection is devoted to introducing the Hölder-Nikol’skii spaces of mixed smoothness under consideration. For univariate functions $f$ on $\mathbb{I}$, the difference operator $\Delta_h$ is defined by

$$\Delta_h f(x) := f(x + h) - f(x),$$

for all $x$ and $h \geq 0$ such that $x, x + h \in \mathbb{I}$. If $u$ is a subset of $\{1, \ldots, d\}$, for multivariate functions $f$ on $\mathbb{I}^d$ the mixed difference operator $\Delta_{h,u}$ is defined by

$$\Delta_{h,u} := \prod_{i \in u} \Delta_{h_i}, \quad \Delta_{h,\emptyset} = \text{Id},$$

for all $x$ and $h$ such that $x, x + h \in \mathbb{I}^d$. Here the univariate operator $\Delta_{h_i}$ is applied to the univariate function $f$ by considering $f$ as a function of variable $x_i$ with the other variables held fixed. If $0 < \alpha \leq 1$, we introduce the semi-norm $|f|_{H^{\alpha}_{\infty}(u)}$ for functions $f \in C(\mathbb{I}^d)$ by

$$|f|_{H^{\alpha}_{\infty}(u)} := \sup_{h > 0} \prod_{i \in u} h_i^{-\alpha} \|\Delta_{h,u}(f)\|_{C(\mathbb{I}^d(h,u))}$$

(in particular, $|f|_{H^{\alpha}_{\infty}(\emptyset)} = \|f\|_{C(\mathbb{I}^d)}$), where $\mathbb{I}^d(h,u) := \{x \in \mathbb{I}^d : x_i + h_i \in \mathbb{I}, i \in u\}$. The Hölder-Nikol’skii space $H^{\alpha}_{\infty}(\mathbb{I}^d)$ of mixed smoothness $\alpha$ then is defined as the set of functions $f \in C(\mathbb{I}^d)$ for which the norm

$$\|f\|_{H^{\alpha}_{\infty}(\mathbb{I}^d)} := \max_{u \subset \{1, \ldots, d\}} |f|_{H^{\alpha}_{\infty}(u)}$$

is finite. From the definition we have that $H^{\alpha}_{\infty}(\mathbb{I}^d) \subset C(\mathbb{I}^d)$. The space $H^{\alpha}_{\infty}(\mathbb{I}^d)$ is a $d$-time tensor product of the space $H^{\alpha}_{\infty}(\mathbb{I})$ in the sense of equivalent norms. For further properties of this space such as embeddings, characterization by wavelets and atoms, we refer the reader to [19, 23, 31] and references there.

Denote by $\hat{C}(\mathbb{I}^d)$ the set of all functions $f \in C(\mathbb{I}^d)$ vanishing on the boundary $\partial \mathbb{I}^d$ of $\mathbb{I}^d$, i.e., the set of all functions $f \in C(\mathbb{I}^d)$ such that $f(x) = 0$ if $x_j = 0$ or $x_j = 1$ for some index $j \in \{1, \ldots, d\}$. Let $\mathcal{U}^{\alpha}_{\infty}$ be the set of all functions $f$ in the intersection $H^{\alpha}_{\infty}(\mathbb{I}^d) \cap \hat{C}(\mathbb{I}^d)$ such that $\|f\|_{H^{\alpha}_{\infty}(\mathbb{I}^d)} \leq 1$.

2.2 Tensorized Faber series and sparse-grid interpolation sampling recovery

In this subsection we describe a representation of functions in $H^{\alpha}_{\infty}(\mathbb{I}^d)$ by tensorized Faber series which plays a central role in the construction of nonlinear methods of noncontinuous manifold approximation.
of functions from the unit ball $\hat{U}_\infty^{\alpha,d}$. We give a dimension-dependent estimate of the approximation error by truncation $R_m(f)$ of the tensorized Faber series for functions $f \in \hat{U}_\infty^{\alpha,d}$. The approximant $R_m(f)$ represents an interpolation sampling recovery on sparse Smolyak grids.

We start with the univariate case. Let $\varphi(x) = (1 - |x - 1|)_+$, $x \in \mathbb{I}$, be the hat function (the piece-wise linear B-spline with knots at 0, 1, 2), where $x_+ := \max(x, 0)$ for $x \in \mathbb{R}$. For $k \in \mathbb{N}_{-1}$ we define the Faber functions $\varphi_{k,s}$ by

$$\varphi_{k,s}(x) := \varphi(2^{k+1}x - 2s), \quad k \geq 0, \quad s \in Z(k) := \{0, 1, \ldots, 2^k - 1\},$$

and

$$\varphi_{-1,s}(x) := \varphi(x - s + 1), \quad s \in Z(-1) := \{0, 1\}.$$

For a univariate function $f$ on $\mathbb{I}$, $k \in \mathbb{N}_{-1}$, and $s \in Z(k)$ we define

$$\lambda_{k,s}(f) := \frac{1}{2}\Delta_2^{2,k-1}f(2^{-k}s), \quad k \geq 0, \quad \lambda_{-1,s}(f) := f(s).$$

Here

$$\Delta_2^{2,k}f(x) := f(x + 2h) - 2f(x + h) + f(x),$$

for all $x$ and $h \geq 0$ such that $x, x + h \in \mathbb{I}$. The functions $\varphi_{k,s}$, $k \in \mathbb{N}_{-1}$, $s \in Z(k)$, constitute a basis for $C(\mathbb{I})$ and every function $f \in C(\mathbb{I})$ can be represented by the Faber series [15]

$$f = \sum_{k \in \mathbb{N}_{-1}} q_k(f), \quad q_k(f) := \sum_{s \in Z(k)} \lambda_{k,s}(f)\varphi_{k,s}$$

(2.1)

converging in the norm of $C(\mathbb{I})$.

For $m \in \mathbb{N}_0$, we define the truncation of the Faber series $R_m(f)$ by

$$R_m(f) := \sum_{k = 0}^m q_k(f).$$

The continuous piece-wise linear function $R_m(f) \in \hat{C}(\mathbb{I})$ possesses a certain interpolatory property. Indeed, one can check that for $f \in \hat{C}(\mathbb{I})$

$$R_m(f) = \sum_{s \in Z_s(m)} f(2^{-m-1}s)\varphi^*_{m,s},$$

(2.2)

where for $k \in \mathbb{N}_0$,

$$\varphi^*_{m,s}(x) := \varphi(2^{m+1}x - s + 1), \quad s \in Z_s(m) := \{1, \ldots, 2^{m+1} - 1\}.$$

Hence one can see that $R_m(f)$ interpolates $f$ at the points $2^{-m-1}s$, $s \in Z_s(m)$, that is,

$$R_m(f)(2^{-m-1}s) = f(2^{-m-1}s), \quad s \in Z_s(m).$$

We next extend the representation (2.1) to functions in $C(\mathbb{I}^d)$ by tensorization of the univariate Faber basis. Putting

$$Z(k) := \prod_{i=1}^d Z(k_i),$$
for \( \mathbf{k} \in \mathbb{N}^d_{-1}, \mathbf{s} \in Z(\mathbf{k}) \), we introduce the tensor product Faber functions

\[
\varphi_{\mathbf{k}, \mathbf{s}}(\mathbf{x}) := \prod_{i=1}^{d} \varphi_{k_i, s_i}(x_i), \quad \mathbf{x} \in \mathbb{I}^d,
\]

and define the linear functionals \( \lambda_{\mathbf{k}, \mathbf{s}} \) for multivariate function \( f \) on \( \mathbb{I}^d \) by

\[
\lambda_{\mathbf{k}, \mathbf{s}}(f) := \prod_{i=1}^{d} \lambda_{k_i, s_i}(f),
\]

where the univariate functional \( \lambda_{k_i, s_i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables held fixed.

**Lemma 2.1** The tensorized functions \( \{ \varphi_{\mathbf{k}, \mathbf{s}} : \mathbf{k} \in \mathbb{N}^d_{-1}, \mathbf{s} \in Z(\mathbf{k}) \} \) are a basis in \( C(\mathbb{I}^d) \). Moreover, every function \( f \in C(\mathbb{I}^d) \) can be represented by the tensorized Faber series

\[
f = \sum_{\mathbf{k} \in \mathbb{N}^d_{-1}} q_{\mathbf{k}}(f), \quad q_{\mathbf{k}}(f) := \sum_{\mathbf{s} \in Z(\mathbf{k})} \lambda_{\mathbf{k}, \mathbf{s}}(f) \varphi_{\mathbf{k}, \mathbf{s}} \tag{2.3}
\]

converging in the norm of \( C(\mathbb{I}^d) \).

The decomposition (2.3) when \( d = 2 \) and an extension for function spaces with mixed smoothness was obtained independently in [28, Theorem 3.10] and in [8, Section 4]. A generalization for the case \( d \geq 2 \) and also to B-spline interpolation and quasi-interpolation representation was established in [8, 9].

When \( f \in \mathring{U}^{a, d}_{\infty} \), \( \lambda_{\mathbf{k}, \mathbf{s}}(f) = 0 \) if \( k_j = -1 \) for some \( j \in \{1, \ldots, d\} \), hence we can write

\[
f = \sum_{\mathbf{k} \in \mathbb{N}^d_{\geq 1}} q_{\mathbf{k}}(f)
\]

with unconditional convergence in \( C(\mathbb{I}^d) \), see [28, Theorem 3.13]. In this case it holds the following estimate

\[
|\lambda_{\mathbf{k}, \mathbf{s}}(f)| = 2^{-d} \prod_{i=1}^{d} \left| \Delta_{2^{-k_i} - 1}^2 f \left(2^{-k_s}\right) \right|
= 2^{-d} \prod_{i=1}^{d} \left| \Delta_{2^{-k_i} - 1} f \left(2^{-k_s} + 2^{-k_i - 1} e_i\right) - \Delta_{2^{-k_i} - 1} f \left(2^{-k_s}\right) \right| \leq 2^{-\alpha d} 2^{-\alpha |\mathbf{k}|}, \tag{2.4}
\]

for \( \mathbf{k} \in \mathbb{N}^d_0, \mathbf{s} \in Z(\mathbf{k}) \). Here \( \{e_i\}_{i=1}^d \) is the standard basis of \( \mathbb{R}^d \).

For \( f \in \mathring{C}(\mathbb{I}^d) \), we define the truncation of Faber series \( R_m(f) \) by

\[
R_m(f) := \sum_{\mathbf{k} \in \mathbb{N}^d_0, |\mathbf{k}| \leq m} q_{\mathbf{k}}(f) = \sum_{\mathbf{k} \in \mathbb{N}^d_0, |\mathbf{k}| \leq m} \sum_{\mathbf{s} \in Z(\mathbf{k})} \lambda_{\mathbf{k}, \mathbf{s}}(f) \varphi_{\mathbf{k}, \mathbf{s}}. \tag{2.5}
\]

The function \( R_m(f) \) belongs to \( \mathring{C}(\mathbb{I}^d) \) and is completely determined by sampled values of \( f \) at the points in the Smolyak grid

\[
G^d(m) := \{ \xi_{\mathbf{k}, \mathbf{s}} = 2^{-k_1 - 1} \mathbf{s} : |\mathbf{k}| = m, \mathbf{s} \in Z(\mathbf{k}) \},
\]
where \( \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d \) and

\[
Z_*(k) := \bigotimes_{j=1}^d Z_*(k_j).
\]

Moreover, \( R_m(f) \) interpolates \( f \) at the points \( \xi \in G^d(m) \).

\[
R_m(f)(\xi) = f(\xi), \quad \xi \in G^d(m).
\]

Thus, the truncation of the Faber series \( R_m(f) \) can be seen as a formula of interpolation sampling recovery on the grids \( G^d(m) \) for \( f \in \tilde{C}(\mathbb{I}^d) \).

Notice that the Smolyak grids \( G^d(m) \) are very sparse. The number of knots in \( G^d(m) \) is smaller than \( \frac{2^d}{(d-1)!} 2^m m^{d-1} \) and is much smaller than \( 2^{dm} \), the number of knots in corresponding standard full grids. However, for periodic functions having mixed smoothness, they give the same error of the sampling recovery on the standard full grids. See [11, Chapter 5] for details.

The following lemma gives a \( d \)-dependent estimate of the error of approximation by the sparse-grid interpolation operators \( R_m(f) \) of functions having mixed smoothness from \( \dot{U}_{\infty}^{\alpha,d} \).

**Lemma 2.2** Let \( d \geq 2, m \in \mathbb{N} \), and \( 0 < \alpha \leq 1 \). Then we have

\[
\sup_{f \in \dot{U}_{\infty}^{\alpha,d}} \| f - R_m(f) \|_{\infty} \leq 2^{-\alpha} B^d 2^{-\alpha m} \left( \frac{m + d}{d - 1} \right), \quad B = (2^\alpha - 1)^{-1}.
\]

**Proof.** For every \( f \in \dot{U}_{\infty}^{\alpha,d} \) and \( k \in \mathbb{N}_0^d \), as the functions \( \varphi_{k,s}, s \in Z(k) \), have disjoint supports, by (2.4) we have

\[
\| f - R_m(f) \|_{\infty} \leq \sum_{k \in \mathbb{N}_0^d : |k|_1 > m} \left\| \sum_{s \in Z(k)} \lambda_{k,s}(f) \varphi_{k,s} \right\|_{\infty} \leq \sum_{k \in \mathbb{N}_0^d : |k|_1 > m} 2^{-d} 2^{-\alpha |k|_1}
\]

\[
= 2^{-\alpha d} \sum_{\ell = m+1}^{\infty} \left( \ell + d - 1 \right) 2^{-\alpha \ell} = 2^{-\alpha d} \sum_{s=0}^{\infty} \left( m + s + d \right) 2^{-\alpha (s+m+1)}
\]

\[
= 2^{-\alpha d} 2^{-\alpha (m+1)} \sum_{s=0}^{\infty} \left( m + s + d \right) 2^{-\alpha s}.
\]

Using

\[
\sum_{s=0}^{\infty} \left( \frac{m + s}{n} \right) t^s \leq (1 - t)^{-n-1} \left( \frac{m}{n} \right), \quad t \in (0,1),
\]

see [12, Lemma 2.2], we finally obtain

\[
\| f - R_m(f) \|_{\infty} \leq 2^{-\alpha d} 2^{-\alpha (m+1)} (1 - 2^{-\alpha} - d) \left( \frac{m + d}{d - 1} \right) = 2^{-\alpha} B^d 2^{-\alpha m} \left( \frac{m + d}{d - 1} \right).
\]

\( \Box \)

### 2.3 Approximation by sets of finite cardinality

In this subsection, we explicitly construct a set of finite cardinality for approximation of \( f \in \dot{U}_{\infty}^{\alpha,d} \) and give an estimate of the approximation error as well as the cardinality of this set.
Again, we start with the univariate case. For \( f \in \dot{U}^{\alpha,1}_\infty \) we explicitly construct the function \( S_f \in \dot{U}^{\alpha,1}_\infty \) by
\[
S_f := \sum_{s \in \mathbb{Z}_\epsilon(m)} 2^{-\alpha(m+1)} l_s(f) \varphi_{m,s}^*,
\]
where we put \( l_0(f) = 0 \) and assign the values \( S_f(2^{-m-1}s) = 2^{-\alpha(m+1)} l_s(f) \) from left to right closest to \( f(2^{-m-1}s) \) for \( s = 1, \ldots, 2^{m+1} - 1 \). If there are two possible choices for \( l_s(f) \) we choose \( l_s(f) \) that is closest to the already determined \( l_{s-1}(f) \). We define
\[
S^\alpha(m) := \{ S_f : f \in \dot{U}^{\alpha,1}_\infty \}.
\]

**Lemma 2.3** Let \( 0 < \alpha \leq 1, \ m \in \mathbb{N}_0 \). Then it holds \( |S^\alpha(m)| \leq 3^{2m+1} \) and for every \( f \in \dot{U}^{\alpha,1}_\infty \) we have
\[
\|R_m(f) - S_f\|_{L^\infty(\mathbb{I})} \leq 2^{-\alpha(m+1)-1}.
\]

**Proof.** In this proof we develop a technique used in [13]. For every \( f \in \dot{U}^{\alpha,1}_\infty \), from the construction of \( S_f \) we have
\[
S_f = \sum_{s \in \mathbb{Z}_\epsilon(m)} S_f(2^{-m-1}s) \varphi_{m,s}^* \]
and
\[
|S_f(2^{-m-1}s) - f(2^{-m-1}s)| \leq 2^{-\alpha(m+1)-1}, \ s = 0, \ldots, 2^{m+1}.
\]
From this, (2.2) and the inequality \( \sum_{s \in \mathbb{Z}_\epsilon(m)} \varphi_{m,s}^*(x) \leq 1 \), we deduce that for every \( x \in \mathbb{I} \),
\[
|R_m(f)(x) - S_f(x)| \leq \sum_{s \in \mathbb{Z}_\epsilon(m)} |f(2^{-m-1}s) - S_f(2^{-m-1}s)| \varphi_{m,s}^*(x)
\]
\[
\leq 2^{-\alpha(m+1)-1} \sum_{s \in \mathbb{Z}_\epsilon(m)} \varphi_{m,s}^*(x) \leq 2^{-\alpha(m+1)-1}.
\]
We have by (2.7) and the inclusion \( f \in \dot{U}^{\alpha,1}_\infty \),
\[
|l_s(f) - l_{s-1}(f)| 2^{-\alpha(m+1)} = |S_f(2^{-m-1}s) - S_f(2^{-m-1}(s-1))|
\]
\[
\leq |S_f(2^{-m-1}s) - f(2^{-m-1}s)| + |f(2^{-m-1}s) - f(2^{-m-1}(s-1))| + |f(2^{-m-1}(s-1)) - S(2^{-m-1}(s-1))| \leq 2^{-\alpha(m+1)+1}.
\]
Hence, \( |l_s(f) - l_{s-1}(f)| \leq 2 \). But the case \( |l_s(f) - l_{s-1}(f)| = 2 \) is not possible since it would imply that \( l_s(f) \) is not closest to \( l_{s-1}(f) \). This means that \( |l_s(f) - l_{s-1}(f)| \leq 1 \). Taking account this inequality and \( l_0(f) = 0 \), we can see that \( |S^\alpha(m)| \leq 3^{2m+1} \).

In the following, we make use the abbreviations: \( \mathbb{x}_j := (x_1, \ldots, x_j) \in \mathbb{R}^j; \ \bar{x}_j := (x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-j} \) with the convention \( \mathbb{x}_0 := 0 \) for \( \mathbb{x} \in \mathbb{R}^d \) and \( j = 0, 1, \ldots, d-1 \). When \( j = 1 \) we denote \( x_1 \) instead of \( \mathbb{x}_1 \).

We now construct a set of finite cardinality for approximation of \( f \in \dot{U}^{\alpha,1}_\infty \). Our strategy is to apply the above result to explicitly construct a set of finite cardinality for approximation of \( R_m(f) \), and show that this set approximates \( f \) as well as \( R_m(f) \). To do this we need a special representation of \( R_m(f) \) in terms of the tensor product of \( \varphi_{\mathbb{k}_1,\mathbb{s}_1}(\bar{x}_1) \) and \( R_{m-|\mathbb{k}_1|} \) of a function in \( \dot{U}^{\alpha,1}_\infty \) of variable \( x_1 \).
Lemma 2.4 Let $d \geq 2$, $0 < \alpha \leq 1$, $m > 1$ and $f \in \hat{U}_{\infty}^\alpha$. It holds the representation

$$R_m(f)(x) = \sum_{|k_1| \leq m} \sum_{\bar{a}_1 \in Z(k_1)} 2^{-\alpha(|k_1| + d - 1)} \varphi_{k_1, \bar{a}_1}(x_1) R_{m-|k_1|1}(K_{k_1, \bar{a}_1}(f)(x_1)),$$  \hspace{1cm} (2.8)

where the univariate function $K_{k_1, \bar{a}_1}(f)$ belongs to $\hat{U}_{\infty}^\alpha$ and is defined by

$$K_{k_1, \bar{a}_1}(f)(x_1) := \prod_{j=2}^{d} \left( - \frac{1}{2} 2^{\alpha(k_j + 1)} \Delta_{2^{-k_j - 1}}^2 f(x_1, 2^{-k_1} \bar{a}_1) \right).$$  \hspace{1cm} (2.9)

Proof. We have that

$$R_m(f)(x) = \sum_{k_2=0}^{m-k_1-1} \sum_{k_3=0}^{m-k_2-1} \cdots \sum_{k_d=0}^{m-k_{d-1}-1} \left[ \sum_{k_1=0}^{m-|k_1|} q_{k_1} \left( \prod_{j=2}^{d} q_{k_j}(f) \right)(x) \right]$$

$$= \sum_{k_2=0}^{m-m-k_1-1} \sum_{k_3=0}^{m-k_2-1} \cdots \sum_{k_d=0}^{m-k_{d-1}-1} R_{m-|k_1|1} \left( \prod_{j=2}^{d} q_{k_j}(f) \right)(x)$$

$$= \sum_{|k_1| \leq m} R_{m-|k_1|1} \left( \sum_{\bar{a}_1 \in Z(k_1)} \prod_{j=2}^{d} \left( - \frac{1}{2} 2^{\alpha(k_j + 1)} \Delta_{2^{-k_j - 1}}^2 f(x_1, 2^{-k_1} \bar{a}_1) \right) \varphi_{k_1, \bar{a}_1}(x_1) \right).$$

Here $R_{m-|k_1|1}$ applies to the function of variable $x_1$. Hence, we can write

$$R_m(f)(x) = \sum_{|k_1| \leq m} \sum_{\bar{a}_1 \in Z(k_1)} \left( \prod_{j=2}^{d} 2^{-\alpha(k_j + 1)} \right) \varphi_{k_1, \bar{a}_1}(x_1) R_{m-|k_1|1}(K_{k_1, \bar{a}_1}(f)(x_1)).$$

Thus, (2.8) is proven. For $x_1 \in \mathbb{I}$ and $x_1 + h_1 \in \mathbb{I}$, is holds the estimates

$$|\Delta_{h_1} K_{k_1, \bar{a}_1}(f)(x_1)| \leq \left( \prod_{j=2}^{d} \frac{1}{2} 2^{\alpha(k_j + 1)} \cdot 2 \cdot 2^{-\alpha(k_j + 1)} \right) |h_1|^\alpha \leq h_1^\alpha$$

which implies that $K_{k_1, \bar{a}_1}(f) \in \hat{U}_{\infty}^\alpha$. \hfill \Box

From the above special representation of $R_m(f)$ and Lemmata 2.2 and 2.3 we derive the following result.

Lemma 2.5 Let $m > 1$, $d \geq 2$ and $0 < \alpha \leq 1$. For $f \in \hat{U}_{\infty}^\alpha$, let the function $S_m(f)$ be defined by

$$S_m(f)(x) := \sum_{|k_1| \leq m} 2^{-\alpha(|k_1| + d - 1)} \sum_{\bar{a}_1 \in Z(k_1)} \varphi_{k_1, \bar{a}_1}(\bar{x}_1) S_{K_{k_1, \bar{a}_1}}(f)(x_1),$$  \hspace{1cm} (2.10)

where $S_{K_{k_1, \bar{a}_1}}(f) \in S^\alpha(m-|k_1|)$ is as in (2.6) for the function $K_{k_1, \bar{a}_1}(f)$. Then it holds the inequality

$$\|f - S_m(f)\|_{\infty} \leq B^d 2^{-\alpha m} \left( \frac{m + d}{d - 1} \right).$$  \hspace{1cm} (2.11)

Moreover, for the set

$$S^\alpha_{m,d}(m) := \{ S_m(f) : f \in \hat{U}_{\infty}^\alpha \},$$

we have

$$N_d(m) := |S^\alpha_{m,d}(m)| \leq 3^{2m+1} \left( \frac{m + d - 1}{d - 1} \right).$$
Proof. We first estimate $N_{d}(m)$. Since the number of $\bar{k}_1$ with $|\bar{k}_1| \leq m$ is $\binom{m+d-1}{d-1}$, and the cardinality of $S^\alpha(m - |\bar{k}_1|)$ is bounded by $3^{2m-|\bar{k}_1|+1}$, we have

$$N_{d}(m) \leq \left(3^{2m-|\bar{k}_1|+1}\right)^{\binom{m+d-1}{d-1}} = 3^{2m(m+d-1)}.$$ 

Lemma 2.2 gives

$$\|f - R_m(f)\|_{\infty} \leq 2^{-\alpha}B^d2^{-\alpha m}\left(\frac{m+d}{d-1}\right).$$

Next we show that

$$\|R_m(f) - S_m(f)\|_{\infty} \leq 2^{-1-\alpha d}2^{-\alpha m}\left(\frac{m+d}{d-1}\right).$$

Since $K_{k_1,\bar{s}_1}(f) \in \mathcal{U}_{\infty}^{\alpha,1}$ by Lemma 2.4, applying Lemma 2.3 we deduce that there is $S_{K_{k_1,\bar{s}_1}(f)} \in S^\alpha(m - |\bar{k}_1|)$ such that

$$\|R_m - |\bar{k}_1| (K_{k_1,\bar{s}_1}(f)) - S_{K_{k_1,\bar{s}_1}(f)}\|_{\infty} \leq 2^{-1-\alpha 2^{-\alpha m - |\bar{k}_1|}}.$$ 

Hence, taking account that the supports of $\varphi_{k_1,\bar{s}_1}$ with $\bar{s}_1 \in Z(\bar{k}_1)$ are disjoint, by the representation (2.8)-(2.9) of $R_m(f)$ and (2.14) we have that for every $x \in \mathbb{R}^d$,

$$|R_m(f)(x) - S_m(f)(x)|$$

$$= \left| \sum_{|\bar{k}_1| \leq m} \sum_{\bar{s}_1 \in Z(\bar{k}_1)} 2^{-\alpha(|\bar{k}_1|+d-1)} \varphi_{k_1,\bar{s}_1}(\bar{x})(R_m - |\bar{k}_1| (K_{k_1,\bar{s}_1}(f)(x_1))) - S_{K_{k_1,\bar{s}_1}(f)}(x_1)) \right|$$

$$\leq \sum_{|\bar{k}_1| \leq m} 2^{-\alpha(|\bar{k}_1|+d-1)} \sup_{\bar{s}_1 \in Z(\bar{k}_1)} \|R_m - |\bar{k}_1| (K_{k_1,\bar{s}_1}(f)) - S_{K_{k_1,\bar{s}_1}(f)}\|_{\infty}$$

$$\leq 2^{-1-\alpha 2^{-\alpha d}} \sum_{|\bar{k}_1| \leq m} 2^{-\alpha|\bar{k}_1|}2^{-\alpha(m - |\bar{k}_1|)} = 2^{-1-\alpha 2^{-\alpha m - |\bar{k}_1|}}$$

This implies (2.13). From (2.12) and (2.13), the triangle inequality and $2^{-\alpha}B^d + 2^{-1-\alpha d} \leq B^d$ prove (2.11).

3 Nonlinear approximation by parametric manifolds

This section aims at constructing nonlinear methods of parametric manifold approximation of functions $f \in \mathcal{U}_{\infty}^{\alpha,d}$. More precisely, we construct such a nonlinear method $R_{N}(f) = G_{N}^{\alpha}(\langle \lambda_{N}(f) \rangle)$ with mappings $\lambda_{N}$ and $G_{N}^{\alpha}$ of the form (1.7), satisfying the upper and lower estimates of approximation error (1.8)–(1.9). In order to do this we use the truncation of the tensorized Faber series $R_{n}(f)$ as an intermediate approximation. We then represent the difference $f - R_{n}(f)$ in a special form and approximate terms in this representation by functions in the set of finite cardinality constructed in the previous section.

For univariate functions $f \in C(\mathbb{R})$, let the operator $T_{k}$, $k \in \mathbb{N}$, be defined by

$$T_{k}(f) := f - R_{k-1}(f)$$
with the operator \( R_k \) defined as in (2.5) and the convention \( R_{-1} := 0 \). From this definition we have \( R_0 \) is the identity operator. Notice that for \( f \in \hat{U}^\omega_{\mathbb{N},d} \), it holds the inequality \( \|T_k(f)\|_{H^2_\mathbb{N}(\mathbb{I}^d)} \leq 2 \). For a multivariate function \( f \in \hat{C}(\mathbb{I}^d) \), the tensor product operator \( T_k, \ k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \), is defined by

\[
T_k(f) := \prod_{j=1}^d T_{k_j}(f),
\]

where the univariate operator \( T_{k_j} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_j \) with the other variables held fixed. It holds that \( \|T_k(f)\|_{H^2_\mathbb{N}(\mathbb{I}^d)} \leq 2^d \).

For \( n \in \mathbb{N} \) we have

\[
f - R_n(f) = \sum_{k \in \mathbb{N}_0^d, \|k\| > n} q_k(f) + \sum_{k_1 = 0}^n q_{k_1} \left( \sum_{\|k\|_1 > n-k_1} q_{k_1}(f) \right)
\]

\[
= T_{(n+1)e_1}(f) + \sum_{k_1 = 0}^n q_{k_1} \left( T_{(n+1-k_1)e_2}(f) + \sum_{k_2 = 0}^{n-k_1} q_{k_2} \left( \sum_{\|k\|_2 > n-k_1} q_{k_2}(f) \right) \right)
\]

\[
= T_{(n+1)e_1}(f) + \sum_{k_1 = 0}^n q_{k_1} T_{(n+1-k_1)e_2}(f) + \sum_{k_2 = 0}^{n-k_1} q_{k_2} \left( \sum_{\|k\|_2 > n-k_1} q_{k_2}(f) \right).
\]

Continuing in this way, we arrive at

\[
f - R_n(f) = T_{(n+1)e_1}(f) + \sum_{k_1 = 0}^n q_{k_1} T_{(n+1-k_1)e_2}(f) + \cdots + \sum_{k_{d-1} = 0}^{n-k_{d-1}} q_{k_{d-1}} T_{(n+1-k_{d-1})e_d}(f)
\]

\[
= T_{(n+1)e_1}(f) + \sum_{k_1 = 0}^n T_{(n+1-k_1)e_2}(q_{k_1}(f)) + \cdots + \sum_{k_{d-1} = 0}^{n-k_{d-1}} T_{(n+1-k_{d-1})e_d}(q_{k_{d-1}}(f)).
\]

Putting

\[
F_{k_j} := T_{(n+1-k_j)e_j+1}(q_{k_j}(f)), \quad j = 0, 1, \ldots, d-1.
\]

we can write

\[
f - R_n(f) = \sum_{j=0}^{d-1} \sum_{\|k_j\| \leq n} F_{k_j}, \quad (3.1)
\]

Let \( f \in \hat{U}^\omega_{\mathbb{N},d} \) be given. We will use this special representation to explicitly construct mappings \( \chi_{\mathbb{N}} \) and \( G_{\mathbb{N}} \) of the form (1.7). To this end, caused by (3.1), we will preliminarily approximate \( T_k(f) \).

Put

\[
I_{k,s} := \bigotimes_{j=1}^d I_{k_j,s_j} = \bigotimes_{j=1}^d [2^{-k_j}s_j, 2^{-k_j}(s_j + 1)], \quad k \in \mathbb{N}_0^d, \quad s \in \mathbb{Z}(k),
\]

and

\[
T_{k,s}(f)(x) := 2\alpha|k| - d(T_k(f)\chi_{I_{k,s}})(2^{-k}(x + s)).
\]

Since \( \text{supp} (T_k(f)\chi_{I_{k,s}}) \subset I_{k,s} \) and \( \|T_k(f)\chi_{I_{k,s}}\|_{H^2_\mathbb{N}(\mathbb{I}^d)} \leq 2^d \), we have that

\[
\text{supp} (T_{k,s}(f)) \subset \mathbb{I}^d, \quad T_{k,s}(f) \in \hat{U}^\omega_{\mathbb{N},d}. \quad (3.2)
\]
This allows us to apply Lemma 2.5 to the functions \( T_{k,s}(f) \). Namely, according to this lemma we explicitly construct the function \( S_m(T_{k,s}(f)) \) by the formula (2.10) so that we have by (2.11)

\[
\|T_{k,s}(f) - S_m(T_{k,s}(f))\|_\infty \leq B^{d-\alpha_m}(m + d)\cdot (d-1).
\] (3.3)

Define

\[
S_{k,m}(f)(x) := 2^{-|k|_1+d} \sum_{s \in Z(k)} S_m(T_{k,s}(f))(2^k \cdot x - s).
\] (3.4)

We then get

\[
\|T_k(f) - S_{k,m}(f)\|_\infty = \left\| \sum_{s \in Z(k)} \left[ T_k(f) \chi_{k,s}(\cdot) - 2^{-|k|_1+d} S_m(T_{k,s}(f))(2^k \cdot \cdot - s) \right] \right\|_\infty
\]

\[
= 2^{-|k|_1+d} \left\| \sum_{s \in Z(k)} \left[ T_{k,s}(f) - S_m(T_{k,s}(f)) \right](2^k \cdot - s) \right\|_\infty,
\]

and, consequently, by the equality \(|Z(k)| = 2^{|k|_1}\) and (3.3), the estimate of the error of the approximation \( T_k(f) \) by \( S_{k,m}(f) \)

\[
\|T_k(f) - S_{k,m}(f)\|_\infty \leq (2B)^d(2^m 2^{|k|_1}) - \alpha (m + d)\cdot (d-1).
\] (3.5)

Let \( \mathcal{F}^d(m) \) be the finite-dimensional subspace in \( \hat{C}(\mathbb{R}^d) \) of the form

\[
g = \sum_{k \in \mathbb{N}^d, |k|_1 \leq m} \sum_{s \in Z(k)} \alpha_k, s \cdot \phi_{k,s}, \quad \alpha_k, s \in \mathbb{R}.
\] (3.6)

It is easy to see that \( R_m(f) \in \mathcal{F}^d(m) \) for \( f \in \hat{C}(\mathbb{R}^d) \) and \( \dim \mathcal{F}^d(m) = \sum_{l=0}^{m-1} 2^d(\ell+d-1) \).

In the following, for any \( N \in \mathbb{N}, \ N \geq N_0 \) we will explicitly construct the maps

\[
\lambda_N^* : \mathcal{U}^{\alpha,d}_\infty \rightarrow \mathbb{R}^N \text{ and } G_N^* : \mathbb{R}^N \rightarrow \mathcal{F}^d(\lfloor \log N \rfloor + \lfloor \log \log N \rfloor + 1)
\]

and estimate the approximation error \( \sup_{f \in \mathcal{U}^{\alpha,d}_\infty} \|f - G_N^*(\lambda_N^*(f))\|_\infty \) in terms of \( N \).

For \( j = 0, 1, \ldots, d - 1 \) we put

\[
M_{d-j}(m) := N_{d-j}(m) \dim \left( \mathcal{F}^{d-j}(m) \right),
\]

where recall, \( N_{d-j}(m) := |S^{\alpha,d-j}(m)| \), see Lemma 2.5. We have

\[
M_{d-j}(m) \leq 3^{2^m+1(m+d-j-1)} \sum_{\ell=0}^{m} 2^\ell \left( \frac{\ell + d - j - 1}{d - j - 1} \right) \leq 3^{2^m+1(m+d-j-1)} 2^{m+1} \left( \frac{m + d - j - 1}{d - j - 1} \right).
\] (3.7)

Let \( \Gamma_j(n) \) be the set of all triples \((k_j, s_j, s_{j+1})\) satisfying the condition

\[
|k_j|_1 \leq n, \quad s_j \in Z(k_j), \quad s_{j+1} = 0, \ldots, 2^{n+1-|k_j|_1} - 1,
\]

in particular, \( \Gamma_0(n) = \{s_1 : 0 \leq s_1 \leq 2^n - 1\} \). We have

\[
|\Gamma_j(n)| = \sum_{|k_j| \leq n} 2^{|k_j|_1} \cdot 2^{n+1-|k_j|_1} = 2^{n+1} \sum_{k=0}^{n} \left( \frac{k + j - 1}{j - 1} \right) = 2^{n+1} \binom{n + j}{j}.
\] (3.8)
for all \( j = 0, \ldots, d - 1 \).

For \( \eta \in [N_{d-j}(m)] := \{1, \ldots, N_{d-j}(m)\} \) and a sequence
\[
\alpha \eta = (\alpha^{\eta}_{\ell}, \theta_{\eta})_{|\ell| \leq m, \ell \in Z(\bar{\ell})} \in \mathbb{R}^{\dim(F^{d-j}(m))},
\]
we put
\[
S_{\alpha \eta}(\bar{x}_j) := \sum_{|\ell| \leq m} \sum_{\ell \in Z(\bar{\ell})} \alpha^{\eta}_{\ell, \ell} \prod_{i = j+1}^{d} \varphi_{\ell, t_i}(x_i).
\]
If
\[
\alpha := (\alpha^{\eta}_{\eta})_{\eta = 1}^{N_{d-j}(m)} \in \mathbb{R}^{M_{d-j}(m)},
\]
and
\[
\theta := (\theta(k_j, s_j, s_{j+1}))_{(k_j, s_j, s_{j+1}) \in \Gamma_j(n)} \in [N_{d-j}(m)]^{\Gamma_j(n)},
\]
that is, elements of \( \theta \) are numbered by indices \((k_j, s_j, s_{j+1}) \in \Gamma_j(n)\), we define the maps
\[
G^{j}_{m,n} : \mathbb{R}^{M_{d-j}(m)} \times [N_{d-j}(m)]^{\Gamma_j(n)} \rightarrow F^d(m + n + 1), \quad j = 0, \ldots, d - 1,
\]
by
\[
\chi^j := (\alpha, \theta) \mapsto G^{j}_{m,n}(\chi^j),
\]
where
\[
G^{j}_{m,n}(\chi^j) := \sum_{\eta = 1}^{N_{d-j}(m)} \sum_{(k_j, s_j, s_{j+1}) \in \theta(k_j, s_j, s_{j+1}) = \eta} \frac{2^{d-j}}{2\alpha(n+1+j)} \varphi_{k_j, s_j}(x_j) S_{\alpha \eta}(2^{n+1-|k_j|} x_{j+1} - s_{j+1}, \bar{x}_{j+1})
\]
if \( j = 1, \ldots, d - 1 \), and
\[
G^{0}_{m,n}(\chi^0) := \sum_{\eta = 1}^{N_{d}(m)} \sum_{s_1, \vartheta, \eta = \eta} 2^{-\alpha(n+1)+d} S_{\alpha \eta}(2^{n+1} x_1 - s_1, \bar{x}_1).
\]
These maps are well-defined in the sense that the functions \( G^{j}_{m,n}(\chi^j), j = 0, \ldots, d - 1 \), belong to \( F^d(m + n + 1) \). We prove this for the case \( j = 1, \ldots, d - 1 \). The case \( j = 0 \) can be carried out similarly.

For the function \( \varphi_{\ell, t_j}(\bar{x}_j) \) with \( |\ell|_1 \leq m, \ell \in Z(\bar{\ell}) \), we have that
\[
\varphi_{\ell, t_j}(2^{n+1-|k_j|} x_{j+1} - s_{j+1}, \bar{x}_{j+1}) = \left( \prod_{i = j+2}^{d} \varphi_{\ell, t_i}(x_i) \right) \varphi_{\ell_{j+1}, t_{j+1}}(2^{n+1-|k_j|} x_{j+1} - s_{j+1})
\]
\[
= \left( \prod_{i = j+2}^{d} \varphi_{\ell, t_i}(x_i) \right) \varphi(2^{j+1+n+2-|k_j|} x_{j+1} - 2^{j+1+s_{j+1}} t_{j+1})
\]
\[
= \left( \prod_{i = j+2}^{d} \varphi_{\ell, t_i}(x_i) \right) \varphi_{\ell_{j+1} + n+1-|k_j|, 2^{j+s_{j+1}+t_{j+1}}}(x_{j+1}) \in F^{d-j}(n + m + 1 - |k_j|).
\]
Since \( S_{\alpha \eta} \) is a linear combination of \( \varphi_{\ell, t_j} \) with \( |\ell|_1 \leq m, \ell \in Z(\bar{\ell}) \), we conclude that
\[
S_{\alpha \eta}(2^{n+1-|k_j|} x_{j+1} - s_{j+1}, \bar{x}_{j+1}) \in F^{d-j}(n + m + 1 - |k_j|)
\]
which implies
\[ \varphi_{k,j,s_j}(x_j)S_{\alpha}(2^{n+1-|k_j|}x_{j+1} - s_{j+1}, x_{j+1}) \in \mathcal{F}^d(m + n + 1). \]

In the following lemma we explicitly construct a preliminary approximation of \( f - R_n(f) \) and estimate the approximation error.

**Lemma 3.1** Let \( \alpha \in (0, 1], \ j = 0, \ldots, d - 1, \) and \( m, n \in \mathbb{N} \). Then we can explicitly construct a map
\[ \lambda^j_{m,n} : \mathcal{U}^{a,d}_\infty \rightarrow \mathbb{R}^{M_{d-j}(m)} \times [N_{d-j}(m)]^{T_j(n)} \]
so that for every \( j \)
\[ \text{all which implies } \]
\[ \sum_{\bar{i} \in \mathcal{Z}(k_j)} (-1)^j 2^{-|k_j| - 1} \prod_{i=0}^{j} \Delta^2_{2^{-|k_j|}} f(2^{-k_j} s_j, \bar{x}_j) \varphi_{k,j,s_j}(x_j) \]
\[ \leq 2^{-\alpha+1}(2B)^d 2^{-\alpha(m+n)} \left( \frac{m + n + d}{d - 1} \right), \]
where \( B \) is given in Lemma 2.2.

**Proof.** Step 1. We auxiliarily construct an approximation and estimate the approximation error for all \( F_{k,j}, \ j = 0, 1, \ldots, d - 1. \) For \( F_{k,0} = T_{(n+1)e^1}(f) \) we take \( S_{(n+1)e^1,m}(f) \) by formula (3.4) and apply (3.5) to obtain the estimate
\[ \| F_{k,0} - S_{(n+1)e^1,m}(f) \|_\infty \leq (2B)^d 2^{m+1} \left( 1 + \frac{m + d}{d - 1} \right). \]

For \( j = 1, \ldots, d - 1 \), we rewrite \( F_{k,j}(x) \) in the form
\[ F_{k,j} (x) = T_{(n+1-|k_j|)e^{j+1}} \left( \sum_{s_j \in \mathcal{Z}(k_j)} (-1)^j 2^{-j} \prod_{i=0}^{j} \Delta^2_{2^{-|k_j|}} f(2^{-k_j} s_j, \bar{x}_j) \right) \varphi_{k,j,s_j}(x_j) \]
\[ = \sum_{s_j \in \mathcal{Z}(k_j)} \varphi_{k,j,s_j}(x_j) \left( T_{k_j^*} \left( (-1)^j 2^{-j} \prod_{i=0}^{j} \Delta^2_{2^{-|k_j|}} f(2^{-k_j} s_j, \bar{x}_j) \right) \right), \]
where \( k_j^* := (n + 1 - |k_j|, 0, \ldots, 0) \in \mathbb{N}_0^{d-j} \). Notice that the functions
\[ f_{k,j,s_j}(\bar{x}_j) := (-1)^j 2^{-j} 2^{\alpha(j+|k_j|)} \prod_{i=0}^{j} \Delta^2_{2^{-|k_j|}} f(2^{-k_j} s_j, \bar{x}_j) \]
are in variable \( \bar{x}_j \in \mathbb{R}^{d-j} \) and their norm in \( H^{\alpha}_{\infty}(\mathbb{R}^{d-j}) \) (with respect to \( \bar{x}_j \)) satisfies the inequality
\[ \| f_{k_j,s_j} \|_{H^{\alpha}_{\infty}(\mathbb{R}^{d-j})} \leq 1. \]

Again, for \( T^*_{k_j^*}(f_{k_j,s_j}) \) with \( k_j^* \in \mathbb{N}_0^{d-j} \) we take \( S_{k_j^*,m}(f_{k_j,s_j}) \) by formula (3.4) and apply (3.5) to have the estimate
\[ \| T^*_{k_j^*}(f_{k_j,s_j}) - S_{k_j^*,m}(f_{k_j,s_j}) \|_\infty \leq (2B)^d 2^{-\alpha j} \left( 1 + \frac{m + d - j}{d - j - 1} \right). \]

For approximation of \( F_{k,j}, \ j = 0, \ldots, d - 1, \) we take the functions \( S_{F_{k,j}} \) which are defined by the explicit formulas
\[ S_{F_{k,0}}(x) := S_{(n+1)e^1,m}(f)(x); \]
\[ S_{F_{k,j}}(x) := \sum_{s_j \in \mathcal{Z}(k_j)} 2^{-\alpha(j+|k_j|)} \varphi_{k,j,s_j}(x_j) S_{k_j^*,m}(f_{k_j,s_j})(x_j), \ j = 1, \ldots, d - 1. \]
We have the estimates by (3.12) for every $j = 1, \ldots, d - 1$ and $x \in \mathbb{I}^d,$

$$\|F_{k_j}(x) - S_{F_{k_j}}(x)\| = \left| \sum_{s_j \in Z(k_j)} 2^{-\alpha|j + |k_j|_1|} \varphi_{k_j,s_j}(x_j) \left( T_{k_j}^* (f_{k_j,s_j}) - S_{k_j,m}(f_{k_j,s_j}) \right)(x_j) \right|$$

$$\leq \sum_{s_j \in Z(k_j)} 2^{-\alpha|j + |k_j|_1|} \varphi_{k_j,s_j}(x_j) \left\| T_{k_j}^* (f_{k_j,s_j}) - S_{k_j,m}(f_{k_j,s_j}) \right\|_\infty$$

$$\leq (2B)^{d-j} \alpha(2(m+1) - \alpha)(m + d - j) \left( \frac{m + d - j}{d - j - 1} \right).$$

From the last estimate and (3.11) we deduce that

$$\|F_{k_j}(x) - S_{F_{k_j}}(x)\|_\infty \leq 2^{-\alpha}(2B)^{d-j} 2^{-\alpha(2\alpha(m+n) - \alpha(m+d-j))} \left( \frac{m + d - j}{d - j - 1} \right), \quad j = 0, 1, \ldots, d - 1.$$

Then we get

$$\left\| f - R_n(f) - \sum_{j=0}^{d-1} \sum_{|k_j| \leq n} S_{F_{k_j}} \right\|_\infty \leq \sum_{j=0}^{d-1} \sum_{|k_j| \leq n} \|F_{k_j}(x) - S_{F_{k_j}}(x)\|_\infty$$

$$\leq 2^{-\alpha} 2^{-\alpha(2(m+n))} \sum_{j=0}^{d-1} (2B)^{d-j} 2^{-\alpha(2(m+n) - \alpha(m+d-j))} \left( \frac{m + d - j}{d - j - 1} \right) \left( \frac{n + j}{j} \right),$$

where we have used $\sum_{|k_j| \leq n} 1 = \binom{n+j}{j}$. By the inequalities $(\frac{m+d-j}{d-j-1}) \binom{n+j}{j} \leq \binom{m+n+d}{d-1}$ for $j = 0, \ldots, d-1$ and $B \geq 1$ the error of the approximation of $f - R_n(f)$ by the function

$$\sum_{j=0}^{d-1} \sum_{|k_j| \leq n} S_{F_{k_j}}$$

can be estimated as

$$\left\| f - R_n(f) - \sum_{j=0}^{d-1} \sum_{|k_j| \leq n} S_{F_{k_j}} \right\|_\infty \leq 2^{-\alpha}(2B)^{d} 2^{-\alpha(2(m+n))} \left( \frac{m + n + d}{d - 1} \right) \sum_{j=0}^{\infty} (2B)^{-j} 2^{-\alpha(j)}$$

$$\leq 2^{-\alpha+1}(2B)^{d} 2^{-\alpha(2(m+n))} \left( \frac{m + n + d}{d - 1} \right).$$

(3.14)

Step 2. Due to (3.14), to complete the proof we explicitly construct a map of the form (3.10) so that

$$\sum_{|k_j| \leq n} S_{F_{k_j}} = G_{m,n}(\lambda_{m,n}^j), \quad j = 0, \ldots, d - 1.$$

We deal with the cases $j \in \{1, \ldots, d - 1\}$. The case $j = 0$ is carried out similarly with slight
Hence we can write
\[
\sum_{|k_j| \leq n} S_{F_{k_j}}(x) = \sum_{|k_j| \leq n} \sum_{s_j \in \mathbb{Z}(k_j)} 2^{-\alpha(j + |k_j|)} \varphi_{k_j, s_j}(x_j) S_{k_j,m}(f_{k_j, s_j})(\bar{x}_j)
\]
\[
= \sum_{|k_j| \leq n} \sum_{s_j \in \mathbb{Z}(k_j)} \frac{2^{-d-j}}{2^{a(n+1+j)}} \varphi_{k_j, s_j}(x_j) \sum_{s_j' \in \mathbb{Z}(k_j')} S_m(T_{k_j', s_j'}(f_{k_j, s_j}))(2^{k_j'} \bar{x}_j - s_j')
\]
\[
= \sum_{|k_j| \leq n} \sum_{s_j \in \mathbb{Z}(k_j)} \frac{2^{-d-j}}{2^{a(n+1+j)}} \varphi_{k_j, s_j}(x_j)
\times \sum_{s_{j+1}=0}^{2^n-1} S_m(T(x_{n+1-|k_j|, 0, 0}, (s_{j+1}, 0, 0))(f_{k_j, s_j}))(2^n-1-|k_j| x_{j+1} - s_{j+1}, \bar{x}_{j+1}).
\]

Notice that by (3.2) the functions \( T(x_{n+1-|k_j|, 0, 0}, (s_{j+1}, 0, 0))(f_{k_j, s_j}) \) belong to \( \hat{U}^{\alpha,d-j} \) and, therefore, by Lemma 2.5, the functions \( S_m(T(x_{n+1-|k_j|, 0, 0}, (s_{j+1}, 0, 0))(f_{k_j, s_j})) \) belong to \( S^{\alpha,d-j}(m) \). Numbering elements of \( S^{\alpha,d-j}(m) \) as
\[
S^{\alpha,d-j}(m) := \{ S_{d-j}^{\alpha,j} \}_{j=1}^{N_{d-j}(m)},
\]
we obtain
\[
\sum_{|k_j| \leq n} S_{F_{k_j}}(x) = \sum_{\eta=1}^{N_{d-j}(m)} \sum_{(k_j, s_j, s_{j+1}) \in \Gamma_{\eta}} \frac{2^{-d-j}}{2^{a(n+1+j)}} \varphi_{k_j, s_j}(x_j) S_{d-j}^{\eta}(2^n-1-|k_j| x_{j+1} - s_{j+1}, \bar{x}_{j+1}),
\]
where
\[
\Gamma_{\eta} = \{ (k_j, s_j, s_{j+1}) \in \Gamma_{\eta}(n) : S_m(T(x_{n+1-|k_j|, 0, 0}, (s_{j+1}, 0, 0))(f_{k_j, s_j})) = S_{d-j}^{\eta} (\bar{x}_{j+1}) \}.
\]

We define the map
\[
\lambda_{m,n}^j : U^{\alpha,d}_\infty \to \mathbb{R}^{M_{d-j}(m)} \times (N_{d-j}(m))_{\Gamma_{\eta}(n)}
\]
by
\[
f \mapsto \lambda_{m,n}^j(f) := (a_\eta, \theta_{(k_j, s_j, s_{j+1})}(f))_{(k_j, s_j, s_{j+1}) \in \Gamma_{\eta}(n)},
\]
where
\[
a_\eta := \left( a_\eta^j, \bar{\ell}_j \right)_{\bar{\ell}_j \in \mathbb{Z}(\bar{\ell}_j)}
\]
are coefficients of \( S_{d-j}^{\eta} \) in Faber series representation and \( \theta_{(k_j, s_j, s_{j+1})}(f) = \eta \) if \( (k_j, s_j, s_{j+1}) \in \Gamma_{\eta} \). Hence we can write
\[
\sum_{|k_j| \leq n} S_{F_{k_j}}(x) = \sum_{\eta=1}^{N_{d-j}(m)} \sum_{(k_j, s_j, s_{j+1}) \in \Gamma_{\eta}} \frac{2^{-d-j}}{2^{a(n+1+j)}} \varphi_{k_j, s_j}(x_j) S_{d-j}^{\eta}(2^n-1-|k_j| x_{j+1} - s_{j+1}, \bar{x}_{j+1})
\]
\[
= G_{m,n}^j(\lambda_{m,n}^j(f)).
\]
This proves (3.15).

Let the map
\[
G_n^R : \mathbb{R}^{|\dim F^d(n)|} \to F^d(n)
\]
be defined by

\[ \lambda^R = (\lambda_{k,s})_{|k|_1 \leq n, s \in Z(k)} \rightarrow G_n^R(\lambda^R) = \sum_{|k|_1 \leq n} \sum_{s \in Z(k)} \lambda_{k,s} \phi_{k,s}. \]

We extend the map \( G_{m,n}^j \) defined in (3.9) as a map

\[ G_{m,n}^j : \mathbb{R}^{M_d-j(m)} \times \mathbb{R}^{[\Gamma_j(n)]} \rightarrow \mathcal{F}^d(m + n + 1) \]

(the extension denoted again by \( G_{m,n}^j \)) by assigning \( G_{m,n}^j(\lambda^j) = 0 \) if \( \lambda^j \notin \mathbb{R}^{M_d-j(m)} \times [N_{d-j}(m)]^{[\Gamma_j(n)]} \).

Denote

\[ N_{m,n} := \dim \mathcal{F}(n) + \sum_{j=0}^{d-1} (M_{d-j}(m) + |\Gamma_j(n)|) \quad (3.16) \]

and

\[ \lambda := (\lambda^R, \lambda^0, \ldots, \lambda^{d-1}) \in \mathbb{R}^{N_{m,n}}, \]

where \( \lambda^R \in \mathbb{R}^{\dim \mathcal{F}(n)} \) and \( \lambda^j \in \mathbb{R}^{M_{d-j}(m)} \times [\Gamma_j(n)]. \) We define the map

\[ G_{m,n} : \mathbb{R}^{N_{m,n}} \rightarrow \mathcal{F}^d(m + n + 1) \quad (3.17) \]

by

\[ G_{m,n}(\lambda) := G_n^R(\lambda^R) + \sum_{j=0}^{d-1} G_{m,n}^j(\lambda^j), \quad (3.18) \]

and put \( K_{m,n} := \dim(\mathcal{F}(m + n + 1)). \)

**Corollary 3.2** Let \( \alpha \in (0, 1], d, m, n \in \mathbb{N}. \) Then we can explicitly construct a map

\[ \lambda_{m,n} : \check{U}_\alpha \rightarrow \mathbb{R}^{N_{m,n}} \]

so that

\[ \sup_{f \in \check{U}_\alpha} \| f - G_{m,n}(\lambda_{m,n}(f)) \|_\infty \leq 2^{-\alpha+1}(2B)^d 2^{-\alpha(m+n)} \left( \frac{m+n+d}{d-1} \right), \quad (3.19) \]

and hence,

\[ d_{K_{m,n}}(\check{U}_\alpha, L_\infty) \leq 2^{-\alpha+1}(2B)^d 2^{-\alpha(m+n)} \left( \frac{m+n+d}{d-1} \right). \]

**Proof.** We define the operator \( \lambda_{m,n} \) by

\[ \lambda_{m,n} := (\lambda^R, \lambda^0_{m,n}, \ldots, \lambda^{d-1}_{m,n}), \]

where the operators \( \lambda^j_{m,n} \) are as in Lemma 3.1 and the operator \( \lambda^R_n \) is defined by

\[ \lambda^R_n(f) := (\lambda_{k,s}(f))_{|k|_1 \leq n, s \in Z(k)}. \]

Then the upper bound is already proved in Lemma 3.1. For the lower bound, by definition of Kolmogorov width we derive that

\[ d_{K_{m,n}}(\check{U}_\alpha, L_\infty) \leq \sup_{f \in \check{U}_\alpha} \| f - G_{m,n}(\lambda_{m,n}(f)) \|_\infty. \quad (3.20) \]
Lemma 3.3 For $n, m, d \in \mathbb{N}$ it holds the inequality

$$N_{m,n} \leq 3^{2m+1} \binom{m+d}{d-1} 2^{m+1} \left( \frac{m+d}{d-1} \right) + 2^{n+2} \left( \frac{n+d}{d-1} \right).$$

Proof. From (3.16), (3.7), and (3.8) we have that

$$N_{m,n} \leq \dim F^d(n) + \sum_{j=0}^{d-1} \left( 3^{2m+1} \binom{m+d-j-1}{d-j-1} 2^{m+1} \left( \frac{m+d-j-1}{d-j-1} \right) + 2^{n+1} \binom{n+j}{j} \right)$$

$$\leq \sum_{\ell=0}^{n} 2^{\ell} \binom{\ell + d - 1}{d - 1} + 3^{2m+1} \binom{m+d-1}{d-1} \sum_{j=0}^{d-1} \left( \frac{m+d-j}{d-j} \right) + 2^{n+1} \sum_{j=0}^{d-1} \binom{n+j}{j}$$

$$\leq 2^{n+1} \binom{n+d-1}{d-1} + 3^{2m+1} \binom{m+d-1}{d-1} 2^{n+1} \binom{n+d}{d-1}$$

where in the third estimate we have used $\sum_{j=k}^{\ell} (\frac{j}{k}) = (\frac{\ell+1}{k+1}).$ \hfill \Box

We now are able to explicitly construct such a method $Q_N(f) = G_N^*((\lambda_N^*(f)))$ with mappings $\lambda_N^*$ and $G_N^*$ of the form (1.7), satisfying the upper and lower estimates of approximation error (1.8)–(1.9). Recall that $F^d(m)$ is the finite-dimensional subspace in $\hat{C}(U^d)$ of the form (3.6) and that $R_m(f) \in F^d(m)$ for $f \in \hat{C}(U^d)$, and $\dim F^d(m) = \sum_{\ell=0}^{d} 2^{\ell}(\frac{d-1}{d-1})$.

Theorem 3.4 Let $\alpha \in (0,1], \ d \in \mathbb{N}$. Then for every

$$N \geq N(d) := 3^{2^{d+2}(d-1)} 2^{d+3} \left( \frac{2d+1}{d-1} \right), \ N \in \mathbb{N}, \quad (3.21)$$

we can explicitly determine a number $m^*(N) \leq \log N + \log \log N + 1, \ m^*(N) \in \mathbb{N},$ and explicitly construct maps

$$\lambda_N^* : \hat{U}_\infty^{\alpha,d} \rightarrow \mathbb{R}^N \quad \text{and} \quad G_N^* : \mathbb{R}^N \rightarrow F^d(\lambda^*(N))$$

so that $N \leq M(N) := \dim F^d(m^*(N))$,

$$\frac{(4d-1)!}{2^d(4d-4)^{d-1}} N \log N \log \log N)^{1-d} \leq M(N) \leq \frac{(12d^3-1)!}{(d-1)!} N \log N \log \log N)^{1-d} \quad (3.22)$$

and

$$\sup_{f \in \mathcal{U}_\infty^{\alpha,d}} \| f - G_N^* (\lambda_N^*(f)) \|_\infty \leq C_\alpha \left( \frac{K^{d-1}}{(d-1)!} \right)^{2\alpha+1} \frac{(\log N)^{(d-1)(\alpha+1)}}{(N \log N)^{\alpha}} (\log \log N)^{(d-1)\alpha}, \quad (3.23)$$

where $K := (4^\alpha 6/(2^\alpha - 1))^{1/(2\alpha+1)}, \ C_\alpha := 2^\alpha + 2/(2^\alpha - 1)$. Moreover, if $\alpha < 1$,

$$\sup_{f \in \mathcal{U}_\infty^{\alpha,d}} \| f - G_N^* (\lambda_N^*(f)) \|_\infty \geq d_{M(N)}(\hat{U}_\infty^{\alpha,d}, L_\infty) \geq C_{d,\alpha} \frac{(\log N)^{(d-1)(\alpha+1)}}{(N \log N)^{\alpha}} (\log \log N)^{(d-1)\alpha}. \quad (3.24)$$

Proof. We prove the case $d \geq 2$. The case $d = 1$ is carried out similarly. Fix a number $N \in \mathbb{N}$ satisfying the condition (3.21). We define

$$m^*(N) := n(N) + m(N),$$
where \( n = n(N) \) and \( m = m(N) \) are chosen so that
\[
2^{n+2} \binom{n + d}{d - 1} \leq \frac{N}{2} < 2^{n+3} \binom{n + d + 1}{d - 1}
\]  
and
\[
3^{2^{m+1} \binom{m+d-1}{d-1}} 2^{m+1} \binom{m + d}{d - 1} \leq \frac{N}{2} < 3^{2^{m+2} \binom{m+d}{d-1}} 2^{m+2} \binom{m + d + 1}{d - 1}.
\]

From this choice of \( m, n \) and Lemma 3.3 we can see that \( N_{m,n} \leq N \). This allows us to define \( \lambda_N := \lambda_{m,n} \) and \( G_N^\alpha \) as an extension of \( G_{m,n} \) from \( \mathbb{R}^{N_{m,n}} \) to \( \mathbb{R}^N \) with the chosen \( m, n \), where
\[
\lambda_{m,n} : U_{\infty}^{\alpha,d} \to \mathbb{R}^{N_{m,n}}
\]

is as in Corollary 3.2 and
\[
G_{m,n} : \mathbb{R}^{N_{m,n}} \to \mathcal{F}^d(m + n + 1)
\]
as in (3.17)–(3.18).

Let us first prove the dimension-dependent upper estimate (3.23). The choice of \( m, n \) and the assumption (3.21) implies \( n \geq m \geq d + 1 \). With \( n \geq d + 1 \) we have the estimate
\[
2^d N^{-1} \frac{n^{d-1}}{(d-1)^{d-1}} \leq 2^{-n} < 2^4 N^{-1} \binom{n + d + 1}{d - 1} < 2^4 N^{-1} \frac{(2n)^{d-1}}{(d-1)!} \quad \text{and} \quad n \leq \log N \leq 4dn.
\]  

With \( m \geq d + 1 \) we deduce
\[
3^{2^{m+1} \binom{m+d-1}{d-1}} 2^{m+1} \binom{m + d}{d - 1} \leq \frac{N}{2} < 3^{2^{m+2} \binom{m+d+1}{d-1}} 2^{m+2} \binom{m + d + 1}{d - 1} \leq \frac{1}{2} 4^{2^{m+2} \binom{m+d}{d-1}},
\]

where we have used \( 3^t \leq \frac{1}{4} 4^t \) for \( t \geq 16 \). Hence,
\[
2^{m+1} \binom{m + d - 1}{d - 1} \log 3 \leq \log N < 2^{m+4} \binom{m + d + 1}{d - 1}
\]
which implies
\[
2 \log 3 (\log N)^{-1} \frac{m^{d-1}}{(d-1)^{d-1}} \leq 2^{-m} < 2^4 (\log N)^{-1} \frac{(2m)^{d-1}}{(d-1)!} \quad \text{and} \quad m \leq \log \log N \leq 4dm.
\]  

Consequently, we obtain
\[
2^{-\alpha(m+n)} \binom{m + n + d}{d - 1} \leq 2^{-\alpha(m+n)} \frac{(3n)^{d-1}}{(d-1)!}
\]
\[
\leq \left( 2^4 N^{-1} \frac{(2 \log N)^{d-1}}{(d-1)!} \right)^{\alpha} \left( 2^4 (\log N)^{-1} \frac{(2 \log N)^{d-1}}{(d-1)!} \right)^{\alpha} \frac{(3 \log N)^{d-1}}{(d-1)!}
\]
\[
\leq \frac{2^{8\alpha (4^\alpha 3)^{d-1}} (\log N)^{(d-1)(\alpha+1)}}{(N \log N)^{\alpha}} \frac{(\log \log N)^{(d-1)\alpha}}{(d-1)!^{2\alpha+1}}.
\]

This together with the upper bound (3.19) proves the upper bound (3.23).
Next, we prove (3.22). With $M(N) = \dim(\mathcal{F}^d(m^*(N)))$, from the choice of $n$ as in (3.25) we obtain

$$N \leq 2^{n+4} \left( \frac{n + d + 1}{d - 1} \right) - 1 \leq \sum_{\ell = 0}^{m+n+1} 2^\ell \left( \frac{\ell + d - 1}{d - 1} \right) = M(N).$$

Moreover, taking account $n \geq m \geq d + 1$ we derive that

$$2^{m+n+1} \left( \frac{n}{d-1} \right)^{d-1} \leq 2^{m+n+1} \left( \frac{m+n+d}{d-1} \right) \leq M(N) \leq 2^{m+n+2} \left( \frac{m+n+d}{d-1} \right) \leq \frac{3^{d-1}}{(d-1)!} 2^{\alpha+m+2} n^{d-1}$$

which from (3.26) and (3.27) implies

$$\frac{(d-1)!^2}{2^{d}(4d-4)^{d-1}} \frac{N \log N}{m^{d-1}} \leq M(N) \leq \frac{3^{d-1}}{(d-1)!} 2^{d-2} \frac{N \log N}{m^{d-1}}$$

and therefore, (3.22).

We finally verify the lower bound (3.24). From the known inequality

$$d_M(\hat{U}^\alpha_{\infty}, L_{\infty}) \gtrsim M^{-\alpha}(\log M)^{(d-1)(\alpha+\frac{1}{2})},$$

(see, e.g., [11, Theorem 4.3.11]) we obtain that

$$d_M(\hat{U}^\alpha_{\infty}, L_{\infty}) \geq C_{\alpha,d}(M(N))^{-\alpha}(\log M)^{(d-1)(\alpha+\frac{1}{2})}$$

$$\geq C_{\alpha,d} \left( \frac{N \log N}{(\log \log N)^{d-1}} \right)^{-\alpha}(\log N)^{(d-1)(\alpha+\frac{1}{2})}$$

$$\geq C_{\alpha,d} \left( \frac{\log N}{N \log N} \right)^{(d-1)(\alpha+\frac{1}{2})}(\log \log N)^{(d-1)\alpha}.$$

Now provided with $M(N) = K_{m(N),n(N)}$ the lower bound (3.24) follows from (3.20).

From the left inequality in (1.4) and Theorem 3.4 we deduce the following upper and lower bounds for $d_{N,M(N)}(\hat{U}^\alpha_{\infty}, L_{\infty})$.

**Corollary 3.5** Let $\alpha \in (0,1)$, $d \in \mathbb{N}$ and $N \geq 4$. With $M(N) = \lfloor N(\log N)(\log \log N)^{-(d-1)} \rfloor$ we have

$$\frac{(\log N)^{(d-1)(\alpha+\frac{1}{2})}}{(N \log N)^{\alpha}\log \log N} \lesssim d_{N,M(N)}(\hat{U}^\alpha_{\infty}, L_{\infty}) \lesssim \frac{(\log N)^{(d-1)(\alpha+1)}}{(N \log N)^{\alpha}}(\log \log N)^{(d-1)\alpha}.$$

In the case when $d = 2$ we get the right asymptotic order of $d_{N,M(N)}(\hat{U}^\alpha_{\infty}, L_{\infty}(I^2))$ as in the following theorem.

**Theorem 3.6** Let $\alpha \in (0,1)$. With $M(N) := \lfloor N(\log N)(\log \log N)^{-(d-1)} \rfloor$ we have

$$d_{N,M(N)}(\hat{U}^\alpha_{\infty}, L_{\infty}(I^2)) \asymp \sup_{f \in \mathcal{U}^\alpha_{\infty}} \| f - G_N^*(\lambda_N^*(f)) \|_{\infty} \asymp N^{-\alpha} \log N(\log \log N)^{\alpha}.$$  \hspace{1cm} (3.28)

**Proof.** From (3.23) we immediately get the upper bound in (3.28):

$$d_{N,M(N)}(\hat{U}^\alpha_{\infty}, L_{\infty}(I^2)) \lesssim \sup_{f \in \mathcal{U}^\alpha_{\infty}} \| f - G_N^*(\lambda_N^*(f)) \|_{\infty} \lesssim N^{-\alpha} \log N(\log \log N)^{\alpha}.$$
To prove the lower bound we use the known asymptotic order of the Kolmogorov $M$-widths for $\alpha \in (0, 1)$
\[
d_M(\hat{U}_\infty^{\alpha, 2}, L_\infty(\mathbb{I}^2)) \asymp M^{-\alpha}(\log M)^{\alpha + 1},
\]
see, e.g., [11, Theorem 4.3.14] and references there. Hence, with $n = n(N)$ and $m = m(N)$ defined as in the proof of Theorem 3.4 for $d = 2$ we derive the lower bound:
\[
\sup_{f \in \hat{U}_\infty^{\alpha, 2}} \|f - G_N^*(\Lambda_N^*(f))\|_\infty \geq d_{N,M}(\hat{U}_\infty^{\alpha, 2}, L_\infty(\mathbb{I}^2)) \geq d_M(\hat{U}_\infty^{\alpha, 2}, L_\infty(\mathbb{I}^2))
\[
\geq C_\alpha N^{-\alpha}(\log M(N))^{\alpha + 1} \geq C_\alpha N^{-\alpha} \log N(\log \log N)^\alpha.
\]

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