Darboux transformation and multi-soliton solutions of discrete sine-Gordon equation

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Abstract

We study a discrete Darboux transformation and construct the multi-soliton solutions in terms of ratio of determinants for integrable discrete sine-Gordon equation. We also calculate explicit expressions of single, double, triple, quad soliton solutions as well as single and double breather solutions of discrete sine-Gordon equation. Dynamical features of discrete kinks and breathers have also been illustrated.

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1 Introduction

The integrable difference-difference, difference-differential equations have attracted considerable attention in recent years. An integrable discrete equation emerges as the superposition formula for Bäcklund transformations of its associated counterpart in continuous regime. Most of the physical systems are described by differential equations which reflect smoothness in the natural processes and we face this situation during investigation of macroscopic phenomena. On the other hand there exist a large number of physical systems that are inherently discrete in nature and are well described by difference equations rather than differential equations. Mostly quantum mechanical systems are governed by difference equations which under the continuum limit (if we take step size infinitesimally small) could be reduced to associated differential equations. Therefore, the discrete systems are more elementary than their continuous counterparts. The discrete systems have rich mathematical structure, actually the difference equations are nonlocal before taking continuum limit. The nonlocality makes such systems more complicated and difficult to explore integrability of such systems. However, most of the established mathematical techniques to explore interesting properties of continuous integrable equations are no more applicable for the discrete systems. For this reason it is necessary to develop classical methods and discover new concepts to explore hidden mathematical structures.

There has been a growing interest in the study of discrete integrable systems. Some well known examples of discrete integrable systems are Toda lattice system, nonlinear Schrödinger equation (also known as Ablowitz and Ladik system), Korteweg-deVries equation (KdV), sine-Gordon, Volterra lattice system, etc. The sine-Gordon (SG) equation has been widely used in different areas of physical and mathematical sciences (for example see [1]-[3] and references therein). First time SG equation was appeared in theory of surfaces in differential geometry [4]. SG equation describes the oscillations of coupled pendulum [5], propagation of magnetic-flux on Josephson array [6], dynamics of crystal dislocation [7], also explain the dynamics of DNA (deoxyribonucleic acid) double helix molecule [8]. SG equation has attracted a great deal of attention in recent decades.

SG equation is an integrable nonlinear equation and was solved by the mean of inverse scattering transform (IST) method [9]. The SG equation is defined as

$$\alpha_{xt} = \gamma \sin \alpha, \quad \alpha = \alpha(x, t),$$

(1)

where $\gamma$ is a real constant and $\alpha$ be a continuous scalar field. The partial derivative are denoted by subscripts. The discrete integrable systems have multiple applications in different fields from pure mathematics to applied sciences. The discrete integrable equations also admit interesting properties such as exactly solvable by IST method, existence of conservation laws, admit multi-soliton solutions and so on [10]-[19].

A semi-discrete sine-Gordon (sd-SG) equation is given by [18]

$$\frac{d}{dt} (\alpha_{n+1} - \alpha_n) = 4\gamma \sin \frac{1}{2} (\alpha_{n+1} + \alpha_n).$$

(2)

SG equation (both continuous and discrete) has a wide range applications in different fields such as mathematics, physics and life sciences. In particular SG equation has been widely used
in the study of nonlinear dynamics of DNA (deoxyribonucleic acid) double helix molecule. In [8], Yomosa has studied the dynamics DNA double helix molecule under continuum limit and reduced the associated problem into SG equation, however, the original governing equation was discrete. So it is worthwhile to study the discrete SG equation. In an earlier work [20], we investigated different solutions of a sd-SG equation (2). In this paper, we would like to explore the dynamics of multi-soliton and multi-breather solutions of discrete sine-Gordon (dSG) equation. The solutions obtained in this article have not reported so far and each expression reduces to the known solutions of semi-discrete and continuous SG equations under continuum limits.

The article is organized as follow. Section 2 deals with a dSG equation and associated discrete linear system. In section 3, we apply Darboux transformation and obtain a determinant representation of multi-soliton solutions. In Section 4, we obtain explicit expressions of soliton solutions in zero background. We also plot single, double, triple and quad soliton solutions as well as we present the dynamics of single and double breather solutions. Last section contains concluding remarks.

2 Integrable discrete sine-Gordon equation

An integrable dSG equation is given as

\[
\sin \left( \frac{\alpha_{n+1,m+1} - \alpha_{n+1,m} - \alpha_{n,m+1} + \alpha_{n,m}}{4} \right) = \gamma \sin \left( \frac{\alpha_{n+1,m+1} + \alpha_{n+1,m} + \alpha_{n,m+1} + \alpha_{n,m}}{4} \right),
\]

where \( \alpha_{n,m} \) real discrete scalar field [15, 18]. Under the continuum limits dSG equation (3) reduces to sd-SG equation (2) and classical SG equation (1). In order to obtain sd-SG equation (2) from (3), we define

\[
\alpha_{i,j} = \alpha_i (t + T_j), \quad t = T m, \quad \text{and} \quad \gamma \to T \gamma.
\]  

(4)

Keeping \( t \) fixed and applying the limit as \( T \to 0 \) and \( m \to \infty \), we get sd-SG equation

\[
\frac{d}{dt} (\alpha_{n+1} - \alpha_n) = 4 \gamma \sin \frac{1}{2} (\alpha_{n+1} + \alpha_n).
\]  

(5)

Similarly, if we define

\[
\alpha_i = \alpha (x + \mathcal{X} i), \quad x = \mathcal{X} n, \quad \text{and} \quad \gamma \to \mathcal{X} \gamma.
\]  

(6)

If we keep \( x \) fixed and apply the limit as \( \mathcal{X} \to 0 \) and \( n \to \infty \), equation (5) reduces to the classical SG equation [11].

\[\text{If we define} \]

\[
\alpha_{i,j} = \alpha (x + \mathcal{X} i, t + T j), \quad x = \mathcal{X} n, \quad t = T m \quad \text{with} \quad \gamma \to \mathcal{X} T \gamma.
\]  

(7)

By keeping \( x, t \) fixed and applying limit as \( \mathcal{X} \to 0, T \to 0 \) and \( n \to \infty, m \to \infty \), equation (3) yields the classical SG equation [11].
Equation (3) may be expressed as integrability condition of following linear difference-difference equations for an auxiliary function $\chi_{n,m}(\lambda)$ [18]:

\begin{align}
\mathcal{N}_n \chi_{n,m} &\equiv \chi_{n+1,m} = P_{n,m} \chi_{n,m} = (\lambda T + U_{n,m}) \chi_{n,m}, \\
\mathcal{M}_n \chi_{n,m} &\equiv \chi_{n,m+1} = Q_{n,m} \chi_{n,m} = (I + \lambda^{-1} V_{n,m}) \chi_{n,m},
\end{align}

where $\mathcal{N}$ and $\mathcal{M}$ are shift operators in $n$ and $m$ respectively. The coefficient matrices are given by

\begin{align}
T &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & U_{n,m} &= \begin{pmatrix} e^{-i(\alpha_{n+1,m} - \alpha_{n,m})/2} & 0 \\ 0 & e^{i(\alpha_{n+1,m} - \alpha_{n,m})/2} \end{pmatrix}, \\
I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & V_{n,m} &= \gamma \begin{pmatrix} 0 & e^{i(\alpha_{n+1,m} + \alpha_{n,m})/2} \\ e^{i(\alpha_{n+1,m} + \alpha_{n,m})/2} & 0 \end{pmatrix},
\end{align}

where $\chi_{n,m} = (X_{n,m} \ Y_{n,m})^T$ and $i = \sqrt{-1}$. The system of linear difference-difference equations (8)-(9) is known as discrete linear system [15] and the associated integrability condition $(\mathcal{M} \mathcal{N} \chi_{n,m} = \mathcal{N} \mathcal{M} \chi_{n,m})$ gives a discrete zero curvature condition (that is, $P_{n,m+1}Q_{n,m} = Q_{n+1,m}P_{n,m}$). Again under continuum limits linear system (8)-(9) reduces into linear eigenvalue problem for sd-SG equation (2) and continuous SG equation (1).

### 3 Darboux transformation

The Darboux transformation is the most powerful method to generate an infinite series of solutions of linear equations (differential as well as difference equations) from a known solution [21]. The one-fold Darboux transformation for the system (8)-(9) can be defined as

\begin{align}
X_{n,m}[1] &= \lambda Y_{n,m} - \Gamma_{n,m}^{(1)} X_{n,m}, \\
Y_{n,m}[1] &= \lambda X_{n,m} - \Omega_{n,m}^{(1)} Y_{n,m},
\end{align}

the unknown coefficients $\Gamma_{n,m}^{(1)}$ and $\Omega_{n,m}^{(1)}$ may be obtained from the following conditions

\begin{align}
X_{n,m}[1]|_{\lambda=\lambda_1} = X_{n,m}[1]|_{\lambda=\lambda_1} = 0, \\
Y_{n,m}[1]|_{\lambda=\lambda_1} = Y_{n,m}[1]|_{\lambda=\lambda_1} = 0.
\end{align}

The above conditions allow us to write one-fold transformation (11)-(12) as

\begin{align}
X_{n,m}[1] &= \lambda Y_{n,m} - \lambda_1 Y_{n,m}[1]|_{\lambda=\lambda_1} X_{n,m} = \frac{\det \begin{pmatrix} \lambda Y_{n,m} & X_{n,m}^{(1)} \\ \lambda_1 Y_{n,m}^{(1)} & X_{n,m}^{(1)} \end{pmatrix}}{X_{n,m}^{(1)}}, \\
Y_{n,m}[1] &= \lambda X_{n,m} - \lambda_1 X_{n,m}[1]|_{\lambda=\lambda_1} Y_{n,m} = \frac{\det \begin{pmatrix} \lambda X_{n,m} & Y_{n,m}^{(1)} \\ \lambda_1 X_{n,m}^{(1)} & Y_{n,m}^{(1)} \end{pmatrix}}{Y_{n,m}^{(1)}},
\end{align}
here $X_{n,m}^{(1)}$ and $Y_{n,m}^{(1)}$ denote the particular solutions to the linear system (8)-(9). The linear system of difference-difference equations (8)-(9) is covariant under the action of Darboux transformation (15)-(16), that is,

\[
\begin{align*}
X_{n+1,m}[1] &= e^{-i(\alpha_{n+1,m}[1]-\alpha_{n,m}[1])/2} X_{n,m}[1] + \lambda Y_{n,m}[1], \\
Y_{n+1,m}[1] &= \lambda X_{n,m}[1] + e^{i(\alpha_{n+1,m}[1]-\alpha_{n,m}[1])/2} Y_{n,m}[1], \\
X_{n,m+1}[1] &= X_{n,m}[1] + \lambda^{-1} \gamma e^{-i(\alpha_{n,m+1}[1]+\alpha_{n,m}[1])/2} Y_{n,m}[1], \\
Y_{n,m+1}[1] &= \lambda^{-1} \gamma e^{i(\alpha_{n,m+1}[1]+\alpha_{n,m}[1])/2} X_{n,m}[1] + Y_{n,m}[1].
\end{align*}
\]

(17)

(18)

By substitution of scalar functions $X_{n,m}[1]$ and $Y_{n,m}[1]$ from (15)-(16) in (17)-(18), we obtain

\[
e^{-i(\alpha_{n+1,m}[1]-\alpha_{n,m}[1])/2} = e^{-i(\alpha_{n+1,m}-\alpha_{n,m})/2} \left( \frac{Y_{n+1,m}^{(1)}}{X_{n+1,m}^{(1)}} \right) \left( \frac{X_{n,m}^{(1)}}{Y_{n,m}^{(1)}} \right),
\]

(19)

which implies,

\[
\alpha_{n,m}[1] = \alpha_{n,m} + 2\lambda \ln \left( \frac{X_{n,m}^{(1)}}{Y_{n,m}^{(1)}} \right).
\]

(20)

The two-fold Darboux transformation is defined as

\[
\begin{align*}
X_{n,m}[2] &\equiv \lambda Y_{n,m}[1] - \Gamma_{n,m}^{(2)}[1] X_{n,m}[1] = \lambda^2 X_{n,m} - f_{n,m}^1 \lambda Y_{n,m} - f_{n,m}^0 X_{n,m}, \\
Y_{n,m}[2] &\equiv \lambda X_{n,m}[1] - \Omega_{n,m}^{(2)}[1] Y_{n,m}[1] = \lambda^2 Y_{n,m} - g_{n,m}^1 \lambda X_{n,m} - g_{n,m}^0 Y_{n,m}.
\end{align*}
\]

(21)

(22)

with $f_{n,m}^1 = \Omega_{n,m}^{(1)} + \Gamma_{n,m}^{(2)}[1]$, $f_{n,m}^0 = -\Gamma_{n,m}^{(2)}[1] \Gamma_{n,m}^{(1)}$, $g_{n,m}^1 = \Gamma_{n,m}^{(1)} + \Omega_{n,m}^{(2)}[1]$, $g_{n,m}^0 = -\Omega_{n,m}^{(2)}[1] \Omega_{n,m}^{(1)}$. The unknown coefficients $f_{n,m}^0, f_{n,m}^1$ and $g_{n,m}^0, g_{n,m}^1$ can be determined from the following conditions

\[
\begin{align*}
X_{n,m}[2]|_{\lambda=\lambda_k, X_{n,m}=X_{n,m}^{(k)}, Y_{n,m}=Y_{n,m}^{(k)}} &= 0, \\
Y_{n,m}[2]|_{\lambda=\lambda_k, X_{n,m}=X_{n,m}^{(k)}, Y_{n,m}=Y_{n,m}^{(k)}} &= 0,
\end{align*}
\]

(23)

(24)

for $k = 1, 2$. The above conditions reduce to following systems of linear equations

\[
\begin{align*}
\begin{pmatrix}
X_{n,m}^{(1)} \\
X_{n,m}^{(2)}
\end{pmatrix} &\begin{pmatrix}
\lambda_1 Y_{n,m}^{(1)} \\
\lambda_2 Y_{n,m}^{(2)}
\end{pmatrix} = \begin{pmatrix}
f_{n,m}^0 \\
f_{n,m}^1
\end{pmatrix}, \\
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)}
\end{pmatrix} &\begin{pmatrix}
\lambda_1 X_{n,m}^{(1)} \\
\lambda_2 X_{n,m}^{(2)}
\end{pmatrix} = \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1
\end{pmatrix}.
\end{align*}
\]

(25)

(26)
On substitution of unknown coefficients the two-fold transformation (21)-(22) reduces to

\[
X_{n,m}[2] = \frac{\det \left( \begin{array}{ccc} \lambda^2 X_{n,m} & \lambda Y_{n,m} & X_{n,m} \\ \lambda^2 X_{n,m} & \lambda Y_{n,m} & Y_{n,m} \\ \lambda Y_{n,m} & \lambda X_{n,m} & Y_{n,m} \end{array} \right)}{\det \left( \begin{array}{ccc} \lambda Y_{n,m} & X_{n,m} \\ \lambda Y_{n,m} & Y_{n,m} \\ \lambda X_{n,m} & Y_{n,m} \end{array} \right)},
\]

(27)

\[
Y_{n,m}[2] = \frac{\det \left( \begin{array}{ccc} \lambda^2 Y_{n,m} & \lambda X_{n,m} \\ \lambda^2 Y_{n,m} & \lambda X_{n,m} \\ \lambda X_{n,m} & \lambda Y_{n,m} \end{array} \right)}{\det \left( \begin{array}{ccc} \lambda X_{n,m} & Y_{n,m} \\ \lambda X_{n,m} & Y_{n,m} \\ \lambda Y_{n,m} & Y_{n,m} \end{array} \right)},
\]

(28)

The two-fold transformed solutions given by (27)-(28) also satisfy the linear system of difference-difference equations (8)-(9)

\[
X_{n+1,m}[2] = e^{-t(\alpha_{n+1,m}[2]-\alpha_{n,m}[2])/2} X_{n,m}[2] + \lambda Y_{n,m}[2],
\]

(29)

\[
Y_{n+1,m}[2] = \lambda X_{n,m}[2] + e^{t(\alpha_{n+1,m}[2]-\alpha_{n,m}[2])/2} Y_{n,m}[2],
\]

\[
X_{n,m+1}[2] = X_{n,m}[2] + \lambda^{-1} e^{-t(\alpha_{n+1,m}[2]+\alpha_{n,m}[2])/2} X_{n,m}[2] + Y_{n,m}[2],
\]

(30)

Using equations (27)-(28) in equations (29)-(30), we obtain

\[
e^{-t(\alpha_{n+1,m}[2]-\alpha_{n,m}[2])/2} = e^{-t(\alpha_{n+1,m}-\alpha_{n,m})/2} \left( \frac{f_{n+1,m}}{f_{n,m}} \right),
\]

(31)

which yields,

\[
\alpha_{n,m}[2] = \alpha_{n,m} + 2t \ln \frac{\det \left( \begin{array}{cc} \lambda Y_{n,m} & X_{n,m} \\ \lambda Y_{n,m} & Y_{n,m} \end{array} \right)}{\det \left( \begin{array}{cc} \lambda X_{n,m} & Y_{n,m} \\ \lambda X_{n,m} & Y_{n,m} \end{array} \right)}.
\]

(32)

Similarly, three-fold Darboux transformation is defined as

\[
X_{n,m}[3] = \lambda Y_{n,m}[2] - \Gamma_{n,m}[2] X_{n,m}[2] = \lambda^3 Y_{n,m} - \gamma^2 X_{n,m} - f_{n,m}^2 \lambda Y_{n,m} - f_{n,m}^1 X_{n,m},
\]

(33)

\[
Y_{n,m}[2] = \lambda X_{n,m}[2] - \Omega_{n,m}[2] Y_{n,m}[2] = \lambda^3 X_{n,m} - \gamma^2 Y_{n,m} - g_{n,m}^2 \lambda X_{n,m} - g_{n,m}^1 Y_{n,m}.
\]

(34)

The unknown coefficients \(f_{n,m}^0, f_{n,m}^1, f_{n,m}^2\) and \(g_{n,m}^0, g_{n,m}^1, g_{n,m}^2\) can be determined from the following conditions

\[
X_{n,m}[3]|_{\lambda=k_1, X_{n,m}=X_{n,m}[k], Y_{n,m}=Y_{n,m}[k]} = 0,
\]

(35)

\[
Y_{n,m}[3]|_{\lambda=k_1, X_{n,m}=X_{n,m}[k], Y_{n,m}=Y_{n,m}[k]} = 0,
\]

(36)
for \( k = 1, 2, 3 \). The above conditions can also be written as

\[
\begin{pmatrix}
X_{n,m}^{(1)} & \lambda_1 Y_{n,m}^{(1)} & \lambda_2 X_{n,m}^{(1)} \\
X_{n,m}^{(2)} & \lambda_1 Y_{n,m}^{(2)} & \lambda_2 X_{n,m}^{(2)} \\
X_{n,m}^{(3)} & \lambda_1 Y_{n,m}^{(3)} & \lambda_2 X_{n,m}^{(3)}
\end{pmatrix}
\begin{pmatrix}
f_0^{(1)} \\
f_1^{(1)} \\
f_2^{(1)}
\end{pmatrix} = 
\begin{pmatrix}
\lambda_1 Y_{n,m}^{(1)} \\
\lambda_2 Y_{n,m}^{(2)} \\
\lambda_3 Y_{n,m}^{(3)}
\end{pmatrix},
\]

(37)

After substitution of unknown coefficients the expression of three-fold transformation (33)-(34) can also be expressed as ratio of determinants

\[
X_{n,m}[3] = \frac{\det \begin{pmatrix}
\lambda^3 Y_{n,m} & \lambda^2 X_{n,m} & \lambda Y_{n,m} & X_{n,m} \\
\lambda_1 Y_{n,m}^{(1)} & \lambda_2 Y_{n,m}^{(2)} & \lambda_3 Y_{n,m}^{(3)} & Y_{n,m} \\
\lambda_2 Y_{n,m}^{(1)} & \lambda_3 Y_{n,m}^{(2)} & \lambda Y_{n,m} & X_{n,m} \\
\lambda_3 Y_{n,m}^{(1)} & \lambda Y_{n,m} & X_{n,m} & Y_{n,m}
\end{pmatrix}}{\det \begin{pmatrix}
\lambda_1 Y_{n,m}^{(1)} & \lambda_2 Y_{n,m}^{(2)} & \lambda_3 Y_{n,m}^{(3)} & Y_{n,m} \\
\lambda_2 Y_{n,m}^{(1)} & \lambda_3 Y_{n,m}^{(2)} & \lambda Y_{n,m} & X_{n,m} \\
\lambda_3 Y_{n,m}^{(1)} & \lambda Y_{n,m} & X_{n,m} & Y_{n,m}
\end{pmatrix}},
\]

(39)

\[
Y_{n,m}[3] = \frac{\det \begin{pmatrix}
\lambda^3 X_{n,m} & \lambda^2 Y_{n,m} & \lambda X_{n,m} & Y_{n,m} \\
\lambda_1 Y_{n,m}^{(1)} & \lambda_2 Y_{n,m}^{(2)} & \lambda_3 Y_{n,m}^{(3)} & Y_{n,m} \\
\lambda_2 Y_{n,m}^{(1)} & \lambda_3 Y_{n,m}^{(2)} & \lambda Y_{n,m} & X_{n,m} \\
\lambda_3 Y_{n,m}^{(1)} & \lambda Y_{n,m} & X_{n,m} & Y_{n,m}
\end{pmatrix}}{\det \begin{pmatrix}
\lambda_1 Y_{n,m}^{(1)} & \lambda_2 Y_{n,m}^{(2)} & \lambda_3 Y_{n,m}^{(3)} & Y_{n,m} \\
\lambda_2 Y_{n,m}^{(1)} & \lambda_3 Y_{n,m}^{(2)} & \lambda Y_{n,m} & X_{n,m} \\
\lambda_3 Y_{n,m}^{(1)} & \lambda Y_{n,m} & X_{n,m} & Y_{n,m}
\end{pmatrix}}.
\]

(40)

Three-fold transformed solutions \( X_{n,m}[3] \) and \( Y_{n,m}[3] \) satisfy the linear system of difference-difference equations (8)-(9) such as

\[
X_{n+1,m}[3] = e^{-\gamma(\alpha_{n+1,m}[3]-\alpha_{n,m}[3])/2} X_{n,m}[3] + \lambda Y_{n,m}[3],
\]

\[
Y_{n+1,m}[3] = \lambda X_{n,m}[3] + e^{\gamma(\alpha_{n+1,m}[3]-\alpha_{n,m}[3])/2} Y_{n,m}[3],
\]

(41)

\[
X_{n,m+1}[3] = X_{n,m}[3] + \lambda^{-1} e^{-\gamma(\alpha_{n,m+1}[3]+\alpha_{n,m}[3])/2} Y_{n,m}[3],
\]

\[
Y_{n,m+1}[3] = \lambda^{-1} e^{\gamma(\alpha_{n,m+1}[3]+\alpha_{n,m}[3])/2} X_{n,m}[3] + Y_{n,m}[3].
\]

(42)

Using equations (39)-(40) in (41)-(42), we obtain

\[
e^{-\gamma(\alpha_{n+1,m}[3]-\alpha_{n,m}[3])/2} = e^{-\gamma(\alpha_{n+1,m}[3]-\alpha_{n,m}[3]/2} \left( \frac{f_0^{(n+1,m)}}{f_0^{n,m}} \right),
\]

(43)
The above equation gives

\[
\alpha_{n,m}[3] = \alpha_{n,m} + 2 t \ln \left| \begin{pmatrix}
\lambda_1^2 Y_{n,m}^{(1)} & \lambda_1 X_{n,m}^{(1)} & Y_{n,m}^{(1)} \\
\lambda_2^2 Y_{n,m}^{(2)} & \lambda_2 X_{n,m}^{(2)} & Y_{n,m}^{(2)} \\
\lambda_3^2 Y_{n,m}^{(3)} & \lambda_3 X_{n,m}^{(3)} & Y_{n,m}^{(3)} \\
\end{pmatrix}
\right|
\]

\[
\det \left( \begin{pmatrix}
\lambda_1^2 X_{n,m}^{(1)} & \lambda_1 Y_{n,m}^{(1)} & X_{n,m}^{(1)} \\
\lambda_2^2 X_{n,m}^{(2)} & \lambda_2 Y_{n,m}^{(2)} & X_{n,m}^{(2)} \\
\lambda_3^2 X_{n,m}^{(3)} & \lambda_3 Y_{n,m}^{(3)} & X_{n,m}^{(3)} \\
\end{pmatrix}
\right). \quad (44)
\]

In what follows, we would like to generalize our results for \(N\)-times iteration of Darboux transformation. For this purpose, first take \(N = 2K\), the Darboux transformation can be factorize as

\[
X_{n,m}[2K] = \lambda^{2K} X_{n,m} - f_{n,m}^{2K-1} \lambda^{2K-1} Y_{n,m} - \cdots - f_{n,m} Y_{n,m} - f_{n,m}^0 X_{n,m}, \quad (45)
\]

\[
Y_{n,m}[2K] = \lambda^{2K} Y_{n,m} - g_{n,m}^{2K-1} \lambda^{2K-1} X_{n,m} - \cdots - g_{n,m} Y_{n,m} - g_{n,m}^0 Y_{n,m}, \quad (46)
\]

where the unknown coefficients can be computed from the following conditions

\[
X_{n,m}[2K] |_{\lambda=\lambda_k, X_{n,m}^{(k)}=X_{n,m}^{(k)}, Y_{n,m}^{(k)}=Y_{n,m}^{(k)}} = 0, \quad (47)
\]

\[
Y_{n,m}[2K] |_{\lambda=\lambda_k, X_{n,m}^{(k)}=X_{n,m}^{(k)}, Y_{n,m}^{(k)}=Y_{n,m}^{(k)}} = 0, \quad (48)
\]

for \(k = 1, 2, \ldots, 2K\). The above conditions reduce to following systems of linear equations

\[
\begin{pmatrix}
X_{n,m}^{(1)} \\
X_{n,m}^{(2)} \\
\vdots \\
X_{n,m}^{(2K-1)} \\
X_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
X_{n,m}^{(1)} \\
X_{n,m}^{(2)} \\
\vdots \\
X_{n,m}^{(2K-1)} \\
X_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
f_{n,m}^0 \\
f_{n,m}^1 \\
\vdots \\
f_{n,m}^{2K-1} \\
f_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K-1)} \\
Y_{n,m}^{(2K)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K} \\
g_{n,m}^{2K}
\end{pmatrix}
\]
Using the values of unknown coefficients the 2\(K\)-fold transformation becomes (51)-(53) as

\[
X_{n,m}[2K] = \begin{vmatrix} 
\lambda^{2K}X_{n,m} & \lambda^{2K-1}Y_{n,m} & \ldots & \lambda Y_{n,m} & X_{n,m} \\
\lambda_1^{2K}X_{n,m} & \lambda_1^{2K-1}Y_{n,m} & \ldots & \lambda_1 Y_{n,m} & X_{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_2^{2K-1}X_{n,m}^{(2K-1)} & \lambda_2^{2K-1}Y_{n,m}^{(2K-1)} & \ldots & \lambda_2 Y_{n,m}^{(2K-1)} & X_{n,m}^{(2K-1)} \\
\lambda_2^{2K}X_{n,m}^{(2K)} & \lambda_2^{2K-1}Y_{n,m}^{(2K)} & \ldots & \lambda_2 Y_{n,m}^{(2K)} & X_{n,m}^{(2K)} \\
\end{vmatrix},
\]

(51)

\[
Y_{n,m}[2K] = \begin{vmatrix} 
\lambda^{2K}Y_{n,m} & \lambda^{2K-1}X_{n,m} & \ldots & \lambda X_{n,m} & Y_{n,m} \\
\lambda_1^{2K}Y_{n,m} & \lambda_1^{2K-1}X_{n,m} & \ldots & \lambda_1 X_{n,m} & Y_{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_2^{2K-1}Y_{n,m}^{(2K-1)} & \lambda_2^{2K-1}X_{n,m}^{(2K-1)} & \ldots & \lambda_2 X_{n,m}^{(2K-1)} & Y_{n,m}^{(2K-1)} \\
\lambda_2^{2K}Y_{n,m}^{(2K)} & \lambda_2^{2K-1}X_{n,m}^{(2K)} & \ldots & \lambda_2 X_{n,m}^{(2K)} & Y_{n,m}^{(2K)} \\
\end{vmatrix},
\]

(52)

\[
2K\text{-fold transformed dynamical variable is given by}
\]

\[
\alpha_{n,m}[2K] = \alpha_{n,m} + 2t \ln \begin{vmatrix} 
\lambda^{2K-1}X_{n,m} & \lambda^{2K-2}Y_{n,m} & \ldots & \lambda X_{n,m} & Y_{n,m} \\
\lambda_1^{2K-1}X_{n,m} & \lambda_1^{2K-2}Y_{n,m} & \ldots & \lambda_1 X_{n,m} & Y_{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_2^{2K-1}X_{n,m}^{(2K-1)} & \lambda_2^{2K-2}Y_{n,m}^{(2K-1)} & \ldots & \lambda_2 X_{n,m}^{(2K-1)} & Y_{n,m}^{(2K-1)} \\
\lambda_2^{2K}X_{n,m}^{(2K)} & \lambda_2^{2K-2}Y_{n,m}^{(2K)} & \ldots & \lambda_2 X_{n,m}^{(2K)} & Y_{n,m}^{(2K)} \\
\end{vmatrix},
\]

(53)
Similarly, for $N = 2K + 1$, we have
\begin{align}
X_{n,m}[2K + 1] &= \lambda^{2K+1}Y_{n,m} - f_{n,m}^{2K}X_{n,m} - \cdots - f_{n,m}^1\lambda Y_{n,m} - f_{n,m}^0X_{n,m}, \\
Y_{n,m}[2K + 1] &= \lambda^{2K+1}X_{n,m} - g_{n,m}^{2K}Y_{n,m} - \cdots - g_{n,m}^1\lambda X_{n,m} - g_{n,m}^0Y_{n,m},
\end{align}
along with following conditions
\begin{align}
X_{n,m}[2K + 1]|_{\lambda = \lambda_k, X_{n,m} = X^{(k)}_{n,m}, Y_{n,m} = Y^{(k)}_{n,m}} &= 0, \\
Y_{n,m}[2K + 1]|_{\lambda = \lambda_k, X_{n,m} = X^{(k)}_{n,m}, Y_{n,m} = Y^{(k)}_{n,m}} &= 0,
\end{align}
for $k = 1, 2, \ldots, 2K + 1$. The above conditions can also be expressed in matrix notation as
\begin{equation}
\begin{pmatrix}
X_{n,m}^{(1)} \\
X_{n,m}^{(2)} \\
\vdots \\
X_{n,m}^{(2K)} \\
X_{n,m}^{(2K+1)}
\end{pmatrix}
\begin{pmatrix}
\lambda_1Y_{n,m}^{(1)} \\
\lambda_2Y_{n,m}^{(2)} \\
\vdots \\
\lambda_{2K}Y_{n,m}^{(2K)} \\
\lambda_{2K+1}Y_{n,m}^{(2K+1)}
\end{pmatrix}
+ \begin{pmatrix}
\lambda_1^2K+1Y_{n,m}^{(1)} \\
\lambda_2^2K+1Y_{n,m}^{(2)} \\
\vdots \\
\lambda_{2K}^2K+1Y_{n,m}^{(2K)} \\
\lambda_{2K+1}^2K+1Y_{n,m}^{(2K+1)}
\end{pmatrix}
= \begin{pmatrix}
f_{n,m}^0 \\
f_{n,m}^1 \\
\vdots \\
f_{n,m}^{2K-1} \\
f_{n,m}^{2K}
\end{pmatrix},
\end{equation}
\begin{equation}
\begin{pmatrix}
Y_{n,m}^{(1)} \\
Y_{n,m}^{(2)} \\
\vdots \\
Y_{n,m}^{(2K)} \\
Y_{n,m}^{(2K+1)}
\end{pmatrix}
\begin{pmatrix}
\lambda_1X_{n,m}^{(1)} \\
\lambda_2X_{n,m}^{(2)} \\
\vdots \\
\lambda_{2K}X_{n,m}^{(2K)} \\
\lambda_{2K+1}X_{n,m}^{(2K+1)}
\end{pmatrix}
+ \begin{pmatrix}
\lambda_1^2K+1X_{n,m}^{(1)} \\
\lambda_2^2K+1X_{n,m}^{(2)} \\
\vdots \\
\lambda_{2K}^2K+1X_{n,m}^{(2K)} \\
\lambda_{2K+1}^2K+1X_{n,m}^{(2K+1)}
\end{pmatrix}
= \begin{pmatrix}
g_{n,m}^0 \\
g_{n,m}^1 \\
\vdots \\
g_{n,m}^{2K-1} \\
g_{n,m}^{2K}
\end{pmatrix}.
\end{equation}

Using the values of unknown coefficients the $(2K+1)$-fold transformation (54)-(56) can be expressed...
\[
\begin{align*}
X_{n,m}[2K + 1] &= \begin{vmatrix}
\lambda^{2K+1}X_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda Y_{n,m} & X_{n,m} \\
\lambda^{2K+1}Y_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda Y_{n,m} & X_{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda^{2K}X_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda Y_{n,m} & X_{n,m} \\
\lambda^{2K+1}X_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda Y_{n,m} & X_{n,m}
\end{vmatrix},
\tag{60}

Y_{n,m}[2K + 1] &= \begin{vmatrix}
\lambda^{2K+1}Y_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda X_{n,m} & Y_{n,m} \\
\lambda^{2K+1}X_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda X_{n,m} & Y_{n,m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda^{2K}X_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda X_{n,m} & Y_{n,m} \\
\lambda^{2K+1}X_{n,m} & \lambda^{2K}X_{n,m} & \cdots & \lambda X_{n,m} & Y_{n,m}
\end{vmatrix},
\tag{61}
\end{align*}
\]

The \((2K + 1)\)-fold transformed solutions given by (60)–(61) satisfy the linear system of difference-difference equations (8)–(9)

\[
\begin{align*}
X_{n+1,m}[2K + 1] &= e^{-i(\alpha_{n+1,m}[2K+1]-\alpha_{n,m}[2K+1])/2}X_{n,m}[2K+1] + \lambda Y_{n,m}[2K + 1], \\
Y_{n+1,m}[2K + 1] &= \lambda X_{n,m}[2K + 1] + e^{i(\alpha_{n+1,m}[2K+1]-\alpha_{n,m}[2K+1])/2}Y_{n,m}[2K + 1], \\
X_{n,m+1}[2K + 1] &= X_{n,m}[2K + 1] + \lambda^{-1} \gamma e^{-i(\alpha_{n,m+1}[2K+1]+\alpha_{m,n}[2K+1])/2}Y_{n,m}[2K + 1], \\
Y_{n,m+1}[2K + 1] &= \lambda^{-1} \gamma e^{i(\alpha_{n,m+1}[2K+1]+\alpha_{m,n}[2K+1])/2}X_{n,m}[2K + 1] + Y_{n,m}[2K + 1].
\end{align*}
\]
Using equations (61)-(62) in (63)-(64), we obtain

\[
\begin{vmatrix}
\lambda_1^{2K} Y_{n,m}^{(1)} & \lambda_2^{2K} Y_{n,m}^{(2)} & \ldots & \lambda_1^{2K-1} X_{n,m}^{(1)} & \ldots & \lambda_1 X_{n,m}^{(1)} & Y_{n,m}^{(1)} \\
\lambda_2^{2K} Y_{n,m}^{(2)} & \lambda_2^{2K-1} X_{n,m}^{(2)} & \ldots & \lambda_2 X_{n,m}^{(2)} & \ldots & Y_{n,m}^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\lambda_2^{2K} Y_{n,m}^{(2K)} & \lambda_2^{2K-1} X_{n,m}^{(2K)} & \ldots & \lambda_2 X_{n,m}^{(2K)} & \ldots & Y_{n,m}^{(2K)} \\
\lambda_2^{2K} Y_{n,m}^{(2K+1)} & \lambda_2^{2K-1} X_{n,m}^{(2K+1)} & \ldots & \lambda_2 X_{n,m}^{(2K+1)} & \ldots & Y_{n,m}^{(2K+1)} \\
\lambda_2^{2K+1} X_{n,m}^{(2K+1)} & \lambda_2^{2K+1} Y_{n,m}^{(2K+1)} & \ldots & \lambda_2 X_{n,m}^{(2K+1)} & \ldots & Y_{n,m}^{(2K+1)} \\
\end{vmatrix}
\]

\[
\alpha_{n,m} [2K + 1] = \alpha_{n,m} + 2 \ln \det \left( \begin{vmatrix}
\lambda_1^{2K} Y_{n,m}^{(1)} & \lambda_2^{2K} Y_{n,m}^{(2)} & \ldots & \lambda_1^{2K-1} X_{n,m}^{(1)} & \ldots & \lambda_1 X_{n,m}^{(1)} & Y_{n,m}^{(1)} \\
\lambda_2^{2K} Y_{n,m}^{(2)} & \lambda_2^{2K-1} X_{n,m}^{(2)} & \ldots & \lambda_2 X_{n,m}^{(2)} & \ldots & Y_{n,m}^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\lambda_2^{2K} Y_{n,m}^{(2K)} & \lambda_2^{2K-1} X_{n,m}^{(2K)} & \ldots & \lambda_2 X_{n,m}^{(2K)} & \ldots & Y_{n,m}^{(2K)} \\
\lambda_2^{2K} Y_{n,m}^{(2K+1)} & \lambda_2^{2K-1} X_{n,m}^{(2K+1)} & \ldots & \lambda_2 X_{n,m}^{(2K+1)} & \ldots & Y_{n,m}^{(2K+1)} \\
\lambda_2^{2K+1} X_{n,m}^{(2K+1)} & \lambda_2^{2K+1} Y_{n,m}^{(2K+1)} & \ldots & \lambda_2 X_{n,m}^{(2K+1)} & \ldots & Y_{n,m}^{(2K+1)} \\
\end{vmatrix} \right) \]

(64)

In the following section, we shall derive explicit expressions of single, double, triple and quad soliton solutions and illustrate our results for different choices of parameters.

### 4 Explicit solutions and their dynamics

In the zero background, that is \( \alpha_{n,m} = 0 \), the linear system of difference-difference equations (8)-(9) reduces to

\[
\begin{align*}
X_{n+1,m} &= X_{n,m} + \lambda Y_{n,m}, & Y_{n+1,m} &= Y_{n,m} + \lambda X_{n,m}, \\
X_{n,m+1} &= X_{n,m} + \lambda^{-1} \gamma Y_{n,m}, & Y_{n,m+1} &= Y_{n,m} + \lambda^{-1} \gamma X_{n,m}.
\end{align*}
\]

(65)-(66)

The solution to the linear system of difference-difference equations (65)-(66) is given by

\[
\begin{align*}
X_{n,m} &= A (1 + \lambda)^n \left(1 + \frac{\gamma}{\lambda}\right)^m + B (1 - \lambda)^n \left(1 - \frac{\gamma}{\lambda}\right)^m, \\
Y_{n,m} &= A (1 + \lambda)^n \left(1 + \frac{\gamma}{\lambda}\right)^m - B (1 - \lambda)^n \left(1 - \frac{\gamma}{\lambda}\right)^m,
\end{align*}
\]

(67)-(68)

where \( A \) and \( B \) are constants also known as plane wave factors. Here

\[
\begin{align*}
X_{n,m}^{(k)} &= A_k (1 + \lambda_k)^n \left(1 + \frac{\gamma}{\lambda_k}\right)^m + B_k (1 - \lambda_k)^n \left(1 - \frac{\gamma}{\lambda_k}\right)^m, \\
Y_{n,m}^{(k)} &= A_k (1 + \lambda_k)^n \left(1 + \frac{\gamma}{\lambda_k}\right)^m - B_k (1 - \lambda_k)^n \left(1 - \frac{\gamma}{\lambda_k}\right)^m,
\end{align*}
\]

(69)-(70)

denotes particular solution set to difference-difference equations (65)-(66) at \( \lambda = \lambda_k \). In order to obtain an explicit expression of one-soliton substitute \( X_{n,m}^{(1)} \) and \( Y_{n,m}^{(1)} \) from (69)-(70) in expression...
we obtain

\[
\alpha_{n,m}[1] = 2 \ln \frac{A_1 (1 + \lambda_1)^n (1 + \frac{\gamma}{\lambda_1})^m - B_1 (1 - \lambda_1)^n (1 - \frac{\gamma}{\lambda_1})^m}{A_1 (1 + \lambda_1)^n (1 + \frac{\gamma}{\lambda_1})^m + B_1 (1 - \lambda_1)^n (1 - \frac{\gamma}{\lambda_1})^m}.
\] (71)

For the choice \( A_1 = 1, B_1 = \iota \), we obtain

\[
\alpha_{n,m}[1] = 2 \ln \frac{X_{n,m}^{(1)*}}{X_{n,m}^{(1)}} = 4 \tan^{-1} \left[ \left( \frac{1 + \lambda_1}{1 - \lambda_1} \right)^n \left( \frac{\lambda_1 + \gamma}{\lambda_1 - \gamma} \right)^m \right],
\] (72)

which is analogous to the expression of one-kink soliton solution obtained in [22] but more general one as it follows continuum limits. The dynamics of one-kink solution is displayed in Figure 1 for \( \lambda_1 = 0.2 \) and \( \gamma = 0.25 \). It is clear from Figure 1, that one-kink soliton (72) has wave span of 2\( \pi \) and inflexion point trace is given by \( n(m) = \frac{\tanh^{-1}(\frac{\gamma}{\lambda_1})}{\tanh^{-1}(-\gamma)} \), whereas the value of \( \alpha_{n,m}[1] \) and slope at inflexion point are \( \pi \) and 4 arctan \( \left( \frac{1+\lambda_1}{1-\lambda_1} \right) \) respectively (For more details on kink dynamics see e.g. [22]-[23]). The one-kink soliton solution (72) may be helpful in describing the nonlinear dynamics of double helix DNA molecule as well as other important physical systems for example propagation of optical soliton in nematic liquid crystals (NLCs).

(a) Propagation of one-kink soliton
(b) Snapshot of one-kink at \( m = 0 \)

Figure 1: Propagation and snapshots of single-soliton solution (71).

Under continuum limit as defined by (1), expression (72) yields

\[
\alpha_n(t)[1] = 4 \tan^{-1} \left[ \left( \frac{1 + \lambda_1}{1 - \lambda_1} \right)^n \exp \left( \frac{2\gamma}{\lambda_1} t \right) \right],
\] (73)

which coincides with expression of one-kink soliton solution for sd-SG equation [20].

Again under continuum limit (6) and \( \lambda_1 \rightarrow X\lambda_1 \), above expression (74) reduces to

\[
\alpha(x,t)[1] = 4 \tan^{-1} \left[ \exp \left( 2\lambda_1 x + \frac{2\gamma}{\lambda_1} t \right) \right],
\] (74)
which represents kink soliton solution for SG equation \( \text{(1)} \).

Now, using (69)-(70) in (32) for \( A_k = 1, B_{2k-1} = B_{2k}^* = \iota \), the explicit form of double-soliton solution can be expressed as

\[
\alpha_{n,m}[2] = 4 \tan^{-1}\left( \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \left( \frac{1 + \lambda_1}{1 - \lambda_1} \right)^n \left( \frac{\lambda_1 + \gamma}{\lambda_1 - \gamma} \right)^m - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \left( \frac{1 + \lambda_2}{1 - \lambda_2} \right)^n \left( \frac{\lambda_2 + \gamma}{\lambda_2 - \gamma} \right)^m \right). \tag{75}
\]

Under continuum limits (7), expression (75) yields the double-kink solution of the continuous SG equation \( \text{(1)} \)

\[
\alpha(x,t)[2] = 4 \tan^{-1}\left( \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right) \frac{\sinh\left( \frac{(\lambda_1 - \lambda_2)(\lambda_1 \lambda_2 x - \gamma t)}{\lambda_1 \lambda_2} \right)}{\cosh\left( \frac{(\lambda_1 + \lambda_2)(\lambda_1 \lambda_2 x + \gamma t)}{\lambda_1 \lambda_2} \right)}. \tag{76}
\]

The dynamics of kink-kink interaction is presented in Figure (2) for \( \lambda_1 = -0.45, \lambda_2 = 0.5 \) and \( \gamma = 0.25 \). It is clear from Figure (2) when two kinks approach each other with certain speed, they tend to repel one another in the vicinity of \( m = 0 \) and then two soliton bounce back with velocities opposite to their initial velocities, as a result kink-kink wave span goes from \(-2\pi\) to \(2\pi\). This is a case of repulsive interaction.

![Figure 2: Propagation and snapshots of double-soliton solution (75)](image)

(a) Propagation of double-kink  
(b) Snapshot of double-kink at \( m = 0 \)

An interaction of kink and anti-kink is depicted in Figure (3) for \( \gamma = 0.25, \lambda_1 = 0.3, \lambda_2 = 0.5 \).
This Figure (3) shows when kink and antikink come close to one another, they begin to accelerate in the neighborhood of \( m = 0 \) and then attempt to annihilate each other which corresponds to the attractive case of interaction. After the interaction they grow and separate as kink and anti-kink and move with same speed and shape.

As \( m \to \pm \infty \) two kinks are well separated and expression (75) for \( \lambda_1 > \lambda_2 \), can be written as

\[
\alpha_{n,m} [2] \approx \frac{4}{\tan^{-1}\left( \frac{1 + \lambda_1}{1 - \lambda_1} \frac{\lambda_1 \gamma}{\lambda_1 - \gamma} \lambda_2 \gamma \right)} \left[ \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \left( 1 + \frac{\lambda_1}{1 - \lambda_1} \frac{\lambda_1 + \gamma}{\lambda_1 - \gamma} \right)^m - \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \left( 1 + \frac{\lambda_2}{1 - \lambda_2} \frac{\lambda_2 + \gamma}{\lambda_2 - \gamma} \right)^m \right], \tag{77}
\]

the asymptotic response yields combination of kink and anti-kink with phase shift that is due to their interaction at \( m = 0 \)

\[
\alpha_{n,m} [2] \approx 4 \tan^{-1} \left[ \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \left( 1 + \frac{\lambda_1}{1 - \lambda_1} \frac{\lambda_1 + \gamma}{\lambda_1 - \gamma} \right)^m - \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \left( 1 + \frac{\lambda_2}{1 - \lambda_2} \frac{\lambda_2 + \gamma}{\lambda_2 - \gamma} \right)^m \right]. \tag{78}
\]

It is important to mention that the dSG equation (3) also admit another type of soliton solution known as doublet or breather, that in general bound solution of soliton-antisoliton pair. We obtain one-breather solution if we take \( \lambda_2 = \lambda_1^* \) in (75)

\[
\alpha_{n,m} [2] = 4 \tanh^{-1} \left( \frac{\Re(\lambda_1)}{\Im(\lambda_1)} \right) \left( 1 + \frac{\lambda_1}{1 - \lambda_1} \frac{\lambda_1 + \gamma}{\lambda_1 - \gamma} \right)^m \left( 1 - \frac{\mu_{n,m}}{1 + \nu_{n,m}} \right), \tag{79}
\]

with

\[
\mu_{n,m} = \left( \frac{1 - 2 i \Im(\lambda_1) - |\lambda_1|^2}{1 + 2 i \Im(\lambda_1) - |\lambda_1|^2} \right)^n \left( \frac{\gamma^2 - 2 i \gamma \Im(\lambda_1) - |\lambda_1|^2}{\gamma^2 + 2 i \gamma \Im(\lambda_1) - |\lambda_1|^2} \right)^m,
\]

\[
\nu_{n,m} = \left( \frac{1 + 2 \Re(\lambda_1) + |\lambda_1|^2}{1 - 2 \Re(\lambda_1) + |\lambda_1|^2} \right)^n \left( \frac{\gamma^2 + 2 \gamma \Re(\lambda_1) + |\lambda_1|^2}{\gamma^2 - 2 \gamma \Re(\lambda_1) + |\lambda_1|^2} \right)^m.
\]
Evolution of one-breather solution (79) for $\lambda_1 = 0.5 + 0.5\iota$, is shown in Figure (4).

![Propagating single-breather solution](image1)

(a) Propagation of single-breather solution

![Snapshot of single-breather solution at $m = 0$](image2)

(b) Snapshot of single-breather solution at $m = 0$

Figure 4: Propagation of single-breather solution

Similarly, the triple-soliton solution can be obtained from (44). Figure (5) represents interaction of two-kink and anti-kink solutions for $\gamma = 0.25$, $A_1 = A_2 = A_3 = 0.1$, $2B_1 = B_2/2 = 2B_3 = \iota$ and $\lambda_1 = 0.45, \lambda_2 = 0.5, \lambda_3 = 0.55$.

![Propagating interaction of two-kink and one-anti kink soliton](image3)

(a) Propagation of interaction of two-kink and one-anti kink soliton

![Snapshot of interaction of two-kink and one-anti kink soliton $m = 0$](image4)

(b) Snapshot of interaction of two-kink and one-anti kink soliton $m = 0$

Figure 5: Propagation and snapshot of triple-soliton solution (44).
Finally, the quad-soliton solution is given as

$$\alpha_{n,m}^{[4]} = 2\iota \ln \det \begin{pmatrix}
\lambda_1 X_{n,m}^{(1)} & \lambda_1 Y_{n,m}^{(1)} & \lambda_1 X_{n,m}^{(1)} & Y_{n,m}^{(1)} \\
\lambda_2 X_{n,m}^{(2)} & \lambda_2 Y_{n,m}^{(2)} & \lambda_2 X_{n,m}^{(2)} & Y_{n,m}^{(2)} \\
\lambda_3 X_{n,m}^{(3)} & \lambda_3 Y_{n,m}^{(3)} & \lambda_3 X_{n,m}^{(3)} & Y_{n,m}^{(3)} \\
\lambda_4 X_{n,m}^{(4)} & \lambda_4 Y_{n,m}^{(4)} & \lambda_4 X_{n,m}^{(4)} & Y_{n,m}^{(4)}
\end{pmatrix}$$

Figure (6) represents dynamics of quad-soliton solutions (80). Figure (6a) shows the interaction of double-kink and double-anti kink solutions for $\gamma = 0.25$, $A_k = 1$, $B_{2k-1} = B_{2k}^{\ast} = \iota$ and $\lambda_1 = 0.4$, $\lambda_2 = 0.5$, $\lambda_3 = 0.6$, $\lambda_4 = 0.7$ and Figure (6b) displays the interaction of double-breather solutions for $\lambda_1 = \lambda_2^{\ast} = 0.7 + 0.7\iota$, $\lambda_3 = \lambda_4^{\ast} = 0.2 - 0.2\iota$. 
5 Conclusions

We have constructed multi-soliton solutions of dSG equation by employing Darboux transformation and expressed our results as ratio of ordinary determinants. We obtained explicit expressions of single, double, triple and quad soliton solutions. We also obtained single and double breather soliton solutions. Finally, we illustrated different interactions of higher order soliton solutions and breather solutions for dSG equation. Under continuum limit results obtained in this paper reduce
to multi-soliton solutions for the classical sine-Gordon equation. It would be interesting to explore
dynamics of higher order degenerate solutions of discrete, semi-discrete sine-Gordon equation. We
shall study these solutions in future.

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