Global Residues and Two-Loop Hepta-Cuts

Mads Søgaard
Niels Bohr International Academy and Discovery Center, Niels Bohr Institute,
University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

E-mail: madss@nbi.dk

Abstract: We examine maximal unitarity in the nonplanar case and derive remarkably compact analytic expressions for coefficients of master integrals with two-loop crossed box topology in massless four-point amplitudes in any gauge theory, thereby providing additional steps towards automated computation of the full amplitude. The coefficients are obtained by assembling residues extracted through integration on linear combinations of higher-dimensional tori encircling global poles of the loop integrand. We recover all salient features of two-loop maximal unitarity, such as the existence of unique projectors for each master integral. Several explicit calculations are provided. We also establish exact equivalence of our results and master integral coefficients recently obtained via integrand-level reduction in any renormalizable gauge theory.
1 Introduction

The initiation of the Large Hadron Collider (LHC) programme at CERN has spawned a new exciting era in experimental high energy physics and generated an acute demand for precision cross section predictions for scattering of elementary particles. Being a hadron collider, LHC experiments are contaminated with a large Quantum Chromodynamics (QCD) background. Discovery of signals of possibly new physics therefore requires a quantitative understanding of all relevant Standard Model processes which necessarily must be subtracted from the observed data. One-loop scattering amplitudes provide Next-to-Leading Order (NLO) estimates, while Next-to-Next-to-Leading order (NNLO) corrections from two loops are needed for a reliable analysis of theoretical uncertainty. Although NNLO calculations form the upcoming frontier, two-loop amplitudes are also relevant already at NLO for processes such as production of diphotons and pairs of electroweak gauge bosons by gluon fusion for which one-loop is the leading order.

Scattering amplitudes have traditionally been computed perturbatively by translating Feynman diagrams into precise mathematical expressions using Feynman rules. This approach gives an invaluable view and interpretation of interaction of subatomic particles, but it inevitably suffers from explosive growth of complexity with multiplicity and order in perturbation theory. Indeed, even in simple problems such as two-by-two gluon scattering an inpracticable computational bottleneck is quickly reached. The origin of this problem is that intermediate states are virtual particles and a vast amount of redundancy is needed for compensation. Catalyzed by Wittens formulation of perturbative gauge theory as a string theory in twistor space [1], new efficient on-shell methods for computing tree-level amplitudes using only physical information rather than off-shell Feynman diagrams have emerged and striking simplicity has been revealed. Most important are the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [2, 3] which remarkably construct all gauge theory and also gravity trees by means of just the Cauchy residue theorem and complex kinematics in three-point amplitudes whose form is actually completely fixed by very general arguments such as scaling properties under little group transformations.

Powerful techniques for computation of one-loop amplitudes exploiting unitarity of the S-matrix were developed from the Cutkosky rules in the early 1990s by Bern, Dixon and Kosower [5, 6] and subsequently studied extensively [7–26]. Unitarity implies that the discontinuity of the transition matrix can expressed in terms of simpler quantities, e.g. trees are recycled for loops. The unitarity method in its original form allows reconstruction of amplitudes from two-particle unitarity cuts that put internal propagators on their mass-shell and constrain parameters in an appropriate ansatz. It has proven extremely useful in a widespread of both theoretical and phenomenological applications in the last two decades, in particular when a proper integral basis of the amplitude is not available. The immediate disadvantage is the need for performing algebra at intermediate stages because many contributions share the same cuts. Generalized unitarity in turn probes the analytic structure of a loop integrand much more deeply by imposing several simultaneous on-shell conditions, thereby rendering selection of single integrals in a basis possible. For instance quadruple cuts isolate a unique box integral [9], whereas other clever projections single out
triangles and bubbles separately \cite{20}, leading to beautifully compact expressions whose simplicity is by no means expected from a Feynman diagram perspective. This method is now fully systematized at one loop with a variety of software libraries of numerical implementations that are vital to phenomenology at the LHC \cite{27–35}.

Using current state-of-the-art unitarity techniques one has been able to compute four-particle processes in massless QCD \cite{36–42}. It is of obvious interest to extend procedures for direct extraction of integral coefficients by generalized unitarity beyond one loop. Indeed, it would be of enormous theoretical and practical value to have closed form expressions for integral coefficients for any two-loop topology such as for instance nonplanar crossed double-triangle, planar penta-bubble and planar sunset. Octa-cuts and hepta-cuts of two-loop amplitudes in maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory were first studied in \cite{43,44}. The major obstacle is however that a complete unitarity compatible integral basis for two-loop amplitudes is not yet known. On the contrary to one-loop integrals whose numerators are trivial, integral basis elements at two loop contain complicated tensors. This problem was recently addressed and steps towards a solution in that direction were taken in \cite{45,46}. Although rather technically complicated, a very interesting method for obtaining planar double box contributions to two-loop amplitudes in any gauge theory using maximal unitarity (i.e. all propagators are placed on-shell) cuts has been reported in \cite{47} and subsequently enhanced and applied in \cite{48–51}. The motivation of our paper is to use this framework to analyze nonplanar amplitude contributions. The continued hope raised by advances along these lines is that scattering amplitudes will generate more fundamental insight in hidden structures underlying quantum field theories.

The above considerations and the remaining part of this paper resemble a perhaps slightly exaggerated, nevertheless quite true, quote by Julian Schwinger: one of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.

1.1 Conventions and Notation

In this paper we consider color-ordered scattering amplitudes at two-loops in gauge theory with $SU(N_c)$ symmetry group in which case decoupling of color and kinematical structures is also important like at tree-level and one-loop. The color-dressed two-loop amplitude with four external particles transforming in the adjoint representation of the gauge group admits color decomposition in terms of single and double traces,

$$A^{2\text{-loop}}_4 = \sum_{\sigma \in S_4/Z_4^1} N_c \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}}) \text{Tr}(T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) A^{(2)}_{4;1,3}(\sigma(1), \sigma(2); \sigma(3), \sigma(4))$$

$$+ \sum_{\sigma \in S_4/Z_4^1} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}) \left[ N^2_c A^{(2),LC}_{4;1,1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) + A^{(2),SC}_{4;1,1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \right], \quad (1.1)$$

where $T^a$ for $a = 1, \ldots, N_c^2 - 1$ are generators of $SU(N_C)$ in the fundamental representation. The color-stripped amplitudes on the right hand side all have expansions as
linear combinations of integrals such as the planar double box, nonplanar crossed box and triangle-pentabox (see fig. 1) with legs permuted appropriately. The complete map is excluded here for brevity, but available in [52]. In this form, color-ordered generalized unitarity cuts can be applied.

Figure 1. The pentabox-triangle and planar double box topologies appearing in the color-decomposition of the two-loop four-point amplitude.

Partial amplitudes are naturally built from antisymmetric Lorentz invariant holomorphic and antiholomorphic inner products of commuting spinors $\lambda_i^\alpha$ and $\bar{\lambda}_j^\dot{\alpha}$ whose components are homogeneous coordinates on complex projective space $\mathbb{CP}^1$. Physically, the spinors are solutions of definite chirality to the massless Dirac equation. We define angle and square brackets by

$$
\langle ij \rangle = -\langle ji \rangle \equiv \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta, \quad [ij] = -[ji] \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}_i^{\dot{\alpha}} \bar{\lambda}_j^{\dot{\beta}}
$$

and identify the corresponding four-dimensional null-momentum $k_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}$. Frequently used momentum invariants can then be written

$$
s_{ij} = \langle ij \rangle [ji] = 2k_i \cdot k_j \tag{1.3}
$$

with Mandelstam variables $s \equiv s_{12}$, $u \equiv s_{13}$ and $t \equiv s_{14}$ such that $s + t + u = 0$. Momenta are by convention outgoing and summed using the notation $K_{i_1 \cdots i_n} = k_{i_1} + \cdots + k_{i_n}$. Our expressions also involve parity-odd contractions between Levi-Civita symbols and momenta in the form

$$
\varepsilon(1, 2, 3, 4) = \sum_{\sigma \in Z_4} (\text{sgn } \sigma) k_{1,\sigma(1)} k_{2,\sigma(2)} k_{3,\sigma(3)} k_{4,\sigma(4)}
$$

$$
\quad = \frac{i}{4}(\langle 12 \rangle [23][34][41] - [12]\langle 23 \rangle [34][41]) \tag{1.4}
$$

2 Generalized Unitarity and Integral Bases

The existence of a finite basis of linearly independent scalar integrals for one-loop gauge theory amplitudes has in recent years established a solid foundation for the success of
the modern formulation of the unitarity method. Using Passarino-Veltmann reduction an
$n$-point amplitude can be written as
\[ A_{\text{1-loop}}^n = \sum_{\text{boxes}} c \Box I_{\Box} + \sum_{\text{triangles}} c \triangle I_{\triangle} + \sum_{\text{bubbles}} c \circ I_{\circ} + \sum_{\text{tadpoles}} c \cdot \circ I_{\cdot \circ} + \text{rational terms} , \]
where scalar bubble, triangle and box integrals are known in dimensional regularization
explicitly and tadpoles are present only in case of massive internal propagators. In a nutshell,
computation of one-loop amplitudes is thus reduced to finding the rational coefficients in
the integral basis. At one-loop, direct extraction procedures exist for all topologies [20]
and even for the rational terms [21].

In this section we describe an approach to maximal unitarity introduced in [43, 44]
and recently systematized for general planar double boxes in [47, 49] using unitarity compatible
integral bases and complex analysis in higher dimensions.

2.1 Multivariate Residue Theorem

The extension of the one-dimensional version of the Cauchy residue theorem to several
complex variables has proven advantageous in order to understand computations of
generalized unitarity cuts of multiloop amplitudes. We therefore now introduce the concept
of global poles and the global residue theorem, and refer the reader to [64] for further
information.

Let the meromorphic function \( \varphi : \mathbb{C}^2 \to \mathbb{C} \) be given by
\[ \varphi(z_1, z_2) = \frac{h(z_1, z_2)}{(az_1 + bz_2 + c)(ez_1 + fz_2 + g)} , \]
and assume regularity of \( h(z_1, z_2) \) where the denominators vanish simultaneously, that is
\( (az_1 + bz_2 + c) = 0 \) and \( (ez_1 + fz_2 + g) = 0 \). Such a point \( (z_1^*, z_2^*) \in \mathbb{C}^2 \) is called a global
pole for \( \varphi \). Then we can consider the multidimensional contour integral of \( \varphi \) on an
infinitesimal two-torus \( T^2_\varepsilon \simeq S^1 \times S^1 \) encircling that global pole. Moreover, we can
shift the global pole to origo by applying the change of variables \( w_1 = az_1 + bz_2 + c \) and
\( w_2 = ez_1 + fz_2 + g \),
\[ \oint_{T^2_\varepsilon(z_1, z_2)} \frac{h(z_1, z_2)dz_1dz_2}{(az_1 + bz_2 + c)(ez_1 + fz_2 + g)} = \oint_{T^2_\varepsilon(0, 0)} \frac{dw_1dw_2 h(z_1(w), z_2(w))}{w_1w_2 \det \left( \frac{\partial (w_1, w_2)}{\partial (z_1, z_2)} \right)} , \]
whence in analogy with the one-dimensional case it is very natural to define the global
residue of \( \varphi \) at \( (z_1^*, z_2^*) \) by
\[ \text{Res}_{(z_1, z_2) = (z_1^*, z_2^*)} \varphi = \frac{h(z_1^*, z_2^*)}{\det \left( \frac{\partial (w_1, w_2)}{\partial (z_1, z_2)} \right)_{(z_1^*, z_2^*)}} . \]
The generalization to meromorphic functions \( \varphi : \mathbb{C}^n \to \mathbb{C} \) of \( n \) complex variables and with
\( m \geq n \) factors in the denominator,
\[ \varphi(z_1, \ldots, z_n) = \frac{h(z_1, \ldots, z_n)}{\prod_{i=1}^{m} p_i(z_1, \ldots, z_n)} , \]
is straightforward. Indeed, we solve \( p_1(z^*_1, \ldots, z^*_n) = \cdots = p_n(z^*_1, \ldots, z^*_n) = 0 \) to determine the global pole \( z^* = (z^*_1, \ldots, z^*_n) \in \mathbb{C}^n \). By assumption \( h \) is regular there and the global residue of \( \varphi \) thus reads

\[
\oint_{T^n} d^m z \frac{h(z_1, \ldots, z_n)}{\prod_{i=1}^m p_i(z_1, \ldots, z_n)} = \frac{h(z^*_1, \ldots, z^*_n)}{\prod_{i \neq (1, \ldots, n)} p_i(z^*_1, \ldots, z^*_n) \det \left( \frac{\partial p_i(z_1, \ldots, z_n)}{\partial z_j} \right)_{(z^*_1, \ldots, z^*_n)}}.
\]

In this way, actually \( \binom{m}{n} \) global residues arise. From now on we will only encounter situations where \( n = m \) so that the integral localizes to a single residue.

Strictly speaking, in order for the global residue to become independent of the orientation of the parametrization of the torus, the integration variables should really be wedged together. However, this point is irrelevant for our purposes as long as the orientation is kept consistent throughout the entire calculation.

2.2 Method of Maximal Cuts

Let us return to the application to generalized unitarity and focus our attention on extraction of the coefficient in front of the four-point one-loop scalar box integral (fig. 2)

\[
\mathcal{I}_{\square}(s, t) \equiv \int_{\mathbb{R}^D} d^D \ell \frac{1}{(2\pi)^D \ell^2(\ell - k_2)^2(\ell - K_{23})^2(\ell + k_1)^2},
\]

with external momenta \( k_1, \ldots, k_4 \). For each such quartet of momenta the solution set \( S \) for the quadruple cut equations formed from the zero locus of the four inverse propagators is a pair of complex conjugates \( S_1 \) and \( S_2 \),

\[
S = \{ \ell \in \mathbb{C}^4 \mid \ell^2 = 0, \ (\ell - k_2)^2 = 0, \ (\ell - K_{23})^2 = 0, \ (\ell + k_1)^2 = 0 \} = S_1 \cup S_2.
\]

The kinematical structure of the solutions is easy to understand since they correspond to the two possible configurations of nonconsecutive holomorphically and antiholomorphically collinear three-vertices in a box.

We now adopt the ideas of \([9]\), later clarified in \([47]\), and define the quadruple cut of a general box integral by shifting integration region from \( \mathbb{R}^4 \) to a surface embedded in \( \mathbb{C}^4 \) formed by a linear combination of the two four-tori encircling the leading singularities \( S_1 \) and \( S_2 \),

\[
\int_{\mathbb{R}^D} d^D \ell \frac{P(\ell)}{\prod_{k=1}^4 p_k^2(\ell)} \rightarrow \sum_{i=1,2} \Lambda_i \int_{T_i} \frac{d^D \ell}{(2\pi)^D \prod_{k=1}^4 p_k^2(\ell)} \frac{P(\ell)}{\prod_{k=1}^4 p_k^2(\ell)}.
\]

Notice that we always strip all expected occurrences of factors of \( 2\pi i \). The contour weights or winding numbers \( \Lambda_1 \) and \( \Lambda_2 \) are a priori unknown, but consistency constraints from integral reduction fix their relative normalization to unity. Applying this recipe to both

\[\text{Technically speaking, identification by complex conjugation presumes reality of momenta.}\]
sides of the master integral equation (2.1) we obtain the augmented quadruple cut

\[
c_{\Box} \sum_{i=1,2} \oint_{T_i} \frac{d^4 \alpha}{(2\pi)^4} \left( \frac{\det \frac{\partial \ell^\mu}{\partial \alpha_j}}{p_{k}^2(\alpha)} \right)^4 \prod_{k=1}^4 \frac{1}{p_{k}^2(\alpha)} A_{(k)}^{\text{tree}}(\alpha), \tag{2.10}
\]

where we absorbed the contour weights into the integrals and also put a tilde on the tree amplitudes to indicate that they are really off-shell until the contour integral is localized onto the cut solutions. Linearity of the loop momentum in \(\alpha_1, \ldots, \alpha_4\) implies that the Jacobian is constant and therefore it can be ignored. We can also cancel common factors on both sides and discard the Jacobian arising from actually evaluating the contour integrals in parameter space and obtain the well-known Britto-Cachazo-Feng formula \[22\]

\[
c_{\Box} = \frac{1}{2} \sum_{i=1,2} \sum_{\substack{\text{helicities} \ \text{particles} \ k=1}}^4 \prod_{k=1}^4 A_{(k)}^{\text{tree}}|_{S_i}. \tag{2.11}
\]

Strikingly simple, it singles out uniquely any one-loop gauge theory scalar box integral coefficient in terms of just a product of four tree amplitudes evaluated at complex momenta arising by promoting all internal lines to on-shell values.

This approach generalizes to two loops and presumably beyond using the following principle \[47\]. We define the maximal cut by continuation of real slice \(L\)-loop integrals into \((\mathbb{C}^4)^{\otimes L}\) by choosing contours that encircle the true global poles of the integrand in such a way that any integral identity in \((\mathbb{R}^D)^{\otimes L}\) is preserved. If necessary, impose auxiliary cut constraints by localizing remaining integrations onto composite leading singularities or poles in tensor integrands to obtain linear algebraic equations that uniquely determine the master integral coefficients from tree-level data.
Consider in brevity the application of this prescription to the primitive amplitude for the four-point planar double box with massless kinematics. The Feynman integral for the diagram shown in fig. 3 reads

\[ I^P[1] \equiv \int_{\mathbb{R}^D} \frac{d^D \ell_1}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{d^D \ell_2}{(2\pi)^D} \frac{1}{\ell_1^2(\ell_1-k_1)^2(\ell_1-K_{12})^2\ell_2^2(\ell_2-k_4)^2(\ell_2-K_{34})^2(\ell_1+\ell_2)^2} . \]  

(2.12)

In general, the integral may have an arbitrary numerator and in that case we write \( I^P[P(\ell_1, \ell_2)] \). Integrals of this type were calculated analytically in \([67, 68]\).

It is now easy to write down and solve the seven on-shell constraints in parameter space using the same parametrization of the loop momenta as for the nonplanar double box below (3.2). Each solution has a free complex parameter \( z \) that parametrizes a Riemann surface of genus 0. Direct evaluation reveals that the localization of the double box scalar integral onto this remaining Riemann sphere yields the same Jacobian for all six solutions, with the very simple result

\[ I^P[1] \big|_{S_i} = -\frac{1}{16s^3_{12}} \oint \frac{dz}{z(z+\chi)} . \]  

(2.13)

We impose an eighth cut condition and freeze the remaining integral completely by choosing linear combinations of contours encircling the Jacobian poles \( z \in \{0, -\chi\} \) and additional tensor poles at \( z = -\chi - 1 \) in integrals with nontrivial numerators. In total we naively find fourteen candidate global poles.

By virtue of integration-by-parts identities among renormalizable Feynman integrals, the double box primitive amplitude may be expanded in an integral basis whose elements are, for instance, \( I^P[1] \) and \( I^P[(\ell_1 \cdot k_4)] \),

\[ A^{2\text{-loop}}_{\text{dbox}} = c_1 I^P[1] + c_2 I^P[(\ell_1 \cdot k_4)] + \cdots . \]  

(2.14)
Integrals with subleading topologies are hidden in the ellipses. All seven-propagator integration-by-parts identities are available in appendix C. The augmented hepta-cut of the master integral equation may then be derived from residue identities between on-shell branches and identification of eight true global poles along the lines of [49]. In particular, the double box primitive amplitude factorizes onto a product of six tree-level amplitudes arranged in six distinct configurations such that no external legs are neither holomorphically nor antiholomorphically collinear for generic momenta.

Requiring that all reduction identities continue to hold after imposing the hepta-cut constraints leads to unique projectors for the two master integral coefficients, up to an irrelevant overall normalization. Following the enumeration of on-shell solutions in [47], one possible minimal representation is the residue expansion

\[ c_1 = +\frac{1}{4} \sum_{i=1,3} \text{Res}_{z=-\chi} \frac{1}{z + \chi} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} \]

\[ +\frac{1}{4} \sum_{i=5,6} \text{Res}_{z=-\chi} \frac{1}{z + \chi} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} \]

\[ -\frac{\chi}{4(1+\chi)} \sum_{i=5,6} \text{Res}_{z=-\chi} \frac{1}{z + \chi} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} , \]  

(2.15)

\[ c_2 = - \frac{1}{2s_{12} \chi} \sum_{i=1,3} \text{Res}_{z=-\chi} \frac{1}{z + \chi} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} \]

\[ +\frac{1}{s_{12} \chi} \sum_{i=5,6} \text{Res}_{z=0} \frac{1}{z} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} \]

\[ -\frac{1}{2s_{12} \chi} \sum_{i=5,6} \text{Res}_{z=-\chi} \frac{1}{z + \chi} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} \]

\[ +\frac{3}{2s_{12}(1+\chi)} \sum_{i=5,6} \text{Res}_{z=-\chi} \frac{1}{z + \chi} \sum_{\text{particles}} \prod_{j=1}^{6} A_{(j)}^{\text{tree}}(z) |_{S_i} , \]  

(2.16)

in which on-shell branches \( S_2 \) and \( S_4 \) are eliminated.

3 Nonplanar Crossed Box

Conventional wisdom and numerous experiences suggest that nonplanar diagrams in general are more complicated to compute than planar ones. In this section we provide additional evidence in favor of the approach to maximal unitarity described above by revealing surprising simplicity in the nonplanar crossed box. In particular, we establish the augmented hepta-cut and derive beautiful formulae for the master integral coefficients from
unique projectors, highlighting differences and similarities to the planar double box in the process.

The dimensionally regularized Feynman integral for the four-point nonplanar double box with massless kinematics and an arbitrary numerator function \( P(\ell_1, \ell_2) \) inserted is

\[
I_{NP}[P(\ell_1, \ell_2)] \equiv \int_{\mathbb{R}^D} \frac{d^D\ell_1}{(2\pi)^D} \int_{\mathbb{R}^D} \frac{d^D\ell_2}{(2\pi)^D} \frac{P(\ell_1, \ell_2)}{\ell_2^2(\ell_1 + k_1)^2(\ell_2 + k_3)^2(\ell_2 - \ell_1 + k_2)^2} ,
\]

following the conventions outlined in fig. 4. In a slight abuse of terminology it is called a tensor integral even though it has no free indices. Explicit expressions for these integrals are available in [69, 70].

\[\begin{align*}
\ell_1^\mu(\alpha_1, \ldots, \alpha_4) &= \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{s_{12} \alpha_3}{2(14)[42]} (1^- | \gamma^\mu | 2^-) + \frac{s_{12} \alpha_4}{2(24)[41]} (2^- | \gamma^\mu | 1^-) , \\
\ell_2^\mu(\beta_1, \ldots, \beta_4) &= \beta_1 k_3^\mu + \beta_2 k_4^\mu + \frac{s_{12} \beta_3}{2(31)[14]} (3^- | \gamma^\mu | 4^-) + \frac{s_{12} \beta_4}{2(41)[13]} (4^- | \gamma^\mu | 3^-) .
\end{align*}\]

The virtue of this form is maximal simplification of the hepta-cut equations and direct exposure of global residues of the integrand. The Jacobians for the change of variables from momenta to parameters are constant and can therefore be disregarded in the augmented hepta-cut below, but for completeness we note that

\[
J_\alpha = \det_{\mu, i} \frac{\partial \ell_1^\mu}{\partial \alpha_i} = -\frac{is_{12}^2}{4\chi(\chi + 1)} , \quad J_\beta = \det_{\mu, i} \frac{\partial \ell_2^\mu}{\partial \beta_i} = -\frac{is_{12}^2}{4\chi(\chi + 1)} ,
\]

3.1 Parametrization of On-Shell Solutions

In order to study the hepta-cut, we exploit slight calculational foresight and choose convenient normalizations in the parametrization of the two independent loop momenta,

\[\begin{align*}
\ell_1^\mu(\alpha_1, \ldots, \alpha_4) &= \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{s_{12} \alpha_3}{2(14)[42]} (1^- | \gamma^\mu | 2^-) + \frac{s_{12} \alpha_4}{2(24)[41]} (2^- | \gamma^\mu | 1^-) , \\
\ell_2^\mu(\beta_1, \ldots, \beta_4) &= \beta_1 k_3^\mu + \beta_2 k_4^\mu + \frac{s_{12} \beta_3}{2(31)[14]} (3^- | \gamma^\mu | 4^-) + \frac{s_{12} \beta_4}{2(41)[13]} (4^- | \gamma^\mu | 3^-) .
\end{align*}\]
where $\chi$ is a ratio of Mandelstam invariants used throughout this calculation,

$$\chi = \frac{s_{14}}{s_{12}}. \quad (3.5)$$

The on-shell equations are maximally degenerate for the kinematical configuration in consideration and rather straightforward to analyze. The solution set $\mathcal{S}$ is the union of eight irreducible branches $\mathcal{S}_i$, each of which is topologically equivalent to a Riemann sphere,

$$\mathcal{S} = \{(\ell_1, \ell_2) \in (\mathbb{C}^4)^{\otimes 2} \mid \ell_1^2 = 0, (\ell_1 + k_1)^2 = 0, \ell_2^2 = 0, (\ell_2 + k_3)^2 = 0, (\ell_2 - k_4)^2 = 0, (\ell_1 - \ell_2 - k_3)^2 = 0, (\ell_1 - \ell_2 - K_{23})^2 = 0\} = \bigcup_{i=1}^{8} \mathcal{S}_i. \quad (3.6)$$

Let us solve the hepta-cut equations using the parametrization of $\ell_1$ and $\ell_2$. We examine the subset of inverse propagators involving only a single loop momentum on the cut, and obtain

$$\ell_1^2 = s_{12} \left( \alpha_1 \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi + 1)} \right) = 0,$$

$$\ell_2^2 = s_{12} \left( \beta_1 \beta_2 + \frac{\beta_3 \beta_4}{\chi(\chi + 1)} \right) = 0,$$

$$(\ell_1 + k_1)^2 = s_{12} \left( (\alpha_1 + 1) \alpha_2 + \frac{\alpha_3 \alpha_4}{\chi(\chi + 1)} \right) = 0,$$

$$(\ell_2 + k_3)^2 = s_{12} \left( (\beta_1 + 1) \beta_2 + \frac{\beta_3 \beta_4}{\chi(\chi + 1)} \right) = 0,$$

$$(\ell_2 - k_4)^2 = s_{12} \left( \beta_1(\beta_2 - 1) + \frac{\beta_3 \beta_4}{\chi(\chi + 1)} \right) = 0. \quad (3.7)$$

These constraints translate into $\alpha_2 = \beta_1 = \beta_2 = 0$, $\alpha_3 \alpha_4 = 0$ and $\beta_3 \beta_4 = 0$ for generic kinematics, and therefore we have to consider four types of solutions. For completeness, we derive equations for the mixed inverse propagators on the hepta-cut whose form is compatible with any kind of solution,

$$(\ell_1 - \ell_2 - k_3)^2|_{\text{cut}} = s_{12} \left[ \alpha_1(1 + \chi - \beta_3 - \beta_4) + \alpha_3 + \alpha_4 - \frac{1}{\chi} (\alpha_3 \beta_3 + \alpha_4 \beta_4) - \frac{1}{\chi + 1} (\alpha_3 \beta_4 + \alpha_4 \beta_3) \right]_{\text{cut}}, \quad (3.8)$$

$$(\ell_1 - \ell_2 - K_{23})^2|_{\text{cut}} = s_{12} \left[ \alpha_1(\chi - \beta_3 - \beta_4) - \frac{1}{\chi} (\alpha_3 \beta_3 + \alpha_4 \beta_4) - \frac{1}{\chi + 1} (\alpha_3 \beta_4 + \alpha_4 \beta_3) + \alpha_3 + \alpha_4 - \beta_3 - \beta_4 + \chi \right]_{\text{cut}}, \quad (3.9)$$

where the cut subscript means $\xi \to 0$ for $\xi \in \{(\alpha_3, \beta_3), (\alpha_3, \beta_4), (\alpha_4, \beta_3), (\alpha_4, \beta_4)\}$. It is trivial to show that the these hepta-cut equations collapse into two classes; for $\alpha_j = \beta_j = 0$
\[ \alpha_i (1 - \beta_i / \chi) + \alpha_1 (1 - \beta_i + \chi) = 0, \]  
\[ (\beta_i - \chi)(\alpha_i + \chi + \alpha_1 \chi) = 0, \]

and \( i \neq j \),

\[ \alpha_i (1 - \beta_i / \chi) + \alpha_1 (1 - \beta_i + \chi) = 0, \]  
\[ (\beta_i - \chi)(\alpha_i + \chi + \alpha_1 \chi) = 0, \]  
\[ (\beta_i - \chi)(\alpha_i + \chi + \alpha_1 \chi) = 0, \]  
\[ (1 + \alpha_1)(\beta_j - \chi) - \alpha_3 (1 - \beta_j / 1 + \chi) = 0. \]

Each set of equations has again two independent branches, whence upon parametrization of the remaining freedom by the complex variable \( z \in \mathbb{C} \) we arrive at the eight solutions listed in Table 1. The appearance of four pairs of complete conjugates is naturally expected in view of the, for generic momenta, valid distributions of internal helicities in the six three-vertices on the hepta-cut, see appendix A.

### 3.2 Composite Leading Singularities

Let us now apply the hepta-cut to the nonplanar double box primitive amplitude. For each solution to the on-shell equations we have to compute the Jacobian associated with the localization of the integral onto a single Riemann sphere. We will work out the case appropriate to the first solution in detail.

Initially we use all constraints involving only either \( \ell_1 \) or \( \ell_2 \),

\[ J_A = \frac{1}{\mathcal{A}_5 S_{12}} \oint_{C_1(0)} d\alpha_2 \oint_{C_1(0)} d\alpha_4 \frac{1}{\alpha_1\alpha_2 + \frac{\alpha_3\alpha_4}{\chi+1}(\alpha_1+1)\alpha_2 + \frac{\alpha_3\alpha_4}{\chi+1} \times \alpha_2} \]
\[ \left( \oint_{C_1(0)} d\beta_1 \oint_{C_1(0)} d\beta_2 \oint_{C_1(0)} d\beta_4 \frac{1}{\beta_1\beta_2 + \frac{\beta_3\beta_4}{\chi+1}(\beta_1+1)\beta_2 + \frac{\beta_3\beta_4}{\chi+1} \beta_2} \right) \left( \frac{1}{\beta_1(\beta_2-1) + \frac{\beta_3\beta_4}{\chi+1}} \right), \]

### Table 1

The eight solutions to the on-shell equations for the maximal cut of the four-point massless nonplanar double box. Each irreducible branch has topology of a genus-0 sphere.

|   | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) | \( \beta_1 \) | \( \beta_2 \) | \( \beta_3 \) | \( \beta_4 \) |
|---|---|---|---|---|---|---|---|---|
| \( S_1 \) | \( \chi - z \) | 0 | \( \chi(z - \chi - 1) \) | 0 | 0 | 0 | \( z \) | 0 |
| \( S_2 \) | \( \chi - z \) | 0 | 0 | \( \chi(z - \chi - 1) \) | 0 | 0 | 0 | \( z \) |
| \( S_3 \) | 0 | 0 | \( z \) | 0 | 0 | 0 | \( \chi \) | 0 |
| \( S_4 \) | 0 | 0 | 0 | \( z \) | 0 | 0 | 0 | \( \chi \) |
| \( S_5 \) | \( \chi - z \) | 0 | 0 | \( (\chi + 1)(z - \chi) \) | 0 | 0 | 0 | \( z \) |
| \( S_6 \) | \( \chi - z \) | 0 | \( (\chi + 1)(z - \chi) \) | 0 | 0 | 0 | 0 |
| \( S_7 \) | -1 | 0 | 0 | \( z \) | 0 | 0 | 0 | \( 1 + \chi \) |
| \( S_8 \) | -1 | 0 | 0 | \( z \) | 0 | 0 | 0 | \( 1 + \chi \) |
and then combine with integrals containing both loop momenta on this support,

\[
J_B = \frac{1}{s_{12}^2} \int_{C_{\alpha}} d\alpha_1 \int_{C_{\beta}} d\alpha_3 \frac{1}{(\chi - \beta_3)(1 + \alpha_1 + \chi^{-1}\alpha_3)\alpha_1(1 + \chi - \beta_3) + \alpha_3(1 - \chi^{-1}\beta_3)},
\]

(3.13)

where we put \(\mu = \chi - \beta_3\) and \(\lambda = \chi(\beta_3 - \chi - 1)\). The seven contour integrals are evaluated as determinants using the multivariate residue theorem and produce the rather simple forms

\[
J_A^{-1} = s_{12}^5 \det \left( \frac{\beta_3}{\beta_2} \right) \frac{\beta_3}{\beta_1} \left( \frac{\beta_3}{\beta_2} \right) \left( \frac{\beta_3}{\beta_1} \right) = -\frac{s_{12}^5}{s_{12}^5}, \quad (3.14)
\]

\[
J_B^{-1} = s_{12}^2 \det \left( 1 + \frac{\chi - \beta_3}{\chi - \beta_3} \right) \left( 1 - \frac{\beta_3}{\chi} \right) = s_{12}^2 \left( 1 - \frac{\beta_3}{\chi} \right). \quad (3.15)
\]

We include previous effects of change of variables (3.4) and derive the full Jacobian

\[
I_{NP}[1]_{S_1} = -\frac{\chi}{16s_{12}^3} \int \frac{d\beta_3}{\alpha_3\beta_3(\beta_3 - \chi)}, \quad (3.16)
\]

which in the specific parametrization of \(\alpha_3\) and \(\beta_3\) becomes

\[
I_{NP}[1]_{S_1} = -\frac{1}{16s_{12}^3} \int \frac{dz}{z(z - \chi)(z - \chi - 1)}. \quad (3.17)
\]

The remaining seven Jacobians follow completely analogously. We repeated the computations and found only three classes of Jacobians,

\[
I_{NP}[1]_{S_{(3,4)}} = -\frac{1}{16s_{12}^3} \int \frac{dz}{z(z + \chi)}, \quad (3.18)
\]

\[
I_{NP}[1]_{S_{(7,8)}} = -\frac{1}{16s_{12}^3} \int \frac{dz}{z(z - \chi - 1)}, \quad (3.19)
\]

\[
I_{NP}[1]_{S_{(1,2,5,6)}} = -\frac{1}{16s_{12}^3} \int \frac{dz}{z(z - \chi)(z - \chi - 1)}, \quad (3.20)
\]

with composite leading singularities or simply Jacobian poles located at \(z \in \{0, -\chi\}\), \(z \in \{0, \chi + 1\}\) and \(z \in \{0, \chi, \chi + 1\}\) respectively. Encircling one of these global poles effectively imposes an eighth condition in addition to the hepta-cut constraints such that the integral localizes completely to a point in \(\mathbb{C}^4 \times \mathbb{C}^4\).

Notice that the overall normalization of the Jacobians is the same for all cut solutions and hence irrelevant in the augmented hepta-cut. In subsequent sections we will frequently refer to integrands without the common prefactor by \(J_i(z)\).

### 3.3 Augmentation of Global Poles

We realize that the product of six tree amplitudes onto which the amplitude integrand factorizes on the hepta-cut for the present parametrization is a holomorphic function of \(z\)
and therefore has no poles, except at complex infinity. This is in contrast to the maximal cut of the planar double box which develops a pole at a finite value of \( z \). Possible nontrivial contributions from poles at infinity in either of the two loop momenta are however safely ignored because the sum of all residues of a meromorphic function on the Riemann sphere must vanish identically.

Therefore we naively consider \( 4 \times 3 + 4 \times 2 = 20 \) residues originating from composite leading singularities. It turns out that only some of these contributions are in fact independent. Indeed, using several nontrivial relations across the on-shell branches we are able to clear out all redundancy and identify only ten true global residues of which the master integral coefficient may be built. For each relation we assume that \( \xi(\ell_1, \ell_2) \) is holomorphic on the two Jacobian poles in question, but otherwise arbitrary. In our calculations, \( \xi \) is of course really just a shorthand for the intermediate state sum of tree amplitudes on the hepta-cut. We list all intersections of the Riemann spheres below and refer to fig. 5 for a graphical depiction.

\[
\begin{align*}
\text{Res}_{z=0} J_1(z) \xi(\ell_1 \ell_2) \big|_{S_1} &= \text{Res}_{z=0} J_6(z) \xi(\ell_1 \ell_2) \big|_{S_6} \\
\text{Res}_{z=0} J_2(z) \xi(\ell_1 \ell_2) \big|_{S_2} &= \text{Res}_{z=0} J_5(z) \xi(\ell_1 \ell_2) \big|_{S_5} \\
\text{Res}_{z=\chi} J_1(z) \xi(\ell_1 \ell_2) \big|_{S_1} &= \text{Res}_{z=-\chi} J_3(z) \xi(\ell_1 \ell_2) \big|_{S_3} \\
\text{Res}_{z=\chi} J_2(z) \xi(\ell_1 \ell_2) \big|_{S_2} &= \text{Res}_{z=-\chi} J_4(z) \xi(\ell_1 \ell_2) \big|_{S_4} \\
\text{Res}_{z=\chi+1} J_1(z) \xi(\ell_1 \ell_2) \big|_{S_1} &= \text{Res}_{z=0} J_7(z) \xi(\ell_1 \ell_2) \big|_{S_7} \\
\text{Res}_{z=\chi+1} J_2(z) \xi(\ell_1 \ell_2) \big|_{S_2} &= \text{Res}_{z=0} J_8(z) \xi(\ell_1 \ell_2) \big|_{S_8} \\
\text{Res}_{z=\chi+1} J_5(z) \xi(\ell_1 \ell_2) \big|_{S_5} &= \text{Res}_{z=\chi+1} J_7(z) \xi(\ell_1 \ell_2) \big|_{S_7} \\
\text{Res}_{z=\chi+1} J_6(z) \xi(\ell_1 \ell_2) \big|_{S_6} &= \text{Res}_{z=\chi+1} J_8(z) \xi(\ell_1 \ell_2) \big|_{S_8} \\
\text{Res}_{z=\chi} J_5(z) \xi(\ell_1 \ell_2) \big|_{S_5} &= -\text{Res}_{z=0} J_3(z) \xi(\ell_1 \ell_2) \big|_{S_3} \\
\text{Res}_{z=\chi} J_6(z) \xi(\ell_1 \ell_2) \big|_{S_6} &= -\text{Res}_{z=0} J_4(z) \xi(\ell_1 \ell_2) \big|_{S_4} .
\end{align*}
\]

(3.21)

It is possible to use intersection labels instead,

\[
\omega_{1\gamma_3}, \omega_{1\gamma_6}, \omega_{1\gamma_7}, \omega_{2\gamma_4}, \omega_{2\gamma_5}, \omega_{2\gamma_6}, \omega_{3\gamma_5}, \omega_{3\gamma_6}, \omega_{4\gamma_7}, \omega_{4\gamma_8},
\]

(3.22)

but contour weights with explicit reference to type of pole are more convenient in actual calculations.

The displayed relations imply major simplifications in the augmented hepta-cut and allow us to cut computation of residues in half. Indeed, we select only solutions 1, 2, 5, 6 and avoid double counting at \( z = 0 \). The global poles may be organized using the following
Figure 5. A view of the global structure of the eight on-shell solutions for the massless two-loop crossed box. The set of solutions has ten intersections and each branch is topologically equivalent to a Riemann sphere. Our convention is to denote holomorphic and antiholomorphic vertices is by \( \oplus \) and \( \ominus \) respectively.

 contours weights or generalized winding numbers,

\[
\begin{align*}
a_{1,j} & \rightarrow \text{encircling } z = 0 \text{ for solution } S_j, \\
a_{2,j} & \rightarrow \text{encircling } z = \chi \text{ for solution } S_j, \\
a_{3,j} & \rightarrow \text{encircling } z = \chi + 1 \text{ for solution } S_j.
\end{align*}
\]

We then have the following ten eight-tori encircling the global poles,

\[
\begin{align*}
T_{1,1} &= T_0 \times C_{\alpha_1}(\chi) \times C_{\alpha_3}(\chi + 1) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \\
T_{1,2} &= T_0 \times C_{\alpha_1}(\chi) \times C_{\alpha_3}(0) \times C_{\alpha_4}(\chi + 1) \times C_{\beta_3}(0) \times C_{\beta_4}(0) \\
T_{2,1} &= T_0 \times C_{\alpha_1}(0) \times C_{\alpha_3}(\chi) \times C_{\alpha_4}(0) \times C_{\beta_3}(\chi) \times C_{\beta_4}(0) \\
T_{2,2} &= T_0 \times C_{\alpha_1}(0) \times C_{\alpha_3}(0) \times C_{\alpha_4}(\chi) \times C_{\beta_3}(0) \times C_{\beta_4}(\chi) \\
T_{2,5} &= T_0 \times C_{\alpha_1}(0) \times C_{\alpha_3}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(\chi) \times C_{\beta_4}(0) \\
T_{2,6} &= T_0 \times C_{\alpha_1}(0) \times C_{\alpha_3}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(\chi) \\
T_{3,1} &= T_0 \times C_{\alpha_1}(\chi + 1) \times C_{\alpha_3}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(\chi + 1) \times C_{\beta_4}(0) \\
T_{3,2} &= T_0 \times C_{\alpha_1}(\chi + 1) \times C_{\alpha_3}(0) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(\chi + 1) \\
T_{3,5} &= T_0 \times C_{\alpha_1}(\chi + 1) \times C_{\alpha_3}(\chi + 1) \times C_{\alpha_4}(0) \times C_{\beta_3}(0) \times C_{\beta_4}(\chi + 1) \\
T_{3,6} &= T_0 \times C_{\alpha_1}(\chi + 1) \times C_{\alpha_3}(\chi + 1) \times C_{\alpha_4}(0) \times C_{\beta_3}(\chi) \times C_{\beta_4}(\chi + 1)
\end{align*}
\] (3.23)
Thus reduces the problem to determination of the rational coefficients
results with those of Badger, Frellesvig and Zhang\cite{52}. This choice of basis
amplitudes allows us to directly compare our evaluation of the primitive
amplitude in this integral basis,\footnote{\(T^{\text{NP}}[1] + c_2 T^{\text{NP}}[(\ell_1 \cdot k_3)] + \cdots\),}
thus reduces the problem to determination of the rational coefficients \(c_1\) and \(c_2\) from the
augmented hepta-cut. This choice of basis integrals allows us to directly compare our
results with those of Badger, Frellesvig and Zhang\cite{52}. All other integrals with fewer than
seven propagators have been suppressed. In general we have
\[
2\ell_1 \cdot k_3 = s_{12} \left(- (1 + \chi) a_1 + \chi a_2 - a_3 - a_4\right)
\]
and therefore,
\[
\ell_1 \cdot k_3|_{S_1} = \ell_1 \cdot k_3|_{S_2} = \frac{s_{12}}{2} z, \quad \ell_1 \cdot k_3|_{S_5} = \ell_1 \cdot k_3|_{S_6} = 0.
\]
The cut master integrals are
\[
T^{\text{NP}}[1]|_{\text{cut}} = -\frac{1}{16s_{12}^2} \left\{ \sum_{j=1,2} \frac{a_{1,j}}{\chi (1 + \chi)} - \sum_{j=1,2,5,6} \left( \frac{a_{2,j}}{\chi} - \frac{a_{3,j}}{1 + \chi} \right) \right\},
\]
\[
T^{\text{NP}}[(\ell_1 \cdot k_3)]|_{\text{cut}} = \frac{1}{32s_{12}^2} \sum_{j=1,2} \left\{ a_{2,j} - a_{3,j} \right\}.
\]
We cancel overall factors and derive the augmented hepta-cut
\[
\sum_{i=1,2,5,6} \oint_{G_i} \frac{dz}{z - \chi} (z - \chi - 1) \sum_{\text{helicities}} \prod_{j=1}^6 A^{\text{tree}}(z)|_{S_i}
= c_1 \left\{ \sum_{j=1,2} \frac{a_{1,j}}{\chi (1 + \chi)} - \sum_{j=1,2,5,6} \left( \frac{a_{2,j}}{\chi} - \frac{a_{3,j}}{1 + \chi} \right) \right\} - \frac{s_{12} c_2}{2} \sum_{j=1,2} \left\{ a_{2,j} - a_{3,j} \right\}.
\]
The intermediate state sum over the product of six tree amplitudes takes the explicit form
\[
\sum_{\text{helicities}} \prod_{j=1}^6 A^{\text{tree}}(z)|_{S_i}
= \sum_{\text{particles}} \sum_{\lambda_i = \pm} A^{\text{tree}}(-p_1^{-\lambda_1}, k_1, p_2^{\lambda_2}) A^{\text{tree}}(-p_6^{-\lambda_6}, k_2, p_7^{\lambda_7}) A^{\text{tree}}(-p_3^{-\lambda_3}, k_3, p_4^{\lambda_4})
\times A^{\text{tree}}(-p_4^{-\lambda_4}, k_4, p_5^{\lambda_5}) A^{\text{tree}}(-p_5^{-\lambda_5}, p_1^{\lambda_1}, p_6^{-\lambda_6}) A^{\text{tree}}(-p_2^{-\lambda_2}, p_3^{\lambda_3}, -p_7^{-\lambda_7})|_{S_i},
\]
where, in this notation, \( p_i \) is the \( i \)th inverse propagator of the crossed box diagram, obtained by following momentum flow with the initial identification \( p_1 = \ell_1 + k_1 \).

### 3.4 Integral Reduction Identities

In order to constrain the integration contours we impose consistency conditions. It is completely clear that vanishing Feynman integrals should have vanishing hepta-cuts. Otherwise the unitarity procedure is not well-defined. Equivalently, we can demand that any integral identity is preserved,

\[
I_1 = I_2 \implies I_{1,\text{cut}} = I_{2,\text{cut}}. \tag{3.32}
\]

We identify the complete variety of Levi-Civita symbols that appears in integral reduction, after using momentum conservation, and require continued vanishing of the following five integrals after pushing loop integration from real slices into C^4 × C^4,

\[
\mathcal{I}^{NP}[\varepsilon(\ell_1, k_2, k_3, k_4)], \quad \mathcal{I}^{NP}[\varepsilon(\ell_2, k_2, k_3, k_4)],
\]

\[
\mathcal{I}^{NP}[\varepsilon(\ell_1, \ell_2, k_1, k_2)], \quad \mathcal{I}^{NP}[\varepsilon(\ell_1, \ell_2, k_1, k_3)], \quad \mathcal{I}^{NP}[\varepsilon(\ell_1, \ell_2, k_3, k_4)]. \tag{3.33}
\]

Let us set the stage and evaluate the first two constraints explicitly. We expand the integral onto the augmented hepta-cut,

\[
0 = \mathcal{I}^{NP}[\varepsilon(\ell_1, k_2, k_3, k_4)]_{\text{cut}} \iff
\]

\[
0 = \oint_{\Gamma_1} dz \frac{\varepsilon \left( (\chi - z) k_1^\mu + \frac{s_{12}(z - \chi)^{-1}}{2[14][42]} (1^- | \gamma^\mu | 2^-), k_2, k_3, k_4 \right)}{z(z - \chi)(z - \chi - 1)}
\]

\[
+ \oint_{\Gamma_2} dz \frac{\varepsilon \left( (\chi - z) k_1^\mu + \frac{s_{12}(z - \chi)^{-1}}{2[24][41]} (2^- | \gamma^\mu | 1^-), k_2, k_3, k_4 \right)}{z(z - \chi)(z - \chi - 1)}
\]

\[
+ \oint_{\Gamma_5} dz \frac{\varepsilon \left( (\chi - z) k_1^\mu + \frac{s_{12}(z - \chi)^{-1}}{2[24][41]} (1^- | \gamma^\mu | 1^-), k_2, k_3, k_4 \right)}{z(z - \chi)(z - \chi - 1)}
\]

\[
+ \oint_{\Gamma_6} dz \frac{\varepsilon \left( (\chi - z) k_1^\mu + \frac{s_{12}(z - \chi)^{-1}}{2[14][42]} (1^- | \gamma^\mu | 2^-), k_2, k_3, k_4 \right)}{z(z - \chi)(z - \chi - 1)} \tag{3.34}
\]

and in virtue of the relation

\[
\varepsilon \left( \frac{1^- | \gamma^\mu | 2^-}{[14][42]}, k_2, k_3, k_4 \right) = -\varepsilon \left( \frac{2^- | \gamma^\mu | 1^-}{[24][41]}, k_2, k_3, k_4 \right) \tag{3.35}
\]

we then obtain the constraint equation,

\[
0 = \mathcal{I}^{NP}[\varepsilon(\ell_1, k_2, k_3, k_4)]_{\text{cut}} = a_{1,1} - a_{1,2} - a_{2,1} + a_{2,2} + a_{3,5} - a_{3,6} = 0. \tag{3.36}
\]

Likewise, the second vanishing identity in question,

\[
0 = \mathcal{I}^{NP}[\varepsilon(\ell_2, k_2, k_3, k_4)]_{\text{cut}} \iff
\]

\[
0 = \oint_{\Gamma_1 + \Gamma_5} dz \frac{\varepsilon \left( \frac{s_{12}(z - \chi)^{-1}}{2[34][14]} (3^- | \gamma^\mu | 4^-), k_2, k_3, k_4 \right)}{z(z - \chi)(z - \chi - 1)}
\]

\[
+ \oint_{\Gamma_2 + \Gamma_6} dz \frac{\varepsilon \left( \frac{s_{12}(z - \chi)^{-1}}{2[41][13]} (4^- | \gamma^\mu | 3^-), k_2, k_3, k_4 \right)}{z(z - \chi)(z - \chi - 1)}, \tag{3.37}
\]
linearity in the contour subscript being implied, becomes

$$0 = \mathcal{I}_{NP}^{\varepsilon} (\ell_2, k_2, k_3, k_4)_{\text{cut}} = a_{2,1} - a_{2,2} - a_{3,1} + a_{3,2} + a_{2,5} - a_{2,6} - a_{3,5} + a_{3,6}, \quad (3.38)$$

where we used the fact that

$$\varepsilon \left( \langle 3^-|\gamma^\mu|4^- \rangle_{(31)(14)} , k_2, k_3, k_4 \right) = -\varepsilon \left( \langle 4^-|\gamma^\mu|3^- \rangle_{(41)(13)} , k_2, k_3, k_4 \right), \quad (3.39)$$

The last three parity vanishing requirements,

$$0 = \mathcal{I}_{NP}^{\varepsilon} (\ell_1, \ell_2, k_i, k_j) \iff$$

$$0 = \oint_{\Gamma_1} d\gamma \left( (\gamma - z)k_1^\mu + \frac{s_{\ell_2}}{2(14)(42)} (1^-|\gamma^\mu|2^-) , \frac{s_{\ell_2}}{2(31)(14)} \langle 3^-|\gamma^\mu|4^- \rangle , k_1, k_j \right) z(z - \gamma)(z - \chi - 1)$$

$$+ \oint_{\Gamma_2} d\gamma \left( (\gamma - z)k_1^\mu + \frac{s_{\ell_2}}{2(24)(41)} (2^-|\gamma^\mu|1^-) , \frac{s_{\ell_2}}{2(31)(14)} \langle 4^-|\gamma^\mu|3^- \rangle , k_1, k_j \right) z(z - \gamma)(z - \chi - 1)$$

$$+ \oint_{\Gamma_3} d\gamma \left( (\gamma - z)k_1^\mu + \frac{s_{\ell_2}}{2(34)(12)} (2^-|\gamma^\mu|1^-) , \frac{s_{\ell_2}}{2(31)(14)} \langle 3^-|\gamma^\mu|4^- \rangle , k_1, k_j \right) z(z - \gamma)(z - \chi - 1)$$

$$+ \oint_{\Gamma_4} d\gamma \left( (\gamma - z)k_1^\mu + \frac{s_{\ell_2}}{2(14)(42)} (1^-|\gamma^\mu|2^-) , \frac{s_{\ell_2}}{2(24)(41)} \langle 4^-|\gamma^\mu|3^- \rangle , k_1, k_j \right) z(z - \gamma)(z - \chi - 1) \quad (3.40)$$

for \((i, j) \in \{(1, 2), (1, 3), (2, 3)\}\) are also rather straightforward to obtain by this strategy, so we will spare the reader for details and just quote the final expressions,

$$0 = \mathcal{I}_{NP}^{\varepsilon} (\ell_1, \ell_2, k_1, k_2)_{\text{cut}} = a_{2,1} - a_{2,2} - a_{3,5} + a_{3,6} = 0,$$

$$0 = \mathcal{I}_{NP}^{\varepsilon} (\ell_1, \ell_2, k_1, k_3)_{\text{cut}} = a_{2,1} - a_{2,2} = 0,$$

$$0 = \mathcal{I}_{NP}^{\varepsilon} (\ell_1, \ell_2, k_2, k_3)_{\text{cut}} = a_{3,1} - a_{3,2} = 0. \quad (3.41)$$

Reduction together with the two single-momentum parity constraints produces the following five linearly independent parity vanishing identities,

$$a_{1,1} - a_{1,2} = 0,$$

$$a_{2,1} - a_{2,2} = 0,$$

$$a_{2,5} - a_{2,6} = 0,$$

$$a_{3,1} - a_{3,2} = 0,$$

$$a_{3,5} - a_{3,6} = 0. \quad (3.42)$$

The displayed equations have a very simple interpretation; they simply translate into the statement that all contours across parity-conjugate solutions \(S_1 \leftrightarrow S_2\) and \(S_5 \leftrightarrow S_6\) must carry weights of equal values, thereby resembling previous observations for both the one-loop box and the planar double box. Actually this feature is expected as its origin can be traced back to the equality of the Jacobians that arise upon localization of the crossed
box integral onto the Riemann spheres parametrized by the four hepta-cut branches in consideration.

We next consider contour constraint equations arising from integration-by-parts identities used for reduction onto master integrals. There are two nonspurious irreducible scalar products parametrizing the general integrand. Gram matrix relations for four-dimensional momenta remove dependent terms and imply that we have the following nineteen naively irreducible tensor integrals in renormalizable theories,

\[
\mathcal{I}^{NP}[1], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)^2], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)^3], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)^4], \mathcal{I}^{NP}[(\ell_2 \cdot k_2)], \mathcal{I}^{NP}[(\ell_2 \cdot k_2)^2], \mathcal{I}^{NP}[(\ell_2 \cdot k_2)^3], \mathcal{I}^{NP}[(\ell_2 \cdot k_2)^4], \mathcal{I}^{NP}[(\ell_2 \cdot k_2)^5], \mathcal{I}^{NP}[(\ell_2 \cdot k_2)^6], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)(\ell_2 \cdot k_2)], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)^2(\ell_2 \cdot k_2)], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)^3(\ell_2 \cdot k_2)], \mathcal{I}^{NP}[(\ell_1 \cdot k_3)^4(\ell_2 \cdot k_2)] \] (3.43)

All identities can be generated with the Mathematica package FIRE and are listed in appendix B. It now just remains to evaluate all tensor integrals on the augmented hepta-cut and enforce continued validity of the integral reduction equations. To this end we compute the tensors using the parametrized loop momenta,

\[
\ell_2 \cdot k_2 = \frac{s_{12}}{2} (\chi \beta_1 - (1 + \chi) \beta_2 - \beta_3 - \beta_4),
\] (3.44)

such that on the relevant on-shell branches,

\[
\ell_2 \cdot k_2|_{s_1} = \ell_2 \cdot k_2|_{s_2} = \ell_2 \cdot k_2|_{s_3} = \ell_2 \cdot k_2|_{s_5} = -\frac{s_{12}}{2} z.
\] (3.45)

Then we can write down the augmented hepta-cuts

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^n]\text{cut} = -\frac{1}{16s_{12}^2} \left( \frac{s_{12}}{2} \right)^n \sum_{i=1,2} \oint_{\Gamma_i} dz \frac{z^{n-1}}{(z - \chi)(z - \chi - 1)},
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^m]\text{cut} = \frac{(-1)^{m+1}}{16s_{12}^2} \left( \frac{s_{12}}{2} \right)^m \sum_{i=1,2,5,6} \oint_{\Gamma_i} dz \frac{z^{m-1}}{(z - \chi)(z - \chi - 1)},
\]

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^n(\ell_2 \cdot k_2)^m]\text{cut} = \frac{(-1)^{m+1}}{16s_{12}^2} \left( \frac{s_{12}}{2} \right)^{n+m} \sum_{i=1,2} \oint_{\Gamma_i} dz \frac{z^{n+m-1}}{(z - \chi)(z - \chi - 1)},
\] (3.46)

and obtain the explicit relations

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^n]\text{cut} = \frac{1}{16s_{12}^2} \left( \frac{s_{12}}{2} \right)^n \sum_{j=1,2} \left\{ \chi^{n-1} a_{2,j} - (1 + \chi)^{n-1} a_{3,j} \right\},
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^m]\text{cut} = \frac{(-1)^m}{16s_{12}^2} \left( \frac{s_{12}}{2} \right)^m \sum_{j=1,2,5,6} \left\{ \chi^{m-1} a_{2,j} - (1 + \chi)^{m-1} a_{3,j} \right\},
\]

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^n(\ell_2 \cdot k_2)^m]\text{cut} = \frac{(-1)^m}{16s_{12}^2} \left( \frac{s_{12}}{2} \right)^{n+m} \sum_{j=1,2} \left\{ \chi^{n+m-1} a_{2,j} - (1 + \chi)^{n+m-1} a_{3,j} \right\}.
\] (3.47)
Insertion into the integration by parts identities yields seventeen linear relations among
the winding numbers. We are able to clarify any redundancy and derive only three
independent constraints,

\[
\begin{align*}
a_{2,1} + a_{2,2} - a_{2,5} - a_{2,6} &= 0, \\
a_{1,1} + a_{1,2} + a_{2,1} + a_{2,2} + a_{3,1} + a_{3,2} &= 0, \\
a_{1,1} + a_{1,2} + a_{2,1} + a_{2,2} + a_{3,5} + a_{3,6} &= 0.
\end{align*}
\]

(3.48)

We further compress these equations together with the parity vanishing identities and find
the final form of the eight constraint equations,

\[
\begin{align*}
a_{1,1} - a_{1,2} &= a_{2,1} - a_{2,2} = a_{2,5} - a_{2,6} = a_{3,1} - a_{3,2} = a_{3,5} - a_{3,6} = 0, \\
a_{2,1} - a_{2,5} &= a_{3,1} - a_{3,5} = 0, \\
a_{1,1} + a_{2,1} + a_{3,1} &= 0.
\end{align*}
\]

(3.49)

In addition to the requirements arising from Levi-Civita integrals we see that winding
numbers of each type of global pole must be uniform across all on-shell solutions, whereas
the last equation states that the sum of weights within a branch vanishes.

### 3.5 Unique Master Integral Projectors

We fix the remaining freedom of the contour weights and derive independent master con-
tours that each project out a single master integral coefficient, for instance we isolate the
scalar master integral by imposing the conditions,

\[
\sum_{j=1,2} \{a_{2,j} - a_{3,j}\} = 0, \quad \sum_{j=1,2} \frac{a_{1,j}}{\chi(1 + \chi)} - \sum_{j=1,2,5,6} \left(\frac{a_{2,j}}{\chi} - \frac{a_{3,j}}{1 + \chi}\right) = 1.
\]

(3.50)

and vice versa for the tensor master integral,

\[
\mathcal{I}^{\text{NP}}[1]\text{cut} = 0, \quad \mathcal{I}^{\text{NP}}[(\ell_1 \cdot k_3)]\text{cut} = -\frac{2}{s_{12}}.
\]

(3.51)

The displayed normalization conditions are chosen purely for convenience in order for the
contour weights to soak up overall prefactors. The cost is loss of an immediate geometrical
interpretation of the contour weights as integral winding numbers. We solve the two set of
equations and find two master contours which we denote \(\mathcal{M}_1\) and \(\mathcal{M}_2\) respectively.

\[
\begin{align*}
\mathcal{M}_1 : \quad & a_{1,1} = a_{1,2} = \frac{1}{2}\chi(1 + \chi) \\
& a_{2,1} = a_{2,2} = a_{2,5} = a_{2,6} = -\frac{1}{2}\chi(1 + \chi) \\
& a_{3,1} = a_{3,2} = a_{3,5} = a_{3,6} = -\frac{1}{2}\chi(1 + \chi)
\end{align*}
\]

\[
\begin{align*}
\mathcal{M}_2 : \quad & a_{1,1} = a_{1,2} = \frac{1 + 2\chi}{2s_{12}} \\
& a_{2,1} = a_{2,2} = a_{2,5} = a_{2,6} = -\frac{1 - 2\chi}{4s_{12}} \\
& a_{3,1} = a_{3,2} = a_{3,5} = a_{3,6} = \frac{3 + 2\chi}{4s_{12}}
\end{align*}
\]

(3.52)
Therefore our final formula for the master integral coefficients can be written in the remarkably compact form

\[
C_i = \oint_{\mathcal{M}_i} \frac{dz}{z(z-\chi)(z-\chi-1)} \sum_{\text{helicities}} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z), \tag{3.53}
\]

or as explicitly as expansions in residues,

\[
c_1 = \frac{1}{4} \sum_{i=1,2} \sum_{z=0} \frac{1}{z} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i}
+ \frac{1+\chi}{8} \sum_{i=1,2,5,6} \sum_{z=\chi} \frac{1}{z-\chi} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i}
- \frac{\chi}{8} \sum_{i=1,2,5,6} \sum_{z=\chi+1} \frac{1}{z-\chi-1} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i}, \tag{3.54}
\]

\[
c_2 = -\frac{1+2\chi}{2s_{12}(\chi+1)} \sum_{i=1,2} \sum_{z=0} \frac{1}{z} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i}
+ \frac{1-2\chi}{4s_{12}\chi} \sum_{i=1,2,5,6} \sum_{z=\chi} \frac{1}{z-\chi} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i}
+ \frac{3+2\chi}{4s_{12}(\chi+1)} \sum_{i=1,2,5,6} \sum_{z=\chi+1} \frac{1}{z-\chi-1} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i}. \tag{3.55}
\]

Although the latter expressions at first sight may look slightly complicated, notice that the number of ingredients really is minimal. Indeed, once the intermediate state sum is computed on the four on-shell branches, which is rather elementary, it is just a matter of plugging in values of \(z\) appropriate to the residues and forming the indicated linear combinations to get both master integral coefficients.

We finally remark that the formulae in this paper are of course compatible with the Bern-Carrasco-Johansson (BCJ) color/kinematics duality \cite{65,66} in the maximally supersymmetric case. Indeed, by absence of triangle subgraphs in \(\mathcal{N} = 4\) we expect the master integral coefficients for the planar and nonplanar double boxes to be equal. The intermediate state sum in \(\mathcal{N} = 4\) Yang-Mills theory is independent of both loop momenta and a standard result in the literature. Anyway, it is easy to rederive for both topologies,

\[
\sum_{\text{multiplet}} \prod_{j=1}^{6} A^{\text{tree}}_{(j)}(z) |_{S_i} = -s_{12}s_{14}A^{\text{tree}}_{4}. \tag{3.56}
\]

Then we readily get \(c^{\text{box}}_{1,\mathcal{N}=4} = c^{\text{box}}_{1,\mathcal{N}=4} = -s_{12}s_{14}A^{\text{tree}}_{4}\) and \(c^{\text{box}}_{2,\mathcal{N}=4} = c^{\text{box}}_{2,\mathcal{N}=4} = 0.\)
4 Examples

In this section we apply the master integral formulae to two-loop four-point gluon amplitudes with specific helicity configurations. We only consider hepta-cuts in the $s$-channel, because contributions from the $t$-channel can be obtained completely analogously. To account for the cyclic permutation we should however substitute $\chi \rightarrow \chi^{-1}$ in (3.55). Our results are valid for supersymmetric theories with $\mathcal{N}$ supersymmetries including QCD.

We track contributions to the intermediate state sums using superspace techniques developed in \cite{62, 63}. In particular, we exploit that the transition from $\mathcal{N} = 4$ to fewer supersymmetries is very straightforward,

$$
\sum_{\mathcal{N}=4}^{\mathcal{N} \leq 4} \prod_{i=1}^{k} A_{\text{tree}}^{(i)} = \Delta^{-1}(A + B + C + \cdots)^4 \rightarrow
$$

$$
\sum_{\mathcal{N} \leq 4}^{\mathcal{N} \leq 4} \prod_{i=1}^{k} A_{\text{tree}}^{(i)} = \Delta^{-1}(A + B + C + \cdots)\mathcal{N} (A^{4-\mathcal{N}} + B^{4-\mathcal{N}} + C^{4-\mathcal{N}} + \cdots).
$$

(4.1)

Here $A, B, C, \ldots$ contain spin factors for each kinematically valid assignment of helicities on the internal lines with only gluons propagating the in loops whereas $\Delta$ is the denominator of the supersum. Let us consider the case of only two gluonic contributions $A$ and $B$ in more detail. This situation is relevant for quadruple cuts of one-loop amplitudes and hepta-cuts at two loops for instance. The trick is to expand the state sum around $A = -B$ such that \cite{47}

$$
\begin{align*}
\sum_{\mathcal{N} \leq 4}^{\mathcal{N} \leq 4} \prod_{j=1}^{6} A_{\text{tree}}^{(j)} &= \frac{A^{4-\mathcal{N}} + B^{4-\mathcal{N}}}{(A + B)^{4-\mathcal{N}}} \left(1 - \frac{\mathcal{N}}{2} \delta_{\mathcal{N},4}\right) \sum_{\mathcal{N}=4}^{\mathcal{N}=4} \prod_{j=1}^{6} A_{\text{tree}}^{(j)} \\
&= \left\{1 - (4 - \mathcal{N}) \left(\frac{A}{A + B}\right) + (4 - \mathcal{N}) \left(\frac{A}{A + B}\right)^2\right\} \sum_{\mathcal{N}=4}^{\mathcal{N}=4} \prod_{j=1}^{6} A_{\text{tree}}^{(j)}.
\end{align*}
$$

(4.2)

In all computations we use the three- and four-gluon MHV amplitudes

$$
A_{-++}^{\text{tree}} = i\frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad A_{-+++}^{\text{tree}} = i\frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \quad A_{-++-}^{\text{tree}} = i\frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle},
$$

(4.3)

together with their parity conjugates obtained by $\langle \rangle \rightarrow [\ ]$.

4.1 Helicities $- - ++$

Our starting point is the tree-level data

$$
\sum_{\mathcal{N}=4}^{\mathcal{N}=4} \prod_{j=1}^{6} A_{\text{tree}}^{(j)} \big|_{S_i} = -s_{12}^2 s_{14} A_{-+++}^{\text{tree}},
$$

(4.4)
which is independent of the loop momenta.

We then compute the ratio of a general state sum relative to that of \( \mathcal{N} = 4 \) explicitly for solution \( S_2 \) as an example. The two valid distributions of internal helicities denoted \( A \) and \( B \) are shown in fig. 6 and the depiction of holomorphic and antiholomorphic vertices by \( \oplus \) and \( \ominus \) follows [62].

![Figure 6. Hepta-cut solution \( S_2 \) allows two distinct assignments of helicities on the internal lines in the \(- - + +\) two-loop crossed box.](image)

The relative sign between gluonic contributions is in general specified by signatures of Grassmann variables in on-shell superspace and by carefully working out directions of all internal momenta and applying analytic continuations appropriately, i.e. \( p_i \rightarrow -p_i \) implies change of sign for the holomorphic spinor while the conjugate is left unchanged. However, for our purposes it is advantageous to cut the calculation short and just infer the sign by matching the expression in \( \mathcal{N} = 4 \) theory, i.e. insisting that

\[
\Delta^{-1}(A + B)^4 = -s_{12}^2 s_{14} A_{-++}^{\text{tree}}. \tag{4.5}
\]

To proceed, label propagators consecutively from \( p_1 = \ell_1 + k_1 \) according to the momentum flow previously outlined in fig. 4. For instance, \( \ell_1 = p_2 \) and \( \ell_2 = p_4 \). Then spinor strings for helicity configurations \( A \) and \( B \) are

\[
A = \langle 2p_7 | [p_7 p_3] [p_5 p_4] [p_4 4] | 1p_1 | p_1 p_5 \rangle, \tag{4.6}
\]

\[
B = - \langle 1p_2 | [p_2 p_3] [p_3 p_4] [p_4 4] | 2p_6 | p_6 p_5 \rangle, \tag{4.7}
\]

whereas the denominator reads

\[
\Delta = \langle p_1 1 | [p_2 p_1] [p_2 p_3] [p_3 p_7] [p_7 p_2] [p_3 3] | 3p_4 | p_4 p_3 \rangle \times [p_4 4 | [4p_5] [p_5 p_4] [p_5 p_1] | p_1 p_6 | p_6 p_5 | 2p_6 | p_6 p_7 | p_7 2 \rangle. \tag{4.8}
\]

We now use momentum conservation several times to cancel common factors and get the compact expression

\[
\frac{A}{A + B} = -\frac{2p_2 | 2}{s_{12}} = z - \chi, \tag{4.9}
\]
where the last equality follows by inserting explicit values for the internal momenta on the hepta-cut branch in question,

$$p_\mu^1 = \ell_\mu^1 + k_\mu^1 = -(z - \chi - 1)k_\mu^1 + \frac{s_{12}z}{2|24|41}
(2^- |\gamma^\mu|1^-)$$  \hspace{1cm} (4.10)

and $p_\mu^2 = p_\mu^1 - k_\mu^1$. The state sum in case of $N \leq 4$ supersymmetries can thus be written as

$$\sum_{N \leq 4} \prod_{j=1}^{6} A_{\text{tree}}^{(j)}(z)|_{S_2} = -s_{12}s_{14}^{A_{--++}}\left\{ 1 - (4 - N)(z - \chi) + (4 - N)(z - \chi)^2 \right\}.$$  \hspace{1cm} (4.11)

**Figure 7.** Hepta-cut solutions $S_1$ and $S_6$ are both singlets for external helicities $--++$ in the sense that only gluons are allowed to propagate in the loops, thereby producing state sums that are independent of the number of supersymmetries.

The treatment is similar for the other on-shell branches. Examples of supported helicity configurations are shown in fig. 7. In the end, inserting the multiplet sums into (3.55) and computing all residues yield the master integral coefficients reconstructed to $O(\epsilon^0)$ in $N = 4, 2, 1, 0$ Yang-Mills theory, with the result

$$A_{xbox}^{--++} = -s_{12}s_{14}^{A_{--++}}\left\{ 1 + (4 - N)\frac{s_{14}}{4s_{12}}\left( 1 + \frac{s_{14}}{s_{12}} \right) I^{\text{NP}}[1]
+ (4 - N)\frac{s_{13}}{2s_{12}} I^{\text{NP}}[(\ell_1 \cdot k_3)] \right\}.$$  \hspace{1cm} (4.12)

**4.2 Helicities $-+--$**

We next turn to the $-+--$ helicity amplitude and work through the contribution to the master integral coefficients due to hepta-cut solution $S_2$. There are two possible assignments $A$ and $B$ of helicities on internal lines, shown in fig. 8.
Again, the product of tree amplitudes is very simple when evaluated in the maximally supersymmetric theory,
\[
\sum_{N=4}^{\text{multiplet}} \prod_{j=1}^{6} A_{\text{tree}}^{(j)}(z) \big|_{\mathcal{S}_2} = -s_{12}^2 s_{14} A_{\text{tree}}^{\text{tree}}_{-+-+} ,
\] (4.13)
whence we need to determine the ratio for $N = 2, 1, 0$ supersymmetries. In the case at hand, single $SU(4)$ factors read
\[
A = \langle 1p_1 | [p_1 p_6] [p_6 p_7] [p_7 p_3] [p_3 3] [p_5 4] \rangle ,
\] (4.14)
\[
B = - \langle 1p_2 | [p_2 p_7] [p_7 p_6] [p_6 p_5] [3p_4] [p_4 4] \rangle .
\] (4.15)
The string of spinor products in the denominator is of course still given by (4.8) as in the previous example. By multiple applications of momentum conservation and insertion of the explicit hepta-cut solutions,
\[
p_3^\mu = \ell_2^\mu - k_3^\mu = \frac{s_{12}^z}{2\langle 41 | 13 \rangle} (4^- | \gamma^\mu | 3^-) + k_3^\mu ,
\] (4.16)
\[
p_5^\mu = \ell_2^\mu + k_3^\mu = \frac{s_{12}^z}{2\langle 41 | 13 \rangle} (4^- | \gamma^\mu | 3^-) - k_3^\mu ,
\] (4.17)
it is not hard to realize that
\[
\frac{A}{A+B} = \frac{\langle 1 | p_5 | 4 \rangle}{\langle 1 | 2 | 4 \rangle} = \frac{z}{1+\chi} .
\] (4.18)
The generically supersymmetric state sum becomes
\[
\sum_{N=4}^{\text{multiplet}} \prod_{j=1}^{6} A_{\text{tree}}^{(j)}(z) \big|_{\mathcal{S}_2} = -s_{12}^2 s_{14} A_{\text{tree}}^{\text{tree}}_{-+-+} \left\{ 1 + (4 - N) \frac{z(z - \chi - 1)}{(1+\chi)^2} \right\} .
\] (4.19)
Then we finally plug the supersymmetric sum together with contributions from the other on-shell branches, which we do not include explicitly, into the master integral formulae and
derive the alternating helicity amplitude

\[ A_{\text{box}}^{\text{tree}} = -s_{12}^2 s_{14} A_{\text{tree}}^{\text{tree}} \left\{ \left( 1 + (4 - N) \frac{s_{14}}{4 s_{13}} \right) \mathcal{I}^{\text{NP}}[1] \right. \]

\[ \left. + (4 - N) \frac{s_{13} + 3 s_{14}}{2 s_{13}} \mathcal{I}^{\text{NP}}[(\ell_1 \cdot k_3)] \right\} , \]  

where the coefficients are valid to \( \mathcal{O}(\epsilon^0) \).

5 Integrand-Level Reduction Methods

Recently other promising methods for two-loop amplitudes such as integrand basis determination by multivariate polynomial division algorithms using Gröbner bases and classification of on-shell solutions by primary decomposition based on computational algebraic geometry have been reported [53–61].

In particular, using hepta-cuts, Gram matrix relations and polynomial fitting techniques Badger, Frellesvig and Zhang [52] were able to obtain master integral coefficients in any renormalizable four-dimensional gauge theory for the planar double box, two-loop crossed box and pentabox-triangle primitive amplitudes, although it turns out that the latter is reducible to simpler topologies that contribute to hexacuts for example. In this section we provide a very brief review of their method and a comparison to that of Kosower and Larsen used here.

5.1 Irreducible Integrand Bases

Let \( \ell_1 \) and \( \ell_2 \) be the loop momenta and that suppose \( \{e_1, e_2, e_3, e_4\} \) spans the space of 4-dimensional momenta, say three external momenta \( \{k_1, k_2, k_4\} \) supplemented with a spurious vector that is orthogonal to those directions and satisfies \( \omega^2 > 0 \). Eliminating contractions that are trivially reducible using proper combinations of inverse propagators and constant terms such as for instance

\[
2(\ell_1 \cdot k_1) = (\ell_1 - k_1)^2 - \ell_1^2 - k_1^2 ,
\]

\[
2(\ell_2 \cdot k_3) = (\ell_2 - k_3)^2 - (\ell_2 - k_3 - k_4)^2 + 2k_3 \cdot k_4 + k_4^2 ,
\]

(5.1)

a completely general integrand can be parametrized with four irreducible scalar products

\[
\{\ell_1 \cdot k_4, \ell_2 \cdot k_1, \ell_1 \cdot \omega, \ell_2 \cdot \omega\} .
\]

(5.2)

Relations from Gram matrix determinants impose further nontrivial constraints on the general form of the integrand. To motivate this we first define for \( 2n \) vectors \( \{l_1, \ldots, l_n\} \) and \( \{v_1, \ldots, v_n\} \) the \( n \times n \) Gram matrix by

\[
G = G^{l_1, \ldots, l_n}_{v_1, \ldots, v_n} , \quad G_{ij} = l_i \cdot v_j .
\]

(5.3)

We will frequently encounter Gram matrices where the two sets are identical. Important properties of the Gram determinant \( \det G \) are linearity and antisymmetry in the vectors
in each row. However the real use owes to the fact that \( \det G \) vanishes if and only if the vectors \( \{l_1, \ldots, l_n\} \) or \( \{v_1, \ldots, v_n\} \) are linearly dependent. Then if \( \ell_1 \) and \( \ell_2 \) are also four-dimensional, by this property,

\[
\det G \left( e_1, e_2, \ell_1, e_1, e_2, \ell_2 \right) = \det G \left( e_1, e_2, e_3, \ell_1, e_1, e_2, e_3, \ell_2 \right) = 0 .
\]

These relations imply that \( (\ell_1 \cdot \omega)^2, (\ell_2 \cdot \omega)^2 \) and \( (\ell_1 \cdot \omega)(\ell_2 \cdot \omega) \) are reducible. Other Gram matrix relations may be derived from combinations of the fundamental three to provide additional constraints on the integrand reducing the number of irreducible scalar products monomials to 32 and 38 for the planar and nonplanar double box respectively. We leave the precise summation ranges implicit and write

\[
\mathcal{N}^P(\ell_1, \ell_2) = \sum_{m,n,\alpha,\beta} c_{mn(\alpha+2\beta)} (\ell_1 \cdot k_4)^m (\ell_2 \cdot k_1)^n (\ell_1 \cdot \omega)^\alpha (\ell_2 \cdot \omega)^\beta .
\]

(5.5)

The crossed box is similar,

\[
\mathcal{N}^{NP}(\ell_1, \ell_2) = \sum_{m,n,\alpha,\beta} c_{mn(\alpha+2\beta)} (\ell_1 \cdot k_3)^m (\ell_2 \cdot k_2)^n (\ell_1 \cdot \omega)^\alpha (\ell_2 \cdot \omega)^\beta .
\]

(5.6)

5.2 Master Integral Coefficients

Badger, Frellesvig and Zhang use the well-known parametrization (3.2) to solve the equations for the hepta-cut, but without normalizations formed by spinor products and momentum invariants in the cross terms. Moreover, the choice of momentum flow and the free parameter differ slightly from ours. Using their parametrization of the two-loop momenta a general form of the integrand at the hepta-cut may be inferred. The cut crossed box has a very simple polynomial form for all on-shell solutions,

\[
\sum_{\text{helicities}} \prod_{j=1}^{6} A_{(j)}^\text{tree}(\tau) = \begin{cases} 
\sum_{n=0}^{6} d_{s,n} \tau^s & s = 1, 2, 5, 6 , \\
\sum_{n=0}^{4} d_{s,n} \tau^s & s = 3, 4, 7, 8 ,
\end{cases}
\]

(5.7)

while the cut planar double box due to poles in tensor integrals also contains terms with inverse powers of the free parameter,

\[
\sum_{\text{helicities}} \prod_{j=1}^{6} A_{(j)}^\text{tree}(\tau) = \begin{cases} 
\sum_{n=0}^{4} d_{s,n} \tau^s & s = 1, 2, 3, 4 , \\
\sum_{n=-4}^{4} d_{s,n} \tau^s & s = 5, 6 .
\end{cases}
\]

(5.8)

Schematically it is now possible to construct a \( 48 \times 38 \) matrix \( M \) for the nonplanar double box relating the coefficients in the integrand to those in the product of tree amplitudes such that

\[
d = Mc \iff c = (M^T M)^{-1} M^T d ,
\]

(5.9)

whereas the matrix in the case of the planar double box has dimensions \( 38 \times 32 \). The matrix \( M \) has full rank and analytical inversion of the hepta-cut matrix equations and subsequent
reduction onto master integrals using the integration-by-parts identities produce the following coefficients for the nonplanar crossed box,

\[
c_1 = c_{000} + \frac{1}{16}s_{14}s_{13}(c_{200} - c_{110} + 2c_{020}) \\
+ \frac{1}{32}s_{14}s_{13}(s_{14} - s_{13})(c_{300} - c_{210} + c_{120} - 2c_{030}) \\
+ \frac{1}{16^2}(3(s_{14} - s_{13})^2 + s_{12}^2)s_{14}s_{13}(c_{400} - c_{310} + c_{220} + 2c_{040}) \\
+ \frac{1}{16^2}((s_{14} - s_{13})^2 + s_{12}^2)s_{14}s_{13}(s_{14} - s_{13})(c_{320} - c_{410} - 2c_{050}) \\
+ \frac{1}{16^3}(5(s_{14} - s_{13})^4 + 10s_{12}^2(s_{14} - s_{13})^2 + s_{12}^4)s_{14}s_{13}(c_{420} + 2c_{060}) ,
\]  
(5.10)

\[
c_2 = c_{100} - 2c_{010} + \frac{3}{8}(s_{14} - s_{13})(c_{200} - c_{110} + 2c_{020}) \\
+ \frac{1}{16}(2(s_{14} - s_{13})^2 + s_{12}^2)(c_{300} - c_{210} + c_{120} - 2c_{030}) \\
+ \frac{2}{16^2}(5(s_{14} - s_{13})^4 + 7s_{12}^2(s_{14} - s_{13})^2 + s_{12}^4)(c_{400} - c_{310} + c_{220} + 2c_{040}) \\
+ \frac{1}{16^2}(3(s_{14} - s_{13})^4 + 8s_{12}^2(s_{14} - s_{13})^2 + s_{12}^4)(c_{320} - c_{410} - 2c_{050}) \\
+ \frac{2}{16^3}(7(s_{14} - s_{13})^4 + 30s_{12}^2(s_{14} - s_{13})^2 + 11s_{12}^4)(s_{14} - s_{13})(c_{420} + 2c_{060}) ,
\]  
(5.11)

and for the planar double box,

\[
c_1 = c_{000} + \frac{s_{12}s_{14}}{8}c_{110} - \frac{s_{12}^2s_{14}}{16}(c_{120} + c_{210}) + \frac{s_{12}^3s_{14}}{32}(c_{130} + c_{310}) \\
- \frac{s_{12}^4s_{14}}{64}(c_{140} + c_{410}) ,
\]  
(5.12)

\[
c_2 = c_{100} + c_{010} - \frac{3s_{12}}{4}c_{110} + \frac{s_{14}}{2}(c_{020} + c_{200}) + \frac{3s_{12}^3}{8}(c_{120} + c_{210}) \\
+ \frac{s_{14}}{4}(c_{030} + c_{300}) - \frac{3s_{12}^3}{16}(c_{130} + c_{310}) + \frac{s_{14}^3}{8}(c_{040} + c_{400}) \\
+ \frac{3s_{14}^4}{32}(c_{140} + c_{410}) .
\]  
(5.13)

Several additional null-space conditions are generated in the process. The structure of these is equivalent to the redundancy of global residues identified previously.

In order to unify the two approaches we have to synchronize the parametrizations of the loop momenta on the hepta-cut. Refer to [52] for the explicit parameters and conventions for the free parameter \( \tau \). It can be shown that this is achieved for the crossed box if \( \mathcal{S}_i \) : \( \tau(z) = -s_{12}z \) ,  
(5.14)

which in particular means that the Jacobians remain simple and uniform across all on-shell solutions. As a consequence of the the additional poles in tensor integrals the situation is
a bit more complicated for the planar double box. We find that the two parametrizations agree everywhere on the hepta-cut if

\[ S_1, \ldots, S_4 : \tau(z) = 1 + \frac{z}{\chi}, \quad S_5, S_6 : \tau(z) = -\frac{1}{z + \chi + 1}. \quad (5.15) \]

with the interchange \( S_2 \leftrightarrow S_3 \) due to the fact that solutions are labeled differently. We can now apply the displayed transformations to the master formulae, carefully keeping track of extra Jacobian factors and how the global poles are mapped. For instance we see that poles in tensor integrands at \( z = -\chi - 1 \) are shifted to \( \tau = \infty \). The contour weights are of course not affected. After all the master integral coefficients for the planar double box can be written

\[
\begin{align*}
\mathcal{C}_1 &= + \frac{1}{4} \sum_{i=1,3} \frac{\text{Res}}{\tau=0} \frac{1}{\tau} \sum_{\text{particles } j=1}^{6} A_{(j)}(z) \bigg|_{S_i} \\
&\quad + \frac{1}{4} \sum_{i=5,6} \frac{\text{Res}}{\tau=-1} \frac{1}{1 + \tau} \sum_{\text{particles } j=1}^{6} A_{(j)}(\tau) \bigg|_{S_i} \\
&\quad - \frac{\chi}{4} \sum_{i=5,6} \frac{\text{Res}}{\tau=\infty} \frac{1}{(1 + \tau)(1 + (1 + \chi)\tau)} \sum_{\text{particles } j=1}^{6} A_{(j)}(\tau) \bigg|_{S_i}, \\
&\quad + \frac{1 + \chi}{s_{12}\chi} \sum_{i=1,3} \frac{\text{Res}}{\tau=0} \frac{1}{\tau} \sum_{\text{particles } j=1}^{6} A_{(j)}(z) \bigg|_{S_i} \\
&\quad + \frac{1 + \chi}{s_{12}\chi} \sum_{i=5,6} \frac{\text{Res}}{\tau=\infty} \frac{1}{1 + (1 + \chi)\tau} \sum_{\text{particles } j=1}^{6} A_{(j)}(\tau) \bigg|_{S_i} \\
&\quad - \frac{1}{2s_{12}\chi} \sum_{i=5,6} \frac{\text{Res}}{\tau=-1} \frac{1}{1 + \tau} \sum_{\text{particles } j=1}^{6} A_{(j)}(\tau) \bigg|_{S_i} \\
&\quad + \frac{3}{2s_{12}} \sum_{i=5,6} \frac{\text{Res}}{\tau=\infty} \frac{1}{(1 + \tau)(1 + (1 + \chi)\tau)} \sum_{\text{particles } j=1}^{6} A_{(j)}(\tau) \bigg|_{S_i}. \quad (5.16)
\end{align*}
\]

Along these lines it is straightforward to obtain expressions for the master integral coefficients formulated in terms of residues that are compatible with any parametrization of the loop momenta.

The explicit mapping between the integrand basis coefficients and the tree-level data is quite complicated. Using Mathematica we are able shuffle around the null-space conditions appropriately in order to establish full analytical equivalence between the two master integral coefficients for both the planar and nonplanar double box prior to any reference to particle content of the gauge theory in consideration.
6 Conclusion

The unitarity method has been applied widely with great success to otherwise unattainable computations of loop corrections to scattering amplitudes. In particular, generalized unitarity provides means for determining one-loop amplitudes from an integral basis whose elements are known explicitly. By imposing multiple simultaneous on-shell conditions on internal propagators, single integrals are projected and their coefficients are expressed in terms of tree-level data.

In this paper we have extended four-dimensional maximal unitarity \[47\] to the nonplanar case. In maximal unitarity computations all propagators are cut by placing them on their mass-shell. The massless four-point two-loop nonplanar double box admits expansion onto two master integrals that are sensitive to hepta-cuts. Maximal cuts are naturally defined by promoting real slice Feynman integrals to multidimensional complex contour integrals encircling the global poles of the loop integrand while requiring continued validity of all integral reduction identities. In order to conform with this principle, each global pole or contour must have a weight. We used this approach to derive unique and strikingly compact formulae for both master integral coefficients. Moreover, we compared our results to coefficients recently computed by integrand-level reduction and found exact agreement in any renormalizable gauge theory with adjoint matter.

We finally mention several interesting directions for future research. It would of course be extremely useful to have formulae for master integral coefficients for subleading topologies. However, these integrals are only accessible using cuts with fewer propagators and thus more complicated to isolate. It is certainly also important to consider \(D\)-dimensional unitarity cuts in order to capture pieces that are not detectable in four dimensions. Indeed, it is possible to establish nonzero linear combinations of tensor integrals whose hepta-cuts vanish identically at \(\mathcal{O}(\epsilon^0)\) \[47\]. However, the most urgent point to address is probably how integration-by-parts identities constrain contours. In particular, a deeper understanding of the unexpected simplicity of contour weights is desirable. A very natural extension of our work is to study nonplanar double boxes with one or more massive external legs or even internal masses. Guided by recent results for planar triple boxes obtained by integrand reconstruction we also expect that the framework of maximal unitarity can be applied beyond two loops. We hope to return to some of these questions soon.

Acknowledgments

It is a pleasure to thank Emil Bjerrum-Bohr, Poul Henrik Damgaard and Yang Zhang for many stimulating discussions. The author is grateful to the theoretical elementary particle physics group at UCLA and in particular Zvi Bern for hospitality during the completion of this work.
A Kinematical Configurations of the Two-Loop Crossed Box

We depict here the eight valid kinematical configurations of the maximally cut two-loop crossed box, labelled according to solutions $S_1, \ldots, S_8$. Holomorphically-collinear and antiholomorphically-collinear three-vertices are represented by $\ominus$ and $\oplus$ respectively.
B Two-Loop Crossed Box Integration-By-Parts Identities

We provide below all four-dimensional integration-by-parts identities used for the reduction onto master integrals of all renormalizable four-point tensor integrals with two-loop crossed box topology. Ellipses denote truncation at the maximal number of propagators.

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^2] = -\frac{1}{16}(1 + \chi)\chi s_{12}^2 \mathcal{I}^{NP}[1] + \frac{3}{8}(1 + 2\chi) s_{12} \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^3] = -\frac{1}{32}(1 + \chi)(1 + 2\chi) s_{12}^3 \mathcal{I}^{NP}[1]
+ \frac{1}{16}(3 + 8\chi(1 + \chi)) s_{12}^2 \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^4] = -\frac{1}{64}(1 + \chi)(1 + 3\chi(1 + \chi)) s_{12}^4 \mathcal{I}^{NP}[1]
+ \frac{1}{32}(1 + 2\chi)(3 + 5\chi(1 + \chi)) s_{12}^3 \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)] = -2 \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^2] = -\frac{1}{8}\chi s_{12}^2(1 + \chi) \mathcal{I}^{NP}[1] + \frac{3}{4}(1 + 2\chi) s_{12} \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^3] = + \frac{1}{16}\chi s_{12}^3(1 + \chi)(1 + 2\chi) \mathcal{I}^{NP}[1]
- \frac{1}{8} s_{12}^2(1 + 2(1 + 2\chi)^2) \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^4] = -\frac{1}{128}\chi s_{12}^4(1 + \chi)(1 + 3(1 + 2\chi)^2) \mathcal{I}^{NP}[1]
+ \frac{1}{64} s_{12}^3(1 + 2\chi)(7 s_{12}^2 + 5(1 + 2\chi)^2) \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^5] = + \frac{1}{128} \chi s_{12}^5(1 + \chi)(1 + 2\chi)(1 + (1 + 2\chi)^2) \mathcal{I}^{NP}[1]
- \frac{1}{128} s_{12}^4(1 + (1 + 2\chi)^2(8 + 3(1 + 2\chi)^2)) \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_2 \cdot k_2)^6] = - \frac{1}{2048} \chi s_{12}^6(1 + \chi)(1 + 2(1 + 2\chi)^2)(10 + 5(1 + 2\chi)^2)) \mathcal{I}^{NP}[1]
+ \frac{1}{1024} s_{12}^5(1 + 2\chi)(11 + (1 + 2\chi)^2(30 + 7(1 + 2\chi)^2)) \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)(\ell_2 \cdot k_2)] = + \frac{1}{16} \chi(1 + \chi) s_{12}^2 \mathcal{I}^{NP}[1] - \frac{3}{8}(1 + 2\chi) s_{12} \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]

\[
\mathcal{I}^{NP}[(\ell_1 \cdot k_3)^2(\ell_2 \cdot k_2)] = + \frac{1}{32} \chi(1 + \chi)(1 + 2\chi) s_{12}^3 \mathcal{I}^{NP}[1]
- \frac{1}{16}(3 + 8\chi(1 + \chi)) s_{12}^2 \mathcal{I}^{NP}[(\ell_1 \cdot k_3)] + \cdots
\]
\[ I^{NP}_{\l_1 \cdot k_3}((l_2 \cdot k_2)) = \frac{1}{64} \chi(1 + \chi)(1 + 3 \chi(1 + \chi))s_{12}^4 I^{NP}_{\l_1 \cdot k_3}[1] \]
\[- \frac{1}{32} (1 + 2 \chi)(3 + 5 \chi(1 + \chi))s_{12}^3 I^{NP}_{\l_1 \cdot k_3] + \cdots \]

\[ I^{NP}_{\l_1 \cdot k_3}((l_2 \cdot k_2)^2) = \frac{1}{128} \chi(1 + \chi)(1 + 2 \chi)(1 + 2 \chi(1 + \chi))s_{12}^4 I^{NP}_{\l_1 \cdot k_3] + \cdots \]
\[- \frac{1}{256} s_{12}^4 (1 + (1 + 2 \chi)^2(8 + 3(1 + 2 \chi)^2)) I^{NP}_{\l_1 \cdot k_3] + \cdots \]

\[ I^{NP}_{\l_1 \cdot k_3}((l_2 \cdot k_2)^2) = - \frac{1}{32} \chi(1 + \chi)(1 + 2 \chi) s_{12}^3 I^{NP}_{\l_1 \cdot k_3] + \cdots \]
\[ + \frac{1}{16} s_{12}^3 (3 + 8 \chi(1 + \chi)) I^{NP}_{\l_1 \cdot k_3] + \cdots \]

\[ I^{NP}_{\l_1 \cdot k_3}((l_2 \cdot k_2)^2) = - \frac{1}{64} \chi(1 + \chi)(1 + 3 \chi(1 + \chi)) s_{12}^4 I^{NP}_{\l_1 \cdot k_3] + \cdots \]
\[- \frac{1}{64} s_{12}^2 (1 + \chi)(1 + 3 \chi(1 + \chi)) I^{NP}_{\l_1 \cdot k_3] + \cdots \]

\[ I^{NP}_{\l_1 \cdot k_3}((l_2 \cdot k_2)^2) = - \frac{1}{128} \chi(1 + \chi)(1 + 2 \chi)(1 + 2 \chi(1 + \chi)) s_{12}^4 I^{NP}_{\l_1 \cdot k_3] + \cdots \]
\[ + \frac{1}{256} s_{12}^4 (1 + (1 + 2 \chi)^2(8 + 3(1 + 2 \chi)^2)) I^{NP}_{\l_1 \cdot k_3] + \cdots \]

\[ I^{NP}_{\l_1 \cdot k_3}((l_2 \cdot k_2)^2) = - \frac{1}{4096} s_{12}^2 (1 + \chi)(1 + 2 \chi)(1 + 2 \chi)(10 + 5(1 + 2 \chi)^2)) I^{NP}_{\l_1 \cdot k_3] + \cdots \]
\[ + \frac{1}{2048} s_{12}^5 (1 + 2 \chi)(1 + 2 \chi)(20 + 7(1 + 2 \chi)^2)) I^{NP}_{\l_1 \cdot k_3] + \cdots \]
C Planar Double Box Integration-By-Parts Identities

For completeness we also include all truncated integration-by-parts identities relevant for the planar double box with four massless external lines.

\[
\begin{align*}
\mathcal{I}^P[(\ell_1 \cdot k_4)^2] &= \frac{1}{2} \chi s_{12} \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)^3] &= \frac{1}{4} \chi^2 s_{12}^2 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)^4] &= \frac{1}{8} \chi^3 s_{12}^3 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_2 \cdot k_1)] &= \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_2 \cdot k_1)^2] &= \frac{1}{2} \chi s_{12} \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_2 \cdot k_1)^3] &= \frac{1}{4} \chi^2 s_{12}^2 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_2 \cdot k_1)^4] &= \frac{1}{8} \chi^3 s_{12}^3 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)(\ell_2 \cdot k_1)] &= \frac{1}{8} \chi s_{12} \mathcal{I}^P[1] - \frac{3}{4} s_{12} \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)^2(\ell_2 \cdot k_1)] &= -\frac{1}{16} \chi s_{12}^2 \mathcal{I}^P[1] + \frac{3}{8} s_{12}^2 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)^3(\ell_2 \cdot k_1)] &= \frac{1}{32} \chi s_{12}^3 \mathcal{I}^P[1] - \frac{3}{16} s_{12}^3 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)^4(\ell_2 \cdot k_1)] &= -\frac{1}{64} \chi s_{12}^4 \mathcal{I}^P[1] + \frac{3}{32} s_{12}^4 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)(\ell_2 \cdot k_1)^2] &= -\frac{1}{16} \chi s_{12}^3 \mathcal{I}^P[1] + \frac{3}{8} s_{12}^3 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)(\ell_2 \cdot k_1)^3] &= \frac{1}{32} \chi s_{12}^4 \mathcal{I}^P[1] - \frac{3}{16} s_{12}^4 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots \\
\mathcal{I}^P[(\ell_1 \cdot k_4)(\ell_2 \cdot k_1)^4] &= -\frac{1}{64} \chi s_{12}^5 \mathcal{I}^P[1] + \frac{3}{32} s_{12}^5 \mathcal{I}^P[(\ell_1 \cdot k_4)] + \cdots 
\end{align*}
\]
References

[1] E. Witten, Commun. Math. Phys. 252, 189 (2004) [hep-th/0312171].
[2] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 715, 499 (2005) [hep-th/0412308].
[3] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94, 181602 (2005) [hep-th/0501052].
[4] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D 78, 085011 (2008) [arXiv:0805.3993 [hep-ph]].
[5] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 435, 59 (1995) [hep-ph/9409265].
[6] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 425, 217 (1994) [hep-ph/9403226].
[7] Z. Bern and A. G. Morgan, Nucl. Phys. B 467, 479 (1996) [hep-ph/9511336].
[8] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B 513, 3 (1998) [hep-ph/9708239].
[9] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 725, 275 (2005) [hep-th/0412103].
[10] R. Britto, F. Cachazo and B. Feng, Phys. Rev. D 71, 025012 (2005) [hep-th/0410179].
[11] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, JHEP 0511, 027 (2005) [hep-ph/0507019].
[12] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, Phys. Lett. B 612, 75 (2005) [hep-th/0502028].
[13] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D 72, 065012 (2005) [hep-ph/0503132].
[14] R. Britto, B. Feng and P. Mastrolia, Phys. Rev. D 73, 105004 (2006) [hep-ph/0602178].
[15] P. Mastrolia, Phys. Lett. B 644, 272 (2007) [hep-th/0611091].
[16] A. Brandhuber, S. McNamara, B. J. Spence and G. Travaglini, JHEP 0510, 011 (2005) [hep-th/0506068].
[17] G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B 763, 147 (2007) [hep-ph/0609007].
[18] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, JHEP 0703, 111 (2007) [hep-ph/0612277].
[19] Z. Bern, L. J. Dixon and D. A. Kosower, Annals Phys. 322, 1587 (2007) [arXiv:0704.2798 [hep-ph]].
[20] D. Forde, Phys. Rev. D 75, 125019 (2007) [arXiv:0704.1835 [hep-ph]].
[21] S. D. Badger, JHEP 0901, 049 (2009) [arXiv:0806.4600 [hep-ph]].
[22] W. T. Giele, Z. Kunszt and K. Melnikov, JHEP 0804, 049 (2008) [arXiv:0801.2237 [hep-ph]].
[23] R. Britto and B. Feng, Phys. Rev. D 75, 105006 (2007) [hep-ph/0612089].
[24] R. Britto and B. Feng, JHEP 0802, 095 (2008) [arXiv:0711.4284 [hep-ph]].
[25] Z. Bern, J. J. Carrasco, T. Dennen, Y. -t. Huang and H. Ita, Phys. Rev. D 83, 085022 (2011) [arXiv:1010.0494 [hep-th]].
[26] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B 645, 213 (2007) [hep-ph/0609191].
[27] R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP 0803, 003 (2008) [arXiv:0708.2398 [hep-ph]].
[28] C. F. Berger, Z. Bern, L. J. Dixon, F. Febres Cordero, D. Forde, H. Ita, D. A. Kosower and D. Maitre, Phys. Rev. D 78, 036003 (2008) [arXiv:0803.4180 [hep-ph]].
[29] G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP 0803, 042 (2008) [arXiv:0711.3596 [hep-ph]].
[30] P. Mastrolia, G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP 0806, 030 (2008) [arXiv:0803.3964 [hep-ph]].
[31] W. T. Giele and G. Zanderighi, JHEP 0806, 038 (2008) [arXiv:0805.2152 [hep-ph]].
[32] C. F. Berger, Z. Bern, L. J. Dixon, F. Febres Cordero, D. Forde, T. Gleisberg, H. Ita and D. A. Kosower et al., Phys. Rev. Lett. 102, 222001 (2009) [arXiv:0902.2760 [hep-ph]].
[33] S. Badger, B. Biedermann and P. Uwer, Comput. Phys. Commun. 182, 1674 (2011) [arXiv:1011.2900 [hep-ph]].
[34] C. F. Berger, Z. Bern, L. J. Dixon, F. Febres Cordero, D. Forde, T. Gleisberg, H. Ita and D. A. Kosower et al., Phys. Rev. Lett. 106, 092001 (2011) [arXiv:1009.2338 [hep-ph]].
[35] V. Hirschi, R. Frederix, S. Frixione, M. V. Garzelli, F. Maltoni and R. Pittau, JHEP 1105, 044 (2011) [arXiv:1103.0621 [hep-ph]].
[36] Z. Bern, J. S. Rozowsky and B. Yan, Phys. Lett. B 401, 273 (1997) [hep-ph/9704242].
[37] Z. Bern, L. J. Dixon and D. A. Kosower, JHEP 0001, 027 (2000) [hep-ph/0001001].
[38] E. W. N. Glover, C. Oleari and M. E. Tejeda-Yeomans, Nucl. Phys. B 605, 467 (2001) [hep-ph/00102201].
[39] Z. Bern, A. De Freitas and L. J. Dixon, JHEP 0203, 018 (2002) [hep-ph/0201161].
[40] C. Anastasiou, E. W. N. Glover, C. Oleari and M. E. Tejeda-Yeomans, Nucl. Phys. B 601, 318 (2001) [hep-ph/0010212].
[41] C. Anastasiou, E. W. N. Glover, C. Oleari and M. E. Tejeda-Yeomans, Nucl. Phys. B 601, 341 (2001) [hep-ph/0011094].
[42] C. Anastasiou, E. W. N. Glover, C. Oleari and M. E. Tejeda-Yeomans, Nucl. Phys. B 605, 486 (2001) [hep-ph/0101304].
[43] E. I. Buchbinder and F. Cachazo, JHEP 0511, 036 (2005) [hep-th/0506126].
[44] F. Cachazo, arXiv:0803.1988 [hep-th].
[45] J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D 83, 045012 (2011) [arXiv:1009.0472 [hep-th]].
[46] R. M. Schabinger, JHEP 1201, 077 (2012) [arXiv:1111.4220 [hep-ph]].
[47] D. A. Kosower and K. J. Larsen, Phys. Rev. D 85, 045017 (2012) [arXiv:1108.1180 [hep-th]].
[48] K. J. Larsen, Phys. Rev. D 86, 085032 (2012) [arXiv:1205.0297 [hep-th]].
[49] S. Caron-Huot and K. J. Larsen, JHEP 1210, 026 (2012) [arXiv:1205.0801 [hep-ph]].
[50] H. Johansson, D. A. Kosower and K. J. Larsen, Phys. Rev. D 87, 025030 (2013) [arXiv:1208.1754 [hep-th]].
[51] H. Johansson, D. A. Kosower and K. J. Larsen, PoS LL 2012, 066 (2012) [PoS LL 2012, 066 (2012)] [arXiv:1212.2132 [hep-th]].

[52] S. Badger, H. Frellesvig and Y. Zhang, JHEP 1204, 055 (2012) [arXiv:1202.2019 [hep-ph]].

[53] P. Mastrolia and G. Ossola, JHEP 1111, 014 (2011) [arXiv:1107.6041 [hep-ph]].

[54] S. Badger, H. Frellesvig and Y. Zhang, JHEP 1208, 065 (2012) [arXiv:1207.2976 [hep-ph]].

[55] Y. Zhang, JHEP 1209, 042 (2012) [arXiv:1205.5707 [hep-ph]].

[56] B. Feng and R. Huang, JHEP 1302, 117 (2013) [arXiv:1209.3747 [hep-ph]].

[57] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, Phys. Lett. B 718, 173 (2012) [arXiv:1205.7087 [hep-ph]].

[58] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, arXiv:1209.4319 [hep-ph].

[59] P. Mastrolia, E. Mirabella, G. Ossola, T. Peraro and H. van Deurzen, PoS LL 2012 (2012) 028 [arXiv:1209.5678 [hep-ph]].

[60] R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, JHEP 1212, 038 (2012) [arXiv:1206.4180 [hep-ph]].

[61] R. Huang and Y. Zhang, JHEP 1304, 080 (2013) [arXiv:1302.1023 [hep-ph]].

[62] Z. Bern, J. J. M. Carrasco, H. Ita, H. Johansson and R. Roiban, Phys. Rev. D 80, 065029 (2009) [arXiv:0903.5348 [hep-th]].

[63] M. Sogaard, Phys. Rev. D 84, 065011 (2011) [arXiv:1106.3785 [hep-th]].

[64] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP 1003, 020 (2010) [arXiv:0907.5418 [hep-th]].

[65] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D 78, 085011 (2008) [arXiv:0805.3993 [hep-ph]].

[66] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. Lett. 105, 061602 (2010) [arXiv:1004.0476 [hep-th]].

[67] V. A. Smirnov, Phys. Lett. B 460, 397 (1999) [hep-ph/9905323].

[68] V. A. Smirnov and O. L. Veretin, Nucl. Phys. B 566, 469 (2000) [hep-ph/9907385].

[69] J. B. Tausk, Phys. Lett. B 469, 225 (1999) [hep-ph/9909506].

[70] C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J. B. Tausk, Nucl. Phys. B 580, 577 (2000) [hep-ph/0003261].