An Invitation to Noncommutative Algebra

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Abstract This is a brief introduction to the world of Noncommutative Algebra aimed for advanced undergraduate and beginning graduate students.

1 Introduction

The purpose of this note is to invite you, the reader, into the world of Noncommutative Algebra. What is it? In short, it is the study of algebraic structures that have a noncommutative multiplication. One’s first encounter with these structures occurs typically with matrices. Indeed, given two $n$-by-$n$ matrices $X$ and $Y$ with $n > 1$, we get that $XY \neq YX$ in general. But this simple observation motivates a deeper reason why Noncommutative Algebra is ubiquitous...

Let’s consider two basic transformations of images in real 2-space: Rotation by 90 degrees clockwise and Reflection about the vertical axis. As we see in the figures below, the order in which these transformations are performed matters.

Fig. 1: The composition of rotation and reflection transformations is noncommutative.

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Since these transformations are linear (i.e., in \( \mathbb{R}^2 \), lines are sent to lines), they can be encoded by 2-by-2 matrices with entries in \( \mathbb{R} \). Namely,

- 90° CW Rotation corresponds to \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), which sends \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) to \( \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} \);
- Reflection about the y-axis is encoded by \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), which sends \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) to \( \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix} \).

The composition of linear transformations is then encoded by matrix multiplication. So, the first row in Figure 1 is corresponds to \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \), which sends \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) to \( \begin{pmatrix} -v_2 \\ -v_1 \end{pmatrix} \). Yet the second row is given by \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), sending \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) to \( \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \). Therefore, the outcome of Figure 1 is a result of the fact that \( \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

One can cook up other, say higher dimensional, examples of the varying outcomes of composing linear transformations by exploiting the noncommutativity of matrix multiplication. This is all part of the general phenomenon that functions do not commute under composition typically. (Think of the myriad of outcomes of composing functions from everyday life— for instance, washing and drying clothes!)

Now let’s turn our attention to special functions that we first encounter as children: Symmetries. To make this concept more concrete mathematically, consider the informal definition and notation below.

**Definition 1.** Take any object \( X \). Then, a symmetry of \( X \) is an invertible, property-preserving transformation from \( X \) to itself. The collection of such transformations is denoted by \( \text{Sym}(X) \).

Historically, the examination of symmetries in mathematics and physics served as one of the inspirations for the defining a group as an abstract algebraic structure (see, e.g., [43, Section 1(c)]). Namely, \( \text{Sym}(X) \) is a group with the identity element \( e \) being the “do nothing” transformation, with composition as the binary operation, and \( \text{Sym}(X) \) is equipped with inverse elements by definition.

Continuing the example above: Take \( X = \mathbb{R}^2 \) and \( \text{Sym}(\mathbb{R}^2) \) to be the collection of \( \mathbb{R} \)-linear transformations from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) (so the origin is fixed). We get that \( \text{Sym}(\mathbb{R}^2) \) is the general linear group \( \text{GL}(\mathbb{R}^2) \), often written as \( \text{GL}_2(\mathbb{R}) \) denoting the group of all invertible 2-by-2 matrices with real entries. Further, this group is nonabelian; thus, composition of \( \mathbb{R} \)-linear symmetries of \( \mathbb{R}^2 \) is noncommutative.

Another concept that is inherently noncommutative is that of a representation. We will see later in Section 3 that this is best motivated by elementary problem of finding matrix solutions to equations (which, in turn, can have physical implications). But for now let’s think about the problem below.

**Question 1.** Which matrices \( M \in \text{Mat}_2(\mathbb{R}) \) satisfy the equation \( x^2 = 1 \)?

Now one could do the chore of writing down an arbitrary matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and solve for entries \( a, b, c, d \) that satisfy

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Not only is this boring, and it can be very tedious to find solutions to more general problems (e.g., taking instead $M \in \text{Mat}_n(\mathbb{R})$ for $n > 2$). For a more elegant approach to Question 1, consider an abstract algebraic structure $T$ defined by the equation $x^2 = 1$, and link $T$ to $\text{Sym}(\mathbb{R}^2)$ via a structure preserving map $\phi$. Then, a solution to Question 1 is produced in terms of an image of $\phi$.

For example, we could take $T$ to be the group $\mathbb{Z}_2$ as its presentation is given by $\langle x \mid x^2 = e \rangle$. An example of a structure-preserving map $\phi$ is given by $\phi : \mathbb{Z}_2 \to \text{Sym}(\mathbb{R}^2)$, $e \mapsto \{\text{Do Nothing}\}$, $x \mapsto \{\text{Reflection about y-axis}\}$. Indeed, $\phi(ge) = \phi(g) \circ \phi(e)$ for all $g, g' \in \mathbb{Z}_2$. For instance, $\phi(x) \circ \phi(x) = \{\text{Ref. about y-axis}\} \circ \{\text{Ref. about y-axis}\} = \{\text{Nothing}\} = \phi(e) = \phi(x^2)$.

Since $\phi(e)$ and $\phi(x)$ correspond respectively to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, these matrices are solutions to Question 1. Further, other reflections of $\mathbb{R}^2$ produce additional solutions to Question 1 (Think about if all solutions to Question 1 can be constructed in this manner.)

Continuing this example, instead of using the abstract group $\mathbb{Z}_2$ we could have used the group algebra $T = \mathbb{R}\mathbb{Z}_2$, as it encodes the same information needed to address Question 1. We will chat more about abstract algebraic structures in Section 2 (see Figure 5); in any case, their representations are defined informally below.

**Definition 2.** Given an abstract algebraic structure $T$, we say that a representation of $T$ is an object $X$ equipped with a property-preserving map $T \to \text{Sym}(X)$.

An example of a representation of a group $G$ is a vector space $V$ equipped with a group homomorphism $G \to \text{GL}(V)$, where $\text{GL}(V)$ is the group of invertible linear transformations from $V$ to itself (e.g., $\text{GL}(\mathbb{R}^2) = \text{GL}_2(\mathbb{R})$ as discussed above). Just as a representation of $G$ is identified as a $G$-module, representations of rings and of algebras coincide with modules over such structures (see Figure 12 below). See also [50, Chapters 1 and 3] for further reading and examples.

Now Representation Theory is essentially a noncommutative area due to the following key fact. Take $A$ to be a commutative algebra over a field $\mathbb{k}$ with a representation $V$ of $A$, that is, a $\mathbb{k}$-vector space $V$ equipped with algebra map $\phi : A \to \text{GL}(V)$. If $(V, \phi)$ is irreducible [Definition 14], then $\dim_\mathbb{k} V = 1$ [50, Section 1.3.2]. Therefore, representations of commutative algebras aren’t so interesting.
Moreover, Representation Theory is a vital subject because the problem of finding matrix solutions to equations is quite natural. Since this boils down to studying representations of algebras that are generally noncommutative, the ubiquity of Noncommutative Algebra is conceivable. (Equations that correspond to representations of groups, like in Question 1, are special.)

To introduce the final notion in Noncommutative Algebra that we will highlight in this paper, observe symmetries and representations both occur under an action of a gadget $T$ on an object $X$, but the difference is that symmetries form the gadget $T$ (what is acting on an object), whereas representations are considered to be the object $X$ (something being acted upon). What happens to these notions if we consider deformations of $T$ and $X$? Consider the following informal terminology.

**Definition 3.** A deformation of an object $X$ is an object $X_{\text{def}}$ that has many of the same characteristics of $X$, possibly with the exception of a few key features. In particular, a deformation of an algebraic structure $T$ is an algebraic structure $T_{\text{def}}$ of the same type that shares a (less complex) underlying structure of $T$.

For example, a deformation of a ring $R$ could be another ring $R_{\text{def}}$ that equals $R$ as abelian groups, possibly with a different multiplication than that of $R$ (see Figure 5).

Now if we deform an object $X$, is there a gadget $T_{\text{def}}$ that acts on $X_{\text{def}}$ naturally? On the other hand, if we deform the gadget $T$, is there a natural deformation $X_{\text{def}}$ of $X$ that comes equipped with an action of $T_{\text{def}}$? These are obvious questions, yet the answers are difficult to visualize. This is because, visually, symmetries of an object $X$ are destroyed when $X$ is altered, even slightly; see Figure 3 below.

So we need to think beyond what can be visualized and consider a larger mathematical framework that handles symmetries under deformation. To do so, it is essential to think beyond group actions, because, many classes of groups, including finite groups, do not admit deformations. However, group algebras or function algebras on groups do admit deformations, so we include these gadgets in the improved framework to study symmetries. We will see later in Section 4 that when symmetries are recast in the setting where they could be preserved under deformation, other interesting and more general algebraic gadgets like bialgebras and Hopf algebras arise in the process. This is crucial in Noncommutative Algebra as some of the most important rings, especially those arising in physics, are noncommutative deformations of commutative rings; the symmetries of such deformations deserve attention.
Symmetries, Representations, and Deformations will play a key role throughout this article, just as they do in Noncommutative Algebra.

The remainder of the paper is two-fold: first, we will review three historical snapshots of how Noncommutative Algebra played a prominent role in mathematics and physics. We will discuss William Rowen Hamilton’s Quaternions in Section 2 and the birth of Quantum Mechanics in Section 3. We will also briefly discuss the emergence of Quantum Groups in Section 4, together with the concept of Quantum Symmetry. In Section 5 we present a couple of research avenues for further investigation. All of the material here is by no means exhaustive, and many references will be provided throughout.

2 Hamilton’s Quaternions (1840s - 1860s)

Can numbers be noncommutative? The best answer is, as always, “Sure, why not!”

In this section we will explore a number system that generalizes both the systems of real numbers \( \mathbb{R} \) and complex numbers \( \mathbb{C} \). The key feature of this new collection of numbers – the quaternions \( \mathbb{H} \) – is that they have a noncommutative multiplication! This feature caused a bit of ruckus for William Rowan Hamilton (1805-1865) after his discovery of the quaternions in the mid-19th century.

“Quaternions came from Hamilton after really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way...”
– Lord Kelvin, 1892

Now what do we mean by a number system? Loosely speaking, it is a set of quantities used to measure or count (a collection of) objects, which is equipped with an algebraic structure [Figure 5].

Since we should be able to add, subtract, multiply and divide numbers, we consider the following terminology.

Definition 4. Fix \( n \in \mathbb{Z}_{\geq 1} \). An \( n \)-dimensional division algebra \( D \) over \( \mathbb{R} \) consists of the set of \( n \)-tuples of real numbers \( a := (a_1, a_2, \ldots, a_n), a_i \in \mathbb{R}, \) with \( \mathbf{0} := (0, 0, \ldots, 0) \) and a unique element designated as \( 1 \) so that

- we can add and subtract two \( n \)-tuples \( a \) and \( b \) component-wise to form \( a + b \) and \( a - b \) in \( D \), respectively;
- we can multiply \( a \) by a scalar \( \lambda \in \mathbb{R} \) component-wise to form \( \lambda \cdot a \) in \( D \);
- there is a rule for multiplying \( a \) and \( b \) to form \( a \cdot b \) in \( D \) (this is not necessarily done component-wise, nor does it need to be commutative); and
- there is a rule for dividing \( a \) by \( b \neq \mathbf{0} \) to form \( a ÷ b \) in \( D \);
in such a way that

(i) \((D, +, -, 0)\) is an abelian group,
(ii) \((D, +, -, 0, *)\) is an \(\mathbb{R}\)-vector space,
(iii) \((D, \cdot, 1)\) is an associative unital ring,
(iv) \((D, +, -, *, 0, 1)\) is an associative \(\mathbb{R}\)-algebra

all in a compatible fashion (e.g., \(\cdot\) distributes over \(+\), etc.).

As one can imagine, there are not many of these gadgets floating around as they have a lot of structure. A \(1\)-dimensional division algebra \(D\) over \(\mathbb{R}\) must be the field \(\mathbb{R}\) itself. Moreover, a \(2\)-dimensional division algebra \(D\) over \(\mathbb{R}\) is isomorphic to the field of complex numbers \(\mathbb{C}\), where the pair \((a_1, a_2)\) is identified with the element \(a_1 + a_2i\) for \(i^2 = -1\). The algebraic structure for the pairs then follows accordingly, e.g., the multiplication of \(\mathbb{C}\) is given by

\[
(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1).
\]

Note that the \(1\)- and \(2\)-dimensional real division algebras, \(\mathbb{R}\) and \(\mathbb{C}\), are commutative rings, and these can be viewed geometrically as in Figure 6.

A natural question is then the following.

**Question 2.** What are the \(n\)-dimensional real division algebras for \(n \geq 3\)?

Hamilton obsessed over this question, especially the \(n = 3\) case, for over a decade. Even his children would routinely ask him, “Papa, can you multiply triplets?”.
His initial ideas were to use two imaginary axes $i$ and $j$ so that the 3-tuples $(a_1,a_2,a_3)$ of a 3-dimensional number system correspond to $a_1 + a_2 i + a_3 j$. However, he could not cook up rules that $i$ and $j$ should obey to make this collection of triples a valid division algebra \[30\] \[55\] \[67\]. According to some mathematicians, this obsession was quite ‘Mad’ \[4\] \[60\].

Finally, on October 16th 1843, on a walk with his wife in Dublin, Hamilton had a moment of Eureka! In his words to his son Archibald,

> “An electric circuit seemed to close; and a spark flashed forth, the herald, as I foresaw immediately, of many long years to come of definitely directed thought and work [...] Nor could I resist the impulse, unphilosophical as it may have been, to cut with a knife on the stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols $i$, $j$, $k$; namely $i^2 = j^2 = k^2 = i j k = -1$, which contains the solution of the problem...”

– W. R. Hamilton, August 5th, 1865

Hamilton had discovered that day a number system generalizing both $\mathbb{R}$ and $\mathbb{C}$, consisting of 4-tuples of real numbers, not constructed from triplets as he had imagined for so long \[30\].

**Definition 5.** The *quaternions* is a 4-dimensional real division algebra, denoted by $\mathbb{H}$, comprised of 4-tuples of real numbers $a := (a_0, a_1, a_2, a_3)$, which are identified as elements of the form $a_0 + a_1 i + a_2 j + a_3 k$, for $a_i \in \mathbb{R}$, where addition, subtraction and scalar multiplication are performed component-wise, and multiplication and division are governed by the rule

\[
i^2 = j^2 = k^2 = i j k = -1.
\]

Observe that $j k = i$, whereas $k j = -i$. Therefore, $\mathbb{H}$ is a noncommutative ring!
In any case, notice that the multiplication rule of $\mathbb{H}$ is a bit complicated:

\[
(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3) = \begin{pmatrix}
a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3, \\
a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2, \\
a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1, \\
a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0
\end{pmatrix}.
\tag{1}
\]

... and let’s not commit this rule to memory. To circumvent this issue Hamilton gave the quaternions a geometric realization that encodes their multiplication. Namely, for $q := a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$, let

- $a_0$ be the “scalar” component of $q$, and
- $\bar{a} := a_1 i + a_2 j + a_3 k$ be the “vector” component of $q$.

Then, the vector components are visualized as points/vectors in $\mathbb{R}^3$, whereas the scalar component is realized as an element of time. See, for instance, the footnote on page 60 and other parts of the preface of [29] for Hamilton’s original thoughts on the connection between the quaternions and the laws of space and time. (Yes, yes, this was all very controversial back then!)

Hamilton then devised two vector operations, now known as the dot product ($\cdot$) and cross product ($\times$) to make the multiplication rule of $\mathbb{H}$ more compact:

\[
a \cdot b = [a_0b_0 - \bar{a} \cdot \bar{b}] + [a_0\bar{b} + b_0\bar{a} + \bar{a} \times \bar{b}], \quad \forall a, b \in \mathbb{H}.
\tag{2}
\]

Not only is formula (2) is easier to retain than (1), the (commutative) dot product and (noncommutative) cross product have appeared in various parts of mathematics and physics throughout the years, including our multi-variable calculus courses.

Geometrically, the operations in $\mathbb{H}$ capture symmetries of $\mathbb{R}^3$ [Definition 1]: Addition/substraction, scalar multiplication, and multiplication/division correspond respectively to translation, dilation, and rotation of vectors of $\mathbb{R}^3$; see, e.g., [29, page 272] and [45] for a discussion of rotation. To see rotations in action, first note that the length of a quaternion $q$ is given by

\[
|q| := \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}.
\]

Next, fix an axis of rotation $n := n_1 i + n_2 j + n_3 k$ with $|n| = 1$, a quaternion of unit-length. Then, rotating a vector $\bar{q}$ about the axis $n$ clockwise by $\theta$ radians (when viewed from the origin) corresponds to conjugating $\bar{q}$ by the quaternion $e^{\frac{\theta}{2} n}$. It’s helpful to use here an extension of Euler’s formula, $e^{\frac{\theta}{2} n} = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})n$, to understand the quaternion $e^{\frac{\theta}{2} n}$. An example is given in Figure 10 below.
Moreover, rotations of $\mathbb{R}^3$ can be encoded as a representation [Definition 2] of the multiplicative subgroup $U(\mathbb{H})$ consisting of unit-length quaternions. Indeed, we have a group homomorphism

$$U(\mathbb{H}) \longrightarrow GL(\mathbb{R}^3) = GL_3(\mathbb{R}),$$

given by

$$a_0 + a_1 i + a_2 j + a_3 k \quad \text{with } |a| = 1 \quad \mapsto \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2a_1a_2 - 2a_0a_3 & 2a_1a_3 + 2a_0a_2 \\ 2a_1a_2 + 2a_0a_3 & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2a_2a_3 - 2a_0a_1 \\ 2a_1a_3 - 2a_0a_2 & 2a_2a_3 + 2a_0a_1 & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}.$$ 

This geometric realization of $\mathbb{H}$ has many modern applications– we refer to the text [44] for a nice self-contained discussion of applications to computer-aided design, aerospace engineering, and other fields.

Returning to Question 2, its answer is now given below.

**Theorem 1.** [20, 62, 45 Theorem 13.12], [8, 42, 71] The answer to Question 2 is Yes if and only if $n = 1, 2, 4, 8$. Such division algebras $D$ are unique up to isomorphism in their dimension with isomorphism class represented by

- the real numbers $\mathbb{R}$ for $n = 1$,
- the complex numbers $\mathbb{C}$ for $n = 2$,
- quaternions $\mathbb{H}$ for $n = 4$,
- octonions $\mathbb{O}$ for $n = 8$.

Here, $D$ is commutative only when $n = 1, 2$, and is associative only when $n = 1, 2, 4$.

So, Hamilton discovered the last associative finite-dimensional real division algebra, but the price that he had to pay (at least mathematically) was the lost of commutativity. Perhaps this was not too high of a price– we are certainly willing to lose ordering to choose to work with $\mathbb{C}$ instead of $\mathbb{R}$. If we are also willing to part with associativity, then the octonions $\mathbb{O}$ is a perfectly suitable number system; see [3] for further reading. And, of course, there are further generalizations of number systems– see [19, 49, 70] to start, and go wild!

We return to the quaternions later in Section 5.1 for a discussion of potential research directions.
3 The Birth of Quantum Mechanics (1920s)

Another period that sparked an interest in Noncommutative Algebra was the birth of Quantum Mechanics in the 1920s. Three of the key figures during this time were Max Born (1882-1970), Werner Heisenberg (1901-1976), and Paul Dirac (1902-1984), who were all curious about the behavior of subatomic particles [7, 32, 20].

Along with their colleagues, Born, Heisenberg, and Dirac believed that important aspects of subatomic behavior are those that could be observed (or measured). However, the tools of classical mechanics available at the time (with observables corresponding to real-valued functions) were not suitable in capturing this behavior properly. A new type of mechanics was needed, leading to the development of quantum mechanics where observables are realized as linear operators. For a great account of how this transition took place (some of which we will recall briefly below), see Part II of the van der Waerden’s text [68]. (For historical context of another figure, Pascual Jordan, who also played a role in these developments, see, e.g., [34].)

The two observables in which Born, Heisenberg, and Dirac were especially interested were the position and momentum of a subatomic particle, and they employed Niels Bohr’s notion of orbits to keep track of these quantities. Mathematically, this boils down to using matrices in order to book-keep data corresponding to the observables under investigation, thus initiating matrix mechanics. The surprising outcome of using this new matrix framework in studying subatomic particles was stated succinctly as follows [33]:

The more precisely the position is determined, the less precisely the momentum is known, and vice versa.
– Heisenberg’s “Uncertainty Principle”, 1927
More precisely, suppose that $P$ and $Q$ are square matrices of the same size representing the observables momentum and position, respectively. The fact that $P$ and $Q$ do not commute typically (as one expects in classic mechanics) led to the discovery of what Born dubbed as *The Fundamental Equation of Quantum Mechanics*:

$$PQ - QP = i \frac{\hbar}{2\pi} \ast I,$$

(3)

Here $i$ is the square root of $-1$, $\hbar$ is Planck’s constant, and $\frac{\hbar}{2\pi} \ast I$ is the scalar multiple of the identity matrix $I$ of the same size as $P$ and $Q$. For physical reasons, it was known early in the theory of quantum mechanics that matrices $P$ and $Q$ that satisfy Equation (3) should be of infinite size, and we will recall a well-known, mathematical proof of this fact later in Proposition 1.

As done in practice by many physicists and mathematicians, through rescaling let’s consider a normalized version of The Fundamental Equation, as this still captures the spirit of Heisenberg’s Uncertainty Principle:

$$PQ - QP = I,$$

(4)

Now with today’s technology, one convenient way of studying matrix solutions $P$ and $Q$ to Equation (4) (or to Equation (3)) is to use the theory of representations of (associative) algebras. To see this connection, first let’s fix a field $k$ and for ease:

**Standing Hypothesis.** We assume in this section that $k$ is a field of characteristic 0.

Then recall from Definition (and Figure 5) that a $k$-algebra $A$ is an $k$-vector space equipped with the structure of an unital ring in a compatible fashion. In this case, $A = \langle A, +, -, \ast, 0, 1 \rangle$ where $\langle A, +, -, \ast, 0 \rangle$ is the $k$-vector space structure where $+$ is the abelian group operation and $\ast$ is scalar multiplication, and $\langle A, \cdot, 1 \rangle$ is an unital ring with $\cdot$ denoting its multiplication. Next, we make our vague notion of representations in Definition 2 more precise in the context of $k$-algebras.

**Definition 6.** Consider the following notions.

1. For a $k$-vector space $V$, the endomorphism algebra $\text{End}(V)$ on $V$ is an $k$-algebra consisting of endomorphisms of $V$ with multiplication being composition $\circ$. (If $V$ is an $n$-dimensional $k$-vector space, then $\text{End}(V)$ is isomorphic to the matrix algebra $\text{Mat}_n(k)$ with matrix multiplication. Here, $n$ could be infinite.)

2. A representation of an associative $k$-algebra $A$ is a $k$-vector space $V$ equipped with a $k$-algebra homomorphism $\phi : A \rightarrow \text{End}(V)$; say $\phi(a) = : \phi_a \in \text{End}(V)$ for $a \in A$. Namely, for all $a, b \in A$, $\lambda \in k$, and $v \in V$, we get that

$$\phi_{a+b}(v) = \phi_a(v) + \phi_b(v), \quad \phi_{\lambda \cdot a}(v) = \lambda \ast \phi_a(v), \quad \phi_{a \ast b}(v) = (\phi_a \circ \phi_b)(v).$$

3. The dimension of a representation $(V, \phi)$ of an associative $k$-algebra $A$ is the $k$-vector space dimension of $V$, which could be infinite.

Representations of associative $k$-algebras $A$ go hand-in-hand with $A$-modules, as illustrated in Figure 12 below.
Now for the purposes of finding matrix solutions of Equation (4), consider the \(k\)-algebra defined below.

**Definition 7.** The (first) Weyl algebra over a field \(k\) is the \(k\)-algebra \(A_1(k)\) generated by noncommuting variables \(x\) and \(y\), subject to relation \(yx - xy = 1\). That is, \(A_1(k)\) has a \(k\)-algebra presentation

\[
A_1(k) = \langle x, y \rangle/(yx - xy - 1),
\]

given as the quotient algebra of the free algebra \(k\langle x, y \rangle\) (consisting of words in variables \(x\) and \(y\)) by the ideal \((yx - xy - 1)\) of \(k\langle x, y \rangle\). (This algebra is sometimes referred to as the Heisenberg-Weyl algebra due to its roots in physics.)

The Weyl algebra is also the first example of an algebra of differential operators—its generators \(x\) and \(y\) can be viewed as the differential operators on the polynomial algebra \(k[x]\) given by multiplication by \(x\) and \(\frac{d}{dx}\), respectively. (Check that \(\frac{d}{dx}x - x \frac{d}{dx}\) is indeed the identity operator on \(k[x]\).)

Returning to the problem of finding \(n\)-by-\(n\) matrix solutions to Normalized Fundamental Equation (4)—this is equivalent to the task of constructing \(n\)-dimensional representations of \(A_1(k)\) as shown in Figure 13 below. In fact, this is why \(A_1(k)\) is known as the ring of quantum mechanics.
Next, with the toolkit of matrices handy, we obtain a well-known result on the size of matrix solutions to (4). We need following facts about the trace of a square matrix $X$ (which is the sum of the diagonal entries of $X$): $\text{tr}(X \pm Y) = \text{tr}(X) \pm \text{tr}(Y)$ and $\text{tr}(XY) = \text{tr}(YX)$ for any $X, Y \in \text{Mat}_n(\mathbb{k})$.

**Proposition 1.** The Normalized Fundamental Equation (4) does not admit finite matrix solutions, i.e., representations of $A_1(\mathbb{k})$ must be infinite-dimensional.

**Proof.** By way of contradiction, suppose that we have matrices $P, Q \in \text{Mat}_n(\mathbb{k})$ with $0 < n < \infty$ so that $PQ - QP = I$. Applying trace to both sides of this equation yields:

$$0 = \text{tr}(PQ) - \text{tr}(PQ) = \text{tr}(PQ) - \text{tr}(QP) = \text{tr}(PQ - QP) = \text{tr}(I) = n,$$

a contradiction as desired. \(\Box\)

On the other hand, the first Weyl algebra does have an infinite-dimensional representation. Take, for instance:

$$P = \begin{pmatrix} 0 & 1 & \cdots \\ 0 & 2 & \cdots \\ 0 & 3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 1 & \cdots \\ 1 & 0 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

...And there are many, many more!

But finding explicit matrix solutions to equations is computationally difficult in general, especially when the most important representations of an algebra are infinite-dimensional. The power of representation theory, however, is centered on its tools to address more abstract algebraic problems that are (perhaps) related to computational goals. For instance, representation theory may address some of the following questions for a given $\mathbb{k}$-algebra $A$, which are all quite natural:

- Do representations of $A$ exist? If so, what are their dimensions?
- When are two representations considered to be the same (or isomorphic)?
- Are (some of) the representations of $A$ parametrized by a geometric object $\mathcal{X}$?
- Do isomorphism classes of representations correspond bijectively to points of $\mathcal{X}$?

We will explore a few of these questions and further notions in Section 5.2 towards a research direction in Representation Theory.

The representation theory of other algebras of differential operators have also been key in modeling subatomic behavior. This includes Dirac’s quantum algebra that addresses the question of how several position observables $(Q_1, \ldots, Q_m)$ and momentum observables $(P_1, \ldots, P_m)$ commute, generalizing Heisenberg’s Uncertainty Principle for $m = 1$ \cite{20}. These days Dirac’s algebra is now known as the $m$-th Weyl algebra $A_m(\mathbb{k})$, which has $\mathbb{k}$-algebra presentation:

$$A_m(\mathbb{k}) = \mathbb{k}\langle x_1, \ldots, x_m, y_1, \ldots, y_m \rangle / (xy_j - y_jx_i, jy_j - y_jy_i, y_ix_j - x_jy_i - \delta_{i,j}). \tag{6}$$
Here, the generators $x_i$ and $y_i$ are viewed as elements of $\text{End}(k[x_1, \ldots, x_m])$ given respectively by multiplication by $x_i$ and partial derivation $\frac{\partial}{\partial x_i}$.

Want more physics?? We’re in luck– the representation theory of numerous non-commutative $k$-algebras play a vital role in several fields of physics. Some of these algebras and a physical area in which they appear are listed below. Happy exploring!

| Noncommutative $k$-Algebras                  | Appearance in Physics      | Reference (Year) |
|---------------------------------------------|----------------------------|------------------|
| $\mathfrak{W}$-algebras                     | Conformal Field Theory     | [9] (1993)       |
| 4-dimensional Sklyanin algebras             | Statistical Mechanics      | [64] (1982)      |
| 3-dimensional Sklyanin algebras             | String Theory              | [6] (2000)       |
| Yang-Mills algebras                        | Gauge Theory               | [13] (2002)      |
| Superpotential algebras                    | String Theory              | [25] (2006)      |
| Various Enveloping Algebras of Lie algebras | * Everywhere *             | Too many to list!|

4 Quantum Groups (1980s - 1990s) and Quantum Symmetries

Let’s begin here with a question mentioned in the introduction on the ties between symmetries [Definition 1] and deformations [Definition 3].

**Question 3.** How do we best handle (i.e., axiomatize, or “make mathematical”, the concept of) symmetries of deformations?

Several answers to this question leads us to use Hopf algebras [Definition 11]. But before we give the precise definition of this structure, we point out that Hopf algebras became prominent in mathematics in a few waves, including: its origins in Algebraic Topology [35], role in Combinatorics [38], and abstraction in Category Theory [40]. One tie to Noncommutative Algebra (in the context of Question 3) first appeared in the 1980s in statistical mechanics, especially in the Quantum Inverse Scattering Method for solving quantum integrable systems. The Hopf algebras that arose this way were coined Quantum Groups by Vladimir Drinfel’d [22], and have been a key structure in Noncommutative Algebra and physics ever since.

Instead of delving further into historical details, let’s now discuss (quantum) symmetries of (deformed) algebras through concrete examples. Fix a field $k$, and recall from Figure 5 that an associative $k$-algebra is a $k$-vector spaces equipped with the structure of a (unital) ring; we consider their deformations below.

**Definition 8.** Fix a $k$-algebra $A$. A $k$-algebra $A_{\text{def}}$ is a deformation of $A$ if $A_{\text{def}}$ and $A$ are the same as $k$-vector spaces, but their respective multiplication rules are not necessarily the same.
Example 1. Our running example of a $k$-algebra throughout this section will be the $q$-polynomial algebra:

$$k_q[x,y] = k(x,y)/(yx - qxy), \quad \text{for } q \in k^\times,$$

which is the quotient algebra of the free algebra $k(x,y)$ by the ideal $(yx - qxy)$. Loosely speaking, $k_q[x,y]$ is a $q$-deformation of $k[x,y]$ as the former structure 'approaches' the latter as $q \to 1$. More explicitly, note that $k_q[x,y]$ and $k[x,y]$ have the same $k$-vector space basis $\{x^iy^j\}_{i,j \geq 0}$, but their multiplication rules differ for $q \neq 1$.

Now we let us examine symmetries of $k_q[x,y]$ for $q \neq 1$ versus those of $k[x,y]$. For this it is enough to consider *degree-preserving symmetries*, i.e., invertible transformations that send the generators $x$ and $y$ to a linear combination of themselves. Namely, let $V = kx \oplus ky$ be the generating space of $k_q[x,y]$ (or $k[x,y]$ with $q = 1$). We want to pin down which invertible matrices in $GL(V) = GL_2(k)$ also induce a symmetry of $k_q[x,y]$, and to do so, we need to rewrite $k_q[x,y]$ using the notion below. (From now on, we need an understanding of tensor products $\otimes$ and a nice discussion of this operation can be found in [15].)

**Definition 9.** Given a $k$-vector space $V$, the tensor algebra $T(V)$ is the $k$-vector space $\bigoplus_{n \geq 0} V^\otimes_{n \geq 0}$ where $V^0 = \mathbb{R}$, and with multiplication given by concatenation, i.e.,

$$(v_1 \otimes \cdots \otimes v_m)(v_{m+1} \otimes \cdots \otimes v_{m+n}) = v_1 \otimes \cdots \otimes v_{m+n}.$$

Ideals $I$ of tensor algebras $T(V)$ are defined as usual, and one can define a quotient $k$-algebra given by $T(V)/I$.

Example 2. The free algebra $k(x,y)$ is identified with the tensor algebra $T(V)$ on $k$-vector space $V = kx \oplus ky$; for the forward direction introduce $\otimes$ between variables, and conversely suppress $\otimes$ between variables. The $q$-polynomial algebra $k_q[x,y]$ is then identified as the quotient algebra of $T(kx \oplus ky)$ by the ideal $(y \otimes x - qx \otimes y)$.

Now take $g \in GL(V)$ for $V = kx \oplus ky$. We want to extend this symmetry on $V$ to a symmetry of $k_q[x,y]$ identified as $T(V)/(y \otimes x - qx \otimes y)$. Let's assume that, as in the case for group actions, $g$ acts on $T(V)$ diagonally:

$$g(v \otimes v') := g(v) \otimes g(v'), \quad \forall v, v' \in V. \quad (7)$$

Now the question is: When is the ideal $(y \otimes x - qx \otimes y)$ preserved under this action? In fact it suffices to show that

$$g(y \otimes x - qx \otimes y) = \lambda (y \otimes x - qx \otimes y), \quad \text{for some } \lambda \in k, \quad (8)$$

since the $g$-action is degree preserving. To be concrete, say $g \in GL(V)$ is given by

$$g(x) = \alpha x + \beta y \quad \text{and} \quad g(y) = \gamma x + \delta y, \quad \text{for some } \alpha, \beta, \gamma, \delta \in k. \quad (9)$$

Then, $g(y \otimes x - qx \otimes y) = [g(y) \otimes g(x)] - q[g(x) \otimes g(y)]$, which is equal to

$$(1 - q)\alpha \gamma (x \otimes x) + (\beta \gamma - q \alpha \delta) (x \otimes y) + (\alpha \delta - q \beta \gamma) (y \otimes x) + (1 - q) \beta \delta (y \otimes y).$$
Remark 1. In fact, to have an action of \( H \) on \( V \) via (7). We can extend this linearly to get that

\[
\text{Question 4. When is } V \text{ correct one!)} \]

But if we are given two vector spaces \( V_1 \) and \( V_2 \) that are \( H \)-modules, we need to think beyond group actions like those in (7). In general, we want to construct symmetries of a \( k \)-algebra \( T(V) \) by (i) considering symmetries of the generating space \( V \), (ii) extending those to symmetries of \( T(V) \), and then (iii) determining which symmetries in (ii) descend to \( T(V) \). For step (i), take \( V \) to be a representation of an algebraic object \( H \), e.g., \( H \) could be a group or a \( k \)-algebra. (We often swap back and forth between using “representations” and “modules”.)

For (ii), one needs to tackle the issue of building a direct sum and tensor product of \( H \)-representations. The former is pretty straightforward– one can always construct \( V_1 \otimes V_2 \) as a module over a group algebra on \( G \). But if \( H \) were an arbitrary algebra, then the diagonal action on \( V_1 \otimes V_2 \) does not necessarily give it the structure of a \( H \)-module (and we will see in Remark 1). In fact, to have an action of \( H \) on \( V_1 \otimes V_2 \) we first need algebra maps

\[
\Delta : H \to H \otimes H, \quad \Delta(h) = \sum h_1 \otimes h_2, \quad \text{and} \quad \varepsilon : H \to k.
\]

Here, we use the Sweedler notation shorthand to denote elements of \( \Delta(H) \). These maps should be compatible in a way that is dual to the manner that the multiplication map \( m : H \otimes H \to H \) and unit map \( \eta : k \to H \) of an algebra are compatible (cf. \( m(\eta \otimes \text{id}_H) = \text{id}_H = m(\text{id}_H \otimes \eta) \)). That is, after identifying \( k \otimes H = H \otimes k \),

\[
(\varepsilon \otimes \text{id}_H) \circ \Delta = \text{id}_H = (\text{id}_H \otimes \varepsilon) \circ \Delta. \tag{10}
\]

**Definition.** [63] Chapter 5] An associative \( k \)-algebra \( H = (H, m, \eta) \) is a \( k \)-bialgebra if it equipped with algebra maps \( \Delta \) (coproduct) and \( \varepsilon \) (counit), so that \( (H, \Delta, \varepsilon) \) is a coassociative \( k \)-coalgebra with the structures \( (H, m, \eta) \) and \( (H, \Delta, \varepsilon) \) being compatible.

---

1 In categorical language, this is the question of whether the category of \( H \)-modules (or of representations of \( H \)) has a monoidal structure.
To answer Question 4[4] if $H$ is a bialgebra, then $H$-module structure on $V_1 \otimes V_2$ is
\[ h(v_1 \otimes v_2) =: \sum h_1(v_1) \otimes h_2(v_2) \quad \forall h \in H \text{ and } v_1, v_2 \in V. \]

We also get that $k$ admits the structure of a trivial $H$-module via $h(1_k) = \varepsilon(h) 1_k$.

**Remark 1.** We cannot always use a diagonal action—sometimes a fancier coproduct is needed to address Question 4. To see this, take $H$ to be the 2-dimensional associative $k$-algebra $k[h]/(h^2)$ (e.g., we are interested in linear operators that are the zero map when composed with itself). If the coproduct of $H$ is $\Delta(h) = h \otimes h$, then $\varepsilon(h) = 1$ by (10). But this implies $0 = \varepsilon(h^2) = \varepsilon(h)^2 = 1$, a contradiction. To “fix” this, check that the coproduct $\Delta(h) = h \otimes 1 + 1 \otimes h$ with the counit $\varepsilon(h) = 0$ gives $k[h]/(h^2)$ the structure of a bialgebra over $k$.

Moreover, one may be interested (in symmetries of) an algebra with generating space $V^*$, the linear dual; this will play a role later in Section 5.2. To get this, we want $V^*$ to have the induced structure of an $H$-module, and we need an anti-algebra-automorphism $S : H \to H$ of $H$ to proceed.

**Definition 11.** [63] Chapters 6-7] A $k$-bialgebra $H = (H, m, \eta, \Delta, \varepsilon)$ is a Hopf algebra over $k$ if there exists anti-automorphism $S : H \to H$ (antipode) so that
\[ m \circ (S \otimes \text{id}_H) \circ \Delta = \eta \circ \varepsilon = m \circ (\text{id}_H \otimes S) \circ \Delta. \]

If $H$ is a Hopf algebra with $H$-module $V$, then the action of $H$ on $V^*$ is given as
\[ [h(f)](v) = f[S(h)(v)], \quad \forall h \in H, \ f \in V^*, \ v \in V. \]

Examples of Hopf algebras are group algebras on finite groups $kG$, function algebras on algebraic groups $O(G)$, and universal enveloping algebras of Lie algebras $U(g)$, which are all considered “classical” in the sense that they are commutative (as an algebra, $m \circ \tau = m$) or cocommutative (as a coalgebra, $\tau \circ \Delta = \Delta$), for $\tau(a \otimes b) = b \otimes a$. Indeed, these Hopf algebras capture the actions of a group on a $k$-algebra by automorphism and actions of a Lie algebra on a $k$-algebra by derivation. Moreover, deformations (or quantized versions) of these structures provide a setting to handle deformations of the aforementioned symmetries (cf. Question 3); refer to [1, 36, 51, 54] for examples of Hopf algebras arising in this fashion. We also recommend the excellent text on (actions of) Hopf algebras by Susan Montgomery [57].

Now we summarize a few frameworks for studying (quantum) symmetries of a $k$-algebra $A$ involving a group $G$ or a Hopf algebra $H$. See [57] for more details.\(^3\)

\(^3\) In this case, the category of $H$-modules is a rigid monoidal category.

\(^3\) For more settings of quantum symmetry, see, e.g., [63] Chapter 11] for a categorical framework.
**[G-ACT]** Group actions on $A$: That is, $A$ is an $G$-module with $G$-action map $G \times A \rightarrow A$ given by $(g, a) \mapsto g(a)$ satisfying $g(ab) = g(a)g(b)$ and $g(1_A) = 1_A$, for all $g \in G$ and $a, b \in A$.

**[G-GRD]** Group gradings on $A$: That is, $A$ is $G$-graded if $A = \bigoplus_{g \in G} A_g$, for $A_g$ a $\mathbb{k}$-vector space, with $A_g \cdot A_h \subseteq A_{gh}$. When $G$ is finite, this is equivalent to $A$ being acted upon by the dual group algebra $(\mathbb{k}G)^*$.

**[H-ACT]** Hopf algebra or bialgebra actions on $A$: That is, $A$ is an $H$-module with $H$-action map $H \times A \rightarrow A$ given by $(h, a) \mapsto h(a)$ with $h(ab) = \sum h_1(a)h_2(b)$ and $h(1_A) = \varepsilon(h)1_A$, for all $h \in H$ and $a, b \in A$, with $\Delta(h) = \sum h_1 \otimes h_2$.

Finally we end with an example of a Hopf algebra action on $\mathbb{k}_q[x, y]$, illustrating a scenario where Question 3 has a possible answer. For other (more general) examples in the literature, we refer to [41, Sections IV.7 and VII.3].

**Example 3.** (A simplified version of [41, Theorem VII.3.3]) For ease, we take $\mathbb{k}$ to be $\mathbb{C}$. Also, let $q$ be a nonzero complex number that’s not a root of unity. We aim to produce an action of a Hopf algebra $H_q$ over $\mathbb{C}$ (whose structure depends on $q$) on the $q$-polynomial algebra $\mathbb{C}_q[x, y] = \mathbb{C}(x, y)/(yx - qxy)$, so that

- the “limit” of $H_q$ as $q \rightarrow 1$ is a “classical” Hopf algebra $H$ (i.e., $H$ is either commutative or cocommutative and $H_q$ is a $q$-deformation of $H$), and

- the “limit” of the $H_q$-action on $\mathbb{C}_q[x, y]$ as $q \rightarrow 1$ is an action of $H$ on $\mathbb{C}[x, y]$.

We begin by defining a Hopf algebra $H_q$ with algebra presentation,

$$H_q = \mathbb{C}\langle g, g^{-1}, h \rangle/(gg^{-1} - 1, g^{-1}g - 1, gh - q^2hg),$$

along with coproduct, counit, and antipode given by

$$\Delta(g) = g \otimes g, \quad \Delta(g^{-1}) = g^{-1} \otimes g^{-1}, \quad \Delta(h) = 1 \otimes h + h \otimes g,$$

$$\varepsilon(g) = 1, \quad \varepsilon(g^{-1}) = 1, \quad \varepsilon(h) = 0, \quad S(g) = g^{-1}, \quad S(g^{-1}) = g, \quad S(h) = -hg^{-1}.$$

Next we define a $q$-number $[\ell]_q := \frac{q^\ell - q^{-\ell}}{q - q^{-1}}$ for any integer $\ell$. Now for any element $p = \sum_{i, j \geq 0} \lambda_{ij}x^iy^j$ in $\mathbb{C}_q[x, y]$, the rule below gives us an action of $H_q$ on $\mathbb{C}_q[x, y]$:

$$g(p) = \sum_{i, j \geq 0} \lambda_{ij}q^{-i}x^iy^j, \quad g^{-1}(p) = \sum_{i, j \geq 0} \lambda_{ij}q^i\ell x^{i+1}y^{j-1}, \quad h(p) = \sum_{i, j \geq 0} \lambda_{ij}[\ell]_q x^{i+1}y^{j-1}.$$

To check this, it suffices to show that (i) the relations of $H_q$ act on $\mathbb{C}_q[x, y]$ by zero, and that (ii) the relation space of $\mathbb{C}_q[x, y]$ is preserved under the rule above. We'll provide some details here and leave the rest as an exercise. We compute:

For (i), $(gh - q^2hg)(p) = g(\sum \lambda_{ij}[\ell]_q x^{i+1}y^{j-1}) - q^2h(\sum \lambda_{ij}q^{-i}x^iy^j) = \sum \lambda_{ij}[\ell]_q q^{-i-j}x^{i+1}y^{j-1} - q^2\sum \lambda_{ij}[\ell]_q q^{-i-j}x^{i+1}y^{j-1} = 0$;

For (ii), $h(xy - qxy) = [1(y)h(x) + h(y)g(x)] - qh(xy) = (x)(qy) - q(x^2) = 0.$
Now the “limit” of $H_q$ as $q \to 1$ is $H = \mathbb{C}[x] \otimes \mathbb{C}Z$, the tensor product of Hopf algebras, for $Z = \langle g \rangle$ (see, e.g., [63, Exercise 2.1.19]); $H$ is both commutative and cocommutative. Also, $H_q = H$ as $\mathbb{C}$-vector spaces. Moreover, as $q \to 1$, the generators $g$ and $g^{-1}$ (resp., $h$) of $H$ act on $\mathbb{C}_q[x,y]$ as the identity (resp., by $\frac{\partial}{\partial y}$).

5 Research directions in Noncommutative Algebra

We highlight a couple of directions for research in Noncommutative Algebra in this section, building on the discussions of Sections 1-4. The material below could also serve as a topic for an undergraduate or Master’s thesis project, or as a reading course topic. Finding a friendly faculty (or advanced graduate student) mentor to help with these pursuits is a good place to start...

5.1 On Symmetries

Continuing the discussions of Section 2 and 4, we propose the following avenue for research: Study of the symmetries of (algebraic structures that generalize) Hamilton’s quaternions [Problem 1]. One such generalization is given below.

**Definition 12.** [12, Section 5.4] Fix a field $k$ along with nonzero scalars $a, b \in k$. Then a quaternion algebra $Q(a,b)_k$ is a $k$-algebra that has an underlying 4-dimensional $k$-vector space with basis $\{1, i, j, k\}$, subject to multiplication rules

\[ i^2 = a, \quad j^2 = b, \quad ij = -ji = k. \]

Note that $k^2 = ijk = -ab$, for instance.

Sometimes $Q(a,b)_k$ is denoted by $(a,b)_k$, by $(a,b;k)$, or even by $(a,b)$ if $k$ is understood. The structure above extends the construction of Hamilton’s quaternions [Definition 5], namely $\mathbb{H} = Q(-1,-1)_\mathbb{R}$. Moreover, split-quaternions, $Q(-1,+1)_\mathbb{R}$, also appear frequently in the literature.

Fun fact: A quaternion algebra is either a 4-dimensional $k$-division algebra [Definition 4], or is isomorphic to the matrix algebra $M_2(k)$! (The latter is called the split case.) Also, these cases are characterized by the norm of elements $Q(a,b)_k$:

\[ N(a_0 + a_1i + a_2j + a_3k) := a_0^2 - a_1^2a_2^2 - ba_2^3 + aba_3^3, \quad \text{for } a_0, a_1, a_2, a_3 \in k. \]

Namely, if $k$ has characteristic not equal to 2, then $Q(a,b)_k$ is a division algebra precisely when $N(a_0 + a_1i + a_2j + a_3k) = 0$ only for $(a_0, a_1, a_2, a_3) = (0,0,0,0)$ [18 Proposition 5.4.3]. For instance, $\mathbb{H} = Q(-1,-1)_\mathbb{R}$ is a $\mathbb{R}$-division algebra since

\[ N(a_0 + a_1i + a_2j + a_3k) = a_0^2 + a_1^2 + a_2^2 + a_3^2. \]
for $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and is 0 if and only if $(a_0, a_1, a_2, a_3) = (0, 0, 0, 0)$.

Quaternion algebras (in the generality of Definition 12 above) have appeared primarily in number theory \[69\] \[56, \text{Chapter 5}\] and in the study of quadratic forms \[47, \text{Chapter III}\]. They have also been used in hyperbolic geometry \[52, \text{Chapter 2}\], and in various parts of physics and engineering; see, e.g., \[5, \text{and 61}\]. For more details about their applications and structure, see \[14\] and the references within.

Recall from Section 4 that there are several frameworks for studying symmetries of a $k$-algebra, including group actions \([G-\text{ACT}]\), group gradings \([G-\text{GRD}]\), and Hopf algebra actions \([H-\text{ACT}]\). Also, the latter symmetries are considered to be quantum symmetries if $H$ is non(co)commutative, as discussed by Figure 14.

**Problem 1.** Study the (quantum) symmetries of quaternion algebras. Namely, pick a setting \([G-\text{ACT}]\), \([G-\text{GRD}]\), \([H-\text{ACT}]\), a collection of structures ($G$ or $H$) in this class, and classify all such symmetries of $G$ or $H$ on $Q(a, b)_k$.

Even if this problem is not addressed in full generality, a collection of examples would be quite useful for the literature. For instance, a group grading of $Q(-1, -1)_\mathbb{R} = \mathbb{H}$ was used in recent work of Cuadra and Etingof as a counterexample to show that their main result on faithful group gradings on division algebras fails when the ground field is not algebraically closed \[18, \text{Theorem 3.1, Example 3.4}\].

There are also other works that partially address Problem 1, such as on group gradings \[17, 58, 59\] and Hopf algebra (co)actions \[21, 66\]. These papers also contain work on (quantum) symmetries of some generalizations of quaternion algebras; **Problem 1 can also be posed for these generalizations of $Q(a, b)_k$ as well.**

Moreover, a second part of Problem 1 could include the study of two algebraic structures formed the symmetries constructed above, namely, the subalgebra of (co)invariants, and the smash product algebra (or, skew group algebra if \([G-\text{ACT}]\) is used). See \[57\] for the definitions, examples, and a discussion of various uses of these algebraic structures. Overall, after one gets comfortable with the terminology, such problems are computational in nature ... and fun to do!

### 5.2 On Representations

In this section, $k$ is a field of characteristic zero.

Towards a research direction in representation theory (continuing the discussion in Section 3) it is natural to think further about the representations of the first Weyl algebra $A_1(k)$. Since there are no finite-dimensional representations of $A_1(k)$ \[Proposition 1\], what are its infinite-dimensional representations? To get one for example, identify $A_1(k)$ as a ring of differential operators on $k[x]$ where the generators $x$ and $y$ act as multiplication by $x$ and by $\frac{d}{dx}$, respectively. So, by fixing basis $\{1, x, x^2, x^3, \ldots \}$ of $k[x]$, we get the (matrix form of) the infinite-dimensional representation in \[5\]. Producing explicit infinite-dimensional representations of $A_1(k)$ is tough in general. But there are many works on the abstract representation theory of
A1(κ) and of other rings of differential operators, and we recommend the student-friendly text of S.C. Coutinho on algebraic D-modules [16] for more information.

Now for a concrete research problem to pursue, we suggest working with deformations of Weyl algebras instead, particularly those that admit finite-dimensional representations (as this is more feasible computationally). One could:

**Problem 2.** Examine the (explicit) representation theory of quantum Weyl algebras (at roots of unity) [Definition 16].

Before we discuss quantum Weyl algebras, we introduce some terminology that will be of use later in order to make the problem above more precise. The text [23] (which, again, is student-friendly) is a nice reference for more details.

**Definition 13.** Take a κ-algebra A with a representation

\[ \phi : A \to \text{Mat}_n(\kappa) (\cong \text{End}(V)) \quad \text{for} \quad V = \kappa^\otimes n. \]

1. We say that \( \phi \) is decomposable if we can decompose \( V \) as \( W_1 \oplus W_2 \) with \( W_1, W_2 \neq 0 \) so that \( \phi|_{W_k} : A \to \text{End}(W_k) \) are representations of \( A \) for \( k = 1, 2 \). Otherwise, we say that \( \phi \) is indecomposable.

2. The representation \( \phi \) is reducible if there exists a proper subspace \( W \) of \( V \) so that \( \phi|_W : A \to \text{End}(W) \) is a representation of \( A \); here, \( \phi|_W \) is called a (proper) subrepresentation of \( \phi \). If \( \phi \) does not have any proper subrepresentations, then \( \phi \) is irreducible; the corresponding \( A \)-module \( V \) is said to be simple (cf. Figure 12).

3. Take another representation \( \phi' : A \to \text{End}(V') \) of \( A \). We say that \( \phi' \) is equivalent (or isomorphic) to \( \phi \) if \( \dim V = \dim V' \) and there exists an invertible \( \kappa \)-linear map \( \rho : V \to V' \) so that \( \rho(\phi_a(v)) = \phi'_a(\rho(v)) \) for all \( a \in A \) and \( v \in V \).

Irreducible representations are indecomposable; the converse doesn’t always hold.

To understand the notions above in terms of matrix solutions of equations (cf. Figure 13), take a finitely presented \( \kappa \)-algebra \( A \), that is, \( A \) has finitely many non-commuting variables \( x_i \) as generators, and finitely words \( f_j(\bar{x}) \) in \( x_i \) as relations:

\[ A = \frac{\kappa\langle x_1, \ldots, x_t \rangle}{(f_1(\bar{x}), \ldots, f_r(\bar{x}))}. \]

Let us also fix an \( n \)-dimensional representation of \( A \), given by

\[ \phi : A \to \text{Mat}_n(\kappa), \quad x_i \mapsto X_i \quad \text{for} \quad i = 1, \ldots, t. \]

**Definition 14.** Retain the notation above. Suppose that we have a matrix solution \( \bar{X} = (X_1, \ldots, X_t) \) to the system of equations \( f_1(\bar{x}) = \cdots = f_r(\bar{x}) = 0 \).

1. If each matrix \( X_i \) can be written as a direct sum of matrices \( X_{i,1} \oplus X_{i,2} \), where
   - \( X_{i,k} \in \text{Mat}_{n_k}(\kappa) \) with \( k = 1, 2 \) for some positive integers \( n_1 \) and \( n_2 \), and
   - \( X_k = (X_{1,k}, \ldots, X_{t,k}) \) is a solution to \( f_1(\bar{x}) = \cdots = f_r(\bar{x}) = 0 \) for \( k = 1, 2 \),
then the matrix solution $X$ is decomposable. Otherwise, $X$ is indecomposable.

2. For $\text{Mat}_n(k)$ identified as $\text{End}(V)$ with $V = k^\oplus n$, suppose that there exists a proper subspace $W$ of $V$ that is stable under the action of each $X_i$. Then we say that $X$ is reducible. Otherwise, $X$ is irreducible.

3. We say that another matrix solution $X' \in \text{Mat}_{n'}(k)^{\times I}$ to the system of equations $f_1(x) = \cdots = f_n(x) = 0$ is equivalent (or isomorphic) to $X$ if $n = n'$ and there exists an invertible matrix $P \in \text{GL}_n(k)$ so that $PX_iP^{-1} = X_i$ for all $i$.

So two representations of $A$ (or, two matrix solutions of $\{f_j(x) = 0\}_{j=1}^r$) are equivalent precisely when they are the same up to change of basis of $V = \bigoplus_{i=1}^n x_i$. Therefore Problem 2 can be refined as follows.

**Precise version of Problem 2** Classify the explicit irreducible representations of the quantum Weyl algebras [Definition 16], up to equivalence.

Let’s define the quantum Weyl algebras now. One way of getting these algebras is by deforming the $m$-th Weyl algebras $A_m(k)$ from [6] via the symmetry discussed below. (The reader may wish to skip to Definition 16 for the outcome of this chat.)

**Definition 15.** Fix a $k$-vector space $V$.

1. A $k$-linear transformation $c : V \otimes V \to V \otimes V$ is a braiding if it satisfies the braid relation, $(c \otimes \text{id}_V) \circ (\text{id}_V \otimes c) \circ (c \otimes \text{id}_V) = (\text{id}_V \otimes c) \circ (c \otimes \text{id}_V) \circ (\text{id}_V \otimes c)$ as maps $V^{\otimes 3} \to V^{\otimes 3}$.

2. A braiding $\mathcal{H} : V \otimes V \to V \otimes V$ is a Hecke symmetry if it satisfies the Hecke condition, $(\mathcal{H} - q \text{id}_{V \otimes V}) \circ (\mathcal{H} + q^{-1} \text{id}_{V \otimes V}) = 0$ as maps $V \otimes V \to V \otimes V$, for some nonzero $q \in k$.

Given a Hecke symmetry $\mathcal{H} \in \text{End}(V \otimes V)$ one can form the $\mathcal{H}$-symmetric algebra $S_{\mathcal{H}}(V) = T(V)/\langle \text{Image}(\mathcal{H} - q \text{id}_{V \otimes V}) \rangle$. For example, when $\mathcal{H} = \text{flip}$ (sending $x_i \otimes x_j$ to $x_j \otimes x_i$) and $q = 1$ we get that $S_{\text{flip}}(V)$ is the symmetric algebra $S(V)$ on $V$; this is isomorphic to the polynomial ring $k[x_1, \ldots, x_m]$ for $V = \bigoplus_{i=1}^m kx_i$.

Summarizing the discussion in [28], we now build a $q$-version of a Weyl algebra using a Hecke symmetry $\mathcal{H}$ as follows. Consider the dual vector space $V^*$ and the induced $k$-linear map $\mathcal{H}^* \in \text{End}(V^* \otimes V^*)$. Then construct the algebra $A_{\mathcal{H},q}(V \oplus V^*)$ on $V \oplus V^*$, which is the tensor algebra $T(V \oplus V^*)$ subject to the relations: $\text{Image}(\mathcal{H} - q \text{id}_{V \otimes V^*})$ and $\text{Image}(\mathcal{H}^* - q^{-1} \text{id}_{V^* \otimes V^*})$, and certain relations intertwining generators from $V$ with those from $V^*$ by using $\mathcal{H}$. The resulting algebra $A_{\mathcal{H},q}(V \oplus V^*)$ is called the quantum Weyl algebra associated to $\mathcal{H}$.

For simplicity, we provide the presentation of $A_{\mathcal{H},q}(V \oplus V^*)$ for the $1$-parameter Hecke symmetry given on [37] page 442 (provided in the form of an R-matrix). Here, $V = \bigoplus_{i=1}^m kx_i$ and $V^* = \bigoplus_{i=1}^m ky_i$ with $y_i := x_i^*$ (linear dual of $x_i$).

**Definition 16.** [37] page 442 [28] Definition 1.4] Take $m \geq 2$. The $1$-parameter quantum Weyl algebra is an associative $k$-algebra $A_{m,q}(k)$ with noncommuting generators $x_1, \ldots, x_m, y_1, \ldots, y_m$ subject to relations
By convention, we define $A^q_H(k)$ to be $k\langle x, y \rangle/(yx - qxy - 1)$. If $q$ is a root of unity then we refer to these algebras as quantum Weyl algebras at a root of unity.

Notice that one gets the Weyl algebras $A^q_1(k)$ [Definition 7] and $A^q_m(k)$ [Equation (6)] by taking the “limit” of $A^q_1(k)$ and $A^q_m(k)$ as $q \to 1$, respectively.

Fun fact: If $q$ is a root of unity, say of order $\ell$, then all irreducible representations of a quantum Weyl algebra $A_{\mathcal{H}, q}(V \oplus V^*)$ are finite-dimensional! Moreover in this case, the dimension of an irreducible representation is $A_{\mathcal{H}, q}(V \oplus V^*)$ is bounded above by some positive integer $N(\ell)$ depending on $\ell$, and this bound is met most of the time. This is part of a general phenomenon for quantum $k$-algebras with scalar parameters: they have infinite-dimensional irreducible representations in the generic case, and in the root of unity case all of their irreducible representations are finite-dimensional. Further, in the root of unity case, most irreducible representations of a quantum algebra $A$ have dimension equal to the polynomial identity (PI) degree of $A$ (See, for instance, the informative text of Brown-Goodearl [11]). For example, the PI degree of $A^q_1(k)$ is equal to $\ell$ when $q$ is a root of unity of order $\ell$.

This leads us to discussion of a partial answer to Problem 2. Indeed, one was achieved for $A^q_1(k)$, for $q$ is a root of unity of order $\ell$, in two undergraduate research projects directed by E. Letzter and L. Wang [10, 31]. The explicit irreducible matrix solutions $(X, Y)$ to the equation $YX - qXY = 1$ were computed in these works (up to equivalence), the majority of which are $\ell$-by-$\ell$ matrices.

Naturally, the next case for Problem 2 is the representation theory of quantum Weyl algebras $A_{\mathcal{H}, q}(V \oplus V^*)$, where $\dim_k V = 2$ and $q$ is a root of unity; this should build on the partial answer above. There are a few routes one could take, such as examining $A^q_m(k)$ for $m \geq 2$, or more generally, addressing Problem 2 for multi-parameter quantum Weyl algebras as in [28, Example 2.1] [11, Definition 1.2.6].

Why care? One reason is that quantum Weyl algebras have appeared in numerous works in mathematics and physics, including Deformation Theory [27, 28, 37, 39], Knot Theory [24], Category Theory [48], Quantum mechanics and Hypergeometric Functions [65] to name a few. Therefore, any (partial) resolution to Problem 2 would be a welcomed addition to the literature. So let’s have a go at this. :)
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EDGE and Me.

I was first introduced to the EDGE program the summer before my first year of graduate school, as some of my mentors suggested that participating in the program would be a great way to build a support network before beginning my studies. At the time, I decided to go with other options to get prepared for graduate school but I kept EDGE in mind for future involvement. Fortunately, during the first year of my post-doctoral position I was granted the opportunity to teach for EDGE. It was fantastic to work with other women, including many women of color, who were about to embark on their graduate school journeys. I was also honored to have the opportunity to work with other faculty who ‘walk the walk’ in efforts to increase diversity, inclusion, and equity of researchers and educators in the mathematical sciences. Fortunately I was able participate as an EDGE instructor for the remainder of my post-doctoral years, and the sisterhood that the EDGE program has provided helped facilitate my path up the academia ladder.

Being able to see myself in others—in students coming after me, in faculty clearing the path for me, and in peers with me along with the way—is a crucial part of my finding happiness and a sense of belonging in this job. This is especially true for women (of color) in general because there are many extra obstacles, major and minor, that we have to confront in order to succeed. For instance, here’s an annoying one: During my literature search for this article I came quotes like,

• “[... developed by the Leningrad School (Ludwig Faddeev, Leon Takhtajan, Evgeny Sklyanin, Nicolai Reshetikhin and Vladimir Korepin) and related work by the Japanese School” [with no Japanese mathematicians listed], and

• “My interview was finished when a dolled-up woman with butterfly-shaped glasses appeared, who informed me that I should rise because a lady has entered the room” [when this woman’s appearance had nothing to do with the topic of the article and no other women were mentioned].

It certainly took extra energy to decide how to address these exclusionary passages (usually being ‘Don’t be distracted by this mess’) and keep moving. Those little, extra efforts add up over time.

But what has kept me going? Loving mathematics, and having a network of people like those in the EDGE program who love mathematics as well and view the field through a similar lens. It is my humble wish to help clear the path so that EDGE program participants and other marginalized folks can see themselves, not through the muddied lens of others’ biases or prejudices, but with the proper view of using one’s talents (mathematics) to find happiness, community, and fulfillment with this work. So when I receive email threads like,

• “Please join me in congratulating two EDGErs on successfully completing their PhDs: Shanise Walker (E’12), who received her PhD in mathematics from Iowa State University in May and Jessica De Silva (E’13), who received her PhD from the University of Nebraska-Lincoln in June! Congrats Dr. Walker! Congrats Dr. De Silva!” – T. Diercks (EDGE program admin.), followed by

• “Wonderful news!!!!!! Warmest congratulations, Shanise and Jessica, and all the best moving forward in your careers. Hugs, Rhonda” – Rhonda Hughes (co-founder of the EDGE program);

• “Congratulations, ladies!” – Chassidy Bozeman (2012 EDGE program participant);

• “BIG CONGRATULATIONS AND LOTS OF JOYFUL NOISE!!!! Awesome. I think we just passed 90 EDGE PhDs !! Bursting with admiration and pride... Ami”

– Ami Radunskaya (EDGE program co-director);

it gives me extra energy to proceed, to not be distracted, and to keep moving. And those meaningful, inspirational boosts add up over time!