Path Integrals in Noncommutative Quantum Mechanics

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Extension of Feynman’s path integral to quantum mechanics of noncommuting spatial coordinates is considered. The corresponding formalism for noncommutative classical dynamics related to quadratic Lagrangians (Hamiltonians) is formulated. Our approach is based on the fact that a quantum-mechanical system with a noncommutative configuration space may be regarded as another effective system with commuting spatial coordinates. Since path integral for quadratic Lagrangians is exactly solvable and a general formula for probability amplitude exists, we restricted our research to this class of Lagrangians. We found general relation between quadratic Lagrangians in their commutative and noncommutative regimes. The corresponding noncommutative path integral is presented. This method is illustrated by two quantum-mechanical systems in the noncommutative plane: a particle in a constant field and a harmonic oscillator.

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I. INTRODUCTION

Already in the thirties of the last century, looking for an approach to solve the problem of ultraviolet divergences, Heisenberg and Schrödinger conjectured that spacetime coordinates may be mutually noncommutative. However, the first papers (see, e.g. [1,2]) on this subject appeared at the end of the forties and after some fifty years later an increased interest in noncommutativity emerged in various quantum theories. In particular, noncommutativity has been recently intensively investigated in string/M-theory, quantum field theory and quantum mechanics (for a review of noncommutative field theory and some related topics, see, e.g. [3]).

Most of the research has been done in noncommutative field theory, including its noncommutative extension of the Standard Model [4]. Since quantum mechanics can be regarded as the one-particle nonrelativistic sector of quantum field theory, it is important to study its noncommutative aspects including connection between ordinary and noncommutative regimes. In order to support some possible phenomenological tests of noncommutativity, noncommutative quantum mechanics (NCQM) of a charged particle in the presence of a constant magnetic and electric fields has been mainly considered on two- and three-dimensional spaces (see, e.g. [5] and references therein).

Recall that description of D-dimensional quantum-mechanical system needs a Hilbert space $L_2(\mathbb{R}^D)$ in which observables are linear self-adjoint operators. In ordinary quantum mechanics (OQM), classical canonical variables $x_k, p_j$ become Hermitian operators $\hat{x}_k, \hat{p}_j$ in $L_2(\mathbb{R}^D)$ satisfying the Heisenberg commutation relations

$$[\hat{x}_k, \hat{p}_j] = i\hbar \delta_{kj}, \quad [\hat{x}_k, \hat{x}_j] = 0, \quad [\hat{p}_k, \hat{p}_j] = 0, \quad (I.1)$$

where $k, j = 1, \ldots, D$.

In a very general NCQM one maintains $[\hat{x}_k, \hat{p}_j] = i\hbar \delta_{kj}$, but some realization of $[\hat{x}_k, \hat{x}_j] \neq 0$ and $[\hat{p}_k, \hat{p}_j] \neq 0$ is assumed. We consider here the most simple and usual NCQM which is based on the following algebra:

$$[\hat{x}_k, \hat{p}_j] = i\hbar \delta_{kj}, \quad [\hat{x}_k, \hat{x}_j] = i\hbar \theta_{kj}, \quad [\hat{p}_k, \hat{p}_j] = 0, \quad (I.2)$$

where $\Theta = (\theta_{kj})$ is the antisymmetric matrix with constant elements.

To find elements $\Psi(x,t)$ of the Hilbert space in OQM, it is usually used the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \hat{H} \Psi(x,t), \quad (I.3)$$

which realizes the eigenvalue problem for the corresponding Hamiltonian operator $\hat{H} = \hat{H}(\hat{p}, x, t)$, where $\hat{p}_k = -i\hbar \partial/\partial x_k$.

However, there is another approach based on the Feynman path integral method [4]

$$\mathcal{K}(x'', t''; x', t') = \int_{(x')}^{(x'')} \exp \left( \frac{i}{\hbar} S[q] \right) \mathcal{D}q, \quad (I.4)$$

where $\mathcal{K}(x'', t''; x', t')$ is the kernel of the unitary evolution operator $U(t)$ acting on $\Psi(x,t)$ in $L_2(\mathbb{R}^D)$.

Functional $S[q] = \int_0^t L(\dot{q}, q, t) \, dt$ is the action for a path $q(t)$ in the classical Lagrangian $L(\dot{q}, q, t)$, and $x'' = q(t'')$, $x' = q(t')$, $x = (x_1, x_2, \ldots, x_n)$ and $q = (q_1, q_2, \ldots, q_n)$. The kernel $\mathcal{K}(x'', t''; x', t')$ is also known as the probability amplitude for a quantum particle to pass from position $x'$ at time $t'$ to another point $x''$ at $t''$, and is closely related to the quantum-mechanical propagator and Green’s...
function. The integral in (1.3) has an intuitive meaning that a quantum-mechanical particle may propagate from $x'$ to $x''$ using infinitely many paths which connect these two points and that one has to sum probability amplitudes over all of them. Thus the Feynman path integral means a continual (functional) summation of single transition amplitudes $\exp \left( \frac{i}{\hbar} S[q] \right)$ over all possible continual paths $q(t)$ connecting $x' = q(t')$ and $x'' = q(t'')$. In Feynman’s formulation, the path integral (1.4) is the limit of an ordinary multiple integral over $N$ variables $q_i = q(t_i)$ when $N \to \infty$. Namely, the time interval $t'' - t'$ is divided into $N + 1$ equal subintervals and integration is performed for every $q_i \in (-\infty, +\infty)$ at fixed time $t_i$.

Path integral in its most general formulation contains integration over paths in the phase space $\mathbb{R}^{2N} = \{(p, q)\}$ with fixed end points $x'$ and $x''$, and no restrictions on the initial and final values of the momenta, i.e.,

$$K(x'', t''; x', t') = \int_{(x')}(x'') \exp \left( \frac{2\pi i}{\hbar} \int_{t'}^{t''} \left[ p_k \dot{q}_k - H(p, q, t) \right] dt \right) Dq \, Dp .$$

However, for Hamiltonians which are polynomials quadratic in momenta $p_k$, one can explicitly perform integration over $-\infty < p_k < +\infty$ using the Gauss integrals and the corresponding path integral is reduced to its form (1.4) (see Appendix).

The Feynman path integral for quadratic Lagrangians can be evaluated analytically (see, e.g., a book [8] and a paper [8]) and the exact expression for the probability amplitude is

$$K(x'', t''; x', t') = \frac{1}{(i\hbar)^N} \det \left( \frac{\partial^2 \bar{S}}{\partial x''_k \partial x''_j} \right) \times \exp \left( \frac{2\pi i}{\hbar} \bar{S}(x'', t''; x', t') \right) ,$$

where $\bar{S}(x'', t''; x', t')$ is the action for the classical trajectory which is the solution of the Euler-Lagrange equation of motion.

$K(x'', t''; x', t')$, as the kernel of unitary evolution operator, can be defined by equation

$$\Psi(x'', t'') = \int K(x'', t''; x', t') \Psi(x', t') \, dx'$$

and then Feynman’s path integral may be regarded as a method to calculate this propagator. Eigenfunctions of the integral equation (1.7) and of the above Schrödinger equation (1.3) are the same for the same physical system. Moreover, Feynman’s approach to quantum mechanics, based on classical Lagrangian formalism, is equivalent to Schrödinger’s and Heisenberg’s quantization of the classical Hamiltonian. Note that the Feynman path integral method, not only in quantum mechanics but also in whole quantum theory, is intuitively more attractive and more transparent in its connection with classical theory than the usual canonical operator formalism. In gauge theories, it is the most suitable method of quantization.

Besides its fundamental role in general formalism of quantum theory, path integral method is an appropriate tool to compute quantum phases. To this end, it has been partially considered in noncommutative plane for the Aharonov-Bohm effect [9–11] and a quantum system in a rotating frame [12]. Our approach contains these and all other possible systems with quadratic Lagrangians (Hamiltonians). For three other approaches to the Feynman path integral in NCQM, see Refs. [13, 14] and [15]. The difficulty of a straightforward generalization of the usual path integral to NCQM is caused by an uncertainty of the end position coordinates $x'$ and $x''$. These coordinates cannot be exactly fixed, since $\Delta x'_{kj} \Delta x''_{kj} \geq \frac{\hbar}{2} \theta_{kj}$ and $\Delta x''_{kj} \Delta x''_{kj} \geq \frac{\hbar}{2} \theta_{kj}$ due to commutation relations (1.2).

In the Section II of this article we start by quadratic Lagrangian which is related to OQM and find the corresponding quadratic Hamiltonian. Then we introduce quantum Hamiltonian with noncommuting position coordinates and transform it to an effective Hamiltonian with the corresponding commuting coordinates. To this end, let us note that algebra (1.2) of operators $\hat{x}_k, \hat{p}_j$ can be replaced by the equivalent one

$$[\hat{q}_k, \hat{p}_j] = i \hbar \delta_{kj}, \quad [\hat{q}_k, \hat{q}_j] = 0, \quad [\hat{p}_k, \hat{p}_j] = 0 ,$$

where linear transformation

$$\hat{x}_k = \hat{q}_k - \frac{\theta_{kj} \hat{p}_j}{2}$$

is used, while $\hat{p}_k$ are retained unchanged, and summation over repeated indices $j$ is assumed. Owing to the transformation (1.9), NCQM related to the classical phase space with points $(p, x)$ can be regarded as an OQM on the other phase space $(p, q)$. Thus, in $q$-representation, $\hat{p}_k = -i \hbar (\partial / \partial \hat{q}_k)$ in the equations (1.7) and (1.8). An alternative way to find the commutative analog of a quantum-mechanical system on noncommutative space is to use the Moyal star product (2) in the Schrödinger equation. Such treatment is equivalent to the change of potential $V(\hat{x})$ by using transformation (1.9). It is worth noting that Hamiltonians $H(\hat{p}, \hat{x}, t) = H(-i \hbar (\partial / \partial \hat{q}_k), \hat{q}_k + (i\hbar \theta_{kj}/2) (\partial / \partial \hat{q}_j), t)$, which are more than quadratic in $\hat{x}$, will induce Schrödinger equations with derivatives higher than second order and even of the infinite order. This leads to a new part of modern mathematical physics of partial differential equations with arbitrary higher-order derivatives. In this paper we restrict our consideration to the case of Lagrangians quadratic in $q$ and $\dot{q}$. Transforming new Hamiltonian to Lagrangian we find an effective Lagrangian related to the system with noncommuting spatial coordinates.

In Sec. III we give a general formula for the path integral expressed in terms of the classical action for the noncommutative quadratic Lagrangians. Summary and a short discussion are presented in Sec. IV. In Appendix
we show the transition from the phase space path integral to the usual Feynman coordinate one for the general case of quadratic Hamiltonians.

II. CLASSICAL DYNAMICS ON NONCOMMUTATIVE SPACES

A. Connection between Lagrangian and Hamiltonian in commutative regime

Let us start with a classical system described by a quadratic Lagrangian which the most general form in three dimensions is:

\[
L(\dot{x}, x, t) = \alpha_{11} \dot{x}_1^2 + \alpha_{12} \dot{x}_1 \dot{x}_2 + \alpha_{13} \dot{x}_1 \dot{x}_3 + \alpha_{22} \dot{x}_2^2 + \alpha_{23} \dot{x}_2 \dot{x}_3 + \alpha_{33} \dot{x}_3^2
\]

where the coefficients \( \alpha_{ij} = \alpha_{ij}(t) \), \( \beta_{ij} = \beta_{ij}(t) \), \( \gamma_{ij} = \gamma_{ij}(t) \), \( \delta_i = \delta_i(t) \), \( \eta_i = \eta_i(t) \) and \( \phi = \phi(t) \) are some analytic functions of the time \( t \).

If we introduce the matrices

\[
\alpha = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix},
\beta = \begin{pmatrix}
\beta_{11} & \beta_{12} & \beta_{13} \\
\beta_{21} & \beta_{22} & \beta_{23} \\
\beta_{31} & \beta_{32} & \beta_{33}
\end{pmatrix},
\gamma = \begin{pmatrix}
\gamma_{11} & \gamma_{12} & \gamma_{13} \\
\gamma_{21} & \gamma_{22} & \gamma_{23} \\
\gamma_{31} & \gamma_{32} & \gamma_{33}
\end{pmatrix},
\delta = (\delta_1, \delta_2, \delta_3),
\eta = (\eta_1, \eta_2, \eta_3),
\]

and assuming that the matrix \( \alpha \) is nonsingular (regular), and vectors

\[
\dot{x}^\tau = (\dot{x}_1, \dot{x}_2, \dot{x}_3),
\quad x^\tau = (x_1, x_2, x_3),
\]

then one can express the Lagrangian (II.1) in the following, more compact, form:

\[
L(\dot{x}, x, t) = \langle \alpha \dot{x}, \dot{x} \rangle + \langle \beta x, \dot{x} \rangle + \langle \gamma x, x \rangle + \langle \delta, \dot{x} \rangle + \langle \eta, x \rangle + \phi,
\]

where index \( \tau \) imposes transpose map and \( \langle \cdot, \cdot \rangle \) denotes standard scalar product. Using the equations

\[
p_j = \frac{\partial L}{\partial \dot{x}_j},
\quad j = 1, 2, 3,
\]

one can express \( \dot{x} \) as

\[
\dot{x} = \frac{1}{2} \alpha^{-1} (p - \beta x - \delta).
\]

Then the corresponding classical Hamiltonian

\[
H(p, x, t) = \langle p, \dot{x} \rangle - L(\dot{x}, x, t)
\]

becomes also quadratic, i.e.

\[
H(p, x, t) = \langle A p, p \rangle + \langle B x, p \rangle + \langle C x, x \rangle + \langle D, p \rangle + \langle E, x \rangle + F,
\]

where:

\[
A = \frac{1}{4} \alpha^{-1},
\quad B = -\frac{1}{2} \alpha^{-1} \beta,
\quad C = \frac{1}{4} \beta^T \alpha^{-1} \beta - \gamma,
\quad D = -\frac{1}{2} \alpha^{-1} \delta,
\quad E = \frac{1}{2} \beta^T \alpha^{-1} \delta - \eta,
\quad F = \frac{1}{4} \delta^T \alpha^{-1} \delta - \phi.
\]

Let us note now that matrices \( A \) and \( C \) are symmetric \( (A^T = A \text{ and } C^T = C) \), since the matrices \( \alpha \) and \( \gamma \) are symmetric. If the Lagrangian \( L(\dot{x}, x, t) \) is nonsingular \( (\det \alpha \neq 0) \) then the Hamiltonian \( H(p, x, t) \) is also nonsingular \( (\det A \neq 0) \).

The above calculations can be considered as a map from the space of quadratic nonsingular Lagrangians \( \mathcal{L} \) to the corresponding space of quadratic nonsingular Hamiltonians \( \mathcal{H} \). More precisely, we have \( \varphi: \mathcal{L} \rightarrow \mathcal{H} \), given by

\[
\varphi(L(\alpha, \beta, \gamma, \delta, \eta, \phi, \dot{x}, x)) = H(\varphi_1(L), \varphi_2(L), \varphi_3(L), \varphi_4(L), \varphi_5(L), \varphi_6(L), \varphi_7(L), \varphi_8(L)) = H(A, B, C, D, E, F, p, x).
\]

From relation (II.10) it is clear that inverse of \( \varphi \) is given by the same relations (II.10). This fact implies that \( \varphi \) is essentially an involution, i.e. \( \varphi \circ \varphi = \text{id} \).

B. Connection between Lagrangian and Hamiltonian in noncommutative regime

In the case of noncommutative spatial coordinates \( [\hat{x}_i, \hat{x}_j] = i \hbar \theta_{ij} \), one can replace these coordinates using the following ansatz

\[
\hat{x} = \hat{q} - \frac{1}{2} \Theta \hat{p},
\]

where, as usually, \( \hat{x}^\tau = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \), \( \hat{q}^\tau = (\hat{q}_1, \hat{q}_2, \hat{q}_3) \), and \( \hat{p}^\tau = (\hat{p}_1, \hat{p}_2, \hat{p}_3) \) are operators and

\[
\Theta = \begin{pmatrix}
0 & \theta_{12} & \theta_{13} \\
-\theta_{12} & 0 & \theta_{23} \\
-\theta_{13} & -\theta_{23} & 0
\end{pmatrix}.
\]
Now, one can easily check that $\dot{q}_k$ for $i = 1, 2, 3$ are mutually commutative operators (but do not commute with operators of momenta, i.e. $[\dot{q}_k, \dot{p}_j] = i\hbar\delta_{kj}$).

If we start with quantization of the nonsingular quadratic Hamiltonian given by the relation (II.8), i.e. $H = H(A, B, C, D, E, F, \dot{\rho}, \dot{\varphi})$ and then apply the change of coordinates (II.11), we will again obtain quadratic quantum Hamiltonian, $\hat{H}_\theta = H_{\theta}(\dot{\rho}, \dot{\varphi}; t)$:

$$\hat{H}_\theta = \langle A_\theta \dot{\rho}, \dot{\rho} \rangle + \langle B_\theta \dot{\varphi}, \dot{\varphi} \rangle + \langle C_\theta \dot{\varphi}, \dot{\varphi} \rangle + \langle D_\theta, \dot{\rho} \rangle + \langle E_\theta, \dot{\varphi} \rangle + F_\theta,$$

(II.12)

where

$$A_\theta = (A - \frac{1}{2} B \Theta - \frac{1}{4} \Theta C \Theta)_{\text{sym}}; \quad B_\theta = B + \Theta C; \quad C_\theta = C; \quad D_\theta = D + \frac{1}{2} \Theta E; \quad E_\theta = E; \quad F_\theta = F,$$

(II.13)

and $\text{sym}$ denotes symmetrization of the corresponding operator. Let us note that for the nonsingular Hamiltonian $H$ and for sufficiently small $\theta_{kj}$ the Hamiltonian $H_{\theta}$ is also nonsingular.

In the process of calculating path integrals for the above systems, we need classical Lagrangians. It is clear that to an arbitrary quadratic quantum Hamiltonian we can associate the classical one replacing operators by the corresponding classical variables. Then, by using equations

$$\dot{q}_k = \frac{\partial H_{\theta}}{\partial p_k}, \quad k = 1, 2, 3,$$

from such Hamiltonian we can come back to the corresponding Lagrangian

$$L_{\theta}(q, \dot{q}, t) = \langle p, \dot{q} \rangle - H_{\theta}(p, q, t),$$

where

$$p = \frac{1}{2} A_\theta^{-1} (\dot{q} - B_\theta q - D_\theta)$$

is replaced in $H_{\theta}(p, q, t)$. In fact, our idea is to find connection between Lagrangians of noncommutative and the corresponding commutative quantum mechanical systems (with $\theta = 0$). This implies to find the composition of the following three maps:

$$L_{\theta} = (\varphi \circ \psi \circ \varphi)(L),$$

(II.14)

where $L_{\theta} = \varphi(H_{\theta}), H_{\theta} = \psi(H)$ and $H = \varphi(L)$ (here we use facts that $\varphi$ is an involution given by formulas (II.9), and $\psi$ is given by (II.13)). More precisely, if

$$L(\dot{x}, x, t) = \langle \alpha \dot{x}, x \rangle + \langle \beta x, \dot{x} \rangle + \langle \gamma x, x \rangle + \langle \delta \dot{x}, x \rangle + \langle \eta x, x \rangle + \phi,$$

(II.15)

and

$$L_{\theta}(\dot{q}, q, t) = \langle \alpha_\theta \dot{q}, q \rangle + \langle \beta_\theta q, \dot{q} \rangle + \langle \gamma_\theta q, q \rangle + \langle \delta_\theta \dot{q}, q \rangle + \langle \eta_\theta q, q \rangle + \phi_\theta,$$

(II.16)

then the connection between their coefficients is given by

$$\alpha_\theta = [\alpha^{-1} - \frac{1}{2} (\Theta \beta^T \alpha^{-1} - \alpha^{-1} \beta \Theta)] + \frac{\Theta \gamma \Theta - \frac{1}{4} \Theta \beta^T \alpha^{-1} \beta \Theta}{}^{-1},$$

$$\beta_\theta = \alpha_\theta (\alpha^{-1} \beta - \frac{1}{2} \Theta \beta^T \alpha^{-1} \beta + 2 \Theta \gamma),$$

$$\gamma_\theta = \frac{1}{4} (\beta^T \alpha^{-1} \beta + \frac{1}{2} \beta^T \alpha^{-1} \beta \Theta - 2 \gamma \Theta) \times \alpha_\theta (\alpha^{-1} \beta - \frac{1}{2} \Theta \beta^T \alpha^{-1} \beta + 2 \Theta \gamma) - \frac{1}{4} \beta^T \alpha^{-1} \beta + \gamma,$$

$$\delta_\theta = \alpha_\theta (\alpha^{-1} \delta - \frac{1}{2} \Theta \beta^T \alpha^{-1} \delta + \Theta \eta),$$

$$\eta_\theta = \frac{1}{4} (\beta^T \alpha^{-1} \beta + \frac{1}{2} \beta^T \alpha^{-1} \beta \Theta - 2 \gamma \Theta) \times \alpha_\theta (\alpha^{-1} \delta - \frac{1}{2} \Theta \beta^T \alpha^{-1} \delta + \Theta \eta) - \frac{1}{2} \beta^T \alpha^{-1} \delta + \eta,$$

$$\phi_\theta = \frac{1}{4} (\delta_\theta, \alpha^{-1} \delta - \frac{1}{2} \Theta \beta^T \alpha^{-1} \delta + \Theta \eta) - \frac{1}{4} (\alpha^{-1} \delta, \delta) + \phi.$$

(II.17)

It is clear that formulas (II.17) are very complicated and that to find explicit exact relations between elements of matrices in general case is a very hard task. However, the relations (II.17) are quite useful in all particular cases. On the above classical analog of Hamiltonian (II.12) and Lagrangian (II.10) one can apply usual techniques of classical mechanics.

### III. NONCOMMUTATIVE PATH INTEGRALS

Although derived for three-dimensional space, the results obtained in the preceding section are valid for an arbitrary spatial dimensionality $D$. If we know Lagrangian (II.15) and algebra (I.2) we can obtain the corresponding effective Lagrangian (II.10) suitable for quantization with path integral in NCQM. Exploiting the Euler-Lagrange equations

$$\frac{\partial L_{\theta}}{\partial q_k} - \frac{d}{dt} \frac{\partial L_{\theta}}{\partial \dot{q}_k} = 0, \quad k = 1, 2, \cdots, D$$

one can obtain classical path $q_k = q_k(t)$ connecting given end points $x' = q(t')$ and $x'' = q(t'')$. For this classical trajectory one can calculate action $S_{\theta}(x'', t''; x', t') = \int_{t'}^{t''} L_{\theta}(\dot{q}, q, t) dt$. Path integral in NCQM is a direct analog of (14) and its exact expression in the form of
quadratic actions \( \tilde{S}_0(x'', t''; x', t') \) is

\[
K_\theta(x'', t''; x', t') = \frac{1}{(i\hbar)^2} \sqrt{\det \left( -\frac{\partial^2 \tilde{S}_0}{\partial x''_k \partial x''_j} \right)} \quad (\text{III.1})
\]
\[
\times \exp \left( \frac{2\pi i}{\hbar} \tilde{S}_0(x'', t''; x', t') \right).
\]

We present here two examples for \( D = 2 \) case.

**A. A particle in a constant field**

In this example the Lagrangian on commutative configuration space is

\[
L(\dot{x}, x) = \frac{m}{2} (\dot{x}^2 + \dot{x}_2^2) - \eta_1 x_1 - \eta_2 x_2. \quad (\text{III.2})
\]

The corresponding data in the matrix form are:

\[
\alpha = \frac{m}{2} I, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0,
\]
\[
\eta^\tau = (-\eta_1, -\eta_2), \quad \phi = 0,
\]  

where \( I \) is a \( 2 \times 2 \) unit matrix.

Using the general composition formula \( \text{III.17} \), one can easily find

\[
\alpha_\theta = \frac{m}{2} I, \quad \beta_\theta = 0, \quad \gamma_\theta = 0, \quad \eta_\theta = \eta,
\]
\[
\delta_\theta = \frac{m \theta}{2} (-\eta_2, \eta_1), \quad \phi_\theta = \frac{m \theta^2}{8} (\eta_1^2 + \eta_2^2).
\]  

In this case, it is easy to find the classical action. The Lagrangian \( L_\theta(\dot{q}, q, t) \) is

\[
L_\theta = \frac{m}{2} (\dot{q}_1^2 + \dot{q}_2^2) + \frac{m \theta}{2} (\eta_1 \dot{q}_2 - \eta_2 \dot{q}_1) - \eta_1 q_1 - \eta_2 q_2 + \frac{m \theta^2}{8} (\eta_1^2 + \eta_2^2). \quad (\text{III.5})
\]

The Lagrangian given by \( \text{III.5} \) implies the Euler-Lagrange equations

\[
m \ddot{q}_1 = -\eta_1, \quad m \ddot{q}_2 = -\eta_2. \quad (\text{III.6})
\]

Their solutions are:

\[
q_1(t) = -\frac{\eta_1 t^2}{2 m} + t C_2 + C_1, \\
q_2(t) = -\frac{\eta_2 t^2}{2 m} + t D_2 + D_1,
\]  

where \( C_1, C_2, D_1 \) and \( D_2 \) are constants which have to be determined from conditions:

\[
q_1(0) = x'_1, \quad q_1(T) = x''_1, \quad q_2(0) = x'_2, \quad q_2(T) = x''_2. \quad (\text{III.8})
\]

After finding the corresponding constants, we have

\[
q_j(t) = x'_j - \frac{\eta_j t^2}{2 m} + t \left( \frac{1}{T} (x''_j - x'_j) + \frac{\eta_j T}{2 m} \right),
\]

\[
q_j(t) = -\frac{\eta_j t}{m} + \frac{1}{T} (x''_j - x'_j) + \frac{\eta_j T}{2 m}, \quad j = 1, 2. \quad (\text{III.9})
\]

Using \( \text{III.8} \) and \( \text{III.9} \), we finally calculate the corresponding action

\[
\tilde{S}_\theta(x'', T; x', 0) = \int_0^T L_\theta(\dot{q}, q, t) \, dt
\]
\[
= \frac{m}{2T} \left[ (x''_1 - x'_1)^2 + (x''_2 - x'_2)^2 \right] - \frac{T}{2} \left[ \eta_1 (x''_1 + x'_1) + \eta_2 (x''_2 + x'_2) \right] \\
+ \frac{m \theta}{2} \left[ \eta_1 (x''_2 - x'_2) - \eta_2 (x''_1 - x'_1) \right] - \frac{T^3}{24m} (\eta_1^2 + \eta_2^2) + \frac{m \theta^2 T}{8} (\eta_1^2 + \eta_2^2). \quad (\text{III.10})
\]

According to \( \text{III.1} \) one gets

\[
K_\theta(x'', T; x', 0) = \frac{1}{i\hbar} \exp \left( \frac{2\pi i}{\hbar} \tilde{S}_\theta(x'', T; x', 0) \right)
\]
\[
= \mathcal{K}_0(x'', T; x', 0) \exp \left( \frac{2\pi i}{\hbar} \frac{m \theta}{2} \left[ \eta_1 (x''_2 - x'_2) - \eta_2 (x''_1 - x'_1) \right] \right), \quad (\text{III.11})
\]

where \( \mathcal{K}_0(x'', T; x', 0) \) is related to the Lagrangian \( \text{III.2} \), for which \( \theta = 0 \). Hence, in this case there is a difference only in the phase factor.

It is easy to see that the following connection holds:

\[
K_\theta(x'', T; x', 0) = \mathcal{K}_0(x'' + \frac{\theta T}{2} J \eta, T; x', 0), \quad (\text{III.12})
\]

where

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

**B. Harmonic oscillator**

The commutative Lagrangian in the question is

\[
L(\dot{x}, x) = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{m \omega^2}{2} (x_1^2 + x_2^2). \quad (\text{III.13})
\]

In this case we have

\[
\alpha = \frac{m}{2} I, \quad \beta = 0, \quad \gamma = -\frac{m \omega^2}{2} I, \quad \Theta = \theta J,
\]
\[
J^2 = -I, \quad \delta = \eta = 0, \quad \phi = 0. \quad (\text{III.14})
\]
Using formulas (III.17), one can easily find

\[
\alpha_\theta = \frac{m}{2\kappa} I, \quad \beta_\theta = \frac{m^2 \omega^2}{2\kappa} \theta J^r, \quad \gamma_\theta = -\frac{m \omega^2}{2\kappa} I, \quad \delta_\theta = \delta_\theta = 0, \quad \phi_\theta = 0,
\]

where \( \kappa = 1 + m^2 \omega^2 \theta^2 \).

The corresponding noncommutative Lagrangian is

\[
L_\theta(q, \dot{q}) = \frac{m}{2\kappa} (q_1^2 + q_2^2) + \frac{m^2 \omega^2}{2\kappa} (\dot{q}_1 - \dot{q}_2)(q_1 - q_2)
- \frac{m \omega^2}{2\kappa} (q_1^2 + q_2^2). \tag{III.16}
\]

From (III.16), we obtain the Euler-Lagrange equations,

\[
\ddot{q}_1 - m \omega^2 \dot{q}_2 + \omega^2 q_1 = 0, \tag{III.17}
\]

\[
\ddot{q}_2 + m \omega^2 \dot{q}_1 + \omega^2 q_2 = 0.
\]

Let us remark that the Euler-Lagrange equations (III.17) form a coupled system of second order differential equations, which is more complicated than in commutative case (\( \theta = 0 \)). One can transform the system (III.17) to

\[
q_1^{(4)} + \omega^2 (2 + m^2 \omega^2 \theta^2) q_1^{(2)} + \omega^4 q_1 = 0,
\]

\[
q_2^{(4)} + \omega^2 (2 + m^2 \omega^2 \theta^2) q_2^{(2)} + \omega^4 q_2 = 0. \tag{III.18}
\]

The solution of the equations (III.17) has the following form

\[
q_1(t) = C_1 \cos(y_1 t) + C_2 \sin(y_1 t) + C_3 \cos(y_2 t) + C_4 \sin(y_2 t),
\]

\[
q_2(t) = D_1 \cos(y_1 t) + D_2 \sin(y_1 t) + D_3 \cos(y_2 t) + D_4 \sin(y_2 t), \tag{III.19}
\]

where \( \omega > 0 \)

\[
y_1 = \frac{m \theta \omega^2 + \omega \sqrt{4 + m^2 \theta^2 \omega^2}}{2} = \frac{m \omega^2 + 2 \omega \sqrt{\kappa}}{2},
\]

\[
y_2 = \frac{m \theta \omega^2 - \omega \sqrt{4 + m^2 \theta^2 \omega^2}}{2} = \frac{m \omega^2 - 2 \omega \sqrt{\kappa}}{2}.
\]

If we impose connections between \( q_1 \) and \( q_2 \) given by (III.17), we obtain the following connection between constants \( C \) and \( D \):

\[
D_1 = C_2, \quad D_2 = -C_1, \quad D_3 = C_4, \quad D_4 = -C_3. \tag{III.20}
\]

The unknown constants \( C_1, C_2, C_3 \) and \( C_4 \) one can find from initial conditions given by (III.18). Then one finally obtains the solutions

\[
q_1 = \frac{1}{2} \left( \left( x_1' + x_2' \cos(\omega \sqrt{\kappa} T) \right) + \left( x_2' \sin(2 \omega \sqrt{\kappa} T) \right) \right) \sin(y_1 t)
\]

\[
\times \left( x_2' \cos \left( \frac{m \theta \omega^2 T}{2} \right) + x_2' \sin \left( \frac{m \theta \omega^2 T}{2} \right) \right) \cos(y_1 t)
\]

\[
+ \left( x_2' - x_2' \cos(\omega \sqrt{\kappa} T) \right) + \left( x_2' \sin(2 \omega \sqrt{\kappa} T) \right) \sin(y_1 t)
\]

\[
\times \left( x_2' \cos \left( \frac{m \theta \omega^2 T}{2} \right) + x_2' \sin \left( \frac{m \theta \omega^2 T}{2} \right) \right) \cos(y_1 t)
\]

\[
+ \left( x_2' + x_2' \cos(\omega \sqrt{\kappa} T) \right) + \left( x_2' \sin(2 \omega \sqrt{\kappa} T) \right) \sin(y_1 t),
\]

\[
q_2 = \frac{1}{2} \left( \left( x_1' + x_2' \cos(\omega \sqrt{\kappa} T) \right) + \left( x_2' \sin(2 \omega \sqrt{\kappa} T) \right) \right) \sin(y_1 t)
\]

\[
\times \left( x_2' \cos \left( \frac{m \theta \omega^2 T}{2} \right) + x_2' \sin \left( \frac{m \theta \omega^2 T}{2} \right) \right) \cos(y_1 t)
\]

\[
- \left( x_1' + x_2' \cos(\omega \sqrt{\kappa} T) \right) - \left( x_2' \sin(2 \omega \sqrt{\kappa} T) \right) \sin(y_1 t)
\]

\[
\times \left( x_2' \cos \left( \frac{m \theta \omega^2 T}{2} \right) + x_2' \sin \left( \frac{m \theta \omega^2 T}{2} \right) \right) \cos(y_1 t)
\]

\[
- \left( x_1' - x_2' \cos(\omega \sqrt{\kappa} T) \right) - \left( x_2' \sin(2 \omega \sqrt{\kappa} T) \right) \sin(y_1 t)
\]

\[
\times \left( x_2' \cos \left( \frac{m \theta \omega^2 T}{2} \right) + x_2' \sin \left( \frac{m \theta \omega^2 T}{2} \right) \right) \sin(y_1 t), \tag{III.21}
\]

where \( \csc u = \frac{1}{\sin u} \). Inserting the above expressions and their time derivatives in (III.16) we find

\[
L_\theta(\dot{q}, q) = \frac{m \omega^2}{2 \sin^2(\omega \sqrt{\kappa} T)}
\]

\[
\times \left( x_1'' x_1'^2 + x_1'' x_2'^2 \right) \cos(2 \omega \sqrt{\kappa} T) + \left( x_1'' x_2'^2 \right) \cos(2 \omega \sqrt{\kappa} T - t)
\]

\[
\times \left( x_1'' x_1'^2 + x_1'' x_2'^2 \right) \cos \left( \frac{m \theta \omega^2 T}{2} \right)
\]

\[
+ \left( x_1'' x_1'^2 - x_1'' x_2'^2 \right) \sin \left( \frac{m \theta \omega^2 T}{2} \right) \left) \right). \tag{III.22}
\]

Using (III.22), we finally compute the corresponding ac-
tion
\[ S_\theta(x'', T; x', 0) = \int_0^T L_0(\dot{q}, q, t) \, dt = \frac{m \omega}{2 \sqrt{\kappa} \sin(\omega \sqrt{\kappa} T)} \]
and consequently
\[ K_\theta(x'', T; x', 0) = \frac{1}{i \hbar} \frac{m \omega}{\sin(\omega \sqrt{\kappa} T)} \times \exp \left( \frac{2 \pi i}{\hbar} S_\theta(x'', T; x', 0) \right) \]
where \( S_\theta(x'', T; x', 0) \) is given by \((\ref{eq:kernel})\).

Let us remark that in this case it is not so easy to find the relation between the kernels of the harmonic oscillators in commutative \((\theta = 0)\) and noncommutative regimes. Namely, in commutative case \((\theta = 0)\) and \(\kappa = 1\), we have
\[ S_\theta(x'', T; x', 0) = \int_0^T L_0(\dot{q}, q, t) \, dt = \frac{m \omega}{2 \sin(\omega T)} \left( (x''_1^2 + x''_2^2 + x''_2') \cos(\omega T) \right. \]
\[ \left. - 2 \left( x'_1 x''_1 + x'_2 x''_2' \right) \right) \]  
\[ (\text{III.23}) \]
and finally
\[ K_\theta(x'', T; x', 0) = \frac{1}{i \hbar} \frac{m \omega}{\sin(\omega T)} \times \exp \left( \frac{2 \pi i}{\hbar} S_\theta(x'', T; x', 0) \right) \]  
\[ (\text{III.24}) \]
where \( S_\theta(x'', T; x', 0) \) is given by \((\ref{eq:kernel})\).

\section*{IV. CONCLUDING REMARKS}

Using transformation \((\ref{eq:transformation})\) for noncommuting coordinates \( \hat{x}_k \) in the quadratic Hamiltonian \((\ref{eq:quadratic_hamiltonian})\), we again obtain quadratic Hamiltonian \((\ref{eq:quadratic_hamiltonian})\) but with commutative space coordinates \( \hat{q}_k \). To employ Feynman’s path integral we derived the corresponding effective quadratic Lagrangian \((\ref{eq:effective_lagrangian})\) and connection of coefficients \((\ref{eq:coefficients})\) in noncommutative regime \((\theta \neq 0)\) with those in the commutative one \((\theta = 0)\). The transition from Hamiltonian to Lagrangian is performed using the usual formula \( L(\dot{q}, q, t) = \langle p, \dot{q} \rangle - H(p, q, t) \). It is worth noting that this transition can be also obtained as a result of integration over momenta in the phase space path integral (see Appendix \((\ref{app:phase_integral})\)). Since our Lagrangian \( L(\dot{q}, q, t) \) is quadratic one can exploit general expression for analytically evaluated Feynman’s path integral \((\ref{eq:feynman_path_integral})\). Probability amplitudes on noncommutative plane are calculated for a particle in a constant field \((\ref{eq:constant_field})\) and for a harmonic oscillator \((\ref{eq:harmonic_oscillator})\).

Note that the algebra of NCQM which is \((\ref{eq:ncqm_algebra})\) in terms of operators \( \hat{x}_a, \hat{p}_b \). In the two-dimensional case one has \( \theta_{ac}, \sigma_{ab} = -\theta \sigma \delta_{ab} \) and it gives \( [\hat{x}_a, \hat{p}_b] = i \hbar (1 + \frac{\theta}{2}) \delta_{ab} \) (see also \((\ref{eq:commutation_relations})\)). For phase space noncommutativity \((\ref{eq:ncqm_algebra})\) and quadratic Lagrangians one can perform a similar procedure described in the preceding sections.

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\section*{APPENDIX A: PATH INTEGRAL ON A PHASE SPACE FOR QUADRATIC HAMILTONIANS}

We here show that a path integral on a phase space can be reduced to the path integral on configuration space when Hamiltonian \( H(p, q, t) \) is any quadratic polynomial in \( p \). Let us start with path integral \((\ref{eq:path_integral})\), where \( H(p, q, t) \) is given by Eq. \((\ref{eq:quadratic_hamiltonian})\) or \((\ref{eq:harmonic_oscillator})\).

By discretization of the time interval \( t'' - t' \) with equal subintervals \( \epsilon = \frac{t'' - t'}{N+1} \) and \( t_n = t' + n \epsilon, n = 0, 1, \ldots, N + 1 \) we have \( q(t_n) \rightleftharpoons q_n = (q_{n,1}, q_{n,2}, \ldots, q_{n,D}) \) and \( p(t_n) \rightleftharpoons p_n = (p_{n,1}, p_{n,2}, \ldots, p_{n,D}) \) as position and momentum at the time \( t_n \). One can write
\[ \mathcal{K}(x''', t'''; x', t') = \lim_{N \to \infty} \int \mathcal{D}^D P_n \prod_{n=1}^{N} d^D q_n \prod_{n=1}^{N+1} d^D p_n \]
\[ (\text{A.1}) \]
where $S_N$ is
\[
S_N = \sum_{n=1}^{N+1} \left( \langle p_n, (q_n - q_{n-1}) \rangle - \epsilon \langle A_n p_n, p_n \rangle \right. \\
+ \left. \langle B_n q_n, p_n \rangle + \langle C_n q_n, q_n \rangle + \langle D_n, p_n \rangle + \langle E_n, q_n \rangle + F_n \right).
\]  
(A.2)

$A_n, B_n, C_n, D_n, E_n$ and $F_n$ are the corresponding matrices (det $A_n \neq 0$) and vectors which elements are taken at the moment $t = t_n$. Employing the Gauss integral
\[
\int \exp \left( \frac{2\pi i}{\hbar} \left[ \langle u, x \rangle + \langle v, x \rangle \right] \right) d^D x = \frac{1}{\sqrt{(4\pi)^D \det(-2u)}} \exp \left( -\frac{2\pi i}{4\hbar} \langle u^{-1} v, v \rangle \right),
\]  
(A.3)

where $u$ is a nonsingular symmetric $D \times D$ matrix and $v$ is a $D$-dimensional vector, one can perform integration over momenta $p_n$ and one obtains
\[
\mathcal{K}(x'', t''; x', t') = \lim_{N \to \infty} \int \prod_{n=1}^{N} \frac{d^D q_n}{\sqrt{(4\pi)^D \det(2\epsilon A_n)}} \\
\times \exp \left( \frac{2\pi i \epsilon}{\hbar} \left[ \langle A_n^{-1} B_n q_n, B_n q_n \rangle - \langle A_n^{-1} q_n, B_n q_n \rangle \right. \\ 
\left. - \langle A_n^{-1} B_n q_n, q_n \rangle + \langle A_n^{-1} q_n, q_n \rangle - 2 \langle A_n^{-1} D_n, q_n \rangle \\
+ 2 \langle A_n^{-1} D_n, B_n q_n \rangle + \langle A_n^{-1} D_n, D_n \rangle - 4 \langle C_n, q_n \rangle \right. \\
\left. - 4 \langle E_n, q_n \rangle - 4 F_n \right] \right).
\]  
(A.4)

Rewriting (A.4) in the form
\[
\mathcal{K}(x'', t''; x', t') = \lim_{N \to \infty} \int \prod_{n=1}^{N+1} \sqrt{\left( \frac{2}{i\hbar \epsilon} \right)^D \det(\alpha_n)} \\
\times \exp \left( \frac{2\pi i \epsilon}{\hbar} \left[ \langle \alpha_n q_n, q_n \rangle + \langle \beta_n q_n, q_n \rangle + \langle \gamma_n q_n, q_n \rangle \\
+ \langle \delta_n, q_n \rangle + \langle \eta_n, q_n \rangle + \langle \phi_n \rangle \right] \right) \prod_{n=1}^{N} d^D q_n,
\]  
(A.5)

and comparing with (A.4) we get
\[
\alpha_n = \frac{1}{4} A_n^{-1}, \quad \beta_n = -\frac{1}{2} A_n^{-1} B_n, \\
\gamma_n = \frac{1}{4} B_n^T A_n^{-1} B_n - C_n, \quad \delta_n = -\frac{1}{2} A_n^{-1} D_n, \\
\eta_n = \frac{1}{2} B_n^T A_n^{-1} D_n - E_n, \quad \phi_n = \frac{1}{4} D_n^T A_n^{-1} D_n - F_n.
\]  
(A.6)

Let us mention that the above formulas are the same as the formulas (I.9) which are obtained on a different way. Now taking $N \to \infty$ one can rewrite (A.4) as (I.4)
\[
\mathcal{K}(x'', t''; x', t') = \int \exp \left( \frac{2\pi i}{\hbar} \int_{t'}^{t''} L(q, p, t) dt \right) Dq,
\]  
(A.7)

where the Lagrangian $L(q, p, t)$ has the form (I.4) and
\[
Dq = \lim_{N \to \infty} \prod_{n=1}^{N} d\tilde{q}_n, \quad d\tilde{q}_n = \left( \frac{2}{i\hbar \epsilon} \right)^D \det(\alpha(t'')) \sqrt{\left( \frac{2}{i\hbar \epsilon} \right)^D \det(\alpha_n)} d q_n.
\]  
(A.8)

Analytic evaluation of the path integral (A.7) yields (I.6) (see, e.g. [7] and [8]).
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