Quantum Ising chains with small-world couplings: No small-world effect at the quantum level; Stability of the quantum critical point

Massimo Ostilli

Instituto de Física, Universidade Federal da Bahia, Salvador, Brazil

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Due to the small-world effect, the critical behavior of finite dimensional classical systems of $N$ spins is known to change radically when an $O(N)$ number of couplings are randomly rewired or superimposed onto the original system. In particular, one-dimensional systems acquire a finite critical temperature while two-dimensional systems get higher critical temperatures and, in both cases, the critical behavior turns out to be mean-field like. Here, we prove that at the quantum level the above scenario does not apply: when an $O(N)$ number of extra ferromagnetic couplings are randomly superimposed onto a quantum Ising chain, its quantum critical point and behavior remain both unchanged. In other words, at zero temperature quantum fluctuations destroy any small-world effect (quantum networks cannot be small-world). This exact result sheds new light on the significance of the quantum critical point as a thermodynamically stable feature of nature and might be crucial for quantum annealing.

In the last decades, the theory of complex systems and complex networks has seen impressive developments [1–10]. At the heart of this classical theory lies the concept of phase transition as a result of the interplay between topology of the underlying network, ranging from completely regular to completely random, and physical process running on it. Two natural questions emerge: To what extent can we apply this theory to quantum systems? How a quantum phase transition [11–15] is affected by the topology of the underlying network? Such questions, far from being purely academic, become crucial in the tentative of solving hard combinatorial optimization problems by using quantum annealing as opposed to the classical thermal annealing: in the search for the global minimum of a cost function characterized by many local minima (such cost functions may be represented by the energy of a classical disordered system of spins, like those in spin glass models [1, 23, 24]), quantum annealing may outperform classical-thermal annealing due to quantum tunneling [16–22]. This appealing idea is already being implemented since a few years, mostly by using qbits simulated by superconducting units, as in the case of the D-Wave system Inc.. However, despite a rapid development of these systems, also for commercial use, it remains not clear if tunneling effects along the quantum annealing path are strong enough to compete against the classical thermal annealing. In practical terms, the failure of the latter scenario would imply that, when facing with the solution of a hard combinatorial problem, a computation based on quantum annealing might not be competitive against classical super computers [25–31]. In this direction, several studies are aiming at finding suitable benchmark combinatorial problems to discriminate whether or not the quantum annealer is outperforming against the classical one [32, 33]. At the core of the issue lies the possibility that the quantum annealer undergoes a quantum phase transition as one or more physical parameters are changed along the quantum annealing path [26, 27, 34]. In particular, it is crucial to understand whether this transition occurs at zero or finite temperature, the latter case usually being realized in correspondence of the hardest and more appealing combinatorial problems [33]. It is in this respect that random graphs and their variants become interesting when applied to a set of $N$ qbits. In fact, the general feature of random graphs and small-world graphs is their small-world character: as opposed to D-dimensional lattices, where the average distance between two randomly chosen nodes scales as $N^{1/D}$, in small-world graphs the distance scales as $\log(N)$ [3]. In terms of phase transitions, this small-world property is responsible for dramatically favoring long range order.

Figure 1. Phase diagram of the quantum Ising chain, Eq. (5), on the plane $(h_0/J_0, T)$, where $J_0$ is the first-neighbor ferromagnetic coupling and $h_0$ the transverse field. The curve $c > 0$ (red), $c$ being the additional mean connectivity, is a qualitative representation of the finite critical temperature $T_c$ as a function of the adimensional parameter $h_0/J_0$. For $c = 0$ this curve collapses to $T_c \equiv 0$ (blue). The points A and B represent the classical and quantum critical points, respectively, and can be exactly calculated: $A \equiv (T_c(h_0 = 0), 0)$, $T_c(h_0 = 0)$ being the solution of Eq. (14); while $B \equiv (1, 0)$. In A, the critical behavior is mean-field classical for any $c > 0$, while for $T = 0$ the critical behavior and the critical point are those of a classical D=2 anisotropic Ising model. As we increase $c$, the point A moves upward but B does not move at all.

Starting from any regular lattice, the small-world property can be attained by rewiring an $O(N)$ number of couplings or by superimposing an equivalent number of extra couplings onto the original system. The latter procedure generates more analytically treatable models and several exact results have been reached [35, 36], also for the $D \geq 2$ case [37, 43]. In particular, for the classical Ising model these studies show that, due to the small-world effect, in the $D=1$ case the system
acquires a finite critical temperature, while in the D=2 case the system gets a higher critical temperatures and, in both cases, the critical behavior turns out to be mean-field like.

Here we prove that this classical scenario does not hold at the quantum level: when an $O(N)$ number of extra ferromagnetic couplings are randomly superimposed onto a D=1 quantum Ising chain, its quantum critical point and behavior remain both unchanged, see Fig. 1. In other words, at zero temperature quantum fluctuations destroy any small-world effect. As we shall show, the ultimate reason for that is the fact that, at zero temperature, an extra dimension - the one associated to the imaginary time evolution - arises that is not covered by the small-world links; the “quantum graph” is never small world. As a consequence, caution is in order before transferring the established knowledge of classical complex systems into the quantum world. This exact result sheds new light into the quantum world. This exact result sheds new light on the meaning of the quantum critical point as a thermodynamically stable feature of systems and might be crucial for determining optimal quantum annealing paths for hard combinatorial problems.

Let us consider a ring of $N$ qbits interacting via a first neighbor ferromagnetic coupling $J_0 > 0$ and subjected to a transverse field $h_0$

$$H_0 = -J_0 \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - h_0 \sum_{i=1}^{N} \sigma_i^x, \quad (1)$$

where $\sigma_i^x$, $\sigma_i^y$, and $\sigma_i^z$ are the Pauli matrices of the $i$-th qbit. This system is known to develop a zero temperature second-order quantum phase transition at the critical point $J_0 = h_0$. A way to see this consists in solving the model via the Jordan-Wigner transformations [38, 39]. Another interesting way consists in applying the quantum classical mapping (QCM) [40]. The QCM evaluates the partition function $Z_0 = \text{Tr} \exp(-\beta H_0)$ of the quantum D=1 model with Hamiltonian (1), as the partition function of an anisotropic classical D=2 discrete torus $[1, \ldots, M] \times [1, \ldots, N]$ and, up to terms $O(1/M^2)$,

$$\beta H_0^{\text{Classic}} = -\beta J_{0x} \sum_{j=1}^{M} \sum_{i=1}^{N} S_{i,j} S_{i+1,j} - \beta J_{0y} \sum_{i=1}^{N} \sum_{j=1}^{M} S_{i,j} S_{i,j+1}, \quad (2)$$

where: the $S_{i,j}$ are $MN$ virtual classical spins arranged on the D=2 discrete torus $[1, \ldots, M] \times [1, \ldots, N]$ and, up to terms $O(1/M^2)$,

$$\beta J_{0x} = \frac{\beta J_0}{M}, \quad \beta J_{0y} = \frac{1}{2} \ln \left( \frac{M}{\beta h_0} \right). \quad (3)$$

Systems (1) and (2) become equivalent in the limit $M \to \infty$, i.e. in the limit in which the Trotter-Suzuki factorization [40], at the base of the QCM, becomes exact. In this limit, $J_{0x} \to 0^+$ while $J_{0y} \to +\infty$ in such a way that the system can have a finite critical point. In fact, by plugging Eqs. (3) into the equation for critical point of the D=2 Ising model (from Kramers-Wannier duality [45], or Onsager’s solution [46]),

$$\sinh(2\beta J_{0x}) \sinh(2\beta J_{0y}) = 1, \quad (4)$$

in the limit $M \to \infty$, regardless of $\beta$, we get the critical point of the original D=1 quantum system (1): $J_0 = h_0$.

The reader should take into account two technical warnings in the following: (i) As first remarked by Suzuki [40], the QCM does not claim that the critical behavior of the D=1 quantum system $H_0$ at finite temperatures is equal to that of the D=2 classical system $H_0^{\text{Classic}}$ since, for any finite $\beta$, the latter, due to Eqs. (3), degenerates when $M \to \infty$ so that, in general, the resulting classical model might be equivalent to a suitable - but rather non obvious - D=1 model; ii) The QCM makes sense and is useful for analytic manipulations also for large but finite $M$, provided that $M$ scales at least proportionally with $N$. As a practical rule one can simply take $M = N$.

Let us now see what happens when we add $cN$ extra interactions between random pairs of qbits via another ferromagnetic coupling $J > 0$; $c > 0$ being the additional mean connectivity. The new quantum Hamiltonian reads

$$H = -J \sum_{i \neq j=1}^{N} c_{i,j} \sigma_i^z \sigma_j^z - J_0 \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - h_0 \sum_{i=1}^{N} \sigma_i^x, \quad (5)$$

where $c_{i,j}$ is the adjacency matrix of the extra couplings, i.e., a random variable taking the values 0 or 1 with probabilities $1 - c/N$ or $c/N$, respectively. The resulting ensemble of graphs generated by the different realizations of $\{c_{i,j}\}$ is known as the Gilbert random graph (a slight variant of the Erdös-Reny random graph), and its properties are well known [4]. In particular, for $N$ large, the connectivity of each node becomes poissonian distributed with mean $c$ and, for any $c > 1$, the graph is percolating and owns the small-world property. In general, for different realizations of $\{c_{i,j}\}$ there correspond different Hamiltonians (5). Yet, in the thermodynamic limit $N \to \infty$, due to the self-averaging character of the random graph, relative fluctuations of the system become negligible. In other words, any extensive observable (like the energy or the magnetization), can be evaluated either via a single realization of a sufficiently large system (which with probability 1 is typical), or as an average over the adjacency matrix realizations. The latter is the usual successful set-up applied to all (quenched [1]) disordered models in classical physics which, thanks to the QCM, we can assume to be valid also in the present quantum case.
On applying the QCM to the quantum system (5) we get
\[ \beta H^{\text{Classic}} = -\beta \frac{J}{M} \sum_{j=1}^{M} \sum_{i, j}^{N} c_{i, j} S_{i} S_{j} \beta H^{\text{Class}}_0, \]
(6)
where \( H^{\text{Class}}_0 \) is defined in Eqs. (2)-(3). From the first term of Eq. (6), we see that the small-world effect on the underlying graph of \( H^{\text{Classic}} \), if any, can realize only along the x-direction; see middle panel of Fig. 2. In other words, in order to cross the D=2 torus \( [1, \ldots, M] \times [1, \ldots, N] \) from one corner to the opposite via random hoppings on the underlying graph of \( H^{\text{Classic}} \), on average we must use a \( O(\log(N)M) \) number of links, while in a small-world graph this would be \( O(\log(MN)) \). As a consequence, we expect that \( H^{\text{Classic}} \), and hence the quantum system governed by \( H \), has not acquired a mean-field character, and that it remains essentially similar to the original quantum system governed by \( H_0 \). In the following we prove that this guess is exact in a extreme sense: not only the quantum critical behaviors, but also the quantum critical points of \( H_0 \) and \( H \) are the same.

Let us denote the averages over the \( \{c_{i, j}\} \) realizations by \( \langle \cdot \rangle \). As explained above, if \( \langle O \rangle = \text{Tr} O \exp(-\beta H)/\text{Tr} \exp(-\beta H) \) denotes the ensemble average of the observable \( O \) associated to a given \( \{c_{i, j}\} \) (quenched) realization, for \( N \) large, we can conveniently identify this average with \( \langle O \rangle \). In turn, all these averages can essentially be derived from the free energy density \( f \) of the quenched model: 
\[ -\beta f = \lim_{N \to \infty} \frac{\ln(Z)}{N} = \lim_{N \to \infty} \lim_{n \to 0} (Z^n - 1)/(N n). \]
The latter identity is at the base of the so called replica-trick that has been used to investigate a large variety of random models, especially spin-glass models [1]. When the replica-trick is used in combination with the high temperature expansion of the free energy of a model built over a random graph, there emerges a general mapping between the random model (or “disordered model”) and a suitable non random model [37, 41, 42]. In the following, we refer to this mapping as the random non random mapping (RNRM). In general, the disorder can be due to the underlying graph structure having a generic random matrix \( \{c_{i, j}\} \) and, more in general, to the random values of the corresponding couplings \( \{J_{i, j}\} \). In both cases, the RNRM consists in the following replacement (here \( \tau \) means average over any kind of disorder)
\[ c_{i, j} \beta J_{i, j} \to c_{i, j} \beta J_{i, j} \tanh(\beta J_{i, j}). \]
(7)
As has been confirmed also via Monte Carlo simulations [43], the RNRM (7) rules out all the difficulties of the random model by providing its exact critical point and behavior while giving effective approximations below the critical temperature. When applied to the adjacency matrix \( \{c_{i, j}\} \) of the random graph with a constant coupling \( J/M \), for \( M \) large but finite, Eq. (7) sends the random classical Hamiltonian (6) to the following non random Hamiltonian
\[ \beta H^{\text{Classic}} = -\frac{\beta J}{MN} \sum_{j=1}^{M} \sum_{i, j}^{N} S_{i} S_{j} + \beta H^{\text{Class}}_0. \]
(8)
The Hamiltonian \( \tilde{H}^{\text{Classic}} \) represents a D=2 Ising model with superimposed fully connected interactions that run only within the rows of the torus \( [1, \ldots, M] \times [1, \ldots, N] \). Its partition function \( Z^{\text{Classic}} \) can be analyzed by a standard technique [44]. By introducing \( M \) auxiliary independent gaussian fields \( \{x_j\} \), up to a \( O(1) \) term in the exponent, and up to an immaterial constant of proportionality, we get
\[ Z^{\text{Classic}} \propto \int \prod_{j=1}^{M} dx_j e^{-N\beta f(x_j)}, \]
(9)
\[ \beta f(\{x_j\}) = \frac{c\beta J}{M} \sum_{j=1}^{M} x_j^2 + \beta f_0 \left( \beta J_{0x}, \beta J_{0y}; \left\{ \frac{c\beta J}{M} x_j \right\} \right), \quad (10) \]

where \( f_0(\beta J_{0x}, \beta J_{0y}; \{\beta h_j\}) \) is the free energy density of the D=2 Ising model with the couplings \( J_{0x} \) and \( J_{0y} \) in the presence of a row-dependent external field \( \{h_j\} \). Notice that, in order to avoid a pedantic notation like \( f_0(\beta J_{0x}, \beta J_{0y}; \{h_j\}; N, M) \) etc., in Eq. (9) and following, the harmless dependencies on finite size effects are left understood but they should be kept in mind for the correct interpretation of the next equations. The steepest descent method applied to Eq. (9) provides the following effective mean-field equations for the row-dependent average magnetizations \( m_j = \frac{1}{N} \sum_{i=1}^{N} S_{i,j} \) of the system.

\[ m_j = m_0 \left( \beta J_{0x}, \beta J_{0y}; \left\{ \frac{c\beta J}{M} m_j \right\} \right), \quad j = 1, \ldots, M, \quad (11) \]

where \( m_0(\beta J_{0x}, \beta J_{0y}; \{h_j\}) \) is the magnetization of the D=2 Ising model with the couplings \( J_{0x} \) and \( J_{0y} \) in the presence of a row-dependent external field \( \{h_j\} \). Equations (11) and following are valid up to \( \mathcal{O}(\log(N)/N) \) corrections. Let us focus on the thermodynamically dominant uniform solution \( \{m_j = m\} \) (making \( f(\{m_j\}) \) a maximum). By deriving Eq. (11) with respect to a uniform external field \( \{\beta h_j = \beta h\} \), we obtain the adimensional susceptibility of the system

\[ \chi = \frac{\chi_0(\beta J_{0x}, \beta J_{0y}; \frac{c\beta J}{M} m)}{1 - \frac{c\beta J}{M} \chi_0(\beta J_{0x}, \beta J_{0y}; \frac{c\beta J}{M} m)}, \quad (12) \]

where \( \chi_0(\beta J_{0x}, \beta J_{0y}; \beta h) \) is the adimensional susceptibility of the D=2 Ising model with the couplings \( J_{0x} \) and \( J_{0y} \) in the presence of a uniform external field \( h \). The paramagnetic solution of Eq. (11) is stable when the denominator of Eq. (12) evaluated at \( m = 0 \) is positive. In other words, for finite \( M \), the paramagnetic solution becomes unstable when

\[ \frac{c\beta J}{M} \chi_0(\beta J_{0x}, \beta J_{0y}; 0) = 1. \quad (13) \]

Taking into account the critical point of the D=2 model, Eq. (4), Eq. (13) tells us that, for \( M \) large but finite, the critical point of the system (shifted at higher temperatures or, in terms of the couplings \( \beta J_{0x} \) and \( \beta J_{0y} \), is such that \( \sinh(2\beta J_{0x}) \sinh(2\beta J_{0y}) < 1 \). Furthermore, for \( M \) large but finite, Eq. (11) tells us that the system is essentially mean-field like, with the classical critical exponents. In fact, from Eq. (12) we see that the susceptibility has the critical exponent \( \gamma = 1 \) and similarly for the other critical exponents. However, the QCM holds true only in the limit \( M \to \infty \). In such a limit, Eq. (13) can be satisfied only at the critical point of the D=2 model (4) (where \( \chi_0(\beta J_{0x}, \beta J_{0y}, 0) \to \infty \)) which, by using explicitly the expression for the couplings, Eqs. (3), implies that the critical point of the quantum system with extra random couplings, Eq. (5), is just equal to the critical point of the quantum system without random couplings, Eq. (1): \( J_0 = h_0 \).

Furthermore, if we choose \( J_0 < h_0 \), so that we are in the paramagnetic region, on sending \( M \to \infty \) in Eq. (12), we get \( \lim_{M \to \infty} \chi = \lim_{M \to \infty} \chi_0(\beta J_{0x}, \beta J_{0y}; 0) \), which implies that the critical exponent of the susceptibility of the system (5) is equal to the critical exponent of the susceptibility of the quantum system (1) (\( \gamma = 7/4 \)), and the same argument applies to the other critical exponents. Notice that our analysis does not imply that the systems (1) and (5) at \( T = 0 \) are identical. As we have mentioned before, inside the ferromagnetic region the RNRM is only an approximation where Eq. (11) turns out to be effective [43].

The above result is limited to zero temperature. The other available exact result concerns the pure classical model, i.e., the case with no transverse field, \( h_0 = 0 \). As discussed in the introduction, in this case, for any \( c > 0 \) the system is effectively mean-field and its critical point can be exactly calculated by the following equation [36, 37]

\[ c \tanh(\beta J) e^{c\beta J} = 1. \quad (14) \]

Equation (14) tells us that the critical temperature is a growing function of \( c \) (linear for large \( c \)). For any other case, i.e., the region \( (h_0 > 0, T > 0) \), there are no exact results, however, as discussed before in warning (i), in this region the D=1 quantum system behaves partially as a D=1 classical system [11, 40]. Combining Eq. (14) with the quantum critical point we get the scenario depicted in Fig. 1, with a line of critical temperatures that grows with \( c \) for any \( h_0 < J_0 \). For \( c = 0 \), two lines of a D=2-D=1 crossover are expected to exist that depart from the quantum critical point \( h_0 = J_0 \) [11, 12, 40]. From the experimental point of view, this crossover region represents the most important part of the problem [12]. Our analysis leads to expect that, for \( c > 0 \), such a cross-over might become D=2-mean-field. This is a rather interesting issue that deserves further investigation.

In conclusion, we have proved that at zero temperature there is no small-world effect, the quantum critical point and behavior of the system remaining those of the finite dimensional model before the addition of the extra links or, equivalently, before the rewiring. Quantum fluctuations destroy any small-world effect and raise the quantum critical point as a thermodynamically robust feature of nature. We expect important applications of this invariance for quantum annealing.

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