DEGREE 1 ELEMENTS OF THE SELBERG CLASS

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In [5] A. Selberg axiomatized properties expected of L-functions and introduced the “Selberg class.” We recall that an element $F$ of the Selberg class $S$ satisfies the following axioms.

**Axiom 1.** In the half-plane $\sigma > 1$ the function $F(s)$ is given by an absolutely convergent Dirichlet series $\sum_{n=1}^{\infty} a(n)n^{-s}$ with $a(1) = 1$ and $a(n) \ll n^\epsilon$ for every $\epsilon > 0$.

**Axiom 2.** There is a natural number $m$ such that $(s - 1)^m F(s)$ extends to an analytic function in the entire complex plane.

**Axiom 3.** There is a function $\Phi(s) = Q^s G(s) F(s)$ where $Q > 0$ and

$$G(s) = \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) \quad \text{with} \quad \lambda_j > 0 \text{ and } \text{Re} \, \mu_j \geq 0$$

such that

$$\Phi(s) = \omega \Phi(1-s),$$

where $|\omega| = 1$ and for any function $f$ we denote $\overline{f}(s) = \overline{f(\overline{s})}$. We let $d := 2 \sum_{j=1}^{r} \lambda_j$ denote the “degree” of $F$.

**Axiom 4.** We may express $\log F(s)$ by a Dirichlet series

$$\log F(s) = \sum_{n=2}^{\infty} \frac{b(n) \Lambda(n)}{n^s \log n}$$

where $b(n) \ll n^\vartheta$ for some $\vartheta < \frac{1}{2}$. Set $b(n) = 0$ if $n$ is not a prime power.

A fundamental conjecture asserts that the degree of an element in the Selberg class is an integer. From the work of H.E. Richert [4] it follows that there are no elements in the Selberg class with degree $0 < d < 1$. This was rediscovered by J.B. Conrey and A. Ghosh [1] who also proved that the Selberg class of degree 0 contains only the constant function 1. Recently J. Kaczorowski and A. Perelli [2] determined the structure of the Selberg class for degree 1 and showed that this consists of the Riemann zeta function and shifts of Dirichlet $L$-functions. Subsequently in [3] they showed that there are no elements of the Selberg class with degree $1 < d < 5/3$. In this note we shall give a short and simple proof of Kaczorowski and Perelli’s beautiful result on the Selberg class for degree 1.

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Theorem. Suppose \( F \) satisfies Axioms 1 to 3 and that the degree of \( F \) is \( 1 \). Then there exists a positive integer \( q \) and a real number \( A \) such that \( a(n) n^{-1} A \) is periodic (mod \( q \)). If in addition \( F \) satisfies Axiom 4 then there is a primitive Dirichlet character \( \chi' \) (mod \( q' \)) such that \( F(s) = L(s + i A, \chi') \).

We remark that Kaczorowski and Perelli obtain their results without assuming the hypothesis \( a(n) \ll n^\epsilon \). We could restructure our proof to avoid this assumption, but have preferred not to do so in the interest of keeping the exposition transparent. Our method may also be modified and combined with the ideas in Kaczorowski and Perelli [3] to give a simplification of their result for \( 1 < d < 5/3 \).

Suppose \( F \) satisfies Axioms 1 to 3 and has degree 1. By Stirling’s formula we see that for \( t \geq 1 \)

\[
\frac{G(1/2 - it)}{G(1/2 + it)} = e^{-it \log \frac{1}{\pi} + i \frac{\pi}{4} + iB - it C - it \left(1 + O\left(\frac{1}{t}\right)\right)},
\]

for some real numbers \( A, B \) and \( C > 0 \). Let \( \alpha \) be positive and \( T \geq 1 \). Define

\[
\mathcal{F}(\alpha, T) = \frac{1}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} F(1/2 + it) e^{it \log \frac{1}{\pi} + i \frac{\pi}{4} - it} \, dt,
\]

and set (it will follow from our proof that the limit below is well defined)

\[
\mathcal{F}(\alpha) = \lim_{T \to \infty} \frac{1}{T^{1+iA}} \mathcal{F}(\alpha, T).
\]

Lemma. For any real number \( t \) and all \( X \geq 1 \) we have

\[
F(\frac{1}{2} + it) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\frac{1}{2} + it}} e^{-n/X} + O((1 + |t|)^{1+\epsilon} X^{-1+\epsilon} + X^{\frac{1}{2}+\epsilon} e^{-|t|}).
\]

Proof. Consider for \( c > \frac{1}{2} \)

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\frac{1}{2} + it + w) X^w \Gamma(w) \, dw.
\]

Expanding \( F(\frac{1}{2} + it + w) \) into its Dirichlet series and integrating term by term we see that this equals \( \sum_n a(n) n^{-\frac{1}{2} - it} e^{-n/X} \). Next we move the line of integration to \( \text{Re}(w) = -1 + \epsilon \). The pole at \( w = \frac{1}{2} - it \) leaves the residue \( F(\frac{1}{2} + it) \). The possible pole at \( w = \frac{1}{2} - it \) leaves a residue \( \ll X^{\frac{1}{2}+\epsilon} (1 + |t|)^{\epsilon} \Gamma(\frac{1}{2} + it) | \ll X^{\frac{1}{2}+\epsilon} e^{-|t|} \) due to the rapid decay of \( \Gamma(\frac{1}{2} + it) \). Note that by the functional equation and Stirling’s formula \( |F(\frac{1}{2} + it + w)| = O(1-2\text{Re} w) |G(\frac{1}{2} - it - w)| |F(\frac{1}{2} - it - w)| \ll (1 + |t| + |w|)^{1+\epsilon} \) for any \( w \) on the line \( \text{Re} w = -1 + \epsilon \).

Hence the integral on the line \( \text{Re}(w) = -1 + \epsilon \) is \( \ll X^{-1+\epsilon} (1 + |t|)^{1+\epsilon} \).
Using the functional equation in (2a) we see that
\[ \mathcal{F}(\alpha, T) = \frac{\omega}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} F(1/2 - it)Q^{-2it}G(1/2 - it)G(1/2 + it) e^{it\log \frac{1}{2\pi e\alpha} - i\frac{\pi}{4}} \, dt \]
and using (1) this is
\[ \frac{\omega e^{iB}}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} F(1/2 - it)(\pi CQ^2\alpha)^{-it}iA(1 + O(1/T)) \, dt. \]

We now input our Lemma above with \( X = T^{\frac{4}{3}} \) to deduce that
\[
\mathcal{F}(\alpha, T) = \frac{\omega e^{iB}}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} \sum_m \frac{a(m)}{\sqrt{m}} e^{-m/X} \left( \frac{m}{\pi CQ^2\alpha} \right)^{it} iA \left( 1 + O\left( \frac{1}{T} \right) \right) \, dt + O(T^{\frac{2}{3} + \epsilon}).
\]

If \( x \neq 1 \) then integration by parts gives that
\[
\int_{\alpha T}^{2\alpha T} x^{it} iA \, dt = \left( \frac{2\alpha T}{i \log x} \right)^{iA} x^{2it} - \left( \frac{\alpha T}{i \log x} \right)^{iA} x^{it} - \int_{\alpha T}^{2\alpha T} \frac{x^{it}}{i \log x} iA \, dt \ll \frac{1}{|\log x|},
\]
while if \( x = 1 \) we have that
\[
\int_{\alpha T}^{2\alpha T} iA \, dt = \frac{(2\alpha T)^{1+iA} - (\alpha T)^{1+iA}}{1+iA}.
\]

Using (4a,b) in (3) we obtain that
\[
\mathcal{F}(\alpha) = \lim_{T \to \infty} T^{-1-iA} \mathcal{F}(\alpha, T) = \omega e^{iB} \delta(\pi CQ^2\alpha) \sum_{\alpha \in \mathbb{N}} \frac{a(\pi CQ^2\alpha)\alpha^{iA}}{\sqrt{\pi CQ}} \frac{2^{1+iA} - 1}{1+iA} + O\left( \lim_{T \to \infty} T^{-1+\epsilon} \sum_m \frac{|a(m)|}{\sqrt{m}} e^{-m/X} \right)
\]
\[
= \omega e^{iB} \delta(\pi CQ^2\alpha) \sum_{\alpha \in \mathbb{N}} \frac{a(\pi CQ^2\alpha)\alpha^{iA}}{\sqrt{\pi CQ}} \frac{2^{1+iA} - 1}{1+iA},
\]
where \( \delta(\pi CQ^2\alpha) = 1 \) if \( \pi CQ^2\alpha \in \mathbb{N} \) and is 0 otherwise.

We now present a different way of evaluating \( \mathcal{F}(\alpha, T) \) which will show that \( \mathcal{F}(\alpha) \) is periodic in \( \alpha \) with period 1. Using our Lemma with \( X = T^{\frac{4}{3}} \) again, we see that
\[
\mathcal{F}(\alpha, T) = \frac{1}{\sqrt{\alpha}} \sum_n \frac{a(n)}{\sqrt{n}} e^{-n/X} \int_{\alpha T}^{2\alpha T} e^{it\log \frac{1}{2\pi e\alpha} - i\frac{\pi}{4}} \, dt + O(T^{\frac{2}{3} + \epsilon}).
\]
The oscillatory integral in (6) above is estimated by familiar techniques, see Lemmas 4.2 and 4.6 of E.C. Titchmarsh [6]. For \(2\pi n > 3T\) we use Lemma 4.2 of Titchmarsh which shows that the integral is \(\ll 1\). Thus the contribution of such \(n\) to (6) is \(\ll T^{\frac{2}{3}} + \varepsilon\). For smaller \(n\) we use Lemma 4.6 of Titchmarsh. In the range \(T \leq 2\pi n \leq 2T\) we obtain that the integral is

\[
2\pi \sqrt{n}ae(-n\alpha) + O\left(T^{\frac{2}{3}} + \min\left(\sqrt{T}, \frac{1}{|\log(T/2\pi n)|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log(T/\pi n)|}\right)\right).
\]

For \(2\pi n\) below \(T\) or between \(2T\) and \(3T\) the integral is bounded by the error terms above. Piecing this together we conclude that

\[
F(\alpha, T) = 2\pi \sum_{T \leq 2\pi n \leq 2T} a(n)e(-n\alpha) + O(T^{\frac{2}{9}} + \varepsilon).
\]

Hence \(F(\alpha) = F(\alpha + 1)\) which leads by (5) to

\[
\delta(\pi CQ^2 \alpha \in \mathbb{N})a(\pi CQ^2 \alpha)\alpha^{iA} = \delta(\pi CQ^2(\alpha + 1) \in \mathbb{N})a(\pi CQ^2(\alpha + 1))(\alpha + 1)^{iA}.
\]

From (7) we deduce immediately that \(\pi CQ^2 = q\) must be a positive integer and further that \(a(n)n^{iA}\) must be periodic \((\text{mod } q)\). This proves the first part of our Theorem.

Suppose now that \(F\) also satisfies Axiom 4 so that the coefficients \(a(n)\) are multiplicative. Periodicity and multiplicativity together imply that for all \(n\) coprime to \(q\), \(a(n)n^{-iA}\) must equal \(\chi(n)\) for a Dirichlet character \(\chi\) \((\text{mod } q)\). Let \(\chi'\) \((\text{mod } q')\) denote the primitive character inducing \(\chi\). Then the ratio \(F(s)/L(s+iA, \chi')\) is an Euler product over the finitely many primes dividing \(q\), and by Axiom (4) the logarithm of this Euler product converges absolutely in the half plane \(\sigma > \vartheta\) (recall that \(\vartheta < \frac{1}{2}\)). Thus, with \(a = (1 - \chi'(-1))/2,\)

\[
H(s) := \frac{Q^*G(s)F(s)}{(q'/\pi)^{2}\Gamma\left(\frac{2+iA+a}{2}\right)L(s + iA, \chi')}
\]

is an entire function in \(\text{Re}(s) > \vartheta\) and \(\overline{H}(s)\) is also entire in this region. Further both functions are free of zeros in this region. The functional equations for \(F\) and \(L\) now show that \(H(s)\) and \(\overline{H}(s)\) are entire and free of zeros in the region \(\text{Re}(s) < 1 - \vartheta\). Since \(\vartheta < \frac{1}{2}\) we deduce that \(H(s)\) and \(\overline{H}(s)\) are entire functions in all of \(\mathbb{C}\) and that they never vanish. Since \(H\) is the ratio of two entire functions\(^1\) of order 1 it follows by Hadamard’s theorem that \(H(s) = ae^{bs}\) for some constants \(a\) and \(b\). The functional equation connecting \(H(s)\) and \(\overline{H}(1-s)\) now mandates that \(b = 0\) and so \(H\) is a constant. Examining the behaviour of \(H(1/2 + it)\) for large \(t\) it follows easily that \(F(s) = L(s+iA, \chi')\), proving our Theorem.

\(^1\)If \(\chi'\) is the trivial character then multiply the numerator and denominator in the definition of \(H\) by \(s(s-1)\) to make them regular at 1.
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