BIFURCATIONS IN AN ECONOMIC MODEL WITH FRACTIONAL DEGREE

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Abstract. A planar ODE system which models the industrialization of a small open economy is considered. Because fractional powers are involved, its interior equilibria are hardly found by solving a transcendental equation and the routine qualitative analysis is not applicable. We qualitatively discuss the transcendental equation, eliminating the transcendental term to polynomialize the expression of extreme value, so that we can compute polynomials to obtain the number of interior equilibria in all cases and complete their qualitative analysis. Orbits near the origin, at which the system cannot be extended differentiably, are investigated by using the GNS method. Then we display all bifurcations of equilibria such as saddle-node bifurcation, transcritical bifurcation and a codimension 2 bifurcation on a one-dimensional center manifold. Furthermore, we prove nonexistence of closed orbits, homoclinic loops and heteroclinic loops, exhibit global orbital structure of the system and analyze the tendency of the industrialization development.

1. Introduction. Industrialization plays a fundamental role in the development of national economy, and every developed country benefited a lot from their industrialization in history. More and more developing countries tried to reduce poverty by industrialization, but some of them did not find a suitable way. These failures attracted great attention from economists ([10, 12, 13, 15]), and some differential dynamical systems were set up to investigate the structure change, the movement of the labor force from the traditional resource-based sector to the modern sector, in small open economies ([1, 2, 9, 11, 13]). Several reasons for the failure have been presented such as expansion of the urban informal sector and low absorption of labor in high productivity sectors ([4, 11, 14]).

Among those differential systems, the one considered in [1] is of special interests because the involved fractional power makes difficulties in computing coordinates of equilibria. The system is the following 2-dimensional system

\[
\frac{dE}{dt} = \begin{cases} 
E(\tilde{E} - E) - \frac{\epsilon K}{(E^{\alpha/(\alpha-1)} + K)^\beta} & \text{for } E > 0, \\
0 & \text{for } E = 0,
\end{cases}
\]

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\[
\frac{dK}{dt} = K \left( \frac{s(1 - \alpha)}{(E^{\beta/(1-\alpha)} + K)^\alpha} - \xi \right),
\]
which is defined on \{ (E, K) \in \mathbb{R}^2 | E \geq 0, K \geq 0 \) and models the time evolutions of \( E \), the stock of the natural renewable resource, and \( K \), the stock of the capital owned by industrial entrepreneurs, in a small open economy. In (1), \( \bar{E} > 0 \) represents the carrying capacity of the natural resource, \( \epsilon > 0 \) measures the environmental impact of industrial production, \( s, \xi \in (0, 1) \) are the marginal propensity to save and the depreciation rate of capital stock respectively, and \( \alpha, \beta \) are parameters such that \( \alpha, \beta > 0 \) and \( \alpha + \beta \leq 1 \). Authors of [1] found that system (1) has two boundary equilibria \( S_1 : (0, \zeta) \) and \( S_2 : (\bar{E}, 0) \), and showed that \( S_1 \) is always attractive and \( S_2 \) is either saddle if \( 0 < \bar{E} < E_M := \zeta^{(1-\alpha)/\beta}, \) where \( \zeta := (\frac{1-\alpha}{\beta})^{1/\alpha}, \) or a stable node if \( \bar{E} > E_M, \) but did not give qualitative properties of \( S_2 \) in the case \( \bar{E} = E_M \) which actually is a degenerate case.

More difficulties come from interior equilibria, which depend on zeros of the following function

\[
g(E) := \epsilon \zeta^{-\alpha} E^{\frac{-2}{\beta}} - E^2 + \bar{E} E - \epsilon \zeta^{1-\alpha}
\]
in the interval \( (0, E_M) \). The function \( g \) is obtained from the first function on the right hand side of (1) substituted with \( K = \zeta - E^{\beta/(1-\alpha)} \), given by the second factor in the second equation of (1) being zero. An effective idea is applying the Intermediate Theorem to the continuous \( g \) by comparing signs of \( g \) at endpoints of the interval \( (0, E_M) \) with the sign of \( g \) at its extreme point as done in [6, 8, 17]. However, one hardly solves a coordinate for extreme point of \( g \) because of the involved fractional powers. The authors of [1] did not compute the extreme value, although they knew it is a single and maximal one, but considered a value above it and a value below it instead. The value above is \( P_2(\bar{E}) := \max_{0 < E \leq \bar{E}} \{ \epsilon \zeta^{-\alpha} E^{\frac{-2}{\beta}} \} + \max_{0 < E \leq \bar{E}} \{ E(\bar{E} - E) \} - \epsilon \zeta^{1-\alpha}, \) an upper but unreachable bound of \( g \); the value below is \( P_1(\bar{E}) := g(\bar{E}/2) \), the value of \( g \) at \( (\bar{E}/2) \) which can be computed conveniently. With those values they presented in Proposition 1 of [1] that two interior equilibria \( S_3 : (E_A, \zeta - E_A^{\beta/(1-\alpha)}) \) and \( S_4 : (E_B, \zeta - E_B^{\beta/(1-\alpha)}) \) exist if \( 0 < \bar{E} \leq E_M \) and \( P_1(\bar{E}) > 0 \), exact one hyperbolic interior equilibrium \( S_3 \) exists if and only if \( \bar{E} > E_M \), and no interior equilibria exist if \( 0 < \bar{E} \leq E_M \) and \( P_2(\bar{E}) < 0 \), where \( E_A \) and \( E_B \) denote the zeros of \( g \) such that \( 0 < E_A < E_B \). Using \( E_1 \) and \( E_2 \) to denote the positive zeros of the increasing functions \( P_1(\bar{E}) \) and \( P_2(\bar{E}) \) respectively, we have \( E_1 > E_2 \) and \( E_1 < E_M \) (resp. \( E_1 = E_M \), \( E_1 > E_M \)) if \( \epsilon < \epsilon \) (resp. \( \epsilon = \epsilon \), \( \epsilon > \epsilon \)), where \( \epsilon := (4 - 2^{2-\frac{1}{\beta}})^{-1} \zeta^{-1/(1-\alpha)(2-\beta)}. \) We summarize their conclusions on interior equilibria in Table 1, where the two ‘unknown’s in cases (C2) and (C6) in the last column are the number of interior equilibria and their qualitative properties related to degeneracy of the system.

In this paper, overcoming their difficulties, we give existence and qualitative properties of equilibria completely and answer their ‘unknown’. Then we discuss bifurcations of those equilibria. Section 2 is devoted to the dynamic properties on boundaries. For the unknown qualitative properties of the boundary equilibrium \( S_2 \) in the case \( \bar{E} = E_M \), we prove that \( S_2 \) is either a saddle-node or a degenerate saddle by the center manifold reduction. In addition, we investigate orbital structure of system (1) near the origin, which was also not considered in [1]. Although system (1) is undefined at the origin, we extend system (1) with a time-rescaling to include
the origin as an isolated equilibrium so that the orbital structure near the origin can be given by discussing the isolated equilibrium qualitatively. However, the routine of eigenvalue analysis is not applicable because the extended system is not differentiable at the origin for the involved fraction powers $\alpha$ and $\beta/(1-\alpha)$. This difficulty is overcome by using the method of GNS ([18]) in exceptional directions. Finally, we prove that system (1) has two equilibria $I_1$ and $I_2$ at infinity, which are both degenerate but their qualitative properties are obtained by the Briot-Bouquet desingularization. Section 3 is devoted to interior equilibria, especially for the ‘unknown’ in cases (C2) and (C6). We give necessary and sufficient conditions for existence of interior equilibria and their qualitative properties in all cases. Our strategy is to determine the sign of the extreme value of $g$, the function defined in (2), without computing the coordinates of the extreme point, which gives accurate conditions for existence and distribution of zeros of $g$ in the interval $(0, E_M)$. For this purpose, we reduce the derivative $g'$ to a form containing the same term of fractional power as in $g$, which can be used to eliminate the fractional power and polynomialize the expression of the extreme value of $g$. The qualitative properties of equilibria is obtained by discussing various cases of signs of the determinant and the trace without knowing a coordinate of equilibria. Moreover, we indicate two errors in Proposition 1 of [1]: the inequality $\tilde{E} > E_M$ is not necessary for exact one hyperbolic interior equilibrium $S_3$ to exist, and the condition that $0 < \tilde{E} \leq E_M$ and $P_1(\tilde{E}) > 0$ is not sufficient for two interior equilibria $S_3$ and $S_4$ to exist. In section 4 we display all bifurcations of equilibria such as saddle-node bifurcation, transcritical bifurcation and a codimension 2 bifurcation on a 1-dimensional center manifold, which occurs at the intersection of two branches of saddle-node bifurcation curves and two branches of transcritical bifurcation curves. In section 5, classifying all possible orbital structures of system (1) into four cases and drawing global phase portraits, we give economic interpretations for those dynamical properties and verify our conclusion by numerical simulation, showing that a sustainable way of development is available only when the carrying capacity of nature resource is modest, the initial environment is well preserved, and the industrial pollution is under control.

2. Boundary dynamics. In this section we discuss the dynamics of system (1) near two axes and equilibria at infinity. As displayed in [1], system (1) has exact two boundary equilibria $S_1 : (0, \zeta)$ and $S_2 : (\tilde{E}, 0)$, where $S_1$ is attractive, and $S_2$ is either a saddle if $0 < \tilde{E} < E_M$ or a stable node if $\tilde{E} > E_M$. So we investigate the qualitative properties of $S_2$ in the remaining case $\tilde{E} = E_M$. Then, we discuss the behaviors of orbits near the origin, which was not considered in [1], because
the origin is an undefined equilibrium. Further, we also consider the tendencies of orbits near the $K$-axis since system (1) is not continuous on the $K$-axis.

**Theorem 2.1.** (i) For $\tilde{E} = E_M$, $S_2$ is either a saddle-node if $\epsilon \neq \epsilon_1 := (1 - \alpha\beta) - (2\alpha - \beta + \alpha\beta)^{1/\beta}$ or a degenerate saddle if $\epsilon = \epsilon_1$. More precisely, $S_2$ has a stable manifold on the $E$-axis and a center manifold in the first quadrant, on which $S_2$ is either stable if $\epsilon < \epsilon_1$ or unstable if $\epsilon \geq \epsilon_1$. (ii) All orbits of system (1) near the origin leave from the origin, one of which lies on the $K$-axis but the others lie in the direction of the $E$-axis. (iii) All orbits near the $K$-axis in the first quadrant reach the $K$-axis in finite time.

**Proof.** First, we investigate qualitative properties of $S_2 : (\tilde{E}, 0)$. Applying the time-scaling $d\tau = (E^{\beta/(1-\alpha)} + K)^\alpha d\tau$, we change system (1) into the form

$$
\begin{align*}
\frac{dE}{d\tau} &= \begin{cases} 
E(\tilde{E} - E)(E^{\beta/\alpha} + K)^\alpha - \epsilon K & \text{for } E > 0, \\
0 & \text{for } E = 0,
\end{cases} \\
\frac{dK}{d\tau} &= \xi\alpha K - \xi K(E^{\beta/\alpha} + K)^\alpha.
\end{align*}
$$

Computing the Jacobian matrix $J(S_2)$ of system (3) at $S_2 : (\tilde{E}, 0)$, we see that the first row is $(-E_M^{1+\beta/\alpha}, -\epsilon)$ and the other entries are both zero. We compute its eigenvalues $\lambda_1 := -E_M^{1+\beta/\alpha} < 0$ and $\lambda_2 = 0$, where $E_M$ is defined before (2). From the second equation of system (3), we see that the function $K(\tau) \equiv 0$ satisfies $dK/d\tau = 0$, showing that the stable manifold of $S_2$ lies on the $E$-axis. On the other hand, applying the invertible linear transformation $E = u - \epsilon E_M^{1+\beta/\alpha} v + E_M$, $K = v$ to translate $S_2$ to the origin $O$ and diagonalize the linear part, we can change system (3) into the form

$$
\begin{align*}
\frac{du}{d\tau} &= \lambda_1 u + A_{20} u^2 + A_{11} uv + A_{02} v^2 + O(|(u,v)|^3), \\
\frac{dv}{d\tau} &= B_{11} uv + B_{02} v^2 + B_{21} u^2 v + B_{12} uv^2 + B_{03} v^3 + O(|(u,v)|^4),
\end{align*}
$$

where the coefficients $A_{ij}$ and $B_{ij}$ are given in Appendix II. By Theorem 3.2.1 of ([7, p.127]), system (4) has a $C^\infty$ center manifold $u = h(v)$ tangent to the $v$-axis at $O$. Clearly it can be presented as $h(v) := v^\beta + O(v^3)$ with an indeterminate constant $\iota$. By its invariance we obtain the equality $\dot{u} = h'(v)v$. Substituting the equations of (4) into the equality and comparing the coefficients, we obtain $\iota = -\lambda_1^{-1} A_{02}$. Thus the restriction of (4) to the manifold is the equation

$$
\frac{dv}{d\tau} = B_{02} v^2 + (\iota B_{11} + B_{03}) v^3 + O(v^4). 
$$

Note in Appendix II that $B_{02} = \frac{\xi\alpha\beta}{\beta - \alpha} E_M^{-2}(\epsilon - \epsilon_1)$ with $\epsilon_1$ defined in Theorem 2.1. If $0 < \epsilon < \epsilon_1$ (or $\epsilon > \epsilon_1$), we have $B_{02} < 0$ (or $> 0$), implying that the origin of system (5) is stable on the positive (or negative) side of the $v$-axis and unstable on the other side. Hence, the origin $O$ of system (4) is a saddle-node, and so is $S_2$. If $\epsilon = \epsilon_1$, we note in Appendix II that $B_{02} = 0$ and $\iota B_{11} + B_{03} = \frac{\xi\alpha(3-3\alpha-\beta)}{2\alpha} E_M^{-2} > 0$, implying that the origin of system (5) is unstable, i.e., $S_2$ is a degenerate saddle. It follows that the center manifold of $S_2$ in the first quadrant is either stable if $\epsilon < \epsilon_1$ or unstable if $\epsilon \geq \epsilon_1$. This completes the proof of (1).
Next, we investigate orbital structure of system (3) near the origin. If $E = 0$, we have $\frac{dE}{dt} = 0$ and $\frac{dK}{dt} = \xi K (\zeta^\alpha - K^\alpha)$, implying that the orbit on the $K$-axis leaves from the origin. If $E > 0$, system (3) is of the form

$$\frac{dE}{d\tau} = E (\tilde{E} - E) (E^{\frac{\alpha}{\beta}} + K)^\alpha - \epsilon K, \quad \frac{dK}{d\tau} = \xi \zeta^\alpha K - \xi K (E^{\frac{\alpha}{\beta}} + K)^\alpha, \quad (6)$$

which can be extended to include the origin $S_0 : (0, 0)$ as an isolated equilibrium. However, the known method of eigenvalue analysis is not available for $S_0$ because the involved fractional powers $\alpha$ and $\beta/(1 - \alpha)$ prevent its vector field from being differentiable at $S_0$, namely one cannot consider its eigenvalues. Our strategy is to find exceptional directions of $S_0$ and determine the number of orbits in each exceptional direction. The so-called “exceptional direction” was raised for degenerate equilibria ([5, 16, 19]). As shown in [18], exceptional directions are zeros of the coefficient in the term of the lowest degree in the equation of $d\theta/dt$, describing the change of polar angle $\theta$. The coefficient is a function of $\theta$ and its zeros give all possible directions in which there are orbits approaching or leaving the equilibrium. Although we cannot refer $S_0$ to a degenerate equilibrium or a nondegenerate one because system (6) is not differentiable, we can discuss (6) in its exceptional directions. Concerning exceptional directions, we reduce system (6) by the substitution $E = r \cos \theta$ and $K = \sigma \sin \theta$ in the polar coordinates to the form

$$\frac{dr}{d\tau} = \sin \theta (\xi \zeta^\alpha \sin \theta - \epsilon \cos \theta) r + r \cos \theta F_1 (r, \theta) + \sin \theta F_2 (r, \theta),$$

$$r \frac{d\theta}{d\tau} = \mathcal{G}(\theta) r - \sin \theta F_1 (r, \theta) + \cos \theta F_2 (r, \theta),$$

where $\mathcal{G}(\theta) := \sin \theta (\xi \zeta^\alpha \sin \theta + \epsilon \sin \theta)$, $F_1 (r, \theta) := \cos \theta (\tilde{E} - r \cos \theta)((\cos \theta)^{\frac{\alpha}{\beta}} + \rho_1 (\cos \theta)^{\frac{\alpha}{\beta}} + \rho_2 (\cos \theta)^{\frac{\alpha}{\beta}} + \rho_3 (\cos \theta)^{\frac{\alpha}{\beta}})$, $F_2 (r, \theta) := \xi \sin \theta ((\cos \theta)^{\frac{\alpha}{\beta}} + \rho_1 (\sin \theta)^{\frac{\alpha}{\beta}} + \rho_2 (\sin \theta)^{\frac{\alpha}{\beta}} + \rho_3 (\sin \theta)^{\frac{\alpha}{\beta}})$, and $F_1, F_2$ are both $o(r)$ as $r \to 0$. As shown above, exceptional directions are determined by zeros of $\mathcal{G}(\theta)$. Since $\mathcal{G}(0) = 0$ and $\mathcal{G}(\theta) > \epsilon (\sin \theta)^2 > 0$ for all $\theta \in (0, \pi/2]$, system (6) has a unique exceptional direction $\theta = 0$ for $0 \leq \theta \leq \pi/2$. Generally, orbits in exceptional directions are considered by the normal sector method ([16, p.217] or [19, section 2.3]). However, every small angular neighborhood of $\theta = 0$, i.e., $\{(r, \theta) : |\theta| < \delta_1, 0 < r < \delta_2\}$ with small $\delta_1$ and $\delta_2$, contains turning points (i.e., $(r_*, \theta_*)$) with $|\theta_*| < \delta_1$ and $0 < r_* < \delta_2$ such that $\frac{d}{d\tau} r (r, \theta) = 0$ because $\frac{d}{d\tau} r = F_1 (r, 0) = (\tilde{E} - r)^{1+\frac{\alpha}{\beta}} > 0$ as $(r, \theta)$ satisfies $0 < r < \delta_2 \ll \tilde{E}$ and $\theta = 0$, i.e., lies on the positive $E$-axis near the origin, but $\frac{d}{d\tau} r = \sin \delta_1 (\xi \zeta^\alpha \sin \delta_1 - \epsilon \cos \delta_1) r + o(r) < 0$ as $r \in (0, \delta_2)$ and $\theta = \delta_1$ are both small enough because for small $\delta_1 > 0$ we have $\xi \zeta^\alpha \sin \delta_1 - \epsilon \cos \delta_1 < 0$. This shows that there are no normal sectors in exceptional direction $\theta = 0$ and the known normal sector method is not applicable.

This difficulty can be overcome by using the method of GNS (abbreviation of Generalized Normal Sector, [18]). Let us construct a curved edging sector with the $E$-axis and the vertical isocline of system (6) in the first quadrant, i.e.,

$$\mathcal{V} := \{(E, K) | E > 0, K > 0 \text{ and } \sigma(E, K) = 0\}, \quad (8)$$

where $\sigma(E, K) := E (\tilde{E} - E)((E^{\frac{\alpha}{\beta}} + K)^\alpha - \epsilon K)$ denotes the right hand side of the first equation of system (6). One hardly uses the well-known Implicit Function Theorem to find a function $K = \varrho(E)$ such that $\sigma(E, \varrho(E)) = 0$ because the function $\sigma(E, K)$ with the fraction power $\alpha$ is not differentiable with respect to $K$ at $K =$
Clearly, $\hat{d}$ positive constant. We discuss separately two situations:

\begin{enumerate}[(S1)]
  \item the proof of $(ii)$ or orbits leaving $\mathcal{I}$. By [18, Lemma 3, p. 1410], system (6) has either no orbits or infinitely many
  \item $r = \hat{d} E = \sigma(E, K) > 0$ and $\frac{d\hat{d}}{d\hat{d}} K = K \sigma(K - (E^{\frac{2}{\alpha}} + K)^{2}) = 0$ on the boundary $\hat{S}_B A$, and $\frac{d\hat{d}}{d\hat{d}} E = \sigma(E, K) > 0$
  \item $r = \hat{d} E = \sigma(E, K) > 0$ on the boundary $\hat{S}_B B$ for small $E, K > 0$.
\end{enumerate}

Clearly, $\hat{S}_B A$ is an orbit of system (6) on the $E$-axis and all semi-orbits of system (6) starting from $\hat{S}_B B$ depart from $\hat{S}_B B$. It follows that $\hat{S}_B B$ is a GNS of class III. By [18, Lemma 3, p. 1410], system (6) has either no orbits or infinitely many orbits leaving $\hat{S}_B$ in $\hat{S}_B B$ as $\tau \to \infty$. Further, we see that the second option is true because $\hat{S}_B A$ is an orbit of system (6) leaving $\hat{S}_B$ as $\tau \to \infty$. This completes the proof of $(ii)$.

Finally, we investigate orbits of system (1) near the $K$-axis. Let $\hat{d}$ be a small positive constant. We discuss separately two situations: $(S1)$: $0 < K \leq \hat{d}$ and $(S2)$: $K > \hat{d}$. In $(S1)$, i.e., near the origin, from the second equation of system (7), we obtain $\frac{d\hat{d}}{d\hat{d}} \theta = \hat{G}(\theta) + o(1)$ where $o(1) \to 0$ as $r \to 0$. Computing that $\hat{G}'(\theta) = \xi \zeta^{\alpha}((\cos \theta)^2 - (\sin \theta)^2) + 2\varepsilon \sin \theta \cos \theta = A \sin(2\theta + B)$ with $A = (\xi^2 \zeta^{2\alpha} + \varepsilon^2)^{1/2} K + B \in (0, \pi/2)$ satisfying $\cos B = \varepsilon/A$, we know $\hat{G}(\theta)$ increases from $\hat{G}(0) = 0$ to the maximum $\hat{G}((\pi - B)/2)$ as $\theta$ increases from $0$ to $(\pi - B)/2$, and then decreases to $\hat{G}(\pi/2) = \varepsilon$ as $\theta$ increases from $(\pi - B)/2$ to $\pi/2$. Thus, the orbit of system (7), starting from an arbitrary point $(r_0, \theta_0)$ in the first quadrant near the origin, rotates counterclockwise and reaches the $K$-axis in finite time, because $\frac{d\hat{d}}{d\hat{d}} \theta = \hat{G}(\theta) + o(1) \geq \max\{\hat{G}(\theta_0), \varepsilon\} + o(1) > \min\{\hat{G}(\theta_0), \varepsilon\}/2 > 0$ for $\theta \in [\theta_0, \pi/2]$ and small $r_0$. In $(S2)$, i.e., $K > \hat{d}$, from the first equation of system (6), we have $\frac{d\hat{d}}{d\hat{d}} E = E(\hat{E} - E)(\hat{E} - \hat{K}) - \hat{\epsilon}K = -\hat{\epsilon}K < -\epsilon\hat{d} < 0$ for $E = 0$. By the continuity, there is $\hat{d}_4 > 0$ such that $\frac{d\hat{d}}{d\hat{d}} E < -\epsilon\hat{d}_4 < 0$ for all $0 < E < \hat{d}_4$ and $K > \hat{d}_3$, which implies that the orbit of system (6), starting from an arbitrary point $(E_0, K_0)$ such that $0 < E_0 < \hat{d}_4$ and $K_0 > \hat{d}_3$, approaches the $K$-axis in finite time. This proves $(iii)$ and therefore the whole theorem.

In the end of this section, we discuss equilibria at infinity.

**Theorem 2.2.** System (1) has two equilibria at infinity $I_1$ and $I_2$ on the positive half $K$-axis and the positive half $E$-axis respectively, and all points at infinity in the first quadrant make up a heteroclinic orbit from $I_2$ to $I_1$. All orbits near $I_2$ leave it, including infinitely many orbits tangent to this heteroclinic orbit and a single orbit in every other direction. All orbits near $I_1$, except for this heteroclinic orbit and one orbit leaving $I_1$ on the $K$-axis, reach the $K$-axis.

**Proof.** We analyze system (3), an orbitally equivalent system of (1) for $E \geq 0$, $K \geq 0$ with $(E, K) \neq (0, 0)$. On the two axes, $dE/d\tau = 0$, $dK/d\tau = \xi K(\zeta^{\alpha} - K^{\alpha}) < 0$ for
\( E = 0 \) and \( K \) large enough and \( dE/d\tau = E^{1+\frac{\alpha}{\beta}}(\dot{E} - E) < 0 \), \( dK/d\tau = 0 \) for \( E \) large enough and \( K = 0 \), showing the tendencies of orbits on the axes.

By the Poincaré transformation \( E = u/z, K = 1/z \) and the time-scaling \( d\tau = z^{1+\alpha}d\tau_1 \), \( \text{(3)} \) is changed into the form

\[
\begin{align*}
\frac{du}{d\tau_1} &= -\epsilon z^{1+\alpha} - \xi \zeta uz^{1+\alpha} + ((\dot{E} + \epsilon)uz - u^2)(u^{1-\alpha}z^{\frac{1-\alpha-\beta}{1-\alpha}} + 1)^\alpha, \\
\frac{dz}{d\tau_1} &= -\xi \zeta^2 z^{2+\alpha} + \epsilon z^{2}(u^{1-\alpha}z^{\frac{1-\alpha-\beta}{1-\alpha}} + 1)^\alpha,
\end{align*}
\]

where all points at infinity of system \( \text{(3)} \), except for the point at infinity on the positive half \( E \)-axis, are changed to the \( u \)-axis, and the \( K \)-axis is changed to the \( z \)-axis. Since \( du/d\tau_1 = -u^2 \) and \( dz/d\tau_1 = 0 \) for \( z = 0 \), system \( \text{(9)} \) has a unique equilibrium \( I'_1 : (0, 0) \) on the \( u \)-axis, namely system \( \text{(3)} \) has an equilibrium at infinity, denoted by \( I_1 \), on the positive half \( K \)-axis and an orbit consisted of all points at infinity in the first quadrant tending to \( I_1 \). For the orbital structure near \( I_1 \), we study qualitative properties of \( I'_1 \). Note that the usual \textit{method of eigenvalue analysis} is not effective here because the right hand side of the second equation in \( \text{(9)} \) is not differentiable with respect to \( u \) for \( u = 0 \). Applying the Briot-Bouquet transformation \( z = \eta u \) and the time-scaling \( d\tau = u^{-\alpha}d\tau_2 \) to \( \text{(9)} \), we get

\[
\begin{align*}
\frac{du}{d\tau_2} &= -\epsilon u^{1+\alpha} - \xi \zeta u^2\eta^{1+\alpha} + ((\dot{E} + \epsilon)u^{2-\alpha}\eta - u^{2-\alpha})(u\eta^{1-\frac{\alpha}{\beta}} + 1)^\alpha, \\
\frac{d\eta}{d\tau_2} &= \epsilon u^{2+\alpha} + (u^{1-\alpha}\eta - \tilde{E}u^{1-\alpha}\eta^2)(u\eta^{1-\frac{\alpha}{\beta}} + 1)^\alpha.
\end{align*}
\]

Since \( du/d\tau_2 = 0, d\eta/d\tau_2 = \epsilon u^{2+\alpha} \) for \( u = 0 \), system \( \text{(10)} \) has a unique equilibrium \( I_{11} : (0, 0) \) on the \( \eta \)-axis. By the fact that \( d\eta/d\tau_2 = \epsilon u^{2+\alpha} + (1 - \tilde{E}\eta)(u\eta^{1-\frac{\alpha}{\beta}} + 1)^\alpha > 0 \) for \( u > 0, 0 < \eta < 1/\tilde{E} \) and \( du/d\tau_2 = -u^{2-\alpha} < 0, d\eta/d\tau_2 = 0 \) for \( \eta = 0 \), we see that the positive half \( u \)-axis is the unique orbit of system \( \text{(10)} \) in connection with \( I_{11} \) in the closure of the first quadrant except for the positive half \( \eta \)-axis. It follows that the positive half \( u \)-axis is the unique orbit of system \( \text{(9)} \) approaching \( I'_1 \) in the closure of the first quadrant as \( \tau_1 \to +\infty \). Since \( du/d\tau_1 = -\epsilon z^{1+\alpha} < 0 \) for \( u = 0 \) and \( z > 0 \), all orbits near \( I'_1 \) in the first quadrant will reach the positive half \( z \)-axis. It follows that all orbits near \( I_1 \), except for the orbit consisted of points at infinity and the orbit leaving \( I_1 \) on the \( K \)-axis, will reach the \( K \)-axis.

For the remaining point \( I_2 \) at infinity on the positive half \( E \)-axis, applying another Poincaré transformation \( \hat{E} = 1/z, K = u/z \) and the time-scaling \( d\tau = z^{1+\alpha}d\tau_1 \) to \( \text{(3)} \), we get

\[
\begin{align*}
\frac{du}{d\tau_1} &= \xi \zeta^2 u^{1+\alpha} + \epsilon u^2 z^{1+\alpha} + u(1 - \xi z - \tilde{E}z)(z^{\frac{1-\alpha-\beta}{1-\alpha}} + u)^\alpha, \\
\frac{dz}{d\tau_1} &= z(1 - \tilde{E}z)(z^{\frac{1-\alpha-\beta}{1-\alpha}} + u)^\alpha - \epsilon uz^{2+\alpha},
\end{align*}
\]

where \( I_2 \) is changed into the origin of system \( \text{(11)} \). Since \( du/d\tau_1 = u^{1+\alpha} \) and \( dz/d\tau_1 = 0 \) for \( z = 0 \), \( \text{(11)} \) has a unique equilibrium \( I'_2 : (0, 0) \) on the \( u \)-axis, while the positive half \( u \)-axis is an orbit leaving \( I'_2 \). Noting that the eigenvalues at \( I'_2 \) are not computable because the right hand side of the second equation in \( \text{(11)} \) is not differentiable with respect to \( u \) for \( u = 0 \), we apply the Briot-Bouquet
transformation $z = \eta u$ and the time-scaling $d\tau_1 = u^{-1-\alpha+\frac{\alpha}{\beta}} d\tau_2$ to (11) and get
\[
\frac{du}{d\tau_2} = \xi \zeta u^{1+\frac{\alpha}{\beta}} \eta^{1+\alpha} + cu^{2+\frac{\alpha}{\beta}} \eta^{1+\alpha} + (1 - (\xi + \hat{E})u\eta)(u^{\frac{\alpha}{\beta}} + \eta^{1-\frac{\alpha}{\beta}})^{\alpha},
\]
\[
\frac{d\eta}{d\tau_2} = -2cu^{1+\frac{\alpha}{\beta}} \eta^{2+\alpha} - \xi \zeta u^{\frac{\alpha}{\beta}} \eta^{2+\alpha} + \xi \eta^2 (u^{\frac{\alpha}{\beta}} + \eta^{1-\frac{\alpha}{\beta}})^{\alpha}.
\]
(12)

For $u = 0$, $du/d\tau_2 = \eta^{\alpha+\frac{\alpha}{\beta}}$ and $d\eta/d\tau_2 = \xi \eta^{2+\alpha-\frac{\alpha}{\beta}}$, implying that the origin is the unique equilibria of system (12) on the $\eta$-axis. Since $du/d\tau_2 > 0$ and $d\eta/d\tau_2 > 0$ for $u = 0$, $\eta > 0$, there is a unique orbit of (12) in the first quadrant connecting the point $(0, \eta_0)$ for all $\eta > 0$. Moreover, since $du/d\tau_2 > 0$ for $u > 0$, $\eta \geq 0$ small enough, there are infinitely many orbits of (12) in the first quadrant leaving the origin tangent to the $u$-axis. Thus, there are infinitely many orbits of (11) leaving $I_2^+$ in the direction of the positive half $u$-axis and a unique orbit leaving $I_2^+$ in each direction $u = kz$ for all $k > 0$. This completes the proof.

3. Interior equilibria. In contrast to boundary equilibria, much more difficulties are involved in discussion on interior equilibria of system (1) because of the involved fractional powers and their unknown coordinates. The authors of [1] gave sufficient conditions for the cases of exact two interior equilibria $S_3$ and $S_4$, exact one interior equilibrium $S_2$ and no interior equilibrium separately by the Intermediate Theorem of continuity. Because of the fractional powers, contained in $g$ (defined in (2)) and its derivative $g'$, it is hard to compute coordinates of extreme points of $g$. So the authors of [1] discussed with an upper bound and a lower bound of the extreme value instead. That is why they could not obtain a necessary and sufficient condition. In this section we deal with the extreme value of $g$ directly. We abandon computation of coordinates for extreme points but take the strategy: Eliminate the term of fractional power in $g$ and polynomialize the expression of the extreme value, so that we can find a simplified function having the same sign as the extreme value. With the simplified function, we are able to give necessary and sufficient conditions for various possible numbers of equilibria. Further, we apply the Center Manifold Theorem to discuss qualitative properties of non-hyperbolic equilibria completely.

For convenience in statement, we need the following notations
\[
\epsilon_0 := 2^{\frac{2-2\alpha}{\beta}} \frac{2-2\alpha - \beta}{\beta} \left( \frac{1-\alpha}{2-2\alpha - \beta} \right)^{\frac{2-2\alpha}{\beta}} \zeta^{-\frac{2\alpha - 3\beta + \alpha\beta}{\beta}},
\]
\[
F(x) := \begin{cases} 
A_1 \left( x + (x^2 + A_2)^{\frac{1}{\beta}} \right)^{\frac{\beta}{\beta - 1}} - A_3(x^2 + A_2)^{\frac{1}{\beta}} + (1 - A_3)x, \\
0 < \beta < 1 - \alpha, \\
\epsilon_1 \zeta^{-\alpha} - 2\epsilon^4 \zeta^{\frac{\alpha}{\beta}} x, \quad \text{as} \beta = 1 - \alpha,
\end{cases}
\]
where $A_1 := \epsilon \beta \zeta^{-\alpha}(1 - \alpha)^{-2}(1 - \alpha - \beta)^{\alpha + \beta - 1} (4 - 4\alpha - 2\beta)^{\frac{1-\alpha-\beta}{\beta}} > 0$, $A_2 := 4\epsilon \beta \zeta^{1-\alpha}(1 - \alpha - \beta)^{-2}(2 - 2\alpha - \beta) > 0$ and $A_3 := \frac{1-\alpha-\beta}{2-2\alpha-\beta} \in (0, 1/2)$.

Lemma 3.1. In the case $0 < \epsilon < \epsilon_0$, $F(x)$ has a unique positive zero, denoted by $Z_F$, and satisfies either $F(x) < 0$ if $0 < x < Z_F$ or $F(x) > 0$ if $x > Z_F$. In the other case $\epsilon \geq \epsilon_0$, $F(x) > 0$ for all $x > 0$.

Lemma 3.2. (1) $0 < Z_F < E_M < E_N := 2\zeta^{\frac{1-\alpha}{\beta}} - \epsilon_0 \frac{\alpha + \beta - 1 - \alpha \beta}{\beta}$ if $0 < \epsilon < \epsilon_1$;
(2) $0 < E_N = E_M = Z_F$ if $\epsilon = \epsilon_1$;
(3) $0 < E_N < Z_F < E_M$ if $\epsilon_1 < \epsilon < 2\epsilon_1$;
(4) $E_N \leq 0 < Z_F < E_M$ if $2\epsilon_1 \leq \epsilon < \epsilon_0$;
(5) $E_N < 0 < E_M$ if $\epsilon \geq \epsilon_0$. 

The above two lemmas, to be proved in Appendix I, will be used to discuss interior equilibria of system (1).

**Theorem 3.3.** In the case $0 < \epsilon < \epsilon_1$, system (1) has no interior equilibria if $0 < \tilde{E} < Z_F$, one interior equilibrium $S_{34} : (\varsigma_0, \zeta - \varsigma_0^{\beta/(1 - \alpha)})$ if $\tilde{E} = Z_F$, two interior equilibria $S_3$ and $S_4$ if $Z_F < \tilde{E} < E_M$, and one interior equilibrium $S_3$ if $\tilde{E} \geq E_M$, where $\epsilon_1$ is defined in Theorem 2.1, $Z_F$ is defined in Lemma 3.1, $\varsigma_0 := ((1 - \alpha - \beta)\tilde{E} + ((1 - \alpha - \beta)^2\tilde{E}^2 + 4\epsilon\beta(2 - 2\alpha - \beta(1 - \alpha)^2)(4 - 4\alpha - 2\beta)^{-1}$, and $E_M$ is defined before (2). In the other case $\epsilon \geq \epsilon_1$, system (1) has no interior equilibria if $0 < \tilde{E} \leq E_M$ and one interior equilibrium $S_3$ if $\tilde{E} > E_M$. Moreover, $S_3$ is a saddle, $S_4$ is a stable node, and $S_{34}$ is a saddle-node.

**Proof.** Assume that $(E, K)$ is an interior equilibrium of system (1). From [1] we know that $E$ is a zero of $g$ in the interval $(0, E_M)$ and $K = \zeta - E^{\frac{\beta}{1 - \alpha}}$, where $E_M$ and $\zeta$ are defined before (2) and $g$ is defined in (2). In order to find all interior equilibria, we discuss the monotonicity of $g$. Compute the derivatives

$$g'(E) = -2E + \frac{\epsilon\beta}{\zeta^{\alpha(1 - \alpha)}}E^\frac{\alpha\beta - 1}{1 - \alpha} + \tilde{E}, \quad g''(E) = \frac{\epsilon\beta(\alpha + \beta - 1)}{\zeta^{\alpha(1 - \alpha)^2}}E^\frac{2\alpha\beta - 2}{1 - \alpha} - 2. \quad (14)$$

Since $\lim_{E \to 0} g'(E) > 0$ and $g''(E) \leq -2$ for all $E > 0$, $g'$ has a unique positive zero, denoted by $\varsigma$, and $g$ increases on $(0, \varsigma)$ but decreases on $(\varsigma, +\infty)$. Since $g(0) = -\epsilon\varsigma^{1 - \alpha} < 0$ and $\lim_{E \to +\infty} g(E) = -\infty$, by the continuity we see that $g$ has no (resp. one, two) positive zeros if $g(\varsigma) < 0$ (resp. $= 0$, $> 0$).

The greatest difficulty in determining the sign of $g(\varsigma)$ comes from the transcendental term of fractional power in $g(\varsigma)$, but we see from the equation $g'(\varsigma) = 0$ that the transcendental term in $g(\varsigma)$ satisfies

$$\epsilon\varsigma^{1 - \alpha} \tilde{E} = \frac{2 - 2\alpha}{\beta} \varsigma^2 - \frac{1 - \alpha - \beta}{\beta} \tilde{E} \varsigma - \epsilon\varsigma^{1 - \alpha},$$

which can be used to eliminate this transcendental term in $g(\varsigma)$ and give

$$g(\varsigma) = \frac{2 - 2\alpha - \beta}{\beta} \varsigma^2 - \frac{1 - \alpha - \beta}{\beta} \tilde{E} \varsigma - \epsilon\varsigma^{1 - \alpha},$$

a quadratic polynomial in $\varsigma$ with a unique positive zero $\varsigma_0$ as defined in Theorem 3.3. Clearly, $g(\varsigma) < 0$ if and only if $0 < \varsigma < \varsigma_0$, which is equivalent to $g'(\varsigma_0) < 0$ by the monotonicity of $g'$. Similarly, $g(\varsigma) = 0$ (or $> 0$) if $g'(\varsigma_0) = 0$ (or $> 0$). Thus, $g$ has no (resp. one, two) positive zeros if $g'(\varsigma_0) < 0$ (resp. $= 0$, $> 0$).

However, because of the complicated expression of $g'(\varsigma_0)$ in $\alpha, \beta, \epsilon, \varsigma$ and $\tilde{E}$, the above obtained condition $g'(\varsigma_0) > 0$ (or $= 0$, $< 0$) does not exhibit an explicit or simple requirement in practice. For explicit and simple requirements, we treat $g'(\varsigma_0)$ as a function of $\tilde{E}$, denoted by $F(\tilde{E})$. A simple computation shows that the function $F$ is exactly the one defined just above Lemma 3.1. By Lemma 3.1, the function $F$ has either a unique positive zero $Z_F$ (if $0 < \epsilon < \epsilon_0$) or no positive zero (if $\epsilon \geq \epsilon_0$), where $\epsilon_0$ is defined in (13). Thus, we discuss separately in the two cases:

**(C1):** $0 < \epsilon < \epsilon_0$ and **(C2):** $\epsilon \geq \epsilon_0$.

In case (C1), we have $F(\tilde{E}) < 0$ (resp. $= 0$, $> 0$) if $\tilde{E} < Z_F$ (resp. $= Z_F$, $> Z_F$) by Lemma 3.1, and further discuss in the three subcases: **(C11):** $0 < \tilde{E} < Z_F$, **(C12):** $Z_F = \tilde{E}$ and **(C13):** $\tilde{E} > Z_F$. In subcase (C11), $g'(\varsigma_0) = F(\tilde{E}) < 0$, implying that $g$ has no positive zero, i.e., system (1) has no interior equilibrium. In subcase (C12), i.e., $\tilde{E} = Z_F$, we have $g'(\varsigma_0) = F(Z_F) = 0$, implying that $g$ has a unique positive zero $\varsigma_0$. Thus system (1) has a unique interior equilibrium $S_{34}$
if $s_0 < E_M$, which is equivalent to $g'(E_M) < 0$ because $g'(s_0) = 0$ and $g''(E) < 0$ for all $E > 0$. Clearly the inequality $g'(E_M) < 0$ can be presented as $E > E_N$, which by Lemma 3.2 is equivalent to $0 < \epsilon < \epsilon_1$ and $\bar{E} = Z_F$ in this subcase, where $E_N$ is defined in Lemma 3.2 and $\epsilon_1$ is defined in Theorem 2.1. Moreover, system (1) has no interior equilibrium in the remaining situation, i.e., $\epsilon_1 \leq \epsilon < s_0$ and $\bar{E} = Z_F$. In subcase (C13), i.e., $\bar{E} > Z_F$, we have $g'(s_0) = F(\bar{E}) > 0$, implying $g$ has two positive zeros $E_A$ and $E_B$ satisfying $E_A < E_B$. Then system (1) has two interior equilibria $S_3$ and $S_4$ if $E_B < E_M$, which is equivalent to $g(E_M) < 0$ and $g'(E_M) < 0$ because $G'(E) < 0$ if and only if $E > \zeta$ and $g(E) < 0$ if and only if $E \in (0, E_A) \cup (E_B, +\infty)$. Clearly the inequalities $g(E_M) < 0$ and $g'(E_M) < 0$ can be presented as $E < \min\{E_M, E_N\}$, which by Lemma 3.2 is equivalent to $0 < \epsilon < \epsilon_1$ and $Z_f < \bar{E} < E_M$ in this subcase. By a similar discussion, system (1) has a unique interior equilibrium $S_3$ if $E_A < E_M \leq E_B$, which is equivalent to either $0 < \epsilon < \epsilon_1$ and $\bar{E} = E_M$ or $0 < \epsilon < s_0$ and $\bar{E} > E_M$. System (1) has no interior equilibrium in the remaining situation, i.e., $\epsilon_1 < \epsilon < s_0$ and $Z_f < \bar{E} \leq E_M$. This completes the discussion in case (C1).

In case (C2), i.e., $\epsilon \geq s_0$, we have $g'(s_0) = F(\bar{E}) > 0$ by Lemma 3.1, implying that $g$ has exact two positive zeros $E_A$ and $E_B$ such that $E_A < E_B$. Then, by a discussion similar to in (C13), system (1) has either a unique interior equilibrium $S_3$ if $\epsilon \geq s_0$ and $\bar{E} > E_M$ or no interior equilibria if $\epsilon \geq s_0$ and $0 < \bar{E} \leq E_M$. This completes the discussion in case (C2).

In what follows, we discuss qualitative properties of those interior equilibria, which has the coordinates $(E, \zeta - \frac{\epsilon}{E^\gamma})$ and $E$ is a zero of the transcendental function $g$. Computing the Jacobian matrix of system (3) at $(E, \zeta - \frac{\epsilon}{E^\gamma})$, we see that the first row is $(\zeta^\alpha \bar{E} - 2E\zeta^\alpha + \frac{\alpha\beta \bar{E}}{1 - \alpha} \zeta^\alpha - \frac{\alpha\beta}{1 - \alpha} \zeta^\alpha - 1 E^{1 + \frac{\beta}{1 - \alpha}}, -\epsilon + \alpha\zeta^\alpha - 1 \bar{E}E - \alpha\zeta^\alpha - 1 E^2)$ and the second row is $(-\frac{\alpha\beta}{1 - \alpha} \zeta^\alpha E^{1 + \frac{\beta}{1 - \alpha}} + \frac{\alpha\beta}{1 - \alpha} \zeta^\alpha E^{1 + \frac{\beta}{3 - \alpha}}, \xi\alpha\zeta^\alpha - 1 E^\alpha - \xi\alpha\zeta^\alpha - 1 E^2).$ One can compute its determinant $D := -\xi\alpha\zeta^\alpha - 1 (\zeta - \frac{\epsilon}{E^\gamma}) g'(E)$, where $g'$ is given in (14), and trace

\[ T := \zeta^\alpha \bar{E} - 2E\zeta^\alpha + \frac{\alpha\beta \bar{E}}{1 - \alpha} \zeta^\alpha - 1 E^\frac{\beta}{1 - \alpha} - \frac{\alpha\beta}{1 - \alpha} \zeta^\alpha - 1 E^{1 + \frac{\beta}{1 - \alpha}} - \xi\alpha\zeta^\alpha + \xi\alpha\zeta^\alpha - 1 E^\frac{\beta}{1 - \alpha}. \]

Since $\zeta - \frac{\epsilon}{E^\gamma} > 0$, we see that $D$ has the same sign as $-g'(E)$. By the monotonicity of $g$ we have $g'(E_A) > 0$, $g'(E_B) < 0$ and $g'(s_0) = 0$. Thus, we have $D < 0$ at the equilibrium $S_3 : (E_A, \zeta - E^{\beta/(1-\alpha)})$, i.e., $S_3$ is a saddle. At $S_4 : (E_B, \zeta - E^{\beta/(1-\alpha)})$ we have $D > 0$, $T = g'(E_B)\zeta^\alpha + \alpha g(E_B)\zeta^\alpha - 1 - (1 - \alpha)\epsilon - \alpha\zeta^\alpha - 1 E^{1 + \frac{\beta}{1 - \alpha}} - \xi\alpha\zeta^\alpha - 1 (\zeta - \frac{\epsilon}{E^\gamma}) < 0$ and $T^2 - 4D = (g'(E_B)\zeta^\alpha + \xi\alpha\zeta^\alpha - 1 (\zeta - \frac{\epsilon}{E^\gamma}))^2 + ((1 - \alpha)\epsilon + \alpha\zeta^\alpha - 1 E^{\beta/(1-\alpha)})^2 + 2(\xi\alpha\zeta^\alpha - 1 (\zeta - \frac{\epsilon}{E^\gamma}) - g'(E_B)\zeta^\alpha)((1 - \alpha)\epsilon + \alpha\zeta^\alpha - 1 E^{\beta/(1-\alpha)}) > 0$, implying that $S_4$ is a stable node.

Consider the equilibrium $S_{34} : (s_0, \zeta - \frac{\epsilon}{s_0^\gamma})$, which appears in the case that $0 < \epsilon < \epsilon_1$ and $\bar{E} = Z_F$. A similar discussion shows that $D = 0$ and $T < 0$, implying that $S_{34}$ is degenerate with a zero eigenvalue and a negative eigenvalue. In order to translate $S_{34}$ to the origin $O$ and diagonalize the linear part, we use the linear transformation $E = (\zeta^\alpha Z_F - 2\alpha \zeta^\alpha + \alpha \beta (Z_F - s_0)(1 - \alpha)\zeta^\alpha - 1 E^{\beta/(1-\alpha)})x - (\zeta\alpha\zeta^\alpha - 1 s_0^{\beta/(1-\alpha)} - \xi\alpha\zeta^\alpha) y + s_0$, $K = (\frac{\alpha \beta}{1 - \alpha} s_0^{\beta/(1-\alpha)} + \frac{\alpha \beta}{1 - \alpha} s_0^{\beta/(1-\alpha)} - 2\beta/(1-\alpha) - 1)(x + \ldots).$
$y + \zeta - S_0^{\beta/(1-\alpha)}$ to change system (3) into the form
\[
\frac{dx}{dt} = T x + c_{20}^{(1)} x^2 + c_{11}^{(1)} xy + c_{02}^{(1)} y^2 + O((x,y)^3),
\]
\[
\frac{dy}{dt} = c_{20}^{(2)} x^2 + c_{11}^{(2)} xy + c_{02}^{(2)} y^2 + O((x,y)^3),
\]
where $c_{20}^{(2)} = -\frac{p_2^2}{2T} S_0^{\beta/(1-\alpha)-1} (Z_F - S_0) S_0^{\alpha-1} \frac{\alpha^2 \beta}{(1-\alpha)} (\alpha + \beta - 1) - 2 \zeta^\alpha - \frac{p_2 \beta}{2T} (\xi (Z_F - S_0^{\beta/(1-\alpha)})^{\alpha-1} \frac{\alpha^2 \beta}{(1-\alpha)} (\alpha + \beta - 1))$, $p_1 := \zeta^\alpha Z_F - 2 S_0^\alpha + \frac{\alpha^2 \beta}{(1-\alpha)} (\alpha-1 (Z_F - S_0)) S_0^{\beta/(1-\alpha)}$, $p_2 := -\frac{\alpha^2 \beta}{(1-\alpha)} S_0^{\beta/(1-\alpha)-1} + \frac{\alpha^2 \beta}{(1-\alpha)} S_0^{\beta/(1-\alpha)-1}$, $p_2 := -\xi \alpha^\alpha + \xi \alpha^\alpha$, and other coefficients of quadratic terms are omitted. By the Center Manifold Theorem ([7, Theorem 3.2.1, p.127]), system (15) has a $C^\infty$ center manifold tangent to the $y$-axis at $O$, which can be expressed as $x = O(y^2)$ near $O$. Thus, the restriction of system (15) to the center manifold is given by
\[
\frac{dy}{dt} = c_{02}^{(2)} y^2 + O(y^3).
\]
We claim that $c_{02}^{(2)} < 0$ in (16). In fact, since $g(S_0) = g'(S_0) = 0$, we have
\[
\epsilon = \frac{(1-\alpha) S_0^2}{(1-\alpha) \zeta^{1-\alpha} - (1-\alpha - \beta) \zeta^{-\alpha} S_0^{\alpha}}.
\]
Thus we can compute
\[
c_{02}^{(2)} = \frac{\xi \alpha^\alpha p_2^2 \zeta^{2\alpha-1} Q(S_0^{\frac{\beta}{1-\alpha}})}{2p_2 T (1-\alpha)^\beta (\zeta - (1 - \frac{\beta}{1-\alpha}) S_0^{\beta/(1-\alpha)}),}
\]
where $Q(x) := -(1-\alpha - \beta)(2 - 2\alpha - \beta) x^2 + \zeta (4 \alpha^2 + 3 \alpha \beta + \beta^2 - 8 \alpha - 3 \beta + 4) x - 2 \zeta^\alpha$. This procedure polynomializes $c_{02}^{(2)}$ as the quadratic $Q$. Since $p_2 < 0$, $T < 0$ and $\zeta - (1 - \frac{\beta}{1-\alpha}) S_0^{\beta/(1-\alpha)} < 0$, $c_{02}^{(2)}$ has the same sign as $Q(S_0^{\beta/(1-\alpha)})$.

Noting $S_0^{\beta/(1-\alpha)} \in (0, \zeta)$, we discuss the sign of $Q$ on the interval $(0, \zeta)$. In the particular case $\beta = 1 - \alpha$, $Q(x) = 2 \zeta (1-\alpha)^2 (x - \zeta) < 0$ for all $0 < x < \zeta$. In the generic case $0 < \beta < 1 - \alpha$, $Q$ has two positive zeros $Z_{Q1} := \zeta$ and $Z_{Q2} := 2 \zeta (1-\alpha)^2 ((1-\alpha - \beta)(2 - 2\alpha - \beta))$. Clearly, $Z_{Q1} < Z_{Q2}$, implying that $Q(x) < 0$ for $0 < x < Z_{Q1} = \zeta$. This proves $Q(S_0^{\beta/(1-\alpha)}) < 0$ and the claim. So $S_{34}$ is a saddle-node. The proof is completed. \hfill $\Box$

4. Bifurcations of equilibria. In this section we discuss bifurcations of system (1) at non-hyperbolic equilibria. By Theorems 2.1 and 3.3, we list all non-hyperbolic equilibria: $S_2 : (\bar{E}, 0)$ is a saddle-node (or degenerate saddle) if $\bar{E} = E_M$ and $\epsilon \neq \epsilon_1$ (or $\epsilon = \epsilon_1$), and $S_{34} : (\zeta - S_0^{\beta/(1-\alpha)})$ is a saddle-node if $0 < \epsilon < \epsilon_1$ and $\bar{E} = Z_F$, where $E_M$ and $\zeta$ are defined before (2), and $\epsilon_1, S_0$ and $Z_F$ are defined in Theorem 2.1, Theorem 3.3 and Lemma 3.1 respectively.

**Theorem 4.1.** For $\epsilon \neq \epsilon_1$, a transcritical bifurcation occurs at $S_2$ as $\bar{E}$ varies across $E = E_M$. More concretely, when $0 < \epsilon < \epsilon_1$, the saddle $S_2$ and the stable node $S_1$ collide to be the saddle-node $S_2$ and then separate to be the stable node $S_2$ and an invalid saddle (outside the first quadrant) as $\bar{E}$ increases from the case below $E_M$ to the case above $E_M$; when $\epsilon > \epsilon_1$, the stable node $S_2$ and the saddle $S_3$ collide to be the saddle-node $S_2$ and then separate to be the saddle $S_2$ and an
invalid stable node (outside the first quadrant) as $\bar{E}$ decreases from the case above $E_M$ to the case below $E_M$.

**Proof.** Using the new parameter $p := \bar{E} - E_M$, we rewrite system (3) as

$$\begin{align*}
\frac{dE}{d\tau} &= E(p + E_M - E)(E^{\alpha_p} + K)^{\alpha} - \epsilon K, \\
\frac{dK}{d\tau} &= \xi^\alpha K - \xi K(E^{1/\alpha} + K)^{\alpha}.
\end{align*}$$

(17)

Applying the invertible linear transformation $E = x - \epsilon E_M^{-1}\frac{\alpha}{1-\alpha} y + p + E_M$, $K = y$ and $p = p$ to translate $S_2$ to the origin $O$ and diagonalize the linear part, we can change system (17) into the form

$$\begin{align*}
\frac{dx}{d\tau} &= P(x, y, p) = \lambda_1 x + O((|x, y, p|^2), \\
\frac{dy}{d\tau} &= Q(x, y, p) = O((|x, y, p|^2), \\
\frac{dp}{d\tau} &= 0
\end{align*}$$

(18)

suspended with $p$, where $P, Q$ are $C^\infty$ with respect to $x, y$ and $p$, and $\lambda_1 = -E_M^{1+\frac{\alpha}{1-\alpha}}$.

As previous theorems, by the Center Manifold Theorem system (18) has a $C^\infty$ center manifold $x = H(y, p)$ tangent to the center invariant subspace $\{(x, y, p) \in \mathbb{R}^3| x = 0\}$ at $O$. Clearly, $H$ is of the form $H(y, p) := h_0(p) + h_1(p)y + O(y^2)$ with two indeterminate functions $h_0$ and $h_1$ such that $h_0(p) = O(p^2)$ and $h_1(p) = O(p)$.

Compute the Taylor expansions $P(x, y, p) = \hat{p}_0(x, p) + \hat{p}_1(x, p)y + O(y^2)$ and $Q(x, y, p) = \hat{q}_1(x, p)y + \hat{q}_2(x, p)y^2 + O(y^3)$, where $\hat{p}_0(x, p) = \hat{p}_01(p)x + O(x^2)$, $\hat{p}_1(x, p) = \hat{p}_{10}(p) + O(x)$, $\hat{q}_1(x, p) = \hat{q}_{10}(p) + \hat{q}_{11}(p)x + O(x^2)$, $\hat{q}_2(x, p) = \hat{q}_{20}(p) + O(x)$, $\hat{p}_{10}(p) = -p + E_M^{\alpha/(1-\alpha)-1}$, $\hat{p}_{11}(p) = -\epsilon + \xi E_M^{1-\alpha} - E_M^{-\alpha(\alpha/(1-\alpha)-1)}(\xi - p - E_M(p + E_M^{\alpha/(1-\alpha)-1}), \hat{q}_{10}(p) = \xi E_M^{1-\alpha} - \xi (p + E_M)^{\alpha/(1-\alpha)} + \hat{q}_{20}(p) = -\xi (p + E_M)^{-\alpha(\alpha/(1-\alpha)-1)} - \xi \alpha (p + E_M)^{-\beta} + \xi E_M^{\alpha/(1-\alpha)-1}(p + E_M)^{\alpha/(1-\alpha)-1}$. Thus, substituting the equations of (18) in the equality $\frac{dx}{d\tau} = \frac{\partial H}{\partial y} \frac{dy}{d\tau}$, obtained by the invariance of the manifold, and comparing the coefficients, we obtain $h_0(p) = 0$ and $h_1(p) = \hat{p}_{10}(p) - \hat{p}_{11}(p))^{-1}$. Thus the restriction of system (18) to the manifold is the equation

$$\frac{dy}{d\tau} = \varpi(y, p) := \varpi_1(p)y + \varpi_2(p)y^2 + O(y^3),$$

(19)

where $\varpi_1(p) := \hat{q}_{10}(p) - \xi E_M^{-1+\frac{\alpha}{1-\alpha}} + O(p^2)$ and $\varpi_2(p) := \hat{q}_{11}(p)h_1(p) + \hat{q}_{20}(p) = \xi E_M^{-2}(\xi - \epsilon_1) + O(p)$. Since $\varpi(y, 0) = y^2(\xi E_M^{-2}(\xi - \epsilon_1) + O(y))$, by the Malgrange Preparation Theorem ([3, Theorem 7.1, p.43]) there are $C^\infty$ functions $\zeta(y, p) := \zeta_0(p) + \zeta_1(p)y + O(y^2)$, $\rho_0(p) := O(p)$ and $\rho_1(p) := O(p)$ with the property that $\zeta(0, 0) = \zeta_0(0) \neq 0$ such that

$$\varpi_1(p)y + \varpi_2(p)y^2 + O(y^3)(\zeta_0(p) + \zeta_1(p)y + O(y^2)) = \rho_0(p) + \rho_1(p)y + y^2.$$
ρ₁(p), we reduce system (19) to its normal form \( \frac{dy}{dx} = ay + y² \), implying a transcritical bifurcation occurring as \( a \) across zero. This completes the proof.

**Theorem 4.2.** For \( 0 < \epsilon < \epsilon_1 \), a saddle-node bifurcation occurs at \( S_{34} \) as \( \tilde{E} \) varies across \( \tilde{E} = Z_F \). Concretely, the saddle \( S_3 \) and the stable node \( S_4 \) collide to be the saddle-node \( S_{34} \) and then vanish as \( \tilde{E} \) decreases from the case above \( Z_F \) to the case below \( Z_F \).

**Proof.** Using the new parameter \( p := \tilde{E} - Z_F \), we rewrite system (3) as

\[
\begin{align*}
\frac{dE}{dT} &= E(p + Z_F - E)(E^{\frac{\alpha}{\beta}} + K) - \epsilon K, \\
\frac{dK}{dT} &= s(1 - \alpha)K - \xi K(E^{\frac{\alpha}{\beta}} + K)^\alpha.
\end{align*}
\]

Applying the linear transformation \( E = (\frac{\alpha \beta Z_F}{1 - \alpha}, \frac{\alpha}{\beta} Z_F^{\alpha - 1}, s') \), \( \zeta \) across zero. This completes the proof.

\[
\begin{align*}
\frac{dx}{dT} &= \tilde{P}(x, y, \tilde{p}) := Tx + O(||(x, y, \tilde{p})||^2), \\
\frac{dy}{dT} &= \tilde{Q}(x, y, \tilde{p}) := \tilde{p} + O(||(x, y, \tilde{p})||^2), \\
\frac{d\tilde{p}}{dT} &= 0
\end{align*}
\]

suspended with \( p \), where \( T = \zeta Z_F - 2s_0 \zeta^\alpha + \frac{\alpha \beta Z_F}{1 - \alpha} \zeta^{\alpha - 1} - \xi \alpha \zeta^\alpha + \xi \alpha Z_F^{\beta - 1} < 0 \) and \( \tilde{P}, \tilde{Q} \) are \( C^\infty \) with respect to \( x, y \) and \( \tilde{p} \). As previous theorems, by the Center Manifold Theorem system (21) has a \( C^\infty \) center manifold \( x = \tilde{H}(y, \tilde{p}) := O(||(y, \tilde{p})||^2) \) and thus the restriction of (21) to the manifold is the equation

\[
\begin{align*}
\frac{dy}{dT} &= \Xi(y, \tilde{p}) := \tilde{Q}(\tilde{H}(y, \tilde{p}), y, \tilde{p}) = \Xi_0(\tilde{p}) + \Xi_1(\tilde{p})y + \Xi_2(\tilde{p})y² + O(y^3),
\end{align*}
\]

where \( \Xi_0(\tilde{p}) := \tilde{p} + O(\tilde{p}²) \), \( \Xi_1(\tilde{p}) := O(\tilde{p}) \) and \( \Xi_2(\tilde{p}) := c_{02}^{(2)} + O(\tilde{p}) \). Since \( \Xi(y, 0) = c_{02}^{(2)} y² + O(y^3) \) and \( c_{02}^{(2)} < 0 \) as defined just below (16), by the Malgrange Preparation Theorem (see [3, Theorem 7.1, p.43]) there are \( C^\infty \) functions \( \hat{\zeta}(y, \tilde{p}) := \hat{\zeta}_0(\tilde{p}) + \hat{\zeta}_1(\tilde{p})y + \hat{\zeta}_2(\tilde{p})y² + O(y^3) \), \( \hat{\rho}_0(\tilde{p}) := O(\tilde{p}) \) and \( \hat{\rho}_1(\tilde{p}) := O(\tilde{p}) \) with the property that \( \zeta(0, 0) = \zeta_0(0) \neq 0 \) such that

\[
\begin{align*}
(\Xi_0(\tilde{p}) + \Xi_1(\tilde{p})y + \Xi_2(\tilde{p})y² + O(y^3))\hat{\zeta}(y, \tilde{p}) = \hat{\rho}_0(\tilde{p}) + \hat{\rho}_1(\tilde{p})y + y². 
\end{align*}
\]

Comparison of the coefficients of the same order on both sides of the above equation gives \( \hat{\rho}_0(\tilde{p}) = \Xi_0(\tilde{p})\hat{\zeta}_0(\tilde{p}), \hat{\rho}_1(\tilde{p}) = \Xi_0(\tilde{p})\hat{\zeta}_1(\tilde{p}) + \Xi_1(\tilde{p})\hat{\zeta}_0(\tilde{p}) + \Xi_1(\tilde{p})\hat{\zeta}_1(\tilde{p}) + \Xi_2(\tilde{p})\hat{\zeta}_0(\tilde{p}), \) which implies that \( \hat{\zeta}_0(0) = 1/\Xi_2(0) = 1/c_{02}^{(2)}, \hat{\rho}_0(\tilde{p}) = \tilde{p}/c_{02}^{(2)} + O(\tilde{p}²) \) and \( \hat{\rho}_1(\tilde{p}) = O(\tilde{p}) \). Applying the time-rescaling \( d\tau = \zeta(y, \tilde{p})d\tau \), the translation \( z = y + \frac{1}{2}\hat{\rho}_1(\tilde{p}) \) and the parameter change \( b := \hat{\rho}_0(\tilde{p}) - 4^{-1}\hat{\rho}_1(\tilde{p})² = \tilde{p}/c_{02}^{(2)} + O(\tilde{p}²) \), we can reduce (22) to its normal form \( \frac{dz}{d\tau} = b + z² \), implying a saddle-node bifurcation occurring as \( b \) across zero. This completes the proof.
Applying the invertible linear transformation $H$ to system (24) has a branch suspended with $p$ and $q$, implying that $TR^-$ and $SN$ meet at $\mathcal{T} : (\epsilon_1, E_M)$. We will show that a codimension-2 bifurcation occurs at $\mathcal{T}$.

**Theorem 4.3.** In a small neighborhood $\mathcal{U}$ of $\mathcal{T}$ in the $(\epsilon, E)$-plane, there are two branches of transcritical bifurcation curves $TR^\pm := TR^\pm \cap \mathcal{U}$ and two branches of saddle-node bifurcation curves $SN^\pm := \{(\epsilon, E) \in \mathcal{U} \mid |(\epsilon - \epsilon_1)| > 0, E = E_M + \rho(\epsilon - \epsilon_1)^2 + O(|\epsilon - \epsilon_1|^3)\}$ (the bifurcation on $SN^+$ is invalid in economic sense since it occurs outside the first quadrant), where $E_M$ is defined before (2), $\epsilon_1$ is defined in Theorem 2.1 and $\rho := -2^{-1}\beta^2(1-\alpha)^{-1}(3-3\alpha - \beta)^{-1}E_M^{2\beta - 3} < 0$, dividing $\mathcal{U}$ into $4$ open regions $\mathcal{R}_1 := \{(\epsilon, E) \in \mathcal{U} \mid E < E_M + \rho(\epsilon - \epsilon_1)^2 + O(|\epsilon - \epsilon_1|^3)\}$, $\mathcal{R}_2 := \{(\epsilon, E) \in \mathcal{U} \mid E > E_M + \rho(\epsilon - \epsilon_1)^2 + O(|\epsilon - \epsilon_1|^3)\} < E < E_M\}$, $\mathcal{R}_3 := \{(\epsilon, E) \in \mathcal{U} \mid E < E_M + \rho(\epsilon - \epsilon_1)^2 + O(|\epsilon - \epsilon_1|^3)\} > E < E_M\}$, $\mathcal{R}_4 := \{(\epsilon, E) \in \mathcal{U} \mid E < \epsilon_1, E_M + \rho(\epsilon - \epsilon_1)^2 + O(|\epsilon - \epsilon_1|^3) < E < E_M\}$ such that in a small neighborhood of $S_2$ system (1) has (i) only the equilibrium $S_2$, being a saddle (resp. degenerate saddle), if $(\epsilon, E) \in \mathcal{R}_1$ (resp. $(\epsilon, E) = (\epsilon_1, E_M)$), (ii) $2$ equilibria, a saddle $S_2$ and an invalid saddle-node (outside the first quadrant), if $(\epsilon, E) \in SN^+$, (iii) $3$ equilibria, a saddle $S_2$ and two equilibria, if $(\epsilon, E) \in \mathcal{R}_2$, (iv) $2$ equilibria, a saddle-node $S_2$ and an invalid saddle, if $(\epsilon, E) \in TR^+$, (v) $3$ equilibria, including a stable node $S_2$, a saddle $S_3$ and an invalid saddle, if $(\epsilon, E) \in \mathcal{R}_3$, (vi) $2$ equilibria, a saddle-node $S_2$ and a saddle $S_3$, if $(\epsilon, E) \in TR^-$, (vii) $3$ equilibria, including two saddles $S_2, S_3$ and a stable node $S_4$, if $(\epsilon, E) \in \mathcal{R}_4$ or (viii) $2$ equilibria, including a saddle $S_2$ and a saddle-node $S_{34}$, if $(\epsilon, E) \in SN^-$.

**Proof.** Using new parameters $p := \tilde{E} - E_M$ and $q := \epsilon - \epsilon_1$, we rewrite (3) as

$$\begin{align*}
\frac{dE}{d\tau} &= E(p + E_M - E)(E^{\frac{\alpha}{\alpha - \beta}} + K)^\alpha - (q + \epsilon_1)K, \\
\frac{dK}{d\tau} &= \xi \zeta^\alpha K - \xi K(E^{\frac{\alpha}{\alpha - \beta}} + K)^\alpha.
\end{align*}$$

Applying the invertible linear transformation $E = x - \epsilon_1 E_M^{1+\frac{\alpha\beta}{1-\alpha}} y + p + E_M, K = y, p = p$ and $q = q$ to translate $S_2$ to the origin $O$ and diagonalize the linear part of system (23), we obtain the system

$$\begin{align*}
\frac{dx}{d\tau} &= X(x, y, p, q) := \lambda_1 x + O(|(x, y, p, q)|^2), \\
\frac{dy}{d\tau} &= Y(x, y, p) := O(|(x, y, p)|^2), \\
\frac{dp}{d\tau} &= 0, \quad \frac{dq}{d\tau} = 0
\end{align*}$$

suspended with $p$ and $q$, where $\lambda_1 := -E_M^{1+\frac{\alpha\beta}{1-\alpha}}$ as given just below (3) and $X, Y$ are both $C^\infty$ functions. As previous theorems, by the Center Manifold Theorem system (24) has a $C^\infty$ center manifold $x = \tilde{H}(y, p, q)$ tangent to the center invariant subspace $\{(x, y, p, q) \in \mathbb{R}^4 | x = 0\}$ at $O$. Clearly, it is of the form $\tilde{H}(y, p, q) := h_0(p, q) + h_1(p, q)y + \tilde{h}_2(p, q)y^2 + \tilde{h}_3(p, q)y^3 + \tilde{h}_4(p, q)y^4 + O(y^5)$ with indeterminate functions $\tilde{h}_i, i = 0, 1, 2, 3, 4$, such that $\tilde{H}(p, q) = O(|(p, q)|^2)$ and $\tilde{H}(p, q) = O(|(p, q)|^3)$. By the invariance we obtain the equality $\frac{dx}{d\tau} = \frac{\partial \tilde{H}}{\partial y} \frac{dy}{d\tau}, \text{i.e.,}$
Thus the restriction of (24) to the manifold is the equation
\[ y = \text{discuss the reduced equation (26) instead. Solving} \]
\[ \text{(required by equilibria)} \text{ and } \tilde{\nu} \text{ which divide the (} \]
\[ \text{y, } 0 \text{ and } 3 \text{ equilibria} \text{), we obtain two branches of transcritical bifurcation curves} \]
\[ \text{branches of saddle-node bifurcation curves} \]
\[ \text{Equation (25) has the normal form} \]
\[ \text{Lemma 4.4. Equation (25) has the normal form} \]
\[ \frac{dy}{dt} = \tilde{F}(y, p, q) := f_1(p)y + \sum_{i=2}^{5} f_i(p, q)y^i + O(|y|^6), \]
\[ \text{where } f_1(p, f_i(p, q), i = 2, ..., 5, \text{ are given in Appendix II.} \]
\[ \text{Lemma 4.4. Equation (25) has the normal form} \]
\[ \dot{y} = \tilde{G}(y, v_1, v_2) := v_1y + \nu_2y^2 + y^3, \]
\[ \text{where } \nu_1 := c_{11}p + c_{111}p^2 + c_{112}pq + O(|(p, q)|^3), \nu_2 := c_{21}p + c_{22}q + O(|(p, q)|^2) \]
\[ \text{and the coefficients } c_{11}, c_{111}, c_{112}, c_{21} \text{ and } c_{22} \text{ are given in Appendix II. Moreover,} \]
\[ \nu_1 \text{ has the same sign as } -p \text{ when } |p| \text{ and } |q| \text{ are small enough.} \]

We leave the proof of Lemma 4.4 to the end of this section. By this lemma, we discuss the reduced equation (26) instead. Solving \( y \) form the equation \( \tilde{G}(y, v_1, v_2) = 0 \) for equilibria of system (26), we obtain three zeros \( y_1 = 0 \) and \( y_2^{\pm} = (-\nu_2 \pm \sqrt{\nu_2^2 - 4\nu_1})/2 \). Further, eliminating \( y \) from the equations \( \tilde{G}(y, v_1, v_2) = 0 \) (required by equilibria) and \( \tilde{G}_y(y, v_1, v_2) = 0 \) (required by the degeneracy of those equilibria), we obtain two branches of transcritical bifurcation curves \( \tilde{TR}^\pm \) and two branches of saddle-node bifurcation curves \( \tilde{SN}^\pm \)

\[ \tilde{TR}^\pm := \{(v_1, v_2) \in \mathbb{R}^2 | v_1 = 0, \pm \nu_2 > 0 \}, \]
\[ \tilde{SN}^\pm := \{(v_1, v_2) \in \mathbb{R}^2 | v_1 = 4^{-1}\nu_2^2, \pm \nu_2 > 0 \}, \]

which divide the \((v_1, v_2)\)-plane into 4 regions \( R_1 := \{(v_1, v_2) \in \mathbb{R}^2 | v_1 > 4^{-1}\nu_2^2 \}, \)
\( R_2 := \{(v_1, v_2) \in \mathbb{R}^2 | 0 < v_1 < 4^{-1}\nu_2^2 \text{ and } \nu_2 > 0 \}, \)
\( R_3 := \{(v_1, v_2) \in \mathbb{R}^2 | v_1 < 0 \} \text{ and } \)
\( R_4 := \{(v_1, v_2) \in \mathbb{R}^2 | 0 < v_1 < 4^{-1}\nu_2^2 \text{ and } \nu_2 < 0 \}. \) It indicates that system (26) has 3 equilibria \( y = y_1 \) and \( y = y_2^\pm \) if \( (v_1, v_2) \in R_3 \), which collide to be one with \( y = 0 \) as \( (v_1, v_2) \) moves from \( R_3 \) to the bifurcation point \((0, 0)\). When \( (v_1, v_2) \)
moves from $R_3$ to $R_2$ (or $R_4$) across $TR^+$ (or $TR^-$), two equilibria $y = y_1$ and $y = y_2^+$ (or $y = y_2^-$) collide to be one $y = 0$ as $(\nu_1, \nu_2)$ reaches $TR^+$ (or $TR^-$), and then separate into $y = y_1$ and $y = y_2^+$ (or $y = y_2^-$) as $(\nu_1, \nu_2)$ moves to $R_2$ (or $R_4$). When $(\nu_1, \nu_2)$ moves from $R_2 \cup R_4$ to $R_1$ across $SN^- \cup SN^+$, two equilibria $y = y_2^+$ collide to be one $y = -\nu_2/2$ as $(\nu_1, \nu_2)$ reaches $SN^- \cup SN^+$, and then vanish as $(\nu_1, \nu_2)$ moves to $R_1$.

Finally, we give expressions of bifurcation curves $\overline{TR}^\pm$ and $SN^\pm$ for the original system (1) according to (27) and (28), so that the expressions of $R_i, i = 1, 2, 3, 4$ are obvious. By Lemma 4.4 the equation $\nu_1 = 0$ is equivalent to $p = 0$, namely $\tilde{E} = E_M$, when $|p|$ and $|q|$ are small enough. If $\nu_1 = 0$, i.e., $p = 0$, we have $\nu_2 = c_{22}q + O(q^2)$ and $c_{22} > 0$, implying that $\nu_2$ has the same sign as $q$. Thus, according to (27) we have

\begin{equation}
\overline{TR}^- = \{(\epsilon, \tilde{E}) \in \mathcal{U}|0 < \epsilon < \epsilon_1, \tilde{E} = E_M\} \quad \text{and} \quad \overline{TR}^+ = \{(\epsilon, \tilde{E}) \in \mathcal{U}|\epsilon > \epsilon_1, \tilde{E} = E_M\}.
\end{equation}

Concerning the saddle-node bifurcation curves, let $\Lambda(p, q) := 4\nu_1 - \nu_2^2 = 4c_{11}p + O(|(p, q)|^2)$. Since $\Lambda(0, 0) = 0$ and $\Lambda_p(0, 0) = 4c_{11} \neq 0$, by the Implicit Function Theorem there is a smooth function $p = \Phi(q) := p_1q + p_2q^2 + O(q^3)$ with two indeterminate $p_1$ and $p_2$ such that $\Lambda(\Phi(q), q) = 0$ for sufficiently small $|p|$ and $|q|$. Comparison of coefficients of the same order on both sides of the equation $\Lambda(\Phi(q), q) = 0$ leads to $p_1 = 0$ and $p_2 = \rho$, where $\rho$ is defined in Theorem 4.3. If $\nu_1 = 4^{-1}\nu_2^2$, i.e., $p = \Phi(q)$, we have $\nu_2 = c_{22}q + O(q^2)$ and $c_{22} > 0$, implying that $\nu_2$ has the same sign as $q$. Thus, by (28) we have $SN^\pm := \{(\epsilon, \tilde{E}) \in \mathcal{U}||\epsilon - \epsilon_1| > 0, \tilde{E} = E_M + \rho(\epsilon - \epsilon_1)^2 + O(|\epsilon - \epsilon_1|^3)\}$. This completes the proof. 

In the end of this section we prove Lemma 4.4. 

**Proof.** Since $f_1(0) = f_2(0, 0) = 0$ and $f_3(0, 0) > 0$, we have $\hat{F}(y, 0, 0) = y^3(f_3(0, 0) + O(y))$, where $f_3$s are given in Appendix II and $\hat{F}$ is defined in (25). By the Malgrange Preparation Theorem (see [3, Theorem 7.1, p.43]), there are $C^\infty$ functions $Q(y, p, q) := \sum_{i=0}^{3}q_i(p, q)y^i + O(y^5)$, $g_1(p, q) := O(|p, q|)$ and $g_2(p, q) := O(|p, q|)$ with the property that $Q(0, 0, 0) = q_0(0, 0) \neq 0$ such that

\begin{equation}
(f_1(p)y + f_2(p, q)y^2 + f_3(p, q)y^3 + f_4(p, q)y^4 + f_5(p, q)y^5 + O(y^6))Q(y, p, q)
= g_1(p, q)y + g_2(p, q)y^2 + y^3.
\end{equation}

(29)

Comparison of coefficients of $y$ and $y^2$ on both sides of equation (29) gives $g_1(p, q) = f_1(p)q_0(p, q)$ and $g_2(p, q) = f_1(p)q_1(p, q) + f_2(p, q)q_0(p, q)$, which implies that the linear coefficients in $g_1$ and $g_2$ as well as the quadratic coefficients in $g_1$ are decided by the constant terms in $q_0$ and $q_1$ as well as the linear coefficients in $q_0$. Comparing coefficients of $y^3, y^4$ and $y^5$ further, we obtain three equations

\begin{align*}
1 = &f_1(p)q_2(p, q) + f_2(p, q)q_1(p, q) + f_3(p, q)q_0(p, q), \\
0 = &f_1(p)q_3(p, q) + f_2(p, q)q_2(p, q) + f_3(p, q)q_1(p, q) + f_4(p, q)q_0(p, q), \\
0 = &f_1(p)q_4(p, q) + f_2(p, q)q_3(p, q) + f_3(p, q)q_2(p, q) + f_4(p, q)q_1(p, q) + f_5(p, q)q_0(p, q),
\end{align*}

(30)

which implies by substitution $p = q = 0$ that $q_0(0, 0) = f_3(0, 0)^{-1}$, $q_1(0, 0) = -f_3(0, 0)^{-2}f_4(0, 0)$ and $q_2(0, 0) = f_3(0, 0)^{-3}f_4(0, 0)^2 - f_3(0, 0)^{-2}f_5(0, 0)$. Comparing the linear terms of the first equation in (30), we obtain the linear terms in $q_0$, denoted by $q_0(p, q) := -f_3(0, 0)^{-1}(f_{11}(p)q_2(0, 0) + f_{21}(p, q)q_1(0, 0) + f_{31}(p, q)q_0(0, 0))$, where $f_{ij}$ $(i = 1, 2, 3)$ denotes the linear terms in $f_i$. Thus we obtain $g_1(p, q) =$
c_{11}p + c_{111}p^2 + c_{112}pq + O(|(p, q)|^3) and \( g_2(p, q) = c_{21}p + c_{22}q + O(|(p, q)|^2) \), where the coefficients \( c_{11}, c_{111}, c_{112}, c_{21} \) and \( c_{22} \) are given in Appendix II. Since

\[
\det \frac{\partial (g_1, g_2)}{\partial (p, q)} \bigg|_{p=0, q=0} = c_{11}c_{22} = -\frac{4\beta^4}{(1-\alpha)^2(3-3\alpha-\beta)^2E_M^{\beta/(1-\alpha)-3}} < 0,
\]

the parameter transformation \( \nu_1 = g_1(p, q) \) and \( \nu_2 = g_2(p, q) \) is locally invertible. Applying the time-rescaling \( d\tau = Q(y, p, q)d\tilde{\tau} \) and the parameter transformation \( (\nu_1, \nu_2) = (g_1, g_2) \), we can reduce system (25) to its normal form (26).

Finally, we consider the sign of \( \nu_1 \) when \(|p|\) and \(|q|\) are small enough. Since

\[
q_0(0, 0) = 2\beta E_M^{\beta(2-\alpha)/(1-\alpha)}/\{\xi\alpha(3-3\alpha-\beta)\} > 0,
\]

we have \( q_0(p, q) > 0 \), implying by the fact \( \nu_1 = f_1(p)q_0(p, q) \) that \( \nu_1 \) has the same sign as \( f_1(p) \). Since \( f_1(0) = 0 \) and \( f_1'(0) < 0 \), \( \nu_1 \) has the same sign as \(-p\). The lemma is proved.

5. Conclusions. In this section, we discuss orbital structures of system (1) in different regions on the first quadrant of \((\epsilon, \tilde{E})\)-parameter plane partitioned by bifurcation curves. Note that the curve \( SN^+ \), one of the 4 bifurcation curves obtained in Theorem 4.3, only serves for invalid equilibria (outside the first quadrant) but the other three \( TR^\pm \) and \( SN^- \) just belong to \( TR^\pm \) and \( SN \), defined before.

![Parameter plane and global phase portrait.](image-url)
Theorem 4.3, respectively. Thus we use $TR^\pm$, $SN$ and their meeting point $\Upsilon$ to divide the first quadrant of the $(\epsilon, E)$-plane into three regions $R_1 := \{(\epsilon, E) \in \mathbb{R}^2 | 0 < \epsilon < \epsilon_1 \text{ and } Z_F < E < E_M \}$, $R_2 := \{(\epsilon, E) \in \mathbb{R}^2 | \epsilon > 0 \text{ and } \dot{E} > E_M \}$, and $R_3 := \{(\epsilon, E) \in \mathbb{R}^2 | \epsilon < \epsilon_1 \text{ and } 0 < E < Z_F \text{ or } \epsilon_1 < \epsilon \text{ and } 0 < \dot{E} < E_M \}$, as shown in Figure 1.

First, we concentrate on orbital structure outside a bounded region

$$\Omega := \{(E, K) | 0 \leq E \leq \hat{E} \text{ and } 0 \leq K \leq \zeta\},$$

which is positively invariant such that every orbit starting from the outside of $\Omega$ either enters $\Omega$ or approaches one of two boundary equilibria $S_1 := (0, \zeta)$ and $S_2 := (E, 0)$ as indicated in Proposition 3 of [1, p.217]. Our Theorem 2.2 implies that every orbit outside $\Omega$ comes from the equilibrium at infinity $I_2$ except for the orbit on the $K$-axis.

Then, we discuss orbital structure inside $\Omega$. The authors of [1] displayed orbital structure of system (1) in $\Omega$ separately in three cases: the case of two interior equilibria $S_3 := (E_A, \zeta - E_A^{(1 - \alpha)})$ and $S_4 := (E_B, \zeta - E_B^{(1 - \alpha)})$, the case of exact one interior equilibrium $S_3$, and the case of no interior equilibrium, where $E_A$ and $E_B$ are zeros of the function (2) such that $0 < E_A < E_B < E_M$. In the following, we summarize the above sections and give a qualitative formulation to orbital structures of system (1) separately in four cases: (A1): $(\epsilon, E) \in R_3 \cup TR^+ \cup \Upsilon$; (A2): $(\epsilon, E) \in \hat{R}_2 \cup TR^-$; (A3): $(\epsilon, E) \in SN$ and (A4): $(\epsilon, E) \in \hat{R}_1$, covering the whole first quadrant of the $(\epsilon, E)$-parameter plane.

In case (A1), by Theorems 3.3, system (1) has no interior equilibria, which implies the nonexistence of closed orbits. By our Theorems 2.1 and Proposition 2 of [1, p.217], system (1) has two boundary equilibria $S_1$ and $S_2$, where $S_1$ is a sink, and $S_2$ is either a saddle if $(\epsilon, E) \in R_3$ or a saddle-node if $(\epsilon, E) \in TR^+$ or a degenerate saddle if $(\epsilon, E) = \Upsilon$ such that $S_2$ has a stable manifold on the $E$-axis and an unstable manifold or a center manifold, on which $S_2$ is unstable in the first quadrant. It follows that $S_1$ attracts the whole positively invariant set $\Omega$ and therefore the whole first quadrant as shown in Figure 1.

In case (A2), i.e., $(\epsilon, E) \in \hat{R}_2 \cup TR^-$, by our Theorems 2.1, 3.3 and Proposition 2 of [1, p.217], system (1) has one interior equilibrium $S_3$ and two boundary equilibria $S_1$ and $S_2$, where $S_1$ and $S_2$ are both sinks and $S_3$ is a saddle. It follows that there is no homoclinic or heteroclinic loops. Since the unique interior equilibrium $S_3$ is a saddle and the positive $E$- and $K$-axes are positively invariant sets, by the index theory of equilibria ([7, pp. 51, Proposition 1.8.4]) there is no closed orbit around $S_3$ (namely no closed orbit in the first quadrant). Hence, $S_1$ and $S_2$ attract the first quadrant except the stable manifold of $S_3$ as shown in Figure 1.

In case (A3), i.e., $(\epsilon, E) \in SN$, by our Theorem 3.3 and Proposition 2 of [1, p.217], system (1) has a unique interior equilibrium $S_{34} := (\varsigma_0, \zeta - \varsigma_0^{(1 - \alpha)})$ and two boundary equilibria $S_1$ and $S_2$, where $S_1$ is a sink, $S_2$ is a saddle and $S_{34}$ is a saddle-node. Consider a set $I_1 := \{(E, K) \in \mathbb{R}^2 | 0 \leq E \leq \varsigma_0, \varrho(E) \leq K \leq \zeta - E^{(1 - \alpha)} \text{ and } (E, K) \neq (0, 0)\}$ surrounded by $S_1$, $S_{34}$, the horizontal isocline $H = \{(E, K) | 0 < E < E_M \text{ and } K = \varsigma - E^{(1 - \alpha)} \}$, the vertical isocline $V = \{(E, K) | 0 < E < \hat{E} \text{ and } K = \varrho(E)\}$ and the $K$-axis, where $\varrho(E)$ is defined in Lemma 5.1. Note that the second coordinate $K$ of the horizontal isocline $H$ decreases as $E$ increases from $0$ to $E_M$, $H$ and $V$ are tangent at $S_{34}$, and $H$ is above $V$ as $0 \leq E \leq \varsigma_0$. Then $dE/dt = \varrho(E) < 0$ and $dK/dt = 0$ as $0 \leq E \leq \varsigma_0$ and $K = \varsigma - E^{(1 - \alpha)}$,
i.e., orbits in \( \mathcal{H} \) between \( S_1 \) and \( S_{34} \) tend to \( I_1 \), because \( g(E) = 0 \) at interior equilibria and the monotonicity of \( g \) was given below (14), and \( dE/dt = 0 \) and \( dK/dt = K(\xi - E^\beta/(1-\alpha) + K)^{-\alpha} - \xi \gamma > 0 \) as \( 0 < E < \bar{E} \) and \( K = g(E) \), i.e., orbits in \( \mathcal{V} \) between the origin and \( S_{34} \) tend to \( I_1 \), because the sign of \( dK/dt \) is the same as \( \xi - E^\beta/(1-\alpha) - K \) with \( \xi := (\alpha/(1-\alpha))^{1/\alpha} \). Moreover, the \( K \)-axis between the origin and \( S_1 \) is invariant obviously. It follows that \( I_1 \) is a positive invariant set next to \( S_{34} \), implying that there is no closed orbit surrounding \( S_{34} \), namely, no closed orbit in the first quadrant. Similarly to cases (A1) and (A2), there is neither homoclinic loop nor heteroclinic loop in the first quadrant. It follows that \( S_1 \) and \( S_{34} \) attract the first quadrant as shown in Figure 1.

In case (A4), i.e., \((\epsilon, \bar{E}) \in \bar{R}_1 \), by our Theorem 3.3 and Proposition 2 of [1, p.217], system (1) has two interior equilibria \( S_3 \) and \( S_4 \) and two boundary equilibria \( S_1 \) and \( S_2 \), where \( S_1 \) is a sink, \( S_2 \) and \( S_3 \) are both saddles and \( S_4 \) is a stable node. Consider three sets \( I_2 := \{(E, K) \in \mathbb{R}^2| 0 \leq E \leq E_A, g(E) \leq K \leq \xi - E^\beta \} \) and \( (E, K) \neq 0 \), \( I_3 := \{(E, K) \in \mathbb{R}^2| E_A \leq E \leq E_B \} \) and \( \xi - E^\beta \leq K \leq g(E) \) and \( I_4 := \{(E, K) \in \mathbb{R}^2| E_B \leq E \leq \bar{E} \} \). Note that \( \mathcal{H} \) and \( \mathcal{V} \) intersect at \( S_3 \) and \( S_4 \), \( \mathcal{H} \) is above \( \mathcal{V} \) as \( E \in (0, E_A) \cup (E_B, \bar{E}) \), and \( \mathcal{H} \) is below \( \mathcal{V} \) as \( E \in (E_A, E_B) \). Similarly to case (A3), we see that \( I_2, I_3 \) and \( I_4 \) are positive invariant sets, implying that there is no closed orbit surrounding \( S_3 \) or \( S_4 \), namely, no closed orbit in the first quadrant. The above mentioned properties of \( S_1, S_2, S_3 \) and \( S_4 \) imply no homoclinic or heteroclinic loops in the first quadrant because two branches of unstable manifold of \( S_3 \) belonging to \( I_1 \) and \( I_3 \) approach \( S_1 \) and \( S_4 \) respectively. It follows that \( S_1 \) and \( S_4 \) attract the first quadrant except for \( S_3 \) and its stable manifold as shown in Figure 1.

Past experiences of successful and failed industrializations tell that industrial development is a necessary but not sufficient condition for overcoming poverty. It is important to find factors that affect the type of developing path for an economy in industrialization. Concerning small open economies with constant labor supply and free intersectoral labor mobility between farming sector and industry sector, Antoci et al. ([1]) set up the model (1) to formulate the time evolution of \( E \) (the stocks of natural renewable resource) and \( K \) (the stocks of capital owned by industrial entrepreneurs). They showed that the initial values of \( E \) and \( K \) and the parameter \( \bar{E} \), which represents the carrying capacity of the nature resource, significantly affect the type of developing path of the economy.

An industrialization is deemed to be successful if and only if the economy takes a sustainable developing path, namely the natural resources coexist with industrial development, which is performed in mathematics by orbits of \((E, K)\) tending to an interior equilibrium. In case (A1), all orbits tend to the boundary equilibrium \( S_1 \) on the \( K \)-axis, implying that the industrial development will destroy the farming sector. In case (A2), orbits tend to one of two boundary equilibria \( S_1 \) and \( S_2 \) except for the orbits on the stable manifold of \( S_3 \), implying that the coexistence of the industry sector and the farming sector is generically impossible. On the contrary, in cases (A3) and (A4), orbits near the \( E \)-axis tend to an interior equilibrium, implying that a successful industrialization is available with sufficient initial nature renewable resources. Additionally, the parameter condition of cases (A3) and (A4), i.e., \( 0 < \epsilon < \epsilon_1 \) and \( Z_F \leq \bar{E} < E_M \), shows that the success of industrialization also
needs the environmental impact of industrial production to be limited by a threshold and the carrying capacity of the nature resource to be bounded.

![Figure 2. Phase portraits of system (1) in a bounded region.](image)

We finally demonstrate our results near the important point \( \Upsilon \) in the \((\epsilon, \tilde{E})\)-parameter plane. This point gives critical values for the parameters such that 4 distinct phenomena may occur nearby. Under the assumption \( \alpha = \beta = 1/3, \xi = 1/2, \zeta = 1 \) and \( \epsilon = 1 \), implementing Matlab Ver.2016b to the four options: (i) \( \tilde{E} = 0.9 \), (ii) \( \tilde{E} = 2 \), (iii) \( \tilde{E} = \tilde{E}^* \), given in Appendix II, and (iv) \( \tilde{E} = 0.95 \), we simulate the phase portraits in Figures 2 (a), (b), (c) and (d) respectively, showing that the choices of initial industrial capital, initial nature resource, the carrying capacity of natural resource and the environmental impact of industrial production provide successful industrialization.

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First, we prove Lemma 3.1.

Proof. We discuss the sign of $F(x)$ by its monotonicity and continuity. In the particular case $\beta = 1 - \alpha$, $F(x) = x + \epsilon \zeta^{-\alpha} - 2\epsilon^{5/2} \zeta^{1/2} - \epsilon^{-1} - \epsilon_0$. If $0 < \epsilon < \epsilon_0$, $F(x)$ has a unique positive zero $Z_F = 2\epsilon^{5/2} \zeta^{1/2} - \epsilon_0$, which is defined in (13). Compute $h(0) = p(\epsilon^{5/2} - \epsilon_0^{5/2})$, where $p := 2\epsilon^{2.5 + \alpha} \zeta^{-\alpha + \beta} > 0$, $\epsilon_0 := (1 - \beta(1 - \alpha)^{-1})^{\alpha/2} > 0$, and $\epsilon_0$ is defined in (13). Then $h(0) < 0$ if $\epsilon > \epsilon_0$ (resp. $\epsilon = \epsilon_0$, $0 < \epsilon < \epsilon_0$). We discuss separately in two cases: (C1) $\epsilon > \epsilon_0$ and (C2) $0 < \epsilon < \epsilon_0$.

In case (C1), we have $h(x) > h(0) > 0$, i.e., $F''(x) > 0$, for all $x > 0$. Since $F'(0) = (A^2_1(1 - \alpha)/(\alpha + \beta - 1))h(0) \leq 0$ and $\lim_{x \to +\infty} F'(x) = -2A_0 > 0$, $F'(x)$ has a unique non-negative zero, denoted by $Z_1$, i.e., $F(x)$ has a minimum $F(Z_1)$ for $x > 0$. Then we simplify the expression of $F(Z_1)$ by the equation $F'(Z_1) = 0$ and obtain $F(Z_1) = \beta(1 - \alpha - \beta)(Z_1^2 + A_2)^{3/2} > 0$, implying that $F(x) \geq F(Z_1) > 0$ for all $x > 0$.

In case (C2), i.e., $0 < \epsilon < \epsilon_0$, we have $h(0) < 0$ and $\lim_{x \to +\infty} h(x) = +\infty$, which implies by the monotonicity of $h$ that $h(x)$ has a unique positive zero, denoted by $Z_h$. It follows that $F''(x) < 0$ (resp. $= 0$, $> 0$) if $0 < x < Z_h$ (resp. $= Z_h$, $> Z_h$), namely $F''(x)$ has a minimum $F'(Z_h)$ for $x > 0$. By the equation $h(Z_h) = 0$, we can eliminate some terms in $F'(Z_h)$ and obtain

$$F'(Z_h) = \frac{\beta Z_h(Z_h^2 + A_2)^{3/2}}{(1 - \alpha - \beta)(Z_h^2 + A_2) + Z_h(Z_h^2 + A_2)^{1/2}} > 0,$$

implying that $F'(x) \geq F'(Z_h) > 0$ for all $x > 0$. Rewriting $F'(0) = \beta(\epsilon^{5/2} - \epsilon_0^{5/2})$, we have $F(0) > 0$ (resp. $= 0$, $< 0$) if $\epsilon_0 < \epsilon < \epsilon_0$ (resp. $\epsilon = \epsilon_0$, $0 < \epsilon < \epsilon_0$), where $\epsilon_0$ is defined in (13).
\[ \hat{p} := \epsilon^2 - 1n^{1-\alpha} - 1/2 \alpha - \beta \frac{\zeta^{\frac{\alpha - \alpha - 1}{2}}}{\zeta^{\frac{\alpha - \alpha - 1}{2}}} \] and \( \epsilon_0 \) is defined in (13). If \( \epsilon_0 \leq \epsilon < \epsilon_0 \), \( F(x) > F(0) \geq 0 \) for all \( x > 0 \). If \( 0 < \epsilon < \epsilon_0 \), \( F(x) \) has a unique positive zero, denoted by \( Z_F \), such that \( F(x) < 0 \) for \( 0 < x < Z_F \) and \( F(x) > 0 \) for \( x > Z_F \). The proof is completed.

Then, we prove Lemma 3.2.

Proof. Since \( E_N - E_M = -\frac{\beta}{2 - \alpha} \zeta^{\frac{1}{2} - \alpha} (\epsilon - \epsilon_1) \), where \( E_M \) is defined before (2), \( E_N = -\frac{\beta}{2 - \alpha} \zeta^{\frac{1}{2} - \alpha} (\epsilon - \epsilon_1) \) by Lemma 3.2, and \( \epsilon_1 \) is defined in Theorem 2.1, we have inequalities: \( E_N < 0 \) (resp. \( = 0 \) if \( \epsilon > 2 \epsilon_1 \) (resp. \( = 2 \epsilon_1 < 2 \epsilon_1 \)) and \( E_N < E_M \) (resp. \( = E_M, > E_M \)) if \( \epsilon > \epsilon_1 \) (resp. \( = \epsilon_1, < \epsilon_1 \)). Then, we only need to discuss the order among \( Z_F, E_M, E_N \) in the case \( 0 < \epsilon < \epsilon_0 \), which by Lemma 3.1 is determined by the signs of \( F(E_M) \) and \( F(E_N) \), where \( \epsilon_0 \) is defined in (13) and satisfies \( \epsilon_0 = 2(1 + \frac{\beta}{2 - \alpha}) \zeta^{\frac{\alpha - \alpha - 1}{2}} \). We claim that \( F(E_M) = 0 \) (or \( > 0 \) if \( \epsilon = \epsilon_1 \) (or \( 0 < \epsilon \neq \epsilon_1 \))), and that \( F(E_N) < 0 \) (resp. \( = 0 \), \( > 0 \)) if \( \epsilon > \epsilon_1 \) (resp. \( \epsilon = \epsilon_1 \), \( 0 < \epsilon < \epsilon_1 \)).

For the claim about \( F(E_M) \), we consider the particular case \( \beta = 1 - \alpha \), where \( F(E_M) = \zeta^{-\alpha}(e^{1/2} - \epsilon_1^{2}/2)^{-2} \). It implies that either \( F(E_M) > 0 \) if \( 0 < \epsilon \neq \epsilon_1 \) or \( F(E_M) = 0 \) if \( \epsilon = \epsilon_1 \). In the generic case \( 0 < \beta < 1 - \alpha \), introducing a new variable \( x := 1 + (1 + c_2 \zeta^{1-\alpha}(\beta-2)/\beta) \zeta^{1/2} \in (2, +\infty) \), we rewrite \( F(E_M) \) as the form \( F(E_M) = (x^2 - 2x)^{-2} \zeta^{-1-\alpha}(1-\alpha)/\beta F_1(x) \), where \( c := \frac{\beta}{2 - \alpha} \in (0, 1) \), \( F_1(x) := c_1 x^{2-1} - c_2 (1 - c_3) x + c_2, c_1 := \epsilon c((4 - 2\epsilon)/(1 - c))^{1-c}, c_2 := 2c((2 - c)/(1 - c))^2, \) and \( c_3 := \frac{1}{2} e^{\epsilon} \). Then the sign of \( F_1(x) \) has the same sign as \( F(E_M) \). Compute the derivative \( F'_1(x) = c_1 x^{1-c} - c_2 (1 - c_3) x + c_2, c_1 := \epsilon c((4 - 2\epsilon)/(1 - c))^{1-c}, c_2 := 2c((2 - c)/(1 - c))^2, \) and \( c_3 := \frac{1}{2} e^{\epsilon} \). Then the sign of \( F'_1(x) \) has the same sign as \( F(E_M) \).

For the claim about \( F(E_N) \), we consider the particular case \( \beta = 1 - \alpha \), where \( F(E_N) = 2 \zeta^{1-\alpha} (e^{1/2} - \epsilon_1^{2}/2)^{-2} \), which satisfies \( F(E_N) < 0 \) (resp. \( = 0 \), \( > 0 \)) if \( \epsilon > \epsilon_1 \) (resp. \( \epsilon = \epsilon_1 \), \( 0 < \epsilon < \epsilon_1 \)). In general case \( 0 < \beta < 1 - \alpha \), we introduce a new variable \( y := z + (z^2 - 4(2 - c)/(1 - c)^{-2} z + 8(2 - c)/(1 - c)^{-2})^2, \zeta := 2 - 2c \zeta^{1-\alpha}(\beta-2)/\beta \zeta^{1/2} \), where \( z := \zeta^{1-\alpha}(\beta-2)/\beta \zeta^{1/2} \in (-\infty, 2) \). Since \( \lim_{z \to -\infty} y = 2(2 - c)/(1 - c)^{-2}, \lim_{z \to 2} y = 4 \) and \( dy/dz = 1 + (z-2(2 - c)/(1 - c)^{-2})/(z-2(2 - c)/(1 - c)^{-2})^2 - 4c((2 - c)/(3 - 2c)/(1 - c)^{-4})^{-2} < 0 \), we obtain \( y \in (4, 2(2 - c)/(1 - c)^{-2}) \). Then we can rewrite \( F(E_N) = \epsilon c y^{-1}(y-4)^{-1} \zeta^{-1-\alpha}(1-\alpha)/\beta G(y) \), where \( G(y) := a_1 y/(1 - c) - a_2 y^2 - a_3 y + a_4, a_1 := ((4 - 2\epsilon)/(1 - c))^{1-c}, a_2 := \frac{\epsilon}{2}, a_3 := \frac{4}{2}, a_4 := 8(2 - c)/(1 - c)^2 \) and \( c = \frac{1}{2-\alpha} < 1 \).

Thus \( F(E_N) \) has the same sign as \( G(y) \). It is easy to check that \( Z_G := 1 + (1 + 4(2 - c)/(1 - c)^{-2})^{1/2} \in (4, 2(2 - c)/(1 - c)^{-2}) \) is a zero of \( G \). Therefore, if we claim the inequality \( G'(y) < 0 \) for \( y \in (4, 2(2 - c)/(1 - c)^{-2}) \), then \( G(y) < 0 \) (or \( > 0 \)) for \( y \in (Z_G, 2(2 - c)/(1 - c)^{-2}) \) (or \( y \in (4, Z_G) \)), i.e., \( F(E_N) < 0 \) (resp. \( = 0 \), \( > 0 \)) if \( \epsilon > \epsilon_1 \) (resp. \( \epsilon = \epsilon_1 \), \( 0 < \epsilon < \epsilon_1 \)), which proves the claim about \( F(E_N) \).

In what follows, we prove the inequality \( G''(y) < 0 \) for \( y \in (4, 2(2 - c)/(1 - c)^{-2}) \) as claimed just now. Compute \( G'(y) = a_1 (c+1) y^{-1} - 4a_1 c y^{c-1} - 2a_2 y - a_3, G''(y) = a_1 c (c+1) y^{-2} - 4a_1 c (c-1) y^{-3} - 2a_2, \) and \( G'''(y) = a_1 c (c+1) y^{-3} - ((c+1)y - 4(c - 2)) < 0 \). Clearly, \( G''(4) = 2c(2 - c)^{-1}(2c(2 - c)^{2-c} - (c - 1)) < 0 \). At the other end-point we have \( G''(2(2 - c)/(1 - c)^{-2}) = c(1 - c)^{-2}(2 - c)^{-2} I(c), \) whose sign is
the same as the sign of $I(c) := (1-c)(4-5c+5c^2-2c^3)-2(1-c)^c$ for $c \in (0,1)$. Since $\ln x - \ln y < 0$ (resp. $= 0$, $> 0$) if $x-y < 0$ (resp. $= 0$, $> 0$), substituting $x = (1-c)(4-5c+5c^2-2c^3)$ and $y = 2(1-c)^c$ in the inequalities, we see that the function $\hat{I}(c) := \ln((1-c)(4-5c+5c^2-2c^3)) - \ln(2(1-c)^c)$ has the same sign as $I(c)$. Compute derivatives $\hat{I}'(c) = -(5-10c+6c^2)(4-5c+5c^2-2c^3)^{-1} - 1$ and $\hat{I}''(c) = (1-c)^{-1}(4-5c+5c^2-2c^3)^{-2}(31-53c+13c^2+24c^3-7c^4-8c^5+4c^6)$. One can check that $\hat{I}''(c) > 0$ for $c \in (0,1)$, $\hat{I}'(0) = -2.25$ and $\lim_{c \to 1} \hat{I}'(c) = +\infty$, implying that $\hat{I}'$ has a unique zero in the interval $(0,1)$, denoted by $Z_{I}'$, and that $\hat{I}'(c) < 0$ (or $> 0$) for $0 < c < Z_{I}'$ (or $Z_{I}' < c < 1$). Further, noting that $\hat{I}(0) = \ln 2 > 0$ and $\lim_{c \to 1} \hat{I}(c) = 0$, we see that $\hat{I}(c)$ has a unique zero in $(0,1)$, denoted by $Z_I$, and that either $\hat{I}(c) < 0$ if $c \in (Z_I,1)$ or $\hat{I}(c) > 0$ if $c \in (0,Z_I)$. Using the software Maple 18 we compute $Z_I \approx 0.38648$. Thus, $G''(2(2-c)/(1-c)^2) < 0$ (resp. $= 0$, $> 0$) if $c \in (Z_I,1)$ (resp. $c = Z_I$, $c \in (0,Z_I)$).

In one case $0 < c \leq Z_I$, we have $G''(y) > G''(2(2-c)/(1-c)^2) \geq 0$ for all $y \in (4, 2(2-c)/(1-c)^2)$. Similarly to the above discussion on $G''(2(2-c)/(1-c)^2)$, we compute $G''(2(2-c)/(1-c)^2) = (1-c)^{-2}(1-c)^{-c}(2c^3-3) < 0$ for $c \in (0,1)$. Thus $G''(y) < G'(2(2-c)/(1-c)^2) < 0$ for $y \in (4, 2(2-c)/(1-c)^2)$.

In the other case that $Z_I < c < 1$, we have $G''(2(2-c)/(1-c)^2) < 0 < G''(4(2-c)/(1-c)^2)$, and $G''(y) < 0$, implying that $G''(y)$ has a unique zero in $(4, 2(2-c)/(1-c)^2)$, denoted by $Z_{G''}$, and that $G''(y) < 0$ (or $> 0$) if $Z_{G''} < y < 2(2-c)/(1-c)^2$ (or $4 < y < Z_{G''}$). Therefore, $G'(y) \leq G'(Z_{G''})$ for all $y \in (4, 2(2-c)/(1-c)^2)$. Using the equation $G''(Z_{G''}) = 0$ to eliminate the term of degree $c$ in $G'(Z_{G''})$, we get $G'(Z_{G''}) = W(Z_{G''})$, where $W(x) := -4a_1x^{-c-1} + 2a_2x^{-c-2}(1-c)x - a_3$. Since $W'(x) = 4a_1(1-c)x^{-c-2} + 2a_2x^{-c-2}(1-c) > 0$ for all $x > 0$, we have $G'(y) \leq G'(Z_{G''}) = W(Z_{G''}) < W(2(2-c)/(1-c)^2) = (1-c)^{-c} < 0$ for $y \in (4, 2(2-c)/(1-c)^2)$, i.e., the claimed inequality is true. The proof is completed. \hfill $\Box$

**Lemma 5.1.** If $E \in (\bar{E}, +\infty)$, then $\sigma(E, K) < 0$ for all $K \geq 0$. If $E \in [0, \bar{E})$, there is a unique $\varrho(E) \geq 0$ such that $\sigma(E, K) < 0$ (resp. $= 0$, $> 0$) if $K > \varrho(E)$ (resp. $= \varrho(E)$, $< \varrho(E)$). $\sigma(E)$ is continuous in $E$. $\varrho(E) = 0$ if and only if $E = 0$ or $\bar{E}$.

**Proof.** If $E > \bar{E}$ then $\sigma(E, K) = E(\bar{E}-E)(E^{\frac{\theta}{-\alpha}} + K)^{\alpha} - c K < 0$ for all $K \geq 0$. If $0 < E < \bar{E}$, we have $\sigma(E, 0) = E^{1+\frac{\theta}{-\alpha}}(\bar{E}-E) > 0$ and $\lim_{K \to +\infty} \sigma(E, K) = -\infty$, which implies by the continuity that there is $\varrho(E) > 0$ such that $\sigma(E, \varrho(E)) = 0$. Furthermore,

$$\partial \sigma(E, K)/\partial K = \alpha(E(\bar{E}-E)(E^{\frac{\theta}{-\alpha}} + K)^{\alpha} - \epsilon \to -\epsilon < 0 \text{ as } K \to +\infty,$$

$$\partial^2 \sigma(E, K)/\partial K^2 = \alpha(\alpha-1)(E(\bar{E}-E)(E^{\frac{\theta}{-\alpha}} + K)^{\alpha} - \epsilon) < 0 \quad \forall E \in (0, \bar{E})$$

because $0 < \alpha < 1$. It follows that $\sigma(E, K)$ either decreases or first increases and then decreases as $K$ increases from zero to infinity. This monotonicity gives the uniqueness of $\varrho(E)$. If $E = 0$ or $\bar{E}$, then $\sigma(E, K) = -c K$, implying that there is a unique $\varrho(E) = 0$ such that $\sigma(E, \varrho(E)) = 0$. Moreover, the continuity of $\sigma(E, K)$ implies the continuity of $\varrho(E)$. The proof is completed. \hfill $\Box$

**Appendix II: Some coefficients**

$$A_{20} = -\frac{\alpha^2 - \alpha + 1}{1-\alpha} E_M^{\frac{\alpha \theta}{\alpha-1}}, A_{11} = -\frac{\xi \alpha \beta}{1-\alpha} E_M^{-2} + \frac{2(1-\alpha + \alpha \beta)}{1-\alpha} E_M^{-1} - \alpha E_M^{1-\beta},$$
\[ A_{02} = \frac{\xi \alpha \beta^2}{1 - \alpha} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{\xi \alpha \beta^2}{1 - \alpha} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) - \xi \alpha \beta E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + \alpha \xi \alpha \beta E_M \left( \frac{\alpha^3}{1 - \alpha} \right), \]
\[ B_{11} = -\frac{\xi \alpha \beta}{1 - \alpha} E_M \left( \frac{\alpha^3}{1 - \alpha} \right), \]
\[ B_{02} = \frac{\xi \alpha \beta}{1 - \alpha} E_M - \xi \alpha E_M, \]
\[ B_{21} = \frac{\xi \alpha \beta(1 - \alpha - \alpha \beta)}{2(1 - \alpha)^2} E_M \left( \frac{\alpha^3}{1 - \alpha} \right), \]
\[ B_{12} = -\frac{\xi \alpha \beta(1 - \alpha - \alpha \beta)}{(1 - \alpha)^2} E_M - \xi \alpha E_M, \]
\[ B_{03} = \frac{\xi \alpha \beta^2(1 - \alpha - \alpha \beta)}{2(1 - \alpha)^2} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) - \xi \alpha \beta E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + 2^{-1} \xi \alpha(1 - \alpha) E_M \left( \frac{\alpha^3}{1 - \alpha} \right), \]
\[ f_1(p) = \frac{\xi \alpha \beta}{1 - \alpha} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) p + \frac{\xi \alpha \beta(1 - \alpha - \alpha \beta)}{2(1 - \alpha)^2} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) p^2 + O(p^3), \]
\[ f_2(p, q) = (\frac{\xi \alpha \beta^2}{1 - \alpha}) E_M - \frac{\xi \alpha(2 - \beta) E_M^{-1} \beta}{1 - \alpha} p + \frac{\xi \alpha \beta}{1 - \alpha} E_M^{-2} q + O(|p, q|^2), \]
\[ f_3(p, q) = \frac{\xi \alpha(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{\xi \alpha \beta(1 - \alpha - \alpha \beta)}{2(1 - \alpha)^2} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{\xi \alpha \beta^2}{(1 - \alpha)^2} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) p + \frac{\xi \alpha \beta(3 - \beta) E_M^{-2} \beta}{1 - \alpha} E_M^{-2} q + O(|p, q|^2), \]
\[ f_4(p, q) = -\frac{\xi \alpha \beta^2(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) - \frac{\xi \alpha \beta(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{\xi \alpha \beta^2(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) p + \frac{\xi \alpha \beta(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) q + O(|p, q|^2), \]
\[ f_5(p, q) = \frac{\xi \alpha \beta^2(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) - \frac{\xi \alpha \beta(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{\xi \alpha \beta^2(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) p + \frac{\xi \alpha \beta(3 - 3 \alpha - \beta)}{2 \beta} E_M \left( \frac{\alpha^3}{1 - \alpha} \right) q + O(|p, q|^2), \]
\[ c_{11} = -\frac{2 \beta^2}{1 - \alpha(3 - 3 \alpha - \beta)} E_M^{\frac{\alpha^3}{1 - \alpha} - 2}, \]
\[ c_{21} = \frac{2 \beta^2}{1 - \alpha(3 - 3 \alpha - \beta)} E_M^{\frac{\alpha^3}{1 - \alpha} - 2}, \]
\[ c_{11} = -\frac{3(1 - \alpha)^3(3 - 3 \alpha - \beta)^3}{4 \xi \alpha \beta^3} \left( \frac{\alpha^3}{1 - \alpha} \right) - \frac{3(1 - \alpha)^3(3 - 3 \alpha - \beta)^3}{4 \xi \alpha \beta^3} \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{3(1 - \alpha)^3(3 - 3 \alpha - \beta)^3}{4 \xi \alpha \beta^3} \left( \frac{\alpha^3}{1 - \alpha} \right) + \frac{3(1 - \alpha)^3(3 - 3 \alpha - \beta)^3}{4 \xi \alpha \beta^3} \left( \frac{\alpha^3}{1 - \alpha} \right), \]
\[ c_{11} = -\frac{4 \xi \alpha \beta^4}{(1 - \alpha)^2(3 - 3 \alpha - \beta)^2 E_M^{\frac{\alpha^3}{1 - \alpha} - 4}} + \frac{4 \xi \alpha \beta^4}{(1 - \alpha)^2(3 - 3 \alpha - \beta)^2 E_M^{\frac{\alpha^3}{1 - \alpha} - 4}}, \]
\[ c_{21} = \frac{4 \beta(1 - 2 \alpha + 3 \beta + 2 \alpha^2 - 3 \alpha \beta - 3 \beta^2)}{3(1 - \alpha)(3 - 3 \alpha - \beta)^2} E_M^{\frac{\alpha^3}{1 - \alpha} - 1}, \]
\[ E^* := -\frac{\sqrt{6}}{72}(-\frac{1}{6} \sqrt{3r^{\frac{1}{3}} - 1296r^{-\frac{1}{3}}} + \frac{1}{2}(-\frac{1}{3}r^{\frac{1}{3}} + 144r^{-\frac{1}{3}} + \frac{18\sqrt{6r^{\frac{1}{3}}}}{\sqrt{3r^{\frac{1}{3}} - 1296}})^{1/2})^3 \]

\[ + \frac{1}{3}(-\frac{1}{6} \sqrt{3r^{\frac{1}{3}} - 1296r^{-\frac{1}{3}}} + \frac{1}{2}(-\frac{1}{3}r^{\frac{1}{3}} + 144r^{-\frac{1}{3}} + \frac{18\sqrt{6r^{\frac{1}{3}}}}{\sqrt{3r^{\frac{1}{3}} - 1296}})^{1/2})^2 - \frac{1}{4} \]

\[ \approx 0.92862, \quad \text{where} \quad r := 729 + 81\sqrt{2369}. \]

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