A note on the computation of Puiseux series solutions of the Riccatti equation associated with a homogeneous linear ordinary differential equation

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Abstract

We present in this paper a detailed note on the computation of Puiseux series solutions of the Riccatti equation associated with a homogeneous linear ordinary differential equation. This paper is a continuation of [1] which was on the complexity of solving arbitrary ordinary polynomial differential equations in terms of Puiseux series.

Introduction

Let \( K = \mathbb{Q}(T_1, \ldots, T_l)[\eta] \) be a finite extension of a finitely generated field over \( \mathbb{Q} \). The variables \( T_1, \ldots, T_l \) are algebraically independent over \( \mathbb{Q} \) and \( \eta \) is an algebraic element over the field \( \mathbb{Q}(T_1, \ldots, T_l) \) with the minimal polynomial \( \phi \in \mathbb{Z}[T_1, \ldots, T_l][Z] \). Let \( \overline{K} \) be an algebraic closure of \( K \) and consider the two fields:

\[
L = \bigcup_{\nu \in \mathbb{N}^*} K((x^{\frac{1}{\nu}})), \quad \mathcal{L} = \bigcup_{\nu \in \mathbb{N}^*} \overline{K}((x^{\frac{1}{\nu}}))
\]

which are the fields of fraction-power series of \( x \) over \( K \) (respectively \( \overline{K} \)), i.e., the fields of Puiseux series of \( x \) with coefficients in \( K \) (respectively \( \overline{K} \)). Each element \( \psi \in L \) (respectively \( \psi \in \mathcal{L} \)) can be represented in the form \( \psi = \sum_{i \in \mathbb{Q}} c_i x^i \), \( c_i \in K \) (respectively \( c_i \in \overline{K} \)). The order of \( \psi \) is defined by \( \text{ord}(\psi) := \min\{i \in \mathbb{Q}, c_i \neq 0\} \). The fields \( L \) and \( \mathcal{L} \) are differential fields with the differentiation

\[
\frac{d}{dx}(\psi) = \sum_{i \in \mathbb{Q}} ic_i x^{i-1}.
\]

Let \( S(y) = 0 \) be a homogeneous linear ordinary differential equation which is written in the form

\[
S(y) = f_n y^{(n)} + \cdots + f_1 y' + f_0 y
\]

where \( f_i \in K[x] \) for all \( 0 \leq i \leq n \) and \( f_n \neq 0 \) (we say that the order of \( S(y) = 0 \) is \( n \)). Let \( y_0, \ldots, y_n \) be new variables algebraically independent over \( K(x) \). We will associate to \( S(y) = 0 \) a non-linear differential polynomial \( R \in K[x][y_0, \ldots, y_n] \) such that \( y \) is a solution of
$S(y) = 0$ if and only if $\frac{y'}{y}$ is a solution of $R(y) = 0$ where the last equation is the ordinary differential equation $R(y, \frac{dy}{dx}, \ldots, \frac{d^{n}y}{dx^{n}}) = 0$. We consider the change of variable $z = \frac{y'}{y}$, i.e., $y' = zy$, we compute the successive derivatives of $y$ and we make them in the equation $S(y) = 0$ to get a non-linear differential equation $R(z) = 0$ which satisfies the above property. $R$ is called the Riccati differential polynomial associated with $S(y) = 0$. We will describe all the fundamental solutions (see e.g. [20, 13]) of the differential equation $R(y) = 0$ in $\mathcal{L}$ by a differential version of the Newton polygon process. There is another way to formulate $R$: let $(r_i)_{i \geq 0}$ be the sequence of the following differential polynomials

$$r_0 = 1, \quad r_1 = y_0, \ldots, \quad r_{i+1} = y_0r_i + Dr_i, \quad \forall i \geq 1,$$

where $Dy_i = y_{i+1}$ for any $0 \leq i \leq n - 1$. We remark that for all $i \geq 1$, $r_i \in \mathbb{Z}[y_0, \ldots, y_{i-1}]$ has total degree equal to $i$ w.r.t. $y_0, y_1, \ldots, y_{i-1}$ and the only term of $r_i$ of degree $i$ is $y_0^i$.

**Lemma 0.1** The non-linear differential polynomial

$$R = f_n r_n + \cdots + f_1 r_1 + f_0 r_0 \in K[x][y_0, \ldots, y_n]$$

is the Riccati differential polynomial associated with $S(y) = 0$.

1 **Newton polygon of $R$**

Let $R$ be the Riccati differential polynomial associated with $S(y) = 0$ as in Lemma 0.1. We will describe the Newton polygon $\mathcal{N}(R)$ of $R$ in the neighborhood of $x = +\infty$ which is defined explicitly in [1]. For every $0 \leq i \leq n$, mark the point $(\deg(f_i), i)$ in the plane $\mathbb{R}^2$. Let $\mathcal{N}$ be the convex hull of these points with the point $(-\infty, 0)$.

**Lemma 1.1** The Newton polygon of $R$ in the neighborhood of $x = +\infty$ is $\mathcal{N}$, i.e., $\mathcal{N}(R) = \mathcal{N}$.

**Proof.** For all $0 \leq i \leq n$, $\deg_{y_0, \ldots, y_{i-1}}(r_i) = i$ and the only term of $r_i$ of degree $i$ is $y_0^i$, then $lc(f_i)x^{\deg(f_i)}y_0^i$ is a term of $R$ and $\mathcal{N} \subset \mathcal{N}(R)$. For any other term of $f_ir_i$ in the form $bx^jy_0^\alpha \cdots y_{i-1}^\alpha$, where $b \in K$, $j < \deg(f_i)$ and $\alpha_0 + \cdots + \alpha_{i-1} < i$, the corresponding point $(j - \alpha_1 - \cdots - (i-1)\alpha_{i-1}, \alpha_0 + \cdots + \alpha_{i-1})$ is in the interior of $\mathcal{N}$. Thus $\mathcal{N}(R) \subset \mathcal{N}$.

**Lemma 1.2** For any edge $e$ of $\mathcal{N}(R)$, the characteristic polynomial of $R$ associated with $e$ is a non-zero polynomial. For any vertex $p$ of $\mathcal{N}(R)$, the indicial polynomial of $R$ associated with $p$ is a non-zero constant. Moreover, if the ordinate of $p$ is $i_0$, then $h_{(R,p)}(\mu) = lc(f_{i_0}) \neq 0$.

**Proof.** By Lemma 1.1 each edge $e \in E(R)$ joints two vertices $(\deg(f_{i_1}), i_1)$ and $(\deg(f_{i_2}), i_2)$ of $\mathcal{N}(R)$. Moreover, the set $N(R, a(e), b(e))$ contains these two points. Then

$$0 \neq h_{(R,e)}(C) = lc(f_{i_1})C^{i_1} + lc(f_{i_2})C^{i_2} + t,$$

where $t$ is a sum of terms of degree different from $i_1$ and $i_2$. For any vertex $p \in V(R)$ of ordinate $i_0$, $lc(f_{i_0})x^{\deg(f_{i_0})}y_0^{i_0}$ is the only term of $R$ whose corresponding point $p$. Then

$$h_{(R,p)}(\mu) = lc(f_{i_0}) \neq 0. \square$$

**Corollary 1.3** For any edge $e \in E(R)$, the set $A_{(R,e)}$ is a finite set. For any vertex $p \in V(R)$, we have $A_{(R,p)} = \emptyset$. 

2
2 Derivatives of the Riccati equation

For each \( i \geq 0 \) and \( k \geq 0 \), the \( k \)-th derivative of \( r_i \) is the differential polynomial defined by

\[
(0)_i := r_i, (1)_i := r'_i := \frac{\partial r_i}{\partial y_0} \quad \text{and} \quad (k+1)_i := (r'_i)^{k+1}_i = \frac{\partial^{k+1} r_i}{\partial y_0^{k+1}}.
\]

**Lemma 2.1** For all \( i \geq 1 \), we have \( r'_i = i r_{i-1} \). Thus for all \( k \geq 0 \), \( r^{(k)}_i = (i)_k r_{i-k} \), where \((i)_0 := 1 \) and \((i)_k := i(i-1) \cdots (i-k+1)\).

**Proof.** We prove the first item by induction on \( i \). For \( i = 1 \), we have \( r'_1 = 1 = 1 \cdot r_0 \). Suppose that this property holds for a certain \( i \) and prove it for \( i + 1 \). Namely,

\[
r'_{i+1} = (y_0 r_i + Dr_i)' = y_0 r'_i + r_i + Dr'_i = i y_0 r_{i-1} + r_i + D(i r_{i-1}) = i(y_0 r_{i-1} + Dr_{i-1}) + r_i = i r_i + r_i = (i + 1)r_i.
\]

The second item is just a result of the first one (by induction on \( k \)). \( \square \)

**Definition 2.2** Let \( R \) be the Riccati differential polynomial associated with \( S(y) = 0 \). For each \( k \geq 0 \), the \( k \)-th derivative of \( R \) is defined by

\[
R^{(k)} := \frac{\partial^k R}{\partial y_0^k} = \sum_{0 \leq i \leq n} f_i r^{(k)}_i.
\]

**Lemma 2.3** For all \( k \geq 0 \), we have

\[
R^{(k)} = \sum_{0 \leq i \leq n-k} (i + k)_k f_{i+k} r_i.
\]

**Proof.** For all \( i < k \), we have \( r^{(k)}_i = 0 \) because \( \deg_{y_0}(r_i) = i \). Then by Lemma 2.1 we get

\[
R^{(k)} = \sum_{k \leq i \leq n} f_i r^{(k)}_i = \sum_{k \leq i \leq n} f_i (i)_k r_{i-k} = \sum_{0 \leq j \leq n-k} (j + k)_k f_{j+k} r_j
\]

with the change \( j = i - k \). \( \square \)

**Corollary 2.4** The \( k \)-th derivative of \( R \) is the Riccati differential polynomial of the following linear ordinary differential equation of order \( n - k \)

\[
S^{(k)}(y) := \sum_{0 \leq i \leq n-k} (i + k)_k f_{i+k} y^{(i)}.
\]

**Proof.** By Lemmas 2.1 and 2.3 \( \square \)
3 Newton polygon of the derivatives of $R$

Let $0 \leq k \leq n$ and $R^{(k)}$ be the $k$-th derivative of $R$. In this subsection, we will describe the Newton polygon of $R^{(k)}$. Recall that $R^{(k)}$ is the $k$-th partial derivative of $R$ w.r.t. $y_{0}$, then by the section 2 of [1], the Newton polygon of $R^{(k)}$ is the translation of that of $R$ defined by the point $(0, -k)$, i.e., $\mathcal{N}(R^{(k)}) = \mathcal{N}(R) + \{(0, -k)\}$. The vertices of $\mathcal{N}(R^{(k)})$ are among the points $(\deg(f_{i+k}), i)$ for $0 \leq i \leq n - k$ by Lemma 2.3 Then for each edge $e_{k}$ of $\mathcal{N}(R^{(k)})$ there are two possibilities: the first one is that $e_{k}$ is parallel to a certain edge $e$ of $\mathcal{N}(R)$, i.e., $e_{k}$ is the translation of $e$ by the point $\{(0, -k)\}$. The second possibility is that the upper vertex of $e_{k}$ is the translation of the upper vertex of a certain edge $e$ of $\mathcal{N}(R)$ and the lower vertex of $e_{k}$ is the translation of a certain point $(\deg(f_{i}), i_{0})$ of $\mathcal{N}(R)$ which does not belong to $e$. In both possibilities, we say that the edge $e$ is associated with the edge $e_{k}$.

**Lemma 3.1** Let $e_{k} \in E(R^{(k)})$ be parallel to an edge $e \in E(R)$. Then the characteristic polynomial of $R^{(k)}$ associated with $e_{k}$ is the $k$-th derivative of that of $R$ associated with $e$, i.e.,

$$H_{(R^{(k)},e_{k})}(C) = H_{(R,e)}^{(k)}(C).$$

**Proof.** The edges $e_{k}$ and $e$ have the same inclination $\mu_{e} = \mu_{e_{k}}$ and $\mathcal{N}(R^{(k)}, e_{k}) = \mathcal{N}(R, e) + \{(0, -k)\}$. Then

$$H_{(R^{(k)},e_{k})}(C) = \sum_{(\deg(f_{i+k}), i) \in \mathcal{N}(R^{(k)}, e_{k})} (i + k)k!c(f_{i+k})C^{i} = \sum_{(\deg(f_{j}), j) \in \mathcal{N}(R, e)} (j)k!c(f_{j})C^{j-k} = H_{(R,e)}^{(k)}(C). \Box$$

**Corollary 3.2** For any edge $e_{k} \in E(R^{(k)})$, the set $A_{(R^{(k)}, e_{k})}$ is a finite set, i.e., $H_{(R^{(k)}, e_{k})}(C)$ is a non-zero polynomial. For any vertex $p_{k} \in V(R^{(k)})$, we have $A_{(R^{(k)}, p_{k})} = \emptyset$.

**Proof.** By Corollaries 2.4 and 1.3 \Box

4 Newton polygon of evaluations of $R$

Let $R$ be the Riccatti differential polynomial associated with $S(y) = 0$. Let $0 \leq c \in \mathbb{R}$, $\mu \in \mathbb{Q}$ and $R_{1}(y) = R(y + cx^{\mu})$. We will describe the Newton polygon of $R_{1}$ for different values of $c$ and $\mu$.

**Lemma 4.1** $R_{1}$ is the Riccatti differential polynomial of the following linear ordinary differential equation of order less or equal than $n$

$$S_{1}(y) := \sum_{0 \leq i \leq n} \frac{1}{i!}R^{(i)}(cx^{\mu})y^{(i)}.$$
Proof. It is equivalent to prove the following analogy of Taylor formula:

\[ R_1 = \sum_{0 \leq i \leq n} \frac{1}{i!} R^{(i)}(c x^\mu) r_i \]

which is proved in Lemma 2.1 of [13]. \(\square\)

Then the vertices of \(\mathcal{N}(R_1)\) are among the points \((\deg(R^{(i)}(c x^\mu), i))\) for \(0 \leq i \leq n\). Thus the Newton polygon of \(R_1\) is given by (Lemma 2.2 of [13]):

**Lemma 4.2** If \(\mu\) is the inclination of an edge \(e\) of \(\mathcal{N}(R)\), then the edges of \(\mathcal{N}(R_1)\) situated above \(e\) are the same as in \(\mathcal{N}(R)\). Moreover, if \(c\) is a root of \(H_{(R,e)}\) of multiplicity \(m > 1\) then \(\mathcal{N}(R_1)\) contains an edge \(e_1\) parallel to \(e\) originating from the same upper vertex as \(e\) where the ordinate of the lower vertex of \(e_1\) equals to \(m\). If \(m = \deg H_{(R,e)}\), then \(\mathcal{N}(R_1)\) contains an edge with inclination less than \(\mu\) originating from the same upper vertex as \(e\).

**Remark 4.3** If we evaluate \(R\) on \(c x^\mu\) we get

\[ R(c x^\mu) = \sum_{0 \leq i \leq n} f_i \times (c^i x^{i \mu} + t), \]

where \(t\) is a sum of terms of degree strictly less than \(i \mu\). Then

\[ \text{lc}(R(c x^\mu)) = \sum_{i \in B} \text{lc}(f_i) c^i = \sum_{(\deg(f_i), i) \in e} \text{lc}(f_i) c^i = H_{(R,e)}(c), \]

where

\[ B := \{0 \leq i \leq n; \deg(f_i) + i \mu = \max_{0 \leq j \leq n} (\deg(f_j) + j \mu; f_j \neq 0)\} \]

\[ = \{0 \leq i \leq n; (\deg(f_i), i) \in e \text{ and } f_i \neq 0\}. \]

**Lemma 4.4** Let \(\mu\) be the inclination of an edge \(e\) of \(\mathcal{N}(R)\) and \(c\) be a root of \(H_{(R,e)}\) of multiplicity \(m > 1\). Then

\[ H_{(R_1,e_1)}(C) = H_{(R,e)}(C + c) \]

where \(e_1\) is the edge of \(\mathcal{N}(R_1)\) given by Lemma 4.2. In addition, if \(e'\) is an edge of \(\mathcal{N}(R_1)\) situated above \(e\) (which is also an edge of \(\mathcal{N}(R)\) by Lemma 4.2) then \(H_{(R_1,e)}(C) = H_{(R,e)}(C)\).

Proof. We have

\[
H_{(R,e)}(C + c) = \sum_{m \leq k \leq n} \frac{1}{k!} H_{(R,e)}^{(k)}(c) C^k
\]

\[
= \sum_{m \leq k \leq n} \frac{1}{k!} H_{(R,e)}^{(k)}(c) C^k
\]

\[
= \sum_{m \leq k \leq n} \frac{1}{k!} \text{lc}(R^{(k)}(c x^\mu)) C^k
\]

\[
= H_{(R_1,e_1)}(C)
\]

where the first equality is just the Taylor formula taking into account that \(c\) is a root of \(H_{(R,e)}\) of multiplicity \(m > 1\). The second equality holds by Lemma 3.1. The third one by Remark 4.3. The fourth one by Lemma 4.1 and by the definition of the characteristic polynomial. \(\square\)
5 Application of Newton-Puiseux algorithm to \( R \)

We apply the Newton-Puiseux algorithm described in [1] to the Riccati differential polynomial \( R \) associated with the linear ordinary differential equation \( S(y) = 0 \). This algorithm constructs a tree \( T = T(R) \) with a root \( \tau_0 \). For each node \( \tau \) of \( T \), it computes a finite field \( K_\tau = K[\theta_\tau] \), elements \( c_\tau \in K_\tau \) \( \mu_\tau \in \mathbb{Q} \cup \{ -\infty, +\infty \} \) and a differential polynomial \( R_\tau \) as above. Let \( \mathcal{U} \) be the set of all the vertices \( \tau \) of \( T \) such that either \( \deg(\tau) = +\infty \) and for the ancestor \( \tau_1 \) of \( \tau \) it holds \( \deg(\tau_1) < +\infty \) or \( \deg(\tau) < +\infty \) and \( \tau \) is a leaf of \( T \). There is a bijective correspondance between \( \mathcal{U} \) and the set of the solutions of \( R(y) = 0 \) in the differential field \( \mathcal{L} \). The following lemma is a differential version of Lemma 2.1 of [4] which separates any two different solutions in \( \mathcal{L} \) of the differential Riccati equation \( R(y) = 0 \).

**Lemma 5.1** Let \( \psi_1, \psi_2 \in \mathcal{L} \) be two different solutions of the differential Riccati equation \( R(y) = 0 \). Then there exist an integer \( \gamma = \gamma_{12}, 1 \leq \gamma < n \), elements \( \xi_1, \xi_2 \in K, \xi_1 \neq \xi_2 \) and a number \( \mu_{12} \in \mathbb{Q} \) such that

\[
\text{ord}(R(\gamma)(\psi_i) - \xi_i x^{\mu_{12}}) < \mu_{12}, \text{ for } i = 1, 2.
\]

**Proof.** By the above bijection, there are two elements \( u_1 \) and \( u_2 \) of \( \mathcal{U} \) which correspond respectively to \( \psi_1 \) and \( \psi_2 \). Let \( i_0 = \max\{i \geq 0; \tau_i(u_1) = \tau_i(u_2) \} \). Denote by \( \tau := \tau_{i_0}(u_1) = \tau_{i_0}(u_2) \) and \( \tau_1 := \tau_{i_0+1}(u_1), \tau_2 := \tau_{i_0+1}(u_2) \). We have \( \tau_1 \neq \tau_2 \) and \( \epsilon := \max(\mu_1, \mu_2) \) is the inclination of a certain edge \( e \) of \( \mathcal{N}(R_\tau) \). There are three possibilities for \( \epsilon \):

- If \( \mu_\tau < \mu_{12} \) then \( \epsilon = \mu_\tau = \mu_2 \). We have \( c_\tau \) is a root of \( H_{(R_\tau,e)} \) of multiplicity \( m_1 \geq 1 \) and \( R_{\tau_1} = R_{\tau}(y + c_\tau x^{\mu_\tau}) \). Then by Lemma [4,2] there is an edge \( e_1 \) of \( \mathcal{N}(R_\tau) \) parallel to \( e \) (so its inclination is \( \epsilon = \mu_\tau \)) originating from the same upper vertex as \( e \) where the ordinate of the lower vertex of \( e_1 \) equals to \( m_1 \). In addition, \( \epsilon \) is also an edge of \( \mathcal{N}(R_{\tau_2}) \) and by Lemma [4,4] we have \( H_{(R_{\tau_2},e_1)}(C) = H_{(R_{\tau_2},e)}(C) \) and

\[
H_{(R_{\tau_1},e_1)}(C) = H_{(R_{\tau},e)}(C + c_\tau). \tag{1}
\]

- If \( \mu_\tau > \mu_{12} \) then \( \epsilon = \mu_{12} \). Then by Lemma [4,2] there is an edge \( e_2 \) of \( \mathcal{N}(R_\tau) \) parallel to \( e \) originating from the same upper vertex as \( e \). By Lemma [4,4] we have \( H_{(R_{\tau_1},e_2)}(C) = H_{(R_{\tau_2},e)}(C) \) and

\[
H_{(R_{\tau_2},e_2)}(C) = H_{(R_{\tau},e)}(C + c_{\tau_2}). \tag{2}
\]

- If \( \mu_\tau = \mu_{12} \) then \( \epsilon \) then \( c_\tau \) and \( c_{\tau_2} \) are two distinct roots of the same polynomial \( H_{(R_{\tau},e)}(C) \). Then equalities of type (1) and (2) hold.

Set \( \gamma = \deg_{C}(H_{(R_{\tau},e)}) - 1 \leq \deg_{y_0,...,y_n}(R) - 1 \leq n - 1 < n \) and \( \gamma \geq 1 \) because that the polynomial \( H_{(R_{\tau},e)}(C) \) has at least two distinct roots \( c_{\tau_1} \) and \( c_{\tau_2} \). Moreover, we have \( \text{ord}(\psi_i - y_{\tau_i}) < \epsilon \) for \( i = 1, 2 \). Let \( \xi_{\tau_1} \in K_{\tau_1} \) and \( \xi_{\tau_2} \in K_{\tau_2} \) be the coefficients of \( C^\gamma \) in the expansion of \( H_{(R_{\tau_1},e_1)}(C) \) and \( H_{(R_{\tau_2},e_2)}(C) \) respectively. There is a point \( (\mu_{12}, \gamma) \) on the edge \( e \) which corresponds to the term of \( H_{(R_{\tau},e)}(C) \) of degree \( \gamma \). We know by Lemma [4,4] that

\[
R(y + \psi_i) = \sum_{0 \leq j \leq n} \frac{1}{j!} R^{(j)}(\psi_i) r_j \text{ for } i = 1, 2.
\]

Then \( \text{ord}(R(\gamma)(\psi_i) - \gamma \xi_{\tau_i} x^{\mu_{12}}) < \mu_{12} \) for \( i = 1, 2 \). This proves the lemma by taking \( \xi_i = \gamma \xi_{\tau_i} \) for \( i = 1, 2 \). □
Let \( \{\Psi_1, \ldots, \Psi_n\} \) be a fundamental system of solutions of the linear differential equation \( S(y) = 0 \) (see e.g. [20, 10, 13]) and \( \psi_1, \ldots, \psi_n \) be their logarithmic derivatives respectively, i.e., \( \psi_1 = \Psi'_1/\Psi_1, \ldots, \psi_n = \Psi'_n/\Psi_n \). Then \( R(\psi_i) = 0 \) for all \( 1 \leq i \leq n \).

**Definition 5.2** Let \( \psi \) be an element of the field \( \mathcal{L} \). We denote by \( \text{span}_r(\psi) \) the \( r \)-differential span of \( \psi \), i.e., \( \text{span}_r(\psi) \) is the \( \mathbb{Z} \)-module generated by \( r_1(\psi), r_2(\psi), \ldots \).

**Lemma 5.3** Let \( \psi \in \mathcal{L} \) be a solution of a Riccatti equation \( R_2(y) = 0 \) where \( R_2 \in \mathbb{Z}[y_0, \ldots, y_n] \) of degree \( n \). Then \( \text{span}_r(\psi) \) is the \( \mathbb{Z} \)-module generated by \( r_1(\psi), \ldots, r_{n-1}(\psi) \).

**Proof.** Write \( R_2 \) in the form \( R_2 = r_n + \alpha_{n-1}r_{n-1} + \cdots + \alpha_1 r_1 + \alpha_0 \) where \( \alpha_i \in \mathbb{Z} \) for all \( 0 \leq i < n \). Then

\[
r_{n+1}(\psi) = \psi r_n(\psi) + Dr_n(\psi) = \sum_{0 \leq i < n} \alpha_i (\psi r_i(\psi) + Dr_i(\psi)) = \sum_{0 \leq i < n} \alpha_i r_{i+1}(\psi) = \sum_{0 \leq i < n} \beta_i r_i(\psi)
\]

for suitable \( \beta_i \in \mathbb{Z} \) using the fact that \( r_n(\psi) = \sum_{0 \leq i < n} \alpha_i r_i(\psi) \). \( \Box \)

Consider a \( \mathbb{Z} \)-module \( M := \text{span}_r(\psi_1, \ldots, \psi_n) \), i.e., \( M \) is the \( \mathbb{Z} \)-module generated by \( r_1(\psi_i), r_2(\psi_i), \ldots \) for all \( 1 \leq i \leq n \). We define now what we call a \( r \)-cyclic vector for \( M \) (this definition is similar to that of the cyclic vectors in [16, 9, 19]).

**Definition 5.4** An element \( m \in M \) is called a \( r \)-cyclic vector for \( M \) if \( M = \text{span}_r(m) \).

The following theorem is called a \( r \)-cyclic vector theorem. It is similar to the cyclic vector theorem of [15, 16, 9, 19].

**Theorem 5.5** Let \( M \) be the \( \mathbb{Z} \)-module defined as above. There is a \( r \)-cyclic vector \( m \) for \( M \).

**Corollary 5.6** Let \( m \in M \) be a \( r \)-cyclic vector for \( M \). Then for any \( 1 \leq i \leq n \), there exists a Riccatti differential polynomial \( R_i \in \mathbb{Z}[y_0, \ldots, y_n] \) such that \( \psi_i = R_i(m) \).

**Lemma 5.7** For each element \( m \in M \), one can compute a Riccatti differential polynomial \( R_m \in K[x][y_0, \ldots, y_n] \) such that \( R_m(m) = 0 \). In addition, there is a positive integer \( s \) such that the order of \( R_m(y) = 0 \) and the degree of \( R_m \) w.r.t. \( y_0, \ldots, y_n \) are \( \leq n^s \).

**Proof.** Each element \( m \in M \) has the form \( m = \alpha_1 \psi_1 + \cdots + \alpha_n \psi_n \) where \( \alpha_1, \ldots, \alpha_n \in \mathbb{Q} \). Then

\[
m = \left( \frac{\Psi_1^{\alpha_1} \cdots \Psi_n^{\alpha_n}}{\Psi_1^{\alpha_1} \cdots \Psi_n^{\alpha_n}} \right)'
\]

is the logarithmic derivative of \( \Psi_1^{\alpha_1} \cdots \Psi_n^{\alpha_n} \). Or Lemma 3.8 (a) of [17] (see also [18, 19]) proves that one can construct a linear differential equation \( S_m(y) = 0 \), denoted by
\[ S^\oplus_{\alpha_1 + \cdots + \alpha_n}(y) = 0 \text{ of order } \leq n^{\alpha_1 + \cdots + \alpha_n} \text{ such that } \Psi_1^{\alpha_1} \cdots \Psi_n^{\alpha_n} \text{ is one of its solutions. The equation } S^\oplus_{\alpha_1 + \cdots + \alpha_n}(y) = 0 \text{ is called the } (\alpha_1 + \cdots + \alpha_n)\text{-th symmetric power of the linear differential equation } S(y) = 0. \] In order to compute the equation \( S_m(y) = 0 \) associated with the linear combination \[ m = \alpha_1 \psi_1 + \cdots + \alpha_n \psi_n \in M, \] we take the change of variable \[ z = y^{\alpha_1 + \cdots + \alpha_n} \] where \( y \) is a solution of \( S(y) = 0 \) and we compute the successive derivatives of \( z \) until we get a linear dependent family over \( K \). The relation between these successive derivatives gives us the linear differential equation \( S_m(z) = 0 \). Let \( R_m \) be the Riccati differential polynomial associated with \( S_m(y) = 0 \), then \( m \) is a solution of the equation \( R_m(y) = 0 \). \( \square \)

**Remark 5.8** For any \( 1 \leq i \leq n \), we can take \( R_{\psi_i} = R \) where \( R_{\psi_i} \) is defined in Lemma 5.7.

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