MOMENTS AND ASYMPTOTICS FOR A CLASS OF SPDES WITH SPACE-TIME WHITE NOISE

LE CHEN, YUHUI GUO, AND JIAN SONG

Abstract. In this article, we consider the nonlinear stochastic partial differential equation of fractional order in both space and time variables with constant initial condition:

\[
\left(\partial_t^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right)u(t, x) = \int_\gamma [\lambda u(t, x)\dot{W}(t, x)] \quad t > 0, \ x \in \mathbb{R}^d,
\]

where \(\dot{W}\) is space-time white noise, \(\alpha > 0, \ \beta \in (0, 2], \ \gamma \geq 0, \ \lambda \neq 0\) and \(\nu > 0\). The existence and uniqueness of solution in the Itô-Skorohod sense is obtained under Dalang’s condition. We obtain explicit formulas for both the second moment and the second moment Lyapunov exponent. We derive the \(p\)-th moment upper bounds and find the matching lower bounds. Our results solve a large class of conjectures regarding the order of the \(p\)-th moment Lyapunov exponents. In particular, by letting \(\beta = 2, \ \alpha = 2, \ \gamma = 0\), and \(d = 1\), we confirm the following standing conjecture for the stochastic wave equation:

\[
t^{-1} \log \mathbb{E}[u(t, x)^p] \asymp p^{3/2}, \quad \text{for } p \geq 2 \text{ as } t \to \infty.
\]

The method for the lower bounds is inspired by a recent work by Hu and Wang [HW21], where the authors focus on the space-time colored Gaussian noise.

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1. Introduction

Let \(\dot{W}\) be a space-time white noise, namely, a centered Gaussian noise with covariance

\[
\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta(t - s)\delta(x - y), \quad (1.1)
\]

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where $\delta(\cdot)$ is the Dirac delta function. The following *stochastic heat equation* 

(SHE) \[
\begin{cases}
\left( \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2} \right) u(t, x) = \lambda u(t, x) \dot{W}(t, x), & t > 0, x \in \mathbb{R}, \\
u u(0, \cdot) = u_0,
\end{cases}
\]  

(1.2)  

and *stochastic wave equation* 

(SWE) \[
\begin{cases}
\left( \frac{\partial^2}{\partial t^2} - \nu \frac{\partial^2}{\partial x^2} \right) u(t, x) = \lambda u(t, x) \dot{W}(t, x), & t > 0, x \in \mathbb{R}, \\
u u(0, \cdot) = u_0, \frac{\partial}{\partial t} u(0, \cdot) = u_1,
\end{cases}
\]  

(1.3)  

with $\lambda \neq 0, \nu > 0, u_0, u_1 \in \mathbb{R}$ are two canonical stochastic partial differential equations. SHE (1.2) has been widely and extensively studied with many fine properties, among which the probability moments enjoy the following explicit asymptotics: 

\[
\lim_{p \to \infty} t^{-3} \log \mathbb{E} [u(t, x)^p] = \frac{\lambda^4}{24}, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R} \text{ and,} 
\]  

(1.4a) 

\[
\lim_{t \to \infty} t^{-1} \log \mathbb{E} [u(t, x)^p] = \frac{1}{24} p(p^2 - 1) \lambda^4, \quad \text{for all } p \geq 2 \text{ and } x \in \mathbb{R}; 
\]  

(1.4b)  

see [Che15] and references therein. One major tool in studying this parabolic equation is the *Feynman-Kac representation* of the moments as being used in *ibid*. Note that the quantity on the right-hand side of (1.4b) is called the $p$-th moment Lyapunov exponent, which characterizes the intermittency property of the solution; see [CM94].

In contrast, much less is known for the hyperbolic counterpart — SWE (1.3). The lack of Feynman-Kac representation for the moments poses a major difficulty in this study. To our best knowledge, only the following upper bound of the $p$-th moment Lyapunov exponent is known: for some constant $C > 0$, 

\[
\limsup_{t \to \infty} t^{-1} \log \mathbb{E} [u(t, x)^p] \leq C p^{3/2} \quad \text{for all } p \geq 2 \text{ and } x \in \mathbb{R}; 
\]  

(1.5)  

see, e.g., [CD15a]. It has been long conjectured that the exponent $3/2$ in (1.5) is sharp; yet there lacks of a rigorous proof. Dalang and Mueller [DM09] studied the three-dimensional SWE with a Gaussian noise that is white-in-time and colored-in-space with a bounded correlation function, namely, 

\[
\mathbb{E} \left[ \dot{W}(t, x) \dot{W}(s, y) \right] = \delta(t-s) f(x-y), 
\]  

(1.6)  

where $f$ is a nonnegative, nonnegative definite and bounded function. Using an earlier developed Feynman-Kac-type formula for moments in [DMT08], they established the following large-time asymptotics: 

\[
C_1 p^{1/3} \leq \liminf_{t \to \infty} \frac{\log \mathbb{E} [u(t, x)^p]}{t} \leq \limsup_{t \to \infty} \frac{\log \mathbb{E} [u(t, x)^p]}{t} \leq C_2 p^{1/3},
\]  

(1.7)  

for all $p \geq 2$ and $x \in \mathbb{R}^3$; see Theorem 1.1 (*ibid*). To obtain the lower bound in (1.7), their arguments crucially depend on the property that one can find a small indicator function below $f(x)$ near the origin, i.e., $c_1 |x|^{3} \leq f(x)$ for all $x \in \mathbb{R}^3$. This requirement prevents the application to the space-time white noise case. Recently, Hu and Wang [HW21] obtained the matching lower and upper $p$-th moment Lyapunov exponents for a wide range of SPDEs with space-time colored Gaussian noise 

\[
\mathbb{E} \left[ \dot{W}(t, x) \dot{W}(s, y) \right] = \gamma(t-s) \Lambda(x-y). 
\]  

(1.8)  

Certain choices or limits of the parameters of the noise in (1.8) may suggest the correct moment asymptotics for SWE (1.3). However, just as the SHE case, the SWE with space-time white noise
needs a separate treatment (see Remark 5.9 for more details). One of the major contributions of this paper is to carry out such arguments and confirm the conjecture about the moment asymptotics of SWE (1.3) by showing that if \( u_0 > 0 \) and \( u_1 \geq 0 \), then
\[
C_1 p^{3/2} \leq \lim_{t \to \infty} \inf_{t \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{t} \leq \lim_{t \to \infty} \sup_{t \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{t} \leq C_2 p^{3/2}, \quad p \geq 2, \tag{1.9a}
\]
\[
C_3 t \leq \lim_{p \to \infty} \inf_{p \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{p^{3/2}} \leq \lim_{p \to \infty} \sup_{p \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{p^{3/2}} \leq C_4 t, \quad t > 0, \tag{1.9b}
\]
where \( C_1, \ldots, C_4 \) are some nonnegative constants that do not depend on \( t \) and \( p \).

It turns out that the method that we use to resolve the above conjecture can be applied to a wider class of stochastic partial differential equations (SPDEs). Indeed, in this paper, we will study the following stochastic fractional diffusion equation with both SHE (1.2) and SWE (1.3) as two special cases:
\[
\begin{cases}
\left( \partial_t^\beta + \frac{\nu}{2} (-\Delta)^{\alpha/2} \right) u(t, x) = I^\beta_1 \left[ \lambda u(t, x) \hat{W}(t, x) \right], \quad t > 0, x \in \mathbb{R}^d, & \\
u \leq 0, & \beta \in (0, 1], \\
u > 0, & \beta \in (1, 2],
\end{cases}
\tag{1.10}
\]
where \( \hat{W} \) is space-time white noise, \((-\Delta)^{\alpha/2} \) is the fractional Laplacian and \( \alpha > 0, \quad \beta \in (0, 2], \quad \gamma > 0, \quad \lambda \neq 0, \quad \nu > 0, \quad u_0, u_1 \in \mathbb{R}. \)

The symbol \( \partial_t^\beta \) denotes the Caputo fractional differential operator of order \( \beta > 0 \):
\[
\partial_t^\beta f(t) := \begin{cases}
\frac{1}{\Gamma(n-\beta)} \int_0^t \frac{f^{(\eta)}(\tau)}{(t-\tau)^{\beta+1-n}} \mathrm{d}\tau, & \text{if } \beta \neq n, \\
\frac{d^n}{dt^n} f(t), & \text{if } \beta = n,
\end{cases}
\]
where \( n = [\beta] \) is the smallest integer that is not smaller than \( \beta \) (i.e., \([\cdot]\) is the ceiling function), and \( \Gamma(x) \) is the gamma function. We use \( I^\beta_1 \) to refer to the Riemann-Liouville integral in the time variable to the right of zero \( I^\beta_0 \); see Definition A.1.

The SPDE (1.10) is interpreted as the following integral equation:
\[
u
u
u
\]
\[
\]
\[
\]
where \( J_0(t, x) \) is the solution to the homogeneous equation (see (3.3) and (3.4) below), \( p(t, x) \) is the underlying fundamental solution (see (3.2)), and the stochastic integral refers to the Walsh or Skorohod integral. Set the following four constants:
\[
\theta := 2(\beta + \gamma) - 2 - \beta d/\alpha, \quad t_p := p^{1+1/(1+\theta)} t, \\
\Theta := (2\pi)^{-d} \int_{\mathbb{R}^d} E_{\beta, \beta+\gamma} (-2^{-1} \nu |\xi|^\alpha) \mathrm{d}\xi, \quad \hat{t} := \Theta \Gamma (\theta + 1) t^{\theta+1},
\tag{1.12}
\]
where the function \( E_{a,b}(z) \) is the two parameter Mittag-Leffler function of two parameters (see, e.g., [KST06, Section 1.8]), i.e., for \( a, b > 0 \),
\[
E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + b)}, \quad z \in \mathbb{C}, \tag{1.13}
\]
We use the convention \( E_a(\cdot) := E_{a,1}(\cdot). \)
We will prove in Theorem 3.3 below that under Dalang’s condition
\begin{equation}
\begin{cases}
d < 2\alpha + \frac{\alpha}{\beta} \min\{2\gamma - 1, 0\}, & \text{if } \beta \in (0, 2), \\
d < \alpha \min\{2, 1 + \gamma\}, & \text{if } \beta = 2,
\end{cases}
\end{equation}
there exists an unique random field solution \( u(t, x) \) with finite \( p \)-th moment for all \( p \geq 2, t > 0 \) and \( x \in \mathbb{R}^d \). It is an easy exercise to check that Dalang’s condition (1.14) implies that \( \theta > -1 \) and \( \Theta < \infty \), so that all constants in (1.12) are well-defined. The aim of this paper is to establish the following theorem, which gives the exact formula for the second moment and the sharp moment asymptotics both in terms of \( t \) and \( p \geq 2 \).

**Theorem 1.1.** Suppose that Dalang’s condition (1.14) is satisfied and let \( u(t, x) \) be the solution to (1.10). Recall that the quantities \( \theta, \Theta, t_p \) and \( t \) are defined in (1.12). The \( p \)-th moment satisfies the following properties:

(a) When \( p = 2 \),
\[
\mathbb{E}[u^2(t, x)] = \begin{cases} 
2u_0^2 E_{\theta+1} (\lambda^2 t) & \text{if } \beta \in (0, 1], \\
2u_0^2 E_{\theta+1} (\lambda^2 t) + 2u_0 u_1 t E_{\theta+1.2} (\lambda^2 t) & \text{if } \beta \in (1, 2], \\
2u_0^2 E_{\theta+1} (\lambda^2 t) + 2u_0^2 t^2 E_{\theta+1.3} (\lambda^2 t) & \text{if } \beta \in (2, 3], \\
\end{cases}
\]
for all \( t > 0 \) and \( x \in \mathbb{R}^d \). As a consequence,
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[u(t, x)^2] = (\lambda^2 \Theta \Gamma(\theta + 1))^{1/(\theta+1)}, \quad \text{for all } x \in \mathbb{R}^d.
\]

(b) For any \( p \geq 2 \),
\[
||u(t, x)||_p^2 = \begin{cases} 
2u_0^2 E_{\theta+1} (8p\lambda^2 t) & \text{if } \beta \in (0, 1], \\
2u_0^2 E_{\theta+1} (8p\lambda^2 t) + 4u_0 u_1 t E_{\theta+1.2} (8p\lambda^2 t) & \text{if } \beta \in (1, 2], \\
2u_0^2 E_{\theta+1} (8p\lambda^2 t) + 4u_0^2 t^2 E_{\theta+1.3} (8p\lambda^2 t) & \text{if } \beta \in (2, 3], \\
\end{cases}
\]
for all \( t > 0 \) and \( x \in \mathbb{R}^d \). As a consequence,
\[
\limsup_{t_p \to \infty} \frac{1}{t_p} \log \mathbb{E}[|u(t, x)|^p] \leq \frac{1}{2} (8\lambda^2 \Theta \Gamma(\theta + 1))^{1/(\theta+1)}. \tag{1.18}
\]
In particular, by freezing \( p > 2 \) or \( t > 0 \), we have the following two asymptotics:
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|u(t, x)|^p] \leq \frac{1}{2} (8\lambda^2 \Theta \Gamma(\theta + 1))^{1/(\theta+1)} \mu^{1+1/(\theta+1)}, \tag{1.19a}
\]
\[
\limsup_{p \to \infty} \frac{-(1+1/(\theta+1))}{(1+1/(\theta+1))} \log \mathbb{E}[|u(t, x)|^p] \leq \frac{1}{2} (8\lambda^2 \Theta \Gamma(\theta + 1))^{1/(\theta+1)} \mu t. \tag{1.19b}
\]

(c) If in addition we have

(1) either \( \beta \in (0, 2) \) and the fundamental function \( p(t, x) \) is nonnegative or \( \alpha = \beta = 2 \) and \( \gamma = 0 \); and

(2) the initial position \( u_0 \) is strictly positive and the initial velocity \( u_1 \) is nonnegative,

then by treating \( t_p \) defined in (1.12) as a function from \( \mathbb{R}_+ \times 2\mathbb{N} \) to \( \mathbb{R}_+ \), for all \( x \in \mathbb{R}^d \),
\[
C := \liminf_{t_p \to \infty} \frac{1}{t_p} \log \mathbb{E}[u(t, x)^p] > 0. \tag{1.20}
\]
In particular, by freezing an even integer \( p \geq 2 \) or \( t > 0 \), with the same constant \( C \) as in (1.20), we have the following two asymptotics:
\[
\liminf_{t \to \infty} t^{-1} \log \mathbb{E} [u(t, x)^p] \geq C t^{1+1/(\theta+1)}, \tag{1.21a}
\]
\[
\liminf_{p \to \infty} p^{-(1+1/(\theta+1))} \log \mathbb{E} [u(t, x)^p] \geq C t. \tag{1.21b}
\]

As applications of Theorem 1.1, in Section 2 we shall revisit some SPDEs which have attracted considerable attention in the literature and calculate sharp moment asymptotics for the solutions.

To conclude the introduction, we highlight some of the contributions of this paper:

1. The conjecture on the moment asymptotics of SWE (1.3) is solved; see (2.13) and Example 2.2.

2. For the solutions of a large class of SPDEs, explicit representations (1.15) for the second moments and (1.16) for the second moment Lyapunov exponents are obtained; the asymptotic behavior of \( p \)-th moments is characterized sharply by the upper bounds (1.18) and the lower bounds (1.20). Regarding the quantities obtained in Theorem 1.1, as will be shown in Section 2, some of them recover known results for SPDEs with some specific parameters \((\alpha, \beta, \gamma, d)\) in the literature, while, to our best knowledge, most of them (in particular the lower bounds for \( p \)-th moments) are new. Moreover, the quantities that characterize the asymptotics of the solutions to (1.10) depend on the parameters \((\alpha, \beta, \gamma, d)\) in an interesting way (see also the figures in Section 2 for an illustration), which may relate to physical phenomena and desire further investigation.

3. For the fundamental solution of (1.10), we extend the results in [CHN19] from \( \alpha \in (0, 2] \) to all \( \alpha > 0 \) (see Appendix C). As a consequence, Dalang’s condition (1.14) allows to consider SPDE (1.10) in high dimension \( d \), if \( \alpha \) is sufficiently big.

The paper is organized as follows: We first list some examples and give some discussions in Section 2. Then in Section 3, we establish the existence and uniqueness of the solution in a slightly more general setting. The second moment formula and the \( p \)-th moment upper bounds are obtained in Section 4; while the lower \( p \)-th moment bounds are derived in Section 5. Some preliminaries about the fractional calculus and Mittag-Leffler functions are given in Appendix A. In Appendix B we prove some technical lemmas used in this paper. Finally, in Appendix C, we derive the fundamental solutions under the settings of \( \alpha > 0, \beta \in (0, 2] \) and \( \gamma \geq 0 \).

Throughout the paper, \( ||\cdot||_p \) denotes the probability \( L^p(\Omega) \)-norm. We use \( B_r(x) \) to denote an open ball centered at \( x \in \mathbb{R}^d \) with radius \( r \), i.e., \( B_r(x) = \{ x \in \mathbb{R}^d : |x| < r \} \), where \( |x| = \sqrt{x_1^2 + \cdots + x_d^2} \). For \( a \in \mathbb{R}, [a] \) (resp. \( \lfloor a \rfloor \)) is the smallest (resp. largest) integer that is not smaller (resp. larger) than \( a \), i.e., the ceiling (reps. floor) function. We use the convention \( \mathbb{N} = \{1, 2, \cdots \} \).

2. Examples and discussions

In this section, we give some concrete examples for the main result Theorem 1.1. We will use \( C_1, \cdots, C_4 \) to denote generic constants that do not depend on \( t \) and \( p \).

**Example 2.1** (SHE). When \( \alpha = 2, \beta = 1, \gamma = 0 \) and \( d = 1 \), equation (1.10) reduces to SHE (1.2). In this case, Dalang’s condition (1.14) is satisfied and
\[
\theta = -1/2, \quad \Theta = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-|\xi|^2} d\xi = \frac{1}{\sqrt{4\pi\nu}}, \quad \hat{t} = \frac{\sqrt{t}}{\sqrt{4\nu}}, \quad \text{and} \quad t_p = p^3 t.
\]
(1) **Second moment formula:** The second moment formula (1.15) reduces to
\[
\mathbb{E}[u^2(t,x)] = u_0^2 E_{1/2} \left( \frac{\lambda^2}{\sqrt{4\nu}t^{1/2}} \right) = 2u_0^2 e^{\frac{\lambda^2}{4\nu}\Phi \left( \frac{\lambda^2t^{1/2}}{\sqrt{2\nu}} \right)},
\] (2.1)
where we have applied (A.5). This formula recovers the one obtained in [CD15b] as a special case; see Corollary 2.5. *ibid.*

(2) **Second moment Lyapunov exponent:** From (2.1), we immediately see that
\[
\lim_{t \to \infty} t^{-1} \log \mathbb{E}[u(t,x)^2] = \frac{\lambda^4}{4\nu}.
\] (2.2)
Results obtained by Balan and Song [BS19] also reduce to this special case with the exact second moment Lyapunov exponent being equal to 1/4 (where \(\lambda = 1\) and \(\nu = 1\)); see Remark 1.6 *ibid.*

(3) **Moment asymptotics:** Because the heat kernel is nonnegative, we can combine the asymptotics in (1.19) and (1.21) to conclude that
\[
C_1 p^3 \leq \lim \inf_{t \to \infty} t^{-1} \log \mathbb{E}[u(t,x)^p] \leq \lim \sup_{t \to \infty} t^{-1} \log \mathbb{E}[u(t,x)^p] \leq C_2 p^3, \quad p \geq 2,
\] (2.3a)
\[
C_3 t \leq \lim \inf_{t \to \infty} p^{-3} \log \mathbb{E}[u(t,x)^p] \leq \lim \sup_{t \to \infty} p^{-3} \log \mathbb{E}[u(t,x)^p] \leq C_4 t, \quad t > 0.
\] (2.3b)
These asymptotics are consistent with the exact asymptotics obtained by X. Chen; see [Che15, Theorem 1.1, Remark 3.1].

**Example 2.2** (SWE). When \(\alpha = 2\), \(\beta = 2\), \(\gamma = 0\) and \(d = 1\), equation (1.10) reduces to SWE (1.3). In this case, \(J_0(t) = u_0 + u_1 t\), Dalang’s condition (1.14) is satisfied, and
\[
\theta = 1, \quad \Theta = \frac{1}{\pi} \int_0^\infty \frac{\sin \left( \frac{\sqrt{\nu/2} \xi}{2} \right)^2}{(\nu/2) \xi^2} d\xi = \frac{1}{\sqrt{2\nu}}, \quad \hat{t} = \frac{t^2}{\sqrt{2\nu}}, \quad \text{and} \quad t_p = p^{3/2} t,
\]
where \(\Theta\) is obtained via Lemma B.3.

(1) **Second moment formula:** The second moment formula (1.15) becomes
\[
\mathbb{E}[u^2(t,x)] = u_0^2 E_2 \left( \frac{\lambda^2}{\sqrt{2\nu}} \right) + 2u_0u_1 t E_{2,2} \left( \frac{\lambda^2}{2\nu} \right) + 2u_1^2 E_{2,3} \left( \frac{\lambda^2}{\sqrt{2\nu}} \right).
\]
Now using (A.6) and the special cases in (A.5), we see that
\[
\mathbb{E}[u^2(t,x)] = -\frac{2^{3/2} \nu^{1/2} u_1^2}{\lambda^2} + \left( u_0^2 + \frac{2^{3/2} \nu^{1/2} u_1^2}{\lambda^2} \right) \cosh \left( \frac{|\lambda| t}{(2\nu)^{1/4}} \right)
+ \frac{2^{5/4} \nu^{1/4} u_0 u_1}{|\lambda|} \sinh \left( \frac{|\lambda| t}{(2\nu)^{1/4}} \right),
\] (2.4)
which recovers [CD15a, Corollary 1.1] \(^1\)

(2) **Second moment Lyapunov exponent:** From (2.4), we immediately see that
\[
\lim_{t \to \infty} t^{-1} \mathbb{E}[[u(t,x)]^2] = \frac{|\lambda|}{(2\nu)^{1/4}},
\] (2.5)
which has also been obtained by Balan and Song in [BS19, Remark 1.6].

\(^1\)There is a typo in the paper [CD15a] where the fundamental solution for the wave kernel should be \(\frac{1}{\lambda} \mathbb{I}_{|\lambda| t,\kappa}(x)\) instead of \(\frac{1}{\lambda} \mathbb{I}_{|\lambda| t,\kappa}(x)\); see the equation after (1.2) *ibid.* If one sets \(\kappa = 1\) *ibid.* or equivalently sets \(\nu = 2\) in the current paper, the results should coincide.
(3) **Moment asymptotics:** Since the fundamental solution is nonnegative, combining the asymptotics in (1.19) and (1.21) shows (1.9). The upper bound in the large-time asymptotics (1.9a) is consistent with [CD15a, Theorem 2.7].

**Example 2.3 (SFHE).** When \( \alpha > 0, \beta = 1, \gamma = 0 \) and \( d = 1 \), equation (1.10) becomes the following one-dimensional stochastic fractional heat equation:

\[
\text{(SFHE)} \quad \begin{cases} 
\left( \frac{\partial}{\partial t} + \frac{\nu}{2} (-\Delta)^{\alpha/2} \right) u(t, x) = \lambda u(t, x) \dot{W}(t, x), & t > 0, x \in \mathbb{R}, \\
u u(0, \cdot) = u_0.
\end{cases}
\]

(2.6)

In this case, Dalang’s condition (1.14) becomes \( \alpha > d = 1 \). For \( \alpha > 1 \), we have

\[
\theta = -\frac{1}{\alpha},
\]

\[
\Theta_{\alpha, \nu} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\nu|\xi|^{\alpha}} d\xi = \frac{\Gamma (1 + 1/\alpha)}{\nu^{1/\alpha} \pi},
\]

\[
\hat{t} = \frac{\Gamma (1 - 1/\alpha) \Gamma (1 + 1/\alpha)}{\nu^{1/\alpha} \pi} t^{1 - 1/\alpha} = \left( \nu^{1/\alpha} \sin (\pi/\alpha) \right)^{-1} t^{1 - 1/\alpha},
\]

\[
t_p = p^{1+\alpha/(\alpha-1)} t,
\]

where in computing \( \hat{t} \) we have used the reflection formula (A.7).

(1) **Second moment formula:** The second moment formula (1.15) reduces to

\[
E[u^2(t, x)] = u_0^2 E_{1-1/\alpha} \left( \frac{\lambda^2}{\nu^{1/\alpha} \sin (\pi/\alpha)} t^{1-1/\alpha} \right).
\]

(2.7)

In [CD15b], this equation with \( \alpha \in (1, 2] \) has been studied with a non-homogeneous initial conditions.

(2) **Second moment Lyapunov exponent:** From (2.7), we immediately see that

\[
\lim_{t \to \infty} t^{-1} E[|u(t, x)|^2] = \left( \frac{\lambda^2}{\nu^{1/\alpha} \sin (\pi/\alpha)} \right)^{\alpha/(\alpha-1)}, \quad \alpha > 1;
\]

(2.8)

see Figure 1 for a plot of this expression as a function of \( \alpha \).

(3) **Moment asymptotics:** If \( \alpha \in (1, 2] \), the fundamental solution is nonnegative (see Remark 5.2), then the asymptotics in (1.19) and (1.21) reduce to

\[
C_1 p^{\frac{\alpha-1}{\alpha}} \leq \liminf_{t \to \infty} \frac{\log E[|u(t, x)|^p]}{t} \leq \limsup_{t \to \infty} \frac{\log E[|u(t, x)|^p]}{t} \leq C_2 p^{\frac{\alpha-1}{\alpha}}, \quad p \geq 2,
\]

(2.9a)

\[
C_3 t \leq \liminf_{p \to \infty} \frac{\log E[|u(t, x)|^p]}{p^{\frac{\alpha-1}{\alpha}}} \leq \limsup_{p \to \infty} \frac{\log E[|u(t, x)|^p]}{p^{\frac{\alpha-1}{\alpha}}} \leq C_4 t, \quad t > 0.
\]

(2.9b)

The upper bound in the large-time asymptotics (2.9a) is consistent with [CD15b, Theorem 3.4]. In [Che+18, Theorem 1.1], Chen et al obtained the exact large-time asymptotics when the noise is colored in the sense of (1.8). Note also that only the lower bounds in (2.9a) and (2.9b) require the nonnegativity of the fundamental solution. The upper bounds still hold true for all \( \alpha > 1 \).

**Example 2.4 (SFWE).** For the stochastic fractional wave equation

\[
\text{(SFWE)} \quad \begin{cases} 
\left( \frac{\partial^2}{\partial t^2} + \frac{\nu}{2} (-\Delta)^{\alpha/2} \right) u(t, x) = \lambda u(t, x) \dot{W}(t, x), & t > 0, x \in \mathbb{R}, \\
u u(0, \cdot) = u_0, \quad \frac{\partial}{\partial t} u(0, \cdot) = u_1,
\end{cases}
\]

(2.10)
Figure 1. Plots of both $\Theta_{\alpha, \nu}$ and the second moment Lyapunov exponents as functions of $\alpha \in (1, \infty)$ with $\nu = \lambda = 1$ for both SFHE in Example 2.3 and SFWE in Example 2.4. For the second moment Lyapunov exponents, two curves intersect at (1.3426, 1.1513) via some numerical solver.

i.e., $\alpha > 0$, $\beta = 2$, $\gamma = 0$ and $d = 1$, Dalang’s condition (1.14) becomes $\alpha > 1$, and the quantities in (1.12) reduce to

$$\theta = 2(1 - 1/\alpha),$$

$$\Theta_{\alpha, \nu} = \frac{1}{\pi} \int_0^\infty \sin^2 \left( \sqrt{\frac{\nu}{2}} \frac{\xi^{\alpha/2}}{\pi^{\alpha/2}} \right) d\xi = \frac{2^{2-1/\alpha} \cos (\pi/\alpha) \Gamma (2(1/\alpha - 1))}{\nu^{1/\alpha} \pi^{1/\alpha}},$$

$$\hat{t} = \frac{2^{2-1/\alpha} \cos (\pi/\alpha) \Gamma (3 - 2/\alpha) \Gamma (2/\alpha - 2) t^{3-2/\alpha}}{\nu^{1/\alpha} \pi^{1/\alpha}} \frac{2^{1-1/\alpha} \nu^{1/\alpha} \sin (2\pi/\alpha) t^{3-2/\alpha}}{\nu^{1/\alpha} \sin (\pi/\alpha)},$$

$$t_p = p^{1+\alpha/(3\alpha-2)} t,$$
where we have applied Lemma B.3 and the reflection formula (A.7) in computing $\Theta$ and $\hat{t}$, respectively.

1) **Second moment formula:** By (1.15), the second moment formula is

$$
\mathbb{E}[u^2(t,x)] = u_0^2 E_{2\alpha} \left( \frac{2^{1-1/\alpha} \lambda^2}{\nu^{1/\alpha} \sin(\pi/\alpha)} t^{3-2/\alpha} \right) + 2 u_0 u_1 t E_{2\alpha,2} \left( \frac{2^{1-1/\alpha} \lambda^2}{\nu^{1/\alpha} \sin(\pi/\alpha)} t^{3-2/\alpha} \right) + 2 u_1^2 t^2 E_{2\alpha,3} \left( \frac{2^{1-1/\alpha} \lambda^2}{\nu^{1/\alpha} \sin(\pi/\alpha)} t^{3-2/\alpha} \right).
$$

(2.11)

2) **Second moment Lyapunov exponent:** From (2.11), we immediately see that

$$
\lim_{t \to \infty} t^{-1} \log \mathbb{E}[u(t,x)^2] = \left( \frac{2^{1-1/\alpha} \lambda^2}{\nu^{1/\alpha} \sin(\pi/\alpha)} \right)^{\alpha/(3\alpha-2)} \alpha/(3\alpha-2), \quad \alpha > 1; \quad (2.12)
$$

see Figure 1 for a plot of this expression as a function of $\alpha$.

3) **Moment asymptotics:** The asymptotics in (1.19) shows that

$$
\limsup_{t \to \infty} t^{-1} \log \mathbb{E}[|u(t,x)|^p] \leq C_2 p \frac{4\alpha-2}{3\alpha-2}, \quad p \geq 2, \quad (2.13a)
$$

$$
\limsup_{p \to \infty} p^{-\frac{4\alpha-2}{3\alpha-2}} \log \mathbb{E}[|u(t,x)|^p] \leq C_4 t, \quad t > 0. \quad (2.13b)
$$

The large-time asymptotics in (2.13a) is consistent with Proposition 4.1 of [SSX20]. Since we don’t know if the fundamental solution is nonnegative, we cannot apply the lower asymptotics in (1.21). To the best of our knowledge, formulas (2.11) and (2.12) and the limit (2.13) are new.

**Example 2.5.** The following one-parameter family of SPDEs

$$
\begin{aligned}
\left\{ \left( \frac{\partial_t^\beta - \nu}{2} \frac{\partial^2}{\partial x^2} \right) u(t,x) = I_t^{[\beta]-\beta} \left[ \lambda u(t,x) \dot{W}(t,x) \right], \quad t > 0, x \in \mathbb{R}, \\
u u(0,\cdot) = u_0, \quad \text{if } \beta \in (0,1], \\
u u(0,\cdot) = u_0, \quad \frac{\partial}{\partial t} u(0,\cdot) = u_1, \quad \text{if } \beta \in (1,2),
\end{aligned}
$$

(2.14)

has been studied in [Che17]. This is the case when $d = 1, \alpha = 2, \beta \in (0,2)$ and $\gamma = [\beta] - \beta$ and the upper bound of the large-time asymptotics was obtained (ibid.). It can be easily checked that Dalang’s condition (1.14) holds true for all $\beta \in (0,2)$ in this case. The quantities in (1.12) reduce to

$$
\theta = 2 ([\beta] - 1) - \beta/2, \quad t_p = p^{1+\frac{[\beta]-2}{2-[\beta]}}, \quad \Theta_{\beta,\nu} = \frac{1}{\pi} \int_0^\infty E_{\beta,[\beta]} \left( \frac{\nu}{2} \xi^2 \right) d\xi, \quad \hat{t} = \Theta_{\beta,\nu} \Gamma(2[\beta]-1-\beta/2) t^{2[\beta]-1-\beta/2},
$$

and hence we have the following:
(1) **Second moment formula:** By (1.15), the second moment formula is
\[
\mathbb{E} [u^2(t, x)] = \begin{cases} 
    u_0^2 E_{1 - \beta/2} \left( \lambda^2 \Theta_{\beta, \nu} \Gamma (1 - \beta/2) t^{1 - \beta/2} \right) & \text{if } \beta \in (0, 1), \\
    u_0^2 E_{3 - \beta/2} \left( \lambda^2 \Theta_{\beta, \nu} \Gamma (3 - \beta/2) t^{3 - \beta/2} \right) & \text{if } \beta \in (1, 2), \\
    + 2u_0 u_1 t E_{3 - \beta/2, 2} \left( \lambda^2 \Theta_{\beta, \nu} \Gamma (3 - \beta/2) t^{3 - \beta/2} \right) & 
\end{cases}
\]  
(2.15)

(2) **Second moment Lyapunov exponent:** From (2.11), we see that
\[
\lim_{t \to \infty} t^{-1} \log \mathbb{E} [u(t, x)]^2 = \left( \lambda^2 \Theta_{\beta, \nu} \Gamma \left( 2 \left[ \beta \right] - 1 - \beta/2 \right) \right)^{4 \beta - 2 - \beta}.
\]  
(2.16)

(3) **Moment asymptotics:** Since the fundamental solution in this case is nonnegative (see Remark 5.2 below), we can combine the asymptotics in both (1.19) and (1.21) to see that
\[
C_1 p^{\frac{2 - \beta}{\beta}} \leq \liminf_{t \to \infty} \frac{\log \mathbb{E} [u(t, x)]^p}{t} \leq \limsup_{t \to \infty} \frac{\log \mathbb{E} [u(t, x)]^p}{t} \leq C_2 p^{\frac{2 - \beta}{\beta}}, \quad p \geq 2, 
\]  
(2.17a)

\[
C_3 t \leq \liminf_{p \to \infty} \frac{\log \mathbb{E} [u(t, x)]^p}{p^{\frac{2 - \beta}{\beta}}} \leq \limsup_{p \to \infty} \frac{\log \mathbb{E} [u(t, x)]^p}{p^{\frac{2 - \beta}{\beta}}} \leq C_4 t, \quad t > 0.
\]  
(2.17b)

The upper bound for the large-time asymptotics in (2.17a) recover the results obtained in [Che17; see Theorems 3.5 and 3.6 (ibid.). In particular, when \( \beta \in (0, 1] \), Mijena and Nane [MN15, Theorem 2] obtained the same upper bound as in (2.17a). Except the upper bound in (2.17a), all the rest results in this example are new.

In Figure 2, we plot the graphs of \( \theta, \Theta_{\beta, \nu}, 1 + 1/(1 + \theta) \), and the second moment Lyapunov exponent as functions of \( \beta \) with \( \lambda \) and \( \nu \) being set to 1 and 2, respectively.

**Example 2.6.** Mijena and Nane [MN15] studied the case when \( \beta \in (0, 1], \alpha \in (0, 2], \gamma = 1 - \beta \), namely,
\[
\begin{cases} 
    \left( \partial_t^\gamma + \frac{\nu}{2} (-\Delta)^\alpha/2 \right) u(t, x) = I_{1 - \beta} \left[ \lambda u(t, x) \hat{W}(t, x) \right], & t > 0, x \in \mathbb{R}^d, \\
    u(0, \cdot) = u_0, 
\end{cases}
\]  
(2.18)

under the condition
\[
d < \alpha \min (2, \beta^{-1}).
\]  
(2.19)

Note that condition (2.19) is the same as (1.14) under this specific setting. In [MN15], the upper bound of the large-time exponent (1.19a) was obtained; see Theorem 2 ibid. Since the fundamental solution in this case is nonnegative (see Remark 5.2), we can apply Theorem 1.1 to have exact formulas for both the second moment and the second moment Lyapunov exponent, and to have matching lower bounds for the moment asymptotics. To be more precise, in this case we have
\[
\theta = -\beta d/\alpha, \quad t_p = p^{\frac{2d - \beta d}{\alpha d}},
\]
\[
\Theta_{\alpha, d, \nu} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} E_\beta \left( -\frac{\nu}{2} \xi^\alpha \right) d\xi, \quad \tilde{t} = \Theta_{\alpha, d, \nu} \Gamma (1 - d/\alpha) t^{1 - d/\alpha}.
\]

and hence we have the following results:

(1) **Second moment formula:**
\[
\mathbb{E} [u^2(t, x)] = u_0^2 E_{1 - \beta d/\alpha} \left( \lambda^2 \Theta_{\alpha, d, \nu} \Gamma (1 - d/\alpha) t^{1 - d/\alpha} \right).
\]  
(2.20)
Figure 2. Plots of the quantities in Example 2.5 with \( \lambda = 1 \) and \( \nu = 2 \). For all these graphs, at the jump point \( \beta = 1 \), one needs to take the left limit.

(2) Second moment Lyapunov exponent:

\[
\lim_{t \to \infty} t^{-1} \log \mathbb{E}[u(t, x)^2] = (\lambda^2 \Theta_{\alpha, \beta, d, \nu} \Gamma (1 - \beta d/\alpha))^{-\frac{\alpha}{\alpha - \beta d}}. \tag{2.21}
\]

(3) Moment asymptotics:

\[
C_1 p^{\frac{2\alpha - \beta d}{\alpha - \beta d}} \leq \liminf_{t \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{t} \leq \limsup_{t \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{t} \leq C_2 p^{\frac{2\alpha - \beta d}{\alpha - \beta d}}, \quad p \geq 2, \tag{2.22a}
\]

\[
C_3 t \leq \liminf_{p \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{p^{\frac{2\alpha - \beta d}{\alpha - \beta d}}} \leq \limsup_{p \to \infty} \frac{\log \mathbb{E}[u(t, x)^p]}{p^{\frac{2\alpha - \beta d}{\alpha - \beta d}}} \leq C_4 t, \quad t > 0. \tag{2.22b}
\]

Note that except the two lower bounds in (2.22a) and (2.22b) require \( \alpha \in (0, 2] \), all the rest formulas/upper bounds in this example hold true for all \( \alpha > 0 \). In particular, this would allow higher dimensions for large \( \alpha \); see (2.19).
Example 2.7. In this example, we study the following one-parameter family of SPDEs with SHE (1.2) (resp. SWE (1.3)) being a special (resp. limiting) case:

\[
\left( \frac{\partial^2_t}{2} - \nu \frac{\partial^2}{\partial x^2} \right) u(t, x) = \lambda u(t, x) \bar{W}(t, x), \quad t > 0, \ x \in \mathbb{R}, \ \beta \in (0, 2),
\]

with the same initial condition as SHE (1.2) (resp. SWE (1.3)) when \( \beta \in (0, 1] \) (resp. \( \beta \in (1, 2) \)). This is the case when \( \alpha = 2, \ \beta \in (0, 2), \ \gamma = 0 \) and \( d = 1 \). Dalang’s condition (1.14) reduces to \( \beta > 2/3 \), and quantities in (1.12) become

\[
\begin{align*}
\theta := -2 + 3\beta/2, \\
\Theta_{\beta, \nu} := \pi^{-1} \int_0^\infty E_{\beta, \beta}^2 (-2^{-1} \nu \xi^2) d\xi, \\
\hat{t} := \Theta_{\beta, \nu} \Gamma (-1 + 3\beta/2) t^{-1+3\beta/2}.
\end{align*}
\]

Note that the fundamental solution is nonnegative (see Remark 5.2). Here we summarize the properties of the solution to (2.23) as follows:

1. **Second moment formula:**

\[
\mathbb{E} [u^2(t, x)] = \left\{ \begin{array}{ll}
u_0^2 E_{-1+3\beta/2} (\lambda^2 \Theta_{\beta, \nu} \Gamma (-1 + 3\beta/2) t^{-1+3\beta/2}) & \text{if} \ \beta \in (0, 1], \\

\nu_0^2 E_{-1+3\beta/2} (\lambda^2 \Theta_{\beta, \nu} \Gamma (-1 + 3\beta/2) t^{-1+3\beta/2}) \\
+ 2u_0 u_1 t E_{-1+3\beta/2, 2} (\lambda^2 \Theta_{\beta, \nu} \Gamma (-1 + 3\beta/2) t^{-1+3\beta/2}) & \text{if} \ \beta \in (1, 2).
\end{array} \right.
\]

2. **Second moment Lyapunov exponent:**

\[
\lim_{t \to \infty} t^{-1} \log \mathbb{E} [u(t, x)^2] = \left( \lambda^2 \Theta_{\beta, \nu} \Gamma (-1 + 3\beta/2) \right)^{2/(3\beta-2)}.
\]

3. **Moment asymptotics:**

\[
C_1 p^{3\beta/2-2} \leq \liminf_{t \to \infty} \frac{\log \mathbb{E} [u(t, x)^p]}{t} \leq \limsup_{t \to \infty} \frac{\log \mathbb{E} [u(t, x)^p]}{t} \leq C_2 p^{3\beta/2-2}, \quad p \geq 2,
\]

\[
C_3 t \leq \liminf_{p \to \infty} \frac{\log \mathbb{E} [u(t, x)^p]}{p^{3\beta/(3\beta-2)}} \leq \limsup_{p \to \infty} \frac{\log \mathbb{E} [u(t, x)^p]}{p^{3\beta/(3\beta-2)}} \leq C_4 t, \quad t > 0.
\]

Thanks to (A.5), all the above quantities when \( \beta \to 2 \) converge to the corresponding ones in Example 2.2 for SWE (1.3); see Figure 3 for some numerical computations.

3. **Existence and uniqueness of the solution for the nonlinear equation**

In this section, we shall establish the well-posedness of (1.10) by working under slightly more general settings as follows. For \( \alpha > 0, \ \beta \in (0, 2], \ \gamma \geq 0, \ \nu > 0, \) and \( \bar{W} \) as in (1.10), consider

\[
\begin{cases}
\left( \frac{\partial^\alpha_t}{\alpha} + \frac{\nu}{2} (-\Delta)^{\alpha/2} \right) u(t, x) = I_t^\alpha \left[ \rho(u(t, x)) \bar{W}(t, x) \right], & t > 0, \ x \in \mathbb{R}^d, \\
\frac{\partial}{\partial t} u(0, x) = u_0(x), & x \in \mathbb{R}^d, \ \text{if} \ \beta \in (0, 1], \\
\frac{\partial}{\partial t} u(0, x) = u_1(x), & x \in \mathbb{R}^d, \ \text{if} \ \beta \in (1, 2],
\end{cases}
\]

where \( \rho(\cdot) \) is Lipschitz continuous and \( u_0, u_1 \in L^\infty(\mathbb{R}^d) \). The fundamental solutions for (3.1), as well as (1.10), consist of three components: \( Z_{\alpha, \beta, \gamma}(t, x) \), \( Z_{\alpha, \beta, \gamma, \delta}(t, x) \) and \( Y_{\alpha, \beta, \gamma, \delta}(t, x) \), which have been studied in [CHN19, Theorem 4.1] for the case when \( \beta \in (0, 2) \) and \( \alpha \in (0, 2] \). The
more general setting, namely, the case when $\alpha > 0$ and $\beta \in (0, 2]$, is proved in Theorem C.1. Throughout the rest of the article, we will write

$$p(t, x) := Y_{\alpha, \beta, \gamma, d}(t, x),$$

whose Fourier transform is given in (C.5).

The solution to the homogeneous equation of (3.1) is given by

$$J_0(t, x) = \begin{cases} \int_{\mathbb{R}^d} Z_{\alpha, \beta, d}(t, x - y)u_0(y)dy, & \text{if } \beta \in (0, 1], \\ \int_{\mathbb{R}^d} Z^*_{\alpha, \beta, d}(t, x - y)u_0(y)dy + \int_{\mathbb{R}^d} Z_{\alpha, \beta, d}(t, x)u_1(y)dy, & \text{if } \beta \in (1, 2]. \end{cases}$$

When the initial conditions $u_0$ and $u_1$ are two constants, then by (C.4) and (C.6), $J_0(t, x)$ does not depend on $x$ and hence is denoted by $J_0(t)$ later on:

$$J_0(t) = \begin{cases} u_0\mathcal{F}Z_{\alpha, \beta, d}(t, \cdot)(0) = u_0, & \text{if } \beta \in (0, 1], \\ u_0\mathcal{F}Z^*_{\alpha, \beta, d}(t, \cdot)(0) + u_1\mathcal{F}Z_{\alpha, \beta, d}(t, \cdot)(0) = u_0 + u_1t, & \text{if } \beta \in (1, 2], \end{cases}$$

where $\mathcal{F}g = \hat{g}$ is the Fourier transform of $g$ in spatial variable, i.e., if $g(t, \cdot) \in L^1(\mathbb{R}^d)$,

$$\mathcal{F}g(t, \xi) = \hat{g}(t, \xi) := \int_{\mathbb{R}^d} g(t, x)e^{-ix\cdot\xi}dx.$$

Let $W = \{W_t(A) : A \in \mathcal{B}_b(\mathbb{R}^d), t \geq 0\}$ be a space–time white noise defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{B}_b(\mathbb{R}^d)$ is the collection of Borel sets with finite Lebesgue measure. Let

$$\mathcal{F}_t = \sigma(W_s(A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}^d)) \cup \mathcal{N}, \quad t \geq 0,$$

be the natural filtration augmented by the $\sigma$-field $\mathcal{N}$ generated by all $\mathbb{P}$-null sets in $\mathcal{F}$.

**Definition 3.1.** A process $u = \{u(t, x) : t > 0, x \in \mathbb{R}^d\}$ is called a random field solution to (3.1) if it is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, jointly measurable with respect to $\mathcal{B}((0, \infty) \times \mathbb{R}^d) \times \mathcal{F}$,
square integrable in the sense that
\[ \int_0^t \int_{\mathbb{R}^d} dy p(t - s, x - y)^2 E \left[ \rho(u(s, y))^2 \right] < \infty, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d, \]
and satisfies the following integral equation a.s.
\[ u(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} p(t - s, x - y) \rho(u(s, y)) W(ds, dy), \quad (3.5) \]
for all \( t > 0 \) and \( x \in \mathbb{R}^d \), where \( J_0(t, x) \) is given by (3.3) and the stochastic integral on the right-hand side is the Walsh integral [Wal86].

The existence and uniqueness of the mild solution to (3.1) with the bounded initial conditions are well covered by the classical Dalang-Walsh theory; see [Wal86; Dal99; Dal+09]. In that theory, Dalang’s condition usually refers to some simplified, but still equivalent, conditions to
\[ \int_0^t \int_{\mathbb{R}^d} p(s, y)^2 dsdy < \infty, \quad \text{for all } t > 0. \quad (3.6) \]
Note that condition (3.6) is the necessary and sufficient condition for the existence and uniqueness of a global solution for the corresponding linear equation, i.e., the case when \( \rho(u) \equiv 1 \) in (3.1). The following lemma finds out the explicit form of Dalang’s condition for (1.10) and (3.1), which extends Lemma 5.3 of [CHN19] from the case \( \alpha \in (0, 2) \) and \( \beta \in (0, 2) \) to the case \( \alpha > 0 \) and \( \beta \in (0, 2) \).

**Lemma 3.2** (Dalang’s condition). For the SPDE (3.1), Dalang’s condition (3.6) is equivalent to (1.14).

**Proof.** By (C.5) and the Parseval–Plancherel identity, we have
\[
\begin{align*}
\int_{\mathbb{R}^d} |p(s, x)|^2 dx &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{p}(s, \xi)|^2 d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} s^{2(\beta+\gamma)-2} E_{\beta,\beta+\gamma}^2(-2^{-1}\nu|\xi|^\alpha s^\beta) d\xi \\
&= s^{2(\beta+\gamma)-2-\beta d/\alpha} \frac{1}{(2\pi)^d} (2^{-1}\nu)^{-d/d} \int_{\mathbb{R}^d} E_{\beta,\beta+\gamma}^2(-|\eta|^\alpha) d\eta, \quad (3.7)
\end{align*}
\]
where in the last step we have used the change of variable \( \xi = (2^{-1}\nu s^\beta)^{-1/\alpha} \eta \). Then clearly, Dalang’s condition (3.6) is equivalent to
\[
\begin{cases}
2(\beta + \gamma) - 2 - \beta d/\alpha > -1, \\
\int_{\mathbb{R}^d} E_{\beta,\beta+\gamma}^2(-|\xi|^\alpha) d\xi < \infty.
\end{cases} \quad (3.8)
\]
To characterize the second condition in (3.8), noting that \( E_{\beta,\beta+\nu}^2(-|\cdot|) \) is locally integrable, it suffices to know the asymptotic behavior of \( E_{\beta,\beta+\gamma}^2(-|\xi|^\alpha) \) as \( |\xi| \to \infty \). By Lemma A.6, as \( |\xi| \to \infty \),
\[
E_{\beta,\beta+\gamma}^2(-|\xi|^\alpha) = \begin{cases}
\frac{1}{\Gamma(\gamma)|\xi|^\alpha} + O(|\xi|^{-2\alpha}), & \beta \in (0, 2), \\
\cos \left( \sqrt{|\xi|^\alpha - \pi(\gamma + 1)/2} \right) + \frac{1}{\Gamma(\gamma)|\xi|^\alpha} + O \left( |\xi|^{-2\alpha} \right), & \beta = 2,
\end{cases} \quad (3.9)
\]
Then, for \( \beta \in (0, 2) \), clearly (3.8) is equivalent to
\[
2(\beta + \gamma) - 2 - \frac{\beta d}{\alpha} > -1 \quad \text{and} \quad 2\alpha > d,
\]
which can be also expressed as \( d < 2\alpha + \frac{\beta}{\beta} \min\{2\gamma - 1, 0\} \). For the case \( \beta = 2 \), by (3.9), we have as \( |\xi| \to \infty \),
\[
E_{\beta, \beta+\gamma}^2 (|\xi|^\alpha) = \frac{\cos^2 \left( \sqrt{|\xi|^\alpha} - \pi (\gamma + 1)/2 \right)}{|\xi|^{\alpha(1+\gamma)}} + \frac{1}{\Gamma^2(\gamma)|\xi|^{2\alpha}} + 2 \frac{\cos \left( \sqrt{|\xi|^\alpha} - \pi (\gamma + 1)/2 \right)}{\Gamma(\gamma)|\xi|^{\alpha(3+\gamma)/2}} + O \left( |\xi|^{-5\alpha/2} \right).
\]
Thus, the second condition in (3.8) is equivalent to, for any \( \varepsilon > 0 \),
\[
\int_{|\xi| > \varepsilon} \frac{\cos^2 \left( \sqrt{|\xi|^\alpha} - \pi (\gamma + 1)/2 \right)}{|\xi|^{\alpha(1+\gamma)}} d\xi < \infty \quad \text{and} \quad \int_{|\xi| > \varepsilon} \frac{1}{|\xi|^{2\alpha}} d\xi < \infty,
\]
where the first condition is equivalent to \( \alpha(1+\gamma) > d \) by Lemma B.1 and the second one is \( 2\alpha > d \). Therefore, when \( \beta = 2 \), we have that (3.8) is equivalent to
\[
\begin{cases}
  d < \alpha(\gamma + \frac{3}{2}), \\
  d < \alpha \min\{2, 1 + \gamma\},
\end{cases}
\]
\[
\iff \quad d < \alpha \min\{2, 1 + \gamma\}.
\]
This completes the proof of Lemma 3.2. \qed

Under Dalang’s condition, it is routine (see, e.g., Theorem 13 of [Dal99] or the proof of Theorem 2.4 of [CD15c]) to establish the following theorem regarding the existence and uniqueness of the solution to (3.1), the proof of which will be left for the interested readers.

**Theorem 3.3.** Under Dalang’s condition (1.14), if the initial conditions are bounded, namely, \( u_0 \) and \( u_1 \in L^\infty(\mathbb{R}^d) \), then there exists a unique (in the sense of versions) random field solution \( u(t, x) \) to (3.1), which is \( L^2(\Omega) \)-continuous with all bounded \( p \)-th moments:
\[
\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[|u(t, x)|^p] < \infty, \quad \text{for all } p \geq 2 \text{ and } T > 0.
\] (3.10)

Before the end of this section, we make some remarks:

**Remark 3.4** (Rough initial data). The main focus of this paper is the exact moment formula with constant initial condition. Theorem 3.3 presents the existence and uniqueness of the solution in a slightly more general setting, which still falls in the classical Dalang-Walsh theory. For the measure-valued initial conditions, such as the Dirac delta initial condition, more efforts are needed and property (3.10) no longer holds; see [CD15c; CD15b; CD15a; CK19; CHN19].

**Remark 3.5** (Hölder regularity). In [CHN19] and [CH22], the space-time Hölder regularity of the solution to (1.10) has been obtained (in the case of \( \alpha \in (0, 2] \) and \( \beta \in (0, 2) \)). It is an interesting open problem to extend the Hölder regularity results in [CHN19] and [CH21] to the more general setting, namely, \( \alpha > 0 \) and \( \beta \in (0, 2] \).

**Remark 3.6** (Second moment comparison for nonlinear SPDEs). Let \( u(t, x) \) be the solution to (3.1) as stated in Theorem 3.3. Suppose that \( \sigma \) is Lipschitz continuous and satisfies the following cone condition with some constants \( 0 \leq \underline{\lambda} \leq \overline{\lambda} \):
\[
\underline{\lambda}|x| \leq |\sigma(x)| \leq \overline{\lambda}|x|, \quad \text{for all } x \in \mathbb{R}.
\]
Then by denoting the right-hand side of (1.15) by \( f_\lambda(t) \), the moment formula in Theorem 1.1 can be extended directly to this case by the following moment comparison principle for the second moment:
\[
f_\lambda(t) \leq \mathbb{E}[u(t, x)^2] \leq f_\overline{\lambda}(t).
\] (3.11)
When the noise is white in time but colored in space (see (1.6)), the moment comparison principle (for \( p \geq 2 \)) or more generally the stochastic comparison principle becomes much more involved and the parabolic nature of the equation will play an important role. Hence, one can in principle only handle the case when \( \beta = 1 \). One may check the work along this line in [CK19; CH19; CK20]. However, for the space-time white noise case, the second moment comparison as in (3.11) comes for free. Note that when the noise is colored in time (see (1.8)), to the best our knowledge, one can only handle the linear case, namely, \( \sigma(u) = \lambda u \). In this case, the moment comparison principle can be easily established by comparing the movements chaos by chaos.

**Remark 3.7** (Wiener chaos expansion). When \( \rho(u) = \lambda u \), instead of using Dalang-Walsh theory, one can equivalently establish the solution to (1.10) using the Wiener chaos expansion specified as follows: Set \( u_0(t, x) = J_0(t) \) and for \( n \geq 1 \),

\[
    u_n(t, x) = J_0(t) + \sum_{k=1}^{n} \lambda^k \int_{[0,t]^{k}} \int_{\mathbb{R}^{2k}} g_k(s_1, \ldots, s_k, x_1, \ldots, x_k; t, x) W(ds_1, dx_1) \cdots W(ds_k, dx_k),
\]

where

\[
    g_k(s_1, \ldots, s_k, x_1, \ldots, x_k; t, x) = p(t - s_k, x - x_k) p(s_k - s_{k-1}, x_k - x_{k-1}) \cdots p(s_2 - s_1, x_2 - x_1) J_0(s_1) \mathbb{I}_{0 < s_1 < \cdots < s_k < t},
\]

(3.12)

where we use the convention \( s_{k+1} = t \) and \( x_{k+1} = x \). Then, the mild solution has the following so-called Wiener chaos representation:

\[
    u(t, x) = J_0(t) + \sum_{k=1}^{\infty} \lambda^k I_k(f_k(\cdot; t, x)),
\]

(3.13)

where \( f_k(\cdot; t, x) \) is the symmetrization of \( g_k(\cdot; t, x) \) given by, denoting by \( \mathcal{P}_k \) the set of all permutations of \( \{1, \ldots, k\} \),

\[
    f_k(s_1, \ldots, s_k, x_1, \ldots, x_k; t, x) = \frac{1}{k!} \sum_{\sigma \in \mathcal{P}_k} g_k(s_{\sigma(1)}, \ldots, s_{\sigma(k)}, x_{\sigma(1)}, \ldots, x_{\sigma(k)}; t, x)
\]

and \( I_k(f_k(\cdot; t, x)) \) denotes the \( k \)-th multiple Wiener-Itô integral. We refer the interested readers to [Hu17] for more details.

**4. Second moment formula and upper bounds for the \( p \)-th moments**

In this section, we shall prove parts (a) and (b) of Theorem 1.1.

**Proof of part (a) of Theorem 1.1.** By the Itô-Walsh isometry, we have

\[
    \mathbb{E}[u^2(t, x)] = J_0^2(t) + \lambda^2 \int_0^t \int_{\mathbb{R}^d} p^2(t - s, x - y) \mathbb{E}[u^2(s, y)] \, ds \, dy,
\]

where \( J_0(t) \) is given by (3.4). Note that due to the choice of the constant initial conditions, the solution to the homogeneous equation does not depend on \( x \), i.e., \( J_0(t, x) = J_0(t) \). Hence, through a standard Picard iteration, one can show that the second moment \( \mathbb{E}(u(t, x)^2) \) does not depend on \( x \). Let \( \eta(t) = \mathbb{E}(u(t, x)^2) \). Invoking (3.7), the above equation can be written as

\[
    \eta(t) = J_0^2(t) + \lambda^2 \Theta \int_0^t (t - s)^\beta \eta(s) \, ds,
\]

(4.1)

where \( \theta \) and \( \Theta \) are given in (1.12). Now we solve the fractional integral equation (4.1) for \( \beta \in (0, 1] \) and for \( \beta \in (1, 2] \) separately.
Case 1. When \( \beta \in (0, 1] \), we have \( J_0(t) = u_0 \) by (C.3) and thus (4.1) is equivalent to

\[
\begin{align*}
(D^\theta_{0+} \eta) (t) &= \lambda^2 \Theta \Gamma (\theta + 1) \eta(t) + (D^\theta_{0+} u_0^2)(t), \\
\eta(0) &= u_0^2 \quad \text{and} \quad \eta^{(k)}(0) = 0, \text{ for } k = 1, 2, \ldots, [\theta],
\end{align*}
\]

where \( D^\theta_{0+} \) is Riemann-Liouville derivative given in Definition A.2. Using the Caputo fractional derivative given in Definition A.5, it can written as

\[
\begin{align*}
(CD^\theta_{0+} \eta) (t) &= \lambda^2 \Theta \Gamma (\theta + 1) \eta(t), \\
\eta(0) &= u_0^2 \quad \text{and} \quad \eta^{(k)}(0) = 0, \text{ for } k = 1, 2, \ldots, [\theta],
\end{align*}
\]

of which the solution is directly given by (A.4):

\[
\eta(t) = u_0^2 E_{\theta,1} \left( \lambda^2 t^\theta \right).
\]

This proves the first part of (1.15).

Case 2. When \( \beta \in (1, 2] \), we have \( J_0(t) = u_0 + u_1 t \) by (C.3) and (4.1) now is

\[
\eta(t) = u_0^2 + 2u_0 u_1 t + u_1^2 t^2 + \lambda^2 \Theta \int_0^t (t-s)^{\theta} \eta(s)ds. \tag{4.2}
\]

Let \( f(t) := 2u_0 u_1 t + u_1^2 t^2 \) then (4.1) can be written as

\[
\begin{align*}
(D^\theta_{0+} \eta)(t) &= \lambda^2 \Theta \Gamma (\theta + 1) \eta(t) + (D^\theta_{0+}[u_0^2 + f(t)])(t), \\
\eta(0) &= u_0^2, \quad \eta^{(1)}(0) = 2u_0 u_1, \quad \eta^{(2)}(0) = 2u_1^2, \\
\eta^{(k)}(0) &= 0, \text{ for } k = 3, \ldots, [\theta].
\end{align*}
\]  

In order to apply the formula (A.4), we will transform (4.3) into a Caputo fractional differential equation. When \( \theta + 1 \in (0, 1) \), by (A.2), we can write (4.3) as

\[
\begin{align*}
(CD^\theta_{0+} \eta)(t) &= \lambda^2 \Theta \Gamma (\theta + 1) \eta(t) + (D^\theta_{0+} f)(t), \\
\eta(0) &= u_0^2.
\end{align*}
\]

The solution now follows directly from (A.4):

\[
\eta(t) = u_0^2 E_{\theta,1} \left( \lambda^2 t^\theta \right) \\
+ \int_0^t (t-s)^{\theta} E_{\theta+1,\theta+1} \left( \lambda^2 \Theta \Gamma (\theta + 1)(t-s)^{\theta+1} \right) (D^\theta_{0+} f)(s)ds. \tag{4.4}
\]

For the integral on the right-hand side, by (1.13) and Lemma A.3 we have

\[
\begin{align*}
&\int_0^t (t-s)^{\theta} E_{\theta+1,\theta+1} \left( \lambda^2 \Theta \Gamma (\theta + 1)(t-s)^{\theta+1} \right) (D^\theta_{0+} f)(s)ds \\
&= \int_0^t (t-s)^{\theta} \sum_{k=0}^{\infty} \frac{(\lambda^2 \Theta \Gamma (\theta + 1))^k}{\Gamma((k+1)(\theta + 1))} (t-s)^k(t-s)^{\theta}(D^\theta_{0+} f)(s)ds \\
&= \sum_{k=0}^{\infty} (\lambda^2 \Theta \Gamma (\theta + 1))^k \left( D^\theta_{0+} f \right)^{(k)}(t) \\
&= \sum_{k=0}^{\infty} (\lambda^2 \Theta \Gamma (\theta + 1))^k \left( k^{\theta+1} D^\theta_{0+} f \right)^{(k)}(t).
\end{align*}
\]
The term \( \left( \int_{0+}^{k(\theta+1)} f \right) (t) = \left( \int_{0+}^{k(\theta+1)} (2u_0u_1s + u_1^2s^2) \right) (t) \) can be computed explicitly noting that Lemma A.4 yields
\[
\left( \int_{0+}^{k(\theta+1)} s \right) (t) = \frac{t^{k(\theta+1)+1}}{\Gamma(k(\theta + 1) + 2)} \quad \text{and} \quad \left( \int_{0+}^{k(\theta+1)} s^2 \right) (t) = \frac{2t^{k(\theta+1)+2}}{\Gamma(k(\theta + 1) + 3)}.
\]
Combining (4.4)–(4.5) and applying (1.13), we arrive at
\[
\eta(t) = u_0^2E_{\theta+1} (\lambda^2t) + 2u_0u_1tE_{\theta+1,2} (\lambda^2t) + 2u_1^2t^2E_{\theta+1,3} (\lambda^2t) .
\]
This proves the second part of (1.15) for \( \theta + 1 \in (0,1) \). For the other two cases \( \theta + 1 \in [1,2) \) and \( \theta + 1 \geq 2 \), one can calculate in a similar way and prove the desired result. Finally, (1.16) is a direct consequence of Lemma A.6. This completes the proof of part (a) of Theorem 1.1. \( \square \)

**Remark 4.1** (Alternative approach). Alternatively, one can also solve (4.1) directly by an application of Lemma B.2 as follows:
\[
\eta(t) = J_0^2 (t) + \int_0^t J_0^2 (s) K(t - s) ds,
\]
where \( J_0(t) \) is given in (C.3) and the resolvent kernel function \( K (\cdot) \) is given by
\[
K(t) = \lambda^2\Theta \Gamma(\theta + 1)t^\theta E_{\theta+1,\theta+1} (\lambda^2t) .
\]
Thus we have, denoting \( A = \lambda^2\Theta \Gamma(\theta + 1) \),
\[
\eta(t) = J_0^2 (t) + A \int_0^t J_0^2 (s)(t - s)^\theta E_{\theta+1,\theta+1} (A(t - s)^{\theta+1}) ds.
\]
When \( J_0(t) = u_0 \), we have by the definition (1.13) of \( E_{a,b} \),
\[
\eta(t) = u_0^2 + u_0^2 \sum_{k=0}^{\infty} \frac{A^{k+1}}{\Gamma((\theta + 1)(k + 1))} \int_0^t (t - s)^{\theta+1}k^\theta ds
\]
\[
= u_0^2 + u_0^2 \sum_{k=0}^{\infty} \frac{A^{k+1}^{(\theta+1)(k+1)}}{\Gamma((\theta + 1)(k + 1) + 1)} = u_0^2 + u_0^2 \sum_{k=1}^{\infty} \frac{A^{k+1}^{(\theta+1)(k+1)}}{\Gamma((\theta + 1)(k + 1))}
\]
\[
= u_0^2E_{\theta+1} (\lambda^2t) .
\]
This proves the equality of (1.15) for \( \beta \in (0,1] \). Applying similar computations to the case \( J_0(t) = u_0 + u_1 t \), we can justify the second part of (1.15) for \( \beta \in (1,2] \). Indeed, the \( p \)-th moment upper bounds will be obtained using this approach in the next proof.

**Proof of part (b) of Theorem 1.1.** Fix an arbitrary \( p \geq 2 \). By (3.5) we have
\[
\| u(t,x) \|_p \leq |J_0(t)| + \left( \mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{R}^d} \rho(t - s, x - y)\lambda u(s,y)W(ds,dy) \right)^p \right] \right)^{1/p} .
\]
Applying the Burkholder–Davis–Gundy inequality, we have
\[
\| u(t,x) \|_p \leq |J_0(t)| + C_p \left( \mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{R}^d} \rho^2(t - s, x - y)\lambda^2 u^2(s,y)dsdy \right)^{p/2} \right] \right)^{1/p} ,
\]
where \( C_p \) is the universal constant in the Burkholder–Davis–Gundy inequality satisfying \( C_p \in (0,2\sqrt{p}) \) and \( C_p = (2 + o(1))\sqrt{p} \) as \( p \rightarrow \infty \) (see [CK91; CK12]). By Minkowski’s inequality, we get
\[
\| u(t,x) \|_p \leq |J_0(t)| + 2\sqrt{p} \left( \int_0^t \int_{\mathbb{R}^d} \lambda^2p^2(t - s, x - y)\| u(s,y) \|_p^2 dsdy \right)^{1/2} .
\]
Denote $\psi(t) = \sup_{x \in \mathbb{R}^d} ||u(t, x)||_p^2$ and recall the definition (1.12) of $\theta$ and $\Theta$. We have
\[
\psi(t) \leq 2 \left( J_0^2(t) + 4 \rho \lambda^2 \Theta \int_0^t (t - s)^\theta \psi(s) ds \right).
\]
Applying Lemma B.2, we have
\[
\psi(t) \leq 2 J_0^2(t) + 2 \int_0^t J_0^2(s) K(t - s) ds,
\]
where
\[
K(t) = 8 \rho \lambda^2 \Theta \Gamma(\theta + 1) t^\theta \mathcal{E}_{\theta+1, \beta+1} \left( 8 \rho \lambda^2 t \right).
\]
Then, one can apply the same computations as those in Remark 4.1 to simplify the above ds integral in order to obtain (1.17). Finally, (1.19a) and (1.19b) follow from Lemma A.6 directly. This proves part (b) of Theorem 1.1.

5. Lower bounds for the $p$-th moments

Compared with the upper bound for the $p$-th moment, the computation for the lower bound is more involved. The methodology used in this section is inspired by the recent work of Hu and Wang [HW21]. Some ideas are originated from Dalang and Mueller [DM09].

5.1. Nondegeneracy and positivity of the fundamental functions. In the next proposition, we prove a nondegeneracy property of the fundamental solutions, which is tailored specially for the spatial white noise. Conditions for the fundamental solutions to be nonnegative are given in Remark 5.2 below.

Proposition 5.1. For all $\varepsilon > 0$ and $c > 2$, if either

1. the fundamental solution $p(\cdot, \varphi)$ is nonnegative and $\beta \in (0, 2)$, or
2. $\alpha = \beta = 2$, $\gamma = 0$, and $d = 1, 2$,

then there exists some constant $C > 0$ independent of $\varepsilon$ such that
\[
\int \int_{B^2(\varepsilon)} p(t, a - y)p(s, b - y')\delta_0(y - y') dy dy' \geq C \varepsilon^{-d}(ts)^{\beta + \gamma - 1} \tag{5.1}
\]
for all $x \in \mathbb{R}^d$, $s, t \in [2e^{\pi}, c \varepsilon^{\pi}]$, and $a, b \in B_{\varepsilon}(x)$.

Proof. Denote the double integral in (5.1) by $I$. We first work under condition (1). In this case, from (C.8), we see that
\[
I = \int_{B_{\varepsilon}(x)} p(t, a - x')p(s, b - x') dx' = \int_{B_{\varepsilon}(x)} \pi^{-d/2}|x' - a|^{-d} t^{\beta + \gamma - 1} h \left( \frac{|x' - a|^\alpha}{2^{\alpha - 1} \nu \epsilon} \right) \times \pi^{-d/2}|x' - b|^{-d} s^{\beta + \gamma - 1} h \left( \frac{|x' - b|^\alpha}{2^{\alpha - 1} \nu \epsilon} \right) dx',
\]
where
\[
h(x) := H_{2, 3}^{1.1} \left( x \mid (1, 1), (\beta + \gamma, \beta); (d/2, \alpha/2), (1, 1), (1, \alpha/2) \right).
\]
Set $I' := \pi^{-d}(ts)^{\beta + \gamma - 1} I$. Notice when $\beta \in (0, 2)$, the fundamental solution $p(t, x)$ is a smooth function for $t > 0$ and $x \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ and the support of $p(t, x)$ is the whole space. Moreover,
under the nonnegative assumption, for $s, t \in [2e^{\mu}, e^{\mu}]$, we have
\[
I' \geq \inf_{c_1, c_2 \in [2e, e]} \int_{B_{r}(x)} |x' - a|^{-d} h \left( \frac{|x' - a|^\alpha}{2^{\alpha - 1} \nu c_1^\beta} \right) \times |x' - b|^{-d} h \left( \frac{|x' - b|^\beta}{2^{\beta - 1} \nu c_2^\alpha} \right) \, dx' \\
= e^{-d} \inf_{c_1, c_2 \in [2e, e]} \int_{B_{r}(x)} |x' - a/e|^d \ h \left( \frac{|x' - a/e|^\alpha}{2^{\alpha - 1} \nu c_1^\beta} \right) \times |x' - b/e|^{-d} h \left( \frac{|x' - b/e|^\beta}{2^{\beta - 1} \nu c_2^\alpha} \right) \, dx' \\
\geq C e^{-d},
\]
where
\[
C = \inf_{a', b' \in B_{r}(x)} \int_{B_{r}(x)} |x' - a'|^{-d} h \left( \frac{|x' - a'|^\alpha}{2^{\alpha - 1} \nu c_1^\beta} \right) \times |x' - b'|^{-d} h \left( \frac{|x' - b'|^\beta}{2^{\beta - 1} \nu c_2^\alpha} \right) \, dx' > 0.
\]
This proves (5.1) under condition (1).

Now we assume condition (2). It suffices to show the case when $\nu = 2$. It is well known that when $\alpha = \beta = \nu = 2$ and $\gamma = 0$,
\[
p(t, x) = \begin{cases} 
\frac{1}{2} \mathbb{I}_{\{|x| < t\}}, & \text{if } d = 1, \\
\frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} \mathbb{I}_{\{|x| < t\}}, & \text{if } d = 2.
\end{cases}
\]
For all $a, b, x' \in B_{r}(x)$ and $s, t \geq 2e$, we have $\mathbb{I}_{\{|x' - a| < t\}} \mathbb{I}_{\{|x' - b| < s\}} \equiv 1$. Hence, when $d = 1$,
\[
I = \int_{B_{r}(x)} \frac{1}{4} \mathbb{I}_{\{|x' - a| < t\}} \mathbb{I}_{\{|x' - b| < s\}} \, dx' = \frac{1}{2} \varepsilon \geq \frac{\varepsilon^{-1}}{2e^2} ts,
\]
where the inequality is due to the fact that $t, s \leq c \varepsilon$. Similarly, when $d = 2$,
\[
I = \int_{B_{r}(x)} \frac{1}{4\pi^2} \frac{1}{\sqrt{t^2 - |x' - a|^2}} \frac{1}{\sqrt{s^2 - |x' - b|^2}} \, dx' \geq \frac{1}{4\pi^2} ts \int_{B_{r}(x)} \, dx' \geq \frac{\varepsilon^2}{4\pi ts} \geq \frac{\varepsilon^{-2}}{4\pi e^4} ts.
\]
This completes the proof of Proposition 5.1. \qed

**Remark 5.2 (Nonnegativity of fundamental solutions).** The nonnegativity of Green’s functions associated with (1.10) was first proved in [Che+17] for the case $\gamma = 0$, and was later extended in [CHN19, Theorem 4.6] to allow $\gamma \geq 0$; see also Remark 1.2 of [CE22]. It is known that the Green’s function is nonnegative in the following three cases:

\[
\begin{align*}
(1) & \quad \alpha \in (0, 2], \beta \in (0, 1], \gamma \geq 0, d \geq 1; \\
(2) & \quad 1 < \beta < \alpha \leq 2, \gamma > 0, 1 \leq d \leq 3; \\
(3) & \quad 1 < \beta = \alpha < 2, \gamma > \frac{d+3}{2} - \beta, 1 \leq d \leq 3.
\end{align*}
\]

### 5.2. Feynman Diagram Formula

In this part, we recall the Feynman Diagram formula, which is useful to compute the expectation of products of multiple Wiener-Itô integrals. We refer interested readers to Section 5.3 of [Hu17] for more details about the multiple Wiener-Itô integrals.

On the lattice $\mathbb{Z}^2$, we use $(k, \ell)$ to denote a vertex, and an ordered pair $[(k_1, \ell_1), (k_2, \ell_2)]$ to denote a directed edge pointing from $(k_1, \ell_1)$ to $(k_2, \ell_2)$.
**Definition 5.3.** Let \( p \geq 1 \) and \( \vec{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p \) with \( |\vec{n}| = n_1 + \cdots + n_p \) be given. A **Feynman diagram** is a directed graph \(\mathcal{D} = (V, E)\) consisting of the set of all vertices

\[
V = \left\{ (k, \ell) : 1 \leq k \leq p, 1 \leq \ell \leq n_k \right\}
\]

and a set \( E \) of directed edges satisfying \( k_1 < k_2 \) if \([(k_1, \ell_1), (k_2, \ell_2)] \in E\). A Feynman diagram \(\mathcal{D} = (V, E)\) is called **admissible** if each vertex is associated with one and only one edge. The set of all admissible diagrams is denoted by \(\mathbb{D} = \mathbb{D}_{\vec{n}}\).

![Feynman diagram](image)

**Figure 4.** Two admissible (red \(\mathcal{D}_1\) and blue \(\mathcal{D}_2\)) Feynman diagrams for the case when \( p = 4 \), \( \vec{n} = (1, 2, 2, 3) \) and \( |\vec{n}| = 8 \); see Example 5.5. Convention (5.7) applies at the gray vertices in the settings of Lemma 5.4.

We shall provide a formula for \( \mathbb{E}[I_{n_1}(h_1) \cdots I_{n_p}(h_p)] \) for square integrable functions

\[
h_i : \left( \mathbb{R}_+ \times \mathbb{R}^d \right)^{n_i} \to \mathbb{R}, \quad i = 1, \cdots, p,
\]

(5.3)

where \( I_{n_i}(h_i) \) refers to the \( n_i \)-th multiple Wiener-Itô integral. In particular, Given an admissible Feynman diagram \(\mathcal{D} \in \mathbb{D}_{\vec{n}}\) for \( h_i \) given in (5.3), denote

\[
F_{\mathcal{D}}(h_1, \ldots, h_p) = \int_{\mathbb{R}_+^{|\vec{n}|}} \int_{\mathbb{R}^{d|\vec{n}|}} dtdx \prod_{i=1}^p h_i \left( t(i, 1), x(i, 1); \ldots; t(i, n_i), x(i, n_i) \right)
\]

\[
\times \prod_{[(k_1, \ell_1), (k_2, \ell_2)] \in E(\mathcal{D})} \delta(t(k_1, \ell_1) - t(k_2, \ell_2)) \delta(x(k_1, \ell_1) - x(k_2, \ell_2)),
\]

(5.4)

where we use the notations and \( dtdx = \prod_{i=1}^p \prod_{n_i=1}^{n_i} dt(i, r_i) dx(i, r_i) \). Then we have (see [HW21, Theorem 5.3] and [Hu17, Theorems 5.7 and 5.8]),

\[
\mathbb{E}[I_{n_1}(h_1) \cdots I_{n_p}(h_p)] = \sum_{\mathcal{D} \in \mathbb{D}_{\vec{n}}} F_{\mathcal{D}}(h_1, \ldots, h_p).
\]

(5.5)

In particular, for any \( t > 0 \) and \( x \in \mathbb{R}^d \), considering the multiple Wiener-Itô integrals \( I_k (f_k (\cdot; t, x)) \) in the chaos expansion (3.13) for the solution \( u(t, x) \) with \( f_k \) given in (3.14) which is a symmetrization of \( g_k \) in (3.12), we have the following result (see [HW21, Theorem 5.4]):
Lemma 5.4. Let $p \geq 1$ and $\vec{n} = (n_1, \cdots, n_p) \in \mathbb{N}^p$ be given. Fix arbitrary $t > 0$ and $x_1, \cdots, x_p \in \mathbb{R}^d$. Recall that $f_n(\cdot; t, x)$ and $g_n(\cdot; t, x)$ be given in (3.14) and (3.12), respectively. Then

$$
\mathbb{E} \left[ \prod_{\ell=1}^{p} f_{n,\ell}(f_{n,\ell}(\cdot; t, x_\ell)) \right] = \sum_{D \in \mathcal{D}} F_{D} \left( g_{n_1}(\cdot; t, x_1), \ldots, g_{n_p}(\cdot; t, x_p) \right)
$$

$$
= \sum_{D \in \mathcal{D}} \int_{[0,t]\times[0,t]} dt dx \left( \prod_{(l(k_1,l_1),(k_2,l_2)) \in E(D)} \delta(t(k_1,l_1) - t(k_2,l_2)) \delta(x(k_1,l_1) - x(k_2,l_2)) \right)
\times \left( \prod_{i=1}^{p} J_0(t_{(i,1)}) \mathbb{I}_{0>t_{(i,1)}<t} p(t_{(i,r_i+1)} - t_{(i,r_i)}; x_{(i,r_i+1)} - x_{(i,r_i)}) \right), \tag{5.6}
$$

where we have used the convention that

$$
(t_{(i,n_i+1)}, x_{(i,n_i+1)}) = (t, x_i) \quad \text{for all } i = 1, \cdots, p. \tag{5.7}
$$

Example 5.5. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) refer to the red (resp. blue) admissible Feynman diagram in Figure 4. Under the setting of Lemma 5.4, let

$$
F_1 = F_{\mathcal{D}_1}, \quad (g_1(\cdot; t, x), g_2(\cdot; t, x), g_2(\cdot; t, x), g_3(\cdot; t, x)) \quad i = 1, 2.
$$

Then we claim that $F_1 = 0$ because its integrand contains the following factor:

$$
\mathbb{I}_{0<t_{(3,2)}<t_{(3,1)}<t} \mathbb{I}_{0<t_{(3,2)}<t_{(3,1)}<t}
$$

which is identically equal to zero. Hence, due to the delta potentials and the simplex conditions in (5.6), edges starting from one column should not cross with each other. This is the case for $F_2$:

$$
F_2 = \int_{[0,t]^4} dt_{(1,1)} dt_{(2,1)} dt_{(2,2)} dt_{(3,2)} \int_{\mathbb{R}^d} dx_{(1,1)} dx_{(2,1)} dx_{(2,2)} dx_{(3,2)}
\times J_0(t_{(1,1)}) \mathbb{I}_{0<t_{(1,1)}<t} p t_{(1,1)} - t_{(1,1)} \left( x - x_{(1,1)} \right)
\times J_0(t_{(2,1)}) \mathbb{I}_{0<t_{(2,1)}<t} p t_{(2,2)} - t_{(2,1)} \left( x - x_{(2,1)} \right)
\times J_0(t_{(1,1)}) \mathbb{I}_{0<t_{(1,1)}<t} p t_{(3,2)} - t_{(1,1)} \left( x - x_{(3,1)} \right)
\times J_0(t_{(2,1)}) \mathbb{I}_{0<t_{(2,1)}<t} p t_{(3,2)} - t_{(2,1)} \left( x - x_{(3,1)} \right)
\times J_0(t_{(1,1)}) \mathbb{I}_{0<t_{(1,1)}<t} p t_{(3,2)} - t_{(1,1)} \left( x - x_{(3,1)} \right)
\times J_0(t_{(2,1)}) \mathbb{I}_{0<t_{(2,1)}<t} p t_{(3,2)} - t_{(2,1)} \left( x - x_{(3,1)} \right)
\times p_{1-t_{(3,2)}} \left( x - x_{(3,2)} \right) p_{t_{(3,2)}-t_{(2,2)}} \left( x - x_{(3,2)} \right) p_{t_{(2,2)}-t_{(1,1)}} \left( x - x_{(2,1)} \right)
$$

Note that the original $2 \times 8$-multiple integral has been collapsed to the above $2 \times 4$-multiple integral. The remaining variables are the roots of all edges in $E(D)$.

**Definition 5.6.** For any $m \in \mathbb{N}$ and $p \in 2\mathbb{N}$, we say that $\vec{n} = (n_1, \cdots, n_p)$ is a balanced partition of $2m$ if

1. $|\vec{n}| = 2m$;
2. $n_i \leq m_p + 1$ for all $i = 1, \cdots, p$, where $m_p := \lfloor 2m/p \rfloor$;
3. $n_1 + \cdots + n_{p/2} = m$;
(4) $r_p \in [0, p)$ is the remainder of $2m/p$.

Moreover, under this setting, an admissible Feynman diagram $D = (V, E)$ is called a balanced diagram provided

$$[(k_1, \ell_1), (k_2, \ell_2)] \in E(D) \implies \ell_1 = \ell_2 \text{ and } k_1 \leq p/2 < k_2.$$ 

The set of all balanced diagrams is denoted by $\mathbb{D}_n^r$. It is clear that $\mathbb{D}_n^r \subset \mathbb{D}_n$.

It is straightforward to show the existence of a balanced partition, which is however not unique in general. Let us check a few examples:

**Example 5.7.** (1) In Figure 4, we have $p = m = 4$. The partition $\vec{n} = (1, 2, 2, 3)$ is not a balanced partition. Indeed, for this example, the only balanced partition is $\vec{n} = (2, 2, 2, 2)$.

(2) If $p = 4$ and $m = 3$, the following partitions are all balanced:

$$(1, 2, 2, 1), \quad (2, 1, 2, 1), \quad (1, 2, 1, 2).$$

However, $(1, 1, 2, 2)$ is not balanced.

(3) If $p = 6$ and $m = 7$, it is easy to check that $\vec{n} = (3, 2, 2, 2, 3, 2)$ is a balanced partition, upon which a balanced diagram is given; see Figure 5.

![Figure 5](image)

**Figure 5.** One example of the balanced partition in case of $m = 7$ and $p = 6$ with a balanced diagram (all edges are horizontal starting from the left half of the vertices pointing to the right half). The grayed-out vertices correspond to the convention (5.7).

5.3. **Proof of the lower bounds.** In this subsection, we derive a lower bound for $\mathbb{E}[u(t, x)^p]$ which is consistent with the upper bound obtained in Theorem 1.1; see also (1.18).

**Theorem 5.8.** Assume that

(1) either $\beta \in (0, 2)$ and the fundamental function $p(t, x)$ is nonnegative or $\alpha = \beta = 2$ and $\gamma = 0$;

(2) the initial position $u_0$ is strictly positive and the initial velocity $u_1$ is nonnegative;

(3) Dalang’s condition (1.14) is satisfied.
Then we have for all \( t > 0, x_1, \ldots, x_p \in \mathbb{R}^d \), and \( p \in 2\mathbb{N} \) such that \( t_p = t p^{1+1/(\theta+1)} \) (see (1.12)) is sufficiently large (recall that \( \theta \) is given in (1.12)), there exist constants \( c_1 \) and \( c_2 \) that do not depend on \( (t, x_1, \ldots, x_p, p) \) such that
\[
E \left[ \prod_{j=1}^{p} u(t, x_j) \right] \geq c_1 \exp \left( c_2 t p^{1+1/(\theta+1)} \right).
\]

Proof. The proof is based on the Feynman diagram formula for the \( p \)-th moments and the non-degeneracy property of the Green’s function – Proposition 5.1, which is inspired by [HW21, Theorem 3.6]. Choose an arbitrary even integer \( p \) and let \( t > 0 \) and \( x_1, \ldots, x_p \in \mathbb{R}^d \) be fixed. By (3.13), we have
\[
E \left[ \prod_{j=1}^{p} u(t, x_j) \right] = E \left[ \prod_{j=1}^{p} \sum_{n_j=0}^{\infty} I_{n_j}(f_{n_j}(\cdot, t, x_j)) \right]
\]
\[
= \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} E \left[ I_{n_1}(f_{n_1}(\cdot, t, x_1)) \cdots I_{n_p}(f_{n_p}(\cdot, t, x_p)) \right]
\]
\[
= \sum_{m=0}^{\infty} \sum_{\vec{n} \in \mathcal{N}^m} \sum_{D \in \mathcal{D}_{\vec{n}}} F_D \left( g_{n_1}(\cdot, t, x_1), \ldots, g_{n_p}(\cdot, t, x_p) \right),
\]
where we have used the convention that \( I_0(f_0(\cdot, t, x)) = J_0(t) \). We will find the lower bound in three steps:

Step 1. We first take care of the three summations in (5.9). Applying the Feynman diagram formula in Lemma 5.4 and noting that \( \inf_{x \in [0,t]} J_0(s) \geq u_0 \) (see (3.4)), we have
\[
E \left[ \prod_{j=1}^{p} u(t, x_j) \right] \geq c_0^p \sum_{m=p/2}^{\infty} \sum_{\vec{n} \in \mathcal{N}^m, |\vec{n}|=2m} \sum_{D \in \mathcal{D}_{\vec{n}}} I_0, \quad \text{with}
\]
\[
I_0 := \int_{[0,t]^{2m}} \int_{\mathbb{R}^{2md}} \left( \prod_{[(k_1,l_1),(k_2,l_2)] \in E(D)} \delta (t_{k_1,l_1} - t_{k_2,l_2}) \delta (x_{k_1,l_1} - x_{k_2,l_2}) \right)
\]
\[
\times \prod_{i=1}^{p} \prod_{0 < t_{(i,1)} < \cdots < t_{(i,n_i)} < t} p(t_{(i,r_i+1)} - t_{(i,r_i)}, x_{(i,r_i+1)} - x_{(i,r_i)}) dt dx,
\]
where we have used the assumption that \( p(t, x) \) is nonnegative and the convention (5.7). Here we emphasize that:

(1) in the first summation of (5.10), we require \( m \geq p/2 \);
(2) in the second summation of (5.10), we only consider the balanced partitions of \( 2m \);
(3) in the third summation of (5.10), we restrict us to the balanced diagrams \( \mathcal{D}_{\vec{n}} \).

One may check Figure 5 for some examples of the selected Feynman diagrams. Recall that \( m_p = \lfloor 2m/p \rfloor \) and \( r_p \) is the remainder of \( 2m/p \) (see Definition 5.6), namely,
\[
2m = m_p \times p + r_p, \quad \text{with } 0 \leq r_p < p.
\]
It is easy to see that $r_p$ has to be an even integer. With these restrictions, for each fixed $m \geq p/2$, one can check that the total number of diagrams satisfying (2) and (3) is at least $((p/2)!)^{m_p} \times (r_p/2)!$.

**Step 2.** Now we proceed to shrink the integral region for the $dt$-integral of $I_0$ in (5.11) as follows: Denote $L = \frac{a}{m_p + 1}$, $t_i = \frac{(2i-1)p}{2(m_p+1)}$, $a_i = t_i - L/4$ and $b_i = t_i + L/4$ for $i = 1, \ldots, m_p + 1$. Let $I_i = [a_i, b_i]$. Then these intervals $I_i$ are disjoint with length $L/2$; See Figure 6 for an illustration.

![Figure 6](image)

**FIGURE 6.** Some illustrations for Step 2 in the proof of Theorem 5.8 with $m_p = 4$.

For the integral with respect to time variables in (1.17), we only integrate on the region where for each $i \in \{1, \ldots, p\}$, $t_{(i,r_i)}$ is in $I_i$ for $1 \leq r_i \leq n_i$, and hence

$$\frac{t}{2(m_p + 1)} = \frac{1}{2}L \leq t_{(i,r_i+1)} - t_{(i,r_i)} \leq \frac{3}{2}L = \frac{3t}{2(m_p + 1)}.$$  

(5.12)

Then, for each integer $m \geq p/2$, by choosing

$$\varepsilon := (p t/16m)^{\beta/\alpha},$$  

(5.13)

we have that

$$t_{(i,r_i+1)} - t_{(i,r_i)} \in \left[ 2\varepsilon^\alpha/\beta, 12\varepsilon^\alpha/\beta \right], \quad i = 1, \ldots, p.$$  

(5.14)

**Step 3.** Now we study the spatial integral portion of $I_0$ in (5.11), which is equal to

$$\int_{\mathbb{R}^m} dx \prod_{i=1}^{p} \prod_{r_i=1}^{n_i} p(t_{(i,r_i+1)} - t_{(i,r_i)}; x_{(i,r_i+1)} - x_{(i,r_i)}) \left( \prod_{(k_1,l_1),(k_2,l_2) \in E(D)} \delta(x_{(k_1,l_1)} - x_{(k_2,l_2)}) \right).$$

It is bounded from below if one replaces the integral region $\mathbb{R}^m$ by $(B^2_{\varepsilon}(x))^m$ for any $\varepsilon > 0$. In particular, by Step 2, we see that $t$ satisfies $t_{(i,r_i)} \in I_i$ for $1 \leq r_i \leq n_i, 1 \leq i \leq p$, i.e., (5.14) holds true. Hence, we can apply Proposition 5.1 with $\varepsilon$ given in (5.13) and $c = 12$ to bound the above integral from be as follows:

$$\geq C^m (pt/m)^{-\beta \alpha/\beta} \prod_{i=1}^{p} \prod_{r_i=1}^{n_i} |t_{(i,r_i+1)} - t_{(i,r_i)}|^{\beta + \gamma - 1}. $$  

(5.15)

Therefore, we can find a lower bound of $I_0$ in (5.11) with only time integral:

$$I_0 \geq C^m (pt/m)^{-\beta \alpha/\beta} \int_{[0,t]^{2m}} dt \prod_{i=1}^{p} \prod_{r_i=1}^{n_i} |t_{(i,r_i+1)} - t_{(i,r_i)}|^{\beta + \gamma - 1} \prod_{(i,r_i) \in I_i} \delta(t_{(i,r_i)} - t_{(k_1,l_1)}) \left( \prod_{(k_1,l_1),(k_2,l_2) \in E(D)} \delta(x_{(k_1,l_1)} - x_{(k_2,l_2)}) \right). $$  

(5.16)
Step 4. Finally, we will carry out the remaining $dt$-integral in (5.16) and complete the proof. We will use $C$ to denote a generic constant that does not depend on $(t,p,m)$ and whose value may change at each appearance. Now denote the integral in (5.16) by $I$, which can be bound from below as follows:

$$I \geq C^m L^{2m(\beta+\gamma-1)} \int_{[0,t]^{2m}} dt \prod_{i=1}^{p} \prod_{r_i=1}^{n_i} \mathbb{I}_{\{t_i,r_i\}\in E\{t\}} \left( \prod_{(i_1,j_1), (i_2,j_2)\in E\{D\}} \delta(t_i,j_1) - t_i,j_2) \right)$$

$$= C^m L^{2m(\beta+\gamma-1)} \left( \frac{L}{2} \right)^m = C^m \left( \frac{t}{m_p} \right)^{m(2\beta+2\gamma-1)}.$$

Replace the space-time integral in (5.10) by the above lower bound, together with the factor in front of the integral in (5.16), to see that

$$\text{E} \left[ \prod_{j=1}^{p} u(t,x_j) \right] \geq c_0^p \sum_{m \geq p/2} \sum_{\mathbb{N} \ni \mathbb{N}} \sum_{D \in \mathbb{D}_{m}^n} C^m \left( \frac{pt}{m} \right)^{-\beta dm/\alpha} \left( \frac{t}{m_p} \right)^{m(2\beta+2\gamma-1)} ((p/2)!)^m \times (r_p/2)!,$$

where we have used the fact that there are at least $((p/2)!)^m \times (r_p/2)!$. terms in the double summations.

Thanks to the following well known bounds to the Gamma function, which is related to the Stirling formula (see, e.g., 5.1.10 on p. 141 of [Olver+10])

$$\sqrt{2\pi n} \left( \frac{n}{e} \right)^n < n! < 2\sqrt{2\pi n} \left( \frac{n}{e} \right)^n,$$

for all $n \geq 1$, (5.17)

we see that up to a constant, one can replace $n!$ by $\sqrt{2\pi n} \left( n/e \right)^n$. Hence, by (5.17) and the fact $(r_p/p)^{r_p/2} \geq e^{-\gamma} \geq C^m$, we have

$$((p/2)!)^m \times (r_p/2)! \geq C^m \left( (p/m_p)^{2} \times (p)^{r_p/2} (r_p/p)^{r_p/2} \right) \geq C^m p^{r_p} = C^m p^{m}.$$

Then bound $t/m_p$ in the above lower bound from below by $pt/(2m)$ and put $c_0^p$ into $C^m$, to see that

$$\text{E} \left[ \prod_{j=1}^{p} u(t,x_j) \right] \geq \sum_{m \geq p/2} C^m \left( \frac{pt}{m} \right)^{-\beta dm/\alpha} \left( \frac{t}{2m} \right)^{m(2\beta+2\gamma-1)} p^m$$

$$\geq \sum_{m \geq p/2} \left( \frac{Cp^{1+\frac{1}{\alpha}} t}{m} \right)^{m(\theta+1)}.$$

Let $n := Cp^{1+\frac{1}{\alpha}} t$, if $n$ is sufficiently large, we have $p/2 \leq n$, then for sufficiently large $n$,

$$\text{E} \left[ \prod_{j=1}^{p} u(t,x_j) \right] \geq \sum_{m \geq p/2} \left( \frac{n^m}{m!} \right)^{\theta+1} \geq \sum_{m \geq n} \left( \frac{n^m}{m!} \right)^{\theta+1} \geq c_1 \exp \left( c_2 p^{1+\frac{1}{\alpha}} t \right)$$

where the third inequality follows from the Lemma B.4. This completes the proof of Theorem 5.8.
Finally, let us explain in the following remark why the space-time white noise case requires a separate treatment.

**Remark 5.9.** The lower bounds for equations with space-time colored noise whose covariance function is given by (1.8) were obtained in Hu-Wang [HW21], by which our methodology is inspired. Here it is important to make a distinction in the treatment between the colored noise case and the white noise case: Firstly, in the white noise case, the balanced Feynman diagrams (see Definition 5.6) make the right contribution to the desired lower bound, and this is different from the colored noise case (see Step 1 in the proof of Theorem 3.6 ibid.). Secondly, Hu-Wang’s proof relies heavily on the assumption $\gamma(t) \geq C|t|^{-\beta}$ and $\Delta(x) \geq C|x|^{-\lambda}$ for small values of $t$ and $x$ (see Step 2 in the proof Theorem 3.5 ibid.), which does not hold for the white noise case. As a consequence, the small ball nondegeneracy property for Green’s function (see Section 3.1 ibid.) which plays a key role in Hu-Wang’s argument does not apply to the white noise case. To resolve this issue, we develop a similar nondegeneracy property for the product of Green’s functions (see Proposition 5.1).

**APPENDIX A. PRELIMINARIES ON FRACTIONAL INTEGRALS AND DERIVATIVES**

In this section, we provide some preliminaries on fractional integrals and derivatives in the sense of Riemann-Liouville and we also recall Caputo fractional derivatives. We refer to [KST06; Pod99] for details.

Let $\alpha \geq 0$ be a constant and $[a, b]$ be a finite interval on $\mathbb{R}$. Let $f(x)$ be a complex-valued function defined on $[a, b]$. We only recall the left-sided integrals/derivatives which will be used in this article, and the right-sided case is similar and thus omitted.

**Definition A.1.** The Riemann-Liouville integral $I^\alpha_{a+} f$ of order $\alpha \geq 0$ is defined by

$$
(I^\alpha_{a+} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x \in [a, b].
$$

**Definition A.2.** The Riemann-Liouville derivative $D^\alpha_{a+} f$ of order $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ is defined by

$$(D^\alpha_{a+} f)(x) := \frac{d^n}{dx^n} (I^{n-\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{n-\alpha}} dt, \quad n = \lceil \alpha \rceil,
$$

and when $\alpha = n \in \mathbb{N}$, $(D^n_{a+} f)(x) = \frac{d^n}{dt^n} f(t)$. We use the convention that $D^\alpha_{a+} f := I^{-\alpha}_{a-} f$, when $\alpha < 0$.

For $1 \leq p \leq \infty$, we denote by $L^p(a, b)$ the set of complex-valued functions $f$ on $[a, b]$ with finite $L^p$-norm $\|f\|_p$, where

$$
\|f\|_p = \begin{cases} 
\left( \left( \int_a^b |f(x)|^p dx \right)^\frac{1}{p} \right), \quad 1 \leq p < \infty, \\
\text{ess sup}_{a \leq x \leq b} |f(x)|, \quad p = \infty.
\end{cases}
$$

**Lemma A.3** (Property 2.2 on p. 74 of [KST06]). For $\alpha > \beta > 0$ and $f(x) \in L^p(a, b), 1 \leq p \leq \infty$, we have

$$
(D^\beta_{a+} I^\alpha_{a+} f)(x) = I^{\alpha-\beta}_{a+} f(x), \quad \text{for } x \in [a, b] \text{ almost everywhere.}
$$

**Lemma A.4** (Property 2.5 on p. 81 of [KST06]). For $\alpha, \beta > 0$, we have

$$
(I^\alpha_{0+} t^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} x^{\beta + \alpha - 1} \quad \text{and} \quad (D^\alpha_{0+} t^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} x^{\beta - \alpha - 1}.
$$
We suppose that
\[ y = \lambda f(x), \quad x \in [a, b] \]
if \( \gamma \) is an integer. For \( \gamma \in [0, 1) \), we define the weighted space \( C_{\gamma}[a, b] \) of continuous functions as follows,
\[ C_{\gamma}[a, b] := \{ f(x) : (x - a)^{\gamma} f(x) \in C[a, b] \}. \]
Consider the following Cauchy Problem, for \( \lambda \in \mathbb{R}, n \in \mathbb{N} \) and \( n - 1 < \beta < n \),
\[
\begin{cases}
(CD_{a+}^\beta f)(x) - \lambda f(x) = y(x), & x \in [0, b], \\
f^{(k)}(0) = b_k, & b_k \in \mathbb{R} \text{ for } k = 0, 1, \ldots, n - 1.
\end{cases}
\tag{A.3}
\]
We suppose that \( y(x) \in C_{\gamma}[0, b] \) with \( 0 \leq \gamma < 1 \) and \( \gamma \leq \beta \). Then (A.3) has a unique solution given by (see \[A.5\]):
\[
f(x) = \sum_{j=0}^{n-1} b_j x^j E_{\beta,j+1}(\lambda x^{\beta}) + \int_0^x (x - t)^{\beta-1}E_{\beta,\beta}(\lambda(x - t)^\beta) y(t) dt,
\tag{A.4}
\]
where \( E_{a,b}(z) \) is the Mittag-Leffler function; see (1.13). One may get more explicit expressions for special values of \( a \) and \( b \), which will be used in this paper:
\[
E_2(z) = 2e^{z^2} \Phi \left( \sqrt{2z} \right), \quad E_1(z) = e^z, \quad E_2(z) = \cosh(\sqrt{z}), \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}},
\tag{A.5}
\]
where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx \) is the cumulative distribution function of standard normal distribution. Another formula that will be useful in this paper is
\[
E_{\alpha,\beta}(|z|) - \frac{1}{\Gamma(\beta)} = |z| E_{\alpha,\beta+1}(|z|),
\tag{A.6}
\]
which can be obtained immediately using the definition of the Mittag-Leffler function in (1.13); see also (1.8.38) on p.45 of \[KST06\]. The asymptotic behavior of the Mittag-Leffler function along the positive and negative real lines plays an important role in the paper, which has been summarized in the following lemma:

**Lemma A.6.** If all \( a > 0 \) and \( b \in \mathbb{C} \), we have that
\begin{itemize}
  \item if \( a < 2 \), as \( z \to +\infty \),
    \[ E_{a,b}(z) = \frac{1}{a} z^{(1-b)/a} \exp(z^{1/a}) - \frac{1}{\Gamma(b-a)} \frac{1}{z} + O(z^{-2}); \]
  \item if \( a \geq 2 \), as \( z \to +\infty \),
    \[ E_{a,b}(z) = \frac{1}{a} \sum_{n \in \mathbb{Z}: |n| \leq a/4} \left( z^{1/a} \exp \left( \frac{2n\pi i}{a} \right) \right)^{1-b} \exp \left( \frac{2n\pi i z}{a} \right) z^{1/a} - \frac{1}{\Gamma(b-a)} \frac{1}{z} + O(|z|^{-2}); \]
  \item if \( a < 2 \), as \( z \to -\infty \),
    \[ E_{a,b}(z) = -\frac{1}{\Gamma(b-a)} \frac{1}{z} + O(z^{-2}); \]
  \item if \( a = 2 \), as \( z \to -\infty \),
    \[ E_{a,b}(z) = |z|^{(1-b)/2} \cos \left( \sqrt{|z|} + \frac{(1-b)\pi}{2} \right) - \frac{1}{\Gamma(b-2)} \frac{1}{z} + O(z^{-2}). \]
\end{itemize}
In particular, for all \( C > 0 \), \( \lim_{t \to \infty} 1/t \log E_{a,b}(Ct^a) = C^{1/a} \).

**Proof.** The case when \( z \to \infty \) is derived from 1.8.27 (resp. 1.8.29) of [KST06] when \( a < 2 \) (resp. \( a \geq 2 \)). The case when \( a < 2 \) (resp. \( a = 2 \)) and \( z \to -\infty \) is a consequence of 1.8.28 (resp. 1.8.31) (ibid.). When \( a < 2 \), the statement for the limit is a direct consequence of the asymptotics at \( +\infty \). When \( a \geq 2 \), denoting \( z = Ct^a \), we have

\[
\lim_{t \to \infty} \frac{1}{t} \log E_{a,b}(z) = \lim_{t \to \infty} \frac{1}{t} \log \sum_{n \in \mathbb{Z} : |n| \leq a/4} \left( z^{\frac{a}{2}} \exp \left( \frac{2n\pi i}{a} \right) \right)^{1-b} \exp \left( \frac{2n\pi}{a} \right) \frac{1}{z^a}
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log \sum_{n \in \mathbb{Z} : |n| \leq a/4} z^{\frac{a}{2}} \cdot \exp \left( \frac{2n\pi}{a} \right) \frac{1}{z^a}
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log \left( z^{\frac{a}{2}} \exp \left( \frac{2n\pi}{a} \right) \right) = C^{1/a}.
\]

\[\square\]

We will use the reflection formula for the Gamma function (see, e.g., [Olv+10; 5.5.3 on p. 138]), namely,

\[\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z), \quad z \neq 0, \pm 1, \ldots. \quad (A.7)\]

**Appendix B. Some miscellaneous lemmas**

In this section, we provide the technical lemmas. Lemma B.1 below will be used used to prove Dalang’s condition (1.14) in Theorem 3.3.

**Lemma B.1.** For all \( \varepsilon > 0 \), \( a, b > 0 \), and \( c \in \mathbb{R} \), it holds that

\[
\int_{B_{\varepsilon}(0)} \frac{\cos^2(|x|^a + c)}{|x|^b} \, dx < \infty \quad \text{if and only if } b > d. \quad (B.1)
\]

**Proof.** We only consider the case \( d \geq 2 \) while the case \( d = 1 \) is similar but easier. Denote the integral in (B.1) by \( I \). Since the integrand is radial,

\[I = C \int_{\varepsilon}^\infty \frac{\cos^2(r^a + c)}{r^b} r^{d-1} \, dr, \quad \text{with } C = \frac{2\pi^{d/2}}{\Gamma(d/2)}.\]

Clearly, \( b > d \) is a sufficient condition for (B.1). To get the necessity, observe that

\[I = \frac{C}{a} \int_{\varepsilon}^\infty \cos^2(s + c) s^{-\frac{1}{2}(b-d)-1} \, ds \]

\[\geq \frac{C}{a} \sum_{n = N}^{\infty} \int_{n\pi - c}^{(n+\frac{1}{2})\pi - c} \cos^2(s + c) s^{-(b-d)/a - 1} \, ds \]

\[\geq \frac{C \pi^{-(b-d)/a}}{8a} \sum_{n = N+1}^{\infty} n^{-(b-d)/a - 1},\]

where \( N = N(\varepsilon^a, c) \) is a finite positive integer. The series on the right-hand side is convergent if and only if \( b > d \) and thus \( b > d \) is also necessary for (B.1).

\[\square\]

The following lemma is a convolution-type Gronwall lemma, which was proved in Lemma A.2 of [CHN21] for \( \theta \in (-1, 0) \). But indeed, the same proof can be extended directly to all \( \theta > -1 \). One can use this lemma to obtain the moment formulas in Theorem 1.1 as pointed out in Remark 4.1.
Lemma B.2. Suppose that $\theta > -1$, $\kappa > 0$ and that $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function. If $f$ satisfies

$$f(t) = g(t) + \kappa \int_0^t (t-s)^\theta f(s) \, ds, \quad \text{for } t \geq 0,$$

then

$$f(t) = g(t) + \int_0^t g(s) K(t-s) \, ds,$$

with $K(t) = \kappa \Gamma(\theta + 1) t^\theta E_{\theta+1,\theta+1}(\kappa \Gamma(\theta + 1) t^{\theta+1})$. Moreover, if we further assume $g(\cdot) \geq 0$ and the equality in (B.2) is replaced by $\leq$ (resp. $\geq$), then the equality in (B.3) is replaced by $\leq$ (resp. $\geq$) accordingly.

The following lemma will be used to obtain the explicit second moment formulas for stochastic wave equation (i.e., $\beta = 2$) in Example 2.4.

Lemma B.3. For $\alpha > 1$ and $b > 0$, it holds that

$$\int_0^\infty \frac{\sin^2 \left( b \frac{\xi^{\alpha/2}}{\xi^\alpha} \right)}{\xi^\alpha} \, d\xi = \begin{cases} 2^{2(1-1/\alpha)} \alpha^{-1} \cos \left( \pi / \alpha \right) \Gamma \left( 2(1/\alpha - 1) \right) b^{2-2/\alpha} & \text{if } \alpha \in (1, 2) \cup (2, \infty), \\ 2^{-1} b \pi & \text{if } \alpha = 2. \end{cases}$$

Proof. Denote the integral by $I$. By change of variable $z = \frac{\xi^{\alpha/2}}{\xi^\alpha}$, we see that

$$I = \frac{2}{\alpha} \int_0^\infty \frac{\sin^2 \left( b z \right)}{z^{3-2/\alpha}} \, dz.$$

Let $f(x) = \mathbb{1}_{[-b,b]}(x)$ and $g(x)$ be an even function defined as, for $x > 0$,

$$g(x) = \frac{\pi}{4 \Gamma \left( 2(1-1/\alpha) \right) \sin \left( \pi / \alpha \right)} \left[ (x+b)^{1-2/\alpha} + |b-x|^{1-2/\alpha} \text{sgn}(b-x) \right].$$

Now we compute the Fourier transforms for these two functions. It is clear that

$$\hat{f}(\xi) = \frac{2 \sin(b \xi)}{\xi}.$$

Let $h(\xi) = |\xi|^{-2(1-1/\alpha)} \sin \left( b |\xi| \right)$. By (2) on p. 19 of [Erd+54], we see that

$$F^{-1} h(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} h(\xi) \, d\xi = \frac{1}{\pi} \int_0^\infty h(\xi) \cos(x\xi) \, d\xi = \frac{1}{\pi} g(x),$$

under the following condition:

$$\left| \frac{2}{\alpha} - 1 \right| < 1 \iff \alpha > 1.$$

Hence, when $\alpha > 1$, we have $\hat{g}(\xi) = \pi h(\xi)$. Then by the Plancherel theorem,

$$\int_{\mathbb{R}} f(x) g(x) \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\xi) \, d\xi = \frac{1}{\pi} \int_0^\infty \frac{2 \sin(b \xi)}{\xi} \pi^{-2(1-1/\alpha)} \sin \left( b \xi \right) \, d\xi = \alpha I.$$
On the other hand,
\[
\int_{\mathbb{R}} f(x)g(x)dx = 2 \int_0^b g(x)dx
\]
\[
= \frac{\pi}{2\Gamma(2(1 - 1/\alpha)) \sin(\pi/\alpha)} \int_0^b \left[ (x + b)^{1-2/\alpha} + (b - x)^{1-2/\alpha} \right] dx
\]
\[
= \frac{\Gamma(2 - 2/\alpha)}{\pi^{2(1 - 2/\alpha)} b^{2 - 2/\alpha}} \sin(\pi/\alpha)
\]= \frac{\Gamma(3 - 2/\alpha) \sin(\pi/\alpha)}{\Gamma(3 - 2/\alpha) \sin(\pi/\alpha)}.
\]
Hence, if \( \alpha = 2 \), the above expression becomes \( b\pi \). This proves the lemma for the case \( \alpha = 2 \).

Now if \( \alpha \neq 2 \), we have
\[
\int_{\mathbb{R}} f(x)g(x)dx = \frac{\pi^{2(1 - 2/\alpha)} b^{2 - 2/\alpha}}{\Gamma(3 - 2/\alpha) \sin(\pi/\alpha)} \times \frac{\Gamma(2(1/\alpha - 1))}{\Gamma(2(1/\alpha - 1))} \sin(2\pi/\alpha)
\]
\[
= \frac{2^{2 - 2/\alpha} b^{2 - 2/\alpha} \Gamma(2(1/\alpha - 1))}{\sin(\pi/\alpha)} \sin(\pi/\alpha)
\]
where we have applied the reflection formula for Gamma function (A.7). This proves the lemma. \( \square \)

The following lemma will be used to prove the lower bound of moment estimates in Theorem 5.8. For two sequence of positive numbers \( a_n, b_n, n \in \mathbb{N} \), we denote \( a_n \sim b_n \) if \( \lim_{n \to \infty} a_n/b_n = 1 \).

**Lemma B.4.** (1) As \( n \to \infty \), we have
\[
\int_{n}^{\infty} t^n e^{-t}dt \sim \frac{n^n}{e^n} \sqrt{\frac{n}{2}} \quad \text{and} \quad \sum_{m=1}^{n-1} \frac{n^m}{m!} \sim \sum_{m=n}^{2n-1} \frac{n^m}{m!} \sim \frac{1}{2} e^n.
\]

(2) Given \( \alpha > 0 \), for \( n \) sufficiently large, there exist two positive constants \( c_1 \) and \( c_2 \) depending only on \( \alpha \) such that
\[
\sum_{m=n}^{\infty} \left( \frac{n^m}{m!} \right)^\alpha \geq c_1 \exp(c_2 n).
\]

**Proof.** (1) Denote the integral in (B.4) by \( I \). By change of variable \( x = t/\sqrt{n} - \sqrt{n} \), we see that
\[
I = \frac{n^n}{e^n} \int_0^{\infty} e^{-\sqrt{n} x} \left( 1 + \frac{x}{\sqrt{n}} \right)^n dx.
\]
Notice that \( \left( 1 + \frac{x}{\sqrt{n}} \right)^n e^{-\sqrt{n} x} \leq (1 + x)e^{-x} \) for all \( t > 0 \) and \( n \geq 1 \) with the upper bound being integrable. Hence, by the dominated convergence theorem and L’Hospital’s rule, we conclude that
\[
\lim_{n \to \infty} I \frac{e^n}{n^n} = \int_0^{\infty} \lim_{n \to \infty} e^{-\sqrt{n} x} \left( 1 + \frac{x}{\sqrt{n}} \right)^n \ dx = \int_0^{\infty} e^{-x^2/2}dx = \sqrt{\frac{\pi}{2}},
\]
which proves (B.4).
To prove (B.5), it suffices to show \( R_n(n) \sim 1/2 e^n \) and \( \lim_{n \to \infty} e^{-n} R_{2n}(n) = 0 \), where \( R_k(x) \) is the remainder function for the Taylor expansion of \( e^x \):

\[
R_k(x) = \sum_{m=k}^{\infty} \frac{x^m}{m!} = \int_0^x \frac{(x-t)^k}{k!} e^t dt.
\]

For \( R_n(n) \), by change of variable we get

\[
R_n(n) = \int_0^n \frac{(n-t)^n}{n!} e^t dt = \frac{e^n}{n!} \int_0^n x^n e^{-x} dx.
\]

By Stirling’s formula \( n! \sim \sqrt{2\pi n} e^{-n} n^n \) and (B.4), we can show

\[
\int_0^n x^n e^{-x} dx = \left( \Gamma(n + 1) - \int_0^\infty x^n e^{-x} dx \right) \sim \frac{1}{2} n!.
\]

(B.6)

For \( R_{2n}(n) \), we have

\[
R_{2n}(n) = \int_0^n \frac{(n-t)^{2n}}{(2n)!} e^t dt \leq \frac{n^n}{(2n)!} \int_0^n (n-t)^n e^t dt = \frac{e^n n^n}{(2n)!} \int_0^n x^n e^{-x} dx.
\]

Thus, by (B.6), we have

\[
\lim_{n \to \infty} \frac{R_{2n}(n)}{e^n} \leq \lim_{n \to \infty} \frac{1}{2} \frac{n^n(n!)}{(2n)!} = \lim_{n \to \infty} \frac{1}{2} \frac{n^{2n} e^{-n} \sqrt{2\pi n}}{(2n)^{2n} e^{-2n} \sqrt{4\pi n}} = 0,
\]

which proves (B.5).

(2) The desired result follows directly from the fact

\[
\sum_{m=n}^{m=n} \left( \frac{n^m}{m!} \right)^\alpha \geq \sum_{m=n}^{m=n} \left( \frac{n^m}{m!} \right)^\alpha \geq \left\{ \begin{array}{ll}
\left( \frac{\sum_{m=n}^{m=n} n^m}{m!} \right)^\alpha, & \text{if } \alpha \in (0, 1), \\
n^{-1} \left( \sum_{m=n}^{m=n} n^m \right)^\alpha, & \text{if } \alpha > 1.
\end{array} \right.
\]

Then an application of (B.5) proves (2).

\( \square \)

**APPENDIX C. Fundamental solutions**

The fundamental solutions to (1.10) in case when \( d = 1, \beta = 1, \gamma = 0, \) and \( \alpha \in 2\mathbb{N} \) (i.e., \( \alpha \) is an even integer) have been studied in [Kry60] and [Hoc78]. In [Deb06], Debbi studied the fundamental solutions to (1.10) when \( d = 1, \beta = 1, \gamma = 0, \) and \( \alpha \in (1, \infty) \backslash \mathbb{N} \) and then, with Dozzi [DD05], they studied the corresponding SPDEs with space-time white noise. This part can be viewed as a generalization of their results to a class of more general of SPDEs. The Fox H-functions [KS04] allow us to study the fundamental solutions to (1.10) with the much more general parameters in a unified way.

The following theorem generalizes Theorem 4.1 of [CHN19] from \( \alpha \in (0, 2] \) and \( \beta \in (0, 2) \) to the case \( \alpha > 0 \) and \( \beta \in (0, 2] \). The statement of the theorem remains almost the same except the conditions on \( \alpha \) and \( \beta \). The proof also follows the same lines of arguments as those in [CHN19]; one may also check the proof of Theorem 3.1 in [Che+17] for the case when \( \gamma = 0 \). The case when \( \beta = 2 \) is new. For the readers’ convenience, we state the theorem below and present its proof to explicitly show why the ranges of \( \alpha \) and \( \beta \) can be generalized.
Theorem C.1. For \( \alpha \in (0, \infty), \beta \in (0, 2], \) and \( \gamma \geq 0, \) the solution to
\[
\begin{cases}
\left( \frac{\partial}{\partial t} + \frac{\nu}{2}(-\Delta)^{\alpha/2} \right) u(t, x) = I_t^\gamma [f(t, x)], \quad t > 0, \ x \in \mathbb{R}^d; \\
\frac{\partial^k}{\partial t^k} u(t, x) \bigg|_{t=0} = u_k(x), \quad 0 \leq k \leq \lceil \beta \rceil - 1, \ x \in \mathbb{R}^d,
\end{cases}
\] (C.1)
is
\[\begin{align*}
u t
\end{align*}\]
\[\begin{align*}
u t
\end{align*}\]
where \( tD_{0+}^{[\beta]-\beta-\gamma} \) denotes the Riemann-Liouville derivative \( D_{0+}^{[\beta]-\beta-\gamma} \) acting on the time variable,
\[
J_0(t, x) := \sum_{k=0}^{\lceil \beta \rceil - 1} \int_{\mathbb{R}^d} u_k(y) \hat{\alpha}^{[\beta]-1-k} Z(t, x - y) \, dy \] (C.3)
is the solution to the homogeneous equation and \( Z(t, x) := Z_{\alpha,\beta,d}(t, x) \) is the corresponding fundamental solution. If we denote
\[
Y(t, x) := Y_{\alpha,\beta,\gamma,d}(t, x) = tD_{0+}^{[\beta]-\beta-\gamma} Z_{\alpha,\beta,d}(t, x),
\]
then we have the following Fourier transforms:
\[
\begin{align*}
u t
\end{align*}\]
\[\begin{align*}
u t
\end{align*}\]
\[\begin{align*}
u t
\end{align*}\]
Moreover, when \( \beta \in (0, 2), \) we have the following explicit expressions:
\[
\begin{align*}
u t
\end{align*}\]
\[\begin{align*}
u t
\end{align*}\]
\[\begin{align*}
u t
\end{align*}\]
\[\begin{align*}
u t
\end{align*}\]
and, if \( \beta \in (1, 2), \)
\[
Z^*(t, x) = \pi^{-d/2} |x|^{\gamma-1} H_{d,2}^{2,1} \left( \frac{|x|^\alpha}{2^{\alpha-1} \nu t^\beta} \right) \left( (1, 1), (\beta+1, \gamma), (d/2, 1, \gamma, 1, 1/2) \right). \] (C.9)

Proof. The proof follows a standard argument using the Fourier and Laplace transforms in the space and time variables, respectively, which are denoted by \( \hat{f} \) and \( \hat{g}. \) Let us apply the Fourier transform to (C.1) first to obtain
\[
\begin{cases}
\frac{\partial}{\partial t} \hat{u}(t, \xi) + \frac{\nu}{2} \xi |\alpha| \hat{u}(t, \xi) = I_t^\gamma \left[ \hat{f}(t, \xi) \right], \quad \xi \in \mathbb{R}^d \\
\frac{\partial^k}{\partial \xi^k} \hat{u}(t, \xi) \bigg|_{t=0} = \hat{u}_k(\xi),\quad 0 \leq k \leq \lceil \beta \rceil - 1, \ \xi \in \mathbb{R}^d.
\end{cases}
\]
Apply the Laplace transform on the Caputo derivative using [Die10, Theorem 7.1 on p. 134]:
\[
L \left[ \frac{\partial}{\partial t} \hat{u}(t, \xi) \right] (s) = s^\beta \hat{u}(0, \xi) - \sum_{k=0}^{\lceil \beta \rceil - 1} s^{\beta-1-k} \hat{u}_k(\xi).
\]
On the other hand, it is known that (see, e.g., [SKM93, (7.14) on p. 140]),
\[
\mathcal{L}^\gamma_i \left[ \hat{f}(t, \xi) \right] = s^{-\gamma} \hat{f}(s, \xi), \quad \Re(\gamma) > 0.
\]
Thus,
\[
\hat{u}(s, \xi) = (s^\beta + \frac{\nu}{2} |\xi|^\alpha)^{-1} \left[ \sum_{k=0}^{[\beta]-1} s^{\beta-1-k} \hat{u}_k(\xi) + s^{-\gamma} \hat{f}(s, \xi) \right].
\]
Notice that (see, e.g., [Pod99, (1.80) on p. 21])
\[
\mathcal{L} \left[ t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) \right] (s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \text{for } \Re(s) > |\lambda|^{1/\alpha}. \tag{C.10}
\]
Hence,
\[
\hat{u}(t, \xi) = \sum_{k=0}^{[\beta]-1} t^k E_{\beta,k+1} \left( -\frac{\nu}{2} |\xi|^\alpha t^\beta \right) \hat{u}_k(\xi) + \int_0^t \int_0^\tau \left( -\frac{\nu}{2} |\xi|^\alpha \tau^\beta \right) \hat{f}(t - \tau, \xi) d\tau.
\]
Now if we denote
\[
U(t, \xi) := t^{[\beta]-1} E_{\beta,|\beta|} \left( -\frac{\nu}{2} |\xi|^\alpha t^\beta \right), \tag{C.11}
\]
using the fact that \( t^k D_{0+}^\gamma = \frac{d^k}{dt^k} \) when \( \gamma \in \mathbb{Z} \) and for all \( \gamma \in \mathbb{R} \) (see [Pod99, (1.82) on p. 21]),
\[
t D_{0+}^\gamma \left( t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \right) = t^{\beta-\gamma-1} E_{\alpha,\beta-\gamma}(\lambda t^\alpha),
\]
we see that
\[
\hat{u}(t, \xi) = \sum_{k=0}^{[\beta]-1} \left( \frac{d^k}{dt^k} U(t, \xi) \right) \hat{u}_k(\xi) + \int_0^t \left( t D_{0+}^{[\beta]-\beta-\gamma} U(\tau, \xi) \right) \hat{f}(t - \tau, \xi) d\tau.
\]
It remains to prove the expressions in (C.7) – (C.9) under the assumption that \( \beta \in (0, 2) \). A key observation is that for \( Z_{\alpha,\beta,d}(t, x) \) defined in (C.7), its Fourier transform is given by \( U(t, \xi) \) in (C.10), namely,
\[
\mathcal{F} Z_{\alpha,\beta,d}(t, \cdot)(\xi) = U(t, \xi), \quad \text{for all } \alpha > 0, \beta \in (0, 2), \text{ and } d \geq 1. \tag{C.11}
\]
Indeed, (C.11) is proved in Lemma 4.2 of [Che+17], but only for the case of \( \alpha \in (0, 2] \). Here we claim that the restriction of \( \alpha \in (0, 2] \) is not necessary. In the proof of this lemma, one needs to consider two cases separately: \( d = 1 \) and \( d \geq 2 \). In the case of \( d = 1 \), the conditions we need are
\[
\frac{2 - \beta}{\alpha} > 0 \quad \text{and} \quad 1 \land \alpha > 0.
\]
For the second case – \( d \geq 2 \), the proof is a direct application of Corollary 2.5.1 of [KS04], where one needs to verify the following conditions:

| Conditions in [KS04] | the corresponding conditions in our setting |
|----------------------|-------------------------------------------|
| \( a^+ > 0 \)        | \( 2 - \beta > 0 \)                       |
| (2.6.8)              | \( \min(\alpha, d) > 0 \)                 |
| (2.6.9)              | \( d > 1 \)                               |
| (2.6.10)             | \( d > 1 \)                               |

Apparently, the above two conditions hold for all \( \alpha > 0 \) and \( \beta \in (0, 2) \). Hence, Lemma 4.2 of [Che+17] is true for all \( \alpha > 0 \) and \( \beta \in (0, 2) \). This proves both (C.4) and (C.7). Once one obtains the expressions for \( Z_{\alpha,\beta,d}(t, x) \) and \( \mathcal{F} Z_{\alpha,\beta,d}(t, \cdot)(\xi) \), it is routine to obtain the corresponding expressions of their fractional or integer derivatives/integrals; see [Che+17] for more details. This completes the proof of Theorem C.1.
Remark C.2. For the case $\beta = 2$, the expression in (C.4) can be simplified using the fourth expression in (A.5).

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References

[BS19] Raluca M. Balan and Jian Song. “Second order Lyapunov exponents for parabolic and hyperbolic Anderson models”. In: Bernoulli 25.4A (2019), pp. 3069–3089 (cit. on p. 6).

[CK91] Eric Carlen and Paul Krée. “$L^p$ estimates on iterated stochastic integrals”. In: Ann. Probab. 19.1 (1991), pp. 354–368 (cit. on p. 18).

[CM94] René A. Carmona and S. A. Molchanov. “Parabolic Anderson problem and intermittency”. In: Mem. Amer. Math. Soc. 108.518 (1994), pp. viii+125 (cit. on p. 2).

[Che17] Le Chen. “Nonlinear stochastic time-fractional diffusion equations on $\mathbb{R}$: moments, Hölder regularity and intermittency”. In: Trans. Amer. Math. Soc. 369.12 (2017), pp. 8497–8535 (cit. on pp. 9, 10).

[CD15a] Le Chen and Robert C. Dalang. “Moment bounds and asymptotics for the stochastic wave equation”. In: Stochastic Process. Appl. 125.4 (2015), pp. 1605–1628 (cit. on pp. 2, 6, 7, 15).

[CD15b] Le Chen and Robert C. Dalang. “Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions”. In: Ann. Probab. 43.6 (2015), pp. 3006–3051 (cit. on pp. 6, 7, 15).

[CD15c] Le Chen and Robert C. Dalang. “Moments, intermittency and growth indices for the nonlinear fractional stochastic heat equation”. In: Stoch. Partial Differ. Equ. Anal. Comput. 3.3 (2015), pp. 360–397 (cit. on p. 15).

[CE22] Le Chen and Nicholas Eisenberg. “Interpolating the stochastic heat and wave equations with time-independent noise: solvability and exact asymptotics”. In: Stoch. Partial Differ. Equ. Anal. Comput. (in press) (2022) (cit. on p. 20).

[CH21] Le Chen and Guannan Hu. “Hölder regularity of the nonlinear stochastic time-fractional slow and fast diffusion equations on $\mathbb{R}^d$”. In: Preprint arXiv:2105.00891 (2021) (cit. on p. 15).

[CH22] Le Chen and Guannan Hu. “Hölder regularity for the nonlinear stochastic time-fractional slow and fast diffusion equations on $\mathbb{R}^d$”. In: Fract. Calc. Appl. Anal. 25.2 (2022), pp. 608–629 (cit. on p. 15).

[Che+17] Le Chen, Guannan Hu, Yaozhong Hu, and Jingyu Huang. “Space-time fractional diffusions in Gaussian noisy environment”. In: Stochastics 89.1 (2017), pp. 171–206 (cit. on pp. 20, 32, 34).

[CHN19] Le Chen, Yaozhong Hu, and David Nualart. “Nonlinear stochastic time-fractional slow and fast diffusion equations on $\mathbb{R}^d$”. In: Stochastic Process. Appl. 129.12 (2019), pp. 5073–5112 (cit. on pp. 5, 12, 14, 15, 20, 32).

[CHN21] Le Chen, Yaozhong Hu, and David Nualart. “Regularity and strict positivity of densities for the nonlinear stochastic heat equation”. In: Mem. Amer. Math. Soc. 273.1340 (2021), pp. v+102 (cit. on p. 29).

[CH19] Le Chen and Jingyu Huang. “Comparison principle for stochastic heat equation on $\mathbb{R}^d$”. In: Ann. Probab. 47.2 (2019), pp. 989–1035 (cit. on p. 16).
[CK19] Le Chen and Kunwoo Kim. “Nonlinear stochastic heat equation driven by spatially colored noise: moments and intermittency”. In: Acta Math. Sci. Ser. B (Engl. Ed.) 39.3 (2019), pp. 645–668 (cit. on pp. 15, 16).

[CK20] Le Chen and Kunwoo Kim. “Stochastic comparisons for stochastic heat equation”. In: Electron. J. Probab. 25 (2020), Paper No. 140, 38 (cit. on p. 16).

[Che15] Xia Chen. “Precise intermittency for the parabolic Anderson equation with an (1+1)-dimensional time-space white noise”. In: Ann. Inst. Henri Poincaré Probab. Stat. 51.4 (2015), pp. 1486–1499 (cit. on pp. 2, 6).

[Che+18] Xia Chen, Yaozhong Hu, Jian Song, and Xiaoming Song. “Temporal asymptotics for fractional parabolic Anderson model”. In: Electron. J. Probab. 23 (2018), Paper No. 14, 39 (cit. on p. 7).

[CK12] Daniel Conus and Davar Khoshnevisan. “On the existence and position of the farthest peaks of a family of stochastic heat and wave equations”. In: Probab. Theory Related Fields 152.3-4 (2012), pp. 681–701 (cit. on p. 18).

[Dal99] Robert C. Dalang. “Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s”. In: Electron. J. Probab. 4 (1999), no. 6, 29 (cit. on pp. 14, 15).

[DM09] Robert C. Dalang and Carl Mueller. “Intermittency properties in a hyperbolic Anderson problem”. In: Ann. Inst. Henri Poincaré Probab. Stat. 45.4 (2009), pp. 1150–1164 (cit. on pp. 2, 19).

[DMT08] Robert C. Dalang, Carl Mueller, and Roger Tribe. “A Feynman-Kac-type formula for the deterministic and stochastic wave equations and other P.D.E.’s”. In: Trans. Amer. Math. Soc. 360.9 (2008), pp. 4681–4703 (cit. on p. 2).

[Dal+09] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. A minicourse on stochastic partial differential equations. Vol. 1962. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha. Springer-Verlag, Berlin, 2009, pp. xii+216. ISBN: 978-3-540-85993-2 (cit. on p. 14).

[Deb06] Latifa Debbi. “Explicit solutions of some fractional partial differential equations via stable subordinators”. In: J. Appl. Math. Stoch. Anal. (2006), Art. ID 93502, 18 (cit. on p. 32).

[DD05] Latifa Debbi and Marco Dozzi. “On the solutions of nonlinear stochastic fractional partial differential equations in one spatial dimension”. In: Stochastic Process. Appl. 115.11 (2005), pp. 1764–1781 (cit. on p. 32).

[Die10] Kai Diethelm. The analysis of fractional differential equations. Vol. 2004. An application-oriented exposition using differential operators of Caputo type. Springer-Verlag, Berlin, 2010, pp. viii+247. ISBN: 978-3-642-14573-5 (cit. on p. 33).

[Erd+54] A. Erdéyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. Tables of integral transforms. Vol. I. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954, pp. xx+391 (cit. on p. 30).

[Hoc78] Kenneth J. Hochberg. “A signed measure on path space related to Wiener measure”. In: Ann. Probab. 6.3 (1978), pp. 433–458 (cit. on p. 32).

[Hu17] Yaozhong Hu. Analysis on Gaussian spaces. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, pp. xi+470. ISBN: 978-981-3142-17-6 (cit. on pp. 16, 20, 21).

[HW21] Yaozhong Hu and Xiong Wang. “Intermittency properties for a large class of stochastic PDEs driven by fractional space-time noises”. In: preprint arXiv:2109.03473 (2021) (cit. on pp. 1, 2, 19, 21, 24, 27).
