Relations Between the Bernstein Polynomials and $q$-monotone Functions

Ulrich Abel, Dany Leviatan, and Ioan Raşa

Abstract. We show that certain inequalities involving differences of the Bernstein basis polynomials and values of a function $f \in C[0,1]$, imply that the function is $q$-monotone. In view of previous results of the authors (see the list of references), the current results provide, among others, a characterization of $q$-monotone functions in $C[0,1]$.

Mathematics Subject Classification. Primary 26D05, 39B62; Secondary 41A17, 41A36.

Keywords. Inequalities for polynomials, functional inequalities, $q$-monotonicity.

1. Introduction and the Main Results

Given a function $f$ defined on $[0,1]$, the classical Bernstein polynomials associated with it are defined by [8],

$$(B_n f)(x) = \sum_{j=0}^{n} p_{n,j}(x)f\left(\frac{j}{n}\right), \quad x \in [0,1],$$

where

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad 0 \leq j \leq n.$$ 

The following theorem was proved by J. Mrowiec, T. Rajba and S. Wąsowicz [9]. It has given an affirmative answer to a conjecture by the third author.
**Theorem A.** If $f \in C[0,1]$ is convex, then for all $n \in \mathbb{N}$,

$$
\sum_{i=0}^{n} \sum_{j=0}^{n} \left[ p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y) \right] f \left( \frac{i + j}{2n} \right) \geq 0.
$$

(1.1)

The proof given by J. Mrowiec, T. Rajba and S. Wąsowicz [9] makes heavy use of probability theory. As a tool they applied stochastic convex orderings (which they proved for binomial distributions) as well as the so-called concentration inequality. This probabilistic approach has been applied in several other papers (see [7] and references therein) connecting (1.1) with the theory of stochastic ordering.

Recently [1], the first author gave an elementary, analytic, proof of Theorem A, and this approach has been followed by several authors (see [5,6] and references therein). Here we continue the analytic line.

Since the Bernstein polynomials preserve convexity, if $f \in C[0,1]$ is convex, then for all pairs $x, y \in [0,1]$,

$$
(B_nf)(x) + (B_nf)(y) \geq 2(B_nf) \left( \frac{x + y}{2} \right).
$$

(1.2)

It follows from Vandermonde’s identity

$$
\sum_{i+j=k}^{n} \binom{n}{i} \binom{n}{j} = \binom{2n}{k},
$$

that,

$$
(B_{2n}f)(u) = \sum_{i=0}^{n} \sum_{j=0}^{n} p_{n,i}(u)p_{n,j}(u) f \left( \frac{i + j}{2n} \right), \quad u \in [0,1].
$$

(1.3)

So, the first and third authors [3] asked whether one may actually prove a stronger inequality than (1.1), namely, for convex $f$, prove that

$$
(B_{2n}f) \left( \frac{x + y}{2} \right) \geq \sum_{i=0}^{n} \sum_{j=0}^{n} p_{n,i}(x)p_{n,j}(y) f \left( \frac{i + j}{2n} \right),
$$

(1.4)

and indeed, they proved it in [3].

A function $f \in C[0,1]$ is called $q$-monotone, $q \in \mathbb{N}$, if $\Delta_q^h(f,x) \geq 0$, for all $x \in [0,1]$ and $h > 0$, where $\Delta_q^h(f,x)$ is the $q$th forward difference with step $h > 0$. In particular, $f$ is 1-monotone or 2-monotone, if it is nondecreasing, respectively, convex in $[0,1]$. It is well known that $f \in C[0,1]$ is $q$-monotone, $q \geq 2$, if and only if $f \in C^{q-2}(0,1)$, and $f^{(q-2)}$ is convex in $(0,1)$.

Recently, Raşa’s conjecture has been extended to $q$-monotone functions, $q > 2$, by the first two authors who proved in [2] that,
Theorem B. Let \( q, n \in \mathbb{N} \). If \( f \in C[0,1] \) is a \( q \)-monotone function, then for all \( x, y \in [0,1] \),

\[
\text{sgn}(x - y)^q \sum_{\nu_1, \ldots, \nu_q = 0}^{n} (-1)^{q-j} {q \choose j} \left( \prod_{i=1}^{j} p_{n, \nu_i}(x) \right) \times \left( \prod_{i=j+1}^{q} p_{n, \nu_i}(y) \right) f \left( \frac{\nu_1 + \cdots + \nu_q}{qn} \right) \geq 0. \quad (1.5)
\]

It is well known (see, e.g., [8, 1.5(2)]) that for \( q \geq 1 \),

\[
\frac{d^q}{dx^q} (B_n f)(x) = q! \left( \frac{n}{q} \right) \sum_{i=0}^{n-q} p_{n-q,i}(x) \Delta_1^q (f, i/n).
\]

Hence, the Bernstein polynomials preserve \( q \)-monotonicity of all orders \( q \geq 1 \), so that if \( f \in C[0,1] \) is \( q \)-monotone, then for any pair \( x, y \in [0,1] \),

\[
\text{sgn}(x - y)^q \sum_{j=0}^{q} (-1)^{q-j} {q \choose j} (B_n f) \left( \frac{jx + (q-j)y}{q} \right) \geq 0. \quad (1.6)
\]

The identity analogous to (1.3) for a general \( q \) is easily proved, applying the Vandermonde’s identity

\[
\sum_{\nu_1, \ldots, \nu_q = 0}^{n} \left( \prod_{i=1}^{q} \left( \frac{n}{\nu_i} \right) \right) = \left( \frac{qn}{k} \right).
\]

Namely,

\[
(B_q^n f)(u) = \sum_{\nu_1, \ldots, \nu_q = 0}^{n} \left( \prod_{i=1}^{q} p_{n, \nu_i}(u) \right) f \left( \frac{\nu_1 + \cdots + \nu_q}{qn} \right), \quad u \in [0,1]. \quad (1.7)
\]

Hence, one may ask whether, for \( q > 2 \), a stronger inequality (analogous to (1.4)) is valid. That is, is it true that for a \( q \)-monotone \( f \),

\[
\text{sgn}(x - y)^q \sum_{j=1}^{q-1} (-1)^{q-j+1} {q \choose j} (B_q^n f) \left( \frac{jx + (q-j)y}{q} \right) \]

\[
\geq \text{sgn}(x - y)^q \sum_{\nu_1, \ldots, \nu_q = 0}^{n} \sum_{j=1}^{q-1} (-1)^{q-j+1} {q \choose j} \times \left( \prod_{i=1}^{j} p_{n, \nu_i}(x) \right) \left( \prod_{i=j+1}^{q} p_{n, \nu_i}(y) \right) f \left( \frac{\nu_1 + \cdots + \nu_q}{qn} \right)? \quad (1.8)
\]

In Sect. 2, we give an affirmative answer to this question. Namely, we prove
Theorem 1.1. If \( f \in C[0,1] \) is \( q \)-monotone, \( q \geq 3 \), then (1.8) holds for all \( n \in \mathbb{N} \).

Since, for \( f \in C[0,1] \), \( \lim_{n \to \infty} (B_n f)(x) = f(x) \), uniformly in \( x \in [0,1] \), evidently, if (1.6) holds for any pair \( x, y \in [0,1] \), for some subsequence \( \{n_k\}_{k=1}^{\infty} \), then \( f \) is \( q \)-monotone in \([0,1]\). Thus, one may ask whether this statement is true also for the weaker inequality (1.5). In Sect. 3, we answer this question affirmatively. Namely, in Sect.3, we prove

Theorem 1.2. Let \( q \geq 1 \) and \( f \in C[0,1] \). If for any pair \( x, y \in [0,1] \), inequality (1.5) holds for a subsequence \( \{n_k\}_{k=1}^{\infty} \), then \( f \) is \( q \)-monotone in \([0,1]\).

In view of the above, we propose the following open question. Prove or disprove that if \( f \in C[0,1] \) and (1.8) holds for all \( n \geq 1 \), then \( f \) is \( q \)-monotone. We note that the answer is unknown even for \( q = 2 \).

Remark 1.3. The above notwithstanding, if (1.8) with \( q = 2 \), that is, (1.4) holds for \( f \in C[0,1] \), which is twice differentiable in \((0,1)\), then, we nevertheless, can prove that \( f \) is convex. In fact, it suffices that for each pair \( x, y \in [0,1] \), (1.4) holds for some subsequence \( \{n_k\} \).

Proof. To this end, we substitute \( x = 0 \), \( y = 2t \), \( 0 < t \leq \frac{1}{2} \), into (1.4), to obtain for \( n = n_k \),

\[
(B_{2n} f)(t) \geq \sum_{j=0}^{n} p_{n,j}(2t) f \left( \frac{j}{2n} \right) = \left( B_n f \left( \frac{1}{2} \right) \right)(2t).
\]

Hence,

\[
2n \left[ (B_{2n} f)(t) - f(t) \right] \geq 2n \left[ \left( B_n f \left( \frac{1}{2} \right) \right)(2t) - f \left( \frac{1}{2} \right)(2t) \right].
\]

By Voronovskaja’s theorem for the classical Bernstein polynomials, passing to the limit on \( \{n_k\} \), yields

\[
\frac{t(1-t)}{2} f''(t) \geq 2 \frac{2t(1-2t)}{4} f''(t),
\]

which in turn implies \( f''(t) \geq 0 \), \( 0 < t \leq \frac{1}{2} \).

Repeating the same with \( g(s) := f(1-s) \), yields that \( f''(t) \geq 0 \), \( \frac{1}{2} \leq t < 1 \).

Hence, we conclude that \( f \) is convex in \([0,1]\). \( \Box \)

Final comment, note that, in our proof, we only used (1.4) with \( x = 0 \) and \( x = 1 \).

2. Proof of Theorem 1.1

We begin with some preparatory lemmas.
Lemma 2.1. If

\[ P(u, v) := \frac{1}{(u - v)^q} \sum_{k=0}^{q} (-1)^{q-k} \left( \frac{q}{k} \right) \left( \frac{ku + (q-k)v}{q} \right)^{qn}, \quad u \neq v, \]  

(2.1)

then it is a homogeneous polynomial of total degree \( q(n-1) \), in the variables \( u \) and \( v \), with non-negative coefficients.

Proof. Denote \( s := qn \) and rewrite

\[ P(u, v) = \frac{1}{(u - v)^q} \sum_{w} (-1)^{\sigma(w)} \left( \frac{w_1 + \cdots + w_q}{q} \right)^s, \]

where the sum is over \( 2^q \) possible \( w = \{w_1, \ldots, w_q\} \), with the variables \( w_1, \ldots, w_q \) taking the values \( u \) and \( v \), and for each \( w, \sigma(w) \) is the number of \( w_i \)'s that take the value \( v \).

Expanding the \( s \) power, applying the multinomial expansion, yields,

\[ P(u, v) = \frac{1}{(u - v)^q} \sum_{m_1+\cdots+m_q=s} \left( \frac{m_1, \ldots, m_q}{q} \right) \sum_{w} (-1)^{\sigma(w)} w_1^{m_1} \cdots w_q^{m_q} \]

(2.2)

Since \( \frac{u^{m_i} - v^{m_i}}{u-v} \) is a homogeneous polynomial in \( u \) and \( v \) of total degree \( m_i - 1 \), the lemma follows. \( \square \)

Thus, we may write

\[ P(u, v) =: \sum_{i=0}^{q(n-1)} a_i u^i v^{q(n-1)-i}, \]

(2.3)

where all \( a_i \geq 0 \), \( 0 \leq i \leq q(n-1) \).

We will compare the coefficients of \( P \) to the coefficients of the homogeneous polynomial of total degree \( q(n-1) \), in the variables \( u \) and \( v \),

\[ Q(u, v) := \left( \frac{u^n - v^n}{u-v} \right)^q =: \sum_{i=0}^{q(n-1)} b_i u^i v^{q(n-1)-i}. \]

(2.4)

We will prove that

\[ 0 \leq a_i \leq b_i \quad \text{for all} \quad i = 0, \ldots, q(n-1). \]

(2.5)

To this end, we observe that, by (2.2), \( P(u, v) \) is a weighted average of the polynomials \( \prod_{m=1}^{q} \frac{u^{m_i} - v^{m_i}}{u-v} \). Hence, it suffices to prove that, coefficient-by-coefficient, they maximize when \( m_1 = \cdots = m_q = n \).

Without loss of generality we may assume that \( v = 1 \) and \( u \neq 1 \).

Lemma 2.2. Let \( 1 \leq m < l-1 \). Then the coefficients of the polynomial \( \frac{(u^{m+1} - v^{m+1})(u^l - 1)}{(u-1)^2} \) are no bigger than those of \( \frac{(w^{m+1} - v^{m+1})(w^l - 1)}{(w-1)^2} \).
Proof. For any $1 \leq m \leq l$,
\[
\frac{(u^m - 1)(u^l - 1)}{(u - 1)^2} = \left( \sum_{j=0}^{l-1} u^j \right) \left( \sum_{i=0}^{m-1} u^i \right) = \sum_{j=0}^{l-1} \sum_{i=0}^{m-1} u^{i+j}
\]
\[
= \sum_{j=0}^{l-1} \sum_{k=j}^{m-1} u^k = \sum_{k=0}^{m+l-2} u^k \left( \sum_{j=\max\{0,k-m+1\}}^{\min\{k,l-1\}} 1 \right)
\]
\[
= \sum_{k=0}^{m-1} (k+1)u^k + \sum_{k=m}^{l-1} u^k \left( \sum_{j=k-m+1}^{l-1} 1 \right)
\]
\[
+ \sum_{k=m}^{m+l-2} u^k \left( \sum_{j=k-m+1}^{l-1} 1 \right)
\]
\[
= \sum_{k=0}^{m-1} (k+1)u^k + \sum_{k=m}^{l-1} u^k + \sum_{k=l}^{m+l-2} (m + l - k - 1)u^k.
\]

If $m < l - 1$, then $m + 1 \leq l - 1$, so we may replace in the above $m$ by $m + 1$ and $l$ by $l - 1$, and obtain
\[
\frac{(u^{m+1} - 1)(u^{l-1} - 1)}{(u - 1)^2} = \sum_{k=0}^{m} (k+1)u^k + (m+1) \sum_{k=m+1}^{l-2} u^k
\]
\[
+ \sum_{k=l-1}^{m+l-2} (m + l - k - 1)u^k
\]
\[
= \sum_{k=0}^{m-1} (k+1)u^k + (m+1) \sum_{k=m}^{l-2} u^k + mu^{l-1}
\]
\[
+ \sum_{k=l}^{m+l-2} (m + l - k - 1)u^k.
\]

Comparing the two equations proves the lemma. □

Lemma 2.3. Let $s = qn$ and $m_1 + \cdots + m_q = s$. Then the biggest coefficients in the expansion of $P(u) := \frac{(u^{m_1} - 1)\cdots(u^{m_q} - 1)}{(u - 1)^n}$ are obtained for $m_1 = \cdots = m_q = n$.

Proof. Suppose that for some $k$, $m_k \neq n$. Then, there exist two indices such that $m_i < n < m_j$. By Lemma 2.2, all coefficients of the polynomial $\frac{(u^{m_1} - 1)\cdots(u^{m_i} - 1)\cdots(u^{m_q} - 1)}{(u - 1)^n}$ are no smaller than those of $P(u)$. As long as there is an $m_k \neq n$, we continue. This completes the proof. □
Corollary 2.4. The difference
\[
D := \sum_{k=1}^{q-1} (-1)^{q-k+1} \binom{q}{k} \left[ \left( \frac{ku + (q-k)v}{q} \right)^{qn} - u^{kn}v^{(q-k)n} \right],
\]
is divisible by \((u - v)^q\), and the resulting polynomial is a homogeneous polynomial of total degree \(q(n-1)\) in the variables \(u\) and \(v\), with nonnegative coefficients.

Proof. Note that
\[
D = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} \left[ u^{kn}v^{(q-k)n} - \left( \frac{ku + (q-k)v}{q} \right)^{qn} \right] = (u - v)^q [Q(u, v) - P(u, v)].
\]
Hence, \(D\) is divisible by \((u - v)^q\), and the assertion that the coefficients are nonnegative follows by (2.5).

Proof of Theorem 1.1. We are ready to prove (1.8).
Using the straightforward representation
\[
p_{n,i}(u) = \frac{1}{i!} \left( \frac{\partial}{\partial z} \right)^i (1 + uz)^n \bigg|_{z=-1},
\]
by [3, Sect. 2], we have the following equations.
\[
\left( B_{qn}f \right) \left( \frac{jq + (q-j)y}{q} \right) = \sum_{k=0}^{q} \frac{1}{k!} \left[ \left( \frac{\partial}{\partial z} \right)^k \left( \frac{j(1+xz) + (q-j)(1+yz)}{q} \right)^{qn} \right] \bigg|_{z=-1} f \left( \frac{k}{qn} \right) =: J_{1,j}, \quad 1 \leq j \leq q - 1,
\]
and
\[
\sum_{\nu_1, \ldots, \nu_q=0}^{n} \left( \prod_{i=1}^{j} p_{n,\nu_i}(x) \right) \left( \prod_{i=j+1}^{q} p_{n,\nu_i}(y) \right) f \left( \frac{\nu_1 + \cdots + \nu_q}{qn} \right) = \sum_{k=0}^{q} \frac{1}{k!} \left[ \left( \frac{\partial}{\partial z} \right)^k \left( (1+xz)^j(1+yz)^{q-j} \right)^n \right] \bigg|_{z=-1} f \left( \frac{k}{qn} \right) =: J_{2,j}, \quad 1 \leq j \leq q - 1.
\]
Thus,
\[
\sum_{j=1}^{q-1} (-1)^j q^{-j+1} \binom{q}{j} (J_{1,j} - J_{2,j}) = \sum_{k=0}^{q^n} \frac{1}{k!} f \left( \frac{k}{qn} \right)
\]
\[
\left( \frac{\partial}{\partial z} \right)^k \left\{ \sum_{j=1}^{q-1} (-1)^j q^{-j+1} \binom{q}{j} \left[ \left( \frac{j(1 + xz) + (q - j)(1 + yz)}{q} \right)^{qn} - \left( (1 + xz)^j (1 + yz)^{q-j} \right)^n \right] \right\} = \sum_{k=0}^{q^n} \frac{1}{k!} \left. \frac{d^k h_{x,y}(z)}{dz^k} \right|_{z=-1} \left. f \left( \frac{k}{qn} \right) \right| =: I. \tag{2.6}
\]

Denoting \(1 + xz := u\) and \(1 + yz := v\), we conclude by Corollary 2.4 that,
\[
h_{x,y}(z) =: (x - y)^q z^q g_{x,y}(z),
\]
where
\[
g_{x,y}(z) = \sum_{j=0}^{q(n-1)-1} c_j (1 + xz)^j (1 + yz)^{q(n-1)-j},
\]
is a polynomial in \(z\) of degree \(q(n-1)\), and \(c_j \geq 0, 0 \leq j \leq q(n-1)\). Note that all derivatives of \(g_{x,y}(z)\), are nonnegative at \(z = -1\).

Indeed, the \(k\)th derivative (with respect to \(z\)), \(k \geq 0\), of the \(j\)th term in the above sum is,
\[
c_j k! \sum_{m=0}^{k} \binom{j}{m} \binom{q(n-1)-j}{k-m} x^m y^{k-m} (1 + xz)^j (1 + yz)^{q(n-1)-j-(k-m)}.
\]
Substituting \(z = -1\), we have
\[
c_j k! \sum_{m=0}^{k} \binom{j}{m} \binom{q(n-1)-j}{k-m} x^m y^{k-m} (1 - x)^j (1 - y)^{q(n-1)-j-(k-m)} \geq 0,
\]

since \(c_j \geq 0\) and \(0 \leq x, y \leq 1\).

Now,
\[
\left. \frac{d^k h_{x,y}(z)}{dz^k} \right|_{z=-1} = (x - y)^q \sum_{j=0}^{q} (-1)^q \binom{q}{j} \frac{q!}{(q-j)!} g_{x,y}^{(k-j)}(-1).
\]
Substituting into (2.6), we obtain for \(f\) which is \(q\)-monotone,
\[
(x - y)^q I = (x - y)^{2q} \sum_{k=0}^{q(n-1)} \frac{1}{k!} g_{x,y}^{(k)}(-1) \Delta^q f \left( \frac{k}{qn} \right) \geq 0,
\]
where $\Delta$ is the forward difference, namely, for the sequence \( \{a_k\} \), $\Delta^1 a_k := a_{k+1} - a_k$, and $\Delta^{m+1} a_k := \Delta(\Delta^m a_k)$.

Therefore, (1.8) is proved, and the proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

We have,

$$I_n := \sum_{\nu_1, \ldots, \nu_q=0}^{n} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \prod_{i=1}^{j} p_{n, \nu_i}(x) \prod_{i=j+1}^{q} p_{n, \nu_i}(y) f\left(\frac{\nu_1 + \cdots + \nu_q}{qn}\right)$$

$$= \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \sum_{k=0}^{qn} \sum_{r+s=k} \left( \sum_{\nu_1 + \cdots + \nu_j = r}^{r+s} \left( \prod_{i=1}^{\nu_j+1} \binom{n}{\nu_i} \right) \right)$$

$$\times x^r (1-x)^{jn-r} y^s (1-y)^{(q-j)n-s} f\left(\frac{r+s}{qn}\right).$$

Applying the generalized Vandermonde identity we obtain

$$\sum_{\nu_1+\cdots+\nu_j=r}^{r+s} \prod_{i=1}^{\nu_j+1} \binom{n}{\nu_i} = \binom{jn}{r} \binom{(q-j)n}{s}. $$

Hence,

$$I_n = \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \sum_{r=0}^{jn} \sum_{s=0}^{(q-j)n} p_{jn, r}(x) p_{(q-j)n, s}(y) f\left(\frac{r+s}{qn}\right).$$

Define

$$g_j(x, y) = f\left(\frac{jx + (q-j)y}{q}\right), \quad 0 \leq j \leq q.$$

Then,

$$g_0(x, y) = f(y) \quad \text{and} \quad g_q(x, y) = f(x),$$

and for $1 \leq j \leq q-1$,

$$g_j\left(\frac{r}{jn}, \frac{s}{(q-j)n}\right) = f\left(\frac{j \frac{r}{jn} + (q-j) \frac{s}{(q-j)n}}{q}\right) = f\left(\frac{r+s}{qn}\right).$$
Hence,

\[ I_n = (B_{qn}f)(x) + (-1)^q(B_{qn}f)(y) + \sum_{j=1}^{q-1} (-1)^{q-j} \binom{q}{j} \sum_{r=0}^{jn} \sum_{s=0}^{(q-j)n} p_{jn,r}(x)p_{(q-j)n,s}(y) g_j \left( \frac{r}{jn}, \frac{s}{(q-j)n} \right) \]

\[ = (B_{qn}f)(x) + (-1)^q(B_{qn}f)(y) + \sum_{j=1}^{q-1} (-1)^{q-j} \binom{q}{j} (B_{jn,(q-j)n}g_j)(x,y), \]

where, for a function \( g(x, y) \) of two variables, \((B_{\mu,\kappa}g)(x, y)\) is the 2-dimensional tensor Bernstein polynomial.

It is well known, see, e.g., [4, p. 122], and actually goes back to Bernstein in Soc. Math. Charkow 13 (1912-13), that if \( g \) is continuous in \([0, 1] \times [0, 1]\), then

\[ \lim_{\min\{\mu, \kappa\} \to \infty} (B_{\mu,\kappa}g)(x, y) = g(x, y) \text{ uniformly in } [0, 1] \times [0, 1]. \]

Since, by the assumptions of Theorem 1.2, inequality (1.5) is valid for a subsequence \( \{n_k\}_{k=1}^\infty \), i.e., \((x - y)^q I_{n_k} \geq 0, k \geq 1\), we get

\[ \lim_{k \to \infty} (x - y)^q I_{n_k} = (x - y)^q \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} g_j (x, y) \]

\[ = (x - y)^q \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} f \left( \frac{jx + (q-j)y}{q} \right) \geq 0, \]

so that \( f \) is \( q \)-monotone on \([0, 1]\). This completes the proof of Theorem 1.2.

**Acknowledgements**

The authors wish to thank Richard Stong for the simpler (and better) proof of Lemma 2.1, which led to his suggestion to prove Lemma 2.2. Thus, enabling us to obtain a more general Theorem 1.1.

**Author contributions** Ulrich Abel, Dany Leviatan and Ioan Raşa contributed equally to this work.

**Funding** Open Access funding enabled and organized by Projekt DEAL. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.
Data Availability Statement Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

[1] Abel, U.: An inequality involving Bernstein polynomials and convex functions. J. Approx. Theory 222, 1–7 (2017)

[2] Abel, U., Leviatan, D.: An extension of Ra˘sa’s conjecture to q-monotone functions. Results Math. 75(180), 1–13 (2020)

[3] Abel, U., Ra˘sa, I.: A sharpening of a problem on Bernstein polynomials and convex functions. Math. Ineq. Appl. 21, 773–777 (2018)

[4] Davis, P.J.: Interpolation & Approximation. Dover publications Inc., New York (1975)

[5] Gavrea, B., Gavrea, I.: An inequality involving Bernstein polynomials and box-convex functions. Med. J. Math. 19(18), 1–15 (2022)

[6] Gavrea, B., Gavrea, I.: An inequality involving some linear positive operators and box-convex functions. Results Math. 77(33), 1–10 (2022)

[7] Komisarski, A., Rajba, T.: On the Ra˘sa inequality for higher order convex functions II. Results Math. 77(88), 1–12 (2022)

[8] Lorentz, G.G.: Bernstein polynomials, Mathematical Expositions, No. 8., University of Toronto Press, Toronto, (1953)

[9] Mrowiec, J., Rajba, T., Wasowicz, S.: A solution to the problem of Rasa connected with Bernstein polynomials. J. Math. Anal. Appl. 446, 864–878 (2017)
Ulrich Abel  
Fachbereich MND  
Technische Hochschule Mittelhessen  
Wilhelm-Leuschner-Straße 13  
61169 Friedberg  
Germany  
e-mail: ulrich.abel@mnd.thm.de

Dany Leviatan  
Raymond and Beverly Sackler School of Mathematical Sciences  
Tel Aviv University  
6139001 Tel Aviv  
Israel  
e-mail: leviatan@tauex.tau.ac.il

Ioan Raşa  
Department of mathematics  
Technical University of Cluj-Napoca, Universitatea Tehnica din Cluj-Napoca  
Str. Memorandumului nr. 28  
400114 Cluj-Napoca  
Romania  
e-mail: Ioan.Rasa@math.utcluj.ro

Received: March 8, 2022.  
Accepted: September 27, 2022.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.