Analytical expressions for greybody factor and dynamic evolution for scalar field in Hořava-Lifshitz black hole

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Abstract

We investigate the propagation and evolution for a massless scalar field in the background of \( \lambda = 1/2 \) Hořava-Lifshitz black hole with the condition of detailed balance. We fortunately obtain an exact solution for the Klein-Gordon equation. Then, we find an analytical expression for the greybody factor which is valid for any frequency; and also exactly show that the perturbation decays without any oscillation. All of these can help us to understand more about the Hořava-Lifshitz gravity.

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Einstein’s general relativity is increasingly important in the modern physics, especially in the frontiers of very large distance scales including astrophysics and cosmology. However, it was proved to be nonrenormalizable by quantum field theories. In other words, the study of the ultra-violet (UV) completion of gravity has been a difficult road for theoretical physics in the past 50 years. The only convincing answer may be string theory, but it works only in perturbation theory and at energies well below the Plank scale. Recently, Hořava [1] proposed a different field theory model for a UV complete theory of gravity which can be a power counting renormalizable gravity theory in four dimensions. The Hořava-Lifshitz theory is based on the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at high energies [2]. So Hořava considered a system whose scaling at short distance exhibits a strong anisotropy between space and time. In the infrared limit, the higher derivative terms do not contribute and it reduces to standard general relativity. Thus, the Hořava-Lifshitz gravity can be regarded as an ultraviolet complete theory of general relativity.

Because of these novel features, the Hořava-Lifshitz gravity has been intensively investigated [3–10] and its cosmological applications have been studied [11–15] since then. Some metrics of the static spherically symmetric black holes with nonvanishing cosmological constant have been obtained in the Hořava-Lifshitz theory [16–20] and the associated thermodynamic properties of those black holes have been investigated [21–24]. It is well known that the Hořava-Lifshitz gravity has several free parameters which should be fixed to obtain general relativity at large scales. According to Ref. [17], the general covariance is restored only for coupling constant $\lambda = 1$, and for $\lambda \neq 1$ the Hořava-Lifshitz gravity seems not to reduce to general relativity even in large scales. Thus, the potentially observable properties of black holes in the deformed Hořava-Lifshitz gravity have been considered by using the gravitational lensing [25, 26], quasinormal modes [27] and the accretion disk [28], et al..

To understand the properties of the Hořava-Lifshitz theory, it is necessary to study other cases in which $\lambda \neq 1$. The main purpose of this paper is to study the propagation and dynamical evolution of a massless scalar field in the Hořava-Lifshitz black-hole with the coupling constant $\lambda = \frac{1}{2}$. The reason we take $\lambda = \frac{1}{2}$ here is that an exact solution for the Klein-Gordon equation in the spacetime can be obtained. Then the greybody factor and quasinormal modes can be worked out analytically.

The paper is organized as follows. In Sec. II we present the black-hole solutions in the Hořava-Lifshitz
gravity. In Sec. III, by exact analytical calculation, we study the greybody of the massless scalar field propagating in the Hořava-Lifshitz black-hole spacetime with the coupling constant $\lambda = 1/2$. In Sec. IV, we study its dynamical evolution of the scalar field. In the last section, we summarize and discuss our conclusions.

II. THE BLACK HOLES IN THE HOŘAVA-LIFSHITZ GRAVITY

The four-dimensional metric in the ADM formalism can be expressed as

$$ds^2_{ADM} = -N^2 dt^2 + g_{ij} \left( dx^i - N^i dt \right) \left( dx^j - N^j dt \right), \quad (2.1)$$

and the action of the nonrelativistic renormalizable gravitational theory proposed by Hořava is given by

$$S_{HL} = \int dt d^3 x \left( L_0 + L_1 \right), \quad (2.2)$$

with

$$L_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) + \frac{\kappa^2 \mu^2 (\Lambda W R - 3 \Lambda W^2)}{8(1 - 3 \lambda)} \right\},$$

$$L_1 = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2 (1 - 4 \lambda)}{32(1 - 3 \lambda)} R^2 - \frac{\kappa^2}{2 \omega^2} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \right\},$$

where $\kappa^2$, $\mu$, $\Lambda$, and $\omega$ are constant parameters, $N^i$ is the shift vector, $K_{ij}$ is the extrinsic curvature, and $C_{ij}$ the Cotton tensor defined by $C^{ij} = \epsilon^{ikl} \nabla_k \left( R^l \ell - \frac{1}{2} R \delta^l \right) = \epsilon^{ikl} \nabla_k R^l \ell - \frac{1}{2} \epsilon^{ikl} \partial_k R$. Taking $N^i = 0$, the spherically symmetric solution is given by

$$ds^2_{SS} = -\tilde{N}^2(r) f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.3)$$

with

$$\tilde{N} = (Rr)^{q_{\pm}(\lambda)}, \quad q_{\pm}(\lambda) = - \frac{1 + 3 \lambda \pm 2 \sqrt{6 \lambda - 2}}{\lambda - 1},$$

$$f = 1 + (Rr)^2 - m(Rr)^{1-q_{\pm}}, \quad (2.4)$$

where $R = \sqrt{-\Lambda W}$, and $m$ is an integration constant related to the mass. Hereafter we choose the negative sign of $q_{\pm}(\lambda)$ because the metric for $q_+(\lambda)$ is no physical meaning. Thus, the line element becomes

$$ds^2 = -\left( Rr \right)^{2 \xi} f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2, \quad (2.5)$$

with

$$f = 1 - m(Rr)^{1-q_{\pm}} + (Rr)^2, \quad (2.6)$$
where $\xi = q_-(\lambda)$, and its Hawking temperature is

$$T_H = \frac{R}{4\pi} \left[ 2(Rr_H)^\xi + 1 - \frac{\xi}{2} m(Rr_H)^{\xi-1}/2 \right]. \tag{2.7}$$

Though the broken of the Lorentz symmetry arises in small scale in the Hořava-Lifshitz theory, the notions of a black hole in the Hořava-Lifshitz theory in the large scale may also be defined as that in the Einstein’s general relativity [30]. Here we calculate its curvature scalar involving cosmological constant

$$R_{\mu\nu\tau\rho}R^{\mu\nu\tau\rho} = 4R^4 \left[ \xi^4 + 4\xi^3 + 8\xi^2 + 8\xi + 6 \right] + \frac{4\xi}{r^4} \left[ \xi(\xi^2 - 2\xi + 3) + 2R^2 r^2(\xi + 1)(\xi^2 + 1) \right]$$

$$-\frac{m}{r^4} (Rr_H)^{(1-\xi)/2} \left[ \xi(3\xi + 1)(\xi^2 - 2\xi + 5) + R^2 r^2(3\xi^4 + 4\xi^3 + 10\xi^2 + 8\xi + 15) \right]$$

$$+ \frac{3m^2 R}{16r^3} (Rr)^{-\xi} (3\xi^2 + 2\xi + 3)(\xi^2 - 2\xi + 9). \tag{2.8}$$

Which shows that $r = 0$ is an intrinsic singularity and $r = r_H$ is a coordinate singularity in the Hořava-Lifshitz spacetime [2.5]. Therefore, we can still define a black hole in which the event horizon is identified as the coordinate singularity.

### III. GREYBODY FACTOR OF THE SCALAR FIELD PROPAGATING IN THE HOŘAVA-LIFSHITZ BLACK-HOLE SPACETIME

Consider a massless scalar field $\Phi$ propagating in the background of the Hořava-Lifshitz black hole and let this mode has definite frequency $\omega$, so we can set

$$\Phi(t, r, \theta, \varphi) = e^{i\omega t} \phi(r) Y_{lm}(\theta, \varphi). \tag{3.1}$$

The equation of a massless scalar field is

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \Phi(t, r, \theta, \varphi) = 0. \tag{3.2}$$

Defining a tortoise coordinate

$$dx = \frac{dr}{F(r)}, \tag{3.3}$$

where $F(r) = (Rr)^\xi f(r)$, we find that the radial equation for the scalar field reads

$$\left( \frac{d^2}{dx^2} + \omega^2 - V \right)(r\phi) = 0, \tag{3.4}$$

with

$$V = F(r) \left[ \frac{1}{r} \frac{dF(r)}{dr} + \frac{l(l+1)}{r} R \right],$$

where $l$ is the angular quantum number.
A. The boundary conditions

In the near-horizon region \((r \simeq r_H\) and \(V(r) \ll \omega^2\)\) the purely ingoing solution can be expressed

\[
\phi(r) = Ae^{i\omega x}.
\] (3.5)

Since Eq. (3.4) takes Schrödinger-like form, the conversation current can be defined as

\[
j = \frac{1}{2i} \left( \phi^* \frac{d\phi}{dx} - \phi \frac{d\phi^*}{dx} \right).
\] (3.6)

The current is nothing but the flux per unit coordinate area. Thus, the flux near the horizon is

\[
J_{\text{hor}} = 4\pi r_H^2 j|_{r_H} = 4\pi r_H^2 \omega |A|^2.
\] (3.7)

Now we consider the asymptotic infinity boundary condition. For large \(r\) (i.e. \(r \gg r_H\)), the metric function in (2.4) becomes \(f(r) \sim (Rr)^{2}\) since \(\frac{1-\xi^2}{2} < 2\) for any \(\lambda\), then Eq. (3.2) reduces to

\[
\left[ r^2 \frac{d^2}{dr^2} + (\xi + 4) r \frac{d}{dr} + \frac{\omega^2}{R^2 (Rr)^{2(\xi+1)}} - \frac{l(l+1)}{R^2 r^2} \right] \phi(r) = 0.
\] (3.8)

We find the equation (3.8) has a pair of exact solutions when \(\xi = 1\) (with coupling constant \(\lambda = 1/2\)). So we focus our attention on the case of \(\xi = 1\). Let

\[
u = \frac{\omega}{2R^3 r^2},
\]
\[
\phi(r(u)) = u\mathcal{F}(u),
\]
\[
\tilde{L}^2 = \frac{l(l+1)}{R^2 r_H^2},
\]
\[
\tilde{\omega} = \frac{\omega}{R^3 r_H^2} = \frac{\omega}{2\pi T_H}.
\] (3.9)

Eq. (3.8) becomes

\[
u^2 \frac{d^2\mathcal{F}}{du^2} + \nu \frac{d\mathcal{F}}{du} + \left[ u^2 - 1 - \frac{\tilde{L}^2}{2\tilde{\omega}} u \right] \mathcal{F} = 0.
\] (3.10)

Eq. (3.10) can be transformed into a confluent hypergeometric equation, and its solutions can be represented by the first and second Kummer’s functions. Then the boundary wave function \(\phi\) is

\[
\phi(r(u)) = e^{-iu} u^2 \left[ \tilde{C}_1 M(\hat{a}, 3, 2iu) + \tilde{C}_2 U(\hat{a}, 3, 2iu) \right],
\] (3.11)

with

\[
\hat{a} = \frac{3}{2} - \frac{i\tilde{L}^2}{4\tilde{\omega}},
\] (3.12)
where \( \hat{C}_1 \) and \( \hat{C}_2 \) are constants. Using the analytical expression of the Kummer’s functions under the condition of \( u \to 0 \)

\[
M(\hat{a}, 3, 2iu) = \sum_{n=0}^{\infty} \frac{(\hat{a})_n}{(3)_n n!} (2iu)^n,
\]

\[
U(\hat{a}, 3, 2iu) = -\frac{1}{4\Gamma(\hat{a})} u^{-2} + \frac{i(\hat{a} - 2)}{2\Gamma(\hat{a})} u^{-1} - \frac{1}{2\Gamma(\hat{a} - 2)} \cdot \sum_{n=0}^{\infty} \frac{(\hat{a})_n (2iu)^n}{(3)_n n!} \left[ \ln(2iu) + \psi(\hat{a} + n) - \psi(1 + n) - \psi(3 + n) \right],
\]

with \( (\rho)_0 = 1 \), \( (\rho)_n = \Gamma(\rho + n)/\Gamma(\rho) \), \( \psi(\rho) = d\ln \Gamma(\rho)/d\rho \), the far-region boundary wave (3.11) becomes

\[
\phi(u) = \hat{C}_1 e^{-iu} u^2 \sum_{n=0}^{\infty} \frac{(\hat{a})_n}{(3)_n n!} (2iu)^n - \hat{C}_2 \frac{4\Gamma(\hat{a})}{\Gamma(\hat{a} - 2)} \sum_{n=1}^{\infty} \frac{(\hat{a})_n (2iu)^n}{(3)_n n!} \left[ \ln(2iu) + \psi(\hat{a}) - \psi(1) - \psi(3) \right] u^2
\]

\[
+ \frac{2e^{-iu} u^2 \Gamma(\hat{a})}{\Gamma(\hat{a} - 2)} \sum_{n=1}^{\infty} \frac{(\hat{a})_n (2iu)^n}{(3)_n n!} \left[ \ln(2iu) + \psi(\hat{a} + n) - \psi(1 + n) - \psi(3 + n) \right] \Bigg] .
\]

B. The exact solution for the Klein-Gordon equation

On the other hand, the equation of the motion for a massless scalar field (3.2) can be expressed as

\[
\frac{1}{r^3} \frac{d}{dr} \left[ r^3 f(r) \frac{d}{dr} \phi \right] + \left[ \frac{\omega^2}{(Rr)^2 f(r)} - \frac{l(l + 1)}{r^2} \right] \phi = 0 .
\]

In order to solve the wave equation exactly, we take

\[
z = 1 - \frac{2u}{\hat{\omega}} = 1 - \frac{r^2_{\rho}}{r^2},
\]

\[
\phi(z) = z^{\hat{\omega}/2} F(z).
\]

Then Eq. (3.15) becomes

\[
z(1 - z) \frac{d^2 F(z)}{dz^2} + (1 + i\hat{\omega} - i\hat{\omega}z) \frac{dF(z)}{dz} + \left[ \frac{1}{2} i\hat{\omega} - \frac{\hat{\omega}^2}{4} \right] F(z) = 0.
\]

It is a hypergeometric differential equation. An exact solution of the equation is

\[
\phi(z) = z^{\hat{\omega}/2} \left[ C_1 F(a, b; c; z) + C_2 z^{1-c} F(a - c + 1, b - c + 1; 2 - c; z) \right],
\]

with

\[
c = 1 + i\hat{\omega},
\]

\[
a = (i\hat{\omega} - 1 + \sqrt{1 - \hat{\omega}^2 - \hat{\omega}^2})/2,
\]

\[
b = (i\hat{\omega} - 1 - \sqrt{1 - \hat{\omega}^2 - \hat{\omega}^2})/2.
\]
C. The greybody factor

In the near horizon region \( r \to r_H \), the asymptotic results of Eq. (3.18) can be written as

\[
\phi(r \to r_H) = C_1 e^{i \omega x} + C_2 e^{-i \omega x}.
\] (3.19)

Comparing with Eq. (3.5), we have \( C_2 = 0 \). Therefore, the solution of the Eq. (3.15) is

\[
\phi(z) = z^{i \omega/2} C_1 F(a; b; c; z).
\] (3.20)

In the far-region, \( z \to 1 \), the hypergeometric function can be expressed as [31]

\[
F(a; b; c; z) = \frac{\Gamma(2) \Gamma(c)}{\Gamma(a+2) \Gamma(b+2) \Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (-1)^n} (1-z)^n - \frac{\Gamma(c)(1-z)^2}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(a+2)_n (b+2)_n}{n!(n+2)!} (1-z)^n
\times \left\{ \psi(a+2+n) + \psi(b+2+n) - \psi(1+2+n) - \psi(1+n) + \ln(1-z) \right\}.
\] (3.21)

Thus, as \( z \to 1 \), by using the expansion in the neighbor of the point \( z = 1 \)

\[
z^{\frac{i \omega}{2}} = 1 - \frac{i \omega}{2} (1-z) - \frac{i \omega^2}{8}(1-z)^2 + \cdots,
\] (3.22)

Eq. (3.20) can be rewritten as

\[
\phi(z) = C_1 \frac{4 \Gamma(c)}{\omega^2 \Gamma(a) \Gamma(b)} \left\{ \frac{4 \omega^2}{4 \omega^2 + L^4} \left[ 1 - \frac{L^2}{4} (1-z) - \frac{\omega^2}{8} (1-z)^2 \left( -1 + \frac{2i}{\omega} - \frac{i L^2}{\omega} \right) \right]
- \frac{\omega^2}{8} (1-z)^2 \left[ \psi(a+2) + \psi(b+2) - \psi(3) - \psi(1) + \ln(1-z) \right] - \frac{\omega^2}{4} \sum_{n=1}^{\infty} \frac{(a+2)_n (b+2)_n}{n!(n+2)!}
\times \frac{\omega^2}{1} (1-z)^n + \cdots \right\}.
\] (3.23)

By matching the far-region boundary condition (3.14) onto Eq. (3.23), we obtain

\[
\hat{C}_1 = -C_1 A(\omega) \left[ \frac{4 \omega^2}{4 \omega^2 + L^4} \left( C(\omega) + \frac{1}{\omega} \right) + \frac{B(\omega)}{2} \right],
\]

\[
\hat{C}_2 = -C_1 A(\omega) \frac{16 \Gamma(\hat{a}) \omega^2}{4 \omega^2 + L^4},
\] (3.24)

with

\[
A(\omega) = \frac{4 \Gamma(c)}{\omega^2 \Gamma(a) \Gamma(b)},
\]

\[
B(\omega) = \psi(a+2) + \psi(b+2) - \psi(3) - \psi(1),
\]

\[
C(\omega) = \frac{2 \Gamma(\hat{a})}{\Gamma(\hat{a}-2)} \left[ \frac{i \pi}{2} + \ln \omega + \psi(\hat{a}) - \psi(1) - \psi(3) \right].
\] (3.25)
We now separate the far-region solution into the parts of outgoing and ingoing waves. Eq. (3.23) becomes

\[
\phi(u) = f_0 \left\{ \left[ 2\hat{C}_1 f_0 + 2i\hat{C}_2 \text{Im}\left( \frac{f_1}{f_0} \right) + 2\hat{C}_2 \text{Re}\left( \frac{f_1}{f_0} \right) \right] u^2 + \hat{C}_1 e^{-iu} \sum_{n=1}^{\infty} \frac{\hat{a}_n^{(2)}(2iu)^n}{(3)_{n!}} \right\},
\]

with

\[
f_0 = -\frac{1}{4\Gamma(\hat{a})},
\]

\[
f_1 = -\frac{1}{4\Gamma(\hat{a})} \left[ C(\omega) + \frac{1}{2} + i\hat{L}^2 \frac{2\omega}{2\omega} \right].
\]

Defining quantities \( D_1 \) and \( D_2 \) as

\[
D_1 + D_2 = 2\hat{C}_1 f_0 + 2i\hat{C}_2 \text{Im}\left( \frac{f_1}{f_0} \right),
\]

\[
i(D_1 - D_2) = 2\hat{C}_2 \text{Im}\left( \frac{f_1}{f_0} \right),
\]

and using the tortoise coordinate in the far-region \( \frac{d}{dx} = -\omega \frac{d}{du} \), the asymptotic flux (\( u \to 0 \)) is given by

\[
J_{\text{asy}} = 4\pi r^2 \cdot \frac{u\omega}{2\text{Im}(\frac{f_0}{f_0})} \left| D_2 \right|^2 - \left| D_1 \right|^2.
\]

Its incoming and outgoing fluxes are

\[
J_{\text{in}} = \frac{\pi\tilde{\omega}^2}{\text{Im}(\frac{f_0}{f_0})} R^3 r_H^4 |D_2|^2,
\]

\[
J_{\text{out}} = \frac{\pi\tilde{\omega}^2}{\text{Im}(\frac{f_0}{f_0})} R^3 r_H^4 |D_1|^2.
\]

Lastly, the greybody factor is

\[
\gamma(\tilde{\omega}) = 1 - \frac{J_{\text{out}}}{J_{\text{in}}} = 1 - \frac{|\frac{\hat{C}_1}{f_0}|^2}{|\frac{\hat{C}_1}{f_0} + 2i\hat{C}_2 \text{Im}\left( \frac{f_1}{f_0} \right)|^2},
\]

which is valid for any frequency. It is well known that for usual black holes in Einstein’s general relativity, one can only calculate the greybody factors analytically in the low frequency region by some approximations as the ones first set out in [32, 33], or calculate them by numerical method [34–36]. Some notable exceptions appear in string theory when considering the propagation of a massless scalar field in the background of an extremal dyonic string in six dimensions [37], and in the background supergravity for the D3-brane [38], where one can obtain analytical expressions for the greybody factor in terms of Mathieu functions. For 3D dilaton
black holes, the greybody factors can be analytically obtained only when frequency $\omega \geq 2$ \cite{39}. Therefore, in four dimensional spacetime our analytical result obtained here is very special one.

In order to have an intuitional picture, we now plot some figures for $\gamma(\tilde{\omega})$ with different angular numbers in following.

i) The greybody factor for $s$-wave, i.e. the angular number $l = 0$:

We now show the greybody for $s$-wave in Figs. (1) and (2), which tell us that the curves of the grey body factor in the $\lambda = 1/2$ Hořava-Lifshitz spacetime are continuous in all frequencies and it approaches to unity when $\tilde{\omega} \geq 5$ which is similar to the numerical results for the black 3-brane in an asymptotically AdS$_5 \times$S$_5$ space \cite{34}.

Fig. (1) shows that the greybody factor $\gamma(\tilde{\omega}) \sim \pi \tilde{\omega}$ in the low frequency region when the frequency of the scalar field $\tilde{\omega} \in (0, 0.01]$. For asymptotically AdS spacetime, the result is $\gamma(\tilde{\omega}) \sim \tilde{\omega}^2$ \cite{40} by using the low frequency approximation approach ($\tilde{\omega} \ll 1$). For the black 3-brane in an asymptotically AdS$_5 \times$S$_5$ space, the result is $\gamma(\tilde{\omega}) \sim \tilde{\omega}^3$ \cite{34} via the low frequency approximation method ($\tilde{\omega} \ll 1/\pi$).

Fig. (2) gives that $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.7}$ when $\tilde{\omega} \in [5, 100]$; $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.4}$ when $\tilde{\omega} \in [100, 1000]$; and $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.25}$ when $\tilde{\omega} \in [1000, 10000]$. For the black 3-brane in an asymptotically AdS$_5 \times$S$_5$ space, the result is $\gamma(\tilde{\omega}) \sim 1 - 2.3\tilde{\omega}^{-8}$ when $\tilde{\omega} \in [8, 15]$ by using numerical scheme \cite{34}.

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.5\textwidth]{fig1.png}
\end{center}
\caption{(Color online) The greybody factor $\gamma(\tilde{\omega})$ of the $\lambda = 1/2$ Hořava-Lifshitz black hole when $\tilde{\omega} \in [0, 0.01]$ for $s$-wave. The solid (red) line present the greybody factor $\gamma(\tilde{\omega})$ described by Eq. (3.32), and the dashed (green) line shows that $\gamma(\tilde{\omega}) \sim \pi \tilde{\omega}$.}
\end{figure}

ii) The greybody factor for the angular number $l \neq 0$:

The curves in Fig. (3) show the greybody factor with different angular number $l$ in the $\lambda = 1/2$ Hořava-Lifshitz black-hole spacetime. It is easy to see that the greybody factor decreases as the angular number $l$ increases, and approaches to unity at large frequencies. These features are similar to usual black holes \cite{35, 36, 42, 43}.
FIG. 2: (Color online) The greybody factor $\gamma(\tilde{\omega})$ of the $\lambda = 1/2$ Hořava-Lifshitz black hole for $s$-wave. The solid (red) line present the greybody factor $\gamma(\tilde{\omega})$ described by Eq. (3.32). The dashed (green) line in the second graph shows that $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.7}$ when $\tilde{\omega} \in [5, 100]$; the dashed line in the third graph shows that $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.4}$ when $\tilde{\omega} \in [100, 1000]$; the last figure shows the logarithm of the greybody factor $\ln \gamma(\tilde{\omega})$, and the dashed line shows that $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.25}$ when $\tilde{\omega} \in [1000, 10000]$.

FIG. 3: (Color online) The greybody factor $\gamma(\tilde{\omega})$ of the $\lambda = 1/2$ Hořava-Lifshitz black hole with different angular number $l$. The lines from left to right in two graphs are corresponding to $l = 0, 1, 2, 3$.

IV. DYNAMIC EVOLUTION OF THE SCALAR FIELD PERTURBATION USING ANALYTICAL METHOD

In Ref. [30], we found that the dynamic evolution of the scalar field perturbation in the $\lambda = 1/2$ Hořava-Lifshitz black-hole spacetime is purely damped modes for $l = 0$ by using numerical method. In this section, we show that we can obtain dynamic evolution modes from Eq. (3.18) by using analytical method for any $l$.

The quasinormal modes of a classical perturbation of black-hole spacetime are defined as the solutions of the wave equations with purely ingoing waves at the horizon, and the amplitude of the ingoing wave has to be zero at the asymptotic region $r \rightarrow \infty$. This leads to $|D_2|^2 = 0$, i.e., $A(\omega) = 0$. Hence, Eq. (3.24) shows us
that the quantity $a + 2$ or $b + 2$ should be a negative integer

$$a + 2 = \frac{1}{2}(i\tilde{\omega} - 1 + \sqrt{1 - \tilde{L}^2 - \tilde{\omega}^2}) + 2 = -n, \quad (n = 0, 1, 2, 3, \cdots). \quad (4.1)$$

Solving it we obtain

$$\tilde{\omega}_n = -i \left[ n + \frac{3}{2} - \frac{1 - \tilde{L}^2}{2(2n + 3)} \right], \quad (4.2)$$

which is showed in Fig. 4. If we take $l = 0$ and use the equation $\tilde{\omega} = \omega/(2\pi T_H^{\lambda=1/2})$, this analytical result coincides with the numerical results $\tilde{\omega}_n = -i(1.009n + 1.333)$ in Ref. 30 (see Fig. 4). These modes are purely damped modes which may be caused by its over-damped potential, and these purely damped modes show that this spacetime is very stable.

![Graph showing the asymptotic behavior of high overtones $\tilde{\omega}_n$ of purely damped frequency of the scalar field perturbation versus overtone number $n$ and different angular number $l$. The right graph shows that the analytical result (solid (red) line) and the numerical result (dashed (blue) line) when $l = 0$.](image)

**V. SUMMARY AND DISCUSSION**

We have studied the propagation and dynamic evolution of the massless scalar field in the background of the $\lambda = 1/2$ Hořava-Lifshitz black hole with the condition of the detailed balance. It is interesting to note that we obtained an exact solution for the Klein-Gordon equation in the whole spacetime. Then, imposing the boundary conditions at the event horizon and infinity, we have found an analytical expression for the greybody factor of the scalar field propagating in the Hořava-Lifshitz black-hole spacetime. The greybody factor approaches to unity when $\tilde{\omega} \geq 5$ and is valid both for any frequency and angular number. To compare with other results, we have also shown behaviors of the greybody factor for the low-frequency and high-frequency for $l = 0$: the greybody factor $\gamma(\tilde{\omega}) \sim \pi\tilde{\omega}$ when the frequency of scalar field $\tilde{\omega} \in (0, 0.01)$; $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.7}$ when $\tilde{\omega} \in [5, 100]$, $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.4}$ when $\tilde{\omega} \in [100, 1000]$, and $\gamma(\tilde{\omega}) \sim 1 - \tilde{\omega}^{-2.25}$ when $\tilde{\omega} \in [1000, 10000]$. 
On the other hand, we have calculated the quasinormal modes of the black hole by using analytical method. The exact result for the modes is given by equation (4.2). If we take \( l = 0 \), then the modes coincide with the numerical results \( \tilde{\omega}_n = -i(1.009n + 1.333) \) in Ref. [30].

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