A Complete Asymptotic Expansion for Bernstein–Chlodovsky Polynomials for Functions on $\mathbb{R}$

Ulrich Abel and Harun Karsli

Abstract. We consider a variant of the Bernstein–Chlodovsky polynomials approximating continuous functions on the entire real line and study its rate of convergence. The main result is a complete asymptotic expansion. As a special case we obtain a Voronovskaja-type formula previously derived by Karsli [11].

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1. Introduction

Let $f$ be a real function on $\mathbb{R}$ which is bounded on each finite interval. For $a, b \in \mathbb{R}$ with $a < b$, define the function $f_{a,b}$ on $[0, 1]$ by $f_{a,b}(t) = f(a + (b - a)t)$. Furthermore, put

$$
\|f\|_{a,b} = \sup_{a \leq t \leq b} |f(t)|.
$$

Obviously, $f_{0,1}$ is the restriction of $f$ to $[0, 1]$ and we have $\|f\|_{a,b} = \|f_{a,b}\|_{0,1}$.

The Bernstein–Chlodovsky operators applied to the function $f$ described above are defined by

$$(C_{n,a,b} f)(x) = (B_n f_{a,b}) \left( \frac{x - a}{b - a} \right), \quad a \leq x \leq b,$$

where $B_n$ denote the Bernstein operators defined by

$$(B_n f)(x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f \left( \frac{\nu}{n} \right), \quad 0 \leq x \leq 1,$$

with Bernstein basis polynomials

$$p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu} (1 - x)^{n - \nu}, \quad 0 \leq \nu \leq n.$$
In the special case \([a, b] = [0, 1]\), we have \(C_{n,0,1} \equiv B_n\).

The symmetric version

\[
(C_{n,-c,c}f)(x) = (B_n f_{-c,c})(x + c)\frac{x + c}{2c}
\]

with \(-a = b = c > 0\) was recently introduced by Kilgore [12, Eq. (7.1)]. Using this Chlodovsky generalization of the Bernstein operators he [12, Theorems 1 and 2] gave a constructive proof for the Weierstrass approximation theorem in weighted spaces of continuous functions defined on \([0, \infty)\) or on \((-\infty, \infty)\).

See also [13].

In the following we suppose that the parameters \(a, b\) are coupled with \(n\), i.e., \(a = a_n\) and \(b = b_n\). Because the difference between two nodes of \(C_{n,a,b}\) is at least \((b - a)/n\) it is clear that the condition \(b_n - a_n = o(n)\) as \(n \to \infty\) is necessary for having convergence of \((C_{n,a_n,b_n}f)(x)\) to \(f(x)\).

In the special case \(a_n = 0 < b_n\), for \(n \in \mathbb{N}\), these polynomials were introduced by I. Chlodovsky [7] in 1937 in order to approximate functions on infinite intervals. He showed that under the condition (1.3), if a function \(f\) satisfies

\[
\lim_{n \to \infty} \exp \left(\frac{-\sigma n}{b_n}\right) \|f\|_{b_n} = 0,
\]

for every \(\sigma > 0\), then

\[
\lim_{n \to \infty} (C_{n,0,b_n}f)(x) = f(x)
\]

at each point \(x\) of continuity of \(f\). Moreover, he proved convergence in each continuity point for the large class of functions \(f\) satisfying the growth condition \(f(t) = O(\exp(t^p))\) as \(t \to +\infty\), if the sequence \((b_n)\) satisfies the condition

\[
b_n = O\left(n^{1/(p+1+\eta)}\right) \quad (n \to \infty),
\]

for an arbitrary small \(\eta > 0\). For more results on Chlodovsky operators see the survey article [9] by Karsli.

Explicit expressions of the coefficients \(c_{[b_n]}^{[a_n]}(f,x)\) in terms of Stirling numbers were given by Karsli [10]. He derived the asymptotic expansion if the function \(f\) satisfies condition (1.1) for every \(\sigma > 0\).

Throughout the paper we assume that the sequences \((a_n)\) and \((b_n)\) satisfy

\[
b_n - a_n > 0, \quad \lim_{n \to \infty} (-a_n) = \lim_{n \to \infty} b_n = +\infty, \quad \text{and} \quad \lim_{n \to \infty} \frac{-a_n b_n}{n} = 0.
\]

The purpose of this note is a pointwise complete asymptotic expansion for the sequence of Bernstein–Chlodovsky operators in the form:

\[
(C_{n,a_n,b_n}f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_{[a_n,b_n]}^{[a_n,b_n]}(f,x) \left(\frac{-a_n b_n}{n}\right)^k + o \left(\left(\frac{-a_n b_n}{n}\right)^q\right)
\]

as \(n \to \infty\), for sufficiently smooth functions \(f\) satisfying \(f(t) = O(\exp(\alpha t^p))\) as \(t \to +\infty\), provided that the sequences \((a_n)\), \((b_n)\) satisfy \((-a_n b_n) = \ldots\)
\(o(n^{1/(p+1)})\) as \(n \to \infty\). The coefficients \(c_k^{[a_n,b_n]}(f,x)\), which depend on \(f\) and \(a_n, b_n\), are bounded with respect to \(n\).

The latter formula means that, for each fixed \(x > 0\) and for all positive integers \(q\):

\[
(C_n,a_n,b_n f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[a_n,b_n]}(f,x) \left( \frac{-a_n b_n}{n} \right)^k + o \left( \left( \frac{-a_n b_n}{n} \right)^q \right)
\]
as \(n \to \infty\).

2. Main Result

For real constants \(\alpha \geq 0\) and \(p \geq 0\), let \(W_{\alpha,p}\) denote the class of functions \(f \in C(\mathbb{R})\) satisfying the growth condition:

\[
f(t) = O\left(\exp(\alpha|t|^p)\right) \quad (|t| \to +\infty).
\]

Note that in the special instance \(p = 0\) the class \(W_{\alpha,0}\) consists of the bounded continuous functions on \(\mathbb{R}\). Since \(W_{0,p}\) and \(W_{\alpha,0}\) coincide we consider only the case \(\alpha > 0\).

Recall that the Stirling numbers \(s(n,k)\) and \(S(n,k)\) of first and second kind, respectively, are defined by the relations:

\[
z^n = \sum_{k=0}^{n} s(n,k) z^k \quad \text{and} \quad z^n = \sum_{k=0}^{n} S(n,k) z^k \quad (z \in \mathbb{C}),
\]

where \(z^0 = 1\) and \(z^n = z(z-1)\cdots(z-n+1)\), for \(n \in \mathbb{N}\), denote the falling factorials.

The following theorem is the main result.

**Theorem 2.1.** Let \(\alpha, p \geq 0\). Suppose that the function \(f \in W_{\alpha,p}\) is \(2q\) times differentiable in the point \(x > 0\). Let \((-a_n)\) and \((b_n)\) be sequences of reals tending to infinity and satisfying the growth condition:

\[
-a_n b_n = o\left(n^{1/(p+1)}\right) \quad (n \to \infty).
\]

Then, for any positive integer \(q\), the Bernstein–Chlodovsky operators \(C_{n,a_n,b_n}\) possess the asymptotic expansion:

\[
(C_n,a_n,b_n f)(x) = f(x) + \sum_{k=1}^{q} c_k^{[a_n,b_n]}(f,x) \left( \frac{-a_n b_n}{n} \right)^k + o \left( \left( \frac{-a_n b_n}{n} \right)^q \right)
\]
as \(n \to \infty\), where

\[
c_k^{[a_n,b_n]}(f,x) = O(1) \quad (n \to \infty).
\]

The coefficients \(c_k^{[a,b]}(f,x)\) have the explicit representation:

\[
c_k^{[a,b]}(f,x) = \sum_{s=k}^{2k} \frac{f(s)}{s!} \sum_{j=0}^{s} A(k,s,j) \frac{(b-a)^j (x-a)^{s-j}}{(-ab)^k} \quad (a < b)
\]
with numbers

\[ A(k, s, j) = \sum_{r=\max\{j,k\}}^{s} (-1)^{s-r} \binom{s}{r} s \cdot (r - j, r - k) S(r, r - j). \quad (2.3) \]

Remark 2.2. Note that the coefficients \( c_k^{[a_n, b_n]}(f, x) \) depend on \( n \) but are bounded with respect to \( n \).

Remark 2.3. Our assumption (2.1) on the sequence \((-a_n, b_n)\) corresponds to Chlodovsky’s condition (1.2). Furthermore, it is related to the assumption in the case \( a_n = 0 \) (see, [3, Theorem 1, Eq. (4)]).

In the special case \( q = 1 \) Theorem 2.1 implies the following Voronovskaja-type result.

**Corollary 2.4.** Let \( \alpha, p \geq 0 \). Suppose that the function \( f \in W_{\alpha, p} \) admits a second derivative at the point \( x > 0 \). Let \((-a_n)\) and \((b_n)\) be sequences of reals tending to infinity and satisfying the growth condition (2.1). Then, the Bernstein–Chlodovsky operators \( C_{n,a_n,b_n} \) satisfy the asymptotic relation:

\[ \lim_{n \to \infty} \frac{n}{-a_n b_n} ((C_{n,a_n,b_n} f)(x) - f(x)) = \frac{1}{2} f^{(2)}(x) \quad (2.4) \]

Remark 2.5. The expansion in Theorem 2.1 is completely different to the (pointwise) complete asymptotic expansion:

\[ (B_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_k^{[0,1]}(f, x) n^{-k} \quad (n \to \infty), \]

for the classical Bernstein polynomials \( B_n \), which is valid for all bounded functions \( f : [0, 1] \to \mathbb{R} \) being sufficiently smooth in \( x \in [0, 1] \). The Voronovskaja formula states that

\[ \lim_{n \to \infty} n ((B_n f)(x) - f(x)) = \frac{1}{2} x (1 - x) f^{(2)}(x). \]

The same is true in the case of the classical Bernstein–Chlodovsky operators \( C_{n,0,b_n} \). Their Voronovskaja-type formula:

\[ \lim_{n \to \infty} \frac{n}{b_n} ((C_{n,0,b_n} f)(x) - f(x)) = \frac{1}{2} x f^{(2)}(x) \]

was derived in 1960 by Albrycht and Radecki [5]. For further history consult the survey article [9].

3. Auxiliary Results and Proof of the Main Theorem

Our starting-point is an explicit representation of the central moments of the Bernstein polynomials in terms of Stirling numbers of the first and second kind. In the following we write \( e_m(x) = x^m, m \in \mathbb{N}_0 \), for the \( m \)-th monomial and \( \psi_x(t) = t - x \) for \( x \in \mathbb{R} \).
Lemma 3.1. The central moments of the Bernstein polynomials possess the representation

\[(B_n \psi^s_x)(x) = \sum_{k=\left[\frac{s+1}{2}\right]}^s n^{-k} \sum_{j=0}^s A(k, s, j) x^{s-j}\]

\[(s = 0, 1, 2, \ldots),\] where the coefficients are given by Eq. (2.3).

For a proof see, e.g., [2].

Lemma 3.2. The central moments of the Bernstein–Chlodovsky operators possess the representation:

\[(C_{n,a,b} \psi^s_x)(x) = \sum_{k=\left[\frac{s+1}{2}\right]}^s n^{-k} \sum_{j=0}^s A(k, s, j) (b - a)^j (x - a)^{s-j}, \]

where the coefficients \(A(k, s, j)\) are given by Eq. (2.3).

Proof. We have

\[(C_{n,a,b} \psi^s_x)(x) = \sum_{\nu=0}^n p_{n, \nu} \left(\frac{x - a}{b - a}\right) \left(a + (b - a) \frac{\nu}{n} - x\right)^s \]

\[= (b - a)^s \sum_{\nu=0}^n p_{n, \nu} \left(\frac{x - a}{b - a}\right) \left(\frac{\nu}{n} - \frac{x - a}{b - a}\right)^s \]

\[= (b - a)^s \left(B_n \psi^s_{\frac{x-a}{b-a}}\right) \left(\frac{x - a}{b - a}\right)\]

and the lemma follows by Lemma 3.1. \(\square\)

As we have seen in the proof of Lemma 3.2 the central moments of the Bernstein–Chlodovsky operators can be expressed in terms of Bernstein polynomials:

\[(C_{n,a,b} \psi^s_x)(x) = (b - a)^s \left(B_n \psi^s_{\frac{x-a}{b-a}}\right) \left(\frac{x - a}{b - a}\right).\]

As a consequence from well-known properties of the Bernstein polynomials we obtain the following result:

Lemma 3.3. For \(a \leq x \leq b\), it holds

\[(C_{n,a,b} \psi^{2s}_x)(x) = \sum_{k=s}^{2s} \left(\frac{x - a}{b - x}\right)^k \times \sum_{i=2s-k}^s d_{2s, i} \left(\frac{x - a}{b - x}\right)^{i-k} (b - a)^{2s-2k},\]

\[(C_{n,a,b} \psi^{2s+1}_x)(x) = (b + a - 2x) \sum_{k=s+1}^{2s+1} \left(\frac{x - a}{b - x}\right)^k \times \sum_{i=2s+1-k}^s d_{2s+1, i} \left(\frac{x - a}{b - x}\right)^{i-k} (b - a)^{2s-2k},\]
where \( d_{s,i} \) are certain real numbers.

Proof. Taking advantage of the well-known formulas for the central moments of the Bernstein polynomials (see [8, Chapt. 10, Theorem 1.1]):

\[
\begin{align*}
(B_n \psi_x^{2s}) (x) &= \sum_{j=0}^{s} n^{j-2s} \sum_{i=j}^{s} d_{2s,i} (x (1-x))^i, \\
(B_n \psi_x^{2s+1}) (x) &= (1 - 2x) \sum_{j=0}^{s} n^{j-(2s+1)} \sum_{i=j}^{s} d_{2s+1,i} (x (1-x))^i,
\end{align*}
\]

where \( d_{s,i} \) are certain real numbers, we obtain

\[
\begin{align*}
(C_n, a, b \psi_x^{2s}) (x) &= (b - a)^{2s} \sum_{k=s}^{2s} \sum_{i=2s-k}^{s} \frac{1}{n^k} d_{2s,i} \left( \frac{(x-a)(b-x)}{(b-a)^2} \right)^i, \\
(C_n, a, b \psi_x^{2s+1}) (x) &= (b - a)^{2s} (b + a - 2x) \sum_{k=s+1}^{2s+1} \frac{1}{n^k} \\
&\quad \times \sum_{i=2s+1-k}^{s} d_{2s+1,i} \left( \frac{(x-a)(b-x)}{(b-a)^2} \right)^i.
\end{align*}
\]

This can be rewritten in the form as stated in the lemma. □

For the sake of brevity, in the following, we write

\[
(C_n, a, b \psi_x^s) (x) = \sum_{k=\left\lfloor \frac{s+1}{2} \right\rfloor}^{s} \left( \frac{-ab}{n} \right)^k Q_{k,s} (a, b, s; x). \quad (3.2)
\]

Note that, for \( a \leq x \leq b \), with \( a < 0 < b \), we have

\[
\begin{align*}
\left| \frac{(x-a)(b-x)}{n} \right| &\leq \left| 1 - \frac{x}{a} \right| \left| 1 - \frac{x}{b} \right| \cdot \frac{|ab|}{n}, \\
\left| \frac{(x-a)(b-x)}{(b-a)^2} \right| &\leq \frac{1}{4}, \\
\left| \frac{b + a - 2x}{b - a} \right| &\leq 1, \\
(b - a)^m &\leq 1, \quad \text{for } m \leq 0, \quad \text{if } b - a \geq 1.
\end{align*}
\]

This immediately implies the following estimate for the central moment of the Bernstein–Chlodovsky operators.

**Lemma 3.4.** Let \((-a_n)\) and \((b_n)\) be sequences of reals tending to infinity and satisfying the condition \(-a_n b_n = o(n)\) as \(n \to \infty\). Then, for \(s = 0, 1, 2, \ldots\), the quantities \(Q_{k,s} (a_n, b_n, s; x) \left( \left\lfloor \frac{s+1}{2} \right\rfloor \leq k \leq s \right)\) are bounded with respect to \(n\), and

\[
(C_n, a_n, b_n \psi_x^s) (x) = O \left( \left( \frac{-a_n b_n}{n} \right)^{\left\lfloor \frac{s+1}{2} \right\rfloor} \right) \quad (n \to \infty).
\]
A crucial tool is the following estimate due to Bernstein (see [14, Theorem 1.5.3, p. 18f]).

**Lemma 3.5.** (Bernstein) For $0 \leq t \leq 1$, the inequality

$$0 \leq z \leq \frac{3}{2} \sqrt{nt(1-t)}$$

implies

$$\sum_{|\nu - nt|\geq 2z\sqrt{nt(1-t)}} p_{\nu \nu}(t) \leq 2 \exp(-z^2).$$

The next lemma presents a form of Lemma 3.5 which is more useful for application to Chlodovsky operators on the real line. It follows the idea of Albrycht and Radecki [5] who proved a similar result for the classical Chlodovsky operators.

**Lemma 3.6.** Let $a < x < b$. If $0 < \delta \leq 3 (x - a) (b - x) / (b - a)$ it holds

$$\sum_{|\nu - n \frac{x-a}{b-a}| \geq \delta / n} p_{\nu \nu}\left(\frac{x-a}{b-a}\right) \leq 2 \exp\left(-\frac{n\delta^2}{4(x-a)(b-x)}\right)$$

**Proof of Lemma 3.6.** Putting $t = (x-a) / (b-a)$ in Lemma 3.5, we have

$$\sum_{|\nu - n \frac{x-a}{b-a}| \geq 2z\sqrt{n^{-1}(x-a)(b-x)}} p_{\nu \nu}\left(\frac{x-a}{b-a}\right) \leq 2 \exp\left(-z^2\right)$$

if $0 \leq z \leq \frac{3}{2} \sqrt{n^{-1}(x-a)(b-x)}$. Choose $\delta = 2z\sqrt{n^{-1}(x-a)(b-x)}$. Then

$$\sum_{|\nu - n \frac{x-a}{b-a}| \geq \delta / n} p_{\nu \nu}\left(\frac{x-a}{b-a}\right) \leq 2 \exp\left(-\frac{n\delta^2}{4(x-a)(b-x)}\right)$$

if $0 \leq \delta / \left(2\sqrt{n^{-1}(x-a)(b-x)}\right) \leq \frac{3}{2} \sqrt{n^{-1}(x-a)(b-x)}$. The latter inequality is equivalent to

$$\delta \leq 3 \frac{(x-a)(b-x)}{b-a}$$

which is a condition of the lemma. □

**Lemma 3.7.** Let $a < x < b$ and $0 < \delta \leq 3 (x - a) (b - x) / (b - a)$. If a bounded function $f : [a,b] \rightarrow \mathbb{R}$ satisfies $f(t) = 0$, for all $t \in (x-\delta, x+\delta) \cap [a,b]$, it follows the estimate

$$| (C_{n,a,b}f) (x) | \leq 2 \exp\left(-\frac{n\delta^2}{4(x-a)(b-x)}\right) \| f \|_{a,b}.$$
Proof. Because of \( f \left( a + (b - a) \frac{\nu}{n} \right) = 0 \) for all \( \nu \in \{0, \ldots, n\} \) with \( \left| a + (b - a) \frac{\nu}{n} - x \right| < \delta \) we have

\[
| (C_{n,a,b} f) (x) | = \left| \sum_{\nu} p_{n,\nu} \left( \frac{x - a}{b - a} \right) f \left( a + (b - a) \frac{\nu}{n} \right) \right|_{ \left| a + (b - a) \frac{\nu}{n} \right| \geq \delta } \leq \| f \|_{a,b} \sum_{\nu} \left| p_{n,\nu} \left( \frac{x - a}{b - a} \right) \right|_{ \left| a + (b - a) \frac{\nu}{n} \right| \geq \delta } \]

and the assertion follows by an application of Lemma 3.6. \( \square \)

A direct consequence is the following localization result for Bernstein–Chlodovsky polynomials which is interesting in itself.

**Proposition 3.8 (Localization theorem).** Let \( \alpha, p \geq 0 \) be fixed constants and propose that \( f \in W_{\alpha,p} \) satisfies the estimate

\[
| f (t) | \leq K \exp \left( \alpha | t |^p \right) \quad (t \in \mathbb{R}).
\]

Furthermore, fix the real number \( x \in (a, b) \) and let \( \delta > 0 \). Then \( f (t) = 0 \), for all \( t \in (x - \delta, x + \delta) \), implies

\[
| (C_{n,a,b} f) (x) | \leq 2K \exp \left( -\frac{n\delta^2}{4 (x - a) (b - x)} \right) \exp \left( \alpha \max \{|a|, |b|\}^p \right).
\]

Suppose that \( \lim_{n \to \infty} (-a_n) = \lim_{n \to \infty} b_n = +\infty \). Then, for sufficiently large values of \( n \), we can assume that \( a_n < -1 < 1 < b_n \), such that \( \max \{|a_n|, |b_n|\} \leq -a_n b_n \). Since \( (x - a_n) (b_n - x) = O (-a_n b_n) \) as \( n \to \infty \), there is as positive constant \( M (x) \) (independent of \( n \)), such that

\[
\frac{1}{4 (x - a_n) (b_n - x)} \geq \frac{M (x)}{-a_n b_n},
\]

for sufficiently large values of \( n \). Hence, we have

\[
\exp \left( -\frac{n\delta^2}{4 (x - a) (b - x)} \right) \leq \exp \left( -M (x) \frac{n\delta^2}{-a_n b_n} \right).
\]

We conclude that, for large \( n \),

\[
| (C_{n,a_n,b_n} f) (x) | \leq 2K \exp \left( -M (x) \frac{n\delta^2}{-a_n b_n} + \alpha (-a_n b_n)^p \right)
= 2K \exp \left( -M (x) \frac{n}{-a_n b_n} \left( \delta^2 - \frac{\alpha}{M (x)} \cdot \frac{(-a_n b_n)^{p+1}}{n} \right) \right).
\]

**Proof of Theorem 2.1.** Suppose that \( f \) is continuous on \( \mathbb{R} \) being \( 2q \) times differentiable at the point \( x \in \mathbb{R} \). Define the function \( h_x \) by

\[
f = \sum_{s=0}^{2q} \frac{f(s) (x)}{s!} \psi_x^s + h_x \psi_x^{2q}
\]
and \( h_x(x) = 0 \). It is a consequence of Taylor’s theorem that \( h_x \) is continuous at \( x \). Hence, \( h_x \in C(\mathbb{R}) \). Applying the operator \( C_{n,a,b} \) to both sides of Eq. (3.3) we obtain

\[
(C_{n,a,b}f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (C_{n,a,b}\psi_x^s)(x) + (C_{n,a,b}(h_x\psi_x^{2q}))(x).
\]

The first sum is equal to

\[
\sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (C_{n,a,b}\psi_x^s)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{k=\lfloor \frac{s+1}{2} \rfloor}^s (-ab)^k \frac{k!}{n^k} Q_{k,s}(a, b, s; x)
\]

\[
= \sum_{k=0}^{2q} (-ab)^k \frac{2k}{n^k} \frac{f^{(s)}(x)}{s!} Q_{k,s}(a, b, s; x)
\]

\[
= \sum_{k=0}^{2q} c_k^{[a,b]}(f, x) (-ab)^k.
\]

Note that \( c_k^{[a,b]}(f, x) = 1 \). Eq. (2.2) is a consequence of Lemma 3.4. We conclude that

\[
\sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (C_{n,a,n, b, \psi_x^s})(x) = \sum_{k=0}^{q} c_k^{[a, b, n]}(f, x) \left( \frac{-a_nb_n}{n} \right)^k + o \left( \left( \frac{-a_nb_n}{n} \right)^q \right)
\]

as \( n \to \infty \). In order to complete the proof we have to show that the remainder can be estimated by

\[
(C_{n,a,n, b, \psi_x^{2q}})(x) = o \left( \left( \frac{-a_nb_n}{n} \right)^q \right) \quad (n \to \infty).
\]

To this end let \( (\delta_n) \) be a sequence of positive numbers such that

\[
\delta_n^2 = -\frac{a_nb_n}{nM(x)} \left( \alpha (-a_nb_n)^p - q \log \frac{-a_nb_n}{n} + \sqrt{\frac{n}{-a_nb_n}} \right) \quad (n \in \mathbb{N}).
\]

Note that the conditions (1.3) and (2.1) imply that \( \delta_n = o(1) \) as \( n \to \infty \). Define

\[
\varepsilon_n = \sup \left\{ |h_x(t)| : t \in (x - \delta_n, x + \delta_n) \right\}.
\]

Because \( h_x \) is continuous with \( h_x(x) = 0 \) we have \( \varepsilon_n = o(1) \) as \( n \to \infty \). We split the remainder into two parts

\[
(C_{n,a,n, b, \psi_x^{2q}})(x) = \sum_{\nu} p_{n, \nu} \left( \frac{x-a_n}{b_n-a_n} \right) (h_x\psi_x^{2q}) \left( a_n + (b_n-a_n) \frac{\nu}{n} \right)
\]

| \( a_n + (b_n-a_n) \frac{\nu}{n} - x | < \delta_n |

\[
+ \sum_{\nu} p_{n, \nu} \left( \frac{x-a_n}{b_n-a_n} \right) (h_x\psi_x^{2q}) \left( a_n + (b_n-a_n) \frac{\nu}{n} \right)
\]

| \( a_n + (b_n-a_n) \frac{\nu}{n} - x | \geq \delta_n |

= \sum_1 + \sum_2,
say. Let us start with the estimate of the first sum:

$$\left| \sum_1 \right| \leq \varepsilon_n \sum_{\nu} p_{n, \nu} \left( \frac{x - a_n}{b_n - a_n} \right)^q \left( a_n + \left( b_n - a_n \right) \frac{\nu}{n} \right)$$

$$\leq \varepsilon_n \left( C_{n, a_n, b_n} \right)^q \left( \frac{x - a_n}{b_n - a_n} \right)^q = o \left( \left( \frac{-a_n b_n}{n} \right)^q \right)$$

as \( n \to \infty \), where we used Lemma 3.4. By the Taylor formula (3.3), the second sum can be rewritten as

$$\sum_2 = \sum_{\nu} p_{n, \nu} \left( \frac{x - a_n}{b_n - a_n} \right)$$

$$\sum_{\nu} p_{n, \nu} \left( \frac{x - a_n}{b_n - a_n} \right) \geq \delta_n$$

$$\times \left( f \left( a_n + \left( b_n - a_n \right) \frac{\nu}{n} \right) - \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} \left( a_n + \left( b_n - a_n \right) \frac{\nu}{n} \right)^s \right)$$

and we obtain

$$\left| \sum_2 \right| \leq 2 \exp \left( -M \left( \frac{n \delta_n^2}{a_n b_n} \right) \right)$$

$$\times \left( \left\| f \right\|_{a_n, b_n} + \sum_{s=0}^{2q} \frac{\left\| f^{(s)}(x) \right\|}{s!} \max \left\{ \left| x - a_n \right|^s, \left| b_n - x \right|^s \right\} \right)$$,

where in the last step Lemma 3.6 was applied. Note that

$$\sum_{s=0}^{2q} \frac{\left\| f^{(s)}(x) \right\|}{s!} \max \left\{ \left| x - a_n \right|^s, \left| b_n - x \right|^s \right\} = O \left( (-a_n b_n)^{2q} \right) \quad (n \to \infty).$$

Hence,

$$\sum_2 = O \left( \exp \left( \alpha \left( -a_n b_n \right)^p - M \left( \frac{n \delta_n^2}{a_n b_n} \right) \right) \right)$$

$$+ O \left( \exp \left( 2q \log \left( -a_n b_n \right) - M \left( \frac{n \delta_n^2}{a_n b_n} \right) \right) \right)$$

as \( n \to \infty \). In the case \( p = 0 \), i.e., \( f \) is bounded on \( \mathbb{R} \), we have

$$\sum_2 = O \left( \exp \left( 2q \log \left( -a_n b_n \right) - M \left( \frac{n \delta_n^2}{a_n b_n} \right) \right) \right) \quad (n \to \infty).$$

We can assume that \( \alpha > 0 \). Therefore, in the case \( p > 0 \), we have

$$\sum_2 = O \left( \exp \left( \alpha \left( -a_n b_n \right)^p - M \left( \frac{n \delta_n^2}{a_n b_n} \right) \right) \right) \quad (n \to \infty).$$

Obviously, it is sufficient to estimate the latter relation. By Eq. (3.4), we infer that

$$\sum_2 = O \left( \exp \left( q \log \frac{-a_n b_n}{n} - \sqrt{\frac{n}{-a_n b_n}} \right) \right)$$

$$= O \left( \left( \frac{-a_n b_n}{n} \right)^q e^{-\sqrt{n/(-a_n b_n)}} \right) \quad (n \to \infty).$$
Finally, we conclude that the remainder can be estimated by

\[(C_{n,a_n,b_n} (h_x \psi_x^{2q})) (x) = o \left( \left( \frac{-a_n b_n}{n} \right)^q \right) \quad (n \to \infty)\]

which completes the proof of the theorem. \( \Box \)

Proof of Corollary 2.4. In the special case \( q = 1 \), Theorem 2.1 states that

\[(C_{n,a_n,b_n} f) (x) = f (x) + c_1^{[a_n,b_n]} (f, x) \frac{-a_n b_n}{n} + o \left( \frac{-a_n b_n}{n} \right) \quad (n \to \infty)\]

which can be rewritten in the form

\[-a_n b_n (f_{n,a_n,b_n} - f) = c_1^{[a_n,b_n]} (f, x) + o (1) \quad (n \to \infty).\]

We have

\[c_1^{[a,b]} (f, x) = \frac{n}{ab} \cdot \frac{f^{(2)} (x)}{2!} \left( C_{n,a,b,\psi_2^x} \right) (x) = \frac{1}{2} f^{(2)} (x) \left( \frac{x - a}{b - x} \right) \]

and the desired formula follows because

\[\lim_{n \to \infty} c_1^{[a_n,b_n]} (f, x) = \frac{1}{2} f^{(2)} (x) \lim_{n \to \infty} \left( \frac{x - a_n}{b_n - x} \right) = \frac{1}{2} f^{(2)} (x).\]

This completes the proof. \( \Box \)

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References

[1] Abel, U.: A Voronovskaya-type result for simultaneous approximation by Bernstein–Chlodovsky polynomials, Results Math. 74 (2019): Paper No. 117, 12 p. https://doi.org/10.1007/s00025-019-1036-5

[2] Abel, U., Ivan, M.: Asymptotic expansion of the multivariate Bernstein polynomials on a simplex. Approx. Theory Appl. 16, 85–93 (2000)
[3] Abel, U., Karsli, H.: Complete Asymptotic Expansions for Bernstein-Chlodovsky Polynomials. In: CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2016 (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), pp. 1–12, Prof. Marin Drinov Academic Publishing House, Sofia, (2017)

[4] Abel, U., Karsli, H.: Asymptotic expansions for Bernstein-Durrmeyer-Chlodovsky polynomials. Results Math. 73 (2018): Paper No. 104, 12 p. https://doi.org/10.1007/s00025-018-0863-0

[5] Albrycht, J., Radecki, J.: On a generalization of the theorem of Voronovskaya. Zeszyty Naukowe UAM, Zeszyt 2, Poznan, (1960), 1–7

[6] Butzer, P.L., Karsli, H.: Voronovskaya-type theorems for derivatives of the Bernstein–Chlodovsky polynomials and the Szász-Mirakyan operator. Comment. Math. 49, 33–58 (2009)

[7] Chlodovsky, I.: Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de M. S. Bernstein. Compositio Math. 4, 380–393 (1937)

[8] DeVore, R. A., Lorentz, G. G.: Constructive approximation. Grundlehren der Mathematischen Wissenschaften 303, Springer, Berlin, Heidelberg (1993)

[9] Karsli, H.: Recent results on Chlodovsky operators. Stud. Univ. Babeş-Bolyai Math. 56, 423–436 (2011)

[10] Karsli, H.: Complete asymptotic expansions for the Chlodovsky polynomials. Numer. Funct. Anal. Optim. 34, 1206–1223 (2013)

[11] Karsli, H.: A complete extension of the Bernstein-Weierstrass Theorem to the infinite interval $(-\infty, +\infty)$ via Chlodovsky polynomials. Submitted

[12] Kilgore, T.: On a constructive proof of the Weierstrass Theorem with a weight function on unbounded intervals. Mediterr. J. Math. 14, No. 6, Paper No. 217, 9 p. (2017)

[13] Kilgore, T., Szabados, J.: On weighted uniform boundedness and convergence of the Bernstein-Chlodowsky operators. J. Math. Anal. Appl. 473, 1165–1173 (2019)

[14] George, G.: Lorentz. University of Toronto Press, Bernstein Polynomials (1953)

Ulrich Abel
Technische Hochschule Mittelhessen, Fachbereich MND
Wilhelm-Leuschner-Straße 13
61169 Friedberg
Germany
e-mail: Ulrich.Abel@mnd.thm.de

Harun Karsli
Faculty of Science and Arts, Department of Mathematics
Bolu Abant Izzet Baysal University
Gölköy-Bolu
Turkey
e-mail: karsli_h@ibu.edu.tr

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