Radiatively induced symmetry breaking and the conformally coupled magnetic monopole in AdS space

1,2 Ariel Edery*, 3 Noah Graham†

1 Department of Physics, Bishop’s University, 2600 College Street, Sherbrooke, QC J1M 1Z7
2 Kavli Institute for Theoretical Physics, University of California, Kohn Hall, Santa Barbara, CA 93106 U.S.A.
3 Department of Physics, Middlebury College, Middlebury, VT 05753

Abstract

We implement quantum corrections for a magnetic monopole in a classically conformally invariant theory containing gravity. This yields the trace (conformal) anomaly and introduces a length scale in a natural fashion via the process of renormalization. We evaluate the one-loop effective potential and extract the vacuum expectation value (VEV) from it; spontaneous symmetry breaking is radiatively induced. The VEV is set at the renormalization scale \( M \) and we exchange the dimensionless scalar coupling constant for the dimensionful VEV via dimensional transmutation. The asymptotic (background) spacetime is anti-de Sitter (AdS) and its Ricci scalar is determined entirely by the VEV. We obtain analytical asymptotic solutions to the coupled set of equations governing gravitational, gauge and scalar fields that yield the magnetic monopole in an AdS spacetime.

*Email: aedery@ubishops.ca
†Email: ngraham@middlebury.edu
1 Introduction

The t’Hooft Polyakov magnetic monopole [1,2] is a topological soliton solution of non-Abelian
gauge theory. When the gauge fields are coupled to scalar fields with the typical symmetry
breaking potential \( V(\phi) = \frac{\lambda}{4}(\phi^4 - a^2 \phi^2) \), the \( SU(2) \) gauge symmetry gets broken down to
the residual \( U(1) \) gauge of the electromagnetic field leading to a magnetic monopole. The
spontaneous breaking of gauge symmetry leaves the 10 parameter Poincaré group intact. A
few years ago, it was shown that one can add the gravitational field to this picture and enlarge
the symmetry group to the 15 parameter conformal group \( SO(2,4) \) by conformally coupling the
scalar fields to the metric [3]. The original gravity sector does not include an Einstein-Hilbert
term because that term is not conformally invariant; instead, one uses the conformally invariant
Weyl squared term. A cosmological constant is also forbidden by conformal invariance. In this
massless model, the usual \( a^2 \phi^2 \) term is replaced by the conformal coupling term \(-R \phi^2/6\),
where \( R \) is the Ricci scalar. The vacuum expectation value (VEV) is now expressed in terms
of \( R \). The theory is classically scale invariant, and one must therefore choose a VEV value by
hand (or alternatively specify the asymptotic value of the Ricci scalar). Magnetic monopole
solutions in a background anti-de Sitter (AdS) space were then found numerically [3]. The static
spherically symmetric solution at large distances (outside the monopole core) is Schwarzschild-
AdS. It should be stressed that the solution in [3] is not a black hole (BH) solution since the
core is non-singular and static. By definition, a BH has a horizon and there exits a region in
the interior which is non-stationary, where all Killing vectors are spacelike [4] (extremal BHs
are the exception since their interiors can be static or stationary, i.e. they possess a timelike
Killing vector [4]).

In this work, we introduce a length scale by considering the effects of quantum fluctuations of
scalar fields at one loop in a classical gravitational and gauge field background. We calculate
the contribution of the trace anomaly through the Schwinger-Dewitt coefficient \( a_2(x) \) of the
theory. To determine the vacuum expectation value (VEV), we derive the effective potential at
one loop for a massless theory with a triplet of scalar fields and \( R \phi^2/6 \) and \( \lambda \phi^4/4! \) interactions.
The coupling constant \( \lambda \) is a running coupling constant which is defined at a renormalization
scale \( M \). We use this fact to trade the dimensionless \( \lambda \) for the dimensionful VEV of the theory,
through dimensional transmutation [5]. The asymptotic (background) spacetime is AdS and
its Ricci scalar is completely determined by the VEV. The effective potential is also used to
evaluate the composite operator \([E]\) that enters the quantum-corrected equations of motion.
Finally, we derive the equations of motion governing the gravitational, gauge and scalar fields
and solve them analytically in the asymptotic region to obtain the relation between the Ricci
scalar and the VEV.

Though conformal invariance is presented here in the context of the magnetic monopole, there is
a more general interest in scale and conformal symmetry as a possible fundamental principle in
physics and cosmology, as discussed for example in [6]. As described there, the classical action of the standard model is already consistent with global scale symmetry if the Higgs mass is dropped. Furthermore, on cosmic scales, the nearly scale-invariant spectrum of primordial fluctuations seems to demand an explanation based on a fundamental symmetry in nature. In short, the authors argue that scale and conformal symmetry may be the clue to fitting observations on very small and very large scales in a coherent theory (see [6] for more details and references). The conformal anomaly has also attracted interest as an approach to the cosmological constant problem [7]. The argument put forward is that the cosmological constant problem may arise from the infrared sector of the effective theory of gravity where the conformal anomaly is expected to be relevant.

2 Quantum corrections

Quantum vacuum fluctuations break explicitly the global scale symmetry and hence the entire conformal symmetry. The energy-momentum tensor develops a nonzero trace that can be evaluated via the Schwinger-Dewitt coefficient $a_2(x)$ in four dimensions. This is the well known trace or conformal anomaly [8, 9]. The effects due to vacuum fluctuations of matter fields can be included via an effective action $W$. It is convenient to split $W$ into two parts: a divergent part $W_{\text{div}}$ and a renormalized finite part $W_{\text{ren}} = W - W_{\text{div}}$. $W_{\text{div}}$ is due to high-frequency fluctuations and is therefore local and state-independent [8]. It is expressed as an integral over $a_2(x)$, which can be calculated via a set of “curvatures” [10]. $W_{\text{div}}$ is incorporated into the original action $S$ by adding the appropriate counterterms. We label this $S_{\text{ren}}$, the renormalized local part of the one loop effective action. $W_{\text{ren}}$ is the finite state-dependent nonlocal part of the effective action due to long wavelength fluctuations that sample the entire geometry. For example, the finite part of the vacuum expectation value of the energy-momentum tensor, $\langle T_{\mu\nu} \rangle$, that will appear on the right hand side of the gravitational equations of motion, can be viewed as a Casimir effect which also arises from long wavelength quantum fluctuations [11].

2.1 Schwinger-Dewitt coefficient for the monopole action and trace anomaly

The conformally invariant action that leads to a magnetic monopole coupled to gravity contains a metric $g_{\mu\nu}$, a triplet of scalar fields $\phi^a$ and nonabelian gauge fields $A^a_\mu$. It is given by [3]

$$S = \int d^4x \sqrt{-g} \left( C_{\mu\nu\sigma\tau} C^{\mu\nu\sigma\tau} - \frac{1}{4e^2} F^a_{\mu\nu} F^{a\mu\nu} + D_\mu \phi^a D^\mu \phi_a + \frac{1}{6} R \phi^a \phi_a - \frac{\lambda}{4!} (\phi^a \phi_a)^2 \right)$$

$$= \int d^4x \sqrt{-g} \left( C^2 - \frac{1}{4e^2} F^2 + (D\phi)^2 + \frac{1}{6} R \phi^2 - \frac{\lambda}{4!} \phi^4 \right)$$

(2.1)
where $C_{\mu\nu\tau}$ is the Weyl tensor, $F_{\mu\nu}^a$ is the gauge field strength defined by $\nabla_\mu A_\nu^a - \nabla_\nu A_\mu^a + \varepsilon_{abc} A_\mu^b A_\nu^c$, and $D_\mu \phi^a$ is the covariant derivative defined by $\nabla_\mu \phi^a + \varepsilon_{abc} A_\mu^b \phi^c$. The square of the Weyl tensor is denoted as $C^2$ and $F^2 \equiv F_{\mu\nu}^a F_{\mu\nu}^a$, $(D\phi)^2 \equiv D_\mu \phi^a D^\mu \phi_a$ and $\phi^2 \equiv \phi^a \phi_a$. The Ricci-Levi-Civita three-dimensional coefficients $\varepsilon_{abc}$ are totally antisymmetric. Raising or lowering of the internal (Latin) indices does not change the sign so that $\phi_a = \phi^a$ and implicit summation is assumed throughout e.g. $\phi_a \phi^a = \phi_1^2 + \phi_2^2 + \phi_3^2$. The action (2.1) is invariant under the conformal transformations $g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}$ and $\phi^a \rightarrow \phi^a / \Omega(x)$ where $\Omega(x)$ is called the conformal factor, an arbitrary positive smooth function.

We consider the vacuum fluctuations of the triplet $\phi^a$ in a gravitational and gauge field classical background. The Euclidean effective action is given by $W = \frac{1}{2} tr (\ln \hat{H})$ where $\hat{H}$ is the hessian of the Euclidean version of the action (2.1). It takes the following form [10]

$$\hat{H} = g^{\mu\nu} D_\mu D_\nu + \hat{P} - \frac{1}{6} R \hat{1}. \tag{2.2}$$

The above operator is a $3 \times 3$ matrix in the internal vector space of the triplet $\phi^a$ and $\hat{1}$ is the identity matrix. Here $\hat{P}$ arises from the self interaction term $\lambda(\phi^a \phi_a)^2 / 4!$, $\frac{1}{6} R \hat{1}$ arises from the conformally coupled scalar field term $\frac{1}{6} R \phi^a \phi_a$ and $g^{\mu\nu} D_\mu D_\nu$ arises from the covariant derivative term $D_\mu \phi^a D^\mu \phi_a$. The gravitational and nonabelian gauge fields are treated classically. One identifies three “curvatures” for the operator (2.2) that enter into the calculation of the effective action [10]. These are the Riemann curvature associated with $g_{\mu\nu}$, the commutator curvature $\mathcal{R}_{\mu\nu}$ associated with the covariant derivative $D_\mu$ and defined by

$$[D_\mu, D_\nu] \phi^a = \mathcal{R}^a_{\; b\mu\nu} \phi^b \text{ with } \mathcal{R}_{\mu\nu} \equiv \mathcal{R}^a_{\; b\mu\nu} \tag{2.3}$$

and the potential $\hat{P}$ which is its own curvature. The divergent part $W_{\text{div}}$ of the Euclidean effective action can be expressed entirely in terms of these curvatures as [10]

$$W_{\text{div}} = \frac{1}{n - 4} \int d^4 x \sqrt{g} \, \hat{\phi}_2(x, x) \quad (n \rightarrow 4) \tag{2.4}$$

$$\hat{\phi}_2(x, x) = \frac{1}{180 \lambda} \left(R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - R_{\mu\nu} R^{\mu\nu} + \Box R \right) \hat{1} + \frac{1}{12} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{1}{2} \hat{P}^2 + \frac{1}{6} \Box \hat{P}. \tag{2.5}$$

We now evaluate the curvatures $\hat{P}$ and $\mathcal{R}_{\mu\nu}$ for the Euclidean version of the action (2.1). The matrix elements of the potential $\hat{P}$ are

$$P_{ij} = -\frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} \frac{\lambda}{4!} (\phi^a \phi_a)^2 = -\frac{\lambda}{6} (\delta_{ij} \phi_a \phi^a + 2 \phi_i \phi_j) \text{ where } i, j = 1, 2, 3. \tag{2.6}$$

To obtain $\mathcal{R}_{\mu\nu}$, we first evaluate the double covariant derivative,

$$D_\mu D_\nu \phi^a = \nabla_\mu (D_\nu \phi^a) + \varepsilon_{abc} A_\mu^b D_\nu \phi^c \quad \nabla_\mu (\nabla_\nu \phi^a + \varepsilon_{de} A_\nu^d \phi^e) + \varepsilon_{abc} A_\mu^b (\nabla_\nu \phi^c + \varepsilon_{fg} A_\nu^f \phi^g)$$

$$= \nabla_\mu \nabla_\nu \phi^a + \varepsilon_{de} A_\mu^e \phi^d + \varepsilon_{de} A_\mu^e \nabla_\nu \phi^d + \varepsilon_{bc} A_\mu^b \nabla_\nu \phi^c + \varepsilon_{bc} A_\mu^b \nabla_\nu \phi^c + \varepsilon_{de} A_\mu^e \phi^d. \tag{2.7}$$
The commutator is then given by

\[
[D_\mu, D_\nu] \phi^a = \varepsilon^{ae}_{\, \, \, df} \left( \nabla_\mu A^d_\nu - \nabla_\nu A^d_\mu \right) \phi^f + \varepsilon^{ae}_{\, \, \, gb} \varepsilon^{\, \, \, f}_c \left( A^b_\mu A^g_f - A^b_f A^g_\mu \right) \phi^c
\]

\[
= \varepsilon^{ae}_{\, \, \, df} \left( \nabla_\mu A^d_\nu - \nabla_\nu A^d_\mu + \varepsilon^{\, \, \, f}_g A^g_\mu \right) \phi^c
\]

\[
= \varepsilon^{ae}_{\, \, \, df} F^d_{\mu\nu} \phi^c
\]

(2.8)

where the contracted epsilon identity

\[
\varepsilon^{jk\ell} \varepsilon_{jlm} = \delta_{km} \delta_{\ell n} - \delta_{kn} \delta_{\ell m}
\]

was used. Comparing (2.8) with (2.3), we obtain the commutator curvature

\[
\hat{\mathcal{R}}_{\mu\nu} \equiv \mathcal{R}^a_{\mu\nu} \varepsilon^a_{\, \, \, df} F^d_{\mu\nu}.
\]

(2.9)

The trace of the operators that appear in \( \hat{a}_2(x, x) \), evaluated using the formulas (2.6) and (2.9) for \( \hat{P} \) and \( \hat{\mathcal{R}}_{\mu\nu} \) respectively, are

\[
\text{tr} \, \hat{1} = 3, \quad \text{tr} \, \hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}_{\mu\nu} = -2 F^2, \quad \text{tr} \, \hat{P}^2 = \frac{22}{3} \frac{\lambda^2}{4!} \phi^4, \quad \text{tr} \, \Box \hat{P} = -\frac{5}{6} \lambda \Box \phi^2.
\]

(2.10)

The quantity of interest that appears in the integrand of \( W_{\text{div}} \) is

\[
a_2(x) \equiv \text{tr} \, \hat{a}_2(x, x) = \frac{1}{60} \left( R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - R_{\mu\nu} R^{\mu\nu} + \Box R \right)
\]

\[
- \frac{1}{6} F^2 \left( \frac{11}{3} \frac{\lambda^2}{4!} \phi^4 - \frac{5}{36} \Box \phi^2 \right).
\]

(2.11)

where the factor of 11/3 in front of \( \lambda^2 \) stems from having a triplet of scalar fields (it is equal to 3 for a single scalar field). To cancel the divergent part \( W_{\text{div}} \), we simply add the appropriate counterterm to the original action (2.1). The renormalized local part of the action at one loop is simply

\[
S_{\text{ren}} = \int d^4 x \sqrt{-g} \left( \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} - \frac{1}{4e^2} F^2 + (D\phi)^2 + \frac{1}{6} R \phi^2 - \lambda \frac{1}{4!} \phi^4 \right).
\]

(2.12)

The total derivatives \( \Box R \) and \( \Box (\phi_a \phi^a) \) appearing in (2.11) are not included in the above action as they lead to boundary terms that have no bearing on the equations of motion. The terms \( C_{\mu\nu\sigma\tau} C^{\mu\nu\sigma\tau} \) and \( R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} \) do not appear in (2.12) since they can be eliminated in favor of \( R_{\mu\nu} R^{\mu\nu} \) and \( R^2 \) using the equality \( C_{\mu\nu\sigma\tau} C^{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - 2 R_{\mu\nu} R^{\mu\nu} + R^2 / 3 \) and the fact that the integral of \( R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \), the Gauss-Bonnet integral, is a topological invariant (the Euler number) in four dimensions. The quantities \( e, \lambda, \alpha, \) and \( \beta \) are renormalized constants. They are running coupling constants governed by a renormalization group equation which we do not need to state here for our purposes (for details see [12, 13]). At one loop, the constants \( \alpha \) and \( \beta \) are related by \( \alpha = -\beta / 3 \) and the sum of the first two terms in (2.12) is conformally invariant. At higher loops, non-conformally invariant terms like \( R^2 \) are generated [12, 13] and the two constants become independent. Our calculations will be at one loop, so we assume the relation \( \alpha = -\beta / 3 \).
2.2 Anomalous trace

To find the trace, we first note that under a conformal transformation the inverse metric transforms according to \( g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu}(x) \) and the scalar field transforms according to \( \phi^a(x) \rightarrow e^{\sigma(x)} \phi^a(x) \) where \( \sigma(x) \) is introduced for convenience and is related to the conformal factor \( \Omega(x) \) (previously introduced) by \( e^{-\sigma(x)} \equiv \Omega(x) \). This yields the functional relation

\[
\frac{\delta}{\delta \sigma(x)} = 2 g_{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} + \phi^a(x) \frac{\delta}{\delta \phi^a(x)}.
\] (2.13)

The functional variation of the one-loop effective action with respect to \( \sigma(x) \) is proportional to the Schwinger-Dewitt coefficient \( a_2(x) \) \([10]\)

\[
\frac{\delta W_{\text{ren}}}{\delta \sigma} = \frac{\sqrt{-g} a_2(x)}{16 \pi^2}
\] (2.14)

and using (2.13) this yields

\[
\frac{1}{\sqrt{-g}} \left( 2 g^{\mu\nu} \frac{\delta W_{\text{ren}}}{\delta g_{\mu\nu}} + \phi^a \frac{\delta W_{\text{ren}}}{\delta \phi^a} \right) = \frac{a_2(x)}{16 \pi^2}.
\] (2.15)

The vacuum expectation value of the energy-momentum tensor is defined as

\[
\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}}{\delta g^{\mu\nu}}.
\] (2.16)

From (2.15), its trace is given by

\[
\langle T^\mu_\mu \rangle = \frac{a_2(x)}{16 \pi^2} - [E]
\] (2.17)

where

\[
[E] = \frac{\phi^a}{\sqrt{-g}} \frac{\delta W_{\text{ren}}}{\delta \phi^a}
\] (2.18)

and

\[
a_2(x) = \frac{1}{60} \left( R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - R_{\mu\nu} R^{\mu\nu} + \Box R \right) - \frac{1}{6} F^2 + \frac{11}{3} \frac{\lambda^2}{4!} \phi^4 - \frac{5}{36} \lambda \Box [\phi^2].
\] (2.19)

The renormalized composite operator \([E]\) can be obtained readily once the effective potential has been calculated.
2.2.1 AdS space

As already discussed, the magnetic monopole solution is obtained when the spacetime is asymptotically anti-de Sitter space. This is a maximally symmetric spacetime with constant Ricci scalar $R$ (positive in our notation). It is the submanifold obtained by embedding a hyperboloid in a flat five-dimensional spacetime of signature$(+,+,−,−,−)$. The universal covering space of AdS space has the topology of $\mathbb{R}^4$ and can be represented by the following metric:

$$ds^2 = (1 + kr^2) dt^2 - \frac{dr^2}{1 + kr^2} - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (2.20)

where $(r, \theta, \phi)$ cover the usual range of spherical coordinates $(r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$ and $-\infty < t < \infty$. The constant $k$ is positive with the Ricci scalar given by $R = 12k$.

The symmetry group (isometry group) of AdS space is the ten parameter group $SO(2,3)$. The only maximally form invariant rank two tensor under this group is the metric $g_{\mu\nu}$ (times a constant) so that the expectation value of the energy-momentum tensor in AdS space can be expressed in terms of its trace:

$$\langle T_{\mu\nu} \rangle_0 = \frac{1}{4} g_{\mu\nu} \langle T^\mu_{\mu} \rangle_0$$ \hspace{1cm} (2.21)

where the zero subscript means that the quantity is evaluated asymptotically in AdS space, the vacuum spacetime.

3 The effective potential for a massless theory with $R\phi^2$ and $\lambda\phi^4$ interactions

In this section we obtain the one loop effective potential by summing all the one loop one-particle irreducible (1PI) Feynman diagrams in the presence of $\lambda\phi^4/24$ and $-R\phi^2/6$ interactions. The coupling constant $\lambda$ is defined at a renormalization scale $M$. We choose the renormalization scale $M$ to coincide with the VEV, the minimum of the effective potential. This in turn fixes the value of $\lambda$; the dimensionless constant $\lambda$ is traded for the dimensionful VEV through dimensional transmutation. The expectation value of the composite operator $[E]$, which appears both in the trace anomaly and in the equations of motion for the scalar field, can be readily obtained from the effective potential. Note that at one loop the effective potential and the expectation value of composite operators do not generate new geometrical curvature terms; they are generated starting only at two loops \cite{12, 13}. In particular, for the calculation of the effective potential, the $\sqrt{-g}$ factor plays no role at one loop. The calculation proceeds in the same manner as in flat space, though of course the Ricci scalar $R$ is non-zero and acts as a vertex for the $R\phi^2$ interaction.
The (classical) potential is given by
\[ U = \frac{\lambda}{4!} \phi^4 - \frac{1}{6} R \phi^2. \]  
(3.22)

We have a triplet of scalar fields and without loss of generality we take the vacuum expectation value to lie along the third component \( \phi_3 \). For \( \phi_3 \) loops, \( U \) generates two vertices: \(-R/3\) and \( \lambda \phi_3^2/2 \). They can be combined into a single vertex given by the second derivative \( U''(\phi_3) = -R/3 + \lambda \phi_3^2/2 \). The vacuum expectation value of \( \phi_1 \) and \( \phi_2 \) are zero but they can still fluctuate in loops. The vertex for both is \( -R/3 + \lambda \phi_3^2/2 \). The vacuum expectation value of \( \phi_1 \) and \( \phi_2 \) in the presence of some external source \( J(x) \)
\[ \phi_c(x) = \frac{\langle 0|\phi_3(x)|0 \rangle}{\langle 0|0 \rangle} |_J. \]  
(3.23)

where \( J \) appears in the action in the usual fashion via the source term \( J \phi \). The effective potential is obtained by summing all the diagrams in Fig. 1. Let the classical field \( \phi_c(x) \) be defined as the vacuum expectation value of \( \phi_3 \) in the presence of some external source \( J(x) \)
\[ V_1 = V + 2 V [\lambda \rightarrow \lambda/3]. \]  
(3.25)

As it stands, the expression \( V_1 \) is divergent in the infinite \( \Lambda \) limit. This is handled in the usual fashion by adding the necessary counterterms and then imposing the appropriate renormalization conditions. The total potential is given by
\[ V_{tot} = \frac{\lambda}{4!} \phi_c^4 - \frac{1}{6} R \phi_c^2 + V_1 + A \phi_c^2 + B \phi_c^4 \]  
(3.26)

where the last two terms are the counterterms. The constants \( A \) and \( B \) are determined via the renormalization conditions
\[ \lambda = \left. \frac{d^4 V_{tot}}{d \phi_c^4} \right|_{\phi_c=M} \text{ and } -\frac{R}{3} = \left. \frac{d^2 V_{tot}}{d \phi_c^2} \right|_{\phi_c=0}. \]  
(3.27)
The renormalization scale $M$ sets the scale for the theory. Substituting $A$ and $B$ back into (3.26), taking the infinite $\Lambda$ limit and then collecting terms into compact expressions, we obtain the one loop effective potential

$$V_{\text{eff}} = \frac{\lambda}{4!} \phi_c^4 - \frac{1}{6} R \phi_c^2 + \frac{1}{1152\pi^2} \left[ 18 \left( -\frac{R}{3} + \frac{\lambda \phi_c^2}{2} \right) \ln \left( \frac{2R - 3\lambda \phi_c^2}{2R} \right) \right.$$  

$$+ 36 \left( -\frac{R}{3} + \frac{\lambda \phi_c^2}{6} \right) ^2 \ln \left( \frac{2R - \lambda \phi_c^2}{2R} \right) + 5R \lambda \phi_c^2 \right.$$  

$$+ \lambda^2 \phi_c^4 \left( \frac{9}{2} \ln \left( \frac{2R}{2R - 3\lambda M^2} \right) + \ln \left( \frac{2R}{2R - \lambda M^2} \right) \right) \right] + \Delta \quad (3.28)$$

where

$$\Delta = \lambda^2 \phi_c^4 \frac{(-1584R^4 + 11904M^2R^3\lambda - 20360M^4R^2\lambda^2 + 12480M^6R\lambda^3 - 2475M^8\lambda^4)}{13824\pi^2(-2R + M^2\lambda)^2(-2R + 3M^2\lambda)^2}. \quad (3.29)$$

Let $\phi_c = v$ be the vacuum expectation value (VEV). It takes on this value in the asymptotic (background) spacetime, which is AdS space. We will see later, in section 4.1, that solving the equations of motion asymptotically yields the relation (4.59) between $R_0$, the Ricci scalar of AdS space, and the VEV: $R_0 = \sqrt{\frac{10}{\Lambda}}\lambda v^2$. The VEV occurs at the minimum of the effective potential in the AdS background spacetime, where

$$\left. \frac{dV_{\text{eff}}}{d\phi_c} \right|_{\phi_c = v} = 0. \quad (3.30)$$

We set the arbitrary scale $M$ to be equal to the VEV i.e. $\phi_c = v = M$. Equation (3.30) then yields a numerical value of $\lambda = C/D = 2519.926$, where the exact expressions for $C$ and $D$ are

$$C = 288\pi^2(197355230\sqrt{\frac{10}{\Lambda}} - 1144879587) \quad \text{and} \quad \lambda v^2$$

$$D = 299795296\sqrt{\frac{10}{\Lambda}} - 3734763307 +$$

$$\left(12 \ln(2\sqrt{10} - 1) + 18 \ln(2\sqrt{10} - 3) - 15 \ln(440))(-96294467\sqrt{\frac{10}{\Lambda}} + 524292560) \right). \quad (3.31)$$

The ratio of the one loop correction to the tree (classical) result for the potential can be readily calculated to be $-0.504$. Such ratios are typical of one loop corrections in massless theories (e.g. in massless $\lambda \phi^4$ theory in flat space with a single scalar field the ratio is close to $-1 \left( \frac{1}{14} \right)$).

We discuss in the conclusions how adding gauge field fluctuations can effect this scenario.
We started with a classical massless theory, a conformally invariant theory with \( \lambda \phi^4 \) and \( R \phi^2 \) interactions. After including one loop quantum corrections, the dimensionless parameter \( \lambda \) has been traded for the dimensionful VEV. An important result is that the Ricci scalar of AdS space is now completely determined by the value of the VEV.

### 3.1 Composite operator \([E]\)

The composite operator \([E(y)]\) is defined via (2.18). It appears in the trace but also as a quantum correction to the equations of motion for the scalar fields (see section 4 below). Inserting this operator into an \( n \)-point Green’s function \( \Gamma_n(x_1, \ldots, x_n) \) and integrating over all \( y \) yields \( n \) times the same Green’s function [13]. The Feynman diagrams are therefore identical to those used to evaluate the effective potential \( V_{\text{eff}} \), namely those of Fig[1](the only difference is that the symmetry factor is multiplied by \( n \)). The upshot is that \([E]\) can be obtained by taking the negative of the derivative of the one loop part of the effective potential (3.28), \( V_{\text{loop}} = V_{\text{eff}} - U \), and then multiplying it by \( \phi_c \),

\[
[E] = -\phi_c \frac{dV_{\text{loop}}}{d\phi_c} = -\frac{1}{1152\pi^2} \left[ 36 \left( -\frac{R}{3} + \frac{\lambda \phi_c^2}{2} \right) \ln \left( \frac{2R - 3\lambda \phi_c^2}{2R} \right) \lambda \phi_c^2 - 108 \left( -\frac{R}{3} + \frac{\lambda \phi_c^2}{2} \right)^2 \frac{\lambda \phi_c^2}{2R - 3\lambda \phi_c^2} \\
+ 24 \left( -\frac{R}{3} + \frac{\lambda \phi_c^2}{6} \right) \ln \left( \frac{2R - \lambda \phi_c^2}{2R} \right) \lambda \phi_c^2 - 72 \left( -\frac{R}{3} + \frac{\lambda \phi_c^2}{6} \right)^2 \frac{\lambda \phi_c^2}{2R - \lambda \phi_c^2} \\
+ 10R \lambda \phi_c^2 + \lambda^2 \phi_c^4 \left( 18 \ln \left( \frac{2R}{2R - 3\lambda M^2} \right) + 4 \ln \left( \frac{2R}{2R - \lambda M^2} \right) \right) \right] - 4\Delta.
\]  

(3.32)

### 4 Equations of motion for the magnetic monopole

The quantum-corrected equations of motion for the metric, scalar and gauge fields are derived in appendix A and are given by equations (A.2), (A.3), and (A.4) respectively. For the magnetic monopole, we seek static spherically symmetric solutions where the spatial symmetry (isometry) and gauge symmetry are both SO(3). These can be viewed as the lowest energy or ground state solution [3]. The metric, scalar triplet, and non-abelian gauge fields take on the
Figure 1: One loop Feynman diagrams for a triplet of scalar fields. The two vertices corresponding to $-R\phi^2/6$ and $\lambda\phi^4/24$ interactions, which are replaced by a single vertex (black dot). The last two sets of diagrams, where the components $\phi_2$ and $\phi_1$ run around the loop, have a vertex with $1/3$ the coupling constant $\lambda$ of the first set of diagrams. The propagator is massless.
The quantities that appear in (4.36) evaluated using Eqs. 4.33-4.35 are following spherically symmetric form [3]:

 metric: $ds^2 = N(r) dt^2 - \psi(r) dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2)$  \hspace{1cm} (4.33)

 scalar: $\phi^a(r) = f(r) \frac{r^a}{r} = f(r) [\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta]$  \hspace{1cm} (4.34)

 gauge: $A^\mu_a = q(r) \xi^\mu_a$ where $\xi^\mu_a$ are the Killing vectors for SO(3), namely

 $\xi^\mu_1 = [0, 0, \cos \varphi, -\sin \varphi \cot \theta], \xi^\mu_2 = [0, 0, -\sin \varphi, -\cos \varphi \cot \theta]$ and $\xi^\mu_3 = [0, 0, 0, 1]$.  \hspace{1cm} (4.35)

 It will be convenient to work with $a(r) \equiv 1 + r^2 q(r)$ instead of $q(r)$. There are four functions of $r$ to determine: the “metric” fields $N(r)$ and $\psi(r)$, the “gauge field” $a(r)$ and the “scalar” field $f(r)$. It is convenient to obtain the equations of motion by direct variation of these functions. The Lagrangian corresponding to $S_{ren}$ is given by

 $L = 4\pi \int_0^\infty dr \sqrt{N \psi} r^2 (\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} - \frac{1}{4e^2} F_{\mu\nu}^a F^{\mu\nu}_a + D_\mu \phi^a D^\mu \phi_a + \frac{1}{6} R \phi^a \phi_a - \frac{\lambda}{4!} (\phi^a \phi_a)^2)$

 $= 4\pi \int_0^\infty \mathcal{L} \, dr$.  \hspace{1cm} (4.36)

 The quantities that appear in (4.36) evaluated using Eqs. 4.33-4.35 are

 $R = \frac{-2\psi'}{r \psi^2} - \frac{N'^2}{4N \psi^2} + \frac{N''}{2N \psi^2} - \frac{2}{r^2} \psi + \frac{2}{r N \psi}$  \hspace{1cm} (4.37)

 $R_{\mu\nu}R^{\mu\nu} = \tilde{R}_j \tilde{R}^j \equiv (\tilde{R}_1)^2 + (\tilde{R}_2)^2 + (\tilde{R}_3)^2 + (\tilde{R}_4)^2$

 where

 $\tilde{R}_1 \equiv N R^{\mu\mu} = \frac{-\psi' N'}{4N \psi^2} - \frac{N'^2}{2N \psi^2} + \frac{N'}{r N \psi}$  \hspace{1cm} (4.38)

 $\tilde{R}_2 \equiv \psi R^{\mu\mu} = \frac{\psi'}{r \psi^2} + \frac{N'^2}{4N \psi^2} - \frac{N''}{2N \psi} + \frac{N'}{4N \psi^2}$  \hspace{1cm} (4.39)

 $\tilde{R}_3 \equiv r^2 R^{\theta\theta} = \frac{\psi'}{2r \psi^2} + \frac{1}{r^2} \psi + \frac{1}{2r N \psi}$  \hspace{1cm} (4.40)

 $\tilde{R}_4 \equiv r^2 \sin^2 \theta R^{\phi\phi} = \tilde{R}_3$  \hspace{1cm} (4.41)

 $F_{\mu\nu}^a F^{\mu\nu}_a = \frac{4a'^2}{r^2 \psi} + \frac{2(a^2 - 1)}{r^4}$  \hspace{1cm} (4.42)

 $D_\mu \phi^a D^\mu \phi_a = -\frac{f'^2}{\psi} - \frac{2a^2 f^2}{r^2}$  \hspace{1cm} (4.43)

 $R \phi^a \phi_a = \frac{f^2}{2 \psi^2 r^2 N^2} \left\{ -4r \psi' N^2 - N'^2 N^2 r^2 + 2N' \psi r^2 N - \psi' N' r^2 N - 4 \psi^2 N^2 + 4N N' r \psi \right\}$  \hspace{1cm} (4.44)

 $(\phi^a \phi_a)^2 = f^4$ and $\sqrt{-g} = \sqrt{N \psi r^2 \sin \theta}$.  \hspace{1cm} (4.45)
Variation with respect to the metric function $N(r)$ yields
\[ \frac{\partial \mathcal{L}}{\partial N} - \frac{\partial}{\partial r} \frac{\partial \mathcal{L}}{\partial N'} + \frac{\partial^2}{\partial r^2} \frac{\partial \mathcal{L}}{\partial N''} = \frac{\sqrt{-g}}{2 \sin \theta} (T_{tt}) \]

The equation of motion for $N$ is
\[ \frac{\sqrt{N \psi} r^2}{2N} (\alpha R^2 + \beta \tilde{R}_j \tilde{R}^j) + \sqrt{N \psi} r^2 \left( 2 \alpha R \frac{\partial R}{\partial N} + 2 \beta \tilde{R}_j \frac{\partial \tilde{R}^j}{\partial N} \right) \]
\[ - \left[ \sqrt{N \psi} r^2 \left( 2 \alpha R \frac{\partial R}{\partial N} + 2 \beta \tilde{R}_j \frac{\partial \tilde{R}^j}{\partial N} \right) \right]'' + \left[ \sqrt{N \psi} r^2 \left( 2 \alpha R \frac{\partial R}{\partial N} + 2 \beta \tilde{R}_j \frac{\partial \tilde{R}^j}{\partial N} \right) \right]'' \]
\[ + \frac{1}{6 \sqrt{N \psi}} \left( f^2 - f'^2 \psi^2 - f' \psi \frac{\partial \psi}{\partial \psi'} - r^2 f f' \psi^2 + 4 r f f' + 2 r f f'' \right) - \frac{\lambda r^2 f^4 \psi}{48 \sqrt{N \psi}} \]
\[ - \frac{(a^2 - 1)^2 \psi + 2 r^2 a'^2}{4 e^2 r^2 \sqrt{N \psi}} - \frac{r^2 f f''}{2 \sqrt{N \psi}} = \frac{\sqrt{N \psi} r^2}{2} (T_{rr}) \quad (4.46) \]

where the Ricci scalar $R$ is given by (4.37), $\tilde{R}_j = \tilde{R}^j$ is given by Eqs. 4.38 and 4.41 and implicit summation over $j = 1, 2, 3, 4$ is assumed. Variation with respect to the metric function $\psi(r)$ yields the equation
\[ \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial r} \frac{\partial \mathcal{L}}{\partial \psi'} + \frac{\partial^2}{\partial r^2} \frac{\partial \mathcal{L}}{\partial \psi''} = -\frac{\sqrt{-g}}{2 \sin \theta} (T_{rr}) \quad (4.47) \]

and we obtain
\[ \frac{\sqrt{N \psi} r^2}{2\psi} (\alpha R^2 + \beta \tilde{R}_j \tilde{R}^j) + \sqrt{N \psi} r^2 \left( 2 \alpha R \frac{\partial R}{\partial \psi} + 2 \beta \tilde{R}_j \frac{\partial \tilde{R}^j}{\partial \psi} \right) \]
\[ - \left[ \sqrt{N \psi} r^2 \left( 2 \alpha R \frac{\partial R}{\partial \psi} + 2 \beta \tilde{R}_j \frac{\partial \tilde{R}^j}{\partial \psi} \right) \right]'' + \left[ \sqrt{N \psi} r^2 \left( 2 \alpha R \frac{\partial R}{\partial \psi} + 2 \beta \tilde{R}_j \frac{\partial \tilde{R}^j}{\partial \psi} \right) \right]'' \]
\[ - \frac{\lambda r^2 f^4 \psi}{48 \sqrt{N \psi}} - \frac{N(a^2 - 1)^2}{4 e^2 r^2 \sqrt{N \psi}} + \frac{N^2 a'^2}{2 e^2 (N \psi)^{3/2}} + \frac{N^2 f^2}{2 (N \psi)^{3/2}} \]
\[ + \frac{r f f'}{6 (N \psi)^{3/2}} + \frac{6 (N \psi)^{3/2}}{6 (N \psi)^{3/2}} = -\frac{\sqrt{N \psi} r^2}{2} (T_{rr}) \quad (4.48) \]

Lagrange’s equations for the gauge field $a$ is given by
\[ \frac{\partial \mathcal{L}}{\partial a} - \frac{\partial}{\partial r} \frac{\partial \mathcal{L}}{\partial a'} = 0 \quad (4.49) \]

which yields the equation of motion
\[ 2 a (a^2 - 1) + 4 a e^2 f^2 r^2 - \frac{2}{\psi} a'' r^2 + \frac{a' r^2}{\psi} \left( \frac{\psi'}{N} - \frac{N'}{N} \right) = 0 \quad (4.50) \]
Lagrange’s equations for the scalar field $f$ is given by
\[
\frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial r} \frac{\partial \mathcal{L}}{\partial f'} = -\frac{\sqrt{N\psi}r^2}{f} [E] \tag{4.51}
\]
which yields the equation of motion
\[
-\frac{4a^2 f^2}{r^2} + \frac{f^2 R}{3} - \frac{\lambda}{6} f^4 + \frac{2f f''}{\psi} + \frac{f f'}{\psi} \left( \frac{4}{r} + \frac{N'}{N} - \frac{\psi'}{\psi} \right) = -[E] \tag{4.52}
\]
where $[E]$ is given by (3.32) and $R$ is the Ricci scalar given by (4.37).

The above equations of motion are for static field configurations. Therefore the higher derivative terms in the metric field equations (4.46) and (4.48) pose no issues as they are spatial not time derivatives. Higher spatial derivatives appear in many branches of physics e.g. in physical acoustics the wave equation is modified by a term with four spatial derivatives when the bending stiffness of a vibrating string is included.

### 4.1 Relation between Ricci scalar of AdS space and the VEV: asymptotic analytical solution

We now solve the equations of motion (4.46), (4.48), (4.50) and (4.52) analytically in the asymptotic region to show that the Ricci scalar of AdS space is determined entirely by the VEV. The vacuum expectation value of the energy momentum tensor in AdS space is given by (2.21) and the non-zero components are
\[
\langle T^{tt} \rangle = \frac{1}{4} \frac{1}{1 + k r^2} \langle T_{\mu}^{\mu} \rangle_0 ; \quad \langle T^{rr} \rangle = -\frac{1}{4} (1 + k r^2) \langle T_{\mu}^{\mu} \rangle_0 \\
\langle T^{\theta\theta} \rangle = -\frac{1}{4 r^2} \langle T_{\mu}^{\mu} \rangle_0 ; \quad \langle T^{\phi\phi} \rangle = -\frac{1}{4 r^2 \sin^2 \theta} \langle T_{\mu}^{\mu} \rangle_0 \tag{4.53}
\]
where the AdS metric (2.20) was used. The trace $\langle T_{\mu}^{\mu} \rangle$ is given by
\[
\langle T_{\mu}^{\mu} \rangle = \frac{a_2(x)}{16\pi^2} - [E] \\
= \frac{1}{16\pi^2} \left\{ \frac{1}{60} \left( R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - R_{\mu\nu} R^{\mu\nu} + \Box R \right) - \frac{1}{6} F^2 + \frac{11}{3} \frac{\lambda^2}{4!} \phi^4 - \frac{5}{36} \lambda \Box \phi |\phi^2| \right\} - [E], \tag{4.54}
\]
where (2.17) and (2.19) were used and $[E]$ is given by (3.32). We now evaluate (4.54) in AdS space, the asymptotic spacetime. Asymptotically, $F_{\mu\nu} \to 0$, $\Box |\phi^2| \to 0$, $R_{\rho\sigma\mu\nu} = k (g_{\mu\nu} g_{\sigma\nu} -$
\( g_{\mu\nu} g_{\sigma\mu} \) and \( R_{\mu\nu} = 3 k g_{\mu\nu} \) so that \( R = 12 k, \quad \Box R = 0 \) and \( R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - R_{\mu\nu} R^{\mu\nu} = -12 k^2 \).
Substituting these values into (4.54), we obtain
\[
\langle T_{\mu\nu} \rangle_0 = -\frac{1}{80} \frac{k^2}{\pi^2} + \frac{11}{152} \frac{\lambda^2}{\pi^2} v^4 - [E]_0 \quad (4.55)
\]
where \([E]_0 \) is \([E] \) evaluated in asymptotic AdS space.

The boundary conditions for the magnetic monopole [3] are that asymptotically, as \( r \to \infty \), the spacetime is AdS where \( N \to 1 + kr^2, \quad \psi \to 1/(1 + kr^2), \quad f \to v, \quad r f' \to 0 \) (\( f' \) drops off faster than \( 1/r \)), \( r^2 f'' \to 0, \quad a \to 0 \) and \( a' \to 0 \). Both \( v \) and \( k \) are positive constants. Substituting these boundary conditions into the gravity equation (4.46) (or (4.48)) and using (4.55) yields the following relation
\[
\frac{v^2}{2} - \frac{\lambda v^4}{48 k} = \frac{1}{8k} \left( -\frac{1}{80} \frac{k^2}{\pi^2} + \frac{11}{152} \frac{\lambda^2}{\pi^2} v^4 - [E]_0 \right). \quad (4.56)
\]
We can eliminate \([E]_0 \) above by solving the scalar equation (4.51) asymptotically. This yields
\[
4 k v^2 - \frac{\lambda v^4}{6} = -[E]_0 \quad (4.57)
\]
where we used that \( R = R_0 = 12 k \) in AdS space. Substituting the above into (4.56) yields the solution
\[
k = \frac{\sqrt{110}}{12} \lambda v^2 \quad (4.58)
\]
so that
\[
R_0 = \sqrt{110} \lambda v^2. \quad (4.59)
\]
The Ricci scalar of AdS space is therefore determined solely by the VEV since the value of \( \lambda \) is known (it is no longer a free parameter having been traded for the dimensionful VEV).
Substituting (4.58) into (4.57), yields \([E]_0 = -\frac{\lambda v^4}{6} (2\sqrt{110} - 1) \). This agrees with expression (3.32) asymptotically i.e. after substituting \( R = \sqrt{110} \lambda v^2 \) and \( \phi_c = v = M \).

The remaining unbroken U(1) is associated with \( F_3^{\mu\nu} \) and the magnetic field is defined via \( F_3^{ij} = \epsilon^{ijk} B^k \). Asymptotically, \( a(r) \to 0 \) and the function \( q(r) \) appearing in (4.35) approaches \(-1/r^2 \). It is easy to verify that one obtains a radial magnetic field that varies as \( 1/r^2 \) at large distances, corresponding to a magnetic monopole.

5 Conclusions

In previous work [3], a magnetic monopole solution in AdS space was obtained without introducing explicitly a mass term. In that calculation, spontaneous symmetry breaking (SSB)
of gauge symmetry responsible for the magnetic monopole occurred via gravitation itself through the coupling term $R\phi^2$ in a conformally invariant action. This works as long as a length scale is introduced by hand because classically there is no length scale. In the present work, we introduced a renormalization scale into the massless theory by considering quantum corrections. Symmetry breaking was radiatively induced à la Coleman-Weinberg [5], albeit in a more complicated massless theory containing gravity where the one loop effective potential must take into account the $R\phi^2$ interaction in addition to the usual $\lambda\phi^4$. The dimensionless $\lambda$, defined at the renormalization scale $M$, was traded for the dimensionful VEV and the Ricci scalar of the background AdS spacetime was determined entirely by the VEV. Though we discussed the quantum corrections of a classical conformal invariant theory in the context of the magnetic monopole, the techniques and results presented here could potentially have wider consequences. For example, in a recent article [6], scale and conformal symmetry are presented as fundamental principles for physics and cosmology. The authors have a model containing a Higgs field $H$, a dilaton field $\phi$, standard model fields as well as gravity. The authors point out that in a conformally invariant theory there is no mechanism at the classical level to set the scale of $\phi_0$, the minimum of the dilaton field $\phi$. They then mention that quantum corrections may alleviate this problem in a fashion that is reminiscent to [5]; our calculation provides a concrete implementation of this proposal.

Our work can now be naturally extended in a few ways. First, one can add gauge field fluctuations in the calculation of the effective potential. Then the coupling constant $\lambda$ can be expressed in terms the electromagnetic coupling constant $e$ as in [5]. In that scenario, the two parameters in the theory become $e$ and the dimensionful VEV. Second, we used the symmetry of AdS space to solve for $\langle T_{\mu\nu}\rangle_0$, the asymptotic value of $\langle T_{\mu\nu}\rangle$. This allowed us to solve the equations of motion analytically in the asymptotic regime and to obtain an expression relating the Ricci scalar of AdS space to the VEV. The interior spacetime obeys spherical symmetry but not the symmetry of AdS space. Therefore, the finite and nonlocal part of $\langle T_{\mu\nu}\rangle$ for the interior would require a more elaborate calculation. One could then obtain numerical solutions of the interior. It is of interest to see how these numerical solutions containing quantum corrections in an AdS background compare with those obtained in the classical context of General Relativity (GR) [17]. Third, we worked with a background AdS spacetime because the Ricci scalar of AdS space had the right sign for classical SSB [8]. However, the VEV here is obtained from the quantum-corrected effective potential. In the presence of quantum corrections, de Sitter (dS) space could well be a viable background spacetime. The Ricci scalar of the background AdS space was determined solely by the VEV, which should apply to dS space as well. Such solutions could be of greater cosmological interest.
Acknowledgments

AE acknowledges support from an NSERC discovery grant. He thanks KITP for a four week stay during the summer of 2012 where part of this work was completed. This research was, supported in part by the National Science Foundation under Grant No. NSF PHY11-25915. NG was supported in part by the National Science Foundation (NSF) through grant PHY-1213456.

References

[1] G. ’t Hooft, Nucl. Phys. B 79, 276, (1974).
[2] A.M. Polyakov, JETP Lett. 20, 194 (1974).
[3] A. Edery, L. Fabbri and M. B. Paranjape, Class. Quant. Grav. 23, 6409 (2006) [hep-th/0603131].
[4] A. Edery and B. Constantineau, Class. Quant. Grav. 28, 045003 (2011) [arXiv:1010.5844].
[5] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
[6] I. Bars, P. Steinhardt and N. Turok, [arXiv:10307.1848].
[7] E.C. Thomas, F.R. Urban and A. R. Zhitnitsky, JHEP 08,043 (2009) [arXiv:0904.3779].
[8] V.F. Mukhanov and S. Winitzki, Introduction to Quantum Effects in Gravity, Cambridge University Press, (2007).
[9] N.D. Birrell and P.C. W. Davies, Quantum Fields in Curved Space, Cambridge University Press, (1982).
[10] A. O. Barvisnsky, Yu. V. Gusev, G.A. Vilkovisky and V.V. Zhytnnikov, Nucl. Phys. B 439, 561 (1995) [hep-th/9404187].
[11] V.M. Mostepanenko and N.N. Trunov, The Casimir Effect And Its Applications, Oxford University Press, (1997), M. Bordag , U. Mohideen and V.M. Mostepanenko, Phys. Rep. 353, 1 (2001), A. Zee, Quantum Field Theory in a Nutshell, Princeton University Press, (2003), A. Edery, J. Phys. A: Math. Gen. 39, 685 (2006) [math-ph/0510056], A. Edery, Phys. Rev. D 75, 105012 (2007) [hep-th/0610173], A. Edery, J. Stat. Mech. P06007 (2006) [hep-th/0510238].
[12] L.S. Brown and J.C. Collins, Ann. Phys. 130, 215 (1980).
A Equations of motion with quantum corrections

The equations of motion for the metric, non-abelian gauge fields, and scalar triplet are obtained by variation of the total action $S = S_{\text{ren}} + W_{\text{ren}}$ with respect to each field. For the metric we obtain

$$
\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S_{\text{ren}} = -\langle T_{\mu\nu} \rangle \tag{A.1}
$$

where we used (2.21). With $S_{\text{ren}}$ given by (2.12) the metric field equations are

$$
\alpha H_{\mu\nu} + \beta K_{\mu\nu} + M_{\mu\nu} = -\langle T_{\mu\nu} \rangle \tag{A.2}
$$

where

$$
H_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int R^2 \sqrt{-g} \ d^4x = 4 \nabla_\mu \nabla_\nu R - 4 g_{\mu\nu} \Box R - g_{\mu\nu} R^2 + 4 R R_{\mu\nu}
$$

$$
K_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int R_{\alpha\beta} R^{\alpha\beta} \sqrt{-g} \ d^4x
= 4 \nabla_\nu \nabla_\alpha R^{\alpha\mu} - 2 \Box R_{\mu\nu} - g_{\mu\nu} \Box R + 4 R^{\alpha\beta} R_{\alpha\beta} - g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta}.
$$

$$
M_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int \left\{ - \frac{1}{4e^2 R} F^2 + (D\phi)^2 + \frac{1}{6} R \phi^2 - \lambda \phi^4 \right\} \sqrt{-g} \ d^4x
= \frac{1}{4e^2 R} g_{\mu\nu} F^2 - \frac{1}{e^2 R} F_{\mu\beta} F^{\beta\nu} + 2 D_\mu \phi^\alpha D_\nu \phi_\alpha - g_{\mu\nu} ((D\phi)^2 - \lambda R \phi^4)
+ \frac{1}{3} (g_{\mu\nu} \nabla_\alpha \phi^2 - \nabla_\mu \nabla_\nu \phi^2 + (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \phi^2).
$$
For the scalar field, we have \( \frac{2}{\sqrt{-g}} \delta \frac{\delta}{\delta \phi} (S_{\text{ren}} + W_{\text{ren}}) = 0 \), which yields the equation

\[
\left( 2 D_\mu D_\mu \phi^a - \left( \frac{R}{3} - 4 \lambda^2 R \phi^2 \right) \phi_a \right) \phi^a = \frac{1}{\sqrt{-g}} \phi^a \frac{\delta W_{\text{ren}}}{\delta \phi^a} = [E]. \tag{A.3}
\]

For the gauge field we simply have \( \frac{2}{\sqrt{-g}} \delta S_{\text{ren}} = 0 \), since \( \frac{2}{\sqrt{-g}} \delta W_{\text{ren}} \) is zero when evaluated in the vacuum state where \( F_{\mu \nu}^a \to 0 \) and \( A_\mu^a \) is a pure gauge which we set to zero. The equation of motion for the gauge field is then

\[
\nabla_\mu F_{\mu \nu}^a + \varepsilon_{a}^{bc} A_{\mu b} F_{\nu c}^a = 2 e R \varepsilon_{a}^{bc} D^\nu \phi_b \phi_c. \tag{A.4}
\]