HODGE CYCLES AND THE LERAY FILTRATION

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Abstract. Given a fibered variety, we pull back the Leray filtration to the Chow group, and use this to give some criteria for the Hodge and Tate conjectures to hold for such varieties. We show that the Hodge conjecture holds for a good desingularization of a self fibre product of a nonisotrivial elliptic surface under appropriate conditions. We also show that the Hodge and Tate conjectures hold for natural families of abelian varieties parameterized by certain Shimura curves.

In [A2], we checked the Hodge and Tate conjectures for the universal family of genus two curves with level structure by showing the absence of these cycles in a critical piece of the Leray spectral sequence. In this article, we will to take this idea further. Given a surjective map of smooth projective varieties \( f: X \to Y \), let \( U \subseteq Y \) be the complement of the discriminant and \( V = f^{-1}U \). Then we define a filtration on the rationalized Chow group such that we have a cycle map

\[
\square^{p,i}: \text{Gr}^i_L CH^p(V) \to H^i(U, R^{2p-i}f_*\mathbb{Q})
\]

By [A1], the right side carries a mixed Hodge structure. We have that \( \square^{p,i} \) lands within the space of Hodge cycles. Let \( m = \dim Y \) and \( r = \dim X - \dim Y \). Our first theorem is that the Hodge conjecture, in Jannsen’s sense, holds for \( V \), if \( \square^{p,i} \) surjects onto the space of Hodge cycles for all \((p, i)\) in the parallelogram bounded by the lines \( i = 0, i = m, i = 2p, \) and \( i = 2p - r \). An analogous statement holds for the Tate conjecture. As a corollary, we show that when \( X \) is a fourfold, the Hodge conjecture holds for \( X \) if \( \square^{2m} \) surjects onto the space of Hodge cycles. A special case of this was used in [A2].

In the rest of this article, we turn our attention to particular classes of examples. Suppose that \( f_\lambda: E_\lambda \to Y_\lambda \) is a family of nonisotrivial elliptic surfaces, such that the “Kodaira-Spencer” map

\[
\kappa_\lambda: \text{Gr}^1_L H^1(U_\lambda, R^1f_*\mathbb{C}) \to \text{Gr}^0_L H^1(U_\lambda, R^1f_*\mathbb{C}) \otimes \Omega^1_\lambda
\]

is injective somewhere. Theorem [2.1] shows that the Hodge conjecture holds for the complement \( V_\lambda \subset E_\lambda \times_{Y_\lambda} \cdots \times_{Y_\lambda} E_\lambda \) of the set of singular fibres, when \( \lambda \) is very general. For a zero dimensional family, injectivity of \( \kappa_\lambda \) holds if and only if the elliptic surface is extremal in the sense that the Picard number is maximal and the Mordell-Weil rank is zero (lemma [2.6]). Elliptic modular surfaces are extremal, so we recover a result of Gordon [G] that theorem [2.1] holds for such surfaces. Our theorem also holds for certain families of nonextremal elliptic K3 surfaces constructed by Dolgachev [Do] and Hoyt [H]. Gordon’s argument for elliptic modular surfaces does not generalize. Our proof is different, and is based on analyzing \( \square^{p,i} \) for \( i = 0, 1 \). The map \( \square^{p,0} \) is shown to surject onto the space of Hodge cycles by the K"unneth

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formula and invariant theory. On the other hand for very general $\lambda$, we will show $\Box^{p,1}$ is trivially surjective in the sense that there are no Hodge cycles on the target. This will be deduced by showing that injectivity of $\kappa_\lambda$ implies injectivity of the Kodaira-Spencer map

$$Gr^p_F H^1(U_\lambda, R^{2p-1} F_{\lambda*} \mathbb{C}) \to Gr^{p-1}_F H^1(U_\lambda, R^{2p-1} F_{\lambda*} \mathbb{C})$$

where $F_\lambda : V_\lambda \to Y_\lambda$ is the structure map.

Suppose that $Y$ is a Shimura curve associated to a quaternion division algebra split at exactly one real place over a totally real field. The curve $Y$ can be viewed as a moduli space of abelian varieties with some extra structure. In particular, for sufficiently high level, $Y$ carries a natural family of abelian varieties $f : A \to Y$. When $i = 0$ or $2$, this follows from invariant theory. When $i = 1$, work of Viehweg and Zuo \cite{VZ} is used to show that in fact

$$Gr^p_F H^1(Y, R^{2p-1} F_* \mathbb{C}) = 0$$

Therefore $\Box^{p,1}$ is again trivially surjective.

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1. Hodge cycles and the Leray filtration

Let MHS denote the category of rational mixed Hodge structures, and MHS$^p$ the full subcategory of polarizable Hodge structures. We note that mixed Hodge structures of geometric origin in fact lie in MHS$^p$, and the category of pure polarizable Hodge structures is semisimple \cite{B}. Given a mixed Hodge structure $H$, set

$$\text{Hodge}(H) := \text{Hom}_{\text{MHS}}(\mathbb{Q}(0), H)$$

The following facts will often be used without comment below.

**Lemma 1.1.**

1. There is an injection

$$\text{Hodge}(H) \hookrightarrow \text{Hodge}(Gr^W_0 H)$$

It is an isomorphism if $H$ has nonnegative weight, i.e. $W_{-1} H = 0$.

2. $\text{Hodge}(-)$ is an exact functor on the category of polarizable mixed Hodge structures of nonnegative weight.

3. If

$$H_1 \to H_2 \to H_3$$

is an exact sequence of mixed Hodge structures, such that $H_1$ is polarizable with nonnegative weight, then

$$\text{Hodge}(H_1) \to \text{Hodge}(H_2) \to \text{Hodge}(H_3)$$

is exact.

4. If $\phi : H_1 \to H_2$ is a morphism of polarizable Hodge structures, such that $Gr^0_F H_1 \to Gr^0_F H_2$ is surjective (resp. injective) then $\text{Hodge}(H_1) \to \text{Hodge}(H_2)$ is surjective (resp. injective).
Proof. The first item follows from strictness of the weight filtration [13, thm 2.3.5]. The second follows from (1) and the semisimplicity of the category of pure polarizable Hodge structures. Statement (3) follows from (1) and (2), and the observation that subobjects and quotients of $H_1$ are also polarizable with nonnegative weight. Given $\phi : H_1 \to H_2$ inducing a surjective map $\text{Gr}_F^0 H_1 \to \text{Gr}_F^0 H_2$, let $H_3 = \text{coker} \phi$. Then
\[
\text{coker}(\text{Hodge}(H_1) \to \text{Hodge}(H_2)) = \text{Hodge}(H_3) \subseteq \text{Gr}^0_F H_3 = 0
\]
The injectivity part of (4) is similar. □

If $X$ is an algebraic variety, let $CH^p(X)$ (resp. $CH^p(X)$) denote the Chow group of $p$ dimensional (resp. codimension $p$) cycles tensored with $\mathbb{Q}$. When $X$ is defined over $\mathbb{C}$, Borel-Moore homology $H_i(X, \mathbb{Q})$ carries a mixed Hodge structure dual to the one on compactly supported cohomology $H^c_i(X, \mathbb{Q})$. We have a cycle map $\bullet^p : CH^p(X) \to H^{2p}(X, \mathbb{Q})$ such that the image lands in $H^p_2(X, \mathbb{Q})(-p)$, c.f. [F1, chap 19], [J, §5, §7]. We will usually write $[Z]$ instead of $\bullet^p(Z)$. Following Jannsen [J], we say that the Hodge conjecture holds for $X$ if the cycle map $CH^p(X) \to \text{Hodge}(H^{2p}_c(X, \mathbb{Q})(-p))$ is surjective for all $p$. When $X$ is smooth, we can apply Poincaré duality to identify this with the cycle map $\square^p : CH^p(X) \to \text{Hodge}(H^{2p}(X, \mathbb{Q})(p))$.

We recall that $\square^*$ is a graded ring homomorphism.

Let us now suppose that $K$ is a finitely generated field of characteristic 0, and let $G_K = Gal(\bar{K}/K)$. Suppose that $X$ is defined over $K$, and let $\bar{X} = X \times_{\text{Spec} K} \bar{K}$. Given a $G_K$-module $H$, define the space of Tate cycles in $H$ by
\[
\text{Tate}(H) = \sum_{L/K \text{ finite}} H^G_L
\]
Fix a prime $\ell$. Again following Jannsen, we say that Tate’s conjecture holds for $X$ if the cycle map $\square_{p, \ell} : CH^p(\bar{X}) \otimes \mathbb{Q}_\ell \to \text{Tate}(H^{2p}_c(\bar{X}_{et}, \mathbb{Q}_\ell)(-p))$ is surjective for each $p$. When $X$ is smooth, this is equivalent to the more familiar statement that the cycle map $\square^p_\ell : CH^p(\bar{X}) \otimes \mathbb{Q}_\ell \to \text{Tate}(H^{2p}(\bar{X}_{et}, \mathbb{Q}_\ell)(p))$ is surjective for each $p$. We note that $\square^p$ and $\square^p_\ell$ are compatible under the Artin comparison isomorphism.

We note that Jannsen’s form of the Hodge conjecture follows from the usual Hodge conjecture, so it is not stronger. More precisely, we have:

**Theorem 1.1** (Jannsen). If the Hodge conjecture holds for a desingularization of a compactification of $X$, then it holds for $X$.

**Proof.** This follows from the proof of [J, thm 7.9]. □

**Lemma 1.2.** Let $U \subset X$ be Zariski open, and $Z = X - U$. If $X$ is projective and the Hodge conjecture holds for $U$ and $Z$, then it holds for $X$. If the Hodge conjecture holds for $X$, then it holds for $U$. 

Proof. We have a localization sequence
\[ H_{2p}(Z) \to H_{2p}(X) \to H_{2p}(U) \]
where \( H_* \) denotes Borel-Moore homology. Projectivity of \( X \) implies that, after twisting by \( \mathbb{Q}(-p) \), the weight of the first term becomes nonnegative. So we deduce from lemma [1.1] that
\[ \text{Hodge}(H_{2p}(Z)(-p)) \to \text{Hodge}(H_{2p}(X)(-p)) \to \text{Hodge}(H_{2p}(U)(-p)) \]
is exact. By [F1 §1.8, chap 19], this fits into a larger commutative diagram with exact rows
\[
\begin{array}{ccc}
CH_p(Z) & \longrightarrow & CH_p(X) & \longrightarrow & CH_p(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Hodge}(H_{2p}(Z)(-p)) & \longrightarrow & \text{Hodge}(H_{2p}(X)(-p)) & \longrightarrow & \beta & & \\
\end{array}
\]
The first part of the lemma follows from the surjectivity of \( \alpha \) and \( \beta \), and the five lemma.

We note that \( Gr^W_{2p}H_{2p-1}(Z) = 0 \), because \( Gr^W_{2p}H^2_{2p-1}(Z) = 0 \). Therefore \( Gr^W_{2p}H_{2p}(X) \to Gr^W_{2p}H_{2p}(U) \) is surjective. Consequently \( \gamma \) is surjective. This implies last part of the lemma.

Now, let us suppose that \( f : X \to Y \) is a surjective morphism with connected fibres between smooth projective varieties. We suppose also that \( \dim Y < \dim X \).

We can analyze the Hodge structure \( H^i(X) \) in terms of the map by applying the decomposition for Hodge modules [Sa1], but we prefer to do this in a more canonical fashion. First, let us choose a nonempty Zariski open subset \( U \subset Y \), such that \( f \) is smooth over \( U \), and let \( V = f^{-1}U \). Then the Leray filtration is defined by
\[
L^i_f H^i(V, \mathbb{Q}) = \text{im}[H^i(U, \tau_{<i-p}\mathbb{R}f_*\mathbb{Q}) \to H^i(U, \mathbb{R}f_*\mathbb{Q}) = H^i(V, \mathbb{Q})]
\]
We usually write \( L^\bullet = L^\bullet_f \). By a theorem of Deligne [D1], the Leray spectral sequence degenerates, so that
\[
Gr^b L^0H^a(V) \cong H^b(U, R^{a-b}f_*\mathbb{Q})
\]
Furthermore, the author [A1 cor 2.6] showed that \( L \) is a filtration by submixed Hodge structures. Therefore \( Gr^b L^aH^a(V) \) carry natural mixed Hodge structures. These can be seen to coincide with the ones constructed by Saito [Sa2]. We define a Leray filtration on the Chow group in the most naive way. Let \( L^\bullet CH^p(V) \) denote the preimage of \( L^\bullet H^{2p}(V, \mathbb{Q}) \) under the cycle map
\[ CH^p(V) \to H^{2p}(V, \mathbb{Q}) \]
Then we have an induced cycle map
\[
\square^{p,i} : Gr^i L^\bullet CH^p(V) \to H^i(U, R^{2p-i}f_*\mathbb{Q})
\]
Since \( \square^{p,i}(\alpha) \equiv \square(\alpha) \mod L^{i+1} \), it follows that the image lies in
\[ \text{Hodge}(H^i(U, R^{2p-i}f_*\mathbb{Q})(p)) \]
When \( f : X \to Y \) is defined over the above field \( K \), one has an entirely parallel story for \( \ell \)-adic cohomology. We have a cycle map
\[
\square^{p,i} : Gr^i L^\bullet CH^p(\tilde{V}) \otimes \mathbb{Q}_\ell \to \text{Tate}(H^i(\tilde{U}_{et}, R^{2p-i}f_*\mathbb{Q}_\ell)(p))
\]
Lemma 1.3.

1. If \( f_1 : V_1 \to U_1 \) are two smooth projective maps, then the exterior product is compatible with \( L^\bullet \) in the sense that

\[
L^p_{f_1} H^i(V_1, \mathbb{Q}) \otimes L^p_{f_2} H^j(V_2, \mathbb{Q}) \subseteq L^{p+q}_{f_1 \times f_2} H^{i+j}(V_1 \times V_2, \mathbb{Q})
\]

2. Furthermore

\[
L^p H^k(V_1 \times V_2, \mathbb{Q}) = \bigoplus_{i+j=k,p+q=r} L^p H^i(V_1, \mathbb{Q}) \otimes L^q H^j(V_2, \mathbb{Q})
\]

3. There is a commutative diagram

\[
\begin{array}{ccc}
Gr^i_L CH^p(V_1) \otimes Gr^j_L CH^q(V_2) & \longrightarrow & Gr^{i+j}_{L^2} CH^{p+q}(V_1 \times V_2) \\
\square^{(p,i) \otimes (q,j)} & & \square^{(p+q, i+j)} \\
H^i(U_1, R^{2p-i} f_1, \mathbb{Q}) \otimes H^j(U_2, R^{2q-j} f_2, \mathbb{Q}) & \longrightarrow & H^{i+j}(U_1 \times U_2, R^{2(p+q)-i-j} (f_1 \times f_2)^\ast \mathbb{Q})
\end{array}
\]

4. When \( U = U_1 = U_2 \), the same statements hold for the fibre product \( V_1 \times_U V_2 \).

Proof. The first two statements are formal consequences of the isomorphism

\[
(\mathbb{R} (f_1 \times f_2)_\ast \mathbb{Q}, \tau_\leq) \cong (\mathbb{R} f_1, \mathbb{Q}, \tau_\leq) \cong (\mathbb{R} f_2, \mathbb{Q}, \tau_\leq)
\]

in the filtered derived category. The third statement is a formal consequence of the first two statements, and the compatibility of \( \square \) with external products. The fourth follows from the previous statements by restricting to the diagonal embedding \( U \subset U \times U \).

Let \( n = \dim X \), \( m = \dim Y \), and \( r = n - m \). Then the groups on the right of (1.1) will vanish for \((p, i)\) outside the closed parallelogram \( P \) in the \((p, i)\)-plane bounded by the lines \( i = 0 \), \( i = 2m \), \( i = 2p \), and \( i = 2p - 2r \). Let \( Q \subset P \) be the closed parallelogram bounded by the lines \( i = 0 \), \( i = m \), \( i = 2p \), and \( i = 2p - r \). This is the lower left quarter of \( P \).

Theorem 1.2. When \( f \) is defined over \( \mathbb{C} \), and the cycle map

\[
\square^p : Gr^p_L CH^p(V) \to Hodge(H^0(U, R^{2p-i} f_\ast \mathbb{Q})(p))
\]

is surjective for all \((p, i)\) in \( Q \), the Hodge conjecture holds for \( V \). When \( f \) is defined over \( \mathbb{K} \), and the cycle map

\[
\square^p : Gr^p_L CH^p(V) \otimes \mathbb{Q}_\ell \to Hodge(H^0(\bar{U}_{et}, R^{2p-i} \bar{f}_{et} \ast \mathbb{Q}_\ell)(p))
\]

is surjective for all \((p, i)\) in \( Q \), the Tate conjecture holds for \( V \).

Proof. We just prove the first statement for the Hodge conjecture. The proof of the second statement is almost identical. Suppose that we know that \( \square^p \) is surjective for all \((p, i) \in P \). By lemma 1.1 Hodge is exact, so we have a noncanonical isomorphism

\[
Hodge(H^{2p}(V, \mathbb{Q})(p)) = \bigoplus_i Hodge(Gr^i_L (H^{2p}(V, \mathbb{Q})(p)))
\]

\[
= \bigoplus_i Hodge(H^i(U, R^{2p-i} f_\ast \mathbb{Q})(p))
\]

\[
= \bigoplus_i \text{im} Gr^i_L CH^p(V)
\]
This implies that the Hodge conjecture holds for \( V \).

It remains to show that surjectivity of \( \square^{p,i} \) for \( (p,i) \in Q \) implies surjectivity for all points in \( P \). We do this in two steps. We first assume \( 2p - i \leq r \), and then we show that the condition of surjectivity of \( \square^{p,i} \) is stable under the reflection \( (p,i) \mapsto (q,i) \), where \( q = r + i - p \). Let \( \mathcal{O}_Y(1) \) and \( \mathcal{O}_X(1) \) be ample divisors on \( Y \) and \( X \) respectively. Cup product with \( c_1(\mathcal{O}_X(1))^{r-2p+i} \) induces an isomorphism of local systems

\[
\xi : R^{2p-i}f_\ast \mathbb{Q} \xrightarrow{\sim} R^{2q-i}f_\ast \mathbb{Q}
\]

Together with lemma 1.3, this gives rise to a commutative diagram

\[
\begin{array}{ccc}
Gr^1_L CH^p(V) & \xrightarrow{\square^{p,i}} & H^i(U, R^{2p-i}f_\ast \mathbb{Q})(p) \\
\downarrow \xi' & & \downarrow \xi'' \\
Gr^1_L CH^q(V) & \xrightarrow{\square^{q,i}} & H^i(U, R^{2q-i}f_\ast \mathbb{Q})(q)
\end{array}
\]

where \( \xi' \) and \( \xi'' \) are products with \( c_1(\mathcal{O}_X(1))^{r-2p+i} \). The map \( \xi'' \) is an isomorphism because \( \xi \) is. If \( \square^{p,i} \) is surjective, then \( \square^{q,i} \) must also be surjective. This allows to extend surjectivity of \( \square^{p,i} \) from \( Q \) to the parallelogram \( P' \) bounded by \( i = 0 \), \( i = m \), \( i = 2p \), and \( i = 2p - 2r \).

To finish the proof, we have to show that if \( \square^{p,i} \) is surjective for \( (p,i) \in P' \), then this holds for \( P \). Let \( i \leq m, i = 2m - i \), and \( q = p + m - i \). The transformation \( (p,i) \mapsto (q,i) \) is a reflection about \( i = m \), which preserves lines of slope 2. Therefore \( P \) is the union of \( P' \) and the image of \( P' \) under this reflection. If \( L \) is a local system defined on \( U \), let \( IH^i(Y, L) = H^i(Y, (j_\ast L[m])[-m]) \) denote intersection cohomology (with the naive indexing convention). We have a commutative diagram of vector spaces

\[
\begin{array}{ccc}
IH^i(Y, R^{2p-i}f_\ast \mathbb{Q}) & \xrightarrow{\pi} & Gr^W_H^i(U, R^{2p-i}f_\ast \mathbb{Q}) \\
\downarrow \eta' & & \downarrow \eta \\
IH^i(Y, R^{2q-i}f_\ast \mathbb{Q}) & \xrightarrow{\pi} & Gr^W_H^i(U, R^{2q-i}f_\ast \mathbb{Q})
\end{array}
\]

where the vertical maps are given by the product with \( c_1(\mathcal{O}_Y(1))^{m-i} \). The map \( \eta' \) is an isomorphism by the hard Lefschetz for intersection cohomology \( \text{[BBD] 6.2.10} \). The maps labelled by \( \pi \) are the natural ones; these are surjective by \( \text{[FS]} \) (plus the comparison theorem in \( \ell \)-adic case). These facts imply that \( \eta \) is surjective. Now consider the diagram

\[
\begin{array}{ccc}
Gr^1_L CH^p(V) & \xrightarrow{\square^{p,i}} & H^i(U, R^{2p-i}f_\ast \mathbb{Q})(p) \\
\downarrow \xi' & & \downarrow \xi'' \\
Gr^1_L CH^q(V) & \xrightarrow{\square^{q,i}} & H^i(U, R^{2q-i}f_\ast \mathbb{Q})(p + m - i)
\end{array}
\]

where the vertical maps are again products with \( c_1(\mathcal{O}(1))^{m-i} \). Since \( \square^{p,i} \) is surjective by assumption, and \( \eta \) is surjective by what we just proved, we can conclude that \( \square^{q,i} \) is surjective. Therefore the surjectivity condition is preserved by the reflection \( (p,i) \mapsto (q,i) \), and this completes the proof.

\( \square \)
Remark 1.3. It is easy to see using a modification of [1.2] that conversely, if the Hodge conjecture holds for $V$, then the $\square^{p,i}$ must surject onto the space of Hodge cycles for all $(p,i) \in P$.

Corollary 1.4. Suppose that $\dim X = 4$, and $\dim Y = m$. If the cycle map

$$\square^{2,m} : Gr^m_{\ell} CH^2(V) \to Hodge(H^m(U, R^{4-m} f_* Q)(2))$$

is surjective, then the Hodge conjecture holds for $X$. In particular, this is the case if $Hodge(H^m(U, R^{4-m} f_* Q)(2)) = 0$.

Proof. It is well known that the Hodge conjecture holds for smooth projective varieties of dimension at most three (by the Lefschetz (1, 1) theorem, and the hard Lefschetz theorem). Therefore by theorem 1.4 it holds for $X - V$. By lemma 1.2 it suffices to prove the conjecture for $V$. Applying theorem 1.2 we have to check the surjectivity for $\square^{p,i}$ for $(p,i) \in Q$. For $p = 0$, this trivial, and for $p = 1$, this follows from the Lefschetz (1, 1) theorem. It is easy to check by plotting $Q$, that in each of the cases $m = 1, 2, 3$, there is exactly one point in $Q$ with $p > 1$, namely $(2, 1), (2, 2), (2, 3)$ respectively.

When $U$ is the moduli space of genus two curves with fine level structure, and $f : X \to Y$ the universal family of curves, then [A2] cor 3.2 shows that

$$Hodge(H^3(U, R^1 f_* Q)(2)) \subset \Gr^3_{\ell} H^3(U, R^1 f_* \mathbb{C}) = 0$$

Therefore the Hodge conjecture holds for $X$. The corollary also implies a result of Conte and Murre [CM] that the Hodge conjecture holds for uniruled fourfolds. Similarly, the conjecture holds for fourfolds fibred by surfaces with trivial geometric genus:

Corollary 1.5. Suppose that $\dim X = 4$, and $\dim Y = 2$, and the general fibre of $f : X \to Y$ is a surface with $p_g = 0$. Then the Hodge conjecture holds for $X$.

Proof. Given an irreducible component $T$ of the relative Hilbert scheme $Hilb_{X/Y}$ [Gr chap IV], let supp($T$) $\subseteq Y$ denote its image in $Y$. Let

$$C = \bigcup_{\text{supp}(T) \subseteq Y} \text{supp}(T)$$

as $T$ ranges over components of $Hilb_{X/Y}$. Since $C$ is a countable union of proper subvarieties $U - C \neq \emptyset$. Choose $y \in U - C$. Since $p_g(X_y) = 0$, by the Lefschetz (1, 1) theorem, there exist irreducible divisors $Z_1, \ldots, Z_N \subseteq X_y$ which span $H^2(X_y, \mathbb{Q})$. Let $T_i$ be a component of $Hilb_{X/Y}$ containing $Z_i$. Then $T_i \to Y$ is surjective. After replacing $T_i$ by an intersection of ample divisors we can assume that $T_i \to Y$ is generically finite. Let $Z_i$ denote the restriction of the universal family over $Hilb_{X/Y}$ to $T_i$. Then $Z_i$ is a relative divisor on $X/Y$ such that $[Z_i]$ is equal to a multiple of $[Z_i]$. It follows that, after a finite base change, $[Z_1], \ldots, [Z_N]$ gives a basis of $R^2 f_* \mathbb{Q}$, i.e. an isomorphism $\mathbb{Q}(-1)^N \cong R^2 f_* \mathbb{Q}$. Therefore, we can identify

$$H^2(U, R^2 f_* \mathbb{Q}) \cong \bigoplus_i H^2(U, \mathbb{Q}) \cup [Z_i]$$

The Lefschetz (1, 1) theorem now shows that

$$Hodge(H^2(U, R^2 f_* \mathbb{Q})(2)) = \bigoplus_i Hodge(H^2(U, \mathbb{Q})(1)) \cup [Z_i]$$
is spanned by algebraic cycles.

Remark 1.6. Define the transcendental part $T \subseteq R^2f_*\mathbb{Q}|_U$ to be the smallest subdivision of Hodge structure such $T_{y}^{20} = H^{20}(X_y)$ for some (hence all) $y \in U$. By essentially the same argument, the Hodge conjecture holds for $X$, if $\text{Hodge}(H^{2}(U,T)(2)) = 0$.

Corollary 1.7. Suppose that $f$ is defined over $K \subset \mathbb{C}$, and that $U = Y$. Let $X_{\mathbb{C}} = X \times_{\text{Spec} K} \text{Spec} \mathbb{C}$ etc. If

\[ \square^{p,i} : \text{Gr}^i \text{CH}^p(X_{\mathbb{C}}) \otimes \mathbb{C} \to \text{Gr}^p H^i(Y_{\mathbb{C}}, R^{2p-i}f_{\mathbb{C},*}\mathbb{C}) \]

is surjective for all $(p,i)$ in $Q$, then the Hodge and Tate conjectures hold for $X_{\mathbb{C}}$ and $X$ respectively.

Proof. The statement for the Hodge conjecture is an immediate consequence of the theorem. The assumption implies that

\[ \text{Gr}^p H^{2p}(X_{\mathbb{C}}, \mathbb{C}) = H^p(\bar{X}, \Omega^p_{\bar{X}}) \otimes \mathbb{C} \]

is generated by a finite number of algebraic cycles on $X_{\mathbb{C}}$, and therefore a finite number of cycles on $\bar{X}$. Tate’s conjecture is now a known consequence of Faltings’ theorem on Hodge-Tate decompositions [F], which implies that

\[ \dim \text{Tate}(H^{2p}(\bar{X}_{et}, \mathbb{Q}_l)(p)) \leq \dim H^p(\bar{X}, \Omega^p_{\bar{X}}) \]

c.f. [T, pp 81-82].

2. Fibre products of elliptic surfaces

We start by summarizing some facts from Hodge theory needed below. Let $\pi : Y \to \Lambda$ be a smooth projective curve over a smooth base $\Lambda$. Let $S \subset Y$ be a divisor étale over $\Lambda$, and $U = Y - S$. We recall that a mixed Hodge module on $Y$, consists a bifiltered regular holonomic left $\mathcal{D}$-module $(M, F, W)$, a filtered perverse sheaf $(L, W)$, and an isomorphism in the derived category

\[ \text{DR}(M) = M \to M \otimes \Omega^1_Y \to M \otimes \Omega^2_Y \to \ldots \cong \mathcal{L} \otimes \mathbb{C} \]

compatible with $W$. These are subject to some axioms that we will not recall [Sa1, Sa2]. The modules of interest to us arise as follows. Let $(\mathcal{L}_U, V_U, F^*_U, \nabla_U)$ be a polarizable variation of Hodge structure on $U$; here $V_U$ denotes a vector bundle, $F^*_U$ a filtration on it, $\nabla_U$ an integrable connection, and $\mathcal{L}$ local system of $\mathbb{Q}$-vector spaces. By Deligne, we have an extension of $(V_U, \nabla_U)$ to an vector bundle with logarithmic connection $(V, \nabla)$ on $Y$ such that the eigenvalues of the residues lie in $[0,1)$. We filter this by $F^p V = V \cap j_* F^p_U$. Then $M = V(S)$ is a regular holonomic $D_V$-module which corresponds under Riemann-Hilbert to the perverse sheaf $\mathcal{L} = \mathcal{R} j_* \mathcal{L}_U[\text{dim} Y]$ tensored with $\mathbb{C}$. When equipped with the Hodge filtration

\[ F^p M = \sum_i F^i D_V \cap F^{p+i} V \]

and an appropriate weight filtration $W$, $(M, \mathcal{L})$ forms a mixed Hodge module $\mathcal{S}$ [Sa2 §3]. We have a filtered quasi-isomorphism

\[ (V \otimes \Omega^*_Y(\text{log} S), F) \cong (\text{DR}(M), F) \]
where the two sides are filtered by
\[ F^p(V \otimes \Omega^i_Y (\log S)) = F^{p-i}V \otimes \Omega^i_Y (\log S) \]
and
\[ F^p(M \otimes \Omega^i_Y ) = F^{p-i}M \otimes \Omega^i_Y . \]
Saito \cite{Sa1, Sa2} has shown that mixed Hodge modules are stable under direct images, and furthermore that \( \mathbb{R} \pi_* \) applied to the second, and therefore also the first, filtered complex of (2.1) is strict. These facts, together with \cite[thm 0.2]{Sa2}, imply that \( R^1(\pi(U)_* \mathcal{L} \) is an admissible variation of mixed Hodge structure such that
\[ \text{Gr}^p F 1(\pi(U)_* \mathcal{L} \otimes \mathbb{C} \cong R^1 \pi_*(\text{Gr}^p F V \xrightarrow{\partial} \text{Gr}^{p-1} F \otimes \Omega^1_{Y/\Lambda}(\log S)) \]
When \( \Lambda \) is point, these results go back to Zucker \cite{Z}, who showed additionally that in the geometric case, the mixed Hodge structure on \( H^1(U, \mathcal{L}) \) coincides with the one coming from the Leray spectral sequence.

Given an admissible variation of mixed Hodge structure \( (V_U, F_U, \nabla_U, \ldots) \) as above, we will describe an associated “Kodaira-Spencer” map following the method of Katz and Oda \cite{KO}. It is convenient to pass to the graded Higgs bundle \( E = \bigoplus E^p \), where \( E^p = \text{Gr}^p F V \), with Higgs field \( \theta : E \rightarrow E \otimes \Omega^1_Y (\log S) \) given by the sum of \( \text{Gr}^p F (\nabla) : \text{Gr}^p F V \rightarrow \text{Gr}^{p-1} F V \), so that \( \theta \) has degree \(-1\). To simplify notation, let us write \( \Omega_Y^p = \text{Gr}^p F \log S \) etc. The pair \((E, \theta)\) is indeed a Higgs bundle in the sense that \( \theta \wedge \theta = 0 \). Therefore it extends to give a complex
\[ (2.2) \quad E \xrightarrow{\theta} E \otimes \Omega_Y^1 \xrightarrow{\theta} E \otimes \Omega_Y^2 \ldots \]
We define a filtration
\[ K^r(V \otimes \Omega^i_Y ) = V \otimes (\Omega^{i-r}_Y \wedge \pi^* \Omega^r_Y) \]
One can check that this is a filtration by subcomplexes of the de Rham complex
\( (V \otimes \Omega^*_Y , \nabla) \). The connecting map associated to
\[ (2.3) \quad 0 \rightarrow K^1/K^2 \rightarrow K^0/K^2 \rightarrow K^0/K^1 \rightarrow 0 \]
induces a Gauss-Manin connection
\[ \mathbb{R}^1 \pi_*(V \otimes \Omega^*_{Y/\Lambda}) \cong \mathbb{R}^2 \pi_* K^1/K^2 \cong \mathbb{R}^1 \pi_*(V \otimes \Omega^*_{Y/\Lambda}) \otimes \Omega^1_{\Lambda} \]
The associated graded of this with respect to \( F \) is the Kodaira-Spencer map \( \kappa(V) = \sum \kappa_p(V) \). We can construct \( \kappa_p(V) \) directly by applying \( \text{Gr}^p F \) to \( (2.3) \) to obtain
\[ 0 \rightarrow [0 \rightarrow E^{p-1} \otimes \Omega^1_Y \ldots] \rightarrow [E^p \rightarrow E^{p-1} \otimes \Omega^1_Y \ldots] \rightarrow [E^p \rightarrow E^{p-1} \otimes \Omega^1_Y \ldots] \rightarrow 0 \]
and then forming the connecting map
\[ \mathbb{R}^1 \pi_*(E^p \xrightarrow{\theta} E^{p-1} \otimes \Omega^1_{Y/\Lambda} \ldots) \xrightarrow{\kappa_p(V)} \mathbb{R}^1 \pi_*(E^{p-1} \otimes \Omega^1_{Y/\Lambda} \ldots) \otimes \Omega^1_{\Lambda} \]
Let \( \kappa_{p,\Lambda}(V) \) denote the fibre of this map at \( \Lambda \).

We will now describe the basic set up, that will be fixed for the remainder of this section We will be interested in families of elliptic surfaces. More precisely, let \( E \xrightarrow{f} Y \xrightarrow{\pi} \Lambda \) be a pair of flat projective morphisms, \( S \subset Y \) a relative divisor and \( \sigma : Y \rightarrow E \) a section, such that:

(1) \( \Lambda \) is smooth.
(2) \( Y \rightarrow \Lambda \) is a smooth relative curve, and the restriction \( S \rightarrow \Lambda \) is étale.
(3) For each \( \lambda \in \Lambda \), the fibre \( f_{\lambda} : E_\lambda \rightarrow Y_\lambda \) is a relatively minimal nonisotrivial elliptic surface.
(4) For each $\lambda \in \Lambda$, $S_\lambda \subset Y_\lambda$ is the discriminant of $f_\lambda$.

(5) The data $E \to Y \supset S$ is topologically locally trivial over $\Lambda$. In particular, the sheaves $R^i(\pi_{|U_\lambda})_* (R^1 f_\lambda_* \mathbb{C})$ are locally constant for all $i$, where $U = Y - S$.

The main results of Saito \cite[thm 0.1]{Sa2} shows that $\mathbb{R}(\pi_{|U_\lambda})_* R^1 f_\lambda_* \mathbb{C}$ can be lifted to the derived category of mixed Hodge modules. Condition (5) above, when combined with \cite[thm 0.2]{Sa2}, ensures that the mixed Hodge module associated to $R^i(\pi_{|U_\lambda})_* R^1 f_\lambda_* \mathbb{C}$ is an admissible variation of mixed Hodge structure. Choosing $\lambda \in \Lambda$, we have the Kodaira-Spencer map

\begin{equation}
\kappa_\lambda = \kappa_{1,\lambda}(R^1 f_\lambda_* \mathbb{Q}) : \text{Gr}_W^1 H^1(U_\lambda, R^1 f_\lambda_* \mathbb{C}) \to \text{Gr}_W^1 H^1(U_\lambda, R^1 f_\lambda_* \mathbb{C}) \otimes \Omega^1_\lambda
\end{equation}

Fix $\lambda \in \Lambda$ and an integer $n > 0$. Let $V_\lambda$ be the preimage of $U_\lambda$ in the $n$-fold fibre product $T_\lambda = \mathcal{E}_\lambda \times_{Y_\lambda} \mathcal{E}_\lambda \times_{Y_\lambda} \ldots \mathcal{E}_\lambda$. Suppose that $f_\lambda$ is semistable, which means that the singular fibres have Kodaira type $I_b$, i.e. they are reduced polygons of rational curves. Then $T_\lambda$ has singularities which are local analytically given by

$$x_1 y_1 = x_2 y_2 = \ldots$$

This can be resolved by blowing up an explicit monomial ideal \cite[lem 5.5]{Du} to obtain a nonsingular variety $X_\lambda$. This is a special case of Mumford's toroidal resolution \cite[p 94]{KKMS}. We let $F_\lambda$ denote the projections $V_\lambda \to Y_\lambda$ and $X_\lambda \to Y_\lambda$ (and this should cause no confusion).

\textbf{Theorem 2.1.} With the notation and assumptions as above, suppose furthermore that for some $\lambda \in \Lambda$, the Kodaira-Spencer map $\kappa_\lambda$, defined in (2.4), is injective. Then the Hodge conjecture holds for $V_\lambda$, when $\lambda$ is very general. In addition, when $f_\lambda$ is semistable, the Hodge conjecture holds for the toroidal resolution $X_\lambda$.

We will give the proof after some preliminaries. Fix $\lambda \in \Lambda$. The local system given by restricting $L = L_\lambda = R^1 f_\lambda_* \mathbb{Q}$ to $U_\lambda$ underlies a polarizable variation of Hodge structures of type $\{(1, 0), (0, 1)\}$. Zucker \cite{Z} constructed mixed Hodge structures on the various cohomologies of $L$. The next lemma shows that the key constituents of these Hodge structures coincide.

\textbf{Lemma 2.1.} We have isomorphisms

\begin{equation}
H^1(Y_\lambda, L) \cong IH^1(Y_\lambda, L) \cong Gr^W_2 H^1(U_\lambda, L)
\end{equation}

and

\begin{equation}
Gr^1_P H^1(Y_\lambda, L) \cong Gr^1_P IH^1(Y_\lambda, L) \cong Gr^1_P H^1(U_\lambda, L)
\end{equation}

\textbf{Proof.} For notational simplicity, we write $U$ and $Y$ instead of $U_\lambda$ and $Y_\lambda$ in the proof. Let $j : U \to Y$ denote the inclusion. We have natural morphisms

$$L \to j_* j^* L \to \mathbb{R} j_* j^* L$$

which are clearly isomorphisms over $U$. We claim that the first map $i$ is an isomorphism at each $s \in S$. Let $s' \in Y$ be a point close to $s$, and $T_s : H^1(X_{s'}) \to H^1(X_s)$ the monodromy about $s$. The map $\tau_s : L_s \to j_* j^* L_s$ can be identified with the specialization map $H^1(X_s) \to H^1(X_{s'})^{T_s}$. One can check that $\tau_s$ is an isomorphism by proceeding case by case through Kodaira tables \cite[p 565, p 604]{K} (or apply semistable reduction to reduce to checking for $I_n$ type). The “cone” of
the second morphism $\iota'$ is $R^1 j_* j^* L[-1]$, which we decompose as $\bigoplus_s V_s[-1]$ where $V_s = H^1(X_s) / \text{im}(T_s - 1)$. Therefore we get an exact sequence

$$0 \to IH^1(Y, L) \xrightarrow{\iota'} H^1(U, L) \to \bigoplus_s V_s$$

A direct calculation using Kodaira’s tables shows that $\dim V_s = 1$ when the fibre is of type $I_n$ (which is semistable), and zero in other cases. To proceed further, we need to recall some of facts about Zucker’s mixed Hodge structure [Z, §13]. The lowest weight of $H^1(U, L)$ is $W_2$, and it coincides with image of $\iota$ in (2.7). This is enough to prove that (2.5) holds.

A careful reading of [Z, §13] or specializing the formulas on [SZ, pp 515-516] to the pure case shows that the spaces $V_s$ are equipped with mixed Hodge structures compatible with (2.7). These Hodge structures are constructed so that for $m > 0$,

$$Gr_{2+m}^W H^1(U, L) = \bigoplus_s Gr_{2+m}^W V_s \cong \bigoplus_s Gr_m^W [\Psi_s/(T_s - I)\Psi_s](-1)$$

where $\Psi_s$ is the limit mixed Hodge structure associated to $L$ at $s$. This implies that $\dim Gr_m^W V_s = \dim V_s = 1$ at a semistable fibre. Therefore for dimension reasons, $V_s = \mathbb{Q}(-2)$. Since $Gr_1^I \mathbb{C}(-2) = 0$, we get (2.0).

Let $M = f_* \omega_{E/\lambda}$. By Grothendieck duality, we have a canonical isomorphism $R^1 f_* \mathcal{O} \cong M^!$. The corresponding graded Higgs bundle $(E, \theta : E \to E \otimes \Omega^1_{\lambda})$, is given by $E^0 = M^!$, $E^1 = M$, $E = E^0 \oplus E^1$,

$$\theta = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$$

where $\phi : E^1 \to E^0 \otimes \Omega^1_{\lambda}$ is given by cupping with the Kodaira-Spencer class associated to $f_\lambda$. Since this map is assumed to be nonisotrivial, $\phi$ is nonzero.

Let us analyze the tensor power $L^{\otimes N}$, which is a variation of Hodge structure of weight $N$. The corresponding graded Higgs bundle is $(E, \theta)^{\otimes N}$. In more explicit terms, the underlying vector bundle is a sum of

$$E^I = E^{i_1} \otimes E^{i_2} \ldots = M^{\otimes 2(i_1 + i_2 + \ldots) - N}$$

where $I = (i_1, i_2, \ldots) \in \{0, 1\}^N$. This is graded by $i_1 + i_2 + \ldots$ If $J = (i_1, \ldots, i_k - 1, \ldots i_N) \in \{0, 1\}^N$, then let

$$\phi_{IJ} : E^I \to E^J \otimes \Omega^1_{\lambda}$$

be given by the tensor product of

$$\phi : E^{i_k} \to E^{i_k - 1} \otimes \Omega^1_{\lambda}$$

and the identity on the other factors. Given two vectors $v = (a_1, a_2, \ldots)$, and $w = (b_1, b_2, \ldots) \in \{0, 1\}^N$, their Hamming distance $d(v, w)$ is the cardinality of the set $\{i \mid a_i \neq b_i\}$. We define $\phi_{IJ} = 0$ if $d(I, J) > 1$. The Higgs bundle

$$(E, \theta)^{\otimes N} = \left( \bigoplus_{I \in \{0, 1\}^N} E^I, \bigoplus_{I, J} \phi_{IJ} \right)$$

These formulas extend to symmetric powers. We note that the symmetric group on $N$ letters $S_N$ acts on the right of $\{0, 1\}^N$ in the obvious way, and on the right
on the tensor power by permuting factors. We have that
\[ E^I \cdot \sigma = E^{I,\sigma} \]
\[ \phi_{I,\sigma,J,\sigma} = \phi_{I,J} \cdot \sigma \]
We define the symmetric power as the invariant part
\[ S^N(E,\theta) = [(E,\theta) \otimes \mathbb{N}]^S_N \cong \left( \bigoplus_{I \text{ incr.}} E^I, \bigoplus_{I,J \text{ incr.}} \phi_{I,J} \right) \]
where \( I,J \) are weakly increasing sequences in \( \{0,1\}^N \). This is the Higgs bundle corresponding to \( S^N L \).

**Lemma 2.2.** If \( \kappa_{1,\lambda}(L) \) is injective, then \( \kappa_{p,\lambda}(S^{2p-1}L) \) is injective for all \( p > 0 \).

*Proof.* Using \( (2.8) \) and \( (2.11) \), one finds that
\[ \text{Gr}_F^p(K^0(S^{2p-1}L)/K^2(S^{2p-1}L)) \cong M \overset{\phi}{\rightarrow} M^{-1} \otimes \Omega_Y^1 \overset{\phi}{\rightarrow} M^{-3} \otimes \Omega_Y^1 \wedge \Omega_Y^1 \ldots \]
and
\[ \text{Gr}_F^1(K^0(L)/K^2(L)) \cong M \overset{\phi}{\rightarrow} M^{-1} \otimes \Omega_Y^1 \]
There is a morphism from the first complex to the second given by projection, and this preserves the subcomplexes induced by \( K^1 \). Therefore, we obtain a commutative diagram with short exact rows
\[
\begin{array}{ccc}
0 \to M^{-1} \otimes \Omega_Y^1 \to \ldots & \to [M \to M^{-1} \otimes \Omega_Y^1 \to \ldots] & \to [M \to M^{-1} \otimes \Omega_Y^1] \\
\downarrow & & \downarrow \\
0 \to M^{-1} \otimes \Omega_Y^1 \to \ldots & \to [M \to M^{-1} \otimes \Omega_Y^1 \to \ldots] & \to [M \to M^{-1} \otimes \Omega_Y^1] \\
\end{array}
\]
The connecting maps fit into a commutative diagram
\[
\begin{array}{ccc}
\text{Gr}_F^p H^1(U_\lambda, S^{2p-1}L) & \xrightarrow{\kappa_{p,\lambda}} & \text{Gr}_F^{p-1} H^1(U_\lambda, S^{2p-1}L) \otimes \Omega_Y^1 \\
\downarrow & & \downarrow \\
\text{Gr}_F^1 H^1(U_\lambda, L) & \xrightarrow{\kappa_{1,\lambda}} & \text{Gr}_F^0 H^1(U_\lambda, L) \otimes \Omega_Y^1 \\
\end{array}
\]
It follows that injectivity of \( \kappa_{1,\lambda}(L) \) on the bottom implies injectivity of \( \kappa_{p,\lambda}(S^{2p-1}L) \) on the top.

Following standard usage, let us say that a point of \( \Lambda \) is very general, if it lies outside a countable union of proper analytic subvarieties.

**Lemma 2.3.** Suppose \( (\mathcal{L}, V, \ldots) \) is an admissible rational variation of mixed Hodge structure over \( \Lambda \) such that for a very general point \( \lambda \in \Lambda \), the Kodaira-Spencer map
\[ K = \kappa_{p,\lambda}(V) : \text{Gr}_F^p V_\lambda \to \text{Gr}_F^{p-1} V_\lambda \otimes \Omega_Y^1 \]
is injective. Then \( V_\lambda \) has no nonzero Hodge cycles of weight \( 2p \) for very general \( \lambda \).

*Proof.* We can replace \( V \) by \( \text{Gr}_F^{2p} V \) and assume \( V \) is pure of weight \( 2p \). Fix \( \lambda_0 \in \Lambda \), and replace \( \Lambda \) by a contractible open neighbourhood of \( \lambda_0 \). Then (the restriction of) \( \mathcal{L} \) is a constant sheaf of \( \mathbb{Q} \)-vector spaces. Therefore given \( \gamma \in \mathcal{L}_{\lambda_0} \), it extends.
to section of $\mathcal{L}$ (denoted by the same symbol) over $\Lambda$. As $\lambda$ varies, $F^pV_\lambda$ varies holomorphically. Therefore the locus

$$NL_\gamma = \{\lambda \in \Lambda \mid \gamma_\lambda \text{ is a Hodge cycle}\}$$

is an analytic subvariety of $\Lambda$. Suppose that $NL_\gamma = \Lambda$ for some $\gamma \neq 0$. Then

$$K(\gamma) = \nabla(\gamma) \mod F^p = 0$$

This contradicts the hypothesis of the lemma. Therefore $NL_\gamma$ is a proper subvariety. Consequently a very general point lies in the complement of $\bigcup_{\gamma \in \mathcal{L}_{70}} NL_\gamma$.

Lemma 2.4. For any $\lambda$, the cycle map

$$\square^{0,\lambda}: \text{Gr}^pL^0(V_\lambda) \to H^0(U, R^{2p}F^*_\lambda \mathbb{Q})$$

is surjective.

Proof. We will suppress $\lambda$ in the argument below. Let us write $L = \mathbb{Q}^2$ with the standard action of $SL_2(\mathbb{Q})$. The local system $R^1f_*\mathbb{Q}$ can be identified with $L$, with monodromy given by a homomorphism $\mu : \pi_1(U) \to SL_2(\mathbb{Z}) \subset SL_2(\mathbb{Q})$. Non-isotriviality implies that $\text{im} \mu$ is Zariski dense. The Künneth formula, allows us to identify

$$(2.12) \quad R^kF_*\mathbb{Q} = \bigoplus_{i_1+\ldots+i_n = k} R^{i_1}f_*\mathbb{Q} \otimes \ldots \otimes R^{i_n}f_*\mathbb{Q}$$

Let us start with the case of $n = k = 2$. Then the decomposition becomes

$$R^2F_*\mathbb{Q} = L^\otimes 2 \oplus (f_*\mathbb{Q} \otimes R^2f_*\mathbb{Q}) \oplus (R^2f_*\mathbb{Q} \otimes f_*\mathbb{Q})$$

So

$$H^0(U, R^2F_*\mathbb{Q}) = H^0(L^\otimes 2) \oplus H^0(f_*\mathbb{Q} \otimes R^2f_*\mathbb{Q}) \oplus H^0(R^2f_*\mathbb{Q} \otimes f_*\mathbb{Q}) \cong \mathbb{Q}^3$$

Let $D \subset E$ denote a section of $f$, and let $\Delta \subset E \times_Y E$ the image of $E$ under the diagonal embedding. Since $[\Delta], p_1^*[D], p_2^*[D] \in H^0(R^2F_*\mathbb{Q})$ are easily seen to be linearly independent, they necessarily span this space. A linear combination of these classes give a generator

$$(2.13) \quad \Delta_0 = [\Delta] - p_1^*[D] - p_2^*[D] \in H^0(L^\otimes 2) = H^0(\lambda^2L) \cong \mathbb{Q}$$

We now turn to the general case. Let $p_i : T \to E$, and $p_{ij} : T \to E \times_Y E$ also denote the projections onto the $i$th and $i'j$th factors. When $k = 2p$, equation (2.12) can be written as

$$R^{2p}F_*\mathbb{Q} = \bigoplus_{q \leq p} \bigoplus_{\sigma} \sigma((f_*\mathbb{Q})^\otimes n-2q-(p-q)) \otimes L^\otimes 2q \otimes (R^2f_*\mathbb{Q})^\otimes(p-q)$$

where $\sigma$ runs over a suitable set of permutations depending on $q$. It follows that

$$(2.14) \quad H^0(U, R^{2p}F_*\mathbb{Q}) = \bigoplus_{q} \bigoplus_{\sigma} H^0(U, L^\otimes 2q) \otimes H^0(U, R^2f_*\mathbb{Q})^\otimes(p-q)$$

Invariant theory [FH, prop F.13] tells us that $H^0(L^\otimes 2q)$ is generated by products of pullbacks of classes from $H^0(L^\otimes 2)$ under various projections $p_{ij}$. This means that $H^0(U, L^\otimes 2p)$ is spanned by products of $p_{ij}^*\Delta_0$. Therefore the summands on the right of (2.14) are generated by classes of the form $p_{ij}^*\Delta_0$ and $p_k^*D$. This proves that $H^0(U, R^{2p}F_*\mathbb{Q})$ is spanned by algebraic cycles.

\[\square\]
Lemma 2.5. For very general $\lambda$,
\[ \text{Hodge}(H^1(U_\lambda, R^{2p-1}F_\lambda^*C))(p) = 0 \]

Proof. We will suppress $\lambda$ again. We can view $GL_2(\mathbb{Q})$ as the (big) Mumford-Tate group of the variation of Hodge structure $L$, or equivalently of a very general fibre of $L$. We have that $S^nL(i) = S^nL \otimes (\det L)^{-i}$ is also a $GL_2$-module. We can decompose
\[ L^{\otimes 2p-1} = S^{2p-1}L \oplus S^{2p-3}L(-1)^{\otimes m_1} \oplus \ldots \]
into a sum of irreducible $GL_2$-modules. Lemmas 2.2 and 2.3 shows that
\[ \text{Hodge}(H^1(S^{2p-1}L))(p) = 0 \]
\[ \text{Hodge}(H^1(S^{2p-3}L(p-1))) = 0 \]
\[ \ldots \]
\[ \Box \]

Proof of theorem 2.1. Choose $\lambda$ very general. Since $f_\lambda$ is nonisotrivial, $U_\lambda \subset Y_\lambda$. Therefore $H^i(U_\lambda, R^j f_\lambda^*Q)(p) = 0$ when $i > 1$. The theorem is now a consequence of theorem 1.2, and lemmas 1.2, 2.4 and 2.5. \[ \Box \]

We will now discuss a few corollaries.

Corollary 2.2. If $\lambda$ is very general, the Hodge conjecture holds for any desingularization of $E_\lambda \times_Y E_\lambda \times_Y E_\lambda$.

Proof. This follows from the theorem, remark 1.3, and corollary 1.4. \[ \Box \]

Now, let us assume that $\Lambda = \{\lambda\}$ is a point. Then $f : E \to Y$ is an elliptic surface. In this case, the assumption that $\kappa_\lambda$ is injective is equivalent to
\[ \text{Gr}^1 H^1(U, R^1 f_\lambda^*C) = 0 \]

This condition occurs in the literature in a different form. An elliptic surface $f : E \to Y$ is said to be extremal (c.f. [Mr, p 75]) if the Picard number $\rho$ equals the Hodge number $h^{11}$, and the Mordell-Weil group $MW(E/Y)$ (which is the group of sections) has rank 0.

Lemma 2.6. We have
\[ \dim \text{Gr}^1 H^1(U, R^1 f_\lambda^*C) = h^{11}(X) - \rho(X) + \text{rank } MW(E/Y) \]

The surface is extremal if and only if
\[ \text{Gr}^1 H^1(U, R^1 f_\lambda^*C) = 0 \]

Proof. By Saito’s version of the decomposition theorem [Sa1, p 857], we can decompose $Rf_*\mathbb{Q}$ as a sum of intersection cohomology complexes up to shift in the constructible derived category, and moreover these complexes underly pure Hodge modules. By restricting this sum to $U$, we can identify some of these components explicitly:
\[ Rf_*\mathbb{Q} \cong H^{1}(X, \mathbb{Q}) \oplus j_\ast R^1 f_\ast \mathbb{Q}[-1] \oplus H^{2}(M) \]
The, as yet undetermined, term $M$ is supported on the finite set $S$. This yields a (noncanonical) decomposition
\[ H^2(X, \mathbb{Q}) \cong f^* H^2(Y, \mathbb{Q}) \oplus IH^1(R^1 f_* \mathbb{Q}) \oplus H^0(Y, j_\ast R^2 f_* \mathbb{Q}) \oplus H^2(M) \]
The first summand on the right is spanned by the fundamental class \([X_t]\) of a fibre. The third summand is spanned by the class \([\sigma]\). To calculate \(M\), we restrict to a point \(s \in S\), and observe that \(H^*([Rf_*Q]_s)\) is the cohomology of the fibre \(X_s = f^{-1}(s)\) by proper base change [13] p 41. Therefore \(M\) gives the excess cohomology not coming from the preceding terms in (2.17). Let

\[
X_s = \sum_{i=1}^{m_s} n_{s,i} X_{s,i}
\]

be the decomposition into irreducible components. Let \(D_s\) be a small disk centered at \(s, t \in D^*_s = D_s - \{s\}\), and \(\gamma_s \in \pi_1(D^*_s, t)\) a generator. We see that

\[
\begin{align*}
H^i(X_s, \mathbb{Q}) &= \begin{cases} 
\mathbb{Q} & \text{if } i = 0 \\
H^1(X_t, \mathbb{Q})^\gamma_s \cong (j_* j^* R^1 f_* \mathbb{Q})_s & \text{if } i = 1 \\
\mathbb{Q}^{m_s} & \text{if } i = 2 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Therefore \(M \cong \bigoplus \mathbb{Q}^{m_s-1}_s[-2]\). From (2.18), we deduce that we have a noncanonical decomposition

\[
(2.19) \quad H^2(X) \cong IH^1(R^1 f_* \mathbb{Q}) \oplus \mathbb{Q}[\sigma] \oplus \mathbb{Q}[X_t] \oplus \bigoplus_s \mathbb{Q}^{m_s-1}_s
\]

We can see that the last two summands are spanned by divisor classes supported on the fibres. Since the decomposition (2.17) can be lifted to the derived category of mixed Hodge modules, we see that (2.19) becomes an isomorphism of Hodge structures provided that all the summands in (2.19) except the first are viewed as sums of the Tate structures \(\mathbb{Q}(-1)\).

Formula (2.19) implies

\[
(2.20) \quad \dim Gr^1 IH^1(R^1 f_* \mathbb{C}) = h^{11}(X) - 2 - \sum (m_s - 1)
\]

When combined with lemma 2.1 and the Shioda-Tate formula [Mr]

\[
(2.21) \quad \text{rank } MW(E/Y) = \rho(X) - 2 - \sum (m_s - 1)
\]

we obtain (2.16). The second part of the lemma characterizing extremal surfaces follows immediately from this and lemma 2.1.

A key set of examples is provided by:

**Proposition 2.1** (Shioda). *Elliptic modular surfaces are extremal.*

*Proof.* The Mordell-Weil rank is zero by [Sho] thm 5.2, and \(\rho = h^{11}\) by [Sho] rmk, 7.8.

We can now state the first corollary of theorem 2.1. This corollary applies to elliptic modular surfaces, so we recover Gordon’s theorem [G].

**Corollary 2.3.** If \(f : E \to Y\) is an extremal elliptic surface then the Hodge conjecture holds for the complement \(V\) of the preimage of \(S\) in \(E \times_Y E \times_Y \ldots E\); it holds for any desingularization of \(E \times_Y E \times_Y E\), and it holds for the toroidal resolution \(X\) when \(f\) is semistable.

*Proof.* When \(\Lambda = \{\lambda\}\), \(\lambda\) is very general since \(\emptyset\) is the only proper analytic subset. 

\[
\square
\]
We next consider families of elliptic K3 surfaces with the following features:

\[ \mathcal{E} \xrightarrow{f} \mathbb{P}^1 \times \Lambda = Y \to \Lambda \]

is a family of elliptic K3 surfaces over a curve \( \Lambda \) such that:

1. **(EK1)** \( \text{rank} \ MW(\mathcal{E}_\lambda/\mathbb{P}^1) = 0 \) for very general \( \lambda \).
2. **(EK2)** There is a fibrewise isogeny between \( \mathcal{E} \) and the family of Kummer surfaces associated to a selfproduct of a nonisotrivial elliptic surface \( g : F \to \Lambda \).

More precisely, there is an algebraic correspondence

\[ Z \in CH^2(\mathcal{E} \times Km(F \times \Lambda \ F/\Lambda)) \]

which induces a fibrewise isomorphism between the transcendental parts

\[ H^2_{tr}(\mathcal{E}_\lambda, \mathbb{Q}) \cong H^2_{tr}(Km(F_\lambda \times F_\lambda), \mathbb{Q}) \cong H^2_{tr}(F_\lambda \times F_\lambda, \mathbb{Q}) \]

as Hodge structures. (Recall that the transcendental part \( H^2_{tr}(\mathcal{E}, \mathbb{Q}) = (NS(\mathcal{E}) \otimes \mathbb{Q})^\perp \).)

Examples satisfying (EK1) and (EK2) include Hoyt’s \( \mathbb{H} \) defined by the affine equation

\[ y^2 = t(t-1)(t-\lambda)x(x-1)(x-t) \]

For \( \lambda \in \Lambda = \mathbb{C} - \{0,1 \} \), this defines an elliptic K3 surface \( \mathcal{E}_\lambda \) over the \( t \)-line. Hoyt shows that this is birational to the quotient of an explicit Kummer surface \( Km(F_\lambda \times F_\lambda) \) by an involution, and that the rank of the Mordell-Weil group \( MW(\mathcal{E}_\lambda/\mathbb{P}^1) \) is zero unless \( F_\lambda \) has complex multiplication.

The families of K3 surfaces constructed by Dolgachev [Do] Ex. 7.8, 7.9, 7.10 can also be seen to satisfy (EK1) and (EK2) above. The first condition is satisfied by construction (see the remarks preceding [Do] Ex. 7.8); for the second use [Da thm 7.6].

**Corollary 2.4** (to thm 2.1). Let \( \mathcal{E} \to \mathbb{P}^1 \times \Lambda \) be a family satisfying (EK1) and (EK2). Suppose that \( \lambda \) is very general. The Hodge conjecture holds for \( V_\lambda \), and for any desingularization of \( \mathcal{E}_\lambda \times \mathbb{P}^1, \mathcal{E}_\lambda \times \mathbb{P}^1, \mathcal{E}_\lambda \).

**Proof.** The correspondence \( Z \) induces an inclusion

\[ H^2_{tr}(\mathcal{E}_\lambda, \mathbb{Q}) \cong H^2_{tr}(F_\lambda \times F_\lambda, \mathbb{Q}) \subseteq S^2H^1(F_\lambda, \mathbb{Q}) \]

Let us choose \( \lambda \) very general. Then \( F_\lambda \) will not have complex multiplication, which implies that \( S^2H^1(F_\lambda) \) will be a simple Hodge structure. Simplicity forces the above inclusion to be an equality. It follows that \( \dim H^2_{tr}(\mathcal{E}_\lambda, \mathbb{Q}) = 3 \), and therefore \( \rho(\mathcal{E}_\lambda) = 19 \). Lemma 2.6 shows that \( \dim Gr^0_{\mathcal{E}} IH^1(R^1f_*\mathcal{O}_\mathcal{E}) = 1 \). The orthogonal complement of \( IH^1(R^1f_*\mathcal{O}_\mathcal{E}) \) is spanned by divisors by (2.16). Therefore \( IH^1(R^1f_*\mathcal{O}_\mathcal{E}) \) must contain the \((2,0)\) and \((0,2)\) parts of \( H^2(\mathcal{E}_\lambda) \). Consequently its dimension is 3 and it meets \( H^2_{tr}(\mathcal{E}_\lambda) \). By simplicity of the latter, we must have that

\[ IH^1(R^1f_*\mathcal{O}_\mathcal{E}) = H^2_{tr}(\mathcal{E}_\lambda) \]

Therefore by lemma 2.14 we have an isomorphism

\[ Gr^2_{\mathcal{E}} H^1(U_\lambda, R^1f_*\mathcal{O}_\mathcal{E}) \cong S^2H^1(F_\lambda, \mathbb{Q}) \]

induced by \( Z \), and this leads to an isomorphism of corresponding variations of Hodge structure, possibly over a nonempty Zariski open subset of \( \Lambda \). Therefore it suffices to check injectivity of the Kodaira-Spencer map

\[ \kappa_{2,\lambda}(S^2R^1g_*\mathcal{O}) : Gr^2_{\mathcal{E}} S^2H^1(F_\lambda) \to Gr^1_{\mathcal{E}} S^2H^1(F_\lambda) \]
Using (2.11) and the formulas preceding it, we can see that $\kappa(S^2R^*g_*\mathbb{Q})$ is the tensor product

$$\kappa_{1,\lambda}(R^*g_*\mathbb{Q}) \otimes \text{id} : H^0(F_\lambda, \Omega^1) \otimes H^1(F_\lambda, \mathcal{O}) \rightarrow H^1(F_\lambda, \mathcal{O}) \otimes H^1(F_\lambda, \mathcal{O})$$

and this is injective for general $\lambda$, because $g$ is nonisotrivial.

\[\square\]

**Remark 2.5.** The conditions (EK1) and (EK2) force $\rho(\mathcal{E}_\lambda) = 19 < h^{11}$ for very general $\lambda$, so these surfaces cannot be extremal. Conversely, using lemma 2.6, we can see that any nontrivial family of elliptic K3 surfaces to which the theorem applies must have $\rho(\mathcal{E}_\lambda) = 19$ generically.

### 3. Families of abelian varieties over a Shimura curve

For our purposes, a Shimura variety $Y$ is a quotient of a Hermitian symmetric domain by an arithmetic group. It can constructed from the following datum: a semisimple algebraic group $G$ over $\mathbb{Q}$, a morphism (or conjugacy class of homomorphisms) $h : U(1) \rightarrow G^{ad}(\mathbb{R})$ satisfying the axioms of [Mi, 4.4], and an arithmetic group $\Gamma \subset G(\mathbb{Q})$. (The group $\Gamma$ is usually required to be a congruence group, but we will not insist on this.) If $K$ is the stabilizer of $h$ in the identity component $G(\mathbb{R})^+$, the corresponding Shimura variety $Y = \Gamma \backslash G(\mathbb{R})^+ / K$. For example, given a rational symplectic vector space $(V, Q)$ of dimension $2g$, let $G = \text{Sp}(V, Q)$. We have a Shimura datum, where the set of homomorphisms $h$ corresponds to the set of Hodge structures of type $\{(-1, 0), (0, -1)\}$ on $V$, polarized by $Q$. If we fix a lattice $L \subset V$, such that $Q|_L$ is principal, and let $\Gamma \subset G(\mathbb{Q})$, be the subgroup preserving $L$, the corresponding Shimura variety is just the moduli space of principally polarized abelian varieties $A_g$. We say that a Shimura variety $Y$ is of Hodge type if there exists a faithful representation $G \rightarrow \text{Sp}_{2g}(\mathbb{Q})$ which extends to a map of Shimura data. Then we get an embedding of $Y$ into the moduli space of abelian varieties with level structure. If level structure is fine, then we can pull back the universal family to get a family of abelian varieties over $Y$.

We now describe the main example of interest to us. Let $K/\mathbb{Q}$ be a totally real number field of degree $d$. Denote the $d$ distinct embeddings by $\sigma_i : K \rightarrow \mathbb{R}, i = 1, \ldots, d$. Fix a quaternion division algebra $D/K$ such that $D \otimes_{\sigma_i} \mathbb{R} \cong M_2(\mathbb{R})$ and $D \otimes_{\sigma_i} \mathbb{R}$ is the Hamilton quaternions $H$ for $i = 2, \ldots, d$. If $\bar{x}$ denotes conjugation in $D$, then

$$G = D_1^* = \{x \in D \mid x\bar{x} = 1\}$$

is an algebraic group over $\mathbb{Q}$ whose group of real points

$$G(\mathbb{R}) \cong \text{SL}_2(\mathbb{R}) \times (SU_2)^{d-1}$$

(3.1)

It follows that

$$G(\mathbb{C}) \cong \text{SL}_2(\mathbb{C})^d$$

(3.2)

and

$$G^{ad}(\mathbb{R}) \cong \text{PSL}_2(\mathbb{R}) \times (PU_2)^{d-1}$$

(3.3)

Let $h : U(1) \rightarrow G^{ad}(\mathbb{R})$ be defined by

$$h(e^{i\theta}) = \left(\begin{array}{cc} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{array}\right) \mod \pm 1, 1\right)$$
with respect to $\text{Cor}(D)$. (For an explanation of $\theta/2$, see [M] ex 1.10.) The pair $(G, h)$ defines a connected Shimura datum. The corresponding symmetric space is the upper half plane $\mathbb{H}$. To complete the description of the Shimura variety, fix a maximal order $\mathcal{O}$ in $D$, and a torsion free finite index subgroup of $\Gamma \subseteq \mathcal{O}_1^\ast = \mathcal{O} \cap D^\ast_1$. Let $Y = \Gamma \backslash \mathbb{H}$, where $\Gamma$ acts through projection to $SL_2(\mathbb{R})$. We fix this notation for the remainder of this section.

We first observe that $Y$ is a projective algebraic curve, c.f. [Sh, chap 9]. We claim that the curve $Y$ is of Hodge type, so it would carry a universal family of abelian varieties $f : A \to Y$. An explicit construction of $A$ was given in a paper of Viehweg-Zuo [VZ]. We outline the construction from section 5 of their paper, because we will need to extract certain facts which were not explicitly stated. The correstriction of $D$ is the central simple $\mathbb{Q}$-algebra

$$\text{Cor}(D) = \bigotimes_{i} D \otimes_{\sigma_i} \overline{\mathbb{Q}}_{\text{Gal}(\overline{\mathbb{Q}}/K)}$$

of dimension $4^d$. In the simplest case, $\text{Cor}(D)$ splits, which means that it is a matrix algebra $M_{2d}(\mathbb{Q})$. Then via the norm homomorphism $\text{Nm} : D^\ast \to \text{Cor}(D)^\ast$, $V = \mathbb{Q}^{2d}$ is a representation of $G$. If $\text{Cor}(D)$ does not split, then it will split over a suitable quadratic field $\mathbb{Q}(\sqrt{b})$. In this case, $V = \mathbb{Q}(\sqrt{b})^{2d}$ becomes a representation of $G$. Under (3.1), $V \otimes \mathbb{Q} \mathbb{R} \cong \mathbb{R}^2 \otimes_{\mathbb{R}} V'$, where $\mathbb{R}^2$ is the standard representation of $SL_2(\mathbb{R})$, and the second factor

$$V' = \begin{cases} \mathbb{R}^{2d-1} & \text{if } \text{Cor}(D) \text{ splits} \\ \mathbb{R}^{2d} & \text{otherwise} \end{cases}$$

is a representation of $SU^{d-1}_2$. The representation $V'$ has no trivial factors, so it carries an $SU^{d-1}_2$-invariant inner product $Q_2$, which unique up to a scalar factor. The complex representation

$$V' \otimes \mathbb{C} \cong \begin{cases} V_2 \otimes \ldots \otimes V_{d-1} & \text{if } \text{Cor}(D) \text{ splits} \\ (V_2 \otimes \ldots \otimes V_{d-1})^\otimes & \text{otherwise} \end{cases}$$

where $V_i$ is the standard representation of $SL_2(\mathbb{C})$ acting through the $i$th factor of (3.2). Let $Q_1$ denote the standard symplectic form on $\mathbb{R}^2$. A multiple of $Q_1 \otimes Q_2$ will be a $G$-invariant $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{b})$-valued symplectic form $Q_3$ on $V$. Setting $Q = Q_3$ in the first case, or $Q = \text{tr}_{\mathbb{Q}(\sqrt{b})/\mathbb{Q}} \circ Q_3$ in the second gives an invariant $\mathbb{Q}$-valued form. Hence, we have an injective homomorphism of algebraic groups $\rho : G \to Sp(V, Q)$. This extends to a map of Shimura data, so $Y$ is of Hodge type. Note that $Sp(V, Q)$ is isomorphic to $Sp_{2d}(\mathbb{Q})$ or $Sp_{2d+1}(\mathbb{Q})$ depending on the cases. We summarize this discussion, along with some additional facts from [VZ] lemma 5.8, p 272.}

**Theorem 3.1** (Viehweg-Zuo). There is an abelian scheme $f : A \to Y$ such that:

1. The relative dimension of $A$ is either $2^{d-1}$, if $\text{Cor}(D)$ splits, or $2^d$ otherwise.
2. The local system $R^1 f_{\ast}Q$ corresponds to a representation

$$\begin{array}{c}
\Gamma \\
\downarrow \rho \\
GL(V)
\end{array}$$

such that:
(a) The representation of $G(\mathbb{R})$ on $V \otimes \mathbb{R}$ is a tensor product $\mathbb{R}^2 \otimes V'$, where $\mathbb{R}^2$ is the standard representation of $SL_2(\mathbb{R})$, and $V'$ is an orthogonal representation of the remaining factor of (3.1).

(b) The representation of $G(\mathbb{C})$ on $V \otimes \mathbb{C}$ is isomorphic to a sum of one (if $\text{Cor}(D)$ splits) or two copies (otherwise) of $V_1 \otimes \ldots \otimes V_d$, where $V_i$ is the standard representation of the $i$th factor of (3.2).

(3) The Zariski closure of $\rho(\Gamma)$ and the special Mumford-Tate group of a very general fibre of $f$ are both equal to $G$.

(4) Corresponding to the decomposition in (2b), the graded Higgs bundle associated to the variation of Hodge structure $R^1 f_* C$ is a tensor product

$$(E^0_1 \oplus E^1_1, \theta) \otimes (E^2_2, 0) \otimes \ldots \otimes (E^d_d, 0)$$

Furthermore, $\theta : E^1_1 \to E^0_1 \otimes \Omega^1_Y$ is an isomorphism.

Let $F : X \to Y$ be the $n$-fold fibre product $A \times_Y \ldots \times_Y A$. We note that $A \to Y$, and therefore $F$, is defined over a number field $k$. This follows from the theory of canonical models [Mi], although we should point out that since we are working with connected Shimura varieties, $k$ might be bigger than the reflex field.

**Theorem 3.2.** The Hodge and Tate conjectures holds for $X$.

The proof follows the same basic strategy as the proof of theorem [21]. We first establish the preliminary results.

**Lemma 3.1.** Let $V_i$ be standard representations of $SL_2(\mathbb{C})$, and let $V = V_1 \otimes \ldots \otimes V_d$ be viewed as an $H = SL_2(\mathbb{C})^d$-module. Then invariant part of the tensor algebra

$$(T^*V)^H$$

is generated by $(V \otimes V)^H$.

**Proof.** We have an isomorphism

$$(T^*V)^H = (T^*V_1)^{SL_2(\mathbb{C})} \otimes \ldots \otimes (T^*V_d)^{SL_2(\mathbb{C})}$$

So the result follows from classical invariant theory [FH] prop. F.13]. □

**Proposition 3.1.** The cycle maps

$$\square^{p,i} : Gr^p L CH^i(X) \to H^i(Y, R^{2p-i} F_* \mathbb{Q})$$

are surjective for $i = 0, 2$ and all $p$.

**Proof.** We first treat the case of $i = 0$. The surjectivity of $\square^{0,0}$ is trivially true, and the surjectivity of $\square^{1,0}$ is a consequence of the Lefschetz (1,1) theorem. We have that $R^i f_* \mathbb{Q} = \wedge^i R^1 f_* \mathbb{Q}$. This together with the Künneth formula and lemma [3.1] implies that the product map

$$S^p H^0(Y, R^2 F_* \mathbb{Q}) \to H^0(Y, R^{2p} F_* \mathbb{Q})$$

is surjective. Therefore

$$S^p Gr^0 L CH^1(X) \to H^0(Y, R^{2p} F_* \mathbb{Q})$$

is surjective. This implies the surjectivity of $\square^{p,0}$. 

We have the commutative diagram
\[
\begin{array}{ccc}
\text{Gr}^p_0 \mathcal{C} \mathcal{H} (X) & \xrightarrow{\phi^0} & H^0 (Y, \mathbb{R}^{2p} F_1) \\
\downarrow & & \downarrow \\
\text{Gr}^{p+1}_2 \mathcal{C} \mathcal{H}^{p+1} (X) & \xrightarrow{\phi^{p+1,2}} & H^2 (Y, \mathbb{R}^{2p} F_1)
\end{array}
\]
where the isomorphism on the right is hard Lefschetz. This implies surjectivity of $\phi^{p+1,2}$.

\[\square\]

**Proposition 3.2.** For all $p$, 
\[
\text{Gr}^p_0 H^1 (Y, \mathbb{R}^{2p-1} F, \mathbb{C}) = 0
\]

**Proof.** The local system $\mathbb{R}^{2p-1} F \mathbb{C}$ corresponds to the $SL_2 (C)$-module $\wedge^{2p-1} (V \otimes n)$. We can decompose it into a sum of irreducible modules, and reduce to the problem of showing that
\[
H^1 (M^q \phi \to M^{q-1} \otimes \Omega^1_k) = 0
\]
where
\[
(M, \phi) = S^{2q-1} (E_1, \theta) \otimes S^{r_2} (E_2, 0) \otimes \ldots \otimes S^{r_d} (E_d, 0) = S^{2q-1} (E_1, \theta) \otimes (M', 0)
\]
and $q, r_1, \ldots$ are arbitrary. Let $I = (1, \ldots, 1, 0, \ldots)$ with $q$ 1’s followed by $q$ 0’s. Then we have an isomorphism of complexes
\[
M^q \phi \to M^{q-1} \otimes \Omega^1_k \cong [E_1^1 \to E_1^0 \otimes \Omega^1_k] \otimes E_1^1 \otimes M'
\]
This implies (3.4).

\[\square\]

**Proof of theorem 3.2.** This is a consequence of corollary 1.7 and the last two propositions.

\[\square\]

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