Transverse rectification of disorder-induced fluctuations in a driven system

Alejandro B. Kolton

1Université de Genève, DPMC, 24 Quai Ernest Ansermet, CH-1211 Genève 4, Switzerland

We study numerically the overdamped motion of particles driven in a two dimensional ratchet potential. In the proposed design, of the so-called geometrical-ratchet type, the mean velocity of a single particle in response to a constant force has a transverse component that can be induced by the presence of thermal or other unbiased fluctuations. We find that additional quenched disorder can strongly enhance the transverse drift at low temperatures, in spite of reducing the transverse mobility. We show that, under general conditions, the rectified transverse velocity of a driven particle fluid is equivalent to the response of a one dimensional flashing ratchet working at a drive-dependent effective temperature, defined through generalized Einstein relations.

PACS numbers: 74.25.Qt,05.40.-a,05.45.-a,05.60.Cd

The idea of generating a directed dissipative transport in a system kept out of thermal equilibrium only by unbiased perturbations has motivated an outburst of experimental and theoretical works in the last years. The ratchet effect is indeed of interest, both for applications and modelling, in very diverse systems, ranging from biological motors, colloidal matter, granular matter, vortex matter in superconductors, Josephson junction arrays, atoms in optical traps, electrons in semiconductor heterostructures to gambling games. One of the simplest models is the “flashing ratchet”, where a directed motion of a Brownian particle (i.e. breaking of the detailed balance condition) is obtained by coupling it to a pulsating asymmetric-periodic potential. The identification of the essential physical ingredients for the effect shows that a large variety of ratchets and rectification mechanisms can be realized.

Recently, there has been a growing interest in the so-called geometrical ratchets since they can be used as continuous “molecular sieves” to separate particles experimentally (such as macromolecules or mesoscopic objects), according to its physical properties. These devices are typically two-dimensional systems containing a periodic array of asymmetric obstacles. By driving the particles through the array an average lateral drift appears, as transverse diffusive motion is rectified by the collisions with the asymmetric obstacles. Different types of geometrical ratchets have been analyzed in the literature, both experimentally and theoretically. The effect of additional quenched disorder in these two-dimensional systems has not been discussed yet, though interesting anomalous transport properties of one-dimensional disordered ratchet systems were reported. Such a study is not only relevant for applications where disorder cannot be avoided, but it is also an interesting and challenging issue. The driven motion of particles in a disordered substrate yields a non-trivial hydrodynamics. The current-driven motion of vortices in type II superconductors is a prominent example, where disorder, apart from reducing dissipation, is responsible for marked non-equilibrium transport and magnetic properties. On the other hand, already the simpler case of driven non-interacting Brownian particles in two dimensions displays complex phenomena. While diffusion is anomalous at equilibrium under a finite drive diffusion becomes normal in the comoving frame, with anisotropic and velocity-dependent diffusion constants and mobilities. Moreover, a disordered substrate can provide alone a local or global ratchet effect, such as the generation of large-scale vorticity in the probability current by driving particles with an uniform alternate drive and the net directed motion produced by driving the particles with crossed ac-drives.

In this paper we investigate the effect of quenched disorder in a simple geometrical ratchet design, under an uniform and constant driving force. We find that disorder can strongly enhance the transverse drift both for non-interacting and interacting particles, thus improving the

FIG. 1: (a) Ratchet potential with disorder. (b) Schematics of the transverse rectification mechanism (top view). Particles move with an average velocity \( V_y \) in the direction of the applied force \( F_y \). In the white regions the interaction with the (attractive or repulsive) centers and with the thermal bath induces diffusion in the non-driven direction In the shaded regions a periodic-asymmetric potential tends to localize particles at its minima. An average transverse shift (see circles) is produced at a rate \( V_t \).
performance of the device for applications. We show that the transverse velocity of a driven fluid is equivalent to the response of a one-dimensional flashing ratchet working at a drive-dependent effective temperature, defined through generalized fluctuation-dissipation relations.

Let us consider the overdamped motion of particles in a two dimensional potential like the one depicted in Fig. 1. The equation of motion of a particle in position \( \mathbf{R}_i \) is:

\[
\eta \frac{d\mathbf{R}_i}{dt} = -\nabla_i \left[ \sum_{j \neq i} V(\mathbf{R}_{ij}) + U(\mathbf{R}_i) \right] + \mathbf{F} + \zeta_i(t),
\]

where \( R_{ij} = |\mathbf{R}_i - \mathbf{R}_j| \) is the distance between particles \( i, j \), \( R_{ip} \) is the distance between the particle \( i \) and a site at \( \mathbf{R}_p \), \( \eta \) is the friction, and \( \mathbf{F} = F_y \hat{y} \) is the driving force. The effect of a thermal bath at temperature \( T \) is given by the stochastic force \( \zeta_i(t) \), satisfying \( \langle \zeta_i(t) \rangle = 0 \) and \( \langle \zeta_i(t) \zeta_j(t') \rangle = 2\eta k_B T \delta(t - t') \delta_i \delta_{jj'} \), where \( \langle \cdot \rangle \) denotes average over the ensemble of \( \zeta_i \). For concreteness we consider a logarithmic repulsive particle-particle interaction \( V(r) = -A_r \ln(r) \) which corresponds for instance to vortex-vortex interaction in 2D thin film superconductors. Particles interact with the quenched potential \( U(\mathbf{R}) = U_R(\mathbf{R}) + U_p(\mathbf{R}) \). \( U_R \) is a ratchet potential with the form, \( U_R(\mathbf{R}) = \frac{R}{2\pi} F_R(Y) G_R(X) \), where \( X \equiv \mathbf{R} \cdot \hat{x} \), \( Y \equiv \mathbf{R} \cdot \hat{y} \), \( G_R(X) = \sin(2\pi X/a) + 0.25 \sin(4\pi X/a) \) and \( F_R(Y) = U_0 \cos(2\pi Y/b) \Theta(\cos(2\pi Y/b)) \), with \( \Theta \) the Heaviside function. This ratchet potential is similar to a periodic array of obstacles, asymmetrical around the \( x - axis \) but symmetrical around the \( y - axis \), as the ones considered in Ref. 11. Disorder is short-range correlated, and it is modeled as a random distribution of centers such that \( U_p(\mathbf{R}_i) = \sum_p A_p e^{-(R_{ip}/r_p)^2} \), where \( R_{ip} = |\mathbf{R}_i - \mathbf{R}_p| \) is the distance between the particle \( i \) and a center at \( \mathbf{R}_p \). Centers can be either attractive \( A_p < 0 \) (wells) or repulsive \( A_p > 0 \) (humps) or a combination of both. We solve Eqn. 1 numerically by using the method of Ref. 11. Length is normalized by \( r_p \), energy by \( 2A_p \), and time by \( \tau = \eta r_p^2 / 2A_p \).

We consider \( N = 60 \) particles and \( N_p \) pinning centers in a rectangular box of size \( L_x \times L_y \) and periodic boundary conditions, with \( L_y = 100 \), \( L_x = 20\sqrt{3}L_y \). We average calculated properties over 500 disorder realizations.

We start by discussing the simplest case of non-interacting particles, \( A_v = 0 \), without disorder, \( N_p = 0 \), and with a ratchet potential of amplitude \( U_0 = 1 \). The dashed-lines of Fig. 2(a) show the transverse drift rate \( V \equiv \langle \frac{1}{T} \sum_i \frac{dX_i}{dt} \rangle \) at \( T = 0.05 \) as a function of the longitudinal velocity \( V_y \equiv \langle \frac{1}{T} \sum_i \frac{dY_i}{dt} \rangle \). We see that the transverse velocity \( V \) increases from zero, has a maximum \( V \sim 0.0075 \) at \( V_y \sim 0.5 \), and decays to zero at large longitudinal velocity. Since \( V = 0 \) at \( T = 0 \), the average directed transverse motion observed is induced by the thermal noise. This rectification effect is easy to understand, and Fig. 2(b) illustrates the mechanism, where the transverse diffusion constant is \( D = 2T \) if there is no disorder nor inter-particle interactions. For our discussion it is useful to make explicit the connection between the type of response shown in Fig. 2(a) and the one of a flashing ratchet. If \( F_y \) is large, the mean velocity in the driven direction \( V_y \equiv \langle \frac{1}{T} \sum_i \frac{dY_i}{dt} \rangle \) is \( V_y \sim F_y - O(F_y^{-1}) \) and longitudinal fluctuations are much smaller (by a factor \( O(F_y^{-1}) \) than transverse fluctuations. At \( T = 0 \), the equation of motion for the coordinate \( X \) of a particle located at \( \mathbf{R} = X \hat{x} + Y \hat{y} \) can be thus written as

\[
\frac{dX}{dt} \approx -F_R(V_y)G_R(X).
\]

Since \( F_R(V_y) = U_0 \cos(2\pi V_y/b) \Theta(\cos(2\pi V_y/b)) \), \( X \) feels the ratchet potential \( G_R(X) \) switching on and off periodically with time periods \( \tau_{\text{on}} = \tau_{\text{off}} = b/2V_y \). At small large the mechanism is the same although \( \tau_{\text{on}} \) becomes increasingly larger than \( \tau_{\text{off}} \) since the wells of \( U_R(X, Y) \) delay the motion in the \( y \) direction. (see Fig. 1). The mapping to a flashing ratchet explains the observed directed transverse motion with \( V > 0 \) when \( T > 0 \) and can be thus used as an effective model to explain all the features of the response shown in Fig. 2(a).

The effect discussed so far is similar to the one described in Ref. 11. Let us now add disorder, by putting \( N_p = 2000 \) randomly located pinning sites. The resulting response is shown in Fig. 2(a) (symbols). As we can see, disorder strongly enhances the rectification at intermediate and large longitudinal velocity and also broaden the range of \( V_y \) where response is appreciable, with respect to the clean case. In addition, we find that at intermediate and large \( V_y \), \( V \) is finite even in the \( T = 0 \) limit, since disorder induces transverse diffusion when \( V_y > 0 \), even in the absence of thermal fluctuations. Finally, let us now turn-on the repulsive interaction between particles. In Fig. 2(b) we show \( V \) as a function of \( V_y \), for different values of the repulsion strength \( A_v \). In the inset of Fig. 2(b), we see that the maximum response \( V_{\text{max}} \) is almost constant with \( A_v \) up to values \( A_v \sim 0.2 \) where a slow decay starts, but it is larger than the response of the clean system up to \( A_v = 2 \). The decay of the response at large \( A_v \) is explained by the decrease of transverse wandering due to increasingly correlated collective motion. In Fig. 4(b), we show that the response for purely attractive pinning centers \( A_p = -1 \) is smaller than for repulsive centers \( A_p > 1 \) for small values of \( V_y \), but indistinguishable for larger values of \( V_y \). This is due to the fact that, at the density of centers considered, attractive centers are more effective to pin particles than humps, since the latter can provide two-dimensional pinning only by forming rare geometric traps. However at a density \( N_p/L_xL_y \sim 1/r_p^2 \) all these differences disappear completely.

In order to understand the rectification characteristics described above it is instructive to study, separately, the motion of particles in the purely disordered case, without the ratchet potential (i.e. \( U_0 = 0 \)). For simplicity we consider only the case of non-interacting particles, but we expect similar results for interacting particles in the dynamical regimes where transverse diffusion is non-zero. We analyze in detail the non-equilibrium transverse fluctuations as a function of \( F_y \), since they
affect directly the rectification in the presence of the ratchet potential. Following Ref.\textsuperscript{18} we define the observables $O(t) = \frac{1}{N} \sum_{i=1}^{N_x} s_i X_i(t)$ and $\bar{O}(t) = \sum_{i=1}^{N_x} s_i X_i(t)$, where $s_i = -1, 1$ are random numbers with $\bar{s}_i = 0$ and $\bar{s}_i s_j = \delta_{ij}$. The quadratic mean displacement can be written as $\Delta(t, t_0) \equiv \frac{1}{N_x} \sum_{i=1}^{N_x} (X_i(t) - X_i(t_0))^2 = C(t, t) + C(t_0, t_0) - 2C(t, t_0)$, with $C(t, t_0) = \langle O(t) \bar{O}(t_0) \rangle$. The integrated response function $\chi$ for the observable $O$ is obtained by applying a perturbative force $f_1 = \epsilon s_i \dot{x}$ at time $t_0$ and keeping it constant for all subsequent times on each particle, $\chi(t, t_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle \Delta(t) \rangle_{\epsilon} - \langle \Delta(t) \rangle_{\epsilon=0}$.

In the steady state $\Delta(t, t_0) = \Delta(t - t_0)$ and $\chi(t, t_0) = \chi(t - t_0)$ and in particular at equilibrium the fluctuation-dissipation theorem (FDT) imposes $\chi(t) = \Delta(t)/2T$. When $F_y > 0$ the system is out of equilibrium and the FDT does not hold. We will show however, that generalized fluctuation-dissipation relations can still be defined for our system. In the long time limit we find $\Delta(t) \sim DT$ and $\chi(t) \sim \mu t$ thus allowing us to define the transverse diffusion constant $D$ and the transverse mobility $\mu$. These two quantities depend on the longitudinal driving force as shown in Fig.\textsuperscript{2}b). $D$ is non-monotonic, has a peak at $F_y \approx 1.5$, and decays approximately as a power law towards the equilibrium value without disorder $2T$ for large forces. This behavior can be understood by considering the effective transverse random walk induced only by the collisions with the pinning centers at $T = 0$, and by simple heuristic arguments is possible to find the asymptotic forms $D \sim n_p r_p^2 V_y$ at small $V_y$ and $D \sim n_p r_p A_y^2 / V_y$ at large $V_y$, indicated in Fig.\textsuperscript{3}b). At large $F_y$ the transverse mobility $\mu$ approaches the equilibrium value without disorder, $\mu = 1$ (independent of $T$). At small $F_y$ $\mu$ decreases due to trapping and its value at the limit $F_y \to 0$ is controlled by $T$. In the inset of Fig.\textsuperscript{3}c) we show the parametric plot of $\chi(t) \Delta(t)/2T$ for $F_y = 0$ (equilibrium) and $F_y = 4.0$ (out of equilibrium). We see that the equilibrium FDT holds for $F_y = 0$ as expected. For $F_y = 4.0$ (and in general for $F_y < 0$) we observe instead that the FDT holds only at very short time scales, $t \lesssim r_p / V_y$, but it is violated at long times. The type of violation observed can be quantified using the notion of time-scale dependent “effective temperatures” introduced by Cugliandolo, Kurchan and Peliti.\textsuperscript{21} Following Ref.\textsuperscript{18} we define a velocity-dependent transverse effective temperature $T_{\text{eff}}$ from the slope shown in the inset of Fig.\textsuperscript{3}c). At long times this implies the generalized Einstein relation $T_{\text{eff}} \sim D/2\mu t$ in the non-driven direction. In Fig.\textsuperscript{3}c) we see that $T_{\text{eff}}$ follows closely $D$ except at low forces where $\mu$ decreases towards the value $\mu \sim 0.5$ at very low forces. For interacting particles similar results for $T_{\text{eff}}$ were obtained at low and intermediate forces, in the plastic and smectic regimes of motion.\textsuperscript{22} At large forces and small temperatures however, the forma-
We see that the transverse drift generated by this model at the length scales of the ratchet potential. In Fig. 2(a) construction, the assumptions of the model are that (i) the averaged fluctuation-dissipation relation shown in the inset of Eq. 3 therefore models transverse motion in a coarse-grained way (in time and space), satisfying the generalized fluctuation-dissipation relation shown in the inset of Fig. 2(b) (in the absence of the ratchet potential). By construction, the assumptions of the model are that (i) transverse forces are small compared with the longitudinal drive \( F_y \), and (ii), the particle motion is incoherent at the length scales of the ratchet potential. In Fig. 2(a) we see that the transverse drift generated by this model is close to the one of the full model, for the parameters analyzed in this paper. The rectification characteristics of the two-dimensional geometrical ratchet are therefore well described by the one-dimensional flashing-ratchet described by Eq. 3 working at the effective temperature \( T_{\text{eff}}(T, V_y) \) and friction \( \mu(T, V_y) \), determined by the disorder and the longitudinal velocity. Using this model the enhancement of the rectification observed in Fig. 2(a) can be simply attributed to the fact that \( T_{\text{eff}} > T \), but with \( T_{\text{eff}} \) still smaller than the optimal temperature for rectification of the effective pulsating ratchet Eq. 3is also expected to work for interacting particles by using the respective \( T_{\text{eff}}(T, V_y) \) and \( \mu(T, V_y) \) except at low \( T \) and large \( F_y \) where condition (ii) can be violated since particles become correlated over long times and distances.

In conclusion, we have studied numerically the effect of quenched disorder in a geometrical ratchet. We find that disorder enhances the transverse rectified velocity of a driven fluid. If particle motion is incoherent at the scale of the ratchet potential the response can be simply described by a one-dimensional flashing ratchet working at a disorder-induced, drive-dependent effective mobility and temperature, satisfying generalized Einstein relations. This effect can be used experimentally to enhance and control the performance of geometrical ratchets.

We acknowledge discussions with L.F. Cugliandolo, D. Domínguez, T. Giamarchi, V.I. Marconi, and A. Rosso. This work was supported in part by the Swiss National Fund under Division II.

---

*Present address: Dept. de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid, 28040 Madrid, Spain.

1. R. D. Astumian and M. Bier, Phys. Rev. Lett. 72, 1766 (1994); R. D. Astumian, Science 276, 917 (1997); Jülicher, A. Adži, and J. Prost, Rev. Mod. Phys. 69, 1269 (1997).
2. C. Marquet et al., Phys. Rev. Lett. 88, 168301 (2002); S. H. Lee et al., Phys. Rev. Lett. 94, 110601 (2005); D. Babic and C. Bechinger, Phys. Rev. Lett. 94, 148303 (2005); A. Libal et al., Phys. Rev. Lett. 96, 188301 (2006).
3. Z. Farkas et al., Phys. Rev. E 60, 7022 (1999); J. F. Wambaugh, C. Reichhardt, and C. J. Olson, Phys. Rev. E 65, 031308 (2002); D. van der Meer et al., Phys. Rev. Lett. 92, 184301 (2004).
4. C. S. Lee et al., Nature (London) 400, 337 (1999); J. F. Wambaugh et al., Phys. Rev. Lett. 83, 5106 (1999); M. B. Hastings, C. J. Olson Reichhardt, and C. Reichhardt, Phys. Rev. Lett. 90, 247004 (2003); C. J. Olson et al., Phys. Rev. Lett. 87, 177002 (2001); B. Y. Zhu et al., Phys. Rev. B 68, 014514 (2003); R. Wördenweber, P. Dymashevski, and V. R. Misko, Phys. Rev. B 69, 184504 (2004); J. E. Villegas et al., 71, 024519 (2005).
5. F. Falo et al., Europhys. Lett. 45, 024519 (1999); E. Trias et al., Phys. Rev. E 61, 2257 (2000); G. Carapella and G. Costabile, Phys. Rev. Lett. 87, 077002 (2001); D. E. Shalóm and H. Pastoriza, Phys. Rev. Lett. 94, 177001 (2005); V. I. Marconi, Physica C 437, 195 (2006); A. Sterek, R. Kleiner, and D. Koelle, Phys. Rev. Lett. 95, 177006 (2005).
6. E. Lundh and M. Wallin, Phys. Rev. Lett. 94, 110603 (2005); R. Gommer, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. 95, 073003 (2005); M. Schiavoni et al., Phys. Rev. Lett. 90, 094101 (2003); R. Gommer, S. Denisov, and F. Renzoni, Phys. Rev. Lett. 96, 240604 (2006).
7. H. Linke et al., Science 286, 2314 (2003).
8. J. M. R. Parrondo and L. Dinis, Contemporary Physics 45, 147 (2004).
9. P. Reimann, Phys. Rep. 361, 57 (2002); R. D. Astumian and P. Hänggi, Physics Today 55, 33 (2002).
10. A. van Oudenaarden and S. G. Boxer, Science 285, 1046 (1999).
11. D. Ertas, Phys. Rev. Lett. 80, 1548 (1998); T. A. J. Duke and R. H. Austin, Phys. Rev. Lett. 80, 1552 (1998); I. Derényi and R.D. Astumian, Phys. Rev. E 58, 7781 (1998); M. Bier et al., M. Kostur, 61, 7184 (2000); M. Kostur and L. Shimansky-Geier, Phys. Lett. A 265, 337 (2000); C. Keller, F. Marquardt, and C. Bruder, Phys. Rev. E 65, 041927 (2002); S. Savel’ev et al., Phys. Rev. B 71, 214303 (2005).
12. T. Hanus and R. Lipowsky, Phys. Rev. Lett. 79, 2895 (1997); F. Marchesoni, Phys. Rev. E 56, 2492 (1997); M. N. Popescu et al., Phys. Rev. Lett. 85, 3321 (2000).
13. T. Giamarchi and S. Bhattacharya, in High Magnetic Fields: Applications in Condensed Matter Physics and
Spectroscopy, edited by C. Berthier et al. (Springer-Verlag, Berlin, 2002), p. 314, cond-mat/0111052.

14 J.-P. Bouchaud and A. Georges, Phys. Rep \textbf{195}, 127 (1990).

15 A. B. Kolton, Physica C \textbf{437}, 026112 (2006).

16 M. A. Makeev, I. Derenyi, and A.-L. Barabasi, Phys. Rev. E \textbf{71}, 026112 (2005).

17 C. Reichhardt and C. J. Olson Reichhardt, Phys. Rev. E \textbf{73}, 011102 (2006).

18 A. B. Kolton \textit{et al.}, Phys. Rev. Lett. \textbf{89}, 227001 (2002).

19 A. B. Kolton, D. Domínguez, and N. Grønbech-Jensen, Phys. Rev. Lett. \textbf{83}, 3061 (1999).

20 A. B. Kolton, to be published.

21 L. F. Cugliandolo, J. Kurchan, and L. Peliti, Phys. Rev. E \textbf{55}, 3898 (1997).