Globalization of Nonlinear FETI-DP Domain Decomposition Methods Using an SQP Approach

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Abstract
The globalization of Nonlinear FETI-DP (Dual Primal Finite Element Tearing and Interconnecting) methods is considered using a Sequential Quadratic Programming (SQP) approach. Nonlinear FETI-DP methods are parallel iterative solution methods for nonlinear finite element problems, based on divide and conquer, using Lagrange multipliers. In these methods, nonlinear elimination is an important ingredient to increase the convergence radius of Newton’s method. We prove standard globalization results for SQP-based globalization of Nonlinear FETI-DP, first for the case that the elimination set is empty. We then show how to combine nonlinear elimination and SQP-based globalization. The globalization preserves the block structure of the FETI-DP operator, which is the basis of the computational parallelism.

Supporting numerical experiments using homogenous and heterogeneous model problems from nonlinear structural mechanics are provided. In the numerical experiments, we consider four standard choices of different elimination sets and different problem setups including stiff or almost incompressible inclusions in every subdomain. The numerical experiments illustrate that a good elimination set is important. However, the use of the SQP-based globalization approach presented here can improve the convergence of Nonlinear FETI-DP methods further, especially, if combined with a good choice of the elimination set.

Keywords Nonlinear domain decomposition · FETI-DP · Globalization · Optimization · SQP

Mathematics Subject Classification (2010) 65N30 · 65N55 · 65F08 · 65F10 · 49M15

This article is dedicated to Professor Alfio Quarteroni on the occasion of his 70th birthday.
This work is part of an ongoing PhD thesis of the first author.

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1 Introduction

Nonlinear FETI-DP (Dual Primal Finite Element Tearing and Interconnection) methods [10, 11] are parallel iterative solution methods for nonlinear finite element problems. Originally introduced in [5, 6], they are nonlinear generalizations of standard FETI-DP domain decomposition methods [22]. Related methods are, e.g., nonlinear BDDC (Balancing Domain Decomposition by Constraints) methods [10, 11], nonlinear FETI-1 methods [18, 19, 21], ASPIN (Additive Schwarz Preconditioned Inexact Newton) methods [3], or ASPEN (Additive Schwarz Preconditioned Exact Newton) methods [4, 9]. Nonlinear FETI-DP domain decomposition methods are robust and scalable [12–15].

In nonlinear domain decomposition solvers, the nonlinear problem is decomposed into parallel nonlinear problems before linearization. This is opposed to more standard Newton–Krylov-domain-decomposition methods, where the nonlinear problem is first linearized and then decomposed into parallel problems. The nonlinear domain decomposition paradigm can help to increase concurrency, to improve solver robustness, and can significantly reduce the number of synchronization points when solving nonlinear finite element problems.

The idea of nonlinear FETI-DP methods is to decompose the global problem into local nonoverlapping nonlinear problems and to interconnect them using Lagrange multipliers. A coarse problem of primal constraints ensures the fast global transport of information. The coarse problem is obtained by assembling the primal variables of each subdomain. As primal variables we can, e.g., choose point, edge average or rotational constraints, see, e.g., [14].

Nonlinear domain decomposition methods are typically used in combination with Newton’s or related method, which are not globally convergent. The most common methods for globalization are Trust-region methods or line search methods. In this paper, we consider nonlinear FETI-DP methods in the context of constrained optimization and will investigate globalization using sequential quadratic programming (SQP) [8].

2 Nonlinear FETI-DP Domain Decomposition

We consider a computational domain \( \Omega \) discretized by finite elements. The corresponding finite element space is denoted by \( \hat{W} \). We consider the minimization problem

\[
\min_{\hat{u} \in \hat{W}} J(\hat{u}). \tag{2.1}
\]

We decompose \( \Omega \) into \( N \) nonoverlapping subdomains \( \Omega_i = 1, \ldots, N \), where the interface is \( \Gamma := \left( \bigcup_{i=1}^N \partial \Omega_i \right) \setminus \partial \Omega \). We denote the associated finite element spaces by \( W(i) \), and the product space by \( W := W^{(1)} \times \cdots \times W^{(N)} \). A function \( u \in W \) can be discontinuous across the interface \( \Gamma \).

To transform the global minimization problem (2.1) into local minimization problems, it is standard to assume [10, 11] that the global energy \( J \) is additive in the subdomains, i.e., there exist local energies \( J(i) \) corresponding to \( \Omega_i \), such that the global energy can be written as \( J(\hat{u}) = \sum_{i=1}^N J(i)(\hat{u}(i)) \), where \( \hat{u} = (\hat{u}(1) \times \cdots \times \hat{u}(N)) \in \hat{W} \). Let us remark that in the finite element context this is not a severe restriction. We can rewrite (2.1) as

\[
\min_{u \in W} J(u) \quad \text{subject to} \quad u \in \hat{W}. \tag{2.2}
\]

In nonlinear FETI–DP, we enforce continuity in (2.2) by subassembly and linear equality constraints. In detail, we partition the variables of a local vector \( u^{(i)} \in W^{(i)} \) into inner...
variables $u^{(i)}$, dual variables $u^{(i)}_\Delta$, and primal variables $u^{(i)}_\Pi$, where the union of the dual and primal variables are the interface variables of $u^{(i)}$. The union of the inner and the dual variables are denoted by $u^{(i)}_B = (u^{(i)}_I, u^{(i)}_\Delta)$, the space of functions continuous at the primal variables by $\tilde{W}$ and the primal part of $\tilde{u}$ by $\tilde{u}_\Pi$. We refer to the assembly operators acting on the primal variables as $(R^{(i)}_\Pi)^T$ and therefore $R^{(i)}_\Pi$ maps the global primal variables to the local ones, i.e., $u^{(i)}_\Pi = R^{(i)}_\Pi \tilde{u}_\Pi$.

By $\tilde{J}(\tilde{u}) := \sum_{i=1}^N J^{(i)}(u^{(i)}_B, R^{(i)}_\Pi \tilde{u}_\Pi)$, we define the corresponding energy on $\tilde{W}$. Hence, the constrained FETI-DP minimization problem (2.2) becomes

$$\min_{\tilde{u} \in \tilde{W}} \tilde{J}(\tilde{u}) \text{ subject to } B\tilde{u} = 0,$$

where $B$ is the standard FETI–DP jump operator as in the linear case, i.e., $B$ is a matrix with exactly one 1 and one $-1$ for each row which corresponds to two dual variables of adjacent subdomains. Therefore, we have $B\tilde{u} = 0$ if and only if $\tilde{u} \in \tilde{W}$. For more details of linear FETI–DP see, e.g., [5, 6] and for nonlinear FETI–DP see, e.g., [10, 16].

The Lagrange function of (2.3) is

$$\mathcal{L}(\tilde{u}, \lambda) = \tilde{J}(\tilde{u}) + \lambda^T B\tilde{u},$$

where $\lambda \in V := \text{range}(B)$. The first order optimality conditions are

$$\nabla_{\tilde{u}} \mathcal{L}(\tilde{u}, \lambda) = \nabla \tilde{J}(\tilde{u}) + B^T \lambda = 0,$$

$$\nabla_\lambda \mathcal{L}(\tilde{u}, \lambda) = B\tilde{u} = 0.$$

Let us remark that, due to our assumptions on the additivity of $\tilde{J}$, the vector $\nabla \tilde{J}(\tilde{u})$ can be obtained by subassembling the vectors $\nabla J^{(i)}(u^{(i)}_B, R^{(i)}_\Pi \tilde{u}_\Pi)$ in the primal variables. The same holds for the Hessian $\nabla^2 \tilde{J}$. Therefore, we can write the Lagrange–Newton equations at a point $(\tilde{u}, \lambda)$ as

$$\begin{bmatrix}
\nabla^2_{BB} \tilde{J}(\tilde{u}) & \nabla^2_{B\Pi} \tilde{J}(\tilde{u}) & B^T_B \\
\nabla^2_{\Pi B} \tilde{J}(\tilde{u}) & \nabla^2_{\Pi \Pi} \tilde{J}(\tilde{u}) & B^T_\Pi \\
B_B & B_\Pi & O
\end{bmatrix}
\begin{bmatrix}
\delta u_B \\
\delta \tilde{u}_\Pi \\
\delta \lambda
\end{bmatrix}
= -
\begin{bmatrix}
\nabla_B \tilde{J}(\tilde{u}) + B^T_B \lambda \\
\nabla_\Pi \tilde{J}(\tilde{u}) + B^T_\Pi \lambda \\
B\tilde{u}
\end{bmatrix},$$

where $\nabla^2_{BB} \tilde{J}$ is a block diagonal matrix consisting of the local matrices $\nabla^2_{BB} J^{(i)}$ and $\nabla^2_{B\Pi} \tilde{J}$ consists of the local blocks $\nabla^2_{B\Pi} J^{(i)}$ assembled in the primal variables, the same holds for $\nabla^2_{\Pi B} \tilde{J}$ and $\nabla^2_{\Pi \Pi} \tilde{J}$; for details, see, e.g., [10].

### 3 Nonlinear Elimination and SQP Methods

In this section, we introduce the notation for nonlinear elimination, a crucial ingredient of the four nonlinear FETI–DP methods presented in [16], we also recall the basics of SQP methods and the combination of both.

To simplify of the notation, we consider the minimization problem

$$\min_{x \in \mathbb{R}^n} J(x) \text{ subject to } c_i(x) = 0, \quad i = 1, \ldots, p,$$

where $J, c_i \in C^3(\mathbb{R}^p), i = 1, \ldots, p, p \leq n$. Let us keep in mind, that in our context $x$ represents a vector $\tilde{u} \in \tilde{W}$ and the constraints $c_i(x) = 0, i = 1, \ldots, p$ represent the FETI–DP continuity constraints $B\tilde{u} = 0$. 
The Lagrange function for (3.1) is

\[ \mathcal{L}(x, \lambda) = J(x) + \lambda^T c(x), \]

where \( c(x) = [c_1(x), \ldots, c_p(x)]^T \). Under sufficient assumptions, a solution \((x^*, \lambda^*)\) of (3.1) fulfills the Karush–Kuhn–Tucker (KKT) conditions:

1. \( \nabla_x \mathcal{L}(x^*, \lambda^*) = \nabla J(x^*) + \nabla c(x^*) \lambda^* = 0 \) \hspace{1cm} (3.2a)
2. \( \nabla_\lambda \mathcal{L}(x^*, \lambda^*) = c(x^*) = 0 \) \hspace{1cm} (3.2b)

In nonlinear elimination, we split the set of equations in (3.2a) into two disjoint subsets \( E, L \) and the related variables \( x_E, x_L \). Furthermore, we want to compute \( x_E \) such that for given \( x_L \) and \( \lambda \) the system

\[ \nabla_{E} \mathcal{L}(x_E, x_L, \lambda) = \nabla_{E} J(x_E, x_L) + \nabla_{E} c(x_E, x_L) \lambda = 0 \] (3.3)

is solved. The \( E \) stands for “elimination” and \( L \) for “linearization”. Let us remark that the solution of (3.3) relies on the implicit function theorem; see below. We denote \( \nabla_{E} \) by \( \nabla_{E} \) and \( \nabla_{L} \) by \( \nabla_{L} \). We explicitly allow \( E = \emptyset \) or \( L = \emptyset \). In such a case the corresponding matrices or vectors are also empty.

The KKT conditions, see (3.1), with respect to the index sets \( E, L \) can be written as

1. \( \nabla_{E} \mathcal{L}(x_E, x_L, \lambda) = 0 \) \hspace{1cm} (3.4a)
2. \( \nabla_{L} \mathcal{L}(x_E, x_L, \lambda) = 0 \) \hspace{1cm} (3.4b)
3. \( \nabla_{\lambda} \mathcal{L}(x_E, x_L, \lambda) = 0 \) \hspace{1cm} (3.4c)

and we denote the partition of \( \mathbb{R}^n \) according to \( E, L \) by \( \mathbb{R}_{E}^n \) and \( \mathbb{R}_{L}^n \).

If there exists a \( x_E^* \), such that (3.3) is fulfilled for \( x_L^* \) and \( \lambda^* \) and if \( \nabla_{EE}^2 \mathcal{L}(x_E^*, x_L^*, \lambda) \) is invertible, then, by the implicit function theorem, there exists a neighborhood \( U \subset \mathbb{R}_{E}^n \) of \( x_E^* \), a neighborhood \( \Lambda \subset \mathbb{R}_{L}^p \) of \( \lambda^* \) and a function \( g_E: U \times \Lambda \to \mathbb{R}_{E}^n \), such that for all \( x_L \in U, \lambda \in \Lambda \)

\[ \nabla_{E} \mathcal{L}(g_E(x_L, \lambda), x_L, \lambda) = 0 \] (3.5)

holds. By \( \nabla_{E} \mathcal{L}(g_E(x_L, \lambda), x_L, \lambda) \) we refer to the evaluation of \( \nabla_{E} \mathcal{L} \) at the point \( (g_E(x_L, \lambda), x_L, \lambda) \), whereas by \( \nabla_{E} \mathcal{L}(g_E(x_L, \lambda), x_L, \lambda) \) we refer to the derivative of \( \mathcal{L}(g_E(x_L, \lambda), x_L, \lambda) \), which includes, by the chain-rule, also the derivative of \( g_E(x_L, \lambda) \). The Jacobian of \( g_E(x_L, \lambda) \) can be expressed using the implicit function theorem as

\[ D g_E(x_L, \lambda) = \left[ -\nabla_{EE}^2 \mathcal{L}\nabla_{EL}^2 \mathcal{L} - \nabla_{EE}^2 \mathcal{L}\nabla_{EL}^2 \mathcal{L} \right]^{-1} \left[ \nabla_{EL}^2 \mathcal{L} \nabla_{EL}^2 \mathcal{L} \right], \] (3.6)

where we dropped the arguments of \( \mathcal{L} \). We assume that for all \( x_L \in \mathbb{R}_{L}^n, \lambda \in \mathbb{R}_{L}^p \) there exists a solution of (3.3), so the implicit function \( g_E \) is defined for all \( x_L, \lambda \).

We drop the arguments for Hessian and the gradient and can write the Lagrange–Newton equations as

\[
\begin{bmatrix}
\nabla_{EE}^2 \mathcal{L} & \nabla_{EL}^2 \mathcal{L} & \nabla_{EL}^2 \mathcal{L} \\
\nabla_{EL}^2 \mathcal{L} & \nabla_{LL}^2 \mathcal{L} & \nabla_{LL}^2 \mathcal{L} \\
\nabla_{EL}^2 \mathcal{L} & \nabla_{LL}^2 \mathcal{L} & \frac{\delta x_E}{\delta x_L} \\
\n\end{bmatrix}
\begin{bmatrix}
\delta x_E \\
\delta x_L \\
\delta \lambda \\
\end{bmatrix}
= -
\begin{bmatrix}
\nabla_{E} \mathcal{L} \\
\nabla_{L} \mathcal{L} \\
\nabla_{\lambda} \mathcal{L} \\
\end{bmatrix}.
\] (3.7)
By taking account of the nonlinear elimination (3.5) the Lagrange–Newton equations at \((g_E(x_L, \lambda), x_L, \lambda)\) are

\[
\begin{bmatrix}
\nabla^2_{EE} \mathcal{L} & \nabla^2_{EL} \mathcal{L} & \nabla^2_{E\lambda} \mathcal{L} \\
\nabla^2_{LE} \mathcal{L} & \nabla^2_{LL} \mathcal{L} & \nabla^2_{L\lambda} \mathcal{L} \\
\nabla^2_{\lambda E} \mathcal{L} & \nabla^2_{\lambda L} \mathcal{L} & \mathcal{O}
\end{bmatrix}
\begin{bmatrix}
\delta x_E \\
\delta x_L \\
\delta \lambda
\end{bmatrix}
= - \begin{bmatrix}
0 \\
\nabla_L \mathcal{L} \\
\nabla_\lambda \mathcal{L}
\end{bmatrix}.
\]

(3.8)

Let us remark that the Hessian and the gradient of (3.7) and (3.8) are different, since in (3.7) we evaluate them at \((x_E, x_L, \lambda)\), while in (3.8) we replace \(x_E\) by the nonlinear elimination \(g_E(x_L, \lambda)\).

Remark 3.1 If the Hessian \(\nabla^2_{EE} \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)}\) is positive definite, then the nonlinear elimination \(g_E(x_L, \lambda)\) is the solution of the minimization problem

\[
\min_{x_E \in U} \mathcal{L}(x_E, x_L, \lambda),
\]

where \(U\) is a neighborhood of \(g_E(x_L, \lambda)\). Otherwise, we can solve the minimization problem \(\min_{x_E \in U} \frac{1}{2} \nabla E \| \mathcal{L}(x_E, x_L, \lambda) \|^2\) to compute \(g_E(x_L, \lambda)\).

In [16], four different selections of elimination sets were introduced: Nonlinear FETI–DP-1 (NL-1), where \(E = \emptyset, L = I \cup \Delta \cup \Pi\), Nonlinear FETI–DP-2 (NL-2), where \(E = I \cup \Delta \cup \Pi, L = \emptyset\), Nonlinear FETI–DP-3 (NL-3), where \(E = I \cup \Delta, L = \Pi\), and Nonlinear FETI–DP-4 (NL-4), where \(E = I, L = \Pi \cup \Delta\). In these methods, the elimination set is static and based on the domain decomposition. In [7], a residual based, dynamic approach was introduced. A corresponding approach has been investigated for Nonlinear FETI–DP in [24].

3.1 Sequential Quadratic Programming with Nonlinear Elimination

An efficient method for the solution of (3.1) is the SQP method. For an approximate solution \(x\) an update \(\delta x\) is computed by the solution of the quadratic program

\[
\min_{d \in \mathbb{R}^n} \nabla J(x)^T d + \frac{1}{2} d^T Hd \quad \text{subject to} \quad c(x) + \nabla c(x)^T d = 0,
\]

(3.9)

where \(H\) is positive definite on \(\ker(\nabla c(x^{(k)})^T)\). The SQP method can be seen as a generalization of Newton’s method in the sense that if \(\nabla^2_{xx} \mathcal{L}\) is positive definite on \(\ker(\nabla c(x^{(k)})^T)\), \(\nabla c(x^{(k)})\) has full rank, and we set \(H = \nabla^2_{xx} \mathcal{L}\), then there exists a solution of (3.9) and this solution is equivalent to the solution of the Lagrange–Newton equation. For the globalization of the SQP method the nondifferentiable penalty function

\[
P_1(x; \mu) = J(x) + \mu \|c(x)\|_1,
\]

where \(\mu > 0\), can be used. The function \(P_1\) is exact in the sense that for each local solution \(x^*\) of (3.1) a penalty parameter \(\mu^* > 0\) exists, such that \(x^*\) becomes a local minimum for \(P_1\) for all \(\mu > \mu^*\), see, e.g., [23].

For completeness we recall some theoretical results about the globalized SQP method and later discuss its combination with nonlinear elimination.
3.1.1 Sequential Quadratic Programming

The penalty function $P_1$ is not differentiable, but the directional derivative exists for every direction $d$. The directional derivative of $P_1$ at $x$ in direction $d$ is denoted by $D P_1(x; d, \mu)$. We have

$$D P_1(x; d, \mu) = \nabla J(x)^T d + \mu \sum_{c_i(x) > 0} \nabla c_i(x)^T d - \mu \sum_{c_i(x) < 0} \nabla c_i(x)^T d + \mu \sum_{c_i(x) = 0} |\nabla c_i(x)^T d|,$$  \hspace{1cm} (3.10)

see, e.g. [23, Satz 19.10].

**Definition 3.1** We say $x^*$ is a critical point of $P_1(\cdot ; \mu)$ if $D P_1(x; d, \mu) \geq 0$ for all $d \in \mathbb{R}^n$.

The next theorem shows the relation between critical points of $P_1$ and KKT points of (3.1).

**Theorem 3.1** ([1, Proposition 4.6]) If $(x^*, \lambda^*)$ is a KKT point for (3.1), there exists $\mu > 0$ such that $x^*$ is critical point of $P_1(\cdot ; \mu)$ for all $\mu \geq \mu$.

Therefore, we can find KKT points of (3.1) by the minimization of $P_1$. Since $P_1$ is nondifferentiable, even gradient related descent directions may fail to permit a convergence result of the form that every limit point $\tilde{x}$ of a sequence of iterates is a critical point of $P_1$. Search directions for which we can expect such a convergence result are related to the solution of (3.9). This was proposed by Han in [8], where an exact line search method is used to show convergence. In [1, Proposition 4.13] it is shown that is sufficient if the Armijo rule is fulfilled.

Let us remark that [1] makes use of the penalty function $P_\infty(x; \mu) = J(x) + \mu \|c(x)\|_\infty$. The use of $P_\infty$ has the advantage that we can slightly modify (3.9) in a way that it is always feasible, but it has the disadvantage that the matrix $H$ needs to be positive definite and not only positive definite on ker($\nabla c(x)^T$). Moreover, for $P_\infty$ it is necessary to modify the quadratic subproblems (3.9). Therefore, we cannot simply solve the Lagrange–Newton equation to compute a search direction of $P_\infty$. In nonlinear FETI–DP this is an algorithmic drawback, since the Lagrange–Newton equation can be solved very efficiently. This is also important when $\nabla^2_{xx} \mathcal{L}$ becomes indefinite, since it may slow down the convergence when we cannot use $\nabla^2_{xx} \mathcal{L}$ as $H$.

For a detailed analysis of nondifferentiable penalty methods, see, e.g. [1, Section 4.1]. Let us remark that the quadratic program (3.9) is strictly convex with affine linear equality constraints, hence there exists a unique solution and a vector of Lagrange multipliers such that the KKT conditions hold, if $\nabla c(x)$ has full rank, see, e.g. [23, Satz 16.26].

First, we show the relation between the solution $d$ of (3.9) at $x$ and $D P_1(x; d, \mu)$.

**Theorem 3.2** ([20, Theorem 18.2]) Let $x \in \mathbb{R}^n$, $H$ symmetric and positive definite on ker($\nabla c(x)^T$), $d$ be the optimal solution for (3.9) and $\tilde{\lambda}$ be corresponding Lagrange multipliers. It follows that

$$D P_1(x; d, \mu) = \nabla J(x)^T d - \mu \|c(x)\|_1.$$  \hspace{1cm} (3.11)
Moreover, we have

\[
DP_1(x; d, \mu) \leq -d^T H d - (\mu - \|\tilde{\lambda}\|_\infty)\|c(x)\|_1.
\] (3.12)

Let us remark that (3.12) shows that, if \( \mu \) is large enough, then we can obtain a descent direction for \( P_1(\cdot; \mu) \) by solving (3.9). Furthermore, we can choose an arbitrary matrix \( H \) in sense of that the only requirement is positive definiteness on \( \ker(\nabla c(x)^T) \).

The next theorem shows that \( P_1 \) is an exact penalty function in the sense that a local solution of (3.1) which is a KKT point is also a local minimum of \( P_1(\cdot; \mu) \), if \( \mu \) is large enough. Let us note that for nonlinear FETI–DP we assume that the jump operator \( B \) has full rank and hence the linear independence constraint qualification (LICQ) holds. This implies that a local solution of (3.1) is a KKT point.

**Theorem 3.3** ([17, Exact Penalty Theorem]) Suppose that \((x^*, \lambda^*)\) is a KKT point for (3.1), \( \nabla c(x^*) \) has full rank and \( \nabla^2_{xx} L(x^*, \lambda^*) \) is positive definite on \( \ker(\nabla c(x^*)^T) \). Furthermore, let \( \mu > \|\lambda^*\|_\infty \). Then \( x^* \) is a local minimum of \( P_1(\cdot; \mu) \).

A globalized SQP algorithm for the computing of a critical point of \( P_1 \) is outlined in Fig. 1.

| Init: \( x^{(0)} \in \mathbb{R}^n, \lambda^{(0)} \in \mathbb{R}^p, \epsilon_{\text{update}}, \epsilon_{\text{tol}}, \mu_0 > 0, \beta \in (0, 1), \eta \in (0, 1) \). |
| for \( k = 0, 1, \ldots \) until convergence do |
| 1. Compute \( \mathcal{L}(x^{(k)}, \lambda^{(k)}) \). |
| if \( \|\nabla \mathcal{L}(x^{(k)}, \lambda^{(k)})\|_\infty \leq \epsilon_{\text{tol}}, \text{STOP} \). |
| 2. (a) Compute a matrix \( H^{(k)} \) symmetric and positive definite. |
| (b) Compute the KKT point \((\delta x^{(k)}, \tilde{\lambda}^{(k)})\) for |
| \[
\min_{d \in \mathbb{R}^n} \nabla J(x^{(k)})^T d + \frac{1}{2} d^T H^{(k)} d |
\]
| subject to \( c(x^{(k)}) + \nabla c(x^{(k)})^T d = 0 \). |
| (c) Set |
| \( \mu_{k+1} = \max \{\mu_k, \|\tilde{\lambda}^{(k)}\|_\infty + \epsilon_{\text{update}}\} \). |
| 3. Compute the largest number \( \alpha_k \in \{\beta^l | l = 0, 1, 2, \ldots\} \) such that the Armijo rule |
| \[
P_1(x^{(k)} + \alpha_k \delta x^{(k)}; \mu_{k+1}) - P_1(x^{(k)}; \mu_{k+1}) |
\]
| \[
\leq \eta \alpha_k D P_1(x^{(k)}; \delta x^{(k)}, \mu_{k+1}) |
\]
| holds. |
| 4. Set \( x^{(k+1)} = x^{(k)} + \alpha_k \delta x^{(k)} \) and \( \lambda^{(k+1)} = \tilde{\lambda}^{(k)} \) or \( \lambda^{(k+1)} = \lambda^{(k)} + \alpha_k \delta \lambda^{(k)} \), |
| where \( \delta \lambda^{(k)} = \tilde{\lambda}^{(k)} - \lambda^{(k)} \). |

**Fig. 1** SQP algorithm for the computation of a critical point of \( P_1 \) (without nonlinear elimination)
Remark 3.2 If $\nabla c(x^{(k)})$ has full rank, then the solution of (3.13) can be computed by solving
\[
\begin{bmatrix}
H^{(k)} & \nabla c(x^{(k)}) \\
\nabla c(x^{(k)})^T & 0
\end{bmatrix}
\begin{bmatrix}
\delta x^{(k)} \\
\delta \lambda^{(k)}
\end{bmatrix} = - \begin{bmatrix}
\nabla J(x^{(k)}) + \nabla c(x^{(k)})\lambda^{(k)} \\
\nabla c(x^{(k)})
\end{bmatrix},
\]
(3.14)
where $\delta \lambda^{(k)} = \tilde{\lambda}^{(k)} - \lambda^{(k)}$.

For the main convergence result of the SQP algorithm in Fig. 1 we need the following assumptions:

Assumption 3.1 The sequence $(x^{(k)}, \lambda^{(k)})_k$ generated by the SQP algorithm in Fig. 1 is contained in a convex set $\Omega$ and the following properties hold:

(a) The functions $J$ and $c_i$, $i = 1, \ldots, p$, their first, and second derivative are bounded on $\Omega$.
(b) The sequence of Lagrange multipliers $(\tilde{\lambda}^{(k)})_k$ related to the solutions of (3.13) is contained in a compact set $A$.
(c) The sequence $(H^{(k)})_k$ is bounded and the matrices are symmetric.
(d) There exists a constant $\gamma > 0$ such that $d^T H^{(k)} d \geq \gamma \|d\|^2$ for all $d \in \mathbb{R}^n$ and for all $k$.

These are standard assumptions for line search methods, see, e.g., [2]. Assumption 3.1(a)–(b) ensures that the estimation of the directional derivative in Theorem 3.2 is valid. If $\nabla c(x^{(k)})$ has full rank, then Assumption 3.1(d) ensures that there exists a unique solution $d^{(k)}$ for (3.13). Assumption 3.1(c) completes the proof of Theorem 3.4, which is the main convergence result for SQP methods and which corresponds to similar results for descent methods for unconstrained minimization problems. Furthermore, it covers the theory for the globalization of Nonlinear FETI–DP–1. The theory for Nonlinear FETI–DP–2, 3, and 4 will be discussed in Section 3.1.2.

Theorem 3.4 ([1, Proposition 4.13]) Let Assumption 3.1 be fulfilled. Then there exists a penalty parameter $\mu^* > 0$ such that for all $\mu \geq \mu^*$ every limit point of the sequence $(x^{(k)})_k$ generated by the algorithm in Fig. 1, where the initial value is $\mu_0 = \mu$, is a critical point of $P_1(\cdot; \mu)$.

Proof We follow the proof presented in [1], which shows the same result for the penalty function $P_\infty$. Here, we introduce all necessary changes to show that the result holds also for the penalty function $P_1(\cdot; \mu)$.

We provide a proof by contradiction. Due to Assumption 3.1(b), there exists a constant $\mu^* > \max_{\lambda \in A} \|\lambda\|_\infty$. Let $\mu \geq \mu^*$ and $x^*$ be a limit point of $(x^{(k)})_k$, which is no critical point of $P_1(\cdot; \mu)$. We assume without loss of generality that
\[
\lim_{k \to \infty} x^{(k)} \to x^*.
\]
Since $(\tilde{\lambda}^{(k)})_k \subset A$, a convergent subsequence exists. We restrict ourselfs to this subsequence and assume $\lambda^{(k)} \to \lambda^*$. Since, $P_1(x^{(k)}; \mu)$ is monotonically decreasing and $P_1$ is continuous, we have $P_1(x^{(k)}; \mu) \to P_1(x^*; \mu)$ and hence also
\[
P_1(x^{(k+1)}; \mu) - P_1(x^{(k)}; \mu) \to 0,
\]
due to Assumption 3.1(a). By the definition of the Armijo rule, we have
\[
P_1(x^{(k+1)}; \mu) - P_1(x^{(k)}; \mu) \leq \eta \alpha_k D P_1(x^{(k)}; \delta x^{(k)}; \mu).
\]
By Theorem 3.2 and the choice of $\mu$ it follows that
\[
DP_1(x^{(k)}; \delta x^{(k)}, \mu) \leq -\delta x^{(k)}^T H^{(k)} \delta x^{(k)} - (\mu - \|\tilde{\lambda}^{(k)}\|_\infty) \|c(x^{(k)})\|_1 \\
\leq -\delta x^{(k)}^T H^{(k)} \delta x^{(k)} < 0. \tag{3.17}
\]
Since $(H^{(k)})_k$ is bounded, a convergent subsequence exists. Again, we restrict ourselves to this subsequence and assume $H^{(k)} \to H$. Due to continuity, it follows that $d^T H d \geq \gamma \|d\|^2$ for all $d \in \ker c(x^*)^T$. Combining (3.16) and (3.17), we have
\[
P_1(x^{(k+1)}; \mu) - P_1(x^{(k)}; \mu) \leq -\eta \alpha_k \delta x^{(k)}^T H^{(k)} \delta x^{(k)} < 0. \tag{3.18}
\]
Hence, by (3.15) and (3.18), it follows that
\[
\alpha_k \delta x^{(k)}^T H^{(k)} \delta x^{(k)} \to 0. \tag{3.19}
\]
Since $H^{(k)} \to H$, there are two possibilities to fulfill (3.19). Either
\[
\liminf_{k \to \infty} \|\delta x^{(k)}\| = 0 \tag{3.20}
\]
or else
\[
\liminf_{k \to \infty} \alpha_k = 0, \quad \liminf_{k \to \infty} \|\delta x^{(k)}\| > 0. \tag{3.21}
\]
If (3.20) holds, then without loss of generality we may restrict ourselves to a convergent subsequence and have $\lim_{k \to \infty} \|\delta x^{(k)}\| = 0$. Therefore, $(\dot{x}^* = 0, \lambda^*)$ is a KKT point for
\[
\min_d \nabla J(x^*)^T d + \frac{1}{2} d^T H d \quad \text{subject to} \quad c(x^*) + \nabla c(x^*)^T d = 0. \tag{3.22}
\]
By the KKT conditions for (3.22), it follows that $(x^*, \lambda^*)$ is a KKT point for (3.1). Therefore, by Theorem 3.1 it follows that $x^*$ is a critical point of $P_1(\cdot; \mu)$ which contradicts the hypothesis made earlier.

If (3.21) holds, then without loss of generality we may restrict ourselves to a convergent subsequence of $(\alpha_k)_k$, and we have
\[
\lim_{k \to \infty} \alpha_k = 0. \tag{3.23}
\]
Furthermore, we assume without loss of generality $\lim_{k \to \infty} \delta x^{(k)} = \delta x^* \neq 0$. By (3.23) and the Armijo rule, it follows that there exists a constant $K \in \mathbb{N}$ such that for all $k \geq K$ the initial step length is reduced at least once, by the constant factor $\beta$, see Fig. 1. Therefore, we have
\[
P_1(x^{(k)} + \alpha_k \delta x^{(k)}; \mu) - P_1(x^{(k)}; \mu) > \overline{\alpha}_k \eta DP_1(x^{(k)}; \delta x^{(k)}, \mu), \tag{3.24}
\]
where $\overline{\alpha}_k = \alpha_k / \beta$. By the definition of the directional derivative it follows that
\[
P_1(x^{(k)} + \alpha_k \delta x^{(k)}; \mu) - P_1(x^{(k)}; \mu) = \overline{\alpha}_k DP_1(x^{(k)}; \delta x^{(k)}, \mu) + o(\overline{\alpha}_k), \tag{3.25}
\]
where
\[
\lim_{k \to \infty} \frac{o(\overline{\alpha}_k)}{\overline{\alpha}_k} = 0. \tag{3.26}
\]
Combining (3.24), (3.25) and (3.17) we obtain
\[
0 < \frac{o(\overline{\alpha}_k)}{\overline{\alpha}_k} + (1 - \eta) DP_1(x^{(k)}; \delta x^{(k)}, \mu) < \frac{o(\overline{\alpha}_k)}{\overline{\alpha}_k} - (1 - \eta) \delta x^{(k)}^T H^{(k)} \delta x^{(k)}. 
\]
This contradicts our hypothesis, since by (3.26), \( \delta x^{(k)}^T H^{(k)} \delta x^{(k)} \geq \gamma \| \delta x^{(k)} \|^2 \) for all \( k \), and \( \delta x^* \neq 0 \) holds.

Since (3.20) and (3.21) lead to a contradiction, it follows that \( x^* \) is a critical point of \( P_1(\cdot; \mu) \).

**Remark 3.3** Due to Assumption 3.1 and the fact that we increase the penalty parameter in Fig. 1 step 2(c) by at least about \( \varepsilon \text{update} \), we make sure there exists \( K \in \mathbb{N} \) such that \( \mu_{k+1} = \mu_k \) for all \( k \geq K \).

### 3.1.2 Combination with Nonlinear Elimination

For the combination of nonlinear elimination (as outlined in Section 3) with the SQP algorithm, we assume from now on that \( \nabla c \) always has full rank. We introduce the objective function \( J : \mathbb{R}^n_L \times \mathbb{R}^p \to \mathbb{R} \) and the constraints \( C : \mathbb{R}^n_L \times \mathbb{R}^p \to \mathbb{R}^p \), which are defined by:

\[
J(x_L, \lambda) := J(g_E(x_L, \lambda), x_L), \quad C(x_L, \lambda) := c(g_E(x_L, \lambda), x_L),
\]

where \( g_E \) denotes the nonlinear elimination defined by (3.7). We denote the rows of \( C \) by \( C_i, i = 1, \ldots, p \). Let us remark that \( J \) and \( C \) are differentiable due to our assumptions.

From (3.6) it follows that \( \nabla J \) is given by:

\[
\nabla_L J(x_L, \lambda) = \nabla_L J \big|_{(g_E(x_L, \lambda), x_L)} - \left( \nabla^2_L E \nabla^2 E - \nabla^2 E \right) \big|_{(g_E(x_L, \lambda), x_L)},
\]

and

\[
\nabla_\lambda J(x_L, \lambda) = - \left( \nabla^2 E \nabla^2 E - \nabla^2 E \right) \big|_{(g_E(x_L, \lambda), x_L)}.
\]

Furthermore, \( \nabla C \) is given by:

\[
\nabla_L C(x_L, \lambda) = \nabla_L C \big|_{(g_E(x_L, \lambda), x_L)} - \left( \nabla^2_L E \nabla^2 E - \nabla^2 E \right) \big|_{(g_E(x_L, \lambda), x_L)}
\]

and

\[
\nabla_\lambda C(x_L, \lambda) = - \left( \nabla^2 E \nabla^2 E - \nabla^2 E \right) \big|_{(g_E(x_L, \lambda), x_L)}.
\]

To shorten the notation we will often drop the arguments for the constraints \( c \), the gradients, and the Hessians; and by \( c, \nabla E, \nabla^2 E \), etc. we will refer to the evaluation at the point \( (g_E(x_L, \lambda), x_L) \) or \( (g_E(x_L, \lambda), x_L, \lambda) \).

Furthermore, we replace the penalty function \( P_1(\cdot; \mu) \) by

\[
P_1(x_L, \lambda; \mu) := J(x_L, \lambda) + \mu \| C(x_L, \lambda) \|_1
\]

and use the relation between the quadratic model (3.9) and the equations (3.14) to compute a search direction \( (\delta x_L, \delta \lambda) \) by the solution of the Schur complement system

\[
\begin{bmatrix}
S_{H, LL} & S_{H, LL} \\
S_{H, LL} & S_{H, \lambda \lambda}
\end{bmatrix}
\begin{bmatrix}
\delta x_L \\
\delta \lambda
\end{bmatrix}
= -\begin{bmatrix}
\nabla_L J + \nabla_L E \lambda \\
c
\end{bmatrix},
\]

(3.30)
where

\[ S_{H,LL} := H_{LL} - H_{LE} H_{EE}^{-1} H_{EL}, \]
\[ S_{H,LL} := \nabla_L c - H_{LE} H_{EE}^{-1} \nabla_E c, \]
\[ S_{H,\lambda L} := \nabla_L c^T - \nabla_E c^T H_{EE}^{-1} H_{EL}, \]
\[ S_{H,\lambda \lambda} := -\nabla_E c^T H_{EE}^{-1} \nabla_E c \]

and \( H_{EE}, H_{EL}, H_{LL}, H_{LE} \) are the related blocks of the matrix \( H \) in (3.9). Let us make some remarks before we show the results corresponding to Section 3.1.1 for \( P_1(\cdot; \mu) \).

**Remark 3.4** The proof of Theorem 3.2 shows that it is crucial for the search direction \( d \), computed by (3.9), to fulfill \( c(x) + \nabla c(x)^T d = 0 \) to be a descent direction for \( P_1 \). This restricts our choices of \( H \). From (3.29) and (3.30), we see that in the case of nonlinear elimination the corresponding equation \( C(x_L, \lambda, \delta x_L, \delta \lambda) + \nabla_L c^T \delta x_L + \nabla_\lambda c^T \delta \lambda = 0 \) holds only if we use the exact Hessian \( \nabla^2 c_{xx} \) for \( H \). This is due to the fact that, in contrast to \( \nabla^2 c_{xx} \), an arbitrary matrix \( H \) will not reflect the change of the nonlinear elimination \( g_E \) if the variables \((x_L, \lambda)\) change. From now on we use \( \nabla^2 c_{xx} \) for \( H \) and omit it in the subscript of the Schur complement blocks in (3.30). The Schur complement is defined by

\[
S(x_L, \lambda) = \begin{bmatrix} S_{LL}(x_L, \lambda) & S_{L\lambda}(x_L, \lambda) \\ S_{\lambda L}(x_L, \lambda) & S_{\lambda\lambda}(x_L, \lambda) \end{bmatrix},
\]

where

\[
S_{LL}(x_L, \lambda) := \left( \nabla^2_{xx} \mathcal{L} - \nabla_{LE}^2 \mathcal{L} \nabla_{EE}^2 \mathcal{L}^{-1} \nabla_{EL}^2 \mathcal{L} \right) \bigg|_{(g_E(x_L, \lambda), x_L, \lambda)},
\]
\[
S_{L\lambda}(x_L, \lambda) := \left( \nabla^2_{Lx} \mathcal{L} - \nabla^2_{LE} \mathcal{L} \nabla_{EE}^2 \mathcal{L}^{-1} \nabla^2_{E\lambda} \mathcal{L} \right) \bigg|_{(g_E(x_L, \lambda), x_L, \lambda)},
\]
\[
S_{\lambda L}(x_L, \lambda) := \left( \nabla^2_{\lambda x} \mathcal{L} - \nabla^2_{LE} \mathcal{L} \nabla_{EE}^2 \mathcal{L}^{-1} \nabla^2_{EL} \mathcal{L} \right) \bigg|_{(g_E(x_L, \lambda), x_L, \lambda)},
\]
\[
S_{\lambda\lambda}(x_L, \lambda) := \left( -\nabla^2_{\lambda E} \mathcal{L} \nabla_{EE}^2 \mathcal{L}^{-1} \nabla^2_{E\lambda} \mathcal{L} \right) \bigg|_{(g_E(x_L, \lambda), x_L, \lambda)}.
\]

**Remark 3.5** The solution of the Schur complement system (3.30), where we use \( H = \nabla^2_{xx} \mathcal{L} \) at the point \((x_L, \lambda)\), and the solution of the Lagrange–Newton equation (3.8) at the point \((g_E(x_L, \lambda), x_L, \lambda)\) are equivalent. If \((\delta x_E, \delta x_L, \delta \lambda)\) is the solution of (3.8), then \((\delta x_L, \delta \lambda)\) is also the solution of (3.30) and vice versa, if \((\delta x_L, \delta \lambda)\) is the solution of (3.31), then \((\delta x_E, \delta x_L, \delta \lambda)\) is the solution of (3.8), where

\[
\delta x_E := -\nabla^2_{EE} \mathcal{L}^{-1} \left( \nabla^2_{EL} \mathcal{L} \delta x_L + \nabla^2_{E\lambda} \mathcal{L} \delta \lambda \right).
\]

Furthermore, by (3.6) we have

\[
g_E(x_L + \delta x_L, \lambda + \delta \lambda) \\ \approx g_E(x_L, \lambda) + D_{xL} g_E(x_L, \lambda) \delta x_L + D_{\lambda} g_E(x_L, \lambda) \delta \lambda \\ = g_E(x_L, \lambda) - \nabla^2_{EE} \mathcal{L}^{-1} \nabla^2_{EL} \mathcal{L} \delta x_L - \nabla^2_{EE} \mathcal{L}^{-1} \nabla^2_{E\lambda} \mathcal{L} \delta \lambda \\ = g_E(x_L, \lambda) - \nabla^2_{EE} \mathcal{L}^{-1} \left( \nabla^2_{EL} \mathcal{L} \delta x_L + \nabla^2_{E\lambda} \mathcal{L} \delta \lambda \right)
\]

Therefore, \( \delta x_E \) is the linearization of \( g_E \) in direction of \((\delta x_L, \delta \lambda)\).
Let us formulate the results of Section 3.1.1 with respect to the penalty function \( P_1(\cdot; \mu) \). As mentioned before, from our assumptions it follows that \( J \) and \( C \) are differentiable and hence, by (3.10) and (3.28), it follows that the derivative of \( P_1(x_L, \lambda; \mu) \) in direction \((\delta x_L, \delta \lambda)\) is given by

\[
DP_1(x_L, \lambda; \delta x_L, \delta \lambda, \mu) = \nabla_L J^T \delta x_L + \nabla_\lambda J^T \delta \lambda + \mu \sum_{C_i(x_L, \lambda) > 0} \left( \nabla_L C_i^T \delta x_L + \nabla_\lambda C_i^T \delta \lambda \right) - \mu \sum_{C_i(x_L, \lambda) < 0} \left( \nabla_L C_i^T \delta x_L + \nabla_\lambda C_i^T \delta \lambda \right) + \mu \sum_{C_i(x_L, \lambda) = 0} \left| \nabla_L C_i^T \delta x_L + \nabla_\lambda C_i^T \delta \lambda \right|.
\]

The following theorem corresponds to Theorem 3.1 but using nonlinear elimination. It shows that a KKT point \((x^*_E, x^*_L, \lambda^*)\) for (3.1) can be found by the minimization of \( P_1 \).

**Theorem 3.5** If \((x^*_E, x^*_L, \lambda^*)\) is a KKT point for (3.1), there exists \( \mu > 0 \) such that \((x^*_L, \lambda^*)\) is critical point of \( P_1(\cdot; \mu) \) for all \( \mu \geq \mu \).

**Proof** Let \((x^*_E, x^*_L, \lambda^*)\) be a KKT point for (3.1). Due to our assumptions, we have \( g_E(x^*_L, \lambda^*) = x^*_E \). Therefore, it follows that

\[
\nabla_E J \big|_{(g_E(x^*_L, \lambda^*), x^*_L)} + \nabla_E C \big|_{(g_E(x^*_L, \lambda^*), x^*_L)} \lambda^* = 0, \\
\nabla_L J \big|_{(g_E(x^*_L, \lambda^*), x^*_L)} + \nabla_L C \big|_{(g_E(x^*_L, \lambda^*), x^*_L)} \lambda^* = 0.
\]

Suppose \( \mu \geq \|\lambda^*\|_\infty \) and that \((x^*_L, \lambda^*)\) is not a critical point of \( P_1(\cdot; \mu) \). There exists a direction \((d, \omega)\) such that \( DP_1(x^*_L, \lambda^*; d, \omega, \mu) < 0 \). By (3.35), (3.28) and (3.29) it follows that

\[
0 = \left( \nabla_L J + \nabla_L C \lambda^* \right)^T d + c^T \omega - \left( \nabla_E J + \nabla_E C \lambda^* \right)^T \nabla_{EE} \mathcal{L}^{-1} \nabla_{EL} \mathcal{L} d - \left( \nabla_E J + \nabla_E C \lambda^* \right)^T \nabla_{EE} \mathcal{L}^{-1} \nabla_{EL} \mathcal{L} \omega = \left( \nabla_L J - \nabla_{EE} \mathcal{L} \nabla_{EE} \mathcal{L}^{-1} \nabla_E J \right)^T d - \lambda^* \nabla_{EE} \mathcal{L} \nabla_{EE} \mathcal{L}^{-1} \nabla_E J \cdot \omega - \lambda^* \nabla_{EE} \mathcal{L} \nabla_{EE} \mathcal{L}^{-1} \nabla_{EE} \mathcal{L} \cdot \omega.
\]
\[ \nabla L J^T d + \nabla \lambda J^T \omega + \lambda^* \left( \nabla L C^T d + \nabla \lambda C^T \omega \right) \]
\[ \leq \nabla L J^T d + \nabla \lambda J^T \omega + \sum_{i=1}^{p} |\lambda_i^*| \left| \nabla L C_i^T d + \nabla \lambda C_i^T \omega \right| \]
\[ \leq \nabla L J^T d + \nabla \lambda J^T \omega + \mu \sum_{i=1}^{p} \left| \nabla L C_i^T d + \nabla \lambda C_i^T \omega \right| \]
\[ = D P_1(x_L^*, \lambda^*; d, \omega, \mu). \]

This is a contradiction to \( D P_1(x_L^*, \lambda^*; d, \omega, \mu) < 0 \). Hence, \((x_L^*, \lambda^*)\) is a critical point of \( P_1(\cdot; \mu) \) if \( \mu \geq \|\lambda^*\|_{\infty} \).

This indicates that we can obtain a KKT point \((x_E^*, x_L^*, \lambda^*)\) of (3.1) by minimizing \( P_1(\cdot; \mu) \).

The next theorem will show how the Schur complement system (3.30) is related to descent directions of \( P_1(\cdot; \mu) \).

**Theorem 3.6** Let \((x_L, \lambda) \in \mathbb{R}^n_L \times \mathbb{R}^p, S_{LL}, S_{\lambda\lambda}\) be positive definite, \( S_{\lambda\lambda}\) be negative semidefinite, and \((\delta x_L, \delta \lambda)\) be the solution of (3.30). It follows that
\[ D P_1(x_L, \lambda; \delta x_L, \delta \lambda, \mu) = \nabla L J^T \delta x_L + \nabla E J^T \delta x_E - \mu \|c\|_1 \quad (3.36) \]
were \( \delta x_E \) is defined by (3.33). Moreover, we have
\[ D P_1(x_L, \lambda; \delta x_L, \delta \lambda, \mu) \leq -\delta x_L^T S_{LL} \delta x_L + \delta \lambda^T S_{\lambda\lambda} \delta \lambda - (\mu - \|\lambda + \delta \lambda\|_{\infty}) \|c\|_1. \quad (3.37) \]

**Proof** The proof follows the same arguments as in the proof of Theorem 3.2. By applying Taylor’s theorem, (3.30), (3.29), (3.28), (3.33), and for \( \alpha \leq 1 \) we obtain
\[ P_1(x_L + \alpha \delta x_L, \lambda + \alpha \delta \lambda; \mu) - P_1(x_L, \lambda; \mu) = J(x_L + \alpha \delta x_L, \lambda + \alpha \delta \lambda; \mu) - J(x_L, \lambda) \]
\[ + \mu \|C(x_L + \alpha \delta x_L, \lambda + \alpha \delta \lambda)\| - \mu \|C(x_L, \lambda)\| \]
\[ \leq \alpha \left( \nabla L J^T \delta x_L + \nabla \lambda J^T \delta \lambda \right) + \alpha^2 \gamma_1 \left( \|\delta x_L\|^2 + \|\delta \lambda\|^2 \right) \]
\[ + \alpha \mu \|C(x_L, \lambda)\| + \nabla L C^T \delta x_L + \nabla \lambda C^T \delta \lambda \| + \alpha^2 \mu \gamma_2 \left( \|\delta x_L\|^2 + \|\delta \lambda\|^2 \right) \]
\[ - \mu \|C(x_L, \lambda)\| \]
\[ = \alpha \left( \nabla L J^T \delta x_L - \nabla E J^T \nabla_{EE}^2 \mathcal{L} \left( \nabla_{EE}^2 \mathcal{L} \delta x_L + \nabla_{EE}^2 \delta \mathcal{L} \right) \right) \]
\[ + \mu \|c\| + \alpha^2 (\gamma_1 + \mu \gamma_2) \left( \|\delta x_L\|^2 + \|\delta \lambda\|^2 \right) \]
\[ \leq \alpha \left( \nabla L J^T \delta x_L - \nabla E J^T \delta x_E - \mu \|c\| \right) + \alpha^2 (\gamma_1 + \mu \gamma_2) \left( \|\delta x_L\|^2 + \|\delta \lambda\|^2 \right), \]
where \( \gamma_1 \) and \( \gamma_2 \) bound the second derivative terms of \( J \) and \( C \). By the same arguments we obtain the lower bound. Hence, (3.36) follows.

From (3.30) we have
\[ -\nabla L J - \nabla L \mathcal{C} \lambda = -\nabla L \mathcal{C} = S_{LL} \delta x_L + S_{L\lambda} \delta \lambda. \]

Hence,
\[ \nabla L J^T \delta x_L = -\delta x_L^T S_{LL} \delta x_L - \delta \lambda^T S_{L\lambda} \delta x_L - \lambda^T \nabla L \mathcal{C}^T \delta x_L. \]
Since $S_{LL}^T = S_{\lambda \lambda}$, we obtain by the last block of equations in (3.30)

$$\nabla_J^T \delta x_L = -\delta x_L^T S_{LL} \delta x_L - \delta \lambda^T ( - S_{\lambda \lambda} \delta \lambda - c) - \lambda^T \nabla_J^T \delta x_L
$$

$$= -\delta x_L^T S_{\lambda \lambda} \delta x_L + \delta \lambda^T S_{\lambda \lambda} \delta \lambda + \delta \lambda^T c - \lambda^T \nabla_J^T \delta x_L. \quad (3.38)$$

From the nonlinear elimination condition $\nabla_E J + \nabla_E c \lambda = 0$, the equation (3.38) and due to the fact that a solution $(\delta x_L, \delta \lambda)$ of (3.30) is also a solution of

$$\begin{bmatrix}
\nabla^2_{xx} E_E \\
\nabla^2_{x \lambda} E_E \\
\nabla^2_{\lambda \lambda} E_E \\
\nabla^2_{\lambda \lambda} \lambda E_E \\
\nabla^2_{\lambda \lambda} \lambda L
\end{bmatrix}
\begin{bmatrix}
\delta x_E \\
\delta x_L \\
\delta \lambda \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
\nabla_J E \\
\nabla_J \lambda \\
\nabla_J \lambda L
\end{bmatrix}(3.39)$$

were $\delta x_E$ is defined in (3.33), it follows that

$$\nabla_J^T \delta x_L + \nabla_E^T \delta x_E
$$

$$= -\delta x_L^T S_{LL} \delta x_L + \delta \lambda^T S_{\lambda \lambda} \delta \lambda + \delta \lambda^T c - \lambda^T \left( \nabla_J^T \delta x_L + \nabla_E^T \delta x_E \right) \quad (3.39)$$

Hence, (3.37) follows by (3.36) and (3.39).

Theorem 3.6 shows that we can compute a descent direction for $P_1(\cdot; \mu)$ by solving (3.30) if $S_{LL}$ is positive definite, $S_{\lambda \lambda}$ is negative semidefinite and if $\mu$ is large enough. Furthermore, (3.36) holds the same structure as (3.11). The only difference is that in (3.36) we need to evaluate $\nabla J$ and $c$ at the point $(g_E(x_L, \lambda), x_L)$, while in (3.11) we evaluate at the point $(x_E, x_L)$.

Like in the case without nonlinear elimination, see Theorem 3.3, the following theorem shows the relation of a local solution of (3.1) and a local minimum of $P_1(\cdot; \mu)$.

**Theorem 3.7** Suppose that $(x^*_E, x^*_L, \lambda^*)$ is a KKT point for (3.1), $\nabla c(x^*_E, x^*_L)$ has full rank and $\nabla^2_{xx} \lambda \mathcal{L}(x^*_E, x^*_L, \lambda^*)$ is positive definite on $\ker(\nabla c(x^*_E, x^*_L)^T)$. Furthermore, let $\mu > \|\lambda^*\|_\infty$. Then $(x^*_L, \lambda^*)$ is a local minimum of $P_1(\cdot; \mu)$.

**Proof** We follow the proof presented in [17, Exact Penalty Theorem] for the case without nonlinear elimination and make the necessary modifications.

We consider the equations

$$\mathcal{F}(x_L, \lambda, w) = \begin{bmatrix}
\nabla_J \mathcal{L}|_{(g_E(x_L, \lambda), x_L, \lambda)} \\
\nabla_J \mathcal{L}|_{(g_E(x_L, \lambda), x_L, \lambda)} - w
\end{bmatrix} \quad (3.40)$$

Since $(x^*_E, x^*_L, \lambda^*)$ is a KKT point, it follows from $x^*_E = g_E(x^*_L, \lambda^*)$ and (3.4) that $\mathcal{F} = 0$ at $(x^*_E, x^*_L, \lambda^*, 0)$. Since $\nabla c(x^*_E, x^*_L)$ has full rank and $\nabla^2_{xx} \mathcal{L}(x^*_E, x^*_L, \lambda^*)$ is positive definite on $\ker(\nabla c(x^*_E, x^*_L)^T)$, it follows that $\nabla^2 \mathcal{L}(x^*_E, x^*_L, \lambda^*)$ is invertible. From (3.6) we have

$$\nabla_J \left( \nabla_J \mathcal{L}|_{(g_E(x_L, \lambda), x_L, \lambda)} \right) = \nabla^2_{LL} \mathcal{L}|_{(g_E(x_L, \lambda), x_L, \lambda)} + \nabla L g_E(x_L, \lambda) \nabla^2_{EL} \mathcal{L}|_{(g_E(x_L, \lambda), x_L, \lambda)}$$

$$= \left( \nabla^2_{LL} \mathcal{L} - \nabla^2_{LE} \mathcal{L} \nabla^2_{EE} \mathcal{L} \nabla^2_{EL} \mathcal{L} \right)|_{(g_E(x_L, \lambda), x_L, \lambda)}$$

$$= \nabla^2_{LL} \mathcal{L}(x_L, \lambda),$$
\[ \nabla_\lambda \left( \nabla_L \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)} \right) = \nabla^2_{xL} \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)} + \nabla_\lambda g_E(x_L, \lambda) \nabla^2_{EL} \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)} \]

\[ = \left( \nabla^2_{xL} \mathcal{L} - \nabla^2_{xL} \mathcal{L} \nabla^2_{EL} \mathcal{L} \nabla^2_{EL} \mathcal{L} \right) \big|_{(g_E(x_L, \lambda), x_L, \lambda)} \]

\[ = \mathbf{S}_{LL}(x_L, \lambda) = \mathbf{S}_{LL}(x_L, \lambda)^T, \]

and

\[ \nabla_\lambda \left( \nabla_\lambda \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)} \right) = \nabla^2_{xL} \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)} + \nabla_\lambda g_E(x_L, \lambda) \nabla^2_{EL} \mathcal{L} \big|_{(g_E(x_L, \lambda), x_L, \lambda)} \]

\[ = \left( -\nabla^2_{xL} \mathcal{L} \nabla^2_{EL} \mathcal{L} \nabla^2_{EL} \mathcal{L} \right) \big|_{(g_E(x_L, \lambda), x_L, \lambda)} \]

\[ = \mathbf{S}_{\lambda\lambda}(x_L, \lambda), \]

see, (3.31) and (3.32). Hence, the Jacobian of \( \mathcal{F} \) with respect to the variables \( x_L \) and \( \lambda \) is invertible. It follows by the implicit function theorem that there exist an open sphere \( S_{0; \epsilon} \) and continuous differentiable functions \( x_L(\cdot), \lambda(\cdot) \), where \( x_L(0) = x^*_L \) and \( \lambda(0) = \lambda^* \), such that \( \mathcal{F}(x_L(w), \lambda(w), w) = 0 \) for all \( w \in S_{0; \epsilon} \). By continuity it follows that there exists an open sphere \( \hat{S}_0 \subset S_{0; \epsilon} \), such that for all \( w \in \hat{S}_0 \) the matrix \( \nabla c^T \big|_{(g_E(x_L(w), \lambda(w)), x_L(w), \lambda(w))} \) has full rank and

\[ \nabla^2_{xx} \mathcal{L} \big|_{(g_E(x_L(w), \lambda(w)), x_L(w), \lambda(w))} \]

is positive definite on

\[ \ker \left( \nabla c^T \big|_{(g_E(x_L(w), \lambda(w)), x_L(w), \lambda(w))} \right). \]

By the definition of \( \mathcal{J} \) and \( \mathcal{C} \) and our assumptions \((x^*_L, \lambda^*)\) is a local solution of \( \min_{(x_L, \lambda) \in S(x^*_L, \lambda^*)} \mathcal{J}(x_L, \lambda) \) subject to \( \mathcal{C}(x_L, \lambda) = 0 \). Hence, it follows that there exists an open sphere \( S(x^*_L, \lambda^*) \) such that for all \( w \in \hat{S}_0 \) the point \( (x_L(w), \lambda(w)) \) is the unique solution of

\[ \min_{(x_L, \lambda) \in S(x^*_L, \lambda^*)} \mathcal{J}(x_L, \lambda) \] subject to \( \mathcal{C}(x_L, \lambda) - w = 0. \]

We consider the primal function

\[ p(w) := \min_{(x_L, \lambda) \in S(x^*_L, \lambda^*)} \{ \mathcal{J}(x_L, \lambda) \mid \mathcal{C}(x_L, \lambda) = w \}. \]

By the arguments outlined above it follows that

\[ p(w) = \mathcal{J}(x_L(w), \lambda(w)) = J(g_E(x_L(w), \lambda(w)), x_L(w)). \] (3.41)

Furthermore, by the chain rule, (3.40), (3.29) and (3.28), we have

\[ 0 = \nabla_w \left( \nabla_\lambda \mathcal{L} \big|_{(g_E(x_L(w), \lambda(w)), x_L(w), \lambda(w))} - w \right) \big|_{w=0} \]

\[ = \nabla_w \left( c(g_E(x_L(w), \lambda(w)), \lambda(w)) - w \right) \big|_{w=0} \]

\[ = \nabla_w \left( \mathcal{C}(x_L(w), \lambda(w)) - w \right) \big|_{w=0} \]

\[ = \nabla_w x_L \big|_{w=0} \nabla_\lambda \mathcal{L} \big|_{(x^*_L, \lambda^*)} + \nabla_w \lambda \big|_{w=0} \nabla_\lambda \mathcal{C} \big|_{(x^*_L, \lambda^*)} - \mathbf{I}_{p \times p} \]

\[ = \nabla_w x_L \big|_{w=0} \left( \nabla^2_{xL} \mathcal{L} - \nabla^2_{LE} \mathcal{L} \nabla^2_{EE} \mathcal{L} \nabla^2_{EL} \mathcal{L} \right) \big|_{(g_E(x^*_L, \lambda^*), x^*_L, \lambda^*)} \]

\[ + \nabla_w \lambda \big|_{w=0} \left( -\nabla^2_{xL} \mathcal{L} \nabla^2_{EL} \mathcal{L} \nabla^2_{EL} \mathcal{L} \right) \big|_{(g_E(x^*_L, \lambda^*), x^*_L, \lambda^*)} - \mathbf{I}_{p \times p} \] (3.42)
We define
\[ (g_E(x^L, \lambda^*), x^L_*, \lambda_*) \]
and
\[ \nabla_w J(x_L(w), \lambda(w)) \big|_{w=0} = \nabla_w x_L \big|_{w=0} \nabla L J \big|_{(x^L_*, \lambda_*)} + \nabla_w \lambda \big|_{w=0} \nabla \lambda J \big|_{(x^L_*, \lambda_*)} \]
\[ = \nabla_w x_L \big|_{w=0} \left( \nabla L J - \nabla^2_{EE} \nabla \lambda \nabla - \nabla_{EE} \lambda \nabla \right) \big|_{(g_E(x^L_*, \lambda_*), x^L_*, \lambda_*)} + \nabla_w \lambda \big|_{w=0} \left( -\nabla^2 \lambda_E \nabla_{EE} \lambda \nabla - \nabla \lambda_E \nabla \right) \big|_{(g_E(x^L_*, \lambda_*), x^L_*, \lambda_*)} \quad (3.43) \]

By the combination of (3.43), the nonlinear elimination condition \(0 = \nabla_E J + \nabla_E c \lambda = \nabla_E J + \nabla_E \lambda \), and (3.42), we obtain
\[ \nabla_w J(x_L(w), \lambda(w)) \big|_{w=0} = \nabla_w x_L \big|_{w=0} \nabla L J \big|_{(g_E(x^L_*, \lambda_*), x^L_*, \lambda_*)} + \nabla_w \lambda \big|_{w=0} \left( -\nabla^2 \lambda_E \nabla_{EE} \lambda \nabla - \nabla \lambda_E \nabla \right) \big|_{(g_E(x^L_*, \lambda_*), x^L_*, \lambda_*)} \quad (3.44) \]

Since \((g_E(x^L_*, \lambda_*), x^L_*, \lambda_*)\) is a KKT point, it follows from (3.44):
\[ \nabla_w p \big|_{w=0} = \nabla_w J(x_L(w), \lambda(w)) \big|_{w=0} = -\lambda^*. \quad (3.45) \]

We define
\[ p_\mu(w) := p(w) + \mu \|w\|_1. \]

Let us recall that due to the assumptions \((x^L_*, \lambda_*)\) is a local solution of \(\min \nabla(x_L, \lambda) \) subject to \(C(x_L, \lambda) = 0\). It follows that
\[
\min_{(x_L, \lambda) \in S(x^L_*, \lambda_*)} \{ \nabla(x_L, \lambda) + \mu \|C(x_L, \lambda)\|_1 \}
= \min_{u \in S^0} \min_{(x_L, \lambda) \in S(x^L_*, \lambda_*)} \{ \nabla(x_L, \lambda) + \mu \|u\|_1 \mid C(x_L, \lambda) = u \}
= \min_{u \in S^0} \min_{(x_L, \lambda) \in S(x^L_*, \lambda_*)} \{ p(u) + \mu \|u\|_1 \mid C(x_L, \lambda) = u \}
= \min_{u \in S^0} \{ p(u) + \mu \|u\|_1 \}
= \min_{u \in S^0} p_\mu(u).
By the mean value theorem, we have

\[ p(w) = p(0) + \nabla_w p(\alpha w)^T w \]

for some \( \alpha, 0 \leq \alpha \leq 1 \). Therefore,

\[ p_{\mu}(w) = p(0) + \nabla_w p(\alpha w)^T w + \mu \|w\|_1. \tag{3.46} \]

Since, \( \nabla_w p \) is continuous at 0, it follows by (3.45) that for every \( \delta > 0 \) there exists a neighborhood \( S_{0;\delta} \) such that \( \|\nabla_w p(w)\|_i < |\lambda^*_i| + \delta, i = 1, \ldots, p \) for all \( w \in S_{0;\delta} \). Thus

\[ \nabla_w p(\alpha w)^T w \geq -\|\nabla_w p(\alpha w)\|_\infty \|w\|_1 \]

\[ \geq -(\|\lambda^*_i\|_\infty + \delta) \|w\|_1. \]

From (3.46) it follows that

\[ p_{\mu}(w) \geq p(0) - (\|\lambda^*_i\|_\infty + \delta) \|w\|_1 + \mu \|w\|_1 \]

\[ \geq p(0) + (\mu - \|\lambda^*_i\|_\infty - \delta) \|w\|_1. \]

Since \( \delta \) is arbitrary and \( \mu > \|\lambda^*_i\|_\infty \), we have

\[ p_{\mu}(w) \geq p(0) + (\mu - \|\lambda^*_i\|_\infty) \|w\|_1 \]

in a neighborhood of 0. Since \( p(0) = p_{\mu}(0) \) it follows that \( p_{\mu} \) has a local minimum in 0. The hypothesis follows from

\[ p_{\mu}(w) = p(w) + \mu \|w\|_1 \leq P_1(x_L, \lambda; \mu), \]

for all \( (x_L, \lambda) \in S(x^*_L, \lambda^*) \), see (3.41), and \( p_{\mu}(0) = P_1(x^*_L, \lambda^*; \mu) \). \[ \square \]

A globalized SQP algorithm with nonlinear elimination is outlined in Fig. 2, where we use the Schur complement in (3.31) to shorten the notation. Furthermore, we define \( S^{(k)} = S(x^{(k)}_L, \lambda^{(k)}) \) and the blocks \( S_{LL}^{(k)} \), etc., respectively. For the right hand side we define

\[ \nabla L^{(k)} = \nabla L_{(g_E(x^{(k)}_L, \lambda^{(k)}), x_L^{(k)}, \lambda^{(k)})}, \]

and \( \nabla \lambda^{(k)} = \nabla \lambda_{(g_E(x^{(k)}_L, \lambda^{(k)}), x_L^{(k)}, \lambda^{(k)})} \).

Remark 3.6 Instead of solving (3.47) in step 2(b), we solve the equivalent system

\[ \begin{bmatrix} \nabla^2 E^{(k)} & \nabla^2 E^{(k)} L^{(k)} & \nabla^2 E^{(k)} L^{(k)} & \nabla^2 E^{(k)} \lambda^{(k)} \\ \nabla^2 L^{(k)} & \nabla^2 L^{(k)} & \nabla^2 L^{(k)} & \nabla^2 L^{(k)} \end{bmatrix} \begin{bmatrix} \delta x_E^{(k)} \\ \delta x_L^{(k)} \\ \delta \lambda^{(k)} \end{bmatrix} = - \begin{bmatrix} \nabla E^{(k)} \\ \nabla L^{(k)} \end{bmatrix}, \tag{3.48} \]

where \( \nabla L^{(k)} := \nabla L_{(g_E(x_L^{(k)}, \lambda^{(k)}), x_L^{(k)}, \lambda^{(k)})} \) and \( \nabla^2 L^{(k)} := \nabla^2 L_{(g_E(x_L^{(k)}, \lambda^{(k)}), x_L^{(k)}, \lambda^{(k)})} \). This has the computational advantage that we do not need to assemble the Schur complement. Furthermore, we can use \( g_E(x_L^{(k)}, \lambda^{(k)}) + \beta \delta x_E^{(k)} \) as a starting point to compute the nonlinear elimination \( g_E(x_L^{(k)} + \beta \delta x_L^{(k)}, \lambda^{(k)} + \beta \delta \lambda^{(k)}) \). This seems to be a good guess, since \( \delta x_E^{(k)} \) is the linearization of \( g_E \) in direction \((\delta x_L^{(k)}, \delta \lambda^{(k)})\), see (3.34).
For the main convergence result of the SQP algorithm in combination with nonlinear elimination in Fig. 2 we need the following assumptions:

Assumption 3.2 The sequence \( \{(x^{(k)}, \lambda^{(k)})\} \) generated by the algorithm in Fig. 2 is contained in a convex set \( \Omega_L \times \Lambda \) and the following properties hold:

(a) The nonlinear elimination \( g_E(x_L, \lambda) \) exists for all \( (x_L, \lambda) \in \Omega_L \times \Lambda \).
(b) The functions \( J \) and \( c_i, i = 1, \ldots, p \), their first, second, and third derivative are bounded on \( g_E(\Omega_L \times \Lambda) \).
(c) The sequence of Lagrange multipliers \( (\lambda^{(k)} + \delta\lambda^{(k)})_k \) related to the solutions of (3.47) is contained in a compact set \( \hat{\Lambda} \).
(d) There exists constants \( \gamma_1, \gamma_2 > 0 \) such that \( d^T S_{LL}^{(k)} d \geq \gamma_1 \|d\|^2 \) for \( d \in \mathbb{R}^{n_L} \) and \( \omega^T \nabla_E^2 \mathcal{L}(k) \omega \geq \gamma_2 \|\omega\|^2 \) for all \( \omega \in \mathbb{R}^{n_E} \) and for all \( k \).

These assumptions are basically the same as Assumption 3.1, except that we need the existence and boundedness of the third derivative of \( J \) and \( c \) for the estimation of the directional derivative of \( \mathcal{P}_1 \).

Remark 3.7 Since \( \nabla c \) has full rank, it follows from (3.32b), (3.32d), and by Assumption 3.2(d) that the Schur complement \( \hat{S}^{(k)} := S_{\lambda\lambda}^{(k)} - S_{\lambda L}^{(k)} S_{LL}^{(k)} S_{L\lambda}^{(k)} \) has only negative eigenvalues. Since \( \hat{S}^{(k)} \) and \( S_{LL}^{(k)} \) are invertible, it follows that \( S^{(k)} \) is also invertible.

Fig. 2  SQP algorithm for the computation of a critical point of \( \mathcal{P}_1 \) (with nonlinear elimination)
We can now show our main result for the combination of nonlinear elimination and a SQP-based method, which corresponds to Theorem 3.4 and covers the theory for nonlinear FETI–DP–2, 3, and 4.

**Theorem 3.8** Let Assumption 3.2 be fulfilled. Then there exists a penalty parameter $\mu^* > 0$ such that for all $\mu \geq \mu^*$ every limit point of the sequence $((x_L^{(k)}, \lambda^{(k)}))_k$ generated by the algorithm in Fig. 2, where the initial value is $\mu_0 = \mu$, is a critical point of $P_1$.

**Proof** The proof follows the structure of the proof of Theorem 3.4. We provide a proof by contradiction. Due to Assumption 3.2(c), there exists a constant $\mu^* > \max_{\lambda \in \hat{\Lambda}} \|\lambda\|_\infty$. Let $\mu \geq \mu^*$ and $(x^*, \lambda^*)$ be a limit point of $((x_L^{(k)}, \lambda^{(k)}))_k$, which is no critical point of $P_1$. We assume without loss of generality that

$$\lim_{k \to \infty} (x_L^{(k)}, \lambda^{(k)}) \to (x^*_L, \lambda^*).$$

Since $(\lambda^{(k)} + \delta \lambda^{(k)})_k \subset \hat{\Lambda}$, a convergent subsequence exists. We restrict ourselves to this subsequence and assume $\lambda^{(k)} \to \lambda^*$. Due to the update rule of $\mu_k$ in 2(c) of the algorithm in Fig. 2 and our choice of $\mu_0$, it follows that $\mu_k = \mu_0$ for all $k$. Hence, $P_1(x_L^{(k)}, \lambda^{(k)}; \mu)$ is monotonically decreasing. By continuity, we have $P_1(x_L^{(k+1)}, \lambda^{(k+1)}; \mu) \to P_1(x^*_L, \lambda^*; \mu)$ and hence also

$$P_1(x_L^{(k+1)}, \lambda^{(k+1)}; \mu) - P_1(x_L^{(k)}, \lambda^{(k)}; \mu) \to 0. \quad (3.49)$$

By the definition of the Armijo rule, we have

$$P_1(x_L^{(k+1)}, \lambda^{(k+1)}; \mu) - P_1(x_L^{(k)}, \lambda^{(k)}; \mu) \leq \eta \alpha_k D P_1(x_L^{(k)}, \lambda^{(k)}; \delta x^{(k)}, \delta \lambda^{(k)}, \mu). \quad (3.50)$$

By Theorem 3.6 and Assumption 3.2 it follows that

$$DP_1(x_L^{(k)}, \lambda^{(k)}; \delta x^{(k)}, \delta \lambda^{(k)}, \mu) \leq -\delta x^{(k)}^T S_{LL} \delta x^{(k)} + \delta \lambda^{(k)}^T S_{\lambda \lambda} \delta \lambda^{(k)} - (\mu - \|\lambda^{(k)} + \delta \lambda^{(k)}\|_\infty) \|c^{(k)}\|_1 \quad (3.51)$$

$$\leq -\delta x^{(k)}^T S_{LL} \delta x^{(k)} + \delta \lambda^{(k)}^T S_{\lambda \lambda} \delta \lambda^{(k)} < 0.$$  

Since $(S^{(k)})_k$ is bounded, a convergent subsequence exists. Again, we restrict ourselves to this subsequence and assume $S^{(k)} \to S^*$. Due to continuity it follows that

$$d^T S_{LL}^* d \geq \gamma_1 \|d\|^2 \quad \text{and} \quad -\omega^T S_{\lambda \lambda}^* \omega \geq 0, \quad (3.52)$$

for all $d \in \mathbb{R}^{n_L}$ and all $\omega \in \mathbb{R}^p$. Combining (3.50) and (3.51), we have

$$P_1(x_L^{(k+1)}, \lambda^{(k+1)}; \mu) - P_1(x_L^{(k)}, \lambda^{(k)}; \mu) \leq \eta \alpha_k \left(\delta \lambda^{(k)}^T S_{L \lambda}^* \delta \lambda^{(k)} - \delta x^{(k)}^T S_{LL}^* \delta x^{(k)}\right) < 0. \quad (3.53)$$

Hence, by (3.49) and (3.53), it follows that

$$\alpha_k \left(\delta \lambda^{(k)}^T S_{L \lambda}^* \delta \lambda^{(k)} - \delta x^{(k)}^T S_{LL}^* \delta x^{(k)}\right) \to 0. \quad (3.54)$$

Since $S^{(k)} \to S^*$ and (3.52), there are two possibilities to fulfill (3.54). Either

$$\liminf_{k \to \infty} \|\delta x_L^{(k)}\| = 0 \quad \text{and} \quad \liminf_{k \to \infty} \|\delta \lambda^{(k)}\| = 0 \quad (3.55)$$

or else

$$\liminf_{k \to \infty} \alpha_k = 0 \quad \text{and} \quad \liminf_{k \to \infty} \|\delta x_L^{(k)}\| > 0, \quad \liminf_{k \to \infty} \|\delta \lambda^{(k)}\| > 0. \quad (3.56)$$
If (3.55) holds, then without loss of generality we may restrict ourselves to a convergent subsequence and have
\[ \lim_{k \to \infty} \| \delta x_L^{(k)} \| = 0 \quad \text{and} \quad \lim_{k \to \infty} \| \delta \lambda^{(k)} \| = 0. \]
As stated in Remark 3.7 the matrix \( S^{(k)} \) is invertible for all \( k \) and we have
\[
\begin{bmatrix}
  \delta x_L^{(k)} \\
  \delta \lambda^{(k)}
\end{bmatrix} = - \left[ \begin{bmatrix}
  S_{LL}^{(k)} & S_{L\lambda}^{(k)} \\
  S_{L\lambda}^{(k)} & S_{\lambda\lambda}^{(k)}
\end{bmatrix} \right]^{-1} \begin{bmatrix}
  \nabla_L L^{(k)} \\
  \nabla_\lambda L^{(k)}
\end{bmatrix}.
\]

By the same arguments which show that \( S^{(k)} \) is invertible, it follows that \( S^* \) is invertible.
Hence, from \( S^{(k)} \to S^* \), it follows that \( (g_E(x^*_L, \lambda^*), x^*_L, \lambda^*) \) is a KKT point for (3.1) and by Theorem 3.5 it follows that \( (x^*_L, \lambda^*) \) is a critical point of \( P_1(\cdot; \mu) \) thus contradicts the hypothesis made earlier.

If (3.56) holds, then without loss of generality we may restrict ourselves to a convergent subsequence of \((\alpha^{(k)})_k\) and we have
\[ \lim_{k \to \infty} \alpha^{(k)} = 0. \]  
(3.57)

Furthermore, we assume without loss of generality \( \lim_{k \to \infty} \delta x_L^{(k)} = \delta x^*_L \neq 0 \) and \( \lim_{k \to \infty} \delta \lambda^{(k)} = \delta \lambda^* \neq 0 \). By (3.57) and the Armijo rule, it follows that there exists a constant \( K \in \mathbb{N} \) such that for all \( k \geq K \) the initial step length is reduced at least once, by the constant factor \( \beta \), see Fig. 2. Therefore, we have
\[
P_1(x_L^{(k)} + \alpha_k \delta x_L^{(k)}, \lambda^{(k)} + \alpha_k \delta \lambda^{(k)}; \mu) - P_1(x^{(k)}, \lambda^{(k)}; \mu) > \frac{\alpha_k}{\alpha_k} D P_1(x^{(k)}, \lambda^{(k)}; \delta x^{(k)}, \delta \lambda^{(k)}, \mu),
\]
(3.58)
where \( \alpha_k = \alpha_L / \beta \). By the definition of the directional derivative it follows that
\[
P_1(x_L^{(k)} + \alpha_k \delta x_L^{(k)}, \lambda^{(k)} + \alpha_k \delta \lambda^{(k)}; \mu) - P_1(x^{(k)}, \lambda^{(k)}; \mu) = \frac{\alpha_k}{\alpha_k} D P_1(x^{(k)}, \lambda^{(k)}; \delta x^{(k)}, \delta \lambda^{(k)}, \mu) + o(\alpha_k).
\]
(3.59)
where
\[
\lim_{k \to \infty} \frac{o(\alpha_k)}{\alpha_k} = 0.
\]
(3.60)
Combining (3.58), (3.59) and (3.51) we obtain
\[
0 < \frac{o(\alpha_k)}{\alpha_k} + (1 - \eta) D P_1(x^{(k)}, \lambda^{(k)}; \delta x^{(k)}, \delta \lambda^{(k)}, \mu)
\]
\[
< \frac{o(\alpha_k)}{\alpha_k} - (1 - \eta) \left( \delta x^{(k)}^T S_{LL}^{(k)} \delta x^{(k)} - \delta \lambda^{(k)}^T S_{\lambda\lambda}^{(k)} \delta \lambda^{(k)} \right).
\]
This contradicts our hypothesis, since (3.60), \( \delta x^{(k)}^T S_{LL}^{(k)} \delta x^{(k)} \geq \gamma_1 \| \delta x^{(k)} \|^2 \), \( - \delta \lambda^{(k)}^T S_{\lambda\lambda}^{(k)} \delta \lambda^{(k)} \geq 0 \) for all \( k \) and \( \delta x^* \neq 0, \delta \lambda^* \neq 0 \) hold.

Since (3.55) and (3.56) lead to a contradiction, it follows that \( (x^*_L, \lambda^*) \) is a critical point of \( P_1(\cdot; \mu) \).

\[ \square \]

4 Numerical Experiments

We consider a two dimensional quasi-static Neo-Hookean benchmark problem using stiff or almost incompressible inclusions embedded in each subdomain, see Fig. 3. The strain energy function for the compressible matrix material and the stiff inclusions, \( J_{\mu}^{\text{comp}} \), is
defined by

$$ J^{\text{comp}}_u(x) = \frac{\mu}{2} \left( \text{tr}(F(x)^T F(x)) - 2 \right) - \mu \log(\psi(x)) + \frac{\lambda}{2} (\log(\psi(x)))^2, $$

where $F(x) = \nabla \varphi(x)$ is the deformation gradient, $\varphi(x) = x + u(x)$ the mapping from the reference to the deformed configuration, $u(x)$ denotes the displacement, and $\psi(x) = \det(F(x))$ the volume change. The Lamé constants are denoted by $\lambda$ and $\mu$. The subscript $u$ indicates that the strain energy density function depends on a given displacement. The nearly incompressible inclusions are modeled by the strain energy function, $J^{\text{incomp}}_u$, defined by

$$ J^{\text{incomp}}_u(x) = \frac{\mu}{2} \left( \frac{1}{J(x)} \text{tr}(F(x)^T F(x)) - 2 \right) + \frac{\kappa}{2} (\psi(x) - 1)^2, $$

where $\kappa = \frac{\lambda(1+\mu)}{3\mu}$, see, e.g., [24]. The Lamé constants are given by $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$, where $E$ is Young’s modulus and $\nu$ Poisson’s ratio. We use zero Dirichlet boundary conditions at the left and right side of the domain and zero Neumann boundary conditions at the remaining boundary, see Fig. 3. Therefore, we search for a displacement $u \in V$ which minimizes the functional

$$ J(u) = \int_{\Omega_{\text{mat}}} J^{\text{comp}}_u(x) dx + \int_{\Omega_{\text{incl}}} J^{\text{incl}}_u(x) dx - \int_{\Omega_{\text{mat}} \cup \Omega_{\text{incl}}} f(x) \cdot u(x) dx, $$

where $\Omega_{\text{mat}}$ is the matrix material, using the strain energy density $J^{\text{comp}}_u(x)$, $\Omega_{\text{incl}}$ denotes the inclusions, using the strain energy density $J^{\text{incl}}_u(x)$, where $J^{\text{incl}}_u(x)$ can be $J^{\text{comp}}_u(x)$ or $J^{\text{incomp}}_u(x)$, $f(x)$ denotes the body force, and $V$ is some function space which imposes the Dirichlet boundary conditions. Hence, we want to solve

$$ \min_{u \in V} J(u). $$

As material parameters, we use $E = 210$ and $\nu = 0.3$ for the matrix material, $E = 210000$ and $\nu = 0.3$ for the stiff inclusions, and, finally, $E = 210$ and $\nu = 0.499$ for the (mildly) almost incompressible inclusions. For the discretization, we use $P_2$ elements, which are not stable for the incompressible case.

Note that, if stiff or almost incompressible inclusions are present in the subdomains, the mechanical behavior will change when changing the number of subdomains. We therefore will solve different mechanical problems when changing the number of subdomains.

### 4.1 Algorithmic Details

Let us explain some technical details for the algorithm in Fig. 2. As parameters, we use $\beta = 0.5$, $\varepsilon_{\text{update}} = 0.9$, $\varepsilon_{\text{tol}} = 10^{-6}$, and $\mu_0 = 10$. As primal variables we use vertex degrees of freedom and furthermore edge and rotational averages, see, e.g. [14].
In step 2(b), instead of solving (3.47), we solve the equivalent system (3.48) at the point \((g_E(x^{(k)}_L, \lambda^{(k)}), x^{(k)}_L, \lambda^{(k)})\), see Remark 3.6. We use conjugate gradients when solving linearized systems.

For each trial step length \(\beta^t\) in step 3, we define \(x^{(k,\ell)}_L = x^{(k)}_L + \beta^t \delta x^{(k)}_L\) and \(\lambda^{(k,\ell)} = \lambda^{(k)} + \beta^t \delta \lambda^{(k)}\). To compute the nonlinear elimination \(g_E(x^{(k,\ell)}_L, \lambda^{(k,\ell)})\), we solve the minimization problem \(\min_{x \in U_L} \frac{1}{2} \|L(x, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|^2\). Let us explain this in detail.

We denote the \(j\)-th iterate in the computation of \(g_E(x^{(k,\ell)}_L, \lambda^{(k,\ell)})\) by \(g^{(k,\ell)}_E,(j)\) and \(\nabla^2_{EE} L(g^{(k,\ell)}_E,(j), x^{(k,\ell)}_L, \lambda^{(k,\ell)})\) by \(\nabla^2_{EE} L^{(k,\ell)}(j)\). During the factorization of \(\nabla^2_{EE} L^{(k,\ell)}(j)\) we detect if it is positive definite; in this case, we compute \(g^{(k,\ell)}_E,(j+1)\) by a Trust-region step for the minimization problem \(\min_{x \in U_L} \frac{1}{2} \|L(x, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|^2\); otherwise we compute \(g^{(k,\ell)}_E,(j+1)\) by a Trust-region step for \(\min_{x \in U_L} \frac{1}{2} \|L(x, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|^2\) and use \(\min_{x \in U_L} \frac{1}{2} \|L(x, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|^2\) in the computation of all following steps, i.e., \(g^{(k,\ell)}_E,(j+i)\), \(i \geq 1\). In the computation of \(g_E(x^{(k,\ell)}_L, \lambda^{(k,\ell)})\) we use a limit of 200 iterations. If the computation does not converge within these 200 iterations, we reject the step length \(\beta^t\) and try \(\beta^{t+1}\). As stopping criterion for the computation of \(g_E(x^{(k,\ell)}_L, \lambda^{(k,\ell)}) =: g^{(k,\ell)}_E\) we use \(\|\nabla L(g^{(k,\ell)}_E, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|_\infty < 10^{-6}\), in the case where we solve \(\min_{x \in U_L} \frac{1}{2} \|L(x, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|^2\), and if we solve \(\min_{x \in U_L} \frac{1}{2} \|L(x, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|^2\), we use \(\|\nabla^2_{EE} L(g^{(k,\ell)}_E, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|\nabla L(g^{(k,\ell)}_E, x^{(k,\ell)}_L, \lambda^{(k,\ell)})\|_\infty < 10^{-6}\).

It is possible that the Hessian of \(\tilde{J}\) is indefinite, although the fully assembled Hessian is positive definite. For our algorithm we must ensure positive definiteness on the kernel of the FETI–DP jump operator \(B\). However, any regularization has to respect the block structure of the Hessian, which is used for the parallelization in FETI–DP. In our implementation, we use regularization by a scaled the mass matrix, where the scaling is determined during the factorization, if it is necessary. This ensures positive definiteness.

There are two drawbacks with this regularization strategy: For a large regularization factor the search direction becomes the steepest descent search direction, which may slow down the convergence. The other drawback is the fact that Theorem 3.8 makes explicitly use of the original Hessian. Hence, it only holds if we use regularization finitely often. In practice this can become a problem, since we do not know a priori how often regularization is needed. Theorem 3.8 covers the theory for nonlinear FETI–DP–2,3,4. This drawback does not effect Theorem 3.4, which covers the theory for nonlinear FETI–DP–1.

### 4.2 Numerical Results

To test whether the SQP-based globalization can be relevant in practice, we provide numerical experiments for Nonlinear FETI-DP–1,2,3, and 4. First without globalization, then with globalization for the inner iteration and, last, the combination of the SQP-based globalization for the outer iteration with the globalization for the inner iteration. The numerical experiments are important since in floating point arithmetic machine precision will limit the convergence, i.e., convergence cannot be obtained if the step size becomes too small.

For space limitations, we present only a small subset of the numerical experiments which we have performed. In Table 1 to Table 6 we report the number of Newton iterations for our
benchmark problem. As a converge criterion, we use $\|\nabla L^{(k)}\|_\infty \leq 1 - 6 - \|\nabla L^{(0)}\|_\infty$. Convergence failure is indicated by $\|\nabla L^{(k)}\|_\infty \geq 1 - 5 - \|\nabla L^{(0)}\|_\infty$ or by a number of globalized Newton iterations $> 100$; see Table 1 to Table 4.

For Tables 5 and 6 globalization is used and, here, we use $\max\left\{\|x^{(k+1)} - x^{(k)}\|_\infty, \|\lambda^{(k+1)} - \lambda^{(k)}\|_\infty\right\}$ as stopping criterion. This indicates that no sufficient progress is reached, and we abort the simulation since we are limited by machine precision. In Tables 1, 3, and 5 we apply a body force of $f = (0, 10)^T$; in Tables 2, 4, and 6 we apply $f = (0, -60)^T$ as body force.

In Tables 1 and 2 we report the results for our benchmark problem without a globalization strategy, i.e., globalization is used neither for the global steps nor for the nonlinear elimination. In Table 1, we observe convergence and numerical scalability for all methods, except for NL-2 if no inclusions are present. For stiff inclusion NL-1 does not converge and also NL-2 in two cases. However, NL-3 and NL-4 converge and are scalable. This illustrates that nonlinear elimination can improve convergence if an appropriate elimination set is chosen. Finally, for the case with almost incompressible inclusions all methods fail to converge.

| d.o.f. | #Sub. | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 |
|-------|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| 16642 | 16    | 5    | 5    | 5    | 5    | 5    | 2    | 4    | 4    | 4    | 4    | 4    | 4    |
| 25922 | 25    | 5    | 5    | 5    | 5    | 5    | 3    | 3    | 4    | 4    | 4    | 4    | 4    |
| 37250 | 36    | 5    | 5    | 5    | 5    | 5    | 3    | 4    | 4    | 4    | 4    | 4    | 4    |
| 50626 | 49    | 5    | 5    | 5    | 5    | 5    | 3    | 4    | 4    | 4    | 4    | 4    | 4    |
| 66050 | 64    | 5    | 5    | 5    | 5    | 5    | 4    | 4    | 4    | 4    | 4    | 4    | 4    |

Table 1: Nonlinear FETI-DP-1,2,3,4 or NL-1,2,3,4; $H/h \approx 8$, see Fig. 3; Newton’s method without globalization; the number of Newton iterations is shown; no conv.: $\|\nabla L^{(k)}\|_\infty \geq 1 - 5 - \|\nabla L^{(0)}\|_\infty$.

Table 2: Nonlinear FETI-DP-1,2,3,4 or NL-1,2,3,4; $H/h \approx 8$, see Fig. 3; Newton’s method without globalization; the number of Newton iterations is shown; no conv.: $\|\nabla L^{(k)}\|_\infty \geq 1 - 5 - \|\nabla L^{(0)}\|_\infty$. 

**Without Globalization**

| d.o.f. | #Sub. | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 |
|-------|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| 16642 | 16    | 5    | 5    | 5    | 5    | 5    | 2    | 4    | 4    | 4    | 4    | 4    | 4    |
| 25922 | 25    | 5    | 5    | 5    | 5    | 5    | 3    | 3    | 4    | 4    | 4    | 4    | 4    |
| 37250 | 36    | 5    | 5    | 5    | 5    | 5    | 3    | 4    | 4    | 4    | 4    | 4    | 4    |
| 50626 | 49    | 5    | 5    | 5    | 5    | 5    | 3    | 4    | 4    | 4    | 4    | 4    | 4    |
| 66050 | 64    | 5    | 5    | 5    | 5    | 5    | 4    | 4    | 4    | 4    | 4    | 4    | 4    |
Table 3  Nonlinear FETI-DP-1,2,3,4 or NL-1,2,3,4; $H/h \approx 8$, see Fig. 3; Newton’s method without globalization only for the nonlinear elimination; the number of Newton iterations is shown; no conv.: $\|\nabla L^{(k)}\|_{\infty} \geq 1e5 \|\nabla L^{(0)}\|_{\infty}$

Globalization only for the nonlinear elimination

| d.o.f. | #Sub. | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 |
|--------|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| 16 642 | 16    | 5    | 3    | 5    | 5    | $\infty$ | $\infty$ | 4    | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 25 922 | 25    | 5    | 3    | 5    | 5    | $\infty$ | $\infty$ | 3    | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 37 250 | 36    | 5    | 3    | 5    | 5    | $\infty$ | $\infty$ | 3    | 4    | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 50 626 | 49    | 5    | 3    | 5    | 5    | $\infty$ | 2    | 3    | 4    | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 66 050 | 64    | 5    | 3    | 5    | 5    | $\infty$ | 3    | 3    | 4    | $\infty$ | $\infty$ | $\infty$ | $\infty$ |

without globalization; see Table 1. In Table 2, using a higher body force, we observe no converge for all methods and for all problem setups.

In Table 3, we use globalization for the nonlinear elimination; see Section 4.1. It is striking that the globalization of the elimination helps NL-2 to converge for the case “no incl.”, where before no convergence was obtained. This illustrates that globalization for the nonlinear elimination can improve the methods. For stiff inclusions, some cases for NL-2 and NL-4 do not converge, because the globalization for the nonlinear elimination does not converge. Finally, we do not obtain convergence for the case with almost incompressible inclusions. In Table 4, for all methods, we do not observe convergence.

Finally, in Tables 5 and 6 we use a globalization strategy for, both, the nonlinear elimination and the global steps; see Fig. 2. In Table 5 all methods converge and are numerical scalable if no inclusions are present. However, NL-3 needs 21 to 36 iterations, which is significantly higher than the other methods. This compares with 5 iterations in NL-1 where no

Table 4  Nonlinear FETI-DP-1,2,3,4 or NL-1,2,3,4; $H/h \approx 8$, see Fig. 3; Newton’s method without globalization only for the nonlinear elimination; the number of Newton iterations is shown; no conv.: $\|\nabla L^{(k)}\|_{\infty} \geq 1e5 \|\nabla L^{(0)}\|_{\infty}$

Globalization only for the nonlinear elimination

| d.o.f. | #Sub. | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 |
|--------|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| 16 642 | 16    |      |      |      |      |      |      |      |      |      |      |      |      |
| 25 922 | 25    |      |      |      |      |      |      |      |      |      |      |      |      |
| 37 250 | 36    |      |      |      |      |      |      |      |      |      |      |      |      |
| 50 626 | 49    |      |      |      |      |      |      |      |      |      |      |      |      |
| 66 050 | 64    |      |      |      |      |      |      |      |      |      |      |      |      |

no convergence was obtained
Table 5  Nonlinear FETI-DP-1,2,3,4 (NL-1,2,3,4); body force \( f = (0, -10)^T \); \( H/h \approx 8 \); globalized SQP method; number of solved quadratic subproblems (Newton iterations) is shown; stopping criterion: \( \| \nabla L^{(k)} \|_\infty \leq 1e-6 \| \nabla L^{(0)} \|_\infty \). In the results marked with * some steps with regularization were necessary. Globalization using the algorithm in Fig. 2

| d.o.f. | #Sub. | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 |
|-------|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| 16 642 | 16    | 5    | 2    | 21   | 4    | 11   | 2    | 10   | 5    | 9*   | 2    | 29   | 6    |
| 25 922 | 25    | 5    | 3    | 26   | 4    | 11   | 3    | 11   | 5    | 9    | 3    | 33   | 6    |
| 37 250 | 36    | 5    | 3    | 30   | 4    | 11   | 2    | 11   | 4    | 9*   | 3    | 46   | 6    |
| 50 626 | 49    | 5    | 3    | 30   | 5    | 11   | 2    | 17   | 4    | 10*  | 3    | 51   | 6    |
| 66 050 | 64    | 5    | 4    | 36   | 5    | 11   | 2    | 18   | 4    | 10*  | 4    | 57   | 6    |

nonlinear elimination is used. NL-2, however converges in only 2 to 4 iterations. For compressible inclusions, we also see convergence for all methods; again, NL-2 is the fastest with respect to the number of iterations, indicating that its elimination set is most appropriate for this problem. For almost incompressible inclusions, also all methods converge, however, for NL-1 a few number of regularized steps were necessary. Again NL-2 needs the lowest number of iterations, whereas NL-3 needs 29 to 57 iterations. Note that without globalization for incompressible inclusions none of the methods converged; see Table 1.

In Table 6 all methods converge, except one case for NL-2 if no inclusions are present. As in Table 5 NL-3 needs the most iterations (11 to 97). NL-2 is, again, the fastest method in terms of iterations, except for the last case where it does not converge. We need a few regularized steps for NL-1 and NL-4. Moreover, for these methods the regularization is

Table 6  Nonlinear FETI-DP-1,2,3,4 (NL-1,2,3,4); body force \( f = (0, -60)^T \); \( H/h \approx 8 \); globalized SQP method; number of solved quadratic subproblems (Newton iterations) is shown; stopping criterion: \( \| \nabla L^{(k)} \|_\infty \leq 1e-6 \| \nabla L^{(0)} \|_\infty \). In the results marked with * some steps with regularization were necessary. In the results marked with ** regularization was necessary until convergence. Globalization using the algorithm in Fig. 2

| d.o.f. | #Sub. | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 | NL-1 | NL-2 | NL-3 | NL-4 |
|-------|-------|------|------|------|------|------|------|------|------|------|------|------|------|
| 16 642 | 16    | 6    | 3    | 11   | 5    | 18   | 2    | 7    | 6    | 26*  | 3    | 12   | 11*  |
| 25 922 | 25    | 6*   | 3    | 12   | 6*   | 16   | 2    | 8    | 7    | 32*  | 3    | 16   | 11*  |
| 37 250 | 36    | 6    | 5    | 14   | 6*   | 16   | 3    | 11   | 6    | 46*  | 4    | 21   | 11*  |
| 50 626 | 49    | 8*   | 6    | 16   | 8*   | 17   | 3    | 12   | 6    | 54*  | 10   | 23   | 14*  |
| 66 050 | 64    | 10** | \infty | 97** | 31** | 16   | 3    | 12   | 7    | 44*  | \infty | 24*  | 11*  |
needed also in the last few iterations. Note, however, that without globalization none of the methods converged; see Table 2.

For compressible inclusions, we also see convergence for all methods. The results are similar to Table 5, although the body force is six times higher. For almost incompressible inclusions, also all methods converge, except NL-2 in the last case. Here NL-1 and NL-4 need a few number of regularized steps. The iteration numbers are significantly higher for NL-1 and NL-3 compared to NL-2 and NL-4. Again NL-2 needs the lowest number of iterations, except the last case, whereas NL-1 needs 26 to 54 iterations.

4.3 Conclusion

We have presented an SQP-based globalization strategy for Nonlinear FETI-DP domain decomposition methods. This strategy preserves the block structure of the FETI-DP operator, which is the basis for the computational parallelism. A first version of this strategy only applies to Nonlinear FETI-DP-1. We then show how this strategy can be combined with nonlinear elimination. We therefore can apply our SQP-based strategy also to nonlinear FETI-DP methods making use of nonlinear elimination, including NL-2, 3, and 4.

We use an exact penalty function $P_1$, i.e., if we have a KKT point for the constrained FETI-DP minimization problem then, for a finite penalty parameter $\mu > \mu^*$, this is also a critical point of $P_1$. We then can show a result of standard form for globalization methods: under sufficient assumptions, we can show that there exists a penalty parameter $\mu^*$ for the SQP-penalty function $P_1$ such that for $\mu > \mu^*$ every limit point is a critical point of $P_1$.

Our numerical experiments then illustrate, again, that a good elimination set is important to obtain fast convergence and that it can help to obtain convergence at all. It is well known that the choice of the elimination set is important for domain decomposition methods which make use of nonlinear elimination. However, we find using our SQP-based globalization strategy in addition, as outlined in this paper, can further improve the convergence of Nonlinear FETI-DP methods, i.e., we obtain convergence where the method otherwise would fail to converge. In certain difficult cases, however, we need additional regularization which is not covered by theory. In certain cases, for certain elimination sets (i.e., NL-3), where convergence was obtained also without globalization, the SQP-based globalization increased the number of iterations. This is, however, not true in general.

The numerical experiments thus show that the SQP-based globalization is helpful especially for challenging problems where no convergence is obtained otherwise. This is illustrated by our numerical experiments with almost incompressible inclusions or the problems with high body force.

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