The uncertainty principle, articulated in 1927 by Heisenberg [1], plays a crucial role in highlighting the non-classical nature of quantum probabilities. It states that the outcomes of two incompatible measurements cannot be predicted simultaneously with certainty. The formal inequality based on the standard deviations of position and momentum was derived by Kennard [2] and Weyl [3], and generalized by Robertson [4] and Schrödinger [5] for general observables. Even though variance-based uncertainty relations play an important role in quantum theory [6,9], later information-theoretic entropy was introduced as a natural way to quantify uncertainty, and the entropic formulation of uncertainty relations for quantum measurements were widely studied [10–26]. However, to fully capture the essence of uncertainty, uncertainty measures in the strictest sense must be monotonically nondecreasing under two classes: randomly chosen symmetry transformations (Dsym) and classical processing via channels followed by recovery (Drec) [27]. In these cases, nonnegative Schur-concave functions [28] are qualified candidates for uncertainty measures, and one can build various uncertainty relations based on majorization relations and nonnegative Schur-concave functions. Moreover, based on the form of joint uncertainty, majorization uncertainty relations can be divided into two major categories: direct-product majorization uncertainty relations, i.e. universal uncertainty relations (UURs) [29,30], and direct-sum majorization uncertainty relations [31].

A significant application of uncertainty relations is to determine the degree of nonlocality, which gives the link to security for quantum cryptography [32]. For instance, Berta et al. [33] derived the uncertainty principle in the presence of quantum memory (UPQM), and provided a lower bound on the uncertainty denoted by conditional von Neumann entropy corresponding to the measurements on the system A given information stored in the system B (i.e., quantum memory) [34]. This bound depends on the degree of entanglement between A and B. Oppenheim and Wehner [35] demonstrated the quantitative connection between uncertainty and nonlocality of quantum games by applying a fine-grained uncertainty relation (FGUR), showing that the amount of nonlocality can determine the strength of uncertainty in measurements. Recently, Jia et al. [36] characterized nonlocal correlations via UURs, Riccardi et al. [37] investigated multipartite steering inequalities by using entropic uncertainty relations, and Wang et al. [38] detected entanglement via direct-sum majorization uncertainty relations.

Each of UURs, UPQM, and FGUR captures different features of uncertainty. Specifically, UURs contain the diversity of uncertainty measures, the UPQM links the uncertainty to the amount of quantum entanglement between subsystems, and FGUR consists of all possible combinations of outcomes for different measurements. The question, thus, naturally arises: can all these uncertainty relations be unified into a general form? We answer the question in the affirmative by revisiting Schrödinger’s concept of probability relations between separated systems [39,40] from a quantum information perspective.

To illustrate the utility of our generalized probability relations, we also apply our formalism to gain an insight into the connection between uncertainty and two subtle forms of nonlocality, Einstein-Podolsky-Rosen (EPR) steering and entanglement. EPR steering is defined in terms of violations of a local hidden state (LHS) model and describes the ability to steer the state of another subsystem (steer) the state of another subsystem [41,42] (note that the concept of ‘steering’ is different from the notion studied in Ref. [35]). It has attracted much attention due to its inherent asymmetry [43,44] which makes it an essential recourse for one-sided device-independent quantum communication [45–52]. However, what leads to this asymmetry still defies a complete understanding.

Here we introduce a simple but universally applicable theory of quantum probability relations (QPRs), namely, quasi-fine-grained uncertainty relations (QFGURs), which unifies UURs, UPQM and FGUR. We show that the functionals...
of EPR steering and entanglement are special cases of our QFGURs, which reveals a fundamental connection between nonlocality and uncertainty. Moreover, our methods clarify that the LHS model itself can be formulated in terms of incompatibility of the available local observables, which sheds light on understanding the intrinsic asymmetry of EPR steering. Our theory is based on the notion of local probability relations from a measured system $A$ and a quantum memory $B$, which obey QPRs. Summing over all outcomes for each measurement in QPRs helps to study the unbounded violation of both quantum steering and entanglement inequalities systematically and efficiently. For illustrative purposes of the general framework, we provide some numerical examples and show that our steering inequality or entanglement inequality is easier to violate to some extent.

First, we generalize Schrödinger’s discussion of probability relations [39] from a quantum information perspective for a task: start with our two protagonists, Alice and Bob. Alice prepares a bipartite quantum state $\rho_{AB}$, holds subsystem $A$ and transmits subsystem $B$ to Bob. This process can be repeated as many times as required. In each round, they measure their own system and communicate classically. Alice chooses one of her measurement settings $x \in \mathbb{N}_N$, where $\mathbb{N}_N := \{1, \ldots, N\}$. She then measures a nondegenerate observable $A_x$ with eigenvectors $|\phi^+_x\rangle$, and receives an outcome $a^{(x)}$ with probability $p(a^{(x)}|x) = \text{Tr} (|\phi^+_x\rangle\langle\phi^+_x| \otimes I) \rho_{AB}$, where $I$ denotes the identity matrix. The corresponding notations for Bob are $B_x, |\phi^+_x\rangle, b^{(x)}$, and $q(b^{(x)}|y)$.

Here we are interested in a general case, where Alice and Bob choose measurements $A, B_x$ according to some joint distribution $p(x, y)$. Then, for each combination of possible outcomes $a = (a^{(1)}, a^{(2)}, \ldots, a^{(N)})$ and $b = (b^{(1)}, b^{(2)}, \ldots, b^{(N)})$ for a fixed set of measurements $x$ and $y$, we define the following QPRs comprising a series of inequalities

$$U_{\text{QPR}} := \left\{ N \sum_{x, y=1} p(x, y) p(a^{(x)}|x) q(b^{(y)}|y) \leq \xi^{\text{QPR}}_{a(b, \omega)} |\forall a, b\right\}.$$  

Here the upper bound $\xi^{\text{QPR}}_{a(b, \omega)}$ restricts the set of allowed probability distributions. To clearly and operationally link QPRs with quantum correlations and uncertainty relations, we introduce a special formalization of QPRs, namely, QFGURs. In the following we show how we can unify UURs, UPQM and FGUR through QFGURs.

In the FGUR approach [35], Alice has access to an unknown quantum state $\rho_A$ with probability $p(a^{(x)}|x) = \text{Tr} (|\phi^+_x\rangle\langle\phi^+_x| \otimes I) \rho_{AB}$. By considering the probability distribution $p(x)$ over the set of measurements $x$, FGUR is

$$U_{\text{FGUR}} := \left\{ N \sum_{x=1} p(x) p(a^{(x)}|x) \leq \xi^{\text{FGUR}}_a |\forall a\right\},$$  

which only focuses on Alice’s system, whereas Eq. 1 considers the probabilities come from both Alice and Bob.

Instead of considering measurements on a single system, now we consider measurements on two space-like separated systems $\rho_{AB}$ and assume that the measured system can have a quantum memory of the other system [33]. When Alice obtains a result $a^{(x)}$ with probability $p(a^{(x)}|x)$, the conditional quantum memory $\sigma^x_{ab} = \text{Tr}_A (|\phi^+_x\rangle\langle\phi^+_x| \otimes I) \rho_{AB}$ is created some distance away at Bob’s location. Bob’s resulting probability to obtain a result $d^{(y)}$ for his observable $B_y$ with eigenvectors $|\phi^+_y\rangle$ is thus quantified by $q(d^{(y)}|x) = \text{Tr} (|\phi^+_y\rangle\langle\phi^+_y| \sigma^x_{ab})$.

To unify these three approaches, we present UPQM, instead of through an entropic function [33], by focusing on the combination of the outcomes’ probability in a fine-grained form

$$U_{\text{QFGUR}} := \left\{ N \sum_{x=1} p(x) p(a^{(x)}|x) q\left(\pi(a^{(x)})|x\right) \leq \xi^{\text{QFGUR}}_a |\forall a\right\},$$  

where $\pi \in \Xi_d$ is a permutation of the outcomes, and $\Xi_d$ is the symmetry group. Assuming each measurement $x$ can result in one of $d$ possible outcomes, i.e. $d^{(x)} \in \mathbb{N}_d$, QFGUR is given by Eq. 3 as the sum of all $d$ outcomes for each measurement $x$ does not equal to one in general; i.e.

$$\sum_{x,i=1}^d p(a^{(x)}|x) q\left(\pi(a^{(x)})|x\right) < 1.$$  

In QFGUR, the combinations of probabilities contain all physical information that is accessible for measured systems and quantum memory, which leads to a generalization of UURs [29–31]. We prove as follows. Suppose two parties share a physical system $\rho_{AB} = \rho \otimes \rho$, and each of which measures $A_x$ and $B_x$ (assuming $p(x)$ equals to 1 for some $x$), respectively. Then we derive the inequalities from Eq. 3:

$$\max_{a(x),\pi} \left\{ p\left(a^{(x)}|x\right) q\left(\pi(a^{(x)})|x\right) \right\} \leq \max_{a(x),\pi} \xi^{\text{QFGUR}}_a := \Omega_1,$$

$$\max_{b(x),\pi} \left\{ \sum_{d^{(x)} \in \{i|j\} \mathbb{N}_d} p\left(a^{(x)}|x\right) q\left(\pi(a^{(x)})|x\right) \right\} \leq \max_{b(x),\pi} \sum_{d^{(x)} \in \{i|j\} \mathbb{N}_d} \xi^{\text{QFGUR}}_a := \Omega_2,$$

$$\ldots$$

$$\max_{\pi} \left\{ \sum_{d^{(x)} = 1}^d p\left(a^{(x)}|x\right) q\left(\pi(a^{(x)})|x\right) \right\} \leq \max_{\pi} \sum_{d^{(x)} = 1}^d \xi^{\text{QFGUR}}_a := \Omega_d.$$  

Note that Eqs. 4 are for a fixed measurement $x$, but choosing the maximum for any one of $d$ outcomes, any two of $d$ outcomes, and all outcomes, respectively. Denoting the probability distribution for each measurement $x$ as $P := (p(a^{(x)}|x))_{d^{(x)}}$ for $A_x$ and $P' := (q(\pi(a^{(x)})|x))_{d^{(x)}}$ for $B_x$, and defining the state-independent vector

$$\omega := (\Omega_1, \Omega_2 - \Omega_1, \ldots, \Omega_d - \Omega_{d-1}) \in \mathbb{R}^d$$

leads to the product-form UURs [29–30]: $P \otimes P' < \omega$, with
“<” standing for majorization [28]. This concludes that the approach of UURs is a special case of QFGURs.

UURs indicate that all nonnegative Schur-concave functions can be used here to measure the uncertainties, and hence the corresponding entropic uncertainty relation reads $H(A_x) + H(B_x) \geq H(A_x)$. On the other hand, one can assume that, if $\rho_{AB} = \rho \otimes \frac{1}{2} \mathbb{1}$, then QFGUR degenerates to FGUR up to a scalar. Without loss of generality, let us choose $A_1 = A_x$ and $A_2 = B_x$; then the direct-sum form $\mathcal{P } \otimes \mathcal{P }'$ [31] is obtained from FGUR.

For simplicity, we now characterize the amount of uncertainty in a physical system while taking a particular permutation $\pi = (1) \in \Xi_d$. We are interested in the values of the upper bound $\varepsilon_{\rho}^{\text{QFGUR}} (\mathcal{M}) = \max_{\rho_{AB} \in \mathcal{M} \in \mathcal{E } Q} \left\{ \sum_p (x)p(a^{(x)}|x)q(a^{(x)}|x) \right\}$, where the maximization is taken over all states within a specific type of quantum states $\rho_{AB} \in \mathcal{M}$. Here, $\mathcal{M}$ can be any quantum states (Q), separable states (S), or the bipartite states allowed for LHS model (E), or even other convex collection of quantum states. The hierarchy relations of the states are sketched in Fig. 1. An important consequence of this is the monotonicity of upper bound $\varepsilon_{\rho}^{\text{QFGUR}} (\mathcal{M})$. Based on the fine-graing probability distributions [31], the feature of quantum correlations can be captured by their upper bound of QFGURs. To simplify the notation, in the following we drop superscripts QFGUR on the upper bound.

**Lemma 1 (Monotonicity).** If the collections of quantum states $\mathcal{S}$ (separable), $\mathcal{E}$ (LHS model), $\mathcal{M}$ and $\mathcal{Q}$ (all quantum states) satisfy $\mathcal{S } \subseteq \mathcal{E } \subseteq \mathcal{M } \subseteq \mathcal{Q}$, then $\varepsilon_{\rho}^{\mathcal{S}} (\mathcal{S}) \leq \varepsilon_{\rho}^{\mathcal{E}} (\mathcal{E}) \leq \varepsilon_{\rho}^{\mathcal{M}} (\mathcal{M}) \leq \varepsilon_{\rho}^{\mathcal{Q}} (\mathcal{Q})$.

In quantum mechanics, if a state $\rho_{AB}$ satisfies $\left( \sum_p (x)p(a^{(x)}|x)q(a^{(x)}|x) \right)_{\rho_{AB}} > \varepsilon_{\rho}^{\mathcal{S}} (\mathcal{S})$, then $\rho_{AB}$ must be entangled. Hence, from our QFGUR, we construct criteria to test entangled states, steerable states (here we always assume steerable of the type “A steers B”). From this viewpoint, we also give criteria to test some unknown types of states $\mathcal{M}$ that possess correlation beyond entanglement and EPR steering.

Next, we derive a general form of inequalities for entanglement and steering, and provide some numerical examples to show the improvement of their experimentally feasible unbounded violation.

The following few paragraphs consider the quantum EPR steering scenario. We use the same notations as appeared in our uncertainty scenario for introducing $U_{\text{QFGUR}}$, where $A$ denotes as a measured system, and regarding Bob’s system as quantum memory. When Alice obtains result $a$ for measurement $x$, the conditional quantum memory at Bob’s place becomes $\sigma^a_x$. The information encoded in conditional quantum memory can be quantified by the quantum functional [53], $S_Q := \frac{1}{N} \sum_{x=1}^{N} \sum_{a=1}^{d} \text{Tr} (\phi _a^x | \phi _a^x | \sigma _a^x )$, where the maximal value of $S_Q$ equals $N$ when the bipartite quantum state $\rho_{AB}$ is maximally entangled and measurements are ideal in mutually unbiased bases.

Within the LHS model, Alice performs a measurement $x$ with an untrusted device, and announces the outcome $a$ with probability $p_1 (a|x)$ involving the local hidden variable $\lambda$. $\Lambda$ represents the possible values a shared classical variable $\Lambda$, also named shared randomness, distributed with the density function $p (\lambda )$. Now Bob’s conditional quantum memory is given by $\tau _a^x = \sum_{\lambda \in \Lambda} p (\lambda ) p_1 (a|x) | \sigma _a^x$, where $\sum_{\lambda \in \Lambda} p (\lambda ) = 1$ and $\sigma _a^x$ is Bob’s local hidden state after Alice’s measurement [42]. Denoting the local response probability function in Bob’s subsystem as $q_{\rho _a^x} (a|x) = \text{Tr} (| \phi _a^x \rangle \langle \phi _a^x | | \sigma _a^x )$ for the given measurement $| \phi _a^x \rangle$, where the function (from $A$ to $B$) functional can be written as

$$S_E := \frac{1}{N} \sum_{x=1}^{N} \sum_{a=1}^{d} \text{Tr} (\phi _a^x | \phi _a^x | \tau _a^x )$$

$$= \sum_{\lambda \in \Lambda} \sum_{x=1}^{N} \sum_{a=1}^{d} p (\lambda ) p_1 (a|x) q_{\rho _a^x} (a|x) . \tag{6}$$

Entanglement or nonseparability is a weaker sort of correlation than steering [42]. Within the quantum separable model $\rho_{AB} = \sum_{\lambda \in \Lambda} p (\lambda ) \rho _a^x \otimes \rho _b^x$, where $\rho _a^x$ and $\rho _b^x$ are some quantum states, we can also give the entanglement functional

$$S_S := \sum_{\lambda \in \Lambda} \sum_{x=1}^{N} \sum_{a=1}^{d} p (\lambda ) p_1 (a|x) q_{\rho _a^x} (a|x) , \tag{7}$$

with $p_1 (a|x) = \text{Tr} (| \phi _a^x \rangle \langle \phi _a^x | \rho _b^x )$ and $q_{\rho _a^x} (a|x) = \text{Tr} (| \phi _a^x \rangle \langle \phi _a^x | | \rho _b^x )$. Note that the local response function $p_1 (a|x)$ of entanglement functional comes from quantum measurements while for steering functional $p_1 (a|x)$ may come from classical measurements.

One immediately sees that $S_E$ and $S_S$ are special forms of combination of left hand side of quasi-fine-grained inequalities given in Eq. [3]. For a given measurement $| \phi _a^x \rangle$, on Bob’s system, the violation of the following inequalities $S_Q \leq \sup_{\rho_{AB} \in \mathcal{S } E} S_E$ and $S_Q \leq \sup_{\rho_{AB} \in \mathcal{S } Q} S_S$ indicates that the quantum state is steerable and entangled, respectively. The maximal degree of violation of the steering and entanglement inequalities is determined by

$$V_E = \frac{\sup_{\rho_{AB} \in \mathcal{S } E} S_Q}{\sup_{\rho_{AB} \in \mathcal{S } E} S_E} > 1 , \quad V_S = \frac{\sup_{\rho_{AB} \in \mathcal{S } Q} S_Q}{\sup_{\rho_{AB} \in \mathcal{S } Q} S_S} > 1 , \tag{8}$$

respectively.

We start introducing our main theorems by some notations. Give an arbitrary number of $N$ measurement settings $x$, we...
denote a set of $d \times (k + 1)$ rectangular matrices, where $k = 0, \ldots, dN - 1$, and define the maximal squares of norms for those matrices as

$$S_k^A := \max \left\{ \sigma_i^2(\{\varphi_{x_i}^{(a)}\}, \ldots, \{\varphi_{x_{k+1}}^{(a)}\}) \right\},$$

$$S_k^B := \max \left\{ \sigma_i^2(\{\varphi_{x_i}^{(b)}\}, \ldots, \{\varphi_{x_{k+1}}^{(b)}\}) \right\},$$

where $\sigma_i(\cdot)$ stands for the maximal singular value, and $|\varphi_{x_i}^{(a)}\rangle$ is the $a$th eigenvectors of measurement $A_x$ ($B_x$).

Our main results are the following.

**Theorem 2.** EPR steering (from A to B) functional $S_E$ satisfies

$$\sup_{\rho_{AB} \in E} S_E \leq S_{AB}^B.$$  \hspace{1cm} (10)

**Corollary 3.** The maximum violation of the EPR steering functional by quantum states is

$$V_E \geq \frac{N}{S_{AB}^B}. \hspace{1cm} (11)$$

**Theorem 4.** Entanglement functional $S_S$ satisfies

$$\sup_{\rho_{AB} \in S} S_S \leq S_{AB}.$$ \hspace{1cm} (12)

with

$$S_{AB} := 1 + \left( S_{k}^{A} - 1 \right) \left( S_{k}^{B} - 1 \right) + \left( S_{k}^{A} - S_{k}^{B} \right) \left( S_{k}^{B} - S_{k}^{A} \right) + \cdots + \left( S_{dN-1}^{A} - S_{dN-1}^{B} \right) \left( S_{dN-1}^{B} - S_{dN-1}^{A} \right).$$ \hspace{1cm} (13)

**Corollary 5.** The maximum violation of the entanglement functional by quantum states is

$$V_S \geq \frac{N}{S_{AB}}. \hspace{1cm} (14)$$

Full proofs of Theorems 1 and 2 are based on the direct-sum majorization uncertainty relation and detailed in the Supplementary Material [54].

Recently, the authors of Ref. [53] focused on the steering functional $\sup_{\rho_{AB} \in E} S_E$ and provided an upper bound by means of the Deutsch-Maassen-Uffink’s entropic relation [14]

$$\sup_{\rho_{AB} \in E} S_E \leq 1 + \sum_{i=1}^{N-1} C_i,$$ \hspace{1cm} (15)

where $C_i := \max_{x_i} C_{x_i(N+N-x_i \mod N)}$ and $C_{xy} := \max_{a,b} |\langle \varphi_x^{(a)}| \varphi_y^{(b)} \rangle|$ is the maximal overlap between these observables. Their inequality shows that the unbounded violation of steering inequality depends on the maximal overlap between incompatible measurements.

Here, our Theorem 1 proves that this unbounded violation does not only depend on the maximal overlap, but on all overlaps between incompatible measurements. Moreover, it is also possible to show that their upper bound is improved by our QFGURs. We give a simple example to show the improvement, which reveals that our method based on QFGURs is more sufficient and relaxed than the one in Eq. (15) [53].

Consider the following observables $M_1, M_2$ and $M_3$

$$M_1 = \left\{ \begin{array}{c}
1 \\
0 \\
0
\end{array} \right\}, \quad M_2 = \left\{ \begin{array}{c}
1 \\
0 \\
0
\end{array} \right\}, \quad M_3 = \left\{ \begin{array}{c}
\cos \theta \\
0 \\
\sin \theta
\end{array} \right\}, \quad M_4 = \left\{ \begin{array}{c}
\sin \theta \\
0 \\
\cos \theta
\end{array} \right\}. \hspace{1cm} (16)$$

Using the uncertainty relations (10), (12) and (15), we obtain three upper bounds $S_{AB}^B$, $S_{AB}$, and $1 + \sum_{i=1}^{N} C_i$, to classify steering, entanglement by our method, and steering by the method used in Ref. [53], respectively. As shown in Fig. 2, it is clear that our method provides tighter bounds such that is easier to violate in this case.

**Examples.** Werner states $[55]$ are the best-known class of mixed entangled state. In the following, we consider various families of Werner states and show some numerical results to compare our criterion with previous works.

(i) $2 \times 2$ Werner states. First, let us consider $\rho_{w} = p|\phi^{-}\rangle\langle\phi^{-}| + (1 - p)I/4$, where $p \in [0, 1]$ and $\langle\phi^{-}\rangle$ is the singlet state. To test its steering and entanglement with Eqs. (10) and (12), we choose two spin measurements $\sigma_x$ and $\sigma_z$, which give us $S_Q = 1 + p$, $S_0 = 1$, $S_{1}^{B}(A) = 1 - 1/\sqrt{2}$ and $S_{2}^{A}(B) = 3/2$, for $N = 2$ and $d = 2$. Therefore, the Werner state is steerable when $p > 1/\sqrt{2}$ based on criterion [10], which coincides with the previous result [12]. On the other hand, the state is entangled when $p > 2 - \sqrt{2}$ based on criterion [12]. It has been proven that two-qubit Werner states are entangled if and only if (iff) $p > 1/3$ [55], so our condition for entanglement is sufficient but not necessary.

(ii) $3 \times 3$ Werner states. We also consider a high dimensional case for qudits with $\rho_{w} = p|\psi^{+}\rangle\langle\psi^{+}| + (1 - p)I/9$, where $|\psi^{+}\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$. For the following three Gell-
We have $S_3 = 1 + 2p$, $S_1^{(A)} = 1$, $S_1^{(B)} = 2$, $S_2^{(A)} = (3 + \sqrt{5})/2$ and $S_2^{(B)} = \cdots = S_8^{(B)} = 3$, for $N = 3$ and $d = 3$. Therefore, our criteria are efficient to classify a two-qutrit Werner state being steerable when $p > 0.809$ and entangled when $p > 0.763$. However, if we turn to the method introduced in Ref. [54], we find the right hand side of Eq. (15) is 3 and a trivial condition of $p > 1$ to verify steering, which means that method is ineffective in this case.

In principle, we wish to find a state-independent bound for uncertainty relations which can reveal the incompatibility between observables. However, evidence suggests that incompatibility provides only partial information for steerability. In terms of the eigenvalues of the reduced system, the bounds of both Eqs. (10) and (12) can be improved and the asymmetry of EPR steering can be revealed:

**Corollary 6.** The EPR steering functional $\mathcal{S}_E$ (from Alice to Bob) satisfies

$$\sup_{\rho_{AB} \in \mathcal{E}} S_E \leq S_N^B(\lambda_B). \quad (18)$$

**Corollary 7.** The EPR steering functional $\mathcal{S}_E$ (from Bob to Alice) satisfies

$$\sup_{\rho_{AB} \in \mathcal{E}} S_E \leq S_N^A(\lambda_A). \quad (19)$$

Note that, above corollaries hold for any positive-operator valued measures. The formal definitions of $S_N^A(\lambda_A)$, $S_N^B(\lambda_B)$ and the proofs of above corollaries are given in Ref. [54]. In actuality, performing the same measurements on each system leads to $S_N^A = S_N^B$, meanwhile $S_N^A(\lambda_A) \neq S_N^B(\lambda_B)$ in general. Similarly, we can also improve the entanglement functional.

**Conclusion.** We revisit Schrödinger’s probability relations and provide an operational definition for quantum probability relations. Based on the notion of local probability relations from a measured system and a quantum memory, we prove a special format, namely a quasi-fine-grained uncertainty relation, which unifies universal uncertainty relations, the fine-grained uncertainty relation, and the uncertainty principle in the presence of quantum memory. Further, we apply our theory to show that the LHS model itself can be formulated in terms of incompatibility of the available local observables. Thus, we derive the experimentally feasible inequalities to test steering and entanglement, and discuss the unbounded violation, which is not only based on the maximal overlap [55] but on all overlaps between incompatible measurements. Here we highlight the role of our framework in tests of entanglement, EPR steering and the uncertainty principle, but it is general and allows us to derive and generalize some results in uncertainty measures. For example, some combinations (like UUR) of our quasi-fine-grained inequalities are monotonic under $\mathcal{D}^{sym}$ and $\mathcal{D}^{rec}$ [27], while some are not (like FGUR), their relations was left open and will be addressed elsewhere. Finally, our method paves the way for deeply understanding uncertainty and nonlocality and the fundamental relations between these two striking aspects of quantum mechanics.

Discussions with Naihuan Jing and Gilad Gour are gratefully acknowledged. B.C.S. acknowledges NSERC support. Y.X. and Q.H. acknowledge support of the National Key R&D Program of China (Grants No. 2016YFA0301302 and No. 2018YFB1107200) and the National Natural Science Foundation of China (Grants No. 11622428, No. 61475006, and No. 61675007).

**Appendix A: Majorization Inequalities**

For completeness we start from the derivation of a majorization inequality: observe that for nonnegative vectors $P$, $Q$, and $W$, i.e. components of the corresponding vector are nonnegative, satisfy $P < W$, then $P \cdot Q < W \cdot Q$. Here the down-arrow notation denotes that the components of the corresponding vector are ordered in decreasing order.

In order to simplify the proof, we apply rearrangement inequality and obtain

$$P \cdot Q \leq P^\downarrow \cdot Q^\downarrow. \quad (A.1)$$

Hence, we only need to prove $P^\downarrow \cdot Q^\downarrow \leq W^\downarrow \cdot Q^\downarrow$.

At first, we assume the length of vectors is 2, i.e. $l(P) = l(Q) = l(W) = 2$, and express them in the following form

$$P^\downarrow = (p_1, p_2), \quad Q^\downarrow = (q_1, q_2), \quad W^\downarrow = (w_1, w_2). \quad (A.2)$$

Thus

$$P^\downarrow \cdot Q^\downarrow = p_1q_1 + p_2q_2 = p_1(q_1 + (p_1 + p_2 - p_1)q_2 = p_1(q_1 - q_2) + (p_1 + p_2)q_2 \leq w_1(q_1 - q_2) + (w_1 + w_2)q_2 = W^\downarrow \cdot Q^\downarrow. \quad (A.3)$$

Assume it is also true for $l(P) = l(Q) = l(W) = n - 1$. Now consider the cases with $l(P) = l(Q) = l(W) = n$, $P^\downarrow = (p_1, p_2, \ldots, p_n)$, $Q^\downarrow = (q_1, q_2, \ldots, q_n)$, $W^\downarrow = (w_1, w_2, \ldots, w_n)$, (A.4)
and then rewrite the product $P^\dagger \cdot Q^\dagger$ as follows,

$$P^\dagger \cdot Q^\dagger = \sum_{i=1}^n p_i q_i = \sum_{i=1}^{n-1} p_i q_i + p_n q_n \leq \sum_{i=1}^{n-2} w_i q_i + (w_{n-1} + w_n - p_n) q_{n-1} + p_n q_n \leq \sum_{i=1}^{n-1} w_i q_i + w_n q_n = W^\dagger \cdot Q^\dagger.$$  \hspace{1cm} (A.5)

Thus, we conclude that

$$P \cdot Q \leq P^\dagger \cdot Q^\dagger \leq W^\dagger \cdot Q^\dagger.$$  \hspace{1cm} (A.6)

Appendix B: Proofs of Theorems 2 and 4

In the main text, we give the notations for measurements, outcomes, corresponding probability distributions, entanglement functional and steering functional. Here for measurement $x$, denoting the probability distributions $(p_i, (a|x)_a)$ as $P_x$ and $(q_i, (a|x)_a)$ as $Q_x$, then assuming

$$P := \bigoplus_x P_x, \quad Q := \bigoplus_x Q_x,$$  \hspace{1cm} (A.7)

and thus we have

$$S_E = \sum_{\lambda} p(\lambda)(P \cdot Q),$$  \hspace{1cm} (A.8)

which is the Eq. (6) of the main text. Note that both probability vectors $P$ and $Q$ are functions of local hidden variable $\lambda$.

We proceed by introducing majorization: a vector $x$ is majorized by another vector $y$ in $\mathbb{R}^n : x \prec y$ if $\sum_{i=1}^k x_i^k \leq \sum_{i=1}^k y_i^k (k = 1, 2, \ldots, n - 1)$ and $\sum_{i=1}^n x_i^1 = \sum_{i=1}^n y_i^1$, where the down arrow denotes that the components are ordered in decreasing order $x_1^1 \geq \cdots \geq x_2^1$. A nonnegative Schur-concave function $\Phi$ on $\mathbb{R}^n$ preserves the partial order in the sense that $x \prec y$ implies that $\Phi(x) \geq \Phi(y)$. We take the conventional expression of a probability distribution vector in short form by omitting the string of zeroes at the end, for instance, $(0.7, 0.3, 0, \ldots, 0) = (0.7, 0.3)$, and therefore the actual dimension of the vector should be clear from the context.

For the probability distributions which come from quantum state under projective measurements $\{|\phi_i^A\rangle\langle\phi_i^A|\}$, it must follow the direct-sum majorization uncertainty relation \[31\]

$$Q \prec W^B,$$  \hspace{1cm} (A.9)

with

$$W^A = (1, S_1^A - 1, S_2^A - S_1^A, \ldots, S_{dn-1}^A - S_{dn-2}^A),$$

$$W^B = (1, S_1^B - 1, S_2^B - S_1^B, \ldots, S_{dn-1}^B - S_{dn-2}^B).$$  \hspace{1cm} (A.10)

Meanwhile, the classical probability distributions, which may not come from the measurements for quantum state, do not have the restriction from quantum uncertainty relation, and they are only majorized by $(1, 0, \ldots, 0)$, i.e.

$$P_x < (1, 0, \ldots, 0) \ \forall x.$$  \hspace{1cm} (A.11)

Furthermore, for the direct-sum of all $P_x$, we have

$$\bigoplus_{N \text{ times}} P_x < \bigoplus_{N \text{ times}} (1, 0, \ldots, 0).$$  \hspace{1cm} (A.12)

In particular, denoting $R$ as

$$R := (1, 1, \ldots, 1, 0, \ldots, 0),$$  \hspace{1cm} (A.13)

implies

$$P \cdot Q \leq R \cdot W^B = S^B_{N-1}.$$  \hspace{1cm} (A.14)

Based on Eq. (A.14), we can construct a simple bound for our QFGURs, which can be used to detect steerability

$$S_E \leq S^B_{N-1}.$$  \hspace{1cm} (A.15)

The violation of Eq. (A.15) indicates that the quantum state is steerable from Alice to Bob. Similarly, the state is entangled if the following inequality

$$S_S \leq W^A \cdot W^B = S^{AB},$$  \hspace{1cm} (A.16)

is violated. Here, $W^i (i = A$ or $B)$ stands for the direct-sum majorization bound for the measurements on $i$‘s system. We complete the proof of our main theorems.

Appendix C: Proofs of corollaries 6 and 7

Quantum states obey the Heisenberg uncertainty principle, meaning that there is an inherent “minimum uncertainty” for incompatible observables. To reveal intrinsic limitations on all quantum states, one need study the bound which is independent of state. For example, given measurements $A_x, B_x$ with $x \in \mathbb{N}_N$, we can formulate the uncertainty relations in the following form

$$P < W^A, \quad Q < W^B.$$  \hspace{1cm} (A.17)

Here we can assume both $A_x$ and $B_x$ are POVMs: $A_x = \{I^{A,\lambda}_{d^{A,\lambda}}\}_{d^{A,\lambda} \in \mathbb{N}_N}$, $B_x = \{I^{B,\lambda}_{d^{B,\lambda}}\}_{d^{B,\lambda} \in \mathbb{N}_N}$. As we will see below, if the quantum state is unknown for us, then the bounds $W^A$ and $W^B$ are of the form

$$W^A = (S_1^A, S_2^A - S_1^A, \ldots, 0),$$

$$W^B = (S_1^B, S_2^B - S_1^B, \ldots, 0).$$  \hspace{1cm} (A.18)
with

\[
S_A^k := \max_{[T_1| \cdots |T_k]|k} \left\| \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{A_{d^{(i)}}} \right\|_{\infty},
\]

\[
S_B^B := \max_{[T_1| \cdots |T_k]|k} \left\| \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{B_{d^{(i)}}} \right\|_{\infty}. \tag{A.19}
\]

Here, \( T_i \) are subsets of distinct indices from \( \mathbb{N}_{N_i} \) or \( \mathbb{N}_{M_i} \), (based on the measurements we choose), and \( \left\| \cdot \right\|_{\infty} \) denotes the infinity operator norm—which, for positive operators, coincides with the maximum eigenvalues of its argument.

It is sometimes useful to consider the eigenvalues of the reduced system \( \rho_A \) and \( \rho_B \), where we will denote their eigenvalues in vector form

\[
spectrum(\rho_A)^{\downarrow} := \lambda_A,
\]

\[
spectrum(\rho_B)^{\downarrow} := \lambda_B, \tag{A.20}
\]

the down arrow notation denotes that the components of the corresponding vector are ordered in decreasing order. Then, for any density matrix \( \rho_{AB} \) with reduced eigenvalues given by \( \lambda_A, \lambda_B \), the following relations hold

\[
P < W^A(\lambda_A), \quad Q < W^B(\lambda_B). \tag{A.21}
\]

After performing \( \{A_i\} \) and \( \{B_i\} \) on Alice’s and Bob’s system respectively, their bounds \( W^A(\lambda_A) \) and \( W^B(\lambda_B) \) are

\[
W^A(\lambda_A) := (S_1^A(\lambda_A), S_2^A(\lambda_A) - S_1^A(\lambda_A), \ldots, 0),
\]

\[
W^B(\lambda_B) := (S_1^B(\lambda_B), S_2^B(\lambda_B) - S_1^B(\lambda_B), \ldots, 0). \tag{A.22}
\]

Quantitatively, we define the eigenvalues of \( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{A_{d^{(i)}}} \) and \( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{B_{d^{(i)}}} \) as

\[
spectrum \left( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{A_{d^{(i)}}} \right)^{\downarrow} := \lambda \left( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{A_{d^{(i)}}} \right),
\]

\[
spectrum \left( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{B_{d^{(i)}}} \right)^{\downarrow} := \lambda \left( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{B_{d^{(i)}}} \right). \tag{A.23}
\]

Finally, \( S_A^k(\lambda_A) \), \( S_B^B(\lambda_B) \) can be given as

\[
S_A^k(\lambda_A) := \max_{[T_1| \cdots |T_k]|k} \lambda_A \cdot \lambda \left( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{A_{d^{(i)}}} \right),
\]

\[
S_B^B(\lambda_B) := \max_{[T_1| \cdots |T_k]|k} \lambda_B \cdot \lambda \left( \sum_{d^{(i)} \in T_i} \Pi_{d^{(i)}}^{B_{d^{(i)}}} \right). \tag{A.24}
\]

Our results of \( W^A(\lambda_A) \) and \( W^B(\lambda_B) \) under any POVMs are generalizations of the bound that appeared in recent work \[56\], which only holds for rank-one projective measurements.

We now have all the necessary ingredients for proving Corollary 6. Denoting

\[
P_x^k := (p_x(a(x)))_a,
\]

\[
Q_x^A := (\text{Tr}(B^A_x\sigma_x))_a,
\]

\[
Q_x^B := (\text{Tr}(B^B_x\rho_B))_a, \tag{A.25}
\]

we observe that

\[
\sum_{x=1}^N p_x \bigoplus Q_x^A = \bigoplus_{x=1}^N Q_x^B. \tag{A.26}
\]

Here the ‘sum’ stands for element-wise sum, for example \((1,0) + (0,1) = (1,1)\). Then, according to direct-sum majorization, we can derive

\[
\bigoplus_{x=1}^N Q_x^B \preceq W^B(\lambda_B). \tag{A.27}
\]

On the other hand, recall steering functional \( S_E \):

\[
S_E = \sum_{x=1}^N p(x) \left( \bigoplus_{x=1}^N P_x^k \cdot \left( \bigoplus_{x=1}^N Q_x^A \right) \right)
\]

\[
\leq \sum_{x=1}^N p(x) \left( R \cdot \bigoplus_{x=1}^N Q_x^A \right)
\]

\[
\leq R \cdot W^B(\lambda_B)
\]

\[
= S^B_N(\lambda_B), \tag{A.28}
\]

where the first equality follows from the majorization inequality which has been proved in next section. The proof of Corollary 6 is complete, and similar method can be applied to the proof of Corollary 7.

Now the connection with the asymmetry of EPR steering: By performing \( \{A_i\} \) on both systems, the steering functional from Alice to Bob is bounded by \( S^A_N(\lambda_A) \), while the steering functional from Bob to Alice is bounded by \( S^B_N(\lambda_B) \). And they do not equal each other in general due to the effect of eigenvalues from different systems. Since \( \lambda_A \neq \lambda_B \), quantum mechanics allows for the possibility of one-way steering.

**Appendix D: Linear EPR steering Inequalities**

To show the universality of our method, we construct linear EPR steering inequality based our framework, which is of the form \[43, 57\]

\[
S_N = \frac{1}{N} \sum_{x=1}^N a^{(x)}(\mathcal{B}_x) \leq \mathcal{B}_N. \tag{A.29}
\]

Here the quantity \( S_N \) stands for the steering parameter for \( N \) measurement settings and \( \mathcal{B}_N \) is the shape bound for \( S_N \). On the other hand, the linear combinations of our QFGURs with coefficients \( a^{(x)} \) and \( b^{(x)} \), involving statistics collected
from an experiment with $N$ measurement settings for each side are
\[
\left\{ a^{(s)} b^{(s)} \sum_{\lambda}(\lambda) p_\lambda (a^{(s)}|x) q_{\sigma_\lambda} (b^{(s)}|x) \right\} \geq \lambda, a, b \right) \}, \tag{A.30}
\]
clearly it may not be a convex combination of our QFGURs. If we make no assumption that Alice’s announcement $a^{(s)}$ is derived from a real quantum measurement, the sum of our combinations forms linear EPR steering inequalities is
\[
\frac{1}{N} \sum_{i=1}^{N} \sum_{b=1}^{d} \sum_{A} a^{(s)} b^{(s)} p (A) p_\lambda (a^{(s)}|x) q_{\sigma_\lambda} (b^{(s)}|x)
\]
\[
= \frac{1}{N} \sum_{i=1}^{N} a^{(s)} (B_i)_{\sigma_\lambda}, \tag{A.31}
\]
where $B_i = \sum b b^{(s)}$. With this, we rewrite the left hand of Eq. (A.29).

Demonstrating the product of accommodation coefficients, i.e. $a^{(s)} b^{(s)}$, in decreasing order and denote it as $C$
\[
C = \left( a^{(s)} b^{(s)} \right)^{1/2}, \tag{A.32}
\]
and the length of vector $C$ is $N d$ since for each measurement $x$, Alice only declares one outcome $a^{(s)}$ while $b^{(s)}$ has $d$ different possibilities in each round of the measurement. Now it is easy to construct a bound from our QFGUR
\[
S_N \leq C \cdot W^B, \tag{A.33}
\]
simply from the fact $0 \leq p_\lambda (a^{(s)}|x) \leq 1$. Any violation of Eq. (A.33) implies the steerability from Alice to Bob.

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[1] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Phys. 43, 172 (1927).
[2] E. H. Kennard, Zur quantenmechanik einfacher bewegungstypen, Z. Phys. 43, 326 (1927).
[3] H. Weyl, Gruppentheorie und Quantenmechanik, Hirzel Leipzig, (1930).
[4] H. P. Robertson, The uncertainty principle, Phys. Rev. 34, 163 (1929).
[5] E. Schrödinger, Über die kraftefreie bewegung in der relativistischen quantenmechanik, Ber. Kgl. Akad. Wiss. Berlin 24, 296 (1930).
[6] L. Maccione and A. K. Pati, Stronger uncertainty relations for all incompatible observables, Phys. Rev. Lett. 114, 039902 (2015).
[7] Y. Xiao, N. Jing, X. Li-Jost, and S.-M. Fei, Weighted uncertainty relations, Sci.Rep. 6, 23201 (2016).
[8] Y. Xiao, C. Guo, F. Meng, N. Jing, and M.-H. Yung, Incompatibility of observables as state-independent bound of uncertainty relations, [arXiv:1706.05650].
[9] H. de Guise, L. Maccone, B. C. Sanders, and N. Shukla, State-independent preparation uncertainty relations, [arXiv:1804.06794].
[10] I. Białynicki-Birula and J. Mycielski, Uncertainty relations for information entropy in wave mechanics, Commun. Math. Phys. 44, 129 (1975).
[11] D. Deutsch, Uncertainty in quantum measurements, Phys. Rev. Lett. 50, 631 (1983).
[12] M. H. Partovi, Entropic formulation of uncertainty for quantum measurements, Phys. Rev. Lett. 50, 1883 (1983).
[13] K. Kraus, Complementary observables and uncertainty relations, Phys. Rev. D 35, 3070 (1987).
[14] H. Maassen and J. B. M. Uffink, Generalized entropic uncertainty relations, Phys. Rev. Lett. 60, 1103 (1988).
[15] I. D. Ivanovic, An in equality for the sum of entropies of unbi ased quantum measurements, J. Phys. A 25, L363 (1992).
[16] J. Sánchez, Entropic uncertainty and certainty relations for complementary observables, Phys. Lett. A 173, 233 (1993).
[17] M. A. Ballester and S. Wehner, Entropic uncertainty relations and locking: tight bounds for mutually unbiased bases, Phys. Rev. A 75, 022319 (2007).
[18] S. Wu, S. Yu, and K. Mølmer, Entropic uncertainty relation for mutually unbiased bases, Phys. Rev. A 79, 022104 (2009).
[19] M. H. Partovi, Majorization formulation of uncertainty in quantum mechanics, Phys. Rev. A 84, 052117 (2011).
[20] Y. Huang, Entropic uncertainty relations in multidimensional position and momentum spaces, Phys. Rev. A 83, 052124 (2011).
[21] M. Tomamichel and R. Renner, Uncertainty relation for smooth entropies, Phys. Rev. Lett. 106, 110506 (2011).
[22] P.J. Coles, R. Colbeck, L. Yu, and M. Zwolak, Uncertainty relations from simple entropic properties, Phys. Rev. Lett. 108, 210405 (2012).
[23] P. J. Coles and M. Piani, Improved entropic uncertainty relations and information exclusion relations, Phys. Rev. A 89, 022112 (2014).
[24] Y. Xiao, N. Jing, S.-M. Fei, T. Li, X. Li-Jost, T. Ma, and Z.-X. Wang, Strong entropic uncertainty relations for multiple measurements, Phys. Rev. A 93, 042125 (2016).
[25] Y. Xiao, N. Jing, S.-M. Fei, and X. Li-Jost, Improved uncertainty relation in the presence of quantum memory, J. Phys. A 49, 49LT01 (2016).
[26] Y. Xiao, N. Jing, and X. Li-Jost, Uncertainty under quantum measures and quantum memory, Quantum Inf. Proc. 16, 104 (2017).
[27] V. Narasimhachar, A. Poostindouz, and G. Gour, Uncertainty, joint uncertainty, and the quantum uncertainty principle, New J. Phys. 18 033019 (2016).
[28] A. W. Marshall, I. Olkin and B. C. Arnold, Inequalities: Theory of Majorization and Its Applications, Springer Series in Statistics, (2011).
[29] S. Friedland, V. Gheorghiu, and G. Gour, Universal uncertainty relations, Phys. Rev. Lett. 111, 230401 (2013).
[30] Z. Puchala, Ł. Rudnicki, and K. Życzkowski, Majorization entropic uncertainty relations, J. Phys. A 46, 272002 (2013).
[31] Ł. Rudnicki, Z. Puchala, and K. Życzkowski, Strong majorization entropy uncertainty relations, Phys. Rev. A 89, 052115 (2014).
[32] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, Rev. Mod. Phys. 89, 015002 (2017).
[33] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, The uncertainty principle in the presence of quantum memory, Nat. Phys. 6, 1734 (2010).
[34] G. Brennen, E. Giacobino, and C. Simon, Focus on Quantum Memory, New J. Phys. 17, 050201 (2015).
[35] J. Oppenheim, and S. Wehner, The uncertainty principle determines the nonlocality of quantum mechanics, Science 330, 1072 (2010).
[36] Z.-A. Jia, Y.-C. Wu, and G.-C. Guo, Characterizing nonlocal correlations via universal uncertainty relations, Phys. Rev. A 96, 032122 (2017).
[37] A. Riccardi, C. Macchiavello, and L. Maccone, Multiparticle steering inequalities based on entropic uncertainty relations, Phys. Rev. A 97, 052307 (2018).
[38] K. Wang, N. Wu, and F. Song, Entanglement detection via direct-sum majorization uncertainty relations, arXiv:1807.02236.
[39] E. Schrödinger, Discussion of probability relations between separated systems, Proc. Cambridge Philos. Soc. 31, 555 (1935).
[40] E. Schrödinger, Probability relations between separated systems, Proc. Cambridge Philos. Soc. 32, 446 (1936).
[41] A. Einstein, B. Podolsky, and N. Rosen, Can quantum mechanical description of physical reality be considered complete?, Phys. Rev. 47, 777 (1935).
[42] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox, Phys. Rev. Lett. 98, 140402 (2007).
[43] X. Deng, Y. Xiang, C. Tian, G. Adesso, Q. Y. He, Q. Gong, X. Su, C. Xie, and K. Peng, Demonstration of monogamy relations for Einstein-Podolsky-Rosen steering in Gaussian cluster states, Phys. Rev. Lett. 118, 230501 (2017).
[44] V. Händchen, T. Eberle, S. Steinlechner, A. Samblowski, T. Franz, R. F. Werner, and R. Schnabel, Observation of one-way Einstein-Podolsky-Rosen steering, Nat. Photonics 6, 596 (2012).
[45] S. Wollmann, N. Walk, A. J. Bennett, H. M. Wiseman, and G. J. Pryde, Observation of genuine one-way Einstein-Podolsky-Rosen steering, Phys. Rev. Lett. 116, 160403 (2016).
[46] K. Sun, X. J. Ye, J. S. Xu, X. Y. Xu, J. S. Tang, Y. C. Wu, J. L. Chen, C. F. Li, and G. C. Guo, Experimental quantification of asymmetric Einstein-Podolsky-Rosen steering, Phys. Rev. Lett. 116, 160404 (2016).
[47] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: security, feasibility, and the connection with steering, Phys. Rev. A 85, 010301 (2012).
[48] T. Gehring, V. Händchen, J. Duhme, F. Furrer, T. Franz, C. Pacher, R. F. Werner, and R. Schnabel, Implementation of continuous-variable quantum key distribution with composable and one-sided-device independent security against coherent attacks, Nat. Commun. 6, 8795 (2015).
[49] N. Walk, S. Hosseini, J. Geng, O. Thearle, J. Y. Haw, S. Armstrong, S. M. Assad, J. Janousek, T. C. Ralph, T. Symul, H. M. Wiseman, and P. K. Lam, Experimental demonstration of Gaussian protocols for one-sided device-independent quantum key distribution, Optica 3, 634 (2016).
[50] S. Armstrong, M. Wang, R. Y. Teh, Q. H. Gong, Q. Y. He, J. Janousek, H. A. Bachor, M. D. Reid, and P. K. Lam, Multiparticle Einstein-Podolsky-Rosen steering and genuine tripartite entanglement with optical networks, Nat. Phys. 11, 167 (2015).
[51] I. Kogias, Y. Xiang, Q. Y. He, and G. Adesso, Unconditional security of entanglement-based continuous-variable quantum secret sharing, Phys. Rev. A 95, 012315 (2017).
[52] Y. Xiang, I. Kogias, G. Adesso, and Q. Y. He, Multiparticle Gaussian steering: Monogamy constraints and quantum cryptography applications, Phys. Rev. A 95, 010101(R) (2017).
[53] A. Rutkowski, A. Buraczewski, P. Horodecki, and M. Stobińska, Quantum steering inequality with tolerance for measurement-setting errors: Experimentally feasible signature of unbounded violation, Phys. Rev. Lett. 118, 020402 (2017).
[54] See Supplemental Material for details of our theoretical proofs of the majorization inequalities, theorems 2, 4, corollaries 6, 7 and constructions of linear EPR steering inequalities. The Supplemental Material contains additional references [56, 57].
[55] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[56] Z. Puchała, Ł. Rudnicki, A. Krawiec, and K. Życzkowski, Majorization uncertainty relations for mixed quantum states, J. Phys. A 51, 175306 (2018).
[57] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Experimental EPR-steering using Bell-local states, Nat. Phys. 6, 1766 (2010).