Cross-over in Scaling Laws:
A Simple Example from Micromagnetics

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Abstract

Scaling laws for characteristic length scales (in time or in the model parameters) are both experimentally robust and accessible for rigorous analysis. In multiscale situations cross-overs between different scaling laws are observed. We give a simple example from micromagnetics. In soft ferromagnetic films, the geometric character of a wall separating two magnetic domains depends on the film thickness. We identify this transition from a Néel wall to an Asymmetric Bloch wall by rigorously establishing a cross-over in the specific wall energy.

1. Introduction

Many continuum systems in materials science display pattern formation. These patterns are characterized by one or several length scales. The scaling of these characteristic lengths in the material parameters and/or in time are usually an experimentally robust feature. These scaling laws, and their characterizing exponents, are of interest to theoretical physics since they express a certain universality. At the same time, scaling laws (rather than more detailed features) are amenable to heuristic and rigorous analysis and thus are a good test for the model and a challenge for mathematics.

Scaling laws and their exponents reflect a scale invariance. In a multiscale model, these scale invariances are broken and only approximately valid in certain parameter and/or time regimes. The cross-over between two scaling laws reflects a change in the dominant physical mechanisms. In studying cross-overs, theoretical analysis may have an advantage over numerical simulation which has to explore many parameter decades and thus has to cope with widely separated length scales.

Together with various collaborators, the author has analyzed scaling laws and their cross-overs in both static (variational) and dynamic models. The dynamic
The examples are:

- The branching of domains in uniaxial ferromagnets [1] (with R. Choksi and R. V. Kohn). Strongly uniaxial ferromagnets have only two favored magnetization directions (“up” and “down”). The width of the corresponding domains decreases towards a sample surface perpendicular to the favored axis. We rigorously establish the scaling of the energy in the sample dimensions in support of this behavior. To leading order, the micromagnetic model behaves like a three-dimensional analogue of the Kohn-Müller [10] model for twin branching.

- The period of cross-tie walls in ferromagnetic films [2] (with A. DeSimone, R. V. Kohn and S. Müller). Cross-tie walls are transition layers between domains in ferromagnetic films. They display a periodic structure in the tangential direction. The experimentally observed scaling of the period in the material parameters is not well-understood [9]. In this paper, we present a combination of heuristic and rigorous analysis which reproduces the experimental scaling and thus identifies the relevant mechanism.

- The rate of capillarity-driven spreading of a thin droplet [6] (with L. Giacomelli). Here, the starting point is the lubrication approximation. The scale invariant version of the model is ill-posed and has to be regularized near the contact line, e.g. through allowing finite slippage. In this paper, we rigorously derive a scaling law for the spreading of the droplet in an intermediate time regime. This scaling law depends only logarithmically on the length scale introduced by the regularization, in agreement with a conjecture of de Gennes [5].

- The first-order correction to the Lifshitz-Slyozov-Wagner theory for Ostwald ripening [7] (with A. Hönig and B. Niethammer). Ostwald ripening describes the late stage of spinodal decomposition in an off-critical mixture (volume fraction of one phase $\phi \ll 1$). The minority phase then consists of several particles immersed in a matrix of the majority phase. The particles are approximately spherical and don’t move—the Lifshitz–Slyozov–Wagner theory describes the evolution of the radii distribution. There is a major interest in identifying the next-order correction term in $\phi$. We rigorously show that there is a cross-over in the correction term from $\phi^{1/3}$ to $\phi^{1/2}$ depending on the system size.

Our method to rigorously analyze these scaling laws in a multiscale model is based on relating integral quantities (energies, average length scales, dissipation rates...). It is different from the more local method of matched asymptotic expan-
Cross-over in Scaling Laws: A Simple Example from Micromagnetics

In particular, it differs from the latter by the absence of a specific Ansatz. In order to relate the integral quantities in our Ansatz-free approach, we need interpolation inequalities. These interpolation inequalities encode the competition of the dominant physical mechanisms in a scale-invariant fashion (e.g. the competition between driving energetics and limiting dissipation or between bulk and surface energy). Hence tools from pure analysis are here employed in a more applied context.

In order to illustrate this set of ideas, we present a simple application.

2. An example from micromagnetics

According to the well-accepted micromagnetic model, the experimentally observed ground-state of the magnetization \( m \) is the minimizer of a variational problem. We are interested in transition layers (“walls”) between domains in a film of thickness \( t \) in the \((x_1, x_2)\)-plane. We assume that the in-plane axis \( m_2 \) is favored by the crystalline anisotropy so that domains of magnetization \( m = (0, 1, 0) \) or \( m = (0, -1, 0) \) form. In order to avoid “magnetic poles”, the walls separating such domains are parallel to the \( x_2 \)-axis. We are interested in their specific energy per unit length in \( x_2 \)-direction. Hence the admissible magnetizations \( m \) are \( x_2 \)-independent and connect the two end-states

\[
m = m(x_1, x_3) \in S^2 \quad \text{for} \quad (x_1, x_3) \in \Omega := (-\infty, \infty) \times (-\frac{t}{2}, \frac{t}{2})
\]

and \( \lim_{x_1 \to \pm \infty} m_2(x_1, x_3) = \pm 1 \). (2.1)

The specific energy, which is to be minimized, is given by

\[
E(m) = d^2 \int_{\Omega} |\nabla m|^2 \, d^2x + Q \int_{\Omega} (m_1^2 + m_3^2) \, d^2x + \int_{\mathbb{R}^2} |\nabla u|^2 \, d^2x,
\]

(2.2)

where \( \nabla \) refers to the variables \( x = (x_1, x_3) \). Here the first term is the “exchange energy”, the second term comes from crystalline anisotropy and favors the \( m_2 \)-axis. The last term is the energy of the stray-field \( h_s = -\nabla u \) determined by the static Maxwell equations

\[
\nabla \times h_s = 0 \quad \text{and} \quad \nabla \cdot (h_s + m) = 0,
\]

which are conveniently expressed in variational form for the potential \( u \)

\[
\int_{\Omega} m \cdot \nabla \zeta \, d^2x = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \zeta \, d^2x \quad \text{for all} \quad \zeta \in C^\infty_0(\mathbb{R}^2).
\]

We see that both “volume charges” (\( \nabla \cdot m \) in \( \Omega \)) and “surface charges” (\( m_3 \) on \( \partial \Omega \)) generate the field \( h_s \) and thus are penalized. Since the energy density, i.e. \( |\nabla u|^2 \), depends on \( m \) through (2.3), the problem is non-local. The constraint of unit length, see (2.1), makes the variational problem nonconvex.

The model is already partially non-dimensionalized: The magnetization \( m \) and the field \( -\nabla u \) are dimensionless, but length is still dimensional. In particular, \( d \) has dimensions of length (the “exchange length”) and \( Q \) is dimensionless (the “quality
factor”). Hence the model has two intrinsic length scales (material parameters), namely \(d\) and \(d/Q^2\), and one extrinsic length scale (sample geometry), namely \(t\). Despite its simplicity, it is an example of a multiscale model and we expect different regimes depending on the two nondimensional parameters \(Q\) and \(t/d\).

We will focus on the most interesting regime of “soft” materials (i.e. with low crystalline anisotropy) and thicknesses \(t\) close to the exchange length \(d\)

\[
Q \ll 1 \quad \text{and} \quad Q \ll (t/d)^2 \ll Q^{-1}.
\]

(2.4)

Numerical simulation suggest a cross-over within this range [9, Chapter 3.6, Fig. 3.81):

- For thin films: “Néel walls” (see [9, Chapter 3.6 (C)]), whose geometry is asymptotically characterized by

\[
\frac{\partial m}{\partial x_3} = 0 \quad \text{and} \quad m_3 = 0 \implies m = (\cos \theta(x_1), \sin \theta(x_1), 0).
\]

(2.5)

- For thick films: “Asymmetric Bloch walls” (see [9, Chapter 3.6 (D)]), whose geometry is asymptotically characterized by

\[
-\nabla u = 0 \implies \nabla \cdot m = 0 \text{ in } \Omega \text{ and } m_3 = 0 \text{ on } \partial \Omega \\
\implies (m_1, m_3) = \left(-\frac{\partial \psi}{\partial x_3}, \frac{\partial \psi}{\partial x_1}\right) \text{ for a } \psi \text{ with } \psi = 0 \text{ on } \partial \Omega.
\]

(2.6)

This cross-over in the wall geometry is reflected by a cross-over in the scaling of the specific wall energy \(E\). Our proposition rigorously captures this cross-over in energy.

**Proposition 1** In the regime (2.4) we have

\[
\min_{m \text{ satisfies } (2.4)} E(m) \sim \begin{cases} 
\frac{d^2}{\ln Q t^2} & \text{for } (t/d)^2 \gtrsim \ln \frac{1}{Q} \\
\frac{t^2}{\ln Q t^2} & \text{for } (t/d)^2 \lesssim \ln \frac{1}{Q}
\end{cases}
\]

(2.7)

By \(>\), \(<\) we mean \(\geq\) resp. \(\leq\) up to a generic universal constant and \(\sim\) stands for both \(\geq\) and \(<\). This scaling qualitatively agrees with the numerical study of the energy cross-over in the thickness \(1\) given in [9, Fig 3.79].

**Upper** bounds are proved by construction. Here we make the Ansatz (2.5), resp. (2.6), and let ourselves be inspired by the physics literature for the details of the construction. The matching lower bound in (2.7) states that one cannot beat the Ansatz—at least in terms of energy scaling—by relaxing the geometry assumptions (2.5) or (2.6). Therefore Proposition 1 is a validation of the predicted cross-over in the geometry. We call this type of analysis *Ansatz-free lower bounds*.

\(^1\)the x-axis corresponds to \(\frac{t}{d}\), the y-axis to \(\frac{Q}{d}\), and \(Q = 0.00025\)
3. Proof

The upper bound in Proposition 1 comes from the following two lemmas. We only sketch their proof since our main focus is on lower bounds.

**Lemma 1** For $(\frac{t}{d})^2 \ll Q^{-1}$ there exists an $m$ of the form (2.6) with
\[ E(m) \sim d^2. \]  

**Lemma 2** For $(\frac{t}{d})^2 \gg Q$ there exists an $m$ of the form (2.5) with
\[ E(m) \sim t^2 \ln^{-1} \frac{t^2}{Qd^2}. \]

For the lower bound we need to estimate the components $m_1$ and $m_3$ by $E$.

**Lemma 3** We have for any $m$ satisfying (2.1)
\[ \int_\Omega m_3^2 d^2 x \leq \left( 1 + \left( \frac{t}{d} \right)^2 \right) E(m). \]  

**Lemma 4** In the regime $(\frac{t}{d})^2 \gg Q$ we have for any $m$ satisfying (2.4)
\[ \sup_{x_1 \in (-\infty, \infty)} m_1^2(x_1) \leq \left( \frac{1}{t^2} \ln \frac{t^2}{Qd^2} + \frac{1}{d^2} \right) E(m). \]  

**Proof of Lemma 1** The construction is due to Hubert [8]. We nondimensionalize length by $t$, i.e. $t = 1$. One can construct a smooth $\psi: \Omega \rightarrow R$ with
\[ |\nabla \psi|^2 \leq 1 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial \Omega \text{ and for } |x_1| \gg 1, \]
such that there exists a curve $\gamma \subset \Omega$ with
\[ \gamma \text{ connects } (0, -\frac{1}{2}) \text{ to } (0, \frac{1}{2}) \text{ and } |\nabla \psi|^2 = 1 \text{ on } \gamma. \]

In line with the Ansatz (2.6), we define $m: \Omega \rightarrow S^2$ via
\[ (m_1, m_3) = \left( -\frac{\partial \psi}{\partial x_3}, \frac{\partial \psi}{\partial x_1} \right), \quad m_2 = \left\{ \begin{array}{ll} - & \sqrt{1 - |\nabla \psi|^2} \left\{ \begin{array}{ll} \text{left} & \text{right} \end{array} \right. \end{array} \right\} \text{ of } \gamma. \]

Indeed, one possible recipe is to start from $\psi(x) = \frac{1}{2} - |x|$ and to modify $\psi$ outside of a neighborhood of the curve $\gamma = \left\{ (\frac{1}{2} \sqrt{\frac{1}{4} - x_2^2}, x_2) \mid x_2 \in [-\frac{1}{2}, \frac{1}{2}] \right\}$.
Only exchange and anisotropy contribute to the energy:

\[ E(m) \sim d^2 + Q, \]

which turns into \( \text{K8} \) in the regime under consideration.

**Proof of Lemma 2** Making the Ansatz \( \text{K2} \), the energy simplifies to

\[
E(m) = d^2 t \int_{-\infty}^{\infty} \frac{dm}{dx_1} d^2 x_1 + Q t \int_{-\infty}^{\infty} m_1^2 dx_1 + \int_{R^2} |\nabla u|^2 d^2 x \tag{3.12}
\]

\[
\leq t^2 \left\{ \frac{d^2}{t} \int_{-\infty}^{\infty} \frac{1}{1 - m_1^2} (\frac{dm}{dx_1})^2 dx_1 + \frac{Q}{t} \int_{-\infty}^{\infty} m_1^2 dx_1 + \int_{R^2} |\nabla U|^2 d^2 x \right\},
\]

where \( U \) is the harmonic extension \( \text{K3} \) of \( m_1 \) from \( \{x_3 = 0\} \) onto \( \mathbb{R}^2 \). Hence \( \text{K3,2} \) holds for any extension \( U \) of \( m_1 \). We now have to construct \( U \) such that its restriction \( m_1 \) satisfies \( m_1^2(0) = 1 \) in order to allow for the sign change of \( m_2 \). \( \int_{\mathbb{R}^2} |\nabla U|^2 d^2 x \)

just fails to control the \( L^\infty \)-norm of \( U \) and thus of \( m_1 \)—the counterexample involves a logarithm which we also use in this construction. The logarithm is cut off at the length scales \( \frac{d^2}{t} \ll \frac{1}{t} \).

\[
U(x) = \ln^{-1} \frac{Q d^2}{t^2} \ln \sqrt{\min\left(\frac{Q |x|}{t} + \left(\frac{Q d^2}{t^2}\right)^2, 1\right)}.
\]

An elementary calculation shows \( \text{K5} \) for \( m_1(x_1) = U(x_1, 0) \). A more detailed analysis of the reduced variational problem \( \text{K12} \) is in \( \text{K4,13} \).

**Proof of Lemma 3** We rewrite \( \text{K3} \) as

\[
\int_{\Omega} m_3 \frac{\partial \zeta}{\partial x_3} d^2 x = \int_{R^2} \frac{\partial u}{\partial x_3} \frac{\partial \zeta}{\partial x_3} d^2 x + \int_{R^2} \frac{\partial u}{\partial x_1} \frac{\partial \zeta}{\partial x_3} d^2 x + \int_{\Omega} \frac{\partial m_1}{\partial x_1} \zeta d^2 x \tag{3.13}
\]

and choose the test function

\[
\zeta(x_1, x_3) = m_3(x_1) \eta(\hat{x}_3) \quad \text{where} \quad x_3 = t \hat{x}_3
\]

and \( \eta \in C^\infty_0(\mathbb{R}) \) is chosen such that \( \frac{\partial \eta}{\partial \hat{x}_3}(\hat{x}_3) = 1 \) for \( \hat{x}_3 \in (-\frac{1}{2}, \frac{1}{2}) \) in order to have

\[
\frac{\partial \zeta}{\partial x_3}(x_1, x_3) = \frac{1}{t} m_3(x_1) \frac{\partial \eta}{\partial \hat{x}_3}(\hat{x}_3) = \frac{1}{t} m_3(x_1) \quad \text{for} \quad x_3 \in (-\frac{t}{2}, \frac{t}{2}).
\]

Hence the term on the l. h. s. of \( \text{K3,13} \) turns into

\[
\int_{\Omega} m_3 \frac{\partial \zeta}{\partial x_3} d^2 x = \int_{-\infty}^{\infty} m_3^2 dx_1 \tag{3.14}
\]

and the first term on the r. h. s. of \( \text{K3,13} \) is estimated as follows

\[
\left| \int_{R^2} \frac{\partial u}{\partial x_3} \frac{\partial \zeta}{\partial x_3} d^2 x \right| \leq \left( \int_{R^2} \left(\frac{\partial u}{\partial x_3}\right)^2 d^2 x \frac{1}{t} \int_{-\infty}^{\infty} m_3^2 dx_1 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{1}{t} E \int_{-\infty}^{\infty} m_3^2 dx_1 \right)^{\frac{1}{2}}. \tag{3.15}
\]

\( \text{K3,13} \)

- The inequality \( \int_{R^2} |\nabla u|^2 d^2 x \leq \int_{R^2} |\nabla U|^2 d^2 x \) can be seen by expressing both integrals in terms of the Fourier transform \( \hat{m}_1(k_1) \) of \( m_1(x_1) \).
The two remaining terms are also easily dominated:

\[
\left| \int_{R^2} \frac{\partial u}{\partial x_1} \frac{\partial \zeta}{\partial x_1} \ d^2x \right| \leq \left( \int_{R^2} \left( \frac{\partial u}{\partial x_1} \right)^2 d^2x \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} \left( \frac{\partial m_3}{\partial x_1} \right)^2 dx_1 \right)^{\frac{1}{2}} \leq \frac{1}{d} E, \quad (3.16)
\]

\[
\left| \int_{R^2} \frac{\partial m}{\partial x_1} \zeta \ d^2x \right| \leq \left( \int_{R^2} \left( \frac{\partial m}{\partial x_1} \right)^2 d^2x \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} \left( \frac{\partial m_3}{\partial x_1} \right)^2 dx_1 \right)^{\frac{1}{2}} \leq \left( \frac{t}{d^2} E \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} m_3^2 \ dx_1 \right)^{\frac{1}{2}}. \quad (3.17)
\]

Collecting (3.14)–(3.17) and using the Cauchy-Schwarz inequality gives

\[
\int_{-\infty}^{\infty} m_3^2 \ dx_1 \leq \left( \frac{1}{t} + \frac{1}{d} + \frac{t}{d^2} \right) E \leq \left( \frac{1}{t} + \frac{t}{d^2} \right) E. \quad (3.18)
\]

On the other hand, we use Poincaré inequality in the \( x_3 \)-direction which we integrate over \( x_1 \in (-\infty, \infty) \)

\[
\int_\Omega (m_3 - \bar{m}_3)^2 \ dx_2 \ d^2x \leq t^2 \int_\Omega \left( \frac{\partial m_3}{\partial x_3} \right)^2 d^2x \leq \left( \frac{t}{d} \right)^2 E. \quad (3.19)
\]

Now (3.18) and (3.19) combine as desired into (3.10).

**Proof of Lemma 4.** In the first step we establish for \( 0 < \rho \ll \ell \) and \( 0 \leq \xi_1 - \tilde{\xi}_1 \leq \ell \)

\[
\left| \frac{1}{\rho} \int_{\xi_1}^{\xi_1 + \rho} m_1 \ dx_1 - \frac{1}{\rho} \int_{\tilde{\xi}_1 - \rho}^{\tilde{\xi}_1} m_1 \ dx_1 \right|^2 \leq \left( \frac{1}{t} \ln \frac{\ell}{\rho} + \frac{1}{\rho t} \right) E. \quad (3.20)
\]

In order to establish (3.20), we construct an appropriate test function \( \zeta \) for \( (2.3) \).

We first define \( \zeta \) on the strip \( \mathbb{R} \times (-\frac{\ell}{2}, \frac{\ell}{2}) \) as piecewise linear

\[
\zeta(x_1, x_3) = \begin{cases} 
0 & \xi_1 + \rho \leq x_1 \\
\frac{1}{\rho} (\xi_1 - x_1 + \rho) & \xi_1 \leq x_1 \leq \xi_1 + \rho \\
\frac{1}{\rho} (x_1 - \tilde{\xi}_1 + \rho) & \xi_1 - \rho \leq x_1 \leq \tilde{\xi}_1 \\
1 & x_1 \leq \xi_1 - \rho \end{cases}.
\]

(3.21)

\( \zeta \) is just defined such that

\[
\int_\Omega m \cdot \nabla \zeta \ d^2x = -\frac{t}{\rho} \int_{\xi_1}^{\xi_1 + \rho} m_1 \ dx_1 + \frac{t}{\rho} \int_{\tilde{\xi}_1 - \rho}^{\tilde{\xi}_1} m_1 \ dx_1. \quad (3.22)
\]

For the r. h. s. of (2.3) we have to extend \( \zeta \) onto all of \( \mathbb{R}^2 \). We harmonically extend \( \zeta \) on the upper and lower half-plane \( \mathbb{R} \times (\frac{\ell}{2}, +\infty) \) resp. \( \mathbb{R} \times (-\infty, -\frac{\ell}{2}) \). We claim

\[
\int_{R^2} |\nabla \zeta|^2 \ d^2x \leq \frac{\ell}{\rho} + \frac{t}{\rho}. \quad (3.23)
\]
This yields the following estimate of the r. h. s. of (2.3)

\[
\left| \int_{\mathbb{R}^2} \nabla u \cdot \nabla \zeta \, d^2x \right| \leq \left( \int_{\mathbb{R}^2} |\nabla u|^2 \, d^2x \int_{\mathbb{R}^2} |\nabla \zeta|^2 \, d^2x \right)^{\frac{1}{2}} \\
\leq \left( E \left( \ln \frac{\ell}{\rho} + \frac{t}{\rho} \right) \right)^{\frac{1}{2}}.
\]

(3.24)

Obviously (3.22) & (3.24) yields (3.20).

We now argue in favor of (3.23). On the strip \( R \times (-\frac{t}{2}, \frac{t}{2}) \) we have

\[
\int_{R \times (-\frac{t}{2}, \frac{t}{2})} |\nabla \zeta|^2 \, d^2x \overset{\text{(3.21)}}{=} 2t \rho \left( \frac{1}{\rho} \right)^2 \sim \frac{t}{\rho}.
\]

(3.25)

The Dirichlet integral of the harmonic extension is estimated in terms of its boundary value as follows

\[
\int_{R \times \left( \frac{-t}{2}, \frac{t}{2} \right)} |\nabla \zeta|^2 \, d^2x \sim \int_0^\infty \frac{1}{x^2} \int_{-\infty}^\infty (\zeta(x_1 + x_3, \frac{t}{2}) - \zeta(x_1, \frac{t}{2}))^2 \, dx_1 \, dx_3,
\]

see [12, Théorème 9.4, Théorème 10.2]. Since

\[
\int_{-\infty}^\infty (\zeta(x_1 + x_3, \frac{t}{2}) - \zeta(x_1, \frac{t}{2}))^2 \, dx_1 \overset{\text{(3.21)}}{=} \begin{cases} \frac{\ell}{x_3} & x_3 \rho \leq \ell \\
\frac{x_3}{\rho} & x_3 \rho \geq \ell \\
\frac{x_3}{\rho} & x_3 \leq \rho \end{cases}
\]

this yields

\[
\int_{R \times \left( \frac{-t}{2}, \frac{t}{2} \right)} |\nabla \zeta|^2 \, d^2x \leq \ln \frac{\ell}{\rho}.
\]

(3.26)

Now (3.25) and (3.26) combine into (3.23).

In the second step, we establish for \( \ell \gg \frac{d^2}{t} \) and \( 0 \leq \xi_1 - \tilde{\xi}_1 \leq \ell \)

\[
|\overline{m}_1(\xi_1) - \overline{m}_1(\tilde{\xi}_1)|^2 \leq \left( \frac{1}{t^2} \ln \frac{\ell t}{d^2} + \frac{1}{d^2} \right) E.
\]

(3.27)

For this, we observe that

\[
\left| \frac{1}{\rho} \int_{\xi_1}^{\xi_1 + \rho} \overline{m}_1(x_1) \, dx_1 - \overline{m}_1(\xi_1) \right|^2 \leq \rho \int_{-\infty}^\infty (\frac{d\overline{m}_1}{dx_1})^2 \, dx_1 \\
\leq \frac{\rho}{t} \int_{\Omega} \left( \frac{\partial \overline{m}_1}{\partial x_1} \right)^2 \, d^2x \leq \frac{\rho}{d^2 t} E,
\]

so that together with (3.20) we obtain

\[
|\overline{m}_1(\xi_1) - \overline{m}_1(\tilde{\xi}_1)|^2 \leq \left( \frac{1}{t^2} \ln \frac{\ell}{\rho} + \frac{1}{\rho t} + \frac{\rho}{d^2 t} \right) E.
\]
We now balance the first and last term by choosing \( \rho = \frac{d^2}{\ell} \lesssim \ell \) and so obtain (3.27).

In the last step, we show (3.11) for \( \frac{t^2}{Qd^2} \gg 1 \). For this we observe that

\[
\int_{-\infty}^{\infty} \overline{m}_1^2 \, dx_1 \leq \frac{1}{t} \int_{\Omega} m_1^2 \, d^2 x \leq \frac{1}{Qt} E.
\]

Hence we obtain together with (3.27) for arbitrary \( \xi_1 \in (-\infty, \infty) \)

\[
\overline{m}_1(\xi_1)^2 \leq \frac{1}{\ell} \int_{\xi_1 - \frac{d}{2}}^{\xi_1 + \frac{d}{2}} \overline{m}_1^2 \, dx_1 + \frac{1}{\ell} \int_{\xi_1 - \frac{d}{2}}^{\xi_1 + \frac{d}{2}} (\overline{m}_1(\xi_1) - \overline{m}_1(x_1))^2 \, dx_1
\]

\[
\lesssim \left( \frac{1}{Qt} + \frac{1}{t^2} \ln \frac{\ell t}{d^2} + \frac{1}{d^2} \right) E.
\]

Choosing \( \ell = \frac{t}{d^2} \gg \frac{d^2}{t} \), we balance the two first terms and so obtain (3.11).

Proof of Proposition 1. It remains to establish the lower bound. For further reference we remark that by Poincaré’s inequality

\[
\int_{(-\frac{t}{2}, \frac{t}{2}) \times (-\frac{t}{2}, \frac{t}{2})} (m_i - \overline{m}_i(0))^2 \, d^2 x \lesssim t^2 \int_{\Omega} |\nabla m_i|^2 \, d^2 x \lesssim \left( \frac{t}{d} \right)^2 E. \tag{3.28}
\]

According to (2.1), we have in particular \( \lim_{x_1 \to \pm \infty} \overline{m}_2(x_1) = \pm 1 \) and thus there exists an \( \xi_1 \) with \( \overline{m}_2(\xi_1) = 0 \). W. l. o. g. we assume \( \xi_1 = 0 \) so that \( \overline{m}_2(0) = 0 \). According to (3.28) we obtain

\[
\int_{(-\frac{t}{2}, \frac{t}{2}) \times (-\frac{t}{2}, \frac{t}{2})} m_2^2 \, d^2 x \lesssim \left( \frac{t}{d} \right)^2 E. \tag{3.29}
\]

Furthermore, we have according to Lemma 3

\[
\int_{(-\frac{t}{2}, \frac{t}{2}) \times (-\frac{t}{2}, \frac{t}{2})} m_3^2 \, d^2 x \lesssim \left( 1 + \left( \frac{t}{d} \right)^2 \right) E. \tag{3.30}
\]

Since \( 1 - m_1^2 = m_2^2 + m_3^2 \), the estimates (3.29) & (3.30) imply

\[
\int_{(-\frac{t}{2}, \frac{t}{2}) \times (-\frac{t}{2}, \frac{t}{2})} (1 - m_1^2) \, d^2 x \lesssim \left( 1 + \left( \frac{t}{d} \right)^2 \right) E.
\]

In view of (3.28), this localizes to

\[
1 - \overline{m}_1(0)^2 \lesssim \left( \frac{1}{t^2} + \frac{1}{d^2} \right) E. \tag{3.31}
\]

On the other hand, we have by Lemma 4

\[
\overline{m}_1(0)^2 \lesssim \left( \frac{1}{t^2} \ln \frac{t^2}{Qd^2} + \frac{1}{d^2} \right) E \tag{3.32}
\]
provided \((\frac{1}{d^2})^2 \gg Q\). Combining (3.31) and (3.32), we obtain

\[
1 \lesssim \left( \frac{1}{t^2} \ln \frac{t^2}{Qd^2} + \frac{1}{d^2} + \frac{1}{t^2} \right) E \sim \left( \frac{1}{t^2} \ln \frac{t^2}{Qd^2} + \frac{1}{d^2} \right) E.
\] (3.33)

Since we have by elementary calculus that

\[
\frac{1}{t^2} \ln \frac{t^2}{Qd^2} \begin{cases} < \frac{1}{d^2} & \text{if } t^2 \gg Qd^2 \\ \gg \frac{1}{d^2} & \text{if } t^2 \ll Qd^2 \end{cases} \iff \ln \frac{1}{Q} \begin{cases} < \frac{1}{d^2} & \text{if } t^2 \gg Qd^2 \\ \gg \frac{1}{d^2} & \text{if } t^2 \ll Qd^2 \end{cases} \left( \frac{t}{d} \right)^2,
\]

is equivalent to the lower bound in (2.7).

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