A uniform Tauberian theorem in dynamic games

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Abstract. Antagonistic dynamic games including games represented in normal form are considered. The asymptotic behaviour of value in these games is investigated as the game horizon tends to infinity (Cesàro mean) and as the discounting parameter tends to zero (Abel mean). The corresponding Abelian-Tauberian theorem is established: it is demonstrated that in both families the game value uniformly converges to the same limit, provided that at least one of the limits exists. Analogues of one-sided Tauberian theorems are obtained. An example shows that the requirements are essential even for control problems.

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§1. Introduction

In this work we generalize the correspondence between Abel summation and Cesàro summation established by Hardy (see [1], for example) to game-theoretic settings. The integral version of Hardy’s result (see [2], §6.8, for example) can be formulated in the following way: for a bounded measurable function \( h \) the limit of its Cesàro mean

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T h(t) \, dt
\]

and the limit of its Abel mean

\[
\lim_{\lambda \to +0} \lambda \int_0^\infty e^{-\lambda t} h(t) \, dt,
\]

coincide if at least one of the two limits exists. Theorems of this kind are usually referred to as Abelian-Tauberian theorems (or just Tauberian theorems). In general, such theorems relate the asymptotic behaviour of the image of an integral transform with the asymptotic behaviour of the preimage. They find applications in a wide range of areas from probability theory to number theory; without meaning to give an exhaustive overview we only mention the references [3]–[6]. In what follows, by Tauberian theorems in game theory we always mean theorems that relate the asymptotic behaviour of values (the optimal means) if the integral cost is averaged by one of the formulae presented above.
Such theorems were originally applied to stochastic games with a finite number of states and actions: a theorem of this kind was employed (see [7]) to prove the existence of a limit value as the game horizon tends to infinity with the use of the corresponding result (see [8]) for games with discounted payoffs. For recent results concerning the asymptotic behaviour in various stochastic settings see [9].

In the study of continuous-time processes in control theory and related areas the limits of optimal means (as the game horizon tends to infinity and/or the discounting parameter tends to zero) have been examined many times: stochastic equations in [10], control with small parameter in [11], the asymptotic problem for Hamilton-Jacobi equations in [12], perturbation theory in [13], infinite horizon control problems in [14] and [15].

The first version (see [16]) of a Tauberian theorem for control problems was obtained in the ergodic case: the mean value as the discounting parameter tends to zero converges uniformly on the invariant compact set to a certain constant if and only if the mean value as the game horizon tends to infinity converges uniformly on the same set to the same constant. In the ergodic case such limits always exist (see [17]–[20]) and their value is also referred to as the Mañé critical value. In the ergodic case a Tauberian theorem was also established for differential games (see [21]). In the general case the game values may converge to a nonconstant function even in control problems related to the simplest situations (see [22]). Some recent results on the existence of such limit values in control problems and differential games can be found in [23] and [24].

Although the general Tauberian theorem for discrete-time controlled processes was obtained long ago (see [25]), it was not until recently that this theorem was carried over to continuous-time control problems (see [26]). The result obtained in [26] (see also the corollary in §7 below) is remarkable for it is established in the most general abstract setting: it is only required that the set of processes be closed under concatenation and that each restriction of an admissible process to a right-infinite interval be an admissible process as well. If under these conditions at least one of the optimal means (as the game horizon tends to infinity or as the discounting parameter tends to zero) converges uniformly (over the invariant set of initial positions) to a limit value, then the other optimal mean also converges to a limit value and these limit values coincide.

For differential games of general form a Tauberian theorem was established in [27] on the basis of iterations of nonanticipating operators (see [28]). A rather general approach suggested in [9] made it possible to generalize a Tauberian theorem to a wide class of discrete-time games; the most powerful tools here are nonexpansive operators on the phase space of the game and their fixed points as solutions to Bellman’s equation.

In this work we derive Tauberian theorems from the conditions that there exists a saddle point and each player has a uniformly optimal rule. As in [26], it is always assumed that the players may switch between the rules as the game goes on. Theorem 1 extends the result in [26] to a class of games represented in the normal form. Theorem 2 is proved for games of a more general form under the additional assumption that there exists a saddle point and the players have uniformly optimal rules.
§ 2. Antagonistic game in normal form

In what follows the symbol \( \triangleq \) is used to denote equality by definition.

Let \( T \triangleq \mathbb{R}_{\geq 0} \) and let there be given:

- a nonempty set \( \Omega \) (the set of states);
- a nonempty set \( K \) of mappings from \( T \) to \( \Omega \) (the set of processes);
- a running cost function \( g: \Omega \to [0, 1] \) for which the mappings \( T \ni t \mapsto g(z(t)) \in [0, 1] \) are Lebesgue measurable for any process \( z \in K \).

For any \( \tau \in T \) and \( z', z'' \in K \) satisfying the condition \( z'((\tau)) = z''(0) \) and for no other triples we define the concatenation \( z' \circ_\tau z'' \), which is also a mapping from \( T \) to \( \Omega \), in the following way:

\[
(z' \circ_\tau z'')(t) \triangleq \begin{cases} 
  z'(t), & 0 \leq t \leq \tau, \\
  z''(t - \tau), & t > \tau.
\end{cases}
\]  

(2.1)

Suppose that for any \( \omega \in \Omega \) nonempty sets \( \mathcal{L}(\omega) \) and \( \mathcal{M}(\omega) \) (the programs of the players) are specified. For these sets we define the sets \( \mathcal{L} \) and \( \mathcal{M} \) of possible selectors \( \Omega \ni \omega \to l(\omega) \in \mathcal{L}(\omega) \) and \( \Omega \ni \omega \to m(\omega) \in \mathcal{M}(\omega) \) (the rules of the players). Suppose that for each state \( \omega \in \Omega \) each pair \( (l, m) \in \mathcal{L} \times \mathcal{M} \) of rules of the players specifies a unique process \( z[\omega, l, m] \in K \) such that \( z[\omega, l, m](0) = \omega \).

In § 5 we prove the following Tauberian theorem for games represented in normal form.

**Theorem 1.** Suppose that for any positive \( \tau > 0 \) both the set \( \mathcal{L} \) and the set \( \mathcal{M} \) are endowed with a binary operation \( \sqcap_\tau \) and that for all \( l', l'' \in \mathcal{L}, m', m'' \in \mathcal{M} \) and \( \omega \in \Omega \) the following conditions are satisfied:

\[
\mathcal{L} = \{ l' \sqcap_\tau l'' \mid l', l'' \in \mathcal{L} \}, \quad \mathcal{M} = \{ m' \sqcap_\tau m'' \mid m', m'' \in \mathcal{M} \},
\]

(2.2)

\[
z[\omega, l' \sqcap_\tau l'', m' \sqcap_\tau m''](\tau) = z[\omega, l', m'](\tau),
\]

(2.3)

\[
z[\omega, l' \sqcap_\tau l'', m' \sqcap_\tau m''] = z[\omega, l', m'] \circ_\tau z[\omega, l', m'](\tau), l'', m''].
\]

(2.4)

Suppose that either the limits

\[
\lim_{T \to \infty} \sup_{l \in \mathcal{L}} \inf_{m \in \mathcal{M}} \frac{1}{T} \int_0^T g(z[\omega, l, m](t)) \, dt
\]

and

\[
\lim_{T \to \infty} \inf_{m \in \mathcal{M}} \sup_{l \in \mathcal{L}} \frac{1}{T} \int_0^T g(z[\omega, l, m](t)) \, dt
\]

exist, are uniform in \( \omega \in \Omega \) and are equal to each other, or the limits

\[
\limsup_{\lambda \uparrow 0} \sup_{l \in \mathcal{L}} \inf_{m \in \mathcal{M}} \lambda \int_0^\infty e^{-\lambda t} g(z[\omega, l, m](t)) \, dt
\]

and

\[
\liminf_{\lambda \downarrow 0} \inf_{m \in \mathcal{M}} \sup_{l \in \mathcal{L}} \lambda \int_0^\infty e^{-\lambda t} g(z[\omega, l, m](t)) \, dt
\]

exist uniformly in \( \omega \in \Omega \) and are equal to each other.

Then all these four limits exist uniformly in \( \omega \in \Omega \) and are equal one to another.
Let us discuss the conditions imposed on $\tau$.

The rule $l' \sqcap_\tau l''$ (as well as the rule $m' \sqcap_\tau m''$) corresponds to ‘switching’ at the instant $\tau$ from the rule $l'$ to $l''$ ($m'$ to $m''$). The conditions of the theorem assume that each player at any instant can switch from any rule to any other rule regardless of the opponent’s actions. Moreover, each rule is applicable for any initial condition $\omega$; in differential games, strategies possessing this property are called universal. Since, in contrast to $\sqcap_\tau$ (switching between the rules of the players), the concatenation of the trajectories $z' \circ_\tau z''$ is not always defined, in the hypothesis of the theorem we impose the condition (2.3), which guarantees the correctness of the rule on the right-hand side in (2.4).

By condition (2.4), based on the trajectory that has been realized at least up to the moment $\tau$ the second player cannot decide which particular rule, $l' \sqcap_\tau l''$ or $l'$, was applied by the first player; in dynamic games such a nonanticipatory requirement is fundamental. Moreover, it follows from (2.4) that by virtue of the rules $l' \sqcap_\tau l''$ and $m' \sqcap_\tau m''$, after switching, the strategies of the players may depend on the choice of $l'$ and $m'$ only through the state $z(\tau)$ observed at the moment of switching; in particular, the strategies do not depend on the history $(z(t))_{t<\tau}$ available at that moment. That the rules in $\mathcal{L}$ and $\mathcal{M}$ satisfy this Markovian property can always be ensured by passing to a wider space of states (for example, where ‘state’ = ‘current state’ + ‘the history realized by the moment’).

Condition (2.2) formalizes the requirement that any rule be representable in the form of switching between two other rules. In the proof of the Tauberian theorem for differential games such a structure was developed; see [27], Remark 3.2. In § 7 we give an example that illustrates that the assumption (2.2) in the conditions of the theorem is essential even for control problems.

§ 3. Abstract setting

3.1. Dynamic system. Consider, as in § 2, a nonempty set $\Omega$, a nonempty set $\mathbb{K}$ of mappings from $\mathbb{T}$ to $\Omega$, and a running cost $g: \Omega \mapsto [0, 1]$. Suppose, as before, that for each $z \in \mathbb{K}$ the mapping $t \mapsto g(z(t))$ is Lebesgue measurable.

For each point $\omega \in \Omega$ we also need the set of all processes starting at that point

$$\Gamma(\omega) \triangleq \{ z \in \mathbb{K} \mid z(0) = \omega \}. $$

A set $A \subset \mathbb{K}$ is called a rule if $A \cap \Gamma(\omega) \neq \emptyset$ for any initial state $\omega \in \Omega$.

Similarly to the concatenation $z' \circ_\tau z'': \mathbb{T} \to \Omega$ introduced above by means of formula (2.1) for any $\tau \in \mathbb{T}$ and $z', z'' \in \mathbb{K}$ satisfying the condition $z'(\tau) = z''(0)$, we can now also introduce ‘switching’ between any two admissible rules: for any moment $\tau > 0$ and any pair of rules $A', A'' \subset \mathbb{K}$ we define the rule

$$A' \sqcap_\tau A'' \triangleq \{ z' \circ_\tau z'' \mid z' \in A', z'' \in A'', z'(\tau) = z''(0) \} = \{ z' \circ_\tau z'' \mid z' \in A', z'' \in A'' \cap \Gamma(z'(\tau)) \}. \quad (3.1)$$

To simplify the calculations, in what follows we everywhere assume that

$$A \sqcap_\tau, A' \sqcap_\tau, A'' \triangleq (A \sqcap_\tau, A') \sqcap_\tau, A''.$$
3.2. Assumptions made of the rules. Let there be given a nonempty family \( \mathcal{A} \) of subsets of the set \( K \). It is assumed everywhere below that \( \mathcal{A} \) satisfies the following conditions.

3.2.1. Each \( A \in \mathcal{A} \) is a rule: for any \( \omega \in \Omega \) we have \( A \cap \Gamma(\omega) \neq \emptyset \).

3.2.2. \( \mathcal{A} \) is closed under \( \triangleleft \): for any two rules \( A', A'' \in \mathcal{A} \) and any positive \( \tau \) we have \( A' \triangleleft A'' \in \mathcal{A} \).

3.3. Lower value. Consider a two-player game with a payoff function \( c: K \to \mathbb{R} \). The first player seeks to maximize this payoff, the second player to minimize it. The first can choose any rule from a nonempty set \( \mathcal{A} \).

Let the first player be awareness-discriminated, that is, the rule that he chooses is known to the second player before the start of the game. Then it may be assumed that the game proceeds in the following way: given \( \omega \in \Omega \), the first player chooses a set \( A \in \mathcal{A} \), and after that the second player chooses a process \( z \in A \cap \Gamma(\omega) \) satisfying both the rule and the initial condition. Then the lower value has the form

\[
\forall^b[c](\omega) \triangleq \sup_{A \in \mathcal{A}} \inf_{z \in A \cap \Gamma(\omega)} c(z) \quad \forall \omega \in \Omega.
\]  

Since for any rule \( A \) it is assumed that \( A \cap \Gamma(\omega) \neq \emptyset \), this definition is consistent in the case when \( c \) is bounded.

Definition 1. For a positive \( \varepsilon \) the first player has an \( \varepsilon \)-optimal rule \( A \in \mathcal{A} \) if

\[
\inf_{z \in A} [c(z) - \forall^b[c](z(0))] \geq -\varepsilon.
\]

3.4. Upper value. Again, consider a two-player game with payoff \( c: K \to \mathbb{R} \). The first player still seeks to maximize this payoff, the second to minimize it. Now suppose that the second player lets the first player know his rule (chosen in a set \( \mathcal{B} \) of rules). Then the value of the corresponding game (upper value) has the form

\[
\forall^u[c](\omega) \triangleq \inf_{B \in \mathcal{B}} \sup_{z \in B \cap \Gamma(\omega)} c(z) \quad \forall \omega \in \Omega.
\]  

Definition 2. For a positive \( \varepsilon \) the second player has an \( \varepsilon \)-optimal rule \( B \in \mathcal{B} \) if

\[
\sup_{z \in B} [c(z) - \forall^u[c](z(0))] \leq \varepsilon.
\]

Fix some nonempty sets of rules \( \mathcal{A} \) and \( \mathcal{B} \) of the first and second player, respectively; then any bounded function \( c: K \to \mathbb{R} \) specifies the lower and upper value of the corresponding games by formulae (3.2) and (3.3).

Definition 3. A game with a payoff \( c: K \to \mathbb{R} \) has a saddle point if \( \forall^b[c](\omega) = \forall^u[c](\omega) \) for any \( \omega \in \Omega \).

3.5. The mean values. Now let us introduce the payoff function as the mean of the running cost \( g \) with the game horizon (Cesàro mean) and the mean of \( g \) with the discounting factor (Abel mean). For any \( z \in K \) define \( v_T(z) \) and \( w_\lambda(z) \) as follows:

\[
v_T(z) \triangleq \frac{1}{T} \int_0^T g(z(t)) \, dt, \quad w_\lambda(z) \triangleq \lambda \int_0^\infty e^{-\lambda t} g(z(t)) \, dt \quad \forall T, \lambda > 0.
\]  

(3.4)
Since the function \( g \) takes values in the interval \([0, 1]\), the definitions introduced above are consistent. The functions \( v_T \) and \( w_\lambda \) are treated below as the payoffs in the corresponding games. For all \( T, \lambda > 0 \) we obtain the upper and lower values
\[
\begin{align*}
\gamma^\sharp_T(\omega) &\triangleq \bigvee^\sharp[v_T](\omega) = \sup_{A \in \mathcal{A}} \inf_{z \in A \cap \Gamma(\omega)} v_T(z), \\
\gamma^\sharp_T(\omega) &\triangleq \bigvee^\sharp[v_T](\omega) = \inf_{B \in \mathcal{B}} \sup_{z \in B \cap \Gamma(\omega)} v_T(z), \\
\gamma^\sharp_\lambda(\omega) &\triangleq \bigvee^\sharp[w_\lambda](\omega) = \sup_{A \in \mathcal{A}} \inf_{z \in A \cap \Gamma(\omega)} w_\lambda(z), \\
\gamma^\sharp_\lambda(\omega) &\triangleq \bigvee^\sharp[w_\lambda](\omega) = \inf_{B \in \mathcal{B}} \sup_{z \in B \cap \Gamma(\omega)} w_\lambda(z).
\end{align*}
\] (3.5)

Note that for all bounded payoffs \( c \) and any \( \omega \in \Omega, \lambda, T > 0 \) we have the equalities
\[
\begin{align*}
1 - \gamma^\sharp[c](\omega) &= \sup_{B \in \mathcal{B}} \inf_{z \in B \cap \Gamma(\omega)} (1 - c(z)), \\
1 - \gamma^\sharp_T(\omega) &= \sup_{B \in \mathcal{B}} \inf_{z \in B \cap \Gamma(\omega)} \frac{1}{T} \int_0^T (1 - g(z(t))) \, dt, \\
1 - \gamma^\sharp_\lambda(\omega) &= \sup_{B \in \mathcal{B}} \inf_{z \in B \cap \Gamma(\omega)} \int_0^\infty \lambda e^{-\lambda t} (1 - g(z(t))) \, dt.
\end{align*}
\] (3.7)

Thus, for the upper value \( V \), payoff \( c \), and running cost \( g \) with the families \( \mathcal{A} \) and \( \mathcal{B} \) for the first and second player the difference \( 1 - V \) coincides with the lower value for payoff \( 1 - c \) and running cost \( 1 - g \) with the families of the rules \( \mathcal{B} \) and \( \mathcal{A} \) for the first and second player, respectively. For this reason, in what follows we formulate all assertions only for the lower value.

With a mapping \( U: \mathbb{R}_{>0} \times \Omega \to \mathbb{R} \) and positive numbers \( h, T, \lambda \) we also associate the payoffs
\[
\begin{align*}
\zeta^U_{h,T}(z) &\triangleq \frac{1}{T+h} \int_0^h g(z(t)) \, dt + \frac{T}{T+h} U_T(z(h)) \quad \forall z \in \mathbb{K} \tag{3.8}
\end{align*}
\]
and
\[
\begin{align*}
\xi^U_{h,\lambda}(z) &\triangleq \lambda \int_0^h e^{-\lambda t} g(z(t)) \, dt + e^{-\lambda h} U_\lambda(z(h)) \quad \forall z \in \mathbb{K}. \tag{3.9}
\end{align*}
\]

To formulate the next result we need to substitute \( U = \gamma^\sharp = \gamma^\sharp \) and \( U = \gamma^\sharp = \gamma^\sharp \) into (3.8) and (3.9), respectively.

3.6. Tauberian theorem for games with saddle point.

**Theorem 2.** Let \( \mathcal{A} \) and \( \mathcal{B} \) satisfy assumptions 3.2.1, 3.2.2.

Suppose that for any \( \lambda, T, h > 0 \) and each of the payoffs \( v_T, w_\lambda \) there exist a saddle point and \( \varepsilon \)-optimal (for each \( \varepsilon > 0 \)) rules in \( \mathcal{A} \) and \( \mathcal{B} \), respectively.
Then if at least one of the limits
\[
\lim_{T \to \infty} \gamma^2_T(\omega), \quad \lim_{T \to \infty} \gamma^2_\lambda(\omega), \quad \lim_{\lambda \to 0} \gamma^2_\lambda(\omega), \quad \lim_{\lambda \to 0} \gamma^2_\lambda(\omega)
\] (3.10)
exists, is uniform in \( \omega \in \Omega \), then all four limits exist uniformly in \( \omega \in \Omega \) and are equal one to another.

The proof of this theorem is presented in §6. The idea of the proof follows the lines of [27]: the existence of a saddle point allows the reduction of the general Tauberian theorem to two inequalities for upper and lower games, respectively—one-sided Tauberian theorems. These one-sided Tauberian theorems are proven in §8 and §9 by means of multiple iteration of Bellman’s optimality principle.

Note that, as is illustrated in [26], even for control problems, the assumption in the Tauberian theorems that the limit of the value be uniform cannot be eliminated. The one-sided Tauberian theorems considered below make it possible to weaken this requirement.

§4. Asymptotic guarantees of the players

4.1. On the guarantees and the suboptimality principle. Let a payoff \( c : \mathbb{K} \to \mathbb{R} \) and a function \( S : \Omega \to \mathbb{R} \) be given.

**Definition 4.** We say that the first player (the second player) has guarantee \( S \) for payoff \( c \) if \( \forall^\gamma [c](\omega) \geq S(\omega) \) \( \langle \forall^\gamma [c](\omega) \leq S(\omega) \rangle \) for any \( \omega \in \Omega \).

We say that the guarantee \( S \) with payoff \( c \) is protected for the first player (the second player) if there is a rule \( A \in \mathfrak{A} \) \( (A \in \mathfrak{B}) \) such that \( c(z) \geq S(z(0)) \) \( (c(z) \leq S(z(0))) \) for any \( z \in A \).

Consider a function \( U : \mathbb{R}_{>0} \times \Omega \to \mathbb{R} \) and suppose that for any positive \( \gamma \) there is a payoff \( \gamma^\gamma : \mathbb{K} \to \mathbb{R} \). Let \( \gamma^\gamma \in \{0, +\infty\} \).

**Definition 5.** We say that for a family of payoffs \( \gamma^\gamma \) the first player has an asymptotic guarantee \( U \) (a protected asymptotic guarantee) as \( \gamma \to \gamma^\gamma \) if, for each \( \varepsilon > 0 \), there exists a neighbourhood of \( \gamma^\gamma \) in \( \mathbb{R}_{>0} \) such that for any \( \gamma \) in this neighbourhood the function \( U_{\gamma - \varepsilon} \), as a function from \( \Omega \) to \( \mathbb{R} \), is a guarantee (a protected guarantee) for payoff \( \gamma^\gamma \).

For the second player the asymptotic guarantee is defined similarly with \( U_{\gamma + \varepsilon} \) replaced with \( U_{\gamma - \varepsilon} \).

Note that since any function \( S : \Omega \to \mathbb{R} \) can be treated as a function from \( \mathbb{R}_{>0} \times \Omega \) to \( \mathbb{R} \) independent of the first argument, such a function \( S \) can be viewed as an asymptotic guarantee as well.

**Definition 6.** We say that \( U : \mathbb{R}_{>0} \times \Omega \to \mathbb{R} \) is a lower (upper) solution for the family of payoffs \( \nu_T \), if for any \( \varepsilon > 0 \) there exists a positive \( T \) such that for all \( h > 0 \) and \( T > T \) the function \( U_{T+h} - \varepsilon \) is a protected guarantee of the first player (the function \( U_{T+h} + \varepsilon \) is a protected guarantee of the second player) for payoff \( \gamma^\gamma_{h,T} \).

Thus, \( U \) is a lower solution for payoffs \( \nu_T \) exactly when for any \( \varepsilon > 0 \) there exists a positive \( T \) such that for all \( h > 0 \) and \( T > T \) there is a rule \( A \in \mathfrak{A} \) such that
\[
U_{T+h}(z(0)) - \varepsilon \leq \zeta^U_{h,T}(z) = \frac{1}{T+h} \int_0^h g(z(t)) \, dt + \frac{T}{T+h} U_T(z(h)) \quad \forall z \in A. \quad (4.1)
\]
We say that a function $U: \mathbb{R}_{>0} \times \Omega \rightarrow \mathbb{R}$ is a lower (upper) solution for the family of payoffs $w_\lambda$ as $\lambda \downarrow 0$ if for any $\varepsilon > 0$ there exists a positive $\bar{\lambda}$ such that for all positive $\lambda < \bar{\lambda}$ and $h > 0$ the function $U_\lambda - \varepsilon$ is a protected guarantee of the first player (the function $U_\lambda + \varepsilon$ is a protected guarantee of the second player) for payoff $\xi_{h,\lambda}$.

Thus, $U$ is a lower solution for payoffs $w_\lambda$ as $\lambda \downarrow 0$ exactly when for any $\varepsilon > 0$ there exists a positive $\bar{\lambda}$ such that for all positive $\lambda < \bar{\lambda}$ and $h > 0$ there exists a rule $A \in \mathcal{A}$ such that

$$U_\lambda(z(0)) - \varepsilon \leq \xi_{h,\lambda}^U(z) = \lambda \int_0^h e^{-\lambda t} g(z(t)) dt + e^{-\lambda h} U_\lambda(z(h)) \quad \forall z \in A. \quad (4.2)$$

Similar definitions can also be found in [29], Definition III.2.31 (the suboptimality principle) and in [30] ($w$, $v$-stability).

**Remark 1.** Let a family of payoffs $\nu_\gamma$ as $\gamma \rightarrow \gamma_*$ (or of $v_T$ as $T \uparrow \infty$, or of $w_\lambda$ as $\lambda \downarrow 0$) be given. Assume that $U_\gamma$ and $U'_\gamma$, as maps from $\Omega$ to $\mathbb{R}$, have a common limit as $\gamma \rightarrow \gamma_*$ and this limit is uniform on $\Omega$. Then

1) $U$ is an asymptotic guarantee for payoffs $\nu_\gamma$ as $\gamma \rightarrow \gamma_*$ if and only if $U'$ is;  
2) $U$ is a protected asymptotic guarantee for these payoffs as $\gamma \rightarrow \gamma_*$ if and only if $U'$ is;  
3) $U$ is a lower (upper) solution if and only if $U'$ is.

**4.2. The bounds obtained.** In §8 we prove the following proposition.

**Proposition 1.** Let $\mathcal{A}$ satisfy assumptions 3.2.1 and 3.2.2.

Suppose that a function $U: \mathbb{R}_{>0} \times \Omega \rightarrow \mathbb{R}$ that is bounded above is a lower solution for payoffs $v_T$, which means that (4.1) is valid.

Also, let $U$ satisfy the condition

$$\liminf_{T \uparrow \infty} \inf_{\omega \in \Omega} (U_{pT}(\omega) - U_T(\omega)) \geq 0 \quad \forall p > 1. \quad (4.3)$$

Then for any $\varepsilon > 0$ there exists a positive integer $N$ such that for all positive $\lambda < 1/N$ the function $U_{1/\lambda} - \varepsilon$ is a protected guarantee of the first player for payoff $w_\lambda$; in particular,

$$\mathcal{W}_\lambda^\gamma(\omega) \geq U_{1/\lambda}(\omega) - \varepsilon \quad \forall \omega \in \Omega, \quad \lambda \in \left(0, \frac{1}{N}\right).$$

The proof of the following proposition is given in §9.

**Proposition 2.** Let $\mathcal{A}$ satisfy assumptions 3.2.1 and 3.2.2.

Suppose that a function $U: \mathbb{R}_{>0} \times \Omega \rightarrow \mathbb{R}$ that is bounded above is a lower solution for payoffs $w_\lambda$, which means that (4.2) is valid.

Also, let $U$ satisfy the condition

$$\liminf_{\lambda \downarrow 0} \inf_{\omega \in \Omega} (U_\lambda(\omega) - U_{p\lambda}(\omega)) \geq 0 \quad \forall p > 1. \quad (4.4)$$

Then for any $\varepsilon > 0$ there exists a positive integer $N$ such that for all positive $T > N$ the function $U_{1/T} - \varepsilon$ is a protected guarantee of the first player for payoff $v_T$; in particular,

$$\mathcal{W}_T^\gamma(\omega) \geq U_{1/T}(\omega) - \varepsilon \quad \forall \omega \in \Omega, \quad T > N.$$
Note that \( U \) satisfies condition (4.4) if either \( U \) decreases monotonically in \( \lambda \) for any \( \omega \in \Omega \) or \( U \) has a limit as \( \lambda \downarrow 0 \) and this limit is uniform on \( \Omega \), or if \( U \) is a sum of such functions. Similarly, \( U \) satisfies (4.3) if either \( U \) increases monotonically in \( T \) for any \( \omega \in \Omega \), or \( U \) has a limit as \( T \uparrow \infty \) and this limit is uniform on \( \Omega \), or \( U \) is a sum of such functions.

Inequalities similar to (4.3) often appear in Tauberian theorems (see, for example, [6], §2.2, [3], Definition 4.1.4 or [2], §6.2). For stochastic games similar inequalities appear, for instance, in [7] (Condition (3*) in Theorem 4.1).

§ 5. Proof of Theorem 1

For all \( l \in \mathcal{L} \) and \( m \in \mathcal{M} \) we introduce \( A_l \triangleq \{ \omega \in \Omega \} \) and \( B_m \triangleq \{ \omega \in \Omega \} \). Let \( A \triangleq \{ A_i | l \in \mathcal{L} \} \) and \( B \triangleq \{ B_m | m \in \mathcal{M} \} \). It is easily seen that \( A_i \cap B_m \cap \Gamma(\omega) = \{ \omega \in \Omega \} \neq \emptyset \) for \( l \in \mathcal{L} \), \( m \in \mathcal{M} \) and \( \omega \in \Omega \). Thus, \( A \) and \( B \) satisfy assumption 3.2.1.

Let us prove that \( B \) satisfies assumption 3.2.2 as well. Indeed, for all \( \tau > 0 \), \( l', l'' \in \mathcal{L} \) and \( \omega \in \Omega \), by the definitions of the rules \( A' \) and \( A'' \) and the properties of the operation \( \square_\tau \) we have

\[
A' \square_\tau A'' \triangleq \left\{ z' \circ_\tau z'' | z' \in A', z'' \in A'' \right\} = \left\{ z[\omega, l', m'] \circ_\tau z[\omega, l'', m''] | \omega \in \Omega, m', m'' \in \mathcal{M}, \omega' = z[\omega, l', m'](\tau) \right\}
\]

(3.1)

(3.2)

(3.3)

(3.4)

In a similar way one can show that assumption 3.2.2 holds for \( B \) as well.

Let \( c: \mathbb{K} \to \mathbb{R} \) be a bounded function. We are going to show that for \( \varepsilon > 0 \) the function \( \nabla^b[c] - \varepsilon \) is a protected guarantee of the first player for payoff \( c \). Indeed, given an \( \varepsilon > 0 \), for each \( \omega \in \Omega \) we can choose \( l^\omega \in \mathcal{L}(\omega) \) in such a way that \( c(z[\omega, l^\omega, m]) > \nabla^b[c](\omega) - \varepsilon \) for \( \omega \in \Omega \) and \( m \in \mathcal{M}(\omega) \). Then there exists an \( l^\star \in \mathcal{L} \) such that \( l^\star(\omega) = l^\omega(\omega) \) for all \( \omega \in \Omega \). Now the inequality \( c(z) > \nabla^b[c](z(0)) - \varepsilon \) holds for all \( z \in A_l^\star \). Thus, the rule \( A_l^\star \) protects the guarantee \( \nabla^b[c] - \varepsilon \) of the first player for this payoff \( c \). Similarly, for the second player there exists a selector \( m^\star \in \mathcal{M} \) such that \( c(z[\omega, l^\star, m^\star]) < \nabla^b[c](\omega) + \varepsilon \) for all \( \omega \in \Omega \), \( l \in \mathcal{L}(\omega) \), and \( B_m^\star \) protects the guarantee \( \nabla^b[c] + \varepsilon \) of the second player for the same payoff \( c \). This also yields \( \nabla^b[c](\omega) - \varepsilon < c(z[\omega, l^\star, m^\star]) < \nabla^b[c](\omega) + \varepsilon \) for any \( \omega \in \Omega \). Since \( \varepsilon > 0 \) has been arbitrarily chosen, we obtain \( \nabla^b[c] \leq \nabla^b[c] \).

Let us show that \( \nabla^b[c] \) is a lower solution for payoffs \( w_\lambda \) as \( \lambda \downarrow 0 \). Indeed, for all \( \lambda, h > 0 \)

\[
\nabla^b[c](\omega) = \sup_{l \in \mathcal{L}} \inf_{m \in \mathcal{M}} \int_0^\infty \lambda e^{-\lambda t} g(z[\omega, l, m](t)) dt
\]

(2.2)

\[
= \sup_{l', l'' \in \mathcal{L}} \inf_{m', m'' \in \mathcal{M}} \int_0^\infty \lambda e^{-\lambda t} g(z[\omega, l', l'', m', m''](h), l'', m''](t)) dt
\]

(2.3)

\[
\nabla^b[c](\omega) = \sup_{l' \in \mathcal{L}} \inf_{l'' \in \mathcal{L}} \int_0^\infty \lambda e^{-\lambda t} g(z[\omega, l', m'], h), l''](t)) dt
\]

(2.4)
(2.1)\[\sup_{T' \in \mathcal{L}} \sup_{l'' \in \mathcal{L}} \inf_{m'' \in \mathcal{M}} \left[ \int_{0}^{h} \lambda e^{-\lambda t} g(z[\omega, l', m'](t)) dt \right.
+ \left. \inf_{m' \in \mathcal{M}} \int_{h}^{\infty} \lambda e^{-\lambda t} g(z[\omega, l', m'](h), l'', m'')(t-h) dt \right]
\leq \sup_{T' \in \mathcal{L}} \inf_{m' \in \mathcal{M}} \left[ \int_{0}^{h} \lambda e^{-\lambda t} g(z[\omega, l', m'](t)) dt \right.
+ \left. \sup_{l'' \in \mathcal{L}} \int_{0}^{\infty} \lambda e^{-\lambda(t+h)} g(z[\omega, l', m'](h), l'', m'')(t) dt \right]
\left. \overset{(3.5)}{=} \sup_{T' \in \mathcal{L}} \inf_{m' \in \mathcal{M}} \left[ \int_{0}^{h} \lambda e^{-\lambda t} g(z[\omega, l', m'](t)) dt + e^{-\lambda h} \mathcal{W}^{\beta}_{\omega, h}(z[\omega, l', m'](h)) \right] \right)
\left. \overset{(3.9)}{=} \sup_{T' \in \mathcal{L}} \inf_{m' \in \mathcal{M}} \mathcal{W}^{\beta}_{\omega, h}(z[\omega, l', m']) = V[\hat{\mathcal{W}}^{\beta}_{h, \lambda}](\omega), \right.

which means that the function $\mathcal{W}^{\beta}_{\omega, h}$ is a guarantee for payoffs $\mathcal{W}^{\beta}_{h, \lambda}$ for any $h > 0$.

By what we proved above, $\mathcal{W}^{\beta} - \varepsilon$ is a protected guarantee for these payoffs for any $\varepsilon > 0$. Consequently, $\mathcal{W}^{\beta}$ is a lower solution for the family of payoffs $v_{T}$. In a similar way we can show that $\mathcal{W}^{\beta}$ is a lower solution for the family of payoffs $v_{T}.$

By (3.2) and (3.3), it is sufficient to show that the functions $V^{\beta}_{T}, V^{\beta}_{1/T}$ and $\mathcal{W}^{\beta}_{1/T}$ converge to a common limit as $T \uparrow \infty$ uniformly on $\Omega$.

By the hypothesis, either $V^{\beta}_{T}$ as $T \uparrow \infty$ or $\mathcal{W}^{\beta}_{\omega, h}$ as $\lambda \downarrow 0$ converges uniformly on $\Omega$ to a limit function $S_{*} : \Omega \rightarrow \mathbb{R}$. Since both $V^{\beta}_{T}$ and $\mathcal{W}^{\beta}_{\omega, h}$ are the lower solutions for their families of payoffs, it follows from Remark 1 that $S_{*}$ is also a lower solution to one of these families of payoffs. Since $S_{*}$ depends only on $\omega \in \Omega$, conditions (4.3) and (4.4) hold a fortiori. Applying either Proposition 1 or Proposition 2 proves that $S_{*}$ is a common asymptotic guarantee of the first player both for payoffs $v_{T}$ as $T \uparrow \infty$ and for payoffs $w_{\lambda}$ as $\lambda \downarrow 0$.

By symmetry (see (3.7)), the function $S_{*}$ is also an asymptotic guarantee of the second player both for payoffs $v_{T}$ as $T \uparrow \infty$ and for payoffs $w_{\lambda}$ as $\lambda \downarrow 0$.

Now for any $\varepsilon > 0$ we can find $T > 0$ such that the following inequalities hold for all $T > T$ and $\omega \in \Omega$:

\[
S_{*}(\omega) - \varepsilon \leq V^{\beta}_{T}(\omega), \quad V^{\beta}_{T}(\omega) \leq S_{*}(\omega) + \varepsilon,
\]

\[
S_{*}(\omega) - \varepsilon \leq \mathcal{W}^{\beta}_{1/T}(\omega), \quad \mathcal{W}^{\beta}_{1/T}(\omega) \leq S_{*}(\omega) + \varepsilon.
\]

As we showed above, $V^{\beta}_{T}(c) \leq V^{\beta}_{1/T}(c)$ for all bounded payoffs $c$, which gives $V^{\beta}_{T}(\omega) \leq V^{\beta}_{1/T}(\omega)$, $\mathcal{W}^{\beta}_{1/T}(\omega) \leq \mathcal{W}^{\beta}_{1/T}(\omega)$ and

\[
S_{*}(\omega) - \varepsilon \leq V^{\beta}_{T}(\omega) \leq V^{\beta}_{1/T}(\omega) \leq S_{*}(\omega) + \varepsilon \quad \forall \omega \in \Omega,
\]

\[
S_{*}(\omega) - \varepsilon \leq \mathcal{W}^{\beta}_{1/T}(\omega) \leq \mathcal{W}^{\beta}_{1/T}(\omega) \leq S_{*}(\omega) + \varepsilon \quad \forall \omega \in \Omega.
\]

Thus, all the functions $V^{\beta}_{T}, V^{\beta}_{1/T}, \mathcal{W}^{\beta}_{T}, \mathcal{W}^{\beta}_{1/T}$ converge to a common limit as $T \uparrow \infty$ uniformly in $\omega \in \Omega$. 
§ 6. Proof of Theorem 2

6.1. Before we proceed with the proof of Theorem 2, we establish a useful lemma; for control systems with payoff \( w_\lambda \) a similar result was, for example, obtained in [29], III.2.7.

**Lemma.** Suppose that \( \mathfrak{A} \) satisfies assumptions 3.2.1, 3.2.2, and let there be given positive numbers \( \varepsilon, T \) and \( \lambda \) and a mapping \( U : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R} \).

Then:
1) if \( U_T - \varepsilon \) is a protected guarantee of the first player for payoff \( v_T \), then \( \forall^b [\zeta^U_{h,T}] - \varepsilon \) is a guarantee of the first player for payoff \( v_{T+h} \) for all \( h > 0 \);
2) if \( U_\lambda - \varepsilon \) is a protected guarantee of the first player for payoff \( w_\lambda \), then \( \forall^b [\zeta^U_{h,\lambda}] - \varepsilon \) is a guarantee of the first player for payoff \( w_{\lambda} \) for all \( h > 0 \).

**Proof.** We prove only the assertion about \( v_T \), since for \( w_\lambda \) the proof follows the same lines.

Let \( U_T - \varepsilon \) be a protected guarantee of the first player for payoff \( v_T \), that is, there exists a rule \( A^\gamma \in \mathfrak{A} \) such that

\[
U_T(z(0)) \leq v_T(z) + \varepsilon \quad \forall z \in A^\gamma. \tag{6.1}
\]

Fix some \( h, \delta > 0 \) and \( \omega \in \Omega \). Then there exists a rule \( A'_\omega \in \mathfrak{A} \) such that

\[
\forall^b [\zeta^U_{h,T}](\omega) - \delta \leq \zeta^U_{h,T}(z) \quad \forall z \in A'_\omega \cap \Gamma(\omega).
\]

Since this inequality is independent of \( z(t) \) for \( t > h \), it also holds for any \( z \in (A'_\omega \sqcap_h A^\gamma) \cap \Gamma(\omega) \). Moreover, for any \( z \in (A'_\omega \sqcap_h A^\gamma) \cap \Gamma(\omega) \) one can find \( z_h \in A^\gamma \) such that \( z = z \circ_h z_h \). Now for all \( z \in (A'_\omega \sqcap_h A^\gamma) \cap \Gamma(\omega) \) we have

\[
\forall^b [\zeta^U_{h,T}](\omega) - \delta \leq \zeta^U_{h,T}(z) = \int_0^h \frac{1}{T+h} g(z(t)) \, dt + \frac{T}{T+h} U_T(z(h))
\]

\[
= v_{T+h}(z) - \frac{T}{T+h} v_T(z_h) + \frac{T}{T+h} U_T(z_h(0))
\]

\[
\leq v_{T+h}(z) + \frac{T}{T+h} \varepsilon. \tag{6.1}
\]

As \( \mathfrak{A} \) is closed under switching, we also have \( A'_\omega \sqcap_h A^\gamma \in \mathfrak{A} \). Now, since the process \( z \) in \( (A'_\omega \sqcap_h A^\gamma) \cap \Gamma(\omega) \) has been chosen arbitrarily, we have

\[
\forall^b [\zeta^U_{h,T}](\omega) - \frac{T}{T+h} \varepsilon - \delta \leq \inf_{z \in (A'_\omega \sqcap_h A^\gamma) \cap \Gamma(\omega)} v_{T+h}(z)
\]

\[
\leq \sup_{A \in \mathbb{R}} \inf_{z \in A \cap \Gamma(\omega)} v_{T+h}(z) = \forall^b [v_{T+h}](\omega) \quad \forall \omega \in \Omega.
\]

Since this inequality is valid for any positive \( \delta \), it follows that \( \forall^b [\zeta^U_{h,T}] - \varepsilon \leq \forall^b [v_{T+h}] \) for all \( h > 0 \).

The proof of the lemma is complete.
6.2. Now let us proceed with the proof of Theorem 2.

By the hypothesis of the theorem \( \mathcal{Y}_T^\flat \equiv \mathcal{Y}_T^\sharp \) and \( \mathcal{W}_\lambda^\flat \equiv \mathcal{W}_\lambda^\sharp \) for all \( \lambda, T > 0 \). Hence, for any \( h, \lambda, T > 0 \) the games with payoffs \( \zeta_{h,T}^\flat \) and the games with payoffs \( \zeta_{h,T}^\sharp \) coincide completely. The same with payoffs \( \xi_{h,\lambda}^\flat \) and \( \xi_{h,\lambda}^\sharp \). Thus, the conditions of the theorem may be regarded as symmetric with respect to upper and lower values.

Suppose that at least one of the limits in (3.10) exists: either the limit of values \( \mathcal{W}_\lambda^\flat = \mathcal{W}_\lambda^\sharp \) as \( \lambda \downarrow 0 \), and that it is uniform on \( \Omega \). Let this be the limit \( S^* \) of values \( \mathcal{V}_T^\flat = \mathcal{V}_T^\sharp \) as \( T \uparrow \infty \).

We shall demonstrate that \( S^* \) is a protected asymptotic guarantee with payoffs \( w_\lambda \) as \( \lambda \downarrow 0 \) both for lower and for upper values. By virtue of symmetry (see (3.7)), we may prove it only for lower values.

Consider arbitrary \( \varepsilon, T > 0 \). By the hypothesis of the theorem there exist \( \varepsilon \)-optimal rules for payoff \( v_T \). Now \( \mathcal{V}_T^\flat + \varepsilon \) is a protected guarantee of the second player for payoff \( v_T \). Applying the lemma established above to the upper values and making use of the existence of a saddle point for payoffs \( \zeta_{h,T}^\sharp \), we obtain

\[
\mathcal{V}_T^\sharp + h - \varepsilon \leq \mathcal{V}_T^\flat \left[ \zeta_{h,T}^\flat \right] - \varepsilon \leq \zeta_{h,T}^\sharp \left( z(0) \right) \quad \forall h > 0.
\]

By the hypothesis, for any \( h > 0 \) the first player has an \( \varepsilon \)-optimal rule \( A^h \in \mathfrak{A} \) for payoff \( \zeta_{h,T}^\flat \); now

\[
\mathcal{V}_T^\flat (z(0)) - 2\varepsilon \leq \mathcal{V}_T^\flat \left[ \zeta_{h,T}^\flat \right] (z(0)) - \varepsilon \leq \zeta_{h,T}^\sharp (z) \quad \forall z \in A^h.
\]

Therefore, we obtain (4.1) with \( U = \mathcal{V}_T^\flat \) for any positive \( h \). Thus, \( \mathcal{V}_T^\flat \) is a lower solution for payoffs \( \zeta_{h,T}^\flat \) as \( T \uparrow \infty \).

By Proposition 1 the function \( \mathcal{V}_T^\flat \) is an asymptotic guarantee for the lower values with payoffs \( w_\lambda \). By Remark 1 \( S^* \) is such a guarantee as well. It is similarly demonstrated that \( S^* \) is also an asymptotic guarantee for the upper values with payoffs \( w_\lambda \).

Thus, we have shown that for any \( \varepsilon > 0 \) and sufficiently small positive \( \lambda \)

\[
\mathcal{W}_\lambda^\flat (\omega) - \varepsilon \leq S^* (\omega) \leq \mathcal{W}_\lambda^\sharp (\omega) + \varepsilon \quad \forall \omega \in \Omega.
\]

Since by the hypothesis of the theorem \( \mathcal{W}_\lambda^\flat \equiv \mathcal{W}_\lambda^\sharp \), the function \( S^* \) is a uniform limit on \( \Omega \) of \( \mathcal{W}_\lambda^\flat = \mathcal{W}_\lambda^\sharp \) as \( \lambda \downarrow 0 \).

In the case when the values \( \mathcal{W}_\lambda^\flat = \mathcal{W}_\lambda^\sharp \) converge uniformly on \( \Omega \) as \( \lambda \downarrow 0 \), the proof follows the same lines. We only have to interchange \( \lambda \) and \( T \) and \( v_T \) and \( w_\lambda \) and to use \( \xi_{h,\lambda} \) and Proposition 2 instead of \( \zeta_{h,T} \) and Proposition 1.

The proof of Theorem 2 is complete.

§7. The case of one player

7.1. In this section we give a proof of the Tauberian theorem for abstract control systems. Our proof is independent of the one in [26]. Note that in the original formulation (see [26]) of this theorem it was only assumed that \( \mathbb{K} \) is closed under concatenation. This condition is insufficient (see [31]), and below we illustrate this by giving a counterexample.
Corollary. Let the sets $\Gamma(\omega) = \{z \in K \mid z(0) = \omega\}$ be nonempty for any $\omega \in \Omega$ and assume that for any positive $\tau > 0$

$$K = \{z' \circ_\tau z' \mid z, z' \in K, z(\tau) = z'(0)\}. \quad (7.1)$$

Suppose that at least one of the limits

$$\lim_{T \to \infty} \sup_{z \in \Gamma(\omega)} \frac{1}{T} \int_0^T g(z(t)) \, dt \quad \text{and} \quad \lim_{\lambda \to 0} \sup_{z \in \Gamma(\omega)} \lambda \int_0^\infty e^{-\lambda t} g(z(t)) \, dt$$

exists and is uniform in $\omega \in \Omega$. Then both limits exist, are uniform in $\omega \in \Omega$, and are equal to each other.

Proof. Let us introduce

$$B \triangleq \{K\}.$$

That $B$ obeys assumption 3.2.1 follows from the fact that $\Gamma(\omega)$ is nonempty for any $\omega \in \Omega$. Assumption 3.2.2 for $B$ is satisfied, as (7.1) implies that $K_{\tau} K :\triangleq \{z' \circ_\tau z' \mid z, z' \in K, z(\tau) = z'(0)\} = K$.

For any $\omega \in \Omega$ we define the set of all selectors $\Omega \ni \omega \to l(\omega) \in \Gamma(\omega)$ and denote it by $L$. With each selector $l \in L$ we associate its image $A_l \triangleq \{l(\omega) \in K \mid \omega \in \Omega\}$. Let

$$A \triangleq \{A_l \subset K \mid l \in L\}.$$

Note that $\{l(\omega)\} = A_l \cap \Gamma(\omega) \neq \emptyset$ by construction; in particular, each $A_l$ is a rule; thus, $A$ satisfies assumption 3.2.1. At the same time for any bounded payoff $c: K \to \mathbb{R}$

$$\sup_{A_l \in A} \inf_{z \in A_l \cap \Gamma(\omega)} c(z) = \sup_{\omega \in \Omega} \inf_{z \in \Gamma(\omega)} c(z) = \inf_{B \in \{K\}} \sup_{z \in B \cap \Gamma(\omega)} c(z).$$

Moreover, for any $\varepsilon > 0$ each player has $\varepsilon$-optimal rules: the first player due to the arbitrariness of the selector from $L$ and the second player due to the uniqueness of the element $B$ from $B = \{K\}$.

Let us show that $A$ satisfies assumption 3.2.2. Indeed, take arbitrary $\tau > 0$, $A_\nu, A_{\nu'} \in A$ and specify $l \in L$ by the rule $l(\omega) \triangleq l'(\omega) \circ_\tau l''(l'(\omega)(\tau))$ for all $\omega \in \Omega$. Then we have

$$A_{\nu} \circ_\tau A_{\nu'} = \{z' \circ z'' \in K \mid z' \in A_{\nu}, z'' \in A_{\nu'}\} = \{l'(\omega) \circ_\tau z'' \in K \mid \omega \in \Omega, z'' \in A_{\nu'}, z''(0) = l'(\omega)(\tau)\} = \{l'(\omega) \circ_\tau l''(l'(\omega)(\tau)) \in K \mid \omega \in \Omega\} = A_l \in K.$$

Thus, $A$ and $B$ satisfy assumptions 3.2.1 and 3.2.2. Now applying Theorem 2 completes the proof of the corollary.
7.2. Example. Let $\Omega = \mathbb{T} \times \mathbb{T} \times \mathbb{T}$. For any $\omega = (x, y, r) \in \Omega$ define $a_\omega : \mathbb{T} \to \Omega$ by

$$a_\omega(t) = a_{(x, y, r)}(t) \triangleq (x, y, t + r) \quad \forall t \geq 0.$$  

In particular, now $a_\omega(0) = \omega$ for all $\omega = (x, y, r) \in \Omega$.

Also, for any $s \in \mathbb{T}$ we define $b_s : \mathbb{T} \to \Omega$ by

$$b_s(t) \triangleq (st, t, 0) \quad \forall t \geq 0.$$  

In particular, for all $s \in \mathbb{T}$ we have $b_s(0) = (0, 0, 0)$.

Now put

$$\mathbb{K} \triangleq \{ a_\omega \mid \omega \in \Omega \} \cup \{ b_s \mid s \in \mathbb{T} \} \cup \{ b_s \circ_\tau a_{b_s(\tau)} \mid s \in \mathbb{T}, \tau > 0 \}.$$  

It is easily seen that $\Gamma(\omega) = \{ a_\omega \}$ for all $\omega \in \Omega \setminus \{ (0, 0, 0) \}$, moreover, $z(\tau) \neq (0, 0, 0)$ for any $z \in \mathbb{K}$, $\tau > 0$. Now $a_{(x, y, r)} \circ_\tau \tau$ is well defined for $z \in \mathbb{K}$ if and only if $z = a_{(x, y, r)}(\tau)$.

Then $a_{(x, y, r)} \circ_\tau a_{(x, y, r)}(\tau)$ coincides with $a_{(x, y, r)}$ for all $\tau > 0$, in particular it also belongs to $\mathbb{K}$. This means that $\mathbb{K}$ is closed under concatenation.

Let us put $g(x, y, r) \triangleq 1$ for $x \in [1, 2]$, $r = 0$ and $g(x, y, r) \triangleq 0$ otherwise.

It is easily verified that $v_T(a_\omega) = w_\lambda(a_\omega) = 0$ for all $\lambda, T > 0, \omega \in \Omega$. This gives

$$\sup_{z \in \Gamma(\omega)} v_T(z) = \sup_{z \in \Gamma(\omega)} w_\lambda(z) = 0 \quad \forall \omega \in \Omega \setminus \{ (0, 0, 0) \}.$$  

Since

$$g(b_s(t)) \geq g((b_s \circ_\tau a_{b_s(\tau)})(t)) \geq g(b_0(t)) \quad = g((b_0 \circ_\tau a_{b_0(\tau)})(t)) = 0 \quad \forall s, t \geq 0, \quad \tau > 0,$$

we have

$$v_T(b_s) \geq v_T(b_s \circ_\tau a_{b_s(\tau)}), \quad w_\lambda(b_s) \geq w_\lambda(b_s \circ_\tau a_{b_s(\tau)}) \quad \forall s \geq 0, \quad \tau > 0.$$  

It is easily seen that $v_T(b_T/2) = 1/2 \geq v_T(b_s)$ for any $s > 0$. Thus,

$$\sup_{z \in \Gamma(0, 0, 0)} v_T(z) = \frac{1}{2} \quad \forall T > 0.$$  

Next, for all $s, \lambda > 0$ we have

$$w_\lambda(b_s) = \lambda \int_0^\infty e^{-\lambda t} 1_{[1, 2]}(st) \, dt = e^{-\lambda/s} - e^{-2\lambda/s} \leq \sup_{x \in \mathbb{R}}(x - x^2) = \frac{1}{4}.$$  

At the same time for $s = \lambda/\ln 2$ we have $e^{-\lambda/s} = 1/2$, $w_\lambda(b_s) = 1/2 - 1/4 = 1/4$. Therefore,

$$\sup_{z \in \Gamma(0, 0, 0)} w_\lambda(z) = w_\lambda(b_{\lambda/\ln 2}) = \frac{1}{4} \quad \forall \lambda > 0.$$  

Thus, although $\mathbb{K}$ is closed under concatenation and both the limits

$$\lim_{T \to \infty} \sup_{z \in \Gamma(\omega)} v_T(z) = \sup_{z \in \Gamma(\omega)} v_1(z) \quad \text{and} \quad \lim_{\lambda \to 0} \sup_{z \in \Gamma(\omega)} w_\lambda(z) = \sup_{z \in \Gamma(\omega)} w_1(z) \quad \forall \omega \in \Omega$$

exist and are uniform on $\Omega$, these limits are distinct at $\omega = (0, 0, 0)$. 

Therefore, in Tauberian theorems for abstract control problems (see the corollary above) it is not sufficient to require that $\mathbb{K} \sqcap_\tau \mathbb{K} \subset \mathbb{K}$. On the other hand, the corollary shows that a sufficient condition here is (7.1), that is, $\mathbb{K} \sqcap_\tau \mathbb{K} = \mathbb{K}$.

The same example may also be used to illustrate the necessity of condition (2.2) in the hypothesis of Theorem 1.

Indeed, set $\mathcal{L}(\omega) \triangleq \Gamma(\omega)$ and $\mathcal{M}(\omega) \triangleq \{1\}$ for all $\omega \in \Omega$ in our example. We denote the sets of selectors $\Omega \ni \omega \rightarrow l(\omega) \in \mathcal{L}(\omega)$ and $\Omega \ni \omega \rightarrow m(\omega) \in \mathcal{M}(\omega)$ by $\mathcal{L}$ and $\mathcal{M}$, respectively, and take $m' \circ_\tau m'' \triangleq m'$ for all $m', m'' \in \mathcal{M}$ and $\tau > 0$. Note that for all $l', l'' \in \mathcal{L}$ and $\tau > 0$ the process $(l' \circ_\tau l'')(\omega) \triangleq l'(\omega) \circ_\tau l''(l'(\omega)(\tau)) \in \Gamma(\omega)$ is well defined. Thus, for any $\tau > 0$ we have introduced binary operations on the sets $\mathcal{L}$ and $\mathcal{M}$.

For any state $\omega \in \Omega$ with each pair $(l, m) \in \mathcal{L} \times \mathcal{M}$ we associate the process $z[\omega, l, m] \triangleq l(\omega) \in \mathbb{K}$. As $l(\omega) \in \Gamma(\omega)$, we have $z[\omega, l, m](0) = \omega$. Now for all $l', l'' \in \mathcal{L}$, $m', m'' \in \mathcal{M}$ and $\tau > 0$ it follows from the above definitions that

\[
\begin{align*}
z[\omega, l' \circ_\tau l'', m' \circ_\tau m''](\tau) &= (l' \circ_\tau l'')(\omega)(\tau) = (l'(\omega) \circ_\tau l''(l'(\omega)(\tau)))(\tau) \\
&= l'(\omega)(\tau) = z[\omega, l', m'](\tau), \\
z[\omega, l' \circ_\tau l'', m' \circ_\tau m''] &= (l' \circ_\tau l'')(\omega) = l'(\omega) \circ_\tau l''(l'(\omega)(\tau)) \\
&= z[\omega, l', m'] \circ_\tau z[\omega, l'', m''],
\end{align*}
\]

which means that conditions (2.3) and (2.4) are fulfilled. Thus, condition (2.2) cannot be omitted from the hypotheses of Theorem 1.

§ 8. Passing from $\mathcal{V}^g$ to $\mathcal{W}^g$. Proof of Proposition 1

8.1. Preliminary constructions and estimates. We introduce one more operation along with the concatenation of trajectories. For all $\tau \in \mathbb{R}_{>0}$ and $z \in \mathbb{K}$ we define a function $z_\tau: \mathbb{T} \mapsto \Omega$ by

\[
z_\tau(t) = z(t + \tau) \quad \forall t \in \mathbb{T}.
\]

(8.1)

Note that if $z_\tau \in \mathbb{K}$, then for any $T > \tau$

\[
\frac{1}{T} \int_0^T g(z(t)) \, dt = v_T(z) - \frac{1}{T} \int_0^{T-\tau} g(z_\tau(t)) \, dt = v_T(z) - \frac{T - \tau}{T} v_{T-\tau}(z_\tau).
\]

(8.2)

Recall that $U$ is bounded above by a positive constant, which we denote by $R$.

It is easily verified that $\ln p < p - 1 < p \ln p$ for $p > 1$. With each positive integer $k > 2$ we associate $p \in (1, 2)$ such that

\[
\frac{1}{k} < \frac{\ln k}{k} < \ln p < p - 1 < p \ln p < \frac{2 \ln k}{k}.
\]

(8.3)

Fix the numbers $k$ and $p$.

Since $U$ is a lower solution, there exists $\hat{T}^{(k)} > 0$ that obeys (4.1): for any positive $T$ and $\delta$ such that $T > 2\hat{T}^{(k)}$ and $\delta < T/2$ we can find a rule $A^{T, \delta} \in \mathcal{A}$ such that for any $z \in A^{T, \delta}$

\[
U_T(z(0)) - \frac{1}{k^2} \geq \frac{1}{T} \int_0^\delta g(z(t)) \, dt + \frac{T - \delta}{T} U_{T-\delta}(z(\delta)).
\]

(8.4)
Moreover, by (4.3) we can also choose $\hat{T}^{(k)}$ as to satisfy the condition
\[ UT(\omega) \geq U_{p^{-1}T}(\omega) - \frac{1}{k^2} \quad \forall \omega \in \Omega, \quad T > 2\hat{T}^{(k)}. \] (8.5)

Fix such a number $T$ and put
\[ \lambda \triangleq \frac{1}{T}, \quad \delta \triangleq \frac{T(p - 1)}{p} \quad \tau_i \triangleq i\delta \quad \forall i \in \{0, \ldots, k\}. \]

This gives
\[ \frac{T - \delta}{T} = p^{-1}, \quad \frac{\lambda \delta}{\ln p} = \frac{p - 1}{p \ln p} \quad \text{(8.3)} \quad \frac{\ln p}{\delta} \geq \frac{1}{T} = \lambda. \] (8.6)

### 8.2. Constructing a near-$w_\lambda$ payoff
Note that if $t \in [0, \tau_k)$, then $t \in [\tau_i, \tau_{i+1})$ for some $i \in \{0, \ldots, k - 1\}$. Define a step function $g: [0, \tau_k) \to [0, 1]$ by
\[ g(t) = p^{-i} \quad \forall t \in [\tau_i, \tau_{i+1}), \quad i \in \{0, \ldots, k - 1\}. \] (8.7)

Then for all $i \in \{0, \ldots, k - 1\}$ and $t \in [\tau_i, \tau_{i+1})$
\[ g(t) = p^{-i} = p^{1-\tau_{i+1}/\delta} \leq p^{1-t/\delta} = pe^{-(t/\delta) \ln p} \leq pe^{-\lambda t}. \] (8.8)

Consider the payoff
\[ c(z) \triangleq \frac{1}{T} \int_0^{\tau_k} g(t)z(t) \, dt + p^{-k}U_{p^{-1}T}(z(\tau_k)) \quad \forall z \in \mathbb{K}. \]

Note that since $U_{p^{-1}T} \leq R$, we have
\[ p^{-k}U_{p^{-1}T} \leq p^{-k}R = e^{-k \ln p}R \leq e^{-k}R = \frac{R}{k} \leq \frac{R \ln k}{k}. \]

Now with regard to the inequalities $0 \leq g \leq 1$ and $0 \leq w_\lambda \leq 1$, for each process $z \in \mathbb{K}$ we obtain
\[ c(z) - w_\lambda(z) \leq \frac{1}{T} \int_0^{\tau_k} pe^{-\lambda t}g(t) \, dt - w_\lambda(z) + p^{-k}U_{p^{-1}T}(z(\tau_k)) \]
\[ = (p - 1)w_\lambda(z) + p^{-k}U_{p^{-1}T}(z(\tau_k)) \leq (R + 2) \frac{\ln k}{k}. \] (8.9)

### 8.3. Constructing the rule $A^*$
Recall that $T > 2\hat{T}^{(k)}$ and $\delta = (1 - p^{-1})T < T/2$ by the choice of $T$ and $p$, respectively. Now (8.4) holds for some $A = A^{T,\delta} \in \mathfrak{A}$.

Note also that since the right-hand side of (8.4) depends only on $z_{|[0,\delta]}$, inequality (8.4) also holds for any rule that can be represented in the form $A \sqcap \delta A'$ for an appropriately chosen $A' \subset \mathbb{K}$. By (8.1), for any $A' \in \mathfrak{A}$ and $z \in A \sqcap \delta A'$ we have $z_{\delta} \in A' \subset \mathbb{K}$. Now it follows from (8.4) that
\[ UT(z(0)) - \frac{1}{k^2} \leq \frac{1}{T} \int_0^{\delta} g(z(t)) \, dt + \frac{T - \delta}{T} U_{T - \delta}(z(\delta)) \]
\[ = v_T(z) - \frac{T - \delta}{T} v_{T - \delta}(z_{\delta}) + \frac{T - \delta}{T} U_{T - \delta}(z(\delta)) \]
\[ = v_T(z) - p^{-1}v_{p^{-1}T}(z_{\delta}) + p^{-1}U_{p^{-1}T}(z(\delta)) \] (8.10)
for any $A' \in \mathfrak{A}$ and $z \in A \sqcap \delta A'$. 
Put
\[ A^* = A \boxtimes \tau_1 A \boxtimes \tau_2 \cdots \boxtimes \tau_{k-1} A \boxtimes A. \]  
(8.11)
Note that \( A^* \) lies in \( \mathfrak{A} \) by assumption 3.2.2. At the same time, by (8.1) we have \( z_{\tau_i} \in K \) for all \( z \in A^* \).

Since \( k \) can be taken arbitrarily large, to prove the proposition it is sufficient to show that
\[ m_{\Lambda}^z(\omega) \geq U_T(\omega) - \frac{(R + 4) \ln k}{k} \quad \forall \omega \in \Omega; \]
by virtue of (8.9) this follows from the inequality
\[ c(z) > U_T(z(0)) - \frac{2 \ln k}{k} \quad \forall z \in A^*. \]  
(8.12)

### 8.4. Proof of the estimate (8.12).

Recall that \( p^{-1} T = T - \delta, \tau_{i+1} = \tau_i + \delta \) and \( g(t) = p^{-i} \) for all \( t \in [\tau_i, \tau_{i+1}), i \in \{0, \ldots, k - 1\} \). Hence, for any \( z \in A^* \) we have
\[
\frac{1}{T} \int_{\tau_i}^{\tau_{i+1}} g(t)g(z(t)) dt = \frac{1}{T} \int_0^\delta g(\tau_i)g(z(t + \tau_i)) dt = \frac{p^{-i}}{T} \int_0^\delta g(z(t + \tau_i)) dt = p^{-i} v_T(\tau_i) - p^{-i-1} v_{p^{-1} T}(\tau_{i+1}).
\]
Now, for any process \( z \in A^* \) we obtain
\[
c(z) = v_T(z) - p^{-1} v_{p^{-1} T}(z_{\tau_i}) + \cdots + p^{-i} v_T(z_{\tau_i}) - p^{-i-1} v_{p^{-1} T}(z_{\tau_{i+1}}) + \cdots + p^{-k+1} v_T(z_{\tau_{k-1}}) - p^{-k} v_{p^{-1} T}(z_{\tau_k}) + p^{-k} U_{p^{-1} T}(z(\tau_k)).
\]
(8.13)

By (8.11), for all \( z \in A^* \) we have \( z_{\tau_{k-1}} \in A \boxtimes A \); then by (8.10), with regard for the equality \( \tau_{k-1} + \delta = \tau_k \) we also obtain
\[
v_T(z_{\tau_{k-1}}) - p^{-1} v_{p^{-1} T}(z_{\tau_k}) + p^{-1} U_{p^{-1} T}(z(\tau_k)) \overset{(8.10)}{\geq} U_T(z(\tau_{k-1})) - \frac{1}{k^2} \overset{(8.5)}{\geq} U_{p^{-1} T}(z(\tau_{k-1})) - \frac{2}{k^2}.
\]

Substituting this relation into (8.13) and taking into account that \( \tau_{k-1} + \delta = \tau_k \) for all \( z \in A^* \) we obtain
\[
c(z) \geq v_T(z) - p^{-1} v_{p^{-1} T}(z_{\tau_i}) + \cdots + p^{-i} v_T(z_{\tau_i}) - p^{-i-1} v_{p^{-1} T}(z_{\tau_{i+1}}) + \cdots + p^{-k+2} v_T(z_{\tau_{k-2}}) - p^{-k+1} v_{p^{-1} T}(z_{\tau_{k-1}}) + p^{-k+1} U_{p^{-1} T}(z(\tau_{k-1})) - \frac{2}{k^2}.
\]
(8.14)

From (8.11) and \( \tau_{k-2} + \delta = \tau_{k-1} \) we also have \( z_{\tau_{k-2}} \in A \boxtimes (A \boxtimes A) \) for all \( z \in A^* \). Thus,
\[
v_T(z_{\tau_{k-2}}) - p^{-1} v_{p^{-1} T}(z_{\tau_{k-1}}) + p^{-1} U_{p^{-1} T}(z(\tau_{k-1})) \overset{(8.10)}{\geq} U_T(z(\tau_{k-2})) - \frac{1}{k^2} \overset{(8.5)}{\geq} U_{p^{-1} T}(z(\tau_{k-2})) - \frac{2}{k^2}.
\]
Substituting this into (8.14) we obtain
\begin{align*}
c(z) & \geq v_T(z) - p^{-1}v_{p-1}T(z_{\tau_1}) + \cdots \\
& \quad + p^{-i}v_T(z_{\tau_i}) - p^{-i-1}v_{p-1}T(z_{\tau_{i+1}}) + \cdots \\
& \quad + p^{-k+3}v_T(z_{\tau_{k-3}}) - p^{-k+2}v_{p-1}T(z_{\tau_{k-2}}) + p^{-k-2}U_{p-1}T(z(\tau_{k-2})) - \frac{4}{k^2} \\
& \forall z \in A^*.
\end{align*}

Proceeding in the same way, namely, using the relation \( \tau_{k-l} + \delta = \tau_{k-l+1} \) and the inequality
\begin{align*}
v_T(z_{\tau_{k-l}}) - p^{-1}v_{p-1}T(z_{\tau_{k-l+1}}) + p^{-1}U_{p-1}T(z(\tau_{k-l+1})) & \geq U_{p-1}T(z(\tau_{k-l+1})) - \frac{2}{k^2},
\end{align*}
which holds for all \( z \in A^* \), at each step, we obtain the following estimate for all \( z \in A^* \):
\begin{align*}
c(z) & \geq v_T(z) - p^{-1}v_{p-1}T(z_{\tau_1}) + p^{-1}U_{p-1}T(z(\tau_1)) - \frac{2k - 2}{k^2} \overset{(8.10)}{=} U_T(z(0)) - \frac{2}{k}.
\end{align*}

As \( \ln k > 1 \), we have (8.12) for all \( z \in A^* \), as was to be proved.

The proof of Proposition 1 is complete.

Remark 2. As follows from the proof presented above, formula (8.11) defines a rule that protects the asymptotic guarantee \( U \) of the first player for payoffs \( w_\lambda \), provided that the rules \( A^{T,h} \) that protect such an asymptotic guarantee for payoffs \( \zeta_h^{U} \) are known.

§ 9. Passing from \( \mathcal{W}^p \) to \( \mathcal{U}^b \). Proof of Proposition 2

9.1. Preliminary constructions and estimates. As in § 8, for all \( \tau \in \mathbb{R}_{>0} \) and \( z \in \mathbb{K} \) we define the function \( z_\tau : \mathbb{T} \mapsto \Omega \) by (8.1). Here, if \( z_\tau \in \mathbb{K} \), then for all \( \lambda > 0 \)
\begin{align*}
\int_0^\tau \lambda e^{-\lambda t}g(z(t)) \, dt & = w_\lambda(z) - \int_0^\infty \lambda e^{-\lambda(t+\tau)}g(z_\tau(t)) \, dt \\
& = w_\lambda(z) - e^{-\lambda \tau}w_\lambda(z_\tau).
\end{align*}

(9.1)

Note that \( U \) is bounded above by a positive number \( R \).

Consider an arbitrary positive integer \( k \); there exist numbers \( M > 1 \) and \( p > 1 \) such that
\begin{align*}
k = M \ln M \quad \text{and} \quad p \overset{\triangle}{=} e^{1/M}.
\end{align*}

Owing to the inequality \( 1 + x < e^x < 1 + x + x^2 \), which holds for all \( |x| \in (0,1) \), it follows from \( M > 1 \) that \( p = e^{1/M} = 1 + 1/M + r'/M^2 \) and \( p^{-1} = e^{-1/M} = 1 - 1/M + r''/M^2 \) for some \( r', r'' \in (0,1) \). Now
\begin{align*}
\frac{1 - p/M}{M(1 - p^{-1})} = \frac{1 - 1/M - 1/M^2 - r'/M^3}{1 - r''/M} < 1.
\end{align*}

(9.2)
By (4.4), given a positive integer \( k \) (and hence a number \( M \) defined above) one can find \( T_0 > k = M \ln M \) such that

\[
U_{pM/T} \leq U_{M/T} + \frac{1}{k^2} \quad \forall T > T_0. \tag{9.3}
\]

Since \( U \) is a lower solution, we can also choose \( T_0 \) in such a way that for any positive \( \lambda < M/T_0 \) and any \( h > 0 \) there exists \( A^{\lambda,h} \in \mathfrak{A} \) such that for any \( z \in A^{\lambda,h} \)

\[
U_{\lambda}(z(0)) - \frac{1}{k^2} \leq \lambda \int_0^h e^{-\lambda t} g(z(t)) \, dt + e^{-\lambda h} U_{\lambda}(z(h)). \tag{9.4}
\]

Fix such numbers \( k, M, p \) and \( T_0 \) and take an arbitrary \( T > T_0 \). Then put

\[
\lambda \triangleq \frac{1}{T}, \quad t_0 \triangleq \frac{T}{M}, \quad \tau_0 \triangleq 0, \quad t_i \triangleq t_0 p^{-i}, \quad \tau_i \triangleq \tau_{i-1} + t_{i-1} \quad \forall i \in \{1, \ldots, k\}.
\]

Now for all \( i \in \{1, \ldots, k\} \) and \( \omega \in \Omega \) we have

\[
p^{-k} = e^{-\ln M} = \frac{1}{M}, \tag{9.5}
\]

\[
\lambda p^i \leq \lambda p^k = \lambda M = \frac{M}{T} \leq \frac{M}{T_0}, \quad U_{\lambda p^i}(\omega) \geq U_{\lambda p^i-1}(\omega) - \frac{1}{k^2}. \tag{9.6}
\]

Note that the \( t_i \) form a monotone decreasing geometric progression; the \( \tau_i \) are their partial sums and

\[
\frac{1 - p/M}{M(1 - p^{-1})} = \frac{1 - p^{-k+1}}{M(1 - p^{-1})} = \frac{\tau_k}{T} \tag{9.2}\leq 1.
\]

**9.2. Constructing a near-\( v_T \) payoff.** We define a scalar function \( g \) on \((0, \tau_k]\) by

\[
g(t) = e^{-\lambda p^i(t-\tau_{i-1})} \quad \forall i \in \{1, \ldots, k\}, \quad t \in (\tau_{i-1}, \tau_i].
\]

Note that in this case for all \( i \in \{1, \ldots, k\} \) and \( t \in (\tau_{i-1}, \tau_i] \) we have

\[
1 \geq g(t) \geq e^{-\lambda p^i(\tau_{i+1} - \tau_i)} = e^{-\lambda p^i t_i} = e^{-\lambda t_0} = e^{-1/M} = p^{-1} > 1 - \frac{1}{M}. \tag{9.8}
\]

Consider the payoff

\[
c(z) \triangleq \frac{1}{T} \int_0^{\tau_k} g(z(t)) \, dt + p^{-k} U_{\lambda p^k}(z(\tau_k)) \quad \forall z \in \mathbb{K}.
\]

Recall that \( 0 \leq g \leq 1 \) and \( U \leq R \); now for any process \( z \in \mathbb{K} \) it follows from

\[
p^{-k} U_{\lambda p^k}(z(\tau_k)) \leq p^{-k} R \tag{9.5} \leq R \tag{9.5} = \frac{R}{M},
\]

that the following inequalities hold true for all \( z \in \mathbb{K} \):

\[
v_T(z) \geq \frac{1}{T} \int_0^{\tau_k} g(z(t)) \, dt \geq \frac{1}{T} \int_0^{\tau_k} \phi(t) g(z(t)) \, dt \geq c(z) - \frac{R}{M}. \tag{9.9}
\]
9.3. Constructing the rule $A^*$. Note that for all $i \in \{0, \ldots, k-1\}$ we have $e^{-\lambda^i t_i} = p^{-1}$ and $\lambda^i < M/T_0$ by (9.8) and (9.6), respectively. Now by (9.4) there exists a rule $A^{(i)} \triangleq A^\lambda, t_i \in \mathfrak{A}$ such that

$$U_{\lambda^i}(z(0)) \leq \lambda^i \int_0^{t_i} e^{-\lambda^i t} g(z(t)) \, dt + e^{-\lambda^i t_i} U_{\lambda^i}(z(t_i)) \quad \forall z \in A^{(i)}. \quad (9.10)$$

Since the right-hand side of inequality (9.10) depends only on $z|_{[0, t_i]}$, the rule $A^{(i)}$ can be replaced by an arbitrary rule that can be represented for an appropriate $A' \in \mathfrak{A}$ in the form $A^{(i)} \sqsubset_{t_i} A'$. Since for any $z \in A^{(i)} \sqsubset_{t_i} A'$ we have $z_{t_i} \in A' \subset \mathbb{K}$, it follows from (9.10) that

$$U_{\lambda^i}(z(0)) \leq \lambda^i \int_0^{t_i} e^{-\lambda^i t} g(z(t)) \, dt + e^{-\lambda^i t_i} U_{\lambda^i}(z(t_i)) + \frac{1}{k^2} \quad (9.11)$$

for all $i \in \{0, \ldots, k-1\}$, $A' \in \mathfrak{A}$ and $z \in A^{(i)} \sqsubset_{t_i} A'$.

By assumption 3.2.2 the following rule belongs to $\mathfrak{A}$:

$$A^* \triangleq A^{(0)} \sqsubset_{\tau_1} A^{(1)} \sqsubset_{\tau_2} \cdots \sqsubset_{\tau_{k-1}} A^{(k)} \in \mathfrak{A}. \quad (9.12)$$

Since $\tau_{i+1} = \tau_i + t_i$, there exists a rule $A' \in \mathfrak{A}$ such that $z_{\tau_i} \in A^{(i)} \sqsubset_{t_i} A'$ for any $z \in A^*$ and $i \in \{0, \ldots, k-1\}$. Now for all $z \in A^*$ and $i \in \{0, \ldots, k-1\}$ it follows from (9.11) that

$$U_{\lambda^i}(z(\tau_i)) \leq w_{\lambda^i}(z_{\tau_i}) - p^{-1} w_{\lambda^i}(z_{\tau_{i+1}}) + p^{-1} U_{\lambda^i}(z(\tau_{i+1})) + \frac{1}{k^2}. \quad (9.13)$$

Finally, by (9.6), for all $z \in A^*$ and $i \in \{0, \ldots, k-1\}$ we obtain

$$U_{\lambda^i}(z(\tau_i)) - p^{-1} U_{\lambda^{i+1}}(z(\tau_{i+1})) \leq w_{\lambda^i}(z_{\tau_i}) - p^{-1} w_{\lambda^i}(z_{\tau_{i+1}}) + \frac{2}{k^2}. \quad (9.13)$$

Recall that $A^* \in \mathfrak{A}$, and then for any $\omega \in \Omega$ we have

$$\mathcal{V}_T^\omega(\omega) \geq \inf_{z \in A^* \cap \Gamma(\omega)} v_T(z).$$

Since $M$ (by the relation $k = M \ln M$) can be chosen arbitrarily large, to prove the proposition it is sufficient to obtain the inequality

$$v_T(z) > U_{\lambda}(z(0)) - \frac{R}{M} - \frac{2}{M \ln M} \quad \forall z \in A^*, \quad (9.9)$$

which, in turn, follows from (9.9) and

$$c(z) \geq U_{\lambda}(z(0)) - \frac{2}{k} \quad \forall z \in A^*. \quad (9.14)$$
9.4. The proof of estimate (9.14). For any \( z \in A^* \) and \( i \in \{0, \ldots, k - 1\} \) we have \( z_{\tau_{i+1}} = (z_{\tau_i})_{\tau_{i+1}-\tau_i} \in \mathbb{K} \). Then we obtain

\[
\lambda p^i \int_{\tau_i}^{\tau_{i+1}} g(t)g(z) \, dt = \lambda p^i \int_{\tau_i}^{\tau_{i+1}} e^{-\lambda p^i (t-\tau_i)} g(z(t)) \, dt
\]

(8.1)

\[
= \lambda p^i \int_0^{\tau_{i+1}-\tau_i} e^{-\lambda p^i t} g(z_{\tau_i}(t)) \, dt
\]

(9.1)

\[
= w_{\lambda p^i} (z_{\tau_i}) - e^{-\lambda p^i (\tau_{i+1}-\tau_i)} w_{\lambda p^i} ((z_{\tau_i})_{\tau_{i+1}-\tau_i})
\]

(9.8)

\[
= w_{\lambda p^i} (z_{\tau_i}) - p^{-1} w_{\lambda p^i} (z_{\tau_i+1})
\]

(9.13)

\[
\geq U_{\lambda p^i} (z(\tau_i)) - p^{-1} U_{\lambda p^{i+1}} (z(\tau_{i+1})) - \frac{2}{k^2}.
\]

Thus, for all \( z \in A^* \) and \( i \in \{0, \ldots, k - 1\} \)

\[
\lambda \int_{\tau_i}^{\tau_{i+1}} g(t)g(z) \, dt \geq p^{-i} U_{\lambda p^i} (z(\tau_i)) - p^{-i-1} U_{\lambda p^{i+1}} (z(\tau_{i+1})) - \frac{2}{k^2}.
\]

Then summation over \( i \in \{0, \ldots, k - 1\} \) gives

\[
c(z) = p^{-k} U_{\lambda p^k} (z(\tau_k)) + \sum_{i=0}^{k-1} \lambda \int_{\tau_i}^{\tau_{i+1}} g(t)g(z_t) \, dt
\]

\[
\geq p^{-k} U_{\lambda p^k} (z(\tau_k)) + \sum_{i=0}^{k-1} \left[ p^{-i} U_{\lambda p^i} (z(\tau_i)) - p^{-i-1} U_{\lambda p^{i+1}} (z(\tau_{i+1})) - \frac{2}{k^2} \right]
\]

\[
= p^{-k} U_{\lambda p^k} (z(\tau_k)) + U_{\lambda} (z(\tau_0)) - p^{-k} U_{\lambda p^k} (z(\tau_k)) - \frac{2}{k}
\]

\[
= U_{\lambda} (z(0)) - \frac{2}{k} \quad \forall \ z \in A^*.
\]

This proves (9.14).

The proof of Proposition 2 is complete.

Remark 3. As follows from the proof of Proposition 2, the rule that protects the asymptotic guarantee \( U \) of the first player for payoffs \( v_T \) can be constructed using (9.12), provided that the rules that protect the same asymptotic guarantee for the payoffs \( \xi^{U}_{\lambda, h} \) are known.

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