A multilayer exponential random graph modelling approach for weighted networks

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November 20, 2018

Abstract

This paper introduces a new modelling approach to analyse weighted networks with ordinal/polytomous edge values. The proposed approach consists in modelling the weighted network generative process as a hierarchical multilayer ERGM process where each network layer represents a different ordinal dyadic category. Each layer is assumed to be generated by an ERGM conditional on the lower layer. A crucial advantage of the proposed method is the possibility of adopting the standard binary ERGM specification to model either the between-layer and across-layer generative processes thus facilitating the interpretation of the parameter estimates of the model. The Bayesian approach provides a natural way to estimate the uncertainty of the parameters associated to the local network effects included in the model. An extension of the approximate exchange algorithm is proposed to sample from the doubly-intractable posterior distribution of the parameters. Finally, applications of the methodology are illustrated on well-known real datasets and a goodness-of-fit diagnostic procedure for model assessment is proposed.

1 Introduction

Statistical network analysis concern modelling relationships defined by edges between nodes [Salter-Townshend et al., 2012]. In many empirical contexts, these relationships
have a strength associated with their edges [Barrat et al., 2004]. The nature of and variation in the strength of an edge between two nodes may be determined by a variety of aspects depending on the application context; for example, the amount of traffic flowing along connections in transportation networks [Opsahl et al., 2008], the functional connectivity levels in brain networks [Bullmore and Sporns, 2009], interactions in cellular and genetic networks [Horvath, 2011].

In the context of social network analysis, one of the most important families of models used to describe the connectivity network structure is represented by the family of exponential random graph models (ERGMs) [Holland and Leinhardt, 1981, Strauss and Ikeda, 1990]. These are generative models for networks postulating an exponential family over the sample space of networks on the fixed set of nodes, specified by a set of sufficient network statistics [Besag, 1974] representing sub-graph configurations of the observed network that are believed to be important to the generative process that is supposed to have produced the observed network structure. ERGMs are very flexible models as they can potentially incorporate any type of network statistic and were originally defined for networks with binary edges encoding the presence or absence of an edge between two nodes. Commonly used network statistics include summaries of density (e.g., number of edges), homophily (e.g., number of edges among nodes with the same nodal attribute), degree-based statistics (e.g., number of k-stars), and triad-based statistics (e.g., number of triangles) [Snijders et al., 2006].

One of the most important limitations in the use of ERGMs has been that the only way to use these exponential-family network models for analysing weighted networks consisted in dichotomising weighted edges according to their positions with respect to a certain threshold value. Although dichotomisation of weighted networks is often able to preserve some community structure [Yan et al., 2018], it generally leads to major issues related to the impact of the choice of the threshold value and consequent loss of information and increase of bias in terms of model estimates [Thomas and Blitzstein, 2011].

Some extensions of ERGMs to specific forms of weighted edges have been proposed by Robins et al. [1999], Pattison and Wasserman [1999]. More recently, several modelling extensions based on ERGMs have been proposed: these include the multi-valued curved ERGMs [Wyatt et al., 2010], the generalised ERGMs for inference on networks with continuous edge values [Desmarais and Cranmer, 2012] and Geometric/Poisson reference ERGMs for ordinal/count networks [Krivitsky, 2012].

In this paper, we propose a Bayesian hierarchical multilayer ERGM approach for
ordinal/polytomous network data in order to simplify model specification and provide a substantial improvement in terms of interpretability by making use of binary ERGMs to capture the dependence structure between ordinal categories (layers) of the weighted network structure. The paper is organised as follows. In Section 2, we review the main features of ERGMs and curved ERGMs and in Section 2.2 we proceed by describing their most recent ERGM extensions for the analysis of weighted networks. In Section 3 we show how multilayer graphs can be used to represent weighted network structures. In Section 4 we introduce the multilayer ERGM approach. In Section 4.3 we focus on the interpretation of the multilayer ERGM framework as a dissolution process which is able to capture the between-layer generative process between the network layers. In Section 5 we generalise the modelling framework using a Bayesian hierarchical modelling approach and we propose to extend the approximate exchange algorithm [Caimo and Friel, 2011] to sample from the doubly-intractable parameter posterior distribution. In Section 6, we demonstrate the usefulness of the multilayer ERGM approach by analysing two very well-known network datasets and assessing model fit by comparing goodness of fit diagnostics based on the weighted degree distribution.

2 Exponential random graph models (ERGMs)

Typically networks consist of a set of actors and relationships between pairs of them, for example social interactions between individuals. The relational structure of a network graph is described by a random adjacency matrix $Y$ of a graph on $n$ nodes (actors) and a set of edges (relationships) $\{Y_{ij} : i = 1, \ldots, n; j = 1, \ldots, n\}$ where:

$$Y_{i,j} = \begin{cases} 1, & i \sim j; \\ 0, & i \not\sim j. \end{cases}$$

The network graph $Y$ may be directed (digraph) or undirected depending on the nature of the relationships between the actors and $y$ a realisation of $Y$.

Exponential random graph models (ERGMs) are a particular class of discrete linear exponential families which represent the probability distribution of $Y$ as:

$$p(y|\theta) = \frac{\exp\{\theta' s(y)\}}{c(\theta)}, \quad (1)$$

where $s(y)$ is a known vector of $p$ sufficient statistics, $\theta \in \mathbb{R}^p$ is the associated parameter vector, and $c(\theta)$ a normalising constant which is difficult to evaluate for all but trivially small graphs. The dependence hypothesis at the basis of these models
is that edges self organize into small structures called configurations. There is a wide range of possible network configurations [Robins et al., 2007] which give flexibility to adapt to different contexts. A positive parameter value for \( \theta_i \) result in a tendency for the certain configuration corresponding to \( s_i(y) \) to be observed in the data than would otherwise be expected by chance.

2.1 Curved ERGMs

The ERGM likelihood defined in Equation 1 can be generalised by assuming that the parameter \( \eta(\theta) \) is nonlinear in the exponential family of distribution

\[
p(y|\theta) = \exp \left\{ \eta(\theta)^t s(y) - c[\eta(\theta)] \right\},
\]

which is therefore a curved exponential family [Hunter, 2007]. Monte Carlo methods [Hunter and Handcock, 2006] can be used to estimate the decay parameter of the geometrically weighted network statistics introduced by Snijders et al. [2006] and currently used to alleviate ERGM degeneracy issues [Handcock, 2003].

2.2 ERGMs for weighted networks

A weighted network can be described by \( N \times N \) weighted adjacency matrix \( Y \) where:

\[
Y_{i,j} = \begin{cases} y_{i,j} > 0, & i \sim j; \\
y_{i,j} = 0, & i \not\sim j. \end{cases}
\]

The edge value of \( y_{i,j} \) can be ordinal, count, bounded continuous and unbounded continuous. We denote with \( D \) and \( E \) the number of dyads and (weighted) edges of the network respectively.

2.2.1 Multi-valued curved ERGMs

Multi-valued curved ERGMs [Wyatt et al., 2010] allow network edges to take discrete, ordinal values representing the observed intensity of the connection between two nodes. To permit comparisons with binary ERGMs models, the edge values are scaled so that the smallest is 0 (empty dyad) and the largest is 1. Standard binary specification of Equation 2 is therefore redefined in a straightforward way: the density statistic of the weighted network (corresponding to the number of edges in the binary case) is captured by the sum of all edge values \( \sum_{i \not\sim j} y_{i,j} \); the nodal degree is corresponding to the sum of the values of the edges incident to that node. More complicated network
statistics involving extra-dyadic configurations require defining the intensity of the configuration. The geometric mean of the edge values composing the configuration is taken as the intensity value for the configuration. For example, a shared partner \(k\) for nodes \(i\) and \(j\) is defined to be a partner of intensity \(\sqrt{y_{ik}y_{jk}}\). This way it is possible to defined the count of shared partners for a dyad \((i, j)\) as the sum of these intensities of each shared partner: \(\sum_k\sqrt{y_{ik}y_{jk}}\). Consequently, it is possible to generalise commonly used statistics [Snijders et al., 2006], such as the geometrically weighted edgewise shared partners, by considering a geometrically weighted function of the intensities corresponding to the overall edgewise shared partner distribution defined above.

### 2.2.2 Generalised ERGMs for continuous data

Generalised ERGMs (GERGMs) [Cranmer and Desmarais, 2011, Desmarais and Cranmer, 2012] consists of two specification steps. In the first step a distribution that captures the dependencies of interest on a restricted continuous weighted network \(X \in [0, 1]^D\):

\[
p(x|\theta) = \frac{\exp\{\theta^t s(x)\}}{\int_{[0, 1]^D} \exp\{\theta^t s(x)\} dx}
\]

(3)

where \(s : [0, 1]^D \to \mathbb{R}^p\) is formulated to represent joint features of \(Y\) in the distribution of \(X\). The second step consists in transforming \(X\) onto the support of \(Y\) by applying parameterised monotone increasing transformations \(G^{-1}(\cdot)\) to the edges of \(X_{i,j}\):

\[y_{i,j} = G^{-1}(x_{i,j}, \lambda_{i,j}).\]

Following the properties of multivariate transformations, we have:

\[p(y|\theta, \lambda) = p(G(y, \lambda)|\theta) |J|
\]

where \(G\) is a cumulative distribution function parameterised to match \(Y\) in terms of support, location, scale, and dependence on covariates and \(J\) is Jacobian matrix of first partial derivatives \(g(y_{i,j}, \lambda_{i,j})\), where \(g\) is the probability density function of \(G\).

The challenge in estimating \(\lambda\) and \(\theta\) is that the intractable integral in Equation 3 can be accomplished by a Markov chain Monte Carlo maximum likelihood estimation (MCMC-MLE) method [Geyer and Thompson, 1992]. In terms of model specification, GERGMs are flexible continuous exponential family that can be adapted to various applications. However, future challenges concern the development of guidelines for
the selection of $G$ and model specifications that are easy to interpret and resistant to model degeneracy.

### 2.2.3 ERGMs for count data

A natural extension of ERGMs for valued networks has been proposed by Krivitsky [2012] and it is defined by the specification of a reference measure for the probability distribution defined in Equation 1. The reference measure represents the distribution relative to which the ERGM form is specified and therefore determines the baseline probability distribution of the dyads of $Y$ and constrains the space of the parameter $\theta$. The reference measure can be derived by the choice of a function $h : Y \rightarrow [0, \infty)$, such that:

$$p(y|\theta) = h(y) \frac{\exp\{\theta s(y)\}}{c(\theta)}.$$ 

For a random graph model where the only network statistic included in the model is the sum of the weighted edges $s(y) = \sum_{i \neq j} y_{ij}$ we have that if $h(y) = 1$ then we have geometric-reference ERGMs, i.e., each dyad is a Geometric random variable $Y_{ij} \sim \text{Geometric}(p = 1 - \exp(\theta))$. If $h(y) = \prod_{i \neq j} (y_{ij})^{-1}$ then we have Poisson-reference ERGMs: i.e., each dyad is a Poisson random variable $Y_{ij} \sim \text{Poisson}(\lambda = \exp(\theta))$. For this latter case, specific network statistics able to represent common network structural properties and distributions of counts have been proposed. In particular, zero-modification parametrisation is proposed in order to account for sparsity and a Conway-Maxwell-Poisson distribution-based network statistic can be incorporated to represent both under- and over-dispersion.

### 3 Weighted networks as multilayer graphs

In this section, we generalise the notation for weighted network in order to obtain a useful representation of their connectivity structure. In particular Robins et al. [1999] showed how weighted networks, and specifically polytomous networks, can be represented by three-dimensional adjacency arrays rather than valued adjacency matrices and how this choice leads to a natural model parameterisation in the context of exponential random graph models by extending the Hammersley-Clifford Theorem for networks where each dyad can take only a finite number of values. This approach is particularly suitable for ordinal networks but it can be easily adapted to a categorisation procedure where multiple thresholding is performed on count or real-valued
networks in order to obtain a binary three-dimensional adjacency array corresponding to what we call a *multilayer* adjacency matrix.

### 3.1 Notation

Suppose to observe an undirected weighed network $Y$ on $N$ nodes. We consider a $N \times N \times W$ adjacency array that we denote with $y_{\{W\}}$, consisting of a set of *network layers* $\{Y_w, w = 1, \ldots, W\}$ representing ordered binary adjacency matrices with dyads defined as follows:

$$Y_{i,j,w} = \begin{cases} 1, & y_{i,j} \geq \tau_w; \\ 0, & y_{i,j} < \tau_w. \end{cases}$$

where $\tau_W = \{\tau_1, \ldots, \tau_W\}$ is a vector of a strictly increasing sequence of (threshold) values. The set of edges observed in a layer $w$, $E_w$, is a subset the set of the edges observed in the lower layers, i.e.: $D_1 \supset E_1 \supset E_2 \supset \cdots \supset E_W$, where $D_1$ is the set of the dyads of the weighted network (Figure 1).

![Figure 1: The multilayer structure of a random weighted network $Y_{\{3\}}$ can be described by a hierarchy of three overlapping binary layers.](image)

### 4 Multilayer ERGMs

The basic assumption of the multilayer random graph model is that the probability of observing a multilayer network corresponds to the product of conditional random graph models:

$$7$$
\[ p(y_{[W]} | \theta) = p(y_1, y_2, \ldots, y_W | \theta) = p(y_1 | \theta) \prod_{w=1}^{W} p(y_{w+1} | y_w, \theta), \quad (4) \]

where \( \theta \) is a vector of parameters, \( p(y_1 | \theta) \) is the unconditional model for the first layer \( y_1 \), and \( p(y_{w+1} | y_w, \theta) \) is the transition probability from layer \( y_w \) to layer \( y_{w+1} \) by assuming first order Markov dependence between layers (this assumption may be generalised to higher order Markov dependence). The multilayer model is essentially the product of stepwise ERGMs across layers.

### 4.1 The multilayer random graph model

Suppose \( p \) is the probability that any two nodes \((i, j)\) are connected in any network layer \( w \) and \( q = 1 - p \) is the probability that any two nodes are not connected in any network layer. The random multilayer graph model can be defined as in Equation 4:

\[
p(y_1 | p) = p^{E_1} q^{D_1 - E_1},
\]
\[
p(y_{w+1} | y_w, p) = p^{E_{w+1}} q^{E_w - E_{w+1}} = p^{E_{w+1}} q^{D_{w+1} - E_{w+1}}, \quad \forall w > 1; \quad (5)
\]

where \( D_1 \) is the overall number of dyads and \( E_w = D_{w+1} \) is the number of edges in the network layer \( y_w \) which is corresponding to the number of random dyads in the network layer \( y_{w+1} \).

The random multilayer graph model corresponds to the weighted random graph model defined by Garlaschelli [2009] which is equivalent to the geometric-reference random graph model defined by Krivitsky [2012]. In fact, merging Equations 4 and 5, we have that:

\[
p(y_{[W]} | p) = \prod_{w=1}^{W} p^{E_w} q^{D_w - E_w} = p^{\sum_{w=1}^{W} E_w} q^{\sum_{w=1}^{W} D_w - \sum_{w=1}^{W} E_w}.
\]

Since:

\[
\sum_{w=1}^{W} D_w = D_1 + \sum_{w=2}^{W} D_w = D_1 + \sum_{w=1}^{W} E_w,
\]

we have that:

\[
p(y_{[W]} | p) = p^{\sum_{w=1}^{W} E_w} q^{D_1} = p^{\sum_{w=1}^{W} \sum_{i,j \in D_1} y_{i,j,w}} q^{D_1} = \prod_{(i,j) \in D_1} p^{\sum_{w=1}^{W} y_{i,j,w}} q.
\]
which corresponds to the product of the probabilities that any two nodes \((i, j)\) are connected by an edge of weight \(\sum_{w=1}^{W} y_{i,j,w} = y_{i,j}\) and therefore each weighted dyad \(Y_{i,j} \overset{i.i.d.}{\sim} \text{Geometric}(q)\). The exponential form of the geometric-reference random graph model defined in Equation 5 is:

\[
p(y_{i|w} | \theta) = \prod_{(i,j) \in \mathcal{D}_1} \frac{\exp \{ \theta y_{i,j} \}}{1 - \exp(\theta)} = \frac{\exp \left\{ \theta \sum_{(i,j) \in \mathcal{D}_1} y_{i,j} \right\}}{c(\theta)},
\]

where \(c(\theta)\) is a normalising constant, \(1 - \exp(\theta) = q\) and \(\exp(\theta) = p\) and therefore \(\theta = \ln(p)\). A direct interpretation of the density parameter \(\theta\) is therefore problematic as \(\theta < 0\).

### 4.2 Multilayer ERGMs as dissolution processes

The multilayer random graph model defined in the previous section can be easily extended in order to incorporate extra-dyadic network effects. In Table 1, we can observe that edges between two nodes in layer \(y_{w}\) may or may not ‘survive’ to the next upper layer \(y_{w+1}\) whereas empty edges in layer \(y_{w}\) remains empty edges in all the upper layers.

| \(Y_{i,j,w} \) | \(Y_{i,j,w+1}\) |
|----------------|-----------------|
| 0              | 0               |
| 1              | 0               |
| 1              | 1               |

The conditional log-odds of an edge between \(i\) and \(j\) in layer \(y_{w+1}\), given the presence of an edge between \(i\) and \(j\) in layer \(y_{w}\) and all the values of the other edges in the network fixed is:

\[
\ln \left( \frac{\Pr(Y_{i,j,w+1} = 1 \mid Y_{i,j,w} = 1, Y_{-(i,j,w+1)})}{\Pr(Y_{i,j,w+1} = 0 \mid Y_{i,j,w} = 1, Y_{-(i,j,w+1)})} \right) = \phi^T \Delta(y)_{i,j,w+1}, \tag{6}
\]

where \(\phi\) is a vector of parameters and \(\Delta(y)_{i,j,w+1} = s(y^+)_{i,j,w+1} - s(y^-)_{i,j,w+1}\) is the vector of change in the value of the network statistics for layer \(y_{w+1}\).

From Table 1, we can notice that the formation process is not allowed when moving from layer \(w\) to layer \(w + 1\), so the conditional log-odds of an edge between \(i\) and \(j\) in layer \(w + 1\), given the absence of an edge between \(i\) and \(j\) in layer \(w\) is:
\[
\ln \left( \frac{\Pr(Y_{i,j,w+1} = 1 \mid Y_{i,j,w} = 0, Y_{-(i,j,w+1)})}{\Pr(Y_{i,j,w+1} = 0 \mid Y_{i,j,w} = 0, Y_{-(i,j,w+1)})} \right) = -\infty,
\]

as \( \Pr(Y_{i,j,w+1} = 1 \mid Y_{i,j,w} = 0, Y_{-(i,j,w+1)}) = 0 \) and \( \Pr(Y_{i,j,w+1} = 0 \mid Y_{i,j,w} = 0, Y_{-(i,j,w+1)}) = 1 \). So the parameter \( \phi \) associated with the network effects expressed by the network statistics \( s(\cdot) \) provides insights about the contribution of each network statistic to edge dissolution between consecutive network layers.

If we consider the multilayer random graph model defined in Section 4.1, where only the number of edges is included in the model and therefore \( \Delta(y)_{i,j,w+1} = 1 \), the relationship between \( \phi \) and \( \theta \) is:

\[
\phi = \ln(p) - \ln(q) = \theta - \ln(1 - \exp(\theta)).
\]

The likelihood of multilayer ERGMs defined in Equation 4, can be written in the log-linear form as:

\[
p(y_{i,W} | \phi) = \frac{\exp\{\phi ^ t s(y_{i,W})\}}{c(\phi, y_{i,W})} = \frac{\exp\{\phi ^ t s(y_{1})\}}{c(\phi, y_{1})} \prod_{w=1}^{W-1} \frac{\exp\{\phi ^ t s(y_{w+1}; y_{w})\}}{c(\phi, y_{w})},
\]

where \( s(y_{w+1}; y_{w}) \) is a vector representing the number of network statistics who did not dissolve between two layers.

The interpretation of the multilayer ERGM as a dissolution process implies that many network statistics developed for ERGMs can be readily used within this modelling framework, retaining much of their interpretation. A positive value for the parameter \( \phi_i \) corresponding to a particular network statistic \( s_i(y) \) increases the probability of observing that network statistic in the next upper layer while a negative value for the parameter \( \phi_i \) decreases the probability of observing the corresponding network statistic \( s_i(y) \) in the next upper layer. So, this means that the dissolution process between two layers can be interpreted exactly like a binary ERGM process. An analogous interpretation of multilayer ERGMs consists in considering a formation process from the layer \( W \) to 1 by just reversing the direction of the dependence structure between layers.

### 4.3 Relaxing homogeneity assumption across layers

We can relax the parameter homogeneity assumption across network layers by considering layer-specific ERGM processes:

\[
p(y_{i,W} | \phi_1, \ldots, \phi_W) = p(y_1 | \phi_1) \prod_{w=1}^{W-1} p(y_{w+1} | y_w, \phi_{w+1}).
\]
It is important to notice that the ERGM for the first layer $p(y_1|\phi_1)$ is a standard binary ERGM that is equivalent to a ERGM dissolution process defined in Section 4.2 conditional on the full binary network graph that we can denote by $y_0$ defined on the same set of nodes, i.e., $y_{ij0} = 1, \forall i, j$. Consequently we have that $p(y_1|\phi_1) = p(y_1|y_0, \phi_1)$, as:

$$\ln \left( \frac{\Pr(Y_{i,j} \geq \tau_1 | \mathbf{Y}_{-ij})}{\Pr(Y_{i,j} = 0 | \mathbf{Y}_{-ij})} \right) = \ln \left( \frac{\Pr(Y_{i,j,1} = 1 | y_{i,j,0} = 1, \mathbf{Y}_{-(i,j,1)})}{\Pr(Y_{i,j,1} = 0 | y_{i,j,0} = 1, \mathbf{Y}_{-(i,j,1)})} \right).$$

The structure of the modelling approach, offers a flexibility in terms of modelling specification. It is in fact possible to specify different network statistics for modelling different network layers.

The conditional log-odds of an edge with weight $w^*$ between $i$ and $j$ is:

$$\ln \left( \frac{\Pr(Y_{i,j} = w^* | \mathbf{Y}_{-ij})}{\Pr(Y_{i,j} = 0 | \mathbf{Y}_{-ij})} \right) = \sum_{w=1}^{w^*} \phi^t_{w^*} \Delta(y)_{i,j,w},$$

where $\Delta(y)_{i,j,w}$ is the vector of change in the value of the network statistics for layer $y_w$.

5 Bayesian inference

The growing interest in Bayesian techniques for the analysis of social networks can be attributed to the development of efficient computational tools [Caimo and Friel, 2011, Alquier et al., 2016, Caimo and Mira, 2015, Bouranis et al., 2017, 2018] and the availability of user-friendly software [Caimo and Friel, 2014]. Bayesian analysis is a promising approach to social network analysis because it yields a rich fully-probabilistic evaluation of uncertainty which is essential when dealing with complex and heterogeneous relational data. Using a Bayesian framework leads directly to the inclusion of prior information about the network effects into the modelling framework, and provides immediate access to the uncertainties by evaluating the posterior distribution of the parameters associated with the network effects. In social network analysis, the Bayesian approach leads to the possibility of specifying informative parameter prior distribution consistent with some a priori expectation, for example, in terms of low density and high transitivity [Caimo and Lomi, 2014, Caimo et al., 2017]. In fact, parameter prior distribution can be concentrated on negative values for the density
parameter and positive values for transitivity parameters and/or even positive correlation between density and transitivity parameters.

In the following sections we will be extending the modelling framework introduced in Section 4.3 and we will focus on parameter estimation by describing an approximate exchange algorithm [Murray and Ghahramani, 2004, Caimo and Friel, 2011] which can sample from the doubly-intractable ERGM posterior distribution of the model parameters.

5.1 A hierarchical framework

The Geometric-reference ERGM approach [Garlaschelli, 2009, Krivitsky, 2012] is not capable of modelling each between-layer dependence in the network and, on the other hand, the multilayer modelling approach defined in Section 4.3 is not capable of capturing overall trends of the network effects across the entire multilayer structure. For this reason, in order to model both the relational processes simultaneously and improve the goodness of fit of our model, we specify a hierarchical multilayer ERGM where layer specific parameters $\phi_1, \ldots, \phi_W$ are coupled through a random variable $\eta$ representing the overall across-layer trend of $p$ network effects of interest with prior distribution $p(\eta|\gamma_0)$ defined by the hyper-parameters $\gamma_0$. In particular, we define the following Bayesian hierarchical model:

$$p(\phi_1, \ldots, \phi_W, \eta|y\{W\}) \propto p(y\{W\}|\phi_1, \ldots, \phi_W) \ p(\phi_1, \ldots, \phi_W|\eta) \ p(\eta|\gamma_0)$$

(10)

and its structure is displayed in Figure 2.

Figure 2: Graphical structure of the hierarchical multilayer ERGM defined in Equation 10.

We choose a $p$-dimensional model assuming that the layer parameters are independent realisations from a normal distribution, i.e.:
\[ p(\phi_1, \ldots, \phi_W|\eta) = \prod_{w=1}^{W} p(\phi_w|\mu, \Sigma), \]

where

\[ \phi_w|\mu, \Sigma \sim \mathcal{N}_p(\mu, \Sigma). \]

In particular, we assume that a Normal-Inverse-Wishart prior setting where:

\[ \Sigma \sim \mathcal{W}^{-1}(\Lambda_0, \nu_0); \quad \mu|\Sigma \sim \mathcal{N}(\mu_0, \Sigma), \]

so that the prior distribution of \( \mu, \Sigma \) is:

\[ \mu, \Sigma \sim \mathcal{NW}^{-1}(\mu_0, \kappa_0, \Lambda_0, \nu_0). \]

The full conditional distribution of \( \mu, \Sigma \) is:

\[ \mu, \Sigma|\phi_1, \ldots, \phi_W \sim \mathcal{NW}^{-1}(\mu_1, \kappa_1, \Lambda_1, \nu_1); \]

with parameters:

\[ \mu_1 = \frac{\kappa_0}{\kappa_0 + W} \mu_0 + \frac{W}{\kappa_0 + W} \bar{\phi}; \quad \kappa_1 = \kappa_0 + W; \]

\[ \Lambda_1 = \Lambda_0 + S + \frac{\kappa_0 W}{\kappa_0 + W} (\bar{\phi} - \mu_0)(\bar{\phi} - \mu_0)^t; \quad \nu_1 = \nu_0 + W; \]

where

\[ \bar{\phi} = \frac{1}{W} \sum_{w=1}^{W} \phi_w; \quad S = \sum_{w=1}^{W} (\phi_w - \bar{\phi})(\phi_w - \bar{\phi})^t. \]

### 5.2 Estimation

To estimate the posterior distribution defined in Equation 10, we extend the approximate exchange algorithm of Caimo and Friel [2011] to sample from the doubly-intractable distribution \( p(\phi_1, \ldots, \phi_W|y_W, \mu, \Sigma) \). This is an asymptotically exact MCMC algorithm which consists in sampling \( \phi_1, \ldots, \phi_W \) using the approximate exchange algorithm described in Algorithm 1 and then drawing \( \mu \) and \( \Sigma \) from the full conditional distribution via Gibbs sampling.
We denote the unnormalised between layer ERGM likelihood as \( q_{\phi_w}(y_w; y_{w-1}) = \exp\{\phi'_{w} s(y_{w+1}; y_{w})\} \) and \( h(\cdot) \) a proposal distribution for updating the model parameters. Adaptive strategies [Caimo and Friel, 2011, Caimo and Mira, 2015] and approximate transition kernels approaches [Alquier et al., 2016] have been successfully implemented in this context.

Algorithm 1 Approximate exchange algorithm for \( p(\phi_1, \ldots, \phi_W | y_{\{W\}}, \mu, \Sigma) \)

Initialise \( \phi_{1}^{(1)}, \ldots, \phi_{W}^{(1)} \)

for \( i = 1, \ldots, N \) do

for \( w = 1, \ldots, W \) do

1) \( \phi'_w \sim h(\cdot | \phi_{w}^{(i)}) \)

2) \( y'_w \sim p(\cdot | \phi'_w, y_{w-1}) \)

3) Set \( \phi_{w}^{(i+1)} = \phi'_w \) with probability:

\[
\min \left( 1, \frac{q_{\phi_w}(y'_w; y_{w-1}) p(\phi_1, \ldots, \phi'_w, \ldots, \phi_W | \mu, \Sigma) q_{\phi'_w}(y'_w; y_{w-1})}{q_{\phi_w}(y_w; y_{w-1}) p(\phi_1, \ldots, \phi_w^{(i)}, \ldots, \phi_W | \mu, \Sigma) q_{\phi'_w}(y'_w; y_{w-1})} \right)
\]

end for

end for

In terms of computational aspects, the approach becomes increasingly expensive as the network size and the number of layers increase. However it is important to notice that the number of MCMC iterations needed for simulating the auxiliary network layer \( y'_w \) (step 2) should be proportional to the number of edges \( E_{w-1} \) in the previous network layer \( y_{w-1} \) as they are the only random dyads \( D_w \) involved in the ERGM simulation process of \( y_w \). For this reason, the number of iterations required to simulate a higher network layer can be set to be smaller than the number of iterations required to simulate a lower network layer.

6 Applications

In this section we present two examples of application of the hierarchical multilayer modelling framework defined in Section 5.1. We will be considering two very well-known datasets: the Bernard and Killworth office network and the Zachary karate club network. We will focus on a polytomous transformation of the networks by thresholding them at 3 different dyadic values so as to obtain a 3-layered network. This procedure has been carried out by arbitrarily defining threshold values, for example based on the quantiles of the edge weight distribution, or that guarantee a ‘sufficient’
difference in terms of density between consecutive network layers or can be guided by some theoretical consideration that allow us to interpret the edge strength intervals. However the issue of selecting the network layers determined by multiple thresholding is beyond the scope of this paper.

6.1 Bernard and Killworth office network

The Bernard and Killworth office network dataset concerns the observed frequency of interactions between $N = 40$ individuals in a small business office.

Figure 3: Weighted graph structure of the Bernard and Killworth office network derived by thresholding at $\tau_1 = 2, \tau_2 = 4, \tau_3 = 8$.

The network graph and the multilayer visualisation of the Bernard and Killworth office network graph obtained by selecting 3 layers of the weighted graph is shown in Figure 3 and Figure 4 respectively.

6.1.1 Model specification

One of the most important aspects of the multilayer modelling approach is that we can include binary network statistics [Snijders et al., 2006] to describe the weighted network topology. For the Bernard and Killworth office network we propose the following model specification:

$s_1$: Edge statistic (edges) is the number of edges in the network: $\sum_{i>j} y_{ij}$. This statistic captures the network density effect.
Figure 4: The multilayer structure of the Zachary karate club weighted network. The number of edges in each layer is: $E_1 = 123, E_2 = 44, E_3 = 15$.

$s_2$: Geometrically weighted degree statistic ($gw\text{degree}$): $\sum_d g_d(\alpha)D_d(y)$, where $g_d(y)$ is an exponential weight function. This statistic captures the tendency towards centralisation in the degree distribution of the network.

$s_3$: Geometrically weighted edgewise shared partner statistic ($gw\text{esp}$) is a function of the edgewise shared partner statistics $EP_d(y)$ defined as the number of unordered connected pairs $(i, j)$ (partners) that are both connected to exactly $d$ other nodes: $\sum_d g_d(\alpha)EP_d(y) = \sum_d g_d(\alpha)\sum_{i<j} y_{ij} 1\{\sum_k y_{ik} y_{jk} = d\}$, where $1\{\cdot\}$ is the indicator function. This statistic captures the tendency towards transitivity, i.e., the tendency of edges to be connected through multiple triadic relations simultaneously.

6.1.2 Posterior analysis

We used adaptive direction sampling based on 6 MCMC chains consisting of 10,000 iterations each for improving the mixing of the approximate exchange by tuning the algorithm parameters in order to get about 20% acceptance rate. In terms of hyperparameters for the hierarchical model, we set $\kappa_0 = 1, \Lambda_0 = I_p, \nu_0 = p + 2$, where $I_p$ is the $p \times p$ identity matrix. Traceplots for the model parameters can be found in the Appendix.

The posterior distribution for $\mu$, displayed in Figure 5 and summarised in Table 2, gives an idea of the general tendencies of the three local effects across the network layers. A large part of the probability density of $\mu_1$ (corresponding to the edgess statistic) is concentrated on negative values. This is compensated by a large part of the probability density of $\mu_3$ (corresponding to the $gw\text{esp}$ statistic) mostly concentrated on positive values. These two tendencies explain the overall increasing sparsity and
transitivity of the multilayer process across the network layers.

![Figure 5: Posterior distribution for μ.](image)

| Parameter (Effect) | Mean | SD  |
|--------------------|------|-----|
| $\mu_1$ (edges)    | -1.10| 1.00|
| $\mu_2$ (gwdegree) | 0.14 | 0.92|
| $\mu_3$ (gwesp)    | 0.38 | 0.78|

Table 2: Posterior estimates for $\mu$.

The predictive posterior for $\phi_1, \phi_2, \phi_3$, displayed in Figure 6 and summarised in Table 3, allows us to understand the between-layer ERGM processes. In particular we can notice that most of the density dissolution process described above is concerning the generation of the first network layer as the credible interval of $\phi_1$ (corresponding to the edges statistic) falls completely on negative values. On the other hand the increase of transitivity captured by $\phi_3$ (corresponding to the gwesp statistic) is observed between each layer meaning that the dissolution ERGM process across layers affects edges that are not embedded into transitive structures. The tendency towards centralisation in the degree distribution represented by $\phi_2$ (corresponding to the gwdegree statistic) does not seem to be important in explaining both the overall and the between-layer weighted structure of the network meaning that the strong edges are not necessarily centralised or dispersed in the degree distribution.

6.2 Zachary karate club network

The Zachary karate club network concerns social relations in a university karate club involving 34 individuals. The network graph in Figure 7 shows the relative strength
Figure 6: Predictive posterior for $\phi_1$ (edges), $\phi_2$ (gwdegree), $\phi_3$ (gwesp).

Table 3: Predictive posterior estimates for $\phi_1$, $\phi_2$, $\phi_3$ for every network layer.

| Parameter (Effect) | Layer 1 | Layer 2 | Layer 3 |
|--------------------|---------|---------|---------|
| $\phi_1$ (edges)   | -2.94   | -0.53   | -0.94   |
| $\phi_2$ (gwdegree)| 1.17    | -0.56   | -0.08   |
| $\phi_3$ (gwesp)   | 0.56    | 0.31    | 0.70    |

of the associations, i.e., the number of situations in and outside the club in which interactions occurred between individuals.

A multilayer visualisation of the karate club network obtained by selecting 3 layers of the weighted graph is shown in Figure 8.

6.2.1 Model specification

We include in the model the edges ($s_1$) and gwesp ($s_2$) statistics as in the previous example but we replace the gwdegree statistic with two different statistics, defined as follows:

$s_3$: Geometrically weighted non-edgewise shared partner statistic (gwnsp) is a function of the non-edgewise shared partner statistics $NP_d(y)$ defined as the number of unordered unconnected pairs $(i, j)$ to exactly $d$ other nodes: $\sum_d g_d(\alpha) DP_d(y) = \sum_d g_d(\alpha) \sum_{i<j} 1\{\sum_k y_{ik} y_{jk} = d\}$. This statistic captures the tendency of non-directly-connected nodes to be connected through multiple others.

$s_4$: Homophily statistic (nodematch) is the number of edges between actors having the same nodal attribute $x$: $\sum_{i>j} y_{ij} 1_{\{x_i = x_j\}}$, where, for this example, $x$ is the...
Figure 7: Weighted graph structure of the Zachary karate club network derived by thresholding at $\tau_1 = 1, \tau_2 = 3, \tau_3 = 4$.

Figure 8: The multilayer structure of the Zachary karate club weighted network. The number of edges in each layer is: $E_1 = 78, E_2 = 48, E_3 = 21$. 
faction alignment of the club members. This statistics captures the density of edges between nodes within the same faction.

6.2.2 Posterior analysis

As for the previous application, we used adaptive direction sampling based on 8 MCMC chains consisting of 10,000 iterations each for improving the mixing of the approximate exchange by tuning the algorithm parameters in order to get about 20% acceptance rate. In terms of hyper-parameters for the hierarchical model, we used the same set-up of the previous application.

The posterior distribution for $\mu$, displayed in Figure 9 and summarised in Table 4, gives an idea of the general tendencies of the 4 local effects across the 3 network layers. A large part of the probability density of $\mu_1$ (corresponding to the edges statistic) is concentrated on negative values. This is compensated by a large part of the probability density of $\mu_2$ (corresponding to the gwesp statistic) and $\mu_4$ (corresponding to the nodematch statistic) mostly concentrated on positive values. These tendencies explain the overall increasing sparsity concerned mainly edges that are not connecting nodes within factions and/or are not embedded in transitive triads.

| Parameter (Effect) | Mean | SD   |
|-------------------|------|------|
| $\mu_1$ (edges)   | -1.20| 1.16 |
| $\mu_2$ (gwesp)   | 0.40 | 0.77 |
| $\mu_3$ (gwnsp)   | 0.02 | 0.76 |
| $\mu_4$ (nodematch)| 0.51 | 0.82 |

Table 4: Posterior estimates for $\mu$.

More specifically the estimated predictive posterior for $\phi_1, \phi_2, \phi_3, \phi_4$, displayed in Figure 10 and summarised in Table 5, indicates that most of the density dissolution process described above and represented by the parameter $\phi_1$ (corresponding to the edges statistic) is stronger in the first and between the second and the third network layer. We can also observe the importance of transitivity captured by $\phi_3$ (corresponding to the gwesp statistic) between all the layers meaning that the dissolution ERGM process across layers affects mainly edges that are not embedded into triadic transitive structures. We can therefore observe that the strongest edges, i.e., the ones surviving the multilayer ERGM dissolution process, are mainly the ones involved in transitive triads. It is interesting to notice how the parameter $\phi_3$ (corresponding to the gwnsp
Figure 9: Posterior distribution for $\mu$. 
statistic) is contributing positively to the generation of the first layer and negatively to the generation of the second layer conditional on the first layer. This can be explained by the fact that connectivity structure of layer 1 is very much influenced by the presence of open transitive triads involving the two faction competing leaders of the karate club (John and Mr. Hi) who are not directly connected but share many partners among the members of the club. However when we consider the transition from layer 1 to layer 2 many connections involved in these open triads dissolve meaning that they are mostly weak edges. The faction density representing homophily between members of the same faction which captured by $\phi_4$ (corresponding to the \texttt{nodematch} statistic) is important in explaining the generation of the first layer but not for explaining the dissolution process between the layers.

![Posterior predictive distributions](image)

Figure 10: Predictive posterior for $\phi_1$ (edges), $\phi_2$ (gwesp), $\phi_3$ (gwnsp), $\phi_4$ (nodematch) for every network layer.

### 6.3 Model assessment

A way to examine the fit of the data to the estimated posterior distribution of the parameters is to implement a graphical Bayesian goodness-of-fit procedure [Hunter
Table 5: Predictive posterior estimates for $\phi_1$, $\phi_3$, $\phi_3$, $\phi_4$.

| Parameter (Effect) | Layer 1 | Layer 2 | Layer 3 |
|--------------------|---------|---------|---------|
| $\phi_1$ (edges)   | -3.96 0.27 | 0.27 0.68 | -1.09 0.63 |
| $\phi_2$ (gwesp)   | 0.53 0.14 | 0.53 0.24 | 0.51 0.25 |
| $\phi_3$ (gwnsp)   | 0.17 0.02 | -0.18 0.09 | 0.06 0.17 |
| $\phi_4$ (nodematch) | 1.29 0.27 | 0.56 0.43 | 0.21 0.52 |

et al., 2008]. In the Bayesian context, simulated network data are simulated from a sample of parameter values randomly drawn from the estimated posterior distribution and compared to the observed data in terms of high-level network characteristics that are not explicitly included as sufficient statistics in the model. Since we are dealing with weighted networks, we focus on the weighted degree distribution. The black solid lines represent the distribution of the weighted degrees in the observed data, the red lines represent the distribution of the weighted degrees calculated on network graphs simulated from the estimated posterior density.

The plots in Figure 11 suggest that both two models are a reasonable fit to their respective data as the black lines representing the observed distributions are lying on the high posterior predictive distribution despite the absence of a degree-based statistic in the Zachary karate club model (this absence is only partially compensated by the presence of the open triadic effect (corresponding to the gwnsp statistic.)

7 Conclusions

This paper has introduced a new Bayesian hierarchical ERGM framework for the analysis of weighted networks with ordinal/polytomous edges which complements the recent advances proposed by [Wyatt et al., 2010], Krivitsky [2012] and Desmarais and Cranmer [2012]. The modelling approach based on multilayer ERGM process is very flexible and is able to describe the ERGM dissolution process which leads to the generation of the strength of the network edges. The multilayer ERGM process is parametrised using binary network statistics and it is therefore providing a natural interpretation of the network effects that are assumed to be at the basis of the generative process.

A fully-probabilistic Bayesian approach has been adopted in order to provide the possibility of specifying prior information of the network effects and analysing their posterior distribution given the observed data. An extension of the approximate ex-
change algorithm [Caimo and Friel, 2011] has been adopted in order to carry out inference on the doubly-intractable posterior distribution. Probabilistic goodness-of-fit diagnostics based on the weighted degree distribution have been proposed to assess two applications based on well-known datasets.

The modelling framework proposed in this paper can be potentially extended to deal with weighted networks with positive and negative edges by considering, for example, two parallel and conditionally independent dissolution processes one for the positive edges and one for the negative edges.

The R code implementing the methodology proposed in this paper is currently available on Github and it will be incorporated into a future public release of the Bergm package [Caimo and Friel, 2014].

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8 Appendix: Traceplots for the model parameters

Figure 12: Bernard and Killworth office network; MCMC traces for $\mu$ (thinning factor = 100).

Figure 13: Bernard and Killworth office network; MCMC traces for $\phi$ (thinning factor = 100).
Figure 14: Zachary karate club network; MCMC traces for $\mu$ (thinning factor = 100).

Figure 15: Zachary karate club network; MCMC traces for $\phi$ (thinning factor = 100).