Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials

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Abstract. The Apéry polynomials are defined by $A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k$ for all nonnegative integers $n$. We confirm several conjectures of Z.-W. Sun on the congruence for the sum $\sum_{k=0}^{n-1} (-1)^k (2k+1) A_k(x)$ with $x \in \mathbb{Z}$.

Keywords: Apéry polynomials, Pfaff-Saalschütz identity, Legendre symbol, Schmidt numbers, Schmidt polynomials

AMS Subject Classifications: 11A07, 11B65, 05A10, 05A19

1 Introduction

The Apéry polynomials [13] are defined by

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \quad (n \in \mathbb{N}).$$

Thus $A_n := A_n(1)$ are the Apéry numbers [2]. Many people (see Chowla et al. [4], Gessel [5], and Beukers [3], for example) have studied congruences for Apéry numbers. Recently, among other things, Sun [13] proved that, for any integer $x$,

$$\sum_{k=0}^{n-1} (2k+1) A_k(x) \equiv 0 \pmod{n},$$

and proposed many remarkable conjectures. The main objective of this paper is to prove the following result, which was conjectured by Sun [13].

Theorem 1.1. Let $x \in \mathbb{Z}$.

(i) If $n \in \mathbb{Z}^+$, then

$$\sum_{k=0}^{n-1} (-1)^k (2k+1) A_k(x) \equiv 0 \pmod{n}.$$
(ii) If $p$ is an odd prime and $\left( \frac{a}{p} \right)$ is the Legendre symbol modulo $p$, then
\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k(2k+1)A_k(x) \equiv \left( \frac{1-4x}{p} \right) \pmod{p}.
\] (1.3)

(iii) If $p > 3$ is a prime, then
\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k(2k+1)A_k \equiv \left( \frac{p}{3} \right) \pmod{p^2}, \quad (1.4)
\]
\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k(2k+1)A_k(-2) \equiv 1 - \frac{4}{3}(2^{p-1} - 1) \pmod{p^2}. \quad (1.5)
\]

We shall establish some preliminary results in Section 2 and prove Theorem 1.1 in Section 3. Then we give some generalizations of the congruences (1.1) and (1.2) in Section 4. Sun [13, 14] also formulated conjectures for the values of $\sum_{k=0}^{p-1} A_k(x)$ modulo $p^2$ with $x = 1, -4, 9$. In particular, he made the following conjecture.

**Conjecture 1.2** (Sun [13, 14]). Let $p$ be an odd prime. Then
\[
\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 
4x^2 - 2p & \text{mod } p^2, \quad \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2, \\
0 & \text{mod } p^2, \quad \text{if } p \equiv 5, 7 \pmod{8}.
\end{cases}
\]

In an effort to prove the above conjecture, we found a single sum formula for $\sum_{k=0}^{p-1} A_k(x)$ modulo $p^2$, as given in the following theorem.

**Theorem 1.3.** Let $p > 3$ be a prime and $x \in \mathbb{Z}$. Then
\[
\sum_{k=0}^{p-1} A_k(x) \equiv \sum_{k=0}^{p-1} \frac{(2k)!^4 p}{(4k+1)!k^4} x^k \equiv \sum_{k=0}^{(p-1)/2} \binom{p+2k}{4k+1} \binom{2k}{k}^2 x^k \pmod{p^2}.
\]

Clearly Theorem 1.3 implies the following result.

**Corollary 1.4.** If $p > 3$ is a prime, then Conjecture 1.2 is equivalent to the following congruence for binomial sums
\[
\sum_{k=0}^{(p-1)/2} \binom{p+2k}{4k+1} \binom{2k}{k}^2 
\equiv \begin{cases} 
4x^2 - 2p & \text{mod } p^2, \quad \text{if } p \equiv 1, 3 \pmod{8} \text{ and } p = x^2 + 2y^2, \\
0 & \text{mod } p^2, \quad \text{if } p \equiv 5, 7 \pmod{8}.
\end{cases}
\]
2 Two preliminary results

**Theorem 2.1.** Let \( n \in \mathbb{Z}^+ \). Then

\[
\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k + 1) A_k(x)
= (-1)^{n-1} \sum_{m=0}^{n-1} \left( \frac{2m}{m} \right) x^m \sum_{k=0}^{m} \left( \frac{m}{k} \right) \left( \frac{m+k}{m+k} \right) \left( \frac{n-1}{m+k} \right) \left( \frac{n+m+k}{m+k} \right).
\]

**Proof.** For \( \ell, m \in \mathbb{N} \), a special case of the Pfaff-Saalschütz identity (See [11, p. 44, Exercise 2.d] and [1]) reads

\[
\left( \frac{\ell}{m} \right) \left( \frac{\ell + m}{m} \right) = \sum_{k=0}^{m} \left( \frac{2m}{m+k} \right) \left( \frac{\ell-m}{k} \right) \left( \frac{\ell + m + k}{m+k} \right). \tag{2.1}
\]

Multiplying both sides of (2.1) by \( \left( \frac{\ell}{m} \right) \left( \frac{\ell + m}{m} \right) \), we obtain another expression of the Apéry polynomials:

\[
A_\ell(x) = \sum_{m=0}^{\ell} \left( \frac{2m}{m} \right) x^m \sum_{k=0}^{m} \left( \frac{m}{k} \right) \left( \frac{m+k}{m+k} \right) \left( \frac{\ell}{m+k} \right) \left( \frac{\ell + m + k}{m+k} \right). \tag{2.2}
\]

Thus,

\[
\sum_{\ell=0}^{n-1} (2\ell + 1)(-1)^\ell A_\ell(x)
= \sum_{m=0}^{n-1} \left( \frac{2m}{m} \right) x^m \sum_{k=0}^{m} \left( \frac{m}{k} \right) \left( \frac{m+k}{m+k} \right) \sum_{\ell=m+k}^{n-1} (-1)^\ell (2\ell + 1) \left( \frac{\ell}{m+k} \right) \left( \frac{\ell + m + k}{m+k} \right).
\]

The result then follows by applying the formula

\[
\sum_{\ell=k}^{n-1} (-1)^\ell (2\ell + 1) \left( \frac{\ell}{k} \right) \left( \frac{\ell + k}{k} \right) = (-1)^{n-1} n \left( \frac{n-1}{k} \right) \left( \frac{n+k}{k} \right), \tag{2.3}
\]

which can be easily verified by induction. \(\square\)

**Theorem 2.2.** Let \( p \) be an odd prime and \( x \in \mathbb{Z} \). Then

\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k + 1) A_k(x) \equiv \sum_{k=0}^{p-1} \left( \frac{2k}{k} \right) x^k \pmod{p^2}.
\]

**Proof.** Suppose that \( 0 \leq k \leq m < p \). If \( 0 \leq m + k \leq p - 1 \), then

\[
\left( \frac{p-1}{m+k} \right) \left( \frac{p+m+k}{m+k} \right) = \prod_{i=1}^{m+k} \left( \frac{p^2 - i^2}{i^2} \right) \equiv (-1)^{m+k} \pmod{p^2}.
\]
If \( m + k \geq p \), then \( \binom{p-1}{m+k} \binom{p+m+k}{m+k} = 0 \). Therefore, by Theorem 2.1, we get

\[
\frac{1}{p} \sum_{\ell=0}^{p-1} (2\ell + 1)(-1)^\ell A_\ell(x) \equiv \sum_{m=0}^{p-1} \binom{2m}{m} x^m \sum_{k=0}^{p-m-1} (-1)^{m+k} \binom{m}{k} \binom{m+k}{k} \quad (\text{mod } p^2).
\]

(2.4)

Note that, for \( 0 \leq k < p - 1 \) and \( p \leq m + k < 2p \), we have

\[
\binom{2m}{m} \equiv \binom{m+k}{k} \equiv 0 \quad (\text{mod } p),
\]

which means that

\[
\binom{2m}{m} \binom{m+k}{k} \equiv 0 \quad (\text{mod } p^2).
\]

(2.5)

Hence, the congruence (2.4) may be written as

\[
\frac{1}{p} \sum_{\ell=0}^{p-1} (2\ell + 1)(-1)^\ell A_\ell(x) \equiv \sum_{m=0}^{p-1} \binom{2m}{m} x^m \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} \binom{m+k}{k} \quad (\text{mod } p^2),
\]

where we have used the Chu-Vandermonde summation formula.

\[\square\]

### 3 Proof of Theorem 1.1

It is clear that the congruence (1.2) is an immediate consequence of Theorem 2.1. To prove (1.3), we first give the following congruence that was implicitly given by Sun and Tauraso [16] (see also Sun [12]).

**Lemma 3.1.** Let \( p \) be an odd prime and let \( a \in \mathbb{Z}^+ \) and \( x \in \mathbb{Z} \). Then

\[
\sum_{k=0}^{p^a-1} \binom{2k}{k} x^k \equiv \left( \frac{1 - 4x}{p} \right)^a \quad (\text{mod } p),
\]

(3.1)

**Proof.** If \( p \mid x \), then (3.1) clearly holds. If \( p \nmid x \), then there is a positive integer \( b \) such that \( bx \equiv 1 \) (mod \( p \)) and

\[
\sum_{k=0}^{p^a-1} \binom{2k}{k} x^k \equiv \sum_{k=0}^{p-1} \binom{2k}{k} b^{-k} \quad (\text{mod } p).
\]
By [16, Theorem 1.1], we have
\[
\sum_{k=0}^{p^a-1} \binom{2k}{k} b^{-k} \equiv \left( \frac{b^2 - 4b}{p} \right)^a \pmod{p}.
\]

Since \( \left( \frac{x^2}{p} \right) = 1 \), we obtain
\[
\left( \frac{b^2 - 4b}{p} \right) = \left( \frac{x^2}{p} \right) \left( \frac{b^2 - 4b}{p} \right) = \left( \frac{1 - 4x}{p} \right).
\]

Combining the above three identities yields (3.1).

\[\square\]

**Proof of (1.3).** Theorem 2.2 implies that
\[
\sum_{k=0}^{p-1} (-1)^k (2k + 1) A_k(x) \equiv p \sum_{k=0}^{p-1} \binom{2k}{k} x^k \pmod{p^3}.
\]

The proof then follows from the \( a = 1 \) case of Lemma 3.1.

\[\square\]

**Proof of (1.4).** Letting \( x = 1 \) in (3.2), we have
\[
\sum_{k=0}^{p-1} (-1)^k (2k + 1) A_k \equiv p \sum_{k=0}^{p-1} \binom{2k}{k} \pmod{p^3}.
\]

The proof then follows from the \( a = 1 \) case of the congruence [17, (1.9)]
\[
\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left( \frac{p^a}{3} \right) \pmod{p^2}.
\]

Note that the condition \( p > 3 \) is not necessary in this case.

\[\square\]

**Proof of (1.5).** Letting \( x = -2 \) in (3.2), we have
\[
\sum_{k=0}^{p-1} (-1)^k (2k + 1) A_k(-2) \equiv p \sum_{k=0}^{p-1} \binom{2k}{k} (-2)^k \pmod{p^3}.
\]

Similarly to (3.1), the first congruence in [12, Theorem 1.1] implies that
\[
\sum_{k=0}^{p^a-1} \binom{2k}{k} x^k \equiv \left( \frac{1 - 4x}{p} \right)^a + \left( \frac{1 - 4x}{p} \right)^{a-1} u_{p-(1-4x)} \pmod{p^2},
\]

where \( p \nmid x \) and the sequence \( \{u_n\}_{n \geq 0} \) is defined as
\[
u_0 = 0, \ u_1 = 1, \ \text{and} \ u_{n+1} = (b - 2)u_n - u_{n-1}, \ n \in \mathbb{Z}^+, \ \text{where} \ bx \equiv 1 \pmod{p^2}.
\]
If \( b^2 - 4b \not\equiv 0 \pmod{p^2} \), then
\[
u_n = \frac{(b - 2 + \sqrt{b^2 - 4b})^n - (b - 2 - \sqrt{b^2 - 4b})^n}{2^n \sqrt{b^2 - 4b}}.
\]

Now suppose that \( p > 3 \). For \( a = 1 \) and \( x = -2 \), the congruence (3.4) reduces to
\[
p - 1 \sum_{k=0}^{p-1} \binom{2k}{k}(-2)^k \equiv 1 + u_{p-1} = 1 - \frac{4^{p-1} - 1}{3 \cdot 2^{p-2}} \equiv 1 - \frac{4}{3} (2^{p-1} - 1) \pmod{p^2}. \quad (3.5)
\]

Combining (3.3) and (3.5), we complete the proof. \( \square \)

**Remark.** The following stronger version of (3.5):
\[
p - 1 \sum_{k=0}^{p-1} \binom{2k}{k}(-2)^k \equiv 1 - \frac{4}{3} (2^{p-1} - 1) \pmod{p^3} \quad \text{for any prime } p > 3,
\]
was independently obtained by Mattarei and Tauraso [7] and Z.-W. Sun [arxiv:0911.5665v52, Remark after Conjecture A69].

### 4 Generalizations of the congruences (1.1) and (1.2)

Following Schmidt [10] we call the numbers \( S_{n}^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^{r} \binom{n+k}{k}^{r} \) the Schmidt numbers, and define the Schmidt polynomials as follows:
\[
S_{n}^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k}^{r} \binom{n+k}{k}^{r} x^k.
\]

As generalizations of (1.1) and (1.2), we have the following congruences for Schmidt polynomials.

**Theorem 4.1.** Let \( n \in \mathbb{Z}^+ \), \( r \geq 2 \) and \( x \in \mathbb{Z} \). Then
\[
\sum_{k=0}^{n-1} (2k+1)S_{k}^{(r)}(x) \equiv 0 \pmod{n}, \quad (4.1)
\]
\[
\sum_{k=0}^{n-1} (-1)^k (2k+1)S_{k}^{(r)}(x) \equiv 0 \pmod{n}. \quad (4.2)
\]

**Remark.** When \( r = 1 \), the polynomial \( D_n(x) := S_n^{(1)}(x) \) is called the Delannoy polynomial. Sun [15, (1.15)] proved that
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)D_k(x) = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} x^k \in \mathbb{Z}[x] \quad \text{for } n = 1, 2, 3, \ldots.
\]

To prove Theorem 4.1, we need the following Lemma.
Lemma 4.2. For $\ell, m \in \mathbb{N}$ and $r \geq 2$, there exist integers $a_{m,k}^{(r)}$ ($m \leq k \leq rm$) such that
\[
\binom{\ell}{m} \binom{\ell + m}{m}^r = \sum_{k=m}^{rm} a_{m,k}^{(r)} \binom{\ell}{k} \binom{\ell + k}{k}^r.
\] (4.3)

Proof. We proceed by mathematical induction on $r$. For $r = 2$, the identity (2.2) gives
\[
a_{m,m+k}^{(2)} = \binom{2m}{m} \binom{m+k}{k}, \quad 0 \leq k \leq m.
\]

Suppose that (4.3) holds for $r$. A special case of the Pfaff-Saalschütz identity [1, (1.4)] reads
\[
\binom{\ell + m}{m+n} \binom{\ell + n}{n} = \sum_{k=0}^{n} \binom{m+n}{n-k} \binom{m}{k} \binom{\ell + m + k}{\ell + m + k}.
\] (4.4)

Now, multiplying both sides of (4.3) by $\binom{\ell}{m} \binom{\ell + m}{m}$ and applying (4.4), we have
\[
\binom{\ell}{m} \binom{\ell + m}{m}^{r+1} = \sum_{k=m}^{rm} a_{m,k}^{(r)} \binom{\ell}{k} \binom{\ell + k}{k} \binom{\ell}{m} \binom{\ell + m}{m}
\]
\[
= \sum_{k=m}^{rm} \sum_{i=0}^{k} \binom{m+k}{k} \binom{m}{k-i} \binom{\ell-m}{i} \binom{\ell + m + i}{i} \binom{\ell}{m} \binom{\ell + m}{m}
\]
\[
= \sum_{k=m}^{rm} \sum_{i=0}^{k} a_{m,k}^{(r)} \binom{m+k}{k} \binom{m}{k-i} \binom{\ell-m}{i} \binom{\ell + m + i}{i} \binom{\ell}{m} \binom{\ell + m + i}{m+i},
\]
which implies that
\[
a_{m,m+i}^{(r+1)} = \sum_{k=m}^{rm} \binom{m+k}{k} \binom{m+i}{k-i} a_{m,k}^{(r)} \in \mathbb{Z}, \quad 0 \leq i \leq rm.
\]

The proof of the inductive step is then completed.

Proof of Theorem 4.1. Applying (4.3) and the easily checked formula (see [12, Lemma 2.1])
\[
\sum_{\ell=k}^{n-k} (2\ell + 1) \binom{\ell}{k} \binom{\ell + k}{k} = n \binom{n}{k+1} \binom{n+k}{k},
\]
we have
\begin{align*}
\sum_{\ell=0}^{n-1} (2\ell + 1) S_{\ell}^{(r)}(x) &= \sum_{\ell=0}^{n-1} (2\ell + 1) \sum_{m=0}^{\ell} \binom{\ell}{m}^r \binom{\ell + m}{m}^r x^m \\
&= \sum_{m=0}^{n-1} x^m \sum_{\ell=m}^{n-1} \sum_{k=m}^{\ell} a_{m,k}^{(r)} (2\ell + 1) \binom{\ell}{k} \binom{\ell + k}{k} \\
&= n \sum_{m=0}^{n-1} x^m \sum_{k=m}^{\ell} a_{m,k}^{(r)} \binom{n}{k} \binom{n+k}{k}.
\end{align*}

Similarly, applying (4.3) and (2.3), we have
\begin{align*}
\sum_{\ell=0}^{n-1} (-1)^{\ell} (2\ell + 1) S_{\ell}^{(r)}(x) &= (-1)^{n-1} n \sum_{m=0}^{n-1} x^m \sum_{k=m}^{\ell} a_{m,k}^{(r)} \binom{n-1}{k} \binom{n+k}{k}.
\end{align*}

Therefore the congruences (4.1) and (4.2) hold.

Furthermore, similarly to the proof of [6, Theorem 1.2], we can prove the following generalization of Theorem 4.1.

**Theorem 4.3.** Let \( n \in \mathbb{Z}^+ \), \( a \in \mathbb{N} \), \( r \geq 2 \) and \( x \in \mathbb{Z} \). Then
\begin{align*}
\sum_{k=0}^{n-1} \varepsilon^k (2k+1)^a (k+1)^\alpha \gamma_{k}^{(r)}(x) &\equiv 0 \pmod{n}, \\
\sum_{k=0}^{n-1} \varepsilon^k (2k+1)^{2a+1} \gamma_{k}^{(r)}(x) &\equiv 0 \pmod{n},
\end{align*}

where \( \varepsilon = \pm 1 \).

We conclude this section with the following conjecture, which is due to Sun [12] in the case \( r = 2 \).

**Conjecture 4.4.** Let \( \varepsilon \in \{ \pm 1 \} \), \( m, n \in \mathbb{Z}^+ \), \( r \geq 2 \), and \( x \in \mathbb{Z} \). Then
\begin{align*}
\sum_{k=0}^{n-1} (2k+1)^\varepsilon \gamma_{k}^{(r)}(x)^m &\equiv 0 \pmod{n}.
\end{align*}

5 **Proof of Theorem 1.3**

From the Lagrange interpolation formula for the polynomial \( \prod_{k=1}^{m} (x-k) \) at the values \(-i\) (\( 0 \leq i \leq m \)) of \( x \) we immediately derive that
\begin{align*}
\sum_{k=0}^{m} \binom{m}{k} \binom{m+k}{k} \frac{(-1)^{m-k}}{x+k} &\equiv \frac{1}{x} \prod_{k=1}^{m} \frac{x-k}{x+k}.
\end{align*}
A proof of (5.1) by using the creative telescoping method was given in [8, Lemma 3.1].

Remark. The identity (5.1) plays an important role in Mortenson’s proof [8] of Rodriguez-Villegas’s conjectures [9] on supercongruences for hypergeometric Calabi-Yau manifolds of dimension \( d \leq 3 \). Letting \( x = m + \frac{1}{2} \) in (5.1), we obtain

\[
\sum_{k=0}^{m} \binom{m}{k} \binom{m+k}{k} \frac{(-1)^{m-k}}{2m+2k+1} = \frac{(2m)!^3}{(4m+1)!m!^2}. \tag{5.2}
\]

Let \( p > 3 \) be a prime. Similarly to the proof of Theorem 2.1, applying (2.2) and the trivial identity

\[
\sum_{\ell=0}^{p-1} \binom{\ell}{m+k} \binom{\ell+m+k}{m+k} = \binom{2m+2k}{m+k} \binom{p+m+k}{2m+2k+1},
\]

we have

\[
\sum_{\ell=0}^{p-1} A_\ell(x) = \sum_{m=0}^{p-1} \binom{2m}{m} x^m \sum_{k=0}^{m} \binom{m+k}{k} \binom{2m+2k}{m+k} \binom{p+m+k}{2m+2k+1}. \tag{5.3}
\]

Note that, if \( 0 \leq m + k < p \), then

\[
\binom{2m+2k}{m+k} \binom{p+m+k}{2m+2k+1} = \frac{(2m+2k)!^p \prod_{i=1}^{m+k} (p^2 - i^2)}{(m+k)!^2 (2m+2k+1)!} \equiv (-1)^{m+k} \frac{p}{2m+2k+1} \pmod{p^2},
\]

while if \( m + k \geq p \), then \( \binom{2m+2k}{m+k} \binom{p+m+k}{2m+2k+1} = 0 \).

From (5.3), we deduce that

\[
\sum_{\ell=0}^{p-1} A_\ell(x) \equiv \sum_{m=0}^{p-1} \binom{2m}{m} x^m \sum_{k=0}^{p-1} \binom{m+k}{k} \frac{(-1)^{m+k} p}{2m+2k+1} \pmod{p^2}. \tag{5.4}
\]

Since \( p > 3 \), we have \( 2m + 2k + 1 \not\equiv 0 \pmod{p^2} \) for any \( 0 \leq k \leq m \leq p - 1 \). Thus, by (2.5) and (5.2), the congruence (5.4) is equivalent to

\[
\sum_{\ell=0}^{p-1} A_\ell(x) \equiv \sum_{m=0}^{p-1} \binom{2m}{m} x^m \sum_{k=0}^{m} \binom{m+k}{k} \frac{(-1)^{m+k} p}{2m+2k+1}
\]

\[
= \sum_{m=0}^{p-1} \frac{(2m)!^4 p}{(4m+1)!m!^4} x^m \pmod{p^2}. \tag{5.5}
\]

On the other hand, it is not difficult to see that

\[
\frac{(2m)!^4 p}{(4m+1)!m!^4} \equiv 0 \pmod{p^2} \quad \text{for } p/2 < m < p,
\]
and
\[
p(2m)!^2 \equiv \frac{p \prod_{i=1}^{2m} (p^2 - i^2)}{(4m + 1)!} \equiv \left( \frac{p + 2m}{4m + 1} \right) (\text{mod } p^2) \quad \text{for } p > 3 \text{ and } 0 \leq m < p/2.
\]
This proves that
\[
\sum_{m=0}^{p-1} \frac{(2m)!^4 p}{(4m + 1)! m!} x^m \equiv \sum_{m=0}^{(p-1)/2} \frac{(2m)!^4 p}{(4m + 1)! m!} x^m \equiv \sum_{m=0}^{(p-1)/2} \left( \frac{p + 2m}{4m + 1} \right) \left( \frac{2m}{m} \right)^2 x^m \quad (\text{mod } p^2).
\]
Combining (5.5) and (5.6), we complete the proof.

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities, Shanghai Rising-Star Program (#09QA1401700), Shanghai Leading Academic Discipline Project (#B407), and the National Science Foundation of China (#10801054).

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