 Modified gravity, Dark Energy and MOND

Ignacio Navarro∗ and Karel Van Acoleyen†

∗DAMTP, University of Cambridge, CB3 0WA Cambridge, UK
†IPPP, University of Durham, DH1 3LE Durham, UK.

Abstract

We propose a class of actions for the spacetime metric that introduce corrections to the Einstein-Hilbert Lagrangian depending on the logarithm of some curvature scalars. We show that for some choices of these invariants the models are ghost free and modify Newtonian gravity below a characteristic acceleration scale given by $a_0 = c\mu$, where $c$ is the speed of light and $\mu$ is a parameter of the model that also determines the late-time Hubble constant: $H_0 \sim \mu$. In these models, besides the massless spin two graviton, there is a scalar excitation of the spacetime metric whose mass depends on the background curvature. This dependence is such that this scalar, although almost massless in vacuum, becomes massive and effectively decouples when one gets close to any source and we recover an acceptable weak field limit at short distances. There is also a (classical) “running” of Newton’s constant with the distance to the sources and gravity is easily enhanced at large distances by a large ratio. We comment on the possibility of building a model with a MOND-like Newtonian limit that could explain the rotation curves of galaxies without introducing Dark Matter using this kind of actions. We also explore briefly the characteristic gravitational phenomenology that these models imply: besides a long distance modification of gravity they also predict deviations from Newton’s law at short distances. This short distance scale depends on the local background curvature of spacetime, and we find that for experiments on the Earth surface it is of order $\sim 0.1\text{mm}$, while this distance would be bigger in space where the local curvature is significantly lower.
1 Introduction

The relative importance of the gravitational interaction increases as we consider larger scales, and it is at the largest scales that we can measure where the observed gravitational phenomena do not agree with our expectations. The Hubble constant measuring the rate of expansion of the Universe does not fall with time as predicted by General Relativity (GR) for a Universe that contains only known forms of matter, and the dynamics of galaxies seem to require much more matter than observed if explained in terms of GR. The most common approach to these problems is to assume the presence of unseen forms of energy that bring into agreement the observed phenomena with GR. The standard scenario to explain the dynamics of galaxies consists in the introduction of an extra weakly interacting massive particle, the so-called Cold Dark Matter (CDM), that clusters at the scales of galaxies and provides the required gravitational pull to hold them together. The explanation of the observed expansion of the universe requires however the introduction of a more exotic form of energy, not associated with any form of matter but associated with the existence of space-time itself: vacuum energy. And while CDM can be regarded as a natural possibility given our knowledge of elementary particle theory, the existence of a non-zero but very small vacuum energy remains an unsolved puzzle for our high-energy understanding of physics. However, the apparent naturalness of the CDM hypothesis finds also problems when one descends to the details of the observations. Increasingly precise simulations of galaxy formation and evolution, although relatively successful in broad terms, show well-known features that seem at odds with their real counterparts, the most prominent of which might be the “cuspy core” problem and the over-abundance of substructure seen in the simulations (see e.g. [1]). But, despite of this, the main problem that the CDM hypothesis faces is probably to explain the correlations of the relative abundances of dark and luminous matter that seem to hold in a very diverse set of astrophysical objects [2]. These correlations are exemplified in the Tully-Fisher law [3] and can be interpreted as pointing to an underlying acceleration scale, below which the Newtonian potential changes and gravity becomes stronger. This is the basic idea of MOND (MOdified Newtonian Dynamics), a very successful phenomenological modification of Newton’s potential proposed in 1983 [4] whose predictions for the rotation curves of spiral galaxies have been realised with increasing accuracy as the quality of the data has improved...
Interestingly, the critical acceleration required by the data is of order $a_0 \sim cH_0$, where $H_0$ is today’s Hubble constant and $c$ the speed of light (that we will set to 1 from now on). The problem with this idea is that MOND is just a modification of Newton’s potential so it remains silent in any situation in which relativistic effects are important. Efforts have been made to obtain MONDian phenomenology in a relativistic generally covariant theory by including other fields in the action with suitable couplings to the spacetime metric [6] (see also [7] for other approaches to galactic dynamics without Dark Matter). But these models do not address in a unified way the Dark Energy and Dark Matter problems, while a common origin is suggested by the observed coincidence between the critical acceleration scale and the Dark Energy density. In this paper we will propose a class of generally covariant actions, built only with the metric, that have the right properties to address these problems in a unified way, and where the relation $a_0 \sim H_0$ finds a natural explanation. The theories we will consider modify gravity in the infrared, making it stronger below a characteristic acceleration scale, but this is not their only characteristic feature. When we are in a situation in which the dominant gravitational field is external, like in table-top experiments on Earth (that measure the gravitational field of some probes embedded in the dominant background gravitational field of the Earth), we can also expect short distance modifications of Newton’s potential.

Regarding the long distance modifications, the source-dependent characteristic distance beyond which Newtonian gravity is modified in these theories, that we shall call $r_c$, is given by

$$\frac{G_NM}{r_c^2} = \frac{\mu}{2},$$

(1)

where $G_N$ is Newton’s constant, $M$ the mass of the source and $\mu$ is a parameter of the model that also determines the late-time Hubble constant: $H_0 \sim \mu$. This makes these models promising candidates to build a theory with a MOND-like Newtonian limit that could address the dynamics of galaxies without the need for Dark Matter. But when measuring the gravitational attraction between two probes in the external dominant gravitational field of a massive object of mass $M$, at a distance $r_d$ from its centre, we can also expect short distance modifications of Newton’s law for distances smaller than

$$r_{SD} \sim \frac{\mu r_d^3}{G_NM^2},$$

(2)
If we plug in this expression the radius and mass of the Earth (with $\mu \sim H_0$), we get that for table top tests of Newton’s law performed on the Earth surface $r_{SD} \sim 10^{-2} cm$. This range is very interesting because it is the range currently being probed by experiments [8]. But notice that the phenomenology of these theories is very different from the one expected from other theoretical considerations that also suggest a deviation from Newton’s law at that scale motivated by the cosmological constant problem$^1$ or the gauge hierarchy problem [10]. The fact that this scale is the same in both cases is just a numerical coincidence. If we performed the same experiment on space, in the neighbourhood of the Earth’s orbit for instance where the dominant gravitational field is that of the Sun, the relevant mass and distance we should use in the previous estimation is the Sun’s mass and the Sun-Earth distance. In this case the “short distance” corrections would be expected in our theory at distances less than about $10^4 m!$. This however does not mean that there should be big modifications to the motion of the planets or other celestial bodies. When the gravitational field we are measuring is that of the Sun, the corrections are suppressed at distances less than $r_c$ that is in this case of the order of $10^5 AU \sim 10^{11} km$.

These characteristic experimental signatures arise in our theory because of the presence of an extra scalar excitation of the spacetime metric besides the massless spin two graviton. But while the graviton remains massless, the extra scalar has a mass that depends on the background curvature. This dependence is such that for the models that we will be interested on, those that modify gravity at large distances, this field becomes massive and effectively decouples when the background curvature is large, and in particular when we approach any source.

In the next section we will briefly review the results we obtained in [11,12] studying models that involve inverse powers of the curvature in the action, giving some general expressions and discussing the generic features of the framework we will use for modifying gravity in the infrared. In particular we will focus on a class of models that had been proposed to address the acceleration of the Universe [13] and also modify gravity at large distances [11,12]. This discussion will enable us to motivate the

$^1$It is well known that naturalness arguments lead to the expectation that new physics associated with electro-weak symmetry breaking, besides the Higgs boson, should be seen in the LHC. Otherwise the electroweak scale becomes unstable under quantum corrections. Applying the same logic to the gravitational sector one would expect new gravitational phenomena to kick in at the vacuum energy scale that would cut-off the quantum divergences contributing to the vacuum energy. This hypothetical new physics should be seen in sub-mm measurements of Newton’s potential [9].
class of actions that we will present in the third section, that depend on the logarithm of some curvature invariants. We will see that the theories that we propose in this section modify gravity at the MOND characteristic acceleration scale, and the gravitational interaction can easily become stronger at large distances. We will explore briefly the characteristic gravitational phenomenology expected in these models, discussing possible tests of these theories. In the fourth section we offer the conclusions. We will comment on further generalisations of the proposed actions and on the generic phenomenological features expected in the class of models that modify gravity at the MOND acceleration scale. We will also comment on the possibility of obtaining these theories as an effective action for the spacetime metric that takes into account strong renormalisation effects in the infrared that might appear in GR.

\section{Modified gravity as an alternative to Dark Energy}

Recently, models involving inverse powers of the curvature have been proposed as an alternative to Dark Energy \cite{13,14}. In these models one generically has more propagating degrees of freedom in the gravitational sector than the two contained in the massless graviton in GR. The simplest models of this kind add inverse powers of the scalar curvature to the action ($\Delta L \propto 1/R^n$), thereby introducing a new scalar excitation in the spectrum. For the values of the parameters required to explain the acceleration of the Universe this scalar field is almost massless in vacuum and one might worry about the presence of a new force contradicting Solar System experiments. However in a recent publication \cite{11} we showed that models that involve inverse powers of other invariants, in particular those that diverge for $r \to 0$ in the Schwarzschild solution, generically recover an acceptable weak field limit at short distances from sources by means of a screening or shielding of the extra degrees of freedom at short distances \cite{12}.

But let us start by discussing the linearisation and vacuum excitations obtained from generic actions built with the Ricci scalar and the scalars

\[ P \equiv R_{\mu \nu} R^{\mu \nu} \quad \text{and} \quad Q \equiv R_{\mu \nu \lambda \rho} R^{\mu \nu \lambda \rho}. \]  \hfill (3)

If the Lagrangian is a generic function $\mathcal{L} = F(R, P, Q)$ the equations of motion for the metric will be of fourth order and we can expect that the particle content of the
theory will have eight degrees of freedom: two for the massless graviton, one in a scalar excitation and five in a ghost-like massive spin two field \[15\]. Expanding the action in powers of the curvature perturbations it can be seen that at the bilinear level the linearisation of the theory over a maximally symmetric spacetime will be the same as that obtained from \[15,16\]

\[
S = \int d^4x \sqrt{-g} \frac{1}{16\pi G_N} \left[ -\Lambda + \delta R + \frac{1}{6m_0^2} R^2 - \frac{1}{2m_2^2} C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma} \right],
\]

where \(C_{\mu\nu\lambda\sigma}\) is the Weyl tensor and we have defined

\[
\Lambda \equiv \langle F - R F_R + R^2 (F_{RR}/2 - F_P/4 - F_Q/6) + R^3 (F_{RP}/2 + F_{RQ}/3) + R^4 (F_{PP}/8 + F_{QQ}/18 + F_{PQ}/6) \rangle_0
\]

\[
\delta \equiv \langle F_R - RF_{RR} - R^2 (F_{RP} + 2F_{RQ}/3) - R^3 (F_{PP}/4 + F_{QQ}/9 + F_{PQ}/3) \rangle_0
\]

\[
m_0^{-2} \equiv \langle (3F_{RR} + 2F_P + 2F_Q) + R (3F_{RP} + 2F_{RQ}) + R^2 (3F_{PP}/4 + F_{QQ}/3 + F_{PQ}) \rangle_0
\]

\[
m_2^{-2} \equiv -\langle F_P + 4F_Q \rangle_0.
\]

Here \(< \ldots >_0\) denotes the value of the corresponding quantity on the background and \(F_R \equiv \partial_R F\), etc... One can see that the situation for the perturbations over vacuum in any modified theory of gravity (built with \(R\), \(P\) and \(Q\)) will be the same as in Einstein gravity supplemented with curvature squared terms. It is well known that for the action \(4\) the mass of the ghost is \(\sim m_2\) and that of the scalar is \(\sim m_0\). So in the case in which \(F(R,P,Q) = F(R,Q-4P)\), \(m_2^{-2} = 0\) and there is no ghost in the spectrum, but there is still the extra scalar. It is easy to check that in the models that explain the acceleration of the Universe by using actions that involve inverse powers of the scalar curvature alone, the mass of the scalar is proportional to some positive power of the scalar curvature \[17\]. And notice that including other terms in \(F\) and fine-tuning the parameters we can make \(m_0^{-2} = 0\) in vacuum, as suggested in \[18\], by including \(R^2\) corrections to the action besides the \(1/R\) or \(\text{Log}(R)\) ones. But the infinite mass (or absence) of the scalar is a property only of a particular background in these models. If we evaluate the expression for \(m_0\) for \(F = R - \mu^4/R + R^2/m_s^2\) for instance, we get

\[
m_0^{-2} = 6 \left( -\frac{\mu^4}{\langle R^3 \rangle_0} + \frac{1}{m_s^2} \right),
\]
and choosing a particular value of \( m_s \), namely \( m_s^2 = 3\sqrt{3}\mu^2 \), we can make \( m_0 \to \infty \) in vacuum, where \( \langle R \rangle_0 = \sqrt{3}\mu^2 \). But we see that when the scalar curvature is bigger than its vacuum value, the scalar mass returns to its natural value, \( m_0 \sim m_s \sim \mu \), which shows that in these models the scalar is still present in general, and its mass is very small in most situations\(^2\). But for the models that include only inverse powers of the curvature, besides the Einstein-Hilbert term, it is however possible that in regions where the curvature is large the scalar has naturally a large mass and this could make the dynamics to be similar to those of GR \(^3\). But the scalar curvature, although bigger than its mean cosmological value, is still very small in the Solar System for instance. So, although a rigorous quantitative analysis of the predictions of these models for observations at the Solar System level is still lacking in the literature, it is not clear that these models constitute a viable alternative to Dark Energy because one can expect that the effects of this extra field should have been observed\(^3\).

But the story is different if we include inverse powers of curvature invariants, like \( Q \), that grow at short distances in the Schwarzschild solution. In this case the mass of the extra scalar field is guaranteed to grow as we approach any source and we recover Einstein gravity at short distances \(^1\). For studying these effects we focused on actions of the type

\[
S = \int d^4x \sqrt{-g} \frac{1}{16\pi G_N} \left[ R - \frac{\mu^{4n+2}}{(aR^2 - 4P + Q)^n} \right].
\]

These actions, with \( \mu \) of the order of the late time Hubble constant \( H_0 \), were proposed in \(^1\) motivated by the Dark Energy problem and have cosmological solutions providing a good fit to the SN data \(^2\). When linearising over vacuum we find, besides the usual massless graviton, a scalar excitation of the metric with a mass of order \( m_0 \sim \mu \sim H_0 \). But in the expanded action higher order non-renormalisable operators appear suppressed by inverse powers of the background curvature. This means that the linearisation will break down when the energy of the fluctuations is extremely small: the strong coupling scale in this theory is \( \Lambda_s \sim \left( M_p H_0^{2n+4}/\mu^{2n+1} \right)^{1/4} \sim (M_p H_0^3)^{1/4} \) \(^2\). This, in turn, means that for a spherically symmetric solution, the linearisation over

\(^2\)This is in contrast with the absence of the ghost for actions in which \( F(R, P, Q) = F(R, Q - 4P) \). In this case \( m_s^{-2} = 0 \) independently of the background curvature.

\(^3\)There is an alternative formulation of these models in which the connection and the metric are varied independently, the so-called Palatini formalism, and in this case these theories might be viable \(^2\).
vacuum will break down at a huge distance from any source, at the so-called Vainshtein radius, that is for these theories $r_V \sim (G_N M \mu^{2n+1}/\mathcal{H}_0^{2n+4}) \sim (G_N M/H_0^3)^{1/4}$ [12]. At smaller distances we can not trust the results obtained using this expansion. It is also important to keep in mind that in this long-distance regime, where we can apply the linearisation of the action over vacuum, the effective Planck mass is given by (see [12])

$$M_{p(\text{eff})}^2 = M_p^2 \left( \delta + \frac{\langle R \rangle_0^2}{3m_0^2} \right) = \left( 1 + \frac{6n(a-1)}{(n+1)(6a-5)} \right) M_p^2 \quad (11)$$

where $M_p^2 = (8\pi G_N)^{-1}$.

As we have said, at distances less than $r_V$ we enter a non-perturbative regime where we can no longer use the linearised action over vacuum because higher order non-renormalisable operators become more important than those involving only two powers of the fluctuations. But we can get some information about the short distance behaviour of the solutions by noticing that the extra term that the modification introduces will be unimportant at short distances in the spacetime of a spherically symmetric mass. The reason is that this term will always be suppressed by inverse powers of $Q$, that for the Schwarzschild solution reads $Q = 48(G_N M)^2/r^6$ and grows at short distances. This tells us that any spherically symmetric solution for which the curvature grows at short radius will converge to the Schwarzschild one with a Planck mass given simply by $M_{p}^2 = (8\pi G_N)^{-1}$. We can then make a different kind of expansion, a weak field expansion over the Schwarzschild solution that shows that in these models Newtonian gravity is modified beyond a distance given by [11]

$$r_c^{3n+2} \equiv \left( \frac{G_N M}{\mu^{2n+1}} \right)^{n+1}. \quad (12)$$

In fact, for these theories, the spherically symmetric solution, in an expansion in powers of $r/r_c$ reads:

$$ds^2 \simeq - \left[ 1 - \frac{2G_NM}{r} \left( 1 - \alpha \left( \frac{r}{r_c} \right)^{6n+4} + \mathcal{O} \left( (r/r_c)^{12n+8} \right) \right) \right] dt^2 \quad (13)$$

$$+ \left[ 1 - \frac{2G_NM}{r} \left( 1 + \frac{\alpha (6n+3)}{2} \left( \frac{r}{r_c} \right)^{6n+4} + \mathcal{O} \left( (r/r_c)^{12n+8} \right) \right) \right]^{-1} dr^2 + r^2 d\Omega_2^2,$$

where $M$ is the mass of the object sourcing the field and $\alpha \equiv \frac{n(1+n)}{(6n+3)^2 n^4}$.

The distance $r_c$ is very large, at least of the order of parsecs for a star like the Sun (assuming $n \geq 1$), so from this expansion we see that the corrections that we
can expect in these theories at the Solar System level are very small. Unfortunately, this short distance expansion breaks down for distances of order $r_c$, and notice that $r_c \ll r_V$. So in these theories we are led to the following picture: at small energies ($E < \Lambda_s$) or large distances ($r > r_V$) we have a scalar tensor theory, with an almost massless scalar and an effective Planck mass given by (11). As we increase the energy of the fluctuations or go to shorter distances we enter a non-perturbative phase, but at even higher energies ($E > \Lambda_{GR}$) or smaller distances ($r < r_c$) we can neglect the effects of the modification and we recover the standard GR dynamics, with a Planck mass given by $M_p$. Here $\Lambda_{GR} = (M_p^{n+1} \mu^{2n+1})^{1/(3n+2)}$ is the energy scale associated with this high energy recovery of Einstein gravity. This situation is depicted in fig.1. Since all the expansions that we have used break down in this non-perturbative intermediate regime, at this point we can only assume that the dynamics are consistent for this range of energies/distances, and that one can obtain a consistent matching between the long distance and short distance solutions. However we regard this as a model dependent issue, and we can extract already a lot of useful information just knowing the dynamics in the low and high energy regimes.

In terms of the particle content of the linearised theory we can get some “non-perturbative” insight in the reasons behind the behaviour of these solutions by evaluating the expression for $m_0$ in them. We see then that we can expect that the mass of the extra degree of freedom that the modification introduces has a contribution in the
spacetime of a spherically symmetric mass that goes like \[12\]

\[\delta_{\text{Source}} m_s(r) \sim \frac{Q^{n+1}}{\mu^{2n+1}} \sim \frac{(G_N M)^{n+1}}{\mu^{2n+1} r^{3n+3}}. \tag{14} \]

This effective mass decouples the scalar at short distances, where \(Q \gg \mu^4\), and we recover Einstein gravity in this domain. It is also worth to point out that one recovers the distance \(r_c\) as the distance at which \(m_s(r) \sim r^{-1}\), with \(m_s(r)\) given by the expression above. Since \(m_s(r)\) grows faster than \(r^{-1}\), at smaller distances the scalar effectively decouples. But the \(r\)-dependent mass of the scalar field is not the only effect of the modification: as we have seen at large distances from sources the effective Planck mass is given by eq.(11) while at short distances it is simply \(M_p\). So there is a rescaling or “running” of the Planck mass with the distance to the sources.

However, one can see that these theories, although capable of explaining the acceleration of the Universe without Dark Energy, are not capable of explaining the dynamics of galaxies without Dark Matter. First, the critical distance at which the modification becomes important \((r_c)\) is too large, since the data consistently indicate that any such theory should have noticeable effects at a distance \(r_c\) given by eq.(11). And second, the effective Planck mass that we obtain at large distances is not significantly reduced with respect to the one that we get at short distances except for a very small range of values of \(a\) that, at least for the \(n = 1\) case, are not phenomenologically acceptable [21]. We need the reduction of the effective Planck mass in vacuum in order to get a large enhancement of the gravitational interaction at large distances as required to have any hope to fit the data without introducing Dark Matter. The critical distance that we obtain in these theories does however suggest that in the \(n \to 0\) limit the modification becomes important at a distance that corresponds to the MOND characteristic acceleration. This motivates the use of Logarithmic actions as a possibility for obtaining a modification of gravity as an alternative to Dark Matter. A first assessment of this possibility is the goal of the next section.

### 3 Logarithmic actions and modified gravity as an alternative to Dark Matter

As we said, the considerations of the previous section motivate the introduction of an action depending on the logarithm of the curvature in order to get a modification of
gravity at the MOND characteristic acceleration scale. In this section we will discuss what appears to be the simplest possibility, namely actions of the type

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi G_N} \left\{ R - \mu^2 \log \left[ f(R, Q - 4P) \right] \right\} ,$$  \hfill (15)

where for the function $f$ we will only assume that

$$f \to 0 \quad \text{for} \quad R_{\mu\nu} \to 0 ,$$  \hfill (16)

and we can approximate

$$f \simeq Q/Q_0 \quad \text{when} \quad Q \gg R^2, P.$$  \hfill (17)

In this case Minkowski spacetime will not be a solution of the theory but there will typically exist de Sitter solutions with $H_0 \sim \mu$. Notice that the addition of a cosmological constant will not change the form of the action\textsuperscript{4}. As we will see we can apply the discussion of the previous section to these theories simply taking $n = 0$ in the relevant formulae. We will study in the next subsection the behaviour of the spherically symmetric solutions of this theory at short distances from sources, and we will show that in this domain the corrections to the Schwarzschild geometry are small. Also, it will be made clear that the Newtonian potential has large corrections at a distance given by $r_c$ in eq. (11). In subsection 3.2 we will study the linearisation of this theory in vacuum. We will give the conditions for the stability of de Sitter space and we will see that the effective Planck mass in vacuum can easily be reduced with respect to the one at short distances by a large ratio, and in this case gravity becomes significantly stronger at large distances. The expansion of the action in vacuum is the relevant one to apply in some dynamical situations at very large distances from sources and in late-time cosmology, but one should keep in mind that this linearisation breaks down when one approaches any source or when the ambient curvature is large (like, for instance, in the early Universe). In subsection 3.3 we will discuss some of the characteristic experimental signatures that these theories imply.

### 3.1 Short distance solution

In this subsection we will follow the same strategy as in \[11\] in order to study the behaviour of the solutions at short distances from sources. The spherically symmetric

\textsuperscript{4} Adding a cosmological constant would simply redefine the function $f$. When writing the action in this form we are of course assuming that the mass scales appearing in $f$ are “reasonable” in terms of $\mu$ so that $H_0 \sim \mu$.\]
solutions for this theory are obtained by solving the equations

\[ G_{\mu\nu} + \mu^2 H_{\mu\nu} = 0, \quad (18) \]

where \( G_{\mu\nu} \) is the usual Einstein tensor and \( \mu^2 H_{\mu\nu} \) is the extra term generated by the logarithmic part of the action. One can see that this extra term, when evaluated in the Schwarzschild solution, is subleading with respect to the terms that appear in the Einstein tensor when \( r \ll r_c \). This indicates that as \( r \to 0 \) the corrections with respect to the Schwarzschild geometry will be small. We can then consider a small perturbation of the black hole geometry and solve at first order in the perturbations. So we take the ansatze

\[
d s^2 = - \left[ 1 - \frac{2G_NM}{r} + \epsilon A(r) \right] dt^2 + \left[ 1 - \frac{2G_NM}{r} + \epsilon B(r) \right]^{-1} dr^2 + r^2 d\Omega^2, \quad (19)
\]

and treat \( \epsilon \) as a small expansion parameter. We can expand the full equations in powers of \( \epsilon \):

\[
G_{\mu\nu} = G^{(0)}_{\mu\nu} + \epsilon G^{(1)}_{\mu\nu} + \epsilon^2 G^{(2)}_{\mu\nu} + \ldots, \\
H_{\mu\nu} = H^{(0)}_{\mu\nu} + \epsilon H^{(1)}_{\mu\nu} + \epsilon^2 H^{(2)}_{\mu\nu} + \ldots. \quad (20)
\]

Since the Schwarzschild solution satisfies the ordinary Einstein equations, we have \( G^{(0)}_{\mu\nu} = 0 \). Treating \( \mu^2 \) as an order \( \epsilon \) parameter, at first order in our expansion the equations for \( A \) and \( B \) become (from now on we set \( \epsilon = 1 \))

\[
G_{\mu\nu}^{(1)} = -\mu^2 H_{\mu}^{(0)}. \quad (21)
\]

For the \( tt \) component of this equation we find:

\[
\frac{B + rB'}{r^2} = -\frac{\mu^2}{2G_NM} \left( 13G_NM - 8r + G_NM \log \left[ \frac{48(G_NM)^2}{r^6 Q_0} \right] \right) \quad (22)
\]

while the \( rr \) component reads:

\[
\frac{(2G_NM - rB)(2G_NM - r)^{-1} + rA'}{r^2} = \frac{\mu^2}{2G_NM} \left( 7G_NM - 2r - G_NM \log \left[ \frac{48(G_NM)^2}{r^6 Q_0} \right] \right). \quad (23)
\]

We can solve the previous equations yielding

\[
B(r) = \left( \frac{2G_NM}{r} \right) \left( \frac{r}{r_c} \right)^4 \left[ 2 - \frac{G_NM}{3r} \left( 15 + \log \left[ \frac{48(G_NM)^2}{Q_0 r^6} \right] \right) \right], \\
A(r) = -\left( \frac{2G_NM}{r} \right) \left( \frac{r}{r_c} \right)^4 \left[ \frac{4}{3} - \frac{G_NM}{3r} \left( 5 - \log \left[ \frac{48(G_NM)^2}{Q_0 r^6} \right] \right) \right], \quad (24)
\]
where $r_c$ is now given by eq. (11). So we see that at short distances the corrections to
Newton’s potential are indeed suppressed by powers of $(r/r_c)^4$. In fact, we can con-
sider our approximate solution as the first order in the Taylor expansion in powers of
$r/r_c$ of the exact solution, where the next order in the expansion would be $O((r/r_c)^3)$. At
distances of order $r_c$ the expansion breaks down, and it is clear that we can ex-
pect a significant modification of Newton’s potential for larger distances (or smaller
accelerations). A difference with respect to the actions involving inverse powers of
the curvature is that now the scalar curvature does not go to zero at short distances.
This is so because the logarithmic part of the action does not go to zero for $r \to 0$,
as happened in the case of the inverse powers in the previous section, but the scalar
curvature (or the extra term in the action) is of the order of the one in vacuum so the
effect is very small.

In the next subsection, we study the linearisation of this theory in vacuum, which
will enable us to obtain the ultra large distance behaviour of the spherically symmetric
solution corresponding to a mass source.

### 3.2 Linearisation in vacuum

In vacuum, without any matter source, the action has de Sitter solutions with curvature
$R = 12H_0^2$ where $H_0$ can be found as a solution of

$$6H_0^2 = \mu^2 \left\langle \log f - 6H_0^2 \left( \frac{f_R - 20H_0^2 f_Q}{f} \right) \right\rangle_0 .$$

(25)

To check the stability of these solutions one can consider the modified Friedmann
equation in vacuum obtained in this theory: $\ddot{H} = h(H, \dot{H})$ and expand for small
perturbations $\dot{H}$ and $\ddot{H} \equiv H - H_0$ of the equation of motion. We find:

$$\ddot{H} \approx h_H(H_0, 0) \dot{H} + h_{\dot{H}}(H_0, 0) \ddot{H} = 16H_0^2 \left( \frac{1}{4} + \frac{C_1}{C_2} \right) \ddot{H} - 3H_0 \dot{H} ,$$

(26)

where we have defined

$$C_1 \equiv \delta + \frac{4H_0^2}{m_0^2} \quad \text{and} \quad C_2 \equiv -\frac{16H_0^2}{m_0^2} ,$$

(27)

$m_0$ and $\delta$ have been defined in the second section. One can easily check that the
fixed point ($\ddot{H} = 0$, $\dot{H} = 0$) of this dynamical system is an attractor (repeller) if the
coefficient in front of $\ddot{H}$ is negative (positive), so de Sitter space will be stable as long
as
\[ 1 + \frac{4C_1}{C_2} < 0. \]  (28)

One would obtain the same result by analysing the equation of motion of the propagating modes, and it is instructive to do so. Because the action is a function of \( Q \) and \( P \) only through the combination \( Q - 4P \), the ghost is absent and there is just a massive scalar field in the spectrum besides the massless graviton. So this field is the only possible source of instability. It was shown in \[12\] that in this case de Sitter spacetime is stable as long as \( m_s^2 > -9H_0^2/4 \), and the mass of the scalar is
\[ m_s^2 \equiv -H_0^2 \left( \frac{25}{4} + 16\frac{C_1}{C_2} \right), \]  (29)
which is consistent with the phase space analysis presented here and also with the results of \[22\], for \( F(R) \) theories. But even when de Sitter space is unstable one should not disregard the model. It has been shown that for the actions involving inverse powers of the curvature de Sitter space is unstable in many cases but the late time background corresponds to a power law FRW cosmology that is nevertheless phenomenologically interesting \[13,21\].

An important feature of this linearisation is that the effective Planck mass that we obtain in vacuum controlling the coupling of the spin two graviton, is now given by
\[ M_{p(eff)}^2 = C_1 M_p^2 = \left\langle 1 - \mu^2 \frac{f_R - 2RfQ}{f} \right\rangle_0 M_p^2. \]  (30)

At short distances however the value of the Planck mass is just \( M_p \), as we saw using the short distance expansion over Schwarzschild geometry. But in vacuum \( R^2 \sim P \sim Q \sim \mu^4 \sim H_0^4 \) and we can expect that \( C_1 \) will depart significantly from 1. This means that when this linearisation is applicable, we get a rescaled Planck mass with respect to the one we would infer at short distances and one can expect an enhancement or suppression of the gravitational interaction at large distances. In fact, applying this linearisation, we get for a spherically symmetric mass the solution \[12\]:
\[ ds^2 \simeq -\left( 1 - \frac{8G_{N(eff)}^2 M}{3r} - H_0^2 r^2 \right) dt^2 + \left( 1 - \frac{4G_{N(eff)}^2 M}{3r} - H_0^2 r^2 \right)^{-1} dr^2 + r^2 d\Omega_2^2 \]  (31)
where \( G_{N(eff)}^2 = G_N/C_1 \) is the effective long-distance Newton’s constant. Remember that the linearisation we are using to get this result breaks down at a huge distance
from any source, at the Vainshtein radius \( r_V \approx (G_N M/H_0^3)^{1/4} \). At shorter distances we can not use the linearised version of the theory, and the solution above is only a good approximation for \( r > r_V \). But we have seen that at even shorter distances, \( r < r_c \), we can use a different expansion: the one that we considered in the previous sub-section that shows that in this regime the solution, eqs. (19, 24), approaches the Schwarzschild one with a Planck mass given by \( M_p^2 \). So also in these theories there is a non-perturbative regime in the range of distances \( r_c < r < r_V \), where one should get an interpolation between the solutions eqs. (19, 24) and eq. (31). In this intermediate range of distances both the linearisation of the theory over the vacuum and the short distance expansion over the Schwarzschild solution break down, but we will assume that the function \( f(R, Q - 4P) \) is such that a consistent matching exists. When \( C_1 < 1 \) and Newton’s constant is enhanced at large distances, the interpolating regime would play the role of a “dark halo” surrounding any source.

Even though we have not solved the gravitational equations in this non-perturbative regime, we would like to stress some important characteristics of this intermediate range of distances that make these theories promising candidates to build a MOND-like modification of gravity that could explain the dynamics of galaxies without recourse to Dark Matter. First, the modification becomes important below a characteristic Newtonian acceleration scale determined by a parameter \( (\mu) \) that also determines the late-time Hubble constant, as consistently indicated by the data [5]. And second, the gravitational interaction can be enhanced at long distances by a ratio that depends on the particular function \( f \) that we choose, but that can easily be of the required magnitude. As an example we plot in fig.2 the ratio \( M_p^2/M_p^{(eff)} = 1/C_1 \), for \( f = (2R^2 - 4P + Q)/Q_0 \), as a function of \( \log[\mu^4/Q_0]/2 \). The apparent mass discrepancy that one would infer cosmologically or at large distances is of the order of this ratio. There are further corrections because the theory at large distances is of the scalar-tensor type, with an almost massless scalar, so we get an extra \( 4/3 \) factor in the Newtonian potential and the deflection of light would also get an analogous correction factor.

These theories make a diverse variety of characteristic experimental predictions that can be used to test or falsify them. In this section we have seen the approximate form of the solution for a spherically symmetric mass at short distances \( (r < r_c) \) and at long distances \( (r > r_V) \) in de Sitter space. Using these approximate solutions and the linearised version of the theory, once we have chosen a particular function \( f \), we can al-
Figure 2: Ratio of the effective Planck mass at short distances \((r < r_c)\) over the one at large distances \((r > r_V)\) for a spherical mass in the model \(f(15)\) with \(f = (2R^2 - 4P + Q)/Q_0\), as a function of \(\log\left(\mu^2/Q_0^{1/2}\right)\). This represents roughly the enhancement of the gravitational interaction at large distances.

ready compare the predictions of the theory with precision Solar System measurements and a wealth of cosmological, astrophysical and large scale structure data. But these are not the only testing grounds for the theory. It is also possible to obtain predictions that would differ from those of GR for laboratory experiments measuring the gravitational interaction of small bodies. The discussion of these experimental implications is the subject of the following subsection.

### 3.3 Experimental implications

The theories we are studying in this section offer a diverse range of characteristic experimental predictions differing from those of GR that would allow their falsification. The most obvious tests would come from the comparison of the predictions of the theory to astrophysical and cosmological observations where the dynamics are dominated by very small gravitational fields. But we would have also some small effects for the motion of the planets or other celestial bodies in the Solar System and as we have said short distance modifications of Newton’s law. In the following we will briefly discuss separately these possible tests, and we will give order-of-magnitude estimations of the expected effects.
3.3.1 Short distance deviations from Newton’s law

Probably the most characteristic experimental signature of our theory corresponds to deviations from Newton’s law that should be seen at short distances, where the precise distance at which the modification is noticeable depends on the local background curvature. While a detailed computation of these corrections will be deferred to a future publication [23], to see why this is so we can use the following arguments. As we have said, for theories of the type (15) there is an extra scalar excitation of the spacetime metric besides the massless graviton. The mass of this field is given in de Sitter space by eq.(29), but it picks up an $r$-dependent contribution in the spacetime of a spherically symmetric mass. In a generic background, if we study a system in a scale that reduces the space-time region of interest to be small enough, we can consider an effective theory in which the extra degree of freedom has a mass given by its local value. Now, if we evaluate the expression for $m_0$ in the spacetime of a spherically symmetric mass we get that at a distance $r_d$ of a massive object of mass $M$, the mass of the scalar is locally of order

$$m_s^2 \sim \frac{Q}{\mu^2} \sim \frac{(G_N M)^2}{r_d^6 \mu^2},$$

(32)

where we are assuming that $Q \gg R^2, P$ so $f \simeq Q/Q_0$. So in the effective gravitational theory that we should apply on the Earth surface there is, besides the massless spin two graviton, an extra scalar field with gravitational couplings and a mass given roughly by the expression above. A peculiar feature of this local effective theory on a Schwarzschild background is that there will be a preferred direction and this will be reflected in an anisotropy of the force that this scalar excitation will mediate [23]. But for the purposes of this section, an estimation of the expected order-of-magnitude of the corrections, we will simply focus on the value of the mass of the scalar. From effective field theory arguments we can then expect short distance modifications of Newton’s law, suppressed by $r_{SD} \sim m_s^{-1} \sim \mathcal{O}(0.1 \, mm)$, when measuring the gravitational field of some probes on the Earth surface. This order of magnitude is very interesting because it is the one currently being probed by experiments [8]. As we said there are also other motivations for expecting short distance modifications of gravity at this scale [9,10], but as we also said the coincidence of these scales is just a coincidence. If one was to measure the gravitational field of a small object in space, in the neighbourhood of the Earth’s orbit but far from the Earth, the relevant mass and distance we should apply in eq.(32) is
the mass of the Sun and the Sun-Earth distance, since the Sun provides the dominant gravitational field in that situation. In this case we get \( m_s^{-1} \sim 10^4 \) m. But notice that to observe a significant modification in the gravitational field of an object we have to measure it at a distance bigger than \( r_c \) for that object, otherwise the self-shielding of the extra scalar excitation induced by the object itself is enough to switch off the modification. This means that locally in the inner Solar System we could only see significant modifications in the gravitational field of objects whose characteristic distance \( r_c \) is smaller than \( 10^4 \) m, so its mass would have to be below \( \sim 10^9 \) kg. As an example we can mention that for an object of mass \( 10^3 \) kg orbiting the Sun at the same distance as the Earth, one can expect modifications of its gravitational field starting at a distance \( r_c \sim 10 \) m (at shorter distances the scalar effectively decouples because of the gravitational field of the object itself) and extending to \( r_{SD} \sim 10^4 \) m (at longer distances the mass induced by Sun’s gravitational field effectively decouples the scalar).

### 3.3.2 Solar System observations

Although the corrections that our theories introduce with respect to GR for a spherically symmetric solution are suppressed by powers of \( (r/r_c)^4 \), and this is a very small number for the Sun within the Solar System, the precision with which the motion of the celestial bodies of the Solar System is known makes it conceivable that one could see the corrections induced by this modification (similarly as happens in other long-distance modifications of gravity as the DGP model [24]). For instance, in the case of the precession of the perihelion of the planets, the anomalous shift \( (\Delta \phi) \) induced by a small correction \( (\delta V) \) to Newton’s potential \( (V_N) \) is given in radians per revolution by (see e.g. [24])

\[
\Delta \phi \simeq \pi r \frac{d}{dr} \left( r^2 \frac{d}{dr} \left( \frac{\delta V}{rV_N} \right) \right),
\]

(33)

so the correction induced by our modification increases with distance as

\[
\Delta \phi \simeq 16\pi \left( \frac{r}{r_c} \right)^4.
\]

(34)

From this expression we see that for the inner planets the perihelion shift induced by our modification is very small, it only becomes of the same order as the correction introduced by GR (with respect to Newtonian gravity) for Jupiter. But for the outer
planets the contribution to the perihelion shift of the modification can be dominant over the standard GR one\(^5\). Unfortunately the only reliable data regarding the planetary perihelion comes from the inner planets of the Solar System \(^{25}\), where our correction is negligible. But the best measured orbit is that of the Moon around the Earth by virtue of the Lunar Laser Ranging (see e.g. \(^{26}\)). Using this the Earth-Moon distance is known with a precision of centimetres. Applying the formula above, for the Moon our theory predicts an anomalous shift of \(\Delta \phi \sim 10^{-12}\), to be compared with the achieved accuracy of \(2.4 \times 10^{-11}\) \(^{24}\). Although these numbers are not far from each other remember that we are just doing an order of magnitude estimation, since the parameter \(\mu\) is related to \(H_0\) only after choosing a particular function \(f\). So although these theories suggest the possibility of surprises in high precision astrometrical measurements, the numbers that we obtained also expose the difficulties of ruling out these theories using these effects, since for this we would need an improvement on the bounds of the anomalous precession of the Moon of at least two orders of magnitude.

### 3.3.3 Cosmology and Astrophysics

In this subsection we will comment on the possible comparisons that one could make of the predictions of these theories to astrophysical and cosmological observations (see for instance \(^{27}\)), although any actual fit to these data is beyond the scope of the current paper. In this respect, once a particular function \(f\) is chosen, one can make unambiguous predictions for the rotation curves of spiral galaxies with the mass-to-light ratio being the only free parameter, since the normalisation of \(\mu\) will be fixed by the Hubble constant, and in fact this issue has been our main motivation for proposing actions of the type \(^{15}\). Our result for Newton’s potential at short distances \(^{24}\) can be parameterised as:

\[
V(r) = \frac{GM}{r} v(x),
\]

(35)

where \(x \equiv r/r_c\) and \(v(x) \approx 1 + \frac{4}{3} x^4 + \ldots\) for small \(x\). The challenge now is to find a form for \(f\) that yields a MOND-like phenomenology in the intermediate region. More specifically we need that \(v(x) \sim -x \ln(x)\) for large \(x\). The potential would then give rise to flat rotation curves that obey the Tully-Fisher law \(^{3}\). But also other aspects of the observations of galactic dynamics can be used to constrain a MOND-like modification

\(^5\)Not only our correction increases with distance, but the GR one \((\Delta \phi \sim 6\pi G_N M/r)\) decreases for larger radius.
of Newton’s potential, see e.g. [28]. And notice also that our theory violates the strong equivalence principle, as expected for any relativistic theory for MOND [4], since locally physics will intrinsically depend on the background gravitational field. If we consider for instance a local system like an open cluster, rotating in the background field of the Milky Way, the solution (35) that we obtained for an isolated system is not necessarily valid anymore. This will be the case if the background curvature dominates the curvature induced by the local system, similarly to the “external field effect” in MOND. Notice that in order to establish the relevance of the external field effect in this framework we have to compare the induced external and internal curvatures, and not the accelerations. This is relevant for the case of the globular clusters orbiting the Milky Way, where MOND (or DM) evidence has been found where the internal acceleration is below \( a_0 \), even if the external acceleration due to the Milky Way is bigger than \( a_0 \) [29]. However, if one compares the values of the curvature (i.e., the invariant \( Q \)) one finds that the curvature is dominated by the internal one, justifying the neglect of the external field effect and the treatment of these globular clusters as isolated objects, into the MOND regime. So the globular cluster data of [29] favors models, like the one presented in this paper, in which the relevance of the external field effect is determined comparing curvatures, and not accelerations.

At larger scales, where one can use the equivalence with a scalar-tensor theory more reliably, we can expect from continuity arguments that such a form for \( f \) (if it exists) will enhance Newton’s constant. One can then compare the theory against the observations of gravitational lensing in clusters, the growth of large scale structure and the fluctuations of the CMB. In fact, it has been pointed out that if GR was modified at large distances, an inconsistency between the allowed regions of parameter space would show up for (non-modified) Dark Energy models when comparing the bounds on these parameters obtained from CMB and large scale structure [30]. This means that although some cosmological observables, like the expansion history of the Universe, can be indistinguishable in modified gravity and Dark Energy models, this degeneracy is broken when considering other cosmological observations and in particular the growth of large scale structure and the Integrated Sachs-Wolfe effect (ISW) have been shown to be good discriminators for models in which GR is modified [31].

Regarding the ISW effect, we would like to mention some characteristics of these theories that point to the possibility of getting a suppression of the low multipoles of
the CMB with respect to a ΛCDM cosmology, a feature shown by the data that cannot be easily accounted for in the ΛCDM model. It has been recently pointed out that the fact that in the DGP model the effective Newton’s constant increases at late times as the background curvature diminishes, causes a suppression of the ISW that brings the theory into better agreement with the CMB data than the ΛCDM model [32]. In our case we can expect an analogous effect, so that the effective Newton’s constant for the cosmic perturbations depends on the background curvature in such a way that it increases at late times as the curvature diminishes. But despite this potentially good features it remains to be seen if one can get a good fit to the CMB data in the absence of Dark Matter in these models.

4 Conclusions

In this paper, motivated by the phenomenological success of MOND fitting the rotation curves of spiral galaxies without requiring Dark Matter, we have proposed a class of actions that modify gravity below the characteristic acceleration scale required by MOND: \( a_0 \sim H_0 \). There are two effects in these theories that are responsible for the infrared modification. First, there is an extra scalar excitation of the spacetime metric besides the massless graviton. The mass of this scalar field is of the order of the Hubble scale in vacuum, but its mass depends crucially on the background over which it propagates. This dependence is such that this excitation becomes more massive as we approach any source, and the extra degree of freedom decouples at short distances in the spacetime of a spherically symmetric mass. This feature makes this excitation to behave in a way that reminds of the chameleon field of [33], but in our case this “chameleon” field is just a component of the spacetime metric coupled to the curvature. But there is a second effect in these theories: the Planck mass that controls the coupling strength of the massless graviton also undergoes a rescaling or “running” with the distance to the sources (or the background curvature). This phenomenon, although a purely classical one in our theory, is reminiscent of the quantum renormalisation group running of couplings. So one might wonder if actions of the type (15) could be an effective classical description of strong renormalisation effects in the infrared that might appear in GR (see e.g. [34] and references therein), as happens in QCD. In fact, corrections depending on the logarithm of the renormalisation scale are ubiquitous in
quantum field theory, and it appears natural to identify the renormalisation scale with a function of the curvature if we want to build an effective classical action for the spacetime metric that takes into account these quantum effects. Indeed, we have seen that these models offer a phenomenology that seems well suited to describe an infrared strongly coupled phase of gravity: at high energies/curvatures we can use the GR action or its linearisation as a good approximation, but when going to low energies/curvatures we find a non-perturbative regime. At even lower energies/curvatures perturbation theory is again applicable, but the relevant theory is of scalar-tensor type in a de Sitter space.

We would like also to emphasise that there are clearly many modifications of the proposed class of actions that would offer a similar phenomenology, such that gravity would be modified below a characteristic acceleration scale of the order of the one required in MOND. For instance if we consider the action

\[ S = \int d^4x \sqrt{-g} \frac{1}{16\pi G_N} \left\{ R - \mu^2 (\log[f])^n \right\}, \]  

(36)

with the same assumptions on \( f \) that we did before we get that now the critical acceleration \( a_0 \) also has a “running” with the mass as

\[ a_0 = \frac{G_N M}{r_c^2} \sim \mu \left( \log \left[ \frac{48a_0^3}{Q_0 G_N M} \right] \right)^{(n-1)/2}. \]  

(37)

To introduce some kind of scale dependence of \( a_0 \) could be interesting since MOND typically gives an overestimation of the amount of visible matter at cluster scales.

Since the expansions we have used break down for some intermediate range of energies/distances one should still show that the dynamics in this “non-perturbative” regime are consistent for some choice of \( f \) and that one recovers an acceptable matching between the high energy/short distance and low energy/large distance regimes. And then one should compute the predictions of these theories in many different situations for which there are experimental data to compare with before any of these models could be considered a viable alternative to a ΛCDM cosmology. This is a non-trivial task but it is worth undertaking it because we have seen that there are reasons to believe that one might explain many aspects of the cosmological and astrophysical observations without introducing Dark Matter in this class of theories. And, as we have seen, these theories also offer the unique possibility of being tested not only through astrophysical observations, but also through well-controlled laboratory experiments.
where the outcome of such experiments is correlated with parameters that can be determined by means of cosmological and astrophysical measurements.

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