Asymptotic expansion of semi-Markov random evolutions

S. Albeverio\textsuperscript{abcd}, V.S. Koroliuk\textsuperscript{e} and I.V. Samoilenko\textsuperscript{e}*

\textsuperscript{a}Institut für Angewandte Mathematik, Universität Bonn, Bonn, Germany; \textsuperscript{b}SFB 611, Bonn, BiBoS, Bielefeld-Bonn, Germany; \textsuperscript{c}CERFIM, Locarno, Switzerland; \textsuperscript{d}USI, Mendrisio, Switzerland; \textsuperscript{e}Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, Ukraine

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Regular and singular parts of asymptotic expansions of semi-Markov random evolutions are given. Regularity of boundary conditions is shown. An algorithm for calculation of initial conditions is proposed.

Keywords: asymptotic expansion; SMREs; singularly perturbed integral equation; boundary layer; estimate of the remainder

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1. Introduction

Many stochastic systems can be described by the abstract mathematical model in the Banach space $\mathcal{B}(\mathbb{R}^d)$ of functions $\varphi(u), u \in \mathbb{R}^d$, called random evolution model, introduced by Griego and Hersh [1–3].

Asymptotic methods in the theory of random evolutions were applied by many mathematicians (see, e.g. Refs. [4,12,16]). Application of these methods to different stochastic systems may be found in Ref. [13]. In Ref. [17] kinetic theory of gases, isotropic transport on manifolds, stability of random oscillators were studied by similar methods.

Among recent works in this area with applications in mathematical biology, we may remind results of Hillen and Othmer [5,14] (see also Ref. [15]).

Namely, transport equations are used in mathematical biology to model movement and growth of populations. Certain bacteria show the following movement pattern: periods of strait runs alternate with periods of random rotations which leads to reorientation of the cells. We may model this movement by a velocity jump process, which leads to a transport equation.

Thus, the linear transport equation

$$\frac{\partial}{\partial t} p(x,v,t) + v \nabla p(x,v,t) = -\lambda p(x,v,t) + \int_{V} \lambda T(v,v') p(x,v',t) dv',$$

in which $p(x,v,t)$ represents the density of particles at spatial position $x \in \mathbb{R}^n$ moving with velocity $v \in V \subset \mathbb{R}^n$ at time $t \geq 0$ arises. The turning rate $\lambda$ may be space- or velocity-dependent. The turning kernel or turn angle distribution $T(v,v')$ gives the probability of a velocity jump from $v'$ to $v$ if a jump occurs.

*Corresponding author. Email: isamoil@imath.kiev.ua
The evolutionary equation studied in our work generalizes the transport equation described above. Analogical generalization of telegraph-type equation is described in Ref. [18].

One more example of application of asymptotic methods is described in the works of Yin and Zhang [23].

They study a model for production planning of a failure-prone manufacturing system, which consists of machines whose production capacity is modelled by a Markov or semi-Markov chain. In a large-scale system, various components may change at different rates. Thus, the system may be decomposed and the states of the chain may be aggregated. The introduction of a small parameter $\varepsilon > 0$ makes the system belong to the category of two-time-scale systems.

The system is described by the equations of the type

$$
\frac{d p^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} p^\varepsilon(t) Q(t).
$$

Here $p^\varepsilon(t)$ is the probability distribution of Markov or semi-Markov chain and $Q(t)$ is the corresponding generator. As a rule, the fast changing process $p^\varepsilon(t)$ in the physical or manufacturing systems is hard to analyze. But using limit properties, obtained with the help of asymptotic expansions, they replace the process by its ‘average’ in the system under consideration.

The problems of minimization of discounted cost function and optimal control are studied in Ref. [23] with the use of asymptotic approximations. Markov decision problems, stochastic control of dynamical systems, numerical methods for control and optimization are also analyzed.

We study the following generalization of the stochastic systems described above.

A semi-Markov random evolution (SMRE) is created by a solution of the evolutionary equation in Euclidean space $\mathbb{R}^d$, $d \geq 1$

$$
\frac{du^\varepsilon(t)}{dt} = v(u^\varepsilon(t); \omega(t/\varepsilon)), \quad t \geq 0
$$

with $u^\varepsilon(t) \in \mathbb{R}^d$, $v$ a given continuous function from $\mathbb{R}^d \times \mathbb{E}$ into $\mathbb{R}$, $(\mathbb{E}, \mathcal{E})$ a standard (polish) space, $\omega$ is a given semi-Markov switching process $\omega(t), t \geq 0$ on $(\mathbb{E}, \mathcal{E})$ given in terms of the semi-Markov kernel [7]

$$
Q(x, B, t) = P(x, B)F_x(t), \quad x \in \mathbb{E}, \ B \in \mathcal{E}, \ t \geq 0,
$$

that defines the transition probabilities of a Markov renewal process $(\omega_n, \tau_n, n \geq 0)$:

$$
Q(x, B, t) = P\{\omega_{n+1} \in B, \theta_{n+1} \leq t|\omega_n = x\} = P\{\omega_{n+1} \in B|\omega_n = x\} P\{\theta_{n+1} \leq t|\omega_n = x\},
$$

with $\theta_n := \tau_{n+1} - \tau_n$.

The stochastic kernel

$$
P(x, B) = P\{\omega_{n+1} \in B|\omega_n = x\}
$$

defines the transition probabilities of the embedded Markov chain $\omega_n = \omega(\tau_n), n \geq 0$; the renewal moments are determined by the distribution function of the sojourn times:

$$
F_x(t) = P\{\theta_{n+1} \leq t|\omega_n = x\} =: P\{\theta \leq t\}, \quad x \in \mathbb{E}.
$$
We assume that the distribution functions \( F_x(t) \), \( x \in E \) satisfy the Cramer condition uniformly on \( x \in E \):

\[
\sup_{x \in E} \int_0^\infty e^{ht} F_x(dt) \leq H < \infty, \quad (1)
\]

for all \( h > 0 \).

We denote by \( Q \) the generator of the associated Markov process \( \alpha^0(t), t \geq 0 \) given by:

\[
Q = q(x)(P - I), \quad (2)
\]

where the operator of transition probabilities \( P \) is defined by

\[
Pf(x) = \int_E P(x, dy)f(y), \quad x \in E, \quad (3)
\]

for all bounded measurable real valued \( f \) defined on \( E \). \( q(x) \) is defined by

\[
q(x) := \frac{1}{m_1(x)},
\]

with \( m_k(x) = \int_0^\infty s^k F_x(ds) \).

We will see later that the equation for the regular part of a random evolution is defined by the generator (2) of a uniformly ergodic associated Markov process \( \alpha^0(t), t \geq 0 \). The Banach space \( B(E) \) is splitted as direct sum of the two subspaces [11]:

\[
B(E) = N_Q \oplus R_Q,
\]

where \( N_Q := \{ \varphi : Q \varphi = 0 \} \) is the null-space of \( Q \), and \( R_Q := \{ \psi : Q \varphi = \psi \} \) is the range of \( Q \).

We define the projector \( \Pi : N_Q := \Pi B(E), R_Q := (I - \Pi) B(E), \Pi \varphi(x) := \hat{\varphi} I, \hat{\varphi} := \int_E \varphi(x) \pi(dx) \), where the stationary distribution \( \pi(B), B \in E \) of the semi-Markov process \( \alpha(t), t \geq 0 \) satisfies the relations [8,10]

\[
\pi(dx) = \frac{\rho(dx)m_1(x)}{\hat{m}},
\]

\[
\hat{m} := \int_E m_1(x) \rho(dx).
\]

\( \rho(B), B \in E \) is by assumption the stationary distribution of the embedded Markov chain \( \alpha_n, n \geq 0 \), given by the formula

\[
\rho(B) = \int_E P(x, B) \rho(dx), \quad \rho(E) = 1.
\]

Let us consider the Banach space \( B(\mathbb{R}^d) \) of real-valued test-functions \( \varphi(u), u \in \mathbb{R}^d \) which are bounded with all their derivatives. We equip \( B(\mathbb{R}^d) \) with sup-norm

\[
\| \varphi \| := \sup_{u \in \mathbb{R}^d} | \varphi(u) | < C_\varphi,
\]

for some \( C_\varphi > 0 \).
The random evolution in $\mathcal{B}(\mathbb{R}^d)$ is given by the relation

$$\Phi^\varepsilon_t := \varphi(u^\varepsilon(t)), \quad \varphi \in \mathcal{B}(\mathbb{R}^d).$$  \hfill (4)

In the present paper, we shall investigate the asymptotic behaviour of SMRE (4) as $\varepsilon \to 0$ under the assumption of uniformly ergodicity of the semi-Markov switching process $\varphi(t)$ described above and under the assumption of the existence of a global solution of the deterministic equations

$$\frac{du_s(t)}{dt} = v(u_s(t); x), \quad x \in E. \hfill (5)$$

Let us first consider the deterministic evolution

$$\Phi_s(t, u) = \varphi(u_s(t)), \quad u_s(0) = u \in \mathbb{R}^d.$$ It generates a corresponding semigroup on $\mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{V}_t(x)\psi(u) := \varphi(u_s(t)), \quad u_s(0) = u,$$ and its generator has the form:

$$\mathcal{V}(x)\psi(u) = v(u; x)\psi'(u) := \sum_{k=1}^{d} v_k(u; x)\psi_k'(u), \quad \varphi(u) \in C^\infty(\mathbb{R}^d).$$ \hfill (6)

Under the assumptions of Ref. [12] by the average principle, the weak convergence

$$u^\varepsilon(t) \Rightarrow \hat{u}(t), \quad \varepsilon \to 0 \hfill (7)$$
takes place. The average limit evolution $\hat{u}(t), t \geq 0$ is defined by a solution of the average equation

$$\frac{d\hat{u}(t)}{dt} = \hat{v}(\hat{u}(t)).$$

The average velocity $\hat{v}(u), u \in \mathbb{R}^d$ is defined by

$$\hat{v}(u) = \int_E v(u; x) \pi(dx)$$

(i.e. by the average of the initial velocity $v(u; x)$ over the stationary distribution $\pi(B), B \in \mathcal{E}$).

The rate of convergence in (7) can be investigated in two directions:

(i) asymptotic analysis as $\varepsilon \downarrow 0$ of the fluctuations

$$\xi^\varepsilon(t) = u^\varepsilon(t) - \hat{u}(t); \hfill (8)$$

(ii) asymptotic analysis of the average deterministic evolution

$$\Phi^\varepsilon_x(u, x) = E[\varphi(u^\varepsilon(t))|u^\varepsilon(0) = u, x^\varepsilon(0) = x]. \hfill (9)$$

The asymptotic analysis of fluctuations (8) leads to the diffusion approximation of the random evolution [9,20].
The asymptotic analysis of the evolution (9) is realized in what follows by constructing the asymptotic expansion in terms of the small parameter \( \varepsilon \) (\( \varepsilon \downarrow 0 \)) in the following form (\( \tau = t/\varepsilon \)):

\[
\Phi_\varepsilon^x(u, x) = U_0(t) + \sum_{k=1}^{\infty} \varepsilon^k [U_k(t) + W_k(\tau)].
\] (10)

The asymptotic expansion (10) contains two parts:

(i) the regular term \( U_\varepsilon^x(t) := U_0(t) + \sum_{k=1}^{\infty} \varepsilon^k U_k(t) \),

(ii) the singular term (boundary layer) \( W_\varepsilon^x(\tau) := \sum_{k=1}^{\infty} \varepsilon^k W_k(\tau) \), \( \tau = t/\varepsilon \).

In addition, the ‘regular degeneration’ (see Refs. \([6,22]\)) of the initial and ‘boundary’ conditions

\[
U_0(0) = \varphi(u)1, \quad U_k(0) + W_k(0) = 0, \quad k \geq 1, \quad W_\varepsilon^x(t) + U_\varepsilon^x(\varepsilon t) - \varphi(u) = 0, \quad t \leq 0 \quad (11)
\]

has to be valid for all \( x \in E, u \in \mathbb{R}^d \).

Remark 1.1. The singular terms in (10) are determined by the boundary condition at infinity:

\[
W_k(\pm \infty) = 0, \quad k \geq 1.
\]

Asymptotic expansions with ‘boundary layers’ were studied by many authors (see Refs. \([6,22]\)). In particular, functionals of Markov and semi-Markov processes are investigated from this point of view in Refs. \([12,18,21]\).

In this work, the asymptotic expansion (10) for the average SMRE (9) is constructed by using the solution of an integral Markov renewal equation. The algorithm for the construction of explicit regular and singular terms and boundary conditions is given in the following theorem, which will be proved in some stages (see the following lemmas).

We use the following notations: let

\[
L_k U(t) := \sum_{n=0}^{k} (-1)^n \nu^n(\mu) U^{(k-n)}(t),
\] (12)

where \( U(t) \) is any smooth function, \( U^{(k)}(t) := \partial^k U(t)/\partial t^k \), \( P \) is defined by (3), \( \nu(\mu) \) is defined by (6),

\[
\mu_k(\mu) := \frac{m_k(\mu)}{k! m_1(\mu)}, \quad \mu_1(\mu) := 1.
\]

We also set:

\[
Q W(\tau) := \int_0^\infty F_x(ds)PW(\tau - s),
\]

\[
\psi^k(\tau) := F^{(k)}(\tau) \nu^k P \varphi(\mu), \quad \psi^k_0(\tau) := \sum_{r=1}^{k-1} Q^r W_{k-r}(\tau),
\]

\[
F^{(k)}(\tau) := \int_0^\infty \frac{s^{k-1}}{(k-1)!} F_x(s)ds, \quad Q^r W(\tau) := \int_0^\tau \frac{s^{r}}{r!} F_x(ds) \nu^r PW(\tau - s),
\]

\[
\bar{F}_x(t) := 1 - F_x(t) \quad \tau = t/\varepsilon.
\]
Theorem 1.2. Under the conditions of uniform ergodicity of the underlying semi-Markov process and the existence of a global solution in $B(R^d)$ of the system (5), the asymptotic expansion of the semi-Markov evolution

$$\Phi^\varepsilon_t(u, x) = E[\varphi(u^\varepsilon(t))|u^\varepsilon(0) = u, x(0) = x]$$

has the form

$$\Phi^\varepsilon_t(u, x) = U^\varepsilon(t) + W^\varepsilon(\tau) = U_0(t) + \sum_{k=1}^{\infty} \varepsilon^k(U_k(t) + W_k(\tau)),$$

where

$$U_0(t) = c_0(t)1.$$

The function $c_0(t)$ satisfies the equation

$$\hat{v}(u) \frac{\partial c_0(t)}{\partial u} - \frac{\partial c_0(t)}{\partial t} = 0,$$

or

$$\frac{\partial c_0(t)}{\partial t} = \hat{\mathcal{V}}c_0(t),$$

where (correspondingly to (6)):

$$\hat{\mathcal{V}}\varphi(u) := \hat{v}(u)\varphi'(u),$$

$\hat{v}(u) := \int_E v(u, x)\pi(dx)$ is the average of the initial velocity $v(u, x)$ over the stationary distribution of the semi-Markov process. The initial condition is

$$c_0(0) = \varphi(u).$$

The regular terms are the following:

$$U_k(t) = R_0 \left( \sum_{n=1}^{k} \mu_n(x)L_n U_{k-n}(t) \right) + c_k(t),$$

where according to Ref. [11], $R_0 = [\mathcal{Q} + \Pi]^{-1} - \Pi$

The functions $c_k(t)$ satisfy the equations

$$\hat{\mathcal{L}}_1 c_k(t) = -\Pi L_k c_0(t) - \cdots - \Pi L_{k-1} c_{k-1}(t), \quad k \geq 1$$

where $\Pi L_k := \sum_{n=1}^{k} \Pi \mu_n(x)L_n R_0 L_{k-n} + \Pi \mu_{k+1}(x)L_{k+1}, \Pi L_0 := \Pi L_1 = \hat{\mathcal{L}}_1.$

The singular terms satisfy the Markov renewal equations ($k \geq 1$):

$$\int_0^\tau F_s(ds)PW_k(\tau - s) - W_k(\tau) = \psi_k(\tau) - \psi_0^k(\tau) - \psi_1^k(\tau), \quad \tau \geq 0.$$
with

\[ \psi^k(\tau) := \int_{\tau}^{\infty} F_x(ds)P\psi^k(\tau - s), \]

\( P \) is given by (3)) and may be explicitly written in the form:

\[ W_1(\tau) = R_0 \left[ \psi^1(\tau) + \tilde{F}_x(\tau)PU(0) + \int_{\tau}^{\infty} (\tau - s)F_x(ds)PU(0) \right], \]

\[ W_k(\tau) = R_0 \left[ \psi^k(\tau) - \psi^k_0(\tau) + \tilde{F}_x(\tau)PU_k(0) + \sum_{n=1}^{k} \int_{\tau}^{\infty} (\tau - s)^nF_x(ds)PU_{k-n}^{(n)}(0) \right], \]

where \( R_0 \) is the Markov renewal operator [19].

The initial conditions are given by:

\[(I - \Pi)[U_k(0) + W_k(0)] = 0,\]

\[ c_k(0) = -\Pi W_k(0), \]

\[ U_k(0) = \sum_{r=0}^{k-1} \int \pi(dx)\nu_{k-r}(x)L_{k-r}U_r(0) \]

\[ - \sum_{r=1}^{k-1} \int \rho(dx) \int_{0}^{\tau} \rho \int_{0}^{s-f} F_x(ds)\psi^r(\tau)PW_{k-r}(\tau - s)d\tau \]

\[ \hat{m}, \]

where \( \nu_k(x) = (-1)^k[m_k(x) - \mu_{k+1}(x)]. \)

**Proof.** The proof is a consequence of the lemmas in Sections 2–6.

**Remark 1.3.** For sufficient conditions for the assumptions to hold see, e.g. Refs. [4,12]. The equation for \( c_0(t) \) corresponds to the one given by the averaging theorem [9]. It states that the limit of the semi-Markov evolution

\[ \Phi^\epsilon_t(u,x) \Rightarrow \hat{\Phi}_t(u), \quad \epsilon \to 0 \]

satisfies the equation

\[ \frac{\partial \hat{\Phi}_t(u)}{\partial t} = \hat{\psi}\hat{\Phi}_t(u). \]

Here \( \hat{\Phi}_t(u) \) is the deterministic evolution

\[ \hat{\Phi}_t(u) = \varphi(u(t)), \quad u(0) = u. \]
The average limit evolution $u(t), t \geq 0$ is defined by a solution of the average equation
\[
\frac{du(t)}{dt} = \dot{v}(u(t)).
\]

2. Markov renewal equation

**Lemma 2.1.** The semi-Markov evolution $\Phi_t^e(u, x)$ satisfies the equation
\[
\int_0^\infty F_\sigma(ds)\mathbb{V}_{es}(x)P\Phi_t^{e}(u, x) - \Phi_t^e(u, x) = e\mathbb{V}(x)\int_\tau^\infty F_\sigma(s)\mathbb{V}_{es}(x)\varphi(u)ds,
\]
where $\tau = t/e$.

**Proof.** Using the first jump moment of the switching process $\theta_x$, we have:
\[
\Phi_t^e(u, x) = E_{u,x}[\varphi(u^e(t)); \theta_x > t/e] + E_{u,x}[\varphi(u^e(t)); \theta_x \leq t/e] = F_\sigma(t/e)\mathbb{V}_{t}(x)P\varphi(u) + \int_0^{t/e} F_\sigma(ds)\mathbb{V}_{es}(x)P\Phi_t^{e}(u, x).
\]

So
\[
\Phi_t^e(u, x) - \int_0^{t/e} F_\sigma(ds)\mathbb{V}_{es}(x)P\Phi_t^{e}(u, x) = F_\sigma(\tau)\mathbb{V}_{\tau}(x)P\varphi(u).
\]

Extending by the continuity, $\Phi_t^{e}(u, x) = \varphi(u), t - es \leq 0$, let us rewrite the latter equation in the form:
\[
\Phi_t^e(u, x) - \int_0^\infty F_\sigma(ds)\mathbb{V}_{es}(x)P\Phi_t^{e}(u, x) = \bar{F}_\sigma(\tau)\mathbb{V}_{\tau}(x)P\varphi(u) - \int_\tau^\infty F_\sigma(ds)\mathbb{V}_{es}(x)P\varphi(u).
\]

So, we have:
\[
\Phi_t^e(u, x) - \int_0^\infty F_\sigma(ds)\mathbb{V}_{es}(x)P\Phi_t^{e}(u, x) = \bar{F}_\sigma(\tau)\mathbb{V}_{\tau}(x)P\varphi(u) - F_\sigma(s)\mathbb{V}_{es}(x)P\varphi(u)ds.
\]

This gives equation (13). \(\square\)
3. Equations for the regular terms

Lemma 3.1. The equation for the regular part of the asymptotics has the form:

\[ QU^\varepsilon(t) = \left[ \sum_{k=1}^{\infty} e^k \mu_k(x) L_k \right] U^\varepsilon(t). \] (14)

Proof. We use the equality:

\[ aPb - 1 = (P - 1) + (a - 1)P + P(b - 1) + (a - 1)P(b - 1), \]

where \( a = \mathbb{V}_{es}(x) = I + \sum_{k=1}^{\infty} e^k (s^k/k!) \psi^k(x), b = \Phi_{t-es}^\varepsilon = \sum_{k=0}^{\infty} (-1)^k e^k (s^k/k!) \Phi_{t}^{(k)}(u,x). \)

Let us rewrite (13) in the following way:

\[
(P - I)\Phi_{t}^{(k)}(u,x) + \int_0^\infty F_s(ds) \left( \sum_{k=1}^{\infty} e^k \frac{s^k}{k!} \psi^k(x) \right)
\]

\[
\times P\Phi_{t}^{(k)}(u,x) + \int_0^\infty F_s(ds)P \left( \sum_{k=1}^{\infty} (-1)^k e^k \frac{s^k}{k!} \Phi_{t}^{(k)}(u,x) \right) + \int_0^\infty F_s(ds) \left( \sum_{k=1}^{\infty} e^k \frac{s^k}{k!} \psi^k(x) \right)
\]

\[
\times P \left( \sum_{k=1}^{\infty} (-1)^k e^k \frac{s^k}{k!} \Phi_{t}^{(k)}(u,x) \right) = \varepsilon \mathbb{V}(x) \int_\tau^\infty \check{F}_s(s) \mathbb{V}_{es}(x)P\varphi(u)ds.
\]

Substituting (10) for the regular part, we have:

\[
(P - I)U^\varepsilon(t) = - \int_0^\infty F_s(ds) \left( \sum_{k=1}^{\infty} e^k \frac{s^k}{k!} \psi^k(x) \right) PU^\varepsilon(t)
\]

\[
- \int_0^\infty F_s(ds)P \left( \sum_{k=1}^{\infty} (-1)^k e^k \frac{s^k}{k!} U^\varepsilon(t) \right)
\]

\[
- \int_0^\infty F_s(ds) \left( \sum_{k=1}^{\infty} e^k \frac{s^k}{k!} \psi^k(x) \right) P \left( \sum_{k=1}^{\infty} (-1)^k e^k \frac{s^k}{k!} U^\varepsilon(t) \right).
\]

Gathering the terms with the same degree of \( \varepsilon \), we obtain:

\[
(P - I)U^\varepsilon(t) = \sum_{k=1}^{\infty} e^k \left[ - \int_0^\infty F_s(ds) \frac{s^k}{k!} \psi^k(x) PU^\varepsilon(t) \right]
\]

\[
- \int_0^\infty F_s(ds) \left( \sum_{n=1}^{k-1} (-1)^n \times \frac{s^k}{(k-n)!} \psi^{(k-n)}(x) PU^{\varepsilon(n)}(t) \right)
\]

\[
- \int_0^\infty (-1)^k F_s(ds)P \frac{s^k}{k!} U^{\varepsilon(k)}(t) \right] = \sum_{k=1}^{\infty} e^k \frac{m_k(x)}{k!} L_k U^\varepsilon(t).
\]

To obtain (14) we should divide the last equality by \( m_1(x) \).
Then the lemma is proved.

If we put into (14), the expansion \( U^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k U_k(t) \) and gather together the terms with the same degree of \( \varepsilon \), we obtain the following corollary.

COROLLARY 3.2. The regular terms of the asymptotics satisfy the following system of equations:

\[
\begin{align*}
QU_0(t) &= 0, \\
QU_k(t) &= \sum_{n=1}^{k} \mu_n(x)L_nU_{k-n}(t), \quad k \geq 1.
\end{align*}
\] (15)

The first regular term \( U_0(t) \) belongs to \( N_Q \), according to the first equation of the system (15). Hence:

\[ U_0(t) = c_0(t)1, \]

where \( c_0(t) \) is a scalar function that does not depend on \( x \). To obtain the equation for \( c_0(t) \) the solvability condition for the following equation is used:

\[ \Pi L_1 c_0(t) = 0, \]

which leads to the average equation

\[ \tilde{L}_1 c_0(t) = 0, \quad \tilde{L}_1 \Pi = \Pi L_1 \Pi. \] (16)

COROLLARY 3.3. The function \( c_0(t) \) satisfies the equation with initial condition:

\[ \begin{align*}
\tilde{v}(u) \frac{\partial c_0(t)}{\partial u} - \frac{\partial c_0(t)}{\partial t} &= 0, \\
c_0(0) &= \varphi(u).
\end{align*} \]

Proof. The explicit form of \( \tilde{L}_1 \) may easily be found using (12):

\[ \tilde{L}_1 c_0(t) := \tilde{v}(u) \frac{\partial c_0(t)}{\partial u} - \frac{\partial c_0(t)}{\partial t}, \quad \tilde{v}(u) := \Pi v(u, x) := \int_E v(u, x) \pi(dx). \]

From this and (16), we obtain the equation in Corollary 3.3. The initial condition follows from the formulae (11).

The corollary is thus proved.

For \( U_1(t) \) we obtain:

\[ U_1(t) = R_0 L_1 U_0(t) + c_1(t) = R_0 L_1 c_0(t) + c_1(t). \]
Using the solvability condition for the third equation of the system (15), we have:

\[ \Pi L_1 R_0 L_1 c_0(t) + \Pi \mu_2(x) L_2 c_0(t) + \hat{L}_1 c_1(t) = 0, \]

or

\[ \hat{L}_1 c_1(t) = -\Pi \Psi_1 c_0(t), \]

where \( \Pi \Psi_1 := \Pi L_1 R_0 L_1 + \Pi \mu_2(x) L_2. \)

For \( U_k(t) \) we obtain by analogy:

\[ U_k(t) = R_0 \left( \sum_{n=1}^{k} \mu_n(x) L_n U_{k-n}(t) \right) + c_k(t), \]

\[ \hat{L}_1 c_k(t) := -\Pi \Psi_k c_0(t) - \cdots - \Pi \Psi_1 c_{k-1}(t), \]

where \( \Pi \Psi_k := \sum_{n=1}^{k} \Pi \mu_n(x) L_n R_0 \Psi_{k-n} + \Pi \mu_{k+1}(x) L_{k+1}, \Pi \Psi_0 := \Pi L_1. \)

The regular terms \( U_k(t), k \geq 1 \) contain two parts: one in the null-space \( c_k(t) \) and one in the space of values \( R_Q. \)

\[ U_k(t) = c_k(t) I + U^R_k(t), \quad U^R_k(t) \in R_Q. \]

The scalar functions \( c_k(t), k \geq 1 \) satisfy the same equations with the operator \( \hat{L}_1 \) but with different right-hand side. Hence, the unique solution can be defined by using the initial condition \( c_k(0). \) Meanwhile the second part \( U^R_k(t) \) is defined uniquely by the relation

\[ U^R_k(t) = R_0 \left( \sum_{n=1}^{k} \mu_n(x) L_n U_{k-n}(t) \right). \]

It is worth noticing that the regular part \( U^R_k(t) \) in \( R_Q \) is determined uniquely for \( t \) on the entire real line \( \mathbb{R} = (-\infty, +\infty). \)

4. Equations for the singular terms

The equation for the singular term \( W(\tau), \tau = t/e (e > 0) \) is constructed in a very simple way, because the shift by \( -e s \) may be transformed into the convolution

\[ \mathcal{L}^e W^e(\tau) = \int_{0}^{\infty} F_s (ds) \mathcal{V}_e(x) PW^e(\tau - s) - W^e(\tau) = e \psi_e(\tau), \]  

where \( e \psi_e(\tau) = e \mathcal{V}(x) \int_{0}^{\infty} \tilde{F}(s) \mathcal{V}_e(x) \psi(u) ds. \)

The algebraic identity

\[ VP - I = P - I + (V - I)P \]

provides the representation

\[ (Q - I) W^e(\tau) + Q^e_t W^e(\tau) = e \psi_e(\tau), \]
where
\[
(Q - I)W^e(\tau) := \int_0^\infty F_x(ds)PW^e(\tau - s) - W^e(\tau)
\]
is the renewal operator on the real line \(-\infty < \tau < +\infty\) and
\[
Q^eW^e(\tau) := \int_0^\infty F_x(ds)[\EN\tau(x) - I]PW^e(\tau - s).
\]

Now the equations for the singular terms are constructed in the usual way:

**Lemma 4.1.** The equations for the singular terms have the following form:

\[
(Q - I)W_1(\tau) = \psi^1(\tau), \quad (Q - I)W_k(\tau) = \psi^k(\tau) - \psi^0_k(\tau),
\]

(18)

**Proof.** If we put the following expansion
\[
W^e(\tau) = \sum_{k=1}^\infty e^kW_k(\tau)
\]
into equation (17) we obtain:

\[
\begin{align*}
\int_0^\infty F_x(ds) \left[ I + \sum_{k=1}^\infty e^{k} \frac{s^k}{k!} \\EN^{k}(x) \right] P \left[ \sum_{k=1}^\infty e^{k}W_k(\tau - s) \right] - \sum_{k=1}^\infty e^{k}W_k(\tau) \\
= \int_\tau^\infty \tilde{F}_x(s) \left[ \sum_{k=1}^\infty e^{k} \frac{s^{k-1}}{(k-1)!} \\EN^{k}(x) \right] P\varphi(u) ds.
\end{align*}
\]

So, we have:

\[
\begin{align*}
e[Q - I]W_1(\tau) + \sum_{k=2}^\infty e^k[Q - I]W_k(\tau) + \sum_{k=2}^\infty e^k \sum_{r=1}^{k-1} Q^{r} \times W_{k-r}(\tau) \\
= \sum_{k=1}^\infty e^{k}F^k(\tau)\EN^{k}(x)P\varphi(u),
\end{align*}
\]

and gathering the terms of the same degree of \(e\), we obtain (18).

The lemma is thus proved. \(\square\)

**Corollary 4.2.** The singular terms of the asymptotics have the following form:

\[
W_1(\tau) = R_0[\psi^1(\tau) - \psi^1_1(\tau)], \quad W_k(\tau) = R_0[\psi^k(\tau) - \psi^0_k(\tau) - \psi^1_k(\tau)], \quad k \geq 2.
\]

Here \(R_0\) is the Markov renewal operator [19].
Proof. The extended Markov renewal equation (18) for the singular terms may be transformed into a standard form as follows

\[
\int_0^\tau F_\varepsilon(ds)PW_k(\tau - s) - W_k(\tau) = \psi_k^\varepsilon(\tau) - \psi_0^\varepsilon(\tau) - \psi_1^\varepsilon(\tau), \quad \tau \geq 0
\]

with

\[
\psi_k^\varepsilon(\tau) := \int_\tau^\infty F_\varepsilon(ds)PW_k(\tau - s).
\]

Following Ref. [9], we may find the solution of standard Markov renewal equation using the Markov renewal operator \( R_0 \).

Hence the corollary is proved. \( \Box \)

From the corollary, we have that the singular term can be defined after continuation to the negative real line for \( \tau < 0 \). The additional boundary condition

\[
W_k(+\infty) = 0
\]

and the ‘regular degeneration’ of boundary conditions (11) provide the unique determination of the singular term.

5. Regularity of boundary conditions
The initial extended Markov renewal equation (13) takes place under the continuously differentiable extension of the regular part \( U^\varepsilon(t) \) on the negative real line for \( t < 0 \). That is the concordance of the boundary conditions (11) have to be valid, or in other form we should have

\[
W^\varepsilon(t) + U^\varepsilon(\varepsilon t) + U_0(\varepsilon t) = 0, \quad t < 0. \tag{19}
\]

The representations (19) and (11) and the Taylor formula for \( \tau < 0 \):

\[
\varphi(u) = \Phi_{\varepsilon u}(u, x)|_{\tau < 0} = U_0(0) + \sum_{k=1}^{\infty} e^k \frac{\tau^k}{k!} U_0^{(k)}(0) + \varepsilon U_1(0) + \varepsilon \sum_{k=1}^{\infty} e^k \frac{\tau^k}{k!} U_1^{(k)}(0) + \cdots + \sum_{k=1}^{\infty} e^k W_k(\tau)
\]

give the boundary conditions for the singular terms in the following form

\[
W_k(\tau) = W_k(0) - \sum_{n=1}^{k} \frac{\tau^n}{n!} U_k^{(n)}(0), \quad \tau < 0, k \geq 1 \tag{20}
\]

and the additional initial condition

\[
W_k(0) + U_k(0) = 0, \quad k \geq 1. \tag{21}
\]

The condition (20) may be used to construct a solution of the Markov renewal equation (18) for the singular terms. The boundary condition for singular terms \( W_k(+\infty) = 0, k \geq 1 \) and the
Markov renewal limit theorem [19] provide by a calculation:

\[ \Pi W_k(0) \in N_Q \]

and, as a result, we obtain an initial condition for the regular part in \( N_Q \)

\[ c_k(0) = -\Pi W_k(0). \]

The last step to verify the regularity of boundary conditions (11) is to establish the following relations:

\[ (P - I)(W_k(0) + U_k(0)) = 0, \quad k \geq 1, \]

which are, in fact, equivalent to (11).

These are the regular and corresponding singular terms in the subspace of values \( R_Q \) which compensate each other without any further assumption. A good formulation of the problem of asymptotic expansion leads to the regularity of the boundary conditions (11).

Lemma 5.1. The regular degeneration equation has the form:

\[ (P - I)[U^e(0) + W^e(0)] = 0. \]

Proof. Let us consider the equation (18). For \( W_1(\tau) \) we have:

\[ \int_0^\infty F_x(ds)PW_1(\tau - s) - W_1(\tau) = \int_0^\infty \bar{F}_x(ds)\bar{\nu}(x)PU_0(0). \]

For \( \tau = 0 \) we obtain:

\[ \left[ \int_0^\infty F_x(ds)PW_1(0) - W_1(0) \right] + \int_0^\infty F_x(ds)P(W_1(-s) - W_1(0)) = \int_0^\infty \bar{F}_x(ds)\bar{\nu}(x)PU_0(0). \]

Using the equality (20), we have:

\[ (P - I)W_1(0) = -\int_0^\infty sF_x(ds)PU_0'(0) + \int_0^\infty sF_x(ds)\bar{\nu}(x)PU_0(0). \quad (22) \]

For the corresponding regular term, we get from (14) and (12):

\[ (P - I)U_1(0) = m_1(x)PU_0'(0) - m_1(x)\bar{\nu}(x)PU_0(0). \]

So, the following equality is true:

\[ (P - I)[U_1(0) + W_1(0)] = 0. \]

By induction, if we have:

\[ (P - I)[U_n(0) + W_n(0)] = 0, \quad n = 1, \ldots, k, \quad (23) \]
then for $U_{k+1}(0)$ and $W_{k+1}(0)$ we obtain:

$$
\int_0^\infty F_x(ds)PW_{k+1}(\tau - s) - W_{k+1}(\tau) = \int_0^\infty \tilde{F}_x(ds) \sum_{n=1}^{k+1} \frac{s^{n-1}}{(n-1)!} \wedge^n(x) PU_0(0)
$$

For $\tau = 0$ we have:

$$
\left[ \int_0^\infty F_x(ds)PW_{k+1}(0) - W_{k+1}(0) \right] + \int_0^\infty F_x(ds)P(W_{k+1}(-s) - W_{k+1}(0)) = \int_0^\infty \tilde{F}_x(ds) \frac{s^k}{k!} \wedge^{k+1}(x) PU_0(0)
$$

By the equality (20) we obtain:

$$(P - I)W_{k+1}(0) = \int_0^\infty F_x(ds)P \sum_{n=1}^{k+1} \frac{(-s)^n}{n!} U_{k-n+1}^{(n)}(0) + \int_0^\infty F_x(ds) \frac{s^{k+1}}{(k+1)!} \wedge^{k+1}(x) PU_0(0)
$$

By induction, as soon as (21) and (23) are true, we may write:

$$(P - I)W_{k+1}(0) = - \int_0^\infty F_x(ds)P \sum_{n=1}^{k+1} \frac{(-s)^n}{n!} U_{k-n+1}^{(n)}(0)
$$

For the corresponding regular term, we have from (14) and (12):

$$(P - I)U_{k+1}(0) = - \int_0^\infty F_x(ds)P \sum_{n=1}^{k+1} \frac{(-s)^n}{n!} U_{k-n+1}^{(n)}(0) - \int_0^\infty F_x(ds) \frac{s^n}{n!} \wedge^n(x) PU_{k-n+1}(0)
$$

It is easy to see that the following equality is true:

$$(P - I)[U_{k+1}(0) + W_{k+1}(0)] = 0.
$$

So, by induction the lemma is proved. □
Corollary 5.2

\[ (I - P)[U^0(0) + W^0(0)] = 0, \]

or, in other words,

\[ (I - \Pi)[U_k(0) + W_k(0)] = 0. \]

The proof is obvious.

So, we can see that in the space of values of the operator \( Q \) the regular and singular parts of the solution fulfil the initial conditions (11).

At the same time, in the null-space of \( Q \) the initial conditions for the regular terms are determined by the initial conditions for the singular terms, so we have

Corollary 5.3

\[ c_k(0) = -\Pi W_k(0), \quad k \geq 1. \]

**Proof.** We obtain obviously: \( \Pi[W_k(0) + U_k(0)] = \Pi W_k(0) + c_k(0) = 0. \)

Corollary 5.4. The singular terms of the asymptotics have the following explicit form:

\[
W_1(\tau) = R_0[\psi^1(\tau) + \bar{F}^1(\tau)PU_1(0) + \int_\tau^\infty (\tau - s)F_x(ds)P U_0'(0)],
\]

\[
W_k(\tau) = R_0[\psi^k(\tau) - \psi^0_k(\tau) + \bar{F}^k(\tau)PU_k(0) + \sum_{n=1}^k \int_\tau^\infty (\tau - s)^n F_x(ds)PU_k^{(n)}(0)],
\]

**Proof.** Using formulae (20) and Corollary 5.3, we easily obtain:

\[
\int_\tau^\infty F_x(ds)PW_k(\tau - s) = \int_\tau^\infty F_x(ds)P \left[ - U_k(0) - \sum_{n=1}^k (\tau - s)^n U_{k-n}^{(n)}(0) \right]
\]

\[
= -F_x(\tau)PU_k(0) - \sum_{n=1}^k \int_\tau^\infty (\tau - s)^n F_x(ds)PU_k^{(n)}(0).
\]

This proves the corollary.

6. **Initial conditions for the regular terms**

We are going to write down an algorithm for the construction of initial conditions (at \( t = 0 \)) for the regular terms using the boundary conditions for the singular terms as \( \tau \to \infty \).
(see Remark 1.1). For the first singular term $W_1(\tau)$, we have the equation (see (18)):

$$
\int_0^\infty Q(ds)W_1(\tau - s) - W_1(\tau) = \tilde{F}_x^{(1)}(\tau)\varphi(x)P\varphi(u),
$$

(25)

where $\tilde{F}_x^{(1)}(\tau) = \int_\tau^\infty \tilde{F}_x(s)ds$.

Separating the first integral into two parts, we obtain:

$$
\int_0^\infty Q(ds)W_1(\tau - s) - W_1(\tau) = \tilde{F}_x^{(1)}(\tau)\varphi(x)P\varphi(u) - \int_0^\infty Q(ds)W_1(\tau - s).
$$

According to the renewal theorem [19], we have for $\tau \to \infty$:

$$
0 = W_1(\infty) = \left(\int \rho(dx)\int_0^\infty \tilde{F}_x(s)ds \varphi(x)P\varphi(u)\right) - \left(\int_0^\infty Q(ds)W_1(\tau - s)d\tau\right)/\hat{m},
$$

(26)

where $\hat{m} = \int \rho(dx)m_1(x)$.

For $\tau < 0$, we have from (20):

$$
W_1(\tau) = W_1(0) - \tau U_0'(0).
$$

(27)

Substituting the correlation (27) into equation (26), we obtain:

$$
0 = \left(\int \rho(dx)\int_0^\infty \tilde{F}_x(s)ds \varphi(x)P\varphi(u)\right) - \left(\int_0^\infty Q(ds)[W_1(0) - (\tau - s)U_0'(0)]d\tau\right)/\hat{m}
$$

$$
= \left(\int \rho(dx)\int_0^\infty F_x(s)[PW_1(0) - \frac{m_2(x)}{2}\varphi(x)]P\varphi(u)\right) - \left(\int_0^\infty Q(ds)(\tau - s)U_0'(0)d\tau\right)/\hat{m}
$$

$$
= \left(\int \rho(dx)\frac{m_2(x)}{2}\varphi(x)P\varphi(u) - \int \rho(dx)m_1(x)PW_1(0) - \int \rho(dx)\frac{m_2(x)}{2}PU_0'(0)\right)/\hat{m}
$$

$$
= \left(-\int \rho(dx)m_1(x)\mu_2(x)L_1(x)\varphi(u) - \int \rho(dx)m_1(x)(P - I)W_1(0) - \int \rho(dx)m_1(x)W_1(0)\right)/\hat{m}
$$

$$
= \left(-\int \rho(dx)m_1(x)\mu_2(x)L_1(x)\varphi(u) - \int \rho(dx)m_1(x)(P - I)W_1(0)\right)/\hat{m} - \Pi W_1(0)
$$

$$
= \left(-\int \rho(dx)m_1(x)\mu_2(x)L_1(x)\varphi(u) - \int \rho(dx)m_1(x)(P - I)W_1(0)\right)/\hat{m} + c_1(0),
$$

(28)

here $\mu_2(x) = m_2(x)/2m_1(x)$. 
We may rewrite (22) in the form:

\[ [P - I]W_1(0) = m_1(x)[\sqrt{v(x)}P\varphi(u) - PU_0'(0)] = L_1(x)\varphi(u). \]

If we put the last equality into (28), we have finally:

\[
0 = \left( -\int \rho(dx)m_1(x)\mu_2(x)L_1(x)\varphi(u) + \int \rho(dx)m_1^2(x)L_1(x)\varphi(u) \right)/\hat{m} + c_1(0),
\]
or

\[
c_1(0) = \Pi U_1(0) = \int \pi(dx)\nu_1(x)L_1(x)\varphi(u)/\hat{m},
\]

where \( \pi(dx) = \rho(dx)m_1(x), \nu_1(x) = \mu_2(x) - m_1(x) = (m_2(x) - 2m_1^2(x))/(2m_1(x)) \).

**Remark 6.1.** It is easily seen that \( \nu_1(x) = 0 \), when \( F_x(t) \) is distributed exponentially. In this case, we obviously have:

\[
c_1(0) = \Pi U_1(0) = 0.
\]

Note, that \( \nu_1(x) \) may be either positive or negative, see, e.g. [9].

We shall describe the algorithm for computing the next terms of the asymptotics in the case of \( W_2(\tau) \):

\[
\int_0^\infty Q(ds)W_2(\tau - s) - W_2(\tau) = \tilde{F}_x^{(2)}(\tau)\sqrt{v}(x)P\varphi(u) - \int_0^\infty s F_x(ds)\sqrt{v}(x)PW_1(\tau - s), \quad (29)
\]

where \( \tilde{F}_x^{(2)}(\tau) = \int_0^\tau s\tilde{F}_x(s)ds \).

Separating the first integral, we obtain:

\[
\int_0^\tau Q(ds)W_2(\tau - s) - W_2(\tau) = \tilde{F}_x^{(2)}(\tau)\sqrt{v}(x)P\varphi(u) - \int_0^\infty s F_x(ds)\sqrt{v}(x)PW_1(\tau - s) - \int_\tau^\infty Q(ds)W_2(\tau - s).
\]

According to the renewal theorem [19], we have for \( \tau \to \infty \):

\[
0 = W_2(\infty) = \left( \int \rho(dx) \int_0^\infty s\tilde{F}_x(s)ds d\tau \sqrt{v}(x)P\varphi(u) - \int \rho(dx) \int_0^\tau s F_x(ds)\sqrt{v}(x)PW_1(\tau - s) d\tau \right)
\]

\[
+ \int_\tau^\infty s F_x(ds)\sqrt{v}(x)PW_1(\tau - s) d\tau \left[ - \int \rho(dx) \int_\tau^\infty Q(ds)W_2(\tau - s) d\tau \right]/\hat{m}.
\]

(30)

For \( \tau < 0 \), we have from (20):

\[
W_2(\tau) = W_2(0) - \tau U_1'(0) - \tau^2 U_0''(0).
\]

(31)
Substituting (31) into (30), we obtain, like in the case of $W_1(\tau)$:

$$0 = \left( \int \rho(dx) \int_0^\tau sF_\tau(s)ds d\tau \sqrt{2}(x)P \varphi(u) - \int \rho(dx) \int_0^\tau \frac{S}{1!} F_\tau(ds) \sqrt{2}(x)PW_1(\tau - s)d\tau \right. $$

$$+ \int_\tau^\infty \frac{S}{1!} F_\tau(ds) \sqrt{2}(x)P \{ W_1(0) - (\tau - s)U_0'(0) \} d\tau $$

$$- \int \rho(dx) \int_0^\tau Q(ds)[W_2(0) - (\tau - s)U_1'(0) - (\tau - s)^2U_0''(0)]d\tau \right) / \hat{m} $$

$$= \left( - \int \rho(dx)m_1(x)\mu_2(x)L_1(x)U_1(0) + \int \rho(dx)m_1(x)\mu_3(x)L_2(x)U_0(0) $$

$$- \int \rho(dx)m_1(x)(P - I)W_2(0) - \int \rho(dx) \int_0^\tau \frac{S}{1!} F_\tau(ds) \sqrt{2}(x)PW_1(\tau - s)d\tau \right) / \hat{m} + c_2(0).$$

We obtain from (24) in case of $k = 1$:

$$[P - I]W_2(0) = m_2(x)\sqrt{2}(x)P \varphi(u) - m_1(x)PU_1'(0) + m_2(x)PU_0''(0) $$

$$\quad + m_1(x)\sqrt{2}(x)PU_1(0) - m_2(x)\sqrt{2}(x)PU_0''(0) $$

$$= m_2(x)L_2(x)U_0(0) - m_1(x)L_1(x)U_1(0).$$

Substituting the last equality into (32), we finally get:

$$c_2(0) = \left[ \int \pi(dx)\nu_2(x)L_2(x)U_0(0) + \int \pi(dx)\nu_1(x)L_1(x)U_1(0) $$

$$\quad - \int \rho(dx) \int_0^\tau \frac{S}{1!} F_\tau(ds) \sqrt{2}(x)PW_1(\tau - s)d\tau \right] / \hat{m},$$

where $\nu_2(x) = m_2(x) - \mu_3(x) = (2m_1(x)m_2(x) - m_3(x))/(3!m_1(x))$.

For the next terms, we have by analogy:

$$c_k(0) = \left[ \sum_{r=0}^{k-1} \int \pi(dx)\nu_2-\nu_1(x)L_{k-r}(x)U_r(0) $$

$$\quad - \sum_{r=0}^{k-1} \int \rho(dx) \int_0^\tau \frac{S}{r!} F_\tau(ds) \sqrt{2}(x)PW_{k-r}(\tau - s)d\tau \right] / \hat{m},$$

where $\nu_2(x) = (-1)^{k+1}[\nu_{k+1}(x) - m_k(x)]$.

Remark 6.2. The last term of this equality may be rewritten without the use of singular terms. This may be done, like in the work [18], using the Laplace transform of the $W_i(\tau)$.

This finishes the proof of Theorem 1.2. \qed
7. Estimate of the remainder

We suppose in addition to the previous assumptions that the function \( v(u;x) \) in (5), (6) satisfies the condition

\[
\sup_{|u| \leq R} \sup_{x \in E} |v(u;x)| \leq C_R, \tag{33}
\]

where \( R > 0 \).

**Lemma 7.1.** If \( \Phi^\varepsilon \) solves

\[
\mathcal{L}^e \Phi^\varepsilon := [Q + \varepsilon Q_1] \Phi^\varepsilon = \varepsilon \psi,
\tag{34}
\]

where \( \psi \) is given, such that \( \|\psi\| \leq C, \|Q^{-1}\| \leq C, \|Q_1\| \leq C \), then

\[
\|\Phi^\varepsilon\| \leq C_1 \varepsilon.
\]

**Proof.** The solution of equation (34) is:

\[
\Phi^\varepsilon = \varepsilon Q^{-1} [\psi - Q_1 \Phi^\varepsilon].
\]

We can make the following iteration: set

\[
\Phi^\varepsilon_{(0)} = \varepsilon \psi,
\]

\[
\|\Phi^\varepsilon_{(0)}\| = \varepsilon \|\psi\| \leq \varepsilon C,
\]

then

\[
\Phi^\varepsilon_{(1)} = \varepsilon Q^{-1} [\psi - Q_1 \Phi^\varepsilon_{(0)}],
\]

as soon as \( Q^{-1} \) and \( Q_1 \) are bounded

\[
\|\Phi^\varepsilon_{(1)}\| \leq \varepsilon C (C + C \|\Phi^\varepsilon_{(0)}\|) \leq C (\varepsilon C + (\varepsilon C)^2).
\]

By induction, we have:

\[
\Phi^\varepsilon_{(N)} \leq C \sum_{k=1}^{N} (\varepsilon C)^k.
\]

We may choose the parameter \( \varepsilon \) small enough, so that \( \varepsilon C < 1 \). Thus, for \( N \to \infty \) we obtain:

\[
\|\Phi^\varepsilon\| \leq C \frac{\varepsilon C}{1 - \varepsilon C} \leq \varepsilon C_1.
\]

The lemma is thus proved. \( \Box \)
Corollary 7.2. If $\Phi^e$ solves

$$
\|\Phi^e\| := \|Q + \epsilon Q_1\| \Phi^e = \epsilon^{N+1}\psi.
$$

where $\|\psi\| \leq C, \|Q^{-1}\| \leq C, \|Q_1\| \leq C$, then $\|\Phi^e\| \leq C_1 \epsilon^{N+1}$.

Proof. Let us suppose

$$
\Phi^e := \epsilon^N \Phi^e.
$$

Then, we have from (35):

$$
\epsilon^N \|\Phi^e\| = \epsilon^{N+1}\psi.
$$

So, we obtain an equation for $\Phi^e$:

$$
\|\Phi^e\| = \epsilon\psi.
$$

By Lemma 7.1, we have

$$
\|\Phi^e\| \leq C_1 \epsilon,
$$

or, for $\Phi^e$:

$$
\|\Phi^e\| \leq C_1 \epsilon^{N+1}.
$$

Hence the corollary is proved.

Let us consider the estimate of the remainder, written in the form:

$$
\Phi^{e,N}(t) = \Phi^e(t, x) - \Phi^e_N(t) = \Phi^e(t, x) - U_0(t) - \sum_{k=1}^{N} \epsilon^k (U_k(t) + W_k(t)).
$$

Theorem 7.3. Under the conditions (1) and (33), the following estimate for the remainder in the expansion of Theorem 1.2 holds:

$$
\|\Phi^{e,N}(t)\| \leq C_1 \epsilon^{N+1},
$$

for some $C$.

Proof. To prove this theorem, we should show that $\Phi^{e,N}(t)$ satisfies the conditions of Corollary 7.2.

Let us first consider $U^{e,N}(t) := U^e(t) - U_0(t) - \sum_{k=1}^{N} \epsilon^k U_k(t)$, that is the regular part of $\Phi^{e,N}(t)$. 

---

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From the equation (13) we obtain:

\[
\int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)PU_{e,N}(t - \epsilon s) - U_{e,N}(t)
\]

\[
= \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)P[U_{e,N}(t - \epsilon s) - U_{e,N}(t)] + \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)PU_{e,N}(t) - U_{e,N}(t)
\]

\[
= QU_{e,N}(t) + \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x) - IPU_{e,N}(t - \epsilon s) - \epsilon \int_0^\infty sF_\epsilon(ds)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))'
\]

\[
+ \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)P[U_{e,N}(t - \epsilon s) - U_{e,N}(t) + \epsilon s(U_{e,N}(t)')]'
\]

\[
= QU_{e,N}(t) - F_\epsilon(s)\mathcal{V}_{e\omega}(x) - IPU_{e,N}(t)\int_0^\infty + \epsilon \mathcal{V}(x)\int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)PU_{e,N}(t)ds
\]

\[
- \epsilon \int_0^\infty sF_\epsilon(ds)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))' + \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)P[U_{e,N}(t - \epsilon s) - U_{e,N}(t) + \epsilon s(U_{e,N}(t)')]'
\]

\[
= [Q + \epsilon Q_1^e]U_{e,N}(t),
\]

where

\[
Q_1^eU_{e,N}(t) := \mathcal{V}(x)\int_0^\infty \bar{F}_\epsilon(s)\mathcal{V}_{e\omega}(x)PU_{e,N}(t)ds - \int_0^\infty sF_\epsilon(ds)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))' + \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)P[U_{e,N}(t - \epsilon s) - U_{e,N}(t) + \epsilon s(U_{e,N}(t)')]'/\epsilon,
\]

(36)

and we have by conditions (1) and (33):

\[
\|Q_1^e\| = \left\|\mathcal{V}(x)\int_0^\infty \bar{F}_\epsilon(s)\mathcal{V}_{e\omega}(x)PU_{e,N}(t)ds - \int_0^\infty sF_\epsilon(ds)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))' + \int_0^\infty F_\epsilon(ds)\mathcal{V}_{e\omega}(x)P[U_{e,N}(t - \epsilon s) - U_{e,N}(t) + \epsilon s(U_{e,N}(t)')]'/\epsilon\right\|
\]

\[
\leq \|\mathcal{V}\|\|U_{e,N}(t)\|\int_0^\infty \mathcal{V}(s)\|\bar{F}_\epsilon(s)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))'\| + \|[(U_{e,N}(t))']\|\int_0^\infty sF_\epsilon(ds)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))'/\epsilon
\]

\[
\leq \|\mathcal{V}\|\|U_{e,N}(t)\|\int_0^\infty \mathcal{V}(s)\|\bar{F}_\epsilon(s)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))'\| + \|[(U_{e,N}(t))']\|\int_0^\infty sF_\epsilon(ds)\mathcal{V}_{e\omega}(x)P(U_{e,N}(t))'/\epsilon
\]

\[
\leq C_1 + C_2 + C_3 = C,
\]

as soon as the functions $U_{e,N}(t)$ are from the space of real-valued functions which are bounded with all their derivatives.

The operator $Q$ has bounded inverse operator $R_0 = [Q + \Pi]^{-1} - \Pi$ in $R_Q$ (see Ref. [11]).
On the other hand, we have from (13):

\[
\int_0^\infty F_\varepsilon(ds)\tilde{F}_\varepsilon(x)P U^{e,N}(t - \varepsilon s) - U^{e,N}(t)
= \left[ \int_0^\infty F_\varepsilon(ds)\tilde{F}_\varepsilon(x)P U^{e,N}(t - \varepsilon s) - U^{e,N}(t) \right] - \left[ Q + \sum_{k=1}^N e^k \mu_k(x)L_k \right] \left[ U_0 + \sum_{k=1}^N e^k U_k \right]
- \frac{(\varepsilon s)^{(N+1)}}{(N+1)!} \int_0^\infty F_\varepsilon(ds) \int_0^{\varepsilon s} \tilde{F}_\varepsilon(x)P U^{e,N}(t - \varepsilon s) d\theta
= - \left[ QU_0 + \sum_{k=1}^N e^k \left( QU_k + \sum_{r=1}^k \mu_r(x)L_r U_{N-r} \right) - e^{N+1} \psi \right] = e^{N+1} \psi,
\]

where the last equality follows from (15), the function \( \psi \) being an integral of the form (36) and may thus be estimated in the same way as the operator \( Q_1^e \).

So, we showed that the function \( U^{e,N}(t) \) satisfies the equation

\[
[Q + eQ_1^e]U^{e,N}(t) = e^{N+1} \psi,
\]

where the conditions of Corollary 7.2 are satisfied for \( Q, Q_1^e, \psi \).

So, we obtain in \( R_Q \):

\[
\|U^{e,N}(t)\| \leq C e^{N+1}.
\]

It is easy to see that equation (37) in \( N_Q \) has the form:

\[
e^N \psi,
\]

From (36) we have:

\[
Q_1^e c^{e,N}(t) = m_1(x)[\tilde{\nu}(x)c^{e,N}(t) - (c^{e,N}(t))^\gamma] + e\tilde{\nu}^2(x)\int_0^\infty F_\varepsilon(x) \int_0^{\varepsilon s} \tilde{F}_\varepsilon(x)P c^{e,N}(t) d\theta ds
- e\tilde{\nu}(x) \int_0^\infty sF_\varepsilon(ds) \int_0^{\varepsilon s} \tilde{F}_\varepsilon(x)P(c^{e,N}(t))^\gamma d\theta + \int_0^\infty F_\varepsilon(ds)\tilde{F}_\varepsilon(x)P[U^{e,N}(t - \varepsilon s) - U^{e,N}(t)]/e.
\]

We obtain (39) in the following form:

\[
[Q_1 + eQ_2^e]c^{e,N}(t) = e^N \psi,
\]

where \( Q_1^{-1} \) is a bounded integral operator, \( Q_2^e \) and \( \psi \) are integrals of the form (36) and may thus be estimated in the same way as the operator \( Q_1^e \).

So the conditions of Corollary 7.2 are satisfied for \( Q_1, Q_2^e, \psi \), and we have in \( N_Q \):

\[
\|c^{e,N}(t)\| \leq C e^N.
\]

Let us now consider \( W^{e,N}(\tau) := W^e(\tau) - \sum_{k=1}^N e^k W_k(\tau) \), that is the singular part of \( \Phi^{e,N}(t) \).
From the equation (13), we have:

\[
\int_0^\infty F_x(ds)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s) - W^{e,N}(\tau) = \int_0^\tau F_x(ds)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s) - W^{e,N}(\tau) + \int_\tau^\infty F_x(ds)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s) + \int_0^\infty F_x(ds)[\mathcal{E}_\varepsilon(x) - I]PW^{e,N}(\tau - s) \\
= [Q(\tau) - I]W^{e,N}(\tau) - \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s)\bigg|_0^\infty + \epsilon\mathcal{V}(x)\int_\tau^\infty \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s)ds \\
- \tilde{F}_x(s)[\mathcal{E}_\varepsilon(x) - I]PW^{e,N}(\tau - s)\bigg|_0^\tau + \epsilon\mathcal{V}(x)\int_0^\tau \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s)ds \\
= [Q(\tau) - I]W^{e,N}(\tau) + \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s) - \tilde{F}_x(\tau)[\mathcal{E}_\varepsilon(x) - I]PW^{e,N}(0) \\
+ \epsilon\mathcal{V}(x)\int_0^\infty \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s)ds = [Q(\tau) - I]W^{e,N}(\tau) + \tilde{F}_x(\tau)PW^{e,N}(0) \\
+ \epsilon\mathcal{V}(x)\int_0^\infty \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s)ds = ([Q(\tau) - I] + \epsilon Q_1)W^{e,N}(\tau),
\]

where due to Lemma 5.1

\[
\epsilon Q_1 W^{e,N}(\tau) := \epsilon\mathcal{V}(x)\int_0^\infty \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)PW^{e,N}(\tau - s)ds + \tilde{F}_x(\tau)PU^{e,N}(0),
\]

and we have by conditions (1), (33) and (38):

\[
\|Q_1\| = \left\|\mathcal{V}\int_0^\infty \tilde{F}_x(s)\mathcal{E}_\varepsilon(x)P\right\| + \|\tilde{F}_x(\tau)PU^{e,N}(0)\| \\
\leq \|\mathcal{V}\|\int_0^\infty \epsilon\mathcal{V}\|\tilde{F}_x(s)ds + \|P^{e,N}(0)\| \leq C_1 + \epsilon^{N+1}C_2 \leq C.
\]

The conditions of Chapter 1, Sections 3 and 4 from [19] are fulfilled for the operator $Q(\tau)$. In fact,

\[
\int_E \int_0^\infty \mathcal{P}(dx)\mathcal{P}(dt)\mathcal{X}(x, dy) = 1 < \infty.
\]

So, the operator $Q(\tau) - I$ has an inverse operator which is bounded.
On the other hand, we have:

\[
\int_0^\infty F_x(ds)\mathcal{V}_x(x)PW^{e,N}(\tau - s) - W^{e,N}(\tau)
= \left[ \int_0^\infty F_x(ds)\mathcal{V}_x(x)PW^{e}(\tau - s) - W^{e}(\tau) \right]
- \left[ (Q - I) + \sum_{k=1}^N e^k \int_0^\infty F_x(ds)\frac{\varphi^k(x)P}{k!} \right] \left[ \sum_{k=1}^N e^kW_k \right]
- \left[ 0_{x} F_x(ds) \frac{e^{N+1}\mathcal{V}^{N+1}(x)}{(N+1)!} \int_0^\infty \mathcal{V}_x(x)PW^{e,N}(\tau - s)d\theta. \right]
\]

As soon as the term \( \left[ \int_0^\infty F_x(ds)\mathcal{V}_x(x)PW^{e}(\tau - s) - W^{e}(\tau) \right] \) satisfies equation (17), we obtain:

\[
\int_0^\infty F_x(ds)\mathcal{V}_x(x)PW^{e,N}(\tau - s) - W^{e,N}(\tau) = e\mathcal{V}(x)\int_0^\tau F_x(s)\mathcal{V}_x(x)\varphi(u)ds
- \sum_{k=1}^N e^k[(Q - I)W_k(\tau) + \psi^k_\theta(\tau)] + O(e^{N+1})
= \sum_{k=1}^N \psi^k(\tau) + O(e^{N+1})
- \sum_{k=1}^N e^k[(Q - I)W_k(\tau) + \psi^k_\theta(\tau)] + O(e^{N+1})
= e^{N+1}\phi,
\]

where we used the equalities (18). The function \( \phi \) is an integral of the form (36) and may be estimated in the same way as the operator \( Q_1 \).

So, we showed that the function \( W^{e,N}(\tau) \) satisfies the equation

\[
([Q(\tau) - I] + eQ_1)W^{e,N}(\tau) = e^{N+1}\phi,
\]

where \( Q(\tau) - I, Q_1 \) and \( \phi \) satisfy the conditions of Corollary 7.2.

So, we obtain:

\[
\|W^{e,N}(\tau)\| \leq Ce^{N+1}.
\]

(41)

For the function \( \Phi^{e,N}(t) \), we have from (38), (40) and (41):

\[
\|
\Phi^{e,N}(t)\| \leq \|U^{e,N}(t)\| + \|c^{e,N}(t)\| + \|W^{e,N}(\tau)\| \leq Ce^{N+1} + Ce^N + Ce^{N+1} = Ce^N.
\]

Finally, if we consider

\[
\Phi^{e,N+1}(t) = \Phi^{e,N}(t) - e^{N+1}[U^{e,N+1}(t) + c^{e,N+1}(t) + W^{e,N+1}(t)],
\]

then

\[
\Phi^{e,N}(t) = \Phi^{e,N+1}(t) + e^{N+1}[U^{e,N+1}(t) + c^{e,N+1}(t) + W^{e,N+1}(t)].
\]
As a result, the following estimation holds:

\[ \| \Phi^{e,N}(t) \| \leq \| \Phi^{e,N+1}(t) \| + O(e^{N+1}) \leq C e^{N+1}. \]

Thus Theorem 7.3 is proved. \( \square \)

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