GENERALIZED DIFFUSION EQUATION WITH NONLOCALITY OF SPACE-TIME: ANALYTICAL AND NUMERICAL ANALYSIS

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Abstract

Based on the non-Markov diffusion equation taking into account the spatial fractality and modeling for the generalized coefficient of particle diffusion $D^{\alpha \beta}(r, r'; t, t') = W(t, t')D^{\alpha \beta}(r, r')$ using fractional calculus the generalized Cattaneo–Maxwell–type diffusion equation in fractional time and space derivatives has been obtained. In the case of a constant diffusion coefficient, analytical and numerical studies of the frequency spectrum for the Cattaneo–Maxwell diffusion equation in fractional time and space derivatives have been performed. Numerical calculations of the phase and group velocities with change of values of characteristic relaxation time, diffusion coefficient and indexes of temporal $\xi$ and spatial $\alpha$ fractality have been carried out.

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1. Introduction

Fractional integrals and derivatives [1–5] are actively used in researches of anomalous diffusion in porous media [5–16], disordered systems [17–30], plasma physics [31–36], turbulent [37–39], kinetic and reaction-diffusion processes [39–50], in quantum mechanics [51–55], viscoelastic [56–59] and biological systems [60–62], etc. [5, 63, 64].

Experimental data on different processes of anomalous diffusion show that not only the distribution law, but also form of diffusion package is
significantly different from the normal diffusion \[5,19,39,63\]. Approaches with variable diffusion coefficients \[65\], on the basis of degree correlations of fractional order \[4\], fractional derivatives \[37-39\], the generalized Fokker-Planck equation \[5,39,66\], generalizations of statistical mechanics (extensive and non-extensive) based on the Tsallis \[67-69\] and Renyi \[67,70\] entropy, and others were developed to describe anomalous diffusion in different physical and chemical systems. Conducted researches show that mathematical basis of anomalous diffusion is equation with fractional derivatives \[5,39\]. In particular, during the study of three-dimensional models of anomalous diffusion \[5,63,71\], basic equations of anomalous diffusion are derived from the general principles of the stochastic theory of random processes based on the Chapman-Kolmogorov integral equations for transition probabilities. Solutions of these equations form a new class of distributions, which are called fractional stable distributions. These distributions are solutions of partial differential equations of fractional order. These equations are generalization of usual diffusion equation to the case of anomalous diffusion. A partial case of the fractional stable distributions is the Gaussian distribution, which corresponds to the normal diffusion. It is important to note that obtained equations for anomalous diffusion with fractional derivatives contain diffusion coefficient, which is a constant in time and space. On the other hand, diffusion coefficients are related to time correlation functions (the Green-Kubo relations), which contain diffusion transfer mechanisms from the perspective of nonequilibrium statistical mechanics.

Currently, together with phenomenological approaches for constructing of the Fokker-Planck equation, the diffusion equation and its generalization — the Cattaneo equation with fractional derivatives, there are two methods of constructing such equations, namely, (1) probabilistic method, which is based on the Chapman-Kolmogorov equation in the stochastic theory of random processes \[5,39,72\], and (2) statistical method, which is based on the method of projection operators (memory functions) in the works \[20-26,44\], and on the Liouville equation with fractional derivatives \[40,73,85\]. In particular, by using this method, the BBGKY hierarchy equations with fractional derivatives \[74,75,81\], transport equation, diffusion equation, and the Heisenberg equation with fractional derivatives \[77,79\] are obtained. This approach is formulated for non-Hamiltonian systems. If the Helmholtz conditions for coordinate and momentum derivatives of fields of velocities and forces, which act on particles, are fulfilled, the Hamiltonian systems with the time-reversible Liouville equation with fractional derivatives are obtained from non-Hamiltonian systems. In Ref. \[86\], time-irreversible equations of motion of Hamilton and Liouville for dynamic of classical particles in space with multifractal
time are offered. By using the definition of fractional derivative and the
Riemann-Liouville integral, the time-irreversible Liouville equation with
fractional derivatives (where the time is given on multifractal sets with
fractional dimensions) is obtained. In Refs. [87, 88], kinetic equations for
systems with fractal structure (in particular, for description of diffusion
processes in space of coordinates and momenta) are obtained within the
Klimontovich approach. A similar approach for constructing of time frac-
tional generalization for the Liouville equation and the Zwanzig equation
(within projection formalism) is proposed in Ref. [89].

An actual problem for description of nonequilibrium processes in
complex systems is construction of generalized diffusion and wave equa-
tions [90 91] using fractional integrals and derivatives. The dispersion
of heat waves in a dissipative environment using the Cattaneo–Maxwell
heat diffusion equation with fractional derivatives has been investigated in
Ref. [92]. On the basis of this equation, the frequency spectrum, phase and
group velocities of propagation of heat waves in a dissipative environment
have been investigated.

It is important to note that, for the first time, in Refs. [20–2 2], Nig-
matullin received diffusion equation with the fractional time derivatives
for the mean spin density [20], the mean polarization [21], and the charge
carrier concentration [22]. In Ref. [23], justification of equations with frac-
tional derivatives is given, and the time irreversible Liouville equation with
the fractional time derivative is provided. Within this approach, some
important results, including microscopic model of a non-Debye dielectric
relaxation, which generalizes the Cole-Cole law [26] and the Cole-Davidson
law [24], are obtained. In Ref. [27], by using the fractal nature of trans-
port processes of charge carriers, low-frequency behavior of conductivity is
studied with taking into account polarization effects of electrode. Results
of this investigation are in good agreement with experimental data.

In our works [28–30,93–100] a statistical approach to obtain general-
ized spatiotemporal nonlocal transfer equations was developed by using
the Zubarev nonequilibrium statistical operator (NSO) method [101,103]
and the Liouville equation with fractional derivatives [40,73]. In particu-
lar, the generalized diffusion equations of Cattaneo [93,95,97], Cattaneo–
Maxwell [98] and electrodiffusion [28,30,99], kinetic equations [100] with
spatiotemporal fractional derivatives were obtained.

In the second section, based on the statistical approach within the Gibbs
statistics [104], we have obtained a generalized diffusion equation with frac-
tional derivatives for the nonequilibrium average value of the number den-
sity of particles. This equation is nonlocal in space and time.
By modeling the temporal and spatial dependence of memory function using fractional calculus [1–5], the generalized Cattaneo–Maxwell–type diffusion equation has been obtained and analyzed.

In the third section, within the Gibbs statistics and approximation of constant diffusion coefficient, the frequency spectrum of the Cattaneo–Maxwell–type diffusion equation for the nonequilibrium average value of the number density of particles has been obtained. The frequency spectrum, phase and group velocities have been calculated, depending on order of the fractional derivative, characteristic relaxation time and value of the diffusion coefficient.

2. Generalized diffusion equations with fractional derivatives

To describe the diffusion processes of particle in heterogeneous environments with fractal structure, one of main parameters of the reduced description is the nonequilibrium density of particle numbers \( n(r; t) = \langle \hat{n}(r) \rangle^t_\alpha \), where \( \hat{n}(r) = \sum_{j=1}^N \delta(r - r_j) \) is the microscopic density of the particle. The corresponding generalized diffusion equation for \( n(r; t) \) can be obtained on the base of approach [94], by using the Zubarev nonequilibrium statistical operator method within the Gibbs statistics for solution of the Liouville equations with fractional derivatives,

\[
\frac{\partial}{\partial t} \langle \hat{n}(r) \rangle^t_\alpha = D_\alpha^\alpha \int \mu \alpha' \int_{-\infty}^t e^{\varepsilon(t' - t)} D^{\alpha\alpha'}(r, r'; t, t') \cdot D_{\alpha' \beta}(r' \beta \nu(r'; t') \, dt', \quad (2.1)
\]

where

\[
D^{\alpha\alpha'}(r, r'; t, t') = \langle \nu^\alpha(r) T(t, t') \nu^{\alpha'}(r') \rangle^t_\alpha,rel
\]

is the generalized coefficient diffusion of the particles within the Gibbs statistics. Averaging in Eq. (2.2) is performed with the power-law Gibbs distribution,

\[
\rho_{rel}(t) = \frac{1}{Z_G(t)} \exp \left[ -\beta \left( H - \int d\mu_\alpha(r) \nu(r; t) \hat{n}(r) \right) \right], \quad (2.3)
\]

where

\[
Z_G(t) = \hat{I}^\alpha(1, \ldots, N) \hat{T}(1, \ldots, N) \exp \left[ -\beta \left( H - \int d\mu_\alpha(r) \nu(r; t) \hat{n}(r) \right) \right]
\]

(2.4)

is the partition function of the relevant distribution function, \( H \) is a Hamiltonian of the system. Parameter \( \nu(r; t) \) is the chemical potential of the particles, which is determined from the self-consistency condition,

\[
\langle \hat{n}(r) \rangle^t_\alpha = \langle \hat{n}(r) \rangle^t_{\alpha,rel}.
\]
\[
\beta = 1/k_B T \quad (k_B \text{ is the Boltzmann constant}), \quad T \text{ is the equilibrium value of temperature}, \quad \hat{v}^\alpha(r) = \sum_{j=1}^N v_j^\alpha \delta(r - r_j) \text{ is the microscopic flux density of the particles.}
\]

In the Markov approximation, the generalized coefficient of diffusion in time and space has the form \( D^{\alpha\alpha'}(r, r'; t, t') \approx D(\delta(t - t') \delta(r - r') \delta_{\alpha\alpha'}). \) And by excluding the parameter \( \nu(r'; t') \) via the self-consistency condition, we obtain the diffusion equation with fractional derivatives from Eq. (2.1)

\[
\frac{\partial}{\partial t} \langle \hat{n}(r) \rangle^t_\alpha = D_D^{2\alpha} \nu(r; t'). \tag{2.6}
\]

The generalized diffusion equation takes into account spatial nonlocality of the system and memory effects in the generalized coefficient of diffusion \( D^{\alpha\alpha'}(r, r'; t, t') \) within the Gibbs statistics. To show the multifractal time in the generalized diffusion equation, we use the following approach for the generalized coefficient of particle diffusion

\[
D^{\alpha\alpha'}(r, r'; t, t') = W(t, t') D^{\alpha\alpha'}(r, r'), \tag{2.7}
\]

where \( W(t, t') \) can be defined as the time memory function. In view of this, Eq. (2.1) can be represented as

\[
\frac{\partial}{\partial t} \langle \hat{n}(r) \rangle^t_\alpha = \int_{-\infty}^t \exp[(t - t') \Psi(r; t')] d t', \tag{2.8}
\]

where

\[
\Psi(r; t') = \int d\mu_{\alpha'}(r') D^{\alpha\alpha'}(r, r') \cdot D^{\alpha'}_D \beta \nu(r'; t'). \tag{2.9}
\]

Further we apply the Fourier transform to Eq. (2.8), and as a result we get in frequency representation

\[
i \omega n(r; \omega) = W(\omega) \Psi(r; \omega). \tag{2.10}
\]

We can represent frequency dependence of the memory function in the following form

\[
W(\omega) = \frac{(i \omega)^{1-\xi}}{1 + (i \omega \tau)^{\xi}}, \quad 0 < \xi \leq 1, \tag{2.11}
\]

where the introduced relaxation time \( \tau \) characterizes of the particle transport processes in system. Then Eq. (2.10) can be represented as

\[
(1 + (i \omega \tau)^{\xi}) i \omega n(r; \omega) = (i \omega)^{1-\xi} \Psi(r; \omega). \tag{2.12}
\]

Further we use the Fourier transform to fractional derivatives of functions,

\[
L(0D^{1-\xi}_t f(t); i \omega) = (i \omega)^{-\xi} L(f(t); i \omega), \tag{2.13}
\]

where

\[
0D^{1-\xi}_t f(t) = \frac{1}{\Gamma(\xi)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t - \tau)^{1-\xi}} d \tau.
\]
is the Riemann–Liouville fractional derivative. By using it, the inverse transformation of Eq. (2.12) to time representation gives the Cattaneo–Maxwell generalized diffusion equation with taking into account spatial fractality, in the expanded form

\[ 0D_t^{2\xi}n(r;t)\tau^\xi + 0D_r^\xi n(r;t) = \int d\mu_{\alpha'}(r') D^{\alpha'}_r(r,r') \cdot D^{\nu'}_r(r';t), \quad (2.14) \]

is the new Cattaneo–Maxwell generalized equation within the Gibbs statistics with time and spatial nonlocality. Eq. (2.14) contains significant spatial heterogeneity in \( D^{\alpha\alpha'}(r,r') \). If we neglect spatial heterogeneity,

\[ D^{\alpha\alpha'}(r,r') = D\delta(r-r')\delta_{\alpha\alpha'}, \quad (2.15) \]

we get the Cattaneo–Maxwell diffusion equation with of space-time nonlocality and constant coefficients of diffusion within the Gibbs statistics

\[ 0D_t^{2\xi}n(r;t)\tau^\xi + 0D_r^\xi n(r;t) = D D_r^{2\alpha} \beta\nu(r;t). \quad (2.16) \]

From the point of view of the analysis of \( n(r;t) \) behavior in space and time when the values of diffusion coefficient, characteristic relaxation time and indexes of spatiotemporal fractality are changed, it is important to investigate the frequency spectrum of this equation. Peculiarities of dispersion relations are to be expected, since there is a characteristic relaxation time taking into account the indexes of spatiotemporal fractality \( \alpha \xi \). It is important to investigate the phase \( v_p(k) \) and group \( v_g(k) \) velocities when the values of wave vector \( k = |k| \) are changed, because for \( v_p(k) < v_g(k) \) we obtain an anomalous diffusion.

### 3. Dispersion relation for the time-space-fractional Cattaneo–Maxwell diffusion equation

Using the self-consistent condition (2.5) and the approved approximations, Eq. (2.16) can be written as

\[ 0D_t^{2\xi}n(r;t)\tau^\xi + 0D_r^\xi n(r;t) - \overrightarrow{D} D_r^{2\alpha} n(r;t) = 0, \quad (3.1) \]

where \( \overrightarrow{D} \) is the renormalized diffusion coefficient. For simplicity, we consider the one-dimensional case and a solution of Eq. (3.1) will be sought in the form of the plane wave, \( n(x;t) \sim e^{-i\omega t + ikx} \), then we get the corresponding frequency spectrum,

\[ \tau^\xi(-i\omega)^{2\xi} + (-i\omega)^\xi - \overrightarrow{D}(ik)^{2\alpha} = 0 \quad (3.2) \]

Equation (3.2) is a quadratic equation in \( (-i\omega)^{\xi} \), with discriminant

\[ \Delta_{\alpha,\xi} = 1 + 4\overrightarrow{D}\tau^\xi(ik)^{2\alpha} \quad (3.3) \]
and the following roots:

\[ (-i \omega)^{\xi} = \frac{-1 \pm \sqrt{\Delta_{\alpha,\xi}}}{2\tau^{\xi}}. \]  

(3.4)

In the next subsections, for different values of parameters \( \alpha \) and \( \xi \) the real \( (\omega_r(k) = \text{Re} \, \omega(k)) \) and imaginary \( (\omega_i(k) = \text{Im} \, \omega(k)) \) parts of complex frequency \( (\omega(k) = \omega_r(k) + i \omega_i(k)) \), as well as the phase \( (v_p(k)) \) and group \( (v_g(k)) \) velocities will be calculated according to the following definitions:

\[ v_p(k) = \frac{\omega_r(k)}{k}, \quad v_g(k) = \frac{\partial \omega_r(k)}{\partial k}. \]  

(3.5)

### 3.1. Limiting case \( \alpha = \xi = 1 \) (absence of spatial and temporal fractality)

In this case, we obtain a dispersion equation for the ordinary Cattaneo–Maxwell equation:

\[ \tau (-i \omega)^2 + (-i \omega) + D^r k^2 = 0, \]  

(3.6)

with discriminant

\[ \Delta = 1 - 4D^r \tau k^2 = 1 - \left(\frac{k}{k_0}\right)^2, \]  

(3.7)

where \( k_0 = 1/\sqrt{4\tau D^r} \).

The solution of equation (3.6), which has a physical meaning, is well known

\[ \omega_r(k) = \begin{cases} 0, & 0 \leq k \leq k_0, \\ \frac{1}{2\tau} \sqrt{(k/k_0)^2 - 1}, & k > k_0, \end{cases} \]  

(3.8)

\[ \omega_i(k) = \begin{cases} -\frac{1}{2\tau} \left(1 + \sqrt{1 - (k/k_0)^2}\right), & 0 \leq k \leq k_0, \\ -\frac{1}{2\tau}, & k > k_0. \end{cases} \]  

(3.9)

Using the definitions (3.5) for phase \( (v_p(k)) \) and group \( (v_g(k)) \) velocities, we obtain that

\[ v_p(k) = \begin{cases} 0, & 0 \leq k \leq k_0, \\ \frac{1}{2\tau} \sqrt{(k/k_0)^2 - 1}, & k > k_0, \end{cases} \]  

(3.10)

\[ v_g(k) = \begin{cases} 0, & 0 \leq k \leq k_0, \\ \frac{k/k_0^2}{2\tau \sqrt{(k/k_0)^2 - 1}}, & k > k_0. \end{cases} \]  

(3.11)

Note that the real (3.8) and imaginary (3.9) parts of complex frequency, as well as the phase velocity (3.10) are continuous functions. Whereas the group velocity (3.11) has a discontinuity of the second kind at the point \( k_0 \).
3.2. Limiting case $\alpha = 1$ (absence of spatial fractality)

In this case, the dispersion equation takes the following form:

$$\tau^\xi (-i\omega)^2 + (-i\omega)^\xi + \overline{D'}k^2 = 0,$$

which solution is

$$(-i\omega)^\xi = \frac{-1 \pm \sqrt{1 - (k/k_{0,\xi})^2}}{2\tau^\xi},$$

where $k_{0,\xi} = 1/\sqrt{4\tau^\xi D'}$.

The real ($\omega_r(k)$) and imaginary ($\omega_i(k)$) parts of complex frequency are:

$$\omega_r(k) = \begin{cases} 
-\frac{1}{2\tau^\xi k} \left[ 1 \pm \sqrt{1 - \left(\frac{k}{k_{0,\xi}}\right)^2} \right] \sin \frac{\pi}{\xi}, & 0 \leq k \leq k_{0,\xi}, \\
\pm \frac{1}{\tau^\xi} \left( \frac{k}{2k_{0,\xi}} \right)^{\frac{1}{\xi}} \sin \left[ -\frac{\pi}{\xi} + \frac{1}{\xi} \arctan \left( \frac{k}{k_{0,\xi}} \right)^2 - 1 \right], & k > k_{0,\xi}, 
\end{cases}$$

$$\omega_i(k) = \begin{cases} 
\frac{1}{2\tau^\xi k} \left[ 1 \pm \sqrt{1 - \left(\frac{k}{k_{0,\xi}}\right)^2} \right] \cos \frac{\pi}{\xi}, & 0 \leq k \leq k_{0,\xi}, \\
\frac{1}{\tau^\xi} \left( \frac{k}{2k_{0,\xi}} \right)^{\frac{1}{\xi}} \cos \left[ -\frac{\pi}{\xi} + \frac{1}{\xi} \arctan \left( \frac{k}{k_{0,\xi}} \right)^2 - 1 \right], & k > k_{0,\xi}, 
\end{cases}$$

According to the definitions \[(3.10)\] for phase ($v_p(k)$) and group ($v_g(k)$) velocities, we obtain that

$$v_p(k) = \begin{cases} 
-\frac{1}{2\tau^\xi k} \left[ 1 \pm \sqrt{1 - \left(\frac{k}{k_{0,\xi}}\right)^2} \right] \sin \frac{\pi}{\xi}, & 0 \leq k \leq k_{0,\xi}, \\
\pm \frac{1}{\tau^\xi} \left( \frac{k}{2k_{0,\xi}} \right)^{\frac{1}{\xi}} \sin \left[ -\frac{\pi}{\xi} + \frac{1}{\xi} \arctan \left( \frac{k}{k_{0,\xi}} \right)^2 - 1 \right], & k > k_{0,\xi}, 
\end{cases}$$

$$v_g(k) = \begin{cases} 
\pm \frac{1}{2\tau^\xi k} \left( \frac{k}{k_{0,\xi}} \right)^{\frac{1}{\xi}} \left[ 1 \pm \sqrt{1 - \left(\frac{k}{k_{0,\xi}}\right)^2} \right] \sin \frac{\pi}{\xi}, & 0 \leq k \leq k_{0,\xi}, \\
\pm \frac{1}{\tau^\xi} \left( \frac{k}{2k_{0,\xi}} \right)^{\frac{1}{\xi}} \sin \left[ -\frac{\pi}{\xi} + \frac{1}{\xi} \arctan \left( \frac{k}{k_{0,\xi}} \right)^2 - 1 \right], & k > k_{0,\xi}, 
\end{cases}$$

$$+ \frac{1}{\sqrt{(k/k_{0,\xi})^2 - 1}} \cos \left[ -\frac{\pi}{\xi} + \frac{1}{\xi} \arctan \left( \frac{k}{k_{0,\xi}} \right)^2 - 1 \right],$$

$k > k_{0,\xi}$. 

(3.17)
Note that
\[ \lim_{k \to k_0, \xi \rightarrow 0} \omega_i(k) = -\frac{1}{2\sqrt[1]{\xi}} \sin \frac{\pi}{\xi}, \quad \lim_{k \to k_0, \xi \rightarrow +0} \omega_i(k) = \frac{1}{2\sqrt[1]{\xi}} \cos \frac{\pi}{\xi}, \]
\[ \lim_{k \to k_0, \xi \rightarrow 0} \omega_r(k) = \lim_{k \to k_0, \xi \rightarrow +0} \omega_r(k) = -\frac{1}{2\sqrt[1]{\xi}} \sin \frac{\pi}{\xi}, \quad \lim_{k \to k_0, \xi \rightarrow 0} v_p(k) = \lim_{k \to k_0, \xi \rightarrow +0} v_p(k) = -\frac{1}{2\sqrt[1]{\xi} \sqrt{k}} \sin \frac{\pi}{\xi}. \]
That is, one branch of the real part of complex frequency is continuous and the other has a discontinuity of the first kind at the point \( k = k_0 \) (and accordingly the phase velocity); whereas the imaginary part of frequency is everywhere continuous function.

Note that in the limit \( \xi \to 1 \) expressions (3.14)–(3.17) must turn into expressions (3.8)–(3.11), respectively. This allows one to define the signs in expressions (3.8)–(3.11). Thus, for the imaginary part of complex frequency (3.15), as well as for the real part of complex frequency (3.14) for \( k > k_0, \xi \), and accordingly for the phase and group velocities, one needs to take the upper sign. Whereas for the real part of complex frequency (3.14) for \( k \leq k_0, \xi \), and accordingly for the phase and group velocities, there are both signs, because this real part, and accordingly the phase and group velocities, becomes zero \( (k \leq k_0, \xi) \) for both signs. As a result, in the case of \( \alpha = 1 \) and \( \xi \neq 1 \) in the real part of complex frequency (and accordingly in the phase and group velocities) a bifurcation appears at the point \( k = k_0, \xi \), to the left of which there are two branches.

### 3.3. Limiting case \( \xi = 1 \) (absence of temporal fractality)

In this case, the dispersion equation takes the following form:

\[ \tau (-i \omega)^2 + (-i \omega) - D'(ik)^{2\alpha} = 0, \quad (3.18) \]

the discriminant of which is:

\[ \Delta_\alpha = 1 + 4D' \tau (ik)^{2\alpha}, \quad (3.19) \]

the roots are:

\[ -i \omega = -\frac{1 \pm \sqrt{\Delta_\alpha}}{2\tau}, \quad (3.20) \]

where the following notations are introduced:

\[ \Delta_\alpha = |\Delta_\alpha| e^{i\omega}, \quad |\Delta_\alpha| = \sqrt{1 + 2 \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \cos(\alpha \pi) + \left( \frac{k}{k_{0,\alpha}} \right)^{4\alpha}}, \]
\[ \psi = \arctan \left( \frac{\left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \sin(\alpha \pi)}{1 + \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \cos(\alpha \pi)} \right), \quad k_{0,\alpha} = \frac{1}{(4D'\tau)^{\frac{1}{2\alpha}}}. \]

If \( 0 \leq \alpha \leq \frac{1}{2} \), then the real and imaginary parts of complex frequency are as follows:

\[ \omega_r(k) = \pm \frac{1}{2\tau} |\Delta_\alpha|^{\frac{1}{2}} \sin \left( \frac{\psi}{2} \right), \quad (3.21) \]
\[ \omega_i(k) = \frac{1}{2\tau} \left( -1 \pm |\Delta_\alpha|^{\frac{1}{2}} \cos \left( \frac{\psi}{2} \right) \right). \quad (3.22) \]

Then, according to the definitions \((3.5)\), the phase and group velocities are as follows:

\[ v_p(k) = \pm \frac{1}{2\tau} |\Delta_\alpha|^{\frac{1}{2}} \sin \left( \frac{\psi}{2} \right), \quad (3.23) \]
\[ v_g(k) = \pm \frac{\alpha}{2\tau |\Delta_\alpha|^{\frac{1}{2}}} \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \left[ \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \sin \left( \frac{\psi}{2} \right) + \sin \left( \alpha \pi + \frac{\psi}{2} \right) \right]. \quad (3.24) \]

If \( \frac{1}{2} < \alpha \leq 1 \), then the real and imaginary parts of complex frequency are as follows:

\[ \omega_r(k) = \begin{cases} 
\pm \frac{1}{2\tau} |\Delta_\alpha|^{\frac{1}{2}} \sin \left( \frac{\psi}{2} \right), & 0 \leq k \leq \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}}, \\
\pm \frac{1}{2\tau} |\Delta_\alpha|^{\frac{1}{2}} \cos \left( \frac{\psi}{2} \right), & k > \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}}. 
\end{cases} \quad (3.25) \]
\[ \omega_i(k) = \begin{cases} 
\frac{1}{2\tau} \left( -1 \pm |\Delta_\alpha|^{\frac{1}{2}} \cos \left( \frac{\psi}{2} \right) \right), & 0 \leq k \leq \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}}, \\
\frac{1}{2\tau} \left( -1 \mp |\Delta_\alpha|^{\frac{1}{2}} \sin \left( \frac{\psi}{2} \right) \right), & k > \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}}. 
\end{cases} \quad (3.26) \]

Then, according to the definitions \((3.5)\), the phase and group velocities are as follows:

\[ v_p(k) = \begin{cases} 
\pm \frac{1}{2\tau} |\Delta_\alpha|^{\frac{1}{2}} \sin \left( \frac{\psi}{2} \right), & 0 \leq k \leq \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}}, \\
\pm \frac{1}{2\tau k_{0,\alpha}} |\Delta_\alpha|^{\frac{1}{2}} \cos \left( \frac{\psi}{2} \right), & \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}} < k < \infty, 
\end{cases} \quad (3.27) \]
\[ v_g(k) = \begin{cases} 
\pm \frac{\alpha}{2\tau |\Delta_\alpha|^{\frac{1}{2}}} \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \left[ \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \sin \left( \frac{\psi}{2} \right) + \sin \left( \alpha \pi + \frac{\psi}{2} \right) \right], & 0 \leq k \leq \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}}, \\
\pm \frac{\alpha}{2\tau |\Delta_\alpha|^{\frac{1}{2}}} \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \left[ \left( \frac{k}{k_{0,\alpha}} \right)^{2\alpha} \cos \left( \frac{\psi}{2} \right) + \cos \left( \alpha \pi + \frac{\psi}{2} \right) \right], & \frac{k_{0,\alpha}}{(-\cos(\alpha \pi))^{\frac{1}{2\alpha}}} < k < \infty. 
\end{cases} \quad (3.28) \]
Note that
\[
\lim_{k \to \frac{k_0}{\cos(\alpha \pi)}} \psi = \lim_{k \to \frac{k_0}{\cos(\alpha \pi)}} \arctan \left( \frac{\left( \frac{k}{k_0} \right)^{2\alpha} \sin(\alpha \pi)}{1 + \left( \frac{k}{k_0} \right)^{2\alpha} \cos(\alpha \pi)} \right) = \pm \frac{\pi}{2}
\]
and
\[
\left| \Delta_\alpha \right| = \lim_{k \to \frac{k_0}{\cos(\alpha \pi)}} \sqrt{1 + 2 \left( \frac{k}{k_0} \right)^{2\alpha} \cos(\alpha \pi) + \left( \frac{k}{k_0} \right)^{4\alpha}} = \left| \tan(\alpha \pi) \right|.
\]
So the real (3.25) and imaginary (3.26) parts of complex frequency, as well as the phase (3.27) and group (3.28) velocities are continuous functions.

Note that in the limit \( \alpha \to 1 \) expressions (3.25)–(3.28) must turn into expressions (3.8)–(3.11), respectively. Since
\[
\lim_{\alpha \to 1} \psi = \lim_{\alpha \to 1} \arctan \left( \frac{\left( \frac{k}{k_0} \right)^{2\alpha} \sin(\alpha \pi)}{1 + \left( \frac{k}{k_0} \right)^{2\alpha} \cos(\alpha \pi)} \right) = 0
\]
and
\[
\lim_{\alpha \to 1} \left| \Delta_\alpha \right| = \lim_{\alpha \to 1} \sqrt{1 + 2 \left( \frac{k}{k_0} \right)^{2\alpha} \cos(\alpha \pi) + \left( \frac{k}{k_0} \right)^{4\alpha}} = \left| 1 - \left( \frac{k}{k_0} \right)^2 \right|,
\]
then
\[
\omega_r(k) = \begin{cases} 
0, & 0 \leq k \leq k_0, \\
\pm \frac{1}{2\tau} \sqrt{(k/k_0)^2 - 1}, & k > k_0,
\end{cases} \tag{3.29}
\]
\[
\omega_i(k) = \begin{cases} 
-\frac{1}{2\tau} \left[ 1 + \sqrt{1 - (k/k_0)^2} \right], & 0 \leq k \leq k_0, \\
-\frac{1}{2\tau}, & k > k_0,
\end{cases} \tag{3.30}
\]
\[
v_p(k) = \begin{cases} 
0, & 0 \leq k \leq k_0, \\
\pm \frac{1}{2\tau} \sqrt{(k/k_0)^2 - 1}, & k > k_0,
\end{cases} \tag{3.31}
\]
\[
v_g(k) = \begin{cases} 
0, & 0 \leq k \leq k_0, \\
\pm \frac{1}{2\tau} \sqrt{k/k_0^2 - 1}, & k > k_0.
\end{cases} \tag{3.32}
\]
It follows, from the comparison of expressions (3.29)–(3.32) with the corresponding expressions (3.8)–(3.11), that one needs to choose the “+” sign in expressions (3.25)–(3.28).

The signs in expressions (3.21)–(3.24) must be chosen so, that in the limit \( \alpha \to 1/2 \) these expressions turn into expressions (3.25)–(3.28) for \( \alpha = \)
To do this, one needs also to choose the “+” sign in expressions (3.21)–(3.24).

Fig. 1 shows the dependencies of the real ($\omega_r(k)$) (3.25) and imaginary ($\omega_i(k)$) (3.26) parts of complex frequency for $\xi = 1$ and $\alpha = 1.0, 0.98, 0.96, 0.94$. Black bold lines represent dependencies (3.8) and (3.9), that is, when $\alpha = 1$. In the case of $\alpha \neq 1$ these dependencies are smoothed, and when approaching $\alpha \to 1$ they converge uniformly to expressions (3.8) and (3.9) for the ordinary Cattaneo–Maxwell equation.

![Figure 1. Frequency spectrum for $\xi = 1$ and $\alpha = 1.0, 0.98, 0.96, 0.94$ ($\tau = 0.2$ and $D' = 1$).](image1)

Fig. 2 shows the dependencies of the phase ($v_p(k)$) (3.27) and group ($v_g(k)$) (3.28) velocities for $\xi = 1$ and $\alpha = 1.0, 0.98, 0.96, 0.94$. In this case the group velocity ($v_g(k)$) has a discontinuity. In the case of $\alpha \neq 1$ these dependencies are smoothed, instead of a discontinuity the maximum is observed, the value of which decreases with decreasing $\alpha$. When approaching $\alpha \to 1$ the phase and group velocities converge uniformly to expressions (3.10) and (3.11) for the ordinary Cattaneo–Maxwell equation.

![Figure 2. Phase ($v_p(k)$) and group ($v_g(k)$) velocities for $\xi = 1$ and $\alpha = 1.0, 0.98, 0.96, 0.94$ ($\tau = 0.2$ and $D' = 1$).](image2)
3.4. General case $0 < \alpha \leq 1$, $0 < \xi \leq 1$

Unfortunately, in this case analytical expressions cannot be found, therefore, the analysis was performed numerically (see Figs. 3, 4).

**Figure 3.** Frequency spectrum for $\tau = 0.2$ and $\mathcal{D} = 1$.

Fig. 3 shows the real ($\omega_r(k)$) and imaginary ($\omega_i(k)$) parts of complex frequency, and Fig. 4 shows the phase ($v_p(k)$) and group ($v_g(k)$) velocities as functions of the wave number $k$ for different values of fractality parameters $\alpha$ and $\xi$.

If $\alpha = 1$ and $\xi = 1$, then we observe the known dependences (3.8)–(3.11) for the ordinary Cattaneo–Maxwell equation. Decreasing the parameter $\alpha$ leads to smoothing of these dependencies.

If $\alpha = 1$ and $\xi \neq 1$, then in the real part of complex frequency a bifurcation and discontinuity of the first kind are observed at the point
Figure 4. Phase \((v_p(k))\) and group \((v_g(k))\) velocities for 
\(\tau = 0.2\) and \(D = 1\).

\(k \leq k_{0,\xi}\), to the left of which two branches exist. Accordingly, the phase velocity also has a bifurcation and discontinuity of the first kind at the point \(k = k_{0,\xi}\), and the group velocity has a singularity at this point and shows the \(\lambda\)-like behavior. The imaginary part of complex frequency is continuous and has an inflection at this point.

If \(\xi \neq 1\) and \(\alpha \neq 1\), then the bifurcation disappears, the real part of complex frequency is continuous. Accordingly, the phase velocity also becomes continuous, and the group velocity doesn’t have any singularities and its lower branch (which was present for \(\alpha = 1\)) disappears. As parameter \(\alpha\) decreases, the dependencies of the real and imaginary parts of complex frequency, phase and group velocities are smoothed.
For $\xi = 2/3$ and $\alpha = 1$ the imaginary part of complex frequency equals zero in the domain $k \leq k_{0,\xi}$, and the real part of complex frequency becomes convex downward in the domain $k > k_{0,\xi}$, this causes a significant change in the group velocity behavior: the $\lambda$-like behavior is lost. The further decreasing of $\xi$ for $\alpha = 1$ leads to an increase in this downward convexity of the real part of complex frequency in the domain $k > k_{0,\xi}$ and a significant change in the group velocity: it becomes $\lambda$-like again, but inverted, moreover, a domain appears where the imaginary part of complex frequency becomes positive. For $\alpha \neq 1$ the bifurcation of the real part of complex frequency disappears, the dependencies of the real and imaginary parts of complex frequency are smoothed.

4. Conclusions

When describing non-Markov diffusion processes with spatiotemporal nonlocality of the diffusion coefficient, problems arise in calculating the time correlation function “velocity–velocity” $(v(r; t)v(r'; t'))$. In general, for such a calculation, one can use the method of Mori projection operators and express it through higher memory functions. But for systems, which have certain characteristic relaxation times or spatial characteristics, fractality, these characteristics must be taken into account. We proceeded from the non-Markov diffusion equation taking into account the spatial fractality and modeled the generalized coefficient of particle diffusion as follows:

$$D^{\alpha\alpha'}(r, r'; t, t') = W(t, t')\overline{D}^{\alpha'}(r, r').$$

Using fractional calculus and the corresponding approximation for time memory function $W(t, t')$ with introduction of the characteristic relaxation time, we have obtained the generalized Cattaneo–Maxwell–type diffusion equation in fractional time and space derivatives. In the case of a constant diffusion coefficient, analytical and numerical studies of the frequency spectrum for the Cattaneo–Maxwell diffusion equation in fractional time and space derivatives have been performed. Numerical calculations of the phase and group velocities with change of values of characteristic relaxation time, diffusion coefficient and indexes of temporal $\xi$ and spatial $\alpha$ fractality have been carried out. Calculations have shown, that in the case of $\alpha = 1$ and $\xi \neq 1$ in the real part of complex frequency (and accordingly in the phase and group velocities) a bifurcation appears at the point $k = k_{0,\xi}$, to the left of which there are two branches.

In the case when $\alpha = 1$ and $\xi = 1$ there is a discontinuity in the group velocity $(v_g(k))$, which corresponds to the ordinary Cattaneo–Maxwell equation. In the case of $\alpha \neq 1$ and $\xi = 1$ these dependencies are smoothed, instead of a discontinuity the maximum is observed, the value of which decreases with decreasing $\alpha$. When approaching $\alpha \to 1$ the phase and group
velocities converge uniformly to expressions (3.10) and (3.11) for the ordinary Cattaneo–Maxwell equation. In the general case, when $\xi \neq 1$ and $\alpha \neq 1$, the bifurcation disappears, the real part of complex frequency is continuous. Accordingly, the phase velocity also becomes continuous, and the group velocity doesn’t have any singularities and its lower branch (which was present for $\alpha = 1$) disappears. As parameter $\alpha$ decreases, the dependencies of the real and imaginary parts of complex frequency, phase and group velocities are smoothed.

The domain is special for $\xi = 2/3$ and $\alpha = 1$, when the imaginary part of complex frequency equals zero in the domain $k \leq k_{0,\xi}$, and the real part of complex frequency becomes convex downward in the domain $k > k_{0,\xi}$. This causes a significant change in the group velocity behavior: the $\lambda$-like behavior is lost. The further decreasing of $\xi$ for $\alpha = 1$ leads to an increase in this downward convexity of the real part of complex frequency in the domain $k > k_{0,\xi}$ and a significant change in the group velocity: it becomes $\lambda$-like again, but inverted. Moreover, a domain appears where the imaginary part of complex frequency becomes positive.

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GENERALIZED DIFFUSION EQUATION WITH NONLOCALITY

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