INTERACTION OF THE PRIMITIVE EQUATIONS WITH SEA ICE DYNAMICS

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Abstract. This article establishes local strong well-posedness and global strong well-posedness close to constant equilibria of a model coupling the primitive equations of ocean and atmospheric dynamics with Hibler’s viscous-plastic sea ice model. In order to treat the coupling conditions, an approach involving the hydrostatic Dirichlet and Dirichlet-to-Neumann operator is developed. Mapping properties of the latter operators are investigated for the first time and are of central importance for showing that the operator associated with the linearized coupled system admits a bounded $\mathcal{H}^\infty$-calculus on suitable $L^q$-spaces. Quasilinear methods allow then to obtain the strong well-posedness results described above.

1. Introduction

The mathematical analysis of the primitive equations of the large-scale atmosphere and ocean dynamics was pioneered by Lions, Temam and Wang in [32, 33]. Moreover, Lions, Temam and Wang introduced and analyzed a coupled atmosphere-ocean model with wind-driven boundary conditions [34, 35]. Ever since then, the study of coupled atmosphere-ocean models has been an active field of research. We refer here for instance to the handbook article of Temam and Ziane [47]. In this article, for the first time, we rigorously investigate the interaction problem of the atmosphere and the ocean when additionally considering sea ice as a thin layer coupled to the atmosphere and the ocean.

In a seminal paper, Cao and Titi [10] established global strong well-posedness of the primitive equations for initial data in $H^1$ by means of energy methods. For related results, we refer to the work of Kukavica and Ziane [29]. A different approach to the primitive equations by means of evolution equations was developed by Hieber and Kashiwabara [26] as well as Giga, Gries, Hieber, Hussein and Kashiwabara [17, 18, 19]. Details of this approach and further references can be found in [24]. We also refer to the article [31] for a survey of results and further references.

In 1979, Hibler [22] proposed the governing equations of large-scale sea ice dynamics. They were investigated numerically by different communities, see e.g. [37, 38, 45, 46]. For surveys on the modeling of sea ice, we refer to [44] and [20]. We also refer here to the recent article of Piersanti and Temam [40] on the modeling and analysis of the dynamics of grounded shallow ice sheets.

This article is concerned with the analysis of a coupled atmosphere-sea ice-ocean model. More precisely, the primitive equations for the ocean and atmosphere dynamics are coupled with the sea ice equations as introduced in [22]. For a precise formulation of the coupling conditions, see (3.1), (3.2) and (3.3) in Section 3. For related results on the primitive equations subject to stochastic wind driven boundary conditions, we refer to the recent work [7].

It is the aim of this article to show that the coupled system of the ocean, the atmosphere and sea ice, see (3.4) below, has a unique, local, strong solution, and that it is globally strongly well-posed for initial data close to constant equilibria.

To this end, we rewrite the system of equations as a quasilinear evolution equation subject to coupling conditions. In order to treat these coupling conditions, we develop an approach involving the hydrostatic Dirichlet and Dirichlet-to-Neumann operator. These operators have been studied intensively in the parabolic situation and also for fluids but not for geophysical flows as the primitive equations. The results on well-posedness of the stationary hydrostatic Stokes problem with inhomogeneous boundary conditions developed in Section 7 will then be of central importance for our approach. Indeed, the hydrostatic Dirichlet and Dirichlet-to-Neumann operators allow to show that the operator associated with the linearized coupled
system admits a bounded \( \mathcal{H}^\infty \)-calculus on suitable \( L^p \)-spaces. The latter property allows to use modern quasilinear methods for solving the coupled system within the strong setting.

With regard to the global strong well-posedness close to constant equilibria, we make use of the so-called generalized principle of linearized stability, see [42] and [41]. We show that constant equilibria are normally stable by means of energy estimates and absorption arguments. More precisely, energy estimates yield that the spectrum of the total linearization \( \mathcal{A}_0 \) besides zero is contained in the left half-plane, that the set of equilibria close to constants is a \( C^1 \)-manifold and that zero is a semi-simple eigenvalue of \( \mathcal{A}_0 \). We also show that the spectrum of \( \mathcal{A}_0 \) is independent of \( q \) and that thus the above properties of \( \mathcal{A}_0 \) obtained via energy estimates within the \( L^2 \)-setting transfer to the \( \mathcal{L}^q \)-setting for \( q \in (1, \infty) \). Note that due to the quasilinear nature of sea ice dynamics, this range of \( q \) is important.

For further information concerning quasilinear evolution equations, maximal \( L^p \)-regularity, bounded \( \mathcal{H}^\infty \)-calculus and decoupling techniques, we refer e. g. to [4], [13], and [41].

Let us note that the rigorous analysis of the single sea ice component in our model began only recently by the article of Brandt, Disser, Haller-Dintelmann and Hieber [9] as well as the article of Liu, Thomas and Titi [36]. The associated system of equations is coupled, degenerate, quasilinear and parabolic-hyperbolic. The interaction problem of sea ice with a rigid body has been studied in [5].

Models describing the coupling of sea ice with certain flows were introduced within the geophysical community e. g. by Hibler and Bryan [23] or by Timmermann, Beckmann and Hellmer [48]. Let us emphasize that these works are concerned with the development of models and not with its analysis. For analysis of related models, we refer to [14]. Coupled models of the atmosphere and the ocean call for an investigation of boundary layers. This is left to future study, and we refer to the work of Dalibard and Gérard-Varet [11] or Dalibard and Saint-Raymond [12] for boundary layer theory in the context of geophysical flows.

The structure of this article is as follows: In Section 2, we introduce the notation and give basic concepts used throughout the paper. Section 3 is dedicated to deriving the coupling conditions as well as the complete coupled atmosphere-sea ice-ocean system, and to stating the two main results on the local strong well-posedness and the global strong well-posedness close to constant equilibria, respectively. We then reformulate the coupled system of equations (3.4) as a quasilinear evolution equation in Section 4. The sea ice equations on the quadratic domain \((0,1) \times (0,1)\) are investigated in Section 5, while we recall properties of the primitive equations on cylindrical domains in Section 6. The central Section 7 deals with the stationary hydrostatic Stokes problem and the hydrostatic Dirichlet-to-Neumann operator. In Section 8, we present the argument to prove the bounded \( \mathcal{H}^\infty \)-calculus of the coupled operator, and we show Lipschitz estimates of the coupled system. Sections 9 and 10 are concerned with the proof of the main results.

2. Preliminaries

We use \( x \) and \( y \) as well as \( x_H := (x,y) \) to denote horizontal coordinates, whereas \( z \) is employed for the vertical coordinate and \( t \) is the time variable. The sub- or superscripts \( \partial_{\text{atm}}, \partial_{\text{ocn}} \) and \( \partial_{\text{ice}} \) are used to indicate whether objects \( o \) in the context of the atmosphere, the ocean or sea ice are considered. For instance, \( u_{\text{atm}}, v_{\text{atm}} \) and \( w_{\text{atm}} \) represent the full, horizontal and vertical velocity of the atmospheric wind, \( u_{\text{ocn}}, v_{\text{ocn}} \) and \( w_{\text{ocn}} \) denote the full, horizontal and vertical velocity of the ocean, and \( v_{\text{ice}} \) is the horizontal velocity of the sea ice. With \( v = (v_{\text{ice}}, h, a) \), the principle variable of the system will be denoted by \( u = (v_{\text{atm}}, v_{\text{ocn}}, v) = (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a) \), while the respective pressure terms for the atmosphere and the ocean will be denoted by \( \pi_{\text{atm}} \) and \( \pi_{\text{ocn}} \). For the sea ice part, with \( G := (0,1) \times (0,1) \), we also introduce the mean ice thickness \( h: G \times [0,T] \to (\kappa_1, \kappa_2) \) and the mean ice compactness \( a: G \times [0,T] \to (0,1) \), defined as the ratio of thick ice per area. We remark that \( \kappa_1 > 0 \) sufficiently small is a parameter indicating the transition to open water, i. e., a value of \( h(x,y,t) \) less than \( \kappa_1 \) means that at \( (x,y) \in G \) at time \( t \) there is open water. In contrast, \( \kappa_2 > 0 \) sufficiently large denotes an upper bound for the mean ice thickness.

Unless stated otherwise, domains will be denoted by \( \Omega \), possibly with a sub- or superscript, while \( \Gamma \) represents boundaries. The domains under consideration are \( \Omega_{\text{atm}} = G \times (\kappa_2, h_{\text{atm}}) \) for the atmosphere, where \( \kappa_2 > 0 \) is the upper bound for the mean ice thickness, as well as \( \Omega_{\text{ocn}} = G \times (-h_{\text{ocn}}, 0) \) for the ocean. The upper boundary is denoted by \( \Gamma_u := G \times \{ h_{\text{atm}} \} \), the lower boundary by \( \Gamma_b := G \times \{ -h_{\text{ocn}} \} \), and the interfaces between the ocean and the sea ice and the sea ice and the atmosphere are denoted by \( \Gamma_o := G \times \{ 0 \} \) and \( \Gamma_i := G \times \{ \kappa_2 \} \), respectively. We remark that these boundaries and interfaces will be identified with \( G \) in the sequel. In addition, \( \Gamma_{\text{latm}} := \partial G \times (\kappa_2, h_{\text{atm}}) \) represents the lateral boundary...
associated to the atmosphere, whereas $\Gamma_{\text{loc}} := \partial G \times (-h_{\text{loc}}, 0)$ denotes the lateral boundary of the ocean. The thickness of the sea ice, expressed by $h$, is relatively small compared to the heights of the ocean and the atmosphere. For simplicity, we assume that the interfaces between the sea ice and the ocean as well as the atmosphere are flat in this article. The more involved consideration of free surfaces is left to future studies.

Furthermore, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, $\nabla = (\partial_x, \partial_y, \partial_z)^\top$ and $\nabla = \partial_x f_1 + \partial_y f_2 + \partial_z f_3$ represent the Laplacian, and the divergence of a vector field $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, while the respective horizontal objects are denoted by $\Delta_\text{H} = \partial_x^2 + \partial_y^2$, $\nabla_\text{H} = (\partial_x, \partial_y)^\top$ and $\nabla_\text{H} g = \partial_x g_1 + \partial_y g_2$ for a vector field $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The trace and the normal derivative on some boundary $\Gamma$ will be denoted by $\text{tr}_\Gamma$ and $\partial_{\nu,\Gamma}$, respectively. As it is only used in the context of the sea ice, the deformation tensor $\varepsilon$ associated to sea ice is denoted by $\varepsilon = \varepsilon(v_{\text{ice}}) = \frac{1}{\varepsilon} \left( (\nabla_h v_{\text{ice}} + (\nabla_h v_{\text{ice}})^\top) \right)$.

Consider a domain $\Omega \subset \mathbb{R}^n$, $p, q \in [1, \infty)$, $k \in \mathbb{N}$ and $s \geq 0$. We denote by $L^q(\Omega)$ the classical $L^q$-spaces, by $W^{k,q}(\Omega)$ or $H^{k,q}(\Omega)$ the classical Sobolev spaces of order $k$, by $H^{s,q}(\Omega)$ the Bessel potential spaces, by $W^{s,q}(\Omega)$ the fractional Sobolev spaces or Sobolev-Slobodeckij spaces and by $B^{s,q}_p(\Omega)$ the Besov spaces. For details on these spaces, we refer e. g. to the monographs [49] or [3]. If $\Omega$ is a bounded domain with sufficiently smooth boundary $\Gamma$, we use the subscript b.c. to indicate that the respective space is considered with boundary conditions.

For $\Omega = G \times (a, b)$, $-\infty < a < b < \infty$, we denote by $\nabla$ the vertical average of $v$, i. e., $\nabla := \frac{1}{b-a} \int_0^b v(\cdot, \xi) \, d\xi$. The primitive equations are modeled on the hydrostatic solenoidal $L^q$-spaces given by the $L^q$-closure of the smooth hydrostatic functions, so following the approach developed in [26, Sections 3 and 4], we set

$$L^q_{\text{per}}(\Omega) := \left\{ v \in C_{\text{per}}(\Omega)^2 : \nabla_\text{H} v = 0 \right\}^{\frac{1}{q}||\cdot||_{L^q(\Omega)}},$$

and horizontal periodicity is incorporated by the function spaces $C_{\text{per}}(\Omega)$ and $C_{\text{per}}(G)$, see also [24, Section 1.5.2] for further details. In that respect, for $p, q \in (1, \infty)$ and $s \in [0, \infty)$, we define the Bessel potential spaces with horizontal periodicity by

$$H^{s,q}_{\text{per}}(\Omega) := \frac{C_{\text{per}}(\Omega)^2}{||\cdot||_{H^{s,q}(\Omega)}},$$

and the Besov spaces with horizontal periodicity $B^{s,q}_{p,\text{per}}(\Omega)$ and $B^{s,q}_{p,\text{per}}(G)$ are defined analogously. The Bessel potential spaces $H^{s,q}(\Omega)$ and Besov spaces $B^{s,q}_{p}(\Omega)$ are defined as restrictions of the respective spaces on the whole space to $\Omega$. We remark that $H^{0,q}_{\text{per}} := L^q_{\text{per}}$.

### 3. Main Results for the Coupled System

The main assumptions with regard to the coupling of the atmosphere and the ocean with sea ice are that the first two exert the forces $\tau_{\text{atm}}$ and $\tau_{\text{ocn}}$ on the sea ice, and that the velocity of the sea ice coincides with the horizontal velocity of the ocean at the water surface. More precisely, we assume that the force exerted by the atmosphere is given by

$$\tau_{\text{atm}} = \rho_{\text{atm}} C_{\text{atm}} |\text{tr}_\Gamma, v_{\text{atm}}| R_{\text{atm}} |\text{tr}_\Gamma, v_{\text{atm}}, \text{ on } G.$$  \hspace{1cm} (3.1)

The force exerted by the ocean on the sea ice is supposed to be proportional to the shear rate, so it is of the form

$$\tau_{\text{ocn}} = -\mu_{\text{ocn}} \partial_{\nu,\Gamma} v_{\text{ocn}}, \text{ on } G.$$  \hspace{1cm} (3.2)

This is in accordance with a plane Couette flow for a Newtonian fluid, where the stress tensor is given by $T = \pi I_3$, with $T$ denoting the usual Cauchy stress tensor, compare e. g. [43, Section 7.2]. The condition on the equality of the velocities on the interface of the ocean and the sea ice can be expressed by

$$\text{tr}_\Gamma v_{\text{ocn}} = v_{\text{ice}}, \text{ on } G.$$  \hspace{1cm} (3.3)

In the above, $\rho_{\text{atm}}$ represents the density for air, $C_{\text{atm}}$ denotes an air drag coefficient, and $R_{\text{atm}}$ is a rotation matrix operating wind vectors, while $\mu_{\text{ocn}} > 0$ is the viscosity associated to the fluid described by the primitive equations of the ocean.

Before providing the complete coupled system, we briefly discuss the internal ice stress. We follow [22] and refer e. g. to [9, Section 2] for a more thorough treatment. In fact, the viscous-plastic rheology is expressed by a constitutive law linking the internal ice stress $\sigma$ and the deformation tensor $\varepsilon$ via an internal ice strength $P$ as well as nonlinear bulk and shear viscosities such that the principal components of the
stress lie on an elliptical yield curve. With $e > 1$ denoting the ratio of major to minor axes, the constitutive law is given by

$$\sigma = \frac{P}{e^2 \Delta(\varepsilon)} \varepsilon + \left(1 - \frac{1}{e^2}\right) \frac{P}{2\Delta(\varepsilon)} \text{tr}(\varepsilon)I_2 - \frac{P}{2} I_2,$$

where for the ice thickness $h$, the ice compactness $a$ and given constants $p^* > 0$ and $c > 0$, the ice strength $P$ takes the shape $P = P(h, a) = p^* h \exp(-c(1-a))$. Moreover, we have

$$\Delta^2(\varepsilon) := (e^2_{11} + e^2_{22}) \left(1 + \frac{1}{e^2}\right) + \frac{4}{e^2} e_{12}^2 + 2 e_{11} e_{22} \left(1 - \frac{1}{e^2}\right).$$

Even though the above law represents an idealized viscous-plastic material, the viscosities become singular for $\Delta$ tending to 0. Following [9,37], for $\delta > 0$, we take the regularization $\Delta_\delta(\varepsilon) := \sqrt{\delta + \Delta^2(\varepsilon)}$ into account and define the regularized ice stress by

$$\sigma_\delta := \frac{P}{e^2 \Delta_\delta(\varepsilon)} \varepsilon + \left(1 - \frac{1}{e^2}\right) \frac{P}{2\Delta_\delta(\varepsilon)} \text{tr}(\varepsilon)I_2 - \frac{P}{2} I_2.$$

We follow [32,33,35] for the incompressible, viscous primitive equations of the atmosphere and the ocean as well as [22] for the sea ice equations. The complete coupled system is then given by

$$\begin{cases}
\partial_t \mathbf{v}_{\text{atm}} - \nabla p_{\text{atm}} + \mathbf{u}_{\text{atm}} \cdot \nabla \mathbf{v}_{\text{atm}} + \nabla H \mathbf{p}_{\text{atm}} = f_{\text{atm}}, & \mathbf{\Omega}_{\text{atm}} \times [0, T], \\
\partial_t \pi_{\text{atm}} = 0, & \mathbf{\Omega}_{\text{atm}} \times [0, T], \\
\text{div} \mathbf{u}_{\text{atm}} = 0, & \mathbf{\Omega}_{\text{atm}} \times [0, T], \\
\partial_t \mathbf{v}_{\text{ocn}} - \nabla p_{\text{ocn}} + \mathbf{u}_{\text{ocn}} \cdot \nabla \mathbf{v}_{\text{ocn}} + \nabla H \mathbf{p}_{\text{ocn}} = f_{\text{ocn}}, & \mathbf{\Omega}_{\text{ocn}} \times [0, T], \\
\partial_t \pi_{\text{ocn}} = 0, & \mathbf{\Omega}_{\text{ocn}} \times [0, T], \\
\text{div} \mathbf{u}_{\text{ocn}} = 0, & \mathbf{\Omega}_{\text{ocn}} \times [0, T], \\
\partial_t v_{\text{ice}} - \frac{1}{m} \text{div} \mathbf{H} \sigma_\delta + (v_{\text{ice}} \cdot \nabla H) v_{\text{ice}} = \frac{1}{m} \tau_{\text{atm}} + \frac{1}{m} \tau_{\text{ocn}} - g \nabla H, & \mathbf{G} \times [0, T], \\
\partial_t h - d_h \Delta h + \text{div} H (v_{\text{ice}} h) = S_h, & \mathbf{G} \times [0, T], \\
\partial_t a - d_a \Delta a + \text{div} H (v_{\text{ice}} a) = S_a, & \mathbf{G} \times [0, T], \\
\text{tr} \gamma_{\text{atm}} \mathbf{v}_{\text{ocn}} = v_{\text{ice}}, & \mathbf{G} \times [0, T].
\end{cases}$$

In the above, $f_{\text{atm}}$ as well as $f_{\text{ocn}}$ represent external forces, $m = \rho_{\text{ice}} h$ denotes the mass of the sea ice, where $\rho_{\text{ice}} > 0$ is the density, $g$ denotes the gravity and $H : \mathbf{G} \times [0, T] \rightarrow [0, \infty)$ the sea surface dynamic height, $S_h$ and $S_a$ are thermodynamic terms discussed in more detail in Section 5, and $d_h, d_a > 0$ are constants. For simplicity, we omit Reynolds numbers and terms associated to the Coriolis force in (3.4), but we emphasize that they can be incorporated easily, see Remark 3.5.

The system is completed by Neumann boundary conditions for the horizontal velocity of the atmospheric wind $\mathbf{v}_{\text{atm}}$, Dirichlet boundary conditions for the horizontal velocity of the ocean $\mathbf{v}_{\text{ocn}}$ on the lower boundary $\Gamma_b$ and Dirichlet boundary conditions for the vertical velocity of the atmospheric wind and the ocean $\mathbf{w}_{\text{atm}}$ and $\mathbf{w}_{\text{ocn}}$, respectively. This can be summarized by

$$\begin{cases}
\partial_{\Gamma_{\text{atm}}} \mathbf{v}_{\text{atm}} = 0 \text{ on } \Gamma_u, & \text{tr} \gamma_{\text{atm}} \mathbf{v}_{\text{atm}} = 0 \text{ on } \Gamma_i, & \text{tr} \gamma_{\text{atm}} h_{\text{atm}} = 0 \text{ on } \Gamma_b, \text{ as well as} \\
\text{tr} \gamma_{\text{atm}} \mathbf{w}_{\text{atm}} = 0 \text{ on } \Gamma_u, & \text{tr} \gamma_{\text{atm}} \mathbf{w}_{\text{atm}} = 0 \text{ on } \Gamma_i, & \text{tr} \gamma_{\text{atm}} \mathbf{w}_{\text{ocn}} = 0 \text{ on } \Gamma_o \text{ and } \text{tr} \gamma_{\text{atm}} \mathbf{w}_{\text{ocn}} = 0 \text{ on } \Gamma_b.
\end{cases}$$

On the lateral boundaries, we assume that all variables are periodic, i.e., $\mathbf{v}_{\text{atm}}$ and $\pi_{\text{atm}}$ periodic on $\Gamma_l, \mathbf{v}_{\text{ocn}}$ and $\pi_{\text{ocn}}$ periodic on $\Gamma_{l,\text{ocn}}$ and $\mathbf{v}_{\text{ice}}$, $h$ and $a$ periodic on $\partial G$. The coupling condition (3.3) is assumed for the horizontal velocity of the ocean on the interface $\Gamma_o$, and it is already included in (3.4).

We also consider initial values $\mathbf{v}_{\text{atm}}(0) = \mathbf{v}_{\text{atm},0}$, $\mathbf{v}_{\text{ocn}}(0) = \mathbf{v}_{\text{ocn},0}$, $\mathbf{v}_{\text{ice}}(0) = \mathbf{v}_{\text{ice},0}$, $h(0) = h_0$ and $a(0) = a_0$.

The ground space is given by

$$X_0 := L^p(\mathbf{\Omega}_{\text{atm}}) \times L^p(\mathbf{\Omega}_{\text{ocn}}) \times L^q(G)^2 \times L^q(G) \times L^q(G).$$
Equipped with a product norm, $X_0$ becomes a Banach space. Moreover, we define

$$X_1 := \{(v_{\text{atm}}, v_{\text{ocean}}, v_{\text{ice}}, h, a) \in (H^{2,q}_{\text{per}}(\Omega_{\text{atm}}) \cap L^q_{\text{per}}(\Omega_{\text{atm}}))^2 \times L^q_{\text{per}}(\Omega_{\text{ocean}}) \times H^2_{\text{per}}(G)^2 \times H^2_{\text{per}}(G) : p_{\text{ocean}} \Delta v_{\text{ocean}} \in L^q_{\text{per}}(\Omega_{\text{ocean}}), \text{tr}_{\Gamma_a} v_{\text{ocean}} = v_{\text{ice}} \text{ on } \Gamma_v, \text{ and (3.5) is satisfied}\},$$

where $p_{\text{ocean}}$ denotes the hydrostatic Helmholtz projection associated to the ocean as made precise in Subsection 4.1, and $w_{\text{atm}}$ and $w_{\text{ocean}}$ can be recovered from $v_{\text{atm}}$ and $v_{\text{ocean}}$, respectively, see also Subsection 4.1.

For the initial data, we introduce the trace space $X_\gamma = (X_0, X_1)_{1-p,p}$, which is discussed in more detail in Section 8. For $p, q \in (1, \infty)$ with $1/p + 1/q < 1/2$, we have $u = (v_{\text{atm}}, v_{\text{ocean}}, v_{\text{ice}}, h, a) \in X_\gamma$ if and only if

$$v_{\text{atm}} \in H^2_{\text{per}}(\Omega_{\text{atm}})^2 \cap L^q_{\text{per}}(\Omega_{\text{atm}}), \quad v_{\text{ocean}} \in H^2_{\text{per}}(\Omega_{\text{ocean}})^2 \cap L^q_{\text{per}}(\Omega_{\text{ocean}}) \quad \text{and} \quad (v_{\text{ice}}, h, a) \in H^2_{\text{per}}(G)^4,$$

subject to (3.5) and $\text{tr}_{\Gamma_a} v_{\text{ocean}} = v_{\text{ice}}$, and so that $h_0 \in (\kappa_1, \kappa_2)$ and $a_0 \in (0, 1)$. Besides, for $T > 0$, the external forcing terms fulfill $f_{\text{atm}} \in L^p(0, T; L^q(\Omega_{\text{atm}})^2)$, $f_{\text{ocean}} \in L^p(0, T; L^q(\Omega_{\text{ocean}})^2)$ and $\nabla_{\text{per}} H \in L^p(0, T; L^q(G)^2)$.

Our first main result on the local existence and uniqueness of a strong solution to (3.4) reads as follows.

**Theorem 3.2.** Let $p, q \in (1, \infty)$ such that $1/p + 1/q < 1/2$, and assume that the initial data $u_0$ and the external forcing terms $f_{\text{atm}}$, $f_{\text{ocean}}$ and $\nabla_{\text{per}} H$ satisfy Assumption 3.1. Then there exist $T' = T'(u_0) \in (0, T]$ and $r = r(u_0)$ with $\overline{B}_{X_\gamma}(u_0, r) \subset V$ such that for each initial value $u_1 \in \overline{B}_{X_\gamma}(u_0, r)$, the coupled system (3.4), subject to the boundary conditions (3.5) and (3.6), admits a unique strong solution

$$u(\cdot, t) = (v_{\text{atm}}, v_{\text{ocean}}, v_{\text{ice}}, h, a)(\cdot, t) \in X_{T'} := W^{1,p}(0, T'; X_0) \cap L^p(0, T'; X_1).$$

We briefly discuss some further properties of the solution from Theorem 3.2.

**Corollary 3.3.**

(a) The solution class satisfies $W^{1,p}(0, T'; X_0) \cap L^p(0, T'; X_1) \to C([0, T'], X_\gamma)$.  

(b) Given an initial value $u_0 \in V$, there exists $C = C(u_0)$ such that for all $u_1, u_2 \in \overline{B}_{X_\gamma}(u_0, r)$, it holds that $\|u(\cdot, u_1) - u(\cdot, u_2)\|_{X_{T'}} \leq C\|u_1 - u_2\|_{X_{T'}}$.

(c) The solution regularizes instantly in time, i.e., $t_0(t) \in W^{1,p}(0, T'; X_0) \cap L^p(0, T'; X_1)$. In particular, $u \in C^1([b, T'); X_\gamma) \cap C^{1-\delta}([b, T'); X_\gamma)$ for every $b \in (0, T')$.

(d) The solution $u = u(u_0)$ exists on a maximal time interval $J(u_0) = [0, t_+(u_0))$, and the latter is characterized by the alternatives

(i) global existence, i.e., it holds that $t_+(u_0) = \infty$, or  
(ii) $\lim_{t \to t_+(u_0)} \text{dist}(u(t), \partial V) = 0$, or  
(iii) $\lim_{t \to t_+(u_0)} u(t)$ does not exist in $X_\gamma$.

For $h_\ast$ and $a_\ast$ constant in space and time, it readily follows that $(0, 0, 0, h_\ast, a_\ast)$ are equilibria to (3.4) provided the external forces $f_{\text{atm}}, f_{\text{ocean}}$ as well as $\nabla_{\text{per}} H$ vanish and the thermodynamic terms $S_h$ and $S_a$ are neglected. Setting $\tau_{\text{atm}} = 0$, we then verify the normal stability of the above equilibria in the trace space $X_\gamma$, resulting in the existence of a unique global strong solution for initial data close to the equilibrium $(0, 0, 0, h_\ast, a_\ast)$. This leads to the second main result of this article.

**Theorem 3.4.** For $h_\ast \in (\kappa_1, \kappa_2)$ and $a_\ast \in (0, 1)$ constant in space and time, $u_\ast = (0, 0, 0, h_\ast, a_\ast)$ is an equilibrium in $X_\gamma$. Moreover, there exists $r > 0$ such that the unique solution $u$ of (3.4) with $f_{\text{atm}} = f_{\text{ocean}} = g\nabla_{\text{per}} H = \tau_{\text{atm}} = 0$, $S_h = S_a = 0$ and $u_0$ fulfilling Assumption 3.1 as well as $\|u_0 - u_\ast\|_{X_\gamma} < r$ exists on $\mathbb{R}_+$ and converges at an exponential rate in $X_\gamma$ to some equilibrium $u_\infty$ of (3.4) as $t \to \infty$.

The following remarks on Theorem 3.2 and Theorem 3.4 are in order at this point.
Theorem 3.2. For simplicity of the presentation we did not introduce time weights \( \mu \in (1/p, 1] \) for the solution space to exploit the parabolic regularization. The corresponding space for the initial data then becomes \( X_{\gamma, \mu} = (X_0, X_1)_{\mu - 1/p, p} \).

Assuming that for \( p, q \in (1, \infty) \) such that \( 1/2 + 1/p + 1/q < \mu \leq 1 \), the initial data \( u_0 \) lie in an open set \( V_{\mu} \subset X_{\gamma, \mu} \) as in (3.7), and supposing that the external forcing terms satisfy the above assumptions, we obtain an analogous statement as in Theorem 3.2, but the solution \( u \) is contained in the class \( u \in \mathcal{W}_{\mu p}^{1,p}(0, T; X_0) \cap L^p_{\mu p}(0, T; X_1) \).  

Remark 3.5. (a) The statements of Theorem 3.2 and Theorem 3.4 remain valid when including Coriolis terms as in [9, Section 2]. They are omitted here for simplicity of the presentation.

(b) Let \( \text{Re}_\text{H, atm}, \text{Re}_\text{H, ocn}, \text{Re}_\text{z, atm} \) and \( \text{Re}_\text{z, ocn} \) denote the horizontal and vertical Reynolds numbers in the context of the atmosphere and the ocean. Instead of \( \Delta v_{\text{atm}} \) and \( \Delta v_{\text{ocn}} \), one can also consider \( 1/\text{Re}_\text{H, atm} \Delta v_{\text{atm}} + 1/\text{Re}_\text{z, atm} \partial_z^2 v_{\text{atm}} \) and \( 1/\text{Re}_\text{H, ocn} \Delta v_{\text{ocn}} + 1/\text{Re}_\text{z, ocn} \partial_z^2 v_{\text{ocn}} \) in (3.4). The statements of Theorem 3.2 and Theorem 3.4 remain valid in this case.

4. Rewriting the Coupled System as a Quasilinear Evolution Equation

The first step in the proof of Theorem 3.2 is to write the coupled system of PDEs (3.4) as a quasilinear evolution equation in the ground space \( X_0 \).

4.1. The hydrostatic Helmholtz projection and the hydrostatic Stokes operator.

For \( a, b \in \mathbb{R} \) with \( a < b \), we consider the cylindrical domain \( \Omega = G \times (a, b) \). The incompressible, isothermal primitive equations for the velocity \( u = (v, w) \) of the fluid and the surface pressure \( \pi \) are given by

\[
\begin{aligned}
\partial_t v - \Delta v + (u \cdot \nabla) v + \nabla H \pi &= f, & \text{on } \Omega \times [0, T], \\
\partial_t \pi &= 0, & \text{on } \Omega \times [0, T], \\
\text{div } u &= 0, & \text{on } \Omega \times [0, T], \\
v(0) &= u_0, & \text{on } \Omega.
\end{aligned}
\]

(4.1)

Here \( f \) is an external force term, and \( v = (v_1, v_2) \) denotes the horizontal velocity, whereas \( w \) represents the vertical velocity. In the sequel, the remaining surface pressure will be denoted by \( \pi_s \). On the upper and bottom parts of the boundary \( \partial \Omega \), \( v \) satisfies Dirichlet or Neumann conditions conditions on \( \Gamma_D \) and \( \Gamma_N \), respectively, while \( w \) is subject to Dirichlet boundary conditions. Both, \( v \) and \( w \), as well as the pressure \( \pi_s \), are assumed to be periodic on the lateral boundary \( \Gamma_l \).

In conjunction with the boundary conditions \( w = 0 \) on \( G \times \{a, b\} \), the incompressibility condition \( \text{div } u = 0 \) yields that the vertical component \( w \) can be expressed by \( w(\cdot, \cdot, z) = -\int_{a}^{z} \text{div}_H v(\cdot, \cdot, \xi) \, d\xi, \ z \in [a, b] \).

Denoting by \( \bar{\pi} \) the vertical average of \( \pi \), the fracturing part is given by \( \bar{\pi} := \pi - \bar{\pi} \). The vertical average commutes with horizontal or tangential derivation and with the horizontal Laplacian for sufficiently smooth vector fields \( v \), i.e., \( \nabla_H \bar{\pi} = \bar{\nabla H v} \) and \( \Delta_H \bar{\pi} = \bar{\Delta_H v} \) in particular, the incompressibility condition can be expressed as \( \text{div}_H \bar{\pi} = 0 \).

We now introduce the associated hydrostatic Helmholtz or Leray projection \( \mathcal{P} : L^q(\Omega)^2 \to L^q(\Omega) \). It is a bounded, linear projection which annihilates the pressure term, i.e., \( \mathcal{P} \nabla_H \pi = 0 \), and the space \( L^q(\Omega) \) can be decomposed into \( L^q(\Omega)^2 = L^q(\Omega) \oplus \{ \nabla_H \pi : \pi \in \bar{\mathcal{W}}^{1,q}(G) \} \) along the hydrostatic Helmholtz projection \( \mathcal{P} \). Furthermore, the Helmholtz projection commutes with the time derivative, i.e., \( \partial_t \mathcal{P} = \mathcal{P} \partial_t \). It is related to the two-dimensional classical Helmholtz projection \( \mathcal{P}_H \), see e.g. [27, Section 2] for further details on the latter one, by

\[
\mathcal{P} : L^q(\Omega)^2 \to L^q(\Omega), \quad \mathcal{P} \nu = \mathcal{P}_H \bar{\nu} + \bar{\nu}.
\]

We define the maximal hydrostatic Stokes operator \( A_m : D(A_m) \subset L^q(\Omega) \to L^q(\Omega) \) by \( A_m v := \mathcal{P} \Delta v \), where \( D(A_m) := \{ v \in L^q(\Omega) : A_m v \in L^q(\Omega) \} \). The hydrostatic Stokes operator with boundary conditions \( A_{b,c} : D(A_{b,c}) \subset L^q(\Omega) \to L^q(\Omega) \) is defined by \( A_{b,c} v := \mathcal{P} \Delta v \), with \( D(A_{b,c}) := H^{2,q}_{\text{per,b,c}}(\Omega)^2 \cap L^q(\Omega) \).
For $s \in (0, 2]$ and $\Gamma_D$ as well as $\Gamma_N$ representing the part of the boundary with Dirichlet or Neumann boundary conditions, respectively, we set

\[
H^{s,q}_{\text{per},b.c.}(\Omega) := \begin{cases} 
\{ v \in H^{s,q}_{\text{per}}(\Omega)^2 : v|_{\Gamma_D} = 0, \partial_z v|_{\Gamma_D} = 0 \}, & \text{for } 1 + \frac{1}{q} < s \leq 2, \\
\{ v \in H^{s,q}_{\text{per}}(\Omega)^2 : v|_{\Gamma} = 0 \}, & \text{for } \frac{1}{q} < s < 1 + \frac{1}{q}, \\
H^{s,q}_{\text{per}}(\Omega)^2, & \text{for } 0 < s < 1/q. 
\end{cases}
\]

Note that $\mathcal{D}(A_m) \subset H^{s,q}_{\text{per},b.c.}(\Omega)$ only for $s < 1/q$ but not for $s > 1/q$. As indicated above, $w$ can be recovered from $v$, so the bilinearity can be written as $(u \cdot \nabla)v = (v \cdot \nabla H)v + w(v) \cdot \partial_z v$. Applying the hydrostatic Helmholtz projection to the bilinearity, we obtain

\[
(4.2) \quad \text{F}(v, v') := \mathcal{P}((v \cdot \nabla H)v' + w(v) \cdot \partial_z v') = \mathcal{P}(v \cdot \nabla H)v' + \mathcal{P}(w(v)\partial_z v'), \quad \text{with } \text{F}(v) := \text{F}(v, v).
\]

4.2. Hibler’s sea ice operator.

Proceeding as in [9, Section 3], we introduce the map $S: \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ with

\[
S = \left( S_{ij}^{kl} \right) = \begin{pmatrix} 1 + \frac{1}{s} & 0 & 0 & 1 - \frac{1}{s} \\
0 & \frac{1}{s} & \frac{1}{s} & 0 \\
0 & \frac{1}{s} & \frac{1}{s} & 0 \\
1 - \frac{1}{s} & 0 & 0 & 1 + \frac{1}{s} \end{pmatrix}
\]

Identifying $\varepsilon \in \mathbb{R}^{2 \times 2}$ with the vector $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22})^T$, we find that the map $S$ corresponds to the positive semi-definite matrix

\[
S = \left( S_{ij}^{kl} \right) = \begin{pmatrix} 1 + \frac{1}{s} & 0 & 0 & 1 - \frac{1}{s} \\
0 & \frac{1}{s} & \frac{1}{s} & 0 \\
0 & \frac{1}{s} & \frac{1}{s} & 0 \\
1 - \frac{1}{s} & 0 & 0 & 1 + \frac{1}{s} \end{pmatrix}
\]

Setting $S_{\delta} = S_{\delta}(\varepsilon, P) := \frac{P}{\Delta_{\delta}(\varepsilon)} S_{i j}^{k l}(\varepsilon)$, we define Hibler’s operator by $A^H_{\text{ice}} := \frac{1}{\rho_{\text{ice}} h} \text{div}_H S_{\delta}(v_{\text{ice}})$ and recall

\[
(A^H_{\text{ice}})_{i} = -\sum_{j,k,l=1}^{2} \frac{1}{2 \rho_{\text{ice}} h} \Delta_{\delta}(\varepsilon) \left( S_{ij}^{kl} - \frac{1}{\Delta_{\delta}(\varepsilon)} (S_{i k}^{l j})(\varepsilon) \right) D_k D_l v_{\text{ice},j} + \frac{1}{2 \rho_{\text{ice}} h} \Delta_{\delta}(\varepsilon) \sum_{j=1}^{2} (\partial_j P)(S_{\delta})_{ij}
\]

from [9, Section 3], where $i = 1, 2$ and $D_m = -i \partial_m$. The principal part of $A^H$ has coefficients given by

\[
a_{ij}^{kl}(\varepsilon, h, a) := -\frac{\rho_{\text{ice}} h}{2 \rho_{\text{ice}} h} \frac{1}{\Delta_{\delta}(\varepsilon)} \left( S_{ij}^{kl} - \frac{1}{\Delta_{\delta}(\varepsilon)} (S_{i k}^{l j})(\varepsilon) \right).\]

For sufficiently smooth initial data $v_0 = (v_{\text{ice},0}, h_0, a_0)$, the linearization of Hibler’s operator is of the form

\[
[A^H(v_0)_{v\text{ice}}] = \sum_{j,k,l=1}^{2} a_{ij}^{kl}(\varepsilon, h_0, a_0) D_k D_l v_{\text{ice},j} + \frac{1}{2 \rho_{\text{ice}} h_0} \Delta_{\delta}(\varepsilon) \sum_{j=1}^{2} (\partial_j P(h_0, a_0))(S_{\delta})_{ij}.\]

The $L^2$-realization $A^H(v_0)$ of $A^H(v_0)$ on $G = (0, 1) \times (0, 1)$ is then defined by $[A^H(v_0)]_{v\text{ice}} := [A^H(v_0)]_{v\text{ice}}$ for $v_{\text{ice}} \in D(A^H(v_0)) = H^2_{\text{per}}(G)^2$.

Comparing this definition of Hibler’s operator to the one presented in [9, Section 3], we observe that the sign is changed and $1/\rho_{\text{ice}} h$ is included in the definition for the sake of consistency with the hydrostatic Stokes operator and to ease notation. We also introduce the lower-order terms

\[
B_h(h_0, a_0) := -\frac{\partial_h P(h_0, a_0)}{2 \rho_{\text{ice}} h_0} \nabla_H h \quad \text{and} \quad B_a(h_0, a_0) := -\frac{\partial_a P(h_0, a_0)}{2 \rho_{\text{ice}} h_0} \nabla_H a,
\]

originating from $\frac{1}{\rho_{\text{ice}}} \text{div}_H \frac{P}{h}$. Finally, we recall that $\Delta_H$ is the horizontal Laplacian on the square $G$ with periodic boundary conditions, and its domain is given by $D(\Delta_H) = H^2_{\text{per}}(G)$.
4.3. A new formulation of the coupled system.

We start the reformulation of (3.4) with the primitive equations of the atmosphere. To this end, we choose \(a = \kappa_2\) and \(b = h_{\text{atm}}\) and denote the atmospheric solenoidal \(L^2\)-space by \(L^2_{\text{per}}(\Omega_{\text{atm}})\). The atmospheric Helmholtz projection \(\mathcal{P}_{\text{atm}}\) and the atmospheric Stokes operator \(A_{\text{atm}}:\ D(A_{\text{atm}}) \subset L^2_{\text{per}}(\Omega_{\text{atm}}) \to L^2_{\text{per}}(\Omega_{\text{atm}})\) are defined as in Subsection 4.1, i. e.,

\[
\begin{align*}
A_{\text{atm}} v_{\text{atm}} &= \mathcal{P}_{\text{atm}} \Delta v_{\text{atm}}, \\
D(A_{\text{atm}}) &= \{ v_{\text{atm}} \in H^2_{\text{per}}(\Omega_{\text{atm}})^2 \cap L^2_{\text{per}}(\Omega_{\text{atm}}) : \partial_{\nu_{\text{atm}}} v_{\text{atm}} = 0 \text{ on } \Gamma_u \text{ and } \partial_{\nu_{\text{atm}}} v_{\text{atm}} = 0 \text{ on } \Gamma_i \}
\end{align*}
\]

in view of (3.5), and the bilinearity \(F_{\text{atm}}\) is defined via (4.2).

For the primitive equations of the ocean, we choose \(a = -h_{\text{ocn}}\) and \(b = 0\). The oceanic solenoidal \(L^2\)-space is denoted by \(L^2_{\text{per}}(\Omega_{\text{ocn}})\), and we define the oceanic Helmholtz projection \(\mathcal{P}_{\text{ocn}}\) as well as the maximal oceanic Stokes operator \(A_{\text{ocn}}:\ D(A_{\text{ocn}}) \subset L^2_{\text{per}}(\Omega_{\text{ocn}}) \to L^2_{\text{per}}(\Omega_{\text{ocn}})\) as in Subsection 4.1, so

\[
A_{\text{ocn}} v_{\text{ocn}} = \mathcal{P}_{\text{ocn}} \Delta v_{\text{ocn}}, \quad D(A_{\text{ocn}}) = \{ v_{\text{ocn}} \in L^2_{\text{per}}(\Omega_{\text{ocn}}) : \mathcal{P}_{\text{ocn}} \Delta v_{\text{ocn}} \in L^2_{\text{per}}(\Omega_{\text{ocn}}) \},
\]

and the bilinearity \(F_{\text{ocn}}\) via (4.2). The oceanic hydrostatic Stokes operator with homogeneous boundary conditions \(A^0_{\text{ocn}}:\ D(A^0_{\text{ocn}}) \subset L^2_{\text{per}}(\Omega_{\text{ocn}}) \to L^2_{\text{per}}(\Omega_{\text{ocn}})\) is given by

\[
\begin{align*}
A^0_{\text{ocn}} v_{\text{ocn}} &= \mathcal{P}_{\text{ocn}} \Delta v_{\text{ocn}}, \\
D(A^0_{\text{ocn}}) &= \{ v_{\text{ocn}} \in H^2_{\text{per}}(\Omega_{\text{ocn}})^2 \cap L^2_{\text{per}}(\Omega_{\text{ocn}}) : \text{tr}_{\nu_{\text{atm}}} v_{\text{ocn}} = 0 \text{ on } \Gamma_b \text{ and } \text{tr}_{\nu_{\text{atm}}} v_{\text{ocn}} = 0 \text{ on } \Gamma_o \}
\end{align*}
\]

with regard to (3.5) and (3.3). Setting \(C_{\text{ice},i}(h) := \mu_{\text{ice}}/\rho_{\text{ice}} h\) so that \((1/\rho_{\text{ice}} h) \partial_{\nu_{\text{atm}}} v_{\text{ocn}}\) for \((v_{\text{ice},0}, h_{\text{0}}, a_{\text{0}}) \in V_{\text{ice}}\), we define \(A(v_{\text{ice},0}, h_{\text{0}}, a_{\text{0}}) : X_1 \subset \mathcal{X}_0 \to \mathcal{X}_0\) by

\[
A(v_{\text{ice},0}, h_{\text{0}}, a_{\text{0}}) := \begin{pmatrix}
A_{\text{atm}} & 0 & 0 & 0 & 0 \\
0 & A^m_{\text{ocn}} & 0 & 0 & 0 \\
0 & 0 & -C_{\text{ice},i}(h_{\text{0}}) \partial_{\nu_{\text{atm}}} & A^H(v_{\text{ice},0}, h_{\text{0}}, a_{\text{0}}) & B_{\text{h}}(h_{\text{0}}, a_{\text{0}}) \\
0 & 0 & 0 & d_{\text{h}} \Delta_{\text{H}} & 0 \\
0 & 0 & 0 & 0 & d_{\text{a}} \Delta_{\text{H}}
\end{pmatrix},
\]

(4.4)

\(X_1 := D(A) := \{ (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a) \in D(A_{\text{atm}}) \times D(A^0_{\text{ocn}}) \times D(A^H) \times D(\Delta_{\text{H}}) : \text{tr}_{\nu_{\text{atm}}} v_{\text{ocn}} = 0 \text{ on } \Gamma_b \text{ and } \text{tr}_{\nu_{\text{atm}}} v_{\text{ocn}} = v_{\text{ice}} \text{ on } \Gamma_o \}.\)

Note that this operator matrix has non-diagonal domain. Further, we define the nonlinearity by

\[
F(v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a) := \begin{pmatrix}
\mathcal{P}_{\text{atm}}((v_{\text{atm}} \cdot \nabla H) v_{\text{atm}} + w(v_{\text{atm}}) \cdot \partial z v_{\text{atm}}) \\
\mathcal{P}_{\text{ocn}}((v_{\text{ocn}} \cdot \nabla H) v_{\text{ocn}} + w(v_{\text{ocn}}) \cdot \partial z v_{\text{ocn}}) \\
(v_{\text{ice}} \cdot \nabla H) v_{\text{ice}} - \frac{1}{\rho_{\text{ice}}} h_{\text{atm}} \\
\text{div}_{\text{H}}(v_{\text{ice}} h) - S_{\text{h}} \\
\text{div}_{\text{a}}(v_{\text{ice}} a) - S_{\text{a}}
\end{pmatrix},
\]

(4.5)

with \(S_{\text{h}}\) and \(S_{\text{a}}\) as in (5.2). Besides, the external force is given by \(f := (P_{\text{atm}} f_{\text{atm}}, P_{\text{ocn}} f_{\text{ocn}}, -g \nabla H, 0, 0)^{\top}\).

The coupled system (3.4) subject to the boundary conditions (3.5) and (3.6) can now be reformulated as an abstract quasilinear Cauchy problem

\[
\begin{align*}
\partial_t u - A(u) u + F(u) &= f, \quad t \in [0, T], \\
u(0) &= u_0,
\end{align*}
\]

(4.6)

on the ground space \(\mathcal{X}_0\).

5. Hibler’s Sea Ice Equations on the Square with Periodic Boundary Conditions

In the first part of this section, we show that Hibler’s operator admits a bounded \(H^\infty\)-calculus, while the second part is concerned with suitable Lipschitz estimates.
5.1. Bounded $\mathcal{H}^\infty$-calculus of Hibler’s operator.

The embedding below is classical, see e. g. [49, Theorem 4.6.1]. Let $p, q \in (1, \infty)$ with $1/p + 1/q < 1/2$. Then

\begin{equation}
B_{qp}^{2-2/n}(G) \hookrightarrow C^1(\overline{G}).
\end{equation}

Making use of (5.1) and proceeding as in [9, Section 4], we obtain the ellipticity of $-A^H(v_0)$. For the notion of strong ellipticity and parameter ellipticity, we refer e. g. to [13, Definition 5.1].

**Lemma 5.1.** Let $p, q \in (1, \infty)$ such that $1/p + 1/q < 1/2$, and let $u_0 = (v_{\text{ice},0}, h_0, a_0) \in V^{\text{ice}}$. Then the principal part of the negative Hibler operator $-A^H(v_0)$ is strongly elliptic and also parameter-elliptic of angle $\varphi_{-A^H(v_0)} = 0$.

We will now derive the bounded $\mathcal{H}^\infty$-calculus for the linearized $L^2$-realization $-A^H(v_0)$.

**Proposition 5.2.** Let $p, q, r, s \in (1, \infty)$ such that $1/p + 1/q < 1/2$, let $v_0 = (v_{\text{ice},0}, h_0, a_0) \in V^{\text{ice}}$, and let $A^H(v_0)$ denote the $L^2$-realization of the linearized Hibler operator. Then there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, $-A^H(v_0) + \omega$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^r(G)^2$ with angle $\Phi^\infty_{-A^H(v_0)+\omega} < \pi/2$.

**Proof.** Identifying functions on the square $G = (0, 1) \times (0, 1)$ endowed with periodic boundary conditions with functions on the torus $T$, exploiting Lemma 5.1 for the ellipticity properties of the negative Hibler operator $-A^H(v_0)$ and using (5.1) in conjunction with the fact that the coefficients of $A^H(v_0)$ depend smoothly on $\nabla Hv_{\text{ice},0}, \nabla Hh_0, \nabla Ha_0, h_0$ and $a_0$, we conclude from [15, Theorem 7.1] that there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, the operator $-A^H(v_0) + \omega$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^r(T)^2 = L^r(G)^2$ with angle $\Phi^\infty_{-A^H(v_0)+\omega} < \pi/2$. \hfill \Box

The following result is a direct consequence of Proposition 5.2, because the latter proposition implies that $-A^H(v_0) + \omega$ has bounded imaginary powers on $L^q(G)^2$, see also the relation (8.7).

**Corollary 5.3.** The linearized operators $B_h$ and $B_a$ are relatively $(-A^H + \omega)^{1/2}$-bounded.

5.2. Estimates of the nonlinearities.

We now show that the nonlinear terms in the sea ice equations satisfy certain Lipschitz estimates. To this end, we set

\[ A^{\text{ice}}_\omega(v_0) = \begin{pmatrix}
0 & -C_{a,i}(h_0) \partial_{v_{\text{ice}}} & A^H(v_0) & B_h(h_0, a_0) & B_a(h_0, a_0) \\
0 & 0 & 0 & d_h \Delta_H & 0 \\
0 & 0 & 0 & 0 & d_a \Delta_H
\end{pmatrix} \]

as well as

\[ F^{\text{ice}}(v) = \begin{pmatrix}
(v_{\text{ice}} \cdot \nabla H)v_{\text{ice}} - \frac{1}{\rho_{\text{ice}}} r_{\text{atm}}(v_{\text{atm}}) \\
\text{div}_H(v_{\text{ice}}h) - S_h \\
\text{div}_H(v_{\text{ice}}a) - S_a
\end{pmatrix}. \]

Further, with regard to Proposition 5.2, it is natural to introduce

\[ A^{\text{ice}}_\omega(v_0) = \begin{pmatrix}
0 & -C_{a,i}(h_0) \partial_{v_{\text{ice}}} & A^H(v_0) - \omega & B_h(h_0, a_0) & B_a(h_0, a_0) \\
0 & 0 & 0 & d_h \Delta_H & 0 \\
0 & 0 & 0 & 0 & d_a \Delta_H
\end{pmatrix} \]

and $F^{\text{ice}}(v) = F^{\text{ice}}(v) + (\omega v_{\text{ice}}, 0, 0)^T$. The thermodynamic terms $S_h$ and $S_a$ are defined by

\[ S_h = f_1(h/a)a + (1 - a)f_1(0) \quad \text{and} \quad S_a = \begin{cases}
\frac{A(0)}{\kappa_1}(1 - a), & \text{if } f_1(0) > 0, \\
0, & \text{if } f_1(0) < 0,
\end{cases} \quad + \begin{cases}
0, & \text{if } S_h > 0, \\
\frac{1}{\nu} S_h, & \text{if } S_h < 0.
\end{cases} \]
where \( f_2 \in C^1([0, \infty); \mathbb{R}) \) is an arbitrary function representing the ice growth rate, see e.g. the one considered by Hibler [22]. Moreover, for \( i \in \{0, 1, \theta, \gamma, \beta\} \), we introduce \( \mathcal{Y}_i = \mathcal{Y}_{i}^{\text{atm}} \times \mathcal{Y}_{i}^{\text{ocn}} \times \mathcal{Y}_{i}^{\text{ice}} \) given by

\[
\begin{align*}
\mathcal{Y}_0 &= \mathcal{L}_p^q(\Omega_{\text{atm}}) \times \mathcal{L}_p^q(\Omega_{\text{ocn}}) \times \mathcal{L}_p^q(\mathbb{G}), \\
\mathcal{Y}_1 &= H^{2,q}_{\text{per}}(\Omega_{\text{atm}})^2 \cap \mathcal{L}_p^q(\Omega_{\text{atm}}) \times H^{2,q}_{\text{per}}(\Omega_{\text{ocn}})^2 \cap \mathcal{L}_p^q(\Omega_{\text{ocn}}) \times H^{2,q}_{\text{per}}(\mathbb{G}), \\
\mathcal{Y}_\gamma &= \mathcal{Y}_{\theta, \gamma}^{\text{atm}} \times \mathcal{Y}_{\theta, \gamma}^{\text{ocn}} \times \mathcal{Y}_{\theta, \gamma}^{\text{ice}} \text{ for } \theta = 1 - \frac{1}{p}, \text{ and} \\
\mathcal{Y}_\beta &= H^{2,q}_{\text{per}, b, c}(\Omega_{\text{atm}})^2 \cap \mathcal{L}_p^q(\Omega_{\text{atm}}) \times H^{2,q}_{\text{per}}(\Omega_{\text{ocn}})^2 \cap \mathcal{L}_p^q(\Omega_{\text{ocn}}) \times H^{2,q}_{\text{per}}(\mathbb{G}).
\end{align*}
\]

In addition, for \( u \in \mathcal{Y}_\gamma \) and \( r > 0 \), we use \( \mathcal{B}_{\mathcal{Y}_\gamma}(u, r) \) to denote the closed ball of center \( u \) with radius \( r \) in \( \mathcal{Y}_\gamma \). Let further \( W = W^{\text{atm}} \times W^{\text{ocn}} \times W^{\text{ice}} \subset \mathcal{Y}_\gamma \) be an open subset such that for \( u_0 \in W \), the properties from (3.7) are satisfied. We then get the following Lipschitz estimates.

**Lemma 5.4.** Let \( p, q \in (1, \infty) \) such that \( \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \), and let \( u_0 = \left(v_{\text{atm}, 0}, v_{\text{ocn}, 0}, v_0\right) \in W \). Then there exists \( r_0 > 0 \) and a constant \( L > 0 \) such that \( \mathcal{B}_{\mathcal{Y}_\gamma}(u_0, r_0) \subset W \), and for all \( v_1, v_2 \in \mathcal{Y}^{\text{ice}} \) with \( u_1, u_2 \in \mathcal{B}_{\mathcal{Y}_\gamma}(u_0, r_0) \) and \( u \in \mathcal{Y}_1 \), it holds that \( \|\mathcal{A}_{\text{ice}}(v_1)u - \mathcal{A}_{\text{ice}}^{\text{loc}}(v_2)u\|_{\mathcal{Y}^{\text{ice}}} \leq L\|v_1 - v_2\|_{\mathcal{Y}^{\text{ice}}} \|\|u\|_{\mathcal{Y}_1} \) as well as \( \|F_{\text{per}}^{\text{loc}}(v_1) - F_{\text{per}}^{\text{loc}}(v_2)\|_{\mathcal{Y}^{\text{ice}}} = L\|v_1 - v_2\|_{\mathcal{Y}^{\text{ice}}} \).

**Proof.** First, choose \( r_0 > 0 \) sufficiently small such that \( \mathcal{B}_{\mathcal{Y}_\gamma}(u_0, r_0) \subset W \) thanks to \( W \) being open, and consider \( v_1, v_2 \) with \( u_1, u_2 \in \mathcal{B}_{\mathcal{Y}_\gamma}(u_0, r_0) \) as well as

\[
u = (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, \gamma, \sigma) \in \mathcal{Y}_1 = H^{2,q}_{\text{per}}(\Omega_{\text{atm}})^2 \cap \mathcal{L}_p^q(\Omega_{\text{atm}}) \times H^{2,q}_{\text{per}}(\Omega_{\text{ocn}})^2 \cap \mathcal{L}_p^q(\Omega_{\text{ocn}}) \times H^{2,q}_{\text{per}}(\mathbb{G}).
\]

As the spaces with horizontal periodicity embed continuously into the ones without, we omit the corresponding subscript in the remainder of the proof for convenience. For most terms, we refer to [9, Lemma 6.2], because the procedure from this lemma carries over to the present situation. Since the velocity of the atmospheric wind now acts as a variable, we verify the respective Lipschitz estimate separately. To this end, we first observe that for suitable \( g_1, g_2 \), it holds that

\[
\|g_1\|_{L^2(G)} - \|g_2\|_{L^2(G)} \leq \|g_1 - g_2\|_{L^2(G)} + \|g_1\|_{L^2(G)}\|g_2\|_{L^2(G)} + \|g_2\|_{L^2(G)}\|g_1\|_{L^2(G)}.
\]

Thanks to \( \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \), there is \( \eta > 0 \) with \( 2 - \frac{2}{p} - \eta - \frac{2}{q} \geq 1 - \frac{2}{2q} \), so [49, Theorem 4.6.1] yields that

\[
B^{2-2/p-\eta}(\mathbb{G}) \Rightarrow B^{2-2/q-\eta}(\mathbb{G}) \Rightarrow H^{1-2q}(\mathbb{G}), \quad B^{2-2/p-\eta}(\Omega_{\text{atm}}) \Rightarrow B^{2-2/q-\eta}(\Omega_{\text{atm}}) \Rightarrow W^{1/4+\gamma/2, 2q}(\Omega_{\text{atm}}).
\]

Thus, using the latter embeddings together with \( v_1 \in W^{\text{ice}} \) and continuity of \( \text{tr}_{\Gamma} \) as an operator from \( W^{1/4+\gamma/2, 2q}(\Omega_{\text{atm}}) \) to \( L^2(G) \), we obtain

\[
\begin{align*}
\left\| \frac{\rho_{\text{atm}} C_{\text{atm}}}{\rho_{\text{ice}} h_1} [\text{tr}_{\Gamma, v_{\text{atm}, 1}}, R_{\text{atm}} \text{tr}_{\Gamma, v_{\text{atm}, 1}} - |\text{tr}_{\Gamma, v_{\text{atm}, 2}}, R_{\text{atm}} \text{tr}_{\Gamma, v_{\text{atm}, 2}}] \right\|_{L^2(G)} \\
\leq C \left( \|v_{\text{atm}, 1}\|_{B^{2-2/q}(\Omega_{\text{atm}})} + \|v_{\text{atm}, 2}\|_{B^{2-2/q}(\Omega_{\text{atm}})} \right) \left( \|v_{\text{atm}, 1} - v_{\text{atm}, 2}\|_{B^{2-2/q}(\Omega_{\text{atm}})} \right) \leq C(r_0 + \|u_0\|_{\mathcal{Y}_\gamma})\|u_1 - u_2\|_{\mathcal{Y}_\gamma}.
\end{align*}
\]

Similarly, we get

\[
\left\| \frac{1}{h_1} - \frac{1}{h_2} \right\|_{L^\infty(G)} \cdot \|\text{tr}_{\Gamma, v_{\text{atm}, 1}}, R_{\text{atm}} \text{tr}_{\Gamma, v_{\text{atm}, 1}} \|_{L^2(G)} \leq C(r_0 + \|u_0\|_{\mathcal{Y}_\gamma})\|u_1 - u_2\|_{\mathcal{Y}_\gamma},
\]

where we employed (5.4). Therefore, a concatenation of the previous two estimates results in

\[
\left\| \frac{1}{\rho_{\text{ice}} h_1} \tau_{\text{atm}}(v_{\text{atm}, 1}) - \frac{1}{\rho_{\text{ice}} h_2} \tau_{\text{atm}}(v_{\text{atm}, 2}) \right\|_{L^2(G)} \leq C \cdot \|u_1 - u_2\|_{\mathcal{Y}_\gamma}
\]

for a suitable constant \( C \) depending on \( r_0 \) and \( \|u_0\|_{\mathcal{Y}_\gamma} \). Recalling the shape of \( \mathcal{Y}_\beta \) from (5.3), we conclude the Lipschitz estimate of \( \mathcal{B}_{\mathcal{Y}_\gamma}^{\text{ice}} \).

Concerning \( \mathcal{A}_{\text{ice}} \), we refer to [9, Lemma 6.2] except for \( C_{\text{a,3}}(h)\partial r_{\text{a}} \) which is new in comparison to the considerations in [9]. The assumption \( \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \) also implies continuity of \( \partial r_{\text{a}} \) from \( H^{2,\beta}(\Omega_{\text{ocn}}) \) to...
L^q(G). A similar argument as above for the atmospheric wind terms exhibits that
\[ \|C_{0,1}(h_1)\partial_{\nu_{\Omega}} v_{ocn} - C_{0,1}(h_2)\partial_{\nu_{\Omega}} v_{ocn}\|_{L^q(\Omega)} \leq \|C_{0,1}(h_1) - C_{0,1}(h_2)\|_{L^\infty(\Omega)} \|\partial_{\nu_{\Omega}} v_{ocn}\|_{L^q(\Omega)} \]
\[ \leq C\|h_1 - h_2\|_{\|v_{ocn}\|_{H^{2,q}(\Omega_{ocn})}} \leq C\|v_1 - v_2\|_{\|v\|_{\Omega_{ocn}}} \cdot \Box \]

6. Primitive Equations on Cylindrical Domains

In this section, we collect properties of the primitive equations \((4.1)\) on cylindrical domains \(\Omega := G \times (a, b)\), where \(G = (0, 1) \times (0, 1)\) and \(-\infty < a < b < \infty\). Apart from recalling that the hydrostatic Stokes operator admits a bounded \(\mathcal{H}^\infty\)-calculus, we provide estimates of the bilinear term appearing in the primitive equations tailored to our setting. Throughout this section, we denote by \(v = (v_1, v_2)\) the principal variable corresponding to the horizontal velocity.

6.1. Bounded \(\mathcal{H}^\infty\)-calculus of the hydrostatic Stokes operator.

We start by recalling the bounded \(\mathcal{H}^\infty\)-calculus of the hydrostatic Stokes operator from Subsection 4.1.

Proposition 6.1 (Theorem 3.1 in [16]). Let \(q \in (1, \infty)\) and \(\mu \geq 0\). Then \(-\Lambda_{atm} + \mu\) has a bounded \(\mathcal{H}^\infty\)-calculus on \(L^q(\Omega_{atm})\) with \(\Phi_{\mathcal{H}^\infty,atm} = 0\), and \(-\Lambda_{ocn} + \mu\) has a bounded \(\mathcal{H}^\infty\)-calculus on \(L^q(\Omega_{ocn})\) with \(\Phi_{\mathcal{H}^\infty,ocn} = 0\), and it holds that \(0 \in \rho(-\Lambda_{ocn})\).

Corollary 6.2. Let \(p, q \in (1, \infty)\) and \(\mu \geq 0\). Then

(a) the operator \(-\Lambda_{atm} + \mu\) admits a bounded \(\mathcal{H}^\infty\)-calculus on \(L^q(\Omega_{atm})\) with \(\Phi_{\mathcal{H}^\infty,atm} = 0\), and
(b) the operator \(-\Lambda_{ocn} + \mu\) admits a bounded \(\mathcal{H}^\infty\)-calculus on \(L^q(\Omega_{ocn})\) with \(\Phi_{\mathcal{H}^\infty,ocn} = 0\), and it holds that \(0 \in \rho(-\Lambda_{ocn})\).

6.2. Estimates of the bilinearity.

We recall the bilinearity \(F\) from \((4.2)\). As a preparation, we briefly introduce anisotropic function spaces, see also [26, Section 5]. For \(s, t \geq 0\) and \(1 \leq p, q \leq \infty\), consider \(W^r,s_{xy}W^{s,p}_{xy} := W^r,s((a, b); W^{s,p}(G))\) equipped with the norms \(\|v\|_{W^r,s_{xy}W^{s,p}_{xy}} := \|v(\cdot, z)\|_{W^r,s_{xy}(a, b)}\) so they become Banach spaces. The same remains valid when considering Bessel potential spaces instead of Sobolev spaces. First, employing Hölder’s inequality independently with respect to \(z\) and \((x, y)\), we derive that for \(p, r, q\) and \(q_1, q_2\) such that \(1/p_1 + 1/p_2 = 1/p\) and \(1/q_1 + 1/q_2 = 1/q\), it holds that
\[
\|f\|_{L^q_{xy} L^p_{zxy}} \leq \|f\|_{L^{q_1}_{zxy} L^{p_1}_{xy}} \|g\|_{L^{q_2}_{zxy} L^{p_2}_{xy}}. \tag{6.1}
\]

Moreover, we will also use embedding relations separately in \(z\) and \((x, y)\), so we observe that
\[
W^r,s_{xy}W^{s,p}_{xy} \hookrightarrow W^r,s_{zxy}W^{s,p}_{zxy} \text{ for } W^r,s((a, b) \hookrightarrow W^r,s_{xy}(a, b), \text{ and }
W^r,s_{xy}W^{s,p}_{xy} \hookrightarrow W^r,s_{zxy}W^{s,p}_{zxy} \text{ for } W^r,s_{xy}(G) \hookrightarrow W^r,s_{zxy}(G).
\]

In addition, we remark that \(W^{r+s,q}(\Omega) \subset W^r,s_{xy}W^{s,q}_{xy}\) is valid provided \(p = q\), and we point out that these relations are also true for Bessel potential spaces.

Analogously, for \(s, t \geq 0\) and \(1 \leq p_0, p_1, q_0, q_1 \leq \infty\), we set \(B^r_{p_0, p_1, xy} := B^r_{p_0, p_1, xy}((a, b); B^s_{q_0, q_1, xy}(G))\) and endow these spaces with the corresponding norms. For \(s, t \geq 0\), the above identity in the case of Sobolev and Bessel potential spaces remains valid, i. e., it holds that \(B^r_{2s, 2s, xy} \subset B^r_{2s, 2s, xy}\).

The next result provides analogous estimates as in [26, Lemma 5.1] or [17, Lemma 6.1] for our setting.

Proposition 6.3. Let \(p, q \in (1, \infty)\) such that \(1/p + 1/q < 1/2\), and consider the bilinear map \(F\) as defined in \((4.2)\). Then for \(F : B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega) \times B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega) \to L^q_{xy}(\Omega)\), we have the following:

(a) For \(v \in B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega)\), there exists a constant \(C > 0\), depending only on \(\Omega\) and \(q\), such that
\[
\|F(v, v)\|_{L^q_{xy}(\Omega)} \leq C\|v\|_{B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega)}^2.
\]
(b) There exists a constant \(C > 0\) such that for \(v, v' \in B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega)\), we have
\[
\|F(v) - F(v')\|_{L^q_{xy}(\Omega)} \leq C(\|v\|_{B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega)} + \|v'\|_{B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega)}) \cdot \|v - v'\|_{B^{2-2/p}_{2p,2p}(\Omega^2) \cap L^q_{xy}(\Omega)}.
\]
Proof: Since $(v \cdot \nabla_H) v' + w(v) \cdot \partial_z v'$ is bilinear, we observe that assertion (b) can be shown in a similar way as (a). More precisely, we obtain

$$\|F(v) - F(v')\|_{L^p(\Omega)} \leq \|F(v, v - v')\|_{L^p(\Omega)} + \|F(v - v', v')\|_{L^p(\Omega)}$$

$$\leq C \left( \|v\|_{B^{0,2/p}_q(\Omega) \cap L^p(\Omega)} + \|v'\|_{B^{0,2/p}_q(\Omega) \cap L^p(\Omega)} \right) \cdot \|v - v'\|_{B^{0,2/p}_q(\Omega) \cap L^p(\Omega)},$$

so it suffices to prove (a). As $P : L^q(\Omega)^2 \to L^p(\Omega)$ is bounded and $B^{0,2/p}_q(\Omega) \cap L^q(\Omega) \subset B^{0,2/p}_q(\Omega)^2$ holds true, it is sufficient to bound the $L^p(\Omega)$-norm of $(v \cdot \nabla_H)v$ and $w(v) \cdot \partial_z v$ separately by $C\|v\|_{B^{0,2/p}_q(\Omega)}$ for some $C > 0$. Next, we deduce from $1/p + 1/q < 1/2$ the existence of $\eta > 0$ such that $2 - 2/p - \eta - 3/2q \geq -3/2q$ and $2 - 2/p - \eta - 3/2q \geq 1 - 2/q$. It then follows by [49, Theorem 4.6.1] that

$$B^{0,2/p}_q(\Omega) \hookrightarrow B^{0,2-q\eta}_q(\Omega) \hookrightarrow L^q(\Omega)$$

and $B^{0,2-q\eta}_q(\Omega) \hookrightarrow L^{3q/(2\eta)}(\Omega)$ and $B^{0,2-q\eta}_q(\Omega) \hookrightarrow B^{0,2-q\eta}_q(\Omega) \hookrightarrow H^{3q/(2\eta)}(\Omega) \hookrightarrow W^{1,3q/(2\eta)}(\Omega),$ where we used that $H^{3q/(2\eta)}(\Omega) \hookrightarrow W^{1,3q/(2\eta)}(\Omega)$ by virtue of $q \geq 2$, see e. g. [49, Chapter 2]. Combining the embeddings from (6.2) with Hölder’s inequality, we obtain

$$\|(v \cdot \nabla H)v\|_{L^p(\Omega)} \leq C\|v\|_{L^{3q/(2\eta)}(\Omega)} \|v\|_{W^{1,3q/(2\eta)}(\Omega)} \leq C\|v\|_{B^{0,2/p}_q(\Omega)}^2,$$

so the proof for the first addend of the bilinearity is complete. For the second one, we use (6.1) for

$$\|w\|_{L^p(\Omega)} \leq \|w\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \|\partial_z v\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)},$$

and we now find estimates for $\|w\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)}$ and $\|\partial_z v\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)}$ separately. Exploiting $W^{1,q}(a, b) \hookrightarrow L^\infty(a, b)$ and Poincaré’s inequality, the embedding

$$B^{0,2-q\eta}_q(\Omega) \hookrightarrow H^{1,2q}(\Omega) \hookrightarrow W^{1,2q}(\Omega), \quad \eta > 0,$$

where as well as the above observations concerning embeddings of anisotropic Sobolev spaces, Bessel potential spaces and Besov spaces, and using the assumption that $\text{div}_H v + \partial_z w = 0$, we infer that

$$\|w\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|w\|_{W^{1,2q}_q} \|\partial_z w\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|\partial_z w\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|\text{div}_H v\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|w\|_{B^{0,2/p}_q(\Omega)}^2.$$

The condition on $p$ and $q$ yields $1 - 2/p - 2\eta - 2q \geq -2/2q$ for some $\eta > 0$, so [49, Theorem 4.6.1] implies

$$B^{0,2-2q\eta}_q(\Omega) \hookrightarrow B^{0,2-2q\eta}_q(\Omega) \hookrightarrow L^{2q}(\Omega).$$

Using the above remarks on the relations of anisotropic Sobolev spaces again and invoking the last embedding, we obtain

$$\|\partial_z v\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|w\|_{W^{1,2q}_q} \|\text{div}_H v\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|w\|_{B^{0,2/p}_q(\Omega)}^2 \|\partial_z v\|_{L^{q/(2\eta)}_q \cap L^p(\Omega)} \leq C\|v\|_{B^{0,2/p}_q(\Omega)}^2.$$
7. The Stationary Hydrostatic Stokes Problem

This section is dedicated to the analysis of the stationary hydrostatic Stokes problem with inhomogeneous boundary conditions. The stationary hydrostatic Stokes problem corresponds to the coupling condition imposed in (3.3) as well as the boundary condition for \( v_{\text{ocn}} \) on the lower boundary as made precise in (3.5), and it is given by

\[
\begin{align*}
A_m v_{\text{ocn}} & = 0, \quad \text{on } \Omega_{\text{ocn}}, \\
\text{tr}_n v_{\text{ocn}} & = \varphi, \quad \text{on } \Gamma_0, \\
\text{tr}_b v_{\text{ocn}} & = 0, \quad \text{on } \Gamma_b,
\end{align*}
\]

(7.1)

on \( L^2(\Omega_{\text{ocn}}) \) with \( \varphi \in L^q(G)^2 \). We denote the solution operator of (7.1) by \( L_0 : L^q(\Gamma_0)^2 \to L^2(\Omega_{\text{ocn}}) \) and call it the hydrostatic Dirichlet operator. In the main part of this section, we show that \( L_0 \) is well-defined and bounded. This operator and its properties are essential for the decoupling approach in Section 8.

7.1. The hydrostatic Dirichlet operator.

We start by solving (7.1) for smooth functions on the boundary. We need the following auxiliary result which guarantees the existence of an extension with vertical average 0. To this end, we construct a smooth solution to the associated extension problem

\[
\begin{align*}
\overline{g} & = 0, \quad \text{on } \Omega_{\text{ocn}}, \\
\text{tr}_n g & = \varphi, \quad \text{on } \Gamma_0, \\
\text{tr}_b g & = 0, \quad \text{on } \Gamma_b.
\end{align*}
\]

(7.2)

The auxiliary result reads as follows.

**Lemma 7.1.** Let \( \varphi \in C^\infty_{\text{per}}(G)^2 \). Then there exists a smooth solution \( g \in C^\infty_{\text{per}}(\Omega_{\text{ocn}})^2 \) of (7.2).

**Proof.** Note that \( \Omega_{\text{ocn}} = G \times (-h_{\text{ocn}}, 0) \). The approach for \( g \) is a splitting into the horizontal and the vertical part \( g(x_H, z) = r(z) \cdot \varphi(x_H) \). Therefore, the problem (7.2) reduces to the construction of a smooth solution \( r \) to the one-dimensional extension problem with mean value 0 given by

\[
\begin{align*}
\overline{r} & = 0, \quad \text{on } (-h_{\text{ocn}}, 0), \\
\tr_n r & = 1, \\
\tr_b r & = 0.
\end{align*}
\]

(7.3)

A solution of (7.3) is given by \( r(z) = \frac{3}{h_{\text{ocn}}} z^2 + \frac{4}{h_{\text{ocn}}} z + 1 \), and it is smooth, i. e., \( g = r \cdot \varphi \in C^\infty_{\text{per}}(\Omega_{\text{ocn}})^2 \). \( \square \)

**Remark 7.2.** (a) In general, the solutions of (7.2) are not unique.

(b) Solving (7.2) guarantees in particular that \( \text{div}_H \overline{g} = 0 \) for all \( \varphi \in C^\infty_{\text{per}}(G)^2 \). Functions of the shape \( g(x_H, z) = r(z) \cdot \varphi(x_H) \) satisfy \( \text{div}_H \overline{g} = 0 \) for all \( \varphi \in C^\infty_{\text{per}}(G)^2 \) if and only if \( \overline{g} \) vanishes.

The regularity and average 0 of the solution \( g \) of (7.2) ensure \( g \in D(A_{m}^{\text{ocn}}) \), so \( v_g := v_{\text{ocn}} - g \in D(A_{m}^{\text{ocn}}) \) satisfies

\[
\begin{align*}
A_m v_g & = f, \quad \text{on } \Omega_{\text{ocn}}, \\
\tr_n v_g & = 0, \quad \text{on } \Gamma_0, \\
\tr_b v_g & = 0, \quad \text{on } \Gamma_b,
\end{align*}
\]

(7.4)

with \( f := -A_{m}^{\text{ocn}} g \in L^2(\Omega_{\text{ocn}}) \). Consequently, (7.1) admits a unique solution if and only if (7.4) does so. Note that (7.4) is equivalent to \( A_0^{\text{ocn}} v_g = f \). The next result follows from \( 0 \in \rho(A_0^{\text{ocn}}) \) in Corollary 6.2(b).

**Lemma 7.3.** For \( \varphi \in C^\infty_{\text{per}}(G)^2 \), there exists a unique solution \( u \in D(A_{m}^{\text{ocn}}) \) of (7.1).

As a consequence of Lemma 7.3, \( L_0 \) exists as a densely defined operator with domain \( C^\infty_{\text{per}}(G)^2 \). This allows us to define a unique adjoint \( L_0^* : D(L_0^*) \subset L^2(\Omega_{\text{ocn}})' \to (L^q(G)^2)' \). Observe that \( L^2(\Omega_{\text{ocn}})' \cong L^2(\Omega_{\text{ocn}}) \) and \( (L^q(G)^2)' \cong L^{q'}(G)^2 \) with \( \frac{1}{q} + \frac{1}{q'} = 1 \).

We denote by \( \partial_r r : D(\partial_r r) \subset L^q(\Omega_{\text{ocn}}) \to L^{q'}(G)^2 \) the (distributional) derivative in the direction of \( \nu r \) with maximal domain \( D(\partial_r r) := \{ f \in L^q(\Omega_{\text{ocn}}) : \partial_r f \in L^{q'}(G)^2 \} \) for \( r \in (1, \infty) \). Moreover, we denote by \( A_{m}^{\text{ocn}} r \) the operator with homogeneous boundary conditions given in (4.3) on \( L^2(\Omega_{\text{ocn}}) \) for \( r \in (1, \infty) \). We remark
Lemma 7.4. Let $q, q' \in (1, \infty)$ such that $1/q + 1/q' = 1$. The adjoint $L'_0$ of $L_0$ satisfies

\begin{equation}
L'_0 = \partial_q^\gamma R(0, A_{0,q}^{\text{ocn}})' = \partial_{q'}^\gamma R(0, A_{0,q'}^{\text{ocn}}).
\end{equation}

Proof. Let $\varphi \in C_0^\infty(\Gamma_\sigma)^2$ and $k \in L_{\text{per}}^q(\Omega_{\text{ocn}})$ with $1/q + 1/q' = 1$, so $(A_{0,q}^{\text{ocn}})' = A_{0,q'}^{\text{ocn}}$. We set $f := L_0 \varphi$ and $g := (A_{0,q'}^{\text{ocn}})^{-1}k$. Then we have $f \in D(A_{0,q}^{\text{ocn}}) \subset L_{\text{per}}^q(\Omega_{\text{ocn}})$, $g \in D(A_{0,q'}^{\text{ocn}}) \subset L_{\text{per}}^q(\Omega_{\text{ocn}})$ with $1/q + 1/q' = 1$, and it follows therefrom, also invoking the periodic boundary conditions on the lateral boundary, that

$$
\langle \nabla H \pi, g \rangle_{L^2(\Omega_{\text{ocn}})} = \int_{\Gamma_\sigma} \nabla H \pi \cdot \nabla H \pi \, d\sigma_{\text{ocn}} = \int_{\Gamma_\sigma} \nabla H \pi \cdot \nabla H \pi \, d\sigma_{\text{ocn}} = 0,
$$

and likewise $\langle f, \nabla H \pi \rangle_{L^2(\Omega_{\text{ocn}})} = 0$. Green’s second identity and the horizontal periodicity then imply that

$$
\langle \Delta f + \nabla H \pi \cdot g \rangle_{L^2(\Omega_{\text{ocn}})} = \langle f, \Delta g + \nabla H \pi \rangle_{L^2(\Omega_{\text{ocn}})}
$$

$$
= \int_{\Omega_{\text{ocn}}} \Delta f + \nabla H \pi \cdot g \, dx_H + \int_{\Omega_{\text{ocn}}} \Delta g + \nabla H \pi \cdot f \, dx_H
$$

$$
= \int_{\Omega_{\text{ocn}}} \Delta f + \nabla H \pi \cdot g \, dx_H + \int_{\Omega_{\text{ocn}}} \Delta g + \nabla H \pi \cdot f \, dx_H
$$

Since $g \in D(A_{0,q'}^{\text{ocn}})$, we have $g(x,0) = g(x,-h_{\text{ocn}}) = 0$, and it also holds that $f(x, -h_{\text{ocn}}) = 0$ as well as $f(x,0) = \varphi$. This yields that

$$
\langle f, \Delta g + \nabla H \pi \rangle_{L^2(\Omega_{\text{ocn}})} = \langle \Delta f + \nabla H \pi \cdot g \rangle_{L^2(\Omega_{\text{ocn}})} + \int_{\Gamma_\sigma} \varphi(x) \partial_2 g(x,0) \, dx_H.
$$

Making use of the latter identity in conjunction with the definition of the hydrostatic Stokes operator as in Subsection 4.1 as well as $A_{0,q}^{\text{ocn}}f = 0$ on $\Omega_{\text{ocn}}$ by construction, we derive that

$$
\langle L_0 \varphi, k \rangle_{L^2(\Omega_{\text{ocn}})} = \langle f, A_{0,q}^{\text{ocn}}g \rangle_{L^2(\Omega_{\text{ocn}})} = \langle f, \Delta g + \nabla H \pi \rangle_{L^2(\Omega_{\text{ocn}})}
$$

$$
= \langle \Delta f + \nabla H \pi \cdot g \rangle_{L^2(\Omega_{\text{ocn}})} + \int_{\Gamma_\sigma} \varphi(x) \partial_2 g(x,0) \, dx_H
$$

$$
= \langle \varphi, (\partial_2 ((A_{0,q'}^{\text{ocn}})^{-1}k)) \rangle_{L^2(\Omega_{\text{ocn}})},
$$

so the claim follows. \hfill \square

The next lemma guarantees that the right-hand side of (7.5) can be extended to a bounded operator.

Lemma 7.5. The normal derivative $\partial_\nu^\gamma$ is relatively $(-A_{0,q}^{\text{ocn}})^\delta$-bounded with $\delta > \frac{1}{2} + \frac{1}{q'}$.

Proof. From [3, Theorem VIII.1.3.1], establishing the result for the half-space, and the observation that horizontally periodic functions can be extended periodically onto a layer and then cut off, we deduce for $r > 1 + 1/q$ the embedding $H_{\text{per}}^q(\Omega_{\text{ocn}})^2 \cap L_{\text{per}}^q(\Omega_{\text{ocn}}) \hookrightarrow D(\partial_\nu^\gamma)$. The claim is then a consequence of Corollary 6.2, implying that $-A_{0,q}^{\text{ocn}}$ has bounded imaginary powers, see also the relation (8.7). \hfill \square

Combining Lemma 7.4 and Lemma 7.5, we conclude that $L'_0$ can be extended to a bounded operator from $L_{\text{per}}^q(\Omega_{\text{ocn}})$ to $L_{\text{per}}^q(G)^2$. We then get the following result.

Proposition 7.6. The operator $L_0$ can be extended to a bounded operator from $L^q(G)^2$ to $L_{\text{per}}^q(\Omega_{\text{ocn}})$ which will be denoted again by $L_0$ for the sake of simplicity. In particular, the unique solution $v$ of (7.1) satisfies the a-priori estimate $\|\varphi_{\text{ocn}}\|_{L^q(\Omega_{\text{ocn}})} \leq C\|\varphi\|_{L^q(G)}$.

As we see in the next corollary, more regularity of the boundary data implies more regularity of the solution.
Corollary 7.7. Let \( q \in (1, \infty) \) and \( r < 1/q \). If \( \varphi \in L^q(G)^2 \), then the unique solution \( u_{ocn} \) of (7.1) satisfies \( u_{ocn} \in H^r_{pere}(\Omega_{ocn})^2 \cap L^q_2(\Omega_{ocn}) \). In addition, for \( s > 0 \) and \( \varphi \in H^s_{pere}(G)^2 \), the unique solution \( u_{ocn} \) of (7.1) satisfies \( u_{ocn} \in H^{s+r,q}_{pere}(\Omega_{ocn})^2 \cap L^q_2(\Omega_{ocn}) \). Furthermore, for \( s > 0 \) and \( \varphi \in B^{s,per}(G)^2 \), the unique solution \( u_{ocn} \) of (7.1) satisfies \( u_{ocn} \in B^{s+r,q}_{pere}(\Omega_{ocn})^2 \cap L^q_2(\Omega_{ocn}) \).

The operator \( L_0 \) is also bounded from \( H^r_{pere}(G)^2 \) to \( H^s_{pere}(\Omega_{ocn})^2 \) and from \( B^{s,per}(G)^2 \) to \( B^{s+r,q}_{pere}(\Omega_{ocn})^2 \), i.e., the solution \( u_{ocn} \) of (7.1) satisfies the a-priori bounds \( \| u_{ocn} \|_{H^r_{pere}(\Omega_{ocn})^2} \leq C \| \varphi \|_{H^s_{pere}(G)^2} \) as well as \( \| u_{ocn} \|_{B^{s+r,q}_{pere}(\Omega_{ocn})^2} \leq C \| \varphi \|_{B^{s,per}(G)^2} \).

Proof. First, we consider \( s \in (0, 2 - 1/q) \) and \( \varphi \in H^r_{pere}(G)^2 \). From Proposition 7.6 it follows that there exists a unique solution \( v \in L^q_2(\Omega_{ocn}) \) to (7.1) such that \( \Delta u_{ocn} = A_{ocn}v_{ocn} = 0 \in L^q_2(\Omega_{ocn}) \). Surjectivity of the hydrostatic Helmholtz projection \( P_{ocn} : L^q_2(\Omega_{ocn}) \rightarrow L^q_2(\Omega_{ocn}) \) yields \( \Delta u_{ocn} \in L^q_2(\Omega_{ocn}) \). In particular, \( u_{ocn} \) solves the inhomogeneous Laplace problem given by

\[
\Delta u_{ocn} = \nabla \cdot \nabla v, \quad \text{on } \Omega_{ocn},
\]

(7.6)

It follows from \( \Delta u_{ocn} \in L^q_2(\Omega_{ocn}) \) that the right-hand side satisfies \( \nabla \cdot \nabla v \in L^q_2(\Omega_{ocn}) \). Standard regularity theory of the Laplacian implies \( v \in H^r_{pere}(\Omega_{ocn})^2 \) with \( r < 1/q \).

Assume now \( s \geq 2 - 1/q \) and \( \varphi \in H^s_{pere}(G)^2 \). By the above argument, we have \( v \in H^s_{pere}(\Omega_{ocn})^2 \cap L^q_2(\Omega_{ocn}) \). Using the hydrostatic solenoidal condition, we obtain by a direct calculation similar to [16, (4.3)] that the pressure term is given by

\[
\nabla \cdot \nabla v = \frac{1}{h_{ocn}} \nabla \cdot \nabla \varphi.
\]

(7.7)

We conclude \( \nabla \cdot \nabla v \in H^{s-1/q}_{pere}(G)^2 \hookrightarrow H^{s-1/q}_{pere}(\Omega_{ocn})^2 \). The general case \( s \in \mathbb{R}_+ \) follows by a bootstrap argument.

Note that for \( s = 0 \), the a-priori bound is shown in Proposition 7.6. We first prove the a-priori bound for \( s \in \mathbb{N} \) with \( s \geq 2 \). The proof uses induction over \( \{ r \in \mathbb{N} : r \geq 2 \} \). Using classical Calderón-Zygmund theory, we obtain from (7.6) that

\[
\| u_{ocn} \|_{H^r_{pere}(\Omega_{ocn})^2} \leq C(\| \varphi \|_{H^s_{pere}(G)^2} + \| \nabla \cdot \nabla v \|_{L^q(G)}).
\]

From (7.7), the boundedness of the normal derivative by Lemma 7.5, the interpolation inequality, Young’s inequality and the a-priori estimate on \( L^q \) as in Proposition 7.6, it follows that for all \( r > 1 + 1/q, \delta = \frac{1}{2} \in (0, 1) \) and \( \eta > 0 \),

\[
\| \nabla \cdot \nabla v \|_{L^q(G)} \leq C(\| \partial_{\nu_{ocn}}u_{ocn} \|_{L^q(G)} + \| \partial_{\nu_{ocn}}u_{ocn} \|_{L^q(G)}).
\]

(7.8)

Plugging this estimate into (7.8), we obtain the desired \( H^2,q \)-a-priori bound by an absorption argument.

Assume that the a-priori bound still holds for \( s = 1 \). Replacing the Calderón-Zygmund a-priori estimate (7.8) by its higher-order analogue and using the higher-order bounds for the normal derivative and the fact that the pseudo-differential operator \( \nabla \cdot \nabla \varphi \) of order 0 is bounded on \( H^s_{pere}(G)^2 \), we deduce the induction step from the same argument as the induction base.

The general case \( s \in \mathbb{R}_+ \) follows by interpolation, and the additional regularity of the solution in the Besov space scale follows from the embeddings \( B^{s,per}(G) \hookrightarrow B^{s+r,q}_{pere}(\Omega_{ocn}) \) and \( B^{s+r,q}_{pere}(\Omega_{ocn}) \hookrightarrow B^{s+1/q}_{pere} \) for \( \delta > 0 \), and the corresponding results for Bessel potential spaces. Finally, the boundedness of \( L_0 \) between Besov spaces follows from the boundedness of \( L_0 \) between \( B^{k,q}_{pere}(G)^2 \) and \( B^{k,q}_{pere}(\Omega_{ocn})^2 \) for \( k \in \mathbb{N}_0 \) by real interpolation.
Combining $0 \in \rho(A_{\text{ocean}}^\text{ocean})$ from Corollary 6.2 with Proposition 7.6 and using the linearity of the hydrostatic Stokes operator, we obtain that the general inhomogeneous hydrostatic Stokes problem given by

$$
\begin{cases}
A_{\text{ocean}}^\text{ocean} v_{\text{ocean}} = f, & \text{on } \Omega_{\text{ocean}}, \\
\tau \gamma_0 v_{\text{ocean}} = \varphi, & \text{on } \Gamma_\sigma, \\
\tau \gamma_0 v_{\text{ocean}} = 0, & \text{on } \Gamma_h,
\end{cases}
$$

for $f \in L^2(\Omega_{\text{ocean}})$ and $\varphi \in L^2(G)^2$, admits a unique solution $v_{\text{ocean}} \in D(A_{\text{m}})$, with $v_{\text{ocean}} = -R(0, A_{\text{ocean}}^\text{ocean})f + L_0\varphi$.

**Remark 7.8.** Analogous results about stationary hydrostatic Stokes problems for pure Dirichlet or pure Neumann boundary conditions can be shown by the same technique. First of all, note that it is sufficient to consider only inhomogeneous boundary conditions, since the general case follows from linearity of the problem. The case of pure (inhomogeneous) Neumann boundary conditions is treated in [7, Section 3].

### 7.2. The Hydrostatic Dirichlet-to-Neumann Operator

The hydrostatic Dirichlet-to-Neumann operator on $L^2(G)^2$ is given by the composition of the normal derivative and the Dirichlet operator, i.e.,

$$N_0 \varphi := \partial_\nu^a L_0 \varphi, \quad D(N_0) := \{ \varphi \in L^2(G)^2 : L_0 \varphi \in D(\partial_\nu^a) \}.$$

Using the regularity theory for the inhomogeneous stationary Stokes problem, we obtain the following result about the domain of the Dirichlet-to-Neumann operator.

**Proposition 7.9.** The domain of the Dirichlet-to-Neumann operator contains $H^{s, \frac{q}{2}}_{\text{per}}(G)^2$ for all $s > 1$.

**Proof.** As in the proof of Lemma 7.5, we infer that $H^{s, \frac{q}{2}}_{\text{per}}(\Omega_{\text{ocean}})^2 \cap L^q(\Omega_{\text{ocean}}) \subset D(\partial_\nu^a)$ for $r > 1 + \frac{1}{q}$. Corollary 7.7 yields $L_0 H^{s, \frac{q}{2}}_{\text{per}}(G)^2 \subset H^{r, \frac{q}{2}}_{\text{per}}(\Omega_{\text{ocean}})^2 \cap L^q(\Omega_{\text{ocean}}) \subset D(\partial_\nu^a)$ for $s > 1$, i.e., $H^{s, \frac{q}{2}}_{\text{per}}(G)^2 \subset D(N_0)$. □

It follows by Proposition 5.2 that $-A^H + \omega$ has bounded imaginary powers, see also (8.7). We deduce therefrom that $D((-A^H)^\delta) \leftrightarrow H^{2s, \frac{q}{2}}_{\text{per}}(G)^2$ and then conclude the following result.

**Corollary 7.10.** The Dirichlet-to-Neumann operator $N_0$ is relatively $(-A^H)^\delta$-bounded for $\delta > \frac{1}{2}$.

### 8. Decoupling

In this section, we combine the results from Section 5, Section 6 and Section 7 to conclude maximal regularity of the operator matrix given in (4.4) as well as Lipschitz estimates of the operator matrix given in (4.4) and the nonlinearity (4.5).

#### 8.1. Maximal Regularity and Bounded $H^\infty$-Calculus of the Linearized Operator Matrix

Fix $(v_{\text{ice}, 0}, h_0, a_0) \in V^{\text{ice}}$ and consider the operator matrix $A(v_{\text{ice}, 0}, h_0, a_0)$. Throughout this section, we omit the fixed variable $(v_{\text{ice}, 0}, h_0, a_0)$ and denote the operator matrix by $A := A(v_{\text{ice}, 0}, h_0, a_0)$. However, we impose the required regularity conditions on $(v_{\text{ice}, 0}, h_0, a_0)$ when necessary. Note that $A$ does not have a diagonal domain. To deal with this issue, we follow the ideas from [6, 8, 21]. We introduce the operator matrix $\hat{A}: D(\hat{A}) \subset \mathcal{X}_0 \to \mathcal{X}_0$ given by

$$\hat{A} := \begin{pmatrix}
A_{\text{atm}}^\text{atm} & 0 & 0 \\
0 & A_{\text{ocean}}^\text{ocean} + L_0 C_{\text{ocean}, i} \partial_\tau^a & -L_0 (A^H - C_{\text{ocean}, i} N_0) \\
0 & -C_{\text{ocean}, i} \partial_\tau^a & A^H - C_{\text{ocean}, i} N_0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & d_0 \Delta_H & 0
\end{pmatrix},$$

$$D(\hat{A}) := D(A_{\text{atm}}) \times D(A_{\text{ocean}}^\text{ocean}) \times D(A^H) \times D(\Delta_H) \times D(\Delta_H).$$

The next lemma investigates the relationship between the operator matrices $A$ and $\hat{A}$.

**Lemma 8.1.** The operator matrices $A$ and $\hat{A}$ are isomorphic.
Proof. The similarity transform is given by

\[ S := \begin{pmatrix} 
\text{Id} & 0 & 0 & 0 \\
0 & \text{Id} & -L_0 & 0 \\
0 & 0 & \text{Id} & 0 \\
0 & 0 & 0 & \text{Id} 
\end{pmatrix}, \quad \text{with inverse} \quad S^{-1} := \begin{pmatrix} 
\text{Id} & 0 & 0 & 0 \\
0 & \text{Id} & L_0 & 0 \\
0 & 0 & \text{Id} & 0 \\
0 & 0 & 0 & \text{Id} 
\end{pmatrix}. \]

From Proposition 7.6 we conclude that \( S \in \mathcal{L}(\lambda_0) \), and that the inverse \( S^{-1} \) is also bounded on \( X_0 \). A direct calculation shows that \( \tilde{A} = S\tilde{A}S^{-1} \). Standard regularity theory implies that \( \tilde{v}_{\text{ocn}} \in D(A_{m-\delta}) \) with homogeneous boundary conditions fulfills \( \tilde{v}_{\text{ocn}} \in D(A_{n-\delta}) \). Consequently, since \( \tilde{v}_{\text{ocn}} := v_{\text{ocn}} - L_0 v_{\text{ice}} \) for \( v \in D(A) = X_1 \) has these properties, we conclude \( D(\tilde{A}) = S D(A) \).

For \( v = (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a) \in X_1 \), we conclude from Lemma 8.1 and its proof that \( v_{\text{ocn}} = \tilde{v}_{\text{ocn}} + L_0 v_{\text{ice}} \), where \( \tilde{v}_{\text{ocn}} \in D(A_{0-\delta}) \) as well as \( v_{\text{ice}} \in W^2_{p；(G)^2} \). Together with Corollary 7.7, this results in the relation \( v_{\text{ocn}} \in W^2_{p；(G)^2} \cap L^\infty(\Omega_{\text{ocn}}) =: D(A_{n-\delta}). \) Therefore, we have

\begin{align}
X_1 &= \{ v \in D(A_{\text{atm}}) \times D(A_{n-\delta}) \times D(A H) \times D(\Delta H) \times D(\Delta H) : \\
& \quad \text{tr}_{\Gamma_b} v_{\text{ocn}} = 0, \text{ on } \Gamma_b, \text{ and } \text{tr}_{\Gamma_\delta} v_{\text{ocn}} = v_{\text{ice}}, \text{ on } \Gamma_\delta \}. 
\end{align}

Note that the operator \( \tilde{A} \) has a diagonal domain and therefore, it is possible to work component-wise. To establish \( \mathcal{H}^{\infty} \)-calculus, we decompose the operator matrix \( \tilde{A} \) into smaller submatrices. More precisely, we use the following representations

\begin{equation}
\tilde{A} = \begin{pmatrix} A_{\text{atm}} & 0 \\
0 & J \end{pmatrix}, \quad D(\tilde{A}) = D(A_{\text{atm}}) \times D(J),
\end{equation}

where the operator matrix \( J \) is split into

\begin{equation}
J = \begin{pmatrix} J_1 & B' \\
0 & J_2 \end{pmatrix}, \quad D(J) = D(J_1) \times D(J_2), \text{ with } B' = \begin{pmatrix} 0 & 0 \\
B_b & B_a \end{pmatrix} \text{ and } J_2 = \text{diag}(d_b \Delta H, d_a \Delta H).
\end{equation}

Moreover, \( J_1 \) is given by

\[ J_1 = \begin{pmatrix} A_0 & -L_0 \partial_z \beta & -L_0 (A H - C_{0,q} I_{n_0}) \\
0 & A_{\partial_z \beta} & -C_{0,q} I_{n_0} \end{pmatrix}, \quad D(J_1) = D(A_{0\text{ocn}}) \times D(A H).
\]

The crucial step is to show that \( J_1 \) admits a bounded \( \mathcal{H}^{\infty} \)-calculus, which requires some preparation.

For convenience of the reader, we recall the notion of diagonal dominance of a block operator matrix from [1, Assumption 3.1]. Let \( X, Y \) be Banach spaces and consider linear operators

\begin{align}
A &: D(A) \subset X \to X, \quad D(D) \subset Y \to Y, \quad B &: D(B) \subset Y \to X \quad \text{and} \quad C &: D(C) \subset X \to Y,
\end{align}

where \( A \) and \( D \) are assumed to be densely defined and closed. For \( Z = X \times Y \), the block operator matrix

\begin{equation}
K = D(K) = D(A) \times D(D) \subset Z \to Z \quad \text{with} \quad K \begin{pmatrix} x \\
y \end{pmatrix} = \begin{pmatrix} A & B \\
C & D \end{pmatrix} \begin{pmatrix} x \\
y \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} x \\
y \end{pmatrix} \in D(K)
\end{equation}

is then said to be diagonally dominant if \( D(D) \subset D(B), D(A) \subset D(C) \), and there exist \( c_A, c_D, L \geq 0 \) with

\begin{align}
\| Cx \|_Y & \leq c_A \| Dx \|_X + L \| x \|_X \quad \text{for all} \quad x \in D(A), \quad \text{and} \\
\| B_y \|_X & \leq c_D \| Dy \|_Y + L \| y \|_Y \quad \text{for all} \quad y \in D(D).
\end{align}

We remark that the relative boundedness of \( C \) is especially implied if there exists \( \gamma \in (0, 1) \) such that \( C \in \mathcal{L}(D(A^{\gamma}), Y) \), see e. g. [1, Corollary 5.7]. For the following result, we also refer to [1, Corollary 5.7].

**Lemma 8.2.** Let \( K \) be a diagonally dominant block operator matrix as in (8.4), suppose in addition that \( A \) and \( D \) are sectorial operators and assume that there exists \( \delta \in (0, 1) \) such that for some constant \( c \geq 0 \)

\begin{align}
C(D(A^{1+\delta})) & \subset D(D^\delta) \quad \text{and} \quad \| D^\delta Cx \|_Y \leq c \| A^{1+\delta} x \|_X \quad \text{for all} \quad x \in D(A^{1+\delta}), \quad \text{and} \\
B(D(D^{1+\delta})) & \subset D(D^\delta) \quad \text{and} \quad \| A^{1+\delta} B_y \|_X \leq c \| D^{1+\delta} y \|_Y \quad \text{for all} \quad y \in D(D^{1+\delta}).
\end{align}

Then if \( A \) and \( D \) admit a bounded \( \mathcal{H}^{\infty} \)-calculus on \( X \) and \( Y \), respectively, it follows that there is \( \omega_0 \in \mathbb{R} \) such that for all \( \omega > \omega_0 \), the block operator matrix \( K + \omega \) admits a bounded \( \mathcal{H}^{\infty} \)-calculus on \( Z \).
We are now in the position to state and prove the $H^\infty$-calculus of $J_1$.

**Lemma 8.3.** Let $p, q \in (1, \infty)$ such that $1/p + 1/q < 1/2$ and $(v_{\text{atm}, 0}, v_{\text{ocn}, 0}, v_0) \in V$. Then there exists a constant $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, $-J_1 + \omega$ admits bounded $H^\infty$-calculus on $L^q_\mathcal{O}(\Omega_{\text{ocn}}) \times L^q(G)^2$.

**Proof.** We set $A := A_0^{\text{ocn}}$, $B := -L_0 A^H$, $C := -C_{o,i} \partial_{\nu_o}$ and $D := A^H$ to mimic the notation from above. First, we consider the operator matrix $\tilde{J}_1$ given by

$$\tilde{J}_1 := \begin{pmatrix} A_0^{\text{ocn}} & -L_0 A^H \\ -C_{o,i} \partial_{\nu_o} & A^H \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with diagonal domain. We now verify the assumptions of Lemma 8.2 for $\tilde{J}_1$. The diagonal dominance of $\tilde{J}_1$ follows from Lemma 7.5, as the latter result also implies that $C \in \mathcal{L}(D(-A)\gamma^1), L^q(G)^2$ for all $\gamma \in (1/2(1 + 1/q), 1)$. It remains to verify (8.5). Let $\delta \in (0, 1/2)$. As the bounded $H^\infty$-calculus of $-A^H + \omega$ and $-A_0^{\text{ocn}}$ from Proposition 5.2 and Corollary 6.2 especially yields the boundedness of the imaginary powers of these operators, see also the relation (8.7) below, it follows that

$$D(A^{1+\delta}) \hookrightarrow H^{2+2\delta,q}_{\text{per}}(\Omega_{\text{ocn}})^2 \cap L^q_\mathcal{O}(\Omega_{\text{ocn}}), \quad D(A^\delta) = H^{2\delta,q}_{\text{per}}(\Omega_{\text{ocn}})^2 \cap L^q_\mathcal{O}(\Omega_{\text{ocn}}),$$

$$D(\Omega^{1+\delta}) = H^{2+2\delta,q}_{\text{per}}(G)^2, \quad D(\Omega^\delta) = H^{2\delta,q}_{\text{per}}(G)^2.$$  

As in the proof of Lemma 7.5, it follows that $CD(A^{1+\delta}) \subset H^{2\delta,q}_{\text{per}}(G)^2 = D(\Omega^\delta)$. From Corollary 7.7 and $B = -L_0 D$ it follows that $BD(\Omega^{1+\delta}) = -L_0 D(\Omega^\delta) = L_0 H^{2\delta,q}_{\text{per}}(G)^2 \subset H^{2\delta,q}_{\text{per}}(G)^2 \cap L^q_\mathcal{O}(\Omega_{\text{ocn}}) = D(A^\delta)$. Since the operators $B$ and $C$ are closed, the estimates in (8.5) follow by the closed graph theorem.

Now, Lemma 8.2 implies that $-\tilde{J}_1 + \omega$, given by (8.6), admits a bounded $H^\infty$-calculus on $L^q_\mathcal{O}(\Omega_{\text{ocn}}) \times L^q(G)^2$ for some constant $\omega > \omega_0$. By Lemma 7.5 and Proposition 7.6, the operator $L_0 C_{o,i} \partial_{\nu_o}$ is relatively $(A^{\infty})^\gamma$-bounded for all $\gamma \in (1/2(1 + 1/q), 1)$. In addition, by Proposition 7.6 and Corollary 7.10, the operators $L_0 C_{o,i} N_0$ and $C_{o,i} N_0$ are relatively $(A^H)^\theta$-bounded for all $\theta \in (1/2, 1)$. Therefore, the claim follows from perturbation theory for the $H^\infty$-calculus, see for instance [30, Proposition 13.1].

We conclude with the main result of this section.

**Proposition 8.4.** Let $p, q \in (1, \infty)$ such that $1/p + 1/q < 1/2$ and $(v_{\text{atm}, 0}, v_{\text{ocn}, 0}, v_0) \in V$. Then there exists a constant $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, the operator matrix $-A + \omega$, with $A$ given in (4.4), admits a bounded $H^\infty$-calculus on $X_0$.

**Proof.** In the sequel, the constants $\omega \in \mathbb{R}$ may change from line to line. It follows from [39, Theorem 8.22] that $-J_2 + \omega$ admits a bounded $H^\infty$-calculus on $L^q(G) \times L^q(G)$ for some $\omega \in \mathbb{R}$. Combining this with Lemma 8.3, we deduce that $-\text{diag}(J_1, J_2) + \omega$ with diagonal domain admits a bounded $H^\infty$-calculus on $L^q_\mathcal{O}(\Omega_{\text{ocn}}) \times L^q(G)^2 \times L^q(G) \times L^q(G)$ for some $\omega \in \mathbb{R}$. Further, we know from Corollary 5.3 that $B'$ is relatively $(A^H + \omega)^{1/2}$-bounded, and it follows from [30, Proposition 13.1] and (8.3) that $-J + \omega$ admits a bounded $H^\infty$-calculus on $L^q_\mathcal{O}(\Omega_{\text{ocn}}) \times L^q(G)^2 \times L^q(G) \times L^q(G)$ for some $\omega \in \mathbb{R}$.

Invoking Corollary 6.2(a), we conclude from (8.2) that $-A + \omega$ admits a bounded $H^\infty$-calculus on $X_0$. The claim finally follows from Lemma 8.1 together with the fact that the bounded $H^\infty$-calculus is preserved under similarity transforms, see for instance [13, Proposition 2.11].

We collect some interesting functional analytic properties. For the spaces $\mathcal{Y}_\beta$ and $\mathcal{Y}_\theta$, see (5.3).

**Corollary 8.5.** Let $p, q \in (1, \infty)$ such that $1/p + 1/q < 1/2$ and $(v_{\text{atm}, 0}, v_{\text{ocn}, 0}, v_0) \in V$. Then there exists $\omega_0 \in \mathbb{R}$ such that for $\omega > \omega_0$, the following properties hold true.

(a) The operator $-A + \omega$ has an $R$-bounded $H^\infty$-calculus on $X_0$ with angle $\Phi_{-A + \omega} < \pi/2$.

(b) It holds that $-A + \omega \in \mathcal{B}^1(\mathcal{Y}_\beta)$, i.e., the operator $-A + \omega$ has bounded imaginary powers.

(c) The operator $-A + \omega$ has the property of maximal $L^q([0, \infty))$-regularity in $X_0$.

(d) For $\beta \in (0, 1)$ with $\beta \notin \{1/2q, 1/2 + 1/2q\}$, the fractional power domains coincide with the complex interpolation spaces, i.e., $D((-A + \omega)^\beta) = [X_0, D(-A)]_\beta \hookrightarrow \mathcal{Y}_\beta$.

(e) For $\theta \in (0, 1)$ with $\theta \notin \{1/2q, 1/2 + 1/2q\}$, the real interpolation spaces satisfy $(X_0, D(-A))_{\theta, p} \hookrightarrow \mathcal{Y}_\theta$.

(f) The Riesz transform $\nabla(-A + \omega)^{1/2}$ is bounded on $X_0$.

(g) The operator matrix $A$ has a compact resolvent, and the spectrum $\sigma(A)$ of $A$ on $X_0$ is $q$-independent.
Proof. For (a), we observe that the space \( X_0 \) has the property (\( \alpha \)) as a Cartesian product of closed subspaces of \( L^p \)-spaces. By [28, Theorem 5.3], \( \mathcal{R} \)-boundedness of the \( \mathcal{H}^\infty \)-calculus is then equivalent to the \( \mathcal{H}^\infty \)-calculus and the angles coincide, showing (a).

The assertions (b) and (c) follow from Proposition 8.4 via the relations
\[
\mathcal{H}^\infty(X_0) \subset \mathcal{B} \mathcal{P}(X_0) \subset \mathcal{R} \mathcal{S}(X_0) \quad \text{with} \quad \phi_A^R \geq \theta_A \geq \phi_A^R,
\]
see for example [13, Section 4.4], and the observation that \( X_0 \) is in particular a UMD-space.

The first part of property (d) follows from [13, Theorem 2.5]. Note that for \( \mathcal{A} \), we obtain from the bounded imaginary powers of Hilber’s operator and the hydrostatic Stokes operators in view of Proposition 5.2 and Corollary 6.2 as well as (8.7) that the complex interpolation spaces are given by
\[
\hat{X}_\beta := [X_0, D(-\hat{A})]_\beta
= (H^{2\beta,q}_{\text{per},b,c}(\Omega_{\text{atm}}) \cap \mathcal{L}_q^G(\Omega_{\text{atm}})) \times (H^{2\beta,q}_{\text{per},b,c}(\Omega_{\text{ocn}}) \cap \mathcal{L}_q^G(\Omega_{\text{ocn}})) \times H^{2\beta,q}(G)^2 \times H^{2\beta,q}(G) \times H^{2\beta,q}(G),
\]
where for the interpolation spaces with periodic boundary conditions, we refer to [25, Section 4]. Now, the isomorphism \( \mathcal{S} \) from Lemma 8.1 is also an isomorphism in the category of Banach couples, see [2, Section 1.2.1]. Thanks to the functoriality of the interpolation, the complex interpolation spaces for \( \mathcal{A} \) are given by
\[
\hat{X}_\beta := [X_0, D(-\hat{A})]_\beta = \{(v_{\text{atm}}, v_{\text{ocn}} + L_0 v_{\text{ice}}, v_{\text{ice}}, h, a) : (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a) \in \hat{X}_\beta \}
\rightarrow (H^{2\beta,q}_{\text{per},b,c}(\Omega_{\text{atm}}) \cap \mathcal{L}_q(\Omega_{\text{atm}})) \times ((H^{2\beta,q}_{\text{per},b,c}(\Omega_{\text{ocn}}) \cap \mathcal{L}_q^G(\Omega_{\text{ocn}})) \times L_0(H^{2\beta,q}(G)^2) + \mathcal{L}_q^G(\Omega_{\text{ocn}}) \times H^{2\beta,q}(G)^4).
\]
Finally, Corollary 7.7 yields \( (H^{2\beta,q}_{\text{per},b,c}(\Omega_{\text{ocn}}) \cap \mathcal{L}_q^G(\Omega_{\text{ocn}})) \times L_0(H^{2\beta,q}(G)^2) \rightarrow H^{2\beta,q}(\Omega_{\text{ocn}})^2 \times \mathcal{L}_q^G(\Omega_{\text{ocn}}).

Property (e) follows from analogous arguments, replacing complex by real interpolation and Bessel potential by Besov spaces.

Assertion (f) is a consequence of the shape of the corresponding fractional power domain obtained in (e). The last assertion follows from the compactness of the resolvent of \( \mathcal{A} \) by the Rellich-Kondrachov theorem as well as the observation that the resolvent set is preserved under the similarity transform and the representation of the resolvent of \( \mathcal{A} \) in terms of \( \mathcal{A} \).

\[ \square \]

8.2. Lipschitz estimates.

We establish Lipschitz estimates of the operator matrix \( \mathcal{A} \) given by (4.4) and the nonlinearity \( \mathcal{F} \) given by (4.5). To account for the fact that we only have maximal regularity up to translation, for \( \omega \in \mathbb{R} \) as in Proposition 8.4 or Corollary 8.5, we set \( \mathcal{A}_\omega := \mathcal{A} - \omega \) as well as \( \mathcal{F}_\omega := \mathcal{F} - \omega \). Thanks to Corollary 8.5(e), we may prove the Lipschitz estimates component-wise.

Proposition 8.6. Let \( p, q \in (1, \infty) \) such that \( \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \) and \( u_0 = (v_{\text{atm},0}, v_{\text{ocn},0}, v_0) \in V \). Then there exists \( r_0 > 0 \) and a constant \( L > 0 \) such that \( B_{X_\lambda}(u_0, r_0) \subset V \) and
\[
\| (\mathcal{A}_\omega(u_1) - \mathcal{A}_\omega(u_2)) \|_{X_\lambda} \leq L \| u_1 - u_2 \|_{X_\lambda}, \quad \| \mathcal{F}_\omega(u_1) - \mathcal{F}_\omega(u_2) \|_{X_\lambda} \leq L \| u_1 - u_2 \|_{X_\lambda},
\]
for all \( u_1, u_2 \in B_{X_\lambda}(u_0, r_0) \) and all \( u \in X_\lambda \).

Proof. We observe that \( u_0 \in V \) especially satisfies that \( u_0 \in W \), and that \( X_0 \) and \( \mathcal{Y}_0 \) coincide. The assertion of this proposition then follows by concatenating Lemma 5.4 and Corollary 6.4, and by exploiting Corollary 8.5(e) with the concrete choice \( \theta = 1 - 1/p \). We remark that the assumptions on \( p \) and \( q \) imply in particular that \( \theta > \frac{1}{2} + \frac{1}{2q} \).

\[ \square \]

9. Local Well-Posedness

This section is dedicated to showing local strong well-posedness of the coupled system.

Proof of Theorem 3.2. It follows from Corollary 8.5(e) that \( -\mathcal{A}_\omega \) has the property of maximal \( \mathcal{L}^p \)-regularity on \( X_0 \), and Proposition 8.6 yields that suitable Lipschitz estimates are valid for \( \mathcal{A}_\omega \) and \( \mathcal{F}_\omega \). The assertion then follows from the local existence theorem for quasilinear evolution equations, see [41, Theorem 5.1.1], and the fact that finding a solution to the quasilinear abstract Cauchy problem (4.6) is equivalent to solving the coupled system of PDEs (3.4) by the reformulation in Section 4.

\[ \square \]
10. Global Well-Posedness close to Constant Equilibria

In this section, we verify that a slightly simplified version of the coupled system (3.4), or, equivalently, (4.6), is globally well-posed for initial data close to constant equilibria. To this end, we employ the so-called generalized principle of linearized stability, see [42] or [41, Section 5.3].

Let $p, q \in (1, \infty)$ such that $1/p + 1/q < 1/2$. We investigate equilibria in the situation that the external forces $f_{atm}, f_{ocn}$ and $g \nabla H$ vanish, the melting and freezing effects $S_h$ and $S_a$ are neglected, and the effect of the atmospheric wind on the sea ice in the form of $\tau_{atm}$ is not present, i.e., $f_{atm} = f_{ocn} = g \nabla H = \tau_{atm} = 0$ and $S_h = S_a = 0$.

Similarly as in Subsection 8.2, see also [9, Sections 6 and 7], we verify that

$$ (A, F_s) \in C^1(V, \mathcal{L}(X_1) \times X_0) $$

for an open subset $V \subset X_\gamma$, where $F_s$ is given by

$$ F_s(v_{atm}, v_{ocn}, v_{ice}, h, a) = \begin{pmatrix} (v_{atm} \cdot \nabla H)v_{atm} + w(v_{atm}) \cdot \partial_z v_{atm} \\ (v_{ocn} \cdot \nabla H)v_{ocn} + w(v_{ocn}) \cdot \partial_z v_{ocn} \\ (v_{ice} \cdot \nabla H)v_{ice} \\ \text{div}_H(v_{ice} h) \\ \text{div}_H(v_{ice} a) \end{pmatrix}. $$

By $E \subset V \cap X_1$, we denote the set of equilibrium solutions to

$$ (10.1) \quad u' - A(u)u + F_s(u) = 0, \quad t > 0, \quad u(0) = u_0, $$

where $u = (v_{atm}, v_{ocn}, v_{ice}, h, a)$ again represents the principal variable of the system. The property of being an equilibrium solution $u \in E$ is equivalent to $u \in V \cap X_1$ as well as $A(u)u = F_s(u)$.

We find that $u_* = (0, 0, 0, h_*, a_*)$, with $h_* \in (\kappa_1, \kappa_2)$ and $a_* \in (0, 1)$ constant in space and time, is an equilibrium solution of (10.1), because it satisfies the coupling condition $\text{tr}_{1,v_{ocn}} = v_{ice}$ on $\Gamma_o$, yielding that $u_* \in V \cap X_1$, and it holds that $A(u_*)u_* = 0 = F_s(u_*)$.

Next, we compute the total linearization of (10.1) at $u_*$ taking the shape

$$ A_0u = A(u_*)u + (A'(u_*)u_*)u_* - F'_s(u_*)u, \quad \text{for } u \in X_1. $$

Abbreviating $P(h_*, a_*)$ by $P_*$, i.e., $P_* = p^* h_* \exp(-c(1 - a_*))$, we find that $A(u_*)u$ is of the form

$$ (10.2) \quad A(u_*)u = \begin{pmatrix} A_{\text{atm}}v_{atm} \\ A_{\text{ocn}}v_{ocn} \\ A_{\text{ice}}v_{ice} \\ \frac{\delta h}{\delta a} \Delta h \\ \frac{\delta a}{\delta a} \Delta a \end{pmatrix}, $$

where

$$ C_{o,i}(h_*) \partial_{\tau_\alpha} v_{ocn} = \frac{\mu_{ocn}}{\rho_{\text{ice}} h_*} \partial_{\tau_\alpha} v_{ocn}, \quad (A_{\text{ice}}v_{ice})_i = \frac{2 P_*}{\rho_{\text{ice}} h_*} \sum_{j,k,l=1}^{2} S_{ijkl} \partial_{\delta h} \partial_{\delta a} v_{ice,j}, $$

$$ B_h(u_*)h = - \frac{\partial_h P_*}{\rho_{\text{ice}} h_*} \nabla H, \quad \text{and} \quad B_a(u_*)a = - \frac{\partial_a P_*}{\rho_{\text{ice}} h_*} \nabla H. $$

Moreover, the shape of $u_*$ yields that $A'(u_*)u_* = 0$ for all $u \in X_1$, and we compute that $F'_s(u_*)u$ is given by $F'_s(u_*)u = (0, 0, 0, h_* \text{div}_H v_{ice}, a_* \text{div}_H v_{ice})^\top$. This results in the total linearization

$$ (10.3) \quad A_0u = A(u_*)u - F'_s(u_*)u = \begin{pmatrix} A_{\text{atm}}v_{atm} \\ A_{\text{ocn}}v_{ocn} \\ A_{\text{ice}}v_{ice} \\ \frac{\delta h}{\delta a} \Delta h - h_* \text{div}_H v_{ice} \\ \frac{\delta a}{\delta a} \Delta a - a_* \text{div}_H v_{ice} \end{pmatrix}, $$

with domain $D(A_0) = X_1$. At this stage, we recall from (8.1) that for $v \in X_1$, it especially follows that $v_{ocn} \in W^{2,2}(\Omega_{ocn})$. In the following, we make use of this improved regularity.

The next step is to verify that an equilibrium $u_*$ of the above shape is normally stable in the sense of [41, Theorem 5.3.1]. To this end, we first investigate the spectral properties of $A_0$. 
Lemma 10.1. If \( u_* = (0, 0, 0, h_*, a_*) \) with \( h_* \in (\kappa_1, \kappa_2) \) and \( a_* \in (0, 1) \) constant in space and time, then the total linearization \( A_0 \) with domain \( D(A_0) = X_1 \) satisfies \( \sigma(A_0) \setminus \{0\} \subset \mathbb{C}_- \).

Proof. As a preparation, we first calculate some integrals for \( u = (v_{atm}, v_{ocn}, v_{ice}, h, a) \in X_1 \). Using an integration by parts, invoking the periodic boundary conditions on the lateral boundary, the homogeneous Neumann boundary conditions on the upper and lower boundary as well as the condition that the horizontal divergence of the vertically averaged \( v_{atm} \) vanishes for \( v_{atm} \in D(A_{atm}) \), we obtain that

\[
-\langle A_{atm} v_{atm}, v_{atm} \rangle_{L^2(\Omega_{atm})} = -\frac{\lambda}{2\rho_{atm} h_*} \int_G \sum_{i,j,k,l=1}^2 S_{ij}^{kl} \partial_i v_{ice,j} \partial_k v_{ice,i} dx_H
\]

(10.4)

Similarly, using that \( \text{tr}_{\Gamma_0} v_{ocn} = \varepsilon \) on \( \Gamma_0 = G \) as well as \( \text{tr}_{\Gamma_0} v_{ocn} = 0 \) on \( \Gamma_b \) and employing Poincaré’s inequality, we infer that

\[
-\langle A_{ocm} v_{ocm}, v_{ocm} \rangle_{L^2(\Omega_{ocm})} \geq C_1 \| v_{ocm} \|^2_{H^1(\Omega_{ocm})} - \langle \partial_{\Gamma_0} v_{ocm}, v_{ice} \rangle_{L^2(G)}
\]

(10.5)

Next, from [9, (4.4) and (7.4)] we recall that

\[
\sum_{i,j,k,l=1}^2 S_{ij}^{kl} \partial_i v_{ice,j} \partial_k v_{ice,i} = \Delta^2 (\nabla_H v_{ice}) \geq \frac{2}{\epsilon^2} |v_{ice}|^2.
\]

Together with an integration by parts, the periodic boundary conditions and Korn’s inequality, the latter estimate yields that

\[
-\langle A^H(u_*) v_{ice}, v_{ice} \rangle_{L^2(G)} = -\frac{P_*}{2\rho_{ice} h_*} \frac{1}{\delta^2} \int_G \sum_{i,j,k,l=1}^2 S_{ij}^{kl} \partial_i v_{ice,j} \partial_k v_{ice,i} dx_H
\]

(10.6)

Using the periodic boundary conditions in another integration by parts, we conclude that

\[
-\langle \Delta_H h, h \rangle_{L^2(G)} = \| \nabla_H h \|^2_{L^2(G)} \quad \text{and} \quad -\langle \Delta_H a, a \rangle_{L^2(G)} = \| \nabla_H a \|^2_{L^2(G)}.
\]

For \( u = (v_{atm}, v_{ocm}, v_{ice}, h, a) \), to determine the spectrum of \( A_0 \), we test the equation \( (A - A_0) u = 0 \) with

\[
\langle v_{atm}, v_{ice} \rangle_{L^2(G)} + c_3 \| v_{ice} \|^2_{L^2(\Omega_{ice})} + \lambda c_4 \| H \|^2_{L^2(G)} + \lambda c_5 \| a \|^2_{L^2(G)}.
\]

(10.7)

(10.8)

Taking into account that \( h_* \in (\kappa_1, \kappa_2) \) and \( a_* \in (0, 1) \), we observe that \( c_3, c_4, c_5 \geq 0 \). Integrating by parts, exploiting the periodic boundary conditions on the lateral boundary and inserting (10.4), (10.5), (10.6) as well as (10.7), we then obtain by virtue of the choice of \( c_3, c_4 \) and \( c_5 \) in (10.8)

\[
\geq \lambda \| v_{atm} \|^2_{L^2(\Omega_{atm})} + \lambda \| v_{ocm} \|^2_{L^2(\Omega_{ocm})} + \lambda \| v_{ice} \|^2_{L^2(\Omega_{ice})} + \lambda c_4 \| H \|^2_{L^2(G)} + \lambda c_5 \| a \|^2_{L^2(G)}.
\]

The above relation can only hold provided \( \lambda \) is real and \( \lambda \leq 0 \). Invoking Corollary 8.5(g) on the \( q \)-independence of the spectrum of \( A \), which carries over to \( A_0 \), we infer that indeed, \( \sigma(A_0) \setminus \{0\} \subset \mathbb{C}_- \). \( \Box \)

Lemma 10.2. Near \( u_* = (0, 0, 0, h_*, a_*) \) with \( h_* \in (\kappa_1, \kappa_2) \) and \( a_* \in (0, 1) \) constant in space and time, the set of equilibria \( E \) is a \( C^1 \)-manifold in \( X_1 \), and the tangent of \( E \) at \( u_* \) is isomorphic to \( N(A_0) \).
Proof. We take into account equilibria \( u = (v_{\text{atm}}, v_{\text{oce}}, v_{\text{ice}}, h, a) \in V \cap X_1 \) with \( \|u - u_*\|_{X_1} < r \) for given \( r > 0 \). Multiplying the sea ice momentum part in the resulting stationary equation \( 0 = -A(u)u + F_s(u) \) by \( \rho_{\text{ice}} h \), we find that \( u \) satisfies the equation

\[
0 = \left( \rho_{\text{ice}} h \left( C_{\text{o,i}} h \partial_{\Gamma_{\text{o}}} v_{\text{oce}} - \Lambda^H (v_{\text{ice}}, h, a) v_{\text{ice}} - B_k(h, a) h - B_a(h, a) a + (v_{\text{ice}} \cdot \nabla H) v_{\text{ice}} \right) \right) .
\]

(10.9)

Again, we require some preparation. First, we consider \( u_i = (v_i, w_i) \) as well as \( \Omega_i \), where \( i = \text{atm} \) or \( i = \text{oce} \) depending on whether we study the terms associated to the atmosphere or the ocean. For simplicity, we drop the index for the following computation, and we observe that \( \text{div} \ u = 0 \) is valid in both cases. Using this condition, the divergence theorem, the periodic boundary conditions on the lateral boundary as well as \( w = 0 \) on the upper and lower boundary, see (3.6), we obtain

\[
\int_{\Omega} (u \cdot \nabla) v \cdot v \, d(x_H, z) = \frac{1}{2} \int_{\Omega} \text{div} \left( |v|^2 u \right) d(x_H, z) = \frac{1}{2} \int_{\partial \Omega} |v|^2 \left( \frac{\nabla}{|\nabla|} \right) \cdot \hat{n} dS = 0,
\]

where \( \hat{n} \) denotes the outer normal vector. For the oceanic term, it is crucial to note that \( \hat{n} = (0, 0, \pm 1)^T \) on the upper and lower boundary, so the coupling condition with sea ice on \( \Gamma_{\text{o}} \) does not come into play. For a similar result, we also refer to [26, Lemma 6.3]. By \( (u \cdot \nabla) v = (v \cdot \nabla H) v + w(v) \cdot \partial_z v \), we then get

\[
\int_{\Omega_{\text{atm}}} ((v_{\text{atm}} \cdot \nabla H) v_{\text{atm}} + w(v_{\text{atm}}) \cdot \partial_z v_{\text{atm}}) \cdot v_{\text{atm}} \, d(x_H, z) = 0, \quad \text{and}
\]

\[
\int_{\Omega_{\text{oce}}} ((v_{\text{oce}} \cdot \nabla H) v_{\text{oce}} + w(v_{\text{oce}}) \cdot \partial_z v_{\text{oce}}) \cdot v_{\text{oce}} \, d(x_H, z) = 0.
\]

(10.10)

We mainly write \( \varepsilon = \varepsilon(v_{\text{ice}}) \) in the sequel for convenience. For \( u \in V \), we especially deduce that \( P(h, a) \geq p^* \kappa \exp(-c) =: P_* \) as well as \( \frac{1}{\mu_1(c)} \geq \frac{1}{\sqrt{\delta + c_e r^2}} \) for a suitable constant \( c_e > 0 \), see also the proof of Lemma 7.2 in [9]. We emphasize that \( P_* \) is independent of \( u, \delta \) and \( r \). Using an integration by parts in conjunction with the periodic boundary conditions, exploiting the previous estimates, invoking \( \varepsilon^T \Sigma \varepsilon = \Delta^2(\varepsilon) \geq 0 \) by virtue of [9, (4.4)] and employing Korn’s inequality, we infer that

\[
-\langle \rho_{\text{ice}} h \Lambda^H (v_{\text{ice}}, h, a) v_{\text{ice}}, v_{\text{ice}} \rangle_{L^2(G)} = -\int_G \text{div}_H \left( \frac{P(h, a)}{2} \frac{\Sigma \varepsilon}{\Delta(\varepsilon)} \right) \cdot v_{\text{ice}} \, d(x_H)
\]

\[
\geq \frac{P_*(1 - 1/\varepsilon)}{2\sqrt{\delta + c_e r^2}} \left( \|\text{div}_H v_{\text{ice}}\|_{L^2(G)}^2 + \|\varepsilon(v_{\text{ice}})\|_{L^2(G)}^2 \right)
\]

\[
\geq \frac{C_K}{\sqrt{\delta + c_e r^2}} \|\nabla_H v_{\text{ice}}\|_{L^2(G)}^2
\]

(10.11)

for a suitable constant \( C_K > 0 \) emerging from Korn’s inequality and incorporating \( P_*(1 - 1/\varepsilon) \).

For \( u \in V \cap X_1 \) with \( \|u - u_*\|_{X_1} < r \), we test the above equation (10.9) with \( (v_{\text{atm}}, v_{\text{oce}}, c_3 h, c_5 a) \), where \( c_3 = \frac{1}{\mu_{\text{ice}}} \), \( c_4 = c_3 \frac{p^* \kappa \exp(-c(1-a))}{\Delta^2} \), and \( c_5 = c_3 \frac{p^* \kappa \exp(-c(1-a))}{\Delta^2} \). Making use of (10.4), (10.5), (10.7), (10.10) as well as (10.11), remarking that

\[
-\langle \partial_{\nu_{\text{oce}}} v_{\text{ice}}, v_{\text{ice}} \rangle_{L^2(G)} + c_3 \mu_{\text{oce}} \langle \partial_{\nu_{\text{oce}}} v_{\text{ice}}, v_{\text{ice}} \rangle_{L^2(G)} = 0
\]

in view of the choice of \( c_3 \) above and integrating by parts, we obtain

\[
0 \geq \|\nabla v_{\text{atm}}\|_{L^2(\Omega_{\text{atm}})}^2 + C_1 \|v_{\text{oce}}\|_{H^1(\Omega_{\text{oce}})}^2 + \frac{c_3 C_K}{\sqrt{\delta + c_e r^2}} \|\nabla_H v_{\text{ice}}\|_{L^2(G)}^2 + c_4 d_h \|\nabla h\|_{L^2(G)}^2
\]

\[
+ c_5 d_a \|\nabla h a\|_{L^2(G)}^2 + c_3 \rho_{\text{ice}} \int_G h \langle (v_{\text{ice}} \cdot \nabla H) v_{\text{ice}} \rangle \cdot v_{\text{ice}} \, d(x_H)
\]

\[
+ \int_G \left( \frac{c_3 \partial_a P(h, a)}{2} - c_4 h \right) \nabla H \cdot v_{\text{ice}} \, d(x_H) + \int_G \left( \frac{c_3 \partial_a P(h, a)}{2} - c_5 a \right) \nabla h a \cdot v_{\text{ice}} \, d(x_H).
\]

(10.12)
It remains to verify that the terms without sign in (10.12) can be absorbed. Arguing as in the proof of Lemma 7.2 in [9], and using Corollary 8.5(e) for an embedding of the trace space $X_\gamma$, we find that there is a constant $C_\ast > 0$ such that

$$\|c_3 \frac{\partial h P(h, a)}{2} - c_4 h\|_{L^\infty(G)} = \frac{c_3}{2} \|p^* \exp(-c(1-a)) - p^* \frac{h}{h_*} \exp(-c(1-a_*))\|_{L^\infty(G)} \leq C_\ast r,$$

and

$$\|c_3 \frac{\partial a P(h, a)}{2} - c_5 a\|_{L^\infty(G)} = \frac{c_3}{2} \|p^* h \exp(-c(1-a)) - p^* \frac{a}{a_*} \exp(-c(1-a_*))\|_{L^\infty(G)} \leq C_\ast r.$$

On the other hand, it follows from $\text{tr}_{\Gamma_o} v_{\text{ocn}} = v_{\text{ice}}$ on $\Gamma_o$ as well as the continuity of the trace from $H^1(\Omega_{\text{ocn}})$ to $L^2(\Gamma_o)$ that $\|v_{\text{ice}}\|_{L^2(G)}^2 = \|\text{tr}_{\Gamma_o} v_{\text{ocn}}\|_{L^2(\Gamma_o)}^2 \leq C_{tr} \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}$. A concatenation of the previous two observations, Hölder’s inequality and Young’s inequality then leads to

$$\int_G \left( c_3 \frac{\partial h h P(h, a)}{2} - c_4 h \right) \nabla H h \cdot v_{\text{ice}} \, dx \geq -C_{\ast\ast} r \left( \|\nabla H h\|_{L^2(G)}^2 + \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}^2 \right),$$

and

$$\int_G \left( c_3 \frac{\partial a P(h, a)}{2} - c_5 a \right) \nabla H a \cdot v_{\text{ice}} \, dx \geq -C_{\ast\ast} r \left( \|\nabla H a\|_{L^2(G)}^2 + \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}^2 \right).$$

for another constant $C_{\ast\ast} > 0$. Similarly, we conclude

$$0 \geq \|\nabla H h\|_{L^2(G)}^2 + (C_{\ast\ast} r + C_{\ast\ast\ast} (r + \mu_* r)) \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}^2,$$

$$+ \left( \frac{c_3 C_K}{\sqrt{\delta + c e r^2}} - C_{\ast\ast\ast} (r + \mu_* r) \right) \|\nabla H h\|_{L^2(G)}^2 + (c_{34} h - C_{\ast\ast\ast} r) \|\nabla H a\|_{L^2(G)}^2,$$

Choosing $r$ sufficiently small, we deduce therefrom that

$$0 \geq C \left( \|\nabla H h\|_{L^2(G)}^2 + \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}^2 + \|\nabla H h\|_{L^2(G)}^2 + \|\nabla H a\|_{L^2(G)}^2 \right)$$

for some constant $C > 0$. Consequently, for $u = (v_{\text{atm}}, v_{\text{ocn}}, v_{\text{ice}}, h, a) \in \mathcal{E}$ with $u \in V \cap \mathcal{X} \ell$ and $\|u - u_\tau\|_{\mathcal{X}_\ell} < r$ for $r > 0$ sufficiently small, it holds that $v_{\text{atm}}, v_{\text{ice}}, h$ and $a$ are constant and $v_{\text{ocn}} = 0$. As $\text{tr}_{\Gamma_o} v_{\text{ocn}} = v_{\text{ice}}$ on $\Gamma_o = G$, it follows that $v_{\text{ice}} = 0$ as well.

On the other hand, setting $\lambda = 0$ in the proof of Lemma 10.1, we find that an element $u_0 \in N(A_0)$, with $u_0 = (v_{\text{atm}}(0), v_{\text{ocn}}(0), v_{\text{ice}}(0), h_0, a_0)$, has to satisfy that $v_{\text{atm}}(0), v_{\text{ice}}(0), h_0$ as well as $a_0$ are constant and $v_{\text{ocn}}(0) = 0$, implying that also $v_{\text{ice}}(0) = 0$ is valid due to the coupling condition. As an element of this shape is especially contained in $N(A_0)$, we find that $\mathcal{E} = N(A_0)$ holds in a neighborhood of $u_\ast$.

In summary, it is in particular valid that near $u_\ast$, the set of equilibria $\mathcal{E}$ is a $C^1$-manifold in $\mathcal{X}_\ell$ of dimension 4 and that the tangent space for $\mathcal{E}$ at $u_\ast$ is isomorphic to $N(A_0)$.

Lemma 10.3. For the total linearization $A_0$ with domain $D(A_0) = \mathcal{X}_\ell$, it holds that $0$ is a semi-simple eigenvalue of $A_0$, meaning that $N(A_0) \oplus R(A_0) = \mathcal{X}_\ell$.

Proof. First, the structure of $A_0$ from (10.3) yields that $A_{\text{atm}}$ can be investigated separately. Using the representation of the hydrostatic Stokes operator with Neumann boundary conditions on the upper and lower boundary $A_{\text{atm}}$ from the proof of Theorem 3.1 in [16], we find that $A_{\text{atm}}$ inherits the property that $0$ is a semi-simple eigenvalue from the respective Neumann-Laplacian operator. It is thus sufficient to consider the remaining matrix $A_{0\text{rem}}$ given by

$$A_{0\text{rem}} \begin{pmatrix} v_{\text{ocn}} \\ v_{\text{ice}} \\ h \\ a \end{pmatrix} = \begin{pmatrix} \frac{A_{\text{m}} v_{\text{ocn}}}{\text{div}_{H} v_{\text{ice}}} \\ -C_{\text{u,i}}(h_a) \partial_{\text{atm}} v_{\text{ocn}} + A^H(u_\ast) v_{\text{ice}} + B_h(u_\ast) h + B_a(u_\ast) a \\ d_h \Delta h - h_a \text{div}_{H} v_{\text{ice}} \\ d_a \Delta a - a_a \text{div}_{H} v_{\text{ice}} \end{pmatrix}$$

on $A_{0\text{rem}} \subseteq L^2(\Omega_{\text{ocn}}) \times L^2(G) \times L^2(G) \times L^2(G)$ with adjusted domain $D(A_{0\text{rem}})$.

In the sequel, we denote by $L^2_0(G)$ the space of all $L^2$-functions on $G$ with mean value zero, i. e.,

$$\frac{1}{|G|} \int_G b \, dx = 0 \quad \text{for} \quad b \in L^2_0(G).$$
We then study $\mathcal{A}_{\text{rem}}$, resulting from restricting $\mathcal{A}_{\text{rem}}$ to $\mathcal{A}_{\text{rem}} = L^2_0(\Omega_{\text{ocn}}) \times L^6(G)^2 \times L^6(G) \times L^6(G)$. As in the proof of Lemma 10.1 for $\lambda = 0$, we obtain

$$0 \geq C \left( \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}^2 + \|Hv_{\text{ice}}\|_{L^2(G)}^2 + \|\nabla h\|_{L^2(G)}^2 + \|\nabla H a\|_{L^2(G)}^2 \right)$$

when testing the equation $\mathcal{A}_{\text{rem}}(v_{\text{ocn}}, v_{\text{ice}}; h, a) = 0$ for $(v_{\text{ocn}}, v_{\text{ice}}, h, a) \in D(\mathcal{A}_{\text{rem}}) = D(\mathcal{A}_{\text{rem}}) \cap \mathcal{A}_{\text{rem}}$ suitably. We deduce therefrom that $v_{\text{ocn}} = 0$ and $v_{\text{ice}}$, as well as $h$ and $a$ are constant so that $\text{tr}_{\Gamma_{\text{ocn}}} = v_{\text{ice}}$ on $\Gamma_0$ and $h, a \in L^6(G)$ yield that $v_{\text{ice}} = 0$ as well as $h = a = 0$. Consequently, 0 is not an eigenvalue of $\mathcal{A}_{\text{rem}}$. The compact resolvent, see Corollary 8.5(g), which carries over to the present setting, then implies that $0 \in \rho(\mathcal{A}_{\text{rem}})$.

By the above argument, this time applied to $\mathcal{A}_{\text{rem}} = L^2_0(\Omega_{\text{ocn}}) \times L^6(G)^2 \times L^6(G) \times L^6(G)$ as the underlying ground space, we get that $N(\mathcal{A}_{\text{rem}}) = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$. Thus, to show that $\mathcal{A}_{\text{rem}} = N(\mathcal{A}_{\text{rem}}) + R(\mathcal{A}_{\text{rem}})$, it is sufficient to verify that $L^2_0(\Omega_{\text{ocn}}) \times L^6(G)^2 \times L^6(G) \times L^6(G) \subset R(\mathcal{A}_{\text{rem}})$. To this end, we consider $f = (f_{\text{ocn}}, f_{\text{ice}}, f_h, f_a) \in L^2_0(\Omega_{\text{ocn}}) \times L^6(G)^2 \times L^6(G) \times L^6(G)$. By virtue of $0 \in \rho(\mathcal{A}_{\text{rem}})$, there exists $(v_{\text{ocn}}, v_{\text{ice}}, h, a) \in D(\mathcal{A}_{\text{rem}})$ such that

$$\mathcal{A}_{\text{rem}}(v_{\text{ocn}}, v_{\text{ice}}, h, a) = f,$$

resulting in $R(\mathcal{A}_{\text{rem}}) \subset \mathcal{A}_{\text{rem}}$ and thus also in $\mathcal{A}_{\text{rem}} = N(\mathcal{A}_{\text{rem}}) + R(\mathcal{A}_{\text{rem}})$ by the above argument.

It remains to check that $N(\mathcal{A}_{\text{rem}}) \cap R(\mathcal{A}_{\text{rem}}) = \{0\}$. For $u = (v_{\text{ocn}}, v_{\text{ice}}, h, a) \in N(\mathcal{A}_{\text{rem}}) \cap R(\mathcal{A}_{\text{rem}})$, it follows from $N(\mathcal{A}_{\text{rem}}) = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$ that $u = (0, 0, c_h, c_a)$ for $c_h$ and $c_a$ constant. On the other hand, by $u \in R(\mathcal{A}_{\text{rem}})$, there exists $u' = (v_{\text{ocn}}, v_{\text{ice}}, h, a) \in D(\mathcal{A}_{\text{rem}})$ such that $\mathcal{A}_{\text{rem}}u' = u$. We then consider

$$u' = \begin{pmatrix} v_{\text{ocn}} \\ v_{\text{ice}} \\ h \\ a \end{pmatrix} = \begin{pmatrix} \frac{1}{|G|} \int_G h \, dx_H \\ \frac{1}{|\hat{G}|} \int_{\hat{G}} h \, d\hat{x}_H \end{pmatrix}, \quad \text{where } \frac{1}{|G|} = \frac{1}{|\hat{G}|} = \frac{1}{|G|},$$

and likewise for $a$. Since $h$ and $\bar{a}$ are constant and $\bar{h}, \bar{a} \in L^6(G)$, we infer that $(0, 0, h, a) \in N(\mathcal{A}_{\text{rem}})$ as well as $(v_{\text{ocn}}, v_{\text{ice}}, h, a) \in D(\mathcal{A}_{\text{rem}})$. Consequently, we have

$$\begin{pmatrix} 0 \\ 0 \\ c_h \\ c_a \end{pmatrix} = u = \mathcal{A}_{\text{rem}}u' = \mathcal{A}_{\text{rem}} \begin{pmatrix} v_{\text{ocn}} \\ v_{\text{ice}} \\ h \\ a \end{pmatrix} + \mathcal{A}_{\text{rem}} \begin{pmatrix} 0 \\ \frac{1}{|G|} \\ \frac{1}{|\hat{G}|} \end{pmatrix} = \mathcal{A}_{\text{rem}} \begin{pmatrix} v_{\text{ocn}} \\ v_{\text{ice}} \\ h \\ a \end{pmatrix} = \mathcal{A}_{\text{rem}} \begin{pmatrix} v_{\text{ocn}} \\ v_{\text{ice}} \\ h \\ a \end{pmatrix},$$

i.e., $u \in R(\mathcal{A}_{\text{rem}}) = L^2_0(\Omega_{\text{ocn}}) \times L^6(G)^2 \times L^6(G) \times L^6(G)$, so it follows that $c_h = c_a = 0$ and therefore, it holds that $u = 0$. In summary, it is valid that $N(\mathcal{A}_{\text{rem}}) \oplus R(\mathcal{A}_{\text{rem}}) = \{0\}$. In conjunction with the above argument concerning the treatment of $A_{\text{atm}}$, the assertion of the lemma follows.

**Lemma 10.4.** For $u_*$ as above, $-A(u_*)$ has the property of maximal regularity on $X_\sigma$.

**Proof.** Combining the triangular structure of $A(u_*)$ as in (10.2) and the maximal regularity of $A_{\text{atm}}$, see [16, Corollary 3.4], and of the horizontal Laplacians on $G$, we find that the task reduces to establishing maximal regularity of

$$A_{\text{ocn},i}(u_*) = \begin{pmatrix} A^\text{ocn}_{\text{atm}} \\ -C_{\text{ocn,i}}(h_*) \partial_{\Gamma_{\text{atm}}} \end{pmatrix},$$

with adjusted domain $A_{\text{ocn},i}(u_*)$, on the resulting ground space $L^2_0(\Omega_{\text{ocn}}) \times L^2(G)^2$.

As a result of the above argument in conjunction with Corollary 8.5(c), there is $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$, it holds that $-A_{\text{ocn},i}(u_*) + \omega$ has the property of maximal regularity. It can be shown that $\omega_0$ can be chosen to be equal to the spectral bound of $s(A_{\text{ocn,i}}(u_*))$ of $A_{\text{ocn},i}(u_*)$. With regard to the $q$-independence of the spectrum, see Corollary 8.5, carrying over to the present situation, it suffices to study the spectrum in the $L^2$-case.

Testing the equation $(\lambda - A_{\text{ocn},i}(u_*))(v_{\text{ocn}}, v_{\text{ice}}) = 0$ with $(v_{\text{ocn}}, c_1 v_{\text{ice}})$, where $c_1 = \frac{1}{v_{\text{ocn}}(u_*)}$, and making use of (10.5) as well as (10.6), we conclude that there is a constant $C > 0$ such that

$$0 \geq \lambda \|v_{\text{ocn}}\|_{L^2(\Omega_{\text{ocn}})}^2 + \lambda c_1 \|v_{\text{ice}}\|_{L^2(G)}^2 + C \left( \|v_{\text{ocn}}\|_{H^1(\Omega_{\text{ocn}})}^2 + \|\nabla H v_{\text{ice}}\|_{L^2(G)}^2 \right),$$
so $s(A_{	ext{ice}}(u_\ast)) < 0$ by virtue of the coupling condition $\text{tr} \Gamma_o v_{\text{ocn}} = v_{\text{ice}}$ on $\Gamma_o$, and the result thus follows by the above arguments. \hfill \Box

Proof of Theorem 3.4: Concatenating Lemma 10.1, Lemma 10.2, Lemma 10.3 and Lemma 10.4, we find that an equilibrium $u_\ast$ of the above shape is normally stable. The generalized principle of linearized stability, see [41, Theorem 5.3.1] and also [42], yields the assertion of Theorem 3.4. \hfill \Box

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