Lowest order covariant averaging of a perturbed metric and of the Einstein tensor II

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Abstract: We generalize and simplify an earlier approach. In three dimensions we present the most general averaging formula in lowest order which respects the requirements of covariance. It involves a bitensor, made up of a basis of six tensors, and contains three arbitrary functions, which are only restricted by their behavior near the origin. The averaging formula can also be applied to the Einstein tensor. If one of the functions is put equal to zero one has the pleasant property that the Einstein tensor of the averaged metric is identical to the averaged Einstein tensor. We also present a simple covariant extension to static perturbations in four dimensions. Unfortunately the result for the Einstein tensor cannot be extended to the four dimensional case.
1 Introduction

The energy-momentum-tensor used in cosmological models is an average over the non homogeneous tensor present in nature. Therefore one needs an averaging prescription for the energy momentum tensor and for the metric. This provides a fundamental problem because of the freedom of choice of coordinates. The averaging problem in general relativity was first raised by Shirkov and Fisher [1] in 1963. The authors of [1] suggested to integrate the metric tensor over a four dimensional volume with the familiar factor $\sqrt{-g}$ in the measure. Such an expression is, however, not covariant due to the freedom of performing local transformations. A covariant averaging prescription can be constructed by introducing a bivector $g^\beta_\alpha(x, x')$ of geodesic parallel displacement, as discussed in the appendix of [2]. This transforms as a vector with respect to coordinate transformations at either $x$ or $x'$ and maps a vector $A_\beta(x')$ to $\bar{A}_\alpha(x) = g^\beta_\alpha(x, x')A_\beta(x')$, analogously for higher order tensors. An averaging with the help of bivectors was also used in the work of Zalaletdinov [3] where the emphasis was on the commutativity of averaging and covariant differentiation. As remarked by Stoeger, Helmi, and Torres [4], the method of using a covariantly conserved bivector is not applicable to the metric, because the covariant derivative of the metric vanishes. The metric is therefore invariant under this averaging procedure. In the thesis of Behrend [5] the metric is represented by tetrads and the averaging performed over the latter. The tetrads are chosen according to a covariant minimalization prescription.

This is only a very brief survey of the literature. For more references, as well as the implications for the fitting problem, back reaction, contributions to dark energy, we refer e.g. to the monograph of Krasinski [6] and to the comprehensive recent reviews of Buchert [7] and of Malik and Wands [8].

In a previous paper [9] we gave an explicit solution. The derivation was rather complicated, furthermore we could not classify the complete set of solutions. The present approach is much more transparent and straight forward. It naturally leads to a classification of all solutions which fulfill the central requirements (1.1) and (1.2) below.

Under a covariant averaging process we understand a prescription which has the following properties. Let two observers describe the same physics in different coordinate systems $S$ and $S'$, with metric tensors $g_{\mu\nu}$ and $g'_{\mu\nu}$. Both of them apply a definite averaging procedure in their respective systems, resulting in the averaged metrics $<g_{\mu\nu}>$ and $<g'_{\mu\nu}>$, respectively. Then the results have to be connected by the same transformation as the original metric, i. e.

$$<g'_\mu\nu> = <g_{\mu\nu}>'. \tag{1.1}$$

In other words, the operations of averaging and of coordinate transformations have to commute.

Furthermore, averaging over a region which is closely located around some point should, of course, reproduce the metric at this point. In this situation only the metric at the origin is relevant. This means that a constant metric has to be reproduced by the averaging process, i. e.

$$<\eta_{\mu\nu}> = \eta_{\mu\nu} \text{ for constant } \eta_{\mu\nu}. \tag{1.2}$$
It is rather obvious that space and time cannot be treated at the same footing in an averaging prescription. Therefore, as usual, we have to assume that a reasonable foliation into space and time can be performed.

In sect. 2 we present the most general lowest order three dimensional covariant averaging formula. The mapping of the metric \( g_{kl}(x') \) to the averaged metric \(< g_{mn} > (x) \) is represented by a bitensor \( K_{mn}^{kl}(x' - x) \) which we will specify in detail. A product of bivectors, as frequently used in the literature, is not sufficient. The bitensor \( K_{mn}^{kl}(x' - x) \) is a superposition of a basis of six tensors, and contains three arbitrary functions \( u(r), v(r), \) and \( w(r) \), which depend upon the distance \( r = |x' - x| \) and are only restricted by a prescribed behavior near the origin. They are necessarily singular there. Our earlier formula in [9] is a special case of our general formula as it should be. In sect. 3 we show that, under the condition \( w(r) = 0 \), we can use the same formula which works for the metric as well for the Einstein tensor. This implies that the Einstein equations for the averaged metric are identical to the averaged equations. In sect. 4 we show why iteration of the averaging formula does not make sense. Therefore one has to choose reasonable functions from the beginning. In sect. 5 we discuss a simple four dimensional covariant generalization for static perturbations. The proof for the covariant averaging of the Einstein tensor “almost” goes through also in this case, but fails at the very end.

We claim that our formulae present the most general framework for a lowest order averaging prescription which respects (1.1) and (1.2).

2 The most general lowest order three dimensional covariant averaging formula

Consider a perturbed flat metric \( g_{kl}(x) = \delta_{kl} + h_{kl}(x) \). In lowest order there must be a linear relation between the original metric and the averaged one. For simplicity, we consider the averaging at the origin for the moment.

\[
< g_{mn} > (0) = \int K_{mn}^{kl}(x) g_{kl}(x) \frac{d^3x}{4\pi}.
\]

The same relation holds for the perturbation \( h_{mn} \), because, as we shall see, averaging of the constant \( \delta_{mn} \) gives back this constant as required by (1.2) (the above \( K_{mn}^{kl}(x) \) is identical to the \( \tilde{K}_{mn}^{kl}(x) \) in [9]).

In order to guarantee covariance with respect to rigid rotations, one has to construct the most general structure of the tensor (tensor in the sense of linear algebra) \( K_{mn}^{kl}(x) \) which is symmetric under the exchange \( m \leftrightarrow n \) and under \( k \leftrightarrow l \). This is represented by a complete set of six independent tensors which are multiplied by functions \( A(r), \cdots, F(r) \), which depend only upon the distance \( r = |x| \):
\[ K^{kl}_{mn}(x) = A(r)[\delta^k_m \delta^l_n + \delta^k_n \delta^l_m] + B(r)\delta_{mn} \delta^{kl} + C(r)\delta_{mn} \frac{x^k x^l}{r^2} + D(r)\delta^{kl} \frac{x_m x_n}{r^2} + E(r)\left[\delta^k_m \frac{x_n x^l}{r^2} + \delta^l_m \frac{x_n x^k}{r^2} + \delta^k_n \frac{x_m x^l}{r^2} + \delta^l_n \frac{x_m x^k}{r^2}\right] + F(r)\frac{x_m x_n x^k x^l}{r^4}. \] (2.2)

We emphasize that this is the most general tensor structure which one can write down. Naive averaging would only make use of \(A(r)\), normalized to \(\int_0^\infty 2A(r)r^2 dr = 1\), while all the other functions \(B(r), \cdots, F(r)\) vanish.

Let us now apply an arbitrary infinitesimal transformation \(x^k = x'^k + \xi^k\), which leads to a change \(\delta g_{kl} = \xi_{k,l} + \xi_{l,k} \rightarrow 2\xi_{k,l}\), when contracted with the symmetrical tensors in (2.2). Covariance according to the requirement (1.1) implies that the change of the rhs of (2.1) must be identical to the change of the lhs, i.e. \(\xi_{mn}(0) + \xi_{m,n}(0)\). This means that the rhs cannot depend upon \(\xi_k(x)\), except at the origin \(x = 0\). To see how this can happen we perform a partial integration with respect to \(x_l\). The change of the integrand on the rhs of (2.1) then becomes \(-2K^{kl}_{mn,l} \xi_k\).

This has to vanish for arbitrary \(\xi_k\), which implies \(K^{kl}_{mn,l} = 0\). Therefore \(K^{kl}_{mn}\) has to be a curl with respect to \(l\), and, by symmetry, also with respect to \(k\), i.e. we can put

\[ K^{kl}_{mn}(x) = \epsilon^{k}_{ab} \epsilon^{l}_{cd} \partial^a \partial^c T^{bd}_{mn}(x). \] (2.3)

The tensor \(T^{bd}_{mn}(x)\) has a decomposition analogous to (2.2), we denote the six radial functions by the corresponding small letters \(a(r), \cdots, f(r)\).

\[ T^{bd}_{mn}(x) = a(r)[\delta^b_m \delta^d_n + \delta^b_n \delta^d_m] + b(r)\delta_{mn} \delta^{bd} + c(r)\delta_{mn} \frac{x^b x^d}{r^2} + d(r)\delta^{bd} \frac{x_m x_n}{r^2} + e(r)\left[\delta^b_m \frac{x_n x^d}{r^2} + \delta^d_m \frac{x_n x^b}{r^2} + \delta^b_n \frac{x_m x^d}{r^2} + \delta^d_n \frac{x_m x^b}{r^2}\right] + f(r)\frac{x_m x_n x^b x^d}{r^4}. \] (2.4)

This can be introduced into (2.3) and, after performing the differentiations, be compared with the general decomposition (2.2). The result is

\begin{align*}
A(r) &= -a''(r) - \frac{d(r)}{r^2} + 2\left[\frac{e'(r)}{r} - \frac{e(r)}{r^2}\right] - \frac{f(r)}{r^2}, \\
B(r) &= 2a''(r) + \left[2b'(r) + \frac{b'(r)}{r^2}\right] - \frac{c'(r)}{r} + 2\frac{d(r)}{r^2} - 4\left[\frac{e'(r)}{r^2} - \frac{e(r)}{r^2}\right] + 2\frac{f(r)}{r^2}, \\
C(r) &= -2[a''(r) - \frac{a'(r)}{r}] - [b''(r) - \frac{b'(r)}{r^2}] + \left[\frac{c'(r)}{r} - \frac{2c(r)}{r^2}\right] + 4\left[\frac{e'(r)}{r^2} - \frac{2e(r)}{r^2}\right] - \frac{2f(r)}{r^2}, \\
D(r) &= -2[a''(r) - \frac{a'(r)}{r}] + [d''(r) + \frac{d'(r)}{r^2}] - \frac{d(r)}{r^2} - 4\left[\frac{e'(r)}{r^2} - \frac{e(r)}{r^2}\right] + 4\left[\frac{f'(r)}{r} + \frac{f(r)}{r^2}\right], \\
E(r) &= [a''(r) - \frac{d'(r)}{r}] - \frac{d(r)}{r} + \frac{d'(r)}{r^2} - 2\left[\frac{e'(r)}{r} - \frac{2e(r)}{r^2}\right] + \frac{f'(r)}{r} + \frac{f(r)}{r^2}, \\
F(r) &= [-d''(r) + 5\frac{d'(r)}{r^2} - 8\frac{d(r)}{r^2}] + \frac{f'(r)}{r} - 4\frac{f(r)}{r^2}. \end{align*}
\[ (2.5) \]
Up to now we have six functions \( a(r), \cdots, f(r) \). But only three of them are relevant. To see this we first put \( a'(r) = \tilde{a}(r)/r \) and \( b'(r) = \tilde{b}(r)/r \). One then finds that \( c(r) \) and \( \tilde{a}(r) \) only appear in the combination \( e(r) - \tilde{a}(r)/2 \), while \( c(r) \) and \( \tilde{b}(r) \) only appear in the combination \( c(r) - \tilde{b}(r) \). One thus can rename these combinations as \( e(r) \) and \( c(r) \), i. e. one can put \( a(r) = b(r) = 0 \) in (2.5). The next step is more subtle. Introduce two new functions \( g(r) \) and \( w(r) \) by \( g(r) = [c(r) + f(r)]/2 \) and \( w(r)/r^2 = C(r) - D(r) \). Eliminate \( c(r) = [-r^2d''(r)/4 - rd'(r)/4 + d(r)] + [rg'(r)/2 + g(r)] - w(r)/4 \), and \( f(r) = [v^2d''(r)/4 + rd'(r)/4 - d(r)] - [rg'(r)/2 - g(r)] + w(r)/4 \), and use the functions \( g(r) \) and \( w(r) \) instead of \( c(r) \) and \( f(r) \). If we further put \( g(r) = rg'(r) - 2g(r) \), and eliminate \( d(r) \) by introducing the function \( u(r) = d(r) - 2c(r) \), we find that \( c(r) \) and \( \tilde{g}(r) \) only appear in the combination \( v(r) = e(r) - \tilde{g}(r) \). Therefore we end up with three relevant functions \( u(r), v(r), w(r) \). (These \( u(r), v(r) \) have, of course, nothing to do with the projective coordinates \( u, v \) introduced in [9].) We thus have found the following representations for the functions \( A(r), \cdots, F(r) \) in (2.2):

\[
A(r) &= -\frac{1}{4}\left[u''(r) + \frac{u'(r)}{r}\right] + 2\left[\frac{v'(r)}{r} - \frac{v(r)}{r^2}\right] - \frac{1}{4}w(r), \\
B(r) &= \frac{1}{4}\left[r^2u''(r) + 5u''(r) - \frac{u'(r)}{r}\right] - 4\left[\frac{v'(r)}{r} - \frac{v(r)}{r^2}\right] + \frac{1}{4}\left[\frac{w'(r)}{r} + 2\frac{w(r)}{r^2}\right], \\
C(r) &= -\frac{1}{4}\left[r^2u''(r) + 3u''(r) - 3\frac{u'(r)}{r}\right] + 4\left[\frac{v'(r)}{r} - 2\frac{v(r)}{r^2}\right] - \frac{1}{4}\frac{w'(r)}{r}, \\
D(r) &= -\frac{1}{4}\left[r^2u''(r) + 3u''(r) - 3\frac{u'(r)}{r}\right] + 4\left[\frac{v'(r)}{r} - 2\frac{v(r)}{r^2}\right] - \frac{1}{4}\frac{w'(r)}{r} + 4\frac{w(r)}{r^2} \\
&= C(r) - \frac{w(r)}{r^2}, \\
E(r) &= \frac{1}{2}\left[u''(r) - \frac{u'(r)}{r}\right] - 2\left[\frac{v'(r)}{r} - 2\frac{v(r)}{r^2}\right] + \frac{1}{2}\frac{w(r)}{r^2}, \\
F(r) &= \frac{1}{4}\left[r^2u''(r) - 5u''(r) + 13\frac{u'(r)}{r}\right] - 16\left[\frac{v'(r)}{r} - 4\frac{v(r)}{r^2}\right] + \frac{1}{4}\frac{w'(r)}{r} - 4\frac{w(r)}{r^2].
\]

We next demand that the averaging over a region which is closely localized around the origin should give back the metric \( g_{mn}(0) \) at the origin. In this case we may put \( g_{kl}(x) = g_{kl}(0) \) in (2.1) and take it out in front of the integral. The angular averages \( \int d\Omega/4\pi \) can be performed using

\[
\frac{x_m x_n}{r^2} \rightarrow \frac{1}{3} \delta_{mn}, \quad \frac{x_m x_n x_k x_l}{r^4} \rightarrow \frac{1}{15} (\delta_{mn}\delta_{kl} + \delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}) \quad \text{etc.} \quad (2.7)
\]

Comparing the factors in front of the terms \( g_{mn}(0) \) and \( g_{kl}'(0) \delta_{mn} \) on both sides one thus obtains

\[
1 = \int_0^\infty [2A(r) + \frac{4}{3}E(r) + \frac{2}{15}F(r)]r^2dr, \quad 0 = \int_0^\infty [B(r) + \frac{1}{3}C(r) + \frac{1}{3}D(r) + \frac{1}{15}F(r)]r^2dr. \quad (2.8)
\]

This derivation also shows the validity of (1.2), i. e. that the averaging of a constant gives back this constant. Therefore our averaging formula may be applied to the metric
as well as to the perturbation $h_{mn}$. The relations can also be rephrased in the property

$$\int K_{mn}^{kl}(x) \frac{d^3 x}{4\pi} = \frac{1}{2}[\delta_m^k \delta_n^l + \delta_m^l \delta_n^k]. \quad (2.9)$$

Inserting the representations (2.6) one finds that the integrands in (2.8) can be written as derivatives. Assuming that there are no boundary terms at infinity one thus obtains

$$1 = \left[ -\frac{1}{30} r^3 u''(r) + \frac{1}{10} r^2 u'(r) + \frac{8}{15} r u(r) - \frac{4}{3} r v(r) - \frac{1}{30} r w(r) \right]_{r=0},$$

$$0 = \left[ -\frac{1}{10} r^3 u''(r) - \frac{11}{30} r^2 u'(r) + \frac{4}{15} r u(r) + \frac{4}{3} r v(r) - \frac{1}{10} r w(r) \right]_{r=0}. \quad (2.10)$$

The derivation above makes clear that the representation (2.1), (2.2), together with the form (2.6) and the boundary conditions (2.10), is the most general first order covariant averaging formula in three dimensions.

Before we discuss the implications of (2.10) we consider the boundary terms at the origin which arise from the partial integration of $2\xi_{k\ell}$, where $\xi_k$ was the shift of an infinitesimal transformation. Restricting the integration to the outside of a small sphere of radius $\epsilon$ these are just the surface terms which arise from Gauß’s theorem. Using $K_{mn\ell}^{kl} = 0$, one has

$$2 \int K_{mn}^{kl}(x) \xi_{k\ell}(x) \frac{d^3 x}{4\pi} = -2 \int_{r=\epsilon} K_{mn}^{kl}(x) x \frac{\xi_k \epsilon^2 d\Omega}{4\pi}. \quad (2.11)$$

Because invariance with respect to translations and to rigid rotations around the point of consideration is manifest, one can restrict to transformations which leave the origin fixed, such that $\xi_k(x) = \xi_k(0)x^i + O(\epsilon^2)$. If we insert this into (2.11) and use the representation (2.2) for $K_{mn}^{kl}(x)$ we can perform the angular averaging. The result has to be identical to the change $\xi_{m,0} + \xi_{n,0}$ of the lhs. Comparing the factors of $\xi_{m,0} + \xi_{n,0}$ and of $2\delta_{mn}\xi_k(0)$ on both sides one obtains

$$1 = -\left[ \frac{2 r^3}{15} [5A(r) + D(r) + 7E(r) + F(r)] \right]_{r=0},$$

$$0 = -\left[ \frac{r^3}{15} [5B(r) + 5C(r) + D(r) + 2E(r) + F(r)] \right]_{r=0}. \quad (2.12)$$

If one introduces the representations (2.6) for $A(r), \cdots, F(r)$, one obtains again the conditions (2.10).

The boundary conditions (2.10) have drastic consequences for the behavior of the functions at the origin. They imply

$$u(r) = u[-1]/r + O(1), \quad v(r) = v[-1]/r + O(1), \quad w(r) = w[-1]/r + O(1), \quad \text{with}$$

$$u[-1] = \frac{5}{4} + \frac{1}{6} w[-1], \quad v[-1] = -\frac{13}{32} + \frac{1}{48} w[-1]. \quad (2.13)$$
This result is unpleasant but unavoidable. The functions $u(r), v(r), w(r)$ are singular and behave like $1/r$ near the origin. This implies that the functions $A(r), \ldots, F(r)$ go like $1/r^3$. At first sight this might look as if the integrand in (2.1) is not integrable at the origin. This is, however, not the case. Expand $g_{kl}(\mathbf{x}) = g_{kl}(0) + O(x)$. The constant term $g_{kl}(0)$ can be taken out of the integral, the angular averaging of the $1/r^3$-term vanishes. This constant term has just been treated in detail. The rest is of order $r/r^3$ and therefore integrable in three dimensions.

Nevertheless one would have preferred functions $A(r), \ldots, F(r)$ in the averaging formula which are smooth at the origin. But covariance definitely prohibits such a smooth behavior. If one considers the change of the integral which arises from an infinitesimal coordinate transformation $x^k = x'^k + \xi^k$, the boundary terms of the partial integration have to reproduce the change of the metric at the origin, i.e. $\xi_{m,n}(0) + \xi_{n,m}(0)$. This enforces the singular behavior of the functions. Smooth functions could not produce boundary terms.

In [9] we presented a special solution for a covariant averaging procedure. It contained a function $f(r)$ (which has nothing to do with the $f(r)$ in (2.4)), normalized to $\int_0^\infty f(r)dr = 1$, as well as two integrals of $f(r)$, namely $F(r) = - \int_r^\infty f(r')dr'$ and $G(r) = - \int_r^\infty f(r')/r'dr'$. This solution must be a special case of our general formula. To demonstrate this we have to introduce another integral $H(r) = - \int_r^\infty f(r')/r'^3dr'$. Then we obtain our old solution if we put

$$u(r) = - \frac{5}{4} \frac{F(r)}{r} + \frac{15}{8} G(r) - \frac{5}{8} r^2 H(r), \quad v(r) = \frac{13}{32} \frac{F(r)}{r} - \frac{5}{16} r^2 H(r). \quad (2.14)$$

When making this comparison one has to take care of the correct factors $r^2$. In [9] we used the integration element $d^3x/4\pi$ because we had to perform several partial integrations with respect to $r$ there, while in (2.1) we use $d^3x/4\pi = r^2drd\Omega/4\pi$.

### 3 Covariant averaging of the Einstein tensor

Besides the metric, the Einstein tensor is the most important object in general relativity because it enters, together with the energy momentum tensor, directly the field equations. It would be highly desirable if one could average the Einstein tensor in exactly the same way as the metric tensor, and if the averaged Einstein tensor would be identical to the Einstein tensor derived from the averaged metric. The problem that the averaged equations are not identical with the equations for the averaged metric would then disappear. We repeat and extend some of the steps of [9] in order to make the paper self-contained.

In first order of the perturbation the Einstein tensor becomes

$$2G_{mn} = h^i_{i,mn} + h_{mn} + h_{im}^i - h_{im}^i - h_{im}^m - h_{ij}^i + \delta_{mn} + h_{ij}^i \delta_{mn}. \quad (3.1)$$

Indices are raised and lowered with $\delta_{ij}$ here, so their position is in fact irrelevant. The averaging formula (2.1) is now used for an arbitrary point $\mathbf{x}$, the integration variables are denoted by a prime, and the tensor $K_{mn}^{kl}(\mathbf{x'} - \mathbf{x})$ depends on the difference $\mathbf{x'} - \mathbf{x}.$
The second possibility is to calculate the Einstein tensor $G$ factor of $\hat{1}$ one calculates the Einstein tensor from (3.1), using the averaged metric on the rhs. Then average it with our formulae in exactly the same way as we averaged the metric the distance $r$ the Einstein tensor near the origin, and expand

\[ h_{ij}(x') = h_{ij}(0) + h_{ij},^a(0)x'_a + \frac{1}{2}h_{ij},^ab(0)x'_a x'_b + \hat{h}_{ij}(x'). \]  

The term $\hat{h}_{ij}(x')$ is of order $x'^3$, no problems with potentially divergent contributions or boundary terms from partial integrations can arise. We begin with this term.

The first possibility is to average the metric in the way described before. Subsequently one calculates the Einstein tensor from (3.1), using the averaged metric on the rhs. The factor of $\hat{h}_{kl}(x')$ in the integrand then becomes

\[ I_{mn}^{kl} = K_{mn}^{i,i} + K_{mn,i}^{i} - K_{mn}^{j,k} - K_{in}^{j,i} - K_{im}^{j,k} - K_{ij}^{k} \delta_{mn} + K_{i}^{k} \delta_{mn}. \]  

The second possibility is to calculate the Einstein tensor $G_{kl}$ with the old metric and then average it with our formulae in exactly the same way as we averaged the metric tensor. Shift the partial derivatives from the perturbation $h$ to $K$, and rename dummy indices where necessary such that $h_{kl}$ appears in all six terms. The factor of $\hat{h}_{kl}(x')$ in the integrand now becomes

\[ J_{mn}^{kl} = K_{mn,i}^{i,j} \delta^{kl} + K_{mn,i}^{i} - K_{mn}^{j,k} - K_{in}^{j,i} - K_{im}^{j,k} - K_{ij}^{k} \delta_{mn} + K_{i}^{k} \delta_{mn}. \]  

Consider the difference $\Delta_{mnkl} = I_{mnkl} - J_{mnkl}$. If one combines the terms appropriately one has

\[ \Delta_{mnkl} = (K_{kl}^{i,mn} - K_{mn}^{i,kl}) - (K_{in}^{i} - K_{mn}^{i,kl}) - (K_{kn}^{i,m} - K_{mn}^{i,kl}) - (K_{kl}^{i,m} - K_{mn}^{i,j} \delta_{kl}) + (K_{ik}^{i,j} \delta_{mn} - K_{mn}^{i,j} \delta_{kl}). \]  

The further investigation can be greatly simplified if one decomposes

\[ K_{mnkl} = K_{mnkl}^{[S]} + K_{mnkl}^{[A]}, \]  

where $K_{mnkl}^{[S]}$ is symmetric against the exchange $(m,n) \leftrightarrow (k,l)$ and $K_{mnkl}^{[A]}$ antisymmetric. Obviously $K_{mnkl}^{[S]}$ consists of the terms with $A(r), B(r), E(r), F(r)$ in (2.2), together with the symmetric combination $[C(r) + D(r)] \delta_{mn} x_k x_l / r^2 + \delta_{kl} x_m x_n / r^2 / 2$, while $K_{mnkl}^{[A]}$ consists of the antisymmetric combination $[C(r) - D(r)] \delta_{mn} x_k x_l / r^2 - \delta_{kl} x_m x_n / r^2 / 2$. From (3.5) it is seen that the symmetry relations in $\Delta_{mnkl}$ are just reversed with respect to $K_{mnkl}$, i.e. the symmetric part $\Delta_{mnkl}^{[S]}$ is obtained from $K_{mnkl}^{[A]}$, while the antisymmetric part $\Delta_{mnkl}^{[A]}$ is obtained from $K_{mnkl}^{[S]}$. Clearly $\Delta_{mnkl}$ has a decomposition analogous to $K_{mnkl}$ with coefficients $A(r), \cdots, F(r)$, say.

Let us first investigate the antisymmetric part $\Delta_{mnkl}^{[A]}$ which arises from $K_{mnkl}^{[S]}$. Knowing the structure $\Delta_{mnkl}^{[A]} = [\hat{C}(r) - \hat{D}(r)] \delta_{mn} x_k x_l / r^2 - \delta_{kl} x_m x_n / r^2 / 2$, we can simplify the investigation by taking the trace $k = l$, thus we only need to calculate $\Delta_{mnk}^{[A]} = [\hat{C}(r) - \hat{D}(r)] \delta_{mn} - 3 x_m x_n / r^2 / 2$. The result is
The functions \( u(r) \) and \( v(r) \) have dropped out completely, the expression vanishes if \( w(r) = 0 \). To see that this condition is also necessary we only need to consider, e.g. the contractions \( m = n, k = l \) of the symmetric part \( \Delta_{mnl}^{[S]} \), which gives

\[
\Delta_{mn}^{[S]} = -2w''(r)/r^2 - 2w'(r)/r^3 + 8w(r)/r^4.
\]

Both (3.7) and (3.8) have to vanish and this is the case if and only if \( w(r) = 0 \). This is equivalent to the equation \( C(r) = D(r) \), i.e. to the symmetry relation \( K_{mnkl}^{[A]} = 0 \).

We have to check that the independence of the order of averaging also holds for the first three terms in the decomposition (3.2). For the constant and linear terms \( h_{ij}(0) + h_{ij,a}(0)x_a \) this is trivial, they do not contribute to the Einstein tensor, irrespective of the order of averaging. In the average of the quadratic term \( h_{ij,ab}(0)x_a'x_b'/2 \), we substitute \( x' = x = y \), such that \( x_a'x_b' = x_ax_b + x/ay_b + y_ax_b + y_ay_b \). In the first term one can take out \( x_ax_b \) in front of the integral, the remaining integral is given by (2.9), therefore \( h_{ij,ab}(0)x_ax_b/2 \) is reproduced. The other three terms which are linear and constant with respect to \( x \) do not contribute to \( G_{mn} \). If, alternatively, we first calculate \( G_{mn} \), which is constant for this contribution, it is as well preserved by the averaging. So we have seen by direct evaluation that the possibly dangerous low order terms in (3.2) do not generate problems.

We finally have shown that the Einstein tensor of the averaged metric is identical to the averaged Einstein tensor of the original metric if and only if \( w(r) = 0 \), which is equivalent to \( C(r) = D(r) \), i.e. the symmetry relation \( K_{mnkl} = K_{klmn} \). In \([9]\) we had assumed this symmetry in order to simplify the discussion, we now have shown that this condition is necessary. In the following we will always assume \( w(r) = 0 \).

An important feature for the understanding of this property are the symmetry relations shared by our averaging formula and by the Einstein tensor. This becomes clear if one writes

\[
2G_{mn} = T_{mn}^{kl}h_{kl},
\]

with the operator

\[
T_{mn}^{kl} = \delta^{kl}\partial_m\partial_n + \frac{1}{2}(\delta^k_m\delta^n_l + \delta^k_n\delta^l_m)\partial^i\partial_i - \frac{1}{2}(\delta^k_m\partial^i_n\partial^j_l + \delta^l_m\partial^i_n\partial^k + \delta^i_m\partial^k_n\partial^j + \delta^l_m\partial^k_n\partial^j_i) - \delta^{kl}\delta_{mn}\partial^i\partial_i + \delta_{mn}\partial^k\partial^j.
\] (3.10)

Both tensors, \( K_{mnkl} \) as well as \( T_{mnkl} \), are symmetric under \( m \leftrightarrow n \), under \( k \leftrightarrow l \), and under \( (m,n) \leftrightarrow (k,l) \). These symmetries were essential in order to show the vanishing of the difference (3.5).
The result for the covariant averaging of the Einstein tensor is certainly not trivial. For the Ricci tensor, which does not fulfill the above symmetry properties, the relation is not valid.

4 Iteration and stability

The general behavior of iterations is most easily first studied in a simple one dimensional toy model. Consider an averaging formula

\[
< g > (x) \equiv g_{[1]}(x) = \int K(x - x')g(x')dx',
\]

with \( K(x) \) real and even, and normalized to \( \int K(x)dx = 1 \). Because (4.1) is a convolution, it is convenient to work with the Fourier transforms \( \tilde{g}(p) = \int g(x)e^{-ipx}dx \), etc. This implies \( \tilde{K}(0) = 1 \) and \( \tilde{g}_{[1]}(p) = \tilde{K}(p)\tilde{g}(p) \). The iteration of order \( n \) of the averaging procedure becomes

\[
\tilde{g}_{[n]}(p) = \tilde{K}^n(p)\tilde{g}(p). \tag{4.2}
\]

From this it is immediately clear that iteration does not make much sense. If \( |\tilde{K}(p)| > 1 \) for some values of \( p \), the iteration will diverge. If \( |\tilde{K}(p)| < 1 \) the iteration will converge to 0 for these \( p \). A stable averaging prescription \( \tilde{K}(p)^2 = \tilde{K}(p) \) will be obtained if and only if \( \tilde{K}(p) \) only takes the values 0 or 1. Thus consider a (finite or infinite) sequence \( a_0 = 0 < b_0 < a_1 < b_1 < \cdots \), and put \( \tilde{K}(p) = \sum_n \Theta(a_n < |p| < b_n) \). This implies the general form \( K(x) = (1/\pi x)\sum_n (\sin b_n x - \sin a_n x) \) for a stable averaging function in this simple toy model.

One can start a similar investigation for our three dimensional averaging formula. It is easy to formulate the iterations in Fourier space, but it appears hard, probably impossible, to fulfill the conditions for a stable solution together with the representation (2.6) and the boundary conditions (2.13). The requirement of covariance prevents a stable averaging procedure.

5 Static perturbations in Minkowski space

Our extension to the four dimensional case is rather modest but practical. We consider a Robertson Walker metric with \( k = 0 \). A substitution \( r = r'/a(t) \), with \( a(t) \) the cosmic scale factor, brings the line element into the form \( ds^2 = dr^2 - dt^2 + \cdots \), where the corrections are small as long as the region of averaging is small compared to the Hubble length. Therefore we can use the Minkowski metric \((1,1,1,-1)\) as the unperturbed metric. We further assume that the perturbation is approximately static. To keep this situation, we restrict the admissible transformations to rigid translations, rigid spatial rotations, and infinitesimal transformations which keep the time unchanged. This means that \( \xi^0 = 0 \), and \( \xi^m \) is independent of \( t \). Furthermore we can drop all time derivatives in the perturbed metric. Under these restrictions the perturbations \( h_{00} \) and \( h_{m0} \) become gauge invariant.

We average the perturbation with the following simple ansatz.
\[ < h_{mn} > (x) = \int K_{mn}^{kl}(x' - x) \frac{d^3x'}{4\pi}, \]  \hspace{1cm} (5.1)

\[ < h_{00} > (x) = \int P(x' - x)h_{00}(x') \frac{d^3x'}{4\pi}, \]  \hspace{1cm} (5.2)

\[ < h_{m0} > (x) = \int Q(x' - x)h_{m0}(x') \frac{d^3x'}{4\pi}. \]  \hspace{1cm} (5.3)

Here \( P(x' - x) \) and \( Q(x' - x) \) have to be rotation invariant and correctly normalized. Equations (5.1) - (5.3) are the simplest generalization of our previous formula. For static perturbations they are covariant in the sense of (1.1) with respect to static transformations.

Let us now consider the (lowest order) Einstein tensor which, in the static case, reads

\[ 2G_{mn} = 2G_{mn}^{(s)} + h_{00, mn}^0 - \delta_{mn}; \]  \hspace{1cm} (5.4)

\[ 2G_{00} = h_{ij, ij}^i - h_{ij, ij}^j; \]  \hspace{1cm} (5.5)

\[ 2G_{m0} = h_{m0, i}^i - h_{i0, i}^m. \]  \hspace{1cm} (5.6)

Here \( G_{mn}^{(s)} \) is the spatial part of the Einstein tensor in (3.1). Under the assumptions above, the additional terms in \( G_{mn} \), as well as \( G_{00} \) and \( G_{m0} \) are invariant under infinitesimal static transformations.

We investigate whether the Einstein tensor of the averaged metric can be identical to the averaged Einstein tensor. Although everything looks promising at the beginning, we will obtain a negative answer at the end. One may therefore skip the rest of this section.

As in the previous section we treat the constant, linear, and quadratic terms which were split off in (3.2) separately. Again for these the averaging is independent of the order in which it is performed. We start with \( G_{mn} \). For \( G_{mn}^{(s)} \) we have shown in the previous section that the result is independent of the order of averaging. We can restrict to the additional contributions in (5.4).

If we average the perturbation \( h_{00}^0 \) according to (5.2) and introduce into (5.4) we obtain the integrand \([P_{mn} - P_{i}^{ij} \delta_{mn}] \hat{h}_{0}^0 \). If, alternatively, we first calculate \( 2G_{mn} \) in (5.4) with the old metric and then average it in the same way as (5.1), i.e. replace \( h_{kl} \) by \( G_{kl} \) there on the rhs, and shift the partial derivatives from the perturbation to the multiplying functions, we obtain the integrand \([K_{mn, kl}^{kl} - K_{mn, kl}^{k} \delta_{kl}] \hat{h}_{0}^0 \).

This gives the condition

\[ P_{mn} - P_{i}^{ij} \delta_{mn} = K_{mn, kl}^{kl} - K_{mn, kl}^{k} \delta_{kl}. \]  \hspace{1cm} (5.7)

We next apply the same procedure to \( G_{00} \). If we average the perturbations \( h_{ij}^i \) and \( h_{ij} \) according to (5.1) and introduce into (5.5) we obtain the integrand \([K_{i}^{kl} - K_{i}^{ij, kl}] \hat{h}_{kl} \). If, alternatively, we first calculate \( 2G_{00} \) in (5.5) with the old metric and then average it in the same way as (5.2), i.e. replace \( h_{00} \) by \( G_{00} \) there, and shift the partial derivatives from the perturbation to the multiplying functions, we obtain the integrand \([P_{ij}^{kl} \delta_{kl} - P_{i}^{kl}] \hat{h}_{kl} \).
The condition \( K^{kl}_{ij} - K^{kl}_{ij} = P^{j}_{ij} \delta^{kl} - P^{kl} \) which now arises is identical to (5.7) if one renames the dummy indices \( k, l \), subsequently replaces \( m, n \) by \( k, l \), and uses the symmetry \( K^{mknl} = K^{kmln} \). We thus only need to consider (5.7) in the following.

An elementary calculation under proper consideration of the singular behavior at the origin gives (of course we demand \( C(r) = D(r) \), i.e. \( w(r) = 0 \) in this section)

\[
K^{kl}_{mn,kl} - K^{k}_{mn,kl} = K_1(r) \left( \frac{x_m x_n}{r^2} + K_2(r) \delta_{mn} - \frac{4\pi}{2} [\partial_m \partial_n - \delta_{mn} \Delta] \delta^{(3)}(r) \right),
\]

(5.8)

with

\[
K_1(r) = 2[A''(r) - \frac{A'(r)}{r}] - 2[C''(r) + \frac{C'(r)}{r} - \frac{C(r)}{r^2}] - 4[\frac{E'(r)}{r} - \frac{2E(r)}{r^2}],
\]

\[
K_2(r) = -2[A''(r) + \frac{A'(r)}{r}] - 2[B''(r) + \frac{B'(r)}{r}] + 2[\frac{C'(r)}{r} - \frac{C(r)}{r^2}] + 4 \frac{E'(r)}{r} - \frac{2F(r)}{r^2}.
\]

(5.9)

The term with the \( \delta \)-function is most conveniently obtained by multiplying (5.8) with \( x^i x^j \) and integrating. If one introduces the representation (2.6) and considers the behavior (2.13) near \( r = 0 \), one finds that the singular terms in \( u(r) \) and \( v(r) \) cancel, \( K_1(r) \) and \( K_2(r) \) behave at most like \( 1/r^4 \) for small \( r \). By a suitable choice of the remaining freedom in \( u(r) \) and \( v(r) \) one could also remove any singularities, but the convergence of the integrals used in the following is guaranteed anyhow. If one equates the factors of \(-[x_m x_n/r^2 + \delta_{mn}]/r\), of \([x_m x_n/r^2 - \delta_{mn}]\), and of \([\partial_m \partial_n - \delta_{mn} \Delta] \delta^{(3)}(r) \) in (5.7) one obtains

\[
P(x) = \hat{P}(r) - \frac{4\pi}{2} \delta^{(3)}(r),
\]

(5.10)

with

\[
\hat{P}'(r) = -\frac{r}{2}[K_1(r) + K_2(r)], \quad \hat{P}''(r) = \frac{1}{2}[K_1(r) - K_2(r)].
\]

(5.11)

This implies the integrability condition

\[
K_1'(r) + K_2'(r) + \frac{2K_1(r)}{r} = 0.
\]

(5.12)

It is fulfilled if one introduces the representations (5.9) and (2.6). The normalization of \( \hat{P} \) becomes

\[
\int_0^\infty r^2 \hat{P}(r) dr = -\frac{1}{3} \int_0^\infty r^3 \hat{P}'(r) dr = \frac{1}{6} \int_0^\infty r^4 [K_1(r) + K_2(r)] dr = \frac{3}{2}.
\]

(5.13)

From (5.10) this implies the correct normalization of \( P \). The last integrand in (5.13) (as well as the integrand with \( K_1(r) \) or \( K_2(r) \) alone) turns out to be a derivative when one
introduces (5.9) and (2.6). Therefore no freedom is left, the integral is determined by the boundary term at zero, and the latter is fixed by the behavior of \( u(r) \) and \( v(r) \) in (2.13).

This result is unpleasant. It fixes the integrals over \( r^4K_1(r) \) and \( r^4K_2(r) \), and leads to the unwanted \( \delta \)-function contribution in (5.8) and (5.10). The \( \delta \)-function in (5.10) would imply that \( < h_0^0 > (x) \) would contain a contribution \(-h_0^0(x)/2\), i.e. a contribution which is not averaged, and furthermore, with opposite sign. Such a contribution cannot be tolerated.

One could use a more general ansatz in (5.1), (5.2), where terms with \( \delta_{kl}h_0^0 \) and \((x_kx_l/r^2)h_0^0\) are inserted into the rhs of (5.1), and a gauge invariant combination of \( h_k^k \) and \((x^kx^l/r^2)h_{kl}\) into (5.2). We found that this does not help to solve the problem. At the end the relevant integrands again turn out to be derivatives, and everything is fixed by the boundary conditions. We conclude that the previous result about the covariant averaging of the Einstein tensor cannot be extended to four dimensions.

Therefore we keep things simple and work with (5.1) - (5.3). We may choose rather arbitrary functions \( P \) and \( Q \) in (5.2), (5.3), of course with the correct normalization and somehow related to the functions in (2.2). Nevertheless we favor the choice \( w(r) = 0 \) also in the four dimensional case.

### 6 Outlook and conclusions

In this paper we presented a covariant averaging prescription which fulfills two essential requirements.

- Under coordinate transformations the averaged metrics are connected by the same transformation as the original metrics, i.e. \( < g'_{\mu\nu} > = < g_{\mu\nu} > \) (eq. (1.1)).
- The averaging of a constant metric reproduces this metric, i.e. \( < \eta_{\mu\nu} > = \eta_{\mu\nu} \) for constant \( \eta_{\mu\nu} \) (eq. (1.2)).

In three dimensions we gave the complete solution of the problem. There is a linear connection between the original perturbation \( h_{kl}(x') \) and the averaged \( < h_{mn} > (x) \), represented by a tensor (tensor in the sense of linear algebra) \( K_{mn}^{kl}(x' - x) \). (Such a connection has also been discussed by Boersma [10], although without going into details.) The tensor \( K_{mn}^{kl}(x' - x) \) is a superposition of a basis of six bitensors which are symmetric with respect to \( m \leftrightarrow n \) and to \( k \leftrightarrow l \). A product of bivectors, as sometimes suggested in the literature, is not sufficient. The representation contains three functions \( u(r), v(r), w(r) \) which depend upon the distance \( r = |x' - x| \). They have to be singular in a definite way at the origin, in order to fulfill the requirement (1.1) of covariance. For \( w(r) = 0 \) we found the welcome property that the Einstein tensor can be averaged in the same way as the metric, and that the Einstein tensor belonging to the averaged metric is identical to the averaged Einstein tensor.

We have further seen that it does not make sense to iterate the averaging procedure. Therefore one has to choose a reasonable ansatz for the functions \( u(r) \) and \( v(r) \) from the beginning, of course with the correct boundary conditions. Three simple choices suggest themselves:
Exponential function:
\[
u_{\text{exp}}(r) = \frac{5}{4} e^{-r/r_0}, \quad v_{\text{exp}}(r) = -\frac{13}{32} e^{-r/r_0/r}.
\] (6.1)

Gaussian:
\[
u_{\text{gauss}}(r) = \frac{5}{4} e^{-(r/r_0)^2}, \quad v_{\text{gauss}}(r) = -\frac{13}{32} e^{-(r/r_0)^2/r}.
\] (6.2)

Averaging over a sphere:
\[
u_{\text{sphere}}(r) = \frac{5}{4} \frac{(r_0 - r)^4}{rr_0^4} \Theta(r_0 - r), \quad v_{\text{sphere}}(r) = -\frac{13}{32} \frac{(r_0 - r)^2}{rr_0^2} \Theta(r_0 - r).
\] (6.3)

In the last case we took care not to get \( \delta \)-like contributions in the derivatives from the boundary at \( r = r_0 \).

The functions \( A(r), \ldots, F(r) \) can be easily obtained from this with the help of (2.6), there is no need to show the explicit expressions here.

The generalization to static perturbations in Minkowski space, for many applications an excellent approximation to the realistic case, is quite simple. The property for the covariant averaging of the Einstein tensor could, however, not be generalized to the four dimensional case.

One can hope to obtain a sufficiently smooth metric after performing the average. But it is important to note that we are still free to perform gauge transformations. By an unfavorable choice of gauge the “smoothed” metric can become wavy and irregular. All one can achieve is that the final metric becomes equivalent to a smooth metric.

The present investigation was already quite elaborate, it gave the mathematical framework for a covariant averaging prescription. Applications have to be postponed to forthcoming work. The central question to be investigated is, of course, how far the averaging of inhomogeneities can mimic the presence of dark energy.

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