Nonclassicality detection from few Fock-state probabilities

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Experimentally certifying the nonclassicality of quantum states in a reliable and efficient way is a challenge that remains both fundamental and daunting. Despite decades of topical research, techniques that can exploit optimally the information available in a given experimental setup are lacking. Here, we propose a novel paradigm to tackle these challenges, that is both directly applicable to experimental realities, and extendible to a wide variety of circumstances. We demonstrate that Klyshko’s criteria, which remained a primary approach to tackle nonclassicality for the past 20 years, is a special case of a much more general class of nonclassicality criteria. We provide both analytical results and numerical evidence for the optimality of our approach in several different scenarios of interest for trapped-ion, superconducting circuits, optical and optomechanical experiments with photon-number resolving detectors. This work represents a significant milestone towards a complete characterisation of the nonclassicality detectable from the limited knowledge scenarios faced in experimental implementations.

I. INTRODUCTION

Many fundamental quantum information protocols rely on the nonclassicality of bosonic systems induced by nonlinear phenomena [1, 2]. Nonclassical statistics are a crucial resource for quantum sensing [3], as demonstrated by the recent experiments with trapped ions [4, 5] and superconducting qubits [6], and more generally for the advancement of quantum information processing [7, 8].

Most existing nonclassicality criteria require knowledge of the statistical moments of the boson-number distribution [9–20], which are hard to estimate accurately in experimental platforms such as superconducting-circuits and trapped-ions [21–25], as well as with photon-number-resolving detectors [26–29]. At the same time, not being tailored to the observables that are directly accessible in a given experimental platform, these tests do not make optimal use of the information available. Nonclassicality criteria relying on photon-click statistics [30–35] have similar shortcomings. We here tackle both issues: on the one hand, we devise improved nonclassicality tests that can be directly applied to finite numbers of estimated boson-number probabilities, which is useful to better resolve the different brands of nonclassicality underlying boson-number distributions [24, 36–42]. On the other hand, we investigate the ultimate limits of any such test.

While deciding the nonclassicality of an unknown input state is fundamentally impossible with finitely many measurements, we find that, remarkably, in at least some cases of interest it is nonetheless possible to devise criteria that are optimal with respect to a given finite amount of information. Such criteria are very useful in providing definitive answers to precisely which states can be certified as nonclassical in a given experimental scenario. While we focus on the nonclassicality detectable from few Fock-state probabilities, our methodology, based on rather general geometric ideas, can be extended to tackle the nonclassicality of different types of measurements. This approach departs considerably from methods relying on quasiprobability phase-space distributions, which typically rely on complete tomographic information to determine the (non)classicality of a state [43–50]. The two approaches not only provide incomparable results, but also rely on fundamentally different assumptions.

A pioneering step in this direction was taken by D.N. Klyshko [51], who developed nonclassicality criteria — in the form of inequalities for the Fock-state probabilities — satisfied by all classical states. These criteria found numerous applications in both theoretical and experimental contexts [39, 52–55]. Similar criteria were also independently formulated in [56]. For many photon and phonon states, Klyshko’s inequalities are however still insufficient to detect nonclassicality, and a thorough analysis of their completeness is lacking.

Here, we strengthen Klyshko’s methodology, developing new criteria to certify nonclassicality from few Fock-state probabilities that are well-suited to experimental implementations. More specifically, we address the following open question: given a vector \( \mathbf{P} \equiv (P_0, P_1, \ldots, P_n) \) of Fock-state probabilities, are these probabilities incompatible with classical states? It is worth stressing that, because knowing the probabilities in a fixed basis is not sufficient to characterize a quantum state, it is possible for a given \( \mathbf{P} \) to correspond to both classical and nonclassical states. Nonetheless, we can assess compatibility with a classical distribution, thus allowing to certify the nonclassicality of a given state. In other words, our approach allows to answer questions of the form: is non-
classicality detectable at all, given the information we are given? Moreover, we prove that in at least some cases our strengthened criteria are already complete, in the sense that all finite sets of Fock-state probabilities corresponding to nonclassical states are detected as such. A significant advantage of our approach over previous endeavours is our working directly on the Fock-state probabilities, rather than using photon-click statistics or statistical moments. This makes the criteria detector-independent and of broader applicability, in particular in the context of recent optical [57], atomic [58], circuit quantum electrodynamics [59], and optomechanical [60] experiments, and in light of the recent progress in photon-number-resolving detection technology [31, 57, 61].

While the focus of this paper is on P-nonclassicality [1–3, 6, 62], that is, on detecting states whose P function cannot be interpreted as a probability distribution, our approach can be extended to different notions of nonclassicality, thus paving the way for a similar characterisation of non-Gaussianity [35], a crucial resource for quantum computing with bosonic systems.

II. RESULTS

In this section, we will discuss the extension of Klyshko’s criteria [51], and how our geometrical approach that allows to provide definitive answers regarding the question of which nonclassical states can be certified as such in a given scenario where a limited number of observables are accessible.

Klyshko showed that, for all $k \geq 1$, the condition $kP_k^2 > (k + 1)P_{k+1}P_{k-1}$ cannot be fulfilled by classical states [51]. Here, we generalize these criteria to make them usable in arbitrary subsets of Fock-state probabilities of the form $\{P_0, P_1, \ldots, P_N\}$. Furthermore, we will show that the nonclassicality criteria involving the unobserved probabilities $\{P_{N+1}, P_{N+2}, \ldots\}$ can be expressed in terms of the observable ones, providing a stronger nonclassicality criterion. We will show that, in the $N = 2$ case, these criteria characterize the set of nonclassical states. This means that our criterion, at least in such special instances, exhausts the amount of information about nonclassicality that can be pieced from Fock-state probabilities. Finally, we will showcase applications of our criteria to several classes of experimentally relevant states whose nonclassicality is impervious to alternative methods. The geometrical approach we use to derive our results has several advantages over alternative methods such as those based on the maximisation of task-dependent functionals [33]. In the appendices, we discuss how some of our results could be derived using this approach, which further highlights how the geometric approach might be a preferable venue to tackle the problems studied here. It is worth stressing that our criteria are not directly comparable with standard phase-space-based approaches which rely on the negativity of Wigner or P-function of the full state. Such approaches require full information about the state [43–50], and cannot be directly compared with the results in the partial-knowledge scenario we consider. Same argument applies for momentum-based approaches [9–20]. All these methods share the shortcomings of relying on information about a state that is not easily estimated in many practical scenarios. For the sake of completeness, we present in the appendices the results obtained using such criteria, in order to highlight the fundamental differences between them.

1. Boundary of nonclassicality

Let $S \subset \mathcal{H}$ be the set of quantum states on a single-mode Hilbert space $\mathcal{H}$, and $\mathcal{C}_{coh} \subset S$ the set of coherent states, that is, of trace-1 projectors of the form $\{|\alpha\rangle\langle\alpha|\}_{\alpha \in \mathbb{C}}$, where $|\alpha\rangle$ denotes a coherent state with average boson number $|\alpha|^2$. Finally, let $\mathcal{C} \subset S$ denote the set of classical states, that is, the convex hull of $\mathcal{C}_{coh}$. This is the set of states $\rho$ that can be written as $\rho = \int d^2\alpha \rho(\alpha)|\alpha\rangle\langle\alpha|$ for some probability distribution $\rho(\alpha)$. Classical states can always be prepared by a classical external driving force on the quantized linear oscillator [63]. As such, their features are explainable via a classical formalism [1].

Given $N \geq 1$, consider the reduced probability space

$$\mathcal{P}_N \equiv \left\{(P_0, P_1, \ldots, P_N) : \sum_{k=0}^N P_k \leq 1 \text{ and } P_k \geq 0\right\}. \quad (1)$$

This is the set of vectors which can be part of some larger probability distribution. Denote with $\pi_N : S \to \mathcal{P}_N$ the natural projection sending each state to its corresponding Fock-state probabilities: $\pi_N(\rho) \equiv (\rho_{kk})_{k=0}^N \in \mathcal{P}_N$. We want to characterize algebraically the projection $\pi_N(C)$ of $\mathcal{C}$ onto the reduced probability space $\mathcal{P}_N$. A crucial observation is that $\pi_N$ is linear. This implies that convex regions in $S$ are mapped into convex regions in $\mathcal{P}_N$, and thus in particular $\pi_N(C)$ is convex. Characterizing its boundary $\partial \pi_N(C)$ is therefore sufficient to characterize the whole of $\pi_N(C)$.

2. Generalizing Klyshko’s inequalities

A first investigation of the $N = 2$ case was presented in [33], where nonclassicality criteria using $(P_0, P_1)$ were derived. We summarize and extend these results, discussing the nonclassicality in general spaces of the form $(P_n, P_m)$. We then extend these considerations to bound the possible Fock-state probabilities in arbitrary probability spaces. In particular, we derive criteria in the form of inequalities relating probability tuples $(P_{I_1}, \ldots, P_{I_t})$ and $(P_{J_1}, \ldots, P_{J_t})$ such that $\sum_{i} I_i = \sum_{j} J_j$ and $I$ and $J$ are comparable via majorization. We say that a tuple $I$ is majorized by $J$, and write $I \preceq J$, if the sum of the
The conclusion then follows from the general identity for products of sums,

$$\prod_{i=1}^{n} I_i! P_{i} \leq \prod_{i=1}^{s} J_i! P_{j_i},$$

(2)

where $s \equiv |I| = |J|$. Each such criterion corresponds to a nonclassicality criterion which can be used when the experimenter is given the corresponding set of Fock-state probabilities.

To prove eq. (2), we start by defining $Q_k \equiv k! P_k$, so that the statement reads $\prod_{i=1}^{s} Q_i \leq \prod_{j=1}^{s} Q_j$. Remembering the general identity for products of sums,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij},$$

(3)

where the last sum ranges over all multi-indices $J$ of length $n$, with $J_i \in \{1, ..., m\}$ for all $i = 1, ..., n$. For any classical state, the probabilities have the form

$$P_k = \sum_{\lambda} p_{\lambda} e^{-\lambda} \frac{\lambda^k}{k!},$$

(4)

and thus

$$\prod_{i} Q_{i} = \prod_{\lambda} p_{\lambda} e^{-\lambda} \lambda^{i} = \sum_{\lambda} p_{\lambda} e^{\lambda} \lambda^{i},$$

(5)

where we used the shorthand notation $p_{\lambda} \equiv \prod_{i} p_{\lambda_{i}}$, $|\lambda| \equiv \sum_{i} \lambda_{i}$, and $\lambda^{i} \equiv \prod_{i} \lambda^{i}_{i}$, and the sum is extended to all possible tuples of values of $\lambda$. From the above expression, we see that

$$\prod_{i} Q_{i} - \prod_{j} Q_{j} = \sum_{\lambda} p_{\lambda} e^{-|\lambda|}(\lambda^{i} - \lambda^{j}).$$

(6)

The conclusion then follows from Miurhead’s inequalities [64, 68]. More details are provided in section A.

In particular, when $I, J$ have length 3, we get inequalities involving triples of probabilities: for all $0 \leq n \leq m \leq k$, classical states are bound to satisfy:

$$(m! P_m)^{k-n} \leq (n! P_n)^{k-m}(k! P_k)^{m-n}.$$  

(7)

Violation of eq. (7) thus certifies nonclassicality. For $n = N - 1, m = N$ and $k \geq N$, defining $Q_k \equiv k! P_k$, we have $Q_{N-1}^{N-1} \leq Q_{N}^{k-N}Q_k$, and thus

$$k! P_k \geq \frac{Q_{N}^{N}}{Q_{N-1}^{k-N}} \left( \frac{Q_{N}}{Q_{N-1}} \right)^{k}, \ \forall k \geq N - 1.$$  

(8)

Using this in conjunction with the normalisation condition $\sum_{k} P_k = 1$ we get

$$\sum_{k=0}^{N-2} P_k + \frac{Q_{N-1}^{N}}{Q_{N}^{k-N}} \sum_{k=1}^{N} \frac{1}{k!} \left( \frac{Q_{N}}{Q_{N-1}} \right)^{k} \leq 1.$$  

(9)

Using the Taylor expansion of the exponential function to write $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x - \sum_{k=0}^{N-2} \frac{x^k}{k!}$, we then conclude that all classical states must satisfy the inequality

$$\sum_{k=0}^{N-2} P_k + \frac{Q_{N-1}^{N}}{Q_{N}^{k-N}} \left[ \frac{e^N - \sum_{k=0}^{N-2} \left( \frac{Q_{N}}{Q_{N-1}} \right)^{k}}{k!} \right] \leq 1.$$  

(10)

Together with the standard Klyshko conditions in the form $Q_k^2 \leq Q_{k-1}Q_{k+1}$, eq. (10), defines a closed region $D_N \subset P_N$ containing $\pi_N(C)$. Any probability vector $P \notin D_N$ is certifiably nonclassical. In the rest of the paper, we will refer to condition (10) as $K_{\infty,N}$, and to the Klyshko condition $Q_k^2 \leq Q_{k-1}Q_{k+1}$ as $K_k$. We will also use $K_{\infty}$ to refer more generally to criteria of the type $K_{\infty,N}$ for some $N$.

Let us remark two additional facts:

1. Having access to a finite set of probabilities $(P_0, ..., P_{N-1})$, there are always nonclassical states that are not detectable by any criterion. For example, consider $\rho^{(N)} \equiv \rho_{cl} + (1-p) |N\rangle \langle N|$ for some classical state $\rho_{cl}$. Having only access to the first $N$ probabilities amounts to working with the reduced distribution $\pi_{N-1}(\rho^{(N)}) = p\pi_{N-1}(\rho_{cl})$. Being $\rho_{cl}$ classical, as we discussed previously, $\pi_{N-1}(\rho_{cl})$ satisfies all the inequalities $K_{\infty}$ and $K_k$. The scaled probability vector $p\pi_{N-1}(\rho_{cl})$ is then also bound to satisfy the Klyshko-like inequalities $K_k$, as these are scale-invariant. It is then also easy to verify that if $K_{\infty,N-1}$ is satisfied for $p\pi_{N-1}(\rho_{cl})$, then it must also be satisfied for $p\pi_{N-1}(\rho_{cl})$ for any $0 < p < 1$.

2. The $K_{\infty}$ inequalities are a necessary addition to fully exploit the knowledge encoded in the Fock-number probability distributions. Indeed, there are always nonclassical states undetected by the Klyshko-like inequalities. For example, knowing any set $(P_0, ..., P_N)$, the Fock state $|N\rangle$ can be seen to satisfy all inequalities of type $K_k$, but it violates $K_{\infty,N}$.

The fundamental question that remains to be addressed is whether the inequalities of the form $K_k$ and $K_{\infty}$ exhaust the information about nonclassicality encoded in Fock-state probabilities. This amounts to asking whether a probability vector satisfying all the inequalities (those usable given a finite set of probabilities), implies the existence of a classical state resulting in said probabilities. In other words, we want to know whether satisfying all relevant inequalities certifies compatibility with some classical state, which is the most one can ask for in this scenario.
3. Nonclassicality in \((P_0, P_1, P_2)\)

To analyze the applicability of the new conditions \(K_{\infty\cdot N}\) beyond [51], we study what nonclassical states can be detected when only the first three Fock-state probabilities are known. We will find that, remarkably, the nonclassicality of a state is completely captured by only two algebraic inequalities.

Let us denote with \(K_1\) the region:

\[
K_1 \equiv \{(P_0, P_1, P_2) \in P_2 : P_1^2 = 2P_0P_2 \},
\]

(11)

and with \(K_{\infty,2}\) the set of points satisfying eq. (10) with \(N = 2\), that is, the probability vectors \((P_0, P_1, P_2) \in P_2\) such that \(P_0 + \frac{P_1^2}{2P_2} \exp\left(\frac{2P_2}{P_1}\right) - 1 = 1\). The associated nonclassicality criteria are then

\[
P_1^2 > 2P_0P_2,
\]

(12)

\[
P_0 + \frac{P_1^2}{2P_2} \exp\left(\frac{2P_2}{P_1}\right) - 1 > 1.
\]

(13)

The notation \(K_{1\bigcap}^\infty\) and \(K_{\infty,2}^\infty\) will be used to denote the sets obtained by replacing the equality in these definitions with the corresponding inequality sign (e.g. \(K_1^\infty\) is the set of points such that \(P_1^2 \geq 2P_0P_2\), while \(K_1^\infty\) is the set of points such that \(P_1^2 < 2P_0P_2\)). We will prove in this section that \(\pi_2(C) = K_1^\infty \cap K_{\infty,2}^\infty\), that is, that eqs. (12) and (13) are necessary and sufficient conditions for a state being detectable as nonclassical when only knowledge of the first three Fock-state probabilities is given. It is worth stressing that these nonclassicality criteria are strictly stronger than previously reported criteria using pairs of Fock-state probabilities [33].

We already showed that all classical states are contained in \(K_{1\bigcap}^\infty \cap K_{\infty,2}^\infty\). To prove that the inequalities provide a necessary and sufficient condition for compatibility with classical states, we need to show that any probability vector in \(K_{1\bigcap}^\infty \cap K_{\infty,2}^\infty\) is compatible with a classical state. For the purpose, we will show that any \(P\) inside this region can be written as convex combination of probability vectors compatible with classical states. In other words, we want to find the mixture of coherent states corresponding to a given triple of probabilities \(P \equiv (P_0, P_1, P_2) \in K_{1\bigcap}^\infty \cap K_{\infty,2}^\infty\). To achieve this, we will (1) show that \(P\) is a convex mixture of the origin and a point \(P_\infty \in K_{1\bigcap}^\infty \cap K_{\infty,2}\); (2) show that \(P_\infty\) can be written as convex combination of \((1, 0, 0)\) (the point corresponding to the vacuum state) and an element of \(K_1 \cap K_{\infty,2}\); (3) show that all vectors in \(K_1 \cap K_{\infty,2}\) are compatible with coherent states. This will allow us to conclude that \(P\) is compatible with a convex combination of coherent states.

Let \(P \in K_{1\bigcap}^\infty \cap K_{\infty,2}^\infty\) be an arbitrary point not satisfying the nonclassicality criteria (12, 13). Define the quantities \(K_1(P) \equiv P_1^2 - 2P_0P_2\) and \(K_{\infty,2}(P) \equiv P_0 + \frac{P_1^2}{2P_2} [\exp(2P_2/P_1) - 1]\). Note that, upon rescaling \(P \to \epsilon P\), the sign of \(K_1\) is invariant, and \(K_{\infty,2} \to \epsilon K_{\infty,2}\). We can therefore always find \(\epsilon \geq 1\) such that \(P' \equiv \epsilon P \in K_1^\infty \cap K_{\infty,2}\). We can thus write \(P\) as a convex combination of \(P'\) and the origin in probability space, \(0 \equiv (0, 0, 0)\), as \(P = 1/\epsilon P' + (1 - 1/\epsilon) 0\). Note that \(0\) is the probability vector generated by a coherent state in the limit of infinite average boson number, and is thus classical. It now remains to prove that \(P'\) is also classical to conclude that \(P\) is.

For the purpose, consider convex combinations of \(P'\) and \(e_0 \equiv (1, 0, 0) \equiv \pi_2((0|0))\), and notice that

\[
K_{\infty,2}^\infty (\rho P' + (1 - \rho)e_0) = p K_{\infty,2}^\infty (P') + (1 - \rho) = 1,
\]

\[
K_1(pP' + (1 - p)e_0) = p^2 K_1^\infty (P') - 2p(1 - p) P_2'.
\]

(14)

Solving for the \(p \neq 0\) such that \(K_1 = 0\), we find

\[
p = \frac{2P_2'}{(P_1')^2} e^{-2P_2'/P_1'} \geq 1,
\]

(15)

where we used \(K_{\infty,2}^\infty (P') = 1\) and \(K_1(P') \leq 0\). This means that there is some \(P'' \in K_1 \cap K_{\infty,2}\) such that \(P' = \frac{1}{1 - p} e_0 + \frac{p}{1 - p} P''\). To conclude, we now only need to show that \(P''\) is compatible with a coherent state. By definition of \(K_1 \cap K_{\infty,2}\), the elements of \(P''\) satisfy

\[
\frac{P_1''}{2P_2''} e^{2P_2''/P_1''} = P_0'' e^{P_1''/P_0'} = 1.
\]

(16)

We finally note that, for any value of \(P_0''\), these conditions uniquely determine \(P_1''\) and \(P_2''\), and that a coherent state with average boson number \(\mu = - \log P_0\) produces these probabilities.

It is worth remarking that the above reasoning not only proves that the given inequalities characterize the boundary of classical states in \(P_2\), but also provides a constructive method to find classical states compatible with an observed (not nonclassical) probability distribution.

III. APPLICATIONS

1. Fock and squeezed states

A class of states benefiting from the \(K_{\infty}\) criteria are Fock states. The Fock state \(|1\rangle \equiv a^\dagger |0\rangle\) clearly satisfies \(P_1^2 > 2P_0P_2\), and is therefore detected as nonclassical by \(K_1\). More generally, convex mixtures of \(|0\rangle\) and \(|1\rangle\) are all detected as nonclassical by \(K_1\) but not by \(K_{\infty,2}\), as also seen in fig. 1. On the other hand, \(|2\rangle = \frac{1}{\sqrt{2}} a^d^2 |0\rangle\) is detected as nonclassical by \(K_{\infty,2}\) but not by \(K_1\). Consider now attenuated Fock states, that is, states of the form \(\mathcal{E}_T(|k|\langle k|)\) with \(|k\rangle\) Fock states and \(\mathcal{E}_T\) the channel corresponding to attenuation through a beamsplitter with transmittivity \(T \in [0, 1]\) (thus, in particular, \(\mathcal{E}_1(\rho) = \rho\) and \(\mathcal{E}_0(\rho) = \mathrm{Tr}(\rho) |0\rangle\langle 0|\) for all \(\rho\). In these
A Fock state such as \( |k \rangle \) is a distribution for \( k \) and is therefore undetectable as nonclassical by both \( \mathcal{K}_1 \) and \( \mathcal{K}_{\infty,2} \). The criteria detect this nonclassicality only using (13).}

Nonclassicality of noisy Fock states — Notation is as in fig. 2. We highlight the regions of nonclassicality detected by \( \mathcal{K}_1 \) (upper light red) and \( \mathcal{K}_{\infty,2} \) (lower dark red) for noisy Fock states statistics \( P^{(k;\mu)} \), for \( k = 1, 2, 3, 4 \). For example, we see that \( \mu = 1.5 \) corresponds to a classical statistics for \( P^{(1;\mu)} \), a nonclassical one detected by \( \mathcal{K}_{\infty,2} \) for \( P^{(2;\mu)} \), and a nonclassical one also for \( P^{(3;\mu)} \) and \( P^{(4;\mu)} \), now detected by \( \mathcal{K}_1 \). Further details can be found in figs. 9 and 10.

\[ \mathcal{K}_1 \] cannot certify the nonclassicality of convex mixtures of \( |2 \rangle \) and \( |0 \rangle \), which is however revealed by \( \mathcal{K}_{\infty,2} \). This suggests squeezed states as another class benefiting from our extended criteria.

In appendix C we show that many squeezed thermal states [69] also require \( \mathcal{K}_{\infty,2} \) to be detected as nonclassical.

2. Boson-added noisy states

Photon- and phonon-added coherent states [70–72] are defined as \( |\alpha;\ell \rangle = C_{\alpha,\ell} a^\dagger \ell |\alpha \rangle \) with \( C_{\alpha,\ell} \) normalisation constants. The associated Fock-state distribution equals that obtained adding single bosons to Poissonian noise with average number \( \mu = |\alpha|^2 \), here denoted \( \rho_{\mu} \). The new criteria provide enhanced predictive power also for these highly noisy states \( \rho_{\mu} \). For example, for \( \ell = 2 \), \( \mathcal{K}_1 \) does not predict nonclassicality with \( P_0, P_1, P_2 \), but \( \mathcal{K}_{\infty,2} \) does. The same holds for probabilistic boson addition. Consider e.g. states of the form \( p a^\dagger |\alpha \rangle + (1 - p) |0 \rangle \). We find that using \( \mathcal{K}_\infty \) criteria allows to detect nonclassicality more efficiently, as shown in fig. 2. More details are found in appendix C.

3. Noisy Fock states

Consider now displaced Fock states, \( |\alpha; k \rangle = D(\alpha) |k \rangle \), obtained applying the displacement operator \( D(\alpha) \) to a Fock state \( |k \rangle \) [73]. Averaging over the phases of \( \alpha \), these produce the same Fock-state probabilities \( \langle P^{(k;\mu)}_j \rangle \) as Fock states with added Poissonian noise, and model
We proved the optimality of our improved criteria with respect to the first three Fock-state probabilities, and discussed a number of example applications of the criteria for several classes of states of interest, including boson-added thermal states, noisy Fock states, and thermal and Fock states. Others, such as thermal states, are also considered and discussed in the Appendix. We remark that even in cases in which the new $\mathcal{K}_\infty$ criteria do not provide additional predictive capabilities, as is the case for example for some classes of noisy Fock states, discussed in sections C2 and III3, and boson-added thermal states, discussed in section C5. We stress that, even in these cases, our analysis is useful allowing to conclude that the nonclassicality certification problem, in some circumstances, is simply unsolvable with the information given.

The optimality of the proposed criteria in higher-dimensional slices of probability space remains a stimulating open question, which if solved would provide further insight into the nonclassicality of boson statistics. Another interesting aspect emerging from a combination of this approach with the methodology of [33], is how adding noise to a state, which is generally an easy operation, can make it easier to detect the nonclassicality of the state from its Fock-state distribution. Our results, paired with modern optimisation techniques, pave the way to a complete characterisation of the nonclassicality accessible from finite sets of measurable quantities.

**DATA AVAILABILITY**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**AUTHOR CONTRIBUTION**

All authors contributed to the conception and development of the idea and to the writing of the manuscript.

**COMPETING INTEREST**

The authors declare no competing interests.

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Appendices

A. GENERALIZED KLYSHKO'S INEQUALITIES

(Section outline) In this section we show how to generalize Klyshko’s criteria. For the purpose, we will start by reviewing and providing a proof for the original result presented in [51], and then show how this is a special case of a more general class of criteria.

(Proof of generalized criteria) Let \( \mathbb{N} \ni k \geq 1 \). We want to prove that, for any classical state,
\[
Q_k^2 \leq Q_{k-1}Q_{k+1},
\]
where \( Q_k \equiv k! P_k \). The proof we present here follows the same ideas given in [51]. For any classical state, the probabilities \( P_k \) have the form
\[
P_k = \sum_{\lambda} p_{\lambda} e^{-\lambda} \lambda^k,
\]
and thus \( Q_k = \sum_{\lambda} p_{\lambda} e^{-\lambda} \lambda^k \). The sum symbol can be replaced with an integral if needed without affecting the calculations. Equation (17) is equivalent to, bringing all the terms on the left hand side,
\[
\sum_{\lambda\mu} p_{\lambda\mu} e^{-\lambda-\mu} [(\lambda\mu)_k - \lambda^{k+1} \mu^{k-1}] = \sum_{\lambda\mu} p_{\lambda\mu} e^{-\lambda-\mu} (\lambda\mu)_k - \lambda^2
\]
\[
= - \sum_{\lambda\mu} p_{\lambda\mu} e^{-\lambda-\mu} (\lambda\mu)_k - \lambda^2,
\]
where in the last step we used the fact that, for any symmetric tensor \( f_{ij} = f_{ji} \) such that \( f_{ii} = 0 \), we have \( \sum f_{ij} = 2 \sum_{i<j} f_{ij} \). The conclusion then follows from the observation that \( (\lambda - \mu)^2 \geq 0 \) for all \( \lambda, \mu \), and that all the sums are extended over only positive \( \lambda, \mu \geq 0 \), by the definition of classical states.

Equation (17) can be further generalized. Let \( L, J \) be arbitrary multi-indices with \( s \) elements for some positive integer \( s \): \( I \equiv (I_1, \ldots, I_s) \) and \( J \equiv (J_1, \ldots, J_s) \), and suppose \( |I| = |J| \), where \( |I| \equiv \sum I_k \). Suppose \( I \preceq J \), that is, that \( I \) is majorized by \( J \). This means that, for all \( 1 \leq k \leq s \), the sum of the first \( k \) greatest elements of \( I \) is smaller than that of the first \( k \) greatest elements of \( J \). We will then prove that, for all classical states, we must have
\[
\prod_{i=1}^s Q_{I_i} \leq \prod_{i=1}^s Q_{J_i}.
\]
We start by remembering the general equality for products of sums:
\[
\prod_{i=1}^n \sum_{j=1}^m a_{ij} = \sum_{j=1}^m \prod_{i=1}^n a_{i,j},
\]
where the last sum is over all multi-indices \( J \) of length \( n \) with \( J_i \in \{1, \ldots, m\} \) for all \( i = 1, \ldots, n \). Applying this to \( Q \) we have
\[
\prod_i Q_{I_i} = \prod_i \sum_{\lambda} p_{\lambda} e^{-|\lambda| I_i} = \sum_{\lambda} \prod_i p_{\lambda} e^{-|\lambda| I_i},
\]
where we used the shorthand notation \( p_\lambda \equiv \prod_i p_{\lambda_i} \), \( |\lambda| \equiv \sum \lambda_i \), and \( \lambda_i \equiv \prod_i \lambda_i \), and the sum is over all tuples of possible values of \( \lambda \). We thus see that
\[
\prod_i Q_{I_i} - \prod_i Q_{J_i} = \sum_{\lambda} p_{\lambda} e^{-|\lambda| (J_i - I_i)}.
\]
The conclusion now follows from Muirhead’s inequalities [64, 68]. To see this, we first notice that \( q_\lambda \equiv p_{\lambda} e^{-|\lambda|} \) is symmetric upon permutations of \( \lambda \). This means that we can separate the sum \( \sum_\lambda \) by summing first over ordered tuples, and then for each ordered tuple sum over all of its possible permutations. For example, if \( \lambda \) has length \( 2 \), \( \lambda = (\lambda_1, \lambda_2) \) with each \( \lambda_i \) taking values in \( \{1, 2, 3\} \), we write
\[
\sum_{\lambda} = \sum_{\gamma} \sum_{\sigma} q_\gamma \lambda^\sigma,
\]
where \( \gamma \in \{(11), (12), (13), (22), (23), (33)\} \) and \( S_\gamma \) denotes the set of permutations of the tuple \( \gamma \). Equation (23) then becomes
\[
\sum_{\gamma} q_\gamma \sum_{\sigma \in S_\gamma} (\gamma^\sigma - \gamma^\sigma),
\]
where \( \gamma^\sigma \) is the tuple obtained by permuting \( \gamma \) according to \( \sigma \). For each \( \gamma \), Muirhead’s inequalities then tell us that, if \( I \preceq J \), and provided \( \gamma_i \leq 0 \), then
\[
\sum_{\sigma \in S_\gamma} \gamma^\sigma \leq \sum_{\sigma \in S_\gamma} \gamma^\sigma.
\]
We conclude that eq. (20) holds whenever \( I \preceq J \).

1. Examples of inequalities

We show here a few special cases of the criteria (20).

In the \( N = 2 \) case, the only significant majorization relation is \( (1, 1) \preceq (2, 0) \), corresponding to the well-known Klyshko criterion \( Q_1^1 \leq Q_0 Q_2 \).

Fo \( N = 3 \), we have the chain of relations:
\[
(1, 1, 1) \preceq (2, 1, 0) \preceq (3, 0, 0),
\]
\[
Q_1^1 \preceq Q_0 Q_1 Q_2 \preceq Q_0^2 Q_2 \preceq Q_3 Q_1 Q_2^2 \leq Q_4 Q_3^2.
\]
Similar chains of criteria are obtained for higher-dimensional spaces. It is interesting to note that it is not however true that all criteria always fall into a single chain of inequalities, like is the case for \( N = 2, 3, 4 \). Indeed, for \( N \geq 6 \) some tuples that are not related via majorization — for example, \((3,1,1,1)\) and \((2,2,2,0)\) are neither majorized by the other.

**B. CHARACTERIZATION OF CLASSICAL SETS IN DIFFERENT PROBABILITY SPACES**

1. Two-dimensional slices

*(Section summary)* We study in this section the projection of the set of classical states in two-dimensional probability spaces of the form \((P_n, P_m)\). Figures 4 and 5 give examples of such sets in \((P_1, P_k)\) and \((P_0, P_k)\) for various \( k \). We seek an algebraic criterion that certifies whether a state is inside or outside such classicality regions.

*(Direct approach)* We first of all note that an algebraic relation characterizing the boundary of any such region can be derived with ease: assuming \( n < m \), we know that \( Q_n = e^{-\mu} \mu^n \), and thus \((Q_m/Q_n)^{1/(m-n)} = \mu\), from which we find

\[
Q_n^{\frac{m-n}{n}} = e^{-(Q_m/Q_n)^{1/(m-n)}} Q_m^{\frac{m-n}{n}}.
\]

(29)

For example, in \((P_0, P_k)\), this gives \( Q_k = Q_0 [\ln(1/Q_0)]^k \). In \((P_1, P_k)\), we instead get the relations

\[
Q_1^{\frac{m-n}{n}} = e^{-(Q_m/Q_1)^{1/(m-n)}} Q_1^{\frac{m-n}{n}},
\]

(30)

which correspond to the shapes given in fig. 4. These two examples show the existence of two possible scenarios: in some cases, coherent states are projected onto a convex curve, as is the case for \((P_0, P_1)\). However, in other cases, the coherent states project onto non-convex curves, as seen in figs. 4 and 5. In the convex instances, making eq. (29) into an inequality is sufficient to get us a tight nonclassicality criterion. However, when this curve is non-convex, this would not work, as the boundary of the classical set is the convex closure of the curve, which differs from the curve itself. To deal with these cases, we have to resort to a different technique, discussed in section D.1.

*(Criteria via Klyshko-like bounds)* This method does not however generalize to spaces of dimension larger than two, in which no individual algebraic relation describes the probabilities corresponding to coherent states. Let us then revisit the problem from a different perspective. Focus on the \((P_0, P_1)\) space. From section A we know that for all \( s \geq 2 \), classical states satisfy \( Q_s \geq Q_1/Q_0^{s-1} \). Combining these with the constraint \( \sum_k P_k = 1 \) we have

\[
P_0 + P_1 + Q_0 \sum_{s=2}^{\infty} \frac{1}{s!} \left( \frac{Q_1}{Q_0} \right)^s \leq 1,
\]

(31)

which simplifies to \( P_0 e^{P_1/P_0} \leq 1 \). Remarkably, we recover the same bound given explicitly in eq. (29). This idea of combining bounds given by Klyshko-like inequalities on the unknown probabilities has the advantages of carrying over to higher dimensions.

*(A case-study: \((P_0, P_2)\))* To highlight a complication sometimes arising with this approach, let us consider the \((P_0, P_2)\) case. This case reveals to be laborious, with the direct approach, due to its non-convexity. Using the inequalities derived in section A, we have for all classical states and integers \( s \geq 3 \) the lower bound \( Q^2_s \geq Q^2_1/Q_0^{s-2} \). However, for \( Q_1 \), we get an upper bound: \( Q^2_1 \leq Q_0 Q_2 \). This is a problem, as we now cannot simply go from \( \sum_k P_k = 1 \) to an inequality in terms of only \( P_0 \) and \( P_2 \). While we could use the trivial lower bound \( Q_1 \geq 0 \), and this would give us a valid nonclassicality criterion, this bound is not tight and the corresponding criterion thus not optimal. To solve this problem we would need to know the optimal lower bound on \( Q_1 \). More precisely, we would need to know the minimum value of \( \rho_{11} \) when \( \rho \) varies over all possible classical states such that \( \rho_{00} = Q_0 \) and \( \rho_{22} = Q_2/2 \). We will obtain this quantity by studying the nonclassicality in \((P_0, P_1, P_2)\), but the nontriviality of this process highlight the complexity that sometimes arises when characterizing the classical region in different probability spaces.

*(Why some cases are harder to analyze)* To gain some physical intuition on the source of such complexity, observe that Klyshko-like criteria characterize sections of the classicality boundary corresponding to (possibly rescaled) coherent states. Therefore, whenever the boundary also contains non-pure states, such criteria will not be able to account for that.
These follow from the relations

\begin{equation}
\frac{Q_{n}}{Q_{n-1}} \geq \frac{Q_{n-1}}{Q_{n-2}} \geq \ldots \geq \frac{Q_{2}}{Q_{1}} \geq \frac{Q_{1}}{Q_{0}}.
\end{equation}

From eq. (32) we see that, if \(Q_{1}Q_{n-1} = Q_{0}Q_{n}\), then the whole sequence of inequalities collapses to the same numerical value. We also have, for all \(s > n\), the inequalities:

\begin{equation}
Q_{s} \geq Q_{n-1}^{s} \left( \frac{Q_{n}}{Q_{n-1}} \right)^{-s}.
\end{equation}

These follow from the relations

\( (N,...,N) < (s,N-1,...,N-1) \). \hspace{1cm} (34)

Putting eq. (33) together with \(\sum_{k} P_{k} = 1\) we get

\begin{equation}
\sum_{k=0}^{n-2} P_{k} \frac{Q_{n-1}^{n-1}}{Q_{n}^{n-2}} \left[ e^{Q_{n-1}^{n-1}} - \sum_{k=0}^{n-2} (Q_{n}/Q_{n-1})^{k} \right] \leq 1. \hspace{1cm} (35)
\end{equation}

For ease of notation, let us introduce the quantities \(K_{n} = Q_{1}Q_{n-1} - Q_{0}Q_{n}\), and \(K_{\infty}\) equal to the left hand-side of eq. (35), so that the nonclassicality criteria take the form \(K_{n} > 0\) and \(K_{\infty} > 1\).

Equality implies coherent state) We will show here \(K_{n} = K_{\infty} - 1 = 0\) implies that the corresponding probability vector is compatible with a coherent state. We know that \(K_{n} = 0\) implies

\begin{equation}
Q_{0} = Q_{1} = \ldots = Q_{n-2} = \frac{Q_{n-1}}{Q_{n}}.
\end{equation}

For all \(s \geq 2\), these imply \(Q_{s} = Q_{0}(Q_{1}/Q_{0})^{s}\). Moreover, using eq. (36) in the definition of \(K_{\infty}\) gives \(K_{\infty} = P_{0} e^{P_{1}/P_{0}} = 1\). Consider a coherent state with average boson number \(\mu \equiv P_{1}/P_{0}\). Then, \(Q_{s} = Q_{0} e^{s \mu}\). Moreover, \(K_{\infty} = 1\) translates into \(P_{0} = e^{-\mu} P_{0} = e^{-\mu}\), allowing us to conclude that \(Q_{s} = e^{-s \mu}\), implying that \(P\) is indeed compatible with a coherent state \(|\alpha\rangle\) such that \(|\alpha|^{2} = \mu\).

C. APPLICATIONS

1. Attenuated Fock states

(Definitions) Let \(E_{T}\) denote the channel corresponding to attenuation through a beamsplitter with transmittivity \(T\). Thus, in particular, we have

\begin{equation}
\rho_{k,T} \equiv E_{T}([|k\rangle\langle k|]) = \sum_{j=0}^{k} \binom{n}{k} T^{k}(1-T)^{n-k} |j\rangle\langle j|,
\end{equation}

for any Fock state \(|k\rangle\), \(k \geq 0\). The corresponding boson-number probabilities are therefore \(P_{j}^{(k,T)} = \binom{n}{j} T^{j}(1-T)^{n-j}\), for all \(j = 0,...,k\).

(Nonclassicality results) To investigate the nonclassicality of \(\rho_{k,T}\), we observe that, for \(k \geq 1\),

\begin{equation}
K_{1}(P^{(k,T)}) = (P_{1}^{(k,T)})^{2} - 2P_{0}^{(k,T)} P_{2}^{(k,T)}
= (k-1)^{2}(1-T)^{2}.
\end{equation}

This shows that \(K_{1}(P^{(k,1)}) = 0\) for all \(k \geq 2\), implying all Fock states \(|k\rangle\) for \(k \geq 2\) are not detected as nonclassical by \(K_{1}\), and that small attenuations of such states still make the nonclassicality hard to detect. Similarly, one can verify that \(K_{\infty}\) also always detects the nonclassicality of these states, but the violation is significantly easier to detect for small attenuations, as \(K_{\infty}(P^{(k,T)})\) goes to \(+\infty\) when \(T \to 1^{-}\). Such behaviour can be clearly seen when plotting the amount with which the two different nonclassicality criteria are violated as a function of \(T\) and \(k\). The results for \(k = 2,3,4\) are reported in fig. 6. The \(k = 1\) case is, on the other hand, well detected as nonclassical by \(K_{1}\), but not by \(K_{\infty}\).

2. Noisy Fock states

(Definition) Displaced Fock states are defined as \(|\alpha;k\rangle = D(\alpha)|k\rangle\). Averaging over \(\alpha\), these correspond to Fock states with added Poissonian noise, which also share the same Fock-state probabilities.

FIG. 5. Curves drawn by the coherent states in the \((P_{0},P_{1},...,P_{N})\) reduced probability spaces, for \(k = 1,2,3,4,5\). A dot marks the inflection point for each curve. Note how the blue (top right) one, corresponding to \(k = 1\), is the only curve without an inflection point, consistently with it being the only convex one. The dashed line is obtained by closing each curve with a line joining the \((1,0)\) point with the point on that curve that is such that the slope of the corresponding interpolating segment equals the tangent of the curve in that same point. This is the region described by the system given in eq. (32).
For $k = 1$, the Fock-state probabilities are

$$P_j^{(1;\mu)} \equiv \langle j | \alpha; 1 \rangle = \frac{e^{-\mu}}{\sqrt{j!}} \mu^{j-1} (j - \mu)^2,$$

(39)

where $\mu \equiv |\alpha|^2$. The corresponding probability vectors in $(P_0, P_1, P_2)$ go from $(0, 1, 0)$ to $(0, 0, 0)$ for $\mu$ going from 0 to $\infty$, as shown in Fig. 7.

(Nonclassicality of noisy Fock states) Explicitly, we have

$$(P_1^{(1;\mu)})^2 - 2P_0^{(1;\mu)} P_2^{(1;\mu)} = e^{-2\mu} (2\mu^2 - 4\mu + 1).$$

(40)

This quantity is non-positive for $\mu \in [\mu_-, \mu_+] \approx [0.29, 1.71]$ where $\mu_\pm \equiv 1 \pm 1/\sqrt{2}$, implying that these states are detected as nonclassical by $K_1$ for small and large Poissonian noise: $\mu < \mu_-$ and $\mu > \mu_+$, but not for $\mu \in [\mu_-, \mu_+]$. On the other hand, one can verify numerically that $K_{\infty,2} > 1$ for $\mu \lesssim 1.35 < \mu_+$. This means that, in the range $\mu \in [0.29, 1.35]$, the nonclassicality of the states can only be certified by $K_{\infty,2}$. At the same time, in the range $\mu \in [1.35, 1.71]$, for which $K_1, K_{\infty,2} - 1 < 0$, we are ensured that the observed probabilities $(P_0, P_1, P_2)$ are compatible with classical states. In Fig. 8 we give the degrees of violation of the different criteria for different values of $\mu$. In Figs. 9 and 10 we show for various values of $\mu$ how nonclassicality is detected by the different criteria, for noisy Fock state statistics $P^{(1;\mu)}$ and $P^{(2;\mu)}$, respectively.

\section{Boson-added coherent states}

(Definition) Boson-added coherent states $|\alpha, \ell \rangle$ are defined as $|\alpha, \ell \rangle \equiv C_{\alpha,\ell} a^{\dagger \ell} |\alpha\rangle$, with $C_{\alpha,\ell}$ normalisation constants. These correspond to the same Fock-state probabilities as photon- and phonon-added Poissonian noise. Denoting with $\mu \equiv |\alpha|^2$ the average boson number of the coherent state, the corresponding Fock-state probabilities $P^{(\mu,\ell)}$ are related to the Poissonian distribution $k! P_k^{(\mu,\ell)} = e^{-\mu} \mu^k$, by

$$P_k^{(\mu,\ell)} = C_{\alpha,\ell} \mu^\ell \frac{(k\ell)^2}{\mu^{k-\ell}},$$

(41)

where $k \ell \equiv k(k - 1) \cdots (k - \ell + 1)$. In particular

$$P_k^{(\mu,1)} = \frac{1}{\mu + 1} \frac{e^{-\mu} \mu^k k^2}{k!},$$

$$P_k^{(\mu,2)} = \frac{1}{\mu^2 + 4\mu + 2} \frac{e^{-\mu} \mu^k [k(k-1)]^2}{k!}.$$

(42)
As discussed in the main text, we again find that having access to more Fock-state probabilities we can certify nonclassicality even for more values of $\mu$. Nonclassicality of states of the form $|\alpha\rangle\langle\alpha|$ follows from positivity of at least one of the lines. We stress how which criteria are available to certify nonclassicality depends on the number of accessible Fock-state probabilities. For example, if only $P_0, P_1, P_2$ are known, the only states that are recognisable as nonclassical are the ones revealed by the blue and orange curves.

The corresponding first Fock-state probabilities are

$$
\begin{align*}
\frac{e^{-\mu}}{\mu + 1} & \quad \text{for } \ell = 1, \\
\frac{2e^{-\mu}}{\mu^2 + 4\mu + 2} & \quad \text{for } \ell = 2.
\end{align*}
$$

(Nonclassicality of boson-added coherent states) As pointed out in the main text, the nonclassicality of $\ell = 1$ follows from $\mathcal{K}_1$. On the other hand, for $\ell = 2$ we have $P_0 = P_1 = 0$, and thus $\mathcal{K}_1$ is inconclusive. If $P_3$ is known, $\mathcal{K}_2$ could be used, but this is not the case when only the first three Fock-state probabilities are known. On the other hand, $\mathcal{K}_{\infty,2}$ allows to certify nonclassicality even without the knowledge of $P_3$.

(Probabilistic boson addition case) Consider now the nonclassicality of states of the form $p a^\dagger |\alpha\rangle\langle\alpha| a + (1 - p) |\alpha\rangle\langle\alpha|$, corresponding to the probabilistic boson addition to a coherent state. These correspond to the Fock-state probabilities

$$
\hat{P}_k^{(\mu,\ell,p)} = \frac{e^{-\mu} \mu^k}{k!} \left[ \frac{k^2}{\mu(\mu + 1)} + (1 - p) \right].
$$

As discussed in the main text, we again find that having access to more Fock-state probabilities we can certify nonclassicality for more values of $\mu$, and that the use of the $\mathcal{K}_{\infty}$ criteria increases our predictive power.
(Nonclassicality of thermally averaged Fock states) Testing the criteria on these probabilities we find that $P_k^{(\mu; \text{th})} \in \mathcal{K}_1^\mu$ for $0 \leq \mu < \sqrt{-1 + \sqrt{2}} \approx 0.64$, thus certifying the nonclassicality in this region. On the other hand, for $\mu \geq \sqrt{-1 + \sqrt{2}}$, we have $P_k^{(\mu; \text{th})} \not\in \mathcal{K}_1^\mu \cap \mathcal{K}_\infty$, and the statistics are therefore compatible with classical states.

5. Boson-added thermal states

(Definition and nonclassicality) A thermal state $\rho^{(\text{th}; \mu)}$ corresponds to Fock-state probabilities

$$P_{\mu}(k; \text{th}) = \frac{\mu^k}{(1 + \mu)^{k+1}}.$$  \hspace{1cm} (46)

These states can be written as convex combinations of coherent states and are thus known to be classical. Let us consider instead boson-added thermal states, that is, states of the form $a_k^\dagger \rho^{(\text{th}; \mu)} a_k$, up to normalisation. These correspond to the Fock-state probabilities

$$P_k^{(\text{th}; 1, \mu)} = \frac{k P_{k-1}^{(\text{th}; \mu)}}{\mu + 1} = \frac{k \mu^{k-1}}{(1 + \mu)^{k+1}}.$$  \hspace{1cm} (47)

These probabilities are certified as nonclassical by $\mathcal{K}_1$ for all $\mu$, thus in this case there is no need for stronger criteria.

(Probabilistic boson addition) Mixing this state with the corresponding thermal state with probability $p$ gives the Fock-state probabilities

$$\left[ (1 - p) \frac{k}{\mu} + p \right] P_k^{(\text{th}; \mu)}.$$  \hspace{1cm} (48)

Depending on the value of $p$, we get nonclassical state for some ranges of $\mu$. For example, for $p = 0.5$, we find that nonclassicality is only detected for $\mu \lesssim 0.4$ by $\mathcal{K}_1$. The $\mathcal{K}_\infty$ criteria do not appear to provide further predictive power in these cases.

6. Squeezed thermal states

(Definition) Squeezed thermal states are defined as $S(\xi) \rho^{(\text{th}; \mu)} S(\xi)^\dagger$, where $S(\xi) \equiv \exp \left[ \frac{1}{2} (\xi a^\dagger - \xi^* a^2) \right]$ is the squeezing operator and $\rho^{(\text{th}; \mu)}$ is a thermal state. The $\mathcal{K}_1$ criterion does not work for these states. At the same time, as shown in fig. 11, $\mathcal{K}_{\infty, 2}$ successfully certifies nonclassicality for some values of $\mu$ and $\xi$.

D. CHARACTERIZING CONVEX HULLS VIA TANGENT PLANES

(Section outline) In this section we outline a general method to compute the convex hull of arbitrary compact regions in $\mathbb{R}^N$, which generalizes the linear functional techniques of [34, 35, 74]. We will then showcase the use of this method to recover some of the results presented in the main text. The mathematical background underlying these methods to handle convex Euclidean spaces can be found e.g. in [75, 76].

1. General procedure

(Convex hulls with tangent planes) Let $A \subset \mathbb{R}^n$ be some bounded region. The convex hull $\mathcal{C}_A$ of $A$ is the smallest convex region containing $A$. Any such $\mathcal{C}_A$ can be characterized by the set of its tangent planes. More precisely, given a unit vector $n_\theta \in \mathbb{R}^n$, the boundary of $\partial \mathcal{C}_A$ is characterized by the tangent planes $T_\theta$, defined as

$$T_\theta \equiv \{ x \in \mathbb{R}^n : \langle n_\theta, x \rangle = \max_{a \in A} \langle n_\theta, a \rangle \},$$  \hspace{1cm} (49)

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the space. Define for future convenience

$$F_{\max}(\theta) \equiv F_{\max}(n_\theta) \equiv \max_{a \in A} \langle n_\theta, a \rangle.$$  \hspace{1cm} (50)

Each such plane $T_\theta$ separates $\mathbb{R}^n$ into two half-planes. Let us denote with $T_\theta^\pm$ the set of $x$ such that $\langle n_\theta, x \rangle \leq F_{\max}(\theta)$. Then $\mathcal{C}_A = \bigcap_\theta T_\theta^{\pm}$.

(How to retrieve convex hull description) This suggests the following general procedure to recover the convex hull of a region $A$: compute the value of $F_{\max}(\theta)$ for
each direction $\theta$ by solving the associated maximisation problem. The region of the boundary of $C_A$ tangent to this direction will then equal the convex hull of the elements $a \in A$ such that $F_{\text{max}}(\theta) = (n_\theta, a)$. This observation will generally be enough to reconstruct the convex hull of the regions we are interested in.

2. An example

Suppose $A \equiv \{(t, 0) : t \in [0, 1]\} \cup \{(0, t) : t \in [0, 1]\}$, and we are interested in finding the convex hull of this set. Clearly, $A$ is the union of two segments, whose convex hull is trivially the triangle touching the points $(0,0), (1,0), (0,1)$. Let us see how this convex hull is retrieved by means of the general procedure given in section D.1. Given an arbitrary unit vector $n_\theta$, we need to find $F_{\text{max}}(\theta) = \max_{a \in A} (n_\theta, a)$. In polar coordinates, $(n_\theta, a) = \cos \theta a_x + \sin \theta a_y$. The maximum thus equals

$$F_{\text{max}}(\theta) = \begin{cases} \cos \theta, & \theta \in [-\pi/2, \pi/4], \\ \sin \theta, & \theta \in [\pi/4, \pi], \\ 0, & \theta \in [\pi, 3\pi/2]. \end{cases}$$

This tells us that, for $\theta \in (-\pi/2, \pi/4)$, the boundary corresponds to the single point $a = (1,0)$. For $\theta = \pi/4$, both $a = (1,0)$ and $a = (0,1)$ achieve $F_{\text{max}}(\pi/4) = 1/\sqrt{2}$, and thus the boundary of the region orthogonal to this direction is the set of points in the convex hull of $(1,0)$ and $(0,1)$, that is, $\{(t,1-t) : t \in [0,1]\}$. For $\theta \in (\pi/4, \pi)$ we again have $a = (0,1)$. For $\theta = \pi$ we have $a = (0,1)$ and $a = (0,0)$, corresponding to $\{(0,t) : t \in [0,1]\}$. Finally, for $\theta \in (\pi, 3\pi/2)$ we have $a = (0,0)$ and $a = (1,0)$, and thus $\{(t,0) : t \in [0,1]\}$.

Overall, this is telling us that the convex hull of $A$ is bounded by the triangle with vertices $(0,0), (1,0)$ and $(0,1)$, as expected from a direct geometric analysis.

E. 2D PROJECTIONS

(Section outline) In this section we apply the method discussed in section D.1 to retrieve the classicality regions for two- and three-dimensional slices of the probability space. Section E.1 focuses on two-dimensional slices of the form $(P_0, P_k)$, and solves the convex hull via direct geometric analysis. The same problem is tackled in section E.2, this time using the procedure given in section D.1, to illustrate the difference between the two methods. Finally, section E.3 applies the same method to find the classicality region in the $(P_0, P_1, P_2)$ space. We thus recover the same results reported in the main text with a completely different approach, that is more methodical and general, but also significantly more convoluted.

1. Direct method for $(P_0, P_k)$

(Nonclassicality in two-dimensional spaces) As we discussed in section B.1, the characterization of nonclassicality in two-dimensional slices is in many cases complicated by the non-convexity of the associated boundary. An explicit example of this is $(P_0, P_k)$ with $k > 1$, as follows from the non-convexity of $P_0 \mapsto P_0(-\ln P_0)^k$, which has an inflection point at $P_0 = e^{1/k}$.

(Closing non-convex curves) In these cases, the convex hull is obtained by “closing” the region with a line connecting $(1,0)$ and the point $P_0 \equiv (P_{01}, f(P_{01}))$ such that $f'(P_{01}) = -f(P_{01})/(1-P_{01})$, where $f(P_0) \equiv P_0(-\ln P_0)^k/k!$. More explicitly, the convex hull of the set of coherent states in the $(P_0, P_k)$ space is therefore defined algebraically by the equations:

$$\begin{cases} 0 \leq P_k \leq f(P_0), & \text{for } P_0 \in [0, P_{01}], \\ 0 \leq P_k \leq \frac{P_0}{\ln P_0} f(P_0), & \text{for } P_0 \in [P_{01}, 1]. \end{cases}$$

The region corresponding to this convex closure can be understood from direct geometric intuition of the corresponding curves, given in fig. 5.

2. Convex hull in 2D spaces

(Section outline) In this section we work out the convex hulls of the classical set in the probability subspaces $(P_0, P_1)$ and $(P_0, P_2)$ using the method of section D.1.

(Convex hull in $(P_0, P_1)$) Define $F_\rho(\theta) = \cos \theta P_0 + \sin \theta P_1$. Then, for all classical states $\rho$, $F_\rho(\theta) \leq F_{\text{max}}(\theta)$, where

$$F_{\text{max}}(\theta) \equiv \max_{\lambda \in [0, \infty]} e^{-\lambda} [\cos \theta + \sin \theta \lambda].$$

We need to find, for each $\theta$, the value of $F_{\text{max}}(\theta)$. There are three distinct regions to consider:

1. When $\theta \in [-\pi/2, \pi/4]$ the maximum is achieved for $\lambda = 0$, and equals $F_{\text{max}}(\theta) = \cos \theta$.
2. When $\theta \in [\pi/4, \pi]$ the maximum is achieved within the interval, for $\lambda = 1 - \cot \theta$, and is $F_{\text{max}}(\theta) = e^{-1+\cot \theta} \sin \theta$.
3. When $\theta \in [\pi, 3\pi/2]$, the maximum is achieved for $\lambda = \infty$, with $F_{\text{max}}(\theta) = 0$.

The boundary of the convex hull is then obtained by joining the points

- $(1,0)$,
- $(P_0(\lambda), P_1(\lambda))$ for $\lambda = \lambda(\theta) = 1 - \cot \theta$ and $\theta \in [\pi/4, \pi]$, which corresponds to $\lambda \in [0, \infty]$,
- $(0,0)$ corresponding to the interval $[\pi, 3\pi/2]$.
This is consistent with what can be seen in fig. 5. Notice how the convexity of the associated curve is reflected into this formalism into the continuity of the first derivative of $\theta \mapsto F_{\max}(\theta)$.

(Convex hull in $(P_0, P_2)$) In $(P_0, P_2)$ we have

$$F_{\max}(\theta) \equiv \max_{\lambda \in [0, \infty]} e^{-\lambda}[\cos \theta + \sin \theta \lambda^2/2].$$

(54)

Define $f_0(\lambda) = e^{-\lambda}(c + s\lambda^2/2)$, using the shorthand notation $c \equiv \cos \theta$ and $s \equiv \sin \theta$. The stationary points are the solutions of $-c - s\lambda^2/2 + s\lambda = 0$ and thus $\lambda^2 - 2\lambda + 2\cot \theta = 0$. This gives $\lambda_\pm = 1 \pm \sqrt{1 - 2\cot \theta}$ for $\cot \theta \leq 1/2$, that is, $\theta \in [\arccot(1/2), \pi] \cup [\pi + \arccot(\pi/2), 3\pi/2]$. Moreover, whereas $\lambda_+$ is always positive (where well-defined), $\lambda_-$ is only positive in the subset $\theta \in [\arccot(1/2), \pi/2] \cup [\pi + \arccot(1/2), 3\pi/2]$. We thus need to consider these cases separately.

- For $\theta \in [0, \arccot(1/2)]$ there are no viable stationary points to check. The values at the extreme of the $\lambda$ intervals are $0$ for $\lambda \rightarrow \infty$ and $\cos \theta$ for $\lambda = 0$. Because $\cos \theta \geq 0$ for the angles considered, the maximum is achieved at the point of contact $\lambda = 0$ (which is the $(1,0)$ in the considered probability space), and $F_{\max}(\theta) = \cos \theta$.

- When $\theta \in [\arccot(1/2), \pi/2]$, it can be checked that $\lambda_+ \geq \lambda_- \geq 0$. We therefore only need to test whether $\lambda_+$ or $\lambda = 0$ give the biggest value for $F_0(\theta)$. Studying the functions, we find that there are two cases. Defining $\theta_0$ as the unique solution in the interval of the equation

$$e^{-1}\sqrt{1-2\cot \theta}(1 + \sqrt{1 - 2\cot \theta}) = \cot \theta,$$

for $\theta \leq \theta_0$ the maximum still corresponds to $\lambda = 0$, but for $\theta \geq \theta_0$ it is achieved by $\lambda = \lambda_+$. Numerically, we find that $\theta_0 \approx 1.26$. Note that this $\theta_0$ corresponds to the line tangent to the classical set at the point of contact between the dashed and the continuous orange line in fig. 5. In summary, we found that

$$\begin{cases}
F_{\max}(\theta) = \cos \theta, & \theta < \theta_0, \\
F_{\max}(\theta) = f_0(\lambda_+), & \theta > \theta_0.
\end{cases}$$

(56)

- Finally, for $\theta \in [3\pi/2, 2\pi]$ we the stationary point $\lambda_-$ is negative and therefore non-physical, while $\lambda_+$ gives $f_0(\lambda_+)$ for $\theta \geq 0$. Because in this region $\cos \theta \geq 0$, we can conclude that the maximum is achieved by $\lambda = 0$ and $F_{\max}(\theta) = \cos \theta$.

We thus showed that

$$F_{\max}(\theta) = \begin{cases}
\cos \theta, & -\pi/2 \leq \theta \leq \theta_0, \\
f_0(\lambda_+), & \theta_0 \leq \theta \leq \pi, \\
0, & \pi \leq \theta \leq 3\pi/2
\end{cases},$$

(57)

where $\theta_0$ is defined as the solution of eq. (55), $\lambda_+(\theta) \equiv 1 + \sqrt{1 - 2\cot \theta}$, and $f_0(\lambda) \equiv e^{-\lambda}(\cos \theta + \sin \theta \lambda^2/2)$. This can be again be seen to be consistent with eq. (52). The shape of $F_{\max}(\theta)$ together with its first derivative is given in fig. 12.

Completely analogous reasoning can be used to derive the convex hull for other two-dimensional subspaces.

3. Recovering Klyshko’s inequalities in $(P_0, P_1, P_2)$

In $(P_0, P_1, P_2)$ we have, for every $\hat{\mathbf{n}} \equiv (a, b, c)$ with $a^2 + b^2 + c^2 = 1$,

$$F_{\max}(\hat{\mathbf{n}}) = \max_{\lambda \in [0, \infty]} e^{-\lambda}(a + b\lambda + c\lambda^2/2).$$

(58)

Define the function $f(\lambda, \hat{\mathbf{n}}) \equiv e^{-\lambda}(a + b\lambda + c\lambda^2/2)$, so that $F_{\max}(\hat{\mathbf{n}}) = \max_{\lambda \geq 0} f(\lambda, \hat{\mathbf{n}})$. The stationary points are the solutions of

$$-c/2\lambda^2 + (c - b)\lambda + (b - a) = 0,$$

(59)

which gives

$$\lambda_\pm = \frac{1}{c} \left[(c - b) \pm \sqrt{c^2 + b^2 - 2ac}\right].$$

(60)
From the second derivative we find that, when both solutions exist, only \( \lambda_+ \) is a local maximum (that is, corresponds to a negative second derivative), and therefore of interest to us. Equation (59) has no real solution if and only if
\[
e^2 + b^2 - 2ac < 0. \tag{61}
\]
Using the spherical coordinates
\[
a = \cos \theta, \quad b = \sin \theta \cos \varphi, \quad c = \sin \theta \sin \varphi, \tag{62}
\]
this condition reads \( \sin \theta (\sin \theta - 2 \cos \theta \sin \varphi) < 0 \). For these angles, the maximum \( F_{\text{max}} \) is achieved by either \( \lambda = 0 \) or \( \lambda = \infty \), corresponding to \( F_{\text{max}} = a \) and \( F_{\text{max}} = 0 \), respectively (as there are no possible stationary points in these directions). The former case corresponds to points on the surface whose tangent plane touches the vacuum state (that is, the point \( (1, 0, 0) \) in the reduced probability space), whereas the latter corresponds to point with tangent plane passing through the origin. We should therefore expect the associated sets of points to match with the surfaces \( K_\infty \) and \( K_1 \), respectively.

These solutions do not, however, give useful information about the nontrivial sectors of the classicality boundary, as they all correspond to flat sections of it. This means that one has to consider directions for which eq. (58) also admits stationary points. This is, however, a rather lengthy procedure: for a given direction \( \mathbf{n} \) one has to check the value of \( F_{\text{max}} \) corresponding to \( \lambda = \lambda_+ \) and \( \lambda = 0, \infty \), finding which one of these corresponds to the largest value and is positive (which is not ensured for \( \lambda_\pm \)). We can simplify this procedure by taking a hint from the results about the \( (P_0, P_1, P_2) \) space obtained in the main text. Indeed, we know that the two sections of the classicality boundary are “partially flat” surfaces, that is, surfaces obtained by connecting a fixed point (either \( (0, 0, 0) \) or \( (1, 0, 0) \)) with the points corresponding to coherent states. In terms of the linear functional approach considered here, this means that the directions corresponding to these surfaces must be such that eq. (58) has a maximum corresponding to two values of \( \lambda \): some \( \lambda \in (0, \infty) \), and then either \( \lambda = 0 \) or \( \lambda = \infty \). By focusing on these directions we can avoid considering the other cases.

Consider then in particular the section of the boundary corresponding to \( P_1^2 = 2P_0P_2 \). The points on this boundary are convex combinations of \( (0, 0, 0) \) with coherent states, that is, points of the form \( (e^{-\lambda}, e^{-\lambda}, e^{-\lambda} \lambda^2/2) \). Consider the directions for which eq. (58) has a maximum for \( \lambda = \infty \) (that is, for which \( F_{\text{max}} = 0 \)) that is also achieved as a local maximum for some finite \( 0 < \lambda < \infty \). This amounts to the condition
\[
a + b\lambda_+ + c\lambda_+^2/2 = 0, \tag{63}
\]
with \( \lambda_+ \) given by eq. (60). Explicitly, this is equivalent to
\[
c + \sqrt{b^2 + c^2 - 2ac} = 0, \] then translates into \( \theta \in [\pi/2, \pi] \) and \( \varphi \) being related to \( \theta \) via one of the following two conditions:
\[
\varphi = 2\pi - \arcsin \cot(\theta/2), \quad \varphi = \pi - \arcsin \cot(\theta/2). \tag{64}
\]
The second one corresponds to \( \lambda_- < 0 \), and is thus not physical. We conclude that the angles \( \theta \in [\pi/2, \pi] \) and \( \varphi = \varphi(\theta) = 2\pi - \arcsin \cot(\theta/2) \) correspond to points of the boundary for which the tangent plane touches both \( (0, 0, 0) \) and the coherent state \( \lambda_+(\theta, \varphi) \). The associated convex hull must then be given, for these angles, by the set of convex combinations \( p(0|0|0) + (1 - p) |\lambda||\lambda| \), for all coherent states with \( \lambda \in \{\lambda_+(\theta, \varphi) : \theta \in [\pi/2, \pi], \varphi = \varphi(\theta)\} \). Moreover, \( \lambda_+(\theta, \varphi) = \sqrt{-\cos \theta \sec(\theta/2)} \), and therefore this set equals the full interval \([0, \infty]\). The section of the classicality boundary we just found corresponds to eq. (12), thus giving further confirmation of the results in the main text.

For the boundary corresponding to eq. (13), we instead look for some \( \lambda \) such that \( F_{\text{max}} \) is achieved both by a stationary point and \( \lambda = \infty \). This amounts to the condition
\[
e^{-\lambda_+} (a + b\lambda_+ + c\lambda_+^2/2) = a. \tag{65}
\]
In this case, however, there seems to be no nice algebraic relation between \( \theta \) and \( \varphi \). To get a better idea of the overall behaviour we then resort to resolving numerically the \( \lambda s \) for which \( f_\mathbf{n}^\star (\lambda) = F_{\text{max}}(\hat{\mathbf{n}}) \) for the different directions \( \mathbf{n} \equiv (\theta, \varphi) \). The result of this is shown in fig. 13. In particular, the directions \( (\theta, \varphi) \) corresponding to the black dashed line, which are the values for which \( F_{\text{max}} \) is achieved by two distinct values of \( \lambda \) at the same time, are the ones that give the most information about the structure of the boundary. The black dashed curve between green and orange regions in the upper right of the figure corresponds to the directions for which \( f_\mathbf{n}^\star (\lambda_+) = f_\mathbf{n}^\star (\lambda_\infty) = 0 \). In each such direction, the boundary is therefore given by the segment joining \( \lambda_+ \) and the origin. This corresponds to the part of the boundary covered by Klyshko’s criterion, as per eq. (12). The black dashed curve between red and green regions in the lower half of the figure corresponds to directions for which \( f_\mathbf{n}^\star (\lambda_+) = f_\mathbf{n}^\star (0) \). Each such direction corresponds to a piece of boundary that is a segment joining \( \lambda_+ \) and \( \lambda = 0 \) (that is, the point \((1, 0, 0)\)). These therefore draw the part of the boundary corresponding to the \( K_\infty \) criterion: eq. (13). Finally, the black dashed line between white and darker white regions, which joins the other two black dashed curves, corresponds to directions for which \( f_\mathbf{n}^\star (0) = f_\mathbf{n}^\star (\lambda_\infty) > f_\mathbf{n}^\star (\lambda_+) \). All of these directions thus correspond to the segment joining the origin with \((1, 0, 0)\).

The above provides us with a complete characterization of the boundary of the classical region in this reduced space: the boundary of the classical set is drawn by segments joining the various coherent states with either the origin or \((1, 0, 0)\), consistently with the results reported in the main text.
F. COMPARISON WITH WIGNER AND P-FUNCTION CRITERIA

In this section we will present an analysis of the classes of states discussed in the main text using the standard methods to certify nonclassicality via the non-positivity of Wigner and P-functions. While the results of such methods are not comparable with those obtained with the techniques we discussed in the main text, in that they rely on completely different assumptions on what information is needed to apply the methods, they can be of interest to compare how restricting the available information can change the capacity of predicting the nonclassicality of a given state.

1. Wigner of Fock and coherent states

In this section we will report the standard expressions for the Wigner functions of Fock and coherent states, for future reference in the later sections. The Wigner function of a Fock state $|k\rangle$ is

$$W_k(x, p) = \frac{1}{\pi} \sum_{|z| = 0} e^{-|z|^2} L_k(2|z|^2),$$

where $z \equiv x + ip$ and $L_k$ are the Laguerre polynomials. If $|\alpha\rangle$, $\alpha \in \mathbb{C}$, is a coherent state, its Wigner function reads

$$W_{\alpha}(z) = \frac{1}{\pi} e^{-|z|\sqrt{2|\alpha|^2}}.$$  \hfill (67)

If instead of a coherent state we consider the corresponding fully depolarized state, i.e. what we referred to as Poissonian noise state, we get a state $\rho_{\mu}$ of the form

$$\rho_{\mu} = \sum_{k=0}^{\infty} \frac{e^{-\mu}|\mu|^k}{k!},$$

whose Wigner is

$$W_{\mu}(z) = \sum_{k=0}^{\infty} \frac{e^{-\mu}|\mu|^k}{k!} \frac{(-1)^k}{\pi} e^{-|z|^2} L_k(2|z|^2).$$  \hfill (69)

2. Wigner of boson-added Poissonian noise states

Consider now boson-added coherent states. To compare with the results in Fig. 2 in the main text, we consider the case of a single boson added. A boson-added coherent state has the form

$$\frac{1}{\sqrt{1 + |\alpha|^2}} a^\dagger |\alpha\rangle,$$

and its Wigner, as shown e.g. in [77], is

$$W_{\alpha,1}(z) = \frac{2|z - \alpha/\sqrt{2}|^2 - 1}{\pi(1 + |\alpha|^2)} e^{-|z - \sqrt{2}|^2}. $$ \hfill (71)

Let $\rho_{\mu}$ denote the states obtained by dephasing a coherent state with average boson number $\mu$. The corresponding boson-added Poissonian noise state is then

$$\rho_{\mu;1} \equiv (a^\dagger \rho_{\mu} a)/(1 + \mu).$$ \hfill (72)

These are diagonal in the Fock basis, so that the corresponding Wigner reads

$$W_{\mu;1}(z) = \sum_{k=0}^{\infty} P^{(1)}_k(z),$$ \hfill (73)
The Wigner function of a thermal state with average boson number $\mu$ is

$$W^{th}_\mu(z) = \frac{1}{\pi(1+2\mu)} e^{-|z|^2/(1+2\mu)},$$

(75)

where $z \equiv x + ip$. Boson-added thermal states have the form

$$\hat{N} \alpha^\dagger \rho_{th} \alpha = \sum_{k=0}^{\infty} \frac{k\mu^{k-1}}{(1+\mu)^{k+1}} P_k$$

(76)

where $N$ is a normalisation constant. The corresponding Wigner function is then

$$W^{th}_{\mu,1}(z) = \sum_{k=0}^{\infty} \frac{k\mu^{k-1}}{(1+\mu)^{k+1}} W_k(z)$$

(77)

$$= \sum_{k=0}^{\infty} \frac{k\mu^{k-1}}{(1+\mu)^{k+1}} \frac{(-1)^k}{\pi} e^{-|z|^2} L_k(2|z|^2).$$

As shown in fig. 17, the corresponding Wigner function is negative for all $\mu$. For comparison, as was shown in the previous sections, Klyshko’s criteria are also capable of detecting the nonclassicality of these states from their first three Fock probabilities.

4. Noisy Fock states

*Noisy Fock states* have states of the form $D(\alpha) \ket{m}$, averaged over the phase of $\alpha$. Let us denote these states with $\rho^{(m,\mu)}$, where $\mu \equiv |\alpha|^2$. To obtain the corresponding expressions, we start from the matrix components of $D(\alpha)$ in the Fock basis:

$$\langle k|D(\alpha)|n\rangle = \frac{1}{\sqrt{n!k!}} e^{-\mu/2} \alpha^{k-n} c_{k,n},$$

$$c_{k,n} \equiv \sum_{i=0}^{n} \frac{(-\mu)^i}{i!} \frac{k!n!}{(n-i)!((k-n+i))!}.$$
Another way to characterize a state is via its P-function. As shown in fig. 18, the corresponding Wigner functions are non-negative, and thus cannot be used to certify nonclassicality. In this case, as shown in fig. 19, the Wigner function is non-positive, and thus certifying nonclassicality.

For $n = 1$, these correspond to the Fock probabilities

$$P_k^{(1;\mu)} = e^{-\mu} \mu^k \frac{(k-\mu)^2}{k!}.$$  \hfill (79)

As shown in fig. 18, the corresponding Wigner functions are non-negative, and thus cannot be used to certify nonclassicality.

For $n = 2$, we get

$$P_k^{(2;\mu)} = e^{-\mu} \mu^k \frac{(k(k-1)-2k\mu+\mu^2)^2}{k!}.$$  \hfill (80)

In this case, as shown in fig. 19, the Wigner function is non-positive, thus certifying nonclassicality.

5. P-function

Another way to characterize a state is via its P-function. This, however, can be highly singular, and thus difficult to manipulate. To explore the P-function of highly nonclassical states such as Fock states, we thus consider the limit of the P-functions of boson-added thermal states $\rho^{th}_{n,k}$ in the limit of the average boson number going to zero. More precisely, we consider the states

$$\rho^{th}_{n,k} \propto (a^\dagger k)^{n} e^{-|a|^2/\bar{n}}.$$  \hfill (81)

where $\bar{n}$ is the average number of photons and $\rho^{th}_{n,k}$ the corresponding thermal state. Note that $\rho^{th}_{n,k} \rightarrow |0\rangle\langle 0|$ for $\bar{n} \rightarrow 0$, and thus also $\rho^{th}_{n,k} \rightarrow |k\rangle\langle k|$ for $\bar{n} \rightarrow 0$. We can thus study the P-function of $|k\rangle\langle k|$ via that of $\rho^{th}_{n,k}$ in the limit of $\bar{n} \rightarrow 0$, effectively regularizing the singular P-functions. The P-function of the thermal state $\rho^{th}_{\bar{n}}$ reads

$$\pi P^{th}_{\bar{n}}(\alpha) = \frac{1}{\bar{n}} e^{-|\alpha|^2/\bar{n}}.$$  \hfill (82)

The $P$-function of $\rho^{th}_{n,k}$ is then

$$\pi P^{th}_{n,k}(\alpha) = N_k \frac{1}{\bar{n}} e^{-|\alpha|^2/\bar{n}} e^{-|\alpha|^2/\bar{n}}.$$  \hfill (83)

where

$$N_k = \left[ \text{Tr} \left( a^\dagger k x^N a^k \right) \right]^{-1}, \quad x = \frac{\bar{n}}{\bar{n} + 1}.$$  \hfill (84)

For example, the $P$-function of $\rho^{th}_{n,1}$ is

$$P^{th}_{n,1}(\alpha) = \frac{1}{\bar{n}^2} (|\alpha|^2(1 + \bar{n}) - \bar{n}) e^{-|\alpha|^2/\bar{n}}.$$  \hfill (85)

which is manifestly negative for $|\alpha|^2 < \bar{n}/(\bar{n} + 1)$. This implies that the $P$ function always lies above the fact that the $P$ function loses its regularity in the limit $\bar{n} \rightarrow 0$. This again certifies the nonclassicality of the Fock state $|1\rangle$.

To verify the negativity of noisy Fock states, obtained applying the displacement operator $D(\sqrt{\bar{n}} e^{i\phi})$ on a Fock state, the Wigner function is observed to remain non-negative for all tested values of $\mu$. As shown in fig. 19, the Wigner function is non-positive for $\mu = 1$, for different values of the average boson number $\mu$. The Wigner function is observed to remain non-negative for all tested values of $\mu$. As discussed previously, for $n = 1$, these correspond to the Fock probabilities

$$P_k^{(1;\mu)} = \frac{e^{-\mu} \mu^k (k-\mu)^2}{k!}.$$  \hfill (79)
state and then averaging over $\phi$, we notice that the $P$-function becomes

$$ P^{\text{th}}_{\bar{n};1;\mu}(\alpha) = \int_0^{2\pi} \frac{d\phi}{2\pi} P_{\bar{n};1}(\alpha - \sqrt{\mu}e^{i\phi}). \quad (86) $$

This is again negative for small $\bar{n}$, as expected. To compare again with the results in Fig. 2 in the main text, as done in section F2, we now consider mixtures of the form

$$ pP^{\text{th}}_{\bar{n};1;\mu} + (1-p)P^{\text{th}}_{\bar{n};0;\mu}. \quad (87) $$

These are characterized by the $P$-function $P_{\bar{n},p,\mu} = pP^{\text{th}}_{\bar{n};1;\mu} + (1-p)P^{\text{th}}_{\bar{n};0;\mu}$, where $P^{\text{th}}_{\bar{n};0;\mu}$ is the $P$-function of a thermal state displaced with a random phase. To observe the negativity for every $p \in (0,1)$, we need to choose sufficiently small $\bar{n}$, since the $P$-function always drops at $|\alpha|^2 = \mu$ without any saturation for decreasing $\bar{n}$. However, very small values of $\bar{n}$ cause failure in numerical evaluation of the integral (83). This limits the numerics in proving the negativity for large $\mu$ and small $p$. We verified that $P_{\bar{n},p,\mu}$ is negative for $p = 0.05$ and $\mu$ up to 19. However, if $p = 0.01$ the negativity could only be verified for $\mu$ up to 1.3. Predictably, we find that the $P$-function is able to capture nonclassicality better than the Wigner function, which instead was found to certify nonclassicality via non-positivity only for $p \geq 0.9$.

**Boson-added Poissonian noise states** Consider now the boson-added state $\frac{1}{\sqrt{\mu}}a^\dagger \rho_\mu a$, with $\rho_\mu$ the Poissonian noise state with average boson number $\mu$. Using the commutation relations between creation and displacement operators we get

$$ a^\dagger \rho_\mu a \propto \int_0^{2\pi} d\phi D(\sqrt{\mu}\exp(i\phi)|q\rangle\langle q|D^\dagger(\sqrt{\mu}\exp(i\phi)), \quad (88) $$

where $|q\rangle \propto |1\rangle - \sqrt{\mu}\exp(-i\phi)|0\rangle$. Using again thermal states to approximate the Fock states in the $\bar{n} \to 0$ limit, we get

$$ P_{\bar{n};p,\mu} \propto e^{|\alpha|^2} \frac{\partial^2}{\partial \alpha \partial \alpha^*} e^{-|\alpha|^2 \frac{1+\bar{n}}{\mu} + \mu e^{-|\alpha|^2}} - e^{|\alpha|^2 \frac{\bar{n}+1}{2\mu}} \sqrt{\mu} \left( e^{-i\phi} \frac{\partial}{\partial \alpha} + e^{i\phi} \frac{\partial}{\partial \alpha^*} \right) e^{-|\alpha|^2 \frac{1+\bar{n}}{2\mu}}. \quad (89) $$

To complete the calculation, we use the transformation rule eq. (86) on the $P$-function in eq. (89).

If the boson-addition happens with probability $p$, the density matrix reads

$$ \frac{p}{1+\mu} a^\dagger \rho_\mu a + (1-p)\rho_\mu. \quad (90) $$

Analysing the corresponding $P$-function presents difficulties similar to the ones for noisy Fock states, since the negativity appears only for small $\bar{n}$ when $p$ is small. The numerics allowed us to observe the negativity only for $\mu$ up to 1.35 when we considered $p = 0.05$. It is worth stressing once more that the observed limits for the negativities in the $P$-functions of the states (87) and (90) are only due to numerical precision, and bear no physical meaning. This is in contrast with the negativities of the Wigner function, which are sensitive to the noise contributions and efficiency of the processes giving the prepared states.