Exact Solution of Gas Dynamics Equations Through Reduced Differential Transform and Sumudu Transform Linked with Pades Approximants

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Abstract. In this paper, we study the analytical method based on reduced differential transform method coupled with sumudu transform through Pades approximants. The proposed method may be considered as alternative approach for finding exact solution of Gas dynamics equation in an effective manner. This method does not require any discretization, linearization and perturbation.

1. Introduction
In the recent years, many researchers have paid much attention to study the numerical and analytical methods for finding the solution of fractional and integer order nonlinear gas dynamics.

In this paper, we consider the nonlinear gas dynamics equation of the following type:

\[ \frac{\partial u}{\partial t} + \frac{1}{2} (u^2)_{x} - u (1 - u) = g(x,t), \quad t > 0 \]

subject to the initial condition \( u(x,0) = f(x) \), where \( g(x,t) \) is a known function.

Gas dynamics equations are mathematical expressions based on the physical laws of conservation of mass, conservation of momentum, conservation of energy etc. The few types of gas dynamics equations in physics have been solved by applying different analytical and approximation methods (see the papers [2] - [3], [6], [20]). Recently several analytical / numerical methods have been developed for the solution of gas dynamic equations. The similar type of fractional / integer order gas dynamics equation have been solved by many numerical methods such as new homotopy perturbation method [14], differential transform method [15], fractional homotopy analysis transform method [16], fractional reduced differential transform method [17], Laplace-homotopy perturbation method [18] and new homotopy perturbation sumudu transform method [19].

2. Analysis of reduced differential transform
The Reduced Differential Transform Method (RDTM) was first envisioned by Keskin (see the papers [10]-[12]) for solving nonlinear partial differential equations. This method based on Taylor series gives approximate analytical solution in the form of convergent series. Later, many researchers have improved and modified this method to obtain rapid convergent series solution of nonlinear problems. The brief analysis of RDTM is as follows:
Consider a function \( u(x, t) \) of two variables and assume that it can be represented as a product of two single variable functions, i.e., \( u(x, t) = f(x) \cdot g(t) \). On the basis of the properties of the one dimensional differential transform, the function \( u(x, t) \) can be represented as

\[
u(x, t) = \sum_{n=0}^{\infty} F(h) x^n \sum_{k=0}^{\infty} G(k) t^k = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} U(h, k) x^n t^k
\]

Where \( U(h, k) = F(h)G(k) \) is called the spectrum of \( u(x, t) \).

The basic definitions and properties of reduced differential transform are introduced below.

The reduced differential transform of \( u(x, t) \) at \( t = 0 \) is defined as

\[
U_k(x) = \frac{1}{k!} \left[ \frac{\partial^n u(x, t)}{\partial t^n} \right]_{t=0}
\]

Where \( u(x, t) \) is the given function and \( U_k(x) \) is the transformed function.

The reduced differential inverse transform of \( U_k(x) \) is defined as

\[
u(x, t) = \sum_{n=0}^{\infty} \frac{1}{k!} \left[ \frac{\partial^n u(x, t)}{\partial t^n} \right]_{t=0} t^k
\]

From definitions (2) and (3), we summarize fundamental properties of two dimensional reduced differential transform.

**Theorems 2.1:** Let \( U_k(x), V_k(x) \) and \( W_k(x) \) be the reduced differential transform of the functions of \( u(x, t), v(x, t) \) and \( W(x, t) \) respectively, then

(a) If \( w(x, t) = u(x, t) + v(x, t) \) then \( W_k(x) = U_k(x) + V_k(x) \)

(b) If \( w(x, t) = au(x, t) \) then \( W_k(x) = aU_k(x) \)

(c) If \( w(x, t) = a\frac{\partial^n u(x, t)}{\partial t^n} \) then \( W_k(x) = a\frac{(k+n)!}{k!} U_{k+n}(x) \)

(d) If \( w(x, t) = x^m t^n \) then \( W_k(x) = x^m \delta(k - n) = \begin{cases} x^m & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} \)

(e) If \( w(x, t) = a\frac{\partial^n u(x, t)}{\partial x^n} \) then \( W_k(x) = a\frac{(k+n)!}{k!} U_{k+n}(x) \)

(f) If \( w(x, t) = u(x, t)v(x, t) \) then \( W_k(x) = \sum_{k_1=0}^{k} U_{k_1}(x)V_{k-k_1}(x) \)

(g) If \( w(x, t) = x^m t^n u(x, t) \) then \( W_k(x) = x^m U_{k-n}(x) \)

(h) If \( w(x, t) = t^n u(x, t)v(x, t) \) then \( W_k(x) = U_{k-n}(x) \)

**3. Background of Sumudu transform**

The concept of Sumudu transform was first demonstrated in [5] for finding the solution of the ordinary differential equations in control engineering problems. The Sumudu transform is defined over the set of functions

\[
A = \left\{ f(t) \mid \exists \tau_1, \tau_2 > 0, |f(t)| < M e^{-1} \text{ if } t \in (-1)^1 \times [0, \infty) \right\}
\]

by the following formula

\[
F(u) = \mathcal{S}[f(t)] = \frac{1}{u} \int_{0}^{\infty} f(t)e^{-\frac{t}{u}} dt, \quad u \in (-\tau_1, \tau_2)
\]

Some important properties of Sumudu transform are derived in [7, 8, 9] and the relation between Sumudu transform and Laplace transform was established in [4].

**4. Pades Approximants**

A pades approximants [1] is the ratio of two algebraic polynomials constructed from the coefficients of Maclaurins series expansion of a function.

Suppose the Maclaurins series of an analytic function \( f(t) \) is given by

\[
f(t) = \sum_{n=0}^{\infty} f_n t^n, \quad 0 < t < T
\]
Then the pades approximants to \( f(t) \) of order \( \left[ \frac{L}{M} \right] \) represented by \( \left[ \frac{L}{M} \right] f(t) \) and is defined as
\[
\left[ \frac{L}{M} \right] f(t) = \sum_{i=0}^{L} a_i t^i
\]
with \( b_0 = 1 \) in which the numerator and denominator have no common factor.

The Maclaurin’s series (7) determines the coefficients of the polynomials in (8) by the equation
\[
f(t) - \left[ \frac{L}{M} \right] f(t) = O(e^{L+M+1})
\]
From (9), we have
\[
f(t) \sum_{n=0}^{M} b_n t^n - \sum_{n=0}^{L} a_n t^n = O(e^{L+M+1})
\]
From the above equation, we get the following system of algebraic equations:
\[
\begin{align*}
f_L b_1 + \cdots + f_{L-M+1} b_M &= -f_{L+1} \\
f_{L+1} b_1 + \cdots + f_{L-M+2} b_M &= -f_{L+2} \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\} 
\end{align*}
\]
From (11), we calculate all the coefficients \( a_n \) for \( 1 \leq n \leq M \) and \( b_n \) for \( 0 \leq n \leq M \).

5. Numerical Applications
In this section, we demonstrate the effectiveness of the proposed method by presenting three nonlinear problems.

Example 5.1 Consider the following homogeneous nonlinear gas dynamics equation [14, 15, 16, 17, 19].
\[
\frac{\partial u}{\partial t} + \frac{1}{2} (u^2) x - u(1 - u) = 0
\]
Subject to the initial condition
\[
u(x, 0) = e^{-x}
\]
Applying the reduced differential transform to the equations (12) and (13), we get
\[
(k + 1) U_{k+1}(x) = U_k(x) - \sum_{k=0}^{k} U_{k-1}(x) U_{k-2}(x) - \sum_{k=1}^{k} U_{k-1}(x) \frac{\partial}{\partial x} U_{k-2}(x)
\]
Substituting the equation (15) in to the equation (14) and by straight ward iterative procedure yields
\[
U_1(x) = e^{-x}, \quad U_2(x) = \frac{e^{-x}}{2}
\]
Therefore, the approximate solution up to second order differential transform is
\[
u(x, t) = U_0 + U_1 t + U_2 t^2 = e^{-x} \left[ 1 + t + \frac{t^2}{2} \right]
\]
Applying Sumudu transform to both sides of above equation, we get
\[
S[u(x, t)] = e^{-x} \left[ 1 + u + u^2 \right]
\]
Its \( \left[ \frac{M}{N} \right] \) pades approximant with \( M \geq 1 \) and \( N \geq 1 \) yields
\[
\left[ \frac{M}{N} \right] e^{-x} \left( 1 - \frac{1}{w} \right) = \frac{e^{-x}}{1 - \frac{1}{w}}
\]
Applying inverse Sumudu transform to \( \left[ \frac{M}{N} \right] \), we obtain the exact solution:
\[
u(x, t) = e^{-x} + t
\]
Example 5.2 Consider the following homogeneous nonlinear gas dynamics equation [16, 17, 18].
\[
\frac{\partial u}{\partial t} + \frac{1}{2} (u^2) x - u(1 - u) \log a = 0
\]
with initial condition
\[ u(x, 0) = a^{-x}, \quad a > 0 \] (22)

Taking known reduced differential transform on both sides of the equations (21) and (22), we get
\[ (k + 1)U_{k+1}(x) = U_k(x) \log a - \sum_{k_1=0}^{k} U_{k_1}(x) U_{k-k_1}(x) \log a - \sum_{k_1=0}^{k} U_{k_1}(x) \frac{\partial}{\partial x} U_{k-k_1}(x) \]
(23)
\[ U_0(x) = a^{-x} \] (24)

Now, substituting the equation (24) in to the equation (23) and by straight ward iterative procedure, yields
\[ U_1(x) = a^{-x} \log a, \quad U_2(x) = \frac{a^{-x}}{2} (\log a)^2 \] (25)

Therefore the approximate solution up to second order differential transform is
\[ u(x, t) = U_0 + U_1 t + U_2 t^2 = a^{-x} \left[ 1 + (\log a) t + \frac{(\log a)^2}{2} t^2 \right] \] (26)

Applying Sumudu transform to both sides of the above equation, we have
\[ S[u(x, t)] = a^{-x} [1 + (\log a) u + (\log a)^2 u^2] \] (27)

Its \[ \frac{M}{N} \] pades approximant with \( M \geq 1 \) and \( N \geq 1 \) yields
\[ \frac{M}{N} = \frac{a^{-x}}{(1-u \log a)} \] (28)

Applying inverse Sumudu transform to \[ \frac{M}{N} \], we obtain the exact solution:
\[ u(x, t) = a^{-x+t} \] (29)

**Example 5.3** Consider the following nonhomogeneous nonlinear gas dynamics equation [14, 15]:
\[ \frac{\partial u}{\partial t} + \frac{1}{2} (u^2) \chi + u(1 - u) = -e^{-x+t} \] (30)

Subject to the initial condition
\[ u(x, 0) = 1 - e^{-x} \] (31)

Using aforesaid reduced differential transform method, we have
\[ (k + 1)U_{k+1}(x) = U_k(x) - \sum_{k_1=0}^{k} U_{k_1}(x) U_{k-k_1}(x) - \sum_{k_1=0}^{k} U_{k_1}(x) \frac{\partial}{\partial x} U_{k-k_1}(x) - \frac{e^{-x}}{k!} \]
(32)
\[ U_0(x) = 1 - e^{-x} \] (33)

Substituting the equation (33) in to the equation (32) we get the first few terms
\[ U_1(x) = -e^{-x}, \quad U_2(x) = \frac{-e^{-x}}{2} \] (34)

Therefore, the approximate solution up to second order differential transform is
\[ u(x, t) = U_0 + U_1 t + U_2 t^2 = 1 - e^{-x} \left[ 1 + t + \frac{t^2}{2} \right] \] (35)

Operating Sumudu transform on both sides of above equation, we get
\[ S[u(x, t)] = 1 - e^{-x} [1 + u + u^2] \] (36)

Its \[ \frac{M}{N} \] pades approximant with \( M \geq 1 \) and \( N \geq 1 \) yields
\[ \frac{M}{N} = 1 - \frac{e^{-x}}{(1-u)} \] (37)

Applying inverse Sumudu transform to \[ \frac{M}{N} \], yields the exact solution:
\[ u(x, t) = 1 - e^{-x+t} \] (38)

6. **Conclusion**

In this paper, we proposed a new hybrid method which is the combination of reduced differential transform method and sumudu transform linked with pades approximants to achieve the exact solution of gas dynamics equations. Results confirm that the present technique is one of the alternative approach for handling solution of nonlinear differential equations.
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