The Worpitzky identity for the groups of signed and even-signed permutations

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Abstract
The well-known Worpitzky identity

\[(x + 1)^n = \sum_{k=0}^{n-1} A_{n,k} \binom{x + n - k}{n}\]

provides a connection between two bases of \(\mathbb{Q}[x]\): the standard basis \((x + 1)^n\) and the binomial basis \(\binom{x + n - k}{n}\), where the Eulerian numbers \(A_{n,k}\) for the symmetric group serve as the entries of the transformation matrix. Brenti has generalized this identity to the Coxeter groups of types \(B_n\) and \(D_n\) (signed and even-signed permutations groups, respectively) using generating functionology. Motivated by Foata–Schützenberger’s and Rawlings’ proof for the Worpitzky identity in the symmetric group, we provide combinatorial proofs for the generalizations of this identity and for their \(q\)-analogues in the Coxeter groups of types \(B_n\) and \(D_n\). Our proofs utilize the language of \(P\)-partitions for the \(B_n\)- and \(D_n\)-posets, introduced by Chow and Stembridge, respectively.

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1 Introduction

The well-known Worpitzky identity involves the Eulerian numbers, the original definition of which was given by Euler in an analytic context [6, §13]. Later, these numbers began to appear in combinatorial problems, and this is the context in which we choose to present them here.

Let $S_n$ be the symmetric group on $n$ elements. For any permutation $\pi \in S_n$, we say that $\pi$ has a descent at position $i$, if $\pi(i) > \pi(i + 1)$, and we denote by $\text{Des}(\pi)$ the set of descents:

$$\text{Des}(\pi) := \{i \in [n - 1] \mid \pi(i) > \pi(i + 1)\}. \quad (1)$$

We denote the number of descents in $\pi$ by $\text{des}(\pi) := |\text{Des}(\pi)|$.

The Eulerian number $A_{n,k}$ counts the number of permutations in $S_n$ having $k$ descents:

$$A_{n,k} = |\{\pi \in S_n : \text{des}(\pi) = k\}|.$$

In this context, the Worpitzky identity states:

$$(x + 1)^n = \sum_{k=0}^{n-1} A_{n,k} \binom{x + n - k}{n}. \quad (2)$$

An excellent overview of the Worpitzky’s identity and the Eulerian numbers can be found in Petersen’s book [12, Chap. 1].

Worpitzky’s identity was generalized to the Coxeter groups of types $B_n$ and $D_n$ by Borowiec and Młotkowski [2], though they used a non-algebraic version of Eulerian numbers.

Generalizations of the Worpitzky identity (and their $q$-analogues), using the algebraic Coxeter definition of the descents in these groups, were introduced by Brenti [3, Theorem 3.4(iii) for $q = 1$ and Corollary 4.11]:

$$(2x + 1)^n = \sum_{k=0}^{n} \binom{x + n - k}{n} B_{n,k} \quad \text{(type $B_n$)},$$

$$(2x + 1)^n - 2^{n-1}(B_n(x + 1) - B(n)) = \sum_{k=0}^{n} \binom{x + n - k}{n} D_{n,k} \quad \text{(type $D_n$)},$$

where $B_{n,k}$ and $D_{n,k}$ are the Eulerian numbers of types $B_n$ and $D_n$, respectively (these notations will be defined in the next section), $B(n)$ is the $n$-th Bernoulli number and $B_n(x)$ is the $n$-th Bernoulli polynomial (see [8] for the definitions of these concepts).

Combinatorial identities usually have more than one possible proof. Some of them are analytic, some algebraic in nature, but the most beautiful ones are combinatorial, meaning that both sides of the identity count the same set of elements in different ways.

In our context, Foata and Schützenberger [7, p. 40] have proved the Worpitzky identity for the Coxeter group of type $A_{n-1}$ in a combinatorial way (see also Rawlings [13]...
and Petersen [12, p. 366]). On the other hand, Brenti’s proofs for the generalizations of Worpitzky’s identity for types $B_n$ and $D_n$ are non-combinatorial and use generating function techniques.

Our contribution in this paper consists of combinatorial proofs for the $q$-analogues of the Worpitzky identity for types $B_n$ and $D_n$:

$$
(1 + (1 + q)x)^n = \sum_{k=0}^{n} \binom{n + x - k}{n} B_{n,k}(q) \quad \text{(type } B_n),
$$
$$
(1 + (1 + q)x)^{n-1} - (1 + q)^{n-1}(B_n(x + 1) - B(n)) = \sum_{k=0}^{n} \binom{n+x-k}{n} D_{n,k}(q) \quad \text{(type } D_n).
$$

These $q$-analogues appear in Brenti [3] (the identity for type $B_n$ is Theorem 3.4(iii), and the identity for type $D_n$ is referred to implicitly before Theorem 4.10).

Our combinatorial proofs are in the spirit of the proof of Foata–Schützenberger [7] for type $A_{n-1}$, though we use the language of $P$-partitions.

Following Chow [5], we look at a specific $B_n$-poset $P$. Then we count the number of $P$-partitions in two ways in order to obtain the Worpitzky identity for type $B_n$.

This idea is then applied to the group $D_n$ using a definition of $D_n$-posets due to Stembridge [15]. In this case, the Worpitzky identity has an extra expression, which we interpret combinatorially in terms of counting $D_n$-permutations.

The paper is organized as follows. In Section 2, we give some preliminaries, including the definitions of the Coxeter groups of types $B_n$ and $D_n$ and the Eulerian numbers associated with them. Section 3 recalls the concept of $P$-partitions of types $A_{n-1}$, $B_n$, and $D_n$. Sections 4 and 5 present the combinatorial proofs of the identities (and their $q$-analogues) for types $B_n$ and $D_n$, respectively.

### 2 Coxeter groups of type $B_n$ and $D_n$

In this section, we provide some background on the Coxeter groups of types $B_n$ and $D_n$. A general reference is Björner–Brenti’s book [4, Chap. 8].

#### 2.1 The Coxeter group of type $B_n$

Let $B_n$ be the group of signed permutations on $\{1, \ldots, n\}$, i.e., the set of permutations $\pi$ on $\{0, \pm 1, \pm 2, \ldots, \pm n\}$, such that $\pi(-i) = -\pi(i)$ for $1 \leq i \leq n$ and $\pi(0) = 0$. We occasionally write $\pi_i$ instead of $\pi(i)$. This is the standard combinatorial realization of the Coxeter group of type $B_n$.

We define some statistics on $B_n$. First, for a permutation $\pi \in B_n$, define:

$$\text{Des}_A(\pi) = \{i : \pi(i) > \pi(i+1), 1 \leq i \leq n-1\},$$

and denote: $\text{des}_A(\pi) = |\text{Des}_A(\pi)|$. 

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Now, for $\pi \in B_n$, we define:

$$\text{Des}_B(\pi) = \begin{cases} \text{Des}_A(\pi) \cup \{0\} & \pi(1) < 0 \\ \text{Des}_A(\pi) & \pi(1) > 0 \end{cases}$$

As before, we denote: $\text{des}_B(\pi) = |\text{Des}_B(\pi)|$.

Note that unlike $\text{Des}_A$, the statistic $\text{Des}_B$ is consistent with the algebraic definition of descent sets for Coxeter groups.

**Example 2.1** Let $\pi = [-1, 2, -5, 4, 3] \in B_5$. Then, $\text{Des}_B(\pi) = \{0, 2, 4\}$ and $\text{des}_B(\pi) = 3$.

Let $B_{n,k} = |\{\pi \in B_n : \text{des}_B(\pi) = k\}|$. The number $B_{n,k}$ is called the *Eulerian number of type $B_n$*. This set of numbers constitutes the sequence A060187 in OEIS [9].

We define another statistic:

$$\text{neg}(\pi) = |\{i : \pi(i) < 0, 1 \leq i \leq n\}|,$$

and a $q$-analogue of $B_{n,k}$ is:

$$B_{n,k}(q) = \sum_{\pi \in B_n, \text{des}_B(\pi) = k} q^{\text{neg}(\pi)}. \quad (3)$$

**Example 2.2** Let $\pi = [-1, 2, -5, 4, 3] \in B_5$. Then, $\text{neg}(\pi) = 2$.

### 2.2 The Coxeter group of type $D_n$

Denote by $D_n$ the group of signed permutations on $\{1, \ldots, n\}$ with an even number of negative elements. This is the standard combinatorial realization of the Coxeter group of type $D_n$.

Following Petersen ([11] and [12, Chap 13.4, Equation 13.14]), we write the elements of $D_n$ as ‘forked permutations’. For example, the even-signed permutation $w = [3, 2, -4, -1] \in D_4$ (in the usual window notation) will be written as a ‘forked permutation’ as follows:

$$w = \left[1, 4, -2, 3, -3, 2, -4, -1\right].$$

Before presenting the Eulerian numbers $D_{n,k}$ for type $D_n$, we need the following definitions. Following Petersen [12, p. 302], we slightly deviate from the usual definition of Coxeter descents of $\pi \in D_n$ (i.e., we use $-1$ instead of 0):

$$\text{Des}_D(\pi) = \begin{cases} \text{Des}_A(\pi) \cup \{-1\} & \pi(1) + \pi(2) < 0 \\ \text{Des}_A(\pi) & \pi(1) + \pi(2) > 0 \end{cases}$$
and denote: \( \text{des}_D(\pi) = |\text{Des}_D(\pi)| \). The statistic \( \text{des}_D \) is consistent with the Coxeter descents, see [4, Prop. 8.2.2].

**Example 2.3** Let

\[
\pi = [-4, -1, 5, -6, -2, -3, 2, 6, -5, 1, 4] \in D_6.
\]

Then, \( \text{Des}_D(\pi) = \{-1, 3\} \), and hence, \( \text{des}_D(\pi) = 2 \).

Let \( D_{n,k} = |\{\pi \in D_n : \text{des}_D(\pi) = k\}| \) be the Eulerian number of type \( D_n \) (sequence A262226 in OEIS [9]). For a \( q \)-analogue, let

\[
D_{n,k}(q) = \sum_{\pi \in D_n, \text{des}_D(\pi) = k} q^{\text{neg}_2(\pi)},
\]

where

\[
\text{neg}_2(\pi) = |\{i \in \{2, \ldots, n\} | \pi(i) < 0\}|.
\]

In the last example, \( \text{neg}_2(\pi) = 1 \) (see Brenti [3, Equation (52)]).

### 3 P-Partitions for Coxeter groups of types \( A_{n-1}, B_{n-1} \) and \( D_{n-1} \)

In this paper, we use \( P \)-partitions of special posets designed for the Coxeter groups of types \( B_n \) and \( D_n \), so we recall their definitions in this section. For the sake of completeness, we add also the definition of a \( P \)-partition for the Coxeter groups of type \( A_{n-1} \).

Note that the definition of \( P \)-partitions for type \( A_{n-1} \) is due to Stanley [14], who defined them to be order-reversing, rather than order-preserving.

#### 3.1 P-Partitions of type \( A_{n-1} \)

Let \( P = \{p_1, \ldots, p_n\} \) be a partially ordered set (poset), labeled by the set \( [n] = \{1, \ldots, n\} \), with the partial order \( \prec_P \). We identify each element in \( P \) with its label. A \( P \)-partition (of type \( A_{n-1} \)) is an order-preserving map \( f : [n] \to \mathbb{Z} \) satisfying:

1. \( f(i) \leq f(j) \), if \( i \prec_P j \).
2. \( f(i) < f(j) \), if \( i \prec_P j \) and \( i > j \) in \( \mathbb{Z} \).

#### 3.2 P-Partitions of type \( B_n \)

We start with the definition of a \( B_n \)-poset (see [5]):
**Definition 3.1** A $B_n$-poset is the set $P = \{0, \pm 1, \pm 2, \ldots, \pm n\}$ with a partial order $\prec_P$ that respects negation, i.e., if $i \prec_P j$, then $-j \prec_P -i$.

**Example 3.2** (a) The following poset $P$ is an example of $B_3$-poset:

(b) A useful general example for a $B_n$-poset with a total order is induced by every signed permutation $\pi \in B_n$. This is done by defining $\pi(i) \prec_P \pi(i + 1)$ for $0 \leq i \leq n - 1$. For instance, the signed permutation $\pi = [-1, 3, 2, -4]$ induces the $B_4$-poset:

$$4 \prec_P -2 \prec_P -3 \prec_P 1 \prec_P 0 \prec_P -1 \prec_P 3 \prec_P 2 \prec_P -4.$$ 

Chow [5] generalized the concept of $P$-partitions to Coxeter groups of type $B_n$ as follows:

**Definition 3.3** Let $P$ be a $B_n$-poset and let $(X, \preceq)$ be a countable totally ordered set. A $P$-partition $P$ of type $B_n$ is an order-preserving map $f : [\pm n] \rightarrow X$ satisfying for all $i, j$:

1. $f(i) \preceq_P f(j)$, if $i \prec_P j$,
2. $f(i) \prec_P f(j)$, if $i \prec_P j$ and $i \succ j$ in $X$,
3. $f(-i) = -f(i)$.

Throughout this paper, we use $X = (\mathbb{Z}, \preceq)$ (with the convention that when $x \preceq y$ and $x \neq y$, we write $x < y$), where the order relation is defined as:

$$0 < -1 < 1 < -2 < 2 < \cdots.$$ 

We will make use of the fact that this order is finer than the order induced by the absolute value: if $a \preceq_P b$ (where $a, b \in \mathbb{Z}$), then $|a| \leq |b|$.

**Example 3.4** Given the $B_3$-poset presented in Example 3.2(a) above, we have the following $P$-partition:

$$f(-3) = -3, f(-2) = 3, f(-1) = 3, f(0) = 0,$$

$$f(1) = -3, f(2) = -3, f(3) = 3.$$
Let $A(P)$ denote the set of all $P$-partitions of type $B_n$ of the poset $P$. In the special case $P = \pi \in B_n$ (as in Example 3.2(b)), it is worth mentioning that $A(\pi)$ is the set of all functions $f : \{\pm n\} \to \mathbb{Z}$ such that for each $0 \leq i \leq n$, $f(-i) = -f(i)$ and

$$f(0) = f(\pi(0)) \preceq f(\pi(1)) \preceq \cdots \preceq f(\pi(n)), \quad (4)$$

where the inequality $f(\pi(i)) \preceq f(\pi(i + 1))$ is strict if $i \in \text{Des}_B(\pi)$.

For a $B_n$-poset $P$, a linear extension of $P$ is a $B_n$-poset which is identical to $P$ as a set, and extends the partial order into a total order. Denote by $L(P)$ the set of all such linear extensions of $P$. Considering signed permutations as $B_n$-posets (as in Example 3.2(b)), it is easy to see that $L(P) \subseteq B_n$ for each $B_n$-poset $B$.

**Example 3.5** Back to Example 3.2(a), we have the following linear extensions of $P$:

$$L(P) = \{[−3, 1, −2, 0, 2, −1, 3], [2, −1, 3, 0, −3, 1, −2]\}.$$ 

Note that by the defining rule of a $B_n$-poset, each linear extension contains 0 in its middle. As signed permutations, we usually omit 0 in its window notation.

We have the following decomposition of $A(P)$, which is called in [10] the fundamental theorem of $P$-partitions of type $B_n$:

**Theorem 3.6** [5, Theorem 2.1.4] Let $P$ be a $B_n$-poset. Then

$$A(P) = \bigsqcup_{\pi \in L(P)} A(\pi).$$

In this paper, we deal with the anti-chain $B_n$-poset:

$$P = \{0, \pm 1, \pm 2, \ldots, \pm n\}$$

with no relations at all. In this case, the set of linear extensions of $P$ coincides with the group $B_n$. We explain now how to associate the appropriate $B_n$-permutation to a given element of $A(P)$.

As in the set of inequalities (4) above, recall that for each $1 \leq i \leq n$, $f(\pi(i))$ is the $i-$th smallest value of $f$ with respect to the order $\preceq$. Moreover, for $i \neq 0$, the sign of $\pi(i)$ is equal to the sign of $f(\pi(i))$. When we come across two identical values of $f$, we read them from left to right if they are non-negative, and from right to left if they are negative.

**Example 3.7** Let $P$ be the anti-chain $B_7$-poset: $P = \{0, \pm 1, \pm 2, \ldots, \pm 7\}$. Let $f : \{1, \ldots, 7\} \to \mathbb{Z}$ be the following $P$-partition:

$$f = (f(1), f(2), \ldots, f(7)) = (-1, 0, 2, 1, -2, 1, -2).$$

We associate with $f$ the following permutation:

$$\pi = [2, -1, 4, 6, -7, -5, 3] \in B_7.$$
It is easy to see that \( f \in A(\pi) \), since

\[
(f(\pi(1)), f(\pi(2)), \ldots, f(\pi(7))) = (0, -1, 1, 1, -2, -2, 2)
\]

is a non-decreasing sequence with respect to the order \( \preceq \). By the construction, as \( f(5) = f(7) = -2 < 0 \), we insert \(-7\) before \(-5\) in the permutation \( \pi \).

Note that if \( 0 \in \text{Des}_B(\pi) \), then the value \( 0 \) does not appear in any \( f \in A(\pi) \), since by definition \( \pi(0) = 0 \). As an example of this situation, take \( \pi = [-1, -2] \in B_2 \). Then,

\[
A(\pi) = \{ f = (f(1), f(2)) = (-i, -j) \mid 0 < i < j \}.
\]

### 3.3 \( P \)-Partitions of type \( D_n \)

In order to obtain the Worpitzky identity for the Coxeter group of type \( D_n \), we recall the concept of \( D_n \)-poset. The original definition is taken from Stembridge [15, Example 5.2(d)].

**Definition 3.8** A \( D_n \)-poset is the set \( P = \{ \pm 1, \pm 2, \ldots, \pm n \} \) (note that now \( 0 \) is missing) with a partial order \( \prec_P \), satisfying the following conditions:

1. If \( i \prec_P j \), then \(-j \prec_P -i\),
2. If \(-i \prec_P i \), then there is some \( j \neq \pm i \) such that \(-i \prec_P j \prec_P i \) (‘fork’ condition).

The second condition means that each (Hasse diagram of a) \( D_n \)-poset must have a “fork” in the middle.

**Example 3.9** Here is an example of a \( D_4 \)-poset:

```
1
  / \  /
-1 -3 -2
  |   |
  2
  |
-4
```

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Here, again, a linear extension is a maximal refined $D_n$-poset. For instance, the set of linear extensions of the $D_4$-poset presented in Example 3.9 is

$$\begin{align*}
&\begin{bmatrix} 1, 4, -2, 3, 2, -4, -1 \end{bmatrix}, & &\begin{bmatrix} -1, 4, -2, -3, 2, -4, 1 \end{bmatrix}, \\
&\begin{bmatrix} 4, 1, -2, 3, 2, -1, -4 \end{bmatrix}, & &\begin{bmatrix} 4, -1, -2, -3, 2, 1, -4 \end{bmatrix}, \\
&\begin{bmatrix} 4, -2, 1, 3, -1, 2, -4 \end{bmatrix}, & &\begin{bmatrix} 4, -2, -1, -3, 1, 2, -4 \end{bmatrix} \\
\end{align*}$$

Note that the order between 3 and $-3$ in these linear extensions is not determined, due to the ‘fork’ condition in Definition 3.8. This phenomenon allows us to consider the linear extensions as elements of the group $D_4$ by agreeing that if the number of negative numbers to the right of the pair $\{3, -3\}$ is even (odd), we place 3 above (below) $-3$, respectively. This idea is easily extended to the general case.

The definition of a $P$-partition of type $D_n$ is identical to the one presented for type $B_n$ (see Definition 3.3 above), and the decomposition theorem reads verbatim as in Theorem 3.6.

Similar to the situation in type $B_n$, we work with the anti-chain $D_n$-poset:

$$P = \{ \pm 1, \pm 2, \ldots, \pm n \}$$

and the linear extensions correspond to the elements of the group $D_n$.

In order to associate an element $\pi \in D_n$ to a $P$-partition $f$, we read the elements of $f$ according to the order $\preceq$, as we did in type $B_n$, but now whenever we get a permutation $\pi \in B_n - D_n$, we switch the sign of $\pi(1)$.

**Example 3.10** Let $f : \{1, \ldots, 7\} \to \mathbb{Z}$ be the following $P$-partition:

$$f = (f(1), f(2), \ldots, f(7)) = (-2, 1, 3, 2, -3, 2, -3)$$

We associate with $f$ the following permutation:

$$\pi = \begin{bmatrix} -3, 5, 7, -6, -4, 1, -2, -1, 4, 6, -7, -5, 3 \end{bmatrix} \in D_7.$$
while the condition $-1 \in \text{Des}_D(\pi)$ forces the strict inequality:

$$f(\pi(-1)) < f(\pi(2)).$$

The elements of the set $A(\pi)$ depend on the intersection $\text{Des}_D(\pi) \cap \{\pm 1\}$, as presented in the following three sub-cases:

(1) $|\text{Des}_D(\pi) \cap \{\pm 1\}| = 1$: This means that either $1 \in \text{Des}_D(\pi)$ but $-1 \notin \text{Des}_D(\pi)$, or vice versa. Assume without loss of generality that $1 \notin \text{Des}_D(\pi)$ but $-1 \in \text{Des}_D(\pi)$. The restrictions on $f \in A(\pi)$ are:

\[(*) \quad f(\pi(1)) \leq f(\pi(2)) \leq \cdots \leq f(\pi(n)).\]

In addition, $f(\pi(-1)) < f(\pi(2))$, and if $i \in \text{Des}_D(\pi)$ (for $1 < i < n$), then $f(\pi(i)) < f(\pi(i+1))$. For instance, taking $\pi = [4, -2, -1, -\frac{3}{3}, 1, 2, -4] \in D_4$, the $P$-partitions $f \in A(\pi)$ must satisfy:

$$f(-3) \leq f(1) \leq f(2) < f(-4)$$

as well as $f(3) < f(1)$, together with the sign conditions: $f(1) > 0$, $f(2) > 0$, and $f(4) < 0$. Since there is no restriction on the sign of $f(3)$, we have three possibilities for the absolute values of $f$:

(a) $0 < f(3) < |f(1)| \leq |f(2)| < |f(4)|$.
(b) $0 < f(-3) < |f(1)| \leq |f(2)| < |f(4)|$.
(c) $0 = f(3) < |f(1)| \leq |f(2)| < |f(4)|$.

(2) $|\text{Des}_D(\pi) \cap \{\pm 1\}| = 2$: In this case, we have that both $1 \in \text{Des}_D(\pi)$ and $-1 \in \text{Des}_D(\pi)$, so in addition to the restrictions $(*)$ above, we have: $f(\pi(-1)) < f(\pi(2))$, $f(\pi(1)) < f(\pi(2))$, and if $i \in \text{Des}_D(\pi)$ (for $1 < i < n$), then $f(\pi(i)) < f(\pi(i+1))$. For instance, taking $\pi = [4, -2, 3, -1, -3, 2, -4] \in D_4$, the $P$-partitions $f \in A(\pi)$ must satisfy:

$$f(1) < f(-3) \leq f(2) < f(-4),$$

as well as $f(1) < f(-3)$, together with the sign conditions: $f(2) > 0$, $f(3) < 0$ and $f(4) < 0$. Since there is no restriction on the sign of $f(1)$, we have three possibilities for the absolute values of $f$:

(a) $0 < f(1) < |f(3)| \leq |f(2)| < |f(4)|$.
(b) $0 < f(-1) < |f(3)| \leq |f(2)| < |f(4)|$.
(c) $0 = f(1) < |f(3)| \leq |f(2)| < |f(4)|$.

(3) $|\text{Des}_D(\pi) \cap \{\pm 1\}| = 0$: In this case, we have that both $1 \notin \text{Des}_D(\pi)$ and $-1 \notin \text{Des}_D(\pi)$, so in addition to the restrictions $(*)$ above, we have that if $i \in \text{Des}_D(\pi)$ (for $1 < i < n$), then $f(\pi(i)) < f(\pi(i+1))$. As an example,
take \( \pi = \left[ 4, -2, -3, -1, 3, 2, -4 \right] \in D_4 \). The \( P \)-partitions \( f \in \mathcal{A}(\pi) \) must satisfy:

\[
f(-1) \preceq f(3) < f(2) < f(-4),
\]

together with the sign conditions: \( f(3) > 0, f(2) > 0 \) and \( f(4) < 0 \). Since there is no restriction on the sign of \( f(1) \), we have three possibilities for the absolute values of \( f \):

(a) \( 0 < f(1) \leq |f(3)| < |f(2)| < |f(4)| \).
(b) \( 0 < f(-1) \leq |f(3)| < |f(2)| < |f(4)| \).
(c) \( 0 = f(1) \leq |f(3)| < |f(2)| < |f(4)| \).

### 4 The proof of Worpitzky identity for type \( B_n \) using \( P \)-partitions

For the Coxeter group \( B_n \), the following identity was proven by Brenti [3, Theorem 3.4(iii)] using generating functions:

**Theorem 4.1**

\[
(2m + 1)^n = \sum_{k=0}^{n} \binom{n + m - k}{n} B_{n,k}.
\]

Here, we use the theory of \( P \)-partitions of type \( B_n \) to obtain a simple combinatorial proof.

**Proof of Theorem 4.1** For each \( m \in \mathbb{N} \), let us denote, where \( P \) is a \( B_n \)-poset:

\[
\mathcal{A}_m(P) = \{ f \in \mathcal{A}(P) \mid \forall i, |f(i)| \leq m \}.
\]

Now, consider the anti-chain \( B_n \)-poset \( P = \{0, \pm 1, \ldots, \pm n\} \) mentioned in Example 3.7. Observe that \( |\mathcal{A}_m(P)| = (2m + 1)^n \). Hence, it suffices to prove that for each \( \pi \in B_n \) we have

\[
|\mathcal{A}_m(\pi)| = \binom{m + n - \text{des}_B(\pi)}{n}, \tag{5}
\]

since, by Theorem 3.6, we then conclude:

\[
(2m + 1)^n = |\mathcal{A}_m(P)| = \sum_{\pi \in B_n} |\mathcal{A}_m(\pi)|
= \sum_{\pi \in B_n} \binom{m + n - \text{des}_B(\pi)}{n}
= \sum_{k=0}^{n} \binom{m + n - k}{n} B_{n,k}.
\]
We prove now Eq. (5). Let \( \pi = [\pi(1), \ldots, \pi(n)] \in B_n \). We have to find the number of \( P \)-partitions, i.e., functions
\[
f = (f(1), f(2), \ldots, f(n)) \in ([0, \pm 1, \ldots, \pm m])^n
\]
such that for each \( j \in \{1, \ldots, n\} \) satisfying \( \pi_j < 0 \), one has \( f(|\pi_j|) < 0 \) and:
\[
0 \leq |f(\pi(1))| \leq |f(\pi(2))| \leq \cdots \leq |f(\pi(n))| \leq m,
\]
with the property that the \( (i + 1)^{th} \) order sign in this sequence of inequalities is strict if \( i \in \text{Des}_B(\pi) \), for \( 0 \leq i \leq n - 1 \). Therefore:
\[
1 \leq |f(\pi(1))| + 1 \leq |f(\pi(2))| + 1 \leq \cdots \leq |f(\pi(n))| + 1 \leq m + 1.
\]
Now, in order to convert to strict order signs, we add 1 to the right-hand side of each non-strict inequality. Since the number of strict order signs in the original sequence of inequalities is \( \text{des}_B(\pi) \), at the end of this process, we have:
\[
1 \leq b_1 < b_2 < \cdots < b_n \leq m + n - \text{des}_B(\pi),
\]
where \( b_i = |f(\pi(i))| + |\{j \in \text{Des}_B(\pi) \mid j < i\}| + 1 \).

The number of integer solutions of this sequence of inequalities is:
\[
\binom{m + n - \text{des}_B(\pi)}{n}.
\]
(6)

Note that for each \( i \), after fixing the value of \( b_i \), the value of \( f(\pi(i)) \) is uniquely determined. \( \square \)

Theorem 4.1 has a \( q \)-version, which requires the following definition:

**Definition 4.2** Let \( P \) be a \( B_n \)-poset and let \( m \in \mathbb{N} \). Define for each \( f \in \mathcal{A}_m(P) \):
\[
\text{neg}(f) = |\{i \in [n] \mid f(i) < 0\}|.
\]

Moreover, we denote:
\[
\Omega_P(q, m) = \sum_{f \in \mathcal{A}_m(P)} q^{\text{neg}(f)}.
\]

We now have the following \( q \)-version:

**Corollary 4.3**
\[
(1 + (1 + q)m)^n = \sum_{k=0}^{n} \binom{n + m - k}{n} B_{n,k}(q).
\]
The proof of this corollary follows from the proof of Theorem 4.1, replacing $|A_m(P)|$ with $\Omega_P(q,m)$ and the fact that for each $f \in A_m(P)$ one has $\text{neg}(f) = \text{neg}(\pi)$, where $\pi$ is the $B_n$-permutation associated with $f$.

5 The proof of Worpitzky identity for Coxeter groups of type $D_n$ using $P$-partitions

The following generalization of Worpitzky identity for type $D_n$ is due to Brenti [3, Coro. 4.11]:

**Proposition 5.1** For $n \geq 2$, we have:

\[(1 + 2x)^n - 2^{n-1}(\mathcal{B}_n(x + 1) - \mathcal{B}(n)) = \sum_{k=0}^{n} \binom{n + x - k}{n} D_{n,k}, \quad (7)\]

where $\mathcal{B}_n(\cdot)$ is the $n^{th}$ Bernoulli polynomial and $\mathcal{B}(n)$ is the $n^{th}$ Bernoulli number.

Mezo [8, Equation (5.12)] states that for $m \in \mathbb{N}$:

\[\mathcal{B}_n(m + 1) - \mathcal{B}(n) = n(1^{n-1} + \cdots + m^{n-1}),\]

and hence, for $x = m \in \mathbb{N}$, Eq. (7) above can also be written as follows:

\[(1 + 2m)^n - 2^{n-1}n(1^{n-1} + \cdots + m^{n-1}) = \sum_{k=0}^{n} \binom{n + m - k}{n} D_{n,k}.\]

As it is a polynomial identity, it suffices to prove the identity for $m \in \mathbb{N}$. For its proof, we need the following Worpitzky-type identity:

**Lemma 5.2** Let $n, m \in \mathbb{N}$. Then,

\[2^{n-1}n(1^{n-1} + \cdots + m^{n-1}) = \sum_{\pi \in D_n} \binom{n + m - \text{des}_D(\pi) - 1 + |\text{Des}_D(\pi) \cap \{\pm 1\}|}{n}.\]

**Proof** The left-hand side of this identity counts the number of elements in the set of vectors of length $n$ over the alphabet

\[\Sigma = \{\pm 1, \ldots, \pm (m + 1)\},\]

satisfying the property that the number of negative entries is even, and the smallest entry in absolute value appears exactly once. We now explain the role of the factors in the left-hand side: The factor $2^{n-1}$ is the number of possibilities to evenly sign the elements and the factor $n$ is the number of choices for the position of the unique smallest element. The sum in brackets counts the number of possibilities to fill in the other $n - 1$ unsigned entries of the vector, depending on the absolute value of the
smallest entry: if the smallest value is 1—we have $m^{n-1}$ possibilities to fill in the remaining $n - 1$ entries of the vector by the values $2, \ldots, m + 1$; if the smallest value is 2—we have $(m - 1)^{n-1}$ possibilities; and so on.

We now show that the right-hand side of this identity counts the same set of vectors by ordering them according to their associated permutation in $D_n$, as follows: For each vector $v \in ((\pm 1, \ldots, \pm (m+1))^n$, we read the entries of $v$ according to the order $\leq$ (defined right after Definition 3.3) and write the locations of each entry in $v$ with positive (negative) sign if the element is positive (negative), respectively, yielding a permutation $\pi \in D_n$, since the vector $v$ has an even number of negative elements. Similar to Example 3.7, when encountering a sequence of identical entries in $v$, we read them from left to right (right to left) in case they are positive (negative), respectively.

For instance, for a vector $v = (-2, -3, 1)$, we associate the permutation $\pi = [3, -1, -2] \in D_3$, while the vectors associated with the permutation $[-2, -3, 1] \in D_3$ are:

$$(2, -1, -2), (3, -1, -2), (3, -1, -3), (3, -2, -3).$$

It remains to prove that for each $\pi \in D_n$, the number of vectors associated with $\pi$, satisfying the property that the smallest entry in absolute value appears exactly once, is:

$$\binom{n + m - \text{des}_D(\pi) - 1 + |\text{Des}_D(\pi) \cap \{\pm 1\}|}{n}.$$ 

The proof is based on the same idea described in the proof of Theorem 4.1. Since we assume that the smallest entry in absolute value of the vector $v$ appears exactly once, its location will be the value of $|\pi(1)|$. The second-smallest entry in absolute value should be strictly larger than the first entry, independent of the descents of $\pi$ in positions $-1$ and 1. This implies two changes in the numerator of the binomial coefficient $\binom{n + m - \text{des}_D(\pi)}{n}$: First, the descents in positions $-1$ and 1 do not contribute to the number of `$\leq$' signs, which might be converted to `$<$' signs, and hence we add them artificially. Second, since the value $|v|_{\pi(1)}|$ is strictly smaller than the value $|v|_{\pi(2)}|$, we have to subtract 1 from the numerator to reflect this. \hfill \Box

**Proof of Theorem 5.1** As in type $B_n$, for each $m \in \mathbb{N}$, denote:

$$A_m(P) = \{ f \in A(P) \mid \forall i, |f(i)| \leq m \}.$$ 

Now, consider the anti-chain $D_n$-poset $P = \{0, \pm 1, \ldots, \pm n\}$ mentioned before Example 3.10.

We have

$$(2m + 1)^n = |A_m(P)| = \sum_{\pi \in D_n} |A_m(\pi)|.$$
Now, using the outline of the proof of Theorem 4.1 and the explanation how to compute the set $\mathcal{A}_m(\pi)$ which appears after Example 3.10, it is easy to see that:

$$\left|\mathcal{A}_m(\pi)\right| = \begin{cases} 
\binom{m+n-\text{des}_D(\pi)}{n} + \binom{m+n-\text{des}_D(\pi)-1}{n-1} & |\text{Des}_D(\pi) \cap \{-1, 1\}| = 1 \\
\binom{m+n-\text{des}_D(\pi)-1}{n} + \binom{m+n-\text{des}_D(\pi)-1}{n-1} & |\text{Des}_D(\pi) \cap \{-1, 1\}| = 0 \\
\binom{m+n-\text{des}_D(\pi)}{n} + \binom{m+n-\text{des}_D(\pi)-1}{n} & |\text{Des}_D(\pi) \cap \{-1, 1\}| = 2 
\end{cases}$$

Note that each summand in each case of the first expression for $\mathcal{A}_m(\pi)$ corresponds to one of the cases in the classification given in the explanation after Example 3.10. The second expression is obtained by using Pascal’s identity.

The first summands in each of the cases now sum up to:

$$\sum_{\pi \in D_n} \binom{m+n-\text{des}_D(\pi)}{n} = \sum_{k=1}^{n} A_D(n,k) \binom{m+n-k}{n}.$$  

The remaining second summands in each of the cases sum up to:

$$\sum_{\pi \in D_n} \binom{m+n-\text{des}_D(\pi)-1}{n} + |\text{Des}_D(\pi) \cap \{-1, 1\}|$$

which is equal by Lemma 5.2 to $2^{n-1}(1^{n-1} + \cdots + m^{n-1})$, yielding the desired identity. $\square$

Remark 5.3 The identity appearing in Lemma 5.2 has a nice algebraic meaning as follows: Let $H$ be the parabolic subgroup of $D_n$ generated by the Coxeter generators $s_2, \ldots, s_{n-1}$. Then, $H$ can be identified with the elements of $S_{n-1} = \{s_2, \ldots, s_{n-1}\}$; hence, we have that $|D_n/H| = \frac{2^{n-1}n!}{(n-1)!} = 2^{n-1}n$. Note that two elements $\pi_1, \pi_2 \in D_n$ satisfy $\pi_1 H = \pi_2 H$ if and only if $|\pi_1(1)| = |\pi_2(1)|$ and

$$\{\pi_1(i) \mid \pi_1(i) < 0, \ i \geq 2\} = \{\pi_2(i) \mid \pi_2(i) < 0, \ i \geq 2\}.$$  

Hence, since

$$\text{des}_{D,2}(\pi) = |\{i \mid \pi(i) > \pi(i + 1), \ 2 \leq i \leq n - 1\}|$$

$$= \text{des}_D(\pi) - |\text{Des}_D(\pi) \cap \{-1, 1\}|$$

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ignores the descents in places $-1$ and $1$, the sum
\[
\sum_{\pi \in \sigma H} \binom{m + n - \text{des}_{D,2}(\pi) - 1}{n}
\]
is independent of the coset $\sigma H$. We conclude that Lemma 5.2 can be written as follows
\[
2^{n-1}n(1^{n-1} + \cdots + m^{n-1}) = \sum_{\pi \in D_n} \binom{n + m - \text{des}_{D,2}(\pi) - 1}{n}.
\]
Hence, one can conclude the following new Worpitzky-like identity:

**Corollary 5.4**

\[
1^{n-1} + \cdots + m^{n-1} = \sum_{\pi \in S_{n-1}} \binom{n + m - \text{des}_{A}(\pi) - 1}{n}.
\]

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