Abstract

We prove the existence of a deformation quantization for integrable Poisson structures on \( \mathbb{R}^3 \) and give a generalization for a special class of three dimensional manifolds.

The program of deformation quantization of the function algebra on a symplectic manifold extends naturally to manifolds with nonregular Poisson structures. In contrast to symplectic manifolds the existence of star products on nonregular Poisson manifolds, even on \( \mathbb{R}^n \), is an open problem. Particular examples of quantizable nonregular Poisson structures were found, e.g. a star product for linear Poisson structures [1, 2] which is induced by a star product on the cotangent bundle of a Lie group, and for quadratic Poisson structures in three dimensions [3]. We will give in this letter a proof for the existence of star products for integrable Poisson structures on \( \mathbb{R}^3 \) and extend this to a class of three dimensional manifolds.

On \( \mathbb{R}^3 \) the components \( P^{ij} \) of a Poisson structure \( P = \frac{1}{2} P^{ij} \partial_i \wedge \partial_j \) can be arranged as a three-dimensional Poisson vector \( \vec{P} = (P^{23}, P^{31}, P^{12}) \). The Jacobi identity for \( P \) is equivalent to the identity \( \vec{P}(\text{rot} \vec{P}) = 0 \). Solutions of this equation are of the form \( \vec{P} = \psi \vec{\nabla} \varphi \) for arbitrary functions \( \psi \) and \( \varphi \). In the following we will concentrate on

\[
\vec{P} = \vec{\nabla} \varphi
\]  

(1)

which are called integrable Poisson structures [4].

The main result of this letter is stated in the following theorem:

**Theorem 1** Let \( P \) be an integrable Poisson structure on \( \mathbb{R}^3 \). Then there exists a star product which is a quantization of this Poisson structure.
Proof: We will construct a star product of Weyl type. We are looking for a product which is a formal power series of bilinear operators $M_k, k \geq 0$, vanishing on constants for $k \geq 1$, such that
\[ f \ast g := \sum_{k=0}^{\infty} \hbar^k M_k(f,g) \]
is an associative product on $C^\infty(\mathbb{R}^3)[[\hbar]]$ with $M_0(f,g) = fg$ and $M_1(f,g) = \frac{\hbar}{2} \{ f, g \}$. The product is of Weyl type if the $M_k$ have the parity
\[ M_k(f,g) = (-1)^k M_k(g,f) \quad (2) \]
Associativity is equivalent to
\[
\delta(M_k) = \frac{1}{2} \sum_{l=1}^{k-1} [M_l, M_{k-l}] =: R_k \quad (3)
\]
with the Hochschild-$\delta$ for a multilinear map $M$ with $m+1$ arguments
\[
\delta(M)(f_0, \ldots, f_{m+1}) := f_0 M(f_1, \ldots, f_{m+1}) - \sum_{i=0}^{m} (-1)^i M(f_0, \ldots, f_i f_{i+1}, \ldots, f_{m+1}) + (-1)^m M(f_0, \ldots, f_m) f_{m+1} \quad (4)
\]
the Gerstenhaber bracket ($N$ multilinear with $n+1$ arguments)
\[
[M,N] := M \circ N - (-1)^{mn} N \circ M \quad (5)
\]
and the Gerstenhaber product
\[
(M \circ N)(f_0, \ldots, f_{m+n}) = \sum_{i=0}^{m} (-1)^{in} M(f_0, \ldots, N(f_i, \ldots, f_{i+n}), \ldots, f_{m+n})
\]
Since $\delta^2 = 0$ solving (3) is a cohomological problem [3]. The necessary condition $\delta R_k$ is satisfied for $M_2 \ldots M_{k-1}$ fulfilling (3) due to the fact that the Gerstenhaber bracket is a graded Lie bracket. Since the cohomology of $\delta$ is nontrivial, we have to show that $R_k$ lies in the trivial cohomology class. The application of the theorem of Hochschild-Kostant-Rosenberg on the algebra of local multi differential operators on the function algebra of a manifold $M$ [6] gives the cohomology classes of $\delta$:
\[
H^k_{\text{Hoch}}(M) \cong \Gamma(\Lambda^{k+1}TM)
\]
The different indices $k$ for the cohomology classes and $k+1$ for the multi vector fields is a consequence of the grading of the operators which is the number of arguments minus one. We will need this theorem for local multilinear maps which are differential operators in each argument, i.e. which vanish on constant functions, the cohomology is the same, however, for the complex of all local multilinear maps [3].

The decomposition $R_k = \delta(M_k) + \alpha_k$ with $\alpha_k \in \Gamma(\Lambda^{3}TM)$ together with the fact that $\delta(M_k)$ is a sum of maps which are symmetric in two consecutive arguments leads to the identity
\[
AR_k = \alpha_k
\]
where $AR_k$ is the antisymmetrization of $R_k$. This can be read in the following way: If $R_k$ is a multilinear map with $\delta(R_k) = 0$ then the antisymmetrization $AR_k$ is a totally antisymmetric differential operator which is 1-differential in each argument.

If $M_2 \ldots M_{k-1}$ were constructed with the symmetry (3) then $R_k(f, g, h) = -(-1)^k R_k(h, g, f)$, i.e. $R_k$ is symmetric for $k$ odd and the condition $AR_k = 0$ is trivially fulfilled. On the other hand for $M(f, g) = \pm M(g, f)$ is $\delta(M)(f, g, h) = \mp \delta(M)(h, g, f)$ and therefore $M_k$ can be chosen with the right symmetry if the integrability condition $AR_k = 0$ is fulfilled.

We will now focus on an important property of $\delta$. Writing out explicitly

$$M(f_0, \ldots, f_m) = \sum_{|I_0|, \ldots, |I_m|} M^{I_0, \ldots, I_m} \partial_{I_0} f_0 \ldots \partial_{I_m} f_m$$

with multiindices $I_0, \ldots, I_m, I_j = (i_{j,1}, \ldots, i_{j,|I_j|}), \partial_{I_j} = \partial_{i_{j,1}} \ldots \partial_{i_{j,|I_j|}}$, and Einstein summation convention for all indices of the multiindices, then

$$(\delta M)(f_0, \ldots, f_{m+1}) = -\sum_{i=0}^{m} (-1)^i \sum_{|I_0|, \ldots, |I_m|} \sum_{J \cup K = I_i} M^{I_0, \ldots, I_m} \partial_{I_0} f_0 \ldots \partial_{I_j} f_i \partial_{K} f_{i+1} \ldots \partial_{I_m} f_{m+1}$$

where $J \cup K = I_i$ is the sum over all partitions of the multiindex $I_i$. We see: The coefficients of the multidifferential operator $\delta(M)$ are linear combinations of the coefficients of $M$. We call this a combinatorial operator. By examination of the proof of the Hochschild-Kostant-Rosenberg theorem in (3) one sees the inverse statement: If $N = \delta(M)$, then there exists an $\tilde{M}$ with $N = \delta(\tilde{M})$ such that the coefficients of $\tilde{M}$ are linear combinations of the coefficients of $N$.

We will recursively prove the existence of bilinear differential operators $M_k$ which satisfy (3) and have the following additional properties:

1. $M_k$ is an ordered polynomial operator (This is defined below).
2. The total number of derivatives of the $M_k$ is greater than two for $k \geq 2$.

**Definition 1** A multidifferential operator $M$ is called an ordered polynomial operator (OPO), if the coefficients of $M$ are polynomials in the coefficients of Poisson structure and in partial derivatives of the coefficients which can be arranged such that the indices of the Poisson structures are contracted with partial derivatives which stand on the right side of the Poisson structure.

Examples:

- The Poisson bracket $\{f, g\} = P^{ij} \partial_i f \partial_j g$ is an OPO.
- The terms of the Jacobi identity $\{\{f, g\}, h\}$ are OPO.
- The operator $\partial_i P^{ij} \partial_j P^{jr} \partial_j f \partial_j g$ is not an OPO.

It is clear that the property of being ordered is well defined for polynomial coefficients since for each term there are only a finite number of arrangements of the factors, if one of these arrangements has the required property then the
term is an OPO, if none of the arrangements has the property then it is not an OPO.

We emphasize that this property is a property of the operator with the given coefficients, it is unfortunately not a property of the operator itself. This is due to the fact that there exist operators which can be written as OPO or not as OPO. An example is given by the operator

\[ \partial_k P^{ij} \partial_i (P^{kr} \partial_r P^{lm} + P^{lr} \partial_r P^{mk} + P^{mr} \partial_r P^{kl}) \partial_j f \partial_m g = 0 \]  

(6)

which vanishes due to the Jacobi identity. There is one term in the expansion of (6), \[ \partial_k P^{ij} P^{kr} \partial_i \partial_r P^{lm} \partial_j f \partial_m g, \] which is an OPO, all other terms are not OPO. Using (6) we can express this OPO as an operator which is not an OPO.

We will now list a few properties of the Gerstenhaber bracket (5) and the Hochschild-δ (4):

- \( M, N \) OPO \( \Rightarrow \) \([M,N]\) OPO. This is clear since in the Gerstenhaber product one operator is inserted in the other which preserves the structure.
- \( M \) OPO \( \Rightarrow \) δ(M) OPO. This is a consequence of the combinatorial property of δ.
- \( N = \delta(M) \), \( N \) OPO \( \Rightarrow \) it is possible to choose \( M \) as an OPO. This is a consequence of the combinatorial property of the inverse of δ.

For \( M_1, \ldots, M_{k-1} \) OPO is \( R_k \) an OPO. If it is possible to show that \( AR_k = 0 \), then it is clear that \( M_k \) can be chosen as an OPO. Since \( R_k \) is a differential operator with 3 arguments it has the total differential degree ≥ 3, therefore \( M_k \) will be a differential operator of degree ≥ 3.

We are now ready to prove \( AR_k = 0 \) for \( k \) even. \( AR_k \) is a sum of totally antisymmetric differential operators of degree (1,1,1) (where the degree of an operator is the sequence of the differential degrees in each argument) These terms emerge from terms in \( R_k = \frac{1}{k} \sum_{l=1}^{k-1} [M_l, M_{k-l}] \) which are of degree (1,1,1) (since the antisymmetrization does not change the degree) This is only possible, if one of \( M_l \) or \( M_{k-l} \) is a differential operator of degree (1,1), this must be \( M_1 \), since all other \( M_l \) have a total degree ≥ 3. Therefore

\[
AR_k(f,g,h) = (A(M_{k-1} \circ M_1)(f,g,h))_{1\text{-diff}}
= \frac{1}{3} (M_{k-1}(M_1(f,g),h) - M_{k-1}(f,M_1(g,h)) + \text{cycl.})_{1\text{-diff}}
= \frac{2}{3} (M_{k-1}(M_1(f,g),h) + \text{cycl.})_{1\text{-diff}}
\]

(7)

We have to take explicitly the 1-differential parts (indicated by the subscript 1-diff) since leaving out terms in \( R_k \) destroys the cocycle property and therefore the property of the antisymmetrization of being 1-differential. The terms of degree (1,1,1) in \( M_{k-1}(M_1(f,g),h) \) come from the terms \( M_{k-1}^{(m,1)} \) which are of degree (m,1). These are of the form

\[
M_{k-1}^{(m,1)} \partial_{i_1} \ldots \partial_{i_m} f \partial_j g
\]

\( M_{k-1} \) is an OPO, therefore the \( M_{k-1}^{(m,1)} \) are polynomials in coefficients of the Poisson structure and partial derivatives thereof. The indices of the last
Poisson structure of this ordered series have to be contracted with partial derivatives standing more right, i.e. partial derivatives of the arguments. Since the indices of the Poisson structure are antisymmetric, it must be partial derivatives of different arguments. Therefore $j$ must be one of the indices, the other is one of $i_1, \ldots, i_m$. Without loss of generality we will assume that $i_m$ is this index and write now
\[
M^{(m,1)}_{k-1}(f,g) = M^{(m,1)Iij}_{k-1}\partial_i f \partial_j g
\]
where $I = (i_1, \ldots, i_{m-1})$ is a multiindex, $M^{(m,1)Iij}_{k-1}$ is antisymmetric in $i$ and $j$. The part of degree $(1,1,1)$ in (7) coming from $M^{(m,1)}_{k-1}$ is then, up to a factor, given by
\[
M^{(m,1)Iij}_{k-1}\partial_i P^{rs} \partial_r f \partial_sg \partial_th + \text{cycl.} \quad (8)
\]
We will show that these terms vanish for each $m$ separately, for this we will now use the fact that the space is $\mathbb{R}^3$ and the special form $\mathbf{1}$ of the Poisson structure.

Since (3) is derivative in its arguments it is sufficient to use coordinate functions to show the vanishing of (3). It is trivially fulfilled if two of the three functions $f, g$ and $h$ are equal. It remains to check the condition for $f = x^1$, $g = x^2$ and $h = x^3$:
\[
\begin{align*}
M^{(m,1)Iij}_{k-1}\partial_i P^{12} + M^{(m,1)I1i}_{k-1}\partial_i P^{13} + M^{(m,1)I2i}_{k-1}\partial_i P^{31} & = M^{(m,1)I1i}_{k-1}\partial_i \partial_1 \phi + M^{(m,1)I2i}_{k-1}\partial_i \partial_2 \phi + M^{(m,1)I3i}_{k-1}\partial_i \partial_3 \phi \\
+ & M^{(m,1)I31}_{k-1}\partial_i \partial_3 \partial_1 \phi + M^{(m,1)I21}_{k-1}\partial_i \partial_2 \partial_1 \phi + M^{(m,1)I13}_{k-1}\partial_i \partial_1 \partial_3 \phi \\
& = 0
\end{align*}
\]
because of the symmetry of the partial derivatives and the antisymmetry of $M^{(m,1)Iij}_{k-1}$ in $i$ and $j$, from line 1 to line 2 we have written out the summation over $i$ explicitly.

Hence we conclude $AR_k = 0$ and we can continue the recursion with an operator with the desired properties. This ends the proof of the existence of a star product for the considered Poisson structures.

It is clear that it is difficult to transfer this proof on other Poisson structures. For this special form of the Poisson structure the Jacobi identity $\mathbf{P}(\text{rot}\mathbf{P}) = 0$ is fulfilled because $\text{rot}\mathbf{P} = 0$, this seems to be the reason for $AR_k = 0$. The proof of the existence of a star product on $\mathbb{R}^3$ for the general case $\mathbf{P} = \psi \nabla \varphi$ in this manner fails, an explicit calculation shows that it is necessary to add a non OPO to $M_3$ in order to get $AR_4 = 0$.

There exist a generalization for special type of three dimensional manifolds given in the following theorem:

**Theorem 2** Let $M$ be an orientable, three dimensional manifold with a flat torsion free connection and a covariant constant volume form $\mu = \frac{1}{6} \epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$. Define an isomorphism
\[
*: \Gamma(\Lambda^2 TM) \rightarrow \Gamma(\Lambda^1 T^*M) \quad Y \mapsto i(Y)\mu
\]
with the injection $i(Y)\mu = \frac{1}{2}Y^{ij}\mu_{ijk}dx^k$ for a bivector $Y = \frac{1}{2}Y^{ij}\partial_i \wedge \partial_j$. Then for all $\alpha \in \Gamma(\Lambda^1T^*M)$ with $d\alpha = 0$ the bivector $\ast^{-1}\alpha$ is a Poisson bivector and there exists a star product for this Poisson structure.

**Proof:** For a flat torsion free connection it is always possible to find a chart around each point such that the Christoffel symbols vanish. In such a chart the coefficients $\mu_{ijk}$ of the covariant constant volume form are constant and proportional to $\epsilon_{ijk}$, without loss of generality we will assume that $\mu_{ijk} = \epsilon_{ijk}$. We will use in the following an atlas containing only such charts. For a one-form $\alpha = \alpha_i dx^i$ the bivector $\ast^{-1}\alpha$ is given by $P^i_j = \epsilon^{ijk}\alpha_k$ and the Jacobi identity for $P$ is then

$$P^{1r}\partial_r P^{23} + P^{2r}\partial_r P^{31} + P^{3r}\partial_r P^{12} = P^{12}(\partial_2 \alpha_1 - \partial_1 \alpha_2) + P^{23}(\partial_3 \alpha_2 - \partial_2 \alpha_3) + P^{31}(\partial_1 \alpha_3 - \partial_3 \alpha_1)$$

$$= 0$$

for $d\alpha = 0$, i.e. $\ast^{-1}\alpha$ is a Poisson bivector.

In the domain of a chart the closed one-form can be written as $\alpha = d\phi$, i.e. the Poisson structure is locally integrable. Using the arguments for $\mathbb{R}^3$ we conclude that there exists a star product in this chart, i.e. the condition $AR_k = 0$ is fulfilled. But for $M_1, \ldots, M_{k-1}$ given on the whole manifold, forming $R_k$ by $\{3\}$ is a coordinate invariant construction, and the vanishing of the total antisymmetrisation is also a coordinate invariant property. Therefore $AR_k = 0$ is fulfilled in every chart and hence on the whole manifold and $M_k$ can be constructed globally, it has the required properties in every chart.

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