**Dynamical friction in finite temperature superfluids, and the Fornax dwarf spheroidal**

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**ABSTRACT**

**Aims.** The aim of the present work is to better understand the gravitational drag forces, so-called dynamical friction, acting on massive objects moving through a finite-temperature superfluid background. This is relevant for models of dark matter consisting of light scalar particles with weak self-interactions that require non-zero temperature, or that have been heated inside galaxies.

**Methods.** Expressions for the dynamical friction are derived using linear perturbation theory, and compared to numerical simulations in which non-linear effects are included. After testing and improving the linear result, it is applied to the dwarf spheroidal galaxy Fornax, and its five gravitationally bound globular clusters. By estimating the rate at which these globular clusters are expected to sink into their host halo due to dynamical friction, limits on the superfluid dark matter parameter space are inferred.

**Results.** The dynamical friction in a finite-temperature superfluid is found to behave very similarly to the zero-temperature limit, even when the thermal contributions are large. However, when a critical velocity for the superfluid flow is included, the friction force can transition from the zero-temperature value to the value in a conventional fluid. Increasing the mass of the perturbing object induces a similar transition as when lowering the critical velocity. When applied to Fornax and its globular clusters, it is found that the parameter space preferred in the literature for a zero-temperature superfluid yields decay times in agreement with observations. However, the present work suggests that increasing the temperature, which is expected to change the preferred parameter space, may lead to too small decay times, and therefore pose a problem for finite-temperature superfluid models of dark matter.

**Key words.** cosmology: theory - dark matter

**1. Introduction**

When a massive object moves through a background medium, it’s gravitational field can cause the background to form an overdensity trailing it, which in turn exerts a gravitational force on the object that produced it. This is known as dynamical friction, and is a purely gravitational phenomenon. Hence, it can also arise in systems in which the constituent components otherwise have very weak or no coupling to one another, or behave as collisionless particles, such as dark matter (DM) and stars. Many important processes in the formation of structure, the evolution of galaxies, and the dynamics of astrophysical systems, such as mergers (Jiang et al. 2008; Boylan-Kolchin et al. 2008), the sinking of satellites into their host halos (Colpi et al. 1999; Cowsik et al. 2009; Cole et al. 2012; Tamfal et al. 2020), the decay of orbiting black holes and binaries (Just et al. 2011; Paní 2015; Dosopoulou & Antonini 2017; Gómez & Rueda 2017), and bar-halo interactions in disk galaxies (Weinberg 1985; Debattista & Sellwood 2000; Sellwood 2014), therefore depend on the nature of this drag force.

The first detailed calculation of dynamical friction was carried out by Chandrasekhar (1943) in the context of stellar dynamics. He considered the varying gravitational forces acting on a star as it moves through it’s stellar neighborhood, and found that it experiences a net average force opposite to its direction of motion, i.e. a kind of friction force, or gravitational drag. He treated the background of stars as an infinite homogeneous gas of collisionless particles following a Maxwell-Boltzmann velocity distribution. The dynamical friction due to a background of collisionless DM can be treated similarly (Mulder 1983; Colpi et al. 1999; Binney & Tremaine 2008). For a collisional medium, however, pressure forces must be taken into account when computing the dynamical friction, and has been done both analytically and numerically for various types of gases, such as ideal (Ostriker 1999; Sánchez-Salcedo & Brandenburg 1999; Lee & Stahler 2011, 2014; Thun et al. 2016), relativistic (Barausse 2007; Katz et al. 2019), and magnetized gases (Sánchez-Salcedo 2012; Shadmehri & Khajenabi 2012).

The nature of the dynamical friction due to dark matter is, of course, related to the nature of dark matter itself. The standard model of the universe, ΛCDM, includes cold and collisionless DM as the predominant matter component, making up about 80% of all matter. While extremely successful at explaining observables such as the microwave background radiation, large scale structure, and the expansion history of the universe (Tegmark et al. 2004; Planck Collaboration et al. 2016; Riess et al. 2016), the identity of dark matter has remained elusive. Furthermore, there are discrepancies between simulations of structure formation at small scales, and observations (Del Popolo & Le Delliou 2017; Bullock & Boylan-Kolchin 2017). A number of extended models for dark matter have therefore been proposed that may explain these discrepancies and provide clues to what kind of particle dark matter is (Hu et al. 2000; Spergel & Steinhardt 2000; Schive et al. 2014; Elbert et al. 2015; Berezhiani & Khoury 2015; Khoury 2016; Schwabe et al. 2016; Mocz et al. 2017; Tulin & Yu 2018). For these reasons there have also been done studies of dynamical friction in various DM models, such as fuzzy dark matter (Hui et al. 2017; Bar-Or et al. 2019; Lancaster et al. 2020), and self-interacting Bose-Einstein con-
densed (BEC) dark matter (Berezhiani et al. 2019), also known as superfluid dark matter.

In this work we extend the analysis of a zero temperature superfluid to finite temperatures, where the fluid is in a mixed state of normal fluid, made up of thermal excitations, and superfluid. This type of system has pressure terms coming from both thermal excitations and self-interactions, and can exhibit unique features due to the separate flow of the superfluid and normal fluid components.

A number of studies have considered finite-temperature effects of interacting superfluid DM (Harko & Mocanu 2012; Slepian & Goodman 2012; Harko et al. 2015; Sharma et al. 2019). Of particular note is the one presented by Berezhiani & Khouri (2015), who suggested that superfluid DM, when provided with a specific Lagrangian structure and coupling to the visible sector, can give rise to modified Newtonian dynamics (MOND) between baryons at galactic scales. This MONDian force is mediated by superfluid phonons, which ceases to be coherent at scales larger than galaxies, resulting in the vanishing of the extra force and the preservation of the large-scale success of CDM. For the fifth force to be MONDian the DM particles need exotic three-body self-interactions, and the DM fluid has to be above a certain temperature to be well-behaved.

Finite temperature DM might arise as processes inside galaxies transfer energy to the DM halo (Pontzen & Governato 2012; Read et al. 2019), possibly heating up the DM fluid.

This paper is organized as follows: In Section 2 the superfluid equations at both zero and finite temperatures are introduced, as well as some basic notions related to superfluidity. In Sections 3 and 4 these equations are used to derive the dynamical friction at linear order, both in a steady-state and a finite time evolution, a non-linear Schrödinger equation with e

\[ \frac{\partial \psi}{\partial t} + \nabla \cdot (\psi \mathbf{v}) = 0, \]

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \left( \frac{\rho_0}{m^2} + Q + V_{\text{ext}} \right) = 0. \]

These are the so-called Madelung equations (Madelung 1926). The first is a continuity equation for mass, and the second is a quantum variant of the momentum equation, with the quantum potential

\[ Q = -\frac{1}{2m^2} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}. \]

coming from the kinetic part of the Schrödinger equation, present even in the absence of interactions. From the definition of the velocity field, we see that it is irrotational, since the curl of a gradient is zero. But there can arise defects in the superfluid, around which the circulation is quantized as

\[ m \int \mathbf{v} \cdot d\mathbf{l} = 2\pi n, \quad n \in \mathbb{Z}, \]

because the complex wave-function must be single-valued. These special structures in superfluids are called quantum vortices.

Both the Schrödinger and Madelung formulations have been used in cosmology as models for dark matter in order to explain the absence of small-scale structure that is predicted in \( N \)-body simulations of CDM (Schive et al. 2014; Mocz et al. 2017; Nori & Baldi 2018, 2020; Mina et al. 2020a,b).

At finite temperatures the hydrodynamic formulation of a superfluid must take into account that the fluid is no longer completely superfluid. There is a thermal cloud of excitations, in addition to the coherent superfluid state, that carries entropy, gives a thermal contribution to the fluid pressure, and can be viscous and rotational. To complicate matters further, as the temperature of the fluid changes, the fraction of the fluid in this thermal cloud changes as well. This property of superfluids, to behave both a fluid with zero viscosity, quantized circulation, and carries no entropy and a conventional fluid, has led to the development of a two-fluid picture of superfluids. The hydrodynamic equations for a finite-temperature superfluid are (neglecting the quantum potential) (Taylor & Griffin 2005; Chapman et al. 2014):

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \]

\[ \frac{\partial S}{\partial t} + \nabla \cdot (S\mathbf{u}_n) = 0, \]

\[ \frac{\partial \mathbf{u}_s}{\partial t} + \nabla (\mu + \frac{1}{2} \mathbf{u}_s^2) = -\nabla \Phi, \]

\[ \frac{\partial \mathbf{j}}{\partial t} + \nabla P + \rho_0 (\mathbf{u}_s \cdot \nabla)\mathbf{u}_s + \rho_0 (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \mathbf{u}_s \left[ \nabla \cdot (\rho \mathbf{u}_s) \right] + \mathbf{u}_n \left[ \nabla \cdot (\rho \mathbf{u}_n) \right] = -\rho \nabla \Phi, \]

This type of system has pressure terms coming from both thermal excitations and self-interactions, and can exhibit unique features due to the separate flow of the superfluid and normal fluid components.

2. Hydrodynamics of finite-temperature superfluids

In the standard treatment, superfluids are often related to BECs, which form when the temperature is sufficiently low and the particle density high enough that the de Broglie wavelengths of identical bosons overlap, creating a coherent state that can be described by a single-particle wave-function. This wave-function is usually associated with the superfluid, and can therefore be regarded as a quantum mechanical effect at macroscopic scales. The wave-function \( \psi \) is governed by the Gross-Pitaevskii equation, a non-linear Schrödinger equation with effective contact interactions parameterized by \( g \):

\[ \frac{\partial \psi}{\partial t} = \left[ -\frac{\nabla^2}{2m} + g|\psi|^2 + m V_{\text{ext}} \right] \psi. \]

The external potential \( V_{\text{ext}} \) can be a trapping potential, as is often used in cold atomic experiments, or a gravitational potential. The amplitude of \( \psi \) is related to the particle number density by \( n = |\psi|^2 \), and mass density \( \rho = m|\psi|^2 \).

By inserting for the wave-function

\[ \psi = \sqrt{n} e^{i \varphi} = \sqrt{\frac{\rho}{m}} \cos \varphi, \]

and defining the velocity field \( \mathbf{v} = \nabla S / m \), the non-linear Schrödinger equation can be reformulated in a hydrodynamic form. The real and imaginary parts of the Schrödinger equation yield the set of equations
The thermal cloud, which we call the "normal fluid", has density $\rho_n$, velocity $u_n$, and transports both mass and thermal energy. The second component is the "superfluid", with density $\rho_s$, a velocity field $u_s$, and carries no entropy. The total mass density is the sum of the two components, $\rho = \rho_n + \rho_s$, and likewise for momentum, $j = \rho_n u_n + \rho_s u_s$. The fluid pressure is $P$, the entropy density $S$, temperature $T$, and $\mu = (P + U - ST - \frac{1}{2} \rho_0 (u_s - u_n)^2) / \rho$.

As previously mentioned, superfluids and BECs are related phenomena, but it is important to stress that they are not equivalent. The formation of a BEC does not automatically imply a superfluid. To see this we must consider the co-called Landau criterion. Landau, in his seminal paper on superfluid liquid helium 4 (Landau 1941), made the following argument: Assume that heating and dissipation in a fluid takes place via the creation of elementary excitations. If these excitations become energetically unfavorable and cannot spontaneously appear, then heating and dissipation ceases, and the fluid becomes superfluid. The criterion for such a condition is for the relative velocity $v$ between the superfluid and a scattering potential, such as an impurity or a container wall, to be smaller than a critical value,

$$v < v_c = \min \frac{\epsilon(p)}{p},$$  \hspace{1cm} (11)

where $\epsilon(p)$ is the energy of an elementary excitation with momentum $p$ (Pitaevskii & Stringari 2016). This criterion shows that an ideal BEC, for which the excitation spectrum is $\epsilon(p) = p^2 / 2m$, has $v_c = 0$ and is therefore not a superfluid. On the other hand, a boson gas with weak interactions has—upon the formation of a BEC—an energy spectrum that is linear at small momentum, $\epsilon(p) = c_s p$. Hence $v_c = c_s$, and weakly interacting BECs are superfluids.

The Landau criterion is usually derived with the velocity relative to an external scatterer in mind, but it also applies to the thermal excitations that make up the normal fluid. The critical value for the relative velocity $w = u_s - u_n$ of the normal fluid and superfluid is smaller than the one determined by Eq. (11), but the difference is small at low temperatures and weak self-interactions (Navez & Graham 2006).

The presence of the relative velocity $w$, due to the partially independent motion of the superfluid and normal fluid components in a finite-temperature superfluid, has important consequences for its behaviour. The superfluid part does not carry heat, while the normal fluid does, allowing mass and entropy to flow separately. This becomes clear if we define the velocity field for the mass flux, $v = j / \rho$, and express the equations for mass and entropy conservation in terms of $w$ and $v$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,$$  \hspace{1cm} (12)

$$\frac{\partial S}{\partial t} + \nabla \cdot (S v) - \nabla \cdot \left( \frac{S \rho_s}{\rho} w \right) = 0.$$  \hspace{1cm} (13)

For a superfluid condensate fraction, the entropy has an additional flux term, and hence entropy and mass can have different flow patterns. This property is called thermal counterflow. The equation for $\partial w / \partial t$ contains a driving term $S \nabla T / \rho_0$, so the counterflow $w$ tends to be directed towards regions of higher temperature, washing out thermal differences in the superfluid. As we will see, this is the property that makes the dynamical friction in a superfluid different from a corresponding fully normal fluid (i.e. a conventional fluid, $\rho_s = 0$, with the same pressure forces).

When the Landau criterion is broken, with $w$ approaching and passing the critical velocity, the superfluid flow starts to decay as a tangle of quantum vortices form and causes a mutual friction between the superfluid and normal fluid components (Skrbek 2011; Skrbek & Sreenivasan 2012; Barenghi et al. 2014). Such a dissipative effect is not present in the superfluid equations, but can be included with additional terms, as has been done in numerical studies of superfluid helium (Doi et al. 2008; Darve et al. 2012; Soulaine et al. 2017). However, to circumvent the need for extra parameters and having to assume the functional form of the mutual friction, we instead follow the same approach used in a previous work (Hartman et al. 2020); the dissipative processes are assumed to take place instantaneously when the relative velocity $w$ exceeds the critical velocity. The velocity field $v$, is changed in such a way that the fluid momentum is conserved, and that only the length of $w$ is altered, not its direction, bringing it to $w = v_c$.

The critical temperature $T_c$ is a central quantity in BEC superfluids. For $T > T_c$, a gas of identical bosons is a normal fluid, but for $T < T_c$, the particles begin accumulating in the ground state, forming a BEC, which in turn can form a superfluid. In the three-dimensional, homogeneous, ideal boson gas, this critical temperature is

$$T_c = \frac{2\pi h^2}{m^{5/3}} \left( \frac{\rho}{\zeta(3/2)} \right)^{2/3},$$  \hspace{1cm} (14)

where $\zeta(x)$ is the Riemann Zeta-function, and is approximately true for weakly interacting gasses as well (Andersen 2004; Sharma et al. 2019).

For the thermodynamic quantities of a weakly interacting boson gas, we again follow the approach used in a previous work (Hartman et al. 2020). The equation of state is approximated by an ideal gas with contributions from two-body interactions,

$$P = \frac{1}{2} \frac{g}{m^2} v^2 + \zeta(5/2) \left( \frac{m}{2\pi} \right)^{3/2} T^{5/2},$$  \hspace{1cm} (15)

$$S = \frac{5}{2} \zeta(5/2) \left( \frac{m}{2\pi} \right)^{3/2} T^{3/2},$$  \hspace{1cm} (16)

valid only for $T < T_c$. The fraction of particles in the condensate $f_0$ and the superfluid $f_s = \rho_s / \rho$ are both taken to be equal the condensate fraction in the ideal case:

$$f_s = f_0 = 1 - \left( \frac{T}{T_c} \right)^{3/2}.$$  \hspace{1cm} (17)

The critical velocity is approximated as

$$v_c = \sqrt{\frac{gn f_0}{m}}.$$  \hspace{1cm} (18)

As long as the temperature is not too close to the transition point, and the interactions are sufficiently weak, these approximations work well.

3. Dynamical friction from steady-state linear perturbation theory

The starting point for computing the dynamical friction acting on an object, or a "perturber", moving through the superfluid are Eqs. (7)-(10). The gravitational potential is sourced by both the
background mass density $\rho$, and the perturber’s mass distribution $\rho_{\text{pert}}$:

$$\nabla^2 \Phi = 4\pi G[\rho + \rho_{\text{pert}}].$$  \hspace{1cm} (19)

The superfluid is assumed to be homogeneous, so the fluid variables are expanded to linear order, $\rho = \rho_0 + \delta \rho$, $S = S_0 + \delta S$, $u = \delta u$, and so on. The linear equations are

$$\frac{\partial \delta \rho}{\partial t} + \nabla \cdot \delta \mathbf{j} = 0, \hspace{1cm} (20)$$

$$\frac{\partial \delta S}{\partial t} + S_0 \nabla \cdot \delta \mathbf{u} = 0, \hspace{1cm} (21)$$

$$\frac{\partial \delta u_s}{\partial t} + \frac{1}{\rho_0} \nabla \delta P - \frac{S_0}{\rho_0} \nabla \delta T = -\nabla \delta \Phi, \hspace{1cm} (22)$$

$$\frac{\partial \delta j}{\partial t} + \nabla \delta P = -\rho_0 \nabla \delta \Phi, \hspace{1cm} (23)$$

$$\delta \mathbf{u}_n = \frac{1}{\rho_0} \delta \mathbf{j} - \frac{\rho_{\text{rot}}}{\rho_0} \delta \mathbf{u}, \hspace{1cm} (24)$$

$$\nabla^2 \delta \Phi = 4\pi G \rho_{\text{pert}}.$$  \hspace{1cm} (25)

These can be combined into two coupled equations for $\delta \rho$ and $\delta S$:

$$\frac{\partial^2 \delta \rho}{\partial t^2} - \left[ \left( \frac{\partial^2 \delta \rho}{\partial \rho_0} \right)_0 \nabla^2 + 4\pi G \rho_0 \right] \delta \rho - \left( \frac{\partial \delta P}{\partial \rho_0} \right)_0 \nabla^2 \delta S = 4\pi G \rho_0 \rho_{\text{pert}}, \hspace{1cm} (26)$$

$$\frac{\partial^2 \delta S}{\partial t^2} - \frac{S_0}{\rho_0} \left( \frac{\partial \delta P}{\partial S_0} \right)_0 \nabla^2 + S_0 \frac{\rho_0}{\rho_{\text{rot}}} \left( \frac{\partial \delta T}{\partial S_0} \right)_0 \nabla^2 + 4\pi G \rho_0 \right] \delta \rho \hspace{1cm} (31)$$

$$- \frac{S_0}{\rho_0} \left( \frac{\partial \delta P}{\partial S_0} \right)_0 \nabla^2 + S_0 \frac{\rho_0}{\rho_{\text{rot}}} \left( \frac{\partial \delta T}{\partial S_0} \right)_0 \nabla^2 \delta S = 4\pi G S_0 \rho_{\text{pert}}, \hspace{1cm} (27)$$

As expected, there are scale dependent pressure terms that inhibit the growth of mass density and entropy perturbations. In the entropy equation there are additional effective pressure terms that further reduce entropy perturbations. These are due to thermal counterflow and depend on the superfluid fraction, vanishing in the fully normal fluid limit. It must be noted that the critical velocity $v_c$ is not included in the present approach, as it is a non-linear effect. This is considered in Section 5 using numerical simulations, where an ad hoc approach to include $v_c$ in the linear theory is motivated, derived, and tested.

Writing $\delta \rho = \alpha \rho_0$, and Fourier transforming into momentum ($k$) and frequency ($\omega$) space, the solutions of the $k$-modes $\alpha_k$ are found,

$$\alpha_k = -4\pi G \rho_{\text{pert},k} \frac{k^2 - A k^2}{(k^2_0 - \omega^2_{+}) (k^2_0 - \omega^2_{-})}. \hspace{1cm} (28)$$

where the dispersion relation is

$$\omega^2_k = C_4 k^2 - C_2 \pm \sqrt{C_3 k^4 - 2C_1 C_2 k^2 + C_2^2}, \hspace{1cm} (29)$$

and,

$$A = \frac{S^2_0 \rho_{\text{rot}}}{\rho_0} \left( \frac{\partial T}{\partial S} \right)_0, \hspace{1cm} (30)$$

The superfluid fraction, $\delta$, is small, and so the gradient terms drop out. Then, in the ground level. The dynamical friction is given by the change in the velocity of the perturber,

$$F_{DF} = \frac{M}{V} \frac{\partial \Phi_n}{\partial t}, \hspace{1cm} (35)$$

where $M$ and $V$ is the mass and velocity of the perturber, and $\Phi_n$ is the gravitational potential of to the background fluid,

$$\nabla^2 \Phi_n = 4\pi G \rho_0 \alpha. \hspace{1cm} (36)$$

This is readily found in $k$-space,

$$\Phi_{n,k} = -\frac{4\pi G \rho_{\text{rot}}}{k^2}, \hspace{1cm} (37)$$

which can be Fourier transformed back into position-space to give the dynamical friction,

$$F_{DF} = \frac{M}{V} \frac{\partial \Phi}{\partial t} \int \frac{dk^4}{(2\pi)^4} e^{ik_{0} \cdot \mathbf{x}} \Phi_{n,k} \approx -\frac{4\pi G M^2 \rho_0}{V} \int \frac{dk^4}{(2\pi)^4} \frac{ik_0}{k^2} e^{ik_{0} \cdot \mathbf{x}} \alpha_k. \hspace{1cm} (38)$$

Approximating the perturber as a point particle moving along the $z$-axis with constant velocity $V$, $\rho_{\text{pert}}(x, t) = M \delta(x) \delta(y) \delta(z - V t)$, or in $k$-space

$$\rho_{\text{pert},k} = 2\pi M \delta(k_0 - V k_z), \hspace{1cm} (40)$$

yields the expression for the dynamical friction as

$$F_{DF} = \frac{32\pi^2 G^2 M^2 \rho}{V} \int \frac{dk^4}{(2\pi)^4} \frac{ik_0}{k^2} e^{ik_{0} \cdot \mathbf{x}} \delta(k_0 - V k_z) \frac{k^2 - A k^2}{(k^2_0 - \omega^2_{+}) (k^2_0 - \omega^2_{-})}. \hspace{1cm} (41)$$

Eq. (41) can be tackled by extending the $k_0$-integral into the complex plane and closing it in the upper half plane (assuming...
\( t > 0 \), so that contour integration can be used. The poles are pushed slightly off the real line by the prescription \( \omega_{kz} \rightarrow \omega_{kz} + i \epsilon \), and only the residual of the poles inside the contour contribute to the integral. Taking the limit \( \epsilon \rightarrow 0^+ \) after integrating gives the dynamical friction as

\[
F_{DF} = -\frac{16\pi^2 G^2 M^2 \rho_0}{V} \int \frac{dk}{k^2} \frac{1}{(2\pi)^3} \omega_k^2 \omega_{k-}^2 - \omega_{k-}^2 \times \left[ e^{i\omega_k \cdot \mathbf{k} \cdot \mathbf{x}} (\omega_{k+}^2 - Ak^2) \delta(\omega_{k+} - V k_+ - \omega_{k-}^2) - e^{i\omega_k \cdot \mathbf{k} \cdot \mathbf{x}} (\omega_{k-}^2 - Ak^2) \delta(\omega_{k-} - V k_+ - \omega_{k-}^2) \right].
\]

(42)

Spherical polar coordinates are adopted for the integral over momentum, with the polar angle \( \theta \) defined as the angle relative to the direction of propagation, the \( z \)-axis, and the force is evaluated at the position of the perturber, \( x = Vt \). The integrand is independent of the azimuthal angle, but depends on the polar angle through \( k = k \cos \theta \). Integrating over the azimuthal angle therefore gives a factor \( 2\pi \), while the polar angle in combination with the \( \delta \)-function fixes the exponentials to one and places upper limits on the momentum, \( k < k_{\text{max}} \), where \( k_{\text{max}} \) satisfies \( kV = \omega_{kz} \). Further constraints are placed on \( k \): The remaining \( k \)-integral is bounded by the finite sizes of the perturber and the cloud if it moves through, \( R_{\text{max}} = R_{\text{cloud}} \) and \( R_{\text{min}} = R_{\text{pert}} \), otherwise both \( UV \)-and \( IR \)-divergences may be encountered, because the perturber is modeled as a point particle, and the background fluid as infinite and homogeneous. We must also have \( k > k_{\text{min}} \), where \( k_{\text{max}} \) is the minimum momentum for which \( \omega_{kz} \) are real. At small \( k \), or equivalently, large scales, where \( \omega_{kz} \) become complex or imaginary the background cloud will be gravitationally unstable and deform. We denote as a general measure the upper limits in \( k \) for the two terms in Eq. (42) by \( k_{\text{max}} \), and the lower limits by \( k_{\text{min}} \). Inserting the expression for \( \omega_{kz} \) and using that \( C_4 - A = C_1 \), the dynamical friction becomes

\[
F_{DF} = -\frac{4\pi G^2 M^2 \rho_0}{V^2} \text{Im} \left\{ \int_{k_{\text{max}}}^{k_{\text{max}}} \frac{dk}{2k} \left( \frac{C_1 k^2 - C_2}{\sqrt{C_1 k^4 - 2C_1 C_2 k^2 + C_2^2}} + 1 \right) \right. \\
- \left. \int_{k_{\text{min}}}^{k_{\text{min}}} \frac{dk}{2k} \left( \frac{C_1 k^2 - C_2}{\sqrt{C_1 k^4 - 2C_1 C_2 k^2 + C_2^2}} - 1 \right) \right\}.
\]

(43)

There is an implicit criterion that \( k_{\text{max}} > k_{\text{min}} > 0 \), otherwise the integral is zero.

Eq. (43) can be solved analytically, but its expression is not very enlightening. Instead we focus on a few limiting cases for which the force is reduced to a simplified form: zero temperature, the fully normal fluid, small velocities, and no self-gravitation.

### 3.1. Zero temperature limit

Taking the limit \( T \rightarrow 0 \) (under the assumption that terms such as \( S^2 \rho_s / \rho_0 \) go to zero as well) yields one band for the dispersion relation,

\[
\omega_k^2 = \frac{c_s^2}{C_4 - \sqrt{C_4}} k^2 - 4\pi G \rho_0,
\]

where the sound speed at zero temperature is

\[
c_s^2 = \left( \frac{\partial P}{\partial \rho} \right)_{T=0}.
\]

(44)

The dynamical friction becomes

\[
F_{DF} = -\frac{4\pi G^2 M^2 \rho_0}{V^2} \ln \left( \frac{k_{\text{max}}}{k_{\text{min}}} \right)
\]

(46)

with

\[
k_{\text{max}} = \min \left\{ 2\pi R_{\text{max}}^{-1}, \left[ \frac{4\pi G \rho_0}{c_s^2} \right] \right\},
\]

(47)

\[
k_{\text{min}} = \max \left\{ 2\pi R_{\text{min}}^{-1}, \left[ \frac{4\pi G \rho_0}{c_s^2} \right] \right\}
\]

(48)

### 3.2. Normal fluid limit

Taking the fully normal fluid limit \( \rho_s \rightarrow 0 \) also gives one band for the dispersion relation,

\[
\omega_k^2 = c_s^2 k^2 - 4\pi G \rho_0,
\]

(49)

with the sound speed in the fully normal fluid

\[
c_s^2 = \left( \frac{\partial P}{\partial \rho} \right)_{T=0} + \frac{S_0}{\rho_0} \left( \frac{\partial P}{\partial S} \right)_{T=0}.
\]

(50)

The dynamical friction is again on the same form as Eq. (46), but with

\[
k_{\text{max}} = \min \left\{ 2\pi R_{\text{max}}^{-1}, \left[ \frac{4\pi G \rho_0}{c_s^2} \right] \right\},
\]

(51)

\[
k_{\text{min}} = \max \left\{ 2\pi R_{\text{min}}^{-1}, \left[ \frac{4\pi G \rho_0}{c_s^2} \right] \right\}
\]

(52)

This is the same as the zero-temperature case, but with a different sound speed.

### 3.3. Small velocity limit

At sufficiently small velocities, \( V^2 \ll C_4 - \sqrt{C_5} \), assuming that the finite sizes of the background cloud and perturber do not set the integral limits in Eq. (43), the dynamical friction becomes

\[
F_{DF} = -\frac{2\pi G^2 M^2 \rho_0}{V^2}
\]

(53)

This is equal to the friction force at \( T = 0 \) in the same limit, as opposed to when \( \rho_s = 0 \);

\[
F_{DF} = -\frac{2\pi G^2 M^2 \rho_0}{c_s^2}.
\]

(54)

The dynamical friction of a superfluid therefore approaches the zero temperature limit even when there is a significant thermal contribution. This happens because counterflow in the superfluid conspires against thermal perturbations, allowing the mass over-density to behave similarly as a zero-temperature fluid. Recall, however, that this result does not include the effect of the critical velocity which would limit this thermal counterflow. In Section 5 we investigate numerically how the critical velocity changes the dynamical friction of a superfluid, and how it can, to some degree, be included in the linear approach.
3.4. Neglecting self-gravitation

The numerical results presented in Section 5 are obtained from simulations where self-gravitation is neglected. It is therefore of interest to see what the steady-state linear theory predicts in this case as well.

Neglecting self-gravitation amounts to setting $C_2 = 0$. The dispersion relation becomes

$$\omega_{\pm}^2 = (C_3 \pm \sqrt{C_3})k^2 = c_{\pm}^2 k^2.$$  \hspace{1cm} (55)

For the equation of state used throughout this work, and $T/T_c < 0.2$, the above superfluid sound speeds can be approximated to a high degree by

$$c_{\pm} = \sqrt{\frac{c_n^2 - c_T^2}{f_n}},$$  \hspace{1cm} (56)

$$c_{-} = c_{T=0}.$$  \hspace{1cm} (57)

Note that for $c_n \gg c_{T=0}$, we have $c_+ \approx c_n / \sqrt{f_n}$. The dynamical friction takes the form

$$F_{DF} = -4\pi G^2 M S_0 \ln \left( \frac{R_{\max}}{R_{\min}} \right) \times \frac{1}{2} \left[ \frac{1}{\sqrt{C_3}} \delta(V - c_-) + \frac{C_3}{\sqrt{C_3}} \delta(V - c_+) \right]$$  \hspace{1cm} (58)

One feature that is clear in this limit is that $F_{DF}$ jumps from zero as $V$ becomes larger than $c_+$ and jumps again as it crosses $c_-$. It seems odd that the force should change value so dramatically when the velocity of the perturber crosses these thresholds, and indeed we find in the numerical simulations in Section 5 that it does not. The problem is that in the steady-state case, as considered in this section, the linear over-density is symmetric upstream and downstream when the perturber moves at subsonic speeds, resulting in a zero net gravitational force at the position of the perturber. In order to overcome this shortcoming of steady-state linear perturbation theory, previous studies have broken this symmetry by switching on the perturber for a finite time (Ostriker 1999; Sánchez-Salcedo 2012), or by going to second order perturbations (Lee & Stahler 2011; Shadmehri & Khajenabi 2012). In the next section the finite-time approach is employed for a superfluid.

4. Dynamical friction from finite-time linear perturbation theory

For the finite-time calculation, Eqs. (26) and (27) are used without self-gravitation, and an approach very similar to the one used by Ostriker (1999) followed.

The equations can be written on matrix form as

$$\frac{\partial^2 Y}{\partial t^2} + A \nabla^2 Y = F_{\text{pert}},$$  \hspace{1cm} (59)

where

$$Y = \left( \frac{\delta \rho}{\delta S} \right),$$  \hspace{1cm} (60)

$$A = \begin{pmatrix} S_0 \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right)_{0} + \sum_{n=0}^{\infty} \frac{\partial \rho}{\partial S} \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right)_{n} + \frac{\partial \rho}{\partial S} \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right) \end{pmatrix}.$$  \hspace{1cm} (61)

and

$$F = \left( \frac{4\pi G \rho_0}{4\pi G S_0} \right).$$  \hspace{1cm} (62)

By diagonalizing the matrix $A$, the coupled set of equations can be transformed into two decoupled wave equations of the form

$$\frac{\partial^2 \chi_{+}}{\partial t^2} - c_{+}^2 \nabla^2 \chi_{+} = f_{+},$$  \hspace{1cm} (63)

$$\frac{\partial^2 \chi_{-}}{\partial t^2} - c_{-}^2 \nabla^2 \chi_{-} = f_{-},$$  \hspace{1cm} (64)

that are solved using the retarded Green’s function for the wave equation in three dimensions;

$$\chi_i(x, t) = \int d^3 x' \int dt' \frac{\delta(t' - (t - |x - x'|/c_i)) f_i(x', t')}{4\pi c_i^2 |x - x'|}.$$  \hspace{1cm} (65)

For a point source switched on at the origin at $t = 0$ and moving at speed $V = V_2$, $f_i(x, t) = K_i \delta(x) \delta(t) \delta(z - V_2 t) H(t)$, where $H(t)$ is the Heaviside function, the solution of $\chi_{i}$ becomes, upon defining $s = z - V_2 t$, $M_i = V_i/c_i$, and $R^2 = x^2 + y^2$,

$$\chi_i(x, t) = \frac{K_i}{4\pi c_i^2} \frac{H_i}{\sqrt{s^2 + R^2(1 - M_i^2)}}$$  \hspace{1cm} (66)

$$H_i = \begin{cases} \frac{1}{2} & \text{for } R^2 + z^2 < (c_i t)^2, \\
1 & \text{for } M_i > 1, R^2 + z^2 > (c_i t)^2, \\
\frac{s}{R} \frac{1}{\sqrt{M_i^2 - 1}} & \text{for } s/R < -\sqrt{M_i^2 - 1}, \text{and } z > c_i t/M_i, \\
0 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (67)

The resulting over-density $\delta \rho$ is a weighted sum of $\chi_{+}$ and $\chi_{-}$, and the dynamical friction is obtained by integrating over the whole volume the gravitational force due to the overdensity.

$$F_{DF} = 2\pi GM \int ds \int dR \left( \frac{s \rho \delta \rho}{(s^2 + R^2)^{3/2}} \right).$$  \hspace{1cm} (68)

In spherical polar coordinates, $s = r \cos \theta = r x$ and $R = r \sin \theta = r \sqrt{1 - x^2}$, we get

$$F_{DF} = -4\pi G^2 M^2 S_0 \left( I_+ + I_- \right) \sqrt{1 - M_i^2 + x^2 M_i^2}.$$  \hspace{1cm} (69)

$$I_i = -D_i \int_{r_{\text{max}}}^{r_{\text{min}}} \frac{dr}{2r} \int_{-1}^{1} \frac{dx}{\sqrt{1 - M_i^2 + x^2 M_i^2}} \cdot$$  \hspace{1cm} (70)

The sound speeds $c_+$ and $c_-$ are the same as the ones given in Eq. (55), and

$$D_+ = \frac{S_0 \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right)_{0} + c_+^2}{\rho_0 (c_+^2 - c_+^2)} \left( \frac{\partial \rho}{\partial S} \right)_{0} \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right)_{0},$$  \hspace{1cm} (71)

$$D_- = \frac{S_0 \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right)_{0} + c_-^2}{\rho_0 (c_-^2 - c_-^2)} \left( \frac{\partial \rho}{\partial S} \right)_{0} \left( \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial S} \right)_{0} + c_-^2.$$

The dynamical friction from the finite-time calculation is compared to the steady-state result in Fig. 1. The discontinuities have been removed, with the force increasing with velocity.
V until it reaches a maximum near the sound speed, after which it decreases with the same $1/V^2$ dependence as in the steady-state result. As time passes the finite-time result approaches the steady-state result.

Both approaches predict $F_{\text{DF}}$ in the superfluid case to be very close to the zero-temperature value, even though thermal pressure dominates over the contribution from self-interactions. We must recall again, however, that the Landau criterion is not included in linear perturbation theory, which will limit the thermal counterflow of the superfluid, making it behave more like a normal fluid, thus decreasing the dynamical friction.

We use the the frame comoving with the perturber, so that its gravitational field is static and centered at the origin, while the background fluid is moving. We take the perturber to be a sphere with uniform mass density $\rho_{\text{pert}} = 3M/4\pi R_{\text{pert}}^3$. The system has rotational symmetry, so cylindrical coordinates are employed; the axial distance $z$ (distance along the axis of rotational symmetry), and the radial distance $r$ (distance from the axis). The simulation volume is therefore a cylinder, and we take its domain to be $-L < z < L$ and $0 < r < L$.

The superfluid is initialized as a uniform fluid moving with velocity $V = -V/2$. More fluid is injected into the simulation volume with the same velocity at the $z = +L$ boundary. The $z = -L$ and $r = L$ boundaries are taken to have zero gradients, while the inner boundary $r = 0$ has a reflecting boundary condition.

To numerically integrate the superfluid equations, a Godunov scheme similar to the one described in Hartman et al. (2020) is used. In the present work, the generation of entropy when the Landau criterion is broken is not included. Also, since we evolve the entropy instead of the energy, and we don’t included any viscosity, the numerical scheme dissipates kinetic energy at shock fronts that is not converted into internal energy. In the absence of this shock heating the total energy is not strictly conserved. We have found, however, that this inaccuracy is by and large negligible for the scenarios we have considered here since the solutions are mostly in or near the linear regime.

The self-gravitation of the superfluid is neglected. The gravitational field it produces is computed only to find the resulting force on the perturber, i.e. the dynamical friction. The initial fluid parameters are $\rho = 2\times10^7 M_\odot kpc^{-3}$, $T = 0.2 T_\odot$, $m = 500 eV$, and $g = 2 \times 10^{-3} eV^{-2}$. These parameters are chosen only to illustrate the basic features of dynamical friction in superfluids while keeping the simulation run times reasonably short. Unless stated otherwise, the size of the perturber is taken to be $R_{\text{pert}} = 2pc$ with mass $M = 0.1 M_\odot$, while the simulation size is $L = 150pc$. The simulation is run until $t = 10pc/V$, i.e. until the background has moved $10pc$. This is small compared to the full simulated length, but is necessary for avoiding boundary effects from interfering with the results. $R_{\text{min}}$ is taken to be the size of the perturber, $R_{\text{min}} = 2pc$, and $R_{\text{max}}$ the radius of the cylindrical simulation volume, $R_{\text{max}} = 150pc$, when compared to linear perturbation theory. The resolution of the simulated volume is $4096 \times 2048$ cells, in the $z$ and $r$ directions, respectively, for the superfluid case. In the zero-temperature and normal fluid limits, for which the numerical scheme was made second-order in time and space using a MINMOD slope-limited MUSCL-Hancock method without stability issues, a lower resolution of $2048 \times 1024$ is used.

An effective critical velocity $v_c^{\text{eff}}$, which is just $v_c$ multiplied by some factor, is used in the simulations to show the effect of varying $v_c$ without to actually changing other parameters such as the particle mass and self-interaction.

### 6. Comparison of perturbation theory and numerical simulation

In Fig. 2 the dynamical friction from the numerical simulations is compared to the linear result. Even with the Landau criterion included, given by Eq. (18), the dynamical friction in the superfluid case is very similar to the zero-temperature fluid, as seen in the linear theory. This similarity can also be seen in the mass density profile, shown in Fig. 6. At $T = 0$, for which the perturber is supersonic with $V = 1.5c_T$, there is a well-defined supersonic cone that trails the perturber. The finite-temperature superfluid has a similar cone, though not as well-defined, and...
Sonics speeds, since \( V \) fully normal fluid case the perturber is instead moving at sub-sonic cone. As \( v_{\text{eff}} \) is decreased, the relative velocity becomes increasingly limited and the thermal counterflow inefficient, causing the superfluid density profile to approach the normal fluid case. The dynamical friction changes accordingly, as shown in Fig. 3.

The superfluid dynamical friction also has a non-trivial dependence on the mass of the perturber, as shown in Fig. 4, which is related to the critical velocity. To see why, consider the linearized equation for the relative velocity,

\[
\frac{\partial \bar{w}}{\partial t} = \frac{S_0}{\rho_0} \nabla \delta T = \frac{S_0}{\rho_0} \left[ \left( \frac{\partial T}{\partial S} \right)_0 \nabla \delta S + \left( \frac{\partial T}{\partial \rho} \right)_0 \nabla \delta \rho \right].
\]

The amplitude of \( \delta \rho \) and \( \delta S \), and hence \( \delta T \), increases with \( M \), driving \( w \) up to the critical value faster, causing the effect of the critical velocity on the system to be more prominent. We therefore expect that increasing \( M \) has a similar effect as lowering \( v_c \) in transitioning the superfluid dynamical friction from the \( T = 0 \) value to the fully normal fluid value.

This suggests an ad hoc approach to incorporate the critical velocity in the linear theory. We assume that for an estimate of the counterflow \( \bar{w} \), there is an interpolating function \( f(\bar{w}, v_c) \) with \( f(\bar{w} \ll v_c) \to 1 \) and \( f(\bar{w} \gg v_c) \to 0 \), and a transitional region around \( \bar{w} \sim v_c \), such that

\[
F_{\text{DF}} = f(\bar{w}, v_c) F_{\text{DF}}^{\text{sf}} + [1 - f(\bar{w}, v_c)] F_{\text{DF}}^{\text{nf}},
\]

where \( F_{\text{DF}}^{\text{sf}} \) and \( F_{\text{DF}}^{\text{nf}} \) is the dynamical friction from linear theory for the superfluid and fully normal fluid, respectively. Using Eq. (73) we can write

\[
\bar{w} = S_0 \frac{\delta T(0)}{\rho_0} \Delta t = S_0 \frac{\delta S(0)}{\rho_0} \frac{\delta \rho(0)}{L} \Delta t.
\]

The length \( L \) and time \( \Delta t \) are characteristic scales over which the fluid attains the mass and entropy overdensity at the origin, \( \delta \rho(0) \) and \( \delta S(0) \). The timescale can be estimated as \( \Delta t = L/v \), where \( v \) is some characteristic velocity in the problem. The first such quantity that comes to mind is the initial average flow \( V \). However, that would suggest that as \( V \) is decreased, \( \bar{w} \) should
be very large, causing \( f(\bar{\omega}, v_c) \to 0 \), in contradiction with the numerical results in Fig. 2. Instead the largest superfluid sound speed, \( c_s \), which is essentially the fastest speed with which the superfluid can respond to disturbances, was found to work.

For the \( \delta \)-function perturbation, the central values for the mass and entropy overdensities diverge, so that \( \delta \rho(0) \) and \( \delta S(0) \) are not well-defined. Instead, they should be evaluated at some point near the origin, as was done for the dynamical friction. With the equation of state used in this work an estimate of the linear entropy contrast at \( R_{\text{min}}/2 \) is

\[
\delta S \left( R_{\text{min}}/2 \right) \approx \frac{2S_0 GM}{c_s^2 R_{\text{min}}}.
\]

The rough estimate of the counterflow is therefore

\[
\bar{\omega} = \frac{S_0}{\rho_0} \left( \frac{\partial T}{\partial S} \right)_{0} \frac{2GM}{c_s^2 R_{\text{min}}}.
\]

Only the form of the interpolating function \( f(\bar{\omega}, v_c) \) remains to be specified. The simple, but rather arbitrary, choice

\[
f(\bar{\omega}, v_c) = \frac{v_c}{v_c + \bar{\omega}} \left[ 1 + \frac{S_0^2}{\rho_0} \frac{\partial T}{\partial S} \right] \left( \frac{2GM}{c_s^2 R_{\text{min}}} \right) v_c^{-1}
\]

was found to work well.

The numerical results of Fig. 2, Fig. 3, and Fig. 4 are re-evaluated in Fig. 5, this time against linear theory with the ad hoc approach to include the \( v_c \). This scheme manages to capture the basic dependence on the perturber mass and critical velocity. However, it appears to fail at low velocities, \( V < c_{T,0} \approx c_s \), which suggests that other factors might come into play at those speeds. We will see in the next section that this does not cause any problems when applied to a realistic system, so we make no further attempt to improve the scheme.

7. Application to Fornax system

So far, only the physics of dynamical friction in superfluids has been discussed, with little reference to the real world. Now, armed with the expressions derived and tested in the previous sections, the parameter space of superfluid DM can be explored. Dwarf spheroidal galaxies (dSph) are particularly well-suited for this purpose. Being poor in visible matter, their dynamical behaviour is dominated by their DM component and therefore provides a testing ground for DM models (Strigari 2018). One such system is the Fornax dSph and its five gravitationally bound globular clusters (GC). The orbital decay times of these GCs due to dynamical friction, using CDM, has been estimated to \( \tau_{\text{DF}} \sim 1 \) Gyr (Oh et al. 2000), much shorter than the supposed age of the host system, \( \tau_{\text{age}} \sim 10 \) Gyr (del Pino et al. 2013; Wang et al. 2019). This apparent mismatch between theoretical prediction and observation suggests one of two scenarios; that we are witnessing all of Fornax’s GCs just as they are about to fall into their host, which seems unlikely; or that there is some mechanism, or property of DM, that stops the GCs from migrating towards the center of the Fornax dSph. This is the so-called timing-problem, and a number of solutions have been proposed, such as massive black holes heating the system (Oh et al. 2000), assuming the CDM profile of Fornax to be cored instead of cuspy (Goerdt et al. 2006; Cole et al. 2012), and extended dark matter models (Hui et al. 2017; Lancaster et al. 2020). The discrepancy may also simply be due to inaccurate modeling of the Fornax system and the rate of the orbital decay (Kaur & Sridhar 2018; Cowsik et al. 2009; Boldrini et al. 2019).

The aim of this section is to explore the parameter space of superfluid DM, estimating what region is allowed and may exhibit superfluidity in the Fornax system, and to compute the resulting orbital decay time-scale due to dynamical friction. This can be defined as the time it takes dynamical friction to reduce the angular momentum \( L \) of the GCs to zero;

\[
\tau_{\text{DF}} = \frac{L}{r|F|_{\text{DF}}} = \frac{MV}{|F|_{\text{DF}}},
\]

where \( M, V, \) and \( r \) is the mass, circular orbital velocity, and the orbital radius of the GCs. Estimates of the masses and projected
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Fig. 6. Density profiles and streamlines for $V = 1.5c_{T,0}$. The mass density profiles are superimposed by the net mass density velocity, $v = j/\rho$, while the entropy density is superimposed by the relative velocity $w = v_s - v_n$. The perturber has mass $M = 5M_\odot$, the simulation volume $L = 75$pc, and the time $t = 50$pc/$V$, so that the features in the various cases becomes clearer.

orbital radii $r_\perp$ are listed in Table 1. As in Lancaster et al. (2020) and Hui et al. (2017), $r = 2r_\perp/\sqrt{3}$ is used as the ”true” radial distance from the Fornax center. The GCs orbital velocities are determined by assuming a near constant core for the Fornax dSph,

$$V = \sqrt{4\pi G \rho / 3r},$$

and their typical size is taken to be $R_{\text{pert}} = 2$pc (Mackey & Gilmore 2003). The DM density of the Fornax dSph is $\rho_0 \approx 4 \times 10^7 M_\odot \text{kpc}^{-3}$, with a core radius of $R_{\text{cloud}} = R \approx 700$pc (Walker et al. 2009; Cole et al. 2012; Read et al. 2019).

Table 1. Projected radial distances and masses of GCs in Fornax, taken from Cole et al. (2012).

| GC label | Projected radial distance $r_\perp$ [kpc] | Mass $M \left[10^5 M_\odot\right]$ |
|----------|----------------------------------------|-------------------------------|
| GC1      | 1.6                                    | 0.37                          |
| GC2      | 1.05                                   | 1.82                          |
| GC3      | 0.43                                   | 3.63                          |
| GC4      | 0.24                                   | 1.32                          |
| GC5      | 1.43                                   | 1.78                          |

There is a limited region of parameter space that is both physically relevant, and may provide a reasonable estimate of $r_{DF}$. This region should satisfy the following:

- The size of the halo obtained from hydrostatic equilibrium should not exceed the observed size of the dSph.

- The DM mass and self-interaction should satisfy constraints from observations.

- The relaxation rate of DM should be higher than the rate of dynamical changes in the dSph, so that the system can thermalize and form a superfluid.

- Perturbation theory is only properly valid for $\delta \rho/\rho \ll 1$.

- The numerical results suggest that the scheme to include the critical velocity into perturbation theory fails for $V < c_c$.

While this list is likely not exhaustive, it provides a minimum set of criteria that should be fulfilled. Due to our ignorance about the general behaviour of superfluid DM in a number of situations, we will enforce relaxed variants of the above constraints. As seen in the previous sections, and shown in an earlier work (Hartman et al. 2020), counterflow can effectively redistribute thermal energy in the superfluid. Hence, it is not clear how the temperature profile of a realistic superfluid DM halo might look. The least constraining assumption is therefore made; that the counterflow has washed out any significant thermal differences, so only the interaction part of the pressure (i.e. the only pressure present at $T = 0$) determines the hydrostatic profile. For a halo of size $R$, defined by $\rho(R) = \rho(0)/2$, this gives

$$g \lesssim \pi G R^2 m^2.$$

Of course, this relaxed constraint is only possible if it is physically feasible for the counterflow to transport a significant portion of the thermal energy away from the halo core. A supplementary criterion can be derived by demanding that the total entropy flux due to thermal counterflow, with $w = v_c$, at the halo edge $R$, must
be of the same order as the total entropy enclosed in $R$. This leads to
\[ g \geq \frac{m^2 R^2}{\delta \rho \Delta t^2}. \tag{82} \]

where $\Delta t$ should be smaller than the age of the dSph, e.g. $\Delta t \sim 1\text{Gyr}$. As we will see, the difference between the $T = 0$ and $T > 0$ treatment of the hydrostatic halo size can be very large, and we do not expect a realistic superfluid halo to be able to completely remove thermal differences, even if upper estimates of the thermal counterflow suggest it could. The region far beyond the finite-temperature limit of the DM halo size, but allowed by the $T = 0$ treatment, should therefore be considered with scepticism.

By measuring the spatial offset of stars, gas, and dark matter in colliding galaxy clusters, a constraint on the self-interaction cross section of dark matter, $\sigma$, can be established (Harvey et al. 2015). The lack of deceleration of DM and its proximity to the collisionless stars in these collisions places an upper limit, $\sigma/m \lesssim 0.5\text{cm}^2/\text{g}$. In terms of the self-interaction parameter $g$, this constraint reads
\[ g = \frac{\sqrt{4\pi \sigma}}{m} \leq 5 \times 10^{-12} \left(\frac{1 \text{eV}}{m}\right)^{1/2} \text{eV}^{-2}. \tag{83} \]

While the above places upper limits on $g$, there is also a lower limit that must be considered, given by the criterion that the DM superfluid should be thermalized across much of the halo. For this we require the relaxation rate of DM, $\Gamma_{\text{DM}}$, to be higher than the rate of dynamical changes in the halo, $\Gamma_{\text{grav}} \sim \sqrt{G\bar{\rho}}$. For two-body interactions, the relaxation rate is $\Gamma \sim n\bar{\sigma}v$, where $\sigma$ is the scattering cross section and $\bar{\sigma}$ the velocity dispersion of the particles. In terms of $g$, as above, the cross section is $\sigma = m^2 g^2/4\pi$. However, for a condensed boson gas the relaxation rate is enhanced, $\Gamma \sim N 4\bar{\sigma}v$, where
\[ N = n \frac{(2\pi)^3}{\sqrt{2\pi (m\bar{\sigma}v)^3}}. \tag{84} \]

because of the high occupation number of the ground state (Sikivie & Yang 2009). Using $\delta v \sim V$, i.e. that the DM velocity dispersion is of the same order as the GC orbital velocity, the criterion $\Gamma_{\text{DM}} > \Gamma_{\text{grav}}$ becomes
\[ g \gtrsim \sqrt{\frac{2}{3\pi}} \frac{m^{3/2}G^{1/4}V}{\rho^{3/4}}. \tag{85} \]

It should be noted that the enhancement factor is included in this criterion, but not in the constraint from cluster collisions. This is another example of a relaxed constraint due to our ignorance of how the superfluid properties might change in the various systems. The characteristic speeds of cluster collisions are $v \sim c_{\text{s}}$, the GC orbital velocity, and $\delta v \sim V$. For $V < c_{\text{s}}$, the DM fluid may not even be condensed throughout most of the cluster, only inside dense structures. We therefore choose the least restrictive constraint by including $N$ inside the dSph DM halo, but not outside.

The remaining constraints due to $\delta \rho/\rho \ll 1$ are readily obtained from perturbation theory and Eq. (57). For computing the dynamical friction, the result from time-dependent perturbation theory, with the ad hoc inclusion of the critical velocity, is used. The characteristic time $t = r/V = \sqrt{3}/4\pi G \bar{\rho}$ is used, but the results are not very sensitive to this particular choice.

The above criteria are illustrated in Fig. 7 for GC4 from Table 1, with the estimated orbital decay time-scale included for reference. The limit due to the largest hydrostatic halo allowed for by the observed size of the Fornax dSph, $g = \pi G R^2 m^2$, and $V = c_{\text{s}}$. Both lie in the region where the time-scale changes from very small to very large. This is no coincidence. At hydrostatic equilibrium, and $T \sim R$, we have $V \sim c_{\text{s}}$, and we know from the previous sections that the dynamical friction quickly goes from its maximum value to zero at velocities below the sound speed. The same is also true in the fully normal fluid case, but instead with the sound speed $c_{\text{s}}$ and hydrostatic equilibrium with thermal pressure included, as we can see for $\bar{w} \gg v_c$ in Fig. 7, where the superfluid behaves like a normal fluid. The figure also shows that in the relevant parameter space where perturbation theory is valid we have $\bar{w} \ll v_c$, hence we do not need to worry about the breakdown of the superfluid-normal fluid interpolation scheme that we observed in Section 6.

![Fig. 7. The criteria listed in the text, and the orbital decay time-scale for GC4 from Table 1 at $T/T_c = 10^{-4}$ for reference. (solid black line) The permitted parameter space: the left side is from the minimum halo size in hydrostatic equilibrium, Eq. (81); the upper right from the constraint from galaxy cluster collisions, Eq. (83); and the lower right from the minimum relaxation rate needed to thermalize the fluid across the halo, Eq. (85). (solid blue) $V = c_{\text{s}}$, with $V < c_{\text{s}}$ on the left side. (dotted blue line) Criterion for linear perturbation theory to be properly valid, with $\delta \rho/\rho < 1$ satisfied on left side. (dashed blue line) The supplementary criterion for the $T = 0$ treatment of the hydrostatic halo size, with Eq. (82) satisfied on left side. (solid red line) $\bar{w} = v_c$, where the superfluid dynamical friction transitions from zero-temperature-like on the left side, to normal-fluid-likethere on the right. (dashed red line) The limit due to a hydrostatic halo with thermal pressure included, with halo sizes smaller than the observed size of the Fornax dSph to the right.

The orbital decay time for a wide range of parameters is shown in Fig. 8 for the two GCs inside the core radius of Fornax, GC3 and GC4. However, only a limited region satisfy the criteria for a sufficiently small halo and weak interactions, can be superfluid, and provides a reasonable estimate for the dynamical friction using perturbation theory. In this region, the decay times are generally very small, $\tau_{\text{DF}} \approx 67\text{Myr}$ and $125\text{Myr}$, except along the leftmost edge, where the halo is largely supported by (zero-temperature) hydrostatic pressure.
By fitting rotation curves in slowly rotating SIBEC-DM halos in 173 nearby galaxies from the Spitzer Photometry & Accurate Rotation Curves (SPARC) data (Lelli et al. 2016), Crăciun & Harko (2020) estimated the properties of SIBEC-DM halos at $T = 0$, and found the preferred values for $g/m^2$ to be between $2.7 \times 10^{-4}\text{eV}^{-4}$ and $5.0 \times 10^{-2}\text{eV}^{-4}$. For reference, the estimated limit from hydrostatic equilibrium, using Eq. 81 and $R \approx 700\text{pc}$, gives $g/m^2 < 2.5 \times 10^{-2}\text{eV}^{-4}$, only marginally smaller than the lower limit, which may be due to our rough estimate. The preferred values obtained by Crăciun & Harko (2020) for zero-temperature SIBEC-DM therefore appear to resolve the timing-problem. This result could also have been found using heuristic arguments; if the halo is supported by hydrostatic pressure, i.e. its Jeans’ length $R_J \sim c_s / \sqrt{G\rho}$ is of the order of the DM halo $R$, then density perturbations on smaller scales inside the halo will be highly suppressed, resulting in very weak dynamical friction.

In a finite-temperatures SIBEC-DM halo—for which we expect the preferred values for $g/m^2$ obtained from fitting rotation curves to be lowered, since it provides extra pressure forces to support DM halos—the present results instead suggest that too large orbital decay rates due to strong dynamical friction may arise. This is counter to what one would naively expect if the superfluid had been treated as a conventional thermal fluid, since an increased pressure generally leads to a smaller maximum dynamical friction. Instead, the superfluid essentially ignores the thermal contribution, and responds to a perturber as if it were at $T = 0$, which can yield a much larger friction force.

### 8. Conclusion

The dynamical friction acting on an object due to a superfluid background has been investigated, starting with steady-state linear perturbation theory. The well-known issue of discontinuities in the friction force as the perturber’s velocity crosses the fluid sound speed was encountered. A finite-time formalism was therefore also employed, which removed these discontinuities, conforming with previous studies that the dynamical friction increases with the perturber’s velocity until the sound speed is reached, after which the force decreases with the same $V^2$ dependence as the steady-state result. Both approaches predict the force in the superfluid case to be very similar to the $T = 0$ limit, even when there are large thermal contributions, yielding a much stronger friction force than one might naively expect from a conventional fluid at the same temperature. This happens because counterflow, which conspires against thermal perturbations, allows the superfluid to behave as if it were at zero temperatures. This result did not include the effect of the critical velocity, which limits the counterflow. Numerical simulations were therefore used to investigate the influence of the critical velocity, finding that as $v_c$ is decreased, it interpolates the superfluid dynamical friction from about the $T = 0$ value to the fully normal fluid value. For the fiducial parameters used in the simulations, the change in friction force was about two orders in magnitude.

The numerical results also revealed a non-trivial dependence on the perturber mass when the critical velocity was included. As the mass is increased, the dynamical friction changes in a manner similar as when the critical velocity is decreased. This motivated

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Fig. 8. The decay time of GC3 and GC4 (as listed in Table 1). (solid line) The permitted parameter space; the left side is from the minimum halo size in hydrostatic equilibrium, Eq. (81); the upper right from the constraint from galaxy cluster collisions, Eq. (83); and the lower right from the minimum relaxation rate needed to thermalize the fluid across the halo, Eq. (85). (dotted line) Criterion for linear perturbation theory to be properly valid, with $\delta p/\rho < 1$ satisfied on left side. (dashed line) The limit due to a hydrostatic halo with thermal pressure included, with resulting sizes smaller than the observed size of the Fornax dSph to the right.
an ad hoc approach to include the effect of the critical velocity in the linear theory based on an estimate of the relative velocity between the superfluid and normal fluid components, which was found to provide the correct qualitative dependence on $M$ and $v_\text{c}$, though it failed at velocities below the smallest superfluid sound speed.

Finally, the superfluid dynamical friction was applied to the Fornax dSph. It was found that the relevant parameter space yields an orbital decay time for Fornax’s GCs much smaller than the age of the dSph, except for a small region preferred in the literature (Cruciuc & Harko 2020). The present work therefore suggests that the timing-problem of Fornax’s GCs is resolved in SIBEC-DM for the values of $g/m^3$ obtained by Cruciuc & Harko (2020) by fitting rotation curves at $T = 0$. For a finite-temperature SIBEC-DM, for which the preferred parameter space of $g/m^3$ is likely lowered, very large decay rates of Fornax’s GCs pose a problem.

The use of linear perturbation theory made it possible to probe a large region of parameter space that is difficult to explore numerically. The main limitations of the numerical scheme used in this work are: the low order of the Godunov scheme used; the absence of entropy production, both when the critical velocity was enforced, and in shock waves, which leads to the total energy not being strictly conserved; and the large difference between the superfluid sound speeds and dynamics, which results in very small time-stepping and hence excessive diffusion of the numerical solution. All these points limit the parameters for which we can be confident that the numerical solution is correct, and therefore limits the range the perturbation results can be tested. Ideally, the dynamical friction would have also been explored using simulations with realistic models for both the DM halo and perturber, as has been done for galaxies with standard CDM and gas (Chapon et al. 2013; Tamfal et al. 2020), but such a study also requires an improved scheme for solving the superfluid hydrodynamics equations.

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References

Andersen, J. O. 2004, Rev. Mod. Phys., 76, 599
Bar-Or, B., Fouvry, J.-B., & Tremaine, S. 2019, The Astrophysical Journal, 871, 28, publisher: American Astronomical Society
Barausse, E. 2007, Monthly Notices of the Royal Astronomical Society, 382, 826, publisher: Oxford Academic
Barenghi, C. F., Skrbek, L., & Sreenivasan, K. R. 2014, Proceedings of the National Academy of Sciences, 111, 4467
Berezhiani, L., Eldar, B., & Khoury, J. 2019, Journal of Cosmology and Astroparticle Physics, 2019, 074
Berezhiani, L., Khoury, J., & Ratra, P. 2015, Phys. Rev. D, 92, 104016
Binney, J. & Tremaine, S. 2008, Galactic Dynamics: Second Edition (Princeton Univ. Press)
Boldrini, P., Mohayaee, R., & Silk, J. 2019, Monthly Notices of the Royal Astronomical Society, 485, 2546, arXiv:1903.03854
Boylan-Kolchin, M., Ma, C.-P., & Quataert, E. 2008, Monthly Notices of the Royal Astronomical Society, 383, 93, publisher: Oxford Academic
Bullock, J. S. & Boylan-Kolchin, M. 2017, Annual Review of Astronomy and Astrophysics, 55, 343
Chandrasekhar, S. 1943, The Astrophysical Journal, 97, 255
Chapman, S., Hoyos, C., & Oz, Y. 2014, Journal of High Energy Physics, 2014, 27
Chapin, D., Mayer, L., & Tevssier, R. 2013, Monthly Notices of the Royal Astronomical Society, 429, 3114, publisher: Oxford Academic
Cole, D. R., Dehnen, W., Read, J. I., & Wilkinson, M. I. 2012, Monthly Notices of the Royal Astronomical Society, 426, 601
Colpi, M., Mayer, L., & Governato, F. 1999, The Astrophysical Journal, 525, 720, publisher: IOP Publishing
Cowsik, R., Wagoner, K., Berti, E., & Sircar, A. 2009, The Astrophysical Journal, 699, 1389
Cruciuc, M. & Harko, T. 2020, arXiv:2007.12222 [astro-ph, physics:gr-qc, physics:sh-ph], arXiv:2007.12222
Darve, C., Bottura, L., Patanak, N. A., & Van Sciver, S. 2012, AIP Conference Proceedings, 1434, 247
Debattista, V. P. & Sellwood, J. A. 2000, The Astrophysical Journal, 543, 704, publisher: IOP Publishing
Del Pino, A., Hidalgo, S. L., Aparicio, A., et al. 2013, Monthly Notices of the Royal Astronomical Society, 433, 1505, publisher: Oxford Academic
Del Popolo, A. & De Delliou, M. 2017, Galaxies, 5
Doi, D., Shirai, Y., & Shibsuto, M. 2008, AIP Conference Proceedings, 985, 648
Dosopoulou, F. & Antonnioni, F. 2017, The Astrophysical Journal, 840, 31, publisher: American Astronomical Society
Elbert, O. D., Bullock, J. S., Garrison-Kimmel, S., et al. 2015, Monthly Notices of the Royal Astronomical Society, 453, 29
Goerd, T., Moore, B., Read, J. I., Stadel, J., & Zemp, M. 2006, Monthly Notices of the Royal Astronomical Society, 368, 1073, publisher: Oxford Academic
Gómez, L. G. & Rueda, J. 2017, Physical Review D, 96, 063001, publisher: American Physical Society
Harko, T., Liang, P., Liang, S.-D., & Mocanu, G. 2015, Journal of Cosmology and Astroparticle Physics, 2015, 027, publisher: IOP Publishing
Harlow, T. & Mocanu, G. 2012, Physical Review D, 85, 084012
Hartman, S. T. H., Winther, H. A., & Mota, D. F. 2020, Astronomy & Astrophysics, 639, A90, publisher: EDP Sciences
Harvey, D., Massey, R., Kitching, T., Taylor, A., & Tittley, E. 2015, Science, 347, 1462
Hu, W., Barkana, R., & Gruzinov, A. 2000, Phys. Rev. Lett., 85, 1158
Hui, L., Ostriker, J. P., Tremaine, S., & Witten, E. 2017, Physical Review D, 95, 043541
Jiang, C. Y., Jing, Y. P., Faltenbacher, A., Lin, W. P., & Li, C. 2008, The Astrophysical Journal, 675, 1095, publisher: IOP Publishing
Just, A., Khan, F. M., Berecz, P., Ernst, A., & Spurzem, R. 2011, Monthly Notices of the Royal Astronomical Society, 411, 653, publisher: Oxford Academic
Katz, A., Kurkela, A., & Soloviev, A. 2019, Journal of Cosmology and Astroparticle Physics, 2019, 017, publisher: IOP Publishing
Kaur, K. & Sridhar, S. 2018, The Astrophysical Journal, 868, 134, publisher: American Astronomical Society
Khoury, J. 2016, Phys. Rev. D, 93, 103533
Lancaster, L., Giovanetti, C., Mocz, P., et al. 2020, Journal of Cosmology and Astroparticle Physics, 2020, 001, publisher: IOP Publishing
Landau, L. 1941, Phys. Rev., 60, 356
Lee, A. T. & Stahler, S. W. 2011, Monthly Notices of the Royal Astronomical Society, 416, 3177, publisher: Oxford Academic
Lee, A. T. & Stahler, S. W. 2014, Astronomy & Astrophysics, 561, A84, publisher: EDP Sciences
Lelli, F., McGaugh, S. S., & Schombert, J. M. 2016, The Astronomical Journal, 152, 157, publisher: American Astronomical Society
Mackey, A. D. & Gilmore, G. F. 2003, Monthly Notices of the Royal Astronomical Society, 340, 175, publisher: Oxford Academic
Madelung, E. 1926, Naturwissenschaften, 14, 1004
Nori, M. & Baldi, M. 2018, Monthly Notices of the Royal Astronomical Society, 453, 29
Nori, M. & Baldi, M. 2020, Monthly Notices of the Royal Astronomical Society, 497, 4559
Mulder, W. A. 1983, A&A, 117, 9
Navez, P. & Graham, R. 2006, Physical Review A, 73, 043612, publisher: American Physical Society
Oh, K. S., Lin, D. N. C., & Richer, H. B. 2000, The Astrophysical Journal, 531, 1462, publisher: IOP Publishing
Ostriker, E. C. 1999, The Astrophysical Journal, 513, 252, publisher: IOP Publishing
Pani, P. 2015, Physical Review D, 92, 123530, publisher: American Physical Society
Piatekiszski, I. P. & Stringani, S. 2016, Bose-Einstein Condensation and Superfluidity (Great Clarendon Street, Oxford, United Kingdom: Oxford University Press)
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2016, A&A, 594, A13
Pontzen, A. & Governato, F. 2012, Monthly Notices of the Royal Astronomical Society, 421, 3464, publisher: Oxford Academic
Read, J. I., Walker, M. G., & Steger, P. 2019, Monthly Notices of the Royal Astronomical Society, 484, 1401, publisher: Oxford Academic
A&A proofs: manuscript no. main

Riess, A. G., Macri, L. M., Hoffmann, S. L., et al. 2016, The Astrophysical Journal, 826, 56, publisher: American Astronomical Society

Sánchez-Salcedo, F. J. & Brandenburg, A. 1999, ApJ, 522, L35

Schive, H.-Y., Chueh, T., & Broadhurst, T. 2014, Nature Physics, 10, 496

Schwabe, B., Niemeyer, J. C., & Engels, J. F. 2016, Phys. Rev. D, 94, 043513

Sellwood, J. A. 2014, Reviews of Modern Physics, 86, 1, publisher: American Physical Society

Shadmehri, M. & Khajenabi, F. 2012, Monthly Notices of the Royal Astronomical Society, 424, 919, publisher: Oxford Academic

Sharma, A., Khoury, J., & Lubensky, T. 2019, Journal of Cosmology and Astroparticle Physics, 2019, 054

Sikivie, P. & Yang, Q. 2009, Physical Review Letters, 103, 111301

Skrbek, L. 2011, Journal of Physics: Conference Series, 318, 012004

Skrbek, L. & Sreenivasan, K. R. 2012, Physics of Fluids, 24, 011301

Slepian, Z. & Goodman, J. 2012, Monthly Notices of the Royal Astronomical Society, 427, 839, publisher: Oxford Academic

Soulaïne, C., Quintard, M., Baudouy, B., & Van Weelden, R. 2017, Physical Review Letters, 118, 074506

Spergel, D. N. & Steinhardt, P. J. 2000, Phys. Rev. Lett., 84, 3760

Strigari, L. E. 2018, Reports on Progress in Physics, 81, 056901, publisher: IOP Publishing

Sánchez-Salcedo, F. J. 2012, The Astrophysical Journal, 745, 135, publisher: IOP Publishing

Tamfal, T., Mayer, L., Quinn, T. R., et al. 2020, Revisiting dynamical friction: the role of global modes and local wakes

Taylor, E. & Giffen, A. 2005, Phys. Rev. A, 72, 8739

Tegmark, M., Blanton, M. R., Strauss, M. A., et al. 2004, The Astrophysical Journal, 606, 702

Thun, D., Kuper, R., Schmidt, F., & Kley, W. 2016, Astronomy & Astrophysics, 589, A10, publisher: EDP Sciences

Tulin, S. & Yu, H.-B. 2018, Physics Reports, 730, 1, dark matter self-interactions and small scale structure

Walker, M. G., Mateo, M., Olszewski, E. W., et al. 2009, The Astrophysical Journal, 704, 1274, publisher: IOP Publishing

Wang, M. Y., Boer, T. d., Pieres, A., et al. 2019, The Astrophysical Journal, 881, 118, publisher: American Astronomical Society

Weinberg, M. D. 1985, Monthly Notices of the Royal Astronomical Society, 213, 451, publisher: Oxford Academic