Analytic regularity for a singularly perturbed reaction-convection-diffusion boundary value problem with two small parameters

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October 31, 2023

Abstract

We consider a second order, two-point, singularly perturbed boundary value problem, of reaction-convection-diffusion type with two small parameters, and we obtain analytic regularity results for its solution, under the assumption of analytic input data. First, we establish classical differentiability bounds that are explicit in the order of differentiation and the singular perturbation parameters. Next, for small values of these parameters we show that the solution can be decomposed into a smooth part, boundary layers at the two endpoints, and a negligible remainder. Derivative estimates are obtained for each component of the solution, which again are explicit in the differentiation order and the singular perturbation parameters.

1 Introduction

Singularly perturbed problems, and the numerical approximation of their solution, have been studied extensively over the last few decades (see, e.g., the books [9], [10], [13] and the references therein). As is well known, a main difficulty in these problems is the presence of boundary layers in the solution, which appear due to fact that the limiting problem (i.e. when the
singular perturbation parameter(s) tend to 0), is of different order than the original one, and the ('extra') boundary conditions can only be satisfied if the solution varies rapidly in the vicinity of the boundary – hence the name boundary layers.

In most numerical methods, high order derivatives of the solution appear in the error estimates, hence one should have a clear picture of how these derivatives grow with respect to the singular perturbation parameter(s). For low order numerical methods, such as Finite Differences (FD) or the $h$ version of the Finite Element Method (FEM), derivatives up to order 3 are usually sufficient. For high order methods such as the $hp$ version of the FEM, derivatives of arbitrary order are needed, thus knowing how these behave with respect to the singular perturbation parameter(s) as well as the differentiation order, is necessary. Usually problems of convection-diffusion or reaction-diffusion type are studied separately and several researchers have proposed and analyzed numerical schemes for the robust approximation of their solution (see, e.g., [13] and the references therein). When there are two singular perturbation parameters present in the differential equation, the problem becomes reaction-convection-diffusion and the relationship between the parameters determines the ‘regime’ we are in (as shown in Table 1 ahead). In [3], the numerical solution to this problem was addressed, using the $h$ version of the FEM as well as appropriate finite differences (see also [1], [2], [4], [12], [14], [17], [18]). Our interest is in high order $hp$ FEM, hence we require information on all derivatives of the solution. In the present article we obtain information about the analytic regularity of the solution, using the method of asymptotic expansions (see also [5]), thus providing the tools for an $hp$ FEM for the approximation of such problems.

The rest of the paper is organized as follows: in Section 2 we present the model problem and the regularity of its solution in terms of classical differentiability. Section 3 contains the asymptotic expansion for the solution, under the assumption that the singular perturbation parameters are small enough. We consider all possible relationships between the singular perturbation parameters, and establish derivative bounds which are explicit in the differentiation order as well as the singular perturbation parameters. We also comment on the transition between the regimes, in the final subsection of Section 3. Finally, in Section 4 we summarize our conclusions.

With $I \subset \mathbb{R}$ an open, bounded interval with boundary $\partial I$ and measure $|I|$, we will denote by $C^k(I)$ the space of continuous functions on $I$ with continuous derivatives up to order $k$. We will use the usual Sobolev spaces $W^{k,m}(I)$...
of functions on $I$ with $0, 1, 2, \ldots, k$ generalized derivatives in $L^m(I)$, equipped with the norm and seminorm $\| \cdot \|_{k,m,I}$ and $| \cdot |_{k,m,I}$, respectively. When $m = 2$, we will write $H^k(I)$ instead of $W^{k,2}(I)$, and for the norm and seminorm, we will write $\| \cdot \|_{k,I}$ and $| \cdot |_{k,I}$, respectively. The usual $L^2(I)$ inner product will be denoted by $\langle \cdot, \cdot \rangle_I$, with the subscript omitted when there is no confusion. We will also use the space $H^1_0(I) = \{ u \in H^1(I) : u|_{\partial I} = 0 \}$.

The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted by $\| \cdot \|_{\infty,I}$. Finally, the notation “$a \lesssim b$” means “$a \leq Cb$” with $C$ being a generic positive constant, independent of any discretization or singular perturbation parameters.

2 The model problem and its regularity

We consider the following model problem (cf. [8]): Find $u$ such that

$$-\varepsilon_1 u''(x) + \varepsilon_2 b(x) u'(x) + c(x) u(x) = f(x), \quad x \in I = (0,1),$$

$$u(0) = u(1) = 0,$$  \quad (1)

where $0 < \varepsilon_1, \varepsilon_2 \leq 1$ are given parameters that can approach zero, and the functions $b, c, f$ are given and sufficiently smooth. In particular, we assume that they are analytic functions satisfying, for some positive constants $\gamma_f, \gamma_c, \gamma_b$ independent of $\varepsilon_1, \varepsilon_2$,

$$\| f^{(n)} \|_{\infty,I} \lesssim n! \gamma_f^n, \quad \| c^{(n)} \|_{\infty,I} \lesssim n! \gamma_c^n, \quad \| b^{(n)} \|_{\infty,I} \lesssim n! \gamma_b^n \quad \forall \ n = 0, 1, 2, \ldots.$$  \quad (3)

In addition, we assume that there exist constants $\beta, \gamma, \rho$, independent of $\varepsilon_1, \varepsilon_2$, such that $\forall \ x \in I = [0,1]$ there holds

$$b(x) \geq \beta > 0, \quad c(x) \geq \gamma > 0, \quad c(x) - \frac{\varepsilon_2}{2} b'(x) \geq \rho > 0.$$  \quad (4)

The solution to (1), (2) satisfies (see, e.g. [3])

$$\| u \|_{\infty,I} \lesssim 1.$$  \quad (5)

We would like to obtain a similar estimate for $u'$. This is achieved in the following.
Lemma 1. Let $u$ be the solution of (1), (2) and assume (3), (4) hold. Then

$$\|u'\|_{\infty, I} \lesssim \varepsilon^{-1}.$$  

Proof. The proof follows [7]. Let

$$A(x) = \frac{\varepsilon_2}{\varepsilon_1} \int_x^1 b(t) dt,$$

and note that $A(1) = 0$ and $A'(x) = -\frac{\varepsilon_2}{\varepsilon_1} b(x)$. Then, multiplying (1) by $e^{A(x)}$ and integrating from $x$ to 1, gives

$$-\varepsilon_1 u'(1) + \varepsilon_1 e^{A(x)} u'(x) + \int_x^1 e^{A(t)} c(t) u(t) dt = \int_x^1 e^{A(t)} f(t) dt.$$  

Multiplying by $\varepsilon^{-1}_1 e^{-A(x)}$ yields

$$u'(x) = e^{-A(x)} u'(1) - \frac{1}{\varepsilon_1} \int_x^1 e^{A(t)-A(x)} c(t) u(t) dt + \frac{1}{\varepsilon_1} \int_x^1 e^{A(t)-A(x)} f(t) dt. \quad (6)$$

Integrating from 0 to 1, we further get

$$0 = u'(1) \int_0^1 e^{-A(x)} dx - \frac{1}{\varepsilon_1} \int_0^1 \int_x^1 e^{A(t)-A(x)} [c(t) u(t) - f(t)] dt dx. \quad (7)$$

Since we wish to first estimate $u'(1)$, we need upper and lower bounds for $\int_0^1 e^{-A(x)} dx$. From (4) we have

$$\int_0^1 e^{-A(x)} dx \leq \int_0^1 \varepsilon_2^{\frac{\beta}{\varepsilon_1}} (1-x) dx \leq \frac{\varepsilon_1}{\varepsilon_2 \beta}. \quad (8)$$

Similarly,

$$\int_0^1 e^{-A(x)} dx \geq \int_0^1 e^{-\frac{\varepsilon_2}{\varepsilon_1} \|b\|_{\infty, I}(1-x)} dx = \frac{\varepsilon_1}{\varepsilon_2 \|b\|_{\infty, I}} \left( 1 - e^{-\frac{\varepsilon_2}{\varepsilon_1} \|b\|_{\infty, I}} \right). \quad (9)$$

Also, to estimate the remaining terms in (7), we consider

$$\frac{1}{\varepsilon_1} \int_0^1 \int_x^1 e^{A(t)-A(x)} dt dx = \frac{1}{\varepsilon_1} \int_0^1 \int_x^1 e^{A'(\zeta)(t-x)} dt dx,$$
for some $\zeta$ between $t$ and $x$. Hence,

$$\frac{1}{\varepsilon_1} \int_0^1 \int_x^1 e^{A(t)-A(x)} dt dx \leq \frac{1}{\varepsilon_1} \int_0^1 \int_x^1 e^{-\frac{\varepsilon_2}{\varepsilon_1} \beta(t-x)} dt dx \leq \frac{1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2}.$$  

Using (7)–(9), we get

$$|u'(1)| \lesssim \frac{1}{\varepsilon_2} \left[ \left( \|c\|_{\infty,I} \|u\|_{\infty,I} + \|f\|_{\infty,I} \right) \left( \frac{1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \right) \right] \lesssim \varepsilon_2 \frac{\|b\|_{\infty,I}}{\varepsilon_1} \left( 1 - e^{-\frac{\varepsilon_2}{\varepsilon_1} \|b\|_{\infty,I}} \right)^{-1} \left( \frac{1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \right) \lesssim \varepsilon_1^{-1}.$$  

Inserting this bound in (6) gives

$$|u'(x)| \lesssim \varepsilon_1^{-1} \left( \|c\|_{\infty,I} \|u\|_{\infty,I} + \|f\|_{\infty,I} \right) \int_x^1 e^{A(t)-A(x)} dt \lesssim \varepsilon_1^{-1} \left( \|c\|_{\infty,I} \|u\|_{\infty,I} + \|f\|_{\infty,I} \right) \int_x^1 e^{-\frac{\varepsilon_2}{\varepsilon_1} \beta(t-x)} dt \lesssim \varepsilon_1^{-1} \left( \frac{\varepsilon_1}{\varepsilon_2 \beta} \right) \lesssim \varepsilon_1^{-1},$$

as desired. \[\square\]

Using an inductive argument we are able to prove the following.

**Theorem 2.** Let $u$ be the solution of (1), (2), and assume $\varepsilon_1 \leq \varepsilon_2$. Then, there exist positive constants $C, K$, independent of $\varepsilon_1, \varepsilon_2$, and $u$, such that for $n = 0, 1, 2, ...$

$$\|u^{(n)}\|_{\infty,I} \leq CK^n \max \{n, \varepsilon_1^{-1}\}.$$  

**Proof.** The proof is by induction on $n$ and follows [5]. Equation (5) and Lemma [1] give the result for $n = 0$, so we assume it holds for $0 \leq \nu \leq n + 1$ and show that it holds for $n + 2$. Differentiating (1) $n$ times gives

$$-\varepsilon_1 u^{(n+2)} = f^{(n)} - \varepsilon_2 (bu')^{(n)} - (cu)^{(n)} = f^{(n)} - \sum_{\nu=0}^n \binom{n}{\nu} \left( \varepsilon_2 b^{(\nu)} u^{(n+1-\nu)} + c^{(\nu)} u^{(n-\nu)} \right).$$
By the induction hypothesis we have
\[ \varepsilon_1 \| u^{(n+2)} \|_{\infty,I} \leq \| f^{(n)} \|_{\infty,I} + \]
\[ + C \sum_{\nu=0}^{n} \binom{n}{\nu} \left[ \varepsilon_2 \gamma_b^\nu \nu! \max \{ n + 1 \nu, \varepsilon_1^{-1} \}^{n+1-\nu} + \right. \]
\[ + \gamma_c^\nu \nu! \max \{ n - \nu, \varepsilon_1^{-1} \}^{n-\nu} \].

Using the estimates below (which follow by standard considerations)
\[ \binom{n}{\nu} \nu! \max \{ n + 1 \nu, \varepsilon_1^{-1} \}^{n+1-\nu} \leq \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1}, \]
\[ \binom{n}{\nu} \nu! \max \{ n - \nu, \varepsilon_1^{-1} \}^{n-\nu} \leq \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1}, \]
\[ \| f^{(n)} \|_{\infty,I} \leq C \gamma_f^n n! \leq C \gamma_f^n \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1}, \]
we obtain
\[ \varepsilon_1 \| u^{(n+2)} \|_{\infty,I} \leq C \gamma_f^n \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1} + \]
\[ + C K^{n+2} \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1} \sum_{\nu=0}^{n} \left[ \frac{1}{K} \left( \frac{\gamma_b}{K} \right)^\nu + \frac{1}{K^2} \left( \frac{\gamma_c}{K} \right)^\nu \right] \]
\[ \leq C K^{n+2} \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1} \left[ \frac{1}{K^2} + \frac{1}{K} \left( 1 - \frac{\gamma_b}{K} \right) + \frac{1}{K^2} \left( 1 - \frac{\gamma_c}{K} \right) \right], \]
where we choose the constant \( K > \max \{ 1, \gamma_f, \gamma_b, \gamma_c \} \) such that the expression in brackets above is bounded by 1. Thus
\[ \varepsilon_1 \| u^{(n+2)} \|_{\infty,I} \leq C K^{n+2} \max \{ n + 1, \varepsilon_1^{-1} \}^{n+1}, \quad (10) \]
and dividing by \( \varepsilon_1 \) gives the desired result. \( \square \)

**Remark 3.** The above result only treats the case \( \varepsilon_1 \leq \varepsilon_2 \), since if \( \varepsilon_2 \) is much smaller than \( \varepsilon_1 \), then we have a ‘regular perturbation’ of reaction-diffusion type. If one considers the limiting case \( \varepsilon_2 = 0 \), then one sees that there are two boundary layers, one at each endpoint, of width \( O \left( \varepsilon_1^{1/2} \right) \). Hence, the result of Theorem 2 should read
\[ \| u^{(n)} \|_{\infty,I} \leq C K^n \max \{ n, \varepsilon_1^{-1/2} \}^n. \]
More details arise if one studies the structure of the solution to (1), which depends on the roots of the characteristic equation associated with the differential operator. For this reason, we let \( \lambda_0(x), \lambda_1(x) \) be the solutions of the characteristic equation and set

\[
\mu_0 = -\max_{x \in [0,1]} \lambda_0(x), \quad \mu_1 = \min_{x \in [0,1]} \lambda_1(x),
\]

or equivalently,

\[
\mu_{0,1} = \min_{x \in [0,1]} \frac{-\varepsilon_2 b(x) + \sqrt{\varepsilon_2^2 b^2(x) + 4\varepsilon_1 c(x)}}{2\varepsilon_1},
\]

with the minus sign associated with \( \mu_0 \) and the plus sign with \( \mu_1 \). The following hold true [14, 17]:

\[
1 \ll \mu_0 \leq \mu_1, \quad \frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_1^{1/2}} \lesssim \varepsilon_2 \mu_0 \lesssim 1, \quad \varepsilon_1^{-1/2} \mu_0 \lesssim 1, \quad \varepsilon_2^{-1}, \mu_0 \lesssim 1
\]

\[
\max \left\{ \epsilon_2 \mu_0, \varepsilon_1 \mu_1 \right\} \lesssim \varepsilon_1 + \varepsilon_2^{1/2}, \quad \varepsilon_2 \lesssim \varepsilon_1 \mu_1
\]

for \( \varepsilon_2 \geq \varepsilon_1 : \varepsilon_1^{-1/2} \lesssim \mu_1 \lesssim \varepsilon_2^{-1/2} \)

for \( \varepsilon_2 \leq \varepsilon_1 : \varepsilon_1^{-1/2} \lesssim \mu_1 \lesssim \varepsilon_2^{-1/2} \)

The values of \( \mu_0, \mu_1 \) determine the width of the boundary layers and since \( |\lambda_0(x)| < |\lambda_1(x)| \) the layer at \( x = 1 \) is stronger than the layer at \( x = 0 \). Essentially, there are three regimes [3]:

| Regime                        | \( \mu_0 \)   | \( \mu_1 \)   |
|-------------------------------|--------------|--------------|
| Convection-diffusion          | \( \varepsilon_1 \ll \varepsilon_2 = 1 \) | \( 1 \)         | \( \varepsilon_1^{-1} \) |
| Convection-reaction-diffusion | \( \varepsilon_1 \ll \varepsilon_2 \ll 1 \) | \( \varepsilon_2^{-1} \) | \( \varepsilon_2 / \varepsilon_1 \) |
| Reaction-diffusion            | \( 1 \gg \varepsilon_1 \gg \varepsilon_2 \) | \( \varepsilon_2^{-1/2} \) | \( \varepsilon_1^{-1/2} \) |

Table 1: Different regimes based on the relationship between \( \varepsilon_1 \) and \( \varepsilon_2 \) [3].

It was shown in [3] (see also [14]) that under the assumptions \( b, c, f \in C^q(I) \) for some \( q \geq 1 \) and \( \frac{\varepsilon_2}{2} q \|b'\|_{\infty,I} \lesssim (1 - \ell) \) for some \( \ell \in (0, 1) \), the solution \( u \) to (1), (2) can be decomposed into a smooth part \( S \), a boundary layer part at the left endpoint \( E_0 \) and a boundary layer part at the right endpoint \( E_1 \), viz.

\[
u = S + E_0 + E_1.
\]
with

\[ |S^{(n)}(x)| \lesssim 1, \quad |E_0^{(n)}(x)| \lesssim \mu_0^n e^{-\ell \mu_0 x}, \quad |E_1^{(n)}(x)| \lesssim \mu_1^n e^{-\ell \mu_1 (1-x)}, \]  

for all \( x \in \Omega \) and for \( n = 0, 1, 2, \ldots, q \). This regularity result is sufficient for proving convergence of a fixed order \( h \) FEM, but not for an \( hp \) FEM – a more refined regularity result is needed for the smooth part that shows how the derivatives grow, with respect to the differentiation order (cf. eq. (32) ahead).

The above considerations suggest the following: if \( \varepsilon_1 \) is small compared to \( \varepsilon_2 \), then it is instructive to consider the limiting case \( \varepsilon_1 = 0 \). There is an exponential layer (of width \( O(\varepsilon_2) \)) at the left endpoint. The homogeneous equation (with constant coefficients) suggests that the different regimes are \( \varepsilon_1 \ll \varepsilon_2, \varepsilon_1 \approx \varepsilon_2 \) and \( \varepsilon_1 \gg \varepsilon_2 \), as discussed below:

1. In the regime \( \varepsilon_1 \ll \varepsilon_2 \), we have \( \mu_0 = O(\varepsilon_2^{-1}) \) and \( \mu_1 = O(\varepsilon_2 \varepsilon_1^{-1}) \). Hence \( \mu_1 \) is much larger than \( \mu_0 \) and the boundary layer in the vicinity of \( x = 1 \) is stronger. Consequently, there is a layer of width \( O(\varepsilon_2) \) at the left endpoint (the one that arises from the analysis of the case \( \varepsilon_1 = 0 \)) and additionally, there is another layer at the right endpoint, of width \( O(\varepsilon_1 / \varepsilon_2) \).

2. In the regime \( \varepsilon_1 \approx \varepsilon_2^2 \) there are layers at both endpoints of width \( O(\varepsilon_2) = O\left(\varepsilon_1^{1/2}\right)\).

3. In the regime \( \varepsilon_2^2 \ll \varepsilon_1 \ll 1 \), there are layers at both endpoints of width \( O\left(\varepsilon_1^{1/2}\right)\).

The above information will be utilized in obtaining regularity estimates for the solution in all three regimes.

3 The asymptotic expansion

We elaborate on (1)–(3) above, and choose an appropriate asymptotic expansion for \( u \), in what follows.

The proofs of each result in the subsequent sections are very similar, hence we will provide the details for Section 3.1 and omit certain proofs in Sections 3.2 and 3.3.
3.1 The regime \( \varepsilon_1 \ll \varepsilon_2 \ll 1 \)

In this case we anticipate a layer of width \( O(\varepsilon_2) \) at the left endpoint and a layer of width \( O(\varepsilon_1/\varepsilon_2) \) at the right endpoint. To deal with this we define the stretched variables \( \tilde{x} = x/\varepsilon_2 \) and \( \hat{x} = (1 - x)\varepsilon_2/\varepsilon_1 \), in order for the differentiation operator to produce the necessary powers of \( \varepsilon \), that yield a balanced (in \( \varepsilon_1, \varepsilon_2 \)) equation.

Since we wish to stay along the lines of (13), we want the solution to be comprised of a smooth part (in the slow variable \( x \)), and two boundary layers (in the fast variables \( \tilde{x}, \hat{x} \)). Hence, we make the formal ansatz

\[
 u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon_2^i (\varepsilon_1/\varepsilon_2)^j \left( u_{i,j}(x) + \tilde{u}_{i,j}^BL(\tilde{x}) + \hat{u}_{i,j}^BL(\hat{x}) \right),
\]

with \( u_{i,j}, \tilde{u}_{i,j}^BL, \hat{u}_{i,j}^BL \) to be determined. Substituting (15) into (1), separating the slow and fast variables, and equating like powers of \( \varepsilon_1 \) and \( \varepsilon_2 \), we get (see [15] for the details)

\[
\begin{align*}
 u_{0,0}(x) &= \frac{f(x)}{c(x)} \\
 u_{i,0}(x) &= -\frac{b(x)}{c(x)} u_{i-1,0}(x), \quad i \geq 1 \\
 u_{i,j}(x) &= u_{1,j}(x) = 0, \quad j \geq 1
\end{align*}
\]

(16),

\[
\begin{align*}
 \tilde{b}_0 \left( \tilde{u}_{0,0}^BL \right)' + \tilde{c}_0 \tilde{u}_{0,0}^BL &= 0 \\
 \tilde{b}_0 \left( \tilde{u}_{i,0}^BL \right)' + \tilde{c}_0 \tilde{u}_{i,0}^BL &= -\sum_{k=1}^{i-1} \left( \tilde{b}_k \left( \tilde{u}_{k-1,0}^BL \right)' + \tilde{c}_k \tilde{u}_{k-1,0}^BL \right), \quad i \geq 1 \\
 \hat{b}_0 \left( \hat{u}_{i,j}^BL \right)' + \hat{c}_0 \hat{u}_{i,j}^BL &= \sum_{k=1}^{j} \left( \hat{b}_k \left( \hat{u}_{i,k-1,j}^BL \right)' + \hat{c}_k \hat{u}_{i,k-1,j}^BL \right), \quad i \geq 1, \quad j \geq 1
\end{align*}
\]

(17),

\[
\begin{align*}
 \left( \tilde{u}_{i,0}^BL \right)' + \hat{b}_0 \left( \tilde{u}_{i,0}^BL \right)' &= 0, \quad i \geq 0 \\
 \left( \tilde{u}_{0,j}^BL \right)' + \hat{b}_0 \left( \tilde{u}_{0,j}^BL \right)' &= \tilde{c}_0 \tilde{u}_{0,j-1}^BL, \quad j \geq 1 \\
 \left( \tilde{u}_{i,1}^BL \right)' + \hat{b}_0 \left( \tilde{u}_{i,1}^BL \right)' &= \tilde{c}_0 \tilde{u}_{i,0}^BL - \hat{b}_1 \left( \tilde{u}_{i-1,0}^BL \right)', \quad i \geq 1 \\
 \left( \hat{u}_{i,j}^BL \right)' + \hat{b}_0 \left( \hat{u}_{i,j}^BL \right)' &= \hat{c}_0 \hat{u}_{i,j-1}^BL - \hat{b}_j \left( \hat{u}_{i-j,0}^BL \right) + \hat{b}_j \left( \hat{u}_{i-j,0}^BL \right) + \hat{c}_0 \hat{u}_{i,j-1}^BL - \hat{c}_j \hat{u}_{i,j-1}^BL + \hat{c}_j \hat{u}_{i,j-1}^BL + \sum_{k=1}^{j-1} \left( \hat{b}_k \left( \hat{u}_{i-k,j-k}^BL \right)' + \hat{c}_k \hat{u}_{i-k,j-k}^BL \right), \quad i \geq 2, \quad j = 2, \ldots, i \\
 \left( \hat{u}_{i,0}^BL \right)' + \hat{b}_0 \left( \hat{u}_{i,0}^BL \right)' &= \hat{c}_0 \hat{u}_{i,1}^BL + \sum_{k=1}^{i-1} \left( \hat{b}_k \left( \hat{u}_{i-k,j-k}^BL \right)' + \hat{c}_k \hat{u}_{i-k,j-k}^BL \right), \quad i \geq 2, \quad j > i
\end{align*}
\]

(18)
where the notation \( \tilde{b}_k(\tilde{x}) = \tilde{x}^k b^{(k)}(0)/k! \), \( \hat{b}_k(\hat{x}) = (-1)^k \tilde{x}^k b^{(k)}(1)/k! \) is used, and analogously for the other terms. (We also adopt the convention that empty sums are 0.) The BVPs (17)–(18) are supplemented with the following boundary conditions (in order for (2) to be satisfied) for all \( i, j \geq 0 \):

\[
\tilde{u}_{i,j}^{BL}(0) = -u_{i,j}(0), \quad \lim_{\tilde{x} \to \infty} \tilde{u}_{i,j}^{BL}(\tilde{x}) = 0
\]

\[
\hat{u}_{i,j}^{BL}(0) = -u_{i,j}(1), \quad \lim_{\hat{x} \to \infty} \hat{u}_{i,j}^{BL}(\hat{x}) = 0.
\]

(19)

Next, we describe the regularity of the functions \( u_{i,j}, \tilde{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL} \), defined by (16)–(19) above. We begin with \( u_{i,j} \), and we have the following.

**Lemma 4.** Let \( u_{i,j} \) be defined by (16) and assume (3) holds. Then there exist positive constants \( C, K \) and a complex neighborhood \( G \) of \( I \) such that the complex extension of \( u \) (denoted again by \( u \)) satisfies

\[
|u_{i,j}(z)| \leq C \delta^{-i} K^i i^i \quad \forall \ z \in G_\delta = \{ z \in G : \text{dist}(z, \partial G) > \delta \}.
\]

**Proof.** The proof is by induction on \( i \). The case \( i = 0 \) holds trivially, so assume the result holds for \( i \) and establish it for \( i + 1 \). Let \( \kappa \in (0,1) \) and let \( K > 0 \) be a constant so that \( \frac{2}{K^2 + \frac{1}{K}} \leq 1 \). We have by (16), the induction hypothesis with \( G_{(1-\kappa)\delta} \supset G_\delta \), and Cauchy’s Integral Theorem for Derivatives (we take as contour a circle of radius \( \kappa \delta \) about \( z_0 \in G_\delta \)),

\[
|u_{i+1,j}(z)| \leq C \left\{ |u''_{i-1,j-1}(z)| + |u'_{i,j}(z)| \right\}
\]

\[
\leq C \left\{ \frac{2}{(\kappa \delta)^2} ((1-\kappa)\delta)^{-i+1} K^{i-1}(i-1)^{i-1} + \frac{1}{(\kappa \delta)} ((1-\kappa)\delta)^{-i} K^i i^i \right\}
\]

\[
\leq C \delta^{-i-1} K^{i+1}(i+1)^{i+1} \left\{ \frac{1}{K^2} \frac{1}{(i+1)^2} \frac{2}{\kappa^2 (1-\kappa)^{-i-1}} \left( \frac{i-1}{i+1} \right)^{i-1} + \frac{1}{K} \frac{1}{i+1} \frac{1}{\kappa (1-\kappa)^i} \left( \frac{i}{i+1} \right)^i \right\}.
\]

Choose \( \kappa = 1/(i+1) \). Then we get

\[
|u_{i+1,j}(z)| \leq C \delta^{-i-1} K^{i+1}(i+1)^{i+1} \left[ \frac{2}{K^2} + \frac{1}{K} \right],
\]

so by the choice of \( K \) the expression in brackets is bounded by 1 and this completes the proof. \( \square \)
Lemma 5. Let \( u_{i,j} \) be defined by (16) and assume (3) holds. Then there exist positive constants \( K_1, K_2 \) such that

\[
\|u_{i,j}^{(n)}\|_{\infty,i} \lesssim n! K_1^n i! K_2^i \forall n \in \mathbb{N}.
\]

Proof. This follows immediately from Lemma 4 and Cauchy’s Integral Theorem for derivatives:

\[
\|u_{i,j}^{(n)}\|_{\infty,i} \lesssim \frac{n!}{(n+1)^n} \delta^{-i} K_1^i e^n \lesssim n! K_1^n i! K_2^i,
\]

with \( K_1 = e, K_2 = K/\delta \).

In order to treat the layer terms \( \tilde{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL} \), we will develop some auxiliary results. The following one will be used in the proof of Lemma 7, and is an analog of Lemma 7.3.6 in [5] (see also Proposition 13 ahead).

Lemma 6. Let \( \lambda, \gamma \in \mathbb{C} \) with \( \text{Re}(\lambda) > 0, \text{Re}(\gamma) > 0 \), and let \( \alpha_1, \alpha_2 \in \mathbb{R}^+ \). Suppose \( F \) is an entire function satisfying, for some \( C_F > 0 \), \( i,j \in \mathbb{N}_0 \),

\[
|F(z)| \leq C_F \gamma^{i+j} e^{-\text{Re}(\lambda z)} (\alpha_1 i + \alpha_2 j + |z|)^{\alpha_1 i + \alpha_2 j} \forall z \in \mathbb{C},
\]

and let \( v_0 \in \mathbb{C} \). Then, the solution \( v : (0, \infty) \to \mathbb{C} \), of the problem

\[
v' + \lambda v = F \text{ on } (0, \infty), \ v(0) = v_0,
\]

can be extended to an entire function (denoted again by \( v \)), which satisfies

\[
|v(z)| \leq \left[ \frac{C_F}{|\lambda|} \gamma^{i+j} (\alpha_1 i + \alpha_2 j + |z|)^{\alpha_1 i + \alpha_2 j + 1} + |v_0| \right] e^{-\text{Re}(\lambda z)} \forall z \in \mathbb{C}.
\]

Proof. Using an integrating factor we find

\[
v(z) = e^{-\lambda z} \left[ v_0 + \int_0^{|z|} e^{\lambda s} F(s) ds \right],
\]

from which we get

\[
|v(z)| \leq e^{-\text{Re}(\lambda z)} \left[ |v_0| + \int_0^{|z|} |e^{\text{Re}(\lambda s)} F(s)| ds \right]
\]

\[
\lesssim e^{-\text{Re}(\lambda z)} \left[ |v_0| + \frac{C_F}{|\lambda|} \gamma^{i+j} \int_0^{|z|} (\alpha_1 i + \alpha_2 j + |s|)^{\alpha_1 i + \alpha_2 j} ds \right],
\]

where we used the assumption on \( F \). The result follows. \( \square \)
Lemma 7. The functions $\tilde{u}_{i,j}^{BL}$ which satisfy (17), (19), are entire and there exist positive constants $C, \tilde{\gamma}$ such that

$$\left| \left( \tilde{u}_{i,j}^{BL} \right)(z) \right| \leq C \frac{\tilde{\gamma}^{i+j}}{i!} (2i+j+|z|)^{2i+j} e^{-\beta \Re(z)}, \quad z \in \mathbb{C}, \quad \Re(z) > 0,$$

where $\beta = \tilde{c}_0 / \tilde{b}_0$.

Proof. We recall that $\tilde{b}_k(x) = \tilde{x}^k b^{(k)}(0)/k!$ and $\tilde{c}_k(x) = \tilde{x}^k c^{(k)}(0)/k!$. Consequently, there exist positive constants $C_b, \gamma_b, C_c, \gamma_c$, depending solely on $b, c$ such that

$$\left| \tilde{b}_k(z) \right| \leq C_b \gamma_b^k |z|^k, \quad |\tilde{c}_k(z)| \leq C_c \gamma_c^k |z|^k.$$

Then, with $K_2$ the constant from Lemma 5, and $\gamma_b, \gamma_c$ given by (21), we choose $\tilde{\gamma} > \max\{K_2, \gamma_b, \gamma_c\}$ so that

$$\left[ \frac{\gamma_b / \tilde{\gamma}}{1 - \gamma_b / \tilde{\gamma}} + \frac{\gamma_c / \tilde{\gamma}}{1 - \gamma_c / \tilde{\gamma}} \right] < 1.$$

Next, we note that from (17) we may calculate $\tilde{u}_{0,0}^{BL}(z) = -u_{0,0}(0)e^{-\frac{\tilde{c}_0}{\tilde{b}_0}z}$. Thus, using Lemma 5 to bound the term $|u_{0,0}(0)|$, we get

$$|\tilde{u}_{0,0}^{BL}(z)| \leq C e^{-\frac{\tilde{c}_0}{\tilde{b}_0}z} \leq C e^{-\beta \Re(z)}, \quad \beta = \tilde{c}_0 / \tilde{b}_0,$$

thus the claim holds for $i, j = 0$. For $j = 0, i > 0$, we proceed with induction on $i$, while keeping $j$ fixed at 0. We have shown the desired result for the case $i = 0$, so we assume it holds for $i > 0$ and we will establish it for $i + 1$.

The function $\tilde{u}_{i+1,0}^{BL}$ satisfies

$$\tilde{b}_0 \left( \tilde{u}_{i+1,0}^{BL} \right)' + \tilde{c}_0 \tilde{u}_{i+1,0}^{BL} = - \sum_{k=1}^{i+1} \left( \tilde{b}_k \left( \tilde{u}_{i+1-k,0}^{BL} \right)' + \tilde{c}_k \tilde{u}_{i+1-k,0}^{BL} \right) =: G_1,$$

as well as $\tilde{u}_{i+1,0}^{BL}(0) = -u_{i+1,0}(0)$. In order to use Lemma 6 we bound the right hand side above as follows:

$$|G_1(z)| \leq \sum_{k=1}^{i+1} \left[ \left| \tilde{b}_k \right| \left( \tilde{u}_{i+1-k,0}^{BL} \right)' + \left| \tilde{c}_k \right| \left( \tilde{u}_{i+1-k,0}^{BL} \right) \right]\left| z \right|^k \leq C \sum_{k=1}^{i+1} \left| z \right|^k \left( \left| \frac{\gamma_b}{\tilde{\gamma}} \tilde{u}_{i+1-k,0}^{BL} \right| + \left| \frac{\gamma_c}{\tilde{\gamma}} \tilde{u}_{i+1-k,0}^{BL} \right| \right),$$

$$\leq C \sum_{k=1}^{i+1} \left| z \right|^k \left( \left| \frac{\gamma_b}{\tilde{\gamma}} \tilde{u}_{i+1-k,0}^{BL} \right| + \left| \frac{\gamma_c}{\tilde{\gamma}} \tilde{u}_{i+1-k,0}^{BL} \right| \right).$$
where we used (21). Cauchy’s Integral Theorem for Derivatives and the induction hypothesis yield

\[
|G_1(z)| \leq C \sum_{k=1}^{i+1} e^{-\beta \Re(z)} |z|^k \left( \gamma_b^k + \gamma_c^k \right) \frac{\tilde{\gamma}^{i+1-k}}{(i+1-k)!} \left( 2(i+1-k) + |z| \right)^{2(i+1-k)}
\]

\[
\leq C e^{-\beta \Re(z)} \frac{\tilde{\gamma}^{i+1}}{i!} \left( 2i + |z| \right)^{2i+1} \sum_{k=1}^{\infty} \left[ \left( \frac{\gamma_b}{\tilde{\gamma}} \right)^k + \left( \frac{\gamma_c}{\tilde{\gamma}} \right)^k \right]
\]

\[
\leq C e^{-\beta \Re(z)} \frac{\tilde{\gamma}^{i+1}}{i!} \left( 2i + |z| \right)^{2i+1},
\]

since the geometric series converges to a quantity bounded by 1, by the choice of \(\tilde{\gamma}\), see eq. (22). Then, Lemma 6 yields

\[
|\tilde{u}_{i+1,0}^{BL}(z)| \leq C \tilde{\gamma}^{i+1} e^{-\beta \Re(z)} \left( \frac{(2(i+1) + |z|)^{2(i+1)}}{(i+1)!} + \frac{|u_{i+1,0}(0)|}{\tilde{\gamma}^{i+1}} \right).
\]

Lemma 5, the choice of \(\tilde{\gamma}\), and Stirling’s formula, further give

\[
|\tilde{u}_{i+1,0}^{BL}(z)| \leq C \tilde{\gamma}^{i+1} e^{-\beta \Re(z)} \left( \frac{(2(i+1) + |z|)^{2(i+1)}}{(i+1)!} + (i+1)^{i+1} \right)
\]

\[
\leq C \tilde{\gamma}^{i+1} e^{-\beta \Re(z)} \frac{(2(i+1) + |z|)^{2(i+1)}}{(i+1)!} \left[ 1 + \frac{(i+1)^{2(i+1)}}{(2(i+1) + |z|)^{2(i+1)}} \right]
\]

\[
\leq C \tilde{\gamma}^{i+1} e^{-\beta \Re(z)} \frac{(2(i+1) + |z|)^{2(i+1)}}{(i+1)!}.
\]

This completes the induction on \(i > 0\) (with \(j = 0\)).

We next consider the case \(i = 0, j > 0\). Assuming

\[
|\tilde{u}_{0,j}^{BL}(z)| \leq C \tilde{\gamma}^j (j + |z|)^j e^{-\beta \Re(z)},
\]

we will establish it for \(j + 1\). The function \(\tilde{u}_{0,j+1}^{BL}(z)\) satisfies for \(j > 0\),

\[
\tilde{b}_0 (\tilde{u}_{0,j+1}^{BL})' + \tilde{c}_0 \tilde{u}_{0,j+1}^{BL} = (\tilde{u}_{0,j+1}^{BL})''.
\]

By Cauchy’s Integral Theorem for Derivatives and the induction hypothesis, we have

\[
\left| (\tilde{u}_{0,j}^{BL}(z))^'' \right| \leq C \tilde{\gamma}^j (j + |z|)^j e^{-\beta \Re(z)},
\]
and by Lemmata 5, 6

\[ \left| (\tilde{u}_{0,j+1}^{BL}(z)) \right| \leq C \tilde{\gamma}^j e^{-\beta \text{Re}(z)} \left\{ \frac{(j + |z|)^{j+1}}{j+1} + \frac{|u_{0,j+1}|}{\tilde{\gamma}^j} \right\} \]

\[ \leq C \tilde{\gamma}^{j+1} e^{-\beta \text{Re}(z)} (j + 1 + |z|)^{j+1} \times \left\{ \frac{(j + |z|)^{j+1}}{\tilde{\gamma}(j + 1)(j + 1 + |z|)^{j+1}} + \frac{\tilde{C}}{\tilde{\gamma}^{j+1}(j + 1 + |z|)^{j+1}} \right\} \]

\[ \leq C \tilde{\gamma}^{j+1} e^{-\beta \text{Re}(z)} (j + 1 + |z|)^{j+1}. \]

This establishes the result for \( i = 0, j > 0. \)

We finally show the case \( i, j > 0. \) We perform induction on \( i > 0, \) while keeping \( j \) fixed (but arbitrary). We assume (20) holds for \( i \geq 1 \) and show it for \( i + 1. \) We note that by (17), \( \tilde{u}_{i+1,j}^{BL} \) satisfies

\[ \tilde{b}_0 (\tilde{u}_{i+1,j}^{BL})' + \tilde{c}_0 \tilde{u}_{i+1,j}^{BL} = (\tilde{u}_{i+1,j-1}^{BL})'' - \sum_{k=1}^{i+1} \left( \tilde{b}_k (\tilde{u}_{i+1,k}^{BL})' + \tilde{c}_k \tilde{u}_{i+1,k}^{BL} \right) =: G_2, \]

as well as \( \tilde{u}_{i+1,j}^{BL}(0) = -u_{i+1,j}(0). \) We bound \( G_2 \) using Cauchy’s Integral Theorem for Derivatives, (22), and the induction hypothesis:

\[ |G_2| \leq \left| (\tilde{u}_{i+1,j-1}^{BL})'' \right| + \sum_{k=1}^{i+1} \left\{ |\tilde{b}_k| \left| (\tilde{u}_{i+1,k-1}^{BL})' \right| + |\tilde{c}_k| \left| \tilde{u}_{i+1,k-1}^{BL} \right| \right\} \]

\[ \leq C e^{-\beta \text{Re}(z)} \tilde{\gamma}^{j+1} \frac{(2(i + 1) + j - 1 + |z|)^{2(i+1)+j-1}}{(i+1)!} + \]

\[ + C e^{-\beta \text{Re}(z)} \tilde{\gamma}^{j+1+2} \sum_{k=1}^{i+1} \frac{(2(i + 1 - k) + j + |z|)^{2(i+1)-k+j}}{((i+1-k)!)^2} \left[ \left( \frac{\tilde{\gamma}_0}{\tilde{\gamma}} \right)^k + \left( \frac{\gamma_e}{\tilde{\gamma}} \right)^k \right] \]

\[ \leq C e^{-\beta \text{Re}(z)} \tilde{\gamma}^{j+1+2} \frac{(2i + 1 + j + |z|)^{2i+1+j}}{i!}, \]

where we argued in a similar fashion as we did for \( G_1. \) Lemmata 5, 6 give

\[ |\tilde{u}_{i+1,j}^{BL}| \leq C e^{-\beta \text{Re}(z)} \tilde{\gamma}^{j+1+2} \left\{ \frac{(2(i + 1) + j + |z|)^{2i+2+j}}{(i+1)!} + \frac{|u_{i+1,j}(0)|}{\tilde{\gamma}^{i+1+j}} \right\} \]

\[ \leq C e^{-\beta \text{Re}(z)} \tilde{\gamma}^{j+1+2} \left\{ \frac{(2(i + 1) + j + |z|)^{2(i+1)+j}}{(i+1)!} + \frac{(i + 1)^{i+1}}{\tilde{\gamma}^{j}} \right\} \]

\[ \leq C e^{-\beta \text{Re}(z)} \tilde{\gamma}^{j+1+2} \frac{(2i + 1 + j + |z|)^{2(i+1)+j}}{(i+1)!}. \]
This completes the proof.

For the other layer term \( \hat{u}_{i,j}^{BL} \), we have a similar result.

**Lemma 8.** The functions \( \hat{u}_{i,j}^{BL} \) which satisfy (18), (19), are entire and there exist positive constants \( C, \hat{\gamma} \) such that

\[
\left| \left( \hat{u}_{i,j}^{BL} \right)(z) \right| \leq C \frac{\hat{\gamma}^{i+j}}{j!} (i + 2j + |z|)^{i+2j} e^{-\beta \text{Re}(z)} \quad , \quad z \in \mathbb{C}, \text{Re}(z) > 0
\]  

where \( \beta = \hat{b}_0 \).

**Proof.** The proof is very similar to that of Lemma 7, utilizing Lemmata 5 and 6 and Cauchy’s Integral Theorem for Derivatives. The details appear in [15]. □

Using the previous two results, we obtain the following.

**Lemma 9.** Let the functions \( \bar{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL} \) satisfy (17), (18) respectively. Then, there exist positive constants \( \tilde{C}, \tilde{\gamma}, \hat{C}, \hat{\gamma}, \tilde{\beta}, \hat{\beta} \), depending only on the data, such that \( \forall \ n \in \mathbb{N} \),

\[
\left| \left( \bar{u}_{i,j}^{BL} \right)^{(n)}(z) \right| \leq \tilde{C} \tilde{\gamma}^{i+j} \frac{(2i + j)^{2i+j}}{i!} e^{-\tilde{\beta} \text{Re}(z)} \quad , \quad z \in \mathbb{C}, \text{Re}(z) > 0,
\]  

\[
\left| \left( \hat{u}_{i,j}^{BL} \right)^{(n)}(z) \right| \leq \hat{C} \hat{\gamma}^{i+j} \frac{(i + 2j)^{i+2j}}{j!} e^{-\hat{\beta} \text{Re}(z)} \quad , \quad z \in \mathbb{C}, \text{Re}(z) > 0.
\]

where \( \tilde{\beta} = \tilde{c}_0 / \tilde{b}_0, \hat{\beta} = \hat{b}_0 \).

**Proof.** We will prove (24), since (25) is similar. Cauchy’s Integral Theorem for Derivatives allows us to infer (24) from (20) as follows:

\[
\left| \left( \bar{u}_{i,j}^{BL} \right)^{(n)}(z) \right| \leq \tilde{C} e^{-\tilde{\beta} \text{Re}(z)} \frac{n! \tilde{\gamma}^{i+j}}{(n+1)^n} \frac{(2i + j + |z|)^{2i+j}}{i!} e^n
\]

\[
\leq \tilde{C} e^{-\tilde{\beta} \text{Re}(z)} \frac{n! \tilde{\gamma}^{i+j}}{(n+1)^n} \frac{(2i + j + n)^{2i+j}}{i!} e^n.
\]

Observing that

\[(2i + j + n)^{2i+j} = (2i + j)^{2i+j} (1 + n/(2i + j))^{2i+j} \leq (2i + j)^{2i+j} e^n, \]  

the result follows. □
We now define, for some $M \in \mathbb{N}$,

$$u_M(x) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_1^i (\varepsilon_1/\varepsilon_2^2)^j u_{i,j}(x), \quad (27)$$

$$\tilde{u}^{BL}_M(\tilde{x}) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j \tilde{u}_{i,j}^{BL}(\tilde{x}), \quad (28)$$

$$\hat{u}^{BL}_M(\hat{x}) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_1^i (\varepsilon_1/\varepsilon_2^2)^j \hat{u}_{i,j}^{BL}(\hat{x}), \quad (29)$$

$$r_M = u - (u_M + \tilde{u}^{BL}_M + \hat{u}^{BL}_M) \quad (30)$$

and we have the following decomposition

$$u = u_M + \tilde{u}^{BL}_M + \hat{u}^{BL}_M + r_M. \quad (31)$$

As the following theorem shows, the estimates on the smooth part $u_M$ in (31), explicitly show the dependence on the differentiation order. Moreover, (32) shows that the smooth part is (real) analytic, hence a high order numerical method could produce exponential rates of convergence (see, e.g. [5]).

**Theorem 10.** Assume (3), (4) hold, and that $\varepsilon_1 \ll \varepsilon_2^2$. Then there exist positive constants $K_1, K_2, K, \tilde{\gamma}, \hat{\gamma}$, independent of $\varepsilon_1, \varepsilon_2$, such that the solution $u$ of (1)–(2) can be decomposed as in (31), with

$$\left\| u_M^{(n)} \right\|_{\infty, I} \lesssim n! K_1^n \quad \forall \ n \in \mathbb{N}_0, \quad (32)$$

$$\left| (\tilde{u}^{BL}_M)^{(n)}(x) \right| \lesssim \tilde{K}^n \varepsilon_2^{-n} e^{-\beta x/\varepsilon_2} \quad \forall \ n \in \mathbb{N}_0, \quad (33)$$

$$\left| (\hat{u}^{BL}_M)^{(n)}(x) \right| \lesssim \hat{K}^n \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-n} e^{-\beta(1-x)\varepsilon_2/\varepsilon_1} \quad \forall \ n \in \mathbb{N}_0, \quad (34)$$

$$\left\| r_M \right\|_{\infty, \partial I} + \left\| r_M \right\|_{0, I} + \varepsilon_1^{1/2} \left\| r_M' \right\|_{0, I} \lesssim \max\{ e^{-\beta \varepsilon_2/\varepsilon_1}, e^{-\beta/\varepsilon_2} \}, \quad (35)$$

where $M$ is chosen so that $\varepsilon_2 e^{2M} \max\{1, K_2, \tilde{\gamma}, \hat{\gamma}\} < 1$ and $\varepsilon_1 e^{2M} \max\{1, \tilde{\gamma}, \hat{\gamma}\} < 1$. The constant $\beta$ is given by $\beta = \min\left\{ \frac{\tilde{c}_0}{\tilde{b}_0}, \frac{h_0}{\hat{b}_0} \right\}$.  

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Proof. We first show (32): from (27) and Lemma 5 we have

\[ \left\| u_M^{(n)} \right\|_{\infty,I} \leq M \sum_{i=0}^{M} M \sum_{j=0}^{M} \varepsilon_i^2 (\varepsilon_1 / \varepsilon_2)^j \left\| u_i^{(n)} \right\|_{\infty,I} \leq M \sum_{i=0}^{M} M \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1 / \varepsilon_2)^j n!K_1^n i!K_2^i \]

\[ \lesssim n!K_1^n \left( \sum_{i=0}^{M} \varepsilon_2^i i!K_2^i \right) \left( \sum_{j=0}^{M} (\varepsilon_1 / \varepsilon_2)^j \right) \]

\[ \lesssim n!K_1^n \left( \sum_{i=0}^{\infty} (\varepsilon_2 MK_2)^i \right) \left( \sum_{j=0}^{\infty} (\varepsilon_1 / \varepsilon_2)^j \right) \]

\[ \lesssim n!K_1^n , \]

since both sums are convergent geometric series due to the assumptions \( \varepsilon_2 MK_2 < 1 \) and \( \varepsilon_1 / \varepsilon_2^2 < 1 \).

Next we show (33): By (28) and Lemma 9, we have

\[ \left| \left( \bar{u}_M^{BL} \right)^{(n)} (\bar{x}) \right| \leq M \sum_{i=0}^{M} M \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1 / \varepsilon_2)^j \left| \left( \bar{u}_i^{BL} \right)^{(n)} (\bar{x}) \right| \]

\[ \lesssim \sum_{i=0}^{M} M \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1 / \varepsilon_2)^j \tilde{K}_n \gamma^{i+j} (2^i + j)^{2i+j} \frac{1}{i!} e^{-\beta \bar{x}} . \]

Since \( (2i + j)^{2i+j} \leq e^{2i(2i)j} e^{j} \) (cf. (26)), we get

\[ \left| \left( \bar{u}_M^{BL} \right)^{(n)} (\bar{x}) \right| \lesssim \tilde{K}_n e^{-\beta \bar{x}} \left( \sum_{i=0}^{M} \frac{i! e^{2i} (\varepsilon_1 / \varepsilon_2)^2}{i!} \right) \left( \sum_{j=0}^{\infty} (\varepsilon_1 / \varepsilon_2)^j \gamma^j e^j \right) \]

\[ \lesssim \tilde{K}_n e^{-\beta \bar{x}} \left( \sum_{i=0}^{\infty} (\tilde{\gamma} e^2 4M \varepsilon_2)^i \right) \left( \sum_{j=0}^{\infty} (\frac{\varepsilon_1}{\varepsilon_2} \tilde{\gamma} e M)^j \right) \]

\[ \lesssim \tilde{K}_n e^{-\beta \bar{x}} , \]

since both sums are convergent geometric series due to the assumptions \( 4\tilde{\gamma} e^2 4M \varepsilon_2 < 1 \), \( \frac{\varepsilon_1}{\varepsilon_2} \tilde{\gamma} e^2 4M < 1 \).
Similarly, we show (34): By (29) and Lemma 9

\[ \left| (\hat{u}_{BL}^{(n)}(\hat{x})) \right| \leq \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^{i} (\varepsilon_1/\varepsilon_2)^{j} \left| (\hat{u}_{i,j}^{BL}^{(n)}(\hat{x})) \right| \]

\[ \lesssim \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^{i} (\varepsilon_1/\varepsilon_2)^{j} \hat{K}^{n+i+j} \frac{(i+2j)^{i+2j}}{j!} e^{-\beta \hat{x}} \]

\[ \lesssim \hat{K}^{n} e^{-\beta \hat{x}} \left( \sum_{i=0}^{\infty} (\hat{\gamma} e M \varepsilon_2)^{i} \right) \left( \sum_{j=0}^{\infty} \left( \frac{\varepsilon_1}{\varepsilon_2} \hat{\gamma} e^{2} 4 M \right)^{j} \right) \]

\[ \lesssim \hat{K}^{n} e^{-\beta \hat{x}}. \]

It remains to show (35). To this end, note that

\[ r_{M}(0) = u(0) - \left[ \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^{i} (\varepsilon_1/\varepsilon_2)^{j} u_{i,j}(0) + \hat{u}_{i,j}^{BL}(0) + \hat{u}_{i,j}^{BL} (\varepsilon_2/\varepsilon_1) \right] \]

\[ = - \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^{i} (\varepsilon_1/\varepsilon_2)^{j} u_{i,j}^{BL} (\varepsilon_2/\varepsilon_1). \]

By (25),

\[ |r_{M}(0)| \leq \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^{i} (\varepsilon_1/\varepsilon_2)^{j} \left| \hat{u}_{i,j}^{BL} (\varepsilon_2/\varepsilon_1) \right| \]

\[ \lesssim \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^{i} (\varepsilon_1/\varepsilon_2)^{j} \hat{\gamma} \hat{x}^{i+j} \frac{(i+2j)^{i+2j}}{j!} e^{-\beta \varepsilon_2/\varepsilon_1} \]

\[ \lesssim e^{-\beta \varepsilon_2/\varepsilon_1} \left( \sum_{i=0}^{\infty} (\hat{\gamma} e M \varepsilon_2)^{i} \right) \left( \sum_{j=0}^{\infty} \left( \frac{\varepsilon_1}{\varepsilon_2} \hat{\gamma} e^{2} 4 M \right)^{j} \right) \]

\[ \lesssim e^{-\beta \varepsilon_2/\varepsilon_1}. \]
Similarly,
\[ |r_M(1)| \leq \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j |\bar{u}_{i,j}(1/\varepsilon_2)| \]
\[ \lesssim \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j \tilde{\gamma}^{i+j} (2i + j)^{2i+j} \frac{i!}{i!} e^{-\beta/\varepsilon_2} \]
\[ \lesssim e^{-\beta/\varepsilon_2} \left( \sum_{i=0}^{\infty} (\tilde{\gamma} e^2 \varepsilon_2 4M)^i \right) \left( \sum_{j=0}^{\infty} \left( (\varepsilon_1/\varepsilon_2^2) \tilde{\gamma} e M \right)^j \right) \]
\[ \lesssim e^{-\beta/\varepsilon_2}. \]

Combining the two results, we have
\[ \|r_M\|_{\infty, \partial I} \lesssim \max \{ e^{-\beta \varepsilon_2/\varepsilon_1}, e^{-\beta/\varepsilon_2} \}. \]

Now, let \( L := -\varepsilon_1 \frac{d^2}{dx^2} + \varepsilon_2 b \frac{d}{dx} + c \) \( \text{Id} \), with \( \text{Id} \) the identity operator, and consider
\[ L (u - u_M) = f(x) - \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^i (\varepsilon_1/\varepsilon_2^2)^j Lu_{i,j}(x), \]
with \( u_{i,j} \) satisfying (16). After some calculations, we find
\[ L (u - u_M) = -\varepsilon_2^{M+1} \sum_{j=1}^{M} \left( \frac{\varepsilon_1}{\varepsilon_2^2} \right)^j b u_{M,j}', \]

hence
\[ \|L (u - u_M)\|_{\infty, I} \leq \varepsilon_2^{M+1} \sum_{j=1}^{M} \left( \frac{\varepsilon_1}{\varepsilon_2^2} \right)^j \|b\|_{\infty, I} \|u_{M,j}'\|_{\infty, I} \]
\[ \lesssim \varepsilon_2^{M+1} \sum_{j=1}^{M} \left( \frac{\varepsilon_1}{\varepsilon_2^2} \right)^j \|u_{M,j}'\|_{\infty, I}. \]

Using Lemma 5, we further obtain
\[ \|L (u - u_M)\|_{\infty, I} \lesssim \varepsilon_2^{M+1} M! K_2^M \sum_{j=1}^{M} \left( \frac{\varepsilon_1}{\varepsilon_2^2} \right)^j \lesssim \varepsilon_2 (\varepsilon_2 M K_2)^M. \]
since the finite sum can be bounded by a converging geometric series.

We also consider the operator $L$ in the stretched variable $\tilde{x}$, and we find, after some calculations,

$$
\tilde{L}\tilde{u}_{i,M} = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^j (\varepsilon_1/\varepsilon_2)^i \tilde{L}\tilde{u}_{i,j}
$$

$$
= \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^j (\varepsilon_1/\varepsilon_2)^i \left\{-\varepsilon_1 \varepsilon_2^{-2} (\tilde{u}_{i,j}^{BL})'' + \sum_{k=0}^{M} \left[ \tilde{b}_k (\tilde{u}_{i,j}^{BL})' + \tilde{c}_k \tilde{u}_{i,j}^{BL} \right] \right\}
$$

$$
= \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{M+1} \sum_{i=0}^{M} \varepsilon_2^i (\tilde{u}_{i,M}^{BL})'',
$$

where (17) was used. Hence, using (24), we have

$$
\|\tilde{L}\tilde{u}_{i,M}^{BL}\|_{\infty,I} \leq \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{M+1} \sum_{i=0}^{M} \varepsilon_2^i \left\| (\tilde{u}_{i,M}^{BL})'' \right\|_{\infty,I}
$$

$$
\lesssim \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{M+1} \sum_{i=0}^{M} \varepsilon_2^i \gamma^{i+M} \frac{(2i+M)^{2i+M}}{i!}
$$

$$
\lesssim \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{M+1} \sum_{i=0}^{M} \varepsilon_2^i \gamma^{i+M} e^{2i(4i)^i} e^M M^M
$$

$$
\lesssim \left( \frac{\varepsilon_1}{\varepsilon_2} \gamma eM \right)^{M+1} \sum_{i=0}^{M} (\varepsilon_2 \gamma e^2 4M)^i
$$

$$
\lesssim \left( \frac{\varepsilon_1}{\varepsilon_2} \gamma eM \right)^{M+1} .
$$

Similarly, in the stretched variable $\hat{x}$ we have with the help of (18),

$$
\hat{L}\hat{u}_{i,M}^{BL} = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^j (\varepsilon_1/\varepsilon_2)^i \hat{L}\hat{u}_{i,j}
$$

$$
= \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_2^j (\varepsilon_1/\varepsilon_2)^i \left\{-\varepsilon_1 \varepsilon_2^{-2} (\hat{u}_{i,j}^{BL})'' - \sum_{k=0}^{M} \left[ \frac{\varepsilon_2^2}{\varepsilon_1} \hat{b}_k (\hat{u}_{i,j}^{BL})' + \hat{c}_k \hat{u}_{i,j}^{BL} \right] \right\}
$$

$$
= \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^M \sum_{i=0}^{M} \varepsilon_2^i (\hat{u}_{i,M}^{BL})'',
$$

20
and thus
\[
\|\hat{L}u^{BL}_{M}\|_{\infty, I} \lesssim \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^M \sum_{i=0}^{M} \varepsilon_2^i \left(\hat{u}^{BL}_{i,M}\right)''\|_{\infty, I} \\
\lesssim \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^M \sum_{i=0}^{M} \varepsilon_2^i \varepsilon_2^i M (i + 2M)^{i+2M} \frac{1}{M!} \\
\lesssim \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^M \hat{\gamma} e M ,
\]
by following the exact same steps as above. Therefore,
\[
\|Lr_M\|_{\infty, I} = \|L(u - u_M - \tilde{u}^{BL}_M - \hat{u}^{BL}_M)\|_{\infty, I} \\
\lesssim \|L(u - u_M)\|_{\infty, I} + \|\tilde{L}u^{BL}_M\|_{\infty, I} + \|\hat{L}u^{BL}_M\|_{\infty, I} \\
\lesssim \varepsilon_2 (\varepsilon_2 M K_2)^M + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^M \hat{\gamma} e M + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^M \hat{\gamma} e M .
\]
Under the assumptions of the theorem, we have shown that the remainder \(r_M\) has exponentially small values at the endpoints of \(I\), and \(Lr_M\) is uniformly bounded by an arbitrarily small quantity on \(I\). By stability (see, e.g., [3]) we have the desired result.

The bounds of the previous theorem are of utmost importance in the design and proof of convergence (independently of \(\varepsilon_1, \varepsilon_2\)) of high order numerical methods, e.g. the \(hp\) Finite Element Method (see, e.g. [16]). The bounds on the boundary layers tell us how to design the mesh for the approximation, so that the negative powers of \(\varepsilon_1, \varepsilon_2\) are eliminated. The bounds on the smooth part, allow us to prove exponential convergence of the numerical method (see, e.g., [5]).

### 3.2 The regime \(\varepsilon_1 \approx \varepsilon_2^2\)
Now there are layers at both endpoints of width \(O(\varepsilon_2)\) and the BVP becomes reaction-diffusion like the one studied in [6]. So with \(\tilde{x} = x/\varepsilon_2, \bar{x} = (1-x)/\varepsilon_2\), we make, analogously as in the previous case, the formal ansatz
\[
u \sim \sum_{i=0}^{\infty} \varepsilon_2^i \left(u_i(x) + \tilde{u}^{BL}_{i}(\tilde{x}) + \bar{u}^{BL}_{i}(\bar{x})\right) ,
\]
with \( u_i, \tilde{u}_{i}^{BL}, \overline{u}_{i}^{BL} \) to be determined. Substituting (36) into (1), separating
the slow and fast variables, and equating like powers of \( \varepsilon_1 \approx \varepsilon_2^2 \) and \( \varepsilon_2 \) we get (see [15] for details)

\[
\begin{align*}
  u_0(x) &= \frac{f(x)}{c(x)}, \quad u_1(x) = -\frac{b(x)}{c(x)} u'_0(x), \\
  u_i(x) &= \frac{1}{c(x)} \left( u_{i-2}''(x) - b(x) u_{i-1}'(x) \right), \quad i \geq 2
\end{align*}
\]

\[
\begin{align*}
  -(\tilde{u}_0^{BL})'' + \tilde{b}_0 \left( \tilde{u}_0^{BL} \right)'' + \tilde{c}_0 \tilde{u}_0^{BL} &= 0 \\
  -\left( \tilde{u}_i^{BL} \right)'' + \tilde{b}_0 \left( \tilde{u}_i^{BL} \right)' + \tilde{c}_0 \tilde{u}_i^{BL} &= \sum_{k=1}^{i} \left( \tilde{b}_k \left( \tilde{u}_{i-k}^{BL} \right)' + \tilde{c}_k \tilde{u}_{i-k}^{BL} \right), \quad i \geq 1
\end{align*}
\]

\[ (37) \]

where the notation \( \tilde{b}_k(\tilde{x}) = \tilde{x}^k b^{(k)}(0) / k! \) etc., is used again. The above equations are supplemented with the following boundary conditions (in order to satisfy (2)):

\[
\begin{align*}
  u_i(0) + \tilde{u}_i^{BL}(0) &= 0 \\
  u_i(1) + \overline{u}_i^{BL}(0) &= 0 \\
  \lim_{\tilde{x} \to \infty} \tilde{u}_i^{BL}(\tilde{x}) &= 0 \\
  \lim_{\tau \to \infty} \overline{u}_i^{BL}(\tau) &= 0
\end{align*}
\]

\[ (40) \]

We then define, for some \( M \in \mathbb{N} \),

\[
\begin{align*}
  u_M(x) &= \sum_{i=0}^{M} \varepsilon_2^i u_i(x), \\
  \tilde{u}_M^{BL}(\tilde{x}) &= \sum_{i=0}^{M} \varepsilon_2^i \tilde{u}_i^{BL}(\tilde{x}), \\
  \overline{u}_M^{BL}(\tau) &= \sum_{i=0}^{M} \varepsilon_2^i \overline{u}_i^{BL}(\tau)
\end{align*}
\]

as well as

\[
\begin{align*}
  u &= u_M + \tilde{u}_M^{BL} + \overline{u}_M^{BL} + r_M
\end{align*}
\]

\[ (41) \]

We have the following theorem.

**Theorem 11.** Assume (3), (4) hold, and that \( \varepsilon_1 \approx \varepsilon_2^2 \). Then there exist positive constants \( K_1, K_2, K, \tilde{K}, \delta \), independent of \( \varepsilon_1, \varepsilon_2 \), such that the solution \( u \) of (7)–(2) can be decomposed as in (41), with

\[
\begin{align*}
  \| u_M^{(n)} \|_{\infty, I} &\lesssim n! K_1^n \quad \forall \ n \in \mathbb{N}_0, \\
  \left( \tilde{u}_M^{BL} \right)^{(n)}(x) &\lesssim \tilde{K}^n \varepsilon_2^{-n} e^{-\beta x / \varepsilon_2} \quad \forall \ n \in \mathbb{N}_0
\end{align*}
\]
We anticipate layers at both endpoints of width \(\varepsilon_2\), analogous to the previous cases, where \(M\) is chosen so that \(\varepsilon_2 K_2 M < 1\), and \(\beta = \min \{\frac{b_0}{\varepsilon_0}, \frac{b_0}{\varepsilon_0}\}\).

Proof. When \(\varepsilon_1 \approx \varepsilon_2\), the BVP \((1)-(2)\) becomes

\[-u''(x) + \varepsilon_2^{-1} b(x) u'(x) + \varepsilon_2^{-2} c(x) u(x) = \varepsilon_2^{-2} f(x),\ x \in I = (0,1),\]
\[u(0) = u(1) = 0.\]

Multiplying the differential equation above by \(e^{-\int_0^x \varepsilon_2^{-1} b(t) dt}\), gives

\[-\left( u'(x) e^{-\int_0^x \varepsilon_2^{-1} b(t) dt}\right)' + e^{-\int_0^x \varepsilon_2^{-1} b(t) dt} c(x) u(x) = e^{-\int_0^x \varepsilon_2^{-1} b(t) dt} f(x),\]

or equivalently, with \(v(x) = e^{-\int_0^x \varepsilon_2^{-1} b(t) dt} u(x)\), \(F(x) = e^{-\int_0^x \varepsilon_2^{-1} b(t) dt} f(x),\)
\[-\varepsilon_2^2 v''(x) + c(x) v(x) = F(x),\ x \in I = (0,1),\]
\[v(0) = v(1) = 0.\]

The above BVP is in the form considered in \cite{6}, with \(c(x)\), \(F(x)\) analytic – the analyticity of \(F(x)\) follows from the analyticity of \(b\) and \(f\). The desired bounds follow from the results in \cite{6}, and the fact that \(|u^{(n)}| < |v^{(n)}|\). \(\square\)

3.3 The regime \(\varepsilon_2^2 \ll \varepsilon_1 \ll 1\)

We anticipate layers at both endpoints of width \(O(\sqrt{\varepsilon_1})\). So we define the stretched variables \(\tilde{x} = x/\sqrt{\varepsilon_1}\) and \(\tilde{x} = (1-x)/\sqrt{\varepsilon_1}\) and make the formal ansatz, analogous to the previous cases,

\[u \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon_1^{i/2} \varepsilon_2^{j/2} \left( u_{i,j}(x) + \hat{u}_{i,j}^{BL}(\tilde{x}) + \hat{u}_{i,j}^{BL}(\tilde{x}) \right),\]

(42)

with \(u_{i,j}, \hat{u}_{i,j}^{BL}, \hat{u}_{i,j}^{BL}\) to be determined. Substituting (42) into (1), separating the slow and fast variables, and equating like powers of \(\varepsilon_1\) and \(\varepsilon_2\) we get (see \cite{15} for the details)

\[
\begin{align*}
    u_{0,0} &= \frac{f(x)}{c(x)}, u_{1,0}(x) = u_{0,1}(x) = 0, j \geq 1 \\
    u_{i,0}(x) &= \frac{1}{c(x)} u''_{i-2,0}(x), i \geq 2 \\
    u_{2i+1,0}(x) &= 0, i \geq 1 \\
    u_{1,1}(x) &= -\frac{b(x)}{c(x)} u_{0,0}(x), u_{1,1}(x) = 0, j \geq 2 \\
    u_{i,j}(x) &= \frac{1}{c(x)} \left( u''_{i-2,j}(x) - b(x) u'_{i-1,j-1}(x) \right), i \geq 2, j \geq 1
\end{align*}
\]

(43)
where the notation $\tilde{b}_k(\tilde{x}) = \tilde{x}^k b^{(k)}(0)/k!$ etc., is used once more. The above equations are supplemented with the following boundary conditions (in order to satisfy (2)):

\[
\begin{aligned}
&\frac{- (\tilde{u}_{i,0}^{BL})'' + \tilde{c}_0 \tilde{u}_{i,0}^{BL} = 0}
&- (\tilde{u}_{i,j}^{BL})'' + \tilde{c}_0 \tilde{u}_{i,j}^{BL} = - \sum_{k=1}^i \tilde{c}_k \tilde{u}_{i-k,0}^{BL}, i \geq 1
&- (\tilde{u}_{0,j}^{BL})'' + \tilde{c}_0 \tilde{u}_{0,j}^{BL} = - b_0 (\tilde{u}_{0,j-1}^{BL})', j \geq 1
&- (\tilde{u}_{i,j}^{BL})'' + \tilde{c}_0 \tilde{u}_{i,j}^{BL} = b_0 (\tilde{u}_{i,j-1}^{BL})' - \tilde{c}_k (\tilde{u}_{i-k,j-1}^{BL})', i \geq 1, j \geq 1
&\sum_{k=1}^i \tilde{b}_k (\tilde{u}_{i-k,j-1}^{BL})' + \tilde{c}_k \tilde{u}_{i-k,j-1}^{BL} = 0, i \geq 1, j \geq 1
\end{aligned}
\]

(44)

where the notation $\tilde{b}_k(\tilde{x}) = \tilde{x}^k b^{(k)}(0)/k!$ etc., is used once more. The above equations are supplemented with the following boundary conditions (in order to satisfy (2)):

\[
\begin{aligned}
\tilde{u}_{i,j}^{BL}(0) &= -u_{i,j}(0) \\
\tilde{u}_{i,j}^{BL}(0) &= -u_{i,j}(1)
\end{aligned}
\]

(46)

The following result is established in a completely analogous way as in the previous cases (cf. Section 3.1).

**Lemma 12.** Let $u_{i,j}$ be defined by (43) and assume (3) holds. Then there exist positive constants $C, K_1, K_2$, independent of $\varepsilon_1, \varepsilon_2$, such that

$$
\|u_{i,j}^{(n)}\|_{\infty, I} \leq C n! K_1^n i! K_2^j \forall n \in \mathbb{N}.
$$

Next we consider the boundary layers. The following result was shown in [5].

**Proposition 13** (5, Lemma 7.3.6). Let $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0, Re(\lambda)^2 > 0$. Let $F$ be an entire function satisfying, for some $C_F > 0$, $j \in \mathbb{N}_0$, $q \geq (j + 1/2)/|\lambda| > 0$,

$$
|F(z)| \leq C_F e^{-Re(\lambda x)} (q + |z|)^j \forall z \in \mathbb{C}.
$$

Let $\alpha \in \mathbb{C}$ and let $v : (0, \infty) \to \mathbb{C}$, be the solution of the problem

$$
-v'' + \lambda^2 v = F \text{ on } (0, \infty), v(0) = g, \lim_{x \to \infty} v(x) = 0.
$$

Let $\alpha \in \mathbb{C}$ and let $v : (0, \infty) \to \mathbb{C}$, be the solution of the problem

$$
-v'' + \lambda^2 v = F \text{ on } (0, \infty), v(0) = g, \lim_{x \to \infty} v(x) = 0.
$$

24
Then $v$ can be extended to an entire function (denoted again by $v$), which satisfies

$$|v(z)| \leq \left[ C_F \frac{1}{|\lambda|} (q + |z|)^{j+1} (j+1)^{-1} + |g| \right] e^{-\Re(\lambda z)} \ \forall \ z \in \mathbb{C}. $$

Using the above we may prove the following.

**Lemma 14.** Let $\hat{u}_{i,j}^{BL}, \hat{\hat{u}}_{i,j}^{BL}$ be defined by (44), (45), respectively. Then there exist positive constants $C, \hat{\gamma}, \hat{K}, \hat{\hat{\gamma}}, \hat{\hat{K}},$ depending only on the data, such that

$$\left| (\hat{u}_{i,j}^{BL})^{(n)} (x) \right| \leq C_{\hat{\gamma}^{i+j}} (i+j)! \hat{K}^{-n} \varepsilon_1^{-n/2} e^{-\beta x/\sqrt{\varepsilon_1}} \ \forall \ i, j \geq 0,$$

$$\left| (\hat{\hat{u}}_{i,j}^{BL})^{(n)} (x) \right| \leq C_{\hat{\hat{\gamma}}^{i+j}} (i+j)! \hat{\hat{K}}^{-n} \varepsilon_1^{-n/2} e^{-\beta(1-x)/\sqrt{\varepsilon_1}} \ \forall \ i, j \geq 0,$$

where $\beta = \min \{ \hat{c}_0, \hat{\hat{c}}_0 \}$.

**Proof.** First we show a similar result as (20), by induction on $i, j$ with the help of Proposition 13. For $n > 0$, the proof is almost identical to that of Lemma 9, utilizing Cauchy’s Integral Theorem for Derivatives. The details appear in [15].\qed

We then define, for some $M \in \mathbb{N},$

$$u_M (x) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j u_{i,j}(x),$$

$$\hat{u}_M^{BL} (\hat{x}) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j \hat{u}_{i,j}^{BL}(\hat{x}),$$

$$\hat{\hat{u}}_M^{BL} (\hat{x}) = \sum_{i=0}^{M} \sum_{j=0}^{M} \varepsilon_1^{i/2} (\varepsilon_2/\sqrt{\varepsilon_1})^j \hat{\hat{u}}_{i,j}^{BL}(\hat{x}),$$

and we have the following decomposition:

$$u = u_M + \hat{u}_M^{BL} + \hat{\hat{u}}_M^{BL} + r_M. \quad (47)$$

The theorem that follows is the analog of Theorem 10 and its proof is almost identical. Nevertheless, it is worth commenting on the fact that $\varepsilon_2$ does not appear in the statement of Theorem 15. In the regime $\varepsilon_2 \ll \varepsilon_1$, the perturbation in the first order term is a regular perturbation, and as such benigne. As a result, its lack of presence in Theorem 15 is not an issue.
Theorem 15. Assume (3), (4) hold. Then there exist positive constants $K_1, \tilde{K}, K_2$ and $\delta$, independent of $\varepsilon_1, \varepsilon_2$, such that the solution $u$ of (1)–(2) can be decomposed as in (47), with

$$
\|u_M^{(n)}\|_{\infty,I} \lesssim n!K_1^n \quad \forall \ n \in \mathbb{N}_0,
$$

$$
\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(x)\right| \lesssim \tilde{K}^n \varepsilon_1^{-n/2} e^{-\beta x/\sqrt{\varepsilon_1}} \quad \forall \ n \in \mathbb{N}_0,
$$

$$
\left|\left(\hat{u}_M^{BL}\right)^{(n)}(x)\right| \lesssim \hat{K}^n \varepsilon_1^{-n/2} e^{-\beta(1-x)/\sqrt{\varepsilon_1}} \quad \forall \ n \in \mathbb{N}_0,
$$

$$
\|r_M\|_{\infty,\partial I} + \|r_M\|_{0,I} + \varepsilon_1^{1/2} \|r'_M\|_{0,I} \lesssim e^{-\delta/\sqrt{\varepsilon_1}},
$$

where $M$ is chosen so that $\sqrt{\varepsilon_1}K_2M < 1$, and $\beta = \min\{\hat{c}_0, \tilde{c}_0\}$.

3.4 On the transition between regimes

As a final question, we would like to see what happens when we fix $\varepsilon_1 \ll 1$ and consider $\varepsilon_2 \in (0, 1]$. In Figure 1 we show the solution of the BVP

$$
-0.005u''(x) + \varepsilon_2u'(x) + u(x) = 1, \ x \in (0, 1) \quad u(0) = u(1) = 0,
$$

for $\varepsilon_2 \in (0, 1]$. The figure shows that the transition between regimes appears “seamless”, in the following sense: as $\varepsilon_2$ takes on values in $(0, 1]$, the solution $u$ “smoothly moves” from one regime to the other, based on the relationship between $\varepsilon_2$ and (the fixed, but small) $\varepsilon_1$. In particular:

- If $\varepsilon_2 = 1$, then we have a convection-diffusion problem, and we have a layer of width $O(\varepsilon_1)$ at the outflow boundary. (This is clearly visible in Figure 1 on the right.)

- If $1 > \varepsilon_2 > \sqrt{\varepsilon_1}$, then we have a reaction-convection-diffusion problem, with layers of different width at each endpoint.

- If $0 < \varepsilon_2 \leq \sqrt{\varepsilon_1}$, then we have a reaction-diffusion problem, with layers of width $O(\sqrt{\varepsilon_1})$ at each endpoint. (See Figure 1 on the left.)

The error bounds of the previous sections allow us to state the following:
Proposition 16. Let $\varepsilon_1 \ll 1$ be fixed, and let $u$ be the solution of (1)–(2) under the assumption (3). Then for any $\varepsilon_2 \in (0, 1)$, there exist positive constants $K_1, K_2, K_3, \delta$, independent of $\varepsilon_1, \varepsilon_2$, such that

$$u = u_S + u_{BL}^+ + u_{BL}^- + u_R.$$

The smooth part $u_S$, satisfies for any $n \in \mathbb{N}$, $\varepsilon_2 \in (0, 1)$,

$$\left\| u_S^{(n)} \right\|_{\infty, I} \lesssim K_1^n n!.$$

The boundary layer parts $u_{BL}^\pm$, satisfy for any $n \in \mathbb{N}$ and

- $\forall$ $\varepsilon_2 \in (0, \sqrt{\varepsilon_1})$, $x \in I$,

$$\left| (u_{BL}^\pm)^{(n)}(x) \right| \lesssim K_2^n n! \varepsilon_2^{-n/2} e^{-\beta\operatorname{dist}(x, \partial I) \sqrt{\varepsilon_1}},$$

- $\forall$ $\varepsilon_2 \in (\sqrt{\varepsilon_1}, 1)$, $x \in I$,

$$\left| (u_{BL}^\pm)^{(n)}(x) \right| \lesssim K_2^n n! \varepsilon_2^{-n} e^{-\beta x/\varepsilon_2}, \quad \left| (u_{BL}^\pm)^{(n)}(x) \right| \lesssim K_2^n n! \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^{-n} e^{-\beta(1-x)/\varepsilon_2/\varepsilon_1}.$$

The remainder $u_R$, satisfies for any $n \in \mathbb{N}$, $\varepsilon_2 \in (0, 1)$,

$$\| u_R \|_{\infty, \partial I} + \| u_R \|_{0, I} + \min\{\varepsilon_2, \sqrt{\varepsilon_1}\} \| u_R' \|_{0, I} \lesssim \max\{e^{-\delta\varepsilon_2/\varepsilon_1}, e^{-\delta/\varepsilon_2}\}.$$
4 Conclusions

We considered a two-point, singularly perturbed, reaction-convection-diffusion problem with analytic input data, and we derived regularity results for its solution. Based on the relationship between the singular perturbation parameters, the problem becomes convection-diffusion, reaction-diffusion or reaction-convection-diffusion, as shown in Table 1. We provided estimates for all three cases (regimes), which reveal the analytic nature of the solution and give derivative bounds which are explicit in the differentiation order as well as the singular perturbation parameters. Such estimates are necessary for the construction and analysis of high order numerical methods, such as hp FEM (see, e.g. [16]).

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