On finding exact and approximate solutions to fractional systems of ordinary differential equations using fractional natural adomian decomposition method

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Abstract
In this work, we present proofs for new theorems that deal with natural transform method (NTM) with Caputo derivative. Also, we give exact and approximate solutions to systems of fractional differential equations along with fractional ordinary and partial differential equations using the fractional natural decomposition method (FNDM). The Caputo derivative is used here to minimize the amount of computational, and this is of great significance for large-scale problems. The work outlines the significant features of the FNDM. Our work can be considered as another technique to existing methods, and will have many applications in variant areas of science and engineering.

Keywords
Fractional Differential Equations, Caputo Derivative, Natural Transform, System Differential Equations, Adomian Method.

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Introduction
Fractional derivatives have proven their capability to describe several phenomena associated with memory and aftereffects due to their non-locality property. Such phenomena are commonplace in physical processes, biological structures, and cosmological phenomena. For instance, the fractional Kelvin-Voigt rheological models have been employed to examine the low applied force frequencies. For this reason, this became necessary to illuminate the solutions of the models that describe these phenomena. Several analytical techniques were presented to achieve their objectives. Actually, all these approaches were accommodation for the existing methods to handle the integer case models which is natural since the fractional derivative generalizes the classical derivative to an arbitrary order.

Recently, fractional calculus and their applications have been treated by many researchers. As fractional derivative models are becoming increasingly popular among the wider scientific community, is the main motivation to study numerical schemes for fractional differential equations. There are many applications of fractional differential equations and just to name few: control systems, elasticity, electric drives, circuits systems, continuum mechanics, heat transfer, quantum mechanics, fluid mechanics, signal analysis, biomathematics, biomedicine, social systems and bio-engineering. Lately, many techniques discussed the way how to explore approximate solutions of FDEs, such as FDTM, the fractional sub-equation method, the fractional natural decomposition method (FNDM), the modified homotopy perturbation method (MHPM), the Conformable Sumudu Decomposition Method (CADM) and the (FADM). The outline of our work

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is as follows: First, in Chapters 2, we give the history of natural transform method, definitions of fractional derivatives. In Chapter 3, we present proofs to theorems related to the Caputo derivatives. Chapter 4 is devoted for applications model of FDEs using the proposed method. In chapter 5, we solve fractional systems of ODEs. Finally, our concluding remarks is presented, in chapter 6, to outline of what we have accomplished in this research.

Related Materials

We explore some definitions terminologies of natural transform that will be needed later on in the proofs of our results, (see for example,22,11,12).

Definition 1: We say a function \( \varphi(y) \in C_a \), where \( v \in \mathbb{R} \), \( y > 0 \), if \( \exists r \in \mathbb{R} \) with \( r > v \), such that \( \varphi(y) = y^r(f(y)) \), and \( f(y) \in C[0, \infty) \), and we say \( \varphi(y) \in C^k \), if \( f(k) \in C_a \) where \( k = 1, 2, 3, \ldots \).

Definition 2: If \( k - 1 < \nu < k \), \( k \in \mathbb{N} \), \( y > 0 \), \( \varphi \in C^k \). We define the Caputo type of \( \nu \) as

\[
\mathcal{C}^k \varphi(y) = y^k \mathcal{D}^k \varphi(y) = \frac{1}{\Gamma(k-\nu)} \int_0^y (y-t)^{k-\nu-1} \varphi(t) dt. \tag{1}
\]

Definition 3:23 The Mittag-Leffler in two parameters is given by

\[
E_{\nu,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + \eta)}, \quad v > 0, \quad \eta > 0, \quad z \in \mathbb{C}.
\]

Definition 4: Let \( H(t) \) denote the Heaviside function, more precisely \( H(t) = 1 \) for \( t > 0 \) and \( H(t) = 0 \) for \( t < 0 \). We introduce a real-valued function, on which \( \mathcal{N}^+ \)-transform can be defined on \( [0, \infty) \), where \( s > cu \), \( c > 0 \). Let \( \varphi(t) \) be continuous on \( \mathbb{R} \). For some \( K \), \( c > 0 \). Consider

\[
\mathbb{B} = \{ \varphi(t) \mid 0 < K e^{\nu t} H(t) + K e^{-\nu t} H(-t) \}. \tag{2}
\]

Note that for any \( \varphi(t) \) in the class \( \mathbb{B} \) we have

\[
\int_{-\infty}^{\infty} e^{\nu t} \varphi(t) dt = K \int_0^\infty e^{\nu t} e^{\nu t} dt + K \int_{-\infty}^0 e^{\nu t} e^{-\nu t} dt
\]

\[
= K \int_0^\infty e^{t(u + c)} dt + K \int_{-\infty}^0 e^{t(c u) dt},
\]

which is convergent provided that \( s > cu \). Then, we define the natural transform

\[
\mathcal{N}^+ (\varphi(t)) = R(s, u) = \int_0^\infty e^{s t} \varphi(t) dt. \tag{3}
\]

Alternatively,

\[
\mathcal{N}^+ (\varphi(t)) = R(s, u) = \frac{1}{u} \int_0^\infty e^{\frac{t}{u}} \varphi(t) dt. \tag{4}
\]

Note that one can obtain the Laplace transform and the Sumudu transform if we plug in \( u = 1, s = 1 \) in the above equations, respectively.

We shall use the well-known gamma function through out this paper

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{-1} dt, \quad z > 0, \tag{5}
\]

where \( \Gamma(z + 1) = z \Gamma(z) \).

Important Properties

Some basic properties of the N-transforms are given as follows22,11,12

Property 1: \( \mathcal{N}^+(1) = \frac{1}{\nu} \).

Property 2: \( \mathcal{N}^+(y^\nu) = \frac{1}{\nu} \mathcal{N}^+(y^\nu)(\nu t)^{\nu - 1}, \quad \nu > -1. \)

Property 3: \( \mathcal{N}^+(^D_{\nu} \varphi(t))(s, u) = \frac{\nu}{s} \mathcal{N}^+(\varphi(t))(s, u) - \frac{\nu}{s} \varphi(0). \)

Natural Caputo Fractional Derivatives

Here we give detailed proofs to some theorems of N-transform of Caputo fractional derivative. The proofs of theorem (1) was given in another published paper by the first author.

Caputo Fractional Derivative

For the sake of readers, we give just some of natural transforms properties. We direct the reader for more properties to see for example,22,11,12

Theorem 1: If \( \nu \in \mathbb{Z}^+ \), where \( k - 1 \leq \nu < k \). Then, the \( \mathcal{N}^+ \)-transform of Caputo derivative of \( \varphi(t) \) is

\[
\mathcal{N}^+(D^\nu_{\nu} \varphi(t)) = \frac{s^\nu}{u^\nu} \mathcal{N}^+(\varphi(t)) - \sum_{m=0}^{k-1} \frac{s^{\nu(m+1)} \varphi(m)}{u^{\nu m} (D^\nu \varphi(t))_{t=0}}. \tag{6}
\]

Theorem 2: The natural transform of the Caputo derivative for \( \varphi(t) = 1 \) is given by

\[
\mathcal{N}^+(D^\nu_{\nu} (1)) = 0. \tag{7}
\]

Proof.: From Eq. (6), we have

\[
\mathcal{N}^+(D^\nu_{\nu} (1)) = \frac{s^\nu}{u^\nu} \mathcal{N}^+(1) - \sum_{m=0}^{k-1} \frac{s^{\nu(m+1)} (D^\nu \varphi(t))_{t=0}}{u^{\nu m} (D^\nu \varphi(t))_{t=0}}
\]

\[
= s^\nu \frac{1}{u^\nu} s^{\nu - 1} \nu [1] - 0
\]

\[
= s^\nu \frac{1}{u^\nu} s^{\nu - 1} \nu [1] - 0
\]

The proof of Theorem (2) is complete. \( \square \)
Theorem 3: (a) The natural transform of the Caputo derivative for $q(t) = t$ with $0 < u \leq 1$ is given by

$$N^+(D^u_0 f(t)) = \frac{s^u - 2}{u^{u-1}}. $$

(b) The natural transform of the Caputo derivative for $q(t) = t$ with $k - 1 < u \leq k$, and $k = 2, 3, 4, \ldots$ is

$$N^+(D^u_k f(t)) = 0. $$

Proof.: First note that $f'(t) = 1$, $f''(t) = 0$, \ldots, $f^{(k-1)}(t) = 0$.

Case 1. $0 < u \leq 1$. From Eq. (6), we have

$$N^+(D^u_0 f(t)) = \frac{s^u}{u^2}N^+(t) - \sum_{m=0}^{k-1} \frac{(s^u - m)(m+1)}{u^{m+1}} (D^m f(t))_{t=0}$$

$$= \frac{s^u}{u^2} - \frac{s^u - 1}{u^3} [0]$$

$$= \frac{s^u - 2}{u^3}. $$

Case 2. $k - 1 < u \leq k$, and $k = 2, 3, 4, \ldots$.

We get from Eq. (6),

$$N^+(D^u_k f(t)) = \frac{s^u}{u^2}N^+(t) - \sum_{m=0}^{k-1} \frac{(s^u - m)(m+1)}{u^{m+1}} (D^m f(t))_{t=0}$$

$$= \frac{s^u}{u^2} - \frac{s^u - 1}{u^3}[0] - \frac{s^u - 2}{u^4}[1] - \cdots - \frac{s^u - n}{u^{n+1}}[0]$$

$$= \frac{s^u - 2}{u^4} - \frac{s^u - 2}{u^5} = 0. $$

The proof of Theorem (3) is complete. $\square$

Theorem 4: The natural transform of the Caputo derivative for $q(t) = \frac{x^u}{a^u}$ is

$$N^+(D^u_{\frac{x^u}{a^u}} f(t)) = \frac{x^u - k}{a^{u-k+1}}. $$

with $k = 3, 4, \ldots$.

Proof.: First note that

$$f'(t) = \frac{(k-2)k^{k-2}}{k!}, f''(t) = \frac{(k-1)(k-2)(k-3)}{k!}, \ldots, f^{(n-1)}(t) = \frac{(k-1)(k-2)\cdots(k-n+1)k^{k-n}}{k!}. $$

One can conclude from Eq. (6),

$$N^+(D^u_{\frac{x^u}{a^u}} f(t)) = \frac{x^u}{u^2}N^+(t) - \sum_{m=0}^{n-1} \frac{(x^u - m)(m+1)}{u^{m+1}} (D^m f(t))_{t=0}$$

$$= \frac{x^u}{u^2} - \frac{x^u - 1}{u^3} [0] - \frac{x^u - 2}{u^4}[1] - \cdots - \frac{x^u - n}{u^{n+1}}[0]$$

$$= \frac{x^u - 2}{u^4} - \frac{x^u - 2}{u^5} = 0. $$

The proof of Theorem (4) is complete. $\square$

Theorem 5: The natural transform of the Caputo derivative for $q(t) = e^{at}$ is

$$N^+(D^u_{e^{at}} f(t)) = \frac{a^u}{u^2}N^+(t) - \sum_{m=0}^{n-1} \frac{(e^{at} - m)(m+1)}{u^{m+1}} (D^m f(t))_{t=0}$$

$$= \frac{a^u}{u^2} - \frac{a^u - 1}{u^3} [0] - \frac{a^u - 2}{u^4}[1] - \cdots - \frac{a^u - n}{u^{n+1}}[0]$$

$$= \frac{a^u - 2}{u^4} - \frac{a^u - 2}{u^5} = 0. $$

The proof of Theorem (5) is complete. $\square$

Theorem 6: The Caputo Fractional Natural Transform of $f(t) = \frac{x^u}{a^u}$, $a \neq b$ is

$$N^+(D^u_{\frac{x^u}{a^u}} f(t)) = \frac{x^u - k}{a^{u-k+1}}. $$

with $k = 3, 4, \ldots$.

Proof.: First note that

$$f'(t) = \frac{(k-2)k^{k-2}}{k!}, f''(t) = \frac{(k-1)(k-2)(k-3)}{k!}, \ldots, f^{(n-1)}(t) = \frac{(k-1)(k-2)\cdots(k-n+1)k^{k-n}}{k!}. $$

$$N^+(D^u_{\frac{x^u}{a^u}} f(t)) = \frac{x^u}{u^2}N^+(t) - \sum_{m=0}^{n-1} \frac{(x^u - m)(m+1)}{u^{m+1}} (D^m f(t))_{t=0}$$

$$= \frac{x^u}{u^2} - \frac{x^u - 1}{u^3} [0] - \frac{x^u - 2}{u^4}[1] - \cdots - \frac{x^u - n}{u^{n+1}}[0]$$

$$= \frac{x^u - 2}{u^4} - \frac{x^u - 2}{u^5} = 0. $$

The proof of Theorem (6) is complete. $\square$
The proof of Theorem (6) is complete.  

**Applications of FNDM for Fractional ODEs and PDEs**

For this section, we shall implement the new scheme to solve two nonlinear fractional ODEs and we present solution to the diffusion fractional differential equations. Finally, we present numerical tables for these examples for multiple values of $\nu$ and $t$.

**Methodology of FDM**

Consider the general nonlinear (FODE)

$$^{\nu}D_{t}^{\alpha} \phi(t) + L(\phi(t)) + F(\phi(t)) = g(t),$$

where $t > 0$ and $0 < \nu \leq 1$, and along with initial condition

$$\phi(0) = h(t),$$

where $^{\nu}D_{t}^{\alpha} \phi(t)$ is the Caputo derivative for $\phi(t)$, $L$ is the linear differential operator and $F$ represents the nonlinear part. Also $g(t)$ is the non-homogeneous part.

Applying Theorem 1 to Eq. (11) one can conclude

$$\mathcal{N}^{+} \left[ ^{\nu}D_{t}^{\alpha} \left( \frac{e^{bt} - e^{at}}{b - a} \right) \right] = \frac{s^{\alpha}}{w^{\alpha}} \left[ \frac{e^{bt} - e^{at}}{b - a} \right] - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{w^{\alpha-k}} [D^{k} f(t)]_{t=0},$$

$$= \frac{s^{\alpha}}{w^{\alpha}} \left[ \frac{u}{(s - au)(s - bu)} \right] - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{w^{\alpha-k}} [D^{k} f(t)]_{t=0},$$

$$= \frac{u^{1-a}\alpha \omega}{(s - au)(s - bu)} - \frac{1}{b-a} \left[ \frac{s^{\alpha-1}}{u^{\alpha}} + \sum_{k=0}^{n-1} \frac{s^{\alpha-n}}{u^{\alpha-(n-k+1)}} (b^{n-1} - a^{n-1}) \right],$$

$$= \frac{u^{1-a}\alpha \omega}{(s - au)(s - bu)} - \frac{1}{b-a} \sum_{k=0}^{n-1} s^{\alpha-(k+1)} (b^{k} - a^{k}),$$

$$= \frac{s^{\alpha}}{u^{\alpha}} \left[ \frac{u - (ub)}{s} \right] - \frac{s^{\alpha-1}}{u^{\alpha}} \left[ \frac{1}{b-a} \sum_{k=0}^{n-1} \left( \frac{ub}{s} \right)^{k} \right] - \frac{s^{\alpha-1}}{u^{\alpha}} \left[ \frac{1}{b-a} \sum_{k=0}^{n-1} \left( \frac{ub}{s} \right)^{k} \right] - \frac{s^{\alpha-1}}{u^{\alpha}} \left[ \frac{1}{b-a} \sum_{k=0}^{n-1} \left( \frac{ub}{s} \right)^{k} \right],$$

$$= \frac{s^{\alpha+1-n} u^{\alpha-a} (b^{\alpha} - a^{\alpha}) + u^{\alpha+1-n} s^{\alpha-n} (b^{\alpha} - a^{\alpha})}{(b-a)(s - au)(s - bu)}.$$
The rest of polynomials can be obtained likewise. Substituting Eq. (15) and Eq. (16) into Eq. (14) one arrive at
\[
\sum_{n=0}^{\infty} q_n(t) = G(t) - \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+\left(L\left(\sum_{n=0}^{\infty} q_n(t) + \sum_{n=0}^{\infty} A_n(t)\right)\right)\right).
\]
(19)

With the help of Eq. (19), one can arrive at
\[
q_0(t) = G(t)
\]
\[
q_1(t) = -\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+\left(L\left(q_0(t) + A_0(t)\right)\right)\right)
\]
\[
q_2(t) = -\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+\left(L\left(q_1(t) + A_1(t)\right)\right)\right)
\]
\[
q_3(t) = -\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+\left(L\left(q_2(t) + A_2(t)\right)\right)\right)
\]
Eventually, we have the general recursive formula as
\[
q_{n+1}(t) = -\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+\left(L\left(q_n(t) + A_n(t)\right)\right)\right), \ n \geq 1.
\]
(20)

Hence, our approximate solution of the form
\[
q(t) = \sum_{n=0}^{\infty} q_n(t).
\]
(21)

Numerical Examples

Example 1: Consider the nonlinear FDE in the form\(^{21}\):
\[
\alpha D^\nu q(t) + \phi^2(t) = 2q(t) + 1, \ 0 < \nu \leq 1,
\]
together with condition
\[
q(0) = 0
\]
(22)

Applying theorem 3.1 to equation (22), one arrive at
\[
\mathcal{N}^{-1}(\phi^2(t)) = \frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+\left(1 + 2\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(q(t)) - \frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi^2(t))\right)
\]
\[
= \frac{u^\alpha}{\alpha^\alpha} + 2\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(q(t)) - \frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi^2(t))
\]
(24)

Taking the \(N\)-transform inverse of Eq. (24) one conclude
\[
q(t) = \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha}\right) + 2\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi^2(t))\right)
\]
\[
- \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi^2(t))\right)
\]
(25)

Suppose a solution exist for \(q(t)\) and the nonlinear term \(\phi^2(t)\) is given as
\[
q(t) = \sum_{n=0}^{\infty} q_n(t), \ \phi^2(t) = \sum_{n=0}^{\infty} A_n(t).
\]
(26)

Note here,
\[
A_0 = (\phi_0)^2
\]
\[
A_1 = 2\phi_0 \phi_1
\]
\[
A_2 = 2\phi_0 \phi_2 + (\phi_1)^2
\]
\[
A_3 = 2\phi_0 \phi_3 + 2\phi_1 \phi_2.
\]

From Eq. (26), then Eq. (25) becomes
\[
q(t) = \sum_{n=0}^{\infty} q_n(t) = \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha}\right) + 2\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi(t))\right)
\]
\[
- \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi^2(t))\right)
\]
(27)

Looking at both sides of Eq. (27), one can conclude
\[
q(t) = \sum_{n=0}^{\infty} q_n(t) = \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha}\right) + 2\mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi(t))\right)
\]
\[
- \mathcal{N}^{-1}\left(\frac{u^\alpha}{\alpha^\alpha} \mathcal{N}^+(\phi^2(t))\right)
\]
(28)

Likewise,
Thus, the approximate solution for \( \phi(t) \) becomes
\[
\phi(t) = \sum_{n=0}^{\infty} \phi_n(t)
= \phi_0(t) + \phi_1(t) + \phi_2(t) + \phi_3(t) + \cdots
\]
(28)

If we choose \( \nu = 1 \), then Eq. (28) becomes
\[
\phi(t) = 1 + \sqrt{2} \tanh \left( \sqrt{2} t + \frac{1}{2} \log \left( \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right).
\]
This exact solution agrees with the one exists in the literature (Figure 1 and Table 1).

**Example 2:** Let us consider one model of the time fractional diffusion of the form \(^\nu D_t^\nu \phi(x, t) = \phi_x(x, t) + \phi(x, t), \ t > 0, \ \ 0 < \nu \leq 1,
(29)

together with initial condition
\[
\phi(x, 0) = \cos(\pi x).
\]
Employ first theorem 1 to equation (29) and see that
\[
\mathcal{N}^\nu \left( \phi(x, t) \right) = \frac{\nu}{\sqrt[\nu]{x}} \sum_{k=0}^{\infty} \frac{\nu^{-(k+1)}}{\nu^k} D_t^\nu \phi(x, 0) + \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi(x, t) \right) + \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi(x, t) \right)
= \frac{1}{\nu^\nu} \phi(x, 0) + \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi(x, t) \right) + \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi(x, t) \right)
\]
(31)

![Diagram](image)

**Figure 1.** These are solutions for example 1 for distinct values of \( \nu \).

Taking the N-transform inverse of equation (31), we arrive at
\[
\phi(x, t) = \cos(\pi x) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi(x, t) \right) \right)
+ \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi(x, t) \right) \right).
\]
(32)

Suppose that our solution for \( \phi(x, t) \) is
\[
\phi(x, t) = \sum_{n=0}^{\infty} \phi_n(x, t).
\]
(33)

From equation (33), one see Eq. (32) becomes
\[
\sum_{n=0}^{\infty} \phi_n(x, t) = \cos(\pi x) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \sum_{n=0}^{\infty} \phi_n(x, t) \right) \right)
+ \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \sum_{n=0}^{\infty} \phi_n(x, t) \right) \right).
\]
(34)

Looking at equation (34), one can get
\[
\phi_0(x, t) = \cos(\pi x)
\]
\[
\phi_1(x, t) = \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_0(x, t) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_0(x, t) \right) \right)
\]
\[
\phi_2(x, t) = \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_1(x, t) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_1(x, t) \right) \right)
\]
\[
\phi_3(x, t) = \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_2(x, t) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_2(x, t) \right) \right)
\]
We follow this direction to obtain
\[
\phi_{n+1}(x, t) = \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_{n+1}(x, t) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_{n+1}(x, t) \right) \right)
\]
Finally, with the help of Eq. (34), one can easily explore the rest of the iterations
\[
\phi_1(x, t) = \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_0(x, t) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \phi_0(x, t) \right) \right)
\]
\[
= \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( -\pi^2 \cos(\pi x) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \cos(\pi x) \right) \right)
\]
\[
= \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \cos(\pi x) \right) \right) + \mathcal{N}^{-\nu} \left( \frac{\nu^\nu}{\sqrt[\nu]{x}} \mathcal{N}^\nu \left( \cos(\pi x) \right) \right)
\]
\[
= \left( 1 - \pi^2 \right) \cos(\pi x) \frac{\nu^\nu}{\sqrt[\nu]{x}} \frac{1}{\left( \Gamma(\nu + 1) \right)}. \]

**Table 1.** Numerical Values for example 1 for different values of \( \nu \).

| \( t \) | \( \nu = 0.45 \) | \( \nu = 0.6 \) | \( \nu = 0.75 \) | \( \nu = 1 \) |
|-------|----------------|----------------|----------------|----------------|
|       | Numerical      | Exact          | Numerical      | Exact          |
| 0.2   | 0.931788       | 0.702985       | 0.471891       | 0.241863       |
| 0.4   | 0.706152       | 1.02876        | 0.897507       | 0.564371       |
| 0.6   | -0.432075      | 0.794303       | 1.11755        | 0.926696       |
| 0.8   | -2.4294        | -0.222495      | 0.873956       | 1.2210187      |
| 1     | -5.21527       | -2.18641       | -0.117103      | 1.2535556      |
Likewise,

$$\phi_2(x, t) = (1 - \pi^2)^2 \cos(\pi x) \frac{t^{2\nu}}{\Gamma(2\nu + 1)}$$
$$\phi_3(x, t) = (1 - \pi^2)^3 \cos(\pi x) \frac{t^{3\nu}}{\Gamma(3\nu + 1)}.$$

Now, our approximate solution for $\phi(x, t)$ is

$$\phi(x, t) = \sum_{n=0}^{\infty} \phi_n(x, t)$$
$$= \phi_0(x, t) + \phi_1(x, t) + \phi_2(x, t) + \phi_3(x, t) + \cdots$$
$$= \cos(\pi x) \left( 1 + (1 - \pi^2) \frac{t^{\nu}}{\Gamma(\nu + 1)} + (1 - \pi^2)^2 \frac{t^{2\nu}}{\Gamma(2\nu + 1)} + (1 - \pi^2)^3 \frac{t^{3\nu}}{\Gamma(3\nu + 1)} + \cdots \right)$$

Therefore, our exact solution is

$$\phi(x, t) = \cos(\pi x) E_\nu((1 - \pi^2)t^\nu).$$

Substitute $\nu = 1$ in Eq. (35) to conclude

$$\phi(x, t) = \cos(\pi x) e^{(1 - \pi^2)t}.$$

This is indeed the intended solution for equation (29) which exists throughout the literature (Figures 2 and 3).

**Remark.** Figure 2 shows that the solution peak is high and one can see that the peak of the solutions of the diffusion equation becomes more and more smooth as the fractional factor $\nu$ increases (Table 2).

### Fractional Systems of Ordinary Differential Equations

Now let us examine two models of systems of FDEs. Then, we present numerical values in tables for some values of $t$. We only used $5^{th}$ order approximate solutions for the two functions.

**Figure 2.** Exact solutions for example 4.2 with $\nu = 0.25, \nu = 0.50$, respectively.

**Figure 3.** Exact solutions for example 4.2 with $\nu = 0.75, \nu = 1$, respectively.
Example 3: Given the system of LFDE in the form:
\[ cD_\nu t x(t) = 2x(t) + y(t), \quad 0 < \nu \leq 1 \]
\[ cD_\eta t y(t) = x(t) + 2y(t), \quad 0 < \eta \leq 1 \]  
(36)
together with value conditions
\[ x(0) = 2, \quad y(0) = 1. \]  
(37)
Applying the natural transform of equations (36) and (37), one conclude
\[ N^+(x(t)) = \frac{2}{s} + 2 \frac{u^\nu}{s^\nu} N^+(x(t)) + \frac{u^\nu}{s^\nu} N^+(y(t)) \]
\[ N^+(y(t)) = \frac{1}{s} + \frac{u^\eta}{s^\eta} N^+(x(t)) + 2 \frac{u^\eta}{s^\eta} N^+(y(t)) \]  
(38)
Using the \( N^{-1} \) on equation (38) to arrive at
\[ x(t) = 2 + 2 N^{-1} \left( \frac{u^\nu}{s^\nu} N^+(x(t)) \right) + N^{-1} \left( \frac{u^\nu}{s^\nu} N^+(y(t)) \right) \]
\[ y(t) = 1 + N^{-1} \left( \frac{u^\eta}{s^\eta} N^+(x(t)) \right) + 2 N^{-1} \left( \frac{u^\eta}{s^\eta} N^+(y(t)) \right) \]  
(39)
Suppose our solutions are of the forms
\[ x(t) = \sum_{n=0}^{\infty} x_n(t), \quad y(t) = \sum_{n=0}^{\infty} y_n(t) \]  
(40)
Then,
\[ \sum_{n=0}^{\infty} x_n(t) = 2 + 2 N^{-1} \left( \frac{u^\nu}{s^\nu} N^+(x(t)) \right) + N^{-1} \left( \frac{u^\nu}{s^\nu} N^+(y(t)) \right) \]
\[ \sum_{n=0}^{\infty} y_n(t) = 1 + N^{-1} \left( \frac{u^\eta}{s^\eta} N^+(x(t)) \right) + 2 N^{-1} \left( \frac{u^\eta}{s^\eta} N^+(y(t)) \right) \]  
(41)
Using equation (41) we obtain
\[ x_0(0) = 2 \quad y_0(0) = 1 \]
\[ x_1(t) = 2 N^{-1} \left( \frac{u^\nu}{s^\nu} N^+(x_0(t)) \right) + N^{-1} \left( \frac{u^\nu}{s^\nu} N^+(y_0(t)) \right) = \frac{5^\nu}{\Gamma(\nu+1)} \]
\[ y_1(t) = -N^{-1} \left( \frac{u^\eta}{s^\eta} N^+(x_0(t)) \right) + N^{-1} \left( \frac{u^\eta}{s^\eta} N^+(y_0(t)) \right) = \frac{4^\nu}{\Gamma(\eta+1)} \]

Table 2. Numerical results for Example 2 for distinct values for \( \nu \).

| \( \nu \) | \( t \) | Numerical | Exact |
|---|---|---|---|
| 0.25 | 0.02 | 0.20104552 | 0.20104552 |
| 0.04 | 0.17414299 | 0.28187077 | 0.28187077 |
| 0.06 | 0.15978332 | 0.23813296 | 0.23813296 |
| 0.08 | 0.1502041 | 0.21016121 | 0.21016121 |
| 0.1 | 0.14310464 | 0.19025412 | 0.19025412 |
| 0.5 | 0.14216061 | 0.25944897 | 0.25944897 |
| 0.04 | 0.12313769 | 0.19931273 | 0.19931273 |
| 0.06 | 0.11298741 | 0.16838543 | 0.16838543 |
| 0.08 | 0.10621033 | 0.14860641 | 0.14860641 |
| 0.1 | 0.10119026 | 0.13452998 | 0.13452998 |
| 0.5 | 0.10052276 | 0.18345812 | 0.18345812 |
| 0.04 | 0.087071497 | 0.14093539 | 0.14093539 |
| 0.06 | 0.079894162 | 0.11906468 | 0.11906468 |
| 0.08 | 0.075102048 | 0.1050806 | 0.1050806 |
| 0.1 | 0.071552321 | 0.09512706 | 0.09512706 |

Figure 4. Approximate solutions \( x(t), y(t) \) for example 1 with some values of \( \nu, \eta \), respectively.
Likewise,
\[ x_2(t) = \frac{10t^{2\nu}}{\Gamma(2\nu + 1)} + \frac{4t^{\nu+\eta}}{\Gamma(\nu + \eta + 1)} \]
\[ y_2(t) = \frac{8t^\eta}{\Gamma(2\eta + 1)} + \frac{5t^{\nu+\eta}}{\Gamma(\nu + \eta + 1)} \]

We proceed in a similar way to get
\[ x_3(t) = \frac{20}{\Gamma(3\nu + 1)} t^{2\nu} + \frac{13}{\Gamma(2\nu + 1)} t^{\nu+2\eta} + \frac{8}{\Gamma(\nu + 2\eta + 1)} t^{\nu+2\eta} \]
\[ y_3(t) = \frac{16}{\Gamma(3\nu + 1)} t^{3\nu} + \frac{28}{\Gamma(2\nu + 1)} t^{\nu+2\eta} + \frac{10}{\Gamma(\nu + 2\eta + 1)} t^{\nu+2\eta} \]
\[ x_4(t) = \frac{40}{\Gamma(4\nu + 1)} t^{4\nu} + \frac{36}{\Gamma(3\nu + 1)} t^{3\nu + 2\eta} + \frac{30}{\Gamma(2\nu + 1)} t^{3\nu + 2\eta} \]
\[ + \frac{16}{\Gamma(\nu + 3\eta + 1)} t^{\nu+3\eta} y_4(t) = \frac{32}{\Gamma(4\nu + 1)} t^{4\nu} + \frac{36}{\Gamma(3\nu + 1)} t^{3\nu + 2\eta} \]
\[ + \frac{20}{\Gamma(2\nu + 2\eta + 1)} t^{2\nu + 2\eta} \]

Finally, the approximate solutions for these functions as
\[ x(t) = \sum_{n=0}^{\infty} x_n(t) \]
\[ y(t) = \sum_{n=0}^{\infty} y_n(t) \]

Thus,
\[ x(t) = 2 + \frac{5t^\nu}{\Gamma(\nu + 1)} + \frac{10t^{2\nu}}{\Gamma(2\nu + 1)} + \frac{4t^{\nu+\eta}}{\Gamma(\nu + \eta + 1)} + \frac{20t^{3\nu}}{\Gamma(3\nu + 1)} \]
\[ + \frac{8t^{\nu+2\eta}}{\Gamma(\nu + 2\eta + 1)} + \frac{14t^{\nu+2\eta}}{\Gamma(2\nu + \eta + 1)} + \frac{5t^{\nu+3\eta}}{\Gamma(3\nu + \eta + 1)} \]
\[ + \cdots y(t) = 1 + \frac{4t^\eta}{\Gamma(\eta + 1)} + \frac{8t^{2\eta}}{\Gamma(2\eta + 1)} + \frac{5t^{\nu+\eta}}{\Gamma(\nu + \eta + 1)} \]
\[ + \cdots \]

Note that when \( \nu = \eta = 1 \), then the exact solutions are
\[ x(t) = e^t + e^{2t}; \quad y(t) = -e^t + e^{3t}. \]

**Example 4:** Suppose we are given a system of LFDE of the form:
\[ {}^cD^\nu x(t) = y(t) - 2x(t), \quad 0 < \nu \leq 1 \]
\[ {}^cD^\eta y(t) = x(t) - 2y(t), \quad 0 < \eta \leq 1 \]

Together with two value conditions
\[ x(0) = 2 \quad y(0) = 1. \]

Apply natural transform to equations (42) and (43), to get
\[ \mathcal{N}^\nu(x(t)) = \frac{2}{s} + \frac{\nu^\nu}{s^\nu} \mathcal{N}^\nu(y(t)) - 2 \frac{\nu^\nu}{s^\nu} \mathcal{N}^\nu(x(t)) \]
\[ \mathcal{N}^\eta(y(t)) = \frac{1}{s} + \frac{\eta^\eta}{s^\eta} \mathcal{N}^\eta(y(t)) - 2 \frac{\eta^\eta}{s^\eta} \mathcal{N}^\eta(x(t)) \]

**Table 3.** The numerical values for \( x(t) \) with some values for \( \nu \) and \( \eta \).

| \( t \) | \( \nu = \eta = 0.5 \) | \( \nu = \eta = 0.6 \) | \( \nu = \eta = 0.75 \) | \( \nu = \eta = 1 \) |
|---|---|---|---|---|
| \( 0.2 \) | 14.4864 | 86.6865 | 5.21454 | 3.04352 |
| \( 0.4 \) | 39.8447 | 22.5925 | 11.4568 | 4.81194 |
| \( 0.6 \) | 81.6907 | 47.9029 | 23.3317 | 7.87177 |
| \( 0.8 \) | 142.325 | 88.08192 | 43.831967 | 13.2487 |
| \( 1 \) | 223.31317 | 146.51415 | 76.394554 | 22.8038 |

**Figure 5.** Approximate solutions \( x(t), y(t) \) for example 2 with some values of \( \nu, \eta \), respectively.
Thus, Eq. (45) become:

\[ x(t) = 2 + \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(y(t)) \right] - 2 \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(x(t)) \right] \]

\[ y(t) = 1 + \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(x(t)) \right] - 2 \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(y(t)) \right] \]

(45)

Suppose our approximate solutions are given as:

\[ x(t) = \sum_{n=0}^{\infty} x_n(t); \quad y(t) = \sum_{n=0}^{\infty} y_n(t). \]

(46)

Thus, Eq. (45) become:

\[ \sum_{n=0}^{\infty} x_n(t) = 2 + \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(y(t)) \right] - 2 \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(x(t)) \right] \]

\[ \sum_{n=0}^{\infty} y_n(t) = 1 + \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(x(t)) \right] - 2 \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(y(t)) \right]. \]

(47)

Using the equations in (47) one concludes:

\[ x_0(0) = 2; \quad y_0(0) = 1 \]

\[ x_1(t) = \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(y_0(t)) \right] - 2 \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(x_0(t)) \right] = -3 t^{\nu} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1)} \]

\[ y_1(t) = \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(x_0(t)) \right] - 2 \mathcal{N}^{-1} \left[ \frac{\partial^{\nu}}{\partial t^{\nu}} N^+(y_0(t)) \right] = 0. \]

Likewise,

\[ x_2(t) = \frac{6 t^{2\nu}}{\Gamma(2\nu + 1)} - \frac{12 t^{3\nu}}{\Gamma(3\nu + 1)} \]

\[ y_2(t) = \frac{-3 t^{2\nu + 3\eta}}{\Gamma(\nu + 3\eta + 1)}. \]

We proceed as before to obtain:

\[ x_3(t) = \frac{12 t^{2\nu + 2\eta}}{\Gamma(2\nu + 2\eta + 1)} \frac{6 t^{3\nu}}{\Gamma(2\nu + 3\nu + 1)} \]

\[ y_3(t) = \frac{12 t^{2\nu + 3\eta}}{\Gamma(2\nu + 2\eta + 1)} \frac{15 t^{2\nu + 2\eta}}{\Gamma(2\nu + 3\nu + 1)} \frac{12 t^{3\nu}}{\Gamma(3\nu + 1)} \frac{24 t^{4\nu}}{\Gamma(4\nu + 1)} \]

Finally, the approximate solutions for these functions are as follows:

\[ x(t) = \sum_{n=0}^{\infty} x_n(t); \quad y(t) = \sum_{n=0}^{\infty} y_n(t). \]

(48)

It follows that,

\[ x(t) = 2 - \frac{3 t^{\nu}}{\Gamma(\nu + 1)} + \frac{6 t^{2\nu}}{\Gamma(2\nu + 1)} - \frac{3 t^{2\nu + 3\eta}}{\Gamma(3\nu + 1)} - \frac{3 t^{3\nu}}{\Gamma(3\nu + 1)} + \frac{6 t^{2\nu + 2\eta}}{\Gamma(2\nu + 2\eta + 1)} + \frac{12 t^{3\nu}}{\Gamma(3\nu + 1)} - \frac{24 t^{4\nu}}{\Gamma(4\nu + 1)} \]

\[ y(t) = 1 - \frac{3 t^{\nu + 3\eta}}{\Gamma(\nu + 3\eta + 1)} + \frac{6 t^{2\nu + 2\eta}}{\Gamma(2\nu + 2\eta + 1)} + \frac{6 t^{2\nu + 3\eta}}{\Gamma(2\nu + 3\nu + 1)} - \frac{15 t^{2\nu + 2\eta}}{\Gamma(2\nu + 2\eta + 1)} - \frac{12 t^{3\nu}}{\Gamma(3\nu + 1)} - \frac{12 t^{3\nu}}{\Gamma(3\nu + 1)} - \frac{12 t^{3\nu}}{\Gamma(3\nu + 1)} \]

Note that when \( \nu = \eta = 1 \), then the exact solutions are

\[ x(t) = e^{-3t} + e^{-t}; \quad y(t) = e^{-3t} - e^{-t}. \]

**Conclusion**

Prior to this work, many techniques were used to handle FDEs. We successfully explore solutions for both linear and nonlinear ordinary FDEs and systems of FDOODEs using the FNDM. We found exact and approximate solutions to systems of ordinary fractional differential equations and fractional diffusion differential equations such as diffusion model using fractional natural decomposition method (FNDM). The results showed that the new scheme is accurate and efficient. We were being able to explore solutions to physical models when \( \nu = \eta = 1 \). The next step for our research is to further apply the new scheme to other FDEs that arises in other areas of scientific fields.
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