Construction of Ricci-type connections by reduction and induction.

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Abstract

Given the Euclidean space $\mathbb{R}^{2n+2}$ endowed with a constant symplectic structure and the standard flat connection, and given a polynomial of degree 2 on that space, Baguis and Cahen \cite{1} have defined a reduction procedure which yields a symplectic manifold endowed with a Ricci-type connection. We observe that any symplectic manifold $(M, \omega)$ of dimension $2n$ ($n \geq 2$) endowed with a symplectic connection of Ricci type is locally given by a local version of such a reduction.

We also consider the reverse of this reduction procedure, an induction procedure: we construct globally on a symplectic manifold endowed with a connection of Ricci-type $(M, \omega, \nabla)$ a circle or a line bundle which embeds in a flat symplectic manifold $(P, \mu, \nabla^1)$ as the zero set of a function whose third covariant derivative vanishes, in such a way that $(M, \omega, \nabla)$ is obtained by reduction from $(P, \mu, \nabla^1)$.

We further develop the particular case of symmetric symplectic manifolds with Ricci-type connections.

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This research was partially supported by an Action de Recherche Concertée de la Communauté française de Belgique.
1 Introduction

A symplectic connection $\nabla$ on a symplectic manifold $(M, \omega)$ of dimension $2n$ is a linear connection which is torsion free and for which $\omega$ is parallel. The space of symplectic connections on $(M, \omega)$, $\mathcal{E}(M, \omega)$ is infinite dimensional.

Selecting some particular class of connections by curvature conditions has, a priori, two interests. The “moduli space” of such particular connections may be finite dimensional; also, on some compact symplectic manifolds which do admit a connection of the chosen class, this connection may be “rigid”.

In this paper, we describe completely the local behaviour of symplectic connections of Ricci-type (see definition below) and give some global description of simply connected symplectic manifolds admitting a connection of Ricci-type.

We denote by $R$ the curvature of $\nabla$ and by $\overline{R}$ the symplectic curvature tensor

$$\overline{R}(X, Y, Z, T) := \omega(R(X, Y)Z, T).$$

For any point $x \in M$, we have the symmetry properties

(i) $R_x(X, Y, Z, T) = -R_x(Y, X, Z, T)$
(ii) $R_x(X, Y, Z, T) = R_x(X, Y, T, Z)$
(iii) $\bigoplus_{X, Y, Z} R_x(X, Y, Z, T) = 0$.

From (i) and (ii), $R_x \in \Lambda^2 T^*_x M \otimes \odot^2 T^*_x M$, where $\odot^k V$ is the symmetrized $k$-tensor product of the vector space $V$.

We denote by $r$ the Ricci tensor of the connection $\nabla$ (i.e. $r_x(X, Y) = \text{tr} [Z \rightarrow R_x(X, Z)Y]$, where $X, Y, Z$ are in $T_x M$); this tensor $r$ is symmetric. We denote by $\rho$ the corresponding endomorphism of the tangent bundle:

$$\omega(X, \rho Y) = r(X, Y)$$
def

so that $\rho_x$ belongs to the symplectic algebra $sp(T_x M, \omega_x)$; in particular $\text{tr} \rho = 0$.

The space $\mathcal{R}_x$ of symplectic curvature tensors at $x$ is

$$\mathcal{R}_x = \ker a \subset \Lambda^2 T^*_x M \otimes \odot^2 T^*_x M$$

where $a$ is the skewsymmetrisation map $a : \Lambda^p T^*_x M \otimes \odot^q T^*_x M \to \Lambda^{p+1} T^*_x M \otimes \odot^{q-1} T^*_x M$

$$a(u_1 \wedge \ldots \wedge u_p \otimes v_1 \ldots v_q) := \sum_{i=1}^q u_1 \wedge \ldots \wedge u_p \wedge v_i \otimes v_1 \ldots \hat{v}_i \ldots v_q.$$
The group $Sp(T_xM,\omega_x)$ acts on $\mathcal{R}_x$. Under this action the space $\mathcal{R}_x$, in dimension $2n \geq 4$, decomposes into two irreducible subspaces \[6\]:

$$\mathcal{R}_x = \mathcal{E}_x \oplus \mathcal{W}_x$$

and the decomposition of the curvature tensor $R_x$ into its $\mathcal{E}_x$ component (denoted $E_x$) and its $\mathcal{W}_x$ component (denoted $W_x$), $R_x = E_x + W_x$, is given by

$$E_x(X, Y, Z, T) = -\frac{1}{2(n+1)} \left[ 2\omega_x(X, Y) r_x(Z, T) + \omega_x(X, Z) r_x(Y, T) 
+ \omega_x(Y, T) r_x(X, Z) - \omega_x(Y, Z) r_x(X, T) - \omega_x(Y, T) r_x(X, Z) \right]$$

A connection $\nabla$ is said to be of Ricci-type if, at each point $x$, $W_x = 0.$ (Let us mention that such connections were called reducible by Vaisman in \[6\]). In dimension 2 ($n = 1$), the space $\mathcal{W}$ vanishes identically; so we shall assume in what follows that the manifold has dimension $m = 2n > 2$.

Let us first recall two interesting features of such connections.

- When a symplectic connection is of Ricci-type, it satisfies the equations:

$$\bigoplus_{X,Y,Z} (\nabla_X r)(Y, Z) = 0.$$

Those are the Euler-Lagrange equations of any natural variational principle whose Lagrangian is a second degree invariant polynomial in the curvature ($r^2$ or $R^2$). Connections which are solutions of those equations are called preferred; they are completely described in dimension 2.

- The condition to be of Ricci-type is the condition on a symplectic connection $\nabla$ to have an integrable almost complex structure $J^\nabla$ on the twistor space over $M$ which is the bundle of all compatible almost complex structures on $M$ (\[2\]).

In this paper, we show that any symplectic manifold $(M, \omega)$ of dimension $2n$ ($n \geq 2$) admitting a symplectic connection of Ricci type has a local model given by a reduction procedure (as introduced by Baguis and Cahen in \[1\]) from the Euclidean space $\mathbb{R}^{2n+2}$ endowed with a constant symplectic structure and the standard flat connection.

We also consider the reverse of this reduction procedure, an induction procedure: we construct globally on a simply connected symplectic manifold endowed with a connection of Ricci-type $(M, \omega, \nabla)$ a circle or a line bundle $N$ which embeds in a flat symplectic manifold $(P, \mu, \nabla^1)$ as the zero set of a function whose third covariant derivative vanishes, in such a way that $(M, \omega, \nabla)$ is obtained by reduction from $(P, \mu, \nabla^1)$.

We finally describe completely the symmetric symplectic manifolds whose canonical connection is of Ricci-type. Those were already studied in \[4\] in collaboration with John Rawnsley.
2 Some properties of the curvature of a Ricci-type connection

Let \((M, \omega)\) be a smooth symplectic manifold of dim \(2n\) \((n \geq 2)\) and let \(\nabla\) be a smooth Ricci-type symplectic connection. The following results follow directly from the definition (and Bianchi’s second identity).

Lemma 2.1 \[\text{[3]}\] The curvature endomorphism reads

\[
R(X,Y) = -\frac{1}{2(n+1)}[-2\omega(X,Y)\rho - \rho Y \otimes X + \rho X \otimes Y - X \otimes \rho Y + Y \otimes \rho X]
\]

where \(X\) denotes the 1-form \(i(X)\omega\) (for \(X\) a vector field on \(M\)) and where, as before, \(\rho\) is the endomorphism associated to the Ricci tensor \(r(U,V) = \omega(U,\rho V)\).

Furthermore:

(i) there exists a vector field \(u\) such that

\[
\nabla_X \rho = -\frac{1}{2n+1}[X \otimes u + u \otimes X];
\]

(ii) there exists a function \(f\) such that

\[
\nabla_X u = -\frac{2n+1}{2(n+1)}\rho^2 X + fX;
\]

(iii) there exists a real number \(K\) such that

\[
tr\rho^2 + \frac{4(n+1)}{2n+1}f = K.
\]

3 Construction by reduction of manifolds with Ricci type connections

Let \(A\) be a nonzero element in the symplectic Lie algebra \(sp(\mathbb{R}^{2n+2}, \Omega')\) where \(\Omega'\) is the standard symplectic structure on \(\mathbb{R}^{2n+2}\). Let \(\Sigma_A\) be the closed hypersurface \(\Sigma_A \subset \mathbb{R}^{2n+2}\) with equation :

\[
\Omega'(x, Ax) = 1;
\]

in order for \(\Sigma_A\) to be non empty we replace, if necessary, \(A\), by \(-A\).

Let \(\tilde{\nabla}\) be the standard flat symplectic affine connection on \(\mathbb{R}^{2n+2}\). If \(X, Y\) are vector fields tangent to \(\Sigma_A\) define:

\[
(\nabla^\Sigma_A X)Y(x) = (\tilde{\nabla}_X Y)(x) - \Omega'(AX,Y)x;
\]
this is a torsion free linear connection on $\Sigma_A$.

The vector field $Ax$ is an affine vector field for this connection; it is clearly complete and we denote by $\phi_t$ the 1-parametric group of diffeomorphisms of $\Sigma_A$ generated by this vector field; clearly this flow is given by the restriction to $\Sigma_A$ of the action of $\exp tA$ on $\mathbb{R}^{2n+2}$.

Since the vector field $Ax$ is nowhere 0 on $\Sigma_A$, for any $x_0 \in \Sigma_A$, there exists:
- a neighborhood $U_{x_0} \subset \Sigma_A$,
- a ball $D \subset \mathbb{R}^{2n}$ of radius $r_0$, centered at the origin,
- a real interval $I = (-\epsilon, \epsilon)$
- and a diffeomorphism $\chi : D \times I \to U_{x_0}$

such that $\chi(0, 0) = x_0$ and $\chi(y, t) = \phi_t(\chi(y, 0))$. We shall denote

$$\pi : U_{x_0} \to D \quad \pi = p_1 \otimes \chi^{-1}.$$ 

If we view $\Sigma_A$ as a constraint manifold in $\mathbb{R}^{2n+2}$, $D$ is a local version of the Marsden-Weinstein reduction of $\Sigma_A$ around the point $x_0$.

If $x \in \Sigma_A$, $T_x \Sigma_A = \langle Ax \rangle^{\perp}$, where $\langle v_1, \ldots, v_p \rangle$ denotes the subspace spanned by $v_1, \ldots, v_p$ and $\perp$ denotes the orthogonal relative to $\Omega'$; let $\mathcal{H}_x (\subset T_x \Sigma_A) = \langle x, Ax \rangle^{\perp}$; then

$$T_x \mathbb{R}^{2n+2} = (\mathcal{H}_x \oplus \mathbb{R}Ax) \oplus \mathbb{R}x$$

and $\pi_x$ defines an isomorphism between $\mathcal{H}_x$ and the tangent space $T_y D$ for $y = \pi(x)$. A vector belonging to $\mathcal{H}_x$ will be called horizontal.

A symplectic form on $D$, $\omega$, is defined by

$$\omega_y (X, Y) = \Omega'(\tilde{X}, \tilde{Y}) \quad y = \pi(x)$$

where $\tilde{X}$ (resp. $\tilde{Y}$) denotes the horizontal lift of $X$ (resp. $Y$). A symplectic connection $\nabla$ on $D$ is defined by

$$\nabla_{\tilde{X}}\tilde{Y}(x) = \nabla^\Sigma_A \tilde{Y}(x) + \Omega'(\tilde{X}, \tilde{Y})Ax$$

**Proposition 3.1** ([1]) *The manifold $(D, \omega)$ is a symplectic manifold and $\nabla$ is a symplectic connection of Ricci-type.*

Furthermore, a direct computation shows that the corresponding $\rho, u$ and $f$ are given
by:

\[
\bar{\rho} \bar{X}(x) = -2(n+1)\bar{A}_x \bar{X} \tag{10}
\]

\[
\bar{u}(x) = -2(n+1)(2n+1)\bar{A}_x x \tag{11}
\]

\[
(\pi^* f)(x) = 2(n+1)(2n+1)\Omega'(A^2 x, Ax) \tag{12}
\]

where \(\bar{A}_x^k\) is the map induced by \(A^k\) with values in \(\mathcal{H}_x\):

\[
\bar{A}_x^k(X) = A^k X + \Omega(A^k X, x)Ax - \Omega(A^k X, Ax)x
\]

4 Local models for symplectic connections of Ricci-type

The properties of a symplectic connection of Ricci-type, as stated in Lemma 2.1, imply in particular that

- the curvature tensor is determined by \(\rho\);
- its covariant derivative is determined by \(u\);
- its second covariant derivative is determined by \(\rho\) and \(f\), hence by \(\rho\) and \(K\) with \(K\) a constant;
- the 3rd covariant derivative of the curvature is determined by \(u, \rho, K\) and similarly for all orders.

Hence

**Corollary 4.1** Let \((M, \omega)\) be a smooth symplectic manifold of dimension \(2n\) \((n \geq 2)\) and let \(\nabla\) be a smooth Ricci-type connection. Let \(p_0 \in M\); then the curvature \(R_{p_0}\) and its covariant derivatives \((\nabla^k R)_{p_0}\) (for all \(k\)) are determined by \((\rho_{x_0}, u_{x_0}, K)\).

**Corollary 4.2** Let \((M, \omega, \nabla)\) (resp. \((M', \omega', \nabla')\)) be two real analytic symplectic manifolds of the same dimension \(2n\) \((n \geq 2)\) each of them endowed with a symplectic connection of Ricci-type.

Assume that there exists a linear map \(b : T_{x_0} M \to T_{x'_0} M'\) such that (i) \(b^* \omega'_{x'_0} = \omega_{x_0}\)
(ii) \(b u_{x_0} = u'_{x'_0}\) (iii) \(b \circ \rho_{x_0} \circ b^{-1} = \rho'_{x'_0}\). Assume further that \(K = K'\).

Then the manifolds are locally affinely symplectically isomorphic, i.e. there exists a normal neighborhood of \(x_0\) (resp. \(x'_0\)) \(U_{x_0}\) (resp. \(U'_{x'_0}\)) and a symplectic affine diffeomorphism \(\varphi : (U_{x_0}, \omega, \nabla) \to (U'_{x'_0}, \omega', \nabla')\) such that \(\varphi(x_0) = x'_0\) and \(\varphi_{**x_0} = b\).
This follows from classical results, see for instance theorem 7.2 and corollary 7.3 in 
Kobayashi-Nomizu volume 1 [5].

Consider now \((M, \omega, \nabla)\) a real analytic symplectic manifold of dimension \(2n (n \geq 2)\) 
endowed with an analytic Ricci-type symplectic connection; denote as before by \(u, \rho, f\) 
and \(K\) the associated quantities (see lemma 2.1).

Let \(p_0\) be a point in \(M\) and choose \(\xi_0\) a symplectic frame of \(T_{p_0}M\), i.e. a linear 
symplectic isomorphism \(\xi_0 : (\mathbb{R}^{2n}, \Omega) \to (T_{p_0}, \omega_{p_0})\), where \(\Omega\) is the standard symplectic form on \(\mathbb{R}^{2n}\).

Denote by \(\tilde{u}(\xi_0)\) the element of \(\mathbb{R}^{2n}\) corresponding to \(u(p_0)\), i.e. 
\[
\tilde{u}(\xi_0) = (\xi_0)^{-1} u(p_0)
\]
and by \(\tilde{\rho}(\xi_0)\) the element of \(\text{sp}(\mathbb{R}^{2n}, \Omega)\) corresponding to \(\rho(p_0)\), i.e. 
\[
\tilde{\rho}(\xi) = (\xi_0)^{-1} \rho(p_0) \xi_0.
\]

Define an element \(A\) of \(\text{sp}(\mathbb{R}^{2n+2}, \Omega')\) as:
\[
A = \begin{pmatrix}
0 & f(p_0) & -\tilde{u}(\xi_0) \\
\frac{1}{2(n+1)(2n+1)} & 2(n+1)(2n+1) & 0 \\
0 & -\tilde{u}(\xi_0) & 2(n+1)(2n+1)
\end{pmatrix}
\]

where \(\tilde{u}(\xi_0) = i(\tilde{u}(\xi_0))\Omega\) and where we have chosen a basis \(\{e_0, e_0', e_1, \ldots, e_{2n}\}\) of the 
symplectic vector space \(\mathbb{R}^{2n+2}\) relative to which the symplectic form has matrix
\[
\Omega' = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & \Omega
\end{pmatrix}, \quad \Omega = \begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}.
\]

Consider the local reduction procedure described in section 3 from the element \(A\) 
defined above around the point \(x_0 = e_0 \in \Sigma_A = \{x \in \mathbb{R}^{2n+2} | \Omega'(x, Ax) = 1\}\).

From what we saw in section 3 this yields a symplectic manifold with a Ricci-type 
connection \((M', \omega', \nabla')\).

Denote by \(\pi'\) the map \(\pi' : U_{e_0} \to M'\) where \(U_{e_0}\) is the neighborhood of \(e_0\) in \(\Sigma_A \subset \mathbb{R}^{2n+2}\) 
considered in section 3 and consider \(y_0 = \pi'(e_0)\). Then \(\mathcal{H}_{e_0} = \{e_0, Ae_0 = e_0' = \}
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This map $b$ is a linear symplectic isomorphism since
\[
\omega'_y(bX, bY) = \Omega'(j\xi_0^{-1}X, j\xi_0^{-1}Y) = \Omega(\xi_0^{-1}X, \xi_0^{-1}Y) = \omega_{p_0}(X, Y).
\]
Furthermore
\[
u'(y_0) = \pi'_{*e_0}(\bar{u}'(x_0)) = \pi'_{*e_0}(-2(n+1)(2n+1)A^2e_0 - \Omega'(A^2e_0, Ae_0)e_0))
= \pi'_{*e_0}(j\bar{u}'(\xi_0)) = \pi'_{*e_0}(\xi_0)^{-1} u(p_0) = bu(p_0)
\]
\[
\rho'(y_0)bX = \pi'_{*e_0}\rho'(y_0)X(e_0) = \pi'_{*e_0}(-2(n+1)A_{e_0}(j\xi_0^{-1}(X))) = \pi'_{*e_0}(j\bar{\rho}(\xi_0)\xi_0^{-1}(X))
\]
so that $\rho'(y_0)b = b\rho(p_0)$
\[
(f')(y_0) = 2(n+1)(2n+1)\Omega'(A^2e_0, Ae_0) = f(p_0).
\]

Hence we have

**Theorem 4.3** Any real analytic symplectic manifold with a Ricci-type connection is locally symplectically affinely isomorphic to the symplectic manifold with a Ricci-type connection obtained by a local reduction procedure around $e_0 = (1, 0, \ldots, 0)$ from a constraint surface $\Sigma_A$ defined by a second order polynomial in the standard flat symplectic manifold $(\mathbb{R}^{2n+2}, \Omega', \dot{\nabla})$.

## 5 Construction of a contact manifold which is a global circle or line bundle over $M$

Consider $(M, \omega, \nabla)$ a smooth symplectic manifold of dimension $2n > 2$ with a smooth Ricci-type connection and let $B(M) \xrightarrow{\pi} M$ be the $Sp(\mathbb{R}^{2n}, \Omega)$ principal bundle of symplectic frames over $M$. (An element in the fiber over a point $p \in M$ is a symplectic isomorphism $\xi : (\mathbb{R}^{2n}, \Omega) \to (T_p M, \omega_p)$).

As before, we consider $\tilde{u} : B(M) \to \mathbb{R}^{2n}$ the $Sp(\mathbb{R}^{2n}, \Omega)$ equivariant function given by
\[
\tilde{u}(\xi) = \xi^{-1}u(x) \text{ where } x = \pi(\xi)
\]
and $\tilde{\rho} : B(M) \to sp(\mathbb{R}^{2n}, \Omega)$ the $Sp(\mathbb{R}^{2n}, \Omega)$ equivariant function given by
\[
\tilde{\rho}(\xi) = \xi^{-1}\rho(x)\xi
\]
and we define the $Sp(\mathbb{R}^{2n}, \Omega)$ equivariant map $\tilde{A} : B(M) \to sp(\mathbb{R}^{2n+2}, \Omega')$

$$\tilde{A}(\xi) = \begin{pmatrix}
0 & (\pi^* f)(\xi) & -\tilde{u}(\xi) \\
1 & 0 & 0 \\
0 & -\tilde{u}(\xi) & -\tilde{\rho}(\xi)
\end{pmatrix}
\begin{pmatrix}
2(n+1)(2n+1) \\
0 \\
2(n+1)(2n+1)
\end{pmatrix}$$  \hspace{1cm} (13)

where $V = i(V)\Omega$ for $V$ in $\mathbb{R}^{2n}$.

We inject the symplectic group $Sp(\mathbb{R}^{2n}, \Omega)$ into $Sp(\mathbb{R}^{2n+2}, \Omega')$ as the set of matrices

$$\tilde{j}(A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix} \quad A \in Sp(\mathbb{R}^{2n}, \Omega).$$

**Lemma 5.1** Define the 1-form $\alpha$ on $B(M)$, with values in $sp(\mathbb{R}^{2n+2}, \Omega')$ by:

$$\alpha_{\xi}(\overline{X}^\text{hor}) = \begin{pmatrix}
0 & -\omega_x(u,X) & -\tilde{\rho}(X)(\xi) \\
0 & 0 & 2(n+1) \\
\tilde{X}(\xi) & -\tilde{\rho}(X)(\xi) & 2(n+1)
\end{pmatrix}
\begin{pmatrix}
2(n+1)(2n+1) \\
0 \\
2(n+1)
\end{pmatrix}$$  \hspace{1cm} (14)

where $X \in T_xM$ with $x = \pi(\xi)$ and $\overline{X}^\text{hor}$ is the horizontal lift of $X$ in $T_\xi B(M)$, and by:

$$\alpha(C^*) = \tilde{j}_*(C)$$  \hspace{1cm} (15)

for all $C \in sp(\mathbb{R}^{2n}, \Omega)$ where $C^*$ denotes the fundamental vertical vector field on $B(M)$ associated to $C$ ($C^*_\xi = \frac{d}{dt}\xi \exp tC_{\xi_0}$).

This form has the following properties:

(i) $R_h^* \alpha = \text{Ad}(\tilde{j}(h^{-1})) \alpha$ \hspace{1cm} $\forall h \in Sp(\mathbb{R}^{2n}, \Omega)$;

(ii) $d\tilde{A} = -[\alpha, \tilde{A}]$;

(iii) $d\alpha + [\alpha, \alpha] = -2\tilde{A}^*\pi^*\omega$

When one has a $G$-principal bundle $P \xrightarrow{p} M$, an embedding of the group $G$ in a larger group $G'$, $j : G \to G'$, and a 1-form $\alpha$ with values in the Lie algebra of $G'$, such that $\alpha(C^*) = j_*(C)$ for all $C$ in the Lie algebra of $G$ and $R_h^* \alpha = \text{Ad}(j(h^{-1})) \alpha$ for all $h$ in $G$, one can build the $G'$-principal bundle $P' = P \times_G G' \xrightarrow{j'} M$ and the unique connection 1-form on $P'$, $\alpha'$ satisfying $i^* \alpha' = \alpha$ where $i : P \to P'; \xi \to [(\xi, 1)]$. 

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In our situation we build the $Sp(\mathbb{R}^{2n+2}, \Omega')$-principal bundle

$$B'(M) = B(M) \times_{Sp(\mathbb{R}^{2n}, \Omega)} Sp(\mathbb{R}^{2n+2}, \Omega')$$

whose elements are equivalence classes of pairs $(\xi, g) \in B(M), g \in Sp(\mathbb{R}^{2n+2}, \Omega')$ with $(\xi, g)$ equivalent to $(\xi h, j(h^{-1})g)$ $\forall h \in Sp(\mathbb{R}^{2n}, \Omega)$.

The projection $\pi': B(M)' \to M$ maps $[(\xi, g)]$ to $\pi(\xi)$.

The connection 1-form $\alpha'$ is characterised by the fact that

$$\alpha'_{[\xi, 1]}([X, 0]) = \alpha(\mathfrak{X}^{hor})$$

and the equations above give:

**Lemma 5.2** The curvature 2-form of the connection 1-form $\alpha'$ is equal to $-2\tilde{\alpha}' \pi'^* \omega$ where $\tilde{\alpha}'$ is the unique $Sp(\mathbb{R}^{2n+2}, \Omega')$-equivariant extension of $\tilde{\alpha}$ to $B'(M)$.

This curvature 2-form is invariant by parallel transport $(d\theta' \text{curv}(\alpha') = 0)$.

Thus the holonomy algebra of $\alpha'$ is of dimension 1.

**Corollary 5.3** Assume $M$ is simply connected. The holonomy bundle of $\alpha'$ is a circle or a line bundle over $M$, $N \xrightarrow{\pi'} M$. This bundle has a natural contact structure $\nu$ given by the restriction to $N \subset B(M)'$ of the 1-form $-\alpha'$ (viewed as real valued since it is valued in a 1-dimensional algebra). One has $d\nu = 2\pi'^* \omega$.

It is enlightening to point out the link between the holonomy bundle $N$ over $M$ and the constraint surface $\Sigma_A$ when one sees $M$ as obtained (locally) by reduction. The link is only local since $\Sigma_A$ is in general not a principal bundle over $M$; in fact in most cases the quotient of $\Sigma_A$ by the action of the group $exp tA$ is at best an orbifold.

Let $A$ be a nonzero element of $sp(\mathbb{R}^{2n+2}, \Omega')$ and let $\Sigma_A = \{ y \in \mathbb{R}^{2n+2} | \Omega'(y, Ay) = 1 \}$; we assume as before that it is not empty. Assume that $(M, \omega, \nabla)$ is obtained by reduction from $\Sigma_A$ (as before, we restrict ourselves to some open set in $\Sigma_A$).

Let $y_0$ be a point in $\Sigma_A$, let $x_0 = \pi(y_0) \in M$ and choose a symplectic frame $\xi_0$ at $x_0$. Let $\gamma(t)$ be a curve in $M$ such that $\gamma(0) = x_0$. Let $\xi(t)$ be the symplectic frame at $\gamma(t)$ obtained by parallel transport along $\gamma$ from $\xi_0$ and let $y(t)$ be the horizontal curve in $\Sigma_A$ lifting $\gamma(t)$ from $y_0$ (i.e. $\pi(y(t)) = \gamma(t)$ and $\Omega'(y(t), y(t)) = 0$). Define the element $C(t)$ of $Sp(\mathbb{R}^{2n+2}, \Omega')$ as the matrix whose columns are

$$C(t) = \begin{pmatrix} y(t) & Ay(t) & \xi(t) \end{pmatrix}$$

where $\xi(t)$ consists of the $2n$ vectors which are the horizontal lifts at the point $y(t)$ of the vectors of the frame $\xi(t)$ (the image under the map $\xi(t)$ of the usual basis of $\mathbb{R}^{2n}$).
Then
\[ \frac{d}{dt}C(t)|_s = C(s) \alpha(s)(\gamma(s)^{\text{hor}}) \]
where \( \alpha \) is the 1-form on \( B(M) \) defined in (14) and where \( \gamma^{\text{hor}} \) is the horizontal lift of \( X \) in \( B(M) \); hence \( \gamma(s)^{\text{hor}} = \dot{\gamma}(s) \).

Let \( B'(M) \) be the \( Sp(\mathbb{R}^{2n+2}, \Omega') \)-principal bundle over \( M \) considered above and let \([([\xi_0, \Lambda_0]) \) (where \( \Lambda_0 \) is an element in \( Sp(\mathbb{R}^{2n+2}, \Omega') \)) be a point of \( B'(M) \) above \( x_0 \). The horizontal lift of \( \gamma(t) \) to \( B'(M) \) starting from \([([\xi_0, \Lambda_0]) \) lives in the holonomy subbundle containing this point; it reads
\[ ([\xi(t), D(t)]) \]
where \( \xi(t) \) has been defined above and where \( D(t) \) obeys the differential equation
\[ \frac{d}{dt}D(t)|_s = -\alpha(s)(\dot{\xi}(s)).D(s) \]
and has initial value \( \Lambda_0 \).

Define the map above \( \gamma \) which sends \( y(t) \) to \([([\xi(t), D(t)]) \) where
\[ D(t) = C^{-1}(t)C(0)\Lambda_0; \]
this map sends elements of \( \Sigma_A \) to elements in the holonomy bundle through \([([\xi_0, \Lambda_0]) \).

The map from the holonomy bundle through \([([\xi_0, \Lambda_0]) \) to \( \mathbb{R}^{2n+2} \) given by:
\[ ([\xi, D]) \mapsto C_0\Lambda_0D^{-1}e_0 \]
where \( C_0 \) is a fixed element in \( Sp(\mathbb{R}^{2n+2}, \Omega') \) has value in the hypersurface \( \Sigma_{A'} \) where \( A' = C_0\tilde{A}(\xi_0)C_0^{-1}. \)

6 Embedding of the contact manifold in a flat symplectic manifold

Let \( (M, \omega) \) be a smooth symplectic manifold of dim \( 2n \ (n \geq 2) \) and let \( \nabla \) be a smooth symplectic connection of Ricci-type. Let \( (N, \alpha) \) be a smooth \( (2n+1) \)-dimensional contact manifold (i.e. \( \alpha \) is a smooth 1-form such that \( \alpha \wedge (d\alpha)^n \neq 0 \) everywhere). Let \( X \) be the corresponding Reeb vector field (i.e. \( i(X)d\alpha = 0 \) and \( \alpha(X) = 1 \)). Assume there exists a smooth submersion \( \pi : N \to M \) such that \( d\alpha = 2\pi^*\omega \). Then at each point \( x \in N \), \( \ker(\pi_*\alpha) = \mathbb{R}X \) and \( \mathcal{L}_X\alpha = 0 \).

Remark that such a contact manifold exists always if \( M \) is simply connected as we saw in the previous section.
If $U$ is a vector field on $M$ we can define its "horizontal lift" $\overline{U}$ on $N$ by:

$$(i) \quad \pi_* U = 0 \quad (ii) \quad \alpha(\overline{U}) = 0.$$ 

Let us denote by $\nu$ the 2-form $\nu = d\alpha = 2\pi^*\omega$ on $N$. Define a connection $\nabla^N$ on $N$ by:

$$\nabla^N U = \nabla U - \nu(U, V) X$$

$$\nabla^N \overline{X} = -\frac{1}{2(n+1)} \rho \overline{U}$$

$$\nabla^N X = -\frac{1}{2(n+1)(2n+1)} \overline{\nu}$$

where $\rho$ is the Ricci endomorphism of $(M, \nabla)$ and where $u$ is the vector field on $M$ appearing in $\nabla \rho$, see lemma 2.1. Then $\nabla^N$ is a torsion free connection on $N$ and the Reeb vector field $X$ is an affine vector field for this connection.

The curvature of this connection has the following form:

$$R^N(U, V)W = \frac{1}{2(n+1)} \left[ \nu(\rho V, W) \overline{U} - \nu(\rho U, W) V \right]$$

$$R^N(U, V)X = \frac{1}{2(n+1)(2n+1)} \left[ \nu(\overline{U}, V) U - \nu(\overline{U}, U) V \right]$$

$$R^N(U, X)V = \frac{1}{2(n+1)(2n+1)} \nu(\overline{U}, \overline{V}) U + \frac{1}{2(n+1)} \nu(\overline{U}, \rho V) X$$

$$R^N(U, X)X = \frac{1}{2(n+1)(2n+1)} \left[ -\pi^* f U + \nu(\overline{U}, \overline{u}) X \right]$$

where $f$ is the function appearing in lemma 2.1.

Consider now the embedding of the contact manifold $N$ into the symplectic manifold $(P, \mu)$ of dimension $2n + 2$, where

$$P = N \times \mathbb{R}$$

and, if we denote by $s$ the variable along $\mathbb{R}$ and let $\theta = e^{2s} p^*_1 \alpha$ ($p_1 : P \to N$), we set

$$\mu = d\theta = 2e^{2s} ds \wedge \alpha + e^{2s} d\alpha$$

and let $i : N \to P$ $x \mapsto (x, 0)$. Obviously $i^* \mu = \nu$.

We now define a connection $\nabla^1$ on $P$ as follows. If $Z$ is a vector field along $N$, we denote by the same letter the vector field on $P$ such that

$$(i) \quad Z_{i(x)} = i_{*x} Z \quad (ii) \quad [Z, \partial_s] = 0.$$
The formulas for $\nabla^1$ are:

$$\nabla^1_Z Z' = \nabla^N Z Z' + \gamma(Z, Z') \partial_s$$

where

$$\gamma(Z, Z') = \gamma(Z', Z)$$

$$\gamma(X, X) = \frac{1}{2(n+1)(2n+1)} \pi^* f$$

$$\gamma(X, U) = -\frac{1}{2(n+1)(2n+1)} \nu(\pi, U)$$

$$\gamma(U, V) = \frac{1}{2(n+1)} \nu(U, \rho V)$$

and

$$\nabla^1_Z \partial_s = \nabla^1_{\partial_s} Z = Z$$

$$\nabla^1_{\partial_s} \partial_s = \partial_s.$$

**Theorem 6.1** The connection $\nabla^1$ on $(P, \mu)$ is symplectic and has zero curvature.

**Proposition 6.2** Let $\psi(s)$ be a smooth function on $P$. Then $\psi$ has vanishing third covariant differential if and only if

$$\partial^2_s \psi - 2 \partial_s \psi = 0. \quad (16)$$

In particular the function $e^{2s}$ has this property.

The procedure described above is called the induction.

Let $(P, \mu, \nabla^1)$ be as above and let $\Sigma = N$ be the constrained submanifold defined by $e^{2s} = 1$. Let $Y$ be the vector field transversal to $\Sigma$ such that $i(Y)\mu = \alpha$, thus $Y = \partial_s$.

Let $H$ be the 1-parametric group generated by $X$. Then $\Sigma/H$ can be identified with $M$ and $(M, \omega)$ is the classical Marsden Weinstein reduction of $(P, \mu)$ for the constraint $\Sigma$.

The connection $\nabla$ on $M$ is obtained from the flat connection $\nabla^1$ on $(P, \mu)$ by reduction. Hence

**Corollary 6.3** Any smooth simply connected symplectic manifold with a Ricci-type connection $(M, \omega, \nabla)$ can be obtained by reduction from an hypersurface $\Sigma$ in a flat symplectic manifold $(P, \mu, \nabla^1)$ defined by the 1—level set of a function $\psi$ on $P$ whose third covariant derivative vanishes.
Corollary 6.4 Any smooth simply connected symplectic manifold with a Ricci-type connection \((M, \omega, \nabla)\) is automatically analytic.

Proof Since \((P, \mu, \nabla^1)\) is locally symmetric, \(P, \mu\) and \(\nabla^1\) are real analytic and the explicit construction given preserves analyticity. \(\square\)

7 Symmetric symplectic spaces with Ricci-type connections

Lemma 7.1 The reduction construction described in section \(\Box\) yields a locally symmetric symplectic space (i.e. such that the curvature tensor is parallel) if and only if the element \(0 \neq A \in \text{sp}(\mathbb{R}^{2n+2}, \Omega')\) satisfies \(A^2 = \lambda I\) for a constant \(\lambda \in \mathbb{R}\).

Proof Indeed the connection \(\nabla\) has parallel curvature tensor if and only if \(\nabla \rho = 0\) hence iff \(u = 0\). From the formulas above, this is true iff
\[
A^2_x(x) = A^2 x - \Omega'(A^2 x, Ax)x = 0
\]
for any \(x \in \Sigma_A\). When \(u = 0\), \(f\) is a constant (cf Lemma 2.1) and it follows from Lemma 3.1 that \(\Omega'(A^2 x, Ax)\) is a constant \(\lambda\). Since \(\Sigma_A\) contains a basis of \(\mathbb{R}^{2n+2}\), this yields \(A^2 = \lambda I\). \(\square\)

Proposition 7.2 If \(0 \neq A \in \text{sp}(\mathbb{R}^{2n+2}, \Omega')\) satisfies \(A^2 = \lambda I\) for a constant \(\lambda \in \mathbb{R}\), the quotient of \(\Sigma_A\) by the action of \(\exp tA\) is a manifold and the natural projection map \(\Sigma_A \to M\) is a submersion which endows \(\Sigma_A\) with a structure of circle or line bundle over \(M\).

Proof Consider \(0 \neq A \in \text{sp}(\mathbb{R}^{2n+2}, \Omega')\) so that \(A^2 = \lambda I\).

Case 1: \(\lambda > 0\), say \(\lambda = k^2\) with \(k > 0\).

Then there exists a basis of \(\mathbb{R}^{2n+2}\) in which
\[
A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]
so that \(\Sigma_A = \{(u, v) \mid u, v \in \mathbb{R}^{n+1} \mid -2ku \cdot v = 1\}\). The flow of the vector field \(Ax\) is given by \(\phi_t = e^{tA}\).

The map \(\pi : \Sigma_A \to TS^n = \{(u', v') \mid u', v' \in \mathbb{R}^{n+1} \mid u' \cdot v' = 1, u' \cdot v' = 0\}\) defined by
\[
\pi(u, v) = \left(\frac{u}{\|u\|}, \|u\|(v + \frac{u}{2k\|u\|^2})\right)
\]
induces a diffeomorphism between $M = \Sigma_A/\phi_t$ and $TS^n$.

$M$ is a non compact simply connected manifold and $\Sigma_A$ is a $\mathbb{R}$–bundle over $TS^n$.

Case 2 : $\lambda < 0$, say $\lambda = -k^2$ with $k > 0$.

One splits $V^C$ ($V = \mathbb{R}^{2n+2}$) into the eigenspaces relative to $A$, $V^C = V_{ik} \oplus V_{-ik}$ and observe that those subspaces are Lagrangian. Choosing a basis $\{z_1, \ldots, z_{n+1}\}$ for $V_{ik}$, consider $\omega_{kl} := \Omega(z_k, \overline{z_l})$; then $i\omega$ is a Hermitian matrix. A change of basis ($z'_j = \sum_{i} z_i U^i_j$) yields $\omega' = t U \omega U$ so we can find a basis for $V_{ik}$ so that $\omega = -2iI_{p,n+1-p} = -2i \begin{pmatrix} I_p & 0 \\ 0 & -I_{n+1-p} \end{pmatrix}$. In the basis of $\mathbb{R}^{2n+2}$ given by $e_j = \frac{1}{2}(z_j + \overline{z}_j)$, $f_j = \frac{1}{2}(z_j - \overline{z}_j)$ we have:

$$A = \begin{pmatrix} 0 & -kI \\ kI & 0 \end{pmatrix}, \quad \Omega' = \begin{pmatrix} 0 & I_{p,n+1-p} \\ -I_{p,n+1-p} & 0 \end{pmatrix}$$

so that $\Sigma_A = \{(u, v) \mid k \sum_{i \leq p} ((u^i)^2 + (v^i)^2) - k \sum_{i > p} ((u^i)^2 + (v^i)^2) = 1\}$. We assume $p \geq 1$ or replace $A$ by $-A$ so that $\Sigma_A \cong S^{2p-1} \times \mathbb{R}^{2n-2p+2}$ is non empty. The flow $\phi_t$ is given by the action of $\exp tA = \begin{pmatrix} \cos ktI & -\sin ktI \\ \sin ktI & \cos ktI \end{pmatrix}$.

Then $M = \Sigma_A/\phi_t = (S^{2p-1} \times \mathbb{R}^{2n-2p+2})/U(1)$, so this reduced manifold is:
- $M = \mathbb{R}^{2n}$ if $p = 1$;
- $M$ is a complex line bundle of rank $q := n + 1 - p$ over the complex projective space $P_{p-1}(\mathbb{C}) = S^{2p-1}/U(1)$ if $1 < p \leq n$;
- $M = P_n(\mathbb{C})$ if $p = n + 1$.

In all those cases, $M$ is simply connected and $\Sigma_A$ is a circle bundle over $M$; the only compact case is $M = P_n(\mathbb{C})$.

Case 3 : $\lambda = 0$, so $A^2 = 0$ with $A \neq 0$. Let us denote by $p$ the rank of $A$. One splits $V = \mathbb{R}^{2n+2}$ into $V = V_0 \oplus V_1 \oplus V_2$ where $V_1 = \text{Im} A$ (dim $V_1 = p$), $V_0 \oplus V_1 = \text{Ker} A$ (so dim $V_0 = 2n + 2 - 2p$ and $V_0$ is symplectic, since $V_0 \oplus V_1 = V_1^\perp$) and $V_2$ is a Lagrangian subspace of $V_0^\perp$ supplementary to $V_1$. Choose a basis of $V_2$ and a corresponding basis (dual for $\Omega'$) in $V_1$ and a symplectic basis of $V_0$ so that in those bases

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A' \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega' = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & I_p \\ 0 & -I_p & 0 \end{pmatrix}$$

and $A'$ is symmetric. Changing the basis of $V_2$ and correspondingly the basis of $V_1$, one can bring $A'$ to the form $A' = I_{r,p-r}$ so that $\Omega'(x, Ax) = \sum_{i \leq r} (w^i)^2 - \sum_{r < i \leq p} (w^i)^2$ if $x = (u, v, w)$.

Hence $\Sigma_A = S^{r-1} \times \mathbb{R}^{2n+2-r}$ if $r > 1$ and $\Sigma_A$ consists of two copies of $\mathbb{R}^{2n+1}$ if $r = 1$. 

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The action of $\phi_t$ on $(u,v,w)$ is given by $\phi_t(u,v,w) = (u,v + tA'w, w)$ so the reduced manifold is:
- two copies of $\mathbb{R}^{2n}$ (if $r = 1$);
- or $M = S^{r-1} \times \mathbb{R}^{2n+1-r}$ if $r > 1$.

In all cases, $M$ is a non compact manifold and $\Sigma_A$ is a line bundle over $M$. \qed

**Proposition 7.3** If $0 \neq A \in \text{sp}(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$, the quotient manifold is a symmetric space and the connection obtained by reduction is the canonical symmetric connection.

**Proof** Any linear symplectic transformation $B$ of $\mathbb{R}^{2n+2}$ which commutes with $A$ obviously induces a symplectic affine transformation $\beta(B)$ of the reduced space $M = \Sigma_A/\phi_t$. If $\pi$ denotes the canonical projection $\pi : \Sigma_A \to M$, then
$$\beta(B) \circ \pi = \pi \circ B.$$ 

In particular the symmetry at the point $x = \pi(y), y \in \Sigma_A$ is induced by
$$B_yu = -u + 2\Omega'(u,y)Ay - 2\Omega'(u,y)Ay.$$

\qed

We shall now describe the tranvection group of $M$ (i.e. the group $G$ of affine transformations of $M$ generated by the composition of two symmetries). Let us denote by $G'$ the group $G' = \{B \in \text{Sp}(\mathbb{R}^{2n+2}, \Omega') \mid BA = AB\}$. The tranvection group of $M$ is clearly included in $\beta(G')$; in fact it is the smallest subgroup of $\beta(G')$ stable under conjugation by a symmetry and which acts transitively on $M$.

Let $x_0 = \pi(y_0)$ be a point in $M$ and let $s_{x_0} = \beta(B_{y_0})$ be the symmetry at this point. Consider the automorphism of $G'$ given by conjugaison by $B_{y_0}$ and denote by $\sigma$ the induced automorphism of the Lie algebra $\mathfrak{g}'$ of $G'$. Let $\mathfrak{p}' = \{C \in \mathfrak{g}' \mid \sigma(C) = -C\}$ and $\mathfrak{f}' = \{C \in \mathfrak{g}' \mid \sigma(C) = C\}$.

The dimension of $\mathfrak{p}'$ is equal to $2n$. Indeed, in a basis $\{e_0, e_{0'}, e_1, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n+2}$ in which $e_0 = y_0$ and $e_{0'} = Ae_0$ and $\Omega' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix}$, one has

$$B_{y_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I_{2n} \end{pmatrix} \quad A = \begin{pmatrix} 0 & \lambda & 0 \\ 1 & 0 & 0 \\ 0 & 0 & A' \end{pmatrix}.$$
$g' = \left\{ \begin{pmatrix} b & \lambda c & A'Z \\ c & -b & -Z \\ Z & A'Z & B \end{pmatrix} \right\}$, \( b, c \in \mathbb{R}; Z \in \mathbb{R}^{2n}; B \in \text{sp}(\mathbb{R}^{2n}, \Omega) \) such that \( BA' = A'B \),

and $p' = \left\{ \begin{pmatrix} 0 & 0 & A'Z \\ 0 & 0 & -Z \\ Z & A'Z & 0 \end{pmatrix} \right\}$, \( Z \in \mathbb{R}^{2n} \).

Hence the Lie algebra of the transvection group is equal to $\beta_*(p' + [p', p'])$.

In all cases the kernel of $\beta$ is given by $\exp \ tA$, and the transvection group is described as follows:

- Case 1: \( \lambda > 0 \), say \( \lambda = k^2 \) with \( k > 0 \).

In the basis of \( \mathbb{R}^{2n+2} \) in which \( A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix} \) and \( \Omega' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \), we have $G' = \left\{ \begin{pmatrix} B & 0 \\ 0 & (tB)^{-1} \end{pmatrix} \right\}$ \( B \in \text{Gl}(n+1, \mathbb{R}) \), and $\beta$ of such an element is the identity iff $B = \lambda I$ with $\lambda > 0$.

The transvection group $G$ is isomorphic to $Sl(n+1, \mathbb{R})$ and

$$TS^n = Sl(n+1, \mathbb{R})/\text{Gl}(n, \mathbb{R}).$$

- Case 2: \( \lambda < 0 \), say $\lambda = -k^2$ with $k > 0$.

In the basis of \( \mathbb{R}^{2n+2} \) in which $A = \begin{pmatrix} 0 & -kI \\ kI & 0 \end{pmatrix}$ and $\Omega' = \begin{pmatrix} 0 & I_{p,n+1-p} \\ -I_{p,n+1-p} & 0 \end{pmatrix}$, we have $G' = \left\{ \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix} | B_1 + iB_2 \in U(p, n+1-p) \right\}$, and $\beta$ of such an element is the identity iff $B_1 + iB_2 = \exp -ikt$.

The transvection group $G$ is isomorphic to $SU(p, n+1-p)$ and

$$M = SU(p, n+1-p)/U(p-1, n+1-p).$$

- Case 3: \( \lambda = 0 \), rank$A = k = p + q$.

In the basis of \( \mathbb{R}^{2n+2} \) in which $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{pq} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\Omega' = \begin{pmatrix} 0 & 0 \\ 0 & -I \\ 0 & I \end{pmatrix}$, we have $g' = \left\{ \begin{pmatrix} D & 0 & C \\ -tC\Omega_1 & -tB & F \\ 0 & 0 & B \end{pmatrix} | D \in \text{sp}(\mathbb{R}^{2n+2-2k}, \omega_1), B \in so(p, q, \mathbb{R}), F \in gl(k, \mathbb{R}), tF = F, C \in \text{Mat}(2n + 2 - 2k, k, \mathbb{R}) \right\}$. 


Then $p'$ is given by the elements of $g'$ for which $D = 0, C = CJ, F = -JFJ, B = -JBJ$ where $J = \begin{pmatrix} 1 & 0 \\ 0 & -I_{k-1} \end{pmatrix}$, so $C = (u 0 \ldots 0)$ for $u \in \mathbb{R}^{2n+2-2k}$, $F = \begin{pmatrix} 0 & t'v \\ v & 0 \end{pmatrix}$

for $v \in \mathbb{R}^{k-1}$ and $B = \begin{pmatrix} 0 & w' \\ w & 0 \end{pmatrix}$ for $w \in \mathbb{R}^{k-1}$ and $w' = I_{p-1,q}w$.

Hence $p' \oplus [p', p']$ is the set of all elements in $g'$ for which $D = 0$.

The transvection group $G$ has algebra $g$ isomorphic to $\{(B, F, C)\}/(0, \mathbb{R}I_{pq}, 0)$ where $B$ is any element in $so(p, q, \mathbb{R})$, $F$ is any symmetric real $k \times k$ matrix, and $C$ is any real $(2n + 2 - 2k) \times k$ matrix and the bracket is defined by

$$\left[ (B, F, C), (B', F', C') \right] = \left[ (B, B'), -t'\Omega_1 C' + t' C' \Omega_1 C - t' BF' + t' B' F, CB' - C'B \right],$$

so when $p + q > 2$, the Levi factor is $so(p, q, \mathbb{R})$ and the radical is a $2$–step nilpotent algebra. If $p = 0$ and $q = 1$ the transvection group is $\mathbb{R}^{2n}$ and the symmetric space is the standard symplectic vector space. If $p = q = 1$ or if $p = 0$ and $q = 2$, the transvection group is solvable but not nilpotent. The two solvable examples are interesting for building exact quantisation.

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