A NEW $A_p$-$A_\infty$ ESTIMATE FOR CALDERÓN-ZYGMUND OPERATORS IN SPACES OF HOMOGENEOUS TYPE

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Abstract. In this note, we study the $A_p$-$A_\infty$ estimate for Calderón-Zygmund operators in terms of the weak $A_\infty$ characteristics in spaces of homogeneous type. The weak $A_\infty$ class was introduced recently by Anderson, Hytönen and Tapiola. Our estimate is new even in the Euclidean space.

1. Introduction and Main Results

Let $T$ be a Calderón-Zygmund operator and $(w, \sigma)$ be a pair of weights. In the Euclidean setting, Hytönen and Lacey [9] proved that if

$$[w, \sigma]_{A_p} := \sup_{Q: \text{cubes in } \mathbb{R}^n} \frac{w(Q)}{|Q|} \left( \frac{\sigma(Q)}{|Q|} \right)^{p-1} < \infty$$

and $w, \sigma \in A_\infty$, then the following estimate holds

$$(1.1) \quad \|T(\cdot, \sigma)\|_{L^p(\sigma) \to L^p(w)} \leq C_{n,p,T}[w, \sigma]_{A_p}^{\frac{1}{p}} [(w)]_{A_\infty}^{\frac{1}{p}} + [\sigma]_{A_\infty}^{\frac{1}{p}}.$$ 

It is well known that (1.1) extends the $A_2$ theorem, which was first proved by Hytönen [6], see also in [12] for a simple proof by Lerner, and in [3], Anderson and Vagharshakyan also gave a proof in the spaces of homogeneous type. Our goal is to extend (1.1) with the weak $A_\infty$ characteristics (which will be introduced below) to the spaces of homogeneous type.

Now let us recall some definitions. By a space of homogeneous type ($SHT$) we mean an ordered triple $(X, \rho, \mu)$, where $X$ is a set, $\rho$ is a quasimetric on $X$, i.e.,

- (1) $\rho(x, y) = 0$ if and only if $x = y$;
- (2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
- (3) $\rho(x, z) \leq \kappa (\rho(x, y) + \rho(y, z))$ for some $\kappa \geq 1$ and all $x, y, z \in X$;

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and \( \mu \) is a nonnegative Borel measure on \( X \) which satisfies the following doubling condition
\[
\mu(B(x, 2r)) \leq D \mu(B(x, r)),
\]
where \( B(x, r) := \{ y \in X : \rho(x, y) < r \} \) and the dilation of a ball \( B := B(x, r) \) denoted by \( \lambda B \) will be understood as \( B(x, \lambda r) \). We point out that the doubling property implies that any ball \( B(x, r) \) can be covered by at most \( N := N_{D, \kappa} \) balls of radius \( r/2 \). Next let us introduce the weak \( A_\infty \) class, which was first introduced by Anderson, Hytönen and Tapiola in [2].

For every \( \delta > 1 \), we say \( w \) belongs to \( \delta \)-weak \( A_\infty \) class \( A_\infty^\delta \) if
\[
[w]_{A_\infty^\delta} := \sup_B \frac{1}{w(\delta B)} \int_B M(1_B w)(y) d\mu(y) < \infty,
\]
where the supremum is taken over all balls \( B \subset X \). We collect some properties of this weak \( A_\infty \) class and refer the readers to [2] for a proof.

**Proposition 1.1.**

1. \( A_\infty^\delta = A_\infty^{\delta'} \) for all \( \delta, \delta' > \kappa \). So hereafter, we denote by \( A_\infty^{weak} := A_\infty^{2\kappa} \) the weak \( A_\infty \) class;
2. For any \( w \in A_\infty^{weak} \), we have \( [w]_{A_\infty^{weak}} \geq 1/(2 \kappa)^{\log_2 N} \);
3. Let \( w \in A_\infty^{weak} \). Then there exists a constant \( \alpha := \alpha(\kappa, D) \) such that for every \( 0 < \epsilon \leq \frac{1}{\alpha [w]_{A_\infty^{weak}}} \),
\[
\left( \frac{\int_B w^{1+\epsilon} d\mu}{\int_{2\kappa B} w d\mu} \right)^{\frac{1}{1+\epsilon}} \lesssim \frac{1}{\epsilon} \int_{2\kappa B} w d\mu.
\]

Now we are ready to state the main result in this paper.

**Theorem 1.2.** Given \( p, 1 < p < \infty \) and an \( SHT(X, p, \mu) \). Let \( T \) be any Calderón-Zygmund operator and \( (w, \sigma) \) be a pair of weights. Then we have
\[
\|T(\cdot, \sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \leq C[w, \sigma]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty^{weak}}^\frac{1}{p} + [\sigma]_{A_\infty^{weak}}^\frac{1}{p}),
\]
where
\[
[w, \sigma]_{A_p} := \sup_{B \text{ balls in } X} \left( \int_B w d\mu \right) \left( \int_B \sigma d\mu \right)^{p-1}
\]
and the constant \( C \) is independent of the weights \( (w, \sigma) \).

**Remark 1.3.** Note that the result is new already in the case that \( X = \mathbb{R}^n \) with Euclidean distance and Lebesgue measure, since the weak \( A_\infty \) class is strictly larger than classical \( A_\infty \) already in this setting.

2. **Proof of the Main result**

In this section, we will give a proof for Theorem [1.2]. First, we introduce the bump conditions. By a Young function \( \phi \), we mean that \( \phi : [0, \infty) \rightarrow [0, \infty) \) is continuous, convex and increasing satisfying \( \phi(0) = 0 \) and \( \phi(t)/t \rightarrow \infty \) as \( t \rightarrow \infty \). Recall that the complementary function of \( \phi \), denoted by \( \bar{\phi} \), is defined by
\[
\bar{\phi}(t) := \sup_{s > 0} \{ st - \phi(s) \}.
\]
Given two Young functions $\Phi, \Psi$, define
\[ [w, \sigma]_{\Phi,p} := \sup_B \left( \int_B w d\mu \right) \frac{1}{\| \sigma^{1/p} \|_{\Phi,B}} , \]
\[ [\sigma, w]_{\Psi,p} := \sup_B \| w^{1/p} \|_{\Psi,B} \left( \int_B \sigma d\mu \right) \frac{1}{\| \sigma^{1/p'} \|_{\Psi,B}} , \]
where the supremum is taken over all balls in $X$ and the Luxemburg norm is defined by
\[ \| f \|_{\Phi,B} := \inf \{ \lambda > 0 : -\int_B \Phi(\lambda^{-1} f) d\mu \leq 1 \} . \]

There is a famous problem named the separated bump conjecture, which states that for any Calderón-Zygmund operator $T$, if
\[(2.1) [w, \sigma]_{\Phi,p} + [\sigma, w]_{\Psi,p} < \infty,\]
where $\Phi \in B_p$ and $\Psi \in B_{p'}$ (the $B_p$ condition is recalled in (2.2) below), then $T(\sigma)$ is bounded from $L^p(\sigma)$ to $L^p(w)$. For the so-called log-bumps, namely, when
\[ \Phi(t) = t^{p'} \log(e + t)^{p'-1+\delta} \quad \text{and} \quad \Psi(t) = t^p \log(e + t)^{p-1+\delta}, \]
this conjecture has been verified in [5] in the Euclidean setting and for the spaces of homogeneous type, see [1]. For more about the separated bump conjecture, see [10, 15] and the references therein. In the rest of this paper, by carefully calculating the constants, we will show the following quantitative estimate for the power bumps.

**Theorem 2.1.** Given $p, 1 < p < \infty$ and an $SHT(X, \rho, \mu)$. Let $T$ be a Calderón-Zygmund operator and $(w, \sigma)$ is a pair of weights satisfying (2.1) for $\Phi(t) = t^{p'}$ and $\Psi(t) = t^p$, where $1 < r, s < 1 + 2(2\kappa)^{\log_2 N / \alpha(\kappa, D)}$. Then we have
\[ \| T(\sigma) \|_{L^p(\sigma) \rightarrow L^p(w)} \leq C_{T,p,D,\kappa} ([w, \sigma]_{\Phi,p} \| \Phi \|_{B_p}^{1/p} + [\sigma, w]_{\Psi,p} \| \Psi \|_{B_{p'}}^{1/p'}), \]
where recall that for a Young function $\phi \in B_p$,
\[(2.2) \| \phi \|_{B_p} := \int_{1/2}^{\infty} \frac{\phi(t) dt}{t^{p'}} . \]

Now with Theorem 2.1 we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** In fact, taking
\[ r = 1 + \frac{1}{\alpha[\sigma]_{A_{\infty}}} . \]
By the reverse Hölder’s inequality (1.2), we have
\[ [w, \sigma]_{B_p} = \sup_B \left( \int_B w d\mu \right) \frac{1}{\| \sigma^r \|_{B_p}} \left( \int_B \sigma^r d\mu \right) \frac{1}{\| \sigma^r \|_{B_p}} . \]
And by taking similar value of $s$, by definition, we know
\[ \Phi(t) = t^{p'} \]

Hence
\[ [\Phi]_{B_p} \approx_{p,k,D} \int_{1/2}^{\infty} \frac{t^{p'}}{t} dt = 2^{(r-1)p} p'^r - 1 \leq C_{p,k,D}[\sigma]_{A_{\infty}}. \]

And by taking similar value of $s$ we can get the result as desired. \qed

In the rest of this paper, we will focus on the proof of Theorem 2.1. We will reduce the estimates for Calderón-Zygmund operators to the so-called sparse operators. So first let us introduce the following result, which can be found in [7], see also in [4]. Here we follow the version used in [2].

**Theorem 2.2.** Let $0 < \eta < 1$ satisfy $96k^6\eta \leq 1$. Then there exists countable sets of points $\{z_{\alpha}^{k,t} : \alpha \in \mathcal{A}_k\}$, $k \in \mathbb{Z}$, $t = 1, 2, \ldots, K = K(k, N, \eta)$, and a finite number of dyadic systems $\mathcal{D} = \{Q_{\alpha}^{k,t} : \alpha \in \mathcal{A}_k, k \in \mathbb{Z}\}$, such that

1. For every $t \in \{1, 2, \ldots, K\}$ we have
   a. $X = \bigcup_{\alpha \in \mathcal{A}_k} Q^{k,t}\left(z_{\alpha}^{k,t}\right)$ (disjoint union) for every $k \in \mathbb{Z}$;
   b. $Q, P \in \mathcal{D}$ implies $Q \cap P = \{\emptyset, Q, P\}$;
   c. $Q_{\alpha}^{k,t} \in \mathcal{D}$ implies $B(z_{\alpha}^{k,t}, c_1\eta^k) \subseteq Q_{\alpha}^{k,t} \subseteq B(z_{\alpha}^{k,t}, C_1\eta^k)$, where $c_1 := (12k^4)^{-1}$ and $C_1 := 4k^2$;

2. For every ball $B = B(x, r)$ there exists a cube $Q_B \in \bigcup_{t} \mathcal{D}^t$ such that $B \subseteq Q_B$ and $l(Q_B) = \eta^{1-k}$, where $k$ is the unique integer such that $\eta^{k+1} < r \leq \eta^k$ and $l(Q_B) = \eta^{k-1}$ means that $Q_B = Q_{\alpha}^{k-1,t}$ for some indices $\alpha$ and $t$.

By the doubling property, we know that $\mu(B(x, r)) \approx \mu(Q_B)$. And if $Q_{\alpha}^{k,t} \subseteq Q_{\beta}^{k-1,t}$, by the doubling property we also know that there exists some constant $C_{k,D}$ such that
\[ \mu(Q_{\beta}^{k-1,t}) \leq \mu(B(z_{\beta}^{k-1,t}, C_1\eta^{k-1})) \leq C_{k,D}[\mu(B(z_{\alpha}^{k,t}, c_1\eta^{k}))] \leq C_{k,D}[\mu(Q_{\alpha}^{k,t})]. \]

Theorem 2.2 characterizes the structure of dyadic system in spaces of homogeneous type. See also in [3] for an exact characterization of which kinds of sets can be dyadic cubes. Now with Theorem 2.2 we can get the following result, which was proved in [1].

**Lemma 2.3.** Given a pair of weights $(w, \sigma)$, and Young functions $\Phi$ and $\Psi$,
\[ [w, \sigma]_{\Phi,p} \approx \max_{t \in \{1, 2, \ldots, K\}} \left[w, \sigma\right]_{\Phi,p}^{\mathcal{D}_t}, \quad [\sigma, w]_{\Psi,p} \approx \max_{t \in \{1, 2, \ldots, K\}} \left[\sigma, w\right]_{\Psi,p}^{\mathcal{D}_t}. \]
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where

$$[w, \sigma]_{\Phi,p}^\varphi := \sup_{Q \in \mathcal{D}^t} \left( \int_Q wd\mu \right)^{\frac{1}{p}} \|\sigma\|_{\Phi,Q}^\varphi,$$

and $[\sigma, w]_{\Phi,p}^\varphi$ is defined similarly.

Now for any fixed $\mathcal{D}^t$, $t \in \{1, 2, \cdots, K\}$, we call a family $\mathcal{S} \subset \mathcal{D}^t$ sparse if for any $Q \in \mathcal{S}$, $\mu(E(Q)) \geq \frac{1}{2} \mu(Q)$, where $E(Q) = Q \setminus \cup_{Q' \in S, Q' \subsetneq Q} Q'$.

Our purpose is to reduce the estimates for Calderón-Zygmund operators to the following so-called sparse operators,

$$T^S(f)(x) := \sum_{Q \in \mathcal{S}} \int_Q f(y) d\mu(y) 1_Q(x),$$

where $\mathcal{S} \subset \mathcal{D}$ is a sparse family in some dyadic system $\mathcal{D}$. In [12], Lerner gave a nice formula which reduces the norm of Calderón-Zygmund operators to such kind of sparse operators. (In the recent book by Lerner and Nazarov [14], it has been shown that Calderón-Zygmund operators can be dominated pointwise by the sparse operators.) In [1], the authors showed that Lerner's formula also holds in spaces of homogeneous type.

**Lemma 2.4.** Given an SHT $(X, \rho, \mu)$ and a Calderón-Zygmund operator $T$, then for any Banach function space $Y$,

$$\|T(f\sigma)\|_Y \leq C_{D,\kappa} \sup_{\mathcal{D}^t, \mathcal{S}} \|T^S(f\sigma)\|_Y,$$

where the supremum is taken over every dyadic system $\mathcal{D}^t$, $t = 1, 2, \cdots, K$ and every sparse family $\mathcal{S}$ in $\mathcal{D}^t$.

In the rest of this paper, we only need to prove Theorem 2.1 for sparse operators. We follow the strategy of [5]. We further reduce the problem to estimate testing condition. To be precise, we have the following, see [11] for a proof.

**Lemma 2.5.** For fixed $t \in \{1, 2, \cdots, K\}$, suppose $\mathcal{S}$ is a sparse family in $\mathcal{D}^t$. Then

$$\|T^S(\sigma)\|_{L^p(\sigma)} \rightarrow L^p(w) \leq \sup_R \frac{\|\sum_{Q \in R} \int_Q \sigma d\mu 1_Q(x)\|_{L^p(w)}}{\sigma(R)^{1/p}} \leq \sup_R \frac{\|\sum_{Q \in R} \int_Q \sigma d\mu w 1_Q(x)\|_{L^{p'}(\sigma)}}{w(R)^{1/p'}} + \sup_R \frac{\|\sum_{Q \in R} \int_Q \sigma d\mu 1_Q(x)\|_{L^{p'}(\sigma)}}{\sigma(R)^{1/p'}}.$$

Before we give further estimates, we introduce the following result. In the Euclidean case this is due to Pérez [16], and in spaces of homogeneous type, see Pérez and Wheeden [17] and Pradolini and Salinas [18]. We give the version used in [1].
Lemma 2.6. Given $p$, $1 < p < \infty$ and an SHT $(X, \rho, \mu)$ and a Young function $\Phi$ such that $\Phi \in B_p$, then

$$\|M_\Phi f\|_{L^p(\mu)} \leq C_{\kappa, D}[\Phi]_{B_p}^{1/p} \|f\|_{L^p(\mu)},$$

where $\mathcal{D}$ is some dyadic system in $X$ and

$$M_\Phi f(x) := \sup_{Q \ni x} \|f\|_{\Phi, Q}.$$

Then by Theorem 2.2 we immediately get

$$\tag{2.4} \|M_\Phi f\|_{L^p(\mu)} \leq C'_\kappa, D[\Phi]_{B_p}^{1/p} \|f\|_{L^p(\mu)}.$$

Now by symmetry we concentrate on the first term of (2.3). We follow the technique introduced in [9], see also in [5]. For convenience, set

$$\langle f \rangle_Q = -\int_Q f \, d\mu$$

and denote

$$S_a := \{Q : 2^a < \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1} \leq 2^{a+1} \text{ and } Q \subset R\}.$$ 

Denote by $\mathcal{P}_0^a$ the collection of maximal cubes in $S_a$. Now we define

$$\mathcal{P}_n^a := \{\text{maximal cubes } P' \subset P \in \mathcal{P}_{n-1}^a \text{ such that } P' \in S_a, \langle \sigma \rangle_{P'} > 2\langle \sigma \rangle_P\}.$$ 

Then denote $\mathcal{P}^a := \cup_n \mathcal{P}_n^a$. For any $P \in \mathcal{P}^a$, set

$$S_a(P) := \{Q \in S_a : \pi(Q) = P\},$$

where $\pi(Q)$ is the minimal cube in $\mathcal{P}^a$ which contains $Q$. We have the following Lemma, which was proved in [9] (see also in [13]) in the Euclidean setting and it is still valid for the spaces of homogeneous type with no change of the proof.

Lemma 2.7. There exists a constant $c$ such that

$$w\{x \in P : T^{S_a(P)}(\sigma) > t\langle \sigma \rangle_P\} \lesssim e^{-ct}w(P),$$

where

$$T^{S_a(P)}(f)(x) := \sum_{Q \in S_a(P)} \int_Q f(y) \, d\mu(y) 1_Q(x).$$

Now we are ready to estimate the first term on the right side of (2.3). We have

$$\left\| \sum_{Q \in S} \langle \sigma \rangle_Q 1_Q(x) \right\|_{L^p(w)} \leq \sum_a \left\| \sum_{Q \in S_a} \langle \sigma \rangle_Q 1_Q(x) \right\|_{L^p(w)} = \sum_a \left\| \sum_{P \in \mathcal{P}^a} T^{S_a(P)}(\sigma) \right\|_{L^p(w)}.$$

Set

$$L_j^a(P) := \left\{ x : T^{S_a(P)}(\sigma)(x) \in [j, j+1)\langle \sigma \rangle_P \right\}. $$
By Lemma 2.7 we have
\[
\left\| \sum_{P \in \mathcal{P}} T_{\sigma}(P)(\sigma) \right\|_{L^p(w)} \leq \sum_{j=0}^{\infty} (j + 1) \left\| \sum_{P \in \mathcal{P}} \langle \sigma \rangle_p \mathbf{1}_{L^p_j(P)}(x) \right\|_{L^p(w)}
\]
\[
\lesssim \sum_{j=0}^{\infty} (j + 1) \left( \sum_{P \in \mathcal{P}} \langle \sigma \rangle_p e^{-c_j w(P)} \right)^{\frac{1}{p}}
\]
\[
\lesssim \left( \sum_{P \in \mathcal{P}} \langle \sigma \rangle_p w(P) \right)^{\frac{1}{p}}.
\]

Therefore,
\[
\left\| \sum_{Q \subset R} \langle \sigma \rangle_{Q} \mathbf{1}_Q(x) \right\|_{L^p(w)} \lesssim \sum_{P \in \mathcal{P}} \left( \sum_{P \in \mathcal{P}} \langle \sigma \rangle_p w(P) \right)^{\frac{1}{p}}.
\]

We follow the idea of [5]. Define \( \Phi_0(t) = t^{-p(r+1)/2} \), set \( \gamma = \frac{1}{2(r+1)} \). Since \( r > 1 \) and it is dominated by some constant that depends only on the structure constant of \( X \), it is easy to check that
\[
[\Phi_0]_{B_p} = \int_{1/2}^{\infty} \Phi_0(t) \frac{dt}{t^p} \leq c_{p,D} \frac{p(r+1) - 2}{p(r-1)} \frac{p^{(r-1)p}}{2^{p(r-1)}},
\]
\[
\leq c_{p,D} \frac{p(r+1) - 2}{p(r-1)} \frac{2^{p(r-1)}}{2^{p(r-1)}} = c_{p,D} [\Phi]_{B_p}.
\]

Now notice that
\[
\frac{2^{r+1}}{r} + \frac{1}{4} = \frac{3}{4} + \frac{1}{4r} < 1.
\]

We have
\[
\int_Q \sigma^{\frac{r+1}{4}} \leq \left( \int_Q \sigma^{\frac{2^{r+1}}{r} - \frac{1}{4}} \right)^{\frac{2^{r+1}}{r}} \cdot \left( \int_Q \sigma^{\frac{1}{4}} \right)^{\frac{1}{4}}.
\]

It follows that
\[
\| \sigma \|_{\Phi_0,Q} \leq \| \sigma \|_{\Phi_0,P} \| \sigma \|_{\Phi_0,P}^{1-\gamma} \| \sigma \|_{\Phi_0,P}^{\gamma}.
\]

Therefore, by the sparseness and Lemma 2.3
\[
\sum_{P \in \mathcal{P}} \langle \sigma \rangle_p w(P) \leq \sum_{P \in \mathcal{P}} w(P) \| \sigma \|_{\Phi_0,P} \| \sigma \|_{\Phi_0,P}^{1-\gamma} \| \sigma \|_{\Phi_0,P}^{\gamma} \| \sigma \|_{\Phi_0,P}^{p} \| \Phi_0,P \|_{P} \mu(P)
\]
\[
\leq \sum_{P \in \mathcal{P}} \langle w \rangle \| \sigma \|_{\Phi_0,P}^{p} \| \sigma \|_{\Phi_0,P}^{1-\gamma} \| \sigma \|_{\Phi_0,P}^{\gamma} \| \sigma \|_{\Phi_0,P}^{p} \| \Phi_0,P \|_{P} \mu(P)
\]
\[
\leq \| w \| \| \sigma \|_{\Phi_0,P}^{p(1-\gamma)} \| \sigma \|_{\Phi_0,P}^{\gamma} \| \sigma \|_{\Phi_0,P}^{p} \| \Phi_0,P \|_{P} \mu(P)
\]
\[
\lesssim \| w \| \| \sigma \|_{\Phi_0,P}^{p(1-\gamma)} \| \sigma \|_{\Phi_0,P}^{\gamma} \int M_{\Phi_0}(1_R \sigma)(x)^p d\mu
\]
< [w, σ]_{p, \Phi, p}^{\gamma(1-\gamma)/2} 2^{(a+1)\gamma} [\Phi_0]_{B_p} \sigma(R)
\leq [w, σ]_{p, \Phi, p}^{\gamma(1-\gamma)/2} 2^{(a+1)\gamma} [\Phi]_{B_p} \sigma(R).

Consequently,

\| \sum_{Q \in S} \langle \sigma \rangle Q \mathbf{1}_Q(x) \|_{L_p(w)} \leq [w, σ]_{p, \Phi, p}^{\gamma(1-\gamma)/2} 2^{(a+1)\gamma/p} \sum_a \langle \Phi \rangle_{B_p} [w, σ]_{p, \Phi, p}^{\gamma/p} \leq C_{p, D, \kappa} [w, σ]_{p, \Phi, p}^{\gamma/p}.

This completes the proof.

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