Complexity for Charged Thermofield Double States

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Abstract: Motivated by recent developments in studying the complexity of thermofield double states and the holographic complexity proposals in charged black hole backgrounds, here we use Nielsen’s geometric approach to examine the complexity of the simple model of a charged thermofield double state constructed from two free complex scalar fields. We show that this state factorizes between positively and negatively charged modes and demonstrate that this can be used to translate the problem into that of two uncharged thermofield double states with shifted temperatures and times. We evaluate the complexity of formation for the charged thermofield double, both numerically and in certain analytic expansions, and present numerical results for the time dependence of complexity. We compare various aspects of these results to those obtained in holography for charged black holes.
1 Introduction and Summary

Quantum computational complexity provides an estimate of the difficulty of constructing quantum states for the purpose of performing quantum computations [1, 2]. Traditionally, complexity is defined for spin/qubit chains, equipped with some universal set of unitary gates chosen such that each gate acts only on a small number of spins/qubits. Appropriately chosen sequences of these gates, i.e., circuits, should be able to reproduce, to a given precision, arbitrary target states of the spin/qubit chain, starting from a simple unentangled reference state. The complexity of a target state is then defined as the minimal number of gates required for such a circuit. The problem of finding the shortest circuits is challenging if one naively attempts an exhaustive check of all possible circuits. An alternative geometric approach, proposed by Nielsen, translates the problem of finding the minimal circuit to a problem of finding geodesics in the manifold of unitaries equipped with the metric that naturally arises from the algebra on its tangent space [3–5].
Outside the context of spin chains, some progress has been made in defining complexity for quantum field theories. The Nielsen approach was extended to Gaussian states of quantum field theories in [6], where the authors studied the complexity of the vacuum state of a free scalar QFT, with extensions following for various classes of Gaussian states including fermions [7–9], the thermofield double state of a free bosonic QFT [10], evolution following a quantum quench [11] and mixed states [12]. These approaches have been compared to the results for analogous systems of complexity in holography at the phenomenological level and surprising similarities have been found. An alternative approach to complexity [13] uses the quantum information metric on the space of quantum states and yields similar results. Despite this progress, very little is known beyond the free theory regime, see however some progress in [14–17].

In holography, the complexity is proposed to be related to certain geometric bulk observables by means of the complexity=volume (CV) proposal [18, 19], relating the complexity to the volume of a maximal bulk slice anchored at the boundary time slice where the state is defined, and the complexity=action (CA) proposal [20, 21], relating the complexity to the action of a certain region in the bulk – the Wheeler-DeWitt (WDW) patch – bounded by light sheets and anchored at the boundary time slice. Complexity comes about as a part of a longer line of research studying the way in which quantum information notions are encoded in gravity, which dates back to the relation between Ryu-Takayanagi surfaces and entanglement entropy, see, e.g., [22, 23]. One setup, which is very often used in holographic studies of complexity is that of the thermofield double (TFD) state. The TFD state is a particularly symmetric purification of the thermal state on a given system (the “left” copy), in terms of an extra identical copy of this system (the “right” copy), equipped with an identical Hamiltonian, see, e.g., [24], i.e.,

$$|\text{TFD}(t_L, t_R)\rangle = \frac{1}{\sqrt{Z_{\beta}}} \sum_n e^{-\beta E_n/2} e^{-iE_n(t_L+t_R)}|E_n\rangle_L|E_n\rangle_R,$$  \hspace{1cm} (1.1)$$

where $t_{L,R}$ are the times on the left and right copies, $\beta = 1/T$ is the inverse temperature, $|E_n\rangle$ are a basis of energy eigenstates with associated eigenvalues $E_n$ and $Z_{\beta}$ is a normalization constant. In holography, the TFD state is dual to an eternal black hole [25] of the corresponding temperature. The complexity of the TFD state using the two holographic proposals is found to increase (approximately) linearly as a function of the time $t_L = t_R = t/2$, for a very long of time, much longer than the typical times it takes for other observables, e.g., the entanglement entropy [26], to saturate. In fact, it was suggested that holographic complexity keeps growing for an exponential amount of time, until the semi-classical gravity approximation breaks down [21]. This unusual time dependence captures the growth of the volume of the wormhole/Einstein-Rosen bridge connecting the two sides of the geometry via the behind horizon region [27] and is also typical of complexity in spin chains of fast scrambling systems, e.g., [28, 29]. This served as one of the original motiva-
tions for proposing the holographic conjectures and similar qualitative evidences follow from
the reaction of complexity to perturbations, including the effects of chaos and scrambling
[19, 28, 33–35]. Outside the context of complexity a number of works have been written on
building the TFD state in the laboratory [36–40] and these further motivate focusing on
the TFD for simple quantum mechanical systems outside the context of holography.

As mentioned earlier, the complexity can be studied for simple systems in QFT, i.e.,
Gaussian states of free field theories, using the Nielsen method and these studies yield some
qualitative similarities with the properties of holographic complexity. First, we should men-
tion that complexity is divergent due to the necessity of establishing short range correlations
in the QFT state. This is similar to what happens for the entanglement entropy and has
to be regulated by the introduction of a UV cutoff, for example, a lattice spacing $\delta$. The
structure of divergences of complexity is found to be similar when comparing the free QFT
results with those found in holography, see [6, 13, 41, 42] and the discussion of [35], with
a leading divergence of the form of a volume law. For subregions of the vacuum state,
one obtains in addition to the leading volume law divergence also a subleading area law
divergence proportional to the entanglement entropy, both in holography and QFT [12].
Furthermore, the structure of divergences is found to match between holography and free
QFT for thermal states too [12]. Another quantity which allows for a qualitative match
between holography and free QFT is the complexity of formation of the thermofield dou-
ble state. The latter is defined as the additional complexity required in order to prepare
the entangled thermofield double state compared to preparing each of the copies of the
field theory in their vacuum state. This quantity is UV finite and is found to be propor-
tional to the entropy with positive coefficient, both in free QFT and in holography [10, 43].
Unfortunately, the simple Gaussian states in free QFTs fail to capture more complicated
properties of chaotic systems such as the time dependence of complexity and its reaction
to perturbations.

The generalization of the TFD state to the grand canonical ensemble is given by the
charged TFD state (cTFD), see, e.g., [44]3

$$|\text{cTFD}(t_L, t_R)\rangle = \frac{1}{\sqrt{Z_{\beta, \mu}}} \sum_{n, \sigma} e^{-\beta(E_n + \mu c_\sigma)/2 - i(E_n + \mu c_\sigma)(t_L + t_R)} |E_n, c_\sigma\rangle_L |E_n, -c_\sigma\rangle_R,$$

(1.2)

where here $c_\sigma$ are eigenvalues of a conserved global $U(1)$ charge and $\mu$ is the associated
chemical potential. The two states in each term were chosen to have opposite charges such
that they are CPT conjugates. It is apparent from this expression that the time evolution
of the charged TFD is governed by the deformed Hamiltonians $H_{L,R} \pm \mu C_{R,L}$ where $C_{R,L}$
are the $U(1)$ charges on the left/right copies, respectively.

In this paper we explore the complexity of Gaussian cTFD states constructed from two
copies of a free complex scalar QFT using Nielsen’s approach.4 The complex scalar factorizes

3We have traded the traditional symbol $Q$ for the charge in favor of $C$, in anticipation for using $Q$ as
the position operator for the harmonic oscillators in the Nielsen construction.

4A previous attempt to study the complexity of charged TFDs was made in [45] using the Fubini-Study
approach. However, in this work, the authors find that the complexity grows linearly for a long time for the
free complex scalar QFT. Their results (in the limit of vanishing chemical potential) stand in contradiction
to a set of complex harmonic oscillators with frequencies \( \omega_k = \sqrt{k^2 + m^2} \) where \( m \) is the mass of the complex scalar and \( k \) is the momentum. Furthermore, each harmonic oscillator can be decomposed using particles and anti-particle modes. A very simple relation then connects the energy and charge to the number operators for the particle and anti-particle modes, \( i.e., \) the energy is proportional to the sum of the numbers of particles and anti-particles and the charge to their difference. It then becomes obvious that we can regard the TFD state as a product of two independent TFD states - one which includes particles on the left copy and anti-particles on the right copy and one which includes particles on the right copy and anti-particles on the left. These two TFD states are associated with effective shifted temperatures and times given by

\[
\beta_{\text{eff}} \equiv \beta \left( 1 \pm \frac{\mu}{\omega} \right), \quad t_{\text{eff}} \equiv t \left( 1 \pm \frac{\mu}{\omega} \right),
\]

respectively, see eq. (3.20)-(3.21), where \( \omega \) is the original frequency of the oscillators. These expressions also imply that the cTFD state is only well defined so long as \( |\mu| < \omega \). As \( \mu \to \omega \), one of the effective temperatures approaches infinity and this means that the associated mode will be infinitely populated, \( i.e., \) a condensate develops. This prevents us from taking the conformal limit \( m \to 0 \) while holding the chemical potential fixed when studying the complexity of the full fledged complex scalar QFT. It would be interesting to verify if this problem can be avoided in a different setup, for example using fermions.

In holography, the charged TFD state is dual to a charged eternal black hole, see, \( e.g., \) [46, 47]. The complexity in this background was studied in holography in [21, 48] and we provide here a summary of those results. Before proceeding, we should mention that a number of other quantities have been studied for these holographic systems including the entanglement entropy, mutual information, and two point functions of charged and uncharged operators [44, 49].

Recall that the complexity of formation was defined as the additional complexity required to prepare the TFD state compared to preparing two copies of the vacuum state.

In the results for the CA proposal we have neglected a term of order \( (\mu/T)^3 \). The coefficient of this term was found in [48] to depend on the normalization constant \( \alpha \) of the null-normals to the boundaries of the WDW patch. Our recent understanding, however, is that one should include a counter term [50] to remove this parametrization-dependence on the null-normals of the WDW patch. It would be interesting to examine the effect of this addition. In the CV results we have neglected a term of order \( (\mu/T)^4 \). In the setup of [48] the chemical potential was assumed to be positive, however, since the definition of the TFD state is invariant under changing \( \mu \to -\mu \) as well as changing the left and right sides, we know that the result for negative chemical potential should take the same form with \( \mu \to |\mu| \) in all the equations below.

\[
\Delta C_A = S \left( c_1 + c_2 \left( \frac{\mu}{T} \right)^3 \ln \left( \frac{\mu}{T} \right) + \ldots \right), \quad \Delta C_V = S \left( \tilde{c}_1 + \tilde{c}_2 \left( \frac{\mu}{T} \right)^2 \ln \left( \frac{\mu}{T} \right) + \ldots \right),
\]
the CA and CV proposals, respectively. These expressions reproduce the neutral results in the limit $\mu \to 0$. Furthermore, the complexity of formation was found to be divergent when the temperature was sent to zero (both using the CV and the CA proposals)\textsuperscript{9}

$$\Delta C_{A,V} \sim S \ln \left( \frac{\mu}{T} \right) + \ldots$$

(1.5)

where again the proportionality coefficients in these relations are order one positive constants. (Note that $S$ remains finite as $T \to 0$ for the charged case.) This curious behavior in the extremal (zero-temperature) limit is a result of an IR divergence due to the infinitely long throat of the wormhole in the near extremal limit. The interpretation suggested in [48] was that of a “third law of complexity”, namely that cTFD states at finite chemical potential and zero temperature are infinitely more complex compared to their finite temperature counterparts and cannot be formed by any physical process during a finite amount of time.\textsuperscript{10}

A number of results concerning the time dependence of complexity for charged black holes are also available [21, 48]. Just like for neutral black holes, also here the complexity exhibits a linear increase at late times. The authors of [21] have proposed that the late time rate of change in the CA complexity should be associated with a modified version of Lloyd’s bound

$$\frac{dC_A}{dt} \leq \frac{2}{\pi} \left[ (M - \mu C) - (M - \mu C)|_{gs} \right],$$

(1.6)

where the subscript ground state (gs) indicates the thermodynamic quantities associated to the state minimizing $(M - \mu C)$ for a given value of the chemical potential.\textsuperscript{11} It was found that large and intermediate charged black holes are inconsistent with this late time rate of change, while small black holes only exhibit smaller violations in the approach to late times. In any event, the rate of change in complexity for charged black holes produced a slightly more regular version of the neutral result using the CA proposal, and was found to approach the correct uncharged limit using both the CA and CV proposals.\textsuperscript{12} The change in complexity at early times obeys $C \sim t^2$ since the geometry is symmetric under $t \leftrightarrow -t$ using both proposals. Finally, the rate of change in complexity was found to vanish in the extremal (vanishing temperature) limit using both the CV and the CA complexity proposals.

Before proceeding we should mention that the gravitational setup with charged black holes is somewhat advantageous compared to the neutral setup especially with regards to evaluating the complexity with the CA proposal since this result is influenced by regions of

\textsuperscript{9}In these expressions we have neglected a constant term. For the case of the CA complexity, this constant term depends on the normalization constant $\alpha$ of the null normals and it would be interesting to review the effect of including the counter term restoring reparametrization invariance [50].

\textsuperscript{10}Note that the third law of thermodynamics takes a slightly different form for the free theory compared to the holographic system; the value of the entropy approached at zero temperature is zero for the free theory, while the entropy approaches a finite constant for the zero temperature limit of the holographic system.

\textsuperscript{11}For small chemical potential this is the vacuum state with a constant gauge field while for larger values of the chemical potential this is an extremal black hole at the same value of $\mu$ [21, 48].

\textsuperscript{12}For the CA approach this limit consist of a vanishing rate of change before a certain critical time.
the WDW patch which go arbitrarily near the singularity of the neutral black hole. This does not happen for charged black holes due to the different causal structure including two horizons. This, in addition to understanding better the influence of a conserved charge on the complexity, consists part of the motivation for studying the complexity in charged backgrounds. Comparison to the holographic charged results, together with the fact that charges are of experimental relevance in various quantum mechanical systems consists our motivation for studying the complexity of charged TFDs in simple QFTs.

Our analysis results in a number of features which can be qualitatively compared to the holographic results. First, all our results obey the symmetry $\mu \to -\mu$ of the cTFD state. Furthermore, we found that the neutral results were recovered upon setting the chemical potential to zero. We found that the complexity of formation increased with the chemical potential, see figures 2a, 2b, 3a and 3b. This is unlike what happens in holography, see figure 26 in [48]. These figures, further supported by an analytic low-temperature expansion, also demonstrate that only in the special case $m = |\mu|$ do we have a “third law of complexity” for our simple scalar system — that is, when $m = |\mu|$, the complexity of formation appears to diverge with decreasing temperature. Note however that in the case of the complex scalar we have to keep the relation $\mu \leq m$ and hence we never really approach the conformal limit at fixed chemical potential, which would be the relevant one for comparison with holography. We have also evaluated the time dependence of complexity and observed that the change from the original value of the complexity increases with the chemical potential. Furthermore, we observed that this change decreases with the temperature. This can be seen from figures 7 and 10. This effect is similar to the holographic observation that the rate of change in complexity vanishes when we approach the zero-temperature limit. Similarly to previous works in the neutral case [10], we found that the time dependence of complexity deviates significantly compared to the results in the holographic systems. This is perhaps not surprising, since the free systems we consider do not enjoy a chaotic behavior similar to the holographic ones. Specifically, we found that the complexity changes linearly (either increases or decreases) at early times and saturates at times of the order of the inverse temperature $\beta$. Furthermore, the complexity oscillates at time scales of the order of the inverse mass.

There are a number of open questions which we leave for future studies. One direction is to conduct an analogous study of the complexity of charged cTFDs for fermions which provide a setup more relevant for experimental studies. In particular it would be interesting to explore if in this setting one can probe the conformal limit. Other obvious generalizations include studying the same problem using different measures of complexity, e.g., different norms, the FS study metric, or including penalty factors. Furthermore, it would be interesting to perform an analysis of the entanglement entropy in the charged TFD state, similar to the one that was conducted in [10] for the uncharged TFD, and compare the dependence on the various parameters – size of the subregion, chemical potential and temperature – to the one found for holographic systems [44, 49]. Finally, it would be interesting to explore if the factorization (1.3) can be used to define a MERA tensor network prescription [51–53] for constructing cTFD states by combining similar ideas to those proposed in the context of uncharged TFD state, e.g., see comments in [54] and appendix E of [43] with some ideas
on the incorporation of symmetries in the MERA tensor network as in [55].

This paper is divided into two main sections. In section 2 we review the various necessary ingredients for our construction; in particular, we generalize parts of the machinery of [10] for circuits using covariance matrices defined with respect to complex phase space coordinates. In section 3 we explain how to evaluate the complexity of the cTFD state. We present our numerical results for the complexity of formation, as well as a number of analytic expansions for low and high temperatures. We then present numerical results for the time dependence of the complexity of the cTFD state. We have left a number of technical details for appendixes. In appendix A we present an explanation for the form of the time evolution of the TFD state. In appendix B we present an example of the generators used to evaluate the complexity as explained in section 2.1. Finally, in appendix C we present an additional basis used for evaluating the low temperature limits in section 3.5.2.

2 Preliminaries

In this section we provide the various preliminary ingredients required to study the complexity of the charged TFD state of a free complex scalar. We start by reviewing Nielsen’s approach for the complexity of Gaussian states using covariance matrices along the lines of [10], generalizing some of the aspects there to account for the possibility of complex phase space coordinates. We then review a number of aspects of the construction in [10] including some results for the covariance matrices of the uncharged TFD, the reference state and the introduction of a gate scale parameter. Finally, we establish our notation for the complex scalar QFT and explain how to put it on the lattice in preparation for the study of the charged TFD state in the next section.

2.1 Complexity from Covariance Matrices

Gaussian states, such as the cTFD studied in this paper, are completely characterized by the one and two-point functions of a set of phase space coordinates $\xi^a = (q_1, \ldots, q_N, p_1, \ldots, p_N)$, where $N$ is the number of degrees of freedom. The phase space coordinates satisfy canonical commutation relations given in terms of the symplectic form $\Omega$ as follows

$$[\xi^a, (\xi^b)^\dagger] = i\Omega_{ab}, \quad \Omega = \begin{bmatrix} 0 & \mathbb{1}_{N \times N} \\ -\mathbb{1}_{N \times N} & 0 \end{bmatrix}. \quad (2.1)$$

In the cases studied in this paper we will sometimes be dealing with complex coordinates, and in this case we will list both the coordinates and their complex conjugates in the list of phase space variables as

$$\xi^a = (q_1, q_1^\dagger, \ldots, q_{N/2}, p_{N/2}, p_1, \ldots, p_{N/2}, q_{N/2}, q_{N/2}^\dagger), \quad (2.2)$$

where we have specifically chosen to invert the order of $p_{i}^\dagger$ and $p_i$ compared to the conjugate coordinates $q_i$ and $q_i^\dagger$ in order to satisfy the commutation relations (2.1). The matrix constructed from these two point functions is given by

$$\langle \psi | \xi^a (\xi^b)^\dagger | \psi \rangle = \frac{1}{2} (G^{ab} + i\Omega^{ab}) \quad (2.3)$$
and its symmetric part
\[ G^{a,b} = G^{(a,b)} = \langle \psi | \xi^a (\xi^b)^\dagger + (\xi^b)^\dagger \xi^a | \psi \rangle \] (2.4)
goes under the name of covariance matrix. Note that the matrix representation of the covariance matrix, i.e., \( G \), is Hermitian, and therefore satisfies \( (G^\dagger)_{ab} = (G_{ba})^\dagger = G_{ab} \). Everywhere in this paper we will only focus on Gaussian states with vanishing one point functions \( \langle \psi | \xi^a | \psi \rangle = 0 \) which will be completely characterized by their covariance matrix. By restricting our attention to Gaussian states, i.e., selecting a reference and target state which are Gaussian and only considering circuits moving through the space of Gaussian states, we are able to make some progress in solving for the optimal circuits in the complexity geometry. The treatment of complexity using covariance matrices was proposed in [10] as an alternative to the wavefunction approach of [6]. This approach proves simpler in cases where the circuit utilizes gates which are quadratic in the momentum operators such as when constructing the TFD and eTFD.

The action of a circuit moving between Gaussian states can be parameterized by the action of Hermitian generators which are quadratic in the canonical coordinates \( \xi \),
\[ | \tilde{\psi} \rangle = \hat{U} | \psi \rangle, \quad \hat{U} = e^{-i\hat{K}}, \] (2.5)
where
\[ \hat{K} = \frac{1}{2} (\xi^a)^\dagger k_{ab} \xi^b. \] (2.6)
Note that \( \xi \) and \( \xi^\dagger \) are not independent from each other but are rather related according to
\[ (\xi^a)^\dagger = A^a_b \xi^b \] (2.7)
where \( A \) is the following matrix
\[ A_R = I_{2N \times 2N}, \quad A_C = \bigoplus_{I=1..N} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \] (2.8)
for the cases of real and complex coordinates, respectively. Note that both these transformations satisfy \( A^2 = 1 \) and \( A \) is symmetric. This means that when enumerating the independent generators acting on our Gaussian states, taking the most general \( k_{ab} \) in eq. (2.6) into account will be over-counting. The independent generators are given by symmetric values of the matrix \( A \cdot k \), namely
\[ k^T = A \cdot k \cdot A \] (2.9)
In addition the requirement that the generator in eq. (2.6) be Hermitian amounts to
\[ k = k^\dagger. \] (2.10)

The action of the unitary in eq. (2.5) on the state can be represented directly as an operation on the covariance matrix. To see this, we start by exploring the effect of conjugating the canonical coordinates with the unitary operation (2.5)
\[ \hat{U}^\dagger \xi^a \hat{U} = \sum_{n=0}^{\infty} \frac{1}{n!} [i\hat{K}, \xi^a]_{(n)} \] (2.11)
where the Taylor expansion is given in terms of the nested commutator defined recursively
\[ [i \hat{K}, \xi^a]_{(n)} \equiv [i \hat{K}, [i \hat{K}, \xi^a]_{(n-1)}] \]
where \([i \hat{K}, \xi^a]_{(0)} = \xi^a\). Using the commutation relations in eq. (2.1) we find
\[
[i \hat{K}, \xi^a] = \frac{1}{2} \left( \Omega \cdot (A \cdot k^T \cdot A + k) \right)^a_b \xi^b = (\Omega \cdot k)^a_b \xi^b \equiv K^a_b \xi^b
\] (2.12)
where we have defined
\[
K \equiv \Omega \cdot k, \quad K^\dagger = -k \cdot \Omega.
\] (2.13)

Resuming eq. (2.11) we obtain
\[
\hat{U}^\dagger \xi^a \hat{U} = S^a_b \xi^b, \quad S = e^K.
\] (2.14)

It is then straightforward to check that this induces the following transformations on the covariance matrix (2.4)
\[
\tilde{G} = S \cdot G \cdot S^\dagger,
\] (2.15)
where \(\tilde{G}\) is the covariance matrix associated with the state \(\tilde{\psi}\). The unitary conjugation above does not modify the commutation relations since
\[
[\tilde{\xi}^a, \tilde{\xi}^b] = S (i \Omega) S^\dagger = (i \Omega)
\] (2.16)
which is satisfied automatically by virtue of the identity \(K \Omega + \Omega K^\dagger = 0\), see eq. (2.13). Note however that this last condition is less restrictive than requiring that \(K\) is defined via \(K = \Omega \cdot k\) where \(k\) satisfies (2.9)-(2.10).

Whether the coordinates are real or complex, the number of independent generators for these transformations between Gaussian states can be counted by counting the number of constraints in eqs. (2.9)-(2.10) and is equal to \(N(2N + 1)\). In fact, this group of all \(S\) in eq. (2.14) where \(K = \Omega \cdot k\) and \(k\) satisfies the conditions (2.9)-(2.10) is simply isomorphic to \(\text{Sp}(2N, \mathbb{R}) = \{ e^K \in M_{2N \times 2N}(\mathbb{R}) \text{ with } K \cdot \Omega + \Omega \cdot K^T \}\). To see this, first note that the generators of \(\text{Sp}(2N, \mathbb{R})\) can be recast as the group of real symmetric matrices defined by \(\tilde{K} = \Omega \cdot \tilde{k}\), since the condition that \(\tilde{k}\) is symmetric is satisfied if and only if \(\tilde{K} \cdot \Omega + \Omega \cdot \tilde{K}^T\).

Further note that we may then relate the generators \(\tilde{k} = R \cdot k \cdot R^\dagger\) where
\[
R = \bigoplus_{I=1,..,N} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix},
\] (2.17)
and these are automatically be real and symmetric using eqs. (2.9)-(2.10) due to the identity \(A \cdot R^T = R^\dagger\).

We will label by \(K_I\) a complete basis of independent generators for the transformations above and will generally assume that they consist an orthonormal basis with respect to the inner product
\[
\frac{1}{2} \text{tr}(K_I K_J^\dagger) = \delta_{IJ}
\] (2.18)
We list those generators for the example of \(\text{Sp}(4, \mathbb{R})\), using both real and complex coordinates in appendix B.
At the next step we will want to construct circuits through the space of Gaussian states. The circuits will act on the covariance matrices according to

\[ G(\sigma) = S(\sigma)G_{\text{ref}}S^\dagger(\sigma), \quad S(\sigma)\Omega S^\dagger(\sigma) = \Omega, \]  

(2.19)

where \( \sigma \in [0, 1] \) is a trajectory parameter. Boundary conditions are imposed such that the circuit \( G(\sigma) \) moves between the covariance matrix of the reference state \( G_{\text{ref}} \) at \( \sigma = 0 \) and ends at the covariance matrix of the target state \( G_{\text{target}} \) at \( \sigma = 1 \). Explicitly, this means that \( S(\sigma = 0) \) acts trivially on \( G_{\text{ref}} \), while

\[ S(\sigma = 1)G_{\text{ref}}S^\dagger(\sigma = 1) = G_{\text{target}}. \]  

(2.20)

The symplectic transformation \( S(\sigma) \) can be decomposed according to

\[ S(\sigma) \equiv \frac{1}{\pi} \exp \int_0^\sigma d\sigma' Y^I(\sigma')K_I, \]  

(2.21)

where \( K_I \) are the generators of the symplectic group, assumed to be orthonormal, \( i.e., \frac{1}{2}\text{tr}(K_IK_J^\dagger) = \delta_{IJ} \). This geometric approach was implemented in [10] using several different cost functions \( F \) to evaluate the length

\[ d[S(\sigma)] = \int_0^1 d\sigma \ F(S(\sigma), Y^I(\sigma)) \]  

(2.22)

of a given circuit. The cost functions considered were:

\[ F_1 = \sum_I |Y^I|, \quad F_2 = \sqrt{\sum_I (Y^I)^2}, \quad D_\kappa = \sum_I |Y^I|^\kappa. \]  

(2.23)

The complexity of the target state is given by the ‘length’ of the shortest circuit, \( i.e., \) path \( S(\sigma) \) through \( \text{Sp}(2N, \mathbb{R}) \), satisfying the boundary condition (2.20) (as well as acting trivially on the reference state at \( \sigma = 0 \)), for a given choice of cost function

\[ \mathcal{C} \equiv \min_{S(\sigma)} d[S(\sigma)]. \]  

(2.24)

Note that the result for the complexity is basis dependent, \( i.e., \) in general the choice of basis \( K_I \) influences this result. However, the \( F_2 \) and \( \kappa = 2 \) cost functions remain unchanged when the two bases are related by an orthogonal transformation on the coordinates and an identical orthogonal transformation on the momenta.

In several cases of interest,\(^\text{13}\) it was found in [10] that the path of minimal cost is the ‘straight line’ circuit, obtained by exponentiating a constant Lie algebra element multiplied by \( \sigma \). By evaluating the integrals of other cost functions for this straight line circuit, one then obtains a non-trivial upper bound for complexity evaluated using other cost functions.

\(^{13}\)This result was proved in [10] for the case of the \( F_2 \) (or, equivalently, \( D_\kappa=2 \)) cost function when a particular relation was chosen between the reference state scale and the gate scale.
$G_{\text{target}}$ and reference state covariance matrix $G_{\text{ref}}$, the relative covariance matrix is defined to be their ratio:

$$\Delta_{\text{target}} \equiv G_{\text{target}} G_{\text{ref}}^{-1}. \quad (2.25)$$

The straight line circuit is then given by

$$S(\sigma) = e^{\sigma K}, \quad \text{where} \quad K = \frac{1}{2} \log \Delta_{\text{target}}. \quad (2.26)$$

Of the cost functions considered in [10], it was found that the $F_1$ cost produced results most similar to the behaviour expected from holographic complexity conjectures. Hence, in this note, we too will focus on $F_1$ complexity, obtaining a upper bound by evaluating it on the straight line circuit. Furthermore, since the $F_1$ cost depends on the choice of generators, two choices were proposed in [10]. The first choice retains the left-right coordinate split between the two sides of the TFD, while the second choice mixes the two into a “diagonal” basis. The first choice was found yield properties more similar to those of holographic complexity in particular in reproducing the proportionality between the complexity of formation and the entropy. We will therefore focus on this choice in this paper.

In order to read the $F_1$ complexity we have to decompose this trajectory according to the expression (2.21) and extract the scalar coefficients $Y^I$, which appear in the cost function. Since we have assumed that our basis of generators $K_I$ is orthogonal we can do this by using the inner product

$$Y^I = \frac{1}{2} \text{tr} \left( KK^\dagger_I \right). \quad (2.27)$$

Finally, an upper bound for the $F_1$ complexity given by integrating $F_1$ along the straight line circuit. But since, the generator of the trajectory was simply constant along the path this yields

$$C_1 \leq \sum_I |Y^I| = \frac{1}{2} \sum_I \text{tr} \left( KK^\dagger_I \right) = \frac{1}{4} \sum_I \text{tr} \left( \log(\Delta_{\text{target}}) \cdot K^\dagger_I \right). \quad (2.28)$$

From here on, we shall be cavalier regarding the fact that (2.28) only produces an upper bound on complexity, replacing the inequality in (2.28) with an equality.

Before we finish this section let us mention that in order to move between different bases of generators it is possible to use a coordinate transformation

$$\xi^a = R^{a_b} \xi^b \quad (2.29)$$

where $R$ is a general complex matrix preserving the commutation relations

$$\Omega = R \cdot \Omega \cdot R^\dagger. \quad (2.30)$$

The covariance matrices, circuit (2.19) and generators(2.14) get rotated according to

$$\tilde{G} = R \cdot G \cdot R^\dagger, \quad \tilde{S} = R \cdot S \cdot R^{-1}, \quad \tilde{K} = R \cdot K \cdot R^{-1}. \quad (2.31)$$
One such useful transformation mentioned earlier is the one moving between the real and complex coordinates, i.e., \( \xi_R = R C \to R \) where \( R \) was defined in eq. (2.17). Note that not all basis transformations can be represented as unitary transformations acting on the state \( |\psi\rangle \) used to define the covariance matrix. Finally, let us point out that the inner product (2.27) has to be evaluated in the basis in which we want to compute the complexity. Alternatively we may compute it in a different basis by using the rotated inner product

\[
Y^I = \frac{1}{2} \text{tr} \left( KGK^\dagger G^{-1} \right). 
\] 

(2.32)

defined in terms of some positive symmetric matrix \( G = RR^\dagger \).

### 2.2 Properties of Uncharged Thermofield Double

Before we consider the charged TFD state, let us briefly summarize some useful results from [10] about the uncharged TFD state of two harmonic oscillators. We will follow the common nomenclature of referring to the two harmonic oscillators as the “right” and “left” copies. We will consider a single mode/oscillator in the left system \((q_L, p_L)\) and a single harmonic oscillator in the right system \((q_R, p_R)\), both taken to have frequency \( \omega \) and mass \( M \). The Hamiltonian for this system is given by

\[
H = \sum_{s \in [L,R]} \left( \frac{1}{2M} p_s^2 + \frac{M \omega^2}{2} q_s^2 \right) = \sum_{s \in [L,R]} \omega (a_s^\dagger a_s + \frac{1}{2} )
\]

(2.33)

where the real phase space coordinates are related to the creation and annihilation operators according to

\[
Q_s = \frac{1}{\sqrt{2M\omega}} (a_s^\dagger + a_s), \quad P = i \sqrt{\frac{M\omega}{2}} (a_s^\dagger - a_s). 
\]

(2.34)

The time dependent TFD state is defined as follows

\[
|\text{TFD}(t)\rangle = Z_\beta^{-1/2} \sum_{n=0}^{\infty} e^{-\frac{\omega}{2} \left( n+\frac{1}{2} \right)} e^{-(n+\frac{1}{2})\omega t} |n\rangle_L |n\rangle_R \\
= Z_\beta^{-1/2} e^{-\frac{\omega}{2} wt} \sum_{n=0}^{\infty} \exp \left[ -\frac{\omega}{2} e^{-\omega t} a_L a_R^\dagger \right] |0\rangle_L |0\rangle_R 
\]

(2.35)

where the normalization factor \( Z_\beta = e^{-\beta \omega/2} (1 - e^{-\beta \omega})^{-1} \).

Although we will not be explicit about it, the gates used in constructing quantum circuits for the cTFD state in this paper, and also those used to construct the TFD state in [10] consist of quadratic combinations of the coordinates and momenta. Since these are dimensionful operators we will have to introduce an additional scale \( \omega_g \) (with inverse length dimensions) in our complexity model, similarly to what was done in section 2.2.3 of [10]. This scale is used to rescale the positions and momentum variables in such a way that they become dimensionless

\[
q_s \equiv \omega_g Q_s, \quad p_s \equiv P_s / \omega_g. 
\]

(2.36)
It will also be useful to define
\[ \lambda \equiv \omega / \mu_g, \quad \text{where} \quad \mu_g \equiv \omega_0^2 / M. \tag{2.37} \]

We will refer to \( \mu_g \) as the gate scale.

The TFD state takes a simpler form in the \( \pm \) basis, defined according to
\[ q_\pm \equiv \frac{1}{\sqrt{2}} (q_L \pm q_R), \quad p_\pm \equiv \frac{1}{\sqrt{2}} (p_L \pm p_R). \tag{2.38} \]

In this basis we are able to write the TFD as (see eqs. (35), (36) and (77) in [10])
\[ |\text{TFD}(t)\rangle = e^{-i\alpha \hat{O}_+} |0\rangle \otimes e^{i\alpha \hat{O}_-} |0\rangle, \tag{2.39} \]
where we have defined
\[ \alpha \equiv \frac{1}{2} \log \left( \frac{1 + e^{-\beta \omega / 2}}{1 - e^{-\beta \omega / 2}} \right). \tag{2.40} \]

The covariance matrices of the TFD state in the \( \pm \) basis is given by eq. (76) of [10], i.e.,
\[ G^\pm_{\text{TFD}}(t, \alpha) \equiv \begin{pmatrix} \lambda \pm 1 & \mp \text{sinh}(2\alpha) \sin(\omega t) \\ \mp \text{sinh}(2\alpha) \cos(\omega t) & \lambda \cos(2\alpha) \mp \text{sinh}(2\alpha) \cos(\omega t) \end{pmatrix}. \tag{2.41} \]

This TFD state consists of a single Harmonic oscillator of frequency \( \omega \). When studying the TFD of the full QFT, [10] have shown that the problem factorizes to evaluating the complexity of a product of different one-mode TFD states, each with a different frequency \( \omega_k = \sqrt{k^2 + m^2} \) where \( k \) is the momentum of the different modes and \( m \) is the QFT mass.

The complexity problem involves also a reference state. This reference state can be seen as a ground state of some Hamiltonian with a different frequency \( \mu \). When moving to the QFT problem, the reference state is considered simple since all modes have the same frequency in the reference state and hence all spatial correlations vanish in the state. The Hamiltonian for the reference state takes the form
\[ H = \sum_{s \in [L,R]} \left( \frac{1}{2M} p_s^2 + \frac{M \mu_{\text{ref}}^2}{2} q_s^2 \right) \tag{2.42} \]
where \( \mu_{\text{ref}} \) goes under the name of reference state scale.\(^{14}\) It was demonstrated in eq. (47) of [10] that the relevant covariance matrix is given by
\[ G^\pm_{\text{ref}} = \begin{pmatrix} \lambda_{\text{ref}}^{-1} & 0 \\ 0 & \lambda_{\text{ref}} \end{pmatrix}, \quad \lambda_{\text{ref}} \equiv \frac{\mu_{\text{ref}}}{\mu_g}. \tag{2.43} \]

Before we proceed we should mention that the optimal trajectories used for studying the complexity in [10] are of the form (2.25)–(2.26) when the complexity is minimized with respect to the \( F_2 \) norm, and when choosing \( \lambda_{\text{ref}} = 1 \), i.e., the reference state scale and gate

\(^{14}\) \( \mu_{\text{ref}} \) was denoted \( \mu \) in [10] which we have change in order to reserve \( \mu \) for the chemical potential.
scale are taken to be equal. In all other cases, the results for the length of the trajectory described in section 2.1 will provide a bound on the complexity.

In a similar way, the covariance matrix for the vacuum state of the Hamiltonian in eq. (2.33) is given by

$$G_{\text{vac}}^\pm = \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix},$$

(2.44)

Here and throughout, we shall use superscripts to identify the basis of phase-space generators in which we write covariance matrices. The rotation to the LR basis will be done according to

$$\begin{bmatrix} q_+ \\ q_- \end{bmatrix} = R_{LR} \begin{bmatrix} q_L \\ q_R \end{bmatrix}, \quad \begin{bmatrix} p_+ \\ p_- \end{bmatrix} = R_{LR} \begin{bmatrix} p_L \\ p_R \end{bmatrix},$$

(2.45)

where

$$R_{LR} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(2.46)

Note that in this section the phase space coordinates were taken to be real.

### 2.3 Complex Scalar Field

In order to build a Gaussian version of the charged TFD state, we will focus on a theory consisting of two copies of a complex scalar field in \(d\)-dimensions. The Hamiltonian for each side of the system is given in terms of the fields \(\phi_s, \phi_s^\dagger\) and conjugate momenta \(\pi_s, \pi_s^\dagger\) according to

$$H_s = \int d^{d-1}x \left( \pi_s^\dagger \pi_s + \vec{\nabla} \phi_s \cdot \vec{\nabla} \phi_s^\dagger + m^2 \phi_s \phi_s^\dagger \right), \quad s \in \{L, R\}.$$

(2.47)

The field and momentum operators obey the equal time commutation relations \([\phi_s(x), \pi_s(y)] = [\phi_s^\dagger(x), \pi_s^\dagger(y)] = i\delta^{d-1}(x-y)\) and similarly for the right copy. The charge for each copy is given by

$$C_s = i \int d^{d-1}x \left( \phi_s^\dagger \pi_s^\dagger - \pi_s \phi_s \right).$$

(2.48)

In the above expression we have chosen our convention such that the fundamental unit of charge is set to one, but of course, this dependence can be recovered later by redefining the chemical potential appropriately. The complex scalar field can be decomposed in terms of the following Fourier modes

$$\phi_s(x) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_p}} \left( a_{s,\vec{p}} e^{-i\omega_p t + i\vec{p} \cdot \vec{x}} + a_{s,\vec{p}}^\dagger e^{i\omega_p t - i\vec{p} \cdot \vec{x}} \right),$$

$$\pi_s(x) = -i \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{\omega_p}{2} \left( a_{s,\vec{p}} e^{-i\omega_p t + i\vec{p} \cdot \vec{x}} - a_{s,\vec{p}}^\dagger e^{i\omega_p t - i\vec{p} \cdot \vec{x}} \right),$$

(2.49)
where \( a_{s,p}, a_{s,p}^\dagger \) and \( \bar{a}_{s,p}, \bar{a}_{s,p}^\dagger \) are annihilation and creation operators for particle and antiparticle modes, respectively, satisfying the following commutation relations
\[
[a_{s,p}, a_{s',p'}^\dagger] = [\bar{a}_{s,p}, \bar{a}_{s',p'}^\dagger] = (2\pi)^{d-1}\delta_{s,s'}\delta(p - p')
\]  
(2.50)
and \( \omega_p \equiv \sqrt{p^2 + m^2} \). In terms of those creation and annihilation operators the Hamiltonian and charge are given by
\[
H_s = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \omega_p \left(a_{s,p}^\dagger a_{s,p} + \bar{a}_{s,p} \bar{a}_{s,p}^\dagger\right), \quad C_s = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left(a_{s,p}^\dagger a_{s,p}^\dagger - \bar{a}_{s,p} \bar{a}_{s,p}\right),
\]  
(2.51)
where as usual we have subtracted the zero point energy. This reflects the fact that particle and antiparticles contribute to the energy of a given state according to the sum of their number operators while contributing to the charge as with inverse signs.

2.4 Normal Mode Decomposition on the Lattice

As explained in section 1, the complexity is divergent and has to be regularized. This was done in [6, 10] by placing the theory on the spatial periodic lattice. Hence, we will start by briefly reviewing how to place the free complex scalar on the spatial periodic lattice. The resulting theory will be a sum of Harmonic oscillators for the different momentum modes.

We will use a periodic lattice of size \( L \) in each space direction with \( N^{d-1} \) sites and lattice spacing \( \delta = L/N \). For convenience we assume that \( N \) is odd. The different sites will be labelled by indices
\[
\vec{a} \equiv (a_1, \ldots, a_{d-1}) \in \{-\vec{N}, \ldots, \vec{N}\}^{d-1}, \quad \text{where and} \quad \vec{N} = \frac{N - 1}{2}.
\]  
(2.52)
The discretized versions of eqs. (2.47) and (2.48) take the form
\[
H_s = \sum_{\vec{a}} \left[ \delta\hat{P}_{s,\vec{a}}^\dagger \hat{P}_{s,\vec{a}} + m^2 \delta^{d-1}\hat{Q}_{s,\vec{a}}^\dagger \hat{Q}_{s,\vec{a}} + \delta^{-3} \sum_j (\hat{Q}_{s,\vec{a}} + e_j) (\hat{Q}_{s,\vec{a}} + e_j) \right]
\]  
(2.53)
\[
C_s = i \sum_{\vec{a}} (\hat{Q}_{s,\vec{a}}^\dagger \hat{P}_{s,\vec{a}} - \hat{P}_{s,\vec{a}}^\dagger \hat{Q}_{s,\vec{a}}),
\]  
(2.54)
where we have defined
\[
\hat{Q}_{s,\vec{a}} \equiv \delta^{d/2}\phi(\delta \cdot \vec{a}), \quad \hat{P}_{s,\vec{a}} \equiv \delta^{d/2-1}\pi(\delta \cdot \vec{a}),
\]  
(2.55)
e\_j denotes the unit vector in the \( j \)-th direction and we denoted the position and momentum variables by hats in order to keep the symbols \( P, Q \) free for later use. These coordinates satisfy the commutation relations
\[
[\hat{Q}_{s,\vec{a}}, \hat{P}_{s,\vec{b}}] = i\delta_{\vec{a},\vec{b}}\delta_{s,s}.
\]  
(2.56)
We see that on the lattice the field theory reduces to a theory of coupled harmonic oscillators.

To decouple these oscillators, we move into Fourier space by defining
\[
\hat{Q}_{s,\vec{n}} \equiv N^{-\frac{d-1}{2}} \sum_{\vec{a}} e^\frac{-2\pi i \vec{a} \cdot \vec{n}}{N} \hat{Q}_{s,\vec{a}}, \quad \hat{P}_{s,\vec{n}} \equiv N^{-\frac{d-1}{2}} \sum_{\vec{a}} e^\frac{-2\pi i \vec{a} \cdot \vec{n}}{N} \hat{P}_{s,\vec{a}}
\]  
(2.57)
n \equiv (n_1, \ldots, n_{d-1}) \in \{-\vec{N}, \ldots, \vec{N}\}^{d-1},
\]
Note that we have chosen opposite conventions for the Fourier transforms of the coordinates and momenta. These coordinates satisfy the commutation relations

\[ [\tilde{Q}_{s,\vec{n}}, \tilde{P}_{r,\vec{k}}] = i\delta_{\vec{n},\vec{k}}\delta_{rs}, \]  

(2.58)

The Hamiltonian and charge take the form\(^{15}\)

\[ H_s = \sum_{\vec{n}} \left( \delta \tilde{P}_{s,\vec{n}}^\dagger \tilde{P}_{s,\vec{n}} + \omega_n^2 \delta^{-1} \tilde{Q}_{s,\vec{n}}^\dagger \tilde{Q}_{s,\vec{n}} \right), \quad \omega_n^2 \equiv m^2 + \frac{4}{\delta^2} \sum_j \sin^2 \left( \frac{n_j \pi}{N} \right), \]  

(2.59)

In order to gain physical intuition for this decomposition, it is instructive to consider the above transformations in terms of creation and annihilation operators. On the lattice, the complex scalar field and its conjugate momentum have mode expansions\(^{16}\)

\[ \phi_s(\delta \cdot \vec{a}, t) = L^{-\frac{d-1}{2}} \sum_{\vec{n}} \frac{1}{\sqrt{2\omega_n}} \left( a_{s,\vec{n}} \mathrm{e}^{-i(\omega_n t - \frac{2\pi \vec{n} \cdot \vec{a}}{N})} + a_{s,\vec{n}}^\dagger \mathrm{e}^{i(\omega_n t - \frac{2\pi \vec{n} \cdot \vec{a}}{N})} \right), \]  

(2.60)

\[ \pi_s(\delta \cdot \vec{a}, t) = -iL^{-\frac{d-1}{2}} \sum_{\vec{n}} \sqrt{\frac{\omega_n}{2}} \left( \bar{a}_{s,\vec{n}} \mathrm{e}^{-i(\omega_n t - \frac{2\pi \vec{n} \cdot \vec{a}}{N})} - a_{s,\vec{n}}^\dagger \mathrm{e}^{i(\omega_n t - \frac{2\pi \vec{n} \cdot \vec{a}}{N})} \right), \]  

(2.61)

where \(\vec{a}, \vec{n}\) take values as indicated by eqs. (2.52) and (2.57), \(\omega_n\) is defined in (2.59), and

\[ [a_{s,\vec{n}}, a_{s,\vec{n}}^\dagger] = [\bar{a}_{s,\vec{n}}, \bar{a}_{s,\vec{n}}^\dagger] = \delta_{\vec{n},\vec{n}'} \]  

(2.62)

with other creation and annihilation commutators vanishing. Using eqs. (2.60), (2.57) and (2.55) we can deduce

\[ \tilde{Q}_{s,\vec{n}} = \sqrt{\frac{\delta}{2\omega_n}} \left( a_{s,\vec{n}} + a_{s,-\vec{n}}^\dagger \right), \quad \tilde{P}_{s,\vec{n}} = -i\sqrt{\frac{\omega_n}{2\delta}} \left( \bar{a}_{s,-\vec{n}} - a_{s,\vec{n}}^\dagger \right) \]  

(2.63)

The \(a_{s,\vec{n}}, \bar{a}_{s,\vec{n}}\) can be regarded as annihilation operators for particles and anti-particles respectively since the Hamiltonian and charge of the field are given by

\[ H_s = \sum_{\vec{n}} \omega_n (N_{s,\vec{n}} + \bar{N}_{s,\vec{n}} + 1), \quad C_s = \sum_{\vec{n}} (N_{s,\vec{n}} - \bar{N}_{s,\vec{n}}), \]  

(2.64)

where the number operators are defined to be

\[ N_{s,\vec{n}} \equiv a_{s,\vec{n}}^\dagger a_{s,\vec{n}}, \quad \bar{N}_{s,\vec{n}} \equiv \bar{a}_{s,\vec{n}}^\dagger \bar{a}_{s,\vec{n}}. \]  

(2.65)

Note that one usually normal-orders the Hamiltonian, removing the last term of the summand in the first equation of (2.64) representing the zero point energy.

\(^{15}\)Throughout the following we will stick to the convention where \(\tilde{Q}_{s,\vec{n}}^\dagger\) is the complex conjugate of \(\tilde{Q}_{s,\vec{n}}\) rather than being the Fourier transform of the coordinate \(\tilde{Q}_{s,\vec{a}}^\dagger\) with conventions as in (2.57).

\(^{16}\)The creation and annihilation operators in this section are dimensionless and are related to ones in the previous section according to \(a_{s,\vec{n}}^{\text{continuous}} = L^{\frac{d-1}{2}} a_{s,\vec{n}}^{\text{lattice}}\) where \(\bar{P}_{\text{continuous}} = \frac{2\pi}{N}\bar{P}_{\text{lattice}}\). 

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Of course, all the above discussion can be phrased in terms of the dimensionless coordinates, see eq. (2.36), i.e.,

\[
\tilde{q}_{s,\bar{n}} \equiv \omega_g \tilde{Q}_{s,\bar{n}} = \sqrt{\frac{1}{2\lambda_n}} \left( a_{s,\bar{n}} + a_{s,\bar{n}}^\dagger \right), \quad \tilde{p}_{s,\bar{n}} \equiv \tilde{P}_{s,\bar{n}} = -i \sqrt{\frac{\lambda_n}{2}} \left( a_{s,\bar{n}} - a_{s,\bar{n}}^\dagger \right). \tag{2.66}
\]

Using these new dimensionless variables we can express the Hamiltonian and charge in eq. (2.59) as

\[
H_s = \sum_{\bar{n}} \omega_n \left( \lambda_n^{-1} \tilde{p}_{s,\bar{n}}^\dagger \tilde{p}_{s,\bar{n}} + \lambda_n \tilde{q}_{s,\bar{n}}^\dagger \tilde{q}_{s,\bar{n}} \right), \quad C_s = i \sum_{\bar{n}} (\tilde{q}_{s,\bar{n}}^\dagger \tilde{p}_{s,\bar{n}} - \tilde{q}_{s,\bar{n}} \tilde{p}_{s,\bar{n}}) \tag{2.67}
\]

where we have defined

\[
\lambda_n \equiv \frac{\omega_n}{\mu_g}, \quad \text{where} \quad \mu_g \equiv \delta \omega_g^2. \tag{2.68}
\]

and we refer to \(\mu_g\) as the gate scale. For later reference, we denote this basis by:

\[
\xi_{s,\bar{n}}^c = [\tilde{q}_{s,\bar{n}} \tilde{q}_{s,\bar{n}}^\dagger \tilde{p}_{s,\bar{n}}^\dagger \tilde{p}_{s,\bar{n}}]^T, \quad s \in [L, R]. \tag{2.69}
\]

The unusual ordering of the variables and their conjugates were chosen such that \([\xi_{s,\bar{n}}^c, \xi_{r,\bar{n}}^c]^T = i\delta_{sr} \Omega\).

The complexity is defined as the minimal distance from a given reference state, usually chosen to be unentangled. For the reference state, we take the ground state of the Hamiltonian

\[
H_{s,\text{ref}} = H_{R,\text{ref}} + H_{L,\text{ref}}; \quad H_{s,\text{ref}} = \int d^{d-1}x \left( \pi_s^\dagger \pi_s + m_{\text{ref}}^2 \phi_s^\dagger \phi_s \right), \quad s \in [L, R]. \tag{2.70}
\]

Note that, due to the lack of the derivative term (compare this to eq. (2.47)), this Hamiltonian only couples spatial degrees of freedom \(\pi(x), \phi(x)\) to themselves. Hence, its vacuum state is spatially unentangled — a desirable property for a ‘simple’ reference state. The Hamiltonian in eq. (2.70) can be expressed in terms of the variables (2.36) yielding

\[
H_{s,\text{ref}} = \sum_{\bar{n}} m_{\text{ref}} \left[ \lambda_{\text{ref}}^{-1} \tilde{p}_{s,\bar{n}}^\dagger \tilde{p}_{s,\bar{n}} + \lambda_{\text{ref}} \tilde{q}_{s,\bar{n}}^\dagger \tilde{q}_{s,\bar{n}} \right] \tag{2.71}
\]

where above we have defined

\[
\lambda_{\text{ref}} \equiv \frac{m_{\text{ref}}}{\mu_g}. \tag{2.72}
\]

### 3 Complexity of the Charged TFD State

The charged TFD state for the complex scalar theory is defined according to eq. (1.2). Note however that this equation only depends on the combination of times \(t_L + t_R\) and so the full time dependence can be captured by setting, e.g., \(t_L = t_R = t/2\) as we do in the following. The relevant state for our construction is therefore give by

\[
|c_{\text{TFD}}(\beta, \mu; t)\rangle \equiv \frac{1}{\sqrt{Z_{\beta, \mu}}} \sum_{E_n, c} e^{-\beta \frac{1}{2}(E_n + \mu c)} - t(E_n + \mu c) |E_n, c\rangle_L |E_n, -c\rangle_R, \tag{3.1}
\]
with \(|E_n, c_\sigma\rangle\) denoting the Hamiltonian and charge eigenstates with eigenvalues \(E_n, c_\sigma\), respectively. The goal of this note is to calculate the complexity of this state for free complex scalar field theory as a toy model.

The decomposition (2.59) together with the commutation relations (2.58) teach us that the charged TFD state in the full-fledged QFT factorizes into charged TFD states of each of the momentum modes, i.e.,

\[
|c\text{TFD}(\beta, \mu; t)\rangle = \bigotimes_n |c\text{TFD}_n(\beta, \mu; t, \omega_n)\rangle.
\]

(3.2)

Below, we explain how each of these factors in the charged TFD state in fact factors further into two uncharged TFDs at inverse temperatures and times offset from \(\beta\) by the chemical potential.

### 3.1 Decoupling the Particles and Anti-Particles

So far, our steps have been nearly identical to those of [10]. Indeed, we have reframed the theory of the complex scalar field as a theory of decoupled harmonic oscillators in the LR basis defined in eq. (2.69). It will be useful to perform an additional transformation to identify degrees of freedom associated with positive and negative charges. This transformation is performed separately on each side (left/right) of the TFD and is given by the explicit expression

\[
\xi^s_\vec{n} \equiv R^s_{sc-\vec{n}} \xi^{sc}_{\vec{n}}, \quad \xi^{sc}_{\vec{n}} \equiv \begin{bmatrix} \hat{q}_{s, \vec{n}} \\ \hat{\bar{q}}_{s, \vec{n}} \\ \hat{p}_{s, \vec{n}} \end{bmatrix}, \quad \xi^s_{\vec{n}} \equiv \begin{bmatrix} q_{s, \vec{n}} \\ \bar{q}_{s, \vec{n}} \\ p_{s, \vec{n}} \end{bmatrix},
\]

(3.3)

This transformation of coordinates does not modify the commutation relations (2.1) since it satisfies the condition (2.30), it is also easy to check explicitly that this transformation generates real coordinates \(\xi^s_{\vec{n}} = \xi^{s\dagger}_{\vec{n}}\). By substituting this coordinate transformation into eq. (2.67) we obtain

\[
H_s = \frac{1}{2} \sum_n \omega_n \left[ \lambda_n^{-1}(p_{s, \vec{n}}^2 + \bar{p}_{s, \vec{n}}^2) + \lambda_n(q_{s, \vec{n}}^2 + \bar{q}_{s, \vec{n}}^2) \right],
\]

\[
C_s = \frac{1}{2} \sum_n \left[ \lambda_n^{-1}(p_{s, \vec{n}}^2 - \bar{p}_{s, \vec{n}}^2) + \lambda_n(q_{s, \vec{n}}^2 - \bar{q}_{s, \vec{n}}^2) \right].
\]

(3.4)

Here we see that the oscillators remain decoupled in the Hamiltonian and we see that the oscillators \((q_{s, \vec{n}}, \bar{p}_{s, \vec{n}})\) have positive charge while \((\bar{q}_{s, \vec{n}}, \bar{p}_{s, \vec{n}})\) have negative charge, cf. eq. (2.64).

In order to gain physical intuition for this decomposition, it is instructive to consider the above transformations in terms of creation and annihilation operators. Written in terms
of creation and annihilation operators, the phase space variables introduced previously read
\[
q_{\mathbf{s}, \mathbf{n}} = \sqrt{\frac{1}{2\lambda_n}} (a_{\mathbf{s}, \mathbf{n}} + a_{\mathbf{s}, \mathbf{n}}^\dagger), \quad p_{\mathbf{s}, \mathbf{n}} = -i \sqrt{\frac{\lambda_n}{2}} (a_{\mathbf{s}, \mathbf{n}} - a_{\mathbf{s}, \mathbf{n}}^\dagger),
\]
(3.5)
\[
\bar{q}_{\mathbf{s}, \mathbf{n}} = \sqrt{\frac{1}{2\lambda_n}} (\bar{a}_{\mathbf{s}, \mathbf{n}} + \bar{a}_{\mathbf{s}, \mathbf{n}}^\dagger), \quad \bar{p}_{\mathbf{s}, \mathbf{n}} = -i \sqrt{\frac{\lambda_n}{2}} (\bar{a}_{\mathbf{s}, \mathbf{n}} - \bar{a}_{\mathbf{s}, \mathbf{n}}^\dagger).
\]
(3.6)
Thus, \((q_{\mathbf{s}, \mathbf{n}}, p_{\mathbf{s}, \mathbf{n}})\) and \((\bar{q}_{\mathbf{s}, \mathbf{n}}, \bar{p}_{\mathbf{s}, \mathbf{n}})\) correspond to the real phase space variables for particles and anti-particles of the field theory, respectively. This explains the signs of the charge contributions in (3.4) from \((q_{\mathbf{s}, \mathbf{n}}, p_{\mathbf{s}, \mathbf{n}})\) and \((\bar{q}_{\mathbf{s}, \mathbf{n}}, \bar{p}_{\mathbf{s}, \mathbf{n}})\). Furthermore, the transformation in eq. (3.3) can be decomposed into a rotation of the complex phases space variables to real phase space variables as in (2.17), followed by a symplectic transformation which separates the particles and anti-particles creation and annihilation operators. Note that the Fourier transform of eq. (3.4) will suffer from some degree of non-locality. This is because we have insisted on decoupling the \(\mathbf{n}\) and \(\mathbf{\bar{n}}\) modes both in eq. (2.67) and in eq. (3.4). There is no conceptual difficulty in restoring locality of the position space Hamiltonian by defining a new set of fields using the \(\mathbf{n}\) and \(\mathbf{\bar{n}}\) modes together. In any event, this set of coordinates allows to directly utilize the covariance matrices from [10] which were reviewed in section 2.2, which is why we chose to use it.

Before moving on, however, let us take a moment to compare the transformations introduced in this section and those introduced for the uncharged TFD problem for a real scalar field [10]. Due to the lack of charge and the reality of their field, the authors of [10] were content to stop at (2.67) — note that, modulo Hermitian conjugation, \(H_s\) is diagonal in the tilde phase space variables there.\(^{17}\) However, we have performed an additional transformation (3.3) so that the charge in eq. (3.4) and the Hamiltonian are simultaneously diagonalized; we will find this crucial later to the decomposition of charged oscillator TFDs to uncharged TFDs. This is somewhat unfortunate as the reference Hamiltonian (2.71) is no longer diagonal following this extra transformation:
\[
H_{s,\text{ref}} = \sum_{\mathbf{n}} m_{\text{ref}} \left[ \lambda_{\text{ref}}^{-1} p_{\mathbf{s}, \mathbf{n}}^\dagger p_{\mathbf{s}, \mathbf{n}} + \lambda_{\text{ref}} q_{\mathbf{s}, \mathbf{n}}^\dagger q_{\mathbf{s}, \mathbf{n}} \right]
\]
\[= \frac{1}{4} \sum_{\mathbf{n}} m_{\text{ref}} \left\{ \lambda_{\text{ref}}^{-1} (p_{\mathbf{s}, \mathbf{n}} + \bar{p}_{\mathbf{s}, \mathbf{n}})^2 + \lambda_{\text{ref}} (p_{\mathbf{s}, \mathbf{n}} - \bar{p}_{\mathbf{s}, \mathbf{n}})^2 \right.\]
\[\left. + \lambda_{\text{ref}} (q_{\mathbf{s}, \mathbf{n}} + \bar{q}_{\mathbf{s}, \mathbf{n}})^2 + \frac{\lambda_n^2}{\lambda_{\text{ref}}} (q_{\mathbf{s}, \mathbf{n}} - \bar{q}_{\mathbf{s}, \mathbf{n}})^2 \right\}.\]
(3.7)

3.2 cTFD of Two Complex Harmonic Oscillators

Next, we consider the charged thermofield double consisting of two complex or four real harmonic oscillators on each side left/right. We will label each oscillator as right or left (R
\[^{17}\text{Actually, the Hamiltonian of [10] had a very similar form to (2.67), but with } n \text{ and } -n \text{ modes mixed. This is because for the case of a real field we have } q_{\mathbf{s}, \mathbf{n}} = q_{\mathbf{s}, -\mathbf{n}} \text{ and } p_{\mathbf{s}, \mathbf{n}}^\dagger = p_{\mathbf{s}, -\mathbf{n}}. \text{ These can be decoupled as in appendix D of [10] by performing a coordinate transformation of the type (2.17). It is the second part of the symplectic transformation described above, which splits the particles and anti-particles degrees of freedom, which was not needed for the case of the uncharged TFD.}
or $L$) and particle or anti-particle (no overbar or overbar). The complete Hamiltonian is given by

$$
H = H_R + H_L;
H_s = \frac{\omega}{2} \left[ \lambda^{-1}(p_s^2 + \bar{p}_s^2) + \lambda(q_s^2 + \bar{q}_s^2) \right], \quad s \in [L, R],
$$

(3.8)

This is to be representative of a single factor in (3.2). Alternatively, in complex coordinates we have

$$
H_s = \omega \left( \lambda^{-1}p_s^2 \bar{p}_s + \lambda q_s^2 \bar{q}_s \right).
$$

We may use similar creation and annihilation operators to those in eq. (3.5) to expand those modes, i.e.,

$$
q_s = \sqrt{\frac{1}{2\lambda}}(a_s + a_s^\dagger), \quad p_s = -i\sqrt{\frac{\lambda}{2}}(a_s - a_s^\dagger),
$$

(3.9)

and define the number operators as in (2.65),

$$
N_s \equiv a_s^\dagger a_s, \quad \bar{N}_s \equiv \bar{a}_s^\dagger \bar{a}_s.
$$

(3.11)

In terms of creation $a_s^\dagger$, $\bar{a}_s^\dagger$, annihilation $a_s$, $\bar{a}_s$, and number $N_s$, $\bar{N}_s$, operators, we have

$$
H_s = \omega (N_s + \bar{N}_s + 1), \quad N_s = a_s^\dagger a_s, \quad \bar{N}_s = \bar{a}_s^\dagger \bar{a}_s.
$$

(3.12)

Similarly, the total charge of a given side is given by

$$
C_s = \frac{1}{2} \left[ \lambda^{-1}(q_s^2 - \bar{q}_s^2) + \lambda(p_s^2 - \bar{p}_s^2) \right] = N_s - \bar{N}_s.
$$

(3.13)

Alternatively, in complex coordinates we have $C_s = i(q_s^\dagger \bar{p}_s - \bar{q}_s^\dagger p_s)$. We denote eigenstates of the number operators $|n, \bar{n}\rangle_s$, satisfying

$$
N_s|n, \bar{n}\rangle_s = n|n, \bar{n}\rangle_s, \quad \bar{N}_s|n, \bar{n}\rangle_s = \bar{n}|n, \bar{n}\rangle_s.
$$

(3.14)

The creation and annihilation operators raise and lower number eigenvalues according to

$$
|n, \bar{n}\rangle_s = \frac{(a_s^\dagger)^n(\bar{a}_s^\dagger)^\bar{n}}{\sqrt{n! \bar{n}!}} |0, 0\rangle_s.
$$

(3.15)

which have energy and charge given by

$$
H_s|n, \bar{n}\rangle_s = E_{n, \bar{n}}|n, \bar{n}\rangle_s, \quad E_{n, \bar{n}} = \omega(n + \bar{n} + 1),
$$

$$
C_s|n, \bar{n}\rangle_s = c_{n, \bar{n}}|n, \bar{n}\rangle_s, \quad c_{n, \bar{n}} = n - \bar{n}.
$$

(3.16)

A single factor of the charged TFD in eqs. (3.1)–(3.2) at a temperature $1/\beta$ and chemical potential $\mu$ is then given in terms of number operator eigenstates (which also have energy and charge eigenvalues $E_{n, \bar{n}}$, $c_{n, \bar{n}}$) as

$$
|\text{cTFD}(\beta, \mu; t, \omega)\rangle = Z_{\beta, \mu}^{-1/2} \sum_{n, \bar{n}=0}^{\infty} \exp \left\{ - \left( \frac{\beta}{2} + it \right) \left( E_{n, \bar{n}} + \mu c_{n, \bar{n}} \right) \right\} |n, \bar{n}\rangle_L |\bar{n}, n\rangle_R
$$

$$
= Z_{\beta, \mu}^{-1/2} e^{-\omega(\frac{\beta}{2} + it)} \sum_{n, \bar{n}=0}^{\infty} \exp \left\{ - \left( \frac{\beta}{2} + it \right) \left[ \omega(n + \bar{n}) + \mu(n - \bar{n}) \right] \right\} |n, \bar{n}\rangle_L |\bar{n}, n\rangle_R.
$$

(3.17)

\(^{18}\)Here $a_s$ is identified with $a_s$, $\bar{a}_s$ while $\bar{a}_s$ is identified with $\bar{a}_s$, $\bar{a}_s$ from the previous derivation.
Note the ordering of $\bar{n}, n$ in writing $|\bar{n}, n\rangle_R$; by this, we mean the $\bar{n}$-th eigenstate of $N_R$ and the $n$-th eigenstate of $\bar{N}_R$. This was done in order to recover the structure in eq. (3.1) with opposite charges on the left and right sides. The normalization constant $Z$ is given by

$$Z_{\beta,\mu} = e^{-\beta \omega} \sum_{n,\bar{n}=0}^{\infty} \exp \{-\beta [\omega(n + \bar{n}) - \mu(n - \bar{n})]\}$$

$$= e^{-\beta \omega} \left[ 1 - e^{-\beta (\omega - \mu)} \right]^{-1} \left[ 1 - e^{-\beta (\omega + \mu)} \right]^{-1}$$

$$= e^{-\beta \omega} \left[ 1 + e^{-2\beta \omega} - 2e^{-\beta \omega} \cosh(\beta \mu) \right]^{-1}.$$

Comparison of (3.17) with (2.35) shows that this charged TFD is just a product of two uncharged TFD states

$$|c\text{TFD}(\beta, \mu; t)\rangle = |\text{TFD}(\beta_{LR}, \mu; t_{LR})\rangle_{LR} \otimes |\text{TFD}(\beta_{L\bar{R}}, \mu; t_{L\bar{R}})\rangle_{L\bar{R}}$$

at temperatures and times shifted by the chemical potential

$$\beta_{LR} \equiv \beta \left(1 - \frac{\mu}{\omega}\right) \quad t_{LR} \equiv t \left(1 - \frac{\mu}{\omega}\right)$$

(3.20)

$$\beta_{L\bar{R}} \equiv \beta \left(1 + \frac{\mu}{\omega}\right) \quad t_{L\bar{R}} \equiv t \left(1 + \frac{\mu}{\omega}\right).$$

(3.21)

Note that this cannot be rephrased purely as a shift in the frequencies $\omega_{LR} \equiv \omega + \mu$, $\omega_{L\bar{R}} \equiv \omega - \mu$ since the states $|n, \bar{n}\rangle_s$ used to construct the TFD in eq. (3.17) are created from the vacuum state of each side $|0, 0\rangle$ with the creation operators defined with respect to the Hamiltonian without the shifted frequency. Therefore in (2.41) it is the case that while $\alpha$ becomes shifted due to the modified temperature above, $\lambda$ in eq. (2.37) is defined with the frequency of the original theory rather than a shifted frequency. Thus, we see that generalizing the problem of evaluating the complexity to the charged TFD is straightforward, given the existing results [10] for the ordinary TFD. However, we already see something interesting: as $|\mu| \rightarrow \omega$, one of the effective temperatures $\beta_{LR}, \beta_{L\bar{R}}$ blows up and as a consequence we form a condensate. This means that for $|\mu| > \omega$, we have an ill-defined negative-temperature state in either $LR$ or $L\bar{R}$. This will force us to take all frequencies in our field theory construction, see eq. (2.59) to be higher than the chemical potential and in particular this implies that in the field theory setup we must have $m > |\mu|$. This means we will not be able to reach the conformal limit for fixed $\mu$ in this work. Moreover, note that in general replacing $\mu \leftrightarrow -\mu$ is equivalent to swapping $LR \leftrightarrow L\bar{R}$, under which complexity is invariant.

3.3 Complexity of the cTFD of Two Complex Harmonic Oscillators

We now have the necessary ingredients to compute the complexity of a charged thermofield double (3.17) of the Two Complex harmonic oscillators system. Recall from the paragraph above eq. (2.27) that the complexity depends on the choice of basis, and that it was found in [10] that the $F_1$ cost function with a choice of basis which does not mix the left and right degrees of freedom was the one which reproduced best a number of qualitative features of
complexity in holography. For this reason we have chosen to focus on a similar choice of basis below. We will consider two different bases, the complex left-right basis $LR_C$ basis corresponding to coordinates

$$
\xi^{LR_C} = \left[ \tilde{q}_L, \tilde{q}_L^\dagger, \tilde{q}_R, \tilde{q}_R^\dagger, \tilde{p}_L, \tilde{p}_L^\dagger, \tilde{p}_R, \tilde{p}_R^\dagger \right] \tag{3.22}
$$

and the real left-right basis $LR$ corresponding to the particles and anti-particles degrees of freedom

$$
\xi^{LR} = \left[ q_L, \bar{q}_L, q_R, \bar{q}_R, p_L, \bar{p}_L, p_R, \bar{p}_R \right]. \tag{3.23}
$$

where $\xi^{LR_C}$ are related to $\xi^{LR}$ by (3.3) and here we simply specify to a single mode. The circuits that we construct in the QFT will factor to circuits which act separately on each value of the lattice momentum $\vec{n}$ and for each value of $\vec{n}$ we will consider the problem of building the cTFD of Two complex oscillators on each side. The relevant circuits will consist of $8 \times 8$ matrices acting on the density matrices according to (2.19)-(2.20) where $S(\sigma) \in \text{Sp}(8)$. We will use the name $K_I$ for the basis elements for $\text{sp}(8)$, orthonormal with respect to the $\text{Sp}(8)$-invariant inner product in the particular basis of interest as discussed in section 2.1.

For the reference state, we take the vacuum of the a single mode of the complex scalar reference Hamiltonian (2.71):

$$
H_{R,\text{ref}} = m_{\text{ref}} \left( \lambda_{\text{ref}}^{-1} \tilde{p}_R \tilde{p}_R + \lambda_{\text{ref}} \tilde{q}_R \tilde{q}_R \right). \tag{3.24}
$$

The covariance matrix of the reference state in the $LR_C$ basis can be evaluated directly from the definition (2.4) by using creation and annihilation operators adapted to the reference state as in eq. (2.66) with the replacement $\lambda_n \rightarrow \lambda_{\text{ref}}$ which yields

$$
G^{LR_C}_{\text{ref}} = G^{R_C}_{\text{ref}} = \text{diag}(\lambda_{\text{ref}}^{-1}, \lambda_{\text{ref}}^\dagger, \lambda_{\text{ref}}, \lambda_{\text{ref}}^\dagger). \tag{3.25}
$$

Using the decomposition (3.19) of the charged TFD into uncharged TFDs at shifted temperatures and times (3.20)-(3.21), we can deduce the (target) covariances of the charged TFD to be given by covariance matrices $G_{\text{TFD}}^\pm(t_{LR,n}, \alpha_{LR,n})$, $G_{\text{TFD}}^\pm(t_{LR,n}, \alpha_{LR,n})$ of uncharged TFDs (2.41), where $\alpha_{LR}, \alpha_{LR}$ are defined by (2.40) in terms respectively of the shifted temperatures $\beta_{LR}, \beta_{LR}$ given in eqs. (3.20)-(3.21). The covariance matrix obtained in this way will be given in the following basis of coordinates

$$
\xi^\pm = \left[ q^+_{LR}, q^+_{LR}, q^-_{LR}, q^-_{LR}, p^+_{LR}, p^+_{LR}, p^-_{LR}, p^-_{LR} \right] \tag{3.26}
$$

obtained from eq. (3.23) by the equivalent change of coordinates to the one in eq. (2.38), see also (2.45), which mixes the $\bar{L}$ and $R$ coordinates separately and the $\bar{R}$ and $L$ coordinates separately, i.e.,

$$
q^\pm_{LR} = \frac{1}{\sqrt{2}}(q_L \pm q_R), \quad p^\pm_{LR} = \frac{1}{\sqrt{2}}(p_L \pm p_R), \tag{3.27}
$$
To combine various components of the theory, it is useful to define the symbol \( \tilde{\oplus} \) to mean direct sum, followed by reordering rows and columns so that positions are listed before momenta. Using this, we can compute the relative covariance matrix (2.25) for the cTFD state in, say the LR basis, as follows

\[
\Delta_{cTFD}^{LR} = R_{\pm \rightarrow LR} \left[ G_{TFD}^+ (t_{LR}, \alpha_{LR}) \tilde{\oplus} G_{TFD}^- (t_{LR}, \alpha_{LR}) \tilde{\oplus} G_{TFD}^+ (t_{\bar{L}R}, \alpha_{\bar{L}R}) \tilde{\oplus} G_{TFD}^- (t_{\bar{L}R}, \alpha_{\bar{L}R}) \right] R_{\pm \rightarrow LR}^\dagger \]

where

\[
\xi_{LR} = R_{\pm \rightarrow LR} \xi_{\pm}, \quad R_{\pm \rightarrow LR} = \text{diag}(R_4, R_4), \quad R_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \in SO(4)
\]

inverts the transformation (3.27) and \( S_{L \rightarrow L} = S_{R \rightarrow R} \) is given in (3.3). We can easily transform \( \Delta_{LR} \) to another basis, say the LR basis as follows

\[
\Delta_{cTFD}^{LR} = (S_{L \rightarrow L} \tilde{\oplus} S_{R \rightarrow R}) \Delta_{\text{cTFD}}^{LR} (S_{L \rightarrow L} \tilde{\oplus} S_{R \rightarrow R})^{-1}.
\]

It only remains to compute (the upper bound for) \( F_1 \) complexity via eq. (2.28) with an appropriate basis \( K_I \) for \( \mathfrak{sp}(8) \).

### 3.4 Integrating Over the Modes

Due to the mode factorization (3.2) of the complex scalar charged TFD, we may compute the complexity of this state by summing the complexities of harmonic oscillator charged TFDs — the calculation of which was described in section 3.3 — according to

\[
\mathcal{C}(|cTFD(\beta, \mu; t)\rangle) = \sum_{\vec{n}} \mathcal{C}(|cTFD_n(\beta, \mu; t, \omega_n)\rangle),
\]

where the frequencies \( \omega_n \) were defined in eq. (2.59). The above equation, with the sum over discrete and finitely-many \( \vec{n} \)-s, gives the complexity for the discretized and compactified theory of the complex scalar. Recovering the continuum limit and decompactified space amounts to taking \( \delta \rightarrow 0 \) and \( L \rightarrow \infty \) respectively. The former is implemented by replacing sums with integrals:\(^{19}\)

\[
\sum_{\vec{n}} \rightarrow \int_{-\tilde{\mathcal{N}}}^{\tilde{\mathcal{N}}} d^{d-1}n = \left( \frac{L}{2\pi} \right)^{d-1} \int_{-\pi/\delta}^{\pi/\delta} d^{d-1}k,
\]

where we have switched to the continuous label

\[
\vec{k} = \frac{2\pi \vec{n}}{L},
\]

cf. footnote 16, and \( \tilde{\mathcal{N}} \) was defined in eq. (2.52). Further taking the continuum limit extends the domain of integration to \( \mathbb{R} \) in (3.32). As already mentioned in the introduction, the

\(^{19}\)There appears to be a typo in (188) of [10] in that the RHS is missing the prefactor we have in (3.32).
complexity in QFT has UV divergences and needs to be regularized. This was the reason why we regulated the theory on the lattice in the first place. The divergences of the complexity for the cTFD state will be the same as those for a product state constructed from two copies of the vacuum state since the UV structure is similar in these two states. When working in the continuum limit, without a lattice, it is possible to regulate the UV divergences by introducing a momentum cutoff $|\vec{k}| < \Lambda$ as the limit of integration in eq. (3.32). In this construction we also have to replace the frequency by the continuous frequency $\omega_k = \sqrt{\vec{k}^2 + m^2}$, see the discussion in section 5.1 of [10] for more details. For our calculations however, we will focus mostly on differences of complexities which yield finite quantities and therefore we will be able to simply take $\Lambda \to \infty$.

3.5 Complexity of Formation

Following along the lines of section 5.3 of [10], we will now investigate the complexity of formation [43] of the complex scalar field in the charged thermofield double state $\Delta_{cTFD}(t = 0) \equiv C_{cTFD}(t = 0) - C_{\text{vac}(L \otimes R)}$ (3.34)

where $C_{\text{vac}(L \otimes R)}$ is the complexity of two copies of the vacuum state of the same complex scalar field theory. This quantity compares how much harder it is to prepare the cTFD state at $t = 0$ compared to preparing to copies of the vacuum state in the same complex scalar theory. This quantity is UV finite.

By setting $t = 0$ to compute the complexity of formation, it is possible to analytically diagonalize the relative covariance matrix (3.28), or (3.30), of each mode. We begin by noting that at $t = 0$, (2.41) simplifies significantly:

$$G_{\pm TFD}(t = 0, \alpha) = \text{diag}(\lambda^{-1} e^{\pm 2\alpha}, \lambda e^{\mp 2\alpha}) .$$

Further, transforming the covariance matrix of the reference state (3.25) also to the $\pm$ basis, we find

$$G_{\pm \text{ref}} = G_{LR}^{LR} R_{LR \rightarrow LR} R_{LR \rightarrow LR}^\dagger = \text{diag}(C, D, D),$$

where the covariance matrix here is given in the $\xi^\pm$ basis from eq. (3.26) and $R_{sc \rightarrow s}$ is given in eq. (3.3). The first equality is due to the fact that the latter is stationary under the change of basis $R_{LR \rightarrow \pm}$, given in (3.29). We can then obtain the relative covariance in the $\pm$ basis as

$$\Delta_{cTFD}^\pm = [G_{TFD}^+ (t_{LR}, \alpha_{LR}) \oplus G_{TFD}^- (t_{LR}, \alpha_{LR}) \oplus G_{TFD}^- (t_{LR}, \alpha_{LR}) \oplus G_{TFD}^+ (t_{LR}, \alpha_{LR})] (G_{\text{ref}}^\pm)^{-1} (3.37)$$

As we show in appendix C, we can in fact analytically diagonalize $\Delta_{cTFD}$ at $t = 0$.

To evaluate the complexity of formation (3.34), we must also evaluate the complexity of the vacuum. The relative covariance matrix of the vacuum is straight-forwardly obtained
since the covariance matrices of both the vacuum and the reference state are diagonal in the $LR_C$ basis:

$$G_{LR_C}^{ref,n} = \text{diag}(\lambda_{ref}^{-1} 1_{4 \times 4}, \lambda_{ref} 1_{4 \times 4})$$  \hspace{1cm} (3.38)

$$G_{LR_C}^{vac,n} = \text{diag}(\lambda_{n}^{-1} 1_{4 \times 4}, \lambda_{n} 1_{4 \times 4})$$  \hspace{1cm} (3.39)

$$\Delta_{LR_C}^{vac,n} = \text{diag} \left( \frac{\lambda_{ref}}{\lambda_{n}} 1_{4 \times 4}, \frac{\lambda_{n}}{\lambda_{ref}} 1_{4 \times 4} \right).$$  \hspace{1cm} (3.40)

Taking $\beta \mu \to 0$ in the $LR$ and $LR_C$ bases gives identical results for $\Delta C(t = 0)$ which match the uncharged TFD [10] — we shall verify this in figure 1.

Transforming (3.37) and (3.40) to the $LR$ and $LR_C$ bases, we numerically evaluate the $F_1$ complexity of formation for the complex scalar charged TFD. In [43], it was found that holographic complexity of formation in both CA and CV is proportional to entropy for planar black holes in $d \geq 3$. Hence, it is natural for us to consider the ratio between complexity of formation and entropy. For the uncharged TFD state of two harmonic oscillators, the entanglement entropy between the two sides, or equivalently thermal entropy of the thermal state obtained on each side after tracing the other, is obtained from the partition function with Bose-Einstein statistics by differentiating with respect to the temperature (cf. eqs. (201)-(202) of [10]). The expression there for the uncharged TFD was

$$S_{TFD} = \text{vol} \int d^{d-1}k \frac{d^{d-1} \beta \omega_k (e^{\beta \omega_k} - 1 - \log(1 - e^{-\beta \omega_k}))}{(2\pi)^{d-1}},$$  \hspace{1cm} (3.41)

where $\omega_k = \sqrt{k^2 + m^2}$ and $\text{vol} = L^{d-1}$ is an IR regulator for the volume of the field theory. For the cTFD state we are dealing with two sets of modified temperatures and times, as indicated in eqs. (3.20)-(3.21). We therefore suggest that it is natural to normalize our result with respect to the following sum of entropies

$$S_{cTFD} = S_{TFD}(\beta \to \beta_{LR}) + S_{TFD}(\beta \to \beta_{LR}^\ast).$$  \hspace{1cm} (3.42)

It will come handy in what follows to work in terms of a set of dimensionless variables $x, y, u, \bar{\gamma}$, defined by

$$x \equiv \beta m, \quad y \equiv \beta \mu, \quad u \equiv \beta k, \quad \bar{\gamma} \equiv \frac{1}{\beta m_{ref}}.$$  \hspace{1cm} (3.43)

In terms of these dimensionless variables, the entropy of one side of the charged TFD of the complex scalar field can be expressed as

$$S_{cTFD} = \frac{\text{vol}}{\beta^{d-1}} [s(\beta m, \beta \mu) + s(\beta m, -\beta \mu)]$$

$$s(x, y) \equiv \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \int_0^\infty du u^{d-2} \left[ \frac{g(u, x, y)}{e^{g(u, x, y)} - 1} - \log(1 - e^{-g(u, x, y)}) \right],$$  \hspace{1cm} (3.44)

where $g$ is given by

$$g(u, x, y) \equiv \sqrt{u^2 + x^2} - y.$$  \hspace{1cm} (3.45)
In most plots below, we will consider the ratio
\[ \frac{\Delta C_1([c\text{TFD}(t = 0)])}{S_{c\text{TFD}}} \]
However, since at low temperatures the entropy goes to zero, in order to study the low temperature limits we will sometimes present un-normalized plots.

### 3.5.1 High Temperature Limit
Taking \( \beta \to 0 \), the dimensionless parameters of the theory \( \beta m, \beta \mu \to 0 \) vanish. Hence, we expect to recover the same results proportional to that of the uncharged TFD for a massless scalar in eq. (206) of [10], namely
\[ \frac{\Delta C_1(c\text{TFD})}{S_{c\text{TFD}}} = 2^{d-1} d \begin{cases} 1 & \text{LR basis} \\ 2^{-1/2} & \text{LR}_C \text{ basis} \end{cases} \quad (\beta m = \beta \mu = 0). \]
Note that in this limit both the entropy and complexity vanish, but their ratio is constant. The extra factor of \( 2^{-1/2} \) in the \( LR_C \) basis can be attributed to the fact that the straight-line circuit is better aligned with the elementary gates of this basis so that the circuit can be generated with fewer gates. Indeed, as we shall numerically verify in figure 1, this proportionality is a generic feature for \( \beta \mu = 0 \).

### 3.5.2 Low Temperature Limit
Here, we consider the low temperature (large \( \beta \)) limit. Focusing first on the neutral case with \( \beta \mu = y = 0 \), see eq. (3.43), and taking \( x = \beta m \gg 1 \), gives the uncharged TFD at low temperatures, which was already treated in [10], see eq. (208) there,
\[ \frac{\Delta C_1(c\text{TFD})}{S_{c\text{TFD}}} \sim 2^{(d+1)/2} x^{d/2} \begin{cases} 1 & \text{LR basis} \\ 2^{-1/2} & \text{LR}_C \text{ basis} \end{cases} \quad (x \gg 1 \text{ and } y = 0). \]
As in (3.53), we have an extra factor of \( 2^{-1/2} \) in the \( LR_C \) basis, with which the straight-line circuit is better aligned with \( y = \beta \mu = 0 \).

Next, we consider the low temperature limit with positive chemical potential, \( \text{i.e., } x = \beta m \gg 1 \text{ and } y = \beta \mu \gg 1 \), where without loss of generality we have chosen \( y \geq 0 \).

Note that, for large \( x \), \( \alpha_{LR} \), defined in eq. (2.40) using the modified temperature (3.20)-(3.21), is exponentially suppressed uniformly for all \( u \):
\[ \alpha_{LR} \sim \exp \left\{ -\frac{1}{2} g(u, x, y) \right\}, \quad (y \geq 0 \text{ and } x \gg 1) \]
For \( x - y \gg 1 \) (e.g., when there is a finite difference between \( m \) and \( \mu \) while \( \beta \) is large), \( \alpha_{LR} \) is also uniformly suppressed:
\[ \alpha_{LR} \sim \exp \left\{ -\frac{1}{2} g(u, x, y) \right\}, \quad (x - y \gg 1) \]

20 Recall that our results are symmetric under the change \( \mu \rightarrow -\mu \) or equivalently \( y \rightarrow -y \).
When \( y \gg 1 \), the suppression of (3.49) is far stronger than the suppression of (3.50), so \( \alpha_{LR} \ll \alpha_{LR} \). Utilizing the diagonalized form of the relative covariance at \( t = 0 \) obtained in appendix C, an analytic expression for the generator (2.26) of the straight-line circuit for the charged TFD can be obtained. Taking \( \alpha_{LR} \to 0 \) and expanding to linear order in \( \alpha_{LR} \), we find, for a single mode of the charged TFD

\[
\mathcal{C}^1_{LR}(cTFD \text{ 1 mode}) \sim 2 \left[ 2 + \alpha_{LR} \frac{4 \text{max}\{\lambda^2, \lambda_{ref}^2\}}{\left| \lambda^2 - \lambda_{ref}^2 \right|} \right] \left| \log \frac{\lambda}{\lambda_{ref}} \right|, \quad (y \gg 1 \text{ and } x - y \gg 1)
\]

\[
\mathcal{C}^{LR_c}(cTFD \text{ 1 mode}) \sim \sqrt{2} \left\{ \left| \log \frac{\lambda}{\lambda_{ref}} \right| + \alpha_{LR} \left[ 1 + \frac{\lambda(1 + \lambda_{ref}^2)}{\left| \lambda^2 - \lambda_{ref}^2 \right|} \left| \log \frac{\lambda}{\lambda_{ref}} \right| \right] \right\}.
\]

The complexity of (two copies of) the vacuum is obtained in the \( \alpha_{LR}, \alpha_{LR} \to 0 \) limit and is proportional to what was found in [10]:

\[
\mathcal{C}^1_{LR}(\text{vac 1 mode}) = 2 \left| \log \frac{\lambda}{\lambda_{ref}} \right|, \quad \mathcal{C}^{LR_c}(\text{vac 1 mode}) = \sqrt{2} \left| \log \frac{\lambda}{\lambda_{ref}} \right|. \quad (3.51)
\]

In the \( LR \) basis, the extra factor of 2 on the RHS compared to [10]. This can be attributed to the fact that each complex field decomposes into two real fields. In the \( LR_c \) basis, this is reduced to a factor of \( \sqrt{2} \), for the reason described below (3.47); indeed, the straight line circuit for the vacuum is generated by two elementary gates, as opposed to the four used in the \( LR \) basis.\(^{21}\) By subtracting these vacuum contributions, we find the complexity of formation for each mode:\(^{22}\)

\[
\Delta\mathcal{C}^1_{LR}(cTFD \text{ 1 mode}) \sim \alpha_{LR} \frac{4 \text{max}\{\lambda^2, \lambda_{ref}^2\}}{\left| \lambda^2 - \lambda_{ref}^2 \right|} \left| \log \frac{\lambda}{\lambda_{ref}} \right|, \quad (y \gg 1 \text{ and } x - y \gg 1)
\]

\[
\Delta\mathcal{C}^{LR_c}(cTFD \text{ 1 mode}) \sim \sqrt{2} \alpha_{LR} \left[ 1 + \frac{\lambda(\lambda_{ref}^2 + 1)}{\left| \lambda^2 - \lambda_{ref}^2 \right|} \left| \log \frac{\lambda}{\lambda_{ref}} \right| \right],
\]

(3.52)

To obtain the total complexity of formation for the cTFD, we integrate over all modes:

\[
\Delta\mathcal{C}^1_{LR}(\text{cTFD}) = \frac{\text{vol}}{\beta^{d-1}} \cdot \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \int_0^\infty du \ u^{d-2} \Delta\mathcal{C}^1_{LR}(cTFD \text{ 1 mode}). \quad (3.53)
\]

Note that the convergence of the integral is guaranteed by the exponential suppression of \( \mathcal{C}(cTFD \text{ 1 mode}) \) by \( \alpha_{LR} \). To continue, we consider applying the expansion

\[
g(u, x, y) = x - y + \frac{u^2}{2x} + x \left( \frac{u^4}{x^7} \right) \quad (3.54)
\]

\(^{21}\)Specifically, in the \( LR_c \) basis, the straight-line generator \( \hat{K} \) for the vacuum is given by \( -\frac{1}{\sqrt{2}} \log \frac{\lambda}{\lambda_{ref}} \), times the sum of the third element in (B.5) written for \( L \) and \( R \). In the \( LR \) basis, the straight-line generator \( \hat{K} \) for the vacuum is \( -\frac{1}{1} \log \frac{\lambda}{\lambda_{ref}} \), times the sum of the fourth and sixth generators in (B.2) both written for \( L \) and for \( R \).

\(^{22}\)Note that the limit \( x - y \gg 1 \) does not necessarily imply \( x \gg y \), for example \( x = 2y \) would satisfy the first condition but not the second.
to (3.50) in the spirit of implementing a saddle-point approximation for (3.53). We find that, in the large $x$ limit, the suppression of $C_1(\text{cTFD})$ by $\alpha_{LR}$ implies that the integral receives dominant contributions only when $u$ is of order at most $\sqrt{x}$. Since

$$\lambda = \lambda_{\text{ref}} \tilde{\gamma} \sqrt{u^2 + x^2}$$  \hspace{1cm} (3.55)$$

is an approximate constant $\lambda \sim \lambda |_{u=0} = \lambda_{\text{ref}} \tilde{\gamma} x$ (with $\tilde{\gamma}$ given by (3.43)) in this region, we obtain

$$\Delta C_1(\text{cTFD}) \sim \frac{\text{vol}}{\beta^{d-1}} \cdot \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \left( \int_0^\infty du \ u^{d-2} \alpha_{LR} \right) \times \begin{cases} 4 \max \{(\tilde{\gamma} x)^2, 1/\tilde{\gamma} \} & \text{LR basis} \\ \sqrt{2} \left[ 1 + \frac{2x(\lambda_{\text{ref}} + \lambda_{\text{ref}}^{-1})/\log(\tilde{\gamma} x)}{\tilde{\gamma} x^2 - 1} \right] & \text{LR}_{\text{C}} \text{ basis} \end{cases} \hspace{1cm} (y \gg 1 \text{ and } x - y \gg 1)$$

We can evaluate the remaining integral using the approximation (3.50) with (3.54). The integral is simplified to a Gaussian moment:

$$\int_0^\infty du \ u^{d-2} \alpha_{LR} \approx \int_0^\infty du \ u^{d-2} e^{-\frac{1}{2}(x-y+\frac{u^2}{x})} = \frac{1}{2} \Gamma \left( \frac{d-1}{2} \right) (4x)^{(d-1)/2} e^{-(x-y)/2}. \hspace{1cm} (3.57)$$

We can apply a similar strategy to approximate the integral giving the entropy (3.44) of the charged TFD. Inserting (3.54), then integrating the logarithmic term by parts, we obtain

$$s(x, y) \sim \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \frac{1}{(d-1)x} \int_0^\infty du \ u^{d-2} e^{-\frac{1}{2}(x-y+\frac{u^2}{x})} = \frac{1}{2} \Gamma \left( \frac{d-1}{2} \right) (4x)^{(d-1)/2} e^{-(x-y)/2}. \hspace{1cm} (3.58)$$

where, in the case $x = y$, the integral was performed directly; and in the case $x - y \gg 1$, after selecting the leading contribution. Note that, in the $y \gg 1$ limit, $s(x, y) \gg s(x, -y)$ is the dominant term in the entropy (3.44); on the other hand, if $y = 0$, then the two terms of (3.44) are equal.

Taking the ratio between complexity of formation (3.56) and entropy (3.44), using the
approximations (3.57) and (3.58), we have

$$\frac{\Delta C_1(\text{cTFD})}{S_{\text{cTFD}}} \sim \frac{2^{(d-1)/2}e^{(x-y)/2}}{x-y} \left\{ \begin{array}{ll}
\frac{4 \max\{\gamma x,1\}\log(\gamma x)}{[(\gamma x)^2-1]} & LR \text{ basis} \\
\sqrt{2} \left[ 1 + \frac{\gamma x(\lambda_{\text{ref}}+\lambda_{\text{ref}}^{-1})\log(\gamma x)}{[(\gamma x)^2-1]} \right] & LR_C \text{ basis (} y \gg 1 \text{ and } x-y \gg 1 \right) \\
\end{array} \right. \quad (y \gg 1 \text{ and } x-y \gg 1)$$

$$= \frac{2^{(d-1)/2}e^{\beta(m-\mu)/2}}{\beta(m-\mu)} \left\{ \begin{array}{ll}
\frac{4 \max\{m^2, m_{\text{ref}}^2\}\log\frac{m}{m_{\text{ref}}}}{|m^2-m_{\text{ref}}^2|} & LR \text{ basis} \\
\sqrt{2} \left[ 1 + \frac{m_{\text{ref}}(\lambda_{\text{ref}}+\lambda_{\text{ref}}^{-1})\log\frac{m}{m_{\text{ref}}}}{|m^2-m_{\text{ref}}^2|} \right] & LR_C \text{ basis } \\
\end{array} \right. \quad (3.59)$$

where, in the last line, we have reverted to the physical quantities of the theory. We will often be selecting $\lambda_{\text{ref}} = m_{\text{ref}}/\mu_g = 1$, see comments below eq. (2.43). Note that in the low temperature limit both the entropy and the complexity are suppressed by exponential factors: $S_{\text{cTFD}} \sim \text{vol} \beta (m-\mu) \left( \frac{m}{\beta} \right)^{d-1} e^{-\beta(m-\mu)}$ and $\Delta C_1(\text{cTFD}) \sim \text{vol} \left( \frac{m}{\beta} \right)^{d-1} e^{-\beta(m-\mu)/2}$ as long as $\beta m - \beta \mu \gg 1$ and $\beta \mu \gg 1$, see eqs. (3.57)-(3.58). So we see that for $\beta m - \beta \mu \gg 1$ and $\beta \mu \gg 1$ we have that both the complexity and entropy are suppressed by exponential factors, but the complexity goes to zero slower than the thermal entropy. Of course as we go away from the large mass limit this conclusion may change, as we will see in the numerics below.

### 3.5.3 Numerical Results

In this section, we present numerical plots of the complexity of formation (3.34) for the complex scalar charged TFD. In all the plots below we have chosen $\lambda_{\text{ref}} = m_{\text{ref}}/\mu_g = 1$, see related comments below eq. (2.43). Of course, all our results will be invariant under the symmetry $\mu \rightarrow -\mu$ of the cTFD and will recover the neutral results (up to possible constants of proportionality) when $\mu = 0$.

We begin in figure 1, where we plot the case with vanishing chemical potential $\mu = 0$ in both the $LR$ (figure 1a) and $LR_C$ (figure 1b) bases. Recall that the $LR_C$ basis was the original basis of complex coordinates (3.22) while the $LR$ basis is the set of coordinates adapted to the particles and anti-particles degrees of freedom (3.23). Note that the vanishing of $\mu$ reduces the charged TFD to two uncharged TFDees. For this case, we see that the $LR$ and $LR_C$ bases give proportional results, with a relative factor of $\sqrt{2}$ — see explanation below (3.47). In [10], the same figure (figure 9 therein) was produced for the uncharged TFD state for a real scalar field, using a basis analogous to $LR_C$. The fact that figure 1b matches figure 9 of [10] provides a check of our numerics.

In figures 2 and 3, we consider the complexity of formation at $t = 0$ for $\mu \neq 0$. We also present results for the complexity normalized by the entropy (3.44). We have added to these figures the low temperature approximations presented in section 3.5.2. In this limit, the ratio (3.46) between complexity of formation and entropy diverges exponentially for $|\mu| < m$ and appears to approach a constant in the special case $|\mu|/m = 1$. We note that for all $|\mu|/m < 1$ the complexity becomes smaller in the low temperature limit, see
**Figure 1:** Complexity of formation, scaled by entropy, as a function of $\beta m$, for the complex scalar charged TFD in the special case $\mu = 0$. Various $d$ are shown. The results for the $LR$ and $LR_C$ bases appear proportional to each other and to figure 9 in [10].

Figures 2a, 2b, 3a and 3b, except for the case $|\mu| = m$ where it appears to increase. The latter case best resembles to what happens in holography where the complexity diverges in the low temperature limit, a phenomena known as the “third law of complexity”. In the scalar theory we see that this effect is not reproduced for $|\mu| < m$. This result can also be seen from the approximation described below eq. (3.59). In general, we observe that the un-normalized complexity of formation increases as $|\mu|/m$ increases for fixed values of $\beta m$ and $\beta m_{\text{ref}}$. Figure 4 explores the dependence of the complexity of formation on the reference scale $\bar{\gamma} = (\beta m_{\text{ref}})^{-1}$ given in (3.43) when $\beta \mu \neq 0$. This dependence is weaker in the $LR_C$ basis than in the $LR$ basis. We observe an approximate symmetry $m_{\text{ref}} \rightarrow 1/m_{\text{ref}}$ in these figures.

### 3.6 Time Dependence

Next, we consider the time dependence of complexity for the complex scalar charged TFD. To calculate complexity at arbitrary times, we evaluate eqs. (2.28)-(2.25) in integral form (3.32), using (2.27) with the relative covariance matrices (3.28) and (3.30) to compute the complexity for single modes. At general times, it is cumbersome to write analytic expressions for these relative covariance matrices, so we immediately resort to numerics. We plot complexity against time in the $LR$ basis in figures 5-7 and in the $LR_C$ basis in figures 8-10.

In the $LR$ basis, taking $\mu = 0$ (figure 5) does not recover the results given for the uncharged TFD of a real scalar in [10]. In particular, note that in figure 5, the late time complexity becomes arbitrarily large as $\bar{\gamma} = \frac{1}{\beta m_{\text{ref}}}$ is taken to be very small or very large. This discrepancy with [10] is due to the fact that the ‘$LR$’ basis used for the real scalar there is more analogous to the $LR_C$ basis here. Indeed, we have verified that taking $\mu = 0$ in the $LR_C$ basis here recovers $\sqrt{2}$ times the complexity of one uncharged TFD, as given in the ‘$LR$’ basis of [10]. In figure 8, we see that taking extreme values of $\bar{\gamma} = \frac{1}{\beta m_{\text{ref}}}$ gives a finite limit for complexity at all times.\footnote{Recall that $|\mu|/m = 1$ is a rather singular case where a condensate would form.}
Figure 2: Complexity of formation in the LR basis, scaled by entropy, for the complex scalar charged TFD in the case $d = 4$. The curves are plotted as functions of $\beta m$ and $\mu/m$ (recall we are constrained to have $|\mu|/m < 1$, see comments below eqs. (3.20)-(3.21)) for various fixed values of $\mu/m$ and $\beta m$ respectively. The dashed curves in figure 2a mark the low temperature limits given in §3.5.2. The neutral limit is obtained on the left hand side of subfigures (a) and (b) since keeping $\mu/m$ fixed and sending $m \to 0$ means we are also decreasing the chemical potential. We see that in this case the dependence on the chemical potential disappears and all the curves approach the same point (alternatively, this can be seen as a large temperature limit, and therefore the chemical potential becomes negligible).

In figures 6-7 and 9-10, we plot the time dependence of complexity in the LR and LR$_C$ bases respectively for $\mu \neq 0$ and various values of $\beta m > 0$. We see that the complexity develops oscillations with a frequency proportional to $m$. This is naively to be expected since,
Figure 3: Complexity of formation in the $LR_C$ basis, scaled by entropy, for the complex scalar charged TFD in the case $d = 4$. The curves are plotted as functions of $\beta m$ and $\mu/m$ for various fixed values of $\mu/m$ and $\beta m$ respectively. The dashed curves in figures 3a and 3b mark the low temperature limit given by (3.48) and (3.59). Note that although the former figure fixes $\beta m_{\text{ref}}$ while and the latter instead fixes $m/m_{\text{ref}}$, the two figures are nearly identical — this is because, as shown in figure 4b, the dependence on $\tilde{\gamma}$ in the $LR_C$ basis is very weak.

In order for the integral over single-mode complexities to be convergent, the contribution of high-frequency modes must necessarily be suppressed. Hence, we expect the oscillations of the total complexity to result from modes of low frequency, which is bounded from below by $\omega_{k=0} = m$.

In general, we note that in the $LR$ basis the complexity plotted in figures 5-7 always
initially increases and peaks at a global maximum, never drops below its initial value, and always saturates to a value fairly close to its global maximum. In the \( LR_C \) basis, on the other hand, complexity of formation does not typically stay above its initial value for all times and indeed sometimes saturates below its initial value, as shown in figures 8-10, in contrast to holographic complexity. Similarly to what was found for the uncharged TFD in [10], we observe that the time dependence of complexity deviates significantly compared to the results in the holographic systems in that it saturates after times of the order of the inverse temperature. This is perhaps not surprising, since the free systems we consider are not chaotic. In figure 7 and 10, we observe that in both bases the time dependence of the (un-normalized) complexity decreases as the temperature is decreased, keeping all the other parameters fixed for all \( |\mu| < m \).\(^{24}\) This effect is similar to the one observed in holography where the rate of computation comes to a halt as the temperature decreases. Further we observe that the fluctuations in complexity as a function of time increase as the chemical potential increases.

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\(^{24}\)We expect that this will be the case for values of \( \beta m \) larger than the value for which the complexity of formation in figure (3a)-(3b) starts decreasing.
Figure 5: Time dependence of complexity in the \( LR \) basis, scaled by entropy, for the complex scalar charged TFD for \( d = 4, \beta m = \beta \mu = 0 \). The curves for various fixed \( \bar{\gamma} = (\beta m_{\text{ref}})^{-1} \) are plotted as functions of \( t/\beta \).

(a) Fixed \( d = 4, \beta m = 1, \bar{\gamma} = 1, \) and various \( \mu/m \).

(b) Fixed \( d = 4, \beta m = 2, \bar{\gamma} = 1, \) and various \( \mu/m \).

(c) Fixed \( d = 4, \beta m = 4, \beta m_{\text{ref}} = 1, \) and various \( \mu/m \).

(d) Fixed \( d = 4, \beta m = 8, \beta m_{\text{ref}} = 1, \) and various \( \mu/m \).

Figure 6: Time dependence of complexity in the \( LR \) basis, scaled by entropy, for the complex scalar charged TFD for \( d = 4 \). The subfigures correspond to different \( \beta m \) values and curves for different fixed \( \mu/m \) are plotted as functions of \( t/\beta \).

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Figure 7: Oscillatory time dependence of complexity for the complex scalar charged TFD for \( d = 4 \) in the \( LR \) basis for large values of \( \beta m \) and even larger values of \( \beta m_{\text{ref}} \). The subfigures 7a and 7b correspond to different \( \beta m = 16, 32 \), respectively, and curves for different fixed \( \mu/m \) are plotted as functions of \( t/\beta \). Note that the vertical axes and values of \( \mu/m \) here differ from those in figure 6.

Figure 8: Time dependence of complexity in the \( LR_C \) basis, scaled by entropy, for the complex scalar charged TFD for \( d = 4 \) with \( \beta m = \beta \mu = 0 \). The curves for various fixed \( \bar{\gamma} = (\beta m_{\text{ref}})^{-1} \) are plotted as functions of \( t/\beta \).

A Time Evolution of the Charged TFD State

The time evolution in eq. (1.2) might at first sight look strange due to the inclusion of the chemical potential. However this is easily understood by coupling the uncharged TFD to...
an external $U(1)$ gauge field capturing the effect of the chemical potential

$$A_\mu dx^\mu = \mu dt.$$  

(A.1)

As a simple example, let us consider the effect of coupling the $U(1)$ gauge field (A.1) to a free complex scalar field

$$\mathcal{L} = -(D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi \phi^\dagger \phi,$$

$$D_\mu = \partial_\mu - i A_\mu,$$

(A.2)

where we have set the elementary charge to one and the metric is in the mostly plus convention. Expanding out, we have

$$\mathcal{L} = - \partial_\mu \phi \dot{\phi} + i \mu \phi \dot{\phi} - i \mu \dot{\phi} \phi - (m^2 - \mu^2) \phi \phi^\dagger \phi,$$

(A.3)

giving the conjugate momenta

$$\pi_\phi = \partial \mathcal{L} \partial \phi = \dot{\phi} - i \mu \phi,$$

$$\pi = \partial \mathcal{L} \partial \phi^\dagger = \dot{\phi}^\dagger + i \mu \phi^\dagger$$

(A.4)

and the electric charge density

$$\mathcal{E} = i (\phi^\dagger \pi_\phi - \pi \phi).$$

(A.5)
We thus find that the effect of introducing the coupling to the $U(1)$ gauge field is to deform the Hamiltonian density by $-\mu C$:}

\[
\mathcal{H} = \dot{\phi}^\dagger \phi + \vec{\nabla} \phi^\dagger \cdot \vec{\nabla} \phi + (m^2 - \mu^2) \phi^\dagger \phi
\]

\[
= \pi^\dagger \pi + \vec{\nabla} \phi^\dagger \cdot \vec{\nabla} \phi - im \phi^\dagger \pi + i \mu \phi^\dagger \phi + m^2 \phi^\dagger \phi
\]

\[
= \pi^\dagger \pi + \vec{\nabla} \phi^\dagger \cdot \vec{\nabla} \phi + m^2 \phi^\dagger \phi - \mu C.
\]

This then explain the sign evolution in eq. (1.2), up to a relative sign which can be attributed to the different conventions for the chemical potential.

\section{B Basis for $\mathfrak{sp}(4,\mathbb{R})$ for Real and Complex Phase Spaces}

Supplementing the discussion in section 2.1, we consider here the example of $\text{Sp}(4,\mathbb{R})$ and its generalization for complex phase space variables, listing orthonormal bases for the corresponding algebra of generators.

Let us begin with real phase space variables

\[\xi = (q, \bar{q}, p, \bar{p}).\]
From this, we may construct unitary operators (2.5) by exponentiating generators which are Hermitian quadratic combinations $\hat{K}$ of these phase space variables, as written in (2.6). One basis for such Hermitian operators is

\[
\frac{q^2}{\sqrt{2}}, \quad q\bar{q}, \quad \frac{qp + pq}{2}, \quad \frac{q^2}{\sqrt{2}}, \quad \frac{q\bar{q} + \bar{q}p}{2}, \quad \frac{p^2}{\sqrt{2}}, \quad \bar{p}p, \quad \frac{p\bar{p}}{\sqrt{2}}. \tag{B.2}
\]

We can equivalently express any generator $\hat{K}$ of the operator representation using a generator $\hat{K}$ in the matrix representation, as given in (2.13). The corresponding matrix representations of the above generators are:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}. \tag{B.3}
\]

These matrix generators are orthonormal with respect to the inner product (2.18).

Next, let us consider an example with complex coordinates,

\[
\tilde{\xi} = (\tilde{q}, \tilde{q}^\dagger, \tilde{p}^\dagger, \tilde{p}). \tag{B.4}
\]

Analogous to (B.2), we may construct a basis of Hermitian quadratic operators $\hat{K}$:

\[
\frac{\tilde{q}\tilde{q}^\dagger}{2}, \quad \frac{\tilde{q}^2 + (\tilde{q}^\dagger)^2}{2\sqrt{2}}, \quad \frac{\tilde{q}\tilde{p} + \tilde{p}\tilde{q} + \tilde{q}^\dagger\tilde{p}^\dagger + \tilde{p}^\dagger\tilde{q}^\dagger}{2\sqrt{2}}, \quad \frac{\sqrt{2}(\tilde{q}\tilde{p}^\dagger + \tilde{q}^\dagger\tilde{p})}{2}, \quad \frac{\tilde{p}\tilde{p}^\dagger + (\tilde{p}^\dagger)^2}{2\sqrt{2}}. \tag{B.5}
\]

The corresponding matrix generators $\tilde{K}$ respectively are:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{bmatrix}, \quad
\begin{bmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}. \tag{B.6}
\]

Again, these are orthonormal with respect to the inner product (2.18).
C Diagonalization of the Relative Covariance Matrix at $t = 0$

Below we present an additional coordinate transformation, used to derive the low temperature limits in section 3.5.2. The reference and target covariance matrices can be diagonalized by a transformation $R_{\pm \to \Delta_{cTFD,n}} \in \text{Sp}(4, \mathbb{R})$ to a basis which we shall call the $\Delta_{cTFD,n}$ basis

$$G_{\text{ref},n}^{\Delta_{cTFD,n}} = R_{\pm \to \Delta_{cTFD,n}} G_{\text{ref},n}^{\pm}{(R_{\pm \to \Delta_{cTFD,n}})^\dagger} = \frac{1}{4\lambda_n \lambda_{\text{ref}}} \text{diag} \left( (a^+_n + d^-_n)e^{\mp 2(\alpha_{LR,n} + \alpha_{LR,n})}, (a^-_n - d^+_n)e^{\mp 2(\alpha_{LR,n} + \alpha_{LR,n})}, a^-_n - d^+_n, a^+_n + d^-_n \right),$$  \hspace{1cm} (C.1)

$$G_{\text{cTFD},n}^{\Delta_{cTFD,n}} = I.$$  \hspace{1cm} (C.2)

where

$$R_{\pm \to \Delta_{cTFD,n}} = \text{diag}(R^{(2)}_{\pm \to \Delta_{cTFD,n}}, R^{(2)}_{\pm \to \Delta_{cTFD,n}})(G_{\text{cTFD},n}^{\pm})^{-1/2}$$

$$R^{(2)}_{\pm \to \Delta_{cTFD,n}} = \begin{bmatrix} \cos \theta^\pm_n - \sin \theta^\pm_n & \sin \theta^\pm_n \\ \sin \theta^\pm_n & \cos \theta^\pm_n \end{bmatrix} \in \text{SO}(2),$$  \hspace{1cm} (C.3)

and

$$a^+_n = (e^{\pm 2\alpha_{LR,n}} + e^{\pm 2\alpha_{LR,n}})(\lambda^2_n + \lambda^2_{\text{ref}}), \quad b^+_n = (e^{\pm 2\alpha_{LR,n}} - e^{\pm 2\alpha_{LR,n}})(\lambda^2_n + \lambda^2_{\text{ref}}),$$

$$c^+_n = 2(\lambda^2_n - \lambda^2_{\text{ref}})e^{\pm(\alpha_{LR,n} + \alpha_{LR,n})}, \quad d^+_n = \sqrt{(b^+_n)^2 + (c^+_n)^2}$$  \hspace{1cm} (C.4)

$$a^-_n = (e^{\pm 2\alpha_{LR,n}} + e^{\pm 2\alpha_{LR,n}})(\lambda^2_n + \lambda^2_{\text{ref}}), \quad b^-_n = (e^{\pm 2\alpha_{LR,n}} - e^{\pm 2\alpha_{LR,n}})(\lambda^2_n + \lambda^2_{\text{ref}}),$$

$$c^-_n = 2(\lambda^2_n - \lambda^2_{\text{ref}})e^{\pm(\alpha_{LR,n} + \alpha_{LR,n})}, \quad d^-_n = \sqrt{(b^-_n)^2 + (c^-_n)^2}$$  \hspace{1cm} (C.5)

$$\theta^\pm_n = \tan^{-1}\left( \frac{c^+_n}{b^+_n - d^+_n} \right).$$  \hspace{1cm} (C.6)

Due to (C.2), in this basis, we in fact have the diagonal relative covariance matrix

$$\Delta_{\text{cTFD},n}^{\Delta_{cTFD,n}} = (G_{\text{ref},n}^{\Delta_{cTFD,n}})^{-1}.$$  \hspace{1cm} (C.7)

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