Casimir densities for a boundary in Robertson-Walker spacetime

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Abstract

For scalar and electromagnetic fields we evaluate the vacuum expectation value of the energy-momentum tensor induced by a curved boundary in the Robertson–Walker spacetime with negative spatial curvature. In order to generate the vacuum densities we use the conformal relation between the Robertson–Walker and Rindler spacetimes and the corresponding results for a plate moving by uniform proper acceleration through the Fulling–Rindler vacuum. For the general case of the scale factor the vacuum energy-momentum tensor is presented as the sum of the boundary free and boundary induced parts.

1 Introduction

The influence of boundaries on the vacuum state of a quantum field leads to interesting physical consequences. Well known example is the Casimir effect \([1, 2, 3, 4]\), when the modification of the zero-point fluctuations spectrum by the presence of boundaries induces vacuum forces acting on the boundaries. It may have important implications on all scales, from cosmological to subnuclear. The particular features of the resulting vacuum forces depend on the nature of the quantum field, the type of spacetime manifold, the boundary geometries and the specific boundary conditions imposed on the field.

The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. Another type of vacuum polarization arises in the case of an external gravitational field. In this paper, we study an exactly solvable problem with both types of sources for the polarization. Namely, we consider the vacuum expectation value of the energy-momentum tensor for both scalar and electromagnetic fields induced by a curved boundary in background of Robertson-Walker (RW) spacetime with negative spatial curvature. In order to generate the vacuum densities we use the well known relation between the vacuum expectation values in conformally related problems (see, for instance, \([5]\)) and the corresponding results for an infinite plane boundary moving with uniform acceleration through the Fulling–Rindler vacuum.

The latter problem for conformally coupled Dirichlet and Neumann massless scalar fields and for the electromagnetic field in four dimensional Rindler spacetime was considered by Candelas and Deutsch \([6]\). These authors consider the region of the right Rindler wedge to the right of

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the barrier. In [7], we have investigated the Wightman function and the vacuum expectation values of the energy–momentum tensor for a massive scalar field with general curvature coupling parameter, satisfying the Robin boundary condition on the infinite plane in an arbitrary number of spacetime dimensions and for the electromagnetic field. We have considered both regions, including the one between the barrier and Rindler horizon. The corresponding surface densities induced on the plate have been considered in [8]. The vacuum expectation values of the energy–momentum tensors for scalar and electromagnetic fields for the geometry of two parallel plates moving by uniform acceleration are investigated in [9]. In particular, the vacuum forces acting on the boundaries are evaluated. In [10] the Casimir energy is evaluated for massless scalar fields under Dirichlet or Neumann boundary conditions, and for the electromagnetic field with perfect conductor boundary conditions on one and two infinite parallel plates moving by uniform proper acceleration through the Fulling–Rindler vacuum in an arbitrary number of spacetime dimension.

A closely related problem for the evaluation of the energy-momentum tensor of a Casimir apparatus in a weak gravitational field recently has been considered in [11, 12]. In particular, it has been shown that the Casimir energy for a configuration of parallel plates gravitates according to the equivalence principle. In [13] the conformal relation between de Sitter and Rindler spacetimes is used to generate the vacuum expectation values of the energy–momentum tensor for a conformally coupled scalar field in de Sitter spacetime in the presence of a curved brane on which the field obeys the Robin boundary condition with coordinate dependent coefficients. The Casimir densities for spherical branes in Rindler-like spacetimes have been investigated in [14].

The organization of the present paper is as follows. In the next section we consider the conformal relation between the problems in the RW spacetime with negative spatial curvature and Rindler spacetime. The geometry of the boundary is specified. In section 3 the vacuum expectation value of the energy-momentum tensor is investigated for a scalar field with Robin boundary condition. The Casimir densities in the case of the electromagnetic field with perfect conductor boundary conditions on the plate are discussed in section 4. The main results are summarized in section 5.

## 2 Conformal relation between the problems in Robertson-Walker and Rindler spacetimes

As a background geometry we shall consider the $k = -1$ RW spacetime with the line element

$$ds^2 = g_{ik}dx^i dx^k = a^2(\eta)(d\eta^2 - \gamma^2 dr^2 - r^2 d\Omega_{D-1}^2),$$

where $\gamma = 1/\sqrt{1 + r^2}$ and $d\Omega_{D-1}^2$ is the line element on the $(D-1)$-dimensional unit sphere in Euclidean space. First of all let us present the RW line element in the form conformally related to the Rindler metric. With this aim we make the coordinate transformation

$$x = (\eta, r, \theta, \ldots, \theta_{D-2}, \phi) \rightarrow x' = (\eta, \xi, x'),$$

with $x' = (x'^2, \ldots, x'^D)$, defined by the relations (see Ref. [5] for the case $D = 3$)

$$\xi = \xi_0 \Omega, \quad x'^2 = \xi_0 r \Omega \sin \theta \cos \theta_2, \ldots, \quad x'^{D-2} = \xi_0 r \Omega \sin \theta \sin \theta_2 \cdots \sin \theta_{D-3} \cos \theta_{D-2},$$

$$x'^{D-1} = \xi_0 r \Omega \sin \theta \sin \theta_2 \cdots \sin \theta_{D-2} \cos \phi, \quad x'^D = \xi_0 r \Omega \sin \theta \sin \theta_2 \cdots \sin \theta_{D-2} \sin \phi.$$
where \( \xi_0 \) is a constant with the dimension of length and we use the notation

\[
\Omega = \gamma/(1 - r \gamma \cos \theta). \tag{4}
\]

Under this transformation the RW line element takes the form

\[
ds^2 = g_{ik}' dx^i dx^k = a^2(\eta)\xi^{-2} \left( \xi^2 d\eta^2 - d\xi^2 - dx'^2 \right). \tag{5}
\]

In this form the RW metric is manifestly conformally related to the metric in the Rindler spacetime with the line element

\[
ds^2_{\text{R}} = a^2(\eta)\xi^{-2} ds^2_{\text{R}}, \quad g_{ik}' = a^2(\eta)\xi^{-2} g_{ik}^{\text{R}}. \tag{6}
\]

By using the standard transformation formula for the vacuum expectation values of the energy–momentum tensor in conformally related problems (see, for instance, [5]), we can generate the results for the RW spacetime from the corresponding results in the Rindler spacetime. First we shall consider the corresponding quantities in the coordinates \((\eta, \xi, x')\) with the line element (5). These quantities are found by using the transformation formula for conformally related problems:

\[
\langle 0_{\text{RW}} | T^k_i \left[ g_{lm}' \right] \varphi \rangle_{\text{RW}} = \left[ \xi/a(\eta) \right]^{D+1} \langle 0_{\text{R}} | T^k_i \left[ g_{lm}^{\text{R}} \right] \varphi \rangle_{\text{R}} + \langle T^k_i \left[ g_{lm}' \right] \varphi \rangle_{\text{an}}, \tag{7}
\]

where the second term on the right is determined by the trace anomaly. In odd spacetime dimensions the conformal anomaly is absent and the corresponding part vanishes: \( \langle T^k_i \left[ g_{lm}' \right] \varphi \rangle_{\text{an}} = 0 \) for even \( D \). The vacuum expectation value of the energy-momentum tensor in coordinates \((\eta, \xi, x')\) is obtained by the standard coordinate transformation formulae. For a second rank tensor \( A_{ik} \), which is diagonal in coordinates \( x'^i = (\eta, \xi, x') \), the transformation to coordinates \( x^i = (\eta, r, \theta, \theta_2, \ldots, \theta_D) \) has the form

\[
A_0^0 = A_0^0, \quad A_1^1 = A_1^1 + \Omega^2 \sin^2 \theta (A_2^2 - A_1^1),
\]

\[
A_1^2 = \Omega^2 \sin \theta \cos \theta \frac{\gamma - r \gamma}{r} (A_2^2 - A_1^1),
\]

\[
A_2^2 = A_2^2 + \Omega^2 \sin^2 \theta (A_1^1 - A_2^2), \quad A_l^i = A_2^i, \quad l = 3, \ldots, D. \tag{8}
\]

In this paper, as a Rindler counterpart we shall take the vacuum energy–momentum tensor induced by an infinite plate moving by uniform proper acceleration through the Fulling–Rindler vacuum. We shall assume that the plate is located in the right Rindler wedge and has the coordinate \( \xi = b \). In coordinates \( x^i \) the boundary \( \xi = b \) is presented by the hypersurface

\[
\sqrt{1 + r^2 - r \cos \theta} = 1/b_0, \quad b_0 = b/\xi_0. \tag{9}
\]

The corresponding normal has the components

\[
n^i = \frac{b_0}{ra(\eta)} (0, \sqrt{1 + r^2}(1 - \sqrt{1 + r^2/b_0}), -\sin \theta, 0, \ldots, 0). \tag{10}
\]

We consider the cases of scalar and electromagnetic fields separately.
3 Vacuum expectation values for the energy-momentum tensor: Scalar field

In this section we consider a conformally coupled massless scalar field $\varphi(x)$ on background of spacetime with the line element $[1]$. The corresponding field equation has the form

$$\left(\nabla_l \nabla^l + \frac{D-1}{4D} R\right) \varphi(x) = 0, \quad (11)$$

where $R$ is the Ricci scalar for the RW spacetime. We assume that the field satisfies the Robin boundary condition

$$(A + B n^l \nabla_l) \varphi(x) = 0, \quad (12)$$
on the hypersurface $[9]$. The expectation value of the energy-momentum tensor induced by the presence of an infinite plane boundary moving with uniform acceleration through the Fulling–Rindler vacuum was investigated in $[6,7]$. For a scalar field $\varphi_R(x')$ it is presented in the decomposed form:

$$\langle 0_R | T^R_l [g_{lm}, \varphi_R] | 0_R \rangle = \langle 0 | T^R_l [g_{lm}, \varphi] | 0_R \rangle + \langle 0 | T^R_{(R)l} \rangle. \quad (13)$$

In this formula, $|0_R\rangle$ are $|\tilde{0}_R\rangle$ are the vacuum states for the Rindler spacetime in presence and absence of the plate respectively and $\langle T^R_{(R)l}\rangle$ is the part of the vacuum energy-momentum tensor induced by the plate. For the part without boundaries one has

$$\langle \tilde{0}_R | T^R_l [g_{lm}, \varphi_R] | \tilde{0}_R \rangle = \frac{a_D \xi^{-D-1}}{2^{D-1} \pi^{D/2} \Gamma(D/2)} \text{diag} (-1, 1/D, \ldots, 1/D), \quad (14)$$

with the notation

$$a_D = \int_0^\infty \frac{\omega^D d\omega}{e^{2\pi \omega} + (-1)^D \prod_{l=1}^{l_m} \left(\frac{D - 1 - 2l}{2\omega}\right)^2 + 1}, \quad (15)$$

where $l_m = D/2 - 1$ for even $D > 2$ and $l_m = (D - 1)/2$ for odd $D > 1$, and the value for the product over $l$ is equal to 1 for $D = 1, 2, 3$.

For a scalar field $\varphi_R(x')$ satisfying the Robin boundary condition

$$\left(A_R + B_R n_R^l \nabla_l \right) \varphi_R(x') = 0, \quad \xi = b, \quad n_R^l = \delta_1^l, \quad (16)$$

with constant coefficients $A_R$ and $B_R$, the boundary induced part in the region $\xi > a$ is given by the formula $[7]$

$$\langle T^R_{(R)l} \rangle = \frac{-2^{1-D} \delta^k b^{-D-1}}{\pi^{(D+1)/2} \Gamma(D/2) a_D} \int_0^\infty dx \int_0^\infty \frac{\tilde{I}_\omega(x)}{\tilde{K}_\omega(x)} F^{(i)}[K_\omega(x\xi/b)]. \quad (17)$$

Here the functions $F^{(i)}[g(z)]$ have the form

$$
\begin{align*}
F^{(0)}[g(z)] &= g^2(z) + \frac{D-1}{z} g(z) g'(z) + \left[1 - (2D - 1) \frac{\omega^2}{z^2}\right] g^2(z), \\
F^{(1)}[g(z)] &= -D g^2(z) - \frac{D-1}{z} g(z) g'(z) + D \left(1 + \frac{\omega^2}{z^2}\right) g^2(z), \\
F^{(i)}[g(z)] &= g^2(z) + \left(\frac{\omega^2}{z^2} - \frac{D+1}{D-1}\right) g^2(z), \quad i = 2, \ldots, D.
\end{align*}
$$
In Eq. (17), $I_\omega(z)$ and $K_\omega(z)$ are the modified Bessel functions and for a given function $f(z)$ we use the notation

$$\tilde{f}(z) = A_R f(z) + (\bar{B}_R/b)zf'(z).$$

(19)

The expression for the boundary part of the vacuum energy-momentum tensor in the region $\xi < a$ is obtained from formula (17) by the replacements $I_\omega \rightarrow K_\omega$.

The formulae given above allow us to present the RW vacuum expectation value in coordinates $x^i$ in the form similar to (13):

$$\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} = \langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} + \langle T^k_i | g_{lm}^{'}, \varphi \rangle_{(b)},$$

(20)

where $\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}}$ is the vacuum expectation value in the RW spacetime without boundaries and the part $\langle T^k_i | g_{lm}^{'}, \varphi \rangle_{(b)}$ is induced by the boundary (9). Conformally transforming the Rindler results one finds

$$\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} = [\xi/a(\eta)]^{D+1}\langle 0_{\text{R}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{R}} + \langle T^k_i | g_{lm}^{'}, \varphi \rangle_{(an)},$$

(21)

$$\langle T^k_i | g_{lm}^{'}, \varphi \rangle_{(b)} = [\xi/a(\eta)]^{D+1}\langle T^k_i | (\text{R}) \rangle_{(b)}.$$  

(22)

Under the conformal transformation $g_{ik}^' = [a(\eta)/\xi^2]g_{ik}^R$, the field $\varphi_R$ is changed by the rule

$$\varphi(x') = [\xi/a(\eta)]^{(D-1)/2}\varphi_R(x').$$

(23)

Now by comparing boundary conditions (12), (13) and taking into account Eq. (23), one obtains the relation between the coefficients in the boundary conditions:

$$\frac{a(\eta)A/B = bA_R/B_R + (1 - D)/2.}$$

(24)

As it is seen from this relation, the Dirichlet boundary condition in the problem on the RW bulk ($B = 0$) corresponds to the Dirichlet boundary condition in the conformally related problem for the Rindler spacetime. For the case of the Neumann boundary condition in the RW bulk ($A = 0$) the corresponding problem in the Rindler spacetime is of the Robin type with $bA_R/B_R = (D - 1)/2$.

As before, we shall present the corresponding components in coordinates $x^i$ in the form of the sum of purely RW and boundary parts:

$$\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} = \langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} + \langle T^k_i | (b) \rangle.$$  

(25)

By using the relations (5) for the purely RW part one finds (for the vacuum polarization in RW spacetimes see [5] [15] [16] and references therein)

$$\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} = \frac{2a_D[a(\eta)]^{-D-1}}{(4\pi)^{D/2}\Gamma(D/2)}\text{diag}(-1, 1/D, \ldots, 1/D) + \langle T^k_i | (an) \rangle.$$  

(26)

In particular, for $D = 3$ we have [5]

$$\langle T^k_i | (an) \rangle = \frac{(3)H^k_{i} - (1)H^k_{i}/6}{2880\pi^2},$$

(27)

where the expressions for the tensors $(j)H^k_{i}$ are given in [5]. Now it can be easily checked that for the static case, $a(\eta) = \text{const}$, one has $\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} = 0$. In the special case of the power-law expansion, $a(t) = \alpha t^c$, with $t$ being the synchronous time coordinate, we find

$$\langle 0_{\text{RW}} | T^k_i | g_{lm}^{'}, \varphi \rangle_{\text{RW}} = \frac{c(c^2 - 6c + 3)(3c - 4)}{2880\pi^2 t^4}\text{diag}(\frac{3c}{3c - 4}, 1, 1, 1).$$

(28)
The corresponding energy density is negative for $|c - 3| < \sqrt{6}$.

For the boundary induced energy-momentum tensor in coordinates $x'^i$ the spatial part is not isotropic and the corresponding part in coordinates $x^i$ is more complicated (no summation over $l$):

$$\langle T^l_l \rangle^{(b)} = \left[\frac{\xi}{a(\eta)}\right]^{D+1} \langle T^l_l \rangle^{(R)}^{(b)}, l = 0, 3, \ldots, D,$$

$$\langle T^l_l \rangle^{(b)} = \left[\frac{\xi}{a(\eta)}\right]^{D+1} \left[\langle T^l_l \rangle^{(R)}^{(b)} + (-1)^l \Omega^2 \sin^2 \theta (\langle T^l_1 \rangle^{(R)}^{(b)} - \langle T^l_2 \rangle^{(R)}^{(b)})\right], l = 1, 2, (29)$$

$$\langle T^{21}_1 \rangle^{(b)} = \left[\frac{\xi}{a(\eta)}\right]^{D+1} \Omega^2 \sin \theta \frac{\cos \theta - r \gamma}{r} (\langle T^2_2 \rangle^{(R)}^{(b)} - \langle T^1_2 \rangle^{(R)}^{(b)}).$$

As we see the resulting energy-momentum tensor is non-diagonal. Note that for the case of the power-law expansion the ratio of the boundary induced and boundary free parts at a given spatial point behaves as $t^{4(1-c)}$ in the model with $D = 3$. Hence, at early stages of the cosmological expansion the boundary induced part dominates for $c > 1$. In figure\[\text{I}\] we have plotted the boundary induced parts in the vacuum energy density ($l = 0$) and $3^3$-stress ($l = 3$) as functions of the ratio $\xi/b$ for $D = 3$ scalar field with Dirichlet boundary condition. The corresponding energy density is positive in the region $\xi < b$ and negative for $\xi > b$. In the case of Robin condition the energy density can be either negative or positive in dependence of the coefficient in the boundary condition.

![Figure 1: Boundary induced parts in the vacuum energy density and $3^3$-stress for $D = 3$ Dirichlet scalar field.](image)

4 Electromagnetic field

The electromagnetic field is conformally invariant in $D = 3$. The vacuum expectation value of the energy-momentum tensor induced by the presence of conducting plate moving with uniform acceleration through the Fulling–Rindler vacuum is investigated in Refs. [6, 7]. We will assume that the plate is a perfect conductor with the standard boundary conditions of vanishing of the normal component of the magnetic field and the tangential components of the electric field, evaluated at the local inertial frame in which the conductor is instantaneously at rest. As in the case of a scalar field, the expectation value of the energy-momentum tensor is presented in the
form (13), where the boundary free part is given by the formula

\[
\langle 0_R | T^R_i [g_{lm}, \varphi_R] | 0_R \rangle = \frac{11}{240\pi^2 \xi^4} \text{diag} \left(-1, 1/3, 1/3, 1/3\right). 
\] (30)

In the region \( \xi > b \) for the boundary induced part one has the expression [6, 7]

\[
\langle T^{(b)}_{(R)} \rangle \approx -\frac{\sigma_b}{4\pi^2 b^4} \int_0^\infty dx \int_0^\infty d\omega \left[ \frac{L_\omega (x)}{K_\omega (x)} + \frac{P_\omega (x)}{K_\omega (x)} \right] F^{(i)}_{\text{em}} [K_\omega (x \xi/b)], 
\] (31)

with the notations

\[
F^{(i)}_{\text{em}} [g(z)] = (-1)^i g^2 (z) + [1 - (-1)^i \omega^2 / z^2] g^2 (z), \quad i = 0, 1,
\]
\[
F^{(2)}_{\text{em}} [g(z)] = F^{(3)}_{\text{em}} [g(z)] = -g^2 (z).
\] (32)

The corresponding formula in the region \( \xi < b \) is obtained from (31) by the replacements \( L_\omega \rightarrow K_\omega \). By taking into account that \( [L_\omega (x) K_\omega (x)]' < 0 \), we see that \( \langle T^{(b)}_{(R)2} \rangle > 0 \) for \( \xi > b \)

\[\text{and}\] \( \langle T^{(b)}_{(R)2} \rangle < 0 \) for \( \xi < b \). For the perpendicular stress one has \( \langle T^{(b)}_{(R)1} \rangle > 0 \) in both regions.

The vacuum expectation value of the energy-momentum tensor in the RW bulk is presented in the form (25), where the boundary free part is given by the expression

\[
\langle 0_R W | T^R_i [g_{lm}, \varphi] | 0_R W \rangle = \frac{11a^{-4}(\eta)}{240\pi^2} \text{diag} \left(-1, 1/3, 1/3, 1/3\right) + \frac{62}{2880\pi^2} H^k_i + \frac{6}{2880\pi^2} H^k_i.
\] (33)

As in the case for a scalar field, this expectation value vanishes for the static RW spacetime. For the power-law expansion, \( a(t) = \alpha t^c \), from (33) we find

\[
\langle 0_R W | T^R_i [g_{lm}, \varphi] | 0_R W \rangle = \frac{c (31c^2 + 54c - 27)(3c - 4)}{1440\pi^2 t^4} \text{diag}(\frac{3c}{3c - 4}, 1, 1, 1)
\]

\[\quad \quad \quad \quad - \frac{c(c - 2)}{18\pi^2 a^2(t) t^2} \text{diag}(\frac{3c}{c - 2}, 1, 1, 1).
\] (34)

For \( c < 1 \) the second term on the right of this formula dominates at late times of the cosmological expansion and the corresponding energy density is negative.

For the boundary induced part we have formulae (29) with \( D = 3 \) and with \( \langle T^{(b)}_{(R)1} \rangle \) given by (31) for the region \( \xi > b \). For points near the boundary the leading terms in the asymptotic expansions for the components of the energy-momentum tensor have the form

\[
\langle T^{(b)}_{0} \rangle \approx -\frac{2}{b^2 \sin^2 \theta} \approx -2(T^{(b)}_{2}) \approx \frac{1 - \xi/b}{30\pi^2 a^4(\eta)},
\]
\[
\langle T^{(b)}_{1} \rangle \approx \frac{r \gamma - \cos \theta}{r} \approx \frac{1326(\xi/b)^4}{8\pi^2 a^4(\eta) \ln(2b/\xi)}.
\] (35)

In the asymptotic term for the off-diagonal component \( r \) and \( \theta \) are related by (9). Near the boundary the total energy-momentum tensor is dominated by the boundary-induced part.

In the limit \( \xi \to 0 \) we have the following asymptotic formulae (no summation over \( l \))

\[
\langle T^{(b)}_{0} \rangle \approx \langle T^{(b)}_{1} \rangle \approx -\langle T^{(b)}_{2} \rangle \approx \frac{1326(\xi/b)^4}{8\pi^2 a^4(\eta) \ln(2b/\xi)}, \quad l = 2, 3,
\]
\[
\langle T^{(b)}_{l} \rangle \approx -b^2 \sin \theta \frac{1326(\xi/b)^6 \sin^2 (\theta/2)}{2\pi^2 a^4(\eta) r \ln(2b/\xi)}.
\] (36)
This limit corresponds to large values of the coordinate \( r \) with the relation \( \xi/\xi_0 \approx [2r \sin^2(\theta/2)]^{-1} \). Now we turn to the limit \( \xi/b \to \infty \). In terms of the coordinates \( r \) and \( \theta \), this limit corresponds to large values of \( r \) and small values of \( \theta \) with \( \xi/\xi_0 \approx 2r/\left(r^2\theta^2 + 1\right) \). To the leading order for the boundary induced part we have

\[
\langle T^k_i \rangle^{(b)} = \frac{a^{-4}(\eta)}{96 \ln^2(\xi/b)} \text{diag}(1, -1/3, -1/3, -1/3). \tag{37}
\]

In this limit the total energy-momentum tensor is dominated by the boundary free part \( \langle T \rangle^{(b)} \).

5 Conclusion

In the investigations of the Casimir effect the calculation of the local densities of the vacuum characteristics is of special interest. In particular, these include the vacuum expectation value of the energy–momentum tensor. In addition to describing the physical structure of the quantum field at a given point, the energy–momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field.

In the present paper we have investigated the vacuum expectation value of the energy-momentum tensor for scalar and electromagnetic fields induced by the boundary, defined by Eq. (9), on background of RW spacetime with negative spatial curvature. For a scalar field the Robin boundary condition is imposed and for the electromagnetic field we have assumed that the boundary is a perfect conductor. In order to obtain the vacuum expectation values we have used the corresponding results for a plate moving with constant proper acceleration through the Fulling–Rindler vacuum and the conformal relation between the \( k = -1 \) RW and Rindler spacetimes.

For the general case of the scale factor we have presented the vacuum energy-momentum tensor as the sum of the boundary free and boundary induced parts. The boundary free parts are given by the standard formulae (26), (27) for a scalar field and by (33) for the electromagnetic field. In the special case of the power-law scale factor the corresponding expressions take the forms (28) and (34), respectively. The boundary induced part in the vacuum expectation value of the energy–momentum tensor is non-diagonal and is given by the expressions (29) with \( \langle T^k_i \rangle^{(b)} \) defined by formulae (17) and (31) for scalar and electromagnetic fields in the region \( \xi > b \). The corresponding formulae for the region \( \xi < b \) are obtained by the replacements \( I_{\omega} \leftrightarrow K_{\omega} \). In the case of the electromagnetic field the boundary induced energy density is positive (negative) in the region \( \xi < b \) \( (\xi > b) \). For the power-law expansion with \( a(t) \propto t^c \), \( c > 1 \), at a given spatial point the ratio of the boundary induced and boundary free parts in the vacuum energy-momentum tensor behaves as \( t^{4(1-c)} \) in the model with \( D = 3 \) and at early stages of the cosmological expansion the boundary induced part dominates.

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