Relaminarization of wall-bounded turbulent flows by means of external static magnetic fields is a long-known phenomenon in the physics of electrically conducting fluids at low magnetic Reynolds numbers. Despite the large literature on the subject, it is not yet completely clear what combination of the Hartmann ($M$) and the Reynolds number has to be used to predict the laminar-turbulent transition in channel or pipe flows fed by upstream turbulent flows free of magnetic perturbations. Relying upon standard phenomenological approaches related to mixing length and structural concepts, we put forward that $M/R_e$, where $R_e$ is the friction Reynolds number, is the appropriate controlling parameter for relaminarization, a proposal which finds good support from available experimental data.

The laminar-turbulent transition of electrolyte or liquid metal flows at low magnetic Reynolds numbers ($R_m \ll 1$), where external magnetic fields are negligibly affected by polarization effects [1–4], is a challenging scientific problem of great technological interest. As it was discovered long ago by Hartmann and Lazarus [5], turbulent flows of electrically conducting fluids can be relaminarized when subject to large enough magnetic fields [6]. Applications are found in flow control [7,8], in semiconductor crystal growth [9,10], in the design of tritium breeding blankets for fusion reactors [11,12], and in steel casting [13,14]. In all of these domains, the essential issue is to understand whether laminar or turbulent flow regimes will take place under the presence of magnetic fields for a variety of boundary conditions.

Flow control by magnetic fields is a promising strategy for the thermal protection of spacecrafts during atmospheric reentry, while laminar flows of melted steel are welcome in casting processes in order to avoid particulate entrainment at the steel-air interface, and also in semiconductor growth technology, for homogeneity enhancement in the production of silicon ingots. In fusion research, in contrast, one is interested to prevent turbulence attenuation of the coolant flow (one of the roles of tritium blankets) associated to the strong magnetic fields produced by the fusion plasma.

Dynamic similarity for incompressible electrically conducting flows is parameterized by two dimensionless quantities, the Hartmann ($M$) and Reynolds ($Re$) numbers [11,14], which are estimates, respectively, of the ratio of magnetic and inertial forces to viscous forces. It turns out that in the asymptotic limit of high magnetic fields, pipe or channel flows become laminar with drag coefficients proportional to $M/Re$. As the magnetic field intensity decreases or the Reynolds number gets larger, flow instabilities come into play and turbulence eventually resumes.

The original discussions presented in the seminal papers [6,15,17] have proposed that $M/Re$ should work as a controlling parameter for the laminar-turbulent transition—an educated guess that still percolates in much of the recent literature. However, broadly acknowledged and careful pipe flow experiments performed by Gardner and Lykoudis [18] almost fifty years ago, established that critical values of $M/Re$ actually depend on the Reynolds numbers at the laminar-turbulent transition point, a fact emphasized by Tsinober [19] and further discussed by Branover [20], who proposed that a more accurate transition criterion would be given by $M/Re^\alpha$, where the exponent $\alpha$ is slightly smaller than unity. Narasimha pointed out, subsequently, that relaminarization could be related to flow regimes where magnetic forces dominate Reynolds stress gradients [21], an observation that is closely related to the arguments we address in this paper.

Our aim is to model the Gardner-Lykoudis measurements of the laminar-turbulent transition, having in mind contemporary ideas about the role of coherent structures in wall-bounded flows.

After a relatively long period of occasional contributions that followed the pioneering works in the structural approach to hydrodynamic turbulence [22–27], there is currently a flurry of research in the field [28–30]. Among many interesting developments, one may single out, for the sake of illustration, the discussion on vortex identification criteria [31] and the associated visualization studies based on optical techniques [32], the modeling of a number of phenomena, as the spatial dependence of statistical moments in boundary layers [33,34], the derivation of pipe friction coefficients for turbulence with Newtonian [35] and non-Newtonian fluids [36], and the scaling structure of attached eddies [37,38]. Coherent structures have, similarly, been the focus of great attention in magnetohydrodynamics, as in the investigation of the dynamo effect [39], particulate deposition in duct flows [40], magnetic reconnection [41], and solar wind heating [42], to name just a few examples out of a myriad of studies.

The dynamic evolution of a neutral fluid of mass density $\rho$, dynamic viscosity $\mu$, and conductivity $\sigma$, which is subject to a static uniform magnetic field $\vec{B}$, is governed, at low $R_m$ (when fluctuations of the magnetic field are quickly damped), by the electromagnetically forced...
Navier-Stokes equations \[1,2,3,4\],

\[
\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla P + \nu \nabla^2 \vec{v} + \frac{\sigma}{\rho} \left(-\nabla \phi + \vec{v} \times \vec{B}\right) \times \vec{B},
\]

where the usual incompressibility constraint, \( \nabla \cdot \vec{v} = 0 \) is imposed, \( \nu = \mu/\rho \) is the kinematic viscosity, and the electric potential \( \phi \) satisfies the Poisson’s equation

\[
\nabla^2 \phi = \nabla \cdot (\vec{v} \times \vec{B}).
\]

Considering a statistically stationary flow in a pipe of diameter \( D \), with bulk velocity \( U \), the dynamic equations can be rewritten in dimensionless form with the help of the following substitutions,

\[
\vec{v} \to U \vec{v}, \quad P \to \frac{U}{L} P, \quad \phi \to U D \phi, \quad \vec{r} \to D \vec{r}, \quad t \to \frac{L^2}{\nu} t.
\]

Equation (1) becomes, then,

\[
\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla \vec{P} + \nabla^2 \vec{v} + M^2 \left(-\nabla \phi + \vec{v} \times \vec{B}\right) \times \vec{B},
\]

where

\[
Re = \frac{UD}{\nu}
\]

is the Reynolds number for the flow, \( \vec{B} \equiv \vec{B}/B \) is the versor parallel to the magnetic field, and

\[
M = BD \sqrt{\frac{\sigma}{\mu}}
\]

is the Hartmann’s number for the magnetohydrodynamic system.

In typical low \( R_m \) experiments which probe the laminar-turbulent transition, a pipe or channel turbulent flow enters a region subject to a uniform magnetic field. Near-wall coherent structures, mainly quasi-streamwise vortices, will lose kinetic energy (and angular momentum, as well) due to the dissipative action of the magnetic field as they are carried by the mean stream. A sketch of this phenomenological picture is shown in Fig. 1. If the magnetic field is strong enough, the quasi-streamwise vortices eventually disappear downstream and, as a consequence, vorticity fluctuations associated to vortical structures cannot be propagated anymore from the wall to the bulk of the flow, which, then, becomes laminar.

In order to get an asymptotic criterion for magnetic relaminarization, we assume, as a reasonable simplification, that quasi-streamwise vortices are perfectly axisymmetric and aligned with the wall (a grounded insulated surface), which is taken to be perpendicular to the magnetic field lines. In this case,

\[
\nabla \cdot (\vec{v} \times \vec{B}) = 0,
\]

implying, from Eq. (2), that the scalar field \( \phi \) vanishes. The power per unit volume provided by the magnetic force is, thus,

\[
P_B = \sigma \vec{v} \cdot \left(\vec{v} \times \vec{B}\right) \times \vec{B} = \sigma \left[(\vec{v} \cdot \vec{B})^2 - \vec{v}^2 \vec{B}^2\right] \leq 0.
\]

The non-positiveness of \( P_B \) tells us that quasi-streamwise vortices are in fact dissipated by the transverse magnetic field. Referring back to Fig. 1, the vortical structures that cross from region I to region II are furthermore supposed to have translation/core rotational velocities and linear dimensions that scale with the friction velocity \( u_r \) and the viscous length \( l \) [43,44]. The total power injected per vortex into region II can, therefore, be straightforwardly estimated as

\[
P_{in} \sim \rho u_r^3 l^2.
\]

On the other hand, the power dissipated per vortex in region II, due to the presence of the magnetic field is analogously expected to be, from (8),

\[
P_{out} \sim \sigma u_r^2 B^2 l^2.
\]

A sufficient condition for relaminarization in pipe flow is naturally written as

\[
P_{out} \gg P_{in},
\]

which leads to

\[
\frac{M}{R_r} \gg 1,
\]

where

\[
R_r = \frac{u_r R}{\nu} = \frac{R}{l}.
\]
is the usual definition of the friction Reynolds number in pipe flows \cite{45}.

Condition (12) can be derived, actually, from the evaluation of the ratio between convective and magnetic dissipative time scales in the very near-wall region. Still keeping an eye on Fig. 1, it is suggested that coherent structures produced by flow instabilities in the viscous layer of region I, near the interface between regions I and II, are transported to upper layers and dissipated by the magnetic field in region II, within the respective time scales

\[ T_C \sim \frac{\ell}{u_\tau} = \frac{\ell^2}{\nu} \tag{14} \]

and

\[ T_B \sim \frac{\rho}{\sigma B^2}. \tag{15} \]

A transition to the laminar flow regime is expected to take place whenever \( T_C/T_B \) goes beyond a critical threshold. Note, in particular, that

\[ \frac{T_C}{T_B} \sim \left( \frac{M}{Re} \right)^2, \tag{16} \]

which yields an alternative interpretation of (12).

The physical picture that emerges here is that relaminarization occurs when magnetic dissipation is strong enough to hamper the growth of small-scale velocity fluctuations in the very near-wall region. As it is clear from the equality relation given in (14), convective and viscous dissipative processes have the same time scale \( T_C \) at the top of the viscous layer. To render the argument more specific, recall the van Driest expression for the mean velocity profile in turbulent boundary layers \cite{45,46},

\[ u^+(y^+) = \int_0^{y^+} \frac{2dy'}{1 + \sqrt{1 + 4\ell_m(y')^2}}, \tag{17} \]

where \( y^+ \equiv y/\ell, \) \( u^+ \equiv u/u_\tau, \) and \( \ell_m(y') \) is the Prandtl mixing length at height \( y' \) (in viscous length units) modulated by the van Driest damping function, viz.,

\[ \ell_m(y') = \kappa y'[1 - \exp(-y'/A)], \tag{18} \]

with \( \kappa = 0.41 \) and \( A = 26. \) Introduce, now, as a way to quantify the relative importance of magnetic and viscous forces, the height-dependent sliding Hartmann number, \( M^*(y^+) \), which in the units of Eq. (1) is

\[ M^*(y^+) = \sqrt{\frac{\| (\vec{\nu}) \times \vec{B} \| \times \vec{B}}{\mu |\nabla^2 (\vec{v})|}} \]

\[ = \frac{M}{2Re} \sqrt{\frac{d^2 u^+(\xi)}{d\xi^2} u^+(\xi)|_{\xi=y^+}}. \tag{19} \]

It turns out, from (17), that the RHS of (19) has a single minimum at \( y^+ = y_0^+ \simeq 5.5, \) for fixed \( M/Re \).

We remark that the viscous layer is usually defined as the region \( y^+ < 5 \simeq y_0^+ \). A condition for the annihilation of coherent structures produced by shear instabilities in the viscous layer – the seeds of bulk turbulence – can be put forward, therefore, as \( M^*(y_0^+) > C, \) for some critical parameter \( C. \) This implies, due to the properties of (19), that \( M^*(y^+) > C \) for any \( y^+, \) which indicates the damping action of the magnetic field over the boundary layer as a whole. We are led, then, to the conjecture that turbulence is suppressed if

\[ \frac{M}{Re} > 2C \sqrt{\frac{d^2 u^+(\xi)}{d\xi^2} u^+(\xi)|_{\xi=y_0^+}}. \tag{20} \]

FIG. 2: Sliding Hartmann numbers \( M^*(y^+) \) for various ratios \( M/Re, \) with \( Re = 10^4. \) The dotted lines give the position of the minimum of \( M^*(y^+) \) for \( M/Re = 1/200. \)

Using the standard definition of the friction factor \( f \) in terms of \( u_\tau \) and \( U \) \cite{45},

\[ f = 8 \left( \frac{u_\tau}{U} \right)^2, \tag{21} \]

Eq. (19) is rewritten as

\[ M^*(y^+) = \frac{M}{Re} \sqrt{\frac{8}{\int \frac{d^2 u^+(\xi)}{d\xi^2} u^+(\xi)|_{\xi=y^+}}} \tag{22} \]

As it is well-known, the friction factor depends uniquely on the Reynolds number (at fixed pipe relative roughness). A useful relation for the smooth pipe case is the Prandtl’s empirical formula

\[ \frac{1}{\sqrt{f}} = 2\log_{10}(Re \sqrt{f}) - 0.8, \tag{23} \]
which yields pragmatically accurate results for a large range of turbulent Reynolds numbers [15, 47]. Taking \( Re = 10^4 \) in Eq. (22), where \( f \) is evaluated from (23), we depict, in Fig. 2, graphs of \( M^*(y^+) \) for various values of \( M/Re \). A close look at experimental data [17, 18] shows that relaminarization is produced, for \( Re = 10^4 \), at \( M/Re \approx 1/200 \). In this case, as it is indicated in Fig. 2, \( M^*(y_0^+) \approx 2/3 \). According to (20), identifying \( C \) to \( M^*(y_0^+) \), we expect to have laminar flow for \[
\frac{M}{Re} > 0.16 .
\] (24)

To check the constancy of \( M/Re \) at the laminar-turbulent transition for various Reynolds numbers, we explore the critical values of \( M/Re \) measured by Gardner and Lykoudis [18]. Their values were obtained through visual inspection of the streamwise velocity signals produced by hot-film anemometry at many different radial and axial angular positions of the pipe’s cross section, up to the minimum distance to the wall \( \Delta^+ = 0.03 R_e \), so that, approximately, \( 10 < \Delta^+ < 100 \), for the covered range of Reynolds numbers. A parabolic fit of the Gardner-Lykoudis data, in monolog scale, is shown in Fig. 3.

The measured critical values of \( M/Re \) change by approximately 20% around their mean, as the Reynolds number is varied from \( 10^4 \) to \( 3 \times 10^5 \). Working instead with critical values of \( M/R_e \), we find that their variations drop to below 5% around the mean, as it is reported in Fig. 4. These results provide strong support to the phenomenological picture of relaminarization by the magnetic suppression of near-wall coherent structures.

It is interesting to note that \( M/R_e \) yields, actually,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Open and closed circles refer, respectively, to the values of \( M/Re \) obtained by Gardner and Lykoudis [18], for the laminar-to-turbulent and turbulent-to-laminar transitions. The solid line is a parabolic fit to the measured data.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{The solid line gives values of \( M/R_e \), normalized by the mean value of the measured sample, as a function of the Reynolds number \( Re \). These values are obtained from the Hartmann numbers extracted from the parabolic fit for \( M/Re \) (Fig. 3), and from the friction Reynolds numbers evaluated through Eqs. (6), (13), (21), and (23). The dashed lines indicate the accuracy range of 5% for the predicted values of \( M/R_e \).}
\end{figure}

a suggestive parameterization of the effects of external magnetic fields on wall-bounded flows: \( M/R_e \) is just the ratio of the viscous length \( \ell \) of the “would-be turbulent boundary layer” at high Reynolds numbers and negligible magnetic fields to the thickness of the “would-be laminar Hartmann boundary layer”, \( R/M \), at high magnetic fields and small Reynolds numbers.

Our analysis has been restricted to the problem of relaminarization where the inlet turbulent flow has been previously produced without magnetic forcing, a boundary condition of practical relevance in many applications. It is not obvious at all, however, if at asymptotic far distances from the inlet the same critical relation (24) between the Hartmann and Reynolds numbers would hold for a prediction of the laminar-turbulent transition. Loop experiments [45], which are likely to be related to this issue, seem to indicate that this is not so. In other words, while (24) would still imply in laminar flow, laminar asymptotic regimes could be induced by the action of magnetic fields of relatively lower intensity.

A main message from the above considerations is that low \( R_m \) magnetohydrodynamics, with particular focus on the laminar-turbulent transition, can be an interesting way to probe and investigate (both experimentally and numerically) the intricate dynamics of near-wall coherent structures in pure hydrodynamics.
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