LINEAR CONNECTIONS ON THE TWO PARAMETER QUANTUM PLANE

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Abstract: We apply a recently proposed definition of a linear connection in non commutative geometry based on the natural bimodule structure of the algebra of differential forms to the case of the two-parameter quantum plane. We find that there exists a non trivial family of linear connections only when the two parameters obeys a specific relation.

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I) Introduction

In the last few years, much attention has been attracted by non commutative differential geometry. Many related attempts focused on generalization of differential forms [1,2,3], as well as covariant derivative [1]. This later generalization of covariant derivative only used a left or a right module structure. However, in order to extend the notion of linear connection to non commutative geometry, one has to deal with one-forms. Then, the bimodule structure of the space of one-forms should be taken into account. This has been done in [4a,b] for the general derivation based differential calculus and in [5] for more general differential calculi. Other examples based on [5] have been worked out in [6] and [7].

In this letter, we adopt this last viewpoint to construct linear connections on the two parameter quantum plane. In section II, we recall the general definition of a linear connection on a non commutative algebra as well as some results already obtained in [6] on the one parameter quantum plane [8]. In section III we present our results for the two parameter quantum plane.

II) Linear connections over a noncommutative algebra

IIa) Basic tools

Let $\mathcal{A}$ and $(\Omega^*(\mathcal{A}), d)$ be respectively a noncommutative algebra and a differential calculus over it ($d$ is the exterior derivative). Let $\Omega^k(\mathcal{A}) (k \geq 0)$ be the algebra of differential forms of degree $k$ on $\mathcal{A}$ and $\pi: \Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$ the projection defined by the product of forms ($\otimes \mathcal{A}$ is the tensor product on $\mathcal{A}$).

We first recall the definition of a linear connection on $\mathcal{A}$ given in [4-6] that will be the main ingredient used in this letter.

**Definition:** A linear connection over $\mathcal{A}$ is determined from two maps $\sigma$ and $D$,

$\sigma: \Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})$, $D: \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A})$, satisfying the following properties: for any $f, g \in \Omega^0(\mathcal{A})$, $\alpha, \beta \in \Omega^1(\mathcal{A})$

$$\sigma \text{ is left and right–linear } \sigma(f \alpha \otimes \beta g) = f \sigma(\alpha \otimes \beta)g$$

$$\pi(\sigma + 1) = 0$$

$$D(f \alpha) = df \otimes \alpha + f D\alpha, \quad D(\alpha f) = D\alpha f + \sigma(\alpha \otimes df)$$

Concerning this definition, some comments are in order.
Basically, the map $\sigma$, which is a bimodule automorphism in $\Omega^1(A) \otimes_A \Omega^1(A)$, can be viewed as a generalization of the permutation map that would appear in the case of a commutative algebra. The map $D$ is a noncommutative extension of the covariant derivative (observe in particular the Leibnitz rules, property (2.3)). Furthermore, it can be easily seen that the right (resp. left) $A$–linearity of $\sigma$, property (2.1), insures that $D((\xi f) g) = D(\xi(f g))$ (resp. $D(d((f g) h)) = D(d(f (g h)))$) for any $\xi \in \Omega^1(A)$, $f, g, h \in \Omega^0(A)$.

As far as the properties (2.2) and (2.3) are concerned, it must be pointed out that not each solution of (2.2) admits a covariant derivative, that is, a map $D$ fulfilling the Leibnitz rules (2.3). In other words, a linear connection for each $\sigma$ does not necessarily exist.

Finally, the above definition is nothing but a noncommutative extension of the definition of a linear connection in term of a covariant derivative introduced by Koszul [9] in the framework of commutative geometry. Indeed, for a commutative algebra $A$, it follows from (2.3) that $\sigma$ reduces to the usual permutation, so that (2.1) and (2.2) are satisfied.

The Koszul definition then follows.

IIb) Application to the Manin quantum plane

Let $\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \Omega^2$ the algebra of forms of the Manin quantum plane [8] whose generators $x^i := (x, y)$, $\xi^i = dx^i := (\xi, \eta)$ obey the following relations

$$x^i x^j - q^{-1} \hat{R}^{ij}_{kl} x^k x^l = 0; \quad x^i \xi^j - q \hat{R}^{ij}_{kl} \xi^k x^l = 0; \quad \xi^i \xi^j + q \hat{R}^{ij}_{kl} \xi^k \xi^l = 0 \quad \text{(2.4a; b; c)},$$

where the tensor $\hat{R}^{ij}_{kl}$ is given in matrix form by

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad \text{(2.5)}.$$  

We recall that $\Omega^*$ is stable under the action of $SL_q(2, C)$ since the following identity

$$\hat{R}^{ij}_{kl} a^k_m a^l_n = a^i_k a^j_l \hat{R}^{kl}_{mn}$$

holds (where $a^i_j$ denote generically the four generators of $SL_q(2, C)$ written again in matrix form with $da^i_j = 0$ and $d$ is the exterior derivative).

Now, the action of $D$ on eqn. (2.4b) which is defined on $\Omega^1$ determines the action of $\sigma$ on $\xi^i \otimes \xi^j$. It is given [6] by

$$\sigma(\xi \otimes \xi) = q^{-2}(\xi \otimes \xi); \quad \sigma(\xi \otimes \eta) = q^{-1}(\eta \otimes \xi) \quad \text{(2.6a; b)},$$

$$\sigma(\eta \otimes \xi) = q^{-1}(\xi \otimes \eta) + (q^{-2} - 1)(\eta \otimes \xi); \quad \sigma(\eta \otimes \eta) = q^{-2}(\eta \otimes \eta) \quad \text{(2.6c; d)},$$
so that $\sigma = q^{-1}R^{-1}$ and can be proven to be left and right linear. Furthermore, $\sigma$ obeys the property (2.2) of the general definition, namely $\pi(\sigma + 1) = 0$, as it can be easily seen by computing the action of $\pi(\sigma + 1)$ on $\xi^i \otimes \xi^j$ and using the relations (2.4c). Notice that $\sigma^2 \neq 1$; however, it satisfies a Hecke relation given by

$$(\sigma + 1)(\sigma - q^{-2}) = 0 \quad (2.7),$$

where the simple (resp. triply degenerate) eigenvalue $-1$ (resp. $q^{-2}$) corresponds to the antisymmetric (resp. symmetric) eigenspaces with eigenvectors $\xi \otimes \eta - q\eta \otimes \xi$ (resp. $\xi \otimes \xi$, $\eta \otimes \eta$, $\eta \otimes \xi + q\xi \otimes \eta$).

Finally, the covariant derivative map $D$ acting on the $\xi^i$'s can be cast into the form

$$D\xi^i = \mu \ x^i \theta \otimes \theta \quad (2.8),$$

where $\mu$ is an arbitrary parameter (which has the dimension of a mass) and $\theta \in \Omega^1$ is the unique 1-form invariant under the coaction of $SL_q(2,\mathbb{C})$ which is given by $\theta = x\eta - qy\xi$ (up to an overall constant) with $\theta^2 = 0$.

**III) The two parameter quantum plane**

The algebraic structure of the two parameter quantum plane is now defined by

$$x^i x^j - q^{-1}\hat{R}_{kl}^{ij}(p,q)x^k x^l = 0; \quad x^i \xi^j - p\hat{R}_{kl}^{ij}(p,q)\xi^k x^l = 0; \quad \xi^i \xi^j + p\hat{R}_{kl}^{ij}(p,q)\xi^k \xi^l = 0 \quad (3.1a; b; c),$$

where again $x^i: = (x, y)$, $\xi^i = dx^i: = (\xi, \eta)$ and the tensor $\hat{R}_{kl}^{ij}(p,q)$ is given by

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - p^{-1} & qp^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad (3.2),$$

and reduces to (2.5) when $q = p$. Recall that $\hat{R}(p,q)$ satisfies the braid equation: $\hat{R}_{12}(p,q)\hat{R}_{23}(p,q)\hat{R}_{12}(p,q) = \hat{R}_{23}(p,q)\hat{R}_{12}(p,q)\hat{R}_{23}(p,q)$ with as usual $\hat{R}_{12}(p,q): = \hat{R}(p,q) \otimes 1$, $\hat{R}_{23}(p,q): = 1 \otimes \hat{R}(p,q)$. For the moment, $p$ and $q$ are independent arbitrary parameters. We will see in a while that a non trivial family of linear connections can be obtained only when $p$ and $q$ obey a supplementary relation.

From the action of $D$ on eqn.(3.1b), we determine the action of $\sigma$ on $\xi^i \otimes \xi^j$. Namely, using the property (2.3) of the general definition, we obtain generically

$$\xi^i \otimes \xi^j = p\hat{R}_{kl}^{ij}(p,q)\sigma(\xi^k \otimes \xi^l); \quad x^i D\xi^j = p\hat{R}_{kl}^{ij}(p,q)(D\xi^k)x^l \quad (3.3a; b).$$
The equation (3.3a) is verified when $\sigma = p^{-1}\hat{R}^{-1}(p,q)$ or equivalently

$$\sigma(\xi \otimes \xi) = p^{-1}q^{-1}(\xi \otimes \xi); \quad \sigma(\xi \otimes \eta) = p^{-1}(\eta \otimes \xi) \quad (3.4a; b),$$

$$\sigma(\eta \otimes \xi) = q^{-1}(\xi \otimes \eta) + (p^{-1}q^{-1} - 1)(\eta \otimes \xi); \quad \sigma(\eta \otimes \eta) = p^{-1}q^{-1}(\eta \otimes \eta) \quad (3.4c; d).$$

It is easy to verify the left and right-linearity of $\sigma$ (property (2.1)). Notice that right-linearity stems from left-linearity, as a mere consequence of the braid equation for $\hat{R}(p,q)$. To see that, it is sufficient to combine $\sigma = p^{-1}\hat{R}^{-1}(p,q)$ with (3.3a) and (3.1b) together with the braid equation; then, the above statement follows. Notice also that $\sigma$ fulfills a braid equation since $\hat{R}(p,q)$ does. Besides, combining (3.4) and (3.1c), we also verify that $\pi(\sigma + 1) = 0$ (property (2.2)) still holds.

Thus, we have determined a suitable $\sigma$ map on the two parameter quantum plane. The remaining eqn. (3.3b) will fix a relation between $p$ and $q$ so that a non trivial family of covariant derivative $D$ can be associated to this $\sigma$. By non trivial family, we mean that $D$ can differ from the trivial case $D\xi = D\eta = 0$ which always verifies (3.3b). Namely, we find after some calculations that (3.3b) is verified provided $D\xi$ and $D\eta$ are given by

$$D\xi = xZ\theta \otimes \theta; \quad D\eta = yZ\theta \otimes \theta; \quad Z = \mu x^{n-1}y^{n-1} \quad (3.5a; b; c),$$

where $\mu$ is an overall complex parameter; (3.5) must be supplemented by

$$p = q^n \quad n \geq 1 \quad (3.6).$$

In (3.5), $\theta = x\eta - qy\xi$ and still verifies $\theta^2 = 0$ and

$$\sigma(\xi \otimes \theta) = q^{-1-2n}\theta \otimes \xi; \quad \sigma(\theta \otimes \xi) = q^n(\xi \otimes \theta) - (1 - q^{-1-n})(\theta \otimes \xi) \quad (3.7a; b),$$

$$\sigma(\eta \otimes \theta) = q^{-2-n}\theta \otimes \eta; \quad \sigma(\theta \otimes \eta) = q(\eta \otimes \theta) - (1 - q^{-1-n})(\theta \otimes \eta) \quad (3.7c; d),$$

$$\sigma(\theta \otimes \theta) = q^{-1-n}(\theta \otimes \theta) \quad (3.7e),$$

where we used (3.6).

It is interesting to observe that a non trivial family of linear connection on the two parameter quantum plane can be consistently found only when the two parameters are related each other through (3.6).

Therefore, the corresponding map $\sigma$ is given in the tensor form by

$$\sigma = \begin{pmatrix}
q^{-1-n} & 0 & 0 & 0 \\
0 & 0 & q^{-n} & 0 \\
0 & q^{-1} & q^{-1-n} - 1 & 0 \\
0 & 0 & 0 & q^{-1-n}
\end{pmatrix} \quad (3.8)$$
and satisfies the Hecke relation \((\sigma + 1)(\sigma - q^{-1-n}) = 0\) where the simple (resp. triply degenerate) eigenvalue \(-1\) (resp. \(q^{-1-n}\)) corresponds to the antisymmetric (resp. symmetric) eigenspace with eigenvectors \(\xi \otimes \eta - q^n \eta \otimes \xi\) (resp. \(\xi \otimes \xi, \eta \otimes \eta, \eta \otimes \xi + q \xi \otimes \eta\)).

Some remarks are in order. Firstly, it happens that \(\sigma\) is actually the only generalized permutation for which there exits a covariant derivative.

Next, one recovers the results of section IIb when \(n = 1\). We point out that the case \(p = q\) is very similar, as far as the structure of the set of linear connections is concerned, to the cases \(p = q^n\). Finally, in (3.6) we have restricted \(n \geq 1\) since working on the quantum plane forces to consider only positive powers in \(x\) and \(y\). However, the case \(pq = 1\) is interesting. In this last situation, the algebra is formally the non commutative torus (for which negative powers of \(x\) and \(y\) are allowed). The differential calculus is then based on derivations in the sense of [4b]. An easy calculation using \(\sigma\) given in (3.4) for \(pq = 1\) leads to an eight complex parameter family of linear connections. This family can also be obtained using the definition of linear connection based on derivations given in [2,4a].

Let \(\pi_{12} = \pi \otimes 1\) and \(\sigma_{12} = \sigma \otimes 1\). The covariant derivative map defined above can be extended to \(\Omega^1 \otimes_A \Omega^1\) by

\[
D(\alpha \otimes \beta) = D\alpha \otimes \beta + \sigma_{12}(\alpha \otimes D\beta)
\]  
(3.9),

for any \(\alpha, \beta \in \Omega^1\). Consider now the following map

\[
\pi_{12}D^2: \Omega^1 \to (\Omega^2/\Theta) \otimes_A \Omega^1
\]  
(3.10),

where \(\Theta\) is a submodule of \(\Omega^2\), called the torsion module, given by the image of \(\Omega^1\) by \(d - \pi D\). This map is left-linear, namely \(\pi_{12}D^2(f\alpha) = f\pi_{12}D^2(\alpha)\) for any \(f \in \Omega^0\) \((\Omega^0 = A)\), \(\alpha \in \Omega^1\) since \(\sigma\) verifies \(\pi(\sigma + 1) = 0\).

By noticing that \((d - \pi D)\xi^i = 0\) holds, thanks to \(\theta^2 = 0\) and combining (3.1), (3.9) and \(\pi(\sigma + 1) = 0\), we obtain

\[
\pi_{12}D^2\theta = 0
\]  
(3.11).

Now, using again (3.1), (3.5), (3.6), (3.8) we find that

\[
\pi_{12}D^2\xi^i = \Omega^i \otimes \theta
\]  
(3.12),

with

\[
\Omega^i = f(q)x^i Z\xi \eta
\]  
(3.13a),
Then, from (3.12) and (3.13), we obtain the 2-form curvature $\Omega^i_j$ given by

$$\pi_{12}D^2\xi^i = -\Omega^i_j \otimes \xi^j$$ (3.14),

$$\Omega^i_j = f(q) \begin{pmatrix} q^{-n-1}xy & -q^{-1-2n}x^2 \\ q^{-1-n}y^2 & -q^{-1-2n}yx \end{pmatrix} Z\xi\eta$$ (3.15).

IV) Conclusion

In this letter, working on the two parameter quantum plane, we have shown that there exists a non trivial family of linear connections only when the parameters are related through $p = q^n, n \geq 1$. In this respect, the usual one parameter quantum plane ($n = 1$) is fully representative of the $p = q^n$ situation. It remains to see whether this specific relation between $p$ and $q$ is purely accidental or reflects a deeper property.

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