CONNECTION WITH PARALLEL TOTALLY TOTALLY
SKEW-SYMMETRIC TORSION ON ALMOST COMPLEX
MANIFOLDS WITH NORDEN METRIC

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Abstract
In the present work we consider an almost complex manifold with Norden metric (i.e. a metric with respect to which the almost complex structure is an anti-isometry). On such a manifold we study a linear connection preserving the almost complex structure and the metric and having a totally skew-symmetric torsion tensor. We consider the case when the manifold admits a connection with parallel totally skew-symmetric torsion and the case when such connection has a Kähler curvature tensor. We get necessary and sufficient conditions for an isotropic Kähler manifold with Norden metric.

Key words: Norden metric, almost complex manifold, indefinite metric, linear connection, Bismut connection, KT connection, totally skew-symmetric torsion, parallel torsion.

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1. INTRODUCTION

There is a strong interest in the metric connections with totally skew-symmetric torsion tensor (3-form). These connections arise in a natural way in theoretical and mathematical physics. For example, such a connection is of particular interest in string theory [1]. In mathematics this connection was used by Bismut to prove the local index theorem for non-Kähler Hermitian manifolds [2]. A connection with totally skew-symmetric torsion tensor is called a KT connection by physicists, and among mathematicians this connection is known as a Bismut connection.

In the present work we continue the investigations from [3] for a connection $\nabla'$ with totally skew-symmetric torsion on non-Kähler quasi-Kähler manifolds with Norden metric. There are proved some necessary and sufficient conditions the curvature tensor of $\nabla'$ to be Kählerian. In the case when this tensor is Kählerian, some relations between its scalar curvature and the scalar curvatures of other curvature-like tensors are obtained. Moreover, conditions for isotropic Kähler manifolds with Norden metric are get.

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Now we consider the case when $\nabla'$ has a parallel torsion. We obtain a relation between the scalar curvatures for $\nabla'$ and the Levi-Civita connection $\nabla$. We establish that the manifold is isotropic Kählerian with Norden metric iff these curvatures are equal. We obtain a necessary and sufficient condition for $\nabla'$ with parallel torsion be with Kähler curvature tensor. Moreover we show that if $\nabla'$ has a parallel torsion and a Kähler curvature tensor, then the manifold is isotropic Kählerian.

2. Preliminaries

Let $(M, J, g)$ be a $2n$-dimensional almost complex manifold with Norden metric, i.e.

$$J^2 x = -x, \quad g(Jx, Jy) = -g(x, y),$$

for all differentiable vector fields $x, y$ on $M$. The associated metric $\tilde{g}$ of $g$ on $M$, given by $\tilde{g}(x, y) = g(x, Jy)$, is a Norden metric, too. The signature of both metrics is necessarily $(n, n)$.

Further, $x, y, z, w$ will stand for arbitrary differentiable vector fields on $M$ (or vectors in the tangent space of $M$ at an arbitrary point $p \in M$). The Levi-Civita connection of $g$ is denoted by $\nabla$. The tensor field $F$ of type $(0, 3)$ on $M$ is defined by

$$(2.1) \quad F(x, y, z) = g((\nabla x) J y, z).$$

It has the following properties:

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \quad F(x, Jy, z) = -F(x, y, Jz).$$

In [4], the considered manifolds are classified into eight classes with respect to $F$: $W_0, W_1, W_2, W_3, W_1 \oplus W_2, W_1 \oplus W_3, W_2 \oplus W_3, W_1 \oplus W_2 \oplus W_3$. The class $W_0$ of the Kähler manifolds with Norden metric is contained in each of the other seven classes. It is determined by the condition $F(x, y, z) = 0$, which is equivalent to $\nabla J = 0$. The class $W_1 \oplus W_2 \oplus W_3$ is the class of all almost complex manifolds with Norden metric.

The condition

$$(2.2) \quad \sum_{x, y, z} F(x, y, z) = 0,$$

where $\sum$ is the cyclic sum over $x, y, z$, characterizes the class $W_3$ of the quasi-Kähler manifolds with Norden metric. This is the only class among the basic classes $W_1, W_2, W_3$ of manifolds with non-integrable almost complex structure $J$.

Let $\{e_i\}$ ($i = 1, 2, \ldots, 2n$) be an arbitrary basis of the tangent space of $M$ at a point $p \in M$. The components of the inverse matrix of $g$, with respect to this basis, are denoted by $g^{ij}$.

Following [5], the square norm $\|\nabla J\|^2$ of $\nabla J$ is defined in [6] by

$$(2.3) \quad \|\nabla J\|^2 = g^{ij} g^{ks} g((\nabla e_i) J) e_k, (\nabla e_j) J) e_s),$$
where it is proven that
\[(2.4) \quad \| \nabla J \|^2 = -2g^{ij} g^{ks} g((\nabla e_i, J) e_k, (\nabla e_j, J) e_j).\]

There, the manifold with \( \| \nabla J \|^2 = 0 \) is called an isotropic-Kähler manifold with Norden metric. It is clear that every Kähler manifold with Norden metric is isotropic-Kähler, but the inverse implication is not always true.

Let \( R \) be the curvature tensor of \( \nabla \), i.e.,
\[ R(x, y) z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z. \]
The corresponding \((0, 4)\)-tensor is determined by \( R(x, y, z, w) = g(R(x, y) z, w) \). The Ricci tensor \( \rho \) and the scalar curvature \( \tau \) with respect to \( \nabla \) are defined by
\[ \rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j). \]

A tensor \( L \) of type \((0,4)\) with the properties
\[(2.5) \quad L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \]
\[(2.6) \quad \mathcal{S}_{x, y, z} L(x, y, z, w) = 0 \quad \text{(the first Bianchi identity)} \]
is called a curvature-like tensor. Moreover, if the curvature-like tensor \( L \) has the property
\[(2.7) \quad L(x, y, J z, J w) = -L(x, y, z, w), \]
it is called a Kähler tensor \([7]\).

Let \( \nabla' \) be a linear connection with a tensor \( Q \) of the transformation \( \nabla \rightarrow \nabla' \) and a torsion tensor \( T \), i.e.
\[(2.8) \quad \nabla'_x y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]. \]
The corresponding \((0,3)\)-tensors are defined by
\[(2.9) \quad Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z). \]
The symmetry of the Levi-Civita connection implies
\[(2.10) \quad T(x, y) = Q(x, y) - Q(y, x), \quad T(x, y) = -T(y, x). \]

A linear connection \( \nabla' \) on an almost complex manifold with Norden metric \((M, J, g)\) is called a natural connection if \( \nabla' J = \nabla' \tilde{g} = 0 \). The last conditions are equivalent to \( \nabla' g = \nabla' \tilde{g} = 0 \). If \( \nabla' \) is a linear connection with a tensor \( Q \) of the transformation \( \nabla \rightarrow \nabla' \) on an almost complex manifold with Norden metric, then it is a natural connection iff the following conditions are valid:
\[(2.11) \quad F(x, y, z) = Q(x, y, J z) - Q(x, J y, z), \]
\[(2.12) \quad Q(x, y, z) = -Q(x, z, y). \]

According to \([8]\), we have
\[(2.13) \quad Q(x, y, z) = \frac{1}{2} \{ T(x, y, z) - T(y, z, x) + T(z, x, y) \}. \]
Let $\nabla'$ be the natural connection with a totally skew-symmetric torsion tensor $T$ on a non-Kähler manifold with Norden metric $(M, J, g)$. According to [3] we have

\begin{equation}
Q(x, y) = \frac{1}{4}\{ (\nabla_x J) J y - (\nabla_y J) y - 2(\nabla_J J) J x \}. \tag{2.14}
\end{equation}

Since $\nabla'$ has a totally skew-symmetric torsion tensor $T$, then

\begin{equation}
T(x, y, z) = -T(y, x, z) = -T(x, z, y) = -T(z, y, x). \tag{2.15}
\end{equation}

From (2.13) and (2.15) it is follows that for the tensor $Q$ it is valid

\begin{equation}
Q(x, y, z) = \frac{1}{2}T(x, y, z). \tag{2.16}
\end{equation}

3. Connection with parallel totally skew-symmetric torsion

Let $\nabla'$ be the connection with totally skew-symmetric torsion tensor $T$ on the quasi-Kähler manifold with Norden metric $(M, J, g)$.

Now we consider the case when $\nabla'$ has a parallel torsion, i.e. $\nabla'T = 0$.

It is known that the curvature tensors $R'$ and $R$ of $\nabla'$ and $\nabla$, respectively, satisfy

\begin{equation}
R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w)
+ Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \tag{3.1}
\end{equation}

Equality (2.16) implies $\nabla'Q = 0$ in the considered case. Then from the formula for covariant derivation with respect to $\nabla'$ it follows that

\begin{equation}
xQ(y, z, w) - Q(\nabla_x y, z, w) - Q(y, \nabla_x z, w) - Q(y, z, \nabla_x w) = 0. \tag{3.2}
\end{equation}

According to the first equality of (2.8) we have

\begin{equation}
\begin{aligned}
Q(\nabla_x y, z, w) &= Q(\nabla_x y, z, w) + Q(Q(x, y), z, w), \\
Q(y, \nabla_x z, w) &= Q(y, \nabla_x z, w) + Q(y, Q(x, z), w), \\
Q(y, z, \nabla_x w) &= Q(y, z, \nabla_x w) + Q(y, z, Q(x, w)).
\end{aligned} \tag{3.3}
\end{equation}

Combining (3.2), (3.3), the first equality of (2.9) and having in mind the formula for covariant derivation with respect to $\nabla$, we obtain

\begin{equation}
(\nabla_x Q)(y, z, w) = Q(Q(x, y), z, w)
-g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)). \tag{3.4}
\end{equation}

From (3.4) and the first equality of (2.10) we have

\begin{equation}
\begin{aligned}
(\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) &= Q(T(x, y), z, w) \\
&- 2g(Q(x, z), Q(y, w)) + 2g(Q(y, z), Q(x, w)).
\end{aligned} \tag{3.5}
\end{equation}
Because of \((3.5)\), equality \((3.1)\) can be rewritten as
\[
R'(x, y, z, w) = R(x, y, z, w) + Q(T(x, y), z, w) - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)).
\]
(3.6)

Since \(Q(e_i, e_j) = -Q(e_j, e_i)\) it follows that \(g^{ij}Q(e_i, e_j) = 0\). Then, from \((3.6)\) after contraction by \(x = e_i, w = e_j\), we obtain the following equality for the Ricci tensor \(\rho'\) of \(\nabla'\):
\[
\rho'(y, z) = \rho(y, z) + 2g^{ij}g(Q(e_i, y), Q(z, e_j)) - g^{ij}g(Q(e_i, z), Q(y, e_j)).
\]
(3.7)

Contracting by \(y = e_k, z = e_s\) in \((3.7)\), we get
\[
\tau' = \tau + g^{ij}g^{ks}g(Q(e_i, e_k), Q(e_s, e_j)),
\]
where \(\tau'\) is the scalar curvature of \(\nabla'\).

By virtue of \((3.8), (2.14), (2.3)\) and \((2.4)\) we have
\[
\tau' = \tau - \frac{1}{8} \|\nabla J\|^2.
\]
(3.9)

Thus we arrive at the following

**Theorem 3.1.** Let \(\nabla'\) be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric \((M, J, g)\). Then for the Ricci tensor \(\rho'\) and the scalar curvature \(\tau'\) of \(\nabla'\) are valid \((3.7)\) and \((3.9)\), respectively.

Equality \((3.9)\) leads to the following

**Corollary 3.2.** Let \(\nabla'\) be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric \((M, J, g)\). Then the manifold \((M, J, g)\) is isotropic Kählerian iff \(\nabla'\) and \(\nabla\) have equal scalar curvatures.

**4. Connection with parallel totally skew-symmetric torsion and Kähler curvature tensor**

Let \(\nabla'\) be a connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric \((M, J, g)\).

We will find conditions for the curvature tensor \(R'\) of \(\nabla'\) to be Kählerian.

From \((2.16)\), having in mind that \(Q\) is a 3-form, we have
\[
Q(T(x, y), z, w) = Q(z, w, T(x, y)) = g(Q(z, w), T(x, y))
\]
\[
= g(T(x, y), Q(z, w)) = 2g(Q(x, y), Q(z, w)).
\]
Then (3.6) obtains the form
\[ R'(x, y, z, w) = R(x, y, z, w) + 2g(Q(x, y), Q(z, w)) - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)). \] (4.1)

From (4.1), identities (2.5) and (2.7) for \( R' \) follow immediately. Therefore \( R' \) is a Kähler tensor iff the first Bianchi identity (2.6) for \( R' \) is satisfied. Since this identity is valid for \( R \), then (4.1) implies that \( R' \) is Kählerian iff
\[ \mathcal{S}_{x,y,z} \{ 2g(Q(x, y), Q(z, w)) - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)) \} = 0. \]

Thus, using that \( Q \) is a skew-symmetric tensor, we arrive the following

**Theorem 4.1.** Let \( \nabla' \) be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric \((M, J, g)\). Then the curvature tensor for \( \nabla' \) is a Kähler tensor iff
\[ \mathcal{S}_{x,y,z} g(Q(x, y), Q(z, w)) = 0. \] \( \square \)

Because of the skew-symmetry of \( Q \), (4.2) implies
\[ g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)) = -g(Q(x, y), Q(z, w)). \]
The last equality and (4.1) lead to the following

**Corollary 4.2.** Let \( \nabla' \) be the connection with parallel totally skew-symmetric torsion and Kähler curvature tensor on the quasi-Kähler manifold with Norden metric \((M, J, g)\). Then
\[ R'(x, y, z, w) = R(x, y, z, w) + g(Q(x, y), Q(z, w)). \] \( \square \)

If \( R' \) is a Kähler tensor then \( R'(x, y, Jz, Jw) = -R'(x, y, z, w) \), and because of (4.3) we have
\[ R(x, y, Jz, Jw) + R(x, y, z, w) = -g(Q(x, y), Q(Jz, Jw)) - g(Q(x, y), Q(z, w)). \] (4.4)
From (2.14) we get
\[ Q(x, Jy) = JQ(x, y) - (\nabla_x J) y. \]
Then we have
\[ Q(Jx, Jy) = -Q(x, y) - (\nabla_{Jx} J) y - (\nabla_{y} J) J x \]
and consequently
\[ g(Q(x, y), Q(Jz, Jw)) = -g(Q(x, y), Q(z, w)) - g(Q(x, y), (\nabla_{Jz} J) w + (\nabla_{w} J) J z). \]
The last equality and (4.4) imply the following

**Corollary 4.3.** Let $\nabla'$ be the connection with parallel totally skew-symmetric torsion and Kähler curvature tensor on the quasi-Kähler manifold with Norden metric $(M, J, g)$. Then

$$R(x, y, Jz, Jw) + R(x, y, z, w) = g(Q(x, y), (\nabla Jz)w + (\nabla w)Jz).$$

□

Contracting by $x = e_i$, $w = e_j$ in (4.5), we obtain

$$g^{ij}R(e_i, y, Jz, Je_j) + \rho(y, z) = g^{ij}g(Q(e_i, y), (\nabla Jz)e_j + (\nabla e_j)Je_s),$$

where $\tau^{**} = g^{ij}g^{ks}R(e_i, e_k, Je_s, Je_j)$.

From (2.14), (2.3) and (2.4) we have

$$g^{ij}g^{ks}g(Q(e_i, e_k), (\nabla Je_s)Je_j + (\nabla e_j)Je_s) = -\frac{1}{8}\|\nabla J\|^2.$$

Then (4.6) can be rewritten as

$$\tau^{**} + \tau = -\frac{1}{8}\|\nabla J\|^2.$$  

On the other hand, according to [6], we have

$$\tau^{**} + \tau = -\frac{1}{2}\|\nabla J\|^2.$$ 

Then $\|\nabla J\|^2 = 0$ and therefore the following is valid.

**Theorem 4.4.** Let $(M, J, g)$ be a quasi-Kähler manifold with Norden metric which admit a connection with parallel totally skew-symmetric torsion and Kähler curvature tensor. Then $(M, J, g)$ is a isotropic Kähler manifold with Norden metric. □

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