Low energy effective theory of Fermi surface coupled with U(1) gauge field in 2+1 dimensions

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We study the low energy effective theory for a non-Fermi liquid state in 2+1 dimensions, where a transverse U(1) gauge field is coupled with a patch of Fermi surface with $N$ flavors of fermion in the large $N$ limit. In the low energy limit, quantum corrections are classified according to the genus of the 2d surface on which Feynman diagrams can be drawn without a crossing in a double line representation, and all planar diagrams are important in the leading order. The emerging theory has the similar structure to the four dimensional SU($N$) gauge theory in the large $N$ limit. Because of strong quantum fluctuations caused by the abundant low energy excitations near the Fermi surface, low energy fermions remain strongly coupled even in the large $N$ limit. As a result, there are infinitely many quantum corrections that contribute to the leading frequency dependence of the Green’s function of fermion on the Fermi surface. On the contrary, the boson self energy is not modified beyond the one-loop level and the theory is stable in the large $N$ limit. The non-perturbative nature of the theory also shows up in correlation functions of gauge invariant operators.

I. INTRODUCTION.

Understanding non-Fermi liquid states is one of the central problems in condensed matter physics. One way of obtaining non-Fermi liquids is to couple a Fermi surface with a massless boson. If a massless boson is associated with criticality achieved by fine tuning of microscopic parameters, a non-Fermi liquid state arises at a quantum critical point. Such non-Fermi liquid states have been observed in heavy fermion compounds near magnetic quantum critical points. On the other hand, a boson can be dynamically tuned to be massless, and a non-Fermi liquid state may occur within a finite parameter space. The latter case may arise in the half filled Landau level of quantum Hall systems and some spin liquid states.

Non-Fermi liquid states in 2+1 dimensions are of particular interest. Experimentally, high temperature superconductors which exhibit non-Fermi liquid behaviors in the strange metallic normal state are quasi two-dimensional. There also exist two dimensional frustrated magnet whose ground states may be related to a non-Fermi liquid state of fermionic spinons which carry only spin half but no charge. In the spin liquid state, fermionic spinons form a Fermi surface which is minimally coupled with an emergent U(1) gauge field. The transverse component of the gauge field remains gapless at low energies because it is not fully screened by particle-hole excitations. The long-range interaction mediated by the transverse gauge field leads to a non-Fermi liquid state.

The same low energy effective theory can arise in various microscopic models, such as frustrated boson systems. Fermi surface of spinless charged fermion coupled with U(1) gauge field has been also proposed for underdoped cuprates. On the theoretical side, two space dimension is special in that it is high enough to have an extended Fermi surface, while it is low enough to support strong quantum fluctuations in the low energy limit. As a result of strong quantum fluctuations and infinitely many gapless excitations on an extended Fermi surface, it is expected that a non-trivial interacting quantum field theory, which is very different from relativistic quantum field theories, can emerge in the low energy limit. Even if the non-Fermi liquid state turns out to be unstable against other more conventional states, the physics within a significant temperature range will be inherited from the unstable non-Fermi liquid state, and it is still important to understand the parent non-Fermi liquid state.

Considerable studies have been devoted to the low energy effective theory of Fermi surface coupled with U(1) gauge field in 2+1D. In this system, there is no controllable parameter other than the number of fermion flavors $N$. Therefore it is natural to attempt to develop a perturbative expansion in terms of $1/N$ ($N$) in the large (small) $N$ limit. Based on the computation of some leading order diagrams, it has been suggested that the $1/N$ expansion is well defined and the low energy limit is described by a stable interacting theory in the large $N$ limit. The purpose of this paper is to study the low energy effective theory of the non-Fermi liquid state more systematically in the large $N$ limit. The key result of the paper is that the theory is not in a perturbative regime even in the large $N$ limit because there are infinitely many leading order quantum corrections for vertex functions of fermions residing on the Fermi surface. This conclusion has been reached by a systematic classification of quantum corrections in the $1/N$ expansion. Strong quantum fluctuations associated with the infinitely many gapless excitations and the absence of the Lorentz symmetry make the classification very different from relativistic quantum field theories. The theory remains strongly coupled in the low energy limit, and even the leading order quantum corrections cannot be summed in a closed Dyson equation which can be truncated with a finite number of vertex corrections.

The paper is organized in the following way. In Sec. II, we start by constructing a minimal local action (given in Eq. 5) that captures the universal low energy physics of the 2+1 dimensional non-Fermi liquid state. The minimal action is a renormalizable theory through which one can probe the universal low energy physics at any (finite) energy scale by sending all UV cut-off and cross-over scales to infinity. This makes the analysis of the low energy physics more transparent, because all non-universal elements of the theory have been stripped away from the minimal action. The peculiar property of the present non-relativistic theory with Fermi surface is that all local time derivative terms are irrelevant in the low energy limit. Nonetheless one can not completely drop
the time derivative terms from the bare action because if one does so, the theory will not have any dynamics. Therefore one needs to consider a special low energy limit to retain non-trivial dynamics while keeping only universal properties of the low energy theory. After we discuss the low energy limit of the minimal theory, in Sec. III we classify quantum corrections in the large $N$ limit. In Sec. III (a), we show that a naive $1/N$ expansion does not work because power of a Feynman graph in $N$ is enhanced in the low energy limit. This is due to strong quantum fluctuations enhanced by abundant gapless particle-hole excitations near the Fermi surface. More specifically, the leading frequency dependence of the fermion propagator is of the order of $1/N$. Due to the suppressed frequency dependence, magnitudes of Feynman diagrams are enhanced whenever there exists a channel for virtual particle-hole excitations to remain on the Fermi surface. Based on this observation, in Sec. III (b), we show that general Feynman diagrams are classified according to the genus of a 2d surface on which Feynman diagrams are drawn without a crossing in a double line representation. Here the double line representation is useful, not because gauge boson or fermions carry a doubled quantum number, but because it allows one to easily count in how many ways particle-hole excitations can remain on the Fermi surface. In particular, when a fermion is on the Fermi surface, the fermion propagator is enhanced to the order of $N$ due to the suppressed frequency dependence. If there are $n$ closed single line loops in the double line representation, all virtual particle-hole excitations can remain right on the Fermi surface no matter what $n$ components of internal momenta are. The abundant low lying excitations give rise to an enhancement factor with a positive power of $N$ in proportion to the number of independent channels via particle-hole excitations remain on the Fermi surface. Due to the enhancement factor, there exist infinitely many leading order quantum corrections for vertex functions of fermions on the Fermi surface. In particular, one has to sum over infinitely many planar diagrams to compute the leading frequency dependence of the fermion propagator. Although fermions on the Fermi surface are strongly coupled, the boson propagator is not modified beyond the one-loop level in the large $N$ limit due to a kinematical constraint.

The genus expansion of Feynman diagrams in the present non-Fermi liquid state is very similar to that of the four dimensional SU(N) gauge theory in the large $N$ limit. In both theories, strong quantum fluctuations make all planar diagrams to contribute to the quantum effective action in the leading order of the $1/N$ expansion. However, the physical origins for strong quantum fluctuations are very different between the two theories. In the SU(N) gauge theory, it is due to fluctuations of color degrees of freedom in the internal space, while in the present theory, it is due to fluctuations of the extended Fermi surface in the momentum space.

In Sec. IV, we study the dynamical properties of the theory in the large $N$ limit. It is shown that there is no UV divergence in individual planar diagrams. As a result, the theory is stable, and there is no quantum correction to the scaling dimension of fermion beyond one-loop order if the summation of individually finite planar diagrams are finite. In Sec. V, we discuss how the non-perturbative nature of the theory manifests itself in correlation functions of a gauge invariant operator.

II. MINIMAL THEORY AND LOW ENERGY LIMIT

A. Minimal local action

We consider $N$ flavors of fermion with Fermi surface coupled with a U(1) gauge boson in 2+1D. In the low energy limit, fermions whose velocities are not parallel or anti-parallel to each other are essentially decoupled because 1) fermions are strongly coupled only with the boson whose momentum is perpendicular to the Fermi velocity for a kinematic constraint, and 2) the angle that parametrizes Fermi surface acquires a positive anomalous scaling dimension, becoming a decom pactified variable which runs from $-\infty$ to $\infty$ in the low energy limit. As a result, two fermions which have different Fermi velocities can not interact with each other through any finite number of scatterings with the boson in the low energy limit. Therefore, in the low energy effective theory, it is justified to focus fermionic excitations locally in the momentum space. In general, one has to consider all patches in which Fermi velocities are parallel or anti-parallel to each other because all of them are strongly coupled with the boson in the same momentum region.

In this paper, we will focus on low energy fermions near one patch in the momentum space. As we will see, understanding low energy dynamics in this simplified case is already non-trivial. At the end, we will comment on the applicability of this restricted theory, and an extension to general cases which include other patches with opposite Fermi velocity. We consider the Lagrangian density:

$$\mathcal{L} = \sum_j \bar{\psi}_j (\partial_\tau - iv_x \partial_x - v_y \partial_y) \psi_j + \frac{e}{\sqrt{N}} \sum_j \alpha \bar{\psi}_j \psi_j + a \left[ -\partial^2 - \partial_x^2 - \partial_y^2 \right] a,$$

where $\psi_j$ is the fermion of flavor $j = 1,2,\ldots,N$. We have chosen the Fermi velocity to be along the $x$-direction at $k = 0$. $v_x$ is the Fermi velocity and $v_y \sim \frac{1}{m}$ determines the curvature.

![FIG. 1: The parabolic Fermi surface of the model in Eq. (1). The shaded region includes negative energy states.](image-url)
of the Fermi surface. The Fermi surface is on \( v_y k_x + v_y k_y^2 = 0 \) as is shown in Fig. 1. This is a ‘chiral Fermi surface’ where the x-component of Fermi velocity is always positive. \( a \) is the transverse component of an emergent U(1) gauge boson in the Coulomb gauge \( \nabla \cdot a = 0 \). We ignore the temporal component of the gauge field which is screened to a short range interaction. The transverse gauge field is massless without a fine tuning due to an emergent U(1) symmetry associated with the dynamical suppression of instantons. \( e \) is the coupling between fermions and the critical boson.

In the one-loop order, singular self energies are generated from the diagrams in Figs. 2 and 3, and the quantum effective action becomes

\[
\Gamma = \sum_j \int dk \left[ \frac{c}{N} \text{sgn}(k_0) k_0^{2/3} + i k_0 \right.
+ v_x k_x + v_y k_y^2 \psi_j^* (k) \psi_j (k) \\
+ \int dk \left[ \frac{c}{N} k_0 \text{sgn}(k_0) + k_0^2 + k_y^2 \right] a^* (k) a (k) \\
+ \frac{e}{\sqrt{N}} \sum_j \int dq \ a(q) \psi_j^* (k + q) \psi_j (k),
\]

where \( c \) and \( \gamma \) are constants of the order of 1. To compute the fermion self energy, the dressed boson propagator has been used because the boson self energy is of the order of 1. In the low energy limit, the leading terms of the quantum effective action are invariant under the scale transformation, \( k_0 = b^{-1} k_0', k_x = b^{-2/3} k_x', k_y = b^{-1/3} k_y' \).

\[
\psi_a (b^{-1} k_0', b^{-2/3} k_x', b^{-1/3} k_y') = b^{1/3} \psi_a (k_0, k_x', k_y'), \\
a (b^{-1} k_0', b^{-2/3} k_x', b^{-1/3} k_y') = b^{1/3} a (k_0', k_x', k_y').
\]

The singular self energies render the terms,

\[
i k_0 \psi_j^* (k) \psi_j (k), \\
\left[ k_0^2 + k_y^2 \right] a^* (k) a (k)
\]

irrelevant in the low energy limit. Usually, it is expected that one restores the same low energy quantum effective action if one drops the irrelevant terms from the beginning. However, this is not true in this case. If one drops the irrelevant terms in the bare action, then the resulting theory becomes completely localized in time and one can not have a propagating mode. If there is no frequency dependence in the bare action, the frequency dependent singular self energies can not be generated either. Therefore, to restore the full low energy dynamics, one has to keep a minimal information that the theory is not completely localized in time. It turns out that the following action given by

\[
\mathcal{L} = \sum_j \psi_j^* (\eta \partial_x - i v_x \partial_y - v_y \partial_y^2) \psi_j \\
+ \frac{e}{\sqrt{N}} \sum_j a \psi_j^* \psi_j + a (- \partial_y^2) a,
\]

is the minimal local theory which restores the one-loop quantum effective action \( \langle \text{2} \rangle \). Here \( \eta \) is a parameter which has the dimension \(-1/3\) according to the scaling \( \langle \text{3} \rangle \).

Since the time derivative term is irrelevant, \( \eta \) will flow to zero in the low energy limit, and the bare value of \( \eta \) does not affect any low energy physics as far as it is nonzero. The role of the nonzero \( \eta \) is to give a non-trivial frequency dependent dynamics by maintaining the minimal causal structure of the theory before it dies off in the low energy limit. For example, in the computation of the one-loop boson self energy (Fig. 2) in

\[
\Pi (q) = e^2 \int d^3k \frac{1}{i \eta (k_0 + q_0) + v_x (k_x + q_x) + v_y (k_y + q_y)^2} \\
\frac{1}{i \eta k_0 + v_x k_x + v_y k_y^2} = \frac{\left| q_0 \right|}{\left| q_y \right|},
\]

the sign of \( \eta \) contains the information on whether the pole is on the upper or lower side in the complex plane for the \( k_x \)
integration. The final result is independent of \( \eta \). As far as the "topological" information on the location of poles is kept for the fermions, it generates the correct frequency dependent self energy for the boson in Eq. (6). Therefore we can completely drop the time derivative term of the boson in the bare action \( \mathcal{A}_0 \). The boson self energy, in turn, generates the frequency dependent fermion self energy through Fig. 5.

B. Low energy limit and large \( N \) limit

At low energy \( k_0 \ll E_\eta \) with \( E_\eta = (N \eta)^{-3} \), the dynamically generated fermion self energy is dominant over the bare term \( i\eta k_0 \). Here \( E_\eta \) is a cross-over energy scale below which physics is described by the scale-invariant universal theory. To study the low energy physics, we will fix our energy scale \( E \ll \Lambda \). The boson self energy, in turn, generates the correct frequency dependent self energy through Fig. 5.

\[
E \ll \Lambda \ll E_\eta. \tag{7}
\]

First, a Feynman diagram with an external energy \( E \) is computed with finite \( \eta \), \( \Lambda \) and \( N \). To maintain (7), we take the \( \eta \to 0 \) limit first and then \( \Lambda \to \infty \) limit later. Finally, we take the large \( N \) limit. This amounts to imposing the condition (7) for all \( N \) as \( N \) is progressively increased in the large \( N \) limit. In this way, we can keep the bare time derivative term to be always smaller compared to the singular self energy at all energy scales. In this limit, not only the IR physics, but also the UV physics is controlled by the same universal theory. This is particularly convenient to study universal low energy dynamics of the theory at the critical dimension which is \( D_c \) in this case. This is because any logarithmic IR divergence is reflected to a UV divergence, and one can read the renormalization group flow by keeping track of UV divergences. We will exploit this property to study dynamical properties of the theory in Sec. IV.

The action (5) has four terms which are marginal at the one-loop level. On the other hand, there are five parameters that set the scales of energy-momentum and the fields. Out of the five parameters, only four of them can modify the coefficients of the marginal terms because the marginal terms remain invariant under the transformation (3). Using the remaining four parameters, one can always rescale the coefficients of the marginal terms to arbitrary values. Therefore, there is no dimensionless parameter in this theory except for the fermion flavor \( N \). In the following, we will set \( v_x = v_y = c \). With this choice, \( c \) and \( \gamma \) in Eq. (2) are automatically of the order of 1. The coefficients of the non-local terms are not independent tunable parameters because those parameters are completely determined from the local terms.

III. \( 1/N \) expansion

A. Failure of a perturbative \( 1/N \) expansion

In the naive counting of power in \( 1/N \), a vertex contributes \( N^{-1/2} \) and a fermion loop contributes \( N^1 \). In this counting, only the fermion RPA diagram (Fig. 2) is of the order of 1, and all other diagrams are of higher order in \( 1/N \). In the leading order, the propagators become

\[
g_0(k) = \frac{1}{i\eta k_0 + k_x + k_y^2},
\]

\[
D(k) = \frac{1}{\gamma [k_0] + k_y^2}. \tag{8}\]

One can attempt to compute the full quantum effective action by including \( 1/N \) corrections perturbatively. However, we will see that this naive \( 1/N \) expansion breaks down in the low energy limit. To see this, let us consider a two-loop vertex correction shown in Fig. 4.

\[
\Gamma(p, p + q) = -N^{-3/2} \int dk dl \ g_0(k)g_0(k + q)g_0(k + l)g_0(p + l)D(l)D(l - q). \tag{9}\]

\[
\Gamma(0, q) = -N^{-3/2} \int d l_0 d l_y d k_0 \ F(l_0, l_y, k_0, q_0, q_y) \frac{1}{l_y q_0 + i\eta q_y q_0} \tag{10}\]

where

\[
F(l_0, l_y, k_0, q_0, q_y) = 4\pi^2 \delta[q_x + q_y] \Theta(l_0 + k_0) - \Theta(l_0) \times \left[ \Theta(k_0 + q_0) - \Theta(k_0) \right] \left[ \Theta(q_y - l_y) - \Theta(q_y) \right] D(l)D(l - q) \tag{11}\]

is a function which is independent of \( \eta \) and \( N \), with \( \Theta(x) \) being a step function, and \( \delta_q = q_x + q_y^2 \) is the 'distance' of \( \mathbf{q} \).
from the Fermi surface. If the final momentum of the fermion is also on the Fermi surface, that is, \( \delta_q = 0 \), the vertex correction becomes

\[
\Gamma(0, q) = -\frac{N^{-3/2}}{\eta_0^{3/3}} f_1(q_{y/g_0}^{1/3}),
\]

where \( f_1(t) \) is a non-singular universal function which is independent of \( N \) and \( \eta \),

\[
f_1(t) = 4\pi^3 \int_{-1}^{0} dx \int_{0}^{x_{|x|}} dy \int_{1}^{\infty} dz \frac{t^2(z - 1)}{(\gamma y + (tz)^3)(\gamma(1 + y) + t^3(z - 1)^3)}.
\]

In the \( \eta \to 0 \) limit, this two-loop vertex correction which connects two fermions on the Fermi surface diverges. This divergence is quite generic: a vertex function which connects fermions on the Fermi surface diverges as \( 1/\eta^n \) for some integer \( n \) in general. The physical reason for this divergence is simple. In the \( \eta \to 0 \) limit, the bare fermion propagator is independent of frequency and the integration over frequencies is ill-defined. This divergence is unphysical in the sense that it disappears once the frequency dependent fermion self energy correction is included. If one includes the one-loop fermion self energy (Fig. 3), the dressed fermion propagator becomes

\[
g(k) = \frac{1}{i \eta k_0 + i \frac{N}{2} \text{sgn}(k_0)|k_0|^{2/3} + k_x + k_y^2},
\]

and the \( 1/\eta \) divergence disappears. Instead, the resulting finite term becomes enhanced by a factor of \( N^n \) for some integer \( n \geq 0 \), because the zero in the denominator (in the \( \eta \to 0 \) limit) is replaced by a term which is proportional to \( 1/N \). As a result, the two-loop vertex correction shown in Fig. 5 becomes

\[
\Gamma(0, q) = -N^{-1/2} f_2(q_{y/g_0}^{1/3}),
\]

where \( f_2(t) \) is a non-singular universal function which is independent of \( N \) and \( \eta \).

\[
f_2(t) = \frac{4\pi^3}{c} \int_{-1}^{0} dx \int_{0}^{x_{|x|}} dy \int_{1}^{\infty} dz \frac{1}{|x + 1|^{2/3} + y^{2/3} + |x + y|^{2/3} + (z - 1)(|x + 1|^{2/3} + |x|^{2/3})} \times \frac{t^2z(z - 1)}{(\gamma y + (tz)^3)(\gamma(1 + y) + t^3(z - 1)^3)}.
\]

The additional factor of \( N \) is from the enhancement factor that arises due to the \( 1/\eta \) divergence when the fermions are on the Fermi surface. With the inclusion of the fermion self energy, the IR divergence in Eq. [12] has been traded with an enhanced power in \( N \) in Eq. [15].

\[
\Sigma(p) = -\frac{c_3}{N} \text{sgn}(p_0)|p_0|^{2/3},
\]

when the external fermion is on the Fermi surface. Here \( c_3 \) is a universal constant of the order of 1.

This discrepancy between the cases with a finite \( \eta \) and an infinitesimally small \( \eta \) can be understood in the following way. With a finite \( \eta \), there is a crossover around the scale \( q_0 \sim E_\eta \). For \( q_0 >> E_\eta \), the \( i\eta k_0 \) is dominant in the fermion propagator and a Feynman diagram obeys the naive counting in \( 1/N \). On the other hand, for \( q_0 << E_\eta \), quantum fluctuations are controlled by the non-local term which is suppressed by \( 1/N \). The enhanced quantum fluctuations at low energies lead to an enhancement factor by a positive power in \( N \). Since we are concerned about the low energy physics, we should consider the latter limit. This correct low energy limit is automatically taken by considering the \( \eta \to 0 \) limit with a fixed energy scale \( q_0 \). This enhancement in the power of \( N \) at IR is a manifestation of the fact that quantum fluctuations become stronger at low energies.

FIG. 5: A three-loop fermion self energy correction.

Similar enhancement factors arise in other diagrams as well. For example, a three-loop fermion self energy correction shown in Fig. [5] is of the order of \( N^{-2} \) according to the naive counting. However, the self energy of fermion on the Fermi surface (\( \delta_p = 0 \)) diverges as \( 1/\eta \) in the \( \eta \to 0 \) limit if the bare fermion propagator in Eq. [8] is used. If one includes the one-loop self energy of fermion, it becomes of the order \( N^{-1} \).
B. Genus Expansion

In the low energy limit, what determines the power of a Feynman diagram in $1/N$? To answer this question, one should understand the origin of the enhancement factor discussed in the previous section more systematically. In the present section, we will develop a simple geometrical way of determining power of general Feynman graphs.

First, we illustrate the basic idea using the example (Fig. 4) considered in the previous section. As we have seen in the previous section, the enhancement factor $N$ is a consequence of the $1/N$ singularity in the $\eta \to 0$ limit. To understand the origin of the $1/N$ singularity, it is useful to examine the way fermions are scattered near the Fermi surface. Suppose both $p$ and $p + q$ are on the Fermi surface in Fig. 4. In the fermion loop with running momentum $k$, the momentum of the fermion consecutively becomes $k, (k + q), (k + l)$ as a result of scatters. For a given external momentum $q$ of the boson, one can always choose the spatial momentum $k$ to make both $k$ and $k + q$ to be on the Fermi surface. There is one unique choice, $k = p$. To make the next momentum $k + l$ to be on the Fermi surface as well, one needs to tune only $l_x$ and there is one remaining free parameter, $l_y$. This is because the Fermi surface has dimension one. As a result, all internal fermions can remain right on the Fermi surface during the scattering process no matter what the values of one momentum component ($l_y$) is if the other three momentum components ($k_x$, $k_y$, $l_z$) are finely tuned. This implies that all four fermion propagators are singular at $\eta = 0$ in an one-dimensional manifold which is embedded in the four dimensional space $k, l$. We refer to this manifold as a ‘singular manifold’. The co-dimension of the singular manifold is $4 - 1 = 3$, and there are only $3$-dimensional integrations which contribute to the phase space volume and cancel the IR divergence. Since the product of the propagators has a singularity whose strength is $4$, the integration over the $3$ parameters can not completely remove the singularity and the remaining singularity becomes of the order of $4 - 3 = 1$. This explains why Fig. 4 has the singularity of $1/N$ when the bare fermion propagator is used, and why it has the enhancement factor $N$ when the one-loop dressed propagator is used. The enhancement factor $N$ for the fermion self energy in Fig. 5 can be understood in the similar way.

What determines the dimension of the singular manifold within which fermions always remain on the Fermi surface? It turns out that the dimension of the singular manifold is given by the number of closed loops when one draws boson propagators using double lines and fermion propagators using single lines. This can be shown by following the scheme used to prove the Migdal’s theorem in the electron-phonon system\(^{3,26}\). First, we restrict momenta of all fermions to be on the Fermi surface. A momentum $k_0$ of fermion on the Fermi surface is represented by an one-dimensional parameter $\theta$. Then, a momentum of the boson $q$ is decomposed into two momenta on the Fermi surface as $q = k_0 - k_0'$, where both $k_0$ and $k_0'$ are on the Fermi surface. This decomposition is unique because there is only one way of choosing such $k_0$ and $k_0'$ near $k = 0$. As far as momentum conservation is concerned, one can view the boson of momentum $q$ as a composite particle made of a fermion of momentum $k_0$ and a hole of momentum $k_0'$. For example, the two-loop vertex correction in Fig. 4 and the three-loop fermion self energy correction in Fig. 5 can be drawn as Fig. 6(a) and (b) respectively in this double line representation. In this representation, each single line represents a momentum on the Fermi surface. Momenta in the single lines that are connected to the external lines should be uniquely fixed to make all fermions stay on the Fermi surface. On the other hand, momenta on the single lines that form closed loops by themselves are unfixed. In other words, all fermions can stay on the Fermi surface no matter what the value of the unfixed momentum component that runs through the closed loop is. Since there is one closed loop in Fig. 6(a), the dimension of the singular manifold is $1$ and the enhancement factor becomes $N^{4 - (4 - 1)} = N$ for the two-loop vertex correction in Fig. 4. In Fig. 6(b), there are two closed loop, and the enhancement factor for Fig. 5 becomes $N^{4 - (6 - 2)} = N$; there are five fermion propagators, six spatial components of internal momenta and two closed loops.

The enhancement factor is a direct consequence of the presence of infinitely many soft modes associated with deformations of the Fermi surface. The extended Fermi surface makes it possible for virtual particle-hole excitations to maneuver on the Fermi surface without costing much energy. As a result, quantum fluctuations becomes strong when external momenta are arranged in such a way that there are sufficiently many channels for the virtual particle-hole excitations to remain on the Fermi surface. This makes higher order processes to be important even in the large $N$ limit. We note that this effect is absent in relativistic quantum field theories where gapless modes exist only at discrete points in the momentum space.

Now we are ready to write down a general formula which tells order of a general Feynman diagram in $1/N$.

- First, draw a Feynman diagram using single lines for fermion propagators and using double lines for boson propagators.
- Second, each vertex contributes $1/\sqrt{N}$.
- Third, each fermion loop contributes $N$. 

![Fig. 6: The double line representations of Fig. 4(a) and Fig. 5(b). Double lines represent propagators of the boson, and the single lines are the propagators of the fermion. The number of single line loops (one in (a) and two in (b)) represents the dimension of the singular manifold (see the text) on which all fermions remain on the Fermi surface in the space of internal momenta.](image)
Finally, the enhancement factor is given by

$$N^{[I_f - 2L + n]}.$$  \hspace{1cm} (18)

Here $I_f$ is the number of (internal) fermion propagators, $L$ is the number of loops (the number of internal momenta) and $n$ is the number of closed single line loops in the double line representation. $[x] = x$ if $x \geq 0$ and $[x] = 0$ if $x < 0$. This enhancement factor can be understood as was illustrated in the previous examples. When all fermions are on Fermi surface, the product of propagators has the singularity of the order of $I_f$. Upon integrating over the internal momenta, the singularity is lowered due to the contribution from phase space volume. There are $2L$ components of spatial momenta, but $n$ of them are degenerate in that all fermions remain on the Fermi surface no matter what the values of the $n$ momentum components are as far as the remaining $2L - n$ components are zero. Therefore, the integration over the internal momenta can remove the singularity only by the power of $(2L - n)$. The power of the remaining IR divergence is $I_f - (2L - n)$ and this results in the enhancement factor, $N^{(I_f - 2L + n)}$. If $(2L - n) > I_f$, the suppression from the phase space of internal momenta is more than enough to suppress the whole singularity, and there is no enhancement factor. In this case, the enhancement factor should be 1, not a negative power of $N$. That is why we use $[I_f - 2L + n]$ which is 0 if $I_f < (2L - n)$. By using the relation between $L$ and $I_f$, $I_f = 2L + \frac{E_f + 2E_b}{2} - 2$, where $E_f$ ($E_b$) is the number of external fermion (boson) lines, one can write the enhancement factor as

$$N^{n + \frac{E_f + 2E_b}{2} - 2}.$$ \hspace{1cm} (19)

As a result, the net order of a Feynman diagram is given by $N^Q$ with

$$Q = \frac{V}{2} + L_f + [n + \frac{E_f + 2E_b}{2} - 2],$$ \hspace{1cm} (20)

where $V$ is the number of vertices and $L_f$ is the number of fermion loops.

Now let us classify Feynman diagrams based on the expression Eq. (20), starting from vacuum diagrams. Classification of non-vacuum diagrams with external lines naturally follows from that of vacuum diagrams, as will be shown shortly. The leading order vacuum diagram is the one fermion loop diagram which is of the order of $N$. In the next order of $N^0$, there are infinitely many diagrams. A typical diagram of the order of $N^0$ is shown in Fig. 7. For the diagram in Fig. 7, we have $V = 38$, $n = 15$, $E_f = E_g = 0$ and $L_f = 6$, which gives

$$Q = -19 + 6 + [15 - 2] = 0.$$ \hspace{1cm} (21)

Actually, there is a simple geometrical way of interpreting the result. First, we turn fermion propagators into double lines as well by drawing additional single line loops for each fermion loop as in Fig. 8. In this way, we can include the factor $N^{L_f}$ from fermion loops by counting the additional closed loops of single lines. We will refer to this way of drawing a ‘full double line representation’. If $n \geq 2$, which is always the case for sufficiently large $V$ if there are not too many crossings, we can remove the bracket in Eq. (20) and the power can be rewritten as

$$Q = V - I + F - 2.$$ \hspace{1cm} (22)

Here we use the identity $3V = 2I$, where $I$ is the number of total internal propagators, $F = n + L_f$ is the total number of single line loops including the additional single line loops added to each fermion loop. In this full double line representation, one can think of a closed 2d surface formed by joining the patches of single line loops. The 2d surface is the...
surface on which a Feynman diagram can be drawn without any crossing in the full double line representation. The factor $\chi = V - I + F$ is nothing but the Euler number of the 2d closed surface and the power $Q$ becomes

$$Q = -2g,$$  \hspace{0.5cm} (23)

where $g$ is the genus of the 2d surface. The diagrams of the $N^0$ order are the planar diagrams which can be drawn on a sphere.

For non-planar diagrams, such as the one shown in Fig. 9, one has to introduce closed surfaces with handles to draw them in the full double line representation without a crossing. This is illustrated in Fig. 10. Contributions from non-planar diagrams are suppressed as the number of genus increases according to Eq. (23).

Power counting of diagrams with external lines can be easily obtained from the counting of vacuum diagrams. To create a boson self energy diagram, one attaches two vertices to fermion propagators. The leading order self energy diagrams can be generated if two vertices are attached to fermion propagators which are parts of one single line loop. A typical leading order boson self energy diagram created from a planar vacuum diagram is shown in Fig. 11. From this procedure, the boson self energy diagram acquires the additional power of $N^{-1}$ from two added vertices ($\Delta V = 2$), $N^2$ from two external boson lines ($\Delta E_b = 2$) and $N^{-1}$ from a lost single line closed loop ($\Delta n = -1$). As a result, the resulting boson self energy diagram has the same power as the parent vacuum diagram which is of the order of $N^0$ for planar diagrams. It is noted that the one-loop boson self energy diagram (Fig. 2), which can be created by attaching two vertices to the vacuum diagram of the order of $N$, is also order of $N^0$. Therefore there are infinitely many planar diagrams which contribute to the boson self energy in the leading order of $N^0$. If one at-
taches two vertices to two fermion propagators which belong to different single line loops, or if one starts from a non-planar vacuum diagram, the resulting boson self energy is down by an additional factor of $1/N$.

Fermion self energy diagrams can be created by cutting a fermion propagator as in Fig. 12. This procedure causes an additional power of $N^{-1}$ from one less single line loop ($\Delta n = -1$), $N^1$ from two external fermion lines ($\Delta E_f = 2$), and $N^{-1}$ from one less fermion loop ($\Delta L_f = -1$). As a result, the fermion self energy diagram is down by $N^{-1}$ from the vacuum diagram. Therefore the leading order fermion self energy corrections are of the order of $N^{-1}$. There are infinitely many planar diagrams that contribute to the leading frequency dependence of the fermion propagator which is of the order of $N^{-1}$.

Leading order three-point vertex functions can be created by cutting a fermion propagator and attaching a vertex to another fermion propagator as in Fig. 13. The resulting diagram is of the order of $N^{-1/2}$: an additional power of $N^{-1/2}$ from $\Delta n = -1$, $N^1$ from $\Delta n = -1$, $N^2$ from $\Delta E_f = 2$ and $\Delta E_b = 1$, and $N^{-1}$ from $\Delta L_f = -1$. All planar vertex corrections are of the same order as the bare vertex.

For the fermion self energy and the 3-point vertex function, the above counting is valid only when the external fermion momenta are sufficiently close to the Fermi surface. If external momenta are far from the Fermi surface, one can not make all internal fermions to be on the Fermi surface without an additional tuning of internal momenta. As a result, the enhancement factor becomes smaller than what is predicted for the case when external fermions are on the Fermi surface. To be more precise, Eq. (20) is valid if

$$|\delta_k| \ll \frac{|k_0|^{2/3}}{N},$$

where $k$ is the momentum of the external fermion. In the opposite limit, $|\delta_k| \gg \frac{|k_0|^{2/3}}{N}$, there are additional factors in $1/N$.

On the other hand, for the boson self energy, Eq. (20) is always valid, irrespective of the energy and momentum of the external boson. This is because any boson momentum can be decomposed into two momenta on the Fermi surface and one can use the double line representation, which leads to the counting in Eq. (20).

The leading contributions come from the planar diagrams where the genus of the underlying 2d surface is zero. In principle, there can be infinitely many diagrams which are order of $N^0 (N^{-1})$ for the boson (fermion) self energy and $N^{-1/2}$ for the three-point vertex function. In particular, the boson self energy has infinitely many leading order terms at any external momentum. This would have made the one-loop boson propagator unreliable even in the large $N$ limit. However, it turns out that all planar diagrams for the boson self energy correction beyond the one-loop level vanish due to a kinematical reason. To see this, let us consider a three-loop boson self energy correction shown in Fig. 14.

\[
\Pi(q) = -N^{-1} \int dk_1dk_2dl \ g(k_1)g(k_1+q)g(k_1+l) \\
g(k_2)g(k_2+q)g(k_2+l)D(l)D(1-q). \quad (25)
\]

This diagram is nominally of the order of $N^0$ due to an enhancement factor $N^{-1}$. Integrating over $\delta_1$, one obtains

\[
\Pi(q) \sim \int d\delta_{k_1}d\delta_{k_2}dk_{1y}dk_{2y}dk_{10}dk_{20}dl_0dl_y D(l)D(l-q) \times \\
\frac{1}{\delta_{k_1} + i\{k_{10}\}} \frac{1}{\delta_{k_2} + \delta_y + 2k_{1y}q_y + i\{k_{10} + q_0\}} \times \\
\frac{1}{\delta_{k_2} + i\{k_{20}\}} \frac{1}{\delta_{k_2} + \delta_y + 2k_{2y}q_y + i\{k_{20} + q_0\}} \times \\
\Theta(k_{10} + l_0) - \Theta(k_{20} + l_0) \\
\frac{\Theta(k_{10} + l_0) - \Theta(k_{20} + l_0) - \{k_{20} + l_0\}}{\{k_{10} + l_0\} - \{k_{20} + l_0\}} \\
\frac{1}{\delta_{k_1} - \delta_{k_2} + 2(k_{1y} - k_{2y})l_y + i\{\{k_{10} + l_0\} - \{k_{20} + l_0\}\}^3}. \quad (26)
\]

with $\delta_k = k_x + k_y^2$ and $\{x\} = \frac{1}{2\pi} x |x|^{-1/3}$. Now we change the integration variables as $k_{1y} = k$ and $k_{2y} = k + k'$. The integration over $k$ has poles at $i\{k_{10} + q_0\}/2q_y$ and $i\{k_{20} + q_0\}/2q_y$. The leading contributions come from the planar diagrams where the genus of the underlying 2d surface is zero. In principle, there can be infinitely many diagrams which are order of $N^0 (N^{-1})$ for the boson (fermion) self energy and $N^{-1/2}$ for the three-point vertex function. In particular, the boson self energy has infinitely many leading order terms at any external momentum. This would have made the one-loop boson propagator unreliable even in the large $N$ limit. However, it turns out that all planar diagrams for the boson self energy correction beyond the one-loop level vanish due to a kinematical reason. To see this, let us consider a three-loop boson self energy correction shown in Fig. 14.

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\Pi(q) \sim \int d\delta_{k_1}d\delta_{k_2}dk_{1y}dk_{2y}dk_{10}dk_{20}dl_0dl_y D(l)D(l-q) \times \\
\frac{1}{\delta_{k_1} + i\{k_{10}\}} \frac{1}{\delta_{k_2} + \delta_y + 2k_{1y}q_y + i\{k_{10} + q_0\}} \times \\
\frac{1}{\delta_{k_2} + i\{k_{20}\}} \frac{1}{\delta_{k_2} + \delta_y + 2k_{2y}q_y + i\{k_{20} + q_0\}} \times \\
\Theta(k_{10} + l_0) - \Theta(k_{20} + l_0) \\
\frac{\Theta(k_{10} + l_0) - \Theta(k_{20} + l_0) - \{k_{20} + l_0\}}{\{k_{10} + l_0\} - \{k_{20} + l_0\}} \\
\frac{1}{\delta_{k_1} - \delta_{k_2} + 2(k_{1y} - k_{2y})l_y + i\{\{k_{10} + l_0\} - \{k_{20} + l_0\}\}^3}. \quad (26)
\]
from Fig. Consider a general boson self energy diagram shown in not to vanish, the external frequency should be the largest of energy in the large plane. This can be seen from the fact that 

\[ |q_0| > \max(|k_{10}|, |k_{20}|, |l_0|). \]  

(27)

If (27) is not true, one of \(|k_{10}|, |k_{20}|, |l_0|\) should be the largest among all frequencies. If \(|k_{10}|\) is the largest, then the integration over \(\delta_{k_1}\) vanishes because all terms dependent on \(\delta_{k_1}\) have poles on the same side in the complex plane. This same argument applies to all other frequencies. Therefore (27) should be satisfied in order for the integrations for \(\delta_{k_1}, \delta_{k_2}\) and \(\delta_{l_1}\) not to vanish. Then \(i(k_{10} + q_0)/2q_y\) and \(i(k_{20} + q_0)/2q_y\) are on the same side in the complex plane, and the integration over \(k\) vanishes. This proves that the Eq. (25) vanishes.

One can easily generalize the previous argument to prove that all planar diagrams for boson self energy correction vanish. Consider a general boson self energy diagram shown in Fig. 11. One can arrange internal momenta so that one internal momentum runs within each single line loop as is shown in Fig. 15. Here \(k_i\)'s are momenta that run within fermion loops, and \(l_i\)'s are momenta that connects different fermion propagators which are not part of any fermion loops (for example the fermion propagator with momentum \(p + l\) in Fig. 4, and the integration over \(k\) (the uniform component of \(k_{ij}\)) can have poles in both sides in the complex plane due to contributions from the isolated fermion propagators. All planar fermion self energy diagrams are of the order of \(N^{-1}\), and planar vertex corrections are order of \(N^{-1/2}\), when external fermions are on the Fermi surface. Since the leading frequency dependence of the fermion propagator is of order of \(N^{-1}\), one has to sum over all planar diagrams to extract dynamical properties of low energy fermions. We note that the set of all planar diagrams is much larger than the set of rainbow diagrams that can be summed in a closed Dyson equation. The fact that the low energy dynamics of fermion is governed by the infinitely many planar diagrams implies that fermions on the Fermi surface remain strongly coupled even in the large \(N\) limit.

A few comments are in order for Eq. (20). First, the counting of power in \(1/N\) is self consistent in that we obtained Eq. (20) based on the assumption (suggested by the one-loop result) that the leading self energy corrections of boson and fermion are of the order of \(N^0\) and \(N^{-1}\) respectively. The fact that Eq. (20) predicts the same conclusion implies that the power counting will not change even though one uses the full propagators obtained by summing over all planar diagrams. Second, some non-planar diagrams may have a higher power in \(1/N\) than nominally predicted in Eq. (20) because they can vanish to the leading order due to an even-odd symmetry and there can be \(\log N\) correction. However, all planar diagrams obey Eq. (20) without \(\log N\) correction.

IV. STABILITY AND ANOMALOUS DIMENSION

The coupling \(e\) in Eq. (5) receives quantum corrections only from the boson self energy due to the Ward identity. The absence of non-vanishing planar diagrams for boson self energy beyond the one-loop level implies that the one-loop beta function is exact in the large \(N\) limit. Since the one-loop boson self energy has no divergence, the beta function is zero and the theory is stable in the large \(N\) limit. The fact that the one-loop result is exact is rather remarkable given that the theory remains strongly coupled even in the large \(N\) limit. This is consistent with an earlier two-loop calculation. The absence of higher order corrections is reminiscent of supersymmetric theories where certain properties are protected from higher-loop corrections due to the non-renormalization theorem.

In contrast to the boson, it is difficult to extract detailed dynamical properties of low energy fermions on the Fermi surface even in the large \(N\) limit because there are infinitely many planar diagrams to be considered. However, one can attempt to study the dynamics of fermion on a general ground. Here we will show that there is no UV divergence in all planar diagrams individually, if one uses the one-loop propagators for the computation of higher-order planar diagrams. Of course, the one-loop propagator is not reliable for fermion on the Fermi surface even in the large \(N\) limit. The present

FIG. 15: The assignment of one internal momentum to each closed single line loop in the boson self energy diagram in Fig. 11.
to survive, one should have

\[ [k_0] = 1, \ [k_x] = \frac{2}{3}, \ [k_y] = \frac{1}{3}. \] (28)

According to Eq. (3), the scaling dimension of \( k \) is given by

\[ D_s = 2L - \frac{2}{3}J = 2 \left(1 - \frac{E}{3}\right), \] (29)

where \( E \) is the number of external lines, \( L \) is the number of loops, and \( I \) is the number of internal propagators. Here we have used the relations, \( 3V = E + 2I \) and \( L = I - V + 1 \). There are three kinds of diagrams which are primitively divergent, that is, diagrams which have generic divergence at UV without divergent sub-diagrams. The primitively divergent diagrams are the 2-point vertex functions (self energies) with \( D_s = 2/3 \) and the 3-point vertex function (\( D_s = 0 \)) which may have power-law and logarithmic divergences, respectively.

Although the superficial degree of divergence suggests that these diagrams are potentially UV divergent, there is actually no divergence for planar diagrams which are dominant in the large \( N \) limit. This can be seen from an argument similar to the one that we used to prove that all planar boson self energy diagrams vanish. For planar diagrams, one can assign an internal momentum \( k_i \) for each single line loop, as we did in Fig. 15 (In contrast to Fig. 15 here we use \( k_i \) for all internal momenta to keep the notation for the following discussion simpler.) If external frequency is zero, the integration over \( \delta_{k_i} \) does not vanish only when

\[ |k_{i0}| < \max_{j\in\{k_{j0}\}}, \] (30)

where \( k_j \)'s are momenta of fermion propagators which are parts of the \( i \)-th loop. For example, for the integration of \( \delta_{k_{i2}} \) in Fig. 15 to survive, one should have

\[ |k_{20}| < \max(|l_{10}|, |l_{20}|, |l_{50}|, |l_{60}|), \] (31)

if \( q_0 = 0 \). These set of constraints can not be simultaneously satisfied for all \( i \)'s. If one of the internal frequencies, say \( k_{m0} \), is the largest, then the \( \delta_{k_{m0}} \) integration vanishes because all poles are on the same side in the complex plane. This implies that the volume of frequency integral vanishes when the external frequency vanishes. Since at least one frequency integral has a UV cut-off at an external frequency, the remaining integrations have a reduced degree of divergence which is at most \( 2/3 - 1 = -1/3 \). Therefore there is no UV divergence in all planar diagrams.

If individually finite planar diagrams can be summed to give a finite result, the theory is UV finite in the large \( N \) limit. This would imply that the scaling dimension of the fermion field given by Eq. (3) will not be modified by higher-loop diagrams in the large \( N \) limit. However, we emphasize that it is not clear whether the summation over planar diagrams is convergent or not. We believe that the true nature of this theory has not been fully understood.

V. CORRELATION FUNCTION OF GAUGE INVARIANT OPERATORS

Since the fermion operator is not gauge invariant, the fermion Green’s function can not be directly measured. Therefore it is of interest to study how correlation functions of gauge invariant operators are affected by the fact underlying fermions remain strongly coupled in the low energy limit. Here we consider correlation functions of the density operator, \( \rho(x) = \psi_j^\dagger(x)\psi_j(x) \). The two-point correlation function \( \langle \rho(q)\rho(-q) \rangle \) is proportional to the self energy of the gauge boson. As is shown in Sec. III, all planar boson self energy diagrams vanish except for the one-loop diagram. Therefore the density-density correlation function shows the usual Fermi liquid-like behavior in the large \( N \) limit. Higher order terms become important only for the \( n \)-point density correlation function with \( n \geq 3 \). Contrary to the two-point correlation function, higher order diagrams for the \( n \)-point function

\[ \text{FIG. 16: The one-loop diagram for the three-point density correlation function. The dots represent the density operators.} \]

\[ \text{FIG. 17: A set of external momenta for which the three-point density correlation function is enhanced. If external momenta are chosen so that all of them connect two points on the Fermi surface, it is possible that all internal fermions stay on the Fermi surface. This gives rise to the divergence } 1/\eta \text{ for the diagram in Fig. 16 if the bare fermion propagator is used. This can be understood following the same argument given in Sec. III. There are three internal propagators which can diverge when } \eta = 0. \text{ Since there are only two integrations for the spatial components of internal momenta, a linear divergence survives. Once the fermion self energy is included, the linear divergence is traded with an enhancement factor } N. \]
with \( n \geq 3 \) do not automatically vanish. Since there are more than one external frequency, the argument used in Sec. III to show that all planar boson self energy diagrams with more than one loop vanish does not apply. For example, let us consider the three-point density correlation function given by

\[
D_3(q_1, q_2) = \langle \rho(q_1) \rho(q_2) \rho(-q_1 - q_2) \rangle. \tag{32}
\]

For generic external momenta, the leading contribution is given by the one-loop diagram shown in Fig. 16 which is of the order of \( N \). However, for a set of external momenta which connect points on the Fermi surface, the magnitude of the diagram is enhanced. If the spatial components of the external momenta can be written as

\[
q_1 = k_2 - k_1, \\
q_2 = k_1 - k_3, \tag{33}
\]

where \( k_i \)'s are momenta residing on the Fermi surface, the diagram is enhanced to the order of \( N^2 \). For these special external momenta, there is a channel for all virtual particle-hole excitations to remain on the Fermi surface, and there is an enhancement factor \( N \). This is illustrated in Fig. 17. For these external momenta, there are infinitely many planar diagrams which are of the same order. For example, the diagram in Fig. 18 is of the order of \( N^0 \) for generic external momenta, but it is enhanced to the order of \( N^2 \) if external momenta satisfy Eq. \( \frac{33}{33} \), as explained in Fig. 19.

VI. DISCUSSIONS

A. Comparison with the SU(N) gauge theory

The genus expansion of the present theory is similar to that of the 3+1D SU(N) gauge theory in the large \( N \) limit. In the SU(N) gauge theory, all Feynman diagrams can be naturally drawn in the double line representation because the gauge field is in the adjoint representation. Although the physical origin is very different from the present case, non-planar diagrams with genus \( g \) are suppressed by the factor \( 1/N^{2g} \). One key difference from the present theory is that there is another dimensionless parameter called \('t Hooft coupling\( \lambda = N g_{YM}^2 \) in the SU(N) gauge theory, where \( g_{YM} \) is the gauge coupling. This allows for a double expansion of the theory in \( 1/N \) and \( \lambda \). For \( \lambda << 1 \), the theory is in the perturbative regime. In the large \( N \) limit with a large but fixed \('t Hooft coupling\( \lambda >> 1 \), planar diagrams with a large number of loops are dominant and the usual perturbative approach breaks down even in the large \( N \) limit. It has been suggested that a more weakly coupled effective description of the theory should be a string theory in an one higher dimensional \((4+1)D \) curved space. The AdS/CFT correspondence is a concrete conjecture of this kind for a supersymmetric SU(N) gauge theory. On the other hand, the present theory with Fermi surface has no \('t Hooft coupling\( that one can tune in addition to \( N \). To put it otherwise, the effective \('t Hooft coupling has been set to be of the order of 1. This is because there is no dimensionless parameter in the theory other than \( N \) as discussed earlier. With \( \lambda \sim 1 \), the theory is still strongly interacting, but it is most likely not in the regime where one can use a dual gravity description in a weakly curved space-time. It would be of great interest to find a more weakly coupled description, which is likely to be a gauge invariant description, for this non-Fermi liquid state.

B. Extension to multiple patches and applicability of the theory with one patch

In this paper, we have focused on low energy fermion on one side of Fermi surface (one patch). If Fermi surface is closed, one should consider multiple patches which include the opposite side of the Fermi surface because fermions whose Fermi velocities are parallel or anti-parallel with each other are all strongly coupled with the boson in the same momentum region. The theory which includes fermions with opposite

![Fig. 18](image1)

**Fig. 18:** A three-loop planar diagram which contributes to the leading order three-point density correlation function when external momenta are chosen so that all of them connect two points on the Fermi surface.

![Fig. 19](image2)

**Fig. 19:** If the external momenta in Fig. 18 satisfy Eq. 33, every external momentum can be decomposed into two momenta on the Fermi surface. This allows one to draw the diagram in the double line representation as is shown in this figure. There is one loop of solid single line in the double line representation, which implies that there is an one-dimensional singular manifold in the space of internal momenta within which all internal fermions stay on the Fermi surface. As a result, the order of the diagram is given by \( Q = -4/2 + 2 + [1 + 3 - 2] = 2 \), where we use \( V = 4, L_f = 2, n = 1, E_f = 0 \) and \( E_b = 3 \) in Eq. 20.
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FIG. 20: A planar diagram which include loops of fermions on both sides of the Fermi surface. The solid (dashed) lines represent
the propagator of fermions on one (the opposite) side of the Fermi surface. In order for fermions to remain on the Fermi surface, two
fermions propagators that face each other should run in the opposite (same) direction if the two fermions are on the same (opposite) side
of the Fermi surface.

Fermi velocities is given by

$$L = \sum_j \sum_{s=\pm} \psi_{js}^\dagger (\eta \partial_\tau - is \nu_{sx} \partial_x - \nu_{sy} \partial_y^2) \psi_{js}$$

$$+ \frac{e}{\sqrt{N}} \sum_{j,s} \alpha \psi_{js}^\dagger \psi_{js} + a(-\partial_y^2 a)$$

(34)

where $\psi_{js}$ ($\psi_{js}$) is the fermion whose velocity is parallel (anti-parallel) to the $x$-direction. It turns out that this theory is
more complicated. For example, vacuum diagrams which in-clude both $\psi_+$ and $\psi_-$ also contribute to planar diagrams. An example is shown in Fig. 20. Those diagrams have the same enhancement factor as those which involve fermionic loops only on one side of the Fermi surface, if the curvatures of the Fermi surfaces are the same, that is, $\nu_{+x}/\nu_{+y} = \nu_{-x}/\nu_{-y}$. A complication arises because there is no constraint on internal frequencies like Eq. (30) in the presence of fermions with opposite velocities. As a result, planar boson self energy diagrams do not vanish in general and there exist UV divergences which are absent in the one patch theory at least in the large N limit. One possible scenarios is that although there are UV divergences in individual diagrams, they cancel with each other and the coupling does not run. This scenario is consistent with the explicit two-loop calculation. If this is the case, we will obtain a similar picture as the one patch theory. The question on how to sum all planar diagrams still remains.

If the curvatures on the opposite sides of the Fermi surface do not match ($\nu_{+x}/\nu_{+y} \neq \nu_{-x}/\nu_{-y}$), diagrams which has mixed fermion loops like the one in Fig. 20 has smaller enhancement factor because all fermions can not stay on the Fermi surface because of the curvature mismatch. In this case one side of the Fermi surface which has a smaller curvature will be dominant and one can focus on one patch as we did in this paper.

VII. ACKNOWLEDGMENT

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