On the sum of the squared multiplicities of the distances in a point set over finite fields

Le Anh Vinh  Dang Phuong Dung
Mathematics Department  International Business School
Harvard University  Brandeis University
vinh@math.harvard.edu  lizdang@brandeis.edu

October 9, 2008

Abstract
We study a finite analog of a conjecture of Erdős on the sum of the squared multiplicities of the distances determined by an $n$-element point set. Our result is based on an estimate of the number of hinges in spectral graphs.

Mathematics Subject Classifications: 05C15, 05C80.
Keywords: finite Euclidean graphs, pseudo-random graphs.

1 Introduction
Let $\mathbb{F}_q$ denote the finite field with $q$ elements where $q \gg 1$ is an odd prime power. For any $x, y \in \mathbb{F}_q^d$, the distance between $x, y$ is defined as $||x - y|| = (x_1 - y_1)^2 + \ldots + (x_d - y_d)^2$. Let $E \subset \mathbb{F}_q^d$, $d \geq 2$. Then the finite analog of the classical Erdős distance problem is to determine the smallest possible cardinality of the set

$$\Delta(E) = \{||x - y|| : x, y \in E\},$$

viewed as a subset of $\mathbb{F}_q$. Bourgain, Katz and Tao ([3]) showed, using intricate incidence geometry, that for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $E \subset \mathbb{F}_q^2$ and $|E| \leq C_\varepsilon q^{2-\varepsilon}$, then $|\Delta(E)| \geq C_\delta q^{1+\delta}$ for some constants $C_\varepsilon, C_\delta$. The relationship between $\varepsilon$ and $\delta$ in their argument is difficult to determine. Going up to higher dimension using arguments of Bourgain, Katz and Tao is quite subtle. Iosevich and Rudnev ([7]) establish the following results using Fourier analytic methods.

Theorem 1.1 ([7]) Let $E \subset \mathbb{F}_q^d$ such that $|E| \geq C q^{d/2}$ for $C$ sufficient large. Then

$$|\Delta(E)| \geq \min \left\{ q, \frac{|E|}{q^{d/2}} \right\}. \quad (1.1)$$
In [11], the author gives another proof of this result using the graph theoretic method. This method also works for many other related problems, see [12, 13, 14, 15]. The advantages of the graph theoretic method are twofold. First, we can reprove and sometimes improve several known results in vector spaces over finite fields. Second, our approach works transparently in the non-Euclidean setting. In this note, we use the same method to study a finite analog of a related conjecture of Erdős.

Let \( \deg_S(p, r) \) denote the number of points in \( S \) at distance \( r \) from \( p \). A conjecture of Erdős [6] on the sum of the squared multiplicities of the distances determined by an \( n \)-element point set states that

\[
\sum_{r > 0} \left( \sum_{p \in S} \deg_S(p, r)^2 \right) \leq O(n^3 (\log n)^\alpha),
\]

for some \( \alpha > 0 \). For this function, Akutsu et al. [1] obtained the upper bound \( O(n^3) \), improving an earlier result of Thiele [10]. If no three points are collinear, Thiele gives the better bound \( O(n^3) \). This bound is sharp by the regular \( n \)-gons [10]. Nothing is known about this function over higher dimensional spaces. The purpose of this note is to study this function in the space \( \mathbb{F}_q^d \). To avoid some null distance pairs (i.e. two distinct points with distance zero), we assume that \( -1 \) is not a square in \( \mathbb{F} \) throughout this note. The main result of this note is the following.

**Theorem 1.2** Let \( E \subset \mathbb{F}_q^d \). For any point \( p \in E \) and a distant \( r \in \mathbb{F}_q^* \), Let \( \deg_E(p, r) \) denotes the number of points in \( E \) at distance \( r \) from \( p \). Let \( f(E) \) denote the sum of the square multiplicities of the distances determined by \( E \):

\[
f(E) = \sum_{r \in \mathbb{F}_q^*} \left( \sum_{p \in E} \deg_E(p, r)^2 \right).
\]

a) Suppose that \( |E| \geq \Omega(q^{d+1}) \) then \( f(E) = \Theta(|E|^3/q) \).

b) Suppose that \( |E| \leq O(q^{d+1}) \) then \( \Omega(|E|^3/q) \leq f(E) \leq O(|E|q^d) \).

The rest of this note is organized as follows. In Section 2, we establish an estimate about the number of hinges (i.e. ordered paths of length 2) in spectral graphs. Using this estimate, we give a proof of Theorem 1.2 in Section 3.

## 2 Number of hinges in an \((n, d, \lambda)\)-graph

We call a graph \( G = (V, E) \) \((n, d, \lambda)\)-graph if \( G \) is a \( d \)-regular graph on \( n \) vertices with the absolute values of each of its eigenvalues but the largest one is at most \( \lambda \). It is well-known that if \( \lambda \ll d \) then an \((n, d, \lambda)\)-graph behaves similarly as a random graph \( G_{n,d/n} \). Precisely, we have the following result (cf. Theorem 9.2.4 in [2]).
Theorem 2.1  ([2]) Let $G$ be an $(n, d, \lambda)$-graph. For a vertex $v \in V$ and a subset $B$ of $V$ denote by $N(v)$ the set of all neighbors of $v$ in $G$, and let $N_B(v) = N(v) \cap B$ denote the set of all neighbors of $v$ in $B$. Then for every subset $B$ of $V$:

$$\sum_{v \in V}(|N_B(v)| - \frac{d}{n}|B|)^2 \leq \frac{\lambda^2}{n}|B|(n - |B|). \quad (2.1)$$

The following result is an easy corollary of Theorem 2.1.

Theorem 2.2  (cf. Corollary 9.2.5 in [2]) Let $G$ be an $(n, d, \lambda)$-graph. For every set of vertices $B$ and $C$ of $G$, we have

$$|e(B, C) - \frac{d}{n}|B||C|| \leq \lambda \sqrt{|B||C|}, \quad (2.2)$$

where $e(B, C)$ is the number of edges in the induced bipartite subgraph of $G$ on $(B, C)$ (i.e. the number of ordered pair $(u, v)$ where $u \in B$, $v \in C$ and $uv$ is an edge of $G$).

From Theorem 2.1 and Theorem 2.2, we can derive the following estimate about the number of hinges in an $(n, d, \alpha)$-graph.

Theorem 2.3  Let $G$ be an $(n, d, \lambda)$-graph. For every set of vertices $E$ of $G$, we have

$$p_2(E) \leq |E| \left( \frac{d|E|}{n} + \lambda \right)^2, \quad (2.3)$$

where $p_2(E)$ is the number of ordered paths of length two in $E$ (i.e. the number of ordered triple $(u, v, w) \in E \times E \times E$ with $uv, vw$ are edges of $G$).

Proof  For a vertex $v \in V$ let $N_E(v)$ denote the set of all neighbors of $v$ in $E$. From Theorem 2.1, we have

$$\sum_{v \in E}(|N_E(v)| - \frac{d}{n}|E|)^2 \leq \sum_{v \in V}(|N_E(v)| - \frac{d}{n}|E|)^2 \leq \frac{\lambda^2}{n}|E|(n - |E|). \quad (2.4)$$

This implies that

$$\sum_{v \in E}N^2_E(v) + \left( \frac{d}{n} \right)^2|E|^2 - 2\frac{d}{n}|E| \sum_{v \in E}N_E(v) \leq \frac{\lambda^2}{n}|E|(n - |E|) \quad (2.5)$$

From Theorem 2.2, we have

$$\sum_{v \in E}N_E(v) \leq \frac{d}{n}|E|^2 + \lambda|E|. \quad (2.6)$$
Putting (2.5) and (2.6) together, we have

\[ \sum_{v \in E} N_E^2(v) \leq \left( \frac{d}{n} \right)^2 |E|^3 + 2 \frac{\lambda d}{n} |E|^2 + \frac{\lambda^2}{n} |E|(n - |E|) \]
\[ < \left( \frac{d}{n} \right)^2 |E|^3 + 2 \frac{\lambda d}{n} |E|^2 + \lambda^2 |E| \]
\[ = |E| \left( \frac{d |E|}{n} + \lambda \right)^2, \]

completing the proof of the theorem. \( \square \)

3 Proof of Theorem 1.2

Let \( \mathbb{F}_q \) denote the finite field with \( q \) elements where \( q \gg 1 \) is an odd prime power. For a fixed \( a \in \mathbb{F}_q^* \), the finite Euclidean graph \( G_q(a) \) in \( \mathbb{F}_q^d \) is defined as the graph with vertex set \( \mathbb{F}_q^d \) and the edge set

\[ E = \{(x, y) \in \mathbb{F}_q^d \times \mathbb{F}_q^d \mid x \neq y, ||x - y|| = a\}, \]

where \( ||.|| \) is the analogue of Euclidean distance \( ||x|| = x_1^2 + \ldots + x_d^2 \). In [8], Medrano et al. studied the spectrum of these graphs and showed that these graphs are asymptotically Ramanujan graphs. They proved the following result.

**Theorem 3.1** ([8]) The finite Euclidean graph \( G_q(a) \) is regular of valency \( (1 + o(1))q^{d-1} \) for any \( a \in \mathbb{F}_q^* \). Let \( \lambda \) be any eigenvalues of the graph \( G_q(a) \) with \( \lambda \neq \) valency of the graph then

\[ |\lambda| \leq 2q^{d-1}. \]

We have the number of ordered triple \((u, v, w)\) \(\in E \times E \times E\) with \(uv\) and \(vw\) are edges of \(G_q(a)\) is \(\sum_{p \in E} \text{deg}_E(p, a)^2\). From Theorem 2.3 and Theorem 3.1, we have

\[ f(E) \leq \sum_{a \in \mathbb{F}_q^*} |E| \left( (1 + o(1)) \frac{|E|}{q} + 2q^{d-1} \right)^2 \leq (q-1)|E| \left( (1 + o(1)) \frac{|E|}{q} + 2q^{d-1} \right)^2. \]

Thus, if \(|E| \geq \Omega(q^{d-1})\) then

\[ f(E) \leq O(|E|^3/q), \]

and if \(|E| \ll O(q^{d-1})\) then

\[ f(E) \leq O(|E|q^d). \]
We now give a lower bound for \( f(E) \). We have

\[
f(E) = \sum_{r \in \mathbb{P}_q} \left( \sum_{p \in E} \deg_E(p, r)^2 \right) \\
\geq \sum_{r \in \mathbb{P}_q} \frac{1}{|E|} \left( \sum_{p \in E} \deg_E(p, r) \right)^2 \\
\geq \frac{1}{(q-1)|E|} \left( \sum_{r \in \mathbb{P}_q} \sum_{p \in E} \deg_E(p, r) \right)^2 \\
\geq \frac{|E|(|E|-1)^2}{(q-1)} = \Omega(|E|^3/q).
\]

(3.5)

Theorem 1.2 follows immediately from (3.3), (3.4) and (3.5).

Remark 3.2 From the above proof, we can derive Theorem 1.1 as follows.

\[
\frac{1}{|\Delta(E)||E|} (|E|(|E| - 1))^2 \leq f(E) \leq |\Delta(E)||E| \left( 1 + o(1) \right) \frac{|E|}{q} + 2q^{d-1} \\
\]

This implies that

\[
|\Delta(E)| \geq \frac{(1 + o(1))q}{1 + 2q^{d-1}/|E|}.
\]

and Theorem 1.1 follows immediately.

References

[1] T. Akutsu, H. Tamaki and T. Tokuyama, Distribution of distances and triangles in a point set and algorithms for computing the largest common point sets, *Discrete Comput. Geom.* 20 (1998), 307–331.

[2] N. Alon and J. H. Spencer, *The probabilistic method*, 2nd ed., Willey-Interscience, 2000.

[3] J. Bourgain, N. Katz, T. Tao, A sum-product estimate in finite fields, and applications, *Geom. Funct. Anal.* 14 (2004), 27-57.

[4] P. Brass, W. Moser and J. Pach, *Research problems in discrete geometry*, Springer, 2005.

[5] P. Erdős, On sets of distances of \( n \) points, *Amer. Math. Monthly* 53 (1946) 248–250.

[6] P. Erdős, Some of my favorite unsolved problems, in: *A Tribute to Paul Erdős*, A. Baker et al., eds., Cambridge Univ. Press 1990, 467–478.
[7] A. Iosevich, M. Rudnev, Erdős distance problem in vector spaces over finite fields, *Transactions of the American Mathematical Society* **359** (12) (2007), 6127-6142.

[8] A. Medrano, P. Myers, H. M. Stark and A. Terras, Finite analogues of Euclidean space, *Journal of Computational and Applied Mathematics*, **68** (1996), 221–238.

[9] L. A. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Comb. Probab. Comput.* **6** (1997), 353–358.

[10] T. Thiele, Geometric selection problems and hypergraphs, Dissertation, Freie Universität Berlin 1995.

[11] L. A. Vinh, Explicit Ramsey graphs and Erdős distance problem over finite Euclidean and non-Euclidean spaces, *Electronic Journal of Combinatorics* **15** (2008), R5.

[12] L. A. Vinh, On the number of orthogonal systems in vector spaces over finite fields, *Electronic Journal of Combinatorics* **15** (2008), N32.

[13] L. A. Vinh, Szemerédi-Trotter type theorem and sum-product estimate in finite fields, *European Journal of Combinatorics*, to appear.

[14] L. A. Vinh, On a Furstenberg-Katznelson-Weiss type theorem over finite fields, preprint (2008).

[15] L. A. Vinh, On kaleidoscopic pseudo-randomness of finite Euclidean and non-Euclidean graphs, preprint (2008).