Construction of dandelin sphere on definition of conics using geogebra classic 5

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Abstract. Conics is a curve formed by the intersection of a plane with a cone. The types of conics depend on the relationship of angle between the axis of the cone and the angle between the cutting plane with one of the generating line. The conics can also be defined in terms of the eccentricity. The problem is when conics are defined as the intersection of a cone and a plane, it does not show the focus and directrix and when conics are defined by the eccentricity, it does not show that the conics are formed by the intersection of a cone and a plane. Germinal Pierre Dandelin finds the way to show that these definitions are related. He uses sphere of certain size and position inscribed inside to the cone. The purpose of this paper is to construct Dandelin sphere to show that the definition of conics are related each other. The construction steps are using GeoGebra Classic 5. The result shows that by dragging the cutting plane the type of the conics are formed and show where the focus point and directrix line are.

1. Introduction

Conic section or conics is a curve formed by the intersection of a plane with a cone. There are four types of conics depending on how the plane intersects the cone: ellipse, hyperbola, parabola, and degenerating curve. The ellipse results if the plane intersects the cone in closed curves, the circle is a special case of the ellipse. The hyperbola results if the plane intersects both nappes of a double cone. The parabola results if the plane is parallel to a line on the cone. The degenerate curve results if the plane passes through the vertex of the cone. There is three type of degenerate curve: intersecting line, single line and a point [1]. More precisely the conics that formed by the intersection of cone and a plane, depend on the angle between the axis of the cone and the angle between the cutting plane with one of the generating line [2].

On the other hand, conics can be defined in terms of the eccentricity or the ratio of the distance from a point on the conic to a fixed point called the focus and the distance from the point to a fixed line called the directrix. An ellipse has eccentricity less than 1, a parabola has eccentricity 1, and a hyperbola has eccentricity greater than 1 [3].

Actually, both definitions define the same object, namely conics. The problem is that when conics are defined as the intersection of a cone and a plane, it does not show where the focus point and directrix line are. On the other hand when conics are defined by the eccentricity, it does not show that the conics are formed by the intersection of a cone and a plane, because only two-dimensional shapes are seen.
In 1822, Germinal Pierre Dandelin finds the way to prove that these definitions are related. He uses sphere of certain size and position inscribed inside to the cone. The sphere is named Dandelin Sphere. Dandelin sphere is a sphere of certain size and position inscribed inside the cone. Dandelin sphere relates many properties of the conics to the cone. To construct Dandelin sphere, this step are used;

1) Inscribe a sphere inside the cone and the cutting plane. Such the sphere will be tangent to the cutting plane at a point and tangent to the cone along a circle. For the case of hyperbola and ellipse, there are two such spheres.

2) The point where the sphere touches the cutting plane is the focus of the conics.

3) The directrix of the conics is the intersection of the cutting plane and the plane of the osculating circle [3].

Dandelin Sphere is usually used by teachers to make it easier to explain the definition of conics. But, the actual object can not be used to explain Dandelin sphere, so software is needed to visualize the Dandelin sphere. Salinas and Pulido [4] used Augmented Reality to visualize Dandelin sphere in 3D. In this way, the picture looks more realistic and student can understand the definition of conics easily. But not a lot of people know how to use this Augmented Reality because the application is not yet familiar to many people.

One of the software that can be used to visualize the geometrical object is GeoGebra. GeoGebra is a dynamic mathematics open source (free) software for learning and teaching mathematics. It was developed by Markus Hohenwarter and an international team of programmers. GeoGebra combines geometry, algebra, statistics, and calculus. On the one hand, GeoGebra is an interactive geometry system. It can construct points, vectors, segments, lines, polygons and conic sections as well as functions while changing them dynamically afterward. On the other hand, equations and coordinates can be entered directly. Thus, GeoGebra has the ability to deal with variables for numbers, vectors, and points[5].

Using the provided geometry tools in the Toolbar can create geometric constructions on the Graphics View with the mouse. At the same time, the corresponding coordinates and equations are displayed in the Algebra View. On the other hand, it can directly enter algebraic input, commands, and functions into the Input Bar by using the keyboard. While the graphical representation of all objects is displayed in the Graphics View, their algebraic numeric representation is shown in the Algebra View. In GeoGebra, geometry and algebra work side by side. Recently, GeoGebra Classic 5 can support to build in 3D, so the 3D images can look more real [5].

The purpose of this paper is to construct Dandelin sphere as a tool to facilitate in explaining the relationship between the definition of conics as the intersection of plane with cones and the definitions of conics in terms of the eccentricity or the ratio of the distance from a point on the conic to the focus and the distance from the point to the directrix. The construction steps is using GeoGebra Classic 5.

2. Discussion

Conic section or conics is a curve formed by the intersection of a plane with a cone. There are four types of conics depending on how the plane intersects the cone: ellipse, hyperbola, parabola, and degenerating curve. Conics also has another definition based on the way its defined. Conics can be defined in terms of their relationship to fixed points called focus or fixed lines called directrices, in terms of the eccentricity or ratio of the distance from a point on the conic to a focus an the distance from the point to the directrices [3].

In term of the definition of conics as the intersection of a plane and the cone, the type of conics depend on the angle between the axis of the cone and the angle between the cutting plane with one of the generating line. Let $\eta$ be the angle between the axis of the cone and one of the generating lines. Let $\theta$ be the angle formed by the axis of the cone and the cutting plane. The relation of $\eta$ and $\theta$ defines the shape of the curve. If $\eta = \theta$ the curve is a parabola, if $\eta < \theta$ the curve is an ellipse and if $\eta > \theta$ the curve is a hyperbola [3].
The relationship between these two angles defines different conics according to the previous explanation. Next is the construction of Dandelin sphere using GeoGebra Classic 5 [6].

*Construction of Dandelin Sphere using GeoGebra Classic 5;*

Construction is carried out by paying attention to the relationships between elements in geometry and written in a coherent way so that it is easier to follow and understand. For several steps, an image will be given as an explanation of the steps.

1) Use point tool on toolbar, then draw any point $K$ as cone vertex.
2) Draw another point, named point $T$, then use line tool on toolbar to draw line $KT$. Line $KT$ becomes axis of the cone.
3) Determine the measure of the angle for the cone angle, we choose $2\alpha$.
4) Use input bar to input the function of double nappes cone. The function is $\text{InfiniteCone}(\text{<Point>}, \text{<Line>}, \text{<Angle } \alpha \text{>})$. It used to creates an infinite cone with given point as vertex, axis of symmetry parallel to given line and apex angle $2\alpha$.

Step 1-4 produce a double nappe cone. Then the next step is drawing a plane that become principal section. Principal section is a plane containing the axis of the cone.

5) Choose any on the cone surfaces except the vertex point, named point $V$. Draw line through $K$ and $V$ named line $KV$, the line was one of the generating lines of the cone.
6) Use plane tool on toolbar, then draw plane through line $KT$ and $KV$ named plane $\beta$. The plane intersects the cone through vertex produce two intersecting lines. This plane was the principal section of the cone.

*Figure 1. Angle $\eta$ and $\theta$ on a cone with cutting plane*

Step 5-6 produce plane $\beta$ as the principle section that contain vertex of the cone. The next step is drawing the cutting plane that contain the conics.

*Figure 2. The construction of double right cone without the base with $\beta$ as principle section*
7) Use circle tool with axis through point on toolbar, then draw circle through axis $\vec{KT}$ and point $V$ named circle $l_1$. Circle $l_1$ intersect the generating line $\vec{KQ}$ on point $Q$.

8) Draw line through point $V$ and $Q$ named line $\vec{VQ}$.

9) Choose any point $N$ on line $\vec{VQ}$.

10) Choose any point $A_1$ on line $\vec{KQ}$.

11) Draw line through point $N$ and $A_1$ named line $\vec{A_1N}$.

12) Draw circle through axis $\vec{KT}$ and point $A_1$ named circle $l_2$.

13) Draw a tangent line of circle $l_2$ through point $A_1$.

14) Draw plane through the tangent line and line $\vec{A_1N}$ named plane $\alpha$. Plane $\alpha$ intersect the cone produce a curve named curve $k$ and intersect the plane $\beta$ on line $\vec{A_1N}$. Plane $\alpha$ intersect line $\vec{KQ}$ on point $A_1$ and intersect line $\vec{KV}$ on point $A_2$.

![Figure 3. The construction of plane $\alpha$](image)

Step 7-14 produce plane $\alpha$ as the cutting plane that contain curve $k$ as the conics. The next step is drawing the first Dandelin Sphere that contain one of the focus point of the conics.

15) Use angle bisector tool on toolbar, then draw angle bisector between line $\vec{KV}$ and $\vec{KQ}$ and angle bisector between line $\vec{A_1N}$ and $\vec{KQ}$. These angle bisector intersect at point $S_1$.

16) Use perpendicular line tool on toolbar, draw line through point $S_1$ perpendicular to line $\vec{A_1N}$. This line intersects line $\vec{A_1N}$ on point $F_1$.

17) Use sphere tool with center through point on toolbar, draw a sphere with center $S_1$ through point $F_1$.

![Figure 4. The Dandelin sphere $B_1$ inside the cone and touch the cone surface and the cutting plane $\alpha$](image)
Sphere $B_1$ is the first Dandelin Sphere as the result of the step 15-17. The tangent point of the sphere $B_1$ and the cutting plane $\alpha$ is the first focus of the conics\([3]\), that is point $F_1$. The next step is drawing plane that contain the directrix line of the conics.

18) Draw line through point $S_1$ perpendicular to line $\vec{KV}$, intersect line $\vec{KV}$ on point $D_1$.

19) Use perpendicular plane tool on toolbar, draw plane through point $D_1$ perpendicular to line $\vec{KT}$ named plane $\gamma$. Plane $\gamma$ intersect the cone on circle $l_3$ and circle $l_3$ intersect the axis $\vec{KQ}$ on point $E_1$. Plane $\gamma$ also intersect plane $\beta$ on line $\vec{D_1E_1}$ and intersect plane $\alpha$ on line $g$. Line $g$ and $\vec{A_1N}$ intersect on point $X_1$.

**Figure 5.** The construction of the tangent circle $l_3$ on sphere $B_1$ and the cone with cutting plane $\gamma$

The intersection of plane $\gamma$ and plane $\alpha$ is the directrix of the conics [3], that is line $g$. The next step is drawing second Dandelin Sphere that contain another focus point of the conics.

20) Draw angle bisector between line $\vec{KV}$ and $\vec{KQ}$ and angle bisector between line $\vec{A_1A_2}$ and $\vec{KA_2}$. These angle bisector intersect at point $S_2$.

21) Draw line through point $S_2$ perpendicular to line $\vec{A_1A_2}$. This line intersects line $\vec{A_1A_2}$ on point $F_2$.

22) Draw a sphere with center $S_2$ through point $F_2$.

**Figure 6.** The Dandelin sphere $B_2$ inside the cone touch the cone surface and the cutting plane $\alpha$

Sphere $B_2$ is the second Dandelin Sphere as the result of the step 20-22. The tangent point of the sphere $B_2$ and the cutting plane $\alpha$ is the second focus of the conics [3], that is point $F_2$. The next step is drawing plane that contain another directrix line of the conics.

23) Draw line through point $S_2$ perpendicular to line $\vec{KV}$, intersect line $\vec{KV}$ on point $E_2$. 

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24) Draw plane through point $E_2$ perpendicular to line $KT$ named plane $\delta$. Plane $\delta$ intersect the cone on circle $l_4$ and circle $l_4$ intersect the axis $KQ$ on point $D_2$. Plane $\delta$ also intersect plane $\beta$ on line $D_2E_2$ and intersect plane $\alpha$ on line $h$. Line $h$ and $A_1A_2$ intersect on point $X_2$.

![Diagram](image)

**Figure 7.** The construction of tangent circle $l_4$ on the sphere $B_2$ and the cone with the cutting plane $\delta$

The intersection of plane $\delta$ and plane $\alpha$ is the directrix of the conics [3], that is line $h$. Figure 7 is the final result of the Dandelin sphere construction using GeoGebra. The three types of conics can be obtained from the final results of this construction by drag the plane $\alpha$ that contain the conics. To drag plane $\alpha$, used the point $N$ in the line $QV$ and shift the point $N$ on the line $QV$. If point $N$ is shifted on the line $QV$, then the angle between the cutting plane and the cone axis will also change, the curve will also change. By the definition of conics [3], if point $N$ is shifted on line $VQ$ so that $\eta = \theta$, then the curve was a parabola, if $\eta < \theta$ the curve is an ellipse and if $\eta > \theta$ the curve is a hyperbola.

![Diagram](image)
Figure 8. The plane $\alpha$ intersect the cone produce angle $\theta$ and $\eta$, the angle between the cone axis and the generating line $K\Vec{V}$ (a)ellipse, (b)parabola, and (c)hyperbola

Figure 8 shows that based on the definition of conics, the form of the conics is depend on the relationship between the angle $\theta$ and $\eta$. It also shows that the point $F_1$ and $F_2$ is the focus points of the curve and line $g$ and $h$ is the directrix lines based on the definition of Dandelin sphere [3]. If plane $\alpha$ is dragged by shifting point $N$ on the line $\Vec{VQ}$, it can show where the focus point and directrix line are. It also can show that parabola has only one focus point and one directrix line, ellipse, and hyperbola have each two focus points and two directrix lines. This result can be used by teacher to explain students that the definition of conics are related each other.

3. Conclusion
The problem that happen when conics are defined as the intersection of a cone and a plane, it does not show where the focus point and directrix line are and when conics are defined by the eccentricity, it does not show that the conics are formed by the intersection of a cone and a plane, because only two-dimensional shapes are seen, can be solved using Dandelin Sphere. Dandelin sphere can show which one is the focus point and the directrix lines. Based on the definition of Dandelin sphere, it is seen that the point which is the tangent point of the sphere with the cutting plane is the conics focus point, in this case it was point $F_1$ and $F_2$. The intersection of the cutting plane and the plane of the osculating circle are directrix lines, in this case it was line $g$ and $h$. To visualize Dandelin Sphere more real, we can use GeoGebra Classic 5 that support to create 3D images. The results of this construction can be used to make it easier for the teacher to explain about conics. GeoGebra is chosen because it is open source (free) software and easy to use so that anyone can use it.
References

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