ON INEXACT ACCELERATED PROXIMAL GRADIENT METHODS  
WITH RELATIVE ERROR RULES

YUNIER BELLO-CRUZ ∗, MAX L. N. GONÇALVES †, AND NATHAN KRISLOCK ‡

Abstract. One of the most popular and important first-order iterations that provides optimal complexity of the classical proximal gradient method (PGM) is the “Fast Iterative Shrinkage/Thresholding Algorithm” (FISTA). In this paper, two inexact versions of FISTA for minimizing the sum of two convex functions are studied. The proposed schemes inexactly solve their subproblems by using relative error criteria instead of exogenous and diminishing error rules. When the evaluation of the proximal operator is difficult, inexact versions of FISTA are necessary and the relative error rules proposed here may have certain advantages over previous error rules. The same optimal convergence rate of FISTA is recovered for both proposed schemes. Some numerical experiments are reported to illustrate the numerical behavior of the new approaches.

Key words. FISTA, inexact accelerated proximal gradient method, iteration complexity, non-smooth and convex optimization problems, proximal gradient method, relative error rule.

AMS subject classifications. 47H05, 47J22, 49M27, 90C25, 90C30, 90C60, 65K10.

1. Introduction. Throughout this paper, we write $p := q$ to indicate that $p$ is defined to be equal to $q$. The nonnegative (positive) numbers will be denoted by $\mathbb{R}_+$ ($\mathbb{R}_+$). Moreover, $\mathbb{E}$ denotes a finite-dimensional real vector space, which is equipped with the inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$.

Consider the following problem

$$
\min_{x \in \mathbb{E}} F(x) := f(x) + g(x),
$$

where $f : \mathbb{E} \to \mathbb{R}$ is a differentiable convex function whose gradient is $L$-Lipschitz continuous and $g : \mathbb{E} \to \mathbb{R} := \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous (lsc) convex function that is not necessarily differentiable. We denote the optimal value of (1) by $F^*$, the set of optimal solutions of (1) by $S_*$, and we assume that $F^* \in \mathbb{R}$ and that $S_*$ is nonempty; thus we have $F(x_*) = F^*$, for all $x_* \in S_*$. It is well-known that (1) contains a wide class of problems arising in applications from science and engineering, including machine learning, compressed sensing, and image processing. There are important examples of this problem such as using $\ell_1$-regularization to obtain sparse solutions with applications in signal recovery and signal processing problems [9, 18, 33], the nearest correlation matrix problem [14, 19, 29], and regularized inverse problems with atomic norms [34].

A plethora of methods has been proposed for solving the aforementioned optimization problem. One of the most studied approaches is the proximal gradient method (PGM) which is a first-order splitting iteration that has been intensively investigated in the literature; see, for instance, [8, 11, 12], PGM iterates by performing a gradient step based on $f$ followed by the evaluation of the proximal (or Prox) operator of $g$, which is defined as $\text{Prox}_g := (\text{Id} + \partial g)^{-1}$ where

$$
\partial g(x) := \{u \in \mathbb{E} \mid g(y) \geq g(x) + \langle u, y - x \rangle, \forall y \in \mathbb{E}\}
$$
is the subdifferential of $g$ at $x \in E$ and $\text{Id}$ is the identity operator. It is well-known that the sequence $(x_k)_{k \in \mathbb{N}}$ generated by PGM has a complexity rate of $O(\rho^{-1})$ to obtain a $\rho$-approximate solution of (1) (that is, a solution $x_k$ satisfying $F(x_k) - F^* \leq \rho$), or equivalently we can say that $F(x_k) - F^* = O(k^{-1})$; see, for instance, [8, 11, 12]. In addition, it is possible to accelerate the proximal gradient method in order to achieve the optimal $O(k^{-2})$ convergence rate by adding an extrapolation step. This scheme, which improved the complexity of the gradient method for minimizing smooth convex functions, was first introduced by Nesterov in 1983 [25] and further extended to constrained problems in 1988 [26, 27]. In the spirit of the work of [25], Nesterov [28] (appeared online in 2007 but published in 2013) and Beck–Teboulle [8] extended Nesterov’s classical iteration to minimizing composite nonsmooth functions.

In this paper, we propose a modification of the “Fast Iterative Shrinkage/Thresholding Algorithm” (FISTA) of [8]. FISTA is described as follows.

**Algorithm 1 (FISTA).** Let $x_0 \in E$, and $L > 0$ be the Lipschitz constant of $\nabla f$. Set $y_1 := x_0$, $t_1 := 1$, and iterate

\begin{align}
(2) \quad x_k &:= \text{Prox}_{\frac{1}{L}g} \left( y_k - \frac{1}{L} \nabla f(y_k) \right), \\
(3) \quad t_{k+1} &:= \frac{1 + \sqrt{1 + 4t_k^2}}{2}, \\
(4) \quad y_{k+1} &:= x_k + \frac{t_k - 1}{t_{k+1}}(x_k - x_{k-1}).
\end{align}

Note that if the update (3) is ignored and $t_k = 1$ for all $k \in \mathbb{N}$, FISTA becomes the (unaccelerated) PGM mentioned before. There are two very popular choices for the sequence $(t_k)_{k \in \mathbb{N}}$ [8, 28] but several different updates are possible for $t_k$ that also achieve the optimal acceleration; see, for instance, [4, 5, 15, 32]. Convergence and complexity results of the sequence generated by FISTA under a suitable tuning of $(t_k)_{k \in \mathbb{N}}$ related to the update (3) can be found in [2, 5, 12, 15]. Many accelerated versions have been proposed in the literature for accelerating the PGM for solving (1). The relaxed case was considered in [6] and error-tolerant versions were studied in [3, 4]. In addition, for results concerning the rate of convergence of function values of (1) with or without minimizers, see [7, 32].

FISTA (and in particular PGM) is an effective and simple choice for solving large scale problems when the Prox operator has a closed-form or there exists an efficient way to evaluate it. Frequently, it could be computationally expensive to evaluate the Prox operator at any point with high accuracy [10]. The theory of convergence for the (accelerated) PGM assumes that the Prox operator can be evaluated at any point; that is, the regularized minimization problem

$$\min_{x \in E} \left\{ g(x) + \frac{1}{2\gamma}||x - z||^2 \right\}, \quad \gamma > 0,$$

can be solved for any $z \in E$. The unique solution of the above problem is actually the Prox operator of $\gamma g$ at $z$, which is the function $\text{Prox}_{\gamma g} : E \rightarrow \text{dom } g$ defined by

$$\text{Prox}_{\gamma g}(z) := \arg\min_{x \in E} \left\{ g(x) + \frac{1}{2\gamma}||x - z||^2 \right\}.$$
This function satisfies the following necessary and sufficient optimality condition:

\[ \frac{1}{\gamma} (z - \text{Prox}_{\gamma g}(z)) \in \partial g(\text{Prox}_{\gamma g}(z)). \]

Therefore, to run FISTA we must compute \( x_k \) by solving the subproblem

\[ (5) \quad \min_{x \in \mathbb{R}} \left\{ g(x) + \frac{L}{2} \| x - \left( y_k - \frac{1}{L} \nabla f(y_k) \right) \|^2 \right\}. \]

That is, we must find the point \( x_k \) that satisfies

\[ 0 \in \partial g(x_k) + L(x_k - y_k) + \nabla f(y_k). \]

A natural question is: What happens if the solution of (5) can not be easily computed? Often in practice in this case the evaluation of the proximal operator is done approximately. However, to guarantee an optimal complexity rate, it is required that the nonnegative sequence of error tolerances be summable. As was shown in [19, 34], with a summable sequence of error tolerances for these approximate solutions, the optimal complexity rate \( O(k^{-2}) \) of FISTA is recovered.

The two works [19, 34] appeared simultaneously around 2013 and proposed inexact variations of FISTA with summable error tolerances for computing the \( \varepsilon \)-approximate solutions of subproblem (5). In [34], given a nonnegative sequence \( (\varepsilon_k)_{k \in \mathbb{N}} \), iterates \( \tilde{x}_k \) are generated such that

\[ (6) \quad 0 \in \partial_{\varepsilon_k} g(\tilde{x}_k) + L(\tilde{x}_k - y_k) + \nabla f(y_k), \]

where

\[ \partial_{\varepsilon_k} g(x) := \{ u \in \mathbb{R} \mid g(y) \geq g(x) + \langle u, y - x \rangle - \varepsilon, \forall y \in \mathbb{R} \} \]

is an enlargement of \( \partial g \). On the other hand, the version in [19] allows the approximate solution \( \tilde{x}_k \) of subproblem (5) such that

\[ (7) \quad F(\tilde{x}_k) \leq g(\tilde{x}_k) + f(y_k) + \langle \nabla f(y_k), \tilde{x}_k - y_k \rangle + \langle \tilde{x}_k - y_k, H_k(\tilde{x}_k - y_k) \rangle + \frac{\varepsilon_k}{2L_k}, \]

\[ v_k \in \partial_{\frac{\varepsilon_k}{2L_k}} g(\tilde{x}_k) + H_k(\tilde{x}_k - y_k) + \nabla f(y_k), \quad \| H_k^{-1/2} v_k \| \leq \frac{\delta_k}{\sqrt{2L_k}}, \]

where \( (\delta_k)_{k \in \mathbb{N}} \) and \( (\varepsilon_k)_{k \in \mathbb{N}} \) are summable sequences of nonnegative numbers, and \( H_k \) is a self-adjoint positive definite operator. If \( H_k = L \text{Id} \), then inequality (7) is trivially satisfied. In both of these approaches, the summability assumption may require us to find \( \tilde{x}_k \) to a level of accuracy that is higher than necessary.

In the spirit of [21, 31], we propose two inexact versions with relative error rules for solving the main subproblem of FISTA. The advantages over the inexact methods given in [19, 34] are the following:

(a) The proposed relative error rules have no summability assumption and the error tolerances naturally depend on the generated iterates. Our first proposed method is a generalization of FISTA and our second proposed method is related to the extra-step acceleration method proposed in [22].

(b) We recover the optimal iteration convergence rate in terms of the objective function value for both proposed inexact methods. Moreover, for a given tolerance \( \rho > 0 \), we also study iteration-complexity bounds for the proposed
algorithms in order to obtain a $\rho$-approximate solution $x$ of the inclusion $0 \in \partial F(x)$ with residual $(r, \varepsilon)$, i.e.,

$$r \in \partial F(x), \quad \max \{||r||, \varepsilon\} \leq \rho.$$  

Since $0 \in \partial F(x_\ast)$, for all $x_\ast \in S_\ast$, the latter condition can be interpreted as an optimality measure for $x$.

The presentation of this paper is as follows. Definitions, basic facts and auxiliary results are presented in section 2. Our inexact criteria with relative error rules are presented in section 3. In sections 4 and 5 we present the inexact algorithms and their convergence rates. Some numerical experiments for the proposed schemes are presented in section 2. Our inexact criteria with relative error rules are reported in section 6. Finally, some concluding remarks are given in section 7.

2. Definitions and auxiliary results. Let $h : E \to \mathbb{R}$ be a proper, convex, and lower semicontinuous (l.s.c.) function. We denote the domain of $h$ by $\text{dom} \ h := \{x \in E \mid h(x) < +\infty\}$. Recall that the proximal operator $\text{Prox}_h : E \to \text{dom} \ g$ is defined by $\text{Prox}_h(x) := (\text{Id} + \partial h)^{-1}(x)$. It is well-known that the proximal operator is single-valued with full domain, is continuous, and has many other attractive properties. In particular, the proximal operator is firmly nonexpansive:

$$||\text{Prox}_h(x) - \text{Prox}_h(y)||^2 \leq ||x - y||^2 - ||(x - \text{Prox}_h(x)) - (y - \text{Prox}_h(y))||^2,$$

for all $x, y \in E$. Moreover,

$$0 \in \partial g(\text{Prox}_{\gamma h}(x)) + \frac{1}{\gamma} (\text{Prox}_{\gamma h}(x) - x), \quad x \in E, \gamma > 0.$$

We let $J : E \times \mathbb{R}^{++} \to \text{dom} \ g$ be the forward-backward operator for problem (1), which is given by

$$J(x, \gamma) := \text{Prox}_{\gamma g}(x - \gamma \nabla h(x)), \quad x \in E, \gamma > 0. \tag{8}$$

It is well known that if $h$ is differentiable and $\nabla h$ is $L$-Lipschitz continuous on $E$, i.e.,

$$||\nabla h(x) - \nabla h(y)|| \leq L||x - y||, \quad x, y \in E,$$

then, for all $x, y \in E$, we have

$$h(y) + \langle \nabla h(y), x - y \rangle \leq h(x) \leq h(y) + \langle \nabla h(y), x - y \rangle + \frac{L}{2}||x - y||^2. \tag{9}$$

The next lemma provides some basic properties of the subdifferential operator.

**Lemma 1.** Let $h, f : E \to \mathbb{R}$ be proper, closed, and convex functions. Then:

(i) $\partial h(x) + \partial \mu f(x) \subseteq \partial_{\varepsilon + \mu}(h + f)(x)$, for all $x \in E$ and $\varepsilon, \mu \geq 0$.

(ii) $w \in \partial h(y)$ implies $w \in \partial h(x)$, where $\varepsilon = h(x) - h(y) + \langle w, x - y \rangle \geq 0$.

The following notion of an approximate solution of problem (1) is used in the complexity analysis of our methods.

**Definition 2.** Given a tolerance $\rho > 0$, a point $x \in \mathbb{R}^n$ is said to be a $\rho$-approximate solution of problem (1) with residues $(v, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^+$ if and only if

$$v \in \partial \varepsilon F(x), \quad \max \{||v||, \varepsilon\} \leq \rho.$$  

We end this section by presenting some elementary properties on the extrapolate sequences used by the proposed methods.
Lemma 3. The positive sequence \((t_k)_{k \in \mathbb{N}}\) generated by (3) satisfies, for all \(k \in \mathbb{N}\),
(i) \(\frac{1}{t_k} \leq \frac{2}{k+1}\),
(ii) \(t_{k+1}^2 - t_k = t_k^2\),
(iii) \(0 \leq \frac{t_k - 1}{t_{k+1}} \leq 1\).

Lemma 4. Let \(\lambda \geq 1\) be given. The sequence \((\tau_k)_{k \in \mathbb{N}}\) recursively defined by
(10) \(\tau_0 := 0\), and \(\tau_{k+1} := \tau_k + \frac{\lambda + \sqrt{\lambda^2 + 4\lambda \tau_k}}{2}\),
satisfies, for all \(k \in \mathbb{N}\),
(i) \(\tau_{k+1} > \tau_k\) and \(\frac{\tau_{k+1} - \tau_k}{(\tau_{k+1} - \tau_k)^2} = \frac{1}{\lambda}\),
(ii) \(\tau_k \geq \frac{\lambda}{4} k^2\).

Proof. The first item follows from definition and the fact that \(\tau_k \geq 0\) for all \(k \in \mathbb{N}\).

To prove the second item, we first note that
\[
\tau_{k+1} = \tau_k + \frac{\lambda + \sqrt{\lambda^2 + 4\lambda \tau_k}}{2} \geq \tau_k + \frac{\lambda + 2\sqrt{\lambda \tau_k}}{2} \geq \left(\sqrt{\tau_k} + \frac{\sqrt{\lambda}}{2}\right)^2,
\]
which implies \(\sqrt{\tau_{k+1}} \geq \sqrt{\tau_k} + \frac{\sqrt{\lambda}}{2}\). Therefore,
\[
\sqrt{\tau_k} \geq \sqrt{\tau_0} + \sum_{i=1}^{k} \frac{\sqrt{\lambda}}{2} = k \frac{\sqrt{\lambda}}{2}.
\]
Squaring both sides, we obtain the second item.

3. Inexact criteria with relative error rules. In this section we present two inexact rules with relative error criteria: the inexact relative rule (IR Rule) and the inexact extra-step relative rule (IER Rule). These rules will be used in the two proposed methods in the following two sections.

Rule 1 (IR Rule). Given \(\tau \in (0,1]\) and \(\alpha \in [0, (1-\tau)L/\tau]\), we define the set-value mapping \(\mathcal{J}^{\alpha, \tau} : \mathbb{E} \times \mathbb{R}_+ \Rightarrow \mathbb{E} \times \mathbb{R} \times \mathbb{R}_+\) as
\[
\mathcal{J}^{\alpha, \tau}(y, \frac{1}{L}) := \left\{ (x, v, \varepsilon) \in \mathbb{E} \times \mathbb{E} \times \mathbb{R}_+ : v = \partial \varepsilon g(x) + \frac{\varepsilon}{L}(x - y) + \nabla f(y), \right. \\
\left. \|v\| + 2\varepsilon L \leq L((1-\tau)L - \alpha \varepsilon)\|x - y\|^2 \right\}.
\]

Note that the IR Rule consists of (possibly many) specific outputs. Next we discuss some particular output possibilities including exact and inexact proximal solutions with relative errors.

Remark 1. By setting \(v = 0\) in the IR Rule, we recover the inexact solution of (6), but with \(L/\tau\) in place of \(L\). In this case, the inclusion is similar to the one in [34]; however, the condition on \(\varepsilon\) is different from the exogenous one in [34]. If \(\tau = 1\) in the IR Rule, then \(\alpha = 0\), \(\varepsilon = 0\), and \(v = 0\), implying that
\[
\mathcal{J}^{0,1}(y, \frac{1}{L}) = \left\{ \left( \text{Prox}_{\frac{1}{L}} g \left( y - \frac{1}{L} \nabla f(y) \right), 0, 0 \right) \right\},
\]
which agrees with the exact prox used in (2).
Given \( \sigma \in [0,1] \) and \( \alpha > 1/L \), we define the set-value mapping \( J_{e}^{\alpha,\sigma} : \mathbb{E} \times \mathbb{R}_+ \rightarrow \mathbb{E} \times \mathbb{E} \times \mathbb{R}_+ \) as

\[
J_{e}^{\alpha,\sigma}(y, \frac{1}{L}) := \left\{ (\tilde{x}, v, \varepsilon) \in \mathbb{E} \times \mathbb{E} \times \mathbb{R}_+ \mid v \in \partial g(\tilde{x}) + L(\tilde{x} - y) + \nabla f(y), \quad \|\alpha v + \tilde{x} - y\|_2^2 + 2\alpha \varepsilon \leq \sigma \|\tilde{x} - y\|_2^2 \right\}.
\]

Remark 2. By fixing \( v = (y - \tilde{x})/\alpha \) in the IER Rule, we recover the inexact solution of (6), but with \( (1 + \alpha L)/\alpha \) in place of \( L \). However, the condition on \( \varepsilon \) is different from the exogenous one in [34]. If \( \sigma = 0 \) in the IER Rule, then \( \varepsilon = 0 \) and \( v = (y - \tilde{x})/\alpha \), where

\[
\tilde{x} = \text{Prox}_{\frac{\alpha}{1+\alpha L}} g \left( y - \frac{\alpha}{1+\alpha L} \nabla f(y) \right),
\]

implying that

\[
J_{e}^{\alpha,0}(y, 1/L) = \left\{ \left( \tilde{x}, \frac{y - \tilde{x}}{\alpha}, 0 \right) \right\}.
\]

It is worth pointing out that the inexact relative rules defined above are nonempty since the inclusions

\[
0 \in \partial g(x) + \frac{L}{\tau}(x - y) + \nabla f(y), \quad 0 \in \partial g(\tilde{x}) + \frac{(1 + \alpha L)}{\alpha}(\tilde{x} - y) + \nabla f(y)
\]

always have solutions, which implies that

\[
\left( \text{Prox}_{\frac{\alpha}{1+\alpha L}} g \left( y - \frac{\alpha}{1+\alpha L} \nabla f(y) \right), 0, 0 \right) \in J_{e}^{\alpha,\tau}(y, 1/L), \quad \tau \in (0, 1], \quad \alpha \in [0, (1 - \tau) L/\tau],
\]

and

\[
\left( \tilde{x}, \frac{y - \tilde{x}}{\alpha}, 0 \right) \in J_{e}^{\alpha,\sigma}(y, 1/L), \quad \alpha > 1/L, \quad \sigma \in [0, 1].
\]

### 4. Inexact accelerated method.

We now formally present our inexact accelerated method.

**Algorithm 2 (I-FISTA).** Let \( x_0 \in \mathbb{E}, \tau \in (0, 1], \) and \( \alpha \in [0, L(1 - \tau)/\tau] \) be given. Set \( y_1 := x_0, t_1 := 1, \) and iterate

\[
\text{(11)} \quad \text{find } (x_k, v_k, \varepsilon_k) \in J_{e}^{\alpha,\tau}(y_k, 1/L),
\]

\[
\text{(12)} \quad t_{k+1} := \frac{1 + \sqrt{1 + 4t_k^2}}{2},
\]

\[
\text{(13)} \quad y_{k+1} := x_k - \left( t_k \frac{\tau}{t_{k+1}} \right) \frac{\tau}{L} v_k + \left( t_k - 1 \right) \frac{1}{t_{k+1}} (x_k - x_{k-1}).
\]

Note that the triple \((x_k, v_k, \varepsilon_k)\) in the iterative step of I-FISTA satisfies

\[
\text{(14)} \quad v_k \in \partial_{\varepsilon_k} g(x_k) + \frac{L}{\tau}(x_k - y_k) + \nabla f(y_k),
\]

\[
\text{(15)} \quad \|\tau v_k\|^2 + 2\tau \varepsilon_k L \leq L[(1 - \tau)L - \alpha \tau]\|x_k - y_k\|^2.
\]
If \( \tau = 1 \), then we have \( \varepsilon_k = 0 \) and \( v_k = 0 \), giving us

\[
0 \in \partial g(x_k) + L(x_k - y_k) + \nabla f(y_k),
\]

\[
y_{k+1} = x_k + \left( \frac{t_k - 1}{t_k + 1} \right) (x_k - x_{k-1});
\]
hence, I-FISTA recovers the classical FISTA.

Next we present a key result for our analysis.

**Proposition 5.** For every \( x \in E \) and \( k \in \mathbb{N} \), we have

\[
F(x) - F(x_k) \geq \frac{L}{2\tau} \left[ \|x_k - x - \frac{\tau}{L} v_k\|^2 - \|y_k - x\|^2 \right] + \frac{\Omega}{2} \|y_k - x_k\|^2.
\]

**Proof.** Let \( x \in E \) and \( k \in \mathbb{N} \). Note first that from (14),

\[
v_k + \frac{L}{\tau} (y_k - x_k) - \nabla f(y_k) \in \partial \varepsilon_k g(x_k).
\]

From the definition of \( \partial \varepsilon_k g \), we have

\[
g(x) - g(x_k) \geq \langle v_k + \frac{L}{\tau} (y_k - x_k) - \nabla f(y_k), x - x_k \rangle - \varepsilon_k.
\]

Moreover, the convexity of \( f \) implies

\[
f(x) - f(y_k) \geq \langle \nabla f(y_k), x - y_k \rangle.
\]

Adding (16) and (17), using \( F = f + g \), and simplifying, we get

\[
F(x) - F(x_k) \geq f(y_k) - f(x_k) + \langle \nabla f(y_k), x_k - y_k \rangle
\]

\[+ \frac{L}{\tau} \langle y_k - x_k, x_k - x \rangle + \langle v_k, x - x_k \rangle - \varepsilon_k.
\]

Combining the above inequality with the following identity

\[
-(y_k - x_k, x_k - x) = \frac{1}{2} \left[ \|y_k - x_k\|^2 + \|x_k - x\|^2 - \|y_k - x\|^2 \right],
\]

we get that

\[
F(x) - F(x_k) \geq f(y_k) - f(x_k) + \langle \nabla f(y_k), x_k - y_k \rangle - \varepsilon_k
\]

\[+ \frac{L}{2\tau} \|y_k - x_k\|^2 + \frac{L}{2\tau} \left[ \|x_k - x\|^2 - \|y_k - x\|^2 \right] + \langle v_k, x - x_k \rangle
\]

\[= f(y_k) - f(x_k) + \langle \nabla f(y_k), x_k - y_k \rangle + \frac{L}{2} \|y_k - x_k\|^2
\]

\[+ \frac{(1 - \tau)L}{2\tau} \|y_k - x_k\|^2 + \frac{L}{2\tau} \left[ \|x_k - x\|^2 - \|y_k - x\|^2 \right]
\]

\[+ \langle v_k, x - x_k \rangle - \varepsilon_k.
\]

Then, using (9) together with the Lipschitz continuity of \( \nabla f \), we have

\[
F(x) - F(x_k) \geq \frac{(1 - \tau)L}{2\tau} \|y_k - x_k\|^2 + \frac{L}{2\tau} \left[ \|x_k - x\|^2 - \|y_k - x\|^2 \right]
\]

\[+ \langle v_k, x - x_k \rangle - \varepsilon_k.
\]
On the other hand, the error condition of IR Rule, given in (15), implies

\[
\frac{(1 - \tau)L}{2\tau} \| y_k - x_k \|^2 - \varepsilon_k \geq \frac{\tau}{2L} \| v_k \|^2 + \frac{\alpha}{2} \| y_k - x_k \|^2.
\]

Hence, combining the last two inequalities, we obtain

\[
F(x) - F(x_k) \geq \frac{L}{2\tau} \left[ \| x_k - x \|^2 - \| y_k - x \|^2 \right] + \langle v_k, x - x_k \rangle + \frac{\tau}{2L} \| v_k \|^2 + \frac{\alpha}{2} \| y_k - x_k \|^2,
\]

which gives us

\[
F(x) - F(x_k) \geq \frac{L}{2\tau} \left[ \| x_k - x - \frac{\tau}{L} v_k \|^2 - \| y_k - x \|^2 \right] + \frac{\alpha}{2} \| y_k - x_k \|^2,
\]

implying that

\[
F(x) - F(x_k) \geq \frac{L}{2\tau} \left[ \| x_k - x - \frac{\tau}{L} v_k \|^2 - \| y_k - x \|^2 \right] + \frac{\alpha}{2} \| y_k - x_k \|^2,
\]

as desired.

**Theorem 6.** Let \((x_k, y_k, t_k)_{k \in \mathbb{N}}\) be the sequence generated by I-FISTA. Then, for all \(k \in \mathbb{N}\),

\[
2\tau \frac{\| t_k^2 (F(x_k) - F^*) - t_{k+1}^2 (F(x_{k+1}) - F^*) \|}{L} \geq \| u_{k+1} \|^2 - \| u_k \|^2 + \frac{\alpha t_{k+1}^2}{L} \| y_{k+1} - x_{k+1} \|^2,
\]

where

\[
u_k := t_k (x_k - x_{k-1}) - \frac{\tau}{L} t_k v_k + (x_{k-1} - x_*)\]

and \(x_* \in S_*\).

**Proof.** Let \(x_* \in S_*\). Using Proposition 5 with \(k + 1\) in place of \(k\) and at \(x = x_*\) and \(x = x_k\), we have

\[
-(F(x_{k+1}) - F^*) \geq \frac{L}{2\tau} \left[ \| x_{k+1} - x_* - \frac{\tau}{L} v_{k+1} \|^2 - \| y_{k+1} - x_* \|^2 \right] + \frac{\alpha}{2} \| y_{k+1} - x_{k+1} \|^2,
\]

\[
F(x_k) - F(x_{k+1}) \geq \frac{L}{2\tau} \left[ \| x_{k+1} - x_k - \frac{\tau}{L} v_{k+1} \|^2 - \| y_{k+1} - x_k \|^2 \right] + \frac{\alpha}{2} \| y_{k+1} - x_{k+1} \|^2.
\]

By multiplying the second inequality by \((t_{k+1} - 1)\) and adding it to the first inequality above, we obtain

\[
(t_{k+1} - 1)(F(x_k) - F^*) - t_{k+1}(F(x_{k+1}) - F^*) \geq \frac{L}{2\tau} \left[ \| x_{k+1} - x_* - \frac{\tau}{L} v_{k+1} \|^2 - \| y_{k+1} - x_* \|^2 + \frac{\alpha t_{k+1}^2}{2} \| y_{k+1} - x_{k+1} \|^2 \right] + \frac{L(t_{k+1} - 1)}{2\tau} \left[ \| x_{k+1} - x_k - \frac{\tau}{L} v_{k+1} \|^2 - \| y_{k+1} - x_k \|^2 \right].
\]
Multiplying now by $2\tau t_{k+1}/L$ in the last inequality and then using part (ii) of Lemma 3 (i.e., $t_{k+1}(t_{k+1} - 1) = t_k^2$), we have

$$\frac{2\tau}{L} \left[ t_k^2(F(x_k) - F^*) - t_{k+1}^2(F(x_{k+1}) - F^*) \right] \geq \left( t_{k+1}^2 - t_k^2 \right) \frac{\|x_{k+1} - x_k - \tau v_{k+1}\|^2}{L}$$

$$- \left( t_{k+1}^2 - t_k^2 \right) \frac{\|y_{k+1} - x_k\|^2}{L} + t_{k+1} \frac{\|x_{k+1} - x_* - \tau v_{k+1}\|^2}{L}$$

$$- t_{k+1} \frac{\|y_{k+1} - x_*\|^2}{L} + \tau \alpha t_{k+1}^2 \frac{\|y_{k+1} - x_{k+1}\|^2}{L},$$

which implies that

$$\frac{2\tau}{L} \left[ t_k^2(F(x_k) - F^*) - t_{k+1}^2(F(x_{k+1}) - F^*) \right]$$

$$\geq \left\| t_{k+1}(x_{k+1} - x_k) - \frac{\tau}{L} t_{k+1} v_{k+1} \right\|^2 - \left\| t_{k+1}(y_{k+1} - x_k) \right\|^2$$

$$+ t_{k+1} \left( \left\| y_{k+1} - x_k \right\|^2 - \left\| x_{k+1} - x_k - \frac{\tau}{L} v_{k+1} \right\|^2 \right)$$

$$+ t_{k+1} \left( \left\| x_{k+1} - x_* - \frac{\tau}{L} v_{k+1} \right\|^2 - \left\| y_{k+1} - x_* \right\|^2 \right) + \frac{\tau \alpha t_{k+1}^2}{L} \frac{\|y_{k+1} - x_{k+1}\|^2}{L}.$$ (20)

Now, from the definitions of $y_{k+1}$ and $u_k$ in (13) and (19), respectively, we have

$$\left\| t_{k+1}(x_{k+1} - x_k) - \frac{\tau}{L} t_{k+1} v_{k+1} \right\|^2 - \left\| t_{k+1}(y_{k+1} - x_k) \right\|^2$$

$$= \left\| u_{k+1} - (x_k - x_*) \right\|^2 - \left\| u_k - (x_k - x_*) \right\|^2$$

$$= \left\| u_{k+1} \right\|^2 - \left\| u_k \right\|^2 + 2 \left\langle u_{k+1} - u_k, x_k - x_* \right\rangle$$

$$= \left\| u_{k+1} \right\|^2 - \left\| u_k \right\|^2 + 2 t_{k+1} \left\langle y_{k+1} - x_{k+1} + \frac{\tau}{L} v_{k+1}, x_k - x_* \right\rangle$$

$$= \left\| u_{k+1} \right\|^2 - \left\| u_k \right\|^2 + 2 t_{k+1} \left[ \left( y_{k+1} - x_{k+1}, x_k - x_* \right) - \left( x_{k+1} - x_k - \frac{\tau}{L} v_{k+1}, x_k - x_* \right) \right]$$

$$= \left\| u_{k+1} \right\|^2 - \left\| u_k \right\|^2 + t_{k+1} \left( \left\| y_{k+1} - x_* \right\|^2 - \left\| y_{k+1} - x_k \right\|^2 \right)$$

$$+ t_{k+1} \left( \left\| x_{k+1} - x_* - \frac{\tau}{L} v_{k+1} \right\|^2 - \left\| x_{k+1} - x_k - \frac{\tau}{L} v_{k+1} \right\|^2 \right).$$

Therefore, (18) now follows from (20) and the last equality.\[\square\]

**Theorem 7.** Let $d_0$ be the distance from $x_0$ to $S_*$. Let $(x_k, y_k, t_k)_{k \in \mathbb{N}}$ be the sequence generated by 1-FISTA. Then, for all $k \in \mathbb{N}$,

$$t_k^2(F(x_k) - F^*) + \frac{\alpha}{2} \sum_{i=1}^{k} t_i \| y_i - x_i \|^2 \leq \frac{L}{2\tau} d_0^2. \tag{21}$$

In particular,

$$F(x_k) - F^* \leq \frac{2L}{\tau(k+1)^2} d_0^2. \tag{22}$$

**Proof.** Summing (18) in Theorem 6 from $k := 1$ to $k := k - 1$, and using the fact that $t_1 = 1$, we obtain

$$\frac{2\tau}{L} t_k^2(F(x_k) - F^*) + \| u_k \|^2 + \frac{\tau \alpha}{L} \sum_{i=2}^{k} t_i \| y_i - x_i \|^2 \leq \frac{2\tau}{L} (F(x_1) - F^*) + \| u_1 \|^2. \tag{23}$$
Now let $x_*$ be the projection of $x_0$ onto $S_\ast$. Then $d_0 = \|x_0 - x_*\|$. From Proposition 5 at $k = 1$ and $x = x_*$, and using the fact that $y_1 = x_0$, $u_1 = x_1 - x_* - \frac{\tau}{L}v_1$, and $t_1 = 1$, we have that

$$\frac{2\tau}{L}(F(x_1) - F^\ast) \leq \|y_1 - x_*\|^2 - \|x_1 - x_* - \frac{\tau}{L}v_1\|^2 - \frac{\tau\alpha}{L}\|y_1 - x_1\|^2 = \|x_0 - x_*\|^2 - \|u_1\|^2 - \frac{\tau\alpha}{L}\|y_1 - x_1\|^2.$$  

This inequality together with (23) imply (21). To prove (22), note that part (i) of Lemma 3 implies $t_k \geq \frac{k+1}{2}$, hence the result follows directly from (21).

Theorem 8. Let $d_0$ be the distance from $x_0$ to $S_\ast$. Let $(x_k, y_k, t_k)_{k \in \mathbb{N}}$ be the sequence generated by I-FISTA. Then, for every $k \in \mathbb{N}$,

$$r_k \in \partial_{x_k}g(x_k) + \nabla f(x_k) \subset \partial_{x_k}F(x_k),$$

where $r_k := v_k + L(y_k - x_k)/\tau + \nabla f(x_k) - \nabla f(y_k)$. Additionally, if $\tau < 1$ and $\alpha \in (0, L(1 - \tau)/\tau]$, then there exists $\ell_k \leq k$ such that

$$(24) \quad \|r_{\ell_k}\| = O\left(d_0 \sqrt{L^3/k^3}\right), \quad \varepsilon_{\ell_k} = O\left(d_0^2 L^2/k^3\right),$$

Proof. The inclusion follows from (14). Now let $x_*$ be the projection of $x_0$ onto $S_\ast$. It follows from (21) that

$$\min_{i=1,...,k} \|y_i - x_i\|^2 \leq \frac{L}{\alpha\tau \sum_{i=1}^k t_i^2} d_0^2,$$

which, when combined with part (i) of Lemma 3, yields

$$\min_{i=1,...,k} \|y_i - x_i\|^2 \leq \frac{4L}{\alpha\tau \sum_{i=1}^k (i + 1)^2} d_0^2.$$  

Since

$$\sum_{i=1}^k (i + 1)^2 = \frac{k(k + 1)(2k + 1)}{6} + k(k + 2) \geq \frac{k^3}{3}, \quad \forall k \geq 1,$$

we obtain

$$\min_{i=1,...,k} \|y_i - x_i\|^2 \leq \frac{12L}{\alpha\tau k^3} d_0^2.$$  

Hence, there exists $\ell_k \leq k$ such that

$$(25) \quad \|y_{\ell_k} - x_{\ell_k}\| \leq 2 \sqrt{\frac{3L}{\alpha\tau k^3}} d_0.$$  

From the definition of $r_k$, condition (15) for $\|v_k\|$ in the IR Rule, and the Lipschitz continuity of $\nabla f$, we have

$$\|r_{\ell_k}\| \leq \|v_{\ell_k}\| + \frac{L}{\tau} \|y_{\ell_k} - x_{\ell_k}\| + \|\nabla f(x_{\ell_k}) - \nabla f(y_{\ell_k})\| \leq \left(\sqrt{L(1 - \tau)L - \alpha\tau} + \frac{L}{\tau} + L\right) \|y_{\ell_k} - x_{\ell_k}\| \leq 2L \left(\frac{\sqrt{1 - \tau} + 1 + \tau}{\tau}\right) \sqrt{\frac{3L}{\alpha\tau k^3}} d_0,$$
which implies the first part of (24). Moreover, it follows from condition (15) for \( \varepsilon_k \) in the IR Rule that
\[
\varepsilon_{t_k} \leq \frac{(1-\tau)L - \alpha \tau}{2\tau} \|x_{t_k} - y_{t_k}\|^2 \leq \frac{6L[(1-\tau)L - \alpha \tau]}{\alpha \tau^2 k^3} \delta_0^2,
\]
which proves the second part of (24).

5. Inexact extragradient accelerated method. We now formally present our inexact accelerated method with an extra-step.

**Algorithm 3 (IE-FISTA).** Let \( x_0, y_0 \in \mathbb{E}, \alpha > 1/L \) and \( \sigma \in [0,1] \) be given, and set \( \lambda := \alpha/(1+\alpha L), \tau_0 := 0, \tilde{x}_0 := x_0 \) and \( k := 0 \).

**Iterative Step.** Compute
\[
\tau_{k+1} := \tau_k + \frac{\lambda + \sqrt{\lambda^2 + 4\lambda \tau_k}}{2},
\]
\[
y_k := \frac{\tau_k - \tilde{x}_k}{\tau_{k+1}} \left( \frac{\tau_{k+1} - \tau_k}{\tau_{k+1}} - x_k \right),
\]
and find a triple
\[
(\tilde{x}_{k+1}, v_{k+1}, \varepsilon_{k+1}) \in J^n_0(y_k, 1/L)
\]
given in IER Rule, and set
\[
x_{k+1} := x_k - (\tau_{k+1} - \tau_k)(v_{k+1} + L(y_k - \tilde{x}_{k+1})).
\]

Note that the triple \((\tilde{x}_{k+1}, v_{k+1}, \varepsilon_{k+1})\) in the iterative step of IE-FISTA satisfies
\[
v_{k+1} \in \partial g(\tilde{x}_{k+1}) + L(\tilde{x}_{k+1} - y_k) + \nabla f(y_k),
\]
\[
\|\alpha v_{k+1} + \tilde{x}_{k+1} - y_k\|^2 + 2\alpha \varepsilon_{k+1} \leq \sigma^2 \|\tilde{x}_{k+1} - y_k\|^2.
\]
If \( \sigma = 0 \), it follows from (30) that \( \varepsilon_{k+1} = 0 \) and \( v_{k+1} = (y_k - \tilde{x}_{k+1})/\alpha \), giving us
\[
\tilde{x}_{k+1} = \arg\min_{x \in \mathbb{E}} \left\{ g(x) + \frac{1}{2\lambda} \|x - (y_k - \lambda \nabla f(y_k))\|^2 \right\},
\]
\[
x_{k+1} = x_k - \frac{(\tau_{k+1} - \tau_k)}{\lambda} (y_k - \tilde{x}_{k+1});
\]
hence, IE-FISTA recovers the exact version proposed in [22, Algorithm I].

We begin the complexity analysis of IE-FISTA by first defining the sequence \((\mu_k)_{k \in \mathcal{N}}\) as
\[
\mu_k := f(\tilde{x}_k) - [f(y_{k-1}) + (\nabla f(y_{k-1}), \tilde{x}_k - y_{k-1})], \quad \forall \ k \in \mathcal{N}.
\]
We also consider the affine maps \( \Psi_k : \mathbb{E} \to \mathbb{R} \) given by
\[
\Psi_k(x) := \tilde{F}(\tilde{x}_k) + (v_k + L(y_{k-1} - \tilde{x}_k), x - \tilde{x}_k) - \mu_k - \varepsilon_k, \quad \forall \ x \in \mathbb{E} \text{ and } k \in \mathcal{N},
\]
and \( \Gamma_k : \mathbb{E} \to \mathbb{R} \) defined as
\[
\Gamma_0(x) \equiv 0, \quad \Gamma_{k+1}(x) := \tau_k \tau_{k+1} \Gamma_k(x) + \tau_{k+1} \Gamma_{k+1}(x), \quad \forall \ x \in \mathbb{E} \text{ and } k \geq 0.
\]

**Lemma 9.** Let \((x_k, \tilde{x}_k, y_k)_{k \in \mathcal{N}}\) be the sequence generated by IE-FISTA. Then the following hold.
(i) For all \( k \in \mathbb{N} \),
\[
\mu_k \leq \frac{L}{2} \|\tilde{x}_k - y_{k-1}\|^2.
\]

(ii) For all \( k \geq 0 \),
\[
x_k = \arg\min_{x \in E} \left\{ \tau_k \Gamma_k(x) + \frac{1}{2} \|x - x_0\|^2 \right\}.
\]

Proof. For part (i), we have that inequality (34) follows from (31) and (9). To prove part (ii), we first observe that (32) and (33) imply that
\[
\tau_k \nabla \Gamma_k(x) = \sum_{i=1}^{k} (\tau_i - \tau_{i-1}) \left( v_i + L(y_{i-1} - \tilde{x}_i) \right), \quad \forall x \in \mathbb{E} \text{ and } k \in \mathbb{N}.
\]
Combining (36) with (28) implies
\[
x_k = x_0 - \sum_{i=1}^{k} (\tau_i - \tau_{i-1}) \left( v_i + L(y_{i-1} - \tilde{x}_i) \right) = x_0 - \tau_k \nabla \Gamma_k(x).
\]
Hence, \( 0 = \tau_k \nabla \Gamma_k(x) + x_k - x_0 \), which proves (35).

Lemma 10. Let \((x_k, \tilde{x}_k, y_k)_{k \in \mathbb{N}}\) be the sequence generated by IE-FISTA. Then the following hold.

(i) For all \( k \in \mathbb{N} \),
\[
\Psi_k(x) \leq F(x), \quad \forall x \in \mathbb{E}.
\]

(ii) For all \( k \geq 0 \),
\[
\tau_k \Gamma_k(x) \leq \tau_k F(x), \quad \forall x \in \mathbb{E}.
\]

Proof. First note that from part (ii) of Lemma 1 and from the definition of \( \mu_k \) (31), we have that \( \nabla f(y_{k-1}) \in \partial \mu_k f(\tilde{x}_k) \). Hence, it follows from (29) and part (i) of Lemma 1 that
\[
v_k + L(y_{k-1} - \tilde{x}_k) \in \partial \varepsilon_k g(\tilde{x}_k) + \nabla f(y_{k-1}) \subset \partial \varepsilon_k + \mu_k F(\tilde{x}_k),
\]
which is equivalent to
\[
F(\tilde{x}_k) + \left( v_k + L(y_{k-1} - \tilde{x}_k), x - \tilde{x}_k \right) - \mu_k - \varepsilon_k \leq F(x), \quad \forall x \in \mathbb{E}.
\]
Thus (37) follows from the definition of \( \Psi_k(x) \) given in (32), which proves part (i).

To prove part (ii), we use (33) and write
\[
\tau_k \Gamma_k(x) = \tau_{k-1} \Gamma_{k-1}(x) + (\tau_k - \tau_{k-1}) \Psi_k(x) = \sum_{i=1}^{k} (\tau_i - \tau_{i-1}) \Psi_i(x).
\]
Then, using item (i), we obtain (38), which concludes the proof.

We next establish a key result for the complexity analysis of IE-FISTA.
Proposition 11. For every $k \geq 0$, let

\begin{equation}
\beta_k := \min_{x \in \mathbb{R}} \left\{ \tau_k \Gamma_k(x) + \frac{1}{2} \|x - x_0\|^2 \right\} - \tau_k F(\tilde{x}_k).
\end{equation}

Then,

\begin{equation}
\beta_{k+1} \geq \beta_k + \frac{(1 - \sigma^2)\tau_{k+1}}{2\alpha} \|\tilde{x}_{k+1} - y_k\|^2.
\end{equation}

Proof. Let $u \in E$. Using the definition of $\Gamma_k$ in (33), we obtain

\begin{equation}
\tau_{k+1} \Gamma_k(x_k) + \frac{1}{2} \|u - x_0\|^2 = \tau_k \Gamma_k(u) + \frac{1}{2} \|u - x_0\|^2 + (\tau_{k+1} - \tau_k) \Psi_{k+1}(u)
\end{equation}

where the last equality is due to the fact that $x_k$ is the minimum point of the quadratic function $\tau_k \Gamma_k(x) + \|x - x_0\|^2/2$ (see part (ii) of Lemma 9). Next, using part (i) of Lemma 10 and the fact that $\Psi_k$ is an affine function, we have

\begin{equation}
(\tau_{k+1} - \tau_k) \Psi_{k+1}(u) \geq (\tau_{k+1} - \tau_k) \Psi_{k+1}(\tilde{x}_k) + \tau_k \Psi_{k+1}(\tilde{x}_k) - \tau_k F(\tilde{x}_k)
\end{equation}

Now we define

\begin{equation}
\tilde{c}(u) := \frac{\tau_k}{\tau_{k+1}} \tilde{x}_k + \frac{\tau_{k+1} - \tau_k}{\tau_{k+1}} u
\end{equation}

and use the definition of $y_k$ in (27) to obtain

\begin{equation}
(\tau_{k+1} - \tau_k) \Psi_{k+1}(u) + \frac{1}{2} \|u - x_k\|^2 \geq \tau_{k+1} \left( \Psi_{k+1}(\tilde{c}(u)) + \frac{\tau_{k+1}}{2(\tau_{k+1} - \tau_k)^2} \|\tilde{c}(u) - y_k\|^2 \right) - \tau_k F(\tilde{x}_k).
\end{equation}

Hence, it follows from (42) and item (i) from Lemma 4 that

\begin{equation}
\tau_{k+1} \Gamma_k(x_k) + \frac{1}{2} \|u - x_0\|^2 \geq \tau_k \Gamma_k(x_k) + \frac{1}{2} \|x_k - x_0\|^2 - \tau_k F(\tilde{x}_k)
\end{equation}

Now, using (35) and the definitions of $\beta_k$ and $\Psi_k$ in (40) and (32), respectively, we have

\begin{equation}
\beta_k + \tau_{k+1} \left( \langle v_{k+1} + L(y_k - \tilde{x}_{k+1}), \tilde{c}(u) - \tilde{x}_{k+1} \rangle - \mu_{k+1} \varepsilon_{k+1} + \frac{1}{2\alpha} \|\tilde{c}(u) - y_k\|^2 \right).
\end{equation}

From part (i) of Lemma 9, we find that

\begin{equation}
L(y_k - \tilde{x}_{k+1}, \tilde{c}(u) - \tilde{x}_{k+1}) - \mu_{k+1} 
\geq -L(y_k - \tilde{x}_{k+1}, \tilde{x}_{k+1} - \tilde{c}(u)) - \frac{L}{2} \|\tilde{x}_{k+1} - y_k\|^2
\end{equation}

\begin{equation}
= -\frac{L}{2} \|y_k - \tilde{c}(u)\|^2 + \frac{L}{2} \|\tilde{x}_{k+1} - \tilde{c}(u)\|
\end{equation}

\begin{equation}
\geq -\frac{L}{2} \|y_k - \tilde{c}(u)\|^2.
\end{equation}
Combining the last two inequalities, we obtain
\[
\tau_{k+1} \Gamma_{k+1}(u) + \frac{1}{2} \|u - x_0\|^2 - \tau_{k+1} F(\tilde{x}_{k+1}) \geq \beta_k + \tau_{k+1} \left( \langle v_{k+1}, \tilde{c}(u) - \tilde{x}_{k+1} \rangle - \varepsilon_{k+1} + \frac{1}{2} \left( \frac{1}{\lambda} - L \right) \|\tilde{c}(u) - y_k\|^2 \right).
\]
(43)

Now, it follows from (30) that
\[
\frac{1 - \sigma^2}{2\alpha} \|\tilde{x}_{k+1} - y_k\|^2 \leq -\varepsilon_{k+1} + \langle v_{k+1}, y_k - \tilde{x}_{k+1} \rangle - \frac{\alpha}{2} \|v_{k+1}\|^2 = -\varepsilon_{k+1} + \langle v_{k+1}, \tilde{c}(u) - \tilde{x}_{k+1} \rangle - \frac{1}{2\alpha} \|\alpha v_{k+1} - (y_k - \tilde{c}(u))\|^2 + \frac{1}{2\alpha} \|y_k - \tilde{c}(u)\|^2,
\]
which implies that
\[
\langle v_{k+1}, \tilde{c}(u) - \tilde{x}_{k+1} \rangle - \varepsilon_{k+1} \geq \frac{1 - \sigma^2}{2\alpha} \|\tilde{x}_{k+1} - y_k\|^2 - \frac{1}{2\alpha} \|\tilde{c}(u) - y_k\|^2.
\]
Combining (43) and the last inequality, we have
\[
\tau_{k+1} \Gamma_{k+1}(u) + \frac{1}{2} \|u - x_0\|^2 - \tau_{k+1} F(\tilde{x}_{k+1}) \geq \beta_k + \tau_{k+1} \left( \frac{1 - \sigma^2}{\alpha} \|\tilde{x}_{k+1} - y_k\|^2 + \left( \frac{1}{\lambda} - L - \frac{1}{\alpha} \right) \|\tilde{c}(u) - y_k\|^2 \right).
\]
Now, using the fact that \(\lambda = \alpha/(1 + \alpha L)\), we obtain
\[
\tau_{k+1} \Gamma_{k+1}(u) + \frac{1}{2} \|u - x_0\|^2 - \tau_{k+1} F(\tilde{x}_{k+1}) \geq \beta_k + \frac{\tau_{k+1}}{2} \left( \frac{1 - \sigma^2}{\alpha} \|\tilde{x}_{k+1} - y_k\|^2 \right).
\]
Since \(u \in \mathbb{E}\) was chosen arbitrarily, this inequality holds for all \(u\). Thus, using (40), we conclude that
\[
\beta_{k+1} \geq \beta_k + \frac{(1 - \sigma^2)\tau_{k+1}}{2\alpha} \|\tilde{x}_{k+1} - y_k\|^2,
\]
which is the desired inequality.\(\square\)

The next result establishes the optimal convergence rate of \(F(\tilde{x}_k) - F^*\).

**Theorem 12.** Let \(d_0\) be the distance from \(x_0\) to \(S_\ast\). Let \((x_k, \tau_k, y_k)_{k \in \mathbb{N}}\) be the sequence generated by IE-FISTA. Then,
\[
\frac{1}{2} \|x_k - x_\ast\|^2 + \tau_k (F(\tilde{x}_k) - F^*) + \frac{1 - \sigma^2}{2\alpha} \sum_{i=1}^k \tau_i \|\tilde{x}_i - y_{i-1}\|^2 \leq \frac{1}{2} d_0^2.
\]
(44)

In particular,
\[
F(\tilde{x}_k) - F^* \leq \frac{2(1 + \alpha L)}{\alpha k^2} d_0^2.
\]

**Proof.** Let \(x_\ast\) be the projection of \(x_0\) onto \(S_\ast\). Using (41) recursively and the fact that \(\beta_0 = 0\), we have
\[
\beta_k \geq \frac{1 - \sigma^2}{2\alpha} \sum_{i=1}^k \tau_i \|\tilde{x}_i - y_{i-1}\|^2.
\]
(45)
From (35) we have that $x_k$ is the minimum point of the quadratic function $\tau_k \Gamma_k(x) + \|x - x_0\|^2/2$ and

$$\tau_k \Gamma_k(x_*) + \frac{1}{2} \|x_* - x_0\|^2 = \min_{x \in E} \left\{ \tau_k \Gamma_k(x) + \frac{1}{2} \|x - x_0\|^2 \right\} + \frac{1}{2} \|x_* - x_k\|^2.$$ 

Combining this with (45) and (40) yields

$$\frac{1}{2} \|x_k - x_*\|^2 + \tau_k (F(\tilde{x}_k) - \Gamma_k(x_*)) + \frac{1 - \sigma^2}{2\alpha} \sum_{i=1}^k \tau_i \|\tilde{x}_i - y_i - 1\|^2 \leq \frac{1}{2} d_0^2.$$ 

Hence, inequality (44) follows from part (ii) of Lemma 10.

The second part of the theorem follows from the first part, and by part (ii) of Lemma 4 and the fact that $\lambda := \alpha/(1 + \alpha L)$.

We now present iteration-complexity bounds for IE-FISTA to obtain approximate solutions of (1) in the sense of Definition 2.

**Theorem 13.** Let $(x_k, r_k, y_k)_{k \in \mathbb{N}}$ be the sequence generated by IE-FISTA. Then,

$$\nu_k \in \partial \nu_k g(\check{x}_k) + \nabla f(y_k - 1) \subset \partial \nu_k + \mu_k F(\check{x}_k), \quad k \in \mathbb{N},$$

where $r_k := v_k + L(y_{k-1} - \check{x}_k)$. Additionally, if $\sigma < 1$, then IE-FISTA generates a $\rho$-approximate solution $\check{x}_k$ of problem (1) with residues $(r_k, \varepsilon_k, \mu_k)$ in the sense of Definition 2 in at most $k = \mathcal{O}((d_0/\rho)^{2/3})$ iterations, where $\rho \in (0, 1)$ is a given tolerance and $d_0$ is the distance from $x_0$ to $S_*$. 

**Proof.** The first statement of the theorem follows from (39) and the definition of $r_k$. It follows from (44) that

$$\min_{i=1, \ldots, k} \|\check{x}_i - y_i - 1\|^2 \leq \frac{\alpha}{1 - \sigma^2} \sum_{i=1}^k \tau_i d_0^2,$$

which, when combined with part (ii) of Lemma 4, yields

$$\min_{i=1, \ldots, k} \|\check{x}_i - y_i - 1\|^2 \leq \frac{4\alpha}{\lambda(1 - \sigma^2)} \sum_{i=1}^k \tau_i d_0^2.$$ 

Since

$$\sum_{i=1}^k \tau_i^2 = \frac{k(k+1)(2k+1)}{6} \geq \frac{k^3}{3}, \quad \forall k \geq 1,$$

we obtain

$$\min_{i=1, \ldots, k} \|\check{x}_i - y_i - 1\|^2 \leq \frac{12\alpha}{\lambda(1 - \sigma^2)k^3} d_0^2.$$ 

Hence, there exists $1 \leq \ell \leq k$ such that

$$\|\check{x}_\ell - y_{\ell - 1}\|^2 \leq \frac{12\alpha}{\lambda(1 - \sigma^2)k^3} d_0^2.$$ 

Since the error condition in (30) implies that

$$\|\alpha v_\ell - \|\check{x}_\ell - y_{\ell - 1}\| \leq \|\alpha v_\ell + \check{x}_\ell - y_{\ell - 1}\| \leq \sigma \|\check{x}_\ell - y_{\ell - 1}\|,$$

we obtain, from the definition of $r_k$, that

$$\|r_\ell\| \leq \|v_\ell\| + L\|y_{\ell - 1} - \check{x}_\ell\| \leq \left(1 + \frac{\sigma}{\alpha} + L\right) \|\check{x}_\ell - y_{\ell - 1}\|. $$
It then follows from (47) that
\[ \|r_\ell\| \leq \left( \frac{1 + \sigma}{\alpha} + L \right) \sqrt{\frac{12\alpha}{\lambda(1 - \sigma^2)} d_0 k^{3/2}}. \]

In addition, from (30), (47), and \( \lambda = \alpha/(1 + \alpha L) \), we have that
\[ \varepsilon_\ell \leq \frac{\sigma^2}{2\alpha} \|\tilde{x}_\ell - y_{\ell-1}\|^2 \leq \frac{6\sigma^2}{\lambda(1 - \sigma^2)k^3} d_0^2 = \frac{6(1 + \alpha L)\sigma^2}{\alpha(1 - \sigma^2)k^3} d_0^2. \]

Moreover, (34), (47), and \( \lambda = \alpha/(1 + \alpha L) \) gives us that
\[ \mu_\ell \leq \frac{L}{2} \|\tilde{x}_\ell - y_{\ell-1}\|^2 \leq \frac{6\alpha L}{\lambda(1 - \sigma^2)k^3} d_0^2 = \frac{6L(1 + \alpha L)}{(1 - \sigma^2)k^3} d_0^2. \]

Combining the last two inequalities, we have
\[ \varepsilon_\ell + \mu_\ell \leq \frac{6(1 + \alpha L)(\sigma^2 + \alpha L)}{\alpha(1 - \sigma^2)k^3} d_0^2. \]

Choosing \( k \) so that
\[ \max \left\{ \left( \frac{1 + \sigma}{\alpha} + L \right) \sqrt{\frac{12\alpha}{\lambda(1 - \sigma^2)} k^{3/2}}, \frac{6(1 + \alpha L)(\sigma^2 + \alpha L)}{\alpha(1 - \sigma^2)k^3} d_0^2 \right\} \leq \rho, \]

gives us
\[ r_\ell \in \partial_{\varepsilon_\ell + \mu_\ell} F(\tilde{x}_\ell), \quad \max\{\|r_\ell\|, \varepsilon_\ell + \mu_\ell\} \leq \rho, \]

which implies that \( \tilde{x}_\ell \) is a \( \rho \)-approximate solution of problem (1) with residues \( (r_\ell, \varepsilon_\ell + \mu_\ell) \).

6. Numerical experiments. In this section we explore the numerical behavior of Algorithm 2 (I-FISTA) and Algorithm 3 (IE-FISTA) and compare them to the inexact method with \( H_k = L I d \) described in [19] that uses the inexact absolute rule (IA Rule),
\[ v_k \in \partial_{\varepsilon_k} g(x_k) + L(x_k - y_k) + \nabla f(y_k), \quad \frac{1}{\sqrt{L}} \|v_k\| \leq \frac{\delta_k}{\sqrt{2t_k}}, \quad \varepsilon_k = \frac{\xi_k}{2t_k^2}, \]

where \( (\delta_k)_{k \in \mathbb{N}} \) and \( (\xi_k)_{k \in \mathbb{N}} \) are summable sequences of nonnegative numbers. In our numerical tests, we use \( \delta_k = t_k^{-2} \); by part (i) of Lemma 3, this choice for \( (\delta_k)_{k \in \mathbb{N}} \) is summable. We explain in detail below how \( \varepsilon_k \) is computed. Based on this choice for \( \varepsilon_k \), we can expect \( \varepsilon_k \) to be quite small; in numerical tests we observed \( \varepsilon_k \) to be approximately machine epsilon. For this reason, we do not explicitly enforce the above condition on \( \varepsilon_k \) in our implementation. We will refer to the algorithm using the IA Rule as IA-FISTA.

We follow [19] by considering the \( H \)-weighted nearest correlation matrix problem for our numerical tests. All algorithms were implemented in the Julia language [13] and all tests were run on a machine with a 2.9 GHz Dual-Core Intel Core i5 processor and 16 GB 1867 MHz DDR3 memory.

It is important to note that the goal of this section is not to demonstrate that the code we developed is state-of-the-art for solving the \( H \)-weighted nearest correlation
matrix problem. Rather our goal is to investigate how three different theoretical algorithms perform in practice, giving us insight beyond the convergence results presented in this paper. This is especially interesting since these three algorithms all have the same optimal rate of convergence. Here we see if they can be distinguished by their numerical performance on a set of test instances of the $H$-weighted nearest correlation matrix problem.

6.1. The nearest correlation matrix problem. Let $S^n$ be the set of $n \times n$ real symmetric matrices. Let $G, H \in S^n$ and define $f : S^n \rightarrow \mathbb{R}$ by

$$f(X) = \frac{1}{2} \| H \circ (X - G) \|_F^2,$$

where $\circ$ is the Hadamard product and $\| \cdot \|_F$ is the Frobenius norm. We seek the minimizer of $f$ over the set $C$ of $n \times n$ correlation matrices, which is defined as the set of $n \times n$ symmetric positive semidefinite matrices with all ones on the diagonal; that is,

$$C := \{ X \in S^n \mid \text{diag}(X) = e, X \succeq 0 \},$$

where $e \in \mathbb{R}^n$ is the vector of all ones and $\text{diag} : S^n \rightarrow \mathbb{R}^n$ is the linear map that returns the vector along the diagonal of the input matrix. The adjoint linear map of $\text{diag}$ is $\text{Diag} : \mathbb{R}^n \rightarrow S^n$ which maps a vector of length $n$ to the $n \times n$ diagonal matrix having that vector along its diagonal; indeed, it is easy to verify that $\langle v, \text{diag}(M) \rangle = \langle \text{Diag}(v), M \rangle$ for all $v \in \mathbb{R}^n$ and $M \in S^n$, where the vector inner-product is $\langle x, y \rangle := x^T y$ for $x, y \in \mathbb{R}^n$ and the symmetric matrix inner-product is $\langle X, Y \rangle := \text{trace}(XY)$ for $X, Y \in S^n$.

Let $g : S^n \rightarrow \mathbb{R} \cup \{ +\infty \}$ be defined by

$$g(X) = \delta_C(X) = \begin{cases} 0, & X \in C, \\ +\infty, & X \notin C. \end{cases}$$

The $H$-weighted nearest correlation matrix (H-NCM) problem is

$$\min_{X \in C} f(X) = \min_{X \in S^n} f(X) + g(X).$$

Note that the gradient of $f$ is given by $\nabla f(X) = H \circ H \circ (X - G)$ and has Lipschitz constant $L := \| H \circ H \|_F$. The KKT optimality conditions for (48) are given by

$$\nabla f(X) - \text{Diag}(y) - \Lambda = 0, \quad \text{diag}(X) = e, \quad X \succeq 0, \quad \Lambda \succeq 0, \quad \langle \Lambda, X \rangle = 0.$$

6.2. The subproblem. The subproblem at $Y \in S^n$ is given by

$$\min_{X \in S^n} f(Y) + \langle \nabla f(Y), X - Y \rangle + \frac{L}{2\tau} \| X - Y \|_F^2 + g(X).$$

The KKT optimality conditions for the subproblem are given by

$$\nabla f(Y) + \frac{L}{\tau} (X - Y) - \text{Diag}(y) - \Lambda = 0, \quad \text{diag}(X) = e, \quad X \succeq 0, \quad \Lambda \succeq 0, \quad \langle \Lambda, X \rangle = 0.$$

The dual objective function of the subproblem is, up to an additive constant and a change in sign, given by

$$\phi(y) := \frac{L}{2\tau} \left\| \left[ Y - \frac{\tau}{L} (\nabla f(Y) - \text{Diag}(y)) \right] \right\|_F^2 - \langle e, y \rangle.$$
Note that \( h \) is a differentiable convex function with gradient
\[
\nabla \phi(y) = \text{diag} \left( \left[ Y - \frac{\tau}{L} (\nabla f(Y) - \text{Diag}(y)) \right]^+ \right) - e.
\]
Suppose that \( y \) solves
\[
\min_{y \in \mathbb{R}^n} \phi(y).
\]
Then \( \nabla \phi(y) = 0 \). We define \( M, X, \) and \( \Lambda \) by
\[
M := Y - \frac{\tau}{L} (\nabla f(Y) - \text{Diag}(y)), \quad X := M_+, \quad \Lambda := \frac{L}{\tau} (X - M) = -\frac{L}{\tau} M_-,
\]
where \( M_+ \) and \( M_- \) are the projections of \( M \) onto the set of positive semidefinite and negative semidefinite matrices, respectively. Note that \( M = M_+ + M_- \) and \( \langle M_+, M_- \rangle = 0 \) by the Moreau decomposition theorem. Thus we have \( X \succeq 0 \), \( \text{diag}(X) = e \), \( \Lambda \succeq 0 \), and \( \langle \Lambda, X \rangle = 0 \). Moreover,
\[
\Lambda = \frac{L}{\tau} (X - M) = \frac{L}{\tau} (X - Y) + \nabla f(Y) - \text{Diag}(y),
\]
which implies that
\[
\nabla f(Y) + \frac{L}{\tau} (X - Y) - \text{Diag}(y) - \Lambda = 0.
\]
Thus, by minimizing the function \( h \), we obtain the optimal solution of the subproblem. Furthermore, letting
\[
\Gamma := -\text{Diag}(y) - \Lambda,
\]
we have \( \Gamma \in \partial g(X) \). Indeed, if \( Z \in C \), then
\[
g(X) + \langle \Gamma, Z - X \rangle = \langle -\text{Diag}(y) - \Lambda, Z - X \rangle
\]
\[
= -\langle y, \text{diag}(Z) \rangle + \langle y, \text{diag}(X) \rangle - \langle \Lambda, Z \rangle + \langle \Lambda, X \rangle
\]
\[
= -\langle \Lambda, Z \rangle \leq 0 = g(Z),
\]
and if \( Z \notin C \), then \( g(Z) = +\infty \), so \( g(Z) \geq g(X) + \langle \Gamma, Z - X \rangle \) as well. Thus, we have shown that
\[
0 \in \nabla f(Y) + \frac{L}{\tau} (X - Y) + \partial g(X).
\]

6.3. Approximately solving the subproblem. In our implementation, we approximately minimize \( \phi(y) \) using the quasi-Newton method L-BFGS-B [23, 35]. Thus, we compute \( y \) such that \( \nabla \phi(y) \approx 0 \), implying that \( \text{diag}(X) \approx e \). Thus, we expect that \( X \notin C \) and \( g(X) = +\infty \). In order to satisfy the requirement that we have an \( \varepsilon \)-subgradient, it is necessary to have a point \( \hat{X} \in C \). As is done in [14, 19], we define \( d := \text{diag}(X) \) and \( D := \text{Diag}(d)^{-1/2} \); since \( d \approx e \), we have that \( D \succ 0 \). We then let
\[
\hat{X} := DXD.
\]
Since \( X \succeq 0 \) and \( D \succ 0 \), we have that \( \hat{X} \succeq 0 \); moreover, \( \text{diag}(\hat{X}) = e \), as required. Next we let
\[
\varepsilon := \langle \Lambda, \hat{X} \rangle \quad \text{and} \quad V := \nabla f(Y) + \frac{L}{\tau} (\hat{X} - Y) + \Gamma = \frac{L}{\tau} (\hat{X} - X).
\]
Note that $\varepsilon \geq 0$ since $\Lambda$ and $\hat{X}$ are both positive semidefinite. We claim that $\Gamma \in \partial_{\varepsilon} g(\hat{X})$. As before, if $Z \notin C$, then $g(Z) = +\infty$, so $g(Z) \geq g(\hat{X}) + \langle \Gamma, Z - \hat{X} \rangle - \varepsilon$ holds. If $Z \in C$, then

\[
g(\hat{X}) + \langle \Gamma, Z - \hat{X} \rangle - \varepsilon = -\langle \text{Diag}(y) - \Lambda, Z - \hat{X} \rangle - \langle \Lambda, \hat{X} \rangle
\]

\[
= -\langle y, \text{diag}(Z) \rangle + \langle y, \text{diag}(\hat{X}) \rangle - \langle \Lambda, Z \rangle + \langle \Lambda, \hat{X} \rangle - \langle \Lambda, \hat{X} \rangle
\]

\[
= -\langle \Lambda, Z \rangle \leq 0 = g(Z).
\]

Therefore, we have

\[
V \in \nabla f(Y) + \frac{L}{\tau} (\hat{X} - Y) + \partial_{\varepsilon} g(\hat{X}).
\]

### 6.4. Computing projections.

Minimizing $\phi(y)$ using a quasi-Newton method like L-BFGS-B requires us to evaluate $\phi(y)$ and its gradient $\nabla \phi(y)$ for each new candidate minimizer $y$. Each time we evaluate $\phi(y)$ and $\nabla \phi(y)$, we compute the projections $M_+$ and $M_-$ in order to compute $X$ and $\Lambda$. We do this by computing the full eigenvalue decomposition of $M$ and obtain $M_+$ (resp. $M_-$) by setting the negative (resp. positive) eigenvalues of $M$ to zero. The choice of eigensolver is important since around 90% of the computation time is spent computing the eigenvalue decomposition of $M$. In our implementation of I-FISTA, IE-FISTA, and IA-FISTA, we compute $M_+$ and $M_-$ using the LAPACK [1] `dsyevd` eigensolver to compute all the eigenvalues and eigenvectors of $M$; see Borsdorf and Higham [14] for more on choice of eigensolver for computing $M_+$ in a preconditioned Newton algorithm for the nearest correlation matrix problem.

### 6.5. Random instances.

For our numerical tests, we generate random $n \times n$ correlation matrices $U$ by sampling uniformly from the set of correlation matrices using the extended onion method [20]. We then generate $n \times n$ symmetric matrices $G$ and $H$ using the following Julia code, based on the parameters $\gamma, p \in [0, 1]$, where $\gamma$ controls the amount of noise in $G$ and $p$ controls the sparsity of $H$.

```julia
# Generate symmetric matrix E with ones on diagonal and off-diagonal entries
# sampled uniformly from the interval [-1, 1].
Etmp = 2 * rand(n, n) .- 1
E = Symmetric(triu(Etmp, 1) + I)

# Matrix G is the convex combination of the matrices U and E, with G = U when
# \gamma = 0 and G = E when \gamma = 1.
Gtmp = (1 - \gamma) .* U + \gamma .* E
G = Symmetric(triu(Gtmp, 1) + I)

# Generate symmetric matrix H with ones on diagonal and off-diagonal entries are
# uniformly sampled from the interval [0, 1] with probability p and are zero
# with probability 1 - p.
Htmp = [rand() < p ? rand() : 0.0 for i = 1:n, j = 1:n]
H = Symmetric(triu(Htmp, 1) + I)
```

For all our tests, we use $p = 0.5$ and we consider $n = 100, 200, \ldots, 800$ and $\gamma = 0.1, 0.2, \ldots, 1.0$, generating a random instance for each combination of $n$ and $\gamma$, giving us a total of eighty test instances.

### 6.6. Numerical tests.

As was done in [19], we obtain a good initial point that is used by all three methods by solving the nearest correlation matrix problem

\[
\min_{X \in C} \frac{1}{2} \| X - G \|_F^2
\]
Fig. 1. Convergence plot for the I-FISTA, IE-FISTA, and IA-FISTA methods on the $n = 400$ and $\gamma = 0.5$ test instance.

using the MATLAB code CorNewton3.m [30] which is based on the quadratically convergent semismooth Newton method in [29].

We also use a similar stopping criterion as the one used in [19]. We let $r_p$ and $r_d$ be the norm of the primal and dual equality constraints for problem (48); that is,

$$r_p := \|\text{diag}(\hat{X}) - e\|_2, \quad r_d := \|\nabla f(\hat{X}) - \text{Diag}(y) - \Lambda\|_F.$$ 

Note that we are guaranteed to have $r_p$ be approximately machine epsilon based on how $\hat{X}$ is computed. We stop each method when

$$\max\{r_p, r_d\} \leq \text{tol}.$$ 

In our tests, we use $\text{tol} = 10^{-1}$ since we found that using a smaller value of $\text{tol}$ results in significantly more function/gradient evaluations and much longer running times for all three methods, but does not alter the main conclusions we draw from our numerical tests.

An example of the typical convergence behavior of the three methods is shown in Figure 1 where the value of $\max\{r_p, r_d\}$ is plotted each time $\phi(y)$ and $\nabla \phi(y)$ are evaluated. Note, however, that during the linesearch procedure of L-BFGS-B, the value of $\max\{r_p, r_d\}$ may vary drastically, so, to obtain a plot without such noise, we replace those intermediate linesearch values with the value obtained at the termination of the linesearch or when the stopping condition for the subproblem is satisfied.

In Tables 1 and 2 we record the number of outer iterations ($k$), the number of function/gradient evaluations ($\text{fgs}$), and the total running time in seconds, but not including the time to compute the initial point. From these results, it is clear that $k$,
Table 1

The number of iterations (k), function/gradient evaluations (fgs), and time in seconds for the I-FISTA, IE-FISTA, and IA-FISTA methods for n = 100, 200, 300, 400.
| n  | γ  | I-FISTA |   | IE-FISTA |   | IA-FISTA |   |
|----|----|---------|---|----------|---|----------|---|
|    |    | k | fgs | time | k | fgs | time | k | fgs | time |
| 500| 0.10| 265 | 513 | 20.2 | 320 | 667 | 26.3 | 257 | 1203 | 45.8 |
|    | 0.20| 499 | 934 | 35.0 | 595 | 2429 | 94.6 | 484 | 9331 | 340.8 |
|    | 0.30| 529 | 924 | 35.3 | 626 | 2716 | 102.8 | 513 | 7012 | 261.6 |
|    | 0.40| 559 | 994 | 37.4 | 654 | 5697 | 214.6 | 539 | 9331 | 340.8 |
|    | 0.50| 573 | 1072 | 40.5 | 669 | 4672 | 175.9 | 553 | 7133 | 260.5 |
|    | 0.60| 591 | 1117 | 42.4 | 679 | 4781 | 182.3 | 569 | 7549 | 279.0 |
|    | 0.70| 598 | 1093 | 41.5 | 700 | 6730 | 256.2 | 578 | 7455 | 275.4 |
|    | 0.80| 607 | 1146 | 43.6 | 711 | 4724 | 180.1 | 594 | 8562 | 318.1 |
|    | 1.00| 626 | 1227 | 46.7 | 721 | 3796 | 144.9 | 604 | 9962 | 369.4 |
| 600| 0.10| 377 | 709 | 46.6 | 434 | 1102 | 72.0 | 366 | 2919 | 185.3 |
|    | 0.20| 586 | 1085 | 68.4 | 678 | 3109 | 194.5 | 567 | 9471 | 576.4 |
|    | 0.30| 629 | 1089 | 67.2 | 723 | 4896 | 303.7 | 610 | 8744 | 525.9 |
|    | 0.40| 662 | 1132 | 66.8 | 750 | 6461 | 383.7 | 636 | 9164 | 526.4 |
|    | 0.50| 683 | 1263 | 76.1 | 777 | 5335 | 318.3 | 660 | 10852 | 629.3 |
|    | 0.60| 696 | 1282 | 78.5 | 792 | 5446 | 332.7 | 673 | 12647 | 751.9 |
|    | 0.70| 710 | 1333 | 82.5 | 801 | 7079 | 439.5 | 685 | 12607 | 758.0 |
|    | 0.80| 729 | 1388 | 84.3 | 811 | 5087 | 307.4 | 703 | 9818 | 576.0 |
|    | 0.90| 733 | 1388 | 84.3 | 832 | 4203 | 257.1 | 708 | 10843 | 640.3 |
|    | 1.00| 739 | 1439 | 89.0 | 840 | 5726 | 354.7 | 715 | 10698 | 645.6 |
| 700| 0.10| 521 | 979 | 89.4 | 586 | 1859 | 169.5 | 498 | 6468 | 571.7 |
|    | 0.20| 673 | 1251 | 109.0 | 762 | 3394 | 298.6 | 651 | 10581 | 896.6 |
|    | 0.30| 725 | 1217 | 105.7 | 819 | 6842 | 596.5 | 703 | 12948 | 1094.2 |
|    | 0.40| 762 | 1424 | 124.1 | 847 | 5549 | 485.3 | 733 | 11215 | 951.6 |
|    | 0.50| 786 | 1380 | 121.6 | 871 | 6919 | 606.9 | 756 | 11574 | 987.7 |
|    | 0.60| 803 | 1524 | 133.3 | 888 | 7032 | 620.5 | 773 | 13455 | 1160.8 |
|    | 0.70| 819 | 1511 | 133.9 | 911 | 8809 | 783.1 | 791 | 15117 | 1301.1 |
|    | 0.80| 828 | 1624 | 142.8 | 932 | 9432 | 829.4 | 813 | 15936 | 1369.9 |
|    | 0.90| 842 | 1624 | 142.8 | 932 | 9432 | 829.4 | 813 | 15936 | 1369.9 |
|    | 1.00| 855 | 1661 | 148.7 | 941 | 9716 | 873.2 | 824 | 17285 | 1509.8 |
| 800| 0.10| 692 | 1286 | 165.3 | 756 | 3108 | 400.4 | 670 | 11408 | 1420.6 |
|    | 0.20| 762 | 1421 | 177.2 | 848 | 3697 | 454.4 | 738 | 14087 | 1682.3 |
|    | 0.30| 822 | 1347 | 173.1 | 907 | 6835 | 878.1 | 794 | 12437 | 1542.9 |
|    | 0.40| 862 | 1573 | 199.0 | 947 | 7783 | 995.7 | 832 | 14689 | 2041.1 |
|    | 0.50| 887 | 1639 | 208.1 | 970 | 7028 | 893.4 | 856 | 16092 | 1985.1 |
|    | 0.60| 910 | 1700 | 213.7 | 995 | 9012 | 1131.0 | 879 | 18385 | 2242.0 |
|    | 0.70| 923 | 1747 | 225.2 | 1013 | 9210 | 1188.9 | 892 | 18666 | 2296.6 |
|    | 0.80| 943 | 1805 | 231.3 | 1023 | 8011 | 1032.9 | 905 | 13242 | 1651.0 |
|    | 0.90| 960 | 1840 | 238.7 | 1036 | 10180 | 1324.2 | 927 | 14950 | 1892.3 |
|    | 1.00| 971 | 1884 | 243.9 | 1056 | 11607 | 1502.1 | 937 | 16054 | 2026.2 |

Table 2

The number of iterations (k), function/gradient evaluations (fgs), and time in seconds for the I-FISTA, IE-FISTA, and IA-FISTA methods for n = 500, 600, 700, 800.
fgs, and time increase for all three methods as $n$ increases and as $\gamma$ increases. However, we also see that although I-FISTA and IE-FISTA require more outer iterations than IA-FISTA, each require fewer total inner iterations (i.e., fgs), and hence less time, than IA-FISTA to solve each instance to the desired tolerance.

Here we include an interesting point. In our numerical tests we observed that L-BFGS-B was always able to satisfy the IR Rule, typically in a small number of iterations. However, we were curious to see that sometimes L-BFGS-B failed to satisfy the IER Rule and only stopped due to a failure of the linesearch or due to having identical function values on two consecutive function evaluations. We would like to investigate this behavior in greater detail in future research.

To see the forest for the trees, in Figure 2 we plot the performance profile [16, 17, 24] of the numerical results from Tables 1 and 2 using the total running time of each solver on each instance. From this plot we clearly see that I-FISTA is the fastest on all test instances and that IE-FISTA also outperforms IA-FISTA on the test instances. Thus, although all three algorithms have the same theoretical rate of convergence, we have demonstrated that the relative error rules and corresponding algorithms proposed in this paper, I-FISTA and, to a lesser extent, IE-FISTA, are potentially valuable to use in situations where IA-FISTA has proved successful in practice.

7. Final Remarks. This paper proposed and analyzed two inexact versions of FISTA for minimizing the sum of two convex functions. Both schemes allow their
subproblems to be solved inexactly subject to satisfying certain relative error rules. Numerical experiments were carried out in order to illustrate the numerical behavior of the methods. They indicate that the proposed methods based on inexact relative error rules are more efficient than those based on the inexact absolute error rule on a set of instances of the $H$-weighted nearest correlation matrix problem.

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