Reflection and refraction of electromagnetic waves at the plane boundary between two chiral media

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This work is concerned with the propagation of electromagnetic waves in isotropic chiral media and with the effects produced by a plane boundary between two such media. In analogy with the phenomena of reflection and refraction of plane electromagnetic waves in ordinary dielectrics, the kinematical and dynamical aspects of these phenomena are studied, such as the intensity of the various wave components and the change in the polarization of the wave as it crosses the boundary. As a prerequisite of this, we show that the plane wave solution must be written as a suitable superposition of the circularly amplitudes on both sides of the interface, we elucidate which is the appropriate set of conditions that the solution must satisfy at the boundary, and we set down the minimal, and complete, set of equations that must be solved for the coefficient amplitudes in order to satisfy the boundary conditions. The equations are solved explicitly for some particular cases and configurations (e.g., normal incidence), the salient features of those solutions are analyzed in some detail, and the general solution to the equations is given as well.

I. INTRODUCTION

The problems related to the propagation of electromagnetic waves in a homogeneous medium, including those of reflection and refraction at the interface between two media, form a traditional set in classical electrodynamics, and are discussed in any textbook on the subject [1]. A less familiar one is the problem of the propagation of an electromagnetic wave in a chiral medium. In such a medium, the dielectric constant has a different value depending on the polarization of the wave, which implies that the two circularly polarized waves propagate with different speeds.

A chiral medium, also called “optically active”, can exist for various reasons. For example, a material medium can become optically active under the influence of an external field, such as a magnetic field (the well-known Faraday effect discussed in many plasma physics textbooks [2]) or a (control) plane electromagnetic wave [3]. Other kind of media exhibit the phenomenon naturally, such as a gas of chiral molecules [4], or a neutrino gas [5, 6] that contains an unequal number of neutrinos and antineutrinos. Recently, man-made composite materials have been produced with similar properties [7].

Various approaches have been proposed in the literature to study the electromagnetic properties of such optically active media [4, 8, 9, 10, 11, 12, 13, 14, 15, 16]. As we showed in Ref. [17], a particularly simple and general method consists in parametrizing the induced charge and current in the most general way consistent with general principles, such as the conservation of the electromagnetic current and, if applicable, the isotropy and homogeneity of the medium.

However, the formalism developed in Ref. [17] concerns a homogeneous medium only. The problems associated with electromagnetic wave propagation in inhomogeneous chiral media were not considered in our earlier work. An important example of an inhomogeneous medium is provided by two semi-infinite homogeneous media, separated by a plane interface. As is well-known, if an electromagnetic wave is incident on this interface, part of it is reflected back into the original medium, and a part is refracted into the other medium. The main object of this paper is to extend our minimal and general parametrization to treat this kind of problem. In particular, here we study the problem of reflection and refraction at the interface in detail, for the case in which one or both of the semi-infinite media on either side of the boundary exhibits the chirality property.

The paper is organized as follows. In Section III we summarize the parametrization of the induced charge and current for a homogeneous chiral media that was introduced in Ref. [17]. We then extended it to treat inhomogeneous media, and we consider in particular the system that consists of two semi-infinite homogeneous media separated by a plane boundary, which is what concerns us in the present work. To set up the stage for the study of wave propagation across the boundary between two chiral media, in Section IV we review briefly the salient features of the propagation of a plane wave in a homogeneous chiral medium. Then in Section V we set up the problem of reflection and refraction at the plane interface between two chiral media, in terms of a set of equations that must be solved for the various coefficient amplitudes that appear in the suitably decomposed plane wave solution of the Maxwell equations. In Section VI we solve those equations to obtain the amplitudes of the reflected as well as refracted waves.
The corresponding solutions for various particular cases (including simpler configurations such as the case of normal incidence), and some of their features, are analyzed in some detail before the solution for the general case is given. Section VI contains some concluding remarks and outlook.

II. PARAMETRIZATION OF THE INDUCED SOURCES

Here, and in the rest of the paper we deal exclusively with materials that are isotropic, and we also assume that the field strengths are such that the material responds linearly. We use the Heaviside-Lorentz system of units throughout.

A. Non-dispersive medium

For the sake of orientation, let us consider first a non-dispersive medium. Our argument, advocated in Ref. [17], was to use the Maxwell equations themselves as a guide and write the induced charge and current densities in the usual form

\[ \rho_{\text{ind}} = -\nabla \cdot \vec{P}, \]
\[ \vec{j}_{\text{ind}} = \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M}, \]

(2.1)

but the relationship between the electric and magnetic polarization \( \vec{P} \) and \( \vec{M} \) being more general than the usual one.

For the example we are considering, the relation is

\[ \vec{P} = (\epsilon - 1)\vec{E}, \]
\[ \vec{M} = (1 - \mu^{-1})\vec{B} - \zeta \vec{E}, \]

(2.2)

which is consistent with all the general principles stated above. In particular, the isotropy of the medium implies that the quantities \( \epsilon, \mu, \zeta \) are scalars rather than tensors, and the non-dispersive nature of the medium implies those same quantities are constants independent of the coordinates. As a consequence of the homogeneous pair of the Maxwell equations,

\[ \nabla \cdot \vec{B} = 0, \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \]

(2.3)

(2.4)

an equivalent parametrization of the induced charge is furnished by writing

\[ \vec{P} = (\epsilon - 1)\vec{E} + \zeta \vec{B}, \]
\[ \vec{M} = (1 - \mu^{-1})\vec{B} - \zeta \vec{E}, \]

(2.5)

instead of Eq. (2.2). In any case,

\[ \rho_{\text{ind}} = -(\epsilon - 1)\nabla \cdot \vec{E}, \]
\[ \vec{j}_{\text{ind}} = (\epsilon - 1)\frac{\partial \vec{E}}{\partial t} + \left(1 - \frac{1}{\mu}\right)\nabla \times \vec{B} + \zeta \nabla \times \vec{E}. \]

(2.6)

A question that arises naturally is whether the expressions for \( \vec{P} \) and \( \vec{M} \) can both contain two terms, one proportional to \( \vec{E} \) and one to \( \vec{B} \), namely,

\[ \vec{P} = (\epsilon - 1)\vec{E} + \zeta_1 \vec{B}, \]
\[ \vec{M} = (1 - \mu^{-1})\vec{B} - \zeta_2 \vec{E}. \]

(2.7)

Indeed, this has been proposed in the literature [20]. However, the discussion above makes it clear that such a parametrization is redundant, since the expression for the induced sources would again be given by Eq. (2.6), with

\[ \zeta = \zeta_1 + \zeta_2, \]

(2.8)

and therefore \( \zeta \) is the only physically relevant quantity and not \( \zeta_{1,2} \) separately.
While these arguments refer to a non-dispersive medium, similar considerations, and the same conclusions, hold for a dispersive, but homogeneous medium as well. In a dispersive medium, the relation between $\vec{P}$, $\vec{M}$ and the fields is non-local. Therefore, expressions like $\epsilon \vec{E}$ should be interpreted in the form

$$\epsilon \vec{E} \rightarrow \hat{\epsilon} \vec{E} = \int d^4x' \epsilon(x, x') \vec{E}(x'),$$

(2.9)

and similarly with the other terms involving $\mu$ and $\zeta$. However, the assumption that the medium is homogeneous implies that the functions $\epsilon$, $\mu$, $\zeta$ depend only on the relative distance $\vec{x} - \vec{x}'$ and, furthermore, the causality principle requires the time dependence to be on the variable $t - t'$ and not on $t$ and $t'$ separately. In this case the relations such as those in Eq. (2.9) become simple algebraic relations in Fourier space and, as we showed in detail in Ref. [17], the argument and the parametrization we have used above continue to hold, if every symbol is taken to be the Fourier transform of the corresponding coordinate space variable, and if we make the identification

$$\vec{\nabla} \rightarrow i\vec{k},$$

$$\frac{\partial}{\partial t} \rightarrow -i\omega.$$

(2.10)

Thus in Fourier space, the induced sources are parametrized as

$$\rho_{\text{ind}}(\omega, \vec{k}) = -i(\epsilon - 1)\vec{k} \cdot \vec{E},$$

$$\vec{j}_{\text{ind}}(\omega, \vec{k}) = -i(\epsilon - 1)\omega\vec{E} + i(1 - \mu^{-1})\vec{B} - i\zeta\vec{k} \times \vec{E},$$

(2.11)

and the corresponding expressions in coordinate space are

$$\rho_{\text{ind}} = -\vec{\nabla} \cdot [(\hat{\epsilon} - 1)\vec{E}],$$

$$\vec{j}_{\text{ind}} = \frac{\partial}{\partial t} [(\hat{\epsilon} - 1)\vec{E}] + \vec{\nabla} \times \left[ \left( 1 - \frac{1}{\mu} \right) \vec{B} - \vec{\nabla} \times (\hat{\zeta}\vec{E}) \right].$$

(2.12)

(2.13)

It should be noted that the last term in Eq. (2.13) can be written in terms of $\vec{B}$ using the relation

$$\vec{\nabla} \times (\hat{\zeta}\vec{E}) = -\frac{\partial(\hat{\zeta}\vec{B})}{\partial t},$$

(2.14)

which as a consequence of Faraday’s law, as can be seen very simply in Fourier space. In coordinate space, it is proven by noticing that, for a homogeneous medium as we are considering, we can use manipulations such as

$$\frac{\partial}{\partial x_i} \int d^4x_1 \zeta(x - x_1') \vec{E}(x') = -\int d^4x \left( \frac{\partial}{\partial x_i} \zeta(x - x') \right) \vec{E}(x')$$

$$= \int d^4x \zeta(x - x') \frac{\partial}{\partial x'_i} \vec{E}(x'),$$

(2.15)

where we have integrated by parts.

Thus, in absence of external sources, the Maxwell equations in a homogeneous chiral medium consist of the pair given in Eqs. (2.3) and (2.4), augmented by

$$\vec{\nabla} \cdot (\hat{\epsilon}\vec{E}) = 0,$$

(2.16)

$$\vec{\nabla} \times \left[ \frac{1}{\mu} \vec{B} + \hat{\zeta}\vec{E} \right] - \frac{\partial}{\partial t} (\hat{\epsilon}\vec{E}) = 0.$$

(2.17)

If the external sources are non-zero, they would appear on the right hand sides of these two equations.

C. An inhomogeneous medium

In an inhomogeneous medium, the function $\epsilon(x, x')$ introduced in the previous section, and similarly $\mu$ and $\zeta$, depend on the coordinates $\vec{x}$ and $\vec{x}'$ separately, and not just on the relative coordinate $\vec{x} - \vec{x}'$. In this case, the language of
the Fourier transforms is not useful. Moreover, whichever way we would like to look at it, the manipulations that led us to Eq. (2.14) no longer hold, and therefore in this case

$$\vec{\nabla} \times (\hat{\zeta} \vec{E}) \neq -\frac{\partial (\hat{\zeta} \vec{B})}{\partial t}. \tag{2.18}$$

The important implication of this for us is that the question that we posed ourselves in Eq. (2.7) comes to haunt us again. In the present context, it amounts to asking whether instead of Eqs. (2.12) and (2.13) we can write

$$\rho_{\text{ind}} = -\vec{\nabla} \cdot [(\hat{\epsilon} - 1) \vec{E}] - \vec{\nabla} \cdot (\hat{\eta} \vec{B}), \tag{2.19}$$

$$\vec{j}_{\text{ind}} = \frac{\partial}{\partial t}[(\hat{\epsilon} - 1) \vec{E}] + \vec{\nabla} \times \left[ \left( 1 - \frac{1}{\hat{\mu}} \right) \vec{B} \right] - \vec{\nabla} \times (\hat{\zeta} \vec{E}) + \frac{\partial}{\partial t}(\hat{\eta} \vec{B}). \tag{2.20}$$

The issue now is not whether the two parameters $\zeta$ and $\eta$ are redundant and whether one can be absorbed in the other. They are not redundant; the last two terms in Eq. (2.20) are not proportional to each other in general, and the second term in Eq. (2.19) is not zero. Thus, $\zeta$ and $\eta$ have a different and separate physical meaning, and they describe different physical effects.

The question of which physical systems are described by this parametrization is one that we cannot consider here. What we can state is that there are no general physical principles that exclude that possibility. Our parametrization is both general and minimal, subject to the restriction of linearity and isotropy that we have assumed. However, as we have argued, such systems must necessarily be inhomogeneous. As interesting as the exploration of these systems would be, further considerations along these lines in a general way is outside our scope.

D. Two homogeneous chiral media with an interface

We finally get to consider the kind of system that concern us in this work, namely, two semi-infinite homogeneous media, separated by a plane interface. As we argue below, the proper way to parametrize this kind of system is by putting $\eta = 0$ in Eqs. (2.19) and (2.20). That is, the Maxwell equations are the same as stated in Eqs. (2.16) and (2.17). The argument is actually subtle, but simple.

Let us take the plane boundary to be the plane defined by $z = 0$. In the idealized limit of taking the plane boundary to be infinitesimally thin, then we can represent $\hat{\eta}$ in the form

$$\hat{\eta} = \eta\theta(-z) + \eta'\theta(z). \tag{2.21}$$

However, on either side of the boundary (that is, for $z \neq 0$), the function is zero since we are taking both semi-infinite media to be homogeneous. Therefore, $\hat{\eta} = 0$ everywhere.

The exception to this argument arises if the situation is such that the plane boundary cannot be taken to be infinitesimally thin, but instead we must take into account the fact that it has some non-zero thickness $\delta$. Certainly, in the transition from one material to the other, the medium is inhomogeneous and in that region $\hat{\eta}$ can be non-zero. If we look at the physical effects that occur in the bulk of either media at distances, say $|z| \sim L$ with $L \gg \delta$, then the effects due to the non-zero value of $\hat{\eta}$ in the transition region are of order $\delta/L$. In other words, they are surface effects. At points that are close to the plane boundary ($|z| \sim \delta$) such effects could be observable. But deep in the bulk of either media the effects become negligible as $\delta/L \rightarrow 0$, as it should be since in that case the transition region can be idealized to be infinitesimally thin and then the representation given in Eq. (2.21) is again a valid one.

III. WAVE PROPAGATION IN AN OPTICALLY ACTIVE HOMOGENEOUS MEDIUM

A. Dispersion relation

For illustrative purposes, let us consider a steady-state (monochromatic) electromagnetic wave propagating along a given direction, which we take to be the $z$ axis. The vector potential is of the form

$$\vec{A}(z, t) = e^{-i\omega t} \vec{a}(z) \tag{3.1}$$

where $\vec{a}(z)$ must be chosen such that the vector potential $\vec{A}(z, t)$ satisfies the Maxwell equations with the appropriate boundary conditions. Putting this into Eqs. (2.16) and (2.17), it follows that the wave equation is satisfied only for
the circularly polarized waves whose polarization vectors are defined by

\[ \hat{e}_\pm \equiv \frac{1}{\sqrt{2}}(\hat{u}_x \pm i\hat{u}_y), \]  

(3.2)

where we denote by \( \hat{u}_x, \hat{u}_y \) and \( \hat{u}_z \) the unit vectors along the \( x, y \) and the \( z \) directions, respectively. Further, these polarization states satisfy the dispersion relations

\[ \frac{K_\sigma(\omega)}{\omega} = n_\sigma(\omega, K_\sigma), \]  

(3.3)

where the refractive index functions \( n_\pm \) that appear in Eq. (3.3) are given in terms of the dielectric function \( (\varepsilon) \), magnetic permeability \( (\mu) \) and activity \( (\zeta) \) constants of the medium by\[17\]

\[ v_\pm^2 \equiv \frac{1}{n_\pm^2} = \frac{1}{\varepsilon \mu} \pm \frac{i\zeta \omega}{\varepsilon K} \].

(3.4)

While the index of refraction can in general be complex, even without the \( \zeta \) term, in this work we consider only systems for which it is real; i.e., the medium is non-absorbing. This implies, in particular, that \( \zeta \) is purely imaginary. This, in turn, has implications on the microscopic properties of the system with respect to the discrete space-time symmetries, which were considered at length in Ref. [17].

B. Linearly polarized wave

For a plane wave moving in the \( z \)-direction that is linearly polarized (i.e., it contains equal admixture of the two circular polarizations), the vector potential is of the form shown in Eq. (3.1), where

\[ \vec{a}(z) = A(e^{izK_+\hat{e}_+} + e^{izK_-\hat{e}_-}). \]  

(3.5)

In writing Eq. (3.5) we have implicitly chosen the origin and orientation of the coordinate system in such a way that, at \( z = 0 \), the linear polarization vector of the wave points along \( \hat{u}_x \), or the \( x \)-direction. By simple algebra, and using Eq. (3.2), \( \vec{A}(z, t) \) can be expressed as

\[ \vec{A}(z, t) = Ae^{-i\omega t}e^{i\Delta(z)}\frac{1}{\sqrt{2}}(\hat{u}_x \cos \theta(z) + \hat{u}_y \sin \theta(z)), \]  

(3.6)

where

\[ \theta(z) \equiv \frac{1}{2}(K_- - K_+)z, \]

\[ \Delta(z) \equiv \frac{1}{2}(K_+ + K_-)z. \]  

(3.7)

Thus, at a given distance \( z = d \), the polarization vector of the wave points at an angle given by \( \theta(d) \) relative to the \( x \)-axis, the phenomenon known as optical rotation.

However, it should be noted that for a fixed point \( z \), the direction of the polarization vector is fixed and does not change with time. This contrasts with what happens for a circularly polarized wave, e.g.,

\[ \vec{A}(z, t) = A_+e^{-i\omega t}e^{izK_+\hat{e}_+}. \]  

(3.8)

In this case the direction of the electric field is given by the vector

\[ \vec{e}(z, t) = \hat{u}_x \cos(\omega t - zK_+) + \hat{u}_y \sin(\omega t - zK_+), \]  

(3.9)

where, for simplicity, we have taken \( A_+ \) to be real, but similar considerations apply otherwise. For a fixed point \( z \), the polarization vector rotates in a circle, in the plane perpendicular to the direction of propagation.
C. Elliptically polarized wave

The obvious generalization of the above special forms is the superposition

$$\vec{A}(z,t) = e^{-i\omega t} \left[ A_+ e^{izK^+} \hat{e}_+ + A_- e^{izK^-} \hat{e}_- \right],$$  \hspace{1cm} (3.10)

which contains an unequal mixture of both circular polarizations. A simple geometric representation of this solution is given by the following construction. At the point $z = 0$, the direction of the electric field is given by the vector

$$\vec{e}(z = 0, t) = (A_+ + A_-)\hat{u}_x \cos \omega t + (A_+ - A_-)\hat{u}_y \sin \omega t.$$  \hspace{1cm} (3.11)

That is, the polarization vector rotates in an ellipse, with the axis of the ellipse lying along the $x$ and $y$ directions. For an arbitrary point $z \neq 0$, we write Eq. (3.10) in the equivalent form

$$\vec{A}(z,t) = e^{-i\omega t} e^{i\Delta(z)} \left[ A_+ e^{-i\theta(z)} \hat{e}_+ + A_- e^{i\theta(z)} \hat{e}_- \right],$$  \hspace{1cm} (3.12)

where $\theta(z)$ and $\Delta(z)$ are defined in Eq. (3.7). The direction of the electric field is easily obtained from this to be given by the vector

$$\vec{e}(z,t) = (A_+ + A_-) \cos(\omega t - \Delta(z))\hat{u}_{x'} + (A_+ - A_-) \sin(\omega t - \Delta(z))\hat{u}_{y'},$$  \hspace{1cm} (3.13)

where we have defined

$$u_{x'} = \cos \theta(z) \hat{u}_x + \sin \theta(z) \hat{u}_y,$$

$$u_{y'} = -\sin \theta(z) \hat{u}_x + \cos \theta(z) \hat{u}_y.$$  \hspace{1cm} (3.14)

Therefore, at a point $z = L$ the polarization is also elliptical, but in contrast to the standard elliptical polarization phenomena in ordinary media, the ellipse itself has rotated by the angle $\theta(L)$. In addition, the polarization angle acquires a negative phase $\Delta(L)$ relative to the $x'$ axis.

In the next sections, we extend these examples to the case in which the wave crosses the plane interface between two media, each of which is individually described as we have explained above. The treatment of that problem must necessarily take into account the kinematical and dynamical aspects of the reflection and refraction of the wave at the plane boundary, among other issues.

IV. REFLECTION AND REFRACTION OF A CIRCULARLY POLARIZED WAVE

A. Kinematics

We consider an electromagnetic wave incident on a plane boundary as shown schematically in Fig. 1. The homogeneity of the space in the $xy$ directions allows us to seek the (plane wave) solutions in the form

$$e^{-i\omega t} e^{ik_{zz}z} \vec{A}(z).$$  \hspace{1cm} (4.1)

For ordinary (non-chiral) media, a suitable ansatz is

$$\vec{A}(z) = \left[ \hat{a} e^{ik_1 z} + \hat{b} e^{-ik_1 z} \right] \Theta(z) + \left[ \hat{c} e^{ik'_1 z} + \hat{d} e^{-ik'_1 z} \right] \Theta(-z).$$  \hspace{1cm} (4.2)

The form chosen in Eq. (4.2) corresponds to the physical situation in which the wave is incident from the left (with no reflection from the far right-hand side, as it should be in a semi-infinite medium), as depicted in Fig. 1.

But here we encounter the first departure from the standard treatment when we consider chiral media. Namely, while we can set the incident component to consist of only one of the two polarizations, the reflected and the refracted ones will in general consist of a superposition of the two propagating modes, which in the present case have different wavelengths. For example, if we decompose the refracted wave vector into its perpendicular and parallel components (relative to the $z = 0$ plane) $\vec{k}' = (k'_\perp, k'_\parallel)$, then for fixed values of $\omega$ and $k'_\perp$ there is actually a different value of the parallel component for each polarization, given by $\sqrt{K'_\parallel^2 - k'_\perp^2}$, so that the two polarization components travel in different directions.
With this in mind, we therefore write our proposed solution corresponding to a wave with definite polarization $\sigma = \pm$ incident from the left, in the form

$$\vec{A}(\vec{x}, t) = e^{-i\omega t} \left[ \vec{A}_I \Theta(\xi) + \vec{A}_{II} \Theta(\xi) \right],$$

where $\vec{A}_I$, which contains the incident and the reflected components, is taken to be

$$\vec{A}_I = a_{\sigma} \hat{e}_{\sigma} e^{i\vec{k}_{\sigma} \cdot \vec{x}} + \sum_{\tau = \pm} a_{\tau} \hat{e}_{\tau} e^{i\vec{k}_{\tau} \cdot \vec{x}},$$

while the refracted component $\vec{A}_{II}$ is

$$\vec{A}_{II} = \sum_{\tau = \pm} a_{\tau} \hat{e}_{\tau} e^{i\vec{k}_{\tau} \cdot \vec{x}}.$$

The wave vectors in the expressions for these waves can be written as

$$\vec{k}_{\sigma} = (\vec{k}_{\perp}, k_{\sigma ||}),$$
$$\vec{k}_{\tau} = (\vec{k}_{\perp}, k_{\tau ||}),$$
$$\vec{k}_{\tau}'' = (\vec{k}_{\perp}, -k_{\tau ||})$$

sharing a common value of the perpendicular component. For the same fixed value of $\omega$, we denote by $K_{\sigma}(\omega)$ and $K_{\tau}'(\omega)$ the solutions of the dispersion relations in the two regions characterized by refractive index functions $n_{\pm}$ and $n'_{\pm}$. Making reference to the angles shown in Fig. 1, the parallel components for the different wave vectors can be written as

$$k_{\sigma ||} = K_{\sigma} \cos \alpha,$$
$$k_{\tau'} || = K_{\tau'} \cos \alpha_{\tau'},$$
$$k_{\tau}'' || = K_{\tau} \cos \alpha_{\tau''},$$

where

$$k_{\tau'} || = \sqrt{K_{\tau'}^2 - k_{\perp}^2},$$
$$k_{\tau}'' || = \sqrt{K_{\tau}^2 - k_{\perp}^2}.$$
In order to specify the polarization vectors uniquely, we define \( \hat{e}_1 \) to be in the direction of \( \vec{k}_\perp \times \hat{u}_z \). Making reference to Fig. 1, it points perpendicular to the page, away from the viewer. We take that to be also the direction of the \( x \) axis, and define the \( y \) axis by

\[
\hat{u}_y = \hat{u}_x \times \hat{u}_z ,
\]

which is the vertical direction in the representation of Fig. 1. Then, for the polarization vectors, we take

\[
\hat{e}_1' = \hat{e}_1'' = \hat{e}_1 = \hat{u}_x ,
\]

while

\[
\begin{align*}
\hat{e}_2 &= \hat{k} \times \hat{e}_1 , \\
\hat{e}_2' &= \hat{k}' \times \hat{e}_1 , \\
\hat{e}_2'' &= (\hat{k}'') \times \hat{e}_1 ,
\end{align*}
\]

(4.11)

where \( \hat{k}, \hat{k}' \) and \( \hat{k}'' \) are the unit vectors along \( \vec{k}_\sigma, \vec{k}'_\tau \) and \( \vec{k}''_\tau \), respectively. Notice that, while the direction of the incident wave is fixed (by us), the reflected and refracted components consist each of two polarized waves that travel in different directions, as we have already remarked. With the choice of axis specified above,

\[
\begin{align*}
\hat{k} &= \cos \alpha \hat{u}_z + \sin \alpha \hat{u}_y , \\
\hat{k}' &= \cos \alpha' \hat{u}_z + \sin \alpha' \hat{u}_y , \\
\hat{k}'' &= - \cos \alpha'' \hat{u}_z + \sin \alpha'' \hat{u}_y ,
\end{align*}
\]

(4.12)

and whence, using Eq. (4.11),

\[
\begin{align*}
\hat{e}_2 &= - \sin \alpha \hat{u}_z + \cos \alpha \hat{u}_y , \\
\hat{e}_2' &= - \sin \alpha' \hat{u}_z + \cos \alpha' \hat{u}_y , \\
\hat{e}_2'' &= - \sin \alpha'' \hat{u}_z + \cos \alpha'' \hat{u}_y .
\end{align*}
\]

(4.13)

The circular polarization vectors that enter in Eq. (4.3) are defined by

\[
\begin{align*}
\hat{e}_\sigma &= \frac{1}{\sqrt{2}} (\hat{e}_1 + i \sigma \hat{e}_2) , \\
\hat{e}'_\tau &= \frac{1}{\sqrt{2}} (\hat{e}_1 + i \tau \hat{e}_2') , \\
\hat{e}''_\tau &= \frac{1}{\sqrt{2}} (\hat{e}_1 + i \tau \hat{e}_2'') ,
\end{align*}
\]

(4.14)

and they satisfy

\[
\begin{align*}
\hat{e}_\sigma \cdot \hat{k}_\sigma &= 0 , \\
\hat{e}'_\tau \cdot \hat{k}'_\tau &= 0 , \\
\hat{e}''_\tau \cdot \hat{k}''_\tau &= 0 .
\end{align*}
\]

Together with Eq. (3.3), in this way it is ensured that the functions

\[ e^{-i\omega t} \vec{A}_X \quad (X = I, II) , \]

(4.15)

satisfy the Maxwell equations in each region.

The fact that the wave vectors have the same transverse component, as indicated in Eq. (4.6), implies the familiar relationship (Snell’s law) between the angles of incidence and refraction,

\[
\frac{\sin \alpha'}{\sin \alpha} = \frac{K_\sigma}{K'_\tau} = \frac{n_\sigma}{n'_\tau} ,
\]

(4.16)

and the analogous relation for the reflected wave

\[
\frac{\sin \alpha''}{\sin \alpha} = \frac{\alpha''}{\alpha} \quad (\tau = \sigma)
\]

\[
\frac{K_\sigma}{K'_\tau} = \frac{n_\sigma}{n'_\tau} \quad (\tau \neq \sigma) .
\]

(4.17)
Thus, for a given value of $\omega$ and $\alpha$, so that the wave vector $\vec{k}_\sigma$ of the incident component is completely specified, the wave vectors $\vec{k}'_{\tau}$ and $\vec{k}_\tau$ of the reflected and refracted components are also completely determined, as well as the polarization vectors. The only quantities yet to be determined are the amplitudes $a'_{\tau}, a''_{\tau}$ in Eq. (4.3).

From Eqs. (4.16) and (4.17) the familiar effects such as total internal reflection can be deduced, in the form that they apply to the present situation. Thus, for example, if the dielectric constants are such that $K_\sigma > K'_\tau$, there is a maximum incident angle $\alpha_{\tau \text{max}}$, determined by

$$\sin \alpha_{\tau \text{max}} = \frac{K'_\tau}{K_\sigma},$$

(4.18)

above which the refracted wave with polarization $\tau$ propagates parallel to the surface ($k_{\tau \parallel} = 0$) but does not penetrate the region II.

In general, the picture that emerges is the following. The incident wave, which has a definite circular polarization, is split at the boundary into a reflected and refracted wave, each of which is a superposition of the two circular polarizations. However, the two circularly polarized modes do not travel in the same direction as a consequence of the fact that their wave vectors have the same transverse component but a different longitudinal one. The determination of the relative admixture of the two polarization modes in the reflected and the refracted wave is precisely one of the dynamical questions that we will address below.

### B. Dynamics

Some of the dynamical issues that we wish to consider involve (i) finding the amplitudes of the reflected and refracted components of the wave for a given incident amplitude; (ii) determining how the polarization of the wave changes as it crosses the boundary. In contrast with the kinematical properties that we considered in the previous section, these dynamical aspects depend on the specific nature of the boundary conditions at the interface. The latter are determined by the requirement that the solution given in Eq. (4.3) satisfies the Maxwell equations at the interface.

#### 1. Boundary Conditions

As we have already mentioned, the dispersion relations given in Eq. (3.3) together with the conditions given in Eqs. (4.15) imply that the function $\vec{A}(\vec{x}, t)$ defined in Eq. (4.3) satisfies the Maxwell equations in each region separately. However, we have yet to ensure that the equations are satisfied at the boundary between the two regions itself. This requirement yields further conditions. Are they the same as for the ordinary equations? Or does the $\zeta$ activity constant term modify them in any way? Answering this question is our next hurdle.

To begin, we calculate the electric and magnetic field associated with the vector potential given in Eq. (4.3). The standard formulas

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t},$$

$$\vec{B} = \nabla \times \vec{A},$$

(4.19)

yield

$$\vec{E} = e^{-i\omega t} \left[ \vec{E}_I \Theta(-z) + \vec{E}_I \Theta(z) \right],$$

(4.20)

$$\vec{B} = e^{-i\omega t} \left[ \vec{B}_I \Theta(-z) + \vec{B}_I \Theta(z) + \delta(z) \hat{u}_z \times (\vec{A}_I - \vec{A}_I) \right],$$

(4.21)

where

$$\vec{E}_X \equiv i\omega A_X,$$

$$\vec{B}_X \equiv \nabla \times \vec{A}_X \quad (X = I, II).$$

(4.22)

Since the fields must be finite everywhere (otherwise the Maxwell equations would be not be satisfied), the last term in Eq. (4.21) must be zero, which yields the condition

$$\hat{u}_z \times \left[ a_\sigma \hat{e}_\sigma + \sum_\tau (a''_{\tau \tau} \hat{e}'_{\tau} - a'_{\tau \tau} \hat{e}'_{\tau}) \right] = 0$$

(4.23)
on the polarization vectors.

The remaining relations follow from the Maxwell equations themselves which were given in Eqs. (2.3), (2.4), (2.10) and (4.17) in the Introduction. The argument to obtain the implied conditions is similar to the one used to arrive at Eq. (4.23). For example, consider Eq. (2.10). Writing

\[ \mathbf{E} = e^{-i\omega t} \left[ e \mathbf{E}_I \Theta(z) + e' \mathbf{E}_{II} \Theta(-z) \right], \]  

we then calculate

\[ \nabla \cdot \mathbf{E} = e^{-i\omega t} \left[ \Theta(z) \nabla \cdot (e \mathbf{E}_I) + \Theta(-z) \nabla \cdot (e' \mathbf{E}_{II}) + \delta(z) \hat{u}_z \cdot (e' \mathbf{E}_{II} - e \mathbf{E}_I) \right]. \]  

The first two terms on the right-hand side are identically zero since there is no free charge in either region. Thus, the condition that the divergence of \( \mathbf{E} \) is zero everywhere requires that the coefficient of \( \delta(z) \) in Eq. (4.26) be zero. This yields the relation

\[ \hat{u}_z \cdot \left[ \epsilon a_\sigma \hat{e}_\sigma + \sum_\tau (\epsilon a'_\tau \hat{e}'_\tau - \epsilon' a'_\tau \hat{e}'_\tau) \right] = 0, \]  

where we have used the expressions for \( \mathbf{E}_I, \mathbf{E}_{II} \) that follow from Eqs. (4.3), (4.4), (4.5) and (4.22).

The conditions that follow from the remaining Maxwell equations are obtained similarly. In general, by means of Eqs. (3.3) and the definition of the corresponding polarization vectors given above, it is ensured that the proposed solution satisfies the equations in each region separately. Thus, when the solution is substituted in the Maxwell equations, only the terms that arise from the derivatives acting on the Theta function, which are proportional to the delta function, are not automatically zero. Demanding that they be zero as well yields the required relations. Proceeding in this way we find that Eq. (2.4) yields Eq. (4.23) again, and the remaining two conditions that complement Eqs. (4.23) and (4.26) are

\[ \hat{u}_z \cdot \left[ \sigma a_\sigma \hat{e}_\sigma K_\sigma + \sum_\tau \tau (K_\tau a''_\tau \hat{e}'_\tau - K'_\tau a'_\tau \hat{e}'_\tau) \right] = 0, \]  

\[ \hat{u}_z \times \left[ \sigma a_\sigma \epsilon v_\sigma \hat{e}_\sigma + \sum_\tau \tau (a''_\tau \epsilon v_\tau \hat{e}'_\tau - a'_\tau \epsilon' v'_\tau \hat{e}'_\tau) \right] = 0. \]  

In writing these conditions, we have used the relation

\[ i\hat{k}_\sigma \times \hat{e}_\sigma = \sigma \hat{e}_\sigma \]  

and the definition of \( v_\sigma \) from Eq. (3.3), as well as the analogous ones for the corresponding primed and doubly-primed quantities. In addition we have used the fact that, for any wave (incident, refracted and reflected), the product of its wave number times its velocity velocity equals \( \omega \), which is the same for all of them.

**V. SOLUTION TO THE BOUNDARY CONDITIONS**

**A. The independent conditions**

Our task is to use the conditions given in Eqs. (4.23), (4.25), (4.27) and (4.28) to obtain, for a given incident amplitude \( a_\sigma \), the reflected and transmitted amplitudes \( a'_\sigma, a''_\sigma \). There are thus a total of four unknown variables to be solved for. On the other hand, at first sight, the number of conditions seems to be six. While Eqs. (4.25) and (4.27) are scalar equations, the relations in Eqs. (4.26) and (4.28) are vectorial. Being a two-dimensional problem, as we have seen, each of the vector relations yields two conditions, for a total of six. Therefore, the solution for \( a'_\sigma, a''_\sigma \) is algebraically overdetermined, and a non-trivial solution exists only if additional auxiliary conditions are satisfied. As we will see, these are just the Snell-like relations given in Eqs. (4.10) and (4.17), and the dispersion relations of Eq. (3.3). Put in another way, when Eq. (3.3) as well as Eqs. (4.10) and (4.17) are satisfied, two of the six relations implied by the set of boundary conditions are redundant, leaving just four independent equations for the four unknowns \( a'_\sigma, a''_\sigma \).
Before going to the details of the problem, it is useful to recollect the corresponding situation in the case of ordinary, non-chiral media. In that case, for a given value of the transverse component of the wave vector and a given frequency, the longitudinal component of the wave vector in each of the reflected and refracted waves has a unique value. Thus, the solution written down in Eq. (4.3) indeed collapses to the form given in Eq. (4.2). Nevertheless, the problem is similar to the one stated above. Namely, for a given incident amplitude $\vec{a}$, we wish to know what are the the coefficients $\vec{b}$ and $\vec{c}$. Being a two-dimensional problem as already argued, then for an arbitrary (but definite) $\vec{a}$ in the $\hat{e}_{1,2}$ plane, there are four unknowns, represented by the two components of $\vec{b}$ in the $\hat{e}_{1,2}$ plane and the two components of $\vec{c}$ in the $\hat{e}_{1,2}$ plane. Further, the problem being a linear one (the boundary conditions are linear equations in the amplitudes), we can proceed to find the particular solutions corresponding to the case $\vec{a} = \hat{e}_1$ and, separately, $\vec{a} = \hat{e}_2$, and the general solution for an arbitrary choice of $\vec{a}$ is obtained by linear superposition of those particular ones. This is actually the procedure followed in Jackson’s book [1]. The fundamental reason why a solution can be obtained at all, is the fact that there are four independent algebraic relations (that follow from the boundary conditions) among the four unknowns the we have identified.

Returning to our problem, these same general principles hold as well. Namely, while some features and details of the solutions are not applicable or relevant, the linearity property, superposition principle and the fact that the problem is algebraically well defined, also apply. The fundamental difference, as far as the algebraic details and manipulations is concerned, is that in the present case it is not useful to build up the general solution by superposing the particular solutions obtained for the cases $\vec{a} = \hat{e}_{1,2}$, and decomposing the reflected and refracted waves in terms of the linear polarization components. In our case, the circular polarization basis is, as we have shown, the appropriate one to use.

For the remaining vector algebra we use the relations

$$
\hat{u}_z \cdot \hat{e}_\sigma = -\frac{1}{\sqrt{2}}i\sigma \sin \alpha,
$$

$$
\hat{u}_z \cdot \hat{e}_\tau' = -\frac{1}{\sqrt{2}}i\tau \sin \alpha',
$$

$$
\hat{u}_z \cdot \hat{e}_\tau'' = -\frac{1}{\sqrt{2}}i\tau \sin \alpha'',
$$

and

$$
\hat{u}_z \times \hat{e}_\sigma = \frac{1}{\sqrt{2}} (\hat{u}_y - i\sigma \cos \alpha \hat{u}_x),
$$

$$
\hat{u}_z \times \hat{e}_\tau' = \frac{1}{\sqrt{2}} (\hat{u}_y - i\tau \cos \alpha' \hat{u}_x),
$$

$$
\hat{u}_z \times \hat{e}_\tau'' = \frac{1}{\sqrt{2}} (\hat{u}_y + i\tau \cos \alpha'' \hat{u}_x).
$$

Using these multiplication rules, Eqs. (4.28), (4.29), (4.30) and (4.31) imply, in that order, the following six relations

$$
\sum_\tau (a_\tau' - a_\tau'') = a_\sigma,
$$

$$
\sum_\tau \tau(a_\tau' \cos \alpha' + a_\tau'' \cos \alpha'') = \sigma a_\sigma \cos \alpha,
$$

$$
\sum_\tau (K_\tau' a_\tau' \sin \alpha' - K_\tau a_\tau'' \sin \alpha'') = a_\sigma K_\sigma \sin \alpha,
$$

$$
\sum_\tau \tau (\epsilon a_\tau' \sin \alpha' - \epsilon a_\tau'' \sin \alpha'') = \sigma a_\sigma \epsilon \sin \alpha,
$$

$$
\sum_\tau \tau (a_\tau' \epsilon' \cdot \epsilon'' - a_\tau'' \epsilon' \cdot \epsilon') = \sigma a_\sigma \epsilon' \cdot \epsilon' \cdot \cos \alpha,
$$

$$
\sum_\tau (a_\tau' \epsilon' \cdot \epsilon'' \cos \alpha' + a_\tau'' \epsilon' \cdot \epsilon'' \cos \alpha'') = a_\sigma \epsilon' \cdot \epsilon' \cdot \cos \alpha.
$$

If we now use Snell’s law [Eqs. (5.39) and (5.40)], it is easy to see that the third equation becomes identical to the first, and the fourth identical to the fifth. Thus, we are left with four independent equations which, for easy reference
in what follows, we recollect below:
\[
\sum_{\tau} (a'_{\tau} - a''_{\tau}) = a_{\sigma},
\]
\[
\sum_{\tau} \tau (a'_{\tau} \cos \alpha'_{\tau} + a''_{\tau} \cos \alpha''_{\tau}) = \sigma a_{\sigma} \cos \alpha,
\]
\[
\sum_{\tau} \tau (a'_{\tau} \epsilon' v'_{\tau} - a''_{\tau} \epsilon v_{\tau}) = \sigma a_{\sigma} \epsilon v_{\tau},
\]
\[
\sum_{\tau} (a'_{\tau} \epsilon' v'_{\tau} \cos \alpha'_{\tau} + a''_{\tau} \epsilon v_{\tau} \cos \alpha''_{\tau}) = a_{\sigma} \epsilon v_{\sigma} \cos \alpha.
\]
These equations must be solved for the four unknowns \(a'_{\tau}\) and \(a''_{\tau}\) (with \(\tau = \pm\)), for any given incident amplitude \(a_{\sigma}\).

Before embarking on the general solution, it is instructive to consider various particular cases.

**B. Non-chiral media as a special limiting case**

If \(\zeta\) is zero in both sides of the boundary, so that we are referring to ordinary, non-chiral, media, the equations for the coefficients simplify. Neither the indices of refraction, nor the angles, depend on the sign of the polarization. In terms of the linear polarization amplitudes
\[
a_1 = \frac{1}{\sqrt{2}} (a_+ + a_-),
\]
\[
a_2 = \frac{i}{\sqrt{2}} (a_+ - a_-),
\]
and similar ones for the primed and doubly primed coefficients, the equations become
\[
a'_1 - a''_1 = \frac{1}{\sqrt{2}} a_{\sigma},
\]
\[
a'_2 \cos \alpha' + a''_2 \cos \alpha = \frac{i}{\sqrt{2}} \sigma a_{\sigma} \cos \alpha,
\]
\[
a'_2 \epsilon' v' - a''_2 \epsilon v = \frac{i}{\sqrt{2}} \sigma a_{\sigma} \epsilon v,
\]
\[
a'_1 \epsilon' v' \cos \alpha' + a''_1 \epsilon v \cos \alpha = \frac{1}{\sqrt{2}} \sigma a_{\sigma} \epsilon v \cos \alpha.
\]
Thus, the set of four equations splits into two \(2 \times 2\) blocks, one for \(a'_1\) and \(a''_1\) and the other for \(a'_2\) and \(a''_2\), which are easily solved to give
\[
\begin{pmatrix}
  a'_1 \\
  a''_1
\end{pmatrix} = \frac{1}{\sqrt{2}} a_{\sigma} E_1
\]
\[
\begin{pmatrix}
  a'_2 \\
  a''_2
\end{pmatrix} = \frac{i}{\sqrt{2}} \sigma a_{\sigma} E_2,
\]
where
\[
E_1 = \frac{1}{\epsilon v \cos \alpha + \epsilon' v' \cos \alpha'} \begin{pmatrix}
  2 \epsilon v \cos \alpha \\
  \epsilon v \cos \alpha' - \epsilon' v' \cos \alpha'
\end{pmatrix},
\]
\[
E_2 = \frac{1}{\epsilon v \cos \alpha' + \epsilon' v' \cos \alpha} \begin{pmatrix}
  2 \epsilon v \cos \alpha \\
  \epsilon' v' \cos \alpha' - \epsilon v \cos \alpha
\end{pmatrix}.
\]
The original coefficients for the circularly polarized amplitudes,
\[
a'_{\sigma} = \frac{1}{\sqrt{2}} (a'_1 - i \tau a'_2)
\]
and similarly for $a''_r$, are then given by

$$\begin{pmatrix} a'_r \\ a''_r \end{pmatrix} = \frac{1}{2} a_\sigma (E_1 + \sigma \tau E_2). \quad (5.10)$$

Regarding the physical interpretation, the formulas in Eq. (5.7) give the solution for the reflected and refracted amplitudes expressed in the linear polarization basis, with the corresponding quantities in the circular polarization basis being obtained by means of Eq. (5.10). It should be noted further that, whichever basis we use to express the solution, we have assumed that the incident wave has a definite circular polarization specified by $\sigma$, which can be $\pm 1$. However, we can easily obtain the solutions for an arbitrary polarization of the incident wave by superposition. If the incident wave is a combination of the two circularly polarized waves, then the solution is generalized to

$$\begin{pmatrix} a'_r \\ a''_r \end{pmatrix} = \frac{1}{2} \sum_\sigma a_\sigma (E_1 + \sigma \tau E_2). \quad (5.11)$$

or equivalently, in the linear basis,

$$\begin{pmatrix} a'_1 \\ a''_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \sum_\sigma a_\sigma E_1 = a_1 E_1$$

$$\begin{pmatrix} a'_2 \\ a''_2 \end{pmatrix} = \frac{i}{\sqrt{2}} \sum_\sigma \sigma a_\sigma E_2 = a_2 E_2. \quad (5.12)$$

This solution in fact reproduces the well known results for the situation we are considering. It is reassuring to confirm, for example, that it embodies the Fresnel formulas when we take the incident wave to be linearly polarized along $\hat{u}_x$ or $\hat{u}_y$. Let us consider the case in which the incident wave is linearly polarized along the $\hat{u}_x$ direction, which implies that the incident wave is an equal admixture of the two circularly polarized waves, with the coefficients satisfying

$$a_+ = a_- \equiv \frac{1}{\sqrt{2}} a_1,$$

$$a_2 = 0. \quad (5.13)$$

Then the solution for this case, namely

$$\begin{pmatrix} a'_1 \\ a''_1 \end{pmatrix} = a_1 E_1,$$

$$\begin{pmatrix} a'_2 \\ a''_2 \end{pmatrix} = 0 \quad (5.14)$$

in the linear basis, is immediately recognized as the so-called *Transverse Electric* solution. Similarly, the solution for linear polarization along $\hat{e}_2$,

$$\begin{pmatrix} a'_1 \\ a''_1 \end{pmatrix} = 0,$$

$$\begin{pmatrix} a'_2 \\ a''_2 \end{pmatrix} = a_2 E_2 \quad (5.15)$$

corresponds to the standard *Transverse Magnetic* solution. The Fresnel formulas are reproduced by writing the above results in terms of the magnetic permeability $\mu$ and the refractive index $n$, instead of quantities $\epsilon$ and $v$. These two sets of quantities are related by Eq. (3.4), which for non-chiral media reads

$$v^2 \equiv \frac{1}{n^2} = \frac{1}{\epsilon \mu}. \quad (5.16)$$
Eliminating \( \epsilon \) and \( v \) in favor of \( \mu \) and \( n \), the transverse electric solution can be written as

\[
\frac{a_1'}{a_1} = \frac{2n \cos \alpha}{n \cos \alpha + \frac{\mu}{\mu'} n' \cos \alpha'},
\]
\[
\frac{a_1''}{a_1} = \frac{n \cos \alpha - \frac{\mu}{\mu'} n' \cos \alpha'}{n \cos \alpha + \frac{\mu}{\mu'} n' \cos \alpha'},
\]

(5.17)

which, after using Snell’s law to eliminate the angle of refraction, is the usual form (given, for example, by Jackson\[1\]). The result for the transverse magnetic case can be reproduced similarly.

C. Incidence from ordinary to chiral media

A particularly simple situation, which reveals some of the features of the general solution, arises by considering the case in which the first medium is an ordinary dielectric, but the second one is not, i.e.,

\( \zeta = 0, \quad \zeta' \neq 0. \)  

(5.18)

In this case,

\( n_+ = n_\equiv n \)

(5.19)

and by Snell’s law

\( \alpha'' = \alpha' = \alpha, \)

(5.20)

so that the equations for the coefficients become

\[
(a_+^\prime + a_-^\prime) - (a_+^{\prime\prime} + a_-^{\prime\prime}) = a_\sigma,
\]
\[
(a_+^\prime \cos \alpha_+ - a_-^\prime \cos \alpha_-) + (a_+^{\prime\prime} - a_-^{\prime\prime}) \cos \alpha = \sigma a_\sigma \cos \alpha,
\]
\[
\epsilon'(a_+^\prime v_+^\prime - a_-^\prime v_-^\prime) - (a_+^{\prime\prime} - a_-^{\prime\prime}) \epsilon v = \sigma a_\sigma \epsilon v,
\]
\[
\epsilon'(a_+^\prime v_+^\prime \cos \alpha_+ + a_-^\prime v_-^\prime \cos \alpha_-) + (a_+^{\prime\prime} + a_-^{\prime\prime}) \epsilon v \cos \alpha = a_\sigma \epsilon v \cos \alpha.
\]

(5.21)

We can eliminate the doubly-primed coefficients in the fourth using the first, and likewise in the third using the second. The resulting equations are easily solved for the primed coefficients, and the doubly primed ones can be obtained by using the first and the second equations. Thus, the solution is easily obtained, and it can be expressed in the form

\[
\frac{a_\tau'}{a_\sigma} = \left( \frac{2\epsilon v \cos \alpha}{T} \right) T_{\tau}^\sigma,
\]
\[
\frac{a_\tau''}{a_\sigma} = -\left( 1 - \frac{\tau \sigma}{2} \right) + \frac{\epsilon v}{T} \sum_{\lambda = \pm} (\cos \alpha - \lambda \tau \cos \alpha'_\lambda) T_{\lambda}^\sigma,
\]

(5.22)

where

\[
T_{\tau}^\sigma = (\epsilon v + \tau \sigma \epsilon' v'_\tau) (\cos \alpha'_{\tau} + \tau \sigma \cos \alpha),
\]

(5.23)

and

\[
T = \frac{1}{2} (T_+ T_- - T_+^- T_-^+ - T_+^+ T_-^-)
\]
\[
= \epsilon \epsilon' (v'_+ + v'_-) (\cos^2 \alpha + \cos \alpha_+ \cos \alpha_-) + (\epsilon^2 v^2 + \epsilon'^2 v'_+ v'_-) \cos \alpha (\cos \alpha'_+ + \cos \alpha'_-).
\]

(5.24)
Normal incidence

A particularly simple solution is obtained if, in addition, we assume that the incident wave travels normal to the plane boundary, \( \alpha = 0 \), in which case the other angles are also all equal to 0 by Snell’s law. The above solution then reduces to

\[
\frac{a'_\tau}{a_\sigma} = \frac{1}{2}(1 + \tau\sigma)F_\tau , \\
\frac{a''_\tau}{a_\sigma} = \frac{1}{2}(1 - \tau\sigma)G_{-\tau} ,
\]

where we have defined

\[
F_\tau = \frac{2\epsilon v}{\epsilon v + \epsilon' v'_\tau} , \\
G_\tau = F_\tau - 1 = \frac{\epsilon v - \epsilon' v'_\tau}{\epsilon v + \epsilon' v'_\tau} .
\]

This solution shows various interesting properties. For example, if we consider either \( \sigma = + \) or \( \sigma = - \), the transmitted wave has the same circular polarization as the incident one, but the reflected wave has the opposite. This is a well known effect even in the case of ordinary media (\( \zeta' = 0 \)). The distinguishing feature in the present case is that the intensity of the reflected (and the transmitted) wave depends on what is the polarization of the incident one.

If the incident wave does not have a definite circular polarization, but it is a linear combination of them, then by superposition the general solution for that case is

\[
\frac{a'_\tau}{a_1} = \frac{1}{2} F_\tau \sum_\sigma (1 + \tau\sigma) , \\
\frac{a''_\tau}{a_1} = \frac{1}{2} G_{-\tau} \sum_\sigma (1 - \tau\sigma) .
\]

Let us consider the case of linear polarization along \( \hat{e}_1 \), as in Eq. (5.13). In that case the solution is

\[
\frac{a'_\tau}{a_1} = \frac{1}{\sqrt{2}} F_\tau , \\
\frac{a''_\tau}{a_1} = \frac{1}{\sqrt{2}} G_{-\tau} .
\]

Thus neither the reflected nor the transmitted waves have a definite linear polarization (e.g., \( a'_1 \neq a'_- \)) in contrast to the situation considered in Section V.B. Furthermore, while the incident wave is represented by

\[
\vec{A}_{I}^{(\text{inc})}(z) = a_1 \hat{e}_1 e^{iKz} ,
\]

the transmitted component is

\[
\vec{A}_{II}(z) = \frac{1}{\sqrt{2}} a_1 \left[ \hat{e}''_+ F_+ e^{iK'_+ z} + \hat{e}''_- F_- e^{iK'_- z} \right] .
\]

That is, while the incident wave is, at the boundary \( z = 0 \), an equal admixture of the positive and negative circular polarizations, given by the \( \hat{e}_1 \) direction, the transmitted wave is split into an unequal admixture of the two circular polarizations. As we saw in Section III.C, such an admixture describes an elliptically polarized wave, with the particular characteristics that are due to the chiral nature of the medium, as we explained there. Moreover, in the region \( z > 0 \), the vector that determines the direction of the electric field in the present case is obtained from the formula given in Eq. (5.13), by making the substitutions

\[
A_\pm \rightarrow F_\pm , \\
\theta(z) \rightarrow \frac{1}{2}(K'_- - K'_+) z , \\
\Delta(z) \rightarrow \frac{1}{2}(K'_+ + K'_-) z .
\]
D. General solution

Other special cases can be treated similarly, such as the case of incidence from chiral to ordinary media, including the particular situation of normal incidence. However, we do not proceed any further along those lines, and go directly to give the solution to Eq. \(5.34\) for the reflected and refracted amplitudes, in the general case. For this purpose, we use the shorthand notation \(c_0 \equiv \cos \alpha_0, \ c_\pm \equiv \cos \alpha_\pm\) and \(c_\pm' \equiv \cos \alpha'_\pm\) for the cosine of various angles. Then, defining

\[
D = (c'_+ c'_- + c''_+ c''_-)(\epsilon' v'_+ + \epsilon' v'_-_)(\sigma v_+ + \sigma v_-)
+ (c'_+ c''_- + c'_- c''_+)(\epsilon' v'_- + \epsilon' v'_+)(\sigma v_+ + \sigma v_-)
+ (c'_+ c''_+ + c'_- c''_-)(\epsilon' v'_+ - \epsilon' v'_-)(\epsilon' v'_- - \epsilon' v'_+),
\]

(5.32)

the solution is given by

\[
\frac{a'_+}{a_\sigma} = (\sigma v_\sigma + \tau \epsilon v'_-)(\epsilon v_+ + \epsilon v_-)(\sigma c'_{-\sigma} + \tau c''_{-\sigma})
+ (\sigma v_\sigma + \epsilon v_-)(\epsilon' v'_+ + \tau v_+)(\sigma c'_{-\sigma} + \tau c''_{-\sigma})
+ (\sigma v_\sigma - \epsilon v_+)(\epsilon v'_- - \tau v_-)(\sigma c'_{-\sigma} + \tau c''_{-\sigma}),
\]

(5.33)

\[
\frac{a''_+}{a_\sigma} = (\sigma v_\sigma + \tau \epsilon v'_-)(\epsilon v'_+ + \epsilon v'_-)(\sigma c'_{-\sigma} - \tau c''_{-\sigma})
+ (\sigma v_\sigma + \tau v_+)(\epsilon' v'_- - \epsilon v'_-)(\sigma c'_{-\sigma} - \tau c''_{-\sigma})
- (\sigma v_\sigma - \tau v_-)(\epsilon v'_- - \epsilon v'_-)(\sigma c'_{-\sigma} - \tau c''_{-\sigma}).
\]

(5.34)

These formulas can be written in several alternate forms, in terms of the angle of incidence and the refractive indices, by using Snell’s law [Eq. \(4.11\)]. In the appropriate limits, they reduce to the special case formulas already seen.

VI. CONCLUSIONS

The electromagnetic properties of a medium that exhibits chirality (also called optical activity), but which is otherwise linear, homogeneous and isotropic, can be described in terms of the usual two parameters \(\epsilon\) and \(\mu\) that represent the dielectric and permeability functions, and an additional parameter \(\zeta\) that is indicative of the chirality property. As we have argued here and in the references cited, such a parameterization in terms of just one additional function, is both complete and minimal. In the case of an infinite (unbounded) medium, an electromagnetic wave exhibits the phenomenon known as ‘natural optical activity’, which is due to the fact that the two circularly polarized states travel with different speed if \(\zeta\) is non-zero.

In this work we have considered in detail various aspects of the propagation of a wave, in a medium that is made of two semi-infinite media, either or both of which may be chiral, which are separated by a plane interface. Mimicking the procedure that is applied to the analogous problem involving ordinary dielectrics, our approach was based on writing the plane wave solution that satisfies the Maxwell equations in each region, supplemented by the appropriate boundary conditions at the interface.

We considered first various kinematical aspects of the solution which are independent of the detailed nature of the boundary conditions. In particular, we obtained the Snell law and the condition for the total internal reflection effect, in the form that it applies to the present situation.

In contrast with the above, the determination of the relative amplitudes of the reflected and refracted components is a dynamical issue that involve the details of the boundary conditions. A significant problem that we had to solve was precisely to elucidate what is the appropriate set of boundary conditions that must be satisfied at the interface. The boundary conditions depend on the Maxwell equations themselves, and since these depend on the parameter \(\zeta\), the boundary conditions necessarily involve that parameter as well.

After finding the appropriate set of independent conditions, we solved them and obtained the amplitudes in various particular cases and configurations, such as one region being chiral and the other one being an ordinary dielectric, and the case of normal incidence. We considered various features of those solutions, and in particular we obtained in various cases the corresponding formula for the angle of the direction of polarization.

In the general case, the boundary conditions form a set of four (linear) equations that must be solved for the four unknown amplitudes in terms of the amplitude of the incident wave. While the explicit form of the solution is not particularly illuminating, we obtained it and for completeness we wrote it down.

Our work opens the way for handling a whole class of related problems involving chiral media, that we can now formulate in a concrete way. We can consider, for example, a slab of one material of finite thickness, inserted between
two semi-infinite media, and in various configurations depending on which ones are chiral or ordinary dielectrics. Possible generalizations include also periodic or semi-periodic arrays of alternating materials, and similar arrangements. From an algebraic point of view, all of them will be ultimately reduced to writing down the plane wave solutions in each region, and imposing the boundary conditions at each interface. In this work we have shown the way for treating and solving all such problems systematically. In addition to the interesting and potentially important physical applications that we have mentioned, some of the ideas exposed here can be useful in other physics problems that have been studied in the literature [18, 19], which have a similar mathematical structure.

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