Symmetry Classes

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Abstract

Physical systems exhibiting stochastic or chaotic behavior are often amenable to treatment by random matrix models. In deciding on a good choice of model, random matrix physics is constrained and guided by symmetry considerations. The notion of 'symmetry class' (not to be confused with 'universality class') expresses the relevance of symmetries as an organizational principle. Dyson, in his 1962 paper referred to as The Threefold Way, gave the prime classification of random matrix ensembles based on a quantum mechanical setting with symmetries. In this article we review Dyson’s Threefold Way from a modern perspective. We then describe a minimal extension of Dyson’s setting to incorporate the physics of chiral Dirac fermions and disordered superconductors. In this minimally extended setting, where Hilbert space is replaced by Fock space equipped with the anti-unitary operation of particle-hole conjugation, symmetry classes are in one-to-one correspondence with the large families of Riemannian symmetric spaces.

1 Introduction

In Chapter 2 of this handbook\textsuperscript{1}, the historical narrative by Bohigas and Weidenmüller describes how random matrix models emerged from quantum physics, more precisely from a statistical approach to the strongly interacting many-body system of the atomic nucleus. Although random matrix theory is nowadays understood to be of relevance to numerous areas of physics, mathematics, and beyond, quantum mechanics is still where many of its applications lie.

\textsuperscript{1}The present article is to be Chapter 3 of the Oxford Handbook of Random Matrix Theory.
Quantum mechanics also provides a natural framework in which to classify random matrix ensembles.

In this thrust of development, a symmetry classification of random matrix ensembles was put forth by Dyson in his 1962 paper *The Threefold Way: algebraic structure of symmetry groups and ensembles in quantum mechanics*, where he proved (quote from the abstract of [Dys62]) “that the most general matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types”. The three types known to Dyson were ensembles of matrices which are either complex Hermitian, or real symmetric, or quaternion self-dual. It is widely acknowledged that Dyson’s Threefold Way has become fundamental to various areas of theoretical physics, including the statistical theory of complex many-body systems, mesoscopic physics, disordered electron systems, and the field of quantum chaos.

Over the last decade, a number of random matrix ensembles beyond Dyson’s classification have come to the fore in physics and mathematics. On the physics side these emerged from work [Ver94] on the low-energy Dirac spectrum of quantum chromodynamics, and also from the mesoscopic physics of low-energy quasi-particles in disordered superconductors [AZ97]. In the mathematical research area of number theory, the study of statistical correlations of the values of Riemann zeta and related $L$-functions has prompted some of the same generalizations [KS99]. It was observed early on [AZ97] that these post-Dyson ensembles, or rather the underlying symmetry classes, are in one-to-one correspondence with the large families of symmetric spaces.

The prime emphasis of the present handbook article will be on describing Dyson’s fundamental result from a modern perspective. A second goal will be to introduce the post-Dyson ensembles. While there seems to exist no unanimous view on how these fit into a systematic picture, here we will follow [HHZ05] to demonstrate that they emerge from Dyson’s setting upon replacing the plain structure of Hilbert space by the more refined structure of Fock space 2. The reader is advised that some aspects of this story are treated in a more leisurely manner in the author’s encyclopedia article [Zir04].

To preclude any misunderstanding, let us issue a clarification of language right here: ‘symmetry class’ must not be confused with ‘universality class’! Indeed, inside a symmetry class as understood in this article various types of physical behavior are possible. (For example, random matrix models for weakly disordered time-reversal invariant metals belong to the so-called Wigner-Dyson symmetry class of real symmetric matrices, and so do Anderson tight-binding models with real hopping and strong disorder. The former are believed to

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2We mention in passing that a classification of Dirac Hamiltonians in two dimensions has been proposed in [BL02]. Unlike ours, this is not a symmetry classification in Dyson’s sense.
exhibit the universal energy level statistics given by the Gaussian Orthogonal Ensemble, whereas the latter have localized eigenfunctions and hence level statistics which is expected to approach the Poisson limit when the system size goes to infinity.) For this reason the present article must refrain from writing down explicit formulas for joint eigenvalue distributions, which are available only in certain universal limits.

2 Dyson’s Threefold Way

Dyson’s classification is formulated in a general and simple mathematical setting which we now describe. First of all, the framework of quantum theory calls for the basic structure of a complex vector space $V$ carrying a Hermitian scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. (Dyson actually argues [Dys62] in favor of working over the real numbers, but we will not follow suit in this respect.) For technical simplicity, we do join Dyson in taking $V$ to be finite-dimensional. In applications, $V \simeq \mathbb{C}^n$ will usually be the truncated Hilbert space of a family of disordered or quantum chaotic Hamiltonian systems.

The Hermitian structure of the vector space $V$ determines a group $U(V)$ of unitary transformations of $V$. Let us recall that the elements $g \in U(V)$ are $\mathbb{C}$-linear operators satisfying the condition $\langle gv, gv' \rangle = \langle v, v' \rangle$ for all $v, v' \in V$.

Building on the Hermitian vector space $V$, Dyson’s setting stipulates that $V$ be equipped with a unitary group action

$$G_0 \times V \to V, \quad (g, v) \mapsto \rho_V(g)v, \quad \rho_V(g) \in U(V). \quad (2.1)$$

In other words, there is some group $G_0$ whose elements $g$ are represented on $V$ by unitary operators $\rho_V(g)$. This group $G_0$ is meant to be the group of joint (unitary) symmetries of a family of quantum mechanical Hamiltonian systems with Hilbert space $V$. We will write $\rho_V(g) \equiv g$ for short.

Now, not every symmetry of a quantum system is of the canonical unitary kind. The prime counterexample is the operation, $T$, of inverting the time direction, called time reversal for short. It is represented on Hilbert space $V$ by an anti-unitary operator $T \equiv \rho_V(T)$, which is to say that $T$ is complex anti-linear and preserves the Hermitian scalar product up to complex conjugation:

$$T(zv) = \overline{z}Tv, \quad \langle Tv, Tv' \rangle = \overline{\langle v, v' \rangle} \quad (z \in \mathbb{C}; \ v, v' \in V). \quad (2.2)$$

Another operation of this kind is charge conjugation in relativistic theories such as the Dirac equation for the electron and its anti-particle, the positron.

Thus in Dyson’s general setting one has a so-called symmetry group $G = G_0 \cup G_1$ where the subgroup $G_0$ is represented on $V$ by unitaries, while $G_1$ (not a group) is represented by anti-unitaries. By the definition of what is meant
by a ‘symmetry’, the generator of time evolution, the Hamiltonian $H$, of the quantum system is fixed by conjugation $gHg^{-1} = H$ with any $g \in G$.

The set $G_1$ may be empty. When it is not, the composition of any two elements of $G_1$ is unitary, so every $g \in G_1$ can be obtained from a fixed element of $G_1$, say $T$, by left multiplication with some $U \in G_0$: $g = TU$. The same goes for right multiplication, i.e., for every $g \in G_1$ there also exists $U' \in G_0$ so that $g = U'T$. In other words, when $G_1$ is non-empty, $G_0 \subset G$ is a proper normal subgroup and the factor group $G/G_0 \simeq \mathbb{Z}_2$ consists of exactly two elements, $G_0$ and $TG_0 = G_1$. For future use we record that conjugation $U \mapsto TUT^{-1} =: a(U)$ by time reversal is an automorphism of $G_0$.

Following Dyson [Dys62] we assume that the special element $T$ represents an inversion symmetry such as time reversal or charge conjugation. $T$ must then be a (projective) involution, i.e., $T^2 = z \times \text{Id}_V$ with $0 \neq z \in \mathbb{C}$, so that conjugation by $T^2$ is the identity operation. Since $T$ is anti-unitary, $z$ must have modulus $|z| = 1$, and by the $\mathbb{C}$-antilinearity of $T$ the associative law

$$z T = T^2 \cdot T = T \cdot T^2 = T z = z T$$

forces $z$ to be real, which leaves only two possibilities: $T^2 = \pm \text{Id}_V$.

Let us record here a concrete example of some historical importance: the Hilbert space $V$ might be the space of totally anti-symmetric wave functions of $n$ particles distributed over the shell-model space of an atom or an atomic nucleus, and the symmetry group $G$ might be $G = O_3 \cup T O_3$, the full rotation group $O_3$ (including parity) together with its translate by time reversal $T$.

In summary, Dyson’s setting assumes two pieces of data:

- a finite-dimensional complex vector space $V$ with Hermitian structure,
- a group $G = G_0 \cup TG_0$ acting on $V$ by unitary and anti-unitary operators.

It should be stressed that, in principle, the primary object is the Hamiltonian, and the symmetries $G$ are secondary objects derived from it. However, adopting Dyson’s standpoint we now turn tables to view the symmetries as fundamental and given and the Hamiltonians as derived objects. Thus, fixing any pair $(V, G)$ our goal is to elucidate the structure of the space of all compatible Hamiltonians, i.e., the self-adjoint operators $H$ on $V$ which commute with the $G$-action. Such a space is reducible in general: the $G$-compatible Hamiltonian matrices decompose as a direct sum of blocks. The goal of classification is to enumerate the irreducible blocks that occur in this setting.

While the main objects to classify are the spaces of compatible Hamiltonians $H$, we find it technically convenient to perform some of the discussion at the integrated level of time evolutions $U_t = e^{-itH/\hbar}$ instead. This change of focus results in no loss, as the Hamiltonians can always be retrieved by linearization.
in $t$ at $t = 0$. The compatibility conditions for $U \equiv U_t$ read

$$U = g_0Ug_0^{-1} = g_1U^{-1}g_1^{-1} \quad (\text{for all } g_\sigma \in G_\sigma) . \quad (2.4)$$

The strategy will be to make a reduction to the case of the trivial group $G_0 = \{\text{Id}\}$. The situation with trivial $G_0$ can then be handled by enumeration of a finite number of possibilities.

2.1 Reduction to the case of $G_0 = \{\text{Id}\}$

To motivate the technical reduction procedure below, we begin by elaborating the example of the rotation group $O_3$ acting on a Hilbert space of shell-model states. Any Hamiltonian which commutes with $G_0 = O_3$ conserves total angular momentum, $L$, and parity, $\pi$, which means that all Hamiltonian matrix elements connecting states in sectors of different quantum numbers $(L, \pi)$ vanish identically. Thus, the matrix of the Hamiltonian with respect to a basis of states with definite values of $(L, \pi)$ has diagonal block structure. $O_3$-symmetry further implies that the Hamiltonian matrix is diagonal with respect to the orthogonal projection, $M$, of total angular momentum on some axis in position space. Moreover, for a suitable choice of basis the matrix will be the same for each $M$-value of a given sector $(L, \pi)$.

To put these words into formulas, we employ the mathematical notions of orthogonal sum and tensor product to decompose the shell-model space as

$$V \simeq \bigoplus_{L \geq 0, \, \pi = \pm 1} V_{(L, \pi)} , \quad V_{(L, \pi)} = \mathbb{C}^{m(L, \pi)} \otimes \mathbb{C}^{2L+1} , \quad (2.5)$$

where $m(L, \pi)$ is the multiplicity in $V$ of the $O_3$-representation with quantum numbers $(L, \pi)$. The statement above is that all symmetry operators and compatible Hamiltonians are diagonal with respect to this direct sum, and within a fixed block $V_{(L, \pi)}$ the Hamiltonians act on the first factor $\mathbb{C}^{m(L, \pi)}$ and are trivial on the second factor $\mathbb{C}^{2L+1}$ of the tensor product, while the symmetry operators act on the second factor and are trivial on the first factor. Thus we may picture each sector $V_{(L, \pi)}$ as a rectangular array of states where the Hamiltonians act, say, horizontally and are the same in each row of the array, while the symmetries act vertically and are the same in each column.

This concludes our example, and we now move on to the general case of any group $G_0$ acting reductively on $V$. To handle it, we need some language and notation as follows. A $G_0$-representation $X$ is a $\mathbb{C}$-vector space carrying a $G_0$-action $G_0 \times X \to X$ by $(g, x) \mapsto \rho_X(g)x$. If $X$ and $Y$ are $G_0$-representations, then by the space $\text{Hom}_{G_0}(X, Y)$ of $G_0$-equivariant homomorphisms from $X$ to $Y$ one means the complex vector space of $\mathbb{C}$-linear maps $\psi : X \to Y$ with the intertwining property $\rho_Y(g)\psi = \psi \rho_X(g)$ for all $g \in G_0$. If $X$ is an irreducible $G_0$-representation, then Schur’s lemma says that $\text{Hom}_{G_0}(X, X)$ is
one-dimensional, being spanned by the identity, $\text{Id}_X$. For two irreducible $G_0$-representations $X$ and $Y$, the dimension of $\text{Hom}_{G_0}(X,Y)$ is either zero or one, by an easy corollary of Schur’s lemma. In the latter case $X$ and $Y$ are said to belong to the same isomorphism class.

Using the symbol $\lambda$ to denote the isomorphism classes of irreducible $G_0$-representations, we fix for each $\lambda$ a standard representation space $R_\lambda$. Note that $\dim \text{Hom}_{G_0}(R_\lambda, V)$ counts the multiplicity in $V$ of the irreducible representation of isomorphism class $\lambda$. In our shell-model example with $G_0 = O_3$ we have $\lambda = (L, \pi)$, $R_\lambda = \mathbb{C}^{2L+1}$, and $\dim \text{Hom}_{G_0}(R_\lambda, V) = m(L, \pi)$.

The following statement can be interpreted as saying that the example adequately reflects the general situation.

**Lemma 2.1** Let $G_0$ act reductively on $V$. Then

$$ \bigoplus_\lambda \text{Hom}_{G_0}(R_\lambda, V) \otimes R_\lambda \rightarrow V; \quad \bigoplus_\lambda (\psi_\lambda \otimes r_\lambda) \mapsto \sum_\lambda \psi_\lambda(r_\lambda) $$

is a $G_0$-equivariant isomorphism.

**Remark.** The decomposition offered by this lemma perfectly separates the unitary symmetry multiplets from the dynamical degrees of freedom and thus gives an immediate view of the structure of the space of $G_0$-compatible Hamiltonians. Indeed, the direct sum over isomorphism classes (or 'sectors') $\lambda$ is preserved by the symmetries $G_0$ as well as the compatible Hamiltonians $H$; and $G_0$ is trivial on $\text{Hom}_{G_0}(R_\lambda, V)$ while the Hamiltonians are trivial on $R_\lambda$.

Next, we remove the time-evolution trivial factors $R_\lambda$ from the picture. To do so, we need to go through the step of transferring all given structure to the spaces $E_\lambda := \text{Hom}_{G_0}(R_\lambda, V)$.

### 2.1.1 Transfer of structure.

We first transfer the Hermitian structure of $V$. In the present setting of a unitary $G_0$-action, the Hermitian scalar product of $V$ reduces to a Hermitian scalar product on each sector of the direct-sum decomposition of Lemma 2.1 by orthogonality of the sum. Hence, we may focus attention on a definite sector $E \otimes R \equiv E_\lambda \otimes R_\lambda$. Fixing a $G_0$-invariant Hermitian scalar product $\langle \cdot, \cdot \rangle_R$ on $R = R_\lambda$ we define such a product $\langle \cdot, \cdot \rangle_E : E \times E \rightarrow \mathbb{C}$ by

$$ \langle \psi, \psi' \rangle_E := \langle \psi(r), \psi'(r) \rangle_V / \langle r, r \rangle_R, \quad (2.6) $$

which is easily checked to be independent of the choice of $r \in R$, $r \neq 0$.

Before carrying on, we note that for any Hermitian vector space $V$ there exists a canonically defined $\mathbb{C}$-antilinear bijection $C_V : V \rightarrow V^*$ to the dual vector space $V^*$ by $C_V(v) := \langle v, \cdot \rangle_V$. (In Dirac’s language this is the conversion from ‘ket’ vector to ‘bra’ vector.) By naturalness of the transfer of Hermitian structure we have the relation $C_{E \otimes R} = C_E \otimes C_R$. 

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Turning to the more involved step of transferring time reversal $T$, we begin with a preparation. If $L : V \to W$ is a linear mapping between vector spaces, we denote by $L^t : W^* \to V^*$ the canonical transpose defined by $(L^t f)(v) = f(Lv)$. Let now $V$ be our Hilbert space with ket-bra bijection $C \equiv C_V$. Then for any $g \in U(V)$ we have the relation $C g C^{-1} = (g^{-1})^t$ because

$$C(gv) = \langle gv, \cdot \rangle = \langle v, g^{-1} \cdot \rangle = (g^{-1})^t C(v) \quad (v \in V). \quad (2.7)$$

Moreover, recalling the automorphism $G_0 \ni g \mapsto a(g) = T g T^{-1}$ of $G_0$ we obtain

$$CT g = a(g^{-1})^t CT \quad (g \in G_0). \quad (2.8)$$

Thus, since $C$ and $T$ are bijective, the $\mathbb{C}$-linear mapping $CT : V \to V^*$ is a $G_0$-equivariant isomorphism interchanging the given $G_0$-representation on $V$ with the representation on $V^*$ by $g \mapsto a(g^{-1})^t$. In particular, it follows that $T$ stabilizes the decomposition $V = \bigoplus \lambda V_\lambda \simeq \bigoplus \lambda (E_\lambda \otimes R_\lambda)$ of Lemma [2.1].

If $T$ exchanges different sectors $V_\lambda$, the situation is very easy to handle (see below). The more challenging case is $TV_\lambda = V_\lambda$, which we now assume.

**Lemma 2.2** Let $TV_\lambda = V_\lambda$. Under the isomorphism $V_\lambda \simeq E_\lambda \otimes R_\lambda$ the time-reversal operator transfers to a pure tensor

$$T = \alpha \otimes \beta, \quad \alpha : E_\lambda \to E_\lambda, \quad \beta : R_\lambda \to R_\lambda,$$

with anti-unitary $\alpha$ and $\beta$.

**Proof.** Writing $E_\lambda \equiv E$ and $R_\lambda \equiv R$ for short, we consider the transferred mapping $CT : E \otimes R \to E^* \otimes R^*$, which expands as $CT = \bigoplus \phi_i \otimes \psi_i$ with $\mathbb{C}$-linear mappings $\phi_i, \psi_i$. Since $CT$ is known to be a $G_0$-equivariant isomorphism, so is every map $\psi_i : R \to R^*$. By the irreducibility of $R$ and Schur’s lemma, there exists only one such map (up to scalar multiples). Hence $CT$ is a pure tensor: $CT = \phi \otimes \psi$. Using $C = C_E \otimes R = C_E \otimes C_R$ we obtain $T = \alpha \otimes \beta$ with $\mathbb{C}$-antilinear $\alpha = C_E^{-1} \phi$ and $\beta = C_R^{-1} \psi$. Since the tensor product lets you move scalars between factors, the maps $\alpha$ and $\beta$ are not uniquely defined.

We may use this freedom to make $\beta$ anti-unitary. Because $T$ is anti-unitary, it then follows from the definition [2.1] of the Hermitian structure of $E$ that $\alpha$ is anti-unitary.

**Remark.** By an elementary argument, which was spelled out for the anti-unitary operator $T$ in Eq. (2.3), it follows that $\alpha^2 = \epsilon_\alpha \text{Id}_E$ and $\beta^2 = \epsilon_\beta \text{Id}_R$ with $\epsilon_\alpha, \epsilon_\beta \in \{ \pm 1 \}$. Writing $T^2 = \epsilon_T \text{Id}_V$, we have the relation $\epsilon_\alpha \epsilon_\beta = \epsilon_T$. Thus when $\epsilon_\beta = -1$ the parity $\epsilon_\alpha = -\epsilon_T$ of the transferred time-reversal operator $\alpha$ is opposite to that of the original time reversal $T$.

This change of parity occurs, e.g., in the case of $G_0 = SU_2$. Indeed, let $R \equiv R_n$ be the irreducible $SU_2$-representation of dimension $n + 1$. It is a
standard fact of representation theory that $R_n$ is $SU_2$-equivariantly isomorphic to $R_n$ by a symmetric isomorphism $\psi = \psi^\beta$ for even $n$ and skew-symmetric isomorphism $\psi = -\psi^t$ for odd $n$. From $(-1)^n\psi^t = \psi = C_R\beta$ and

$$\psi(v)(v') = \langle \beta v, v' \rangle_R = \langle \beta^2 v, \beta^2 v' \rangle_R = \psi(v')(\beta^2 v),$$

we conclude that $\beta^2 = (-1)^n\text{Id}_{R_n}$.

### 2.2 Classification

By the decomposition of Lemma 2.1, the space $Z_{U(V)}(G_0)$ of $G_0$-compatible time evolutions in $U(V)$ is a direct product of unitary groups,

$$Z_{U(V)}(G_0) \simeq \prod_\lambda U(E_\lambda).$$

We now fix a sector $V_\lambda \simeq E_\lambda \otimes R_\lambda$ and run through the different situations (of which there exist three, essentially) due to the absence or presence of a transferred time-reversal symmetry $\alpha : E_\lambda \rightarrow E_\lambda$.

#### 2.2.1 Class A

The first type of situation occurs when the set $G_1$ of anti-unitary symmetries is either empty or else maps $V_\lambda \simeq E_\lambda \otimes R_\lambda$ to a different sector $V_{\lambda'}$, $\lambda \neq \lambda'$. In both cases, the $G$-compatible time-evolution operators restricted to $V_\lambda$ constitute a unitary group $U(E_\lambda) \simeq U_N$ with $N = \dim E_\lambda$ being the multiplicity of the irreducible representation $R_\lambda$ in $V$. The unitary groups $U_N$ or to be precise, their simple parts $SU_N$, are symmetric spaces (cf. Section 3.4) of the $A$ family or $A$ series in Cartan’s notation – hence the name Class A. In random matrix theory, the Lie group $U_N$ equipped with Haar measure is commonly referred to as the Circular Unitary Ensemble, CUE$_N$\cite{Dys62a}.

The Hamiltonians $H$ in Class A are represented by complex Hermitian $N \times N$ matrices. By putting a $U_N$-invariant Gaussian probability measure

$$d\mu(H) = c_0 e^{-Tr H^2/2\sigma^2} dH, \quad dH = \prod_{i=1}^N dH_{ii} \prod_{j<k} dH_{jk} dH_{kj},$$

on that space, one gets what is called the GUE – the Gaussian Unitary Ensemble – defining the Wigner-Dyson universality class of unitary symmetry. An important physical realization of that class is by electrons in a disordered metal with time-reversal symmetry broken by the presence of a magnetic field.

#### 2.2.2 Classes AI and AII

We now turn to the cases where $T$ is present and $TV_\lambda = V_\lambda \simeq E_\lambda \otimes R_\lambda$. We abbreviate $E_\lambda \equiv E$. From Lemma 2.1 we know that $T = \alpha \otimes \beta$ is a pure tensor with anti-unitary $\alpha$, and we have $\alpha^2 = \epsilon_\alpha \text{Id}_E$ with parity $\epsilon_\alpha = \epsilon_T \epsilon_\beta$. 

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Using conjugation by $\alpha$ to define an automorphism

$$\tau : U(E) \to U(E), \quad u \mapsto \alpha u \alpha^{-1},$$

(2.12)

we transfer the conditions (2.4) to $V_\lambda$ and describe the set $Z_{U(E)}(G)$ of $G$-compatible time evolutions in $U(E)$ as

$$Z_{U(E)}(G) = \{ x \in U(E) \mid \tau(x) = x^{-1} \}.$$  

(2.13)

Now let $U \equiv U(E)$ for short and denote by $K \subset U$ the subgroup of $\tau$-fixed elements $k = \tau(k) \in U$. The set $Z_U(G)$ is analytically diffeomorphic to the coset space $U/K$ by the mapping

$$U/K \to Z_U(G) \subset U, \quad uK \mapsto u\tau(u^{-1}),$$

(2.14)

which is called the Cartan embedding of $U/K$ into $U$. The remaining task is to determine $K$. This is done as follows.

Recalling the definition $C_E \alpha = \phi$ and using $C_E k = (k^{-1})^t C_E$ we express the fixed-point condition $k = \tau(k) = \alpha k \alpha^{-1}$ as $(k^{-1})^t = \phi k \phi^{-1}$ or, equivalently, $\phi = k^t \phi k$, which means that the bilinear form associated with $\phi : E \to E^*$,

$$Q_\phi : E \times E \to \mathbb{C}, \quad (e, e') \mapsto \phi(e)(e'),$$

(2.15)

is preserved by $k \in K$. By running the argument around Eq. (2.9) in reverse order (with the obvious substitutions $\psi \to \phi$ and $\beta \to \alpha$), we see that the non-degenerate form $Q_\phi$ is symmetric if $\epsilon_\alpha = +1$ and skew if $\epsilon_\alpha = -1$. In the former case it follows that $K = O(E) \simeq O_N$ is an orthogonal group, while in the latter case, which occurs only if $N \in 2\mathbb{N}$, $K = USp(E) \simeq USp_N$ is unitary symplectic. In both cases the coset space $U/K$ is a symmetric space (cf. Section 3.4) – a fact first noticed by Dyson in [Dys70].

Thus in the present setting of $TV_\lambda = V_\lambda$ we have the following dichotomy for the sets of $G$-compatible time evolutions $Z_{U(E)}(G) \simeq U/K$:

Class AI: $U/K \simeq U_N/O_N \quad (\epsilon_\alpha = +1),$  
Class AII: $U/K \simeq U_N/USp_N \quad (\epsilon_\alpha = -1, \ N \in 2\mathbb{N}).$

Again we are referring to symmetric spaces by the names they – or rather their simple parts $SU_N/SO_N$ and $SU_N/USp_N$ – have in the Cartan classification. In random matrix theory, the symmetric space $U_N/O_N$ (or its Cartan embedding into $U_N$ as the symmetric unitary matrices) equipped with $U_N$-invariant probability measure is called the Circular Orthogonal Ensemble, COE, while the Cartan embedding of $U_N/USp_N$ equipped with $U_N$-invariant probability measure is known as the Circular Symplectic Ensemble, CSE$_N$ [Dys62a]. (Note the confusing fact that the naming goes by the subgroup which is divided out.)
Examples for Class A1 are provided by time-reversal invariant systems with symmetry \( G_0 = (SU_2)_{\text{spin}} \). Indeed, by the fundamental laws of quantum physics time reversal \( T \) squares to \((-1)^{2S} \) times the identity on states with spin \( S \). Such states transform according to the irreducible \( SU_2 \)-representation of dimension \( 2S + 1 \), and from \( \beta^2 = (-1)^{2S} \) (see the Remark after Lemma 2.1) it follows that \( \epsilon_\alpha = \epsilon_T \epsilon_\beta = (-1)^{2S}(-1)^{2S} = +1 \) in all cases. A historically important realization of Class A1 is furnished by the highly excited states of atomic nuclei as observed by neutron scattering just above the neutron threshold.

By breaking \( SU_2 \)-symmetry (i.e., by taking \( G_0 = \{\text{Id}\} \)) while maintaining \( T \)-symmetry for states with half-integer spin (say single electrons, which carry spin \( S = 1/2 \)), one gets \( \epsilon_\alpha = \epsilon_T = (-1)^{2S} = -1 \), thereby realizing Class AII. An experimental observation of this class and its characteristic wave interference phenomena was first reported in the early 1980’s [Ber84] for disordered metallic magnesium films with strong spin-orbit scattering caused by gold impurities.

The Hamiltonians \( H \), obtained by passing to the tangent space of \( U/K \) at unity, are represented by Hermitian matrices with entries that are real numbers (Class A1) or real quaternions (Class AII). The simplest random matrix models result from putting \( K \)-invariant Gaussian probability measures on these spaces; they are called the Gaussian Orthogonal Ensemble and Gaussian Symplectic Ensemble, respectively. Their properties delineate the Wigner-Dyson universality classes of orthogonal and symplectic symmetry.

3 Symmetry Classes of Disordered Fermions

While Dyson’s Threefold Way is fundamental and complete in its general Hilbert space setting, the early 1990’s witnessed the discovery of various new types of strong universality, which were begging for an extended scheme:

- The introduction of QCD-motivated chiral random matrix ensembles (reviewed by Verbaarschot in Chapter 32 of this handbook) mimicked Dyson’s scheme but also transcended it.

- Number theorists had introduced and studied ensembles of \( L \)-functions akin to the Riemann zeta function (see the review by Keating and Snaith in Chapter 24 of this handbook). These display random matrix phenomena which are absent in the classes A, A1, or AII.

- The proximity effect due to Andreev reflection, a particle-hole conversion process in mesoscopic hybrid systems involving metallic as well as superconducting components, was found [AZ97] to give rise to post-Dyson mechanisms of quantum interference (cf. Chapter 35 by Beenakker).

By the middle of the 1990’s, it had become clear that there exists a unifying mathematical principle governing these post-Dyson random matrix phenomena.
This principle will be explained in the present section. We mention in passing that a fascinating recent development [Kit08, SRF09] uses the same principle for a homotopy classification of topological insulators and superconductors.

3.1 Fock space setting

We now describe an extended setting, which replaces the Hermitian vector space $V$ by its exterior algebra $\wedge(V)$ but otherwise retains Dyson’s setting to the fullest extent possible. In physics language we say that we pass from the (single-particle) Hilbert space $V$ to the fermionic Fock space $\wedge(V)$ generated by $V$. The $\mathbb{Z}$-grading $\wedge(V) = \oplus_n \wedge^n(V)$ by the degree $n$ has the physical meaning of particle number. Thus $\wedge^0(V) \equiv \mathbb{C}$ is the vacuum, $\wedge^1(V) \equiv V$ is the one-particle space, $\wedge^2(V)$ is the two-particle space, and so on. We adhere to the assumption of finite-dimensional $V$. Particle number $n$ then is in the range $0 \leq n \leq N := \dim V$. Note that $\dim \wedge^n(V) = \binom{N}{n}$.

The $n$-particle subspace $\wedge^n(V)$ of the Fock space of a Hermitian vector space $V$ carries an induced Hermitian scalar product defined by

$$\langle u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_n \rangle_{\wedge^n(V)} = \text{Det} \begin{pmatrix} \langle u_1, v_1 \rangle_V & \cdots & \langle u_1, v_n \rangle_V \\ \vdots & \ddots & \vdots \\ \langle u_n, v_1 \rangle_V & \cdots & \langle u_n, v_n \rangle_V \end{pmatrix}. \quad (3.1)$$

Relevant operations on $\wedge(V)$ are exterior multiplication (or particle creation) $\varepsilon(v) : \wedge^n(V) \to \wedge^{n+1}(V)$ by $v \in V$ and contraction (or particle annihilation) $\iota(f) : \wedge^n(V) \to \wedge^{n-1}(V)$ by $f \in V^*$. The standard physics convention is to fix some orthonormal basis $\{e_k\}_{k=1,\ldots,N}$ of $V$ and write $a_k^\dagger := \varepsilon(e_k)$ for the particle creation operators and $a_k := \iota(e_k, \cdot)_V$ for the particle annihilation operators. This notation reflects the fact that Hermitian conjugation $\dagger$ in Fock space relates $\varepsilon(v)$ and $\iota(f)$ by $\varepsilon(v)^\dagger = \iota(f)$ where $f = \langle v, \cdot \rangle_V$. The operators $a_k^\dagger$ and $a_k$ satisfy the so-called canonical anti-commutation relations

$$a_k a_l + a_l a_k = 0 = a_k^\dagger a_l^\dagger + a_l^\dagger a_k^\dagger, \quad a_k^\dagger a_l + a_l a_k^\dagger = \delta_{kl}. \quad (3.2)$$

These represent the defining relations of the Clifford algebra $\text{Cl}(W)$ of the vector space $W = V \oplus V^*$ with quadratic form $(v \oplus f, v' \oplus f') \mapsto f(v') + f'(v)$.

Having introduced the basic Fock space structure, we now turn to what is going to be our definition of a symmetry group $G$ acting on Fock space $\wedge(V)$.

As before, we assume that we are given a normal subgroup $G_0 \subset G$. The action of the elements $g \in G_0$ is defined by unitary operators on $V$ which are extended to $\wedge(V)$ by

$$g(v_1 \wedge \cdots \wedge v_n) := (gv_1) \wedge \cdots \wedge (gv_n). \quad (3.3)$$

Similarly, the anti-unitary operator of time reversal $T$ is defined on $V$ and is extended to $\wedge(V)$ by $T(v_1 \wedge \cdots \wedge v_n) := (Tv_1) \wedge \cdots \wedge (Tv_n)$.
Now the \( \mathbb{Z} \)-grading of Fock space offers the natural option of introducing another kind of anti-unitary operator, which is a close cousin of the Hodge star operator for the deRham complex: particle-hole conjugation, \( C \), transforms an \( n \)-particle state into an \( (N-n) \)-particle state or a state with \( n \) holes. (Note the change of meaning of the symbol \( C \) as compared to Section 3.3.)

**Definition 3.1** Fix a generator \( \Omega \in \wedge^N(V) \), \( N = \dim V \), with normalization \( \langle \Omega, \Omega \rangle_{\wedge^N(V)} = 1 \). Then particle-hole conjugation \( C : \wedge^n(V) \to \wedge^{N-n}(V) \) is the anti-unitary operator defined by
\[
(C\psi) \wedge \psi' = \langle \psi, \psi' \rangle_{\wedge^n(V)} \Omega.
\]

Thus the definition of the operator \( C \) uses the Hermitian scalar product of Fock space and a choice of fully occupied state \( \Omega \). An elementary calculation shows that \( C^2 |_{\wedge^n(V)} = (-1)^{n(N-n)} \).

What are the commutation relations of \( C \) with \( T \) and \( g \in G_0 \)? To answer this question, we observe that by \( \dim \wedge^N(V) = 1 \) we may always choose \( \Omega \) to be \( T \)-invariant (i.e., \( T\Omega = \Omega \)). Then from the following computation,
\[
(TC\psi) \wedge T\psi' = T \langle (C\psi) \wedge \psi' \rangle = T \langle \psi, \psi' \rangle \Omega = \langle T\psi, T\psi' \rangle \Omega = (CT\psi) \wedge T\psi',
\]
we have \( CT = TC \). Also, making the natural assumption that both the vacuum space \( \wedge^0(V) \) and the fully occupied space \( \wedge^N(V) \) transform as the trivial \( G_0 \)-representation (i.e., \( g\Omega = \Omega \) for \( g \in G_0 \)), a similar calculation gives \( Cg = gC \).

In order to enlarge the set of possible symmetries and hence the scope of the theory, we now introduce a ‘twisted’ variant of particle-hole conjugation. Let \( S \in \text{U}(V) \) be an involution \( (S^2 = \text{Id}) \) and extend \( S \) to \( \wedge(V) \) by Eq. (3.3). To obtain an extension of the group \( G_0 \), we require that \( S \) commutes with \( T \), satisfies \( S\Omega = \Omega \), and normalizes \( G_0 \), i.e., \( SG_0S^{-1} = G_0 \). (Here we identify \( G_0 \) with its action on Fock space.) By twisted particle-hole conjugation we then mean the operator \( \tilde{C} = CS = SC \). Note that \( \tilde{C}G_0\tilde{C}^{-1} = G_0 \) and \( \tilde{C}T = T\tilde{C} \).

**Definition 3.2** On the Fock space \( \wedge(V) \) over a Hermitian vector space \( V \), let there be the action of a group \( G = G_0 \cup TG_0 \cup \tilde{C}G_0 \cup \tilde{C}TG_0 \) with \( G_0 \) a normal subgroup and \( \tilde{C}T = T\tilde{C} \). We call this a minimal extension of Dyson’s setting if \( G_0 \) acts by unitary operators defined on \( V \), time reversal \( T \) acts as an anti-unitary operator also defined on \( V \), and twisted particle-hole conjugation \( \tilde{C} \) is an anti-unitary bijection \( \wedge^n(V) \to \wedge^{N-n}(V) \).

### 3.2 Classification goal

The simplest question to ask now is this: what is the structure of the set of Hamiltonians that operate on Fock space \( \wedge(V) \) and commute with the given
$G$-action on $\wedge(V)$? Since this question ignores the grading of Fock space by particle number, it takes us back to Dyson’s setting and the answer is, in fact, provided by Dyson’s Threefold Way. (Note that in the absence of restrictions, the most general Hamiltonian in Fock space involves $n$-body interactions of arbitrary rank $n = 1, 2, 3, \ldots, N$.) So there is nothing new to discover here.

We shall, however, be guided to a new and interesting answer by asking a somewhat different question: what is the structure of the set of one-body time evolutions of $\wedge(V)$ which commute with the given $G$-action? Here by a one-body time evolution we mean any unitary operator obtained by exponentiating a self-adjoint Hamiltonian $H$ which is quadratic in the particle creation and annihilation operators:

$$H = \sum_{kl} W_{kl} a_k^+ a_l + \frac{1}{2} \sum_{kl} (Z_{kl} a_k^+ a_l + \bar{Z}_{kl} a_l a_k).$$

These operators $U = e^{-iH/\hbar} \in U(\wedge V)$ form what is called the spin group, $\text{Spin}(W_\mathbb{R})$, of the $2N$-dimensional Euclidean $\mathbb{R}$-vector space $W_\mathbb{R}$ spanned by the Majorana operators $a_k + a_k^\dagger, ia_k - ia_k^\dagger$ ($k = 1, \ldots, N$). $\text{Spin}(W_\mathbb{R}) \simeq \text{Spin}_{2N}$ is a double covering of the real orthogonal group $\text{SO}(W_\mathbb{R}) \simeq \text{SO}_{2N}$. The spin group of most prominence in physics is $\text{Spin}_3 \equiv \text{SU}_2$, a double covering of $\text{SO}_3$. (This double covering is known to physicists by the statement that a rotation by $2\pi$, which acts as the neutral element of $\text{SO}_3$, changes the sign of a spinor.)

Thus, our interest is now in the set

$$Z_{\text{Spin}}(G) := Z_{U(\wedge V)}(G) \cap \text{Spin}(W_\mathbb{R})$$

of $G$-compatible time evolutions in $\text{Spin}(W_\mathbb{R}) \subset U(\wedge V)$. By adaptation of the earlier definition, the $G$-compatibility conditions are

$$U = g_0 U g_0^{-1} = g_1 U^{-1} g_1^{-1}$$

for all $g_0 \in G_0 \cup \tilde{CT}G_0$ and $g_1 \in TG_0 \cup \tilde{CG}_0$.

### 3.3 Reduction to Nambu space

To investigate the set $Z_{\text{Spin}}(G)$ we use the following fact. Any invertible element $A \in \text{Cl}(W)$ determines an automorphism $\gamma \mapsto A\gamma A^{-1}$ of the Clifford algebra $\text{Cl}(W)$ by conjugation. This conjugation action restricts to a representation

$$\tau(g)w := gwg^{-1}$$

of $\text{Spin}(W_\mathbb{R}) \subset \text{Cl}(W)$ on $W_\mathbb{R} \subset \text{Cl}(W)$. Phrased in physics language, the set of Majorana field operators $w = \sum_k (z_k a_k^\dagger + \bar{z}_k a_k) \in W_\mathbb{R}$ is closed under conjugation $w \mapsto gwg^{-1}$ by one-body time evolution operators $g \in \text{Spin}(W_\mathbb{R})$. In fact, by elementary considerations one finds that $\tau(g) : w \mapsto gwg^{-1}$ for
$g \in \text{Spin}(W_R)$ is an orthogonal transformation $\tau(g) \in \text{SO}(W_R)$ of the Euclidean vector space $W_R$. The mapping $\tau : \text{Spin}(W_R) \to \text{SO}(W_R)$, $g \mapsto \tau(g) = \tau(-g)$ is two-to-one. It is a covering map, which amounts to saying that any path in $\text{SO}(W_R)$ lifts uniquely to a path in $\text{Spin}(W_R)$. Note also that the linear mapping $\tau(g) : W_R \to W_R$ extends to a linear mapping $\tau(g) : W \to W$ by $\mathbb{C}$-linearity.

Thus, instead of studying $\text{Spin}(W_R)$ as a group of operators on the full Fock space $\wedge(V)$, we may simplify our work by studying its representation $\tau : \text{Spin} \to \text{SO}$ on the smaller space $W = V \oplus V^*$, here referred to as Nambu space. Of course the object of interest is not $\text{Spin}(W_R)$ but its intersection with the $G$-compatibility conditions. To keep track of the latter conditions, we now transfer the $G$-action from $\wedge(V)$ to $W = V \oplus V^*$.

It is immediately clear how to transfer the actions of $G_0$ and $T$, as these are defined on $V$. In the case of the twisted particle-hole conjugation operator $\tilde{C}$, we do the following computation. Let $\psi \in \wedge^n(V)$ and $\psi' \in \wedge^{n+1}(V)$. Then

$$\langle \tilde{C}a_k^\dagger \psi, \psi' \rangle = \langle S a_k^\dagger \psi, \psi' \rangle \Omega = \langle S \psi, S a_k S^{-1} \psi' \rangle \Omega = \langle \tilde{C} \psi, (S a_k S^{-1}) \psi' \rangle = (-1)^{N-n+1} \langle S a_k S^{-1} \tilde{C} \psi, \psi' \rangle.$$  

Thus the twisted particle-hole conjugate of $a_k^\dagger \in V \subset \text{Cl}(W)$ is $\tilde{C} a_k^\dagger \tilde{C}^{-1} = \pm S a_k S^{-1} \in V^* \subset \text{Cl}(W)$ where the sign alternates with particle number. Note that the operation $a_k^\dagger \mapsto \pm S a_k S^{-1}$ is anti-unitary. Note also that the untwisted particle-hole conjugation $a_k^\dagger \mapsto a_k$ is none other than the $\mathbb{C}$-antilinear bijection $C_V : V \to V^*, v \mapsto \langle v, \cdot \rangle_V$.

To sum up, we have the following induced structures on Nambu space:

- One-body time evolutions $g \in \text{Spin}(W_R)$ act on $W = V \oplus V^*$ by orthogonal transformations $\tau(g) \in \text{SO}(W_R)$.
- $G_0$ is defined on $V$ and acts on $W = V \oplus V^*$ by $g(v \oplus f) = gv \oplus (g^{-1})^t f$. The same goes for time reversal $T$.
- The operator $\tilde{C}$ of twisted particle-hole conjugation induces on $W = V \oplus V^*$ an anti-unitary involution $V \leftrightarrow V^*$.

The goal of symmetry classification now is to characterize the set $Z_{\text{SO}}(G)$ of elements in $\text{SO}(W_R)$ which are compatible with the induced $G$-action on $W$. This problem was posed and solved in [HHZ05], by using an elaboration of the algebraic tools of Section 2 to make a reduction to the case of the trivial group $G_0 = \{\text{Id}\}$. (The involution $V \leftrightarrow V^*$ given by twisted particle-hole conjugation is called a mixing symmetry in [HHZ05].) The outcome is as follows.

**Theorem 3.1** The space $Z_{\text{SO}}(G)$ is a direct product of factors each of which is a classical irreducible compact symmetric space. Conversely, every classical irreducible compact symmetric space occurs in this setting.
There is no space to reproduce the proof here, but in order to turn the theorem into an intelligible statement we now record a few basic facts from the theory of symmetric spaces [Hel78, CM04].

### 3.4 Symmetric spaces

Let $M$ be a connected $m$-dimensional Riemannian manifold and $p$ a point of $M$. In some open subset $N_p$ of a neighborhood of $p$ there exists a map $s_p : N_p \to N_p$, the geodesic inversion with respect to $p$, which sends a point $x \in N_p$ with normal coordinates $(x_1, \ldots, x_m)$ to the point with normal coordinates $(-x_1, \ldots, -x_m)$. The Riemannian manifold $M$ is called locally symmetric if the geodesic inversion is an isometry (i.e., is distance-preserving), and is called globally symmetric if $s_p$ extends to an isometry $s_p : M \to M$, for all $p \in M$. A globally symmetric Riemannian manifold is called a symmetric space for short.

The Riemann curvature tensor of a symmetric space is covariantly constant, which leads to three distinct cases: the scalar curvature can be positive, zero, or negative, and the symmetric space is said to be of compact type, Euclidean type, or non-compact type, respectively. In random matrix theory each type plays a role: the first one provides us with the scattering matrices and time evolutions, the second one with the Hamiltonians, and the third one with the transfer matrices. Our focus here will be on compact type, as it is this type that houses the unitary time evolution operators of quantum mechanics.

Symmetric spaces of compact type arise in the following way. Let $U$ be a connected compact Lie group equipped with a Cartan involution, i.e., an automorphism $\tau : U \to U$ with the involutive property $\tau^2 = \text{Id}$. Let $K \subset U$ be the subgroup of $\tau$-fixed points $u = \tau(u)$. Then the coset space $U/K$ is a compact symmetric space in a geometry defined as follows. Writing $u := \text{Lie}(U)$ and $\mathfrak{k} := \text{Lie}(K)$ for the Lie algebras of the Lie groups involved, let $u = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into positive and negative eigenspaces of the involution $d\tau : u \to u$ induced by linearization of $\tau$ at unity. Fix on $p$ a Euclidean scalar product $\langle \cdot, \cdot \rangle$ which is invariant under the adjoint $K$-action $\text{Ad}(k) : \mathfrak{p} \to \mathfrak{p}$ by $X \mapsto kXk^{-1}$. Then the Riemannian metric $g_{uK}$ evaluated on vectors $v, v'$ tangent to the coset $uK$ is $g_{uK}(v, v') := \langle dL_{u^{-1}}v, dL_{u^{-1}}v' \rangle_\mathfrak{p}$ where $dL_u$ denotes the differential of the operation of left translation on $U/K$ by $u \in U$.

It is important for us that the coset space $U/K$ can be realized as a subset

$$M := \{ x \in U \mid \tau(x) = x^{-1} \}$$

(3.8)

by the Cartan embedding $U/K \to M \subset U$, $uK \mapsto u\tau(u^{-1})$. The metric tensor in this realization is given (in a self-explanatory notation) by $g = \text{Tr} \, dx \, dx^{-1}$. It is invariant under the $K$-action $M \to M$ by twisted conjugation $x \mapsto ux\tau(u^{-1})$. The geodesic inversion with respect to $y \in M$ is $s_y : M \to M$, $x \mapsto yx^{-1}y$. 

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| family | compact type | Euclidean type |
|--------|--------------|----------------|
| $A$    | $U_N$        | $H$ complex Hermitian |
| $AI$   | $U_N/O_N$    | $H$ real symmetric |
| $AII$  | $U_{2N}/USp_{2N}$ | $H$ quaternion self-dual |
| $C$    | $USp_{2N}$  | $Z$ complex symmetric |
| $CI$   | $USp_{2N}/U_N$ | $Z$ complex sym., $W = 0$ |
| $B, D$ | $SO_N$      | $Z$ complex skew |
| $DIII$ | $SO_{2N}/U_N$ | $Z$ complex skew, $W = 0$ |
| $AIII$ | $U_{p+q}/(U_p \times U_q)$ | $Z$ complex $p \times q$, $W = 0$ |
| $BDI$  | $O_{p+q}/(O_p \times O_q)$ | $Z$ real $p \times q$, $W = 0$ |
| $CII$  | $USp_{2p+2q}/(USp_{2p} \times USp_{2q})$ | $Z$ quaternion $2p \times 2q$, $W = 0$ |

Table 1: The Cartan table of classical symmetric spaces

We note that special examples of compact symmetric spaces are afforded by compact Lie groups $K$. For these examples, one takes $U = K \times K$ with flip involution $\tau(k, k') = (k', k)$ leading to $U/K = (K \times K)/K \simeq K$. Cartan’s list of classical (or large families of) compact symmetric spaces is presented in Table 1. The form of the generator $H$ of time evolutions $u = e^{-itH/\hbar}$ is indicated in the third column, where the notation $W, Z$ refers to the Fock space expression (3.4) which translates to $H = \left( \begin{array}{cc} W & Z \\ Z^\dagger & -W^t \end{array} \right)$ by the covering map $\tau : \text{Spin} \to \text{SO}$.

### 3.5 Post-Dyson classes

We now run through the symmetry classes beyond those of Wigner-Dyson. As was mentioned before, these appear in various areas of physics and in the random matrix theory of $L$-functions. For brevity we concentrate on their physical realization by quasi-particles in disordered metals and superconductors.

#### 3.5.1 Class $D$

Consider a superconductor with no symmetries in its quasi-particle dynamics, so $G = \{\text{Id}\}$. (Some concrete physical examples follow below.) The time evolutions $u = e^{-itH/\hbar}$ in this case are constrained only by the condition $u \in \text{Spin}(W_\mathbb{R})$ in Fock space and $\tau(u) \in \text{SO}(W_\mathbb{R})$ in Nambu space. The orthogonal group
SO(W_R) \simeq SO_{2N} is a symmetric space of the $D$ family – hence the name class $D$. In a basis of Majorana fermions $a_k + a_k^\dagger$, $ia_k - ia_k^\dagger$, the matrix of $iH \in \mathfrak{so}_{2N}$ is real skew, and that of $H$ is imaginary skew.

Concrete realizations are found in superconductors where the order parameter transforms under rotations as a spin triplet in spin space and as a $p$-wave in real space. A recent candidate for a quasi-2d (or layered) spin-triplet $p$-wave superconductor is the compound Sr$_2$RuO$_4$. (A non-charged analog is the $A$-phase of superfluid $^3$He.) Time-reversal symmetry in such a system may be broken spontaneously, or else can be broken by an external magnetic field creating vortices in the superconductor.

The simplest random matrix model for class $D$, the SO-invariant Gaussian ensemble of imaginary skew matrices, is analyzed in Mehta’s book [Meh04]. From the expressions given there it is seen that the level correlation functions at high energy coincide with those of the Wigner-Dyson universality class of unitary symmetry (Class $A$). The level correlations at low energy, however, show different behavior defining a separate universality class. This universal behavior at low energies has immediate physical relevance, as it is precisely the low-energy quasi-particles that determine the thermal transport properties of the superconductor at low temperatures.

### 3.5.2 Class DIII

Now, let time reversal $T$ be a symmetry: $G = \{\text{Id}, T\}$. Physically speaking this implies the absence of magnetic fields, magnetic impurities and other agents which distinguish between the forward and backward directions of time. Our physical degrees of freedom are spin-1/2 particles, so $T^2 = -\text{Id}_V$.

According to (3.6) we are looking for the intersection $Z_{SO}(G)$ of the condition $u^{-1} = TuT^{-1}$ with Spin($W_R$), or after transfer to Nambu space, SO($W_R$). By introducing the involution $\tau(u) := TuT^{-1}$ we express the wanted set as

$$Z_{SO}(G) = \{u \in \text{SO}(W_R) \mid u^{-1} = \tau(u)\}. \quad (3.9)$$

Following the discussion around Eq. (3.3) we have $Z_{SO}(G) \simeq U/K$ where $U = \text{SO}(W_R)$ and $K \subset U$ is the subgroup of $\tau$-fixed points in $U$.

In order to identify $K$ we note that time reversal $T : W \rightarrow W$ preserves the real subspace $W_R$ of Majorana operators $a_k + a_k^\dagger$, $ia_k - ia_k^\dagger$. Because $T^2 = -\text{Id}$, the $\mathbb{R}$-linear operator $T : W_R \rightarrow W_R$ is a complex structure of the real vector space $W_R \simeq \mathbb{R}^{2N} \simeq \mathbb{C}^N$. In other words, there exists a basis \{e_{1,j}, e_{2,j}\}_{j=1,...,N} of $W_R$ such that $Te_{1,j} = e_{2,j}$ and $Te_{2,j} = -e_{1,j}$. Now the $\tau$-fixed point condition $k = \tau(k)$ says that $k \in K$ commutes with the complex linear extension $J : W \rightarrow W$ of $T : W_R \rightarrow W_R$ by $Je_{\pm,j} = \pm ie_{\pm,j}$ where $e_{\pm,j} = e_{1,j} \pm ie_{2,j}$. The general element $k$ with this property is a $U_N$-transformation which acts
on \( \text{span}_C \{e_{+}, 1, \ldots, e_{+}, N \} \) as \( k \) and on \( \text{span}_C \{e_{-}, 1, \ldots, e_{-}, N \} \) as \( \overline{k} = (k^{-1})^t \). Hence \( K = U_N \) and

\[
Z_{\text{SO}}(G) \simeq U/K \simeq \text{SO}_2/N/U_N , \tag{3.10}
\]

which is a symmetric space in the DIII family.

Known realizations of this symmetry class exist in gapless superconductors, say with spin-singlet pairing, but with a sufficient concentration of spin-orbit impurities to break spin-rotation symmetry. In order for quasi-particle excitations to exist at low energy, the spatial symmetry of the order parameter should be different from s-wave. A non-charged realization occurs in the B-phase of \(^3\)He, where the order parameter is spin-triplet without breaking time-reversal symmetry. Another candidate are heavy-fermion superconductors, where spin-orbit scattering often happens to be strong owing to the presence of elements with large atomic weights such as uranium and cerium.

### 3.5.3 Class C

Next let the spin of the quasi-particles be conserved, but let time-reversal symmetry be broken instead. Thus magnetic fields (or some equivalent T-breaking agent) are now present, while the effect of spin-orbit scattering is absent. The symmetry group of the physical system then is \( G = G_0 = \text{Spin}_3 = \text{SU}_2 \). Such a situation is realized in spin-singlet superconductors in the vortex phase. Prominent examples are the cuprate superconductors, which are layered and exhibit an order parameter with \( d \)-wave symmetry in their copper-oxide planes.

The symmetry-compatible time evolutions are identified by going through the process of eliminating the unitary symmetries \( G_0 = G \). For that, we decompose the Hilbert space as \( V = E \otimes R, \, E = \text{Hom}_G(R, V) \), where \( R := \mathbb{C}^2 \) is the fundamental representation of \( G = \text{SU}_2 \). Now there exists a skew-symmetric \( \text{SU}_2 \)-equivariant isomorphism (known in physics by the name of spin-singlet pairing) between the vector space \( R \) and its dual \( R^* \). Therefore we have \( W = V \oplus V^* \simeq (E \oplus E^*) \otimes R \), and elimination of the conserved factor \( R \) transfers the canonical symmetric form of \( W = V \oplus V^* \) to the canonical alternating form \( (e \oplus f, e' \oplus f') \mapsto f(e') - f'(e) \) of \( E \oplus E^* \). On transferring also the Hermitian scalar product from \( V \oplus V^* \) to \( E \oplus E^* \), one sees that the \( G \)-compatible time evolutions form a unitary symplectic group,

\[
Z_{\text{SO}}(G) = \text{SO}(W_R)^G \simeq \text{USp}(E \oplus E^*) , \tag{3.11}
\]

which is a compact symmetric space of the \( C \) family.

### 3.5.4 Class CI

The next class is obtained by taking spin rotations as well as the time reversal \( T \) to be symmetries of the quasi-particle system. Thus the symmetry group
now is $G = G_0 \cup TG_0$ with $G_0 = \text{Spin}_3 = SU_2$. Like in the previous symmetry class, physical realizations are provided by the low-energy quasi-particles of unconventional spin-singlet superconductors. The superconductor must now be in the Meissner phase where magnetic fields are expelled by screening currents.

To identify the relevant symmetric space, we again transfer from $V \oplus V^* = (E \oplus E^*) \otimes R$ to the reduced space $E \oplus E^*$. By this reduction, the canonical form undergoes a change of type from symmetric to alternating as before. We must also transfer time reversal; because the fundamental representation $R = \mathbb{C}^2$ of $SU_2$ is self-dual by a skew-symmetric isomorphism, the parity of the time-reversal operator changes from $T^2 = -\text{Id}_{E \oplus E^*}$ to $T^2 = +\text{Id}_{E \oplus E^*}$ by the mechanism explained at the end of Section 2.2.

We have $Z_{SO}(G) \simeq U/K$ where $U = \text{USp}(E \oplus E^*)$ and $K$ is the subgroup of elements fixed by conjugation with $T$. Because the reduced $T$ squares to $+1$, we may realize it on matrices as the complex conjugation operator by working in a basis of $T$-fixed vectors of $E \oplus E^*$. The Lie algebra elements $X \in \text{usp}(E \oplus E^*)$ have the form $X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with anti-Hermitian A and complex symmetric $B$. They commute with the operation of complex conjugation if $A$ is real skew and $B$ real symmetric. Matrices $X$ with such $A$ and $B$ span the Lie algebra $\mathfrak{u}_N$, $N = \dim(E)$. At the Lie group level it follows that $K = U_N$. Hence $Z_{SO}(G) \simeq \text{USp}_{2N}/U_N$ – a symmetric space in the CI family.

3.5.5 Class AIII

So far, we have made no use of twisted particle-hole conjugation $\tilde{C}$ as a symmetry, but now let the symmetry group be $G = G_0 \cup \tilde{C}G_0$ where $G_0 = U_1$ acts on $W = V \oplus V^*$ by $v \oplus f \mapsto zv \oplus z^{-1}f$ (for $z \in \mathbb{C}$, $|z| = 1$).

In order for the elements $u \in \text{SO}(W_R)$ to commute with the $G_0$-action, they must be of the block-diagonal form $u = k \oplus (k^{-1})^t$, $k \in U(V)$. Therefore $Z_{SO}(G_0) \simeq U(V)$. The wanted set then is $Z_{SO}(G) \simeq U/K$ with $U \equiv U(V)$ and $K$ the subgroup of elements which are fixed by conjugation with $\tilde{C}$.

Recall from Section 3.4 that $\tilde{C}|_V = CS$ where $S \in U$, $S^2 = \text{Id}$, and untwisted particle-hole conjugation $C$ coincides (up to an irrelevant sign) with the canonical bijection $C_V : V \to V^*$, $C_V(v) = \langle v, \cdot \rangle_V$. The condition for $u = k \oplus (k^{-1})^t$ to belong to $K$ reads

$$(k^{-1})^t = \tilde{C}k\tilde{C}^{-1}.$$  

Since $k^{-1} = k^\dagger$ and $C^{-1}k^tC = k^\dagger$, this condition is equivalent to $k = SkS$.

Now let $V = V_+ \oplus V_-$ where $V_\pm$ are orthogonal subspaces with projection operators $\Pi_\pm$. Then if $S = \Pi_+ - \Pi_-$ we have $K = U(V_+) \times U(V_-)$ and hence

$$Z_{SO}(G) \simeq U(V)/(U(V_+) \times U(V_-))$$  

(3.13)
The space (3.13) is a symmetric space of the $A_{III}$ family. Its symmetry class is commonly associated with random-matrix models for the low-energy Dirac spectrum of quantum chromodynamics with massless quarks \cite{Ver94}. An alternative realization exists \cite{ASZ02} in $T$-invariant spin-singlet superconductors with $d$-wave pairing and soft impurity scattering.

### 3.5.6 Classes $BDI$ and $CII$ 

Finally, let the symmetry group $G$ have the full form of Definition 3.2 with $G_0 = U_1$ and $\tilde{C}$ as before (Class $A_{III}$) and a time-reversal symmetry $T$, $T^2 = \pm \text{Id}$. We recall that the elements of $Z_{SO}(G_0)$ are $u = k \oplus (k^{-1})^t$, $k \in U(V)$. The requirement of commutation with the product $\phi := \tilde{C}T : V \to V^*$ of anti-unitary symmetries is equivalent to the condition $\phi = k^t \phi k$.

Let $U := Z_{SO}(G_0 \cup \tilde{C}TG_0)$. To identify $U$, we use that $\phi(v)(v') = \langle STv, v' \rangle$ and $ST = TS$. By the computation of (2.9), it follows that the parity of $T$ equals the parity of the isomorphism $\phi : V \to V^*$. In other words, if $T^2 = \epsilon \text{Id}$ then $\phi = \epsilon \phi$. Thus the condition $\phi = k^t \phi k$ singles out an orthogonal group $U = O(V) \simeq O_N$ in the symmetric case ($\epsilon = +1$) and a unitary symplectic group $U = USp(V) \simeq USp_N$ in the alternating case ($\epsilon = -1$).

In both cases, the wanted set is $Z_{SO}(G) = U/K$ with $K$ the subgroup of fixed points $k = \tilde{C}^{-1}(k^{-1})^t \tilde{C} = SkS$. In the former case we have $K \simeq O_p \times O_{N-p}$, and in the latter case $K \simeq USp_p \times USp_{N-p}$ (with even $N$, $p$). Thus we arrive at the final two entries of Cartan’s list:

- **Class $BDI$**: $U/K \simeq O_N/(O_p \times O_{N-p})$ \quad ($T^2 = +1$),
- **Class $CII$**: $U/K \simeq USp_N/(USp_p \times USp_{N-p})$ \quad ($T^2 = -1$).

These occur as symmetry classes in the context of the massless Dirac operator \cite{Ver94}. Class $BDI$ is realized by taking the gauge group to be $SU_2$ or $USp_{2n}$, Class $CII$ by taking fermions in the adjoint representation or gauge group $SO_n$.

### 4 Discussion 

Given the classification scheme for disordered fermions, it is natural to ask whether an analogous scheme can be developed for the case of bosons. Although there exists no published account of it (see, however, \cite{LSZ06}), we now briefly outline the answer to this question.

The mathematical model for the bosonic Fock space is a symmetric algebra $S(V)$. It is still equipped with a canonical Hermitian structure induced by that of $V$. The real form $W_\mathbb{R}$ of Nambu space $W = V \oplus V^*$ for bosons has an interpretation as a classical phase space spanned by positions $q_j = (a_j + a_j^\dagger)/\sqrt{2}$ and momenta $p_j = (a_j - a_j^\dagger)/\sqrt{2}i$. At the level of one-body unitary time
evolutions in Fock space, the role of the spin group $\text{Spin}(W_R)$ for fermions is handed over to the metaplectic group $\text{Mp}(W_R)$ for bosons.

By the quantum-classical correspondence, a one-parameter group of time evolutions $u_t = e^{-itH/\hbar} \in \text{Mp}(W_R)$ in Fock space gets assigned to a linear symplectic flow $\tau(u_t) \in \text{Sp}(W_R)$ in classical phase space. This correspondence $\tau : \text{Mp}(W_R) \to \text{Sp}(W_R)$ is still two-to-one (reflecting, e.g., the well-known fact that the sign of the harmonic oscillator wave function is reversed by time evolution over one period). An important difference as compared to fermions is that the classical flow $\tau(u_t) \in \text{Sp}(W_R)$ is not unitary in any natural sense.

In nuclear physics, the differential equation of the flow $\tau(u_t)$ is called the RPA equation. For example, in the case without symmetries this equation reads

$$\frac{d}{dt} a_k^\dagger = \sum_j (a_j^\dagger A_{jk} + a_j B_{jk}) , \quad \frac{d}{dt} a_k = \sum_j (a_j^\dagger C_{jk} + a_j D_{jk}) , \quad (4.1)$$

where one requires $B = B^\dagger$, $C = C^\dagger$, and $D = -A^\dagger$ in order for the canonical commutation relations of the boson operators $a^\dagger, a$ to be conserved. Unitarity of the flow (as a time evolution in Fock space) requires $A = -A^\dagger$ and $C = B^\dagger$.

This should be compared with the fermion problem in Class $C$, where one has exactly the same set of equations but for a single sign change: $C = -B^\dagger$. Thus the corresponding generator of time evolution is $X = \begin{pmatrix} A & B \\ \pm B & A \end{pmatrix}$ where the plus sign applies to bosons and the minus sign to fermions. In either case $X$ belongs to the same complex Lie algebra, $\mathfrak{sp}(W)$. The difference is that the generator for fermions lies in a compact real form $\mathfrak{usp}(W_R) \subset \mathfrak{sp}(W)$, whereas the generator for bosons lies in a non-compact real form $\mathfrak{sp}(W_R) \subset \mathfrak{sp}(W)$.

This remains true in the general case with symmetries. Thus if the word ‘symmetry class’ is understood in the complex sense, then the bosonic setting does not lead to any new symmetry classes; it just leads to different real forms of the known symmetry classes viewed as complex spaces. The same statement applies to the non-Hermitian situation. Indeed, all of the spaces of \cite{Mag08} are complex or non-compact real forms of the symmetric spaces of Cartan’s table. Here we must reiterate that the notion of symmetry class is an algebraic one whose prime purpose is to inject an organizational principle into the multitude of possibilities. It must not be misunderstood as a cheap vehicle to produce immediate predictions of eigenvalue distributions and universal behavior!

Let us end with a few historical remarks. The disordered harmonic chain, a model in the post-Dyson Class $BDI$, was first studied by Dyson \cite{Dys53}. The systematic field-theoretic study of models with sublattice symmetry (later recognized as members of the chiral classes $AIII, BDI$) was initiated by Oppermann, Wegner and Gade \cite{OW79, Gad93, GW91}. Gapless superconductors were the subject of numerous papers by Oppermann; e.g., \cite{Opp90} computes the one-loop beta function of the non-linear sigma model for Class $CI$. 

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The 10-way classification of Section 3 was originally discovered by a very different reasoning: the mapping of random matrix problems to effective field-theory models [Zir96] combined with the fact that closure of the renormalization group flow takes place for non-linear sigma models where the target is a symmetric space. A less technical early confirmation of the 10-way classification came from Wegner’s flow equations [Weg94]. These take the form of a double-commutator flow for Hamiltonians $H$ belonging to a matrix space $\mathfrak{p}$; if the double commutator $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]]$ closes in $\mathfrak{p}$, so does Wegner’s flow. The closure condition is satisfied precisely if $\mathfrak{p}$ is the odd part of a Lie algebra $\mathfrak{u} = \mathfrak{f} \oplus \mathfrak{p}$ with involution, i.e., the infinitesimal model of a symmetric space.

Last but not least, let us mention the viewpoint of Volovik (see, e.g., [Vol03]) who advocates classifying single-particle Green’s functions rather than Hamiltonians. That viewpoint in fact has the advantage that it is not tied to non-interacting systems but offers a natural framework in which to include (weak) interactions.

References

[ASZ02] A. Altland, B.D. Simons, and M.R. Zirnbauer, “Theories of low-energy quasi-particle states in disordered $d$-wave superconductors” Phys. Rep. 359 (2002) 283-354

[AZ97] A. Altland and M.R. Zirnbauer, “Non-standard symmetry classes in mesoscopic normal-/superconducting hybrid systems”, Phys. Rev. B 55 (1997) 1142-1161

[Ber84] G. Bergmann, “Weak localization in thin films, a time-of-flight experiment with conduction electrons”, Phys. Rep. 107 (1984) 1-58

[BL02] D. Bernard and A. LeClair, “A classification of random Dirac fermions”, J. Phys. A 35 (2002) 2555-2567

[CM04] M. Caselle and U. Magnea, “Random matrix theory and symmetric spaces”, Phys. Rep. 394 (2004) 41-156

[Dys53] F.J. Dyson, “The dynamics of a disordered linear chain”, Phys. Rev. 92 (1953) 1331-1338

[Dys62] F.J. Dyson, “The threefold way: algebraic structure of symmetry groups and ensembles in quantum mechanics”, J. Math. Phys. 3 (1962) 1199-1215

[Dys62a] F.J. Dyson, “Statistical theory of energy levels of complex systems”, J. Math. Phys. 3 (1962) 140-156
[Dys70] F.J. Dyson, “Correlations between eigenvalues of a random matrix”, Commun. Math. Phys. 19 (1970) 235-250

[Gad93] R. Gade, “Anderson localization for sublattic models”, Nucl. Phys. B 398 (1993) 499-515

[GW91] R. Gade and F. Wegner, “The $n = 0$ replica limit of $\text{U}(n)$ and $\text{U}(n)/\text{SO}(n)$ models”, Nucl. Phys. B 360 (1991) 213-218

[HHZ05] P. Heinzner, A.H. Huckleberry, and M.R. Zirnbauer, “Symmetry classes of disordered fermions”, Commun. Math. Phys. 257 (2005) 725-771

[Hel78] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York 1978

[Kit08] A. Kitaev, “Periodic table for topological insulators and superconductors”, AIP Conf. Proc. 1134 (2009) 22-30 [arXiv:0901.2686]

[KS99] N. Katz and P. Sarnak, Random matrices, Frobenius eigenvalues, and monodromy, American Mathematical Society, Providence, R.I., 1999

[LSZ06] T. Lueck, H.-J. Sommers, and M.R. Zirnbauer, “Energy correlations for a random matrix model of disordered bosons”, J. Math. Phys. 47 (2006) 103304

[Mag08] U. Magnea, “Random matrices beyond the Cartan classification”, J. Phys. A 41 (2008) 045203

[Meh04] M.L. Mehta, Random Matrices, 3rd Edition, Academic Press, London 2004

[Opp90] R. Oppermann, “Anderson localization problems in gapless superconducting phases”, Physica A 167 (1990) 301-312

[OW79] R. Oppermann and F. Wegner, “Disordered system with $n$ orbitals per site – $1/n$ expansion”, Z. Phys. B 34 (1979) 327-348

[SRF09] A.P. Schnyder, S. Ryu, A. Furusaki, A.W.W. Ludwig, “Classification of topological insulators and superconductors”, AIP Conf. Proc. 1134 (2009) 10-21 [arXiv:0905.2029]

[Ver94] J. Verbaarschot, “Spectrum of the QCD Dirac operator and chiral random-matrix theory”, Phys. Rev. Lett. 72 (1994) 2531-2533

[Vol03] G.E. Volovik, The universe in a helium droplet, Clarendon Press, Oxford 2003
[Weg94] F. Wegner, “Flow equations for Hamiltonians”, Annalen d. Physik 3 (1994) 77-91

[Zir96] M.R. Zirnbauer, “Riemannian symmetric superspaces and their origin in random matrix theory”, J. Math. Phys. 37 (1996) 4986-5018

[Zir04] M.R. Zirnbauer, “Symmetry classes in random matrix theory”, Encyclopedia of Mathematical Physics, vol. 5, pp. 204-212, Academic Press, Oxford 2006 [arXiv:math-ph/0404058]