On some Godbillon-Vey classes of a family of regular foliations

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Abstract

The aim of the paper is to construct some Godbillon-Vey classes of a family of regular foliations, defined in the paper. These classes are cohomology classes on the manifold or on suitable open subsets. Some examples are also considered.

Keywords: family of regular foliations, singular foliation, test function, differential form, basic form, cohomology class, Godbillon-Vey class.

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1 Introduction

The families of regular foliations considered in the paper are regular foliations on open subsets such that all the induced leaves on an intersection set give a system of subfoliations as in [1, 7] (i.e. the induced larger-sized leaves are saturated with smaller-sized ones; see conditions (F1)–(F3) in the next section). The resulting geometric distribution, given by the tangent subspaces to leaves of maximal dimension, is a singular one (1. of Proposition 1). Assuming that any intersection is saturated by whole leaves, particular classes of Stefan-Sussmann foliations are obtained (2. of Proposition 1), called here singular foliations that are locally regular.

A tool used to extend Godbillon-Vey forms, on a stratum with a non-minimal dimensional leaves, is the existence of a basic test function on the complement of the stratum. We call a test function, according to a closed subset $M_0 \subset M$, a smooth real function that has $M_0$ as its set of zeros. The existence of a general test function follows from a classical results of Whitney and some properties of extension of smooth sections on closed subsets (see [8, 11, 16], but in a slight different form). Using the line of [3, Section 4], we give a proof in Proposition 2.

The main constructions in the paper are performed in the fourth section. The most important one is that of the Godbillon-Vey class of leaves of minimal dimension in $M$ and in $\Sigma_{\geq r_i}$ (Theorem 1), where we prove that the Godbillon-Vey form of the leaves extends to a global cohomology class $GV_{\text{min}}(F) \in H^{1+2d_{\text{max}}}(M)$ (for the leaves of minimal dimension on $U_0$) and to some Godbillon-Vey classes $GV_{\text{min}}(F_{\Sigma_{\geq r_i}}) \in H^{2(m-r_i)+1}(\Sigma_{\geq r_i})$ (for the other leaves on $U_i$, $i > 0$). In the
case when there is a basic test function of $M \setminus U_i$, then one get a cohomology class on $M$ (Proposition \ref{prop1}).

Two cases are considered in the last section. First, given a regular foliation $F_0$ on $M$, one can easily construct a family of regular foliations on $M$ (for example, adding in a suitable open set a trivial foliation with one leaf), such that its Godbillon-Vey class $GV_{\text{min}}(F)$ is the same as $GV(F_0)$, the usual Godbillon-Vey class of $F_0$ (Proposition \ref{prop2}). Thus if the the Godbillon-Vey class of $F_0$ is non-trivial, also is that of the family $F$. Second, we prove that if $0$ is a regular value for the (weak) test basic function $\varphi$, then the cohomology class $[\nu] \in H^{2n+1}(M)$ vanishes (Proposition \ref{prop3}).

Looking at the first example, it seems likely to find a non-trivial family of regular foliations, maybe a singular foliation, that is locally regular, having a non-trivial Godbillon-Vey class. The second example shows that a non-trivial Godbillon-Vey class can be found not for a regular (weak) test function, possible for a strong one. We let it as an open problem.

## 2 Families of regular foliations

Let $M$ be a differentiable manifold. Let us suppose that there is an open cover $\{U_i\}_{i \in I}$ of $M$ such that the following three conditions hold:

- (F1) - on every $U_i$ there is a regular foliation $F_i$ having $r_i$ as dimension of leaves,

- (F2) - if $i \neq j$ then $r_i \neq r_j$ and

- (F3) - if $U_i \cap U_j \neq \emptyset$, $r_i < r_j$, then $U_i \cap U_j$ is saturated by open subsets of leaves of $F_j$ and every such open set is saturated to its turn by open subsets of leaves of $F_i$.

We can consider a stronger condition than (F3) as:

- (F3') - if $U_i \cap U_j \neq \emptyset$, $r_i < r_j$, then $U_i \cap U_j$ is saturated by leaves of $F_j$ and every such leaf of $F_j$ is saturated to its turn by leaves of $F_i$.

It is easy to see that $I$ is a finite set, $I = 0, k$. The rank of a point $x \in M$ is $r(x) = \max\{r_i : x \in U_i\}$; if $r(x) = r_i$, then and we denote by $D_x$ the tangent space to the leaf of $F_i$. We denote by $\mathcal{R} = \{r(x) = \dim D_x : x \in M\}$. If $S \subset M$ and by $D_S = \bigcup_{x \in S} D_x$ the restriction of $D$ to $S$. Let $\mathcal{R} = \{r_i\}_{i=0 \text{ to } k}$, where $r_{\min} = r_0 < r_1 < \cdots < r_k = r_{\max}$. For $r_i \in \mathcal{R}$, we denote by $\Sigma_{r_i} = \{x \in M : \dim D_x = r_i\}$, $\Sigma_{\leq r_i} = \{x \in M : \dim D_x < r_i\}$, $\Sigma_{\geq r_i} = \{x \in M : \dim D_x > r_i\}$, $\Sigma_{< r_i} = \{x \in M : \dim D_x < r_i\}$, $\Sigma_{> r_i} = \{x \in M : \dim D_x > r_i\}$. We say that the subset $\Sigma_{r_{\min}}$ is the minimal set and $\Sigma_{r_{\max}}$ is the maximal set. The subsets $\Sigma_{< r_i}$ and $\Sigma_{\leq r_i}$ are closed subsets and their complements, the sets $\Sigma_{\geq r_i}$ and $\Sigma_{> r_i}$ are open closed subsets in $M$. The subset $\Sigma_{r_i} \subset \Sigma_{\geq r_i}$ is the minimal subset of $D_{\Sigma_{\geq r_i}}$, and $\Sigma_{> r_i}$ is void if $i = k$ and is equal to $\Sigma_{\geq r_{i+1}}$ if $0 \leq i < k$. We say also that the leaves of $F_i$ are leaves of minimal dimension.

The assignment of a vector subspace $D_x \subset T_x M$, $(\forall) x \in M$, gives a singular
distribution $\mathcal{D}$ on $M$, $\mathcal{D} = \bigcup_{x \in M} \mathcal{D}_x \subset TM$. We denote by $\Gamma_{loc}(\mathcal{D})$ the set of local smooth vector fields tangent to $\mathcal{D}$ in every point where they are defined. One say that $\mathcal{D}$ is:

- smooth, if $\mathcal{D}_x$ is spanned by some restrictions to $x$ of some smooth local vector fields from $\Gamma_{loc}(\mathcal{D})$, $(\forall)x \in M$;
- (completely) integrable, if $\mathcal{D}$ is smooth and there is a partition of $M$ in immersed submanifolds $L \subset M$ such that if $x \in L$, then $\mathcal{D}_x = T_xL$.

(See, for example [2, 15] for more details.)

**Proposition 1**.

1. Assuming the conditions (F1), (F2) and (F3), then $\mathcal{D}$ is a smooth singular distribution on $M$.

2. Assuming the conditions (F1), (F2) and (F3'), then the singular distribution $\mathcal{D}$ is integrable.

**Proof.** Let $x \in M$ and a regular foliate chart of the leaf $F_i$ of $\mathcal{F}_i$ that contain $x$, where $r(x) = r_i$. The condition (F3) implies that the canonical tangent vectors to $F_i$ belong to $\Gamma_{loc}(\mathcal{D})$ and their restrictions to $x$ generate $T_xF_i = \mathcal{D}_x$. Assuming supplementary the condition (F3'), then this local chart is also one corresponding to a singular Stefan-Sussmann foliation on $M$ (according for example to [15]) that is tangent to $\mathcal{D}$. □

We say that

- the conditions (F1), (F2) and (F3) define a *family of regular foliations* and

- the conditions (F1), (F2) and (F3') define a *singular foliation that is locally regular*.

For a family regular foliations, we can define the *leaf* of $x \in M$ as the leaf $F_i$ of $\mathcal{F}_i$ that contains $x$, of maximal dimension $r(x) = r_i$. Moreover, in general a non-ambiguous leaf can be defined only for totally integrable foliations.

Notice that the conditions (F1), (F2) and (F3) does not always assure that $\mathcal{D}$ (defined as above) is integrable. Indeed, consider the open cover of $\mathbb{R}^2$ given by $U_1 = \{(x,y) \in \mathbb{R}^2, \ x > 0\}$ and $U_2 = \{(x,y) \in \mathbb{R}^2, \ x < 1\}$. Let us consider the foliation $\mathcal{F}_1$ by one leaf on $U_1$ and the foliation $\mathcal{F}_2$ by horizontal lines $y = const.$ on $U_2$. The conditions (F1)-(F3) are fulfilled, but the condition (F3') is not fulfilled. It generates a singular smooth distribution $\mathcal{D}$ that is not integrable, generated by the vector fields $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \varphi(x)\frac{\partial}{\partial y}$, where $\varphi$ vanishes for $x \leq 0$ and $\varphi(x) = e^{-\frac{1}{x}}$ for $x > 0$.

Let us consider some other examples.

- Given a family of regular foliations (or a singular foliation that is locally regular), the open set $\Sigma_{\geq r}$ is saturated by leaves of $\mathcal{F}_i$, where $r_i \geq r$, thus a family of regular foliations (or a singular foliation that is locally regular) $\mathcal{F}_{\geq r}$ is induced. In particular $\mathcal{F}_{\geq r_k} = \mathcal{F}_{r_k}$ on $\Sigma_{\geq r_k} = \Sigma_{r_{\text{max}}}$ is regular.

- A regular foliation on $M$ is an singular foliation that is locally regular, when all the points have the same rank, equal to the dimension of the leaves (i.e. of the foliation).
A non-trivial example the foliation of $\mathbb{R}^n$ by concentric spheres (as leaves of dimension $n-1$) and the origin (as a leaf of dimension 0) is a singular foliation that is locally regular. An other non-trivial example is a singular foliation having as leaves concentric spheres, as in the previous example (of dimension $n-1$), outside a compact ball $B(0, \rho) \subset \mathbb{R}^n$, $\rho > 0$, while $B(0, \rho)$ is a union of points (as leaves of dimension 0).

A singular Stefan-Sussmann foliation on $M$ that has $\mathcal{R} = \{0, r\}$, where $0 < r \leq m = \dim M$ is locally regular. In general, consider a regular foliation $\mathcal{F}_U$ on an open subset $U \subset M$, such that the dimension of fibers is $r$, where $0 < r \leq m$. The partition of $M$ by the leaves of $U$ and by the points of $\Sigma_0 = M \setminus U$ gives a locally regular Stefan-Sussmann foliation on $M$. The singular distribution has $\mathcal{R} = \{0, r\}$. Notice that any singular Stefan-Sussmann foliation on $M$ that has $\mathcal{R} = \{0, r\}$ can be obtained in this way.

Consider a regular foliation $\mathcal{F}_U$ on an open subset $U \subset M$, such that the dimension of fibers is $r$, where $0 \leq r < m$. Let $\Sigma_0 \subset U$ be a closed subset of $M$, saturated or not by leaves of $\mathcal{F}_U$. The partition of $M$ by the leaves of $\mathcal{F}_\Sigma_0$ and the leaf $\Sigma_1 = M \setminus \Sigma_0$ gives a family of regular foliations. This is a singular foliation that is locally regular only if $\Sigma_0$ is saturated by the leaves of $\mathcal{F}_U$, when it gives a locally regular Stefan-Sussmann foliation on $M$. This singular distribution has $\mathcal{R} = \{r, m\}$.

Consider some open subsets $U_1, U_2 \subset M$ and a regular foliation $\mathcal{F}_1$ on $U_1$; we suppose that $U_1 \cap U_2 \neq \emptyset$ and $U_1 \cap U_2 \neq M$. Denote by $\Sigma_0 = M \setminus (U_1 \cup U_2)$ and let $U_0 \supset \Sigma_0$ be an open set. We consider on $U_0$ and $U_2$ the trivial foliations $\mathcal{F}_0$ and $\mathcal{F}_2$ respectively, where $\mathcal{F}_0$ has points as leaves and $\mathcal{F}_2$ has one leaf. It follows a family of regular foliations. If $U_1 \cap U_2$ is saturated by leaves of $\mathcal{F}_1$, then the family of regular foliations is a singular foliation that is locally regular.

The suspension constructed for regular foliations (as, for example, in [5, 2.7, 2.8]) can be extended to a family of regular foliations, as follows. Let $B$ and $M$ be two manifolds and $\mathcal{F}$ be a family of regular foliations or a singular foliation that is locally regular. Let us suppose that $\rho : \pi_1(B) \to Diff(M)$ is a representation (i.e. a group morphism) such that every diffeomorphism $\rho(g) \in Diff(M)$ invariate an open neighborhood $U_k$ of $\Sigma_k$, as well as the leaves of the foliation $\mathcal{F}_k$ on $U_k$ that restricts to the leaves on $\Sigma_k$. If we denote by $\tilde{B}$ the universal simple connected cover of $B$, then the suspension space is the quotient space $S = (\tilde{B} \times M)/\sim$ of the equivalence relation $(b, m)^\sim (bg, \rho(g)^{-1}m)$, $g \in \pi_1(B)$, on $\tilde{B} \times M$. As in the classical case, one can first consider on $\tilde{B} \times M$ the product foliations $\mathcal{F}_0$ of the foliation by one leaf on $B$ and the foliations $\mathcal{F}_i$ on $M$. An family of regular foliations or a singular foliation that is locally regular (accordingly to that on $M$) is induced on the quotient space $S$; the leaves, the sets $\Sigma_{k'}$ of the leaves of a same dimension $k'$ and the open neighborhoods $U_{k'}$ of $\Sigma_{k'}$ are naturally induced.

As a particular case, consider an open subset $U \subset M$, a regular foliation $\mathcal{F}_U$ on $U$ and $f \in Diff(M)$ such that $f(U) = U$ and $f$ invariate $\mathcal{F}_U$. We can consider an open neighborhood $W$ of the closed set $M \setminus U$ (for example $W = M$) and the trivial foliation $\mathcal{F}_W$ by points on $W$. The leaves of $\mathcal{F}_U$ and the points of $M \setminus U$ as 0-dimensional leaves give a locally regular Stefan-Sussmann foliation.
on $M$. The suspension of $f$ is considered for $B = S^1, \hat{B} = \mathbb{R}, \pi_1(S^1) = \mathbb{Z}$ and the actions $\mathbb{R} \times \mathbb{Z} \to \mathbb{R}, (x,n) \to x - n$ and $\mathbb{Z} \times M \to M, (n,m) \to f^n(m)$.

For example, consider the natural central symmetry $\sigma : S^n \to S^n \subset \mathbb{R}^{n+1}, \sigma(x) = -x$. Consider also two open spherical caps $C_1 \subset C_2$ centred in the same point $A$ of the sphere $S^n$ and let $C'_1 = \sigma(C_1) \subset C'_2 = \sigma(C_2)$ the symmetric spherical caps centred in $A' = \sigma(A)$, such that $C_2 \cap C'_2 \neq \emptyset$. Denote by $U_1 = S^n \setminus (C_1 \cup C'_1)$ and by $U_2 = C_2 \cup C'_2$. Consider the trivial foliation $\mathcal{F}_2$ on $U_2$ by points and a $k$-regular foliation $\mathcal{F}_1$ on $U_1$ obtained by intersection of $U_1$ by $k+1-$ parallel planes that can be parallel or not with the support $n-$hyperplanes of the spherical caps. Obviously the open sets $U_1$ and $U_2$, as well as the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are invariant by $\sigma$. One can consider a quotient locally regular foliation on $\mathbb{R}P^n$, as well as a suspension locally regular foliation on $S = (\mathbb{R} \times S^n)/\sim$, given by the $\mathbb{Z}$–action $n \cdot (\alpha, \bar{x}) = (\alpha - n, \sigma^n(\bar{x}))$.

3 Test functions

We consider now test functions, that allow us to extend smooth functions and vector fields.

Let $M_0 \subset M$ be a closed subset. We say that a real function $\varphi \in \mathcal{F}(M)$ is a weak test function for $M_0$ if $M_0 = \varphi^{-1}(0)$ (i.e. $\varphi(x) = 0$ iff $x \in M_0$). We say that a weak test function is a strong test function for $M_0$ if, additionally, its values are in $[0, 1]$ and all its differentials vanish in every $x \in M_0$. The existence of test functions is an important tool used in the sequel.

The following simple Lemma shows that the existence of a weak test function gives a strong one.

**Lemma 1** Let $\psi_0 : \mathbb{R} \to [0, 1]$ be smooth such that $\psi_0(t) = 0$ iff $t = 0$ and all the derivatives of $\psi_0$ vanish in $t = 0$. Then for every function $f : M \to \mathbb{R}$ the function $F = \psi_0 \circ f$ has the same zeros as $f$ and all the differentials of $F$ vanish in its zeros.

Notice that a function $\psi_0$ as in Lemma 1 is

$$\psi_0(t) = \begin{cases} e^{-\frac{t}{1+e^{-\frac{1}{t}}}} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \quad (1)$$

A first fact is the existence of a weak test function $\varphi_{M_0}$ for any closed subset $M_0 \subset M$, i.e. a positive smooth real function on $M$, having the set of zeros exactly $M_0$. The existence follows from a classical results of Whitney and some properties of extension of smooth sections on closed subsets (see [8, 11, 16]), but in a slight different form. We give a proof below, in line of [3, Section 4].

**Proposition 2** Let $M$ be a differentiable manifold and $M_0 \subset M$ be a closed subset. Then there is a weak test function for $M_0$. 

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Proof. We can proceed as in [3, Section 4] reducing the problem to the case when \( M = \mathbb{R}^n \) and considering \( M \) properly embedded in \( \mathbb{R}^k \) for some \( k \). Then \( M_0 \subset \mathbb{R}^k \) is also closed. A test function on \( \mathbb{R}^k \) for \( M_0 \) reduces to \( M \) also to a test function for \( M_0 \). Since \( M_0 \) is a closed set, then \( M_1 = \mathbb{R}^k \setminus M_0 \) is an open subset of \( \mathbb{R}^k \). For any point \( p \in M_1 \) there is a ball \( B_p = B(p, 2r) \subset M_1 \). We denote by \( B'_p = B(p, r) \) such that its support is \( \bar{B}_p = B(p, 2r) \), its values are 0 outside \( B_p \) (i.e. on \( \mathbb{R}^n \setminus B_r \)), 1 on \( B'_p = B(p, r) \) and all the other values are in the open interval \((0, 1)\). We can consider an at most countable cover of \( M_1 \) with such balls \( B_p \). In the case when the cover of \( M_1 \) is a finite set \( \{B_i\}_{i=1}^r \), we can consider \( \psi = \sum_{i=1}^r \psi_i \), that is obviously a test function for \( M_0 \). In the case when the cover of \( M_1 \) is a finite set \( \{B_i\}_{i=1}^r \), we can proceed as in [3, Section 4]. For each \( i \in \mathbb{N} \) consider the constants \( c_i \) such that \( c_i \|\psi_i\| \leq 1/2^i \), where the norms are in \( BC^\infty(\mathbb{R}^n, \mathbb{R}) \), then denote \( \varphi_i = c_i \psi_i \) and finally

\[
\varphi = \sum_{i=1}^\infty \varphi_i.
\]

As in the proof of [3, Proposition 4.3], \( \varphi \) is a smooth function and the set of its zeros is \( \mathbb{R}^n \setminus M_1 = M_0 \). Using Lemma [1] with \( \psi_0 \) given by the formula (1), we obtain a test function for \( M_0 \). □

The existence of a weak test function that is not a strong one depends on the zero set (i.e. the closed set). For example, the singular foliation of \( \mathbb{R}^n \) by concentric spheres (as leaves of dimension \( n-1 \)) and the origin (as a leaf of dimension 0) is locally regular and the square of the euclidian norm is a weak test function that is not a strong one. But the singular foliation having as leaves concentric spheres, as in the previous example (of dimension \( n-1 \)), outside a compact ball \( \bar{B}(\overline{0}, \rho) \subset \mathbb{R}^n, \rho > 0 \), while \( \bar{B}(\overline{0}, \rho) \) is a union of points (as leaves of dimension 0) is also locally regular, but every test function of \( \bar{B}(\overline{0}, \rho) \) is always a strong one.

4 The construction of Godbillon-Vey forms and classes

Integrability conditions for a regular foliation are given by Frobenius theorem. It can be expressed using differential forms, as, for example, in [14, Ch. 2. and Ch. 3]. We use this in a similar way as in [12]. If a differentiable \( q \)-form \( \nu \) on \( M \) has locally the form \( \nu = \omega_1 \wedge \cdots \wedge \omega_q \), where \( \omega_1, \ldots, \omega_q \) are local one-forms, we say that \( \nu \) has rank \( q \).

A regular foliation of co-dimension \( q \) on a differentiable manifold \( M \) is given by a non-singular global form \( \nu \in \Omega^q(M) \) of rank \( q \) and, in the locally form \( \nu = \omega_1 \wedge \cdots \wedge \omega_q \), the local one-forms \( \omega_1, \ldots, \omega_q \) are sections of the transverse bundle of the foliations, that generate the \( \mathcal{F}(M) \)-module of transverse one-forms...
One briefly say that the foliation (or its tangent bundle) is given by \( \nu = 0 \), or by vanishing \( \nu \).

Let us consider now two regular foliations \( \mathcal{F}_U \) and \( \mathcal{F}_V \), \( \mathcal{F}_{U|U\cap V} \subset \mathcal{F}_{V|U\cap V} \), such that the tangent bundles of the foliations \( \mathcal{F}_U \) and \( \mathcal{F}_V \) are given vanishing the differential forms \( \omega_1 \in \Omega^{\nu_1+q_2}(U) \) and \( \omega_2 \in \Omega^\nu(V) \) respectively.

**Proposition 3** Denoting by \( \omega'_1 \in \Omega^\nu(U \cap V) \) and \( \omega'_2 \in \Omega^{\nu_1+q_2}(U \cap V) \) the restrictions to \( U \cap V \) of \( \omega_1 \) and \( \omega_2 \) respectively, where \( q_1 > 0 \), then there is a differentiable form \( \theta \in \Omega^\nu(U \cap V) \) such that

\[
\omega'_1 = \omega'_2 \wedge \theta.
\]

**Proof.** First, let us suppose that \( U = V = U \cap V \) is a domain of coordinates \( \{x^u, \bar{x}^\alpha, \bar{x}^\bar{\beta}\}, u = 1, \ldots, q_1 \) and \( \bar{u} = \bar{1}, \ldots, q_2 \) such that \( \{x^u\} \) and \( \{x^\alpha, \bar{x}^\bar{\beta}\} \) are coordinates on the leaves of \( \mathcal{F}_{U|U\cap V} \) and \( \mathcal{F}_{V|U\cap V} \) respectively. Then \( \omega'_1 = h_1 \bar{d}x^1 \wedge \cdots \wedge \bar{d}x^{q_1} \wedge \bar{d}^\alpha \wedge \cdots \wedge \bar{d}^\bar{\beta} \) and \( \omega'_2 = h_2 \bar{d}x^1 \wedge \cdots \wedge \bar{d}x^{q_2} \) with \( h_1, h_2 \in \mathcal{F}(U \cap V) \) having no zeros, thus relation (2) holds for \( \theta = \frac{h_2}{h_1} \bar{d}x^1 \wedge \cdots \wedge \bar{d}x^{q_2} \).

Returning to the general case, let us consider a partition of unity \( \{v_\alpha\}_{\alpha \in A} \) on \( U \cap V \) subordinated to a cover with open domain of local foliated charts, as above, where \( A \) is finite or \( A = N \). Then define \( \theta = \sum_{\alpha \in A} v_\alpha \theta_\alpha \in \Omega^\nu(U \cap V) \).

Since \( \omega'_1 = \omega'_2 \wedge \theta_\alpha \) and \( \sum_{\alpha \in A} v_\alpha = 1 \), then relation (2) holds. \( \square \)

In order to avoid coordinates, we consider in the sequel the ideals \( \mathcal{I}(\mathcal{F}_1) \subset \Omega^\nu(U) \) and \( \mathcal{I}(\mathcal{F}_2) \subset \Omega^\nu(V) \) of differential forms that vanish when evaluated when all vectors are tangent to the leaves of \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) respectively. The two ideals are finitely generated, each homogeneous term containing at least one of the local forms that on \( U \cap V \) can be taken of the form \( \{\bar{w}^\alpha, \bar{w}^\bar{\beta}\}_{\alpha = 1, \ldots, q_1, \bar{\beta} = 1, \ldots, q_2} \) and \( \{\bar{w}^\alpha\}_{\alpha = 1, \ldots, q_1} \) respectively. Notice that \( d\bar{w}^\alpha = \sum_{\nu = 1}^{q_2} \bar{w}^\nu \wedge \nu^\alpha_\nu \) and \( d\bar{w}^\bar{\beta} = \sum_{\nu = 1}^{q_2} \bar{w}^\nu \wedge \nu^\bar{\beta}_\nu \), with \( \nu^\alpha_\nu \) and \( \nu^\bar{\beta}_\nu \) \( \in \Omega^1(U \cap V) \). Then \( \omega_2 \) has the local form

\[
\omega_2 = h_2 \bar{w}^\alpha \wedge \cdots \wedge \bar{w}^{q_1}.
\]

The Frobenius theorem used for \( \mathcal{F}_U \) and \( \mathcal{F}_V \) reads that there are \( \mu_1 \in \Omega^\nu(U) \) and \( \mu_2 \in \Omega^\nu(V) \) such that

\[
d\omega_1 = \omega_1 \wedge \mu_1, \quad d\omega_2 = \omega_2 \wedge \mu_2.
\]

A product of \( q_1 + q_2 + 1 \) forms in \( \mathcal{I}(\mathcal{F}_1) \) as well as of \( q_2 + 1 \) forms in \( \mathcal{I}(\mathcal{F}_2) \) are null. This enables to consider the closed *Godbillon-Vey forms* \( \mu_1 \wedge (d\mu_1)^{\nu_1+q_2} \in \Omega^{2(q_1+q_2)+1}(U) \) and \( \mu_2 \wedge (d\mu_2)^{q_2} \in \Omega^{2q_2+1}(U) \) and the *Godbillon-Vey classes* of the foliations \( \mathcal{F}_U \) and \( \mathcal{F}_V \) as the cohomology classes \( [\mu_1 \wedge (d\mu_1)^{\nu_1+q_2}] \in H^{2(q_1+q_2)+1}(U) \) and \( [\mu_2 \wedge (d\mu_2)^{q_2}] \in H^{2q_2+1}(U) \).

Let us look closely to \( U \cap V \), when the relation (2) holds. For sake of simplicity, we use notations \( \omega_1 \) and \( \omega_2 \) instead of \( \omega'_1 \) and \( \omega'_2 \) respectively.
Differentiating by \( d \), then using (4) and the usual properties of the exterior product, we obtain
\[
\omega_2 \wedge ((-1)^{q_2} d\theta - \theta \wedge (\mu_1 - (-1)^{q_1} q_2 \mu_2)) = 0.
\]
Taking into account (3), then
\[
d\theta - (-1)^{q_2} \theta \wedge (\mu_1 - (-1)^{q_1} q_2 \mu_2) = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^\bar{v} \wedge \eta_\bar{v}, \tag{5}
\]
with \( \eta_\bar{v} \in \Omega^{q_2+1}(\bar{U} \cap \bar{V}) \). Thus the left side of equality (5) belongs to \( \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \subset \Omega^{q_2+1}(U \cap V) \). Denote by
\[
\mu_3 = (-1)^{q_2} (\mu_1 - (-1)^{q_1} q_2 \mu_2). \tag{6}
\]
Differentiating by \( d \) and using again the same relation (5), we obtain
\[
\theta \wedge d\mu_3 = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^\bar{v} \wedge \bar{\eta}_\bar{v}, \tag{7}
\]
with \( \bar{\eta}_\bar{v} \in \Omega^{q_2+1}(U \cap V) \), i.e. \( \theta \wedge d\mu_3 \in \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \). But using local coordinates as in the proof of Proposition 3 we have that, on a domain \( U' \) of such coordinates, there is a local function \( h_3 \) such that \( \theta - h_3 d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^{q_1} \in \mathcal{I}(\mathcal{F}_2)|_{U'} \). Using this fact in (7), it follows that
\[
d\mu_3 = \sum_{\bar{v}=1}^{q_2} \bar{\omega}^\bar{v} \wedge \bar{\eta}_\bar{v}
\]
with \( \bar{\eta}_\bar{v} \in \Omega^1(U \cap V) \), i.e. \( d\mu_3 \in \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \). But \( d\mu_2 \in \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \), thus using (6) it follows that \( d\mu_1 \in \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \).

**Proposition 4** Assuming \( q_1 > 0 \), then the following assertions hold true:

1. The restriction \( d\mu_1|_{U \cap V} \) belongs to the ideal \( \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \).
2. The Godbillon-Vey form of \( d\mu_1|_{U \cap V} \) and its cohomology class according to the foliation \( \mathcal{F}_U|_{U \cap V} \), both vanish.
3. If \( \mathcal{F}' \subset \mathcal{F}'' \), \( \mathcal{F}' \neq \mathcal{F}'' \), are regular foliations on \( M \) and the foliation \( \mathcal{F}'' \) has not a null co-dimension, then the Godbillon-Vey class of \( \mathcal{F}' \) vanishes.

**Proof.** Taking into account (6), \( d\mu_3 \in \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \) and since \( d\mu_2 \in \mathcal{I}(\mathcal{F}_2)|_{U \cap V} \), then the first assertion holds true. If \( q_1 > 0 \), then \( q_1 + q_2 \geq q_1 + 1 \), thus \( (d\mu_1)^{q_1+q_2} = 0 \), because \( (d\mu_1)^{1+q_2} = 0 \); it follows that \( \mu_1 \wedge (d\mu_1)^{q_1+q_2} = 0 \), as well as its cohomology class, thus 2. follows. Then 3. is a simple consequence of 2. □

The result in this Proposition allows to consider the Godbillon-Vey class of the foliation \( \mathcal{F}_U \) having the maximal co-dimension \( q_{\text{max}} = m - r_{\text{min}} \), on
the open subset \( U_{r_{\min}} \subset M \); the foliation has the leaves of minimal dimension. The Godbillon-Vey class is the class \([\mu_{r_{\min}} \wedge (d\mu_{r_{\min}})^{r_{\min}}]\). The differential form \( GV_{r_{\min}} = \mu_{r_{\min}} \wedge (d\mu_{r_{\min}})^{m-r_{\min}} \in \Omega^{1+2q_{\max}}(U_{\max}) \) is null on any intersection \( U_{r_{\min}} \cap U_0 \neq \emptyset \), where \( U_0 \) is an open subset corresponding to a foliation \( F_{U_0} \) of codimension \( q_0 = m - r_0 < q_{\max} = m - r_{\min} \). Thus, extending \( GV_{r_{\min}} \) as null outside \( U_{r_{\min}} \), we obtain a global closed form that gives \( GV_{r_{\min}}(F) \in H^{1+2q_{\max}}(M) \); we call it as the Godbillon-Vey class on leaves of minimal dimension.

In the general case, let us consider the ascending sequence of open sets \( \Sigma_{r_k} \subset \Sigma_{r_{k-1}} \subset \cdots \subset \Sigma_{r_1} \subset \Sigma_{r_0} = M \). Denote by \( F_{\Sigma_{r_i}} \) the restriction of \( F \) to the open set \( \Sigma_{r_i} \), \( i = 0, \cdots, k \); notice that the set \( \Sigma_{r_i} \) is saturated by the leaves of \( F = F_{\Sigma_{r_0}} \). The subset \( \Sigma_{r_i} \subset \Sigma_{r_{i-1}} \) is that of minimal dimensions of leaves. We can consider the Godbillon-Vey classes \( GV_{\min}(F_{\Sigma_{r_i}}) \in H^{2(q_{r_i}+1)}(\Sigma_{r_i}) \). In particular, \( GV_{\min}(F) = GV_{\min}(F_{\Sigma_{r_0}}) \in H^{2(q_{r_0}+1)}(\Sigma_{r_0}) = H^{2(q_{r_0}+1)}(M) \).

**Theorem 1** A Godbillon-Vey form of the leaves extends to a global cohomology class \( GV_{\min}(F) \in H^{1+2q_{\max}}(M) \) (for the leaves of minimal dimension) and to some Godbillon-Vey classes \( GV_{\min}(F_{\Sigma_{r_i}}) \in H^{2(q_{r_i}+1)}(\Sigma_{r_i}) \) (for the leaves on the other \( U_i \), \( i > 0 \)).

In order to obtain global cohomology classes on \( M \), the construction on the Godbillon-Vey class on the leaves of minimal dimension can be extended to the other strata, provided that there is a foliated test function according to that stratum. We perform below this construction.

Let us suppose that the foliation \( F_{r_i} \) on \( U_i \subset M \) has the dimension \( r_i \) of leaves and it is defined on \( U_i \) by the equation \( \omega_i = 0 \), where \( \omega_i \in \Omega^{q_i}(U_i) \), \( q_i = m - r_i \). Then

\[
\text{d} \omega_i = \omega_i \wedge \mu_i
\]

with \( \mu_i \in \Omega^{1}(U_i) \). We suppose below that there is a test function \( \varphi_i \in \mathcal{F}(M) \) for \( M \setminus U_i \) that restricts to a basic function for the foliation \( F_{r_i} \) on \( U_i \); we suppose also that \( \bar{\mu}_i = \varphi_i \mu_i \) (where \( \mu_i \) is defined by zero on \( M \setminus U_i \)) is differentiable on \( M \), i.e. \( \bar{\mu}_i \in \Omega^{1}(M) \); this is always true if \( \varphi_i \) is a strong test function.

**Proposition 5** Let us suppose that the test function \( \varphi_i \) is basic and \( \bar{\mu}_i = \varphi_i \mu_i \) is differentiable on \( M \). Then the differential form \( \bar{\nu}_i = \bar{\mu}_i \wedge (d\bar{\mu}_i)^{q_i} \) is closed, giving a cohomology class \( [\bar{\nu}_i] \in H^{2q_i+1}(M) \).

**Proof.** We have \( \bar{\nu}_i = \bar{\mu}_i \wedge (d\bar{\mu}_i)^{q_i} = \varphi_i^{1+q_i} \mu_i \wedge (d\mu_i)^{q_i} \). If \( \varphi_i \) is basic, then \( \psi_i = \varphi_i^{1+q_i} \) is also basic and \( d\psi_i \wedge \mu_i \wedge (d\mu_i)^{q_i} = 0 \). Thus \( d\bar{\nu}_i = 0 \) and the conclusion follows. \( \square \)

Notice that if the maximal stratum has the dimension \( r_k = m \), then its Godbillon-Vey form vanishes, as well as its Godbillon-Vey class. In particular, if a family of regular foliations has \( R = \{r_0, r_1\} \) and \( r_1 = m \), then the only possible non-null is the Godbillon-Vey class of the leaves of minimal dimension.
5 Two cases

First, we prove that the usual Godbillon-Vey class of a regular foliation is the same with the Godbillon-Vey class of leaves of minimal dimension of a suitable non-trivial family of regular foliations. Let \((M, \mathcal{F}_0)\) be a regular foliation of codimension \(q_0\) defined by a \(q_0\)-differential form \(\omega_0 = 0\), such that \(d\omega_0 = \omega_0 \wedge \mu_0\). Let us consider two open and non-void subsets \(W, U_2\) having the properties that \(W \subset U_2\) and \(\varphi \in \mathcal{F}(M)\) a Uryson function such that \(supp \varphi = M \setminus W = U_1\). Consider on \(U_1\) the foliation \(\mathcal{F}_{U_1}\) as being the restriction to \(U_1\) of foliation \(\mathcal{F}\). Let us suppose that there is on \(U_2\) a non–trivial foliation \(\mathcal{F}_{U_2}\) such that its leaves are saturated by leaves of \(\mathcal{F}_{0|U_2}\) (for this we can take \(U_2\) the domain of a \(\mathcal{F}_0\)–foliate simple chart and then take as \(\mathcal{F}_{U_2}\) a proper foliation having as subfoliation \(\mathcal{F}_{0|U_2}\) (for example, a trivial foliation with one leaf). The foliation \(\mathcal{F}_{U_2}\) is defined by the \(q_0\)-form \(\bar{\omega} = \varphi \omega_0\), that has the same support as \(\varphi\). The foliations \(\mathcal{F}_{U_1}\) and \(\mathcal{F}_{U_2}\) give a non-trivial family of regular foliations on \(M\). The Godbillon-Vey class \(GV_{min}(\mathcal{F}) \in H^{2q_0+1}(M)\) is given extending naturally (using Proposition\(\[1]\) a form that gives the Godbillon-Vey class of \(\mathcal{F}_{U_1}\).

Proposition 6 The Godbillon-Vey class \(GV_{min}(\mathcal{F})\) is the same as \(GV(\mathcal{F}_0)\), the usual Godbillon-Vey class of \(\mathcal{F}_0\).

Proof. The Godbillon-Vey class of \(\mathcal{F}_0\) is given by a differential form \(\eta \wedge (\partial \eta)^{[q_0]}\), such that \(d\varphi = \omega \wedge \eta\), where the definition does not depend of \(\omega\) and \(\eta\) (see \[13\] Theorem 3.11). It can be easy proved that we can take the restriction of \(\varphi\) to \(U_2\) having the form \(f \partial x_1 \cdots \partial x_{q_0}\), where \(\{\bar{x}^i\}_{\alpha = 1, q_0}\) are transverse coordinates for \(\mathcal{F}_0\) on \(U_2\), thus \(\eta|_{U_2} = (-1)^{q_0}df\) and \(d\eta|_{U_2} = 0\). Thus the restriction of the differential form \(\eta \wedge (\partial \eta)^{[q_0]}\) to \(U_2\) vanishes and it extends the differential form on \(U_1\) that gives the Godbillon-Vey class of \(\mathcal{F}_{0|U_2}\), thus it gives \(GV_{min}(\mathcal{F})\). It follows that \(GV_{min}(\mathcal{F}) = GV(\mathcal{F}_0)\). □

We consider below a non-trivial case when the Godbillon-Vey class vanishes. More specifically, we prove that for a regular (weak) test function \(\varphi_i \in \mathcal{F}(M)\) for \(M \setminus U_i\) that restricts to a basic function for the foliation \(\mathcal{F}_{r_i}\) on \(U_i\) the cohomology class \([\varphi_i]\) \(\in H^{2q_0+1}(M)\) vanishes.

Firstly we shall need some preliminary notions about singular forms and cohomology attached to a function, for more see \[2\] [10]. Accordingly, for a smooth function \(f \in \mathcal{F}(M)\) and \(U \subset M\) a \(p\)-form \(\omega \in \Omega^p(U)\) is called a singular \(p\)-form if the form \(f^p \omega\) can be extended to a smooth form on \(M\), that is \(f^p \omega \in \Omega^p(M)\). We denote the space of singular \(p\)-forms with respect to \(f\) by \(\Omega^p_f(M)\). We notice that if \(\omega \in \Omega^p_f(M)\) then \(d\omega \in \Omega^{p+1}_f(M)\) and so we have a differential complex \((\Omega^*_{M}, d)\). The cohomology of this differential complex is isomorphic with the cohomology attached to the function \(f\), denoted by \(H^*_f(M)\), which is defined as cohomology of the differential complex \((\Omega^*(M), d_f)\), where the coboundary operator \(d_f : \Omega^p(M) \rightarrow \Omega^{p+1}(M)\) is defined by \(d_f \omega = f d\omega - pdf \wedge \omega\). The mentioned isomorphism is produced by the map of chain complexes \(\phi : (\Omega^*_{M}, d) \rightarrow (\Omega^*_{M}, d_f)\) given by \(\phi^p : \Omega^p_f(M) \rightarrow \Omega^p(M), \phi(\omega) = f^p \omega\), see \[10\].
Now, let us return to our study. As well as we seen from the above discussion $\mu_i \in \Omega_{p_i}^1(M)$ and, accordingly $d\mu_i \in \Omega_{p_i}^2(M)$. We have then that $\mu_i \wedge (d\mu_i)^q_i \in \Omega_{p_i}^{2q_i + 1}(M)$. Since $\mu_i \wedge (d\mu_i)^q_i$ is closed, from the above isomorphism we have that $\varphi_i^{2q_i + 1} \mu_i \wedge (d\mu_i)^q_i$ is $d\varphi_i$-closed. Thus, if $\varphi_i$ is basic function for the foliation $\mathcal{F}_{r_1}$ on $U_i$ then $d\varphi_i(\varphi_i^q \pi_i) = 0$ which leads to the cohomology class $[\varphi_i^q \pi_i] \in H_{2q_i + 1}(M)$. Let us consider now the regular case for the test function $\varphi_i$, that is $\varphi_i$ does not have singularities in a neighborhood of its zero set (i.e., $0$ is a regular value). The subsets $S_i = \varphi_i^{-1}(\{0\}) = M \setminus U_i$ are then embedded submanifolds of $M$. We also assume that $S_i$ are connected.

We consider some useful notations. Let $V_i \subset V_i'$ be tubular neighborhoods of $S_i$. We may assume that $V_i = S_i \times [-\epsilon_i, \epsilon_i]$ and $V_i' = S_i \times [-\epsilon_i', \epsilon_i']$, with $\epsilon_i' > \epsilon_i$, and that $\varphi_i|_{V_i'} : S_i \times [-\epsilon_i', \epsilon_i'] \to \mathbb{R}$, $(x, t) \mapsto t$. We denote by $\pi_i$ the projections $V_i' \to S_i$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function which is $1$ on $[-\epsilon_i, \epsilon_i]$ and has support contained in $[-\epsilon_i', \epsilon_i']$. Note that the function $\rho \circ \varphi_i$ is $1$ on $V_i$, and we can assume that the function $\rho \circ \varphi_i$ vanishes on $M \setminus V_i'$. If $\omega$ is a form on $S_i$, we will denote by $\tilde{\omega}$ the form $\rho(\varphi_i)\pi_i^*\omega$ and notice that $d\varphi_i \wedge d\tilde{\omega} = d\varphi_i \wedge d\omega$, see [10].

According to Theorem 4.1 from [10], if $0$ is a regular value of $\varphi_i$ then, for each $p \geq 1$, there is an isomorphism

$$H_{\varphi_i}^p(M) \cong H_{dR}^p(M) \oplus H_{dR}^{p-1}(S_i),$$

(8)

given by $\Phi : \Omega^p(M) \oplus \Omega^{p-1}(S_i) \to \Omega^p(M)$ defined by $\Phi(\alpha, \beta) = \varphi_i^p \alpha + \varphi_i^{p-1} d\varphi_i \wedge \beta$.

Now, taking into account the isomorphism (8) it follows that there exist $\alpha_i \in \Omega_{p_i}^{2q_i + 1}(M)$ and $\beta_i \in \Omega_{p_i}^{2q_i}(S_i)$ with $d\alpha_i = d\beta_i = 0$ such that

$$\varphi_i^q \pi_i = \varphi_i^{1+2q_i} \alpha_i + \varphi_i^{2q_i} d\varphi_i \wedge \beta_i.$$

(9)

Thus we obtain that $\alpha_i = \varphi_i^{-q} \pi_i - \frac{d\alpha_i}{\varphi_i} \wedge \beta_i$ and by differentiation and taking into account $d\pi_i = d\alpha_i = d\beta_i = 0$, one get

$$(-1 - q_i)\varphi_i^{-2q_i - 1} d\varphi_i \wedge \pi_i = 0,$$

where we have used $d\varphi_i \wedge d\beta_i = d\varphi_i \wedge d\beta_i = 0$.

Now, since $d\pi_i = 0$ and $d\varphi_i \wedge \pi_i = 0$, by Proposition 3.4 from [9] there exist $\pi_i \in \Omega_{p_i}^{2q_i - 1}(M)$ such that $d\pi_i = d\varphi_i \wedge d\pi_i$ and so $\pi_i = d(\varphi_i d\pi_i)$. Thus, we obtain the announced result:

**Proposition 7** If $0$ is a regular value for the (weak) test function $\varphi_i$ that is also basic, then the cohomology class $[\pi_i] \in H_{2q_i + 1}(M)$ vanishes.

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