BCJ, worldsheet quantum algebra and KZ equations

Chih-Hao Fu$^a$ and Yihong Wang$^b$

$^a$School of Physics and Information Technology, Shaanxi Normal University,
No. 620 West Chang’an Avenue, Xi’an 710119, P.R. China
$^b$Department of Physics, National Taiwan University,
No. 1 Sec. 4 Roosevelt Road Taipei 10617, Taiwan (R.O.C.)

E-mail: chihhaofu@snnu.edu.cn, yihongwang@phys.ntu.edu.tw

ABSTRACT: We exploit the correspondence between twisted homology and quantum group to construct an algebra explanation of the open string kinematic numerator. In this setting the representation depends on string modes, and therefore the cohomology content of the numerator, as well as the location of the punctures. We show that quantum group root system thus identified helps determine the Casimir appears in the Knizhnik-Zamolodchikov connection, which can be used to relate representations associated with different puncture locations.

KEYWORDS: Scattering Amplitudes, Quantum Groups, Bosonic Strings

ArXiv ePrint: 2005.05177
1 Introduction

In their original formulation, Bern, Carrasco and Johansson (BCJ) proposed a new symmetry feature between the colour and kinematic factors of Yang-Mills amplitude at generic $n$ points and helicity configurations [1, 2]. Different from the standard Feynman rules construct that defines the amplitude, the new formulation states that the Yang-Mills amplitude is expressible as a double copy.

$$\mathcal{M}_{YM} = \sum_{\text{cubic graph } i} \frac{c_i n_i}{D_i}$$  \hspace{1cm} (1.1)

The above formula resembles the structure-wise simpler bi-adjoint $\phi^3$ theory in the sense that the amplitude is organised as a sum of all possible cubic graphs divided by appropriate propagators $D_i$, except that one copy of the gauge group contribution is replaced by a momentum dependent factor known as the kinematic numerator $n_i$. The $n_i$'s are assumed to satisfy all the Jacobi and anti-symmetry relations as the cubic graphs prior to the replacement, for example at four points the three numerators associated with each channel sum up to zero,

$$n_s + n_t + n_u = 0 \leftrightarrow f^{a,b}_\sigma f^{\sigma,c,d} + \text{cyclic perm of } a,b,c = 0$$  \hspace{1cm} (1.2)

and this correspondence between colour and kinematics is assumed to hold for all loop levels provided we integrate (1.1) over loop momenta. In addition it was conjectured when
both sets of numerators of a $\phi^3$ are replaced by kinematic numerators, the result correctly reproduces the amplitude of gravitons coupled to dilatons and $B$ fields. The BCJ double copy structure has detectable consequences on the whole scattering amplitude. In physically realised theories the gauge group is $SU(N)$ and its dependence factorises, whereas each ordering of the Chan-Paton factor defines a partial (or colour-ordered) amplitude. At tree level it was realised that the number of independent kinematic numerators, counted each as a degree of freedom, is fewer than the number of partial amplitudes. As a consequence the partial amplitude needs to satisfy the following BCJ amplitude identity.

\[
\begin{align*}
    k_1 \cdot k_2 A(1, 2, 3, \ldots, n) + (k_1 + k_3) \cdot k_2 A(1, 3, 2, \ldots, n) + \ldots \\
    + (k_1 + \cdots + k_{n-1}) \cdot k_2 A(1, 3, \ldots n - 1, 2, n) &= 0.
\end{align*}
\]  

(1.3)

The aforementioned duality structure originally observed in Yang-Mills amplitudes was later realised to be a common feature of a wide range of quantum field theories including a variety of supersymmetric gravity and gauge theories [1–16] as well as effective field theories such as nonlinear sigma model [17], Dirac-Born-Infeld, special-Galilean theory [18], and perhaps most importantly the physical theory that incorporates fermions [7, 19, 20]. In addition it is known to present in the classical solution of Einstein’s equations [21, 22]. The generic feature is a double copy of colour and/or kinematics whereas replacing one or both copies reproduces amplitude of one theory from another. The compatibility of gauge invariance among these theories with double copy has been shown to imply supersymmetry and diffeomorphism invariance and is proved to be a powerful tool in the understanding of analytic behaviour of generic quantum field theory. For more details we refer the readers to the very readable recent review [23] and the references therein. It is perhaps worth emphasise that the interest in further investigation of BCJ duality is more than an academic one. Indeed, some of the most cutting-edge higher loop level corrections to the gluon scattering amplitudes were obtained through double copy [2, 5, 8, 12, 24–41], as it vastly reduces the number of independent diagrams required in the computation [42]. When applied to perturbative gravity the double copy construction provides more drastic boost in efficiency, as it allows alternative calculation using gauge field kinematic numerators, thereby circumventing the otherwise inconceivable infinite number of vertices present in the amplitude computations.

Despite its importance, a systematic construct of the kinematic numerator for generic BCJ satisfying amplitude or the field theory all loop level understanding of the duality remains to a large extent an open question. At the time of writing loop level verification of the duality among field theory amplitudes has acquired a great number of supporting evidence through case by case explicit numerator solving with the help of generalised unitarity techniques and in specific asymptotic limit verified to all loop orders [43, 44]. One of the puzzles lies at the heart of the understanding of the duality is to find an explanation to the apparent Lie algebra-like feature presented by kinematic numerators and has inspired numerous research from a wide range of interesting perspectives [45–52]. An insight of great importance to the current paper is the string theory explanation. The concept is to exploit the fact that string amplitudes coincide with their field theory
In a previous paper [62] we extended the string theory explanation one step further and suggested a natural string generalisation of kinematic numerator that plays a similar role in the analogue of (1.1) and presents the expected algebra-like structure. The idea was to introduce a technique extensively used to determine basis kinematic numerators (or master numerators) in terms of partial amplitudes [63–68]. It is known that once a pair of reference legs are chosen ((1, n) in this case), the set of (n − 2)! consecutive products of structure constants known as the Del Duca-Dixon-Maltoni [69] half-ladders

\[ f^{(1)} \sigma^{(2)} \sigma^{(3)} \ldots f^{(n−1)} \sigma \]

and therefore the set of kinematic numerators associated with the same graphs, reproduces all n point cubic tree graphs through repeatedly imposing Jacobi identities, where \( \sigma(2), \sigma(3), \ldots, \sigma(n−1) \) refers to the \( S_{n−2} \) permutations of the (n − 2) non-reference legs. Or formally speaking, there is a homomorphism from the set of n point BCJ numerators to the order \( n−1 \) Lie polynomials [74]. The half-ladder basis consists of the BCJ numerators whose images are in the Lyndon-Shirshov basis under this homomorphism. Keeping this in mind, note additionally that the (n − 2)! partial amplitudes

\[ A(1, \sigma^{(2)}, \sigma^{(3)}, \ldots, \sigma^{(n−1)}, n) \]

being stripped of one copy of Chan-Paton factor, depends only on propagator \( D_i \’s \) and planar kinematic numerators with the specific ordering, which in turn can be spanned by half-ladders. The rest of the partial amplitudes can be solved using other amplitude identities known to be compatible with the BCJ duality. The (n − 2)! by (n − 2)! coefficient matrix that relates partial amplitudes and half-ladders is known to coincide with the momentum kernel [70, 71] \( S_{\sigma^T | \sigma'} \) originally appeared in the gravity-gauge theory amplitude relation discovered by Kawai, Lewellyn, and Tye (KLT) [72]. As a matter of fact from the BCJ perspective the KLT relation can be understood as the double copy formulation of a graviton amplitude with its two copies of kinematic numerators being translated back to gauge field partial amplitudes.\(^1\) In light of the amplitude-numerator relation just described, in [62] we examine a natural generalisation of this relation in the context of open string theory and wrote down the string analogue of half-ladder numerators,

\[ n(1, \sigma^{(2)}, \sigma^{(3)}, \ldots, \sigma^{(n−1)}, n) = \lim_{k \to 0} \frac{1}{k^n} \sum_{\sigma' \in S_{n−2}} S_{\alpha'} [\sigma^T | \sigma'] A(1, \sigma', n), \quad (1.4) \]

with the momentum kernel also replaced by its string analogue \( S_{\alpha'} [\sigma^T | \sigma'] \) defined in [73]

\(^1\)A subtle technical issue actually arises because the coefficient propagator matrix in the (n − 2)! half-ladder basis problem turns out to be singular and prevent us from solving the numerators by direct inversion [65]. In [62] we bypassed this issue by analytic continuing one of the legs and taking the invert while the coefficient matrix in non-singular, at the cost of breaking the \( SL(2, R) \) invariance of the string amplitude temporarily. The analytic continuation results in an overall \( 1/k^n \) factor appears in the expression (1.4).
as a product of sinusoidal factors determined by the two sets of ordering $\sigma$ and $\sigma'$.

$$S_{\alpha'}[\sigma_1, \ldots, \sigma_k|\sigma'_1, \ldots, \sigma'_k] := \left(\frac{\pi \alpha'}{2}\right)^{-k} \prod_{t=1}^{k} \sin \left(\pi \alpha' (k \cdot k_{\sigma_t} + \sum_{q>t} \theta(\sigma_t, \sigma_q) k_{\sigma_t} \cdot k_{\sigma_q})\right)$$  \hspace{1cm} (1.5)

Following similar argument used in [73] we showed that when multiplied with open string partial amplitudes, which are Selberg integrals over ordered domains, the sinusoidal factors lift contours to appropriate branches of the Riemann surface, so that the half-ladder numerator (1.4) is given by the same integral as partial amplitude, but with the integration domain combined into the multi-layer C-shaped contours (figure 1(a)), which explains the expected half-ladder-like structure $f^{1, \sigma(2)}_{\rho} f^{\rho, \sigma(3)}_{\rho'} \ldots f^{\rho', \sigma(n-1)}_{\rho}$ when expressed as consecutive commutators of vertex operators integrated over line contours along two sides of the branch cut.

$$\lim_{k^2 \to 0} \int_0^1 \prod_{i=2}^{n-2} \frac{dz_i}{z_i} \langle f \left[[[V(z_1), V(z_2)]_{\alpha'}, V(z_3)]_{\alpha'} \ldots , V(z_{n-1})]_{\alpha'} \right| 0 \rangle$$  \hspace{1cm} (1.6)

This consecutive quantum deformed commutator formula (1.6) we arrived at suggests that BCJ numerators could be naturally derived within the framework of an underlying quantum algebra. The existence of such quantum algebraic structure is hinted by the twisted homological representation of string amplitudes: as pointed out in [79–81], an n-point string amplitude can be viewed as (twisted) topological invariants in the configuration space $\{(z_2, \ldots, z_{n-2})| z_i \neq 0, 1, \ z_i \neq z_j\}$ with a twist identical to the Koba-Nielson factor: open string amplitudes are pairings of one twisted cycle and one twisted cocycle, and the closed string amplitudes are pairings of two twisted cocycles. Meanwhile, the isomorphism between twisted cycles and quantum algebra modules is well known to the quantum algebra community (Detailed discussions can be found in monographs such as [82–84]). For example, consider a representation of the quantum algebra $U_q(sl(2))$ defined in (2.20), (2.20) and (2.22), which is a tensor product of two Verma modules. Such representation is generated by the tensor of the two highest weight vectors $v_m \otimes v_l$. In this representation, the vectors with weight $m+l-2$ are isomorphic to the twisted cycles in the configuration $C - \{0, 1\}$ with twist factor $z^{-m} (z - 1)^{-l}$. (The notations used here will be explained in section 2.1) Under this isomorphism, $Ev_l \otimes v_m$ and $v_l \otimes E v_m$ are mapped to the twisted cycle encircling the branch points 0 and 1 respectively, and the boundary operator on the twisted cycles is equivalent to the action of $F$ on the vectors. This isomorphism

![Figure 1](image_url)
between twisted homology and representation of quantum algebra leads to the Drinfeld-Kohno theorem \cite{75, 76}: the solutions of Knizhnik-Zamolodchikov (KZ) equations \cite{77}, as a representation of the homology group, is isomorphic to the corresponding $R$-matrix representation of the quantum algebra. This theorem is another hint for the quantum algebraic structure in string amplitudes: at tree level, string amplitude takes the form of generalised Selberg integrals, which appear as coefficients in solutions for KZ equations \cite{78}. Following the same spirit as Drinfeld-Kohno theorem, by matching string amplitudes with KZ coefficients, we can read off the root system of the Lie algebra underlying KZ equations, then the quantum deformation of this Lie algebra will naturally characterise the behaviour of string amplitudes.

In this paper we explore the relation between the basic concept of quantum algebra and string amplitudes by finding the root system for the kinematic algebra associated with specific string amplitudes, showing how the defining structure of quantum algebra is represented by contour integrals of the open string vertex operator, known as the screening operator in the quantum algebra literature \cite{85–87}. In addition we articulate how string amplitude is related to the solution of KZ equations with the same root system. We organise this paper as the following. In section 2, we begin with a minimal review of the quantum algebra, followed by an introduction of two slightly different versions of the screening operators relevant to the discussion. Then by identifying spacetime momentum of the external legs as roots of the quantum algebra and deforming screening contours, we verified in section 3 that the screening of the vertex operators reproduce all defining relations for representation of such quantum algebra, such as the quantum deformed Lie brackets and the coproduct for the quantum universal enveloping algebra. In section 4, we discussed the relation between $Z$-amplitude and the solutions to KZ equations with a brief introduction to the KZ equation in 4.1 followed by the explicit 4 point and 5 point examples. Finally we conclude our paper in section 5.

2 Preliminaries

2.1 Quantum algebra: a quick summary

In this section we briefly review the notion of quantum groups. Historically quantum groups first appeared in the inverse scattering problem of integrable systems \cite{88–90} and was independently discovered by Drinfeld \cite{91} and Jimbo \cite{92} when generalising classical Lie algebra through deformations. Numerous equivalent formulations are known \cite{91–97}, each emphasise on different aspects of the algebra. We adopt the approach taken by Faddeev, Reshetikhin and Takhtajan (FRT) \cite{93} as it is formulated in a language that is more familiar to the readers with physics background.

The modern definition of the quantum groups is a Hopf algebra $\mathcal{H}$ that satisfies additionally the quasi-triangular condition.

$$ R \Delta(x) = \Delta'(x) R, \quad \text{for all } x \in \mathcal{H}, \quad (2.1) $$

where $\Delta' = \sigma \circ \Delta$ is the coproduct with the two factors of its result permuted, and $R \in \mathcal{H} \otimes \mathcal{H}$ is a tensor element that plays the special role in the definition of the algebra called the
universal $R$-matrix. To motivate the definition, consider the matrix elements of classical Lie group representation $T_{ij}(g)$, regarded as complex functions of group element $g$. The polynomials of $T_{ij}$’s is known to be a Hopf algebra, albeit a relatively trivial one, with the coproduct, counit, antipode defined as

\begin{align}
\Delta(T_{ij})(g,g') &= T_{ik}(g)\otimes T_{kj}(g'), \\
\epsilon(T_{ij}) &= \delta_{ij}, \\
S(T_{ij}) &= T_{ij}^{-1}.
\end{align}

The matrix elements are themselves $C$-numbers, therefore commutative, even though the matrix as a whole is not. The idea of FRT is to generalise $T_{ij}$ to incorporate non-abelian algebra (operators) by introducing a deformation through what is usually called the RTT relation,

\begin{equation}
R_{ij;k\ell}T_{km}T_{\ell n} = T_{j\ell}T_{ik}R_{k\ell;m n}.
\end{equation}

For example in the simplest case where the universal $R$-matrix is holomorphic in the indeterminate $h$, $R(h) = 1 + hR^{(1)} + h^2R^{(2)} + \ldots$, the RTT relation (2.5) reads

\begin{equation}
T_{im}T_{jn} - T_{jn}T_{im} = h \cdot \left[ R_{ij;k\ell}T_{km}T_{\ell n} + T_{j\ell}T_{ik}R_{k\ell;m n} \right] + O(h^2).
\end{equation}

Assuming the deformed $T_{ij}$ is also holomorphic, the above relation implies its $h \to 0$ limit is commutative, which is consistent with the complex number representation of the classical Lie group. The commutation relations of $T_{ij}$’s can be solved order by order once an explicit $R$-matrix is chosen, starting with the classical matrix element as its zeroth order $T_{ij}^{(0)}$. The $T_{ij}$’s thus obtained was proved to satisfy the quasi-triangular condition (2.1) and at the same time preserves the Hopf algebra structure (2.2), (2.3), (2.4) at every order [93]. Note from this perspective all classical Lie group algebras are actually (trivial) quantum groups with the $R$-matrix being simply the identity. Generically $T_{ij}$ can be a formal Laurent series in $h$, or sometimes deformed through $q = e^h$, so that $T_{ij} \in \mathcal{H}[[q,q^{-1}]]$. The Yangian $Y(g)$, in particular, familiar to the amplitude community as the symmetry structure of the $N = 4$ super Yang-Mills amplitude [98] can be regarded as a special type of quantum group defined by specific $R$-matrix.

In the literature very often the RTT relation is abbreviated with the help of introducing a convenient basis where $e^{i,j} \in \text{End} \mathbb{C}^N$ stands for the $N \times N$ complex matrix whose $i,j$-th entry is 1 and otherwise zero. Denote

\begin{equation}
T_{a} := \sum_{i,j} 1 \otimes \cdots \otimes 1 \otimes e^{i,j} \otimes 1 \otimes \cdots \otimes 1 \otimes T_{ij}
\end{equation}

as the algebra-coefficient matrix acting on $m$-tuple tensor of complex vectors, $T_{a} \in (\text{End} \mathbb{C}^N)^{\otimes m} \otimes \mathcal{H}[[q,q^{-1}]]$, where the coefficient is contracted with the $a$-th factor, the RTT relation can be written in this language as

\begin{equation}
R T_{1}T_{2} = T_{2}T_{1}R.
\end{equation}

The same spirit is applied when describing matrices with tensor algebra coefficients. For example $R_{0[13]}$, or more often written simply as $R_{13}$ when no confusion can occur, refers to
the 0-tuple matrix (namely a scalar, 1) with its coefficient in $H \otimes 1 \otimes H$. As consistency to the quasi-triangular condition (2.1), it is understood that the $R$-matrix needs satisfy the quantum Yang-Baxter equation,

\begin{equation}
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\end{equation}

Additionally, it is known that the center of the algebra is generated by deformation parameter expansion of its quantum determinant $\det_q T := \sum_{\sigma \in S_N} (-q)^{\text{length}(\sigma)} T_{1\sigma(1)} \cdots T_{n\sigma(N)}$, which was elegantly used by Chicherin, Derkachov and Kirschner in [99] to prove Yangian symmetry of the $\mathcal{N} = 4$ super Yang-Mills amplitude recursively.

The quantum groups obtained from classical Lie group representation through the deformation described above are called the quantum matrix group. The simplest example being the deformed $\text{SL}(2)$, or $\text{SL}_q(2)$. In this case the $R$-matrix satisfying the Yang-Baxter equation (2.9) is given by $R_{ij;k\ell} = \delta_{i\ell}\delta_{jk} - q^{-2}\left(-q^1_{ij} \right)_{k\ell}$, and the RTT relation (2.5) when read off component by component gives

\begin{align}
ab &= q ba, & cd &= q dc, & ac &= q ca, \\
bd &= q db, & bc &= cb, & ad &= da + (q - q^{-1})bc,
\end{align}

where $a$, $b$, $c$, $d$ are the entries of the $\text{SL}_q(2)$ matrix, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The $R$-matrix corresponding to deforming $N \times N$ matrices has been worked out so that $G\text{L}_q(N)$ can be derived similarly. (See [100] for example.)

The quantum groups considered in this paper is actually the dual of the quantum matrix group, $\mathcal{H}^* = \text{Hom}(\mathcal{H}, \mathbb{C})$, in the same spirit that Lie algebra is dual to the group manifold, and with the multiplication, unit, coproduct, counit defined as the pullback of the coproduct, counit, multiplication, unit respectively: for all $x, y \in \mathcal{H}^*$, $a, b \in \mathcal{H}$ we have

\begin{align}
\langle xy, a \rangle &= \langle x \otimes y, \Delta(a) \rangle, & \langle I, a \rangle &= \epsilon(a), \\
\langle \Delta(x), a \otimes b \rangle &= \langle x, ab \rangle, & \epsilon(x) &= \langle x, I \rangle, \\
\langle S(x), a \rangle &= \langle x, S(a) \rangle.
\end{align}

It follows from the properties of a Hopf algebra that its dual $\mathcal{H}^*$ is also a Hopf algebra. Conventionally $\mathcal{H}^*$ is arranged into an upper and a lower triangular matrix $L^\pm_{ij}$ and normalised with the help of the $R$-matrix (in the case of quantum matrix groups happens to contain only $\mathbb{C}$-numbers)

\begin{equation}
\langle L^\pm_{ij}, T_{k\ell} \rangle = R^\pm_{ik,j\ell},
\end{equation}

where $R^+_{ik,j\ell} := q^{-1/2}R_{ik,\ell j}$ and $R^-_{ik,j\ell} := q^{1/2}R^{-1}_{ik,j\ell}$. It follows from the definition that $L^\pm_{ij}$’s also satisfy RTT relation (which in this case is sometimes referred as the RLL relations)

\begin{align}
RL^+_2 L^+_1 &= L^+_1 L^+_2 R, \\
RL^-_2 L^-_1 &= L^-_1 L^-_2 R,
\end{align}
and therefore themselves is also a quantum group. For $\text{SL}_q(2)$ (2.10), its dual $L^\pm_{ij}$’s are conventionally written as the following.

\[
L^+ = \begin{pmatrix} q^{-\frac{1}{2}H} & q^\frac{1}{2}(q - q^{-1})X^+ \\ 0 & q^\frac{1}{2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^\frac{1}{2} & 0 \\ q^\frac{1}{2}(q^{-1} - q)X^- & q^{-\frac{1}{2}H} \end{pmatrix},
\]

and their commutation relations are read off from the $RLL$ relations (2.15), (2.16), similarly to the case of quantum matrix group.

\[
[X^+, X^-] = [H]_q, \quad [H, X^\pm] = \pm 2 X^\pm,
\]

where in the above equations $[H]_q$ stands for the abbreviation $[H]_q := \frac{q^H - q^{-H}}{q - q^{-1}}$ called a $q$-number, as the same combinations appear frequently in quantum group related calculations. The similarity between the relation of the Lie group and Lie algebra with that of their corresponding $q$-deforms extends actually beyond an analogy. In the classical limit $q \to 1$ (namely, $h \to 0$) equations (2.18) and (2.19) reduce to the familiar commutation relations of the classical $\text{sl}(2)$ Lie algebra. The associative algebra over $\mathbb{C}$ generated by $X^\pm$ and $H$ modulo (2.18) and (2.19) is called the quantum universal enveloping algebra (QUEA) $U_q(\text{sl}(2))$ for $\text{sl}(2)$, or quantum algebra for shortness. Another related formulation of the quantum algebra due to Jimbo [92] is expressed in terms of the generators $E \sim X^-q^{H/2}$, $F \sim X^+q^{-H/2}$ and $K^\pm = q^{\pm H}$, properly normalised so as to satisfy the following commutation relations.

\[
[E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad KEK^{-1} = q^{-2}E, \quad KFK^{-1} = q^2 F,
\]

which also coincide with the classical $\text{sl}(2)$ Lie algebra commutation relations in the $q \to 1$ limit. For our purposes we can simply take the above as the definition of the QUEA as it is exactly the starting point taken by Drinfeld and Jimbo [91, 92]. For arbitrary algebra root systems the commutation relations are given by (2.32), (2.33), (2.34) and (2.35).

The Hopf algebra relations of the QUEA can be worked out from the FRT prescription (2.11), (2.12), (2.13). In the case of $U_q(\text{sl}(2))$ these are given by

\[
\Delta E = 1 \otimes E + E \otimes K, \quad S(E) = -EK^{-1}, \quad \epsilon(E) = 0, \quad (2.23)
\]

\[
\Delta F = K^{-1} \otimes F + F \otimes 1, \quad S(F) = -KF, \quad \epsilon(F) = 0, \quad (2.24)
\]

\[
\Delta K = K \otimes K, \quad S(K) = K^{-1}, \quad \epsilon(K) = 1. \quad (2.25)
\]

**Remark.** Note that the Hopf algebra structure naturally arises from the symmetry algebra of multi-particle quantum system represented as tensor of Hilbert space vectors $|1\rangle \otimes |2\rangle$ or complex vectors for spin states. A rotation generator for example, satisfies

\[
\Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i, \quad \text{or very often written as } J^{\text{total}}_i = J^{(1)}_i + J^{(2)}_i \quad (i = +, -, z) \in \mathbb{C}.
\]
physics literature. The defining conditions of the Hopf algebra such as homomorphism of
the coproduct ensures that when exponentiated, generators produce the appropriate two
particle rotation operator \( \Delta(e^{-iJ_0}) = e^{-iJ_0} \otimes e^{-iJ_0} \), and that the cocommutative con-
dition ensures a unique \( n \)-particle operator \( \Delta^{(n)}(e^{-iJ_0}) \) and so on. From this perspective
the quantum algebra, in particular its Hopf algebra relations (2.23), (2.24), (2.25), which
also reduce to their classical counterparts in the \( q \to 1 \) limit, can be seen as a non-trivial
generalisation of the symmetry algebra of multi-particle system. Indeed, as we should see
in the following discussions that the tensor of a string Hilbert state with an \( \text{SL}(2, \mathbb{R}) \) fixed
vertex operator \( V(z) \) (also known as an evaluation module [83]) serves as a non-trivial
two-particle representation of the QUEA \(|1 \rangle \otimes |2 \rangle \) with the deformation parameter given
by braiding factor \( q = e^{-i\pi\alpha'} \) produced by string monodromy, while the rest of the string
insertions, being integrated over \( C \)-shaped contours in the kinematic numerator, act on the
two particle state as symmetry algebra generators \( E_i \).

In this paper we do not assume particular value for the inverse string tension \( \alpha' \),
especially we rely on \( \alpha' \) being able to adjust freely to obtain field theory amplitudes from
string theory ones as their point particle limit, so that generically \( q \) not being a root
of unity, in which case the representation of the quantum algebra is known to be given
by the Verma module \( M'_{\lambda} = \text{Ind}_{H \otimes \mathbb{C}}^{U_q(\mathfrak{sl}(2))} C_{\lambda} \): starting with a highest weight, namely highest
eigenvalue, 1-dimensional complex vector \( v_{\lambda}, H v_{\lambda} = \lambda v_{\lambda}, F v_{\lambda} = 0 \),
the irreducible representation of the QUEA \( U_q(\mathfrak{sl}(2)) \) contains all vectors freely generated
by \( E \): \( E v_{\lambda}, E^2 v_{\lambda}, \ldots \). We distinguish vectors by actions of \( E \)'s instead of labeling vectors
by eigenvalues like in the case of angular momentum algebra, so that no confusion occurs
when generalised beyond \( \mathfrak{sl}(2) \) where multiple non-mutually commuting \( E_i \)'s are present.
The representation generically is infinite dimensional and only becomes finite when \( \lambda \) is an integer multiple of the deformation parameter \( h \).

2.2 (i) C-shaped contour screening operators
Suppose if we denote the open string Hilbert space as \( \mathcal{H} \), the vertex operators are algebra-
valued distributions \( \mathcal{V} \in \text{End}(\mathcal{H})[[\tau, \tau^{-1}]] \) satisfying the state-field correspondence prin-
ciple. For the purpose of discussion let us for the moment focus on bosonic open strings, so
that a typical vertex operator takes the explicit form
\[
V_p(z) = u e^{i p \cdot X(z)}, \quad u \in \{1, \epsilon \cdot \partial X, (\epsilon \cdot \partial X)^2, \ldots \},
\]
with \( z = e^{i \tau} \). A screening operator \( E_i \in \text{End}(\mathcal{V}) \) associated with any vertex operator
\( S_i(t) = u e^{i k_i \cdot X(t)} \) of interest is defined as the following integral over the C-shaped contour
along both sides of the branch cut \([85, 86], \)
\[
E_i V_p(z) := \int_C dt \, V_p(z) S_i(t).
\]
\[
= \int_C dt \, (z-t)^{i p \cdot k_i} f(t) : V_p(z) S_i(t) :.
\]
\[
\]
In this paper we follow the same conventions as in [102] and [73]. Variables appear on the left are assumed to start with a larger value on the real line than those on the right, correspondingly shifted by a larger $i\epsilon$ on the complex plane, and then analytically continued to their designated values. Note however that in terms of figures, conventionally the real line points to the right instead of left, so that everything illustrated in figures will be the mirror image to what appears in the equation. For example the action of $E_i$ is represented by figure 1(b). The analytic continuation we use here leads to the following braiding relations for $z_1 > z_2$.

\[
S_i(z_1) S_j(z_2) = e^{i\pi\alpha' k_i k_j} S_j(z_2) S_i(z_1) \quad (2.29)
\]

\[
S_i(z_1) V_p(z_2) = e^{i\pi\alpha' k_i p} V_p(z_2) S_i(z_1). \quad (2.30)
\]

In the presence of successive actions, the contour associated with the operator that comes later is defined so as to surround the pre-existing contours (figure 1(a)).

The action of an operator $F_i$ is defined to annihilate the contour integral created by $E_i$ using the conformal property \[L_{-1}, \int_C dt S_i(t)\] = $\int_C dt \partial S_i(t)$. Explicitly this is defined to carry a normalisation factor so that

\[
S_i(1) F_i (\ldots E_i \ldots V_p(z)) := \frac{1}{e^{i\pi\alpha' k_i^2} - e^{-i\pi\alpha' k_i^2}} \left( V_p(z) \ldots [L_{-1}, \int_C dt S_i(t)] \ldots \right) \quad (2.31)
\]

\[
= \frac{e^{-i\pi\alpha' k_i p} - e^{i\pi\alpha' k_i p}}{e^{i\pi\alpha' k_i^2} - e^{-i\pi\alpha' k_i^2}} S_i(1) (\ldots V_p(z)).
\]

The screenings thus defined together with the fixed point vertex operator $V_p(z)$, which serves as the highest weight Verma module, provides a representation of the QUEA $U_q(g)$ [92, 103–105],

\[
E_i E_j - F_j E_i = \delta_{ij} K_i - K_i^{-1} \quad K_i^{-1} = q_i K_i^{-1}
\]

\[
K_i E_j = q_i^{-k_i k_j / k_i^2} E_j K_i \quad (2.33)
\]

\[
K_i F_j = q_i^{k_i k_j / k_i^2} F_j K_i \quad (2.34)
\]

\[
K_i K_j = K_j K_i \quad (2.35)
\]

where $q_i := e^{-i\pi\alpha' k_i^2}$ and $K_i$ is the operator that measures momentum, or charge in the original settings [85–87], so that $E_i$'s and $F_i$'s were supposed to lower or raise the background charge produced by the module, hence the name screenings.

\[
K_i := \exp - \oint k_i \cdot \partial X
\]

A natural representation for tensor of modules $v_p \otimes v_s$ can be obtained by simply taking the product of vertex operator at distinct fixed points $V_p(z_1) V_s(z_2)$. The coproduct of a screening $\Delta(E_i)$ is then defined by the corresponding action on this product followed by

\[
\text{The operator } L_{-1} \text{ used here refers to the Virasoro generator } L_m = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n.
\]
integration over a contour that surrounds both vertices (figure 2), which in turn can be translated into the actions on individual modules by breaking the original contour into two smaller ones surrounding each modules and then swap the ordering using braiding relation (2.30).

$$\Delta E_i \left( V_p(z_1) V_s(z_2) \right) = \int_C dt \, V_p(z_1) \, V_s(z_2) \, S_i(t)$$

$$= V_p(z_1) \left[ \int_C dt \, V_s(z_2) \, S_i(t) \right] + e^{-i\pi k_i} \left[ \int_C dt \, V_p(z_1) \, S_i(t) \right] \, V_s(z_2).$$

The above result is the same as the following tensors of screenings

$$\Delta E_i = 1 \otimes E_i + E_i \otimes K_i,$$

as was expected for a quantum group. The action of antipode and counit are represented by reversing and removing the contour of a screening respectively.

### 2.2.1 The R-matrix

In the screening representation of quantum groups the universal R-matrix is defined as the composition of a plain permutation $\sigma$ that swaps modules along with their screenings, together with the application of braiding relations (2.29) and (2.30) that eventually restores modules back to their original order.

$$v_p \otimes v_s \xrightarrow{\sigma} v_s \otimes v_p \xrightarrow{R} v_p \otimes v_s$$

Explicitly $\sigma$ maps for example $V_p(z_1) \, (E_i V_s(z_2))$ to $(E_i V_s(z_2)) \, V_p(z_1)$ as is illustrated in figure 3. The result can be re-expressed as integrals over segments $t > z_1 > z_2$ and $z_1 > t > z_2$, which in turn can be spanned by $V_p(z_1) \, (E_i V_s(z_2))$ and $(E_i V_p(z_1)) \, V_s(z_2)$ once braiding relations were used. (The last procedure is denoted as $R$ in (2.39) so that the
Figure 4. Universal $R$-matrix acting on a coproduct.

$R$-matrix is the composite $R = R \circ \sigma$.

$$V_p(z_1) \left( E_i V_s(z_2) \right) \rightarrow e^{-i\pi \alpha' p_{k1}} (e^{i\pi s_{k1}} - e^{-i\pi s_{k1}}) \int_{t>z_1>z_2} S_i(t) V_p(z_1) V_s(z_2)$$

$$+ (e^{i\pi s_{k1}} - e^{-i\pi s_{k1}}) \int_{z_1>t>z_2} V_p(z_1) S_i(t) V_s(z_2)$$

$$= V_p(z_1) \left( E_i V_s(z_2) \right) + (e^{-i\pi \alpha' s_{k1}} - e^{i\pi \alpha' s_{k1}}) (E_i V_p(z_1)) V_s(z_2)$$

The above effect is the same as the action that successively removes screenings from one of the modules within the tensor product and then reapply them onto the other.\footnote{For simplicity we have neglected here the braiding factor produced by swapping modules, which will result in an overall $e^{-\frac{i\pi \alpha R_{12}}{2}}$ in the $R$-matrix.}

$$R \sim 1 \otimes 1 + (q_i - q_i^{-1}) E_i \otimes F_i + \ldots$$

Generically the complete formula for $R$ can be derived term by term following similar reasoning \cite{91, 106}. The quasi-triangular condition $R \Delta(E_i) = \Delta'(E_i) R$ (equation (2.1)) can be seen from the fact that braiding two fixed vertices $V_p(z_1) V_s(z_2)$ does not alter the contour of a screening that encompasses them both (figure 4). In particular that Yang-Baxter equation $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ is indeed satisfied can be seen from the fact that $R$ derives from braiding.

2.3 (ii) Line interval screenings

An alternative version of the screening operator \cite{107} that turns out to be also relevant to our BCJ problem is defined as the line integral over a fixed interval along the real line, for example over $[0, 1]$,

$$E_i V_p(z) := \int_0^1 dt V_p(z) S_i(t),$$

whereas the charge operator is defined by the same closed integral as before (2.36). In the case of line interval screenings it is sometimes convenient to restrict our considerations to only the positive Borel subalgebra $U_q^+(g) = H_i \oplus E_i$ of the full quantum universal enveloping algebra generated by Cartan subalgebra and positive root vectors but without $F_i$’s, because the boundary of a line interval is less symmetric than the C-shaped contour, making it less natural to define the action of $F_i$ using conformal generator as in (2.31), even though one can simply define it as manually removing one line screening. As we will see in section 3 that the positive part of the full algebra will be enough as far as numerators and amplitudes are concerned. In these cases the screening operator is to be interpreted
as the insertion of an external particle and there is no physical reason one must define an operator action that removes particles. In the settings of line screenings tensors of modules \( v_p \otimes v_s \) are represented by products of vertex operators \( V_p(z_1) V_s(z_2) \) as before. Action of an \( E_i \) on individual module is defined similar to (2.42) but with the integral carried out only to a manually fixed point \( a \) between \( z_1 \) and \( z_2 \), whereas in the coproduct it is carried out over the full segment \([0, 1]\). The antipode and counit are represented by reversing and removing the contour respectively as before. More details regarding line interval screenings can be found for example in [107].

For the purpose of discussions it is useful to consider right action of a line screening \( V_p(z) E_i \), defined the same as (2.42) but with the ordering of the two vertex operators swapped. According to our convention this corresponds to a line integral starting with a point \( t_2 \) on the real line that is larger than \( z \), and then analytically continued, so that the contours associated with left and right actions corresponds to the blue and lilac lines illustrated in figure 5 respectively. From this perspective a C-shaped screening can be identified as the \( q \)-deformed adjoint action of line screening,\(^4\)

\[
ad_q(E_i) = (m_L \otimes m_R) \circ (S \otimes 1) \circ \Delta(E_i),
\]

where \( m_L \) and \( m_R \) represent taking the left and right actions respectively and \( S \) is the antipode, \( S(E_i) = -E_i K_i^{-1} \). Explicitly we have

\[
ad_q(E_i) V_p(z) = V_p(z) E_i - E_i K_i^{-1} V_p(z) K_i,
\]

\[
= (-1) e^{i\pi p k_i} \int_0^1 dt V_p(z) S_i(t) - e^{-i\pi p k_i} \int_0^1 dt S_i(t) V_p(z),
\]

which is the same as (2.28) up to an overall factor that we will discard through redefinition.

### 3 String BCJ numerators

In a previous paper [62] we showed that the on-shell limit of the multiple C-shaped contour integrals derived originally from KLT in [73] serves as a natural string theory generalisation of the \((n - 2)!\) basis BCJ numerator.

\[
n_{(n-2)!-basis}(1, 2, 3, \ldots, n) = \int_{C_i} \prod_{i=2}^{n-2} dt_i \left\langle f \left| V_1(z_1) \cdot V_2(t_2) V_3(t_3) \cdots V_{n-1}(t_{n-1}) \right| 0 \right\rangle,
\]

\(^4\)There are actually two different adjoint actions, \( ad_q^- \) and \( ad_q^+ \), that one can define for a quantum algebra, through taking the antipode of the first and second tensor in (2.43), respectively. The two definitions correspond to the \( q \) and \( q^{-1} \)-deformation of the Lie bracket. In this paper the adjoint of interest is the \( ad_q^- \) and we shall drop the minus sign to avoid excessive notation.
where $\langle f \rangle$ is the asymptotic state at infinity, associated with leg $n$ through the state-field correspondence principle $\langle f \rangle = \lim_{z \to \infty} \langle 0 | V_n(z) | z \rangle$. In the equation above we have slightly abused the notation. To every vertex operator $V_i(z_i)$ we assumed that a specific string excitation mode (not necessarily tachyon) is implicitly assigned along with the momentum $k_i$. Additionally let us also recall that the $(n-3)!$ field theory basis numerator was identified in the context of Cachazo-He-Yuan (CHY) formulation of amplitudes in [112] as the product of momentum kernel with partial amplitudes,$^5$

$$
n(1, \gamma(2), \gamma(3), \ldots, \gamma(n-1), n) = \begin{cases} 
\sum_{\beta \in \mathbb{Z}_{n-3}} S[\gamma^T | \beta] \tilde{A}_n((1, \beta, n, n-1), \gamma(n-1) = n-1
0, \gamma(n-1) \neq n-1
\end{cases}
$$

(3.2)

When applied to strings, with both the momentum kernel and amplitudes substituted by their string theory generalisations, the $(n-3)!$ basis numerator can be expressed also as multiple C-shaped contour integrals [73], but with three of the vertices $(z_1, z_2, z_n)$ fixed instead of two,$^6$

$$
n_{(n-3)!-basis}(1, 2, 3, \ldots, n) \sim \int \prod_{i=2}^{n-2} C_i \langle 0 | V_n(z_n) \cdot V_2(z_2) V_1(z_1) \cdot V_3(t_3) \ldots V_{n-1}(t_{n-1}) | 0 \rangle.
$$

(3.3)

For a specific string theory, the integrands in (3.1) and (3.3) can be computed by normal ordering vertex operators. The explicit formula for such integrands is the product of Koba-Nielsen factor $\prod (z_i - z_j)^a k_i k_j$ and a linear combination rational functions composed of factors of the form $\frac{\epsilon_i \epsilon_j}{(z_i - z_j)^2}$. Generically the result of integration will be a linear combination of hypergeometric functions of $k_i \cdot k_j$ dressed with rational factors depending on $\epsilon_i \cdot k_j$ and $\epsilon_i \cdot \epsilon_j$. Suppose if we focus on the vertex operators in (3.1) and (3.3), ignoring for the moment a common final leg $V_n(z_n)$ that is frequently pushed to infinity, the multiple C-shaped contour integrals appear in $(n-2)!$ and $(n-3)!$ basis numerators can be identified as screenings acting on single and tensor modules respectively. Explicitly, for the $(n-3)!$ basis numerator this is

$$\Delta(E_{n-1}) \ldots \Delta(E_4) \Delta(E_3) v_{k_2} \otimes v_{k_1}.
$$

(3.4)

The above settings naturally defines a representation of quantum group $U_q(g)$ with the $E_3, E_4, \ldots, E_{n-1}$ identified as the simple root vectors. Comparing with the definition of a screening (2.28) and equations (2.32) to (2.34) we see that the corresponding simple roots are identified with the momenta $k_3, k_4, \ldots, k_{n-1}$ carried by external legs. In the infinite string tension limit $\alpha' \to 0$, therefore $q_i \to 1$ and the QUEA $U_q(g)$ reduces to the classical Lie (or Kac-Moody) algebra $g$ with the (symmetrised) Cartan matrix defined by

---

$^5$The $\gamma$ on the right is understood to contain only the first $(n-3)$ indices, $\gamma^T = \gamma(n-2), \ldots, \gamma(2)$ so that when all permutations are considered, the momentum kernel $S[\gamma^T | \beta]$ is an $(n-3)! \times (n-3)!$ matrix.

$^6$For the purpose of illustration we used here the antisymmetry of the numerator and reversed its ordering, $n(1, 2, \ldots, n-1, n) = (-1)^{n-2} n(n, n-1, \ldots, 2, 1)$. 

the same roots,

\[ [e_i, f_j] = \delta_{ij} h_j \]  \hspace{1cm} (3.5)
\[ [h_i, e_j] = (k_i \cdot k_j) e_j \]  \hspace{1cm} (3.6)
\[ [h_i, f_j] = -(k_i \cdot k_j) f_j \]  \hspace{1cm} (3.7)

so that in the field theory limit the BCJ kinematic algebra should be isomorphic to the algebra determined by external leg momenta. Note that we have used lower case letters for generators of classical Lie (Kac-Moody) algebra, and will keep using this convention in later discussions involving classical Lie (Kac-Moody) algebra.

Starting with \( E_3, E_4, \ldots, E_{n-1} \) as building blocks the QUEA thus defined contains non-simple root vectors generated by \( q \)-commutators \( E_{k_1+k_2} \sim [E_2,E_1]_q = \text{ad}_q(E_2)E_1 \). These are root vectors in the sense that they satisfy similar commutation relations (2.33), (2.34) and \( [E_\gamma,F_\gamma] = C_\gamma[H_\gamma]_q \) up to a normalisation that depends on its root \( \gamma \). (Here \([H_\gamma]_q \) stands for the \( q \)-number (2.18) defined in section 2.1.) The result of the \( q \)-commutator can be seen from figure 6 to be by itself a screening operator, but with its vertex operator calculated from the following operator product.

\[
\int_{C_2} dt_2 e^{ik_1 \cdot X(t_1)} e^{ik_2 \cdot X(t_2)} = \sin \pi \alpha' k_1 \cdot k_2 : e^{i(k_1+k_2) \cdot X(t_1)} \left( \frac{1}{k_1 \cdot k_2 + 1} + \frac{k_2 \cdot \partial X(t_1)}{k_1 \cdot k_2 + 2} + \frac{1}{2} \frac{k_2 \cdot \partial^2 X(t_1)}{k_1 \cdot k_2 + 3} + \ldots \right) : \]  \hspace{1cm} (3.8)

where we have chosen \( E_1, E_2 \) to be tachyons as an example. This process continues generating new root vectors indefinitely until it is terminated by quantum Serre relations \((\text{ad}_q(E_i))^{1-(k_i \cdot k_j)} E_j = 0\), which in turn are determined by roots. For example the result of two consecutive adjoint actions \( \text{ad}_q(E_3)\text{ad}_q(E_2) E_1 \) is to replace the vertex operator with the following.

\[
\int_{C_2,C_3} dt_2 dt_3 S_1(t_1) S_2(t_2) S_3(t_3) = (2i)^3 \sin \pi \alpha' k_1 \cdot k_2 \left[ \sin \pi \alpha' k_1 \cdot k_3 I(k_1,k_3,k_2) + \sin \pi \alpha' (k_1 + k_2) \cdot k_3 I(k_1,k_2,k_3) \right], \]  \hspace{1cm} (3.9)

where \( I \)'s are ordered operator product line integrals,

\[ I(k_1,k_2,k_3) = \int_{t_1 < t_2 < t_3} (t_3 - t_2)^{k_3-k_2} (t_3 - t_1)^{k_3-k_1} (t_2 - t_1)^{k_2-k_1} : S_1(t_1) S_2(t_2) S_3(t_3) : \]  \hspace{1cm} (3.10)

and likewise for the other ordering. When the momenta \( k_i \)'s are identified with for example, the simple roots \( \alpha \) and \( \beta \) of the \( U_q(sl(3)) \), namely if we choose \( k_1 = \beta \) and \( k_2 = k_3 = \alpha \), we see that equation (3.9) becomes

\[ \text{ad}_q(E_\alpha)^2 E_\beta \sim \sin \pi \alpha'(\beta \cdot \alpha) I(\beta,\alpha,\alpha) + \sin \pi \alpha'(\beta + \alpha \cdot \alpha) I(\beta,\alpha,\alpha) \]  \hspace{1cm} (3.11)
which indeed vanishes because for $U_q(sl(3))$ the roots satisfy $\alpha \cdot (\alpha + \beta) + \alpha \cdot \beta = 0$, and therefore $\sin \pi \alpha'((\alpha + \beta) \cdot \alpha) = -\sin \pi \alpha'(\alpha \cdot \beta)$. For generic momentum configuration there is no such identity to stop new non-simple roots being generated and the algebra $g$ is therefore infinite dimensional. To make the algebra finite one can chose to work with compactified spacetime such that all momenta live on a rational lattice [113], in this case all $\alpha' k_1 \cdot k_j$ eventually become integers after being superposed large enough number of times and the overall sinusoidal factor appears in (3.8) vanishes. Note that the BCJ amplitude relation can be regarded as a special type of the Serre relations (3.9) even though it eliminates only the root vector that carries zero root constrained by momentum conservation $k_1 + \cdots + k_n = 0$, especially that roots containing multiple copies of the same momentum $n_1 k_1 + n_2 k_2 + \cdots$ remain and the algebra is infinite unless rest of the constraints just described were imposed.

In the settings of string theory asymptotic states provides a natural definition for a bilinear form that can be used to normalise modules and root vectors. For example when both states are gluons, from state-field correspondence we have

$$
(v_p, v_s) := \lim_{z_1 \to 0, z_n \to \infty} z_n \langle V_p(z_n)V_s(z_1) \rangle \frac{1}{z_1} \langle 0 | \epsilon_n \cdot \alpha_1 e^{-ip \cdot x} \epsilon_1 \cdot \alpha_{-1} e^{is \cdot x} | 0 \rangle
$$

Note in particular from this perspective the $(n-3)!$ basis numerator can be regarded as a quantum Clebsch-Gordan coefficient. Different choices of vertex operator screenings in this setting correspond to representations associated with different modes. Generically the bilinear form can be similarly defined for arbitrary values of $z_1$ and $z_n$ and different choices of bilinear form are related by KZ equations. We leave this part of the discussion to section 4.1.

**Remark.** Before we proceed, we would like to clarify a subtle issue related to the bilinear form defined above. The Hilbert space and its dual inherited from string settings are Verma modules of the string modes, which by themselves comprised a Heisenberg algebra. From this perspective the screening operators are actually a representation of the quantum alge-
bra built on top of the Heisenberg algebra (referred to as the basic representation in [108]).

While the string Hilbert space and its dual together defines a involutive (Hermitian in this case) bilinear form $\langle \alpha^p_n v_p, v_s \rangle = \langle v_p, \alpha^p_nv_s \rangle$, the involution feature generically does not pass on to the screening operators. The action of a quantum algebra root vector in this bilinear form is not the same as the annihilation of the same root vector on its dual space. Namely, we do not have the identity $\langle E_kv_p, v_s \rangle = \langle v_p, F_kv_s \rangle$, so that the bilinear form (and therefore the kinematic numerator) is not directly computed from simple algebraic manipulations as angular momentum algebra. As a matter of fact when the C-shaped contour extends to infinity the action of a quantum root vector actually equals its antipode acting on the dual module, which can be seen by flipping the contour surrounding one module to the other (figure 7).

$$\langle E_kv_p, v_s \rangle = -\int_{\text{surrounding } z_1} dt_1 z_n \langle 0 | V_p(z_n) S_k(t_1) V_s(z_1) | 0 \rangle \frac{1}{z_1}$$  \hspace{1cm} (3.13)

(An extra phase factor $K^{-1}$ was due to swapping $S_k(t_1)$ and $V_s(z_1)$)

### 3.1 Jacobi-like identities

In light of the original idea of BCJ duality it is perhaps more or less expected that the numerator is expressible as successive adjoint actions that mimics the colour dependence of the amplitude, and indeed it was realised in [62, 110, 111] that such structure is accounted for in string theory by deformed brackets. In addition we note that because the C-shaped screening can be regarded as the $q$-deformed adjoint action of line screenings, the $(n-2)!$ basis numerator (3.1) can be recast into the following BCJ manifest form.

$$n_{(n-2)!-\text{basis}}(1, 2, 3, \ldots, n) = \left\langle f | \text{ad}_q(E_{n-1}) \ldots \text{ad}_q(E_2) \text{ad}_q(E_2) V_1(t_1) | 0 \right\rangle,$$  \hspace{1cm} (3.14)

where the $E_i$’s above are line screenings, and the numerator is therefore the successive adjoint actions of QUEA determined by external leg momenta. If our only purpose is to explain the BCJ duality originally observed in field theory amplitudes, it is not strictly necessary to come up with a $q$-deformed analogue of the Jacobi-like identity satisfied by string numerators. However we do actually have an identity that explains the expected relation at quantum level,

$$\text{ad}_q(E_i) \text{ad}_q(E_j) \circ E_\ell = \text{ad}_q([E_i, E_j]_q) \circ E_\ell = \text{ad}_q([E_i, E_j]_q) \circ E_\ell.$$  \hspace{1cm} (3.15)
Surprisingly the above identity does not directly come from the perhaps seemingly more natural candidate implied by the definition of $q$-commutator, $[E_a, [E_b, E_c]_{q^a-b}]_{q^c} = [[E_a, E_b]_{q^a-b}, E_c]_{q^c}$, as careful inspection would quickly show that the $q$-deformed factors mismatch, but rather is the consequence of braiding relations (2.29) and (2.30). The Jacobi-like identity (3.15) can be verified by explicit translating screenings into ordered line integrals and cancel.

4 Relation to the KZ equations

Recall that the KZ equations [77] is a set of differential equations for Lie algebra (or more generally Kac-Moody algebra)-module $\phi$ with coordinate dependence,

$$\frac{\partial \phi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \phi. \quad (4.1)$$

For a Kac-Moody algebra with simple roots $\{k_i\}$ and Cartan subalgebra $\{h_i\}$, satisfying the following (classical) commutation relations,

$$[h_i, e_{k_j}] = -(k_j)_i e_{k_j}, \quad [h_i, f_{k_j}] = (k_j)_i f_{k_j},$$

$$[e_{k_j}, f_{k_j}] = \delta_{ij} (k_j)_i h_i, \quad [h_i, h_j] = 0. \quad (4.2)$$

Its Verma module $\tilde{V}_\Lambda$ is generated by the lowest weight module $\tilde{v}_\Lambda$, and root vectors $f_{k_i}$,

$$\tilde{v}_{nk_i+\Lambda} = (f_{k_i})^n \tilde{v}_\Lambda. \quad (4.3)$$

The module $\phi$ that appears in the KZ equation is a map from the space $U = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \}$ to the space of tensor modules $\tilde{V}_{\Lambda_1} \otimes \tilde{V}_{\Lambda_2} \otimes \cdots \otimes \tilde{V}_{\Lambda_n}$. For example when $n = 2$,

$$\phi_0 = I_0 \tilde{v}_{p_1} \otimes \tilde{v}_{p_2}, \quad (4.4)$$

$$\phi_1 = I_{(1,0)} f_k \tilde{v}_{p_1} \otimes \tilde{v}_{p_2} + I_{(0,1)} \tilde{v}_{p_1} \otimes f_k \tilde{v}_{p_2}, \quad (4.5)$$

On the other hand the operator $\Omega$ in the KZ is the Casimir,

$$\Omega = \sum_{h_i \in \text{Cartan subalgebra}} h_i \otimes h_i + \sum_{k_i \in \text{all roots}} (f_{k_i} \otimes e_{k_i} + e_{k_i} \otimes f_{k_i}). \quad (4.6)$$

Note that the second summation runs over all root vectors including non-simple ones, and $\Omega_{ij}$ is understood to act only on the $i$-th and $j$-th factor of the tensor. By construction $\Omega_{ij}$ commutes with all coproducts in the algebra, in particular $[\Omega, \Delta(e_i)] = 0$, so that it only mixes tensors with the same overall weights. In light of this the solutions $\phi$ can be assorted into ground modules, 1-level raised modules and so on, as was shown by equations (4.4) and (4.5). Explicitly the coefficient functions are given by

$$I_0 = (z_2 - z_1)^{p_1-p_2}, \quad I_{(1,0)} = \int_\gamma dt \frac{1}{t - z_1} \Phi_\kappa, \quad I_{(0,1)} = \int_\gamma dt \frac{1}{z_2 - t} \Phi_\kappa, \quad (4.7)$$

$$\int_\gamma dt \frac{1}{t - z_1} \Phi_\kappa, \quad I_{(0,1)} = \int_\gamma dt \frac{1}{z_2 - t} \Phi_\kappa, \quad (4.7)$$
Figure 8. A braiding defined by the Knizhnik-Zamolodchikov connection.

where $\Phi_n = (t - z_1)^{k_{p_1}/\kappa} (z_2 - t)^{k_{p_2}/\kappa} (z_2 - z_1)^{p_1 p_2/\kappa}$ and $\gamma$ is any closed contour. Generically the solution corresponding to an $m$-level lowered module is given by integrals of $m$-forms $\Phi_n A_m dt_{i_1} \wedge dt_{i_2} \cdots \wedge dt_{i_n}$ over a loop $\gamma$ in the punctured space.

The settings of KZ has a natural geometry interpretation, where $\phi$ can be identified as the horizontal section, $d = 0$, determined by the KZ connection $\Gamma = \sum_{i,j} \Omega_{i,j} z_i z_j (dz_i - dz_j)$. Indeed if we consider a bundle with base $U$ and $\tilde{V}_1 \otimes \tilde{V}_2 \otimes \cdots \otimes \tilde{V}_n$ as the fibre, starting with $\phi(z_0^0, z_0^0, \ldots, z_0^0)$ at a specific value of $z_i$’s, the solution to KZ equations at generic point $\phi(z'_1, z'_2, \ldots, z'_n)$ can be obtained through a unique lift. The coefficients $I_i$’s on the other hand, defines a pairing between the twisted cohomology group $H^m(C_{n,m}(z), \Phi_n)$ and homology group $H_m(C_{n,m}(z), \Phi_n)$ on the punctured space\footnote{The integration variables $t_i$’s appear in (4.7) and their higher level generalisations live in the space of discriminantal arrangement $C_{n,m}(z)$ [84]. When there is only one variable this space is simply the punctured space $\mathbb{C} - \{z_1, z_2, \ldots, z_n\}$. In the cases of multiple $t_i$’s the integral generically would contain $(t_i - t_j)^{k_i,k_j}$ and we must impose additionally that $t_i \neq t_j$.} $C_{n,m}(z)$, identified as the $m$-form and the closed contour $\gamma$ respectively. An $I_i$ depends only on the (twisted) homology once we have chosen a particular $m$-form, so that when $z_i$’s vary along a path in the base space the contour $\gamma$ deforms continuously, and the action of the KZ provides a Gauss-Manin connection on the bundle with base $U$ and twisted homology group $H_m(C_{n,m}(z), \Phi_n)$ as its fibre. In particular when the end point $\{z_1, z_2, \ldots, z_n\}$ is a permutation of the starting point $(z_0^0, z_0^0, \ldots, z_0^0)$, the action of KZ braids $\gamma$ (figure 8) and defines an $R$-matrix on $H_m(C_{n,m}(z), \Phi_n)$. The twisted homology group is known to be isomorphic to the quantum group $U_q(g)$ that corresponds to the $q$-deformation of (4.2) [78].

4.1 Correlators, bilinear forms and KZ solutions

In this section we temporarily remove all integrals present in an amplitude or BCJ numerators (3.1), (3.3) and focus exclusively on their correlator $\langle 0 | V_{p_n}(z_n) \cdots V_{p_2}(z_2) V_{p_1}(z_1) | 0 \rangle$. When all vertices are tachyons apparently the correlator is the 0-form coefficient $I_0 = \prod_{i,j} (z_i - z_j)^{\alpha_i k_i, k_j}$ of the top highest weight module $\phi_0 = I_0 v_{p_n} \otimes \cdots \otimes v_{p_2} \otimes v_{p_1}$ and the
KZ equations (4.1) in this case translate to the differential equations of $I_0$, whereas the action of Casimir $\Omega_{ij}$ can be read off directly from the module, giving

$$\frac{\partial}{\partial z_i} I_0 = \frac{1}{\kappa} \sum_{j \neq i} p_i \cdot p_j I_0.$$  

(Assuming that we identify $\alpha' = 1/\kappa$.) Starting with $\phi_0(z_1^0, z_2^0, \ldots, z_n^0)$ at a specific set of $z_i$’s the KZ connection uniquely determines the value $\phi(z_1', z_2', \ldots, z_n')$ through parallel transport, and therefore $I_0(z_1', z_2', \ldots, z_n')$ at any set of $z_i$’s. Especially when the number of punctures $n$ is restricted to 2 we see the bilinear form $\langle v_p, v_s \rangle := \langle V_{-p}(z_2) V_s(z_1) \rangle$ for all $(z_1, z_2)$ including the asymptotics $(0, \infty)$ are related to each other in the same manner.

In the case where one gluon is present, suppose instead of direct substitution with a gluon vertex operator $V_{p_1}^{\text{gluon}}(z_1) = \epsilon \cdot \hat{X} e^{ip_1 \cdot X(z_1)}$ we choose to represent gluon by the Del Giudice-Di Vecchia-Fubini (DDF) constructed vertex [115],

$$V_{p_1}^{\text{gluon}}(z_1) = \int d\tau e^{ip_1 \cdot X(z_1)} S_{k_0}(\tau),$$

where $S_{k_0}(t) = \epsilon \cdot \hat{X} e^{ik_0 \cdot X(t)}$. The polarisation is taken to be in the orthogonal direction $\epsilon \cdot p_0 = 0$, and $p_0 + k_0 = p_1$ is the original gluon momentum. Note the settings of DDF demands $\alpha' p_0 \cdot k_0 = -1$ so that equation (4.9) can be regarded as a special case of the screening $E_{k_0}$ acting on a module $V_{p_0}(z_1)$, where the branch cut vanishes because of the integer exponent so that we can safely close the C-shaped contour (since now its boundaries are on the same sheet) to form a closed loop $\ell_1$ around $z_1$. From this perspective the one gluon correlator is the pairing of the cycle $\ell_1$ with a 1-form derived from OPE, which in turn can be easily identified term by term with the 1-forms generated by KZ coefficients $I_{(0,1,...,0)} = \int d\tau \frac{1}{\tau - z_i} \Phi_\kappa$.

$$\langle 0 | V_{p_0}(z_n) \ldots V_{p_2}(z_2) (E_{k_0} V_{p_0}(z_1)) | 0 \rangle = \sum_{i=2}^n \epsilon \cdot p_i I_{(0,1,...,0)}^{i-\text{th entry}}.$$  

In light of the 1-level lowered KZ solution is given by the following sum of tensor modules,

$$\phi_1 = I_{(0,1,...,0)} E_{k_0} \tilde{v}_{p_0} \circ \tilde{v}_{p_2} \circ \cdots \circ \tilde{v}_{p_n} + I_{(0,1,...,0)} \tilde{v}_{p_0} \circ E_{k_0} \tilde{v}_{p_2} \circ \cdots \circ \tilde{v}_{p_n} + \cdots + I_{(0,0,...,1)} \tilde{v}_{p_0} \circ \tilde{v}_{p_2} \circ \cdots \circ E_{k_0} \tilde{v}_{p_n},$$

the one gluon correlator can be expressed as $\phi_1$ projected onto a dual vector $0 w_{(0,1,...,0)} + \epsilon \cdot p_2 w_{(0,1,...,0)} + \cdots + \epsilon \cdot p_n w_{(0,...,0,1)}$. The action of Casimir $\Omega_{ij}$ can be again read off directly from the module, yielding a slightly more complicated set of differential equations that relates correlators at different $z_i$’s. As a quick consistency check of the relations just described, recall that in the zero string tension limit $\alpha' \to \infty$, the correlator should only have support on the Gross-Mende saddle points [116]. Suppose if we fix the values of $(p_0, p_2, \ldots, p_n)$ while maintaining the condition $\alpha' p_0 \cdot k_0 = -1$, so that $k_0 \sim 1/\alpha' \to 0$ and the action of root vector $e_{k_0}$ on modules becomes negligible, the KZ equations of $\phi_1$ implies

$$\frac{\partial}{\partial z_i} \langle V_{p_n}(z_n) \ldots V_{p_2}(z_2) V_{p_1}^{\text{gluon}}(z_1) \rangle = \frac{1}{\kappa} \sum_{j \neq i} p_i \cdot p_j \frac{1}{z_i - z_j} \langle V_{p_n}(z_n) \ldots V_{p_2}(z_2) V_{p_1}^{\text{gluon}}(z_1) \rangle,$$

(4.12)
therefore we see that scattering equations indeed must be satisfied if the $z_i$’s are to localise. It is straightforward to generalise the above reasoning to incorporate more higher modes in the string spectrum, for example an $n$-gluon correlator is given by the pairing of an $n$-cycle with the $n$-form derived from the OPE, whereas the $n$-cycle, when visualised on the punctured plane, is given by the $n$ independent closed loops $\ell_1, \ldots, \ell_n$ surrounding each puncture $z_i$. The corresponding $n$-form, on the other hand, can be spanned by the $n$-forms generated by KZ coefficients through straightforward term by term identifications.

### 4.2 KZ solutions and $Z$-amplitudes

We return to amplitudes and numerators. In the previous section we directly identified the KZ coefficients needed to span a correlator. Generically a correspondence is known as the Drinfeld-Kohno theorem [75, 76] which identifies given quantum algebra $U_q(g)$ behaviour with the monodromy of $\phi$’s, which lives in the representation space of classical Kac-Moody algebra $g$ with the same roots as $U_q(g)$. In view of the discussions in section 3 we see that an $SL(2,\mathbb{R})$ fixed $n$-point amplitude or numerator is described by the QUEA with simple roots $f_{k_2};k_3;\ldots;k_n$ read off from its external legs as in (3.4), it is therefore natural to look for KZ coefficients in the $P_n$ $k_2$ weight lowered level subspace of the classical tensor module $\tilde{V}_{k_1} \otimes \tilde{V}_{k_{n-1}}$.

For example at four points, suppose if we fix $(z_1, z_3, z_4)$ at two arbitrary points and infinity respectively, the associated algebra is then given by (4.2) with only one (simple) root $k_2$. For this algebra, the Casimir operator (4.6) reduces to:

$$\Omega = \sum_{i=0}^{D-1} h_i \otimes h_i + f_{k_2} \otimes e_{k_2} + e_{k_2} \otimes f_{k_2}$$

The KZ equation is solved on the maps from $(z_1, z_3) \in \{ \mathbb{C}^2 | z_1 \neq z_3 \}$ to vectors in $\tilde{V}_{k_1} \otimes \tilde{V}_{k_3}$, which has the following form.

$$\phi_4 = I_{\{k_2, \emptyset\}} f_{k_2} \tilde{v}_{k_1} \otimes \tilde{v}_{k_3} + I_{\emptyset,\{k_2\}} \tilde{v}_{k_1} \otimes f_{k_2} \tilde{v}_{k_3}. \quad (4.14)$$

Here we denote the coefficients as $I_{\{k_2, \emptyset\}}$ to emphasise generically they should be labeled by an ordered set that clarifies in which order the root vector $f_{k_i}$’s are applied to the corresponding tensor vector. In terms of this notation the solution to the KZ equations is given by the following.

$$I_{\{k_2, \emptyset\}} = \int_\gamma dt \frac{1}{t - z_1} \Phi_\kappa, \quad I_{\emptyset,\{k_2\}} = \int_\gamma dt \frac{1}{z_3 - t} \Phi_\kappa, \quad (4.15)$$

where

$$\Phi_\kappa = (t - z_1)^{k_2,k_1/\kappa} (z_3 - t)^{k_3,k_2/\kappa} (z_3 - z_1)^{k_1,k_3/\kappa}, \quad (4.16)$$

and $\gamma$ is any closed contour, for example the Pochhammer encircling $z_1$ and $z_3$. A $Z$-theory amplitude $A_{P\{1,2,3,4\}}$ [109, 110, 114] for example is known to be expressible (up to proportionality factors produced when translating between ordered integrals and Pochammer) as the linear combination $I_{\{k_2, \emptyset\}} + I_{\emptyset,\{k_2\}}$ and therefore satisfies the KZ equations, in the
sense that it can be expressed as scalar product of $\phi_4$ and a $z_i$ independent dual module, assuming the orthonormal duals to the two basis tensor vectors in (4.5) are $w_{(k_2, \emptyset)}$ and $w_{(\emptyset, (k_2))}$:

$$A_p(1, 2, 3, 4) = (w_{(k_2, \emptyset)} + w_{(\emptyset, (k_2))}, \phi_4) = \int_{\gamma_P} dt \frac{z_3 - z_1}{t - z_1} \frac{1}{z_3 - t} \Phi_\kappa, (\alpha_1). \quad (4.17)$$

Similarly for the five-point $Z$-amplitude, the algebra contains two simple roots $k_2$ and $k_3$, and the target space of KZ equation is the $\sum_{i=2}^3 k_i$-lowered level of $V_{k_1} \otimes V_{k_4}$. (Namely, the weight $\sum_{i=1}^3 k_i$ subspace.) For generic value of $k_2$ and $k_3$ the Casimir operator in (4.6) is a infinite sum, as any positive integer sum of $k_2$ and $k_3$ will appear as a root in the summation. However, in the space of vectors of the form:

$$\phi_5 = I_{(\{k_2, k_3\}, \emptyset)} f_{k_2} f_{k_3} v_{k_1} \otimes v_{k_4} + I_{(\{k_2\}, \{k_3\})} f_{k_2} v_{k_1} \otimes f_{k_3} v_{k_3} + I_{(\emptyset, \{k_2, k_3\})} v_{k_1} \otimes f_{k_2} f_{k_3} v_{k_4} + (k_2 \leftrightarrow k_3). \quad (4.18)$$

the Casimir operator is effectively

$$\Omega = \sum_{i=0}^D h_i \otimes h_i + \sum_{i=2,3} (e_{k_i} \otimes f_{k_i} + e_{k_i} \otimes f_{k_i}) \quad (4.19)$$

$$+ \frac{1}{k_2 \cdot k_3} [e_{k_2}, e_{k_3}] \otimes [f_{k_2}, f_{k_3}] + \frac{1}{k_2 \cdot k_3} [f_{k_2}, f_{k_3}] \otimes [e_{k_2}, e_{k_3}],$$

as all terms involving higher orders of $e_{k_i}$ will vanish in the subspace we are considering here. The solution for the first three $I$'s takes the following form,

$$I_{(\{k_2, k_3\}, \emptyset)} = \int_{\gamma} dt_2 dt_3 \frac{1}{t_3 - t_2} \frac{1}{t_3 - z_1} \frac{1}{t_3 - z_1} \Phi_\kappa, \quad (4.20)$$

$$I_{(\{k_2\}, \{k_3\})} = \int_{\gamma} dt_2 dt_3 \frac{1}{t_2 - z_2} \frac{1}{t_4 - z_3} \frac{1}{t_3} \Phi_\kappa, \quad (4.21)$$

$$I_{(\emptyset, \{k_2, k_3\})} = \int_{\gamma} dt_2 dt_3 \frac{1}{t_2 - z_2} \frac{1}{z_4 - t_3} \frac{1}{t_3} \Phi_\kappa, \quad (4.22)$$

with the Koba-Nielsen factor $\Phi_\kappa$ given by

$$\Phi_\kappa = (z_4 - z_1)^{k_1 \cdot k_4 / \kappa} (t_3 - t_2)^{k_2 \cdot k_3 / \kappa} \prod_{i=2,3} (t_i - z_1)^{k_i \cdot k_1 / \kappa} (z_4 - t_i)^{k_i \cdot k_4 / \kappa}. \quad (4.23)$$

The rest three coefficients can be obtained by permutations of $t_2$ and $t_3$. Note that the solution $\phi_5$ explicitly depends on the homology class of the integration domain. For this reason we shall write the coefficients $I$ and $\phi_5$ as a function of the homology class $\gamma$ of $\gamma$.

Similarly to the four-point case, the $Z$-amplitude at 5 points can be written as (signed) sum of the KZ coefficients $I$'s, for example,

$$A_p(1, 2, 3, 4, 5) = I_{(k_3, k_2), \emptyset} (|\gamma_P|) + I_{(k_2), \{k_3\}} (|\gamma_P|) + I_{(\emptyset, (k_2, k_3))} (|\gamma_P|) \quad (4.24)$$

where the integration domain $\gamma_P$ is the 2-dimension generalization of the Pochammer contour, which is a 2-cycle in $C_{2,2}(t) = \{ t \in \mathbb{C} | t_2, t_3 \neq z_1, z_4, t_2 \neq t_3 \}$ that can be identify
with the integration domains correspond to the relative order $P[2,3]$: $z_1 \leq t_2 \leq t_3 \leq z_4$ for $P[2,3] = \{2,3\}$, and $z_1 \leq t_3 \leq t_2 \leq z_4$ for $P[2,3] = \{3,2\}$. Therefore the $Z$-theory amplitudes $A_P(1,Q\{2,3\},4,5)$ can be constructed from $\phi_5$ in the following way.

$$A_P(1,Q\{2,3\},4,5) = (w_{Q\{2,3\}},\phi_5([\gamma_{P\{2,3\}}]))$$  \hspace{1cm} (4.25)

where the two dual vectors read

$$w_{\{2,3\}} = (f_{k_3}^* f_{k_2}^* v_k \otimes v_k)^* + (f_{k_2}^* v_k \otimes f_{k_3}^* v_k)^* + (v_k \otimes f_{k_2}^* f_{k_3}^* v_k)^*$$  \hspace{1cm} (4.26)

and

$$w_{\{3,2\}} = (f_{k_2}^* f_{k_3}^* v_k \otimes v_k)^* + (f_{k_3}^* v_k \otimes f_{k_2}^* v_k)^* + (v_k \otimes f_{k_2}^* f_{k_3}^* v_k)^*$$  \hspace{1cm} (4.27)

Notably, the KZ equation was introduced earlier to the study of string amplitudes by Broedel, Schlotterer, Stieberger and Terasoma in [101] to construct the Drinfeld associator that relates higher point $Z$-integrals to lower point ones. (See also [117–119].) The KZ equation used in [101] is a normalised version of KZ equation from Kac-Moody algebra with simple roots $\{k_2,k_3,\ldots,k_{n-2}\}$ solving for $\phi$ on the weigh-$\sum_{i=2}^{n-2} k_i$ lowered submodule a tri-tensor space $V_{k_i} \otimes V_{k_0} \otimes \tilde{V}_{n-1}$ with coordinates $(z_1,z_0,z_{n-1})$.

$$\frac{\partial}{\partial z_0} \phi = \left(\frac{\Omega_{1,0}}{z_0} + \frac{\Omega_{0,n-1}}{z_0 - z_{n-1}}\right) \phi$$  \hspace{1cm} (4.28)

matrix representation of (4.28) acting on $Z$-integrals can be used to build the Drinfeld associator. And the specific form of matrices used in [101] can be achieved by suitable linear transformations. Note that when the basis of twisted cocycles is already given, the matrix representation of KZ equations as differential relation for twisted cocycles can be directly written down by taking derivatives, removing exact terms and expanding on basis. Such matrix representations have been discussed in detail by Mizera in [81].

5 Conclusions

In this paper we showed, with the help of screening vertex operators, that the string generalisation of the BCJ numerators previously derived in [62, 73] have a natural quantum group explanation. The associated algebra structure depends on the specific root system, which in turn is entirely defined by external leg momenta. The definition of a screening involves a string vertex operator followed by a contour integration. For this setting to be interpreted as a representation of the quantum group, a screening operator only needs to create a 1-chain on the punctured plane $\mathbb{C} - \{z_1,z_2,\ldots,z_n\}$ while various choices for the vertex operator in the string spectrum leads to different cohomology contents and corresponds to different representations. Generically the representation depends on string modes as well as on the exact location $z_i$ of the modules. We showed that modules built from the same string modes (and therefore lead to the same cohomology structures) but located at different $z_i$’s are related to each other by the flat KZ connection. From this perspective the action of a universal $R$-matrix has an explicit graphical interpretation as the braiding of two punctures. In other words, quantum algebraic structure of string amplitude can be
represented by sections of a local system over configuration spaces of $z_i$s, with modules of the kinematic algebra as its fibre and KZ connection as its flat connection. This local system can be isomorphically mapped to the local system used in discussing twisted homology for string amplitude, with each element in the module mapped to a class of twisted forms and the KZ connection mapped to the Gauss-Manin connection. In fact this identification is known to the quantum groups community as part of a broader structure built from hyperplane arrangements. While recent years have definitely witnessed substantial progress already made from analytic and geometry observations, we feel optimistic that more can be added along this direction.

Note particularly that BCJ numerator has been calculated in the context of field theory using fusion rules in [52] up to the next-to-MHV level. At the momentum of writing it is not clear whether the field theory results are identical to those derived from the screenings. It would be interesting to see if any of the two could be used to improve calculation efficiency.

Finally we would like to briefly remark on a subtle, but important issue regarding screening representation of the quantum algebra. The vast majority of the discussions in the literature start with tachyon vertex operators as the representation for simple root vectors. For $sl(N)$ this construction is known to be particularly simple, as the root system $k_i \cdot k_j = \pm 1, \pm 2$ combined with residue theory ensures that all root vectors are tachyons. On the other hand for root systems other than $sl(N)$ the screenings for non-simple root are known to inevitably involve various string excitation modes or even an infinite expansion such as (3.8). A question naturally arises is how to interpret gluon in this quantum algebraic setting. From a pragmatic viewpoint, the integrand of screening only needs to satisfy the braiding relations (2.29), (2.30) for the purpose of serving as a representation of the quantum algebra, and indeed this was the perspective taken for example in [113]. An alternative interpretation can be obtained through RNS strings. Note that a $bc$ system-like screening was discussed in [83] and can be used to explain gluons subject to some modest notation matching. In addition the gluon can be understood as a composite object built from tachyons only, in a manner similar to the DDF but slightly modified approach, or as various residues similar to the viewpoint taken in [108]. We leave this part of the discussion to future work as it involves much details and is perhaps better expanded in a separate paper.

Acknowledgments

We would like to thank Song He, Pei-Ming Ho, Yu-Tin Huang, Kirill Krasnov and Pierre Vanhove for valuable discussions and comments during various stages of this work. CF would like to thank National Taiwan University for their hospitality, where a substantial part of this work was made. CF is supported by the Fundamental Research Funds for the Central Universities (GK201803018). YW is supported by MoST grant 106-2811-M-002-196.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.
References

[1] Z. Bern, J.J.M. Carrasco and H. Johansson, New Relations for Gauge-Theory Amplitudes, *Phys. Rev. D* **78** (2008) 085011 [arXiv:0805.3993] [nSPIRE].

[2] Z. Bern, J.J.M. Carrasco and H. Johansson, Perturbative Quantum Gravity as a Double Copy of Gauge Theory, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476] [nSPIRE].

[3] Z. Bern, C. Boucher-Veronneau and H. Johansson, $N \geq 4$ Supergravity Amplitudes from Gauge Theory at One Loop, *Phys. Rev. D* **84** (2011) 105035 [arXiv:1107.1935] [nSPIRE].

[4] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, The Ultraviolet Behavior of $N = 8$ Supergravity at Four Loops, *Phys. Rev. Lett.* **103** (2009) 081301 [arXiv:0905.2326] [nSPIRE].

[5] Z. Bern, S. Davies and T. Dennen, Enhanced ultraviolet cancellations in $N = 5$ supergravity at four loops, *Phys. Rev. D* **90** (2014) 105011 [arXiv:1409.3089] [nSPIRE].

[6] Z. Bern, S. Davies, T. Dennen, A.V. Smirnov and V.A. Smirnov, Ultraviolet Properties of $N = 4$ Supergravity at Four Loops, *Phys. Rev. Lett.* **111** (2013) 231302 [arXiv:1309.2498] [nSPIRE].

[7] H. Johansson and A. Ochirov, Pure Gravities via Color-Kinematics Duality for Fundamental Matter, *JHEP* **11** (2015) 046 [arXiv:1407.4772] [nSPIRE].

[8] M. Chiodaroli, M. Günyaydin, H. Johansson and R. Roiban, Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity, *JHEP* **01** (2015) 081 [arXiv:1408.0764] [nSPIRE].

[9] M. Chiodaroli, M. Günyaydin, H. Johansson and R. Roiban, Complete construction of magical, symmetric and homogeneous $N = 2$ supergravities as double copies of gauge theories, *Phys. Rev. Lett.* **117** (2016) 011603 [arXiv:1512.09130] [nSPIRE].

[10] M. Ben-Shahar and M. Chiodaroli, One-loop amplitudes for $\mathcal{N} = 2$ homogeneous supergravities, *JHEP* **03** (2019) 153 [arXiv:1812.00402] [nSPIRE].

[11] A. Anastasiou, L. Borsten, M.J. Duff, A. Marrani, S. Nagy and M. Zoccali, Are all supergravity theories Yang-Mills squared?, *Nucl. Phys. B* **934** (2018) 606 [arXiv:1707.03234] [nSPIRE].

[12] J.J.M. Carrasco, M. Chiodaroli, M. Günyaydin and R. Roiban, One-loop four-point amplitudes in pure and matter-coupled $N \geq 4$ supergravity, *JHEP* **03** (2013) 056 [arXiv:1212.1146] [nSPIRE].

[13] A. Anastasiou, L. Borsten, M.J. Hughes and S. Nagy, Global symmetries of Yang-Mills squared in various dimensions, *JHEP* **01** (2016) 148 [arXiv:1502.05359] [nSPIRE].

[14] P.H. Damgaard, R. Huang, T. Sondergaard and Y. Zhang, The Complete KLT-Map Between Gravity and Gauge Theories, *JHEP* **08** (2012) 101 [arXiv:1206.1577] [nSPIRE].

[15] H. Johansson and J. Nohle, Conformal Gravity from Gauge Theory, arXiv:1707.02965 [nSPIRE].

[16] H. Johansson, G. Mogull and F. Teng, Unraveling conformal gravity amplitudes, *JHEP* **09** (2018) 080 [arXiv:1806.05124] [nSPIRE].

[17] G. Chen and Y.-J. Du, Amplitude Relations in Non-linear $\sigma$-model, *JHEP* **01** (2014) 061 [arXiv:1311.1133] [nSPIRE].
[18] F. Cachazo, S. He and E.Y. Yuan, Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM, JHEP 07 (2015) 149 [arXiv:1412.3479] [InSPIRE].

[19] H. Johansson and A. Ochirov, Color-Kinematics Duality for QCD Amplitudes, JHEP 01 (2016) 170 [arXiv:1507.00332] [InSPIRE].

[20] H. Johansson and A. Ochirov, Double copy for massive quantum particles with spin, JHEP 09 (2019) 040 [arXiv:1906.12292] [InSPIRE].

[21] R. Monteiro, D. O’Connell and C.D. White, Black holes and the double copy, JHEP 12 (2014) 056 [arXiv:1410.0239] [InSPIRE].

[22] A. Luna et al., Perturbative spacetimes from Yang-Mills theory, JHEP 04 (2017) 069 [arXiv:1611.07508] [InSPIRE].

[23] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, The Duality Between Color and Kinematics and its Applications, arXiv:1909.01358 [InSPIRE].

[24] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes, Phys. Rev. D 85 (2012) 105014 [arXiv:1201.5366] [InSPIRE].

[25] Z. Bern, S. Davies and T. Dennen, The Ultraviolet Structure of Half-Maximal Supergravity with Matter Multiplets at Two and Three Loops, Phys. Rev. D 88 (2013) 065007 [arXiv:1305.4876] [InSPIRE].

[26] C.R. Mafra and O. Schlotterer, Multiparticle SYM equations of motion and pure spinor BRST blocks, JHEP 07 (2014) 153 [arXiv:1404.4986] [InSPIRE].

[27] C.R. Mafra and O. Schlotterer, Towards one-loop SYM amplitudes from the pure spinor BRST cohomology, Fortsch. Phys. 63 (2015) 105 [arXiv:1410.0668] [InSPIRE].

[28] C.R. Mafra and O. Schlotterer, Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace, JHEP 10 (2015) 124 [arXiv:1505.02746] [InSPIRE].

[29] J.J.M. Carrasco and H. Johansson, Five-Point Amplitudes in N= 4 Super-Yang-Mills Theory and N=8 Supergravity, Phys. Rev. D 85 (2012) 025006 [arXiv:1106.4711] [InSPIRE].

[30] N.E.J. Bjerrum-Bohr, T. Dennen, R. Monteiro and D. O’Connell, Integrand Oxidation and One-Loop Colour-Dual Numerators in N = 4 Gauge Theory, JHEP 07 (2013) 092 [arXiv:1303.2913] [InSPIRE].

[31] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang and J. Nohle, Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops, Phys. Rev. D 92 (2015) 045041 [arXiv:1303.6605] [InSPIRE].

[32] J. Nohle, Color-Kinematics Duality in One-Loop Four-Gluon Amplitudes with Matter, Phys. Rev. D 90 (2014) 025020 [arXiv:1309.7416] [InSPIRE].

[33] A. Ochirov and P. Tourkine, BCJ duality and double copy in the closed string sector, JHEP 05 (2014) 136 [arXiv:1312.1326] [InSPIRE].

[34] M. Chiodaroli, Q. Jin and R. Roiban, Color/kinematics duality for general abelian orbifolds of N = 4 super Yang-Mills theory, JHEP 01 (2014) 152 [arXiv:1311.3600] [InSPIRE].
[36] E.Y. Yuan, *Virtual Color-Kinematics Duality: 6-pt 1-Loop MHV Amplitudes*, JHEP 05 (2013) 070 [arXiv:1210.1816] [inSPIRE].

[37] G. Mogull and D. O’Connell, *Overcoming Obstacles to Colour-Kinematics Duality at Two Loops*, JHEP 12 (2015) 135 [arXiv:1511.06652] [inSPIRE].

[38] S. He, R. Monteiro and O. Schlotterer, *String-inspired BCJ numerators for one-loop MHV amplitudes*, JHEP 05 (2013) 070 [arXiv:1210.1816] [inSPIRE].

[39] G. Mogull and D. O’Connell, *Overcoming Obstacles to Colour-Kinematics Duality at Two Loops*, JHEP 12 (2015) 135 [arXiv:1511.06652] [inSPIRE].

[40] S. He, O. Schlotterer and Y. Zhang, *New BCJ representations for one-loop amplitudes in gauge theories and gravity*, Nucl. Phys. B 930 (2018) 328 [arXiv:1706.00640] [inSPIRE].

[41] S. He, O. Schlotterer and Y. Zhang, *New BCJ representations for one-loop amplitudes in gauge theories and gravity*, Nucl. Phys. B 930 (2018) 328 [arXiv:1706.00640] [inSPIRE].

[42] Y. Geyer and R. Monteiro, *Gluons and gravitons at one loop from ambitwistor strings*, JHEP 03 (2018) 068 [arXiv:1711.09923] [inSPIRE].

[43] Y. Geyer, R. Monteiro and R. Stark-Muchão, *Two-Loop Scattering Amplitudes: Double-Forward Limit and Colour-Kinematics Duality*, JHEP 12 (2019) 049 [arXiv:1908.05221] [inSPIRE].

[44] J.J.M. Carrasco, *Gauge and Gravity Amplitude Relations*, in Theoretical Advanced Study Institute in Elementary Particle Physics: Journeys Through the Precision Frontier: Amplitudes for Colliders, Boulder U.S.A. (2014), pg. 477 [arXiv:1506.00974] [inSPIRE].

[45] R. Saotome and R. Akhoury, *Relationship Between Gravity and Gauge Scattering in the High Energy Limit*, JHEP 01 (2013) 123 [arXiv:1210.8111] [inSPIRE].

[46] S. Oxburgh and C.D. White, *BCJ duality and the double copy in the soft limit*, JHEP 02 (2013) 127 [arXiv:1210.1110] [inSPIRE].

[47] R. Monteiro and D. O’Connell, *The Kinematic Algebra From the Self-Dual Sector*, JHEP 07 (2011) 007 [arXiv:1105.2565] [inSPIRE].

[48] N.E.J. Bjerrum-Bohr, P.H. Damgaard, R. Monteiro and D. O’Connell, *Algebras for Amplitudes*, JHEP 06 (2012) 061 [arXiv:1203.0944] [inSPIRE].

[49] M. Tolotti and S. Weinzierl, *Construction of an effective Yang-Mills Lagrangian with manifest BCJ duality*, JHEP 07 (2013) 111 [arXiv:1306.2975] [inSPIRE].

[50] R. Monteiro and D. O’Connell, *The Kinematic Algebras from the Scattering Equations*, JHEP 03 (2014) 110 [arXiv:1311.1151] [inSPIRE].

[51] N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard and B. Feng, *Manifesting Color-Kinematics Duality in the Scattering Equation Formalism*, JHEP 09 (2016) 094 [arXiv:1608.00006] [inSPIRE].

[52] P.-M. Ho, *Generalized Yang-Mills Theory and Gravity*, Phys. Rev. D 93 (2016) 044062 [arXiv:1501.05378] [inSPIRE].

[53] C.-H. Fu and K. Krasnov, *Colour-Kinematics duality and the Drinfeld double of the Lie algebra of diffeomorphisms*, JHEP 01 (2017) 075 [arXiv:1603.02033] [inSPIRE].

[54] G. Chen, H. Johansson, F. Teng and T. Wang, *On the kinematic algebra for BCJ numerators beyond the MHV sector*, JHEP 11 (2019) 055 [arXiv:1906.10683] [inSPIRE].

[55] N.E.J. Bjerrum-Bohr, P.H. Damgaard and P. Vanhove, *Minimal Basis for Gauge Theory Amplitudes*, Phys. Rev. Lett. 103 (2009) 161602 [arXiv:0907.1425] [inSPIRE].

[56] S. Stieberger, *Open & Closed vs. Pure Open String Disk Amplitudes*, arXiv:0907.2211 [inSPIRE].
[55] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, Monodromy and Jacobi-like Relations for Color-Ordered Amplitudes, *JHEP* **06** (2010) 003 [arXiv:1003.2403] [INSPIRE].

[56] P. Srisangyingcharoen and P. Mansfield, Plahte Diagrams for String Scattering Amplitudes, arXiv:2005.01712 [INSPIRE].

[57] E. Plahte, Symmetry properties of dual tree-graph n-point amplitudes, *Nuovo Cim. A* **66** (1970) 713 [INSPIRE].

[58] P. Tourkine and P. Vanhove, Higher-loop amplitude monodromy relations in string and gauge theory, *Phys. Rev. Lett.* **117** (2016) 211601 [arXiv:1608.01665] [INSPIRE].

[59] S. Hohenegger and S. Stieberger, Monodromy Relations in Higher-Loop String Amplitudes, *Nucl. Phys. B* **925** (2017) 63 [arXiv:1702.04963] [INSPIRE].

[60] A. Ochirov, P. Tourkine and P. Vanhove, One-loop monodromy relations on single cuts, *JHEP* **10** (2017) 105 [arXiv:1707.05775] [INSPIRE].

[61] E. Casali, S. Mizera and P. Tourkine, Monodromy relations from twisted homology, *JHEP* **12** (2019) 087 [arXiv:1910.08514] [INSPIRE].

[62] C.-H. Fu, P. Vanhove and Y. Wang, A Vertex Operator Algebra Construction of the Colour-Kinematics Dual numerator, *JHEP* **09** (2018) 141 [arXiv:1806.09584] [INSPIRE].

[63] Y.-J. Du, B. Feng and C.-H. Fu, BCJ Relation of Color Scalar Theory and KLT Relation of Gauge Theory, *JHEP* **08** (2011) 129 [arXiv:1105.3503] [INSPIRE].

[64] M. Kiermaier, Gravity as the Square of Gauge Theory, talk at Amplitudes 2010, London U.K. (2010), http://www.strings.ph.qmul.ac.uk/~theory/Amplitudes2010/Talks/MK2010.pdf.

[65] R.H. Boels and R.S. Isermann, On powercounting in perturbative quantum gravity theories through color-kinematic duality, *JHEP* **06** (2013) 017 [arXiv:1212.3473] [INSPIRE].

[66] Y.-J. Du and C.-H. Fu, Explicit BCJ numerators of nonlinear sigma model, *JHEP* **09** (2016) 174 [arXiv:1606.05846] [INSPIRE].

[67] C.-H. Fu, Y.-J. Du, R. Huang and B. Feng, Expansion of Einstein-Yang-Mills Amplitude, *JHEP* **09** (2017) 021 [arXiv:1702.08158] [INSPIRE].

[68] C.R. Mafra and O. Schlotterer, Berends-Giele recursions and the BCJ duality in superspace and components, *JHEP* **03** (2016) 097 [arXiv:1510.08846] [INSPIRE].

[69] V. Del Duca, L.J. Dixon and F. Maltoni, New color decompositions for gauge amplitudes at tree and loop level, *Nucl. Phys. B* **571** (2000) 51 [hep-ph/9910563] [INSPIRE].

[70] N.E.J. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, Gravity and Yang-Mills Amplitude Relations, *Phys. Rev. D* **82** (2010) 107702 [arXiv:1005.4367] [INSPIRE].

[71] N.E.J. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, Proof of Gravity and Yang-Mills Amplitude Relations, *JHEP* **09** (2010) 067 [arXiv:1007.3111] [INSPIRE].

[72] H. Kawai, D.C. Lewellen and S.H.H. Tye, A Relation Between Tree Amplitudes of Closed and Open Strings, *Nucl. Phys. B* **269** (1986) 1 [INSPIRE].

[73] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, The Momentum Kernel of Gauge and Gravity Theories, *JHEP* **01** (2011) 001 [arXiv:1010.3933] [INSPIRE].
[74] H. Frost and L. Mason, Lie Polynomials and a Twistorial Correspondence for Amplitudes, arXiv:1912.04198 [insPIRE].

[75] V. Drinfeld, Quasi Hopf algebras, Alg. Anal. 1 (1989) 114 [Leningrad Math. J. 1 (1990) 1419].

[76] T. Kohno, Monodromy representations of braid groups and Yang-Baxter equations, Ann. Inst. Fourier 37 (1987) 139.

[77] V.G. Knizhnik and A.B. Zamolodchikov, Current Algebra and Wess-Zumino Model in Two-Dimensions, Nucl. Phys. B 247 (1984) 83 [insPIRE].

[78] V. Schechtman and A. Varchenko, Arrangements of hyperplanes and Lie algebra homology, Invent. Math. 106 (1991) 139.

[79] S. Mizera, Combinatorics and Topology of Kawai-Lewellen-Tye Relations, JHEP 08 (2017) 141601 [arXiv:1711.04659] [insPIRE].

[80] S. Mizera, Aspects of Scattering Amplitudes and Moduli Space Localization, other thesis, 6, 2019, 10.1007/978-3-030-53010-5 [arXiv:1906.02099] [insPIRE].

[81] A. Varchenko, Multidimensional Hypergeometric Functions The Representation Theory Of Lie Algebras And Quantum Groups, Adv. Ser. Math. Phys. 21 (1995) 1.

[82] V.S. Dotsenko and V.A. Fateev, Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models, Nucl. Phys. B 240 (1984) 312 [insPIRE].

[83] V.S. Dotsenko and V.A. Fateev, Four Point Correlation Functions and the Operator Algebra in the Two-Dimensional Conformal Invariant Theories with the Central Charge $c < 1$, Nucl. Phys. B 251 (1985) 691 [insPIRE].

[84] L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtajan, Quantum completely integral models of field theory, Sov. Sci. Rev. C 1 (1980) 107 [insPIRE].

[85] V.G. Drinfeld, Quantum groups, J. Sov. Math. 41 (1988) 898 [insPIRE].

[86] M. Jimbo, A q difference analog of $U(g)$ and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985) 63 [insPIRE].

[87] L. Fadeev, N.Y. Reshetikhin and L.A. Takhtajan, Quantumization of Lie groups and Lie algebras, Alg. Anal. 1 (1989) 178.
[94] Yu. I. Manin, *Quantum Groups and Non-commutative Geometry*, Centre de recherches mathématiques, Université de Montréal, Montréal Canada (1988).

[95] Yu. I. Manin, *Multiparametric quantum deformation of the general linear supergroup*, Commun. Math. Phys. **123** (1989) 16.

[96] S.L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. **111** (1987) 613 [SPIRE].

[97] S.L. Woronowicz, *Twisted SU(2)* group: An Example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci. Kyoto **23** (1987) 117.

[98] J.M. Drummond, J.M. Henn and J. Plefka, *Yangian symmetry of scattering amplitudes in N = 4 super Yang-Mills theory*, JHEP **05** (2009) 046 [arXiv:0902.2987] [SPIRE].

[99] D. Chicherin, S. Derkachov and R. Kirschner, *Yang-Baxter operators and scattering amplitudes in N = 4 super-Yang-Mills theory*, Nucl. Phys. B **881** (2014) 467 [arXiv:1309.5748] [SPIRE].

[100] V.K. Dobrev, *Invariant Differential Operators. Vol. 2: Quantum Groups*, De Gruyter Stud. Math. Phys. **39** (2017) 1.

[101] J. Broedel, O. Schlotterer, S. Stieberger and T. Terasoma, *All order α'-expansion of superstring trees from the Drinfeld associator*, Phys. Rev. D **89** (2014) 066014 [arXiv:1304.7304] [SPIRE].

[102] M.B. Green, J. Schwarz and E. Witten, *Superstring Theory. Vol. 1: Introduction*, Cambridge University Press, Cambridge UK. (1987).

[103] V.G. Drinfeld, *Hamiltonian structures of lie groups, lie bialgebras and the geometric meaning of the classical Yang-Baxter equations*, Sov. Math. Dokl. **27** (1983) 68 [SPIRE].

[104] V.G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Dokl. **32** (1985) 254 [SPIRE].

[105] M. Jimbo, *A q Analog of U (Gl(n + 1)), Hecke Algebra and the Yang-Baxter Equation*, Lett. Math. Phys. **11** (1986) 247 [SPIRE].

[106] A. Kirillov and N. Reshetikhin, *Representations of the algebra U_q(sl(2)), q-orthogonal polynomials and invariants of links*, in Advanced Series in Mathematical Physics. Vol. 11: New Developments in the Theory of Knots, World Scientific, Singapore (1990), pg. 202.

[107] A.M. Semikhatov and I. Tipunin, *The Nichols algebra of screenings*, Commun. Contemp. Math. **14** (2012) 1250029 [arXiv:1101.5810] [SPIRE].

[108] I.B. Frenkel and V.G. Kac, *Basic Representations of Affine Lie Algebras and Dual Resonance Models*, Invent. Math. **62** (1980) 23 [SPIRE].

[109] J.J.M. Carrasco, C.R. Mafra and O. Schlotterer, *Abelian Z-theory: NLSM amplitudes and α'-corrections from the open string*, JHEP **06** (2017) 093 [arXiv:1608.02569] [SPIRE].

[110] J.J.M. Carrasco, C.R. Mafra and O. Schlotterer, *Semi-abelian Z-theory: NLSM+ϕ^3 from the open string*, JHEP **08** (2017) 135 [arXiv:1612.06446] [SPIRE].

[111] Q. Ma, Y.-J. Du and Y.-X. Chen, *On Primary Relations at Tree-level in String Theory and Field Theory*, JHEP **02** (2012) 061 [arXiv:1109.0685] [SPIRE].

[112] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, JHEP **07** (2014) 033 [arXiv:1309.0885] [SPIRE].
[113] S.D. Lentner, *Quantum groups and Nichols algebras acting on conformal field theories*, arXiv:1702.06431 [nSPIRE].

[114] C.R. Mafra and O. Schlotterer, *Non-abelian Z-theory: Berends-Giele recursion for the $\alpha'$-expansion of disk integrals*, JHEP 01 (2017) 031 [arXiv:1609.07078] [nSPIRE].

[115] E. Del Giudice, P. Di Vecchia and S. Fubini, *General properties of the dual resonance model*, Annals Phys. 70 (1972) 378 [nSPIRE].

[116] D.J. Gross and P.F. Mende, *String Theory Beyond the Planck Scale*, Nucl. Phys. B 303 (1988) 407 [nSPIRE].

[117] P. Vanhove and F. Zerbini, *Closed string amplitudes from single-valued correlation functions*, arXiv:1812.03018 [nSPIRE].

[118] G. Puhlürst and S. Stieberger, *Differential Equations, Associators, and Recurrences for Amplitudes*, Nucl. Phys. B 902 (2016) 186 [arXiv:1507.01582] [nSPIRE].

[119] A. Kaderli, *A note on the Drinfeld associator for genus-zero superstring amplitudes in twisted de Rham theory*, J. Phys. A 53 (2020) 415401 [arXiv:1912.09406] [nSPIRE].