Center vortices, the functional Schrödinger equation, and CSB

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Abstract

The functional Schrödinger equation (FSE) for QCD gives a unique perspective on generation of a gluon mass $m$, as required for center vortices. The FSE, which yields a special $d=3$ gauge action, combined with lattice calculations strictly in $d=3$ give a value for the dimensionless ratio of $d=3$ coupling to mass $g^2_3/m$. From this we infer a reasonably accurate value for the $d=4$ running coupling $g^2(0)$ in the region of low momentum where it is nearly constant. The result, consistent with other estimates, is too low to drive chiral symmetry breaking (CSB) for quarks in a standard gap equation that has no explicit confinement effects. We recall and improve on old work showing that confinement implies CSB for quarks, and consider CSB for test (that is, quenched) Dirac fermions in the adjoint representation. Here the previously-found value of $g^2(0)$ is large enough to drive CSB in a gap equation, which we relate to the presence of center vortices (non-confining, for the adjoint) and nexuses that drive fermionic zero modes. We discuss the extension of adjoint CSB to finite temperature.

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1 Introduction

I discuss two related topics. The first is an approximate evaluation of the QCD coupling at zero momentum, $\alpha_s(0)$, using the functional Schrödinger equation (FSE). The second is the beginning of a program, not completed, for relating chiral symmetry breakdown (CSB) properties for both quarks and (hypothetical) adjoint Dirac fermions in QCD to $\alpha_s(0)$. This program can be looked on as using known CSB results (from the lattice) to constrain the allowed range of $\alpha_s(0)$ or as using a theoretically-determined value of $\alpha_s(0)$ to predict the CSB properties. The present accuracy of some preliminary investigations of these issues is not high enough to be definitive, but it does suggest that $\alpha_s(0)$ in the range $0.6 \pm 0.2$ is consistent with known properties of CSB.

The FSE has long been of interest to me \cite{1}; more recently, motivated by a talk \cite{2} of Š. Oleník at the 2006 Oberwölz symposium, I wrote a paper last year \cite{3}, one conclusion of which is an estimate of the $d = 4$ strong coupling constant $g^2$ at zero (or small) momentum, $\alpha_s(0) \equiv g^2(0)/(4\pi) \simeq 0.5$. This estimate is somewhat unusual since this estimate is based on one-loop gluon gap equations and lattice numerics in $d = 3$, not $d = 4$, plus some theory that attempts to relate $d = 3$ QCD to $d = 4$ QCD through the FSE. The $d = 3$ studies \cite{12,13,14,15,16,17,18,19,20,21} vary somewhat, but in my view (Ref. \cite{16} argues that the one-loop gap-equation results for $m$ are too small) the most reliable value for the ratio $g_3^2/m$ is $\simeq 6.3/N$ (for gauge group $SU(N)$), where $m$ is the dynamical gluon mass and $g_3^2$ is the $d = 3$ gauge coupling. FSE theoretical estimates suggest $g^2(0)/(4\pi) \simeq (2.9/4\pi)(g_3^2/m) \simeq 0.5$. There are, of course, other estimates of $\alpha_s(0)$: Phenomenology sensitive to infrared properties of QCD gives $\alpha_s(0) \simeq 0.7 \pm 0.3$ \cite{22}. Other pinch-technique calculations \cite{9,10} suggest $\alpha_s(0) \simeq 0.5$, just as I use here.

Recently, lattice simulations \cite{23} have been reported for an interesting question with a long lineage \cite{24,25,26}: How does CSB work in QCD with adjoint fermions $^3$ instead of the usual fundamental-representation quarks?

Please note that this coupling is the scheme- and process-independent coupling defined in the pinch technique $^4$ and not the process-dependent running coupling that is related to our coupling by a process-dependent transformation $^5$. The pinch technique is an all-order way of extracting from Feynman graphs for the S-matrix new graphical structures for off-shell proper vertices that are completely locally gauge-invariant.

$^3$The lattice work is, of course, in Euclidean space, where there are no Majorana fermions and hence adjoint fermions plus gauge fields are not a supersymmetric theory.
These authors’ works state that there is CSB for quenched adjoint fermions not only at temperatures below the deconfinement phase transition but also up to a temperature exceeding the deconfinement phase transition. In contrast, for quarks there is no CSB above the deconfinement phase transition, consistent with lattice findings [27] that center vortices, the standard confinement machinery of today, are both necessary and sufficient for CSB with quarks. Adjoint fermions, blind to the long-range parts of center vortices, are not confined and the argument is, therefore, that there are different mechanisms for CSB for fundamental and adjoint fermions. I do not claim to have definitively answered this question, but present numerical estimates based both on estimates of $\alpha_s(0)$ and on fermion gap equations that have appropriate kernels for small momentum suggest an answer that, given fairly large uncertainties, seems to accord with present-day knowledge from the lattice. What I report here is at best the beginning of a program of refining our quantitative understanding of QCD by comparing estimates of $\alpha_s(0)$ with lattice and theoretical studies of adjoint CSB.

Theoretical papers from long ago [28, 29, 30] argue that various confinement mechanisms produce CSB for quarks. (Later I will give a brief update of some of these ideas, based on the role of a condensate of center vortices and their close relatives, nexuses, in confinement and CSB). These confining mechanisms depend only indirectly on $\alpha_s(0)$. In principle it could be (although the lattice data say otherwise) that there would be CSB for quarks even with no confinement, a possibility that does depend on $\alpha_s(0)$. Generally, CSB for quarks and adjoint fermions should be sensitive to three couplings: (1) the standard QCD coupling $g^2(0)/(4\pi)$ at zero momentum transfer; (2) the critical coupling $g_c^2(0; \text{fund})/(4\pi)$ above which CSB occurs for fundamental-representation fermions (quarks) as found from a gap equation that does not contain confinement effects; and (3) $g_c^2(0; \text{adj})/(4\pi)$, the same coupling for (quenched) adjoint fermions. Because these last two couplings differ from the first only by Casimir factors, the critical couplings for gap equations are inversely proportional to $C_2$, the quadratic Casimir eigenvalue for the fermions in the gap equation, and so for QCD, $g_c^2(0; \text{fund})/(4\pi) = (9/4)g_c^2(0; \text{adj})/(4\pi)$. However, just knowing this is not enough to settle the issue of whether CSB can or cannot take place through a standard gap equation for quarks or adjoint fermions; we need not just the ratios but also the values of the critical couplings to compare with estimates from other QCD models of the couplings.
There are in principle three possibilities:

\[ g^2(0) < g^2(0;\text{adj}) \]
\[ g^2(0;\text{adj}) < g^2(0) < g^2(0;\text{fund}) \]
\[ g^2(0;\text{fund}) < g^2(0) \]

In the first case only quarks can show CSB, and confinement is necessary for quark CSB. In the second case (which is favored both from the lattice data and from the estimates I give here) there is CSB for both adjoint and fundamental fermions. Confinement is again necessary for quark CSB, whose transition temperature must be rather near that of the deconfinement transition; the gap equation is largely irrelevant, although it may account for some separation in these two transition temperatures. CSB for adjoint fermions comes solely from non-confining effects as summarized in a gap equation, and adjoint-fermion CSB may or may not extend above the deconfinement transition. In the third case, there is CSB for both kinds of fermions, and this ought to persist even if confiners such as center vortices are removed from the lattice simulations. Since there is more than a factor of two between adjoint and fundamental critical couplings, the inequality in the second case is a fairly broad one and great accuracy is not needed to single out this case.

Fermion gap equations have a long history, beginning with the JBW equation [31], and nearly all previous work that does not address confinement issues makes three approximations: 1) Gluons are massless; 2) Landau gauge is used; 3) because to one-loop order in this gauge vertex corrections are not ultraviolet-divergent, vertex corrections are ignored. Non-perturbative phenomena of low-energy QCD require a more careful treatment that I will sketch here. In the first place, infrared slavery implies dynamic gluon mass generation, which can be only studied effectively in the gauge-invariant pinch technique [6, 9]. In the second place, because fermion mass generation is an infrared effect there may be important low-energy fermion-gluon vertex corrections. Finally, the possibility of such corrections requires a more careful study of gauge invariance of the gap equation. To some extent these issues have been addressed before [32, 33, 34], but not in a context particularly useful here. In the present work I include a dynamical gluon mass in the gluon propagator; give a sketch of the derivation of a gauge-invariant gap equation using the pinch technique (which seems not to have been addressed in detail before); and use the gauge technique to infer approximate low-momentum vertex corrections that satisfy the correct Ward identities of the pinch tech-
nique. The simplest application of these principles, all that I report here, yields a gap equation much like the JBW \[31\] equation except that the gluon is massive.

The work reported here is still very much in progress, and needs considerable sharpening. However, I believe it is already at the point where one can qualitatively see and explain the differences between CSB for quarks and for adjoint fermions.

2 The functional Schrödinger equation for QCD

This work has already been published \[3\] so I will be brief here. The vacuum wave functional of QCD is a gauge-invariant functional that I write in the form:

\[ \psi\{A^a_i(x)\} = e^{-S_3(A^a_i(x))} \]  

(2)

in which \( 2S_3 \) is a real gauge-invariant \( d = 3 \) effective action (a factor of two because \( |\psi|^2 \) is the weight function for constructing vacuum expectation values). It is constructed to satisfy

\[ H\psi = E_{\text{vac}}\psi \]  

(3)

with Hamiltonian

\[ H = \int \left\{ -\frac{1}{2g^2}(\delta A^a_i)^2 + \frac{1}{4g^2}(G^a_{ij})^2 \right\} \equiv \int \left[ \frac{1}{2}(\Pi^a_i)^2 \right] + V. \]  

(4)

The functional \( S_3 \) has infinitely many terms:

\[ g^2S_3 = \frac{1}{2!} \int \int A_i^a \Omega_{ij} A_j^b + \frac{1}{3!} \int \int \int A_i^a A_j^b A_k^c \Omega_{ijk}^{abc} + \ldots \]  

(5)

Gauge invariance requires \[1\] that \( \Omega_{ij} \) be conserved, and any two successive terms in the expansion are related by ghost-free Ward identities. Even the simplest of these terms for a free gauge theory is not familiar as an effective action, because it has a square root:

\[ S_{3\text{free}} = \frac{1}{2g^2} \int A_i^a \sqrt{-\nabla^2} P_{ij} A_j^a. \]  

(6)

Here \( P_{ij} \) is the usual transverse projector:

\[ P_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}. \]  

(7)
My approach to the FSE (and later to the fermion gap equations) begins with the fact that QCD, because of infrared slavery, undergoes dynamical gluon mass generation \[4, 5, 6, 7, 13, 9, 10\]. The FSE must display this fact; how does it do so? It is easy to describe in a toy model, an Abelian gauge theory with gauge-invariant mass \(M\) put in by hand. The Hamiltonian is:

\[
H_{Abel} = \int \{ -\frac{1}{2} g^2 \left( \frac{\delta}{\delta A_i} \right)^2 + \frac{1}{4 g^2} \left[ (F_{ij})^2 + 2m^2 A_i P_{ij} A_j \right] \} \tag{8}
\]

The corresponding \(S_3\) that exactly satisfies this Hamiltonian is:

\[
S_{3Abel} = \frac{1}{2g^2} \int A_i \sqrt{m^2 - \nabla^2} P_{ij} A_j. \tag{9}
\]

Once I add the mass, a nonlocality appears from the transverse projector. But this is easily remedied, by introducing a scalar field in the mass term:

\[
S_{3mAbel} = \frac{m}{2g^2} \int [A_i - \partial_i \phi]^2 \tag{10}
\]

and functionally integrating over not only the gauge potentials but also over \(\phi\) when constructing vacuum expectation values. This is entirely equivalent to using the non-local transverse projector. Note that the mass term by itself satisfies a FSE with the \(F_{ij}^2\) term missing from the Hamiltonian of Eq. \(8\).

Since we are interested in infrared effects, it is reasonable to make an expansion in inverse powers of the mass \(m\) (or equivalently in powers of the gradient operator). The leading term is \(\mathcal{O}(m)\) and is just the mass term \(S_{3mAbel}\) itself. However, a naive expansion runs into a little bit of trouble, as we see by expanding the square root in the exact Abelian solution:

\[
S_{3Abel} \rightarrow \frac{1}{2g^2} \int A_i \left[ m - \frac{\nabla^2}{2m} + \ldots \right] P_{ij} A_j. \tag{11}
\]

(Observe that the second and succeeding terms in the expansion are local; in fact, the second term is, up to a factor \(1/m\), the usual \(F_{ij}^2\) term.) If only these two terms are kept, the field has mass \(\sqrt{2}m\) and not \(m\). Higher-order terms not written must correct for this discrepancy. Of course, no such expansion

\[^4\text{Or one can begin with a simple gauge-dependent mass term } \int mA^2 \text{ and project out its gauge-independent part by integrating over all gauge transformations of the gauge potential.}\]
in $1/m$ is necessary for the Abelian case, but it is for the non-Abelian case, and requires \cite{3} a somewhat better approximation to the square root which is usable for momenta whose components are comparable in size to $m$.

How does this go for the non-Abelian case? I have argued for decades \cite{35} that one describes locally gauge-invariant gauge-boson mass generation through a gauged non-linear sigma model (GNSM), analogous to the local mass action $S_{3M}$ above. To simplify the notation I use the anti-Hermitean matrix gauge potential

$$A_i = \frac{1}{2i} \lambda_a A^a_i,$$  \hspace{1cm} (12)

where the $\lambda_a$ are the standard Gell-Mann matrices, and covariant derivative

$$D_i = \partial_i + A_i.$$  \hspace{1cm} (13)

Introduce a unitary matrix $U$, with the gauge transformation properties $U \rightarrow VU$ when the gauge potential is transformed by:

$$A_i \rightarrow VA_i V^{-1} + V \partial_i V^{-1}.$$  \hspace{1cm} (14)

Then the locally gauge-invariant GNSM mass term is $\cite{3}$.

$$S_{3m} = \frac{-m}{g^2} \int Tr[U^{-1} D_i U]^2.$$  \hspace{1cm} (15)

The non-covariant derivative $U \partial_i U^{-1}$ is the non-Abelian generalization of the Abelian scalar $\partial_i \phi$; in fact, the GNSM action can be written as:

$$S_{3m} = \frac{-m}{g^2} \int Tr[A_i - U \partial_i U^{-1}]^2.$$  \hspace{1cm} (16)

One can, just as in the Abelian case, eliminate $U$ through its equations of motion (that is, minimize $S_{3m}$), which are (after some non-trivial algebra):

$$[D_i, A_i - U \partial_i U^{-1}] = 0.$$  \hspace{1cm} (17)

The perturbative solution has infinitely many terms, of which a few are $\cite{35}$:

$$U = e^\omega; \quad \omega = -\frac{1}{\nabla^2} \partial \cdot A + \frac{1}{\nabla^2} \left\{ [A_i, \partial_i \frac{1}{\nabla^2} \partial \cdot A] + \frac{1}{2} [\partial \cdot A, \frac{1}{\nabla^2} \partial \cdot A] + \cdots \right\}.$$  \hspace{1cm} (18)

\footnote{Since I am only interested in infrared effects I interpret the coupling as being evaluated at zero momentum.}
The linear term simply generates the transverse projector I have already used in the Abelian case. In addition, there are non-perturbative solutions relevant for center vortices.

I claim that this GNSM mass term is the leading term in the $1/m$ expansion of an effective action that capture the leading non-perturbative effect of infrared slavery, which is dynamic gluon mass generation. (It also captures the structure of massless poles in the pinch-technique Schwinger-Dyson equation yielding the mass dynamically.) It is a good candidate for the leading term of the effective action $S_3$ because it is gauge-invariant; indeed, just as in the Abelian case, it comes from projecting out the non-Abelian gauge invariant from the simple $A^2$ mass term. It is almost evident without calculation\footnote{Ref. 3 contains details about the calculation, using the pinch technique and the gauge technique.} that the next-leading term should be the usual Yang-Mills term, which is just the non-Abelian gauge completion of the Abelian term shown in Eq. (11):

$$S_3 = \frac{-m}{g^2} \int Tr[A_i - U \partial_i U^{-1}]^2 - \frac{1}{4g^2m} \int TrG^2_{ij} + \ldots$$  \hspace{1cm} (19)$$

where $G_{ij}$ is the usual Yang-Mills field strength.

The normalization of the second term follows from the fact that the quadratic term in $S_3$ is just the Abelian action of Eq. (11), one copy for each gauge boson. But just as in the Abelian case this wrongly yields a free-field mass of $\sqrt{2}m$ instead of $m$. I have proposed\footnote{Ref. 3 contains details about the calculation, using the pinch technique and the gauge technique.} to cure this approximately by choosing a renormalization factor $Z$ that best approximates the actual square root operator of the FSE with a two-term expansion; the result for the approximate two-term action $I_{d=3} = 2S_3$ is:

$$-2S_3 \equiv -I_{d=3} = \frac{2mZ}{g^2} \int d^3x Tr[U^{-1} D_i U]^2 \} \int d^3x TrG^2_{ij} + O(m^{-3})$$

with $Z \simeq 1.1 - 1.2$.

The final step is to compare this to the standard $d = 3$ form of gauge-invariant massive QCD for the given mass $m$, which is:

$$I_{d=3} = -\int d^3x \left\{ \frac{1}{2g_3^2} TrG^2_{ij} + \frac{m^2}{g_3^2} Tr[U^{-1} D_i U]^2 \right\}.$$  \hspace{1cm} (21)$$

Here the $d = 3$ coupling $g_3^2$ has the dimensions of mass. Comparing the two
forms of \( I_{d=3} \) yields:

\[
g^2 = \frac{2Z g_3^2}{m}.
\]

Several authors \[12, 13, 14, 15, 16, 17, 18, 19, 20, 21\] have given either lattice or theoretical estimates of the dimensionless ratio \( m/N g_3^2 \) for gauge group \( SU(N) \) with \( N = 2, 3 \). The results are quite consistent with an average value of \( g_3^2/m \simeq 6.3/N \). One can roughly convert the no-quark coupling in Eq. (22) to three light flavors by multiplying the right side of Eq. (22) by \( 11/9 \), and the resulting value of the strong coupling at zero momentum is:

\[
\frac{g^2(0)}{4\pi} \simeq 0.5.
\]

This estimate and other quite similar pinch-technique estimates \[9, 10\], combined with phenomenology \[22\], that gives somewhat higher values, suggests that \( \alpha_s(0) \simeq 0.6 \pm 0.2 \). I will now see how this range of values fits into fermion gap equations for CSB.

### 3 Confinement, soliton condensates, and gap equations

The first step is to understand the difference between gap equations that purport to show the effects of confinement and those that do not. I will not do that in any detail here, but simply draw a few conclusions from the fact that confinement comes from the long-range pure-gauge parts of center vortices, which are quantum solitons of an effective action of the type given in Eq. (21) or its \( d = 4 \) extension. Because of complications having to do with integrating over center-vortex collective coordinates, it is easiest to present the argument in \( d = 2 \), where I will use not the familiar action of \( d = 2 \) Yang-Mills theory but rather the effective action, with a mass term, of Eq. (21) in two dimensions \[36\]. (I have argued \[3\] that this is (possibly up to an overall factor) the correct action for the \( d = 2 + 1 \) FSE vacuum wave functional, and that leaving out the mass term cannot be right; the reason is that without the mass term Wilson loops of all representations show an area law, while with it only \( N \)-ality \( \neq 0 \) representations are confined, which is correct for \( d = 2 + 1 \). But all this is irrelevant to the present argument.)
The simplest center vortex is a soliton solution to the equations of motion of the effective \( d = 2 \) action:

\[
A_j(x - a; K) = (2\pi Q_K / i)\epsilon_{jk}\partial_k\{\Delta_m(x - a) - \Delta_0(x - a)\}. \tag{24}
\]

Here \( Q_K, K = 1 \ldots N - 1 \) is a generator of the Cartan subalgebra of \( SU(N) \), normalized so that \( \exp[2\pi i Q_K] \) is an element of the center, and \( \Delta_{m,0} \) are free propagators of mass \( m,0 \). The vector \( a \) is a collective coordinate for translations, and I do not indicate collective coordinates for group rotations. A gluon propagator can be defined by integrating the product of two soliton potentials over their common collective coordinates:

\[
\langle A_i^a(x)A_j^b(y) \rangle = \frac{\delta_{ab}}{N^2 - 1} \sum_a \sum_{K=1}^{[N/2]} (-2)Tr(Q^2_j)A_i(x - a; K)A_j(y - a; K). \tag{25}
\]

This propagator has a long-range part coming from the \( \Delta_0 \) term, and the remainder is short-range. The full gluon propagator, with both terms, can be used in the gap equation for the pinch-technique fermion propagator. This gap equation is derived from an S-matrix element, or equivalently from some complicated functional of a Wilson loop. The long-range pure-gauge parts are detected by their linkage with the Wilson loop. If a vortex is inside the loop it gives a non-trivial center element; otherwise it gives unity. Since there are no non-trivial elements for \( N \)-ality zero representations, such as the adjoint, the adjoint fermion is completely blind to them and sees only the short-range parts. So for quarks the gap equation should be described with a gluon propagator containing the long-range pure-gauge terms, which gives the long-range gluon propagator:

\[
\langle A_i^a(x)A_j^b(y) \rangle|_{long} = const.\delta_{ab}(\Delta_0)_{ij}(x - y) \tag{26}
\]

where

\[
(\Delta_0)_{ij}(x - y) = \frac{1}{(2\pi)^2} \int d^2k (\delta_{ij} - k_i k_j / k^2) e^{ik \cdot x} \frac{e^{ik \cdot x}}{k^2} \tag{27}
\]

is the gauge propagator of \( d = 2 \) QCD. This propagator, singular at large distances, not only confines quarks, it breaks CSB. (Generally in \( d \) dimensions the propagator behaves like \( k^{-d} \), which I used in \( d = 4 \) in an earlier discussion of confinement and CSB.)

The remaining short-range part couples to all fermions with strength proportional to the quadratic Casimir \( C_2 \), and has range \( 1/m \). In fact, these
short-range soliton parts must sum to a standard massive gauge propagator
of the form (omitting the group labels):

\[ \Delta_{ij}(k) = P_{ij}(k) \frac{1}{k^2 + m^2} + \frac{\xi_{ik}k_j}{k^4}. \]

(28)

I have written this propagator in the form given by the pinch technique, where
the physical part is both gauge-invariant (unlike conventional propagators)
and transverse. The last term on the right is a necessary but inert term de-
pending on the chosen gauge that cannot enter any pinch-technique physical
prediction. In particular, it must cancel out in pinch-technique fermion gap
equations.

4 Fermion gap equations without confinement

In this section I briefly mention the older gap equations, which are oriented
toward ultraviolet behavior. Then I go onto newer equations that treat the
infrared regime of QCD more accurately, including a quick discussion of a
pinch-technique gap equation. More details on these newer gap equations
will be published elsewhere.

The history of gap equations, from the Johnson-Baker-Willey (JBW)
equation of the sixties [31] to work of the nineties, can be traced from various
specializations of an approximate gap equation for the CSB-breaking running
fermion mass \( M(p^2) \):

\[ M(p^2) = 3C_2 \int \frac{d^4k}{(2\pi)^4} \frac{g^2(k^2)M(k^2)}{(k^2 + M(k^2)^2)((p - k)^2 + m^2)}. \]

(29)

An often-studied variant, and the only one I will consider explicitly, drops
the non-linear fermion mass terms in the denominator of the fermion propa-
gator on the right side of the equation, replacing \( k^2 + M(k^2)^2 \) by \( k^2 \). Trouble
arises with any linearized gap equation that has a massless gluon propagator,
because it is impossible to have a finite fermion mass \( M(0) \) at zero momen-
tum. Removing this problem by keeping the full non-linear equation with a
massless gluon and a fermion mass in the denominator is not really a good
solution, since the only mass scale would have to come from the running
charge, which is essentially constant in the low-momentum regime where the
non-linear fermion mass is needed as an infrared cutoff. In fact the funda-
mental infrared cutoff is the gluon mass \( m \) and I expect that \( M(0) \sim m \).
JBW used the linearized equation for QED, where $C_2 = 1$, the charge does not run, the photon mass $m$ vanishes, and $M$ is the electron mass, supposed to be generated spontaneously in massless QED. The idea was to show that this equation leads to a running mass vanishing at large momentum, hence requiring no bare mass counterterm. When this equation is used for QCD, $C_2$ is the quadratic Casimir eigenvalue, $m$ is the gluon mass, and $g^2(k^2)$ is the running QCD charge. (In principle the dynamical gluon mass $m^2$ must run too [6], but since the most important effect of this mass is in the infrared region I will not include such running.)

The master gap equation is a simplified form of the Schwinger-Dyson equations for the fermion propagator $S(p)$, which has the form:

$$S^{-1}(p) = \frac{p}{A(p^2)}[1 + iM(p^2)].$$

(30)

As defined, the fermion “mass” is essentially gauge-invariant, at least in the sense that its ultraviolet anomalous dimension is gauge-invariant. In practice, most workers specialize to the Landau gauge because large-momentum radiative corrections to the fermion-gluon vertex are absent in one-loop order, so it is argued that in this gauge it should be a decent approximation to ignore vertex corrections and set (using the QED Ward identity) $A(p^2) = 1$ for all momenta. I will also use Landau gauge and assume $A(p^2)$ does not change much with momentum, although this requires a few words of justification for the pinch technique / gauge technique approach given later.

The linearized fermion gap equation has the generic matrix form

$$M = g^2 K M$$

(31)

where $K$ is the kernel, derived from the single skeleton graph for the inverse fermion propagator. There is always a chiral-symmetry-preserving solution $M \equiv 0$, but we seek CSB-breaking solutions with $M \neq 0$. If the kernel is a well-behaved (finite-dimensional, bounded) matrix it is clear that CSB can only occur if $g^2$ is sufficiently large; otherwise the determinant $\det(1 - g^2 K)$ will not vanish. Actually, $K$ (from, for example, Eq. (29)) is not that well-behaved, but in the equations we use there is a critical coupling $g^2_c$ marking the boundary between CSB and chiral symmetry preservation; most students of gap equations give rather similar values for this critical coupling.

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7In QED, with its small coupling, ignoring higher-order effects could well be justified. In QCD, with its strong coupling, the justification is that we are looking for infrared-dominated effects, so large-momentum contributions should not be important.
There are two forms of criticality: The first is based on a differential equation derived from the gap equation. Its solution may be well-behaved for sufficiently small coupling, and then show unphysical features, such as alternation of signs of the running fermion mass, for larger coupling. The second is based on the original linearized integral equation, which imposes a boundary condition equivalent to consistency between the left-hand and right-hand sides of the gap equation evaluated at zero momentum. Consistency fails if the coupling is too small. Whether either or both criticality criteria hold depends on the gap equation, as I will show by an explicit example.

4.1 Ultraviolet behavior: The JBW equation and variants

Now I give some simple special cases, the first of which is well-known, of the master gap equation of Eq. (29). The first is the original JBW equation used to study possible dynamical generation of the electron mass in QED. The JBW equation sets $g^2$ to a constant and has no mass terms in the propagators on the right side. With the aid of

$$\Box \frac{1}{(p - k)^2} = -4\pi^2 \delta(p - k)$$

the JBW equation (now for QCD, so $C_2$ is reinstated) becomes the differential equation

$$M'' + \frac{2M'}{p^2} + \frac{\lambda M}{p^4} = 0$$

where

$$\lambda = \frac{3C_2 g^2}{16\pi^2}$$

and the primes indicate derivatives with respect to $p^2$. There are two linearly-independent solutions:

$$M_{\pm}(p^2) = \text{const.} (p^2)^{\nu_{\pm}}, \quad \nu_{\pm} = \frac{1}{2} \{-1 \pm [1 - 4\lambda]^{1/2}\}$$

For small coupling the $\nu_{-}$ solution decreases roughly at large momentum like $1/p^2$ and is commonly called infrared-dominated; the $\nu_{+}$ solution corresponds to the original JBW solution, which falls off very slowly at large momentum.
and is called ultraviolet-dominated. Although we may term one of the solutions infrared-dominated, this does not mean that it can be used for small momenta; in fact, both \( M_{\pm}(0) \) diverge. The infrared-dominated solution is only useful in the ultraviolet.

These solutions \( M_{\pm} \) are not appropriate for finding the ultraviolet behavior in QCD, where the charge runs at large momentum. This issue was first clarified by Lane [37], who showed that the master gap equation with a running charge and no masses exactly captures the ultraviolet behavior of the fermion running mass. At large momentum the gluon mass \( m \) can be dropped in the denominators, and the running coupling has its usual behavior (at leading order)

\[
g^2(k^2) \approx \frac{1}{b \ln(k^2/\Lambda^2)}
\]

where \( b \) is the leading term in the \( \beta \)-function (\( \beta = -bg^3 + \ldots \)) and \( \Lambda \) is the QCD mass. With the aid of the renormalization group Lane showed that the ultraviolet behavior appropriate to CSB is:

\[
M(p^2) \sim \frac{(\ln p^2)^a}{p^2}; \quad a = \frac{3C_2}{16\pi^2 b}.
\]

I will recover this behavior later in a JBW-like equation with both a gluon mass and a running charge.

### 4.2 Critical couplings

The simple JBW equation, with a non-running charge evaluated at zero momentum, has a critical coupling at \( \lambda = 1/4 \), corresponding to a critical coupling \( g_c^2 \) of value

\[
g_c^2 = \frac{\pi}{3C_2}.
\]

For couplings larger than critical the exponents \( \nu_{\pm} \) become complex, with \( \nu_+ = \nu_- \), and the asymptotic solutions both decay and oscillate, for example, like \( \sim p^{-1} \cos[\ln p] \). There is certainly no reason to accept as physical a running fermion mass that alternates in sign. I have already noted that for a well-behaved kernel \( g^2 \) must exceed a certain value for CSB to take place, yet for the simple JBW equation criticality marks the onset of apparently unphysical behavior. So is criticality in the JBW equation at all related to
CSB? The general answer is yes, although CSB may require a somewhat different value from $g_c^2$.

There are in fact a few reasons to believe that $g_c^2$ above is indeed close to the true critical coupling for CSB. For example, Miransky [38] and his collaborators [39] have studied the positronium Bethe-Salpeter equation and find that if $g^2/(4\pi) \geq \pi/4$ tachyonic levels appear. They relate these to vacuum rearrangement and scale-breaking phenomena, of which CSB is an example, and compare the oscillatory behavior of equations like the JBW equation with supercritical coupling to the quantum-mechanical problem of fall into the center or to the behavior of solutions to the Dirac equation with supercritical QED charge $Z\alpha > 1$. In fact, the JBW differential equation is nothing but the radial Schrödinger equation at zero angular momentum and energy for an attractive potential $V(p^2 \equiv r)$:

$$V(r) = -\frac{\lambda}{r^2}.$$  

This potential shows fall into the center if $\lambda \geq 1/4$, which is just the critical coupling given above. The QCD critical coupling of Refs. [38, 39] would be $\pi/(4C_2)$, not much different from the JBW critical value. Others (see [49] and references therein) claim that the JBW value is the critical coupling for the pion Bethe-Salpeter equation to admit a massless pion. So I will assume that a critical coupling deduced from the gap equation is close to, if somewhat below, the critical coupling above which there is CSB.

5 Infrared gap equations, gauge invariance, and the pinch technique

The renormalization group cannot say anything about the behavior of $M(p^2)$ at low momentum, where (among other things) it becomes necessary to include the effects of the gluon mass $m$ not only on the propagator but also on the running coupling. I and others (see [40], which has many references to other works) claim that in QCD there is a quasi-conformal infrared regime where the running charge $g^2(k^2)$ is only slowly changing with momentum. Long ago it was argued [6] that a decent approximation to the running charge at both low (Euclidean) and high momentum, with the right sort of two-gluon
threshold, is:
\[ g^2(k^2) \simeq \frac{1}{b \ln\left(\frac{k^2 + 4m^2}{\Lambda^2}\right)} . \] (40)

This quasi-conformal coupling runs very slowly at \( k \ll m \). Because higher-order terms are neglected this expression cannot be more than perhaps 10% accurate. In the ultraviolet region a wide range of values of \( \Lambda \) does not change the coupling very much, but in the infrared regime dimensional transmutation has taken place, with the zero-momentum coupling determined by the ratio \( 2m/\Lambda \). The reader can verify that the choice \( m = 0.5 \text{GeV}, \Lambda = 0.3 \text{ GeV} \) is within 10% of a recent evaluation from data \[41\] of the strong coupling at the masses of the \( \tau \) and \( Z \), and (for three light flavors) gives \( \alpha_s(0) \equiv \frac{g^2(0)}{(4\pi)} \simeq 0.6 \).

Now consider the linear gap equation keeping the gluon mass, but the running charge is replaced by the fixed-point value \( g^2(0) \). There is no simple differential equation, but one can do the angular integrals. The resulting integral equation is:
\[ M(p^2) = \int_0^\infty dk^2 \frac{3g^2(0)C_2}{8\pi^2} M(k^2)K(p,k) \] (41)
where
\[ K(p,k) = K(k,p) = \frac{1}{p^2 + k^2 + m^2 + [(p^2 + k^2 + m^2)^2 - 4p^2k^2]^{1/2}} . \] (42)

These equations can only be solved numerically, but they are closely related to a solvable differential equation with a dominating kernel \( \tilde{K} \), such that \( K \leq \tilde{K} \). The new kernel is:
\[ K(p,k) \rightarrow \tilde{K} = \frac{1}{2} \left[ \frac{\theta(p^2 - k^2)}{p^2 + m^2} + \frac{\theta(k^2 - p^2)}{k^2 + m^2} \right] . \] (43)

The new kernel \( \tilde{K} \) is exactly equal to \( K \) for \( p > 0, k = 0 \) (or \( k > 0, p = 0 \)), and \( K, \tilde{K} \) are asymptotically the same at large momentum. When both momenta are non-zero, \( \tilde{K} \) is greater, by a maximum factor of about 1.3 at \( k^2 = p^2 = m^2 \). It should therefore be that the critical value \( g_c^2 \) for \( K \) is greater than that for \( \tilde{K} \). I expect a further (modest) increase in the critical coupling because if I had included a properly-running charge in the equation with kernel \( K \) it would further reduce this compared to the presently-considered kernel \( \tilde{K} \) with a fixed charge.
The approximate version of the original (linearized) master gap equation, using the kernel $\tilde{K}$ and fixed charge, is:

$$M(p^2) = \frac{3\zeta_K C_g g^2(0)}{16\pi^2} \left\{ \frac{1}{p^2 + m^2} \int_0^{p^2} dk^2 M(k^2) + \int_0^\infty \frac{dk^2}{k^2 + m^2} M(k^2) \right\}$$  \hspace{1cm} (44)

where $\zeta_K$, a positive number between zero and one, measures the discrepancy between using the kernel $g^2(k^2)K$ and the kernel $g^2(0)\tilde{K}$. One should think of $\zeta_K$ as roughly measuring some sort of momentum average of the form

$$\zeta_K \simeq \langle K(p,k) \ln(4m^2/\Lambda^2) \rangle. \hspace{1cm} (45)$$

I estimate roughly that $\zeta_K \simeq 0.7 - 0.8$.

The differential equation emerging from Eq. (44) is exactly the same as Eq. (33) with no gluon mass, except that the independent variable is changed from $p^2$ to $p^2 + m^2$ and $\lambda$ of Eq. (34) is changed to $\zeta_K \lambda$:

$$M'' + \frac{2M'}{p^2 + m^2} + \frac{\zeta_K \lambda M}{(p^2 + m^2)^2} = 0.$$ \hspace{1cm} (46)

The solutions are:

$$M_{\pm}(p^2) = \text{const.}(p^2 + m^2)^{\nu_{\pm}}, \hspace{0.5cm} \nu_{\pm} = \frac{1}{2}\left\{ -1 \pm [1 - 4\zeta_K \lambda]^{1/2} \right\}$$ \hspace{1cm} (47)

and, unlike the massless-gluon solutions of Eq. (35), these are finite at $p^2 = 0$.

Criticality occurs now at $\zeta_K \lambda = 1/4$, or

$$\frac{g_c^2(0)}{4\pi} \simeq \frac{\pi}{3\zeta_K C_g^2}. \hspace{1cm} (48)$$

As mentioned above, there is another criticality criterion. I consider evaluating the integral in Eq. (44) at zero momentum:

$$1 = \frac{\zeta_K \lambda}{4} \int_0^\infty dk^2 \tilde{K}(k,0)M(k^2)/M(0). \hspace{1cm} (49)$$

This sets a boundary condition on the linear combination of solutions $M_{\pm}$ of the differential equation, from which the solution of the integral equation must be formed. One might expect that if $\lambda$ is too small this criterion can never be satisfied, so the differential equation can be satisfied but not the
corresponding integral equation. However, because both \( M_+ \) and \( M_- \) are finite at the origin and integrable over the kernel \( \tilde{K} \) at infinity, it is in fact always possible to find a linear combination that satisfies Eq. (49). The problem is to find two numbers \( \alpha_\pm \) such that
\[
\alpha_+ M_+(0) + \alpha_- M_-(0) = 1 \tag{50}
\]
and
\[
1 = \frac{\zeta_{K'}}{4} \int_0^\infty dk \tilde{K}(k, 0)[\alpha_+ M_+(k^2) + \alpha_- M_-(k^2)]. \tag{51}
\]
These 2×2 linear equations are soluble for \( \alpha_\pm \) except for at most one value of \( \zeta_{K'} \). (In the case at hand, \( \alpha_\pm = \nu_\pm / (\nu_\pm - \nu_\pm) \); even the limit \( \zeta_{K'} = 1/4 \) exists.) So the criterion of Eq. (49) is not useful whenever the two solutions to the differential equation are both finite at the origin and integrable over the kernel at infinity.

The final change to be made in the gap equation is to replace \( g^2(0) \) by \( g^2(k^2) \) in the \( \tilde{K} \)-equation given as Eq. (44). As long as the momentum dependence of the running charge is on the integration variable \( k^2 \) alone, a differential equation can be found for any \( k \)-dependence. This differential equation is:
\[
M'' + \frac{2M'}{p^2 + m^2} + \frac{3C_2\zeta'_{K'} g^2(p^2) M}{16\pi^2(p^2 + m^2)^2} = 0 \tag{52}
\]
Here \( \zeta'_{K'} \) accounts for the average difference between \( K \) and \( \tilde{K} \) but not for the running charge (cf. Eq. (45)). I estimate \( \zeta'_{K'} \approx 0.9 \). Using the running charge of Eq. (40) gives the equation:
\[
M'' + \frac{2M'}{p^2 + m^2} + \frac{a\zeta'_{K'} M}{(p^2 + m^2)^2 \ln[(p^2 + 4m^2)/\Lambda^2]} = 0 \tag{53}
\]
where \( a \) is Lane’s constant, from Eq. (37).

To my knowledge this is not a form of any standard differential equation, but it reduces to one in two cases. For small momentum if \( p^2 \) is dropped compared to \( 4m^2 \), the result is Eq. (46), already solved. An equation that is infrared-finite and asymptotically-exact for large momentum results from replacing \( p^2 + 4m^2 \) by \( 4(p^2 + m^2) \), which yields a confluent hypergeometric equation.

Unfortunately, this is not very accurate for small momentum, but since it is asymptotically-exact for large momentum I will consider it briefly. Replace
the running charge by:

\[ g^2(k^2) = \frac{1}{b \ln[4(k^2 + m^2)/\Lambda^2]} \quad (54) \]

As one should insist, \( g^2(0) \) is unchanged by this modification, and the large-momentum behavior is insensitive to \( m \).

In addition to the correction factor \( \zeta_K' \), the last term of Eq. (53) should be multiplied by a factor \( \zeta_g \geq 1 \) that attempts to correct for this mutilation of the running charge. Think of \( \zeta_g \) as an average of the type:

\[ \zeta_g \approx \langle \ln[4(p^2 + m^2)/\Lambda^2] \rangle \quad (55) \]

Now change to the new independent variable

\[ t = \ln\left[\frac{4(p^2 + m^2)}{\Lambda^2}\right] \quad (56) \]

With this factor, Eq. (53) becomes:

\[ \ddot{M} + \dot{M} + \frac{a \zeta_g \zeta_K' M}{t} = 0 \quad (57) \]

where dots indicate \( t \) derivatives. A simple estimate suggests that \( \zeta_g \approx 1.1 - 1.2 \), so the product \( \zeta_g \zeta_K' \) is essentially unity within the accuracy to which I aspire, and I drop this product.

Equation (56) is a confluent hypergeometric equation (see, for example, [42]). The two linearly-independent solutions corresponding to Whittaker functions have different asymptotic behaviors. At large momentum the infrared-dominated solution goes like

\[ M_{IR}(p^2) \to \frac{(\ln p^2)^a}{p^2}[1 + \mathcal{O}(\frac{1}{\ln p^2})] \quad (58) \]

which is just the behavior found by Lane [37]. The ultraviolet-dominated solution goes like

\[ M_{UV}(p^2) \to (\ln p^2)^{-a}[1 + \mathcal{O}(\frac{1}{\ln p^2})]. \quad (59) \]

Both of these solutions are finite at \( p^2 = 0 \) and both are integrable over the kernel (including the running charge), so the same situation arises as
with the closely-related Eq. (44): Except for at most one value of $a$, there is always a linear combination of Whittaker functions that satisfies the zero-momentum consistency condition analogous to Eq. (49) that holds when the charge does not run.

Our other criterion for criticality is the onset of zeros in $M(p^2)$. The confluent hypergeometric equation, Eq. (57), shows critical behavior at $a = a_c = 1$—not in the sense of singularities in the solution, but, as for the original JBW equation, if $a > a_c$ zeros of the mass function $M(p^2)$ set in.

The criticality condition $a_c = 1$ is not really dynamical; it depends only on the particle spectrum and gauge group (see Eq. (37)). In QCD with three light flavors I find for quarks that $a = 4/9$, but for (quenched) adjoint fermions, $a = 1$. It appears unlikely that the modified JBW equation could lead to CSB for quarks, but it might well do so for adjoint fermions, in view of all the approximations and uncertainties of our development.

I can, for what it is worth, convert the condition $a_c = 1$ into a criterion for the critical coupling, combining Eqs. (37,40) by eliminating the $\beta$-function coefficient $b$. The result is:

$$\frac{g_s^2(0)}{4\pi} = \frac{\pi}{3C_2} \left( \frac{4}{\ln(4m^2/\Lambda^2)} \right).$$

This estimate differs from that coming from the ultraviolet JBW equation by the factor in parentheses. This factor plausibly varies from 1 (at $m \simeq 3.7\Lambda$) to 2 (at $m \simeq 1.4\Lambda$), given uncertainties in both $m$ and the effective value of $\Lambda$ that works best for a one-loop approximation.

### 5.1 The pinch technique/gauge technique gap equation

My final infrared gap equation is based on the gauge technique (Ref. [44] and references therein), in which one “solves” the Ward identity for the fermion-gluon vertex in terms of the fermion propagator. The gauge technique is combined with the pinch technique\(^8\) which is a way (already mentioned) of finding off-shell Schwinger-Dyson equations, such as the fermion gap equation, that are locally gauge-invariant. These are not, of course, the usual Schwinger-Dyson equations; pinch-technique proper vertices and self-energies

\(^8\)See [45] for a thorough discussion of the two-loop pinch-technique fermion self-energy.
contain contributions from parts of graphs naively unrelated to the proper vertex under study. As a result, the Ward identities of a non-Abelian gauge theory are modified; they are just the naive Ward identities of a QED-like theory.

The pinch technique begins by setting up an on-shell S-matrix elements containing the off-shell Green’s function of interest—see Fig. 1.

Parts of numerators of some of these graphs contain longitudinal momenta which, when they strike an elementary vertex, trigger an elementary Ward identity of the form

\[ k_\mu \gamma_\mu = S^{-1}(k) - S^{-1}(p - k). \]  

(61)

When an inverse propagator hits an external quark line it annihilates it, and
the inverse propagator with momentum of an adjoining internal quark line simply removes that propagator, replacing it by unity. The resulting “pinch” yields graphs such as Fig. 1(h). A pinch can change part of a vertex graph to a propagator graph, resulting in new terms that must be added to the conventional fermion propagator. Since the whole S-matrix element is gauge-invariant, it turns out that the new fermion propagator is gauge-invariant; gauge-dependent terms in the original Feynman graphs are cancelled by these other pinch terms. It is already known [5, 6, 8] that this procedure yields, in an $R_\xi$ gauge, a gauge-invariant gluon propagator of the type:

$$\hat{\Delta}_{\mu\nu}(k) = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\hat{d}(k) + \frac{\xi k_\mu k_\nu}{k^4}$$

(62)

where $\hat{d}(k)$ is completely gauge-invariant (independent of $\xi$) and the $\xi$ term receives no radiative corrections (other than wave-function renormalization). Not only do the longitudinal terms $\sim k_\mu$ cancel out of the S-matrix, they also cancel out of the fermion pinch-technique propagator. (Because of this cancellation proper vertices obey naive ghost-free Ward identities.) Although I have illustrated the pinch technique only to one loop in the figure, it is possible to extend it to all orders and to non-perturbative phenomena (see [46] and references therein). For the gap equation the essential non-perturbative phenomenon is a gluon mass, and I use $\hat{d}(k) = 1/(k^2 + m^2)$.

As is well-known [8], the pinch technique leads to Schwinger-Dyson equations identical to those of the background-field method in the Feynman gauge ($\xi = 1$). However, one can formulate the pinch technique in any gauge [17], in the sense that ghost-free Ward identities and certain other structural elements of Schwinger-Dyson equations important for the pinch technique in the Feynman gauge still hold. I will, in Pilafsis’ [17] sense of the pinch technique, use Landau gauge as all other workers do, because ultraviolet corrections to the vertex are unimportant to one loop in this gauge. Because the Ward identity relating the vertex to the fermion propagator has no ghost terms, the same is true (as in QED) for the coefficient $\mathcal{A}(p^2)$ of $p^\mu$ in the fermion propagator [see Eq. (30)]. In the infrared regime $\mathcal{A}(p^2)$ should, like the running charge, not run much. So I expect $\mathcal{A}(p^2) \approx \mathcal{A}(0)$ over a large momentum range, and will set it to unity. (With the gauge technique vertex, any constant $\mathcal{A}(p^2)$ cancels out from the gap equation.)

The gauge-technique solution for a pinch-technique form factor is an infrared-valid approximation to the form factor that is asymptotically exact as the gluon momentum vanishes, and its ultraviolet inaccuracies can
be compensated for \[48\], although I will not attempt that here. Because its Ward identity is ghost-free, just as in QED, all necessary formulas can be read off with straightforward modifications from the QED work in [48]. One pinch technique/gauge technique improper form factor is (omitting irrelevant group factors and indices):

\[
F_\mu(p, p') = S(p)\Gamma_\mu(p, p')S(p') = \frac{1}{p^2 - p'^2}[(\not{\gamma} \gamma_\mu + \gamma_\mu \not{\gamma})S(p') - S(p)(\gamma_\mu \gamma + \gamma_\mu p')]
\]

and obeys the QED-like Ward identity

\[
(p - p')_\mu F_\mu(p, p') = S(p') - S(p).
\]

Eq. (63) is not a unique choice for the form factor, but it has the advantage of being an identity for a free massive theory, with \(\Gamma_\mu(p, p') = \gamma_\mu\) and \(S^{-1}(p) = \gamma - iM\). All other choices for the pinch-technique form factor, as well as corrections needed for the ultraviolet behavior at loop level, are identically conserved and therefore differ from Eq. (63) only by terms that vanish with at least one more power of \(p - p'\) at small values of \(p - p'\).

The Schwinger-Dyson (SD) equation is:

\[
S_0^{-1}(p)S(p) = 1 + \frac{g^2 C_2}{(2\pi)^4} \int d^4k \gamma_\nu \hat{\Delta}_\nu(\gamma) F_\mu(k, p)
\]

and with the gauge-technique form factor it is linear in the fermion propagator. In the pinch technique the gauge-boson propagator, which shows dynamical mass generation, has the form given in Eq. (62), except that the gauge-dependent term is dropped.

To study CSB I take \(S^{-1}(p) = \gamma\) and extract the coefficients of \(\gamma\) in the SD equation; this yields the gap equation. Because the form factor of Eq. (63) is identically equal to that of a free massive field theory and because the running fermion mass is not running very fast in the infrared, the resulting pinch-technique equation at small momentum (including small integration momentum \(k\)) is really the same as I started with in Eq. (29), except that for small momenta I ignore the running of masses and the coupling. It is plausible that a nearly-correct modification of the infrared pinch-technique equation that accounts for ultraviolet effects is simply to let the masses and coupling run—cases I have already reviewed or analyzed. Actually, it is somewhat more complicated than this, but the final analysis is much too elaborate to discuss here.
The gauge technique incorporates infrared-important vertex corrections and the pinch technique yields a unique gauge-invariant gap equation in an arbitrary gauge for the underlying Feynman graphs. At the present rather simple level of approximation to the pinch technique/gauge technique gap equation, the deep-infrared gap equation, with plausible ultraviolet corrections, is the same as found by many others over decades by ignoring vertex corrections and working in the Landau gauge (of course, self-consistent at one-loop level).

6 Finite temperature CSB

The full finite-temperature \( (T) \) gap equation, especially for the pinch technique, has a number of complications that I have not dealt with as of this writing, but will save for a more detailed work. These include separating the gluon propagator into space-space, space-time, and time-time components, which are no longer related by Lorentz invariance and which have two mass scales, the magnetic and electric (Debye). This complicates the application of pinch-technique cancellation mechanisms [12]. I will only present here the crudest initial attempts at understanding finite-temperature CSB with massive-gluon exchange, using what I call the superconductor approximation, because it is in the same spirit as used in the original BCS paper on superconductivity. It amounts to saying that the gluon propagator is relatively unchanged by temperature effects as long as the temperature is not too large compared to the phase-transition temperature \( T_c \approx 170 \text{ MeV} \). The reason is that the gluon already has a large \( T = 0 \) mass of some 600 MeV, which is not changed drastically by thermal effects at \( T \sim T_c \). However, the fermion mass steadily decreases from its \( T = 0 \) value, eventually vanishing at the CSB phase transition. The difference between fermionic and gluonic dynamical mass generation at finite \( T \) is that the effective coupling strength for fermions is decreased as \( T \) increases, by a factor something like \( \tanh(\beta \omega/2) \) where \( \beta = 1/T \) and \( \omega \) is a characteristic energy scale; ultimately, the fermionic mass has to vanish as the coupling diminishes. But the gluonic mass is increased by a factor roughly \( \coth(\beta \omega/2) \) (so the mass grows like \( T \) at large \( T \)).

In the superconductor approximation I convert the original zero-temperature

9By mass I mean a quantity that violates chiral symmetry, which the usual perturbative finite-temperature fermion “mass” does not.
gap equation, Eq. (29), to finite temperature by the usual replacement of the integral over the (Euclidean) time momentum $k_4$ by a discrete frequency sum:

$$\int \frac{dk_4}{2\pi} \rightarrow \sum_{N=-\infty}^{\infty} \delta[k_4 - 2\pi T (N + \frac{1}{2})].$$

(66)

As mentioned above it is not quite correct simply to make this substitution in an equation such as (29) where the vector-propagator kinematics have been worked out at zero temperature. A further approximation is to average certain momentum-dependent quantities that vary fairly slowly by averages, which allows us to make contact with an already-studied zero-$T$ equation, Eq. (44). The resulting gap equation is then Eq. (44) with the modified zero-momentum but finite-$T$ coupling:

$$g^2(k = 0, T = 0) \rightarrow g^2(k = 0, T) \langle \tanh(\frac{\beta \omega}{2}) \rangle \equiv G^2(T)$$

(67)

with the finite-$T$ coupling determined from the zero-momentum form of Eq. (44) by using a plausible form for the temperature dependence of the finite-$T$ gluon mass:

$$g^2(k = 0, T) = \{b \ln[\frac{4m^2(T)}{\Lambda^2}]\}^{-1}; \quad m(T) = m(T = 0) \coth(\frac{\beta \omega_G}{2})$$

(68)

where I choose for the average gluon frequency

$$\omega_G = \frac{4\pi m(0)}{Ng^2(k = 0, T = 0)}$$

(69)

so as to give a simple but fairly accurate high-$T$ limit (cf the value used in connection with the FSE of $m(T) = 0.16Ng_3^2$, with $g_3^2 \simeq g^2(0)T$). The fermion frequency $\omega_F$ is of the form:

$$\omega_F = \sqrt{k^2 + M^2(\vec{k})}.$$  

(70)

At the level of the simple approximations used here, it is not possible to predict with any accuracy the actual ratio of the temperature $T_{\chi A}$, at which adjoint chiral symmetry breaking is restored, to the usual deconfinement transition temperature $T_d$. However, after some not very interesting numerics involving plausible ranges of unknown and approximated quantities, and using the superconductor approximation given here, it appears possible that
If this possibility survives more detailed numerics then adjoint chiral symmetry breaking survives deconfinement, as the lattice simulations show.

Of course, there is much more to learn about QCD in general from adjoint CSB at finite temperature beyond our interests here, as the literature shows [23, 24, 25]. I hope to take up these questions in more detail later.

7 Summary

I use the FSE to estimate, using \( d = 3 \) dynamical mass calculations, the usual \( d = 4 \) strong coupling at zero momentum: \( \alpha_s(0) \simeq 0.5 \) (for three light flavors), a value in accord with other estimates using the pinch-technique Schwinger-Dyson equations. This is somewhat smaller than phenomenological estimates of around 0.75. I then consider modern versions of the fermion gap equation both for quarks and for adjoint fermions in QCD—made modern by adding dynamical gluon mass effects and by consideration of the gauge-invariant pinch technique/gap technique gap equation. The final results, although not impressively accurate, suggest that confinement is essential to break CSB for quarks, but that standard fermion gap physics can explain CSB for adjoint fermions, for some (uncalculated) range of temperatures, possibly reaching above the deconfinement temperature. These results are consistent with present-day lattice simulations.

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