Yang-Mills and Born-Infeld actions on finite group spaces

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Abstract. Discretized nonabelian gauge theories living on finite group spaces $G$ are defined by means of a geometric action $\int \text{Tr} \, F \wedge *F$. This technique is extended to obtain a discrete version of the Born-Infeld action.\textsuperscript{‡}

1. Introduction

The base space in a gauge theory is usually a manifold; we consider here a finite group $G$ as base space, and the Lie group $G = U(N)$ as fiber. One can associate to $G$ a “manifold” structure by constructing on $G$ a differential calculus. This can be done following the general procedure first studied by Woronowicz \cite{1} in the noncommutative context of quantum groups. The differential calculus on $G$ not only determines the “manifold” $G$, but has been proven to be a very useful tool in the formulation of gauge and gravity theories with base space $G$ \cite{2, 3, 4, 5, 6}. Following \cite{5}, we here show that given a differential calculus on $G$, a Yang-Mills action can be naturally constructed using just geometric objects: differential forms, invariant metric, \textasteriskcentered-Hodge operator, Haar measure. We similarly construct a discretized version of Born-Infeld theory. These actions can be generalized \cite{5} to the case where the base space is $M^D \times G$, with $M^D$ a continuous $D$-dimensional manifold. Then one can study gauge theories on $M^D$ that are “Kaluza-Klein” reduced from those on $M^D \times G$: one has just to reinterpret the $M^D \times G$ gauge action as a new $M^D$ one. For example, approximating the circle $S_1$ with the cyclic group $\mathbb{Z}_n$ we obtain on $M^D$ a gauge action plus extra scalar fields from a pure gauge action on $M^D \times \mathbb{Z}_n$. This provides an alternative and different procedure from the usual Kaluza-Klein massive modes truncation in $M^D \times S_1 \rightarrow M^D$. In some cases one can obtain a Higgs field and potential; a similar mechanism is present in the Connes-Lott standard Model \cite{7}.

Noncommutative structures in string/brane theory have emerged in the last years and are the object of intense research. Our study is in this general context. The noncommutativity we discuss is mild, in the sense that fields commute between themselves (in the classical theory), and only the commutations between fields and differentials, and of differentials between themselves, are nontrivial. This noncommutativity is in a certain sense complementary to the one of \textasteriskcentered (Moyal)-deformed theories, where (for constant $\theta$) only functions do not commute. It is tempting to interpret the product space $M^D \times G$ as a bundle of $n$ (D-1)-dimensional branes evolving in time, $n$ being the dimension of the finite group $G$.

The paper is organized as follows. In Section 2 we recall some basic facts about the differential geometry of finite groups, and construct those geometric tools needed in Section

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3 for the study of Yang-Mills theories. In Section 4 an action for the Born-Infeld theory is presented.

2. Differential geometry on finite groups

Let $G$ be a finite group of order $n$ with generic element $g$ and unit $e$. Consider $\text{Fun}(G)$, the set of complex functions on $G$. The left and right actions of the group $G$ on $\text{Fun}(G)$ read $\mathcal{L}_g f|_{g'} = f(g g')$, $\mathcal{R}_g f|_{g'} = f(g' g)$ for all $f \in \text{Fun}(G)$. The vectorfields $t_g$ on $G$ are defined by their action on $\text{Fun}(G)$: $t_g f|_{g'} = f(g g') - f(g') = (\mathcal{R}_g f - f)|_{g'}$, with $g \neq e$. They are nonlocal and left-invariant: $\mathcal{L}_g (t_g f) = t_g (\mathcal{L}_g f)$. We also have $\mathcal{R}_g (t_g f) = t_{g' g^{-1}} (\mathcal{R}_g f)$.

The differential of an arbitrary function $f \in \text{Fun}(G)$ is then given by

$$df = \sum_{g \neq e} \left( t_g f \right) \theta^g ,$$

where $\theta^g$ are one-forms; by definition they span the linear space dual to that of the vectorfields $t_g$: $\langle t_g, \theta^g \rangle = \delta^g_e$. From the Leibniz rule for the differential $d(fh) = (df)h + f(dh)$, $\forall f, h \in \text{Fun}(G)$, we have that functions do not commute with one-forms:

$$\theta^g f = (\mathcal{R}_g f) \theta^g \quad (g \neq e) .$$

Similarly, left and right invariance of the differential $\mathcal{L}_g df = d(\mathcal{L}_g f)$, $\mathcal{R}_g df = d(\mathcal{R}_g f)$ implies that $\theta^g$ are left-invariant one-forms $\mathcal{L}_g (\theta^g) = \theta^g$ and that $\mathcal{R}_g (\theta^g) = \theta^{g' g^{-1}}$.

A generic one form $\rho$ can always be written in the $\theta^g$ basis as $\rho = \sum_{g \neq e} f_g \theta^g$, with $f_g \in \text{Fun}(G)$. We have here described the so-called universal differential calculus on $G$. Smaller calculi can be obtained setting $\theta^g = 0$ for some $g \in G$. Because of (3), the new differential is still left and right invariant iff, given $\theta^g = 0$, also $\theta^{g' g^{-1}} = 0$. In other words, (bicovariant) differential calculi are in 1-1 correspondence with unions of conjugacy classes of $G$.

The algebra $\text{Fun}(G)$ has a natural *-conjugation: $f^*(g) \equiv \overline{f(g)}$, where $\overline{\cdot}$ is complex conjugation. This *-conjugation can be extended to the differential algebra, the reality condition $(df)^* = d(f^*)$ then implying $(\theta^g)^* = -\theta^{g^{-1}}$. If we set $\theta^g = 0$ then the one-forms in both the $\{\theta^g\}$ and the $\{\theta^{g^{-1}}\}$ conjugacy classes should be set to zero.

An exterior product, compatible with the left and right actions of $G$, can be defined as (here $\otimes$ by definition satisfies $\rho f \otimes \rho' = \rho \otimes f \rho'$ with $\rho, \rho'$ generic one-forms)

$$\theta^g \wedge \theta^{g'} = \theta^g \otimes \theta^{g'} - \theta^{g' g^{-1}} \otimes \theta^g = \theta^g \otimes \theta^{g'} - (\mathcal{R}_g \theta^{g'}) \otimes \theta^g \quad (g, g' \neq e).$$

The metric $\langle \cdot, \cdot \rangle$ maps couples of 1-forms $\rho, \sigma$ into $\text{Fun}(G)$, and is required $\langle \cdot, \cdot \rangle$ to satisfy the properties $\langle f \rho, \sigma h \rangle = f \langle \rho, \sigma \rangle h$, $\langle \rho f, \sigma \rangle = \langle \rho, f\sigma \rangle$, and to be symmetric on left-invariant one-forms. Up to a normalization the above conditions imply

$$g^{g^*} \equiv \langle \theta^g, \theta^g \rangle \equiv \delta^{g^*}_{g^{-1}} .$$

We can generalize $\langle \cdot, \cdot \rangle$ to tensor products of left-invariant one-forms as follows:

$$\langle \theta^{i_1}, \ldots \theta^{i_k}, \theta^{j_1}, \ldots \theta^{j_l} \rangle \equiv \langle \theta^{i_1}, \theta^{j_1}, \ldots, \theta^{i_k}, \theta^{j_l} \rangle \equiv \langle \theta^{i_1}, \theta^{j_1}, \ldots, \theta^{i_k}, \theta^{j_l} \rangle \equiv \langle \theta^{i_1}, \theta^{j_1}, \ldots, \theta^{i_k}, \theta^{j_l} \rangle \equiv \langle \theta^{i_1}, \theta^{j_1}, \ldots, \theta^{i_k}, \theta^{j_l} \rangle$$

(6)

where $f_{\rho'} = (i_{p+1}, i_{p+2}, \ldots, i_k)f_p(i_{p+1}, i_{p+2}, \ldots, i_k)^{-1}$, i.e. $\theta^{i_{p+1}} = R_{i_{p+1}, i_{p+2}, \ldots, i_k} \theta_{i_p}$. The pairing (6) is extended to all tensor products by $\langle f \rho, \sigma h \rangle = f \langle \rho, \sigma \rangle h$ where now $\rho$ and $\sigma$ are generic tensor products of same order. Then, using (2) and (6), one can prove that $\langle \rho f, \sigma \rangle = \langle \rho, f \sigma \rangle$ for any function $f$. Moreover $\langle \cdot, \cdot \rangle$ is left and right invariant.

Finally, if there exists a form $vol$ of maximal degree, then it can be chosen left-invariant, right-invariant and real. The Hodge dual is then defined by $\rho \wedge \ast \sigma = \langle \rho, \sigma \rangle vol$. It is left linear; if $vol$ is central it is also right linear: $\ast (f \rho h) = f(\ast \rho) h$. 

3. Gauge theories on finite groups spaces

The gauge field of a Yang-Mills theory on a finite group $G$ is a matrix-valued one-form $A(g) = A_h(g)\theta^h$. The components $A_h$ are matrices whose entries are functions on $G$, $A_h = (A_h)_{\alpha\beta}^\gamma$, $\alpha, \beta = 1, \ldots, N$. As in the usual case, $G$ gauge transformations are given by

$$A' = -(dT)T^{-1} + TAT^{-1},$$

where $T(g) = T(g)^\gamma_\beta$ is an $N \times N$ representation of a $G$ group element; its matrix entries belong to $\text{Fun}(G)$. The 2-form field strength $F$ is given by the familiar expression

$$F = dA + A \wedge A,$$

and satisfies the Bianchi identity: $dF + A \wedge F - F \wedge A = 0$. Note that $A \wedge A \neq 0$ even if the gauge group $G$ is abelian. Thus $U(1)$ gauge theory on a finite group space looks like a nonabelian theory, a situation occurring also in gauge theories with $\ast$-product noncommutativity. Under the gauge transformations (7), $F$ varies homogeneously: $F' = TFT^{-1}$. We also have

$$F = U \wedge U,$$

where $U_h = 1 + A_h$, $U = \sum_{h \neq e} U_h\theta^h = \sum_{h \neq e} \theta^h + A$. From (7) we see that also $U$ transforms covariantly: $U' = TUT^{-1}$. Defining the components $F_{h,\gamma}$ as:

$$F \equiv F_{h,k} \theta^h \otimes \theta^k$$

eq. (8) yields:

$$F_{h,k} = U_h (\mathcal{R}_h U_k) - U_k (\mathcal{R}_k U_{h^{-1}h}).$$

We now have all the geometric tools needed to construct a Yang-Mills action. The Yang-Mills action is the geometrical action quadratic in $F$ given by

$$A_{YM} = -\int Tr(F \wedge F^*) = -\int Tr < F, F > vol = -\sum_{s \in G} Tr < F, F >$$

the right-hand side being just the Haar measure of the function $Tr < F, F >$. Recalling the properties of the pairing $< , >$, the proof of gauge invariance of $Tr < F, F >$ is immediate: $Tr < F', F' > = Tr < TFT^{-1}, TFT^{-1} > = TrT < F, F > T^{-1} = Tr < F, F >$.

The metric (5) is an euclidean metric (as is seen using a real basis of one-forms) and as usual we require (11) to be real and positive definite. This restricts the gauge group $G$ and imposes reality conditions on the gauge potential $A$. Positivity of (11) requires $- (Tr < F, F >) \geq 0$. Explicitly $< F, F > = F_{r,s} < \theta^r \otimes \theta^s, \theta^m \otimes \theta^n > \mathcal{R}_{n^{-1}m^{-1}} F_{m,n} = F_{r,s} \mathcal{R}_{r,s} F_{s^{-1}r^{-1}s^{-1}},$ and therefore

$$-Tr < F, F > = -(F_{r,s})^\alpha_{\beta} (\mathcal{R}_{r,s} F_{s^{-1}r^{-1}s^{-1}})^{\alpha}_{\beta};$$

we see that (12) is positive if $(\mathcal{R}_{r,s} F_{s^{-1}r^{-1}s^{-1}})^{\alpha}_{\beta} = -(F^*_{r,s})^\alpha_{\beta}$. This holds if (use (10))

$$U = -U^\dagger \quad \text{i.e.} \quad A^\dagger = -A;$$

here hermitionian conjugation on matrix valued one forms $A$ (or $U$) is defined as:

$$A^\dagger = (A_h \theta^h)^\dagger = (\theta^h)^* A_h^\dagger = -\theta^{-h} A_h^\dagger = -\theta^h A_{h^{-1}}^\dagger = -(\mathcal{R}_h A_{h^{-1}}^\dagger) \theta^h.$$
$A^\dagger = -A$ is equivalent to $A_{ab}(g)^b_a = A_{-ah}(hg)^b_a$ and is thus not local (not fiberwise). It follows that $A$ has values in $M_{N^2 \times N}(C)$ and not in the Lie algebra of $U(N)$. Nevertheless $A^\dagger = -A$ is a good reality condition because it cuts by half the total number of components of $A$. This can be seen counting the real components of the antihermitian field $A$: they are $N^2 \times n \times d$, where $n$ is the number of points of $G$, and the “dimension” $d$ counts the number of independent left-invariant one-forms. In conclusion, when $\mathcal{G} = U(N)$, we have a bona fide pure gauge action, where the number of components of $A$ is consistent with the dimension of the gauge group.

The action (11) can be expressed in terms of the link fields $U_h$: substituting (9), (10) into (11) leads to: $A_{YM} = 2 \sum_G Tr[U_h U_h^\dagger U_h^\dagger - U_h^\dagger U_h(R_h U_k)(R_k U_{k-1}^\dagger)]$. We could as well start with a Yang-Mills action on $M^D \times G$ and then obtain ($\mu, \nu = 1, ... D$)

$$A_{YM} = -\int_{M^D \times G} Tr F \wedge \ast F = -\int_{M^D} d^Dx \sum_G Tr[F_{\mu\nu} F_{\mu\nu}$$

$$+ \frac{1}{2} D_{\mu} U_h (D_{\mu} U_h)^\dagger + 2 U_h U_h^\dagger U_h U_h^\dagger - 2 U_h U_h (R_h U_k)(R_k U_{k-1}^\dagger)]$$

This action describes a Yang-Mills theory on $M^D$ minimally coupled to the scalar fields $U_g$, with a nontrivial quartic potential.

4. Born-Infeld theory on finite group spaces

Due to space limitations we only consider here a Born-Infeld action on abelian finite group space $G$. In the commutative case we have $A_{BI} = \int_{M^D} d^Dx \sqrt{\det(\delta_{\mu\nu} + F_{\mu\nu})}$. The analogue of $\delta_{\mu\nu} + F_{\mu\nu}$ becomes simply $E_{g,h} \equiv \delta_{g,h} + 1 + i F_{g,h}$ (the $i$ is because a constant $F_{g,h}$ is an antihermitian matrix if $g = g^{-1}$ and $h = h^{-1}$). The matrix $E_{g,h}$ transforms under $U(N)$ gauge variations in the same way as $F_{g,h}$: $E'_{g,h} = T E_{g,h} R_{g,h} T^{-1}$. We need now a gauge covariant definition of determinant for a matrix transforming as $E_{g,h}$. We define

$$\det_G E_{g,h} = \epsilon^{g_1 ... g_p} E_{g_1 h_1} (R_{g_1 h_1} E_{g_2 h_2}) (R_{g_1 h_1 g_2 h_2} E_{g_3 h_3}) ...$$

$$\cdots (R_{g_1 h_1 g_2 h_2 ... g_{p-1} h_{p-1}} E_{g_p h_p}) \epsilon^{h_1 ... h_p},$$

where $\epsilon^{h_1 ... h_p}$ is the usual antisymmetric epsilon tensor (this definition can be generalized to the case of nonabelian $G$ [5]). Then one can prove that $\det_G E'_{g,h} = T \det_G E_{g,h} T^{-1}$. This determinant is also real and the discrete Born-Infeld action reads

$$A_{BI}^G = \int_G Tr \sqrt{\det_G (\delta_{g,h} + 1 + i F_{g,h})} vol = \sum_G Tr \sqrt{\det_G (\delta_{g,h} + 1 + i F_{g,h})}.$$

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