Star clusters with primordial binaries – II. Dynamical evolution of models in a tidal field

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ABSTRACT
We extend our analysis of the dynamical evolution of simple star cluster models, in order to provide comparison standards that will aid in interpreting the results of more complex realistic simulations. We augment our previous primordial-binary simulations by introducing a tidal field, and starting with King models of different central concentrations. We present the results of $N$-body calculations of the evolution of equal-mass models, starting with primordial binary fractions of 0–100 per cent, and $N$ values from 512 to 16 384. We also attempt to extrapolate some of our results to the larger number of particles that are necessary to model globular clusters.

We characterize the steady-state ‘deuterium main-sequence’ phase in which primordial binaries are depleted in the core in the process of ‘gravitationally burning’. In this phase, we find that the ratio of the core to half-mass radius, $r_c/r_h$, is similar to that measured for isolated systems. In addition to the generation of energy due to hardening and depletion of the primordial binary population, the overall evolution of the star clusters is driven by a competing process: the tidal dissolution of the system. If the primordial binary fraction is greater than 5 per cent and the total number of particles $N \geqslant 8192$, we find that primordial binaries are not fully depleted before tidal dissolution, in systems initially described by a King model with a self-consistent tidal field.

We compare our findings, obtained by means of direct $N$-body simulations but scaled, where possible, to larger $N$, with similar studies carried out by means of Monte Carlo methods. We find significant qualitative and quantitative differences with the results in the earlier paper. Some of these differences are explicable by the different treatment of the tidal field in the two studies. Others, however, confirm the conclusion of Fregeau et al. that the efficiency of binary burning in the earlier Monte Carlo runs was too high. There remain unexplained differences, however. In particular, the binary population appears to be depleted too quickly, even in the most recent Monte Carlo results.

Key words: stellar dynamics – methods: $N$-body simulations – globular clusters: general.

1 INTRODUCTION
Several observational surveys of globular clusters have highlighted the presence of binaries, which are found to constitute up to 50 per cent of the mass of the core (e.g. see Rubenstein & Bailyn 1997; Albow et al. 2001; Bellazzini et al. 2002; Pulone et al. 2003). Such a high number of objects is extremely unlikely to have been formed dynamically, as, at the density required to create a signif-
The evolution of star clusters with primordial binaries still presents open issues, mainly due to the intrinsic complexity of numerical simulations of a system where the local dynamical timescale may be many orders of magnitude smaller than the global relaxation timescale. (Hard binaries have an orbital period of a few hours, while the half-mass relaxation time $t_{\text{rh}}$ may be of the order of a few billion years.) Numerical simulations have thus been performed either using approximate algorithms such as Fokker Planck or Monte Carlo methods (Gao et al. 1991; Giersz & Spurzem 2000; Fregeau et al. 2003), which often have to rely on a knowledge of the relevant interaction cross-sections; or using direct $N$-body codes but employing only a modest number of particles of the order of $N \approx 10^{3}$ (McMillan, Hut & Makino 1990; Heggie & Aarseth 1992; McMillan & Hut 1994). Both approaches could potentially lead to misleading results. On the one hand, Monte Carlo methods rely on input physics that may not accurately reflect the realistic interactions in a densely populated and fluctuating core (e.g. the velocity distribution may be assumed to be isotropic), while interaction cross-sections in the case of unequal masses are not well known. On the other hand, extrapolation from direct simulations may prove to be non-trivial, as it is not clear how to scale the results obtained with the number of particles.

We have thus recently started (Heggie, Trenti & Hut 2006, hereafter Paper I) a programme to survey the basic properties of the evolution of an idealized star cluster with a population of primordial binaries, with the aim of placing a stepping stone between simplified models, such as those of Fokker–Planck type, and realistic complex numerical simulations, such as those recently performed by Portegies Zwart & McMillan (2004). In the first paper of our series, we have focused on the evolution of isolated, equal-mass models, with a primordial binary fraction in the range $0$–$100$ per cent and with a number of particles $N$ in the range from $256$ to $16384$. We have compared our results not only with those of the study carried out by Gao et al. (1991) (who used a Fokker Planck code), and also to some extent with Giersz & Spurzem (2000) and with Fregeau et al. (2003), but also with a theoretical model which predicts the evolution of the core radius (Vesperini & Chernoff 1994). We have shown that significant differences arise between direct $N$-body simulations and the Fokker Planck calculations of Gao et al. (1991). In particular, we found that, starting from the same initial conditions, the number of binaries in the core after the core collapse is significantly lower in our simulations than the number reported by Gao et al. (1991). This could have important consequences if one attempted to infer the original primordial binary fraction by observations of the current number of binaries in the cores of globular clusters. Another interesting finding is that the ratio of core radius $r_{c}$ to half-mass radius $r_{h}$ in the post-collapse phase seems not to follow the theoretical expectation from the work of Vesperini & Chernoff (1994), as, by varying the number of particles $N$, we observed a steeper decrease of $r_{c}/r_{h}$ than that expected $[x1/\log(0.4N)]$.

In this paper, we extend our simulations to include the effect of the tidal field of the galaxy. We consider the evolution of King models with different values of $N$, the concentration index $W_{0}$ and the primordial binary ratio $f$. We compare our direct simulations to the Monte Carlo simulations performed by Fregeau et al. (2003) and Fregeau, Gürkan & Rasio (2005), hereafter Fregeau et al. They studied the evolution of isolated and tidally truncated star clusters with a population of primordial binaries in the range $2$–$20$ per cent and a realistic particle number ($N = 3 \times 10^{5}$). Their two-dimensional Monte Carlo Method is expected to offer significant advantages over the one-dimensional code used by Gao et al. (1991) as the influence of anisotropy in the velocities can be taken into account, and it is unnecessary to assume that the distributions of energy and position (for binaries) are independent. The work of Fregeau et al. led to some important conclusions. For tidally limited clusters, they noted for the first time the possibility of an initial expansion of the core radius of the cluster in the presence of a significant population of primordial binaries, when starting from models with a high central concentration (i.e. King profiles with $W_{0} \gtrsim 7$). Fregeau et al. also showed that, in general, primordial binaries delay the deep core collapse phase that is observed in clusters with only single stars, so that the system can be tidally dissolved before collapsing. The comparison of our results with their simulations is mainly focused on fundamental quantities like the disruption rate of the binaries, the core radius and the dissolution timescale. It must be noted, however, that Fregeau et al. improved their code between the 2003 and 2005 papers by the inclusion of direct numerical integration of encounters. They note (Fregeau et al. 2005) that this alters some of their earlier results, as we mention at appropriate points in the present paper. In particular, Fregeau et al. (2005) note that binary burning was too efficient in their earlier models.

The paper is organized as follows. In the next Section, we present the parameters of our simulation data set and the numerical method used. In Section 3, we give a physical picture of the evolution of a star cluster with primordial binaries and tidal field that will guide our interpretation of the simulations. In Sections 4 and 5, we present our results for runs with a tidal field and a tidal cut-off, respectively, including comparison with the results of Fregeau et al. The last section of the paper provides a summary and discussions.

2 SIMULATIONS: SETUP AND ANALYSIS

2.1 Specification of the models

The models considered in this paper are tidally limited, with stars of equal mass $m$. The initial distribution is a King model with scaled central potential $W_{0} = 3, 7, 11$. Our standard models have a primordial binary ratio of $10$–$20$ per cent, although we have also performed some runs with fewer or no primordial binaries ($0, 2, 5$ per cent) as well as with higher ratios, $50$ and $100$ per cent. As in Paper I, we define the primordial binary fraction as

$$ f = n_{b}/(n_{s} + n_{b}), $$

with $n_{s}$ and $n_{b}$ being the number of singles and binaries, respectively. This implies that the fraction of the total mass in binary stars, in the case of equal masses, is larger in the following way:

$$ f_{m} = 2n_{b}/(n_{s} + 2n_{b}). $$

For example, for a run with $10$ per cent primordial binaries, $f = 10$ per cent whereas $f_{m} = 2/11 \sim 18$ per cent.

All our results are presented using standard units (Heggie & Mathieu 1986) in which

$$ G = M = -4E_{T} = 1, $$

where $G$ is the gravitational constant, $M$ is the total mass and $E_{T}$ is the total energy of the system of bound objects. In other words, $E_{T}$ does not include the internal binding energy of the binaries, only the kinetic energy of their centre-of-mass motion and the potential energy contribution where each binary is considered to be a point mass. We denote the corresponding unit of time by

$$ t_{a} = GM^{5/2}/(-4E_{T})^{3/2}, $$

in general. For the relaxation time, we use the following expression

$$ t_{\text{rh}} = 0.138N r_{h}^{3/2}/\ln(0.11N). $$
where \( r_h \) is the half-mass radius and \( N \) denotes the number of original objects, i.e. \( N = n_1 + n_2 \). When we discuss a run with \( N = 4096 \) and 50 per cent primordial binaries, we are dealing with 6144 stars. We have considered runs with \( N \) in the range 512–16 384.

Our runs follow the evolution of the system until the number of particles drops below 10. However, to avoid possible biases introduced by long-living small-N configurations, we define the dissolution time \( t_{\text{diss}} \), as the moment when 98 per cent of the initial mass of the system is no longer bound.

Following Fregeau et al., our initial distribution of the binaries’ binding energies is flat in log scale in the range from \( \epsilon_{\text{min}} \) to 133\( \epsilon_{\text{min}} \), with \( \epsilon_{\text{min}} = m \sigma_i(0)^2 \). Here, \( \sigma_i(0) \) is the initial central velocity dispersion and this choice, if applied to an isolated Plummer model, corresponds approximately to the standard range adopted in Paper I, i.e. to \( \epsilon_{\text{min}} \approx 5.1 \kappa T \) and 133\( \epsilon_{\text{min}} \approx 680 \kappa T \), where the mean kinetic energy per particle (over the entire system, with binaries replaced by their centres of mass) is \( 3 \kappa T/2 \).

To follow the evolution of the system, we have used Aarseth’s N BODY6 (Aarseth 2003), which has been slightly modified to provide additional runtime diagnostics on the spatial distributions of single and binary stars. A King model with \( W_0 = 7 \) required almost one month of CPU time on a Pentium 4, 3-GHz PC before tidal dissolution for \( N = 16 \) 384 and \( f = 10 \) per cent; the use of GRAPE hardware would not help significantly, as the computational bottleneck is in the treatment of the local dynamics of binaries (however, see Makino & Hut 1990, for a discussion on the scaling of the computational time with the number of particles in the presence of primordial binaries).

The galactic tidal field is treated in our standard sample of simulations as that due to a point mass, and the tidal force acting on each particle is computed using a linear approximation of the field. The tidal radius \( r_t \) is defined as (Spitzer 1987)

\[
r_t = \frac{M}{3M_{\text{Gal}}} R_{\text{Gal}},
\]

where \( M_{\text{Gal}} \) is the galaxy mass and \( R_{\text{Gal}} \) is the distance of the centre of the star cluster from the centre of the galaxy. We assume \( R_{\text{Gal}} = \text{constant} \), i.e. the cluster describes a circular orbit. Thus the equations of motion of a star in the cluster are

\[
\begin{align*}
x' + 2\omega y + (\kappa^2 - 4\omega^2)x &= -\phi, \\
y' - 2\omega x &= -\phi, \\
z' + \nu^2 z &= -\phi,
\end{align*}
\]

where the axes are orientated conventionally with respect to the direction to the centre of the galaxy and the galactic motion of the star cluster, \( \omega \) is its angular velocity, \( \kappa \) is its epicyclic frequency, \( \nu \) is its vertical frequency and the right-hand sides are the components of the gravitational acceleration due to the other stars. The value of \( r_t(0) \) is set at the beginning of the simulation to match the cut-off given by the adopted initial King model. In our standard runs, particles are removed from the system when they reach a distance from the density centre greater than twice the instantaneous tidal radius \( r_t \), and the value of \( r_t \) is updated during the simulation according to the decrease of \( M \), the total mass of the system.

Note that this method of treating the influence of the galactic field is different from the one adopted by Fregeau et al., as they imposed a radial ‘cut-off’, i.e. they did not consider the tidal force acting on individual particles, and simply removed particles whose distance exceeded \( r_t \) (see the description of their code in Joshi, Rasio & Portegies Zwart 2000). At variance with our tidal field treatment, the simulations with tidal cut-off do not include the Coriolis and centrifugal contributions to the particles acceleration. Therefore, the two approaches have important physical differences. The effects of this different treatment are discussed in Section 5, where we describe a subset of simulations (\( W_0 = 7 \) and \( W_0 = 3, f = 10 \) per cent, with \( N \) from 512 to 16 384) using the tidal radial cut-off treatment adopted by Fregeau et al.

Our results, unless otherwise noted, are presented by applying a triangular smoothing filter with width 2.5\( t_{\text{ah}}(0) \) (see fig. 3 and section 3.2 in Paper I for further details).

A summary of the runs is presented in Table 1.

2.2 Core radius: definition

In Paper I, we adopted the following definition for the core radius:

\[
r_c = \frac{\sqrt{\sum_{i=1}^{N} \rho_i^2 r_i^3}}{\sum_{i=1}^{N} \rho_i^3},
\]

where the sum is made over the particles within the half-mass radius and the density \( \rho_i \) around each particle is computed from the distance to the fifth nearest neighbour (Casertano & Hut 1985). This definition presents some systematic differences (see Fig. 1) from the one adopted by Fregeau et al.:

\[
r_c = \frac{3\sigma_r^2}{4\pi G \rho_0},
\]

where \( \sigma_r^2 \) is the central velocity dispersion (mass weighted as in Vesperini & Chernoff 1994) and \( \rho_0 \) is the central density (Spitzer 1987). For a proper comparison between our set of simulations and the runs discussed by Fregeau et al., it is important to refer to the same quantities, and in this paper we adopt as definition for the core radius \( r_c \) the choice taken by Fregeau et al., i.e. equation (7).

We estimated the velocity dispersion \( \sigma_r^2 \) among all the particles within the radius defined by equation (6), while \( \rho_0 \) has been obtained from the Lagrangian radius containing 1 per cent of the instantaneous mass of the system. For the purpose of comparison with Paper I, we also report in Table 3 the value of the alternative definition for the core radius, \( r_c^\star \), at the end of the core collapse.

2.3 Time-scales

In the presence of primordial binaries and a tidal field, several dynamical processes operate, each with its own time-scale. Here, we summarize them as follows:

(i) _Dynamical time_ \( t_d \) (cf. equation 2): within a small numerical factor, this is the typical time-scale for a particle to cross the system. It also represents the time-scale on which approximate dynamical equilibrium is established in the case of initial conditions starting out of equilibrium. (Note, however, that all our simulations start from approximate dynamical equilibrium.)

(ii) _Half-mass relaxation time_ \( t_{\text{bb}} \) (cf. equation 3): this is the time-scale for energy diffusion in phase space, i.e. for establishing thermal equilibrium. It is of the order of \( N/\log (0.11 \ N) \) \( t_d \).

(iii) _Total mass relaxation time_ \( t_{\text{tot}} \): this is the time-scale for relaxation of the cluster as a whole and is defined using the tidal radius as characteristic length-scale for estimating the relaxation time. \( t_{\text{tot}} \) is proportional to \( t_{\text{bb}} \) (defined above): \( t_{\text{tot}}/t_{\text{bb}} \propto (r_t/r_h)^{3/2} \).

(iv) _Time for initial core contraction_ \( t_{\text{cc}} \): this is the time-scale on which the core of the cluster contracts (or expands) to reach a quasi-steady configuration in which the energy released in the core fuels the expansion of the half-mass radius. The typical value of \( t_{\text{cc}} \) is of the order of 10\( t_{\text{ah}}(0) \).
(v) Tidal dissolution time ($t_{\text{dis}}$): time needed to lose 98 per cent of the initial mass of the system due to escape across the tidal boundary. Depending on the tidal field strength, the initial concentration of the system and the initial number of particles, this time-scale ranges from a few relaxation times to a few tens of relaxation times. (vi) Tidal half-mass dissolution time ($t_{\text{dis,bu}}$): similar to $t_{\text{dis}}$, but defined as the time needed for 50 per cent of the mass of the system to escape. (vii) Binary burning time ($t_{\text{burn}}$): time needed to deplete 80 per cent of the initial number of binaries. Around this time,
3 THE PHYSICAL PICTURE: PRIMORDIAL BINARIES AND THE TIDAL FIELD

The evolution of an isolated star cluster made of equal mass stars is driven initially by the gravitational instability. Two-body encounters drive a heat flow from the central region of the star cluster, which behaves like a self-gravitating system with a specific heat, to the halo. This triggers a thermal ‘collapse’ on the time-scale of the heat flow process.

The ‘collapse’ phase lasts until an efficient form of energy generation in the centre can stop the process by providing an energy production rate equal to the energy-loss rate by two-body relaxation from the region of the core. In the case of systems consisting of single stars only, the collapse is stopped only after one or more binaries are formed due to three-body encounters in the dense central region. The collapse typically takes a time $t_{\text{cc}}$ of the order of $15 t_{\text{hr}}(0)$, where $t_{\text{hr}}(0)$ is the initial half-mass relaxation time. At this point, i.e. at the end of core collapse (sometimes called ‘core bounce’), the central density can exceed $10^4$ times the average density inside the half-mass radius $r_h$ (for $N = 16,384$, or far more for larger $N$). After core bounce, the generation of energy in the core is accompanied by a steady expansion of the half-mass radius. If $N$ is large enough, however, binary activity in the core may cause a temporary temperature inversion which drives the gravitational instability in reverse. This suppresses binary activity, and further collapses may recur, in a succession of ‘gravothermal oscillations’ (Sugimoto & Bettwieser 1983).

This picture is strongly modified when a population of primordial binaries is present, as the generation of energy due to existing binaries is more efficient than that due to dynamically formed pairs (see Section 2 of Paper I). For this reason, if the primordial binary fraction is above a few per cent, the initial contraction of the core is more gentle, and it is halted at a lower core density (and a larger core radius). The time-scale for initial core contraction ($t_{\text{cc}}$) remains of the same order of magnitude as in the case where no binaries are initially present (see fig. 17 in Paper I). From then until the time when the population of binaries becomes heavily depleted, the evolution of the system proceeds with an almost constant ratio of core radius to half-mass radius. This phase lasts for a time proportional to the initial binary ratio, and thereafter a phase of gravothermal oscillations sets in, much as for the case where there are no primordial binaries (see figs 5 and 6 in Paper I). For an isolated system, provided that $N$ is large enough (i.e. $N \gtrsim 10^4$), the onset of gravothermal oscillations occurs when the mass of binaries has fallen to roughly 10–15 per cent of its initial value (Fregeau et al. 2003, fig. 4; Paper I, figs 5 and 7). These are empirical results, however, and may change for primordial binary fractions smaller than the lowest used in these simulations, i.e. 2 per cent.

The presence of a tidal field introduces a new time-scale for the evolution of the system, i.e. $t_{\text{dis}}$. In the presence of a tidal field, the star cluster steadily loses mass at a rate that is almost independent of the properties of the central parts of the system, where binaries accumulate and act as an energy source for the system. In principle, the tidal dissolution can be so fast ($t_{\text{dis}} < t_{\text{hr}}$) that the system does not have enough time to deplete its reservoir of primordial binaries before dissolving. On the other hand, if the initial primordial binary fraction is low or if the tidal field is weak, then the system can burn almost all of the binaries before being dissolved, and it then undergoes a phase of gravothermal oscillations, as already described for an isolated system, provided that $N$ is large enough.

In an isolated system with an initial binary ratio of 10 per cent, approximately 100 $t_{\text{hr}}(0)$ is the typical time-scale $t_{\text{hr}}$ required for disrupting 80 per cent of the binary population considered in our runs (e.g. see fig. 7 in Paper I). Note, however, that the range of binding energy in these runs is $5–700\, kT$, where $3/2\, kN\epsilon$ is the total kinetic energy of the cluster (when the binaries are replaced by their barycentres). The time to disrupt most of the binaries is certainly dependent on the initial distribution.

The time to disrupt most of the binaries can also be estimated with the following argument. A primordial binary in the range of $5–700\, kT$ gives off, on average, around $100\, kT$ before leaving the system (or being dissolved; see e.g. Hut et al. 1992). Thus if we start with a 10 per cent primordial binary population each single star receives around $10\, kT$. The energy loss at the half-mass radius due to the temperature gradient is of the order of $0.1\, kT$ per star per relaxation time, so we have to wait an order of 100 relaxation times before depleting the binaries. The same argument suggests that the time taken to burn most of the binaries is proportional to $f$.

4 TIDAL FIELD RUNS: RESULTS AND COMPARISONS

In this section, we discuss the properties of our simulations, our main purpose being to set out the essential empirical facts about the evolution of systems with primordial binaries in a galactic tidal field. We shall also compare our results with those obtained by Fregeau et al., though they treated the tidal field as a cut-off. To elucidate this comparison, we were motivated to carry out further $N$-body runs with the same treatment (i.e. a cut-off). We defer presentation of those calculations to the next section, however; in the present section
Table 2. Comparative results on the dissolution time-scale \( t_{\text{dis}}/t_{\text{rh}(0)} \) for simulations with \( f = 10 \) per cent.

\[
\begin{array}{cccc}
W_0 & \text{THH field} & \text{THH cut-off} & \text{Fregeau et al. cut-off} \\
3 & 5 & 15 & 13 \\
7 & 18 & 20–40 & \geq 38 \\
11 & 11 & \text{N/A} & 27 \\
\end{array}
\]

Notes. The values for \( t_{\text{dis}} \) for our runs (THH) have been extrapolated to \( N = 3 \times 10^5 \) from \( N = 16384 \) (see discussion in the text). In the second column, we give \( t_{\text{dis}} \) for our standard tidal field runs, extrapolated by using equation (8).

In the third column, we report our results from runs with a tidal cut-off, with constant extrapolation for \( W_0 = 3 \) and a range of values for \( W_0 = 7 \) (see Section 5). The fourth column contains the results from Fregeau et al.

we consider only runs that include our most realistic treatment of the tidal force (equation 5).

4.1 Total mass and dissolution time-scale

Our results are summarized by the values of \( t_{\text{dis}} \) in Tables 1 and 2, and details for a subset of the runs are presented in the top panel in Figs 2–5; these are intended also to facilitate comparison with corresponding plots in Fregeau et al., as specified in the captions. The evolution of the total mass in our different runs depends on three initial parameters: (i) the initial fraction of primordial binaries, \( f \); (ii) the scaled central potential of the initial King model, \( W_0 \); and (iii) the initial number of objects, \( N \). We consider the dependence on each in turn.

4.1.1 Dependence on \( f \)

The initial fraction of primordial binaries has only a small influence on the rate of mass loss of the system and on \( t_{\text{dis}} \). The dependence is,

![Figure 2](https://example.com/fig2.png)

Figure 2. Time dependence of the total mass and of the mass in binaries (upper panel) and of the half-mass and core radius (bottom panel). The dotted line is the half-mass radius for singles while the dashed line in the lower panel is the half-mass radius for binaries. The lowest curve is the core radius of the system (in units of the initial half-mass radius). The simulation has been performed with 16 384 particles and 10 per cent of binaries starting from \( W_0 = 3 \). It is the equivalent of fig. 13 in Fregeau et al. (2003).

![Figure 3](https://example.com/fig3.png)

Figure 3. Same as Fig. 2 for a simulation starting from \( W_0 = 7 \) with 16 384 particles and 10 per cent of binaries. It is the equivalent of fig. 10 in Fregeau et al. (2003).

![Figure 4](https://example.com/fig4.png)

Figure 4. Same as Fig. 2 for a simulation starting from \( W_0 = 7 \) with 16 384 particles and 20 per cent of binaries. It is the equivalent of fig. 11 in Fregeau et al. (2003) or fig. 3 in Fregeau et al. (2005). The results from a new, unpublished run by Fregeau (Fregeau, private communication) are overplotted in red.

however, systematic, as can be seen from Fig. 6. (Though this figure shows runs with \( W_0 = 7 \), the trend shown is representative of all concentration indexes that we have studied.) The simulation with the longest dissolution time is the one starting with 100 per cent of primordial binaries. This may not be surprising, as this run has the highest total number of stars and so, after disruption of a given proportion of binaries, it has the longest relaxation time. On the other hand, it is closely followed by the simulations with low binary content (2 and 5 per cent) and then by the run with single stars only, which is dissolved in 90 per cent of the time needed in the 100 per cent case. This may be due to the fact that simulations with \( f \lesssim 5 \) per cent deplete their primordial binaries before dissolution...
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Figure 5. Same as Fig. 2 for a simulation starting from \( W_0 = 11 \) with 16 384 particles and 10 per cent of binaries. It is the equivalent of fig. 12 in Fregeau et al. (2003).

Figure 6. Evolution of the total mass for a King models with \( W_0 = 7 \) and different ratios of primordial binaries. The simulations have been performed with 4096 particles.

\( (t_{\text{bb}} < t_{\text{dis}}) \); this leads to a deep core collapse, which creates a strongly bound core which is rather resilient to dissolution. Curiously, simulations with 20 per cent of primordial binaries are subject to the fastest tidal dissolution, and this could be due to effects related to mass segregation, which is absent if \( f = 0 \) or 1, and which tends to expel single stars.

4.1.2 Dependence on \( W_0 \)

Consider, as an example, runs with \( N = 4096 \) and \( f = 10 \) per cent. A low concentration King model with \( W_0 = 3 \) is tidally dissolved in approximately \( 13 t_{\text{dd}}(0) \) (Table 1), while the system can survive for approximately \( 50 t_{\text{dd}}(0) \) for \( W_0 = 7 \); but \( W_0 = 11 \) is intermediate between the two \( (t_{\text{dd}} \approx 25 t_{\text{dd}}(0)) \). This non-monotonic dependence on concentration index \( W_0 \) is a general feature, independent of the number of particles. It may be understood in terms of the non-monotonic dependence of the tidal radius on \( W_0 \), at fixed total mass and total energy (Table 1; see also fig. 8.3 in Heggie & Hut 2003). (Another way of looking at this is that, in the units we adopt, the strength of the tidal field varies non-monotonically with \( W_0 \).) Though the difference in \( r_t \) between the cases \( W_0 = 7 \) and 11 may not seem great, two further factors may be relevant: (i) \( r_t \) also increases slightly between \( W_0 = 7 \) and 11 (Table 3), which would increase the vulnerability of the more concentrated system, even if the tidal field were the same; (ii) high-concentration models begin with a short period of core expansion (Fig. 5), and the associated burst of energy generation enhances tidal overflow.

If the tidal dissolution time is expressed in units of the total-mass relaxation time \( (t_{\text{rt}}) \), then the differences between the different King models are greatly reduced. By accounting for the factor \( (r_t/r_h)^3/2 \), so that the time is in units of \( t_{\text{rt}} \), the difference in tidal dissolution time between the various King models considered here is within 20 per cent for \( N = 4096 \) and within 10 per cent for \( N = 16 384 \).

4.1.3 Dependence on \( N \)

In our set of simulations, we observe a marked dependence of the dissolution time-scale \( t_{\text{dd}} \) on the number of particles used (see Table 1 and Fig. 7). This is not surprising, as a similar effect is well known in simulations with the tidal force modelled as in our

Table 3. Initial tidal and half-mass radius \( r_t(0), r_h(0) \) (in natural units) as functions of the scaled central potential \( W_0 \).

| \( W_0 \) | \( r_t(0) \) | \( r_h(0) \) |
|---------|---------|---------|
| 3       | 3.15    | 0.84    |
| 7       | 7.00    | 0.81    |
| 11      | 6.62    | 1.01    |
equation (5), though using single particles only: the time for the loss of half of the mass of a cluster is approximately proportional to $t_{\text{ dissolution}}(0)^{1/4}$, i.e.

$$t_{\text{ dissolution}} \propto \left( \frac{\ln (0.11N)}{N} \right)^{1/4} t_{\text{ dissolution}}(0)$$

(Baumgardt 2001). Besides empirical support from $N$-body simulations, this formula can be understood on the basis of the combined effect of the tidal field and of two-body relaxation (Baumgardt 2001). We find (Fig. 7) that it also provides a reasonable fit to the dissolution time $t_{\text{ dissolution}}$ of models with primordial binaries (for the parameters specified in the caption of the figure), though the observed dependence on $N$ is slightly stronger. Therefore, we can take advantage of equation (8) for the approximate extrapolation of results on the dissolution time $t_{\text{ dissolution}}$ to larger numbers of particles $N$. Note, however, that equation (8) is derived in Baumgardt (2001) using simulations with $N \leq 16384$; it is expected to be valid up to $N \lesssim 10^7$ at most, as for larger $N$ the equation eventually leads to $t_{\text{ dissolution}} < t_{\text{ dissolution}}(0)$, a clearly unphysical result.

Assuming, then, that this scaling applies to the dissolution time $t_{\text{ dissolution}}$, we can compare our results with those in Fregeau et al. If we extrapolate to $N = 3 \times 10^5$, we find that $t_{\text{ dissolution}}$ decreases by a factor of $\approx 2.6$ from the results obtained for the simulations with $N = 4096$ and by a factor of $\approx 1.9$ from the runs with $N = 16384$. This would mean that a simulation starting from $W_0 = 7$ with 10 per cent binaries and $N = 3 \times 10^5$ would take $\approx 18 t_{\text{ dissolution}}(0)$ to dissolve. For comparison, Fregeau et al. measure $t_{\text{ dissolution}} \gtrsim 38 t_{\text{ dissolution}}(0)^{1/2}$ for these initial conditions. This and other comparisons are given in Table 2 (second and fourth columns). An understanding of such differences requires consideration of the treatment of the tide, and so we defer further discussion to Section 5.

4.2 Core and half-mass radius

In the presence of a tidal field, the system evolves towards conditions in the centre very similar to those that we observed for isolated systems. The core radius evolves, usually in a few $t_{\text{ 1/2}}$, to reach a ratio $r_c/r_h$ that is close to that attained at the end of core collapse during the evolution of isolated Plummer models (see section 4.1 of Paper I) with a similar fraction of primordial binaries. As is depicted in Fig. 8, the evolution of the core to half-mass radius, after a first adjustment phase, is largely independent of the initial conditions considered. Thereafter this ratio is almost constant, as long as there are binaries to burn or until tidal dissolution has reduced the total mass below 10 per cent of the initial value.

We discuss the value of the core radius in further detail below, but summarize here the behaviour of the half-mass radius. This depends on the strength of the tidal field, which depends on $W_0$ (Section 4.1). For $W_0 = 7$, $r_h$ remains almost constant, up to the final stages of the life of the system, when it eventually falls to zero. For stronger tidal fields (models with $W_0 = 3$ and 11), the half-mass radius decreases steadily. Our interpretation is that, for $W_0 = 7$, the tendency towards expansion (due to the energy generated in the core) is balanced by the mass loss due to the tidal field, while the tide dominates in the other two cases.

Now we return to the evolution of the core radius, and consider first the initial phase of contraction or expansion. The evolution in the first few relaxation times depends on how the initial value of $r_c$ compares with the quasi-equilibrium value attained in the intermediate steady binary burning phase. Thus runs starting initially with a relatively big core, such as $W_0 = 3$ [$r_c(0) \approx 0.6 r_h(0)$, Fig. 2] have a rather deep collapse; after about $t_{\text{ 1/2}}(0)$ in our simulation with $N = 16384$ and $f = 10$ per cent, we measure $r_c/r_h(0) \approx 0.05$. For $W_0 = 7$, $f = 10$ per cent and $N = 16384$ (see Fig. 3), the duration of the initial contraction is shorter, and thereafter the core radius is somewhat larger. On the other hand, runs that start with a small, concentrated core ($W_0 = 11$), have a first rapid expansion of the core radius (see also Fig. 5) up to $r_c/r_h(0) \approx 0.09$ for our $N = 16384$ run with $f = 10$ per cent. This can be understood in terms of the energy balance of the core: if the density is too high, the binary heating exceeds the cooling by heat transport to the halo, so that the core has to expand to lower density to reduce the rate of energy generation. Incidentally, though we have focused on numerical values for runs with $f = 10$ per cent, we find that the effect of primordial binaries saturates by around $f = 10$ per cent (see Fig. 9), much as we have noted in the evolution of isolated clusters in Paper I (see also Vesperini & Chernoff 1994).

Now we turn to the value of the core radius during the phase of steady binary burning, and, in particular, its dependence on the number of particles. Our results are summarized in Fig. 10. At variance with Paper I, here the Vesperini & Chernoff (1994) model provides an excellent fit to the dependence on $N$ of the ratio of the core to half-mass radius, provided that in the Coulomb logarithm the value 0.11 is used for the coefficient of $N$. Thus the formula used for the fit is (see Vesperini & Chernoff 1994, and Paper I)

$$\frac{r_c}{r_h} = \frac{\alpha}{\log_{10}(0.11N)} \left( \frac{\phi_b(1 - \phi_b)\mu_{r_h} + \phi_c^2 \mu_{r_c}}{(1 + \phi_b)^2} \right),$$

where the various quantities are defined as in Paper I. Thus the quantity $\alpha$ is a parameter depending on $v_c/v_h$, where $v_c$ is

![Figure 8. Evolution of the ratio of the half-mass to core radius (upper panel) and of the concentration parameter $c = \log_{10}(r_c/r_h)$ for different King models with $f = 10$ and 20 per cent. The solid line refers to simulations starting from $W_0 = 7$, the dotted line to $W_0 = 11$ and the dashed line to $W_0 = 3$; the number of particles used is 16384. The evolution quickly decreases differences due to initial conditions, and during binary burning $c$ and $r_c/r_h$ evolve very similarly. This figure can be directly compared with fig. 16 in Fregeau et al. Note that we kept the same range in the $y$-axis to highlight the differences between our runs and theirs.](https://academic.oup.com/mnras/article-abstract/374/1/344/961831)
Finally, $\mu_{bb}$ and $\mu_{bs}$ are coefficients for the efficiency of binary–single and binary–binary burning and depend on the distribution of binding energy of the binaries. They have been computed as in Paper I assuming a flat distribution in the logarithm of the binding energy from $12\, kT$ to $300\, kT$. $\alpha$, $\mu_{bb}$ and $\mu_{bs}$ do not significantly depend on the number of particles $N$ (see section 5.1.2 in Paper I).

With regard to the core radius, there are many quantitative and even some qualitative differences between our results and the earlier results of Fregeau et al. (2003). However, they point out (Fregeau et al. 2005) that the core radius is one parameter which has turned out to behave rather differently following the code improvements summarized in their later paper. Indeed the one case for which a direct comparison with their most recent results is possible is shown in Fig. 4 ($W_0 = 7, f = 20$ per cent). Both models (our $N$-body model and their Monte Carlo model) show an initial contraction followed by a phase of steady binary burning. The comparison with the new results by Fregeau et al. (2005) is quantitatively satisfactory, except that, in the phase of steady binary burning, our core radius is about twice as large as theirs. However, it is known that the core radius in this phase is $N$-dependent (see above and Paper I), and the sense of the difference is consistent with this.

As already mentioned in Section 3, the late evolution of the core radius depends on whether the binary population can be heavily depleted by the time of dissolution, $t_{\text{dis}}$. The evolution of the binary population is discussed in the following section, and so we defer further discussion until then.

### 4.3 Evolution of the primordial binary population

The destruction of the primordial binary population is depicted in the top panel of Figs 2–5. The number of binaries decreases due to both destruction of the softer pairs in the centre of the system and the ejection of stars across the tidal boundary. The changing relative contribution of these two processes can be inferred from Fig. 11, upper panel. At first the ratio of binaries to singles decreases, as at the beginning of the simulation relatively many soft binaries (i.e. those
with $E_b \lesssim 15kT$ are present. However, at later times the relative number of binaries to singles starts to rise, as most of the surviving binaries are hard to disrupt in three- or four-body encounters. In addition, tidal ejection is more probable for singles, as binaries have sunk towards the core of the cluster by mass segregation (see bottom panel in Figs 2–5 and Fig. 12).

There are some quantitative differences between our results and those of Fregeau et al., even after the recent improvements to their technique. For example they point out (Fregeau et al. 2005) that, for a model with $W_0 = 7$ and $f = 20$ per cent, the binary fraction shrinks to about 10 per cent before recovering to about its initial value shortly before $t_{\text{disk}}$. This differs considerably from our result in Fig. 11 (upper panel). Another example is illustrated in the upper panel of Fig. 4, where it is clear that the disruption of binaries in the most recent available run by Fregeau et al. appears to be rather faster than in our run starting from similar initial conditions.

Now we consider models in which the population of binaries can be strongly depleted before dissolution of the cluster, i.e. $t_{\text{bin}} < t_{\text{disk}}$. We have already remarked (Section 4.1) that $t_{\text{bin}}$ increases as $N$ decreases, and is longest for $W_0 = 7$ (out of the three values we adopted). It has also been argued (Section 3) that the time taken to burn most binaries is roughly proportional to $f$. Based on these considerations, it may be expected that the only runs that may burn most of their binaries within the lifetime of the system are those in which $f$ or $N$ is sufficiently small, and that intermediate values of $W_0$ are most favourable. Indeed, we found that the only runs which did so were those with $W_0 = 7$ and $N \lesssim 4096$ and $f \lesssim 5$ per cent. (Note, however, that $W_0 = 7$ is the only value of $W_0$ for which we carried out runs with $N < 4096$, see Table 2.) At the point where the binaries become heavily depleted, there is a deep core contraction, similar to what happens in runs with single stars only; however, here $N$ is too low for gravothermal oscillations to occur. In Fig. 13, we depict the evolution of a $W_0 = 7$ model starting from $f = 5$ per cent and $N = 4096$: by about $30t_{\text{bin}}(0)$ the fraction of binaries has fallen below 20 per cent of its initial value and the core radius contracts sharply.

Next we comment in particular on the evolution of the binary population in the core. At the end of our simulations starting from $f = 10–20$ per cent, we find that the fraction of binaries in the core has increased by at least a factor of 4. In general, the density of binaries in the core is in broad agreement with the numbers observed in Paper I and does not depend significantly on the details of the initial conditions if $f$ is in this range (see bottom panel of Fig. 11, where runs starting from $W_0 = 7$ and $W_0 = 11$ show a rather similar behaviour). However, there are differences in detail in the time evolution. In an isolated cluster, the proportion of binaries in the core, after the sharp initial increase caused by mass segregation, declines slowly; in these tidally limited models, on the other hand, there is a slow increase. The $W_0 = 3$ King model is an extreme example, as there is an enhanced rate of ejection of singles relative to the rate of disruption and ejection of binaries, because of the relatively small tidal radius; in this situation, the binary fraction in the core increases rapidly throughout the evolution.

The evolution of the internal binding energy (see Fig. 14) is similar to that observed for isolated systems, with a preferential decrease in the number of less bound binaries. Furthermore, as expected, there is a significant correlation of the internal binding energy with the radius (see Fig. 15). After approximately $15t_{\text{bin}}(0)$ the survival probability in the core for binaries with binding energy below $\approx 20kT$ is low, as they have been either destroyed or hardened. We thus see that softer binaries are mainly present around the half-mass radius and in the halo, while the core is dominated by hard binaries (with occasionally some short-lived low-energy binaries that have been formed dynamically). At later stages of the evolution (see bottom right-hand panel of Fig. 15), the softer binaries have almost completely disappeared.

This picture is in qualitative agreement with the results already known in the literature (see e.g. McMillan & Hut 1994; Giersz & Spurzem 2000 and Fregeau et al.). As our resolution is limited by the modest number of binaries that are present in our runs, it is hard to attempt any quantitative comparison.

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**Figure 12.** Evolution of the distribution of binary radius in units of the instantaneous core radius. The ordinate is the number of binaries per bin, which are of length 0.5 in log $r$. Curves have unit normalization, and are shown for $t/t_{\text{bin}}(0) = 0, 8, 16, 24$. The dashed line is the position of the half-mass radius. The simulation has been performed with 16,384 particles and 20 per cent of binaries starting from a King model with $W_0 = 7$.

**Figure 13.** Same as Fig. 2 for a simulation starting from $W_0 = 7$ with 4096 particles and 5 per cent of binaries.
5 TIDAL CUT-OFF RUNS: RESULTS AND COMPARISON

It was pointed out in Section 4.1.3 that our result on the lifetime differed significantly (at least in the one case considered) from that of Fregeau et al. This conclusion, however, required a knowledge of the $N$ dependence of the lifetime, and it is known that this in turn depends on the treatment of the tide, which differs between the work of Fregeau et al. and the $N$-body simulations which we have presented so far. For the case of a tidal cut-off, as adopted by Fregeau et al., it was found by Baumgardt (2001) that the half-mass dissolution time-scale $t_{\text{Half}}$ (expressed in relaxation times) does not depend on the number of particles of the system (as we assumed for the purpose of our comparison) if $N \gtrsim 4096$. Note, however, that Baumgardt (2001) used a fixed tidal radius and, as we have mentioned, measured the time-scale of mass loss by the half-mass time $t_{\text{Half}}$, whereas our discussion has focused on the time for loss of 98 per cent of the mass, $t_{\text{dis}}$.

Because of these complications, for a better comparison with Fregeau et al. we have run a second series of simulations, in which the tidal field we have used hitherto has been replaced by a tidal cut-off. We have ensured that the cut-off radius decreases in the correct manner (equation 4) as the mass inside this radius decreases, exactly as adopted by Fregeau et al. The initial parameters of these models are $W_0 = 7$ and $W_0 = 3$, $f = 10$ per cent, with $N$ in the range 512–16384.

For the series starting from $W_0 = 3$ with $f = 10$ per cent (i.e. with a strong tidal field; Section 4.1.2), we do not observe a significant $N$ dependence of the half-mass dissolution time $t_{\text{Half}}$, except for a modest ($\approx 10$ per cent) variation from $N = 4096$ to 8192. This is consistent with the experiments (without primordial binaries) by Baumgardt (2001), despite his use of a constant tidal field. Moreover, it is also consistent with unpublished results by one of us using the initial conditions of the collaborative experiment summarized in Heggie (2003), except for replacement of the tidal field by a tidal cut-off; in this case, the cut-off was evolved in the correct manner as mass was lost. For the $W_0 = 3$ runs with tidal cut-off, the agreement with the dissolution time $t_{\text{dis}}$ of Fregeau et al. (2003) is good: we extrapolate $t_{\text{dis}} \approx 15 t_{\text{rh}}(0)$ and they have $t_{\text{dis}} = 13 t_{\text{rh}}(0)$.

By contrast, for the series with $W_0 = 7$ and $f = 10$ per cent and a cut-off, we observe an $N$ dependence similar to that found when
a tidal field is used, as in our standard runs (see Fig. 16). On the other hand, it is not known why the N dependence should differ for the cases $W_0 = 3$ and 7 when a cut-off is used. One possibility is that it is a 'small-N' effect which is more important in a system with a small core; if so, the N dependence of the dissolution time $t_{\text{dis}}$ will eventually flatten out for larger N. The presence of primordial binaries does not seem to be the issue; we have run a few simulations with a tidal cut-off, starting from a $W_0 = 7$ model and no binaries, and find that the half-mass dissolution time $t_{\text{dis}}$ is almost equal to that measured when binaries are present. Tankinawa & Fukushige (2005) have also observed differences (in the scaling of the half-mass time $t_{\text{dis}}$ with N) between models with $W_0 = 3$ and 7, though their models adopted a tidal field, and the potential was softened. They tentatively attribute the difference to the time taken for a particle to escape.

Whatever the explanation, this (unexpected) N dependence of $t_{\text{dis}}$ in our tidal cut-off runs with $W_0 = 7$ makes the extrapolation of our results obtained with the tidal cut-off prescription highly uncertain. While the dissolution time of our model with $W_0 = 7$, $f = 10$ per cent and $N = 16 384$ in a tidal cut-off is $t_{\text{dis}} \approx 41.2 t_{\text{rh}}(0)$ (Table 3), the inferred dissolution time for a $W_0 = 7$, $f = 10$ per cent model with $N = 3 \times 10^5$ may be in the range of 20–40 $t_{\text{rh}}(0)$ (Table 2, Column 3), depending on whether $t_{\text{rh}}$ is proportional to $t_{\text{rh}}(0)$ or scales in accordance with equation (8). If the actual value is close to the upper limit, then the tidal dissolution time $t_{\text{dis}}$ in our $W_0 = 7$ cut-off models would compare marginally well with Fregeau et al., as they find $t_{\text{dis}} \geq 38$. Furthermore, recent improvements to their code appear to have reduced the lifetime slightly for a model with $W_0 = 7$, $f = 20$ per cent, and a similar reduction for $f = 10$ per cent would improve the agreement between our N-body and their Monte Carlo results.

We close this section with a brief comparison between other properties of our runs with a tidal field and those with a tidal cut-off. Because of the limited scope of our runs with a tidal cut-off, we are concerned less here with general trends; but these results will be useful for comparison with simplified treatments in which it is necessary to use a cut-off, such as Fokker–Planck models; they also illustrate the kinds of results which may be sensitive to the manner in which the tidal boundary conditions are treated.

For our run with $W_0 = 3$, $f = 10$ per cent, $N = 16 384$ and a cut-off (Fig. 17), the initial contraction of the core radius is slower than that observed in the run with a tidal field (see Fig. 2): after $8 t_{\text{rh}}(0), r_c$ is approximately twice as large in the cut-off run. However, just before tidal dissolution the values of the core radius in the two runs are quite similar. Actually, if the time is normalized by the dissolution time $t_{\text{dis}}$, the evolution of $r_c$ looks very similar, except at the very last moment, where $r_c$ decreases for the run with a cut-off.

By contrast, the run with $W_0 = 7$, $f = 10$ per cent, $N = 16 384$ and a cut-off (Fig. 18) is much more similar to the corresponding run with a tidal field (Fig. 3). The dissolution time $t_{\text{dis}}$ scales for the two tidal treatments differ by only $\approx 25$ per cent, and the core radius evolves in a similar way. However, the half-mass radius for the run with a cut-off initially increases in much the same way as in the run reported by Fregeau et al., while in our run with a tidal field it remains almost constant.

6 DISCUSSION AND CONCLUSIONS

In this paper, we have continued our investigation of the dynamical evolution of stellar systems with primordial binaries. In Paper I, we considered simple isolated models with equal mass stars and a primordial binary population in the range 0–100 per cent. Then the main parameters that control the evolution of the system are the primordial binary fraction $f$ and the number of bodies $N$. However, for $f \gtrsim 10$ per cent the efficiency of energy generation due to three- and four-body encounters saturates, so that little difference was observed in the size of the core radius for runs starting with the same number of particles and $f \gtrsim 10$ per cent. This behaviour had already been predicted by the theoretical model of Vesperini & Chernoff (1994), and our runs confirmed this expectation. Interestingly, however, the scaling of the core radius with the number of particles was not quite the same as that predicted by Vesperini & Chernoff (1994).

Here, we have added an important new ingredient to our simulations: a tidal field. The presence of a tidal field introduces a new time-scale in the simulation, the time needed for the dissolution of...
the system, which is usually shorter than the time needed to deplete
the primordial binary population for simulations starting from \( f \gtrsim 
10 \) per cent and \( N \gtrsim 4096 \). We have considered runs starting
from King models with initial central potential \( W_0 = 3.7, 11 \) (and thus a
different intensity of the tidal field, in the units adopted) with 0–100
per cent primordial binaries, and \( N \) from 512 to 16 384.

By comparison with the results of Paper I, our Fig. 8 clearly
confirms that the evolution of the core to half-mass radius proceeds
much as for isolated models, after an initial transient; the models
starting with \( W_0 = 7 \) and 11 evolve very similarly, for initial binary
fractions of \( f = 10 \) and 20 per cent. The King model with \( W_0 = 3 \),
however, nicely illustrates the effects of a strong tidal field: the
evolution of \( r_c/r_h \) tries to go towards the steady value reached by the
other runs with higher \( W_0 \), but the tidal field manages to dissolve
the system just when this common \( r_c/r_h \) value is about to be attained.

One objective of this paper was also to compare the results
obtained by means of our direct \( N \)-body simulations with the outcome
of simulations able to employ realistic number of particles by using
approximate methods, like Monte Carlo and/or Fokker Planck
approaches. Just as in Paper I, we compared our work with the mile-
stone study of Gao et al. (1991), here we took as comparison the
recent work carried out by Fregeau et al. There are considerable
differences with the results reported in Fregeau et al. (2003), but
they have since reported that several results change significantly
following an improved treatment of binary interactions (Fregeau
et al. 2005). As a result of these improvements, they have discov-
ered that the emission of energy in encounters with binaries was too
high in the earlier models. Therefore, we have focused our compar-
ison on one case which they illustrated in some detail in their later
paper. While it is promising to report that most differences have
been cleared up by the improvements, there are exceptions. One is
that the binaries still appear to be depleted too rapidly, even in the
most recent runs (Fig. 11; cf. Fregeau et al. 2005, fig. 3). There
remain uncertainties also about the lifetime, but this depends on
perplexing problems in the scaling with \( N \): we find evidence that the
tidal cut-off scaling derived by Baumgardt (2001) for single stars
with a fixed cut-off radius does not seem to be applicable to the self-consistent prescription of a cut-off radius prescription used by
us (for some simulations) and by Fregeau et al. Therefore, the best
approach to understanding the remaining differences between our
runs and those of Fregeau et al. would be to directly compare a set
of Monte Carlo simulations starting from the same initial conditions
as we have adopted.

We finally note that, in the presence of a tidal field, the prediction
of the Vesperini & Chernoff (1994) model, on the size of the core
as a function of the number of particles used in the simulation, is in
detailed quantitative agreement with the values measured in our set
of simulations with \( W_0 = 7 \). This fact comes as a surprise, because,
as we discussed in Paper I, the \( N \) dependence given by Vesperini
& Chernoff (1994) was not verified for isolated clusters: however,
the model does not make any assumption about the presence or
absence of a tidal field. The presence of a tidal field seems thus to
introduce a difference in the scaling with \( N \) of the ratio \( r_c/r_h \) in the
steady burning phase, which it would be interesting to understand
better.

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