Intersection of two quadrics with no common hyperplane in $\mathbb{P}^n(\mathbb{F}_q)$

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Abstract

Let $Q_1$ and $Q_2$ be two arbitrary quadrics with no common hyperplane in $\mathbb{P}^n(\mathbb{F}_q)$. We give the best upper bound for the number of points in the intersection of these two quadrics. Our result states that $|Q_1 \cap Q_2| \leq 4q^{n-2} + \pi_{n-3}$. This result inspires us to establish the conjecture on the number of points of an algebraic set $X \subset \mathbb{P}^n(\mathbb{F}_q)$ of dimension $s$ and degree $d$: $|X(\mathbb{F}_q)| \leq dq^s + \pi_{s-1}$.

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1 Introduction

In 1954, L. Carlitz [4, Theorem 5, p.144] gave a quite formula to compute the number of points in the intersection of two quadrics. His formula depended mainly on the capacity to write the first quadric as a sum of non-degenerate quadrics $Q_i$, the number of indeterminates of each quadric $Q_i$, and the number of quadrics $Q_i$ with odd indeterminates. Secondly on the capacity to write the second quadrics as linear combination of the quadrics $Q_i$.
In the same year A. Weil [17, p.348] extend the result of L. Carlitz, to two arbitrary quadrics. His formula depended on the ability to diagonalize

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the forms in the pencil defined by the two quadrics. We think that these sophisticated methods are concretely difficult to apply, in order to find an interesting upper bound for the number of points in the intersection of two quadrics.

In 1975, W. M. Schmidt [14., Lemma 3C, p. 175], was the first to give an explicit upper bound for the number of intersection of two hypersurfaces by using some results on the theory of resultants. Therefore for two quadrics, his estimate leads to the following result:

\[ |Q_1 \cap Q_2| \leq 2(4q^{n-2} + 4\pi_{n-3}) + \frac{7}{q-1}. \]

But he was convinced that its estimate was not the best possible.

In 1986, A. A. Bruen and J. W. P. Hirschfeld [3, pp. 218-220] computed the number of points in the intersection of two quadrics only for non-degenerate quadrics in a projective space of odd dimension. Even in these cases, theirs results don’t satisfy totally our hope.

In 1992, Y. Aubry [1, Theorem 3, p. 11] inspired by the techniques of W. M. Schmidt and the upper bound for hypersurfaces of Tsfasman-Serre-Sørensen [15], [16, chap.2,pp.7-10], improved the above upper bound. We prefer to write his result in this way:

\[ |Q_1 \cap Q_2| \leq 2(4q^{n-2} + \pi_{n-3}) + \frac{1}{q-1}. \]

In 1999, D. B. Leep and L. M. Schueller [13, p.172] improved the result of Y. Aubry under the condition that the pair of quadrics have full order. This is a too much restrictive condition. These are their bounds: \(|Q_1 \cap Q_2| \leq LS(n, q)\) with

\[
LS(n, q) = \begin{cases} 
2q^{n-2} + \pi_{n-3} + 2q^{\frac{n-1}{2}} - q^{\frac{n-3}{2}} & \text{if } n + 1 \geq 4 \text{ and even} \\
2q^{n-2} + \pi_{n-3} + q^{\frac{n}{2}} & \text{if } n + 1 \geq 5 \text{ and odd.}
\end{cases}
\]

In 2008, F. A. B. Edoukou [6] gave the best upper bound for the number of points in the intersection of two quadrics with no common plane in \(\mathbb{P}^3(F_q)\), when one of them is non-degenerate (hyperbolic or elliptic) or degenerate of rank 3. He deduced that \(4q + 1\) is the optimal bound.

In 2009, F. A. B. Edoukou [7] study the same problem in \(\mathbb{P}^4(F_q)\) when one of the two quadrics is non-degenerate quadric (i.e. parabolic), degenerate of rank 4 or 3. He deduced that \(4q^2 + q + 1\) is the best upper bound and he formulated the following conjecture:
Conjecture [7, p.31]: Let $Q_1$ and $Q_2$ two quadrics in $\mathbb{P}^n(\mathbb{F}_q)$ with no common hyperplane. Then
\[ |Q_1 \cap Q_2| \leq 4q^{n-2} + \pi_{n-3} \]
And this bound is the best possible.

In this paper we will give a proof of this conjecture. The paper has been organized as follows. First of all we recall some generaties on quadrics. Secondly, based on what has been done for the study of intersection of quadrics in the projective of three and four dimension, and section of two quadrics with full order in higher dimension, we prove the conjecture. Finally we will formulate another conjecture for the intersection of quadrics, and a generalization of the above conjecture for a projective algebraic set of dimension $s$ and degree $d$ in a conjectural way:
\[ |X(\mathbb{F}_q)| \leq dq^s + \pi_{s-1}. \]
We will discuss on this conjecture.

2 Generalities

We denote by $\mathbb{F}_q$ the field with $q$ elements and $\mathbb{P}^n(\mathbb{F}_q) = \Pi_n$ the projective space of n-dimension over the field $\mathbb{F}_q$. Then
\[ \pi_n = \#\mathbb{P}^n(\mathbb{F}_q) = q^n + q^{n-1} + ... + 1. \]

Let $Q$ be a quadric in $\mathbb{P}^n(\mathbb{F}_q)$ (i.e. $Q = Z(F)$ where $F$ is a form of degree 2). The rank of $Q$, denoted $r(Q)$, is the smallest number of indeterminates appearing in $F$ under any change of coordinate system. The quadric $Q$ is said to be degenerate if $r(Q) < n + 1$; otherwise it is non-degenerate. For $Q$ a degenerate quadric and $r(Q)=r$, $Q$ is a cone $\Pi_{n-r}Q_{r-1}$ with vertex $\Pi_{n-r}$ (the set of the singular points of $Q$) and base $Q_{r-1}$ in a subspace $\Pi_{r-1}$ skew to $\Pi_{n-r}$.

Definition 2.1 For $Q = \Pi_{n-r}Q_{r-1}$ a degenerate quadric with $r(Q) = r$, $Q_{r-1}$ is called the non-degenerate quadric associated to $Q$. The degenerate quadric $Q$ will be said to be of hyperbolic type (resp. elliptic, parabolic) if its associated non-degenerate quadric is of that type.
Definition 2.2 [13, p.158] Let \( Q_1 \) and \( Q_2 \) be two quadrics. The order \( w(Q_1, Q_2) \) of the pair \((Q_1, Q_2)\), is the minimum number of variables, after invertible linear change of variables, necessary to write \( Q_1 \) and \( Q_2 \). When \( w(Q_1, Q_2) = t + 1 \), we assume that \( Q_1 \) and \( Q_2 \) are defined with the first \( t + 1 \) indeterminates \( x_0, \ldots, x_t \) and we define \( E_t(F_q) = \{ x \in \mathbb{P}^n(F_q) \mid x_{t+1} = \ldots = x_n = 0 \} \).

Lemma 2.3 [5, pp.70-71] Let \( Q_1 \) and \( Q_2 \) be two quadrics in \( \mathbb{P}^n(F_q) \) and \( l \) an integer such that \( 1 \leq l \leq n - 1 \). Suppose that \( w(Q_1, Q_2) = n - l + 1 \) (i.e. there exists a linear transformation such that \( Q_1 \) and \( Q_2 \) are defined with the indeterminates \( x_0, x_1, \ldots, x_{n-l} \)) and \( |Q_1 \cap Q_2 \cap E_{n-l}(F_q)| \leq m \). Then
\[
|Q_1 \cap Q_2| \leq mq^l + \pi_{l-1}.
\]
This bound is optimal as soon as \( m \) is optimal in \( E_{n-l}(F_q) \).

3 Resolution of the conjecture

Let us state the following result which will be very useful in the proof of the below theorem.

Lemma 3.1 For \( n \geq 3 \), \( LS(n, q) \leq 4q^{n-2} + \pi_{n-3} \).

Proof: An easy computation.

Theorem 3.2 Let \( Q_1 \) and \( Q_2 \) be two quadrics in \( \mathbb{P}^n(F_q) \) with no common hyperplane. Then
\[
|Q_1 \cap Q_2| \leq 4q^{n-2} + \pi_{n-3}
\]
And this bound is the best possible.

Proof: It has been proved in [6, 7] that the above conjecture is true for \( n = 3 \) and \( 4 \). We will make an induction reasoning on the dimension of the projective space.

Let us suppose that the conjecture is true for \( i, \ 4 \leq i \leq n - 1 \); that means for two quadrics \( Q_1 \) and \( Q_2 \) in \( \mathbb{P}^i(F_q) \) with no common hyperplane we have
\[
\text{for } \ 4 \leq i \leq n - 1, \quad |Q_1 \cap Q_2| \leq 4q^{i-2} + \pi_{i-3}.
\]

Suppose now that \( Q_1 \) and \( Q_2 \) are two quadrics in \( \mathbb{P}^n(F_q) \) with no common hyperplane. Let us consider the order \( w(Q_1, Q_2) \) of the two quadrics. We have either \( w(Q_1, Q_2) = n + 1 \) or \( w(Q_1, Q_2) \leq n \).

If \( w(Q_1, Q_2) = n + 1 \), from the result of D. B. Leep and L. M. Schueller,
we deduce that $|Q_1 \cap Q_2| \leq LS(n, q)$. Therefore from the above Lemma we deduce that the conjecture is true.

If $w(Q_1, Q_2) \leq n$, then there is an integer $l$, $1 \leq l \leq n - 1$ such that $w(Q_1, Q_2) = n - l + 1$. Therefore there exists a linear transformation such that $Q_1$ and $Q_2$ are defined with the indeterminates $x_0, x_1, ..., x_{n-l}$. And from the above hypothesis we deduce that:

$$|Q_1 \cap Q_2 \cap E_{n-l}(F_q)| \leq 4q^{n-l-2} + \pi_{n-l-3}.$$  

Then by applying Lemma 2.3 with $m = 4q^{n-l-2} + \pi_{n-l-3}$, we deduce that

$$|Q_1 \cap Q_2| \leq 4q^{n-2} + \pi_{n-3}.$$  

Let now $Q_1$ and $Q_2$ be two quadrics of $\mathbb{P}(F_q)$ defined by the equations $f_1(x_0, x_1, ..., x_n) = x_0^2 + x_1^2 - x_2^2$, and $f_2(x_0, x_1, ..., x_n) = x_0x_1$. In [Edoukou IEEE], it has been proved that (with $q$ odd), $|Q_1 \cap Q_2 \cap E_3(F_q)| = 4q + 1$. Therefore by using the above Lemma, we deduce that, we have $|Q_1 \cap Q_2| = 4q^{n-2} + \pi_{n-3}$. If $Q_1$ and $Q_2$ are defined the equations $f_1(x_0, x_1, ..., x_n) = (x_0 + x_1)x_2 + x_2^2$, and $f_2(x_0, x_1, ..., x_n) = (x_2 + x_0)x_1 + x_1^2$, we have $|Q_1 \cap Q_2 \cap E_2(F_q)| = 4$. And by the preceding reasoning we have $|Q_1 \cap Q_2| = 4q^{n-2} + \pi_{n-3}$.

**Remark 3.3** It has been proved in [8] that when one of the two quadrics $Q_1$ or $Q_2$ is non-degenerate (hyperbolic, elliptic, parabolic), we have

$$|Q_1 \cap Q_2| \leq EH(n, q)$$

with

$$EH(n, q) = \begin{cases} 2q^{n-2} + \pi_{n-3} + 2q^{\frac{n-1}{2}} - q^{\frac{n-3}{2}} & \text{if } n + 1 \geq 4 \text{ and even} \\ 2q^{n-2} + \pi_{n-3} + q^{\frac{n-1}{2}} & \text{if } n + 1 \geq 4 \text{ and even} \\ 2q^{n-2} + \pi_{n-3} + 2q^{\frac{n-2}{2}} & \text{if } n + 1 \geq 5 \text{ and odd}. \end{cases}$$

These bounds are the best possible. Therefore, in this particular case, these bounds are better than the one of D. B. Leep and L. M. Schueller [13, p.172].

**4 Conjectures on the number of points of algebraic sets**

Here we will establish two conjectures concerning the number of points in the intersection of two degenerate quadrics with no common hyperplane and
secondly for the number of points of a general algebraic set.
If $\mathcal{X} = \Pi_{n-1}P_0$ is the degenerate quadric of rank $r=1$ (i.e. a repeated hyperplane) and $Q$ any quadric not containing $\mathcal{X}$ we can easily prove that $|\mathcal{X} \cap Q| \leq 2q^{n-2} + \pi_{n-3}$. And this upper bound is optimal if we take for example $\mathcal{X}$ and $Q$ defined by the equations $f_{\mathcal{X}}(x_0, x_1, ..., x_n) = x_2^2$, and $f_Q(x_0, x_1, ..., x_n) = x_0x_1 + x_2^2$.

If $\mathcal{X} = \Pi_{n-2}E_1$ is the degenerate quadric of rank $r=2$ of elliptic type, for any quadric $Q$ we have $|\mathcal{X} \cap Q| \leq \pi_{n-2}$. And this upper bound is reached if we take for example $\mathcal{X}$ and $Q$ defined by the equations $f_{\mathcal{X}}(x_0, x_1, ..., x_n) = f(x_0, x_1)$ ($f$ is irreducible), and $f_Q(x_0, x_1, ..., x_n) = x_0x_1$.

We also know from the examples in the previous section that when $\mathcal{X} = \Pi_{n-r}Q_{r-1}$ is a degenerate quadric of rank $r=2$ of hyperbolic type or rank $r=3$, we can find another degenerate quadric such that $|\mathcal{X} \cap Q| = 4q^{n-2} + \pi_{n-3}$.

Let us now suppose that $r \geq 4$. We have the following conjecture

**Conjecture 4.1** Let $\mathcal{X} = \Pi_{n-r}Q_{r-1}$ be a degenerate quadric of rank $r(\mathcal{X}) = r$ and $Q$ another degenerate quadric. If $r \geq 4$, then

$$|\Pi_{n-r}Q_{r-1} \cap Q| \leq EH(r-1, q)q^{n-r+1} + \pi_{n-r}.$$ 

And this bound is the best possible.

Let us remark that if the conjecture is true, that means for every functional code of order 2 defined on a degenerate quadric $\mathcal{X} = \Pi_{n-r}Q_{r-1}$, we know exactly its minimum distance.

Based on the authors experiences on the study of the number of points in the intersection of hypersurfaces, we would like to propose the following conjecture.

**Conjecture 4.2** Let $X \subset \mathbb{P}^n(\mathbb{F}_q)$ be a projective algebraic set of degree $d$ and dimension $s$. Then,

$$|X(\mathbb{F}_q)| \leq dq^s + \pi_{s-1}. \quad (1)$$

**Remark 4.3** Let us remark that this conjecture is true if $X$ is of codimension 1. This is the Tsfasman-Serre-Sørensen’s upper bound for hypersurfaces [15], [16, chap.2, pp.7-10].

The Theorem 3.2 is a particular case of this conjecture. In fact, the intersection of the two quadrics with no common hyperplane is an algebraic set of dimension $n-2$ and degree 4.
Remark 4.4 There is also an upper bound concerning the number of points of a projective algebraic set which was already known by G. Lachaud since 1993 and proved by Lachaud[12, p.80], Boguslavsky[2, pp.288, 293-294] and Gorpade-Lachaud [9,pp.627-630]:

$$|X(F_q)| \leq d \pi_s.$$  

In 1995, G. Lachaud [2, p. 292], [9, p.629] proposed the following conjecture on the upper bound for the number of points of a projective algebraic set:

$$|X(F_q)| \leq d(\pi_s - \pi_{2s-n}) + \pi_{2s-n}. \quad (2)$$  

Our bound (1) is more better than the one of G. Lachaud (2). In [5, 6, 7, 8] we can find several cases where the Conjecture 4.2 is true and where the upper bound of G. Lachaud (2) seems to be too large.

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