Applications of amenable semigroups in operator theory

by

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Abstract. The paper deals with continuous representations \( S \ni s \mapsto T_s \in \mathcal{L}(E) \) of amenable semigroups \( S \) into the algebra \( \mathcal{L}(E) \) of all bounded linear operators on a Banach space \( E \). For a closed linear subspace \( F \) of \( E \), sufficient conditions are given under which there exists a projection \( P \in \mathcal{L}(E) \) onto \( F \) that commutes with all \( T_s \). And when \( E \) is a Hilbert space, sufficient conditions are given for the existence of an invertible operator \( L \in \mathcal{L}(E) \) such that all \( LT_sL^{-1} \) are isometries. Also some results on extending intertwining operators, on renorming and on operators on hereditarily indecomposable Banach spaces are offered.

1. Introduction. A (semi)group \( \mathcal{S} \) is amenable if there exists an invariant mean on a suitably chosen vector space of bounded real-valued functions defined on \( \mathcal{S} \). This mean is a positive linear functional invariant under (left, right or both left and right) translations of the semigroup. The so-called Banach limits on \( \ell_\infty \) (see e.g. [10], II.4.22, p. 73) are the most classical examples of invariant means on a semigroup that is not a group. The above (sketch of a) definition of an amenable semigroup deals with an intrinsic property of the semigroup. However, amenability is equivalent to a strong (and powerful) property of a fixed point. Namely, a semigroup \( \mathcal{S} \) is amenable iff for every affine action (with certain additional properties related to continuity and corresponding to the version of amenability we deal with) \( K \ni x \mapsto \phi_s(x) \in K \ (s \in \mathcal{S}) \) of \( \mathcal{S} \) on a compact convex non-empty subset \( K \) of a locally convex topological vector space there is a point \( a \in K \) such that \( \phi_s(a) = a \) for all \( s \in \mathcal{S} \). This property makes amenability widely applicable and thus amenable structures (which include locally
compact topological groups, abstract semigroups, extremely amenable Polish groups, Banach algebras, $C^*$-algebras) are still widely investigated and looked for. A well-known sufficient condition for a topological group (or a semigroup) to be amenable is the commutativity of its binary action. However, there are a number of amenable groups (in particular, all compact topological groups) that are non-abelian. The reader interested in the classical notion of amenability (for topological groups or discrete semigroups) is referred to, e.g., [12], [21], [22] or [26, Chapter 1].

In this paper we deal with two versions of amenability (see Definition 3.1 below). Because of applications offered in this paper, we focus on right amenability. Our considerations cover all semigroups equipped with arbitrary topologies, which is a much wider class than topological semigroups. This is motivated by a growing interest in semitopological semigroups (or Ellis semigroups, related to ultrafilter techniques) in dynamical systems and ergodic theory (see, e.g., [1, Chapter 6]). We offer certain results on the existence of projections commuting with a collection of bounded operators on a Banach space (Propositions 4.2 and 4.5, Theorems 4.6 and 4.7 and Corollary 4.8); extending intertwining operators (Theorem 4.10 and Corollary 4.11); renorming the space so that all operators from a given family become isometric (Theorem 4.13 and Corollary 4.14); and joint similarity to isometries in a Hilbert space (Proposition 4.16 and Corollary 4.17). For clarity and simplicity, below we formulate our main results in the case of amenable (discrete) semigroups of operators, where amenability has the classical meaning:

**Theorem 1.1.** Let $\mathcal{T} \subset \mathcal{L}(E)$ be a multiplicative semigroup of operators on a Banach space $E$ such that $\mathcal{T}$ is right amenable in the discrete topology, and let $F$ be a closed linear subspace of $E$ such that:

1. For any $T \in \mathcal{T}$, $T(F) = F$ and the restriction $T|F$ of $T$ to $F$ is an isomorphism;
2. $F$ is a dual Banach space;
3. For any $T \in \mathcal{T}$, $T|F$ is weak* continuous on $B_F$;
4. $\sup_{T \in \mathcal{T}} \|T\| \cdot \|(T|F)^{-1}\| < \infty$.

Then there exists a projection $P \in \mathcal{L}(E)$ onto $F$ that commutes with all $T \in \mathcal{T}$ iff $F$ is complemented in $E$. More specifically, if $F$ is complemented in $E$, then there exists a projection $Q \in \mathcal{L}(E)$ onto $F$ that commutes with all operators in $\mathcal{T}$ and satisfies:

- $\|Q\| \leq \sup_{T \in \mathcal{T}} (\|T\| \cdot \|(T|F)^{-1}\|) \cdot \lambda(E, F)$, where $\lambda(E, F) = \inf\{\|P\| : P \in \mathcal{L}(E) \text{ a projection onto } F\}$,
- $Q$ is minimal; that is, $\|Q\| \leq \|P\|$ for any other projection $P$ onto $F$ that commutes with all operators in $\mathcal{T}$. 
The above result for compact groups (where conditions (2)–(4) can be dropped and the conclusion is just the existence of a projection commuting with the operators from the given group) is due to Rudin [24].

**Theorem 1.2.** Let $\mathcal{I}$ be a right amenable discrete semigroup, $X$ a Banach space and $Y$ a dual Banach space. Let $\mathcal{I} \ni s \mapsto A_s \in \mathcal{L}(X)$ and $\mathcal{I} \ni s \mapsto B_s \in \mathcal{L}(Y)$ be two representations, with $B_s$ invertible and weak* continuous on the closed unit ball of $Y$ for all $s \in \mathcal{I}$. Further, let $E$ be a closed linear subspace of $X$ and $T_0 : E \to Y$ a bounded linear operator such that for any $s \in \mathcal{I}$, $A_s(E) \subset E$ and $B_sT_0 = T_0A_s|E$. Then the following conditions are equivalent:

(i) $T_0$ extends to a bounded linear operator $T : X \to Y$ such that $B_sT = TA_s$ for all $s \in \mathcal{I}$;

(ii) there exists a bounded linear operator $T' : X \to Y$ that extends $T_0$ and satisfies $\sup_{s \in \mathcal{I}} \|B_s^{-1}T'A_s\| < \infty$.

**Theorem 1.3.** Let $E$ be a Banach space and $\mathcal{T} \subset \mathcal{L}(E)$ a bounded multiplicative semigroup of operators that is right amenable in the discrete topology. Put $N(\mathcal{T}) := \{x \in E : Tx = x \ (T \in \mathcal{T})\}$ and $R(\mathcal{T}) := \overline{\text{lin}\{Tx - x : T \in \mathcal{T}, \ x \in E\}}$. Then:

- $N(\mathcal{T}) \cap R(\mathcal{T}) = \{0\}$;
- the subspace

\[
D(\mathcal{T}) := N(\mathcal{T}) + R(\mathcal{T})
\]

is closed in $E$;
- the projection $P : D(\mathcal{T}) \to N(\mathcal{T})$ induced by the decomposition (1.1) has norm not greater than $\sup_{T \in \mathcal{T}} \|T\|$.

**Theorem 1.4.** Let $H$ be a Hilbert space and $\mathcal{T} \subset \mathcal{L}(H)$ a bounded multiplicative semigroup of operators that is right amenable in the discrete topology. Suppose there are positive real constants $m$ and $M$ such that for any $T \in \mathcal{T}$ and $x \in H$,

\[
m\|x\| \leq \|Tx\| \leq M\|x\|.
\]

Then there exists an invertible positive operator $A \in \mathcal{L}(H)$ such that $mI \leq A \leq MI$ and $ATA^{-1}$ is an isometry for each $T \in \mathcal{T}$.

A full version of the above theorem (see Proposition 4.16) generalises the classical results of Sz.-Nagy [28], Dixmier [9] and Day [5] on bounded amenable groups of Hilbert space operators.

Except the aspects of the so-called extreme amenability, so far amenability (in categories related to groups and semigroups) was studied mainly in the realms of locally compact topological groups and abstract semigroups.
(that is, semigroups without topologies or, in other words, with discrete topologies). In both these classes weak and strong amenabilities (see Definition 3.1) coincide. Roughly speaking, weak amenability deals with jointly continuous actions of semigroups, whereas its strong version with separately continuous ones. In the present work almost all results are formulated simultaneously for these two versions of amenability to emphasize when the weak version is insufficient to get the conclusion. Our results can be divided into four thematic areas. They deal with:

• Projections commuting with a semigroup of operators (Theorems 1.1, 4.6, Propositions 4.2, 4.5 and Corollary 4.3). These results generalise the aforementioned Rudin theorem [24] for compact groups. Our settings are significantly more general, but need additional assumptions that compensate for the lack of compactness. As such, they use different arguments than those presented by Rudin.

• Intertwining operators between representations (Theorems 1.2, 4.10 and Corollary 4.11). To the best knowledge of the authors, all these results are new.

• Bounded partial projections induced by bounded representations (Theorem 1.3, Corollaries 4.8 and 4.9). Also these results appear to be new. The first two actually give an interesting characterisation of amenability.

• Representations that are similar or quasi-similar to isometric representations (Theorems 1.4, 4.13, Propositions 4.16, 4.17). These results are natural generalisations of theorems due to Sz.-Nagy [28], Dixmier [9] and Day [5] (Hilbert space settings) as well as to Koehler and Rosenthal [17] (Banach space settings). We present here a different approach to the Hilbert space case (classically, one defines a new scalar product by means of an invariant mean; here we apply Day’s fixed point property for a certain set of norms or positive operators). A result on quasi-similarity of a representation of a weakly amenable semigroup to an isometric representation (Proposition 4.16) appears to be new.

We also show that the free group on two (or, more generally, finitely or countably many) generators can be given a structure of a weakly right amenable topological group (Theorem 5.2 and Example 3.4(E)).

The paper is organised as follows. In the next section we introduce basic notions and fix the notation. Section 3 is a preliminary part on two variants of amenability we will deal with further. The fourth, main section is devoted to applications of amenability. We prove and formulate there all results dealing with amenable semigroups and operators in Banach spaces. This section is concluded by the proofs of Theorems 1.1, 1.3. In Section 5 we give more information on the two kinds of amenability introduced in Section 3.
2. Notation and terminology. In this paper all semigroups are non-empty and Banach spaces are real or complex. We use multiplicative notation to denote binary actions of abstract semigroups (excluding vector spaces). The abbreviation SGT means a semigroup with topology. So, a statement \( \mathcal{S} \) is an SGT means that \( \mathcal{S} \) is both a semigroup and a (possibly non-Hausdorff) topological space, and no compatibility of the two structures is assumed. In particular, in an SGT the semigroup operation can be separately discontinuous. Although SGT’s need not be Hausdorff, for us topological groups, by definition, are. Important examples of non-topological SGT’s are Ellis semigroups in which the semigroup operation is continuous only in the left variable (according to the definition given in [1, Chapter 6]). Closed unit balls of Banach spaces of the form \( \mathcal{L}(X^*) \) where \( X^* \) is the dual Banach space of a Banach space \( X \) (considered as multiplicative semigroups with the weak* operator topologies) are classical examples of compact Ellis semigroups that widely appear in functional analysis (multiplication in these SGT’s is separately continuous only when \( X \) is reflexive).

An SGT need not have a neutral element, and if it does have, it is said to be unital (otherwise it is non-unital). Similarly, a homomorphism between two unital SGT’s is called unital if it sends the unit of the source semigroup to the unit of the target. We adopt analogous terminology for representations.

By a right action of a semigroup \( \mathcal{S} \) on a set \( X \) we mean any function \( X \times \mathcal{S} \ni (x, s) \mapsto \phi_s(x) = x.s \in X \) such that for all \( s, t \in \mathcal{S} \) and \( x \in X \), \((x.s).t = x.(st)\). If, in addition,

- \( \mathcal{S} \) is unital and \( x.1 = x \) for any \( x \in X \), the action is called unital;
- \( X \) is a convex set (in a real vector space) and \( \phi_s \) is affine—that is,

\[(1 - \alpha)x + \alpha y).s = (1 - \alpha)(x.s) + \alpha(y.s)\]

for any \( x, y \in X \) and \( \alpha \in [0, 1] \), then the action is called affine.

A point \( a \in X \) is said to be a fixed point for the action if \( a.s = a \) for any \( s \in \mathcal{S} \).

If \( (x, s) \mapsto x.s \) is a right action of \( \mathcal{S} \) on a set \( X \) and \( u: X \to Y \) is an arbitrary function, then for any \( s \in \mathcal{S} \), \( u_s: X \to Y \) is defined by \( u_s(t) = u(ts) \). This, in particular, applies to the natural right action of \( \mathcal{S} \) on itself.

For a topological space \( X \), \( C_b(X) \) stands for the algebra of all bounded continuous real-valued functions on \( X \), equipped with the sup-norm. If \( X \) is compact, we write \( C(X) \) in place of \( C_b(X) \).

For any Banach space \( E \), \( \mathcal{L}(E) \) denotes the algebra of all bounded linear operators from \( E \) into \( E \). When \( F \) is a closed linear subspace of \( E \), a statement “\( P: E \to F \) is a projection” means that \( P \) is a bounded linear operator such that \( Pf = f \) for any \( f \in F \). If \( F \) is complemented in \( E \), then \[ \lambda(E, F) = \inf \{ \|P\| : P: E \to F \text{ a projection} \} \].
A representation of an SGT $\mathcal{S}$ is a (possibly discontinuous) homomorphism of $\mathcal{S}$ into the multiplicative semigroup $L(E)$ for some Banach space $E$.

A dual Banach space is a Banach space that is linearly isometric to the dual space $X^*$ of some Banach space $X$. Every dual Banach space $Y$ can be endowed with an (abstract) weak* topology (transferred from the classical weak* topology of $X^*$). Recall that the closed unit ball of $Y$ is weak* compact.

3. Two kinds of amenability. In the literature there are two classes of SGT’s in which amenability is well studied—the classes of locally compact topological groups (consult, e.g., [12, 21, 22] or [26, Chapter 1]) and of abstract semigroups (that is, semigroups without any topology, or equivalently discrete semigroups; see, e.g., [21] or [12, Section 1]). Amenability was also generalised to Banach algebras (see, e.g., [26, Chapter 2] or [14] and references therein). Amenability in $C^*$-algebras also has a special interest (consult, e.g., [18]).

A topic closely related to amenable groups is extremely amenable Polish groups, which are now intensively studied. According to Definition 3.1 below, all such groups are weakly right amenable. The literature on extremely amenable Polish groups is still growing; we just mention [13, 16, 23, 11, 27, 20] and references therein.

Because of further applications, we will deal with two versions of right amenability. Of course, in a similar manner one can introduce their counterparts for left amenability.

From now on, let $\mathcal{S}$ be an SGT and let $C_r(\mathcal{S})$ consist of all functions $f \in C_b(\mathcal{S})$ such that $f_s$ is continuous for any $s \in \mathcal{S}$.

Let $\mathcal{F}$ be a linear subspace of $C_r(\mathcal{S})$. A mean on $\mathcal{F}$ is a linear functional $\phi: \mathcal{F} \to \mathbb{R}$ such that $\inf f(\mathcal{S}) \leq \phi(f) \leq \sup f(\mathcal{S})$ for any $f \in \mathcal{F}$. (If $\mathcal{F}$ contains a function $j$ constantly equal to 1, a linear functional $\phi$ on $\mathcal{F}$ is a mean iff $\|\phi\| = \phi(j) = 1$.) The space $\mathcal{F}$ is said to be right invariant if $u_s \in \mathcal{F}$ for any $u \in \mathcal{F}$ and $s \in \mathcal{S}$. (It is easily seen that $C_r(\mathcal{S})$ is right invariant.)

A mean $\phi$ on a right invariant linear subspace $\mathcal{F}$ of $C_r(\mathcal{S})$ is right invariant if $\phi(f_u) = \phi(f)$ for any $f \in \mathcal{F}$ and $s \in \mathcal{S}$.

Right amenability deals with right invariant means on suitable right invariant linear spaces $\mathcal{F}$. The bigger $\mathcal{F}$ is, the stronger version of amenability we get. Below we propose two such versions.

Definition 3.1. An SGT $\mathcal{S}$ is said to be weakly right amenable if there is a right invariant mean on

$$C_{\text{norm}}(\mathcal{S}) := \{f \in C_r(\mathcal{S}) : s \mapsto f_s \text{ and } s \mapsto f_{st} \text{ are continuous}
\text{in the norm topology of } C_r(\mathcal{S}) \text{ for all } t \in \mathcal{S}\}.$$
\( \mathcal{S} \) is said to be strongly right amenable if there is a right invariant mean on

\[
C_{\text{weak}}(\mathcal{S}) := \{ f \in C_r(\mathcal{S}) : s \mapsto f_s \text{ and } s \mapsto f_{st} \text{ are continuous in the weak topology of } C_r(\mathcal{S}) \text{ for all } t \in \mathcal{S} \}.
\]

Since \( C_{\text{norm}}(\mathcal{S}) \subset C_{\text{weak}}(\mathcal{S}) \), a strongly right amenable SGT is weakly right amenable.

The counterparts of the spaces \( C_{\text{norm}}(\mathcal{S}) \) and \( C_{\text{weak}}(\mathcal{S}) \) for an SGT \( \mathcal{S} \) with separately continuous multiplication were introduced by Paterson [21] (denoted \( U_r(\mathcal{S}) \) and \( WU_r(\mathcal{S}) \), respectively [21, paragraph (2.10), p. 56]). In his book, the space \( WU_r(\mathcal{S}) \) has an auxiliary role and was considered mainly as a maximal left introverted subspace of \( C_b(\mathcal{S}) \). Although he proved Theorem 3.2 (see below) for such SGT’s (see [21, Theorems 2.23 and 2.24]), he did not introduce a new kind of amenability.

The following result, proved by Day [7, Theorem 3], [8] for amenable abstract semigroups and by Paterson [21] for SGT’s with separately continuous multiplication, is a powerful tool that makes amenable SGT’s useful. Actually, it is the main tool of the present paper.

**Theorem 3.2.** For an SGT \( \mathcal{S} \) the following conditions are equivalent:

1. (SG1) \( \mathcal{S} \) is weakly [resp. strongly] right amenable;
2. (SG2) every jointly [resp. separately] continuous affine right action of \( \mathcal{S} \) on a non-empty compact convex set in a Hausdorff locally convex topological vector space has a fixed point.

If, in addition, \( \mathcal{S} \) is unital, then the above conditions are equivalent to

1. (SG2') every jointly [resp. separately] continuous unital affine right action of \( \mathcal{S} \) on a non-empty compact convex set in a Hausdorff locally convex topological vector space has a fixed point.

In the proof we shall apply the following known result on linear functionals on the space of all continuous affine functions on a compact convex set:

**Lemma 3.3.** Let \( K \) be a compact convex non-empty set in a locally convex space and \( CA(K) \) be the Banach space (equipped with the sup-norm) of all real-valued continuous affine functions defined on \( K \). Every continuous linear functional \( \psi \) on \( CA(K) \) such that

\[
\| \psi \| = \psi(1_K) = 1
\]

is of the form \( \psi(f) = f(a) \) (\( f \in CA(K) \)) where \( a \in K \) is uniquely determined by \( \psi \).

The above folklore result is formulated in the introduction of [3], without a proof. However, a full proof can be found in unpublished notes Affine...
functions on compact convex sets} by Anthony W. Wickstead, available online (see Corollary I.1.9 therein). For the reader’s convenience, we sketch the proof: Thanks to the Krein–Milman theorem, it is sufficient to prove the conclusion for those \( \psi \) that are extreme points of the set of all functionals satisfying \((3.1)\). Since this set is a face in the closed unit ball \( B \) of \( CA(K)^* \), \( \psi \) is an extreme point of \( B \) as well. Hence, it extends to a functional \( \mu \in C(K)^* \) that is an extreme point of the closed unit ball of \( C(K)^* \). Since \( \mu(1_K) = 1 \), it follows from the characterisation of extreme functionals on \( C(\Omega) \) for compact \( \Omega \) that \( \mu(f) = f(a) \) (\( f \in C(K) \)) for some \( a \in K \).

**Proof of Theorem 3.2.** Since the result can be proved in almost the same way as Day’s classical fixed point theorem, we only give a sketch.

Let \( \mathcal{I} := C_{\text{norm}}(\mathcal{I}) \) [resp. \( \mathcal{I} := C_{\text{weak}}(\mathcal{I}) \)].

First assume (SG2) (or (SG2')) holds. Equip the set \( \Delta \) of all means on \( \mathcal{I} \) with the weak* topology. For \( \phi \in \Delta \) and \( s \in \mathcal{I} \) we define \( \phi.s \in \Delta \) by \( (\phi.s)(f) = \phi(f_s) \). Then \( (\phi,s) \mapsto \phi.s \) is a (unital if \( \mathcal{I} \) is unital) affine right action that is jointly [resp. separately] continuous. So, an application of (SG2) (or (SG2')) yields the existence of a right invariant mean.

Now assume \( \mathcal{I} \) is right amenable and let \( K \) and \( K \times \mathcal{I} \supseteq (x,s) \mapsto x.s \in K \) be, respectively, a non-empty compact convex set in a Hausdorff locally convex topological vector space and a jointly [resp. separately] continuous affine right action. Let \( CA(K) \) be as in Lemma 3.3. Fix a right invariant mean \( \phi \) on \( \mathcal{I} \) and a point \( b \in K \). For each \( u \in CA(K) \) let \( \hat{u} \in \mathcal{I} \) be given by \( \hat{u}(s) = u(b,s) \). Observe that \( \psi: CA(K) \ni u \mapsto \phi(\hat{u}) \in \mathbb{R} \) is a linear functional satisfying \((3.1)\). It follows from Lemma 3.3 that there is \( a \in K \) for which \( \psi(u) = u(a) \) for any \( u \in CA(K) \). In order to check that \( a \) is a fixed point for the action, it suffices to verify that \( u(a.s) = u(a) \) for any \( u \in CA(K) \). To this end, note that

\[ u_s(a) = \psi(u_s) = \phi(\hat{u}_s) = \phi(\hat{u}) = \psi(u) = u(a) \]

and we are done.

**Example 3.4.** (A) Let \( \mathcal{I} \) be a compact topological group. Then

\[ C_{\text{weak}}(\mathcal{I}) = C_{\text{norm}}(\mathcal{I}) = C(\mathcal{I}) \]

and \( \mathcal{I} \) is strongly right amenable. The functional induced by the Haar measure of \( \mathcal{I} \) is a (unique) invariant mean on \( C(\mathcal{I}) \).

(B) Every abelian semigroup \( \mathcal{I} \) equipped with the discrete topology is strongly right amenable and thus \( \mathcal{I} \) is so when equipped with any topology. This is a consequence of the well-known Markov–Kakutani fixed point theorem [19], [15]. The fact that abelian SGT’s are strongly right amenable will be used in the proofs of all results formulated in Section 1.

(C) When the topology of an SGT \( \mathcal{I} \) is discrete, we have \( C_{\text{weak}}(\mathcal{I}) = C_{\text{norm}}(\mathcal{I}) = \) the algebra of all bounded real-valued functions on \( \mathcal{I} \) and thus
\( \mathcal{S} \) is strongly right amenable iff it is weakly right amenable, iff it is right
amenable as an abstract semigroup (considered without topology) \[26\].

(D) Let \( \mathcal{G} \) be a locally compact topological group and \( UC(\mathcal{G}) \) the algebra
of all those functions \( f \in C_b(\mathcal{G}) \) that are uniformly continuous with respect
to both left and right uniformities (cf. \[26\] Definition A.2.1). It can be easily
shown that \( UC(\mathcal{G}) \subset C_{\text{norm}}(\mathcal{G}) \subset C_{\text{weak}}(\mathcal{G}) \subset C_b(\mathcal{G}) \) and thus, thanks
to \[26\] Theorem 1.1.9, \( \mathcal{G} \) is strongly right amenable iff it is weakly right
amenable, iff \( \mathcal{G} \) is amenable in the classical sense (that is, if there is an
invariant mean on \( L^\infty(\mathcal{G}) \)).

(E) It is well known that the free (non-abelian) group \( F_2 \) on two gener-
ators is non-amenable as an abstract group—or equivalently is not weakly
right amenable when equipped with the discrete topology. It is also well
known that there are two orthogonal matrices \( U, V \in O_3 \) that generate a
group \( \mathcal{G} \) isomorphic to \( F_2 \). Since the closure \( \overline{\mathcal{G}} \) of \( \mathcal{G} \) (in \( O_3 \)) is compact, \( \mathcal{G} \) is
(strongly and thus) weakly right amenable. In Section 5 we will show that a
dense subgroup of a weakly right amenable topological group is weakly right
amenable as well (see Theorem 5.2). We conclude that \( F_2 \) admits a separa-
ble metrizable topology that makes it a weakly right amenable topological
group.

At this time we are unable to give an example of a weakly right amenable
SGT that is not strongly right amenable. We conjecture that the two notions
are not equivalent.

More on right amenability can be found in Section 5.

4. Applications. In this section we show how right amenability can be
applied in operator theory. Although our first result has nothing to do with
amenability, the method of proof was one of our motivations.

**Proposition 4.1.** Let \( Y \) be a closed linear subspace of a Banach space \( X \)
and let \( T \in \mathcal{L}(X) \) be such that there is a unique minimal projection \( P \) of \( X \)
onto \( Y \), \( T(Y) = Y \), \( T|Y \) is an isometry and \( \|T\| = 1 \). Then \( TP = PT \).

**Proof.** Observe that \( Q := (T|Y)^{-1}PT \) is a projection onto \( Y \) such that
\( \|Q\| \leq \|P\| \). Since \( P \) is unique minimal, we have \( Q = P \) and hence \( TP = PT \). \( \blacksquare \)

The idea of the above proof will be applied in the proof of

**Proposition 4.2.** Let \( \mathcal{S} \) be a strongly \[resp. weakly\] right amenable
SGT, \( X \) a Banach space and \( \Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X) \) a representation.
Suppose \( Y \) is a closed linear subspace of \( X \) and:

(p0) \( \Phi \) is continuous in the strong operator topology \[resp. operator norm
topology\] of \( \mathcal{L}(X) \);

(p1) for any \( s \in \mathcal{S} \), \( T_s(Y) = Y \) and \( T_s|Y \) is an isomorphism;
(p2) $Y$ is a dual Banach space;
(p3) for any $s \in \mathcal{S}$, $T_s|Y$ is weak* continuous on $B_Y$;
(p4) the function $\mathcal{S} \ni s \mapsto \|(T_s|Y)^{-1}\| \in \mathbb{R}$ is locally bounded.

Then the following conditions are equivalent:

(i) there exists a projection $P: X \to Y$ such that $PT_s = T_sP$ for all $s \in \mathcal{S}$;
(ii) there is a projection $Q: X \to Y$ such that

\[
\sup_{s \in \mathcal{S}} \|(T_s|Y)^{-1}QT_s\| < \infty.
\]

Moreover, if (ii) holds, then there exists a projection $P_0: X \to Y$ such that $P_0$ commutes with all operators $T_s$ and $\|P_0\| \leq \|P\|$ for any other projection $P: X \to Y$ commuting with all $T_s$.

The part concerning $P_0$ generalises a classical theorem due to Cheney and Morris [4] which asserts that any complemented subspace (of a Banach space) that is a dual Banach space admits a minimal projection. Indeed, one obtains the Cheney and Morris theorem by applying Proposition 4.2 to the trivial group $\mathcal{S}$ and its unital representation on $X$.

Note that condition (p4) is automatically fulfilled (thanks to (p0)) in the version of this proposition for weakly right amenable SGT’s.

Proof of Proposition 4.2. First of all, (p3) combined with the Kreĭn–Shmul’yan theorem (that characterises weak* closed convex sets) implies that

(p3′) for any $s \in \mathcal{S}$, $T_s|Y$ is adjoint to some bounded operator on the predual of $Y$ and $(T_s|Y)^{-1}$ is weak* continuous (on the whole $Y$).

For a projection $P: X \to Y$ and $s \in \mathcal{S}$ we write $P.s$ for $(T_s|Y)^{-1}PT_s$. Observe that $P.s: X \to Y$ is again a projection and

\[
(P, s) \mapsto P.s
\]

is an affine right action of $\mathcal{S}$ on the set of all projections from $X$ onto $Y$.

Assume $Q$ is as specified in (ii). Put $M := \sup_{s \in \mathcal{S}} \|Q.s\| < \infty$. Equip the set $\Delta$ of all projections $P: X \to Y$ with $\|P\| \leq M$ and $\sup_{s \in \mathcal{S}} \|P.s\| \leq M$ with the topology of pointwise weak* convergence (we call it the weak* operator topology). It is easily seen that $\Delta$ is convex, $Q.s \in \Delta$ and $P.s \in \Delta$ for any $P \in \Delta$ and $s \in \mathcal{S}$. Moreover, it follows from (p3′) that $\Delta$ is compact. (Indeed, closed balls in the Banach space $\mathcal{L}(X,Y)$ of all bounded linear operators from $X$ to $Y$ are compact in the weak* operator topology; and if $\{P_\sigma\}_{\sigma \in \Sigma} \subseteq \Delta$ is a net that converges in this topology to $P: X \to Y$, then $Py = y$ for any $y \in Y$, and for $s \in \mathcal{S}$, $P_\sigma.s$ converges in the weak* operator topology to $P.s$, since $(T_s|Y)^{-1}$ is weak* continuous. Consequently, $\|P.s\| \leq M$ and $P \in \Delta$.) So, by Theorem 3.2, it suffices to show that the action (4.2) is separately [resp. jointly] continuous.
It follows from (p3') that $\Delta \ni P \mapsto P.s \in \Delta$ is continuous for each $s \in \mathcal{S}$. Further, if $x$ is an arbitrary vector of $X$ and $(s_{\sigma})_{\sigma \in \Sigma}$ is a net in $\mathcal{S}$ convergent to $s \in \mathcal{S}$, then $\lim_{\sigma \in \Sigma} T_{s_{\sigma}} x = T_s x$ and hence (thanks to (p4)) the nets $(T_{s_{\sigma}} x)_{\sigma \in \Sigma}$ and $(\| (T_{s_{\sigma}} | Y)^{-1}) \|_{\sigma \in \Sigma}$ are eventually bounded. This implies that $(T_{s_{\sigma}} | Y)^{-1}$ converges in the strong operator topology to $(T_s | Y)^{-1}$ and $\lim_{\sigma \in \Sigma} (P.s_{\sigma}) x = (P.s) x$ for every $P \in \Delta$. So, the action is separately continuous. Finally, assume that, in addition, $\mathcal{S}$ is weakly right amenable; then—by (p0)—we get

$$
\lim_{\sigma \in \Sigma} \| T_{s_{\sigma}} - T_s \| = 0
$$

and $\lim_{\sigma \in \Sigma} \| (T_{s_{\sigma}} | Y)^{-1} - (T_s | Y)^{-1} \| = 0$. Now if $(P_\sigma)_{\sigma \in \Sigma}$ is a net in $\Delta$ convergent (in the topology of $\Delta$) to $P \in \Delta$, then the net $((T_{s_{\sigma}} | Y)^{-1} P_{\sigma})_{\sigma \in \Sigma}$ is bounded and converges in the weak* operator topology to $(T_s | Y)^{-1} P$ (thanks to (p3') and the boundedness of $\Delta$), which—combined with (4.3)—implies that $P_{\sigma}.s_{\sigma}$ converges to $P.s$ in the topology of $\Delta$.

To complete the proof, observe that the “moreover” claim of the proposition follows from the compactness of $\Delta$ in the weak* operator topology, and from the lower semicontinuity of the operator norm in this topology. ■

**Corollary 4.3.** Let $\mathcal{S}$ be a weakly right amenable SGT, $Z$ a Banach space and $\Psi: \mathcal{S} \ni s \mapsto V_s \in \mathcal{L}(Z)$ a representation that is continuous in the operator norm topology of $\mathcal{L}(Z)$. Suppose $W$ is a closed linear subspace of $Z$ such that:

(a) for any $s \in \mathcal{S}$, $V_s(W) = W$ and $V_s|W$ is an isomorphism;
(b) there is a projection $Q: Z \to W$ with $\sup_{s \in \mathcal{S}} \| (V_s|W)^{-1} Q V_s \| < \infty$.

Then there exists a projection $P: Z^{**} \to W_{\perp\perp}$ such that $P V_s^{**} = V_s^{**} P$ for all $s \in \mathcal{S}$.

Recall that $W_{\perp\perp} = (W_{\perp})_{\perp} \subset Z^{**}$ is the annihilator of $W_{\perp} \subset Z^*$ (where $W_{\perp}$ consists of all functionals in $Z^*$ that vanish on $W$).

**Proof.** Observe that $Q^{**}: Z^{**} \to Z^{**}$ is a projection onto $W_{\perp\perp}$, set $X := Z^{**}$, $T_s := V_s^{**}$ ($s \in \mathcal{S}$), $\Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$, $Y := W_{\perp\perp}$ and apply Proposition 4.2. ■

**Definition 4.4.** When $Y$ is a closed linear subspace of a Banach space $X$ and $\Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ is a representation, a projection $P_0: X \to Y$ is called $\Phi$-minimal if $P_0$ commutes with all $T_s$, and $\| P_0 \| \leq \| P \|$ for any other projection $P: X \to Y$ that commutes with all $T_s$.

For finite-dimensional subspaces $Y$, Proposition 4.2 can be strengthened as follows.

**Proposition 4.5.** Let $\mathcal{S}$ be a strongly [resp. weakly] right amenable SGT, $X$ a Banach space and $\Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ a representation.
Suppose $Y$ is a finite-dimensional linear subspace of $X$ and:

(fp0) $\Phi$ is continuous in the weak operator topology [resp. strong operator topology] of $\mathcal{L}(X)$;

(fp1) for any $s \in \mathcal{S}$, $T_s(Y) = Y$.

Then the following conditions are equivalent:

(i) there exists a projection $P : X \to Y$ with $PT_s = T_sP$ for all $s \in \mathcal{S}$;

(ii) there is a projection $Q : X \to Y$ such that

$$\sup_{s \in \mathcal{S}} \| (T_s|Y)^{-1} QT_s \| < \infty.$$ 

Moreover, if (ii) holds, then there exists a $\Phi$-minimal projection $P_0 : X \to Y$.

The proof is quite similar to the previous one and we skip it. (Use the fact that under the assumptions of Proposition 4.5, both the functions

$\mathcal{S} \ni s \mapsto T_s|Y \in \mathcal{L}(Y)$

and

$\mathcal{S} \ni s \mapsto (T_s|Y)^{-1} \in \mathcal{L}(Y)$

are continuous in the operator norm topology of $\mathcal{L}(Y)$.)

Consequences of Propositions 4.2 and 4.5 follow.

**Theorem 4.6.** Let $\mathcal{S}$ be a strongly [resp. weakly] right amenable SGT, $X$ a Banach space and $\Phi : \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ a representation. Suppose $Y$ is a closed linear subspace of $X$ such that conditions (p0)–(p3) of Proposition 4.2 are fulfilled and

(p4') $m_Y(\Phi) := \sup_{s \in \mathcal{S}} (\|T_s\| \cdot \|(T_s|Y)^{-1}\|)$ is finite.

Then there exists a $\Phi$-minimal projection $P_0 : X \to Y$ iff $Y$ is a complemented subspace of $X$. Moreover, if $Y$ is complemented, each $\Phi$-minimal projection $P : X \to Y$ satisfies

(4.4) $\|P\| \leq m_Y(\Phi) \lambda(X,Y)$.

A similar (but different) result for compact topological groups can be found in [24] (see also [25, Theorem 5.18]).

**Proof of Theorem 4.6.** First of all, observe that (p4') implies (p4) (from Proposition 4.2): Otherwise we would have a net $(s_\sigma)_{\sigma \in \Sigma} \subset \mathcal{S}$ convergent to some $s \in \mathcal{S}$ such that

(4.5) $\lim_{\sigma \in \Sigma} \|(T_{s_\sigma}|Y)^{-1}\| = \infty$.

Then (p4') would imply that $\lim_{\sigma \in \Sigma} \|T_{s_\sigma}\| = 0$ and consequently $T_{s_\sigma} = 0$, and $Y = \{0\}$ (by (p1)), which contradicts (4.5).

Now thanks to Proposition 4.2, we only need to prove that a $\Phi$-minimal projection $P : X \to Y$ satisfies (4.4) provided $Y$ is complemented. For any $\varepsilon > 0$ there exists a projection $Q_\varepsilon : X \to Y$ such that $\|Q_\varepsilon\| \leq \lambda(X,Y) + \varepsilon$. Now repeat the proof of Proposition 4.2 with $M_\varepsilon := m_Y(\Phi)(\lambda(X,Y) + \varepsilon)$ to infer the existence of a projection $P_\varepsilon : X \to Y$ that commutes with all $T_s$ and
satisfies $\|P_\varepsilon\| \leq M_\varepsilon$. It follows that any $\Phi$-minimal projection $P: X \to Y$ satisfies $\|P\| \leq \|P_\varepsilon\|$ and thus (4.4) holds. ■

The proof of the next result is similar and therefore omitted. In the next two results, we use $m_Y(\Phi)$ to denote the quantity introduced in Theorem 4.6 (see (p4′) there).

**Theorem 4.7.** Let $\mathcal{S}$ be a strongly [resp. weakly] right amenable SGT, $X$ a Banach space and $\Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ a representation. Suppose $Y$ is a finite-dimensional linear subspace of $X$ such that conditions (fp0)–(fp1) (of Proposition 4.5) and (p4′) (of Theorem 4.6) hold. Then there exists a $\Phi$-minimal projection $P: X \to Y$, and $\|P\| \leq m_Y(\Phi)\lambda(X,Y)$.

For simplicity, for any representation $\Phi: \mathcal{S} \to \mathcal{L}(X)$ (where $X$ is a Banach space) we denote by $\mathcal{N}(\Phi)$ and $\mathcal{R}(\Phi)$ the intersection of the kernels and, respectively, the closed linear span of the ranges of all operators of the form $\Phi(s) - I$ where $s \in \mathcal{S}$ and $I$ is the identity operator on $X$. When $\Phi$ is the identity function (and $\mathcal{S} \subset \mathcal{L}(X)$), we write $\mathcal{N}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ instead of $\mathcal{N}(\Phi)$ and $\mathcal{R}(\Phi)$.

**Corollary 4.8.** For an SGT $\mathcal{S}$ the following conditions are equivalent:

(a) $\mathcal{S}$ is strongly [resp. weakly] right amenable;

(b) for any representation $\Phi: \mathcal{S} \to \mathcal{L}(X)$ continuous in the weak [resp. strong] operator topology with $M(\Phi) := \sup_{s \in \mathcal{S}} \|T_s\|$ finite one has:

- $\mathcal{N}(\Phi) \cap \mathcal{R}(\Phi) = \{0\}$,
- the subspace

$$D(\Phi) := \mathcal{N}(\Phi) + \mathcal{R}(\Phi)$$

is closed in $X$, and

- the projection $P: D(\Phi) \to N(\Phi)$ induced by the decomposition (4.6) has norm not greater than $M(\Phi)$.

**Proof.** Assume (b) holds, put $X = C_{\text{norm}}(\mathcal{S})$ (resp. $X = C_{\text{weak}}(\mathcal{S})$) and define $\Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ by $T_s f = f_s$. Note that $\Phi$ is a representation continuous in the weak (resp. strong) operator topology such that $M(\Phi) = 1$. So, it follows from (b) that the projection $P: D(\Phi) \to N(\Phi)$ specified therein has norm not exceeding 1. Observe that the function $j: \mathcal{S} \to \mathbb{R}$ constantly equal to 1 belongs to $\mathcal{N}(\Phi)$ and thus $1 = \|j\| = \|P(j + z)\| \leq \|j + z\|$ for any $z \in \mathcal{R}(\Phi)$. Equivalently, dist$(j, \mathcal{R}(\Phi)) = 1$. Now the Hahn–Banach theorem implies that there is a linear functional $\phi: X \to \mathbb{R}$ that vanishes on $\mathcal{R}(\Phi)$ and satisfies $\phi(j) = 1 = \|\phi\|$. This means that $\phi$ is an invariant mean of $X$ and we are done.

Now assume (a) holds and let $X$ and $\Phi: \mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ be as specified in (b). Fix a non-zero vector $y \in N(\Phi)$ and consider the linear subspace $Y$ generated by $y$. Since $Y$ is one-dimensional, $\lambda(X,Y) = 1$. Observe
that conditions (fp0) and (fp1) hold, and \( m_Y(\Phi) = M(\Phi) \). So, Theorem 4.7 gives us a \( \Phi \)-minimal projection \( P_y : X \to Y \) with \( \|P_y\| \leq M(\Phi) \). For any \( s \in \mathcal{S} \) and \( x \in X \) we have

\[
P_y(T_s x - x) = T_s(P_y x) - P_y x = 0,
\]
as \( P_y x \in Y \subset N(\Phi) \). So, \( T_s x - x \in \ker(P_y) \) and hence \( \mathcal{R}(\Phi) \subset \ker(P_y) \).

We conclude that for any \( z \in \mathcal{R}(\Phi) \), \( \|y\| = \|P_y(y + z)\| \leq M(\Phi)\|y + z\| \).

It follows from the arbitrariness of \( y \) and \( z \) that \( N(\Phi) \cap \mathcal{R}(\Phi) = \{0\} \) and \( \|P\| \leq M(\Phi) \) (where \( P \) is as in (b)). The closedness of \( N(\Phi) + \mathcal{R}(\Phi) \) follows from the continuity of \( P \) and the closedness of \( N(\Phi) \) and \( \mathcal{R}(\Phi) \). ■

Recall that a Banach space \( X \) is HI (hereditarily indecomposable) if for every projection \( P : Y \to Z \) where \( Z \subset Y \subset X \) the kernel or the range of \( P \) is finite-dimensional.

As an immediate consequence of Corollary 4.8 (see Example 3.4(B)) we obtain

**Corollary 4.9.** Let \( \mathcal{S} \subset \mathcal{L}(E) \) be a bounded abelian multiplicative semigroup of operators on a HI Banach space \( E \). Then either \( N(\mathcal{S}) \) or \( \mathcal{R}(\mathcal{S}) \) is finite-dimensional.

Now we offer a result on extending intertwining operators.

**Theorem 4.10.** Let \( \mathcal{S} \) be a strongly [resp. weakly] right amenable SGT, let \( X \) and \( Y \) be Banach spaces, and let \( \Phi : \mathcal{S} \ni s \mapsto A_s \in \mathcal{L}(X) \) and \( \Psi : \mathcal{S} \ni s \mapsto B_s \in \mathcal{L}(Y) \) be two representations. Suppose \( E \) is a closed linear subspace of \( X \) and \( T_0 : E \to Y \) is a bounded linear operator such that:

(i0) \( \Phi \) and \( \Psi \) are continuous in the strong operator topologies [resp. operator norm topologies] of \( \mathcal{L}(X) \) and \( \mathcal{L}(Y) \), respectively;

(i1) for any \( s \in \mathcal{S} \), \( A_s(E) \subset E \) and \( B_s \) is an isomorphism;

(i2) \( Y \) is a dual Banach space;

(i3) for any \( s \in \mathcal{S} \), \( B_s \) is \( \text{weak}^* \) continuous on the closed unit ball \( B_Y \);

(i4) the function \( \mathcal{S} \ni s \mapsto \|B_s^{-1}\| \in \mathbb{R} \) is locally bounded;

(i5) \( B_sT_0 = T_0A_s \) \( E \) for any \( s \in \mathcal{S} \).

Then the following conditions are equivalent:

(i) \( T_0 \) extends to a bounded linear operator \( T : X \to Y \) that intertwines \( \Phi \) and \( \Psi \), that is, \( B_sT = TA_s \) for all \( s \in \mathcal{S} \);

(ii) there exists a bounded linear operator \( T' : X \to Y \) that extends \( T_0 \) and satisfies \( \sup_{s \in \mathcal{S}} \|B_s^{-1}T'A_s\| < \infty \).

**Proof.** The proof is similar to that of Proposition 4.2 and thus we only sketch it. For any bounded linear operator \( L : X \to Y \) and \( s \in \mathcal{S} \) we denote \( Ls := B_s^{-1}LA_s \). This defines an affine right action of \( \mathcal{S} \). Next, assume
$T'$ is as in (ii) and set $M := \sup_{s \in \mathcal{S}} \|T'.s\|$. Equip the set $\Delta$ of all linear extensions $T: X \to Y$ of $T_0$ such that $\|T\| \leq M$ and $\sup_{s \in \mathcal{S}} \|T.s\| \leq M$ with the weak* operator topology. Now it suffices to repeat the arguments in the proof of Proposition 4.2 to conclude that the action defined above is suitably continuous on $\Delta \times \mathcal{S}$ and thus has a fixed point $T$ in $\Delta$. Then $B_s T = T A_s$ for any $s \in \mathcal{S}$ and we are done.

The following result is a special case of Theorem 4.10 and we skip its proof.

**Corollary 4.11.** Suppose all assumptions of Theorem 4.10 (on $\Phi$, $\Psi$, $Y$ and $T_0$) hold. If, in addition, $\sup_{s \in \mathcal{S}} (\|B^{-1}_s \| \cdot \|A_s\|) < \infty$, then $T_0$ extends to a bounded linear operator $T: X \to Y$ intertwining $\Phi$ and $\Psi$ iff it extends to a bounded linear operator from $X$ into $Y$.

**Example 4.12.** As the following simple example shows, the assumption in Theorem 4.6 (and hence also in Proposition 4.2) that $T_s(Y) = Y$ for all $s \in \mathcal{S}$ (see (p1)) cannot be dropped in general. Let $X$ be a separable Hilbert space with an orthonormal basis $e_0, e_1, e_2, \ldots$ and $S: X \to X$ be the shift—that is, $Se_n = e_{n+1}$ ($n \geq 0$). Further, let $\mathcal{S}$ be the additive semigroup of all natural numbers, $\Phi: \mathcal{S} \ni n \mapsto S^n \in \mathcal{L}(X)$ and $Y$ be the range of $S$. Then all assumptions of Theorem 4.6 except for “$S(Y) = Y$” are fulfilled and $Y$ is complemented. However, there is no projection onto $Y$ that commutes with $S$.

The above example can also be employed to show that the assumption in Corollary 4.11 that all the operators $B_s$ are isomorphisms (see (i1)) cannot be dropped in general. Indeed, it suffices to set, in addition to the above, $T_0$ to be the identity operator on $E := Y \subset X$, and $\Psi = \Phi$.

Further results of this section deal with renorming-like issues. The first of them generalises results of Koehler and Rosenthal [17] (see therein for the definition of a semi-inner product on a Banach space).

**Theorem 4.13.** Let $\mathcal{S}$ be a weakly right amenable SGT, $(X, \| \cdot \|_X)$ a Banach space and $\mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(X)$ a representation that is continuous in the strong operator topology of $\mathcal{L}(X)$. Further, let $m$ and $M$ be positive real constants such that

\[
\tag{4.7} m \|x\|_X \leq \|T_s x\|_X \leq M \|x\|_X \quad (x \in X, s \in \mathcal{S}).
\]

Then there exists a norm $\| \cdot \|_*$ on $X$ such that for any $x \in X$ and $s \in \mathcal{S}$, $m \|x\|_X \leq \|x\|_* \leq M \|x\|_X$ and $\|T_s x\|_* = \|x\|_*$. If, in addition, the topology of $\mathcal{S}$ is discrete, then there exists a semi-inner product $[\cdot, -]_*$ inducing the norm $\| \cdot \|_*$ such that $[T_s x, T_s y]_* = [x, y]_*$ for all $x, y \in X$ and $s \in \mathcal{S}$.
Proof. For any seminorm \( \| \cdot \| \) on \( X \) and \( s \in \mathcal{I} \) we denote by \( \| \cdot \|_s \) the seminorm on \( X \) that assigns to a vector \( x \) the number \( \| T_s x \| \). It is easy to check that
\[
(\| \cdot \|_s, s) \mapsto \| \cdot \|_s
\]
is an affine right action of \( \mathcal{I} \) on the set of all seminorms on \( X \).

Firstly, let \( \Delta \) be the set of all norms \( \| \cdot \| \) on \( X \) such that \( m\|x\|_X \leq \|x\| \leq M\|x\|_X \) and \( m\|x\|_X \leq \|T_s x\| \leq M\|x\|_X \) for all \( x \in X \) and \( s \in \mathcal{I} \). We equip \( \Delta \) with the pointwise convergence topology. Observe that \( \Delta \) is convex and compact (by the Tikhonov theorem), and \( \| \cdot \|_s \in \Delta \) (by (4.7)) and \( \| \cdot \|_s \in \Delta \) for all \( \| \cdot \| \in \Delta \) and \( s \in \mathcal{I} \). One also easily verifies that the action (4.8) is jointly continuous on \( \Delta \times \mathcal{I} \). Hence, by Theorem 3.2 there is a norm \( \| \cdot \|_* \in \Delta \) which is a fixed point for this action. This means that all \( T_s \) are isometries with respect to this norm.

Secondly, let \( \Gamma \) consist of all semi-inner products on \( X \) that induce the norm \( \| \cdot \|_* \). As before, \( \Gamma \) is convex, compact (in the pointwise convergence topology) and non-empty. For \([ \cdot, -] \in \Gamma \) and \( s \in \mathcal{I} \), we define \([ \cdot, -]_s \in \Gamma \) by \([x, y]_s := [T_s x, T_s y] \). As before, one easily checks that in this way we have defined an affine right action of \( \mathcal{I} \) on \( \Gamma \). So, if the topology of \( \mathcal{I} \) is discrete, the fixed point property in Theorem 3.2 finishes the proof.

Corollary 4.14. Let \( \mathbb{I} \) be an SGT, \((X, \| \cdot \|)\) a Banach space and \( \mathbb{I} \ni s \mapsto T_s \in \mathcal{L}(X) \) a representation that is continuous in the strong operator topology of \( \mathcal{L}(X) \). Further, assume \( \mathcal{I}_0 \) is a weakly right amenable subsemigroup of \( \mathcal{I} \) (in the topology inherited from \( \mathcal{I} \)) such that

\((\star)\) \( \mathcal{I} \) is a unique subsemigroup \( \mathcal{Z} \) of \( \mathcal{I} \) that is closed, contains \( \mathcal{I}_0 \) and has the following property: if \( x, y \in \mathcal{I} \) and \( x, xy \in \mathcal{Z} \), then \( y \in \mathcal{Z} \).

If \( m \) and \( M \) are positive real constants such that
\[
(4.9) \quad m\|u\| \leq \|T_s u\| \leq M\|u\| \quad (u \in X, s \in \mathcal{I}_0),
\]
then for any \( s \in \mathcal{I} \) and \( u \in X \),
\[
\frac{m}{M}\|u\| \leq \|T_s u\| \leq \frac{M}{m}\|u\|.
\]

Proof. It follows from Theorem 4.13 that there exists a norm \( \| \cdot \|_* \) on \( X \) such that \( m\|u\| \leq \|u\|_* \leq M\|u\| \) and \( \|T_s u\|_* = \|u\|_* \) (that is, \( T_s \) is isometric with respect to \( \| \cdot \|_* \) for any \( s \in \mathcal{I}_0 \) and \( u \in X \). Now let \( \mathcal{Z} \) consist of all \( s \in \mathcal{I} \) such that \( T_s \) is isometric with respect to \( \| \cdot \|_* \). We conclude from (\( \star \)) that \( \mathcal{Z} = \mathcal{I} \). But then, for any \( s \in \mathcal{I} \) and \( u \in X \), \( \|T_s u\| \leq \frac{1}{m}\|T_s u\|_* = \frac{1}{m}\|u\| \leq \frac{M}{m}\|u\| \) and similarly \( \|T_s u\| \geq \frac{1}{M}\|T_s u\|_* = \frac{1}{M}\|u\|_* \geq \frac{m}{M}\|u\| \). ■

Remark 4.15. The above result has an interesting consequence: if \( \mathcal{I}_0 \) is a weakly right amenable SGT that generates a topological group \( \mathcal{I} \) and \( \mathcal{I} \ni g \mapsto T_g \in \mathcal{L}(X) \) is a unital representation that is continuous in the
strong operator topology and satisfies (4.9) (see Corollary 4.14), then the operators of the form $T_{s_1}^{\varepsilon_1} \cdots T_{s_k}^{\varepsilon_k}$ (where $s_1, \ldots, s_k \in S_0$, $\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}$ and $k > 0$ are arbitrary) are uniformly bounded.

Sz.-Nagy [28] proved that an invertible operator on a Hilbert space whose powers (both positive and negative) are uniformly bounded is similar to a unitary operator. Equivalently, bounded representations on Hilbert spaces of the additive group of integers are unitarizable. A few years later this last result was generalised to the context of all amenable locally compact groups by Dixmier [9] and Day [5]. Our next result generalises both the Sz.-Nagy and Dixmier–Day theorems. Below we use $I$ to denote the identity operator on a Hilbert space $H$.

**Proposition 4.16.** Let $\mathcal{S}$ be a weakly right amenable SGT, $(H, \langle \cdot, \cdot \rangle)$ a Hilbert space and $\mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(H)$ a representation that is continuous in the strong operator topology of $\mathcal{L}(H)$. Further, suppose there exist an operator $A_0 \in \mathcal{L}(H)$ and functions $m, M : H \to (0, \infty)$ such that for any $x \in H$,

\begin{equation}
4.10 \quad m(x) \|x\| \leq \|A_0 T_s x\| \leq M(x) \|x\| \quad (s \in \mathcal{S}).
\end{equation}

Then there exist a positive operator $A \in \mathcal{L}(H)$ with trivial kernel and a representation $\mathcal{S} \ni s \mapsto V_s \in \mathcal{L}(H)$ continuous in the strong operator topology such that:

- $m(x) \|x\| \leq \|Ax\| \leq M(x) \|x\|$ for any $x \in H$;
- $AT_s = V_s A$ for all $s \in \mathcal{S}$;
- $V_s$ is an isometry for any $s \in \mathcal{S}$.

**Proof.** It follows from (4.10) and the Uniform Boundedness Principle that

$$\sup_{s \in \mathcal{S}} \|A_0 T_s\| < \infty.$$  

Consequently, we may and do assume that

\begin{equation}
4.11 \quad \sup_{x \in H} M(x) < \infty.
\end{equation}

For any $B \in \mathcal{L}(H)$ and $s \in \mathcal{S}$ let $B.s$ stand for the operator $T_s^* B T_s$. It is easily seen that $(B, s) \mapsto B.s$ is an affine right action of $\mathcal{S}$ on $\mathcal{L}(H)$.

Denote by $\Delta$ the set of all positive operators $B \in \mathcal{L}(H)$ such that

\begin{equation}
4.12 \quad m(x)^2 \|x\|^2 \leq \langle Bx, x \rangle \leq M(x)^2 \|x\|^2
\end{equation}

and $m(x)^2 \|x\|^2 \leq \langle (B.s)x, x \rangle \leq M(x)^2 \|x\|^2$ for any $x \in H$ and $s \in \mathcal{S}$, and equip $\Delta$ with the weak operator topology. It is clear that $\Delta$ is convex and compact, and that $(A_0^* A_0).s \in \Delta$ (by (4.10)) and $B.s \in \Delta$ for any $B \in \Delta$ and $s \in \mathcal{S}$. We claim that the action just defined is jointly continuous on $\Delta \times \mathcal{S}$. To see this, assume $(B_{\sigma})_{\sigma \in \Sigma} \subset \Delta$ and $(s_{\sigma})_{\sigma \in \Sigma} \subset \mathcal{S}$ are nets that converge to, respectively, $B \in \Delta$ and $s \in \mathcal{S}$. Then $T_{s_{\sigma}}$ converges to $T_s$ in
the strong operator topology. Since the operators $B_\sigma$ are uniformly bounded (by (4.11) and (4.12)), we conclude that $B_\sigma T_{s_\sigma}$ converges to $BT_s$ in the weak operator topology. So, when $x, y \in H$ are fixed, the net $(T_{s_\sigma}x)_{\sigma \in \Sigma}$ is eventually bounded and thus
\[
\lim_{\sigma \in \Sigma} \langle T_{s_\sigma}^* B_\sigma T_{s_\sigma} x, y \rangle = \lim_{\sigma \in \Sigma} \langle B_\sigma T_{s_\sigma} x, T_{s_\sigma} y \rangle = \langle BT_s x, T_s y \rangle = \langle T_s^* BT_s x, y \rangle
\]
(because the vectors $B_\sigma T_{s_\sigma} x$ are uniformly bounded and converge weakly to $BT_s x$, and $T_{s_\sigma} y$ norm converges to $T_s y$).

Now the fixed point property from Theorem 3.2 gives us an operator $B \in \Delta$ such that
\[
(4.13) \quad T_s^* BT_s = B \quad (s \in \mathcal{S}).
\]
We define $A$ as the (positive) square root of $B$. For any $x \in H$, $\|Ax\|^2 = \langle Bx, x \rangle$ and thus $m(x)\|x\| \leq \|Ax\| \leq M(x)\|x\|$ (thanks to (4.12)), which in turn implies that the kernel of $A$ is trivial.

Fix $s \in \mathcal{S}$ for a moment and let $V_s$ be the partial isometry that appears in the polar decomposition $AT_s = V_s|AT_s|$ (where $|AT_s| = \sqrt{(AT_s)^*(AT_s)}$). Formula (4.13) gives $(AT_s)^*(AT_s) = A^2$. By the uniqueness of the positive square root we obtain $|AT_s| = A$ and consequently $AT_s = V_sA$. Since $A$ has trivial kernel, $V_s$ is an isometry.

It remains to check that $s \mapsto V_s$ is a representation continuous in the strong operator topology. For $s, t \in \mathcal{S}$ we have $V_{st}A = AT_{st} = AT_sT_t = V_sAT_t = V_sV_tA$. Since the range of $A$ is dense in $H$, we get $V_{st} = V_sV_t$. Finally, if $s_\sigma \in \mathcal{S}$ converges to $s \in \mathcal{S}$, then for any $x \in H$, $\lim_{\sigma \in \Sigma} V_{s_\sigma}(Ax) = \lim_{\sigma \in \Sigma} A(T_{s_\sigma}x) = AT_s x = V_s(Ax)$, and again the density of the range of $A$ in $H$ implies that $V_{s_\sigma}$ converges to $V_s$ in the strong operator topology.

As an immediate consequence of the above result, we obtain a counterpart of the Dixmier–Day theorem for weakly right amenable SGT’s:

**Corollary 4.17.** Let $\mathcal{S}$ be a weakly right amenable SGT, $H$ a Hilbert space and $\mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(H)$ a representation that is continuous in the strong operator topology of $\mathcal{L}(H)$. There exists an invertible positive operator $A \in \mathcal{L}(H)$ such that $AT_sA^{-1}$ is an isometry for each $s \in \mathcal{S}$ iff there are positive real constants $m$ and $M$ such that
\[
(4.14) \quad m\|x\| \leq \|T_s x\| \leq M\|x\| \quad (s \in \mathcal{S}, \ x \in H).
\]
Moreover, if (4.14) holds, then $A$ can be chosen so that $mI \leq A \leq MI$.

The proof is left to the reader.

**Remark 4.18.** Corollary 4.17 combined with Corollary 4.14 leads to a stronger generalisation of the Sz.-Nagy theorem that reads as follows:

If $\mathcal{S}$ is an SGT, $H$ is a Hilbert space and $\mathcal{S} \ni s \mapsto T_s \in \mathcal{L}(H)$ is a representation that is continuous in the strong operator topology and
there are a weakly right amenable subsemigroup \( \mathcal{S}_0 \) of \( \mathcal{S} \) and positive real constants \( m \) and \( M \) such that condition (⋆) of Corollary 4.14 is fulfilled and \( m\|x\| \leq \|T_s x\| \leq M\|x\| \) for any \( s \in \mathcal{S}_0 \) and \( x \in H \), then there exists an invertible positive operator \( A \in \mathcal{L}(H) \) such that \( AT_s A^{-1} \) is an isometry for each \( s \in \mathcal{S} \).

In particular, if \( G \) is a topological group and a weakly right amenable semigroup \( \mathcal{S} \subset G \) generates \( G \), then each unital representation \( \Phi: G \ni g \rightarrow T_g \in \mathcal{L}(H) \) that is continuous on \( \mathcal{S} \) in the strong operator topology and satisfies (4.14) has image \( \Phi(G) \) “similar” to a subgroup of the unitary group of \( H \).

Proofs of Theorems 1.1–1.4. Since all right amenable discrete semigroups are strongly right amenable, the results in question are special cases of, respectively, Theorem 4.6, Theorem 4.10, Corollary 4.8 and Corollary 4.17.

5. More on amenability. To formulate our next result, let us introduce some elementary notion. For any (unital or non-unital) SGT \( \mathcal{S} \) we use \( \mathcal{S} \sqcup \{1\} \) to denote its unitization, that is, \( \mathcal{S} \sqcup \{1\} \) is a unital SGT such that \( \mathcal{S} \sqcup \{1\} = \mathcal{S} \sqcup \{1\} \) where \( 1 \notin \mathcal{S} \) is the neutral element of \( \mathcal{S} \sqcup \{1\} \) and an isolated point, the multiplication of \( \mathcal{S} \sqcup \{1\} \) extends that in \( \mathcal{S} \), \( \mathcal{S} \) is both closed and open in \( \mathcal{S} \sqcup \{1\} \) and the topology of \( \mathcal{S} \) coincides with the one inherited from \( \mathcal{S} \sqcup \{1\} \).

Note that even if \( \mathcal{S} \) is unital, the neutral element of \( \mathcal{S} \sqcup \{1\} \) differs from the one of \( \mathcal{S} \).

Below we list basic properties of the class of right amenable SGT’s. They are likely well-known and more or less straightforward (however, we have been unable to find them in the literature). For the reader’s convenience, we give some hints on how to prove some of them.

**Proposition 5.1.**

(A) An SGT \( \mathcal{S} \) is weakly [resp. strongly] right amenable iff so is \( \mathcal{S} \sqcup \{1\} \).

(B) Let \( h: \mathcal{T} \rightarrow \mathcal{S} \) be a continuous onto homomorphism between SGT’s. If \( \mathcal{T} \) is weakly [resp. strongly] right amenable, so is \( \mathcal{S} \).

(C) Let \( \mathcal{T} \) be a subsemigroup of an SGT \( \mathcal{S} \) such that \( \mathcal{S} \) is a unique subsemigroup \( \mathcal{Z} \) of \( \mathcal{S} \) that is closed, contains \( \mathcal{T} \) and has the following property:

(⋆⋆) if \( x, y \in \mathcal{Z} \) and \( x, xy \in \mathcal{S} \), then \( y \in \mathcal{Z} \).

Then, if \( \mathcal{T} \) is weakly [resp. strongly] right amenable, so is \( \mathcal{S} \).

(D) Let \( \{ \mathcal{S}_b \}_{b \in B} \) be a family of subsemigroups of an SGT \( \mathcal{S} \) such that \( \mathcal{S} \) coincides with the smallest closed subsemigroup that contains \( \bigcup_{b \in B} \mathcal{S}_b \). Assume \( ss' = s's \) for any \( s \in \mathcal{S}_b \) and \( s' \in \mathcal{S}_{b'} \) (\( b, b' \in B \) with \( b \neq b' \)). If all \( \mathcal{S}_b \) are weakly [resp. strongly] right amenable, so is \( \mathcal{S} \).

(E) The (topological) product of a family of weakly [resp. strongly] right amenable unital SGT’s is weakly [resp. strongly] right amenable.
We do not know whether the assumption in (E) that SGT’s are unital can be dropped (even when the family of SGT’s is finite). It is worth noting here that a counterpart of (E) for abstract groups (that is, groups without topologies or—equivalently—if their full product is equipped with the discrete topology) is false even for countable families of countable groups (see, e.g., [7, paragraph preceding condition (K), p. 517]).

Proof of Proposition 5.1. Item (A) is trivial, (E) is a special case of (D) (each of the factor semigroups can be naturally embedded in the product), whereas all other items easily follow from Theorem 3.2. Indeed, given a suitably (that is, jointly or separately) continuous affine right action of the SGT in question on a compact convex set $K$, use the following properties:

(B) $K \times \mathcal{I} \ni (x, t) \mapsto x.h(t) \in K$ is an affine action.

(C) $\mathcal{Z} := \{s \in \mathcal{I} : a.s = a\}$ is a closed subgroup of $\mathcal{I}$ which satisfies (**).

(D) The family $\{F_b\}_{b \in B}$ with $F_b := \{x \in K : x.s = x \text{ for any } s \in \mathcal{I}_b\}$ has the finite intersection property (that is, $\bigcap_{b \in B_0} F_b \neq \emptyset$ for any finite non-empty $B_0 \subset B$) and each of these sets is invariant for the action restricted to all $\mathcal{I}_b$ ($b \in B$).

The next property (in the class of topological groups), with which we end the paper, is less elementary.

Theorem 5.2. If an SGT $\mathcal{I}$ contains a dense subsemigroup that is weakly [resp. strongly] right amenable, then $\mathcal{I}$ itself is weakly [resp. strongly] right amenable.

A dense subgroup of a weakly right amenable topological group is weakly right amenable as well.

Proof. The first claim of the theorem is a special case of Proposition 5.1 (C).

To prove the second claim, we will make use of so-called Raïkov-complete topological groups (see, e.g., [2, Section 3.6]). We need the following two facts about them:

(RC1) If $u : H_0 \to G$ is a continuous homomorphism of a dense subgroup $H_0$ of a topological group $H$ into a Raïkov-complete topological group $G$, then $u$ (uniquely) extends to a continuous homomorphism from $H$ into $G$.

(RC2) The homeomorphism group (equipped with the compact-open topology) of a compact Hausdorff space is Raïkov-complete.

Now assume $H_0$ is a dense subgroup of a weakly right amenable topological group $H$. Let $K \times H_0 \ni (x, h) \mapsto x.h \in K$ be a (jointly) continuous affine right action of $H_0$ on a non-empty compact convex set $K$. For any $h \in H_0$ denote by $f_h : K \to K$ the homeomorphism $x \mapsto x.h^{-1}$. It is clear that $\Phi_0 : h \mapsto f_h$ is a homomorphism of $H_0$ into the homeomorphism group $G$. 
of $K$. What is more, the joint continuity of the action is equivalent to the continuity of this homomorphism. So, (RC1) combined with (RC2) implies that $\Phi_0$ extends to a continuous homomorphism $\Phi : H \to G$. We conclude that the given action (of $H_0$ on $K$) extends to a jointly continuous right action $(x, h) \mapsto \Phi(h^{-1})(x)$ of $H$ on $K$. It is easy to verify that this action is affine and hence has a fixed point, which finishes the proof.

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