A new class of Pseudo-Hermitian Hamiltonians with Real Positive Spectra

Abouzeid. M. Shalaby

Theoretical Physics Group, Physics Department, Faculty of Science,
Mansoura University, Mansoura P.O. Box 35516, Egypt

Abstract

We introduce a new class of non-Hermitian Hamiltonians which possesses both $\mathcal{PT}$-symmetric and non-$\mathcal{PT}$-symmetric members. We calculated the corresponding class of positive definite metric operators in a closed form. We showed that the energy spectrum for each member in the class is positive ($E_n \geq 0$). To check the physical acceptability of a Hamiltonian in the class, we obtained the ground state functions for each member in a closed form to test their square integrability and conclude that only $\mathcal{PT}$-symmetric members out of the class can have ground state functions belong to the Hilbert space $L^2(\mathbb{R})$. Since the Hamiltonians introduced have an interaction term of the form of a pure function of the position operator times the momentum operator, we reintroduced a closely related class of Hamiltonians for which the ordering ambiguity does not exist. As the metric operators are quasi-gauge transformations and the class introduced has a potential term due to an imaginary magnetic field, they shift the action by a terminal term which will add nothing to the amount of any Physical quantity when calculated in the path integral formulation of the problem.

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I. INTRODUCTION

Hermiticity was introduced at the early stages of quantum mechanics as a mathematical constraint to secure a real spectrum for a Hamiltonian model. This constraint limited us to consider a sub-class of the physically acceptable Hamiltonians which has its drawback on the status of finding a complete understanding of matter interactions. For instance, the obligation to use Hermitian scalar field Hamiltonian leaded to the famous problematic Higgs mechanism while it is easy to show that the non-Hermitian scalar field can solve such kind of problems [1]. However, one has to prove that the energy spectrum is real for a non-Hermitian theory to be a physically acceptable candidate. In fact, there exist old examples in which non-Hermitian Hamiltonians can have real spectra. A very clear example is the Hermite differential equation given by

\[-\frac{d^2\psi}{dx^2} + 2x \frac{d\psi}{dx} = 2n\psi, \tag{1}\]

or in a Hamiltonian form $2H\psi = 2E\psi$, with $H = \frac{p^2}{2} + ip$ and $E = n\hbar\omega$ with $\hbar = \omega = 1$. It is clear that this Hamiltonian is not Hermitian though the energy is real and has the spectrum of a harmonic oscillator.

The picture has been declared rigorously after the appearance of the pioneering article of Carl Bender and Stefan Boettcher [2] where they showed that the energy spectra of a class of non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonians are real and positive. Their $\mathcal{PT}$-symmetric class has the form;

$H = p^2 + x^2 (ix)^n$, \hspace{1em} n \geq 0. \tag{2}$

The spectra of such class have been shown, numerically, to be real and positive [3] even in the case of $n = 2$. The reality of the spectrum of such kind of theories is proved to be due to the existence of an unbroken $\mathcal{PT}$ symmetry for such models. However, one can easily realize that, as we will show in this work, the model in Eq.(1) is a member of a class of an infinite number of Hamiltonians which are all non-Hermitian, not necessarily $\mathcal{PT}$-symmetric, and having positive real spectra as well. Accordingly, sticking to the requirement that a physically acceptable theory has to be either Hermitian or $\mathcal{PT}$-symmetric will exclude a huge number of acceptable theories. Mostafazadeh showed that the reality of the spectrum of a Hamiltonian is not limited either to Hermiticity or the existence of $\mathcal{PT}$-symmetry [4, 5]. Instead, he showed that if a Hamiltonian model $H$ has the property that $\eta_+ H \eta_+^{-1} = H^\dagger$, then the spectrum of $H$ is real. Here, $\eta_+$ is a Hermitian linear invertible operator and is a
positive definite operator as well. This formulation of the problem can be used for Hermitian
and $\mathcal{PT}$-symmetric theories as well as for any pseudo-Hermitian Hamiltonian with respect
to a positive definite metric operator $\eta$.

We know from the theory of orthogonal polynomials that if we have a differential equation
of the form

$$Q(x) \frac{d^2 \psi}{dx^2} + L(x) \frac{d \psi}{dx} + \lambda x = 0,$$

where $Q$ is a given quadratic (at most) polynomial, and $L$ is a given linear polynomial with
$\lambda$ is a constant, then

$$\int_a^b w(x) \psi_n \psi_m dx = 0 \quad \forall m \neq n,$$

where the weight function $w(x) = \frac{R(x)}{Q(x)}$ with $R(x) = \exp(\int \frac{L(x)}{Q(x)} dx)$. For the case at hand,
$a = -b = \infty$ and $w(x) = \exp(\int -x dx) = e^{-\frac{1}{2}x^2}$. In fact, the weight function $w(x)
represents the positive definite metric operator because $wH w^{-1} = H^\dagger$. Also,

$$\rho H \rho^{-1} = h = p^2 + \frac{1}{4}x^2 - \frac{1}{2},$$

where $h$ is the equivalent Hermitian Hamiltonian with $E = n \left(\sqrt{\frac{1}{4} + \frac{1}{2}}\right) = n$. Note also
that the eigen functions of $h$ are $\sqrt{\eta} \psi = \rho \psi$ which are the parabolic cylindrical functions.
According to the above analysis, the non-Hermitian Hermite equation is in every respect
equivalent to the Hermitian Hamiltonian of a harmonic oscillator which represents the mem-
ber with $n = 0$ in the $\mathcal{PT}$-symmetric class in Eq.(2). Relying on that, one may conjecture
that the new class of infinite number of Hamiltonians which are non-Hermitian, with non-
$\mathcal{PT}$-symmetric members, of the form;

$$H = H_0 + g H_I,$$

$$H_0 = p^2,$$

$$H_I = i x^\epsilon p,$$

has members equivalent to the non-real line members in the class in Eq.(2). To elucidate
this, one has to show first that each member in the new class in Eq.(6) has a real spectrum.
One can do that by finding out a positive definite metric operator $\eta$ by which one can map
$H$ to $h$ via the relation $h = \rho H \rho^{-1}$. 

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In the literature, reality of the eigen values of the non-Hermitian Hamiltonians are always concerned. However, the eigen functions should pass certain conditions like continuity and square integrability for the theory to be an acceptable one. Our class, however, is quasi solvable and one can , at least, get the ground state functions in a closed form to check its square integrability.

The paper is organized as follows. In Section II we introduce the class of non-Hermitian Hamiltonians and calculate its corresponding class of exact metric operators. In section III we analyze the conditions required by the class parameter to fulfill Hermiticity and stability of the corresponding equivalent Hermitian Hamiltonians. In this section, we showed that the energy spectrum for any member of the class is positive. Also, in this section, we obtained the ground state wave functions for each member in the class and showed that only \( \mathcal{PT} \)-symmetric Hamiltonians out of the class have square integrable wave functions and thus can have bound states. In section IV we showed that the form of the class introduced resembles a particle motion in an imaginary magnetic field while metric operators represent gauge transformations which leaded to the disappearance of the \( Q \) operator in any path integral calculation of a physical quantity. Finally, the conclusions follow in section V.

II. EXACT METRIC OPERATORS FOR THE PSEUDO-HERMITIAN igx\( \epsilon \) POTENTIALS

The calculations of the metric operator are always done perturbatively except for very rare cases. What is interesting in our class is that it is a real line theory, has non-\( \mathcal{PT} \)-symmetric members and one can get the exact metric operator for each member in a somehow simple manner. Moreover, the ground state of each member in the class can be calculated in a closed form as we will show in this work.

To start, consider the class of non-Hermitian Hamiltonians in the form:

\[
H = H_0 + gH_I,
\]

\[
H_0 = p^2,
\]

\[
H_I = ix^\epsilon p,
\]

where \( \epsilon \) is a real parameter. Since \( \rho H \rho^{-1} = h \) with \( \rho = \sqrt{\eta_+} = \exp \left( -\frac{Q}{2} \right) \) and \( \eta_+ H \eta_+^{-1} = H^\dagger \) where \( \eta_+ = \exp (-Q) \), we can get
\[ H^\dagger = \exp(-Q) H \exp(Q) = H + [-Q, H] + [-Q, [-Q, H]] \]
\[ + [-Q, [-Q, [-Q, H]]] + .... \]

Also, one has a similar expansion for the Hermitian Hamiltonian \( h = \exp(-\frac{Q^2}{2}) H \exp(\frac{Q^2}{2}) \), which will result in a perturbative expansion for \( h \) as

\[ h = h_0 + gh_1 + g^2 h_2 + ..... \]

Now, we have for \( H^\dagger \) the expansion;

\[ \exp(-Q) H \exp(Q) = H_0 + g H_I + [-Q, H_0] + [-Q, g H_I] + [-Q, [-Q, H_0]] + [-Q, [-Q, [-Q, H_0]]] + [-Q, [-Q, [-Q, g H_I]]] + [-Q, [-Q, [-Q, [Q, H_0]]]][\]
\[ = H_0 + g H_I^\dagger, \]

with

\[ Q = Q_0 + g Q_1 + g^2 Q_2 + + g^3 Q_3 + ..... \]

Thus, we get a set of coupled equations for the operators \( Q_n \), where the first few equations are given by

\[ 0 = [-Q_0, H_0] \Rightarrow Q_0 = 0 \text{ is a good choice,} \]
\[ H_I^\dagger - H_I = \frac{1}{2} [-Q_1, H_0], \]
\[ 0 = \frac{1}{2} [-Q_2, H_0] + \frac{1}{2} [-Q_1, H_I] + \frac{1}{3!} [Q_1, [Q_1, H_0]], \]
\[ 0 = \frac{1}{2} [-Q_3, H_0] + \frac{1}{2} [-Q_2, H_I] + \frac{1}{3!} [Q_2, [Q_1, H_0]] \]
\[ + \frac{1}{3!} [Q_1, [Q_2, H_0]] + \frac{1}{4!} [-Q_1, [-Q_1, [-Q_1, H_0]]] \]
\[ + \frac{1}{3!} [-Q_1, [-Q_1, H_I]], \]
\[ 0 = \frac{1}{2} [-Q_4, H_0] + \frac{1}{4} [-Q_3, H_I] + \frac{1}{3!} [-Q_2, [-Q_2, H_0]] \]
\[ + \frac{1}{5!} [Q_1, [Q_1, [Q_1, [Q_1, H_0]]]] + \frac{1}{3!} [-Q_2, [-Q_1, H_I]] \]
\[ + \frac{1}{3!} [-Q_1, [-Q_2, H_I] + \frac{1}{4!} [-Q_1, [-Q_1, [-Q_1, H_I]]] \]
\[ + \frac{1}{8 \times 4!} [-Q_1, [-Q_1, [-Q_2, H_0]]] \]
\[ + [-Q_1, [-Q_2, [-Q_1, H_0]]] + [-Q_2, [-Q_1, [-Q_1, H_0]]]. \]
To simplify this set of equations, one can use the realization that the Hermitian representation for the model $H = H_0 + gH_I$ can be obtained by the search for transformations which are able to kill the non-Hermitian interaction term $H_I$. In fact, assuming that $Q(x,p)$ is a function of $x$ only can do the job because then the transformation of $H_0$ with a suitable choice of $Q(x)$ will result in another function of $x$ times $p$. This piece of information can greatly simplify the above set of operator coupled equations. To show this, consider the transformation of $H_I$;

$$\exp (-Q) H_I \exp (-Q),$$

since $H_I$ is linear in the momentum operator then the commutators $[Q_n, H_I]$ are all functions of $x$ only. Accordingly, the above set has a miraculous simplification such that

$$Q_0 = 0,$$

$$H_I^\dagger - H_I = \frac{1}{2}[-Q_1, H_0],$$

$$Q_2 = Q_2 = Q_3 = ... = 0.$$

Now, one can try a polynomial in $x$ for $Q_1$ and start from lower order terms in the polynomial till we get the correct $H_I$. Following this, we find that the suitable choice for $Q$ is

$$Q = gQ_1 = g\frac{x^{\epsilon+1}}{\epsilon + 1}.$$

Accordingly, we get the result

$$h = p^2 + \frac{1}{4}g^2x^{2\epsilon} - \frac{1}{2}gx^{\epsilon-1}\epsilon.$$(10)

**Case 1:** $\epsilon = 1$

We have the result $h = p^2 + \frac{1}{4}g^2x^2 - \frac{1}{2}g$, which represents a harmonic oscillator.

**Case 2:** $\epsilon = 2$

Then, $h = p^2 + \frac{1}{4}g^2x^4 - gx$, which represents a quartic anharmonic oscillator with anomaly. Note that one can consider the non-Hermitian Hamiltonian $H = p^2 + gix^2p + \frac{g}{2}x$ to get the equivalent Hermitian Hamiltonian $h = p^2 + \frac{1}{4}g^2x^4 - \frac{g}{2}x$ which has the same form of the equivalent Hermitian Hamiltonian for the $\mathcal{PT}$-symmetric $-gx^4$ potential.

**Case 3:** $\epsilon = 3$
In this case, we get \( h = p^2 + \frac{1}{4}g^2x^6 - \frac{3}{2}gx^2. \)

One can take negative \( \epsilon \) values. For instance, in case of \( \epsilon = -1 \), we have the result \( h = p^2 + \frac{g^2 + 2a}{4x^2} \), which is an exactly solvable potential. Note that in the cases we considered here \( \epsilon \) has been chosen to be integer. In the following section, we discuss the constraints on the parameter \( \epsilon \) due to stability and Hermiticity requirements.

### III. SPECTRA POSITIVITY, WAVE FUNCTIONS SQUARE INTEGRABILITY AND CONSTRAINTS ON THE PARAMETER \( \epsilon \)

Consider the equivalent Hermitian class of Hamiltonians given by

\[
h = p^2 + \frac{1}{4}g^2x^{2\epsilon} - \frac{1}{2}gx^{\epsilon-1}\epsilon. \tag{11}
\]

For \( h \) to be Hermitian everywhere, \( \epsilon \) has to choose integer values. In fact, this constraint also secure the stability of the Hamiltonian \( h \) as the leading power of the position variable is always even.

Since the first two members (for positive \( \epsilon \) values), with the addition of a suitable \( x \)-dependent term to the potential, can reproduce the first two even \( n \) members of the \( \mathcal{PT} \)-symmetric class of the form;

\[
H = p^2 + x^n(ix)^n, \quad n = 0, 2, \ldots.
\tag{12}
\]

one can conjecture that, for \( \epsilon \) is a positive integer, our class is equivalent to the above class for \( n \) is an even positive integer. However, to assure this equivalence one has to get the positive definite metric operators for the \( \mathcal{PT} \)-symmetric class which is not fully known except for \( n = 0 \) and \( n = 2 \).

For the class in Eq.\((2)\), numerical calculations showed that the spectra are all positive which led to the conjecture that all the complex Hamiltonians (with real spectra ) have positive energies \[3\]. However, for the class in this work one can show, in an analytic way, that the spectra are positive with the possibility of a ground state energy \( E_0 = 0 \).

To show this, consider the state function

\[
|\chi\rangle = |\chi_1\rangle + i|\chi_2\rangle, \tag{13}
\]

the Hermitian conjugate is then

\[
\langle \chi | = \langle \chi_1 | - i\langle \chi_2 |, \tag{14}
\]

\( \]
clearly $\langle \chi | \chi \rangle \geq 0$ and thus we get the identity

$$\langle \chi | \chi \rangle = \langle \chi_1 | \chi_1 \rangle + \langle \chi_2 | \chi_2 \rangle - i \langle \chi_2 | \chi_1 \rangle + i \langle \chi_1 | \chi_2 \rangle \geq 0,$$

(15)

or

$$\langle \chi_1 | \chi_1 \rangle + \langle \chi_2 | \chi_2 \rangle - 2 \text{Im} \langle \chi_1 | \chi_2 \rangle \geq 0.$$

(16)

Now consider the the expectation value of the Hamiltonian $h = p^2 + \frac{1}{4} q^2 x^{2\epsilon} - \frac{1}{2} g x^{\epsilon-1} \epsilon$ with respect to any state $\phi_n$;

$$E_n = \langle \phi_n | p^2 + \frac{1}{4} q^2 x^{2\epsilon} - \frac{1}{2} g x^{\epsilon-1} \epsilon | \phi_n \rangle,$$

(17)

But

$$\frac{1}{2} g x^{\epsilon-1} \epsilon = -i \left[ \frac{g x^{\epsilon}}{2}, p \right],$$

(18)

then

$$E_n = \langle p \phi_n | p \phi_n \rangle + \langle \frac{g x^{\epsilon}}{2} \phi_n | \frac{g x^{\epsilon}}{2} \phi_n \rangle + i \langle \phi_n | \left[ \frac{g x^{\epsilon}}{2}, p \right] | \phi_n \rangle,$$

(19)

which has the same form in Eq.(15) with $\chi_1 = \frac{g x^{\epsilon}}{2} \phi_n$ and $\chi_2 = p \phi_n$. Accordingly, the spectra of the whole class are all positive and can have the eigen value $E_0 = 0$, which, if it exists, resembles the ground state of a Hamiltonian $h_\epsilon$ out of the class.

To test the physical acceptability of the members in the class, let us consider Shrödinger equation of the class in Eq.(6);

$$- \frac{d^2 \psi}{dx^2} + g x^\epsilon \frac{d \psi}{dx} = E \psi,$$

(20)

For the ground state, $E_0 = 0$, we have the solution $\psi_0 = C$, where $C$ is a constant. Accordingly, the wave function $\phi_0$ of the Hermitian class $h$ is then given by

$$\phi_0 = \rho \psi_0 = C \exp \left( -g \frac{x^{\epsilon+1}}{(\epsilon + 1)} \right).$$

To get the value of the constant $C$ we use;

$$\langle \phi_0 | \phi_0 \rangle = C^2 \left( \left( \frac{1}{\epsilon + 1} \right)^{-1/(\epsilon+1)} + (\epsilon + 1)^{1/(\epsilon+1)} \right) \Gamma \left( 1 + \frac{1}{\epsilon + 1} \right),$$

(21)
while the normalized ground state wave function takes the form

$$
\phi_0 = \left( \left( \frac{(-1)^{\epsilon+1}}{\epsilon + 1} \right)^{-1/(\epsilon+1)} + (\epsilon + 1)^{1/\epsilon} \right) \frac{1}{\Gamma \left( 1 + \frac{1}{\epsilon + 1} \right)} \exp \left( -g \frac{x^{\epsilon+1}}{(\epsilon + 1)} \right),
$$

however, this formula does not admit $\epsilon$ to take the value $-3$. Moreover, the ground state wave function is square integrable only for odd $\epsilon$ values which means that the ground state function exists only for $\mathcal{PT}$-symmetric members out of the class. We assert that our results are in complete agreement with the analytic calculations of Ref. [7].

The normalizability of the wave functions means that the associated Hamiltonians leads to true Physical states. We have shown that the ground state is not normalizable for even $\epsilon$ values. One may then conclude that these theories may possess partial normalizability instead of full normalizability. To check this, consider the set of wave functions $\psi^\epsilon_i$, which are the solutions of the non-Hermitian set of Hamiltonians in Eq.(6). Accordingly, the set $\phi^\epsilon_i = \rho \psi^\epsilon_i$ is the corresponding wave functions of the equivalent Hermitian Shrödinger Equation (Eq.(11)). In fact, the Hamiltonian operator in Eq.(6) is a quasi-solvable operator as one can write it in a Lie algebraic form [8, 9];

$$
H = \sum_{a,b} C_{a,b} J^a J^b + \sum_a C_a J^a,
$$

where $J$ is a set of first order differential operators which generate a finite-dimensional Lie algebra. Accordingly, the solutions $\psi^\epsilon_i$ are polynomials of finite order $m_\epsilon$. Thus, the true wave functions $\phi^\epsilon_i = \rho \psi^\epsilon_i$ are either fully normalizable (odd $\epsilon$) or fully non-normalizable (even $\epsilon$). Thus, only the wave functions corresponding to $\mathcal{PT}$-symmetric members out of the class lie in the Hilbert space $L^2(\mathbb{R})$. The class of Hamiltonians in Eq.(6) depends on the ordering of the dynamical variables $x$ and $p$. However, one can get rid of the ordering ambiguity by considering the class

$$
H = H_0 + gH_I,
$$

$$
H_0 = p^2,
$$

$$
H_I = i \frac{1}{2} \{x^\epsilon, p\},
$$

where $\{A, B\}$ is the anticomutator of the operators $A$ and $B$. Since the two classes in Eqs.(6&24) differs only by the term $[p, x^\epsilon]$ which is a function of $x$ only, then the class of
metric operators \( \eta_+ = \exp(-Q) \), with \( Q = g \frac{x^{e+1}}{e+1} \), works well for the two classes. Accordingly, the equivalent Hermitian Hamiltonians of the class in Eq. (24) take the form

\[
h = p^2 + \frac{1}{4}g^2x^{2e}.
\]  

(25)

IV. QUASI-GAUGE TRANSFORMATIONS AND DISAPPEARANCE OF THE \( Q \) OPERATOR

In Ref. [10], it has been shown that, for the Swanson Hamiltonian, the \( Q \) operator plays no role in the calculation of expectation values in path integral formulation of the problem. In this case, the \( Q \) operator is a pure function in the position variable \( x \). Accordingly, one may expect that, for the class introduced in this work, working with \( H, H^\dagger \) or \( h \) the end result stays the same. We will look to the problem from a different point of view to show that the \( Q \) operator has no effect on the calculations in path integral formulation.

Now, consider the Hamiltonian \( H \) of the form

\[
H = p^2 + igx^e p,
\]

\[
= p^2 + ig \{ x^e, p \} + \frac{ig}{2} [ x^e, p ]
\]

\[
= \left( p + \frac{igx^e}{2} \right)^2 - \frac{eg}{2}x^{e-1} - \frac{g^2x^{2e}}{4}.
\]  

(26)

This form reminds us with an old problem in Physics, namely, a particle moving in an electromagnetic field which has a Hamiltonian of the form

\[
H = (p - eA)^2 + e\phi,
\]  

(27)

where \( A \) and \( \phi \) are the vector and scalar potentials, respectively. Comparing the two forms of \( H \), we see that the quasi-Hermitian models introduced in this work resemble a particle moving in an imaginary magnetic field but real electric field. Accordingly, the classical action has the form

\[
S = \int dt \left( p^2 + igx^e_x + \frac{eg}{2}x^{e-1} + \frac{g^2x^{2e}}{4} \right).
\]  

(28)

Now, consider the probability amplitude

\[
K(x,t|x_0,t_0) = \int Dx(t) \exp(iS).
\]  

(29)
It is well known that electromagnetic problems are always associated with gauge transformations which add a terminal term to the time integration (total time derivative) and thus is path independent. What plays the role of gauge transformations are the metric operators. In fact, they are quasi-gauge transformations as they shift the derivative by an imaginary quantity. Moreover, the metric operators leaves the measure invariant while they shift $S$ by a total time derivative which can be integrated out and cancels out in any practical calculations.

V. CONCLUSIONS

We introduced a new class of infinite number of real-line quasi-Hermitian models which has $\mathcal{PT}$-symmetric as well as non-$\mathcal{PT}$-symmetric members. We were able to obtain the exact metric operators for each member in the class. For the energy spectra of the whole class, we showed that each member has an energy spectrum which is real and positive (with $E_0 = 0$). Moreover, the closed form ground state functions for the whole class have been obtained and found that only the $\mathcal{PT}$-symmetric members can have bound state solutions. However, when $\epsilon$, the parameter in the model that characterizes each member, takes large values, the non-$\mathcal{PT}$-symmetric members tend to have square integrable ground states. We assert that our results coincide with previous results.

We have shown that the class introduced in this work resembles a particle motion in an imaginary magnetic field while the metric operators represent quasi-gauge transformations. This picture explains the disappearance of the $Q$ operator in the path integral calculations of physical quantities since gauge transformations add total time derivatives to the classical action which contribute nothing to the physical quantities.

Our class can be adapted to include non-Hermitian Hamiltonians which are equivalent to the non-real line members of the Benders class of Hamiltonians. Besides, our class is real line and written in a quasi-solvable form which makes it easy to obtain the ground state functions as well as the metric operators of the whole class. Accordingly, working with our class turns things simpler and can be used to simplify calculations in quantum mechanics as it transforms a Hermitian Hamiltonian model with a polynomial potential of degree $n$ into a non-Hermitian-quasi-solvable potential with a degree $\frac{n}{2}$ in the position variable.
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[1] Abouzeid shalaby, Will the $\mathcal{PT}$-symmetric and Non-Hermitian $\phi^4$ Theory Solve the Hierarchy and Triviality Problems in the Standard Model? hep-th/0712.2521.

[2] Carl Bender and Stefan Boettcher, Phys.Rev.Lett.80:5243-5246 (1998).

[3] C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. 40, 2201 (1999).

[4] A. Mostafazadeh, J. Math. Phys., 43, 3944 (2002).

[5] A. Mostafazadeh, J. Math. Phys. 43, 205 (2002).

[6] Carl. M. Bender, International Journal of Modern Physics A20, No.19 2646 (2005).

[7] Jing-Ling Chen, L.C. Kwek and C.H. Oh, Phys. Rev. A 67, 012101 (2003).

[8] Artemio González-López, Niky Kamran and Peter J. Olver, Math. Phys. 153, No. 1 (1993).

[9] A.V. Turbiner, Comm. Math. Phys. 118, 467 (1988).

[10] H. F. Jones and R. J. Rivers, Phys.Rev.D75:025023 (2007).