On exact and perturbation solutions to nonlinear equations for heat transfer models

Francisco M. Fernández

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica,
Bvd. 113 y 64 (S/N), Sucursal 4, Casilla de Correo 16,
1900 La Plata, Argentina

Abstract

We analyze some exact and approximate solutions to nonlinear equations for heat transfer models. We prove that recent results derived from a method based on Lie algebras are either trivial or wrong. We test a simple analytical expression based on the hypervirial theorem and also discuss earlier perturbation results.

1 Introduction

In a recent paper Moitsheki et al.[1] argued that a method based on Lie algebras is suitable for obtaining the solution to nonlinear ordinary differential equations that appear in simple models for heat transfer. They compared the analytical solutions with other results coming from perturbation approaches like homotopy perturbation method (HPM) and homotopy analysis method (HAM)[2][3][4][5]. It is worth noticing that there is an unending controversy between the users of those fashionable perturbation approaches that arose some time ago[6][7].

1 e–mail: fernande@quimica.unlp.edu.ar
The purpose of this paper is to determine the usefulness of the results for the heat transfer systems provided by the Lie algebraic method and those perturbation approaches. In Sec. 2 we analyze the exact solutions arising from Lie algebras, in Sec. 3 we outline the application of the well known Taylor–series approach, in Sec. 4 we derive a simple accurate analytical expressions for one of the models and in Sec. 5 we summarize our results and draw conclusions.

2 Exact solutions

The first example is the nonlinear ordinary differential equation

\[ (1 + \epsilon u(x))u''(x) + \epsilon uu'(x)^2 = 0 \]
\[ u(0) = 1, \quad u(1) = 0 \]  

(1)

where the prime denotes differentiation with respect to the variable \( x \). This equation is trivial if one rewrites it in the following way \([(1 + \epsilon u(x))u]' = 0]\)

and the solution is

\[ u(x) = \frac{\sqrt{(1 + \epsilon)^2 + [1 - (1 + \epsilon)^2]} x - 1}{\epsilon} \]  

(2)

Moitsheki et al.\[\text{[1]}\] derived exactly this result by means of a rather lengthy algebraic procedure. It is clear that in this case the Lie algebraic method gives us the same answer that we can obtain in a simpler way.

For the second example

\[ u''(x) - \epsilon u(x)^4 = 0 \]
\[ u'(0) = 0, \quad u(1) = 1 \]  

(3)

the authors derived the simple analytical expression[1]

\[ u(x) = \left( \frac{\sqrt{9\epsilon}}{10} x + 1 - \frac{\sqrt{9\epsilon}}{10} \right)^{-2/3} \]  

(4)
They argued correctly that it satisfies $u(1) = 1$ but they were wrong when they stated that “However, $u'(0) = 0$ only if $\epsilon = 10/9$”. Notice that the function $u(x) = x^{-2/3}$ that comes from such value of $\epsilon$ does not have the correct behaviour at $x = 0$. Therefore, in this case the Lie algebraic approach led to a wrong result.

Other authors have applied HPM and HAM to the equation

$$[1 + \epsilon u(x)]u'(x) + u(x) = 0$$

$$u(0) = 1$$

with the trivial solution

$$\ln u(x) + \epsilon [u(x) - 1] + x = 0$$

In the following two sections we discuss some of these problems from different points of view.

3 Taylor series

If the variable of the nonlinear equation is restricted to a finite interval, one can try a straightforward power–series solution $u(x) = u_0 + u_1 x + u_2 x^2 + \ldots$ and obtain the unknown model parameter from the boundary conditions. In the case of the example (2) the radius of convergence of this series is $(\epsilon + 1)^2/[\epsilon(\epsilon + 2)]$ and therefore the approach will be useful for small and moderate values of $\epsilon$. As $\epsilon$ increases the rate of convergence of the Taylor–series method decreases because the radius of convergence approaches unity from above. However, this example is trivial and of no interest whatsoever for the application of a numerical or analytical method. This reasoning also applies to example (5) although in this case we do not have an explicit solution $u(x)$ but $x(u)$.

The example (3) is more interesting because there appears to be no exact
solution, and for this reason we discuss it here. The unknown parameter is $u(0) = u_0$ and the partial sums for the Taylor series about $x = 0$

$$u^{[N]}(x) = \sum_{j=0}^{N} u_j(u_0)x^j \quad (7)$$

enable one to obtain increasingly accurate estimates $u_0^{[N]}$ as $N$ increases. Such estimates are roots of $u^{[N]}(1) = 1$. Although the rate of convergence decreases as $\epsilon$ increases it is sufficiently great for most practical purposes. Notice that the HAM perturbation corrections for this model are polynomial functions of $x[2]$ whereas the HPM has given polynomial functions of either $x[3]$ or $e^{-x}[4]$. However, there is no doubt that the straightforward power–series approach is simpler and does not require fiddling with adjustable parameters[2,5].

4 The hypervirial theorem

The analysis of the nontrivial equations for heat transfer models may be easier if we have simple approximate analytical solutions instead of accurate numerical results or cumbersome perturbation expressions. In the case of the models (1) and (5) there is no doubt that the exact analytical expressions should be preferred. For that reason, in what follows we concentrate on the seemingly nontrivial model (3).

We have recently shown that the well known virial theorem may provide simple analytical solutions for some nonlinear problems[8,9]. In particular, we mention the analysis of a bifurcation problem that appears in simple models for combustion[8]. The only nontrivial problem outlined above is a particular case of nonlinear ordinary differential equations of the form

$$u''(x) = f(u(x))$$

$$0 \leq x \leq 1$$

$$\quad (8)$$
The hypervirial theorem is a generalization of the virial one. If \( w(u) \) is an arbitrary differentiable weight function, the hypervirial theorem provides the following suitable expression for our problem (8):

\[
\int_0^1 [w(u)u']' \, dx = w(u(1))u'(1) - w(u(0))u'(0)
\]

\[
= \int_0^1 \left[ \frac{dw}{du}(u')^2 + w(u)f(u) \right] \, dx
\]  

(9)

In the particular case of the example (3) we have

\[
w(1)u'(1) = \int_0^1 \left[ \frac{dw}{du}(u')^2 + \epsilon w(u)u^4 \right] \, dx
\]  

(10)

When \( w(u) = u \) we obtain the virial theorem. Here we also consider the even simpler choice \( w(u) = 1 \) that we will call hypervirial although it is just a particular case.

Since \( u''(x) > 0 \) we try the ansatz

\[ u_{\text{app}}(x) = \frac{\cosh(bx)}{\cosh(b)} \]  

(11)

that satisfies the boundary conditions in equation (3). It follows from equation (10) that the adjustable parameter \( b \) is a root of

\[
3e^{10b}(5b^2 - 2\epsilon) + 5e^{8b}(12b^3 + 9b^2 - 10\epsilon) + 30e^{6b}(6b^3 + b^2 - 10\epsilon) + 30e^{4b}(12b^3 - 9b^2 + 10\epsilon) - 3(5b^2 - 2\epsilon) = 0
\]  

(12)

when \( w(u) = u \) and

\[
3e^{10b}(5b^2 - \epsilon) + 5e^{8b}(9b^2 - 5\epsilon) + 30e^{6b}(b^2 - 5\epsilon) + 30e^{4b}(5\epsilon - b^2) + 5e^{2b}(9b^2 - 3(5b^2 - \epsilon) = 0
\]  

(13)

when \( w(u) = 1 \).
Fig. 1 shows \( u_{app}(0) \) for some values of \( \epsilon \) and also the accurate result obtained from the Taylor series discussed in Sec. 3. We appreciate that the accuracy of the analytical expression (11) decreases as \( \epsilon \) increases. However, if one takes into account the simplicity of equation (11) the agreement is remarkable. Besides, the hypervirial theorem with \( w = 1 \) proves to be more accurate than the virial theorem. It is curious that there is no such test for the HPM or HAM2,3.

As a particular example we consider \( \epsilon = 0.7 \) (the preferred parameter value for both HAM and HPM calculations2,3). From the partial sums of the Taylor–series with \( N \leq 30 \) we obtain \( u_0 = 0.8186424785 \). The analytical function (11) yields \( b \approx 0.70, \ u_{app}(0) \approx 0.80 \) for \( w = u \) and \( b = 0.657, \ u_{app}(0) \approx 0.817 \) for \( w = 1 \) that is a reasonable estimate of the unknown parameter. Again we see that the hypervirial approach is better than the virial one. Fig. 2 shows accurate values of \( u(x) \) given by the Taylor series with \( N = 30 \), our approximate analytical virial expression \( u_{app}(x) \) and equation (4) for \( 0 \leq x \leq 1 \). It seems that the accuracy of \( u_{app}(x) \) is somewhat between the HAM results of 5th and 10th order2. On the other hand, the equation (4) derived by the Lie algebraic method1 exhibits a wrong behaviour.

Finally, in Fig. 3 we compare the numerical, virial (\( w = u \)) and hypervirial (\( w = 1 \)) approaches to the function \( u(x) \) in a wider scale. We conclude that the virial theorem is not always the best choice for obtaining approximate solutions to nonlinear problems.

5 Conclusions

The purpose of this paper has been the discussion of some recent results for the nonlinear equations arising in heat transfer phenomena. The oversimplified models considered here may probably be of no utility in actual physical or
engineering applications. Notice that the authors did not show any sound application of those models and the only reference is a pedagogical article cited by Rajabi et al[4]. However, it has not been our purpose to discuss this issue but the validity of the methods for obtaining exact and approximate solutions to simple nonlinear equations.

It seems that the particular application of the Lie algebraic method by Moitsheki et al[1] has only produced the exact result of a trivial equation and a wrong result for a nontrivial one. Therefore, we believe that the authors failed to prove the utility of the technique and it is not surprising that they concluded that their results did not agree with the HAM ones[2] (see Fig. 2).

We have also shown that under certain conditions the well known straightforward Taylor–series method is suitable for the accurate treatment of such nontrivial equations. It is simpler than both HAM and HPM[2,3] and as accurate as the numerical integration routine built in a computer algebra system[3].

Finally, we have shown that the well known hypervirial theorem may provide simple analytical expressions that are sufficiently accurate for a successful analysis of some of those simple models for heat transfer systems. It is surprising that our results suggest that the virial theorem[8,9] may not be the best choice.

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Fig. 1. Numerical (circles) virial (solid line) and hypervirial (dashed line) values of \( u_0 \).

Fig. 2. Accurate values of \( u(x) \) for \( \epsilon = 0.7 \) (circles) and approximate results from equations (11) (solid line) and (4) (dashed line).
Fig. 3. Accurate values of $u(x)$ for $\epsilon = 0.7$ (circles), virial (solid line) and hypervirial (dashed line) approximate results from equation (11).