1. Introduction and convention

1.1. Introduction. N. Katz proved the following theorem.

**Theorem 1.1.** Let \( X \) be a connected separated scheme of finite type over \( \mathbb{F}_p \) and \( F : X \to X \) the absolute Frobenius map. Then there exists an equivalence of categories between

1. the category \((LS_{\mathbb{F}_p}/X_{et})\) of smooth étale \( \mathbb{F}_p \) sheaves on \( X \) of finite rank, and
2. the category \((UR_{\mathbb{F}_p}/X)\) of pairs \((\mathcal{F}, \varphi)\) consisting of
   (a) a locally free \( \mathcal{O}_X \) module \( \mathcal{F} \) of finite rank and
   (b) an isomorphism \( \varphi : F^* \mathcal{F} \to \mathcal{F} \) of locally free \( \mathcal{O}_X \) modules.
In this theorem, the functor from \((UR_{F_p}/X)\) to \((LS_{F_p}/X_{et})\) is given as follows. Let \(h : W \to X\) be an etale morphism and \(F_X : X \to X\) and \(F_W : W \to W\) the absolute Frobenius maps. Then the relative Frobenius map \(\Phi\) is defined by the following commutative diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{\Phi} & F_W \\
\downarrow h & & \downarrow h \\
W_{X,F} & \xrightarrow{\Phi} & W \\
\downarrow F_X & & \downarrow F_X \\
X & & X
\end{array}
\]

The map \(\Phi : W \to W_{X,F} X\) induces a map \(\varphi^* : F_*\mathcal{F}(U) = \mathcal{F}(U_{X,F}) \to \mathcal{F}(U)\), and we have the composite map \(\Phi^* \circ a^* : \mathcal{F}(U) \to \mathcal{F}(U)\) of \(\Phi^*\) and the adjoint \(a^* : \mathcal{F}(U) \to F_*\mathcal{F}(U)\) of the map \(\varphi\). Then the functor \(W \mapsto \ker(\Phi^* \circ a^* - 1) \subset \mathcal{F}(W)\) becomes a smooth \(F_p\) etale sheaf of finite rank on \(X_{et}\). The construction of the inverse functor is based on the etale descent theory of locally free \(\mathcal{O}_X\) sheaves of finite rank.

An object in the category \((UR_{F_p}/X)\) is sometimes called a unit root \(F_p\)-F-crystal on \(X\). A smooth etale \(F_p\) sheaf \(V\) is called unipotent if there is a filtration \(\{F^i V\}\) such that the associated graded sheaf \(Gr^i_F (V)\) is a constant sheaf on \(X\). The full subcategory of unipotent \(F_p\) etale local systems is denoted by \((NLS_{F_p}/X_{et})\). The full subcategory of \((UR_{F_p}/X)\) corresponding to \((NLS_{F_p}/X_{et})\) is denoted by \((NUR_{F_p}/X)\). For the moment, we assume that there exists an \(F_p\)-valued point \(x\) in \(X\). By taking a geometric point \(\bar{x}\) over \(x\), we have the fundamental group of the category \((NLS_{F_p}/X_{et})\) with respect to the base point \(\bar{x}\), which is called the \(F_p\)-completion \(\pi_1(X, \bar{x})^F_p\) of the etale fundamental group of \(X\). Note that \(\pi_1(X, \bar{x})^F_p\) is isomorphic to the pro-p completion of \(\pi_1(X, \bar{x})\) (See Section 7).

Let \(F_p[[\pi_1(X, \bar{x})]]^*\) be the topological dual of the completion of \(F_p[[\pi_1(X, \bar{x})]]\) with respect to the topology defined by the augmentation ideal \(I\). Then it is equipped with a structure of Hopf algebra. The \(F_p\)-completion \(\pi_1(X, \bar{x})^{F_p}\) is isomorphic to the group of group-like elements in the Hopf algebra \(F_p[[\pi_1(X, \bar{x})]]^*\). In this paper, we show that a Hopf algebra \(F_p[[\pi_1(X, \bar{x})]]^*\) is isomorphic to the cohomology of the bar complex of the Artin-Schreier DGA of \(X\) in Section 4.

The bar construction is a standard construction to get a coalgebra out of an associative differential graded algebra \(A^\bullet\) (DGA for short). If the DGA \(A^\bullet\) is graded commutative, then the bar complex \(B(A^\bullet, \epsilon)\) has a multiplicative structure defined by the shuffle product. Using this shuffle product, \(H^0(B_{red}(A^\bullet, \epsilon))\) has a structure of Hopf algebra. Unfortunately Artin-Schreier DGA’s are not graded commutative in general. Therefore the shuffle product construction does not work as it is. In Section 5 we define the shuffle product on \(B_{red}(A, \epsilon)\) up to homotopy.
Here we give the definition of Artin-Schreier DGA in the case of affine scheme \( X = \text{Spec}(R) \), which is the easiest case. As a complex, \( A^\bullet \) is a complex of length two, defined by the Artin-Schreier map

\[
A^0 = R \xrightarrow{d} A^1 = R : x \mapsto x^p - x.
\]

For an element \( x \in R \), \((x)_i\) denote the element in \( A^i \) corresponding to \( x \). We introduce a multiplication structure \( \cup \) for an element \( x \in A^i \). Actually, by this multiplication, \( A^\bullet \) becomes a DGA, by the direct calculation:

\[
d(((x)_0 + (\xi)_1) \cup ((y)_0 + (\eta)_1)) = (x y)_0 + (x \eta + y^p \xi)_1.
\]

Actually, by this multiplication, \( A^\bullet \) becomes a DGA, by the direct calculation:

\[
d(((x)_0 + (\xi)_1) \cup ((y)_0 + (\eta)_1)) = (x^p y^p - x y)_1 = (y^p (x^p - x))_1 + (x (y^p - y))_1
\]

\[
= (x^p - x)_1 \cup ((y)_0 + (\eta)_1) + ((x)_0 + (\xi)_1) \cup (y^p - y)_1
\]

\[
= d((x)_0 + (\xi)_1) \cup ((y)_0 + (\eta)_1) + (x)_0 \cup d((y)_0 + (\eta)_1) - (\xi)_1 \cup d((y)_0 + (\eta)_1))
\]

We give a Cech theoretic interpretation of the category \( (\text{NUR}_F/X) \) and the Artin-Schreier DGA \( A^\bullet \) in Section 2. To understand the relation between Artin-Schreier DGA and the category \( (\text{NUR}_F/X) \), we will recall the patching of algebras and patching of DG categories.

In Section 5, we treat the product structure on the cohomology of the bar complex. Though the Artin-Schreier DGA \( A^\bullet \) is not graded commutative, it is commutative up to homotopy. In this section, we define a shuffle product up to higher homotopy.

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1.2. **Convention.** Let \( k \) be a field. For a \( k \)-linear abelian tensor category \( C \), the category of complexes in \( C \) is denoted as \( KC \). By considering morphisms of complexes, \( KC \) becomes a \( k \)-linear abelian category.

An object \( A \) in \( KK C \) is a complex \( \{(A^i, d)\}_i \) in \( KC \). The differential \( d \) of \( A^i \) is called the inner differential and the differential \( \delta : A^i \to A^{i+1} \) is called the outer differential. We define the associated simple complex \( s(A) \) of \( A \) by \( s(A) = \oplus_i A^i \otimes k[-i] = \oplus_i A^i[-i] \). The total differential \( D \) is defined as \( d \otimes 1 + 1 \otimes t \), where \( t : k[-i] \to k[-i - 1] \) is a degree one map defined by \( k[-i] \to k[-i - 1] \) with \( 1 \mapsto 1 \). The degree \(-i\) element corresponding to 1 in \( k[i] \) is denoted as \( e_i \). The symbol \( e^1 \) is also denoted as \( e \). Using this notation, \( s(A) \) can be written as \( \oplus_i A^i e^{-i} \).
For objects $A = A^\bullet$ and $B = B^\bullet$ in $KC$, we define the tensor product $A \otimes B$ as an object in $KC$ by the rule $(A \otimes B)^p = \oplus_{i+j=p} A^i \otimes B^j$ and $d(a \otimes b) = da \otimes b + (-1)^{\deg(a)} a \otimes db$ for homogeneous “elements” $a$ and $b$ in $A$ and $B$. We have a canonical switching isomorphism

$$\sigma: A \otimes B \simeq B \otimes A$$

defined by $\sigma(a \otimes b) = (-1)^{\deg(a) \deg(b)} b \otimes a$. This switching rule can be extended to tensors of arbitrary number:

$$\sigma: A_1 \otimes \cdots \otimes A_n \rightarrow A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}$$

for $\sigma \in S_n$. We always use an identification $A[p] \otimes B[q] \simeq A \otimes B[p + q]$ by the following composite:

$$A[p] \otimes B[q] \simeq A \otimes k[p] \otimes B \otimes k[q]$$

$$\simeq A \otimes B \otimes k[p] \otimes k[q]$$

$$\simeq A \otimes B \otimes k[p + q].$$

Here $\sigma$ is the switching homomorphism.

Let $A = A^\bullet, B^\bullet$ be complexes in $C$ and $p$ be an integer. A system of morphisms $A^i \rightarrow B^{i+p}$ indexed by $i \in \mathbb{Z}$ is called a homogeneous morphism of degree $p$ from $A$ to $B$. Let $A_1, A_2, B_1$ and $B_2$ be complexes in the category $C$ and $\varphi: A_1 \rightarrow A_2$ and $\psi: B_1 \rightarrow B_2$ be homogeneous morphisms of degree $p$ and $q$, respectively. Then the tensor product

$$\varphi \otimes \psi: A_1 \otimes B_1 \rightarrow A_2 \otimes B_2$$

of $\varphi$ and $\psi$ is defined by the rule $(\varphi \otimes \psi)(a \otimes b) = (-1)^q \deg(a) \varphi(a) \otimes \psi(b)$. Using this convention, the differential $d \otimes 1$ on $A[p] = A \otimes k[p]$ is $(d \otimes 1)(x \otimes e^p) = dx \otimes e^p$. We remark that some references prefer sign convention $d_A[p](x) = (-1)^{\deg(x)} d_A(x)$ for homogeneous element $x \in A^*[p]$, which is denoted $k[p] \otimes A^\bullet = e^p A^\bullet$ in this paper to avoid confusing convention “$[p]A^\bullet$”. The differential of a complex $A$ is a homogeneous map from $A$ to $A$ of degree one. Therefore the differential of the tensor complex $A \otimes B$ defined as above is written as $d \otimes 1 + 1 \otimes d$. The differential $d$ of $A$ can be considered as a homogeneous map of degree zero $d \otimes t^{-1}$ from $A$ to $A[1]$ by setting

$$x = x \otimes 1 \mapsto (-1)^{\deg x} dx \otimes e.$$

For homogeneous maps $\varphi_1: A_1 \rightarrow A_2, \varphi_2: A_2 \rightarrow A_3$ and $\psi_1: B_1 \rightarrow B_2, \psi_2: B_2 \rightarrow B_3$, we have

$$(\varphi_2 \otimes \psi_2) \circ (\varphi_1 \otimes \psi_1) = (-1)^{\deg \psi_2 \cdot \deg \varphi_1} (\varphi_2 \circ \varphi_1) \otimes (\psi_2 \circ \psi_1).$$

A homogeneous map $L \otimes M \rightarrow N$ is extended to $L[p] \otimes M[q] \rightarrow N[p + q]$ by the composite

$$L \otimes k[p] \otimes M \otimes k[q] \simeq L \otimes M \otimes k[p] \otimes k[q] \simeq L \otimes M \otimes k[p + q] \rightarrow N[p + q].$$

For objects $A = A^\bullet \cdot \cdot \cdot, B^\bullet \cdot \cdot \cdot$ in $KKC$, the tensor product $A \otimes B$ is defined as an object in $KKC$. Then we have a natural isomorphism in $KC$

$$\nu: s(A) \otimes s(B) \simeq s(A \otimes B)$$
defined by $\nu(a \otimes b) = (-1)^{ji} a \otimes b$ for $a \in A^{ij}, b \in B^{i'j'}$. This isomorphism is compatible with the switching of tensor and natural associativity isomorphism.

The set $\{m, m+1, \ldots, n\}$ is denoted by $[m, n]$ and $[0, n]$ is denoted by $[n]$. We define the category of simplexes $\Delta$ by setting the set of objects to be the set of simplexes $\{[n] \mid n \geq 0\}$ and the set of morphisms to be the set of weakly increasing maps between them.

We use copies $\mathbf{1}, \ldots, \pi$ of numbers $1, \ldots, n$ which is totally ordered in the natural way.

**2. $(UR_{\mathbf{F}_p}/X)$ and Quotient by Relation**

In this subsection, we give an interpretation of objects in $(UR_{\mathbf{F}_p}/X)$ as locally free sheaves on the quotient “space” by a relation diagram.

**Definition 2.1.** (1) The following diagram $R$ of schemes

\[
\begin{array}{ccc}
R & \xrightarrow{f_0} & Y \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{f_1} & \text{Spec}(k[t])
\end{array}
\]

is called a relation diagram.

(2) For a relation diagram $R$, we define a (locally free) $\mathcal{O}_R$-module of finite rank as a pair $(M, \varphi)$ consisting of (locally free) $\mathcal{O}_Y$ module $M$ of finite rank and an isomorphism $f_0^* M \simeq f_1^* M$. Since we have $\text{can} : f_0^* \mathcal{O}_Y \simeq \mathcal{O}_X \simeq f_1^* \mathcal{O}_Y$, $(\mathcal{O}_Y, \text{can})$ is an $\mathcal{O}_R$ module. It is written $\mathcal{O}_R$ for simplicity.

(3) Let $R$ be a relation diagram (2.1) and $M = (M, \varphi)$ be a locally free $\mathcal{O}_R$-module of finite rank. A filtration $F_M^\bullet$ on $M$ is called a nilpotent filtration if

(a) the filtrations $f_1^* F_M^\bullet$ and $f_2^* F_M^\bullet$ coincide under the isomorphism $\varphi$, and

(b) the associated graded $\mathcal{O}_R$ modules are sum of copies of $\mathcal{O}_R$.

A locally free $\mathcal{O}_R$-module of finite rank with a nilpotent filtration is called a nilpotent $\mathcal{O}_R$-module.

**Example 2.2.** Let

\[
\begin{array}{ccc}
R & \xrightarrow{f_0} & \text{Spec}(k[t]) \\
\downarrow & & \downarrow \\
\mathbb{A} & \xrightarrow{f_1} & \mathbb{A}
\end{array}
\]

be a relation diagram, where $f_0$ and $f_1$ are the evaluation map at 0 and 1, respectively. Then the category of $\mathcal{O}_R$-modules is naturally equivalent to the category of $\mathcal{O}_C$-modules where $C = \text{Spec}(\text{Ker}(k[t]^{	ext{ev}_0 \to \text{ev}_1} k))$ is the nodal affine line.

In the same way, a diagram

\[
\begin{array}{ccc}
A^{\bullet,0} & \xrightarrow{f_0} & A^{\bullet,1} \\
\downarrow & & \downarrow \\
\mathbb{A} & \xrightarrow{f_1} & \mathbb{A}
\end{array}
\]
of DGA’s is called a relation diagram of DGA’s. For a relation diagram \((2.2)\), we denote \(s(A)\) by the associated simple complex of

\[
\begin{array}{c}
\begin{array}{c}
A_{\text{degree 0}} \\
A^0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f_1-f_0 \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A_{\text{degree 1}} \\
A^1
\end{array}
\end{array}
\end{array}
\]

Then \(s(A) = A^0 \oplus A^1 e^{-1}\) is equipped with a structure \(\cup\) of DGA by setting

\[
a \cup b = a_0 b_0 + a_1 b_1 + f_0 (a_0) e^{-1} b_01 + a_{01} f_1 (b_1) e^{-1}
\]

\[
= a_0 b_0 + a_1 b_1 + (-1)^{\deg(b_01)} f_0 (a_0) b_{01} e^{-1} + a_{01} f_1 (b_1) e^{-1}
\]

where \(a = a_0 + a_1 + a_{01} e^{-1}, b = b_0 + b_1 + b_{01} e^{-1}\), where \(a_0, b_0 \in A_0, a_1, b_1 \in A_1\) and \(a_{01}, b_{01} \in A_{01}\) are homogeneous elements. Note that \(\deg(b_{01} e^{-1}) = \deg(b_{01}) + 1\).

Since the unipotencies for two categories \((LC_{F_p/X_{et}})\) and \((UR_{F_p/X})\) correspond to each other by the functor stated just after the statement of Theorem \(1.1\), the following proposition is a consequence of this theorem.

**Proposition 2.3.** Let \(X\) be a scheme of finite type over \(F_p\) and \(R\) be the relation diagram

\[
\begin{array}{c}
\begin{array}{c}
\text{id}_X \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X \\
\rightarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F_X
\end{array}
\end{array}
\end{array}
\]

Then the category of locally free sheaves of finite rank on \(O_R\) is equivalent to the category of \(F_p\)-smooth etale sheaves of finite rank on \(X\). Moreover the category of nilpotent \(O_R\)-modules is equivalent to the category of nilpotent \(F_p\) smooth sheaves on \(X\).

### 3. Bar complex and Čech complex

3.1. **The bar complex and nilpotent connections.** We recall the definition of the reduced bar complex and the simplicial bar complex of a DGA \(A^\bullet\) over a field \(k\) with an augmentation \(\epsilon : A^\bullet \to k\). The multiplication of \(A^\bullet\) is written as “.”. Let \(I = Ker(\epsilon : A^\bullet \to k)\) be the augmentation ideal of \(\epsilon\).

We define the double complex

\[
B_{\text{red}}(A^\bullet, \epsilon) : \cdots \xrightarrow{d_B} I^\otimes 2 \xrightarrow{d_B} I^\otimes 1 \xrightarrow{\epsilon} \cdots \xrightarrow{d_B} I^\otimes p \xrightarrow{d_B} I^\otimes (p-1) \xrightarrow{\epsilon} \cdots
\]

where \(d_B : I^\otimes p \to I^\otimes (p-1)\) is given by

\[
d_B(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = (-1)^p \epsilon(x_1) \otimes x_2 \otimes \cdots \otimes x_p
\]

\[
+ \sum_{i=1}^{p-1} (-1)^{p-i} x_1 \otimes \cdots \otimes (x_i \cdot x_{i+1}) \otimes \cdots \otimes x_p
\]

\[
+ x_1 \otimes x_2 \otimes \cdots \otimes x_{p-1} \otimes \epsilon(x_p).
\]
The associated simple complex \( B_{\text{red}}(A^\bullet, \epsilon) = \oplus_i B_{\text{red}}(A^\bullet, \epsilon)^{-i} \) is called Chen’s reduced bar complex of \( (A^\bullet, \epsilon) \).

We define the simplicial bar complex of \( A^\bullet \). For a sequence of integers \( \alpha = (\alpha_0 < \cdots < \alpha_n) \), we set \( B_{\text{simp}, \alpha}(A^\bullet, \epsilon) = A^\bullet^{\otimes n} \). The length of \( \alpha \) is by definition \( n \). We write this tensor product as

\[
\kappa \otimes A \otimes \cdots \otimes A \otimes k
\]

to distinguish the index \( \alpha \).

For indices \( \alpha = (\alpha_0, \ldots, \alpha_n) \) and \( \beta = (\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n) \), we define a homomorphism of complexes \( \partial_{\text{bar}} : B_{\text{simp}, \alpha}(A, \epsilon) \rightarrow B_{\text{simp}, \beta}(A, \epsilon) \) by

\[
d_{\text{bar}} : 1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1 \mapsto \begin{cases} 
(-1)^n \epsilon(x_1) \otimes x_2 \cdots x_n \otimes 1 & (i = 0) \\
(-1)^{n-i} 1 \otimes \cdots \otimes x_i \cdot x_{i+1} \otimes \cdots \otimes 1 & (0 < i < n) \\
1 \otimes x_1 \cdots x_{n-1} \otimes \epsilon(x_n) & (i = n)
\end{cases}
\]

We set \( B_{\text{simp}}^{-n}(A, \epsilon) = \bigoplus_{|\alpha| = n} B_{\text{simp}, \alpha}(A, \epsilon) \). Using \( d_{\text{bar}} \), we have the following double complex:

\[
\cdots \rightarrow B_{\text{simp}}^{-2}(A, \epsilon) \rightarrow B_{\text{simp}}^{-1}(A, \epsilon) \rightarrow B_{\text{simp}}^0(A, \epsilon) \rightarrow 0
\]

The associated simple complex \( B_{\text{simp}}(A^\bullet, \epsilon) \) is called the simplicial bar complex of \( (A^\bullet, \epsilon) \). By Theorem 5.2 of \([T]\), we have the following proposition.

**Proposition 3.1.** Let \( \alpha = (\alpha_0 < \cdots < \alpha_n) \) be a sequence of integers. The map

\[
B_{\text{simp}, \alpha} \rightarrow I^{\otimes n} : 1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1 \mapsto \pi(x_1) \otimes \cdots \otimes \pi(x_n)
\]

defines a homomorphism of double complexes \( B_{\text{simp}}(A, \epsilon) \rightarrow B_{\text{red}}(A, \epsilon) \), which induces a quasi-isomorphism \( B_{\text{simp}}(A, \epsilon) \rightarrow B_{\text{red}}(A, \epsilon) \).

We introduce a differential graded coalgebra structure on the double complex \( B_{\text{red}}(A^\bullet, \epsilon) \):

\[
\mu : B_{\text{red}}(A^\bullet, \epsilon) \rightarrow B_{\text{red}}(A^\bullet, \epsilon) \otimes B_{\text{red}}(A^\bullet, \epsilon)
\]
defined by

\[
\mu(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n).
\]

The homomorphism \( \mu \) defines a comultiplication on the associated simple complex \( B_{\text{red}}(A^\bullet, \epsilon) \) and its cohomology \( H^0(B_{\text{red}}(A^\bullet, \epsilon)) \):

\[
\mu : H^0(B_{\text{red}}(A^\bullet, \epsilon)) \rightarrow H^0(B_{\text{red}}(A^\bullet, \epsilon)) \otimes H^0(B_{\text{red}}(A^\bullet, \epsilon))
\]

Via this comultiplication, \( H^0(B_{\text{red}}(A^\bullet, \epsilon)) \) becomes a coalgebra over \( k \).

We define nilpotent \( A^\bullet \)-connections for a connected DGA.

**Definition 3.2.** Let \( A^\bullet \) be a DGA over a field \( k \).

1. \( A^\bullet \) is said to be connected if \( H^i(A^\bullet) = 0 \) for \( i < 0 \) and \( H^0(A^\bullet) = k \).

(This condition is sometimes called the cohomological connectedness.)
Let $M$ be a finite dimensional $k$ vector space. A map $\nabla : M \to A^1 \otimes M$ is called a nilpotent $A^\bullet$-connection if there exists a finite decreasing filtration $\{F^i M\}_i$ such that $\nabla(F^i M) \subset A^1 \otimes F^{i+1}$ and $F^p M = 0$, $F^0 M = M$ for some $p, q \in \mathbb{Z}$. A nilpotent $A^\bullet$-connection is said to be integrable if the following diagram is commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{\nabla} & A^1 \otimes M \\
\nabla \downarrow & & \downarrow d \otimes 1_M \\
A^1 \otimes M & \xrightarrow{(\mu \otimes 1)_0(1_A \otimes \nabla)} & A^2 \otimes M.
\end{array}
$$

A pair $(M, \nabla)$ of a vector space $M$ and an integrable nilpotent $A^\bullet$-connection $\nabla$ on $M$ is called an integrable nilpotent $A^\bullet$-connection for short.

Let $M_1 = (M_1, \nabla_1)$ and $M_2 = (M_2, \nabla_2)$ be integrable nilpotent connections. A $k$ linear map $\varphi : M_1 \to A^0 \otimes M_2$ is called a homomorphism of integrable nilpotent connections from $M_1$ to $M_2$ if the following diagram commutes:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\nabla_1} & A^1 \otimes M_1 \\
\varphi \downarrow & & \downarrow (\cup \otimes 1_{M_2})(1_A \otimes \nabla_2) + d \otimes 1_{M_2} \\
A^0 \otimes M_2 & \xrightarrow{1_A \otimes \varphi} & A^1 \otimes A^0 \otimes M_2 \xrightarrow{\cup \otimes 1_{M_2}} A^1 \otimes M_2.
\end{array}
$$

(4) Let $M_1 = (M_1, \nabla_1)$ and $M_2 = (M_2, \nabla_2)$ be two integral nilpotent $A^\bullet$-connections. Let $\varphi_0$ and $\varphi_1$ be two homomorphisms from $M_1$ to $M_2$. The homomorphisms $\varphi_0$ and $\varphi_1$ are homotopy equivalent if there exists a $k$ linear map $h : M_1 \to A^{-1} \otimes M_2$ such that the difference $\varphi_1 - \varphi_0$ is equal to the sum of two maps

$$
M_1 \xrightarrow{h} A^{-1} \otimes M_2 \xrightarrow{(\cup \otimes 1_{M_2})(1_A \otimes \nabla_2) + d \otimes 1_{M_2}} A^0 \otimes M_2
$$

$$
M_1 \xrightarrow{\nabla_1} A^1 \otimes M_1 \xrightarrow{1_A \otimes h} A^1 \otimes A^{-1} \otimes M_2 \xrightarrow{} A^0 \otimes M_2.
$$

A $k$ linear map $h$ satisfying the above condition is called a homotopy of $\varphi_0$ and $\varphi_1$.

The category of integrable nilpotent connections is denoted as $(INC_A)$. It is easy to see that the set of null homotopic morphisms in $(INC)$ forms an ideal in the set of morphisms and we get a quotient category by replacing the morphism by their homotopy equivalence classes. This quotient category is denoted by $(HINC_A)$. We refer the following theorem in [1].

**Theorem 3.3** (Theorem 7.6 in [1]). Let $A^\bullet$ be a connected DGA. The category $(HINC_A)$ is equivalent to the category of $H^0(B(A^\bullet, \epsilon))$-comodules.

### 3.2. Cech complexes.

We recall Cech complexes for a coverings and surjective morphisms.
3.2.1. Let $I$ be a finite ordered set and $\mathcal{P}(I)$ (resp. $\mathcal{P}^+(I)$) be the set of non-empty subsets (resp. subsets) of $I$. Then $\mathcal{P}(I)$ and $\mathcal{P}^+(I)$ become categories by inclusions. For a category $\mathcal{C}$, a covariant functor from $\mathcal{P}(I)$ (resp. $\mathcal{P}^+(I)$) to $\mathcal{C}$ is called a (resp. an augmented strict cosimplicial object) strict cosimplicial object of $\mathcal{C}$ indexed by $I$.

For an element $i = (i_0 < \cdots < i_p) \in \mathcal{P}(I)$, we define $|i| = p$. We set $D_k(i) = (i_0, \ldots, \widehat{i_k}, \ldots, i_p)$ for $i = (i_0, \ldots, i_p)$. We denote $i_1 * i_2 \vdash i$ if $i_1 = (i_0, \ldots, i_k), i_2 = (i_k, \ldots, i_p)$ and $i = (i_0, \ldots, i_k, \ldots, i_p)$. Let $\mathcal{C}$ be an abelian category and $\{A_J\}_{J \in \mathcal{P}(I)}$ a strict cosimplicial object of $\mathcal{C}$. Then we define the Čech complex $\check{\mathcal{C}}(I, A)$ by

$$0 \to \prod_{|J|=0} A_J \to \prod_{|J|=1} A_J \to \cdots,$$

where the differential $d$ is given by the formula

$$(da)_i = \sum_{k=0}^{p} (-1)^k a_{D_k(i)}.$$

3.2.2. Let $I_0, \ldots, I_m$ be finite non-empty ordered sets. The sequence $(I_0, \ldots, I_m)$ is denoted as $I$. The covariant functor $A$ from $\mathcal{P}(I_1) \times \cdots \times \mathcal{P}(I_m)$ to a category $\mathcal{C}$ is called a strict polycosimplicial object in $\mathcal{C}$ indexed by $I$. For a strict polycosimplicial object in an abelian category $\mathcal{C}$, we define Čech complex $\check{\mathcal{C}}(I, A) \in K\mathcal{C}$ of $A$ by

$$0 \to \prod_{|i_0|+\cdots+|i_m|=0} A_{i_0,\ldots,i_m} \to \prod_{|i_0|+\cdots+|i_m|=1} A_{i_0,\ldots,i_m} \to \cdots,$$

Here the differential $d$ is defined by

$$(da)_{i_1,\ldots,i_m} = \sum_{0 \leq l \leq m, 0 < |i_l|} \sum_{k=0}^{|i_l|}(-1)^{|i_0|+\cdots+|i_{l-1}|+l+k} a_{i_0,\ldots,D_k(i_l),\ldots,i_m}.$$

3.2.3. Let $\Delta$ be the category of simplexes and $\mathcal{C}$ be an abelian category. A covariant functor from $\Delta$ to $\mathcal{C}$ is called a cosimplicial object in $\mathcal{C}$. If $\mathcal{C}$ is an abelian category, then for a cosimplicial object $\{A^i\}_{i \geq 0}$, we define the associated total complex $\check{t}(A)$ as an object of $K\mathcal{C}$ by

$$0 \to A^0 \to A^1 \to \cdots.$$

**Definition 3.4** (Alexander Whitney products). (1) Let $A$ be a strict cosimplicial DGA indexed by $I$. We define the product $\mu$

$$\mu : \check{\mathcal{C}}(I, A) \otimes \check{\mathcal{C}}(I, A) \to \check{\mathcal{C}}(I, A)$$

by

$$\mu(a \otimes b)_i = \sum_{i_1 * i_2 = i} a_{i_1} \cdot b_{i_2}.$$
The associated simple complex $s(\tilde{C}(I, A))$ of the Čech complex of $A$ is denoted as $\tilde{C}(I, A)$. Then $\tilde{C}(I, A)$ becomes an associative DGA by the multiplication $\mu$.

(2) Let $A$ be a strict polyicosimplicial DGA indexed by $I = (I_0, \ldots, I_m)$. We define the product $\mu$

$$\tilde{C}(I, A) \otimes \tilde{C}(I, A) \to \tilde{C}(I, A)$$

by

$$\mu(a \otimes b)_{i_0, \ldots, i_m} = \sum_{j_0 \cdots k_m \vdash i_0 \cdots j_m} a_{j_0, \ldots, j_m} \cdot a_{k_0, \ldots, k_m}$$

the associated simple complex $\tilde{C}(I, A)$ becomes an associative DGA.

(3) Let $\{A^i\}_i$ be a cosimplicial DGA. We define the product

$$\mu : A^l \otimes A^m \to A^{l+m},$$

by

$$\mu(a \otimes b) = i_{l+m,l,*}(a) \cdot j_{l+m,m,*}(b),$$

where $i_{l+m,l,*}$ and $j_{l+m,m,*}$ are morphism of DGA’s induced by

$$i_{l+m,l} : [l] \to [l+m] : s \mapsto s \text{ for } s \in [l]$$

$$j_{l+m,m} : [m] \to [l+m] : s \mapsto s + l \text{ for } s \in [m].$$

By this product, the associated simple complex of $t(A^\bullet)$ becomes an associative DGA. It is denoted by $t(A^\bullet)$.

The products defined as above are called Alexander-Whitney products.

3.2.4. We set $I = I_0 \coprod \cdots \coprod I_m$ and introduce a total order on $I$. For simplicity, we set $I_0 = [0, i_1 - 1], I_1 = [i_1, i_2 - 1], \ldots, I_m = [i_m, k]$ and $I = [0, k]$. For a non-empty subset $j = (j_0, \ldots, j_l) \in \mathcal{P}(J)$ of $J = [m]$, we set $I_j = I_{j_0} \coprod \cdots \coprod I_{j_l}$ and $I_j = (I_{j_0}, \ldots, I_{j_l})$. Then we have an inclusion $\mathcal{P}(I_{j_0}) \times \cdots \times \mathcal{P}(I_{j_l}) \to \mathcal{P}(I)$ obtained by taking the union of subsets of $I_{j_0}, \ldots, I_{j_l}$. Let $A$ be a strict cosimplicial object indexed by $I$ in $\mathcal{C}$. By restricting the functor $A$ to $\mathcal{P}(I_{j_0}) \times \cdots \times \mathcal{P}(I_{j_l})$, we have a strict polyicosimplicial object $A_j$ indexed by $I_j$. Thus we have a Čech complex $\tilde{C}(I_j, A_j) \in KC$.

Let $j = (j_0, \ldots, j_l) \in \mathcal{P}(J)$ and $0 \leq q \leq l$ and set $D_q(j) = (j_0, \ldots, \hat{j}_q, \ldots, j_l)$. We define a homomorphism $\sigma_{j,q} : \tilde{C}(I_{D_q(j)}, A_{D_q(j)}) \to \tilde{C}(I_j, A_j)$ in $KC$ by

$$\sigma(a)_{i_{j_0}, \ldots, i_{j_l}} = \begin{cases} (-1)^{p-i_q} a_{i_{j_0}, \ldots, \hat{i}_q, \ldots, i_{j_l}} & \text{if } i_{j_q} = \{p\}, p \in I_{j_q} \\ 0 & \text{otherwise} \end{cases}$$

As a consequence, the system $\{\tilde{C}(I_j, A_j)\}_j$ forms a strict cosimplicial object in $KC$ indexed by $J = [m]$. It is denoted as $\tilde{C}(I/J, A)$ and called the relative Čech complex for $I/J$. Thus we have the Čech complex $\tilde{C}(J, \tilde{C}(I/J, A))$ in $KKC$. The associated simple complex in $KC$ is denoted as $\tilde{C}(J, \tilde{C}(I/J, A))$.

**Proposition 3.5.** The complex $\tilde{C}(J, \tilde{C}(I/J, A))$ in $KC$ is isomorphic to $\tilde{C}(I, A)$.
Let \( A \) be a strict cosimplicial DGA indexed by \( I \). For an element \( j \in \mathcal{P}(J) \), \( \check{C}(I_j, A_j) \) is a DGA under the multiplication defined in Definition 3.3[2] and this correspondence gives rise to a strict cosimplicial DGA indexed by \( J \) which is denoted as \( \check{C}(I/J, A) \).

**Proposition 3.6.** The DGA \( \check{C}(J, \check{C}(I/J, A)) \) is isomorphic to \( \check{C}(I, A) \).

Let \( I \) and \( J \) be finite ordered sets and \( \varphi : I \to J \) be a non-decreasing map. Let \( \mathcal{C} \) be an abelian category and \( A \) and \( B \) be strict cosimplicial objects in \( \mathcal{C} \) indexed by \( I \) and \( J \), respectively. A \( \varphi \) morphism from \( B \) to \( A \) is a set of morphisms \( \tau_{i,j} : B_j \to A_i \) for \( i \in \mathcal{P}(I), j \in \mathcal{P}(J), \varphi(i) \subset j \) such that the following diagram commutes

\[
\begin{array}{ccc}
B_j & \to & A_i \\
\downarrow & & \downarrow \\
B_{j'} & \to & A_{i'}
\end{array}
\]

for all \( i \subset i' \in \mathcal{P}(I), j \subset j' \in \mathcal{P}(J) \) such that \( \varphi(i) \subset j, \varphi(i') \subset j' \). For a \( \varphi \) morphism from \( B \) to \( A \), we define a morphism of complexes

\[
\varphi_* : \check{C}^p(B) = \prod_{|j|=p} B_j \to \check{C}^p(A) = \prod_{|i|=p} A_i
\]

by

\[
\varphi_*(a)_{i_0,\ldots,i_p} = \begin{cases} 
\tau(a_{\varphi(i_0),\ldots,\varphi(i_p)}) & \text{if } \varphi(i_0) < \cdots < \varphi(i_p) \\
0 & \text{otherwise.}
\end{cases}
\]

4. **Bar complex of Artin-Schreier DGA for \( \mathbf{F}_p \)-scheme**

4.1. **Čech complex of Artin-Schreier DGA.** Let \( X = \text{Spec}(A) \) be an irreducible affine \( \mathbf{F}_p \)-scheme of finite type with an \( \mathbf{F}_p \)-valued point \( x \) of \( X \). By the diagram (2.3) in Proposition 2.3, we have a diagram (2.2), since the commutative algebra \( R \) is a DGA over \( \mathbf{F}_p \). The total complex \( AS(A) = (A \xrightarrow{F_A - id_A} A) \) has a structure of DGA by the last paragraph and is called the Artin-Schreier DGA of \( A \). The \( \mathbf{F}_p \)-valued point \( x \) defines an augmentation \( \epsilon \) of \( AS(A) \). For a scheme \( U \), by attaching \( AS(\Gamma(U, \mathcal{O}_U)) \), we get a presheaf complex on \( (\text{Sch}/\mathbf{F}_p) \) which is denoted as \( AS \).

Let \( \mathcal{W} = \{W_i \to X\}_{i \in I} \) be a family of \( X \)-schemes indexed by a totally ordered set \( I \). We set \( W_J = W_{j_0} \times_X \cdots \times_X W_{j_n} \) for a subset \( J = (j_0, \ldots, j_n) \) of \( I \). Then by attaching \( AS(W_J) \) to an element \( J \in \mathcal{P}(I) \), we get a strict cosimplicial DGA indexed by \( I \), which is denoted as \( AS(\mathcal{W}) \). Then the Čech complex \( \check{C}(I, AS(\mathcal{W})) \) defined in Definition 3.4[1] is a DGA.

Let \( U \to X \) be a morphism of \( \mathbf{F}_p \)-scheme. Let \( U_n \) be the \( n+1 \) times fiber product \( U \times_X \cdots \times_X U \) of \( U \) over \( X \). For a presheaf \( \mathcal{A} \) of DGA on \( X_{et} \), we have a cosimplicial DGA \( \{\mathcal{A}(U_n)\}_n \), which is denoted as \( \mathcal{A}(U/X) \). By Definition 3.3[3], we have a DGA \( t(\mathcal{A}(U/X)) \). By considering the constant sheaf \( \mathbf{F}_p \) on \( X \), the associate simple complex \( t(\mathbf{F}_p(U/X)) \) of

\[
\mathbf{F}_p(U) \to \mathbf{F}_p(U \times_X U) \to \mathbf{F}_p(U \times_X U \times_X U) \to \cdots
\]
becomes a DGA. Similarly, we have a DGA \( t(AS(U/X)) \) arising from the sheaf \( AS \) of Artin-Schreier DGA.

### 4.2. Bar complex of Artin-Schreier DGA

Let \( X \) be a separated irreducible scheme of finite type over \( F_p \) with an \( F_p \)-valued point \( x \) of \( X \). Let \( \mathcal{W} = \{ W_i \to X \}_{i \in I} \) be a finite affine covering of \( X \) indexed by \( I \). We choose \( i \in I \) such that the base point \( x \) factors through \( Spec(F_p) \to W_i \subset X \). By using this \( x \), we get an augmentation \( \epsilon : \hat{C}(I, AS(W)) \to F_p \).

**Theorem 4.1.** Let \( X \) be a separated irreducible scheme of finite type over \( F_p \) and \( x \) be an \( F_p \)-valued point of \( X \). The category of \( H^0(B_{red}(\hat{C}(I, AS(W)), \epsilon)) \)-comodules is equivalent to the category of nilpotent smooth etale \( F_p \)-sheaves of finite rank on \( X \).

**Corollary 4.2.** Let \( A \) be an \( F_p \)-algebra of finite type. Assume that \( Spec(A) \) is irreducible. The category of \( H^0(B_{red}(AS(A), \epsilon)) \)-comodules is equivalent to the category of smooth etale \( F_p \)-sheaves on \( Spec(A) \).

**Proof.** Let \( U_i \to W_i \) be an etale covering of \( W_i \). Then for a subset \( J \) of \( I \), \( U_J = U_{j_0} \times_X \cdots \times_X U_{j_n} \) is an etale covering of \( W_J \). The union \( U = \coprod_{i \in I} U_i \) is an etale covering of \( X \). For a commutative diagram

\[
\begin{array}{ccc}
S' & \to & S \\
\downarrow & & \downarrow \\
T' & \to & T
\end{array}
\]

where \( S' \to T' \) and \( S \to T \) are etale coverings, we have a homomorphism of DGA

\[ t(A(S/T)) \to t(A(S'/T')). \]

Therefore by attaching \( t(A(U_J/W_J)) \) to \( J \), we obtain a strict cosimplicial DGA indexed by \( I \), which is denoted as \( t(A(U/W)) \). Therefore we have a homomorphism of complex

\[ t(A(U/X)) \to \hat{C}(I, t(A(U/W))). \]

By taking a inductive limit on \( U \) for the diagrams

\[
\begin{array}{ccc}
U_i' & \to & U_i \\
\downarrow & & \downarrow \\
W_i, \end{array}
\]

we have the following quasi-isomorphism for an etale sheaf \( A \) on \( X_{et} \):

\[
\lim_U t(A(U/X)) \to \lim_U \hat{C}(I, t(A(U/W))). \tag{4.1}
\]

Thus we have the following commutative diagram of quasi-isomorphisms:

\[
\begin{array}{ccc}
\lim_U t(F_p(U/X)) & \to & \lim_U \hat{C}(I, t(F_p(U/W))) \\
\downarrow & & \downarrow \\
\lim_U t(AS(U/X)) & \to & \lim_U \hat{C}(I, t(AS(U/W)))
\end{array}
\]
The horizontal quasi-isomorphisms is those from the quasi-isomorphism \((4.1)\) and the vertical quasi-isomorphism on the left comes from the Artin-Schreier exact sequence for etale topology.

\[
0 \to F_p \to \mathcal{O}_X \xrightarrow{F - id} \mathcal{O}_X \to 0.
\]

By the descent theory for coherent sheaves for etale topology, we have an exact sequence

\[
0 \to \Gamma(W_J, \mathcal{O}_{W_J}) \to \Gamma(U_J, \mathcal{O}_{U_J}) \to \Gamma(U_J \times W_J, \mathcal{O}_{U_J \times W_J}) \to \cdots
\]

and as a consequence, the morphism \(\mathcal{A}S(W_J) \to \mathcal{A}S(U_J/W_J)\) is a quasi-isomorphism. Therefore we have a quasi-isomorphism of DGA’s:

\[
\check{C}(I, \mathcal{A}S(W)) \to \lim_{\longrightarrow} \check{C}(I, t(\mathcal{A}S(U/W))).
\]

Let \(\bar{x}\) be a geometric point over \(x\) and \(\bar{\epsilon}\) be a lift \(\bar{x} \to \lim_{\longrightarrow} U_i\) of \(x\). Then we have a homomorphism of algebras \(\lim F_p(U_i) \to F_p\) and it defines an augmentation of \(t(F_p(U/X))\), which is also denoted as \(\bar{\epsilon}\). Using the following diagram,

\[
\begin{array}{ccc}
\lim F_p(U_i/W_i) & \to & \mathcal{A}S(U_i/W_i) \\
\downarrow & & \downarrow \quad \text{quasi-iso} \\
\mathcal{A}S(F_p/F_p) & \to & \mathcal{A}S(F_p) \\
\bar{\epsilon} & \quad \text{quasi-iso} & \quad \epsilon \\
F_p & \to & \mathcal{A}S(W_i)
\end{array}
\]

we have an isomorphism of coalgebras:

\[
\lim_{\longrightarrow} H^0(B(t(F_p(U/X)), \bar{\epsilon})) \simeq H^0(B(\check{C}(I, \mathcal{A}S(W)), \epsilon)).
\]

By the following proposition, we have the theorem. \(\square\)

**Proposition 4.3.** The homotopy category of integrable unipotent \(\lim_{\longrightarrow} t(F_p(U/X))\)-connections is equivalent to the category of nilpotent smooth etale \(F_p\) sheaves.

**Proof.** For a scheme \(X\), the constant sheaf on \(X\) with values in \(M\) is denoted by \(M_X\). Let \((M, \nabla)\) be a \(\lim_{\longrightarrow} t(F_p(U/X))\)-connection, and \(F\) be a nilpotent filtration of \(M\) for the connection. We construct an etale sheaf on \(X\) by descending a constant sheaf on an etale covering of \(X\).

Since \(M\) is finite dimensional (see the Definition \[3.2(2)]\), there exists an etale covering \(U \to X\) and a map

\[
\nabla : M \to A^1 \otimes M,
\]

where \(A^* = t(F_p(U/X))\). Let \(\text{End}(M)^{\text{nil}}\) be the set of nilpotent endomorphism with respect to the filtration \(F\). The map \(\nabla\) defines an element of \(A^1 \otimes \text{End}(M)^{\text{nil}} = F_p(U \times_X U) \otimes \text{End}(M)^{\text{nil}}\), which is also denoted by \(\nabla\).
For an element $\varphi \in A^p \otimes \text{End}(M)^{\text{nil}}$ and $\psi \in A^q \otimes \text{End}(M)^{\text{nil}}$, the composite $\varphi \circ \psi$ is defined by the following composite map

$$A^p \otimes \text{End}(M)^{\text{nil}} \otimes A^q \otimes \text{End}(M)^{\text{nil}} \to A^{p+q} \otimes \text{End}(M)^{\text{nil}} : \omega \otimes \alpha \otimes \eta \otimes \beta \mapsto (\omega \cup \eta) \otimes (\varphi \circ \psi).$$

By the integrability condition, we have

$$pr_{01}^*(\nabla) \circ pr_{12}^*(\nabla) = pr_{12}^*(\nabla) - pr_{02}^*(\nabla) + pr_{01}^*(\nabla).$$

We construct a descent data for $M_U$. Via the isomorphism,

$$F_p(U \times_X U) \otimes \text{End}(M) \simeq \text{End}(M_{U \times_X U}),$$

the element $\rho = 1 - \nabla$ in $\text{Aut}(M_{U \times_X U})$ gives an isomorphism $\rho : pr_0^*M_U \to pr_1^*M_U$. Since

$$(pr_{01}^*(\rho)) \circ (pr_{12}^*(\rho)) = (pr_{01}^*(1 - \nabla)) \circ (pr_{12}^*(1 - \nabla))$$

$$= 1 - pr_{01}^*(\nabla) - pr_{12}^*(\nabla) + pr_{01}^*(\nabla) \circ pr_{12}^*(\nabla)$$

$$= 1 - pr_{02}^*(\nabla) = pr_{02}^*(\rho).$$

Therefore $\rho = 1 - \nabla$ satisfies the 1-cycle condition and it defines a descent data for $M_U$. We see that a homomorphism of connections defines a morphism of the descended sheaves. This gives an equivalence of the categories. \qed

By the theory of Tannakian categories, we have the following theorem.

**Corollary 4.4.** The space $H^0(B_{\text{red}}(\tilde{C}(I, AS(W)), \epsilon))^*$ is isomorphic to $F_p[[\pi_1(X, \bar{x})]]$, where

$$F_p[[\pi_1(X, \bar{x})]] = \lim_n F_p[\pi_1(X, \bar{x})]/I^n$$

as algebras.

4.3. **A variant for geometric base points.** Let $X$ be a connected separated scheme of finite type over $F_p$ and $\mathcal{W} = \{W_i\}$ be an affine covering of $X$. Let $\bar{x} : \text{Spec}(\overline{F_p}) \to X$ be an $\overline{F_p}$-valued geometric point of $X$. By choosing a morphism $\text{Spec}(\overline{F_p}) \to W_i$ for some $i$, we have a homomorphism $\tilde{\epsilon} : AS(W_i) \to AS(\overline{F_p})$ and the induced DGA homomorphism $\tilde{C}(I, AS(W)) \to AS(\overline{F_p})$ is also denoted as $\tilde{\epsilon}$. Since the natural map $F_p \to AS(\overline{F_p})$ is a quasi-isomorphism, we can consider the bar complex of $\tilde{C}(I, AS(W))$ for an augmentation map $\tilde{\epsilon}$. We can consider the same diagram (1.2) except for $\epsilon$. As a consequence, we have the following proposition.

**Proposition 4.5.** The reduced bar complex $B(\tilde{C}(I, AS(W)), \tilde{\epsilon})$ is quasi-isomorphic to the bar complex $\lim_U B(t(F_p(U/X)), \epsilon)$, and $H^0(B(\tilde{C}(I, AS(W)), \tilde{\epsilon}))$ is isomorphic to $\lim_U H^0(B(t(F_p(U/X)), \tilde{\epsilon}))$ as a coalgebra.
4.4. **Universal pro-\(p\) covering of schemes.** In this section, we give a description of the universal object which gives an equivalece of the category of \(H^0(B(AS(W/X), \epsilon))\)-comodules and that of \(F_p\) local systems on a connected \(F_p\)-scheme \(X\) of finite type. We set \(A^\bullet = AS(W/X)\).

Let \(H\) be a coalgebra and \(L\) and \(M\) be a right and left comodules over \(H\). The coaction is denoted as

\[
L\Delta : L \rightarrow L \otimes H, \quad \Delta_M : M \rightarrow H \otimes M.
\]

The cotensor product of \(\cotor_H(L, M)\) is defined by the kernel of the map

\[
L \otimes M \rightarrow L \otimes H \otimes M : l \otimes m \mapsto L\Delta(l) \otimes m - l \otimes \Delta_M(m).
\]

Let

\[
M = (M \xrightarrow{\nabla} A^1 \otimes M)
\]

be an \(A\)-connection with a right \(H^0(B(A, \epsilon))\) comodule structure

\[
M\Delta : M \rightarrow M \otimes H^0(B(A, \epsilon))
\]

such that

\[
\begin{align*}
M & \rightarrow M \otimes H^0(B(A, \epsilon)) \\
\nabla_M & \downarrow \downarrow \nabla_M \otimes 1 \\
A^1 \otimes M & \rightarrow A^1 \otimes M \otimes H^0(B(A, \epsilon))
\end{align*}
\]

is commutative. Then for a left \(H^0(B(A, \epsilon))\) comodule \(F\), \(\cotor_H(M, F)\) becomes an \(A\)-connection since the following diagram is commutative:

\[
\begin{align*}
M \otimes F & \rightarrow M \otimes H^0(B(A, \epsilon)) \otimes F \\
\nabla_M \otimes 1 & \downarrow \downarrow \nabla_M \otimes 1 \otimes 1 \\
A^1 \otimes M \otimes F & \rightarrow A^1 \otimes M \otimes H^0(B(A, \epsilon)) \otimes F.
\end{align*}
\]

We apply the definition of \(\cotor_H(M, F)\) if \(M\) is an inductive limit of (finite dimensional) integrable nilpotent connection with a left \(H^0(B(A, \epsilon))\)-coaction.

**Definition 4.6 (Universal connection).** The pair \((M, \Delta_M)\) is called the universal connection if the functor

\[
(H - \text{comod}) \rightarrow (HINC_A) : F \mapsto \cotor_H(M, F)
\]

is an equivalence of categories.

It is easy to see that the universal connection is unique up to isomorphism if it exists. In this section, we construct the universal connection.

Since \(X\) is a connected \(F_p\) scheme, \(A^\bullet\) is a connected DGA over \(F_p\). We choose \(A^1 \subset A^1\) such that \(A^1 \oplus dA^0 \simeq A^1\). Then

\[
A' = F_p \oplus A^1 \oplus \bigoplus_{i \geq 2} A^i
\]

is a sub-DGA of \(A\) and is quasi-isomorphic to \(A\). Also \(B_{red}(A', \epsilon)^i = 0\) for \(i < 0\) and we have \(i : H^0(B_{red}(A', \epsilon)) \subset B_{red}(A', \epsilon)\). We consider the comultiplication

\[
H^0(B_{red}(A', \epsilon)) \rightarrow H^0(B_{red}(A', \epsilon)) \otimes H^0(B_{red}(A', \epsilon))
\]
defined in [32]. By composing the inclusion $i$, and the projection $B_{red}(A', \epsilon) \to A^1$, we have a map

$\nabla_B : H^0(B_{red}(A', \epsilon)) \to B_{red}(A', \epsilon) \otimes H^0(B_{red}(A', \epsilon)) \to A^1 \otimes H^0(B_{red}(A', \epsilon))$.

Then the pair $(H^0(B_{red}(A', \epsilon)), \nabla_B)$ is an integrable nilpotent $A$-connection. By Theorem 7.6 of [T], we have the following proposition.

**Proposition 4.7.** The connection $(\nabla_B, H^0(B_{red}(A', \epsilon)))$ is the universal connection.

**Example 4.8 (Affine case).** Let $R$ be an integral domain over $\mathbb{F}_p$. We choose $V$ such that $\mathrm{Im}(x) \mapsto x^p - x \oplus V \simeq R$. Then we have $H^0(B_{red}(A', \epsilon)) \simeq \bigoplus_{i=0}^{\infty} V^\otimes i$ and the universal connection is given as

$\nabla : H^0(B_{red}(A', \epsilon)) \to V \otimes H^0(B_{red}(A', \epsilon)) : [a_1 | a_2 | \cdots | a_n] \mapsto a_1 \otimes [a_2 | \cdots | a_n]$.

The corresponding étale sheaf $F_{univ}$ is given by

$F(R)_{univ} = \ker \left[ R \otimes H^0(B_{red}(A', \epsilon)) \overset{\partial_{AS} \otimes 1 - (1 \otimes 1)(1 \otimes \nabla)}{\longrightarrow} R \otimes H^0(B_{red}(A', \epsilon)) \right]$.

for an étale $A$-algebra $R$. Here the differential $d_{AS}$ is given by $d_{AS}(r) = r^p - r$.

5. Homotopy shuffle product for strict cosimplicial DGA

5.1. Homotopy shuffle system. In Corollary 4.4 we showed the isomorphism

$$H^0(B_{red}(\hat{C}(I, AS(W)), \epsilon)^* \simeq \mathbb{F}_p[[\pi_1(X, \bar{x})]]$$

as coalgebras. The algebra $\mathbb{F}_p[[\pi_1(X, \bar{x})]]$ has a coproduct and the $\mathbb{F}_p$-completion of $\pi_1(X, \bar{x})$ (actually it is isomorphic to the pro-$p$ completion of $\pi_1(X, \bar{x})$, see Appendix.) is defined to be the set of group like elements in $\mathbb{F}_p[[\pi_1(X, \bar{x})]]$. The coproduct on $\mathbb{F}_p[[\pi_1(X, \bar{x})]]$ corresponds to a product on $H^0(B_{red}(\hat{C}(I, AS(W)), \epsilon)$. In the theory of real homotopy type, the cohomology of bar complex the $C^\infty$ differential forms is equipped with a product structure obtained from the shuffle product.

If the characeristic of the base field $k$ is zero, the associate simple complex of a strict cosimplicial commutative DGA has a natural structure of commutative DGA by Thom-Whitney construction. In our case, char($k$) = $p > 0$ and the Artin-Schreier DGA is not graded commutative and as a consequence, the shuffle product is not available as it is. Thus, the shuffle product is not graded commutative, it is graded commutative up to homotopy. In this section, we discuss higher homotopy for commutativity, which is necessary to define the homotopy shuffle product on the bar complex.

Let $I$ be a finite ordered set and $A$ be a strict cosimplicial graded commutative DGA indexed by $I$. In this section, we construct a homotopy shuffle product

$$B_{red}(\hat{C}(I, A) \otimes, \epsilon) \otimes B_{red}(\hat{C}(I, A) \otimes, \epsilon) \to B_{red}(\hat{C}(I, A) \otimes, \epsilon),$$

which induces a natural coproduct on $\mathbb{F}_p[[\pi_1(X, \bar{x})]]$ via the linear isomorphism (5.1).
Let $A$ be a complex. Let $\sigma$ be an element of $\mathfrak{S}_n$. A homogeneous element $a_1 e_1 \otimes \cdots \otimes a_n e_n$ of the tensor product $(\mathbf{A}[1])^{\otimes n} = \mathbf{A}[1] \otimes \cdots \otimes \mathbf{A}[1]$ of $n$-copies of $\mathbf{A}[1]$ goes to $\pm a_{\sigma(1)} e_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} e_{\sigma(n)}$ by the rule mentioned in (1.2). Here $e_i$ is the canonical generator of $\mathcal{K}[1]$ of degree $-1$. The element $\sigma$ also acts on $\mathbf{A}^{\otimes n}[n]$ via the identification

\begin{equation}
(\mathbf{A}[1])^{\otimes n} \simeq \mathbf{A}^{\otimes n}[n] : a_1 e_1 \otimes \cdots \otimes a_n e_n \mapsto (a_1 \otimes \cdots \otimes a_n) e_1 \cdots e_n = (a_1 \otimes \cdots \otimes a_n) e^n.
\end{equation}

For example, let $a_1$ and $a_2$ be elements of degree $\alpha_1$ and $\alpha_2$ in $\mathbf{A}$, respectively. Via this identification, we have

\[ a_1 e_1 \otimes a_2 e_2 = (-1)^{\alpha_2}(a_1 \otimes a_2) e_1 e_2 \]

and the permutation of first and the second components becomes the following map.

\[
\begin{align*}
(a_1 \otimes a_2)e^2 &= (a_1 \otimes a_2)e_1 e_2 = (-1)^{\alpha_2} a_1 e_1 \otimes a_2 e_2 \\
&= (-1)^{\alpha_2 + (\alpha_1 + 1)(\alpha_2 + 1)} a_2 e_2 \otimes a_1 e_1 \\
&= (-1)^{\alpha_1 \alpha_2 + 1} a_2 \otimes a_1 e^2.
\end{align*}
\]

Let $\mathbf{A}$ be a DGA and $\mu : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ be the multiplication morphism. The morphism $\mu$ induces the morphism $\mathbf{A}[p] \otimes \mathbf{A}[q] \to \mathbf{A}[p + q]$ by the composite defined in (1.2) (1.3). If $\mathbf{A}$ is graded commutative, then $\mu + \mu \circ \sigma : \mathbf{A}[1] \otimes \mathbf{A}[1] \to \mathbf{A}[2]$ is the zero map.

**Definition 5.1** (Homotopy shuffle system). Let $\mathbf{A}^\bullet$ be an associative DGA. A homotopy shuffle system consists of the system of degree preserving $\mathcal{K}$ linear map

\[ h^{l,m} : \mathbf{A}^{\otimes [l]} \otimes \mathbf{A}^{\otimes [m]} \to \mathbf{A}[1] \]

for $l \geq 0, m \geq 0, l + m > 0$ with the following properties. The map $\mathbf{A}^{\otimes [l + 1]} \otimes \mathbf{A}^{\otimes [m]} \to \mathbf{A}[2]$ and $\mathbf{A}^{\otimes [l]} \otimes \mathbf{A}^{\otimes [m + 1]} \to \mathbf{A}[2]$ induced by $h^{l,m}$ are also denoted as $h^{l,m}$ using the rule (1.3).

1. $h^{l,0} = h^{0,1} = 1_A$, and $h^{0,p} = h^{p,0} = 0$ for $p > 1$.
2. Let $\mu : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ be the multiplication for the DGA. The induced map $\mathbf{A}[1] \otimes \mathbf{A}[1] \to \mathbf{A}[2]$ is also denoted as $\mu$. Let $\sigma$ be the switching homomorphism defined in (1.7). Then we have

\[ \mu + \mu \circ \sigma = -dh^{1,1} + h^{1,1}d : \mathbf{A}[1] \otimes \mathbf{A}[1] \to \mathbf{A}[2]. \]
(3) Let \( l + m > 2, l > 0 \) and \( m > 0 \). We define the following composite linear map \( \alpha \).

\[
\alpha : A^{\otimes l}[l] \otimes A^{\otimes m}[m] \xrightarrow{\sigma} \bigoplus_{l'+l''=l, \atop m'+m''=m, \atop l',m' \geq 1, \atop l'',m'' \geq 1} (A^{\otimes l'[l']} \otimes A^{\otimes m'[m']}) \otimes (A^{\otimes l''[l'']} \otimes A^{\otimes m''[m'']}) \\
\sum (h_{l','m'} \otimes h_{l'',[m'']}) A[1] \otimes A[1] \xrightarrow{\mu} A[2]
\]

Here the morphism \( \sigma \) is given by the permutation of components. We define the following linear maps \( \beta \) and \( \gamma \).

\[
\beta : A^{\otimes l}[l] \otimes A^{\otimes m}[m] \xrightarrow{-dB \otimes 1} A^{\otimes (l-1)[l]} \otimes A^{\otimes m}[m] \\
\xrightarrow{h_{l-1,m}} A[2]
\]

\[
\gamma : A^{\otimes l}[l] \otimes A^{\otimes m}[m] \xrightarrow{-1 \otimes dB} A^{\otimes l}[l] \otimes A^{\otimes (m-1)[m]} \\
\xrightarrow{h_{l,m-1}} A[2]
\]

Then \( h_{l,m} \) satisfies the relation:

\[
-dh_{l,m} + h_{l,m}d = \alpha + \beta + \gamma : A^{\otimes l}[l] \otimes A^{\otimes m}[m] \to A[2].
\]

For the augmentation ideal \( I \), a homotopy shuffle system \( h_{l,m} : I^{\otimes l}[l] \otimes I^{\otimes m}[m] \to I[1] \) is defined similarly. In the rest of this subsection, we define the homotopy shuffle product

\[
B_{red}(A, \epsilon) \otimes B_{red}(A, \epsilon) \to B_{red}(A, \epsilon)
\]

by assuming the existence of a homotopy shuffle system \( \{h_{l,m}\}_{l,m} \) for \( I \). We define the set \( S^{(k)}_{l,m} \) by

\[
S^{(k)}_{l,m} = \{ f : [1, l] \cup [\bar{l}, \bar{m}] \to [1, l + m - k] \mid (1) \text{ } f \text{ } \text{is surjective,} \\
(2) \text{ } f(i) \leq f(j) \text{ for } i \leq j, \text{ and (3) } f(\bar{i}) \leq f(\bar{j}) \text{ for } i \leq j \}.
\]

For \( f \in S^{(k)}_{l,m} \) and \( p \in [1, l + m - k] \), we define \( S(f,p) = f^{-1}(p) \) and \( S'(f,p) = S(f,p) \cap [1,l], S''(f,p) = S(f,p) \cap [\bar{l}, \bar{m}] \). The cardinality of \( S(f,p), S'(f,p) \) and \( S''(f,p) \) is denoted by \( s(f,p), s'(f,p) \) and \( s''(f,p) \), respectively. We define a linear map \( H(f) : I^{\otimes l}[l] \otimes I^{\otimes m}[m] \to I^{l+m-k}[l+m-k] \) by the composite
We define

\[ I^\otimes [l] \otimes I^\otimes m [m] \]

\[
\sigma \left( I^{s'(f,1)}[s'(f,1)] \otimes I^{s''(f,1)[s''(f,1)]} \right) \otimes \ldots \\
\otimes \left( I^{s'(f,l+m-k)[s'(f,l+m-k)]} \otimes I^{s''(f,l+m-k)[s''(f,l+m-k)]} \right)
\]

\[ \theta \rightarrow I[1] \otimes \ldots \otimes I[1] \]

\[ \simeq I^{\otimes l+m-k}[l+m-k], \]

where \( \theta = h^{s'(f,1),s''(f,1)} \otimes \ldots \otimes h^{s'(f,l+m-k),s''(f,l+m-k)} \) and \( \sigma \) is the permutation homomorphism. We define a linear map \( H_{l,m,k} \) by

\[ H_{l,m,k} = \sum_{f \in S^{(k)}_{l,m}} H(f) : I^{\otimes [l]} \otimes I^{\otimes m}[m] \rightarrow I^{\otimes (l+m-k)[l+m-k]}. \]

**Proposition 5.2.** The linear map \( H = \sum_{l,m,n} H_{l,m,n} \) defines a homomorphism of complexes

\[ B(A, \epsilon) \otimes B(A, \epsilon) \rightarrow B(A, \epsilon). \]

**Proof.** The differential of the bar complex is the sum of inner differential

\( d_I : I^{\otimes n}[n] \rightarrow I^{\otimes n}[n+1] \) and bar differential \( d_B : I^{\otimes n}[n] \rightarrow I^{\otimes (n-1)}[n] \), where

\[ d_I = \sum_{i=1}^{n} d_{I,i}, \quad d_{I,i} = 1 \otimes \cdots \otimes d \otimes \cdots \otimes 1, \]

\[ d_B = \sum_{i=1}^{n-1} d_{B,i}, \quad d_{B,i}(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes (x_i \cup x_{i+1}) \otimes \cdots \otimes x_n. \]

Here we used the identification (5.2) and the definition of sign (3.1).

We consider the following diagram

\[
\begin{array}{c}
I^{\otimes [l]} \otimes I^{\otimes m}[m] \\
\downarrow d_B \otimes 1 + 1 \otimes d_B \\
I^{\otimes (l-1)}[l] \otimes I^{\otimes m}[m] \oplus I^{\otimes [l]} \otimes I^{\otimes (m-1)}[m] \\
\downarrow H_{l,m,k} + H_{l,m,k-1} \\
I^{\otimes (l+m-k-1)[l+m-k]} \\
\downarrow d_B
\end{array}
\]

It is enough to show that

\[ (5.6) \quad d_B \circ H_{l,m,n} - H_{l-1,m,k} \circ (d_B \otimes 1) - H_{l,m,k-1} \circ (1 \otimes d_B) = -d_I H_{l,m,k+1} + H_{l,m,k+1} d_I. \]

Let \( \tilde{f} \in S^{(k+1)}_{l,m} \). For \( i \in [1, l+m-k-1] \), we define a surjective non decreasing map \( \sigma_i : [l+m-k] \rightarrow [l+m-k-1] \) by setting \( \sigma_i(i) = \sigma_i(i+1) \). We set

\[ S^{(k)}_{l,m}(\tilde{f}) = \{(i, f) \in [1, l+m-k-1] \times S^{(k)}_{l,m} | \sigma_i \circ f = \tilde{f}\}. \]

We define \( T^{(k)}_{l-1,m}(\tilde{f}) \) and \( T^{(k)}_{l,m-1}(\tilde{f}) \) by

\[ T^{(k)}_{l-1,m}(\tilde{f}) = \{(j, g) \in [1, l-1] \times S^{(k)}_{l-1,m} | g \circ (\sigma_j \times 1) = \tilde{f}\}, \]

\[ T^{(k)}_{l,m}(\tilde{f}) = \{(j', g') \in [1, m-1] \times S^{(k)}_{l,m-1} | g' \circ (1 \times \sigma_{j'}) = \tilde{f}\}. \]
To show the equality (5.6), it is enough to show the following identity by considering $\bar{f}$-component.

\[(5.7) \sum_{(i, f) \in S^{(k)}_{l, m}(\bar{f})} d_{B, i} \circ H(f) - \sum_{(j, g) \in T^{(k)}_{l-1, m}(\bar{f})} H(g) \circ (d_{B, j} \otimes 1) - \sum_{(j', g') \in T^{(k)}_{l, m-1}(\bar{f})} H(g') \circ (1 \otimes d_{B, j'}) = - d_I H(\bar{f}) + \sum_{j \in S(\bar{f}, i)} H(\bar{f})d_{I, j}.\]

We fix $\bar{f}$ and $i$. The set

\[S^{(k)}_{l, m}(\bar{f}, i) = \{ f \mid (i, f) \in S(\bar{f}) \} = \{(P', P'') \mid P' \coprod P'' = S(\bar{f}, i) \text{ is a non empty partition such that } p' < p'' \text{ for } p' \in P' \text{ and } p'' \in P'' \}\]

and

\[T^{(k)}_{l-1, m}(\bar{f}, i) = \{(j, g) \in [1, l-1] \times S^{(k)}_{l-1, m} \mid g \circ (\sigma_j \times 1) = \bar{f}, \bar{f}(j) = i \}, \]

\[T^{(k)}_{l, m-1}(\bar{f}, i) = \{(j', g') \in [1, m-1] \times S^{(k)}_{l, m-1} \mid g' \circ (1 \times \sigma_{j'}) = \bar{f}, \bar{f}(j') = i \}\].

Then by the condition for homotopy shuffle system, we have

\[\sum_{(i, f) \in S^{(k)}_{l, m}(\bar{f}, i)} d_{B, i} \circ H(f) - \sum_{(j, g) \in T^{(k)}_{l-1, m}(\bar{f}, i)} H(g) \circ (d_{B, j} \otimes 1) - \sum_{(j', g') \in T^{(k)}_{l, m-1}(\bar{f}, i)} H(g') \circ (1 \otimes d_{B, j'}) = - d_I H(\bar{f}) + \sum_{j \in S(\bar{f}, i)} H(\bar{f})d_{I, j}.\]

By taking the summation over $i$, we have the equality (5.7). \(\square\)

5.2. Construction of homotopy shuffle system. In this section, we construct a homotopy shuffle system for a total DGA of a strict cosimplicial graded commutative DGA’s over the set $I = [0, n]$. If $n = 0$, since the total DGA $t(A) = A_0$ is graded commutative, we define $h^{l, m} = 0$ for $l + m > 1$.

We define $h^{l, m}$ by the induction on $n$.

5.2.1. The case for $n = 1$. The total DGA $A$ is written as $A_0 \oplus A_1 \oplus A_{0,1}[-1]$ and the multiplication can be written as

\[\begin{array}{l}
(a_0 + a_1 + a_{0,1}) \cup (b_0 + b_1 + b_{01}) \\
= (a_0 \cdot b_0) + (a_1 \cdot b_1) + (i_0(a_0) \cdot b_{01} + a_{01} \cdot i_1(b_1)) \\
= (a_0 \cdot b_0) + (a_1 \cdot b_1) + (i_0(a_0) \cdot (b_{01}e) + (-1)^{\deg(b_1)}(a_{01}e) \cdot i_1(b_1))e^{-1}.
\end{array}\]
for \(a_0, b_0 \in A_0, a_1, b_1 \in A_1, a_{01}, b_{01} \in A_{01}[-1]\). Here we used the DGA homomorphism \(i_0 : A_0 \to A_{01}, i_1 : A_1 \to A_{01}\) and \(\cdot\) for the multiplication of \(A_0, A_1\) and \(A_{01}\).

We set \(h^{1,1}(a \otimes b) = -(a_{01} e) \cdot (b_{01} e) \in A_{01} \subset A[1]\) for elements \(a = (a_0 + a_1 + a_{01}) e\) and \(b = (b_0 + b_1 + b_{01}) e\) in \(A[1]\), and \(h^{l,m} = 0\) for \(l + m > 2\). We check the conditions for a homotopy shuffle system. To avoid the sign complexity, we only check for degree zero elements.

**Proposition 5.3.** Let \(x, y\) be degree zero elements in \(A[1]\). We have the following formula.

\[
\begin{align*}
\text{(5.8)} & \quad x \cup y + y \cup x = -d_A h^{1,1}(x \otimes y) + h^{1,1}(d_{A \otimes A}(x \otimes y)) \\
\text{(5.9)} & \quad h^{1,1}((x_1 \cup x_2) \otimes y) = x_1 \cup h^{1,1}(x_2 \otimes y) + h^{1,1}(x_1 \otimes y) \cup x_2 \\
\text{(5.10)} & \quad h^{1,1}(x \otimes (y_1 \cup y_2)) = y_1 \cup h^{1,1}(x \otimes y_1) + h^{1,1}(x \otimes y_1) \cup y_2
\end{align*}
\]

**Proof.** We set \(x = (x_0 + x_1 + x_{01}) e\) and \(y = (y_0 + y_1 + y_{01}) e\). Since \(A_0, A_1\) are graded commutative, to prove the equality (5.8), it is enough to show the following equality.

\[
- \left[ (i_0(x_0) \cdot y_{01} + x_{01} \cdot i_1(y_1)) + (i_0(y_0) \cdot x_{01} + y_{01} \cdot i_1(x_1)) \right] e^2 = (-d_A h^{1,1} + h^{1,1} d_{A \otimes A})(x \otimes y)
\]

The left hand side is equal to

\[
- \left[ i_0(x_0) \cdot y_{01} e - x_{01} e \cdot i_1(y_1) + i_0(y_0) \cdot x_{01} e - y_{01} e \cdot i_1(x_1) \right] e.
\]

Let \(d_0, d_1\) and \(d_{01}\) be the differentials of \(A_0, A_1\) and \(A_{01}\), respectively. Using expression of (1.2), we have \(d_A(x) = (d_0 x_0 + d_1 x_1 + d_{01} x_{01}) e^2 + (i_0 x_0 - i_1 x_1) e\) and by the definition of extension rule defined in (1.3), the right hand side of the above equality is equal to

\[
\begin{align*}
d_0 & (x_{01} e \cdot y_{01} e) \\
& + h^{1,1}((d_0 x_0 + d_1 x_1 + d_{01} x_{01}) e^2 + (i_0(x_0) - i_1(x_1)) e) \otimes (y_0 + y_1 + y_{01}) e \\
& + h^{1,1}(x_0 + x_1 + x_{01}) e \otimes ((d_0 y_0 + d_1 y_1 + d_{01} y_{01}) e^2 + (i_0 y_0 - i_1 y_1)) e \\
& = [i_1(x_1) - i_0(x_0)] \cdot (y_{01} e) e + [(x_{01} e) \cdot (i_1 y_1) - i_0(y_0)] e \\
& = [y_{01} e \cdot i_1(x_1) - i_0(x_0) \cdot y_{01} e - i_0(y_0) \cdot (x_{01} e) + x_{01} e \cdot i_1(y_1)] e
\end{align*}
\]

Thus we have (5.8).

As for the equality (5.9), we have

\[
\begin{align*}
h^{1,1}((x_1(1) \cup x_1(2)) \otimes y) &= -(i_0(x_0(1)) \cdot x_{01}^{(2)} + x_{01}^{(1)} \cdot i_1(x_1^{(2)})) \cdot y_{01}
\end{align*}
\]

and

\[
\begin{align*}
x^{(1)} \cup h^{1,1}(x^{(2)} \otimes y) + h^{1,1}(x^{(1)} \otimes y) \cup x^{(2)} \\
&= - i_0(x_0^{(1)}) \cdot x_{01}^{(2)} \cdot y_{01} - x_{01}^{(1)} \cdot y_{01} \cdot i_1(x_1^{(2)}).
\end{align*}
\]

The equality (5.9) is similar. \(\square\)
We will check the condition of Definition 5.1 (3) for the homotopy shuffle system. In the case of $l + m = 3$, the condition is nothing by the equality (5.9), (5.10). If $l + m > 3$, then the homomorphisms $\alpha$ is equal to zero. In fact, unless $l' = l'' = m' = m'' = 1$, then either $l', l'', m'$ or $m''$ is greater than one. The term for $l' = l'' = m' = m'' = 1$ is also zero by the relation

$$h^{1,1}(x^{(1)} \otimes y^{(1)}) \cup h^{1,1}(x^{(2)} \otimes y^{(2)}) = 0.$$  

$\beta$ and $\gamma$ are also zero.

5.2.2. The case for $n > 1$. Let $B$ is a commutative DGA and $A_\bullet$ a strict cosimplicial graded commutative DGA. The map $A_k \to A_{jk}$ is denoted as $i_{jk,k}$. A set of homomorphisms of DGA’s $i_k : B \to A_k$ is called central if $i_{jk,k} \circ i_k = i_{jk,j} \circ i_j$. The map $H : A \otimes A \to A[-1]$ is called central if for any central homomorphism $i : B \to A$, $H(i(b) \otimes a) = H(a \otimes i(b)) = 0$.

We construct $h^{l,m}$ by the induction of $n$. Let $A_\bullet$ be a strict cosimplicial DGA. Let $A_0$ be the strict Cech complex of $\{A_I\}_{I \subset [1,n-1]}$. It is easy to see that the system $\{A_{I \cup n}\}_{I \subset [1,n-1]}$ also becomes a strict cosimplicial graded commutative DGA and the homomorphism $A_I \to A_{I \cup n}$ defines a homomorphism of strict cosimplicial DGA. The Cech complex of $\{A_{I \cup n}\}_{I \subset [1,n-1]}$ is denoted as $A_{01}$ and the homomorphism $A_0 \to A_{01}$ is denoted as $i_0$. We set $A_1 = A_n$. The homomorphisms $A_n \to A_{I \cup n}$ defines a homomorphism of DAG $A_1 \to A_{01}$, which is denoted as $i_1$. Thus we have a strict cosimplicial DGA

$$A_0 \oplus A_1 \to A_{01}.$$  

The total DGA $A = A_0 \oplus A_0 \oplus A_{01}[-1]$ of the above diagram is equal to the Cech DGA of $A_\bullet$. By the inductive construction, we already have $k$ linear maps

$$h_{\star}^{l,m} = h_{\star}^{l,m} : A_{\star}^\otimes[l] \otimes A_{\star}^\otimes[m] \to A_{\star}[1]$$

for $\star = 0, 1, 01$ compatible with $i_0$ and $i_1$. (Note that $h_1^{1,1} = 0$, since it is strict cosimplicial on the set $\{n\}$.) We construct central $k$ linear map

$$h_{\bullet}^{l,m} : A^\otimes[l] \otimes A^\otimes[m] \to A[1].$$

The case $(l, m) = (1, 1)$. We construct $h^{1,1}_A$ so that the relation Definition 5.1 (2) holds. Let $x = (x_0 + x_1 + x_{01})e, y = (y_0 + y_1 + y_{01})e$ be elements in $A[1]$, where $x_0, y_0 \in A_0$, $x_1, y_1 \in A_1$ and $x_{01}, y_{01} \in A_{01}[-1]$. Since the map of $i_1 : A_n \to (A_{01})_k = A_{kn}$ is central and $A_1$ is commutative, We set

$$h_A^{1,1} : A[1] \otimes A[1] \to A[1]$$

by

$$h_A^{1,1}(x \otimes y) = h_0^{1,1}(x_0 e \otimes y_0 e) - h_1^{1,1}(x_{01} e \otimes i_0(y_0) e) - (x_{01} e) \cdot (y_0 e),$$

and will show the equality:

$$x \cup y + y \cup x = -d_A h_A^{1,1}(x \otimes y) + h_A^{1,1}(d_A \otimes A(x \otimes y)).$$
For simplicity, we assume that the degree of $x$ and $y$ is zero. We have

$$x \cup y + y \cup x$$

$$(x_0 + x_1 + x_{01})e \cup (y_0 + y_1 + y_{01})e + (y_0 + y_1 + y_{01})e \cup (x_0 + x_1 + x_{01})e$$

$$= - (x_0 \cdot y_0 + y_0 \cdot x_0)e^2 + (-i_0(x_0) \cdot y_{01}e^2 + x_{01}e \cdot i_1(y_1)e)$$

$$+ (-i_0(y_0) \cdot x_{01}e^2 + y_{01}e \cdot i_1(x_1)e)$$

$$= - (x_0 \cdot y_0 + y_0 \cdot x_0)e^2 + (-i_0(x_0) + i_1(x_1)) \cdot y_{01}e^2 - x_{01}e \cdot (i_0(y_0) - i_1(y_1))e$$

$$+ (x_{01}e \cdot i_0(y_0)e - i_0(y_0) \cdot x_{01}e^2)$$

On the other hand,

$$d_{A \otimes A}(x \otimes y) = ((d_0x_0 + d_1x_1 + d_{01}x_{01})e^2 + (i_0(x_0) - i_1(x_1))e) \otimes (y_0 + y_1 + y_{01})e$$

$$+ ((x_0 + x_1 + x_{01})e \otimes ((d_0y_0 + d_1y_1 + d_{01}y_{01})e^2 + (i_0(y_0) - i_1(y_1))e).$$

We have

$$- d_A h_{A_1}^{1,1}(x \otimes y) + h_{A_1}^{1,1}(d_{A \otimes A}(x \otimes y))$$

$$= - d_A h_{A_1}^{1,1}(x_0e \otimes y_0e)e - i_0(h_{A_1}^{1,1}(x_0e \otimes y_0e)) + d_{01}h_{A_1}^{1,1}(x_{01}e \otimes i_0(y_0)e) + d_0(x_{01}e \cdot y_{01}e)e$$

$$+ h_0^{1,1}(d_0(x_0e \otimes y_0)e)e - h_0^{1,1}((d_0x_{01}e^2 + (i_0(x_0) - i_1(x_1))e) \otimes i_0(y_0)e)$$

$$- h_0^{1,1}(x_{01}e \otimes i_0(d_0y_0)e^2)$$

$$+ (i_1(x_1) - i_0(x_0)) \cdot y_{01}e^2 - (x_{01}e) \cdot (i_0(y_0) - i_1(y_1))e + d_0(x_{01}e \cdot y_{01}e)e$$

using the central assumption $h_{A_1}^{1,1}(i_1(x_1) \otimes i_0(y_0)) = 0$ and inductive hypotheses. Therefore the right hand side is equal to

$$- (x_0 \cdot y_0 + y_0 \cdot x_0)e^2 + (-i_0(x_0) + i_1(x_1)) \cdot y_{01}e^2 - x_{01}e \cdot (i_0(y_0) - i_1(y_1))e$$

$$+ (x_{01}e \cdot i_0(y_0)e - i_0(y_0) \cdot x_{01}e^2).$$

It is easy to see that $h_{A_1}^{1,1}$ is also central.

**The case** $(l, m) = (1, m)$ for $m > 1$.

Assume that $h_{A_1}^{1, m} : A_1 \otimes (A_1)^\otimes m \rightarrow A_1[1]$ is defined for $* = 0, 1, 01$.

Let $(x = x_0 + x_1 + x_{01})e, y^{(i)} = (y_0^{(i)} + y_1^{(i)} + y_{01}^{(i)})e$ be elements in $A_1[1]$, where $x_0, y_0^{(i)} \in A_0, x_1, y_1^{(i)} \in A_1$ and $x_{01}, y_{01}^{(i)} \in A_0[-1]$. We define $h^{1, m} : A_1 \otimes (A_1)^\otimes m \rightarrow A_1[1]$ by

$$h^{1, m}(x \otimes y^{(1)} \otimes \cdots \otimes y^{(m)})$$

$$= h_{01}^{1, m}(x_0e \otimes y_0^{(1)}e \otimes \cdots \otimes y_0^{(m)}e) - h_{01}^{1, m}(x_{01}e \otimes i_0(y_0^{(1)}e \otimes \cdots \otimes i_0(y_0^{(m)}e))$$

$$- \left[h_{01}^{1, m-1}(x_{01}e \otimes i_0(y_0^{(1)}e \otimes \cdots \otimes i_0(y_0^{(m-1)}e)) \cdot (y_0^{(m)}e).$$

The condition

$$\alpha + \beta + \gamma = -d_{A_1[1]}h^{1, m} + h^{1, m}d_{A_1[1] \otimes A_1[1]}^\otimes m$$
in Definition 5.1 (3) is equal to

\[(5.11)\]
\[(-1)^{\deg(x)\deg(y)} y(1) \cup h^{1,m-1}(x \otimes y^{(2)} \otimes \ldots \otimes y^{(m)})
+ h^{1,m-1}(x \otimes y^{(1)} \otimes \ldots \otimes y^{(m-1)}) \cup y^{(m)}
- h^{1,m-1}(x \otimes (y^{(1)} \cup y^{(2)}) \otimes \ldots \otimes y^{(m)}) - \ldots
- h^{1,m-1}(x \otimes y^{(1)} \otimes \ldots \otimes (y^{(m-1)} \cup y^{(m)}))
= (-d_A h^{1,m} + h^{1,m} d_A \otimes A \otimes \ldots) (x \otimes y^{(1)} \otimes \ldots \otimes y^{(m)}).\]

We show this equality by assuming that the degrees of \(x\) and \(y^{(i)}\) are zero. The terms of the left hand side of (5.11) are as follows.

\[(5.12)\]
\[y(1) \cup h^{1,m-1}(x \otimes y^{(2)} \otimes \ldots \otimes y^{(m)})
= y_0^{(1)} e \cdot h^{1,m-1}_0 (x_0 e \otimes y_0^{(2)} e \otimes \ldots \otimes y_0^{(m)} e)
- i_0(y_0^{(1)}) e \cdot h^{1,m-1}_0 (x_0 e \otimes i_0(y_0^{(2)}) e \otimes \ldots \otimes i_0(y_0^{(m)}) e)
- i_0(y_0^{(1)}) e \cdot [h^{1,m-2}_0 (x_0 e \otimes i_0(y_0^{(2)}) e \otimes \ldots \otimes i_0(y_0^{(m-1)}) e) \cdot y_0^{(m)} e],\]

\[(5.13)\]
\[h^{1,m-1}_0 (x \otimes y^{(1)} \otimes \ldots \otimes y^{(m-1)}) \cup y^{(m)}
= h^{1,m-1}_0 (x_0 e \otimes y_0^{(1)} e \otimes \ldots \otimes y_0^{(m-1)} e) \cdot y_0^{(m)} e
+ i_0(h^{1,m-1}_0 (x_0 e \otimes y_0^{(1)} e \otimes \ldots \otimes y_0^{(m-1)} e)) \cdot y_0^{(m)} e
- h^{1,m-1}_0 (x_0 e \otimes i_0(y_0^{(1)}) e \otimes \ldots \otimes i_0(y_0^{(m-1)}) e) \cdot i_1(y_1^{(m)}) e
- h^{1,m-2}_0 (x_0 e \otimes i_0(y_0^{(1)}) e \otimes \ldots \otimes i_0(y_0^{(m-2)}) e) \cdot y_0^{(m-1)} e \cdot i_1(y_1^{(m)}) e,\]

\[(5.14)\]
\[- h^{1,m-1}_0 (x \otimes (y^{(1)} \cup y^{(2)}) \otimes \ldots \otimes y^{(m)}) - \ldots
- h^{1,m-1}_0 (x \otimes y^{(1)} \otimes \ldots \otimes (y^{(m-1)} \cup y^{(m)}))
= - h^{1,m-1}_0 (x_0 e \otimes (y_0^{(1)} e \cdot y_0^{(2)} e) \otimes \ldots \otimes y_0^{(m)} e) - \ldots
- h^{1,m-1}_0 (x_0 e \otimes y_0^{(1)} e \otimes \ldots \otimes (y_0^{(m-1)} e \cdot y_0^{(m)} e))
+ h^{1,m-1}_0 (x_0 e \otimes i_0(y_0^{(1)}) e \cdot y_0^{(2)} e) \otimes \ldots \otimes i_0(y_0^{(m)}) e) - \ldots
+ h^{1,m-1}_0 (x_0 e \otimes i_0(y_0^{(1)}) e \otimes \ldots \otimes i_0(y_0^{(m-1)}) e) \cdot y_0^{(m)} e - \ldots
+ h^{1,m-2}_0 (x_0 e \otimes i_0(y_0^{(1)}) e \cdot y_0^{(2)} e) \otimes \ldots \otimes i_0(y_0^{(m-2)}) e) \cdot y_0^{(m-1)} e \cdot y_0^{(m)} e
+ h^{1,m-2}_0 (x_0 e \otimes i_0(y_0^{(1)}) e \otimes \ldots \otimes i_0(y_0^{(m-2)}) e)
\cdot (i_0(y_0^{(m-1)}) e \cdot y_0^{(m)} e + y_0^{(m-1)} e \cdot i_1(y_1^{(m)}) e).\]
By the inductive hypotheses for $h^{(1,m)}_0$, $h^{(1,m)}_{01}$, the left hand side of (5.11) is equal to

$$(-d_0 h^{1,m}_0 + h^{1,m}_0 d_0) (x_0 e \otimes y_0^{(1)} e \otimes \cdots \otimes y_0^{(m)} e)$$

$$+ (d_0 h^{1,m}_{01} - h^{1,m}_{01} d_0) (x_0 e \otimes i_0 (y_0^{(1)} e \otimes \cdots \otimes i_0 (y_0^{(m)} e))$$

$$+ h^{1,m-1}_{01} (x_0 e \otimes i_0 (y_0^{(1)} e \otimes \cdots \otimes i_0 (y_0^{(m-1)} e)) \cdot (i_1 (y_1^{(m)} e) - i_0 (y_0^{(m)} e))$$

$$+ (-d_0 h^{1,m-1}_{01} + h^{1,m-1}_{01} d_0) (x_0 e \otimes i_0 (y_0^{(1)} e \otimes \cdots \otimes i_0 (y_0^{(m-1)} e) \cdot y_0^{(m)} e)$$

$$- h^{1,m-1}_{01} ((i_0 (x_0) e - i_1 (x_1) e) \otimes i_0 (y_0^{(1)} e \otimes \cdots \otimes i_0 (y_0^{(m-1)} e) \cdot y_0^{(m)} e)$$

and this is equal to the right hand side of (5.11).

By the induction on $n$, we can prove the following proposition.

**Proposition 5.4.** Let $A$ be the total DGA of a strict cosimplicial graded commutative DGA’s and $h^{(1,m)}$ be the homomorphism defined as above. Let $B$ be a graded commutative DGA’s and $j_k : B \to A_k$ be a family of central morphisms of DGA’s. Let $j : B \to A$ be the induced morphisms of DGA’s. Then we have

$$h^{(1,m)} ((x \cdot j(x')) \otimes y_1 \otimes \cdots \otimes y_m) = (-1)^{\deg j(x')} \sum_k \deg y_k h^{(1,m)} (x \otimes y_1 \otimes \cdots \otimes y_m) \cdot j(x').$$

for $x, y_1, \ldots, y_m \in A[1]$ and $x' \in B$.

**The case $(l,m)$ for $l > 1$.**

We define $h^{l,m} = 0$ for $l > 1$. To show the condition Definition 5.1 (3), it is enough to consider the case where $l = 2$. In the case of $l = 2$, we prove this condition by the induction on $m$. By the inductive hypotheses, we have

$$h^{1,m} (x^{(1)} \otimes x^{(2)} \otimes y^{(1)} \otimes \cdots \otimes (y^{(i)} \cup y^{(i+1)}) \otimes \cdots \otimes y^{(m)}) = 0,$$

and $\alpha + \beta + \gamma$ in Definition 5.1 (3) is equal to

$$x^{(1)} \cup h^{1,m} (x^{(2)} \otimes y^{(1)} \otimes \cdots \otimes y^{(m)})$$

$$+ (-1)^{\deg x^{(2)} \deg y^{(1)}} h^{1,1} (x^{(1)} \otimes y^{(1)}) \cup h^{1,m-1} (x^{(2)} \otimes y^{(2)} \otimes \cdots \otimes y^{(m)}) \pm \cdots$$

$$\pm h^{1,m-1} (x^{(1)} \otimes y^{(1)} \otimes \cdots \otimes y^{(m-1)}) \cup h^{1,1} (x^{(2)} \otimes y^{(m)})$$

$$\pm h^{1,m} (x^{(1)} \otimes y^{(1)} \otimes \cdots \otimes y^{(m)}) \cup x^{(2)}$$

$$- h^{1,m} ((x^{(1)} \cup x^{(2)}) \otimes y^{(1)} \otimes \cdots \otimes y^{(m)}).$$
Again we consider the case where the degrees of $x^{(i)}$ and $y^{(j)}$ are zero. By the inductive definition of $h^{1,m}$, it is equal to

$$x_0^{(1)} e \cdot h_0^{1,m} (x_0^{(2)} e \otimes y_0^{(1)} e \otimes \cdots \otimes y_0^{(m)} e)$$

$$\quad + h_0^{1,1} (x_0^{(1)} e \otimes y_0^{(1)} e) \cdot h_0^{1,m-1} (x_0^{(2)} e \otimes y_0^{(2)} e \otimes \cdots \otimes y_0^{(m)} e) + \cdots$$

$$\quad + h_0^{1,m-1} (x_0^{(1)} e \otimes y_0^{(1)} e \otimes \cdots \otimes y_0^{(m-1)} e) \cdot h_0^{1,1} (x_0^{(2)} e \otimes y_0^{(m)} e)$$

$$\quad + h_0^{1,m} (x_0^{(1)} e \otimes y_0^{(1)} e \otimes \cdots \otimes y_0^{(m)} e) \cdot x_0^{(2)} e$$

$$- h_0^{1,m} ((x_0^{(1)} e \cdot x_0^{(2)} e) \otimes y_0^{(1)} e \otimes \cdots \otimes y_0^{(m)} e)$$

$$- i_0 (x_0^{(1)} e) \cdot h_0^{1,m} (x_0^{(2)} e \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m)} e))$$

$$\quad - i_0 (h_0^{1,1} (x_0^{(1)} e \otimes y_0^{(1)} e)) \cdot h_0^{1,m-1} (x_0^{(2)} e \otimes i_0 (y_0^{(2)} e) \otimes \cdots \otimes i_0 (y_0^{(m)} e)) + \cdots$$

$$\quad - i_0 (h_0^{1,m-1} (x_0^{(1)} e \otimes y_0^{(1)} e \otimes \cdots \otimes y_0^{(m-1)} e)) \cdot h_0^{1,1} (x_0^{(2)} e \otimes i_0 (y_0^{(m)} e))$$

$$\quad + h_0^{1,m} ((i_0 (x_0^{(1)} e) \cdot x_0^{(2)} e) + x_0^{(1)} e \cdot i_0 (x_0^{(1)} e) \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m-1)} e)) \cdot y_0^{(m)} e$$

$$\quad - h_0^{1,m} (x_0^{(1)} e \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m)} e)) \cdot i_0 (x_0^{(1)} e)$$

$$\quad - h_0^{1,m-1} ((x_0^{(1)} e) \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m-1)} e)) \cdot y_0^{(m)} e \cdot i_1 (x_0^{(2)} e).$$

By a simple computation and Proposition 5,4, it is equal to

$$h_0^{1,m} ((x_0^{(1)} e \cdot i_1 (x_0^{(2)} e) \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m)} e))$$

$$\quad + h_0^{1,m-1} ((x_0^{(1)} e \cdot i_1 (x_0^{(2)} e) \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m-1)} e)) \cdot y_0^{(m)} e$$

$$\quad - h_0^{1,m} (x_0^{(1)} e \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m)} e)) \cdot i_1 (x_0^{(2)} e)$$

$$\quad - h_0^{1,m-1} ((x_0^{(1)} e) \otimes i_0 (y_0^{(1)} e) \otimes \cdots \otimes i_0 (y_0^{(m-1)} e)) \cdot y_0^{(m)} e \cdot i_1 (x_0^{(2)} e)$$

$$\quad = 0$$

Thus we have the following theorem.

**Theorem 5.5.** For a strict cosimplicial graded commutative DGA, there exists a functorial homotopy shuffle system for the total DGA. As a consequence, we have the following homomorphism of complexes:

$$\mu : B_{red} (\tilde{C}(I, A) \otimes, e) \otimes B_{red} (\tilde{C}(I, A) \otimes, e) \rightarrow B_{red} (\tilde{C}(I, A) \otimes, e).$$

This map is called the homotopy shuffle product.
5.3. Artin-Schreier sheaf and homotopy shuffle product. We apply the construction of the last section to construct homotopy shuffle products of the Artin-Schreier DGA's. For an $F_p$-algebra, we set

$$AS^*(R) = (AS^*_0(R) \oplus AS^*_1(R) \to AS^*_01(R)),$$

where $AS^*_0(R) = R, AS^*_1(R) = R, AS^*_01(R) = R \oplus R$ and $i_0 : AS^*_0 \to AS^*_01$ and $i_1 : AS^*_1 \to AS^*_01$ is given by $i_0(x) = (x, x)$ and $i_1 = (x, x^p)$. Then the morphism of complexes $AS(R) \to AS^*(R)$ given by

$$(5.16) \quad AS^0 \to AS^*_0 \oplus AS^*_1 : x \mapsto x \oplus x$$

$$(5.17) \quad AS^1 \to AS^*_01 : y \mapsto 0 \oplus y$$

is a quasi-isomorphism of DGA’s. Then the DGA $AS^*$ is a total DGA of a strict cosimplicial graded commutative DGA. By this fact and the result of the last section, we have a homotopy shuffle product on the bar complex of $AS^*$ for an affine scheme. We extend this construction to separated irreducible schemes. The sheaf on $X_{et}$ associated to $AS^*$ is denoted as $AS^*$.

Let $W = \{W_i \to X\}_{i \in I}$ be a finite set of morphisms to $X$ indexed by $I = [n]$. For an element $i = (i_0, \ldots, i_l)$ in $P(I)$, we set $W_i = W_{i_0} \times_X \cdots \times_X W_{i_l}$. We consider a set $I = \{0^+, 0^-; \ldots, n^+, n^-\}$ and $I_i = \{i^+, i^-\}$. We introduce a total order on $I$ by $i^+ < i^-$ for $i = 0, \ldots, n$ and $i^- < (i + 1)^+$ for $i = 0, \ldots, n - 1$. Let $\pi$ be the projection $\bar{I} \to I$. We define a strict cosimplicial graded commutative DGA $AS^*(W)$ indexed by $\bar{I}$ by

$$AS^*(W)_{\bar{i}} = \begin{cases} AS_0^*(W_{\pi(i)}) & \text{if } \bar{i} \subset I^+ \\ AS_1^*(W_{\pi(i)}) & \text{if } \bar{i} \subset I^- \\ AS_{01}^*(W_{\pi(i)}) & \text{otherwise.} \end{cases}$$

Since the homomorphism $AS^*(W)_{\bar{i}}$ is a quasi-isomorphism, we have the following proposition.

Proposition 5.6. The morphism $AS(W) \to \check{C}(\bar{I}/I, AS^*(W))$ indexed by $I$. Moreover the map $AS(W)_{\bar{i}} \to \check{C}(\bar{I}/I, AS^*(W))_{\bar{i}}$ is a quasi-isomorphism of DGA.

The Cech complex $\check{C}(\bar{I}, AS^*(W))$ is a strict cosimplicial graded commutative DGA. By the last subsection, we have a product on $H^0(B_{red}(\check{C}(\bar{I}, AS^*(W))))$ arising from homotopy shuffle product.

Theorem 5.7. The multiplication

$$H^0(\mu) : H^0(B_{red}(\check{C}(\bar{I}, AS^*(W)))) \otimes H^0(B_{red}(\check{C}(\bar{I}, AS^*(W)))) \to H^0(B_{red}(\check{C}(\bar{I}, AS^*(W))))$$

obtained from the homotopy shuffle product $\otimes$ in Theorem 5.3 coincides with the product on $F_p[[\pi_1(X, \bar{x})]]^*$ via the isomorphism in Corollary 4.4. As a consequence, the above product $\mu$ is commutative.
Proof. Let $U \to X$ be an etale covering of $X$ and we set $I = \{m\}$. We consider the family $\mathcal{U} = \{U_i \to X\}_{i \in I}$ of covering of $X$ indexed by $I$ by setting $U = U_i$. We have a strict cosimplicial DGA $AS^*(\mathcal{U})$ indexed by $\tilde{I}$. Then we have the following quasi-isomorphisms:

$$\lim_{\overrightarrow{i, l}} \tilde{C}(I, F_p(\mathcal{U})) \to \lim_{\overrightarrow{i, l}} \tilde{C}(I, AS(\mathcal{U})) \to \lim_{\overrightarrow{i, l}} \tilde{C}(\tilde{I}, AS^*(\mathcal{U}))$$

Since the homomorphism $\tilde{C}(I, F_p(\mathcal{U})) \to \tilde{C}(\tilde{I}, AS^*(\mathcal{U}))$ comes from the homomorphism of strict cosimplicial graded commutative DGA, the homotopy shuffle product on $B_{red} = B_{red}(\tilde{C}(I, F_p(\mathcal{U})), \epsilon)$ and that on $B_{red}(\tilde{C}(\tilde{I}, AS^*(\mathcal{U})), \epsilon)$ are compatible. It is enough to show that the homotopy shuffle product on $B_{red}(\tilde{C}(I, F_p(\mathcal{U})), \epsilon)$ is compatible with the tensor structure on nilpotent $F_p$ smooth local systems. Let $B_{simp} = B_{simp}(\tilde{C}(I, F_p(\mathcal{U})), \epsilon)$ be the simplicial bar complex and we assume that $M$ comes form a $B_{simp}$ comodule. We set $A^\bullet = \tilde{C}(I, F_p(\mathcal{U}))$. Then $M$ has a direct sum decomposition $M = \oplus_\alpha M_\alpha$ and the comodule structure defines a map

$$\nabla_{\alpha_0, \alpha_1} : M_{\alpha_0} \to A^1 \otimes M_{\alpha_1} = \oplus_{i_0 < i_1} F_p(U_{i_0} \times X U_{i_1}) \otimes M_{\alpha_1}$$

for $\alpha_0 < \alpha_1$. We set $\nabla_{\alpha_0, \alpha_1} = \oplus_{i_0 < i_1} \nabla_{\alpha_0, \alpha_1, i_0, i_1}$ and

$$R_{i_0, i_1} = id - \sum_{\alpha_0 < \alpha_1} \nabla_{\alpha_0, \alpha_1, i_0, i_1} : M \to F_p(U_{i_0} \times X U_{i_1}) \otimes M.$$

By the comodule condition, we have $R_{i_0, i_1} R_{i_1, i_2} = R_{i_0, i_2}$, the set $\{R_{i_0, i_1}\}$ of automorphism defines a local system on $X$. Here we used the multiplication structure

$$F_p(U_{i_0} \times X U_{i_1}) \otimes F_p(U_{i_1} \times X U_{i_2}) \to F_p(U_{i_0} \times X U_{i_1} \times X U_{i_2})$$

induced by the diagonal map. Now we consider two comodules $M_1$ and $M_2$. Then we have two systems of automorphisms

$$R^{(1)}_{i_0, i_1} : M_1 \to F_p(U_{i_0} \times X U_{i_1}) \otimes M_1,$$

$$R^{(2)}_{i_0, i_1} : M_2 \to F_p(U_{i_0} \times X U_{i_1}) \otimes M_2.$$

And the direct sum decompositions $M_1 = M_{1, \alpha_0}$ and $M_2 = M_{2, \beta_0}$. The tensor product of the local systems associated to $\{R^{(1)}_{i_0, i_1}\}$ and $\{R^{(2)}_{i_1, i_2}\}$ is equal to that associated to $R^{(1)}_{i_0, i_1} \otimes R^{(2)}_{i_0, i_1}$. Here we used the algebra structure on $F_p(U_{i_0} \times X U_{i_1})$.

On the other hand, we compute the comodule structure on the tensor product $M_1$ and $M_2$ using the homotopy shuffle product by the following diagram:

(5.17)

$$M_1 \otimes M_2 \to B_{red} \otimes M_1 \otimes B_{red} \otimes M_2 = B_{red} \otimes B_{red} \otimes M_1 \otimes M_2 \mu^{1 \otimes 1} \to B_{red} \otimes M_1 \otimes M_2$$

We use the map $h^{1,1} : A^1_{\alpha_0, \alpha_1} \otimes A^1_{\beta_0, \beta_1} \to A^1_{\alpha_0 + \beta_0, \alpha_1 + \beta_1}$ which is compatible with the homotopy shuffle system for $B_{red}$. Note that the terms $h^{i,j}$ for
$i+j > 2$ vanishes by the construction of homotopy shuffle product. Therefore the $(i_0, i_1)$ part of (5.17) is equal to the direct sum of
\[- \sum_{\alpha_0+\beta_0=\gamma_0, \alpha_1+\beta_1=\gamma_1} \nabla^{(1)}_{\alpha_0,\alpha_1} \otimes \nabla^{(1)}_{\beta_0,\beta_1} + \nabla^{(1)}_{\gamma_0,\gamma_1} \otimes 1 + 1 \otimes \nabla^{(2)}_{\gamma_0,\gamma_1}\]

Here we used the algebra structure (commutative) of $F_p(U_{i_0} \times_X U_{i_1})$. Therefore the comodule structure defined by the map (5.17) defines the tensor of two local systems. □

We expect the following.

**Conjecture 5.8.** More strongly, it is expected that the homotopy shuffle product (5.15) is homotopy commutative for any strict simplicial graded commutative DGA $A^\bullet$ indexed by $I$. If it is homotopy commutative, the product on cohomology
\[H^\bullet(\mu) : H^\bullet(B_{red}(\hat{C}(I, A^\bullet))) \otimes H^\bullet(B_{red}(\hat{C}(I, A^\bullet))) \to H^\bullet(B_{red}(\hat{C}(I, A^\bullet)))\]
of homotopy shuffle product is graded commutative. The author does not know whether the homotopy shuffle product is homotopy commutative nor the commutativity of $H^\bullet(\mu)$ for any strictly simplicial graded commutative DGA.

If the base field is of characteristic zero, there is a construction of commutative DGA out of strict simplicial graded commutative DAG due to Thom-Whitney (see $[A]$).

6. DESCENT THEROY UP TO HOMOTOPY

In this section, we treat $F_q$-schemes with an $F_q$-valued point $x$, where $q = p^e$. We recover $F_p$-Malcev completion from $F_q$-Malcev completion by the descent theory.

Let $R$ be an algebra over $F_q$. We define $q$-Artin-Schreier DGA $AS(R)^{(e)}$ by the total complex of the relation diagram
\[\begin{array}{ccc}
R & \xrightarrow{id} & R \\
\xrightarrow{F^e} & & \\
\end{array}\]

Let $X$ be a separated scheme of finite type over $F_q$ and $\mathcal{W} = \{W_i\}_{i \in I}$ be a finite affine open covering of $X$. Then the correspondence $J \mapsto AS(W_J)^{(e)}$ defines a strict cosimplicial DGA indexed by $I$, which is denoted as $AS(\mathcal{W})^{(e)}$. The Cech complex is denoted as $\check{C}(I, AS(\mathcal{W})^{(e)})$. Since the Frobenius map $F : R \to R : x \mapsto x^p$ is a ring homomorphism, we have a $F$-linear complex homomorphism $F : AS(R) \to AS(R)$. The $e$-th power $F^e$ of $F$ is $F_q$-linear and it is homotopic to the identity map by the homotopy $h_{Spec(R)} : AS^1(R)^{(e)} \to AS^0(R)^{(e)} : x \mapsto x$. Since the Frobenius map is functorial for $F_q$-algebras, it induces a $F$-linear homomorphism of DGA’s
\[\varphi : \check{C}(I, AS(\mathcal{W})^{(e)}) \to \check{C}(I, AS(\mathcal{W})^{(e)})\].
Since the homotopy \( h \) is functorial on \( \mathbb{F}_q \)-algebras,

\[
\prod_{|J|=p} h_{W_j} : \prod_{|J|=p} AS^1(W_J) \to \prod_{|J|=p} AS^0(W_J)
\]
defines a null homotopy of

\[
\mathbb{F}_q - id : \mathcal{C}(I, AS(W)^{(e)}) \to \mathcal{C}(I, AS(W)^{(e)}).
\]

Assume that we have an \( \mathbb{F}_q \)-valued point on \( X \). Then we have an augmentation map \( \epsilon : \mathcal{C}(I, AS(W)^{(e)}) \to \mathbb{F}_q \). Since \( \mathcal{C}(I, AS(W)) \) is a DGA over \( \mathbb{F}_q \), we can define the bar complex \( B(\mathcal{C}(I, AS(W)^{(e)}), \epsilon)_{\mathbb{F}_q} \) over \( \mathbb{F}_q \). The homomorphism \( \varphi \) is \( \mathbb{F} \)-linear. Therefore we have the following \( \mathbb{F} \)-linear homomorphism.

\[
\varphi : H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_{\mathbb{F}_q} \to H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_{\mathbb{F}_q}.
\]

Similarly as in (4) the degree zero cohomology of the bar complex of \( \mathcal{C}(AS(W)^{(e)}) \) over \( \mathbb{F}_q \) is isomorphic to \( \mathbb{F}_q \)-completion of \( \pi_1(X, \bar{x}) \). To recover the \( \mathbb{F}_p \)-completion of \( \pi_1(X, \bar{x}) \), we need the descent data for \( H^0(B(AS(W/X)^{(e)}, \epsilon))_{\mathbb{F}_q} \).

In this section, we construct the descent data for \( \mathcal{C}(I, AS(W)) \).

**Proposition 6.1.** (1) \( \varphi^e \) is an identity. As a consequence, there is a unique \( \mathbb{F}_p \) subspace \( H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_0 \) called the standard \( \mathbb{F}_p \) structure in \( H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_{\mathbb{F}_q} \) such that

\[
H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_{\mathbb{F}_q} = \mathbb{F}_q \otimes_{\mathbb{F}_p} H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_0
\]

and the action of \( \varphi \) is equal to \( F \otimes id \) under this identification.

(2) The standard \( \mathbb{F}_p \) structure is stable under multiplication and comultiplication.

(3) \( H^0(B(\mathcal{C}(I, AS(W)), \epsilon))_0 \) is isomorphic to the dual of \( \mathbb{F}_p[[\pi_1(X, \bar{x})]] \).

**Proof.** We construct a null homotopy of

\[
\mathbb{F}_q - id : B(\mathcal{C}(I, AS(W)), \epsilon)_{\mathbb{F}_q} \to B(\mathcal{C}(I, AS(W)), \epsilon)_{\mathbb{F}_q}.
\]

In the proof, the tensor product “\( \otimes \)” means a tensor product over \( \mathbb{F}_q \) unless otherwise mentioned. Let \( A \) be the Artin-Schreier DGA \( \mathcal{C}(I, AS(W)) \). For \( \alpha_0 < \cdots < \alpha_n \), we define a \( \mathbb{F}_p \)-linear map

\[
H_{\alpha_0, \ldots, \alpha_n} : \mathbb{F}_q \otimes A \otimes \cdots \otimes A \otimes \mathbb{F}_q[1] \to \mathbb{F}_q \otimes A \otimes \cdots \otimes A \otimes \mathbb{F}_q
\]

by

\[
H_{\alpha_0, \ldots, \alpha_n} = 1_{\mathbb{F}_q} \otimes h_A \otimes F_A^e \otimes \cdots \otimes F_A^{\alpha_{n-1}} \otimes 1_{\mathbb{F}_q} + 1_{\mathbb{F}_q} \otimes 1_A \otimes h_A \otimes F_A^{\alpha_0} \otimes \cdots \otimes F_A^{\alpha_{n-1}} \otimes 1_{\mathbb{F}_q} + \cdots
\]

\[
+ 1_{\mathbb{F}_q} \otimes 1_A \otimes \cdots \otimes 1_A \otimes h_A \otimes 1_{\mathbb{F}_q}.
\]
Then $H = \sum_{\alpha_0, \ldots, \alpha_n} H_{\alpha_0, \ldots, \alpha_n}$ defines a homotopy of $F^e - id$ on $B(\tilde{C}(I, AS(W)), \epsilon)_{F_q}$.

To show that $H$ defines a homotopy of $1 - F^e$ on $B(\tilde{C}(I, AS(W)), \epsilon)_{F_q}$, it is enough to show that the following diagram commutes:

$$
\begin{align*}
A \otimes A & \xrightarrow{\mu} A \\
1 \otimes h_A + h_A \otimes F^e & \downarrow \\
A \otimes A[-1] & \xrightarrow{\mu} A[-1]
\end{align*}
$$

Let $(x)_{J,i}$ be an element $x \in R_J$ considered in $AS^i(R_J)$ and $J = (j_0, \ldots, j_k), K = (j_k, \ldots, j_{k+l}), L = (j_0, \ldots, j_{k+l})$. Then we have

$$
h_A \circ \mu(((x_0)_{J,0} + (x_1)_{J,1}) \otimes ((y_0)_{K,0} + (y_1)_{K,1})) = (x_0 \cdot y_1 + x_1 \cdot y_0^q)_{L,0}
$$

and

$$
\mu \circ (1 \otimes h_A - h_A \otimes (F^e))(((x_0)_{J,0} + (x_1)_{J,1}) \otimes ((y_0)_{K,0} + (y_1)_{K,1})) = (x_0 \cdot y_1)_{L,0} + (x_1 \cdot y_0^q)_{L,1} + (x_1 \cdot y_1)_{L,0} - (x_1 \cdot y_1^q)_{L,1}.
$$

Therefore $H$ defines a homotopy of $F^e - id$.

\section*{7. Appendix Remarks on pro $p$ completion}

In this section, we recall that $F_p$ completion of a group is isomorphic to its pro-$p$ completion.

**Proposition 7.1.** Let $G$ be a finit $p$ group. Then $F_p[G]$ is a Artin local field. Especially, Let $\epsilon : F_p[G] \to F_p$ be the augmentation map and $I = \text{Ker}(\epsilon)$ be the augmentation ideal. Then $I^n = 0$ for sufficiently large $n$. (See also \cite{S}, §15.6)

**Proof.** We prove the theorem by the induction of the length of upper central series. If $G = \mathbb{Z}/q_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/q_n \mathbb{Z}$ is abelian where $q_1, \ldots, q_n$ is a power of $p$, then $F[G] \cong F[x_1, \ldots, x_n]/(x_1^{q_1} - 1, \ldots, x_n^{q_n} - 1)$ is a local ring whose maximal ideal is the augmentation ideal $I$. Let $G$ is a $p$ group and $P$ be a center of $G$. Then $P$ is non-trivial. Since the augmentation ideal $I$ is generated by $g - 1$, it is enough to prove that there exists an $N$ such that $(g_1 - 1) \cdots (g_N - 1) = 0$ for any $g_1, \ldots, g_N \in G$. By the induction of the length of upper central series, there exists a positive integer $M$ such that $(g_1 - 1) \cdots (g_M - 1)$ is contained in $\text{Ker}(F_p[G] \to F_p[G/P]) \cap I$ for any $g_1, \ldots, g_M \in G$. Therefore $(g_1 - 1) \cdots (g_M - 1)$ can be expressed as a sum of $\gamma(p - q)$, where $p, q \in P$. Moreover there exists $K$ such that

$$
\gamma_1(p_1 - q_1)\gamma_2(p_2 - q_2) \cdots \gamma_K(p_K - q_K)
$$

$$
= \gamma_1 \gamma_2 \cdots \gamma_K (p_1 - q_1)(p_2 - q_2) \cdots (p_K - q_K)
$$

$$
= 0
$$

for any $\gamma_i \in G, p_i, q_i \in P$. Therefore $I^{MK} = 0$. This proves the theorem. \hfill $\Box$

Thus we have the following corollary.

**Corollary 7.2.** Let $G$ be a $p$-group.
(1) Then the group like element in $\mathbb{F}_p[G]$ is equal to $G$.
(2) $G$ is equal to the $\mathbb{F}_p$ Malcev completion of $G$.
(3) The pro-$p$ completion of $G$ is equal to $\mathbb{F}_p$ Malcev completion.

**Proof.** (1) Let $a = \sum_{g \in G} a_g g$ be a group like element in $\mathbb{F}_p[G]$. Then by the equality $\Delta(a) = a \otimes a$, we have

$$a_g a_h = \begin{cases} a_g & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}$$

Since $a \neq 0$, there is an element $g \in G$ such that $a_g \neq 0$. By the above equality, we have $a_g = 1$. Therefore $a_h = 0$ for $h \neq g$. Thus we have $a = [g]$. The second and the third statement follows from the first by the last proposition.

□

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