Robust 1-Bit Compressed Sensing via Hinge Loss Minimization

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Abstract. This work theoretically studies the problem of estimating a structured high-dimensional signal \( x_0 \in \mathbb{R}^n \) from noisy 1-bit Gaussian measurements. Our recovery approach is based on a simple convex program which uses the hinge loss function as data fidelity term. While such a risk minimization strategy is very natural to learn binary output models, such as in classification, its capacity to estimate a specific signal vector is largely unexplored. A major difficulty is that the hinge loss is just piecewise linear, so that its “curvature energy” is concentrated in a single point. This is substantially different from other popular loss functions considered in signal estimation, e.g., the square or logistic loss, which are at least locally strongly convex. It is therefore somewhat unexpected that we can still prove very similar types of recovery guarantees for the hinge loss estimator, even in the presence of strong noise. More specifically, our non-asymptotic error bounds show that stable and robust reconstruction of \( x_0 \) can be achieved with the optimal oversampling rate \( O(m^{-1/2}) \) in terms of the number of measurements \( m \). Moreover, we permit a wide class of structural assumptions on the ground truth signal, in the sense that \( x_0 \) can belong to an arbitrary bounded convex set \( K \subset \mathbb{R}^n \). The proofs of our main results rely on some recent advances in statistical learning theory due to Mendelson. In particular, we invoke an adapted version of Mendelson’s small ball method that allows us to establish a quadratic lower bound on the error of the first order Taylor approximation of the empirical hinge loss function.

Key words. 1-bit compressed sensing, structured empirical risk minimization, hinge loss, Gaussian width, Mendelson’s small ball method

1 Introduction

This paper considers the problem of estimating an unknown signal vector \( x_0 \in \mathbb{R}^n \) from 1-bit observations of the form

\[
y_i = f_i(\langle a_i, x_0 \rangle) \in \{-1, +1\}, \quad i = 1, \ldots, m,
\]

(1.1)

where \( a_1, \ldots, a_m \in \mathbb{R}^n \) is a collection of known measurement vectors and \( f_i : \mathbb{R} \to \{-1, +1\}, i = 1, \ldots, m, \) are binary-valued output functions. The number of samples \( m \) is typically much smaller than the ambient dimension \( n \), so that the equation system of (1.1) is highly underdetermined. Such types of recovery tasks have recently caught increasing attention in various research areas, most importantly in the field of compressed sensing, which has become a state-of-the-art approach in signal processing during the last decade. It builds upon a novel paradigm according to which many reconstruction problems can be efficiently solved by linear or convex programming, if the low-dimensional structure of the ground truth signal is explicitly taken into account; see [FR13] for a comprehensive introduction.

In contrast to traditional compressed sensing, the setup of (1.1) does also involve a non-linear component. More specifically, each output function \( f_i \) plays the role of a quantizer that distorts the linear observation rule \( a_i \mapsto \langle a_i, x_0 \rangle \). Such a quantization step is of particular interest to real-world sensing schemes in which only a finite number of bits can be (digitally) processed during transmission. In fact, the model of (1.1) assumes the most extreme case of quantization where only 1-bit information is available per measurement — that is why we speak of 1-bit compressed sensing in this context [BB08; Jac+13; PV13a]. Let us emphasize that the quantizers \( f_i \) can be completely deterministic, e.g., \( f_i = \text{sign}, \) but they could be also contaminated by noise in the form of random bit flips.

A large class of signal estimation methods can be formulated as an optimization problem of the form

\[
\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^m \mathcal{L}(\langle a_i, x \rangle, y_i) \quad \text{subject to} \quad x \in K, \quad (P_{\mathcal{L}, K})
\]

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1Hereafter, we make the convention that \( \text{sign}(v) = +1 \) for \( v \geq 0 \) and \( \text{sign}(v) = -1 \) for \( v < 0 \).
where $\mathcal{L}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a convex loss function and $K \subset \mathbb{R}^n$ defines a convex constraint set, usually called the signal set. The purpose of $\mathcal{L}$ is to assess how well the candidate model $a_i \mapsto \langle a_i, x \rangle$ matches with the true outputs $y_i$. With regard to our initial recovery task, we may therefore hope that a minimizer $\hat{x} \in \mathbb{R}^n$ of $(P_{L,K})$ provides an accurate approximation of the signal vector $x_0$. On the other hand, the signal set $K$ encodes certain structural hypotheses about $x_0$, e.g., sparsity. A prototypical choice is an $\ell^1$-constraint, i.e., $K = RB^n_1$ for some $R > 0$, which indeed often serves as a powerful sparsity prior if $m \ll n$.

In this work, we will focus on a special instance of $(P_{L,K})$ that is based on the so-called hinge loss given by $\mathcal{L}_{\text{hng}}(v) := \max\{0, 1 - v\} := 0$ for $v \in \mathbb{R}$. Using $\mathcal{L}(v, \nu^i) := \mathcal{L}_{\text{hng}}(v \cdot \nu^i)$ as loss function, the program of $(P_{L,K})$ now reads as follows:

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^m \max\{0, 1 - y_i \langle a_i, x \rangle\} \quad \text{subject to} \quad x \in K.$$  

$(P_{\text{hng},K})$

This estimator is specifically tailored to deal with binary observations: Intuitively, by minimizing the objective functional of $(P_{\text{hng},K})$, one tries to select $x \in K$ in such a way that the sign of $\langle a_i, x \rangle$ equals $y_i \in \{-1, +1\}$ for most of the samples $i \in \{1, \ldots, m\}$. With other words, a solution $\hat{x}$ to $(P_{\text{hng},K})$ yields a predictor $a_i \mapsto \text{sign}(\langle a_i, \hat{x} \rangle)$ of the true outputs $y_i$. While this simple heuristic explains the success of hinge loss minimization in many classification tasks, it also indicates a certain capability to tackle the 1-bit compressed sensing problem stated by (1.1). However, let us point out that the latter challenge does not just ask for finding any good predictor but actually aims at retrieving the ground truth signal $x_0$. Compared to reliable prediction, successful signal estimation usually relies on relatively strong model assumptions, and in fact, the performance of $(P_{\text{hng},K})$ is only poorly understood on this matter. The key concern of this paper is therefore to establish theoretical recovery guarantees for $(P_{\text{hng},K})$ under the hypothesis of (1.1) with Gaussian measurement vectors. In particular, we intend to address the following issues:

How many sample pairs $\{(a_i, y_i)\}_{i=1,\ldots,m} \subset \mathbb{R}^n \times \{-1, +1\}$ are required to accurately estimate the ground truth signal $x_0$ via hinge loss minimization $(P_{\text{hng},K})$? How is this related to the structural assumptions on $x_0$ and the choice of the signal set $K$?

### 1.1 Main Contributions and Overview

In Section 2, we will make the above model setting more precise and provide several definitions of complexity measures for signal sets, in particular the concepts of (local) Gaussian width and effective dimension. Our first main result (Theorem 2.10) is then presented in Subsection 2.2, considering signal sets in the Euclidean unit ball, i.e., $K \subset B^n_2$. In contrast, our second main results in Subsection 2.3 (Theorem 2.14 and Theorem 2.16) drop this condition and allow for arbitrary bounded convex signal sets, but they are merely restricted to the perfect 1-bit case with $f_i = \text{sign}$ for all $i = 1, \ldots, m$. To give a first glance at these theoretical findings, let us state the following informal recovery guarantee.

**Theorem 1.1 (Informal)** Let the measurement rule (1.1) hold true with i.i.d. standard Gaussian measurement vectors $a_1, \ldots, a_m \sim \mathcal{N}(0, I_n)$. If $\|x_0\|_2 = 1$ and $x_0 \in K$, then with high probability, any minimizer $\hat{x}$ of $(P_{\text{hng},K})$ satisfies

$$\frac{\|x_0 - \hat{x}\|_2}{\|\hat{x}\|_2} \leq C(f_i) \cdot \left(\frac{\delta(K, x_0)}{m}\right)^{1/2},$$

where $C(f_i) > 0$ only depends on the quantizers $f_1, \ldots, f_m$ and $\delta(K, x_0)$ is a measure of complexity for $K$ (with respect to $x_0$) that may change in model situations of Subsection 2.2 and Subsection 2.3.

The non-asymptotic error bound of (1.2) shows that 1-bit compressed sensing via hinge loss minimization is feasible for a large class of measurement schemes and structural hypotheses. In particular, the parameter $C(f_i)$ can be regarded as a model-dependent constant that is well-behaved under very mild correlation conditions on the (noisy) quantizers. Our guarantees significantly improve a recent
result from Kolleck and Vybíral [KV17], whose analysis of the hinge loss estimator is just limited to $\ell^1$-constraints and a quite restrictive noise pattern. Apart from that, they do only achieve an oversampling rate of $O(m^{-1/4})$, which is clearly worse than the optimal rate of $O(m^{-1/2})$ promoted by (1.2).

All proofs are postponed to Section 6. Our key arguments are based on tools from statistical learning theory, more specifically, on some recent uniform lower and upper bounds for empirical stochastic processes established by Mendelson [Men15; Men16]. For this reason, we will also frequently use the common terminology of statistical learning throughout this paper — for example, the estimators of (common terminology of statistical learning throughout this paper — for example, the estimators of statistical learning throughout this paper rather fits into the context of signal processing than into machine learning (see also Subsection 2.3.3 for a connection to setup of this paper). Particularly, the second derivative of the hinge loss function $L_{hng}$ does not meet the prerequisites of known signal estimation results. This in turn requires an extension of many arguments, especially in the case of general signal sets where the standard notion of local Gaussian width needs to be carefully adapted. A major technical difficulty is that the second derivative of the hinge loss $L_{hng}$ does not exist in a distributional sense (due to its piecewise linearity), so that it is not even locally strongly convex. Consequently, the proofs of this work are of independent interest because they may provide a template to establish important properties of loss functions, such as restricted strong convexity [Gen17; Neg+09; Neg+12], under fairly mild regularity assumptions.

In order to illustrate the general recovery framework of Section 2, we present some specific applications of our main results in Section 3. On the one hand, this concerns the choice of the signal set $K$ in (common terminology of statistical learning throughout this paper rather fits into the context of signal processing than into machine learning (see also Subsection 2.3.3 for a connection to support vector machines). Finally, some concluding remarks as well as several open issues can be found in Section 5.

1.2 Notation

Let us fix some notations and conventions that will be frequently used in the remainder of this paper:

1. For an integer $k \in \mathbb{N}$, we set $[k] := \{1, \ldots, k\}$. Vectors and matrices are denoted by lower- and uppercase boldface letters, respectively. Unless stated otherwise, their entries are indicated by subscript indices and lowercase letters, e.g., $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ for a vector and $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ for a matrix.

2. The support of $x \in \mathbb{R}^n$ is the set of its non-zero components, $\text{supp}(x) := \{j \in [n] \mid x_j \neq 0\}$, and we set $\|x\|_0 := |\text{supp}(x)|$. In particular, $x$ is called $s$-sparse if $\|x\|_0 \leq s$. For $p \geq 1$, the $\ell^p$-norm of $x$ is given by

$$
\|x\|_p := \begin{cases} 
(\sum_{j=1}^n |x_j|^p)^{1/p}, & p < \infty, \\
\max_{j \in [n]} |x_j|, & p = \infty.
\end{cases}
$$

The associated unit ball is denoted by $B^*_p := \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$ and the Euclidean unit sphere is $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$.

3. The positive part of a number $t \in \mathbb{R}$ is given by $[t]_+ := \max\{0, t\}$. For a subset $A \subset \mathbb{R}^n$, we denote the associated step function (or characteristic function) by

$$
\chi_A(x) := \begin{cases} 
1, & x \in A, \\
0, & \text{otherwise,}
\end{cases} x \in \mathbb{R}^n.
$$
(4) Let $K, K' \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. We denote the linear hull of $K$ by $\text{span}(K)$ and its convex hull by $\text{conv}(K)$. The Minkowski difference between $K$ and $K'$ is defined as $K - K' := \{x - x' \mid x \in K, x' \in K'\}$ and we simply write $K - x_0$ instead of $K - \{x_0\}$. The descent cone of $K$ at $x_0$ is given by

$$C(K, x_0) := \{\tau(x - x_0) \mid x \in K, \tau \geq 0\}.$$  

Furthermore, $\text{rad}(K) := \sup_{x \in K} ||x||_2$ is the radius of $K$ (around $0$).

If $E \subset \mathbb{R}^n$ is a linear subspace, the associated orthogonal projection onto $E$ is denoted by $P_E \in \mathbb{R}^{n \times n}$. Then, we have $P_{E^\perp} = I_n - P_E$, where $E^\perp \subset \mathbb{R}^n$ is the orthogonal complement of $E$ and $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix. Moreover, if $E = \text{span}\{x\}$, we use the short notations $P_x := P_E$ and $P_{x^\perp} := P_{E^\perp}$.

(5) For the expected value of a random variable $X$, we write $\mathbb{E}[X]$. The probability of an event $A$ is denoted by $\mathbb{P}[A]$ and the corresponding indicator function is $\mathbb{I}_A$. We write $g \sim \mathcal{N}(0, I_n)$ for an $n$-dimensional standard Gaussian random vector, and similarly, $g \sim \mathcal{N}(0, v^2)$ is a mean-zero Gaussian variable with variance $v^2$. We call a random variable $X$ sub-Gaussian if its sub-Gaussian norm $\|X\|_{\psi_2} := \inf\{t > 0 \mid \mathbb{E}[e^{X^2/t^2}] \leq 2\}$ is finite, see also [Ver12, Def. 5.22].

(6) The letter $C$ is always reserved for a (generic) constant, whose value could change from time to time. We refer to $C > 0$ as a numerical constant if its value does not depend on any other involved parameter. If an (in-)equality holds true up to a numerical constant, we sometimes simply write $A \lesssim B$ instead of $A \leq C \cdot B$, and if $C_1 \cdot A \leq B \leq C_2 \cdot A$ for numerical constants $C_1, C_2 > 0$, we use the abbreviation $A \asymp B$.

## 2 Main Results

This part presents our main theoretical findings on signal estimation via hinge loss minimization. In Subsection 2.1, we start with a formal definition of the 1-bit measurement model that was stated in (1.1). Moreover, we introduce the notions of (local) Gaussian width and effective dimension, which will serve as measures of complexity for the signal set $K$ in $(P_{\text{class},K})$. Subsection 2.2 then deals with recovery in the Euclidean unit ball, i.e., $K \subset B^n_2$, whereas our second main result in Subsection 2.3 considers general signal sets with perfect 1-bit observations. Note that the exposition below frequently uses the terminology of statistical learning theory, on which the proofs in Section 6 are based. For a very brief overview of this field, see also Subsection 4.2.

### 2.1 Model Setup and Gaussian Width

Let us first give a precise definition of the observation model that was informally introduced in (1.1):

**Assumption 2.1 (Measurement Model)** Let $f : \mathbb{R} \to \{-1, +1\}$ be a (random) quantization function and let $a \sim \mathcal{N}(0, I_n)$ be a standard Gaussian random vector which is independent of $f$. We consider a noisy 1-bit Gaussian measurement model of the form

$$y := f(\langle a, x_0 \rangle) \in \{-1, +1\}$$  

(2.1)

where $x_0 \in \mathbb{R}^n$ is the (unknown) ground truth signal. Each of the $m$ samples $\{(a_i, y_i)\}_{i \in [m]} \subset \mathbb{R}^n \times \{-1, +1\}$ is then drawn as an independent copy from the random pair $(a, y)$. Consequently, the binary observations are given by

$$y_i = f_i(\langle a_i, x_0 \rangle), \quad i = 1, \ldots, m, \tag{2.2}$$

where $f_i$ is an independent copy of $f$.

As already mentioned in the introduction, a prototypical example of a 1-bit quantizer is the sign-function, that is, $f = \text{sign}$. We refer to this (noiseless) observation scheme as the perfect 1-bit model. Note that all information on the magnitude of $x_0$ is lost in this case, implying that it is impossible to determine $||x_0||_2$ from the measurements. For that reason, we will always assume that the signal vector


Assumption 2.2 is not fulfilled. But the Cauchy-Schwarz inequality implies that both noise particularly allows us to study different sources of noise, such as random bit flips or additive Gaussian noise. See Subsection 3.2 for more details on these types of distortions.

In many scenarios of interest, there is some prior knowledge about the unknown signal \( x_0 \) available. The hinge loss estimator \( (P_{\text{hinge}, K}) \) encodes such additional structural assumptions by means of a convex constraint set \( K \subset \mathbb{R}^n \). Hence, we supplement Assumption 2.1 with the following signal model:

**Assumption 2.2 (Signal Model)** We assume that \( \| x_0 \|_2 = 1 \) and \( x_0 \in K \) for a certain subset \( K \subset \mathbb{R}^n \), which is called the signal set. Furthermore, we require that \( K \) is convex, bounded, and \( 0 \in K \).

Perhaps the most prominent (low-dimensional) signal structure is sparsity, for instance, if we have \( \| x_0 \|_0 \leq s \) for some \( s \ll n \). The set of all \( s \)-sparse vectors (in the unit ball) is however not convex, so that Assumption 2.2 is not fulfilled. But the Cauchy-Schwarz inequality implies that both \( K = \sqrt{s} B^n_1 \) and \( K = \sqrt{s} B^n_1 \cap B^n_2 \) may serve as admissible convex relaxations, which meet the conditions of Assumption 2.2. For more examples of signal sets, see Subsection 3.1.

A key issue in signal estimation concerns the number of measurements \( m \) that a certain recovery procedure requires to accurately approximate the target vector \( x_0 \). With regard to empirical risk minimization in \((P_{L, K})\), we may hope that, by restricting the solution space to an appropriate signal set \( K \subset \mathbb{R}^n \), reconstruction even succeeds in high-dimensional situations where \( m \ll n \). This desirable behavior obviously relies on a good choice of \( K \) that is supposed to capture low-dimensional “features” of \( x_0 \). In this context, the so-called Gaussian width has turned out to be a very useful complexity parameter. Indeed, many recent results — including ours below — show that the square of the Gaussian width often determines the (minimal) number of samples to ensure recovery via convex optimization [Ame+14; Cha+12].

In the following, we briefly introduce all notions of complexity that are required to formulate our main results. For a more extensive discussion on the role of the Gaussian width and related quantities in signal estimation, the reader is referred to [PV13b; PVY16; Ver15; Ver18]. Apart from that, some basic properties and specific examples are presented in Subsection 3.1.

**Definition 2.3 (Gaussian Width)** Let \( g \sim \mathcal{N}(0, I_n) \) be a standard Gaussian random vector. The Gaussian width of a bounded set \( K \subset \mathbb{R}^n \) is defined as

\[
  w(K) := \mathbb{E}[\sup_{x \in K} \langle g, x \rangle].
\]

See Figure 1(a) for an illustration of the Gaussian width and its geometric meaning. Since our task is to estimate a fixed vector \( x_0 \), it is natural to measure the complexity of the signal set \( K \) in a small neighborhood of \( x_0 \), rather than computing the “global” width \( w(K) \). To this end, let us consider the following localized version: For \( t > 0 \), the local Gaussian width of a subset \( K \subset \mathbb{R}^n \) at scale \( t > 0 \) is given by

\[
  w_t(K) := w(K \cap tB^n_2).
\]

In particular, we call \( w_t(K - x_0) \) the local Gaussian width of \( K \) in \( x_0 \) at scale \( t > 0 \); see also Figure 1(b). Note that the local Gaussian width is always bounded from above by its global counterpart:

\[
  w_t(K) = \mathbb{E}[\sup_{x \in K \cap tB^n_2} \langle g, x \rangle] \leq \mathbb{E}[\sup_{x \in K} \langle g, x \rangle] = w(K). \tag{2.3}
\]

In order to relate the (local) Gaussian width to the number of samples \( m \), it is very convenient to work with a scaling invariant complexity parameter. This property is achieved by the notion of effective dimension,

\[
  d(K) := \frac{w(K)^2}{\text{diam}(K)^2},
\]

where \( \text{diam}(K) \) denotes the Euclidean diameter of the subset \( K \subset \mathbb{R}^n \). If \( K \subset B^n_2 \) (as in Subsection 2.2), it particularly follows that \( d(K) \propto w(K)^2 \).
Combining the above concepts of localization and scaling invariance, we now define the local effective dimension, which originates from the recent works of [Gen17; PV16; PVY16] and in fact forms a crucial ingredient of our first recovery result in Theorem 2.10.

**Definition 2.4 (Local Effective Dimension)** The local effective dimension of a subset $K \subset \mathbb{R}^n$ in $x_0$ at scale $t > 0$ is given by

$$d_t(K - x_0) := \frac{w_t((K - x_0) \cap \mathbb{B}^n)}{t^2}.$$ 

Since $w_t(K - x_0) = w((K - x_0) \cap t\mathbb{B}^n)$, the scaling parameter $t$ essentially captures the diameter of the local neighborhood of $x_0$ and therefore normalizes the local Gaussian width. More precisely, from

$$d_t(K - x_0) \propto d((K - x_0) \cap t\mathbb{S}^{n-1}),$$

we can conclude that the local effective dimension is equivalent to the effective dimension of the shifted and localized signal set $(K - x_0) \cap t\mathbb{S}^{n-1}$.

To better understand the role of the scale $t$ in $d_t(K - x_0)$, let us consider the limit case $t \to 0$, or with other words, what happens if the neighborhood of $x_0$ becomes infinitesimal small. For this purpose, we first rewrite the local effective dimension as follows:

$$d_t(K - x_0) = \frac{1}{t} w((K - x_0) \cap t\mathbb{B}^n)^2 = w(\frac{1}{t}(K - x_0) \cap \mathbb{B}^n)^2.$$

Hence, if $K$ is convex, it is not hard to see that

$$w(\frac{1}{t}(K - x_0) \cap \mathbb{B}^n)^2 \to w(\mathcal{C}(K, x_0) \cap \mathbb{B}^n)^2, \quad \text{as } t \to 0,$$

where $\mathcal{C}(K, x_0)$ is the descent cone of $K$ at $x_0$. The limit on the right-hand side of (2.4) is well-known as conic effective dimension or statistical dimensional in the literature [Ame+14; Cha+12]. Let us provide a formal definition of this important quantity because it will reappear in the hypotheses of Theorem 2.14 and Theorem 2.16 below:

**Definition 2.5 (Conic Effective Dimension)** The conic effective dimension of a subset $K \subset \mathbb{R}^n$ in $x_0$ is defined as

$$d_0(K - x_0) := w(\mathcal{C}(K, x_0) \cap \mathbb{B}^n)^2.$$
From a geometric viewpoint, \( d_0(K - x_0) \) measures the size (narrowness) of the cone generated by \( K - x_0 \). While this complexity parameter is conceptually simple, it leads to problems if \( x_0 \) lies in the interior of the signal set \( K \) and not exactly on its boundary. Supposed that \( \text{span} K = \mathbb{R}^n \), we would then have \( C(K, x_0) = \mathbb{R}^n \), which in turn implies \( d_0(K - x_0) \approx n \). In contrast, the local effective dimension does not suffer from such an “unstable” behavior, since it avoids to take the conic hull of \( K - x_0 \). Indeed, due to
\[
d_t(K - x_0) \leq d_t(C(K, x_0)) = \frac{w_t(C(K, x_0))^2}{t^2} = w_t(C(K, x_0))^2 = d_0(K - x_0),
\]
we conclude that the local effective dimension can be (much) smaller than its conic counterpart, e.g., in the situation of Figure 1(b). With regard to our specific recovery problem, Theorem 2.10 below shows that, by tolerating a reconstruction error of order \( t \), the sampling rate is actually determined by \( d_t(K - x_0) \), rather than \( d_0(K - x_0) \). For a more extensive discussion of the local and conic effective dimension, see also [Gen17, Sec. III.D].

### 2.2 Recovery in Subsets of the Unit Ball

This subsection investigates signal recovery under Assumption 2.1 and 2.2 but with the restriction that the signal set belongs to the Euclidean unit ball, i.e., \( K \subset B^n_2 \). The proofs of all results stated below are postponed to Subsection 6.2. Let us first recall the hinge loss estimator:

**Definition 2.6** (Hinge Loss Minimization) Let the model conditions of Assumption 2.1 and 2.2 hold true. A hinge loss estimator \( \hat{x} \in \mathbb{R}^n \) is defined as a solution of the convex program

\[
\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^m \mathcal{L}_{\text{hng}}(y_i \langle a_i, x \rangle) \quad \text{subject to} \quad x \in K,
\]

where \( \mathcal{L}_{\text{hng}}(v) := [1 - v]_+ = \max\{0, 1 - v\} \) denotes the hinge loss function.

Since \((P_{\mathcal{L}_{\text{hng}, K}})\) is a particular instance of empirical risk minimization, we also introduce the notion of risk function, which is very common in statistical learning theory:

**Definition 2.7** (Empirical and Expected Risk Function) The objective functional of \((P_{\mathcal{L}_{\text{hng}, K}})\) is called empirical risk function and denoted by

\[
\hat{\mathcal{R}}(x) := \frac{1}{m} \sum_{i=1}^m \mathcal{L}_{\text{hng}}(y_i \langle a_i, x \rangle).
\]

Its expected value is given by

\[
\mathcal{R}(x) := \mathbb{E}[\hat{\mathcal{R}}(x)] = \mathbb{E}[\mathcal{L}_{\text{hng}}(y \langle a, x \rangle)],
\]

which is referred to as the expected risk function.

For a better understanding of the formal arguments below, let us briefly outline the basic idea behind signal recovery via empirical risk minimization: The first step is to verify that any minimizer of the expected risk,

\[
\min_{x \in K} \mathcal{R}(x) = \min_{x \in K} \mathbb{E}[\mathcal{L}_{\text{hng}}(y \langle a, x \rangle)],
\]

is well-behaved in the sense that it is sufficiently close to \( \text{span}\{x_0\} \). Indeed, Lemma 2.8 below states that the expected risk minimizer takes the form \( \mu x_0 \) for some \( 0 < \mu \leq 1 \), which in turn is a consequence of the unit-ball assumption \( K \subset B^n_2 \). According to the law of large numbers, one can therefore expect that \( \mathcal{R}(\mu x_0) \approx \mathcal{R}(\hat{x}) \) as \( m \) gets sufficiently large. The second key step is then to exploit the convexity of \( \mathcal{R}(\cdot) \) to show that one even has \( \mu x_0 \approx \hat{x} \). More specifically, it will turn out that \( \mathcal{R}(\cdot) \) satisfies restricted strong convexity on a certain neighborhood of \( \mu x_0 \) (see Proposition 6.6 and Remark 6.7). A simple normalization
step eventually leads to an error bound in the form of (1.2). The interested reader is referred to [Gen17; Men17] for a more detailed discussion of this strategy, which in fact also applies to many other convex loss functions.

According to this roadmap, our first task is to study the expected risk minimizer of (2.6):

**Lemma 2.8** Let \( g \sim \mathcal{N}(0, 1) \) be a standard Gaussian random variable. Moreover, we set

\[
\mu := \mu_f := \arg \min_{s \in [0, 1]} \mathbb{E}[\mathcal{L}^{\text{log}}(sf(g)g)],
\]

where \( f : \mathbb{R} \to \{-1, +1\} \) is the 1-bit quantizer from Assumption 2.1. Assuming that \( \mathbb{E}[f(g)g] > 0 \), we have that \( \mu > 0 \) and \( \mu x_0 \in K \) is the minimizer of (2.6), i.e.,

\[
\mathcal{R}(\mu x_0) = \min_{x \in K} \mathcal{R}(x).
\]

Let us emphasize that \( \mathbb{E}[f(g)g] > 0 \) is a very natural assumption because it ensures that the linear signal \( g = (a, x_0) \) and the output variable \( y = f(g) \) are positively correlated. Otherwise, if \( \mathbb{E}[f(g)g] = 0 \), all information on \( x_0 \) would be completely buried in noise so that there is no hope for recovery in any case. The correlation parameter \( \lambda_f := \mathbb{E}[f(g)g] \) therefore essentially captures the signal-to-noise ratio of the measurement rule (2.1). In particular, a large value of \( \lambda_f \) implies that the expected risk minimizer \( \mu x_0 \) is not too close to 0 (see Lemma 6.4). Besides this correlation assumption, we need a second mild condition on the quantizer \( f \) in order to formulate our main recovery result, Theorem 2.10:

**Assumption 2.9** (Correlation Conditions) Let \( g \sim \mathcal{N}(0, 1) \) be a standard Gaussian random variable and let \( f : \mathbb{R} \to \{-1, +1\} \) be the 1-bit quantizer from Assumption 2.1. Assume that the following two model conditions hold true:

(C1) \( \lambda := \lambda_f := \mathbb{E}[f(g)g] > 0 \),

(C2) \( \mathbb{E}[f(g) \text{ sign}(g) | |g||] \geq 0 \) (a.s.).

We call \( \lambda \) the correlation parameter of the quantizer \( f \).

The statistical meaning of (C1) and (C2) is studied in greater detail in Subsection 3.2. In this context, we will also show that both conditions are easily satisfied for two prototypical sources of model corruptions, namely random bit flip noise (after quantization) and additive Gaussian noise (before quantization).

We are now ready to state the main result of this subsection:

**Theorem 2.10** (Signal Recovery in Unit Ball) Let the model conditions of Assumption 2.1, 2.2, and 2.9 be satisfied, assume that \( K \subset B^n_0 \), and let \( \mu \) be defined according to (2.7). For every \( t \in (0, \mu) \) and \( \eta \in (0, \frac{1}{2}) \), the following holds true with probability at least \( 1 - \eta \): If the number of samples obeys

\[
m \gtrsim \lambda^{-2} \cdot t^{-2} \cdot \max\{d_1(K - \mu x_0), \log(\eta^{-1})\},
\]

then any minimizer \( \hat{x} \) of \((P_{\text{log}, K})\) satisfies

\[
\left\|x_0 - \frac{\hat{x}}{||\hat{x}||_2}\right\|_2 \leq \frac{t}{\mu} \lesssim t \cdot \sqrt{\log(\lambda^{-1})}.
\]

A remarkable feature of Theorem 2.10 is that the impact of the underlying 1-bit measurement model is completely controlled by the correlation parameter \( \lambda \). Since \( \lambda \) can be regarded as a constant scaling factor, recovery via \((P_{\text{log}, K})\) is still possible when the specific output rule is unknown and the signal-to-noise ratio is very low. With other words, under the hypothesis of Assumption 2.9, \( \hat{x} / ||\hat{x}||_2 \) constitutes a consistent estimator of \( x_0 \) even if \( \lambda \) is relatively close to 0. Note that the above recovery statement strongly resembles the non-uniform results of [Gen17; PV16], where the uncertainty about the non-linear output function \( f \) is also captured by a few model-dependent parameters.
We wish to emphasize that the error tolerance $t > 0$ needs to be fixed in advance. It is therefore quite convenient to rewrite (2.8) as a condition on $t$:

$$t \gtrsim \lambda^{-1} \cdot \left( \frac{d_t(K - \mu x_0)}{m} \right)^{1/2}.$$  

(2.10)

Thus, if $m$ is adjusted such that (2.10) holds true with equality (up to numerical constants), the actual error bound (2.9) can be directly related to the number of samples:

$$\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim \mu^{-1} \cdot \lambda^{-1} \cdot \left( \frac{d_t(K - \mu x_0)}{m} \right)^{1/2}. $$

(2.11)

While this expression already exhibits the optimal oversampling rate of $O(m^{-1/2})$, the local effective dimension on the right-hand side still involves the parameter $t$. But as the following corollary shows, one can easily get rid of the dependence on $t$ by applying (2.3) or (2.5).

**Corollary 2.11** The assertion of Theorem 2.10 still holds true if the condition (2.8) is replaced by

$$m \gtrsim \lambda^{-2} \cdot t^{-2} \cdot \max\{d_0(K - \mu x_0), \log(\eta^{-1})\},$$

or by

$$m \gtrsim \lambda^{-2} \cdot t^{-4} \cdot \max\{w(K)^2, \log(\eta^{-1})\}.$$  

Adjusting $m$ similarly to (2.11), we now obtain the non-asymptotic error bounds

$$\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim \left( \frac{d_0(K - \mu x_0)}{m} \right)^{1/2},$$

(2.12)

and

$$\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim \left( \frac{w(K)^2}{m} \right)^{1/4},$$

(2.13)

where the constants hide the dependence on the model parameters $\mu$ and $\lambda$. The error estimate promoted by (2.12) is desirable due to the optimal approximation rate of $O(m^{-1/2})$, but as already pointed out at the end of Subsection 2.1, using the conic effective dimension as complexity measure has several downsides. This particularly concerns the issue of stable recovery in situations where $x_0$ is only close to the boundary of $K$ or if the boundary of $K$ is (locally) smooth. In either of these scenarios, the recovery statement of Theorem 2.10 is significantly stronger because it relates the error accuracy to the local effective dimension at the right scale.

Compared to this, the bound of (2.13) does not even depend on the ground truth signal $x_0$ and therefore, in principle, holds true for every $x_0 \in K$. Apart from that, there exist explicit upper bounds for the global Gaussian width in many cases of interest, see [Ver15] for example. But these appealing features of (2.13) clearly come along with a much slower oversampling rate of $O(m^{-1/4})$.

### 2.3 Recovery in General Convex Sets

The crucial assumption of the previous subsection was that the signal set belongs to the Euclidean unit ball, meaning that $K \subset B^n_2$. According to Lemma 2.8, we were able to ensure that the minimizer of the expected risk (2.6) takes the form $\mu x_0$ with $\mu \in (0,1]$. Consequently, Theorem 2.10 states that the normalized empirical risk minimizer $\hat{x}/\|\hat{x}\|_2$ of $(P_{\text{hinge},K})$ constitutes a consistent estimator of $x_0$. On the other hand, it can be computationally appealing to drop the unit-ball assumption and to allow for “larger” convex signal sets. A very common example is an $\ell^1$-penalty, for which $(P_{\text{hinge},K})$ can be reformulated as a linear program (cf. [KV17, Sec. VI.A]).

---

1But note that our recovery results are non-uniform, i.e., the ground truth signal needs to be fixed in advance.
This motivates us to investigate the recovery performance of hinge loss minimization under arbitrary convex constraints. More precisely, we will make the following model assumptions throughout this subsection:

**Assumption 2.12 (General Signal Sets)** Let \( x_0 \in \mathbb{S}^{n-1} \) be a unit vector in \( \mathbb{R}^n \) and let \( a \sim \mathcal{N}(0, I_n) \) be a standard Gaussian. We consider perfect 1-bit Gaussian measurements

\[
y := \text{sign}(\langle a, x_0 \rangle) \in \{-1, +1\}.
\]

Each of the \( m \) samples \( \{(a_i, y_i)\}_{i \in [m]} \) is then drawn as an independent copy from the random pair \( (a, y) \), implying that

\[
y_i = \text{sign}(\langle a_i, x_0 \rangle), \quad i = 1, \ldots, m.
\]

We assume that \( x_0 \in K \) for a fixed signal set \( K \subset \mathbb{R}^n \) which is convex, bounded, and closed.

Note that this model includes a special case of Assumption 2.1 with \( f = \text{sign} \) and also revives the requirements of Assumption 2.2, except from \( \emptyset \in K \). Although one might generalize the recovery guarantees below to noisy 1-bit measurements, we will only analyze the noiseless case in this part. This restriction simplifies the exposition significantly and allows us to highlight the geometric aspects of our recovery approach.

### 2.3.1 Recovery via Scalable Signal Sets

One of the major difficulties in the general setup of Assumption 2.12 is that an expected risk minimizer of the hinge loss is not necessarily a scalar multiple of \( x_0 \) anymore:

\[
\arg\min_{x \in K} \mathcal{R}(x) \not\subset \text{span}\{x_0\}.
\]

Indeed, by an orthogonal decomposition \( x = P_{x_0}(x) + P_{x_0}^\perp(x) = \langle x, x_0 \rangle x_0 + P_{x_0}^\perp(x) \), the expected risk at \( x \in K \) takes the form (cf. Definition 2.7)

\[
\mathcal{R}(x) = \mathbb{E}[\mathcal{L}^\text{hinge}(y\langle a, x \rangle)] = \mathbb{E}[\mathcal{L}^\text{hinge}(\langle x, x_0 \rangle \langle a, x_0 \rangle + y\langle a, P_{x_0}^\perp(x) \rangle)]
\]

\[
= \mathbb{E}\left[\left[1 - \nu_x |g| + v_x^\perp g^\perp\right]_+\right]
\]

(2.15)

where \( \nu_x := \langle x, x_0 \rangle, v_x^\perp := \|P_{x_0}^\perp(x)\|_2, \) and \( g, g^\perp \sim \mathcal{N}(0, 1) \) are independent. Hence, we can conclude that, in order to minimize \( \mathcal{R}(\cdot) \), it is beneficial to select \( x \in K \) such that \( \nu_x \) is large and \( v_x^\perp \) is small at the same time. As long as \( K \subset B_2^n \), Lemma 2.8 simply states that this trade-off is satisfied for \( x = x_0 \), i.e., \( \nu_x = \langle x_0, x_0 \rangle = 1 \) and \( v_x^\perp = 0 \). But without the unit-ball assumption, there might exist \( x \in K \) with \( \nu_x > 1 \) and \( v_x^\perp > 0 \) such that \( \mathcal{R}(x) < \mathcal{R}(x_0) \). See Figure 2 for an illustration. While this situation might appear somewhat artificial in two dimensions, it is actually characteristic for high-dimensional convex sets, which implies that a minimizer \( \hat{x} \) of \( \mathcal{P}_{x_0}^\text{hinge}K \) is not expected to be close to \( x_0 \), even if \( m \to \infty \).

To come up with an improved estimation strategy, let us make a simple observation about the expected risk in (2.15): Rescaling the vector \( x \in K \) by a factor of \( \mu \geq 1 \) yields

\[
\mathcal{R}(\mu x) = \mathbb{E}[\mathcal{L}^\text{hinge}(y\langle a, \mu x \rangle)] = \mathbb{E}\left[\left[1 - \mu v_{x}\|g\| + \nu_{x} v_{x}^\perp g^\perp\right]_+\right]
\]

(2.16)

with the same notation as in (2.15). There is no closed form expression for the integral in (2.16), but from the definition of \( [\cdot]_+ \), we can at least make the following informal conclusion: The “contribution” of \( \mu v_{x}\|g\| \) to minimizing \( \mathcal{R}(\cdot) \) becomes smaller as \( \mu \) grows (if \( \nu_x > 0 \)), whereas the variance of the symmetric variable \( \mu v_{x}^\perp g^\perp \) increases. Hence, if \( x^* \) denotes an expected risk minimizer on \( \mu K \), the ratio \( \nu_{x^*} / v_{x^*}^\perp \) must

\[\text{The proof of Lemma 2.8 particularly makes use of the fact that } |\langle x, x_0 \rangle| \leq 1 \text{ for all } x \in K \subset B_2^n.\]
By the law of large numbers, we may hope again that 
\( \hat{x} \) (see Definition 2.7). In conclusion, there must be a certain relationship between the values of 
\( \mu \) and \( m \) such that \( \hat{x} \) becomes highly non-unique and any 
\( \hat{x} \in \mu K \) with \( \bar{R}(\hat{x}) = 0 \) is a minimizer, no matter how distant it is from 
span \{x_0\}. Note that the letter \( \mu \) is already used in (2.7) in order to define the expected risk minimizer in subsets of the unit ball. It will be clear from the context which definition is meant. However, this ambiguous notation is no coincidence because we will analyze the excess risk functional with respect to \( \mu x_0 \) in both cases (see beginning of Section 6).

\[ \begin{align*}
\text{For every fixed } \mu > 0 \text{ and } \eta \in (0, \frac{1}{2}) \text{, the following holds true with probability at least } 1 - \eta: \text{ If } \mu \geq 1 \text{ and the}
\end{align*} \]

\[ \]
number of samples obeys
\[ m \gtrsim \mu^4 \cdot \max \{d_0(K - x_0), \log(\eta^{-1})\}, \quad (2.17) \]

then any minimizer \( \hat{x} \) of \( \text{(PL}_{\log, \mu K} \) satisfies
\[ \|x_0 - \|\hat{x}\|_2 \|_2 \lesssim \frac{1}{\mu}. \quad (2.18) \]

The same assertion holds true if (2.17) is replaced by
\[ m \gtrsim \mu^4 \cdot \max \{w(K)^2, \log(\eta^{-1})\}. \quad (2.19) \]

The proof of Theorem 2.14 is postponed to Subsection 6.3. It relies on similar statistical tools as the proof of Theorem 2.10, but the actual arguments are quite different. In fact, the role of the underlying empirical processes may change significantly if \( K \) is not contained in the unit ball anymore.

This also indicates why the above recovery statement is conceptually somewhat different from Theorem 2.10: The hypothesis of Theorem 2.14 merely relies on the “coarser” complexity measure of conic efficient dimension (or Gaussian width), while the scaling parameter \( \mu^{-1} \) now mimics the role of the scale \( t \) of the local effective dimension in Theorem 2.10. Indeed, \( \mu \) can be regarded as an oversampling factor that controls the recovery accuracy, even though it also affects the size of the constraint set in \( \text{(PL}_{\log, \mu K} \). Adjusting \( \mu \) such that (2.17) holds true with equality (up to numerical constants), the error bound of (2.18) can be rewritten in terms of the sample count \( m \):
\[ \|x_0 - \|\hat{x}\|_2 \|_2 \lesssim \left( \frac{d_0(K - x_0)}{m} \right)^{1/4}. \quad (2.20) \]

Thus, by appropriately upscaling the signal set \( K \), the normalized minimizer \( \hat{x}/\|\hat{x}\|_2 \) turns into a consistent estimator of \( x_0 \) with an approximation rate of \( O(m^{-1/4}) \).

We wish to emphasize that using the conic effective dimension as complexity parameter is not as problematic as in the setting of Subsection 2.2, where the expected risk minimizer \( \mu x_0 \) could lie in the interior of \( K \). Due to the flexible scaling parameter of \( \text{(PL}_{\log, \mu K} \), one can basically assume that \( x_0 \) lies on the boundary of \( K \): If \( x_0 \) would belong to the interior of \( K \), there exists a factor \( \nu \in (0, 1) \) such that \( x_0 \in \partial(\nu K) \). Then the constraint of \( \text{(PL}_{\log, \mu K} \) takes the form
\[ x \in \mu K = \nu \cdot \frac{\nu K'}{\mu'} = K', \]

and we may apply Theorem 2.14 with \( \mu' \) and \( K' \) instead of \( \mu \) and \( K \), respectively. In that way, the error accuracy is just affected by an additional (unknown) scalar factor \( \nu \). This particularly explains why the illustrations of Figure 2 and Figure 3 assume that \( x_0 \) lies on the boundary of \( K \).

However, there are still cases in which the conic effective dimension is badly behaved, e.g., if \( x_0 \) is just compressible and not exactly sparse. Then we may replace the condition (2.17) by (2.19) to use the Gaussian width as complexity measure, which does only take global features of \( K \) into account. For more details, see the discussion subsequent to Corollary 2.11 as well as Subsection 3.1.

**Remark 2.15** (1) Theorem 2.14 with (2.19) contains a recent result from Kolleb and Vybíral [KV17] as special case. Their work was the first one that theoretically studied the recovery performance of hinge loss minimization under 1-bit measurements. These results however do only focus on a global analysis of \( \ell^1 \)-constraints, adapting concentration inequalities from Plan and Vershynin in [PV13b]. In contrast, our approach is rather local and relies on refined bounds for empirical processes. This eventually allows us to extend the restrictive model setup of [KV17] into various directions and to improve their approximation rates (with respect to \( m \)), as we will see below in Theorem 2.16.
(2) The variance-bias problem. This issue is well-known from statistical learning theory. In general, the variance-bias problem states that there is a fundamental trade-off between the sample error (variance) and the approximation error (bias) in empirical risk minimization if the size of the hypothesis set is varied (and the sample count \(m\) remains fixed); see also [CZ07, Sec. 1.5]. Interestingly, the above estimation strategy fits very well into this situation: The “hypothesis set” \(K\) in \((P_{\text{Chen}, \mu K})\) is enlarged by increasing the scaling parameter \(\mu\), and (2.16) indicates that an expected risk minimizer \(x^*\) aligns better with \(\text{span}\{x_0\}\) as \(\mu\) grows. With other words, the approximation error gets smaller. But on the other hand, an empirical risk minimizer \(\hat{x}\) of \((P_{\text{Chen}, \mu K})\) might be still a poor approximation of \(x^*\) if \(\mu\) is too large, meaning that the sample error is large. This issue of overfitting can be fortunately resolved by increasing the sampling rate according to (2.17). In this light, Theorem 2.14 just claims that the variance-bias problem can be handled by carefully balancing the scaling parameter \(\mu\) and the sample size \(m\). 

\[\diamond\]

### 2.3.2 Localized Approximation Rates

A downside of Theorem 2.14 is the relatively slow error decay of \(O(m^{-1/4})\) in (2.20). In fact, this approximation rate is obviously much worse than the decay of \(O(m^{-1/2})\) achieved by Theorem 2.10 in (2.11), which in turn is essentially optimal (see [PVY16, Sec. 4]). Our third main result, Theorem 2.16, shows that the factor of \(\mu^4\) in condition (2.17) can be replaced by \(\mu^2 \cdot t_0^2\), where \(t_0\) is an additional geometric parameter that depends on \(x_0\) and \(K\). Formally, it is defined by

\[t_0 := \max\{1, \text{rad}((\partial(\mu K) \cap \text{Cyl}(x_0, \mu)) - \mu x_0)\}\]

where \(\text{Cyl}(x_0, \mu)\) denotes the following cylindrical tube around \(\text{span}\{x_0\}\):

\[\text{Cyl}(x_0, \mu) := \{x \in \mathbb{R}^n \mid \|x - \langle x, x_0 \rangle x_0\|_2 \leq 1, \langle x, x_0 \rangle \geq 2\}\]

see Figure 3 for a visualization. We wish to emphasize that the intersection \(\partial(\mu K) \cap \text{Cyl}(x_0, \mu)\) can be significantly smaller than \(\mu K \cap \text{Cyl}(x_0, \mu)\), whose radius around \(\mu x_0\) is basically of order \(\mu\). Intuitively, if \(\mu\) is sufficiently large, the boundary \(\partial(\mu K)\) does only intersect with the side of \(\text{Cyl}(x_0, \mu)\), which has diameter 2. The value of \(t_0\) then becomes (almost) independent of \(\mu\) and is determined by the “local” geometry of \(K\) in a neighborhood of \(x_0\) (of size \(1/\mu\)). In this situation, the sampling rate would only scale quadratically in \(\mu\), leading to the desired rate of \(O(m^{-1/2})\).

Before we further discuss the impact of \(t_0\), let us state the actual recovery guarantee:
Theorem 2.16 (Signal Recovery in Convex Sets – Local Version) Let the model conditions of Assumption 2.12 be satisfied and let $t_0$ be defined according to (2.21). For every fixed $\mu > 0$ and $\eta \in (0, \frac{1}{2})$, the following holds true with probability at least $1 - \eta$: If $\mu \gtrsim t_0$ and the number of samples obeys

$$m \gtrsim \mu^2 \cdot t_0^2 \cdot \max \{ d_0(K - x_0), \log(\eta^{-1}) \},$$

then any minimizer $\hat{x}$ of $(P_{\text{cyl}}(\mu K))$ satisfies

$$\| x_0 - \hat{x} \|_2 \lesssim \frac{1}{\mu}.$$  

As before, we can adjust $\mu$ in (2.23) to obtain a convenient error bound depending on $m$:

$$\| x_0 - \hat{x} \|_2 \lesssim \left( \frac{t_0^2 \cdot d_0(K - x_0)}{m} \right)^{1/2}.$$  

While this oversampling rate certainly resembles the unit-ball case in (2.12), let us point out an important difference: Roughly speaking, the geometric conclusion of Theorem 2.10 is that every minimizer of $(P_{\text{cyl}}(\mu K))$ must lie within the spherical intersection $K \cap (tB_2^0 + \mu x_0)$; see Figure 5. In contrast, the proof of Theorem 2.16 argues that every minimizer of $(P_{\text{cyl}}(\mu K))$ must belong to the cylindrical intersection $\mu K \cap \text{Cyl}(x_0, \mu)$; see Figure 3. Rescaling the latter set by a factor of $1/\mu$ with $\mu = 1/t \gg 1$, it is not hard to see that the shapes of both intersections are fundamentally different. More specifically, due to the anisotropy of $\text{Cyl}(x_0, \mu)$, the conic effective dimension $d_0(K - x_0)$ needs to be scaled by an extra factor of $t_0^2$ in (2.24).

Without any further assumptions on the geometric arrangement of $K$ and $x_0$, it is difficult to make precise statements about the order of $t_0$. For example, if the boundary of $K$ is almost orthogonal to $\text{span}\{x_0\}$ in a small neighborhood of $x_0$, we can expect that $t_0 \approx 1$. But as $\partial K$ gets more “tangent” to $\text{span}\{x_0\}$, as in Figure 3, $t_0$ may become significantly larger. If $t_0 \approx \mu$, the condition of (2.23) particularly degenerates to (2.17). Moreover, the situation of $t_0 \gg \mu$ is excluded by Theorem 2.16 in any case, whereas Theorem 2.14 would still provide meaningful error bounds. However, as long as $t_0$ is considered as a (possibly large) signal-dependent constant, we can always achieve the optimal non-asymptotic rate of $O(m^{-1/2})$.

2.3.3 Geometric Interpretation and Classification Margins

We conclude this subsection with a brief discussion on the geometric aspects of the above recovery problem. To this end, let us consider the following interpretation of the perfect 1-bit model from Assumption 2.12: The measurement vectors $C := \{ a_1, \ldots, a_m \} \subset \mathbb{R}^n$ form a “cloud” of random points that is generated by the standard Gaussian distribution in $\mathbb{R}^n$. According to the observation rule (2.14), each point is then endowed with a binary label depending on which side of the hyperplane $H(x_0) := \{ x \in \mathbb{R}^n : \langle x, x_0 \rangle = 0 \}$ the respective point resides. With other words, $H(x_0)$ is a separating hyperplane that divides $C$ into two classes, say $C^+$ and $C^-$; see also Figure 4(a).

The actual recovery task can be now translated into finding the hyperplane $H(x_0)$ — or equivalently, its normal vector $x_0$ — where only the labeled point clouds $C^+$ and $C^-$ are available. It should be emphasized that this problem is somewhat different from traditional classification, where one is already satisfied with any separating hyperplane $H(x)$. If $m$ is sufficiently large, such as in Figure 4(b), one can expect that $x$ and $x_0$ are however close, in the sense that

$$\left\| \frac{x}{\|x\|_2} - x_0 \right\|_2 = \sqrt{-2 - 2\langle \frac{x}{\|x\|_2}, x_0 \rangle}$$

is small. From a statistical perspective, this conclusion essentially follows from the law of large numbers, which implies that the margin between $C^+$ and $C^-$ “contracts” to $H(x_0)$ as $m \to \infty$. On the other hand,
if \( m \) is too small, the margin between both classes is typically large, so that there are many separating hyperplanes and the error in (2.25) is not necessarily small.

The major concern of this work is to face this challenge by minimizing the empirical hinge loss \( \bar{R}(\cdot) \) on a certain signal set (see Definition 2.7). It is well-known from the literature that the associated estimator \((P_{\text{hng},K})\) can be identified with a support vector machine (SVM), whose actual purpose is to maximize the classification margin between labeled data points,\(^1\) see [KV17, Sec. II] for example. The scenario of Figure 4(b) is therefore somewhat undesirable because the margin between \( C^+ \) and \( C^- \) is very narrow. Consequently, hinge loss minimization via \((P_{\text{hng},K})\) can be inappropriate if \( m \) is too large, even though all separating hyperplanes do almost align with \( H(x_0) \). This precisely corresponds to the observations made at the beginning of Subsection 2.3.1, where we investigated the expected risk minimizer. Indeed, if \( K \) does not belong to the unit ball, a vector \( x^* \) could minimize \( R(\cdot) \) on \( K \) but still induce a hyperplane \( H(x^*) \) that is not separating as \( m \to \infty \).

Interestingly, we were able to resolve this issue by introducing a scalable hinge loss estimator \((P_{\text{hng},\mu K})\) in Definition 2.13. Due to the identity

\[
\min_{x \in \mu K} \frac{1}{m} \sum_{i=1}^{m} L_{\text{hng}}(y_i \langle a_i, x \rangle) = \min_{x \in K} \frac{1}{m} \sum_{i=1}^{m} L_{\text{hng}}(y_i \langle \mu a_i, x \rangle),
\]

(2.26)

solving \((P_{\text{hng},\mu K})\) is equivalent to solving \((P_{\text{hng},K})\) with measurement vectors upscaled by a factor of \( \mu \). In the situation of Figure 4(b), this implies that the dense point cloud and the separating margin are both enlarged. The geometric arrangement of the transformed data then resembles Figure 4(a) and hinge loss minimization becomes feasible. Our main results, Theorem 2.14 and Theorem 2.16, confirm this heuristic reasoning and particularly show how \( m \) and \( \mu \) need to scale (non-asymptotically) in order to obtain a consistent estimator of \( x_0 \).

Finally, it is worth mentioning that the geometric viewpoint of SVMs was also the initial motivation of [KV17] to study the performance of the scalable hinge loss estimator as it is stated on the right-hand side of (2.26). While the upscaling strategy indeed allows us to mimic a well-posed classification problem for SVMs, this perspective nevertheless appears a bit artificial in the light of 1-bit observation models. In real-world applications, the data \( C \) is very unlikely to follow a standard Gaussian distribution, but

\(^1\)In general, there does not necessarily exist a separating hyperplane. For this reason, one rather speaks of soft margin classification where a certain amount of misclassified points is permitted. The purpose of SVMs is therefore to find a hyperplane that maximizes the soft margin. This perspective is also taken when dealing with noisy 1-bit measurements in Subsection 2.2.
rather exhibits anisotropic features, as it is the case for Gaussian mixture models. For that reason, the focus of this work is mainly on 1-bit compressed sensing, where Gaussian measurements often serve as a benchmark to analyze recovery algorithms.

3 Examples and Applications

In this section, we illustrate the general framework of Section 2 by a few applications and examples. Subsection 3.1 continues our discussion on signal sets from Subsection 2.1 and conveys more intuition regarding how their complexity is measured by means of the Gaussian width and the effective dimension. A particular emphasis is on the case of sparse vectors and related convex relaxations, which will also be applied to our main results. Subsection 3.2 then focuses on two specific noisy 1-bit observation models in the unit-ball setup of Subsection 2.2, namely random bit flips and additive Gaussian noise. In this context, we will demonstrate that hinge loss minimization even yields a consistent estimator if the signal-to-noise ratio is very low.

3.1 The Gaussian Width and Sparse Recovery

Let us begin with a few examples of structured signal sets $K$ that are widely used in signal estimation theory and satisfy Assumption 2.2:

Example 3.1 (1) (Effectively) Sparse signals. The prototypical example of low-dimensional structures studied in compressed sensing is sparsity. We denote the set of all $s$-sparse vectors on the Euclidean unit sphere by

$$\Sigma^u_s := \{ x \in \mathbb{R}^n \mid \|x\|_0 \leq s, \|x\|_2 = 1 \}.$$  

Since $\Sigma^u_s$ is obviously non-convex, it falls out of the scope of our main results. However, by the Cauchy-Schwarz inequality, each vector $x_0 \in \Sigma^u_s$ satisfies

$$\|x_0\|_1 \leq \sqrt{\|x_0\|_0 \cdot \|x_0\|_2} \leq \sqrt{s} \cdot \|x_0\|_2 = \sqrt{s}, \quad (3.1)$$

which implies that $K_{n,s} := \sqrt{s}B^n_1 \cap B_n^2$ is a convex signal set that contains all normalized $s$-sparse vectors. Interestingly, this set essentially equals the convex hull of $\Sigma^u_s$ in the sense that (cf. [PV13a, Lem. 3.1])

$$\text{conv}(\Sigma^u_n) \subset K_{n,s} \subset 2\text{conv}(\Sigma^u_n).$$

Consequently, $K_{n,s}$ forms a natural convex relaxation of $\Sigma^u_s$. Since $K_{n,s}$ particularly contains compressible vectors that are almost $s$-sparse, it is also referred to as the set of effectively $s$-sparse vectors.

(2) Scaled $\ell^1$-balls. Another straightforward convex relaxation of sparsity are scaled $\ell^1$-balls, i.e., $K = RB^n_1$ with an appropriately chosen scaling parameter $R > 0$. Indeed, according to (3.1), we can immediately conclude that $K = \sqrt{s}B^n_1$ is a superset of $\Sigma^u_n$. Such types of $\ell^1$-constraints became very popular in practice because they often allow for sparse approximation and linear programming at the same time. Unfortunately, this example does not meet the unit-ball assumption of Theorem 2.10, so that it is only admissible for Theorem 2.14 and Theorem 2.16.

(3) Subspaces. Perhaps the simplest example of a structured signal set is a linear subspace. If $x_0 \in S^{n-1}$ belongs to a known subspace $E \subset \mathbb{R}^n$ of low dimension, one may just consider the signal set $K = E \cap B_2^n$ as prior.

(4) Polytopes. If $x_0$ is known to be a convex combination of finitely many vectors $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$, one could simply choose the polytope $K = \text{conv}\{0, x_1, \ldots, x_k\}$ as signal set.

\[\Box\]

Next, we collect some basic yet important properties of the Gaussian width and effective dimension. The proofs are omitted, since most statements are direct consequences of the definition. For a more extensive discussion, see also [PV13b; Ver15; Ver18].
**Proposition 3.2** The Gaussian width \( w(K) \) of a bounded subset \( K \subset \mathbb{R}^n \) (see Definition 2.3) and the effective dimension

\[
d(K) = \frac{w(K)^2}{\text{diam}(K)^2},
\]
satisfy the following properties:

(i) If \( K \subset K' \subset \mathbb{R}^n \), it follows that \( w(K) \leq w(K') \).

(ii) \( w(K) = \frac{1}{2} w(K - K) \).

(iii) \( w(K + x) = w(K) \) for every \( x \in \mathbb{R}^n \).

(iv) \( w(K) = w(\text{conv } K) \).

(v) \( d(K) \leq \text{dim}(\text{span } K) \).

(vi) If \( K \) is the Euclidean ball of a \( d \)-dimensional subspace of \( \mathbb{R}^n \), then \( d(K) \approx d \).

(vii) If \( K \) is finite, we have \( d(K) \lesssim \log(\left| K \right|) \).

Let us now focus on the important case of sparse recovery. More precisely, we intend to apply our main results from Section 2 to the set of effectively \( s \)-sparse vectors as defined in Example 3.1(1) as well as to scaled \( \ell^1 \)-balls as considered in Example 3.1(2). In either of these cases, there exist (sharp) bounds on the Gaussian width in literature (see [PV13b, Sec. 2 and 3]):

\[
w(\sqrt{s}B_1^n) \lesssim \sqrt{s \log(n)},
\]

\[
w(\sqrt{s}B_1^n \cap B_2^n) \lesssim \sqrt{s \log\left(\frac{2n}{s}\right)}.
\]

Using (3.2), we now obtain the following sparse recovery result from Theorem 2.14: Set \( K = \sqrt{s}B_1^n \) and assume that \( x_0 \in \sqrt{s}B_1^n \cap S^{n-1} \). Provided that \( m \gtrsim \mu^4 \cdot s \log(n) \), any minimizer \( \hat{x} \) of \( (P_{\text{L}^1,\mu K}) \) satisfies with high probability

\[
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim \frac{1}{\mu}.
\]

Combined with (3.3), Corollary 2.11 yields a similar statement: Set \( K = \sqrt{s}B_1^n \cap B_2^n \) and assume that \( x_0 \in \sqrt{s}B_1^n \cap S^{n-1} \). If \( m \gtrsim t^{-4} \cdot s \log(2n/s) \), then any minimizer \( \hat{x} \) of \( (P_{\text{L}^1,\mu K}) \) satisfies with high probability

\[
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim t.
\]

We would like to point out that these assertions resemble the findings of [KV17, Thm. II.3, Thm. IV.1]. While the number of required measurements is essentially optimal with respect to the signal complexity in both cases (cf. [FR13, Chap. 11]), the dependence on the respective oversampling factor is clearly sub-optimal. Indeed, the fourth power of \( \mu^4 \) and \( t^{-4} \) leads to an error decay rate of \( O(m^{-1/4}) \). In order to achieve the optimal oversampling rate of \( O(m^{-1/2}) \), we need to investigate the conic effective dimension of sparse vectors. Invoking a well-known bound on the conic effective dimension from [Cha+12, Prop. 3.10], it turns out that an \( s \)-sparse vector \( x_0 \) that lies on the boundary of an \( \ell^1 \)-ball satisfies

\[
d_0(\|x_0\|_1 B_1^n - x_0) \lesssim s \log\left(\frac{2n}{s}\right).
\]

Together with Theorem 2.16 (and (2.24)), this gives the following: Set \( K = RB_1^n \) and assume that \( x_0 \) is \( s \)-sparse with \( x_0 \in \partial K \cap S^{n-1} \). Then any minimizer \( \hat{x} \) of \( (P_{\text{L}^1,\mu K}) \) satisfies with high probability

\[
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim \left(\frac{\mu^2 \cdot s \log\left(\frac{2n}{s}\right)}{m}\right)^{1/2},
\]
where \( t_0 \) is the signal-dependent constant defined in (2.21). An analogous result follows from Corollary 2.11 (and (2.12)) in the unit-ball case by considering the signal set \( K = \|\mu x_0\|_1 B_1^\mu \cap B_2^\mu \). In this situation, one can even remove the factor \( t_0 \) in (3.5), meaning that any solution \( \hat{x} \) to \( (P_{\text{log}, K}) \) obeys

\[
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim \left( \frac{s \log \left( \frac{2n}{s} \right)}{m} \right)^{1/2}.
\]

(3.6)

While the error bound of (3.6) is essentially optimal (see [PVY16, Sec. 4]), it is however only of limited practical interest because the signal set \( K = \left\{ \|x\|_1 \leq 1 \right\} \) depends on the unknown vector \( x_0 \). Indeed, if we would just select \( K = RB_1^\mu \cap B_2^\mu \) for some \( R > \|x_0\|_1 \), the conic effective dimension would drastically increase to \( d_0((RB_1^\mu \cap B_2^\mu) - \mu x_0) \asymp n \), which in turn leads to a very pessimistic sampling rate. A similar problem would occur if \( x_0 \) is just compressible rather than exactly \( s \)-sparse. In this situation, \( x_0 \) may reside in a higher dimensional face of a scaled \( \ell^1 \)-ball but is still relatively close to a low-dimensional face.

Fortunately, the refined concept of local effective dimension is able to resolve these issues and allows for stable recovery. Roughly speaking, the idea is as follows: If the “anchor vector” \( \mu x_0 \in K = \sqrt{s}B_1^\mu \cap B_2^\mu \) in Theorem 2.10 is not exactly \( s \)-sparse, select a nearby \( s \)-sparse vector \( x^* \in \partial K \) and set \( t^* = \|\mu x_0 - x^*\|_2 \). Then, the local effective dimension \( d_t(K - \mu x_0) \) at any scale \( t \geq t^* \) behaves as if we would consider its conic counterpart in \( x^* \), i.e., \( d_t(K - x^*) \). This geometric argument is formalized by the following proposition, which is a consequence of [GKM17, Lem. A.2]:

**Proposition 3.3** Let \( \mu x_0 \in K := \sqrt{s}B_1^\mu \cap B_2^\mu \) and set \( K_{n,s} := \{ x \in K \mid \|x\|_1 \leq s, \|x\|_2 = \sqrt{s} \} \). Moreover, let \( x^* \in \mathbb{R}^n \) be a best \( s \)-term approximation of \( \mu x_0 \) in \( K_{n,s} \), i.e., \( x^* = \arg\min_{x \in K_{n,s}} \|\mu x_0 - x\|_2 \). Then for every \( t \geq t^* := \|\mu x_0 - x^*\|_2 \), we have that

\[
d_t((\sqrt{s}B_1^\mu \cap B_2^\mu) - \mu x_0) \lesssim d_t((\sqrt{s}B_1^\mu \cap B_2^\mu) - x^*),
\]

(3.4)

An application of Theorem 2.10 now leads to stable sparse recovery in the following sense: Let \( x_0 \in S^{n-1} \) be effectively \( s \)-sparse, i.e., \( x_0 \in K = \sqrt{s}B_1^\mu \cap B_2^\mu \), and let \( t^* \) be defined as in Proposition 3.3. Fix any \( t \geq t^* \) and assume that the number of samples obeys \( m \gtrsim t^{-2} \cdot s \log \left( \frac{2n}{s} \right) \). Then any minimizer \( \hat{x} \) of \( (P_{\text{log}, K}) \) satisfies with high probability

\[
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \lesssim t.
\]

The best possible accuracy is achieved for \( t = t^* \), which basically reflects the degree of compressibility of \( x_0 \). In particular, \( t^* \) is supposed to be very small as long as there exists a good \( s \)-term approximation for \( x_0 \). On the other hand, if \( t \) exceeds this “base level”, the sampling rate precisely behaves as if \( x_0 \) would be \( s \)-sparse and \( K \) is perfectly tuned such that \( \mu x_0 \in \partial K \).

### 3.2 Correlation Conditions and Noisy Quantization Models

Let us begin with a brief discussion on the correlation conditions of Assumption 2.9 that are required for Theorem 2.10. For the remainder of this section, we assume that the hypotheses of Subsection 2.2 hold true, in particular, \( K \subset B_2^\mu \). Moreover, we agree on the following terminology: According to Assumption 2.1, denote the linear and quantized sampling rules by \( y_{\text{lin}} = \langle a, x_0 \rangle \) and \( y = f(\langle a, x_0 \rangle) \), respectively. Analogously, \( y_{\text{sign}} = \text{sign}(\langle a, x_0 \rangle) \) is associated with the perfect 1-bit model, which arises from \( f = \text{sign} \).
As already mentioned in the course of Lemma 2.8 in Subsection 2.2, the parameter \( \lambda \) in condition (C1) simply corresponds to the covariance between quantized and linear measurements:

\[
\text{Cov}(y, y^\text{lin}) = \text{Cov}(f(\langle a, x_0 \rangle), \langle a, x_0 \rangle) = \mathbb{E}[f(\langle a, x_0 \rangle) \langle a, x_0 \rangle] = \mathbb{E}[f(g)g] = \lambda,
\]

where \( g = \langle a, x_0 \rangle \sim \mathcal{N}(0,1) \) is due to \( \|x_0\|_2 = 1 \). Thus, demanding \( \lambda > 0 \) in (C1) means that the true output \( y \) is positively correlated with the linear signal \( y^\text{lin} \).

In contrast, the condition of (C2) requires that the covariance between the noisy observation model \( y \) and its “perfect” counterpart \( y^\text{sign} \) is non-negative while conditioning on the magnitude of the underlying linear signal \( y^\text{lin} \). Indeed, it holds that

\[
\mathbb{E}[y y^\text{sign} | y^\text{lin}] = \mathbb{E}[f(\langle a, x_0 \rangle) \text{sign}(\langle a, x_0 \rangle) | \|\langle a, x_0 \rangle\|] = \mathbb{E}[f(g) \text{sign}(g) | |g|],
\]

and

\[
\mathbb{E}[y^\text{sign} | y^\text{lin}] = \mathbb{E}[\text{sign}(g) | |g|] = \mathbb{E}[\text{sign}(g)] = 0 \quad \text{(a.s.)},
\]

where we have used that \( \text{sign}(g) \) and \(|g|\) are independent. Consequently, (C2) is equivalent to

\[
\text{Cov}(y, y^\text{sign} | y^\text{lin}) = \mathbb{E}[y y^\text{sign} | y^\text{lin}] - \mathbb{E}[y | y^\text{lin}] \mathbb{E}[y^\text{sign} | y^\text{lin}] \geq 0 \quad \text{(a.s.)}.
\]

This indicates that both (C1) and (C2) are mild and natural conditions which are fulfilled for many types of noisy 1-bit quantization models, e.g., random bit flips (after quantization) and additive Gaussian noise (before quantization).

Before studying these two prototypical examples in greater detail, let us make the following general observation: An ideal scenario for 1-bit compressed sensing is that the signal vector \( x_0 \) is first linearly measured by \( y^\text{lin} = \langle a, x_0 \rangle \) and then quantized in a noiseless fashion via \( y^\text{sign} = \text{sign}(\langle a, x_0 \rangle) \). This heuristic is particularly reflected by the size of the correlation parameter \( \lambda \): If \( f : \mathbb{R} \to \{-1, +1\} \) is an arbitrary quantization function, we have \( f(g)g \leq |g| \), implying that

\[
\lambda_f = \mathbb{E}[f(g)g] \leq \mathbb{E}[\text{sign}(g)g] = \lambda_{\text{sign}} = \sqrt{\frac{2}{\pi}}.
\]

Hence, \( \lambda_f \) is maximized for \( f = \text{sign} \), which eventually leads to the best possible error bound and sampling rate in Theorem 2.10. However, it is difficult to build perfect 1-bit quantizers in practice, so that the measurement process is usually disturbed by noise. In this context, one roughly distinguishes between two sources of measurement errors: firstly, noise that corrupts the linear signal \( y^\text{lin} = \langle a, x_0 \rangle \) before quantization, and secondly, noise that affects the actual quantization step. The following example considers each of these two scenarios:

**Example 3.4**  
1. **Additive Gaussian noise.** A typical example of corruptions before quantization is additive Gaussian noise. More precisely, we consider Assumption 2.1 for a 1-bit quantization function

\[
\langle f(v) = f_{\sigma}(v) := \text{sign}(v + \tau)\rangle
\]

where \( \tau \sim \mathcal{N}(0, \sigma^2) \) is independent from \( a \). Consequently, the samples in (2.2) take the form

\[
y_i = \text{sign}(\langle a_i, x_0 \rangle + \tau_i), \quad i = 1, \ldots, m,
\]

with \( \tau_i \sim \mathcal{N}(0, \sigma^2) \) being an independent copy of \( \tau \). If \( \sigma = 0 \), we are in the situation of perfect 1-bit measurements, whereas all information about the signal \( x_0 \) is lost as \( \sigma \to \infty \).

2. **Random bit flips.** An example of noise during quantization are independent sign flips of \( y^\text{sign} = \text{sign}(\langle a, x_0 \rangle) \). This can be easily modeled by setting

\[
f(v) = f_{\epsilon}(v) := \epsilon \cdot \text{sign}(v)
\]
in Assumption 2.1 for an independent Bernoulli random variable $\epsilon \in \{-1, +1\}$ with $\mathbb{P}[\epsilon = 1] = p > \frac{1}{2}$. Then, (2.2) looks as follows:

$$y_i = \epsilon_i \cdot \text{sign}(\langle a_i, x_0 \rangle), \quad i = 1, \ldots, m,$$

where $\epsilon_i \in \{-1, +1\}$ is an independent copy of $\epsilon$. Note that $p = 1$ again corresponds to perfect 1-bit measurements, whereas $p \to \frac{1}{2}$ leads to a complete loss of information.

The next proposition verifies that both types of noise are compatible with Assumption 2.9. Its proof is postponed to Subsection 6.4.

**Proposition 3.5** The noisy 1-bit models from Example 3.4(1) and (2) do both satisfy Assumption 2.9. Moreover, the associated correlation parameters are given by

$$\lambda_{f_\sigma} \asymp \frac{1}{1 + \sigma} \quad \text{for additive Gaussian noise, and}
$$

$$\lambda_{f_p} = (2p - 1)\sqrt{\frac{2}{\pi}} \quad \text{for random bit flips.}$$

Recalling the assertions of Theorem 2.10 and Corollary 2.11, we obtain the following conditions on the number of required measurements:

$$m \gtrsim (1 + \sigma)^2 \cdot C_{t,K} \quad \text{for additive Gaussian noise in Example 3.4(1), and}
$$

$$m \gtrsim \frac{1}{(2p - 1)^2} \cdot C_{t,K} \quad \text{for random bit flips in Example 3.4(2),}
$$

where the constant $C_{t,K} > 0$ hides the dependence on the oversampling factor $t$ and the signal complexity. This indicates that signal recovery is still feasible in the presence of strong noise, but when approaching the limiting cases of $\sigma \to \infty$ or $p \to \frac{1}{2}$, we clearly need to take more and more samples to ensure accurate estimates of $x_0$.

### 4 Related Literature

In this part, we give a brief overview of some recent approaches from the literature that are closely related to the problem setup considered in this work. While our focus is clearly on a challenge in (non-linear) compressed sensing, as discussed in Subsection 4.1, we will also point out connections to statistical learning in Subsection 4.2, which is particularly useful to understand the proof strategy of our main results.

#### 4.1 Signal Processing and Compressed Sensing

As already mentioned before, our measurement model in Assumption 2.1 fits well into the framework of 1-bit compressed sensing, or more generally, non-linear compressed sensing. In fact, there is an increasing interest in this subject in the recent literature; we refer the interested reader to [Bou+15, DJR17] and the references therein for an overview. Perhaps the most related branch of research is by Plan, Vershynin, and collaborators [Bar+17, PV13a, PV13b, PV14, PV16, PVY16], whose model assumptions are very similar to ours. Indeed, [PV16] deals with the estimation of a structured index vector $x_0 \in K \subset \mathbb{R}^n$ from single-index observations

$$y_i = f_i(\langle a_i, x_0 \rangle), \quad i = 1, \ldots, m,$$

where the $f_i$ are independent copies of an unknown function $f : \mathbb{R} \to \mathbb{R}$ that could be non-linear and random. In particular, if $f$ is binary-valued, we precisely end up with the sampling rule of (2.2). Under the hypothesis of i.i.d. Gaussian measurement vectors, [PV16] investigates the performance of the generalized Lasso

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} (\langle a_i, x \rangle - y_i)^2 \quad \text{subject to} \quad x \in K,$$

where the $f_i$ are independent copies of an unknown function $f : \mathbb{R} \to \mathbb{R}$ that could be non-linear and random. In particular, if $f$ is binary-valued, we precisely end up with the sampling rule of (2.2). Under the hypothesis of i.i.d. Gaussian measurement vectors, [PV16] investigates the performance of the generalized Lasso

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} (\langle a_i, x \rangle - y_i)^2 \quad \text{subject to} \quad x \in K,$$

where $Y$ is the measurement output and $A$ is the measurement matrix.
which simply corresponds to \((P_{L,K})\) using the square loss \(\mathcal{L}(v,v') := \mathcal{L}^\text{sq}(v-v') := (v-v')^2\). While the Lasso was originally designed to solve linear regression problems, the recovery results of [PV16] reveal that \((P_{L,K})\) is surprisingly robust against non-linear distortions, even if the output function \(f\) is completely unknown. More technically, it turned out that, for an appropriate scaling parameter \(\mu \in \mathbb{R}\), the model mismatch \(y_i - \langle a_i, \mu x_0 \rangle\) is uncorrelated to the measurement vector \(a_i\). With other words, the Lasso with non-linear inputs essentially works as well as if the inputs would follow a noisy linear model.

Despite the universal applicability of the Lasso, practitioners however often choose different types of loss functions for \((P_{L,K})\), which are specifically tailored to their model hypotheses, e.g., if the output variables \(y_i\) are discrete. This issue particularly motivated the first author in [Gen17] to extent the framework of Plan and Vershynin to other choices of \(\mathcal{L}\). A key finding of [Gen17] is that, in many situations of interest, restricted strong convexity (RSC) is a crucial property of an empirical risk function to ensure successful signal recovery via \((P_{L,K})\). The criterion of RSC is indeed satisfied for a large class of loss functions, for instance, all those \(\mathcal{L} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) which are twice differentiable in the first variable and locally strongly convex in a neighborhood of the origin (cf. [Gen17, Thm. 2.5]). While this includes popular choices of \(\mathcal{L}\), such as the logistic loss, the hinge loss \(\mathcal{L}(v,v') = \mathcal{L}^\text{hinge}(v-v') = \max\{0, 1 - v \cdot v'\}\) unfortunately does not meet these sufficient conditions at all. Therefore, it is a somewhat surprising observation of this work that the hinge loss function still satisfies RSC (see Proposition 6.6 and Remark 6.7) and similar recovery statements as in [Gen17; PV16] remain valid. Apart from that, we wish to emphasize that the proofs of our results improve some of the techniques used in [Gen17; PV16]. For example, the linear multiplier term of the hinge loss is now handled by a more sophisticated concentration inequality due to Mendelson (see Theorem 6.2), which eventually leads to an enhanced probability of success in our reconstruction guarantees.

At this point, it is again worth mentioning the recent work of Kolleck and Vybíral [KV17] whose analysis of hinge loss minimization is directly related to ours. We have already presented several details of their approach in Subsection 1.1 and Remark 2.15(1), as well as a link to support vector machines, which served as their original motivation (see Subsection 2.3.3). Let us briefly recap to what extent our results improve the recovery guarantees of [KV17]: Firstly, we go beyond \(\ell^1\)-based signal sets and allow for arbitrary convex bodies as structural constraints. Secondly, the error bounds in [KV17] do only achieve an oversampling rate of \(O(m^{-1/4})\), while Theorem 2.10 and Theorem 2.16 exhibit the (optimal) rate of \(O(m^{-1/2})\). And finally, the noise patterns considered in [KV17] are far more restrictive than what is permitted by Assumption 2.1 and Assumption 2.9. The latter issue is probably an artifact of the proof techniques adapted from [PV13b], where only a constrained linear estimator is investigated. This is substantially different from our statistical framework, in which the “quadratic” part of the hinge loss is explicitly taken into account (see Proposition 6.6).

### 4.2 Statistical Learning Theory

Since our proof strategy heavily relies on tools from statistical learning theory, let us also briefly discuss how our approach relates to this field of research. In that context, the model of Assumption 2.1 is regarded as a sampling procedure according to which the sample set \(\{(a_i,y_i)\}_{i \in [m]}\) is independently drawn from a random pair \((a,y)\) that obeys a (partially) unknown probability distribution on \(\mathbb{R}^n \times \{-1,+1\}\). The measurement vectors \(a_i\) do usually play the role of data (or feature) vectors, whereas \(y_i\) denotes a class label that depends on these features in some way.

One of the key goals in statistical learning is then to specify a (deterministic) prediction function \(F : \mathbb{R}^n \to \mathbb{R}\) that minimizes the risk of wrongly predicting the true label \(y\) by \(F(a)\). Since only a finite collection of samples is given instead of \((a,y)\), this is in fact a challenging problem and one typically restricts the set of predictors to a convex hypothesis set \(\mathcal{H}\) (a subset of measurable functions), encoding one’s beliefs in the underlying observation model. Due to the specific form of our output rule, i.e., \(y = f(\langle a, x_0 \rangle)\), it is quite natural to consider a linear hypothesis set

\[
\mathcal{H} = \mathcal{H}_K = \{v \mapsto \langle v, x \rangle \mid x \in K\},
\]
where $K \subset \mathbb{R}^n$ is convex. Identifying $\mathcal{H}_K$ with $K$, this precisely reflects Assumption 2.2.

The purpose of our main results in Section 2 is to study the capability of the associated empirical risk minimizer $\hat{x}$ of $(P_{\text{chng}, k})$ to approximate the ground truth vector $x_0$. In the literature, such types of statements are often referred to as estimation, but note that, somewhat unusually, the expected risk minimizer does not necessarily belong to $\text{span}\{x_0\}$, see Subsection 2.3.1. This stands in contrast to the above problem of prediction in which one is rather interested in controlling the so-called sample error

$$
\mathbb{E}[\mathcal{L}_{\text{chng}}(y(a, \hat{x}))] - \min_{x \in K} \mathbb{E}[\mathcal{L}_{\text{chng}}(y(a, x))].
$$

Indeed, a small sample error does not automatically imply that the normalized minimizer $\hat{x}/\|\hat{x}\|_2$ is also close to $x_0$. For more details on estimation and prediction, we refer to [Men17], and for a comprehensive overview of statistical learning theory, one may consider [CZ07; Vap98].

Of particular relevance to our approach are the works of Mendelson [Men15; Men17] on learning without concentration. His estimation results for empirical risk minimization establish very general principles that relate geometric properties of the hypothesis set to the sampling rate. While these statements bear a certain resemblance to ours, the actual goals of Mendelson are somewhat different: [Men15; Men17] consider a very abstract model setting, where the output variable $y$ is left unspecified and the hypothesis set is not just restricted to linear functions. The key concern of Mendelson’s small ball method developed in [Men15; Men17] is to allow for heavy tailed feature variables, for which concentration inequalities fail to hold true. Following this strategy, it is still possible to prove powerful estimation guarantees under very mild assumptions on the underlying probability measure.

In contrast, we investigate a specific 1-bit output rule with Gaussian data. This enables us to prove much more explicit error bounds and to precisely quantify the recovery behavior of the hinge loss estimator. But let us emphasize that the results of this work are not implicitly contained in the framework of [Men15; Men17] because the hinge loss does by far not satisfy the required assumptions, especially local strong convexity (cf. Remark 6.7). The very recent work of [ACL17] follows an alternative path to tackle this issue: Based on regularized empirical risk minimization, the authors prove estimation bounds for Lipschitz loss functions, which in principle also includes the hinge loss. Their theoretical findings again hold true in a fairly general learning setting, but the actual statements rely on an abstract Bernstein condition that the loss function needs to fulfill. While the lower bound on the excess risk in (6.15) actually resembles such a Bernstein condition, it is still unclear whether the framework of [ACL17] would apply to our setup. Indeed, the proof of (6.15) turns out to be highly non-trivial, so that verifying the general assumptions of [ACL17] might take a lot of effort in a specific model situation. This observation manifests once more that, despite obvious overlaps, the fields of statistical learning and signal processing address different types of problems.

### 5 Conclusion and Outlook

Our main results show that 1-bit compressed sensing via hinge loss minimization is indeed feasible under fairly general model conditions. This particularly includes a wide class of noisy bit flip patterns (see Theorem 2.10) as well as arbitrary convex constraint sets that encode structural hypotheses, such as sparsity (see Theorem 2.14 and Theorem 2.16). While comparable recovery guarantees were recently established for different loss functions [Gen17; Men17], it is somewhat astonishing that these assertions essentially remain valid for the hinge loss, since it is neither differentiable nor locally strongly convex. The proofs of our results however strongly rely on the specific form of 1-bit observations and require several sophisticated adaptations of previous arguments. For this reason, we do not expect that empirical hinge loss minimization is as universally applicable as the Lasso (cf. [GJ17; Men15; PV16]). On the other hand, the special ability of the hinge loss to deal with binary outputs also implies computational advantages. For example, the estimator $(P_{\text{chng}, k})$ can be recasted as a linear program in the case of $\ell^1$-constraints (cf. [KV17, Sec. VI.A]), which in turn is appealing for practical purposes.

---

1 Compared to the program $(P_{\ell^1, k})$, a regularized estimator takes the form $\min_{x \in K} \frac{1}{m} \sum_{i=1}^m \mathcal{L}(\langle a_i, x \rangle, y_i) + \lambda \|x\|_1$. 

Let us conclude with some potential extensions and open issues that might be investigated in future works:

- **Relaxing the model assumptions.** Although the technical details are not elaborated here, we suppose that — by adapting known proof strategies — the following points are relatively straightforward generalizations of our model setup: adversarial bit flips (see [Gen17]), unnormalized signal vectors, and anisotropic sub-Gaussian measurements (see [GJ17; GK16; Men15]).

A probably more challenging problem is to unify the respective hypotheses of our main results from Section 2, ultimately leading to a recovery guarantee that allows for general convex signal sets and noisy observations at the same time. In fact, there are several significant differences in the argumentation of Subsection 6.2 and Subsection 6.3. This particularly concerns the slightly different role of signal complexity in both parts, which is not even fully understood in the situation of general convex constraints (see Remark 6.14).

- **Different loss functions.** The hinge loss is actually a prototypical example of a piecewise linear loss function. Since our analysis shows that the associated empirical risk function satisfies restricted strong convexity under certain conditions (cf. Remark 6.7), one could expect that this important property holds true for a larger class of piecewise linear losses. Our proof techniques may serve as a template at this point, but adapting the individual steps would certainly require some care and technical effort.

- **Optimal choice of the loss.** An issue that is closely related to the previous one is the following: Supposed we have (partial) knowledge of the true observation model, what is a good or even optimal choice of loss in empirical risk minimization (PLhng, K)? More specifically, when is a loss function, e.g., the hinge loss, superior to others? What practical rules-of-thumb can be derived from this study? These questions are of course quite vaguely formulated. One of the major difficulties is to come up with a quantitative measure to assess the recovery performance of a loss. Such a benchmark would also involve sharp lower bounds on the recovery error, which we consider as a very challenging problem on its own.

- **Regularized estimation.** From an algorithmic perspective, it can be very useful to solve a regularized optimization problem of the form

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^m \mathcal{L}(\langle a_i, x \rangle, y_i) + \lambda \|x\|$$

instead of (PLhng, K). Here, the norm \(\| \cdot \|\) encourages structured solutions, similarly to the constraint \(x \in K\) in (PLhng, K). An adaption of our results to such types of estimators is by far not obvious and might rely on rather different arguments in the proofs. See also [ACL17; LM16; LM17] for recent achievements for regularized empirical risk minimization in statistical learning.

### 6 Proofs of the Main Results

Let us start with a brief roadmap of the proof strategy pursued in this section. The common recovery approach of Subsection 2.2 and Subsection 2.3 is to estimate the ground truth signal \(x_0 \in S^{n-1}\) via constrained empirical risk minimization. More specifically, we invoke the program of (PLhng, K) if the signal set \(K \subset \mathbb{R}^n\) is contained in the Euclidean unit ball, whereas the scalable estimator (PLhng, µK) is used for general convex constraints. In both cases, it will turn out that, with high probability, the respective minimizer \(\hat{x}\) resides in a certain (local) neighborhood of \(\mu x_0\) for an appropriately chosen scaling parameter \(\mu > 0\). In order to make this localization argument more precise, let us introduce the so-called excess risk functional

$$\mathcal{E}(x) := \bar{R}(x) - \bar{R}(\mu x_0), \quad x \in \mathbb{R}^n,$$
Figure 5: If $\mathcal{E}(\cdot)$ is positive on a spherical intersection $K \cap (tS^{n-1} + \mu x_0)$ (red arc), then Fact 6.1 implies that every minimizer $\hat{x}$ of $(P_{\mathcal{P}_{\text{Lin},K}})$ must belong to $K \cap (tB^2 + \mu x_0)$ (dark gray region), that means, we have $\|\mu x_0 - \hat{x}\|_2 \leq t$.

where $\mathcal{R}(x) = \frac{1}{m} \sum_{i=1}^{m} \mathcal{L}^\text{hng}(y_i \langle a_i, x \rangle)$ denotes the empirical risk (cf. Definition 2.7), which serves as objective functional in both recovery programs. The following simple observation shows that positivity of the excess risk allows us to reduce the set of potential minimizers:

**Fact 6.1** Let $\mathcal{E}(x) > 0$ for some $x \in \mathbb{R}^n$ and consider $\text{Ray}(x) := \{\mu x_0 + \tau(x - \mu x_0) \mid \tau \geq 0\}$, which is the ray starting at $\mu x_0$ and passing through $x$. Then, we have $\mathcal{E}(\mu x_0 + \tau(x - \mu x_0)) > 0$ for all $\tau \geq 1$. Moreover, if a minimizer $\hat{x}$ of $(P_{\mathcal{P}_{\text{Lin},K}})$ or $(P_{\mathcal{P}_{\text{Lin},K}})$ belongs to $\text{Ray}(x)$, it must be contained in the line segment between $\mu x_0$ and $x$, i.e., $\hat{x} \in \text{conv}\{\mu x_0, x\}$. This particularly implies $\|\mu x_0 - \hat{x}\|_2 \leq \|\mu x_0 - x\|_2$.

This claim directly follows from the convexity of the excess risk and constraint set as well as the fact that $\mathcal{E}(\mu x_0) = 0$ and $\mathcal{E}(\hat{x}) \leq 0$. Note that the last inequality holds true because $\hat{x}$ minimizes the empirical risk on a certain signal set and $\mu x_0$ is a feasible vector. Figure 5 demonstrates how one can use Fact 6.1 to derive an error bound for $\hat{x}$ in the case of spherical intersections.

Consequently, the actual key challenge is to verify that the excess risk is positive on the boundary of an appropriate neighborhood of $\mu x_0$, e.g., a small Euclidean ball. For this purpose, it is useful to consider the first order Taylor expansion of $x \mapsto \mathcal{R}(x)$ at $\mu x_0$. The approximation error is then given by

$$Q(x, \mu x_0) := \mathcal{R}(x) - \mathcal{R}(\mu x_0) - \frac{1}{m} \sum_{i=1}^{m} z_i \langle a_i, x - \mu x_0 \rangle,$$

where $\mathcal{M}(\cdot, \mu x_0)$ is the “linearization” of $\mathcal{R}(\cdot)$ at $\mu x_0$ with

$$z_i := y_i \cdot \mathcal{L}^\text{hng}[y_i \langle a_i, \mu x_0 \rangle] = -y_i \cdot \chi_{[-\infty,0]}(y_i \langle a_i, \mu x_0 \rangle), \quad i = 1, \ldots, m.$$

We will call $\mathcal{M}(\cdot, \mu x_0)$ the multiplier term in the following because the mapping $x \mapsto \mathcal{M}(x, \mu x_0)$ indeed forms a multiplier empirical process (cf. [Men15; Men16; Men17]). In contrast, $Q(\cdot, \mu x_0)$ is referred to as the quadratic term of the excess risk $\mathcal{E}(\cdot)$.

Note that the convexity of $\mathcal{R}(\cdot)$ implies that the quadratic term $Q(\cdot, \mu x_0)$ is always non-negative. Hence, in order to achieve

$$\mathcal{E}(x) = \mathcal{R}(x) - \mathcal{R}(\mu x_0) = \mathcal{M}(x, \mu x_0) + Q(x, \mu x_0) > 0 \quad (6.1)$$

for all $x$ in a fixed subset of $\mathbb{R}^n$, it suffices to show that $Q(\cdot, \mu x_0)$ uniformly dominates $\mathcal{M}(\cdot, \mu x_0)$ on that specific set. To this end, we will treat both terms independently and apply different tools from empirical process theory. The multiplier term can be easily handled by a recent result of Mendelson [Men16],

This is a slight abuse of terminology because $\mathcal{L}^\text{hng}$ is not twice differentiable, so that $Q(x, \mu x_0)$ is actually not a quadratic function. However, $x \mapsto Q(x, \mu x_0)$ mimics the role of a quadratic empirical process, as considered in [Gen17; Men17] for example, and we will prove similar quadratic lower bounds for it.
Theorem 6.2, which concerns the uniform deviation of multiplier empirical processes from their mean. For the quadratic process on the other hand, we will employ Mendelson’s small ball method as stated by Theorem 6.3 below.

We conclude this overview part with deriving a lower bound for \( \mathcal{Q}(\cdot, \mu x_0) \) that is more convenient to work with. For this, let us rewrite the quadratic term as follows:

\[
\mathcal{Q}(x, \mu x_0) = \frac{1}{m} \sum_{i=1}^{m} \left( (1 - y_i(a_i, x)) \chi_{(-\infty,1]}(y_i(a_i, x)) - (1 - y_i(a_i, \mu x_0)) \chi_{(-\infty,1]}(y_i(a_i, \mu x_0)) + (y_i(a_i, x) - y_i(a_i, \mu x_0)) \chi_{(-\infty,1]}(y_i(a_i, \mu x_0)) \right)
\]

Setting \( h := x - \mu x_0 \), the following estimates hold true for every \( \xi > 0 \):

\[
(1 - y_i(a_i, x)) \cdot \chi_{[1,\infty)}(y_i(a_i, \mu x_0)) \cdot \chi_{(-\infty,1]}(y_i(a_i, x)) = (-y_i(a_i,h) - (y_i(a_i, \mu x_0) - 1)) \cdot \chi_{[1,\infty)}(y_i(a_i, \mu x_0)) \cdot \chi_{(-\infty,1]}(y_i(a_i, \mu x_0)) \geq (-y_i(a_i,h) - (y_i(a_i, \mu x_0) - 1)) \cdot \chi_{[1,\infty)}(y_i(a_i, \mu x_0)) \cdot \chi_{(-\infty,1]}(y_i(a_i, h)) \geq (-y_i(a_i,h) - 2\xi) \cdot \chi_{[1,\infty)}(y_i(a_i, \mu x_0)) \cdot \chi_{(-\infty,1]}(y_i(a_i, h)) =: F_-(a_i,h)
\]

and

\[
(y_i(a_i, x) - 1) \cdot \chi_{(-\infty,1]}(y_i(a_i, \mu x_0)) \cdot \chi_{[1,\infty)}(y_i(a_i, x)) = (y_i(a_i,h) - (1 - y_i(a_i, \mu x_0))) \cdot \chi_{(-\infty,1]}(y_i(a_i, \mu x_0)) \cdot \chi_{[1,\infty)}(y_i(a_i, h) + y_i(a_i, \mu x_0)) \geq (y_i(a_i,h) - (1 - y_i(a_i, \mu x_0))) \cdot \chi_{[1,\infty)}(y_i(a_i, \mu x_0)) \cdot \chi_{[2,\infty)}(y_i(a_i, h)) \geq (y_i(a_i,h) - 2\xi) \cdot \chi_{[1,\infty)}(y_i(a_i, \mu x_0)) \cdot \chi_{[2,\infty)}(y_i(a_i, h)) =: F_+(a_i,h).
\]

Therefore, the resulting non-negative empirical process

\[
h \mapsto \frac{1}{m} \sum_{i=1}^{m} F(a_i, h) \quad \text{with} \quad F(a_i, h) := F_+(a_i, h) + F_-(a_i, h)
\]

satisfies

\[
\mathcal{Q}(x, \mu x_0) \geq \frac{1}{m} \sum_{i=1}^{m} F(a_i, h)
\]

for all \( h = x - \mu x_0 \) and \( \xi > 0 \).

6.1 Tools From Empirical Process Theory

This subsection provides two important tools from empirical process theory which we will apply to control the multiplier and the quadratic term of the excess risk in (6.1). The following concentration inequality by Mendelson investigates the uniform deviation of multiplier processes. Note that this result even holds true in a more general setting, see [Men16, Thm. 4.4].
Theorem 6.2 ([Men16]) Let \( L \subseteq tB_2^d \). For every \( i \in [m] \), assume that \( a_i \) is an independent copy of a standard Gaussian random vector \( a \sim \mathcal{N}(0, I_n) \), and \( z_i \) is an independent copy of a sub-Gaussian random variable \( z \) which is not necessarily independent of \( a \). There exist numerical constants \( C_1, C_2 > 0 \) such that for every \( u > 0 \), the following holds true with probability at least \( 1 - 2 \exp(-C_1 \cdot u^2) - 2 \exp(-C_1 \cdot m) \):

\[
\sup_{h \in L} \left| \frac{1}{m} \sum_{i=1}^{m} (z_i \langle a_i, h \rangle - \mathbb{E}[z_i \langle a_i, h \rangle]) \right| \leq C_2 \cdot \|z\|_{\psi_2} \cdot \frac{w(L) + u \cdot t}{\sqrt{m}}.
\]

Our second ingredient is Mendelson’s small ball method, which is a powerful concept to establish lower bounds for non-negative empirical processes. We state an adaption of Tropp’s version in [Tro15, Prop. 5.1] below, but it should be emphasized that the original idea is due to Mendelson [Men15, Thm. 5.4].

Theorem 6.3 (Mendelson’s small ball method) Let \( L \subseteq \mathbb{R}^n \) be a subset, \( a \sim \mathcal{N}(0, I_n) \) be a standard Gaussian random vector, and \( F : \mathbb{R} \to \mathbb{R} \) be a non-negative (random) contraction that fixes the origin.\(^1\) For every \( i \in [m] \), assume that \( a_i \sim \mathcal{N}(0, I_n) \) is an independent copy of \( a \) and \( F_i \) is an independent copy of \( F \). Then, for every \( \xi > 0 \) and \( u > 0 \), the following holds true with probability at least \( 1 - \exp(-\frac{u^2}{2}) \):

\[
\inf_{h \in L} \left\{ \frac{1}{m} \sum_{i=1}^{m} F_i(\langle a_i, h \rangle) \geq \xi \cdot \left( Q_{2\xi}(L) - \frac{2}{\xi} \cdot w(L) + u, \frac{1}{\sqrt{m}} \right) \right. \]

where

\[
Q_{2\xi}(L) := \inf_{h \in L} \mathbb{P}[F(\langle a, h \rangle) \geq 2\xi]
\]

denotes the small ball function that is associated with \( F \).

6.2 Proof of Theorem 2.10 (Subsets of the Unit Ball)

Throughout this subsection, we assume that the hypotheses of Theorem 2.10 are satisfied, in particular \( K \subseteq B_2^d \) and the model conditions of Assumption 2.1, 2.2, and 2.9. Following our proof sketch from the beginning of Section 6, our goal is to show that the excess risk functional \( \mathcal{E}(\cdot) = \mathcal{R}(\cdot) - \mathcal{R}(\mu_{x_0}) \) is uniformly positive on the boundary of a small Euclidean ball centered at \( \mu_{x_0} \), i.e.,

\[
\inf_{x \in K \cap (tS^{n-1} + \mu_{x_0})} \mathcal{E}(x) > 0
\]

for \( t > 0 \) small enough. For the sake of readability, we denote by

\[
K_t := K \cap (tS^{n-1} + \mu_{x_0})
\]

the set of all points in \( K \) with distance \( t \) to \( \mu_{x_0} \), and by

\[
L_t := K_t - \mu_{x_0} = (K - \mu_{x_0}) \cap tS^{n-1}
\]

its counterpart that arises from a parallel shift of \( \mu_{x_0} \) to the origin. Thus, every point \( x \in K_t \) is associated with a directional vector \( h := x - \mu_{x_0} \in L_t \).

We start by proving Lemma 2.8. For this purpose, let us define the convex function

\[
R : \mathbb{R} \to \mathbb{R}, \quad s \mapsto R(s) = \mathbb{E}[\mathcal{E}^{\text{hinge}}(s \exp(g(s) g))], \quad g \sim \mathcal{N}(0, 1),
\]

which corresponds to the expected risk function restricted to the span of \( x_0 \). Indeed, for every \( s \in \mathbb{R} \), it

---

\(^1\)That means, \( F \) is 1-Lipschitz and \( F(0) = 0 \).
holds that (cf. Definition 2.7)
\[ R(sx_0) = \mathbb{E}[L_{\text{hng}}(y(a, sx_0))] = \mathbb{E}[L_{\text{hng}}(sy(a, x_0))] = \mathbb{E}[L_{\text{hng}}(sf(g)g)] = R(s). \tag{6.4} \]

**Proof of Lemma 2.8.** We first show that, by convexity of the hinge loss, every expected risk minimizer belongs to the span of \( x_0 \): For \( x \in K \), the orthogonal decomposition
\[ x = P_{x_0}(x) + P_{x_0}^\perp(x) = \langle x, x_0 \rangle x_0 + P_{x_0}^\perp(x), \]
allows us to rewrite the expected risk as follows:
\[ R(x) = \mathbb{E}[L_{\text{hng}}(y(a, x))] = \mathbb{E}[L_{\text{hng}}(y(x, x_0)\langle a, x_0 \rangle + y(a, P_{x_0}^\perp(x)))]. \]

Since the projections of a standard Gaussian random vector onto orthogonal vectors are independent, we conclude that \( \langle a, P_{x_0}^\perp(x) \rangle \) is in fact independent of both \( y \) and \( \langle a, x_0 \rangle \). Therefore, Fubini’s theorem and Jensen’s inequality imply
\[ R(x) = \mathbb{E}[L_{\text{hng}}(y(a, x))] \geq \mathbb{E}_{y, a, x_0} E_{\langle a, P_{x_0}^\perp(x) \rangle} [L_{\text{hng}}(y(x, x_0)\langle a, x_0 \rangle + y(a, P_{x_0}^\perp(x)))] = \mathbb{E}[L_{\text{hng}}(y(x, x_0)\langle a, x_0 \rangle)] = R(\langle x, x_0 \rangle x_0). \]

Since \( K \subset B_2^\perp \), it holds that \( |\langle x, x_0 \rangle| \leq 1 \) for all \( x \in K \). Consequently, the minimum of the expected risk function on \( K \) is bounded from below by the minimum of \( R \) on the compact interval \([-1, 1]\), i.e.,
\[ \min_{x \in K} R(x) \geq \min_{s \in [-1, 1]} R(sx_0) = \min_{s \in [-1, 1]} R(s). \]

Computing the first (weak) derivative of \( R \),
\[ R'(s) = \mathbb{E}[[L_{\text{hng}}]'(sf(g)g)f(g)g] = -\mathbb{E}[\chi_{(-\infty,1]}(sf(g)g)f(g)g], \tag{6.5} \]
and using the correlation assumption \( \mathbb{E}[f(g)g] > 0 \) in (C1), we observe that
\[ R'(0) = -\mathbb{E}[f(g)g] < 0. \]

The convexity of \( R \) therefore implies that \( R \) attains its minimum on the interval \((0, 1]\), which yields
\[ \min_{x \in K} R(x) \geq \min_{s \in (0,1]} R(s) = \min_{s \in (0,1]} R(sx_0). \]

Since \( \text{conv}\{0, x_0\} \subset K \) by Assumption 2.2, it even follows that
\[ \min_{x \in K} R(x) = \min_{s \in (0,1]} R(sx_0). \]

Hence, if \( \mu > 0 \) is a minimizer of \( R \) on \((0, 1]\), then we have
\[ \min_{x \in K} R(x) = R(\mu x_0). \]

\[ \]
Our next auxiliary result states the relationship between $\mu$ and $\lambda = E[f(g)g]$ that was used in the error bound (2.9) of Theorem 2.10.

**Lemma 6.4** We have the following upper bound on $\mu^{-1}$:

$$
\mu^{-1} \lesssim \sqrt{\log(\lambda^{-1})}.
$$

More specifically, if $R'(1) \geq 0$, it holds that

$$
\mu^{-1} \leq \sqrt{2 \log(\lambda^{-1})}.
$$

**Proof.** Set $X := f(g)g$ and $s_* := \sqrt{2 \log(\lambda^{-1})}$. Since $|X| \leq |g|$, we observe that

$$
0 \overset{(C1)}{<} \lambda = E[X] \leq E[|g|] = \sqrt{\frac{2}{\pi}} < 1, \quad (6.6)
$$

and in particular, that $s_*$ is well-defined and positive. Moreover, we have

$$
E[\chi_{(s_*, \infty)}(X) \cdot X] \leq E[\chi_{(s_*, \infty)}(|g|) \cdot |g|] = \int_{s_*}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}} e^{-s_*^2/2} = \sqrt{\frac{2}{\pi}} \lambda.
$$

Hence,

$$
E[\chi_{(-\infty, s_*]}(X) \cdot X] = E[X] - E[\chi_{(s_*, \infty)}(X) \cdot X] \geq \lambda - \sqrt{\frac{2}{\pi}} \lambda > 0,
$$

which gives

$$
R'(\frac{1}{s_*}) = -E[\chi_{(-\infty, s_*]}(X) \cdot X] < 0. \quad (6.7)
$$

Let us make a case distinction for the sign of $R'(1)$: If $R'(1) \geq 0$, we have $R'(\mu) = 0$ because $\mu$ is a minimizer of $R$ on $[0,1]$ and $R'(0) < 0$. In particular, $R'(\mu) > R'(\frac{1}{s_*})$ due to (6.7). Since $R'$ is non-decreasing, this implies $\mu \geq \frac{1}{s_*}$, which is the claim of Lemma 6.4. Finally, if $R'(1) < 0$, the convexity of $R$ implies that $\mu = 1$ is the minimizer of $R$ on $[0,1]$. And from $\lambda \leq \sqrt{\frac{2}{\pi}}$, it follows that

$$
\sqrt{\log(\lambda^{-1})} \geq \sqrt{\log(\frac{2}{\pi})} \gtrapprox 1 = \mu^{-1}.
$$

\[ \blacksquare \]

The proof of Lemma 6.4 reveals the following two important facts about the minimizer $\mu$, depending on the sign of $R'(1)$:

- if $R'(1) < 0$, it follows that $\mu = 1$, and
- if $R'(1) \geq 0$, it follows that $R'(\mu) = 0$. \quad (6.8) \quad (6.9)

The next lemma shows that, as long as $K \subset B_{\frac{n}{2}}^{F_2}$, the multiplier term is always non-negative in expectation. This is very different from the case of general signal sets, where the expected value could become negative (see Remark 6.8).

**Lemma 6.5** For every $t > 0$, it holds that

$$
\inf_{x \in K_t} E[M(x, \mu x_0)] \geq 0.
$$

According to (6.8) and (6.9), we can distinguish between two cases:
Proposition 6.6. There exist numerical constants $C, C' > 0$ such that for every $t \leq \mu$ and $\eta \in (0, \frac{1}{2})$, the following holds true with probability at least $1 - \eta$:

$$\inf_{x \in K_t} Q(x, \mu x_0) \geq C' \cdot \lambda \cdot t^2 - C \cdot t \cdot \frac{\sqrt{d_1(K - \mu x_0)} + \sqrt{\log(\eta^{-1})}}{\sqrt{m}}.$$  (6.12)

Remark 6.7. Combining (6.12) with the condition (2.8), we obtain a lower bound of the form

$$Q(x, \mu x_0) \geq t^2 = \|x - \mu x_0\|_2^2 \quad \text{for all } x \in K_t = K \cap (tS^{n-1} + \mu x_0).$$

According to [Gen17, Def. 2.2], this means that the empirical risk function $\bar{R}(\cdot)$ satisfies restricted strong convexity (RSC) at scale $t$ with respect to $\mu x_0$ and $K$. This observation is quite remarkable because the hinge loss $L_{\text{hng}}$ does by far not fulfill sufficient criteria known from the literature, e.g., in [Gen17, Sec. II.C] or [Men17, Sec. 4]. More specifically, since $L_{\text{hng}}$ is piecewise linear, its second derivative does only exist in a distributional sense. Proposition 6.6 therefore indicates that one can even expect RSC (at certain scales $t$) if the “curvature energy” of the loss function is concentrated in a single point.
Proof. According to (6.3), the quadratic term can be uniformly bounded from below by a simplified non-negative empirical process (6.2), i.e.,

$$\inf_{x \in K_t} Q(x, \mu x_0) \geq \inf_{h \in L_t} \frac{1}{m} \sum_{i=1}^{m} F(a_i, h),$$

where $L_t = K_t - \mu x_0$. We now apply Theorem 6.3 with $\xi = \frac{L}{t}$, $L = L_t$, $u = \sqrt{2 \log(\eta^{-1})}$, and

$$F(s) = (-ys - \frac{L}{3}) \cdot X_{[1, 1]}(y(a, \mu x_0)) \cdot X_{(-\infty, -\frac{L}{3}]}(ys)$$

$$+ (ys - \frac{L}{3}) \cdot X_{[1, 1]}(y(a, \mu x_0)) \cdot X_{[\frac{L}{3}, \infty)}(ys).$$

It is not hard to see that $F$ is indeed a non-negative contraction with $F(0) = 0$. Since $t^{-1} w(L_t) \leq \sqrt{d_t(K - \mu x_0)}$ and $F_t(\langle a_i, h \rangle) = F(a_i, h)$, the assertion of Theorem 6.3 implies that, with probability at least $1 - \eta$, it holds that

$$\inf_{h \in L_t} \frac{1}{m} \sum_{i=1}^{m} F(a_i, h) \geq \frac{t}{6} \cdot \left( Q_{1/3}(L_t) - \frac{12 \sqrt{d_t(K - \mu x_0) + 2 \log(\eta^{-1})}}{\sqrt{m}} \right).$$

It remains to show that the small ball function associated with $F$ satisfies the lower bound

$$Q_{1/3}(L_t) = \inf_{h \in L_t} \mathbb{P}[F(a, h) \geq \frac{t}{6}] \geq t \cdot \lambda.$$

For this purpose, we divide $L_t$ into two disjoint subsets in the following way:

$$L_t = (K - \mu x_0) \cap tS^{-1} = L_t^+ \cup L_t^-,$$

where $L_t^+ := \{ h \in L_t \mid \langle h, x_0 \rangle \geq 0 \}$ and $L_t^- := \{ h \in L_t \mid \langle h, x_0 \rangle < 0 \}$. Let us first consider the case of $h \in L_t^+$ and bound $\mathbb{P}[F(a, h) \geq \frac{t}{6}]$ from below: There exist $d, e \in [0, t]$ and $x' \in \mathbb{R}^n$ such that

$$h = \langle h, x_0 \rangle x_0 + P_{x_0} \cdot (h) = dx_0 + ex',$$

with $d^2 + e^2 = t^2$, $\langle x_0, x' \rangle = 0$, and $\|x_0\|_2 = \|x'\|_2 = 1$. Hence, we obtain

$$\mathbb{P}[F(a, h) \geq \frac{t}{6}] = \mathbb{P}[y(a, h) \geq \frac{t}{6}, y(a, \mu x_0) \in [1 - \frac{t}{3}, 1]]$$

$$= \mathbb{P}[dy(a, x_0) + ey(a, x') \geq \frac{t}{6}, y(a, x_0) \in [\frac{1}{3} - \frac{t}{3p}, \frac{1}{3}]]$$

$$\geq \mathbb{P}[d(\frac{1}{3} - \frac{t}{3p}) + ey(a, x') \geq \frac{t}{6}, y(a, x_0) \in [\frac{1}{3} - \frac{t}{3p}, \frac{1}{3}]]$$

$$= \mathbb{P}[ey(a, x') \geq \frac{2t}{3} - d(\frac{1}{3} - \frac{t}{3p})] \cdot \mathbb{P}[y(a, x_0) \in [\frac{1}{3} - \frac{t}{3p}, \frac{1}{3}]].$$

where we have again used that the components of an orthogonal decomposition of a standard Gaussian vector are independent. Next, we show that

$$\mathbb{P}[ey(a, x') \geq \frac{2t}{3} - d(\frac{1}{3} - \frac{t}{3p})] \geq \mathbb{P}[g \geq 1] \quad \text{for } g \sim \mathcal{N}(0, 1).$$

(6.13)

From $t \leq \mu \leq 1$, it follows that $\frac{2t}{3} + \frac{t}{3} \leq 1$, which is equivalent to $1 - \frac{1}{3} + \frac{t}{3p} \leq \frac{1}{3}$. Since $0 \leq d \leq t$, this leads to $d - d(\frac{1}{3} - \frac{t}{3p}) \leq \frac{t}{3}$, or equivalently, $\frac{2t}{3} - d(\frac{1}{3} - \frac{t}{3p}) \leq t - d$. Due to $t - d \leq \sqrt{t^2 - d^2} = e$, we therefore obtain $\frac{2t}{3} - d(\frac{1}{3} - \frac{t}{3p}) \leq e$, so that
\[ P[ey(a, x') \geq 2 \frac{t}{2} - d(\frac{1}{\mu} - \frac{1}{2\mu'})] \geq P[ey(a, x') \geq e] \geq P[y(a, x') \geq 1] \]
\[ = P[(a, x') \geq 1] = P[g \geq 1], \]
where we have particularly used that the binary variable \( y \in \{ -1, +1 \} \) is independent of \( g = \langle a, x' \rangle \sim \mathcal{N}(0, 1) \). In conclusion, we have
\[ P[F(a, h) \geq \frac{t}{2}] \geq P[F_+(a, h) \geq \frac{t}{2}] \geq P[g \geq 1] \cdot P[y(a, x_0) \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]], \]
for all \( h \in L_t^+ \).

To handle the case of \( h \in L_t^- \), let us consider the decomposition
\[ -h = (-h, x_0)x_0 + P_{x_0}(-h) = dx_0 + ex', \]
with \( d, e \in [0, t] \) such that \( d^2 + e^2 = t^2 \), \( \langle x_0, x' \rangle = 0 \), and \( \|x_0\|_2 = \|x'\|_2 = 1 \). We proceed as before:
\[ P[F_-(a, h) \geq \frac{t}{2}] = P[-y(a, h) \geq \frac{2t}{3}, y(a, \mu x_0) \in [1, 1 + \frac{t}{3}]] = P[dy(a, x_0) + ey(\langle a, x' \rangle \geq \frac{2t}{3}, y(a, x_0) \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] \geq P[\frac{t}{2} + ey(\langle a, x' \rangle \geq \frac{2t}{3}, y(a, x_0) \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]]. \]

By (6.13) again, it holds that
\[ P[ey(\langle a, x' \rangle \geq \frac{2t}{3} - \frac{d}{\mu}] \geq P[ey(\langle a, x' \rangle \geq \frac{2t}{3} - d(\frac{1}{\mu} - \frac{1}{2\mu'})] \geq P[g \geq 1], \]
and consequently
\[ P[F(a, h) \geq \frac{t}{2}] \geq P[F_-(a, h) \geq \frac{t}{2}] \geq P[g \geq 1] \cdot P[y(a, x_0) \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]], \]
for every \( h \in L_t^- \).

To this point, we have shown that
\[ Q_{t/3}(L_t) \geq P[g \geq 1] \cdot \min \{ P[f(g)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]], P[f(g)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] \}, \]
where \( g = \langle a, x_0 \rangle \sim \mathcal{N}(0, 1) \). Note that the correlation condition (C2) of Assumption 2.9 is equivalent to
\[ P[f(g)g \geq 0 | |g|] \geq P[f(g)g \leq 0 | |g|] \quad (\text{a.s.}). \]
In particular, this means that
\[ P[f(g)g \geq 0, |g| \in I] \geq P[f(g)g \leq 0, |g| \in I] \]
for every interval \( I \subset \mathbb{R} \). Selecting \( I = [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}] \), this implies
\[ P[f(g)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] = P[f(g)g \geq 0, |g| \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] \geq P[f(g)g \leq 0, |g| \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] = P[-f(g)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]], \]
and therefore
\[ P[|g| \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] = P[|f(g)g| \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]] \leq 2P[f(g)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{1}{2\mu'}]]. \]
Similarly, for $I = [\frac{1}{\mu}, \frac{1}{\mu} + \frac{t}{3\mu}]$, we obtain
\[
P[|z| \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{t}{3\mu}]] \leq 2P[f(z)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{t}{3\mu}]]
\]
Elementary estimates now lead to
\[
P[f(z)g \in [\frac{1}{\mu} - \frac{t}{3\mu}, \frac{1}{\mu}]] \geq \frac{1}{2}P[|z| \in [\frac{1}{\mu} - \frac{t}{3\mu}, \frac{1}{\mu}]] = \int_{\frac{1}{\mu} - \frac{t}{3\mu}}^{\frac{1}{\mu}} \frac{t}{\sqrt{2\pi}} e^{-s^2/2} ds
\]
\[
\geq \frac{1}{\sqrt{2\pi}} \cdot \frac{t}{3\mu} \cdot e^{-1/(2\mu^2)} \geq \frac{1}{\sqrt{2\pi}} \cdot \frac{t}{3} \cdot e^{-1/(2\mu^2)}
\]
and
\[
P[f(z)g \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{t}{3\mu}]] \geq \frac{1}{2}P[|z| \in [\frac{1}{\mu}, \frac{1}{\mu} + \frac{t}{3\mu}]] = \int_{\frac{1}{\mu}}^{\frac{1}{\mu} + \frac{t}{3\mu}} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds
\]
\[
\geq \int_{\frac{1}{\mu}}^{\frac{1}{\mu} + \frac{t}{3\mu}} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds \geq \frac{1}{\sqrt{2\pi}} \cdot \frac{t}{3} \cdot e^{-(\frac{1}{\mu} + \frac{t}{3})^2/2} \geq \frac{1}{\sqrt{2\pi}} \cdot \frac{t}{3} \cdot e^{-1/(2\mu^2)} \cdot e^{-1/2},
\]
where the assumption $t \leq \mu$ was used in the last step. Finally, we show that $e^{-1/(2\mu^2)} \geq \lambda$, which would imply the claim $Q_{1.3}(L_t) \geq t \cdot \lambda$. If $R'(1) \geq 0$, this bound directly follows from Lemma 6.4. On the other hand, if $R'(1) < 0$, (6.8) states that $\mu = 1$. Combined with the estimate of (6.6), it again follows that $e^{-1/(2\mu^2)} \geq \sqrt{\frac{2}{\pi}} \geq \lambda$.

We are ready to prove Theorem 2.10.

Proof of Theorem 2.10. Using the triangle inequality and the decomposition of (6.1), we can derive the following lower bound for the excess risk:
\[
\mathcal{E}(x) \geq \mathbb{E}[\mathcal{M}(x, \mu x_0)] - |\mathcal{M}(x, \mu x_0) - \mathbb{E}[\mathcal{M}(x, \mu x_0)]] + \mathcal{Q}(x, \mu x_0).
\]
Next, we take the infimum over the localized signal set $K_t = K \cap (tS_{n-1}^\mu + \mu x_0)$ and obtain a uniform lower bound:
\[
\inf_{x \in K_t} \mathcal{E}(x) \geq \inf_{x \in K_t} \mathbb{E}[\mathcal{M}(x, \mu x_0)] - \sup_{x \in K_t} |\mathcal{M}(x, \mu x_0) - \mathbb{E}[\mathcal{M}(x, \mu x_0)]] + \inf_{x \in K_t} \mathcal{Q}(x, \mu x_0).
\]
(6.14)
Let us treat each of the three summands in (6.14) separately. According to Lemma 6.5, the first term in (6.14) is non-negative. Recalling that $z_i = -y_i \cdot \chi_{(-\infty, 1]} (y_i \langle a_i, \mu x_0 \rangle)$ and $L_t = K_t - \mu x_0 \subset tB_{2}^n$, the second term takes the form
\[
\sup_{x \in K_t} |\mathcal{M}(x, \mu x_0) - \mathbb{E}[\mathcal{M}(x, \mu x_0)]] = \sup_{h \in L_t} \left| \frac{1}{m} \sum_{i=1}^{m} [z_i \langle a_i, h \rangle - \mathbb{E}[z_i \langle a_i, h \rangle]] \right|.
\]
Since $|z_i| \leq 1$, the multipliers $z_i$ are sub-Gaussian random variables with $\|z_i\|_2 \leq (\log(2))^{-1/2}$. Hence, Theorem 6.2 can be applied with $L = L_t$, and due to $m \geq \log(\eta^{-1})$, it states that
\[
\sup_{h \in L_t} \left| \frac{1}{m} \sum_{i=1}^{m} [z_i \langle a_i, h \rangle - \mathbb{E}[z_i \langle a_i, h \rangle]] \right| \leq C \cdot \frac{w(L_t) + t \cdot \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \leq C \cdot t \cdot \sqrt{d_t(K - \mu x_0) + \sqrt{\log(\eta^{-1})}}.
\]
with probability at least \(1 - \frac{\eta}{2}\), where \(\hat{C} > 0\) is an appropriate numerical constant. In order to bound the third term in (6.14), we simply invoke Proposition 6.6, after which

\[
\inf_{x \in K_t} Q(x, \mu x_0) \geq C' \cdot \lambda \cdot t^2 - C'' \cdot t \cdot \sqrt{d_t(K - \mu x_0) + \sqrt{\log(\eta^{-1})}} \sqrt{m}
\]

with probability at least \(1 - \frac{\eta}{2}\) for appropriate numerical constants \(C', C'' > 0\).

In total, the following holds true with probability at least 1:

\[
\inf_{x \in K_t} \mathcal{E}(x) \geq C' \cdot \lambda \cdot t^2 - (C'' + \hat{C}) \cdot t \cdot \sqrt{d_t(K - \mu x_0) + \sqrt{\log(\eta^{-1})}} > 0,
\]

(6.15)

where the positivity directly follows from the initial assumption (2.8), that is,

\[
m \geq \lambda^{-2} \cdot t^{-2} \cdot \max\{d_t(K - \mu x_0), \log(\eta^{-1})\}.
\]

The lower bound of (6.15) now implies that every minimizer \(\hat{x}\) of \((P_{\text{Ling}, K})\) satisfies \(\|x_0 - \hat{x}\|_2 \leq t\). Indeed, if we would have \(\|x_0 - \hat{x}\|_2 > t\), the line segment between \(\mu x_0\) and \(\hat{x}\) must intersect \(K_t\), since the signal set \(K\) is convex. But this would contradict the conclusion of Fact 6.1.

Finally, we observe that \(\|\hat{x}\|_2 - \mu\) = \(\|\hat{x}\|_2 - \|\mu x_0\|_2\) \leq \(\|\hat{x} - \mu x_0\|_2 \leq t\). In particular, \(0 < \mu - t \leq \|\hat{x}\|_2\), so that we can estimate

\[
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \leq \left\| x_0 - \frac{\hat{x}}{\mu} \right\|_2 + \left\| \frac{\hat{x}}{\|\hat{x}\|_2} - \frac{\hat{x}}{\mu} \right\|_2 \leq \frac{t}{\mu} + \frac{\|\hat{x}\|_2 - \mu \|\hat{x}\|_2}{\mu \|\hat{x}\|_2} \cdot \|\hat{x}\|_2 \leq \frac{2t}{\mu}.
\]

Rescaling in \(t\) precisely yields the claim of Theorem 2.10. ■

Remark 6.8 A careful study of the above proofs reveals that we have used the unit-ball assumption \(K \subset B^q_\mu\) at two points: firstly, in the proof of Lemma 2.8 to show that the expected risk minimizer lies on the span of \(x_0\), and secondly, in (6.11) to verify the non-negativity of the multiplier term on \(K_t\) in expectation. Especially the latter conclusion was crucial to ensure that the excess risk is positive on \(K_t\), since the first term of (6.14) can be ignored in this way.

Interestingly, as long as \(\mu = 1\), one could even apply the quadratic lower bound of (6.11) to establish positivity of the excess risk on \(K_t\), while completely disregarding the quadratic term in (6.14). With other words, the linear part of the hinge loss is “sufficient” for signal recovery and RSC is not needed at all (cf. Remark 6.7). A similar observation was already made by Plan and Vershynin in [PV13b], where they analyzed the performance of a simple linear estimator in 1-bit compressed sensing.

Such a simple argumentation however does not work out below in the case of general convex signal sets. Indeed, \(\mathbb{E}[\mathcal{M}(x, \mu x_0)]\) may become negative at certain points \(x\). Consequently, in order to still ensure that \(\mathcal{E}(x) > 0\), the adverse impact of the multiplier process needs to be “compensated” by the size of the quadratic term \(Q(x, \mu x_0)\), which in turn requires a refined error analysis. ◊

6.3 Proofs of Theorem 2.14 and Theorem 2.16 (General Convex Sets)

Throughout this part, we assume that the model hypotheses of Subsection 2.3 hold true, in particular Assumption 2.12. According to our roadmap from the beginning of Section 6, our proof builds upon analyzing the excess risk functional

\[
x \mapsto \mathcal{E}(x) = \tilde{R}(x) - \tilde{R}(\mu x_0) = \mathcal{M}(x, \mu x_0) + Q(x, \mu x_0)
\]

(6.16)

on a certain local neighborhood of \(\mu x_0\). In contrast to the strategy of Subsection 6.2, an exclusion of a spherical intersection \(\tilde{K} := \mu K \cap (S^{n-1} + \mu x_0)\) as potential minimizers of \((P_{\text{Ling}, \mu K})\) turns out to be too restrictive. More specifically, if \(x = \mu x_0 + h \in \tilde{K}\) with \(|(x_0, h)| \approx 1\), the quadratic term \(Q(x, \mu x_0)\) would
become too small compared to $|\mathcal{M}(x, \mu x_0)|$ (cf. Remark 6.8). Therefore, we cannot expect that $\mathcal{E}(x)$ is strictly positive in these cases.

This issue can be resolved by carefully enlarging the set of “feasible” points in $\mu K$, i.e., those points at which the excess risk is not guaranteed to be positive. More precisely, we will show that, with high probability, every minimizer $\hat{x}$ of $(P_{\text{Cyl}, \mu K})$ belongs to the cylindrical tube defined in (2.22), namely

$$\text{Cyl}(x_0, \mu) = \{x \in \mathbb{R}^n \mid \|x - (x, x_0)x_0\|_2 \leq 1, \langle x, x_0 \rangle \geq \frac{\mu}{2}\}.$$ See Figure 3 for more details on the geometric shape of $\text{Cyl}(x_0, \mu)$. The following lemma collects some basic properties of $\text{Cyl}(x_0, \mu)$.

**Lemma 6.9**

1. $\text{Cyl}(x_0, \mu)$ is a convex set and $\mu x_0$ is an interior point.
2. Every $x \in \text{Cyl}(x_0, \mu)$ satisfies
   $$\left\|x_0 - \frac{x}{\|x\|_2}\right\|_2 \leq \frac{3}{\mu}.$$
3. The boundary of $\text{Cyl}(x_0, \mu)$ can be written as the union of its side and base:
   $$\partial \text{Cyl}(x_0, \mu) = \left\{x \in \mathbb{R}^n \mid \|x - (x, x_0)x_0\|_2 = 1, \langle x, x_0 \rangle \geq \frac{\mu}{2}\right\} = \text{Cyl}_1(x_0, \mu) \text{ "side"},
   \cup \left\{x \in \mathbb{R}^n \mid \|x - (x, x_0)x_0\|_2 \leq 1, \langle x, x_0 \rangle = \frac{\mu}{2}\right\} = \text{Cyl}_2(x_0, \mu) \text{ "base"}.$$

**Proof.** The statements of (1) and (3) are elementary geometric properties. Let us verify (2). For every $x \in \text{Cyl}(x_0, \mu)$, the definition of $\text{Cyl}(x_0, \mu)$ implies that $\|x\|_2^2 \leq 1 + \langle x, x_0 \rangle^2$ as well as $\|x\|_2 \geq \langle x, x_0 \rangle \geq \mu/2$. Hence,

$$\left\|x_0 - \frac{x}{\|x\|_2}\right\|_2^2 = 2 \left(1 - \frac{\langle x, x_0 \rangle}{\|x\|_2}\right) = 2 \left(\frac{\|x\|_2 - \langle x, x_0 \rangle}{\|x\|_2}\right) \leq 2 \left(\frac{\sqrt{1 + \langle x, x_0 \rangle^2} - \langle x, x_0 \rangle}{\langle x, x_0 \rangle}\right) \leq \frac{2}{\langle x, x_0 \rangle^2} \leq \frac{8}{\mu^2}.$$

According to Lemma 6.9(2), every minimizer $\hat{x}$ of $(P_{\text{Cyl}, \mu K})$ that is contained in $\text{Cyl}(x_0, \mu)$ already satisfies the error bound of Theorem 2.14 and Theorem 2.16. Therefore, it suffices to show that every point $x \in \mu K$ that does not belong to $\text{Cyl}(x_0, \mu)$ cannot solve $(P_{\text{Cyl}, \mu K})$.

The next lemma forms the main technical component of our proof. It provides a uniform lower bound for the excess risk $\mathcal{E}(\cdot)$ on cylindrical intersections with $\partial \text{Cyl}(x_0, \mu)$.

**Lemma 6.10** Let $L \subset \mathbb{R}^n$ be a bounded subset. Moreover, we define

$$\tilde{K} := L \cap \partial \text{Cyl}(x_0, \mu) \quad \text{and} \quad \tilde{L} := \tilde{K} - \mu x_0 \subset t_0 B_2^n$$

where $t_0 := \max\{1, \text{rad}((L \cap \text{Cyl}(x_0, \mu)) - \mu x_0)\}$. There exist numerical constants $C_1, C_2 > 0$ such that for every $\eta \in (0, \frac{1}{2})$, the following holds true with probability at least $1 - \eta$: If $m \geq \log(\eta^{-1})$ and $\mu \geq 1$, then the excess risk satisfies

$$\mathcal{E}(x) \geq t_0 \cdot \left(\frac{1}{t_0 \mu} - C_1 \cdot \frac{\max\{0, \text{sign}({\theta}_x)\}}{\mu^2} - C_2 \cdot \frac{\frac{1}{t_0} w(\tilde{L}) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}}\right) \quad (6.17)$$

for every $x \in \tilde{K}$, where $\theta_x := \langle x - \mu x_0, x_0 \rangle$. 
Proof. Let us first analyze the multiplier term of the decomposition in (6.16). Analogously to the proof of Theorem 2.10 in Subsection 6.2, we apply Theorem 6.2 with $z = -y \cdot \chi_{(-\infty,0]}(y(a, \mu x_0))$ and $L = L \subset t_0 B_2^n$. Combined with the assumption $m \geq \log(\eta^{-1})$, Theorem 6.2 then implies that the following bound holds true with probability at least $1 - \frac{n}{m}$:

$$
\mathcal{M}(x, \mu x_0) \geq \mathbb{E}[\mathcal{M}(x, \mu x_0)] - C_2 \cdot \|z\|_{\psi_2} \cdot \frac{w(L) + t_0 \cdot \sqrt{\log(\eta^{-1})}}{\sqrt{m}}
$$

$$
= t_0 \cdot \left( \frac{1}{t_0} \mathbb{E}[\mathcal{M}(x, \mu x_0)] - C_2 \cdot \|z\|_{\psi_2} \cdot \frac{1}{t_0} w(L) + \sqrt{\log(\eta^{-1})} \right)
$$

for every $x \in \tilde{K}$. Since $y = \text{sign}(\langle a, x_0 \rangle)$, the expected multiplier term simplifies as follows (cf. (6.10)):

$$
\mathbb{E}[\mathcal{M}(x, \mu x_0)] = \mathbb{E}[\chi_{(-\infty,1]}(y(a, \mu x_0)) \cdot y(a, \mu x_0 - x)]
$$

$$
= \mathbb{E}[\chi_{(-\infty,1]}(y(a, \mu x_0)) \cdot y(a, \mu x_0 - \langle x, x_0 \rangle x_0 - P_{x_0} (x))]
$$

$$
= \langle x_0 - x, x_0 \rangle \cdot \mathbb{E}[\chi_{(-\infty,1]}(\|a, \mu x_0\|)) \cdot |\langle a, x_0 \rangle|]
$$

$$
= \langle x_0 - x, x_0 \rangle \cdot \frac{1}{\sqrt{\pi}} \cdot (1 - e^{-x_0^2})
$$

where we have again used the decomposition $x = \langle x, x_0 \rangle x_0 + P_{x_0} (x)$. Due to $\mu x_0 - x \in -L \subset t_0 B_2^n$, it holds that $\langle x - \mu x_0, x_0 \rangle \leq t_0 \cdot \max\{0, \sin(\theta_x)\}$, so that we finally end up with

$$
\mathcal{M}(x, \mu x_0) \geq -t_0 \cdot \left( \frac{\langle x - \mu x_0, x_0 \rangle}{t_0} \cdot \frac{1}{\sqrt{\pi}} \cdot (1 - e^{-x_0^2}) + C_2 \cdot \|z\|_{\psi_2} \cdot \frac{t_0}{t_0} w(L) + \sqrt{\log(\eta^{-1})} \right)
$$

$$
\geq -t_0 \cdot C_M \cdot \left( \frac{\max\{0, \sin(\theta_x)\}}{\mu^2} + \frac{1}{t_0} w(L) + \sqrt{\log(\eta^{-1})} \right)
$$

(6.18)

for an appropriate numerical constant $C_M > 0$.

Next, we derive a lower bound for the quadratic term in (6.16). For this purpose, we apply Mendelson’s small ball method similarly to the proof of Proposition 6.6: For $L = \tilde{L}$ and $u = \sqrt{2 \log(\frac{m}{2})^{-1}}$, Theorem 6.3 implies that

$$
\inf_{x \in \tilde{K}} Q(x, \mu x_0) \geq \inf_{h \in L} \frac{1}{m} \sum_{i=1}^{m} F(a_i, h) \geq \zeta \cdot \left( Q_{2\tilde{L}}(\hat{L}) - \frac{2}{m} w(\hat{L}) + \sqrt{2 \log(\frac{m}{2})^{-1}} \right)
$$

with probability at least $1 - \frac{n}{m}$, where the constant $\zeta > 0$ is specified later on.

Thus, it remains to find an appropriate lower bound for the small ball functional $Q_{2\tilde{L}}(\hat{L})$. Let $h = x - \mu x_0 \in \hat{L}$ and consider the orthogonal decomposition

$$
h = \langle h, x_0 \rangle x_0 + P_{x_0} (h)
$$

with $x' \in S^{n-1}$ and $e \in \mathbb{R}$. First, assume that $h \in \text{Cyl}_1(x_0, \mu) - \mu x_0$ and $\langle h, x_0 \rangle \geq 0$ (see also Lemma 6.9(3)). Then, $e = 1$ and therefore
By a simple union bound argument (similarly to the proof of Proposition 6.6), we conclude that

\[ P[F_+(a, h) \geq 2\zeta] = P[y(a, h) \geq 4\zeta, |\langle a, \mu x_0 \rangle| \in [1 - 2\zeta, 1]] \]
\[ = P\{h, x_0\} \cdot |\langle a, x_0 \rangle| + ey(a, x') \geq 4\zeta, |\langle a, x_0 \rangle| \in [\frac{1}{16}, 1]\} \]
\[ \geq P[y(a, x') \geq 4\zeta, |\langle a, x_0 \rangle| \in [\frac{1}{16}, 1, 1 - 2\zeta, 1]] \]

We now set \( \zeta := 1/16 \) and use the independence of \( \langle a, x' \rangle \) and \( \langle a, x_0 \rangle \) to obtain

\[ P[F_+(a, h) \geq 2\zeta] \geq P[y\langle a, x' \rangle \geq \frac{1}{4}, |\langle a, x_0 \rangle| \in [\frac{7}{8}, \frac{1}{4}]] \]
\[ = P[y\langle a, x' \rangle \geq \frac{1}{4}] \cdot P(|\langle a, x_0 \rangle| \in [\frac{7}{8}, \frac{1}{4}]] \]
\[ = P[g \geq \frac{1}{4}] \cdot P[|g| \in [\frac{7}{8}, \frac{1}{4}]) \geq \frac{1}{\mu}, \]

where \( g \sim \mathcal{N}(0, 1) \), and the last estimate is due to the assumption \( \mu \geq 1 \). Similarly, if \( h \in \text{Cyl}_1(x_0, \mu) - \mu x_0 \) and \( \langle h, x_0 \rangle \leq 0 \), we have

\[ P[F_-(a, h) \geq 2\zeta] = P[-y\langle a, h \rangle \geq 4\zeta, |\langle a, \mu x_0 \rangle| \in [1, 1 + 2\zeta]] \]
\[ = P[-y\langle h, x_0 \rangle \cdot |\langle a, x_0 \rangle| - ey\langle a, x' \rangle \geq 4\zeta, |\langle a, x_0 \rangle| \in [\frac{1}{4}, \frac{1}{4} + 2\zeta]] \]
\[ \geq P[-y\langle a, x' \rangle \geq \frac{1}{4}, |\langle a, x_0 \rangle| \in [\frac{1}{4}, \frac{1}{2}]] \]
\[ = P[g \geq \frac{1}{4}] \cdot P[|g| \in [\frac{1}{4}, \frac{1}{2})] \geq \frac{1}{\mu}, \ g \sim \mathcal{N}(0, 1). \]

Finally, we need to handle the base of \( \text{Cyl}_1(x_0, \mu) \): For every \( h \in \text{Cyl}_1(x_0, \mu) - \mu x_0 \), we have that \( \langle h, x_0 \rangle = -\frac{\mu}{2} \) and \( e \leq 1 \), which implies

\[ P[F_-(a, h) \geq 2\zeta] = P[-y\langle a, h \rangle \geq 4\zeta, |\langle a, \mu x_0 \rangle| \in [1, 1 + 2\zeta]] \]
\[ = P[y\langle a, x_0 \rangle | - ey\langle a, x' \rangle \geq 4\zeta, |\langle a, x_0 \rangle| \in [\frac{1}{2}, 1 + 2\zeta]] \]
\[ = P[y\langle a, x_0 \rangle | - ey\langle a, x' \rangle \geq \frac{1}{2}, |\langle a, x_0 \rangle| \in [\frac{1}{2}, \frac{1}{2}]] \]
\[ \geq P[|ey\langle a, x' \rangle| \leq \frac{1}{2}, \frac{1}{2} |\langle a, x_0 \rangle| \geq \frac{1}{2}, |\langle a, x_0 \rangle| \in [\frac{1}{2}, \frac{1}{2}]] \]
\[ = P[|ey\langle a, x' \rangle| \leq \frac{1}{2}] \cdot P[|\langle a, x_0 \rangle| \in [\frac{1}{2}, \frac{1}{2}]] \geq \frac{1}{\mu}, \ g \sim \mathcal{N}(0, 1). \]

By a simple union bound argument (similarly to the proof of Proposition 6.6), we conclude that

\[ Q_{1/8}(\bar{L}) \geq \frac{1}{\mu}. \]

Hence, with probability at least \( 1 - \frac{1}{2} \), the quadratic term satisfies the lower bound

\[ Q(x, \mu x_0) \geq \frac{C}{\mu} - C_Q \cdot \frac{w(\bar{L}) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \]
\[ \geq t_0 \cdot \left( \frac{C}{t_0 \mu} - C_Q \cdot \frac{\frac{1}{2} w(\bar{L}) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right) \tag{6.19} \]

for all \( x \in \bar{K} \) and appropriate numerical constants \( C, C_Q > 0 \); note that we have also used that \( \eta \leq \frac{1}{2} \) and \( t_0 \geq 1 \) here.
Combining our lower bounds from (6.18) and (6.19), the following holds true with probability at least $1 - \eta$ for all $x \in \hat{K}$:

$$
\mathcal{E}(x) = \mathcal{M}(x, \mu x_0) + \mathcal{Q}(x, \mu x_0) \\
\geq t_0 \cdot \left( \frac{C}{t_0 \mu} - C_M \cdot \frac{\max\{0, \text{sign}(\beta_x)\}}{\mu^2} - (C_M + C_G) \cdot \frac{\frac{1}{t_0} w(\tilde{L}) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right),
$$

which is the claim of Lemma 6.10.

We are now ready to prove Theorem 2.14. For this purpose, we will apply Lemma 6.10 to different subsets of $\mu K$ and show that the excess risk is positive on their respective cylindrical intersections with $\partial \text{Cyl}(x_0, \mu)$. Applying Fact 6.1 separately to each of these subsets then yields the desired error bound.

**Proof of Theorem 2.14. Part 1:** We first apply Lemma 6.10 to

$$
L^+ := L := \{ x = \mu x_0 + h \in \mu K \mid 0 \leq \langle x_0, \frac{h}{\|h\|_2} \rangle \leq \sqrt{1 - \frac{1}{D^2 \mu^2}} \},
$$

where $D > 0$ is a numerical constant that is specified later on. Note that this set is well-defined due to the assumption $\mu \geq 1$, or more precisely, $\mu \geq D^{-1}$. Let us now estimate the radius $t_0 = \max\{1, \text{rad}(L^+ \cap \text{Cyl}(x_0, \mu)) - \mu x_0 \}$. To this end, let $h \in (L^+ \cap \text{Cyl}(x_0, \mu)) - \mu x_0$. Since $x = \mu x_0 + h \in \text{Cyl}(x_0, \mu)$, we particularly have

$$
\|h\|^2 = \|\mu x_0 - x\|^2 = \|\mu - \langle x, x_0 \rangle \| x_0 - \langle x, x_0 \rangle x_0 \|^2 = \|\langle h, x_0 \rangle \|^2 + \|x - \langle x, x_0 \rangle x_0\|^2 \leq \|\langle h, x_0 \rangle \|^2 + 1. \quad (6.20)
$$

And by the definition of $L^+$, it holds that

$$
\|h\|^2 \leq \|\langle h, x_0 \rangle \|^2 + 1 \leq \|h\|^2 \cdot \left(1 - \frac{1}{D^2 \mu^2}\right) + 1 \\
\Rightarrow \|h\|^2 \leq D^2 \mu^2 \Rightarrow \|h\| \leq D \mu,
$$

which implies $t_0 \leq D \mu$. From (6.17), we can conclude that, with probability at least $1 - \frac{\eta}{2}$, the excess risk satisfies

$$
\mathcal{E}(x) \geq t_0 \cdot \left( \frac{1}{t_0 \mu} - \frac{C_1}{\mu^2} - C_2 \cdot \frac{\frac{1}{t_0} w(\tilde{L}) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right) \\
\geq t_0 \cdot \left( \frac{1}{D \mu^2} - \frac{C_1}{\mu^2} - C_2 \cdot \frac{\frac{1}{t_0} w(\tilde{L}) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right) \quad (6.21)
$$

for all $x \in \hat{K} = L^+ \cap \partial \text{Cyl}(x_0, \mu)$. Adjusting $D$ (depending on $C_1$) and observing that

$$
\frac{1}{t_0} w(\tilde{L}) \leq \frac{1}{t_0} w((L^+ - \mu x_0) \cap t_0 B_2^n) \\
\leq w\left(\frac{1}{t_0}(\mu K - \mu x_0) \cap B_2^n\right) \quad (2.5) \leq \sqrt{\frac{d_0(K - x_0)}{m}},
$$

the assumption (2.17) implies that $\mathcal{E}(x) > 0$ for all $x \in \hat{K}$. In conclusion, Fact 6.1 states that every minimizer $\hat{x}$ of $(P_{\text{LOG}, \mu K})$ that belongs to $L^+$ must be also contained in $\text{Cyl}(x_0, \mu)$. Lemma 6.9(2) finally yields

$$
\left\| x_0 - \frac{\hat{x}}{\|\hat{x}\|_2} \right\|_2 \leq \frac{1}{\mu}.
$$
Part 2: Analogously to Part 1, we now invoke Lemma 6.10 with
\[ L^- := L := \{ x = \mu x_0 + h \in \mu K \mid \langle x_0, \frac{h}{\|h\|_2} \rangle \leq 0 \}. \]

For \( h \in (L^- \cap \text{Cyl}(x_0, \mu)) - \mu x_0 \), the definition of \( \text{Cyl}(x_0, \mu) \) implies that \( -\frac{\mu}{2} \leq \langle h, x_0 \rangle \). Hence,
\[ \|h\|_2^2 \leq \langle h, x_0 \rangle^2 + 1 \leq \frac{\mu^2}{4} + 1 \lesssim \mu^2, \]
and therefore \( t_0 = \max\{1, \text{rad}((L^- \cap \text{Cyl}(x_0, \mu)) - \mu x_0)\} \lesssim \mu \). By (6.17) and (6.22) again, we obtain
\[ \mathcal{E}(x) \gtrsim t_0 \cdot \left( \frac{1}{t_0 \mu} - C_2 \cdot \frac{\frac{1}{t_0} w(L) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right) \]
\[ \gtrsim t_0 \cdot \left( \frac{1}{\mu^2} - C_2 \cdot \frac{\sqrt{d_0(K - x_0)} + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right)^{(2.17)} > 0 \]
for all \( x \in \bar{K} = L^- \cap \partial \text{Cyl}(x_0, \mu) \) with probability at least \( 1 - \frac{\mu}{2} \). As in the first part, we can conclude that every minimizer \( \hat{x} \) of \( (P_{L^2 \mu, \mu K}) \) that belongs to \( L^- \) satisfies
\[ \|x_0 - \|x\|_2\| \lesssim \frac{1}{\mu}. \]

Part 3: It remains to deal with those vectors \( x = \mu x_0 + h \in \mu K \) with \( \langle x_0, \frac{h}{\|h\|_2} \rangle > \sqrt{1 - \frac{1}{D^2 \mu^2}} \). In fact, such points satisfy the desired error bound (2.18), no matter if they solve \( (P_{L^2 \mu, \mu K}) \) or not. To verify this claim, observe that
\[ \|x_0 - \|x\|_2\| \lesssim \lim_{\|h\|_2 \to \infty} \left[ 2 \left( 1 - \frac{\mu + \|h\|_2 \langle x_0, v \rangle}{\sqrt{\mu^2 + 2 \mu \|h\|_2 \langle x_0, v \rangle + \|h\|_2^2}} \right) \right]^{1/2}, \]
where \( v := h/\|h\|_2 \in S^{n-1} \). It is not hard to see that the expression on the right-hand side is monotonically increasing in \( \|h\|_2 \). Thus, by taking the limit \( \|h\|_2 \to \infty \), we obtain
\[ \|x_0 - \|x\|_2\| \leq \sqrt{2(1 - \langle x_0, v \rangle)} \leq \sqrt{2(1 - \langle x_0, v \rangle^2)} < \frac{\sqrt{2}}{D \mu} \lesssim \frac{1}{\mu}. \]

This proves the claim under the hypothesis of (2.17).

The same argument works out if (2.19) is satisfied instead. Indeed, the only estimate that needs to be changed in Part 1 is (6.21):
\[ \mathcal{E}(x) \gtrsim t_0 \cdot \left( \frac{1}{t_0 \mu} - C_1 \mu^2 - C_2 \cdot \frac{\frac{1}{t_0} w(L) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \right) \]
\[ = \frac{1}{\mu} - C_1 t_0 \mu^2 - C_2 \cdot \frac{w(L) + t_0 \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \]
\[ \gtrsim \frac{1}{\mu} - C_1 D - C_2 \cdot \frac{w(\mu(K - x_0)) + D \mu \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \]
\[ \gtrsim \frac{1}{\mu} - C_2 \cdot \frac{w(K) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}}. \]
Since the same lower bound can be achieved in Part 2, the positivity of $E(\cdot)$ is a consequence of (2.19) in both cases.

\begin{remark}
A key step of the above proof is to control the radii $t_0$ of the respective cylindrical intersections. While this strategy works out nicely in Part 1 and Part 2 with $t_0 \lesssim \mu$, the remaining vectors of Part 3 need to be treated very differently. In fact, the statistical argument of Lemma 6.10 would completely fail in this case. But fortunately, every vector $x = mx_0 + h \in \mu K$ that satisfies $\langle x_0, \frac{h}{\|h\|_2} \rangle > \sqrt{1 - \frac{1}{D^2 \mu^2}}$ already lies in a “narrow” conic segment around $\text{span}(x_0)$, whose projection onto the sphere is sufficiently close to $x_0$. But one should be aware of the fact that these $x$ do not necessarily belong to $\text{Cyl}(x_0, \mu)$.
\end{remark}

\hfill \Box

The proof of Theorem 2.16 is slightly more involved. A careful study of the proof of Theorem 2.14 reveals that the set $(L^c \cap \text{Cyl}(x_0, \mu)) - mx_0$ considered in Part 2 is simply too large and its radius $t_0$ may scale in the order of $\mu$. In order to ensure positivity of the excess risk, we therefore have to accept a suboptimal factor of $\mu^4$ in (2.17). However, this drawback can be avoided by showing that any minimizer of $(P_{\text{theo}, \mu K})$ actually lies on the boundary of $\mu K$ with high probability. For this purpose, we first verify that the empirical risk $x \mapsto \mathcal{R}(x)$ does not vanish in a neighborhood of $mx_0$, i.e., $mx_0$ is not too far behind the classification margin (cf. Subsection 2.3.3). This is precisely what is claimed by the following lemma, which is based on a standard concentration argument:

\begin{lemma}
Let $L \subset tB_2^\mu$ be a subset and assume that $\mu \gtrsim \max \{1, t\}$. For every $\eta \in (0, \frac{1}{2})$, the following holds true with probability at least $1 - \eta$: Supposed that
\begin{equation}
m \gtrsim \mu^2 \cdot \max \{w(L)^2, \log(\eta^{-1})\},
\end{equation}
the empirical risk is positive on $L + mx_0$, i.e.,
\begin{equation}
\mathcal{R}(mx_0 + h) > 0 \quad \text{for all } h \in L.
\end{equation}
\end{lemma}

\begin{proof}
The monotony of the hinge loss function yields the following lower bound for the empirical risk function:
\begin{equation}
\mathcal{R}(mx_0 + h) = \frac{m}{m} \sum_{i=1}^{m} \left[ 1 - y_i \langle a_i, mx_0 + h \rangle \right]_+ \gtrsim \frac{m}{m} \sum_{i=1}^{m} \left[ 1 - \left| \langle a_i, mx_0 + h \rangle \right| \right]_+ \quad =: Z_i(h)
\end{equation}
for all $h \in \mathbb{R}^n$. Let us establish a concentration inequality for the empirical process on the right-hand side of this estimate. Since $Z_i(\cdot) \in [0, 1]$, the bounded difference inequality [BLM13, Thm. 6.2] implies that
\begin{equation}
\sup_{h \in L} \frac{1}{m} \sum_{i=1}^{m} \left( \mathbb{E}[Z_i(h)] - Z_i(h) \right) \leq \mathbb{E} \left[ \sup_{h \in L} \frac{1}{m} \sum_{i=1}^{m} \left( \mathbb{E}[Z_i(h)] - Z_i(h) \right) \right] + \frac{\sqrt{2 \log(\eta^{-1})}}{\sqrt{m}}
\end{equation}
with probability at least $1 - \eta$.

In order to bound $E$ from above, we first apply a standard symmetrization argument (cf. [LT91, Pf. of Lem. 6.3]):
\begin{equation}
E = \mathbb{E} \left[ \sup_{h \in L} \frac{1}{m} \sum_{i=1}^{m} \left( 1 - Z_i(h) - \mathbb{E}[1 - Z_i(h)] \right) \right] \leq 2\mathbb{E} \left[ \sup_{h \in L} \frac{1}{m} \sum_{i=1}^{m} e_i(1 - Z_i(h)) \right],
\end{equation}
where $e_i$ are independent Rademacher variables. Next, we define $\psi(v) := 1 - [1 - [v]]_+$ for $v \in \mathbb{R}$ and observe that $\psi(\langle a_i, mx_0 + h \rangle) = 1 - [1 - \langle a_i, mx_0 + h \rangle]_+ = 1 - Z_i(h)$. Since $\psi$ is a contraction that fixes
\begin{equation}
\end{equation}
the origin, the contraction principle [LT91, Eq. (4.20)] finally leads to
\[ E \leq 2 \mathbb{E}\left[ \sup_{h \in L} \frac{1}{m} \sum_{i=1}^{m} c_i \langle a_i, \mu x_0 + h \rangle \right] = 2 \mathbb{E}\left[ \sup_{h \in L} \frac{1}{\sqrt{m}} (a, \mu x_0 + h) \right] = \frac{2}{\sqrt{m}} \mathbb{E}\left[ \sup_{h \in L} \langle g, h \rangle \right] = \frac{2w(L)}{\sqrt{m}}, \]
where \( g \sim \mathcal{N}(0, I_n) \). Therefore, under the hypothesis of (6.24), we have
\[ \hat{\mathcal{R}}(\mu x_0 + h) \geq \frac{1}{m} \sum_{i=1}^{m} Z_i(h) \succcurlyeq \mathbb{E}[Z_1(h)] - \frac{w(L) + \sqrt{\log(\eta^{-1})}}{\sqrt{m}} \]
for all \( h \in L \). To conclude the proof, it remains to estimate \( \mathbb{E}[Z_1(h)] \) from below:
\[ \mathbb{E}[Z_1(h)] = \mathbb{E}\left[ |\langle a, \mu x_0 + h \rangle| + \right] \succcurlyeq \|\mu x_0 + h\|_2 \geq \frac{1}{\mu + t} \succcurlyeq \frac{1}{\mu}, \]
where we have also used that \( \mu \succcurlyeq \max\{1, t\} \). The claim now follows from (6.25) and the assumption of (6.23).

We are now ready to prove Theorem 2.16.

**Proof of Theorem 2.16.** **Part 1:** For \( x \in \mathbb{R}^n \), we denote by \( x^\dagger \) the ray that starts at \( x \) and proceeds in the direction of \( x_0 \) (parallel to span \( \{x_0\} \) ), i.e.,
\[ x^\dagger := \{ x + \tau x_0 \mid \tau \geq 0 \} \subset \mathbb{R}^n. \]
If \( x \in \mu K \), then \( x^\dagger \) intersects the boundary of \( \mu K \) in a point \( \partial_0 x := x + \tau_0 x_0 \in x^\dagger \cap \partial(\mu K) \) with\(^1\)
\[ \tau_0 = \sup\{ \tau \geq 0 \mid x + \tau x_0 \in \mu K \}. \]

Since we consider perfect 1-bit observations, i.e., \( y_i = \text{sign}(\langle a_i, x_0 \rangle) \), it is not hard to see that for every \( x \in \mathbb{R}^n \), there exists \( x^\dagger \in x^\dagger \cap (\mu K)^c \) such that \( \hat{\mathcal{R}}(x^\dagger) = 0 \); note that the choice of \( x^\dagger \) is highly non-unique and may strongly depend on the measurement vectors \( a_1, \ldots, a_m \). Hence, similarly to the assertion of Fact 6.1, the convexity of \( \hat{\mathcal{R}}(\cdot) \) and \( \mu K \) implies that every minimizer \( \hat{x} \) of \( \mathcal{P}_{\text{Cyl}(x_0, \mu)} \) satisfies the following property:

**Fact 6.13** If \( \hat{x} \neq \partial_0 \hat{x} \), then we have \( \hat{\mathcal{R}}(\hat{x}^\dagger) = \{0\} \). Or equivalently, \( \hat{\mathcal{R}}(\hat{x}) > 0 \) implies that \( \hat{x} = \partial_0 \hat{x} \).

A particular consequence of Fact 6.13 is that every point in \( \hat{x}^\dagger \cap \mu K \) is also a minimizer of \( \mathcal{P}_{\text{Cyl}(x_0, \mu)} \). The proof strategy of the following two parts is visualized in Figure 6.

**Part 2:** Let us assume that \( \hat{x} \) is a minimizer of \( \mathcal{P}_{\text{Cyl}(x_0, \mu)} \) such that \( \partial_0 \hat{x} \in \text{Cyl}(x_0, \mu) \). If we could show that \( \hat{x} = \partial_0 \hat{x} \), the desired error bound would follow again from Lemma 6.9(2). Towards a contraction, let us therefore assume that \( \hat{x} \neq \partial_0 \hat{x} \). Then Fact 6.13 yields \( \hat{\mathcal{R}}(\partial_0 \hat{x}) = 0 \). In order to show that this event does not occur with high probability, we apply Lemma 6.12 with
\[ L = (\partial(\mu K) \cap \text{Cyl}(x_0, \mu)) - \mu x_0 \subset t B_2^n, \]
where \( t = t_0 = \max\{1, \text{rad}(L)\} \); see also (2.21). Note that the condition (6.23) is fulfilled by (2.23) and
\[ \max\{w(L)^2, \log(\eta^{-1})\} \overset{t_0 \geq 1}{\lesssim} t_0^2 \cdot \max\{\frac{1}{t_0}w(L)^2, \log(\eta^{-1})\} \]
\[ \overset{(2.5)}{\lesssim} \overset{(2.5)}{\lesssim} \max\{d_0(K - x_0), \log(\eta^{-1})\}. \]

\(^1\)While the intersection of \( x^\dagger \) and \( \partial(\mu K) \) is not necessarily a single point, the boundary point \( \partial_0 x \) is uniquely defined due to the definition of \( \tau_0 \).
Remark 6.14  Carefully reviewing the estimates on the Gaussian width in the above proofs indicates the convenient and well-known notion of conic effective dimension. Examples of interest in Subsection 3.1. Therefore, we have decided to state our main results by means of (maybe even optimal) sampling rates, it is by far not clear how to compute these quantities for our independence from \( g \) in Proposition 3.5.

Proof of Proposition 3.5. Let us first consider the additive Gaussian noise model introduced in Example 3.4(1), i.e., \( f(v) = f_e(v) = \text{sign}(v + \tau) \) with \( \tau \sim \mathcal{N}(0, \sigma^2) \). Using the symmetry of \( \tau \) and the independence from \( g = \langle a, x_0 \rangle \sim \mathcal{N}(0, 1) \), we observe that
Consequently, which implies that \( f \) which leads to

\[
\text{In this way, we can compute the correlation parameter:}
\]

\[
\lambda_{f_o} = \mathbb{E}[\text{sign}(g + \tau)|g] = \mathbb{E}[|g| \cdot (1_{\text{sign}(g) = \text{sign}(g+\tau)} - 1_{\text{sign}(g) \neq \text{sign}(g+\tau)})]
\]

\[
= \mathbb{E}[|g| \cdot \mathbb{E}_\tau[1_{|\tau| \leq |g|}]]
\]

\[(6.26)\]
\begin{align*}
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty x \cdot \left( \frac{2}{\sqrt{2\pi}\sigma^2} \int_0^x e^{-y^2/(2\sigma^2)} \, dy \right) \cdot e^{-x^2/2} \, dx \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty x \cdot \text{erf}\left(\frac{x}{\sqrt{2\sigma}}\right) \cdot e^{-x^2/2} \, dx,
\end{align*}

where \( \text{erf}(\cdot) \) denotes the error function. From the asymptotic equivalence
\[
\text{erf}\left(\frac{x}{\sqrt{2\sigma}}\right) \approx \min\{\frac{x}{\sigma}, 1\}, \quad x \geq 0,
\]
we can conclude that
\[
\lambda_{f_r} \approx \int_0^\infty x \cdot \min\{\frac{x}{\sigma}, 1\} \cdot e^{-x^2/2} \, dx.
\]

This immediately implies the desired upper bound:
\[
\lambda_{f_r} \lesssim \min\left\{ \frac{1}{\sigma} \cdot \int_0^\infty x^2 e^{-x^2/2} \, dx, \int_0^\infty xe^{-x^2/2} \, dx \right\} \lesssim \min\{\frac{1}{\sigma}, 1\} \approx \frac{1}{1+\sigma}.
\]

In order to see that this bound is tight, we finally make a case distinction in \( \sigma \):
\[
\begin{align*}
\sigma \geq 1 & \quad \Rightarrow \quad \min\{\frac{x}{\sigma}, 1\} \geq \frac{1}{\sigma} \cdot \min\{x, 1\} \quad \Rightarrow \quad \lambda_{f_r} \gtrsim \frac{1}{\sigma}, \\
\sigma < 1 & \quad \Rightarrow \quad \min\{\frac{x}{\sigma}, 1\} \geq \min\{x, 1\} \quad \Rightarrow \quad \lambda_{f_r} \gtrsim 1,
\end{align*}
\]
so that \( \lambda_{f_r} \gtrsim \min\{\frac{1}{\sigma}, 1\} \approx \frac{1}{1+\sigma} \) for every \( \sigma > 0 \). In particular, the correlation condition of (C1) is fulfilled.

Let us now consider the bit flip model from Example 3.4(2), i.e., \( f(v) = f_p(v) = \varepsilon \cdot \text{sign}(v) \) where \( \varepsilon \in \{-1, +1\} \) is a Bernoulli random variable with \( \mathbb{P}[\varepsilon = 1] = p > \frac{1}{2} \). The condition (C2) directly follows from the independence of \( \varepsilon \) and \( g \):
\[
\mathbb{E}[f(g) \cdot \text{sign}(g) \mid |g|] = \mathbb{E}[\varepsilon \mid |g|] = \mathbb{E}[\varepsilon] = p - (1 - p) = 2p - 1 > 0 \quad (a.s.).
\]

Moreover, we observe that
\[
\lambda_{f_p} = \mathbb{E}[f_p(g) | g] = \mathbb{E}[\varepsilon \cdot |g|] = \mathbb{E}[\varepsilon] \cdot \mathbb{E}[|g|] = (2p - 1) \sqrt{\frac{2}{\pi}} > 0,
\]
which shows that the condition (C1) is satisfied as well.

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