Generalized curvature and the equations of $D = 11$ supergravity

Igor A. Bandos$^\dagger$*, José A. de Azcárraga$^\dagger$, Moisés Picón$^{\dagger,1}$ and Oscar Varela$^{\dagger,2}$

$^\dagger$Departamento de Física Teórica, Univ. de Valencia and IFIC (CSIC-UVEG),
46100-Burjassot (Valencia), Spain

*Institute for Theoretical Physics, NSC “Kharkov Institute of Physics and Technology”,
UA61108, Kharkov, Ukraine

$^1$Department of Physics and Astronomy, University of Southern California, Los Angeles,
CA 90089-2535, USA

$^2$Michigan Center for Theoretical Physics, Randall Laboratory, Department of Physics,
University of Michigan, Ann Arbor, MI 48109-1120, USA

Abstract

It is known that, for zero fermionic sector, $\psi^\alpha_\mu(x) = 0$, the bosonic equations of Cremmer–Julia–Scherk eleven–dimensional supergravity can be collected in a compact expression, $\mathcal{R}_{\alpha\beta} \Gamma^{\beta} = 0$, which is a condition on the curvature $\mathcal{R}_{\alpha\beta}$ of the generalized connection $w$. In this letter we show that the equation $\mathcal{R}_{\beta\alpha} \Gamma^{\alpha\beta} = 4i((\bar{D}\psi)_{\alpha} \Gamma^{\beta\alpha})$, where $\bar{D}$ is the covariant derivative for the generalized connection $w$, collects all the bosonic equations of $D = 11$ supergravity when the gravitino is nonvanishing, $\psi^\alpha_\mu(x) \neq 0$. 
1 Introduction

Recently, the notion of generalized connection and generalized holonomy has been applied to the analysis of supersymmetric solutions of $D = 10, 11$ dimensional supergravity \cite{1, 2, 3, 4, 5, 6, 7}. The generalized connection (see \cite{8})

$$w_\beta^\alpha := \omega L_\beta^\alpha + t_1^\beta^\alpha = \frac{1}{4} \omega^{ab} \Gamma_{ab}^\beta^\alpha + t_1^\beta^\alpha$$

includes, in addition to the true Lorentz (or spin) connection, the Lorentz covariant part

$$t_1^\beta^\alpha = \frac{i}{8} E^a \left( F_{a [3]} \Gamma^{[3]}_{\beta^\alpha} + \frac{1}{8} F^{[4]} \Gamma_{a [4]}^\beta^\alpha \right),$$

constructed from the tensor $F_{abcd}$, the ‘supersymmetric’ field strength of the antisymmetric tensor field $A_{\mu \nu \rho}(x)$ (see Eqs. (19)). In Eq. (2) $F_{a [3]} \Gamma^{[3]}_{\beta^\alpha} = F_{b_1 b_2 b_3} \Gamma_{a b_1 b_2 b_3}$, $F^{[4]} \Gamma_{a [4]}^\beta^\alpha = F_{b_1 \cdots b_4} \Gamma_{a b_1 \cdots b_4}$ and we have denoted the vielbein one–form $dx^\mu e_\mu(x)$ by $E^a, E^a = dx^\mu e_\mu^a(x)$.

The generalized connection allows for a simple expression of the supersymmetric transformation rules for the gravitino (hence the name of ‘supersymmetric’ connection frequently used). Denoting the gravitino one–form by $\psi^\alpha = dx^\alpha \psi^\alpha(x)$, this variation is given by

$$\delta \varepsilon \psi^\alpha = \mathcal{D} \varepsilon^\alpha(x) := D \varepsilon^\alpha(x) - \varepsilon^\beta(x) t_1^\beta^\alpha(x) = \frac{i}{8} E^a \left( F_{a [3]} \Gamma^{[3]}_{\beta^\alpha} + \frac{1}{8} F^{[4]} \Gamma_{a [4]}^\beta^\alpha \right).$$

It was already noticed in \cite{8} that the gravitino equation of motion has also a compact form (see Eq. (25)) in terms of its generalized covariant (or ‘supercovariant’) derivative

$$\hat{D} \psi^\alpha := d \psi^\alpha - \psi^\beta \wedge w_\beta^\alpha \equiv D \psi^\alpha - \psi^\beta \wedge t_1^\beta^\alpha$$

defined for the generalized connection (1). Then the following observation (see \cite{4, 7}) holds: when the fermionic sector is set to zero, all the bosonic equations of the Cremmer–Julia–Scherk (CJS) eleven–dimensional supergravity can be collected in the simple expression

$$N_{a \beta}^\gamma := R_{ab}^c \Gamma^b_{\gamma^\alpha} = 0$$

or, equivalently, $i_b R_{a}^\gamma \Gamma^b_{\gamma^\alpha} \equiv E^a R_{ba}^\gamma \Gamma^b_{\gamma^\alpha} = 0$, in terms of the generalized curvature $R$ (see, e.g. \cite{3})

$$R_{a \beta}^\alpha := dw_\beta^\alpha - w_\beta^\gamma \wedge w_\gamma^\alpha$$

$$= \frac{1}{4} R^{ab} (\Gamma_{ab})_\alpha^\beta + Dt_1^\alpha \gamma \wedge t_1^\gamma \beta,$$

which takes values in the Lie algebra of the generalized holonomy (holonomy of the generalized connection) group $\mathfrak{h}_1^1$. A similarly concise equation in the case of the purely bosonic limit of massive type IIA supergravity was given recently in \cite{13}.

We present here the generalization of Eq. (5), to the case of nonzero gravitino, $\psi^\alpha \neq 0$. It reads

$$R_{b \alpha}^\gamma \wedge E_{abc} \Gamma_{\gamma^\alpha} = -i \hat{D} \psi^\delta \wedge \psi^\gamma \wedge E_{a_1 \cdots a_4}^{\delta \alpha} \Gamma_{\alpha}^{a_1 a_2 a_3 a_4} \Gamma_{\beta^\gamma}^\delta$$

$$\Rightarrow R_{b \alpha}^\gamma \Gamma_{\gamma^\alpha} = 4i((\hat{D} \psi)^b \psi^a \Gamma_{\gamma^\alpha})_\beta \Gamma_{\beta^\gamma}^\delta (\psi^d \Gamma^d)_{\alpha},$$

\footnote{See \cite{3} for further discussion on the generalized holonomy.}
where $\tilde{D}\psi^\alpha = 1/2E^a \wedge E^b(\tilde{D}\psi)_{ba}^\alpha$ is defined in [14] and

$$E^{\wedge(11-k)}_{a_1...a_k} := \frac{1}{(11-k)!}\varepsilon_{a_1...a_k b_1...b_{11-k}}E^{b_1} \wedge \ldots \wedge E^{b_{11-k}}.$$  

Eq. 7 (or 5) collects all the bosonic equations when the gravitino is not zero, $\psi^\alpha \neq 0$.

Although the final result is formulated as a statement about dynamical equations of motion and, in this sense, refers to the second order approach to supergravity, we find it convenient to use the first order supergravity action of [10, 11]. Our notation (which is explained in the text) is close to that in [11] and the same of [7, 12].

## 2 First order action for $D = 11$ supergravity

The first order action for $D = 11$ supergravity [10, 11],

$$S = \int_{M^{11}} L_{11}[E^a, \psi^\alpha, \omega^{ab}, A_3, F_{a_1 a_2 a_3 a_4}],$$

is the integral over eleven-dimensional spacetime $M^{11}$ of the eleven form $L_{11}$ which can be written as [10, 11]

$$L_{11} = \frac{1}{4} R^{ab} \wedge E^{\wedge 9}_{ab} - D\psi^\alpha \wedge \psi^\beta \wedge \Gamma^{(8)}_{\alpha\beta} + \frac{1}{4} \psi^\alpha \wedge \psi^\beta \wedge (T^a + i/2 \psi \wedge \psi \Gamma^a) \wedge E_a \wedge \Gamma^{(6)}_{\alpha\beta} +$$

$$+ \ (dA_3 - a_4) \wedge (*F_4 + b_7) + \frac{1}{2} a_4 \wedge b_7 - \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{3} A_3 \wedge dA_3 \wedge dA_3.$$  

(11)

Following [11] (see also [12]), we have introduced the notation

$$a_4 := \frac{1}{2} \psi^\alpha \wedge \psi^\beta \wedge \Gamma^{(2)}_{\alpha\beta}, \quad b_7 := \frac{i}{2} \psi^\alpha \wedge \psi^\beta \wedge \Gamma^{(5)}_{\alpha\beta}$$

(12)

for the bifermionic 4- and 7-forms and

$$F_4 := \frac{1}{3!} E^{a_4} \wedge \ldots \wedge E^{a_1}F_{a_1...a_4},$$

$$*F_4 := -\frac{1}{3!} E^{\wedge 7}_{a_1...a_4}F_{a_1...a_4}.$$  

(13)

for the purely bosonic forms constructed from the antisymmetric tensor zero-form $F_{abcd}$. We also use the compact notation

$$\tilde{\Gamma}^{(k)}_{\alpha\beta} := \frac{1}{k!} E^{a_k} \wedge \ldots \wedge E^{a_1} \Gamma_{a_1...a_k a_\alpha a_\beta}$$

(14)

and Eq. 3 [to be compared with the notation of [11], $E^{\wedge(11-k)}_{a_1...a_k} = \Sigma_{a_1...a_k}, \tilde{\Gamma}^{(k)}_{\alpha\beta} = (-)^{(k-1)/2}(\Sigma^{(k)})_{\alpha\beta}$].

The action (11) is invariant under the local supersymmetry transformations $\delta_\varepsilon$, which are given by

$$\delta_\varepsilon E^a = -2i\psi^\alpha \Gamma_{a\alpha\beta}^{\beta} \varepsilon^\beta,$$

$$\delta_\varepsilon \psi^\alpha = D\varepsilon^\alpha(x) = D\varepsilon^\alpha(x) - \varepsilon^\beta(x)t_1\varepsilon^\alpha(x),$$

$$\delta_\varepsilon A_3 = \psi^\alpha \wedge \Gamma^{(2)}_{\alpha\beta} \varepsilon^\beta,$$

(15)

(16)

(17)

plus more complicated expressions for $\delta_\varepsilon \omega^{ab}$ and $\delta_\varepsilon F_{abcd}$, which can be found in [11] and that will not be needed below. Let us stress that, as shown in [11], the supersymmetry transformation rules of the physical fields are the same in the second and in the first order formalisms.
3 Equations of motion

In the first order action (10) one distinguishes between the true equations of motion and the algebraic (or nondynamical) equations

\[ T^a = -i \psi^\alpha \wedge \psi^\beta \Gamma^a_{\alpha\beta} , \quad (18) \]

\[ dA_3 = a_4 + F_4 \quad (19) \]

(see Eqs. (12), (13) for the notation) which follow, respectively, from the variation with respect to the spin connection \( \omega^a_{\alpha\beta} \) and the antisymmetric tensor \( F_{abcd} \)

\[ \delta_\omega L_{11} = \frac{1}{4} E^{abc} \wedge (T^a + i \psi^\alpha \wedge \psi^\beta \Gamma^a_{\alpha\beta}) \wedge \delta \omega^{bc} + d(\ldots) , \]

\[ \delta_F L_{11} = -\frac{1}{4!} (dA_3 - a_4 - F_4) \wedge E^a_{a_1\ldots a_4} \delta F^{a_1\ldots a_4} . \quad (20) \]

Notice that Eqs. (18), (19) are the counterparts of the super space constraints of \( D = 11 \) supergravity (see [12] for a discussion and references), but for forms on eleven–dimensional spacetime. As far as the dynamical bosonic equations are concerned, one finds that the variation with respect to \( A_3 \),

\[ \delta_A L_{11} = d(*F_4 + b_7 - A_3 \wedge dA_3) \wedge \delta A_3 + d(\ldots) \quad (21) \]

results in

\[ G_8 := d(*F_4 + b_7 - A_3 \wedge dA_3) = 0 , \quad (22) \]

which becomes the standard CJS three–form gauge field equations of motion once the algebraic equations (constraints) (18), (19) are taken into account. The more complicated variation \( \delta_E \) with respect to the vielbein form, \( \delta_E L_{11} = \ldots \), as well as the full expression of the Einstein equations

\[ M_{10a} := \frac{1}{4} R^{bc} \wedge E^a_{abc} + \ldots = 0 , \quad (23) \]

which follows from that variation, will not be needed here (see [11]).

After some algebra, the fermionic variation \( \delta_\psi \) of the Lagrangian form \( L_{11} \), Eq. (11), reads (cf. [11])

\[ \delta_\psi L_{11} = -2 \hat{D} \psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} \wedge \delta \psi^\beta + i (dA_3 - a_4 - F_4) \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} \wedge \psi^\alpha \wedge \delta \psi^\beta + \\
+ \left( i a \bar{\Gamma}^{(8)}_{\alpha\beta} + 1/2 E_a \wedge \bar{\Gamma}^{(6)}_{\alpha\beta} \right) \wedge (T^a + i \psi^\alpha \wedge \psi^\beta \Gamma^a_{\alpha\beta}) \wedge \psi^\alpha \wedge \delta \psi^\beta - \\
- d \left[ \psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} \wedge \delta \psi^\beta \right] , \quad (24) \]

where \( \hat{D} \psi^\alpha \) is given by Eq. (1). Taking into account the algebraic equations (18), (19), and ignoring the (last) total derivative term in Eq. (24) one finds the gravitino equation of [8] written, as in [11], in the suggestive differential form

\[ \Psi_{10\beta} := \hat{D} \psi^\alpha \wedge \bar{\Gamma}^{(8)}_{\alpha\beta} = 0 . \quad (25) \]
4 Bosonic equations of $D = 11$ supergravity as a condition on the generalized curvature

4.1 A concise form of the bosonic equations from self-consistency of the gravitino equations

It is important that the above gravitino equation, $\Psi_{10,\beta} = 0$, is expressed in terms of the covariant derivative $\hat{D}$, Eqs. (1), (1), (2). As a result, the integrability/selfconsistency condition for Eq. (25) may be written in terms of the generalized curvature (6). Using $\hat{D}\hat{D}\psi = -\psi^\beta \wedge R^\alpha_\beta$ and $t_{1}[\gamma \wedge \bar{\Gamma}^{(8)}_{a1...a4}]_{\beta\gamma} = 0$ which implies $\hat{D}\bar{\Gamma}^{(8)}_{\beta\alpha} = D\bar{\Gamma}^{(8)}_{\beta\alpha} = T^a \wedge \bar{i}^a \bar{\Gamma}^{(8)}_{\beta\alpha}$, we obtain

$$\hat{D}\Psi_{10,\alpha} = \hat{D}\psi^\beta \wedge \left(T^a + i\psi \wedge \psi^\Gamma^a\right) \wedge \bar{i}^a \bar{\Gamma}^{(8)}_{\beta\alpha} - \frac{i}{6}\psi^\beta \wedge \left[R^\gamma_\beta \wedge E^\Lambda_{abc} \Gamma^{abc}_{\gamma\alpha} + i\hat{D}\psi^\delta \wedge \psi^\gamma \wedge E^\Lambda_{a1...a4} \Gamma^{[a1a2a3]}_{\delta\alpha} \Gamma^{a4}_{\beta\gamma}\right] = 0 \quad (26)$$

The first term in the second part of Eq. (26) vanishes due to the algebraic (constraint) equation (18). Hence on the surface of constraints the selfconsistency of the gravitino equation is guaranteed when

$$M_{10,\alpha\beta} := R^\gamma_\beta \wedge E^\Lambda_{abc} \Gamma^{abc}_{\gamma\alpha} + i\hat{D}\psi^\delta \wedge \psi^\gamma \wedge E^\Lambda_{a1...a4} \Gamma^{[a1a2a3]}_{\delta\alpha} \Gamma^{a4}_{\beta\gamma} = 0 \quad (27)$$

Our main observation is that Eq. (27) (see Eq. (7) or (8)) collects all the bosonic equations of motion (22), (23) and the corresponding Bianchi identities for the $A_3$ gauge field and for the Riemann curvature tensor. Let us stress that we distinguish between the algebraic equations or constraints, Eqs. (13) and (19), from the true dynamical equations (22), (23), and that our statement above refers to the dynamical equations; thus it is also true for the second order formalism.

To show this it is not necessary to make an explicit calculation. It is sufficient to use the second Noether theorem and/or the fact that the purely bosonic limit of (27) implies Eq. (5) (see Sec. 4.3 below), which is equivalent to the set of all bosonic equations and Bianchi identities when $\psi^\alpha = 0$.

4.2 Proof using the Noether identities for supersymmetry

In accordance with the second Noether theorem, the local supersymmetry under (15), (17) reflects (and is reflected by) the existence of an interdependence among the bosonic and fermionic equations of motion; such a relation is called a Noether identity. Furthermore, as the local supersymmetry variation of the gravitino is given by the covariant derivative $\hat{D}\epsilon^\alpha$ with generalized connection, Eq. (16), the gravitino equation $\Psi$ should enter the corresponding Noether identity through $\hat{D}\Psi$. Thus, $\hat{D}\Psi$ should be expressed in terms of the equations of motion for the bosonic fields, in our case including the algebraic equations for the auxiliary fields. Hence, in the light of (26), (18) the l.h.s. of Eq. (27) vanishes when all the bosonic equations are taken into account.

---

2This follows e.g., from direct calculation of $t_{1}^a \gamma \wedge \bar{\Gamma}^{(8)}_{\gamma\beta} = -\frac{1}{2} F_4 \wedge \bar{\Gamma}^{(5)}_{\alpha\beta} + \frac{1}{2} \star F_4 \wedge \bar{\Gamma}^{(2)}_{\alpha\beta}$. 

---

5
Indeed, schematically, ignoring for simplicity the purely algebraic equations and neglecting the boundary contributions, the variation of the action (10), (11) (considered now in the second order formalism) reads

$$\delta S = \int \left( -2\Psi_{10a} \wedge \delta\psi^a + \mathcal{G}_8 \wedge \delta A_3 + M_{10a} \wedge \delta E^a \right).$$

(28)

For the local supersymmetry transformations $\delta \epsilon$, Eqs. (16), (15) and (17), one finds integrating by parts

$$\delta \epsilon S = \int \left( -2\Psi_{10a} \wedge D\epsilon^a + \mathcal{G}_8 \wedge \delta A_3 + 2i M_{10a} \wedge \psi^a \Gamma_{\beta\alpha} \right) \epsilon^\beta = 0.$$

(29)

As $\delta \epsilon S = 0$ is satisfied for an arbitrary fermionic function $\epsilon^\alpha(x)$, it follows that

$$D\Psi_{10a} = -\frac{1}{2} \psi^\beta \wedge \left( -2i \Gamma_{\beta\alpha}^a M_{10a} + \mathcal{G}_8 \wedge \Gamma_{\beta\alpha}^{(2)} \right).$$

(30)

In the light of Eqs. (26) and (30), and after the algebraic equations (18), (19) are taken into account,

$$\mathcal{M}_{10 a\beta} := \mathcal{R}_{\beta\gamma} \wedge \mathcal{E}^{abc}_{\gamma\delta} \Gamma_{\delta\gamma}^{a\beta} + iD\psi^\delta \wedge \psi^\gamma \wedge \mathcal{E}^{ab}_{\gamma\delta} \Gamma_{\delta\gamma}^{[a\beta} \Gamma_{\gamma\delta]}^a =$$

$$= -3i \left( -2E_{10a} \wedge \mathcal{G}_8 \wedge \Gamma_{\beta\alpha}^{(2)} \right).$$

(31)

It then follows that $\mathcal{M}_{10 a\beta} = 0$, Eq. (27), is satisfied after the dynamical equations (23), (22) are used. Moreover, Eq. (31) also shows what Lorentz–irreducible parts of the concise bosonic equations $\mathcal{M}_{10 a\beta} = 0$ coincide with the Einstein and with the 3–form gauge field equations. These are given, respectively, by

$$M_{10a} = -\frac{1}{192} \text{tr}(\Gamma_\alpha M_{10}) ,$$

(32)

$$\mathcal{G}_8 \wedge \mathcal{E}^a \wedge \mathcal{E}^b = \frac{i}{96} \text{tr}(\Gamma_{ab} M_{10}) .$$

(33)

It is clear that all other Lorentz–irreducible parts in Eq. (27), $\mathcal{M}_{10 a\beta} = 0$, are satisfied either identically or due to the Bianchi identities that are the integrability conditions for the algebraic equations (18), (19) used in the derivation of (31).

Thus, we have proven that Eq. (27) collects all the dynamical bosonic equations of motion in the second order approach to supergravity. To see that it collects all the Bianchi identities as well, one may either perform a straightforward calculation or study the pure bosonic limit of Eq. (27). The latter way is simpler and it also provides an alternative proof of the above statement as we now show below.

4.3 Proof using the purely bosonic limit of the equations

For bosonic configurations, $\psi^a = 0$, Eq. (27) takes the form

$$\psi^a = 0 , \quad \mathcal{R}_{\beta\gamma} \wedge \mathcal{E}^{abc}_{\gamma\delta} \Gamma_{\delta\gamma}^{a\beta} = 0 .$$

(34)

We show here that this equation is another form of Eq. (5)–(7),

$$\psi^a = 0 , \quad i_a \mathcal{R}_{\beta\gamma} \Gamma_{\gamma\delta}^a \equiv E_{ab} \mathcal{R}_{ab\delta} \Gamma_{\gamma\delta}^a = 0 .$$

(35)
As the above equation (35) collects all the bosonic equations of standard CJS supergravity as well as all the Bianchi identities in the purely bosonic limit [4, 7], the equivalence of Eqs. (35) and (34) will imply that $\mathcal{M}_{10\alpha\beta} = 0$, Eq. (27), does the same for the case of nonvanishing fermions, $\psi^\alpha \neq 0$.

Decomposing $R_{\alpha\beta}$ on the basis of bosonic vielbeins, $R_{\alpha\beta} = 1/2 E^a \wedge E^b R_{ba\alpha\beta}$, one finds that Eq. (24) implies

$$R_{ab\beta\gamma} \Gamma_{abc} \gamma_a = 0.$$ (36)

Contracting (36) with $\Gamma_{\alpha\delta}^c$ one finds

$$R_{ab\beta\gamma} \Gamma_{ab\gamma\delta} = 0.$$ (37)

Then, contracting again with the Dirac matrix $\Gamma_{\alpha\delta}^d$ and using $\Gamma^{ab} \Gamma_d \Gamma_{ab} + 2 \Gamma^{[a} \delta_{db]}$ as well as Eq. (36), one recovers Eq. (5), $\mathcal{N}_{a\beta\alpha} = 0$, which is an equivalent form of Eq. (35).

The Bianchi identities $R_{(ab)c} \equiv 0$ and $dF_4 \equiv 0$ appear as the irreducible parts $tr(\Gamma_{c1\ldots c5} N_a)$ and $tr(\Gamma_{c1\ldots c5} N_a)$ of Eq. (5) [more precisely, in the last case the relevant part in $N_a$ is proportional to $dF_4_{b_1\ldots b_5} (\Gamma^{a_1\ldots a_5} + 10 \delta_{a_1}^{[b_1} \Gamma^{a_2\ldots a_5]}$, but the two terms in the brackets are independent]. Knowing this, one may also reproduce the terms that include the Bianchi identities in the concise equation (7) (equivalent to (8) or (31)) with a nonvanishing gravitino.

5 Conclusion

We have shown that all the bosonic equations of $D=11$ supergravity can be collected in a single equation, Eq. (27), written in terms of the generalized curvature (6) which takes values in the algebra of the generalized holonomy group. In the first proof we used the shortcut provided by the second Noether theorem, which implies the Noether identity (30) for the local supersymmetry (15)–(17) relating the bosonic and fermionic equations. The second proof uses the purely bosonic limit (35) of the desired equation (27). This is simpler since the properties of the purely bosonic Eq. (35) are known [1, 7] and may also be used to simplify the extraction of Bianchi identities from Eq. (27), although we do not do it here.

The concise form (27) of all the bosonic equations is obtained by factoring out the fermionic one–form $\psi^\beta$ in the selfconsistency (or integrability) conditions $\mathcal{D} \Psi_{10\beta} = 0$ [Eqs. (20)], for the gravitino equations $\Psi_{10\alpha} = 0$, Eqs. (25). In this sense, one can say that in (the second order formalism of) $D=11$ CJS supergravity all the equations of motion and Bianchi identities are encoded in the fermionic gravitino equation $\Psi_{10\beta} := \mathcal{D} \Psi^\alpha \wedge \hat{\Gamma}_{\alpha\beta}^{(8)} = 0$ [Eq. (25)].

Actually this should be expected for a supergravity theory including only one fermionic field, the gravitino, and whose supersymmetry algebra closes on shell. As we have discussed, the basis for such an expectation is provided by the second Noether theorem.

We hope that the explicit form (27) of the equation collecting all the bosonic equations of motion and Bianchi identities may be useful in a further understanding of the properties of $D = 11$ supergravity and in the analysis of its supersymmetric solutions, including those with nonvanishing fermionic sector (see [14] and refs therein).

Acknowledgments. The authors thank Dima Sorokin for useful comments and Paul de Medeiros for correspondence. This work has been partially supported by the
research grant BFM2002-03681 from the Ministerio de Educación y Ciencia and from EU FEDER funds, the Generalitat Valenciana (Grupos 03/124), the grant N 383 of the Ukrainian State Fund for Fundamental Research, the INTAS Research Project N 2000-254 and the EU network MRTN–CT–2004–005104 ‘Forces Universe’. M.P. and O.V. wish to thank the Ministerio de Educación y Ciencia and the Generalitat Valenciana, respectively, for their FPU and FPI research grants, and I. Bars (M.P.) and M. Duff (O.V.) for their hospitality at the USC and the University of Michigan.

References

[1] M. J. Duff and J. T. Liu, *Hidden spacetime symmetries and generalized holonomy in M-theory*, Nucl. Phys. B674, 217–230 (2003) [arXiv:hep-th/0303140], M. J. Duff, *Erice lectures on The status of local supersymmetry*, arXiv:hep-th/0403160. An early reference is M. J. Duff and K. S. Stelle, *Multimembrane solutions of D = 11 Supergravity*, Phys. Lett. B 253, 113–118 (1991).

[2] C. Hull, *Holonomy and symmetry in M-theory*, arXiv:hep-th/0305039.

[3] J. Figueroa O’Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten and eleven-dimensional supergravities*, JHEP 0303, 048 (2003) [arXiv:hep-th/0212008].

[4] J. P. Gauntlett and S. Pakis, *The geometry of D = 11 Killing spinors*, JHEP 0304, 039 (2003) [arXiv:hep-th/0212008].

[5] G. Papadopoulos and D. Tsimpis, *The holonomy of the supercovariant connection and Killing spinors*, JHEP 0307, 018 (2003) [arXiv:hep-th/0306117].

[6] A. Batrachenko, M.J. Duff, J.T. Liu and W.Y. Wen, *Generalized holonomy of M-theory vacua*, arXiv:hep-th/0312165.

[7] I. A. Bandos, J. A. de Azcárraga, J. M. Izquierdo, M. Picón and O. Varela, *On BPS preons, generalized holonomies and D=11 supergravities*, Phys. Rev. D69, 105010 (2004) [arXiv:hep-th/0312266].

[8] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in eleven dimensions*, Phys. Lett. B76, 409–412 (1978).

[9] A. Batrachenko, J. T. Liu, O. Varela and W. Y. Wen, *Higher order integrability in generalized holonomy*, arXiv:hep-th/0412154.

[10] R. D’Auria and P. Fré, *Geometric supergravity in D = 11 and its hidden supergroup*, Nucl. Phys. B201, 101–140 (1982) [Erratum-ibid. B206, 496 (1982)].

[11] B. Julia and S. Silva, *On first order formulations of supergravities*, JHEP 0001, 026 (2000) [arXiv:hep-th/9911035].

[12] I. A. Bandos, J. A. de Azcárraga, M. Picón and O. Varela, *On the formulation of D = 11 supergravity and the composite nature of its three-form field*, Ann. Phys. 317, 238–279 (2005) [arXiv:hep-th/0409100].

[13] D. Lust and D. Tsimpis, *Supersymmetric AdS(4) compactifications of IIA supergravity*, JHEP 0502, 027 (2005) [arXiv:hep-th/0412250].

[14] C. M. Hull, *Exact pp wave solutions of 11–dimensional supergravity*, Phys. Lett. B 139, 39–41 (1984).