Purity as a witness for initial system-environment correlations in open-system dynamics

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We study the dynamics of a two-level atom interacting with a Lorentzian structured reservoir considering initial system-environment correlations. It is shown that under strong system-reservoir coupling the dynamics of purity can determine whether there are initial correlations between system and environment. Moreover, we investigate the interaction of two two-level atoms with the same reservoir. In this case, we show that besides determining if there are initial system-environment correlations, the dynamics of the purity of the atomic system allows the identification of the distinct correlated initial states. In addition, the dynamics of quantum and classical correlations is analyzed.

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I. INTRODUCTION

The preparation of the initial state plays a key role in the implementation of any experiment. As recently noted, different procedures of the preparation of initial states can lead to nontrivial differences in the experimental results due to interaction with the environment. These differences arise from initial correlations between the system and its environment. Although the assumption of uncorrelated system-reservoir initial states is widely used, it is not always well justified, specially when the system strongly interacts with the environment. Thus, the influence of initial correlations on the dynamics of open quantum systems has been intensively studied. In Ref. [12], the authors verified that the linear transient response of a two-level system coupled to a reservoir differs significantly if the initial state is correlated or not. Another important issue concerns about the possibility of associating a completely positive map to a quantum dynamics, which is possible only when there are no initial quantum correlations between system and reservoir. Moreover, the effects of the initial correlations were recently observed experimentally by using an all-optical apparatus to probe the evolution of the two-qubit polarization entangled state.

The usual way of measuring correlations in quantum systems is by mutual information. Although it is a well-established measure, the mutual information does not distinguish between quantum and classical aspects of correlation, which is an important issue of quantum information theory. In recent years, the study of measures that can differentiate among these two distinct aspects of correlation has been intensified, specially after the discovery that mixed unentangled states can also have non-classical correlation. The more widespread measure of total quantum correlation in a bipartite quantum system is the quantum discord, which was proposed by Ollivier and Zurek. A related quantity concerning classical correlations was also proposed by Henderson and Vedral. A possible way to determine the existence of correlated initial states was suggested by E.-M. Laine et al and it is based on the trace distance, a measure of distinguishability between two quantum states. They have shown that if the trace distance increases over its initial value, the system-reservoir state was initially correlated. However, this approach does not identify the nature of the initial system-environment correlations, that is, the quantum and classical aspects.

In this work we study the dynamics of a system consisting of a two-level atom (qubit) interacting with a zero-temperature environment without the Markov and Born approximations. We show that the system purity dynamics can be used as a witness for initial correlations between system and environment. Furthermore, by introducing a second atom (probe qubit) interacting with the same environment, we are able to determine the different initial system-environment correlations through the dynamics of the purity between the qubit and the probe qubit. We also study the dynamics of entropy and classical and quantum correlations of the atomic system. In this case, we show that these dynamics can be useful to identify and to classify the initial correlations between system and environment.

II. THEORETICAL MODEL

In this section, we first describe theoretically some important quantities that are employed to quantify quantum and classical correlations within quantum systems. These quantities are useful in our work because they exhibit signatures of initial system-environment correlations, as demonstrated by our results. Moreover, we present the model used to describe two atoms coupled to a common environment at zero temperature. The approach to solve the dynamics of the system-environment is discussed as well.
A. Quantum and Classical Correlations

Quantum Discord: Quantum correlations have played an important role in quantum information and communication theory [13]. If we consider for instance a bipartite quantum state described by the density matrix $\rho_{AB}$, the total correlations between the subsystems $A$ and $B$ can be computed by using the mutual information [15]:

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

where $\rho_i = \Tr_j (\rho_{ij})$ and $S(\rho) = -\Tr (\rho \log_2 \rho)$ is the von-Neumann entropy. A measure of classical correlations present in the quantum state $\rho_{AB}$ is defined as [19]:

$$\mathcal{J}(\rho_{AB}) = S(\rho_A) - S(\rho_{A|i}),$$

where $S(\rho_{A|i}) = \min \{M_b\} \sum_b p_b S(\rho_{A|i})$ is the quantum conditional entropy. The minimization is given over generalized measurements $\{M_b\}$ [15], with $\sum_b p_b M_b = 1_B$, $M_b \geq 0$ for all $b$, and

$$\rho_{A|i} = \frac{(I_A \otimes M_b)\rho_{AB}(I_A \otimes M_b)}{\Tr [(I_A \otimes M_b)\rho_{AB}(I_A \otimes M_b)]},$$

is the reduced density operator of $A$ after obtaining the outcome $b$ in $B$. For two-qubit systems, the minimization over generalized measurements is equivalent to a minimization over projective measurements [22]. In this work, we use numerical minimization to compute the classical correlations.

Using the results above, the quantum discord [17] is defined as

$$D(\rho_{AB}) \equiv \mathcal{I}(\rho_{AB}) - \mathcal{J}(\rho_{AB}),$$

and it assumes equal values, irrespective of whether the measurement is performed on the subsystem $A$ or $B$, only when $S(\rho_A) = S(\rho_B)$ [19].

Entanglement: A special class of quantum correlated states are the entangled states. A quantum bipartite state $\rho_{AB}$ is said entangled if and only if it cannot be written as a separable state $\rho_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j$, where $\sum_j p_j = 1$ [23]. In this work we use the entanglement of formation (EOF) [24] as a measure of entanglement, which is defined as $\text{EOF}(\rho_{AB}) = \min \{\sum_j p_j S(|\psi_j\rangle)\}$, where the minimization is over pure ensembles of pure states $\{p_j, |\psi_j\rangle\}$ such that $\rho_{AB} = \sum_j p_j |\psi_j\rangle \langle \psi_j|$. For a pair of qubits, the EOF is a monotonically increasing function of the concurrence $C$ [22].

$$\text{EOF}(\rho_{AB}) = -f(C) \log_2 f(C) - (1 - f(C)) \log_2 (1 - f(C)),$$

where $f(C) = (1 + \sqrt{1 - C^2}) / 2$ and $C = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ with $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ the square roots of the eigenvalues, in decreasing order, of the matrix $R = \rho (\sigma_1^x \otimes \sigma_2^y) \rho^* (\sigma_1^y \otimes \sigma_2^x)$. Here $\rho^*$ denotes the complex conjugation of the matrix $\rho$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. While all pure quantum correlated states are entangled states, there are quantum mixed states with null entanglement and a positive value of quantum discord. This result exemplifies that not all quantum correlations can be described by entanglement.

B. Dissipative dynamics of two atoms coupled to a common environment

In this subsection, we describe the dynamics of two non-interacting two-level atoms coupled to a zero-temperature common environment - the damped Jaynes-Cummings model. The environment is represented by a bath of harmonic oscillators with the Hamiltonian given by $(h = 1) H_R = \sum_k \omega_k a_k^\dagger a_k$, where $a_k^\dagger$ ($a_k$) is the creation (annihilation) operator for the field mode $k$ with frequency $\omega_k$. Our system consists of two two-level atoms whose Hamiltonian (in computational basis) is $H_0 = \omega_0 (\sigma_1^+ \sigma_1^- + \sigma_2^+ \sigma_2^-)$, where $\omega_0$ is the Bohr frequency and $\sigma_1^+$ and $\sigma_1^-$ are the Pauli raising and lowering operators of the $j$-atom, respectively. The Hamiltonian of the system plus environment is $H = H_S + H_R + H_{SR}$ with

$$H_{SR} = \left((\sigma_1^+ + \sigma_2^+) \sum_k g_k a_k + h.c.,\right)$$

where $g_k$ is the coupling constant.

The dynamics can be solved using auxiliary variables defined from the properties of the spectral distribution through the pseudo-mode approach [26]. Here we assume that the two atoms interact resonantly with a Lorentzian structured reservoir, resulting in the following pseudo-mode master equation in the interaction picture [26]

$$\frac{d\rho}{dt} = -i[H_S, \rho] + \frac{\Gamma}{2} (2a a^\dagger a - a^\dagger a a^\dagger - \rho - \rho^\dagger a a^\dagger),$$

where $\rho$ is the density operator for the two atoms and the pseudo-mode, $a (a^\dagger)$ is annihilation (creation) operator of the pseudo-mode and $V$ is the interaction Hamiltonian of the atomic system and the pseudo-mode. The spectral distribution associated with the pseudo-mode is

$$J(\omega) = \frac{\Omega^2}{\pi} \frac{\Gamma}{(\omega - \omega_0)^2 + \Gamma^2/4},$$

where $\Gamma$ is the pseudo-mode decay rate and $\Omega$ is the coupling constant between the pseudo-mode and the two atoms. In this model, there are two regimes to be considered [26]: (i) a strong-coupling regime that occurs when $\Gamma < 2\Omega$, and (ii) a weak-coupling regime for $\Gamma > 2\Omega$. As discussed in Ref. [26], for strong-coupling regime, the atomic dynamics presents non-Markovian features and when the coupling becomes weak, the Markovian behavior is recovered. In all cases studied in this work, we assume that the total system has at most one excitation, so that the Hamiltonian $V$ can be written as [26]

$$V = \Omega (\sigma_1^+ + \sigma_2^+) a + h.c.$$
This approach does not rely on either the Born or the Markov approximation and therefore it is possible to solve the dynamics without performing any further approximation.

III. RESULTS AND DISCUSSION

One important issue in the theory of open quantum systems is the ability of determining the origin of the decoherence of a qubit [27]. For instance, if we start with the single qubit mixed state

\[ \rho_S(0) = \alpha^2 |g\rangle \langle g| + (1 - \alpha^2) |e\rangle \langle e|, \tag{9} \]

where \( \alpha \in (0, 1) \), we are not able to determine how this state was initially prepared. For example, the qubit state described by Eq. (9) can be obtained from different situations:

a) the state was prepared without any correlation with the environment

\[ \rho_a = \rho_S(0) \otimes |0\rangle_E \langle 0|, \tag{10} \]

where \( |n\rangle_E \) means the state of the environment with \( n \) excitations;

b) the system was classically correlated with the environment

\[ \rho_b = \alpha^2 |g\rangle \langle g| \otimes |1\rangle_E \langle 1| + (1 - \alpha^2) |e\rangle \langle e| \otimes |0\rangle_E \langle 0|; \tag{11} \]

c) the system had quantum correlations and null entanglement with the environment

\[ \rho_c = \alpha^2 |g\rangle \langle g| \otimes \phi \langle \phi| + (1 - \alpha^2) |e\rangle \langle e| \otimes |0\rangle_E \langle 0|, \tag{12} \]

with \( |\phi\rangle_E = (|0\rangle_E + |1\rangle_E)/\sqrt{2} \);

d) the system had non-null entanglement with the environment

\[ \rho_d(0) = |\Psi(0)\rangle \langle \Psi(0)|, \tag{13} \]

being \( |\Psi(0)\rangle = \alpha |g\rangle |1\rangle_E + \sqrt{1 - \alpha^2} |e\rangle |0\rangle_E \).

In all states above, tracing over the environment variables, we end up with the same density operator given in Eq. (9). Then, as the information about the environment is not directly accessible, any measure carried out on the qubit state can not distinguish whether the system was initially correlated to the environment or not. According to Ref. [20], the dynamics of the qubit can be used to identify the presence of initial correlations among system and environment. However, there is only the possibility of determining if the system was initially correlated to the environment or not. Based on this approach, we might say if the qubit initial state described by Eq. (9) was prepared in that way or it was a result from the interaction with the environment.

In this work, we employ the dynamics of the system purity to witness the system-environment initial correlation. The purity \( P \) of a quantum state \( \rho \) is defined as

\[ P(\rho) = \text{Tr} (\rho^2) \] and a quantum state is pure if and only if the density operator \( \rho \) is idempotent, that is, \( \rho^2 = \rho \).

We begin our analysis by studying the dynamic behavior of the purity \( P \) of a qubit due to the interaction with the environment. We numerically calculate the dynamics of \( P \) for the initial states given in Eqs. (10)-(13), for both strong and weak coupling regimes. In Fig. 1, one can notice that the behavior of the dynamics of the system purity for the uncorrelated state is different from the correlated ones. However, the dynamics of the entangled initial state \( \rho_d \) coincides with the classically correlated state \( \rho_b \). These states differ from each other by two extra terms that appear in \( \rho_d \). For our type of interaction, these extra terms evolve in time not affecting the dynamics of the reduced system, where the environment has been traced out. In this situation, the dynamics of the system purity for both \( \rho_b \) and \( \rho_d \) becomes indistinguishable. Based on our results, we see that the dynamics of the system purity can be used as a measure to determine the system-environment initial correlation, however the kind of initial system-reservoir correlation is not well determined. Moreover, we observe in Fig. 1(b) that the curves become indistinguishable when the system-environment coupling is weak, that is, the initial correlation does not interfere too much in the system dynamics, so the supposition of a non-correlated initial state is well justified in this case. In addition, the purity asymptotically assumes the same value \( (P = 1) \), regardless the initial state. This is expected since we are assuming a zero-temperature reservoir and thus the asymptotic state of the qubit is the ground state \( |g\rangle \). Although the Fig. 1 refers to a particular value of the parameter \( \alpha = 0.5 \), our results were checked for different values of \( \alpha \) between 0 and 1 and the same characteristics were observed.

By employing a second qubit (probe) interacting with the same environment and considering the initial states as \( \rho = |g\rangle \otimes \rho_i \) (\( i = a, b, c, d \)), we numerically evaluate the dynamics of the purity for the two qubits as shown in Fig. 2. For this situation, the dynamics of \( P \) have different behavior for each initial state. Therefore, we are able to determine the initial correlation between system and environment by observing the system purity dynamics. To understand this phenomenon, we need to analyze the role of a common environment. Although both qubits do not interact with each other directly, they are correlated because both qubits interact with the same reservoir. Such behavior has been observed in different physical systems. Moreover, the common reservoir produces correlations between the qubits and these correlations depend on the initial state of the system-reservoir. Such a dependency is more evident when the interaction between the qubits and the reservoir becomes stronger. Thus, the stronger the interaction, the better the distinction among the dynamics for each initial state. Besides, the initial correlation of the original qubit and the environment can be dynamically transferred to the second qubit in such a way that the purity of both qubits can produce a signature of the kind of initial correlation among the first
qubit and the environment. In other words, the two extra terms of the initial state \( \rho_d \) induce correlations that are captured by the whole system (two atoms) purity. Again, in the asymptotic regime, the values of these quantities do not depend on the initial state. In Fig. 3 we plot the (a) mutual information, (b) classical correlation, (c) entanglement of formation, and (d) quantum discord for two-qubits as a function of dimensionless time \( \Omega t \) for the same initial states of Figs. 1 and 2. As already seen in the purity dynamics, each initial state leads to a particular dynamic behavior of the correlation measurements. Another interesting aspect about these quantities is their usefulness to improve the identification of a particular state. For instance, the dynamics of the purity for the states \( \rho_c \) and \( \rho_d \) are very similar (see Fig. 2), which makes difficult the determination of the initial correlated state. On the other hand, the dynamics of the other quantities shown in Fig. 3 for the same states \( \rho_c \) and \( \rho_d \) are very distinct, thereby improving the ability of specifying the initial correlated state.

To determine if there is or not initial correlation between the qubit and its environment, it would be enough to compare the dynamics of the qubit purity obtained experimentally with the theoretical result. If it is necessary to obtain a more precise information about the type of the initial correlation, we must employ a probe qubit interacting with the same environment and then compare the dynamics of one of those physical quantities discussed above with the theoretical results. In order to reduce errors between experimental outcomes and theoretical results, the purity would be the best choice at first glance, since it is just obtained by matrix multiplication \( \text{Tr} \left[ \rho_s^2(t) \right] \) instead of diagonalization or maximization processes, which must be performed to calculate the other physical quantities. If the results obtained from the purity are not decisive, one should perform an analysis of the other physical quantities to be able to determine
the initial correlated state. Thus, by employing such a procedure described above, one is able to infer if there is initial system-environment correlation and which is its nature.

It is interesting to remark what would occur if other kinds of interactions in the Hamiltonian were considered. For example, an important class of such Hamiltonians is found when the system is coupled to a dephasing environment, where such a coupling is described by the Pauli matrix $\sigma_z$. This kind of interaction is very common to model the dynamics of solid states systems, e.g., quantum dots [29] and Josephson junctions [30]. If we consider this type of interaction together with the initial conditions treated in this work, we are not able to identify the initial correlation between the system and the environment. This shortcoming occurs because the initial states of Eqs. (10)-(13) are written as an ensemble of eigenvectors of $\sigma_z$, thus no difference on the dynamics will be observed for this type of interaction. However, if other initial conditions were considered, e.g., initial states represented by an ensemble of eigenvectors of $\sigma_x$, the same methodology could have been applied.

IV. CONCLUSION

In summary, we studied a way to determine the existence of initial correlation between a two-level atom and a Lorentzian structured reservoir at zero temperature. We found that the dynamics of quantum and classical correlations have signatures of the initial system-environment correlations. Moreover, we showed that it is possible to identify the distinct initial correlated states by considering a probe qubit interacting with the same reservoir. We verified that, in our case, the purity might be the best choice to witness the initial system-environment correlations because this quantity reduces errors when the experimental outcomes are compared to the theoretical results, although the purity is not always decisive to determine the nature of the initial correlations. In such a situation, the dynamics of classical or quantum correlations measures for the bipartite system (two atoms) can be used to distinguish among different initial states. Finally, as we have only presented a numerical study, we hope that our work stimulates new researches in this area, specially in order to obtain analytical results that probes the efficiency of purity and the measures of correlations as witness to initial system-environment correlations for more general states and Hamiltonians. We also observed that both the purity and the measures of correlations become less efficient when the interaction between the qubits and the reservoir is weakened, so that it is necessary to find a more precise witness to the initial qubit-reservoir correlation in the weak coupling regime.

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[1] K. Modi and E. C. G. Sudarshan, Phys. Rev. A 81, 052119 (2010); K. Modi, arXiv:0903.2027 (2010).
[2] P. Pechukas, Phys. Rev. Lett. 73, 1060 (1994); Ibid 75, 3021 (1995).
[3] R. Alicki, Phys. Rev. Lett. 75, 3020 (1995).
[4] P. Stelmachovic and V. Buzek, Phys. Rev. A 64, 062106 (2001); Ibid 67, 029902 (2003).
[5] T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, Phys. Rev. A 70, 052110 (2004).
[6] H. A. Carteret, D. R. Terno, and K. Zyczkowski, Phys. Rev. A 77, 042113 (2008).
[7] Y. J. Zhang, X. B. Zou, Y. J. Xia, G. C. Guo, Phys. Rev. A 82, 022108 (2010).
[8] A. G. Dijkstra and Y. Tanimura, Phys. Rev. Lett. 104, 250401 (2010).
[9] Y. Zhang, X. Zou, Y. Xia, and G. Guo, J. Phys. B: At. Mol. Opt. Phys. 44, 035503 (2011).
[10] C. A. Rodríguez-Rosario, K. Modi, and A. Aspuru-Guzik, Phys. Rev. A 81, 012313 (2010).
[11] A. R. Usha Devi, A. K. Rajagopal, and Sudha, Phys. Rev. A 83, 022109 (2011).
[12] C. Uchiyama and M. Aihara, Phys. Rev. A 82, 044104 (2010).
[13] A. Shaban and D. A. Lidar, Phys. Rev. Lett 102, 100402 (2009).
[14] A. Smirne et al., arXiv:1105.0174 (2011).
[15] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[16] J. Niset and N. J. Cerf, Phys. Rev. A 74, 052103 (2006); M. Horodecki et al., Phys. Rev. A 71, 062307 (2005); C. H. Bennett et al., Phys. Rev. A 59, 1070 (1999).
[17] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
[18] S. Luo, Phys. Rev. A 77, 022301 (2008); B. Dakic, V. Vedral, and C. Brukner, Phys. Rev. Lett. 105, 190502 (2010); D. Girolami, M. Paternostro, and G. Adesso, arXiv:1008.4136 (2010).
[19] L. Henderson and V. Vedral, J. Phys. A: Math. Gen. 34, 6899 (2001).
[20] E.-M. Laine, J. Piilo, and H.-P. Breuer, Europhys. Lett. 92, 60010 (2010).
[21] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett 103, 210401 (2009).
[22] S. Hamieh, R. Kobes, and H. Zaraket, Phys. Rev. A 70, 052325 (2004).
[23] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[24] C. H. Bennett , D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996).
[25] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[26] B. M. Garraway, Phys. Rev. A 55, 2290 (1997); B. J. Dalton, S. M. Barnett, and B. M. Garraway, Phys. Rev. A 64, 053813 (2001); L. Mazzola, B. Bellomo, R. Lo Franco, and G. Compagno, Phys. Rev. A 81, 052116 (2010).
[27] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford,
[28] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda and A. O. Caldeira, Phys. Rev. A 81, 052107 (2010); L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway, Phys. Rev. A 79, 042302 (2009); D. Z. Rossatto, T. Werlang, E. I. Duzzioni and C. J. Villas-Boas, Phys. Rev. Lett. (in press). [29] F. F. Fanchini, L. K. Castelano, and A. O. Caldeira, New J. Phys. 12, 073009 (2010). [30] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).