Dynamics of helical vortices and helical-vortex rings

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Abstract – The letter considers dynamics of helical vortices and helical-vortex rings either solving directly the equations of motion of a vortex line or using canonical relations following from the Hamiltonian equations of motion. An analytical solution in elliptical integrals was found for helical-vortex rings in the local-induction approximation. The analysis based on the canonical Hamilton relation provides a clear physical explanation for anomalous velocities of helical-vortex rings, i.e., for suppression of the velocity and even inversion of its direction at sufficiently large amplitude of the helical distortion. The extended local-induction approximation is suggested, which provides an exact solution for the equations of motion of helical vortices and helical-vortex rings in the limit when the small-pitch helical vortex reduces to a cylindric sheet of uniform vorticity.

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Introduction. – Helical vortices were intensively studied in classical hydrodynamics [1,2]. They appear in wakes of propellers and other spinning bodies. A helical vortex is a straight vortex line with a circularly polarized Kelvin wave of arbitrary amplitude propagating along it. One may roll up the helical vortex into a ring keeping its helical deformation. Then it will be a helical-vortex ring. A vortex ring is a ubiquitous object in vortex dynamics in general and in superfluid turbulence in particular and has already been studied for about two centuries. Its dynamics is well established. Recently attention was focused on the dynamics of helical-vortex rings with large-amplitude Kelvin waves propagating around the ring [3–5]. This problem being interesting itself, may have important implications for superfluid turbulence. An interesting outcome of studies of this object was that for large enough Kelvin wave amplitudes the vortex ring may move in a direction opposite to its moment. This phenomenon was first revealed by Kiknadze and Mamaladze [3] in numerical calculations and the perturbation theory within the local-induction approximation. Later it was confirmed by numerical calculations based on the Bio-Savart law [4] and the Gross-Pitaevskii equation [5]. The effect seemed to be mysterious and was called anomalous vortex ring velocity [4,6].

This letter addresses dynamics of helical vortices and helical-vortex rings in the local-induction approximation either solving directly the equations of motions of a vortex line or using simple canonical relations following from the Hamiltonian equations of motion. An analytical solution in elliptical integrals was found for helical-vortex rings with periodic helical distortion of arbitrary amplitude. It confirms the approach based on the canonical Hamilton relations, which provides a transparent physical explanation for the origin of the anomalous vortex ring velocity. Suppression of the ring velocity and the eventual inversion of its direction are related with conserved angular momentum of the flow around the ring axis. Agreement of the canonical approach with the full analytical solution (within the area of validity for the former) is a proof of its effectivity. A slowdown of a vortex ring with swirl is also known in classical hydrodynamics [7]. In quantum hydrodynamics a swirl leading to azimuthal flow requires vortices piercing the ring, but in our case azimuthal flow was generated by helical deformation of the vortex ring.

The local-induction approximation assumes that the energy of a vortex line is proportional to its length and a velocity of a fixed point on the vortex line is induced only by the closest segment of the line and is determined by its curvature radius. In the case of a helical vortex the closest segment is restricted by one turn of the helix. The presented analysis goes beyond the strict local-induction approximation and suggests the extended local-induction approximation, which takes into account also the velocity induced by other segments (turns) of the vortex line. In order to approximately estimate the effect of the other
turns, they are replaced by a cylindric vortex sheet with uniform distribution of vorticity. This approach gives an exact solution of the Bio-Savart law of vortex motion in the limit of a helix of very low pitch compared to the helix radius. Being also accurate in the opposite limit of high pitch, where the local-induction approximation is valid, the extended local-induction approximation provides a reliable interpolation model for a broad spectrum of vortex configurations.

**Helical vortex.** – Let us consider a Kelvin wave of arbitrary amplitude propagating along a straight vortex line. In the Cartesian coordinate frame with the z-axis coinciding with the unperturbed vortex line, the transverse displacements in the circularly polarized Kelvin wave are

$$x = a \cos(kz - \omega t), \quad y = a \sin(kz - \omega t). \quad (1)$$

So in the Kelvin mode the vortex line forms a helix moving along a cylinder coaxial with the z-axis with the pitch equal to the Kelvin wavelength \( \lambda = 2\pi/k \) (fig. 1).

In the local-induction approximation the equation of motion for a curved vortex line is

$$-\rho_s \kappa \frac{d\mathbf{s}(\mathbf{R}_L)}{dt} \times \frac{d\mathbf{R}_L}{dt} = -\frac{\mathbf{R}}{R^2} \epsilon. \quad (2)$$

Here \( \rho_s \) is the superfluid mass density, \( \kappa = h/m \) is the circulation quantum, \( \mathbf{R}_L \) the position vector for points on the vortex line, \( \mathbf{s}(\mathbf{R}_L) \) is the unit vector tangent to the vortex line at the point with the position vector \( \mathbf{R}_L \), \( \mathbf{R} \) is the curvature radius vector, which is determined by the relation

$$\frac{\mathbf{R}}{R^2} = \frac{d\mathbf{s}}{dl}. \quad (3)$$

where \( dl \) is the element of the vortex line length, and \( \epsilon \) is the vortex line tension equal to the vortex line energy per unit length:

$$\epsilon = \frac{\rho_s \kappa^2}{4\pi} \ln \frac{r_m}{r_c} = \rho_s \kappa \nu_s. \quad (4)$$

Here \( r_c \) is the core radius, \( r_m \) is the scale to which the velocity field \( v \sim 1/r \) penetrates, and \( \nu_s = \kappa \ln(r_m/r_c)/4\pi \) is the line tension parameter. The curvature radius vector at any point of the helix is directed along the radius of the cylinder while its absolute value is \( R = (1 + k^2 a^2)/k^2 a \). In the cylindrical coordinate system \( (z, r, \phi) \) the unit vector has components \( s_z = 1/\sqrt{1 + k^2 a^2} \), and \( s_r = \pm a/\sqrt{1 + k^2 a^2} \). We look for an automodel solution of the equation (2) of motion, which can be presented in two forms:

$$\phi(z, t) = k z - \omega t = k(z - v_z t) \quad (5)$$

or

$$z(\phi, t) = \frac{\phi - \Omega_z t}{k}. \quad (6)$$

Thus the helix motion may be described either as pure vertical translation with the velocity \( v_z \), or pure rotation with the angular velocity \( \Omega_z \). According to eq. (2)

$$v_z = \frac{\omega}{k} = \frac{\nu_s \kappa}{\sqrt{1 + k^2 a^2}}, \quad (7)$$

$$\Omega_z = -\frac{\nu_s \kappa^2}{\sqrt{1 + k^2 a^2}}. \quad (8)$$

In the limit of small \( a \) this yields the dispersion relation \( \omega = \nu_s k^2 \) for the linear Kelvin wave. But the frequency decreases with increasing amplitude \( a \) of the Kelvin wave.

Expressions (7) and (8) can be also derived from the general canonic Hamilton equations,

$$v_z = \frac{\partial E}{\partial P_z} = \frac{\partial E/\partial a}{\partial P_z/\partial a}, \quad \Omega_z = \frac{\partial E}{\partial M_z} = \frac{\partial E/\partial a}{\partial M_z/\partial a} \quad (9)$$

where \( P_z \) and \( M_z \) are the linear and the angular momenta along the z-axis,

$$E = \frac{\rho_s \kappa^2 L}{4\pi} \left( \sqrt{1 + k^2 a^2} \ln \frac{1}{k r_c} + \ln \frac{k R_c}{\sqrt{1 + k^2 a^2}} \right), \quad (10)$$

is the energy of the vortex line, and \( L \) and \( R_c \) are the height and the radius of the container with the helical vortex. Partial derivatives in eq. (9) take into account that the only varying parameter of the helix is its radius \( a \).

Indeed, imposing the periodic boundary conditions in the box of the height \( L \) the pitch \( 2\pi/k \) of the helix becomes a topological invariant. The first term in the expression (10) is the kinetic energy at the distances \( r < \sqrt{1 + k^2 a^2}/k \), where the helical deformation increases the length of the vortex line by the factor \( \sqrt{1 + k^2 a^2} \), while the second term is the kinetic energy at distances \( r > \sqrt{1 + k^2 a^2}/k \), where helical deformation does not affect the velocity \( v = \kappa/2\pi r \) determined only by the circulation quantum.
In general a momentum of a vortex line is given by [1]

\[ P = \frac{\rho_s k}{2} \int [R_s \times dl]. \]  

(11)

Here \( dl = s \, ds \), and \( ds \) is the element of the vortex line length. The expression leaves the vortex momentum undefined (similar to an undefined dipole moment of a single electron). But physically only the difference of the momenta of a distorted and a straight vortex along the chosen vertical axis is of importance. Then

\[ P_z = \frac{\rho_s k L}{4\pi} \int s \varphi \left( 1 + \left( \frac{dz}{r d\varphi} \right)^2 \right) r^2 d\varphi \Bigg|_{r=a} = \frac{\rho_s k L}{2} a^2 k. \]  

(12)

This momentum is nothing else as the momentum of the constant vertical velocity field \( \kappa k/2\pi \) inside the cylinder of radius \( a \) around which the vortex line is winding. The winding vortex line with tilted circulation vector produces a jump of the vertical velocity equal to \( \kappa k/2\pi \) in average, and the latter is equal to the vertical velocity inside the helix under natural assumption that there is no velocity outside the helix. Although the vertical velocity determines the whole momentum, the kinetic energy of the vertical flow is ignored in the local-induction approximation since it has not a large logarithmic factor. But we shall take the kinetic energy of the vertical flow into account studying the case when the strict local-induction approximation becomes invalid (see below). Similarly, calculating the \( z \) component \( M_z \) of the angular momentum of the velocity field induced by the helical vortex, only the difference between the angular momenta of a distorted and a straight vortex is of importance, and

\[ M_z = -\frac{\rho_s k L}{2} a^2. \]  

(13)

This follows from the general expression for an angular momentum of a curved vortex line (see sect. 3.5 in the book by Saffman [1]). Note a negative sign in this expression. Indeed, the angular momentum of a vortex line displaced from the container axis is less than the angular momentum of a vortex line exactly along the container axis. The latter is very large and depends on the container radius, but does not affect the outcome of the analysis determined only by momentum difference.

**Helical-vortex ring: canonical formalism.** – Now let us roll up the helical vortex into a ring of the radius \( R = L/2\pi \). The helix makes \( n \) turns around the vortex ring. It is assumed that the radius \( a \) of the helix is much smaller than \( R \) but may be of the same order or larger than the pitch \( 2\pi R/n \) of the helix (the Kelvin wavelength). We shall derive the velocity of the ring in the absence of dissipation using the canonical relation for the ring velocity, which takes into account that the coordinate along the ring axis and the linear momentum along the same axis constitute a pair of canonically conjugate Hamiltonian variables:

\[ v_r = \frac{\partial E}{\partial P} = \frac{\partial E/r}{\partial P/\partial R}. \]  

(14)

The momentum \( P = \pi\rho_s k R^2 \) related with the ring is not affected by helical deformation of the vortex line. The expression (10) for the energy given in the previous section can be used now after replacing \( L \) by \( 2\pi R \) and \( k \) by \( n/R \). So the energy is

\[ E = \frac{\rho_s k^2}{2} \left( \sqrt{R^2 + n^2 R^2} \ln \frac{R}{n c} + R \ln \frac{n R}{\sqrt{R^2 + n^2 R^2}} \right). \]  

(15)

But now the system has two degrees of freedom, and one should decide what is going on with the vortex helix if the ring radius varies. Certainly the number \( n \) of turns cannot vary being a topological invariant. Another invariant is the angular momentum along the ring axis. For large ring radius \( R \gg a \), it can be estimated from the component \( P_z \) of the straight helical vortex, eq. (12):

\[ M_{ax} = \frac{\pi \rho_s k a^2 n}{2}. \]  

(16)

Variation of the energy with respect to \( P \) must be performed at fixed \( M_{ax} \). So the growth of \( R \) is accompanied by a decrease of the helix radius \( a \): \( da/dR = -a/2R \). Keeping the most important logarithmic terms the ring velocity is

\[ v_r = v_s \frac{1 - n^2 a^2/2R^2}{\sqrt{R^2 + n^2 a^2}}, \]  

(17)

where \( v_s = \kappa \ln(\sqrt{R^2 + n^2 a^2}/\pi c)/4\pi \). So at large number of helix turns \( n > 2R/a \) the sign of the velocity changes, and the ring starts to move in the direction opposite to its momentum. This is a phenomenon of the anomalous vortex ring velocity, which is here directly explained by the canonical equations of motion and the angular-momentum conservation law. Expansion of eq. (17) in \( na/R \) yields the perturbation theory result by Kiknadze and Mamaladze [3]. The critical value of \( na/R = \sqrt{2} \) at which the vortex ring stops is larger than \( na/R = 1 \) given by the perturbation theory and is in reasonable agreement with the numerical calculations with the Bio-Savart law giving the critical value \( na/R = 1.7 \) at large \( n \) [4].

The angular velocity of the helical-vortex ring also follows from the canonical relation:

\[ \Omega = \frac{\partial E}{\partial M_{ax}} = \frac{v_s n}{R \sqrt{R^2 + n^2 a^2}}. \]  

(18)

Now taking the partial derivative the linear momentum \( P \) (i.e., the average radius \( R \)) must be fixed. Rotation with angular velocity \( \Omega \) corresponds to translation of the straight helical vortex, so \( \Omega = v_z/R \), where \( v_z \) is given by eq. (7) but with \( k \) replaced with \( n/R \).
The second integration yields expressions for functions $E$, $B$, $S$, and $n$ of the curvature vector in the cylindrical coordinate system. The line tension forces proportional to the components of the curvature vector, applied to an element of the vortex line: The left-hand sides are the line tension forces proportional to the components of the curvature vector in the cylindrical coordinate frame. The first integration of the equations is straightforward:

$$\frac{d}{d\phi}\left[\frac{\nu_r d\phi}{r} \left(1 + (d/r d\phi)^2 + (dz/r d\phi)^2\right)^{1/2}\right] = 0,$$

$$-\nu_r \frac{d^2 z}{d\phi^2} = \frac{\nu_z}{r} \left(1 + (d/r d\phi)^2 + (dz/r d\phi)^2\right),$$

For integration two boundary conditions were used: i) $d z / d\phi = 0$ at $r = R_0$ (the maximum of the function $z(\phi)$), and ii) $d r / d\phi = 0$ at $r = R_1 < R_0$ (the minimum of the function $r(\phi)$). The maximum of the function $r(\phi)$ (the vortex line position most distant from the ring axis) is at the distance $r = R_2 > R_0$ given by

see eq. (22) above

The second integration yields expressions for functions $E(r)$ and $z(\phi)$ via elliptic integrals:

$$\phi(r) = \frac{v_r (R_2^2 + R_3^2)}{\Omega} \cdot \left(\frac{F(k, m)}{R_3 \sqrt{R_2^2 + R_3^2}}\right),$$

$$z(\phi) = \frac{(R_2^2 + R_3^2) F(k, m)}{\Omega} \sqrt{R_2^2 + R_3^2}.$$
Dynamics of helical vortices and helical-vortex rings

Fig. 2: (Colour on-line) The ring velocity vs. the ratio \(2\pi a/\lambda = na/R\). The horizontal dotted line points out the velocity of an unperturbed vortex ring. The dashed line shows the velocity obtained from the canonical formalism, eq. (17). The thick lines show the velocities for \(a/R = 0.9, 0.7, \) and \(0.5\).

For not large wave numbers \(n\) and large amplitudes \(a = R - R_1\) one should use values of unexpanded elliptic integrals in calculations. Figure 3 shows helical-vortex rings of wave number \(n = 4\) for various ratios \(R_1/R\). The vortex line is winding around a surface of a toroid, which changes from an ideal torus at \(R_1 \to R\) with circular cross-section to a toroid obtained by revolution of a figure of eight at \(R_1 \to 0\) (fig. 4).

**Beyond the local-induction approximation.** – All the results received above were obtained strictly within the local-induction approximation, which takes into account only the vortex velocity induced by the close segment of the vortex line and determined by its curvature. Meanwhile, in a helical vortex of low pitch \(2\pi/k \ll a\) (fig. 1(b)) distant parts of the same vortex line (distant in the sense of distance measured along the vortex line) approach to a given point of the vortex line to distances much shorter than the curvature radius. Then the local-induction approximation in its strict meaning becomes invalid: one must take into account velocity induced by other turns of the spiral. The most rigorous way to deal with such cases is using the Bio-Savart law. In classical hydrodynamics an analytical solution of the Bio-Savart equations for helical vortices was obtained in the form of infinite series of Bessel functions (Kapteyn series) [2]. On the other hand, one may suggest a much simpler but still reasonably accurate approach, which takes into account the velocity induced by distant parts of the vortex line by modification or extension of the local-induction approximation.

We start from the straight vortex shown in fig. 1(b). At low pitch the surface of the cylinder, around which the vortex line is winding, can be considered as a continuous vortex sheet with the jump of the vertical velocity component \(\kappa k/2\pi\). Assuming that there is no velocity outside the helix, but only inside, one obtains the kinetic energy of axial (vertical) motion per unit length: \(\rho_s \kappa^2 k^2 a^2/8\pi\).

Adding this energy to the energy in the local-induction approximation given by eq. (10), one obtains

\[
E = \frac{\rho_s \kappa^2 L}{4\pi} \left( \sqrt{1 + k^2 a^2} \ln \frac{1}{kR_c} + \ln \frac{kR_c}{\sqrt{1 + k^2 a^2} + \frac{k^2 a^2}{2}} \right).
\]

(26)

The added kinetic energy (the last term in the parentheses) is related to the velocity induced by all turns of the vortex line around the cylinder of radius \(a\). It is beyond the strict local-induction approximation since it does not contain a large logarithm factor. But it provides the most
essential contribution at \(ak \gg |\ln(kr_c)|\). In this limit the approach coincides with that obtained from the Bio-Savart law. So the expression (26) is a reliable interpolation between two limits \(ak \ll |\ln(kr_c)|\) (local-induction limit) and \(ak \gg |\ln(kr_c)|\) (continuous-vortex-sheet limit). The extended local-induction approximation does not change expressions for the linear momentum \(P_z\) (eq. (12)) and the angular momentum \(M_z\) (eq. (13)) since at their derivation the local-induction approximation was not used.

Using the expression (26) for the energy in the canonical expressions for the translational and angular velocities (eq. (9)) one obtains

\[
v_z = \frac{\omega}{k} = \frac{\nu_s k}{\sqrt{1 + k^2 a^2}} + \frac{\kappa k}{4 \pi},
\]

\[
\Omega_z = -\frac{\nu_s k^2}{\sqrt{1 + k^2 a^2}} - \frac{\kappa k^2}{4 \pi}.
\]  

(27)

Note that the continuous-vortex-sheet contribution \(\kappa k/4\pi\) to the translational velocity is an average between the zero velocity outside and the velocity \(\kappa k/2\pi\) inside the cylindrical vortex sheet like in the case of plane vortex sheets well known in hydrodynamics.

A similar modification of the local-induction approximation is possible for helical-vortex rings. The added continuous velocity is now an azimuthal velocity inside the helix formed along the ring. This leads to the modification of the expression (17) for the ring translational velocity:

\[
v_r = \frac{\kappa}{4\pi} \frac{1 - n^2 a^2/2R^2}{\sqrt{R^2 + n^2 a^2}} \ln \frac{\sqrt{R^2 + n^2 a^2}}{nr_c} - \frac{\kappa n^2 a^2}{8\pi R^3}.\]

(29)

A negative sign of the new non-logarithmic term demonstrates that the reverse motion of the vortex ring due to its helical deformation remains also in the continuous-vortex-sheet limit.

The principle, at which the extended local-induction approximation is based, is not new. The well-known Hall-Vinen-Belarevich-Khalatnikov theory for rotating superfluids [8,9] was based on the same principle: The logarithmically large self-induction line tension contribution to the vortex line velocity is combined with the velocity induced by other vortices in the vortex array, which is approximated by continuous vorticity. In our case there is no other vortices, but the continuous vorticity is introduced as an approximation for the effect of other turns of the same vortex line.

**Discussion and conclusions.** – The letter studies dynamics of helical vortices and helical-vortex rings in the local-induction approximation either analytically solving directly the equations of motions of a vortex line, or using canonical relations following from the Hamiltonian equations of motion. The analytical solution of equations of motions fully confirms the results following from simple canonical relations. The analysis based on these relations explains the origin of the anomalous vortex ring velocity. Suppression of the ring velocity and eventual inversion of its direction are related with conserved angular momentum around the ring axis, which is generated by the helical deformation of the vortex ring. The analysis goes beyond the local-induction approximation and suggests the extended local-induction approximation, which agrees with the exact solution for helical vortices and helical-vortex rings in the limit when the vortex line winding around a cylinder (helical vortex) or a torus (helical-vortex ring) reduces to a sheet with uniformly distributed vorticity.

Our analysis addressed the case of a single Kelvin mode with fixed wave number. A natural extension of the analysis is the case of the ensemble of Kelvin modes assuming that the effect of various modes is additive [6], i.e., replacing \(n^2 a^2\) by \(\sum_n n^2 a_n^2\). Then \(L = 2\pi \sqrt{R^2 + \sum_n n^2 a_n^2}\) is the total length of the vortex line increased by the vortex line fluctuations. This is important for the theory of superfluid turbulence, where the dynamics of vortex rings is widely exploited. However, one may not expect that in a vortex tangle with many interacting vortex segments the conservation law of the momentum along a particular vortex is applicable, and the slowdown of vortex line motion discussed here needs a special investigation.

We have left an interesting question of stability beyond the scope of this work. However, observation of propagating helical-vortex rings in numerical simulations [4,5] is experimental evidence of their stability at least in some area of parameters. Instability is expected first of all in the limit \(R_1 \to 0\) (fig. 4). The latter corresponds to a bifurcation point, where the local-induction approximation fails and reconnection to a few separate loops may occur. This requires a more involved analysis.

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**REFERENCES**

[1] Saffman P. G., *Vortex Dynamics* (Cambridge University Press) 1995.
[2] Alekseenko S. V., Kubin P. A. and Okulov V. L., *Theory of Concentrated Vortices* (Springer) 2007.
[3] Kiknadze L. and Mamaladze Y., *J. Low Temp. Phys.*, 126 (2002) 321.
[4] Barenghi C. F., Hänninen R. and Tsubota M., *Phys. Rev. E*, 74 (2006) 046303.
[5] Helm J. L., Barenghi C. F. and Youd A. J., *Phys. Rev. A*, 83 (2011) 045601.
[6] Krstulovic G. and Brachet M., *Phys. Rev. B*, 83 (2011) 132506.
[7] Cheng M., Lou J. and Lim T. T., *Phys. Fluids*, 22 (2010) 097101.
[8] Donnelly R. J., *Quantized Vortices in Helium II* (Cambridge University Press) 1991.
[9] Sonin E. B., *Rev. Mod. Phys.*, 59 (1987) 87.