Neutron star collapse and gravitational waves with a non-convex equation of state

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ABSTRACT
The thermodynamical properties of the equation of state (EoS) of high-density matter (above nuclear saturation density) and the possible existence of exotic states such as phase transitions from nuclear/hadronic matter into quark-gluon plasma, may critically influence the stability and dynamics of compact relativistic stars. From a theoretical point of view, establishing the existence of those states requires the analysis of the “convexity” of the EoS. There are indications of the existence of possible regions in the dense-matter EoS where the thermodynamics may actually be non-convex. Flows subject to non-convex dynamics, dubbed Bethe-Zel’dovich-Thompson (or BZT) flows, are characterized by the appearance of compound waves (rarefaction shocks) during the evolution. Moreover, these fluids may develop a non-monotonic dependence of the sound speed with the rest-mass density. In this paper we investigate the effects of a phenomenological, non-convex EoS on the equilibrium structure of stable compact stars and on the dynamics of unstable neutron stars that collapse gravitationally to black holes, both for spherically symmetric and uniformly-rotating configurations. We show how the dynamics of the collapse with a non-convex EoS greatly departs from the convex case, leaving distinctive imprints on the gravitational waveforms. The astrophysical significance of these results for microphysical EoSs is briefly discussed.

Key words: dense matter – equation of state – gravitational waves – hydrodynamics – shock waves – stars: neutron

1 INTRODUCTION
A long-standing, fundamental, and still open issue in relativistic astrophysics is the knowledge of the equation of state (EoS) describing the thermodynamical properties of high-density matter, i.e. matter at densities above nuclear-matter. Such extreme conditions are achieved in the cores of neutron stars. Theoretical progress towards the understanding of this issue relies on electromagnetic observations and heavy-ion experiments (see Glendenning (2000); Heiselberg & Hjorth-Jensen (2000); Glendenning (2001); Weber (2005); Camenzind (2007); Haensel et al. (2007) and references therein). With the recent observations of gravitational waves from mergers of binary black holes and binary neutron stars (Abbott et al. 2016a,b, 2017a,b,c,d) a new channel to collect complementary information and improve our understanding of the dense-matter EoS will soon be opened. However, despite the ongoing efforts, the issue has not been fully addressed thus far.

There are many reasons why this matter must be worked out. On the one hand, the properties of the dense-matter EoS and the possible existence of exotic states such as phase transitions to quark-gluon plasma or associated with the presence of hadrons in the core of neutron stars, may critically influence the stability and dynamics of these objects. Furthermore, a third family of compact stars, more compact and denser than neutron stars, and originated by the appearance of quark phases in the core of neutron stars, has been long suggested (Schertler et al. 2000; Glendenning & Kettner 2000). More recently, the observations of two high-mass pulsars, PSR J1614-2230 (Demorest et al. 2010) and PSR J0348-0432 (Antoniadis et al. 2013), has also placed severe constraints on the dense-matter EoS. In particular, the softening of the EoS due to the presence of hyperons or phase transitions to quark matter or boson condensates is prone to affect the stability of neutron stars (Bednarek et al. 2012; Zdunik & Haensel 2013). The possibility of different types of phases, i.e. neutrons and quarks, coexisting in dense matter is currently under intense scrutiny (see Buballa et al. (2014) and references therein). Moreover, the possible existence of hybrid stars has recently been considered by Bejger et al. (2017).

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The dense-matter EoS also plays a fundamental role in the evolution (on a hydrodynamical timescale) of archetypal scenarios of relativistic astrophysics such as core-collapse supernovae, short- and long-duration progenitors of gamma-ray bursts, the cooling of proto-neutron stars, the formation of stellar-mass black holes (BHs), or the merger of compact-binary systems. In particular, and in the context of phase transitions, the dynamics of neutron star cores collapsing to BHs has been analyzed numerically in spherical symmetry by Abdikamalov et al. (2009) and by Peres et al. (2013a). In the former work the collapse is induced by a phase-transition from hadronic matter to deconfined quark matter, while in the latter the collapse is induced by a phase transition to hyperonic matter. The corresponding extensions of these works including rotation can be found in Bejger et al. (2012) and Peres (2013).

From a theoretical point of view, the existence of such exotic states of matter in the dense-matter EoS also requires the analysis of the “convexity” of the EoS. Relevant contributions towards the knowledge of the properties of non-convex thermodynamics induced by some EoS were made in the pioneering works of Bethe (1942), Zel’dovich (1946), and Thompson (1971). In particular Thompson (1971) introduced the concept of fundamental derivative in gas dynamics. Nowadays, fluids which display a region of negative values of the fundamental derivative are called Bethe-Zel’dovich-Thompson fluids, or BZT fluids (see Voss (2005) and references therein). A classical example is provided by a Van der Waals EoS. In this EoS, besides the mixing regime where different phases coexist, there is a region of non-convexity, where the fundamental derivative is negative (Menikoff & Plohr 1989).

BZT fluids have drawn some attention in the last fifteen years due to their potential applications in industry (see, e.g. Cinnella 2008; Guardone et al. 2010). BZT fluids show a non-convex dynamics in which compound waves as, for example, rarefaction shocks, can develop during their evolution (see Argrow 1996; Guardone & Vigevano 2002; Voss 2005; Cinnella & Corre 2006; Serna & Marquina 2014). Unlike a regular fluid, a BZT fluid might condense on isentropic compression.

The extension to relativistic fluid dynamics of previous studies on BZT fluids in the framework of classical fluid dynamics was accomplished by Ibáñez et al. (2013). This work presented the conditions under which the hyperbolic system of relativistic Euler equations is convex. The authors considered a perfect fluid obeying a causal EoS and the results were obtained by analyzing the properties of the characteristic fields of the relativistic hydrodynamics equations. Following Thompson (1971) the conditions were given in terms of the so-called (classical) fundamental derivative, $\Theta_{CG}$. As we show below, a non-convex region in the space of thermodynamical parameters appears where the adiabatic exponent, $\Gamma_1$, satisfies $\partial \Gamma_1 / \partial \rho_s |_{s} < 0$, $\rho$ and $s$ being the density and the entropy, respectively. This is equivalent to the following condition on the classical local sound speed, $\partial c_{\text{loc}} / \partial \rho_s |_{s} < 0$.

A classical, and somewhat academic, example of a thermodynamical system in which the adiabatic index displays a non-monotonous behaviour with the density is the region around the neutron-drip point in cold catalyzed dense matter (Shapiro & Teukolsky 1983). Similar regions appear also in EoSs derived from a field-theoretical model for nuclear and neutron matter, both at zero temperature (Diaz Alonso 1985) and at finite temperature (Martí et al. 1988). Moreover, the most popular EoS used in recent hydrodynamical simulations of compact stars display again, in some regions of the space of thermodynamical parameters, a non-monotonous behaviour of the adiabatic exponent with the density (as can be seen in the fittings reported by Haensel et al. 2002; Haensel & Potekhin 2004; Haensel et al. 2007; Bauswein et al. 2010). Those regions are good candidates to develop non-convex thermodynamics.

Other examples can be found at densities much higher than nuclear saturation density ($n_0 \approx 0.16 \text{ fm}^{-3}$) at which nuclear/hadronic matter undergoes a transition into a quark-gluon plasma (QGP). The nature of the finite-temperature QCD transition (namely, first-order, second-order or analytic crossover) remains ambiguous (Aoki et al. 2006). Using QCD lattice techniques, the HotQCD Collaboration (Bazavov et al. 2014) and the Wuppertal-Budapest Collaboration (Borsányi et al. 2014), have recently reported results about the EoS characterizing the transition from the hadronic phase into the QGP phase. Their results, which favor the crossover nature of the transition, in the continuum extrapolated EoS and in the phenomenologically relevant range of temperature, 130 – 400 MeV, show similarities regarding the trace anomaly, pressure, energy density and entropy density. The energy density in the crossover region, $145 \lesssim T \text{ (MeV)} \lesssim 163$, is a factor of about 1.2 to 3.1 times the nuclear saturation density. The square of the speed of sound has a minimum at $T \approx 145 – 150$ MeV, being up to about 200 MeV smaller than the ideal gas value of 1/3. Following a different strategy, which combines the knowledge of the EoS of hadronic matter at low densities with the observational constraints on the masses of neutron stars, Bedaque & Steiner (2015) conclude that the speed of sound of dense matter is not a monotonous function of the energy density, with local maximum and minimum above and below $1/\sqrt{3}$, respectively.

Motivated by the above indications of the existence of possible regions in the dense-matter EoS where the thermodynamics can be non-convex, we present in this paper a numerical study of the structure and dynamics of compact stellar configurations described by a BZT fluid. We choose a particularly simple form of the EoS, namely an ideal gas EoS with an adiabatic index which depends on the density (Ibáñez et al. 2018). While this phenomenological EoS can only be regarded as a toy-model, it serves nonetheless to exemplify the particularities that appear when the EoS is non-convex. For our study we consider two different situations, firstly, the equilibrium structure of stable compact stars and, secondly, the dynamics of unstable neutron stars that collapse gravitationally to BHs, both for spherically symmetric and uniformly-rotating initial configurations. A future study using actual microphysical EoS from nuclear physics will be presented elsewhere.

This paper is organized as follows: Section 2 shows that non-convex thermodynamics may exist in microphysical EoS of common use in astrophysical scenarios such as massive stellar core collapse. Section 3 describes our toy-model, non-convex EoS. This EoS is employed to obtain the results presented in the following three sections. Section 4 discusses the structure of spherically-symmetric relativistic stellar equilibrium configurations, while Sections 5 and 6 analyse the dynamics of unstable configurations which promptly collapse producing a central BH in spherical and axial symmetry, respectively. Since, as we show below, the effects of convexity loss are bound to very compact collapsing cores, the observational signature of this anomalous thermodynamics may potentially be best noticed on the gravitational-wave signature. Thus, from the collapsing, axisymmetric, rotating cores we present in Section 6, we calculate their gravitational-wave emission aiming at identifying features that differentiate convex dynamics from non-convex ones. Finally, the conclusions of our work are presented in Section 7.
2 NON-CONVEXITY IN MICROPHYSICAL EOS EMPLOYED IN STELLAR CORE COLLAPSE

In classical fluid dynamics, the convexity of a thermodynamical system is determined by the EoS (Menikoff & Plohr 1989) and, more specifically, by the so-called fundamental derivative, \( \mathcal{G}_C \)

\[
\mathcal{G}_C := \frac{1}{V} \frac{\partial^2 p}{\partial V^2} \bigg|_s,
\]

where \( V := 1/\rho \) the specific volume, \( \rho \) the rest mass density, \( p \) the pressure and \( s \) is the specific entropy. The fundamental derivative measures the convexity of the isentropes in the \( p - V \) plane. If \( \mathcal{G}_C > 0 \) then the isentropes in the \( p - V \) plane are convex and the rarefaction waves are expansive.

The relationship between the classical and the relativistic fundamental derivatives was found in Ibáñez et al. (2013) and is given by

\[
\mathcal{G}_R = \mathcal{G}_C - \frac{3}{2} c_\text{ss}^2,
\]

where \( c_\text{ss} \) is the relativistic sound speed, related to the classical definition of the sound speed,

\[
c_\text{ss}^2 = \frac{\partial p}{\partial \rho} \bigg|_s,
\]

through the relation \( c_\text{ss}^2 = \hbar c_\text{so}^2 \), where \( h = 1 + \varepsilon/p \rho \) is the specific enthalpy and \( \varepsilon \) the specific energy.

We have performed a survey of a few nuclear-matter EoS which can be found in the Compostar Online Supernovae Equations of State (CompOSE) looking for regions of the parameter space in which either the relativistic or the classical fundamental derivative become negative. We do not aim to exhaustively check all the possible dens-matter EoSs. Instead, we shall see that some of the EoSs we consider here (all of which have been used in the context of stellar core collapse) display regions where the thermodynamics is non-convex. The EoS from the CompOSE database have been included to sample cases in which baryons are treated as non-relativistic particles or, alternatively, they are included within a suitably relativistic theory. Also, we have considered variants of the latter cases where different parameter sets of a relativistic mean-field (RMF) theory are available and employed in astrophysical simulations. Finally, different variants of the EoS account for the possibility of transitions to quark matter or include more exotic particles such as hyperons. We note that the tables employed to compute the fundamental derivatives are exactly the same as in the CompOSE database. This is an important point, since high-order derivatives of the thermodynamic variables (like those needed for the calculation of the fundamental derivatives) may display small amplitude, high-frequency oscillations associated to the discretization of the EoS table. While this problem is minor in regions where the relativistic fundamental derivative is positive and significantly different from zero, it may affect the determination of a “physically sound” non-convex region when the fundamental derivatives are close to zero. We warn the reader on the “numerical” loss of convexity associated to insufficiently fine thermodynamic discretization of some tabulated EoS when the adiabatic index is non-constant (Vaidya et al. 2015).

The EoS of Lattimer & Swesty (1991) with compression modulus \( K = 220 \text{ MeV} \) (LS220) is considered here in two variants: LS220n1 and LS220n1. The difference between both cases is the introduction of \( \Lambda \)–hyperons in the second one (following Oertel et al. 2012; Gulminelli et al. 2013). The hyperon-nucleon interaction is taken from the model by Balberg & Gal (1997). The Lattimer & Swesty (1991) EoS assumes that the nuclear interaction is an effective non-relativistic Skyrme type model without momentum dependence. Nucleons are treated as non-relativistic particles; \( \alpha \)–particles as hard spheres forming an ideal Boltzmann gas. As the density increases, nuclei dissolve into homogeneous nuclear matter above saturation density. The transition to bulk nuclear matter is treated by a Maxwell construction. Photons and elecetrons/positrons are included as a free gas. The low density extension, below the validity range of the original Lattimer and Swesty EoS is based on a nuclear statistical equilibrium model by Oertel et al. (2012). A first application of this EoS in the supernova context is described in Peres et al. (2013b). As for the LS220n1 EoS, it has been broadly employed in the literature (e.g. Obergaulinger et al. 2014; Obergaulinger & Aloy 2017).

The EoS of Shen et al. (1998a,b) (dubbed STOS, hereafter) uses the Thomas-Fermi and variational approximations with a RMF model. It has been considered here in two variants. The first one STOS_npeaNR, with only baryonic contributions (no leptons or photons included). The second variant of the Shen et al. EoS, STOS_B165_e_npeaNRQGQ, includes a transition to quark matter (Sagert et al. 2009, 2010; Fischer et al. 2011). The transition from the hadronic to the quark phase is done via a Gibbs construction and employing a bag model for the quark phase with a bag constant of \( B^{1/4} = 165 \text{ MeV} \) and \( \alpha_s = 0.3 \). In both cases the EoS employs a non-linear RMF model with the TM1 parametrization (Sugahara & Toki 1994) of the effective interaction. Only neutrons, protons, alpha particles and a single heavy nucleus as well as electrons/positrons and photons are considered.

We have also included cases in which the hadronic EoS is based on the statistical model of Hempel & Schaffner-Bielich (2010) (HS) and with RMF interactions of different types. These are the EoS tagged with BHB_LP_npeBSN, HS_TMA_npeN, HS_TM1_npeN, HS_NL3_npeN, SFHQ_npeN, SFHX_npeN. The first one, BHB_LP_npeBSN (Banik et al. 2014), assumes RMF interactions DD2 (Typel et al. 2010) and includes \( \Lambda \)–hyperons, neutrinos, anti-neutrinos, protons, anti-protons, lambda, anti-lambdas, and nuclei. The EoS tagged with HS_TMA_npeN, HS_TM1_npeN, HS_NL3_npeN, SFHQ_npeN and SFHX_npeN, include RMF interactions with parametrizations TM1 (Toki et al. 1995), TM1 (Sugahara & Toki 1994), NL3 (Lalazissis et al. 1997), SHFo (Hempel & Schaffner-Bielich 2010) and SFHx (Hempel & Schaffner-Bielich 2010), respectively, and contributions from neutrons, anti-neutrons, protons, anti-protons, electrons, positrons, photons, and nuclei. Applications of HS EoS for various different RMF interactions in supernova simulations can be found in Hempel et al. (2012) and Steiner et al. (2012).

The EoS of Shen et al. (2011a), to which we will refer as GSHen in the following, is based on a RMF model to self-consistently calculate non-uniform matter at intermediate density and uniform matter at high density. At low densities, a virial expansion for a non-ideal gas of nucleons and nuclei is used to obtain the EoS. Three variants of the GSHen EoS are included in our sample: GSHen_FSU1_npeaNR, GSHen_FSU2_npeaNRN, GSHen_NL3_npeaNRN. The differences between them are due to the distinct approximations employed within the RMF model. They employ either the FSUGold
(FSU1; Todd-Rutel & Piekarewicz 2005) or FSU2 parameter sets of 
Shen et al. (2011b), in the first two cases, or the NL3 parameter 
set of Lalazissis et al. (1997).

The SU(3) Chiral Mean Field EoS (CMF; Dexheimer & 
Schramm 2008; Schürhoff et al. 2010; Dexheimer et al. 2015) is 
a non-linear realization of the sigma model which includes pseudo-
scalar mesons as the angular parameters for the chiral transform-
ation. In the particular variant of this EoS we have chosen, 
CMF_npeBs_hypermns, it includes nucleons and hyperons as 
degrees of freedom (and in the case we consider here also free le-
ptons). Within the model, baryons are mediated by vector-iso scalar, 
vector-isovector, scalar-iso scalar, and scalar-isovector mesons (in-
cluding strange quark-antiquark states). At low densities and/or 
temperatures, the nuclear liquid-gas first-order phase transition is 
reproduced very smoothly, as can be seen in Figs. 1 and 2.

None of the two variants of the Lattimer & Swesty (1991) 
EoS are causal at high enough densities. This has the implication 
that the sound speed predicted by these EoSs is larger than the 
speed of light. Consistently, the relativistic fundamental derivative 
becomes negative (see Figs. 1 and 2 at baryonic number densities 
\( n \gtrsim 2 \text{ fm}^{-3} \)). Interestingly, the GSHen EoS displays different behaviors 
depending on the RMF parameterization. While the original 
FSUGold or FSU1 parameterization included in the 
GSHen_FSU1_npeaNNR EoS shows large amplitude oscillations 
where both the relativistic and classical fundamental derivatives be-
come negative at high density, the FSU2 and NL3 parameter sets 
(GSHen_FSU2_npeaNNR and GSHen_NL3_npeaNNR) seem 
to be convex up to the highest baryon density at which they are 
tabulated. The final fall off the black and orange lines in Figs. 1 and 
2 are due to numerical artifacts in the computation of high-order 
thermodynamic derivatives near the table boundaries.

We note that the presence of a phase transition to hyperons in 
the BHB_LP_npeBSN EoS generates a non-convex thermody-
namics if the temperature is large enough at baryonic number den-
sities \( n \sim 0.05 \text{ fm}^{-3} \) (Fig. 2). In contrast, this is not the case for 
the CMF model implemented in the CMF_npeBs_hypermns 
EoS, which also incorporates hyperons. Another location where the 
EoS convexity can be lost is close to the transition to uniform nu-
clear matter (\( n \gtrsim 0.1 \text{ fm}^{-3} \)). We emphasize that the negative values 
atained by the classical and relativistic derivatives may result 
from the treatment of the phase transition in the EoS. However, we 
cautiously note that alternative treatments of this phase transi-
tion may yield a convex thermodynamic behaviour. We outline 
the sampled hadronic EoSs based on the HS statistic model 
which do not contain hyperons (HS_TMA_npeN, HS_TM1_npeN, 
HS_NL3_npeN, SFHO_npeN and SFHX_npeN) seem to 
loose convexity in a narrow range of baryonic number densities with 
a typical width \( \Delta n \sim 0.02 \text{ fm}^{-3} \).

### 3. A PHENOMENOLOGICAL NON-CONVEX EOS

The traditional (simple) way to mimic the complex thermodynam-
ical processes taking place inside a collapsing stellar core in simu-
lations of hydrodynamical supernovae leading to the formation of 
compact objects, or during the merger of neutron stars in a com-
 pact binary system, is to consider EoSs of polytropic-type. Some 
examples include (i) a polytropic EoS where ‘gamma’ is a discon-
tinuous function of the density (van Riper 1978), (ii) the piecewise-
polytropic approximation (Müller & Eriguchi 1985), and (iii) the 
hybrid polytropic EoS, in which the pressure is composed of a cold 
component, \( p_c \), described by a polytropy of adiabatic index \( \Gamma_c \), and 
an ideal-gas component which incorporates the thermal effects, \( p_t \) (see e.g. Maione et al. 2016). We name the latter EoS ‘PolyTh’ and 
present a detailed analysis of its properties in Appendix A.

In order to explore the fundamental traits of a relativistic non-
convex dynamics induced by a non-convex thermodynamics we use a 
phenomenological EoS introduced in Ibáñez et al. (2018). Here 
we recap the essentials of the analysis performed on Ibáñez et al. (2018) 
regarding the non-convex properties that this EoS possess. We begin 
by the expression of the pressure \( p \), which obeys an ideal-
gas-like EoS of the form
\[
p = (\gamma - 1) \rho \epsilon,
\]
where \( \gamma \) depends on the density according to the following law:
\[
\gamma := \gamma_0 + \chi \exp \left( -\frac{x^2}{\sigma^2} \right), \quad \chi := \gamma_0 - \chi_0, \quad x := \rho - \rho_1,
\]
and where \( \epsilon \) and \( \rho \) are, respectively, the specific internal energy 
and the rest-mass density. Ibáñez et al. (2018) proposal for \( \gamma \) in 
Eq. (5) can be considered as a generalization of the classical pre-
scription used in early studies of core-collapse supernovae (see e.g. 
van Riper 1978). The function \( \gamma(\rho) \) in Eq. (5) reaches a maximum 
at \( \rho = \rho_1 \), a value we designate as \( \gamma_1 = \gamma(\rho_1) \). Let us notice that \( \rho_1 \) plays the role of a simple scale factor for the density, if we 
express the \( \rho \) of the Gaussian law (\( \sigma \)) in units of \( \rho_1 \) too, 
convention we adopt in the following. The EoS defined by Eqs. (4) 
and (5) will be named hereafter ‘GGL-EoS’ (for Gaussian Gamma 
Law).

The reference parameters we chose to analyse its properties are 
\( \gamma_0 = 4/3, \gamma_1 = 1.9, \sigma = 1.1, \) and \( \rho_1 = 10^{15} \text{ cm}^{-3} \). The 
values of \( \gamma_0, \gamma_1 \) and \( \gamma_l \) mimic the behaviour of collapsing dense matter 
during the standard prompt mechanism of hydrodynamical supernu-
voe, before and after core bounce (see, e.g. Janka et al. 2012).

Applying the definition of the classical speed of sound (Eq. 3) 
to our GGL-EoS, we obtain
\[
c_{\text{s}}(\rho) = \gamma \left( \frac{p}{\rho} + \epsilon \frac{d \ln \gamma}{d \ln \rho} \right) = \gamma \epsilon \left( \gamma_1 - 1 + \frac{d \ln \gamma}{d \ln \rho} \right).
\]
For later reference, we also write the specific enthalpy for the GGL-
EoS:
\[
h = 1 + \gamma \epsilon.
\]
Figure 3 shows, in logarithmic scale, the relativistic speed of sound 
(in units of the speed of light \( c \)) as a function of the density, param-
eterized by the specific internal energy. We note that the parameter-
ization used in the GGL-EoS avoids non-causality (i.e., yields 
\( c_{\text{s}}(\rho) < c \)) and leads to very low values of \( c_{\text{s}}(\rho) \) for densities much 
higher than \( \rho_1 \). As the legend of Fig. 3 indicates, \( c_{\text{s}}(\rho) \) is an increasing 
function of \( \epsilon \).

The explicit expressions for the adiabatic index, \( \Gamma_1 \) (see e.g. 
Chandrasekhar 1939), and the fundamental derivatives for the 
GGL-EoS have been obtained in Ibáñez et al. (2018). The adiabatic 
index, which is in general \( \Gamma_1 \neq 1 \), reads
\[
\Gamma_1 := \frac{\partial \ln p}{\partial \ln \rho} |_{\epsilon} = \left( \frac{p}{\rho} \right) c_{\text{s}}^2(\rho), \quad c_{\text{s}}^2(\rho) = 1 + \left( \frac{\rho \epsilon}{\rho} \right) \frac{d \ln \gamma}{d \ln \rho}.
\]

The classical fundamental derivative for our GGL-EoS is:
\[
\theta(\rho) = \gamma_{\text{cf}} + \frac{\gamma_{\text{cf}}}{2 c_{\text{s}}^2(\rho)} \left( \frac{\gamma}{d \ln \rho} + \frac{c_{\text{s}}^2(\rho)}{d \ln \rho} \right)^2,
\]
where
\[
\gamma_{\text{cf}} := \frac{1}{2} \left( 1 + \gamma^2 \left( \frac{d \ln \gamma}{d \ln \rho} \right)^2 \right).
\]
Neutron star collapse and gravitational waves with a non-convex equation of state

Figure 1. Classical (left) and relativistic (right) fundamental derivative of a few selected dense-matter EoS from CompOSE as a function of the baryon number density $n$. For all EoS we fix the value of the temperature to $T = 1.05 \text{MeV}$ and of the charge fraction $Y_q = n_q/n = 0.30$, where $n_q$ is the charge density of strongly interaction particles. The legends GSHen_FSU2_npeaNNR, GSHen_FSU1_npeaNNR, GSHen_NL3_npeaNNR refer to the GSHen EoS (Shen et al. 2011a) including different parameterizations of the RMF. Models dubbed with LS220Lnl and LS220nl correspond to the LS220 EoS (Lattimer & Swesty 1991) including hyperons or not including them, respectively. The tags STOS_npeaNR and STOS_B165_e_npeaNRQQs, refer to the STOS EoS (Shen et al. 1998a,b), the latter one including a transition to a quark matter. BHB_LP_npeBSN, HS_TMA_npeN, HS_TM1_npeN, HS_NL3_npeN, SFHO_npeN and SFHX_npeN correspond to the hadronic EoS based on the HS statistical model and implementing RMF interactions of different types. Finally, CMF_npe_Bs_hyperons, corresponds to the hadronic CMF model.

From the above equation and the expression for $c_{x,\nu}^2$ given by Eq. (6) it is easy to conclude that $\mathcal{G}(C)$ is independent of $\varepsilon$.

Figure 4 shows the regions of the $P - \rho$ plane in which the GLL-EoS is divided in terms of the character of the thermodynamics. We observe that with the exception of a small region around and above $\rho_1$, the EoS is convex, i.e., the classical and relativistic fundamental derivatives, satisfy $\mathcal{G}_R > 0$ and $\mathcal{G}_C > 0$ (blue region in Fig. 4). In the green region $\mathcal{G}_C \leq 0$, and, as a result, so is $\mathcal{G}_R \leq 0$. This is a non-convex classical (NCC) region of the EoS. There is also a non-convex relativistic (NCR) region in which only $\mathcal{G}_R < 0$, while $\mathcal{G}_C > 0$ (red region in Fig. 4).

Figure 5 shows the two fundamental derivatives, classical (red thick line) and relativistic, as a function of density, being $\mathcal{G}_C$ parameterized by the specific internal energy. The inset shows a detail of the region around $\rho = \rho_1$ in order to highlight that, according to Eq. (2), $\mathcal{G}_C$ is an upper bound of $\mathcal{G}_R$. Furthermore, the inset clearly displays the existence of regions for which $\mathcal{G}_R < 0$ and $\mathcal{G}_C \geq 0$ simultaneously. We point out the qualitative similarity between the behaviour of the fundamental derivatives of the GGL-EoS around $\rho = \rho_1$ compared with that of a number of microphysical EoSs at high enough number density (see Figs. 1 and 2). Note that, in the case of the GLL-EoS, the convexity-loss around $\rho = \rho_1$ is not related to any phase transition, which does not exist in the phenomenological EoS. This point is relevant inasmuch as the convexity loss of several microphysical EoSs at baryon densities around the phase transition to uniform nuclear matter may result from the explicit construction employed to deal with the mixed phase in a thermodynamically consistent way.

4 EQUILIBRIUM CONFIGURATIONS

We turn next to analyze spherically-symmetric configurations of relativistic stars in equilibrium that satisfy the GGL-EoS introduced in the previous section. The relationship between the specific internal energy and the rest-mass density follows from the first law of
Figure 2. Same as Fig. 1 but computed at a temperature $T = 50.1 \text{MeV}$.

Figure 3. Relativistic speed of sound versus density for the GLL-EoS ($\gamma = 4/3, \gamma = 1.9, \alpha = 1.1, \rho_1 = 10^{13} \text{g cm}^{-3}$). The curves are parameterized by the specific internal energy, using the particular values indicated in the legend (in units of $c^2$).

thermodynamics for adiabatic processes. The corresponding ordinary differential equation, for our GGL-EoS, can be written as

$$\frac{d \ln \varepsilon}{d \ln \rho} = \gamma(p) - 1. \quad (11)$$

The integration constant in the above equation can be defined from the polytropic form ($p = \kappa_{\text{rad}} \rho^{2+b}$) of the EoS for a degenerate ideal Fermi gas of electrons at very low densities, i.e. the degenerate non-relativistic regime (dnr), where $\kappa_{\text{nr}} = 1.0036 \times 10^{13} \sqrt[5]{e} \, (\text{in CGS units, and } \rho = 1/2)$. In practice, and in order to obtain values of the maximum gravitational mass (see below) compatible with current observational data, we have verified that an optimal value is $\kappa_{\text{nr}} = 2.0072 \times 10^{12} \sqrt[5]{e}$ (i.e. a reduction factor of $1/5$).

The resulting tabulated relationship between the specific internal energy and the rest-mass density is fitted with a potential law

$$\varepsilon = \kappa_{\text{ad}} \rho^b, \quad (12)$$

where $\varepsilon$ is given in units of $c^2$. The fitting parameters are $\kappa_{\text{ad}} = 4.2266 \times 10^{-10}$ and $b = 0.58584$ for a fitting interval $\rho \in [\rho_1^{-1}, 10 \rho_1]$. Equations (4), (5) and (12) define completely our GGL-EoS. Figure 6 shows the GGL-EoS used to construct the static equilibrium models in this section and employed in the dynamical evolutions of rotational collapse of neutron stars to BHs in the next one.
Fundamental Derivatives

-20 -10 0 10 20
ρ / ρ_1
0.5 1 1.5 2

Figure 5. Classical and relativistic fundamental derivatives as a function of density for the GGL-EoS. The classical fundamental derivative is indicated by a red thick line. The relativistic fundamental derivative is parameterized by the specific internal energy, with the particular values indicated in the legend. The inset shows a detail of the region around ρ = ρ_1.

In order to obtain spherically-symmetric relativistic equilibrium configurations that obey the GGL-EoS we solve the Tolman-Oppenheimer-Volkoff (TOV) equation. The gravitational mass M_G of the equilibrium configurations, parameterized by the central density ρ_c, is shown in Fig. 7 as a function of the radius. It reaches a maximum M_G^{max} = 2.536M_☉, at a central density ρ_c^{crit} ≈ 1 (units of ρ_1), being the corresponding radius R ≈ 16.6 km. The inset displays M_G versus the central density. Models with central densities in the interval ρ_c^{crit} ≤ ρ_c / ρ_1 ≤ 1.4, define a small plateau in the M_G(ρ_c) curve where this function is strictly decreasing, i.e. there is no local maximum. By construction, the models are isentropic and therefore they satisfy the well-known static stability criterion against radial oscillations (Bardeen et al. 1966): the stability region is the one at central densities below the critical one, ρ_c^{crit}, at which the gravitational mass has an absolute maximum. Also shown in Fig. 7 is the region bounded by the Schwarzschild radius (black dots in the upper-left corner of the figure).

Moreover, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. Therefore, the specific internal energy and the specific enthalpy are, by definition, increasing functions of the density. 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for, their maxima (in radius) are reached at the centre of the configuration. For the critical model, the central values of $\varepsilon$ and $h$ are, respectively, $\varepsilon = 0.26$ and $h = 1.49$. Thus, the critical model is, from a thermodynamical point of view, only moderately relativistic. Consistent with our GGL-EoS, the relativistic speed of sound at the centre of the equilibrium configurations is not a monotonic function of the central density. Its central value for the critical model is $c_{s(R)} = 0.546$. This value is an upper bound for all the equilibrium models. Let $\varepsilon := \max(2Gm/(rc^2))$ be the maximum value of the compactness parameter, in radius, for each model. For our GGL-EoS the models reach an absolute maximum of $\varepsilon = 0.59$ at $\rho_c = 2\rho_1$, being its value at the critical central density $\varepsilon_c = 0.53$.

Figure 8 shows the two fundamental derivatives, $\theta_{(R)}$ and $\theta_{(C)}$, as a function of the radius, for an equilibrium model obeying the GGL-EoS and with central density $\rho_c/\rho_1 = 1.4$. Due to the particular form of our GGL-EoS, equilibrium models with central densities larger than the critical one will develop non-convex thermodynamics. For the sequence of equilibrium models we computed, there exists a small interval of central densities, namely $1.3 < \rho_c/\rho_1 < 1.4$, in which $\theta_{(R)} < 0$ and $\theta_{(C)} > 0$. This is shown in Fig. 8 for the particular case $\rho_c/\rho_1 = 1.4$. As a result, in such a narrow region of central-density values, the innermost cores of our models can develop non-convex thermodynamics induced by purely relativistic effects. The dynamical collapse of these objects, if perturbed, would rapidly trigger the presence of compound waves induced by such non-convex thermodynamics. Alternatively, the presence of non-convex relativistic regions may also induce a non-standard dynamics as a result of the non-monotonic dependence of the sound speed with the rest-mass density. We investigate the aforementioned possibilities in the next sections.

5 GRAVITATIONAL COLLAPSE IN SPHERICAL SYMMETRY

Within the framework of the GGL-EoS, the most promising scenario for encountering non-convex effects is the collapse of a star with a central density, $\rho_c$, similar to or above of $\rho_1$. We explore this possibility first in spherically symmetric simulations of toy models for neutron stars, comparing two equations of state. Four models were simulated with the GGL-EoS with different parameters and two with a modified version thereof, which we call Semi-GGL-EoS (SGGL-EoS). It consists of the GGL-EoS, but with a flat rather than decaying adiabatic index above $\rho_1$:

$$\gamma_{SGGL} = \gamma_{GGL}(\min(\rho, \rho_1))$$

$$= \gamma_0 + \xi \exp \left(\frac{-\min(\rho, \rho_1) - \rho_1}{\sigma^2}\right).$$

This EoS maintains the stiffening of the GGL-EoS at $\rho = \rho_1$, but its avoidance of the non-convexity at high densities allows us to gauge the importance of non-convex dynamics.

We consider three different initial models, all of which have been constructed by solving the TOV equation, albeit using different EoSs:

(i) The first one is a star with a polytropic EoS, $p = \rho^\gamma$, with a single adiabatic index $\gamma = 2$ for all densities and $\kappa = 8.422 \times 10^4$ in CGS units.

(ii) The second model was computed with the GGL-EoS, following the prescription developed in Sec. 4, but with the following parameters: $\gamma_0 = 4/3$, $\gamma_1 = 1.9$, $\sigma = 1.1$, $\rho_1 = 10^{15}$ g cm$^{-3}$.

(iii) In the third type of model, we use the SGGL-EoS (again following the prescription developed in Sec. 4) with the same parameters as in the point (ii).

For numerical reasons, we endow our initial configurations with a power-law damping atmosphere for values of the rest-mass density $\rho \lesssim 10^{-10}\rho_c$, where $\rho_c$ is the central rest-mass density. This atmosphere possesses a dynamically negligible mass. Irrespective of the type of initial model, we simulate the models with the GGL-EoS or SGGL-EoS.

All initial models have the central rest-mass density in the range $\rho_c \approx 2.05 \times 10^{15} - 2.18 \times 10^{15}$ g cm$^{-3}$, which is about twice the parameter $\rho_1$. The mass of the initial configurations are either $M_0 \approx 1.39M_\odot$ or $M_0 \approx 1.98M_\odot$ (see Table 1). These two masses roughly bracket the mass of the iron cores of massive stars (from which collapse a neutron star remnant may result) with main sequence masses in the range $10M_\odot - 120M_\odot$ and solar metallicity (Woosley & Heger 2007). They are initially in equilibrium, but an ad hoc reduction of the pressure will trigger their collapse. Indeed, the reason to choose three different EoS to construct the initial model is that we aim to assess the dynamical effects of the ad hoc initialization of the collapse on the subsequent dynamics. In the polytropic initial models, the reduction of the pressure is the result of the switch to the (SG)GGL-EoS, while in the initial models with GGL-EoS it is brought about by a uniform reduction of the internal energy density by 15%. Following O’Connor & Ott (2011), we define a compactness parameter as

$$\xi := \frac{M_0/M_\odot}{R_0/10\text{km}}$$

where $M_\odot$ and $R_0$ are the initial mass and radius, respectively, of the equilibrium configuration. According to this definition, the initial

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Note the difference in the definition of compactness, $\varepsilon$, used in Sec. 4.
Neutron star collapse and gravitational waves with a non-convex equation of state

Table 1. List of spherically symmetric, non-rotating models. For each model (name in the first column), the second column states the initial conditions (P, G, and S standing for the polytropic model and the ones computed with the GGL-EoS and SGGL-EoS, respectively). The further columns characterize the EoS used in the simulation: the parameters $\gamma_i$ and $\rho_i$ are given in the third column and fourth columns, while in the fifth column we list the variant of the GGL-EoS employed in the run. Finally, in the last four columns, we provide the mass, the radius and the compactness of the initial configuration, as well as the time of formation of the BH, respectively.

| model name | IC | $\gamma_i$ | $\rho_i$ | $EoS$ | $\rho_i$ | $M_i$ | $R_0$ | $\xi$ | $t_{BH}$ |
|------------|----|------------|---------|-------|---------|------|-------|------|---------|
| P-1.9G     | P  | 1.9        | 1.0     | GGL   | 2.18    | 1.39 | 10.0  | 1.39 | 0.066   |
| P-1.9S     | P  | 1.9        | 1.0     | SGGL  | 2.18    | 1.39 | 10.0  | 1.39 | 0.186   |
| P-1.6G     | P  | 1.6        | 1.0     | GGL   | 2.18    | 1.39 | 10.0  | 1.39 | 0.063   |
| P-1.45G    | P  | 1.45       | 1.0     | GGL   | 2.18    | 1.39 | 10.0  | 1.39 | 0.036   |
| G-1.9G     | G  | 1.9        | 1.0     | GGL   | 2.18    | 1.28 | 10.6  | 1.21 | 0.117   |
| S-1.9S     | S  | 1.9        | 1.0     | SGGL  | 2.18    | 1.33 | 11.3  | 1.18 | 0.166   |
| P-1.9G1    | P  | 1.9        | 0.9354  | GGL   | 2.046   | 1.98 | 11.8  | 1.68 | 0.068   |
| P-1.9G2    | P  | 1.9        | 1.00    | GGL   | 2.046   | 1.98 | 11.8  | 1.68 | 0.073   |
| P-1.9G3    | P  | 1.9        | 1.31    | GGL   | 2.046   | 1.98 | 11.8  | 1.68 | 0.093   |
| P-1.9G4    | P  | 1.9        | 1.559   | GGL   | 2.046   | 1.98 | 11.8  | 1.68 | 0.111   |

models built with a polytropic EoS are more compact than models constructed with the (SG)GL-EoS. The models having larger mass ($M_0 = 1.98 M_0$; Table 1) feature the largest compactness. Consistent with the large compactness of our models, we do not expect them to develop supernova explosions, even if a detailed neutrino physics and energy transport or magnetic fields were included in our simulations. Certainly, both of these effects may slightly change the dynamics, but for the purpose of assessing exclusively the impact on the dynamics of the convexity loss, we may neglect them.

The simulations were performed with a version of the code AENUS employed in Obergaulinger & Aloy (2017) and Obergaulinger et al. (2018), but restricted to special relativistic hydrodynamics, using fifth-order monotonicity preserving reconstruction schemes and an HLL Riemann solver. Gravity was incorporated using the pseudo-relativistic TOV potential of Marek et al. (2006), which provides a very good approximation to full GR in spherically symmetry. Once the center collapses to a BH, we excise the innermost region. The excision is undertaken by following the evolution of the lapse function, $\alpha$, related to the pseudo-relativistic TOV potential, $\Phi$, by $\alpha \approx \exp(\Phi/\sqrt{2})$. Numerical cells that develop $\alpha \leq \alpha_n := 0.018$ in the course of the evolution are frozen, except for a gauge transformation, which shifts their position from their location when they hit the condition $\alpha = \alpha_n$. This property allows the outer boundary to shift by a suitable radial shift, $\beta_\alpha$, on a time scale $r_{AH}/\dot{\beta}_\alpha$. The latter shift greatly diminishes spurious reflections at the apparent horizon location. The simulation grid consists of 3200 zones logarithmically spaced up to an outer radius of 180 km. The large extent of the grid, much larger than the radius of the initial equilibrium configuration, reduces any potential contamination by boundary effects. The minimum grid resolution is $\Delta r = 100$ m.

We present an overview of the time evolution of the six models holding the smaller total mass in the spacetime diagrams shown in Fig. 9. All models collapse quickly, as we see in the high negative velocities (brown shades in Fig. 9) and the contracting iso-density contours. Black holes are formed promptly, between 0.06 and 0.19 ms as can be seen from the growth of the white regions for $r < 3$ km in the aforementioned figure and from the last column of Table 1. This time scale can be compared with the light-crossing time of the initial configurations, which range between 0.033 and 0.038 ms. We note that the surface of the neutron star, visible as a large concentration of iso-density (dark-green) contours initially at about 10 km Fig. 9, falls towards the center. In models with the standard GGL-EoS, the whole neutron star is accreted, whereas it avoids this fate for the model initially built and later evolved with the SGGL-EoS, where a shock wave is launched at the surface and ejects parts of the matter (blue shades in the lower right panel of Fig. 9). This effect is, however, only circumstantial to our analysis since it is not connected to the appearance of non-convex regions in the star. It is and artefact due to the artificial atmosphere that surrounds the initial configuration, which is necessary for numerical reasons. Instead, we turn our attention to the central regions before the formation of the BH.

Model P-1.9G (top left panel of Fig. 9) possesses regions where the EoS is non-convex right from the beginning: all gas inside radii of $r_{	ext{tot}} \approx 4.8$ km and $r_{	ext{csc}} \approx 4.0$ km is relativistically and classically non-convex (cf. the orange and red lines). As the collapse accelerates, velocities become supersonic and sonic points form at $t \approx 0.01$ ms and $r_{	ext{csc}} \approx 2.1$ km (white lines). Note that different from a standard collapse developed with a fully convex EoS (of which model P-1.9S is an example), two separated sonic points form relatively close to the stellar center (i.e. detached from the -artificial- dynamics of the nearly free-falling stellar surface). This is because of the non-monotonicity of the sound speed dependence with density (e.g. Fig. 3). Shortly afterwards, the density increases sufficiently for the central regions to become convex. At $t \approx 0.03$ ms, the inner sonic point and the boundary between convex and non-convex regions merge. At this point, a shock wave forms at this transition (in the spacetime diagram, it appears as a transition from darker to lighter brown in the radial velocity maps starting at $r \approx 2.8$ km, and following the innermost white line). Since the compactness of our cores is so large, the collapse is too violent for the shock wave to propagate outwards or explode the star. Instead, it remains an accretion shock through which gas falls towards the center. Furthermore, it is rather short-lived and disappears at $t \approx 0.065$ ms inside the nascent BH. After BH formation at $t \approx 0.066$ ms the sonic point initially located closer to the center falls on the growing apparent horizon, which becomes a transonic point thereafter. The second sonic point, initially located further off center, soon follows the same fate and touches the apparent horizon at $t \approx 0.013$ ms. Meanwhile, the collapsing outer stellar shells speed up and become supersonic, first close to the infalling surface and a bit later closer to the apparent horizon. The formation of another sonic point right at the location where the initial atmosphere is set up (a point that is also free falling with the rest of the star) is an
Figure 9. Evolution of six different spherically symmetric models, from top left to bottom right P-1.9G, P-1.6G, P-1.45G, P-1.9S, G-1.9G, and S-1.9S. The diagrams show the velocity in units of the speed of light as a function of time and radius. In addition, contours of density (dark green lines) and the boundaries of the regions of classical and relativistic non-convexity are displayed (classical: dark red, relativistic: orange lines; models with the SGGL-EoS do not exhibit such regions) and the locations of the sonic point are marked by white lines. The white region at the bottom of each panel is the excised BH. The black-blue-white, triangular region displayed in the lower right panel corresponds to parts of the self-gravitating configuration that bounce and acquire positive radial speeds.
Neutron star collapse and gravitational waves with a non-convex equation of state

artefact of the atmospheric initialization. After most of the mass falls onto the BH ($r \approx 0.14$ ms) this artificial sonic point remains steady at a distance of $r \approx 3.6$ km. From there on, the dynamics ceases and a steady accretion of the artificial atmosphere goes on. We stress again that the mass in the atmosphere is totally negligible with respect to the initial mass of the model.

Reducing $\gamma_1$ to a value of $\gamma_1 = 1.6$ (model P-1.6G, top right panel of Fig. 9) leads to a faster collapse and reduces the extent of non-convex regions. Although sonic points form as in the previous model, they do not align with the border of the non-convex region. A shock forms, but it is much weaker than before, hardly noticeable in the space-time diagram. A further reduction to $\gamma_1 = 1.45$ (model P-1.45G, middle left panel) entirely removes the non-convex region. No shock can be observed, and the collapse proceeds smoothly. This statement does not hold for model P-1.9S (middle right panel), where we use the SGGL-EoS with $\gamma_1 = 1.9$. In this case, the absence of non-convex regions is not due to the low value of $\gamma_1$, but to the constant adiabatic index above $\rho_1$. This case demonstrates that the appearance of a shock wave is not solely connected to the value of $\gamma_1$, but to the non-convexity. This remark is relevant in view of the fact that virtually all EoS of nuclear matter yield values of $\gamma$ significantly larger than $\gamma_1 = 1.45$ for rest-mass densities above $\rho_0 \approx 10^{14}$ g cm$^{-3}$.

The two models with the initial data constructed for the (S)GGL-EoS (G-1.9G and S-1.9S, bottom panels of Fig. 9) confirm the findings obtained for the polytropic initial models. For the standard GGL-EoS, a shock is formed at the inner border of the convex region, where the inner sonic point is situated. Similarly to the polytropic initial model, it does not suffice to explode the star and ultimately ends up in the BH. The model with the SGGL-EoS, on the other hand, does not develop a shock wave in the vicinity of the BH.

Models P-1.9G and G-1.9G are evolved with the same GGL-EoS, but differ in the initial configuration, which is polytropic (with $\gamma = 2$) for the former and constructed according to the GGL-EoS with $\gamma_{\text{max}} = \gamma_1 = 1.9$ for the latter. This difference in the initial configuration yields a temporal shift to the overall dynamics, which otherwise is qualitatively the same. We observe a delayed BH formation in model G-1.9G compared to model P-1.9G (see Table 1). Furthermore, the formation of the shock associated with the existence of two sonic points in the collapsing fluid is also present (though delayed) in model G-1.9G. Thus, we conclude that building a polytropic initial model and then evolving it with the GGL-EoS does not introduce major differences either in the qualitative dynamics, nor in the final fate of the collapsing core.

We have also run a series of models having relatively large masses of nearly 2$M_\odot$. This series is formed by models P-1.9G1, P-1.9G2, P-1.9G3 and P-1.9G4, which have all the same initial polytropic model ($\gamma = 2$ and $\kappa = 100$ in the same units we employ later in Sec. 6, or, equivalently $\kappa = 3.46 \times 10^5$ in CGS units) but the evolution is followed employing the GGL-EoS with different values of $\rho_1$ and $\sigma$ (see Table 1). For the latter, we fix $\sigma = 1.10$ in the former four cases. These models are the non-rotating analogs of the models D1 that we will introduce in the next section (see Tab. 3). Spacetime diagrams of the logarithm of the rest-mass density of all these models are displayed in Fig. 10. We observe that all of them show the same qualitative behaviour as described for the reference case P-1.9G. From this series of models we observe that BH formation time increases with $\rho_1$ (see Table 1). In the spacetime evolution of the rest-mass density we observe the much smaller density of the surrounding (rarefied) atmosphere (red shades in all panels of Fig. 10). We also point out that the BH-excised region displays a density gradient from the values in the atmosphere to the highest densities in the domain (note the regions below the inner sonic point displayed with a white contour, which is nearly horizontal for $t > 0.15$ ms, in all the panels of the figure). This gradient is the result of the radial velocity shift we apply inside of the excised region (see above) to concentrate effectively all the mass in a volume around $r \approx 0$. There is, however, a small quantitative difference among the P-1.9G1 to P-1.9G4 series of models in the time of shock formation, which is associated with the loss of convexity of the EoS, as in the previous models of lower mass. The region of non-convexity does not appear from the very beginning in model P-1.9G4. There is, first a small region surrounding the center of the star where the relativistic fundamental derivative becomes negative during a brief and transitory episode ($0.02 \text{ ms} \lesssim t \lesssim 0.036 \text{ ms}$). Due to the adjustment of the central region to the loss of convexity, a small oscillation happens and the core slightly expands. Since the collapse is ongoing, the oscillation is very quickly dumped and, once the density in the vicinity of the stellar center grows again above $\rho_1$, the non-convex region begins to grow from the center (at $t \approx 0.04$ ms) until it reaches a maximum radial extend of $r \approx 4.5$ km at $t \approx 0.07$ ms. As the shock forms so close to the BH formation time, it is even very difficult to detect it as a shock in our numerical simulations. We observe that the region where the classical fundamental derivative is negative does not appear from the beginning in model P-1.9G3. Instead, it appears at $t \approx 0.01$ ms at $r \approx 2.8$ km, moves radially outwards a few hundred meters (up to $r \lesssim 3$ km) and then falls back onto the BH. Also in the latter model the shock formation is slightly delayed with respect to the initiation of the core collapse, though not so much as in model P-1.9G4. In model P-1.9G3 the shock forms sufficiently early to be clearly captured in our simulations.

We point out that the numerical code employed in this section is different from the one used in the next one for reasons we discuss in Section 7. We have repeated the experiments presented in this section with the same fully general relativistic hydrodynamics code with which we obtain the results of Section 6 finding that the qualitative results as well as the quantitative details are nearly the same. This result is reassuring from the methodological point of view since the algorithms implemented in both codes are significantly different. We also consider the independence of the results with respect to the numerical details as a clear hint of their robustness.

6 GRAVITATIONAL COLLAPSE OF ROTATING NEUTRON STARS

In order to study the effects of using our non-convex GGL-EoS in a fully dynamical situation, we consider uniformly rotating neutron star models that are dynamically unstable to axisymmetric perturbations and, hence, collapse to BHs on a dynamical timescale. In the previous section, we have chosen the spherically-symmetric and non-rotating initial data for this purpose. Here, we rather consider the more interesting rotating case since it allows to identify the influence of the non-convex EoS not only on the dynamics of the collapse but also on the gravitational-wave signals produced in the process. In particular, we use as initial data two uniformly rotating relativistic star models, dubbed D1 and D4, that have been previously used in a number of numerical-relativity simulations of neutron star collapse (Font et al. 2002a; Baiotti et al. 2005a; Baiotti et al. 2007; Giacomazzo et al. 2011). We construct our initial rotating stellar models for a polytropic EoS, $p = \kappa \rho^\gamma$, where $\kappa = 100$
(in code units, where $G = c = M_\odot = 1$) is the polytropic constant and $\gamma = 2$ is the adiabatic index, using the RNS open-access code (Stergioulas & Friedman 1995). The main characteristics of our two models are reported in Table 2. Model D1 is slowly rotating and thus almost spherical, with a ratio of polar-to-equatorial coordinate radii of $r_p/r_e = 0.95$. Correspondingly, model D4 is rotating almost at the mass-shedding limit, with $r_p/r_e = 0.65$. BU2 is a stable model with $r_p/r_e = 0.90$.

The numerical evolution of the initial data entails solving the coupled system of equations given by Einstein’s equations, governing the dynamics of the gravitational field, and by the hydrodynamics equations, governing the dynamics of the matter. This is done using the numerical-relativity code in spherical-polar coordinates described in Baumgarte et al. (2013); Montero et al. (2014) and that we have used in previous works (see e.g. Sanchis-Gual et al. 2015, 2017). The Einstein equations are formulated in the so-called BSSN formulation (Baumgarte & Shapiro 1998; Shibata & Nakamura 1995). The evolution equations are integrated using the second-order PIRK method (Cordero-Carrión & Cerdá-Durán 2012; Cordero-Carrión & Cerdá-Durán 2014) which allows to handle singular terms associated with the choice of curvilinear coordinates. The derivatives in the spacetime evolution are computed using fourth-order finite-differences, including fourth-order Kreiss-Oliger dissipation terms to avoid high-frequency noise. The equations of hydrodynamics are formulated in the so-called Valencia formulation (Banyuls et al. 1997) and solved using the second-order MC reconstruction scheme and the HLLE approximate Riemann solver (Montero & Cordero-Carrion 2012). Despite the initial data are built using a polytropic EoS, they are evolved in our code using the GGL-EoS, Eqs. (4-5). As we have tested in the Sec. 5, building a polytropic initial model and then evolving it with the GGL-EoS does not introduce major differences either in the qualitative dynamics, or in the final fate of the collapsing core. It simply results in a delayed dynamics, including the time of BH formation.

The gravitational radiation produced during the collapse of the neutron stars is computed using the Newman-Penrose for-
Neutron star collapse and gravitational waves with a non-convex equation of state

Figure 11. (Top panel): L1-norm of the difference between the evolved rest-mass density and the initial one as a function of time for model BU2. Three different resolutions have been employed. In the legends, we show the minimum radial grid spacings of each of them (Δr, Δθ) = [(0.05, π/16), (0.071, π/12), (0.10, π/8)]. The results corresponding to the finer resolutions are multiplied by the factors of 4 and 2 to show clearly the second order convergence of the method. As in all evolution plots, time is given in “code units”, corresponding to G = c = M = 1. (Bottom panel): Radial profile of the Hamiltonian constraint for model D1 with ρ1 = 1.5 × 10⁻³ for three different resolutions (Δr, Δθ) = [(0.035, π/44), (0.05, π/32), (0.10, π/16)] rescaled to second order convergence. All models have been evolved for a dimensionless time t = 50. The snapshot corresponds to the dashed curve in the right panel in Fig. 14. The vertical cyan line signals the location of the shock wave in model D1 with ρ1 = 1.5 × 10⁻³ (Tab. 3). Around the shock location is where the largest (absolute value) violations of the Hamiltonian constraint occur in our models.

Figures 12 and 13. (Top panel): Initial radial distribution of Γ1 along the equatorial plane for model D1. Bottom panel: Time evolution of the central rest-mass density ρ for model D1. The vertical dashed lines indicate the time at which the apparent horizon forms for each value of ρ1. The legend is the same for the two panels.

Figure 13. Spins of the final BHs after the collapse of model D1 (solid lines) and model D4 (dashed lines). Each set of four curves corresponds to the four values of ρ1 in the same colour code as in Fig. 12.

More precisely, we compute the so-called Newman-Penrose scalar Ψ4, defined by Ψ4 ≡ C_{αβγδ} n^α m^β n^γ m^δ, where C_{αβγδ} is the conformal Weyl tensor associated with the spacetime metric g_{αβ} and n, m are part of a null-tetrad. We use the definition of the electric and magnetic parts of the Weyl tensor, E_{ij} and B_{ij}, as a function of the 3+1 variables evolved by the code, to rewrite the Weyl Ψ4 scalar as Ψ4 = Q_{ij} n^i m^j with Q_{ij} ≡ E_{ij} - B_{ij}. We then compute the l = 2, m = 0 multipole (which is the dominant mode since the collapse is essentially axisymmet
ric) from
\[
\Psi_4(t, \theta, \phi) = \sum_{\ell m} \Psi_4^{lm}(t) \cdot Y_{\ell m}(\theta, \phi),
\]
\[
\Psi_4^{lm}(t) = \int \Psi_4(t, \theta, \phi) \cdot Y_{\ell m}(\theta, \phi) d\Omega,
\]
where \( Y_{\ell m} \) are the \((s=-2)\) spin-weighted spherical harmonics.

In order to test the convergence and the gravitational-wave extraction properties of our code we first evolve the stable rotating neutron star model BU2 in Table 2 (Stergioulas et al. 2004). Following Font et al. (2002b), we perturb the velocity of the initial model according to
\[
u_0(t = 0) = 0.02 \sin \left( \frac{\pi r}{R_e} \right) \sin \theta \cos \theta,
\]
where \( R_e \) is the circumferential equatorial radius.

The top panel of Fig. 11 shows the time evolution of the L1-norm of the difference between the evolved rest-mass density and its initial value computed for all the grid points inside the star. The
Figure 15. Same as Fig. 14 but for model D4.

Table 4. Central properties of various models used in the rotating neutron star collapse simulations. $\gamma_0 = 4/3$ and $\gamma_1 = 1.9$ for all models.

| Model | $\sigma/\rho_1$ | $\rho_1$ | $\rho_{\gamma}/\rho_1$ | $\rho_{\gamma}/\rho_1$ |
|-------|----------------|----------|------------------------|------------------------|
| D1    | 1.10           | $1.5 \times 10^{-3}$ | 2.187                  | 0.366                  |
| D1    | 1.10           | $1.7 \times 10^{-3}$ | 1.929                  | 0.387                  |
| D1    | 1.10           | $2.0 \times 10^{-3}$ | 1.562                  | 0.394                  |
| D1    | 1.10           | $2.5 \times 10^{-3}$ | 1.312                  | 0.368                  |
| D4    | 1.10           | $1.5 \times 10^{-3}$ | 2.077                  | 0.356                  |
| D4    | 1.10           | $1.7 \times 10^{-3}$ | 1.833                  | 0.373                  |
| D4    | 1.10           | $2.1 \times 10^{-3}$ | 1.484                  | 0.370                  |
| D4    | 1.10           | $2.5 \times 10^{-3}$ | 1.246                  | 0.339                  |

We start by fixing the value of the Gaussian width to $\sigma = 1.1$ and study the effects of varying the parameter $\rho_1$. In the top panel of Figure 12 we show the initial radial profile of $\Gamma_1$ along the equator for model D1 and for the different values of $\rho_1$ we are considering.
For later reference, we point out that the set of models D1 with 
\( \rho_1 = \{0.9356, 1.06, 1.31, 1.559\} \times 10^{-3} \text{ g cm}^{-3} \) (Table 3) can be regarded as 2D rotating counterparts of models P-1.9G1, P-1.9G2, P-1.9G3 and P-1.9G4 of Sec. 5 (see Table 1). The non-convex region of the EoS becomes – in radius – larger as \( \rho_1 \) becomes smaller, as can be seen from the larger region of non-monotonicity of \( \Gamma_1 \) in the top panel of Fig. 12. The time evolution of the central density of model D1 for the four different values of \( \rho_1 \) is shown in the bottom panel. Note that the radius and the time is given in these two panels in code units. The time evolution shows that the smaller the value of \( \rho_1 \), the faster the collapse takes place. This happens because \( \Gamma_1 \) (and also \( \gamma \)) is significantly smaller near the central regions of the star as \( \rho_1 \) is reduced (cf. top panel of Fig. 12) and, therefore, the pressure becomes smaller. The time of the formation of the apparent horizon of the BH is indicated in the figure by the vertical dashed lines. We note that BH formation time for models of the series D1 and different values of \( \rho_1 \) are about a factor two longer than the values found for models P-1.9G1 to P-1.9G4. The BH formation times in the D1 series range from \( t_{\text{BH}} \approx 27 \) to \( t_{\text{BH}} \approx 45 \) code units, or equivalently, \( t_{\text{BH}} \approx 0.13 \) ms to \( t_{\text{BH}} \approx 0.22 \) ms. We attribute the small differences to the rotation present in the 2D models rather than to the approximate treatment of the general relativistic gravitational potential in the AENUS code.

The final outcome is in all cases a rotating Kerr BH whose spin parameter is plotted in Fig. 13. This figure shows that for all the unstable models, the final value of the BH spin is fairly independent of \( \rho_1 \). The spin is computed using the expression

\[
\frac{a}{M_{\text{BH}}} = \sqrt{1 - (-1.55 + 2.55C_\gamma)^2},
\]

where \( C_\gamma \) is the ratio of polar-to-equatorial proper circumference and \( M_{\text{BH}} \) is the mass of the horizon, which coincides with \( M \) when the spacetime has become axisymmetric and stationary. This expression has an accuracy of \( \approx 2.5\% \) (Brandt & Seidel 1995; Baiotti et al. 2005b). The values for the spin and for the irreducible mass with our GGL-EoS differ with those obtained employing a polytropic EoS (Baiotti et al. 2005b) by less than 1%.

In Figs. 14 and 15 we plot the radial profiles of the velocity of the fluid and of the rest-mass density (shown in the insets) for models D1 and D4, respectively. The profiles are plotted at the equatorial plane (\( \theta = \pi/2 \)). The different curves indicate different times during the evolution. The four panels in each of the two figures correspond to the four values of \( \rho_1 \), as indicated in the caption of Fig. 14. We note that for convex EoS, as a polytrope or a gamma-law, the dynamics of the collapse proceeds smoothly towards the formation of a BH, as discussed in Font et al. (2002a); Baiotti et al. (2005a) and we have show in Sec. 5. The larger the centrifugal support of the initial model, the more it takes for the model to collapse. As shown in Baiotti et al. (2005a), the collapse of the rapidly-rotating model D4 goes through a short-lived centrifugal hang-up when the stellar surface slows its inward motion and stalls, although ultimately it shrinks to a volume smaller than that of the radially-increasing event horizon that forms at the central regions. During the evolution of these models, a shock develops at the edge between the homologous inner core of the star and the outer core, which falls supersonically. Consistent with the dynamics observed in the 1D models of Sec. 5, this shock is eventually engulfed by the growing BH that forms as a result of the collapse. For the nearly-spherical D1 model this process is much faster than for the rapidly-rotating model D4. We have also performed the evolutions using an ideal gas EoS, in order to qualitatively compare our findings on the dynamics and on the gravitational-wave emission with the results from these previous works.

It is important to highlight that the formation of the former shock is entirely due to the non-convex dynamics. In the case with \( \rho_1 = 2.5 \times 10^{-3} \) (bottom-right panels of Figs. 14 and 15), there is no such shock because for that value of \( \rho_1 \) the sound speed in the non-convex region is (much) larger than that of other models.
with smaller values of $\rho_1$. This fact prevents reaching a supersonic regime in the convex region and avoids the formation of the shock. We also point out that for the case with $\rho_1 = 2.1 \times 10^{-3}$ (bottom-left panels of Figs. 14 and 15), and contrary to the two cases displayed in the top panels of both figures, the shock propagates outwards. Furthermore, the flow speed ahead of the shock location is slightly positive. This is due to the borderline behaviour of this model, which develops a tiny supersonic region right to the inner radial boundary where the classical fundamental derivative is negative (green triangles in Figs. 14 and 15). This supersonic region persists for a relatively short time an along its inner boundary is where the shock forms. We note that the behaviour described for the models D1 with $\rho_1 = 2.1 \times 10^{-3}$ and $\rho_1 = 2.5 \times 10^{-3}$ bears qualitative similarities with the 1D models P-1.9G3 and P-1.9G4, respectively. In model P-1.9G3, we also observe a tiny radial outwards displacement of the shock and the shock formation is significantly delayed with respect to other models of the same series in the case of model P-1.9G4. Thus, we conclude that there is a gross qualitative agreement between the 2D models D1 and their non-rotating counterparts in 1D. The small quantitative differences are almost exclusively induced by the rotation of the former models.

Figure 16 displays the gravitational-wave signals $\Psi^{20}$ for models D1 and D4 and for all values of $\rho_1$. For the sake of comparison, the three panels of this figure also include additional curves which correspond to a polytropic (convex) EoS. The waveforms are extracted at a radius $r_{GW} = 200$. For convex EoS, gravitational waveforms of the collapse of these two models have been reported before by Giacomazzo & Perna (2012). The signal is of the burst-type, i.e. it is characterized by an exponential increase of the amplitude and by a short-duration burst at the moment of BH formation (which coincides with the largest positive peak, see Dietrich & Bernuzzi (2015)) followed by the subsequent quasinormal mode ringdown of the BH. Our comparison with the results of Giacomazzo & Perna (2012) for convex EoS shows good agreement in the waveform morphology and amplitude, particularly for model D1 (for model D4 we obtain a few times larger amplitude; note the difference in the vertical scales between the upper and lower panels of Fig. 16).

The non-convex dynamics leaves an imprint in the gravitational waveforms produced during the process. The smallest amplitudes at the moment of BH formation are obtained for the polytropic EoS, specially in the case of model D1 (top panel of Fig. 16). For the GGL-EoS, the frequency of the various signals is quite close to each other, while their amplitudes are different depending on the value of $\rho_1$. This is more apparent for model D1 than for model D4. In the case of model D1 in particular, the largest gravitational-wave amplitude is obtained for $\rho_1 = 1.7 \times 10^{-3}$ (red curve in the top panel of Fig. 16). The maximum amplitude is about twice that attained in the polytropic case. For model D4 the maximum amplitude is also achieved for the same value of $\rho_1$, but the differences among the various simulated models are not as apparent as for model D1. This means that the faster the rotation of the initial neutron star, the smaller the imprint the loss of convexity leaves on the gravitational-wave signal after the BH has been formed.

The radially outwards propagation of the shock in model D1 with $\rho_1 = 2.1 \times 10^{-3}$ (Fig. 14) translates in slightly higher gravitational-wave amplitudes at the time of collapse, but slightly smaller in the instants preceding the BH formation. The speed of this outgoing shock is smaller than the speed at which the BH horizon grows and eventually all neutron star matter will be in-

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3 Notice that Giacomazzo & Perna (2012) associate the first negative peak to the moment of BH formation due to a global sign difference in the expression of $\Psi^4$ they use compared to ours and Dietrich & Bernuzzi (2015).
The waves shown in the bottom panel of Fig. 16 correspond to model D4. In this case, the maximum amplitudes of the burst signals are significantly larger than in the case of model D1, and for all values of \( \rho_1 \), the reason being the increased deviation from spherical symmetry of this rapidly-rotating model. While model D4 displays more similar gravitational waveforms for all values of \( \rho_1 \) than model D1, when comparing with the polytropic EoS there is still a visible change in frequency associated with the non-convexity properties of the GGL-EoS. This is particularly evident in the first part of the signal associated with the collapsing phase before the BHs form.

Additionally, we also study the effects of varying the width \( \sigma \) of the Gaussian used in the definition of the GGL-EoS, fixing \( \rho_1 = 2.1 \times 10^{-3} \). We analyze the dynamics of the collapse for four different values of \( \sigma \), namely \( \{1.10, 1.15, 1.20, 1.50\} \). The results for models D1 and D4 are displayed in Fig. 17, which depicts the radial profile of the fluid velocity at the equatorial plane. As we have shown before, for this value of \( \rho_1 \) the shock located in the region \( 0.2 < r < 0.9 \) attains a slightly positive speed if \( \sigma \lesssim 1.20 \). The jumps at the latter shock become gradually smaller when \( \sigma \) increases from 1.10 (see Fig. 14) to 1.20. For \( \sigma = 1.50 \) the shock is no longer visible and the dynamics resembles that of a convex EoS. This trend is the same for both models, i.e. it does not depend on the initial rotation of the unstable neutron star.

The corresponding gravitational waveforms are shown in Fig. 18. The waveforms look remarkably similar irrespective of the value of \( \sigma \), with minor differences in the peak amplitudes among all models. As in the cases previously analyzed, the waveforms of the most rapidly rotating models D4 are less sensitive to the changes in \( \sigma \) than in models D1.

7 SUMMARY AND OUTLOOK

A number of microphysical EoSs of high-density matter contain regions in which the thermodynamics may be non-convex. These EoSs, commonly used in a tabular form, may develop non-convex thermodynamics either as a result of phase transitions, or non-consistent treatment of the matter constituents (Newtonian instead of relativistic), or poor parameterizations of the RMF theoretical framework. In the first group we find EoSs where transitions from nuclear hadronic matter into quark-gluon plasma or into matter phases containing exotic particles (e.g. hyperons) are included employing suitable Gibbs constructions. The second group gathers EoSs in which baryons are treated as non-relativistic ideal gases. A prototype example of the latter group is the LS220 EoS. To the third group belong the GSHen EoS with a parameter set FSUGold (or FSU1) in the RMF treatment. However, other parameterizations of the RMF (e.g. FSU2 or NL3) seem to be fully convex even at high density. We cautiously warn that the loss of convexity of the GSHen EoS with the FSU1 parameterization may result from an insufficiently fine discretization of the hadronic table at baryonic number densities \( n \gtrsim 1 \text{fm}^{-3} \).

In any instance, a good number of the studied microphysical EoSs display a sensitive reduction of the relativistic fundamental derivative as the baryon number density grows above \( n \gtrsim 1 \text{fm}^{-3} \). Unfortunately, most available microphysical EoSs are only tabulated up to baryon number densities \( n \lesssim 3 \text{fm}^{-3} \), making it difficult to assess whether convexity will be lost at high enough baryon number density. Hopefully the present work will spark an interest in this question, by pointing the phenomenological consequences that such a non-convex regime would have. Adding to these arguments, we point out the non-monotonic behaviour of the sound speed in dense matter found by Bedaque & Steiner (2015), which is a strong hint on the non-convex character of matter at high densities. Remarkably, Bedaque & Steiner (2015) found that the more abrupt the sound speed changes with density (above nuclear satu-
Neutron star collapse and gravitational waves with a non-convex equation of state

The existence of regions where the fundamental derivatives (heat capacity, speed of sound) are non-zero is of importance in understanding the dynamics of neutron star mergers and the formation of black holes (BHs). The collapse of neutron stars can be studied numerically to understand the evolution of the system.

In this paper, we present a numerical study of the structure, dynamics, and gravitational-wave signature of compact stellar configurations described by a BZT fluid. The numerical simulations of collapsing stars have shown that the non-convexity of the dynamics is not achieved in the course of the collapse, and hence, they may wash out any prominent effect of the convexity loss in the course of the collapse dynamics. A future study using actual microphysical EOSs from nuclear physics and a suitable neutrino transport is opportune and will be presented elsewhere.

APPENDIX A: ANALYSIS OF THE ‘PolyTh’ EOS

A simple way to mimic the complex thermodynamical processes taking place inside a collapsing stellar core in simulations of hydrodynamical supernovae and in the formation of compact objects, considers an EOS for which the pressure has two components, namely a polytropic component (the cold one, \( p_c \)), and an ideal-gas component which incorporates the thermal effects \( p_t \). This EOS, that we call ‘PolyTh’ reads as (see, e.g. Maione et al. 2016)

\[
p = p_c + p_t, \quad p_c = K \rho^{\Gamma_c}, \quad p_t = (\Gamma_t - 1) \rho \epsilon_t,
\]

(A1)

where

\[
\epsilon = \epsilon_c + \epsilon_t, \quad \epsilon_c = \epsilon_0 + \frac{K}{\Gamma_c - 1} \rho^{\Gamma_c - 1}.
\]

(A2)

The total specific internal energy \( \epsilon \), or its thermal component \( \epsilon_t \), can be considered as an independent thermodynamical variable, being the cold component \( \epsilon_c \) a function of density given by Eq. (A2). In practice, we take \( \epsilon_0 \), in Eq. (A2), equal to zero. The free parameters of the PolyTh-EoS are \( K, \Gamma_c \) and \( \Gamma_t \).

Let us define

\[
a^2_{\alpha} := \Gamma_{\alpha} \left( \frac{\rho_\alpha}{\rho} \right) = \Gamma_{\alpha} (\Gamma_{\alpha} - 1) \epsilon_{\alpha}
\]

(A3)

where \( \alpha = (c, t) \) stands, respectively, for the cold and thermal components of pressure. Hence, the classical definition of the local speed of sound can be written

\[
c^2_{\alpha} = a^2_{\alpha} + a^2_t.
\]

(A4)

The specific enthalpy is given by

\[
h := 1 + \epsilon + \frac{p}{\rho} = 1 + \Gamma_t \epsilon_c + \Gamma_c \epsilon_t,
\]

(A5)

or, alternatively,

\[
h = 1 + \frac{\epsilon_c^2}{\Gamma_c - 1} + \frac{a^2_t}{\Gamma_t - 1}.
\]

(A6)

The relativistic definition of the speed of sound is related to the classical one according to:

\[
c_{\alpha}^{(R)} = h^{-1} c_{\alpha} = \frac{\Gamma_c (\Gamma_c - 1) \epsilon_c + \Gamma_t (\Gamma_t - 1) \epsilon_t}{1 + \Gamma_c \epsilon_c + \Gamma_t \epsilon_t}.
\]

(A7)

From this equation we obtain the following constraint that the PolyTh EOS has to satisfy in order to be causal:

\[
c_{\alpha}^{(R)} \leq 1 \quad \Rightarrow \Gamma_c (\Gamma_c - 2) \epsilon_c + \Gamma_t (\Gamma_t - 2) \epsilon_t \leq 1.
\]

(A8)

Hence, assuming that both \( \epsilon_c \) and \( \epsilon_t \) are non-negative, a sufficient condition for causality is

\[
\Gamma_c \leq 2 \quad \text{and} \quad \Gamma_t \leq 2
\]

(A9)

These values of \( \epsilon_c^{\text{crit}} \) and \( \epsilon_t^{\text{crit}} \), at the stationary point of \( c_{\alpha}^{(R)} \):

\[
\frac{\partial^2 c_{\alpha}^{(R)}}{\partial \epsilon_\alpha} = 0 \quad \Rightarrow \epsilon_\alpha^{\text{crit}} = \frac{\Gamma_{\alpha} - 1}{\Gamma_{\beta} (\Gamma_{\beta} - \Gamma_{\alpha})} (\alpha \neq \beta).
\]

(A10)
By definition, the adiabatic exponent, $\Gamma_1$, is

$$\Gamma_1 = \left(\frac{\rho}{p}\right) = \left(\frac{\rho}{p}\right)^2 (a_c^2 + a_t^2), \quad \text{(A11)}$$

or, alternatively

$$\Gamma_1 = \Gamma_c \beta + \Gamma_t (1 - \beta), \quad \text{(A12)}$$

where $\beta := p_c / p$. According to Eq. (A12), $\Gamma_1$ can be considered as just the average of the cold and thermal ‘gammas’ weighted with their relative components of pressure.

The classical fundamental derivative, $\theta(C)$, for the PolyTh EoS is

$$\theta(C) = \frac{1}{2} (1 + \Gamma) \quad \text{(A13)}$$

where

$$\Gamma = \frac{\rho \alpha_c^2 + \rho \alpha_t^2}{\alpha_c^2 + \alpha_t^2} \quad \text{(A14)}$$

or, alternatively

$$\Gamma = \frac{\Gamma_c^2 \beta + \Gamma_t^2 (1 - \beta)}{\Gamma_t} \quad \text{(A15)}$$

which can be interpreted as the ratio between the mean of both $\Gamma_c^2$ and $\Gamma_t^2$ and the adiabatic exponent $\Gamma_1$. By construction, the quantity $\Gamma$ varies between the values of $\Gamma_c$ and $\Gamma_t$.

The relativistic fundamental derivative, $\theta(R)$, for the PolyTh EoS is

$$\theta(R) = \theta(C) - \frac{3}{2} \sigma_{s}^2 = \frac{1}{2} (1 + \Gamma - 3 \sigma_{s}^2) \quad \text{(A16)}$$

Some comments are in order:

1) From Eq. (A16), one concludes that the PolyTh EoS can develop, due to relativistic effects, non-convex regions there where the following relationships are satisfied:

$$\frac{1 + \Gamma}{3} \leq \sigma_{s}^2 \leq 1 \quad \text{and} \quad 1 \leq \Gamma \leq 2. \quad \text{(A17)}$$

where the lower bound on $\Gamma$ comes from its definition, assuming that: $\Gamma_{\text{crit}} \geq 1 \forall a_c = c, t$.

2) The analysis of the particular cases $\beta = 1$ and $\beta = 0$ can shed light on the previous conclusion. These cases are easily covered by taking $\Gamma = \Gamma_c$ and $\Gamma = \Gamma_t$, respectively, in Eqs. (A13), (A16) and (A17). Let us consider, e.g. $\beta = 1$. From Eqs. (A7), (A4) and (A6) we obtain

i) $\lim_{\epsilon_c \to \infty} \sigma_{s}^2 = \Gamma_c - 1, \quad \text{(A18)}$

ii) $\lim_{\epsilon_t \to \infty} \theta(R) = 2 - \Gamma_t, \quad \text{(A19)}$

and, therefore, the thermodynamics is convex for a causal EoS, if and only if $1 \leq \Gamma_t \leq 2$, as it happens for an ideal-gas EoS.

3) The above two comments help us to give the conditions to be satisfied by the PolyTh EoS in order to be both causal and convex:

$$\sigma_{s}^2 \leq -1 \quad \text{and} \quad 1 \leq \Gamma \leq 2. \quad \text{(A20)}$$

As an example, let us complete the analysis by taking for the PolyTh EoS one of the set of parameters used in the binary neutron star merger simulations of Maione et al. (2016), namely $\Gamma_c = 3.005$ and $\Gamma_t = 1.8$. We take $\epsilon_c$ and $\epsilon_t$ as the independent thermodynamical variables. Figure A1 shows the relativistic speed of sound, $c_{s}^2$, defined in Eq. (A7). It is an increasing function, in both $\epsilon_c$ and $\epsilon_t$, up to some value of $\epsilon_{c}^{\text{crit}}$ given by Eq. (A10). In our example, this value is $\epsilon_{c}^{\text{crit}} = 0.22$. For $\epsilon_c \geq \epsilon_{c}^{\text{crit}}$ (depending on $\epsilon_c$) the PolyTh EoS becomes non-causal. On the other hand, Figure A1 also shows the relativistic fundamental derivative, $\theta(R)$, given by Eq. (A16). It is a decreasing function in both $\epsilon_c$ and $\epsilon_t$. For $\epsilon_t \geq \epsilon_{t}^{\text{crit}}$ (depending on $\epsilon_t$) the PolyTh EoS becomes non-convex.

As a summary, from the above example and from our previous analysis, we conclude that the PolyTh EoS is convex in those regions of the space of parameters in which it is causal. The non-convex regions are associated with the non-causal ones and, therefore, the corresponding subset of parameters has no physical meaning.

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