On the cavity evolution and the Rayleigh–Plesset equation in superfluid helium

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On the basis of the two-fluid hydrodynamics, an analogue of the famous Rayleigh–Plesset equation for the dynamics of a spherical bubble in superfluid helium is obtained. The mass flow velocity \( v \) and the velocity of the normal component \( v_n \) were chosen as independent variables. Due to the two-fluid nature of He II, the cross terms in the evolution equation for the boundary position \( R(t) \) appeared, which were absent in classical Rayleigh–Plesset equation in ordinary fluids. One of them renormalizes the coefficient in front of \( (dR/dt)^2 \). Another additional term formally coinciding with the viscous term, describes the attenuation of the boundary oscillations. This "extra-damping" term, greatly exceeding the usual viscous term, leads to a significant difference in the dynamics of cavity compared to He I. In particular, this results in the interesting effect of abnormal suppression of oscillations of the vapor–liquid boundary observed in many works. There is also an additional term proportional to the squared velocity of the normal component, which is independent of the derivative \( dR/dt \), and can be included in the pressure drop. Its physical meaning is that it describes a ”Bernoulli”-like pressure created by the flow of a normal component. The obtained result declares that some results on the dynamics of the cavity in superfluid helium should be reviewed.

I. INTRODUCTION AND SCIENTIFIC BACKGROUND

The study of the cavity dynamics is an important part of the problems of continuum mechanics, including the hydrodynamics of superfluids. These may be problems related to the evolution of bubbles created by electrons, (see, e.g., \[1, 2, 3\]), bubbles caused by sound (see, e.g., \[4\]), cavitation due to negative pressure (see, e.g., \[5\]), collapse of bubbles (see, e.g., \[6, 7\]), sonoluminescence (see, e.g., \[8\]), etc. Another series of examples is related to the heat transfer in He II and to the possibility of utilizing superfluid helium as a coolant in cryogenic systems, which has been discussed extensively recently (see, e.g., book \[9\]). Knowledge of the laws governing the formation and development of vapor films on the surfaces of heaters is important for solving corresponding problems (see, e.g., \[10, 11, 12, 13\]).

Studying the dynamics of the cavity, the authors of the above works appeal to the Rayleigh–Plesset problem on the evolution and oscillations of an air or vapor bubble, elaborated initially for an ordinary fluid (see, e.g., book \[14\] and references therein). Such treatment, however, is justified in the case when superfluid helium behaves as an ordinary fluid and moves as a whole with a mass velocity \( v = j/\rho \) (see the notations below). This situation occurs when helium is driven in the motion under the action of a pressure gradient (or gravity). However, whenever the heat fluxes occur, a flow of the normal component appears with a velocity \( v_n \), different from the mass velocity \( v \) and the problem requires a fundamentally different treatment related to the two-fluid nature of He II.

In the present paper the problem of evolution of a spherically symmetric vapor bubble in the superfluid helium is considered. The equation playing the role of the famous Rayleigh–Plesset equation in an ordinary fluid is derived. The obtained equation results in a number of effects absent in ordinary fluids. They include effects such as additional pressure caused by the movement of the normal component or an extra-damping term. These effects essentially influence the dynamics of cavities and should be taken into account in the relevant works. To our knowledge, the analog of the classical Rayleigh–Plesset equation for the two-fluid hydrodynamics of superfluid helium have been not previously considered.

Loosely speaking the dynamics of bubble is determined by inertia (mass) of ambient liquid and by the elastic properties due to pressure inside cavity. The latter is result of many factors including such as surface tension, Coulomb pressure, hydrostatic pressure, viscous contribution into stress tensor etc. The corresponding task is a complex problem requiring careful analysis of many factors.

In this work, we will concentrate on hydrodynamics processes inside the fluid, not considering the involved phenomena inside the bubble. In addition, for clarity and for deductive purposes, we take a pure two-fluid Landau-Khalatnikov model, excluding more complex phenomena, such as quantum turbulence (see, e.g., \[15, 16\]) or existence of tiny thermal boundary layer near the interface boundary liquid-vapor (see, e.g., \[17\]). Realizing the importance of the above processes on the whole evolution of the cavity, we nonetheless omit them in order to highlight the fascinated features of the two-fluid Landau-Khalatnikov model of superfluid hydrodynamics in the process under consideration.

The structure of the paper is as follows: The second and third sections can be considered as the introductory. They are useful to introduce basic ideas and methods, notations and terminology. In Sec. II we shortly describe the Rayleigh problem in ordinary (nonsuperfluid) fluid and in Sec. III we write down and comment a set of the
motion equation of superfluid hydrodynamics. Sec. IV is devoted to a detailed derivation of the Rayleigh–Plesset like equation for the evolution of the boundary position of a spherical bubble in superfluid helium. The main features of the derived equation are also described there. In Conclusion the remarks on the obtained results are made and possible development is discussed.

II. THE VAPOR FILM DYNAMIC IN ORDINARY (NONSUPERFLUID) FLUIDS

Before starting our study on the bubble dynamic in superfluid helium it seems instructive to briefly describe the corresponding Rayleigh problem in the ordinary (nonsuperfluid) fluid (see, e.g. book [14]). In spherical coordinates the continuity equation for velocity $v$ reads (the fluid is supposed to be incompressible)

$$\frac{\partial}{\partial r}(r^2v) = 0.$$ (1)

The obvious solution of equation (1) is

$$v(r, t) = \frac{R^2}{r^2} \frac{dR}{dt}.$$ (2)

Here $R = R(t)$ is the time dependent radius of the bubble. Solution (2) is a consequence of that the rate of change of the bubble radius $dR/dt$ coincides with the velocity $v(r = R, t)$, which justifies Eq. (2). Here we neglect the motion of the boundary arising from the mass flux density at the interface liquid-vapor due to evaporation (or condensation) [13].

The momentum equation in spherical coordinates reads

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$ (3)

Substituting the solution (2) into expression (3) and integrating over $r$ in limits from from the bubble boundary $r = R(t)$ to $\infty$ we have.

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left( \frac{dR}{dt} \right)^2 \left( \frac{dR}{dt} \right)^2 = \frac{(p(R) - p_{\infty})}{\rho}.$$ (4)

Here $p_{\infty}$ is the pressure far from the bubble, often the hydrostatic pressure $\rho gh$. Thus, as it was described above, the evolution of the vapor bubble occurs due to inertia of fluids and elastic properties of pressure on the bubble boundary. The latter appears from many factors, viscosity, surface tension etc (see, e.g. book [14]). When the fluid viscosity and the surface tension are taken into account, equation (4) is converted into the classical Rayleigh–Plesset equation

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left( \frac{dR}{dt} \right)^2 + \frac{4}{R} \frac{dR}{dt} = \frac{(p(R) - p_{\infty} - 2\gamma)}{\rho},$$ (5)

where $\nu$ is the kinematic viscosity and $\gamma$ is the surface tension. The Rayleigh–Plesset equation (5) is intended to describe the evolution of the bubble, to determine the size of the bubbles, the oscillation frequency, etc.

The principle problem is to determine the pressure $p(R)$ at the interface. For that it is necessary to use some additional considerations. In the simplest case, it can be assumed that an adiabatic process occurs inside the bubble. However, in general case, the pressure should be extracted from resolution of adjoint thermal problem. For instance in series of works on the boiling helium ([13], [10], [13]), the vapor pressure $p(R)$ is determined from the Boltzmann kinetic equation for problems of evaporation and condensation. Thus, "mechanical" and "thermal" problems are tied already in classical fluids. As we will see later, this statement is more actual in case of superfluid helium.

III. HYDRODYNAMICS OF QUANTUM FLUIDS. TWO-FLUID MODEL

Before we consider the dynamics of the vapor bubble in the superfluid helium, it is useful to recall shortly the hydrodynamics of superfluid helium. From the viewpoint of hydrodynamics, He II can be viewed as a mixture of two components. One of them, a superfluid liquid with density $\rho_s(p, T)$ ($p$ and $T$ are the pressure and the temperature, correspondingly), moves with velocity $v_s$. The superfluid component has no shear viscosity, it cannot be subjected
to torsion ($\nabla \times \mathbf{v}_s = 0$), and also cannot absorb and carry heat. Another part with the density $\rho_n(p, T)$, moving with velocity $\mathbf{v}_n$, is the normal component, it behaves as usual classical viscous fluid. From a deeper point of view, the flow of normal component is just the drift of the thermal excitations (phonons and rotons), which appeared in the background coherent state The motion of the two components is thermodynamically reversible and consequently independent. The superfluid component $\rho_s(T)$ appears at below $T_\lambda \approx 2.1768$ K at saturated vapor pressure, growing with the decrease of the temperature and reaching the full density $\rho = \rho_s(T) + \rho_n(T)$ at zero temperature. The full flux of mass is just

$$j = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$$

(6)

The equations of motion of such a liquid can be obtained on the basis of the laws of conservation [19] (see also [20], [17]). We shall write out and explain these equations, here we restrict ourselves with the dissipationless case:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

(7)

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0,$$

(8)

$$\frac{\partial S}{\partial t} + \nabla \cdot [S \mathbf{v}_n] = 0,$$

(9)

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla (\mu + \frac{\mathbf{v}_s^2}{2}) = 0.$$  

(10)

Equations (7), (8) are the usual laws of conservation of mass and momentum density.

The momentum density consist of two parts - superfluid and normal

$$j_i = \rho_s v_{si} + \rho_n v_{ni},$$

(11)

The momentum flux tensor $\Pi_{ik}$ has also two ingredients and is equal to

$$\Pi_{ik} = \rho_s v_{si} v_{sk} + \rho_n v_{mi} v_{nk} + \delta_{ik} \rho$$

(12)

The subscripts $i, k$ denote the coordinates $x, y, z$; $\delta_{ik}$ is the unit tensor. As can be seen from (12), the complete momentum-flux tensor can be decomposed into a normal part and a superfluid part, and Eq. (8) has an obvious structure. Equation (9) is also obvious. This is the law of conservation of density of entropy $S(p, T)$. Here we see reflected the fact that entropy is carried over only by the normal component. The expression (10) for the velocity of the superfluid component is new, in contrast with the expression for an ordinary liquid. It contains the information that the superfluid component cannot be subjected to torsion because it has no shear viscosity. Therefore, $\nabla \times \mathbf{v}_s$, and consequently, the convection term is $(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = \nabla (\mathbf{v}_s^2/2)$. The driving force for the superfluid part is the chemical potential $\mu(p, T)$.

We write an expression for the energy flux $W$ which follows from Eqs. (7)-(10) and which will be useful in our further discussion:

$$W = (\mu + \frac{\mathbf{v}_s^2}{2})j + STv_n + \rho_n \mathbf{v}_n ((\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{v}_s) + W_{irr}$$

(13)

Here $W_{irr}$ stands for the irreversible fluxes caused by the dissipative effects, which are negligibly small for all real cases. It should be noted at once that we observe a macroscopic energy flux $ST\mathbf{v}_n$ even in the case when the total mass flow $j$ is equal to zero (the so called counterflow). Neglecting the nonlinear effects of third order and taking that the energy flux $W$ in this case is just the heat flux $Q$, we arrive at formula

$$Q = ST \mathbf{v}_n$$

(14)

In case $v_n = v$ (and in accordance with Eq. (11)) $\mathbf{v}_n = \mathbf{v}_s = \mathbf{v}$, that is, the fluid moves as a whole (the so called co-flow) the energy flux $W$ has a form

$$W = (\mu + \frac{\mathbf{v}_s^2}{2})j + ST\mathbf{v} = (\mu + \frac{\mathbf{v}_s^2}{2})\mathbf{v} + ST\mathbf{v} = (\mu + \frac{\mathbf{v}_s^2}{2})\mathbf{v}.$$  

(15)

(here $h = \mu + TS$ is the enthalpy) as it should be in the one-phase fluid or in case of co-flow (see, e.g. the handbook [21]).
IV. RAYLEIGH–PLESSET PROBLEM IN SUPERFLUID HELIUM.

A. Treatment of the momentum flux tensor $\Pi_{ik}$

From the solution of the problem in classical fluid it is seen that the master variable, which controls the whole process is the mass flow velocity $\mathbf{v}(\mathbf{r}, t)$, for the simple reason that in spherical case the variable $v(r, t)$ at the boundary points coincides with quantity $dR/dt$. The master Rayleigh–Plesset equation is derived from equation for momentum density, which, due to non compressibility condition is just the Euler equation (3) for velocity $v(r, t)$.

In the superfluid helium situation is more complicated. The reason is that the equation (8) for momentum density $\Pi_{ik}$, does not include directly the mass flow velocity $v$ (see Eq. (3)). In other words the following inequity takes place

$$\frac{\partial}{\partial r_k} \Pi_{ik} \neq \rho v_k \frac{\partial v_i}{\partial r_k}$$

and as it is frequently used or understood in the relevant works. On the contrary, because of the two fluid hydrodynamics, the structure of the momentum flux density tensor is more involved and should reflect the presence of two ingredients - the superfluid and normal parts.

In fact, the superfluid and normal velocities $v_s = \mathbf{v}_s, v_n = \mathbf{v}_n$ are not convenient for solving our problems. More suitable are variables the mass flow velocity $v(r, t) = j/\rho = (\rho_s v_s + \rho_n v_n)/\rho$ and the velocity $v_n$ of normal component. Indeed, the mass flow velocity $v(r, t)$ is responsible for inertia of fluid and the normal velocity $v_n$ is tightly related to thermal processes in accordance to relation (14). Further we take that the total density $\rho$, as well as the superfluid and normal densities $\rho_s$ and $\rho_n$ separately, are constant.

The next, crucial step is to treat equation for momentum density (8). We have to express the momentum flux density tensor $\Pi_{ik}$ via quantities $v(r, t)$ and $v_n(r, t)$, which were selected as the primary variables. To get rid of the superfluids velocity entering in equation (12) for $\Pi_{ik}$ we use relation, known from classical superfluid hydrodynamics (see books [20], [17], and also Eq. (11)).

$$\mathbf{v}_s = \frac{\rho v}{\rho_s} \mathbf{v}_n$$

The momentum flux tensor $\Pi_{ik}$ (see Eq. (12)) can be rewritten in the chosen variables $v(r, t)$ and $v_n(r, t)$ as

$$\Pi_{ik} = \rho_s (\frac{\rho v}{\rho_s} \mathbf{v}_n) (\frac{\rho v}{\rho_s} \mathbf{v}_n) = \rho_n v_n v_{nk} + \delta_{ik} p$$

Now we have to transform the momentum flux density tensor $\Pi_{ik}$ (13) into spherical coordinates. The simplest way to do this is as follows. We can represent expressions of type $A \partial B / \partial x_k$ as a i component of combination $(\mathbf{A} \nabla) \mathbf{B}$. In accordance with well known mathematical relation $(\mathbf{A} \nabla) \mathbf{B} = \nabla (\mathbf{A} \cdot \mathbf{B}) / 2$ (see, e.g., [22]). The latter operation is possible, since due to spherical symmetry and incompressibility of both components, $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$ and the same for vector $\mathbf{B}$. Using this rule we rewrite the radial component of the momentum flux density tensor $\Pi_{rr}$ (18) in spherical coordinates as

$$\Pi_{rr} (r) = \frac{1}{2 \rho_s} \left( v^2 - 2 \rho \rho_n v v_n + \rho_n v_n^2 \right)$$

Then, the equation for momentum $j(r, t)$, or, better for the mass velocity $v(r, t) = j(r, t)/\rho$, has the form:

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\rho}{\rho_s} \frac{\partial \mathbf{v}}{\partial r} + \frac{\rho}{\rho_s} \frac{\partial \mathbf{v}}{\partial r} = \frac{\rho_n}{\rho_s} v_n \frac{\partial \mathbf{v}}{\partial r} + v \frac{\partial v_n}{\partial r}$$

If $v_n = v$ (the co-flow case) the momentum flux density tensor $\Pi_{ik}$ (19) transforms to

$$\Pi_{rr} = \frac{1}{2} \frac{v^2}{\rho_s} (\rho - \rho_n) = \frac{1}{2} \rho v^2$$

as it should be in the ordinary fluid.
B. Treatment of $v_n$, Flux of energy

Of course, the direct resolution of a set of equations (7)-(10) with substitutions (17) and (19) is the most general way to study the problem of the cavity dynamics in the superfluid helium. There is, however, the simpler approach which allows to get rid of the normal velocity $v_n$ with the use of the expression for the energy flux $W$ (13). Variant of this way was used in paper [13] where the authors used relation (14) to express $v_n$ via heat flux $Q$, where $Q$ is energy released by heater. Moreover, referring to the Gorter -Mellink regime they added the mass flow velocity $v$ to quantity $v_n$ (see Eq. (33) in paper [13] ). Since, however, the total mass flow $j$ is not equal to zero, part of total energy released by heater is converted into mechanical energy, which associated with motion of helium as a whole. Thus, the flow is not pure counterflow (and not co-flow, either), the normal velocity $v_n$ is not determined unambiguously by the energy flux and the situation requires more thorough investigation. We have to use the full expression for the energy flux (13). Neglecting again the nonlinear effects of third order and irreversible heat fluxes $W_{irr}$ we arrive at formula (since we work for pure spherically symmetric case we omit the vector notations)

$$v_n = \frac{W}{ST} - \frac{\mu \rho}{ST} v$$  

(22)

The origin and nature of the energy flow can be different, for example, it can be a spherical heater making up a (spherical) vapor region around itself or it can be some additional external pressure that causes the vortex cavity to either collapse or oscillate.

Substituting $v_n$ from (22) in expression for the momentum flux tensor $\Pi_{rr}$ (19) in spherical case we obtain

$$\Pi_{rr} = \frac{1}{2} \frac{1}{\rho_s} \left( \frac{\rho}{\rho_0} v^2 - 2 v \rho_0 \rho \rho \rho \left( \frac{W}{ST} \right) - \rho_0 \rho \rho \rho \rho \left( \frac{\mu \rho}{ST} v \right) \right)$$

$$= \frac{1}{2} \frac{1}{2} \frac{1}{\rho_s} \left( S^2 T^2 v^2 - 2 \rho_0 S T v^2 - 2 \rho_0 S T v^2 W + \rho_0 v^2 \rho^2 - 2 \rho_0 v \mu \rho W + \rho_0 W^2 \right)$$  

(23)

Further we will use instead of the chemical potential $\mu$ the enthalpy $h = \mu \rho + ST$.The enthalpy $h$ is more reliably measured and tabulated. After that the expression for $\Pi_{rr}$ take a form.

$$\Pi_{rr} = \left( \frac{1}{2} \frac{1}{2} \frac{1}{\rho_s} \left( h^2 \rho_0 + S^2 T^2 \rho - S^2 T^2 \rho_0 \right) \right) v^2 +$$

$$- \left( \frac{\rho h}{2 S^2 T^2 \rho \rho} \right) W v + \frac{\rho_0}{2 S^2 T^2 \rho} W^2$$

(24)

We grouped the terms as follows: the first term contains the squared velocity $v$; the second term contains the cross term $W v$; and, finally, the third term contains an expression that does not contain the mass velocity $v$ at all. coefficients Then, substituting (24) into equation (20) we get

$$\frac{\partial v}{\partial \rho} + \frac{1}{\rho} \frac{\partial p}{\partial v} + \frac{1}{2} B v \frac{\partial W}{\partial v} = \frac{1}{2} B W \frac{\partial v}{\partial W} + C \frac{1}{2} W^2 = 0.$$  

(25)

$A, B$ and $C$ are equal

$$A = \frac{1}{2 S^2 T^2 \rho} \left( h^2 \rho_0 + S^2 T^2 \rho - S^2 T^2 \rho_0 \right)$$

$$B = - \frac{\rho h}{S^2 T^2 \rho}$$

$$C = \frac{1}{2 S^2 T^2 \rho} \rho_0$$

The physical meaning of such a grouping we will discuss further.
C. Equation for the boundary position $R(t)$

We proceed to derive equations for the boundary position evolution $R(t)$. Just as it had been done in the case of classical fluid, we start with the continuity equation (7). It includes only quantity $j(r, t)$ and it has the obvious solution, exactly as in ordinary classical fluids.

$$j(r, t) = \rho \frac{R^2}{r^2} \frac{dR}{dt}$$  \hspace{1cm} (26)

Again, due to incompressibility condition, the mass flow $v$ velocity is $v = j/\rho$ and coincides with the classical solution [2].

To move further we have, just as in the classical case, to work with the equation for the momentum flux $j(r, t)$. For concretization, we consider a purely thermal problem: the development of a vapor film created by a spherical heater, and the energy flux $W(r)$ into the surrounding space has the form

$$W = \frac{R_H^2}{r^2} Q.$$  \hspace{1cm} (27)

Here $Q$ is the heat flux density, released on the surface of the heater. Combining (27) with the expression for mass velocity $v(r, t) = \frac{R^2}{r^2} \frac{dR}{dt}$ following from (26) and accomplishing differentiation with respect to $r$ we rewrite (25)

$$\frac{R^2}{r^2} \frac{d^2 R}{dt^2} + \frac{2R}{r^2} \frac{dR}{dt} \frac{dR}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial r} + A(-2) R^2 \frac{dR}{dt} \frac{dR}{dt} (r^3) + B((\frac{R_H^2}{r^2} Q) (-2) \frac{R^2}{r^2} \frac{dR}{dt} + \frac{R^2}{r^2} \frac{dR}{dt} - 2) \frac{R_H^2}{r^2} Q) + C \frac{R_H^2}{r^2} Q (-2) \frac{R_H^2}{r^2} Q = 0.$$

Integrating this equation over $r$ in limits from $R$ to $\infty$ we have:

$$R \frac{d^2 R}{dt^2} + (2 - \frac{A}{2}) (\frac{dR}{dt})^2 - B(\frac{R_H^2}{R^2} Q) \frac{dR}{dt} - C(\frac{R_H^2}{R^2} Q)^2 \frac{1}{2} + \frac{1}{\rho}(Q(\infty) - Q(R)) = 0$$  \hspace{1cm} (28)

Just as in case of a classical fluid, it is possible to add terms into stress tensor (pressure) arising due to viscosity of the normal component and surface tension. So the final form is:

$$R \frac{d^2 R}{dt^2} + (2 - \frac{A}{2}) (\frac{dR}{dt})^2 = -4 \nu_n \frac{dR}{R} \frac{dR}{dt} + B(\frac{R_H^2}{R^2} Q) \frac{dR}{dt} + \frac{1}{2} C(\frac{R_H^2}{R^2} Q)^2 + \frac{1}{\rho}(p(\infty) - p(R) - 2\gamma)$$  \hspace{1cm} (29)

Here $\nu_n$ is the shear viscosity of the normal component. The master equation for the boundary position of the film [29], plays the role of Rayleigh–Plesset equation for superfluid helium. We again note that some processes such as quantum turbulence (see, e.g., [13]) or the thermal boundary layer (see, e.g., [17]) were not included into consideration of the whole problem. In this sense, the equation (29) can be considered as the first step.

D. Analysis of the Rayleigh–Plesset equation

Equation (29) differs from the Rayleigh–Plesset equation for ordinary fluids [5]. It includes additional terms, absent in the classical case. This difference arises due to two fluid model and specific form of the momentum flux tensor $\Pi_{rr}$ [12]. In the case when $v_n = v$ (and in accordance with Eq. (11)) $v_n = v_s = v$, that is, the fluid moves as a whole, the co-flow case Eq. (29) is reduced to the classical equation [5]. Indeed, if to use $W = h v$ (neglecting the nonlinear effects of third order), then $\Pi_{rr}$ [24] transforms as

$$\frac{1}{2ST^2 \rho_s} \frac{\nu^2}{2} (S^2 T^2 - 2\rho_n S T h + 2\rho_n S T (h - ST) + \rho_n h^2 - 2\rho_n h (h - ST) + \rho_n (h - ST)^2) = \frac{1}{2} \nu^2,$$

that is, it acquires its classical value and equation (29) is converted into equation [5]. As a result, the well-known problems such as isothermal oscillations of the gas bubble, or the collapse of empty cavity, fully coincides with the classical solution (see, e.g., book [14]). The real and essential difference appear for non-zero heat transfer.
The terms in the momentum flux tensor $\Pi_{rr}$ and, hence, in equation (29) are combined into groups with different physical meanings. So the terms containing derivatives $dR/dt$ are important for non-stationary processes, such as a transient process, or oscillatory motion. The remaining terms not containing derivative $dR/dt$ determine the stationary solution $R(t \to \infty)$, e.g. the thickness of the vapor film (or size of the vapor bubble). So, the third term in the right hand side of Eq. (29), which is independent on the velocity of the boundary position $dR/dt$, can be unified with the pressure term $p(R)$. That can be additionally justified by the fact that this term is proportional to squared velocities and, hence, it is related to dependence of pressure on the velocity (see also books [20, 17]). Further we will name it as "Bernoulli" -like pressure. In many cases, this additional pressure is small, for example, in experiments (18, 10) this "Bernoulli" -like pressure is of the order of 10% of the hydrostatic pressure $\rho gh$, however for smaller values of $h$, and, especially, under microgravity conditions it can be extremely important.

The second term in the right hand side of Eq. (29) is of the particular interest. It has the same structure as the viscous term with shear viscosity for normal velocity, which is responsible for the attenuation of bubble oscillations. At the same time it essentially (by several orders of magnitude) exceeds the usual viscous damping. For this reason we will name it as "extra-damping" term. The described term can be the reason for the strong attenuation of the bubble oscillations, observed in many works (see, e.g., (18, 10)). The authors called it as abnormal "Suppression of oscillations of the vapor–liquid phase boundary in superfluid helium" (see [11]). To our knowledge, the authors could not explain this phenomena and referred to pure experimental obstacles.

Finally the second term in the in the left hand side of Eq. (29) differs from classical case in that the coefficient $3/2$ is changed with the quantity $2 - A/2$. Preliminary numerical analysis of the solution of Eq. (29) shows that this term does not affect greatly the final results.

V. CONCLUSION

The problem of the cavity dynamics in the superfluid helium is considered on the basis of Landau-Khalatnikov two-fluid hydrodynamics. The equation, which play the role of the classical Rayleigh–Plesset equation (29), significantly differs from its classical analogue. This difference appears from special treatment of the momentum flux tensor, which, due to the two-fluid nature of superfluids generates several new effects, such as a "Bernoulli" -like pressure term or an extra-damping term. These terms essentially affect the dynamics of cavities compared to ordinary fluids and can influence results and conclusions made in the relevant works.

Equation (29) is intended to investigate problems associated with the evolution of a cavity in superfluids. We again would like to emphasize that this hydrodynamic description is part of the general problem and, maybe, not the primary part. Probably more important ingredient is the correct analysis of the pressure drop, due to the involved thermal or/and electric processes inside the bubble.

Thus, in works on the boiling helium (see, e.g., [18, 10]), the authors determine vapor pressure $p(R)$ from the Boltzmann kinetic equation for evaporation and condensation processes. Studying multielectron bubbles in liquid helium (see, e.g., [1, 2, 3]), the authors find the electron density and its contribution into the pressure inside the bubble with the use of the Poisson equation.

The study of according processes is a separate involved problem, that goes beyond the scope of this work. Therefore, we deliberately limited ourselves to the hydrodynamic part, since our goal was to emphasize the role of two-fluid hydrodynamics. Moreover, consequently pursuing that goal, we simplified situation by taking the pure two-fluid Landau-Khalatnikov model and omitting other phenomena such as quantum turbulence [15] or existence of tiny thermal boundary layer near the interface liquid-vapor [17]. These topics are supposed to be study in future.

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