Abstract. Let $A$ be a non-projectively-pluripolar set in a Fréchet space $E$. We give sufficient conditions to ensure the convergence on some zero-neighbourhood in $E$ of a (sequence of) formal power series of Fréchet-valued continuous homogeneous polynomials provided that the convergence holds at a zero-neighbourhood of each complex line $\ell_a := \mathbb{C}a$ for every $a \in A$.

1. Introduction

In mathematics, a formal power series is a generalization of polynomials as a formal object, where the number of terms is allowed to be infinite. The theory of formal power series has drawn attention of mathematicians working in different branches because of their various applications. One can find applications of formal power series in classical mathematical analysis and in the theory of Riordan algebras. Specially, this theory lays the foundation for substantial parts of combinatorics and real and complex analysis.

A formal power series $f(z_1, \ldots, z_n) = \sum c_{\alpha_1, \ldots, \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ in $\mathbb{C}^n$, $n \geq 2$, with coefficients in $\mathbb{C}$ is said to be convergent if it converges absolutely in a zero-neighborhood in $\mathbb{C}^n$. A classical result of Hartogs states that a series $f$ converges if and only if $f_z(t) = f(tz_1, \ldots, tz_n)$ converges, as a series in $t$, for all $z \in \mathbb{C}^n$. This can be interpreted as a formal analog of Hartogs theorem on separate analyticity. Because a divergent power series still may converge in certain directions, it is natural and desirable to consider the set of all $z \in \mathbb{C}^n$ for which $f_z$ converges. Since $f_z(t)$ converges if and only if $f_w(t)$ converges for all $w \in \mathbb{C}^n$ on the affine line through $z$, ignoring the trivial case $z = 0$, the set of directions along which $f$ converges can be identified with a subset of the projective space $\mathbb{P}^{n-1}$. The convergence set $\text{Conv}(f)$ of a divergent power series $f$ is defined to be the set of all directions $\xi \in \mathbb{P}^{n-1}$ such that $f_z(t)$ is convergent for some $z \in \rho^{-1}(\xi)$ where $\rho : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the natural projection. In the two-variables case, Lelong [12] proved that $\text{Conv}(f)$ is an $F_{\sigma}$-polar set (i.e. a countable union of closed sets of vanishing logarithmic capacity) in $\mathbb{P}^1$, and moreover, every $F_{\sigma}$-polar subset of $\mathbb{P}^1$ is contained in the $\text{Conv}(g)$ of some formal power series $g$. The optimal result was later obtained by Sathaye [13] who showed that the class of convergence sets of divergent power series in two-variables is precisely the class of $F_{\sigma}$-polar sets in $\mathbb{P}^1$. Levenberg and Molzon, in [13], showed that if the restriction of $f$ on sufficiently many (non-pluri)polaring sets of complex line passing through the origin is convergent on small neighborhood of $0 \in \mathbb{C}$ then $f$ is...
actually represent a holomorphic function near \( 0 \in \mathbb{C}^n \). By using delicate estimates on volume of complex varieties in projective spaces, Alexander [11, Theorem 6.1] showed that if the restriction of a sequence \((f_m)_{m \geq 1}\) of formal power series on every complex line passing through the origin in \( \mathbb{C}^n \) is convergent on compact sets (of the unit disk \( \Delta \subset \mathbb{C} \)) then \((f_m)_{m \geq 1}\) is the series of holomorphic function on the unit ball \( \Delta_n \subset \mathbb{C}^n \) which is convergent uniformly on compact sets. By considering the class \( PSH_\omega(\mathbb{P}^n) \) of \( \omega \)-plurisubharmonic functions on \( \mathbb{P}^n \) with respect to the form \( \omega := dd^c \log |Z| \) on \( \mathbb{P}^n \), Ma and Neelon proved that a countable union of closed complete pluripolar sets in \( \mathbb{P}^n \) belongs to \( \text{Conv}(\mathbb{P}^n) \). This generalizes the results of Lelong [12], Levenberg and Molzon [13], and Sathaye [18]. In the same work, they also showed that each convergence set (of divergent power series) is a countable union of projective hulls of compact pluripolar sets. Recently, based on an investigation on a projectively pluripolar subset of \( \mathbb{C}^n \) (via logarithmically homogeneous plurisubharmonic function) Long and Hung [14] have shown that a sequence \((f_m)_{m \geq 1}\) of formal power series in \( \mathbb{C}^n \) converges uniformly on compact subsets of the ball \( \Delta_n(r_0) \subset \mathbb{C}^n \) for some \( r_0 > 0 \) if for each \( n \geq 1 \), the restriction of \( f_m \) to the complex line \( \ell_a := Ca \) is holomorphic on the disk \( \Delta(r_0) \subset \mathbb{C} \) for every \( a \in A \), a non-projectively-pluripolar set in \( \mathbb{C}^n \).

The main goal of this paper is to study the convergence of a (sequence of) formal power series of Fréchet-valued continuous homogeneous polynomials. To prepeare for the proofs of the main results, with the help of techniques of pluripotential theory, we investigate the Hartogs lemma for sequence of plurisubharmonic functions for the infinite dimensional case (Theorem 3.2). The first main result, Theorem 3.3, gives a condition on a non-projectively-pluripolar set \( A \) in a Fréchet space \( E \) such that a formal power series \( f \) of Fréchet-valued continuous homogeneous polynomials of degree \( n \) on \( E \) converges in a neighbourhood of \( 0 \in E \) provided that the restriction of \( f \) on the complex line \( \ell_a \) is convergent for every \( a \in A \). Theorem 3.2 also allows us to treat the problem on the extension to a entire function from the unit ball \( \Delta_n \subset \mathbb{C}^n \) of a \( C^\infty \)-smooth function \( f \). Naturally, the condition “for a non-projectively-pluripolar set \( A \), the restriction of \( f \) on every \( \ell_a \), \( a \in A \), is entire function” will be considered here (Theorem 3.7). This result may be considered as a Fréchet-valued version of the Forelli theorem [19].

The problem considered in the last result, Theorem 3.8, is giving the conditions under which a sequence of formal power series of Fréchet-valued continuous homogeneous polynomials on \( \mathbb{C}^n \) converges on a zero-neighbourhood. Another expression of this theorem will show that this is an extension of Alexander’s theorem for the Fréchet-valued case.

2. Preliminaries

The standard notation of the theory of locally convex spaces used in this note is presented as in the book of Jarchow [10]. A locally convex space is always a complex vector space with a locally convex Hausdorff topology. For a locally convex space \( E \) we use \( E'_{\text{bor}} \) to denote \( E' \) equipped with the bornological topology associated with the strong topology \( \beta \).

The locally convex structure of a Fréchet space is always assumed to be generated by an increasing system \( \{ \| \cdot \|_k \}_{k \geq 1} \) of seminorms. For an absolutely convex subset \( B \) of \( E \), by \( E_B \) we denote the linear hull of \( B \) which becomes a normed space in a canonical way if \( B \) is bounded (with the norm \( \| \cdot \|_B \) is the gauge functional of \( B \)).

Let \( D \) be a domain in a locally convex space \( E \). An upper-semicontinuous function \( \varphi : D \to [-\infty, +\infty) \) is said to be plurisubharmonic, and write \( \varphi \in PSH(D) \), if \( \varphi \) is subharmonic on every one dimensional section of \( D \).
A subset $B \subset D$ is said to be pluripolar in $D$ if there exists $\varphi \in PSH(D)$ such that $\varphi \not\equiv -\infty$ and $\varphi|_B = -\infty$.

A function $\varphi \in PSH(E)$ is called homogeneous plurisubharmonic if

$$\varphi(\lambda z) = \log |\lambda| + \varphi(z) \quad \forall \lambda \in \mathbb{C}, \forall z \in E.$$  

We denote by $HPSH(E)$ the set of homogeneous plurisubharmonic functions on $E$. We say that a subset $A \subset E$ is projectively pluripolar if $A$ is contained in the $-\infty$ locus of some element in $HPSH(E)$ which is not identically $-\infty$. It is clear that projective pluripolarity implies pluripolarity. The converse is not true (see [14, Proposition 3.2 b]).

Some properties, examples and counterexamples of projectively pluripolar sets may be found in [14]. We introduce below a few examples in locally convex spaces.

**Example 2.1.** Let $E$ be a metrizable locally convex space. Fix $a \in E$. Then, the complex line $\ell_a$, hence, every $A \subset \ell_a$, is projectively pluripolar in $E$.

Indeed, let $d$ be the metric defining the topology on $E$. Consider the function

$$\varphi(z) = -\log d(z, \ell_a) := -\log \inf_{w \in \ell_a} d(z, w).$$

It is easy to check that $\varphi \in HPSH(E)$, $\varphi \not\equiv -\infty$ and $\ell_a \subset \varphi^{-1}(-\infty)$.

**Example 2.2.** Let $E$ be a Fréchet space which contains a non-pluripolar compact balanced convex subset $B$. By the same proof as in Example 2.1, the set $\partial B$ is pluripolar. However, $\partial B$ is not projectively pluripolar in $E$.

Otherwise, we can find a function $\varphi \in HPSH(E)$, $\varphi \not\equiv -\infty$ and $\partial B \subset \varphi^{-1}(-\infty)$. For every $z \in B$ we can write $z = \lambda y$ for some $y \in \partial B$ and $|\lambda| < 1$. Then

$$\varphi(z) = \varphi(\lambda y) = \log |\lambda| + \varphi(y) = -\infty \quad \forall z \in B.$$  

It is impossible because $B$ is non-pluripolar.

Hence, by Theorem 9 of [7], a nuclear Fréchet space having the linear topological invariant ($\tilde{\Omega}$) which is introduced by Vogt (see [21]) contains a non-projectively-pluripolar set.

We use throughout this paper the following notations:

$$\Delta_n(r) = \{ z \in \mathbb{C}^n : \|z\| < r \}; \quad \Delta_n := \Delta_n(1); \quad \Delta(r) = \Delta_1(r); \quad \Delta := \Delta_1;$$

and $\ell_a$ is the complex line $\mathbb{C}a$. For further terminology from complex analysis we refer to [6].

3. The results

First we investigate the Hartogs lemma for sequence of plurisubharmonic functions for the infinite dimensional case. This is essential to our proofs.

**Lemma 3.1.** Let $(P_n)_{n \geq 1}$ be a sequence of continuous homogeneous polynomials on a Baire locally convex space $E$ of degree $\leq n$. Assume that

$$\limsup_{n \to \infty} \frac{1}{n} \log |P_n(z)| \leq M$$

for each $z \in E$ and some constant $M$. Then for every $\varepsilon > 0$ and every compact set $K$ in $E$ there exists $n_0$ such that

$$\frac{1}{n} \log |P_n(z)| < M + \varepsilon \quad \forall n > n_0, \forall z \in K.$$
Proof. Without loss of generality we may suppose that $M \leq 0$. Since
\[
\limsup_{n \to \infty} |P_n(z)|^{1/n} \leq 1 \quad \forall z \in E
\]
the formula
\[
f(z)(\lambda) = \sum_{n \geq 1} P_n(z)\lambda^n
\]
defines a function $f : E \to H(\Delta)$, the Fréchet space of holomorphic functions on the open unit disc $\Delta \subset \mathbb{C}$.

Let us check $f$ is holomorphic on $E$. Given $z \in E \setminus \{0\}$ and consider $f(\cdot \cdot z) : \mathbb{C} \to H(\Delta)$ with
\[
f(\xi z)(\lambda) = \sum_{n \geq 1} P_n(z)\lambda^n \xi^k \]
where $k_n = \deg P_n \leq n$. Then $f(\cdot \cdot z)$ is holomorphic because for $0 < r < 1$ we have
\[
\lim_{n \to \infty} \sup_{|\lambda| \leq r} (|P_n(z)|^{1/n})^{\frac{1}{k_n}} = \lim_{n \to 0} \sup_{|r| = 1} (|P_n(z)|^{1/r})^{\frac{1}{k_n}} \leq \lim_{n \to \infty} |P_n(z)|^{\frac{1}{r}} r \leq 1.
\]
This means $f$ is Gâteaux holomorphic on $E$.

Now for each $k \geq 1$ we put
\[
A_k := \{z \in E : |P_n(z)| \leq k^k, \quad \forall n \geq 1\}.
\]
By the continuity of $P_n$, the sets $A_k$ are closed in $E$. Moreover, $E = \bigcup_{k \geq 1} A_k$. Since $E$ is a Baire space, there exists $k_0 \geq 1$ such that $\text{Int} A_{k_0} \neq \emptyset$. Then $f$ is holomorphic on $\frac{1}{k_0} A_{k_0}$ because
\[
\sum_{n \geq 1} |P_n(z)||\lambda|^n \leq \sum_{n \geq 1} \frac{k_0^k}{k_n^k} r^n = \sum_{n \geq 1} r^n < \infty \quad \text{for } 0 < r < 1.
\]
Hence, by Zorn’s Theorem [16, Theorem 1.3.1], $f$ is holomorphic on $E$.

Now given $K \subset E$ a compact set and $\varepsilon > 0$. Take $0 < r < 1$ and denote
\[
C := \sup\{|f(z)(\lambda)| : z \in K, |\lambda| \leq r\} < \infty.
\]
Then we have
\[
|P_n(z)| = \left| \frac{1}{2\pi i} \int_{|\xi| = r} \frac{f(z)(\xi)d\xi}{\xi^{n+1}} \right| \leq \frac{C}{r^n} \quad \forall z \in K,
\]
i.e.,
\[
|P_n(z)|^{\frac{1}{n}} \leq \frac{C^{\frac{1}{n}}}{r}.
\]
Choose $n_0$ sufficiently we obtain
\[
|P_n(z)|^{\frac{1}{n}} \leq \frac{C^{\frac{1}{n}}}{r} < e^\varepsilon \quad \forall n > n_0.
\]
The lemma is proved. $\square$

The Proposition 5.2.1 in [11] says that a non-empty family $(u_\alpha)_{\alpha \in I}$ of plurisubharmonic functions from the Lelong class such that the set $\{z \in \mathbb{C}^n : \sup_{\alpha \in I} u_\alpha(z) < \infty\}$ is not $\mathcal{L}$-polar is locally uniformly bounded from above.

The next is similar to the above result in the infinite dimensional case.
Theorem 3.2. Let $B$ be a balanced convex compact subset of a Fréchet space $E$ and $(P_n)_{n \geq 1}$ be a sequence of continuous homogeneous polynomials on $E$ of degree $\leq n$. Assume that the set

$$\left\{ z \in E_B : \sup_{n \geq 1} \frac{1}{n} \log |P_n(z)| < \infty \right\}$$

is not projectively pluripolar in $E_B$. Then the family $(\frac{1}{n} \log |P_n|)_{n \geq 1}$ is locally uniformly bounded from above on $E_B$.

Proof. Suppose that the family $(\frac{1}{n} \log |P_n|)_{n \geq 1}$ is not locally uniformly bounded from above on $B$. Then there exists a sequence $(u_j)_{j \geq 1} = (\frac{1}{n} \log |P_{n_j}|)_{j \geq 1}$ such that

$$M_j := \sup_{z \in B} u_j(z) \geq j \quad \forall j \geq 1.$$ 

Take $w \in E_B \setminus B$ and for each $j \geq 1$ consider the function

$$v_j(\zeta) := u_j(\zeta^{-1} w) - M_j - \log^+ (|\zeta|^{-1} \|w\|_B), \quad \zeta \in \Delta(\|w\|_B) \setminus \{0\}.$$ 

Obviously, $v_j$ is subharmonic and, it is easy to see that $v_j(\zeta) \leq O(1)$ as $\zeta \to 0$. Hence, in view of Theorem 2.7.1 in [11], $v_j$ extends to a subharmonic function, say $\tilde{v}_j$, on $\Delta(\|w\|_B)$. Now, by the maximum principle, $\tilde{v}_j \leq 0$ on $\Delta(\|w\|_B)$. In particular,

$$v_j(1) = \tilde{v}_j(1) = u_j(w) - M_j - \log^+ \|w\|_B \leq 0.$$ 

Hence

$$u_j(z) - M_j \leq \log^+ \|z\|_B \quad \text{for} \quad z \in E_B, \forall j \geq 1.$$ 

Then there exists $z_0 \in E_B$ such that

$$\limsup_{j \to \infty} \exp(u_j(z_0) - M_j) =: \delta > 0.$$ 

For otherwise we would have $\limsup_{j \to \infty} \exp(u_j(z) - M_j) \leq 0$ at each point $z \in E_B$.

Note that, by Lemma 3.1, the sequence $(u_j(z) - M_j)_{j \geq 1}$, hence, $(\exp(u_j(z) - M_j))_{j \geq 1}$ is bounded from above on any compact set in $E_B$. This would imply from [16] Lemma 1.1.12 that $\exp(u_j(z) - M_j) < \frac{1}{2}$ for all $z \in B$ and all sufficiently large $j$. But then the last estimate would contradict the definition of the constants $M_j$.

Now we choose a subsequence $(u_{j_k})_{k \geq 1} \subset (u_j)_{j \geq 1}$ such that

$$\lim_{k \to \infty} \exp(u_{j_k}(z_0) - M_{j_k}) = \delta \quad \text{and} \quad M_{j_k} \geq 2^k$$

for all $k \geq 1$. Consider the function

$$w(z) := \sum_{k \geq 1} 2^{-k}(u_{j_k} - M_{j_k}), \quad z \in E_B.$$ 

In view of Lemma 3.1 we have the estimate

$$w_k(z) := 2^{-k}(u_{j_k} - M_{j_k}) - 2^{-k} \log R \leq 0$$

for $z \in E_B, \|z\|_B \leq R$ and $R \geq 1$. Thus $w_k$ is plurisubharmonic on $\{ z \in E_B : \|z\|_B < R \}$ and $w_k \leq 0$. Hence, the function $\sum_{k \geq 1} w_k = w - \log R, R > 1$, is either plurisubharmonic on $\{ z \in E_B : \|z\|_B < R \}$ or identically $-\infty$. Consequently, as $R$ can be chosen arbitrarily large, $w$ is either plurisubharmonic or identically $-\infty$. Therefore, since $w(z_0) > -\infty$, $w \in PSH(E_B)$. It is easy to see that $w \in HPSH(E_B)$.

If $z \in E_B$, $\sup_{n \geq 1} \frac{1}{n} \log |P_n(z)| < \infty$ then $\sum_{k \geq 1} 2^{-k} u_{j_k}(z) < \infty$ and, hence

$$w(z) \leq \sum_{k \geq 1} 2^{-k} u_{j_k}(z) - \sum_{k \geq 1} 1 = -\infty$$

Corollary 3.3. Let $B, E$ and $(P_n)_{n \geq 1}$ be as in Theorem 3.2, in addition assume that $B$ contains a non-projectively-pluripolar subset. Then the family $(\frac{1}{n} \log |P_n|)_{n \geq 1}$ is locally uniformly bounded from above on $E$.

Proof. It suffices to prove that $E_B$ is dense in $E$. Indeed, if the closure of the subspace $E_B$ is not equal to $E$ then, by the Hahn-Banach theorem, there exists $\varphi \in E'$, $\varphi \neq 0$, such that $\varphi(E_B) = 0$. Then it is easy to see that $v := \log |\varphi| \in HPSH(E)$, $v \neq 0$, $B \subset E_B \subset \{z : v(z) = -\infty\}$. This contradicts the fact that $B$ contains a non-projectively-pluripolar subset.

It is known that a subset with non-empty interior in a Fréchet space is not pluripolar, hence it is not projectively pluripolar. Then by Corollary 3.3 we have the following.

Corollary 3.4. Let $B$ be a balanced convex compact subset of a Fréchet space $E$ which contains a non-projectively-pluripolar subset and $(P_n)_{n \geq 1}$ be a sequence of continuous homogeneous polynomials on $E$ of degree $\leq n$. If the set

$$\left\{ z \in E_B : \sup_{n \geq 1} \frac{1}{n} \log |P_n(z)| < \infty \right\}$$

has the non-empty interior in $E$ then the family $(\frac{1}{n} \log |P_n|)_{n \geq 1}$ is locally uniformly bounded from above on $E$.

Lemma 3.5. A regular inductive limit $E = \lim_{{n \to \infty}} E_n$ of countable family of locally convex spaces satisfies the countable boundedness condition (c.b.c), i.e. if given a sequence $\{B_n\}$ of bounded subsets of $E$, there are $\lambda_n > 0$, $n \geq 1$, such that $\bigcup_{{n \geq 1}} \lambda_n B_n$ is bounded.

Proof. Given a sequence $(B_n)_{n \geq 1}$ of bounded subsets of $E$. By the regularity of $E$, for each $n \geq 1$, there exists $k_n \geq 1$ such that $B_n$ is bounded in $E_{k_n}$. Without loss of generality we may assume that $k_n = n$. Hence we can find a sequence of positive numbers $(\lambda_n)_{n \geq 1}$ such that $\lambda_n B_n \subset U_n$, a zero-neighbourhood in $E_n$, for all $n \geq 1$. Obviously, $\bigcup_{{n \geq 1}} \lambda_n B_n \subset U := \bigcup_{{n \geq 1}} U_n$, a zero-neighbourhood in $E$. Lemma is proved.

Theorem 3.6. Let $A$ be a non-projectively-pluripolar set which is contained in a balanced convex compact subset of a Fréchet space $E$ and $f = \sum_{{n \geq 1}} P_n$ be a formal power series where $P_n$ are continuous homogeneous polynomials of degree $n$ on $E$ with values in a Fréchet space $F$. If for each $a \in A$, the restriction of $f$ on the complex line $\ell_a$ is convergent then $f$ is convergent in a neighbourhood of $0 \in E$.

Proof. We divide the proof into two steps:

(i) Step 1: We consider the case where $F = \mathbb{C}$. It follows from the hypothesis that

$$\limsup_{{n \to \infty}} |P_n(z)|^{\frac{1}{n}} < \infty \quad \forall z \in A.$$

Then, by Corollary 3.3 there exists a zero-neighbourhood $U$ in $E$ such that

$$\sup\{|P_n(z)|^{\frac{1}{n}} : z \in U, n \geq 1\} =: M < \infty.$$

This implies that $f$ is uniformly convergent on $(2M)^{-1} U$. 

(ii) **Step 1:** We consider the case where \( F \) is Fréchet. By the step 1 we can define the linear map
\[
T : F'_{\text{bor}} \to H(0_{E})
\]
by letting
\[
T(u) = \sum_{n \geq 1} u(P_n)
\]
where \( H(0_{E}) \) denotes the space of germs of scalar holomorphic functions at \( 0 \in E \). Suppose that \( u_\alpha \to u \) in \( F'_{\text{bor}} \) and \( T(u_\alpha) \to v \) in \( H(0_{E}) \) as \( \alpha \to \infty \). This implies, in particular, that \( [T(u_\alpha)](z) \to v(z) \) for all \( z \) in some zero-neighbourhood \( U \) in \( E \). However, for \( z \in U \) we have
\[
[T(u_\alpha - u)](z) = \sum_{n \geq 1} (u_\alpha - u)(P_n(z)) = \lim_{n \to \infty} \sum_{k=1}^{n} (u_\alpha - u)(P_n(z))
\]
\[
= (u_\alpha - u)\left( \lim_{n \to \infty} \sum_{k=1}^{n} P_n(z) \right) = (u_\alpha - u)\left( \sum_{n \geq 1} P_n(z) \right).
\]
Then \( [T(u_\alpha)](z) \to [T(u)](z) \) for all \( z \in U \). This implies that \( v = T(u) \). Hence \( T \) has a closed graph.

Meanwhile, since \( F \) is Fréchet, by [10, Theorem 13.4.2] we have \( \beta(F', F)_{\text{bor}} = \eta(F', F) \) on \( F' \). This implies that \( F'_{\text{bor}} \) is ultrabornological. On the other hand, because \( E \) is metrizable, we have
\[
H(0_{E}) = \lim_{n \to \infty} (H^\infty(V_n), \| \cdot \|_n)
\]
where \( (V_n)_{n \geq 1} \) is a countable fundamental neighbourhood system at \( 0 \in E \), and \( \| \cdot \|_n \) is the norm on the Banach space \( H^\infty(V_n) \) given by \( \| f \|_n = \sup_{z \in V_n} |f(z)| \). Hence, by the closed graph theorem of Grothendieck [8, Introduction, Theorem B], \( T \) is continuous.

Now, by [5, Lemma 4.33] and Lemma [5.5] \( H(0_{E}) \) satisfies (c.b.c). Using Proposition 1.8 in [5] we deduce that there exists a neighbourhood \( V \) of \( 0 \in E \) such that \( T : F'_{\text{bor}} \to H^\infty(V) \) is continuous linear.

Now we define the map \( \hat{f} : V \to F''_{\text{bor}} \) by the formula
\[
\hat{f}(z)(u) = [T(u)](z), \quad z \in V, \ u \in F'_{\text{bor}}.
\]
Since \( T \) is continuous and point evaluations on \( H(V)_{\text{bor}} \) (see [6, Proposition 3.19]) are continuous it follows that \( \hat{f}(z) \in F''_{\text{bor}} \) for all \( z \in V \). Moreover, for each fixed \( u \in F'_{\text{bor}} \) the mapping
\[
z \in V \mapsto [T(u)](z)
\]
is holomorphic, that is
\[
\hat{f} : V \to (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))
\]
is a continuous mapping. For all \( a \in V, b \in E \) and all \( u \in F'_{\text{bor}} \) the mapping
\[
\{ t \in \mathbb{C} : a + tb \in V \} \ni \lambda \mapsto u \circ \hat{f}(a + \lambda b)
\]
is a Gâteaux holomorphic mapping and hence
\[
\hat{f} : V \to (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))
\]
is holomorphic.
By [10] 8.13.2 and 8.13.3, \( F'_{\text{bor}} \) is a complete locally convex space. Hence by [9] Theorem 4, p.210 applied to the complete space \( F'_{\text{bor}} \) we see that \( (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}})) \) and \( (F'_{\text{bor}})'_\beta \) have the same bounded sets. An application of [15, Proposition 13] shows that

\[
\hat{f}: V \rightarrow (F'_{\text{bor}})'_\beta
\]

is holomorphic.

Let \( j \) denote the canonical injection from \( F \) into \( F'' \). If \( z \in B := V \cap \{ta : t \in \mathbb{C}, a \in A\} \) and \( \hat{f}(z) \neq j(f(z)) \) then there exists \( u \in F' \) such that

\[
\hat{f}(z)(u) \neq j(f(z))(u) = u(f(z)).
\]

This, however, contradicts the fact that for all \( z \in B \) we have

\[
\hat{f}(z)(u) = [T(u)](z) = \sum_{n \geq 1} u(P_n)(z) = u(f(z)).
\]

We now fix a non-zero \( z \in B \). Then there exists a unique sequence in \( F'' \), \( (a_{n,z})_{n=1}^{\infty} \), such that for all \( \lambda \in \mathbb{C} \)

\[
\hat{f}(\lambda z) = \sum_{n=0}^{\infty} a_{n,z} \lambda^n.
\]

Since \( \hat{f}(0) = f(0) = a_{0,z} \) it follows that \( a_{0,z} \in F \). Now suppose that \( (a_{j,z})_{j=0}^{n} \subset F \). When \( |\lambda| \leq 1 \), \( \hat{f}(\lambda z) = f(\lambda z) \in F \). Hence, if \( \lambda \in \mathbb{C}, 0 < |\lambda| < 1 \), then

\[
\frac{\hat{f}(\lambda z) - \sum_{j=0}^{n} a_{j,z} \lambda^j}{\lambda^{n+1}} = \sum_{j=n+1}^{\infty} a_{j,z} \lambda^{j-n-1} \in F.
\]

Since \( F \) is complete we see, on letting \( \lambda \) tend to 0, that \( a_{n+1,z} \in F \). By induction \( a_{n,z} \in F \) for all \( n \) and hence \( \hat{f}(\lambda z) \in F \) for all \( \lambda \in \mathbb{C} \) and all \( z \in B \). Since \( \hat{f} \) is continuous and \( F \) is a closed subspace of \( (F''_{\text{bor}})'_\beta \) (see [17, Lemma 2.1]) we have shown that \( \hat{f}: V \rightarrow F \) is holomorphic.

Hence, the series \( \sum_{n \geq 1} P_n \) is convergent on \( V \) to \( f \). \( \square \)

**Theorem 3.7.** Let \( F \) be a Fréchet space, \( f : \Delta_n \rightarrow F \) be a function which belongs to \( C^k \)-class at \( 0 \in \mathbb{C}^n \) for \( k \geq 0 \) and \( A \subset \mathbb{C}^n \) be a non-projectively-pluripolar set. If the restriction of \( f \) on each complex line \( \ell_{a}, a \in A \), is holomorphic then there exists an entire function \( \hat{f} \) on \( \mathbb{C}^n \) such that \( \hat{f} = f \) on \( \ell_{a} \) for all \( a \in A \).

**Proof.** By the hypothesis, for each \( k \geq 0 \) there exists \( r_k \in (0,1) \) such that \( f \) is a \( C^k \)-function on \( \Delta_n(r_k) \). We may assume that \( r_k \searrow 0 \). Put

\[
P_k(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(\lambda z) d\lambda}{\lambda^{k+1}}, \quad z \in \Delta_n(r_k).
\]

Then, for each \( k \geq 0 \) and \( p \geq k \), \( P_m \) is a bounded \( C^p \)-function on \( \Delta_n(r_p) \). Since \( \lambda \mapsto f(\lambda a) \) is holomorphic for all \( a \in A \) we deduce that

\[
P_k(\lambda a) = \lambda^k P_k(a) \quad \text{for} \ a \in A, \lambda \in \mathbb{C}.
\]

By the boundedness of \( P_k \) on \( \Delta_n(r_k) \) we have

\[
P_k(w) = O(|w|^k) \quad \text{as} \ w \rightarrow 0.
\]
On the other hand, since $P_k \in C^{k+1}(\Delta_n(r_{k+1}))$, the Taylor expansion of $P_k$ at $0 \in \Delta_n(r_{k+1})$ has the form

$$P_k(z) = \sum_{\alpha+\beta=k} P_{k,\alpha,\beta}(z) + |z|^k \varrho(z)$$

(3.4)

where $P_{k,\alpha,\beta}$ is a polynomial of degree $\alpha$ in $z$ and degree $\beta$ in $\overline{z}$ and $\varrho(z) \to 0$ as $z \to 0$.

In (3.3), replacing $z$ by $\lambda z$, $|\lambda| < 1$, from (3.3) we obtain

$$\sum_{\alpha+\beta=k} P_{k,\alpha,\beta}(z) \lambda^\alpha \overline{\lambda}^\beta + |\lambda|^k |z|^k \varrho(\lambda z) = \sum_{\alpha+\beta=k} P_{k,\alpha,\beta}(z) \lambda^k + \lambda^k |z|^k \varrho(z)$$

(3.5)

for $z \in r_{k+1}A$.

This yields that $\varrho(\lambda z) = \varrho(z)$ for $\lambda \in [0,1)$, and hence, $\varrho(z) = \varrho(0) = 0$ for $z \in r_{k+1}A$.

Thus

$$P_{k,\alpha,\beta}(z) = 0 \quad \text{for } \beta > 0 \text{ and } z \in r_{k+1}A.$$ Note that $r_{k+1}A$ is also not projectively pluripolar. It is easy to check that

$$P_{k,\alpha,\beta} = 0 \quad \text{for } \beta > 0.$$ Indeed, for every $\varphi \in F'$, the function

$$u(w) = \frac{1}{\deg P_{k,\alpha,\beta}} \log |(\varphi \circ P_{k,\alpha,\beta})(w)|$$

is homogeneous plurisubharmonic on $\mathbb{C}^n$, $u \equiv -\infty$ on $r_{k+1}A$. Since $r_{k+1}A$ is not projectively pluripolar, it implies that $u \equiv -\infty$ and hence $\varphi \circ P_{k,\alpha,\beta} \equiv 0$ on $\mathbb{C}^n$ for every $\varphi \in F'$. It implies that $P_{k,\alpha,\beta} \equiv 0$ on $\mathbb{C}^n$ for $\beta > 0$.

Thus, from (3.4) we have

$$P_k(z) = P_{k,k,0}(z) = \sum_{|\alpha| = k} c_\alpha z^\alpha$$

for $z \in \Delta_n(r_{k+1})$ and $P_k$ is a homogeneous holomorphic polynomial of degree $k$.

Now, let $(\| \cdot \|_m)_{m \geq 1}$ be an increasing fundamental system of continuous semi-norms defining the topology of $F$. By the hypothesis, for every $m \geq 1$

$$\limsup_{k \to \infty} \frac{1}{k} \log \|P_k(z)\|_m = -\infty \quad \text{for } z \in A.$$ Then, by Corollary [5.3] the sequence $(\frac{1}{k} \log \|P_k(z)\|_m)_{k \geq 1}$ is locally uniformly bounded from above on $\mathbb{C}^n$ for all $m \geq 1$. Thus we can define

$$u_m(z) = \limsup_{k \to \infty} \frac{1}{k} \log \|P_k(z)\|_m, \quad z \in \mathbb{C}^n.$$ By [20] the upper semicontinuous regularization $u_m^*$ of $u_m$ belongs to the Lelong class $L(\mathbb{C}^n)$ of plurisubharmonic functions with logarithmic growth on $\mathbb{C}^n$. Moreover, by Bedford-Taylor’s theorem [3]

$$S_m := \{z \in \mathbb{C}^n : u_m^*(z) \neq u_m(z)\}$$

is pluripolar for all $m \geq 1$.

On the other hand, by [14], $A^* := \{ta : t \in \mathbb{C}, a \in A\}$ is not pluripolar. This yields that $u_m \equiv -\infty$ for all $m \geq 1$ because $u_m^* = u_m = -\infty$ on $A^* \setminus S_m$ and $A^* \setminus S_m$ is non-pluripolar. Since $u_m^* \geq u_m$ we have $u_m \equiv -\infty$ for $m \geq 1$. Hence the series $\sum_{k \geq 0} P_k(z)$ is convergent for $z \in \mathbb{C}^n$ and it defines a holomorphic extension $\tilde{f}$ of $f|_{\ell_a}$ for every $a \in A$. □
Theorem 3.8. Let $A \subset \mathbb{C}^n$ be a non-projectively-pluripolar set and $(f_\alpha)_{\alpha \geq 1}$ be a sequence of formal power series of continuous homogeneous polynomials on $\mathbb{C}^n$ with values in a Fréchet space $F$. Assume that there exists $r_0 \in (0,1)$ such that, for each $\alpha \in A$, the restriction of $(f_\alpha)_{\alpha \geq 1}$ on $\ell_\alpha$ is a sequence of holomorphic functions which is convergent on the disk $\Delta(r_0)$. Then there exists $r > 0$ such that $(f_\alpha)_{\alpha \geq 1}$ is a sequence of holomorphic functions that converges on $\Delta_n(r)$.

The proof of this theorem requires some extra results concerning to Vitali’s theorem for a sequence of Fréchet-valued holomorphic functions.

Remark 3.1. In exactly the same way, Theorem 2.1 in \cite{[4]} is true for the Fréchet-valued case.

Lemma 3.9. Let $E, F$ be Fréchet spaces, $D \subset E$ be an open set. Let $f : D \to F$ be a locally bounded function such that $\varphi \circ f$ is holomorphic for all $\varphi \in W \subset F'$, where $W$ is separating. Then $f$ is holomorphic.

The proof of Lemma runs as in the proof of Theorem 3.1 in \cite{[2]}, but here we use Vitali’s theorem in (\cite{[4]}, Prop. 6.2) which states for a sequence of holomorphic functions on an open connected subset of a locally convex space.

Lemma 3.10. Let $D$ be a domain in a Fréchet space $E$ and $f : D \to F$ be holomorphic, where $F$ is a barreled locally convex space. Assume that $D_0 = \{z \in D : f(z) \in G\}$ is not rare in $D$, where $G$ is a closed subspace of $F$. Then $f(z) \in G$ for all $z \in D$.

Proof. (i) We first consider the case $G = \{0\}$. On the contrary, suppose that $f(z^*) \neq 0$ for some $z^* \in D \setminus D_0$. By the Hahn-Banach theorem, we can find $\varphi \in F'$ such that $(\varphi \circ f)(z^*) \neq 0$. Let $z_0 \in (\text{int} \overline{D_0}) \cap D$ and let $W$ be a balanced convex neighbourhood of $0 \in E$ such that $z_0 + W \subset D_0$. Then by the continuity of $f$ we deduce that $f = 0$ on $z_0 + W$. Hence, it follows from the identity theorem (see \cite{[4]}, Prop. 6.6) that $f = 0$ on $D$. This contradicts above our claim $(\varphi \circ f)(z^*) \neq 0$.

(ii) For the general case, consider the quotient space $F/G$ and the holomorphic function $\omega \circ f : D \to F/G$ where $\omega : F \to F/G$ is the canonical map. Then $\omega \circ f \equiv 0$ on $D_0$. By the case (i), $\omega \circ f \equiv 0$ on $D$. This means that $f(z) \in G$ for all $z \in D$. \hfill \Box

Proposition 3.11. Let $E, F$ be Fréchet spaces and $D \subset E$ a domain. Assume that $(f_i)_{i \in \mathbb{N}}$ is a locally bounded sequence of holomorphic functions on $D$ with values in $F$. Then the following assertions are equivalent:

(i) The sequence $(f_i)_{i \in \mathbb{N}}$ converges uniformly on all compact subsets of $D$ to a holomorphic function $f : D \to F$;

(ii) The set $D_0 = \{z \in D : \lim \limits_i f_i(z) \text{ exists} \}$ is not rare in $D$.

Proof. It suffices to prove the implication (ii) $\Rightarrow$ (i) because the case (i) $\Rightarrow$ (ii) is trivial. Define $\tilde{f} : D \to \ell^\infty(\mathbb{N}, F)$ by $\tilde{f}(z) = (f_i(z))_{i \in \mathbb{N}}$, where $\ell^\infty(\mathbb{N}, F)$ is the Fréchet space with the topology induced by the system of semi-norms

$$
\|x\|_k = \|(x_\alpha)_{\alpha \in \mathbb{N}}\| = \sup \|x_\alpha\|, \forall k, \forall x = (x_\alpha) \in \ell^\infty(\mathbb{N}, F).
$$

For each $k \in \mathbb{N}$ we denote $pr_k : \ell^\infty(\mathbb{N}, F) \to F$ is the $k$-th projection with $pr_k((w_i)_{i \in \mathbb{N}}) = w_k$. Obviously

$$
W = \{\varphi \circ pr_k ; \varphi \in F', k \in \mathbb{N}\} \subset \ell^\infty(\mathbb{N}, F)'
$$

is separating and

$$
\varphi \circ pr_k \circ \tilde{f} = \varphi \circ pr_k \circ (f_i)_{i \in \mathbb{N}} = \varphi \circ f_k
$$
is holomorphic for every $k \in \mathbb{N}$. Then by Lemma 3.9 $\tilde{f}$ is holomorphic.

Since the space $G = \{(w_i)_{i \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, F) : \lim_{i \to \infty} w_i \text{ exists}\}$ is bounded in $H$.

Proof of Theorem 3.8.

By the continuity of $f$, Proposition 3.11 it follows that the sequence $(f_i)_{i \in \mathbb{N}}$ is equicontinuous in $\Delta$. Since for each $i \in \mathbb{N}$ the sequence $(f_i(z))_{z \in \Delta}$ is contained and bounded in $H^\infty(\Delta_n(r))$. This yields that $(f_n)_{n \geq 1}$ is contained and bounded in $H^\infty(\Delta_n(r), F)$. Since for each $z \in \Delta_n(r)$ the sequence $(f_n(z))_{n \geq 1}$ is convergent in $\Delta_1(r_0) \subset \ell_2$, by Remark 3.11 the sequence $(f_n(z))_{n \geq 1}$ is convergent for every $z \in \Delta_n(r)$. On the other hand, because $(f_n)_{n \geq 1}$ is bounded in $H^\infty(\Delta_n(r), F)$, by Proposition 3.11 it follows that the sequence $(f_n)_{n \geq 1}$ is convergent in $H(\Delta_n(r), F)$.
4. Discussion and open question.

From Proposition 3.1 in [14] it is clear that, in $\mathbb{C}^n$, the following are equivalent:

a) $A$ is projectively pluripolar;

b) $A^A := \{tz: t \in \mathbb{C}, |t| < \lambda, z \in A\}$ is pluripolar for each $\lambda > 0$;

c) $\mu(A^A) = 0$ where $\mu$ is the Lebesgue measure;

d) $\nu(A^A) = 0$ where $\nu$ is the invariant measure on the projective space $\mathbb{CP}^{n-1}$ and $g: \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ is the natural projection.

Thus, we can restate Theorem 3.8 in an alternative form as follows to obtain an extension of Hartogs’ result (cf. [1, Corollary 6.3]) that is an immediate consequence Alexander’s theorem from the scalar case to the Fréchet-valued one.

**Theorem 4.1.** Let $(f_\alpha)_{\alpha \geq 1}$ be a sequence of Fréchet-valued holomorphic functions on $\Delta_n \subset \mathbb{C}^n$. Let $B$ be a subset of $\Delta_n$ such that $\nu(g(B)) = 0$ where $\nu$ is the invariant measure on the projective space $\mathbb{CP}^{n-1}$ and $g: \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ is the natural projection. Assume that for some $r_0 \in (0,1)$, the restriction of the sequence $(f_\alpha)_{\alpha \geq 1}$ on each complex line $\ell$ through $0 \in \Delta_n$ with $\ell \cap B = \{0\}$ is convergent in $\Delta_1(r_0)$. Then $(f_\alpha)_{\alpha \geq 1}$ is the sequence of holomorphic functions which converges on a zero-neighbourhood in $\Delta_n$.

One question still unanswered is whether we can obtain a truly generalization of Alexander’s theorem (cf. [1] Theorem 6.2) for Fréchet-valued version? In other word, whether Theorem 3.8 is true or not if the uniform convergence of the family $(f_\alpha)_{\alpha \geq 1}$ on the disk $\Delta_1(r_0) \subset \ell_a$ for each $a \in A$ is replaced by normality of this family on $\Delta_1(r_0)$?

**References**

[1] H. Alexander, *Volumes of images of varieties in projective space and in Grassmannians*, Trans. Amer. Math. Soc., 189 (1974), 237-249.

[2] W. Arendt and N. Nikolski, *Vector-valued holomorphic functions revisited*, Math. Z., 234 (2000), 777-805.

[3] E. Bedford and B. A. Taylor, *A new capacity of plurisubharmonic functions*, Acta. Math., 149 (1982), 1-40.

[4] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math., 39 (1971), 77-112.

[5] J. Bonet, A. Galbis, *The identity $L(E, F) = LB(E, F)$, tensor products and inductive limits*, Note Mat., 9(2) (1989), 195-216.

[6] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer, New York, (1999).

[7] S. Dineen, R. Meise, D. Vogt, *Characterization of nuclear Fréchet spaces in which every bounded set is polar*, Bull. Soc. Math. France., 112 (1984), 41-68.

[8] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., 16 (1955).

[9] J. Horvath, *Topological Vector Spaces and Distributions*, Vol. 1, Addison Wesley, 1966.

[10] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart (1984).

[11] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford, (1991).

[12] P. Lelong, *On a problem of M.A. Zorn*, Proc. Amer. Math. Soc., 2(1951), 11-19.

[13] N. Levenberg, R. Molzon, *Convergent sets of a formal power series*, Math. Z., 197(1988) 411-420.

[14] T. V. Long, L. T. Hung, *Sequences of formal power series*, J. Math. Anal. Appl., 452(1)(2017), 218-225.

[15] L.Nachbin, *A Glimpse at Infinite Dimensional Holomorphy*, Proc. on Infinite Dimensional Holomorphy, Lecture Notes in Math. 364 (1974), 69–79.

[16] Ph. Noverraz, *Pseudo-conevezite, Conevezite Polynomiale et Domaines d’Holomorphie en Dimension Infinie*, North-Holland Math. Stud., 3(1973).

[17] T. T. Quang, D. T.Vy, L. T. Hung and P. H. Bang, *The Zorn property for holomorphic functions*, Ann. Polon. Math., 120(2)(2017), 115-133.
[18] A. Sathaye, *Convergence sets of divergent power series*, J. Reine Angew. Math., 283(1976), 86-98.

[19] B. V. Shabat, *An Introduction to Complex Analysis*, Vol II, Nauka, Moscow, 1985 (in Russian).

[20] J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Ann. Polon. Math., 39 (1981), 175-211.

[21] D. Vogt, *Frecheträume zwischen denen jede stetige linear Abbildung beschränkt ist*, J. reine angew. Math., 345 (1983), 182-200.

Department of Mathematics, Quy Nhon University, 170 An Duong Vuong, Quy Nhon, Binh Dinh, Vietnam.

E-mail address: thaithuanquang@qnu.edu.vn