ASYMPTOTIC EXPANSION OF THE WAVELET TRANSFORM FOR SMALL $a$

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Abstract

Asymptotic expansion of the wavelet transform for small values of the dilation parameter $a$ is obtained using asymptotic expansion of the Mellin convolution technique of Wong. Asymptotic expansions of Morlet wavelet transform, Mexican hat wavelet transform and Haar wavelet transform are obtained as special cases.

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1 Introduction

The wavelet transform of $f$ with respect to the wavelet $\psi$ is defined by

$$ (W_{\psi}f)(b,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \psi \left( \frac{t-b}{a} \right) dt, \quad b \in \mathbb{R}, a > 0, $$

provided the integral exists [1].

For fixed parameter $b \in \mathbb{R}$ in (1) the family $\{ \psi \left( \frac{\cdot-a}{b} \right) : a > 0 \}$ zooms in every detail of in a neighborhood of $b$ as long as $a$ is sufficiently small. The frequency resolution is controlled by the parameter $a$ and for small $a$, $(W_{\psi}f)(b,a)$ represents the frequency components of the signal $f$. Therefore, it is highly desirable to know the asymptotic behavior of $(W_{\psi}f)(b,a)$ for small values of $a$. Using Fourier transform (1) can also be expressed as

$$ (W_{\psi}f)(b,a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} e^{ib\omega} \hat{f}(\omega) \hat{\psi}(a\omega) d\omega, $$

where

$$ \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t) dt. $$

Putting $a = \frac{1}{c}$, from (2) we have

$$ (W_{\psi}f)(b,a) = \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} e^{ibcu} \hat{f}(cu) \hat{\psi}(u) du, $$

Asymptotic expansion with explicit error term for the general integral

$$ I(x) = \int_{0}^{\infty} g(t) h(xt) dt, $$

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as $x \to \infty+$, was obtained by Wong [5], [6] under different conditions on $g$ and $h$. The asymptotic expansion for (3) can be obtained by setting $h(t) = e^{ibt} \hat{f}(t)$ for fixed $b \in \mathbb{R}$. Let us recall basic result from [6] that will be used in the present investigation. Here we assume that $g(t)$ has an expansion of the form
\[
g(t) \sim \sum_{s=0}^{n-1} c_s t^{s+\lambda-1}, \quad \text{as} \quad t \to 0^+,
\]
where $0 < \lambda \leq 1$. Regarding the function $h$, we assume that as $t \to 0^+$,
\[
h(t) = O(t^\rho), \quad \rho + \lambda > 0,
\]
and that as $t \to +\infty$
\[
h(t) \sim \exp(i\tau t^p) \sum_{s=0}^{\infty} b_s t^{-s-\beta},
\]
where $\tau \neq 0$ is real, $p \geq 1$ and $\beta > 0$. Let $M[h; z]$ denote the generalized Mellin transform of $h$ defined by
\[
M[h; z] = \lim_{\epsilon \to 0^+} \int_0^\infty t^{z-1} h(t) \exp(-\epsilon t^p) \, dt.
\]
This together with (4) and [6, p.217], gives
\[
I(x) = \sum_{s=0}^{n-1} c_s M[h; s+\lambda] x^{-s-\lambda} + \delta_n(x),
\]
where
\[
\delta_n(x) = \lim_{\varepsilon \to 0^+} \int_0^\infty g_n(t) h(x t) \exp(-\varepsilon t^p) \, dt.
\]
If we now define recursively $h^\circ(t) = h(t)$ and
\[
h^{(-j)}(t) = - \int_t^\infty h^{(-j+1)}(u) du, \quad j = 1, 2, \ldots,
\]
then conditions of validity of aforesaid results are given by the following [6, Theorem 6, p.217].

**Theorem 1.** Assume that (i) $g^{(m)}(t)$ is continuous on $(0, \infty)$, where $m$ is a non-negative integer; (ii) $g(t)$ has an expansion of the form (4), and the expansion is $m$ times differentiable; (iii) $h(t)$ satisfies (6) and (7) and (iv) and as $t \to \infty$, $t^{-\beta} g^{(j)}(t) = O(t^{-1-\varepsilon})$ for $j = 0, 1, \ldots, m$, and for some $\varepsilon > 0$. Under these conditions, the result (9) holds with
\[
\delta_n(x) = \frac{(-1)^m}{x^m} \int_0^\infty g_n^{(m)}(t) h^{(-m)}(x t) \, dt,
\]
where $n$ is the smallest positive integer such that $\lambda + n > m$.

The asymptotic expansion for the wavelet transform (2) for large values of dilation parameter $a$ has already been obtained in [3]. The aim of the present paper is to derive asymptotic expansion of the wavelet transform given by (2) for small values of $a$, using formula (9).
2 ASYMPTOTIC EXPANSION FOR SMALL $a$

In this section using aforesaid technique, we obtain asymptotic expansion of $(W_\psi f)(b, a)$ for small values of $a$, keeping $b$ fixed. We have

\[(W_\psi f)(b, a) = \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} e^{ibcu} \overline{\psi}(u) \hat{f}(cu) du\]

\[= \frac{\sqrt{c}}{2\pi} \left\{ \int_0^\infty e^{ibcu} \overline{\psi}(u) \hat{f}(cu) du + \int_0^\infty e^{-ibcu} \overline{\psi}(-u) \hat{f}(-cu) du \right\} \]

\[= \frac{\sqrt{c}}{2\pi} (I_1 + I_2), \quad \text{(say).} \tag{12}\]

Let us set

\[h(u) = e^{ibu} \hat{f}(u). \tag{13}\]

Assume that

\[\hat{f}(u) \sim \sum_{r=0}^{\infty} b_r u^{-r-\beta}, \quad \beta > 0, \quad u \to \infty;\]

so that

\[h(u) \sim e^{ibu} \sum_{r=0}^{\infty} b_r u^{-r-\beta}, \quad \beta > 0, \quad u \to \infty, b \neq 0. \tag{14}\]

\[\overline{\psi}(u) \sim \sum_{s=0}^{\infty} c_s u^{s+\lambda-1}, \quad \text{as } u \to 0. \]

For $n \geq 1$, we write

\[\overline{\psi}(u) \sim \sum_{s=0}^{n-1} c_s u^{s+\lambda-1} + \psi_n(u), \tag{15}\]

where $0 < \lambda \leq 1$. Also assume that

\[h(u) = O(u^\rho), \quad u \to 0, \quad \rho + \lambda > 0. \tag{16}\]

The generalized Mellin transform of $h$ is defined by

\[M[h; z] = \lim_{\varepsilon \to 0^+} \int_0^\infty u^{z-1} h(u) e^{-\varepsilon u} du. \tag{17}\]

Then by (9),

\[I_1(c) = \sum_{s=0}^{n-1} c_s M[h; s + \lambda] c^{-s-\lambda} + \delta_1(c), \tag{18}\]
where
\[
\delta^3_n(c) = \lim_{\varepsilon \to 0+} \int_0^{\infty} \hat{\psi}_n(u) h(cu)e^{-\varepsilon u} du,
\] (19)
and, from (17)
\[
M[h(-u); z_1] = \lim_{\varepsilon \to 0+} \int_0^{\infty} u^{z_1-1} h(-u) e^{-\varepsilon u} du.
\]
Hence
\[
I_2(c) = \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda+1} M[h(-u); s + \lambda] c^{-s-\lambda} + \delta_n^2(c),
\] (20)
where
\[
\delta_n^2(c) = \lim_{\varepsilon \to 0+} \int_0^{\infty} \hat{\psi}_n(-u) h(-cu)e^{-\varepsilon u} du.
\] (21)
Finally, from (12), (18) and (20) we get the asymptotic expansion:
\[
(W_\psi f)(b, a) = \frac{\sqrt{c}}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s \left( M[h(u); s + \lambda] + (-1)^{s+\lambda+1} M[-h(u); s + \lambda] \right) \right.
\[
\times c^{-s-\lambda} + \delta_n^1(c) + \delta_n^2(c) \right\},
\]
Finally, setting \( c = \frac{1}{a} \) we get the asymptotic expansion for small values of \( a \):
\[
(W_\psi f)(b, a) = \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s \left( M[h(u); s + \lambda] + (-1)^{s+\lambda+1} M[-h(u); s + \lambda] \right) \right.
\[
\times a^{s+\lambda-1/2} + \delta_n(a) \right\},
\] (22)
where
\[
\delta_n(a) = \frac{1}{\sqrt{a}} \lim_{\varepsilon \to 0+} \left( \int_0^{\infty} \hat{\psi}_n(u) h(u/a)e^{-\varepsilon u} du \right.
\[
+ \int_{-\infty}^{\infty} \hat{\psi}_n(u) h(-u/a)e^{-\varepsilon u} du \right). \] (23)
Using Theorem 1, we get the following existence theorem for formula (22).

**Theorem 2.** Assume that (i) \( \hat{\psi}^{(m)}(u) \) is continuous on \((-\infty, \infty)\), where \( m \) is a nonnegative integer; (ii) \( \hat{\psi}(u) \) has asymptotic expansion of the form (15) and the expansion is \( m \) times differential; (iii) \( h(u) \) satisfies (14) and (16) and (iv) as \( u \to \infty \) \( u^{-\beta} \hat{\psi}^{(m)}(u) = O(u^{-1-\epsilon}) \) for \( j = 0, 1, 2, ..., m \) and for some \( \epsilon > 0 \). Under these conditions, the result (22) holds with
\[
\delta_n(a) = (-1)^m a^{m-1/2} \int_{-\infty}^{\infty} \hat{\psi}_n^{(m)}(u) h^{(-m)}(u/a)e^{-\varepsilon u} du.
\]
where \( n \) is the the smallest positive integer such that \( \lambda + n > m \).
In the following sections we shall obtain asymptotic expansions for certain special cases of the general wavelet transform.

3  MORLET WAVELET TRANSFORM

In this section we shall exploit the following result [4, eq.(12) p.57] for series manipulation

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) t^{n+2k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k,n-2k) t^{n-2k}. \tag{24}
\]

We choose

\[
\psi(t) = \sqrt{2\pi} e^{iu_{0} t - t^{2}/2}. \tag{25}
\]

Then from [1, p.373]

\[
\hat{\psi}(u) = \sqrt{2\pi} e^{-(u-u_{0})^{2}/2}. \tag{26}
\]

Now, using (24) we can write \(\hat{\psi}(u)\) in form of (15)

\[
\hat{\psi}(u) = \sqrt{2\pi} e^{-u_{0}^{2}/2} e^{u_{0} u} e^{-u^{2}/2} \sum_{s=0}^{\infty} \frac{(u_{0} u)^{s}}{s!} \sum_{p=0}^{\infty} \frac{(-1)^{p} u^{2p}}{2^{p} p!}
\]

\[
= \sqrt{2\pi} e^{-u_{0}^{2}/2} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{p} u_{0}^{s+2p}}{s! 2^{p} p!}
\]

\[
= \sqrt{2\pi} e^{-u_{0}^{2}/2} \sum_{s=0}^{\infty} \sum_{p=0}^{[s/2]} \frac{(-1)^{p} u_{0}^{s-2p}}{p! (s-2p)!} 2^{p}
\]

\[
= \sum_{s=0}^{\infty} c_{s} u^{s} + \hat{\psi}(u), \tag{27}
\]

where

\[
c_{s} = \sqrt{2\pi} e^{-u_{0}^{2}/2} \sum_{p=0}^{[s/2]} \frac{(-1)^{p} u_{0}^{s-2p}}{p! (s-2p)!} 2^{p}. \tag{28}
\]

Thus \(\hat{\psi}(u)\) possesses asymptotic expansion (15) with \(\lambda = 1\) and \(c_{s}\) given by (26). Hence, using (22) and (26) we get the following asymptotic expansion of \((W_{\psi} f ) (b,a)\) for small values of \(a\).

\[
(W_{\psi} f ) (b,a) = \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_{s} (M[h(u) : s + 1] + (-1)^{s} M[h(-u) : s + 1])
\]

\[
\times a^{s+1/2} + \delta_{n}(a) \right\}, \tag{29}
\]
where
\[
\delta_n(a) = \frac{1}{\sqrt{a}} \lim_{\epsilon \to 0^+} \left( \int_0^\infty \hat{\psi}_n(u) h(u/a) e^{-\epsilon u} du + \int_0^\infty \hat{\psi}_n(-u) h(-u/a) e^{-\epsilon u} du \right).
\]  
(30)

Using Theorem 2, we get the following existence theorem for formula (29).

**Theorem 3.** Assume that \( h(u) \) satisfies (14) and (16). Under these conditions, the result (29) holds with
\[
\delta_n(a) = (-1)^m a^{m-1/2} \int_{-\infty}^\infty \hat{\psi}_m(u) h^{(-m)}(u/a) e^{-\epsilon u} du.
\]

where \( n \) is the smallest positive integer such that \( 1 + n > m \).

### 4 MEXICAN HAT WAVELET TRANSFORM

In this section we choose
\[
\psi(t) = (1 - t^2)e^{-t^2/2}.
\]
Then from [11 p. 372]
\[
\hat{\psi}(u) = \sqrt{2\pi u^2} e^{-u^2/2}.
\]

Now,
\[
\hat{\psi}(u) = \sqrt{2\pi} u^2 \sum_{r=0}^{\infty} \frac{(-1)^r u^{2r}}{2^r r!} = \sqrt{2\pi} \sum_{l=1}^{\infty} \frac{(-1)^{l-1} u^{2l}}{2^{l-1} (l-1)!} = \sum_{s=0}^{\infty} c_s u^s,
\]
(31)

where
\[
c_s = \begin{cases} \sqrt{2\pi} \frac{(-1)^{l-1}}{2^{l-1} (l-1)!} & \text{if } s = 2l, l = 1, 2, 3... \\ 0 & \text{otherwise}. \end{cases}
\]
(32)

Then, from (22) with \( \lambda = 1 \) and \( c_s \) given by (32) yields the asymptotic expansion:
\[
(W_{\psi}f)(b, a) = \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s (M[h(u) : s + 1] + (-1)^s M[h(-u) : s + 1]) \times a^{s+1/2} + \delta_n(a) \right\},
\]
(33)
where

\[
\delta_n(a) = \frac{1}{\sqrt{a}} \lim_{\varepsilon \to 0^+} \left( \int_0^\infty \hat{\psi}_n(u) h(u/a) e^{-\varepsilon u} \, du + \int_0^\infty \hat{\psi}_n(-u) h(-u/a) e^{-\varepsilon u} \, du \right).
\]  

(34)

Using Theorem 2, we get the following existence theorem for formula (33).

**Theorem 4.** Assume that \( h(u) \) satisfies (2.3) and (2.5). Under these conditions, the result (32) holds with

\[
\delta_n(a) = (-1)^m a^{m-1/2} \int_{-\infty}^\infty \hat{\psi}_n^{(m)}(u) h^{(-m)}(u/a) e^{-\varepsilon u} \, du,
\]  

(35)

where \( n \) is the smallest positive integer such that \( 1 + n > m \).

## 5 HAAR WAVELET TRANSFORM

Let us choose

\[
\psi(t) = \begin{cases} 
1, & 0 \leq t < 1/2 \\
-1, & 1/2 \leq t < 1 \\
0, & \text{otherwise},
\end{cases}
\]

Then from [1, p. 368],

\[
\overline{\psi}(au) = 4ie^{-iu/2} \sin^2 u/4 
\]

\[
= \frac{i}{u} (1 - 2e^{iu/2} + e^{iu})
\]

\[
= \frac{i}{u} \left( 1 - 2 \sum_{r=0}^\infty (iu)^r + \sum_{r=0}^\infty \frac{(iu)^r}{r!} \right)
\]

\[
= \sum_{r=1}^\infty \frac{i^{r+1}u^{r-1}}{r!} \left( 1 - \frac{1}{2^{r-1}} \right)
\]

\[
= \sum_{s=0}^\infty \frac{i^{s+2}u^s}{(s+1)!} \left( 1 - \frac{1}{2^s} \right)
\]

\[
= \sum_{s=0}^\infty c_s u^s,
\]  

(36)

where

\[
c_s = \frac{i^{s+2}}{(s+1)!} \left( 1 - \frac{1}{2^s} \right).
\]  

(37)
Then, from (22) with \( \lambda = 1 \) and \( c_s \) given by (37) we get

\[
(W_\psi f)(b,a) = \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s (M[h(u); s + 1] + (-1)^s M[h(-u); s + 1]) \right.
\times a^{s+1/2} + \delta_n(a) \bigg\},
\]

where

\[
\delta_n(a) = \frac{1}{\sqrt{a}} \lim_{\varepsilon \to 0^+} \left( \int_0^\infty \overline{\psi_n(u)} h(u/a) e^{-\varepsilon u} du - \int_0^\infty \overline{\psi_n(-u)} h(-u/a) e^{-\varepsilon u} du \right).
\]

\[
\text{(38)}
\]

6 ASYMPTOTIC EXPANSION FOR SMALL \( a \) CONTINUED

In this section we obtain asymptotic expansion of the wavelet transform given in the form (1) when \( a \to 0^+ \). Naturally, in this case we have to impose conditions on \( f \) and \( \psi \) instead of \( \hat{f} \) and \( \hat{\psi} \).

Now, let us write (1) in the form:

\[
(W_\psi f)(b,a) = c^{1/2} \int_{-\infty}^\infty f(t + b) \overline{\psi(ct)} dt,
\]

where \( c = 1/a \to +\infty \) and \( b \) is assumed to be a fixed real number. Then setting \( g(t) = f(t+b) \) and \( h(t) = \psi(t) \), we have

\[
(W_\psi f)(b,a) = c^{1/2} \left[ \int_0^\infty g(t)h(ct) dt + \int_{-\infty}^0 g(t)h(ct) dt \right]
\]

\[
= c^{1/2} [I_1 + I_2] \ \ (\text{say}).
\]

\[
\text{(41)}
\]

Assume that \( g(t) \) satisfies (5) and \( h(t) \) satisfies (6) and (7). Then from (9) it follows that

\[
I_1 = \sum_{s=0}^{n-1} c_s M[\bar{\psi}; s + \lambda] c^{-s-\lambda} + \delta^1_{n}(a),
\]

where

\[
\delta^1_{n}(a) = \lim_{\varepsilon \to 0^+} \int_0^\infty g_n(t)\overline{\psi(t/a)} e^{-\varepsilon t^\varphi} dt;
\]

\[
\text{(43)}
\]

and

\[
I_2 = \sum_{s=0}^{n-1} c_s (-1)^{s+1}\lambda M[\bar{\psi}(-t); s + \lambda] + \delta^2_{n}(a),
\]

\[
\text{(44)}
\]
where
\[ \delta_n^2(a) = \lim_{\varepsilon \to 0^+} \int_0^\infty g_n(t) \psi(-t/a) e^{-\varepsilon t} dt. \] (45)

From (41), (42) and (44) we get
\[ (W_\psi f)(b, a) = \sum_{s=0}^{n-1} c_s M[\tilde{\psi}; s + \lambda] a^{s+\lambda-1/2} \]
\[ + \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda-1} M[\psi(-t); s + \lambda] \]
\[ \times a^{s+\lambda-1/2} + \delta_n(a), \] (46)

where
\[ \delta_n(a) = a^{-1/2} \lim_{\varepsilon \to 0^+} \left\{ \int_0^\infty g_n(t) \psi(t/a) e^{-\varepsilon t} dt \right. \]
\[ + \int_0^\infty g_n(-t) \psi(-t/a) e^{-\varepsilon t} dt \right\}. \] (47)

7 Example

Let us find again asymptotic expansion of Morlet wavelet transform for small \( a \), using the above technique. Here
\[ \psi(t) = e^{i\omega_0 t - t^2/2}. \]

Suppose that \( g(t) = f(t + b) \) satisfies (5). Then from (42), using formula [2, eq.(21), p.16], we get
\[ I_1 = \sum_{s=0}^{n-1} c_s M[\tilde{\psi}; s + \lambda] e^{-s-\lambda} + \delta_n^1(a) \]
\[ = \sum_{s=0}^{n-1} c_s \int_0^\infty e^{i\omega_0 t - t^2/2} t^{s+\lambda-1} dt a^{s+\lambda} + \delta_n^1(a) \]
\[ = \sum_{s=0}^{n-1} c_s \Gamma(s + \lambda) e^{-\omega_0^2/4} D_{-s-\lambda}(-i\omega_0) a^{s+\lambda} + \delta_n^1(a). \] (48)

where \( D_\nu(z) \) denoted parabolic cylinder function. Similarly,
\[ I_2 = \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda-1} \Gamma(s + \lambda) e^{-\omega_0^2/4} D_{-s-\lambda}(i\omega_0) a^{s+\lambda} + \delta_n^2(a). \] (49)
Therefore,

\[
(W_{\psi})(b,a) = \sum_{s=0}^{n-1} c_s \Gamma(s + \lambda) e^{-i\omega_0^2/4} (D - s - \lambda)(-i\omega_0)
+ (-1)^{s+\lambda-1} D - s - \lambda(i\omega_0) \times a^{s+\lambda-1/2} + \delta_n(a),
\]

where

\[
\delta_n(a) = a^{-1/2} \left\{ \int_0^\infty g_n(t) e^{-i\omega_0(t/a) - (t/a)^2/2} dt + \int_0^\infty g_n(-t) e^{i\omega_0(t/a) - (t/a)^2/2} dt \right\}.
\]

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