MUTUAL ENTROPY IN QUANTUM INFORMATION AND INFORMATION GENETICS

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Abstract
After Shannon, entropy becomes a fundamental quantity to describe not only uncertainty or chaos of a system but also information carried by the system. Shannon’s important discovery is to give a mathematical expression of the mutual entropy (information), information transmitted from an input system to an output system, by which communication processes could be analyzed on the stage of mathematical science. In this paper, first we review the quantum mutual entropy and discuss its uses in quantum information theory, and secondly we show how the classical mutual entropy can be used to analyze genomes, in particular, those of HIV.

1 Introduction
The study of mutual entropy (information) and capacity in classical system was extensively done after Shannon by several authors like Kolmogorov [16] and Gelfand [10]. In quantum systems, there have been several definitions of the mutual entropy for classical input and quantum output [5,11,12,17]. In 1983, the author defined [23] the fully quantum mechanical mutual entropy by means of the relative entropy of Umegaki [34], and it has been used to compute the capacity of quantum channel for quantum communication process; quantum input-quantum output [27,28].
A correlated state in quantum systems, so-called quantum entangled state or quantum entanglement, are used to study quantum information, in particular, quantum computation, quantum teleportation, quantum cryptography [6, 7, 8, 9, 14, 15, 28, 31, 32]. Recently Belavkin and Ohya [6] characterized the entangled states and introduced the mutual entropy for entangled states to measure the degree of the entanglement.

In part I of this paper, we mainly discuss the following two topics; (1) the quantum mutual entropy, the capacity of quantum channel and their uses in quantum communication; (2) the quantum mutual entropy for entangled states.

Genome sequences is considered to carry information, and the information is stored in base or amino acid sequences so that it originates the life itself. In part II of this paper, we present how information theory is used to investigate the ”information” stored in DNA. In particular, we shall discuss the uses of several informations (entropies) and the artificial codes to analyze genomes of, for instance, HIV.

Part I
Quantum Information

2 Quantum Mutual Entropy

The quantum mutual entropy was introduced in [23] for a quantum input and quantum output, namely, for purely quantum channel, and it was generalized for a general quantum system described by C*-algebraic terminology [25]. We here review the quantum mutual entropy in usual quantum system described by a Hilbert space.

Let $\mathcal{H}$ be a Hilbert space for an input space, $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $\mathcal{S}(\mathcal{H})$ be the set of all density operators on $\mathcal{H}$. An output space is described by another Hilbert space $\tilde{\mathcal{H}}$, but often $\mathcal{H} = \tilde{\mathcal{H}}$. A channel from the input system to the output system is a mapping $\Lambda^*$ from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\tilde{\mathcal{H}})$ [22]. A channel $\Lambda^*$ is said to be completely positive if the dual map $\Lambda$ satisfies the following condition: $\Sigma_{k,j=1}^{n} A_k^* \Lambda(B_k^*B_j)A_j \geq 0$ for any $n \in \mathbb{N}$ and any $A_j \in B(\mathcal{H}), B_j \in B(\tilde{\mathcal{H}})$. 

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An input state \( \rho \in \mathcal{S}(\mathcal{H}) \) is sent to the output system through a channel \( \Lambda^* \), so that the output state is written as \( \tilde{\rho} \equiv \Lambda^* \rho \). Then it is important to ask how much information of \( \rho \) is correctly sent to the output state \( \Lambda^* \rho \). This amount of information transmitted from input to output is expressed by the mutual entropy in Shannon’s theory.

In order to define the quantum mutual entropy, we first mention the entropy of a quantum state introduced by von Neumann\[20\]. For a state \( \rho \), there exists a unique spectral decomposition

\[
\rho = \sum_k \lambda_k P_k,
\]

where \( \lambda_k \) is an eigenvalue of \( \rho \) and \( P_k \) is the associated projection for each \( \lambda_k \). The projection \( P_k \) is not one-dimensional when \( \lambda_k \) is degenerated, so that the spectral decomposition can be further decomposed into one-dimensional projections. Such a decomposition is called a Schatten decomposition, namely,

\[
\rho = \sum_k \lambda_k E_k,
\]

where \( E_k \) is the one-dimensional projection associated with \( \lambda_k \) and the degenerated eigenvalue \( \lambda_k \) repeats \( \text{dim} P_k \) times. This Schatten decomposition is not unique unless every eigenvalue is non-degenerated. Then the entropy (von Neumann entropy) \( S(\rho) \) of a state \( \rho \) is defined by

\[
S(\rho) = -\text{tr} \rho \log \rho,
\]

which equals to the Shannon entropy of the probability distribution \( \{ \lambda_k \} :\)

\[
S(\rho) = -\sum_k \lambda_k \log \lambda_k.
\]

The quantum mutual entropy was introduced on the basis of the above von Neumann entropy for purely quantum communication processes. The mutual entropy depends on an input state \( \rho \) and a channel \( \Lambda^* \), so it is denoted by \( I(\rho; \Lambda^*) \), which should satisfy the following conditions:

1. The quantum mutual entropy is well-matched to the von Neumann entropy. Furthermore, if a channel is trivial, i.e., \( \Lambda^* = \) identity map, then the mutual entropy equals to the von Neumann entropy: \( I(\rho; \text{id}) = S(\rho) \).

2. When the system is classical, the quantum mutual entropy reduces to classical one.

3. Shannon’s fundamental inequality \( 0 \leq I(\rho; \Lambda^*) \leq S(\rho) \) is held.
Before mentioning the quantum mutual entropy, we briefly review the classical mutual entropy. Let \((\Omega, \mathcal{F}), (\overline{\Omega}, \overline{\mathcal{F}})\) be an input and output measurable spaces, respectively, and \(P(\Omega), P(\overline{\Omega})\) are the corresponding set of all probability measures (states). A channel \(\Lambda^*\) is a mapping from \(P(\Omega)\) to \(P(\overline{\Omega})\) and its dual \(\Lambda\) is a map from the set \(B(\Omega)\) of all Baire measurable functions on \(\Omega\) to \(B(\overline{\Omega})\). For an input state \(\mu \in P(\Omega)\), the output state \(\overline{\mu} = \Lambda^*\mu\) and the joint state (probability measure) \(\Phi\) is given by

\[
\Phi (Q \times \overline{Q}) = \int_Q \Lambda (1_Q) d\mu, \quad Q \in \mathcal{F}, \quad \overline{Q} \in \overline{\mathcal{F}},
\]

where \(1_Q\) is the characteristic function on \(\Omega\): \(1_Q(\omega) = \begin{cases} 1 & (\omega \in Q) \\ 0 & (\omega \notin Q) \end{cases}\). The classical entropy, relative entropy and mutual entropy are defined as follows:

\[
S(\mu) = \sup \left\{ -\sum_{k=1}^{n} \mu(A_k) \log \mu(A_k); \{A_k\} \in \mathcal{P}(\Omega) \right\},
\]

(6)

\[
S(\mu, \nu) = \sup \left\{ \sum_{k=1}^{n} \mu(A_k) \log \frac{\mu(A_k)}{\nu(A_k)}; \{A_k\} \in \mathcal{P}(\Omega) \right\},
\]

(7)

\[
I(\mu; \Lambda^*) = S(\Phi, \mu \otimes \Lambda^*\mu),
\]

(8)

where \(\mathcal{P}(\Omega)\) is the set of all finite partitions on \(\Omega\), that is, \(\{A_k\} \in \mathcal{P}(\Omega)\) iff \(A_k \in \mathcal{F}\) with \(A_k \cap A_j = \emptyset (k \neq j)\) and \(\bigcup_{k=1}^{n} A_k = \Omega\).

In order to define the quantum mutual entropy, we need the joint state (it is called "compound state" in the sequel) describing the correlation between an input state \(\rho\) and the output state \(\Lambda^*\rho\) and the quantum relative entropy. A finite partition of \(\Omega\) in classical case corresponds to an orthogonal decomposition \(\{E_k\}\) of the identity operator \(I\) of \(\mathcal{H}\) in quantum case because the set of all orthogonal projections is considered to make an event system for a quantum system. It is known [26] that the following equality holds

\[
\sup \left\{ -\sum_{k} tr \rho E_k \log tr \rho E_k; \{E_k\} \right\} = -tr \rho \log \rho,
\]

and the supremum is attained when \(\{E_k\}\) is composed of the Schatten decomposition of \(\rho\). Therefore the Schatten decomposition is used to define the compound state and the quantum mutual entropy.
The compound state $\theta_E$ (corresponding to joint state in CS) of $\rho$ and $\Lambda^*\rho$ was introduced in [23, 24], which is given by

$$\theta_E = \sum_k \lambda_k E_k \otimes \Lambda^* E_k,$$

(9)

where $E$ stands for a Schatten decomposition of $\rho$, so that the compound state depends on how we decompose the state $\rho$ into basic states (elementary events), in other words, how to see the input state.

The relative entropy for two states $\rho$ and $\sigma$ is defined by Umegaki [34] and Lindblad [18], which is written as

$$S(\rho, \sigma) = \begin{cases} \text{tr} \rho (\log \rho - \log \sigma) & \text{when } \text{ran}\rho \subset \text{ran}\sigma \\ \infty & \text{otherwise} \end{cases}$$

(10)

Then we can define the mutual entropy by means of the compound state and the relative entropy [23], that is,

$$I(\rho; \Lambda^*) = \sup \left\{ S(\theta_E, \rho \otimes \Lambda^* \rho) ; E = \{E_k\} \right\},$$

(11)

where the supremum is taken over all Schatten decompositions. Some computations reduce it to the following form:

$$I(\rho; \Lambda^*) = \sup \left\{ \sum_k \lambda_k S(\Lambda^* E_k, \Lambda^* \rho) ; E = \{E_k\} \right\},$$

(12)

This mutual entropy satisfies all conditions (1)~(3) mentioned above.

When the input system is classical, an input state $\rho$ is given by a probability distribution or a probability measure, in either case, the Schatten decomposition of $\rho$ is unique, namely, for the case of probability distribution $\rho = \{\lambda_k\}$,

$$\rho = \sum_k \lambda_k \delta_k,$$

(13)

where $\delta_k$ is the delta measure, that is,

$$\delta_k(j) = \delta_{k,j} = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}, \forall j.$$

(14)

Therefore for any channel $\Lambda^*$, the mutual entropy becomes

$$I(\rho; \Lambda^*) = \sum_k \lambda_k S(\Lambda^* \delta_k, \Lambda^* \rho),$$

(15)
which equals to the following usual expression of Shannon when it is well-defined:

\[ I(\rho; \Lambda^*) = S(\Lambda^*\rho) - \sum_k \lambda_k S(\Lambda^*\delta_k), \]  

which has been taken as the definition of the mutual entropy for a classical-quantum(-classical) channel [4, 5, 11, 12, 17].

Note that the above definition of the mutual entropy (2.12) is written as

\[ I(\rho; \Lambda^*) = \sup \left\{ \sum_k \lambda_k S(\Lambda^*\rho_k, \Lambda^*\rho) ; \rho = \sum_k \lambda_k \rho_k \in F_o(\rho) \right\}, \]

where \( F_o(\rho) \) is the set of all orthogonal finite decompositions of \( \rho \).

More general formulation of the mutual entropy for general quantum systems was done [25, 13] in C* dynamical system by using Araki’s or Uhlmann’s relative entropy [3, 33, 26]. This general mutual entropy contains all other cases including measure theoretic definition of Gelfand and Yaglom [10].

### 3 Communication Processes

The information communication process is mathematically set as follows: M messages are sent to a receiver and the \( k \)th message \( \omega^{(k)} \) occurs with the probability \( \lambda_k \). Then the occurrence probability of each message in the sequence \( (\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(M)}) \) of M messages is denoted by \( \rho = \{ \lambda_k \} \), which is a state in a classical system. If \( \xi \) is a classical coding, then \( \xi(\omega) \) is a classical object such as an electric pulse. If \( \xi \) is a quantum coding, then \( \xi(\omega) \) is a quantum object (state) such as a coherent state. Here we consider such a quantum coding, that is, \( \xi(\omega^{(k)}) \) is a quantum state, and we denote \( \xi(\omega^{(k)}) \) by \( \sigma_k \). Thus the coded state for the sequence \( (\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(M)}) \) is written as

\[ \sigma = \sum_k \lambda_k \sigma_k. \]  

This state is transmitted through a channel \( \gamma \), which is expressed by a completely positive mapping \( \Gamma^* \) from the state space of \( X \) to that of \( \tilde{X} \), hence
the output coded quantum state \( \tilde{\sigma} \) is \( \Gamma^* \sigma \). Since the information transmission process can be understood as a process of state (probability) change, when \( \Omega \) and \( \tilde{\Omega} \) are classical and \( X \) and \( \tilde{X} \) are quantum, the process (3.1) is written as

\[
P (\Omega) \xrightarrow{\Xi^*} \mathcal{S} (\mathcal{H}) \xrightarrow{\Gamma^*} \mathcal{S} (\tilde{\mathcal{H}}) \xrightarrow{\bar{\Xi}^*} P (\tilde{\Omega}),
\]

where \( \Xi^* \) (resp. \( \bar{\Xi}^* \)) is the channel corresponding to the coding \( \xi \) (resp. \( \tilde{\xi} \)) and \( \mathcal{S} (\mathcal{H}) \) (resp. \( \mathcal{S} (\tilde{\mathcal{H}}) \)) is the set of all density operators (states) on \( \mathcal{H} \) (resp. \( \tilde{\mathcal{H}} \)).

We have to be care to study the objects in the above transmission process (3.1) or (3.3). Namely, we have to make clear which object is going to study. For instance, if we want to know the information capacity of a quantum channel \( \gamma \) (= \( \Gamma^* \)), then we have to take \( X \) so as to describe a quantum system like a Hilbert space and we need to start the study from a quantum state in quantum space \( X \) not from a classical state associated to a message. If we like to know the capacity of the whole process including a coding and a decoding, which means the capacity of a channel \( \tilde{\xi} \circ \gamma \circ \xi \) (= \( \tilde{\Xi}^* \circ \Gamma^* \circ \Xi^* \)), then we have to start from a classical state. In any case, when we concern the capacity of channel, we have only to take the supremum of the mutual entropy \( I (\rho; \Lambda^*) \) over a quantum or classical state \( \rho \) in a proper set determined by what we like to study with a channel \( \Lambda^* \). We explain this more precisely in the next section.

4 Channel Capacity

We discuss two types of channel capacity in communication processes, namely, the capacity of a quantum channel \( \Gamma^* \) and that of a classical (classical-quantum-classical) channel \( \bar{\Xi}^* \circ \Gamma^* \circ \Xi^* \).

(1) Capacity of quantum channel: The capacity of a quantum channel is the ability of information transmission of a quantum channel itself, so that it does not depend on how to code a message being treated as classical object and we have to start from an arbitrary quantum state and find the supremum of the quantum mutual entropy. One often makes a mistake in this point. For example, one starts from the coding of a message and compute the supremum of the mutual entropy and he says that the supremum is the
capacity of a quantum channel, which is not correct. Even when his coding is a quantum coding and he sends the coded message to a receiver through a quantum channel, if he starts from a classical state, then his capacity is not the capacity of the quantum channel itself. In his case, usual Shannon’s theory is applied because he can easily compute the conditional distribution by a usual (classical) way. His supremum is the capacity of a classical-quantum-classical channel, and it is in the second category discussed below.

The capacity of a quantum channel $\Gamma^*$ is defined as follows: Let $S_0(\subset S(\mathcal{H}))$ be the set of all states prepared for expression of information. Then the capacity of the channel $\Gamma^*$ with respect to $S_0$ is defined by

$$C^{S_0}(\Gamma^*) = \sup \{ I(\rho; \Gamma^*) ; \rho \in S_0 \}. \quad (19)$$

Here $I(\rho; \Gamma^*)$ is the mutual entropy given in (2.11) or (2.12) with $\Lambda^* = \Gamma^*$. When $S_0 = S(\mathcal{H})$, $C^{S(\mathcal{H})}(\Gamma^*)$ is denoted by $C(\Gamma^*)$ for simplicity. The capacity $C(\Gamma^*)$ is written as

$$C(\Gamma^*) = \sup \{ I(\rho; \Gamma^*) ; \rho \in S(\mathcal{H}) \}, \quad (20)$$

where the supremum is taken over all states $\rho$ with its orthogonal pure decomposition $\sum_k \lambda_k \rho_k$ of $\rho$. In [27, 21], we also considered the pseudo-quantum capacity $C_p(\Gamma^*)$ defined by (4.1) with the pseudo-mutual entropy $I_p(\rho; \Gamma^*)$ where the supremum is taken over all finite decompositions instead of all orthogonal pure decompositions:

$$I_p(\rho; \Gamma^*) = \sup \left\{ \sum_k \lambda_k S(\Gamma^* \rho_k, \Gamma^* \rho) ; \rho = \sum_k \lambda_k \rho_k, \\text{finite decomposition} \right\}. \quad (21)$$

However the pseudo-mutual entropy is not well-matched to the conditions explained in Sec.2, and it is difficult to be computed numerically. The relation between $C(\Gamma^*)$ and $C_p(\Gamma^*)$ was discussed in [27]. From the monotonicity of the mutual entropy [26], we have

$$0 \leq C^{S_0}(\Gamma^*) \leq C^{S_0}(\Gamma^*) \leq \sup \{ S(\rho) ; \rho \in S_0 \}. \quad (2)$$

(2) Capacity of classical-quantum-classical channel: The capacity of C-Q-C channel $\tilde{\Xi}^* \circ \Gamma^* \circ \Xi^*$ is the capacity of the information transmission process starting from the coding of messages, therefore it can be considered as the
capacity including a coding (and a decoding). As is discussed in Sec.3, an input state $\rho$ is the probability distribution $\{\lambda_k\}$ of messages, and its Schatten decomposition is unique as (2.9), so the mutual entropy is written by (2.11):

$$I\left(\rho; \tilde{\Xi} \circ \Gamma^* \circ \Xi^*\right) = \sum_k \lambda_k S\left(\tilde{\Xi} \circ \Gamma^* \circ \Xi^* \delta_k, \tilde{\Xi} \circ \Gamma^* \circ \Xi^* \rho\right).$$ (22)

If the coding $\Xi^*$ is a quantum coding, then $\Xi^* \delta_k$ is expressed by a quantum state. Let denote the coded quantum state by $\sigma_k$ and put $\sigma = \Xi^* \rho = \sum_k \lambda_k \sigma_k$. Then the above mutual entropy is written as

$$I\left(\rho; \tilde{\Xi} \circ \Gamma^* \circ \Xi^*\right) = \sum_k \lambda_k S\left(\tilde{\Xi} \circ \Gamma^* \sigma_k, \tilde{\Xi} \circ \Gamma^* \sigma\right).$$ (23)

This is the expression of the mutual entropy of the whole information transmission process starting from a coding of classical messages. Hence the capacity of C-Q-C channel is

$$C_{P0}^c\left(\Xi^* \circ \Gamma^* \circ \Xi^*\right) = \sup\{I\left(\rho; \tilde{\Xi} \circ \Gamma^* \circ \Xi^*\right); \rho \in P_0\},$$ (24)

where $P_0(\subset P(\Omega))$ is the set of all probability distributions prepared for input (a-priori) states (distributions or probability measures). Moreover the capacity for coding free is found by taking the supremum of the mutual entropy (4.4) over all probability distributions and all codings $\Xi^*$:

$$C_{cP0}^c\left(\Xi^* \circ \Gamma^*\right) = \sup\{I\left(\rho; \tilde{\Xi} \circ \Gamma^* \circ \Xi^*\right); \rho \in P_0, \Xi^*\}.\quad (25)$$

The last capacity is for both coding and decoding free and it is given by

$$C_{cdP0}^c\left(\Gamma^*\right) = \sup\{I\left(\rho; \tilde{\Xi} \circ \Gamma^* \circ \Xi^*\right); \rho \in P_0, \Xi^*, \tilde{\Xi}^*\}.\quad (26)$$

These capacities $C_{cP0}^c, C_{cdP0}^c$ do not measure the ability of the quantum channel $\Gamma^*$ itself, but measure the ability of $\Gamma^*$ through the coding and decoding.
Remark that $\sum_k \lambda_k S(\Gamma^* \sigma_k)$ is finite, then (4.4) becomes

$$I \left( \rho; \Xi^* \circ \Gamma^* \circ \Xi^* \right) = S(\Xi^* \circ \Gamma^* \sigma) - \sum_k \lambda_k S(\Xi^* \circ \Gamma^* \sigma_k).$$  (27)

Further, if $\rho$ is a probability measure having a density function $f(\lambda)$ and each $\lambda$ corresponds to a quantum coded state $\sigma(\lambda)$, then $\sigma = \int f(\lambda) \sigma(\lambda) d\lambda$ and

$$I \left( \rho; \Xi^* \circ \Gamma^* \circ \Xi^* \right) = S(\Xi^* \circ \Gamma^* \sigma) - \int f(\lambda) S(\Xi^* \circ \Gamma^* \sigma(\lambda)) d\lambda.$$  (28)

This is bounded by

$$S(\Gamma^* \sigma) - \int f(\lambda) S(\Gamma^* \sigma(\lambda)) d\lambda,$$

which is called the Holevo bound and is computed in several occasions [36, 27].

The above three capacities $C^{P_0}$, $C^{P_0}_c$, $C^{P_0}_{cd}$ satisfy the following inequalities

$$0 \leq C^{P_0} \left( \Xi^* \circ \Gamma^* \circ \Xi^* \right) \leq C^{P_0}_c \left( \Xi^* \circ \Gamma^* \right) \leq C^{P_0}_{cd} (\Gamma^*) \leq \sup \{ S(\rho); \rho \in P_0 \}$$

where $S(\rho)$ is not the von Neumann entropy but the Shannon entropy: $-\sum \lambda_k \log \lambda_k$.

The capacities (4.1), (4.6), (4.7) and (4.8) are generally different. Some misunderstandings occur due to forgetting which channel is considered. That is, we have to make clear what kind of the ability (capacity) is considered, the capacity of a quantum channel itself or that of a classical-quantum(-classical) channel. The computation of the capacity of a quantum channel was carried in several models in [27, 28].

5 Quantum Entanglements

Recently the quantum entangled state has been mathematically studied [8, 19, 31], in which the entangled state is defined by a state not written as a form $\sum_k \lambda_k \rho_k \otimes \sigma_k$ with any states $\rho_k$ and $\sigma_k$. A state written as above is called a
separable state, so that an entangled state is a state not belonge d to the set of all separable states. However it is obvious that there exist several correlated states written as separable forms. Such correlated states have been discussed in several contexts in quantum probability such as quantum filtering \[4\], quantum compound state \[23\], quantum Markov state \[1\] and quantum lifting \[2\]. In \[6\], we showed a mathematical construction of quantum entangled states and gave a finer classification of quantum states.

For the (separable) Hilbert space \( \mathcal{K} \) of a quantum system, let \( A \equiv B \left( \mathcal{K} \right) \) be the set of all linear bounded operators on \( \mathcal{K} \). A normal state \( \varphi \) on \( A \) can be expressed as \( \varphi \left( A \right) = \text{tr} G \kappa^\dagger A \kappa, \ A \in A \), where \( \mathcal{G} \) is another separable Hilbert space, \( \kappa \) is a linear Hilbert-Schmidt operator from \( \mathcal{G} \) to \( \mathcal{K} \) and \( \kappa^\dagger \) is the adjoint operator of \( \kappa \) from \( \mathcal{K} \) to \( \mathcal{G} \). The (unique) density operator \( \sigma \in A \) associated to the state \( \varphi \) is written by \( \kappa \) such as \( \sigma = \kappa \kappa^\dagger \). This \( \kappa \) is called the amplitude operator, and it is called just the amplitude if \( \mathcal{G} \) is one dimensional space \( \mathbb{C} \), corresponding to the pure state \( \varphi \left( A \right) = \kappa^\dagger A \kappa \) for a \( \kappa \in \mathcal{K} \) with \( \kappa \kappa^\dagger = \| \kappa \|^2 = 1 \). In general, \( \mathcal{G} \) is not one dimensional, the dimensionality \( \text{dim} \mathcal{G} \) must be not less than \( \text{dim} \mathcal{K} \).

Since \( \mathcal{G} \) is separable, \( \mathcal{G} \) is realized as a subspace of \( l^2 \left( \mathbb{N} \right) \) of complex sequences (i.e., \( \zeta^* = \left( \zeta^n \right), \ z^n \in \mathbb{C}, \ n \in \mathbb{N} \) with \( \sum |z^n|^2 < +\infty \)), so that any vector \( \zeta^* = \left( \zeta^n \right) \) represents a vector \( \zeta = \sum \zeta^n |n\rangle \) in the standard basis \( \{ |n\rangle \} \in \mathcal{G} \) of \( l^2 \left( \mathbb{N} \right) \).

Given the amplitude operator \( \kappa \), one can define not only the states \( \sigma \equiv \kappa \kappa^\dagger \) and \( \rho \equiv \kappa^\dagger \kappa \) on the algebras \( \mathcal{A} (= B \left( \mathcal{K} \right)) \) and \( \mathcal{B} (= B \left( \mathcal{G} \right)) \) but also an entanglement state \( \Theta \) on the algebra \( \mathcal{B} \otimes \mathcal{A} \) of all bounded operators on the tensor product Hilbert space \( \mathcal{G} \otimes \mathcal{K} \) by

\[
\Theta \left( B \otimes A \right) = \text{tr}_G B \kappa^\dagger A \kappa = \text{tr}_K A \kappa B \kappa^\dagger
\]

for any \( B \in \mathcal{B} \). This state is pure as it is the case of \( \mathcal{F} = \mathbb{C} \) in the theorem below, and it satisfies the marginal conditions: For any \( B \in \mathcal{B}, A \in \mathcal{A}, \)

\[
\Theta \left( B \otimes I \right) = \text{tr}_G B \rho, \quad \Theta \left( I \otimes A \right) = \text{tr}_K A \sigma.
\]

**Theorem 1** \[6\] Let \( \Theta : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathbb{C} \) be a state

\[
\Theta \left( B \otimes A \right) = \text{tr}_F \psi^\dagger \left( B \otimes A \right) \psi,
\]

defined by an amplitude operator \( \psi \) on a separable Hilbert space \( \mathcal{E} \) into the tensor product Hilbert space \( \mathcal{G} \otimes \mathcal{K} \); \( \psi : \mathcal{E} \rightarrow \mathcal{G} \otimes \mathcal{K} \) with \( \text{tr}_F \psi^\dagger \psi = 1 \). Then
there exists an amplitude operator $\kappa: \mathcal{G} \to \mathcal{F} \otimes \mathcal{K}$ such that the state $\Theta$ can be achieved by an entanglement

$$\Theta (B \otimes A) = tr_{\mathcal{G}} B \kappa^\dagger (I \otimes A) \kappa = tr_{\mathcal{F} \otimes \mathcal{K}} (I \otimes A) \kappa B \kappa^\dagger$$  \hspace{1cm} (30)$$

The entangling operator $\kappa$ is uniquely defined up to a unitary transformation of the minimal space $\mathcal{F}$.

The entangled state (5.2) is written as

$$\Theta (B \otimes A) = tr_{\mathcal{G}} B \phi (A) = tr_{\mathcal{K}} A \phi_* (B),$$  \hspace{1cm} (31)$$

where $\phi (A) \equiv \kappa^\dagger (I \otimes A) \kappa$ is in the predual space $\mathcal{B}_* \subset \mathcal{B}$ of all trace-class operators in $\mathcal{G}$, and $\phi_* (B) \equiv tr_{\mathcal{F} \otimes \mathcal{K}} B \kappa^\dagger$ is in $\mathcal{A}_* \subset \mathcal{A}$. The map $\phi$ is the Steinspring form of the general completely positive map $A \to \mathcal{B}_*$, written in the eigen-basis $\{|n\rangle\}$ of $\mathcal{G} \subseteq L^2 (\mathbb{N})$ of the density operator $\rho = \phi (I)$ as

$$\phi (A) = \sum_{m,n} |m\rangle \kappa^\dagger_m (I \otimes A) \kappa_n \langle n|, \hspace{0.5cm} A \in \mathcal{A}$$  \hspace{1cm} (32)$$

where $\kappa_n$ is the vector in $\mathcal{F} \otimes \mathcal{K}$ such that $\kappa = \sum_n \kappa_n \langle n|$. The dual operation $\phi_*$ is the Kraus form of the general completely positive map $\mathcal{B} \to \mathcal{A}_*$, given in this basis as

$$\phi_* (B) = \sum_{n,m} \langle n| B |m\rangle tr_{\mathcal{F} \otimes \mathcal{K}} \kappa_n \kappa^\dagger_m, \hspace{0.5cm} B \in \mathcal{B}.$$  \hspace{1cm} (33)$$

It corresponds to the general form of the density operator

$$\theta_\phi = \sum_{m,n} |n\rangle \langle m| \otimes tr_{\mathcal{F} \otimes \mathcal{K}} \kappa_n \kappa^\dagger_m$$  \hspace{1cm} (34)$$

for the entangled state $\Theta$ with the weak orthogonality property

$$tr_{\mathcal{F} \otimes \mathcal{K}} \kappa_n \kappa^\dagger_m = p_n \delta^m_n = \kappa^\dagger_m \kappa_n.$$  \hspace{1cm} (35)$$

**Definition 1** The dual map $\phi_* : \mathcal{B} \to \mathcal{A}_*$ to a completely positive map $\phi: \mathcal{A} \to \mathcal{B}_*$, normalized as $tr_{\mathcal{G}} \phi (I) = 1$, is called the quantum entanglement of the state $\rho = \phi (I)$ on $\mathcal{B}$ to the state $\sigma = \phi_* (I)$ on $\mathcal{A}$. The entanglement by $\phi (A) = \sigma^{1/2} A \sigma^{1/2}$ of the state $\rho = \sigma$ on the algebra $\mathcal{B} = \mathcal{A}$ given by the standard entangling operator $\kappa = \sigma^{1/2}$ is called standard.
A compound state, playing the similar role as the joint input-output probability measures in classical systems, was introduced in [23] as explained in Sec.2. It corresponds to a particular diagonal type

$$\theta_\phi = \sum_n |n\rangle \langle n| \otimes tr_\mathcal{F} \kappa_n \kappa_n^\dagger$$

of the entangling map (33) in the eigen-basis (Schatten decomposition) of the density operator $\rho = \sum p_n |n\rangle \langle n|$. Therefore the entangled states, generalizing the compound state, also play the role of the joint probability measures.

The diagonal entanglements can be considered as a quantum correspondences of symbols $\{1, \cdots, n, \cdots\}$ to quantum states. The general entangled states $\Theta$ are described by the density operators $\theta_\phi$ of the form (34) which is not necessarily diagonal in the eigen-representation of the density operator $\rho = \sum p_n |n\rangle \langle n|$. Such nondiagonal entangled states were called in [25] the quasicompound (q-compound) states, so we can call also the nondiagonal entanglement the quantum quasi-correspondence (q-correspondence) in contrast to the d-correspondences, described by the diagonal entanglements, giving rise to the d-compound states.

Take $tr_\mathcal{F} \kappa_n \kappa_n^\dagger \equiv v_n v_n^\dagger$, $v_n \in \mathcal{K}$. The density operator

$$\theta = \sum_n |n\rangle \langle n| \otimes \sigma_n, \quad \sigma_n = p_n v_n v_n^\dagger$$

(36)
define the compound states on $\mathcal{B} \otimes \mathcal{A}$, giving the quantum correspondences $n \mapsto |n\rangle \langle n|$ with the probabilities $p_n$. The entanglement with (36) is a diagonal entanglement such as

$$\phi_* (B) = \sum_n p_n |n\rangle \langle B| \langle n| v_n v_n^\dagger$$

(37)

whose dual is

$$\phi (A) = \sum_n p_n |n\rangle v_n^\dagger A \langle n|.$$ (38)

These entanglements has the stronger orthogonality

$$tr_\mathcal{F} \kappa_n \kappa_m^\dagger = p_n v_n v_m^\dagger \delta_m^n,$$ (39)

for the amplitudes $\kappa_n \in \mathcal{F} \otimes \mathcal{K}$ of the decomposition $\kappa = \sum_n \kappa_n |n\rangle$ in comparison with the weak orthogonality of $\kappa_n$ in (34).
Definition 2 The positive diagonal map
\[ \phi_* (B) = \sum_n \langle n | B | n \rangle \sigma_n \] (40)
into the subspace of trace-class operation \( K \) with \( tr_G \phi_* (I) = 1 \), is called quantum d-entanglement with the input probabilities \( p_n = tr_K \sigma_n \) and the output states \( \omega_n = p_n^{-1} \sigma_n \), and the corresponding compound state \( \Theta \) is called d-compound state. The d-entanglement is called c-entanglement and compound state is called c-compound if all density operators \( \sigma_n \) commute: \( \sigma_m \sigma_n = \sigma_n \sigma_m \) for all \( m \) and \( n \).

Note that due to the commutativity of the operators \( B \otimes I \) with \( I \otimes A \) on \( G \otimes K \), one can treat the correspondences as the nondemolition measurements in \( B \) with respect to \( A \). So, the compound state is the state prepared for such measurements on the input \( G \). It coincides with the mixture of the states, corresponding to those after the measurement without reading the sent message. The set of all d-entanglements corresponding to a given Schatten decomposition of the input state \( \rho \) on \( A \) is obviously convex with the extreme points given by the pure elementary output states \( \omega_n \) on \( A \), corresponding to a not necessarily orthogonal decompositions \( \sigma = \sum_n \sigma_n \) into one-dimensional density operators \( \sigma_n = p_n \omega_n \).

The orthogonal Schatten decompositions \( \sigma = \sum_n p_n \omega_n \) correspond to the extreme points of c-entanglements which also form a convex set with mixed commuting \( \omega_n \) for a given Schatten decomposition of \( \sigma \). The orthogonal c-entanglements were used in [2] to construct a particular type of Accardi’s transition expectations [1] and to define the entropy in a quantum dynamical system via such transition expectations [6].

Thus we classified the entangled states into three categories, namely, q-entangled state, d-entangled state and c-entangled state, and their rigorous expressions were given.

6 Mutual Entropy via Entanglements

Let us consider the entangled mutual entropy by means of the above three types compound states. We denote the quantum mutual entropy of the compound state \( \Theta \) achieved by an entanglement \( \phi_* : B \rightarrow A_* \) with the marginals
\[ \Theta (B \otimes I) = tr_G B \rho, \; \Theta (I \otimes A) = tr_K A \sigma \] (41)
by $I_\phi (\rho, \sigma)$ or $I_\phi (A, B)$ and it is given as

$$I_\phi (\rho, \sigma) = \text{tr} \theta_\phi (\log \theta_\phi - \log (\rho \otimes \sigma)).$$  \hfill (42)

Besides this quantity describes an information gain in a quantum system $(A, \sigma)$ via an entanglement $\phi_*$ with another system $(B, \rho)$, it is naturally treated as a measure of the strength of an entanglement, having zero the value only for completely disentangled states (41), corresponding to $\theta_\phi = \rho \otimes \sigma$.

**Definition 3** The maximal quantum mutual entropy for a fixed state $\sigma$

$$H_\sigma (A) = \sup \{I_\phi (A, B) ; \phi_* (I) = \sigma\}$$ \hfill (43)

is called q-entropy of the state $\sigma$. The differences

$$H_\phi (B|A) = H_\sigma (A) - I_\phi (A, B),$$

$$D_\phi (B|A) = S (\sigma) - I_\phi (A, B)$$

are respectively called the q-conditional entropy on $B$ with respect to $A$ and the degree of disentanglement for the compound state $\phi$.

$H_\phi (B|A)$ is obviously positive, however $D_\phi (B|A)$ has the positive maximal value $S (\sigma) = \sup \{D_\phi (B|A) ; \phi_* (I) = \sigma\}$ and can achieve also a negative value

$$\inf \{D_\phi (B|A) ; \phi_* (I) = \sigma\} = S (\sigma) - H_\sigma (A)$$ \hfill (44)

for the entangled states [6], which is called the chaos degree in [13].

Let us consider $G$ as a Hilbert space describing a quantum input system and $K$ as its output Hilbert space. A quantum channel $\Lambda^*$ sending each input state defined on $G$ to an output state defined on $K$. A deterministic quantum channel is given by a linear isometry $\Upsilon : G \rightarrow K$ with $\Upsilon^\dagger \Upsilon = I_0$ ($I_0$ is the identity operator in $G$) such that each input state vector $\eta \in G$, $\|\eta\| = 1$ is transmitted into an output state vector $\Upsilon \eta \in K$, $\|\Upsilon \eta\| = 1$. The mixtures $\rho = \sum_n p_n \omega_n$ of the pure input states $\omega_n = \eta_n \eta_n^\dagger$ are sent into the mixtures $\sigma = \sum_n p_n \sigma_n$ with pure states $\sigma_n = \Upsilon \omega_n \Upsilon^\dagger$. A noisy quantum channel sends pure input states $\omega$ into mixed ones $\sigma = \Lambda^* \omega$ given by the dual of the following completely positive map $\Lambda$

$$\Lambda (A) = \Upsilon^\dagger (I_1 \otimes A) \Upsilon, \quad A \in A$$ \hfill (45)
where $\Upsilon$ is a linear isometry from $G \otimes K$, $\Upsilon^\dagger (I_1 \otimes I) \Upsilon = I_0$, and $I_1$ is the identity operator in a separable Hilbert space $\mathcal{F}_1$ representing the quantum noise. Each input mixed state $\rho \in B(G)$ is transmitted into the output state $\sigma = \Lambda^* \rho$ on $A \subseteq B(K)$, which is given by the density operator

$$\sigma = tr_{\mathcal{F}_1} \Upsilon \rho \Upsilon^\dagger \equiv \Lambda^* \rho \in \mathcal{A}.$$  \hfill (46)

We apply the proceeding discussion of the entanglement to the above situation containing a channel $\Lambda^*$. For a given Schatten decomposition $\rho = \sum p_n |n\rangle \langle n|$, and the state $\sigma \equiv \Lambda^* \rho$, we can construct three entangled states of the proceeding section:

(1) q-entanglement $\phi^q_0$ and q-compound state $\theta^q_0$ are given as

$$\phi^q_0(B) = \sum_{n,m} \langle n | B | m \rangle \ tr_{\mathcal{F}_1} \kappa_n \kappa_m^\dagger,$$

$$\theta^q_0 = \sum_{m,n} |n\rangle \langle m| \otimes tr_{\mathcal{F}_1} \kappa_n \kappa_m^\dagger,$$

with the marginals $\rho = \sum p_n |n\rangle \langle n|$, $\sigma \equiv \Lambda^* \rho = tr_{\mathcal{G}} \theta^q_0$ and $tr_{\mathcal{K}} \kappa_n \kappa_m^\dagger = p_n \omega_n \delta_n^m = \kappa_n^\dagger \kappa_n$ for $\omega_n = \Lambda^* |n\rangle \langle n|$. Let $\mathcal{E}_q$ be the convex set of all completely positive maps $\phi^q$.

(2) d-entanglement $\phi^d_0$ and d-compound state $\theta^d_0$ are given as

$$\phi^d_0(B) = \sum_n \langle n | B | n \rangle \ tr_{\mathcal{F}_1} \kappa_n \kappa_n^\dagger,$$

$$\theta^d_0 = \sum_n |n\rangle \langle n| \otimes tr_{\mathcal{F}_1} \kappa_n \kappa_n^\dagger,$$

with the same marginal conditions as (1). Let $\mathcal{E}_d$ be the convex set of all completely positive maps $\phi^d$.

(3) c-entanglement $\phi^c_0$ and c-compound state $\theta^c_0$ are same as those of (2) with commuting $\{\omega_n\}$. Let $\mathcal{E}_c$ be the convex set of all completely positive maps $\phi^c$.

Now, let us consider the entangled mutual entropy and the capacity of quantum channel by means of the above three types of compound states.
Definition 4 The mutual entropy $I_q(\rho, \Lambda^*)$ and the $q$-capacity $C_q(\Lambda^*)$ for a quantum channel $\Lambda^*$ are defined by

\[
I_q(\rho, \Lambda^*) = \sup \left\{ S(\theta^q_\phi, \rho \otimes \Lambda^* \rho) ; \phi^q \in \mathcal{E}_q \right\}, \\
C_q(\Lambda^*) = \sup \left\{ I_q(\rho, \Lambda^*) ; \rho \right\}.
\] (47)

The $d$-mutual entropy, the $d$-capacity and the $c$-mutual entropy, the $c$-capacity are defined as above using $\theta^d_\phi$ and $\theta^c_\phi$, respectively.

Note that due to $\mathcal{E}_c \subseteq \mathcal{E}_d \subseteq \mathcal{E}_q$, we have the inequalities

\[
I_q(\rho, \Lambda^*) \geq I_d(\rho, \Lambda^*) \geq I_c(\rho, \Lambda^*), \\
C_q(\Lambda^*) \geq C_d(\Lambda^*) \geq C_c(\Lambda^*)
\]

for a deterministic channel ($\Lambda^* = id$), the two lower mutual entropies coincide with the von Neumann entropy:

\[
I_d(\rho, id) = -tr\rho \log \rho = I_c(\rho, id).
\]

The capacity for such a channel is finite if $\mathcal{A}$ has a finite rank, $C_d(\Lambda^*) \leq \dim \mathcal{K}$. On the other hand, the $q$-mutual entropy can achieve the $q$-entropy

\[
I_q(\rho, id) = -2tr\rho \log \rho
\]

and its capacity is bounded by the dimension of the algebra $\mathcal{A}$, $C_q(\Lambda^*) \leq \dim \mathcal{A}$ which doubles the $d$-capacity $\dim \mathcal{K}$ when $\mathcal{A} = B(\mathcal{K})$. These equalities will be related to the work on entropy by Voiculescu [35].

\title{Part II
Information Genetics

7 Entropy Evolution Rate}

Genome sequence carries information as an order of four bases, and the information is transmitted to m-RNA, which makes a protein as a sequence of amino acids by a help of t-RNA.
In information theory, the concept of information has two aspects, one of which expresses the amount of complexity of a whole system like a sequence itself and another does the structure of the system (or message) such as the rule stored in the order of sequence. From Shannon’s philosophy, a system has the larger complexity, the system carries the larger information, from which the information of a whole system has been expressed by the entropy. The structure of the system is studied in the field named “coding theory”, that is, how to code the messages is essential in communication of information.

Pioneering works for application of information theory to genome sequence were done by Smith and Gatlin, since then few works have been appeared along this line. In 1989, I introduced a measure representing the difference of two genome or amino acid sequences, which is called the entropy evolution rate and has been used to make phylogenetic trees. The coding theory was applied to the study of genome sequences in order to examine the coding structure of several species.

Let A and B be amino acid or base sequences. When they are considered to be close each other, for instance, they specify an identical protein, we first have to align these sequences by inserting a gap “*”, whose arrangement is called the alignment of sequences. As an example, take two sequences A and B given as

\[
\begin{align*}
A &= a \ c \ b \ a \ c \ d \\
B &= a \ d \ b \ c \ a \ c \ b
\end{align*}
\]

Then the aligned sequences become

\[
\begin{align*}
A &= a \ c \ b \ * \ a \ c \ d \\
B &= a \ d \ b \ c \ a \ c \ b
\end{align*}
\]

After the alignment, two sequences have the same length. Take two aligned sequences A and B having the length n given by \(A=(a_1, a_2, \ldots, a_n)\), \(B=(b_1, b_2, \ldots, b_n)\), where \(a_i, b_i\) are the gap “*” or an amino acid for an amino acid sequence or a base for a base sequence. There are 21 events (20 amino acids and “*”) in an amino acid sequence and 5 events (4 bases and “*”) in a base sequence. Therefore, in an aligned sequence, the occurrence probability of each amino acid (resp. base) is associated, and it is denoted by \(p_k\) for k-th amino acid (resp. base), where \(0 \leq k \leq 20\) (resp. \(0 \leq k \leq 4\)) and “0” corresponds to the gap. Then the entropy (information) carried by the amino acid (resp. base) sequence A is defined as

\[18\]
\[ S(A) \text{ (or } S(p)) = - \sum_k p_k \log p_k \]

where \( p \) denotes the probability distribution \((p_k)\). Similarly, there exists the event system \((B, q \equiv (q_k))\) for the amino acid (or base) sequence \(B\), and its entropy is denoted \(S(B)\) or \(S(q)\). Through the alignment, we can find the correspondence between the amino acid (resp. base) of \(A\) and that of \(B\), which enables to make the compound event system \((A \times B, r)\) of \(A\) and \(B\). Here \(r\) is the joint probability distribution between \(A\) and \(B\), so that it satisfies \(\sum_k r_jk = p_j\) and \(\sum_k r_jk = q_k\).

The most important information measure in Shannon’s communication theory is the mutual entropy (information) expressing the amount of information transmitted from \((A, p)\) to \((B, q)\), which is defined as follows:

\[ I(A, B) = \sum_{j,k} r_{jk} \log \frac{r_{jk}}{p_j q_k} \]

Using the entropy and the mutual entropy, an quantity measuring the similarity between \(A\) and \(B\) was introduced as

\[ r(A, B) = \frac{1}{2} \left\{ \frac{I(A, B)}{S(A)} + \frac{I(A, B)}{S(B)} \right\}, \]

which was called the symmetrized entropy ratio or the entropy evolution rate in [43] and it takes the value 0 when \(A\) and \(B\) are completely different and 1 when they are identical. The minus of this rate from 1 indicates the difference between \(A\) and \(B\). We here call it the entropy evolution rate, and it is denoted by \(\rho(A, B)\):

\[ \rho(A, B) = 1 - r(A, B). \]

Using this rate, we can construct a genetic matrix and write a phylogenetic tree of species [43, 41]. Note that a similar measure providing the difference between \(A\) and \(B\) can be defined as

\[ \rho'(A, B) = 1 - \frac{I(A, B)}{S(A) + S(B) - I(A, B)}, \]

but this does not have a precise meaning from the information theoretical point of view.
An application of this rate to the variation of HIV virus for six patients reported by [51, 38, 40, 39] is discussed in [49].

8 Code Structure of Genes

When we send an information (a series of messages), we have to process the messages in proper forms so as to correctly and quickly send the information to a receiver. It is the coding theory that teaches us how to process the messages properly. There are many ways to encode the messages in communication processes. We shall explain some of such codings and their use to the study of genome sequences.

Let \( i = (i_1, i_2, \cdots, i_k) \) be a properly processed information sequence. In order to send the symbol \( i \) to a receiver correctly, that is, to avoid some noise and loss in the course of information transmission, we have to add some redundancy (parity check symbol) \( p = (p_1, p_2, \cdots, p_{n-k}) \) to the information symbol \( i \). This redundancy \( p \) detects or corrects the errors in the communication process. The whole code-word now becomes

\[
x = (i_1, i_2, \cdots, i_k, p_1, p_2, \cdots, p_{n-k}).
\]

The above \( x \) is called a systematic code, and to make the systematic code \( x \) from the information symbol \( i \) is called a coding. A coding is realized by a Galois group \( GF(q) \) with a primary number \( q \) and a certain parity check \( p \). When the relation between \( i \) and \( p \) is linear, the code so obtained is called a linear code. Among the linear codes, there are the block code such as cyclic code and BCH code and the convolutional code such as self-orthogonal code and Iwadare code. Each code has its own parity check correcting the error such as random error, burst error and bite error. We do not go into the details of the coding theory here, but we explain how to use the coding technique to examine the code structure of genome sequences.

When we like to know the code structure of a species, an organism, a special part of a genome sequence indicating a protein or a set of these objects, we rewrite a base sequence of an object into the sequence of the symbols of \( GF(2^2) \) because we have four bases, and we apply several coding methods to the symbol sequence and get the coded symbol sequence (systematic code), then we write it back the coded base sequence. This process is written as
follows:

Base sequence $A \implies$ Symbol sequence $A_s$
$\implies$ Coded symbol sequence $A_s^C \implies$ Coded base sequence $A^C$

In order to know the common code structure of the sequences $A_1, A_2, \cdots, A_n$, we use the following index obtained from the entropy evolution rate and a coding $C$ applied to the sequences:

$$D_C = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\rho(A_i, A_j) - \rho(A_i^C, A_j^C)|}{nC_2},$$

where $nC_2$ is the combination 2 out of $n$, that is, $nC_2 = \binom{n}{2}$ and $A$ is an amino acid sequence or a base sequence. Note that when $A$ is originally an amino acid sequence, we first translate it the corresponding base sequence and take the above procedure, then we convert the coded base sequence to the coded amino acid sequence. If this index $D_C$ is close to 0, then a common code structure of the group $\{A_1, A_2, \cdots, A_n\}$ is close to the structure of the code $C$ used.

We studied the code structure of Vertebrate, Onco virus and HIV virus by means of the structure index $D_C$. We used some parts of the base sequence for each organisms; MDH, LDH, hemoglobin $\alpha, \beta$ for Vertebrate; pol, env, gag for Onco and HIV virus. Then we obtained the following results:

1. Vertebrate has a similar code structure of the convolutional code with high ability correcting the burst errors like the codes named UI, ZI, and the code structure of hemoglobin $\alpha$ is closest to that of the artificial codes.

2. Onco virus has a similar code structure of the cyclic code with the burst error correction (C2) or the self-orthogonal code (TB,VD), so that it does not have so high ability correcting the errors.

3. HIV virus has a similar code structure of the cyclic code (C1) or the self-orthogonal code with the random error correction (TA), so that the ability correcting the errors is low.

4. In Onco and HIV virus, the pol protein has the closest code structure of the artificial codes.

The structure index is applied to the study of the variation and the condition of the patients having the HIV infection in [50].
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