Dynamical characterization of Weyl nodes in Floquet Weyl semimetal phases

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Due to studies in nonequilibrium (periodically-driven) topological matter, it is now understood that some topological invariants used to classify equilibrium states of matter do not suffice to describe their nonequilibrium counterparts. Indeed, in Floquet systems the additional gap arising from the periodicity of the quasienergy Brillouin zone often leads to unique topological phenomena without equilibrium analogues. In the context of Floquet Weyl semimetal, Weyl points may be induced at both quasienergy zero and $\pi/T$ ($T$ being the driving period) and these two types of Weyl points can be very close to each other in the momentum space. Because of their momentum-space proximity, the chirality of each individual Weyl point may become hard to characterize in both theory and experiments, thus making it challenging to determine the system’s overall topology. In this work, inspired by the construction of dynamical winding numbers in Floquet Chern insulators, we propose a dynamical invariant capable of characterizing and distinguishing between Weyl points at different quasienergy values, thus advancing one step further in the topological characterization of Floquet Weyl semimetals. To demonstrate the usefulness of such a dynamical topological invariant, we consider a variant of the periodically kicked Harper model (the very first model in studies of Floquet topological phases) that exhibits many Weyl points, with the number of Weyl points rising unlimitedly with the strength of some system parameters. Furthermore, we investigate the two-terminal transport signature associated with the Weyl points. Theoretical findings of this work pave the way for experimentally probing the rich topological band structures of some seemingly simple Floquet semimetal systems.

I. INTRODUCTION

There has been a great surge in research on topological phases of matter after the discovery of Quantum Hall effect 1. In addition to topological insulators 2–12 and superconductors 13,14, which are characterized by gapped bulk bands, topological semimetal 15–22 phases with gapless bulk bands have also been reported. The latter exhibits band touching that may occur at isolated points 20,21,23,24, along a line 25–28, or a closed loop 29,30. Depending on which of these various band touching structures is featured, such topological semimetal (SM) phases can further be categorized as topological Weyl, nodal line and nodal loop semimetals, respectively. These topological semimetal phases can be characterized in terms of valence band Chern numbers (Weyl SM) 24,31 and certain winding numbers or Berry phases along a momentum space structure (Nodal line-loop SM) 25,30.

Isolated band touching points appearing in Weyl semimetals (Weyl points) are particularly interesting due to their linear dispersion along all three quasimomenta (thus resembling relativistic particles) and their robustness against generic perturbations 23. Such Weyl points act as the equivalent of magnetic monopoles in the momentum space, whose associated magnetic charge is equal to their chirality 24,32. Due to the fermion doubling theorem 32, Weyl nodes always appear in pairs with opposite magnetic charges. Their topological signature is further evidenced by the existence of surface states with zero dispersion along the line connecting such pairs of Weyl nodes (Fermi arcs 23,33,34) in finite size systems. The aforementioned features of Weyl points lead to various exotic transport properties such as chiral anomaly 35–39, negative magneto-resistance 40, and anomalous Hall effect 41, to name a few. For these reasons, studies of Weyl semimetal materials and how to engineer them have remained an active research topic up to this date.

Since the last decade, the use of periodic driving has emerged as one attractive method to engineer topological materials. It leads to a variety of novel topological phases such as Floquet topological insulators 42–53, superconductors 54–58 and semimetals 59–65. In such systems, energy is no longer a conserved quantity and is replaced by a quantity termed quasienergy, which is only defined modulo the driving frequency ($\omega$). The latter feature gives rise to the formation of quasienergy Brillouin zone (BZ), where in-gap or gapless topological edge states may emerge not only around the BZ center (quasienergy zero), but also around the BZ edge (quasienergy $\omega/2$) 66. Consequently, the definition of new dynamical invariants 47,48,59,67 is often necessary to faithfully capture all the possible edge states of Floquet topological matter under open boundaries. Finally, on a more practical side, periodic driving naturally offers an extra tunable parameter which allows the realization of distinct topological phases within the same platform.

In the context of Weyl semimetals, periodic driving enables the formation of Weyl nodes and Fermi arcs at both quasienergies zero and $\omega/2$, whose signatures may no longer be uniquely captured by the Chern number of some 2D slice of the system 64,65. In Ref. 59, quantum adiabatic charge pumping was proposed to capture the
Hamiltonian the one period unitary evolution operator, usually re-
ing in Sec. V.

the two-terminal conductance associated with the Weyl
quantum chaos, to further demonstrate the usefulness of
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due to the quantum Weyl nodes with quasienergy values, resulting in a zero net band
Chern number. On the other hand, dynamical winding number calculations still yield nontrivial values which ad-
dress two such Weyl points individually. Moreover, we
study two-terminal transport signatures associated with
the Weyl nodes of opposite chirality. We shall reveal that
the two-terminal conductance captures the total chirality
of the Weyl nodes at quasienergy zero and \( \omega/2 \).

The article is structured in the following way. In
Sec. II, we briefly review the literature on the Floquet
theory and dynamical winding number. This is to make
this work more self consistent. To explicitly demonstrate the correlation between dynamical winding number sur-
rounding Weyl points and their chirality, we then con-
sider a simple Floquet four band toy model exhibiting two
Weyl nodes of different quasienergy values at the same
quasimomenta. In Sec. III, we employ the kicked Harper
model, a celebrated dynamical model in the literature of
quantum chaos, to further demonstrate the usefulness of
dynamical winding number calculations in systems with
potentially high Weyl node density. In Sec. IV, we study the
two-terminal conductance associated with the Weyl
nodes of opposite chirality. Finally, we conclude our find-
ings in Sec. V.

II. CHIRALITY OF WEYL NODES

A. Floquet Theory: A Review

The Floquet theory is a powerful tool to study time periodic systems whose dynamics is governed by the one period unitary evolution operator, usually referred to as the Floquet operator. For a time-periodic Hamiltonian \( H(k, t) \) with \( H(k, t) = H(k, t + T) \), where \( k \) is the set of system parameters (e.g., quasimomenta) and \( t \) is time, the Floquet operator is denoted by \( U(k) \) and given as, \( U(k) = e^{-i\int_0^T H(k, t)dt} \), where \( i \) is time ordering operator and \( T = \frac{2\pi}{\omega} \) is the time-period (\( \omega \) = driving frequency) after which the Hamiltonian repeats itself. It satisfies the Floquet eigenvalue equation \( e^{-i\int_0^T H(k, t)dt} | \Psi_n(k) \rangle = e^{-i\epsilon_n(k)T/\hbar} | \Psi_n(k) \rangle \), where \( \Omega_n(k) \) is called quasienergy, which replaces the role of energy in such non-equilibrium systems.

The quasienergy is defined modulo \( w = \frac{2\pi}{T} \), which in this paper is taken \( \in (-\frac{\pi}{T}, \frac{\pi}{T}) \). As a consequence of this periodicity, quasienergy bands may close not only at quasienergy zero, but also at \( \pi/T \). In the context of Floquet Weyl semimetals, this enables the formation of Weyl nodes at either quasienergy zero or \( \pi/T \).

B. Dynamical Winding number: A Review

Dynamical winding number \( W^\epsilon \) has been introduced in to characterize the net chirality of edge states crossing a gap around \( \epsilon \). In Floquet topological insulators, it can uniquely characterize systems with arbitrary number of co-propagating edge states. Together with some additional invariants, it can further count the number of counter-propagating edge states, thus recovering the notion of bulk-boundary correspondence in Floquet systems. The general applicability of dynamical winding number, as well as its ability to characterize a variety of Floquet topological phases with no static counterparts, has led us to think of more possibilities where it can play a significant role. As will be demonstrated in the next few sections, such an invariant can in fact be utilized to separately probe the chirality of the Weyl nodes at zero and \( \pi/T \) quasienergy. To this end, we will first review the theory of dynamical winding number to develop some intuitions.

In order to calculate dynamical winding number, cyclic evolution is introduced by employing a modified time-evolution operator in momentum representation which is denoted by \( \tilde{U}_\epsilon(\Theta, t) \) and given as \( \tilde{U}_\epsilon(\Theta, t) = \left\{ \begin{array}{ll} U(\Theta, 2t) & \text{if } 0 \leq t < T/2 \\ e^{-iH_{\text{eff}}(2T-2t)} & \text{if } T/2 \leq t < T \end{array} \right. \) \( (1) \), where \( T = 1 \) is the period of drive and \( \Theta \) is the set of continuous parameters which can form a closed surface. In 2D systems, \( \Theta = (k_x, k_y) \) simply represents a set of quasi-momenta in two spatial directions, whereas in three dimensions (3D), \( \Theta = (\theta, \phi) \) can be taken as comprising the polar and azimuthal angles that form a closed 2D spherical or toroidal surface in 3D BZ.

\( H_{\text{eff}} = -\frac{\hbar}{\epsilon} \log[U(\Theta, T)] \) is the effective Hamiltonian, with \( \epsilon \) being the branch cut of logarithm function, such that its eigenvalues \( \Omega \in (-2\pi, \epsilon) \). The operator during the second half of the drive is a return map, which sends the modified time-evolution operator to identity at the end of one period, i.e., \( \tilde{U}_\epsilon(\Theta, t = 0) = \tilde{U}_\epsilon(\Theta, t = T) = 1 \).

With the above notations, we are now ready to define the dynamical winding number \( W^\epsilon \) with respect to
quasienergy \( e^{\text{qg}} \). By focusing in particular to 3D systems, it is defined as

\[
W^\tau = \frac{1}{8\pi^2} \int_0^T dt \int_S d\theta d\phi \times \text{Tr} \left( \tilde{U}_e^{-1} \partial_t \tilde{U}_e \left[ \tilde{U}_e^{-1} \partial_\theta \tilde{U}_e, \tilde{U}_e^{-1} \partial_\phi \tilde{U}_e \right] \right),
\]

where \( \theta \) and \( \phi \) are the polar and azimuthal angles respectively, which together parameterize a 2D spherical or toroidal surface \( S \) in the 3D BZ. From Eq. (1), we can observe that the modified Floquet operator during first half of the period depends on the driving protocol of the periodically driven system, whereas during the second half of the period, modified Floquet operator depends on the full period time-evolution operator along with the choice of branch cut of the logarithm function.

**C. Toy model and Weyl nodes**

In order to illustrate how the dynamical winding number defined above works in capturing the chirality of Weyl nodes, we consider a simple four band toy model. In particular, it possesses two Weyl nodes, one with quasienergy zero and the other \( \pi/T \), located at the same point \((k_x, 0, k_z)\) in the 3D BZ. The Hamiltonian of the system is defined as \( H(k_x, k_y, k_z, t) \) and given as

\[
H(k_x, k_y, k_z, t) = \begin{cases} \tau_+ \otimes H^0 & 0 \leq t < T/2 \\ \tau_- \otimes H^\pi & T/2 \leq t < T \end{cases}
\]

where \( \tau_+ \) is the set of Pauli matrices, \( \tau_+ = \frac{\tau_0 + \tau_z}{2} \), and \( \tau_- = \frac{\tau_0 - \tau_z}{2} \) form the upper and lower diagonal matrices. Moreover, \( H^0 \) [\( H^\pi \)] is the Weyl Hamiltonian at \( 4\pi/[2\pi] \) energy and given as,

\[
H^0(k_x, k_y, k_z) = 4\pi \sigma_0 + 2J(k_x \sigma_x + k_y \sigma_y - k_z \sigma_z),
\]

\[
H^\pi(k_x, k_y, k_z) = 2\pi \sigma_0 + 2J(k_x \sigma_x + k_y \sigma_y + k_z \sigma_z),
\]

where \( \sigma_i \) is the set of Pauli matrices associated with the internal degree of freedom of the effective Weyl Hamiltonian. Given a Weyl Hamiltonian \( H = f_x k_x \sigma_x + f_y k_y \sigma_y + f_z k_z \sigma_z \), the chirality of its associated Weyl node is given as \( \chi = \text{sgn}[f_x f_y f_z]^{24} \). In this case, the chirality of the two Weyl nodes associated with Eq. (4) is then given as \( \chi^0 = -1 \) and \( \chi^\pi = +1 \) for the effective Weyl Hamiltonian \( H^0 \) and \( H^\pi \) respectively.

The Floquet operator associated with the effective Hamiltonian of the system [Eq. (3)] for time period \( T \), where \( T = \hbar = 1 \) is given by

\[
U(k_x, k_y, k_z) = \tau_+ \otimes U^0 + \tau_- \otimes U^\pi,
\]

where \( U^0 \) and \( U^\pi \) can be regarded as the time evolution operators of some effective Hamiltonian possessing a single Weyl node at quasienergy zero and \( \pi/T \), respectively and they are given by,

\[
U^0(k_x, k_y, k_z) = e^{-i(2\pi \sigma_0 + Jk_x \sigma_x + Jk_y \sigma_y - Jk_z \sigma_z)},
\]

\[
U^\pi(k_x, k_y, k_z) = e^{-i(\pi \sigma_0 + Jk_x \sigma_x + Jk_y \sigma_y + Jk_z \sigma_z)},
\]

Fig. 1 depicts the quasienergy band structure associated with the above model. There, two Weyl nodes with quasienergy zero and \( \pi/T \) are clearly observed at the same point \((0, 0, 0)\) in the 3D BZ. In order to directly compute the dynamical winding number on a spherical surface enclosing the Weyl nodes, we carry out coordinate transformation from Cartesian to spherical polar coordinates: \( k_x = r \sin \theta \cos \phi, k_y = r \sin \theta \sin \phi \) and \( k_z = r \cos \theta \), where \( r \) is the radius of the sphere and it is taken to be small such that some kind of first-order approximation in our analytical treatment holds i.e; \( r^2 \approx 0 \).

**FIG. 1:** The quasienergy spectrum of the toy model has been shown for fixed \( k_z = 0 \). Weyl nodes at quasienergy zero and \( \pi/T \) can be observed to exist at the same momenta value in the Brillouin zone.

Let \( |\Psi_i\rangle \) and \( \Omega_i \in [\epsilon - 2\pi, \epsilon] \) be the \( i \)-th band eigenvectors and the associated quasienergy of the Floquet operator [Eq. (5)]. We may then construct the modified Floquet operator in the spirit of Eq. (1) which is given as,

\[
\hat{U}_i(\Theta, t) = \begin{cases} \mathcal{T} e^{-i \int_0^t dt' H(\theta, \phi, t')} & \text{if } 0 \leq t < T/2 \\ \sum_{i=1}^4 e^{-i \Omega_i t} & \text{if } T/2 \leq t < T \end{cases}
\]

which is unitary such that \( \hat{U}_i \hat{U}_i^{-1} = \hat{U}_i^{-1} \hat{U}_i = 1 \). The dynamical winding number can be determined by dividing the time integral into two parts from \( t \in [0, T/2] \) and \( t \in [T/2, T] \). The Eq. (2) during the time interval
$t \in [0, T/2]$ will be given as,

$$W^r(t_0 \to \frac{T}{2}) = \frac{1}{8\pi^2} \int_0^{T/2} dt \int d\theta \, d\phi \times Tr \left( \hat{U}_{\epsilon}^{-1} \partial_t \hat{U}_{\epsilon} \left[ \hat{U}_{\epsilon}^{-1} \partial_{\phi} \hat{U}_{\epsilon}, \hat{U}_{\epsilon}^{-1} \partial_{\theta} \hat{U}_{\epsilon} \right] \right)$$

which leads to $W^r(t_0 \to \frac{T}{2}) = \frac{2r - 2\cos(\epsilon \sin(\epsilon))}{\pi} = 0$ as $\cos(\epsilon) \approx 1$ and $\sin(\epsilon) \approx r$ under our “first-order” approximation. That is, for cases with sufficiently small $r$, it becomes clear that the dynamical winding number is only contributed by $\hat{U}$, during the time interval $t \in [T/2, T)$. The modified Floquet during this interval is given as,

$$\hat{U}_{\epsilon}(\theta, \phi, t_{T/2 \to T}) = \tau_+ \otimes \hat{U}_{\epsilon}^{1,2} + \tau_- \otimes \hat{U}_{\epsilon}^{3,4}, \quad (7)$$

where

$$\hat{U}_{\epsilon}^{1,2} = \left( e^{-i\Omega_1^* [2T - 2t]} \sin^2 \left( \frac{\phi}{2} \right) + e^{-i\Omega_1^* [2T - 2t]} \cos^2 \left( \frac{\phi}{2} \right) \right) \frac{e^{-i\epsilon \sin(\theta)}}{2} \left( e^{-i\Omega_1^* [2T - 2t]} - e^{-i\Omega_2^* [2T - 2t]} \right)$$

$$\hat{U}_{\epsilon}^{3,4} = \left( e^{-i\Omega_1^* [2T - 2t]} \sin^2 \left( \frac{\phi}{2} \right) + e^{-i\Omega_1^* [2T - 2t]} \cos^2 \left( \frac{\phi}{2} \right) \right) - \frac{e^{-i\epsilon \sin(\theta)}}{2} \left( e^{-i\Omega_1^* [2T - 2t]} - e^{-i\Omega_2^* [2T - 2t]} \sin^2 \left( \frac{\phi}{2} \right) \right).$$

The dynamical winding number during time interval $t \in [T/2, T)$ is then given as,

$$W^r(t_{T/2 \to T}) = \frac{1}{8\pi^2} \int_0^{\pi} d\theta \int_0^{T/2} d\phi \int_0^T dt \times Tr \left( \hat{U}_{\epsilon}^{-1} \partial_t \hat{U}_{\epsilon} \left[ \hat{U}_{\epsilon}^{-1} \partial_{\phi} \hat{U}_{\epsilon}, \hat{U}_{\epsilon}^{-1} \partial_{\theta} \hat{U}_{\epsilon} \right] \right) \quad (8)$$

$$= -\frac{1}{2\pi} \left[ -\Omega_1 + \Omega_2 - \Omega_3 + \Omega_4 \right] \right].$$

where $\epsilon$ is the choice of the branch cut of logarithmic function and it is taken as either 0 or $\pi/T$. Moreover, $\Omega_i^*(i \in [1, 2, 3, 4])$ is the quasienergy of the $i^{th}$ band and depends on the choice of the branch cut $\epsilon$ of the logarithmic function. For $\epsilon = 0$, the quasienergy is taken $\Omega \in [-2\pi, 0]$ and we will have $\Omega_1^* = -2\pi + \tan^{-1}(Jr), \Omega_2^* = -\tan^{-1}(Jr), \Omega_3^* = -\pi - \tan^{-1}(Jr)$ and $\Omega_4^* = -\pi + \tan^{-1}(Jr)$ which then produces $\chi^0 = W^0 = -1$ from Eq. (8). Similarly, for the quasienergy gap or the branch cut $\epsilon = \pi$, the quasienergy is taken in the period of $-\pi$ to $\pi$ and the quasienergy of the bands are given as $\Omega_1^* = \tan^{-1}(Jr), \Omega_2^* = -\tan^{-1}(Jr), \Omega_3^* = \pi - \tan^{-1}(Jr)$ and $\Omega_4^* = -\pi + \tan^{-1}(Jr)$ which then produces $\chi^\pi = W^\pi = +1$ from Eq. (8). These results are in full agreement with the chirality determined from the effective Hamiltonian Eq. (4) at zero and $\pi$ quasienergy gaps.

The above analysis illustrates the mechanism in which dynamical winding number captures the chirality of Weyl nodes at zero and $\pi/T$ quasienergy located at a shared single point in 3D BZ. Due to the system’s simplicity, the calculated dynamical winding number can be directly compared to the Weyl points’ chirality obtained from inspecting the Hamiltonian Eq. (3). In the following section, we apply the above dynamical winding number calculation to a more physical system, which may potentially host as many Weyl nodes as we wish.

### III. KICKED HARPER MODEL

In this section, we investigate a variant of the so-called kicked Harper model as a rich model of Floquet topological matter. Note that the kicked Harper model was a seminal dynamical model in the literature of quantum chaos and it is actually the first model ever used to examine topological phase transitions in Floquet quasiequency bands. The Hamiltonian in the lattice basis can be written as

$$\hat{H}_{KHM} = \sum_{n=1}^{N-1} \sum_{j} V \cos(2\pi \beta_2 n + \alpha_z) | n \rangle \langle n | \delta(t - jT)$$

$$+ \sum_{n=1}^{N-1} \left[ J + \lambda \cos(2\pi \beta_1 n + \alpha_y) \right] | n + 1 \rangle \langle n | + H.c.$$
sentation is then given as,
\[ H_{KHM}(k_x, \alpha_y, \alpha_z, t) = 2J \cos(k_x)\sigma_x + 2\lambda \sin(k_x)\cos(\alpha_y)\sigma_y + V \cos(\alpha_z)\sigma_z \delta(t - jT) \]
where \( \sigma_j \) are the Pauli matrices in the sublattice degree of freedom and \( k_x \) is the momentum along the physical dimension.

We can easily write the system’s Floquet operator as (by considering the time interval \( t \in \{0, T\} \))
\[ U_{KHM}(k_x, \alpha_y, \alpha_z) = e^{-i[2J \cos(k_x)\sigma_x + 2\lambda \sin(k_x)\cos(\alpha_y)\sigma_y]T} e^{-iV \cos(\alpha_z)\sigma_z T} \]
where we have again fixed \( \hbar = T = 1 \). It is worth mentioning that the detailed analysis of the above model has been studied in Ref. [59], with Weyl and line nodes, as well as nodal loops explicitly identified at certain parameter values.

In this paper, we focus on the regime for which Weyl nodes exist and calculate the dynamical winding number and valence band Chern number surrounding these points. To this end, we first note that \( U_{KHM} \) can be easily diagonalized, which yields two quasienergies \( \Omega_{\pm} = \pm \cos^{-1}[\cos(f_1) \cos(f_2)] \), where \( f_1 = V \cos(\alpha_z) \) and \( f_2 = \sqrt{4J^2 \cos^2(k_x) + 4\lambda^2 \cos^2(\alpha_y) \sin^2(k_x)} \). It thus follows that band touching at zero \( \pi/T \) quasienergy occurs at \( f_1 = 2\pi/\ell \) \( f_1 = (2\ell + 1)\pi \) and \( f_2 = 0 \), where \( \ell \in \mathbb{Z} \). Reference [59] further found that, following such band touching events, a new set of Weyl nodes at quasienergy \( \pi/T \) emerges at \( \{k_{x_0}, \alpha_{y_0}, \alpha_{z_0}\} = \{\pm \pi/2, \pm \pi/2, \pm \cos^{-1}[\sqrt{2\pi/\ell}]\} \). In particular, such a model can host as many Weyl points as we wish by tuning the parameter \( V \).

Let us now take \( (2\ell + 1)\pi < V < (2\ell + 2)\pi \). The system then hosts \( \ell + 1 \) quartets of Weyl points with zero energy at \( \{k_{x_0}, \alpha_{y_0}, \alpha_{z_0}\} = \{\pm \pi/2, \pm \pi/2, \pm \cos^{-1}[\sqrt{2\pi/\ell}]\} \) and \( \ell + 1 \) quartets of Weyl points at \( \pi/T \) quasienergy \( \{k_{x_0}, \alpha_{y_0}, \alpha_{z_0}\} = \{\pm \pi/\ell, \pm \pi/2, \pm \cos^{-1}[\sqrt{(2\pi + 1)/\ell}]\} \). We may now write the effective Weyl Hamiltonian around these Weyl points. For example, by expanding \( U_{KHM} \) at \( \pi/2 + \delta_x, \pi/2 + \delta_y, \pm \cos^{-1}[\sqrt{(2\pi + 1)/\ell}] + \delta_z \) and \( \pi/2 + \delta_x, \pi/2 + \delta_y, \pm \cos^{-1}[\sqrt{(2\pi + 1)/\ell}] + \delta_z \), we obtain the effective Hamiltonians
\[ H^0_{eff} = 2q \pi \sigma_0 - 2J \delta_x \sigma_x - 2\lambda \delta_y \sigma_y + \zeta_0 \delta_z \sigma_z, \]
\[ H^{\pi/2q}_{eff} = (2q + 1)\pi \sigma_0 - 2J \delta_x \sigma_x - 2\lambda \delta_y \sigma_y + \zeta_1 \delta_z \sigma_z, \]
where \( \zeta_0 = \sqrt{1 - 4\frac{\pi^2}{2}} \) and \( \zeta_1 = \sqrt{1 - \frac{(2q + 1)\pi^2}{2}} \). The chirality of these Weyl nodes can again be deduced from the effective Hamiltonian [24] and are given as \( \chi^0, \pi = \mp 1 \) at zero and \( \pi/T \) quasienergy. In Ref. [59], it has been shown through quantum adiabatic pumping that when multiple Weyl nodes with quasienergy zero are enclosed in a closed surface, the total charge pumped during the adiabatic cycle captures their net chirality. On the other hand, if some enclosed Weyl nodes are of quasienergy \( \pi/T \), then the total charge pumped may no longer correlate with the Weyl points’ net chirality. In the following, we verify that the dynamical winding number always yields the correct net chirality in both cases.

| \( \Delta_i \) | \( (k_x, \alpha_y, \alpha_z) \) | \( (R, r) \) | \( (\chi^0, \chi^\pi) \) | \( (W^0, W^\pi) \) |
|---|---|---|---|---|
| \( \Delta_1 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_2 + 2\pi) \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}) \) | \( (\mp 1, 1) \) | \( (\mp 1, 0) \) |
| \( \Delta_2 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_3 + 2\pi) \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}) \) | \( (0, \mp 1) \) | \( (0, \mp 1) \) |
| \( \Delta_3 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_4 + 2\pi) \) | \( (\mu_4, \frac{\pi}{\ell}) \) | \( (0, \mp 2) \) | \( (0, \mp 2) \) |
| \( \Delta_4 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_5 + 2\pi) \) | \( (\mu_5, \frac{\pi}{\ell}) \) | \( (0, \mp 2) \) | \( (0, \mp 2) \) |
| \( \Delta_5 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_6 + 2\pi) \) | \( (\mu_6, \frac{\pi}{\ell}) \) | \( (0, \mp 2) \) | \( (0, \mp 2) \) |
| \( \Delta_6 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_7 + 2\pi) \) | \( (\mu_7, \frac{\pi}{\ell}) \) | \( (0, \mp 2) \) | \( (0, \mp 2) \) |
| \( \Delta_7 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_8 + 2\pi) \) | \( (\mu_8, \frac{\pi}{\ell}) \) | \( (0, \mp 2) \) | \( (0, \mp 2) \) |
| \( \Delta_8 \) | \( (\frac{\pi}{\ell}, \frac{\pi}{\ell}, \pm \mu_9 + 2\pi) \) | \( (\mu_9, \frac{\pi}{\ell}) \) | \( (0, \mp 2) \) | \( (0, \mp 2) \) |

We summarize our results in Table I while the analytical calculations are presented in Appendix-A. The dynamical winding number and valence band Chern number are determined over a closed surface enclosing the Weyl node(s). We have considered the torus geometry which is parametrized such that \( \delta_x = \pi \theta \), \( \delta_y = \pi \phi \), \( \delta_z = \pi \theta \), and \( \theta, \phi \in [0, 2\pi] \). We have labelled various cases by \( \Delta_i \) in Table I which we will discuss in detail. Let us
first focus on the points where surface encloses a single Weyl node at zero or $\pi/T$ quasienergy which corresponds to the case $\Delta_1$ and $\Delta_2$ respectively in Table I. There, while the dynamical winding number correctly captures the chirality of each Weyl node, the Chern number instead predicts the opposite chirality of the Weyl node at quasienergy $\pi/T$.

Secondly, we turn our attention to the situation where the surface encloses more than one Weyl nodes at a given quasienergy which is shown in Fig. 2 (a,b). Two Weyl nodes at quasienergy $\pi/T$ [0] are shown in Fig. 2 (a b) which correspond to the case $\Delta_3$ $\Delta_4$ in Table I. In this case, both the dynamical winding number and Chern number yield the expected net chiralities when the two Weyl nodes are of quasienergy zero. On the other hand, if the two Weyl nodes are of quasienergy $\pi/T$, the Chern number results in the wrong sign, whereas the dynamical winding number continues to faithfully produce the correct net chirality.

Next, we turn our attention to the point labelled as $\Delta_5$ in Table I, which corresponds to a surface enclosing two Weyl points with different quasienergy Fig. 2(c), but of the same chirality. In this case, the dynamical winding number correctly captures the net chirality of both Weyl nodes, whereas the Chern number instead gives zero. Similarly, the point labelled as $\Delta_6$ correspond to a surface enclosing two Weyl nodes at zero $\pi/T$, which have the same chirality. While dynamical winding number determines the net chirality of Weyl nodes at zero and $\pi/T$ quasienergy, the valence band Chern number $C = W^0 - W^\pi 47$ itself has no information about the chirality which can be observed from the results presented in Table I.

Finally, we consider a spherical surface such that $R = 0$ and $\theta \in [0, \pi)$ which encloses an odd number of Weyl nodes such that there is an imbalance between number of Weyl nodes at zero and $\pi/T$ quasienergy, see $\Delta_7 - \Delta_8$ in Table I. The sphere in Fig. 2(e) depicts the situation where the surface encloses one [two] Weyl node at quasienergy zero $\pi/T$ and refers to point $\Delta_7$ in the Table I. Similarly, a surface encloses one [two] Weyl node at $\pi/T$ [zero] quasienergy has been shown in Fig. 2(f) which...
FIG. 3: Parameter values is taken as $J = \lambda = 1$ and $V = 16$. Weyl nodes under periodic boundary conditions in all directions for fixed (a) $\alpha_{z_0} = \pi/2$ (b) $\alpha_{z_0} = \cos^{-1}(\pi/V)$ and (c) $\alpha_{y_0} = \pi/2$ has been shown. The Fermi arcs under open (periodic) boundary conditions along $x$ ($\alpha_y$, $\alpha_z$)-direction connecting the two Weyl nodes for fixed (d) $\alpha_{z_0} = \pi/2$ (e) $\alpha_{z_0} = \cos^{-1}(\pi/V)$ have been shown. The counter-propagating surface states for fixed $\alpha_{y_0} = \pi/4$ has been shown in panel (f). Red (Green) color represent the states localized at left (right) edge of open lattice in $x$–direction.

correspond to point $\Delta_8$ in Table I. The dynamical winding number captures the net chirality while the Chern number once again provide the difference of Weyl nodes at zero and $\pi/T$ quasienergy. The above analysis emphasizes on the dynamical winding number characterization of the Weyl nodes in Floquet Weyl semimetals.

Before ending this section, we verify the presence of Fermi arcs in the system when OBC are applied in one direction. In particular, we focus on a parameter regime for which many Weyl points at quasienergy zero and $\pi/T$ coexist, which are hence very close to each other in 3D Brillouin zone. Our results are summarized in Fig. 3. By plotting the quasienergy spectrum at two different $\alpha_{z_0} = \pi/2$ and $\alpha_{z_0} = \cos^{-1}(\pi/V)$ values, Fermi arcs at quasienergy zero and $\pi/T$ can be observed in panels (d) and (e) respectively. The Fermi arcs connect the two band touching points through both the BZ center and edge (e.g., degenerate edge states are present both at $\alpha_y = 0$ and $\alpha_y = \pi$). This is possible due to the fact that each band touching point observed in Fig. 3(d) or (e) corresponds to the projection of two Weyl points at $k_x = \pm \pi/2$ in Fig. 3(a) or (b) respectively to the system’s surfaces, where each pair of Weyl points thus contributes to each of the two Fermi arcs that together span the whole $\alpha_y$ BZ. Moreover, since the system hosts Weyl points that appear in quartets due to the presence of time-reversal symmetry, the Chern number on any fixed $\alpha_z$ plane is zero. This is further evidenced in Fig. 3(f) that the system’s quasienergy spectrum at a fixed $\alpha_y = \pi/4$ plane yields counter-propagating chiral
edge states at both quasienergy zero and $\pi/T$. These counter-propagating chiral edge states can be captured through two-terminal conductance\textsuperscript{52} which signals that the Weyl nodes of opposite chirality might have the same transport response which is studied in the next section.

The above results further demonstrate the application of dynamical winding number in categorizing the Floquet Weyl semimetal phases. In particular, the cases $\Delta_3 - \Delta_8$ in Table I represent the scenario for which dynamical winding number calculation is truly necessary for probing the presence of coexisting Weyl nodes at quasienergy zero and $\pi/T$. Strictly speaking, in two-band systems, it is impossible for two Weyl nodes at zero and $\pi/T$ quasienergy to coincide at the same quasimomenta. However, certain systems, such as that considered in this section, are capable of hosting a large number of Weyl nodes. Consequently, due to the limited size of the 3D BZ, these Weyl nodes may necessarily be very close to one another [which can be observed in Fig. 3(c)]. In this case, considering a small enough closed surface that encloses only a single Weyl point will be difficult to achieve in practice. We expect that this is the scenario for which the proposed dynamical winding number calculation will be most useful.

IV. TWO-TERMINAL CONDUCTANCE AND TOTAL CHIRALITY OF WEYL NODES

In the previous section, we have studied that the dynamical winding number efficiently captures the net chirality of Weyl nodes enclosed by a surface. It is evident that the dynamical winding number will not capture the total chirality of the Weyl nodes, i.e., the total number of Weyl points. Furthermore, though the dynamical winding number determines the net chirality of the Weyl nodes, it cannot distinguish between a single Weyl node and three Weyl nodes, two of which having opposite chirality. Such subtleties require the information regarding the total number of Weyl nodes for a thorough characterization of topological entities, i.e., Weyl nodes in this case. Indeed, this may be understood as another interesting aspect of nonequilibrium topological matter.

In this section, we attempt to capture the total chirality of the Weyl nodes through conductance signatures in two-terminal transport. For this purpose, we use the Floquet scattering matrix approach\textsuperscript{50}, which can be applied in a straightforward manner. We consider a finite lattice, with $q$ orbital degrees of freedom, in the physical axis for some fixed $\alpha_y$ and $\alpha_z$ as tunable parameters. Moreover, we apply point-like absorbing terminals at the ends of the lattice as shown in Fig. 4. The projector on to the absorbing leads is chosen as,

$$P = \begin{cases} 1 & \text{if } n_x \in \{1, N_x\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $n_x$ is the lattice site index. The projector acts stroboscopically. That is to say that, the absorbing terminals only act at the beginning and end of each period. The unitary scattering matrix of dimension $2q \times 2q$ is then defined as $S^r$ and given by,

$$S^r = P \left[ 1 - e^{i\epsilon} \hat{U} (1 - P T) P \right]^{-1} e^{i\epsilon} \hat{U} P T,$$

where $T$ denotes the matrix transpose, $\epsilon$ is the quasienergy gap $\epsilon \in \{0, \pi\}$ and $\hat{U}$ being the Floquet operator under the boundary conditions defined above. The resulting $2q \times 2q$ scattering matrix becomes the following:

$$S^r = \begin{pmatrix} r & t \\ t^* & r^* \end{pmatrix},$$

where $^*$ corresponds to the complex conjugation, $r$ and $t$ are the $q \times q$ blocks of reflection and transmission amplitudes respectively. The two-terminal conductance is then given as a function of quasienergy as $G^r = \text{Trace}(t t^*)$, where $\epsilon$ is taken in either zero or $\pi$ gap. In actual scattering experiments, an incoming state cannot be prepared at a given quasienergy $\epsilon$ value. For this reason, the quantized conductance obtained in our calculations corresponds to a sum of conductance over all the cases with different energy values associated with a given quasienergy value. This understanding, the so-called Floquet sum rule\textsuperscript{73}, was also confirmed in Refs.\textsuperscript{74,75}.

As we change periodic boundary conditions to open boundary conditions, the Weyl nodes at $(k_x, \alpha_y, \alpha_z) = (\pm \pi/2, \alpha_y, \alpha_z)$ at quasienergy $\epsilon$ project themselves at the surface of the system. First of all, we choose $(\alpha_y, \alpha_z) = (\pm \pi/2, \pm \cos^{-1}(2\pi m))$ such that two Weyl nodes of opposite chirality exist at zero quasienergy $\epsilon = 0$. The two-terminal conductance is found to yield $(G_0^r, G^\pi) = (2, 0)$, which captures the total chirality of the Weyl nodes at zero quasienergy whereas zero value of $G^\pi$ indicates that there is no Weyl node at $\pi$ quasienergy. Secondly, we consider the tunable parameter of artificial dimension such that Weyl nodes occur at $\pi$ quasienergy for $(\alpha_y, \alpha_z) = (\pm \pi/2, \pm \cos^{-1}(2\pi + 1/2))$. The two-terminal conductance of the system is found to be $(G_0^r, G^\pi) = (0, 2)$ where it predicts the total chirality of the Weyl nodes at $\pi$ quasienergy.

In the light of above results, it can be inferred that the two-terminal conductance $G^\pi$ nicely captures the total chirality of the Weyl nodes at quasienergy $\epsilon$. Indeed,
the chiral\textsuperscript{21} or even counter-propagating\textsuperscript{22} surface states associated with the Weyl nodes all can contribute to the two-terminal conductance, thus providing the necessary information regarding the total chirality of the Weyl nodes. In short, our two-terminal conductance calculations have confirmed that the transport signature can reveal the total chirality, i.e., the total number of Weyl nodes at \( \epsilon \) quasienergy, whereas the net chirality can be characterized by the dynamical winding number.

V. SUMMARY

In this paper, we have proposed the use of dynamical winding number to characterize the Weyl points in Floquet Weyl semimetal phases. Using a simple four band toy model, we demonstrate how dynamical winding number can separately address Weyl points at quasienergy zero and \( \pi/T \) when they are located at the same point in the 3D BZ. To further compare the usefulness of dynamical winding number with that of Chern number in the context of probing Weyl points, we analyse a variant of the seminal kicked Harper model as a Floquet Weyl semimetal. Our investigation reveals that the dynamical winding number over a closed 2D surface (which has been chosen to be either of spherical or toroid shape) always correctly determines the net chirality of all the Weyl points enclosed (regardless of their quasienergy).

By contrast, when such a surface encloses multiple Weyl points of different quasienergy values, the Chern number does not reflect the net chirality of the multiple Weyl points under investigation. Moreover, we have studied the two-terminal transport signature associated with the Weyl points of opposite chirality. It is found that the two-terminal conductance captures the total magnitude of the chirality of Weyl nodes at zero and \( \pi/T \) quasienergy.

As a possible future study, it would be interesting to look into the dynamical characterization of other Floquet Weyl semimetal phases. Some of these possibilities are the Floquet type-II Weyl semimetal and Floquet multi Weyl semimetal phases where dynamical winding number is expected to capture the higher monopole charges and chiralities associated with each Weyl node. Secondly, it would also be of much interest to study whether the dynamical winding number can characterize phases of higher-order Weyl semimetal phases.

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Appendix A: Dynamical Winding number calculation

In this section, we carry out the simplest analytical calculation of dynamical winding number by considering a variant of the kicked Harper model presented in section III. We consider the Weyl node in the three dimensional Brillouin zone \( (k_z_0, \alpha_y, \alpha_z) = (\pi/2, \pi/2, \pi/2) \) such that \( (k_x, \alpha_y, \alpha_z) = (k_x+\delta_x, \alpha_y+\delta_y, \alpha_z+\delta_z) \) where \( \delta_x, \delta_y \) and \( \delta_z \) are the deviations from the Weyl point in three spatial directions. Moreover, we consider \( \delta_y = r \sin(\theta) \cos(\phi), \delta_y = r \sin(\theta) \sin(\phi) \) and \( \delta_z = r \cos(\theta) \) which form a closed 2D surface around the Weyl point in the form of a sphere, where \( r \) is taken small such that we may use some kind of first-order approximation to obtain the dynamical winding number with convenience.

To this end we expand the time-dependent Hamiltonian around this point for \( 2J = 2\lambda = V = f \) (for simplicity) which result in,

\[ H(\theta, \phi, t) = -f r \sin(\theta) \cos(\phi) \sigma_x - f r \sin(\theta) \sin(\phi) \sigma_y - f r \cos(\theta) \sigma_z \delta(t - j T), \]

The Floquet operator is then given as,

\[ U(\theta, \phi) = e^{|\Omega_0 + f r \sin(\theta) \cos(\phi) \sigma_x + f r \sin(\theta) \sin(\phi) \sigma_y + f r \cos(\theta) \sigma_z |} \]

where \( U(\theta, \phi) \) is the Floquet operator with \( \Omega_1 \) and \( | \Psi_i \rangle \) being the quasienergy and eigenvectors which are given as,

\[ |\Psi_1 \rangle = \left( \begin{array}{c} -e^{i\phi} \cos(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{array} \right) \quad |\Psi_2 \rangle = \left( \begin{array}{c} e^{-i\phi} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{array} \right) \]

with quasienergy \( \Omega^0_1 = -2\pi + \tan^{-1}(fr), \quad \Omega^0_2 = -\tan^{-1}(fr) \) which is defined in the range \( \Omega \in [-2\pi, 0] \) with branch cut of logarithmic function \( e = 0 \). Similarly for branch cut \( \epsilon = \pi \), the quasienergy are given as \( \Omega^0_1 = +\tan^{-1}(fr), \quad \Omega^0_2 = -\tan^{-1}(fr) \) which is defined in the range \( \Omega \in [-\pi, \pi] \). The modified Floquet operator during the time interval \( t \in [0, T/2] \) is then given as,

\[ \tilde{U}(\theta, \phi, 2t) = e^{i|f r \sin(\theta) \cos(\phi) \sigma_x + f r \sin(\theta) \sin(\phi) \sigma_y + f r \cos(\theta) \sigma_z |} \]

which leads to \( W(r_0, 2T/2) = 0 \), up to first order in \( r \).

The modified Floquet operator during the time interval \( t \in [T/2, T] \) is given as \( \tilde{U}(\theta, \phi, t_{T/2-T}) = \sum_{i=1}^{2} e^{-i\Omega^0_1(2T-2t)} |\Psi_1 \rangle \langle \Psi_i | \) which results in,

\[ \tilde{U}(\theta, \phi, t_{T/2-T}) = \left( \begin{array}{c} e^{-i\Omega^0_1(2T-2t)} \sin^2(\frac{\theta}{2}) + e^{-i\Omega^0_2(2T-2t)} \cos^2(\frac{\theta}{2}) \\ e^{i\phi} \sin(\theta) \cos(\frac{\theta}{2}) \cos^2(\frac{\theta}{2}) + e^{-i\phi} \sin(\theta) \sin^2(\frac{\theta}{2}) + e^{-i\phi} \sin(\theta) \cos(\frac{\theta}{2}) \sin^2(\frac{\theta}{2}) \end{array} \right) \]
which leads to the dynamical winding number $W^r(t_{\frac{\pi}{2}} \rightarrow T)$ given as,

$$W^r(t_{\frac{\pi}{2}} \rightarrow T) = \frac{\Omega_1^r - \Omega_2^r - \sin(\Omega_1^r - \Omega_2^r)}{2\pi}, \quad (A2)$$

where $\Omega_1^0 = -2\pi + \tan^{-1}(fr)$ and $\Omega_2^0 = -\tan^{-1}(fr)$. This results in $W^0 = -1$ for $r$ being a small number and captures the chirality of the Weyl node. On the other-hand, for the branch cut $\epsilon = \pi$, the quasienergy is found to be $\Omega_1^\pi = +\tan^{-1}(fr)$ and $\Omega_2^\pi = -\tan^{-1}(fr)$, yielding $W^\pi = 0$ and hence that the $\pi$ quasienergy gap does not have a Weyl node.

In summary, we have shown that up to some kind of first-order approximation in treating a small 2D closed surface, the dynamical winding number can be directly calculated and it is found to capture the chirality of the chosen Weyl nodes here at zero quasienergy. Similar calculation can be carried out for the Weyl node at quasienergy $\pi/T$.

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