Subtraction terms at NNLO

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Abstract

Perturbative calculations at next-to-next-to-leading order for multi-particle final states require a method to cancel infrared singularities. I discuss the subtraction method at NNLO. As a concrete example I consider the leading-colour contributions to $e^+e^- \rightarrow 2$ jets. This is the simplest example which exhibits all essential features. For this example, explicit subtraction terms are given, which approximate the four-parton and three-parton final states in all double and single unresolved limits, such that the subtracted matrix elements can be integrated numerically.

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1 Introduction

Present and future collider experiments will provide a large sample of multi-particle final states. In order to extract information from this data, precise theoretical calculations are necessary. This implies to extend perturbative calculations from next-to-leading order (NLO) to next-to-next-to-leading order (NNLO). Due to a large variety of interesting jet observables it is desirable not to perform this calculation for a specific observable, but to set up a computer program, which yields predictions for any infra-red safe observable relevant to the process under consideration. Compared to already existing NNLO calculations for specific (inclusive) observables like Drell-Yan or inclusive Higgs production [1]-[5], this requires to work with fully differential cross-sections. This is the major complication of the project. The three major challenges to accomplish this task are: the calculation of two-loop amplitudes, a method for the cancellation of infrared divergences and stable and efficient Monte Carlo techniques.

The last three years have witnessed a tremendous progress in the calculation of two-loop amplitudes [6]-[9]. In particular the two-loop amplitudes for $e^+e^- \rightarrow 3$ jets have been calculated [10, 11].

Up to now, less is known for the cancellation of infrared divergences at NNLO. Loop amplitudes, calculated in dimensional regularization, have explicit poles in the dimensional regularization parameter $\varepsilon = 2 - D/2$, arising from infrared singularities. These poles cancel with similar poles arising from amplitudes with additional partons, when integrated over phase space regions where two (or more) partons become “close” to each other. However, the cancellation occurs only after the integration over the unresolved phase space has been performed and prevents thus a naive Monte Carlo approach for a fully exclusive calculation. It is therefore necessary to cancel first analytically all infrared divergences and to use Monte Carlo methods only after this step has been performed. Infrared divergences occur already at next-to-leading order. In this case, general methods to handle the problem are known. Examples are the phase-space slicing method [12, 13] and the subtraction method [14]-[18]. As already mentionend, similar methods at NNLO are not yet known. The major difficulty arises from so-called double unresolved configurations, where three partons become degenerate simultaneously. A phase-space slicing approach to double unresolved configurations has been used in the calculation of the photon + 1-jet rate in electron-positron annihilation [19]. The calculational complexity of this process is somewhere in between a NLO calculation and a full NNLO calculation, as the relevant two-loop amplitude vanishes.

In this paper I discuss the general setup for the subtraction method at NNLO. To illustrate the method, I use as an example the leading-colour NNLO contributions to $e^+e^- \rightarrow 2$ jets. This is the simplest example which exhibits all essential features. For this example I give all necessary subtraction terms. These subtraction terms approximate the leading-colour Born contribution of $e^+e^- \rightarrow qgq\bar{q}$ in all double and single unresolved limits, and the leading-colour one-loop contribution of $e^+e^- \rightarrow qg\bar{q}$ in all single unresolved limits. The subtracted matrix elements can therefore be integrated numerically over the appropriate phase space. Valuable information for the construction of the subtraction
terms is provided by the known behaviour of the tree and one-loop amplitudes in singular limits. The double unresolved limits of tree amplitudes have been considered in [19]-[26]. Single unresolved limits of one-loop amplitudes have been considered in [27]-[33]. However, for the subtraction method this information is not yet sufficient. (It is sufficient for a phase space slicing approach.) Since the subtraction terms are subtracted over the complete phase space, they must have the correct singular behaviour not only in the double unresolved cases, but also in the single unresolved ones. This requires to extend the calculations of the singular behaviour of tree amplitudes and to include also subleading single unresolved singularities.

The method presented here is general and not restricted to the example of $e^+e^- \rightarrow 2\text{ jets}$. Subtraction terms for other splittings (like $g \rightarrow ggg$) and other kinematical configurations (e.g. with partons in the initial state) can be worked out along the same lines.

The subtraction terms still need to be integrated analytically over the appropriate unresolved phase space. Although some of the occurring integrals are quite involved, recently developed integration techniques [34, 35] seem to indicate that these integrals can be done analytically.

The knowledge of the subtraction terms will also be useful for an improvement of parton showering algorithms in event generators [36].

This paper is organized as follows: In the following section I discuss the general setup of the subtraction method at NNLO. Sect. 3 gives the relevant amplitudes for the example $e^+e^- \rightarrow 2\text{ jets}$. Sect. 4 reviews the singular limits of amplitudes. In sect. 5 I present the NNLO subtraction terms and in sect. 6 I discuss the mechanism of cancellations between the various terms. Finally sect. 7 contains the conclusions and an outlook.

2 General idea

In this section I outline the general setup for the subtraction method at NNLO. The NNLO contribution to an observable $\mathcal{O}$ whose LO contribution depends on $n$ partons is given by

$$\langle \mathcal{O} \rangle_n^{NNLO} = \int \mathcal{O}_{n+2}(p_1,\ldots,p_{n+2}) \ d\sigma_n^{(0)} + \int \mathcal{O}_{n+1}(p_1,\ldots,p_{n+1}) \ d\sigma_n^{(1)} + \int \mathcal{O}_n(p_1,\ldots,p_n) \ d\sigma_n^{(2)}.$$  

The various contributions are symbolically shown in fig. 1. The observable $\mathcal{O}$ has to be infrared safe. This requires that whenever a $n+1$ parton configuration $p_1,\ldots,p_{n+1}$ becomes kinematically degenerate with a $n$ parton configuration $p'_1,\ldots,p'_n$ we must have

$$\mathcal{O}_{n+1}(p_1,\ldots,p_{n+1}) \rightarrow \mathcal{O}_n(p'_1,\ldots,p'_n).$$  

In addition, we must have in the double unresolved case (e.g. when a $n+2$ parton configuration $p_1,\ldots,p_{n+2}$ becomes kinematically degenerate with a $n$ parton configuration $p'_1,\ldots,p'_n$)

$$\mathcal{O}_{n+2}(p_1,\ldots,p_{n+2}) \rightarrow \mathcal{O}_n(p'_1,\ldots,p'_n).$$  

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Figure 1: Modelling of jets at next-to-next-to-leading order. The cone labeled with $y_{\text{cut}}$ represents the experimental cuts. At next-to-next-to-leading order a jet is modelled either by one, two or three partons.

The subscript $n$ for $\mathcal{O}_n$ and $d\sigma_n^{(l)}$ indicates that this contribution is integrated over a $n$-parton phase space. The various contributions $d\sigma_n^{(l)}$ are given by

$$d\sigma_{n+2}^{(0)} = \left( A_{n+2}^{(0)} \ast A_{n+2}^{(0)} \right) d\phi_{n+2},$$

$$d\sigma_{n+1}^{(1)} = \left( A_{n+1}^{(0)} \ast A_{n+1}^{(1)} + A_{n+1}^{(1)} \ast A_{n+1}^{(0)} \right) d\phi_{n+1},$$

$$d\sigma_n^{(2)} = \left( A_n^{(0)} \ast A_n^{(2)} + A_n^{(2)} \ast A_n^{(0)} + A_n^{(1)} \ast A_n^{(1)} \right) d\phi_n,$$

where $A_n^{(l)}$ denotes an amplitude with $n$ external partons and $l$ loops. $d\phi_n$ is the phase space measure for $n$ partons. Taken separately, each of the individual contributions $d\sigma_n^{(0)}$, $d\sigma_{n+1}^{(1)}$ and $d\sigma_n^{(2)}$ gives a divergent contribution. Only the sum of all contributions is finite. However, since one would like to perform the integration numerically, one must first cancel analytically the divergences, before numerical integration over the different phase spaces can be envisaged. This is not a new phenomena at NNLO, it already occurs at NLO. We can consider an observable, whose LO contribution depends on $n + 1$ partons. The NLO contribution is given by

$$\langle \mathcal{O} \rangle_{n+1}^{\text{NLO}} = \int \mathcal{O}_{n+2}(p_1, \ldots, p_{n+2}) \, d\sigma_{n+2}^{(0)} + \int \mathcal{O}_{n+1}(p_1, \ldots, p_{n+1}) \, d\sigma_{n+1}^{(1)}.$$  \hspace{1cm} (5)

To render the two contributions finite, one adds and subtracts a suitable term [15]

$$\int \mathcal{O}_{n+2} \, d\sigma_{n+2}^{(0)} + \int \mathcal{O}_{n+2} \, d\sigma_{n+1}^{(1)} = \int \left( \mathcal{O}_{n+2} \, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\sigma_{n+1}^{(0,1)} \right) + \int \left( \mathcal{O}_{n+1} \, d\sigma_{n+1}^{(1)} + \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} \right).$$  \hspace{1cm} (6)

The approximation $d\alpha_{n+1}^{(0,1)}$ has to fulfill the following requirements: First, $d\alpha_{n+1}^{(0,1)}$ must be a proper approximation of $d\sigma_{n+2}^{(0)}$ such as to have the same pointwise singular behaviour in $D$ dimensions as $d\sigma_{n+2}^{(0)}$ itself. Thus, $d\alpha_{n+1}^{(0,1)}$ acts as a local counterterm for $d\sigma_{n+2}^{(0)}$ and
one can safely perform the limit \( \varepsilon \to 0 \). Secondly, \( \sigma_{n+1}^{(0,1)} \) must be analytically integrable in \( D \) dimensions over the one-parton subspace leading to soft and collinear divergences. The integral

\[
\int_1 d\alpha_{n+1}^{(0,1)}
\]

contains then explicit poles in \( 1/\varepsilon^2 \) and \( 1/\varepsilon \) which exactly cancel the poles from the loop amplitude in \( \sigma_{n+1}^{(1)} \). Note that the observable \( O \) in the approximation term is evaluated with a \( n+1 \) parton configuration. This allows the integration over the one-parton subspace to be performed analytically, but requires a mapping of four-momenta from the \( n+2 \) parton configuration to the \( n+1 \) parton configuration. This mapping has to satisfy momentum conservation and also has to keep all partons massless. In addition it has to have the right behaviour in the singular limits. Several choices for such a mapping exist [15, 37]. The notation \( O_{n+1} \circ d\alpha_{n+1}^{(0,1)} \) is a reminder, that in general the approximation is a sum of terms

\[
O_{n+1} \circ d\alpha_{n+1}^{(0,1)} = \sum O_{n+1} d\alpha_{n+1}^{(0,1)}
\]

and the mapping used to relate the \( n+2 \) parton configuration to a \( n+1 \) parton configuration differs in general for each summand.

In a similar way, I rewrite the NNLO contribution as

\[
\langle O \rangle_{n}^{\text{NNLO}} = \int \left( O_{n+2} \sigma_{n+2}^{(0)} - O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(0,2)} \right) + \int \left( O_{n+1} \sigma_{n+1}^{(1)} + O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(1,1)} \right) + \int \left( O_n \sigma_n^{(2)} + O_n \circ d\alpha_n^{(0,2)} + O_n \circ d\alpha_n^{(1,1)} \right).
\]

Here \( d\alpha_n^{(0,2)} \) acts as a local counterterm for double unresolved configurations to \( \sigma_n^{(0)} - d\alpha_n^{(0,1)} \) in \( D \) dimensions. Furthermore, it is integrable over a two-parton subspace. \( d\alpha_n^{(1,1)} \) is a local counterterm to \( \sigma_n^{(1)} + d\alpha_n^{(0,1)} \) in \( D \) dimensions, whenever one of the \( n+1 \) partons becomes unresolved. It is integrable over a one-parton subspace. This setup extends to higher orders and in general, \( d\alpha_n^{(l,k)} \) is an approximation to

\[
d\sigma_{n+k}^{(l)} + d\alpha_{n+k}^{(0,l)} + d\alpha_{n+k}^{(1,l-1)} + ... + d\alpha_{n+k}^{(l-1,1)} - d\alpha_{n+k-1}^{(l,1)} - d\alpha_{n+k-2}^{(l,2)} - ... - d\alpha_{n+1}^{(l,k-1)},
\]

e.g. a contribution with at most \( n+k \) partons and at most \( l \) loops in the case when \( k \) partons become unresolved.

The task at NNLO is to give the explicit forms of \( d\alpha_n^{(0,2)} \) (and of \( d\alpha_n^{(0,1)} \) and \( d\alpha_n^{(1,1)} \)) and to integrate these approximations over a two-parton phase space (or a one-parton phase space in the case of \( d\alpha_n^{(0,1)} \) and \( d\alpha_n^{(1,1)} \)).
To illustrate the cancellations for double unresolved contributions, I consider a simple toy model, where the double real emission contribution to a 1-jet observable is given by

$$O_3 \, d\sigma^{(0)}_3 = \theta_3(x_1, x_2, x_3) F_3(x_1, x_2, x_3).$$  \hspace{1cm} (11)

The singular behaviour of the functions $F_3$ is assumed to be

$$\lim_{x_1 \to 0} F_3(x_1, x_2, x_3) = \frac{1}{x_1} F_2(x_2, x_3),$$

$$\lim_{x_1 \to 0} F_2(x_1, x_2) = \frac{1}{x_1} F_1(x_2).$$  \hspace{1cm} (12)

The function $\theta_3$, defining the observable satisfies

$$\lim_{x_1 \to 0} \theta_3(x_1, x_2, x_3) = \theta_2(x_2, x_3),$$

$$\lim_{x_1 \to 0} \theta_2(x_1, x_2) = \theta_1(x_2).$$  \hspace{1cm} (13)

Similar relations are assumed to hold for the singular limits in the other variables. The NLO subtraction term is given by

$$O_2 \circ d\alpha^{(0,1)}_2 =
\frac{1}{x_1} F_2(x_2, x_3) \theta_2(x_2, x_3) + \frac{1}{x_2} F_2(x_1, x_3) \theta_2(x_1, x_3) + \frac{1}{x_3} F_2(x_1, x_2) \theta_2(x_1, x_2).$$  \hspace{1cm} (14)

Let us first consider the cancellation of single unresolved contributions to a 2-jet observable. I consider the $x_1 \to 0$ limit. Here, $O_3 \, d\sigma^{(0)}_3 - O_2 \circ d\alpha^{(0,1)}_2$ consists of two contributions: The first part

$$\lim_{x_1 \to 0} \left( F_3(x_1, x_2, x_3) \theta_3(x_1, x_2, x_3) - \frac{1}{x_1} F_2(x_2, x_3) \theta_2(x_2, x_3) \right)$$

is finite by construction. The second part is given by

$$\lim_{x_1 \to 0} \left( -\frac{1}{x_2} F_2(x_1, x_3) \theta_2(x_1, x_3) - \frac{1}{x_3} F_2(x_1, x_2) \theta_2(x_1, x_2) \right) =
- \frac{1}{x_1 x_2} F_1(x_3) \theta_1(x_3) - \frac{1}{x_1 x_3} F_1(x_2) \theta_1(x_2).$$  \hspace{1cm} (16)

This part is proportional to $\theta_1(x_2)$ or $\theta_1(x_3)$ and therefore does not contribute to a 2-jet observable ($\theta_1$ is zero for a two-jet observable). However, this argument no longer holds if we move to NNLO and consider double unresolved contributions to a 1-jet observable. Here, $\theta_1$ does not vanish and the r.h.s of eq. (16) gives a divergent contribution already in the single unresolved case $x_1 \to 0$, and $x_2, x_3 \neq 0$. At NNLO these singular pieces have to be taken properly into account and subtracted out. To summarize, the major complication for the subtraction method is not to find subtraction terms which match the
matrix element in all double unresolved limits, but to find subtraction terms, which match in all double unresolved limits and all single unresolved limits. For the toy example the NNLO subtraction term is given by

\[ O_1 \circ d\alpha_2^{(0,2)} = -\frac{1}{x_1 x_2} F_1(x_3) \theta_1(x_3) - \frac{1}{x_1 x_3} F_1(x_2) \theta_1(x_2) - \frac{1}{x_2 x_3} F_1(x_1) \theta_1(x_1). \] (17)

Note that in this toy example the NNLO subtraction term can be obtained solely from the singular behaviour of the NLO subtraction term. In general however, the limits do not commute, e.g. taking the singular limit of the single unresolved approximation does not equal the double unresolved approximation of the matrix element.

3 An example

To make the method for the double unresolved contributions as transparent as possible, I consider as a simple example the leading \( N_c \)-contribution to the tree-level amplitude \( e^+e^- \rightarrow qgq\bar{q} \). This example is complicated enough to expose all essential features.

3.1 Double real emission

The singular behaviour of the kinematical structure of QCD amplitudes is entangled with colour factors. For a clear understanding of the factorization properties in soft and collinear limits it is desirable to disentangle the colour factors from the rest. This can be done either by introducing colour charge operators \([15]\) or by decomposing the full amplitude into partial and primitive amplitudes. In this paper I follow the second approach, since this method is closer related to the way several one-loop amplitudes were calculated. Throughout this paper I use the following normalization for the colour matrices:

\[ \text{Tr} \ T^a T^b = \frac{1}{2} \delta^{ab}. \] (18)

Tree-level amplitudes are decomposed into colour factors and partial amplitudes. Each partial amplitude is gauge-invariant and has usually a simpler structure. For example, the colour decomposition of the the tree amplitudes for \( e^+e^- \rightarrow q_1 g_2 g_3 \bar{q}_4 \) amplitude reads

\[ A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) = e^2 g^2 c_0 \left[ (T^2 T^3)_{14} A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) + (T^3 T^2)_{14} A_4^{(0)}(q_1, g_3, g_2, \bar{q}_4) \right]. \] (19)

Here, \( c_0 \) denotes a factor from the electro-magnetic coupling:

\[ c_0 = -Q^2 + v^e v^u P_Z(s), \]

\[ P_Z(s) = \frac{s}{s - M_Z^2 + i \Gamma_Z M_Z}. \] (20)
The electron – positron pair can either annihilate into a photon or a Z-boson. The first term in the expression for \( c_0 \) corresponds to an intermediate photon, whereas the last term corresponds to a Z-boson. The left- and right-handed couplings of the Z-boson to fermions are given by

\[
v_L = \frac{I_3 - Q \sin^2 \theta_W}{\sin \theta_W \cos \theta_W}, \quad v_R = \frac{-Q \sin \theta_W}{\cos \theta_W},
\]

where \( Q \) and \( I_3 \) are the charge and the third component of the weak isospin of the fermion.

The short-hand notation for the colour-matrices corresponds to \( T_{14}^{a} = T_{i,j,q}^{a} \). Figure (2) shows the diagrams contributing to the colour-ordered partial amplitude \( A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) \).

Squaring the full amplitude \( A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) \) one obtains

\[
A_4^{(0)\ast}A_4^{(0)} = \left(4\pi\alpha\right)^2 \left(4\pi\alpha_s\right)^2 |c_0|^2 \\
\times \frac{1}{4} \left(N_c^2 - 1\right) N_c \left[ |A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4)|^2 + |A_4^{(0)}(q_1, g_3, g_2, \bar{q}_4)|^2 + O \left(\frac{1}{N_c^2}\right)\right].
\]

Note that interference terms between \( A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) \) and \( A_4^{(0)}(q_1, g_3, g_2, \bar{q}_4) \) are subleading in \( N_c \). In four dimensions compact expressions for the partial amplitudes are obtained by calculating helicity amplitudes. For the partial amplitude \( A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) \) there are only three independent helicity configurations, which can be taken to be

\[
q_1^+, g_2^+, g_3^+, \bar{q}_4^-; p_5^-, p_6^+, \quad q_1^+, g_2^+, g_3^-, \bar{q}_4^-; p_5^+, p_6^-, \quad q_1^+, g_2^-, g_3^+, \bar{q}_4^-; p_5^+, p_6^-.
\]

The helicity amplitudes are expressed in term of spinor products and read [38]:

\[
A_4^{(0)}(q_1^+, g_2^+, g_3^+, \bar{q}_4^-, p_5^-, p_6^+) = -4i \frac{\langle 4 5 \rangle^2}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 5 6 \rangle},
\]

Figure 2: Diagrams contributing to the colour-ordered partial amplitude \( A_4^{(0)}(q_1, g_2, g_3, \bar{q}_4) \).
Collinear limits

For the last two helicity amplitudes there are alternate forms, which have a clearer singular structure in the one loop. The colour decomposition of the tree amplitudes and the one-loop amplitudes are

\[ A_4^{(0)}(q_1^+, g_2^+, g_3, q_4^-, p_5^+, p_6^+) = 4i \left[ \frac{\langle 31 \rangle \langle 12 \rangle \langle 45 \rangle \langle 3(1+2) \rangle 6}{\langle 12 \rangle s_{23}s_{123}s_{56}} - \frac{\langle 34 \rangle \langle 42 \rangle \langle 16 \rangle \langle 5(3+4) \rangle 2}{\langle 34 \rangle s_{23}s_{234}s_{56}} \right], \]

\[ A_4^{(0)}(q_1^+, g_2^-, g_3^+ g_4^-, p_5^-, p_6^+) = 4i \left[ \frac{[13]^2 \langle 45 \rangle \langle 2(1+3) \rangle 6}{[12] s_{23}s_{123}s_{56}} + \frac{(24)^2 \langle 16 \rangle \langle 5(2+4) \rangle 3}{(34) s_{23}s_{234}s_{56}} + \frac{[13] \langle 24 \rangle \langle 16 \rangle \langle 45 \rangle}{[12] [34] s_{23}s_{56}} \right]. \]

The notation of spinor products follows [38]. For the singular limits it is sufficient to know that spinor products scale as the square root of invariants \( s_{ij} = (p_i + p_j)^2 \):

\[ |\langle i \mid j \rangle| = |[i \mid j] \rangle = \sqrt{s_{ij}} \] (25)

For the last two helicity amplitudes there are alternate forms, which have a clearer singular structure in the \( p_2||p_3 \) limit, at the expense of a more obscure behaviour in the triple collinear limits \( p_1||p_2||p_3 \) and \( p_2||p_3||p_4 \):

\[ A_4^{(0)}(q_1^+, g_2^+, g_3, q_4^-, p_5^+, p_6^+) = 4i \left[ \frac{\langle 54 \rangle \langle 42 \rangle \langle 12 \rangle \langle 5(3+4) \rangle 2}{\langle 23 \rangle [34] s_{123}s_{234} \langle 56 \rangle} + \frac{\langle 31 \rangle \langle 16 \rangle \langle 34 \rangle \langle 3(1+2) \rangle 6}{\langle 12 \rangle [23] s_{123}s_{234} \langle 56 \rangle} - \frac{\langle 3 \rangle \langle 1+2 \rangle \langle 6 \rangle \langle 5(3+4) \rangle 2}{\langle 12 \rangle [34] s_{123}s_{234}} \right], \]

\[ A_4^{(0)}(q_1^+, g_2^-, g_3^+ g_4^-, p_5^-, p_6^+) = 4i \left[ \frac{[13]^2 \langle 45 \rangle \langle 5(2+4) \rangle 3}{[12] [23] s_{123}s_{234} \langle 56 \rangle} - \frac{(24)^2 \langle 16 \rangle \langle 2(1+3) \rangle 6}{(23) [34] s_{123}s_{234} \langle 56 \rangle} + \frac{[13] \langle 24 \rangle \langle 16 \rangle \langle 45 \rangle}{[12] [34] s_{123}s_{234}} \right]. \]

As can be seen from these explicit formulae there are simultaneous singularities in single collinear limits \( s_{ij} \to 0 \) and triple collinear limits \( s_{ijk} \to 0 \). These are nested singularities. It should be noted that the helicity amplitudes are sufficient for the derivation of the subtraction terms when working in a four-dimensional regularization scheme [39]-[41]. In conventional dimensional regularization [42] the squared matrix elements has \( O(\varepsilon) \)-terms, which also have to be taken into account.

### 3.2 One-loop amplitudes with one unresolved parton

For a NNLO analysis of \( e^+e^- \to 2 \) jets one needs also the amplitude for \( e^+e^- \to qgq \) up to one loop. The colour decomposition of the tree amplitudes and the one-loop amplitudes are

\[ A_3^{(0)}(q_1, g_2, \bar{q}_3) = e^2 g_c (T^2)_{13} A_3^{(0)}(q_1, g_2, \bar{q}_3), \]

\[ A_3^{(1)}(q_1, g_2, \bar{q}_3) = e^2 g_c (T^2)_{13} \left( \frac{\alpha_s}{2\pi} \right) A_{3,\text{partial}}^{(1)}(q_1, g_2, \bar{q}_3). \] (27)
Figure 3: Examples of one-loop diagrams, contributing to different primitive amplitudes:
Diagram 2a contributes to $A^{(1),L,\{1\}}_3$, diagram 2b contributes to $A^{(1),R,\{1\}}_3$ and diagram 2c contributes to $A^{(1),L,\{1/2\}}_3$.

For both the tree amplitude and the one-loop amplitude there is only one partial amplitude. However, to obtain a clear factorization in singular kinematical limits, it is convenient to decompose the one-loop partial amplitude further into primitive amplitudes [31]. Primitive amplitudes can be defined as the sum of all Feynman diagrams with a fixed cyclic ordering of the QCD partons and a definite routing of the external fermion lines through the diagram. For the case at hand we have

$$A^{(1)}_{3,\text{partial}}(q_1, g_2, \bar{q}_3) = \frac{1}{2} N_c A^{(1),L,\{1\}}_3(q_1, g_2, \bar{q}_3) - \frac{1}{2} N_c A^{(1),R,\{1\}}_3(q_1, g_2, \bar{q}_3) + \frac{1}{2} N_f A^{(1),L,\{1/2\}}_3(q_1, g_2, \bar{q}_3). \quad (28)$$

The superscript $L$ or $R$ indicates whether the external fermion line turns left or right when entering the loop, while the superscripts $\{1\}$ and $\{1/2\}$ indicate the spin of the particle circulating in the loop. Examples of one-loop diagrams corresponding to different primitive amplitudes are shown in fig. (3). Since we are here only interested in the leading-colour contribution, only the primitive amplitude $A^{(1),L,\{1\}}_3(q_1, g_2, \bar{q}_3)$ is relevant. In later sections I will drop the superscript “$L,\{1\}$” for the leading-colour primitive amplitude. There is only one independent helicity configuration, which we can take to be

$$q_1^+, g_2^+, \bar{q}_3^-, p_4^+, p_5^+. \quad (29)$$

The helicity amplitudes are expressed in terms of spinorproducts and read:

$$A^{(0)}_3(q_1^+, g_2^+, \bar{q}_3^-, p_4^+, p_5^+) = 2\sqrt{2i} \frac{\langle 34 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 54 \rangle},$$

$$A^{(1),L,\{1\}}_3(q_1^+, g_2^+, \bar{q}_3^-, p_4^+, p_5^+) = V^{(1)}_3 A^{(0)}_3(q_1^+, g_2^+, \bar{q}_3^-, p_4^+, p_5^+)$$

$$+ 2\sqrt{2i} \frac{\langle 13 \rangle \langle 15 \rangle \langle 34 \rangle}{\langle 12 \rangle \langle 23 \rangle s_{45}} F_0 + 2\sqrt{2i} \frac{\langle 13 \rangle^2 \langle 15 \rangle^2 \langle 45 \rangle}{\langle 12 \rangle \langle 23 \rangle s_{45}^2} F_1. \quad (30)$$

The renormalized coefficients $V^{(1)}_3$, $F_0$, and $F_1$ are given in the HV scheme by

$$V^{(1)}_3 = -\frac{2}{\varepsilon^2} + \left( \ln(x_1) + \ln(x_2) - \frac{10}{3} \right) \frac{1}{\varepsilon} - R(x_1, x_2) - \frac{1}{2} \ln^2(x_1) - \frac{1}{2} \ln^2(x_2) + \frac{7}{6} \pi^2.$$
\( + \frac{3}{2} \ln(x_2) - \frac{7}{2} + O(\varepsilon) + \text{imaginary parts}, \)

\[
F_0 = \frac{\ln(x_2)}{1 - x_2} + O(\varepsilon), \\
F_1 = \frac{1}{2} \frac{(1 - x_2) + \ln(x_2)}{(1 - x_2)^2} + O(\varepsilon). 
\] (31)

Here, \( x_1 = s_{12}/s_{123}, \) \( x_2 = s_{23}/s_{123} \) and the symmetric function \( R(x_1, x_2) \) is defined by

\[
R(x_1, x_2) = \left( \frac{1}{2} \ln(x_1) \ln(x_2) - \ln(x_1) \ln(1 - x_1) + \frac{1}{2} \zeta_2 - \text{Li}_2(x_1) \right) + (x_1 \leftrightarrow x_2). \] (32)

The amplitude in the FDH scheme is obtained by replacing \( V^{(1)}_3 \) by

\[
V^{(1)}_{3,FDH} = V^{(1)}_3 + \frac{1}{2}. \] (33)

### 3.3 The two-loop amplitude

For a NNLO calculation one also needs the amplitudes for \( e^+ e^- \to q \bar{q} \) up to two loops. The colour decompositions of the tree amplitudes, the one-loop amplitudes and the two-loop amplitudes are

\[
A_2^{(0)}(q_1, \bar{q}_2) = e^2 c_0 \delta_{12} A_2^{(0)}(q_1, \bar{q}_2), \\
A_2^{(1)}(q_1, \bar{q}_2) = e^2 c_0 \delta_{12} \left( \frac{\alpha_s}{2\pi} \right) A_2^{(1)}_{\text{partial}}(q_1, \bar{q}_2), \\
A_2^{(2)}(q_1, \bar{q}_2) = e^2 c_0 \delta_{12} \left( \frac{\alpha_s}{2\pi} \right)^2 A_2^{(2)}_{\text{partial}}(q_1, \bar{q}_2). \] (34)

There is only one independent helicity configuration, which we can take to be

\( q_1^+, \bar{q}_2^-, p_3^+, p_4^+. \) (35)

The helicity amplitude expressed in terms of spinor products reads:

\[
A_2^{(0)}(q_1^+, \bar{q}_2^-, p_3^+, p_4^+) = 2i \frac{[14](32)}{s_{12}}. \] (36)

The leading colour contributions to \( A_2^{(1)} \) and \( A_2^{(2)} \) are:

\[
A_{2,\text{partial}}^{(1)}(q_1^+, \bar{q}_2^-, p_3^+, p_4^+) = \frac{1}{2} N_c V_2^{(1)} A_2^{(0)}(q_1^+, \bar{q}_2^-, p_3^+, p_4^+) + O \left( \frac{1}{N_c^2} \right), \\
A_{2,\text{partial}}^{(2)}(q_1^+, \bar{q}_2^-, p_3^+, p_4^+) = \frac{1}{4} (N_c^2 - 1) V_2^{(2)} A_2^{(0)}(q_1^+, \bar{q}_2^-, p_3^+, p_4^+) + O \left( \frac{1}{N_c} \right). \] (37)

The leading-colour primitive amplitudes are therefore

\[
A_2^{(1)} = V_2^{(1)} A_2^{(0)}, \\
A_2^{(2)} = V_2^{(2)} A_2^{(0)}. \] (38)
The renormalized coefficients \( V^{(1)}_2 \) and \( V^{(2)}_2 \) are given in the \( HV \) scheme by [43]-[45]

\[
V^{(1)}_2 = -\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + \frac{7}{12}\pi^2 + \left(-8 + \frac{7}{8}\pi^2 + \frac{7}{3}\zeta_3\right)\varepsilon + \left(-16 + \frac{7}{3}\pi^2 + \frac{7}{2}\zeta_3 - \frac{73}{1440}\pi^4\right)\varepsilon^2 \\
+ i\pi \left[-\frac{1}{\varepsilon} - \frac{3}{2} + \left(-4 + \frac{\pi^2}{4}\right)\varepsilon + \left(-8 + \frac{3}{8}\pi^2 + \frac{7}{3}\zeta_3\right)\varepsilon^2\right] + O(\varepsilon^3),
\]

\[
V^{(2)}_2 = \frac{1}{2\varepsilon^4} + \frac{17}{4\varepsilon^3} + \left(\frac{433}{72} - \pi^2\right)\frac{1}{\varepsilon^2} + \left(\frac{4045}{432} - \frac{83}{24}\pi^2 + \frac{7}{6}\zeta_3\right)\frac{1}{\varepsilon} - \frac{9083}{2592} - \frac{2153}{432}\pi^2 \\
+ \frac{26}{9}\zeta_3 + \frac{263}{720}\pi^4 + O(\varepsilon) + \text{imaginary parts}. \tag{39}
\]

Note that for a NNLO calculation of \( e^+e^- \rightarrow 2 \text{ jets} \) one needs the real part of \( V^{(1)}_2 \) to order \( O(\varepsilon^2) \) and the imaginary part of \( V^{(1)}_2 \) to order \( O(\varepsilon) \).

### 3.4 Ultraviolet renormalization

The amplitudes in the examples above are the renormalized ones, i.e. the ultraviolet subtraction has been performed. To obtain the renormalized amplitudes in the \( \overline{\text{MS}} \) scheme from the bare ones, one replaces the bare coupling \( \alpha_0 \) with the renormalized coupling \( \alpha_s(\mu^2) \) evaluated at the renormalization scale \( \mu^2 \):

\[
\alpha_0 = \alpha_s S_\varepsilon^{-1} \mu^{2\varepsilon} \left[ 1 - \beta_0 \left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{2\pi^2} - \frac{\beta_1}{2\pi}\right) \left(\frac{\alpha_s}{2\pi}\right)^2 + O(\alpha_s^3) \right], \tag{40}
\]

where

\[
S_\varepsilon = (4\pi)^\varepsilon e^{-\varepsilon\gamma_E}, \tag{41}
\]

is the typical phase-space volume factor in \( D = 4 - 2\varepsilon \) dimensions, \( \gamma_E \) is Euler’s constant, and \( \beta_0 \) and \( \beta_1 \) are the first two coefficients of the QCD \( \beta \)-function:

\[
\beta_0 = \frac{11}{6} C_A - \frac{2}{3} T_R N_f, \quad \beta_1 = \frac{17}{6} C_A^2 - \frac{5}{3} C_A T_R N_f - C_F T_R N_f, \tag{42}
\]

with the color factors

\[
C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_R = \frac{1}{2}. \tag{43}
\]

Let the expansion in the strong coupling of the unrenormalized amplitude be

\[
A_{n,\text{bare}} = (4\pi\alpha) (4\pi\alpha_0)^\frac{\varepsilon}{2}\ c_0\ C \left[ A_{n,\text{bare}}^{(0)} + \left(\frac{\alpha_0}{2\pi}\right) A_{n,\text{bare}}^{(1)} + \left(\frac{\alpha_0}{2\pi}\right)^2 A_{n,\text{bare}}^{(2)} + O(\alpha_s^3) \right] \\
+ \text{subleading colour structures}. \tag{44}
\]
where $\mathcal{C}$ is a colour factor. Then, the renormalized two-loop amplitude can be expressed as

$$A_{n,\text{ren}} = (4\pi \alpha) (4\pi \alpha_s)^{\frac{n-2}{2}} \epsilon_0 \mathcal{C} \left( S_{\varepsilon}^{-1} \mu^2 \varepsilon \right)^{\frac{n-2}{2}} \left[ A_{n,\text{ren}}^{(0)} + \left( \frac{\alpha_s}{2\pi} \right) A_{n,\text{ren}}^{(1)} + \left( \frac{\alpha_s}{2\pi} \right)^2 A_{n,\text{ren}}^{(2)} + O(\alpha_s^3) \right] + \text{subleading colour structures.} \quad (45)$$

The relations between the renormalized and the bare amplitudes are given by

$$A_{n,\text{ren}}^{(0)} = A_{n,\text{bare}}^{(0)},$$

$$A_{n,\text{ren}}^{(1)} = S_{\varepsilon}^{-1} \mu^2 \varepsilon A_{n,\text{bare}}^{(1)} - \frac{(n-2)}{2} \beta_0 \varepsilon A_{n,\text{bare}}^{(0)},$$

$$A_{n,\text{ren}}^{(2)} = S_{\varepsilon}^{-2} \mu^4 \varepsilon A_{n,\text{bare}}^{(2)} - \frac{n}{2} \beta_0 S_{\varepsilon}^{-1} \mu^2 \varepsilon A_{n,\text{bare}}^{(1)} + \frac{(n-2)}{2} \left( n \beta_0^2 - \frac{n^2}{4} \beta_1 \right) A_{n,\text{bare}}^{(0)}. \quad (46)$$

Below I will discuss the factorization of one-loop amplitudes in singular (soft and collinear) limits. The general structure is

$$A_{n}^{(1)} = \text{Sing}^{(0)} \cdot A_{n-1}^{(1)} + \text{Sing}^{(1)} \cdot A_{n-1}^{(0)}, \quad (47)$$

where the function Sing corresponds either to the eikonal factor (soft limit) or a splitting function (collinear limit). Eq. (47) holds both for bare and renormalized quantities. The relation between the renormalized and the bare expression for the singular function is

$$\text{Sing}_{\text{ren}}^{(1)} = S_{\varepsilon}^{-1} \mu^2 \varepsilon \text{Sing}_{\text{bare}}^{(1)} - \frac{\beta_0}{2\varepsilon} \text{Sing}_{\text{bare}}^{(0)}. \quad (48)$$

## 4 Singular limits

In this section I review the behaviour of amplitudes in singular limits, e.g. when two or three partons become degenerate. The knowledge of the singular limits serves as a starting point for the construction of the subtraction terms. Most of the material in this section is summarized from the literature \[19]-[33], although some results, like the triple collinear splitting function for colour-ordered amplitudes, are new. When considering singular limits it is convenient to work in a physical gauge. In this case, only diagrams where the splitting occurs at external lines contribute \[46, 24]. In this paper I use the axial gauge $n \cdot A = 0$. In this gauge the gluon propagator reads

$$\frac{i}{k^2} d^{\mu\nu}(k, n) = \frac{i}{k^2} \left( -g^\mu\nu + \frac{k^\mu n^\nu + n^\mu k^\nu}{k \cdot n} - n^2 \frac{k^\mu k^\nu}{(k \cdot n)^2} \right) = -\frac{i}{k^2} \left( g^\mu\rho - \frac{k^\rho n^\mu}{k \cdot n} \right) \left( g^\nu_\rho - \frac{n_\rho k^\nu}{k \cdot n} \right). \quad (49)$$
4.1 Soft gluons

If a single gluon becomes soft, the partial tree amplitude factorizes according to

$$A_n^{(0)}(p_1, p_2, p_3, ...) = \text{Eik}^{(0,1)}(p_1, p_2, p_3) A_{n-1}^{(0)}(p_1, p_3, ...)$$

(50)

where the eikonal factor is given by

$$\text{Eik}^{(0,1)}(p_1, p_2, p_3) = \frac{p_1^\nu F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_3^\tau}{s_{12}s_{23}} \varepsilon^\mu(p_2)$$

(51)

and $F_{\rho\mu\sigma}(p)$ is defined by

$$F_{\rho\mu\sigma}(p) = g^\rho\mu (p^\sigma - p^\rho g^{\mu\sigma}).$$

(52)

The square of the eikonal factor is given by

$$\left| \text{Eik}^{(0,1)}(p_1, p_2, p_3) \right|^2 = 4 \frac{s_{13}}{s_{12}s_{23}}.$$  

(53)

In a similar way the partial tree amplitude factorizes, when two gluons become soft [20]

$$A_n^{(0)}(p_1, p_2, p_3, p_4, ...) = \text{Eik}^{(0,2)}(p_1, p_2, p_3, p_4) A_{n-2}^{(0)}(p_1, p_4, ...),$$

(54)

with

$$\text{Eik}^{(0,2)}(p_1, p_2, p_3, p_4) = 8 \left[ \frac{p_1^\nu F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_3^\tau}{s_{12}s_{23}s_{34}} - \frac{p_1^\nu F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_1^\tau}{s_{12}s_{23}(s_{12} + s_{13})} \right]$$

$$- \frac{p_1^\nu F_{\rho\mu\sigma}(p_2) F_{\sigma\nu\tau}(p_3) p_4^\tau}{s_{23}s_{34}(s_{24} + s_{34})} \varepsilon^\mu(p_2) \varepsilon^\nu(p_3).$$

(55)

The square of the double soft eikonal factor is given by

$$\left| \text{Eik}^{(0,2)}(p_1, p_2, p_3, p_4) \right|^2 = 8 \left[ (1 - \rho \varepsilon) \frac{(s_{123}s_{34} + s_{12}s_{234} - s_{123}s_{234})^2}{s_{123}s_{23}s_{234}} \right]$$

$$+ \frac{s_{14}^2}{s_{12}s_{23}s_{123}s_{234}} + \frac{s_{14}}{s_{23}} \left[ \frac{1}{s_{12}s_{23}} + \frac{1}{s_{12}s_{234}} + \frac{1}{s_{123}s_{23}} - \frac{4}{s_{123}s_{234}} \right].$$

(56)

I introduced the parameter $\rho$, which specifies the variant of dimensional regularization: $\rho = 1$ for the CDR/HV schemes and $\rho = 0$ for the FD scheme. In the soft-gluon limit, a one-loop primitive amplitude factorizes according to

$$A_n^{(1)}(p_1, p_2, p_3, ...) = \text{Eik}^{(0,1)}(p_1, p_2, p_3) A_{n-1}^{(1)}(p_1, p_3, ...)$$

$$+ \text{Eik}^{(1,1)}(p_1, p_2, p_3) A_{n-1}^{(0)}(p_1, p_3, ...),$$

(57)

For the example considered here, the relevant one-loop eikonal function is given by [31]

$$\text{Eik}^{(1,1)}(p_1, p_2, p_3) =$$

$$- \frac{S_{\epsilon}}{\varepsilon^2} \Gamma(1 + \varepsilon) \Gamma(1 - \varepsilon) \left( \frac{\mu^2 (-s_{13})}{(-s_{12})(-s_{23})} \right) \varepsilon - \frac{11}{6\varepsilon} \text{Eik}^{(0,1)}(p_1, p_2, p_3).$$

(58)
Here,
\[ c_{\Gamma} = (4\pi)^{\frac{1}{2}} \frac{\Gamma(1 + \varepsilon)\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)}. \] (59)

As in the tree-level case with one soft gluon, the eikonal factor is independent of the variant of the dimensional regularization scheme used.

### 4.2 Collinear particles

In the collinear limit tree amplitudes factorize according to
\[ A^{(0)}_n(..., p_i, p_j, ...) = \sum_\lambda \text{Split}^{(0,1)}(p_i, p_j) A^{(0)}_{n-1}(..., p, ...). \] (60)

where \( p_i \) and \( p_j \) are the momenta of two adjacent legs and the sum is over all polarizations. In the collinear limit we parametrize the momenta of the partons \( i \) and \( j \) as [15]
\[ p_i = zp + k_\perp - \frac{k_\perp^2}{2vn}, \]
\[ p_j = (1 - z)p - k_\perp - \frac{k_\perp^2}{1 - 2vn}. \] (61)

Here \( n \) is a massless four-vector and the transverse component \( k_\perp \) satisfies \( 2pk_\perp = 2nk_\perp = 0 \). The collinear limits occurs for \( k_\perp^2 \to 0 \). The splitting amplitudes \( \text{Split}^{(0,1)} \) are universal, they depend only on the two momenta becoming collinear, and not upon the specific amplitude under consideration. The splitting functions \( \text{Split}^{(0,1)} \) are given by
\[ \text{Split}^{(0,1)}_{q\to qg} = \frac{1}{s_{ij}} \bar{u}(p_i) \not{\epsilon}(p_j) u(p), \]
\[ \text{Split}^{(0,1)}_{g\to gg} = \frac{2}{s_{ij}} \left[ \not{\epsilon}(p_i) \cdot \not{\epsilon}(p_j) p_i \cdot \not{\epsilon}(p) + \not{\epsilon}(p_j) \cdot \not{\epsilon}(p) p_j \cdot \not{\epsilon}(p_i) - \not{\epsilon}(p_i) \cdot \not{\epsilon}(p) p_i \cdot \not{\epsilon}(p_j) \right], \]
\[ \text{Split}^{(0,1)}_{g\to q\bar{q}} = \frac{1}{s_{ij}} \bar{u}(p_i) \not{\epsilon}(p_j) u(p_j). \] (62)

Due to the sum over spins in eq. (60), spin correlations are retained in the collinear limit of squared tree amplitudes. With
\[ P^{(0,1)} = \sum_{\lambda, \lambda'} u(p) \text{Split}^{(0,1)} \ast \text{Split}^{(0,1)} \bar{u}(p) \] for quarks,
\[ P^{(0,1)} = \sum_{\lambda, \lambda'} \not{\epsilon}(p)^\ast \text{Split}^{(0,1)} \ast \text{Split}^{(0,1)} \not{\epsilon}(p) \] for gluons, (63)

and the parametrization eq. (61) one finds
\[ P^{(0,1)}_{q\to qg} = \frac{2}{s_{ij}} \hat{p} \left[ \frac{2z}{1 - z} + (1 - \rho\varepsilon)(1 - z) \right], \]
\[ P_{g \rightarrow gg}^{(0,1)} = \frac{2}{s_{ij}} \left[ -g^{\mu \nu} \left( \frac{2z}{1-z} + \frac{2(1-z)}{z} \right) - 4(1-\rho\varepsilon)z(1-z) \frac{k_1^\mu k_1^\nu}{k_1^2} \right], \]

\[ P_{g \rightarrow qg}^{(0,1)} = \frac{2}{s_{ij}} \left[ -g^{\mu \nu} + 4z(1-z) \frac{k_1^\mu k_1^\nu}{k_1^2} \right]. \] (64)

One-loop primitive amplitudes factorize according to [27]-[31]

\[ A_n^{(1)}(..., p_i, p_j, ...) = \sum_\lambda \text{Split}^{(0,1)}(p_i, p_j) A_{n-1}^{(1)}(..., p, ...) + \sum_\lambda \text{Split}^{(1,1)}(p_i, p_j) A_{n-1}^{(0)}(..., p, ...). \] (65)

Here, the splitting amplitudes Split\(^{(1,1)}\) occur as a new structure. For the one-loop amplitude \(A_3^{(1)}\) for \(e^+e^- \rightarrow qg\bar{q}\) we only need the \(q \rightarrow gg\) splitting function. This function is given by [30, 31]

\[ \text{Split}^{(1,1)}_{q \rightarrow gg} = S_{\varepsilon}^{-1} c_T \left( \frac{-s_{12}}{\mu^2} \right)^{-\varepsilon} \left( f_1(z) + f_2 \right) \text{Split}^{(0,1)}_{q \rightarrow qg} - f_2 \left( \frac{2p \cdot \varepsilon(p_j)}{s_{ij}^2} \bar{u}(p_i) \slashed{p} u(p) \right) \]

\[ \frac{11}{6}\varepsilon \text{Split}^{(0,1)}_{q \rightarrow qg}, \] (66)

where the coefficients \(f_1(z)\) and \(f_2\) are

\[ f_1(z) = -\frac{1}{\varepsilon^2} 2F_1 \left( 1, -\varepsilon, 1-\varepsilon; \frac{-z}{1-z} \right), \]

\[ f_2 = \frac{1-\rho\varepsilon}{2(1-\varepsilon)(1-2\varepsilon)}. \] (67)

The \(\varepsilon\)-expansion of the hypergeometric function is

\[ 2F_1 \left( 1, -\varepsilon, 1-\varepsilon; \frac{-z}{1-z} \right) = 1 - \sum_{k=1}^{\infty} \varepsilon^k \text{Li}_k \left( \frac{-z}{1-z} \right). \] (68)

The hypergeometric function can also be written as

\[ 2F_1 \left( 1, -\varepsilon, 1-\varepsilon; \frac{-z}{1-z} \right) = \Gamma(1+\varepsilon)\Gamma(1-\varepsilon) \left( \frac{z}{1-z} \right)^\varepsilon + 1 - z^\varepsilon 2F_1 \left( \varepsilon, \varepsilon, 1+\varepsilon; 1-z \right), \] (69)

which is useful when studying the soft limit. The interference term of \(\text{Split}^{(0,1)}\) with \(\text{Split}^{(1,1)}\) is given by

\[ P_{q \rightarrow qg}^{(1,1)} = \sum_\lambda u(p) \left( \text{Split}^{(0,1)}_{q \rightarrow qg} \right)^* \text{Split}^{(1,1)}_{q \rightarrow qg} u(p) + c.c. = \]

\[ S_{\varepsilon}^{-1} c_T \left( \frac{-s_{12}}{\mu^2} \right)^{-\varepsilon} \left\{ f_1(z) P_{q \rightarrow qg}^{(0,1)} + f_2 \left( \frac{2}{s_{ij}} \bar{p} \left[ 1-\rho\varepsilon(1-z) \right] \right) \right\} - \frac{11}{6\varepsilon} P_{q \rightarrow qg}^{(0,1)} + c.c. \] (70)
Figure 4: Diagrams contributing to the triple collinear limit $q \rightarrow qgg$.

Here “c.c.” denotes the complex conjugate.

For tree amplitudes we also have to consider the case, where three partons become collinear simultaneously. In the triple collinear limit we parametrize the momenta as follows [24]:

$$p_i = z_i p + k_{\perp i} - \frac{k_{\perp i}^2 n}{z_i 2pn},$$  \hspace{1cm} (71)

where $n^2 = 0$, $2k_{\perp i} p = 2k_{\perp i} n = 0$, and the $z_i$ and $k_{\perp i}$ satisfy

$$\sum z_i = 1, \quad \sum k_{\perp i} = 0.$$  \hspace{1cm} (72)

The following formulae are useful:

$$z_i = \frac{2p_i n}{2pn}, \quad s_{ij} = -z_i z_j \left( \frac{k_{\perp i}}{z_i} - \frac{k_{\perp j}}{z_j} \right)^2, \quad k_{\perp i} = z_i \left( s_{jk} - (1-z_i) s_{ijk} \right).$$  \hspace{1cm} (73)

Here, $p_{ijk} = p_i + p_j + p_k$. For the example of the leading $N_c$ contributions to $e^+ e^- \rightarrow qgg \bar{q}$ we have to consider only the triple collinear limit $q \rightarrow qgg$. In the triple collinear limit the tree amplitude factorizes according to

$$A^{(0)}_n(p_1, p_2, p_3, ...) = \sum_{\lambda, \lambda^\prime} \text{Split}^{(0,2)}_{\lambda \rightarrow qgg}(p_1, p_2, p_3) A^{(0)}_{n-2}(p, ...).$$  \hspace{1cm} (74)

Fig. (4) shows the Feynman diagrams contributing to the triple collinear limit $q \rightarrow qgg$. We are interested in the square of the splitting function $\text{Split}^{(0,2)}_{q \rightarrow qgg}$, defined by

$$P^{(0,2)} = \sum_{\lambda, \lambda^\prime} u(p) \text{Split}^{(0,2)}_{\lambda \rightarrow qgg} \ast \text{Split}^{(0,2)}_{\lambda^\prime \rightarrow qgg} \bar{u}(p) \quad \text{for quarks.}$$  \hspace{1cm} (75)

For the splitting $q \rightarrow qgg$ I find

$$P^{(0,2)}_{q \rightarrow qgg} =$$

$$\frac{4}{s^2_{123}} \dot{\rho} \left( 2(1 - \rho \varepsilon) \frac{(z_3 - (1 - x_1) z_{23})^2}{x_2^2 z_{23}^2} + 4(1 - \rho \varepsilon) \frac{x_1}{x_2} + (1 - \rho \varepsilon)^2 \frac{x_2}{x_1} + 3 - 4 \rho \varepsilon + \rho^2 \varepsilon^2 \right)$$

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Here $x_1 = s_{12}/s_{123}$, $x_2 = s_{23}/s_{123}$ and $z_{23} = z_2 + z_3$. Note that eq. (76) gives the triple collinear splitting function for colour-ordered amplitudes. This implies a fixed cyclic ordering for the two gluons. In contrast, the triple collinear splitting functions given in ref. [22] and ref. [24] are the ones for the full matrix element squared, e.g. they include a sum over the permutations of the two gluons.

4.3 Factorizable double unresolved contributions

For the double unresolved contributions there are also two configurations, which have quite a simple factorizable structure in the singular limit. These two cases are: a) the emission of two independent pairs of collinear particles and b) the emission of a soft gluon together with one pair of collinear particles. Let us first consider the case of the emission of two independent pairs of collinear particles. I consider the case $p_1||p_2$ and $p_3||p_4$. The amplitude factorizes according to

$$A_n^{(0)}(p_1, p_2, p_3, p_4) = \sum_{\lambda, \lambda'} \text{Split}^{(0,1)}(p_1, p_2) \text{Split}^{(0,1)}(p_3, p_4) A_{n-2}^{(0)}(p, q),$$

where $p = p_1 + p_2 + O(k^2)$ and $q = p_3 + p_4 + O(k^2)$. For the leading-colour contributions to $e^+ e^- \rightarrow qg \bar{q}g$ only the splittings $q \rightarrow qg$ and $\bar{q} \rightarrow \bar{q}g$ are relevant. Squaring the splitting functions in eq. (77) one obtains

$$P_{q \rightarrow qg, \bar{q} \rightarrow \bar{q}g}^{(0,2)} = P_{q \rightarrow qg}^{(0,1)} P_{\bar{q} \rightarrow \bar{q}g}^{(0,1)}. \quad (78)$$

In the case of the emission of a soft gluon together with a pair of collinear partons it is for our purposes sufficient to consider the case of a soft gluon ($p_3$) together with a collinear splitting $q \rightarrow qg$ of the partons $p_1$ and $p_2$. I simply quote the singular factor for the squared matrix element [22, 24]:

$$P_{q \rightarrow qg, q_3 \text{soft}}^{(0,2)} = 4 \frac{s_{p4}}{s_{p3}s_{34}} P_{q \rightarrow qg}^{(0,1)}. \quad (79)$$

Here, $p = p_1 + p_2 + O(k^2)$ and the singular factor factorizes into an eikonal factor times a splitting function.
5 Subtraction terms

In this section I first review the subtraction terms for single unresolved contributions. I then give the subtraction terms for double unresolved contributions and for contributions with one virtual and one real unresolved parton.

5.1 Subtraction terms for single unresolved contributions

The approximation term $d\alpha^{(0,1)}_{n+1}$ is given as a sum over dipoles [15]:

$$d\alpha^{(0,1)}_{n+1} = \sum_{\text{topologies } T} D^{(0,1)}_{n+1}(T). \tag{80}$$

For leading colour contributions the sum reduces to combinations where the emitting parton $i$ and the emitted parton $j$ are adjacent, and the spectator $k$ is either adjacent to $i$ or $j$. It is also convenient to denote by $k'$ the other parton adjacent to the pair $i$ and $j$. Therefore the cyclic ordering is either $k', i, j, k$ or $k, j, i, k'$. The relevant topologies for the example of the leading $N_c$ contributions to $e^+e^- \rightarrow qgqg$ are shown in fig. (5). A dipol is constructed from amputated amplitudes, evaluated with mapped momenta and dipole splitting functions. By an amputated amplitude I mean an amplitude where a polarization vector (or external spinor for fermions) has been removed. I denote amputated amplitudes by $|A\rangle$. If the amputated parton is a gluon, one has

$$|A^{(0)}_{n+1}(..., p, ...)\rangle = \frac{\partial}{\partial \varepsilon(p)} A^{(0)}_{n+1}(..., p, ...). \tag{81}$$

If the amputated parton is a quark, one has

$$|A^{(0)}_{n+1}(..., p, ...)\rangle = \frac{\partial}{\partial \bar{u}(p)} A^{(0)}_{n+1}(..., p, ...). \tag{82}$$

The dipole subtraction terms are obtained by sandwiching a dipole splitting function in between these amputated amplitudes:

$$D^{(0,1)}_{n+1}(T) = c \left\langle A^{(0)}_{n+1}(..., p_e, ...) \right| \mathcal{P}^{(0,1)}(T) \left| A^{(0)}_{n+1}(..., p_e, ...) \right\rangle d\phi_{n+2}. \tag{83}$$

Figure 5: Splitting topologies for NLO subtraction. At each splitting the emitted particle is directed towards the spectator.
Here, $c$ denotes an overall prefactor. For the example $e^+e^- \rightarrow qg\bar{q}$, it is given by
\[ c = (4\pi\alpha)^2 (4\pi\alpha_s)^2 |c_0|^2 \times \frac{1}{4} \left( N_c^2 - 1 \right) N_c. \] (84)

The dipole splitting functions read
\[
\mathcal{P}^{(0,1)}_{g \rightarrow qg}(p'_k, p_i, p_j, p_k) = \frac{2}{s_{ijk} y} \frac{1}{s_{ij}} \left[ \frac{2}{1 - z(1 - y)} - 2 + (1 - \rho \varepsilon)(1 - y)(1 - z) \right],
\]
\[
\mathcal{P}^{(0,1)}_{g \rightarrow gg}(p'_k, p_i, p_j, p_k) = \frac{2}{s_{ijk} y} \left[ -g^{\mu\nu} \left( \frac{2}{1 - z(1 - y)} - 2 \right) + (1 - \rho \varepsilon) \frac{4r(1 - y)^2}{s_{ij}} k_\perp^{\mu} k_\perp^{\nu} \right],
\]
\[
\mathcal{P}^{(0,1)}_{g \rightarrow q\bar{q}}(p'_k, p_i, p_j, p_k) = \frac{2}{s_{ijk} y} \left[ -\frac{1}{2} g^{\mu\nu} - \frac{4r(1 - y)^2}{s_{ij}} k_\perp^{\mu} k_\perp^{\nu} \right],
\] (85)

where
\[
r = \frac{2p_k p_e}{2p_e p_e + 2p_e p_s}. \] (86)

These splitting functions differ by finite terms from the original choice of Catani and Seymour. The new set of splitting functions in eq.(85) is introduced for later convenience with the NNLO subtraction terms. Although arbitrary non-singular terms may be added to the NLO subtraction terms, the NNLO subtraction terms depend on the actual form of the NLO subtraction terms.

The dipole splitting functions in eq. (85) have also the correct singular behaviour in the soft limit. The term $1/(1 - z(1 - y))$ equals $s_{ijk}/(s_{ij} + s_{jk})$ and combining this term with the one from the dipole, where the roles of emitter and spectator are exchanged, one obtains
\[
\frac{1}{s_{ij} (s_{ij} + s_{jk})} + \frac{1}{s_{jk} (s_{ij} + s_{jk})} = \frac{s_{ijk}}{s_{ij}s_{jk}},
\] (87)

which is the eikonal factor in the soft limit.

For the $q \rightarrow gg$ splitting, the quark polarization matrix $\hat{p}_e$ is included in the definition of the dipole splitting function. For the $g \rightarrow gg$ and $g \rightarrow q\bar{q}$ splittings, the full splitting function is shared between the dipoles with spectators $k$ and $k'$, where $k$ and $k'$ are the partons adjacent to the ($i, j$) pair in the colour ordered amplitude $A^{(0)}(\ldots, k', i, j, k, \ldots)$. For the $g \rightarrow gg$ splitting the dipole splitting function is chosen in such a way that the singularity when parton $j$ becomes soft resides in $\mathcal{P}_{g \rightarrow gg}(p_i; p_j; p_k)$, whereas the singularity when parton $i$ becomes soft resides in $\mathcal{P}_{g \rightarrow gg}(p_j; p_i; p_k)$. The amputated amplitudes are evaluated with mapped momenta. The mapping of momenta relates a $n + 1$ parton configuration to a $n$ parton configuration:
\[
P_{n+1 \rightarrow n} : (p_1, \ldots, p_{n+1}) \rightarrow (p'_1, \ldots, p'_n).
\] (88)

Such a mapping has to satisfy momentum conservation and the on-mass-shell conditions. Furthermore it must have the right behaviour in the singular limits. Several choices for
such a mapping exist [15, 37]. One possible choice relates three parton momenta of the $n + 1$ parton configuration to two parton momenta of the $n$ parton configuration, while leaving the remaining $n - 2$ parton momenta unchanged. This mapping is given by

$$
p_e = p_i + p_j - \frac{y}{1 - y} p_k,\n$$

$$
p_s = \frac{1}{1 - y} p_k.\n$$

(89)

$p_i$, $p_j$ and $p_k$ are the momenta of the $n + 1$ parton configuration, and $p_e$ and $p_s$ are the resulting momenta of the of the $n$ parton configuration. The variables $y$ and $z$ are given by

$$
y = \frac{s_{ij}}{s_{ijk}}, \quad z = \frac{s_{ik}}{s_{ik} + s_{jk}}.\n$$

(90)

It is easily verified that this mapping satisfies momentum conservation

$$
p_e + p_s = p_i + p_j + p_k,\n$$

(91)

and the on-shell conditions

$$
p_e^2 = 0, \quad p_s^2 = 0.\n$$

(92)

The singular limit corresponds to $y \to 0$ and in this limit we have

$$
p_e - (p_i + p_j) = O(y), \quad p_s - p_k = O(y).\n$$

(93)

It is also useful to introduce the vector $k_\perp$, defined by

$$
k_\perp = (1 - z)p_i - zp_j - (1 - 2z) \frac{y}{1 - y} p_k.\n$$

(94)

With this vector and the value of the longitudinal momentum fraction $z$, the mapping may be inverted to yield

$$
p_i = z p_e + k_\perp + y (1 - z) p_s,\n$$

$$
p_j = (1 - z) p_e - k_\perp + y z p_s,\n$$

$$
p_k = (1 - y) p_s,\n$$

(95)

where $y$ is now given by

$$
y = \frac{-k_\perp^2}{z(1 - z)(p_e + p_s)^2}.\n$$

(96)

The singular limit occurs now for $k_\perp \to 0$. The $g \to gg$ and $g \to q\bar{q}$ splittings also involve spin correlation through the spin correlation tensor $k_\perp^\mu k_\perp^\nu$. Note that the spin correlation tensor $k_\perp^\mu k_\perp^\nu$ is orthogonal to $p_e$ and $p_s$:

$$
2p_e k_\perp = 2p_s k_\perp = 0.\n$$

(97)

Further note that the contraction of $p_e$ into an amputated amplitude vanishes due to gauge invariance.

The parameter $\rho$ specifies the variant of dimensional regularization: $\rho = 1$ corresponds to the CDR/HV schemes and $\rho = 0$ to a four-dimensional scheme.
5.2 Subtraction terms for double unresolved contributions

In this subsection I give the subtraction terms for double unresolved contributions. To be more precise, these subtraction terms have to approximate

\[ \mathcal{O}_{n+2} d\sigma^{(0)}_{n+2} - \mathcal{O}_{n+1} \circ d\alpha^{(0,1)}_{n+1} \]

in all single and double unresolved limits. Eq. (98) is integrable over single unresolved regions for observables, which vanish on \( n \)-parton configurations, e.g. \( (n+1) \)-jet observables. However, as discussed in the example of eq. (16), for a generic \( n \)-jet observable, eq. (98) is in general not integrable over single unresolved regions. Therefore, not only the singularities corresponding to double unresolved limits, but also the singularities corresponding to single unresolved limits have to be subtracted.

The subtraction term \( d\alpha^{(0,2)}_{n} \) at NNLO is also written as a sum over topologies:

\[ d\alpha^{(0,2)}_{n} = \sum_{\text{topologies } T} \mathcal{D}^{(0,2)}_{n}(T). \]

These topologies should be thought of as pictorial representations of the pole structure of the subtraction terms \( \mathcal{D}^{(0,2)}_{n}(T) \) and the way the momenta of the \( n \)-parton configurations are constructed from the momenta of the \( (n+2) \)-parton configurations. The intuitive picture of “iterated dipoles” is only justified for the approximations of \( d\alpha^{(0,1)}_{n+1} \).

The relevant topologies for the leading \( N_c \) contributions to \( e^+ e^- \rightarrow qgq\bar{q} \) are shown in fig. (6) to fig. (7). They fall into two classes: two single splittings from two different branches (topologies \( T^{(2)}_a \) and \( T^{(2)}_b \)) and two sequential splittings from the same branch (topologies \( T^{(2)}_c - T^{(2)}_h \)). It is sufficient to consider topologies \( T^{(2)}_a \) and \( T^{(2)}_e - T^{(2)}_e \) only. The remaining topologies are related by symmetry to the ones mentioned above. The subtraction term is given by

\[ \mathcal{D}^{(0,2)}_{n}(T) = c \mathcal{J}(T) \left\langle A^{(0)}_{n}(\ldots, p^{(2)}_e, \ldots, p^{(2)}_s, \ldots) \right| \mathcal{P}^{(0,2)}(T) \left| A^{(0)}_{n}(\ldots, p^{(2)}_e, \ldots, p^{(2)}_s, \ldots) \right\rangle d\phi_{n+2}. \]
Figure 7: Splitting topologies for NNLO subtraction: Two sequential splittings from the same branch. At each splitting the emitted particle is directed towards the spectator.

$J$ is a prefactor not singular in any limit, which can be used to absorb Jacobians into the definition of the subtraction terms, thus rendering the analytical integration over the unresolved phase space simpler. Here, I simply take $J = 1$. The splitting function $P^{(0,2)}(T)$ is sandwiched between double-amputated amplitudes for the topologies $T^{(2)}_a$ and $T^{(2)}_b$, and is sandwiched between single-amputated amplitudes for the topologies $T^{(2)}_c$-$T^{(2)}_h$. It is convenient to write

$$P^{(0,2)}(T) = P^{(0,2)}_{(0,0)}(T) - P^{(0,2)}_{(0,1)}(T), \quad (101)$$

where $P^{(0,2)}_{(0,0)}$ is an approximation to $d\sigma^{(0)}_{n+2}$ and $P^{(0,2)}_{(0,1)}$ is an approximation to $d\alpha^{(0,1)}_{n+1}$. The notation with super- and subscripts is as follows:

$$P^{(l,k)}_{(l',k')}(T) \quad (102)$$

contributes to $d\alpha^{(l,k)}$ and approximates a term of $d\alpha^{(l',k')}$, (with $d\sigma^{(l)}$ identified as $d\alpha^{(l,0)}$). It should be noted that the subtraction terms are not unique. For example, the complete triple collinear splitting function $P_{q \rightarrow qgg}$ is shared between the topologies $T^{(2)}_c$, $T^{(2)}_d$ and $T^{(2)}_e$. Therefore, a term of the form

$$\frac{c}{s_{123}^2}, \quad (103)$$

which is only singular in the triple collinear limit and finite in all other cases, can be freely
moved between these topologies. On the other hand, a term of the form

\[ \frac{c'}{s_{12}s_{13}}, \]

which is singular in the triple collinear limit and in the single unresolved limit \( s_{12} \to 0 \) is uniquely associated with topology \( T_c^{(2)} \), as the pictorial representation suggests. Below I will give one possible set for the subtraction terms.

I briefly mention how the \( n \)-parton momentum configuration is constructed from the \((n + 2)\)-parton configuration. As an example I discuss topology \( T_c^{(2)} \). For the outermost splitting, particle 1 is the emitter, particle 2 is the emitted soft or collinear particle and particle 3 is the spectator. Particle 4 is not involved in this splitting. From this 4-parton momentum configuration \((p_1, p_2, p_3, p_4)\) a 3-parton momentum configuration \((p_e^{(1)}, p_s^{(1)}, p_4)\) is first constructed, using the Catani-Seymour reconstruction function given in eq. (89). From the 3-parton momentum configuration a 2-parton momentum configuration \((p_e^{(2)}, p_s^{(2)})\) is then constructed, using the symmetric reconstruction functions given by Kosower [37]. This symmetric choice is necessary to ensure that topologies \( T_c^{(2)} \) and \( T_b^{(2)} \) are evaluated with the same \( n \)-parton momentum configuration. This is necessary for a cancellation of (subleading) single unresolved singularities in \( s_{12} \to 0 \) and is discussed in detail in sect. 6.

To present the splitting functions for double unresolved configurations I use the following notation:

\[ \bar{y} = 1 - y, \quad \bar{z} = 1 - z, \]

and \( p_{ij} = p_i + p_j \).

### 5.2.1 Topology A

This topology corresponds to two independent single splittings and the splitting function \( \mathcal{P} \) is inserted into a double-amputated amplitude. The splitting functions read:

\[
\mathcal{P}_{(0,0)}^{(0,2)}_{q \rightarrow gq, \bar{q}g, \bar{q}g} \left( T_c^{(2)} \right) = \left( \rho_e^{(2)} \right)_{ij} \left( \rho_s^{(2)} \right)_{ij} \frac{4}{s_{1234}} \frac{1}{y_1 x_2 (\bar{y}_1 x_1 + y_1 x_2)} \\
\times \left\{ \left[ 1 + \frac{\bar{y}_1 \bar{z}_1}{u} \right] \frac{2}{\bar{y}_1 x_1 + (y_1 + \bar{y}_1 \bar{z}_1) x_2} + \left[ \frac{1}{y_1 x_2} + \frac{1}{y_1 + \bar{y}_1 \bar{z}_1} \right] [-2 + (1 - \rho \varepsilon) x_2] \\
+ \left[ \frac{1}{\bar{y}_1 x_1 + x_2} + \frac{u}{\bar{y}_1 x_1 + \bar{y}_1 \bar{z}_1 x_2} \right] [-2 + (1 - \rho \varepsilon) u] + [-2 + (1 - \rho \varepsilon) x_2] [-2 + (1 - \rho \varepsilon) u] \\
- \rho \varepsilon (\bar{y}_1 x_1 + y_1 x_2) \left[ \frac{2}{y_1 + \bar{y}_1 \bar{z}_1} - 2 + (1 - \rho \varepsilon) \bar{y}_1 \bar{z}_1 \right] \right\},
\]

\[
\mathcal{P}_{(0,1)}^{(0,2)}_{q \rightarrow gq, \bar{q}g, \bar{q}g} \left( T_c^{(2)} \right) = \left( \rho_e^{(2)} \right)_{ij} \left( \rho_s^{(2)} \right)_{ij} \frac{4}{s_{1234}} \frac{1}{x_1 x_2} \left[ \frac{2}{x_1 + x_2} - 2 + (1 - \rho \varepsilon) x_2 - \rho \varepsilon x_1 \right] \\
\times \frac{1}{y_1} \left[ \frac{2}{1 - z_1(1 - y_1)} - 2 + (1 - \rho \varepsilon)(1 - y_1)(1 - z_1) \right].
\]
As already noted, this is inserted into a double-amputated amplitude. The variables \( x_1, x_2 \) and \( x_3 \) are given by

\[
x_1 = y_2, \quad x_2 = (1 - y_2)(1 - z_2), \quad x_3 = (1 - y_2)z_2.
\] (107)

The variables \( y_1, z_1, y_2 \) and \( z_2 \) are given by

\[
y_1 = \frac{s_{34}}{s_{234}}, \quad z_1 = \frac{s_{24}}{s_{23} + s_{24}}, \quad y_2 = \frac{1}{(1 - y_1)} \frac{s_{12}}{s_{1234}}, \quad z_2 = 1 - \frac{1}{(1 - y_2)} \frac{s_{234}}{s_{1234}}.
\] (108)

The remaining variable \( u \) is given by

\[
u = (y_1 - \bar{y}_1\bar{z}_1)x_1 + \bar{y}_1\bar{z}_1x_2 + \bar{z}_1x_3 + 2(1 - 2w)\sqrt{y_1\bar{z}_1x_1x_3},
\]

\[
w = \frac{1}{2} (1 - c_2),
\]

\[
c_2 = \frac{s_{12}s_{23}^2 + s_{24}((s_{23} + s_{24})(s_{13} + s_{14}) - s_{12}s_{23}) - s_{14}(s_{23} + s_{24})^2}{2s_{23}\sqrt{s_{12}s_{24}((s_{23} + s_{24})(s_{13} + s_{14}) - s_{12}s_{23})}}.
\] (109)

The amputated amplitude is evaluated with the mapped momenta

\[
p_e^{(2)} = \frac{1}{2} \left[ 1 + q + \frac{y_2\bar{z}_2 (1 + q - 2r)}{1 - y_2\bar{z}_2} \right] p_1 + rp_s^{(1)} + \frac{1}{2} \left[ 1 - q + \frac{y_2(1 - q - 2r)}{1 - y_2} \right] p_e^{(1)},
\]

\[
p_s^{(2)} = \frac{1}{2} \left[ 1 - q - \frac{y_2\bar{z}_2 (1 + q - 2r)}{1 - y_2\bar{z}_2} \right] p_1 + (1 - r)p_s^{(1)} + \frac{1}{2} \left[ 1 + q - \frac{y_2(1 - q - 2r)}{1 - y_2} \right] p_e^{(1)},
\] (110)

where

\[
p_e^{(1)} = p_{34} - \frac{y_1}{1 - y_1}p_2, \quad p_s^{(1)} = \frac{1}{1 - y_1}p_2,
\] (111)

and the variables \( r \) and \( q \) are given by

\[
r = \frac{\bar{y}_2\bar{z}_2}{y_2 + \bar{y}_2\bar{z}_2}, \quad q = \sqrt{1 + 4r(1 - r)y_2\frac{1 - z_2}{z_2}}.
\] (112)

### 5.2.2 Topology C

The splitting functions read:

\[
P_{(0,0)}^{(0,2)} g g g (T_c^{(2)}) = p_e^{(2)} \frac{4}{s_{1234} x_1^2} \left\{ \frac{1}{y_1(y_1 + \bar{y}_1\bar{z}_1)} \left[ \frac{2}{(y_1 + \bar{y}_1\bar{z}_1)x_1 + \bar{y}_1x_2} + \frac{2}{y_1x_1 + u} \right] \right\}
\]

\[
+ \frac{u}{(y_1 + \bar{y}_1\bar{z}_1)x_1 + \bar{y}_1x_2} \left( -2 + (1 - \rho \varepsilon)u \right) + \frac{\bar{y}_1x_2}{u} \left( -2 + (1 - \rho \varepsilon)\bar{y}_1x_2 \right)
\]
The amputated amplitude is evaluated with the mapped momenta

\[ p^{(2)}_{e} \] 
\[
\frac{1}{y_{1}} \left[ \frac{2}{u} \left( y_{1} \bar{y}_{1} \bar{z}_{1} x_{1} + \bar{y}_{1} x_{2} \right) + \frac{1}{x_{1} + \bar{y}_{1} x_{2}} \left( -2 + (1 - \rho \varepsilon)u \right) - \frac{4}{u} + (1 - \rho \varepsilon) \right] \frac{\bar{y}_{1} x_{2}}{u} \]
\[
- \frac{\bar{y}_{1} x_{2}}{u} \left( -2 + (1 - \rho \varepsilon) \bar{y}_{1} x_{2} \right) + \rho \varepsilon \left( 1 - \rho \varepsilon \right) u - \rho \varepsilon \left( 1 - \rho \varepsilon \right) \bar{y}_{1} x_{2} - 1 + 3 \rho \varepsilon \right] \]
\[ + \left( 1 - \rho \varepsilon \right)^{2} \frac{y_{1} \bar{y}_{1} \bar{z}_{1}}{y_{1}} + 3 - 4 \rho \varepsilon + \rho \varepsilon \varepsilon^{2} + x_{1} \frac{1}{y_{1}} \left[ - \rho \varepsilon \left( \frac{2}{y_{1} + \bar{y}_{1} \bar{z}_{1}} - 2 + (1 - \rho \varepsilon) \bar{y}_{1} \bar{z}_{1} \right) \right] + \frac{2}{u} \]
\[ - \left( 1 - \rho \varepsilon \right) \left( \frac{x_{1}}{u} - 2 \bar{y}_{1} z_{1} + (1 - \rho \varepsilon) \bar{y}_{1} z_{1} x_{1} \right) + \frac{1}{u \left( y_{1} + \bar{y}_{1} \bar{z}_{1} \right)} \left( -2 + (1 - \rho \varepsilon) x_{1} \right) \right] \right),
\]
\[ p_{(0,1)}^{(2)} q_{ggg} \left( T_{c}^{(2)} \right) = p_{e}^{(2)} \frac{4}{s_{123}^{2} x_{1}^{2}} \left[ \frac{2}{x_{1} + x_{2}} - 2 + (1 - \rho \varepsilon) x_{2} - \rho \varepsilon x_{1} \right] \]
\[ \times \frac{1}{y_{1}} \left[ \frac{2}{1 - z_{1} \left( 1 - y_{1} \right)} - 2 + (1 - \rho \varepsilon) \left( 1 - y_{1} \right) \left( 1 - z_{1} \right) \right]. \] (113)

The variables \( x_{1}, x_{2} \) and \( x_{3} \) are given by
\[ x_{1} = y_{2}, \quad x_{2} = (1 - y_{2}) \left( 1 - z_{2} \right), \quad x_{3} = (1 - y_{2}) z_{2}. \] (114)

The variables \( y_{1}, z_{1}, y_{2} \) and \( z_{2} \) are given by
\[ y_{1} = \frac{s_{12}}{s_{123}}, \quad z_{1} = \frac{s_{13}}{s_{13} + s_{23}}, \]
\[ y_{2} = \frac{s_{123}}{s_{1234}}, \quad z_{2} = 1 - \frac{1}{(1 - y_{1}) \left( 1 - y_{2} \right) s_{1234}}. \] (115)

The remaining variable \( u \) is given by
\[ u = \bar{y}_{1} \bar{z}_{1} x_{1} + (z_{1} + \bar{y}_{1} \bar{z}_{1}) x_{2} + \bar{z}_{1} x_{3} + 2(1 - 2w) \sqrt{y_{1} z_{1} \bar{z}_{1} x_{2} x_{3}}, \]
\[ w = \frac{1}{2} \left( 1 - c_{2} \right), \]
\[ c_{2} = \frac{s_{12}s_{23}s_{34} + s_{13} \left( (s_{13} + s_{23})(s_{14} + s_{24}) - s_{12}s_{34} \right) - s_{14}(s_{13} + s_{23})^{2}}{2 \sqrt{s_{12}s_{23}s_{34} \left( (s_{13} + s_{23})(s_{14} + s_{24}) - s_{12}s_{34} \right)}}. \] (116)

The amputated amplitude is evaluated with the mapped momenta
\[ p^{(2)}_{e} = \frac{1}{2} \left[ 1 + q - \frac{\bar{y}_{2} \bar{z}_{2} \left( 1 - q - 2r \right)}{1 - \bar{y}_{2} \bar{z}_{2}} \right] p^{(1)}_{e} + \left( 1 - r \right) p^{(1)}_{s} + \frac{1}{2} \left[ 1 - q - \frac{y_{2} \left( 1 + q - 2r \right)}{1 - y_{2}} \right] p_{4}, \]
\[ p^{(2)}_{s} = \frac{1}{2} \left[ 1 - q + \frac{\bar{y}_{2} \bar{z}_{2} \left( 1 - q - 2r \right)}{1 - \bar{y}_{2} \bar{z}_{2}} \right] p^{(1)}_{e} + r p^{(1)}_{s} + \frac{1}{2} \left[ 1 + q + \frac{y_{2} \left( 1 + q - 2r \right)}{1 - y_{2}} \right] p_{4}, \] (117)

where
\[ p^{(1)}_{e} = p_{12} - \frac{y_{1}}{1 - y_{1}} p_{3}, \quad p^{(1)}_{s} = \frac{1}{1 - y_{1}} p_{3}, \] (118)
and the variables \( r \) and \( q \) are given by
\[ r = \frac{y_{2}}{y_{2} + \bar{y}_{2} \bar{z}_{2}}, \quad q = \sqrt{1 + 4r(1 - r) y_{2} \frac{1 - z_{2}}{z_{2}}}. \] (119)
5.2.3 Topology D

The splitting functions read:

\[
\mathcal{P}_{(0,0)}^{(2)} T^{(2)}_{d} = \mathcal{P}_{e}^{(2)} \frac{4}{s_{1234}^2} \frac{1}{x_1^2} \left\{ \frac{1}{y_1 (y_1 + \bar{y}_1 \bar{z}_1)} \left[ \frac{2}{(y_1 + \bar{y}_1 \bar{z}_1)} (y_1 x_1 + y_1 x_3) \right] + \frac{2}{y_1 x_1 + x_2 + y_1 x_3} [2 + (1 - \rho \varepsilon) (y_1 x_1 + x_2 + y_1 x_3)] \right\}
\]

\[
\mathcal{P}_{(0,1)}^{(2)} T^{(2)}_{d} = \mathcal{P}_{e}^{(2)} \frac{4}{s_{1234}^2} \frac{1}{x_1^2} \left\{ \frac{1}{y_1 (1 - z_1)} \left[ \frac{2}{1 - z_1 (1 - y_1)} (y_1 x_1 + x_2 + y_1 x_3) \right] \right\} \times \frac{1}{y_1} \left\{ \frac{2}{1 - z_1 (1 - y_1)} \left[ \frac{2}{x_1 + x_2} [1 + 2 \left( \frac{1}{x_1} + \frac{1}{x_2} \right) (1 - 2w) \frac{1}{y_1} (1 - y_1)^2 z_1 (1 - z_1) \right] \right\}.
\]

The variables \( x_1, x_2 \) and \( x_3 \) are given by

\[
x_1 = y_2, \quad x_2 = (1 - y_2) (1 - z_2), \quad x_3 = (1 - y_2) z_2.
\]

The variables \( y_1, z_1, y_2 \) and \( z_2 \) are given by

\[
y_1 = \frac{s_{23}}{s_{123}}, \quad z_1 = \frac{s_{13}}{s_{12} + s_{13}}, \quad y_2 = \frac{s_{123}}{s_{1234}}, \quad z_2 = \frac{1}{(1 - y_1) (1 - y_2) s_{1234}}.
\]

The remaining variable \( u \) is given by

\[
u = z_1 x_2 + y_1 \bar{z}_1 x_3 - 2 (1 - 2w) \sqrt{y_1 \bar{z}_1 \bar{z}_1 x_2 x_3},
\]

\[
w = \frac{1}{2} (1 - c_2),
\]

\[
c_2 = \frac{s_{123} s_{23} s_{14} + s_{13} ((s_{12} + s_{13}) (s_{24} + s_{34}) - s_{23} s_{14}) - s_{34} (s_{12} + s_{13})^2}{2 \sqrt{s_{123} s_{13} s_{14} (s_{12} + s_{13}) (s_{24} + s_{34}) - s_{23} s_{14}}}. \quad (123)
\]
The amputated amplitude is evaluated with the mapped momenta

\[
p^{(2)}_e = \frac{1}{2} \left[ 1 + q - \frac{y_2 \bar{z}_2 (1 - q - 2r)}{1 - y_2 \bar{z}_2} \right] p^{(1)}_s + (1 - r)p^{(1)}_e + \frac{1}{2} \left[ 1 - q - \frac{y_2 (1 + q - 2r)}{1 - y_2} \right] p_4,
\]

\[
p^{(2)}_s = \frac{1}{2} \left[ 1 - q + \frac{y_2 \bar{z}_2 (1 - q - 2r)}{1 - y_2 \bar{z}_2} \right] p^{(1)}_s + rp^{(1)}_e + \frac{1}{2} \left[ 1 + q + \frac{y_2 (1 + q - 2r)}{1 - y_2} \right] p_4,
\]

where

\[
p^{(1)}_e = p_{23} - \frac{y_1}{1 - y_1} p_1, \quad p^{(1)}_s = \frac{1}{1 - y_1} p_1,
\]

and the variables \( r \) and \( q \) are given by

\[
r = \frac{y_2}{y_2 + \bar{y}_2 \bar{z}_2}, \quad q = \sqrt{1 + 4r(1 - r)y_2 \frac{1 - z_2}{z_2}}.
\]

5.2.4 Topology E

The splitting functions read:

\[
P^{(0,2)}_{(0,0) q \rightarrow qgg} (T_e^{(2)}) = \frac{4}{s_{1234}^2} \left\{ \frac{1}{(x_1 + y_1 x_2 + y_1 x_3) y_1 x_2} \right. \\
\times \left[ -\frac{4}{x_1 + (1 + y_1) x_2 + y_1 x_3} + \frac{1}{y_1 y_2} \frac{[-2 + (1 - \rho \varepsilon) x_2] + 4 - 2(1 - \rho \varepsilon) x_2}{y_1 x_2 [(y_1 + \bar{y}_1 \bar{z}_1) x_2 + ul (y_1 + \bar{y}_1 \bar{z}_1)]} \right] \\
+ \frac{1}{y_1 x_2} \left[ -\frac{2}{y_1 + \bar{y}_1 \bar{z}_1} - 2 + \bar{y}_1^2 \bar{z}_1 \bar{z}_1 \right] \left\} \right.,
\]

\[
P^{(0,2)}_{(0,1) q \rightarrow qgg} (T_e^{(2)}) = \frac{4}{s_{1234}^2} \frac{1}{x_1 x_2} \left[ \frac{2}{x_1 + x_2} - 2 + (1 - \rho \varepsilon) x_2 - \rho \varepsilon x_1 \right] \\
\times \frac{2}{y_1 \frac{1}{1 - z_1 (1 - y_1) - 2}}.
\]

The variables \( x_1, x_2 \) and \( x_3 \) are given by

\[
x_1 = y_2, \quad x_2 = (1 - y_2)(1 - z_2), \quad x_3 = (1 - y_2)z_2.
\]

The variables \( y_1, z_1, y_2 \) and \( z_2 \) are given by

\[
y_1 = \frac{s_{23}}{s_{234}}, \quad z_1 = \frac{s_{24}}{s_{24} + s_{34}}, \quad y_2 = \frac{1}{(1 - y_1) \frac{s_{123} s_{234} - s_{23} s_{1234}}{s_{234} s_{1234}}}, \quad z_2 = 1 - \frac{1}{(1 - y_2) \frac{s_{234}}{s_{1234}}},
\]

\[28\]
The remaining variable \( u \) is given by
\[
\begin{align*}
u &= z_1 x_1 + y_1 \bar{z}_1 x_3 - 2(1 - 2w) \sqrt{y_1 z_1 \bar{z}_1 x_1 x_3}, \\
w &= \frac{1}{2} (1 - c_2), \\
c_2 &= \frac{s_{23}s_{34}s_{14} + s_{24}((s_{12} + s_{13})(s_{24} + s_{34}) - s_{23}s_{14}) - s_{12}(s_{24} + s_{34})^2}{2\sqrt{s_{23}s_{34}s_{14}(s_{12} + s_{13})(s_{24} + s_{34}) - s_{23}s_{14}}}. \quad (130)
\end{align*}
\]

The amputated amplitude is evaluated with the mapped momenta
\[
\begin{align*}
p_e^{(2)} &= \frac{1}{2} \left[ 1 + q + \frac{y_2 \bar{z}_2 (1 + q - 2r)}{1 - y_2 \bar{z}_2} \right] p_1 + r p_e^{(1)} + \frac{1}{2} \left[ 1 - q + \frac{y_2 (1 - q - 2r)}{1 - y_2} \right] p_s^{(1)}, \\
p_s^{(2)} &= \frac{1}{2} \left[ 1 - q - \frac{y_2 \bar{z}_2 (1 + q - 2r)}{1 - y_2 \bar{z}_2} \right] p_1 + (1 - r) p_e^{(1)} + \frac{1}{2} \left[ 1 + q - \frac{y_2 (1 - q - 2r)}{1 - y_2} \right] p_s^{(1)}, \quad (131)
\end{align*}
\]
where
\[
\begin{align*}
p_e^{(1)} &= p_{23} - \frac{y_1}{1 - y_1} p_4, \quad p_s^{(1)} = \frac{1}{1 - y_1} p_4, \quad (132)
\end{align*}
\]
and the variables \( r \) and \( q \) are given by
\[
r = \frac{y_2 \bar{z}_2}{y_2 + y_2 \bar{z}_2}, \quad q = \sqrt{1 + 4r(1 - r)y_2 \frac{1 - z_2}{z_2}}. \quad (133)
\]

### 5.3 Subtraction terms for contributions with one virtual and one real unresolved parton

In this section I give the subtraction term \( d\alpha_n^{(1,1)} \), which approximates
\[
\begin{align*}
d\sigma_{n+1}^{(1)} + d\alpha_{n+1}^{(0,1)} &= \frac{4\pi\alpha_s}{2\pi} \frac{\alpha_s}{c_0} \frac{1}{4} \left( N_c^2 - 1 \right) N_c \\
&\times \left[ A_{n+1}^{(0)} \ast A_{n+1}^{(1)} + A_{n+1}^{(1)} \ast A_{n+1}^{(0)} + A_{n+1}^{(0)} \ast I A_{n+1}^{(0)} \right] d\phi_{n+1} \quad (134)
\end{align*}
\]
\( I \) is obtained from integrating \( d\alpha_{n+1}^{(0,1)} \) over the unresolved phase space and is given for our example by
\[
I = S_{\epsilon}^{-1} \frac{(4\pi)^\epsilon}{\Gamma(1 - \epsilon)} \left\{ \left( \frac{s_{12}}{\mu^2} \right)^{-\epsilon} \left[ \nu_{q \to gq}^{(0,1)} + \nu_{g \to gg}^{(0,1)} \left( \frac{s_{23}}{s_{12} + s_{23}} \right) \right] + \left( \frac{s_{23}}{\mu^2} \right)^{-\epsilon} \left[ \nu_{q \to gq}^{(0,1)} + \nu_{g \to gg}^{(0,1)} \left( \frac{s_{12}}{s_{12} + s_{23}} \right) \right] \right\}. \quad (135)
\]
Figure 8: Splitting topologies for the subtraction terms for the one-loop amplitude with one unresolved parton.

The explicit forms of the functions $V^{(0,1)}_{q \rightarrow qg}$ and $V^{(0,1)}_{g \rightarrow qg}$ are given in eq. (142). The subtraction term $d\alpha^{(1,1)}_n$ is written as a sum over topologies:

$$d\alpha^{(1,1)}_n = \sum_{\text{topologies } T} \mathcal{D}^{(1,1)}_n(T).$$

The relevant topologies for the leading $N_c$ contributions to $e^+ e^- \rightarrow qgg\bar{q}$ are shown in fig. (8). It is sufficient to consider the topology $T_a^{(1,1)}$. The topology $T_b^{(1,1)}$ is related by symmetry to $T_a^{(1,1)}$. The subtraction term is given by

$$\mathcal{D}^{(1,1)}_n(T) = \frac{c}{8\pi^2} \left[ \langle A^{(0)}_n | \mathcal{P}^{(0,1)}(T) | A^{(1)}_n \rangle + \langle A^{(1)}_n | \mathcal{P}^{(0,1)}(T) | A^{(0)}_n \rangle + \langle A^{(1)}_n | \mathcal{P}^{(1,1)}(T) | A^{(0)}_n \rangle \right] d\phi_{n+1}.$$

Here, in the first and second term the NLO splitting function $\mathcal{P}^{(0,1)}$ is sandwiched between $\mathcal{A}^{(0)}_n$ and $\mathcal{A}^{(1)}_n$. The third term contains as a new structure the one-loop splitting function $\mathcal{P}^{(1,1)}$. It is convenient to write

$$\mathcal{P}^{(1,1)}(T) = \mathcal{P}^{(1,1)}_{(1,0)}(T) + \mathcal{P}^{(1,1)}_{(0,1)}(T),$$

where $\mathcal{P}^{(1,1)}_{(1,0)}$ is (part of ) the approximation to $d\sigma^{(1)}_{n+1}$ and $\mathcal{P}^{(1,1)}_{(0,1)}$ is an approximation to $d\alpha^{(0,1)}_{n+1}$. The splitting functions read:

$$\mathcal{P}^{(1,1)}_{(1,0)}\,_{q \rightarrow gg} (T_a^{(1,1)}) =$$

$$S^{(1,1)}_\varepsilon c_T \left( \frac{-s_{123}}{\mu^2} \right)^{-\varepsilon} y^{-\varepsilon} \left\{ g_1(y, z) \mathcal{P}^{(0,1)}_{q \rightarrow gg} + g_2 \frac{1}{s_{123} y} \frac{1 - \rho \varepsilon(1 - y)(1 - z)}{1 - \varepsilon} \right\},$$

$$-\frac{11}{6\varepsilon} \mathcal{P}^{(0,1)}_{q \rightarrow gg} + c.c.,$$

$$\mathcal{P}^{(1,1)}_{(0,1)}\,_{q \rightarrow gg} (T_a^{(1,1)}) =$$

$$S^{(1,1)}_\varepsilon \left( \frac{4\pi \varepsilon}{\Gamma(1 - \varepsilon)} \right) \left( \frac{s_{123}}{\mu^2} \right)^{-\varepsilon} \left\{ y^{-\varepsilon} \left[ \mathcal{V}^{(0,1)}_{q \rightarrow gg} + \mathcal{V}^{(0,1)}_{g \rightarrow gg} \left( \frac{(1 - y)(1 - z)}{1 - z(1 - y)} \right) \right] \right\}.$$
\[(1 - y)^{-\varepsilon} (1 - z)^{-\varepsilon} \left[ \gamma^{(0,1)}_{q \to qg} + \gamma^{(0,1)}_{g \to gg} \left( \frac{y}{1 - z(1 - y)} \right) \right] \}

P^{(0,1)}_{q \to qg}. \tag{139}\]

The variables \(y\) and \(z\) are given by
\[
y = \frac{s_{12}}{s_{123}}, \quad z = \frac{s_{13}}{s_{13} + s_{23}}. \tag{140}\]

The functions \(g_1\) and \(g_2\) are given by
\[
g_1(y, z) = -\frac{1}{\varepsilon^2} \left[ \Gamma(1 + \varepsilon) \Gamma(1 - \varepsilon) \left( \frac{z}{1 - z} \right)^{\varepsilon} + 1 - (1 - y)^{\varepsilon} z^{\varepsilon} \right] 2F_1(\varepsilon, \varepsilon, 1 + \varepsilon; (1 - y)(1 - z)) \],
\[
g_2 = \frac{1 - \rho \varepsilon}{2(1 - \varepsilon)(1 - 2\varepsilon)}. \tag{141}\]

The sum of the integrated dipoles reads
\[
\gamma^{(0,1)}_{q \to qg} + \gamma^{(0,1)}_{g \to gg}(r) = \frac{\Gamma(-\varepsilon) \Gamma(-2\varepsilon) \Gamma(1 - \varepsilon)^2}{\Gamma(1 - 2\varepsilon) \Gamma(1 - 3\varepsilon)} \left\{ 4 + \frac{8\varepsilon}{1 - 3\varepsilon} - 2(1 - \rho \varepsilon) \frac{\varepsilon(1 - \varepsilon)}{(1 - 3\varepsilon)(2 - 3\varepsilon)} \right\}.
\tag{142}\]

The amputated amplitudes are evaluated with the momenta
\[
p_e = p_1 + p_2 - \frac{y}{1 - y} p_3, \quad p_s = \frac{1}{1 - y} p_3. \tag{143}\]

For the numerical integration over the \((n + 1)\)-parton phase space we need the \(\varepsilon\)-expansion of the subtraction terms:
\[
\langle A^{(0)}_n \ | \ P^{(1,1)}_n \ (T) \ | A^{(0)}_n \rangle = A^{(0)*}_2 A^{(0)}_2
\]
\[
\times \frac{2}{s_{12}} \left\{ \left[ \frac{2}{\varepsilon^2} + (3 - 2L) \frac{1}{\varepsilon} + 2 \operatorname{Li}_2 \left( (1 - y)(1 - z) \right) + 2 \left( \ln(1 - y) \right)^2 \right.ight.
\]
\[
+ 2 \ln(1 - y) \ln(z) + 2 \ln(1 - y) \ln(1 - z) - 2 \ln(y) \ln(1 - z) + 2 \ln(z) \ln(1 - z)
\]
\[
- 2 \ln(y) \ln(1 - y) - \frac{3}{2} \varepsilon^2 - \frac{19}{6} \ln(1 - z) - \frac{19}{6} \ln(1 - y) - \frac{7}{2} \ln(y) + \frac{58}{3}
\]
\[
+ \frac{1}{3} \left( \frac{(1 - y)(1 - z)}{1 - z(1 - y)} \ln(y) - \ln(1 - y) - \ln(1 - z) \right) + L^2 - \frac{20}{3} L + \rho \}
\]
\[
\times \left[ \frac{2}{1 - z(1 - y)} - 2 + (1 - y)(1 - z) \right] + \left[ \frac{2}{\varepsilon} - 3 + 2L \right] \rho(1 - y)(1 - z) + 1 \}
+ O(\varepsilon). \tag{144}\]
Here, $L = \ln(s_{123}/\mu^2)$. The $\varepsilon$-expansion of the insertion of $\mathcal{P}^{(0,1)}$ into the interference term with the loop amplitude $A_2^{(1)}$ is given by

\[
\langle A_n^{(0)} | \mathcal{P}^{(0,1)}(T) | A_n^{(1)} \rangle + \langle A_n^{(1)} | \mathcal{P}^{(0,1)}(T) | A_n^{(0)} \rangle = A_2^{(0)} s_{12} \times \frac{2}{s_{12}} \times \\
\left\{ \left\{ \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} - 8 + 3L - L^2 + \frac{7}{6} \pi^2 \right\} \left\{ \frac{2}{1 - z(1 - y)} - 2 + (1 - y)(1 - z) \right\} + O(\varepsilon) \right\}
\]

We see that the poles in $\varepsilon$ cancel in the sum of the two contributions, as it should be.

6 Cancellations

In this section I discuss how singularities cancel between different terms for the various singular limits. The most intricate part is the one which is integrated over the $n+2$ parton phase space:

\[
\int \left( \mathcal{O}_{n+2} \, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \circ d\alpha_{n+1}^{(0,1)} - \mathcal{O}_n \circ d\alpha_{n}^{(0,2)} \right),
\]

where

\[
da_{n+1}^{(0,1)} = \sum_{T=T_d^{(1)},...,T_d^{(1)}} \mathcal{D}_{n+1}^{(0,1)}(T), \quad da_{n}^{(0,2)} = \sum_{T=T_d^{(2)},...,T_d^{(2)}} \mathcal{D}_{n}^{(0,2)}(T).
\]

Although individual terms in the integrand of eq. (146) are singular, the singularities cancel in the sum and the integrand is integrable. Therefore the integration can be performed by Monte Carlo methods. It is instructive to see in detail, how the cancellations of singularities take place. Critical region in phase space are regions where one or two particles are unresolved. Double unresolved contributions can be divided into the following classes:

a) Two pairs of separately collinear particles.

b) Three particles collinear.

c) Two particles collinear and a third soft particle.

d) Two soft particles.

e) Coplanar degeneracy.

The last item occurs only in the subtraction terms and is an artefact of our choice for the NLO subtraction terms. It is included here for completeness. The single unresolved contributions are divided into two classes:
f) Two particles collinear.

g) One soft particle.

For the leading-colour contributions to $e^+e^- \rightarrow qggq$ it is sufficient to check the following cases: Particles 1 and 2 collinear and particles 3 and 4 collinear (case a); particles 1, 2 and 3 collinear (case b); particles 1 and 2 collinear and particle 3 soft (case c); particles 2 and 3 soft (case d); the sum of the four-momenta of particles 2 and 3 linear dependent with the four-momenta of particles 1 and 4 (case e); particles 1 and 2 collinear (case f); particles 2 and 3 collinear (also case f); particle 2 soft (case g). All other cases are related by symmetry to the ones above.

All limits have been checked using the symbolic algebraic manipulation program “FORM” [47].

6.1 Double unresolved regions

In the double unresolved cases, the singularities in the NLO subtraction terms $d\sigma^{(0,1)}_{n+2}$ are cancelled by the subtraction terms $d\sigma^{(0,2)}_{n+1}$. In particular we always have

\[
O_{n+1} D^{(0,1)}_{n+1} (T_a^{(1)}) - O_n D^{(0,2)}_{(0,1)} n (T_b^{(2)}) - O_n D^{(0,2)}_{(0,1)} n (T_c^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D^{(0,1)}_{n+1} (T_b^{(1)}) - O_n D^{(0,2)}_{(0,1)} n (T_d^{(2)}) - O_n D^{(0,2)}_{(0,1)} n (T_e^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D^{(0,1)}_{n+1} (T_c^{(1)}) - O_n D^{(0,2)}_{(0,1)} n (T_f^{(2)}) = \text{integrable},
\]

(148)

for all double unresolved cases. Here, “integrable” stands for “integrable over the double unresolved region”. It remains to discuss the cancellations between $d\sigma^{(0)}_{n+2}$ and $d\sigma^{(0,2)}_{(0,0)} n$ in the double unresolved regions. It should be noted that in specific double unresolved limits some subtraction terms in eq. (148) may be integrable by themselves. This will also be briefly discussed below case by case.

6.1.1 Case 1 and 2 collinear and 3 and 4 collinear.

This case corresponds to the emission of two independent pairs of collinear particles ($p_1 || p_2$ and $p_3 || p_4$). For each pair of collinear particles we write the corresponding momenta according to eq. (61). We are interested in contributions, which scale in the double collinear limit as $|k_\perp|^{-4}$. Cancellations of these terms occurs in the following combination:

\[
O_{n+2} d\sigma^{(0)}_{n+2} - O_n D^{(0,2)}_{(0,0)} n (T_a^{(2)}) - O_n D^{(0,2)}_{(0,0)} n (T_b^{(2)}) = \text{integrable}. \tag{149}
\]

For completeness I also state the cancellations for terms related to $d\sigma^{(0,1)}_{n+1}$:

\[
O_{n+1} D^{(0,1)}_{n+1} (T_a^{(1)}) - O_n D^{(0,2)}_{(0,1)} n (T_b^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D^{(0,1)}_{n+1} (T_b^{(1)}) - O_n D^{(0,2)}_{(0,1)} n (T_a^{(2)}) = \text{integrable}. \tag{150}
\]

All other subtraction terms are separately integrable in this limit.
6.1.2 Case 1, 2 and 3 collinear.

This case corresponds to the triple collinear limit \((p_1||p_2||p_3)\). We parameterize the momenta of the three collinear particles according to eq. (71). We are interested in contributions, which scale in the double collinear limit as \(|k_-|^4\). Cancellations of these terms occurs in the following combination:

\[
O_{n+2} d\sigma_{n+2}^{(0)} - O_n D_{(0,0)}^{(0,2)} n(T_c^{(2)}) - O_n D_{(0,0)}^{(0,2)} n(T_d^{(2)}) - O_n D_{(0,0)}^{(0,2)} n(T_e^{(2)}) = \text{integrable.}
\] (151)

For completeness I also state the cancellations for terms related to \(d\alpha_{n+1}^{(0,1)}\):

\[
O_{n+1} D_{n+1}^{(0,1)} (T_a^{(1)}) - O_n D_{(0,1)}^{(0,2)} n(T_c^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D_{n+1}^{(0,1)} (T_b^{(1)}) - O_n D_{(0,1)}^{(0,2)} n(T_d^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D_{n+1}^{(0,1)} (T_c^{(1)}) - O_n D_{(0,1)}^{(0,2)} n(T_e^{(2)}) = \text{integrable}.
\] (152)

All other subtraction terms are separately integrable in this limit.

6.1.3 Case 1 and 2 collinear and 3 soft.

This case corresponds to the emission of a collinear pair \((p_1||p_2)\) together with a soft gluon \((p_3)\). In this limit, the invariants \(s_{12}, s_{23}, s_{34}\) and \(s_{123}\) become small. We are interested in contributions, which have three of these four invariants in the denominator. Cancellations of these terms occurs in the following combination:

\[
O_{n+2} d\sigma_{n+2}^{(0)} - \sum_{T=T_a^{(2)},...,T_g^{(2)}} O_n D_{(0,0)}^{(0,2)} n(T) = \text{integrable.}
\] (153)

For completeness I also state the cancellations for terms related to \(d\alpha_{n+1}^{(0,1)}\):

\[
O_{n+1} D_{n+1}^{(0,1)} (T_a^{(1)}) - O_n D_{(0,1)}^{(0,2)} n(T_c^{(2)}) - O_n D_{(0,1)}^{(0,2)} n(T_d^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D_{n+1}^{(0,1)} (T_b^{(1)}) - O_n D_{(0,1)}^{(0,2)} n(T_e^{(2)}) = \text{integrable},
\]

\[
O_{n+1} D_{n+1}^{(0,1)} (T_c^{(1)}) - O_n D_{(0,1)}^{(0,2)} n(T_e^{(2)}) = \text{integrable}.
\] (154)

All other subtraction terms are separately integrable in this limit.

6.1.4 Case 2 and 3 soft.

This case corresponds to the emission of two soft gluons (with momenta \(p_2\) and \(p_3\)). If we scale the momenta of two soft particles as

\[
p_2 \to \lambda p_2, \quad p_3 \to \lambda p_3,
\] (155)
we are interested in contributions, which scale in the double soft limit as $\lambda^{-4}$. Cancellations of these terms occurs in the following combination:

$$ O_{n+2} \, \delta \sigma_{n+2}^{(0)} - \sum_{T=T_{s}^{(2)},..,T_{h}^{(2)}} O_{n} \, D_{n}^{(0,2)}(T) = \text{integrable}. \quad (156) $$

For completeness I also state the cancellations for terms related to $d\alpha_{n+1}^{(0,1)}$:

$$ O_{n+1} \, D_{n+1}^{(0,1)}(T_{b}^{(1)}) - O_{n} \, D_{n}^{(0,2)}(T_{d}^{(2)}) - O_{n} \, D_{n}^{(0,2)}(T_{h}^{(2)}) = \text{integrable}, $$

$$ O_{n+1} \, D_{n+1}^{(0,1)}(T_{c}^{(1)}) - O_{n} \, D_{n}^{(0,2)}(T_{e}^{(2)}) - O_{n} \, D_{n}^{(0,2)}(T_{g}^{(2)}) = \text{integrable}. \quad (157) $$

All other subtraction terms are separately integrable in this limit.

### 6.1.5 Coplanar case

For completeness I briefly discuss singularities occuring in coplanar configurations. The matrix element squared $\delta \sigma_{n+2}^{(0)}$ is not singular in this limit. This singularity occurs only in the subtraction terms. An example is given by the subtraction term $D_{n+1}^{(0,1)}(T_{b}^{(1)})$. This subtraction term has a singularity in

$$ 2p_{4}p_{e}^{(1)} = \frac{s_{123}s_{234} - s_{23}s_{1234}}{s_{123} - s_{23}}. \quad (158) $$

The numerator is a Gram determinant and can also be written as

$$ s_{123}s_{234} - s_{23}s_{1234} = \langle 1 - |2 + 3|4\rangle\langle 4 - |2 + 3|1\rangle. \quad (159) $$

This expression vanishes, whenever the sum of the four-momenta $p_{2} + p_{3}$ becomes linear dependent with $p_{1}$ and $p_{4}$. Therefore, $2p_{4}p_{e}^{(1)}$ becomes small, whenever this condition is fulfilfilled and whenever $p_{2} + p_{3}$ is not linear dependent with $p_{1}$. (In the later case also the denominator in eq. (158) vanishes.) This type of singularity occurs only in the subtraction terms and the cancellation takes place in the following combination:

$$ O_{n+1} \, D_{n+1}^{(0,1)}(T_{b}^{(1)}) - O_{n} \, D_{n}^{(0,2)}(T_{h}^{(2)}) = \text{integrable}. \quad (160) $$

### 6.2 Single unresolved regions

The expression in eq. (148) is integrated over the complete $(n+2)$-parton phase space. This implies in particular an integration over regions which correspond to single unresolved configurations. It is clear that in this case we cannot expect a cancellation of singularities between terms, which involve $O_{n+2}$ and $O_{n}$, e.g. similar to that what we had in the double unresolved case. In the single unresolved case the $(n+2)$-parton configuration used in the calculation of the observable $O_{n+2}$ will go smoothly to a $(n+1)$-parton configuration. The numerical value for the observable $O$ evaluated with this $(n+1)$-parton configuration
will however in general have no relationship with the numerical value obtained from the evaluation with the \( n \)-parton configuration of the subtraction terms. A similar argument applies to the factorization of the matrix element. Both effects spoil a cancellation of the singularities. The cancellation of the singularities has to occur therefore either between terms involving \( O_{n+2} \) and \( O_{n+1} \), or between terms involving \( O_{n+1} \) and \( O_n \), or between terms involving only \( O_n \). In the following, “integrable” stands for “integrable over the single unresolved region”.

### 6.2.1 Case 1 and 2 collinear.

This case corresponds to the collinear limit \( p_1 |\parallel p_2 \). We parameterize the momenta of the two collinear particles according to eq. (61). We are interested in contributions, which scale in the collinear limit as \( |k_\perp|^2 \). A cancellations of these terms occurs in the following combination:

\[
O_{n+2} \, d\sigma^{(0)}_{n+2} - O_{n+1} \, D^{(1)}_{n+1}(T_a^{(1)}) = \text{integrable.} \tag{161}
\]

This is just the combination which already occurs at NLO. At NNLO we also have to take into account the singularity in \( D^{(0,1)}_{n+1}(T_d^{(1)}) \). This one is cancelled in the combination

\[
O_{n+1} \, D^{(0,1)}_{n+1}(T_d^{(1)}) - O_n \, D^{(0,2)}_{(0,1)}(T_a^{(2)}) = \text{integrable.} \tag{162}
\]

Finally, there is a third equation, describing the cancellations among the NNLO subtraction terms:

\[
O_n \, D^{(0,2)}_{(0,0)}(T_b^{(2)}) + O_n \, D^{(0,2)}_{(0,0)}(T_c^{(2)}) - O_n \, D^{(0,2)}_{(0,1)}(T_b^{(2)}) - O_n \, D^{(0,2)}_{(0,1)}(T_c^{(2)}) = \text{integrable.} \tag{163}
\]

All other subtraction terms are separately integrable in this limit. Note that the pieces

\[
O_n \, D^{(0,2)}_{(0,0)}(T_b^{(2)}) - O_n \, D^{(0,2)}_{(0,1)}(T_b^{(2)}) \tag{164}
\]

and

\[
O_n \, D^{(0,2)}_{(0,0)}(T_c^{(2)}) - O_n \, D^{(0,2)}_{(0,1)}(T_c^{(2)}) \tag{165}
\]

are with our choice of subtraction terms in general not integrable separately. Only the sum of the two pieces is integrable. In order for the cancellations to occur, the \( n \)-parton configuration of topology \( T_b^{(2)} \) must approach in the limit \( s_{12} \to 0 \) the same configuration as the \( n \)-parton configuration of topology \( T_c^{(2)} \). With a symmetric choice for the reconstruction functions in eq. (110) and eq. (117) the \( n \)-parton configurations for topologies \( T_b^{(2)} \) and \( T_c^{(2)} \) are identical for any value of \( s_{12} \) and this requirement is fulfilled trivially.

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6.2.2 Case 2 and 3 collinear.

Here we discuss the case, where particles 2 and 3 become collinear. Again, we parameterize the momenta of the two collinear particles according to eq. (61). We are interested in contributions, which scale in the collinear limit as \(|k_\perp|^2\). The NLO relation for the cancellation of singularities reads:

\[
\mathcal{O}_{n+2} \, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \, D_{n+1}^{(0,1)}(T_b^{(1)}) - \mathcal{O}_{n+1} \, D_{n+1}^{(0,1)}(T_c^{(1)}) = \text{integrable.} \quad (166)
\]

In addition, there is one relation among the subtraction terms with \(n\)-parton configurations:

\[
\mathcal{O}_n \, D_{(0,0)} n (T_d^{(2)}) + \mathcal{O}_n \, D_{(0,0)} n (T_h^{(2)}) - \mathcal{O}_n \, D_{(0,1)} n (T_d^{(2)}) - \mathcal{O}_n \, D_{(0,1)} n (T_h^{(2)}) + \mathcal{O}_n \, D_{(0,0)} n (T_c^{(2)}) - \mathcal{O}_n \, D_{(0,0)} n (T_g^{(2)}) - \mathcal{O}_n \, D_{(0,1)} n (T_c^{(2)}) - \mathcal{O}_n \, D_{(0,1)} n (T_g^{(2)}) = \text{integrable.} \quad (167)
\]

All other subtraction terms are separately integrable in this limit. Again we have to require that the \(n\)-parton configurations of the topologies \(T_d^{(2)}, T_c^{(2)}, T_g^{(2)}\) and \(T_h^{(2)}\) all approach the same \(n\)-parton configuration in the limit \(s_{23} \to 0\). With the choice for the reconstruction functions in eq. (124) and eq. (131) this requirement is fulfilled.

6.2.3 Case 2 soft.

This case corresponds to the emission of a single soft gluon (particle 2). If we scale the momentum of the soft gluon as

\[
p_2 \to \lambda p_2, \quad (168)
\]

we are interested in contributions, which scale in the soft limit as \(\lambda^{-2}\). Cancellations of these terms occurs in the following combination:

\[
\mathcal{O}_{n+2} \, d\sigma_{n+2}^{(0)} - \mathcal{O}_{n+1} \, D_{n+1}^{(0,1)}(T_a^{(1)}) - \mathcal{O}_{n+1} \, D_{n+1}^{(0,1)}(T_b^{(1)}) = \text{integrable.} \quad (169)
\]

This is the usual NLO relation. In addition, we have

\[
\mathcal{O}_n \, D_{(0,0)} n (T_b^{(2)}) + \mathcal{O}_n \, D_{(0,0)} n (T_c^{(2)}) - \mathcal{O}_n \, D_{(0,0)} n (T_b^{(2)}) - \mathcal{O}_n \, D_{(0,1)} n (T_c^{(2)}) = \text{integrable},
\]

\[
\mathcal{O}_n \, D_{(0,0)} n (T_d^{(2)}) + \mathcal{O}_n \, D_{(0,0)} n (T_h^{(2)}) - \mathcal{O}_n \, D_{(0,0)} n (T_d^{(2)}) - \mathcal{O}_n \, D_{(0,1)} n (T_h^{(2)}) = \text{integrable}. \quad (170)
\]

All other subtraction terms are separately integrable in this limit.

6.3 Cancellations for contributions with one virtual and one real unresolved parton

The cancellation mechanism for contributions with one virtual and one real unresolved parton are quite similar to those already encountered at NLO. The only difference is the functional form of the subtraction terms (as given in eq. (139)), which approach smoothly the appropriate singular limit (as given in eq. (70)).
7 Outlook and conclusions

In this paper I considered the extension of the subtraction method to next-to-next-to-leading order. Such a method is needed to extend fully differential perturbative calculations in quantum field theories from NLO to NNLO. As a particular example I considered the leading-colour contributions to $e^+e^- \rightarrow 2$ jets and derived the subtraction terms at NNLO. The subtracted matrix elements can be integrated numerically over the appropriate phase space. The method presented here is general and not restricted to the example of $e^+e^- \rightarrow 2$ jets. Subtraction terms for other splittings (like $g \rightarrow ggg$) and other kinematical configurations (e.g. with partons in the initial state) can be worked out along the same lines.

The subtraction terms still have to be integrated analytically over the unresolved phase space. With the advance of integration techniques witnessed in the past years, this seems feasible. The completion of this program will open the door to fully differential NNLO Monte Carlo programs.

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