ASYMPTOTIC REPRESENTATION OF MINIMAL POLYNOMIALS ON SEVERAL INTERVALS

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Abstract. Asymptotic representation of minimal polynomials on several intervals is given. The last modifications and corrections of this manuscript were done by the author in the two months preceding his passing away in November 2009. The manuscript remained unsubmitted and is not published elsewhere

1. Introduction

Let \( E = \bigcup_{k=1}^{l} E_k \), where \( E_k = [a_{2k-1}, a_{2k}] \), \( a_1 < a_2 < \ldots < a_{2l} \), be a system of intervals and let \( W \in C(E) \) be a positive weight function on \( E \). It is a classical problem to find the minimal polynomial on \( E \) with respect to a given norm and weight function \( W \), that is in case of the maximum norm, to find the unique monic polynomial \( \hat{M}_n(x; W) := \hat{M}_n(x) = x^n + \ldots \) such that

\[
||\hat{M}_n(:, W)||_{\infty} := \max_{x \in E} \frac{\hat{M}_n(x; W)}{W(x)} = \min_{a_i} \max_{x \in E} \left| \frac{x^n + a_{n-1}x^{n-1} + \ldots + a_0}{W(x)} \right|
\]

and respectively, to find the normalized minimal polynomial

\[
M_n(x; W) := \frac{\hat{M}_n(x; W)}{E_{n,\infty}(W)},
\]

where

\[
E_{n,\infty}(w) := ||\hat{M}_n(:, W)/W||_{\infty}
\]

denotes the minimum deviation. In the case of a single interval, say \([-1, 1]\), and weight function \( W(x) \equiv 1 \), it is well known that the Chebyshev polynomial

\[
2T_n(z) = (\phi(z, \infty, [-1, 1]))^n + (\phi(z, \infty, [-1, 1]))^{-n}
\]

is the normalized minimal polynomial, where

\[
\phi(z, \infty, [-1, 1]) = z + \sqrt{z^2 - 1}
\]

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is the complex Green’s function of $\mathbb{C}\setminus[-1,1]$ with pole at infinity, while, even for the single interval case, asymptotic representations with respect to a weight function have been proved only recently [15].

For several intervals not much is known with respect to the $L_\infty$-norm (in contrast to the $L_2$-norm, that is, for orthogonal polynomials, whose asymptotic behavior is well understood nowadays thanks to the papers [4, 5, 28, 35, 45, 47], see also the forthcoming book [38]). Indeed in the thirties of the last century for two intervals Achieser [1] derived an asymptotic representation of the minimum deviation $E_{n,\infty}(W)$ with the help of elliptic functions and in the late sixties of the last century Widom [47] found an asymptotic representation of the minimum deviation for several intervals. But (up to the very special case that $E$ is an inverse image of $[-1,1]$ under a polynomial mapping, see [30]) the main points of interest, the explicit or asymptotic representations of the minimal polynomials remained open, though these open problems have been pointed out in [47, p. 128, 205].

To state our main results we need some notations. By $\phi(z, z_0)$ we denote a so-called complex Green’s function for $\mathbb{C}\setminus E$ uniquely determined up to a multiplication constant of absolute value one (chosen conveniently below), that is, $\phi(z, z_0)$ is a multiple valued function which is analytic on $\mathbb{C}\setminus E$ up to a simple pole at $z = z_0$, has no zeros on $\mathbb{C}\setminus E$ and satisfies $|\phi(z, z_0)| \to 1$ for $z \to x \in E$ quasi-everywhere; or in other words $\log|\phi(z, z_0)|$ is the Green’s function with pole at $z = z_0 \in \mathbb{C}\setminus E$, as usual denoted by $g(z, z_0)$. In the case under consideration, as it is known [47, 14], a complex Green’s function may be represented as

$$\phi(z, \infty) = \exp\left(\int_0^z r_\infty(x) \frac{dx}{\sqrt{H(x)}}\right)$$

where

$$H(x) = \prod_{j=1}^{2l} (x - a_j)$$

and $r_\infty(x) = x^{l-1} + \ldots$ is the unique polynomial such that

$$\int_{a_{2j}}^{a_{2j+1}} r_\infty(x) \frac{dx}{\sqrt{H(x)}} = 0 \text{ for } j = 0, \ldots, l - 1$$

and that for $x \in \mathbb{C}\setminus E$

$$\phi(z, x_0) = \exp\left(\int_0^z \frac{r_{x_0}(x)}{x - x_0} \frac{dx}{\sqrt{H(x)}}\right)$$

where $r_{x_0} \in \mathbb{P}_{l-1}$ is such that $r_{x_0}(x_0) = -\sqrt{H(x_0)}$ and

$$\int_{a_{2j}}^{a_{2j+1}} \frac{r_{x_0}(x)}{x - x_0} \frac{dx}{\sqrt{H(x)}} = 0 \text{ for } j = 1, \ldots, l - 1.$$
Recall that the so-called capacity of \( E \) is given by
\[
\text{cap}(E) = \lim_{z \to \infty} \left| \frac{z}{\phi(z, \infty, E)} \right|.
\] (8)

By \( \omega(z, B; \mathbb{C} \setminus E) \) we denote the harmonic measure of \( B \subseteq E \) with respect to \( \mathbb{C} \setminus E \) at \( z \), which is that harmonic and bounded function on \( \mathbb{C} \setminus E_i \) which satisfies for \( \xi \in E_i \) that \( \lim_{z \to \xi} \omega(z, B, \mathbb{C} \setminus E_i) = i_B(\xi) \), where \( i_B \) denotes the characteristic function of \( B \). For abbreviation we put
\[
\omega(z, E_k; \mathbb{C} \setminus E) = \omega_k(z).
\]

Furthermore let us recall that for a given weight function \( W \) which is Lipschitz continuous on \( E \), there is a unique multi-valued analytic function \( \mathcal{W}(z) \) with \( \mathcal{W}(\infty) > 0 \) which has no zeros or poles in \( \mathbb{C} \setminus E \) and is such that the limiting values \( \mathcal{W}^\pm(x) = \lim_{\pm \infty, z \to 0} \mathcal{W}(z) \) from the upper and lower half-plane are Lipschitz-continuous on \( E \) with the property that
\[
\sqrt{\mathcal{W}^+(x)\mathcal{W}^-(x)} = W(x) \text{ for } x \in E.
\] (9)

Note that
\[
\mathcal{W}(\infty) = \exp \left\{ \frac{1}{2\pi} \int_E \log W(x) \frac{\partial g(\xi; \infty)}{\partial n^+} |d\xi| \right\}.
\]

**Notation 1.1.** For given \( E \) and weight function \( W \) put for \( k = 1, \ldots, l - 1 \)
\[
\gamma_{k,n}(W) := n \omega_k(\infty) + \frac{1}{2\pi} \int_E \log W(x) \frac{\partial \omega_k(\xi)}{\partial n^+} |d\xi| + \sigma_{k,n}
\] (10)

where \( \sigma_{k,n} \in \{0, 1\} \) is such that \( \gamma_{k,n}(W) \in [0, 1] \) modulo 2. Note that \( \sigma_{k,n} \) is uniquely determined.

For abbreviation set \( \mathbf{\omega}(\infty) = (\omega_1(\infty), \ldots, \omega_{l-1}(\infty)) \) and \( \gamma_n(W) = (\gamma_{1,n}(W), \ldots, \gamma_{l-1,n}(W)) \), i.e., \( \gamma_n(W) \in [0, 1]^{l-1} \) modulo 2.

**Theorem 1.2.** Let \( W \in C^{1+\alpha}(E) \), \( \alpha > 0 \), be positive on \( E \). Let \( c_{j,n} \in [a_{2j}, a_{2j+1}] \), \( j = 1, \ldots, l - 1 \), be the unique points such that
\[
\sum_{j=1}^{l-1} \omega_k(c_{j,n}) = \gamma_{k,n}(W) \text{ modulo } 2 \text{ for } k = 1, \ldots, l - 1,
\] (11)

and suppose that \( \lim_{n \in \mathbb{M}} c_{j,n} = c_j \) with \( c_j \in (a_{2j}, a_{2j+1}) \) for \( j = 1, \ldots, l - 1 \), where \( \mathbb{M} \) is an infinite subset of \( \mathbb{N} \). Then the normalized minimal polynomial \( M_n(x; W) \) has for \( n \in \mathbb{M} \) the following uniform asymptotic representation on \( E \):
\[
2M_n(x; W) = \psi_n^+(x) + \psi_n^-(x) + o(1),
\] (12)

where
\[
\psi_n(z) = \frac{\phi(z; \infty)^n}{\prod_{j=1}^{l-1} \phi(z; c_j)} \mathcal{W}(z)
\] (13)
and the constant in the $o(\cdot)$ term is independent of $n$ and $x$, $x \in E$.

Furthermore on any compact subset $K$ of $\mathbb{C} \setminus E$

\begin{equation}
2M_n(z;W) \frac{\phi(z;\infty)^n}{\prod_{j=1}^{l-1} \phi(z;c_j)} = \mathcal{W}(z) + o(1)
\end{equation}

uniformly on $K$.

Remark 1.3. We note that condition (11) implies Widom’s condition [47, Theorem 5.4], that is, that there exists a unique $l' \leq l$, depending on $n$, and unique points $c_{1,n}, ..., c_{l'-1,n}$ from the open gaps such that

\begin{equation}
\sum_{j=1}^{l'-1} \omega_k(c_{j,n}) = n\omega_k(\infty) + \frac{1}{2\pi} \int_E \log W(x) \frac{\partial\omega_k(\xi)}{\partial n^k} |d\xi| \mod 1
\end{equation}

In fact we will show that (11) and (15) are equivalent. (11) has the big advantage that it can be written as a Jacobi-inversion problem from which many informations can be extracted.

The asymptotic representation (14) on compact subsets of $\mathbb{C} \setminus E$ was conjectured by Widom in [47, p. 205]. In fact he conjectured that (14) holds for arbitrary arcs in the complex plane, but this does not hold in general [?]. Letting $z \to \infty$ in relation (14) we obtain by (8) immediately the following expression for the minimum deviation due to Widom [47, Theorem 11.5], derived by him in a completely different way and for a more general class of weight functions.

Corollary 1.4. For $n \in \mathbb{M}$,

\begin{equation}
2||M_n(\cdot;W)||_{(\text{cap}E)^n} = \prod_{j=1}^{l-1} \phi(c_j;\infty) + \mathcal{W}(\infty) + o(1)
\end{equation}

We mention that in the case of the three intervals a formula for the capacity in terms of Theta functions has been given by T. Falliero and A. Sebbar [10] recently.

Theorem 1.2 enables us to obtain a precise description of the location of the zeros of the minimal polynomial in the gaps $(a_{2k}, a_{2k+1})$, $k = 1, ..., l - 1$ and to give the number of zeros in the intervals $E_k$, denoted by $Z(M_n, E_k)$, as in the gaps. By the way it is known that $M_n$ has at most one zero in each gap, which can be proved by Kolmogorov’s criteria for the best approximation.

Corollary 1.5. Under the assumptions of Theorem 1.2 the following statements about the zeros hold:

a) $M_n(z;W)$ has exactly one zero in each interval $[c_j - \varepsilon, c_j + \varepsilon]$, $j = 1, ..., l - 1$, $\varepsilon > 0$ for $n \in \mathbb{M}$, $n \geq n_0$. 
Theorem 1.6. The following statements are equivalent:

a) The solutions $c_n = (c_{1,n}, \ldots, c_{l-1,n})$ of (11) satisfy

$$\lim_{\nu} c_{n,\nu} = c$$

b) $\lim_{\nu} \gamma_{n,\nu}(W) = \tilde{A}(c)$ modulo 2 with $\tilde{A}(c) \in (0,1)^{l-1}$.

c) $c \in \mathcal{X}_{j=1}^{l-1}(a_{2j}, a_{2j+1})$ is an accumulation point of zeros of $(M_{n,\nu}(c; W))_{\nu \in \mathbb{N}}$.

Furthermore, the set of accumulation points of the sequence of solutions $(c_{n,\nu})_{\nu \in \mathbb{N}}$ of (11) as well as the set of accumulation points of zeros of $(M_{n}(c; W))_{n \in \mathbb{N}}$ in the gaps is dense in $\mathcal{X}_{j=1}^{l-1}(a_{2j}, a_{2j+1})$, if $1, \omega_1(\infty), \ldots, \omega_{l-1}(\infty)$ are linearly independent over $\mathbb{Q}$.

If the harmonic measures of the intervals are rational, that is,

$$\omega_k(\infty) = \frac{m_k}{N}, \quad m_k \in \{1, \ldots, N-1\} \text{ for } k = 1, \ldots, l$$

then the points $c_{j,n}$ appear periodically with respect to $n$. More precisely we have

Corollary 1.7. Suppose that (16) holds and let $b \in \{0, \ldots, N-1\}$.

a) The points $c_{j,n}$ solving (11) satisfy

$$c_{j,b} = c_{j,b+kN} \text{ for all } k \in \mathbb{N}$$

b) Each $c_{j,b}$ located in the open gap $(a_{2j}, a_{2j+1})$ is an accumulation point of zeros of $(M_{b+kN}(c; W))_{k \in \mathbb{N}}$ and there are no other accumulation points of zeros of $(M_{n}(c; W))_{n \in \mathbb{N}}$ in the open gaps.
We mention that the case when the intervals have rational harmonic measure is covered by Theorem 1.2 also taking into consideration the following remark and relation (17).

**Remark 1.8.** Theorem 1.2 holds true, if some of the \( c_{j,n} \)'s, \( j \in \{1, \ldots, l-1\} \) coincide with a boundary point \( a_{2j} \) or \( a_{2j+1} \) of \( E \) for each \( n \in M \). In this case for the corresponding \( c_j \) one has to put \( \phi(z, c_j) \equiv 1 \) in (14), respectively, in Corollary 1.3. Furthermore Corollary 1.5 a) about the zeros holds only with respect to such \( c_j \)'s which do not coincide with a boundary point of \( E \).

Finally we conjecture that the asymptotics from Theorem 1.2 (with the above modifications) hold true when a boundary point of \( E \) is a limit point of the \( c_{j,n} \)'s.

There is also an interesting connection between the \( L_\infty \)-minimum deviation and minimal polynomials (for that see the Remark at the end of the paper) and that ones with respect to suitable weighted \( L_2 \)-norm. First we need some notation. For convenience we set

\[
1/h(x) := \begin{cases} 
(-1)^{l-k}/\sqrt{|H(x)|}, & \text{for } x \in \text{int}(E_k) \\
0, & \text{elsewhere} 
\end{cases}
\]

Let \( \mathcal{E} = \{ R : R(x) = x^{l-1} + \ldots, R \text{ vanishes either at } a_{2j} \text{ or } a_{2j+1} \text{ for } j = 1, \ldots, l-1 \} \) and let \( \mathcal{E}^{(1-x)} = \{ (1-x)R : R \in \mathcal{E} \} \). Note that for \( R \in \mathcal{E} \) we have that

\[
\frac{R}{h} = \frac{1}{\sqrt{(x-a_1)(a_{2l}-x)}} \prod_{j=1}^{l-1} \left( \frac{x-a_{2j+1}}{x-a_{2j}} \right)^{\varepsilon_j/2} > 0 \text{ on int}(E)
\]

and

\[
\frac{(1-x)R}{h} = \left[ \frac{1-x}{1+x} \right]^{l-1} \prod_{j=1}^{l-1} \left( \frac{x-a_{2j+1}}{x-a_{2j}} \right)^{\varepsilon_j/2} > 0 \text{ on int}(E)
\]

where \( \varepsilon_j \in \{\pm 1\} \). The minimum deviation of \( x^n \) on \( E \) with respect to the \( L_2 \)-norm squared is denoted by

\[
E_{n-1,2}(x^n;W) = \min_{q \in \mathbb{P}_{n-1}} \int_E |x^n - q(x)|^2 W(x) dx
\]

and by \( \sharp Z(f;A) \) we denote the number of zeros of \( f \) on \( A \).

**Theorem 1.9.** Suppose that the assumptions of Theorem 1.2 hold.

a)

\[
E_{2n-1,\infty}(x^{2n}, W) \sim \frac{1}{2} \max_{\varepsilon_j \in \{\pm 1\}} E_{n-1,2} \left( x^n; \frac{W(x)}{\sqrt{(x-a_1)(x-a_{2l})}} \prod_{j=1}^{l-1} \left( \frac{x-a_{2j+1}}{x-a_{2j}} \right)^{\varepsilon_j/2} \right)
\]
where $\sigma_n$ is given by (??) and $R(\sigma_n) \in \mathcal{E}$ is uniquely determined by the condition

$$\sharp Z(R(\sigma_n), [a_{2j-1}, a_{2j}]) = \sigma_{j,n} \text{ modulo } 2 \text{ for } j = 1, \ldots, l - 1$$

b)

$$E_{2n,\infty}(x^{2n}, W) \sim \frac{1}{2} \max_{\varepsilon_j \in \{\pm 1\}} E_{n-1,2} \left( x^n; W(x) \sqrt{\frac{1-x}{1+x}} \prod_{j=1}^{l-1} \left( \frac{x-a_{2j+1}}{x-a_{2j}} \right)^{\varepsilon_j/2} \right)$$

$$= \frac{1}{2} E_{n-1,2}(x^n; W(x)((1-x)R(\sigma_n)/h)$$

where $\sigma_n$ is given by (??) and $((1-x)R(\sigma_n)) \in \mathcal{E}(1-x)$ is uniquely determined by the condition

$$\sharp Z((1-x)R(\sigma_n), [a_{2j-1}, a_{2j}]) = \sigma_{j,n} \text{ modulo } 2 \text{ for } j = 1, \ldots, l - 1$$

We note that the asymptotic representation [14] and [12] may be given in terms of Theta functions also in the following way.

**Corollary 1.10.** The function $\psi_n/W$ from Theorem [14] may also be written in the form

$$e^\psi_n(z)/W(z) = \exp \left\{ - \sum_{k=1}^{l-1} \left( \frac{1}{2\pi} \int \log W(\xi) \frac{\partial \omega_k(\xi)}{\partial \xi_j} |d\xi| \right) \int_z^\infty \varphi_k \right\}$$

$$\left( \frac{\vartheta(z; \int_{a_1}^\infty \varphi_k)}{\vartheta(z; \int_{a_1}^{\infty+} \varphi_k)} \right)^n \frac{\vartheta(z; \sum_{j=1}^{l-1} \int_{jz_0}^{\infty+} \varphi_k)}{\vartheta(z; \sum_{j=1}^{l-1} \int_{jz_0}^{\infty+} \varphi_k)}$$

$\vartheta$ denotes the Riemann Theta function, i.e., for given constants $b_j$, $j = 1, \ldots, l - 1$ and a given point $z_0$ on the Riemann surface $\mathcal{R}$ of $y^2 = H$

$$\vartheta(z; b_k) := \vartheta(z; b_1, \ldots, b_{l-1}) := \vartheta(\int_{z_0}^z \varphi_1 - b_1 - k_1, \ldots, \int_{z_0}^z \varphi_{l-1} - b_{l-1} - k_{l-1})$$

where $k_1, \ldots, k_{l-1}$ are the so-called Riemann-constants and $\varphi_j$, $j = 1, \ldots, l - 1$, is a basis of normalized differentials of first kind, see the beginning of the next Section.

Briefly and roughly speaking we derive the above statements in the following way. First we consider the case when the weight function is a polynomial, which includes the particular interesting case $W(x) \equiv 1$ and, what is important, can be treated with the help of rational functions on the Riemann surface $y^2 = H$. More precisely, we write the system of equations [11] in terms of Abelian integrals of first kind, see [30] below. The transformed system of equations guarantees by Abel’s Theorem the existence of a rational function $\mathcal{R}_n$ on the Riemann surface which is the keystone in obtaining asymptotics for the minimal polynomials. In fact it turns out that $R_n := \mathcal{R}_n^+ + \mathcal{R}_n^-$, is a rational function on $\mathbb{C}$, which equioscillates $n+1$ times.
on $E$ and thus has zero as best approximation with respect to any linear subspace of dimension $n$. Showing that the rational function is asymptotically a polynomial of degree $n$ we obtain, using the Lipschitz continuity of the operator of best approximation and deriving a so-called strong unicity constant, that this polynomial is asymptotically the minimal polynomial.

To get the asymptotic representation for general weight functions we approximate the weight function $W(x)$ by polynomials where it is important that the approximating sequence of polynomials $(\rho_\nu)$ can be chosen such that the boundary value problem $I^+(x)I^-(x) = W(x)/\rho_\nu(x)$ can be solved uniquely by a function $I(z)$ which is a nonvanishing, single valued, analytic function on $\mathbb{C}\setminus E$.

### 2. Asymptotics with respect to rational weights

In this section we derive an asymptotic formula for the minimal polynomial with respect to rational weight functions of the form $1/\rho_\nu$, where $\rho_\nu$ is a polynomial of degree $\nu$ which is positive on $E$, which on the one hand contains the particular interesting case $W(x) \equiv 1$ (with geometric convergence of the error even) and on the other hand is the basis to obtain asymptotics with respect to general weight functions roughly speaking by approximating the weight function by a sequence of $1/\rho_\nu$’s. Hence in the following let

\begin{equation}
\rho_\nu(x) = \pm \prod_{j=1}^{\nu^*}(x - w_j)^{\nu_j} \text{ with } \rho_\nu > 0 \text{ on } E.
\end{equation}

Let $\mathcal{R}$ denote the hyperelliptic Riemann surface of genus $l - 1$ defined by $y^2 = H(z)$ with branch cuts $[a_1, a_2], [a_2, a_3], \ldots, [a_{2l-1}, a_2]$. The two sheets of $\mathcal{R}$ are denoted by $\mathcal{R}^+$ and $\mathcal{R}^-$, where on $\mathcal{R}^+$ the branch of $\sqrt{H(z)}$ is chosen for which $\sqrt{H(x)} > 0$ for $x > a_2$. To indicate that $z$ lies on the first resp. second sheet we write $z^+$ and $z^-$. Furthermore let the cycles $\{\alpha_j, \beta_j\}_{j=1}^{l-1}$ be the usual canonical homology basis on $\mathcal{R}$, i.e., the curve $\alpha_j$ lies on the upper sheet $\mathcal{R}^+$ of $\mathcal{R}$ and encircles there clockwise the interval $E_j$ and the curve $\beta_j$ originates at $a_2$ arrives at $a_{2l-1}$ along the upper sheet and turns back to $a_2$ along the lower sheet. $\mathcal{R}'$ denotes now the simple connected canonical dissected Riemann surface. Let $\{\varphi_1, \ldots, \varphi_{l-1}\}$, where

\begin{equation}
\varphi_j = \sum_{s=1}^{l-1} d_{j,s} \frac{z^s}{\sqrt{H(z)}}dz, d_{j,s} \in \mathbb{C},
\end{equation}

be a base of the normalized differential of the first kind i.e.

\begin{equation}
\int_{\alpha_j} \varphi_k = 2\pi i \delta_{jk} \quad \text{and} \quad \int_{\beta_j} \varphi_k = B_{jk} \quad \text{for } j, k = 1, \ldots, l - 1
\end{equation}

where $\delta_{jk}$ denotes the Kronecker symbol here. Note that the $d_{j,s}$’s are real since $\sqrt{H(z)}$ is purely imaginary on $E_j$ and since $\sqrt{H(z)}$ is real on $\mathbb{R}\setminus E$ the $B_{jk}$’s are also real. In the following $\eta(P, Q)$ denotes the differential of the third kind which has simple poles at $P$ and $Q$ with residues 1 and -1,
respectively, and is normalized such that

$$\int_{\alpha_j} \eta(P,Q)dz = 0 \quad \text{for} \quad j = 1, \ldots, l - 1$$

Next we are going to show that the system of equations (11) has a unique solution. This is done by writing (11) in terms of Abelian-integrals and considering the resulting system as a (real) Jacobi inversion problem, see Lemma 2.4 a).

We need some preliminaries concerning the connection of harmonic measures and Abelian integrals more or less known.

Lemma 2.1. a) Let \( c \in \mathbb{C} \setminus E \) and \( \bar{c} \) its conjugate number. Then the following relation holds:

$$\int_{c^+} \phi_j + \int_{c^-} \phi_j = \sum_{k=1}^{l-1} (\omega_k(c) + \omega_k(\bar{c})) B_{jk} \mod (2\pi)$$

b) Let \( \log |W| \) be integrable. Then

$$\sum_{\kappa=1}^{l-1} \left( \frac{1}{2\pi} \int_E \log |W(\xi)| \frac{\partial \omega_k(\xi)}{\partial n_+} |d\xi| \right) B_{k\kappa} = -\frac{2}{\pi i} \int_{E+} \varphi_k^+ \log |W|$$

c) $$\frac{2}{\pi i} \int_{E+} \varphi_k^+ \log |\rho| = \sum_{j=1}^{\nu_j} \varphi_j$$

Proof. a) Let us represent the Green’s functions \( G(z;c) = \ln \phi(z;c) \), extended analytically to the Riemann surface by \( \tilde{\phi}(\tilde{z};c) = 1/\phi(z;c) \) in the form

$$dG(z;c) = \sum_{k=1}^{l-1} \mu_k \varphi_k + \eta(z;c^+,c^-)$$

which implies by integrating along the \( \alpha_j \)-cycles

$$-2\pi i \omega_j(c) = \int_{\alpha_j} dG(z;c) = \sum_{k=1}^{l-1} \mu_k \delta_{jk} 2\pi i$$

where the first equality follows by the representation of the harmonic measure in terms of Green’s function, see e.g. [21], and the last equality by the normalizations of the differentials.

Integrating along the \( \beta_j \)-cycle and using the bilinear relation for abelian differentials of the first and third kind, i.e.,

$$\int_{\beta_j} \eta(X,Y) = \int_X Y \varphi_j$$
we obtain
\[ (25) \quad \oint_{\beta_j} dG(z; c) = -\sum_{k=1}^{l-1} \omega_k(c) B_{jk} + \int_{c^-}^{c^+} \varphi_j \]

Next let us note that the polynomial \( r_c(\xi) \) from (7) can be represented in the form
\[ r_c(\xi) = (\xi - c)(r_\infty(\xi) - M_c(\xi)) - \sqrt{H(c)} \]
where \( M_c(\xi) = \xi^{l-1} + \ldots \in \mathbb{P}_{l-1} \) is the unique polynomial which satisfies
\[ \int_{a_2}^{a_2+1} \frac{\sqrt{H(c)}}{c-x} \frac{dx}{\sqrt{H(x)}} = \int_{a_2}^{a_2+1} M_c(x) \frac{dx}{\sqrt{H(x)}} \text{ for } j = 1, \ldots, l \]
Hence we obtain that
\[ M_c(\xi) = \overline{M_c(\xi)} \]
and
\[ d(G(z, c) + G(z, \overline{c})) = \frac{r_c(\xi)}{\xi - c} + \frac{\overline{r_c(\xi)}}{\xi - \overline{c}} \frac{d\xi}{\sqrt{H(\xi)}} \]
which implies that the integral \( \oint_{\beta_j} d(G(z, c) + G(z, \overline{c})) \) is purely imaginary, since the integrand becomes purely imaginary when \( \xi \) approaches \( E \) and the integrals over the gaps cancel out by the opposite opposite direction of integration. On the other hand it follows by (25) that the integral is real and thus relation (21) is proved.

b) Denote by
\[ w_k(z) = \omega_k(z) + i\omega_k^*(z) \]
the analytic extension of the harmonic measure on \( \mathbb{C} \) and, as usual, let
\[ (26) \quad w_k = -\omega_k + i\omega_k^* \]
be the extension to the second sheet. Since \( w_k, k = 1, \ldots, l-1 \), is just another basis of Abelian differentials of first kind we may represent \( \int \varphi_j \) as linear combinations of the \( w_k \)'s. Integrating along the \( \beta_k \) cycle, \( k \in \{1, \ldots, l-1\} \), and recalling the fact that the integral along a \( \beta_k \)-cycle is the difference of values along the \( \alpha_k \) cycle (which is \( E_k \)) we obtain with the help of (26) that
\[ (27) \quad \int \varphi_j = -\frac{1}{2} \sum_{k=1}^{l-1} w_k B_{jk} \]
Using again the fact that \( \omega_k(z) = 1 \) on \( E_k, k \in \{1, \ldots, l\} \), we obtain that
\[ \frac{1}{i} w_k' dz = \frac{1}{i} d\omega_k(z) = \frac{\partial \omega_k}{\partial n} ds \]
when we approach from the upper half plane to \( E \), hence by (27)
\[ (28) \quad -\frac{2}{\pi i} \varphi_j^+ = \frac{1}{\pi} \sum_{k=1}^{l-1} \left( \frac{\partial \omega_k}{\partial n^+} ds \right) B_{kj} \]
from which part b) follows.
Concerning part c) recall that, consider the periods of the harmonic function $g(\xi, z) - g(\xi, \infty) + \log |\xi - z|$

$$\omega_k(z) - \omega_k(\infty) = \frac{1}{2\pi} \int_E \frac{\partial \omega_k(\xi)}{\partial n^k_\xi} \log |\xi - z| \, d|\xi|$$

Using (21) and (28) the equality follows.

**Notation 2.2.** By Jac $\mathcal{R}$ we denote the Jacobi variety of $\mathcal{R}$, that is, the quotient space $\mathbb{C}^{l-1}/(2\pi i \vec{n} + \vec{B}\vec{m})$, $B = (B_{jk})$ the matrix of periods, $\vec{n}, \vec{m} \in \mathbb{Z}^{l-1}$. Jac $\mathcal{R}/\mathbb{R} := \mathbb{R}^{l-1}/\mathbb{B}\mathbb{m}$ denotes the Jacobi variety restricted to the reals. Finally, let $[a_{2j}, a_{2j+1}]^\pm$ denote the two copies of $[a_{2j}, a_{2j+1}]$, $j = 1, \ldots, l - 1$, in $\mathcal{R}^\pm$. Note that $[a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-$ is a closed loop on $\mathcal{R}$.

The following important statement holds, see e.g., [14, Theorem 5.12] and [32, Lemma 3.5 (a)]:

The restricted Abel map

$$\mathcal{A} : X_{j=1}^{l-1}([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-) \to \text{Jac} \mathcal{R}/\mathbb{R}$$

$$(\zeta_1, \ldots, \zeta_{l-1}) \mapsto \frac{1}{2} \left( \sum_{j=1}^{l-1} \int_{\zeta_j^+} \varphi_1, \ldots, \sum_{j=1}^{l-1} \int_{\zeta_j^-} \varphi_{l-1} \right)$$

is a holomorphic bijection. Moreover the so called real Jacobi inversion problem has a unique solution, that is, for given $(\eta_1, \ldots, \eta_{l-1}) \in \mathbb{R}^{l-1}$ there exists a unique $(\zeta_1, \ldots, \zeta_{l-1}) \in X_{j=1}^{l-1}([a_{2j}, a_{2j+1}]^+ \cup [a_{2j}, a_{2j+1}]^-)$ such that

$$\frac{1}{2} \sum_{j=1}^{l-1} \int_{\zeta_j^+} \varphi_k = \eta_k \mod (B).$$

**Lemma 2.3.** a) Let $s \in \mathbb{R}^{l-1}$ and $\delta \in (\{-1, 1\})^{l-1}$ be given. Then there exists a unique $(\kappa_1, \ldots, \kappa_{l-1}) \in X_{j=1}^{l-1} \mathcal{R}^\delta$ and a unique $\vec{\sigma} \in (\{0, 1\})^{l-1}$ such that

$$\frac{1}{2} \sum_{j=1}^{l-1} \int_{\kappa_j^+} \varphi_k = s_k + \sum_{j=1}^{l-1} \frac{\sigma_j}{2} B_{kj} \mod (B) \text{ for } k = 1, \ldots, l - 1$$

b) Choosing in a) $\delta_j = 1$ for $j = 1, \ldots, l - 1$ the representation

$$\frac{1}{2} \sum_{j=1}^{l-1} \int_{\kappa_j^+} \varphi_k = s_k + \sum_{j=1}^{l-1} \frac{\sigma_j}{2} B_{kj} = \sum_{j=1}^{l-1} t_j B_{kj} \mod (B)$$

with

$$t_\kappa = \frac{1}{2} \sum_{j=1}^{l-1} \omega_k(\text{pr}(\kappa_j)) \in [0, 1/2] \text{ for } \kappa = 1, \ldots, l - 1$$

holds.
Choosing in a) \( \delta_j = -1 \) for \( j = 1, \ldots, l - 1 \) then \( t_j \) has a representation of the form (30) with \( t_\kappa \in [-\frac{1}{2}, 0] \) for \( \kappa = 1, \ldots, l - 1 \).

Proof. The simplest way to prove part a) is to change the model of the Riemann surface by choosing now as in [14, Section 5.5.2] the canonical homology basis \( \{ \alpha_j', \beta_j' \}_{j=1}^{l-1} \), where the curve \( \alpha_j \) originates at \( a_{2j} \), arrives at \( a_{2j+1} \) along the upper sheet and turns back to \( a_{2j} \) along the lower sheet and \( \beta_j' \) lies in the upper sheet and encircles the interval \([a_1, a_{2j}]\) clockwise. Note that the \( \alpha' \) and \( \beta' \) periods can be expressed easily in terms of the \( \alpha \) and \( \beta \) periods from (19). More precisely, the period matrix of the new model is obtained by a linear transformation with entire coefficients of the original model. Thus modulo periods it does not matter which model we take.

We know that there exists an unique solution \( \vec{\xi} \in X^{l-1}_{-1} \mathcal{R} \), see [14, Theorem 5.12] and [32, Lemma 3.5 (a)], such that

\[
\frac{1}{2} \sum_{j=1}^{l-1} \int_{\xi_j}^{\xi_j^*} \phi_k = s_k \quad \text{for} \quad k = 1, \ldots, l - 1 \mod B'
\]

If \( \vec{\xi} \in X^{l-1}_{-1} \mathcal{R}^{\delta} \) we are done. If \( \xi_p \in \mathcal{R}^{-\delta_p} \) then we consider the inversion problem for \( (s_1 + B'_{21}, \ldots, s_{l-1} + B'_{l-1}) \) that is, we add the half period \( (B'_p) \). Then for \( j \neq p \) all points \( \xi_j \) from (31) remain in place and \( \xi_p \) will be displaced by \( \xi_p^* \) if \( \xi_p \notin \{ a_j \}_{j=1}^{2l} \). If \( \xi_p \in \{ a_j \}_{j=1}^{2l} \) then this point is displaced from one end of \( (a_{2j}, a_{2j+1}) \) to the other. Thus part a) follows by going back to our model.

b) By (??) we know that (29) holds with \( t_j \in [-1/2, 1/2] \). Taking into consideration the fact that \( \kappa_j = x_j^+ \) and \( \kappa_j^* = x_j^- \) and writing the LHS of (29) in terms of harmonic measures with the help of (21) it follows that

\[
t_k = \frac{1}{2} \sum_{j=1}^{l-1} \omega_k(\text{pr}(\kappa_j)) \geq 0
\]

which proves part b).

\[\square\]

Lemma 2.4. a) The unique solution \( (\zeta_{1,n}, \ldots, \zeta_{l-1,n}) \) of the Jacobi inversion problem, \( \kappa = 1, \ldots, l - 1 \),

\[
\sum_{j=1}^{l-1} \int_{\zeta_{j,n}}^{\zeta_{j,n}^*} \varphi_\kappa = n \int_{\infty^-}^{\infty^+} \varphi_\kappa + \frac{2}{\pi i} \int_{E^+} \varphi_\kappa^- \log |W| + \sum_{j=1}^{l-1} \sigma_{j,n}(W) B_{\kappa j} \mod 2(B)
\]

where \( \sigma_{j,n}(W) \) is given by (10), has the property that \( \zeta_{j,n} \in \mathcal{R}^+ \) for \( j = 1, \ldots, l - 1 \). Moreover, putting \( \zeta_{j,n} = c_{j,n}^+ \), hence \( \zeta_{j,n}^* = c_{j,n}^- \) and using Lemma 2.1 the system of equations (32) becomes (11).
b) The sequence of solutions \((c_n^+)_{n \in \mathbb{N}}\) of (32) converges if and only if \((\gamma_n(W))_{n \in \mathbb{N}}\) converges modulo 2, where \(\gamma_n(W)\) is given in (10). Furthermore, the transformed Abel map \(\tilde{A} = B^{-1} \circ A\) is a real analytic homeomorphism between the sets of accumulation points of \((c_n^+)_{n \in \mathbb{N}}\) and \((\gamma_n(W))_{n \in \mathbb{N}}\) modulo 2.

Proof. Assume that not all \(\zeta_{j,n}\)’s are from \(R^+\). Then by Lemma 2.3 there is a \(\tilde{\sigma} \in \{0,1\}^{l-1}, \tilde{\sigma} \neq \sigma_n(W)\), and a \(\tilde{\zeta} \in X_{\gamma_n(W)}\) which solves the correspondingly modified Jacobi inversion problem (32). Applying Lemma 2.1 to the RHS of (32) and using representation (29) with property (30) it follows that \(\gamma_n(W) - \sigma_n(W) + \tilde{\sigma} \in [0,1]^{l-1}\) modulo 2. But this implies by the definition of \(\sigma_n(W)\) and its uniqueness, recall (10), that \(\sigma_n(W) = \tilde{\sigma}\), which is a contradiction.

With the help of Lemma 2.1 it follows by straightforward calculation that (32) is equivalent to (11).

b) Writing relation (32) in the form

\[
2(A((c_n^+)_{n \in \mathbb{N}}))^t = B\gamma_n^t
\]

where \(t\) denotes the transpose of the vectors, the assertion follows immediately by the bijectivity and the other properties of the Abel map \(A\).

\[\square\]

Theorem 2.5. Let \(c_{j,n} \in [a_{2j}, a_{2j+1}]\), \(j = 1, ..., l-1\), be such that

\[
\sum_{j=1}^{l-1} \omega_k(c_{j,n}) = \gamma_n(1/\rho_n)
\]

and suppose that for \(n \in M \subseteq \mathbb{N}\) there holds \(c_{j,n} \in [a_{2j} + \delta, a_{2j+1} - \delta]\), \(\delta > 0\), \(j = 1, ..., l-1\). Then for \(n \in M\) the following asymptotic representation holds uniformly on \(E\)

\[
\frac{2M_n(x; \rho_n)}{\rho_n(x)} = \left(\mathcal{R}_n^+(x; \rho_n) + \frac{1}{\mathcal{R}_n^+(x; \rho_n)}\right) + O(q^n),
\]

where

\[
\mathcal{R}_n^+(x; \rho_n) = \left(\phi^+(x; \infty)\right)^{n-\nu} \prod_{j=1}^{l-1} (\phi^+(x; w_j))^{\nu_j} \prod_{j=1}^{l-1} \phi^+(x; c_{j,n})
\]

and where \(q \in (-1, 1)\) and the constant in the \(O\)-term does not depend on \(n\) and \(x, x \in E\).

Proof. For simplicity of writing let us assume that \(\nu_j = 1\) for \(j = 1, ..., \nu\). First let us transform condition (33) into the equivalent condition on Abelian differentials of first kind. Multiplying each equation from (33) by \(B_{nk}\) and
summing up we obtain that \( (33) \) is equivalent to

\[
\sum_{j=1}^{l-1} \int_{c_{j,n}}^{c_{j,n}^+} \varphi_\kappa = n \int_{-\infty}^{\infty} \varphi_\kappa - \sum_{j=1}^{\nu} \left( \int_{-\infty}^{\infty} \varphi_\kappa - \int_{w_j^+}^{w_j^-} \varphi_\kappa \right) + \sum_{k=1}^{l-1} \sigma_{kn}(1/\rho_\nu)B_{nk} \quad \text{for} \quad \kappa = 1, \ldots, l-1.
\]

Thus by Abel’s Theorem, see e.g. [23, Theorem 1], there is a rational function \( R_n \) on the Riemann surface \( y^2 = H \) such that

\[
x = \infty^\pm \text{ is a pole (zero) of } R_n \text{ with multiplicity } n - \nu,
\]

\[
x = w_j^\pm \text{ is a simple pole (zero) of } R_n
\]

\[
x = c_{j,n}^\pm \text{ is a simple zero (pole) of } R_n
\]

Thus \( R_n \) is of the form

\[
R_n = \frac{P_{n+l-1} + \sqrt{H}Q_{n-l-1}}{g_{(n)}(z)^{\rho_\nu}},
\]

where

\[
g_{(n)}(z) = \prod_{j=1}^{l-1} (x - c_{j,n}),
\]

\( P_{n+l-1} \) and \( Q_{n+l-1} \) are polynomials of degree \( n + l - 1 \) and \( n - l - 1 \) which are such that the numerator in \( (38) \) has the properties that

\[
P_{n+l-1} + \sqrt{H}Q_{n-l-1} \text{ has a double zero at } c_{j,n} \text{ for } j = 1, \ldots, l-1
\]

and

\[
P_{n+l-1} - \sqrt{H}Q_{n-l-1} \text{ has a double zero at } w_j \text{ for } j = 1, \ldots, \nu.
\]

By the way, since the points \( c_{j,n} \) and \( w_j \) are real the rational function \( \overline{R_n(x)} \)

has the same properties \( (37) \) as \( R_n(x) \), i.e. \( R_n(x) = \overline{R_n(x)} \), or in other words the coefficients of \( P_{n+l-1}(x) \) and \( Q_{n-1}(x) \) are real. Taking involution (denoted by \( \tilde{x} \)), in \( (37) \), which corresponds to multiplication of relation \( (36) \) by \(-1\), we obtain that

\[
\frac{1}{R_n(z)} = R_n(z) = \frac{P_{n+l-1}(z) - \sqrt{H(z)}Q_{n-l-1}(z)}{g_{(n)}(z)^{\rho_\nu(z)}}
\]

Moreover, putting

\[
R_n := R := \frac{P_{n+l-1}}{\rho_\nu g_{(n)}} \quad \text{and} \quad S_n := S := \frac{Q_{n-l-1}}{\rho_\nu g_{(n)}}
\]

i.e.

\[
R_n = R + \sqrt{H}S \quad \text{and} \quad \frac{1}{R_n} = R - \sqrt{H}S
\]
it follows that

\begin{equation}
R^2 - HS^2 = 1
\end{equation}

Note that for \( x \in E \)

\begin{equation}
\mathcal{R}_n^+(x) = R(x) \pm i \sqrt{-H(x)}S(x)
\end{equation}

hence for \( x \in E \)

\begin{equation}
2R_n(x) = \mathcal{R}_n^+(x) + \mathcal{R}_n^-(x) = \mathcal{R}_n^+(x) + \frac{1}{\mathcal{R}_n^+(x)},
\end{equation}

where the last equality follows by (43), and moreover

\begin{equation}
|\mathcal{R}_n^+(x)|^2 = \mathcal{R}_n^+(x)\mathcal{R}_n^-(x) = 1
\end{equation}

Now we claim that

\begin{equation}
\mathcal{R}_n(z) = (\phi(z; \infty))^{n-\nu} \prod_{j=1}^{l} \phi(z; w_j) \prod_{j=1}^{\nu} \phi(z; c_{j,n})
\end{equation}

Indeed, since the function

\[ f(z) = \mathcal{R}_n(z)(\phi(z; \infty))^{-\nu} \prod_{j=1}^{l} \phi(z; c_{j,n}) \prod_{j=1}^{\nu} \phi(z; w_j) \]

has by (37) neither zeros nor poles on \( \bar{\mathbb{C}} \setminus E \) and satisfies by the definition of \( \phi \) and by (33) that \( |f^\pm| = 1 \) on \( E \). Thus \( \log |f| \) is a harmonic bounded function on \( \mathbb{C} \setminus E \) which has a continuous extension to \( E \) and thus \( f \equiv 1 \), which proves the claim (47).

Next let us demonstrate that \( |R| \leq 1 \) on \( E \) and that \( R \) has \( n+1 \) alternation points on \( E \) i.e. there exist \( n+1 \) points \( y_i \) from \( E \), \( y_1 < y_2 < \ldots < y_{n+1} \), such that

\begin{equation}
(-1)^{n+1-i} = R(y_i) = \frac{P_{n+l-1}(y_i)}{\rho_{\nu}(y_i)g_{(n)}(y_i)}
\end{equation}

which implies by the Alternation Theorem that

\begin{equation}
0 \text{ is a best approximation to } R \text{ with respect to } L\{x^j / \rho_{\nu} \}_{j=0}^{n-1}
\end{equation}

Since \(-H > 0\) on \( \tilde{E} \) the property that \( |R| \leq 1 \) on \( E \) follows immediately by (43). Furthermore, \( |R| = 1 \) on \( E \) if and only if \( HQ_{n-1-l} = 0 \). Thus if we are able to show that the zeros of \( HQ_{n-1-l} \) and \( P_{n+l-1} \) are simple and strictly interlacing on \( E \) and that \( Q_{n-l} \) has exactly two zeros in each open gap
(a_{2j}, a_{2j+1}), j = 1, ..., l - 1, the alternation property (48) and thus (49) will follow. First let us recall that

$$R = \frac{1}{2} \left( R_n + \frac{1}{R_n} \right) = \cosh \ln R_n \quad \text{and} \quad \sqrt{HS} = \frac{1}{2} \left( R_n - \frac{1}{R_n} \right) = \sinh \ln R_n$$

Hence

$$HQ_{n-l-1}(z) = 0 \quad \text{if and only if} \quad R_n(z) = \pm 1$$

where for $z \in E$ one has to take the limiting value $R_n^+$. Note, that by representation (47), (4) and (7)

$$R^+_n(x) = e^{i\chi_n(x)}$$

where

$$\chi_n(x) = -(n - \nu)\pi \int_{a_1}^x \frac{r_\infty(t)}{h(t)} \, dt$$

$$+ \int_{a_1}^x \left( \text{bounded function with respect to } x \right) \frac{dt}{h(t)}$$

where we have used the fact that

$$\lim_{z \to \pm \infty} \frac{r_\infty(z)}{\sqrt{H(z)}} = -\frac{i\pi r_\infty(x)}{h(x)},$$

where

$$\frac{1}{h(x)} = \begin{cases} 
\frac{(-1)^{l-k}}{\pi \sqrt{-H(x)}} & \text{for } x \in E_k \\
0 & \text{elsewhere}
\end{cases}$$

Since, by (17), the polynomial $r_\infty(x)$ has exactly one zero in each open gap $(a_{2j}, a_{2j+1}), j = 1, ..., l - 1$, it follows that

$$|r_\infty(x)| \geq \text{const} > 0 \quad \text{on } E \quad \text{and} \quad r_\infty(x)/h(x) > 0 \quad \text{on } \bar{E}.$$ 

Thus $\chi_n$ is strictly monotone on $E$ for sufficiently large $n$. Now by (50) and (52)

$$R(x) = \cos \chi_n(x) \quad \text{and} \quad i\sqrt{H(x)}S(x) = \sin \chi_n(x),$$

hence it follows that on $E$ the zeros of $P_{n+l-1}$ and $HQ_{n-l-1}$ are simple and strictly interlace.

Next let us prove that $Q_{nk-l-1}$ has exactly two zeros in $(c_j - \varepsilon, c_j + \varepsilon) \subset (a_{2j}, a_{2j+1})$ for $j = 1, ..., l - 1$, if $lim_{k} c_{j,n_k} = c_j$.

By (17) $\mathcal{R}_n$ has a simple zero at $c_{j,n}$. Since $\mathcal{R}_n$ is real on $\mathbb{R} \setminus E$ and since $|\phi(z; \infty)| > 1$ on $\mathbb{C} \setminus E$, $\mathcal{R}_n$ is unbounded with respect to $n$ on compact subsets of $(a_{2j}, a_{2j+1}) \setminus \{c_j\}, j = 1, ..., l - 1$, it follows that $\mathcal{R}_n$ takes the values $-1$ and $+1$ on $(c_j - \varepsilon, c_j + \varepsilon) \subset (a_{2j}, a_{2j+1})$ for $j = 1, ..., l - 1$. Hence $Q_{nk-l-1}$ has at least two zeros in $(c_j - \varepsilon, c_j + \varepsilon)$. 
To show that \( Q_{n}^{k-l-1} \) has exactly two zeros in \((c_{j} - \varepsilon, c_{j} + \varepsilon), \ j = 1, ..., l-1,\) we derive first an explicite formula for the number of zeros of \( P_{n+l-1} \) in \( E_{j} \) which is of interest by itself. Indeed by (47)

\[
d ln \mathcal{R}_{n}(z) = (n - \nu) \eta(z; \infty^-, \infty^+) + \sum_{\kappa = 1}^{\nu} \eta(z; w_{\kappa}^-, w_{\kappa}^+) \\
- \sum_{\kappa = 1}^{l-1} \eta(z; c_{\kappa,n}^-, c_{\kappa,n}^+) + \sum_{j = 1}^{l-1} e_{j,n} \varphi_{j}
\]

where we claim that

\[
e_{j,n} = \sharp Z(P_{n+l-1}; E_{j}) := \text{number of zeros of } P_{n+l-1} \text{ in } E_{j}.
\]

and that

\[
\sharp Z(HQ_{n-l-1}; E_{j}) - 1 = \sharp Z(P_{n+l-1}; E_{j})
\]

\[
= n \omega_{j}(\infty) - \sum_{\kappa = 1}^{l-1} \omega_{j}(c_{\kappa,n}) + \frac{1}{\pi} \int_{E} \log |\rho_{\nu}(\xi)| \left| \frac{\partial \omega_{k}}{\partial \eta_{\xi}} \right| d\xi
\]

Indeed, by (47) it follows that \( d ln \mathcal{R}_{n} \) has a representation of the form (57) with \( e_{j,n} \in \mathbb{C}. \) Now by (normalization)

\[
2 \pi i e_{j,n} = \int_{\alpha_{j}} d ln \mathcal{R}_{n}(z)
\]

and on the other hand by shrinking \( \alpha_{j} \) to \( E_{j} \) and (52), (41) and (56) we obtain

\[
\int_{\alpha_{j}} d ln \mathcal{R}_{n}(z) = -i \Delta_{E_{j}} \text{arg} \mathcal{R}_{n} = \chi_{n}(a_{2j}) - \chi_{n}(a_{2j-1})
\]

\[
= -2 \pi i \sharp Z(P_{n+l-1}; E_{j})
\]

where the last equality follows by (59), recalling the strictly interlacing property of \( P_{n+l-1} \) and \( HQ_{n-l-1}. \)

Considering the \( \beta_{k} \)-cycles gives with the help of (58) and Riemann’s bilinear relation (24) that

\[
\int_{\beta_{k}} d ln \mathcal{R}_{n}(z) = (n - \nu) \int_{-\infty}^{+} \varphi_{k} + \sum_{\kappa = 1}^{\nu} \int_{w_{k,n}^-}^{w_{k,n}^+} \varphi_{k} - \sum_{\kappa = 1}^{l-1} \int_{c_{\kappa,n}^-}^{c_{\kappa,n}^+} \varphi_{k}
\]

\[
- \sum_{j = 1}^{l-1} \sharp Z(P_{n+l-1}; E_{j})
\]

Let us consider the integral at the LHS. By the direction of integration the integrals along the gaps cancel out and along the \( E_{k} \)'s the real part of the differential becomes zero because of (17). Hence the real part of the integral at the LHS (62) is zero and thus zero, since the RHS is real. Writing the integrals with the help of the formulas from Lemma (24) a) and b) it follows that (59) holds.
Since \( \sum_{k=1}^{l} \omega_k(z) = 1 \) we obtain by summing up \([59]\) that

\[
n - (l - 1) = Z(HQ_{n-l}; E) - l
\]
i.e. \( HQ_{n-l} \) has \( n + 1 \) zeros in \( E \) (which implies that \( Q_{n-k-l} \) has at most and thus exactly two zeros in each interval \((c_j - \varepsilon, c_j + \varepsilon), j = 1, \ldots, l - 1\)) and thus \( R \) has \( n + 1 \) alternation points on \( E \) which proves statement \([49]\).

Next let us show that \( P_{n+l-1}/g(n) \) is asymptotically equal on \( \Omega \backslash \{ \bigcup_{j=1}^{l-1} U_{c_j} \} \) to a polynomial \( \tilde{M}_n \) of degree \( n \). Indeed partial fraction expansion gives

\[
(63) \quad \frac{P_{n+l-1}(z)}{g(n)(z)} = \tilde{M}_n(z) + \sum_{j=1}^{l-1} \frac{\lambda_{j,n}}{z - c_{j,n}}
\]

Thus

\[
(64) \quad \lambda_{j,n} = \lim_{z \to c_{j,n}} \left( z - c_{j,n} \right) \frac{P_{n+l-1}(z)}{g(n)(z)} = \lim_{z \to c_{j,n}} \left( z - c_{j,n} \right) \frac{\phi(z; c_{j,n}, E)}{\phi(z; \infty, E)^n} \frac{\rho_{\nu}(z)}{\prod_{j=1}^{\nu} \phi(z; w_j)}
\]
i.e.

\[
(65) \quad \lambda_{j,n} = O(q^n), \quad q < 1,
\]
where we used the facts that \( |\phi(z, \infty)| \geq 1/q \) on \( \Omega \) and that the \( c_{j,n} \)'s stay away from this set.

Finally let us show that asymptotically 0 is a best approximation to \( M_n(\cdot; \rho_{\nu})/\rho_{\nu} \) by demonstrating that the best approximation of \( R_n = P_{n+l-1}/\rho_{\nu}g(n) \) and \( M_n(\cdot; \rho_{\nu})/\rho_{\nu} \) differ only slightly from each other for \( n \) sufficiently large.

To prove this fact we use the so-called strong unicity constant defined for a function \( f \in C(E) \) with respect to a linear space \( G_n \) by

\[
(66) \quad \gamma(f; G_{\rho}) := \gamma(f) := \sup_{p \in G_{\rho}} \frac{||p - p^*(f)||}{||f - p|| - ||f - p^*(f)||}
\]

If \( G_n \) is a Haar system on \( E \) of dimension \( n \) and \( f - g^* \) has the alternation points \( y_1, \ldots, y_{n+1} \), then it is known that

\[
(67) \quad \gamma(f) \leq \max_{1 \leq k \leq n+1} ||p_k||
\]
where the \( p_k \)'s from \( G_n \) are uniquely defined by

\[
(68) \quad p_k(y_k) = \text{sgn}(f - p^*(f))(y_k) =: \sigma_k \quad \text{and} \quad p_k(y_j) = 0 \quad \text{for} \quad j \neq k
\]
Furthermore, let us recall that the operator of best approximation is Lipschitz continuous that is, if \( h \in C(E) \) is another function, then

\[
||p^*(f) - p^*(h)|| \leq 2\gamma(f)||f - h||.
\]
In the case under consideration we put $f = R_n$, $G_n = L(x^j/\rho_\nu(x))_{j=0}^{n-1}$, $h = \tilde{M}_n/\rho_\nu$ which yields by (49) and (64) that

$$||p^*(\tilde{M}_n)|| \leq 2\gamma(R_n)O(q^n)$$

Since we will prove in the final step that (69)

$$\gamma(R_n) = O(n)$$

it follows by (64), recall that $||R_n|| = 1$, that

$$||\frac{\tilde{M}_n}{\rho_\nu} - p^*(\tilde{M}_n)|| = 1 + O(q^n)$$

and thus

$$\frac{M_n(x)}{\rho_\nu(x)} = \frac{\tilde{M}_n(x) - p^*(x; \tilde{M}_n)}{\rho_\nu(x)||\frac{M_n}{\rho_\nu} - p^*(\tilde{M}_n)||} = \frac{\tilde{M}_n(x)}{\rho_\nu(x)} + O(q^n) = R_n(x) + O(q^n)$$

which is by (45) the assertion of the theorem.

Thus let us prove (69). First we note that in the case under consideration

$$p_k(y) = \sigma_k\rho_\nu(y)\frac{l_{k,n}(y)}{\rho_\nu(y)}$$

where $l_{k,n}$ is the fundamental Lagrange polynomial with respect to the nodes $y_1, \ldots, y_{n+1}$ which, as we have proved above, are by (48), (43) and (41) the zeros of

$$H\tilde{Q}, \text{ where } Q_{n-l-1} = \tilde{Q}v_{2l-2,n}$$

Recall that we have shown that

$$v_{2l-2,n}(x) \rightarrow \prod_{j=1}^{l-1}(x - c_j)^2 \text{ and that } g_{(n)}(x) \rightarrow \prod_{j=1}^{l-1}(x - c_j).$$

Thus by (67)

$$\gamma(R_n) \leq \text{const} \max_{1 \leq k \leq n+1} ||l_{k,n}(y)||$$

and therefore it suffices to show that

$$||l_{k,n}(y)|| = ||\frac{(H\tilde{Q})(y)}{(y - y_k)(H\tilde{Q})'(y_k)}|| = O(n)$$

With the help of the mean value and Markov’s inequality we get that

$$||\frac{H\tilde{Q}(y)}{y - y_k}|| \leq n^2\text{const}||H\tilde{Q}|| \leq n^2\text{const}||\sqrt{H}\tilde{Q}_{n-l-1}|| = O(n^2)$$

where we took into consideration (72) and (43). Finally let us show that at the zeros $y_k$ of $H\tilde{Q}$

$$||(H\tilde{Q})'(y_k)|| \geq \text{const } n$$
Using (41), (56) and (71) it follows that at the zeros $y_k$ of $\tilde{Q}$

$$(HQ_{n-l-1})' = \pm (\sqrt{-H} \rho \nu g(n) \chi_n'(y_k))$$

where we used the fact that $\cos \chi_n(y_k) = \pm 1$. By (53), (55) and (71) inequality (74) follows at the zeros of $\tilde{Q}$.

At a boundary point of $E$, say $a_j$, we have

$$(75) \quad (HQ_{n-l-1})' = \rho \nu g(n) H'Q_{n-l-1}$$

Now by (53) and (56), recall that $\sin \chi_n(a_j) = 0$,

$$\lim_{x \to a_j} x \in E Q_{n-l-1} = \lim_{x \to a_j} \sqrt{-H(x)} \sin \chi_n(x) - H(x)$$

which implies, with the help of $|r_\infty(a_j)| > 0$, that (74) holds at the zeros of $H$ also. $\square$

Corollary 2.6. For the minimum deviation the following asymptotics hold for $n \in M$

$$E_{n-1}(x^n; \rho) = \frac{\prod_{j=1}^{l-1} \phi(c_{j,n}; \infty)}{\prod_{j=1}^{\nu^*} \phi(w_j; \infty)^{\nu_j} n^n} (1 + O(q^n))$$

Proof. Recalling that

$$R_n = \frac{P_{n+l-1}}{\rho \nu g(n)} = \frac{1}{2} \left( \frac{R_n + 1}{R_n} \right)$$

it follows by (47) and $|\phi| > 1$ on $C \setminus E$ that

$$2lc(P_{n+l-1}) = 2 \lim_{z \to \infty} \frac{P_{n+l-1}}{z^{n-\nu} \rho \nu g(n)} = \lim_{z \to \infty} z^{n-\nu} + O(q^n)$$

$$(77) \quad \frac{\nu^*}{l-1} \prod_{j=1}^{\nu^*} \phi(w_j; \infty)^{\nu_j} \prod_{j=1}^{l-1} \phi(c_{j,n}; \infty)$$

Since by (48) at the zeros $y_i$ of $H\tilde{Q}$, $\tilde{Q}$ defined in (71),

$$(-1)^{n+1-i} = R_n(y_i) = \frac{\tilde{M}_n(y_i)}{\rho \nu(y_i)} + O(q^n)$$

it follows by Vallée-Poussin’s Theorem, see [],

$$(78) \quad lc(\tilde{M}_n)E_{n-1}(x^n; \rho) = 1 + O(q^n)$$

Because of (63) we have that

$$(79) \quad lc(P_{n+l-1}) = lc(\tilde{M}_n)$$

which gives by (77) and (78) the assertion. $\square$
3. Proof of Theorem 1.2

The link with the weights of the form $1/\rho$ is given by the following two Lemmata. For the next lemma compare [15].

**Lemma 3.1.** Let $W \in C(E)$ and $\rho_\nu$ be positive on $E$. Then

\[ ||M_n(x; W)/W(x) - M_n(x; \rho_\nu)/\rho_\nu(x)|| = O(n) \left( O(||1 - \rho_\nu/W||) + O(q^n) \right) \]

where $q \in (-1, 1)$.

**Proof.** Put

\[ a_n = 1/E_{n-1}(x^n; W) \quad \text{and} \quad b_n = 1/E_{n-1}(x^n; \rho_\nu) \]

that is, $M_n(x; W) = a_n x^n + ...$ and $M_n(x; \rho_\nu) = b_n x^n + ...$. Using the extremal property of $M_n(\cdot; W)$ and $M_n(\cdot; \rho_\nu)$ we obtain that

\[ \frac{1}{a_n} \leq ||M_n(x; \rho_\nu)/W|| \leq \frac{1}{b_n} ||\rho_\nu/W|| \]

and an analogous estimate for $M_n(x; W)/\rho_\nu$ yields

\[ \frac{1}{||\rho_\nu/W||} \leq \frac{a_n}{b_n} \leq \frac{||W||}{||\rho_\nu||} \]

which implies using

\[ ||W/\rho_\nu - 1|| \leq ||W|| ||\rho_\nu/W - 1|| \quad \text{and} \quad 1 \leq ||\rho_\nu/W|| ||W/\rho_\nu|| \]

that

\[ \left| \frac{a_n}{b_n} - 1 \right| \leq ||W|| ||\rho_\nu/W - 1|| \]

Let

\[ R_n(x; \rho_\nu) := R_n(x) = \frac{P_{n+t-1}}{\rho_\nu g(n)} \]

be given by [11]. Since $||M_n(x; W)/W|| = 1$ and by (70)

\[ ||M_n(x; \rho_\nu)/\rho_\nu - R_n(x; \rho_\nu)|| = O(q^n) \]

it suffices to show that

\[ ||M_n(x; W)/\rho_\nu - R_n(x; \rho_\nu)|| = O(n) \left( O(||1 - \rho_\nu/W||) + O(q^n) \right) \]

We may write

\[ M_n(x; W) - \frac{P_{n+t-1}(x)}{g(n)(x)} = (1 - \frac{a_n}{b_n}) M_n(x; W) + M_n(x; \rho_\nu) - \frac{P_{n+t-1}(x)}{g(n)(x)} + h(x) \]

where

\[ h(x) = \frac{a_n}{b_n} M_n(x; W) - M_n(x; \rho_\nu) \in \mathbb{P}_{n-1} \]
Thus by (41)
\[ \| - \frac{h}{\rho} \| \leq \gamma(R_n) \left( \| R_n + \frac{h}{\rho} \| - \| R_n \| \right) \]
\[ = \gamma(R_n) \left( \| R_n - \frac{M_n(x, \rho_x)}{\rho} \| - \left( 1 - \frac{a_n}{b_n} \right) \frac{M_n(x; W)}{W} \frac{W}{\rho} \| - 1 \right) \]
\[ = \gamma(R_n) \left( O(q^n) + O(\| \frac{\rho}{W} - 1 \|) \right) \]
which gives by (80) the assertion.

The following lemma is due to Achieser and Tomcuk [4] and shows that the RHS in (80) tends to zero, if \( W \in C^{1+\alpha}, \alpha > 0 \).

**Lemma 3.2.** Let \( W \in C^m(E) \) be positive on \( E \) with \( \lim_{n \to \infty} \omega_2(\frac{1}{n}) \log n = 0 \) where \( \omega_2 \) denotes the modulus of continuity of second order. Then there is a sequence of polynomials \( \rho_\nu \) of degree \( \nu \), \( \rho_\nu \) positive on \( E \), such that

(83) \quad \left| \frac{\rho_\nu(x)}{W(x)} - 1 \right| \leq \frac{\text{const}}{\nu^m} \omega_2(\frac{1}{\nu})

and

(84) \quad \int_E \log |\rho_\nu(\xi)| \frac{\partial \omega_k(\xi)}{\partial n^2_\xi} d\xi = \int_E \log |W(\xi)| \frac{\partial \omega_k(\xi)}{\partial n^2_\xi} d\xi \quad \text{for} \quad k = 1, \ldots, l - 1

**Proof.** In [4] instead of relation (84) the relation

(85) \quad \int_E x^j \log \frac{\rho_\nu(x)}{W(x)} \frac{dx}{h(x)} = 0 \quad j = 0, \ldots, l - 1

is given which obviously is equivalent to

(86) \quad \int_E \varphi_+^j \log \frac{\rho_\nu(x)}{W(x)} = 0 \quad j = 0, \ldots, l - 1

Now (85) follows by (28).

**Proof of the asymptotic representation** (12) **of the minimal polynomial** \( M_n(x; W) \) **on** \( E \). Let \( \rho_\nu \) be such a sequence of polynomials whose existence is guaranteed by Lemma 3.2. Then in conjunction with Lemma 3.1 and Theorem 2.5 it follows that for \( \nu \geq \nu_0 \), uniformly on \( E \)

(87) \quad 2 \frac{M_n(x; W)}{W(x)} = \mathcal{R}_n^+(x; \rho) + \mathcal{R}_n^-(x; \rho) + o(1)

where \( o(1) \) is uniformly bounded with respect to \( n \) and \( \nu \) and \( \mathcal{R}_n(z; \rho) := \mathcal{R}_n(z) \) is given by (47). First let us demonstrate that for \( \nu \geq \nu_0 \) uniformly on \( E \)

(88) \quad \mathcal{R}_n^\pm(x; \rho) = \frac{\phi^\pm(x; \infty)^n}{\prod_{j=1}^{l-1} \phi^\pm(x, c_{j,n})} \sqrt{\frac{W^\pm(x)}{W^\pm(x)}(1 + o(1))}. 
Recall, see [47], that there exists a function $\Omega_\nu(z)$ which has neither zeros nor poles, is analytic outside $E$, and is such that $\Omega_\nu^+$ and $\Omega_\nu^-$ extend continuously to $E$ and satisfies for $\xi \in E$

\begin{equation}
|\Omega_\nu(x)| = \sqrt{\Omega_\nu^+(x)\Omega_\nu^-(x)} = \rho_\nu(x) > 0
\end{equation}

Since

$$f(z) = \frac{\rho_\nu(z)}{\omega_\nu(z)} \prod_{j=1}^\nu \frac{\phi(z; w_{j,\nu})}{\phi(z; \infty)}$$

has neither zeros nor poles on $\bar{\mathbb{C}} \setminus E$ and satisfies $|f^\pm| = 1$ on $E$ it follows as above that $\log |f|$ is a harmonic bounded function on $\bar{\mathbb{C}} \setminus E$ which has a continuous extension to $E$ and therefore $f \equiv 1$, that is,

\begin{equation}
\Omega_\nu(z) = \prod_{j=1}^\nu \frac{\phi(z; w_{j,\nu})}{\phi(z; \infty)}
\end{equation}

Next let us consider the function

\begin{equation}
I_\nu(z) = \exp\left\{ \frac{\sqrt{H(z)}}{2\pi} \int_E \frac{1}{z - x} \log \frac{W(x)}{\rho_\nu(x)} h(x) \right\}
\end{equation}

Because of (85) $I_\nu(z)$ is analytic and nonzero on $\bar{\mathbb{C}} \setminus E$. Since $W/\rho_\nu \in C^{1+\alpha}$ we may apply the Sochozki-Plemelj formula which yields that for $x \in E$

\begin{equation}
I_\nu^\pm(x) = \exp\left\{ \frac{1}{2} \log \left| \frac{W(x)}{\rho_\nu(x)} \right| \pm i\Psi(x) \right\},
\end{equation}

where

\begin{equation}
\Psi(x) = \frac{\sqrt{H(x)}}{2\pi} \int_E \frac{\log \left| W(t)/\rho_\nu(t) \right|}{x - t} \frac{dt}{\sqrt{H(t)}}.
\end{equation}

Moreover

\begin{equation}
|I_\nu(x)| = \sqrt{I_\nu^+(x)I_\nu^-(x)} = \frac{W(x)}{\rho_\nu(x)}
\end{equation}

Hence, recalling (89), the unique function $W(z)$ with property (9), introduced in the introduction, is given by

\begin{equation}
W(z) = I_\nu(z)\Omega_\nu(z).
\end{equation}

Since by (83), (93) and (94)

\begin{equation}
I_\nu(z) \xrightarrow{\nu \to \infty} 1 \text{ uniformly on compact subsets of } \mathbb{C} \setminus E \text{ as well as}
\end{equation}

\begin{equation}
I_\nu^\pm(x) \xrightarrow{\nu \to \infty} 1 \text{ uniformly on } E
\end{equation}
it follows that
\[ W^\pm(x) = I^\pm_v(x) \Omega^\pm_v(x) = \Omega^\pm_v(x)(1 + o(1)) \]
hence
\[ \sqrt{\frac{W^\pm(x)}{W^\pm(x)}} = \sqrt{\frac{\Omega^\pm_v(x)}{\Omega^\pm_v(x)}} (1 + o(1)) \]

Thus by (47) and (91) relation (88) is proved.

Finally let us recall that because of (83)
\[ c_{j,n}(\rho_v) = c_{j,n}(W) \]
where the \( c_{j,n}(\rho_v) \)'s are the points \( \rho_v \) and the \( c_{j,n}(W) \)'s the points satisfying \( \Omega \), using the fact that the associated Jacobi-inversion problem is uniquely solvable. Since
\[ \prod_{j=1}^{l-1} \phi^\pm(x; c_{j,n}) = \prod_{j=1}^{l-1} \phi^\pm(x; c_j)(1 + o(1)) \]
we obtain by (88), (87) and (9) the asymptotic representation (12) on \( E \). \( \square \)

The asymptotic representation (14) on \( \mathbb{C} \setminus E \) and the asymptotic value of the minimum deviation will be derived after the Lemma.

**Proof of the asymptotic representation (14) outside of \( E \):** Put \( M_n(\cdot; W) = M_n \), let \( \tilde{M}_n \) be the polynomial from (64) and set
\[ d_n = \frac{lc(M_n)}{lc(M_n)} = \frac{1}{lc(M_n)E_n(x^n; W)} = 1 + o(1) \]
where the last equality follows by the fact, see (79), that
\[ \frac{1}{lc(M_n)} = E_{n-1}(x^n; \rho_v) + O(q^n) \]
and (81) and (82).

Now let us consider
\[ \frac{M_n(z) - c_n \tilde{M}_n(z)}{\phi^n(z)} \]
Since \( (M_n - d_n \tilde{M}_n)/\phi^n \) is a single valued function which vanishes at \( z = \infty \) we may apply Plemelj-Sochozki’s formula and obtain,
\[ \left| \frac{M_n - d_n \tilde{M}_n(z)}{\phi^n(z)} \right| = \left| \int_E \frac{(M_n - d_n \tilde{M}_n)(\xi)}{\phi^+(\xi)^n - \phi^-(\xi)^n} d\xi \right| = o(1) \text{ uniformly on compact subsets of } \mathbb{C} \setminus E \]
using the fact that by (87) and (11), (41) and (63)
\[ M_n(\xi) - d_n \tilde{M}_n(\xi) = o(1) \text{ uniformly on } E. \]
Since $||\tilde{M}_n||_E$ is bounded we know by the so-called Bernstein-Walsh Lemma
that $|\tilde{M}_n(z)| \leq \text{const} |\phi^n(z)|$ for $z \in \mathbb{C} \setminus E$, hence

$$\frac{M_n(z)}{\phi^n(z)} = \frac{\tilde{M}_n(z)}{\phi^n(z)} + o(1) = \frac{\Omega(z)}{\prod_{j=1}^{l-1} \phi(c_{j,n}; \infty)} + o(1)$$

which, by (99) and (97), proves (14). $\square$

**Proof of Corollary 1.5:**

a) Since $1/\prod_{j=1}^{l-1} \phi(z; c_j)$ has a simple zero at $c_j$, $j = 1, ..., l-1$, and is bounded from below on compact subsets of $(a_{2j}, a_{2j+1}) \setminus [c_j - \varepsilon, c_j + \varepsilon]$ it follows by (14) that $M_n(x; 1/W)$ has different sign on $(a_{2j} + \varepsilon, c_j - \varepsilon)$ and $(c_j + \varepsilon, a_{2j} - \varepsilon)$, $j = 1, ..., l-1$, and thus at least one zero in each $(c_j - \varepsilon, c_j + \varepsilon)$, and therefore exactly one zero.

b) The assertion follows by (59), (84), (99), (63) and the relations

$$\sharp Z(P_n, E_k) = \sharp Z(\tilde{M}_n, E_k) = \sharp Z(M_n, E_k)$$

recalling the fact that at the boundary points $|\tilde{M}_n/\rho|$ tends to one. $\square$

**Proof of Theorem 1.6.** The equivalence of statement a) and b) follows by Lemma 2.4 b) and the equivalence of a) and c) by Corollary 1.5. $\square$

### 4. Proof of Theorem 1.9

**Lemma 4.1.** Let $P_n$ be orthonormal on $E$ with respect to the weight function $R/\rho_\nu = R/\sqrt{-H} \rho_\nu$, $R/\rho_\nu > 0$ on $\text{int}(E)$ and $\rho_\nu > 0$ on $E$ and assume for simplicity of writing that the zeros $w_j$ of $\rho_\nu$ are simple. Then the following relation holds

$$RP_n^2 - SQ_m^2 = 2\rho_\nu g(n)$$

with

$$(RP_n)(w_j) = (\sqrt{H}Q_m)(w_j),$$

$$g(n)(x) = \prod_{j=1}^{l-1} (x - x_{j,n}), \text{ where } x_{j,n} \in [a_{2j}, a_{2j+1}] \text{ for } j = 1, ..., l-1$$

and

$$RP_n(x_{j,n}) = \delta_{j,n} \sqrt{H}Q_m(x_{j,n})$$

where $\delta_{j,n} \in \{\pm 1\}$. Furthermore putting

$$R_1 = \frac{RP_n^2}{\rho_\nu g(n)} - 1$$

and

$$\sqrt{H}R_2 = \frac{\sqrt{H}Q_m P_n}{\rho_\nu g(n)}$$

we obtain

$$R_1^2 - HR_2^2 = 1$$
\[ R_1(x_{j,n}) = \delta_{j,n} \sqrt{H} R_2(x_{j,n}) \]

and

\[ R_1(w_j) = \sqrt{H} R_2(w_j) \]

and there holds for \( n \geq n_0, k = 1, \ldots, l - 1 \)

\[ \sum_{j=1}^{l-1} \int_{\kappa_{j,n}} \varphi_k = \sum_{j=1}^{l-1} \int_{x_{j,n}^+} \varphi_k = -(2n + 1 + \partial R - (\nu + l)) \int_{\infty^-} \varphi_k - \sum_{j=1}^{l-1} \int_{w_j^+} \varphi_k + \sum_{j=1}^{l-1} (2\#Z(P_n, E_j) + \#Z(R, E_j)) B_{kj} \]

where \( \text{pr}(\kappa_{j,n}) = x_{j,n} \) and \( \kappa_{j,n} \in \mathcal{R}_{\delta_{j,n}} \).

The \( L_2 \)-minimum deviation is given by

\[ \left( \int p_n^2 \right) l c(\rho_\nu) = 2(\text{cap } E)^{2n+\partial R-(\nu+l)} \frac{\prod_{j=1}^{l-1} \phi(\infty; x_{j,n})}{\prod_{j=1}^{l-1} \phi(\infty; w_j)} \delta_{j,n} + O(q^n), \]

where \( 0 < q < 1 \).

**Proof.** The first statements follow by [25] and relation (105) follows by [32, Lemma 3.1]. It has been shown in [32, Lemma 2.3] that

\[ \mathcal{R}_1 = \frac{1}{2} \left( \psi_n + \frac{1}{\psi_n} \right) \quad \text{and} \quad \sqrt{H} \mathcal{R}_2 = \frac{1}{2} \left( \psi_n - \frac{1}{\psi_n} \right) \]

where

\[ \psi_n(z) = \phi(z, \infty)^{2n+\partial R-(\nu+l)} \prod_{j=1}^{l-1} \phi(z; w_j) \prod_{j=1}^{l-1} \phi(z, x_{j,n})^{\delta_{j,n}} \]

Thus we obtain by (105) that

\[ \frac{1}{(\int p_n^2) l c(\rho_\nu)} = \lim_{z \to \infty} \frac{1}{z^{2n+\partial R-(\nu+l)}} \left( \frac{R D_n^2}{\rho_\nu g(n)} - 1 \right) \]

\[ = \lim_{z \to \infty} \frac{1}{z^{2n+\partial R-(\nu+l)}} \frac{1}{2} \left( \psi_n + \frac{1}{\psi_n} \right) \]

\[ = \frac{1}{2} (\text{cap } E)^{-2n+\partial R-(\nu+l)} \prod_{j=1}^{\nu} \phi(z, w_j) \prod_{j=1}^{l-1} \phi(z, x_{j,n})^{\delta_{j,n}} + o(q^n), \]

\( 0 < q < 1 \), where we used the fact that \( \lim_{z \to \infty} 1/(\psi_n z^{2n+\partial R-(\nu+l)}) = o(q^n) \). \( \square \)

**Lemma 4.2.** For every \( \tilde{R}, R \in \mathcal{E} \), respectively, \( \tilde{R}, R \in \mathcal{E}^{(1-z)} \) the solutions of (105) satisfy

\[ x_{j,n}(R) = \text{pr}(\kappa_{j,n}(R)) = \text{pr}(\kappa_{j,n}(\tilde{R})) = x_{j,n}(\tilde{R}) \quad j = 1, \ldots, l - 1 \]
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if $\kappa_n(R), \kappa_n(\tilde{R}) \in X_{j=1}^{l-1}(a_{2j}, a_{2j+1}) - (a_{2j}, a_{2j+1})^+$. In particular

$$g(n)(x; R) = g(n)(x; \tilde{R})$$

Proof. Since $\sum_{j=1}^{l-1} \#Z(R, E_j)B_{kj}/2$ and $\sum_{j=1}^{l-1} \#Z(\tilde{R}, E_j)B_{kj}/2$ differ modulo 2 by half-periods only statement (107) follows, see the proof of Lemma 2.3, since the solutions of the Jacobi inversion problem (105) with respect to $R$ and $\tilde{R}$ differ with respect to the sheet only. □

Proof of Theorem 1.9. a) For $\sigma_n := \sigma_n(1/\rho_\nu)$ given by (10) there exists a $R(\sigma_n)$ such that

$$\sigma_n = \#Z(R(\sigma_n)) \text{ mod } 2$$

that is,

$$c_{j,n} = x_{j,n} \text{ and } -1 = \delta_{j,n} \text{ for } j = 1, \ldots, l - 1$$

where $\kappa_n(\sigma_n)$ are the solutions from (105) and $c_n$ the solution from (11). By Lemma 4.2 and $\Phi(\infty, x) > 1$ for $x \notin E$ it follows that the RHS takes its maximum for $\delta_{j,n} = -1$, $j = 1, \ldots, l - 1$, hence by (108) for $R(\sigma_n)$.

Lemma 4.2 and Corollary ?? yield

$$\frac{1}{2} \left( \int p_n^2 \frac{R(\sigma_n)}{\tilde{\rho}_\nu h} \right) = \frac{1}{2} E_{n-1,2}(x^n; R(\sigma_n)/\tilde{\rho}_\nu h) = E_{2n-1,\infty}(x^{2n}; 1/\tilde{\rho}_\nu)$$

where \text{`} \text{ means monic.}

Now by the assumptions on the weight function $W$, see e.g. [4], there is a sequence of $\rho_\nu$'s positive on $E$ such that on $E$

$$\left| \frac{\rho_\nu(x)}{W(x)} - 1 \right| < \frac{\text{const}}{\nu} \omega_2 \left( \frac{1}{\nu} \right)$$

which implies, as we have demonstrated above,

$$E_{n-1,\infty}(x^n, 1/\rho_\nu) = E_{n-1,\infty}(x^n, W)(1 + o(1))$$

and, see [47] or [4], that

$$E_{n-1,2}(x^n; W R(\sigma_n)/h) = E_{n-1,2}(x^n; R(\sigma_n)/h \rho_\nu)$$

which gives the assertion.

b) Replacing $R(\sigma_n)$ from a) by $(1 - x)R(\sigma_n)$ the assertion follows analogously. □

It can be shown quite similarly as in the second part of the proof of Theorem 2.5 that the normalized minimal polynomial $M_{2n}(\cdot; 1/\rho_\nu)$ with
\( ||M_{2n}(1/\rho_n)|| = 1 \) is given asymptotically by the orthogonal polynomial as follows

\[
R(x; \sigma_n)P_n^2(x; R(\sigma_n)/\rho_n h) - \rho_n(x)g_n(x; \sigma_n)
\]

\[
= \frac{M_{2n}(x; 1/\rho_n)g_n(x; \sigma_n) + q(x)}{\rho_n(x)g_n(x; \sigma_n)} + O(r^n),
\]

where \( 0 < r < 1 \), uniformly on compact subsets of \( \mathbb{C}\{c_1, ..., c_{l-1}\} \) where \( c_1, ..., c_{l-1} \) are the accumulation points of zeros of \( g_n(x; \sigma_n) \). For odd \( n \)'s the assertion holds analogously.

**Proof of Corollary 1.10**. By (22) and (23)

\[
d\ln \phi(z; c) = - \sum_{k=1}^{l-1} \omega_k(c)\varphi_k + \eta(z; c^+, c^-),
\]

hence by (35)

\[
\frac{\psi_n(z)}{W(z)} = \exp\left\{ \sum_{k=1}^{l-1} (-n\omega_k(\infty) + \sum_{j=1}^{l-1} \omega_k(c_j)) \int_a^z \varphi_k \right\}
\]

\[
\cdot \frac{(\exp \int \eta(z; \infty^+, \infty^-))^n}{\prod_{j=1}^{l-1} e^{\int \eta(z; c_j^+, c_j^-)}}
\]

Note that the first factor can be written by (11) in terms of integrals of \( \log W \). Now

\[
\prod_{j=1}^{l-1} e^{\int \eta(z; c_j^+, c_j^-)} = \frac{\vartheta(z; \sum_j \int_{a_j}^{c_j^+} \varphi_k)}{\vartheta(z; \sum_j \int_{a_j}^{c_j^-} \varphi_k)}
\]

since the functions at the LHS and RHS have the same zeros and poles and the same \( \beta \)-periods and coincide at the point \( a_1 \). Analogously the representation for \( e^{\int \eta(z; \infty^+, \infty^-)} \) follows, using (11) with \( W \equiv 1 \), or by taking a look at [10, Proposition 2.1]. The assertion follows by (11). \( \square \)

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