$q$-poly-Bernoulli numbers and $q$-poly-Cauchy numbers with a parameter by Jackson’s integrals

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Abstract
We define $q$-poly-Bernoulli polynomials $B^{(k)}_{n,\rho,q}(z)$ with a parameter $\rho$, $q$-poly-Cauchy polynomials of the first kind $c^{(k)}_{n,\rho,q}(z)$ and of the second kind $\hat{c}^{(k)}_{n,\rho,q}(z)$ with a parameter $\rho$ by Jackson’s integrals, which generalize the previously known numbers and polynomials, including poly-Bernoulli numbers $B^{(k)}_n$ and the poly-Cauchy numbers of the first kind $c^{(k)}_n$ and of the second kind $\hat{c}^{(k)}_n$. We investigate their properties connected with usual Stirling numbers and weighted Stirling numbers. We also give the relations between generalized poly-Bernoulli polynomials and two kinds of generalized poly-Cauchy polynomials.

1 Introduction
Let $n$ and $k$ be integers with $n \geq 0$, and let $\rho$ be a real number parameter with $\rho \neq 0$. Let $q$ be a real number with $0 \leq q < 1$. Define $q$-poly-Bernoulli polynomials $B^{(k)}_{n,\rho,q}(z)$ with a parameter $\rho$ by

$$\frac{\rho}{1-e^{-\rho t}} \text{Li}_{k,q}(1-e^{-\rho t}) e^{-tz} = \sum_{n=0}^{\infty} B^{(k)}_{n,\rho,q}(z) \frac{t^n}{n!}, \tag{1}$$

where $\text{Li}_{k,q}(z)$ is the $q$-polylogarithm function defined by

$$\text{Li}_{k,q}(z) = \sum_{n=1}^{\infty} \frac{z^n}{[n]_q^k}.$$ 

Here,

$$[x]_q = \frac{1-q^x}{1-q}$$

$q$ was used as a parameter in [15] [16], but in this paper we use $\rho$ in order to avoid confusions with $q$-integral.
is the \( q \)-number with \([0]_q = 0\) (see e.g. [1] (10.2.3), [9]). Note that \( \lim_{q \to 1} [x]_q = x \).

Notice that

\[
\lim_{q \to 1} B_{n,\rho,q}^{(k)}(z) = B_{n,\rho}^{(k)}(z),
\]

which is the poly-Bernoulli polynomial with a \( \rho \) parameter ([5]), and

\[
\lim_{q \to 1} \text{Li}_{k,q}(z) = \text{Li}_k(z),
\]

which is the ordinary polylogarithm function, defined by

\[
\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}.
\]  

(2)

In addition, when \( z = 0 \), \( B_{n,\rho}^{(k)}(0) = B_{n,\rho}^{(k)} \) is the poly-Bernoulli number with a \( \rho \) parameter. When \( z = 0 \) and \( \rho = 1 \), \( B_{n,1}^{(k)}(0) = B_n^{(k)} \) is the poly-Bernoulli number ([12]) defined by

\[
\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},
\]  

(3)

The poly-Bernoulli numbers are extended to the poly-Bernoulli polynomials ([2, 7]) and to the special multi-poly-Bernoulli numbers ([8]).

Let \( n \) and \( k \) be integers with \( n \geq 0 \), and let \( \rho \) be a real number parameter with \( \rho \neq 0 \). Let \( q \) be a real number with \( 0 \leq q < 1 \). Define \( q \)-poly-Cauchy polynomials of the first kind \( c_{n,\rho,q}^{(k)}(z) \) with a parameter \( \rho \) by

\[
c_{n,\rho,q}^{(k)}(z) = \rho^n \int_0^1 \cdots \int_0^1 \left( \frac{x_1 \cdots x_k - z}{\rho} \right)_n \ d_q x_1 \cdots d_q x_k = \rho^n n! \int_0^1 \cdots \int_0^1 \left( \frac{(x_1 \cdots x_k - z)/\rho}{n} \right)_n \ d_q x_1 \cdots d_q x_k,
\]  

(4)

where \((x)_n = x(x - 1) \cdots (x - n + 1) \) \((n \geq 1)\) with \((x)_0 = 1\).

Jackson’s \( q \)-derivative with \( 0 < q < 1 \) (see e.g. [1] (10.2.3), [9]) is defined by

\[
D_q f = \frac{d_q f}{d_q x} = \frac{f(x) - f(qx)}{(1 - q)x}
\]

and Jackson’s \( q \)-integral ([1] (10.1.3), [9]) is defined by

\[
\int_0^x f(t)d_q t = (1 - q)x \sum_{n=0}^{\infty} f(q^n x) q^n.
\]
For example, when \( f(x) = x^m \) for some nonnegative integer \( m \),

\[
D_q f = \frac{x^m - q^m x^m}{(1 - q)x} = [m]_q x^{m-1}
\]

and

\[
\int_0^x t^m d_q t = (1 - q)x \sum_{n=0}^{\infty} q^{mn} x^n q^n = (1 - q)x^{m+1} \sum_{n=0}^{\infty} q^{n(m+1)} = \frac{x^{m+1}}{[m+1]_q}.
\]

Notice that

\[
\lim_{q \to 1} c_{n,\rho,q}^{(k)}(z) = c_{n,\rho}^{(k)}(z),
\]

which is the poly-Cauchy polynomial with a \( \rho \) parameter ([15]). In addition, when \( z = 0 \), \( c_{n,\rho}^{(k)}(0) = c_{n,\rho}^{(k)} \) is the poly-Cauchy number with a \( \rho \) parameter ([15]). When \( z = 0 \) and \( \rho = 1 \), \( c_{n,1}(0) = c_{n}^{(k)} \) is the poly-Cauchy number (of the first kind) ([14]), defined by the integral of the falling factorial:

\[
c_{n}^{(k)} = \int_0^1 \cdots \int_0^1 (x_1 \cdots x_k)_n dx_1 \cdots dx_k
\]

\[
= n! \int_0^1 \cdots \int_0^1 \binom{x_1 \cdots x_k}{n} dx_1 \cdots dx_k.
\]

If \( k = 1 \), then \( c_{n}^{(1)} = c_{n} \) is the classical Cauchy number ([6, 20]). The number \( c_{n}/n! \) is sometimes referred to as the Bernoulli number of the second kind ([3, 10, 21]). The poly-Cauchy numbers of the first kind \( c_{n}^{(k)} \) can be expressed in terms of the Stirling numbers of the first kind.

\[
c_{n}^{(k)} = \sum_{m=0}^{n} \frac{(-1)^{n-m} S_1(n, m)}{(m+1)^k} \quad (n \geq 0, \ k \geq 1)
\]

([14, Theorem 1]), where \( S_1(n, m) \) is the (unsigned) Stirling number of the first kind, see [6], determined by the rising factorial:

\[
x(x+1) \cdots (x+n-1) = \sum_{m=0}^{n} S_1(n, m)x^m.
\]

(5)
The generating function of the poly-Cauchy numbers $c_n^{(k)}$ is given by

$$L_{\text{if}}(\ln(1 + t)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!},$$

(6) [14, Theorem 2]), where $L_{\text{if}}(z)$ is called polylogarithm factorial function (or simply, polyfactorial function) defined by

$$L_{\text{if}}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m + 1)^k}.$$  

(7)

By this definition, $k$ is not restricted to a positive integer in $c_n^{(k)}$. Similarly, define the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ (14) by

$$\hat{c}_n^{(k)} = \int_0^1 \cdots \int_0^1 (-x_1 \cdots x_k)_n dx_1 \cdots dx_k = n! \int_0^1 \cdots \int_0^1 \left(-x_1 \cdots x_k \frac{1}{n}\right) dx_1 \cdots dx_k.$$ 

If $k = 1$, then $\hat{c}_n^{(1)} = \hat{c}_n$ is the classical Cauchy number of the second kind (6 [20]). The poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind.

$$\hat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{S_1(n, m)}{(m + 1)^k} \quad (n \geq 0, \ k \geq 1)$$

(14, Theorem 4). The generating function of the poly-Cauchy numbers of the second kind $\hat{c}_n^{(k)}$ is given by

$$L_{\text{if}}(-\ln(1 + t)) = \sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{t^n}{n!}$$

(8) [14, Theorem 5].

The poly-Cauchy numbers have been considered as analogues of poly-Bernoulli numbers $B_n^{(k)}$. The poly-Cauchy numbers (of both kinds) are extended to the poly-Cauchy polynomials (14), and to the poly-Cauchy numbers with a $q$ parameter (15). The corresponding poly-Bernoulli numbers with a $q$ parameter can be obtained in [5]. A different direction of generalizations of Cauchy numbers is about Hypergeometric Cauchy numbers (17). Arithmetical and combinatorial properties including sums of products have been studied (16 [18 19]).

In this paper, by using Jackson’s $q$-integrals, as essential generalizations of the previously known numbers, including poly-Bernoulli numbers $B_n^{(k)}$, the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\hat{c}_n^{(k)}$, we introduce the concept about
q-analogues or extensions of the poly-Bernoulli polynomials $B^{(k)}_{n,\rho,q}(z)$ with a parameter, the poly-Cauchy polynomials of the first kind $c^{(k)}_{n,\rho,q}$ and of the second kind $\hat{c}^{(k)}_{n,\rho,q}$ with a parameter. We investigate their properties connected with usual Stirling numbers and weighted Stirling numbers. We also give the relations between generalize poly-Bernoulli polynomials and two kinds of generalized poly-Cauchy polynomials.

2 q-poly-Bernoulli polynomials with a parameter

Carlitz [4] defined the weighted Stirling numbers of the first kind $S_1(n, m, x)$ and of the second kind $S_2(n, m, x)$ by

$$
(1 - t)^{-x}(-\ln(1 - t))^m m! = \sum_{n=0}^{\infty} S_1(n, m, x) t^n n!
$$

and

$$
e^{xt}(e^t - 1)^m m! = \sum_{n=0}^{\infty} S_2(n, m, x) t^n n!,
$$

respectively. Note that Carlitz [4] used the notation $R_1(n, m, x)$ and $R(n, m, x)$ instead of $S_1(n, m, x)$ and $S_2(n, m, x)$, respectively. When $x = 0$, $S_1(n, m, 0) = S_1(n, m)$ and $S_2(n, m, 0) = S_2(n, m)$ are the (unsigned) Stirling number of the first kind and the Stirling number of the second kind, respectively.

The q-poly-Bernoulli polynomials with a parameter $\rho$ can be expressed in terms of the weighted Stirling numbers of the second kind.

**Theorem 1** We have

$$
B^{(k)}_{n,\rho,q}(z) = \sum_{m=0}^{n} S_2\left(n, m, \frac{z}{\rho}\right) \left(-\rho\right)^{n-m} \frac{m!}{[m+1]^k_q}.
$$

**Proof.** From [1], and using [10], we have

$$
\sum_{n=0}^{\infty} B^{(k)}_{n,\rho,q}(z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-\rho)^{-m}}{[m+1]^k_q} e^{-\rho t} \left(e^{-\rho t} - 1\right)^m
$$

$$
= \sum_{m=0}^{\infty} \frac{(-\rho)^{-m} m!}{[m+1]^k_q} \sum_{n=m}^{\infty} S_2\left(n, m, \frac{z}{\rho}\right) \left(-\rho t\right)^n n!
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_2\left(n, m, \frac{z}{\rho}\right) \left(-\rho\right)^{n-m} \frac{m!}{[m+1]^k_q}\right) \frac{t^n}{n!}.
$$

Comparing the coefficients on both sides, we get the result. Notice that $S_2(n, m, x) = 0$ for $n < m$. ■
Corollary 1 For $q$-poly-Bernoulli numbers with a parameter $\rho$, we have

$$B_{n,\rho,q}^{(k)} = \sum_{m=0}^{n} S_{2}(n, m) \frac{(-\rho)^{n-m} m!}{m + 1}_{1}^{k}. \quad (12)$$

3 $q$-poly-Cauchy polynomials of the first kind with a parameter

The $q$-poly-Cauchy polynomials of the first kind $c_{n,\rho,q}^{(k)}(z)$ with a parameter can be expressed in terms of the weighted Stirling numbers of the first kind $S_{1}(n, m, x)$. In this expression, $k$ is not restricted to a positive integer.

Theorem 2 For integers $n$ and $k$ with $n \geq 0$, we have

$$c_{n,\rho,q}^{(k)}(z) = \sum_{m=0}^{n} S_{1}(n, m) (-\rho)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^{i}}{m - i + 1}_{q}^{k}$$

$$= \sum_{m=0}^{n} S_{1}(n, m, z) \frac{(-\rho)^{n-m}}{m + 1}_{q}^{k}. \quad (13)$$

Proof. From (4) and (5), we have

$$c_{n,\rho,q}^{(k)}(z) = \rho^{n} \sum_{m=0}^{n} (-1)^{n-m} S_{1}(n, m) \int_{0}^{1} \cdots \int_{0}^{1} \left( \frac{x_{1} \cdots x_{k} - z}{\rho} \right)^{m} d_{q}x_{1} \cdots d_{q}x_{k}$$

$$= \sum_{m=0}^{n} (-\rho)^{n-m} S_{1}(n, m) \sum_{i=0}^{m} \binom{m}{i} (-z)^{m-i} \int_{0}^{1} \cdots \int_{0}^{1} x_{1}^{i} \cdots x_{k}^{i} d_{q}x_{1} \cdots d_{q}x_{k}$$

$$= \sum_{m=0}^{n} (-\rho)^{n-m} S_{1}(n, m) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^{m-i}}{i + 1}_{q}^{k}$$

$$= \sum_{m=0}^{n} (-\rho)^{n-m} S_{1}(n, m) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^{i}}{m - i + 1}_{q}^{k}.$$  

By using the relation, see [4, Eq. (5.2)],

$$S_{1}(n, m, x) = \sum_{i=0}^{n} \binom{m + i}{i} x^{i} S_{1}(n, m + i).$$


we obtain

\[
c^{(k)}_{n,\rho,q}(z) = \sum_{i=0}^{n} \sum_{m=0}^{n} S_1(n, m) (-\rho)^{n-m} \binom{m}{i} \frac{(-z)^i}{[m-i+1]_q^k}
\]

\[
= \sum_{i=0}^{n} \sum_{m=0}^{n+i} S_1(n, m) (-\rho)^{n-m} \binom{m}{i} \frac{(-z)^i}{[m-i+1]_q^k}
\]

\[
= \sum_{i=0}^{n} \sum_{m=0}^{n} S_1(n, m+i) (-\rho)^{n-m-i} \binom{m+i}{i} \frac{(-z)^i}{[m+1]_q^k}
\]

\[
= \sum_{m=0}^{n} (-\rho)^{n-m} \frac{1}{[m+1]_q^k} \sum_{i=0}^{m} \binom{m+i}{i} \left(\frac{z}{\rho}\right)^i S_1(n, m+i)
\]

\[
= \sum_{m=0}^{n} S_1 \left(n, m, \frac{z}{\rho}\right) (-\rho)^{n-m} \frac{1}{[m+1]_q^k}.
\]

**Corollary 2** For \(q\)-poly-Cauchy numbers of the first kind with a parameter \(\rho\), we have

\[
c^{(k)}_{n,\rho,q} = \sum_{m=0}^{\infty} S_1(n, m) (-\rho)^{n-m} \frac{1}{[m+1]_q^k}.
\]

Define the \(q\)-polyfactorial functions \(L_{k,q}(z)\) by

\[
L_{k,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n![n+1]_q^k}.
\]

Notice that

\[
\lim_{q \to 1} L_{k,q}(z) = L_k(z),
\]

which is the ordinary polyfactorial function in (7). The generating function of \(c^{(k)}_{n,\rho,q}\) is given by the following theorem.

**Theorem 3** We have

\[
\frac{1}{(1+\rho t)^{\frac{z}{\rho}}} L_{k,q} \left(\frac{\ln(1+\rho t)}{\rho}\right) = \sum_{n=0}^{\infty} c^{(k)}_{n,\rho,q}(z) \frac{t^n}{n!}.
\]
Proof. By the first identity of Theorem 2

\[ \sum_{n=0}^{\infty} c_{n,\rho,q}^{(k)}(z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} S_1(n,m)(-\rho)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)_q} \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\ln(1 + \rho t)}{\rho} \right)^n \sum_{m=0}^{n} \binom{m}{i} \frac{(-z)^i}{(m-i+1)_q} \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{1}{(n+1)_q \rho^n} \binom{m}{i} \frac{(-z)^i}{(m-i+1)_q} \frac{t^n}{n!} \]

\[ = \frac{1}{(1 + \rho t)^z/\rho} \sum_{n=0}^{\infty} \frac{\ln(1 + \rho t)^n}{n!} \frac{1}{(n+1)_q} \frac{t^n}{n!} \]

\[ = \frac{1}{(1 + \rho t)^z/\rho} \text{Lif}_{k,q} \left( \frac{\ln(1 + \rho t)}{\rho} \right). \]

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} S_1(n,m)(-\rho)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)_q} \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\ln(1 + \rho t)}{\rho} \right)^n \sum_{m=0}^{n} \binom{m}{i} \frac{(-z)^i}{(m-i+1)_q} \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{1}{(n+1)_q \rho^n} \binom{m}{i} \frac{(-z)^i}{(m-i+1)_q} \frac{t^n}{n!} \]

\[ = \frac{1}{(1 + \rho t)^z/\rho} \sum_{n=0}^{\infty} \frac{\ln(1 + \rho t)^n}{n!} \frac{1}{(n+1)_q} \frac{t^n}{n!} \]

\[ = \frac{1}{(1 + \rho t)^z/\rho} \text{Lif}_{k,q} \left( \frac{\ln(1 + \rho t)}{\rho} \right). \]

4 \textbf{ $q$-poly-Cauchy polynomials of the second kind with a parameter}

Let $n$ and $k$ be integers with $n \geq 0$ and $k \geq 1$, and let $\rho$ be a real number parameter with $\rho \neq 0$. Define $q$-poly-Cauchy polynomials of the second kind \( \hat{c}_{n,\rho}(z) \) with a parameter $\rho$ by

\[ \hat{c}_{n,\rho}(z) = \rho^n \int_0^1 \cdots \int_0^1 \left( -x_1 \cdots x_k + z \right) \left( \frac{-x_1 \cdots x_k + z}{\rho} \right) d_q x_1 \cdots d_q x_k \]

\[ = \rho^n n! \int_0^1 \cdots \int_0^1 \left( -x_1 \cdots x_k + z \right) \left( \frac{-x_1 \cdots x_k + z}{\rho} \right) d_q x_1 \cdots d_q x_k. \]  \hspace{1cm} (16)

Notice that

\[ \lim_{q \to 1} \hat{c}_{n,\rho,q}(z) = \hat{c}_{n,\rho}(z), \]

which is the poly-Cauchy polynomial of the second kind with a $\rho$ parameter (15). In addition, when $z = 0$, $\hat{c}_{n,\rho}(0) = \hat{c}_{n,\rho}$ is the poly-Cauchy number of the second kind with
a \rho parameter \((12)\). When \(z = 0\) and \(\rho = 1\), \(\hat{c}_{n,1}(0) = c_{n}^{(k)}\) is the poly-Cauchy number \((14)\) given in \((8)\).

The \(q\)-poly-Cauchy polynomials of the first kind \(c_{n,\rho,q}^{(k)}(z)\) with a parameter can be expressed in terms of the weighted Stirling numbers of the first kind \(S_{1}(n, m, x)\). In this expression, \(k\) is not restricted to positive integers.

**Theorem 4** For integers \(n\) and \(k\) with \(n \geq 0\), we have

\[
\hat{c}_{n,\rho,q}^{(k)}(z) = (-1)^{n} \sum_{m=0}^{n} S_{1}(n, m) \rho^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^{i}}{[m-i+1]_{q}^{k}}.
\]

**Proof.** From \((4)\) and \((5)\), and using the relation \([4\text{ Eq. (5.2)}]\), similarly to the proof of Theorem 2 we obtain the result.

**Corollary 3** For \(q\)-poly-Cauchy numbers of the second kind with a parameter \(\rho\), we have

\[
\hat{c}_{n,\rho,q}^{(k)} = (-1)^{n} \sum_{m=0}^{\infty} S_{1}(n, m) \rho^{n-m} [m+1]_{q}^{k}.
\]

**Proof.** Putting \(z = 0\) in Theorem 4 we immediately get the result.

The generating function of \(c_{n,\rho,q}^{(k)}\) is given by using the \(q\)-polyfactorial functions \(L_{k,q}(z)\).

**Theorem 5** We have

\[
(1 + \rho t)^{z/\rho} L_{k,q} \left( -\frac{\ln(1 + \rho t)}{\rho} \right) = \sum_{n=0}^{\infty} c_{n,\rho,q}^{(k)}(z) \frac{t^{n}}{n!}.
\]

**Proof.** Similarly to the proof of Theorem 3 by the first identity of Theorem 2 we obtain the result.

5 Several relations of \(q\)-poly-Bernoulli polynomials and \(q\)-poly-Cauchy polynomials

There exist orthogonality and inverse relations for weighted Stirling numbers \((14)\). Namely, from the orthogonal relations

\[
\sum_{l=m}^{n} (-1)^{n-l} S_{2}(n, l, x) S_{1}(l, m, x) = \sum_{l=m}^{n} (-1)^{l-m} S_{1}(n, l, x) S_{2}(l, m, x) = \delta_{m,n},
\]
where \( \delta_{m,n} = 1 \) if \( m = n \); \( \delta_{m,n} = 0 \) otherwise, we obtain the inverse relations

\[
f_n = \sum_{m=0}^{n} (-1)^{n-m} S_1(n, m, x) g_m \quad \iff \quad g_n = \sum_{m=0}^{n} S_2(n, m, x) f_m.
\]

**Theorem 6** For \( q \)-poly-Bernoulli and \( q \)-poly-Cauchy polynomials with a parameter, we have

\[
\sum_{m=0}^{n} S_1\left(n, m, \frac{z}{\rho}\right) \rho^{-m} \binom{k}{m, \rho, q}(z) = \frac{n!}{\rho^n [n + 1]_q^k}, \quad (19)
\]

\[
\sum_{m=0}^{n} S_2\left(n, m, \frac{z}{\rho}\right) \rho^{-m} \hat{c}_{m, \rho, q}(z) = \frac{1}{\rho^n [n + 1]_q^k}, \quad (20)
\]

\[
\sum_{m=0}^{n} S_2\left(n, m, -\frac{z}{\rho}\right) \rho^{-m} \hat{c}_{m, \rho, q}(z) = \frac{(-1)^n}{\rho^n [n + 1]_q^k}. \quad (21)
\]

**Remark.** If \( q \to 1 \), then Theorem 6 is reduced to Theorem 3.2 in [5].

**Proof.** By Theorem 1 applying (18) with

\[
f_m = \frac{m!}{(-\rho)^m [m + 1]_q^k} \quad \text{and} \quad g_n = \frac{B_{n, \rho, q}(z)}{(-\rho)^n}
\]

and \( x \) is replaced by \( z/\rho \), we get the identity (19). Similarly, by Theorem 2 and Theorem 4 we have the identities (20) and (21), respectively.

There are relations between two kinds of \( q \)-poly-Cauchy polynomials with a parameter.

**Theorem 7** For \( n \geq 1 \) we have

\[
(-1)^n \frac{c_{n, \rho, q}(z)}{n!} = \sum_{m=1}^{n} \left(\frac{n-1}{m-1}\right) \frac{\hat{c}_{m, \rho, q}(z)}{m!}, \quad (22)
\]

\[
(-1)^n \frac{\hat{c}_{n, \rho, q}(z)}{n!} = \sum_{m=1}^{n} \left(\frac{n-1}{m-1}\right) \frac{\hat{c}_{m, \rho, q}(z)}{m!}. \quad (23)
\]

**Remark.** Since \( c_{n,1,q}(z) \to c_n(z) \) and \( \hat{c}_{n,1,q}(z) \to \hat{c}_n(z) \) as \( q \to 1 \), Theorem 7 is reduced to Theorem 4.2 in [11].
Proof. We shall prove identity (23). The identity (22) can be proven similarly. By the definition of \( \hat{c}_{n,q}^{(k)}(z) \)

\[
(-1)^n \frac{\hat{c}_{n,q}^{(k)}(z)}{n!} = (-\rho)^n \int_0^1 \cdots \int_0^1 \frac{(-x_1 \cdots x_k + z)/\rho}{n} d_q x_1 \cdots d_q x_k 
\]

\[
= \rho^n \int_0^1 \cdots \int_0^1 \frac{(x_1 \cdots x_k - z)/\rho + (n - 1)}{n} d_q x_1 \cdots d_q x_k .
\]

We use the well-known identity, see [22, p. 8],

\[
\binom{x + y}{n} = \sum_{l=0}^{n} \binom{x}{l} \binom{y}{n - l}
\]

as \( x = (x_1 \cdots x_k - z)/\rho \) and \( y = n - 1 \). Then

\[
(-1)^n \frac{\hat{c}_{n,q}^{(k)}(z)}{n!} = \rho^n \int_0^1 \cdots \int_0^1 \sum_{m=0}^{n} \frac{(-x_1 \cdots x_k - z)/\rho}{m} \binom{n - 1}{n - m} d_q x_1 \cdots d_q x_k
\]

\[
= \sum_{m=1}^{n} \binom{n - 1}{m - 1} \frac{\hat{c}_{m,q}^{(k)}(z)}{m!}.
\]

Note that \( \binom{n-1}{-1} = 0 \). 

\[\]

**Theorem 8** For any \( x \) and \( y \) we have

\[
B_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! \rho^{n-l} S_2 \left( n, m, \frac{x}{\rho} \right) S_2 \left( m, l, \frac{y}{\rho} \right) c_{l,q}^{(k)}(y),
\]

\[
B_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! \rho^{n-l} S_2 \left( n, m, \frac{x}{\rho} \right) S_2 \left( m, l, \frac{y}{\rho} \right) c_{l,q}^{(k)}(y),
\]

\[
c_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} \rho^{n-l} S_1 \left( n, m, \frac{x}{\rho} \right) S_1 \left( m, l, \frac{y}{\rho} \right) B_{l,q}^{(k)}(y),
\]

\[
\hat{c}_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{m!} \rho^{n-l} S_1 \left( n, m, \frac{x}{\rho} \right) S_1 \left( m, l, \frac{y}{\rho} \right) B_{l,q}^{(k)}(y).
\]

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Remark. If $\rho = 1$ and $q \to 1$, then Theorem 8 is reduced to Theorem 3.3 in [5]. If $\rho = 1$, $x = y$ and $q \to 1$, then Theorem 8 is reduced to Theorem 4.1 in [19]. A different generalization without Jackson’s integrals is discussed in [18].

Proof. We shall prove the first and the fourth identities. Others can be proven similarly. By (20) in Theorem 6, and using (11), we have

$$
\sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{n-m} m! \rho^{n-l} S_2 \left( n, m, \frac{x}{\rho} \right) S_2 \left( m, l, \frac{y}{\rho} \right) c_{l,\rho,q}^{(k)}(y) 
$$

$$
= \sum_{m=0}^{n} (-1)^{n-m} m! \rho^{n} S_2 \left( n, m, \frac{x}{\rho} \right) \sum_{l=0}^{m} S_2 \left( m, l, \frac{y}{\rho} \right) \rho^{-l} c_{l,\rho,q}^{(k)}(y) 
$$

$$
= \sum_{m=0}^{n} S_2 \left( n, m, \frac{x}{\rho} \right) (-\rho)^{n-m} m! \frac{1}{[m+1]^k} 

= B_{n,\rho,q}^{(k)}(x).
$$

By (19) in Theorem 6, and using (17), we have

$$
\sum_{l=0}^{n} \sum_{m=0}^{n} \frac{(-1)^n}{m!} \rho^{n-l} S_1 \left( n, m, -\frac{x}{\rho} \right) S_1 \left( m, l, \frac{y}{\rho} \right) B_{l,\rho,q}^{(k)}(y) 
$$

$$
= \sum_{m=0}^{n} \frac{(-1)^n}{m!} \rho^{n} S_1 \left( n, m, -\frac{x}{\rho} \right) \sum_{l=0}^{m} S_1 \left( m, l, \frac{y}{\rho} \right) \rho^{-l} B_{l,\rho,q}^{(k)}(y) 
$$

$$
= (-1)^n \sum_{m=0}^{n} S_1 \left( n, m, -\frac{x}{\rho} \right) \frac{\rho^{n-m}}{[m+1]^k} 

= c_{n,\rho,q}^{(k)}(x). 
$$

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