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Analytical and numerical relaxation results for models in soil mechanics

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Abstract A variational model of pressure-dependent plasticity employing a time-incremental setting is intro-
duced. A novel formulation of the dissipation potential allows one to construct the condensed energy in a
variationally consistent manner. For a one-dimensional model problem, an explicit expression for the quasi-
convex envelope can be found which turns out to be essentially independent of the original pressure-dependent
yield surface. The model problem can be extended to higher dimensions in an empirical manner. Numerical
simulation exhibits well-posed behavior showing mesh-independent results.

Keywords Pressure-dependent plasticity · Relaxation · Convex envelope · Microstructures

1 Introduction

Classical models for soils and granular materials are formulated in the framework of linearized plasticity based
on pressure-dependent flow rules. While this approach works very well in engineering applications, it cannot
explain complex structures seen in experiments conducted in granular media like clay, silt or sand, see in
particular [1, Figure 12]. The aim of this paper is both the investigation and the modeling of these complex
structures via relaxation methods, a well-established approach based on variational analysis. In a first step,
the analysis will be based on a time discretization scheme which involves a variational formulation for the
problems at given finite time increments. The corresponding variational problems will be studied concerning
their stability properties and the main features of minimizing microstructures, which are closely related to
the complexity of the experimental data. Moreover, the dependence of the main features of the minimizing

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structures on the given yield functions will be investigated. The goal is to employ the variational approach in order to describe the essential model features with as few parameters as possible, thus establishing this type of models for plasticity in soil mechanics.

Starting with [2–4], it was realized that experimentally observed microstructures may be explained based on energy minimization. This observation made it possible to use variational methods in order to understand complex material behavior. For example, the shape memory effect is related to an accommodation of macroscopic deformations on a microscopic scale through complex patterns. These patterns are determined through mathematical compatibility conditions which require in the nonlinear theory that two adjacent elastic deformation gradients $A$ and $B$ be rank-one connected in the sense that $A - B = a \otimes n$. The vector $n$ provides geometric information since it represents the orientation of planes that can connect the two states $A$ and $B$. Minimization of the original energy with respect to such compatible microstructures leads to the relaxed energy. This approach allows one on the one hand to formulate well-posed boundary-value problems that lend themselves to numerical solutions. The minimizing microstructures, on the other hand, will provide information on the expected deformation patterns. Applications of this approach specifically to elastoplastic materials can be found in [5,6].

Since the explicit construction of microstructures requires to determine rank-one connections, which is algebraically a challenging problem, it is not surprising that there are only a few examples with applications to elastic materials for which a closed form of $W^{qc}$ has been found [7–13]. In the linearized setting, the compatibility condition concerns symmetric stains $A$ and $B$ whose difference is symmetrized rank-one convex, i.e., $A - B = a \otimes b$ with $a \otimes b = \frac{1}{2}(a \otimes b + b \otimes a)$. Examples for relaxation results in this context can be found in [14–17]. Numerical relaxation schemes are employed in [18]. A combination of an analytical approximation of the relaxed energy density combined with a finite element simulation can be found in [19–21]. A novel algorithmic approach for the computation of the relaxed energy density with applications to problems in phase transformations and models in plasticity was recently proposed in [22–24], see also [25,26].

The paper is organized as follows: In Sect. 2, standard models of pressure-dependent plasticity are reviewed. In Sect. 3, we bring these models into a variational setting suitable for treatment within relaxation theory. In Sect. 4, this setting is specialized in order to obtain a one-dimensional model problem for which analytical results can be obtained. Section 5 contains the main results of this paper from the mathematical point of view. An explicit expression for the relaxed energy is derived in the one-dimensional case, i.e., the quasiconvex envelope of the energy density, which, in the given situation with one spatial variable, is equal to the convex envelope. In Sect. 6, the one-dimensional relaxed energy is, in a heuristic manner, extended to three dimensions, and in Sect. 7 numerical results are presented for both models, first for the relaxed energy derived for the one-dimensional problem, then for an extension of this energy to three dimensions, for which no rigorous mathematical results are available yet. More extensive results will be given in [27]. We will close with conclusion and outlook in Sect. 8.

## 2 Pressure-dependent plasticity

The most important class of materials exhibiting pressure-dependent plastic behavior are soils and granular media. They are usually heterogeneous mixtures of fluids (usually air and water) and particles (usually clay, silt, sand, and gravel) that have little to no cementation [28]. The shear strength is provided by friction and interlocking of the particles.

The understanding of deformations of solid materials is of key importance in applications. In contrast to elastic materials, which return to their original configuration upon unloading, elastoplastic materials may undergo a permanent change of their configuration if sufficiently large deviatoric strains occur in the material. On the other hand, it is frequently assumed that they may support arbitrary hydrostatic pressures. Usually this material behavior is modeled in a linearized setting via a convex set $\mathbb{K} \subset \mathbb{R}^{n \times n}_{\text{sym}}$ which describes the set of admissible stresses in the material which do not lead to permanent deformations. Only for stresses on $\partial \mathbb{K}$, permanent plastic deformation occurs and this deformation is formulated via a flow rule which determines the evolution of the plastic part of the total elastoplastic deformation. Consequently, the set $\mathbb{K}$ is an infinite cylinder

$$\mathbb{K} = \bigcup_{\pi \in \mathbb{R}} \pi \text{Id} \oplus K = \bigcup_{\pi \in \mathbb{R}} \{\pi \text{Id} + E, E \in K\}.$$  \hfill (2.1)
where $\text{Id}$ denotes the identity matrix, $\mathcal{K}$ is a compact subset in $\mathbb{R}_D^{n \times n}$ and

$$\mathbb{R}_D^{n \times n} = \{ E \in \mathbb{R}_D^{n \times n} : E = E^T, \text{tr}(E) = 0 \}$$

(2.2)
is the set of all symmetric and trace free matrices. Here and in the following we will use boldface symbols for vector-valued or matrix-valued objects.

Thus, the basic problem in the theory of linear elastoplasticity can be formulated as follows. Given a material body described in its reference configuration by a domain $\Omega \subset \mathbb{R}^n$, find a deformation $y : \Omega \rightarrow \mathbb{R}^n$ and a decomposition of the symmetrized displacement gradient $\varepsilon$ defined by

$$y(x) = \text{Id}(x) + u(x), \quad \varepsilon(x) = \frac{1}{2}(Du(x) + Du(x)^T)$$

(2.3)

into an elastic and a plastic part, $\varepsilon = \varepsilon_e + \varepsilon_p$, $\varepsilon_p \in \mathbb{R}_D^{n \times n}$, such that the following system of equations holds:

(i) constitutive equation $\sigma = C \varepsilon_e$,

(ii) balance of momentum $u_{tt} - \text{div}\sigma = f$ or the equilibrium condition $-\text{div}\sigma = f$,

(iii) stress constraint $\sigma \in \mathcal{K}$,

(iv) associated flow rule $(\xi - \sigma) : \dot{\varepsilon}_p \leq 0$ for every $\xi \in \mathcal{K}$.

It is important to note that the assumption (iv) is not realistic for materials in which frictional effects dominate as in clay, sand, gravel or other granular materials. Here, pressure dependence needs to be taken into account and this leads to a set of admissible stresses that cannot be modeled by an infinite cylinder; instead one is led to consider cones in the space of principle stresses, classical models being the Drucker–Prager and Mohr–Coulomb criteria [29]. To be specific, let $\pi = -1/3 \text{tr}\sigma$ denote the pressure, then the convex elastic domain is given as

$$\mathcal{K} = \bigcup_{\pi = \pi_{\min}}^{\pi_{\max}} \pi \text{Id} \oplus \mathcal{K}_\pi = \bigcup_{\pi = \pi_{\min}}^{\pi_{\max}} \{ \pi \text{Id} + E, E \in \mathcal{K}_\pi \},$$

(2.4)

where $\mathcal{K}_\pi$ is a compact subset in $\mathbb{R}_D^{n \times n}$ and $-\infty \leq \pi_{\min} < \pi_{\max} \leq \infty$. In order to comply with experimental observations, plastic deformation should still be volume preserving. For this purpose, the flow rule (iv) is then replaced by the non-associated flow rule and the full system of equations in its quasistatic form is given by

(i) constitutive equation $\sigma = C \varepsilon_e$,

(ii) the equilibrium condition $-\text{div}\sigma = f$,

(iii) stress constraint $\sigma \in \mathcal{K}$,

(iv') non-associated flow rule $(\xi - \sigma) : \dot{\varepsilon}_p \leq 0$ for every $\xi \in \mathcal{K}_\pi$.

Note that $\dot{\varepsilon}_p$ is now normal to the boundary of $\mathcal{K}_\pi$ as subset of $\mathbb{R}_D^{n \times n}$ but not normal to $\mathcal{K}$ as subset of $\mathbb{R}_D^{n \times n}$ (Fig. 1). This observation lies at the heart of the problems concerning a variational formulation to be discussed in the following.
3 Variational model

In order to be able to apply relaxation theory, we have to reformulate the setting given in the previous section. For this purpose, we will employ a variational approach for the description of inelastic processes, see [30–35].

Let us consider a physical system described by (sets of) external, i.e., controllable, state variables, in our case given by the total strain $\varepsilon$, and internal state variables $\mathbf{z}$. Here $\mathbf{z} = \{\varepsilon_p, p\}$, where $\varepsilon_p$ denotes plastic strain and $p \geq 0$ a hardening variable. We assume that the system behavior may be defined using only two scalar potentials: free energy $\psi(\varepsilon, \mathbf{z})$ and dissipation potential $\Delta(\dot{\varepsilon}, \mathbf{z}, \dot{\mathbf{z}})$.

The evolution of the internal variables is described by

$$
0 \in \frac{\partial \psi}{\partial \mathbf{z}} + \frac{\partial \Delta}{\partial \dot{\mathbf{z}}}. \tag{3.1}
$$

Note that Eq. (3.1) may be written as a stationarity condition of the minimization problem

$$
\dot{\mathbf{z}} = \arg \inf_{\dot{\mathbf{z}}} \left\{ \dot{\psi} + \Delta \right\}. \tag{3.2}
$$

We assume a free energy of the form

$$
\psi = K \frac{1}{2} \operatorname{tr}\varepsilon^2 + \mu \left| \operatorname{dev}\varepsilon - \varepsilon_p \right|^2 + \rho(\operatorname{tr}\varepsilon) p + \frac{\beta}{2} |\varepsilon_p|^2. \tag{3.3}
$$

Here, $\rho$ is an auxiliary internal variable, $K$ the bulk modulus, $\mu$ the shear modulus, and $\beta$ a hardening modulus. The function $\rho(\operatorname{tr}\varepsilon)$ governs pressure dependence. For constant $\rho(\operatorname{tr}\varepsilon)$, Eq. (3.3) corresponds to the free energy of an isotropic elastoplastic material with kinematic hardening. In this work, we will assume $\rho$ to be a concave function inside an interval $[\xi_{\text{min}}, \xi_{\text{max}}]$ and zero outside of it. The function $\rho(\operatorname{tr}\varepsilon)$ for a typical model, e.g., capped Drucker–Prager, see [28], can be seen in Fig. 2. The stress is given as

$$
\sigma = \frac{\partial \psi}{\partial \varepsilon} = (K \operatorname{tr}\varepsilon + \rho'(\operatorname{tr}\varepsilon) p) I + 2\mu \left( \operatorname{dev}\varepsilon - \varepsilon_p \right). \tag{3.4}
$$

Note that the pressure is given by the relation $\pi = -K \operatorname{tr}\varepsilon - \rho'(\operatorname{tr}\varepsilon) p$. Here, the second term may be interpreted as a consolidation pressure due to an inelastic volume change governed by the parameter $p$. A variationally more consistent approach would be to replace Eq. (3.3) by an energy of the form

$$
\psi_{\text{consol}} = K \frac{1}{2} (\operatorname{tr}\varepsilon - \theta_p)^2 + \mu \left| \operatorname{dev}\varepsilon - \varepsilon_p \right|^2 + \rho(\theta_p) p + \frac{\beta}{2} |\varepsilon_p|^2, \tag{3.5}
$$

where $\theta_p$ now represents the inelastic volume change explicitly. However, we would like to keep our formulation simple within the present work and plan to discuss this formulation in a subsequent paper.

The dissipation potential will be assumed as

$$
\Delta(\dot{\varepsilon}_p, \dot{p}) = \begin{cases} 
0 & \text{for } |\dot{\varepsilon}_p| \leq \dot{p} \\
\infty & \text{otherwise}
\end{cases}. \tag{3.6}
$$
This procedure differs from the standard one, as for example implemented in [31]. In the standard approach, Eq. (3.3) is replaced by

$$\psi_{alt} = \frac{K}{2} \text{tr}\varepsilon^2 + \mu |\text{dev}\varepsilon - \varepsilon_p|^2 + \frac{\beta}{2} |\varepsilon_p|^2,$$

(3.7)

and Eq. (3.6) by

$$\Delta_{alt}(\text{tr}\varepsilon, \dot{\varepsilon}_p) = \rho(\text{tr}\varepsilon)|\dot{\varepsilon}_p|.$$

(3.8)

However, the formulation in Eq. (3.8) would be problematic with respect to the variational formulation which does not allow the dissipation potential to depend on an external variable. Moreover, it would lead to complications when it comes to integrating it up in order to obtain a dissipation distance, as will be done later in the text. Minimizing trajectories connecting different states will be non-monotone leading to behavior that is questionable from a physical point of view.

Minimization in Eq. (3.2) gives, taking into account the expressions in Eqs. (3.3) and (3.6),

$$\text{dev}\sigma - \beta\varepsilon_p \in \rho(\text{tr}\varepsilon)\text{sign}\dot{\varepsilon}_p,$$

(3.9)

and

$$\dot{p} = |\dot{\varepsilon}_p|.$$

(3.10)

Note that Eq. (3.9) is equivalent to the yield condition

$$\Phi := |\text{dev}\sigma - \beta\varepsilon_p| - \rho(\text{tr}\varepsilon) \leq 0,$$

(3.11)

and the flow rule

$$\dot{\varepsilon}_p = \kappa (\text{dev}\sigma - \beta\varepsilon_p),$$

(3.12)

where $\kappa \geq 0$ is a consistency parameter satisfying the Kuhn–Tucker conditions

$$\kappa \geq 0, \quad \Phi \leq 0, \quad \kappa \Phi = 0.$$

(3.13)

Note that Eq. (3.10) defines $p$ as equivalent plastic strain.

Having established a variational scheme for pressure-dependent plasticity now, we will employ a time-incremental approach as proposed in [31,33] in order to investigate the evolution of material microstructures. For this purpose, we consider a time increment $\Delta t = [t_n, t_{n+1}]$. Given the external forces and boundary conditions at time $t_{n+1}$ and the values of the internal variables $z_n$ at time $t_n$, we seek the values $\varepsilon_{n+1}$ and $z_{n+1}$ at time $t_{n+1}$.

A time-incremental version of the dissipation-potential is given by the dissipation distance

$$D(\varepsilon_{p,0}, p_0, \varepsilon_{p,1}, p_1)$$

$$= \inf_{\varepsilon_{p}, p} \left\{ \int_0^1 \Delta(\dot{\varepsilon}_p(t), \dot{p}(t)) \, dt \mid \varepsilon_p(0) = \varepsilon_{p,0}, p(0) = p_0, \varepsilon_p(1) = \varepsilon_{p,1}, p(1) = p_1 \right\}$$

$$= \begin{cases} 0 & \text{for } |\varepsilon_{p,1} - \varepsilon_{p,0}| \leq p_1 - p_0, \\ \infty & \text{otherwise}. \end{cases}$$

(3.14)

The extremal principle Eq. (3.2) can now be replaced by

$$\{\varepsilon_{p,n+1}, p_{n+1}\} = \arg \inf_{\varepsilon_{p}, p} \left\{ \psi(\varepsilon, \varepsilon_p, p) + D(\varepsilon_{p,n}, p_n, \varepsilon_p, p) \right\}.$$  

(3.15)

Minimization with respect to $p$ gives immediately

$$p - p_n = |\varepsilon_p - \varepsilon_{p,n}|.$$  

(3.16)
Then, substitution of Eq. (3.16) into Eq. (3.15) and minimization with respect to \( \varepsilon_p \) yields
\[
2 \mu (\text{dev} \varepsilon - \varepsilon_p) - \beta \varepsilon_p \in \rho(\text{tr} \varepsilon)(\varepsilon_p - \varepsilon_{p,n}).
\] (3.17)
Solving for \( \varepsilon_p \) in Eq. (3.17) yields
\[
\varepsilon_p = \varepsilon_{p,n} + \frac{1}{2 \mu + \beta} \left[ 2 \mu |\text{dev} \varepsilon - \varepsilon_{p,n}| - \rho(\text{tr} \varepsilon) \right]_+ \text{sign}(\text{dev} \varepsilon - \varepsilon_{p,n}),
\] (3.18)
where \([x]_+ = \max\{0, x\}\). This allows to introduce the condensed energy, see [31,33], via
\[
\psi_{\text{cond}}(\varepsilon, \varepsilon_{p,n}, p_n) = \inf_{\varepsilon_p} \{ \psi(\varepsilon, \varepsilon_p, p) + D(\varepsilon_{p,n}, p, \varepsilon_p, p) \}. 
\] (3.19)
Substitution of Eqs. (3.16) and (3.18) into Eq. (3.19) gives, up to an irrelevant constant term of the form \( \frac{\beta}{2} |\varepsilon_p|^2 \),
\[
\psi_{\text{cond}}(\varepsilon, \varepsilon_{p,n}, p_n) = \frac{K}{2} \text{tr} \varepsilon^2 + \mu \left|\text{dev} \varepsilon - \varepsilon_{p,n}\right|^2 + \rho(\text{tr} \varepsilon)p_n - \frac{1}{2} \frac{1}{2 \mu + \beta} \left[ 2 \mu |\text{dev} \varepsilon - \varepsilon_{p,n}| - \rho(\text{tr} \varepsilon) \right]_+^2.
\] (3.20)
The condensed energy can now be considered as formally elastic and dependent on the parameters \( \varepsilon_{p,n}, p_n \), governing the behavior during the time increment \( \Delta t = [t_n, t_{n+1}] \). At the end of the time step, the internal variables can be updated using Eqs. (3.16) and (3.18). This way, the inelastic behavior can be approximated by solving a sequence of elastic problems.

4 A one-dimensional model problem

It is well known that quasiconvex envelopes can be calculated analytically only in rare special cases. In order to circumvent this difficulty, we restrict ourselves to a one-dimensional problem for which the quasiconvex envelope coincides with the convex envelope. We consider a state of specific deformation corresponding to a slender bar of material under uniform tension or compression and shear. We assume the bar to be supported by stiff fibers in a way that prevents bending deformations. The nonvanishing displacements and strains are in Cartesian coordinates
\[
\begin{align*}
  u_x &= u(x), \quad u_y = v(x) + y u'(x), \quad u_z = z u'(x) \\
  \varepsilon_{xx} &= \varepsilon_{yy} = \varepsilon_{zz} = u', \quad \varepsilon_{yx} = \varepsilon_{xy} = \frac{1}{2} v',
\end{align*}
\] (4.1)
where we neglected terms of the form \( y u'' \) and \( z u'' \), which are small due to the slenderness of the bar. For the plastic strains, we assume that only the components \( \varepsilon_p := \varepsilon_{pyx} = \varepsilon_{pxy} \) are different from zero. The free energy assumes the form
\[
\psi = \frac{K}{2} \varepsilon_{xx}^2 + \mu \left( (\varepsilon_{yx} - \varepsilon_{pyx})^2 + (\varepsilon_{xy} - \varepsilon_{pxy})^2 \right) + \rho(\varepsilon_{xx}) p + \beta \varepsilon_p^2.
\] (4.3)
Substitution of Eq. (4.2) into Eq. (4.3) gives
\[
\psi = \frac{K}{2} \left( u' \right)^2 + 2 \mu \left( \frac{1}{2} v' - \varepsilon_p \right)^2 + \rho \left( u' \right) p + \beta \varepsilon_p^2.
\] (4.4)
As before, minimization with respect to \( \varepsilon_p \) and \( p \) once again leads to the condensed energy

\[
\psi_{\text{cond}}(u', v', \varepsilon_{p,n}, p_n) = \frac{K}{2} (u')^2 + 2\mu \left( \frac{1}{2} v' - \varepsilon_{p,n} \right)^2 + \rho \left( u' \right) p_n
\]

\[
- \frac{1}{2} \frac{1}{2\mu + \beta} \left[ 2\mu \left| \frac{1}{2} v' - \varepsilon_{p,n} \right| - \rho \left( u' \right) \right]^2.
\]

(4.5)

For the remainder of the paper, we will focus on the case \( p_n = 0 \). This will simplify the mathematical treatment significantly and will allow to keep the derivations stringent and instructive. In mechanical terms, this means that we will describe the initiation of microstructures rather than their evolution. However, the general case will remain subject of future investigations.

It is convenient for the mathematical treatment following in the subsequent section to introduce a dimensionless formulation. For this purpose, we define modified variables and quantities by

\[
y_1 = \sqrt{\frac{K}{2\mu}} u', \quad y_2 = \frac{1}{2} v', \quad r(y_1) = \frac{1}{2\mu} \rho \left( u' \right), \quad b = \frac{\beta}{2\mu},
\]

(4.6)

giving, using \( p_n = 0 \),

\[
\psi_{\text{cond}}(y_1, y_2, \varepsilon_{p,n}, p_n) = 2\mu \left[ \frac{1}{2} y_1^2 + \frac{1}{2} (y_2 - \varepsilon_{p,n})^2 - \frac{1}{2} \frac{1}{b+1} \left[ |y_2 - \varepsilon_{p,n} - r(y_1)| \right]^2 \right].
\]

(4.7)

Finally, we may drop the multiplicative constant \( 2\mu \) and write \( z_n = \varepsilon_{p,n} \). Since \( \psi_{\text{cond}} \) is the free energy density in the variational functional that needs to be minimized in the first time step, it is crucial to understand the convexity properties of \( \psi_{\text{cond}} \) in order to employ the direct method in the calculus of variations. It turns out that the concavity of \( r \) leads to a lack of convexity of \( \psi_{\text{cond}} \) and the approach via relaxation [36] requires one to characterize the quasiconvex envelope of \( \psi_{\text{cond}} \), which, in the scalar case, reduces to the convex envelope. This task is accomplished in the next section.

5 Characterization of the relaxed energy for the one-dimensional model

For any function \( f : \mathbb{R}^2 \to \mathbb{R} \), we denote the convex envelope of \( f \) by \( f_c \) and, if \( f \) is differentiable at \( y \in \mathbb{R}^2 \), the linear Taylor polynomial of \( f \) about the point \( y \) by \( T_y f \). Fix \( b > 0 \) and \( y_{\min}, y_{\max} \in \mathbb{R} \) with \( y_{\min} < y_{\max} \). Define the set of admissible dissipation functions by

\[
\mathcal{R} := \{ r \in C(\mathbb{R}, [0, \infty)) \mid \text{supp}(r) = [y_{\min}, y_{\max}], r|_{[y_{\min}, y_{\max}]} \text{ concave} \}.
\]

(5.1)

Fix \( y_{\text{mid}} := (y_{\min} + y_{\max})/2 \) and define the piecewise affine function \( r_0 \in \mathcal{R} \) by

\[
r_0(y_1) = \sqrt{b} \cdot \left( \frac{y_{\max} - y_{\min}}{2} - |y_1 - y_{\text{mid}}| \right) \cdot \chi_{[y_{\min}, y_{\max}]}(y_1),
\]

(5.2)

see Fig. 2 for a sketch with \( b = 0.095 \). Note that the slope of \( r_0 \) is only related to the hardening parameter \( b \) and, depending on the relative size of \( r_{\max} = \max r \) and \( b \), the two curves may have intersections or not. Our theory covers the case of small values of \( b \) for which the two curves do not intersect. For \( r \in \mathcal{R} \), the condensed energy is given by

\[
f^{(r)}_{z_n}(y) = \frac{1}{2} (y_1^2 + (y_2 - z_n)^2) - \frac{(|y_2 - z_n| - r(y_1))^2}{2(b+1)}.
\]

(5.3)
Since Eq. (5.6) implies for any \( f \) the following important property. For a lower semicontinuous function \( f \) the inequality \( f(Y) \) sets \( \bar{Y} \) is equivalent to \( f(\bar{Y}) \), respectively, and the set \( \bar{Y} \) is the complement of these three sets in \( \mathbb{R}^2 \).

![Diagram](image_url)

**Fig. 3** Definition of the condensed energy for the model problem. The set \( Y_0 \) in Eq. (5.4) is enclosed by the blue curves \((\pm r)\), the sets \( Y_{\pm} \) correspond to the sets above and below \( Y_0 \), respectively, and the set \( \bar{Y} \) is the complement of these three sets in \( \mathbb{R}^2 \).

Since the energy depends only on \( y_2 - z_n \), it is sufficient to calculate the convex envelope \( f_c^{(r)} \) of \( f^{(r)} := f_0^{(r)} \) with \( z_n = 0 \); the convex envelope of \( f_c^{(r)}(y_1, y_2) \) is given by \( f_c^{(r)}(y_1, y_2 - z_n) \). If the dissipation function is uniquely defined from the context, we simply write \( f \) instead of \( f^{(r)} \). Define the disjoint sets

\[
\begin{align*}
\bar{Y} &= \{ y \in \mathbb{R}^2 : y_1 \leq y_{\text{min}} \text{ or } y_1 \geq y_{\text{max}} \}, \\
Y_0 &= \{ y \in \mathbb{R}^2 : y_1 \in (y_{\text{min}}, y_{\text{max}}), |y_2| < r(y_1) \}, \\
Y_{\pm} &= \{ y \in \mathbb{R}^2 : y_1 \in (y_{\text{min}}, y_{\text{max}}), \pm y_2 \geq r(y_1) \},
\end{align*}
\]

whose union is \( \mathbb{R}^2 \), see Fig. 3. These sets describe the regions in which the condensed energy \( f^{(r)} \) is given by different algebraic expressions, depending on the sign of \( |y_2| - r(y_1) \). The first lemma describes a monotonicity property of the map \( r \mapsto f^{(r)} \).

**Lemma 1** Consider two dissipation functions \( r, \tilde{r} \in \mathcal{R} \). If \( y_1 \in \mathbb{R} \) and \( r(y_1) \leq \tilde{r}(y_1) \), then for all \( y_2 \in \mathbb{R} \) the inequality \( f^{(r)}(y_1, y_2) \leq f^{(\tilde{r})}(y_1, y_2) \) holds. In particular, if \( r \leq \tilde{r} \), then \( f^{(r)} \leq f^{(\tilde{r})} \) and \( g : \mathbb{R}^2 \rightarrow \mathbb{R}, g(y) = \frac{1}{2} y_1^2 + \frac{1}{2} \frac{r}{\bar{r}} y_2^2 \) is a convex lower bound of \( f^{(r)} \), which coincides on \( \bar{Y} \) with \( f^{(\tilde{r})} \).

**Proof** For any \( y \in \mathbb{R}^2 \) with \( r(y_1) \leq \tilde{r}(y_1) \), it follows that \( |y_2| \geq |y_2| - r(y_1) \geq |y_2| - \tilde{r}(y_1) \). Therefore, we have \( y_2^2 \geq (|y_2| - r(y_1))^2 \geq (|y_2| - \tilde{r}(y_1))^2 \) and the inequality

\[
g(y) \leq f^{(r)}(y_1, y_2) \leq f^{(\tilde{r})}(y_1, y_2) \tag{5.5}
\]

is obtained from Eq. (5.3). □

Since Lemma 1 establishes a lower bound on \( f_c \), we may use the results in [37, Theorem 2.1] which state the following important property. For a lower semicontinuous function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \), which satisfies the growth condition \( \lim_{|y| \rightarrow \infty} f(y)/|y| = \infty \), the convex envelope of \( f \) is a real-valued convex function and for any \( y \in \mathbb{R}^d \) the value \( f_c(y) \) is a convex combination of \( q \) function values \( f(y^{(i)}) \), \( 1 \leq i \leq q \leq d + 1 \). Moreover, the subgradient \( \partial f_c(y) \) in the sense of convex analysis, see [38], is nonempty and for any \( v \in \partial f_c(y) \) the hyperplane \( z \mapsto f(y) + \langle v, z - y \rangle \) is a supporting hyperplane at all the points \( y^{(i)} \), that is,

\[
f(y^{(i)}) - \langle v, y^{(i)} \rangle = f(y) - \langle v, y \rangle, \quad i \in \{1, \ldots, q\} \tag{5.6}
\]

and

\[
\forall \tilde{y} \in \mathbb{R}^d : f(\tilde{y}) - \langle v, \tilde{y} \rangle \geq f(y^{(i)}) - \langle v, y^{(i)} \rangle, \quad i \in \{1, \ldots, q\}. \tag{5.7}
\]

Since Eq. (5.6) implies \( f(y^{(i)}) - \langle v, y^{(i)} \rangle = f(y^{(j)}) - \langle v, y^{(j)} \rangle \) for all \( i, j \in \{1, \ldots, q\} \) and Eq. (5.7) is equivalent to \( v \in \partial f(y^{(i)}) \), a natural approach to calculate the convex envelope of a continuous function
Fig. 4 Characterization of the convex envelope (thick, dashed) of a nonconvex function $f$ (thick). The value $f_c(y)$ is calculated from two points $y^{(1)}$ and $y^{(2)}$ on the graph of $f$ and the derivatives of $f_c$ at the three points $y^{(1)}, y^{(2)}$ and $y$ coincide. The convex hull coincides on $[y^{(1)}, y^{(2)}]$ with the tangent to $f$ at both the points $y^{(i)}$.

$f : \mathbb{R}^d \to \mathbb{R}$ is the strategy to find pairwise distinct points $y^{(1)}, \ldots, y^{(q)} \in \mathbb{R}^d$ with $q \leq d + 1$ that satisfy the subgradient condition

$$\exists v \in \bigcap_{i=1}^{q} \partial f(y^{(i)}): \forall i, j \in \{1, \ldots, q\}: f(y^{(i)}) - \langle v, y^{(i)} \rangle = f(y^{(j)}) - \langle v, y^{(j)} \rangle$$

(5.8)

and to define, for all $y \in \text{Conv}([y^{(1)}, \ldots, y^{(q)}])$, the convex hull of the points $y^{(1)}, \ldots, y^{(q)}$, a candidate for the convex envelope of $f$ by

$$y = \sum_{i=1}^{q} \lambda_i y^{(i)}, \quad f_c(y) = \sum_{i=1}^{q} \lambda_i f(y^{(i)}), \quad \lambda_i \in (0, 1], \quad \sum_{i=1}^{q} \lambda_i = 1. \quad (5.9)$$

It is useful to note that the representation (5.9) is related to the tangent plane of $f$ in one of the points $y^{(i)}$ in case that $f$ is differentiable at $y^{(i)}$. In fact, if for some direction $w \in \mathbb{R}^d$ and some $y \in \mathbb{R}^d$ the directional derivative $D_w f(y)$ exists, then $v \in \partial f(y)$ implies $\langle v, w \rangle = D_w f(y)$. If $f$ is even differentiable in $y$, then $\partial f(y) = \{\nabla f(y)\}$. This leads to the observation that, if $y^{(1)}, \ldots, y^{(q)} \in \mathbb{R}^d$ is a solution of Eq. (5.8) and if $f$ is differentiable in some $y^{(i)}$, then for all $y \in \text{Conv}([y^{(1)}, \ldots, y^{(q)}])$ the function $f_c$ is given by

$$f_c(y) = (T_{y^{(i)}} f)(y) = f(y^{(i)}) + \langle \nabla f(y^{(i)}), y - y^{(i)} \rangle, \quad (5.10)$$

see Fig. 4. The calculation of the subgradient involves an evaluation of the values of $f$ on the whole space, and the solution of Eq. (5.8) is in general challenging and the necessary calculations are presented in “Appendix B.”

**Theorem 2** Recall the definition of $\widetilde{Y}$ in (5.4) and define the disjoint sets

$$Y_1 = \left\{ y \in \mathbb{R}^2, \ |y_2| < r_0(y_1) \right\}, \quad (5.11)$$

$$Y_2 = \left\{ y \in (y_{\text{min}}, y_{\text{max}}) \times \mathbb{R}, \ r_0(y_1) \leq |y_2| < \frac{b + 1}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} - \frac{r_0(y_1)}{b} \right\}, \quad (5.12)$$

$$Y_3 = \left\{ y \in (y_{\text{min}}, y_{\text{max}}) \times \mathbb{R}, \ \frac{b + 1}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} - \frac{r_0(y_1)}{b} \leq |y_2| \leq \frac{b + 1}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} \right\}, \quad (5.13)$$

$$Y_4 = \left\{ y \in (y_{\text{min}}, y_{\text{max}}) \times \mathbb{R}, \ \frac{b + 1}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} < |y_2| \right\}, \quad (5.14)$$

(5.9)
Fig. 5 Construction of the relaxed energy. The sides of the rhombus $Y_1$ have slope $\pm \sqrt{b}$ and correspond to the dashed dissipation function in Fig. 2.

as shown in the right panel in Fig. 5. Then, the convex envelope $f_c$ of $f$ is given by

$$f_c(y) = \begin{cases} 
\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 & \text{on } \tilde{Y}, \\
\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 & \text{on } Y_1, \\
\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - \frac{((y_2) - r_0(y_1))^2}{2(b+1)} & \text{on } Y_2, \\
\frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - \frac{((y_2) - r_0(y_1))^2}{2(b+1)} & \text{on } Y_2, \\
- \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{((y_2) - r_0(y_1))^2}{2(b+1)} & \text{on } Y_3, \\
\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{((y_2) - r_0(y_1))^2}{2(b+1)} & \text{on } Y_4.
\end{cases}$$

(5.15)

The proof is presented in detail in Appendix B and is based on the verification of the geometry for the calculation of $f_c$ in the left panel in Fig. 5 which formalizes the intuitive picture for the convex hull of a function in Fig. 4. In fact, $f_c = f$ on $Y_1$ and for $y \in Y_2$ the convex hull $f_c(y)$ is obtained as an average of two function values where the red line segments intersect the green and the blue line, respectively. Finally, for $y \in Y_3$ the value of $f_c(y)$ is an average of three points, namely of the function values in the three corners of the triangle $Y_3$. The convex hull coincides with the tangent plane to $f$ at $y_{1+}^{+}$ as indicated in (5.10).

**Corollary 3** For all $r \in \mathcal{R}$ with $\sqrt{b} \cdot \frac{y_{\max} - y_{\min}}{2} \leq r \left( \frac{y_{\min} + y_{\max}}{2} \right)$ the equality

$$f_c^{(r)} = f_c^{(r_0)}$$

holds.

**Proof** In the proof of the previous theorem, we already showed the inequality

$$f_c^{(r)} \leq f^{(r_0)} \leq f^{(r)},$$

which implies the claimed equality by the monotonicity in taking convex envelopes. \qed
Remark 4 More generally, one can even show that without any restrictions on \( r \) the convex envelope of \( f^{(r)} \) does not change while replacing \( r \) by \( \min[r, r_0] \). Corollary 3 is a special case of this statement, since Eq. (B1) implies \( \min[r, r_0] = r_0 \) by concavity of \( r \) on \( (y_{\min}, y_{\max}) \).

6 Extension of the results to three dimensions

The convex envelope in the one-dimensional case for the density \( f(y_1, y_2, \varepsilon_{p,n}, p_n) \) given in Theorem 2 may be generalized in a natural way to three dimensions by employing a substitution inspired by the transformation of variables defined in Eq. (4.6). In this sense, we define a formally relaxed energy as

\[
\psi_{\text{rel}}(\varepsilon, \varepsilon_{p,n}, p_n) = f\left(\sqrt{\frac{K}{2\mu}}(\text{tr}\varepsilon, |\text{dev}\varepsilon|, \varepsilon_{p,n}, p_n)\right).
\]  

(6.1)

Observe that, when applying the same substitution to the one-dimensional condensed energy given in Eq. (4.7), we recover the original three-dimensional condensed energy given in Eq. (3.20).

The relations of the relaxed energies according to Fig. 5, and their derivatives with respect to the strain tensor, i.e., the corresponding relaxed stress tensor, are given with

\[
\rho_0(\varepsilon) = \sqrt{\beta K}\left(\frac{\text{tr}\varepsilon_{\text{max}} - \text{tr}\varepsilon_{\text{min}}}{2} - \left|\text{tr}\varepsilon - \frac{\text{tr}\varepsilon_{\text{max}} + \text{tr}\varepsilon_{\text{min}}}{2}\right|\right)
\]  

(6.2)

on the domains \( Y_i, i = 1, \ldots, 4 \) by the following expressions. Domain \( Y_1 \):

\[
\psi_{1,\text{rel}} = \frac{K}{2}(\text{tr}\varepsilon)^2 + \mu |\text{dev}\varepsilon|^2, \quad \sigma_{1,\text{rel}} = K(\text{tr}\varepsilon) I + 2\mu \text{dev}\varepsilon.
\]

(6.3)

Domain \( Y_2 \):

\[
\psi_{2,\text{rel}} = \frac{K}{2}(\text{tr}\varepsilon)^2 + \mu |\text{dev}\varepsilon|^2 - \frac{4\mu^2}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right)^2,
\]

(6.4)

\[
\sigma_{2,\text{rel}} = K(\text{tr}\varepsilon) I + 2\mu \text{dev}\varepsilon - \frac{4\mu^2}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right)\frac{\text{dev}\varepsilon}{|\text{dev}\varepsilon|} - \frac{2\mu}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right) I.
\]

(6.5)

Domain \( Y_3 \):

\[
\psi_{3,\text{rel}} = \frac{K}{2}(\text{tr}\varepsilon)^2 + \mu |\text{dev}\varepsilon|^2 - \frac{4\mu^2}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right)^2
\]

\[
- \frac{2\mu}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right) - \frac{2\mu + \beta}{2\mu} \sqrt{\frac{K}{\beta}} \frac{\text{tr}\varepsilon_{\text{max}} - \text{tr}\varepsilon_{\text{min}}}{2}
\]

(6.6)

\[
\sigma_{3,\text{rel}} = K(\text{tr}\varepsilon) I + 2\mu \text{dev}\varepsilon - \frac{4\mu^2}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right)\frac{\text{dev}\varepsilon}{|\text{dev}\varepsilon|}
\]

\[
- \frac{2\mu}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right) I
\]

\[
- \frac{2\mu}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right) - \frac{2\mu + \beta}{2\mu} \sqrt{\frac{K}{\beta}} \frac{\text{tr}\varepsilon_{\text{max}} - \text{tr}\varepsilon_{\text{min}}}{2} \right) I
\]

\[
- \frac{2\mu}{2(2\mu + \beta)}\left(|\text{dev}\varepsilon| - \frac{1}{2\mu}\rho_0(\text{tr}\varepsilon)\right) - \frac{2\mu + \beta}{2\mu} \sqrt{\frac{K}{\beta}} \frac{\text{tr}\varepsilon_{\text{max}} - \text{tr}\varepsilon_{\text{min}}}{2} \right) \frac{\text{dev}\varepsilon}{|\text{dev}\varepsilon|}.
\]

(6.7)
yield surface of a modified Drucker–Prager model employing a piecewise quadratic function given by

\[ \psi_{4,\text{rel}} = \frac{K}{2} (\text{tr} \varepsilon)^2 + \frac{2\mu\beta}{2(2\mu + \beta)} |\text{dev} \varepsilon|^2 + \frac{K}{2} \left( \text{tr} \varepsilon - \text{tr} \varepsilon_{\text{min}} \right) \left( \text{tr} \varepsilon_{\text{max}} - \text{tr} \varepsilon \right). \]  

(6.8)

\[ \sigma_{4,\text{rel}} = K (\text{tr} \varepsilon) I + \frac{2\mu\beta}{2\mu + \beta} \text{dev} \varepsilon + K \left( \frac{\text{tr} \varepsilon_{\text{max}} + \text{tr} \varepsilon_{\text{min}}}{2} - \text{tr} \varepsilon \right) I. \]  

(6.9)

7 Numerical results

7.1 The one-dimensional model problem

We make use of the dimensionless formulation introduced in Eq. (4.6). The function \( r(y_1) \) is selected to fit the yield surface of a modified Drucker–Prager model employing a piecewise quadratic function given by

\[ r(y_1) = r_{\text{max}} \begin{cases} 
1 - \frac{(y_1 - y_0)^2}{(y_0 - y_{\text{min}})^2}, & \text{for } y_{\text{min}} \leq y_1 \leq y_0, \\
1 - \frac{(y_1 - y_0)^2}{(y_{\text{max}} - y_0)^2}, & \text{for } y_0 \leq y_1 \leq y_{\text{max}}, \\
0, & \text{otherwise},
\end{cases} \]  

(7.1)

where \( y_{\text{min}} = -0.058, y_{\text{max}} = 0.00107, y_0 = -0.0385 \) and \( r_{\text{max}} = 0.016 \), see Fig. 6. We choose a hardening parameter \( b = 0.095 \) to insure that we are within the regime of small values of \( b \) treated in Sect. 5.

We model a one-dimensional specimen of length \( L = 1 \), represented by a domain \( \Omega = (0, 1) \), employing a discretization using \( n = 80 \) standard linear finite elements. The specimen is fixed at the left-hand side, \( u(0) = v(0) = 0 \), and subjected to given displacements \( u(L) = u_{\text{ext}} \) and \( v(L) = v_{\text{ext}} \) in \( x \)- and \( y \)-direction, respectively, at the right-hand side. The affine solution is given by \( x \mapsto (u(x), v(x)) = x(u_{\text{ext}}, v_{\text{ext}}) \) with constant derivative \( (u_{\text{ext}}, v_{\text{ext}}) \). We fix \( u_{\text{ext}} \) throughout our calculations and vary \( v_{\text{ext}} \) exploring four different regions in the phase diagram Fig. 5 which correspond to the four panels in Fig. 7: \( (u_{\text{ext}}, v_{\text{ext}}) \) in (a) \( Y_1 \), (b) \( Y_2 \cap \{ y_2 \leq r(y_1) \} \), (c) \( Y_2 \setminus \{ y_2 \leq r(y_1) \} \) and (d) \( Y_3 \), respectively.

The total energies are then functions of the vectors of nodal displacements \( \mathbf{u} = \{ u_1, \ldots, u_{n+1} \} \), \( \mathbf{v} = \{ v_1, \ldots, v_{n+1} \} \) with \( u_1 = v_1 = 0 \) and \( u_{n+1} = u_{\text{ext}}, v_{n+1} = v_{\text{ext}} \), given as

\[ W_{\text{cond,rel}}(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \psi_{\text{cond,rel}} \left( \frac{u_{j+1} - u_j}{L/n}, \frac{v_{j+1} - v_j}{L/n} \right) \frac{L}{n}, \]  

(7.2)

for the condensed and the relaxed energy, respectively. The minimization is performed using the Mathematica function \textit{FindMinimum}, which employs a gradient-based local search. Note that, for the relaxed energy, the minimum is obtained for the affine displacements \( u_i = \frac{1}{n} u_{\text{ext}}, v_i = \frac{1}{n} v_{\text{ext}} \).
We add a random perturbation of amplitude $\alpha$ to the exact solution given above as an initial guess for the minimization procedure. Hence, let $\text{rand}(i)$ be a random function possessing a uniform distribution within the interval $[-1, 1]$. Then, the initial guess is for $i = 2, \ldots, n$ given by
Fig. 8 A plate with a hole is fixed on the left and subject to a given displacement in x-direction on the right-hand side. The figure to the left shows the coarse mesh with 1793 elements and that to the right the fine mesh with 4917 elements. The elements employed for the stress plots are marked in red and cyan (color figure online)

\[ u_i^{\text{ini}} = \frac{i-1}{n}u_{\text{ext}} + \frac{L}{n}\text{rand}(i) , \]

\[ v_i^{\text{ini}} = \frac{i-1}{n}v_{\text{ext}} + \frac{L}{n}\text{rand}(n+i) . \]

The results are depicted in Fig. 7. The numerical simulations reflect the predictions based on the phase diagram in Fig. 5. We expect the calculations employing the condensed energy to return microstructures consisting of up to three distinct points in \(y_1-y_2\)-space. These microstructures will approximate the actual minimum more or less successfully depending on the quality of the initial guess. This behavior is precisely reflected by the points in red color corresponding to the condensed energy in Fig. 7. The calculations employing the relaxed energy, on the other hand, are expected to return the actual minimum up to round-off errors, which is precisely what is found. Due to the fact that the graph of the relaxed energy contains affine (flat) areas, the variational problem with the relaxed energy does have a unique affine minimizer, the displacement gradients are not expected to take only values in the local minima of the nonconvex free energy leading to the formation of microstructure, and can in fact take any value on the affine parts of the energy. The only restriction is the global restriction, that the total energy of the displacement is equal to the minimum. This behavior is reflected once again by the points in green color corresponding to the relaxed energy in Fig. 7. In the following, we comment in Fig. 7 in detail.

In case (a) with \(v_{\text{ext}} \in [0, 0.0036]\), both the relaxed and the condensed energy are convex with \(f = f_c\) and the corresponding minimizers are affine with gradient \((u_{\text{ext}}, v_{\text{ext}})\). For larger values of \(v_{\text{ext}} \in [0.0036, 0.014]\), the function \(f\) is locally convex but different from its relaxation \(f_c\) which is obtained by a mixture of two states and which is affine on lines parallel to the boundary of the region \(Y^d\), see the red lines in the left panel in Fig. 5. Thus, the calculation with the condensed energy is expected to discover the microstructure supported on two points, while the calculation with \(f_c\) can use all the gradients on the affine part. In fact, the second panel in Fig. 7 shows exactly this behavior. The case (c) corresponds to \(v_{\text{ext}} \in [0.014, 0.063]\) and shows the same behavior of case (b), the only difference being that \(f\) is not locally convex at the point \((u_{\text{ext}}, v_{\text{ext}})\). Finally, for \(v_{\text{ext}} \in [0.0163, 0.15]\) the condensed energy \(f\) is not locally convex and the relaxed energy \(f_c\) is affine on the triangle \(Y_3\). Therefore, the range of the gradient of the numerical solution for the calculation with the condensed energy is expected to be located in the three corners of the triangle, while the gradient of the solution with the relaxed energy can explore the full triangle. The last panel in Fig. 7 confirms this prediction.

The procedure employing the condensed energy does not recover the exact minimum, but produces values which are between 10 and 20% higher. However, the values are lower than the energy of the condensed energy evaluated at the affine solution. The approximate minimizer attempts to mimic the construction of the relaxed energy by concentrating points at the corresponding positions within the \(y_1-y_2\)-plane. The procedure employing the relaxed energy returns the exact minimum with working precision in all cases.
Analytical and numerical relaxation results

Fig. 9 Square plate with circular hole: from top left to bottom right: stresses $\sigma_{xx}$, at the marked elements in cyan, and $\sigma_{yy}$, $\sigma_{xy}$ and plastic strain $\varepsilon_{yy}$, at the marked elements in red, as a function of the external boundary displacement $u_{x,ext}$, path of average stress $\sigma_{yy}$ versus average strain $\varepsilon_{yy}$, path of average stress $\sigma_{xy}$ versus average strain $\varepsilon_{xy}$ (color figure online)

7.2 Three-dimensional model

The generalization to a three-dimensional model derived in Sect. 6 is tested through a two-dimensional boundary value problem, namely the square plate with a circular hole. The plate is fixed at the left-hand side and subject to displacements in compression in $x$-direction along the right-hand side increasing in time. Two unstructured meshes are computed, a coarse one with 1793 elements and a finer mesh with 4917 elements. The meshes are shown in Fig. 8.
Finite element computations are performed using the finite element analysis program FEAP [39]. Standard hexahedral (8-node) elements are used, they are restricted in the third direction employing plane strain conditions, which apply throughout all examples. Stresses are calculated as analytical derivatives of the energies with respect to the strains. Tangent operators are calculated as second derivatives of the energies with respect to the strains, numerically via perturbation. Equilibrium states are obtained by standard Newton–Raphson procedure.

For the computations, we use the same material parameters as in the one-dimensional case taken from the material model with bulk modulus $K = 3.9$ GPa, shear modulus $\mu = 2.8$ GPa, internal friction angle $\phi = 32^\circ$, cohesion $c = 25$ MPa, Poisson’s ratio $\nu = 0.25$, and tensile strength $\sigma_t = 5$ MPa, introduced in [40]. Under increasing load, the elements in the neighborhood of the hole start to plastify first. Therefore, we compare the results for the model employing the relaxed energy at the marked elements adjacent to the hole for both meshes. The values are taken for a specific integration point at these elements. We would like to stress that a comparison with the model employing the condensed energy is not possible, as the lack of convexity causes severe numerical instabilities. Therefore, only the behavior of the relaxed model can be inferred from the results below.
The final external displacement is reached employing 100 load steps. The corresponding strains switch from domain $Y_1$ into domain $Y_2$, inducing an initiation of microstructure. In the top two rows of Fig. 9, we show the stresses and plastic strains (calculated as the difference of total and elastic strains) at the elements marked in Fig. 8 as a function of the external displacements. The point marked in cyan color in Fig. 8 corresponds to the Gauss integration point closest to the vertical axis of symmetry. The point marked in red color in Fig. 8 correspond to the Gauss integration point closest to the diagonal. In order to compare global quantities as well, we display in the bottom row average stresses versus average strains, where the averages are calculated along the path marked in magenta color in Fig. 8.

The distribution of the stress $\sigma_{yy}$ can be seen in Fig. 10. The relaxed model shows similar stress distribution for the coarse and fine mesh. Microstructure is initiating at the same positions and load step despite the different spatial discretization. Contours of the plastic strains are shown in Fig. 11 for a state shortly after microstructure initiation at $u_{\text{ext}} = 0.00678$ mm. For comparison, the stresses at the same instant are shown in Fig. 12. It can be seen that for both meshes the results agree very well for local and global quantities. This demonstrates that the relaxed model is mesh independent and capable of capturing the material behavior in different zones of relaxation.

8 Conclusion and outlook

We established a variational model for pressure-dependent plasticity as common in models of soil mechanics. Our approach employed a novel aspect by using a characteristic function as dissipation potential. That way, it was possible to establish a dissipation distance as well as a condensed energy connected to a time-incremental setting in a consistent manner. This allowed us to investigate the initiation of microstructure via relaxation theory by calculating the quasiconvex envelope of the condensed energy. For a one-dimensional model problem, for which the quasiconvex envelope coincides with the convex envelope, we succeeded in arriving at a closed-form expression. Interestingly, the quasiconvex envelope turned out to be largely independent of the yield surface of the original model for small values of the hardening parameter $b$. The proof of this surprising fact illustrates that even the microstructure can be chosen independently of the yield surface. Numerical simulations confirmed the predictions of the analytical results and illustrated the qualitatively different behavior of the variational models using the condensed and the relaxed energy, respectively. In particular, the simulations employing the condensed energy succeeded in finding the necessary oscillations. It was possible to extend the one-dimensional formulation to higher dimensions in an empirical manner. Numerical simulations for a paradigmatic model problem provided strong evidence for the superior predictive power of the relaxed model.

However, problems left for future investigations are manifold. The characterization of the relaxed energy for large values of $b$ is a challenging task since the affine curves in Fig. 5 bounding the domain with $f = f_c$ intersect the graphs of the functions $\pm r$ and become nonlinear. It is expected that the qualitative behavior for
the relaxation involving at most three points in the phase diagram remains unchanged for an intermediate range of values for \( b \) and that for sufficiently large values of \( b \) two points will suffice.

From the mathematical side, the treatment of the higher-dimensional case is completely open. In fact, our heuristic model provides a guaranteed lower bound for the energy in the system but may, in general, fail to be the largest lower bound and thus may not be optimal. Concerning variational modeling, there are still problems to be solved connected to the non-associated character of pressure-dependent plasticity. Finally, the treatment of several successive time increments is still open as well as establishing fully evolutionary models.

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**Declarations**

**Competing interests** The authors declare that there are no competing interests.

### Appendix A Convex analysis

The construction of \( f_c \) in Theorem 2 is based on the construction of a candidate \( f_a \) and a decomposition of \( \mathbb{R}^2 \) into disjoint regions with the property that \( f_a \) is globally continuous and convex on the interior of the different regions. This appendix collects the results that are needed in order to prove that \( f_a \) is convex on \( \mathbb{R}^2 \). The proofs follow from classical results in convex analysis, see [41] for more information.

**Definition 5** Suppose that \( \Omega \subset \mathbb{R}^d \) is open and that \( f : \Omega \to \mathbb{R} \). Then, \( f \) is said to be locally convex, if for all \( x \in \Omega \) there exists an \( r_x > 0 \) with \( B(x, r_x) \subset \Omega \) and \( f \) convex on \( B(x, r_x) \). A vector \( v \in \mathbb{R}^d \) is called a subgradient for \( f \) at \( x \in \Omega \) if there exists an \( r_x > 0 \) with \( B(x, r_x) \subset \Omega \) and if for all \( y \in B(x, r_x) \) the inequality \( f(y) \geq f(x) + \langle v, y-x \rangle \) holds. The set of all local subgradients is denoted by \( \partial_{\text{loc}} f(x) \) and referred to as the local subdifferential.

The subdifferential in the sense of convex analysis is denoted by \( \partial f(x) \). Then, by definition, \( \partial f(x) \subset \partial_{\text{loc}} f(x) \) and if \( f \) is differentiable, then \( \partial_{\text{loc}} f(x) \subset \{ f'(x) \} \); however, the local subdifferential may be empty if \( f \) the graph of \( f \) does not lie above the tangent plane in a sufficiently small neighborhood of \( x \).

The construction of the convex envelope of the condensed energy relies on a local construction and the verification that the local construction leads to a convex function. The proof uses the following results.

**Lemma 6** Fix \( a, b \in \mathbb{R} \), \( a < 0 < b \), and \( f_a \in C^1([a, 0]; \mathbb{R}) \), \( f_b \in C^1([0, b]; \mathbb{R}) \) convex with \( f_a(0) = f_b(0) \), \( f_a'(0) = f_b'(0) \). Then, \( f = f_a\chi_{(a, 0)} + f_b\chi_{[0, b]} \in C^1((a, b]; \mathbb{R}) \) is convex.

**Lemma 7** Suppose that \( f \in C(\mathbb{R}^d) \), \( B, C \subset \mathbb{R}^d \) open and disjoint, \( C \) convex and \( f \) locally convex on \( B \cup C \), \( f \in C^1(\text{int}(B \cup C)) \). Then, \( f \) is locally convex on \( \text{int}(B \cup C) \).

**Lemma 8** Suppose that \( d \geq 2 \), \( \Omega \subset \mathbb{R}^d \) is open and \( f : \Omega \to \mathbb{R} \) is continuous. The following statements are equivalent:

(i) \( f \) is locally convex in \( x \in \Omega \);

(ii) for \( x \in \Omega \) there exists an \( r_x > 0 \) with \( B(x, r_x) \subset \Omega \) such that \( \partial_{\text{loc}} f(y) \neq \emptyset \) for all \( y \in B(x, r_x) \).

**Lemma 9** Suppose that \( a_i, b_i \in \mathbb{R} \), \( i = 1, 2 \) with \( a_1 < b_1 < a_2 < b_2 \) and that \( f \in C^0([a_1, b_2]) \) with \( f|_{[a_1, a_2]} \), \( f|_{[b_1, b_2]} \) convex. Then, \( f \) is convex.
Lemma 10 Suppose that $\Omega \subset \mathbb{R}^n$ is open and that $f \in C(\Omega)$ is locally convex. Then, for any pair $a, b \in \Omega$ with $\{\alpha a + (1 - \alpha) b : \alpha \in [0, 1]\} \subset \Omega$, the convexity inequality holds on $[a, b]$. In particular, if $\Omega$ is convex, then $f$ is convex on $\Omega$.

Lemma 11 Suppose $\Omega \subset \mathbb{R}^d$ is convex and $f \in C(\mathbb{R}^d)$. If $f|_\Omega$ is convex and if there is a convex function $g : \mathbb{R}^d \to \mathbb{R}$ with $g \leq f$ on $\mathbb{R}^d$ and $g = f$ on $\mathbb{R}^d \setminus \Omega$, then $f$ is convex.

Appendix B Proof of Theorem 2

Based on the discussion at the beginning of Sect. 5, for the construction of the convex envelope of the condensed energy $f^{(r)}$, we first derive pairs and triples of points $y^{(1)}, \ldots, y^{(q)} \in \mathbb{R}^2$ ($q = 2, 3$), in which $f^{(r)}$ is differentiable in the variable $y_2$ and in which $\partial_2 f^{(r)}$ is equal. Then, we derive a candidate for the direction $v \in \mathbb{R}^d$ satisfying the equation in Eq. (5.8). However, at this point we do not know whether this $v$ in Eq. (5.8) is indeed a subgradient of $f$ in all the points $y^{(1)}, \ldots, y^{(q)}$. For those pairs and triples for which we expect this to be true, we do not show explicitly that the first part in Eq. (5.8) is satisfied, rather we apply the construction described in Eqs. (5.9) and (5.10) and prove in Theorem 2 that this construction leads to the convex envelope. It is a remarkable result of the subsequent analysis that the explicit formulas for the relaxed energy show a very different behavior with respect to the dependence on the dissipation function $r$. In fact, for small values of $b$ relative to $r$, i.e., for

$$\sqrt{b} \cdot \frac{y_{\max} - y_{\min}}{2} \leq r \left( \frac{y_{\min} + y_{\max}}{2} \right),$$

(B1)

which is the case of relevance in soil mechanics, the relaxed energy is in fact independent of $r$. Therefore, we restrict the analysis in this paper to the case of small $b$, the general case will be treated in [41]. In fact, this assumption implies $r_0 \leq r$ by concavity of $r$. In the following we consider $r$ as fixed, write $f = f^{(r)}$ and assume without loss of generality that $r_0 < r$ on $(y_{\min}, y_{\max})$.

In order to implement the strategy just outlined, we construct a candidate $f_a$ for $f_c$. In a first step, we search for points with common supporting planes in $Y_0$ and $\bar{Y}$, see Fig. 3. Assume thus that $(y, \bar{y}) \in Y_0 \times \bar{Y}$ satisfy Eq. (5.8). Since $f$ is differentiable in $y$, $v = \nabla f(y)$ is the only possible choice for $v$ in Eq. (5.8). If $\bar{y} \in \text{int}(\overline{Y})$, then $f$ is differentiable in $\bar{y}$ and $\nabla f(y) = v \in \partial f(\bar{y}) \subset \{\nabla f(\bar{y})\}$ implies $\tilde{y}_1 = \partial_1 f(\bar{y}) = \partial_1 f(y) = y_1$, a contradiction. Consequently we assume that $\bar{y} \in \partial \bar{Y}$, that is, $\bar{y}_1 \in (y_{\min}, y_{\max})$. The existence of $\partial_2 f(\bar{y})$ implies $\partial_2 f(\bar{y}) = v_2 = \partial_2 f(y)$ and equivalently

$$\frac{b}{b + 1} \bar{y}_2 = y_2 \equiv \bar{y}_2 - y_2 = \frac{1}{b} y_2.$$

(B2)

Equation (5.8) is equivalent to $f(\bar{y}) = (T_y f)(\bar{y})$. By definition of $f$ and since $f|_{Y_0}$ is a quadratic function, these conditions lead to

$$\frac{1}{2} \bar{y}_1^2 + \frac{b}{2 b + 1} \bar{y}_2^2 = f(\bar{y}) = (T_y f)(\bar{y}) = \frac{1}{2} \bar{y}_1^2 + \frac{1}{2} \bar{y}_2^2 - \frac{1}{2} \nabla^2 f(y)[\bar{y} - y, \bar{y} - y].$$

(B3)

In view of $\nabla^2 f(y) = \text{Id}$, the algebraic condition can be simplified to

$$(\bar{y}_1 - y_1)^2 + (\bar{y}_2 - y_2)^2 = \frac{1}{b + 1} \bar{y}_2^2.$$

(B4)

One defines $s = |\bar{y}_1 - y_1|$ and substitutes the relation between $\bar{y}_2$ and $y_2$ to obtain $y_2 = \pm \sqrt{b} s$. Since $\bar{y}_1 \in (y_{\min}, y_{\max})$, one obtains four one-parameter families of pairs of points,

$$\alpha_{\min}^+(s) = \left( \frac{y_{\min} + s}{\pm \sqrt{b} \cdot s} \right), \quad \beta_{\min}^+(s) = \left( \frac{y_{\min}}{\pm \frac{b + 1}{\sqrt{b}} s} \right).$$

(B5)

$$\alpha_{\max}^+(s) = \left( \frac{y_{\max} - s}{\pm \sqrt{b} \cdot s} \right), \quad \beta_{\max}^+(s) = \left( \frac{y_{\max}}{\pm \frac{b + 1}{\sqrt{b}} s} \right).$$

(B6)
where $s$ can take all values in $(\gamma_{\text{min}}, \gamma_{\text{max}})$ with $\alpha_{\text{min}}^{\pm}(s) \in Y_0$ (or $\alpha_{\text{max}}^{\pm}(s) \in Y_0$). For the construction of the convex envelope only the values $0 < s < (\gamma_{\text{max}} - \gamma_{\text{min}})/2 =: s_*$ will be relevant, since the curves $\alpha_{\text{min}}^{\pm}$ and $\alpha_{\text{max}}^{\pm}$ intersect in the point

$$y_* := \alpha_{\text{min}}^{\pm}(s_*) = \alpha_{\text{max}}^{\pm}(s_*) = (y_{\text{mid}}, \pm \sqrt{b}(\gamma_{\text{max}} - \gamma_{\text{min}})/2) \in Y_0.$$  \hspace{1cm} (B7)

For each $y \in \mathbb{R}^2$, which lies on a connecting straight line of two corresponding points of the curves $\alpha_{\text{min}}^{\pm}$ and $\beta_{\text{min}}^{\pm}$, i.e., $y = t \cdot \alpha_{\text{min}}^{\pm}(s) + (1 - t) \cdot \beta_{\text{min}}^{\pm}(s)$ with $s \in (0, s_*)$ and $t \in [0, 1]$, we define $f_a(y) = t \cdot f(\alpha_{\text{min}}^{\pm}(s)) + (1 - t) \cdot f(\beta_{\text{min}}^{\pm}(s))$ according to Eq. (5.9) and we use the same construction for $\alpha_{\text{max}}^{\pm}$ and $\beta_{\text{max}}^{\pm}$. Note that $f_a$ is affine along these connecting lines and the area, which is covered by those lines consists of four triangular shaped sets denoted by $Y_2$ in Fig. 5. The fact that $\alpha_{\text{min}}^{\pm}$ and $\alpha_{\text{max}}^{\pm}$ intersect in $y_*$ means that the tangential plane of $f$ in $y_*$ touches the graph in two other points, namely

$$y_{\text{min}}^{\pm} := \beta_{\text{min}}^{\pm}(s_*) = \left( y_{\text{min}}, \pm \frac{b + 1}{\sqrt{b}}(\gamma_{\text{max}} - \gamma_{\text{min}})/2 \right),$$  \hspace{1cm} (B8)

$$y_{\text{max}}^{\pm} := \beta_{\text{max}}^{\pm}(s_*) = \left( y_{\text{max}}, \pm \frac{b + 1}{\sqrt{b}}(\gamma_{\text{max}} - \gamma_{\text{min}})/2 \right).$$  \hspace{1cm} (B9)

Since the pairs $(\alpha_{\text{min}}^{\pm}(s_*), \beta_{\text{min}}^{\pm}(s_*))$ and $(\alpha_{\text{max}}^{\pm}(s_*), \beta_{\text{max}}^{\pm}(s_*))$ satisfy the equality in Eq. (5.8) for all $s \in (0, s_*)$, the triples $(y_*^+, y_{\text{min}}^{\pm}, y_{\text{max}}^{\pm})$ also satisfy Eq. (5.8). Therefore, we define $f_a$ for $y$ in one of the triangles Conv($y_*^+, y_{\text{min}}^{\pm}, y_{\text{max}}^{\pm}$) $\cup$ Conv($y_*^-, y_{\text{min}}^{\pm}, y_{\text{max}}^{\pm}$) ($Y_3$ in Fig. 5) according to Eq. (5.10) as the unique affine function which coincides with $f$ in the vertices of each of these triangles,

$$f_a(y) = \begin{cases} (T_{y_*^+}f)(y) & \text{on Conv}((y_*^+, y_{\text{min}}^{\pm}, y_{\text{max}}^{\pm})), \\ (T_{y_*^-}f)(y) & \text{on Conv}((y_*^-, y_{\text{min}}^{\pm}, y_{\text{max}}^{\pm})). \end{cases}$$  \hspace{1cm} (B10)

Finally, consider a pair of two different points $(\tilde{y}_1^{(1)}, \tilde{y}_2^{(1)}) \in \partial \tilde{Y}$ satisfying Eq. (5.8). For the existence of a supporting hyperplane of $f$, which touches the graph of $f$ in $\tilde{y}_1^{(1)}$ and $\tilde{y}_2^{(2)}$, we require equal partial derivatives in the second component, i.e.,

$$\tilde{y}_2^{(1)} = \frac{b + 1}{b} \partial_2 f(\tilde{y}_1^{(1)}) = \frac{b + 1}{b} \partial_2 f(\tilde{y}_2^{(2)}) = \tilde{y}_2^{(2)}.$$  \hspace{1cm} (B11)

With $\tilde{y}_1^{(1)} \in [\gamma_{\text{max}}, \gamma_{\text{max}}]$ and $\tilde{y}_2^{(1)} \neq \tilde{y}_2^{(2)}$ we get $\tilde{y}_1^{(1)} = \gamma_{\text{max}}$ and $\tilde{y}_1^{(2)} = \gamma_{\text{max}}$ and obtain for $s > 0$ the pairs of curves

$$\alpha_{\infty}^{\pm}(s) := (\gamma_{\text{min}}, \pm s), \quad \beta_{\infty}^{\pm}(s) := (\gamma_{\text{max}}, \pm s).$$  \hspace{1cm} (B12)

Since we already constructed our candidate $f_a$ for the convex envelope on the set $(\gamma_{\text{min}}, \gamma_{\text{max}}) \times [-\sqrt{b}s, \sqrt{b}s]$, we restrict $s$ to have values $s > \frac{b + 1}{\sqrt{b}s_\infty}$. For each $y \in \mathbb{R}^2$, which lies on a connecting straight line of two corresponding points of the curves $\alpha_{\infty}$ and $\beta_{\infty}$, i.e., $y = t \cdot \alpha_{\infty}^{\pm}(s) + (1 - t) \cdot \beta_{\infty}^{\pm}(s)$ with $s \in (\frac{b + 1}{\sqrt{b}s_\infty}, \infty)$ and $t \in [0, 1]$ (represented by the two unbounded sets forming $Y_4$ in Fig. 5), we define $f_a(y) = t \cdot f(\alpha_{\infty}^{\pm}(s)) + (1 - t) \cdot f(\beta_{\infty}^{\pm}(s))$ according to Eq. (5.9).

After these preparations, we turn to the proof of the theorem and divide it into several steps. Denote by $f_c$ the formula on the right-hand side in the assertion of the theorem in (5.15). We are going to show in Step 1 that $f_a$ is differentiable on $\mathbb{R}^2 \setminus \{(\gamma_{\text{min}}, \gamma_{\text{max}}) \times \mathbb{R}\}$ and convex. In Step 2 we verify that $f_a$ is indeed the expression we obtain from the previously described construction and $f_c \leq f_a$. Finally in Step 3 we prove $f_a \leq f$ and conclude that $f_a$ coincides with $f_c$ by $f_c \leq f_a \leq f$ and $f_a$ being convex.
Step 1: Differentiability and convexity of $f_a$. The differentiability of $f_a$ on $\text{int}(\tilde{Y}) \cup (y_{\min}, y_{\max}) \times \mathbb{R}$ follows from

$$
\nabla f_a(y) = \begin{cases}
(y_1, \frac{b}{b+1} y_2) & \text{on int}(\tilde{Y}), \\
(y_1, y_2) & \text{on } Y_1, \\
(y_1 + \frac{\text{sign}(y_{\mid mid} - y_1)\sqrt{b}}{b+1} (y_2 - r_0(y_1))) & \text{on } Y_2, \\
\left(\frac{b}{b+1} y_2 + \frac{\text{sign}(y_{\mid mid} - y_1)}{b+1} y_2\right)^T & \text{on } Y_3, \\
\left(\frac{\text{sign}(y_{\mid mid} - y_1)\sqrt{b}}{b+1} \frac{y_{\max} - y_{\min}}{2} \frac{y_2}{2}\right) & \text{on } Y_4,
\end{cases}
$$

and verifying the continuity of $\nabla f_a$ along the boundaries of the different regions. With the notation $e_1 \otimes e_2 := \frac{1}{2} (e_1 \otimes e_2 + e_2 \otimes e_1)$, the second derivatives are given by

$$
\nabla^2 f_a(y) = \begin{cases}
e_1 \otimes e_1 + \frac{b}{b+1} e_2 \otimes e_2 & \text{on int}(\tilde{Y}), \\
e_1 \otimes e_1 + e_2 \otimes e_2 & \text{on int}(Y_1), \\
\frac{1}{b+1} \cdot \left(e_1 \otimes e_1 + \frac{2\sqrt{b}\text{sign}(y)}{\text{sign}(y_{\mid mid} - y_1)} e_1 \otimes e_2 + b e_2 \otimes e_2\right) & \text{on int}(Y_2), \\
0 & \text{on int}(Y_3), \\
\frac{b}{b+1} e_2 \otimes e_2 & \text{on int}(Y_4).
\end{cases}
$$

In fact, $\nabla^2 f_a$ is positively semidefinite on the interior of each domain $\tilde{Y}$, $Y_1$, $Y_2$, $Y_3$, $Y_4$, consequently $f_a$ is locally convex on those domains. Since $f_a$ is differentiable on $(y_{\min}, y_{\max}) \times \mathbb{R}$, an iterative application of Lemma 7 gives us the local convexity (and hence the convexity by Lemma 10) of $f_a$ on $(y_{\min}, y_{\max}) \times \mathbb{R}$. A short calculation shows that $g : \mathbb{R}^2 \to \mathbb{R}$, $g(y) = \frac{1}{2} y_1^2 + \frac{1}{b+1} y_2^2$ is a convex lower bound of $f_a$, and since $f_a$ and $g$ coincide on $\tilde{Y}$ and $f_a$ is convex on $\mathbb{R} \setminus \tilde{Y} = (y_{\min}, y_{\max}) \times \mathbb{R}$, the convexity of $f_a$ follows by Lemma 11.

Step 2: $f_c \leq f_a$. Since the calculation of the explicit formula of the candidate $f_a$ derived from the previously described construction is extensive, we just prove that the given formula is the one, which is obtained by this construction. Refer to Fig. 5 of the various domains.

On $\tilde{Y} \cup Y_1$: Since by assumption $r_0 \leq r$ and hence $Y_1 \subset Y_0$, $f_a$ coincides with $f$ in $Y_1$ and in $\tilde{Y}$ and $f_c \leq f = f_a$, as asserted.

On $Y_2$: The set $Y_2$ is exactly the set of points, which are covered by the straight line segments connecting the corresponding pairs of points on the four pairs of curves $(a_{\min}^+, \beta_{\min}^-(s))$ and $(a_{\max}^+, \beta_{\max}^+(s))$. Assume $y \in Y_2$ with $y_1 < y_{\mid mid}$ and $y_2 > 0$ (the other cases are analogous). There exists $s \in (0, (y_{\min} - y_{\mid mid})/2)$ and $t \in (0, 1)$ with $y = t \cdot a_{\min}^+(s) + (1 - t) \cdot \beta_{\min}^+(s)$. To show that the given expression for $f_a(y)$ results from the construction described above, we notice that $f_a(a_{\min}^+(s)) = f(a_{\min}^+(s))$ and $f_a(\beta_{\min}^+(s)) = f(\beta_{\min}^+(s))$ and that $f_a$ is affine along the connecting line between $a_{\min}^+(s)$ and $\beta_{\min}^+(s)$. This can be seen by $\beta_{\min}^+(s) - a_{\min}^+(s) = (s - y_{\min}) \cdot (-1, \sqrt{b}^{-1})^T$ and using $y_2, y_{\mid mid} - y_1 > 0$ to obtain

$$
(1 - \sqrt{b}^{-1})^T \nabla^2 f_a(y) \left(\begin{array}{c} -1 \\ -1 \end{array}\right) = \frac{1}{b+1} \left(\begin{array}{c} -1 \\ -1 \end{array}\right) \left(\begin{array}{c} 1 \\ \sqrt{b}^{-1} \end{array}\right) \left(\begin{array}{c} 1 \\ \sqrt{b}^{-1} \end{array}\right) = 0.
$$

Since $f_c$ must lie below the linear interpolation, we obtain $f_c \leq f_a$.

On $Y_3$: By construction, $f_a$ and $f$ have the same values in $y_{\pm}^+, y_{\min}^+, y_{\max}^+$ and on $\text{Conv}(y_{\pm}^+, y_{\min}^+, y_{\max}^+)$. On $\text{Conv}(y_{\pm}^+, y_{\min}^+, y_{\max}^+)$ the function $f_a$ is affine due to $\nabla^2 f_a = 0$ on int($Y_3$). By $y_{\pm}^+ \in Y_0$, $f$ is differentiable in $y_{\pm}^+$ with $\nabla f(y_{\pm}^+) = \nabla f_a(y_{\pm}^+)$. Now it follows in view of Eq. (B10) that $f_a$ is the above constructed function.
On $Y_4$: Similar to the argument on $Y_2$, we recognize that for any $y \in Y_4$ with $\pm y_2 > 0$ there is an $s \in (0, 1)$ with $r \in (0, 1)$ with $y = t \cdot \alpha \pm(s) + (1 - t) \cdot \beta \pm(s)$. We have $f_a(\alpha \pm(s)) = f(\alpha \pm(s))$ and $f_a(\beta \pm(s)) = (\beta \pm(s))$, while $f_a$ is affine along the connecting line between $\alpha \pm(s)$ and $\beta \pm(s)$ since $\beta \pm(s) - \alpha \pm(s) = (y_{\text{max}} - y_{\text{min}}, 0)$ and in the formula of $f_a$ on $Y_4$ the quadratic term in $y_1$ cancels out. Finally, we get $f_c \leq f_a$ since any value of $f_a$ is a convex combination of two or three function values of $f$.

Step 3: The inequality $f_a \leq f$. We show $f_a \leq f^{(n)} \leq f$, where the second inequality is an immediate consequence of Lemma 1. For the first inequality, recognize that $f_a = f^{(n)}$ holds on $Y \cup Y_1 \cup Y_2$ and the inequality only has to be shown on $Y_3$ and $Y_4$.

On $Y_3$, the estimate follows by

$$f_a(y) = f^{(n)}(y) = \frac{1}{2} \left( |y_2| - \frac{b + 1}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} + \frac{r_0(y_1)}{b} \right)^2. \tag{B16}$$

On $Y_4$, we can calculate

$$f^{(n)}(y) - \left( \frac{1}{2} y_1^2 + \frac{b}{2b + 1} y_2^2 \right) = \frac{y_2^2}{2(b + 1)} - \frac{(|y_2| - r_0(y_1))^2}{2(b + 1)} \geq - \frac{r_0(y_1)^2}{2(b + 1)} \geq \frac{r_0(y_1)}{b} |y_2| \geq - \frac{r_0(y_1)}{2b} \frac{y_{\text{max}} - y_{\text{min}}}{\sqrt{b}} \tag{B17}$$

For $y_1 \in (y_{\text{min}}, y_{\text{mid}})$, we have

$$- \frac{r_0(y_1)}{2b} + \frac{r_0(y_1)}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} = \frac{-b(y_1 - y_{\text{min}})^2}{2b} + \frac{\sqrt{b}(y_1 - y_{\text{min}})}{\sqrt{b}} \cdot \frac{y_{\text{max}} - y_{\text{min}}}{2} \tag{B18}$$

and for $y_1 \in (y_{\text{mid}}, y_{\text{max}})$ we have

$$- \frac{r_0(y_1)}{2b} + \frac{r_0(y_1)}{\sqrt{b}} \frac{y_{\text{max}} - y_{\text{min}}}{2} = \frac{-b(y_{\text{max}} - y_1)^2}{2b} + \frac{\sqrt{b}(y_{\text{max}} - y_1)}{\sqrt{b}} \cdot \frac{y_{\text{max}} - y_{\text{min}}}{2} \tag{B19}$$

which gives us the desired estimate on $Y_4$.

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