Deformations of annuli on Riemann surfaces and the generalization of Nitsche conjecture

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Abstract

Let $A$ and $A'$ be two circular annuli and let $\rho$ be a radial metric defined in the annulus $A'$. Consider the class $\mathcal{H}_\rho$ of $\rho$-harmonic mappings between $A$ and $A'$. It is proved recently by Iwaniec, Kovalev and Onninen that, if $\rho = 1$ (that is, if $\rho$ is Euclidean metric), then $\mathcal{H}_\rho$ is not empty if and only if there holds the Nitsche condition (and thus is proved the J. C. C. Nitsche conjecture). In this paper, we formulate a condition (which we call $\rho$-Nitsche conjecture) with corresponds to $\mathcal{H}_\rho$ and define $\rho$-Nitsche harmonic maps. We determine the extremal mappings with smallest mean distortion for mappings of annuli with respect to the metric $\rho$. As a corollary, we find that $\rho$-Nitsche harmonic maps are Dirichlet minimizers among all homeomorphisms $h : A \rightarrow A'$. However, outside the $\rho$-Nitsche condition of the modulus of the annuli, within the class of homeomorphisms, no such energy minimizers exist. This extends some recent results of Astala, Iwaniec and Martin (ARMA, 2010) where it is considered the case $\rho = 1$ and $\rho = 1/|z|$.

1. Introduction

1.1. Mappings of finite distortion

A homeomorphism $w = f(z)$ between planar domains $\Omega$ and $D$ has finite distortion if

(a) $f$ lies in the Sobolev space $W^{1,1}_{loc}(\Omega, D)$ of functions whose first derivatives are locally integrable, and

(b) $f$ satisfies the distortion inequality

$$|f_z| \leq \mu(z)|f_z|,$$

$0 \leq \mu(z) < 1$ almost everywhere (a.e.) in $\Omega$. Such mappings are generalizations of quasiconformal homeomorphisms, where one works with the stronger assumption $\mu(z) \leq k < 1$. Mappings of finite distortion have found considerable interest in geometric function theory and the mathematical theory of elasticity. A comprehensive overview of the theory of mappings of finite distortion in two-dimensions can be found in [2]. The Jacobian determinant of a mapping $f$ of finite distortion is non-negative a.e., since

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = (1 - \mu(z)^2)|f_{\bar{z}}|^2 \geq 0.$$

The distortion function of particular interest to us in this article is defined by the rule

$$K(z, f) = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} = \frac{\|Df(z)\|^2}{J_f(z)}$$

if $J_f(z) > 0$. Here

$$\|A\|^2 = \frac{1}{2} \text{Tr}(A^T A)$$
is the square of mean Hilbert–Schmidt norm. We conveniently set $\mathcal{K}(z, f) = 1$ if $f_z = f_{\bar{z}} = 0$. Note that then $\mathcal{K}(z, f) = 1$ and we have the equality $\mathcal{K}(z, f) = 1$ if and only if $f$ is conformal, by the Looman Menchoff theorem.

1.2. Radial metrics

By $U$ we denote the unit disk $\{z : |z| < 1\}$, by $\overline{\mathbb{C}}$ is denoted the extended complex plane. Let $\Sigma$ be a Riemannian surface over a domain $C$ of the complex plane or over $\overline{\mathbb{C}}$ and let $p : C \rightarrow \Sigma$ be a universal covering. Let $\rho_\Sigma$ be a conformal metric defined in the universal covering domain $C$ or in some chart $D$ of $\Sigma$. It is well known that $C$ can be one of three sets: $U$, $\mathbb{C}$ and $\overline{\mathbb{C}}$. Then the distance function is defined by

$$d(a, b) = \inf_{a, b \in \gamma} \int_0^1 \rho_\Sigma(\tilde{\gamma}(t))|\tilde{\gamma}'(t)| \, dt,$$

where $\tilde{\gamma}$, $\tilde{\gamma}(0) = 0$, is a lift of $\gamma$, that is, $p(\tilde{\gamma}(t)) = \gamma(t)$, $\gamma(0) = a$, $\gamma(1) = b$.

The Gauss curvature of the surface (and of the metric $\rho_\Sigma$) is given by

$$K = -\frac{\Delta \log \rho_\Sigma}{\rho_\Sigma^2}.$$

In this paper, we will consider those surfaces $\Sigma$, whose metric have the form

$$\rho_\Sigma(z) = h(|z|^2),$$

defined in some chart $A' = \{z : \tau < |z| < \sigma\}$ of $\Sigma$ (not necessarily in the whole universal covering surface). Here $h$ is a positive twice differentiable function. We call these metrics radial symmetric.

**Definition 1.1.** The radial metric $\rho$ is called a regular metric if

$$\inf_{\tau < s < \sigma} s \rho(s) = \lim_{s \rightarrow \tau + 0} s \rho(s)$$

and has bounded Gauss curvature $K$.

Euclidean metric $\rho(z) = 1$, and the metric $\rho(z) = 1/|z|$ are regular metrics which are considered by Astala, Iwaniec and Martin in the paper [3]. The authors of [3] settled corresponding problems of deformations of annuli with smallest mean distortion with respect to these two metrics. The aim of this paper is to extend the results in [3] for all regular metrics (see Theorems 3.1, 3.3, Corollaries 3.4 and 3.6). This paper has been 'published' in arXiv on 2010 (arXiv:1005.5269). In media time precisely on 2012, Martin and McKubre-Jordens published a paper, which partly contains a result from my paper (cf. Theorem 3.1 and [19, Section 5.1]), nevertheless the largest novelty of this paper is that it solves two different minimization problems concerning the mappings of finite distortion and harmonic minimizers on the same time (Theorems 3.1, 3.3, Corollaries 3.4 and 3.6).

In the appendix, it is shown that most of known metrics with constant Gauss curvatures defined in annuli are regular metrics. Such metrics are, for example, hyperbolic metric

$$\lambda(z) = \frac{2}{1 - |z|^2}, \quad K = -1$$

defined in the unit disk $U$ and Riemann metric

$$\eta(z) = \frac{2}{1 + |z|^2}, \quad K = 1$$

defined in the Riemann sphere $S^2 := \overline{\mathbb{C}}$. 
1.3. Harmonic mappings between Riemann surfaces

Let \((M, \sigma)\) and \((N, \rho)\) be Riemann surfaces with metrics \(\sigma\) and \(\rho\), respectively. If a mapping \(f : (M, \sigma) \to (N, \rho)\) is \(C^2\), then \(f\) is said to be harmonic (to avoid the confusion we will sometimes say \(\rho\)-harmonic) if

\[
f_z \sigma + (\log \rho^2) w \circ f \cdot f_z f_{\bar{z}} = 0, \tag{1.2}
\]

where \(z\) and \(w\) are the local parameters on \(M\) and \(N\), respectively. Also \(f\) satisfies (1.2) if and only if its Hopf differential

\[
\Psi = (\rho^2 \circ f) f_z f_{\bar{z}} \tag{1.3}
\]
is a holomorphic quadratic differential on \(M\).

For \(g : M \mapsto N\), the energy integral is defined by

\[
E_\rho[g] = \int_M (|\partial g|^2 + |\bar{\partial} g|^2) \, dV_\sigma, \tag{1.4}
\]

where \(\partial g\) and \(\bar{\partial} g\) are the partial derivatives taken with respect to the metrics \(g\) and \(\sigma\), and \(dV_\sigma\) is the volume element on \((M, \sigma)\). Assume that energy integral of \(f\) is bounded. Then \(f\) is harmonic if and only if \(f\) is a critical point of the corresponding functional, where the homotopy class of \(f\) is the range of this functional. For this definition and some important properties of harmonic maps, see [23].

It follows from the definition the following lemma.

**Lemma 1.2.** If \(a\) is holomorphic and \(f\) is harmonic, then \(f \circ a\) is harmonic.

Using the fact that the function defined in (1.3) is holomorphic, the following well-known lemma can be proved (see, for example, [17]).

**Lemma 1.3.** Let \((S_1, \rho_1)\), \((S_2, \rho_2)\) and \((R, \rho)\) be three Riemann surfaces. Let \(g\) be an isometric transformation of the surface \(S_1\) onto the surface \(S_2\):

\[
\rho_1^2(|\omega|dw|^2 = \rho_2^2(w)|dw|^2, \quad w = g(\omega).
\]

Then \(f : R \mapsto S_1\) is \(\rho_1\)-harmonic if and only if \(g \circ f : R \mapsto S_2\) is \(\rho_2\)-harmonic. In particular, if \(g\) is an isometric self-mapping of \(S_1\), then \(f\) is \(\rho_1\)-harmonic if and only if \(g \circ f\) is \(\rho_1\)-harmonic.

**Example 1.4.** Let \(\rho\) be the Riemann metric

\[
\rho = \frac{2}{1 + |z|^2}.
\]

Equation (1.2) becomes

\[
u_{z\bar{z}} - \frac{2\bar{u}}{1 + |u|^2} u_z \cdot u_{\bar{z}} = 0. \tag{1.5}
\]

Note this important example. The Gauss map of a surface \(\Sigma\) in \(\mathbb{R}^3\) sends a point on the surface to the corresponding unit normal vector \(n \in \mathbb{C} \cong S^2\). In terms of a conformal coordinate \(z\) on the surface, if the surface has constant mean curvature, its Gauss map \(n : \Sigma \mapsto \mathbb{C}\), is a Riemann harmonic map [22].

**Example 1.5.** If \(u : U \mapsto \mathbb{U}\) is a harmonic mapping with respect to the hyperbolic metric

\[
\lambda = \frac{2}{(1 - |z|^2)}.
\]
then Euler–Lagrange equation of $u$ is

$$u_{\bar{z}z} + \frac{2\bar{u}}{1-|u|^2} u_z \cdot \bar{u}_z = 0. \quad (1.6)$$

An important example of hyperbolic harmonic mapping is the Gauss map of a space-like surface with constant mean curvature $H$ in the Minkowski 3-space $M^{2,1}$ (see [7, 20, 24]).

2. Radial $\rho$-harmonic mappings and $\rho$-Nitsche conjecture

The conjecture in question concerns the existence of a harmonic homeomorphism between circular annuli $A(r, 1)$ and $A(\tau, \sigma)$, and is motivated in part by the existence problem for doubly connected minimal surfaces with prescribed boundary. In 1962, Nitsche [21] observed that the image annulus cannot be too thin, but it can be arbitrarily thick (even a punctured disk). Then he conjectured that for such a mapping to exist we must have the following inequality, now known as the Nitsche bound:

$$\frac{\sigma}{\tau} \geq \frac{1}{2} \left( \frac{1}{r} + r \right).$$

For some results concerning the partial solution of Nitsche conjecture, see papers [14, 18, 25]. For the generalization of this conjecture to $\mathbb{R}^n$ and some related results, we refer the reader to [15]. For the case of hyperbolic harmonic mappings, we refer the reader to [8]. Some other generalization has been done in [14] (see Proposition 2.3). The Nitsche conjecture for Euclidean harmonic mappings is settled recently in [12] by Iwaniec, Kovalev and Onninen, showing that, only radial harmonic mappings

$$h(\zeta) = C \left( \zeta - \omega \frac{\zeta}{\zeta} \right),$$

$C \in \mathbb{C}$, $\omega \in \mathbb{R}$, $|C|(1-\omega) = \sigma$, which inspired the Nitsche conjecture, make the extremal distortion of rounded annuli.

In this section, we will state a similar conjecture with respect to $\rho$-harmonic mappings. In order to do this, we will find all examples of radial $\rho$-harmonic maps between annuli. We put

$$w(z) = g(s) e^{it}, \quad z = se^{it},$$

where $g$ is an increasing or a decreasing function to be chosen. This will include all radial harmonic mappings.

Direct calculations yield

$$w_{\bar{z}z} = \frac{1}{4} \Delta w = \frac{1}{4s^2} (s^2 w_{ss} + sw_s + w_{tt}) \quad (2.1)$$

and

$$w_z w_{\bar{z}} = \frac{1}{4s^2} (s^2 w_s^2 - w_t^2). \quad (2.2)$$

Inserting this into harmonic equation (1.2), because

$$\partial_w \log \rho^2(|w|) = \frac{\rho'(|w|)\bar{w}}{|w|^2},$$

we obtain

$$s^2 g'' + sg' - g + \frac{\rho'(g(s))}{\rho}(s^2 g'^2 - g^2) = 0.$$ 

Let $s = e^x$ and

$$\varrho = \frac{1}{\rho}, \quad (2.3)$$
Put $y(x) = g(e^x)$. Then the corresponding differential equation is

$$y'' - y = \frac{g'(y)}{g(y)} (y'^2 - y^2).$$

After some changes we obtain that, the general solution to this equation is

$$x + c_1 = \int \frac{dy}{\sqrt{y^2 + c_2 y^2}},$$

where $c$ and $c_1$ are certain constants. The mapping $w$ given by

$$w(se^{it}) = q^{-1}(s) e^{it}, \quad (2.4)$$

where

$$q(s) = \exp(\varphi(s)) = \exp \left( \int_s^\tau \frac{dy}{\sqrt{y^2 + c_2 y^2}} \right), \quad \tau \leq s \leq \sigma, \quad (2.5)$$

and $c$ satisfies the condition:

$$y^2 + c_2 y^2(y) \geq 0 \quad \text{for} \quad \tau \leq s \leq \sigma, \quad (2.6)$$

is a $\rho$-harmonic mapping between annuli $A = A(r, 1)$ and $A' = A(\tau, \sigma)$, where

$$r = \exp \left( \int_\tau^\sigma \frac{dy}{\sqrt{y^2 + c_2 y^2}} \right). \quad (2.7)$$

The harmonic mapping $w$ is normalized by

$$w(e^{it}) = \sigma e^{it}.$$

The mapping $w = h^c(z)$ is a diffeomorphism, and we will call it $\rho$-Nitsche map. From now on, we will assume that the metric $\rho$ is regular in the sense of Definition 1.1. Then (2.6) is equivalent to

$$\tau^2 + c_2(\tau) \geq 0. \quad (2.8)$$

Accordingly, for $c = -\tau^2 \rho^2(\tau)$, we have well-defined function

$$q^\#(s) = \exp \left( \int_s^\tau \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(\tau) y^2}} \right), \quad \tau \leq s \leq \sigma.$$

The mapping $h^\#: A \to A'$ defined by $h^\#(se^{it}) = (q^\#)^{-1}(s) e^{it}$ is called the critical Nitsche map.

For $\tau \leq s \leq \sigma \leq 1$, we have:

$$s^2 \varphi'(s)^2 - 1 = \frac{-c}{s^2 \rho^2 + c} \begin{cases} 
\leq 0 & \text{if} \quad c \geq 0; \\
\geq 0 & \text{if} \quad -\tau^2 \rho^2(\tau) \leq c < 0.
\end{cases} \quad (2.9)$$

Note that, the mapping

$$f^c(se^{it}) = q(s) e^{it} : A' \to A$$

is the inverse of the harmonic diffeomorphism $w$.

**Conjecture 2.1.** Let $\rho$ be a regular metric. If $r < 1$, and there exists a $\rho$-harmonic mapping of the annulus $A' = A(r, 1)$ onto the annulus $A = A(\tau, \sigma)$, then

$$r \geq \exp \left( \int_\tau^\sigma \frac{\rho(y) dy}{\sqrt{y^2 \rho^2(y) - \tau^2 \rho^2(\tau)}} \right). \quad (2.10)$$
Note that if $\rho = 1$, then this conjecture coincides with standard Nitsche conjecture.

The following example asserts that, in the settings of the previous conjecture an upper bound for $r$ does not hold. In other words, the image domain it can be arbitrarily thick (even a punctured disk). This differs harmonic mappings from conformal mappings and quasiconformal mappings.

**Example 2.2.** Let $\rho$ be a metric defined on the unit disk $U$, and take $A' = A(0,1)$ ($0 = \tau < \sigma = 1$). Let $c > 0$ and 

$$q(s) = \exp \left( \int_1^s \frac{\rho(s) \, ds}{\sqrt{c + s^2 \rho^2(s)}} \right).$$

Define $w(z) = q^{-1}(s) e^{it}$. Then $w$ is a $\rho$-harmonic diffeomorphism between annuli $A(r_c, 1)$ and the degenerated annulus $A(0,1)$. Here

$$0 < r_c = \exp \left( - \int_0^1 \frac{\rho(s) \, ds}{\sqrt{c + s^2 \rho^2(s)}} \right) < 1$$

and

$$\lim_{c \to +\infty} r_c = 1.$$

We call the inequality (2.10) $\rho$-Nitsche condition. The converse inequality

$$r < \exp \left( \int_\tau^\sigma \frac{\rho(y) \, dy}{\sqrt{y^2 \rho^2(y) - \tau^2 \rho^2(\tau)}} \right), \quad (2.11)$$

we will call the fatness condition.

If $r < 1$ satisfies the condition (2.10), then by continuity argument, there exists a $c$ satisfying

$$c \geq -\tau^2 \rho^2(\tau)$$

such that

$$r = \exp \left( \int_\tau^\sigma \frac{\rho(y) \, dy}{\sqrt{y^2 \rho^2(y) + c}} \right). \quad (2.12)$$

The following theorem is a partial result in solving the previous conjecture.

**Proposition 2.3 ([16]).** Assume that $\rho$ is a metric defined in the disk $\{z : |z| \leq \sigma\}$ with positive or negative Gauss curvature $K(z)$ and let

$$h(x) = \int_0^x \rho(t) \, dt, \quad 0 \leq x \leq \sigma.$$

If there exists a $\rho$-harmonic diffeomorphism between the annuli $A = A(r, 1) = \{z \in \mathbb{C} : r < |z| < 1\}$ and $A' = \{z \in \mathbb{C} : \tau < |z| < \sigma\}$, then

$$\frac{h(\sigma)}{h(\tau)} \geq 1 + \frac{\tau}{2h(\tau)} \log^2 r \begin{cases} (t\rho(t))'|_{t=\tau} & \text{if } K \text{ is negative;} \\ (t\rho(t))'|_{t=\sigma} & \text{if } K \text{ is positive.} \end{cases} \quad (2.13)$$

3. Statement of the main results

The classical formulations of the extremal Grötsch and Teichmüller problems are concerned with finding mappings $\Omega \to D$ in some class (for instance, with free or prescribed boundary values) which have smallest $L^\infty$-norm of the distortion function, thus ‘extremal quasiconformal
mappings’. In this article, we shall investigate mappings \( f : \Omega \to D \) in some class which minimize integral means

\[
K_\rho[f] = \int_\Omega \mathbb{K}(z, f) \rho^2(z) \, dx \, dy, \quad z = x + iy,
\]

with respect to appropriate metrics of the distortion function \( \mathbb{K}(z, f) \). The case of bounded simply connected domains, without boundary data, is well known; the extremals are the conformal mappings of \( \Omega \) onto \( D \) asserted to exist by the Riemann mapping theorem. Namely, if \( f_0 : \Omega \to D \) is a conformal mapping, then

\[
A_\rho(\Omega) = K_\rho[f_0] = \int_\Omega \rho^2(z) \, dx \, dy \leq K_\rho[f]
\]

because \( \mathbb{K}(z, f) \geq 1 \), and \( \mathbb{K}(z, f) \equiv 1 \) if and only if \( f \) is conformal.

The simply connected case where the boundary data are prescribed is solved in [4]. For the free boundary problem, Astala, Iwaniec and Martin [3] considered the first non-trivial case where there are conformal invariants; namely doubly connected domains and, in particular, annuli. Given two annuli

\[
A' = \{ w : \tau < |w| < \sigma \}, \quad A = \{ z : r < |z| < 1 \},
\]

they consider homeomorphisms of finite distortion \( f : A' \to A \) with respect to the Euclidean metric and the metric \( \rho = 1/|z| \). We shall consider the same problem, but for an arbitrary regular metric. Here note that \( |f| \) extends continuously to \( \overline{A} \), with values \( r \) and 1 on the boundary of \( A \). We shall normalize our mappings in the way so that

\[
|f(z)| = r \quad \text{for } |z| = \tau \quad \text{and} \quad |f(z)| = \sigma \quad \text{for } |z| = 1.
\]

Let \( \mathcal{F} = \mathcal{F}(A', A) \) denote the family of all normalized homeomorphisms \( f : A' \to A \) of finite distortion. Since \( A' \) and \( A \) are certainly diffeomorphic \( \mathcal{F} \neq \emptyset \).

Let \( z = x + iy = s e^{it} \) and

\[
dm(z) = dx \, dy = s \, ds \, dt
\]

be the usual Lebesgue measure on the complex plane \( \mathbb{C} \). The integral mean of the distortion function \( K(z, f) \) which concern us in this work is

\[
K_\rho[f] = \int_{A'} \mathbb{K}(z, f) \rho^2(z) \, dm(z),
\]

where \( \rho \) is a given regular metric. The minimization problem we are concerned with here is to evaluate the following infimum:

\[
\inf\{ K_\rho[f] : f \in \mathcal{F}(A', A) \}.
\]

(3.1)

Further, we should decide if the infimum is attained and, in that case, prove uniqueness (up to the obvious rotational symmetry of the annuli). The concept of the conformal modulus will prove useful in proving our results. It is convenient to take the following definition of the modulus of an annulus \( A(p, q) := \{ z : p < |z| < q \} \):

\[
\operatorname{Mod}(A(p, q)) = 2\pi \log \frac{q}{p} = \int_{A(p, q)} \frac{dm(z)}{|z|^2}.
\]

(3.2)

Note that, the standard definition of modulus is indeed \( \text{mod}(A(p, q)) = (1/2\pi) \log(q/p) \). Every topological annulus \( R \) is conformally equivalent to a round annulus \( A \), and we can set \( \operatorname{Mod}(R) = \operatorname{Mod}(A) \).
The main theorems of this paper are the following.

**Theorem 3.1.** Let \( \rho \) be a regular metric. Let \( A \) and \( A' \) be annuli satisfying the condition (2.10). Among all mappings \( f \in \mathcal{F}(A, A') \), the infimum of

\[
\int_{A'} \mathcal{K}(z, f) \rho^2(z) \, dm(z)
\]  

(3.3)

is attained by the function

\[
f^c(z) = \exp \left( i(t + \alpha) + \int_{\sigma}^s \frac{\rho(y) \, dy}{\sqrt{y^2 \rho^2(y) + c}} \right), \quad z = se^{it}, \quad \alpha \in [0, 2\pi),
\]  

(3.4)

where \( c \) is given by (2.12). Its inverse \( h^c \) is \( \rho \)-harmonic between annuli \( A' \) and \( A \).

**Remark 3.2.** In the special cases where \( \rho(z) = 1 \), we easily obtain that

\[
f^c(z) = C(c) \frac{z}{|z|} (|z| + \sqrt{|z|^2 + c}),
\]

which is the inverse of the Nitsche’s map

\[
h^c(\zeta) = C'(c) \left( \zeta - \frac{\omega(c)}{\zeta} \right),
\]

where \( \omega(c) \) is a positive constant and \( C(c), C'(c) \in \mathbb{C} \).

If \( \rho(z) = |z|^{-1} \), and \( z = se^{it} \), then \( f^c \) is a power function

\[
f^c(z) = \sigma s^{-\alpha} |z|^\alpha - 1 z \quad \text{where} \quad \alpha = \alpha(c) = \frac{\text{Mod}(A')}{\text{Mod}(A)}.
\]

See [3, Theorems 1 and 2] for the same conclusion.

**Theorem 3.3.** Let \( \rho \) be a regular metric. Under the fatness condition (2.11), the infimum of (3.3) is not attained by any homeomorphism \( f \in \mathcal{F}(A, A') \) (more generally, it is not attained by any continuous mapping of finite distortion of \( A \) onto \( A' \)). Moreover, for the inverse of critical Nitsche map \( f^\#(se^{it}) = e^{\varphi'(s)+it} \), where

\[
\varphi'(s) = \int_{\sigma}^s \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(y) \rho^2(y)}}, \quad \tau \leq s \leq \sigma,
\]  

(3.5)

there holds the sharp strict inequality

\[
\int_{A'} \mathcal{K}(z, f) \rho^2(s) \, dm(z) > \int_{A'} \mathcal{K}(z, f^\#) \rho^2(s) \, dm(z) + \frac{\tau^2 \rho^2(\tau)}{2} \text{Mod}A(r, r'),
\]  

(3.6)

where \( f \in \mathcal{F}(A, A') \)

\[
r' = \exp \left( \int_{\sigma}^\tau \frac{\rho(y) \, dy}{\sqrt{y^2 \rho^2(y) - \tau^2 \rho^2(\tau)}} \right) \quad (r < r' < 1).
\]

The minimization of the integral means of the distortion functions of homeomorphisms \( f : A' \to A \) turns to be equivalent to the Dirichlet-type problem for the inverse mapping \( h = f^{-1} : A \to A' \). If a homeomorphism \( f \in W^{1,1}_{\text{loc}}(A', A) \) has integrable distortion, then \( h \in W^{1,2}(A, A') \) and we can consider the energy functional

\[
E_\rho[h] = \int_A \| Dh(\zeta) \|^2 \rho^2(h(\zeta)) \, dm(\zeta) = \int_{A'} \mathcal{K}(z, f) \rho^2(z) \, dm(z).
\]  

(3.7)
In general, the converse of (3.8) is not true, because the inverse of a homeomorphism $h \in W^{1,2}(A, A')$ need not belong to the Sobolev class $W^{1,1}_{loc}(A, A')$. It has bounded variation, but fails to be absolutely continuous on lines, see [9] for related results. Thus

$$\{K_\rho[f] : f \in \mathcal{F}(A', A) \} \subset \{E_\rho[h] : h \in W^{1,2}(A, A')\}. \tag{3.8}$$

As in [3], we prove the correction lemma and show that the infimum of the left-hand side (3.8) is equal to the infimum of the right-hand side of (3.8).

Indeed, for every homeomorphism $h \in W^{1,2}(A, A')$, we can construct a homeomorphism $\tilde{h} \in W^{1,2}(A, A')$, with $E_\rho[\tilde{h}] \leq E_\rho[h]$, whose inverse $\tilde{f}_h$ lies in $\mathcal{F}(A', A)$ (Lemma 5.1). Thus $K_\rho[\tilde{f}_h] \leq E_\rho[h]$. As a consequence, the minimization problem for $K_\rho[f]$ is equivalent to the minimization problem for $E_\rho[h]$. Namely from (3.8) and

$$\inf \{K_\rho[f] : f \in \mathcal{F}(A', A) \} \leq \inf \{K_\rho[\tilde{f}_h] : h \in W^{1,2}(A, A') \} = \inf \{E_\rho[\tilde{h}] : \tilde{h} \in W^{1,2}(A, A') \} = \inf \{E_\rho[h] : h \in W^{1,2}(A, A') \},$$

we obtain

$$\inf \{K_\rho[f] : f \in \mathcal{F}(A', A) \} = \inf \{E_\rho[h] : h \in W^{1,2}(A, A') \}.$$  

The energy integral defined in (1.4) coincides with the energy integral in (3.7). We should point out that, the image surface $M$ is indeed the annulus $A$ with metric $\rho$ and make use of the formulas

$$|\partial g|^2 = \frac{\sigma^2 |g|_2^2}{\sigma^2(z)}, \quad |\partial \sigma|^2 = \frac{\sigma^2 |g|_2^2}{\sigma^2(z)}$$

and $dV_\sigma = \sigma^2(z) \, dm(z)$.

From Theorem 3.1 and Lemma 5.1, we deduce the following corollary.

**Corollary 3.4.** Let $\rho$ be a regular metric. Within the Nitsche range (2.10), for the annuli $A$ and $A'$, the absolute minimum of the energy integral

$$h \longrightarrow E_\rho[h], \quad h \in W^{1,2}(A, A')$$

is attained by a $\rho$-Nitsche map

$$h^\tau(z) = q^{-1}(s) e^{i(t + \beta)}, \quad z = se^{it}, \quad \tau \in [0, 2\pi),$$

where

$$q(s) = \exp \left( \int_\sigma^s \frac{dy}{\sqrt{y^2 + cg^2}} \right), \quad \tau < s < \sigma.$$  

**Remark 3.5.** If $R$ is a doubly connected surface, conformal to a given annulus $A$, and $(R', \rho')$ another annulus isometric to a given annulus $(A', \rho)$, then by Lemmas 1.2 and 1.3, the minimization of the energy integral

$$h \longrightarrow E_\rho[h], \quad h \in W^{1,2}(A, A')$$

is equivalent to the minimization of the energy integral

$$k \longrightarrow E_{\rho'}[k], \quad k \in W^{1,2}(R, R').$$

Thus Corollary 3.4 can be formulated in terms of not-necessarily rounded annuli.
From Theorem 3.3 and Lemma 5.1, we deduce the following corollary.

**Corollary 3.6.** Let $\rho$ be a regular metric. Outside the $\rho$-Nitsche range (2.10) for the annuli $A$ and $A'$, the infimum of the energy functional $E_\rho[h]$ is not attained by any homeomorphism $h' \in W^{1,2}(A, A')$.

**Remark 3.7.** Theorems 3.1, 3.3, Corollaries 3.4 and 3.6 are generalizations of corresponding [3, Theorems 1, 2, 3, Corollaries 2 and 3].

4. **Proof of Theorems 3.1 and 3.3**

We need the following elementary formulas in the sequel. Let $z = se^{it}$. Then

$$|f_z|^2 + |f_{\bar{z}}|^2 = \frac{1}{2}(|f_s|^2 + s^{-2}|f_t|^2) \quad (4.1)$$

and

$$J_f(z) = \frac{1}{s} \Im(f_t \bar{f}_s). \quad (4.2)$$

From (4.1) and (4.2), we obtain

$$K(z, f) = \frac{s|f_s|^2 + s^{-1}|f_t|^2}{2\Im(f_t \bar{f}_s)} \quad (4.3)$$

If a mapping $f$ is radial stretching between annuli $A'$ and $A$, then for some increasing function $P(s), \tau < s < \sigma$ there holds the formula

$$f( se^{it}) = P(s) e^{it}.$$  

If $\Phi(s) = \log P(s)$, then we can express the distortion function as

$$K(z, f) = \frac{1}{2}\left(s\Phi'(s) + \frac{1}{s\Phi'(s)}\right). \quad (4.4)$$

**Lemma 4.1.** Let $f$ be a mapping of finite distortion and $\varphi$ be a differentiable monotonic function. For $z \in A^+ := \{z : J_f(z) > 0\}$, we have the following equivalent inequalities:

$$K(z, f) \geq s\varphi'(s) + \frac{1 - s^2(\varphi'(s))^2}{2s^2 J_f(z)} |f_t|^2 \quad (4.5)$$

and

$$K(z, f) \geq \frac{1}{s\varphi'(s)} + \frac{s^2 \varphi'(s)^2 - 1}{\varphi'(s)^2} \frac{1}{2s^2 J_f(z)} |f_s|^2. \quad (4.6)$$

In both inequalities, the equality is attained a.e. if and only if $f_s = -i\varphi'(s)f_t$ for $z \in A^+$.

**Proof.** In view of (4.3), it is easily to verify that, the following trivial inequality:

$$|f_t - \frac{if_s}{\varphi'(s)}| \geq 0 \quad (4.7)$$

is equivalent with both inequalities (4.5) and (4.6).  

4.1. **Proof of inequalities**

**Lemma 4.2 (The main lemma).** Let $\rho$ be a regular metric and let $f : A' \to A$ be a homomorphism of finite distortion between annuli $A' = A(\tau, \sigma)$ and $A = A(r, 1)$.  

In view of this fact, we divide the proof into two cases.

(a) Assume that there holds the \( \rho \)-Nitsche condition (2.10) and let \( c \) be defined by (2.12). For \( f^c(s e^{it}) = e^{c(s)+it} \), where

\[
\phi(s) = \int_{\sigma}^{s} \frac{dy}{\sqrt{y^2 + c\rho^2(y)}}, \quad \tau \leq s \leq \sigma,
\]

there holds the inequality

\[
\int_{A'} K(z, f)\rho^2(s) \, dm(z) \geq \int_{A'} K(z, f^c)\rho^2(s) \, dm(z).
\]

(b) Assume that there holds the fatness condition (2.11). Now we make use of inverse of critical Nitsche map. For \( f^\#(s e^{it}) = e^{c^\#(s)+it} \), where

\[
\phi^\#(s) = \int_{\sigma}^{s} \frac{dy}{\sqrt{y^2 - \tau^2\rho^2(\tau)\rho^2(y)}}, \quad \tau \leq s \leq \sigma,
\]

there holds the strict inequality

\[
\int_{A'} K(z, f)\rho^2(s) \, dm(z) > \int_{A'} K(z, f^\#)\rho^2(s) \, dm(z) + \frac{\tau^2\rho^2(\tau)}{2} \text{ Mod } A(r, r'),
\]

where

\[
r' = \exp \left( \int_{\sigma}^{\tau} \frac{\rho(y) \, dy}{\sqrt{y^2 - \rho^2(\rho^2(y) - \tau^2\rho^2(\tau))}} \right) \quad (r' > r).
\]

Proof. Let \( z = s e^{it}, \ s = |z|, \ t \in [0, 2\pi) \) and \( A^+ = \{ z : J_f(z) > 0 \} \). We will apply Lemma 4.1 with

\[
\phi(s) = \int_{\sigma}^{s} \frac{dy}{\sqrt{y^2 + c\rho^2}}, \quad \tau \leq s \leq \sigma.
\]

Proof of (a). Since \( g = 1/\rho \), we have

\[
1 - s^2\phi'(s)^2 = \frac{c}{s^2 \rho^2 + c}.
\]

In view of this fact, we divide the proof into two cases.

(a) The case \( c \geq 0 \). Observe first that, for a.e. \( z = s e^{it} \in A' \)

\[
s\phi'(s) = \frac{s}{\sqrt{s^2 + c\rho^2(s)}} \leq 1 \leq K(z, f).
\]

According to (4.5) and (4.8), we have

\[
\int_{A'} K(z, f)\rho^2(s) \geq \int_{A' \setminus A^+} s\rho^2(s)\phi'(s) + \int_{A^+} s\rho^2(s)\phi'(s)
\]

\[
+ \int_{A^+} \rho^2(s) \frac{1 - s^2(\phi'(s))^2}{2s^2 J_f(z)} |f_1|^2 
\]

\[
= \int_{A'} s\rho^2(s)\phi'(s) + \int_{A^+} \frac{c}{2s^2(s^2 + c\rho^2(s))} J_f(z) |f_1|^2.
\]

By Hölder inequality, we obtain

\[
\int_{A^+} \frac{|f_1|}{s|f| \sqrt{2(s^2 + c\rho^2(s))}} \leq \left( \int_{A^+} \frac{1}{2s^2(s^2 + c\rho^2(s))} |f_1|^2 J_f(z) \right)^{1/2} \left( \int_{A^+} \frac{J_f(z)}{|f|^2} \right)^{1/2}.
\]
By [3, Lemma 1], it follows that
\[ \int_{A^+} \frac{J_f(z)}{|J|^2} \leq \int_{A^v} \frac{J_f(z)}{|J|^2} \leq \text{Mod}(A). \] (4.11)
Since
\[ \int_0^{2\pi} \frac{|f_t(s e^{it})|}{|f(s e^{it})|} \geq \left| \int_0^{2\pi} \frac{f_t(s e^{it})}{f(s e^{it})} \, dt \right| = 2\pi, \]
from (4.11), we obtain
\[ \int_{A^v} \frac{1}{2s^2(s^2 + cg^2(s))} \frac{|f_t|^2}{J_f(z)} \geq \frac{1}{\text{Mod}(A)} \left( \int_{A^v} \frac{|f_t|}{s|\sqrt{2(s^2 + cg^2(s))}|} \right)^2 \geq \frac{4\pi^2}{\text{Mod}(A)} \left( \int_{\tau}^\sigma \frac{ds}{\sqrt{2(s^2 + cg^2(s))}} \right)^2. \]
On the other hand,
\[ \int_{A^v} s\rho^2(s) \varphi'(s) \, dm(z) = 2\pi \int_{\tau}^\sigma \frac{s^2 \rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds. \]
Therefore,
\[ \int_{A^v} K(z,f)\rho^2(s) \, dm(z) \geq 2\pi \int_{\tau}^\sigma \frac{s^2 \rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds + \frac{4\pi^2}{\text{Mod}(A)} \left( \int_{\tau}^\sigma \frac{ds}{\sqrt{2(s^2 + cg^2(s))}} \right)^2. \]
Since
\[ \text{Mod}(A) = 2\pi \log \frac{1}{r} = 2\pi \int_{\tau}^\sigma \frac{ds}{\sqrt{s^2 + cg^2(s)}}, \] (4.12)
it follows that
\[ \int_{A^v} K(z,f)\rho^2(s) \, dm(z) \geq 2\pi \int_{\tau}^\sigma \frac{s^2 \rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds + \pi c \int_{\tau}^\sigma \frac{ds}{\sqrt{s^2 + cg^2(s)}} \]
\[ = \pi \int_{\tau}^\sigma \frac{s^2 \rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds + \pi \int_{\tau}^\sigma \frac{s^2 \rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds + \pi c \int_{\tau}^\sigma \frac{ds}{\sqrt{s^2 + cg^2(s)}} \]
\[ = \pi \int_{\tau}^\sigma \frac{s^2 \rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds + \pi \int_{\tau}^\sigma \rho^2(s) \sqrt{s^2 + cg^2(s)} \, ds. \]
For \( f^c(s e^{it}) = e^{\varphi(s)+it} \), where
\[ \varphi(s) = \int_{\tau}^s \frac{dy}{\sqrt{y^2 + cg^2}}, \quad \tau \leq s \leq \sigma, \]
by making use of formula (4.4), we have
\[ \int_{A^v} K(z,f^c)\rho^2(s) \, dm(z) = \frac{2\pi}{2} \int_{\tau}^\sigma s\rho^2(s) \left( \varphi'(s)s + \frac{1}{s \varphi'(s)} \right) \, ds \]
\[ = \pi \int_{\tau}^\sigma s\rho^2(s) \left( \frac{s}{\sqrt{s^2 + cg^2(s)}} + \frac{\sqrt{s^2 + cg^2(s)}}{s} \right) \, ds. \]
Thus
\[ \int_{A'} \mathbb{K}(z, f) \rho^2(s) \, dm(z) \geq \int_{A'} \mathbb{K}(z, f^c) \rho^2(s) \, dm(z). \]

(b) The case \(-\tau \rho^2(\tau) \leq c \leq 0\). In this case, we make use of (4.6). Observe also that, for a.e. \( z \in A' \)
\[ \frac{1}{s \varphi'(s)} = \frac{\sqrt{s^2 + cg^2(s)}}{s} \leq 1 \leq \mathbb{K}(z, f). \] (4.13)

For \( z \in A^+ \), we obtain
\[ \mathbb{K}(z, f) \rho^2(s) \geq \frac{\rho^2(s)}{s \varphi'(s)} + \frac{-c |f_s|^2}{2s^2 \rho^2(s) J_f}. \]

Hence by (4.13)
\[ \int_{A'} \mathbb{K}(z, f) \rho^2(s) \geq \int_{A' \setminus A^+} \frac{\rho^2(s)}{s \varphi'(s)} + \int_{A^+} \frac{\rho^2(s)}{s \varphi'(s)} + \int_{A^+} \frac{-c |f_s|^2}{2s^2 J_f} \]
\[ = \int_{A'} \frac{\rho^2(s)}{s \varphi'(s)} + \int_{A^+} \frac{-c |f_s|^2}{2s^2 J_f}. \] (4.14)

By [3, Lemma 3] and (4.12), we have
\[ \int_{A^+} \frac{1}{s^2} \frac{|f_s|^2}{J_f} \geq \text{Mod}(A) = 2\pi \int_{\frac{d}{\sqrt{s^2 + cg^2(s)}}}. \] (4.15)

On the other hand,
\[ \int_{A'} \frac{\rho^2(s)}{s \varphi'(s)} = 2\pi \int_{A^+} \frac{\rho^2(s)}{\sqrt{s^2 + cg^2(s)}} \, ds. \] (4.16)

Let
\[ k(s) := \sqrt{s^2 + cg^2(s)}. \]

From (4.14)–(4.16), we obtain
\[ \int_{A'} \mathbb{K}(z, f) \rho^2(s) \geq -c \pi \int_{\tau} \frac{ds}{k(s)} + 2\pi \int_{\tau} \frac{\rho^2(s)}{k(s)} ds \]
\[ = \pi \int_{\tau} \rho^2(s) k(s) ds + 2\pi \int_{\tau} s^2 \rho^2(s) k(s) ds \]
\[ - c \pi \int_{\tau} \frac{ds}{k(s)} + \pi \int_{\tau} \rho^2(s) k(s) ds - \pi \int_{\tau} s^2 \rho^2(s) k(s) ds \]
\[ = \int_{A'} \mathbb{K}(z, f^c) \rho^2(s) \, dm(z) + X, \] (4.17)

where
\[ X = \pi \int_{\tau} \left( -\frac{c}{k(s)} + \rho^2(s) k(s) - \frac{s^2 \rho^2(s)}{k(s)} \right) ds = 0. \]

This concludes the proof of (a).

The proof of (b). We proceed similarly as in the proof of (a). For \( \varphi = \varphi^# \), according to (4.6), for \( z \in A^+ \) we have
\[ \mathbb{K}(z, f) \rho^2(s) \geq \frac{\rho^2(s)}{s \varphi'(s)} + \frac{\tau^2 \rho^2(\tau) |f_s|^2}{2s^2 J_f}. \]

In this case, for \( c^# = -\tau^2 \rho^2(\tau) \), instead of (4.15) we have
\[ \int_{A^+} \frac{1}{s^2} \frac{|f_s|^2}{J_f} \geq \text{Mod}(A) = 2\pi \log \frac{1}{\tau} > 2\pi \int_{\frac{d}{\sqrt{s^2 + c^# g^2(s)}}}. \] (4.18)
On the other hand, the relation (4.16) holds. Hence
\[
\int_{A'} K(z, f) \rho^2(s) \, dm(z) \geq \int_{A'} K(z, f^\#) \rho^2(s) \, dm(z) + Y,
\]
where
\[
Y = -\frac{c^\#}{2} \left( 2\pi \log \frac{1}{r} - 2\pi \log \frac{1}{r'} \right) = \frac{\tau^2 \rho^2(\tau)}{2} \text{ Mod } A(r, r').
\]
This completes the proof of the lemma.

**Proof of Theorem 3.1.** In view of Lemma 4.2(a), it remains to prove the equality statement. By [3, Lemma 3], the equality in (4.15) occurs if and only if a.e. on \( A^+ := \{ z \in A' : J_f(z) > 0 \} \) we have
\[
\Re \left( \frac{f_s}{f_t} \right) = \left| \frac{f_s}{f_t} \right| \quad (4.20)
\]
and
\[
\Im \left( \frac{f_s}{f_t} \right) = k \quad (4.21)
\]
for some constant \( k > 0 \). On the other hand, by Lemma 4.1 we have the equality
\[
f_s = -i \varphi'(s) f_t, \quad (4.22)
\]
for \( z \in A^+ \). Combining (4.20)–(4.22), we arrive at the following system of PDE’s:
\[
\frac{f_t}{f} = i k \quad \text{and} \quad \frac{f_s}{f} = k \varphi'(s). \quad (4.23)
\]
Integrating the first equality over the unit circle gives \( k = 1 \). Let \( g(s, t) = \log f(s e^{it}) \). Then \( g \) is well-defined function from \([\tau, \sigma] \times [0, 2\pi]\) onto \([\log r, 0] \times [0, 2\pi]\). Then from (4.23), we obtain
\[
g = it + \phi(s) \quad \text{and} \quad g = \varphi(s) + \psi(t),
\]
for some functions \( \phi \) and \( \psi \). We obtain that, the general solution of system of PDE’s (4.23) is
\[
f(s e^{it}) = C e^{\phi(s) + it}. \quad \Box
\]

**Proof of Theorem 3.3.** The second part of Theorem 3.3 follows by Lemma 4.2(b). It remains to show that the inequality (3.6) is sharp and the infimum of (3.3) is not attained by any homeomorphism. Assume for a moment that the inequality (3.6) is sharp. Show that the infimum of (3.3) is not attained. Otherwise some \( f \in \mathcal{F}(A', A) \) would satisfy (3.6) with equality. This is a contradiction. To show that the inequality is sharp, we construct the minimizing sequence.

4.2. The minimizing sequence

We are now given two round annuli \( A \) and \( A' \) with inner and outer radii \( r, 1 \) and \( \tau, \sigma \), respectively. Moreover, we assume that the fatness condition (2.11) is satisfied. To show that the inequality (3.6) is sharp, we construct a family \( f_n : A' \to A \) of mappings of finite distortion. Let
\[
r' = \exp \left( \int_{\sigma}^{\tau} \frac{\rho(y) \, dy}{\sqrt{y^2 \rho^2(y) - \tau^2 \rho^2(\tau)}} \right).
\]
Then \( r < r' < 1 \). Let \( z = s e^{it} \) and \( n \in \mathbb{N} \) such that
\[
1 - r \left( \frac{\sigma}{\tau} \right)^n < 0.
\]
Define

\[ f_n(z) = \begin{cases} 
\exp \left( it + \int_{s_n}^{s} \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(y) \varrho^2(y)}} \right) & \text{if } s_n \leq s \leq \sigma, \\
12 \left( \frac{n+1}{n} \right) & \text{if } \tau \leq s \leq s_n.
\end{cases} \]

where \( s_n \) is a solution of the equation

\[ \exp \left( \int_{s_n}^{s} \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(y) \varrho^2(y)}} \right) = r \left( \frac{s_n}{\tau} \right)^n. \tag{4.24} \]

To show that \( s_n : \tau < s_n < \sigma \) exists satisfying the condition (4.24) take

\[ p(s) = \exp \left( \int_{s_n}^{s} \frac{dy}{\sqrt{y^2 - \tau^2 \rho^2(y) \varrho^2(y)}} \right) - r \left( \frac{s}{\tau} \right)^n. \]

Then

\[ p(\tau) p(\sigma) = (r' - r) \left( 1 - r \left( \frac{\sigma}{\tau} \right)^n \right) < 0, \]

if \( n \) is big enough. From (4.24), we easily get the relation

\[ \lim_{n \to \infty} n(s_n - \tau) = \tau \log \frac{r'}{r}. \tag{4.25} \]

Further, by a similar analysis as in [3, Section 9] we obtain that

\[ \mathcal{K}(z, f_n) = \begin{cases} 
\left( \frac{s}{\sqrt{s^2 - \tau^2 \rho^2(\tau) \varrho^2(s)}} + \frac{\sqrt{s^2 - \tau^2 \rho^2(\tau) \varrho^2(s)}}{s} \right) & \text{if } s_n \leq s \leq \sigma, \\
\frac{1}{2} \left( \frac{n+1}{n} \right) & \text{if } \tau \leq s \leq s_n.
\end{cases} \]

Finally, by (4.25)

\[
\lim_{n \to \infty} \int_{A'(\tau,\sigma)} \mathcal{K}(z, f_n) \rho^2(s) = \int_{A'(\tau,\sigma)} \mathcal{K}(z, f^\#) \rho^2(s) + \lim_{n \to \infty} \int_{A'(\tau, s_n)} \mathcal{K}(z, f_n) \rho^2(s) \\
= \int_{A'} \mathcal{K}(z, f^\#) \rho^2(s) + 2\pi \lim_{n \to \infty} n \int_{\tau}^{s_n} \rho^2(s)s \, ds \\
= \int_{A'} \mathcal{K}(z, f^\#) \rho^2(s) + \pi \rho^2(\tau) \lim_{n \to \infty} \frac{n}{2} (s_n^2 - \tau^2) \\
= \int_{A'} \mathcal{K}(z, f^\#) \rho^2(s) + \frac{\tau^2 \rho^2(\tau)}{2} \Mod A(r, r').
\]

This finishes the proof of Theorem 3.3. \( \square \)

5. Proof of Corollaries 3.4 and 3.6

In order to apply Theorems 3.1 and 3.3 to energy minimization problems for the inverse mappings (see (3.7)), proving in this way Corollaries 3.4 and 3.6, we need to establish the following lemma.

**Lemma 5.1 (The correction lemma).** Let \( D \) be doubly connected domain in \( \mathbb{C} \) and let \( A' = \{ z : \tau < |z| < \sigma \} \). Let \( h : D \to A' \), be a homeomorphism of finite Dirichlet energy,

\[ E_\rho[h] = \int_D \rho^2 \circ h(|h|^{-2} + |h|^{-2}) \, dm(z) < \infty. \]
Assume that $\rho$ is a radial metric with bounded Gauss curvature satisfying
\[ 0 < \inf_{w \in A'} \rho^2(w) \leq \sup_{w \in A'} \rho^2(w) < \infty. \] (5.1)

Then there exists a homeomorphism $\tilde{h} : D \to A'$ such that
\[ E_\rho[\tilde{h}] \leq E_\rho[h]. \]

The inverse $\tilde{f} = \tilde{h}^{-1}$ belongs to $W^{1,1}(A', D)$ and has finite distortion. We have the identity
\[ K_\rho[\tilde{f}] = E_\rho[\tilde{h}]. \]

**Proof.** We proceed as in [3, Lemma 7], however due to the different intrinsic geometry, we should provide a different proof. A domain $G$ is said to be convex with respect to the metric $\rho$, if for $z, w \in G$, there exists a geodesic line $l[z, w] \subset G$, $z, w \in l[z, w]$ with respect to the metric $\rho$. By following an argument of Buser ([5, pp. 116–121], see also [6, The proof of Theorem 1]), there exists a triangulation of the Riemann surface $(A', \rho)$ by a locally finite set of triangles $\Delta_j, j = 1, \ldots, m$ with pairwise disjoint interiors, such that the edges of triangles are geodesic arcs or part of boundary (such domains are lipschitz and convex domains (if the diameter is small enough)). Also we can assume that
\[ \text{diam}_\rho(\Delta_j) := \sup\{d_\rho(z, w) : z, w \in \Delta_j\} < \varepsilon, \]
where $\varepsilon$ is a positive constant to be chosen later.

Let us sketch the construction of these geodesic triangles. Let $\mathcal{P}$ be a finite set of points in $A$ such that any two points of $\mathcal{P}$ lie at $\rho$-distance $\leq \varepsilon$ and $\geq \varepsilon/2$ from each other.

Such a set $\mathcal{P}$ is easily obtained by successively marking points in $A'$ at pairwise distances $\geq \varepsilon/2$ until there is no more room for such points. The set $\mathcal{P}$ is finite because the $\rho$-diameter of $A'$ is finite. We add to the set $\mathcal{P}$ a maximal finite set of boundary points at pairwise distance between $\varepsilon/2$ and $\varepsilon$. For arbitrary triple of points $a, b, c$ from $\mathcal{P}$, we can construct a (possible degenerated) triangle $\Delta_i, i = 1, \ldots, n, n = |\mathcal{P}|$ defined as follows. If $a, b, c$ are inside $A'$, then the wedges are geodesic. If, for example, $a, b$ belong to the same connected component of $\partial A'$, then the wedge $ab$ is the smaller boundary arc bounded by $a$ and $b$. We exclude all degenerated triangles, those who contain more than three points inside and those whose diameter is larger than $\varepsilon$. Thus we get the family $\Delta_j, j = 1, \ldots, m$. Then each $\Delta_j$ is contained in a geodesic disk $B_\varepsilon(p)$ with center at $p \in \Delta_j$ and radius $\varepsilon$.

Through the homeomorphism $h$, we have a decomposition
\[ D = \bigcup_{k=1}^m D_k, \]
where $D_k = h^{-1}(\Delta_k)$.

Let the Gauss curvature of the metric $\rho$ satisfy $K \leq \kappa^2$. Let $C$ be a positive constant, smaller or equal to the minimal distance of a given point $p \in A'$ from its cut-locus. Because the metric is radial, the cut-locus of a point $p \in A'$ is the arc $[-\tau p, -\sigma p]$. Since the metric satisfies the condition (5.1), it follows that
\[ C = \inf\{d_\rho(x, y) : \tau \leq x, y \leq \sigma\} > 0. \]

Take
\[ \varepsilon < \min\left\{\frac{\pi}{2\kappa}, C\right\}. \] (5.2)
Consider the Dirichlet problem of finding a \( \rho \)-harmonic map \( h_j : D_j \to A' \) with the given boundary values: \( h_j|_{\partial D_j} = h \). Since \( h_j(\partial D_j) \) is contained in a geodesic disk \( B_\varepsilon(p) \) with:

(a) radius \( \varepsilon < \pi/(2\kappa) \);
(b) the cut-locus of the center \( p \) disjoint from \( B_\varepsilon(p) \);

by a result of Hildebrandt, Kaul and Widman \([11]\), this Dirichlet problem has a solution contained in \( B_\varepsilon(p) \).

Moreover, by a result of Jost \([13]\) we obtain that, since \( h_j : \partial D_j \to A' \) is a homeomorphism onto a Lipschitz convex curve \( \partial \Delta_j \), then the above solution \( h_j \) is a homeomorphism.

Let 
\[
\tilde{h} = \sum_{j=1}^{m} h_j(z) \chi_{D_j}(z).
\]

Then \( \tilde{h} \) is a homeomorphism by construction. By using the well-known energy estimates, we have
\[
\int_{D_j} \|Dh_j\|^2 \rho^2(z) \, dm(z) \leq \int_{D_j} \|D\tilde{h}\|^2 \rho^2(z) \, dm(z).
\]

Let \( \tilde{f} = \tilde{h}^{-1} \). Proceeding as in \([3, \text{Lemma } 7]\), we obtain
\[
\mathcal{K}_\rho[\tilde{f}] \leq E_\rho[h].
\]

To continue observe that
\[
E[h] = \int_D (|h_1|^2 + |h_2|^2) \, dm(z) \leq \int_D \frac{\rho^2 \circ h}{\inf_{w \in A'} \rho^2(w)} (|h_1|^2 + |h_2|^2) \, dm(z) < \infty.
\]

By using a recent result of Hencl, Koskela and Onninen \([10]\), we obtain that the homeomorphism \( f : A' \to D \) of integrable distortion in the Sobolev class \( W^{1,1}(A', D) \) has its inverse \( h \in W^{1,2}(D, A') \). By introducing the change of variables \( w = \tilde{h}(z) \), proceeding as in \([10]\), by making use of formulas
\[
\|B\| = \|B^T\|, \quad (A^T)^{-1} \det A = \text{adj } A, \quad \|\text{adj } A\| = \|A\|
\]
we obtain the identity
\[
\mathcal{K}_\rho[\tilde{f}] = E_\rho[\tilde{h}].
\]

This completes the proof of Lemma 5.1.

Remark 5.2. We think that some of the conditions of Lemma 5.1 concerning the metric \( \rho \) are superfluous.

Appendix

In this section, we will give some examples of regular metrics.

A.1. Hyperbolic metrics

For every hyperbolic Riemann surface, the fundamental group is isomorphic to a Fuchsian group, and thus the surface can be modeled by a Fuchsian model \( U/\Gamma \), where \( U \) is the unit disk and \( \Gamma \) is the Fuchsian group \([1]\). If \( \Omega \) is a hyperbolic region in the Riemann sphere \( \mathbb{C} \), that is, \( \Omega \) is open and connected with its complement \( \Omega^c := \mathbb{C} \setminus \Omega \) possessing at least three points. Each such \( \Omega \) carries a unique maximal constant curvature \( -1 \) conformal metric \( \lambda |dz| = \lambda_\Omega(z)|dz| \) referred to as the Poincaré hyperbolic metric in \( \Omega \). The domain monotonicity property, that larger regions have smaller metrics, is a direct consequence of Schwarz’s Lemma. Except for a
short list of special cases, the actual calculation of any given hyperbolic metric is notoriously difficult.

By the formula
\[ \rho_G(z) = h(|z|^2), \]
we obtain that the Gauss curvature is given by
\[ K = \frac{4(|z|^2 h^2 - |z|^2 hh'' - hh')}{h^4}. \]
Setting \( t = |z|^2 \), we obtain that
\[ K = -\frac{1}{h^2} \left( \frac{4h'(t)}{h} \right)'. \]  
(A.1)

As \( K \leq 0 \), it follows that
\[ \left( \frac{4h'(t)}{h} \right)' \geq 0. \]
Therefore, the function
\[ \frac{4h'(t)}{h} \]
is increasing, that is,
\[ t \geq s \Rightarrow \frac{4h'(t)}{h(t)} \geq \frac{4sh'(s)}{h(s)}, \]  
(A.2)
and in particular
\[ t \geq 0 \Rightarrow \frac{4h'(t)}{h(t)} \geq 0. \]  
(A.3)

In this case, we obtain that \( h \) is an increasing function.

The examples of hyperbolic surfaces are as follows.

(a) The Poincaré disk \( \mathbb{U} \) with the hyperbolic metric
\[ \lambda = \frac{2}{1 - |z|^2}. \]
(b) The punctured hyperbolic unit disk \( \Delta = \mathbb{U} \setminus \{0\} \). The linear density of the hyperbolic metric on \( \Delta \) is
\[ \lambda_\Delta = \frac{1}{|z| \log(1/|z|)}. \]
(c) The hyperbolic annulus \( A(1/R, R) \), \( R > 1 \). The hyperbolic metric is given by
\[ h_R(|z|^2) = \lambda_R(|z|) = \frac{\pi}{|z| \log R} \sec \left( \frac{\pi |z|}{2 \log R} \right). \]

In all these cases, the Gauss curvature is \( K = -1 \).

A.2. Riemann metrics

In the case of the Riemann sphere, the Gauss–Bonnet theorem implies that a constant-curvature metric must have positive curvature \( K \). It follows that the metric must be isometric to the sphere of radius \( 1/\sqrt{K} \) in \( \mathbb{R}^3 \) via stereographic projection.

(d) In the \( z \)-chart on the Riemann sphere, the metric with \( K = 1 \) is given by
\[ ds^2 = h_R^2(|z|^2)|dz|^2 = \frac{4|dz|^2}{(1 + |z|^2)^2}. \]
Another important case is Hamilton cigar soliton or in physics is known as Wittens’s black hole. It is a Kähler metric defined on $\mathbb{C}$ by

$$ds^2 = h^2(|z|^2)|dz|^2 = \frac{|dz|^2}{1 + |z|^2}.$$  

The Gauss curvature is given by

$$K = \frac{2}{1 + |z|^2}.$$  

In both these cases, $K > 0$. This means that

$$\left(\frac{4th'(t)}{h}\right)' \leq 0.$$  

Therefore, the function

$$\frac{4th'(t)}{h}$$

is decreasing, that is,

$$t \geq s \Rightarrow \frac{4th'(t)}{h(t)} \leq \frac{4sh'(s)}{h(s)}.$$  

(A.4)

In this case, we obtain that $h$ is a decreasing function.

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