Global existence and stabilization in a forager-exploiter model with general logistic sources

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Abstract. We study a forager-exploiter model with generalized logistic sources in a smooth bounded domain with homogeneous Neumann boundary conditions. A new boundedness criterion is developed to prove the global existence and boundedness of the solution. Under some conditions on the logistic degradation rates, the classical solution exists globally and remains bounded in the high dimensions. Moreover, the large time behavior of the obtained solution is investigated in the case of the nutrient supply is a positive constant or has fast decaying property.

Keywords: Forager-exploiter model; Chemotaxis; Logistic sources; Boundedness; Stabilization.

AMS subject classifications (2010): 35A09, 35B40, 35K55, 35Q92, 92C17.

1 Introduction

We consider the forager-exploiter system with generalized logistic sources

$$\begin{aligned}
    \frac{\partial u}{\partial t} &= \Delta u - \chi \nabla \cdot (u \nabla w) + a_1 u - b_1 u^\alpha, & & x \in \Omega, \ t > 0, \\
    \frac{\partial v}{\partial t} &= \Delta v - \xi \nabla \cdot (v \nabla u) + a_2 v - b_2 v^\beta, & & x \in \Omega, \ t > 0, \\
    \frac{\partial w}{\partial t} &= \Delta w - (u + v) w - \mu w + r, & & x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), & & x \in \Omega,
\end{aligned}$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $\frac{\partial}{\partial \nu}$ denotes the homogeneous Neumann boundary condition, and $\chi, \xi, \mu$ are positive constants. We use $u, v$ and $w$ to denote population densities of the forager, exploiter and nutrient, respectively. This model includes two basic settings. Firstly, the forager and exploiter pursue nutrient as their common food. Secondly, except the random diffusions, the forager move towards the increasing nutrient gradient direction (corresponding to the first taxis dynamic: $-\nabla \cdot (u \nabla w)$), while the exploiter follows the forager to find nutrients (corresponding to the second taxis mechanism: $-\nabla \cdot (v \nabla u)$). It has been indicated in [18, 25] that, as a doubly cross-diffusive parabolic system, (1.1) possess more complex dynamics than the single-taxis model. Although problem (1.1) has attracted many attentions recently, only a few results are available. The mathematical results are summarized as follows:

Without kinetic source terms: $a_1 = a_2 = b_1 = b_2 = 0$. The classical solution exists globally and remain bounded in one space dimension ([19]). Whereas, in the high dimensions, smallness conditions on the initial data and production rate or weak taxis effects are required to

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ensure the global solvability of (1.1) (23). This differs from the chemotaxis-consumption or prey taxis model whose global solvability in two dimensional case is well-known (10, 20). Even for the generalized solutions, the global existence relies on some smallness conditions on the initial data $w_0$ and production rate of $w$. In [6, 14], it is found that the full saturation or limited saturation can exclude blow up phenomenon. For the study of the forager-exploiter model with singular sensitivities, nonlinear resource consumption or proliferation, refer to [5, 13, 15].

With generalized logistic source: $a_1, a_2, b_1, b_2 > 0$. In [4], the global generalized solutions are obtained in two dimensional setting provided $\alpha > \sqrt{2} + 1$ and $\min\{\alpha, \beta\} > (\alpha + 1)/(\alpha - 1)$, and the eventual smoothness of the generalized solution after some waiting time are provided under some more restrict conditions. When $2 \leq \alpha < 3$ and $\beta \geq 3\alpha/(2\alpha - 3)$ or $\alpha, \beta \geq 3$, the global existence and boundedness of the classical solution in two dimensional setting are obtained in [23]. Later on, the recent work [26] claims that $2 \leq \alpha < 3, \beta \geq 3$ suffices to ensure the global boundedness of the solution. It is noted that these results only hold in two dimensional setting, and the conditions for $\alpha, \beta$ seem too strong and may be far from optimal. Moreover, no results on the global solvability of (1.1) with $a_1, a_2, b_1, b_2 > 0$ in high dimensions ($N > 2$) are available. This manuscript has two motivations. The first one is to establish the global solvability of (1.1) in the high dimensions ($N > 2$) and relax the conditions in two dimensional case. The other one is to establish the large time behavior of the solutions obtained.

Before stating the main results, we give basic assumptions for the initial data and nutrient supply $r$. The initial data $u_0, v_0, w_0$ satisfy

$$u_0, v_0, w_0 \in W^{2,\infty}(\Omega), \quad u_0, v_0, w_0 \geq 0, \neq 0 \text{ on } \bar{\Omega}, \quad \text{and} \quad \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega.$$  

The production rate function $r$ is nonnegative and satisfies

$$r \in C^1(\bar{\Omega} \times [0, \infty)) \cap L^\infty(\Omega \times (0, \infty)). \quad (1.2)$$

We can hence denote

$$r_* := \|r\|_{L^\infty(\Omega \times (0, \infty))}. \quad (1.3)$$

Throughout this article, we always set $a_1, a_2, b_1, b_2 > 0$. The first result concerns the global solvability of (1.1) in high dimensions (i.e., $N > 2$).

**Theorem 1.1.** Let $N > 2$. Suppose that $\alpha, \beta > \frac{N}{2} + 1$. Then the problem (1.1) admits a unique nonnegative and global solution $(u, v, w) \in (C^{2,1}(\bar{\Omega} \times (0, \infty)))^3$ which is bounded in $\bar{\Omega} \times (0, \infty)$.

In two dimensional setting, we have

**Theorem 1.2.** Let $N = 2$ and $\alpha \geq 2$ with $\beta > 2$. Then there exists a solution $(u, v, w) \in (C^{2,1}(\bar{\Omega} \times (0, \infty)))^3$ solving (1.1) uniquely and remain bounded in $\bar{\Omega} \times (0, \infty)$.

We remark that, Theorem 1.1 gives the first result of global solvability of forager-exploiter model with generalized logistic sources in high dimensions, and Theorem 1.2 improves the previous results in [23, 26]. The proofs of Theorem 1.1 and Theorem 1.2 are based on a boundedness criterion in Proposition 3.1. This criterion is different from the known one in [3]. We use the nutrient-taxi...
model as an example to see the difference of these two criteria. Dropping $v$ and its equation, system (1.1) becomes a nutrient-taxis model

$$
\begin{aligned}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla w) + a_1 u - b_1 u^\alpha, \quad x \in \Omega, \quad t > 0, \\
    w_t &= \Delta w - uw - \mu w + r, \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega.
\end{aligned}
$$

(1.4)

The boundedness criterion in [3, Lemma 3.2] says that the local solution of system (1.4) exists globally and remain bounded provided the $L^p$ regularity of $u$ with $p > \frac{N}{2}$ is uniformly bounded. Whereas, our boundedness criterion says that the condition can be replaced by the space-time $L^p$ regularity of $u$ with $p > \frac{N}{2} + 1$.

For the Keller-Segel chemotaxis model, the classical logistic source can prevent blow-up in two space dimension ([17, 27]). Is this still true for (1.1)? Unfortunately, the conclusion in Theorem 1.2 holds for $\alpha = 2$ and $\beta > 2$. For the case $\beta = 2$, we have to leave this problem in the future.

We next investigate the large time behavior of the solution obtained in the above theorems and see how the taxis mechanisms and logistic damping rates affect the large time behavior. Clearly, the nutrient supply $r$ is critical for the large time behavior of $w$. We shall consider two cases:

- **Fast decaying resupply of nutrient**, i.e.,
  $$
  \int_0^\infty \int_\Omega r(x, t)dxdt < \infty. \tag{1.5}
  $$

- **$r$ is a positive constant**.

We define

$$
\bar{u} = \left( \frac{a_1}{b_1} \right)^{\frac{1}{\alpha-1}}, \quad \bar{v} = \left( \frac{a_2}{b_2} \right)^{\frac{1}{\beta-1}}
$$

and

$$
\bar{w} = \frac{r}{\bar{u} + \bar{v} + \mu} \text{ if } r \text{ is a positive constant.}
$$

In the case that $r$ has fast decaying property, without any additional restrictions on the parameters, we show the large time behavior.

**Theorem 1.3.** Let $\alpha, \beta > \frac{N}{2} + 1$ when $N > 2$, and $\alpha \geq 2$ with $\beta > 2$ when $N = 2$. Suppose that $r$ satisfies (1.2) and (1.5). Then

$$
\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.
$$

Furthermore, if $r$ satisfies (1.2) and

$$
\int_\Omega r(\cdot, t) \leq Ke^{-\delta t} \text{ for all } t > 0
$$

for some $K, \delta > 0$, then there exist $\lambda, C > 0$ such that

$$
\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \text{ for all } t > 0.
$$
If \( r \) is a positive constant, the large time behavior reads as follows.

**Theorem 1.4.** Let \( \alpha, \beta > \frac{N}{2} + 1 \) when \( N > 2 \), and \( \alpha \geq 2 \) with \( \beta > 2 \) when \( N = 2 \). Assume that \( r \) is a positive constant. Then there exist \( \tilde{\chi}, \tilde{\xi}, \tilde{b}_1, \tilde{b}_2 > 0 \) such that, once either

\[
\chi < \tilde{\chi}
\]
or
\[
b_1 > \tilde{b}_1 \quad \text{and} \quad b_2 > \tilde{b}_2
\]
or
\[
b_1 > \tilde{b}_1 \quad \text{and} \quad \xi < \tilde{\xi},
\]

one has

\[
\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \quad \text{for all } t > 0
\]

for some \( C, \lambda > 0 \).

The above theorem shows that, the solution will eventually reach a co-existence homogeneous steady state provided that \( \chi \) is small enough, or the logistic damping rates \( b_1 \) and \( b_2 \) are sufficiently large, or the logistic damping rate \( b_1 \) is large and \( \xi \) is small enough. It seems a little surprising that the large time behavior holds for small \( \chi \), without any small condition for \( \xi \).

## 2 Existence and uniqueness of local solutions, some preliminaries

The following lemma asserts the local-in-time existence of the classical solutions to (1.1).

**Lemma 2.1.** Let \( N \geq 1 \) and \( \alpha, \beta > 1 \). Then there exist \( T_{\text{max}} \in (0, \infty) \) and nonnegative functions

\[
u, v, w \in \bigcap_{q>n} C^0([0,T_{\text{max}}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\text{max}}))
\]

which solves (1.1) in \((0,T_{\text{max}})\) in the classical sense. Moreover, if \( T_{\text{max}} < \infty \), then

\[
\limsup_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{W^{1,p}(\Omega)} + \|v(\cdot, t)\|_{W^{1,p}(\Omega)} + \|w(\cdot, t)\|_{W^{1,p}(\Omega)}) = \infty \quad \text{for all } p > n.
\]

**Proof.** The local existence of the classical solution can be obtained by using Amann’s theory, and the nonnegativity of the solution is deduced by the maximum principle (cf. [19, Lemma 2.1]). \( \square \)

We set from now on that \( \tau = \min\{T_{\text{max}}/2, 1\} \) and \( T_{\text{max}} = T_{\text{max}} - \tau \). The following gives the \( L^\infty \) bound for \( w \) and some basic regularities for \( u \) and \( v \).

**Lemma 2.2.** Let \( N \geq 1, \alpha, \beta > 1 \) and \( M = \|w_0\|_{L^\infty(\Omega)} + \frac{r}{\mu} \). Then

\[
\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq M \quad \text{for all } t \in (0,T_{\text{max}})
\]

and

\[
\int_{\Omega} u \leq M_1 \quad \text{and} \quad \int_{\Omega} v \leq M_2 \quad \text{for all } t \in (0,T_{\text{max}})
\]
Proof. The inequality (2.1) has been proven in [19, Lemma 2.2].

A simple use of Hölder’s inequality provides $|\Omega|^{1-\alpha}\langle \int_{\Omega} u \rangle^\alpha \leq \int_{\Omega} u^\alpha$. We then integrate the first equation in (1.1) over $\Omega$ to get

$$\frac{d}{dt} \int_{\Omega} u = a_1 \int_{\Omega} u - b_1 \int_{\Omega} u^\alpha \leq a_1 \int_{\Omega} u - b_1 |\Omega|^{1-\alpha}(\int_{\Omega} u)^\alpha \quad \text{for all } t \in (0, T_{\text{max}})$$

which implies

$$\int_{\Omega} u \leq \int_{\Omega} u_0 + |\Omega| \left( \frac{a_1}{b_1} \right)^{\frac{1}{\alpha-1}} \quad \text{for all } t \in (0, T_{\text{max}}). \quad (2.5)$$

Integrating (2.4) upon $(t, t+\tau)$ for $t \in (0, T_{\text{max}})$, and using (2.5), we have

$$\int_{t}^{t+\tau} \int_{\Omega} u^\alpha \leq C_1 b_1^{-1} (b_1^{\frac{1}{\alpha-1}} + 1) \quad \text{for all } t \in (0, T_{\text{max}})$$

for some $C_1 = C_1(|\Omega|, \int_{\Omega} u_0, a_1, \alpha) > 0$. We thus obtain the regularity properties for $u$ in (2.2) and (2.3). The statements for $v$ in (2.2) and (2.3) can be derived by the same way.

The following lemma claims that, once we get the uniform boundedness of $u$ (resp. $v$), the estimation of $\nabla u$ (resp. $\nabla v$) relies on $\Delta w, \nabla w$ (resp. $\Delta u, \nabla u$).

**Lemma 2.3.** Let $N \geq 1$ and $\alpha, \beta > 1$.

(i) Suppose that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq k_1$ for all $t \in (0, T_{\text{max}})$. Then for $p > 1$, there exists $C = C(N, \alpha, \beta, k_1, a_1, \chi) > 0$ such that

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^{2p} + \int_{\Omega} |\nabla u|^{2p} \leq C \left( \int_{\Omega} |\nabla w|^{2(p+1)} + \int_{\Omega} |\Delta w|^{p+1} + 1 \right) \quad \text{for all } t \in (0, T_{\text{max}}). \quad (2.6)$$

(ii) Suppose that $\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq k_2$ for all $t \in (0, T_{\text{max}})$. Then for $p > 1$, there exists $C_* = C_*(N, \alpha, \beta, k_2, a_2, \xi) > 0$ such that

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \int_{\Omega} |\nabla v|^{2p} \leq C_* \left( \int_{\Omega} |\nabla u|^{2(p+1)} + \int_{\Omega} |\Delta u|^{p+1} + 1 \right) \quad \text{for all } t \in (0, T_{\text{max}}). \quad (2.7)$$

**Proof.** By direct computations (cf. [23]), we have

$$\frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla u|^{2p} + \int_{\Omega} |\nabla u|^{2p} = \int_{\Omega} |\nabla u|^{2(p-1)} \nabla u \cdot \nabla u_t + \int_{\Omega} |\nabla u|^{2p} = \int_{\Omega} |\nabla u|^{2(p-1)} \nabla u \cdot (\Delta u - \chi \nabla \cdot (u \nabla w) + a_1 u - b_1 u^\alpha) + \int_{\Omega} |\nabla u|^{2p} \leq \int_{\Omega} |\nabla u|^{2(p-1)} \nabla u \cdot \nabla \Delta u + \chi \int_{\Omega} \nabla \cdot (|\nabla u|^{2(p-1)} \nabla u)(\nabla \cdot (u \nabla w)) + (a_1 + 1) \int_{\Omega} |\nabla u|^{2p} =: I(t) + J(t) + (a_1 + 1) \int_{\Omega} |\nabla u|^{2p} \quad \text{for all } t \in (0, T_{\text{max}}). \quad (2.8)$$
Following the derivation of [23] (3.21), one can find $C_1 = C_1(N, p, |\Omega|, k_1, \chi) > 0$ fulfilling
\[
\frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla u|^{2p} + \int_{\Omega} |\nabla u|^{2p} + \frac{1}{16} \int_{\Omega} |\nabla u|^{2(p-1)} D^2 w|^2 \\
\leq C_1 \int_{\Omega} |\nabla w|^{2(p+1)} + C_1 \int_{\Omega} |\Delta w|^{p+1} + (C_1 + a_1) \int_{\Omega} |\nabla u|^{2p} \quad \text{for all } t \in (0, T_{\text{max}}). \tag{2.9}
\]

We use the third term in the left hand side of (2.9) to absorb $(C_1 + a_1) \int_{\Omega} |\nabla u|^{2p}$ (cf. [23] (3.22)), it then arrives at
\[
\frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla u|^{2p} + \int_{\Omega} |\nabla u|^{2p} \\
\leq C_2 \int_{\Omega} |\nabla w|^{2(p+1)} + C_2 \int_{\Omega} |\Delta w|^{p+1} + C_2 \quad \text{for all } t \in (0, T_{\text{max}})
\]
for some $C_2 = C_2(N, p, |\Omega|, k_1, a_1, \chi) > 0$. The deduction of (2.7) is similar.

We then construct a relationship between $\nabla w$ and $u, v$.

**Lemma 2.4.** Let $N \geq 1$. For $p \geq 1$, one can find $C = C(N, p, |\Omega|, \|w_0\|_{L^\infty(\Omega)}, r_*, \mu) > 0$ such that
\[
\frac{d}{dt} \int_{\Omega} |\nabla w|^{2p} + \mu \int_{\Omega} |\nabla w|^{2p} \leq C \int_{\Omega} u^{p+1} + C \int_{\Omega} v^{p+1} + C \quad \text{for all } t \in (0, T_{\text{max}}). \tag{2.10}
\]

**Proof.** By straightforward computations, we have
\[
\frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla w|^{2p} + \mu \int_{\Omega} |\nabla w|^{2p} \\
= \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla w_t + \mu \int_{\Omega} |\nabla w|^{2p} \\
= \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla (\Delta w - (u + v)w - \mu w + r) + \mu \int_{\Omega} |\nabla w|^{2p} \\
= \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \Delta w - \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla (uw) - \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla (vw) \\
+ \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla r \\
=: I_1(t) + I_2(t) + I_3(t) + I_4(t) \quad \text{for all } t \in (0, T_{\text{max}}). \tag{2.11}
\]

In view of [12] Lemma 2.2 and (2.1), there holds
\[
\int_{\Omega} |\nabla w|^{2(p+1)} \leq 2(N + 4p^2)\|w\|_{L^\infty(\Omega)}^{2p} \int_{\Omega} |\nabla w|^{2(p-1)} D^2 w|^2 \leq \tilde{k} \int_{\Omega} |\nabla w|^{2(p-1)} D^2 w|^2 \tag{2.12}
\]
for all $t \in (0, T_{\text{max}})$ with $\tilde{k} := 2(N + 4p^2)M^2$ where $M$ is given by Lemma 2.2.

Recalling (2.1), similar to the derivation of [23] (3.15) (or [23] (4.14)), by adjusting some parameters, one can find $C_1 = C_1(N, p, |\Omega|, \|w_0\|_{L^\infty(\Omega)}, r_*, \mu) > 0$ such that, for all $t \in (0, T_{\text{max}})$,
\[
I_1(t) \leq -\frac{3}{4} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2 - \frac{p - 1}{4} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla |\nabla w|^2|^2 + C_1. \tag{2.13}
\]
Using integration by parts, Young’s inequality, (2.12) and the known inequality: $|\Delta w| \leq \sqrt{N}|D^2 w|$, the second term on the right hand side of (2.11) can be estimated as:

$$I_2(t) = - \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla (uw)$$

$$= (p - 1) \int_{\Omega} u w |\nabla w|^{2(p-2)} |\nabla w|^2 \cdot \nabla w + \int_{\Omega} u w |\nabla w|^{2(p-1)} \Delta w$$

$$\leq M(p - 1) \int_{\Omega} u |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w| + M \sqrt{N} \int_{\Omega} u |\nabla w|^{2(p-1)} |D^2 w|$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{8} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2$$

$$+ M^2 (4(p - 1) + 2N) \int_{\Omega} u^2 |\nabla w|^{2(p-1)}$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{8} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2$$

$$+ \frac{1}{8k} \int_{\Omega} |\nabla w|^{2(p-1)} + C_2 \int_{\Omega} u^{p+1}$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{4} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2 + C_2 \int_{\Omega} u^{p+1} \quad (2.14)$$

for some $C_2 = C_2(N, p, \|w_0\|_{L^\infty(\Omega)}, r_*, \mu) > 0$. Similarly, $I_3$ can be estimated as:

$$I_3(t) = - \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla (vw)$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{4} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2 + C_2 \int_{\Omega} u^{p+1} \quad (2.15)$$

For the last term $I_4$, similar to the deduction of (2.14), we find

$$I_4(t) = \int_{\Omega} |\nabla w|^{2(p-1)} \nabla w \cdot \nabla r$$

$$= -(p - 1) \int_{\Omega} r |\nabla w|^{2(p-2)} |\nabla w|^2 \cdot \nabla w - \int_{\Omega} r |\nabla w|^{2(p-1)} \Delta w$$

$$\leq r_* (p - 1) \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w| - r_* \sqrt{N} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{8} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2 + r_*^2 (4(p - 1) + 2N) \int_{\Omega} |\nabla w|^{2(p-1)}$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{8} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2 + \frac{1}{8k} \int_{\Omega} |\nabla w|^{2(p-1)} + C_3$$

$$\leq \frac{p - 1}{16} \int_{\Omega} |\nabla w|^{2(p-2)} |\nabla w|^2 |\nabla w|^2 + \frac{1}{4} \int_{\Omega} |\nabla w|^{2(p-1)} |D^2 w|^2 + C_3 \quad (2.16)$$

for all $t \in (0, T_{\max})$ for some $C_3 = C_3(N, p, \|w_0\|_{L^\infty(\Omega)}, r_*, \mu) > 0$. Plugging (2.13), (2.14)-(2.16) into (2.11), one can find $C_4 = C_4(N, p, |\Omega|, \|w_0\|_{L^\infty(\Omega)}, r_*, \mu) > 0$ such that

$$\frac{1}{2p} \frac{d}{dt} \int_{\Omega} |\nabla w|^{2p} + \mu \int_{\Omega} |\nabla w|^{2p} \leq C_4 \int_{\Omega} u^{p+1} + C_4 \int_{\Omega} u^{p+1} + C_4$$

for all $t \in (0, T_{\max})$.

This completes the proof. \[\square\]

We collect an important lemma from [23, 16, 4].
Lemma 2.5. Let \( N \geq 1, T > 0 \) and \( \theta = \min\{\frac{T}{2}, \frac{1}{2}\} \). Suppose that for some \( p > 1 \) and \( K, H > 0 \),
\[
\int_t^{t+\theta} \int_{\Omega} |f|^p \leq K \quad \text{and} \quad \int_t^{t+\theta} \int_{\Omega} |z|^p \leq H \quad \text{for all} \ t \in (0, T - \theta),
\]
and \( z \in C^{2,1}(\Omega \times (0,T)) \) solves
\[
\begin{cases}
  z_t = \Delta z + f(x,t), & x \in \Omega, \ 0 < t < T, \\
  \frac{\partial z}{\partial \nu} = 0, & x \in \partial \Omega, \ 0 < t < T, \\
  z(x,0) = z_0(x), & x \in \Omega,
\end{cases}
\]
where \( z_0 \in W^{2,\infty}(\Omega) \) with \( z_0 \geq 0 \) and \( \frac{\partial z_0}{\partial \nu} = 0 \) on \( \partial \Omega \). Then, there is \( C > 0 \) fulfilling
\[
\int_t^{t+\theta} \int_{\Omega} |z_t|^p_{L^p(\Omega)} + \int_t^{t+\theta} \int_{\Omega} |z|^p_{W^{2,p}(\Omega)} \leq C(\|z_0\|_{W^{2,p}(\Omega)}^p + K + H) \quad \text{for all} \ t \in (0, T - \theta). \tag{2.18}
\]
Especially, one can find \( C_\ast > 0 \) such that
\[
\int_t^{t+\theta} \int_{\Omega} |\Delta z|^p \leq C(\|z_0\|_{W^{2,p}(\Omega)}^p + K + H) \quad \text{for all} \ t \in (0, T - \theta).
\]

Proof. We used the ideas in [4, 16] to show this lemma. The proof will be split into two cases.

Case I: \( T \leq 2 \). It is easy to see that \( \theta = T/2 \), i.e., \( T = 2\theta \). Then, we use the maximal Sobolev regularity properties of the Neumann heat semigroup \((e^{t\Delta})_{t \geq 0}\) \([9]\) and the first inequality in (2.17) to get \( C_1 > 0 \) fulfilling
\[
\int_0^{2\theta} \int_{\Omega} |z_t|^p_{L^p(\Omega)} + \int_0^{2\theta} \int_{\Omega} |z|^p_{W^{2,p}(\Omega)} \leq C_1 \|z_0\|_{W^{2,p}(\Omega)}^p + C_1 \int_0^{2\theta} \int_{\Omega} |f|^p_{L^p(\Omega)} \leq C_1 \|z_0\|_{W^{2,p}(\Omega)}^p + 2C_1 K. \tag{2.19}
\]
Thus, we get (2.18) directly.

Case II: \( T > 2 \). Clearly, we have \( \theta = 1 \). Hence, (2.18) can be rewritten as
\[
\int_t^{t+1} \int_{\Omega} |z_t|^p_{L^p(\Omega)} + \int_t^{t+1} \int_{\Omega} |z|^p_{W^{2,p}(\Omega)} \leq C(\|z_0\|_{W^{2,p}(\Omega)}^p + K + H) \quad \text{for all} \ t \in (0, T - 1). \tag{2.20}
\]

Let \( \rho \in C^\infty(\mathbb{R}) \) be an increasing function satisfying
\[
0 \leq \rho \leq 1 \ \text{in} \ \mathbb{R}, \quad \rho \equiv 0 \ \text{in} \ (-\infty, 0], \quad \rho \equiv 1 \ \text{in} \ (1, \infty).
\]
It is easy to see that \( \|\rho\|_{L^\infty(\mathbb{R})} = \|\rho\|_{L^\infty([0,1])} \). For arbitrary fixed \( \sigma \in (0, T - 2) \), we define \( \rho_\sigma(t) = \rho(t - \sigma) \). Then, \( \rho_\sigma \in [0, 1] \) and \( \|\rho_\sigma\|_{L^\infty(\mathbb{R})} = \|\rho\|_{L^\infty([0,1])} \). By direct computations, \( \rho_\sigma z \) solves
\[
\begin{cases}
  (\rho_\sigma z)_t = \Delta (\rho_\sigma z) + \rho_\sigma f(x,t), & x \in \Omega, \ t \in (\sigma, T), \\
  \frac{\partial (\rho_\sigma z)}{\partial \nu} = 0, & x \in \partial \Omega, \ t \in (\sigma, T), \\
  (\rho_\sigma z)(x, \sigma) = 0, & x \in \Omega.
\end{cases}
\]
Making use of the maximal Sobolev regularity properties of the Neumann heat semigroup \((e^{t\Delta})_{t\geq 0}\) (9), there exist \(C_2 > 0\) such that
\[
\int_{\sigma}^{\sigma+2} \| (\rho_{\sigma} z)_t \|_{L^p(\Omega)}^p + \int_{\sigma}^{\sigma+2} \| \rho_{\sigma} z \|_{W^{2,p}(\Omega)}^p \leq C_2 \int_{\sigma}^{\sigma+2} \| \rho'_{\sigma} z + \rho_{\sigma} f \|_{L^p(\Omega)}^p \quad \text{for all } \sigma \in (0, T-2).
\]

Thanks to (2.17) and the boundedness properties of \(\rho_{\sigma}\), we have from the above inequality that
\[
\int_{\sigma}^{\sigma+2} \| \rho_{\sigma} z \|_{L^p(\Omega)}^p + \int_{\sigma}^{\sigma+2} \| \rho_{\sigma} z \|_{W^{2,p}(\Omega)}^p \leq C_2 \rho'_{\sigma} \| z \|_{L^p(\Omega)}^p + C_2 \int_{\sigma}^{\sigma+2} \| f \|_{L^p(\Omega)}^p \quad \text{for all } \sigma \in (0, T-2),
\]
which combined with \(\rho_{\sigma} = 1\) in \((\sigma+1, \sigma+2)\) implies
\[
\int_{\sigma+1}^{\sigma+2} \| z_t \|_{L^p(\Omega)}^p + \int_{\sigma+1}^{\sigma+2} \| z \|_{W^{2,p}(\Omega)}^p \leq C_3 (K + H) \quad \text{for all } \sigma \in (0, T-2),
\]
i.e.,
\[
\int_t^{t+1} \| z_t \|_{L^p(\Omega)}^p + \int_t^{t+1} \| z \|_{W^{2,p}(\Omega)}^p \leq C_3 (K + H) \quad \text{for all } t \in (1, T-1).
\]

For \(t \in (0, 1]\), we have \((2.20)\) from \((2.19)\) with \(\theta = 1\). So, we get \((2.18)\) in the case of \(T_{\text{max}} > 2\). The proof is finished.

Before ending this section, we recall from \([23]\) a generalized version of Gronwall’s inequality.

**Lemma 2.6.** Let \(c, k > 0\). Assume that for some \(\hat{T} \in (0, \infty]\) and \(\hat{\tau} = \min\{1, \frac{\hat{T}}{T}\}\), the nonnegative functions \(y \in C([0, \hat{T}]) \cap C^1((0, \hat{T}))\), \(z \in L^1_{\text{loc}}([0, \hat{T}])\) and satisfy
\[
y'(t) + cy(t) \leq z(t), \quad t \in (0, \hat{T}),
\]
\[
\int_t^{t+\hat{\tau}} z(s) ds \leq k, \quad t \in (0, \hat{T} - \hat{\tau}).
\]
Then
\[
y(t) \leq y(0) + 2k + \frac{k}{c}, \quad t \in (0, \hat{T}).
\]

3 A criteria governing boundedness of solutions

It is known that, the uniform-in-time \(L^\infty\) boundedness of \(u\) can be ensured by \((2.2)\) and \(L^p\) regularity of \(\nabla w\) with \(p > N\). And, Lemma 2.4 and Lemma 2.6 tell us that, we can use the space-time integral property of \(u, v\) to estimate the \(L^p\) regularity of \(\nabla w\). Hence, we obtain the uniform boundedness of \(u\) provided suitable space-time regularity for \(u\) and \(v\). The upper bound for \(u\) in the coming lemma is independent of \(\chi, \xi, b_1, b_2\), which is critical for the construction of the global stability of the positive equilibrium.
Lemma 3.1. Let $N \geq 2$ and $\alpha, \beta > 1$. Suppose that there exist $\bar{p}, \bar{q} > \frac{N}{2} + 1$ and $K_1, K_2 > 0$ such that
\begin{equation}
\int_{t}^{t+\tau} \int_{\Omega} u^\bar{p} \leq K_1 \quad \text{and} \quad \int_{t}^{t+\tau} \int_{\Omega} v^\bar{q} \leq K_2 \quad \text{for all} \ t \in (0, \tilde{T}_{\max}).
\end{equation}
Then, there exist $C > 0$, $\theta > 1$ and $\eta =: \min\{\bar{p}, \bar{q}\} > \frac{N}{2} + 1$ independent of $\chi, \xi, b_1, b_2, K_1, K_2$ fulfilling
\begin{equation}
\|\nabla w(\cdot, t)\|_{L^{2(n-1)}(\Omega)} \leq C(K_1 + K_2 + 1) \quad \text{for all} \ t \in (0, T_{\max})
\end{equation}
and
\begin{equation}
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(b_1^{-\frac{1}{n-1}} + 1) + C\chi^\theta M_1(K_1 + K_2 + 1)^\theta \quad \text{for all} \ t \in (0, T_{\max}).
\end{equation}

Proof. We use $C_i$ to denote the general constants appeared in the proof, which are independent of $t$ and $\chi, \xi, b_1, b_2, K_1, K_2$. Let $\kappa := 2(\eta - 1)$. Clearly, $\kappa > N$. We have from (3.1) that
\begin{equation}
\int_{t}^{t+\tau} \left( \int_{\Omega} (u^\bar{p} + v^\bar{q}) + 1 \right) \leq C_1(K_1 + K_2 + 1) \quad \text{for all} \ t \in (0, \tilde{T}_{\max})
\end{equation}
for some $C_1 > 0$. By Lemma 2.4 there are $C_2, C_3 > 0$ such that
\begin{equation}
\frac{d}{dt} \int_{\Omega} |\nabla w|^\kappa + C_2 \int_{\Omega} |\nabla w|^\kappa \leq C_3 \left( \int_{\Omega} (u^\bar{p} + v^\bar{q}) + 1 \right) \quad \text{for all} \ t \in (0, T_{\max}).
\end{equation}
Then, in light of Lemma 2.6 we get
\begin{equation}
\int_{\Omega} |\nabla w|^\kappa \leq \int_{\Omega} |\nabla w_0|^\kappa + \left( 2 + \frac{1}{C_2} \right) C_1 C_5(K_1 + K_2 + 1)
\leq C_4(K_1 + K_2 + 1) \quad \text{for all} \ t \in (0, T_{\max})
\end{equation}
where $C_4 = \int_{\Omega} |\nabla w_0|^\kappa + \left( 2 + \frac{1}{C_2} \right) C_1 C_3 + 1$. We thus get (3.2). Since $C_4 > 1$ and $K_1 + K_2 + 1 > 1$, we have from (3.1) that
\begin{equation}
\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4(K_1 + K_2 + 1) \quad \text{for all} \ t \in (0, T_{\max}).
\end{equation}
Let us take $N < q < \kappa$ and set $\theta := \frac{\kappa q}{\kappa - q} > N$. We denote
\begin{equation}
H(T) = \sup_{t \in (0, T)} \|u(\cdot, t)\|_\infty \ < \infty \quad \text{for} \ T \in (0, T_{\max}).
\end{equation}
On the basis of a variation-of-constants representation of $u$ and the known regularized properties of $(e^{t\Delta})_{t \geq 0}$, (8.24), one can find $\lambda_1, C_5 > 0$ such that
\begin{equation}
\|u(\cdot, t)\|_\infty \leq \|u_0\|_{L^\infty(\Omega)} + \chi \int_{0}^{t} \|e^{(t-s)\Delta} \nabla \cdot (u \nabla w)\|_{L^\infty(\Omega)} ds + \int_{0}^{t} \|e^{(t-s)\Delta} (a_1 u - b_1 u^\alpha)\|_{L^\infty(\Omega)} ds
\leq \|u_0\|_{L^\infty(\Omega)} + \chi \int_{0}^{t} \|e^{(t-s)\Delta} \nabla \cdot (u \nabla w)\|_{L^\infty(\Omega)} ds + \int_{0}^{t} \|e^{(t-s)\Delta} (a_1 u - b_1 u^\alpha)\|_{L^\infty(\Omega)} ds
\leq C_5 + C_5 \chi \int_{0}^{t} \left( 1 + (t-s)^{-\frac{1}{2} - \frac{N}{2q}} \right) e^{-\lambda_1(t-s)} \|u \nabla w\|_{L^q(\Omega)} ds
+ \int_{0}^{t} \|e^{(t-s)\Delta} (a_1 u - b_1 u^\alpha)\|_{L^\infty(\Omega)} ds \quad \text{for all} \ t \in (0, T).
\end{equation}
Making use of Hölder’s inequality, (3.5) and the first inequality in (2.2), we infer that, for all \( t \in (0, T) \),

\[
\| u \nabla w \|_{L^q(\Omega)} \leq \| u \|_{\theta} \| \nabla w \|_{L^\infty(\Omega)} = \left( \int_{\Omega} u^\theta \right)^{\frac{1}{\theta}} \| \nabla w \|_{L^\infty(\Omega)} \\
\leq \left( \int_{\Omega} u \right)^{\frac{2}{\theta}} H^{\frac{\theta-1}{\sigma}}(T) \| \nabla w \|_{L^\infty(\Omega)} \\
\leq C_4 M_1^{\frac{1}{\beta}} (K_1 + K_2 + 1) H^{\frac{\theta-1}{\sigma}}(T)
\]

(3.7)

Letting \( f(u) = a_1 u - b_1 u^\alpha \), there exists \( C_6 > 0 \) such that

\[
(a_1 u - b_1 u^\alpha)_+ \leq f \left( \left( \frac{a_1}{b_1 \alpha} \right)^{\frac{1}{1-\alpha}} \right) \leq C_6 b_1^{-\frac{1}{1-\alpha}}.
\]

In conjunction with the known regularized properties of \((e^{t\Delta})_{t \geq 0}\) (2.4), this shows that

\[
\int_0^t \| e^{(t-s)\Delta} (a_1 u - b_1 u^\alpha)_+ \|_{L^\infty(\Omega)} ds \\
\leq C_7 \int_0^t e^{-\lambda_1(t-s)} \| (a_1 u - b_1 u^\alpha)_+ \|_{L^\infty(\Omega)} ds \\
\leq C_8 b_1^{-\frac{1}{1-\alpha}} \quad \text{for all } t \in (0, T_{\max})
\]

(3.8)

for some \( C_7, C_8 > 0 \). Inserting (3.7) and (3.8) into (3.6) yields that, for some \( C_9 > 0 \),

\[
\| u(\cdot, t) \|_{\infty} \leq C_9 (b_1^{-\frac{1}{1-\alpha}} + 1) + C_9 \chi M_1^{\frac{1}{\beta}} (K_1 + K_2 + 1) H^{\frac{\theta-1}{\sigma}}(T) \int_0^t \left( 1 + (t-s)^{-\frac{1}{2} - \frac{N}{2}} \right) e^{-\lambda_1(t-s)} ds,
\]

\[
\leq C_9 (b_1^{-\frac{1}{1-\alpha}} + 1) + C_{10} \chi M_1^{\frac{1}{\beta}} (K_1 + K_2 + 1) H^{\frac{\theta-1}{\sigma}}(T) \quad \text{for all } t \in (0, T),
\]

which implies

\[
H(T) \leq C_9 (b_1^{-\frac{1}{1-\alpha}} + 1) + C_{10} \chi M_1^{\frac{1}{\beta}} (K_1 + K_2 + 1) H^{\frac{\theta-1}{\sigma}}(T) \quad \text{for all } T \in (0, T_{\max}).
\]

Hence, thanks to Young’s inequality, we have

\[
H(T) \leq C_{11} (b_1^{-\frac{1}{1-\alpha}} + 1) + C_{11} \chi^\theta M_1 (K_1 + K_2 + 1)^\theta
\]

for all \( T \in (0, T_{\max}) \). This combined with the definition of \( H(T) \) finishes the proof. \( \square \)

We proceed to derive the uniform-in-time \( L^\infty \) boundedness of \( v \). Similar to the situation of \( u \), the boundedness of \( v \) relies on \( \nabla u \). Since we have obtained the \( L^\infty \) boundedness of \( u \), Lemma 2.3(i) can be applied to estimate \( \nabla u \). To achieve this, we need to establish the space-time \( L^p \) integral property of \( \Delta w \) which is ensured by the Sobolev maximal regularity asserted in Lemma 2.5.

**Lemma 3.2.** Let \( N \geq 2 \) and \( \alpha, \beta > 1 \). Assume that for some \( \tilde{p}, \tilde{q} > \frac{N}{2} + 1 \), there exist \( K_1, K_2 > 0 \) fulfilling

\[
\int_t^{t+\tau} \int_{\Omega} u^\theta \leq K_1 \quad \text{for all } t \in (0, T_{\max})
\]
and
\[ \int_t^{t+\tau} \int_\Omega v^\beta \leq K_2 \quad \text{for all } t \in (0, \hat{T}_{\max}). \] (3.9)

Then there exists \( C > 0 \) such that
\[ \int_\Omega |\nabla u|^{2(\beta-1)} \leq C \quad \text{for all } t \in (0, T_{\max}), \] (3.10)

and
\[ \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \] (3.11)

Proof. Let \( f(x, t) := -(u + v)w - \mu w + r \). By (1.3), (2.1), (3.3) and (3.9), there exists \( C_1 > 0 \) such that
\[ \int_t^{t+\tau} \int_\Omega |f|^\beta \leq C_1 \quad \text{for all } t \in (0, \hat{T}_{\max}). \] (3.12)

It follows from (1.1) that \( w \) satisfies
\[
\begin{cases}
 w_t = \Delta w + f(x, t), & x \in \Omega, \ t > 0, \\
 \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
 w(x, 0) = w_0(x), & x \in \Omega.
\end{cases}
\] (3.13)

Thanks to (2.1) and (3.12), we apply Lemma 2.5 to (3.13) to find \( C_2 > 0 \) such that
\[ \int_t^{t+\tau} \int_\Omega |\Delta w|^{\beta} \leq C_2 \quad \text{for all } t \in (0, \hat{T}_{\max}). \] (3.14)

It follows from (3.3) and Lemma 2.3(i) that
\[
\frac{d}{dt} \int_\Omega |\nabla u|^{2(\beta-1)} + \int_\Omega |\nabla u|^{2(\beta-1)} \leq C_3 \left( \int_\Omega |\nabla w|^{2\beta} + \int_\Omega |\Delta w|^{\beta} + 1 \right) \quad \text{for all } t \in (0, T_{\max}).
\] (3.15)

where \( C_3 > 0 \). In view of the Gagliardo-Nirenberg inequality and (2.1), there are \( C_4, C_5 > 0 \) such that
\[ \int_\Omega |\nabla w|^{2\beta} = \|\nabla w\|^{2\beta}_{L^{2\beta}(\Omega)} \leq C_4 \|\Delta w\|_{L^{\beta}(\Omega)}^{\beta} \|w\|_{L^\infty(\Omega)}^{\beta} + C_4 \|w\|_{L^\infty(\Omega)}^{2\beta} \leq C_5 \int_\Omega |\Delta w|^{\beta} + C_5.
\]

Inserting this into (3.15) yields \( C_6 > 0 \) fulfilling
\[ \frac{d}{dt} \int_\Omega |\nabla u|^{2(\beta-1)} + \int_\Omega |\nabla u|^{2(\beta-1)} \leq C_6 \int_\Omega |\Delta w|^{\beta} + C_6 \quad \text{for all } t \in (0, T_{\max}). \] (3.16)

By using (3.16), (3.14) and Lemma 2.6, there exists \( C_7 > 0 \) such that
\[ \int_\Omega |\nabla u|^{2(\beta-1)} \leq C_7 \quad \text{for all } t \in (0, T_{\max}). \] (3.17)

We thus obtain (3.10).

It follows from \( \tilde{\beta} > \frac{N}{2} + 1 \) that \( \tilde{\beta} := 2(\beta-1) > N \). By (3.17), there is \( C_8 > 0 \) such that
\[ \int_\Omega |\nabla u|^{\beta} \leq C_8 \quad \text{for all } t \in (0, T_{\max}). \] (3.18)

By standard arguments paralleled to Lemma 3.1, one can deduce (3.11) by using (3.18) and (2.2). \( \Box \)
The following provides a criterion for the global existence and boundedness of the solution.

**Proposition 3.1.** Let $N \geq 2$ and $\alpha, \beta > 1$. Suppose that there exist $\bar{p}, \bar{q} > \frac{N}{2} + 1$ and $K_1, K_2 > 0$ fulfilling

$$
\int_t^{t+\tau} \int_{\Omega} u^\bar{q} \leq K_1 \quad \text{for all } t \in (0, \tilde{T}_{\max})
$$

and

$$
\int_t^{t+\tau} \int_{\Omega} v^\bar{p} \leq K_2 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
$$

Then $T_{\max} = \infty$, and there exist $\theta \in (0, 1)$ and $C > 0$ fulfilling

$$
\|u\|_{C^{2+\theta,1+\theta}_{t,x}([t, t+1], \Omega)} + \|v\|_{C^{2+\theta,1+\theta}_{t,x}([t, t+1], \Omega)} + \|w\|_{C^{2+\theta,1+\theta}_{t,x}([t, t+1], \Omega)} \leq C \quad \text{for all } t \in (0, \infty).
$$

(3.19)

**Proof.** Let us take $p > \frac{N}{2}$ throughout this proof. Thanks to (3.3) and (3.11), we can use Lemma 2.4 and Gronwall’s inequality to find $C_1 > 0$ fulfilling

$$
\int_{\Omega} |\nabla u|^{2(p+2)} \leq C_1 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
$$

(3.20)

Again by (3.3) and (3.11), similar to the derivation of (3.14), there is $C_2 > 0$ such that

$$
\int_t^{t+\tau} \int_{\Omega} |\Delta u|^{p+2} \leq C_2 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
$$

(3.21)

Based on (3.3), (3.20) and (3.21), we apply Lemma 2.3(i) and Lemma 2.6 to get $C_3 > 0$ fulfilling

$$
\int_{\Omega} |\nabla u|^{2(p+1)} \leq C_3 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
$$

(3.22)

Rewriting the equation of $u$ in (1.1) as

$$
\begin{cases}
  u_t = \Delta u + F(x, t), & x \in \Omega, \quad t \in (0, T_m), \\
  \frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega, \quad t \in (0, T_m), \\
  u(x, 0) = u_0, & x \in \Omega,
\end{cases}
$$

where $F(x, t) = -\chi(\nabla u \cdot \nabla w + u \Delta w) + a_1 u - b_1 u^\alpha$. Making use of Young’s inequality, (3.3), (3.20), (3.21) and (3.22), there exists $C_4 > 0$ fulfilling

$$
\int_t^{t+\tau} \int_{\Omega} |F(x, s)|^{p+1} \, dx \, ds \leq C_4 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
$$

This combined with (3.3) enables us to apply Lemma 2.5 to get $C_5 > 0$ such that

$$
\int_t^{t+\tau} \int_{\Omega} |\Delta u|^{p+1} \leq C_5 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
$$

(3.23)

With (3.11) at hand, we use Lemma 2.3(ii) to get $C_6 > 0$ such that

$$
\frac{d}{dt} \int_{\Omega} |\nabla v|^{2p} + \int_{\Omega} |\nabla v|^{2p} \leq C_6 \left( \int_{\Omega} |\nabla u|^{2(p+1)} + \int_{\Omega} |\Delta u|^{p+1} + 1 \right) \quad \text{for all } t \in (0, T_{\max}).
$$
Then, by \((3.22)\), \((3.23)\) and Lemma 2.6, we have
\[
\int_\Omega |\nabla v(\cdot, t)|^{2p} \leq C_7 \quad \text{for all } t \in (0, T_{\max})
\] (3.24)
for \(C_7 > 0\).

From \((2.1)\), \((3.3)\), \((3.11)\), \((3.2)\), \((3.22)\) and \((3.24)\), one can find \(q > N\) and \(C_8 > 0\) such that
\[
\|u(\cdot, t)\|_{W^{1,q}(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C_8 \quad \text{for all } t \in (0, T_{\max}).
\] (3.25)
This deduces that \(T_{\max} = \infty\) due to Lemma 2.1. Based on \((3.25)\), the regularities in \((3.19)\) can be derived by a standard reasoning involving the known parabolic regularity theory in [11] (cf. [21, Theorem 2.1]).

**Remark 3.1.** By some minor changes, it can be shown that the criterion in Proposition 3.1 also holds for the cases that (i) \(a_1 = b_1 = 0\) and \(b_2 > 0\) with \(\beta > 1\), (ii) \(a_2 = b_2 = 0\) and \(b_1 > 0\) and \(\alpha > 1\), (iii) \(a_i = b_i = 0\) for \(i = 1, 2\).

## 4 Boundedness for the forager-exploiter model with logistic sources

**Proof of Theorem 1.1.** Since \(\alpha, \beta > \frac{N}{2} + 1\), the conditions of Proposition 3.1 are satisfied according to \((2.3)\). Hence, we can get Theorem 1.1 from Proposition 3.1.

The following lemma shows that one can improve the space-time regularity of \(u\) provided \(\alpha \geq 2\) and \(\beta \geq 2\) in two dimensional setting. We also establish an explicit upper bound so that it can be used in the stability arguments.

**Lemma 4.1.** Let \(N = 2\). Suppose that \(\alpha \geq 2\), \(\beta \geq 2\). Then there exists \(C > 0\) independent of \(\chi, \xi, b_1, b_2\) such that
\[
\int_{t}^{t+\tau} \int_\Omega u^3 \leq C \left( M_1H_1(H_1 + H_2)e^{C(\chi^2 + M_1^4 + 1)(H_1 + H_2)} + M_1^3 \right) \quad \text{for all } t \in (0, \tilde{T}_{\max}),
\] (4.1)
where \(H_1 = b_1^{-1}(b_1^{-\frac{1}{\alpha-1}} + 1) + 1\) and \(H_2 = b_2^{-1}(b_2^{-\frac{1}{\beta-1}} + 1) + 1\).

**Proof.** The derivation of \((4.1)\) follows the ideas of [11, Lemma 2.5] and [23, Lemma 4.1]. However, to see how \(\chi, \xi, b_1, b_2\) influence the upper bound of \(\int_{t}^{t+\tau} \int_\Omega u^3\), delicate analysis will be processed in the following. Keeping in mind that \(H_1, H_2 > 1\). Let \(f(x,t) = -(u + v)w - \mu w + r\). For the simplicity, we set
\[
\Sigma := \left\{ |\Omega|, \int_\Omega u_0, \int_\Omega u_0^2, \int_\Omega v_0, \|w_0\|_{W^{2,\infty}(\Omega)}, a_1, a_2, \alpha, \beta, r, \mu \right\}
\]
By \((2.3)\) with \(\alpha, \beta \geq 2\), there exists \(C_1 = C_1(|\Omega|, \int_\Omega u_0, \int_\Omega v_0, a_1, a_2, \alpha, \beta) > 0\) such that
\[
\int_{t}^{t+\tau} \int_\Omega u^2 \leq C_1 H_1 \quad \text{and} \quad \int_{t}^{t+\tau} \int_\Omega v^2 \leq C_1 H_2 \quad \text{for all } t \in (0, \tilde{T}_{\max}).
\] (4.2)
We deduce from \((1.3)\), \((2.1)\) and \((4.2)\) that
\[
\int_{t}^{t+\tau} \int_\Omega |f|^2 \leq C_2 (H_1 + H_2), \quad t \in (0, \tilde{T}_{\max})
\]
for some $C_2 = C_2(\Sigma) > 0$. Recalling (2.1), an application of Lemma 2.5 yields $C_3 = C_3(\Sigma) > 0$ such that
\[
\int_t^{t+\tau} \int_{\Omega} |\Delta w|^2 \leq C_3(\mathcal{H}_1 + \mathcal{H}_2) \quad \text{for all } t \in (0, \tilde{T}_{\max}). \tag{4.3}
\]

Testing the first equation of (1.1) by $u$ and using (2.2), the Gagliardo-Nirenberg inequality and Young’s inequality, one can find $C_4, C_5 > 0$ depending upon $\Sigma$ fulfilling (cf. [23 Lemma 4.1])
\[
\frac{d}{dt} \int_{\Omega} u^2 + 2 \int_{\Omega} |\nabla u|^2 \\
\leq C_4(\|\nabla u\|_2\|u\|_2 + M_1^2)\|\Delta w\|_2 + 2\alpha_1 \int_{\Omega} u^2 \\
\leq \|\nabla u\|^2_2 + C_5(\chi^2 + M_1^4 + 1)(\|u\|^2_2\|\Delta w\|^2_2 + \|u\|^2_2 + \|\Delta w\|^2_2 + 1) \quad \text{for all } t \in (0, T_{\max}),
\]
i.e.,
\[
z'(t) + \int_{\Omega} |\nabla u|^2 \leq C_5(\chi^2 + M_1^4 + 1)z(t)h(t) \quad \text{for all } t \in (0, T_{\max}), \tag{4.4}
\]
where
\[
z(t) = \int_{\Omega} |u(\cdot, t)|^2 + 1, \quad h(t) = \int_{\Omega} |\Delta w(\cdot, t)|^2 + 1.
\]

We next claim that, there is $C_* = C_*(\Sigma) > 0$ such that
\[
z(t) = \int_{\Omega} |u(\cdot, t)|^2 + 1 \leq C_*\mathcal{H}_1 e^{C_*(\chi^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \quad \text{for all } t \in (0, T_{\max}). \tag{4.5}
\]
For any $0 \leq \tilde{t} \leq t < T_{\max}$, we have from (4.4) that
\[
z(t) \leq z(\tilde{t}) e^{C_5(\chi^2 + M_1^4 + 1)\int_{\tilde{t}} h(s)ds}. \tag{4.6}
\]
By (4.6) and (4.3), there is $C_6 = C_6(\Sigma) > 0$ such that
\[
z(t) \leq z(0) e^{C_5(\chi^2 + M_1^4 + 1)\int_0 h(s)ds} \leq C_6 e^{C_5(\chi^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \quad \text{for } t \in [0, \tau]. \tag{4.7}
\]
If $T_{\max} < 2$, then $T_{\max} = 2\tau$ by the definition of $\tau$. Thanks to (4.6), (4.3) and (4.7), one can find $C_7 = C_7(\Sigma) > 0$ fulfilling
\[
z(t) \leq z(\tau) e^{C_5(\chi^2 + M_1^4 + 1)\int_0 h(s)ds} \leq C_7 e^{C_5(\chi^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \quad \text{for } t \in (\tau, T_{\max}).
\]
This combined with (4.7) gives $C_8 = C_8(\Sigma) > 0$ such that
\[
z(t) = \int_{\Omega} |u(\cdot, t)|^2 + 1 \leq C_8 e^{C_5(\chi^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \quad \text{for all } t \in (0, T_{\max}). \tag{4.8}
\]
We thus obtain (4.3) due to $\mathcal{H}_1 > 1$.

When $T_{\max} \geq 2$, we have $\tau = 1$. For $t \in (1, T_{\max})$, by the known mean value theorem and (4.2), there exists $t_0 \in [t - 1, t]$ such that
\[
z(t_0) = \int_{\Omega} u^2(\cdot, t_0) + 1 \leq C_1 \mathcal{H}_1 + 1 \leq (C_1 + 1)\mathcal{H}_1
\]
Making use of (4.3), we have
\[
\int_{t_0}^{t} h(s) \, ds = \int_{t_0}^{t} \left( \int_{\Omega} |\Delta w|^2 \, dx + 1 \right) \, ds \leq \int_{t-1}^{t} \left( \int_{\Omega} |\Delta w|^2 \, dx + 1 \right) \, ds \leq C_3(\mathcal{H}_1 + \mathcal{H}_2) + 1 \leq (C_3 + 1)(\mathcal{H}_1 + \mathcal{H}_2).
\]
Hence, we have from (4.6) that
\[
z(t) \leq z(t_0) e^{C_5(x^2 + M_1^4 + 1) \int_{t_0}^{t} h(s) \, ds} \leq C_9 \mathcal{H}_1 e^{C_5(x^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \quad \text{for all } 1 < t < T_{\max}.
\]
for some \( C_9 = C_9(\Sigma) > 0 \). In conjunction with (4.7), this shows (4.3).

Inserting (4.5) into (4.3) yields
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq C_s \mathcal{H}_1 e^{C_s(x^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \left( \int_{\Omega} |\Delta w|^2 + 1 \right) \quad \text{for all } t \in (0, T_{\max}),
\]
which by an integration upon \((t, t + \tau)\) for \( t \in (0, T_{\max} - \tau) \) implies that, for all \( t \in (0, T_{\max}) \),
\[
\int_{t}^{t+\tau} \int_{\Omega} |\nabla u|^2 \leq \int_{t}^{t+\tau} u^2(-, t) + C_s \mathcal{H}_1 e^{C_s(x^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \left( \int_{t}^{t+\tau} \int_{\Omega} |\Delta w|^2 + 1 \right).
\]
By means of (4.3) and (4.5) again, one can find \( C_{10} = C_{10}(\Sigma) > 0 \) such that
\[
\int_{t}^{t+\tau} \int_{\Omega} |\nabla u|^2 \leq C_{10} \mathcal{H}_1 (\mathcal{H}_1 + \mathcal{H}_2) e^{C_{10}(x^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} \quad \text{for all } t \in (0, T_{\max}). \tag{4.9}
\]

By the Gagliardo-Nirenberg inequality, we obtain from (2.2) that
\[
\|u\|_{L_2(\Omega)}^3 \leq C_{11} \left( \|\nabla u\|_{L_2(\Omega)}^2 \|u\|_{L_1(\Omega)} + \|u\|_{L_1(\Omega)}^3 \right) \leq C_{11} \left( M_1 \|\nabla u\|_{L_2(\Omega)}^2 + M_1^3 \right) \quad \text{for all } t \in (0, T_{\max})
\]
for some \( C_{11} = C_{11}(\Sigma) > 0 \). This combined with (4.9) provides
\[
\int_{t}^{t+\tau} u^3 \leq C_{11} \left( C_{10} M_1 \mathcal{H}_1 (\mathcal{H}_1 + \mathcal{H}_2) e^{C_{10}(x^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} + M_1^3 \right) \quad \text{for all } t \in (0, T_{\max}).
\]
This shows (4.1). The proof is end. \( \square \)

**Proof of Theorem 1.2** From (4.11) and the second inequality in (2.3) with \( \beta > 2 \), it is easy to see that the conditions of Proposition 3.1 are satisfied for \( N = 2 \). Then, we get Theorem 1.2 from Proposition 3.1. \( \square \)

## 5 Large time behavior of the solution

The following explicit \( L^\infty \)-boundedness of \( u \) plays an important role in the stability analysis in the case that \( r \) is a positive constant.

**Lemma 5.1.** (i) Let \( N > 2 \) and \( \alpha, \beta > \frac{N}{2} + 1 \). Then there exist \( C > 0 \) and \( \theta > 1 \) independent of \( \chi, \xi, b_1, b_2 \) such that
\[
M_u := C(\chi^\theta + 1)(b_1^{-\frac{1}{\theta - 1}} + 1) \left( b_1^{-\frac{1}{\alpha - 1}} + 1 \right) \left( b_2^{-\frac{1}{\beta - 1}} + 1 \right) + 1)^\theta
\]
\[
M_u := C(\chi^\theta + 1)(b_1^{-\frac{1}{\theta - 1}} + 1) \left( b_1^{-\frac{1}{\alpha - 1}} + 1 \right) \left( b_2^{-\frac{1}{\beta - 1}} + 1 \right) + 1)^\theta
\]
satisfying

\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_u \quad \text{for all } t \in (0, \infty). \] (5.1)

(ii) Let \( N = 2 \) and \( \alpha \geq 2 \) with \( \beta > 2 \). Then there exist \( C > 0 \) and \( \theta > 1 \) independent of \( \chi, \xi, b_1, b_2 \) such that

\[ M_u := C(b_1^{\frac{1}{\alpha-1}} + 1) + C\chi^\theta M_1 \left( M_1 \mathcal{H}_1(\mathcal{H}_1 + \mathcal{H}_2)e^{C(\chi^2 + M_1^4 + 1)(\mathcal{H}_1 + \mathcal{H}_2)} + M_1^3 + \mathcal{H}_2 \right)^\theta \]

fulfilling (5.1), where \( \mathcal{H}_1 = b_1^{-1}(b_1^{\frac{1}{\alpha-1}} + 1) + 1 \) and \( \mathcal{H}_2 = b_2^{-\frac{1}{\beta-1}}(b_2^{\frac{1}{\beta-1}} + 1) + 1 \).

Proof. For \( N > 2 \), we combine (2.2), (4.1), the second inequality in (2.3) and (3.3) to get (5.1).

For \( N = 2 \), we use (2.2), (4.1), the second inequality in (2.3) and (3.3) to get (5.1). \( \square \)

The following lemma provides an inequality which does not require any additional restrictions for \( r \).

Lemma 5.2. Let \( N \geq 2 \) and

\[ \alpha, \beta > \frac{N}{2} \text{ when } N > 2; \quad \alpha \geq 2, \beta > 2 \text{ for } N = 2. \] (5.2)

Then,

\[
\frac{d}{dt} \left\{ \int_\Omega \left[ u - \bar{u} - \bar{u} \ln \frac{u}{\bar{u}} \right] + \frac{\bar{u}}{\xi^2 v M_\bar{u}} \int_\Omega \left( v - \bar{v} - \bar{v} \ln \frac{v}{\bar{v}} \right) \right\} \\
\leq \frac{\chi^2 \bar{u}}{2} \int_\Omega |\nabla w|^2 - b_1 \bar{u}^\alpha - 2 \int_\Omega (u - \bar{u})^2 - \frac{b_2 \bar{u} \bar{v}^{\beta-2}}{\xi^2 M_\bar{u}} \int_\Omega (v - \bar{v})^2 \quad \text{for all } t \in (0, \infty). \] (5.3)

Proof. By standard calculations,

\[
\frac{d}{dt} \int_\Omega \left( u - \bar{u} - \bar{u} \ln \frac{u}{\bar{u}} \right) \\
= -\bar{u} \int_\Omega \frac{[\nabla u]^2}{u^2} - \chi \bar{u} \int_\Omega \nabla u \cdot \nabla w - b_1 \int_\Omega (u - \bar{u})(u^{\alpha-1} - \bar{u}^{\alpha-1}) \\
\leq -\bar{u} \int_\Omega \frac{[\nabla u]^2}{u^2} + \chi \bar{u} \int_\Omega |\nabla w|^2 - b_1 \bar{u}^\alpha - 2 \int_\Omega (u - \bar{u})^2 \\
\leq -\frac{\bar{u}}{2 M_\bar{u}} \int_\Omega |\nabla u|^2 + \frac{\chi \bar{u}}{2} \int_\Omega |\nabla w|^2 - b_1 \bar{u}^\alpha - 2 \int_\Omega (u - \bar{u})^2, \] (5.4)

where we have used Young’s inequality, (5.1) and a basic inequality (cf. [7]) (4.14]). Similarly,

\[
\frac{d}{dt} \int_\Omega \left( v - \bar{v} - \bar{v} \ln \frac{v}{\bar{v}} \right) \\
\leq -\bar{v} \int_\Omega \frac{[\nabla v]^2}{v^2} + \frac{\xi \bar{v}}{2} \int_\Omega |\nabla u|^2 - b_2 \bar{v} \beta - 2 \int_\Omega (v - \bar{v})^2 \\
\leq \xi \bar{v} \int_\Omega |\nabla u|^2 - b_2 \bar{v} \beta - 2 \int_\Omega (v - \bar{v})^2. \] (5.5)
Multiplying (5.5) by $\frac{\ddot{u}}{\xi^2 \dot{v} M^2}$ and adding the obtained result to (5.4), we get

$$\frac{d}{dt} \left\{ \int_{\Omega} \left( u - \ddot{u} - \dddot{u} \ln \frac{u}{\ddot{u}} \right) + \frac{\ddot{u}}{\xi^2 \dot{v} M^2} \int_{\Omega} \left( v - \ddot{v} - \dddot{v} \ln \frac{v}{\ddot{v}} \right) \right\}$$

$$\leq \frac{\chi^2 \ddot{u}}{2} \int_{\Omega} |\nabla w|^2 - b_1 \dddot{u} - 2 \int_{\Omega} (u - \ddot{u})^2 - \frac{b_2 \ddot{u} \dddot{v}^{\beta - 3}}{\xi^2 M^2} \int_{\Omega} (v - \ddot{v})^2 + \frac{\chi^2 \ddot{u} \mu}{2} \int_{\Omega} w^2 + \frac{\chi^2 \ddot{u} M}{2} \int_{\Omega} \bar{r}.$$

This shows (5.3). \hfill \Box

### 5.1 Global stability of $(\ddot{u}, \ddot{v}, 0)$

With fast decaying property (1.5), the large time behavior of the solution reads as follows.

**Lemma 5.3.** Suppose that (1.2) and (1.5) hold as well as

$$\alpha, \beta > \frac{N}{2} \text{ when } N > 2; \quad \alpha \geq 2, \beta > 2 \text{ for } N = 2.$$  

Then,

$$\|u(\cdot, t) - \ddot{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \ddot{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \quad (5.6)$$

**Proof.** Using (2.1), it is easy to get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 = -\int_{\Omega} |\nabla w|^2 - \int_{\Omega} (u + v)w^2 - \mu \int_{\Omega} w^2 + \int_{\Omega} rw$$

$$\leq -\int_{\Omega} |\nabla w|^2 - \mu \int_{\Omega} w^2 + M \int_{\Omega} r \quad \text{for all } t \in (0, \infty). \quad (5.7)$$

Multiplying (5.7) by $\frac{\chi^2 \ddot{u}}{2}$ and adding the obtained result to (5.3) to get

$$\frac{d}{dt} \left\{ \int_{\Omega} \left( u - \ddot{u} - \dddot{u} \ln \frac{u}{\ddot{u}} \right) + \frac{\ddot{u}}{\xi^2 \dot{v} M^2} \int_{\Omega} \left( v - \ddot{v} - \dddot{v} \ln \frac{v}{\ddot{v}} \right) \right\}$$

$$\leq -b_1 \dddot{u} - 2 \int_{\Omega} (u - \ddot{u})^2 - \frac{b_2 \ddot{u} \dddot{v}^{\beta - 3}}{\xi^2 M^2} \int_{\Omega} (v - \ddot{v})^2 - \frac{\chi^2 \ddot{u} \mu}{2} \int_{\Omega} w^2 + \frac{\chi^2 \ddot{u} M}{2} \int_{\Omega} \bar{r} \quad (5.8)$$

for all $t \in (0, \infty)$. Let

$$\mathcal{E}(t) := \int_{\Omega} \left( u - \ddot{u} - \dddot{u} \ln \frac{u}{\ddot{u}} \right) + \frac{\ddot{u}}{\xi^2 \dot{v} M^2} \int_{\Omega} \left( v - \ddot{v} - \dddot{v} \ln \frac{v}{\ddot{v}} \right) + \frac{\chi^2 \ddot{u}}{4} \int_{\Omega} w^2,$$

and

$$\mathcal{F}(t) := b_1 \dddot{u} - 2 \int_{\Omega} (u - \ddot{u})^2 + \frac{b_2 \ddot{u} \dddot{v}^{\beta - 3}}{\xi^2 M^2} \int_{\Omega} (v - \ddot{v})^2 + \frac{\chi^2 \ddot{u} \mu}{2} \int_{\Omega} w^2.$$

Then it follows from (5.8) that

$$\mathcal{E}'(t) \leq -\mathcal{F}(t) + \frac{\chi^2 \ddot{u} M}{2} \int_{\Omega} r \quad \text{for all } t \in (0, \infty). \quad (5.9)$$

Noting that $\mathcal{E}(t) \geq 0$ for any $t > 0$, integrating (5.9) over $(1, \infty)$ and using (1.5), we have

$$\int_1^\infty \mathcal{F}(t) dt \leq \mathcal{E}(1) + \frac{\chi^2 \ddot{u} M}{2} \int_1^\infty \int_{\Omega} r < \infty.$$
By the regularity of \((u, v, w)\) in (3.19), we know that \(\mathcal{F}(t)\) is uniformly continuous in \([1, \infty)\). Making use of Barbalat’s Lemma (cf. [1, 2]), it follows that \(\mathcal{F}(t) \to 0\) as \(t \to 0\), i.e.,

\[
\int (u(\cdot, t) - \bar{u})^2 + \int (v(\cdot, t) - \bar{v})^2 + \int w^2 \to 0 \quad \text{as} \quad t \to 0.
\]  

(5.10)

Again by the regularities in \((3.19)\), the \(W^{1,\infty}(\Omega)\)-norm of \((u, v, w)\) is bounded. Hence, we can apply the Gagliardo-Nirenberg inequality to get

\[
\|u - \bar{u}\|_{L^\infty(\Omega)} \leq C_1 \|u - \bar{u}\|_{W^{1,\infty}(\Omega)}^{N/(N+2)} \|u - \bar{u}\|_{L^2(\Omega)}^{2/(N+2)} \leq C_2 \|u - \bar{u}\|_{L^2(\Omega)}^{2/(N+2)} \quad \text{for all} \quad t \in (0, \infty)
\]

for some \(C_1, C_2 > 0\). This combined with \((5.10)\) gives the \(L^\infty\) convergence statement of \(u\) in \((5.6)\). By the same arguments, one can get the convergence statements of \(v\) and \(w\). The proof is finished.

\[\square\]

**Lemma 5.4.** Let \(\alpha, \beta > \frac{N}{2} + 1\) for \(N > 2\) and \(\alpha \geq 2, \beta > 2\) for \(N = 2\). Suppose that \(r\) satisfies (1.2) and

\[
\int r(\cdot, t) \leq Ke^{-\lambda t} \quad \text{for all} \quad t > 0
\]

(5.11)

for some \(K, \lambda > 0\). Then, there exist \(\kappa, C > 0\) such that

\[
\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{-\kappa t} \quad \text{for all} \quad t > 0.
\]

(5.12)

**Proof.** The following formula for \(f(x) = x - a \ln x\) with \(a > 0\) is a consequence deduced by using L’Hôpital’s rule

\[
\lim_{x \to a} \frac{f(x) - f(a)}{(x - a)^2} = \lim_{x \to a} \frac{f'(x)}{2(x - a)} = \frac{1}{2a}.
\]

Recalling \((5.6)\), there is \(t_0 > 0\) such that, for all \(t > t_0\),

\[
\frac{1}{4\bar{u}} \int (u - \bar{u})^2 \leq \frac{1}{\bar{u}} \int (u - \bar{u} - \bar{u} \ln \frac{u}{\bar{u}}) \leq \frac{1}{\bar{u}} \int (u - \bar{u})^2,
\]

(5.13)

\[
\frac{1}{4\bar{u}} \int (v - \bar{v})^2 \leq \frac{1}{\bar{v}} \int (v - \bar{v} - \bar{v} \ln \frac{v}{\bar{v}}) \leq \frac{1}{\bar{v}} \int (v - \bar{v})^2.
\]

(5.14)

Thanks to the right inequalities in \((5.13)\) and \((5.14)\), one can find \(C_1 > 0\) fulfilling

\[
\mathcal{E}(t) \leq \frac{1}{C_1} \mathcal{F}(t) \quad \text{for all} \quad t > t_0,
\]

where \(\mathcal{E}(t)\) and \(\mathcal{F}(t)\) are given in the proof of Lemma 5.3. Plugging this into \((5.9)\) gives

\[
\mathcal{E}'(t) \leq -\mathcal{F}(t) + \frac{\lambda^2 \bar{u}M}{2} \int r \leq -C_1 \mathcal{E}(t) + \frac{\lambda^2 \bar{u}M}{2} \int r \quad \text{for all} \quad t > t_0
\]

Hence, we have from this that

\[
\mathcal{E}(t) \leq e^{-C_1(t-t_0)} \mathcal{E}(t_0) + \frac{\lambda^2 \bar{u}M}{2} \int_{t_0}^{t} \left( e^{C_1s} \int_{\Omega} r(x, s) dx \right) ds \leq \mathcal{E}(t_0) e^{C_1(t-t_0)} + \frac{K\lambda^2 \bar{u}M}{2} e^{-C_1t} \int_{t_0}^{t} e^{(C_1-\lambda)s} ds \\
\leq C_2 e^{-\theta t} \quad \text{for all} \quad t > t_0
\]

(5.15)
for some $C_2, \theta_1 > 0$ where we used (5.11). According to the left inequalities in (5.13) and (5.14), we use (5.15) to find $C_3 > 0$ fulfilling

$$\mathcal{F}(t) \leq C_3 e^{-\theta_1 t},$$

which implies

$$\int_{\Omega} (u - \bar{u})^2 + \int_{\Omega} (v - \bar{v})^2 + \int_{\Omega} \bar{w}^2 \leq C_4 e^{-\theta_1 t}$$

for some $C_4 > 0$. Again by the uniform $W^{1,\infty}(\Omega)$-boundedness of $(u, v, w)$ and the Gagliardo-Nirenberg inequality, we infer (5.12).

Proof of Theorem 1.3. We combine Lemma 5.3 and Lemma 5.4 to get the desired conclusion.

\[\square\]

5.2 Global stability of $(\bar{u}, \bar{v}, \bar{w})$

Lemma 5.5. Let $r$ be a positive constant and suppose that (5.2) holds. Then there exist $\bar{\chi}, \bar{\xi}, \bar{b}_1, \bar{b}_2 > 0$ such that, once either $\chi < \bar{\chi}$ or $b_1 > \bar{b}_1$ and $b_2 > \bar{b}_2$ or $b_1 > \bar{b}_1$ and $\xi < \bar{\xi}$, one has

$$\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \quad (5.16)$$

Proof. First of all, we claim that

$$b_1 \bar{u}^{\alpha - 2} - \frac{\chi^2 \bar{u} M^3}{2r} > 0 \quad \text{and} \quad \frac{b_2 \bar{u} \bar{v}^{\beta - 3}}{2r M^3} - \frac{\chi^2 \bar{u} M^3}{2r} > 0 \quad (5.17)$$

holds for small $\chi$ or large $b_1$ and $b_2$ or large $b_1$ and small $\xi$. Since the discussion of case $N = 2$ is same with the case $N > 2$, we only get the arguments of $N > 2$. We recall from Lemma 5.1(i) that

$$M_u := C_1(\chi^\theta + 1)(b_1^{-\frac{1}{\alpha - 1}} + 1) \left( b_1^{-1}(b_1^{-\frac{1}{\alpha - 1}} + 1) + b_2^{-1}(b_2^{-\frac{1}{\beta - 1}} + 1) + 1 \right)^\theta$$

for some $C_1 > 0, \theta > 1$ independent of $\chi, \xi, b_1, b_2$. Inserting this into (5.17) and using the definitions of $\bar{u}$ and $\bar{v}$, one can find $C_2, C_3, C_4, C_5 > 0$ independent of $\chi, \xi, b_1, b_2$ such that (5.17) can be written as

$$C_2 b_1^{-\frac{1}{\alpha - 1}} - C_3 \chi^2 b_1^{-\frac{1}{\alpha - 1}} > 0 \quad (5.18)$$

and

$$\frac{C_4 b_1^{-\frac{1}{\alpha - 1}} b_2^{-\frac{2}{\beta - 1}}}{\xi^2 (\chi^\theta + 1)(b_1^{-\frac{1}{\alpha - 1}} + 1) \left( b_1^{-1}(b_1^{-\frac{1}{\alpha - 1}} + 1) + b_2^{-1}(b_2^{-\frac{1}{\beta - 1}} + 1) + 1 \right)^\theta} - C_5 \chi^2 b_1^{-\frac{1}{\alpha - 1}} > 0. \quad (5.19)$$

The inequalities (5.18) and (5.19) can be satisfied in the following three cases:

- **Small $\chi$:** There is $\bar{\chi} = \bar{\chi}(\xi, b_1, b_2) > 0$ such that (5.18) and (5.19) hold for $\chi < \bar{\chi}$.

- **Large $b_1$ and $b_2$:** There exist $\bar{b}_1 = \bar{b}_1(\chi) > 0$ and $\bar{b}_2 = \bar{b}_2(\chi, \xi, b_1) > 0$ such that (5.18) and (5.19) hold for $b_1 > \bar{b}_1$ and $b_2 > \bar{b}_2$.

- **Large $b_1$ and small $\xi$:** One can find $\tilde{b}_1 = \tilde{b}_1(\chi) > 0$ and $\tilde{\xi} > 0$ depending on $\chi, b_1, b_2$ such that (5.18) and (5.19) hold for $b_1 > \tilde{b}_1$ and $\xi < \tilde{\xi}$. 
Therefore, (5.17) holds under the conditions of this lemma.

By honest computations, we have

\[ \frac{d}{dt} \int_{\Omega} \left( w - \bar{w} - \bar{w} \ln \frac{w}{\bar{w}} \right) \]

\[ = -\bar{w} \int_{\Omega} \frac{\nabla w}{w^2} - \int_{\Omega} \left( w - \bar{w} \right) \left( -(u + v) - \mu + \frac{r}{w} \right) \]

\[ \leq -\frac{\bar{w}}{M^2} \int_{\Omega} |\nabla w|^2 + \int_{\Omega} \left( w - \bar{w} \right) \left[ (\bar{u} - u) + (\bar{v} - v) + \frac{r(\bar{w} - w)}{w} \right] \]

\[ \leq -\frac{\bar{w}}{M^2} \int_{\Omega} |\nabla w|^2 - \int_{\Omega} (u - \bar{u})(w - \bar{w}) - \int_{\Omega} (v - \bar{v})(w - \bar{w}) - \int_{\Omega} \frac{r}{w} (w - \bar{w})^2 \]

\[ \leq -\frac{\bar{w}}{M^2} \int_{\Omega} |\nabla w|^2 - \int_{\Omega} (u - \bar{u})(w - \bar{w}) - \int_{\Omega} (v - \bar{v})(w - \bar{w}) - \frac{r}{wM} \int_{\Omega} (w - \bar{w})^2 \]

\[ \leq -\frac{\bar{w}}{M^2} \int_{\Omega} |\nabla w|^2 - \frac{\bar{w}M}{r} \int_{\Omega} (u - \bar{u})^2 + \frac{\bar{w}M}{r} \int_{\Omega} (v - \bar{v})^2 - \frac{r}{2wM} \int_{\Omega} (w - \bar{w})^2 \quad (5.20) \]

for all \( t \in (0, \infty) \). We multiply (5.20) with \( \frac{\chi^2 uM^2}{2w} \) and add the obtained result to (5.3) to eliminate \( \int_{\Omega} |\nabla w|^2 \), i.e.,

\[ \frac{d}{dt} \left\{ \int_{\Omega} (u - \bar{u} - \bar{u} \ln \frac{u}{\bar{u}}) + \frac{\bar{u}}{\xi^2 \bar{v} M^2} \int_{\Omega} (v - \bar{v} - \bar{v} \ln \frac{v}{\bar{v}}) + \frac{\chi^2 \bar{u}M^2}{2w} \int_{\Omega} (w - \bar{w} - \bar{w} \ln \frac{w}{\bar{w}}) \right\} \]

\[ \leq -\left( b_1 \bar{u}^{\alpha-2} - \frac{\chi^2 \bar{u}M^3}{2r} \right) \int_{\Omega} (u - \bar{u})^2 - \left( \frac{b_2 \bar{u}^{\beta-3}}{\xi^2 M^2} - \frac{\chi^2 \bar{u}M^3}{2r} \right) \int_{\Omega} (v - \bar{v})^2 \]

\[ - \frac{r\chi^2 \bar{u}M}{4w^2} \int_{\Omega} (w - \bar{w})^2 \quad \text{for all } t \in (0, \infty), \quad (5.21) \]

where the coefficients in the right side are all positive according to (5.17). By letting

\[ \mathcal{E}_1(t) = \int_{\Omega} (u - \bar{u} - \bar{u} \ln \frac{u}{\bar{u}}) + \frac{\bar{u}}{\xi^2 \bar{v} M^2} \int_{\Omega} (v - \bar{v} - \bar{v} \ln \frac{v}{\bar{v}}) + \frac{\chi^2 \bar{u}M^2}{2w} \int_{\Omega} (w - \bar{w} - \bar{w} \ln \frac{w}{\bar{w}}) \]

and

\[ \mathcal{F}_1(t) = \left( b_1 \bar{u}^{\alpha-2} - \frac{\chi^2 \bar{u}M^3}{2r} \right) \int_{\Omega} (u - \bar{u})^2 + \left( \frac{b_2 \bar{u}^{\beta-3}}{\xi^2 M^2} - \frac{\chi^2 \bar{u}M^3}{2r} \right) \int_{\Omega} (v - \bar{v})^2 \]

\[ + \frac{r\chi^2 \bar{u}M}{4w^2} \int_{\Omega} (w - \bar{w})^2, \]

we have from (5.21) that

\[ \mathcal{E}_1(t) \leq -\mathcal{F}_1(t) \quad \text{for all } t \in (0, \infty). \quad (5.22) \]

Clearly, \( \mathcal{E}_1(t), \mathcal{F}_1(t) \geq 0 \) for any \( t > 0 \). Parallel to the arguments performed in Lemma 5.3, one can show (5.16).

\[ \square \]

Proof of Theorem 1.4. With (5.16) and (5.22) at hand, similar to the arguments in Lemma 5.4 (cf. [1] [2] [21] [22]), one can easily show the conclusion.

\[ \square \]

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