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Measures of information for concomitants of generalized order statistics from subfamilies of Farlie–Gumbel–Morgenstern distributions

Abstract In this paper, we study Shannon’s entropy and Fisher information number for concomitants of generalized order statistics from subfamilies of Farlie–Gumbel–Morgenstern when the marginal distributions are Weibull, exponential, Pareto and power function. Also, we provide some numerical results of Shannon entropy and Fisher information number for concomitants of order statistics.

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1 Introduction

Kamps [9] has introduced GOS’s as a unified model of ordered random variables such as ordinary order statistics, sequential order statistics, progressive type-II censoring, record values and Pfeifers records. The joint density function of the GOS’s $X(1, n, m, k), X(2, n, m, k), \ldots, X(n, n, m, k)$ is given by:

$$f_{X(1,n,m,k),\ldots,X(n,n,m,k)}(x_1, \ldots, x_n) = k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} \left( 1 - F(x_i) \right)^m f(x_i) \right) \times (1 - F(x_n))^{k-1} f(x_n),$$

with parameters $n \in \mathbb{N}, k > 0, m \in \mathbb{R}$, such that $\gamma_r = k + (n - r)(m + 1) > 0$, for all $1 \leq r \leq n$. 

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Morgenstern [12] has introduced FGM distributions; Gumbel [8] has studied FGM for exponential distribution. Farlie [5] has considered this family in the general form. Let \( F_X(x) \) and \( F_Y(y) \) be the distribution functions of the random variables \( X \) and \( Y \), respectively. Then the probability density function (pdf) of the bivariate FGM distributions is given by:
\[
f_{X,Y}(x, y) = f_X(x)f_Y(y)[1 + \alpha(2F_X(x) - 1)(2F_Y(y) - 1)], \quad -1 \leq \alpha \leq 1. \tag{1.1}
\]

Here, \( f_X(x) \) and \( f_Y(y) \) are the marginal pdf’s of \( X \) and \( Y \), respectively. The parameter \( \alpha \) is known as the dependence parameter of the random variables \( X \) and \( Y \). If \( \alpha \) is zero, then \( X \) and \( Y \) are independent. For the FGM family with pdf given by (1.1), the density function of the concomitant of \( r \)-th GOS’s \( Y_{[r,n,m,k]} \), \( 1 \leq r \leq n \), is given by Beg and Ahsanullah [1], as follows:
\[
g_{[r,n,m,k]}(y) = f_Y(y)\left[1 + \alpha C^* (r, n, m, k)(2F_Y(y) - 1)\right], \tag{1.2}
\]
where \( C^*(r, n, m, k) = 1 - \frac{2\prod_{i=1}^{n-k}y_i}{\prod_{i=1}^{n}y_i} \). Entropy is an index that is used to measure dispersion, volatility risk and uncertainty. This concept was formerly introduced by Shannon [13] in the information theory literature. The Shannon entropy of a random variable \( X \) is a mathematical measure of information which measures the average reduction of uncertainty of \( X \). Tahmasebi and Behboodian [15] have introduced the Shannon entropy for concomitants of GOS’s of FGM family, the Shannon entropy for a continuous random variable \( X \) with pdf \( f_X(x) \) is defined as:
\[
H(X) = -E(\ln f_X(X)) = -\int_{-\infty}^{\infty} f_X(x) \ln f_X(x) \, dx. \tag{1.3}
\]

Tahmasebi and Jafari [16] have introduced the Fisher information number for concomitants of GOS’s of FGM family, the Fisher information number for a continuous random variable \( X \) with pdf \( f_X(x) \) is defined as:
\[
I(X) = \int_{-\infty}^{\infty} \left[ \frac{\partial \ln f_X(x)}{\partial x} \right]^2 f_X(x) \, dx, \tag{1.4}
\]
this is Fisher information number for location parameter, and also called shift-invariant Fisher information number. Furthermore, it has been used to develop a unifying theory physical law called the principle of “extreme physical information” (see Frieden [6, 7]). Noting that, it is different than what was introduced by BuHamra and Ahsanullah [2].

**Remark 1.1** In the computation of this paper, we use some important formulas as follow:

1. Let \( T(t) = \int_{-\infty}^{\infty} [f(x)]^t \, dx \). Then \( \left[-\frac{\partial T(t)}{\partial t}\right]_{t=1} = H(X), t \geq 1 \).
2. Let \( A(X) = \int_{-\infty}^{\infty} u(x)f(x) \, dx, U(t) = \int_{-\infty}^{\infty} u(x)[f(x)]^t \, dx \). Then \( \left[\frac{\partial U(t)}{\partial t}\right]_{t=1} = A(X), t \geq 1 \).

The rest of this article is organized as follows. In Sect. 2, we derive Shannon entropy for concomitants of GOS’s of FGM family for some well-known distributions such as Weibull, Pareto and power function distributions. In Sect. 3, we develop Fisher information number for concomitants of GOS’s of FGM family for some well-known distributions such as exponential, Pareto and power function distributions. In Sect. 4, we compute numerical values of our results for concomitants of order statistics.

## 2 Shannon entropy for concomitants of GOS’s from subfamilies of FGM family

Tahmasebi and Behboodian [15] have introduced the Shannon entropy for concomitants of GOS’s of FGM family by the following theorem:

**Theorem 2.1** If \( Y_{[r,n,m,k]} \) is the concomitant of \( r \)-th GOS’s from (1.1), then, from (1.3), the Shannon entropy of \( Y_{[r,n,m,k]} \) for \( 1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1 \) is given by:
\[
H(Y_{[r,n,m,k]}) = W(r, \alpha, n, m, k) + H(Y)(1 - \alpha C^*(r, n, m, k)) - 2\alpha C^*(r, n, m, k)\phi_f(y), \tag{2.1}
\]
where
\[
W(r, \alpha, n, m, k) = \frac{1}{4\alpha C^*(r, n, m, k)} \left[ (1 - \alpha C^*(r, n, m, k))^2 \ln(1 - \alpha C^*(r, n, m, k)) - (1 + \alpha C^*(r, n, m, k))^2 \ln(1 + \alpha C^*(r, n, m, k)) \right] + \frac{1}{2}, \tag{2.2}
\]
\[
\phi_f(y) = \int_{-\infty}^{\infty} F_Y(y) f_Y(y) \ln f_Y(y) dy.
\tag{2.3}
\]

In the following subsections, we will apply the last theorem to some subfamilies of FGM family such as Weibull, Pareto and power function distributions.

2.1 Weibull distribution

The pdf and cdf for Weibull distribution are given by, respectively:

\[
f(y) = cy^{c-1} e^{-y^c},
\tag{2.4}
\]

\[
F(y) = 1 - e^{-y^c}, \quad 0 \leq y < \infty, \quad c > 0.
\tag{2.5}
\]

**Theorem 2.2** If \(Y_{r,n,m,k}\) is the concomitant of \(r\)-th GOS's for Weibull distribution from (1.1) and (2.5) then, from (2.1), the Shannon entropy of \(Y_{r,n,m,k}\) for \(1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1\) is given by:

\[
H(Y_{r,n,m,k}) = W(r, \alpha, n, m, k) + (1 + \alpha C^*(r, n, m, k)) \left[ \frac{c-1}{c} v - \ln c + 1 \right] - \alpha C^*(r, n, m, k) \left[ \frac{c-1}{c} (v + \ln 2) - \ln c + \frac{1}{2} \right],
\tag{2.6}
\]

where \(v = -\Gamma'(1)\) is the Euler’s constant and \(W(r, \alpha, n, m, k)\) is defined by (2.2).

**Proof** From (2.1), (2.4) and (2.5), we have

\[
H(Y_{r,n,m,k}) = W(r, \alpha, n, m, k) - (1 - \alpha C^*(r, n, m, k)) \int_0^\infty cy^{c-1} \exp(-y^c) \ln(cy^{c-1} \exp(-y^c)) dy
\]

\[
- 2\alpha C^*(r, n, m, k) \int_0^\infty cy^{c-1} \exp(-y^c)(1 - \exp(-y^c)) \ln(cy^{c-1} \exp(-y^c)) dy,
\]

\[
= W(r, \alpha, n, m, k) - (1 + \alpha C^*(r, n, m, k)) \int_0^\infty cy^{c-1} \exp(-y^c) \ln(cy^{c-1} \exp(-y^c)) dy
\]

\[
+ 2\alpha C^*(r, n, m, k) \int_0^\infty cy^{c-1} \exp(-y^c) \ln(cy^{c-1} \exp(-y^c)) dy.
\]

\[
= W(r, \alpha, n, m, k) - (1 + \alpha C^*(r, n, m, k)) I + 2\alpha C^*(r, n, m, k) II.
\tag{2.7}
\]

To find

\[
I = -H(Y) = \int_0^\infty cy^{c-1} \exp(-y^c) \ln(cy^{c-1} \exp(-y^c)) dy,
\]

first, we want to obtain

\[
T(t) = \int_0^\infty [f(y)]' dy = \int_0^\infty c'y^{(c-1)} \exp(-ty^c) dy
\]

\[
= e^{\gamma(t)} - \frac{(c-1+1)}{c} \Gamma \left( \frac{t(c-1)+1}{c} \right).
\]

\[
\Longrightarrow \frac{\partial T(t)}{\partial t} = T'(t) = Qe^{\gamma(t)} \Gamma \left( \frac{t(c-1)+1}{c} \right) + Q' e^{\gamma(t)} \Gamma' \left( \frac{t(c-1)+1}{c} \right)
\]

\[
+ Qe^{\gamma(t)} \left( \frac{c-1}{c} \right) \Gamma' \left( \frac{t(c-1)+1}{c} \right).
\]

\[
\tag{2.8}
\]
where \( Q = t^{\left(\frac{c-1}{c}\right)} \), \( Q' = Q \left( -\left(\frac{t(c-1)+1}{tc}\right) - \left(\frac{c-1}{c}\right) \ln t \right) \).

\[ \implies T'(1) = I = \ln c - \frac{c-1}{c} v - 1. \] (2.8)

To find

\[ I I = \int_{0}^{\infty} cy^{c-1} \exp(-2y^c) \ln(cy^{c-1} \exp(-y^c)) dy, \]

first, we want to obtain

\[ U(t) = \int_{0}^{\infty} \exp(-y^c)[f(y)]' dy = \int_{0}^{\infty} c' y^{c'(c-1)} \exp(-t(c-1)y^c) dy \]

\[ = c'^{-1}(t+1)^{-\left(\frac{tc-1}{c}\right)} \Gamma \left( \frac{t(c-1)+1}{c} \right). \]

\[ \implies U'(1) = II = \ln c - \frac{c-1}{c} \left( \frac{v + \ln 2}{2} \right) - \frac{1}{4}. \] (2.9)

where \( v = -\Gamma'(1) = 0.57722 \) is the Euler’s constant. By substituting (2.8) and (2.9) in (2.9), the result follows.

From Weibull distribution, we can get the Shannon entropy of other related distributions such as exponential and Rayleigh distributions by changing the parameters.

2.2 Pareto distribution

The pdf and cdf for Pareto distribution are given by, respectively:

\[ f(y) = cy^{-(c+1)}, \] (2.10)

\[ F(y) = 1 - y^{-c}, \quad y \geq 1, \quad c > 0. \] (2.11)

**Theorem 2.3** If \( Y_{[r,n,m,k]} \) is the concomitant of \( r \)-th GOS’s for Pareto distribution from (1.1) and (2.11) then, from (2.1), the Shannon entropy of \( Y_{[r,n,m,k]} \) for \( 1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1 \) is given by:

\[ H(Y_{[r,n,m,k]}) = W(r, \alpha, n, m, k) + (1 + \alpha C^n(r, n, m, k)) \left[ \ln \frac{1}{c} + \frac{1}{c} + 1 \right] \]

\[ - \alpha C^n(r, n, m, k) \left[ \frac{e + 1}{2c} - \ln c \right], \] (2.12)

where \( W(r, \alpha, n, m, k) \) is defined by (2.2).

**Proof** The proof is similar to the proof of Theorem 2.2. \( \square \)

2.3 Power distribution function

The pdf and cdf for Power distribution function are given by, respectively:

\[ f(y) = cy^{c-1}, \] (2.13)

\[ F(y) = y^c, \quad 0 \leq y \leq 1, \quad c > 0. \] (2.14)
Theorem 3.1 If \( Y_{[r,n,m,k]} \) is the concomitant of \( r \)-th GOS’s for Power distribution function from (1.1) and (2.14) then, from (2.1), the Shannon entropy of \( Y_{[r,n,m,k]} \) for \( 1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1 \) is given by:

\[
H(Y_{[r,n,m,k]}) = W(r, \alpha, n, m, k) - (1 - \alpha C^*(r, n, m, k)) \left( \ln c - \frac{c - 1}{c} \right)
\]

where \( W(r, \alpha, n, m, k) \) is defined by (2.2).

Proof The proof is similar to the proof of Theorem 2.2. \( \square \)

3 Fisher information number for concomitants of GOS’s from subfamilies of FGM family

Tahmasebi and Jafari [16] have introduced the Fisher information number for concomitants of GOS’s of FGM family by the following theorem:

Theorem 3.2 If \( Y_{[r,n,m,k]} \) is the concomitant of \( r \)-th GOS’s from (1.1), then, from (1.4), the Fisher information number of \( Y_{[r,n,m,k]} \) for \( 1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1 \) is given by:

\[
I(Y_{[r,n,m,k]}) = \int_{-\infty}^{\infty} \left[ \frac{d \ln f_Y(y)}{dy} \right]^2 f_Y(y) \left[ 1 + \alpha C^*(r, n, m, k)(2F_Y(y) - 1) \right] dy
\]

\[
+ 4\alpha C^*(r, n, m, k) \int_{-\infty}^{\infty} f_Y(y) f_Y(y) dy
\]

\[
+ 4[\alpha C^*(r, n, m, k)]^2 \int_{-\infty}^{\infty} \frac{f_Y^3(y)}{1 + \alpha C^*(r, n, m, k)(2F_Y(y) - 1)} dy. \tag{3.1}
\]

In the following subsections, we will apply the last theorem for some subfamilies of FGM family such as exponential, Pareto and power function distributions.

3.1 Exponential distribution

The pdf and cdf for exponential distribution are given by, respectively:

\[
f(y) = e^{-y}, \tag{3.2}
\]

\[
F(y) = 1 - e^{-y}, \quad 0 \leq y < \infty. \tag{3.3}
\]

Theorem 3.2 If \( Y_{[r,n,m,k]} \) is the concomitant of \( r \)-th GOS’s for exponential distribution function from (1.1) and (3.3) then, from (3.1), the Fisher information number of \( Y_{[r,n,m,k]} \) for \( 1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1 \) is given by:

\[
I(Y_{[r,n,m,k]}) = \frac{(1 + \alpha C^*(r, n, m, k))^2}{2\alpha C^*(r, n, m, k)} \left[ \ln(-1 + \alpha C^*(r, n, m, k)) - \ln(-1 + \alpha C^*(r, n, m, k)) \right]
\]

\[-4\alpha C^*(r, n, m, k). \tag{3.4}
\]

Proof Let \( d = \alpha C^*(r, n, m, k) \). From (3.1), (3.2) and (3.3), we have

\[
I(Y_{[r,n,m,k]}) = \int_{0}^{\infty} \left[ e^{-y} \left[ 1 + d(1 - 2e^{-y}) \right] - 4de^{-2y} + 4d^2 \frac{e^{-3y}}{1 + d(1 - 2e^{-y})} \right] dy
\]

\[
= \int_{0}^{\infty} \left[ -8de^{-2y} + \frac{(1 + d)^2}{-2d + (1 + d)e^y} \right] dy
\]

\[
= -4d + \frac{(1 + d)^2}{2d} [\ln(-1 - d) - \ln(-1 + d)]. \tag{3.5}
\]

\( \square \)
3.2 Pareto distribution

**Theorem 3.3** If \( Y_{r,n,m,k} \) is the concomitant of \( r \)-th GOS's for Pareto distribution from (1.1) and (2.11) then, from (3.1), the Fisher information number of \( Y_{r,n,m,k} \) for \( 1 \leq r \leq n, \alpha \neq 0, -1 \leq \alpha \leq 1 \) is given by:

\[
I(Y_{r,n,m,k}) = c(c+1)^2 \left[ \frac{1}{c+2} + \alpha C^*(r, n, m, k) \left( \frac{1}{c+2} - \frac{1}{c+1} \right) \right] - 2\alpha C^*(r, n, m, k)c^2
- \frac{\alpha C^*(r, n, m, k)c^3}{c+1} - (1 + \alpha C^*(r, n, m, k))c^3
+ \frac{1 + \alpha C^*(r, n, m, k)^2c^3}{4\alpha C^*(r, n, m, k)} \left[ 2 F_1 \left( \frac{2}{c}, 1; 1 + \frac{2}{c}; \frac{2\alpha C^*(r, n, m, k)}{1 + \alpha C^*(r, n, m, k)} \right) - 1 \right].
\] (3.5)

**Proof** Let \( d = \alpha C^*(r, n, m, k) \). From (3.1), (2.10) and (2.11), we have

\[
I(Y_{r,n,m,k}) = c(c+1)^2 \int_1^\infty \left[ y^{-c-3} + d(y^{-c-3} - 2y^{-2c-3}) \right] dy - 4dc^2(c+1) \int_1^\infty y^{-2c-3} dy
+ 4d^2c^3 \int_1^\infty \frac{y^{-3c-3}}{1 + d(1 - 2y^{-c})} dy
= c(c+1)^2 \left[ \frac{1}{c+2} + d \left( \frac{1}{c+2} - \frac{1}{c+1} \right) \right] - 2dc^2 + 4d^2c^3 \int_1^\infty \frac{y^{-3c-3}}{1 + d(1 - 2y^{-c})} dy
= c(c+1)^2 \left[ \frac{1}{c+2} + d \left( \frac{1}{c+2} - \frac{1}{c+1} \right) \right] - 2dc^2 + 4d^2c^3 III.
\] (3.6)

To find

\[
III = \int_1^\infty \frac{y^{-3c-3}}{1 + d(1 - 2y^{-c})} dy = \int_1^\infty \left[ \frac{y^{-2c-3}}{2d} - \frac{1 + d}{4d^2} \frac{y^{-c-3}}{y^{-3}} - \frac{(1+d)^2}{8d^3} \frac{y^{-3}}{y^{-3}} + \frac{1}{8d^3} \frac{1}{y^{3}(1+d) - 2dy^{-c}} \right] dy
- \frac{1}{4d(1+c)} - \frac{(1+d)^2}{16d^3} + \frac{(1+d)^3}{8d^3} IV,
\] (3.7)

we want to obtain

\[
IV = \int_1^\infty \frac{1}{y^{3}(1+d) - 2dy^{-c}} dy = -\int_1^\infty y^{-3}(2dy^{-c} - (1+d))^{-1} dy,
\]

let \( z = y^{-c} \) \( \Rightarrow \) \( dy = -\frac{1}{c} z^{-\frac{1}{c} - 1} dz \). Then

\[
IV = \frac{1}{c} \int_1^\infty z^{-\frac{1}{c} - 1} (2dz - (1+d))^{-1} dz
= \frac{1}{c} \int_1^\infty z^{-\frac{1}{c} - 1} \sum_{n=0}^\infty \left( -\frac{1}{n} \right) (2dz)^n (1+d)^{-1-n} dz
= \frac{1}{c} \sum_{n=0}^\infty \left( -\frac{1}{n} \right) \left( \frac{2d}{1+d} \right)^n (1+d)^{-1} \int_1^\infty z^{n+\frac{2}{c} - 1} dz
= \left[ \frac{-1}{2(1+d)} \sum_{n=0}^\infty \left( -\frac{1}{n} \right) \frac{1}{n+\frac{2}{c}} \left( \frac{2d}{1+d} \right)^n \right]_1^\infty
= \frac{1}{2(1+d)} \left( 2 F_1 \left( \frac{2}{c}, 1; 1 + \frac{2}{c}; \frac{2d}{1+d} \right) \right),
\] (3.8)

where \( 2 F_1 (a, b; c; z) \) is the Gaussian or ordinary hypergeometric function. By substituting (3.8) in (3.7) and (3.7) in (3.6), the result follows. \( \square \)
Table 1 $H(Y_{r,n,m,k})$ for Weibull distribution and $I(Y_{r,n,m,k})$ for exponential distribution based on order statistics with $c = 20$

| $n$ | $r$ | $H(Y_{r,n,m,k})$ | $I(Y_{r,n,m,k})$ |
|-----|-----|------------------|------------------|
|     |     | $\alpha = -1$   |                  |
|     |     | $-0.5$           |                  |
|     |     | $-0.25$          |                  |
|     |     | $0.25$           |                  |
|     |     | $0.5$            |                  |
|     |     | $1$              |                  |
| 5   | 1   | $-1.63092$       | $0.686329$       |
|     |     | $-1.51894$       | $0.707262$       |
|     |     | $-1.42561$       | $0.71556$        |
|     |     | $-1.31438$       | $1.36765$        |
|     |     | $-1.196$         | $1.79543$        |
| 5   | 2   | $-1.51894$       | $0.515059$       |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 5   | 3   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 10  | 4   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 10  | 5   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 10  | 6   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 10  | 7   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 10  | 8   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 10  | 9   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 1   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 2   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 3   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 4   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 5   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 6   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| 15  | 7   | $-1.51894$       | $1.79543$        |
|     |     | $-1.42561$       | $0.843006$       |
|     |     | $-1.31438$       | $1.17556$        |
|     |     | $-1.196$         | $1.36765$        |
|     |     | $-1.03093$       | $1.79543$        |
| $n$ | $r$ | $H(Y_{r,n,m,k})$ | $I(Y_{r,n,m,k})$ |
|-----|-----|-----------------|-----------------|
|     |     | $\alpha = -1$  | $\alpha = -1$  |
| 15  | 10  | -1.41824        | 1.57468         |
|     |     | -1.43017        | 1.26965         |
|     |     | -1.43812        | 1.13005         |
|     |     | -1.45793        | 0.88038         |
|     |     | -1.4698         | 0.772279        |
|     |     | -1.49748        | 0.59633         |
| 15  | 11  | -1.41172        | 1.91065         |
|     |     | -1.42354        | 1.41806         |
|     |     | -1.43399        | 1.19871         |
|     |     | -1.4637         | 0.824812        |
|     |     | -1.48297        | 0.677039        |
|     |     | -1.53059        | 0.48757         |
| 15  | 12  | -1.41092        | 2.27465         |
|     |     | -1.41824        | 1.57468         |
|     |     | -1.43017        | 1.26965         |
|     |     | -1.4698         | 0.772279        |
|     |     | -1.49748        | 0.59633         |
| 15  | 13  | -1.41631        | 2.64296         |
|     |     | -1.41429        | 1.73901         |
|     |     | -1.42669        | 1.34279         |
|     |     | -1.47622        | 0.722956        |
|     |     | -1.51335        | 0.532266        |
|     |     | -1.61442        | 0.597637        |
| 15  | 14  | -1.42867        | 3.08108         |
|     |     | -1.41172        | 1.91065         |
|     |     | -1.42354        | 1.41806         |
|     |     | -1.48297        | 0.677039        |
|     |     | -1.53059        | 0.48757         |
|     |     | -1.6664         | 0.9729          |
| 15  | 15  | -1.4494         | 3.52418         |
|     |     | -1.41059        | 2.08929         |
|     |     | -1.42072        | 1.49538         |
|     |     | -1.49006        | 0.634748        |
|     |     | -1.54926        | 0.465824        |
|     |     | -1.72675        | 1.94028         |
Table 2 $H(Y_{n,m,k})$ and $I(Y_{n,m,k})$ for Pareto distribution based on order statistics with $c = 20$

| $n$ | $r$ | $H(Y_{n,m,k})$ | $I(Y_{n,m,k})$ |
|-----|-----|----------------|----------------|
|     |     | $\alpha = -1$ | $\alpha = -1$ |
| 5   | 1   | -1.67362       | 253.395        |
| 5   | 2   | -1.78946       | 198.525        |
| 5   | 3   | -2.13946       | -              |
| 5   | 4   | -2.37362       | 1126.36        |
| 5   | 5   | -3.5250        | 253.395        |
| 10  | 1   | -1.63728       | 723.581        |
| 10  | 2   | -1.88225       | 521.317        |
| 10  | 3   | -2.14229       | 227.992        |
| 10  | 4   | -2.85044       | 174.074        |
| 10  | 5   | -3.89938       | 1126.36        |
| 10  | 6   | -1.99484       | 479.452        |
| 10  | 7   | 2.10141        | 658.292        |
| 10  | 8   | -2.21956       | 861.79         |
| 10  | 9   | -2.35043       | 1086.86        |
| 10  | 10  | -2.49637       | 1332.46        |
| 15  | 1   | -1.62706       | 741.537        |
| 15  | 2   | -1.65214       | 363.34         |
| 15  | 3   | -1.6856        | 220.019        |
| 15  | 4   | -1.72026       | 175.259        |
| 15  | 5   | -1.77264       | 186.16         |
| 15  | 6   | -1.82497       | 206.063        |
| 15  | 7   | -1.88272       | 233.453        |
| 15  | 8   | -2.01397       | 307.16         |
| 15  | 9   | -1.9792        | 307.16         |
| $n$ | $r$ | $H(Y_{r,n,m,k})$ | $I(Y_{r,n,m,k})$ |
|-----|-----|-----------------|-----------------|
|     |     | $\alpha = -1$   | $\alpha = -1$   |
|     |     | $-0.5$          | $-0.5$          |
|     |     | $-0.25$         | $-0.25$         |
|     |     | $0.25$          | $0.25$          |
|     |     | $0.5$           | $0.5$           |
|     |     | $1$             | $1$             |
| 15  | 10  | $-2.08747$      | 634.502         |
| 15  | 11  | $-2.16639$      | 769.972         |
| 15  | 12  | $-2.25102$      | 916.111         |
| 15  | 13  | $-2.34185$      | 1072.19         |
| 15  | 14  | $-2.43964$      | 1237.92         |
| 15  | 15  | $-2.54581$      | 1413.59         |
Table 3 $H(Y_{r,n,m,k})$ and $I(Y_{r,n,m,k})$ for power function distribution based on order statistics with $c = 20$

| $n$ | $r$ | $H(Y_{r,n,m,k})$ | $I(Y_{r,n,m,k})$ |
|-----|-----|-----------------|-----------------|
|     |     | $\alpha = -1$  | $\alpha = -1$  |
| 5   | 1   | -2.44028        | 1115.59         |
| 5   | 2   | -2.2228         | 741.253         |
| 5   | 3   | -1.90613        | 545.156         |
| 5   | 4   | -1.86095        | 288.058         |
| 5   | 5   | -1.97121        | 216.343         |
| 10  | 1   | -2.55546        | 1324.38         |
| 10  | 2   | -2.41862        | 1075.83         |
| 10  | 3   | -2.29683        | 850.93          |
| 10  | 4   | -2.18777        | 650.124         |
| 10  | 5   | -2.09029        | 476.291         |
| 10  | 6   | -2.00393        | 335.081         |
| 10  | 7   | -1.92868        | 236.693         |
| 10  | 8   | -1.86502        | 200.68          |
| 10  | 9   | -1.81407        | 270.952         |
| 10  | 10  | -1.77819        | 586.307         |
| 15  | 1   | -2.60206        | 1407.29         |
| 15  | 2   | -2.50214        | 1228.31         |
| 15  | 3   | -2.4106         | 1061.09         |
| 15  | 4   | -2.32602        | 904.959         |
| 15  | 5   | -2.24764        | 759.996         |
| 15  | 6   | -2.17497        | 626.834         |
| 15  | 7   | -2.10772        | 506.166         |
| 15  | 8   | -1.98897        | 321.954         |
| 15  | 9   | -1.93747        | 246.118         |
| 15  | 10  | -1.89139        | 206.835         |
| 15  | 11  | -1.89139        | -2.09173        |
Table 3 continued

| $n$ | $r$ | $H(Y_{r,m,k})$ | $I(Y_{r,m,k})$ |
|-----|-----|----------------|----------------|
|     |     | $\alpha = -1$ | $-0.5$ | $-0.25$ | $0.25$ | $0.5$ | $1$ | $\alpha = -1$ | $-0.5$ | $-0.25$ | $0.25$ | $0.5$ | $1$ |
| 15  | 12  | -1.85102      | -1.93747 | -1.98897 | -2.10772 | -2.17497 | -2.32602 | 205.559 | 246.118 | 312.954 | 506.616 | 626.834 | 904.959 |
| 15  | 13  | -1.81685      | -1.91373 | -1.97559 | -2.12403 | -2.21061 | -2.4106  | 261.821 | 222.513 | 294.051 | 535.373 | 691.882 | 1061.09 |
| 15  | 14  | -1.78964      | -1.89139 | -1.96255 | -2.14067 | -2.24764 | -2.50214 | 419.166 | 206.835 | 276.55  | 565.013 | 759.996 | 1228.31 |
| 15  | 15  | -1.77081      | -1.87047 | -1.94984 | -2.15765 | -2.28609 | -2.60206 | 815.38  | 200.531 | 260.539 | 595.509 | 831.054 | 1407.29 |
3.3 Power distribution function

Theorem 3.4 If \( Y_{[n,m,k]} \) is the concomitant of \( r \)-th GOS’s for Power distribution function from (1.1) and (2.14) then, from (3.1), the Fisher information number of \( Y_{[n,m,k]} \) for \( 1 \leq r \leq n \), \( \alpha \neq 0 \), \(-1 \leq \alpha \leq 1 \) is given by:

\[
I(Y_{[n,m,k]}) = c(c - 1)^2 \left[ \frac{1}{c - 2} - \alpha C^*(r, n, m, k) \left( \frac{1}{(c - 1)(c - 2)} \right) \right] + 2\alpha C^*(r, n, m, k)c^2
\]

\[
+ \frac{2\alpha C^*(r, n, m, k)}{c - 2} \left[ \frac{2}{c} \right] - \frac{2\alpha C^*(r, n, m, k)}{c - 1} \left[ \frac{2}{c} \right]
\]

\[
+ \frac{2\alpha C^*(r, n, m, k)}{c - 2} \left[ \frac{2}{c} \right] 2F1 \left( \frac{-2}{c}, 1; 1 - \frac{2}{c}, 1 - \alpha C^*(r, n, m, k) \right)
\]

(3.9)

\( c \neq 1, 2 \).

Proof The proof is similar to the proof of Theorem 3.3.

4 Numerical results

Tables 1, 2 and 3 provide \( H(Y_{[n,m,k]}) \) and \( I(Y_{[n,m,k]}) \) values of the concomitants of order statistics \( (\gamma_i = n - i + 1) \) when the marginal distributions are Weibull, exponential, Pareto and power function for \( 1 \leq r \leq n \), \( n = 5, 10, 15 \) and \( \alpha = -1, -0.5, -0.25, 0.25, 0.5, 1 \). We can find some properties from the numerical results as follow:

1. For \( n \) is odd, \( r = \frac{n+1}{2} \) we find that \( H(Y_{[n,m,k]}) \) and \( I(Y_{[n,m,k]}) \) are indeterminate.
2. Let \( H(Y_{[n,m,k]}) = H_a(Y_{[r]}) \) and \( I(Y_{[n,m,k]}) = I_a(Y_{[r]}) \). Then we have \( H_a(Y_{[r]}) = H - \alpha(Y_{[n-r+1]}) \) and \( I_a(Y_{[r]}) = I - \alpha(Y_{[n-r+1]}) \), \( 1 \leq r \leq n \).
3. For fixed \( n, r \), at \( c > n, r < \frac{n+1}{2} (r > \frac{n+1}{2}) \) then we have, for Pareto distribution \( H(Y_{[r,m,k]}) \) is decreasing (increasing) in \( \alpha \), for power function distribution \( H(Y_{[r,m,k]}) \) is increasing (decreasing) in \( \alpha \).
4. For fixed \( n, \alpha \), at \( c > n, r \neq n \), \(-1 \leq \alpha < 0 \) (\( 0 < \alpha \leq 1 \)) then we have, for Weibull distribution \( H(Y_{[r,m,k]}) \) is increasing (decreasing) in \( r \), for Pareto distribution \( H(Y_{[r,m,k]}) \) is decreasing (increasing) in \( r \), for power function distribution \( H(Y_{[r,m,k]}) \) is increasing (decreasing) in \( r \).
5. For fixed \( n, \alpha \), at \( c > n \), \(-0.5 \leq \alpha < 0 \) (\( 0 < \alpha \leq 0.5 \)) then we have, for Weibull distribution \( I(Y_{[r,m,k]}) \) is increasing (decreasing) in \( r \), for Pareto distribution \( I(Y_{[r,m,k]}) \) is increasing (decreasing) in \( r \), for power function distribution \( I(Y_{[r,m,k]}) \) is increasing (decreasing) in \( r \).
6. For fixed \( r, \alpha \), at \( c > n \), \(-1 \leq \alpha < 0 \) (\( 0 < \alpha \leq 1 \)) then we have, for Weibull distribution \( H(Y_{[r,m,k]}) \) is decreasing (increasing) in \( n \), for Pareto distribution \( H(Y_{[r,m,k]}) \) is increasing (decreasing) in \( n \), for power function distribution \( H(Y_{[r,m,k]}) \) is decreasing (increasing) in \( n \).
7. For fixed \( r, \alpha \), at \( c > n \), \(-0.5 \leq \alpha < 0 \) (\( 0 < \alpha \leq 0.5 \)) then we have, for Weibull distribution \( I(Y_{[r,m,k]}) \) is decreasing (increasing) in \( n \), for Pareto distribution \( I(Y_{[r,m,k]}) \) is decreasing (increasing) in \( n \), for power function distribution \( I(Y_{[r,m,k]}) \) is increasing (decreasing) in \( n \).
8. For GOS’s, we find that \(-1 \leq C^*(r, n, m, k) \leq 1 \) for all possible values of \( r, n, m, k \).

5 Conclusion

We derived an analytical expression of Shannon entropy and Fisher information number from subfamilies of FGM family such as Weibull, exponential, Pareto and power distributions, based on concomitants of GOS’s. Applications of these results are applied based on order statistics as a special case of GOS’s. We find some important relations in entropy and Fisher information number at some values of the parameters. Conditions for decreasing (increasing) uncertainty and Fisher information number are obtained. We also observed the limit values of the constant \( C^*(r, n, m, k) \) for GOS’s. The proposed procedures may be considered for other models (such as dual generalized order statistics and case-II of generalized order statistics which are introduced by Burkschat et al. [3] and Kamps and Cramer [10], respectively), and for some other distributions.
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