SEMI-INVARIANTS OF 2-REPRESENTATIONS OF QUIVERS

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ABSTRACT. In this work we obtain a version of the Procesi-Rasmyslov Theorem for the algebra of semi-invariants of representations of an arbitrary quiver with dimension vector \((2,2,\ldots,2)\).

1. Introduction

We work over a base field \(k\) of characteristic zero. A quiver \(Q\) is a directed graph, determined by two finite sets \(Q_0\) (the set of “vertices”) and \(Q_1\) (the set of “arrows”) with two maps \(h, t : Q_1 \to Q_0\) which indicate the vertices at the head and tail of each arrow. A representation \((W, \varphi)\) of \(Q\) consists of a collection of finite dimensional \(k\)-vector spaces \(W_v\) for each \(v \in Q_0\), together with linear maps \(\varphi_a : W_{ta} \to W_{ha}\), for each \(a \in Q_1\). The dimension vector \(\alpha \in \mathbb{Z}^{Q_0}\) of such a representation is given by \(\alpha_v = \dim_k W_v\). A morphism \(f : (W_v, \varphi_a) \to (U_v, \psi_a)\) of representations consists of linear maps \(f_v : W_v \to U_v\), for each \(v \in Q_0\), such that \(f_{ha} \varphi_a = \psi_a f_{ta}\), for each \(a \in Q_1\). Evidently, it is an isomorphism if and only if each \(f_v\) is.

Having chosen vector spaces \(W_v\) of dimension \(\alpha_v\), the isomorphism classes of representations of \(Q\) with dimension vector \(\alpha\) are in natural one-to-one correspondence with the orbits of the group

\[
GL(\alpha) := \prod_{v \in Q_0} GL(W_v)
\]

in the representation space

\[
\text{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \text{Hom}(W_{ta}, W_{ha}).
\]

This action is given by \((g \cdot \varphi)_a = g_{ha} \varphi_a g_{ta}^{-1}\), where \(g = (g_v)_{v \in Q_0} \in GL(\alpha)\). Note that the one-parameter subgroup \(\Delta = \{(tE, \ldots, tE)\}\) acts trivially.

One can also consider the action of the smaller group \(SL(\alpha) = \prod_{v \in Q_0} SL(W_v)\) on \(\text{Rep}(Q, \alpha)\); \(SL(\alpha)\)-invariant functions on \(\text{Rep}(Q, \alpha)\) are usually called semi-invariants. In this work we study generators of the algebra \(k[\text{Rep}(Q, \alpha)]^{SL(\alpha)}\). Recall that generators of \(k[\text{Rep}(Q, \alpha)]^{GL(\alpha)}\) are given by the Procesi-Rasmyslov Theorem.

Theorem 1. \([5\text{, Theorem 3.4}, 6\] For an arbitrary quiver \(Q\) and dimension vector \(\alpha\) the algebra \(k[\text{Rep}(Q, \alpha)]^{GL(\alpha)}\) is generated by the traces of oriented cycles of length not greater than \((\sum_v \alpha_v)^2\). All relations among them can be deduced from Cayley-Hamilton polynomials.

As for the algebra \(k[\text{Rep}(Q, \alpha)]^{SL(\alpha)}\), we only have several descriptions of its spanning sets; the main approaches are presented in \([3, 2, 7]\). In this work we describe a generating set for the algebra of 2-representations of an arbitrary quiver, i.e., of representations with dimension vector \((2,2,\ldots,2)\), in the spirit of the Procesi-Rasmyslov Theorem.

For a matrix \(A\) we will, as usually, denote by \(A^T\) its adjoint matrix, i.e., the matrix consisting of cofactors to the elements of \(A^T\). If \(A_F\) is the matrix of a linear map \(F : U \to V\), then it is convenient to assume that its adjoint matrix defines a linear map from \(V\) to \(U\).

Consider a quiver \(Q\). Let \(Q_0 = \{1, \ldots, n\}\) and \(Q_1 = \{a_1, \ldots, a_s\}\). To \(Q\) we associate the quiver \(\bar{Q}\) with \(\bar{Q}_0 = Q_0\) and \(\bar{Q}_1 = \{a_1, \ldots, a_s\} \cup \{b_1, \ldots, b_t\}\), where \(hb_i = ta_i, tb_i = ha_i\). For each representation \((W, \varphi)\) of \(Q\), consider the associated representation of \(\bar{Q}\) with the same spaces \(W_v\) and maps \(\varphi_{a_i}\) and...
\(\varphi_b = \widehat{\varphi}_a.\) By a route in \(Q\) we mean an oriented cycle in \(\widehat{Q}\). For example, any cycle in \(Q\) is a route. We say that a route is simple, if no edge appears twice in the corresponding oriented cycle. The trace of a route is the trace of the corresponding cycle in \(Q\) in the associated representation.

**Theorem 2.** For a quiver \(Q\) and a dimension vector \(\alpha = (2, 2, \ldots, 2)\) the algebra \(k[\text{Rep}(Q, \alpha)]^{SL(\alpha)}\) is generated by the traces of simple routes.

**Example 1.** Let \(Q\) be the quiver \(1 \overset{a}{\rightarrow} b \overset{b}{\rightarrow} 2\). Denote by \(|X|\) the determinant of a matrix \(X\). By Theorem 2 the algebra \(k[\text{Rep}(Q, (2, 2))]^{SL(2) \times SL(2)}\) is generated by \(\text{tr} \varphi_a \widehat{\varphi}_a = 2|\varphi_a|\), \(\text{tr} \varphi_b \widehat{\varphi}_b = 2|\varphi_b|\) and \(\text{tr} \varphi_a \varphi_b\) (note that \(\varphi_a = (\frac{x_{11}}{x_{21}}, \frac{x_{12}}{x_{22}}), \varphi_b = (\frac{y_{11}}{y_{21}}, \frac{y_{12}}{y_{22}})\), then 
\[
k[\text{Rep}(Q, (2, 2))]^{SL(2) \times SL(2)} = k[2x_{11}x_{22} - 2x_{12}x_{21}, 2y_{11}y_{22} - 2y_{12}y_{21}, x_{11}y_{22} - x_{12}y_{21} - x_{21}y_{12} + x_{22}y_{11}].
\]

Our arguments use the description of a spanning set of the algebra \(k[\text{Rep}(Q, \alpha)]^{SL(\alpha)}\) devised by Domokos and Zubkov [3]: we briefly recall it in Section 2. In Sections 3 and 4 we obtain a formula expressing the determinant of a 2-block matrix as a polynomial in the traces of associated routes. In Section 5 we prove Theorem 2.

The author hopes to use the information received about the basic semi-invariants to generalize King’s construction [4] in the way similar to [4].

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2. The Domokos-Zubkov Theorem

Let us recall the results of [3]. Let \(Q_0 = \{1, \ldots, n\}\). Fix a dimension vector \(\alpha\) and two tuples \(\tau = (i_1, \ldots, i_k)\) and \(\bar{\tau} = (j_1, \ldots, j_l)\) of integers from 1 to \(n\) (possibly repeating) such that \((\alpha_{i_1} + \ldots + \alpha_{i_k}) = (\alpha_{j_1} + \ldots + \alpha_{j_l})\), and consider all possible matrices of size \((\alpha_{i_1} + \ldots + \alpha_{i_k}) \times (\alpha_{j_1} + \ldots + \alpha_{j_l})\) of form

\[
A_{\tau\bar{\tau}} := \begin{pmatrix}
i_1 & \cdots & i_s & \cdots & i_k \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
j_1 & \cdots & j_s & \cdots & j_l \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
y_{i_1}F_{i_1} & \cdots & y_{i_s}F_{i_s} & \cdots & y_{i_k}F_{i_k} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
y_{j_1}F_{j_1} & \cdots & y_{j_s}F_{j_s} & \cdots & y_{j_l}F_{j_l} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\end{pmatrix},
\]

where \(y_{rs}\) are formal variables, and each matrix \(F_{rs}\) may be either 0, or the matrix of a map, that corresponds to an arrow going from the \(i_s\)-th vertex to the \(j_r\)-th one, or an identity matrix if \(i_s = j_r\). For a fixed representation \((W, \varphi)\) a matrix of form (3) defines a map from \(W_{i_1} \oplus \ldots \oplus W_{i_k}\) to \(W_{j_1} \oplus \ldots \oplus W_{j_l}\). Moreover, its determinant is a polynomial in variables \(y_{rs}\) with \(SL(\alpha)\)-invariant coefficients: \(|A_{\tau\bar{\tau}}| = \sum_{\mu} y^\mu \overline{\varphi}(x^r_{pq})\), where \(\overline{\varphi} = (\mu_{ij})_{i,j=1}^{k,l}\) are multidegrees, and \(x^r_{pq}\) are matrix elements of the matrices \(F_{rs}\).

**Theorem 3.** [3 Thm. 4.1] The algebra \(k[\text{Rep}(Q, \alpha)]^{SL(\alpha)}\) is spanned by semi-invariants \(\overline{\varphi}(x^r_{pq})\).

Thus all basic semi-invariants of \(Q\) are given by coefficients of monomials in variables \(y_{rs}\) in determinants of block matrices of form (3). We will describe them precisely for \(\alpha = (2, 2, \ldots, 2)\).

3. Block Matrices and Associated Routes

For a matrix \(X = (\frac{x_{11}}{x_{21}}, \frac{x_{12}}{x_{22}})\) its adjoint matrix \(\widehat{X}\) equals \((\frac{x_{22}}{-x_{21}}, \frac{-x_{12}}{-x_{11}})\). Recall that for \(X, Y \in \text{Mat}_{2 \times 2}(k)\) we have \(\text{tr} XY = \text{tr} \widehat{X}Y = \text{tr} X \text{tr} Y - \text{tr} XY\), \(\text{tr} \widehat{X} = \text{tr} X\) and \(\text{tr} XY = \text{tr} \widehat{Y}X\).

**Lemma 1.** Let \(X, Y, Z \in \text{Mat}_{2 \times 2}(k)\). Then

1. \(\text{tr} X^2Y = \text{tr} X \text{tr} XY - |X| \text{tr} Y\);
2. \(\text{tr} XYXZ = \text{tr} XY \text{tr} XZ - |X| \text{tr} Y \widehat{Z}\).
Proof. It suffices to prove a polynomial identity for the elements of a Zariski open subset. Hence, we may assume that all matrices are invertible. By the Cayley-Hamilton Theorem for two by two matrices we have
\[ X^2 - (\text{tr } X)X + |X|E = 0. \]

Multiplying this equality by \( Y \) and taking the trace of the product received, we get (1). If \( Y \) is invertible, it follows that
\[
\text{tr } XYZ = \text{tr } ((XY)^2 Y^{-1} Z) = \text{tr } XY \text{tr } (XY \cdot Y^{-1} Z) - |XY| \text{tr } Y^{-1} Z =
\]
\[ = \text{tr } XY \text{tr } (XZ) - |X| \text{tr } \hat{Y}Z. \]
\[ \square \]

Consider a matrix \( Z \in \text{Mat}_{2k \times 2k}(k) \) divided into blocks \( X_{ij} \) of size 2 \( \times \) 2. To \( Z \) we associate a quiver \( \Gamma \) with \( \Gamma_0 = \{1, \ldots, k\} \cup \{-1, \ldots, -k\} \), \( \Gamma_1 = \{a_{ij} \mid i, j = 1, \ldots, k\} \), \( ha_{ij} = i \) and \( ta_{ij} = -j \). A route in \( \Gamma \) defines a sequence of matrices: if an arrow goes from \(-j\) to \( i\), we take \( X_{ij} \), and if it goes from \( k \) to \(-l\), we take \( X_{kl}^{-1} \). Thus constructed sequences we call \emph{routes} in \( Z \). The \emph{adjoint} for a given route \( P = (X_1, \ldots, X_N) \) is \( \hat{P} = (\hat{X}_N, \ldots, \hat{X}_1) \). Observe that any cyclic permutation of factors as well as taking the adjoint route does not change the trace of the product of matrices along a route. We say that two routes are \emph{equivalent} if there is such a transformation turning one of them into another. We claim that the determinant of \( Z \) is a polynomial in traces of its routes. To prove this we need to introduce the following construction.

**Construction.** The determinant of \( Z \) equals \( |Z| = \sum_{\sigma \in S_2k} (-1)^{\sigma} z_{\sigma(1)} \cdot z_{\sigma(2)} \cdot \ldots \cdot z_{\sigma(2k)}. \) To each summand \( z_\sigma := z_{\sigma(1)} \cdot z_{\sigma(2)} \cdot \ldots \cdot z_{\sigma(2k)} \) we assign the \emph{associated route set} \( \mathcal{P}_\sigma \) constructed as follows. Let \( z_{\sigma(1)} \) be in a block \( X_{1s_1}. \)

1a) if \( z_{\sigma(2)} \) is in the same block as \( z_{\sigma(1)} \), then add to \( \mathcal{P}_\sigma \) the route \((\hat{X}_{1s_1}, X_{1s_1})\) and consider the submatrix \( Z_{\sigma(1) \sigma(2)}^{12} \) associated to \( X_{1s_1}. \)

1b) let \( z_{\sigma(2)} \) be in some other \( X_{1s_2}. \). In this case we consider the factor \( z_{t_{s_2}} \), which lies in the same block column as \( z_{t_{s_2}} \), i.e., in some \( X_{r_{s_2}t_{s_2}} \).

2) continue this process as long as it is possible, alternating horizontal and vertical shifts; 3) there comes a moment when we can not make another, some \( N \)-th shift. If it were the time to move vertically (respectively horizontally), then we have arrived to some \( p \)-th block column (respectively to a \( p \)-th block row) for the second time. But in each block column (except the \( s_1 \)-th) and in each block row we have already chosen two blocks (and hence, two factors of \( z_\sigma \), so it is impossible to get there again. Therefore just before the algorithm failed we had arrived to the \( s_1 \)-th block column. It follows that \( N \) is even (because the last shift had been horizontal) and moreover, the last considered block was some \( X_{rNs_1}. \)

5) consider the route \( P = (\hat{X}_{1s_1}, X_{1s_2}, \hat{X}_{rNs_2}, X_{rNs_4}, \ldots, X_{rNs_1}) \) (all odd factors are adjoint). We add it to \( \mathcal{P}_\sigma \) and pass to the submatrix matrix \( Z_{\mathcal{P}_\sigma} \) of size \((k - N) \times (k - N)\) containing all the factors of \( z_\sigma \) that we have not used yet.

Note that the construction is not unique: we may take various starting elements and choose one of two possible directions of circuit. These transformations correspond to cyclic permutations of the routes and/or taking the adjoint routes, hence not changing their equivalence classes.

**Example 2.** We illustrate the construction for the block matrix \( Z = (X_{rs})_{r,s=1}^3 \) and the permutation \( \sigma = (1\ 2\ 3\ 4\ 5\ 6) \).
The summand $z_{\sigma}$ equals $z_{13}z_{24}z_{31}z_{45}z_{56}z_{61}$. Starting at $z_{13}$, we next take the factor $z_{24}$. Since both of them are in the same block $X_{12}$, we add to $P_\sigma$ the route $(\widehat{X}_{12}, X_{12})$ and consider the submatrix $Z_{34}^{12} = (X_{21}, X_{23})$. Here we start at $z_{31}$, then take $z_{45}$. This element is in the second block column of $Z_{34}^{12}$; we should take another factor lying in this column, that is $z_{66}$, and then the factor from the same block row as $z_{66}$; it is $z_{52}$. Thus we add to $P_\sigma$ the route $(\widehat{X}_{21}, \widehat{X}_{23}, X_{33}, X_{31})$. Finally, $P_\sigma = \{(\widehat{X}_{12}, X_{12}), (\widehat{X}_{21}, X_{23}, \widehat{X}_{33}, X_{31})\}$

Observe that different permutations may correspond to the same associated route sets. Namely, for $k = 1$ there are two permutations and only one route. We say that two route sets $P$ and $S$ are equivalent if their elements are pairwise equivalent. Now, take a representative from each equivalence class of route sets received by our construction; denote the collection by $P_k$.

4. The determinant of a 2-block matrix

Define the length function $l(P)$ as the number of factors of $P$, and the index given by

$$\nu(P) = \begin{cases} 1, & \text{if } P \text{ is of form } (\widehat{X}_{ij}, X_{ij}) \text{ or } (X_{ij}, \widehat{X}_{ij}), \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 1.** Assume that $Z = (X_{rs})_{r,s=1}^k \in \text{Mat}_{2k \times 2k}(k)$ is a 2-block matrix, i.e., it is divided into blocks $X_{rs}$ of size $2 \times 2$. Then the following equality holds:

$$|Z| = \sum_{P \in P_k} \prod_{P \in P} (-1)^{l(P)-1} 2^{-\nu(P)} \text{tr } P. \quad (4)$$

**Proof.** Both sides of (4) there are polynomials in matrix elements of $X_{ij}$. We claim that each $z_{\sigma}$ from the left side occurs in the right side with the same coefficient.

**Lemma 2.** The product $\prod_{P \in P} 2^{-\nu(P)} \text{tr } P$ equals a sum of monomials $\pm z_{\sigma}$, where $\sigma$ run through such permutations that every $z_{p,\sigma(p)}$ lies in an element of some route $P \in P$.

**Proof.** First prove the lemma for $P = \{P\}$, $\nu(P) = 0$.

Note that a transposition of block rows or block columns changes neither $|Z|$, nor the right side of (4). Hence we may assume that $P = (\widehat{X}_{11}, X_{12}, \widehat{X}_{22}, X_{23}, \ldots, \widehat{X}_{kk}, X_{kk})$.

The polynomial $\text{tr } P$ is a sum of monomials in matrix elements of blocks $X_{ij}$ of the following form

$$(\widehat{X}_{11})_{p_1} p_2 (X_{12})_{p_3} (X_{22})_{p_4} (X_{23})_{p_5} \cdots (\widehat{X}_{kk})_{p_{2k-1}} p_{2k} (X_{kk})_{p_{2k+1}} = (X_{q,q+1})_{p,q+1} (X_{q+1,q+1})_{p,q+2}. \quad (5)$$

Such a product equals some $z_{\sigma}$ whenever each pair of its factors does not lie in one row or in one column. By construction of $P$ only some $(X_{q,q+1})_{p,q+1}$ and $(X_{q+1,q+1})_{p,q+2} = (X_{q+1,q+1})_{p,q+2}$
may lie in the same block column. They are in the same column if and only if \( q = 2q + 1 \). But for \( X, Y \in \text{Mat}_{2 \times 2}(k) \) we have
\[
X\hat{Y} = \begin{pmatrix} x_{11}y_{22} - x_{12}y_{21} & -x_{11}y_{12} + x_{12}y_{11} \\ x_{21}y_{22} - x_{22}y_{21} & -x_{21}y_{12} + x_{22}y_{11} \end{pmatrix},
\]
and \( b \neq d \) holds in all the products \( x_{ab}y_{cd} \) that occur here. The same argument may be used to prove that every two factors of (5) do not lie in the same row.

If \( \nu(P) = 1 \), i.e., \( P = (\tilde{X}_r, X_s) \), we have \( trP = 2|X_s| \). This gives rise to the factors \( 2^{-\nu(P)} \) in (4).

Assume that there is more than one route in \( P \). Each route \( P = \{P_1, \ldots, P_m\} \) determines a collection of block submatrices \( Z_{P_1}, \ldots, Z_{P_m} \) such that each \( P_i \) is entirely situated in \( Z_{P_i} \) and each block row and each block column of \( Z \) interacts with a unique \( Z_{P_i} \). By transpositions of block rows and columns we can make all the blocks \( Z_{P_i} \) diagonal. Now it is evident that \( \prod_{P \in P} 2^{-\nu(P)} trP \) equals a sum of products \( z_{\sigma_1} \cdots z_{\sigma_m} \), where \( z_{\sigma_q} \) are summands of \( |Z_{P_q}| \) and \( \sigma_j \) is associated to \( P_j \).

Thus the right side of (4) is a linear combination of \( z_{\sigma} \).

**Lemma 3.** Each monomial \( z_{\sigma} \) occurs in \( \prod_{P \in P_{\tau}} 2^{-\nu(P)} trP \) with nonzero coefficient.

**Proof.** Without loss of generality, we may assume that \( P_{\sigma} = \{P\} \) with \( P = (\tilde{X}_{r_1}, X_{r_2}, \tilde{X}_{r_3}, \ldots, X_{r_{2s_1}}) \). Denote by \( z_{\sigma, q} \) the factor of \( z_\sigma \) lying in the \( q \)-th element of \( P \). Fixing the row containing \( z_{\sigma, 1} \), we determine the row, in which \( z_{\sigma, 2} \) lies: it is the second row of a pair \( (2r_1 - 1, 2r_1) \). Similarly, the choice of the column containing \( z_{\sigma, 2} \) determines in which column \( z_{\sigma, 3} \) lies, and so on. Finally, knowing the column containing \( z_{\sigma, 2k} \) we learn in which column \( z_{\sigma, 1} \) lies. Consequently, all the permutations \( \tau \) with \( P_{\tau} = P_{\sigma} \) are parametrised by tuples \( (\xi_1, \ldots, \xi_k) \in (\mathbb{Z}_2)^{2k} \). Further, if we interchange the \( (2r_1 - 1) \)-th and the \( 2r_1 \)-th rows of \( Z \), then \( \xi_1 \) becomes \( 1 - \xi_1 \) and the other elements of a tuple do not change. Thus there exists a sequence of such permutations sending a product \( z_{\tau} \) to any \( z_{\tau} \) with the same associated set. It is only left to understand, how these transformations change \( trP \).

If we interchange the \( (2s_q - 1) \)-th and the \( 2s_q \)-th columns, it only influences the fragment \((X_{r_q - 2s_q}, \tilde{X}_{r_q s_q}) \). But for \( X, Y \in \text{Mat}_{2 \times 2}(k) \) we have
\[
X\hat{Y} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} = \begin{pmatrix} x_{11}y_{22} - x_{12}y_{21} & -x_{11}y_{12} + x_{12}y_{11} \\ x_{21}y_{22} - x_{22}y_{21} & -x_{21}y_{12} + x_{22}y_{11} \end{pmatrix},
\]
and after the permutation:
\[
\begin{pmatrix} x_{12} & x_{11} \\ x_{22} & x_{21} \end{pmatrix} \begin{pmatrix} y_{12} & y_{11} \\ y_{22} & y_{21} \end{pmatrix} = \begin{pmatrix} x_{12}y_{21} - x_{11}y_{22} & -x_{12}y_{12} + x_{11}y_{11} \\ x_{22}y_{21} - x_{21}y_{22} & -x_{22}y_{12} + x_{21}y_{11} \end{pmatrix}.
\]
Comparing the results we see that both products contain the same members, yet with different signs. So a transposition multiplies \( trP \) by \((-1)\). The same argument works for transformations of columns. Hence all \( z_{\tau} \) with \( P_{\tau} = \{P\} \) occur in \( trP \) with nonzero coefficient, which in fact equals \pm 1. □

It is clear that the same \( z_{\sigma} \) cannot occur as a summand in \( \prod_{S \in S} 2^{-\nu(S)} trS \) and \( \prod_{P \in P} 2^{-\nu(P)} trP \) for two different sets \( P \) and \( S \) in \( P_k \). Therefore each \( z_{\sigma} \) occurs in both sides of (4) with coefficient \pm 1.

It remains to prove that these signs are the same.

Assume as before that \( P_{\sigma} = P \), where \( P = (\tilde{X}_{11}X_{12} \tilde{X}_{22} \ldots X_{l(\hat{P}),l}) \). We know that if \( P_{\tau} = P_{\sigma} \), then there exists a permutation of rows and columns of \( Z \) multiplying both sides of (4) by the same number 1 or \((-1)\) and transforming \( z_{\sigma} \) into \( z_{\tau} \). So the question is whether the coefficients of a fixed \( z_{\sigma_0} \) are the same. Consider \( \sigma_0 = (1, 2, 3, \ldots, 2l(P) - 1, 2l(P)) \). Then the coefficient of \( z_{\sigma_0} \) in \( |Z| \) equals \((-1)^{\sigma_0} = -1 \).

As for the right side, multiply the summand
\[
\pm z_{12}z_{23} \cdots z_{2l(P)-1,2l(P)} z_{2l(P),1} = (\tilde{X}_{11})_{21} (X_{12})_{21} (\tilde{X}_{22})_{21} (X_{23})_{21} \ldots (\tilde{X}_{l(\hat{P}),l(\hat{P})})_{21} (X_{l(\hat{P}),1})_{21}
\]
by \((-1)^{\frac{l}{2}(P)-1}\). We have \((X_{q,q+1})_{21} = z_{2q,2q+1} \) and \((\tilde{X}_{qq})_{21} = -z_{2q-1,2q} \), so the total sign is \((-1)^{\frac{l}{2}(P)-1}(-1)^{\frac{l}{2}(P)} = -1 \). This completes the proof of Proposition 1. □
5. Proof of Theorem 2

Recall that by Theorem 3 the algebra $k[\text{Rep}(Q, \alpha)]^{SL(\alpha)}$ is generated by the coefficients of monomials in $y_{rs}$ in determinants of block matrices of form (3). So it is only left to prove that these are precisely the traces of routes associated to those matrices.

It is easy to see that in the determinant of $(y_{rs}X_{rs})_{r,s=1}^k$ the coefficient of $y_{r_1s_1} \cdots y_{r ks}$ equals the alternating sum of all such $z_\sigma$ that there exists a $\tau \in S_{2k}$ for which every $z_{p \sigma(p)}$ is in $X_{r_{\tau(p)}s_{\tau(p)}}$. Observe that each block row and each block column contains precisely two blocks $X_{rs}y_{rs}$ and hence these blocks are elements of a route from some set $P$. Now from the proof of Proposition 1 it is clear that the coefficient of $y_{r_1s_1} \cdots y_{r ks}$ equals $\prod_{P \in \mathcal{P}} (-1)^{s(P)-1} 2^{-v(P)} \text{tr} P$.

Now prove that traces of the routes under consideration are semi-invariants. If $\hat{X}_1, \ldots, \hat{X}_N$ are all the adjoint factors of a route $P$ (with multiplicities), then

$$\text{tr} P = \text{tr} \hat{X}_1 P_1 \cdots \hat{X}_N P_N = |X_1| \cdots |X_N| \text{tr} X_1^{-1} P_1 \cdots X_N^{-1} P_N,$$

where $P_i$ are products of the remaining elements of $P$. Since $\text{tr} X_1^{-1} P_1 \cdots X_N^{-1} P_N$ is a rational invariant, $\text{tr} P$ and $|X_1| \cdots |X_N|$ have the same weight.

Further, if a factor $X$ appears twice in a route $P$, then by Lemma 1 we have

$$\text{tr} P = \text{tr} XP_1 XP_2 = \text{tr} X P_1 \text{tr} X P_2 - \frac{1}{2} \text{tr} XX \text{tr} P_1 P_2.$$

The sequence $(P_1, P_2)$ is a route yielding that $\text{tr} P_1 P_2$ is a semi-invariant. So the algebra of semi-invariants is generated by traces of routes. Theorem 2 is proved.

Observe that the trace of a route containing a pair $\hat{\varphi}_a, \hat{\varphi}_b$ with $ha = tb, ta = hb$ may be excluded from the generating set. Indeed, such a route is of form $(\hat{\varphi}_a, P, \hat{\varphi}_b, S)$, where $P$ and $S$ are subroutes. But for $X, Y \in \text{Mat}_{2x2}(k)$ Lemma 1 implies that $|X| \text{tr} YZ = \text{tr} XY \text{tr} XZ - \text{tr} XYZ$. Therefore,

$$\text{tr} \hat{\varphi}_a P \hat{\varphi}_b S = \frac{1}{|\varphi_a|} |\varphi_a| \text{tr} \hat{\varphi}_a P \hat{\varphi}_b S = \frac{1}{|\varphi_a|} (\text{tr} \hat{\varphi}_a P \hat{\varphi}_b - \text{tr} \hat{\varphi}_a P \hat{\varphi}_b S) =$$

$$= \frac{1}{|\varphi_a|} (|\varphi_a| \text{tr} P (\text{tr} \hat{\varphi}_a P S - \text{tr} \hat{\varphi}_b P \varphi_a S)) =$$

$$= \text{tr} \varphi_a \varphi_b P S - \text{tr} P \text{tr} \varphi_a \varphi_b P S - \text{tr} S \text{tr} \varphi_a \varphi_b P + \text{tr} \varphi_a \varphi_b P S.$$

If the initial product $\hat{\varphi}_a P \hat{\varphi}_b S$ were assigned to a route, then $P$ and $S$ should map from $W_{ha}$ to $W_{ha}$ and from $W_{ta}$ to $W_{ta}$ respectively:

$$W_{ta} \xleftarrow{\hat{\varphi}_a} W_{ha} \xrightarrow{P} W_{ha} \xleftarrow{\hat{\varphi}_b} W_{ta}.$$

So the products whose traces we take in the final formula are also associated to routes.

Finally, note that for 3-representations Theorem 2 does not hold. Indeed, consider the quiver

```
1  \rightarrow \rightarrow 2
|     |     |
d  \rightarrow  \rightarrow  \rightarrow
4  \rightarrow  \rightarrow 3
```

and the dimension vector $(3,3,3)$. The coefficient $F$ of $\prod_{i,j} y_{ij}$ in the determinant of the block matrix

$$Z = \begin{pmatrix}
y_{11} \hat{\varphi}_a & y_{12} E & y_{13} \hat{\varphi}_b \\
y_{21} \hat{\varphi}_a & y_{22} \varphi_h & y_{23} E \\
y_{31} \hat{\varphi}_d & y_{32} \varphi_v & y_{33} \varphi_c
\end{pmatrix}$$

is not invariant, and its total degree with respect to the variables of each $\varphi$ equals 1. On the other hand, the trace of a route is not invariant if and only if the route contains an adjoint matrix. The total degree of a trace with respect to the variables from this matrix is at least 2. Thus the subalgebra generated by the traces of routes does not contain $F$. 

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