A Mixed Generalized Multifractal Formalism
For Vector Valued Measures

Anouar Ben Mabrouk

Computational Mathematics Laboratory, Department of Mathematics
Faculty of Sciences, 5019 Monastir, Tunisia.

Abstract
We introduce a mixed generalized multifractal formalism which extends the mixed
multifractal formalism introduced by L. Olsen based on generalizations of the Haus-
dorff and packing measures. The validity of such a formalism is proved in some
special cases.

Key words: Hausdorff and packing measures and dimensions, Multifractal
formalism.

PACS: 28A78, 28A80.

1 Introduction and main results

Dom(B) = \{-\nabla B_\mu(q); \nabla B_\mu \exists \} and \ f_\mu

2 Hausdorff and packing measures and dimensions

Given a subset \ E \subseteq \mathbb{R} \, and \, \epsilon > 0, \, we \, call \, an \, \epsilon \,-\text{covering} \, of \, E, \, any \, countable
set \ (U_i)_i \, of \, non-empty \, subsets \, U_i \subseteq \mathbb{R} \, satisfying

\[ E \subseteq \bigcup_i U_i \quad \text{and} \quad |U_i| = diam(U_i) \leq \epsilon, \]

Email address: anouar.benmabrouk@issatso.rnu.tn (Anouar Ben Mabrouk).

Preprint submitted to Elsevier 6 May 2014
where for any subset $U \subseteq \mathbb{R}$, $|U| = \text{diam}(U)$ is the diameter defined by $|U| = \text{diam}(U) = \sup_{x,y \in U} |x - y|$. Remark here that for $\epsilon_1 < \epsilon_2$, any $\epsilon_2$-covering of $E$ is obviously an $\epsilon_2$-covering of $E$. This implies that the quantity

$$
\mathcal{H}_\epsilon^s(E) = \inf\{\sum_i |U_i|^s ; (U_i) \text{ satisfying } [1]\}
$$

is a non increasing function in $\epsilon$. It’s limit

$$
\mathcal{H}^s(E) = \lim_{\epsilon \downarrow 0} \mathcal{H}_\epsilon^s(E)
$$

defines the so-called $s$-dimensional Hausdorff measure of $E$. It holds that for any set $E \subseteq \mathbb{R}$ there exists a critical value $s_E$ in the sense that

$$
\mathcal{H}^s(E) = 0, \forall s < s_E \quad \text{and} \quad \mathcal{H}^s(E) = +\infty, \forall s > s_E,
$$

or otherwise,

$$
s_E = \sup\{s > 0 ; \mathcal{H}^s(E) = 0\} = \inf\{s > 0 ; \mathcal{H}^s(E) = +\infty\}.
$$

Such a value is called the Hausdorff dimension of the set $E$ and is usually denoted by $\text{dim}_H E$ or simply $\text{dim} E$. When $U_i = B(x_i, r_i)$ is a ball centered at $x_i \in E$ and with diameter $r_i < \epsilon$, the covering $(B(x_i, r_i))_i$ is called an $\epsilon$-centered covering of $E$. However, surprisingly, the quantity $\mathcal{H}^s$ restricted only on centered coverings does not define a measure. To obtain a good measure with centered coverings one should do more. Denote

$$
\mathcal{C}_\epsilon^s(E) = \inf\{\sum_i |2r_i|^s ; (B(x_i, r_i))_i \text{ an } \epsilon - \text{centered covering of } E\}
$$

and similarly as above,

$$
\mathcal{C}^s(E) = \lim_{\epsilon \downarrow 0} \mathcal{C}_\epsilon^s(E).
$$

As stated previously, this is not a good measure. Indeed, ............

So, to obtain a good candidate, we set for $E \subseteq \mathbb{R}$,

$$
\mathcal{C}^s(E) = \sup_{F \subseteq E} \mathcal{C}^s(F).
$$

It is called the centered Hausdorff $s$-dimensional measure of $E$. But, although a fascinating relation to the Hausdorff measure exists. It holds that

$$
2^{-s} \mathcal{H}^s(E) \leq \mathcal{C}^s(E) \leq \mathcal{C}^s(E); \forall E \subseteq \mathbb{R}^d.
$$

Indeed, let $F \subseteq E$ be subsets of $\mathbb{R}^d$. It follows from the definition of $\mathcal{H}^s$ and $\mathcal{C}^s$ that $\mathcal{H}^s(F) \leq \mathcal{C}^s(F)$. Next, from the fact that $\mathcal{H}^s$ is an outer metric measure on $\mathbb{R}^d$, and the definition of $\mathcal{C}^s$, it results that $\mathcal{H}^s(E) \leq \mathcal{C}^s(E)$. Next,
let \( \{U_j\}_j \) be an \( \epsilon \)-covering of \( F \) and \( r_j = diam(U_j) \). For each \( i \) fixed, consider a point \( x_i \in U_i \cap F \). This results in a centered \( \epsilon \)-covering \( \{B(x_i, r_i)\}_i \) of \( F \).

Consequently,
\[
\overline{C}_s^*(F) \leq \sum_i (2r_i)^s = 2^s \sum_i (diam(U_i))^s.
\]

Hence,
\[
\overline{C}_s^*(F) \leq 2^s \mathcal{H}_s^*(F).
\]

Next, as \( \epsilon \downarrow 0 \), we obtain
\[
\overline{C}_s^*(F) \leq 2^s \mathcal{H}_s^*(F), \ \forall \ F \subseteq E.
\]

which guarantees that
\[
C_s^*(E) \leq 2^s \mathcal{H}_s^*(E).
\]

It holds that these measures give rise to some critical values in the sense that, for any set \( E \subseteq \mathbb{R} \) there exists a critical value \( h_E \) and \( c_E \) for which
\[
\mathcal{H}_s^*(E) = 0, \ \forall s < h_E \quad \text{and} \quad \mathcal{H}_s^*(E) = +\infty, \ \forall s > h_E
\]

and similarly
\[
C_s^*(E) = 0, \ \forall s < c_E \quad \text{and} \quad C_s^*(E) = +\infty, \ \forall s > c_E.
\]

But using equation 2 above, it proved that \( h_E = c_E \) and otherwise,
\[
h_E = \sup\{s > 0 ; \mathcal{H}_s^*(E) = 0\} = \inf\{s > 0 ; \mathcal{H}_s^*(E) = +\infty\}.
\]

Such a value is called the Hausdorff dimension of the set \( E \) and is usually denoted by \( \dim_H E \) or simply \( \dim E \).

Similarly, we call a centered \( \epsilon \)-packing of \( E \subseteq \mathbb{R}^d \), any countable set \( \{B(x_i, r_i)\}_i \) of disjoint balls centered at points \( x_i \in E \) and with diameters \( r_i < \epsilon \). The packing measure and dimension are defined as follows.

\[
\mathcal{P}_s^*(E) = \lim_{\epsilon \downarrow 0} \left( \sup\left\{ \sum_i (2r_i)^s ; \ \{B(x_i, r_i)\}_i \ \epsilon \ - \ \text{packing of } E \right\} \right),
\]

\[
\mathcal{P}_s^*(E) = \inf\left\{ \sum_i \mathcal{P}_s^*(E_i) ; \ E \subseteq \cup_i E_i \right\}.
\]

It holds as for the Hausdorff measure that there exists critical values \( \Delta_E \) and \( p_E \) satisfying respectively
\[
\mathcal{P}_s^*(E) = +\infty \text{ for } s < \Delta(E) \quad \text{and} \quad \mathcal{P}_s^*(E) = 0 \text{ for } s > \Delta(E)
\]

and respectively
\[
\mathcal{P}_s^*(E) = \infty \text{ for } s < p_E \quad \text{and} \quad \mathcal{P}_s^*(E) = 0 \text{ for } s > p_E.
\]
The critical value $\Delta(E)$ is called the logarithmic index of $E$ and $p_E$ is called the packing dimension of $E$ denote by $\text{Dim}_P(E)$ or simply $\text{Dim}(E)$. These quantities may be shown as

$$\Delta(E) = \sup\{s; \mathcal{P}^s(E) = 0\} = \inf\{s; \mathcal{P}^s(E) = +\infty\}.$$ 

and respectively

$$\text{Dim}(E) = \sup\{s; \mathcal{P}^s(E) = 0\} = \inf\{s; \mathcal{P}^s(E) = +\infty\}.$$ 

Usually, we have the inequality

$$\text{dim}(E) \leq \text{Dim}(E) \leq \Delta(E), \forall E \subseteq \mathbb{R}^d.$$ 

**Definition 2.1** A set $E \subseteq \mathbb{R}^d$ is said to be fractal in the sense of Taylor iff $\text{dim}(E) = \text{Dim}(E)$.

### 3 Multifractal generalizations of Hausdorff and packing measures

Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$, and a nonempty set $E \subseteq \mathbb{R}^d$ and $\epsilon > 0$. Let also $q$, $t$ be real numbers. We will recall hereafter the steps leading to the multifractal generalizations of the Hausdorff and packing measures due to L. Olsen in [9]. Denote

$$\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(E) = \inf\{\sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t\},$$

where the inf is taken over the set of all centered $\epsilon$-coverings of $E$, and for the empty set, $\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(\emptyset) = 0$. As for the preceding cases of Hausdorff and packing measures, it consists of a non increasing quantity as a function of $\epsilon$. We then consider its limit

$$\overline{\mathcal{H}}^{q,t}_{\mu}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(E) = \sup_{\delta > 0} \overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(E)$$

and finally, the multifractal generalization of the $s$-dimensional Hausdorff measure

$$\overline{\mathcal{H}}^{q,t}_{\mu}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu}(F).$$

Similarly, we define the multifractal generalization of the packing measure as follows.

$$\overline{\mathcal{P}}^{q,t}_{\mu,\epsilon}(E) = \sup\{\sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t\}$$
where the sup is taken over the set of all centered $\varepsilon$-packings of $E$. For the empty set, we set as usual $\mathcal{P}_{\mu,\varepsilon}^{\rho,t}(\emptyset) = 0$. Next,

$$\mathcal{P}_{\mu}^{\rho,t}(E) = \lim_{\varepsilon \downarrow 0} \mathcal{P}_{\mu,\varepsilon}^{\rho,t}(E) = \inf_{\delta > 0} \mathcal{P}_{\mu,\varepsilon}^{\rho,t}(E)$$

and finally,

$$\mathcal{P}_{\mu}^{\rho,t}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \mathcal{P}_{\mu}^{\rho,t}(E_i).$$

In [9], it has been proved that the measures $\mathcal{H}_{\mu}^{\rho,t}$, $\mathcal{P}_{\mu}^{\rho,t}$ and the pre-measure $\mathcal{P}_{\mu}^{\rho,t}$ assign in a usual way a dimension to every set $E \subseteq \mathbb{R}^d$ as resumed in the following proposition.

**Proposition 3.1** [9] Given a subset $E \subseteq \mathbb{R}^d$,

1. There exists a unique number $dim_{\mu}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{H}_{\mu}^{\rho,t}(E) = \begin{cases} +\infty & \text{for } t < dim_{\mu}^q(E) \\ 0 & \text{for } t > dim_{\mu}^q(E) \end{cases}$$

2. There exists a unique number $Dim_{\mu}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{P}_{\mu}^{\rho,t}(E) = \begin{cases} +\infty & \text{for } t < Dim_{\mu}^q(E) \\ 0 & \text{for } t > Dim_{\mu}^q(E) \end{cases}$$

3. There exists a unique number $\Delta_{\mu}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{P}_{\mu}^{\rho,t}(E) = \begin{cases} +\infty & \text{for } t < \Delta_{\mu}^q(E) \\ 0 & \text{for } t > \Delta_{\mu}^q(E) \end{cases}$$

The quantities $dim_{\mu}^q(E)$, $Dim_{\mu}^q(E)$ and $\Delta_{\mu}^q(E)$ define the so-called multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set $E$. More precisely, one has

$$dim_{\mu}^0(E) = dim(E), \quad Dim_{\mu}^0(E) = Dim(E) \quad \text{and} \quad \Delta_{\mu}^0(E) = \Delta(E).$$

The characteristics of these functions have been studied completely by L. Olsen. He proved among author results that $dim_{\mu}^q$ and $Dim_{\mu}^q$ are monotones and $\sigma$-stables. Furthermore, if $E = \text{support}(\mu)$ is the support of the measure $\mu$, one obtains

a. The functions $q \mapsto Dim_{\mu}^q(E)$ and $q \mapsto \Delta_{\mu}^q(E)$ are convex non increasing.

b. $q \mapsto dim_{\mu}^q(E)$ is non increasing.

c. i. For $q < 1$; $0 \leq dim_{\mu}^q(E) \leq Dim_{\mu}^q(E) \leq \Delta_{\mu}^q(E)$.

ii. $dim_{\mu}^1(E) \leq Dim_{\mu}^1(E) \leq \Delta_{\mu}^1(E) = 0$.

iii. For $q > 1$; $dim_{\mu}^q(E) \leq Dim_{\mu}^q(E) \leq \Delta_{\mu}^q(E) \leq 0$. 

5
4 Mixed multifractal generalizations of Hausdorff and packing measures and dimensions

The purpose of this section is to present our ideas. As it is noticed from the literature on multifractal analysis of measures, this latter always considered a single measure and studies its scaling behavior as well as the multifractal formalism associated. Recently, many works have been focused on the study of simultaneous behaviors of finitely many measures. In [15], a mixed multifractal analysis is developed dealing with a generalization of Rényi dimensions for finitely many self similar measures. This was one of the motivations leading to our present paper. Secondly, we intend to combine the generalized Hausdorff and packing measures and dimensions recalled in section 3 with Olsen’s results in [15] to define and develop a more general multifractal analysis for finitely many measures by studying their simultaneous regularity, spectrum and to define a mixed multifractal formalism which may describe better the geometry of the singularities’ sets of these measures especially simultaneous singularities.

Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) a vector valued measure composed of probability measures on \( \mathbb{R}^d \). We aim to study the simultaneous scaling behavior of \( \mu \) which we denote

\[
\lim_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} = (\lim_{r \downarrow 0} \frac{\log \mu_1(B(x, r))}{\log r}, \ldots, \lim_{r \downarrow 0} \frac{\log \mu_k(B(x, r))}{\log r}).
\]

In this paper, we apply the techniques of L. Olsen especially in [9] and [15] with the necessary modifications to give a detailed study of computing general mixed multifractal dimensions of simultaneously many finite number of measures and try to project our results for the case of a single measure to show the genericity of our’s. Let a \( E \subseteq \mathbb{R}^d \) be a nonempty set and \( \epsilon > 0 \). Let also \( q = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k \) and \( t \in \mathbb{R} \). The mixed generalized multifractal Hausdorff measure is defined as follows. Denote

\[\mu(B(x, r)) \equiv (\mu_1(B(x, r)), \ldots, \mu_k(B(x, r)))\]

and the product

\[(\mu(B(x, r)))^q \equiv (\mu_1(B(x, r)))^{q_1} \ldots (\mu_k(B(x, r)))^{q_k}.\]

Denote next,

\[\overline{H}_{\mu, \epsilon}^{q, t}(E) = \inf\{\sum_i (\mu(B(x_i, r_i)))^{q} (2r_i)^t\},\]

where the inf is taken over the set of all centered \( \epsilon \)-coverings of \( E \), and for the empty set, \( \overline{H}_{\mu, \epsilon}^{q, t}(\emptyset) = 0 \). As for the single case, of Hausdorff measure, it consists of a non increasing function of the variable \( \epsilon \). So that, its limit as
\[ \epsilon \downarrow 0 \text{ exists. Let} \]
\[ \mathcal{H}^{q,t}_{\mu}(E) = \lim_{\epsilon \downarrow 0} \mathcal{H}^{q,t}_{\mu,\epsilon}(E) = \sup_{\delta > 0} \mathcal{H}^{q,t}_{\mu,\epsilon}(E). \]

Let finally
\[ \mathcal{H}^{q,t}_{\mu}(E) = \sup_{F \subseteq E} \mathcal{H}^{q,t}_{\mu}(F). \]

**Lemma 4.1** \( \mathcal{H}^{q,t}_{\mu} \) is an outer metric measure on \( \mathbb{R}^d \).

**Proof.** We will prove firstly that \( \mathcal{H}^{q,t}_{\mu} \) is an outer measure. This means that

i. \( \mathcal{H}^{q,t}_{\mu}(\emptyset) = 0 \).

ii. \( \mathcal{H}^{q,t}_{\mu} \) is monotone, i.e. \( \mathcal{H}^{q,t}_{\mu}(E) \leq \mathcal{H}^{q,t}_{\mu}(F) \), whenever \( E \subseteq F \subseteq \mathbb{R}^d \).

iii. \( \mathcal{H}^{q,t}_{\mu} \) is sub-additive, i.e. \( \mathcal{H}^{q,t}_{\mu}(\bigcup A_n) \leq \sum_n \mathcal{H}^{q,t}_{\mu}(A_n) \).

The first item is obvious. Let us prove (ii). Let \( E \subseteq F \) be nonempty subsets of \( \mathbb{R}^d \). We have
\[ \mathcal{H}^{q,t}_{\mu}(E) = \sup_{A \subseteq E} \mathcal{H}^{q,t}_{\mu}(A) \leq \sup_{A \subseteq F} \mathcal{H}^{q,t}_{\mu}(A) = \mathcal{H}^{q,t}_{\mu}(F). \]

We next prove (iii). If the right hand term is infinite, the inequality is obvious. So, assume that it is finite. Let \( (E_n)_n \) be a countable family of subsets \( E_i \subseteq \mathbb{R}^d \) for which \( \sum_n \mathcal{H}^{q,t}_{\mu}(E_n) < \infty \). Let also \( \epsilon, \delta > 0 \) and \( (B(x_{ni}, r_{ni}))_i \), a centered \( \epsilon \)-covering of \( E_n \) satisfying
\[ \sum_i \mu(B(x_{ni}, r_{ni}))q(2r_{ni})^t \leq \mathcal{H}^{q,t}_{\mu,\delta}(E_n) + \frac{\delta}{2^n}. \]

The whole set \( \bigcup_{n,i} B(x_{ni}, r_{ni}) \) is a centered \( \epsilon \)-covering of the whole union \( \bigcup_{n} E_n \).

As a consequence,
\[ \mathcal{H}^{q,t}_{\mu,\epsilon}(\bigcup_{n} E_n) \leq \sum_n \sum_i \mu(B(x_{ni}, r_{ni}))q(2r_{ni})^t \]
\[ \leq \sum_n \left( \mathcal{H}^{q,t}_{\mu,\epsilon}(E_n) + \frac{\delta}{2^n} \right) \]
\[ \leq \sum_n \left( \mathcal{H}^{q,t}_{\mu}(E_n) + \frac{\delta}{2^n} \right) \]
\[ \leq \sum_n \mathcal{H}^{q,t}_{\mu}(E_n) + \delta. \]

Having \( \epsilon \) and \( \delta \) going towards 0, we obtain
\[ \mathcal{H}^{q,t}_{\mu}(\bigcup_{n} E_n) \leq \sum_n \mathcal{H}^{q,t}_{\mu}(E_n). \]
Let next a set $F$ covered with the countable set $(A_n)_n$. That is $F \subseteq \bigcup_n A_n$. We have
\[ \mathcal{H}_{\mu}^{q,t}(F) = \mathcal{H}_{\mu}^{q,t}\left(\bigcup_n (A_n \cap F)\right) \leq \sum_n \mathcal{H}_{\mu}^{q,t}(A_n \cap F) \leq \sum_n \mathcal{H}_{\mu}^{q,t}(A_n). \]

Taking the sup on $F$, we obtain
\[ \mathcal{H}_{\mu}^{q,t}(F) \leq \sum_n \mathcal{H}_{\mu}^{q,t}(A_n). \]

We now prove that $\mathcal{H}_{\mu}^{q,t}$ is metric. Let $A, B$ subsets of $\mathbb{R}^d$ where the distance $d(A, B)$ is defined by $d(A, B) = \inf\{|x - y|; \ x \in A, y \in B\} > 0$ and $\mathcal{H}_{\mu}^{q,t}(A \cup B) < \infty$. Let next $0 < \delta < d(A, B)$, $\varepsilon > 0$, $F_1 \subseteq A$, $F_2 \subseteq B$ and $(B(x_i, r_i))_i$ a centered $\delta$-covering of the set $F_1 \cup F_2$ and such that
\[ \mathcal{H}_{\mu,\delta}^{q,t}(F_1 \cup F_2) \leq \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t \leq \mathcal{H}_{\mu,\delta}^{q,t}(F_1 \cup F_2) + \varepsilon. \]

This is always possible from the definition of $\mathcal{H}_{\mu,\delta}^{q,t}(F_1 \cup F_2)$. Denote next the index sets
\[ I = \{ i; \ B(x_i, r_i) \cap F_1 \neq \emptyset \} \quad \text{and} \quad J = \{ i; \ B(x_i, r_i) \cap F_2 \neq \emptyset \}. \]

Hence, the countable sets $(B(x_i, r_i))_{i \in I}$ and $(B(x_i, r_i))_{i \in J}$ are centered $\delta$-coverings of $F_1$ and $F_2$ respectively. Consequently,
\[ \mathcal{H}_{\mu,\delta}^{q,t}(F_1) + \mathcal{H}_{\mu,\delta}^{q,t}(F_2) \leq \sum_{i \in I} (\mu(B(x_i, r_i)))^q (2r_i)^t + \sum_{i \in J} (\mu(B(x_i, r_i)))^q (2r_i)^t \leq \mathcal{H}_{\mu,\delta}^{q,t}(F_1 \cup F_2) + \varepsilon. \]

As a result,
\[ \mathcal{H}_{\mu}^{q,t}(F_1) + \mathcal{H}_{\mu}^{q,t}(F_2) \leq \mathcal{H}_{\mu}^{q,t}(F_1 \cup F_2) + \varepsilon \leq \mathcal{H}_{\mu}^{q,t}(A \cup B) + \varepsilon. \]

When $\varepsilon \downarrow 0$ and taking the sup on the sets $F_1 \subseteq A$ and $F_2 \subseteq B$, we obtain
\[ \mathcal{H}_{\mu}^{q,t}(A \cup B) \geq \mathcal{H}_{\mu}^{q,t}(A) + \mathcal{H}_{\mu}^{q,t}(B). \]

The inequality
\[ \mathcal{H}_{\mu}^{q,t}(A \cup B) \leq \mathcal{H}_{\mu}^{q,t}(A) + \mathcal{H}_{\mu}^{q,t}(B). \]

follows from the sub-additivity property of the measure $\mathcal{H}_{\mu}^{q,t}$. 

8
Definition 4.1 The restriction of $H_{q,t}^\mu$ on Borel sets is called the mixed generalized Hausdorff measure on $\mathbb{R}^d$.

Now, we define the mixed generalized multifractal packing measure. We use already the same notations as previously. Let

$$P_{q,t}^\mu,\epsilon(E) = \sup \{ \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t \}$$

where the sup is taken over the set of all centered $\epsilon$-packings of $E$. For the empty set, we set as usual $P_{q,t}^\mu(\emptyset) = 0$. Next, we consider the limit as $\epsilon \downarrow 0$,

$$P_{q,t}^\mu(E) = \lim_{\epsilon \downarrow 0} P_{q,t}^\mu,\epsilon(E) = \inf_{\delta > 0} P_{q,t}^\mu(E)$$

and finally,

$$P_{q,t}^\mu(E) = \inf_{E \subseteq \cup \{ E_i \}} \sum_i P_{q,t}^\mu(E_i).$$

Lemma 4.2 $P_{q,t}^\mu$ is an outer metric measure on $\mathbb{R}^d$.

The proof of this lemma uses the following result.

$$P_{q,t}^\mu(A \cup B) = P_{q,t}^\mu(A) + P_{q,t}^\mu(B), \quad \text{whenever } d(A, B) > 0. \quad (3)$$

Indeed, let $0 < \epsilon < \frac{1}{2} d(A, B)$ and $(B(x_i, r_i))_i$ be a centered $\epsilon$-packing of the union $A \cup B$. It can be divided into two parts $I$ and $J$,

$$(B(x_i, r_i))_i = (B(x_i, r_i))_{i \in I} \cup (B(x_i, r_i))_{i \in J}$$

where

$$\forall i \in I, \ B(x_i, r_i) \cap B = \emptyset \quad \text{and} \quad \forall i \in J, \ B(x_i, r_i) \cap A = \emptyset.$$ 

Therefore, $(B(x_i, r_i))_{i \in I}$ is a centered $\epsilon$-packing of $A$ and $(B(x_i, r_i))_{i \in J}$ is a centered $\epsilon$-packing of the union $B$. Hence,

$$\sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t = \sum_{i \in I} (\mu(B(x_i, r_i)))^q (2r_i)^t + \sum_{i \in J} (\mu(B(x_i, r_i)))^q (2r_i)^t$$

Consequently,

$$P_{q,t}^\mu(A \cup B) \leq P_{q,t}^\mu(A) + P_{q,t}^\mu(B)$$

and thus the limit for $\epsilon \downarrow 0$ gives

$$P_{q,t}^\mu(A \cup B) \leq P_{q,t}^\mu(A) + P_{q,t}^\mu(B).$$
The converse is more easier and it states that $\mathcal{P}_{\mu,\epsilon}^{q,t}$ and next $\mathcal{P}_{\mu}^{q,t}$ are sub-additive. Let $(B(x_i, r_i))_i$ be a centered $\epsilon$-packing of $A$ and $(B(y_i, r_i))_i$ be a centered $\epsilon$-packing of $B$. The union $(B(x_i, r_i))_i \cup (B(y_i, r_i))_i$ is a centered $\epsilon$-packing of $A \cup B$. So that

$$\mathcal{P}_{\mu,\epsilon}^{q,t}(A \cup B) \geq \sum_i (\mu(B(x_i, r_i)))^q(2r_i)^t + \sum_i (\mu(B(y_i, r_i)))^q(2r_i)^t.$$ 

Taking the sup on $(B(x_i, r_i))_i$ as a centered $\epsilon$-packing of $A$ and next the sup on $(B(y_i, r_i))_i$ as a centered $\epsilon$-packing of $B$, we obtain

$$\mathcal{P}_{\mu,\epsilon}^{q,t}(A \cup B) \geq \mathcal{P}_{\mu,\epsilon}^{q,t}(A) + \mathcal{P}_{\mu,\epsilon}^{q,t}(B)$$

and thus the limit for $\epsilon \downarrow 0$ gives

$$\mathcal{P}_{\mu}^{q,t}(A \cup B) \geq \mathcal{P}_{\mu}^{q,t}(A) + \mathcal{P}_{\mu}^{q,t}(B).$$

**Proof of Lemma 4.2.** We shall prove as previously

i. $\mathcal{P}_{\mu}^{q,t}(\emptyset) = 0.$

ii. $\mathcal{P}_{\mu}^{q,t}$ is monotone, i.e. $\mathcal{P}_{\mu}^{q,t}(E) \leq \mathcal{P}_{\mu}^{q,t}(F)$, whenever $E \subseteq F \subseteq \mathbb{R}^d$.

iii. $\mathcal{P}_{\mu}^{q,t}$ is sub-additive, i.e. $\mathcal{P}_{\mu}^{q,t}(\bigcup_n A_n) \leq \sum_n \mathcal{P}_{\mu}^{q,t}(A_n)$.

The first item is immediate from the definition of $\mathcal{P}_{\mu,\epsilon}^{q,t}(\emptyset) = 0$. Let $E \subseteq F$ be subsets of $\mathbb{R}^d$. We have

$$\mathcal{P}_{\mu}^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \mathcal{P}_{\mu}^{q,t}(E_i) \leq \inf_{F \subseteq \bigcup_i E_i} \sum_i \mathcal{P}_{\mu}^{q,t}(E_i) = \mathcal{P}_{\mu}^{q,t}(F).$$

So is the item ii. Let next $(A_n)_n$ a countable set of subsets of $\mathbb{R}^d$, $\epsilon > 0$ and for each $n$, $(E_{ni})_i$ be a covering of $A_n$ such that

$$\sum_i \mathcal{P}_{\mu}^{q,t}(E_{ni}) \leq \mathcal{P}_{\mu}^{q,t}(A_n) + \frac{\epsilon}{2n}.$$ 

It follows for all $\epsilon > 0$ that

$$\mathcal{P}_{\mu}^{q,t}(\bigcup_n A_n) \leq \sum_n \sum_i \mathcal{P}_{\mu}^{q,t}(E_{ni}) \leq \sum_n \mathcal{P}_{\mu}^{q,t}(A_n) + \epsilon.$$ 

Hence,

$$\mathcal{P}_{\mu}^{q,t}(\bigcup_n A_n) \leq \sum_n \mathcal{P}_{\mu}^{q,t}(A_n).$$

So is the item iii. We now prove that $\mathcal{P}_{\mu}^{q,t}$ is metric. Let $A, B$ subsets of $\mathbb{R}^d$ be such that $d(A, B) > 0$. We shall prove that

$$\mathcal{P}_{\mu}^{q,t}(A \cup B) = \mathcal{P}_{\mu}^{q,t}(A) + \mathcal{P}_{\mu}^{q,t}(B).$$
Since \( \mathcal{P}_{\mu}^{q,t} \) is an outer measure, it suffices to show that
\[
\mathcal{P}_{\mu}^{q,t}(A \cup B) \geq \mathcal{P}_{\mu}^{q,t}(A) + \mathcal{P}_{\mu}^{q,t}(B).
\]

Of course, if the left hand term is infinite, the inequality is obvious. So, suppose that it is finite. For \( \varepsilon > 0 \), there exists a covering \( (E_i)_i \) of the union set \( A \cup B \) such that
\[
\sum_i \mathcal{P}_{\mu}^{q,t}(E_i) \leq \mathcal{P}_{\mu}^{q,t}(A \cup B) + \varepsilon.
\]

By denoting \( F_i = A \cap E_i \) and \( H_i = B \cap E_i \), we get countable coverings \( (F_i)_i \) of \( A \) and \( (H_i)_i \) for \( B \) respectively. Furthermore, \( F_i \cap H_j = \emptyset \) for all \( i \) and \( j \). Consequently,
\[
\mathcal{P}_{\mu}^{q,t}(A) + \mathcal{P}_{\mu}^{q,t}(B) \leq \sum_i \left( \mathcal{P}^{q,t}_{\mu}(F_i) + \mathcal{P}^{q,t}_{\mu}(H_i) \right).
\]

Since \( d(A,B) > 0 \), \( F_i \subset A \) and \( H_i \subset B \), it follows that \( d(F_i, H_j) > 0 \) for all \( i \) and \( j \). Hence, claim 3 affirms that
\[
\mathcal{P}^{q,t}_{\mu}(E_i) = \mathcal{P}^{q,t}_{\mu}(F_i \cup H_i) = \mathcal{P}^{q,t}_{\mu}(F_i) + \mathcal{P}^{q,t}_{\mu}(H_i).
\]

Hence,
\[
\mathcal{P}_{\mu}^{q,t}(A) + \mathcal{P}_{\mu}^{q,t}(B) \leq \sum_i \mathcal{P}^{q,t}_{\mu}(E_i) \leq \mathcal{P}_{\mu}^{q,t}(A \cup B) + \varepsilon
\]
and the result is obtained by having \( \varepsilon \downarrow 0 \).

**Definition 4.2** The restriction of \( \mathcal{P}_{\mu}^{q,t} \) on Borel sets is called the mixed generalized Hausdorff measure on \( \mathbb{R}^d \).

It holds as for the case of the multifractal analysis of a single measure that the measures \( \mathcal{H}_{\mu}^{q,t} \), \( \mathcal{P}_{\mu}^{q,t} \) and the pre-measure \( \mathcal{P}^{q,t}_{\mu} \) assign a dimension to every set \( E \subseteq \mathbb{R}^d \).

**Proposition 4.1** Given a subset \( E \subseteq \mathbb{R}^d \),

(1) There exists a unique number \( \dim_{\mu}^{q}(E) \in [-\infty, +\infty] \) such that
\[
\mathcal{H}_{\mu}^{q,t}(E) = \begin{cases} 
+\infty & \text{for } t < \dim_{\mu}^{q}(E) \\
0 & \text{if } t > \dim_{\mu}^{q}(E)
\end{cases}
\]

(2) There exists a unique number \( \text{Dim}_{\mu}^{q}(E) \in [-\infty, +\infty] \) such that
\[
\mathcal{P}_{\mu}^{q,t}(E) = \begin{cases} 
+\infty & \text{for } t < \text{Dim}_{\mu}^{q}(E) \\
0 & \text{for } t > \text{Dim}_{\mu}^{q}(E)
\end{cases}
\]
There exists a unique number $\Delta^q_{\mu}(E) \in [-\infty, +\infty]$ such that
\[
\mathcal{F}^{q,t}_{\mu}(E) = \begin{cases} 
+\infty & \text{for } t < \Delta^q_{\mu}(E) \\
0 & \text{for } t > \Delta^q_{\mu}(E) 
\end{cases}
\]

**Definition 4.3** The quantities $\dim^q_{\mu}(E)$, $\Dim^q_{\mu}(E)$ and $\Delta^q_{\mu}(E)$ defines the so-called mixed multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set $E$.

Remark that if we denote $1_i = (0, 0, \ldots, q_i, 0, \ldots, 0)$ the vector with zero coordinates except the $i$th one which equals 1, we obtain the multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set $E$.

**Definition 4.3** The quantities $\dim_{\mu}^{1_i}(E)$, $\Dim_{\mu}^{1_i}(E)$ and $\Delta_{\mu}^{1_i}(E)$ defines the so-called mixed multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set $E$ for the single measure $\mu_i$.

Similarly, for the null vector of $\mathbb{R}^k$, we obtain
\[
\dim^0_{\mu}(E) = \dim(E), \quad \Dim^0_{\mu}(E) = \Dim(E) \quad \text{and} \quad \Delta^0_{\mu}(E) = \Delta(E).
\]

**Proof of Proposition 4.1.**
1. We claim that $\forall t \in \mathbb{R}$ such that $\mathcal{H}^{q,t}_{\mu}(E) < \infty$ it holds that $\mathcal{H}^{q,t'}_{\mu}(E) = 0$ for any $t' > t$. Indeed, let $\epsilon > 0$, $F \subseteq E$ and $(B(x_i, r_i))_i$ be a centered $\epsilon$-covering of $F$. We have
\[
\mathcal{H}^{q,t}_{\mu}(F) \leq \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t \leq \delta^{-t} \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t.
\]
Consequently,
\[
\mathcal{H}^{q,t}_{\mu}(F) \leq \epsilon^{-t} \mathcal{H}^{q,t}_{\mu}(F).
\]
Hence,
\[
\mathcal{H}^{q,t}_{\mu}(F) = 0, \quad \forall F \subseteq E.
\]
As a result, $\mathcal{H}^{q,t}_{\mu}(E) = 0$. We then set
\[
\dim^q_{\mu}(E) = \inf \{ t \in \mathbb{R}; \mathcal{H}^{q,t}_{\mu}(E) = 0 \}.
\]
One can proceed otherwise by claiming that $\forall t \in \mathbb{R}$ such that $\mathcal{H}^{q,t}_{\mu}(E) > 0$ it holds that $\mathcal{H}^{q,t'}_{\mu}(E) = +\infty$ for any $t' < t$. Indeed, proceeding as previously, we obtain for $\epsilon > 0$,
\[
\epsilon^{-t} \mathcal{H}^{q,t}_{\mu}(F) \leq \mathcal{H}^{q,t}_{\mu}(F).
\]
Hence,
\[
\mathcal{H}^{q,t}_{\mu}(F) = +\infty, \quad \forall F \subseteq E.
\]
As a result, $\mathcal{H}^{q,t}_{\mu}(E) = +\infty$. We then set
\[
\dim^q_{\mu}(E) = \sup \{ t \in \mathbb{R}; \mathcal{H}^{q,t}_{\mu}(E) = +\infty \}.
\]
2. Similarly to the previous case, let $t \in \mathbb{R}$ be such that $\mathcal{P}^{q,t}_\mu(E) < \infty$. There exists $(E_i)_i$ subsets of $\mathbb{R}^d$ satisfying

$$E \subseteq \bigcup_i E_i, \quad \text{and} \quad \mathcal{P}^{q,t}_\mu(E_i) < \infty, \quad \text{for any } i.$$ 

Let next $t' > t$, $\epsilon > 0$ and $(B(x_{ni}, r_{ni}))_n$ be a centered $\epsilon$-packing of the set $E_i$. Then

$$\sum_n (\mu(B(x_{ni}, r_{ni})))^q (2r_{ni})^{t'} \leq \epsilon^{t'-t} \sum_n (\mu(B(x_{ni}, r_{ni})))^q (2r_{ni})^t$$

which implies that

$$\mathcal{P}^{q,t'}_{\mu,\epsilon}(E_i) \leq \epsilon^{t'-t} \mathcal{P}^{q,t}_{\mu,\epsilon}(E_i). \quad (4)$$

Hence, $\mathcal{P}^{q,t'}_{\mu}(E_i) = 0$ for all $i$ and consequently $\mathcal{P}^{q,t'}_{\mu}(E) = 0$. we set as previously

$$\text{Dim}^q_\mu(E) = \inf \{ t \in \mathbb{R}; \mathcal{P}^{q,t}_\mu(E) = 0 \}.$$

3. It follows from equation 4 that for any $t \in \mathbb{R}$ such that $\mathcal{P}^{q,t}_\mu(E) < \infty$, we have $\mathcal{P}^{q,t'}_{\mu}(E) = 0$ for any $t' > t$. We then set

$$\Delta^q_\mu = \inf \{ t \in \mathbb{R}; \mathcal{P}^{q,t}_\mu(E) = 0 \}.$$

Next, we aim to study the characteristics of the mixed multifractal generalizations of dimensions. To do this we will adapt the following notations. For $q = (q_1, \ldots, q_k) \in \mathbb{R}^k$,

$$b_{\mu,E}(q) = \text{dim}^q_\mu(E), \quad B_{\mu,E}(q) = \text{Dim}^q_\mu(E) \quad \text{and} \quad \Lambda_{\mu,E}(q) = \Delta^q_\mu(E).$$

When $E = \text{support}(\mu)$ is the support of the measure $\mu$, we will omit the indexation with $E$ and denote simply

$$b_\mu(q), \quad B_\mu(q) \quad \text{and} \quad \Lambda_\mu(q).$$

The following propositions resumes the characteristics of these functions and extends the results of L. Olsen [9] for our case.

**Proposition 4.2**

a. $0 \leq b_{\mu,E}(q)$, whenever $q_i \leq 1$ and $\mu_i(E) > 0$ for all $i = 1, 2, \ldots, k$.

b. $\Lambda_{\mu,E}(q) \leq 0$, whenever $q_i \geq 1$ for all $i = 1, 2, \ldots, k$.

c. $b_\mu(q)$ and $B_\mu(q)$ are non decreasing with respect to the inclusion property in $\mathbb{R}^d$.

d. $b_\mu(q)$ and $B_\mu(q)$ are $\sigma$-stable.
Proof. a. ...

b. For \( q \leq 1 \), it holds that

\[
\mathcal{P}_{\mu,\epsilon}^{q,t}(E) \leq \epsilon^t, \, \forall \, t > 0.
\]

Hence,

\[
\mathcal{P}_{\mu,\epsilon}^{q,t}(E) = 0, \, \forall \, t > 0
\]

which means that

\[
\Lambda_{\mu,E}(q) \leq t, \, \forall \, t > 0 \iff \Lambda_{\mu,E}(q) \leq 0.
\]

c. Let \( E \subseteq F \) be subsets of \( \mathbb{R}^d \). We have

\[
\mathcal{H}_{\mu}^{q,t}(E) = \sup_{A \subseteq E} \mathcal{H}_{\mu}^{q,t}(A) \leq \sup_{A \subseteq F} \mathcal{H}_{\mu}^{q,t}(A) = \mathcal{H}_{\mu}^{q,t}(F).
\]

So for the monotony of \( b_{\mu,\cdot}(q) \). Next, since \( \mathcal{P}_{\mu}^{q,t} \) is an outer measure,

\[
\mathcal{P}_{\mu}^{q,t}(E) \leq \mathcal{P}_{\mu}^{q,t}(F) = 0, \, \forall \, t > B_{\mu,F}(q).
\]

Consequently,

\[
\mathcal{P}_{\mu}^{q,t}(E) = 0, \, \forall \, t > B_{\mu,F}(q).
\]

Therefore,

\[
B_{\mu,E}(q) \leq t, \, \forall \, t > B_{\mu,F}(q).
\]

So that

\[
B_{\mu,E}(q) \leq B_{\mu,F}(q).
\]

d. Let \( (A_n) \) be a countable set of subsets \( A_n \subseteq \mathbb{R}^d \) and denote \( A = \bigcup_{n} A_n \). It holds from the monotony of \( b_{\mu,\cdot}(q) \) that

\[
b_{\mu,A_n}(q) \leq b_{\mu,A}(q), \, \forall \, n.
\]

Hence,

\[
\sup_{n} b_{\mu,A_n}(q) \leq b_{\mu,A}(q).
\]

Next, for any \( t > \sup_{n} b_{\mu,A_n}(q) \), there holds that

\[
\mathcal{H}_{\mu}^{q,t}(A_n) = 0, \, \forall \, n.
\]

Consequently, from the sub-additivity property of \( \mathcal{H}_{\mu}^{q,t} \), it holds that

\[
\mathcal{H}_{\mu}^{q,t}(\bigcup_{n} A_n) = 0, \, \forall \, t > \sup_{n} b_{\mu,A_n}(q).
\]

Which means that

\[
b_{\mu,A}(q) \leq t, \, \forall \, t > \sup_{n} b_{\mu,A_n}(q).
\]
Hence,

\[ b_{\mu,A}(q) \leq \sup_n b_{\mu,A_n}(q). \]

We shall now prove the \( \sigma \)-stability of \( B_{\mu,}(q) \). Consider as previously a countable set \( (A_n)_n \) of subsets of \( \mathbb{R}^d \). The following inequality is immediate.

\[ \sup_n B_{\mu,A_n}(q) \leq B_{\mu,}(q). \]

Next, for any \( t > \sup_n B_{\mu,A_n}(q) \), we have

\[ \mathcal{P}_t^q(A) = \sum_n \mathcal{P}_t^q(A_n) = 0. \]

So that

\[ \mathcal{P}_t^q(A) = 0, \quad \forall t > \sup B_{\mu,A_n}(q). \]

Consequently,

\[ B_{\mu,A}(q) \leq t, \quad \forall t > \sup B_{\mu,A_n}(q). \]

Which means that

\[ B_{\mu,A}(q) \leq \sup_n B_{\mu,A_n}(q). \]

Next, we continue to study the characteristics of the mixed generalized multifractal dimensions. The following result is obtained.

**Proposition 4.3**  

\begin{itemize}
  \item[\textbf{a.}] The functions \( q \mapsto -B_{\mu}(q) \) and \( q \mapsto -\Lambda_{\mu}(q) \) are convex.
  \item[\textbf{b.}] For \( i = 1, 2, \ldots, k \), the functions \( q_i \mapsto -b_{\mu}(q) \), \( q_i \mapsto -B_{\mu}(q) \) and \( q_i \mapsto -\Lambda_{\mu}(q) \), \((\tilde{q}_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_k) \text{ fixed})\), are non increasing.
\end{itemize}

**Proof.**  

\textbf{a.} We start by proving that \( \Lambda_{\mu,E} \) is convex. Let \( p, q \in \mathbb{R}^k, \alpha \in [0,1], s > \Lambda_{\mu,E}(p) \) and \( t > \Lambda_{\mu,E}(q) \). Consider next a centered \( \epsilon \)-packing \( (B_i = B(x_i, r_i))_i \) of \( E \). Applying Hölder’s inequality, it holds that

\[ \sum_i (\mu(B_i))^{\alpha q + (1-\alpha)p} (2r_i)^{\alpha t + (1-\alpha)s} \leq \left( \sum_i (\mu(B_i))^{q}(2r_i)^t \right)^{\alpha} \left( \sum_i (\mu(B_i))^p(2r_i)^s \right)^{1-\alpha}. \]

Hence,

\[ \mathcal{P}_{\mu,E}^{\alpha q + (1-\alpha)p, \alpha t + (1-\alpha)s}(E) \leq \left( \mathcal{P}_{\mu,E}^{q,t}(E) \right)^{\alpha} \left( \mathcal{P}_{\mu,E}^{p,s}(E) \right)^{1-\alpha}. \]

The limit on \( \epsilon \downarrow 0 \) gives

\[ \mathcal{P}_{\mu}^{\alpha q + (1-\alpha)p, \alpha t + (1-\alpha)s}(E) \leq \left( \mathcal{P}_{\mu}^{q,t}(E) \right)^{\alpha} \left( \mathcal{P}_{\mu}^{p,s}(E) \right)^{1-\alpha}. \]

Consequently,

\[ \mathcal{P}_{\mu}^{\alpha q + (1-\alpha)p, \alpha t + (1-\alpha)s}(E) = 0 \quad \forall s > \Lambda_{\mu,E}(p) \text{ and } t > \Lambda_{\mu,E}(q). \]
It results that
\[ \Lambda_{\mu,E}(\alpha q + (1 - \alpha)p) \leq \alpha \Lambda_{\mu,E}(q) + (1 - \alpha)\Lambda_{\mu,E}(p). \]

We now prove the convexity of \( B_{\mu,E} \). We set in this case \( t = B_{\mu,E}(q) \) and \( s = B_{\mu,E}(p) \). We have
\[
\mathcal{P}_{\mu}^{q,t+\varepsilon}(E) = \mathcal{P}_{\mu}^{p,s+\varepsilon}(E) = 0.
\]
Therefore, there exists \((H_i)_i\) and \((K_i)_i\) coverings of the set \( E \) for which
\[
\sum_{i} \mathcal{P}_{\mu}^{q,t+\varepsilon}(H_i) \leq 1 \quad \text{et} \quad \sum_{j} \mathcal{P}_{\mu}^{p,s+\varepsilon}(K_j) \leq 1.
\]
Denote for \( n \in \mathbb{N} \), \( E_n = \bigcup_{1 \leq i,j \leq n} (H_i \cap K_j) \). Thus, \((E_n)_n\) is a covering of \( E \). So that,
\[
\sum_{i,j=1}^{n} \mathcal{P}_{\mu}^{q,t+\varepsilon}(H_i \cap K_j) \leq n \quad \text{et} \quad \sum_{i,j=1}^{n} \mathcal{P}_{\mu}^{p,s+\varepsilon}(H_i \cap K_j) \leq n \cdot \frac{n}{\alpha} \quad \text{et} \quad \frac{n-1}{\alpha} = n < \infty.
\]
Consequently,
\[
B_{\mu,n}(\alpha q + (1 - \alpha)p) \leq \alpha t + (1 - \alpha)s + \varepsilon, \quad \forall \varepsilon > 0.
\]
Hence,
\[
B_{\mu,E}(\alpha q + (1 - \alpha)p) \leq \alpha B_{\mu,E}(q) + (1 - \alpha)B_{\mu,E}(p).
\]

b. For \( i = 1, 2, \ldots, k \), let \( \tilde{q}_i \) fixed and \( p_i \leq q_i \) reel numbers. Denote next \( q = (q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_k) \) and \( p = (q_1, \ldots, q_{i-1}, p_i, q_{i+1}, \ldots, q_k) \). Let finally \( A \subseteq E \). For a centered \( \varepsilon \)-covering \((B(x_i, r_i))_i\) of \( A \), we have immediately
\[
\mu(B(x_i, r_i))^q(2r_i)^t \leq \mu(B(x_i, r_i))^p(2r_i)^t, \quad \forall t \in \mathbb{R}.
\]
Hence,
\[
\mu^{q,t}(A) \leq \mu^{p,t}(A).
\]
When \( \varepsilon \downarrow 0 \), we obtain
\[
\mathcal{H}_{\mu}^{q,t}(A) \leq \mathcal{H}_{\mu}^{p,t}(A).
\]
Therefore,
\[
\mathcal{H}_{\mu}^{q,t}(E) = \sup_{A \subseteq E} \mathcal{H}_{\mu}^{q,t}(A) \leq \sup_{A \subseteq E} \mathcal{H}_{\mu}^{p,t}(A) = \mathcal{H}_{\mu}^{p,t}(E).
\]
This induces the fact that
\[ H_{\mu}^{t}(E) = 0, \quad \forall t > b_{\mu,E}(p). \]
Consequently
\[ b_{\mu,E}(q) < t, \quad \forall t > b_{\mu,E}(p). \]
Hence,
\[ b_{\mu,E}(q) \leq b_{\mu,E}(p). \]

We shall now prove the monotony \( \Lambda_{\mu,E} \). With the same notations as above and using a centered \( \epsilon \)-packing of the set \( E \), we obtain
\[ \mathcal{P}_{\mu}^{q,t}(E) \leq \mathcal{P}_{\mu}^{p,t}(E). \]
As a consequence,
\[ \mathcal{P}_{\mu}^{p,t}(E) = 0, \quad \forall t > \Lambda_{\mu,E}(p). \]
Therefore,
\[ \Lambda_{\mu,E}(q) < t, \quad \forall t > \Lambda_{\mu,E}(p). \]
Hence,
\[ \Lambda_{\mu,E}(q) \leq \Lambda_{\mu,E}(p). \]

We now prove that \( B_{\mu,E} \) is non increasing. For \( i = 1,2,\ldots,k \), let \( \hat{q}_i \) fixed and \( q_i \leq p_i \) reel numbers. Denote next \( q = (q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_k) \) and \( p = (q_1, \ldots, q_{i-1}, p_i, q_{i+1}, \ldots, q_k) \). Let next \( (E_i)_i \) be a covering of the set \( E \). It results from the previous case that
\[ \sum_i \mathcal{P}_{\mu}^{p,t}(E_i) \geq \sum_i \mathcal{P}_{\mu}^{q,t}(E_i). \]
Which means that \( \mathcal{P}_{\mu}^{p,t}(E) \geq \mathcal{P}_{\mu}^{q,t}(E) \). Consequently,
\[ \mathcal{P}_{\mu}^{q,t}(E) = 0, \quad \forall t > B_{\mu,E}(p) \]
and thus
\[ B_{\mu,E}(q) < t, \quad \forall t > B_{\mu,E}(p). \]
Hence,
\[ B_{\mu,E}(q) \leq B_{\mu,E}(p). \]

**Proposition 4.4**

a. \( 0 \leq b_{\mu}(q) \leq B_{\mu}(q) \leq \Lambda_{\mu}(q) \), whenever \( q_i < 1 \) for all \( i = 1,2,\ldots,k \).

b. \( b_{\mu}(\mathfrak{q}_i) = B_{\mu}(\mathfrak{q}_i) = \Lambda_{\mu}(\mathfrak{q}_i) = 0 \), where \( \mathfrak{q}_i = (0,0,\ldots,1,0,\ldots,0) \).

c. \( b_{\mu}(q) \leq B_{\mu}(q) \leq \Lambda_{\mu}(q) \leq 0 \) whenever \( q_i > 1 \) for all \( i = 1,2,\ldots,k \).

The proof of this results reposes on the following intermediate ones.
Lemma 4.3 There exists a constant $\xi \in [0, +\infty[$ satisfying for any $E \subseteq \mathbb{R}^d$,
\[
\mathcal{H}^{q,t}_\mu(E) \leq \xi \mathcal{P}^{q,t}_\mu(E) \leq \xi \overline{\mathcal{P}}^{q,t}_\mu(E), \quad \forall q, t.
\]
More precisely, $\xi$ is the number related to the Besicovitch covering theorem.

Theorem 4.1 There exists a constant $\xi \in \mathbb{N}$ satisfying: For any $E \in \mathbb{R}^d$ and $(r_x)x \in E$ a bounded set of positive real numbers, there exists $\xi$ sets $B_1, B_2, \ldots, B_\xi$, that are finite or countable composed of balls $B(x, r_x), x \in E$ such that

- $E \subseteq \bigcup_{1 \leq i \leq \xi} \bigcup_{B \in B_i} B$.
- each $B_i$ is composed of disjoint balls.

Proof of Lemma 4.3 It suffices to prove the first inequality. The second is always true for all $\xi > 0$. Let $F \subseteq \mathbb{R}^d$, $\epsilon > 0$ and $\mathcal{V} = \{ B(x, \frac{\epsilon}{2}); \ x \in F \}$. Let next $((B_{ij})_{j})_{1 \leq i \leq \xi}$ be the $\xi$ sets of $\mathcal{V}$ obtained by the Besicovitch covering theorem. So that, $(B_{ij})_{i,j}$ is a centered $\epsilon$-covering of the set $F$ and for each $i$, $(B_{ij})_{j}$ is a centered $\epsilon$-packing of $F$.

Therefore,
\[
\overline{\mathcal{H}}^{q,t}_{\mu,\epsilon}(F) \leq \sum_{i=1}^{\xi} \sum_{j} (\mu(B_{ij}))^q (2r_{ij})^t \leq \sum_{i=1}^{\xi} \overline{\mathcal{P}}^{q,t}_{\mu,i}(F) = \xi \overline{\mathcal{P}}^{q,t}_{\mu}(F).
\]
Hence, $\overline{\mathcal{H}}^{q,t}_{\mu}(F) \leq \xi \overline{\mathcal{P}}^{q,t}_{\mu}(F)$. Consequently, for $E \subseteq \bigcup_{i} E_i$, we obtain
\[
\mathcal{H}^{q,t}_\mu(E) = \mathcal{H}^{q,t}_\mu\left(\bigcup_{i} (E_i \cap E)\right) \leq \sum_{i} \mathcal{H}^{q,t}_\mu(E_i \cap E) \leq \sum_{i} \sup_{F \subseteq E_i \cap E} \overline{\mathcal{H}}^{q,t}_\mu(F) \leq \xi \sum_{i} \sup_{F \subseteq E_i \cap E} \overline{\mathcal{P}}^{q,t}_\mu(F) \leq \xi \sum_{i} \overline{\mathcal{P}}^{q,t}_\mu(E_i).
\]
So as Lemma 4.3

Proof of Proposition 4.4 It follows from Proposition 4.2, Proposition 4.3 and Lemma 4.3

5 Mixed multifractal generalization of Bouligand-Minkowski’s dimension

In this section, we propose to develop mixed multifractal generalization of Bouligand-Minkowski’s dimension. Such a dimension is sometimes called the
box-dimension or the Renyi dimension. Some mixed generalizations are al-
ready introduced in [15]. We will see hereafter that the mixed generalizations
to be provided resemble to those in [15]. We will prove that in the mixed
case, these dimensions remain strongly related to the mixed multifractal gen-
eralizations of the Hausdorff and packing dimensions. In the case of a single
measure $\mu$, the Bouligand-Minkowski dimensions are introduced as follows.
For $E \subseteq \text{Support}(\mu)$, $\delta > 0$ and $q \in \mathbb{R}$, let

$$ T_{\mu,\delta}^q(E) = \inf \left\{ \sum_i (\mu(B(x_i, \delta)))^q \right\} $$

where the inf is over the set of all centered $\delta$-coverings $(B(x_i, \delta))_i$ of the set
$E$. The Bouligand-Minkowski dimensions are

$$ T_{\mu}^q(E) = \limsup_{\delta \downarrow 0} \frac{\log(T_{\mu,\delta}^q(E))}{-\log \delta} $$

for the upper one

$$ L_{\mu}^q(E) = \liminf_{\delta \downarrow 0} \frac{\log(T_{\mu,\delta}^q(E))}{-\log \delta} $$

for the lower. In the case of equality, the common value is denoted $L_{\mu}^q(E)$ and
is called the Bouligand-Minkowski dimension of the set $E$. We can equivalently
define these dimensions via the $\delta$-packings as follows. For $\delta > 0$ and $q \in \mathbb{R}$, we set

$$ S_{\mu,\delta}^q(E) = \sup \left\{ \sum_i (\mu(B(x_i, \delta)))^q \right\} $$

where the sup is taken over all the centered $\delta$-packings $(B(x_i, \delta))_i$ of the set
$E$. The upper dimension is

$$ C_{\mu}^q(E) = \limsup_{\delta \downarrow 0} \frac{\log(S_{\mu,\delta}^q(E))}{-\log \delta} $$

and the lower is

$$ c_{\mu}^q(E) = \liminf_{\delta \downarrow 0} \frac{\log(S_{\mu,\delta}^q(E))}{-\log \delta} $$

and similarly, when these are equal, the common value will be denoted $C_{\mu}^q(E)$
and it defines the dimension of $E$. We now introduce the mixed multifractal
generalization of the Bouligand-Minkowski dimensions. As we have noticed,
our ideas here is quite the same as the one in [15]. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be
a vector valued measure composed of probability measures on $\mathbb{R}^d$. Denote as
previously

$$ \mu(B(x, r)) \equiv (\mu_1(B(x, r)), \ldots, \mu_k(B(x, r))) $$

and for $q = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$,

$$ (\mu(B(x, r)))^q \equiv (\mu_1(B(x, r)))^{q_1} \ldots (\mu_k(B(x, r)))^{q_k}. $$
Next, for a nonempty subset $E \subseteq \mathbb{R}^d$ and $\delta > 0$, we will use the same notations for $\mathcal{T}^q_{\mu,\delta}(E)$, $\mathcal{C}^q_{\mu}(E)$ and $\mathcal{C}^q_{\mu}(E)$ but without forgetting that we use the new product for the measure $\mu$. Similarly for $\mathcal{S}^q_{\mu,\delta}(E)$, $\mathcal{T}^l_{\mu}(E)$ and $\mathcal{L}^l_{\mu}(E)$.

**Definition 5.1** For $E \subseteq \text{Support}(\mu)$ and $q = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$, we will call

a. $\mathcal{C}^q_{\mu}(E)$ and $\mathcal{T}^l_{\mu}(E)$ the upper mixed multifractal generalizations of the Bouligand Minkowski dimension of $E$.

b. $\mathcal{C}^q_{\mu}(E)$ and $\mathcal{L}^l_{\mu}(E)$ the lower mixed multifractal generalizations of the Bouligand Minkowski dimension of $E$.

c. $\mathcal{C}^q_{\mu}(E)$ and $\mathcal{L}^l_{\mu}(E)$ the mixed multifractal generalizations of the Bouligand Minkowski dimension of $E$.

**Remark 5.1** We stress the fact that each quantity defines in fact a mixed generalization that can be different from the other. That is, we did not mean that $\mathcal{C}^q_{\mu}(E)$ and $\mathcal{T}^l_{\mu}(E)$ are the same (equal) and similarly for the lower ones. We will prove in the contrary that as for the single case, they can be different.

**Theorem 5.1** (1) For all $q \in \mathbb{R}^k$, we have

\[
\mathcal{L}^l_{\mu}(E) \leq \mathcal{C}^q_{\mu}(E) \quad \text{and} \quad \mathcal{L}^l_{\mu}(E) \leq \mathcal{C}^q_{\mu}(E).
\]

(2) For any $q \in \mathbb{R}^*_k$, we have

i. $b_{\mu,E}(q) \leq \mathcal{L}^l_{\mu}(E) = \mathcal{C}^q_{\mu}(E)$.

ii. $\mathcal{T}^l_{\mu,E}(q) = \mathcal{C}^q_{\mu}(E) = \Lambda_{\mu,E}(q)$.

(3) For any $q \in \mathbb{R}^*_k$, we have

\[
\mathcal{T}^l_{\mu,E}(q) \leq \mathcal{C}^q_{\mu}(E) \leq \Lambda_{\mu,E}(q).
\]

**Proof.** 1. Using Besicovitch covering theorem we get

\[
\mathcal{T}^q_{\mu,\delta}(E) \leq C \mathcal{S}^q_{\mu,\delta}(E),
\]

with some constant $C$ fixed. So as 1. is proved.

2. We firstly prove that

\[
\mathcal{L}^q_{\mu}(E) \geq \mathcal{C}^q_{\mu}(E) \quad \text{and} \quad \mathcal{T}^l_{\mu}(E) \geq \mathcal{C}^q_{\mu}(E).
\]

Indeed, let $(B(x_i, \delta))_i$ be a centered $\delta$-packing of $E$ and $(B(y_i, \frac{\delta}{2}))$ be a centered $\frac{\delta}{2}$-covering of $E$. Consider for each $i$, the integer $k_i$ such that $x_i \in B(y_{k_i}, \frac{\delta}{2})$. It is straightforward that for $i \neq j$ we have $k_i \neq k_j$. Consequently, for $q \in \mathbb{R}^*_k,$
there holds that
\[ \sum_i (\mu(B(x_i, \delta)))^q = \sum_i \left( \frac{\mu(B(x_i, \delta))}{\mu(B(y_k, \delta/2))} \right)^q (\mu(B(y_k, \delta/2)))^q \leq \sum_i (\mu(B(y_k, \delta/2)))^q. \]

Which means that
\[ S_{q, \delta}(E) \leq T_{q, \delta}(E) \]
and thus, for any \( q \in \mathbb{R}^* \),
\[ L_q^q(E) \geq C_q^q(E) \quad \text{and} \quad L_q^t(E) \geq C_q^t(E). \]

Using the assertion 1., we obtain the equalities
\[ L_q^q(E) = C_q^q(E) \quad \text{and} \quad L_q^t(E) = C_q^t(E) \]
for all \( q \in \mathbb{R}^* \). Therefore, to prove 2.i., it remains to prove the inequality of the left hand side. So, let \( t > L_q^q(E) \) and \( F \subseteq E \). Consider next a sequence \( (\delta_n)_n \subseteq ]0, 1[ \) to be \( \downarrow \), 0, and satisfying
\[ t > \frac{\log(T_{q, \delta_n}(E))}{-\log \delta_n}, \quad \forall n \in \mathbb{N}. \]

This means that for each \( n \in \mathbb{N} \), there exists a centered \( \delta_n \)-covering \( (B(x_{ni}, \delta_n))_i \) of \( E \) such that
\[ \sum_i (\mu(B(x_{ni}, \delta_n)))^q < \delta_n^{-t}. \]

There balls may be considered to be intersecting the set \( F \). Next, for each \( i \), choose an element \( y_i \in B(x_{ni}, \delta_n) \cap F \). This results on a centered \( 2\delta_n \)-covering \( (B(y_i, 2\delta_n))_i \) of \( F \). Therefore,
\[ H_{\mu, 2\delta_n}^{q, t}(F) \leq \sum_i (\mu(B(x_{ni}, \delta_n)))^q (4\delta_n)^t \]
\[ = 4^t \sum_i \left( \frac{\mu(B(y_i, 2\delta_n))}{\mu(B(x_{ni}, \delta_n))} \right)^q (\mu(B(x_{ni}, \delta_n)))^q \delta_n^t \]
\[ \leq 4^t \sum_i (\mu(B(x_{ni}, \delta_n)))^q \delta_n^t \]
\[ \leq 4^t \delta_n^{-t} \delta_n^t = 4^t. \]

Hence,
\[ H_{\mu}^{q, t}(F) \leq 4^t, \quad \forall F \subseteq E, \ t > L_q^q(E). \]
So that,
\[ H_{\mu}^{q, t}(E) \leq 4^t < \infty, \quad \forall t > L_q^q(E). \]
Consequently,
\[ b_{\mu,E}(q) \leq t, \quad \forall \ t > L^\mu_q(E) \Rightarrow b_{\mu,E}(q) \leq L^\mu_q(E). \]

We now prove the remaining part of 2.ii. We will prove firstly that
\[ \overline{C}^q_{\mu}(E) \leq \Lambda_{\mu,E}(q), \forall \ q \in \mathbb{R}^k. \] (5)

This is of course obvious when the right hand term is infinite. So, without loss of the generality, we assume that it is finite. Denote \( t = \Lambda_{\mu,E}(q) \), and consider \( \varepsilon > 0 \) and \( 0 < \delta_\varepsilon < 1 \) be such that \( \overline{P}^{t, \varepsilon}_{\mu, \delta}(E) < 1 \) for all \( 0 < \delta < \delta_\varepsilon \). This is possible because of the fact that \( \overline{P}^{t, \varepsilon}_{\mu, \delta}(E) = \lim_{\delta \downarrow 0} \overline{P}^{t, \delta + \varepsilon}_{\mu, \delta}(E) = 0 \). Consequently, for a centered \( \delta \)-packing \( (B(x_i, \delta)) \) of \( E \), we obtain
\[
\sum_i (\mu(B(x_i, \delta)))^q = (2\delta)^{-(t+\varepsilon)} \sum_i (\mu(B(x_i, \delta)))^q (2\delta)^{t+\varepsilon} \\
\leq (2\delta)^{-(t+\varepsilon)} \overline{P}^{t, \delta + \varepsilon}_{\mu, \delta}(E) \\
\leq (2\delta)^{-(t+\varepsilon)}.
\]

Hence, \( S^q_{\mu, \delta}(E) \leq (2\delta)^{-(t+\varepsilon)} \) and consequently, equation (5.1) holds. We now prove the converse
\[ \Lambda_{\mu,E}(q) \leq \overline{C}^q_{\mu}(E), \forall \ q \in \mathbb{R}^*_k. \]

Let \( t = \Lambda_{\mu,E}(q), \varepsilon > 0 \) and \( 0 < \delta_0 < 1 \). It holds that
\[ \infty = \overline{P}^{t-\varepsilon/2}_{\mu}(E) \leq \overline{P}^{t-\varepsilon/2}_{\mu, \delta_0}(E). \]

This means that there exists a centered \( \delta_0 \)-packing \( (B(x_i, r_i)) \) of \( E \) such that
\[ 1 < \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^{t-\varepsilon/2}. \]

Next, denote for \( n \in \mathbb{N} \),
\[ I_n = \{ i \in \mathbb{N}; \frac{\delta_0}{2^{n+1}} \leq r_i < \frac{\delta_0}{2^n} \} \quad \text{and} \quad \nu_n = \sum_{i \in I_n} (\mu(B(x_i, r_i)))^q. \]

A straightforward computation yields that
\[ C \sup_m \left( \nu_m \left( \frac{\delta_0}{2^m} \right)^{t-\varepsilon} \right) > 1 \]
for an appropriate constant \( C > 0 \) depending only on \( t \) and \( \varepsilon \). Consequently, for \( N \in \mathbb{N} \) such that \( 1 < C\nu_N \left( \frac{\delta_0}{2^N} \right)^{t-\varepsilon} \) and \( \delta = \frac{\delta_0}{2^{N+1}} \), the set \( (B(x_i, \delta)) \) forms a centered \( \delta \)-packing of \( E \). Observing that \( q \in \mathbb{R}^*_k \), it results that
\[ S^q_{\mu, \delta}(E) \geq C^{-1} \delta^{-(t-\varepsilon)}. \]
Consequently, $\overline{C}_\mu^q(E) \geq \Lambda_{\mu,E}(q)$ for all $q \in \mathbb{R}_+^k$.

3. It follows from 1. and equation (5).

Next we need to introduce the following quantities which will be useful later. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector valued measure composed of probability measures on $\mathbb{R}^d$. For $j = 1, 2, \ldots, k$, $a > 1$ and $E \subseteq \text{Support}(\mu)$, denote

$$T_a^j(E) = \limsup_{r \downarrow 0} \left( \sup_{x \in E} \frac{\mu_j(B(x, ar))}{\mu_j(B(x, r))} \right)$$

and for $x \in \text{Support}(\mu)$, $T_a^j(x) = T_a^j(\{x\})$. Denote also

$$P_0(\mathbb{R}^d, E) = \{ \mu; \ \exists a > 1; \ \forall x \in E, \ T_a^j(x) < \infty, \ \forall j \},$$

$$P_1(\mathbb{R}^d, E) = \{ \mu; \ \exists a > 1; \ T_a^j(E) < \infty, \ \forall j \},$$

$$P_0(\mathbb{R}^d) = P_0(\mathbb{R}^d, \text{Support}(\mu)) \quad \text{and} \quad P_1(\mathbb{R}^d) = P_1(\mathbb{R}^d, \text{Support}(\mu)).$$

**Theorem 5.2**  (1) For $\mu \in P_0(\mathbb{R}^d)$ and $q \in \mathbb{R}_+^k$, there holds that

$$b_{\mu,E}(q) \leq L_\mu^q(E).$$

(2) For $\mu \in P_1(\mathbb{R}^d)$ and $q \in \mathbb{R}_+^k$, there holds that

i. $L_\mu^q(E) = C_\mu^q(E)$.

ii. $\overline{L}_{\mu,E}(q) = \overline{C}_\mu^q(E) = \Lambda_{\mu,E}(q)$.

**Proof.** 1. The vector valued measure $\mu \in P_0(\mathbb{R}^d)$ yields that

$$E = \bigcup_{m \in \mathbb{N}} E_m$$

where

$$E_m = \{ x \in E; \ \frac{\mu_j(B(x, 4r))}{\mu_j(B(x, r))} < m, \ 0 < r < \frac{1}{m}, \ \forall j \}.$$  

Next, remark that for $t > T_\mu^q(E)$ and $F \subseteq E_m$, there exists a sequence $(\delta_n)_n \in ]0,1[ \downarrow 0$ for which

$$t < \frac{\log(T_\mu^q(F))}{-\log \delta_n}, \quad \forall n \in \mathbb{N}.$$

Therefore, there exists a centered $\delta_n$-covering $(B(x_{ni}, \delta_n))_i$ of $F$ satisfying

$$\sum_i (\mu(B(x_{ni}, \delta_n)))^q < \delta_n^{-t}.$$
Let next $y_{ni} \in B(x_{ni}, \delta_n)$. Then, $(B(x_{ni}, 2\delta_n))_i$ is a centered $2\delta_n$-covering of $F$. Hence,

$$
\mathcal{H}_{\mu,2\delta_n}^{q,t}(F) \leq \sum_i (\mu(B(y_{ni}, 2\delta_n)))^q(4\delta_n)^t \\
\leq 4^t \sum_i \left(\frac{\mu(B(y_{ni}, 2\delta_n))}{\mu(B(x_{ni}, \delta_n))}\right)^q (\mu(B(x_{ni}, \delta_n)))^q\delta_n^t \\
\leq 4^t \sum_i \left(\frac{\mu(B(x_{ni}, 4\delta_n))}{\mu(B(x_{ni}, \delta_n))}\right)^q (\mu(B(x_{ni}, \delta_n)))^q\delta_n^t \\
\leq 4^t m^{q|q|} \sum_i (\mu(B(x_{ni}, \delta_n)))^q\delta_n^t \\
\leq 4^t m^{q|q|}
$$

where $|q| = q_1 + q_2 + \ldots + q_k$. Thus,

$$
\mathcal{H}_{\mu}^{q,t}(F) \leq 4^t m^{|q|}, \; \forall m, \text{ and } F \subseteq E_m.
$$

Which means that

$$
\mathcal{H}_{\mu}^{q,t}(E_m) \leq 4^t m^{|q|} < \infty, \; \forall m, \text{ and } t > L_{\mu}^q(E).
$$

Consequently,

$$
b_{\mu,E_m}(q) \leq t, \; \forall m, \text{ and } t > L_{\mu}^q(E).
$$

Using the $\sigma$-stability of $b_{\mu,}(q)$ (See Proposition 4.2 c.), it results that

$$
b_{\mu,E}(q) \leq t, \; \forall t > L_{\mu}^q(E) \Rightarrow b_{\mu,E}(q) \leq L_{\mu}^q(E).
$$

2. i. From Theorem 5.1 it remains to prove that $L_{\mu}^q(E) \geq C_{\mu}^q(E)$. Let $C, \delta_0 > 0$ such that

$$
\sup_{x \in E} \frac{\mu_j(B(x, 4r))}{\mu(B(x, r))} < C, \; \forall 0 < r < \delta_0, \; \forall j = 1, 2, \ldots, k.
$$

Let next $0 < \delta < \delta_0$, $(B(x_i, \delta))_i$ be a centered packing and $(B(y_i, \delta/2))_i$ be a centered covering of $E$. For each $i \in \mathbb{N}$, denote $k_i$ the unique integer such that
\( x_i \in B(y_k, \delta/2) \). It holds that

\[
S_{\mu,\delta}^q(E) \leq \sum_i (\mu(B(x_i, \delta)))^q \\
= \sum_i \left( \frac{\mu(B(x_i, \delta))}{\mu(B(y_k, \delta/2))} \right)^q (\mu(B(y_k, \delta/2)))^q \\
\leq \sum_i \left( \frac{\mu(B(y_k, 2\delta))}{\mu(B(y_k, \delta/2))} \right)^q (\mu(B(y_k, \delta/2)))^q \\
\leq C^{[q]} \sum_i (\mu(B(y_k, \delta/2)))^q \\
\leq C^{[q]} \sum_i (\mu(B(y_i, \delta/2)))^q.
\]

This yields that

\[
S_{\mu,\delta}^q(E) \leq C^{[q]} T_{\mu,\delta/2}^q(E).
\]

Consequently,

\[
C_{\mu}^q(E) \leq L_{\mu}^q(E) \quad \text{and} \quad \overline{C}_{\mu}^q(E) \leq T_{\mu}^q(E)
\]

Using Theorem 5.1, we obtain the equalities.

\[
C_{\mu}^q(E) = L_{\mu}^q(E) \quad \text{and} \quad \overline{C}_{\mu}^q(E) = T_{\mu}^q(E), \quad \forall \ q \in \mathbb{R}^k_+, \ \mu \in P_1(\mathbb{R}^d) \quad (6)
\]

ii. Using equations (5) and (6), it remains to prove that

\[
\overline{C}_{\mu}^q(E) \geq \Lambda_{\mu,E}(q).
\]

The vector measure \( \mu \) lies in \( P_1(\mathbb{R}^d), E \). So that, there exists as above \( C > 0 \), and \( 0 < r_0 < 1 \) such that

\[
\frac{\mu_j(B(x, 2r))}{\mu_j(B(x, r))} \leq C, \quad \forall \ 0 < r < \delta_0, \ x \in E, \ \text{and} \ j = 1, 2, \ldots, k.
\]

Denote \( t = \Lambda_{\mu,E}(q), \ \varepsilon > 0 \) and \( 0 < \delta_0 < r_0 \). Then \( P_{\mu,\delta_0/2-E}^q(E) = \infty \). Which means that there exists a centered \( \delta_0 \)-packing of the set \( E \), \( (B(x_i, r_i))_i \) for which

\[
1 < \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^{t-\varepsilon/2}.
\]
By considering the set $I_N$, $\nu_N$ and $\delta$ as above, we obtain

$$S_{\mu,\delta}(E) \geq \sum_{i \in I_N} \left( \mu(B(x_i, \delta)) \right)^q$$

$$\geq \beta^{-q} \sum_{i \in I_N} \left( \frac{\mu(B(x_i, \delta_0/2^{N+1}))}{\mu(B(x_i, r_i))} \right)^q \left( \mu(B(x_i, r_i)) \right)^q$$

$$\geq C^{-\vert q \vert} \sum_{i \in I_N} \left( \mu(B(x_i, r_i)) \right)^q$$

$$\geq C^{-\vert q \vert} \nu_N > \beta^{-q} C^{-1} \left( \frac{\delta_0}{2^{N}} \right)^{-(t-\varepsilon)}.$$ 

Hence,

$$\overline{C}_{\mu}^q(E) \geq \Lambda_{\mu,E}(q).$$

We now recall re-introduce the mixed multifractal generalization of the $L^q$-dimensions called also Renyi dimensions based on integral representations. See [15] for more details and other results. For $q \in \mathbb{R}^{+k}$, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ and $\delta > 0$, we set

$$I_{\mu,\delta}^q = \int_{S_{\mu}} \left( \mu(B(t, \delta)) \right)^q d\mu(t),$$

where, in this case,

$$S_{\mu} = \text{Support}(\mu_1) \times \text{Support}(\mu_2) \times \ldots \times \text{Support}(\mu_k),$$

$$\left( \mu(B(t, \delta)) \right)^q d\mu(t) = \left( \mu_1(B(t_1, \delta)) \right)^{q_1} \left( \mu_2(B(t_2, \delta)) \right)^{q_2} \ldots \left( \mu_k(B(t_k, \delta)) \right)^{q_k}$$

and

$$d\mu(t) = d\mu_1(t_1) d\mu_2(t_2) \ldots d\mu_k(t_k).$$

The mixed multifractal generalizations of the Renyi dimensions are

$$\mathcal{T}_{\mu}^q = \limsup_{\delta \downarrow 0} \frac{\log I_{\mu,\delta}^q}{-\log \delta}, \quad \text{and} \quad \mathcal{L}_{\mu}^q = \liminf_{\delta \downarrow 0} \frac{\log I_{\mu,\delta}^q}{-\log \delta}.$$

We now propose to relate these dimensions to the quantities $C_{\mu}^q$, $\overline{C}_{\mu}^q$, $L_{\mu}^q$, $\overline{L}_{\mu}^q$ introduced previously.

**Proposition 5.1** The following results hold.

a. $\forall q \in \mathbb{R}^{+,-k}$,

$$C_{\mu}^{q+\xi}(\text{Support}(\mu)) \geq \mathcal{L}_{\mu}^q \quad \text{and} \quad \overline{C}_{\mu}^{q+\xi}(\text{Support}(\mu)) \geq \mathcal{T}_{\mu}^q.$$

b. $\forall q \in \mathbb{R}^{+,-k}$,

$$C_{\mu}^{q+\xi}(\text{supp}(\mu)) \leq \mathcal{L}_{\mu}^q \quad \text{and} \quad \overline{C}_{\mu}^{q+\xi}(\text{supp}(\mu)) \leq \mathcal{T}_{\mu}^q.$$
c. \( \forall q \in \mathbb{R}^{*} \), \( \mu \in P_{1}(\mathbb{R}^{d}) \),
\[ C_{\mu}^{q+\delta}(\text{supp}(\mu)) = L_{\mu}^{q} \quad \text{and} \quad \overline{C}_{\mu}^{q+\delta}(\text{supp}(\mu)) = T_{\mu}^{q}. \]

d. \( \forall q \in \mathbb{R}^{*} \),
\[ L_{\mu}^{q} \leq L_{\mu}^{q+\delta}(\text{supp}(\mu)) \quad \text{and} \quad T_{\mu}^{q} \leq T_{\mu}^{q+\delta}(\text{supp}(\mu)). \]

**Proof.** a. For \( \delta > 0 \), let \( (B(x_{i}, \delta))_{i} \) be a centered \( \delta \)-covering of \( \text{Support}(\mu) \) and let next \( (B(x_{ij}, \delta))_{j} \), \( 1 \leq i \leq \xi \) the \( \xi \) sets defined in Besicovitch covering theorem. It holds that
\[
\sum_{i,j} (\mu(B(x_{ij}, \delta)))^{q+\delta} = \sum_{i,j} (\mu(B(x_{ij}, \delta)))^{q} \int_{B(x_{ij}, \delta)^{k}} d\mu(t) \\
\geq \sum_{i,j} \int_{B(x_{ij}, \delta)^{k}} (\mu(B(t, 2\delta)))^{q} d\mu(t) \\
\geq \int_{S_{\mu}} (\mu(B(t, 2\delta)))^{q} d\mu(t).
\]
As a results,
\[
\xi S_{\mu,\delta}^{q+\delta}(\text{Support}(\mu)) \geq I_{\mu,2\delta}^{q}.
\]
Which implies that
\[
C_{\mu}^{q+\delta}(\text{Support}(\mu)) \geq L_{\mu}^{q} \quad \text{and} \quad \overline{C}_{\mu}^{q+\delta}(\text{Support}(\mu)) \geq T_{\mu}^{q}.
\]
b. Let \( \delta > 0 \) and \( (B(x_{i}, \delta))_{i} \) a centered \( \delta \)-packing of \( \text{Support}(\mu) \). It holds that
\[
\sum_{i} (\mu(B(x_{i}, \delta)))^{q+\delta} = \sum_{i} (\mu(B(x_{i}, \delta)))^{q} \int_{B(x_{i}, \delta)^{k}} d\mu(t) \\
\leq \sum_{i} \int_{B(x_{i}, \delta)^{k}} (\mu(B(t, 2\delta)))^{q} d\mu(t) \\
\leq \int_{S_{\mu}} (\mu(B(t, 2\delta)))^{q} d\mu(t).
\]
Therefore,
\[
S_{\mu,\delta}^{q+\delta}(\text{Support}(\mu)) \leq I_{\mu,2\delta}^{q}
\]
and thus,
\[
C_{\mu}^{q+\delta}(\text{Support}(\mu)) \leq L_{\mu}^{q} \quad \text{and} \quad \overline{C}_{\mu}^{q+\delta}(\text{Support}(\mu)) \leq T_{\mu}^{q}.
\]
c. Assume firstly that \( q \in \mathbb{R}^{*} \). Observing assertion a., it suffices to prove that
\[
C_{\mu}^{q+\delta}(\text{Support}(\mu)) \leq L_{\mu}^{q} \quad \text{and} \quad \overline{C}_{\mu}^{q+\delta}(\text{Support}(\mu)) \leq T_{\mu}^{q}.
\]
Since the measure \( \mu \in P_1(\mathbb{R}^d) \), there exists a constant \( C > 0 \) and \( r_0 > 0 \) such that
\[
\frac{\mu_j(B(x, 2r))}{\mu_j(B(x, r))} < C; \quad \forall x \in \text{Support}(\mu), \quad 0 < r < r_0, \quad j = 1, 2, \ldots k.
\]

Next, consider for \( 0 < \delta < r_0 \) a centered \( \delta \)-packing \( (B(x_i, \delta))_i \) of \( \text{Support}(\mu) \).
It holds that
\[
\sum_i \left( \mu(B(x_i, \delta)) \right)^{q+\delta} = \sum_i \left( \mu(B(x_i, \delta)) \right)^q \int_{B(x_i, \delta)^k} d\mu(t) \\
\leq C^{-2|q|} \sum_i \int_{B(x_i, \delta)^k} \left( \mu(B(t, 2\delta)) \right)^q d\mu(t) \\
\leq C^{-2|q|} \int_{S_\mu} \left( \mu(B(t, 2\delta)) \right)^q d\mu(t).
\]

Consequently,
\[
S_{\mu, \delta}^{q+\delta}(\text{Support}(\mu)) \leq C^{-2|q|} I_{\mu, 2\delta}^q.
\]

Hence,
\[
\overline{C}_{\mu}^{q+\delta}(\text{Support}(\mu)) \leq I_{\mu}^q \quad \text{and} \quad \overline{C}_{\mu}^{q+\delta}(\text{Support}(\mu)) \leq T_{\mu}^q.
\]

So the equality for \( q \in \mathbb{R}_+^*k \).
Assume now that \( q \in \mathbb{R}_-^*k \). Observing assertion \textit{b.}, it remains to prove that
\[
\underline{C}_{\mu}^{q+\delta}(\text{supp}(\mu)) \geq I_{\mu}^q \quad \text{and} \quad \underline{C}_{\mu}^{q+\delta}(\text{supp}(\mu)) \geq T_{\mu}^q.
\]

To do so, we use the fact that \( \mu \in P_1(\mathbb{R}^d) \), which means that there exists \( C > 0 \) and \( r_0 > 0 \) satisfying
\[
\frac{\mu_j(B(x, 2r))}{\mu_j(B(x, r))} < C, \quad \forall x \in \text{Support}(\mu), \quad 0 < r < r_0, \quad j = 1, 2, \ldots k.
\]

Let next, \( 0 < \delta < r_0 \), \( (B(x_i, \delta))_i \) a centered \( \delta \)-covering of \( \text{Support}(\mu) \) and as previously, \( (B(x_{ij}, \delta))_j \), \( 1 \leq i \leq \xi \) the \( \xi \) sets defined in Besicovitch covering theorem. We have
\[
\sum_{i,j} \left( \mu(B(x_{ij}, \delta)) \right)^{q+\delta} = \sum_{i,j} \left( \mu(B(x_{ij}, \delta)) \right)^q \int_{B(x_{ij}, \delta)^k} d\mu(t) \\
\geq C^{-2|q|} \sum_{i,j} \int_{B(x_{ij}, \delta)^k} \left( \mu(B(t, 2\delta)) \right)^q d\mu(t) \\
\geq C^{-2|q|} \int_{S_\mu} \left( \mu(B(t, 2\delta)) \right)^q d\mu(t).
\]
Hence, \( \xi S_{\mu,\delta}^q(\text{Support}(\mu)) \geq C^{-2|q|}\mathcal{I}_{\mu,2\delta}^q \).

Consequently, \( C_{\mu}^{q+\frac{d}{2}}(\text{Support}(\mu)) \geq I^q_\mu \) and \( \overline{C}_{\mu}^{q+\frac{d}{2}}(\text{Support}(\mu)) \geq \mathcal{T}^q_\mu \).

Hence, the equality for \( q \in \mathbb{R}^{*,k} \).

\[ d. \text{ Let } \delta > 0 \text{ and } \left( B(x_i, \delta) \right)_i \text{ be a centered } \delta\text{-covering of } \text{Support}(\mu). \text{ We have} \]
\[
\sum_i \left( \mu(B(x_i, \delta)) \right)^{q+\frac{d}{2}} = \sum_i \left( \mu(B(x_i, \delta)) \right)^q \int_{B(x_i, \delta)^k} d\mu(t) \\
\geq \sum_i \int_{B(x_i, \delta)^k} \left( \mu(B(t, 2\delta)) \right)^q d\mu(t) \\
\geq \int_{S_\mu} \left( \mu(B(t, 2\delta)) \right)^q d\mu(t). 
\]

As a result, \( T_{\mu,\delta}^{q+1}(\text{Support}(\mu)) \geq I^q_{\mu,2\delta} \).

Consequently, \( L_{\mu}^{q+\frac{d}{2}}(\text{supp}(\mu)) \geq I^q_\mu \) and \( \overline{L}_{\mu}^{q+\frac{d}{2}}(\text{supp}(\mu)) \geq \mathcal{T}^q_\mu \).

6 A mixed multifractal formalism for vector valued measures

Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) be a vector valued probability measure on \( \mathbb{R}^d \). For \( x \in \mathbb{R}^d \) and \( j = 1, 2, \ldots, k \), we denote
\[
\alpha_{\mu_j}(x) = \liminf_{r \downarrow 0} \frac{\log(\mu_j(B(x,r)))}{\log r} \\
\overline{\alpha}_{\mu_j}(x) = \limsup_{r \downarrow 0} \frac{\log(\mu_j(B(x,r)))}{\log r}
\]
respectively the local lower dimension and the local upper dimension of \( \mu_j \) at the point \( x \) and as usually the local dimension \( \alpha_{\mu_j}(x) \) of \( \mu_j \) at \( x \) will be the common value when these are equal. Next for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}_+^k \), let
\[
\mathcal{X}_\alpha = \{ x \in \text{Support}(\mu); \alpha_{\mu_j}(x) \geq \alpha_j, \forall j = 1, 2, \ldots, k \},
\]
\[
\overline{\mathcal{X}}_\alpha = \{ x \in \text{Support}(\mu); \overline{\alpha}_{\mu_j}(x) \leq \alpha_j, \forall j = 1, 2, \ldots, k \}
\]
and
\[
X(\alpha) = \mathcal{X}_\alpha \cap \overline{\mathcal{X}}_\alpha.
\]
The mixed multifractal spectrum of the vector valued measure \( \mu \) is defined by
\[
\alpha \mapsto \dim X(\alpha) 
\]
where \( \text{dim} \) stands for the Hausdorff dimension.

In this section, we propose to compute such a spectrum for some cases of measures that resemble to the situation raised by Olsen in [9] but in the mixed case. This will permit to describe better the simultaneous behavior of finitely many measures. We intend precisely to compute the mixed spectrum based on the mixed multifractal generalizations of the Haudorff and packing dimensions \( b_\mu, B_\mu \) and \( \Lambda_\mu \). We start with the following technic results.

**Lemma 6.1** 1. \( \forall \delta > 0, t \in \mathbb{R} \) and \( q \in \mathbb{R}_+, \alpha \in \mathbb{R}_+^k \) such that \( \langle \alpha, q \rangle + t \geq 0 \), we have

\[ H^{(\alpha, q)}_t + t + k \delta (X^\alpha_m) \leq 2^{(\alpha, q) + k \delta} H^q_{\mu_\alpha}(X^\alpha). \]

ii. \[ P^{(\alpha, q)}_t + t + k \delta (X^\alpha_m) \leq 2^{(\alpha, q) + k \delta} P^q_{\mu}(X^\alpha). \]

2. \( \forall \delta > 0, t \in \mathbb{R} \) and \( q \in \mathbb{R}_+, \alpha \in \mathbb{R}_+^k \) such that \( \langle \alpha, q \rangle + t \geq 0 \), we have

\[ H^{(\alpha, q)}_t + t + k \delta (X^\alpha_n) \leq 2^{(\alpha, q) + k \delta} H^q_{\mu_\alpha}(X^\alpha_n). \]

ii. \[ P^{(\alpha, q)}_t + t + k \delta (X^\alpha_n) \leq 2^{(\alpha, q) + k \delta} P^q_{\mu_\alpha}(X^\alpha_n). \]

**Proof.** 1. i. We prove the first part. For \( m \in \mathbb{N}^* \), consider the set

\[ X^\alpha_m = \{ x \in X^\alpha ; \frac{\log(\mu_j(B(x, r)))}{\log r} \leq \alpha_j + \frac{\delta}{q_j}; 0 < r < \frac{1}{m}, 1 \leq j \leq k \}. \]

Let next \( 0 < \eta < \frac{1}{m} \) and \( (B(x_i, r_i))_i \) a centered \( \eta \)-covering of \( X^\alpha_m \). It holds that

\[ (\mu(B(x_i, r_i)))^q \geq r_i^{(\alpha, q) + k \delta}. \]

Consequently,

\[ H^{(\alpha, q)}_t + t + k \delta (X^\alpha_m) \leq \sum_{i} (2r_i)^{(\alpha, q) + t + k \delta} \leq 2^{(\alpha, q) + k \delta} \sum_{i} (\mu(B(x_i, r_i)))^q(2r_i)^t. \]

Hence, \( \forall \eta > 0 \), there holds that

\[ H^{(\alpha, q)}_t + t + k \delta (X^\alpha_m) \leq 2^{(\alpha, q) + k \delta} P^q_{\mu_\eta}(X^\alpha_m). \]

Which means that

\[ H^{(\alpha, q)}_t + t + k \delta (X^\alpha_m) \leq 2^{(\alpha, q) + k \delta} P^q_{\mu}(X^\alpha_m) \leq 2^{(\alpha, q) + k \delta} H^q_{\mu_\alpha}(X^\alpha). \]

Next, observing that \( X^\alpha = \bigcup_{m} X^\alpha_m \), we obtain

\[ H^{(\alpha, q)}_t + t + k \delta (X^\alpha) \leq 2^{(\alpha, q) + k \delta} P^q_{\mu, \eta}(X^\alpha). \]

ii. For \( q \in \mathbb{R}_+^k \) and \( m \in \mathbb{N}^* \), consider the set \( X^\alpha_m \) defined previously and let \( E \subseteq X^\alpha_m \), \( 0 < \eta < \frac{1}{m} \) and \( (B(x_i, r_i))_i \) a centered \( \eta \)-packing of \( E \). We have

\[ \sum_{i} (2r_i)^{(\alpha, q) + t + k \delta} \leq 2^{(\alpha, q) + k \delta} \sum_{i} (\mu(B(x_i, r_i)))^q(2r_i)^t \leq 2^{(\alpha, q) + k \delta} P^q_{\mu, \eta}(E). \]
Consequently, $\forall \eta > 0,$
\[
\mathcal{P}_{\eta}^{(\alpha,q) + t + k\delta}(E) \leq 2^{(\alpha,q) + k\delta}\mathcal{P}_{\mu,\eta}^{q,t}(E).
\]

Hence, $\forall E \subseteq \mathcal{X}_m^\alpha,$
\[
\mathcal{P}^{(\alpha,q) + t + k\delta}(E) \leq 2^{(\alpha,q) + k\delta}\mathcal{P}_{\mu}^{q,t}(E).
\]

Let next, $(E_i)_i$ be a covering of $\mathcal{X}_m^\alpha.$ Thus,
\[
\mathcal{P}^{(\alpha,q) + t + k\delta}(\mathcal{X}_m^\alpha) = \mathcal{P}^{(\alpha,q) + t + k\delta}(\bigcup_i (\mathcal{X}_m^\alpha \cap E_i)) = \sum_i \mathcal{P}^{(\alpha,q) + t + k\delta}(\mathcal{X}_m^\alpha \cap E_i) \leq \sum_i \mathcal{P}_{\mu}^{(\alpha,q) + t + k\delta}(\mathcal{X}_m^\alpha \cap E_i) \leq 2^{(\alpha,q) + k\delta}\sum_i \mathcal{P}_{\mu}^{q,t}(E_i).
\]

Hence, $\forall, m,$
\[
\mathcal{P}^{(\alpha,q) + t + k\delta}(\mathcal{X}_m^\alpha) \leq 2^{(\alpha,q) + k\delta}\mathcal{P}_{\mu}^{q,t}(\mathcal{X}_m^\alpha).
\]

Consequently,
\[
\mathcal{P}^{(\alpha,q) + t + k\delta}(\mathcal{X}^\alpha) \leq 2^{(\alpha,q) + k\delta}\mathcal{P}_{\mu}^{q,t}(\mathcal{X}^\alpha).
\]

2. i. and ii. follow similar arguments and techniques as previously.

**Proposition 6.1** Let $\alpha \in \mathbb{R}_+^k$ and $q \in \mathbb{R}^k.$ The following assertions hold.

**a.** Whenever $\langle \alpha, q \rangle + b_\mu(q) \geq 0,$ we have
i. $\dim \mathcal{X}^\alpha \leq \langle \alpha, q \rangle + b_\mu(q), \quad \forall q \in \mathbb{R}_+^k.$
ii. $\dim X^\alpha \leq \langle \alpha, q \rangle + b_\mu(q), \quad \forall q \in \mathbb{R}^k.$

**b.** Whenever $\langle \alpha, q \rangle + B_\mu(q) \geq 0,$ we have
i. $\dim \mathcal{X}^\alpha \leq \langle \alpha, q \rangle + B_\mu(q), \quad \forall q \in \mathbb{R}_+^k.$
ii. $\dim X^\alpha \leq \langle \alpha, q \rangle + B_\mu(q), \quad \forall q \in \mathbb{R}^k.$

**Proof. a. i.** It follows from Lemma 6.1 assertion 1. i.,
\[
\mathcal{H}^{(\alpha,q) + t + k\delta}(\mathcal{X}^\alpha) = 0, \quad \forall t > b_\mu(q), \quad \delta > 0.
\]

Consequently,
\[
\dim \mathcal{X}^\alpha \leq \langle \alpha, q \rangle + t + k\delta, \quad \forall t > b_\mu(q), \quad \delta > 0.
\]

Hence,
\[
\dim X^\alpha \leq \langle \alpha, q \rangle + b_\mu(q).
\]
a. ii. It follows from Lemma 6.1 assertion 2. i., as previously, that
\[ H^{(\alpha,q)+t+k\delta}(X^\alpha) = 0, \quad \forall \ t > b_\mu(q), \ \delta > 0. \]
Hence,
\[ \dim X_\alpha \leq \langle \alpha, q \rangle + t + k\delta, \quad \forall \ t > b_\mu(q), \ \delta > 0 \]
and finally,
\[ \dim X_\alpha \leq \langle \alpha, q \rangle. \]

b. i. observing Lemma 6.1 assertion 1. ii., we obtain
\[ P^{(\alpha,q)+t+k\delta}(X^\alpha), \quad \forall \ t > B_\mu(q), \ \delta > 0. \]
Consequently,
\[ \dim X^\alpha \leq \langle \alpha, q \rangle + t + k\delta, \quad \forall \ t > B_\mu(q), \ \delta > 0. \]
Hence,
\[ \dim X^\alpha \leq \langle \alpha, q \rangle + B_\mu(q). \]

b. ii. observing Lemma 6.1 assertion 2. ii., we obtain
\[ P^{(\alpha,q)+t+k\delta}(X^\alpha) = 0, \quad \forall \ t > B_\mu(q), \ \delta > 0. \]
Hence,
\[ \dim X_\alpha \leq \langle \alpha, q \rangle + t + k\delta, \quad \forall \ t > B_\mu(q), \ \delta > 0 \]
and finally,
\[ \dim X_\alpha \leq \langle \alpha, q \rangle + B_\mu(q). \]

**Lemma 6.2** \( \forall q \in \mathbb{R}^k \) such that \( \langle \alpha, q \rangle + b_\mu(q) < 0 \) or \( \langle \alpha, q \rangle + B_\mu(q) < 0 \), we have \( X(\alpha) = \emptyset \).

**Proof.** It is based on

Claim 1. For \( q \in \mathbb{R}^k \) with \( \langle \alpha, q \rangle + b_\mu(q) < 0 \) or \( \langle \alpha, q \rangle + B_\mu(q) < 0 \), \( X_\alpha = \emptyset \).
Claim 2. For \( q \in \mathbb{R}_+^k \) with \( \langle \alpha, q \rangle + b_\mu(q) < 0 \) or \( \langle \alpha, q \rangle + B_\mu(q) < 0 \), \( X_\alpha = \emptyset \).

Indeed, let \( q \in \mathbb{R}^k \) and assume that \( X_\alpha \neq \emptyset \). This means that there exists at least one point \( x \in \text{Support}(\mu) \) for which \( \omega_{\alpha_j}(x) \geq \alpha_j \), for \( 1 \leq j \leq k \). Consequently, for all \( \epsilon > 0 \), there is a sequence \( (r_n)_n \downarrow 0 \) and satisfying
\[ 0 < r_n < \frac{1}{n} \quad \text{and} \quad \mu_j(B(x,r_n)) < r_n^{\alpha_j - \epsilon}, \quad 1 \leq j \leq k. \]

Hence,
\[ \left( \mu(B(x,r_n)) \right)^q (2r_n)^t > 2^t r_n^{\langle \alpha - \epsilon \mathbf{1}, q \rangle + t}. \]
Choosing $t = ((e^t - \alpha), q)$, this induces that $H_{\mu}^{q,t}(\{x\}) > 2^t$ and consequently,

$$b_{\mu}(q) \geq \dim_{\mu}^q(\{x\}) \geq t, \quad \forall \varepsilon > 0.$$  

Letting $\varepsilon \downarrow 0$, it results that $b_{\mu}(q) \geq -\langle \alpha, q \rangle$ which is impossible. So as the first part of Claim 1. The remaining part as well as Claim 2 can be checked by similar techniques.

**Theorem 6.1** Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector-valued Borel probability measure on $\mathbb{R}^d$ and $q \in \mathbb{R}^k$ fixed. Let further $t_q \in \mathbb{R}$, $r_q > 0$, $K_q, \overline{K}_q > 0$, $\nu_q$ a Borel probability measure supported by $\text{Support}(\mu)$, $\varphi_q : \mathbb{R}_+ \to \mathbb{R}$ be such that $\varphi_q(r) = o(\log r)$, as $r \to 0$. Let finally $(r_{q,n})_n \subset [0,1]$, 0 and satisfying

$$\frac{\log r_{q,n+1}}{\log r_{q,n}} \to 1 \quad \text{and} \quad \sum_n r_{q,n}^\varepsilon < \infty, \quad \forall \varepsilon > 0.$$  

Assume next the following assumptions.

**A1.** $\forall x \in \text{Support}(\mu)$ and $r \in \mathbb{R}$, $r_q$,

$$K_q \leq \frac{\nu_q(\mu(B(x,r)))}{(\mu(B(x,r)))^q(2r)^{t_q} \exp(\varphi_q(r))} \leq \overline{K}_q.$$  

**A2** $C_q(p) = \lim_{n \to +\infty} C_{q,n}(p)$ exists and finite for all $p \in \mathbb{R}$, where

$$C_{q,n}(p) = \frac{1}{-\log r_{q,n}} \log \left( \int_{\text{supp}(\mu)} \left( \mu(B(x,r_{q,n})) \right)^p d\nu_q(x) \right).$$  

Then, the following assertions hold.

**i.**

$$\dim(X_{-\nabla C_q(0)} \cap X_{-\nabla -C_q(0)}) \geq$$

$$\begin{cases}
-\nabla C_q(0)q + \Lambda_{\mu}(q) \geq -\nabla C_q(0)q + B_{\mu}(q) \geq -\nabla C_q(0)q + b_{\mu}(q), \quad q \in \mathbb{R}^k, \\
-\nabla + C_q(0)q + \Lambda_{\mu}(q) \geq -\nabla + C_q(0)q + B_{\mu}(q) \geq -\nabla + C_q(0)q + b_{\mu}(q), \quad q \in \mathbb{R}^k.
\end{cases}$$

**ii.** Whenever $C_q$ is differentiable at 0, we have

$$f_{\mu}(-\nabla C_q(0)) = b_{\mu}^{*}(-\nabla C_q(0)) = B_{\mu}^{*}(-\nabla C_q(0)) = \Lambda_{\mu}^{*}(-\nabla C_q(0)).$$

**Theorem 6.2** Assume that the hypotheses of Theorem 6.1 are satisfied for all $q \in \mathbb{R}^k$. Then, the following assertions hold.

**i.** $\alpha_{\mu} = -B_{\mu}, \quad \nu_q \text{ a.s. whenever } B_{\mu} \text{ is differentiable at } q.$

**ii.** $\text{Dom}(B) \subseteq \alpha_{\mu}(\text{supp}(\mu))$ and $f_{\mu} = B_{\mu}^{*}$ on $\text{Dom}(B)$.

The proof of this result is based on the application of a large deviation formalism. This will permit to obtain a measure $\nu$ supported by $X_{-\nabla C_q(0)} \cap X_{-\nabla -C_q(0)}$. 

33
To do this, we re-formulate a mixed large deviation formalism to be adapted to the mixed multifractal formalism raised in our work.

**Theorem 6.3 The mixed large deviation formalism.** Consider a sequence of vector-valued random variables \((W_n = (W_{n,1}, W_{n,2}, \ldots, W_{n,k}))_n\) on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and \((a_n)_n \subset ]0, +\infty[\) with \(\lim_{n \to +\infty} a_n = +\infty\). Let next the function

\[
C_n : \mathbb{R}^k \to \mathbb{R} \\
t \mapsto C_n(t) = \frac{1}{a_n} \log \left( E(\exp(\langle t, W_n \rangle)) \right).
\]

Assume that

**A1.** \(C_n(t)\) is finite for all \(n\) and \(p\).

**A2.** \(C(t) = \lim_{n \to +\infty} C_n(t)\) exists and is finite for all \(t\).

There holds that

**i.** The function \(C\) is convex.

**ii.** If \(\nabla_- C(t) \leq \nabla_+ C(t) < \alpha\), for some \(t \in \mathbb{R}^k\), then

\[
\limsup_{n \to +\infty} \frac{1}{a_n} \log \left( e^{-a_n C(t)} E \left( \exp(\langle t, W_n \rangle) 1_{\{W_n \leq \alpha\}} \right) \right) < 0.
\]

**iii.** If \(\sum_n e^{-\varepsilon a_n} < \infty\) for all \(\varepsilon > 0\), then

\[
\limsup_{n \to +\infty} \frac{W_n}{a_n} \leq \nabla_+ C(0) \quad \mathbb{P} \text{ a.s.}
\]

**iv.** If \(\alpha < \nabla_- C(t) \leq \nabla_+ C(t)\), for some \(t \in \mathbb{R}^k\), then

\[
\limsup_{n \to +\infty} \frac{1}{a_n} \log \left( e^{-a_n C(t)} E \left( \exp(\langle t, W_n \rangle) 1_{\{W_n \leq \alpha\}} \right) \right) < 0.
\]

**v.** If \(\sum_n e^{-\varepsilon a_n}\) is finite for all \(\varepsilon > 0\), then

\[
\nabla_- C(0) \leq \limsup_{n \to +\infty} \frac{W_n}{a_n} \quad \mathbb{P} \text{ a.s.}
\]

**Proof.**

**i.** It follows from Holder’s inequality.
ii. Let $h \in \mathbb{R}^*_+^k$ be such that $C(t) + \langle \alpha, h \rangle - C(t + h) > 0$. We have

$$\frac{1}{a_n} \log \left[ e^{-a_nC(t)} \mathbb{E} \left( \exp(\langle t, W_n \rangle) 1 \left\{ \frac{W_n}{a_n} \geq \alpha \right\} \right) \right]$$

$$= \frac{1}{a_n} \log \left[ e^{-a_nC(t)} \int_{\left\{ \frac{W_n}{a_n} \geq \alpha \right\}} e^{\langle t, W_n \rangle} d\mathbb{P} \right]$$

$$\leq \frac{1}{a_n} \log \left[ e^{-a_nC(t) + \langle \alpha, h \rangle} \int_{\left\{ \frac{W_n}{a_n} \geq \alpha \right\}} e^{\langle t+h, W_n \rangle} d\mathbb{P} \right]$$

$$\leq \frac{1}{a_n} \log \left[ e^{-a_n(C(t) + \langle \alpha, h \rangle)} \mathbb{E}(\exp(\langle t + h, W_n \rangle)) \right]$$

$$= \frac{1}{a_n} \log \left[ e^{-a_n(C(t) + \langle \alpha, h \rangle - C_n(t+h))} \right]$$

$$= - (C(t) + \langle \alpha, h \rangle - C_n(t + h)).$$

Next, by taking the limsup as $n \to +\infty$, the result follows immediately.

References

[1] A. Ben Mabrouk, A note on Hausdorff and packing measures, Interna. J. Math. Sci., 8(3-4) (2009), 135-142.

[2] A. Ben Mabrouk, A higher order multifractal formalism, Stat. Prob. Lett. 78 (2008), pp. 1412-1421.

[3] F. Ben Nasr, Analyse multifractale de mesures, C. R. Acad. Sci. Paris, 319(I) (1994), 807-810.

[4] F. Ben Nasr et I. Bhouri, Spectre multifractal de mesures boréliennes sur $\mathbb{R}^d$, C. R. Acad. Sci. Paris, 325(I) (1997), 253-256.

[5] F. Ben Nasr, I. Bhouri and Y. Heurteaux, The validity of the multifractal formalism: results and examples, Adv. Math. 165 (2002), 264-284.

[6] I. Bhouri, On the projections of generalized upper $L^q$-spectrum, Chaos, Solitons and Fractals, 42 (2009), 14511462

[7] P. Billingsley, Ergodic theory and information, J. Wiley & Sons Inc., New York, 1965.

35
[8] G. Brown, G. Michon and J. Peyriere, On the multifractal analysis of measures, J. Stat. Phys., 66(3/4) (1992), 775-790.

[9] L. Olsen, A multifractal formalism, Adv. Math., 116 (1995), 82-196.

[10] L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$, Pacific J. Math., 183(1) (1998), 143-199.

[11] L. Olsen, Dimension inequalities of multifractal Hausdorff measures and multifractal packing measures, Math. Scand., 86 (2000), 109-129.

[12] L. Olsen, Integral, probability, and fractal measures, by G. Edgar, Springer, New York, 1998, Bull. Amer. Math. Soc., 37 (2000), 481-498.

[13] L. Olsen, Divergence points of deformed empirical measures, Math. Resear. Letters, 9 (2002), 701-713.

[14] L. Olsen, Mixed divergence points of self-similar measures, Indiana Univ. Math. J., 52 (2003), 1343-1372.

[15] L. Olsen, Mixed generalized dimensions of self-similar measures, J. Math. Anal. and Appl., 306 (2005), 516-539.

[16] J. Peyriere, Multifractal measures, in Probabilistic and Stochastic methods in analysis, Proceedings of the NATO ASI, II Ciocco 1991, J. Bymes Bd., Keuwer Academic Publisher, 1992.

[17] Y.-L. Ye, Self-similar vector-valued measures, Adv. Appl. Math., 38 (2007), 7196.

[18] M. Dai and Z. Liu, The Quantization Dimension and Other Dimensions of Probability Measures. International Journal of Nonlinear Science, 5 (2008), 267-274.

[19] Y. Shi and M. Dai, Typical Lower $L^q$-dimensions of Measures for $q \leq 1$, International Journal of Nonlinear Science, 7 (2009), 231236.

[20] Q. Guo, H. Jiang and L. Xi, Hausdorff Dimension of Generalized sierpinski Carpet. International Journal of Nonlinear Science, 6 (2006), 153158.

[21] X. Wang, M. Dai, Mixed Quantization Dimension Function and Temperature Function for Conformal Measures, International Journal of Nonlinear Science, 10(1) (2010), 24-31.

[22] Y. Pesin, Dimension Theory in Dynamical Systems, University of Chicago Press, 1997.

[23] S. J. Taylor, The fractal analysis of Borel measures in $\mathbb{R}^d$, J. Fourier. Anal. and Appl., Kahane Special Issue, (1995), 553-568.