TRANSPARENT PAIRS

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Abstract. Let \( M \) be a closed orientable Riemannian surface. Consider an \( SO(3) \)-connection \( A \) and a Higgs field \( \Phi : M \to \mathfrak{so}(3) \). The pair \((A, \Phi)\) naturally induces a cocycle over the geodesic flow of \( M \). We classify (up to gauge transformations) cohomologically trivial pairs \((A, \Phi)\) with finite Fourier series in terms of a suitable Bäcklund transformation. In particular, if \( M \) is negatively curved we obtain a full classification of \( SO(3) \)-transparent pairs.

1. Introduction

Let \((M, g)\) be a closed oriented Riemannian surface and let \( A \) be a \( g \)-valued 1-form, where \( g \) is \( \mathfrak{su}(n) \) or \( \mathfrak{so}(n) \). We think of \( A \) as a smooth map \( A : TM \to g \) which is linear in \( v \in T_xM \) for all \( x \in M \). The 1-form \( A \) defines a unitary/orthogonal connection \( d + A \) on the trivial bundle \( M \times \mathbb{F}^n \), where \( \mathbb{F} = \mathbb{R}, \mathbb{C} \). Suppose we are given in addition a smooth map \( \Phi : M \to g \) which we call the Higgs field.

The pair \((A, \Phi)\) naturally induces a cocycle over the geodesic flow \( \phi_t \) of the metric \( g \) acting on the unit sphere bundle \( SM \) with projection \( \pi : SM \to M \). The cocycle takes values in the group \( G = SU(n), SO(n) \) and is defined as follows: let \( C : SM \times \mathbb{R} \to G \) be determined by

\[
\frac{d}{dt} C(x, v, t) = -(A(\phi_t(x, v)) + \Phi(\pi \circ \phi_t(x, v)))C(x, v, t), \quad C(x, v, 0) = \text{Id}.
\]

The function \( C \) is a cocycle:

\[
C(x, v, t + s) = C(\phi_t(x, v), s)C(x, v, t)
\]

for all \((x, v) \in SM\) and \( s, t \in \mathbb{R} \). The cocycle \( C \) is said to be cohomologically trivial if there exists a smooth function \( u : SM \to G \) such that

\[
C(x, v, t) = u(\phi_t(x, v))u^{-1}(x, v)
\]

for all \((x, v) \in SM\) and \( t \in \mathbb{R} \). We call \( u \) a trivializing function and note that two trivializing functions \( u_1 \) and \( u_2 \) (for the same cocycle) are related by \( u_2w = u_1 \) where \( w : SM \to G \) is constant along the orbits of the geodesic flow. In particular, if \( \phi_t \) is transitive (i.e. there is a dense orbit) there is a unique trivializing function up to right multiplication by a constant matrix in \( G \).

Definition 1.1. We will say that a pair \((A, \Phi)\) is cohomologically trivial if \( C \) is cohomologically trivial.
Observe that the gauge group given by the set of smooth maps \( r : M \to G \) acts on pairs as follows:

\[
(A, \Phi) \mapsto (r^{-1}dr + r^{-1}Ar, r^{-1}\Phi r).
\]

This action leaves invariant the set of cohomologically trivial pairs: indeed, if \( u \) trivializes the cocycle \( C \) of a pair \((A, \Phi)\), then it is easy to check that \( r^{-1}u \) trivializes the cocycle of the pair \((r^{-1}dr + r^{-1}Ar, r^{-1}\Phi r)\).

In the present paper we will classify up to gauge equivalence all cohomologically trivial \( SO(3) \)-pairs which admit a trivializing function with finite Fourier series. By finite Fourier series we mean the following. Consider a smooth map \( u : SM \to \mathbb{M}_n(\mathbb{C}) \), where \( \mathbb{M}_n(\mathbb{C}) \) is the set of \( n \times n \) complex matrices (we think of maps into \( SO(3) \) also as maps into \( \mathbb{M}_3(\mathbb{C}) \)). Let \( \rho_t \) be the flow of the vertical vector field \( V \) determined by the principal circle fibration \( \pi : SM \to M \). We shall say that \( u \) has finite Fourier series if there exists a non-negative integer \( N \) such that

\[
\enspace
\enspace u_m(x, v) := \frac{1}{2\pi} \int_0^{2\pi} u(\rho_t(x, v)) e^{-imt} \, dt
\]

vanishes identically for all \( m \) with \( |m| \geq N + 1 \). This means that we can expand \( u \) as a finite sum \( u = \sum_{m=-N}^{m=N} u_m \).

The classification of cohomologically trivial \( SO(3) \)-pairs with finite Fourier series comes in terms of a suitable Bäcklund transformation. Basically this means that there is an explicit algebraic way of constructing new cohomologically trivial pairs from known ones and it works as follows. Given a cohomologically trivial pair \((A, \Phi)\) with a trivializing function \( b \) we seek a smooth map \( a : SM \to SO(3) \) such that \( u := ab \) is a trivializing function for the new pair. The map \( a \) will be special, of degree one in the velocities and such that \( g := a^{-1}V(a) \) is a map from \( M \) into \( \mathfrak{so}(3) \) with norm one, i.e. taking values in \( S^2 \subset \mathfrak{so}(3) \). We will show that \( u \) is a trivializing function of a new pair if \( g \) satisfies the equation \( -\star d_A g = [d_A g, g] \), where \( \star \) is the Hodge star operator of the metric and \( d_A \) is the covariant derivative on endomorphisms associated with the connection \( A \). This will form the basis of our classification and for a detailed statement we refer to Theorem 6.4 below. A similar procedure was employed in [12] to classify cohomologically trivial \( SU(2) \)-connections. However if one wishes to include a Higgs field \( \Phi \) the method in [12] has to be modified. Perhaps, the main contribution of this paper is the observation that in order to consider Higgs fields (even in the \( SU(2) \) case) it is better to work with \( SO(3) \) rather \( SU(2) \). Roughly speaking, the \( SO(3) \) Bäcklund transformation of this paper acts as the “square root” of the \( SU(2) \) Bäcklund transformation from [12], this is explained in detail in Section 6. Note that a classification of \( SO(3) \) cohomologically trivial pairs includes a classification of \( SU(2) \) pairs, since any \( SU(2) \) cocycle can be converted into an \( SO(3) \) cocycle just by composing with the 2-1 homomorphism \( SU(2) \to SO(3) \).

The motivation for understanding cohomologically trivial pairs comes from the related weaker notion of transparent pairs. The pair \((A, \Phi)\) is said to be transparent if \( C(x, v, T) = Id \) every time that \( \phi_T(x, v) = (x, v) \). In other words the linear transport
equation associated with $A + \Phi$ has trivial holonomy along closed geodesics\footnote{To avoid cumbersome notation we will often write $A + \Phi$ instead of $A + \Phi \circ \pi$.}. These pairs are invisible from the point of view of the closed geodesics of the Riemannian metric and it is a natural inverse problem to try to determine them. This inverse problem can also be considered in the case of manifolds with boundary or $\mathbb{R}^d$ with appropriate decay conditions at infinity, and in this context, it has been addressed by several authors, see for instance [1, 2, 10, 13, 15]. The inclusion of a Higgs field is natural and transparent pairs on $S^2$ are related to solutions of the Bogomolny equations $D\Phi = \ast F$ in (2 + 1)-dimensional Minkowski space [16, 17]. The inverse problem with a Higgs field is also related with the attenuated X-ray transform as explained in [10, Section 8].

Obviously a cohomologically trivial pair is transparent. There is one important situation in which both notions agree. If $\phi_t$ is Anosov, then the Livsic theorem [6, 7] together with the regularity results in [9] imply that a transparent pair is also cohomologically trivial. The Anosov property is satisfied, if for example $(M,g)$ has negative curvature.

In Theorem 4.1 we will show that if $(M,g)$ is negatively curved, then the trivializing function of a cohomologically trivial pair (unique up to multiplication on the right by a constant) must have finite Fourier series. We do not know if this result holds true just assuming that the geodesic flow is Anosov.

In summary we end up proving (see Theorem 7.3):

**Theorem.** Let $M$ be a closed orientable surface of negative curvature. Then any transparent $SO(3)$-pair can be obtained by successive applications of Bäcklund transformations as described in Theorem 6.4.

Our results also give a classification of transparent $SO(3)$ pairs with finite Fourier series over an arbitrary Zoll surface (i.e. a Riemannian metric on $S^2$ all of whose geodesics are closed). This is because we show that in this case (cf. Corollary 2.7) any transparent $SO(3)$ pair is cohomologically trivial. Of course in the Zoll case there may be many other cohomologically trivial pairs which do not have a finite Fourier series. A full classification of $U(n)$ transparent connections ($\Phi = 0$) for the case of the round metric on $S^2$ has been obtained by L. Mason (unpublished) using methods from twistor theory as in [8].

**Acknowledgements:** I am very grateful to the MSRI and to the organizers of the program Inverse Problems and Applications for hospitality while this work was being completed. I am also very grateful to Maciej Dunajski for several useful conversations related to this paper and to Will Merry for comments and corrections to a previous draft.
Let \( G \) be a compact Lie group; for the purposes of this paper it is enough to think of \( G \) as \( SU(n) \) or \( SO(3) \).

**Definition 2.1.** A \( G \)-valued cocycle over the flow \( \phi_t \) is a map \( C : N \times \mathbb{R} \to G \) that satisfies

\[
C(x, t + s) = C(\phi_t x, s) C(x, t)
\]

for all \( x \in N \) and \( s, t \in \mathbb{R} \).

In this paper the cocycles will always be smooth. In this case \( C \) is determined by its infinitesimal generator \( B : N \to \mathfrak{g} \) given by

\[
B(x) := -\frac{d}{dt} \bigg|_{t=0} C(x, t).
\]

The cocycle can be recovered from \( B \) as the unique solution to

\[
\frac{d}{dt} C(x, t) = -dR_{C(x,t)}(B(\phi_t x)), \quad C(x, 0) = \text{Id},
\]

where \( R_g \) is right translation by \( g \in G \). We will indistinctly use the word “cocycle” for \( C \) or its infinitesimal generator \( B \).

**Definition 2.2.** The cocycle \( C \) is said to be cohomologically trivial if there exists a smooth function \( u : N \to G \) such that

\[
C(x, t) = u(\phi_t x) u^{-1}(x)
\]

for all \( x \in N \) and \( t \in \mathbb{R} \).

Observe that the condition of being cohomologically trivial can be equivalently expressed in terms of the infinitesimal generator \( B \) of the cocycle by saying that there exists a smooth function \( u : N \to G \) that satisfies the equation

\[
d_x u(X(x)) + d_{Id} R_{u(x)}(B(x)) = 0
\]

for all \( x \in N \). If \( G = SU(n) \) or \( SO(3) \) we can write this more succinctly as

\[
X(u) + Bu = 0
\]

where it is understood that differentiation and multiplication is in the set of complex \( n \times n \)-matrices for \( SU(n) \) or in the set of real \( 3 \times 3 \)-matrices for \( SO(3) \).

**Definition 2.3.** A cocycle \( C \) is said to satisfy the periodic orbit obstruction condition if \( C(x, T) = \text{Id} \) whenever \( \phi_T x = x \).

Obviously a cohomologically trivial cocycle satisfies the periodic orbit obstruction condition. The converse turns out to be true for transitive Anosov flows: this is one of the celebrated Livsic theorems \([6, 7, 9]\).

**Theorem 2.4** (The smooth Livsic periodic data theorem). Suppose \( \phi_t \) is a smooth transitive Anosov flow. Let \( C \) be a smooth cocycle such that \( C(x, T) = \text{Id} \) whenever \( \phi_T x = x \). Then \( C \) is cohomologically trivial.
There is another situation of interest to us in which a cocycle that satisfies the periodic orbit obstruction condition is cohomologically trivial. Suppose $N$ is a closed orientable 3-manifold which admits a free circle action with infinitesimal generator $V$ and flow $\phi_t$. The quotient of $N$ by the circle action is a closed orientable surface $M$; let $e(N) \in H^2(M, \mathbb{Z}) = \mathbb{Z}$ be the Euler class of the circle bundle $p : N \to M$.

**Proposition 2.5.** Any cocycle $B : N \to \mathfrak{su}(n)$ over $\phi_t$ which satisfies the periodic orbit obstruction condition is cohomologically trivial. The same holds for any cocycle $B : N \to \mathfrak{so}(3)$ provided that $e(N)$ is even.

**Proof.** Consider first the case of $B : N \to \mathfrak{su}(n)$. Let $U$ be a neighbourhood of $M$ such that $p^{-1}(U)$ is trivialized as $U \times S^1$ via $\psi_U : p^{-1}(U) \to U \times S^1$. In this trivialization we may write $V = \partial/\partial \theta$, where $\theta \in \mathbb{R}/2\pi\mathbb{Z} = S^1$. The fact that $B$ satisfies the periodic orbit obstruction condition means that we can find a smooth $u_U : U \times S^1 \to SU(n)$ such that

$$\frac{\partial u_U}{\partial \theta}(x, \theta) + B(x, \theta)u_U(x, \theta) = 0$$

for all $(x, \theta) \in U \times S^1$ and $u_U(x, 0) = \text{Id}$. We will show that the functions $u_U$ can be glued to determine a globally defined $u : N \to SU(n)$ which solves $V(u) + Bu = 0$.

Observe first that if $g : U \to SU(n)$ is any smooth map, then $u_Ug$ also satisfies (1).

Take now another set $U'$ which overlaps with $U$ and over which we can trivialize $p : N \to M$. We obtain a transition function $\psi_{UU'} : U \cap U' \to S^1$. To have a globally defined $u$ we need the existence of smooth functions $g_{U'} : U \to SU(n)$ such that

$$u_U(x, \psi_{UU'}(x))g_{U'}(x) = g_{U'}(x)$$

for all $x \in U \cap U'$ and all overlaps of a covering of $M$ by trivializing sets. Observe that by construction

$$(2) \quad u_{U'}(x, \theta) = u_{U'}(x, \theta + \psi_{UU'}(x))[u_{U'}(x, \psi_{UU'}(x))]^{-1} \quad \text{on } (U \cap U') \times S^1.$$

The key observation now is that $\varphi_{UU'} : U \cap U' \to SU(n)$ given by

$$\varphi_{UU'}(x) := u_U(x, \psi_{UU'}(x))$$

is an $SU(n)$-cocycle in the sense of principal bundles. Indeed, the cocycle property $\varphi_{UU'}(x) = \varphi_{UU''}(x)\varphi_{UU'}(x)$ follows right away from the fact that $\psi_{UU'}$ is an $S^1$-cocycle and (2). But any principal $SU(n)$-bundle over a surface is trivial, thus there exist smooth functions $g_U : U \to SU(n)$ such that

$$\varphi_{UU'}(x) = g_{U'}(x)(g_U(x))^{-1}$$

which says precisely that $u$ can be defined globally.

The argument for $SO(3)$ is pretty much the same except that we now have two different $SO(3)$-bundles over the surface, the non-trivial one having non-zero Stiefel-Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$. Let $E$ denote the principal bundle determined by $\varphi_{UU'}$ and let us check that $E$ is trivial if $e(N)$ is even. Perhaps the easiest way to see
this is to argue as follows. Consider the pull-back bundle \( p^*E \) and observe that it is trivial. Indeed, its transition functions are given by

\[ \varphi_{p^{-1}(U)p^{-1}(U')}^{-1}(y) = \varphi_{UU'}(py) \]

and if we set

\[ g_{p^{-1}(U)}^{-1}(y) = [u_U \circ \psi_U(y)]^{-1} \]

then

\[ \varphi_{p^{-1}(U)p^{-1}(U')}^{-1}(y) = g_{p^{-1}(U')} \left( g_{p^{-1}(U)}^{-1} \right)^{-1}. \]

It follows that \( w_2(p^*E) = p^*w_2(E) = 0 \). But using the Gysin sequence of the circle bundle \( p : N \to M \), we see that \( p^*w_2(E) \) vanishes iff \( w_2(E) \) is in the image of the map \( H^0(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2) \) given by the cup product with the Euler class \( e(N) \). Hence \( w_2(E) = 0 \) as desired.

\[ \square \]

**Remark 2.6.** So far we have considered *right* cocycles. However one can easily go from a right cocycle \( C \) to a left cocycle simply by considering \( C^{-1} \) and thus Proposition 2.5 applies to them as well.

Let us now describe the two main consequences of this result.

2.1. **Zoll surfaces.** Recall that a Zoll surface is a Riemannian metric on \( S^2 \) all of whose geodesics are closed. By [3] all the closed geodesics are simple and with the same length and thus the action of the geodesic flow on the unit circle bundle \( SM \) fits the setting above. In fact, by [14] the Euler class of the circle bundle determined by the action of the geodesic flow is even.

Observe that by definition, saying that a pair \((A, \Phi)\) is transparent is the same as saying that the cocycle determined by \((A, \Phi)\) satisfies the periodic orbit obstruction condition. Thus Proposition 2.5 gives right away the following:

**Corollary 2.7.** Let \((S^2, g)\) be a Zoll surface. Then any transparent \( SU(n) \) or \( SO(3) \)-pair \((A, \Phi)\) is cohomologically trivial.

2.2. **An important step to set up the \( SO(3) \) Bäcklund transformation.** In the next application of Proposition 2.5, our circle bundle is the unit tangent bundle \( \pi : SM \to M \) and the circle action is given by the flow of the vertical vector field \( V \). Since \( M \) is orientable, the Euler class of this bundle is obviously even (the Euler characteristic of \( M \) is even). We consider the left cocycle over \( V \) determined by \( f : M \to \mathfrak{su}(n) \) or \( f : M \to \mathfrak{so}(3) \) which we also regard as a function over \( SM \) by composing it with \( \pi \). In other words, if \( \rho_t \) is the flow of \( V \), we consider the cocycle determined by

\[ \partial_t C(x, v, t) = C(x, v, t) f(\pi \circ \rho_t(x, v)), \quad C(x, v, 0) = \text{Id}. \]

Since \( f \) only depends on \( x \in M \) we can write \( C(x, v, t) = \exp(t f(x)) \).

There are two situations of interest to us. The first is \( f : M \to \mathfrak{su}(2) \) with \( f^2 = -\text{Id} \). Under this condition for any \( x \in M \), \( f(x) \) has eigenvalues \( \pm i \) and it clearly satisfies the periodic orbit obstruction condition with respect to \( V \), thus we have:
Corollary 2.8. Let $f : M \to \mathfrak{su}(2)$ be a smooth map with $f^2 = -\text{Id}$. Then there exists a smooth $u : SM \to SU(2)$ such that $uf = V(u)$.

This corollary was proved with slightly different methods in [12, Lemma 4.1]. Observe that any two solutions $u$ and $v$ of $uf = V(u)$ are related by $u = rv$, where $r : M \to SU(2)$. Consider local coordinates $(x, y)$ on a neighbourhood $U$ of $M$. This gives coordinates $(x, y, \theta)$ on $SM$ where $\theta$ is the angle between a unit vector $v$ and $\partial/\partial x$. In these coordinates, $V = \partial/\partial \theta$ and any solution $u$ of $uf = V(u)$ can be expressed locally as

$$u(x, y, \theta) = r(x, y)(\cos \theta \text{Id} + \sin \theta f(x, y)),$$

for some smooth $r : U \to SU(2)$.

The next situation of interest to us is that of a smooth $f : M \to \mathfrak{so}(3)$ with $f^3 + f = 0$. Again, under this condition $f$ defines an $SO(3)$-cocycle over $V$ satisfying the periodic orbit obstruction condition and thus by Proposition 2.5 we have:

Corollary 2.9. Let $f : M \to \mathfrak{so}(3)$ be a smooth map with $f^3 + f = 0$. Then there exists a smooth $u : SM \to SO(3)$ such that $uf = V(u)$.

This last corollary will be essential to run the Bäcklund transformation for transparent pairs.

As above, if we consider local coordinates any solution $u$ to $uf = V(u)$ can be written locally as

$$u(x, y, \theta) = r(x, y)(\text{Id} + f(x, y)^2 + \sin \theta f(x, y) - \cos \theta f^2(x, y))$$

for some smooth $r : U \to SO(3)$. Note that

$$\text{Id} + f^2 + \sin \theta f - \cos \theta f^2 = -\frac{e^{-i\theta}}{2}f(f - i\text{Id}) + (\text{Id} + f^2) - \frac{e^{i\theta}}{2}f(f + i\text{Id}).$$

From these local expressions it is clear that $u = u_{-1} + u_0 + u_1$ with

$$\text{Ker } u_0 = E_i \oplus E_{-i},$$
$$\text{Ker } u_1 = \text{Ker}(f) \oplus E_{-i},$$
$$\text{Ker } u_{-1} = \text{Ker}(f) \oplus E_i,$$

where $E_{\pm i}$ is the eigenspace of $f$ corresponding to the eigenvalue $\pm i$.

3. The set up

In this section we summarise the setting in [11] and the results needed for the subsequent sections.

Let $M$ be an oriented surface with a Riemannian metric and let $SM$ be its unit tangent bundle. Recall that $SM$ has a canonical framing $\{X, H, V\}$, where $X$ is the geodesic vector field, $V$ is the vertical vector field and $H = [V, X]$ is the horizontal vector field.
Let $\mathbb{M}_n(\mathbb{C})$ be the set of $n \times n$ complex matrices. Given functions $u, v : SM \to \mathbb{M}_n(\mathbb{C})$ we consider the inner product

$$\langle u, v \rangle = \int_{SM} \text{trace}(uv^*) \, d\mu,$$

where $\mu$ is the Riemannian measure associated with the Sasaki metric of $SM$ which makes $\{X, H, V\}$ into an orthonormal frame. The space $L^2(SM, \mathbb{M}_n(\mathbb{C}))$ decomposes orthogonally as a direct sum

$$L^2(SM, \mathbb{M}_n(\mathbb{C})) = \bigoplus_{m \in \mathbb{Z}} H_m$$

where $-iV$ acts as $m \text{Id}$ on $H_m$.

Following Guillemin and Kazhdan in [4] we introduce the following first order elliptic operators

$$\eta_+, \eta_- : C^\infty(SM, \mathbb{M}_n(\mathbb{C})) \to C^\infty(SM, \mathbb{M}_n(\mathbb{C}))$$

given by

$$\eta_+ := (X - iH)/2, \quad \eta_- := (X + iH)/2.$$

Clearly $X = \eta_+ + \eta_-$. Let $\Omega_m := C^\infty(SM, \mathbb{M}_n(\mathbb{C})) \cap H_m$. We have

$$\eta_+ : \Omega_m \to \Omega_{m+1}, \quad \eta_- : \Omega_m \to \Omega_{m-1}, \quad (\eta_+)^* = -\eta_-.$$

If a $U(n)$ pair $(A, \Phi)$ is cohomologically trivial there exists a smooth $u : SM \to U(n)$ such that $C(x, v, t) = u(\phi_t(x, v))u^{-1}(x, v)$. Differentiating with respect to $t$ and setting $t = 0$, this is equivalent to $X(u) + (A + \Phi)u = 0$, where we now regard $A$ and $\Phi$ as functions $A, \Phi : SM \to u(n)$. To deal with this equation, we introduce the “twisted” operators

$$\mu_+ := \eta_+ + A_1, \quad \mu_- := \eta_+ + A_1,$$

where $A = A_1 + A_1$, and

$$A_1 := \frac{A - iV(A)}{2} \in H_1,$$
$$A_{-1} := \frac{A + iV(A)}{2} \in H_{-1}.$$

Observe that this decomposition corresponds precisely with the usual decomposition of $u(n)$-valued 1-forms on a surface:

$$\Omega^1(M, u(n)) \otimes \mathbb{C} = \Omega^{1,0}(M, u(n)) \oplus \Omega^{0,1}(M, u(n)),$$

where $\ast = -i$ on $\Omega^{1,0}$ and $\ast = i$ on $\Omega^{0,1}$ (here $\ast$ is the Hodge star operator of the metric).

We also have

$$\mu_+ : \Omega_m \to \Omega_{m+1}, \quad \mu_- : \Omega_m \to \Omega_{m-1}, \quad (\mu_+)^* = -\mu_-.$$

The equation $X(u) + (A + \Phi)u = 0$ is now $\mu_+(u) + \mu_-(u) + \Phi u = 0$.

For future use, it is convenient to write the operators $\eta_-$ and $\mu_-$ in local coordinates. Consider isothermal coordinates $(x, y)$ on $M$ such that the metric can be written as $ds^2 = e^{2\lambda}(dx^2 + dy^2)$ where $\lambda$ is a smooth real-valued function of $(x, y)$. This gives
coordinates \((x, y, \theta)\) on \(SM\) where \(\theta\) is the angle between a unit vector \(v\) and \(\partial/\partial x\).

In these coordinates, \(V = \partial/\partial \theta\) and the vector fields \(X\) and \(H\) are given by:

\[
X = e^{-\lambda} \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} + \left( -\frac{\partial \lambda}{\partial x} \sin \theta + \frac{\partial \lambda}{\partial y} \cos \theta \right) \frac{\partial}{\partial \theta} \right);
\]

\[
H = e^{-\lambda} \left( -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} - \left( \frac{\partial \lambda}{\partial x} \cos \theta + \frac{\partial \lambda}{\partial y} \sin \theta \right) \frac{\partial}{\partial \theta} \right).
\]

Consider \(u \in \Omega_m\) and write it locally as \(u(x, y, \theta) = h(x, y)e^{im\theta}\). Using these formulas a simple, but tedious calculation shows that

\[
\eta_-(u) = e^{-(1+m)\lambda} \bar{\partial}(he^{m\lambda})e^{i(m-1)\theta},
\]

where \(\bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)\). In order to write \(\mu_-\) suppose that \(A(x, y, \theta) = a(x, y) \cos \theta + b(x, y) \sin \theta\). If we also write \(A = A_x dx + A_y dy\), then \(A_x = ae^\lambda\) and \(A_y = be^\lambda\). Let \(A_\bar{z} := \frac{1}{2}(A_x + iA_y)\). Using the definition of \(A_-\) we derive

\[
A_- = \frac{1}{2}(a + ib)e^{-i\theta} = A_\bar{z}d\bar{z}.
\]

Putting this together with (3) we obtain

\[
\mu_-(u) = e^{-(1+m)\lambda} \left( \bar{\partial}(he^{m\lambda}) + A_\bar{z}he^{m\lambda} \right)e^{i(m-1)\theta}.
\]

Note that \(\Omega_m\) can be identified with the set of smooth sections of the bundle \((M \times M_n(\mathbb{C})) \otimes K^\otimes m\) where \(K\) is the canonical line bundle. The identification takes \(u = h e^{im \theta}\) into \(he^{m \lambda}(dz)^m\) \((m \geq 0)\) and \(u = h e^{-im \theta} \in \Omega_{-m}\) into \(he^{m \lambda}(d\bar{z})^m\). The second equality in (4) should be understood using this identification.

4. Finite Fourier series in negative curvature

Given an element \(u \in C^\infty(SM, M_n(\mathbb{C}))\), we write \(u = \sum_{m \in \mathbb{Z}} u_m\), where \(u_m \in \Omega_m\). We will say that \(u\) has degree \(N\), if \(N\) is the smallest non-negative integer such that \(u_m = 0\) for all \(m\) with \(|m| \geq N + 1\). The following finiteness result will be important for us.

**Theorem 4.1.** If \(M\) has negative curvature every solution \(u\) of \(X(u) + (A + \Phi)u = 0\) has finite degree.

This theorem was proved in [11, Theorem 5.1] for the case \(\Phi = 0\) and the extension to include a Higgs field is not entirely straightforward due to the fact that the commanding recurrence relation has a more complicated nature.

**Proof.** We shall use the following equality proved in [11, Corollary 4.4]. Given \(u \in C^\infty(SM, M_n(\mathbb{C}))\) we have

\[
|\mu_+ u|^2 = |\mu_- u|^2 + \frac{i}{2} \langle KV(u), u \rangle + \langle (*F_A)u, u \rangle,
\]
where $K$ is the Gaussian curvature of the metric and $F_A$ is the curvature of $A$. Hence for $u_m \in \Omega_m$ we have

$$|\mu_+ u_m|^2 = |\mu_- u_m|^2 + \frac{1}{2}((i \ast F_A - mK \text{Id})u, u).$$

Hence if $K < 0$, there exist constants $c_m > 0$ with $c_m \to \infty$ and a positive integer $\ell$ such that

$$|\mu_+ u_m|^2 \geq |\mu_- u_m|^2 + c_m|u_m|^2$$

for all $m \geq \ell$. We know that for all $m \in \mathbb{Z}$

$$(m \geq \ell - 1)$$

Using (6) and (7) we can write ($m \geq \ell - 1$)

$$|\mu_+ (u_{m+1})|^2 \geq |\mu_- (u_{m+1})|^2 + c_{m+1}|u_{m+1}|^2$$

$$= |\mu_+ (u_{m-1}) + \Phi u_m|^2 + c_{m+1}|u_{m+1}|^2$$

$$= |\mu_+ (u_{m-1})|^2 + 2\Re\langle \mu_+ (u_{m-1}), \Phi u_m \rangle + |\Phi u_m|^2 + c_{m+1}|u_{m+1}|^2$$

$$= |\mu_+ (u_{m-1})|^2 - 2\Re\langle u_{m-1}, \mu_- (\Phi u_m) \rangle + |\Phi u_m|^2 + c_{m+1}|u_{m+1}|^2.$$ 

Let us compute $\mu_- (\Phi u_m)$:

$$\mu_- (\Phi u_m) = \eta_- (\Phi u_m) + A_1 \Phi u_m$$

$$= \eta_- (\Phi) u_m + \Phi \eta_- (u_m) + A_1 \Phi u_m$$

$$= \Phi \mu_- (u_m) + \bar{\partial}_A (\Phi) u_m$$

$$= \Phi (-\mu_+ (u_{m-2}) - \Phi u_{m-1}) + \bar{\partial}_A (\Phi) u_m,$$

where $\bar{\partial}_A (\Phi) := \eta_- (\Phi) + [A_1, \Phi]$. Therefore using that $\Phi^* = -\Phi$ we have

$$\Re\langle u_{m-1}, \mu_- (\Phi u_m) \rangle = \Re\langle u_{m-1}, -\Phi \mu_+ (u_{m-2}) - \Phi^2 u_{m-1} + \bar{\partial}_A (\Phi) u_m \rangle$$

$$= \Re\langle \Phi u_{m-1}, -\mu_+ (u_{m-2}) \rangle + |\Phi u_{m-1}|^2 + \Re\langle u_{m-1}, \bar{\partial}_A (\Phi) u_m \rangle$$

$$= -\Re\langle u_{m-2}, \mu_- (\Phi u_{m-1}) \rangle + |\Phi u_{m-1}|^2 + \Re\langle u_{m-1}, \bar{\partial}_A (\Phi) u_m \rangle$$

and thus

$$\Re\langle u_{m-1}, \mu_- (\Phi u_m) \rangle + \Re\langle u_{m-2}, \mu_- (\Phi u_{m-1}) \rangle = |\Phi u_{m-1}|^2 + \Re\langle u_{m-1}, \bar{\partial}_A (\Phi) u_m \rangle.$$ 

If we now set

$$a_m := |\mu_+ (u_m)|^2 + |\mu_+ (u_{m-1})|^2$$

then we derive ($m \geq \ell + 1$)

$$a_{m+1} \geq a_{m-1} + |\Phi u_m|^2 - |\Phi u_{m-1}|^2 + c_{m+1}|u_{m+1}|^2 + c_m|u_m|^2 - 2\Re\langle u_{m-1}, \bar{\partial}_A (\Phi) u_m \rangle$$

$$\geq a_{m-1} + |\Phi u_m|^2 - |\Phi u_{m-1}|^2 + c_{m+1}|u_{m+1}|^2 + c_m|u_m|^2 - |u_{m-1}|^2 - |\bar{\partial}_A (\Phi) u_m|^2$$

$$\geq a_{m-1} - |\Phi u_{m-1}|^2 + c_{m+1}|u_{m+1}|^2 + c_m|u_m|^2 - |u_{m-1}|^2 - |\bar{\partial}_A (\Phi) u_m|^2.$$
Since $M$ is compact there exist positive constants $B$ and $C$ such that
\[|\Phi f|^2 \leq (B - 1)|f|^2\]
\[|\bar{\partial}_A(\Phi) f|^2 \leq C|f|^2\]
for any $f \in C^\infty(SM, M_n(\mathbb{C}))$. Therefore
\[a_{m+1} \geq a_{m-1} + r_m\]
where
\[r_m := -B|u_{m-1}|^2 + c_{m+1}|u_{m+1}|^2 + (c_m - C)|u_m|^2.\]
Now choose a positive integer $N_0 \geq \ell$ large enough so that for $m \geq N_0$ we have
\[c_m > \max\{B, C\}.$
Let $m = N + 1 + 2k$, where $k$ is a non-negative integer and $N$ is an integer with $N \geq N_0$. Note that from the definition of $r_m$ and our choice of $N$ we have
\[r_m + r_{m-2} + \cdots + r_{N+1} \geq -B|u_N|^2.\]
Thus
\[a_{m+1} \geq a_N + r_m + r_{m-2} + \cdots + r_{N+1} \geq a_N - B|u_N|^2.\]
From the definition of $a_m$ and \([\square]\) we know that $a_N \geq c_N|u_N|^2$ and hence
\[a_{m+1} \geq (c_N - B)|u_N|^2.\]
Since the function $u$ is smooth, $\mu_+(u_m)$ must tend to zero in the $L_2$-topology as $m \to \infty$. Hence $a_{m+1} \to 0$ as $k \to \infty$ which in turns implies that $u_N = 0$ for any $N \geq N_0$. A similar argument shows that $u_m = 0$ for all $m$ sufficiently large and negative thus concluding that $u$ has finite degree as desired.

\[\square\]

An inspection of the proof above gives the following:

**Corollary 4.2.** Let $M$ be a closed oriented surface of negative curvature and let $(A, \Phi)$ be a transparent pair, where $A$ is a flat connection. Then $\Phi \equiv 0$ and $A$ is gauge equivalent to the trivial connection.

**Proof.** Indeed, if $A$ is flat and $K < 0$, the equality
\[|\mu_+ u|^2 = |\mu_- u|^2 + \frac{i}{2}(\langle KV(u), u \rangle),\]
implies that $\mu_+$ is injective on $\Omega_n$ for $n \geq 1$ (and $\mu_-$ is injective on $\Omega_n$ for $n \leq -1$). Any $u : SM \to U(n)$ solving $X(u) + (A + \Phi) u = 0$ must have a finite Fourier series and thus if we write $u = \sum_{-N}^{N} u_m$, then $\mu_+(u_N) = \mu_-(u_{-N}) = 0$ which in turn implies $u_N = u_{-N} = 0$ if $N \geq 1$. Arguing inductively, it follows that $u = u_0$ and $X(u) + (A + \Phi) u = 0$ may be rewritten as $du_0 + (A + \Phi) u_0 = 0$. This clearly implies $\Phi \equiv 0$ and $A$ gauge equivalent to the trivial connection.

\[\square\]
5. A general correspondence for pairs

The purpose of this section is to describe a general classification result for cohomologically trivial pairs on any surface similar to [12, Theorem 3.1]. This correspondence is important for our approach since it will help us expose the relation of the problem at hand with the underlying complex structure of $M$. In what follows we assume that $G$ is $SU(n)$ or $SO(n)$ with lie algebra $\mathfrak{g}$.

Let $M$ be an oriented surface with a Riemannian metric and let $SM$ be its unit tangent bundle. Let

$$A := \{ A : SM \to \mathfrak{g} : V^2(A) = -A \}.$$  

The set $A$ is identified naturally (after fixing a metric) with $\Omega^1(M, \mathfrak{g})$. An element in $\Omega^1(M, \mathfrak{g})$ is a smooth map $A : TM \to \mathfrak{g}$ such that for each $x \in M$ it is linear in $v \in T_x M$ and the bijection with $A$ is obtained by restriction to $SM$. To see that this is a bijection note that a function $A : SM \to \mathfrak{g}$ satisfying $V^2(A) + A = 0$ is a function that locally (in isothermal coordinates) can be written as $A(x, y, \theta) = a(x, y) \cos \theta + b(x, y) \sin \theta$ and we recover the element in $\Omega^1(M, \mathfrak{g})$ locally by setting $A = A_x dx + A_y dy$ where $A_x = ae^\lambda$ and $A_y = be^\lambda$. It is straightforward to check that this defines a global element in $\Omega^1(M, \mathfrak{g})$. Under this identification the star operator $\star : \Omega^1(M, \mathfrak{g}) \to \Omega^1(M, \mathfrak{g})$ is just $-V : A \to A$.

A Higgs field $\Phi : M \to \mathfrak{g}$ can also be regarded as a function $\Phi : SM \to \mathfrak{g}$ such that $V(\Phi) = 0$ and we denote this set as $C_0^\infty(SM, \mathfrak{g})$.

Recall from the introduction that a pair $(A, \Phi) \in A \times C_0^\infty(SM, \mathfrak{g})$ is said to be cohomologically trivial if there exists a smooth $u : SM \to G$ such that $C(x, v, t) = u(\phi_t(x, v))u^{-1}(x, v)$. Differentiating with respect to $t$ and setting $t = 0$ this is equivalent to

$$X(u) + (A + \Phi)u = 0.$$  

Let $\mathcal{P}_0$ be the set of all cohomologically trivial pairs, that is, the set of all $(A, \Phi) \in A \times C_0^\infty(SM, \mathfrak{g})$ such that there exists $u : SM \to G$ for which (3) holds.

Given a vector field $W$ in $SM$, let $G_W$ be the set of all $u : SM \to G$ such that $W(u) = 0$, i.e. first integrals of $W$. Note that $G_V$ is nothing but the group of gauge transformations of the trivial bundle $M \times \mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

We wish to understand $\mathcal{P}_0/G_V$. Now let $\mathcal{H}_0$ be the set of all pairs $f, \Psi : SM \to \mathfrak{g}$ such that

$$H(f) + VX(f) - [X(f), f] + \Psi = 0,$$

and there is $u : SM \to G$ such that $f = u^{-1}V(u)$. It is easy to check that $G_X$ acts on $\mathcal{H}_0$ by

$$(f, \Psi) \mapsto (a^{-1}f a + a^{-1}V(a), a^{-1}\Psi a).$$

where $a \in G_X$. 

**Theorem 5.1.** There is a 1-1 correspondence between $\mathcal{P}_0/G_V$ and $\mathcal{H}_0/G_X$.

*Proof.* Forward direction: a cohomologically trivial pair $(A, \Phi)$ comes with a $u$ such that $X(u) + (A + \Phi)u = 0$. Let us set $f := u^{-1}V(u)$ and $\Psi := u^{-1}\Phi u$. We need to check that $(f, \Psi) \in \mathcal{H}_0$, i.e., the pair $(f, \Psi)$ satisfies the PDEs:

\begin{align}
H(f) + V X(f) - [X(f), f] + \Psi &= 0, \\
V(\Psi) + [f, \Psi] &= 0 \tag{9}
\end{align}

Using $u$ we may define a connection on $SM$ gauge equivalent to $\pi^*A$ by setting $B := u^{-1}du + u^{-1}\pi^*Au$, where $\pi : SM \to M$ is the foot-point projection. Using $X(u) + (A + \Phi)u = 0$ we derive

\begin{align}
B(X) &= -u^{-1}\Phi u = -\Psi. \tag{11}
\end{align}

Note also that $B(V) = f$. Since $V(\Phi) = 0$, the equation $V(u\Psi u^{-1}) = 0$ gives

\begin{align}
V(u)\Psi u^{-1} + uV(\Psi)u^{-1} + u\Psi V(u^{-1}) &= 0. \tag{10}
\end{align}

Using that $-f = V(u^{-1})u$ we obtain

\begin{align}
V(\Psi) + [f, \Psi] &= 0
\end{align}

which is (10). To derive (9) we first note that since $\pi^*A$ is the pull-back of a connection on $M$, the curvature $F_B$ of $B$ must vanish when one of the entries is the vertical vector field $V$. Using that $F_B = dB + B \wedge B$ and (11) we compute

\begin{align}
0 = F_B(X, V) = dB(X, V) + [B(X), B(V)] = dB(X, V) - [\Psi, f].
\end{align}

But

\begin{align}
dB(X, V) &= XB(V) - VB(X) - B([X, V]) = XB(V) + V(\Psi) + B(H),
\end{align}

and combined with (10) this gives

\begin{align}
B(H) &= -XB(V) = -X(f). \tag{12}
\end{align}

We also compute

\begin{align}
0 = F_B(H, V) = dB(H, V) + [B(H), B(V)],
\end{align}

and

\begin{align}
dB(H, V) &= HB(V) - VB(H) - B([H, V]) = HB(V) - VB(H) - B(X),
\end{align}

hence

\begin{align}
HB(V) - VB(H) + [B(H), B(V)] = B(X). \tag{13}
\end{align}

Combining (12) and (13) gives:

\begin{align}
H(f) + V X(f) - [X(f), f] + \Psi &= 0
\end{align}

which is (9).

Backward direction: Given a pair $(f, \Psi)$ with $fu = V(u)$ satisfying (9) and (10) set $\Phi := u\Psi u^{-1}$ and $A := -X(u)u^{-1} - \Phi$. We need to check that $(A, \Phi) \in \mathcal{A}_0 \times C^\infty_0(SM, g)$, i.e. $V^2(A) = -A$ and $V(\Phi) = 0$. Checking that $V(\Phi) = 0$ is easy, simply use $V(\Psi) + [f, \Psi] = 0$ and $fu = V(u)$. After this, checking that $V^2(A) = -A$ is
completely analogous to a calculation done in the proof of [11, Theorem B] and so we omit the details.

Now there are two ambiguities here. Going forward, we may change $u$ as long as we solve $X(u) + (A + \Phi)u = 0$. This changes $(f, \Psi)$ by the action of $G_X$. Going backwards we may change $u$ as long as $fu = V(u)$, this changes $(A, \Phi)$ by a gauge transformation, i.e. an element in $G_V$.

\[ \square \]

**Remark 5.2.** Note that if the geodesic flow is transitive (i.e. there is a dense orbit) the only first integrals are the constants and thus $G_X = U(n)$ acts simply by conjugation. If $M$ is closed and of negative curvature, the geodesic flow is Anosov and therefore transitive.

**Remark 5.3.** There is an alternative way of writing equations (12) and (13) which reveals a bit of their structure.

A connection on $SM$ is determined as long as we specify the values of a $g$-valued 1-form $\tau$ on $SM$. Let $\tau$ be given by

\[ \tau(X) = 0, \quad \tau(H) = -X(f), \quad \tau(V) = f. \]

We compute the curvature $F_\tau$ of $\tau$ at $(H, V)$ and we find

\[ F_\tau(H, V) = H(f) + VX(f) - [X(f), f] \]

thus using (9) we derive

\[ (14) \quad \Psi = -F_\tau(H, V). \]

Recall that a connection induces a covariant derivative on endomorphisms and for $\tau$ we denote it by $D^\tau$. Hence we may write (10) as

\[ (15) \quad D^\tau_V(\Psi) = 0. \]

Thus equations (14) and (15) can be seen as the master equations for pairs. In fact there is obviously just one equation for $f$:

\[ (16) \quad D^\tau_V(F_\tau(H, V)) = 0, \]

and once $f$ is found we obtain $\Psi$ from (14).

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**6. THE BäCKLUND TRANSFORMATION FOR $SO(3)$-PAIRS**

In this section we restrict to the case in which the structure group is $SO(3)$.

Suppose there is a smooth map $b : SM \to SO(3)$ such that if we let $f := b^{-1}V(b)$ and

\[ -\Psi := H(f) + VX(f) - [X(f), f], \]

then $\Psi$ and $f$ are related by the PDE:

\[ (17) \quad V(\Psi) + [f, \Psi] = 0. \]
Then, by Theorem 5.1, the pair \((f, \Psi)\) determines a cohomologically trivial pair \((A, \Phi)\) with \(A + \Phi := -X(b)b^{-1}\) and \(\Phi = b\Psi b^{-1}\). Using that \([V, X] = H\) we derive:

\[
V(A)b = -VX(b) - X(b)V(b^{-1})b \\
= -XV(b) - H(b) + X(b)f \\
= -H(b) - bX(f)
\]

and thus

\[
\text{(18)} \quad -\star A = V(A) = -bX(f)b^{-1} - H(b)b^{-1}.
\]

Now suppose we are given a smooth function \(g : M \to \mathfrak{so}(3)\) with \(g^3 + g = 0\). By Corollary 2.9 there exists a smooth \(a : M \to SO(3)\) such that \(ag = V(a)\). In what follows we shall assume that \(g\) is not identically zero. The equality \(g^3 + g = 0\) means that \(g\) has norm one with respect to the canonical inner product \(\langle \cdot, \cdot \rangle\) in \(\mathfrak{so}(3)\) defined by \(\langle g, h \rangle = \text{trace}(gh^t)/2\).

Let us set \(u := ab : SM \to SO(3)\), \(F := u^{-1}V(u) = b^{-1}gb + f\) and

\[
-\Lambda := H(F) + VX(F) - [X(F), F].
\]

**Question.** When does \((F, \Lambda)\) satisfy (17)?

If it does, then it defines (via Theorem 5.1) a new cohomologically trivial pair given by

\[
A_g + \Phi_g = -X(ab)(ab)^{-1} = -X(a)a^{-1} + a(A + \Phi)a^{-1}, \\
\Phi_g = (ab)\Lambda(ab)^{-1},
\]

where \((A, \Phi)\) is the cohomologically trivial pair associated to \((f, \Psi)\).

Recall that the connection \(A\) defines a covariant derivative \(d_A g = dg + [A, g]\).

**Lemma 6.1.** \((F, \Lambda)\) satisfies (17) if and only if

\[
(19) \quad -\star d_A g = [d_A g, g].
\]

**Proof.** Starting with \(F = b^{-1}gb + f\) and using that \(A + \Phi = -X(b)b^{-1} = bX(b^{-1})\) we compute

\[
X(F) = b^{-1} ([A + \Phi, g] + X(g)) b + X(f).
\]

Similarly, using \(H(b) = -(V(A))b - bX(f)\) (cf. \(\text{(18)}\)) we find

\[
H(F) = b^{-1} ([V(A), g] + H(g)) b + [X(f), b^{-1}gb] + H(f).
\]

Now we compute \(VX(F)\); here we use that \(V(g) = 0\). We obtain

\[
VX(F) = [b^{-1}([A + \Phi, g] + X(g))b, f] + b^{-1}([V(A), g] + VX(g))b + VX(f).
\]

The last term we need in order to compute \(\Lambda\) is:
\[ [X(F), F] = b^{-1}[[A + \Phi, g] + X(g), g] b + b^{-1}([A + \Phi, g] + X(g)) b + [X(f), b^{-1}g b] + [X(f), f]. \]

Note that \( X(g) = dg, H(g) = -\star dg \) and \( V(A) = -\star A \). Putting everything together, using the definition of \( \Psi \) in terms of \( f \) and simplifying we obtain

\[ -\Lambda = b^{-1} (\star dA g, g) + \star dA g - \Phi \]

We can simplify this further as

\[ -\Lambda = b^{-1} (\star dA g, g) - \Phi \]

Let us set

\[ T := [dA g, g] + \star dA g + \Phi, \]

then we see that \( (F, \Lambda) \) satisfies \( \mathcal{L} \) iff \( T \) satisfies

\[ V(T) + [g, T] = 0. \]

Recall that the Lie bracket in \( \mathfrak{so}(3) \) satisfies \([a, [b, c]] = b\langle a, c \rangle - c\langle a, b \rangle\). Thus since \( g \) has norm one (we are assuming \( g^3 + g = 0 \) with \( g \) non-zero) we see that

\[ [[\Phi, g], g] = -\Phi + g\langle g, \Phi \rangle, \]

and since \( \langle dA g, g \rangle = 0 \) we also see that

\[ [g, [dA g, g]] = dA g. \]

Therefore

\[ V(T) = -[\star dA g, g] + 2dA g \]

\[ [g, T] = dA g + 2[\star dA g]. \]

Thus \( (F, \Lambda) \) satisfies \( \mathcal{L} \) iff \( g \) satisfies:

\[ -[\star dA g, g] + dA g = 0. \]

and applying \( \star \) we see that the last equation is equivalent to \( \mathcal{L} \).

From the proof above we can derive a fairly explicit form for \( \Phi_g \) and \( A_g \). Since \( \Phi_g = (ab)\Lambda(ab)^{-1} \) and

\[ b\Lambda b^{-1} = [dA g, g] + g\langle g, \Phi \rangle + 2\star dA g \]

we obtain:

\[ \Phi_g = a([dA g, g] + g\langle g, \Phi \rangle + 2\star dA g) a^{-1}, \]

and using that \([dA g, g] = -\star dA g\) we have

\[ \Phi_g = a(g\langle g, \Phi \rangle + \star dA g) a^{-1}. \]

And from this and \( A_g + \Phi_g = -X(ab)(ab)^{-1} = -X(a)a^{-1} + a(A + \Phi)a^{-1} \) we derive the following formula for \( A_g \):

\[ A_g = -X(a)a^{-1} + a(A + \Phi - g\langle g, \Phi \rangle - \star dA g) a^{-1}. \]
6.1. New features. In this subsection we explain the relationship between the transformation we just introduced and the one described in [12] for cohomologically trivial connections (no Higgs field present). We also explain how to obtain cohomologically trivial $SU(2)$-pairs.

**Lemma 6.2.** Let $q := aga^{-1}$. Then $V(q) = 0$ and $d_{A_q}q = a[\Phi, g]a^{-1} = [a\Phi a^{-1}, q]$. Moreover

$$d_{A_q}q = -[*d_{A_q}g, q].$$

**Proof.** Since $V(g) = 0$ we have $V(q) = V(a)ga^{-1} + agV(a^{-1})$. But $ag = V(a)$ and taking transposes $-ga^{-1} = V(a^{-1})$ (recall that $g^t = -g$ and $a^{-1} = a^t$). Therefore $V(q) = ag^2a^{-1} - ag^2a^{-1} = 0$.

Since $V(q) = 0$ we may identify $dq$ with $X(q)$, so we compute using (21):

$$dq = X(q) = X(a)ga^{-1} + aX(g)a^{-1} + agX(a^{-1})$$

$$= X(a)a^{-1}q + adg a^{-1} + qaX(a^{-1})$$

$$= [X(a)a^{-1}, q] + adg a^{-1}$$

$$= [-A_g, q] + a[A + \Phi - *d_{A_g}g, g]a^{-1} + a d g a^{-1}$$

$$= [-A_g, q] + a(d_{A_g} - [*d_{A_g}g])a^{-1} + a[\Phi, g]a^{-1}$$

and the first part of the lemma follows from the fact that $d_{A_g} - [*d_{A_g}g, g] = 0$.

To prove the equation displayed in the statement of the lemma we compute using that $d_{A_q}q = a[\Phi, g]a^{-1}$ and $ag = V(a)$:

$$-[*d_{A_q}g, q] = V(d_{A_q}q) = V(a)[\Phi, g]a^{-1} + a[\Phi, g]V(a^{-1})$$

$$= qa[\Phi, g]a^{-1} - a[\Phi, g]a^{-1}q$$

$$= [q, [a\Phi a^{-1}, g]]$$

$$= a\Phi a^{-1} - q\langle q, a\Phi a^{-1} \rangle,$$

where in the last equation we used that $|q| = 1$ and that $[a, [b, c]] = b\langle a, c \rangle - c\langle a, b \rangle$. Thus

$$-[*d_{A_q}g, q] = [a\Phi a^{-1}, q] = d_{A_q}q$$

as desired. □

It follows from the lemma that if we let $g' := -q = -aga^{-1}$, then $g'$ satisfies equation (19). Also, since $a^{-1}g' = -ga^{-1} = V(a^{-1})$ it follows that if we run the Bäcklund transformation on $(A_g, \Phi_g)$ using $g'$ and $a^{-1}$ we get back to the pair $(A, \Phi)$.

There is an interesting case which arises from the lemma. Suppose we start with a cohomologically trivial pair of the form $(A, 0)$, i.e. $\Phi = 0$ and we run the Bäcklund transformation with $(g, a)$. Using (20) we see that

$$\Phi_g = a(*d_A g)a^{-1}.$$
But if $\Phi = 0$, then $d_A q = 0$ and thus $q$ satisfies (19). Choose $b : SM \to SO(3)$ such that $bq = V(b)$ and run the Bäcklund transformation again on the pair $(A_q, \Phi_q)$ using $(q, b)$. We obtain a new cohomologically trivial pair $(A_q, \Phi_q)$. Using (20) we see that

$$\Phi_q = b(q(\Phi_q) + \ast d_A q)b^{-1}.$$  

But $d_A q = 0$ and

$$\langle q, \Phi_q \rangle = \langle aga^{-1}, a(\ast d_A g)a^{-1} \rangle = \langle g, \ast d_A g \rangle = 0$$

and hence $\Phi_q = 0$. Thus doing this special 2-step Bäcklund transformation on the cohomologically trivial connection $A$ produces a new cohomologically trivial connection $A_q$.

We claim that this 2-step $SO(3)$ Bäcklund transformation coincides with the $SU(2)$ Bäcklund transformation for connections introduced in [12]. In this way we have introduced a "square root" which requires the intermediate step to have a non-trivial Higgs field.

Here is a way to see this. The 2-step process is implemented by $c := a_q a$ so let us compute

$$c^{-1} V(c) = a^{-1} a_q^{-1} (V(a_q) a + a_q V(a)) = a^{-1} q a + g = 2g.$$  

Consider the isomorphism $\ell : \mathfrak{so}(3) \to \mathfrak{su}(2)$ given by

$$\ell(\begin{pmatrix} 0 & t & x \\ -t & 0 & y \\ -x & -y & 0 \end{pmatrix}) \mapsto \frac{1}{2} \begin{pmatrix} -it & -x - iy \\ x - iy & it \end{pmatrix}.$$  

Given $g : M \to \mathfrak{so}(3)$ with norm one, it is easy to check that the map $h : M \to \mathfrak{su}(2)$ defined as $h := \ell \circ 2g$ has the property that $h^2 = -\text{Id}$. Clearly we can reverse this process and obtain $g$ given $h$. Now observe that $g$ satisfies $\ast d_A g = [d_A g, g]$ if and only if $h$ satisfies $-2 \ast d_{\ell \circ h} h = [d_{\ell \circ h} h, h]$. The last equation is precisely what is needed to run the $SU(2)$ Bäcklund transformation on cohomologically trivial connections (i.e. with $\Phi = 0$).

We conclude this subsection with the following remark. Note that a cohomologically trivial $SO(3)$-pair defines a cohomologically trivial $SU(2)$-pair if and only if there is a trivializing function $u : SM \to SO(3)$ such that the induced homomorphism $u_* : \pi_1(SM) \to \pi_1(SO(3)) = \mathbb{Z}_2$ is trivial. Recall that multiplying a trivializing function on the left by a gauge transformation of $M$ gives a trivializing function of a gauge equivalent pair. We claim that if we start with $u$ such that $u_* = 0$ (e.g. we start with $(A, \Phi) = 0$) and we apply the $SO(3)$ Bäcklund transformation an even number of times we obtain a cohomologically trivial $SU(2)$-pair. Observe first that

$$[0, 2\pi] \ni \theta \mapsto \text{Id} + g^2 + \sin \theta g - \cos \theta g^2$$

where $g$ has norm one is an explicit non-trivial loop in $SO(3)$, thus the map $a$ with $g = a^{-1} V(a)$ is such that $a_*$ is non-zero. In fact since $(ab)_* = a_* + b_* \pmod{2}$ we see that after applying two Bäcklund transformations with $a$ and $b$, the morphism $(ab)_*$ has the element generated by the fibres of $SM$ in its kernel and thus it induces a morphism $\rho : \pi_1(M) \to \mathbb{Z}_2$. But it is always possible to choose a smooth gauge
map \( r : M \to SO(3) \) such that \( r_* = \rho \). Indeed, express the surface \( M \) as a wedge of circles to which a 2-cell is attached and use for example [5, Lemma 4.3] to obtain a continuous \( r : M \to SO(3) \) with \( r_* = \rho \). Now approximate the continuous map by a smooth one. Hence there is always a smooth \( r : M \to SO(3) \) such that \((rab)_* = 0\). Hence after applying an even number of \( SO(3) \) Bäcklund transformations we obtain a trivializing function which lifts to \( SU(2) \).

6.2. Holomorphic interpretation of (19). We will now rephrase equation (19) in terms of holomorphic line bundles. There are two ways in which we can do this. First consider as above the isomorphism \( \ell : \mathfrak{so}(3) \to \mathfrak{su}(2) \) and \( h : M \to \mathfrak{su}(2) \) defined as \( h := \ell \circ g \) satisfying \( h^2 = -\text{Id} \). We pointed out already that \( g \) satisfies \(-\star d_A g = [d_A g, g] \) if and only if \( h \) satisfies \(-2 \star d_{\ell \circ A} h = [d_{\ell \circ A} h, h] \). Now, the meaning of this last equation was analysed in detail in [12]. If we let \( L \) denote the line bundle given by the eigenspace associated with the eigenvalue \( i \) of \( h \), it turns out (cf. [12, Lemma 4.3]) that \( h \) satisfies \(-2 \star d_{\ell \circ A} h = [d_{\ell \circ A} h, h] \) if and only if the line bundle \( L \) is holomorphic with respect to the complex structure in \( M \times \mathbb{C}^2 \) induced by \( \ell \circ A \) (we will prove something quite similar below). Thus maps \( g : M \to \mathfrak{so}(3) \) with norm one satisfying (19) are in 1-1 correspondence with holomorphic line subbundles of \( M \times \mathbb{C}^2 \) with respect to the complex structure induced by \( \ell \circ A \). It is well known that holomorphic line subbundles always exist (there are always non-zero meromorphic sections).

The second way to interpret (19) is closely related and is needed to complete the classification in the next section. For this we consider the monomorphism \( SO(3) \hookrightarrow SU(3) \) and the corresponding monomorphism of Lie algebras \( \mathfrak{so}(3) \hookrightarrow \mathfrak{su}(3) \). We will thus think of our \( SO(3) \)-connection also as an \( SU(3) \)-connection.

Recall that an \( SU(3) \)-connection \( A \) induces a holomorphic structure on the trivial bundle \( M \times \mathbb{C}^3 \) and on the endomorphism bundle \( M \times \mathbb{M}_3(\mathbb{C}) \). We have an operator \( \bar{\partial}_A = (d_A - i \star d_A)/2 = \bar{\partial} + [A_{-1}, \cdot] \) acting on sections \( f : M \to \mathbb{M}_3(\mathbb{C}) \).

Set \( \pi := -g(g + \text{Id})/2 \) and \( \pi^\perp = \text{Id} + g(g + \text{Id})/2 \) so that \( \pi + \pi^\perp = \text{Id} \). For each \( x \in M \), the map \( \pi(x) \) is the Hermitian orthogonal projection over the 1-dimensional subspace \( E_i(x) \) given by the eigenvectors of \( g(x) \) with eigenvalue \( i \). The map \( \pi^\perp(x) \) is the Hermitian orthogonal projection onto \( \text{Ker}(g(x)) \oplus E_{-i}(x) \).

**Lemma 6.3.** Let \( g : M \to \mathfrak{so}(3) \) be a smooth map with \( \langle g(x), g(x) \rangle = 1 \) for all \( x \in M \). The following are equivalent:

1. \(-\star d_A g = [d_A g, g] \);
2. \( \bar{\partial}_A g = i[\bar{\partial}_A g, g] \);
3. \( E_i \) is a \( \partial_A \)-holomorphic line bundle;
4. \( \pi^\perp \bar{\partial}_A = 0 \).

**Proof.** Suppose that (1) holds. Apply \( \star \) to obtain: \( d_A g = [\star d_A g, g] \). Thus \( d_A g - i \star d_A g = i[\bar{\partial}_A g, g] \). In other words \( \bar{\partial}_A g = i[\bar{\partial}_A g, g] \) which is (2). Conversely, if (2) holds we recover (1) just by taking real and imaginary parts.
Let us show that (1) and (4) are equivalent. We make some preliminary observations. Since \( g^3 + g = 0 \) we derive
\[
(d_A g)g^2 + d_A g + g(d_A g)g + g^2(d_A g) = 0
\]
which together with \([g, [d_A g, g]] = d_A g\) shows that \( g(d_A g)g = 0 \) and
\[
-d_A g = (d_A g)g^2 + g^2(d_A g).
\]
Consider the equation \( \pi^\perp \partial_A \pi = 0 \). It is equivalent to saying that that the image of \( \bar{\partial}_A \pi \) is contained in \( \ker \pi^\perp = \text{Im} \pi \). Since the image of \( \pi \) is \( E_i \) the equation \( \pi^\perp \partial_A \pi = 0 \) is equivalent to
\[
g(\bar{\partial}_A \pi) = i \bar{\partial}_A \pi.
\]
But
\[
\bar{\partial}_A \pi = -\frac{1}{2}((\bar{\partial}_A g)g + g(\bar{\partial}_A g) + i \bar{\partial}_A g)
\]
and thus (4) is equivalent to
\[
g(\bar{\partial}_A g)g + g^2(\bar{\partial}_A g) + \bar{\partial}_A g = i (\bar{\partial}_A g)g.
\]
Using the observations above this can be further simplified to
\[
(22) \quad i (\bar{\partial}_A g)g = -(\bar{\partial}_A g)g^2.
\]
We claim that (22) is equivalent to (1). First note that (22) is equivalent to
\[
(23) \quad -(\star d_A g)g = (d_A g)g^2.
\]
Clearly one can go from (1) to (23) by multiplying (1) by \( g^2 \) and using that \( g^3 + g = 0 \) and \( g(d_A g)g = 0 \). To go from (23) to (1) take transposes in (23) to obtain
\[
g(\star d_A g) = g^2(d_A g)
\]
and if we add this to (23) we get
\[
-d_A g = (d_A g)g^2 + g^2(d_A g) = -[\star d_A g, g]
\]
which is (1).

Finally we will prove that (3) and (4) are equivalent. Using the condition \( \pi^2 = \pi \), we see that \( \bar{\partial}_A \pi = (\bar{\partial}_A \pi)\pi + \pi(\bar{\partial}_A \pi) \). Recall that \( \pi^\perp \partial_A \pi = 0 \) is equivalent to saying that the image of \( \bar{\partial}_A \pi \) is contained in \( \ker \pi^\perp = \text{Im} \pi \) which in turn is equivalent to \( \pi(\bar{\partial}_A \pi) = \bar{\partial}_A \pi \). Hence \( \pi^\perp \partial_A \pi = 0 \) if and only if \( (\bar{\partial}_A \pi)\pi = 0 \).

The line bundle \( E_i \) is holomorphic iff given a local section \( \xi \) of \( E_i \), we have \( \bar{\partial}_A \xi \in E_i \). Applying \( \bar{\partial}_A \) to \( \pi \xi = \xi \) we see that \( \bar{\partial}_A \xi \in E_i \) iff \( (\bar{\partial}_A \pi)\xi = 0 \). Clearly, this happens iff \( (\bar{\partial}_A \pi)\pi = 0 \) and thus (3) holds iff (4) holds.

The next theorem summarises the Bäcklund transformation introduced in this section and it follows directly from Lemma 6.1 and Theorem 5.1.
Theorem 6.4. Let \((A, \Phi)\) be a cohomologically trivial pair and let \(g : M \to \mathfrak{so}(3)\) be a smooth map with \(|g(x)| = 1\) for all \(x \in M\) such that \(-\ast d_A g = [d_A g, g] \). Consider \(A : SM \to SO(3)\) with \(g = a^{-1}V(a)\) as given by Corollary \(\text{Corollary 2.9}\). Then

\[
\Phi_g = a(g, g) + \ast d_A g a^{-1},
\]

\[
A_g = -X(a) a^{-1} + a(A + \Phi - g, g) - \ast d_A g a^{-1}
\]

defines a cohomologically trivial pair \((A_g, \Phi_g)\).

7. The classification result

Let \((A, \Phi)\) be a cohomologically trivial pair with a trivializing function of degree \(N \neq 0\). We will show that \((A, \Phi)\) can be obtained as a Bäcklund transformation of a cohomologically trivial pair which admits a trivializing function of degree \(N - 1\). Thus by induction we end up showing that all cohomologically trivial pairs which admit a trivializing function with finite Fourier series are Bäcklund transformations of cohomologically trivial pairs of degree zero, i.e., those pairs \((A, \Phi)\) with \(\Phi = 0\) and \(A\) gauge equivalent to the trivial connection. This will be our classification result and its consequence for negative curvature is Theorem \(7.3\) below.

Let \((A, \Phi)\) be a cohomologically trivial pair with \(A + \Phi = -X(b)b^{-1}\) and \(f = b^{-1}V(b)\), where \(b : SM \to SO(3)\) is a trivializing function. We think of \(b\) as a map \(b : SM \to SO(3) \hookrightarrow SU(3) \hookrightarrow \mathbb{M}_3(\mathbb{C})\) and as such we make the following:

Assumption. Suppose \(b\) has a finite Fourier expansion, i.e., \(b = \sum_{k=-N}^{k=N} b_k\), where \(N \geq 1\). By Theorem \(4.1\) we know that this holds if \(M\) has negative curvature.

Let us assume also that \(N\) is the degree of \(b\) and thus both \(b_N\) and \(b_{-N} = \bar{b}_N\) are not identically zero.

The orthogonality condition \(bb^t = b\bar{b} = \text{Id}\) implies that \(b_N b_N^t = b_N b_N = 0\). Since \(b_N^t\) and \(b_N^t\) must have the same rank, \(\text{Im} b_N \subset \ker b_N^t\) and \(\text{Im} b_N^t \subset \ker b_N\), it follows that \(\dim \text{Im} b_N \leq 1\) and \(\dim \ker b_N \geq 2\) with a similar statement holding for \(b_N^t\). Since \(b_N\) is not identically zero it follows that the rank of \(b_N\) is one on an open set, which, as we will see shortly, must be all of \(M\) except for perhaps a finite number of points.

Consider now a fixed vector \(\xi \in \mathbb{C}^3\) such that \(s(x, v) := b_{-N}(x, v)\xi \in \mathbb{C}^3\) is not zero identically. Clearly \(s\) can be seen as a section of \((M \times \mathbb{C}^3) \otimes K_{-N}^\infty\). We may write \(b_{-N}\) in local isothermal coordinates as \(b_{-N} = he^{-iN\theta}\), using the notation from Section \(3\). We can thus write \(s\) locally as \(s = e^{N\lambda}h \xi(dz)^N\).

Lemma 7.1. The local section \(e^{-2N\lambda}s\) is \(\bar{\partial}_A\)-holomorphic.

Proof. Using the operators \(\mu_{\pm}\) introduced in Section \(3\) we can write \(X(b) + (A + \Phi)b = 0\) as

\[
\mu_+(b_{k-1}) \mu_-(b_{k+1}) \Phi b_k = 0
\]
for all \(k\). This gives \(\mu_+(b_N) = \mu_-(b_N) = 0\). But \(\mu_-(b_N) = 0\) is saying that \(e^{-2N\lambda}s\) is \(\bar{\partial}_A\)-holomorphic. Indeed, using (5), we see that \(\mu_-(b_N) = 0\) implies

\[
\bar{\partial}(he^{-\lambda}) + A\bar{z}he^{-\lambda} = 0
\]

which in turn implies

\[
\bar{\partial}(e^{-\lambda}h\xi) + A\bar{z}e^{-\lambda}h\xi = 0.
\]

This equation says that \(e^{-2N\lambda}s = e^{-\lambda}h\xi(d\bar{z})^N\) is \(\bar{\partial}_A\)-holomorphic. Indeed, using (5), we see that \(\mu_-(b_N) = 0\) implies

\[
\bar{\partial}(he^{-\lambda}) + A\bar{z}he^{-\lambda} = 0
\]

which in turn implies

\[
\bar{\partial}(e^{-\lambda}h\xi) + A\bar{z}e^{-\lambda}h\xi = 0.
\]

This equation says that \(e^{-2N\lambda}s = e^{-\lambda}h\xi(d\bar{z})^N\) is \(\bar{\partial}_A\)-holomorphic. □

The section \(s\) spans a line bundle \(L\) over \(M\) which by the previous lemma is \(\bar{\partial}_A\)-holomorphic. The section \(s\) may have zeros, but at a zero \(z_0\), the line bundle extends holomorphically. Indeed, in a neighbourhood of \(z_0\) we may write \(e^{-N\lambda(z)}s(z) = (z - z_0)^k w(z)\), where \(w\) is a local holomorphic section with \(w(z_0) \neq 0\). The section \(w\) spans a holomorphic line subbundle which coincides with the one spanned by \(s\) off \(z_0\). Therefore \(L\) is a \(\bar{\partial}_A\)-holomorphic line bundle that contains the image of \(b_N\). We summarise this in a lemma (recall that we are assuming \(N \neq 0\)):

**Lemma 7.2.** The line bundle \(L\) determined by the image of \(b_N\) is \(\bar{\partial}_A\)-holomorphic.

We now wish to use the line bundle \(L\) to construct an appropriate \(g : M \rightarrow \mathfrak{so}(3)\) with norm one such that when we run the Bäcklund transformation from the previous section we obtain a cohomologically trivial pair of degree \(\leq N - 1\).

We would also like to find \(a : SM \rightarrow SO(3)\) such that \(u := ab\) has degree \(\leq N - 1\). The map \(a\) should be related to \(g\) by \(ag = V(a)\) and \(a = a_{-1} + a_0 + a_1\). For \(u = ab\) to have degree \(\leq N - 1\) we need

\[
a_1b_{N-1} + a_0b_N = 0,
\]

\[
a_1b_N = 0.
\]

Note that by conjugating the previous two equations we obtain the corresponding relations involving \(b_N\) which ensure that \(u\) has degree \(\leq N - 1\).

Write \(b_N = C + iD\), where \(C\) and \(D\) are \(3 \times 3\) real matrices. Since \(\text{Ker}b_N\) has (complex) dimension two, there must exist a real vector \(0 \neq x \in \mathbb{R}^3 \subset \mathbb{C}^3\) such that \(C^tx = D^tx = 0\). The relations \(b_Nb_N^t = b_N^tb_N = 0\) imply that for any real vector \(y \in \mathbb{R}^3\) we have:

\[
|C^ty| = |D^ty|,
\]

\[
|Cy| = |Dy|
\]

\[
\langle Cy, Dy \rangle = 0.
\]

Consider now \(y \in \mathbb{R}^3\) such that \(Cy \neq 0\). Then \(\{x, Cy, Dy\}\) is an orthogonal basis of \(\mathbb{R}^3\) such that \(|Cy| = |Dy|\). Define \(g \in \mathfrak{so}(3)\) with norm one as follows:

\[
g(x) = 0,
\]

\[
g(Cy) = Dy,
\]

\[
g(Dy) = -Cy.
\]
Obviously the eigenspace $E_i$ of $g$ corresponding to the eigenvalue $i$ is spanned by $Cy - iDy$ and $E_{-i}$ is spanned by $Cy + iDy$. Thus $\text{Im} \, b_N = E_{-i}$ and $\text{Ker} \, b_N^t = E_{-i} \oplus \text{Ker}(g)$ and $L = E_i = \text{Im} \, b_{-N}$.

For this choice of $g$, Corollary 2.9 gives a smooth $a : SM \to SO(3)$ with $ag = V(g)$ and such that

\[
\begin{align*}
\text{Ker} \, a_0 &= E_i \oplus E_{-i}, \\
\text{Ker} \, a_1 &= \text{Ker}(g) \oplus E_{-i}, \\
\text{Ker} \, a_{-1} &= \text{Ker}(g) \oplus E_i.
\end{align*}
\]

Hence $\text{Im} \, b_N \subset \text{Ker} \, a_1$ and $\text{Im} \, b_N \subset \text{Ker} \, a_0$ and thus $a_1 b_N = a_0 b_N = 0$. This gives (24) and to get (25) we need to show that $a_1 b_{N-1} = 0$.

Since $b \in SO(3)$, $b_N^t b_{N-1} + b_{N-1}^t b_N = 0$ which says that the complex matrix $b_N^t b_{N-1}$ is antisymmetric. Since the kernel of $b_N$ is two dimensional, it follows that the kernel of $b_N^t b_{N-1}$ is at least two dimensional which combined with the fact that it is antisymmetric forces $b_N^t b_{N-1} = 0$. In other words $\text{Im} \, b_{N-1} \subset \text{Ker} \, b_N^t = \text{Ker} \, a_1$ which gives $a_1 b_{N-1} = 0$.

Finally, Lemmas 7.2 and 6.3 tell us that $- \star d_A g = [d_A g, g]$ and therefore by Theorem 6.4 $a$ gives rise to a cohomologically trivial pair. Combining this with Theorem 4.1 we have proved:

**Theorem 7.3.** Let $M$ be a closed orientable surface of negative curvature. Then any transparent $SO(3)$-pair can be obtained by successive applications of Bäcklund transformations as described in Theorem 6.4.

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