Oscillations in turbulence-condensate system

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\textsuperscript{Dated: February 26, 2013}

We consider developed turbulence in the Gross-Pitaevsky model where condensate appears due to an inverse cascade. Despite being fully turbulent, the system demonstrates non-decaying periodic oscillations around a steady state, when turbulence and condensate periodically exchange a small fraction of waves. We show that these collective oscillations are not of a predator-prey type, as was suggested earlier; they are due to phase coherence and anomalous correlations imposed by the condensate.

PACS numbers: 47.27.Gs; 03.75.Hh; 42.65.Sf

Understanding the interaction of turbulence and a coherent flow is an important problem in turbulence studies in fluid mechanics and beyond, both from fundamental and practical perspectives. In fluids, coherent flows are system-size vortices or zonal flows of different profiles \cite{1,2}, which are known to diminish the turbulence level, change its nature, and make its statistics more non-Gaussian \cite{1,3}. Here we consider arguably the simplest case of a turbulent system with a condensate: the coherent part is expected to be a constant and turbulence to consist of weakly interacting waves.

The Nonlinear Schrödinger equation (NSE), also known as the Gross-Pitaevsky equation, provides a universal description of the evolution of any nonlinear, spectrally narrow wave packet \cite{4,5}:

\begin{equation}
\imath \dot{\psi} + \nabla^2 \psi - |\psi|^2 \psi = 0. \tag{1}
\end{equation}

As such, it can provide a model of nonlinear behavior in a wide range of physical systems, from locally interacting bosons to plasmas and fluids. The addition of damping and driving causes the NSE to exhibit turbulence, and its universality makes it a major subject of interest in the study of wave turbulence \cite{5,9}. Because the equation conserves both total energy and wave action,

\begin{align*}
\mathcal{H} &= \int \left( |\nabla \psi|^2 + \frac{1}{2} |\psi|^4 \right) \, d\mathbf{r}, \quad \mathcal{N} = \int |\psi|^2 \, d\mathbf{r},
\end{align*}

it has the potential to form both a direct and inverse turbulent cascade. In a system with external forcing, the inverse cascade leads to the growth of a single coherent mode known as the spectral condensate, which comes to dominate the dynamics of the system. Understanding the interaction between a large condensate and other spatial modes of a system governed by the NSE is crucial to a deeper understanding of wave turbulence in such a system.

Numerical simulations of wave turbulence in the NSE suggest the existence of collective oscillations of the turbulence-condensate system \cite{7,10}. During these oscillations, a small fraction of wave action is periodically converted from the condensate to the turbulent part of the spectrum, with total wave action unchanged. Such oscillations were predicted to take place when broad turbulent spectra coexist with a sharp spectral peak \cite{11}. For three-wave interaction, a simple model of a predator-prey type that describes the evolution of the total numbers of waves in the two groups has the form:

\begin{align*}
dN_0/\, dt &= -bN_0 + N_0n, \\
dn/\, dt &= \gamma n - N_0n,
\end{align*}

which gives the oscillations with the frequency $\sqrt{b} = \sqrt{nN_0}$ \cite{11}. For the oscillations in NSE with a condensate a similar model was suggested \cite{7}:

\begin{align*}
dN_0/\, dt &= -bN_0 + N_0n^2, \\
dn/\, dt &= \gamma n - N_0n.
\end{align*}

The most evident defect of this model is that it does not conserve the total number of waves. Moreover, its predictions for the steady-state values and the frequency of oscillations are in disagreement with the data, see \cite{12}. It was also pointed out in \cite{12} that the account of phase coherence (or anomalous correlations) is needed to describe the oscillations (for the account of anomalous correlations, see also \cite{13}). Unfortunately, the model with anomalous correlations was not treated properly in \cite{12}. Here we derive the simplest model of this type, based on anomalous correlations in the three-wave system of two counter-propagating waves interacting with the condensate. We study the model analytically and compare the results with the oscillations of individual modes extracted from the direct numerical simulations of the turbulence in the NSE. We demonstrate that the three-wave model provides reasonable explanation for collective oscillations.

To start, we show that an important element of turbulence against the background of a condensate must be anomalous correlations, i.e. phase coherence between waves running in opposite directions. Let us denote $N = |\psi|^2$ and $N_0 = |\psi_0|^2$. The simplest condensate is a spatially-uniform field, $\Psi = \sqrt{N_0} \exp(-iN_0t)$, which is an exact solution of (1). Small over-condensate fluctuations satisfy

\begin{equation}
\imath \dot{\psi}_k = (k^2 + 2N_0)\psi_k + \Psi^2 \psi^*_{-k}. \tag{2}
\end{equation}

This equation gives the Bogolyubov dispersion relation,

\begin{equation}
\Omega_k^2 = 2N_0k^2 + k^4, \tag{3}
\end{equation}

for a pair of counter-propagating waves $\psi_{\pm k} \propto \exp(-iN_0t \mp i\Omega_k t)$. 

\[\int (\nabla \psi^2 + \frac{1}{2} |\psi|^4) \, d\mathbf{r} \quad \mathcal{N} = \int |\psi|^2 \, d\mathbf{r},\]

\[\{ |\nabla \psi|^2 + \frac{1}{2} |\psi|^4 \} \, d\mathbf{r}, \quad \int |\psi|^2 \, d\mathbf{r},\]
To look into the effective dynamics of both the number of waves and the condensate, let us assume for simplicity that the condensate interacts with only two counterpropagating waves with amplitudes

\[ \psi_{\pm k} = \sqrt{n} \exp(-i N_0 t \pm i kx + i\phi_{\pm k}) \]

and phase difference \( \theta = 2\phi_0 - \phi_k - \phi_{-k} \).

\[ H = 2k^2 n + \frac{1}{2} N^2 + 2n(N - 2n)(1 + \cos \theta) + n^2. \]  

That gives the equations [4]:

\[ \dot{n} = 2n(N - 2n) \sin \theta, \]  
\[ \dot{\theta} = 2k^2 + 2(N - 4n)(1 + \cos \theta) + 2n. \]  

Combining (4) and (5) one can obtain the equation for wave amplitude, \( \dot{n} = \sqrt{f(n)} \), where \( f(n) = 7n^4 + (12k^2 - 4N)n^3 + (3N^2 - 6H - 4k^2 - 8Nk^2)n^2 + (4Hk^2 + 4HN - 2k^2 N^2 + (H^2 - N^2 - \frac{N^2}{4}) \). The solution of this equation can be expressed via the elliptic functions:

\[ n(t) = n_0 + \frac{1}{4} f'(n_0) (P(t, g_2, g_3) - \frac{1}{27} f''(n_0))^{-1}. \]  

Here \( n_0 = -k^2 + \sqrt{H + k^4 - \frac{N^2}{4}} \) is the solution of \( f(n) = 0 \), \( P \) is the Weierstrass elliptic function, and \( g_2 \) and \( g_3 \) are two invariants of \( P \) determined by coefficients in \( f(n) \).

The typical solutions of system [5]-[6] are shown in Fig. [1] for a wide range of initial conditions. Motivated by application with very large condensate, we are mostly interested in limit of \( n/N \ll 1 \) (solid line in Fig. [1]). This limit is characterized by longer periods of oscillations, cusped shape of \( a(t) \) curves (where \( a \equiv \sqrt{n} \)), and by open trajectories in the phase space. The system spends most of its time around \( \theta = \pi \) state, avoiding both stable points: \( \theta = 0, n = (4N + k^2)/14 \) with its unrealistically high \( n/N \) ratio, and the unphysical \( \theta = \pi, n = \frac{1}{2}k^2 \).

In the limit of \( n \ll N \), the system [5]-[6] reduces to

\[ \dot{n} = 2nN \sin \theta, \quad \dot{\theta} = 2k^2 + 2N(1 + \cos \theta) \]

resulting in

\[ n(t) = n(0) \left( 1 + \frac{2N}{k^2} \sin^2 \Omega_k t \right), \]  
\[ \theta(t) = 2 \arctan \left( \frac{\Omega_k}{k^2} \tan \Omega_k t \right), \]  
\[ n(\theta) = n(0) \left( \frac{2N + k^2}{N(1 + \cos \theta + k^2)} \right). \]

Here, \( \Omega_k = \sqrt{2Nk^2 + k^4} \) is the Bogolubov frequency, and \( n(0) = n(\theta = 0) \) is the constant of integration. The second constant of integration shifts the solution in time; it is selected so that \( \theta(0) = 0 \). The solution [7]-[9] has the frequency of oscillations \( 2\Omega_k \), the stair-like time dependence of phase, and the cusped shape of amplitude, \( a = \sqrt{n} \propto |\sin \Omega_k t| \), as suggested by the numerical solution of [5]-[6] shown in Fig. [1]. Next, we demonstrate that \( n \ll N \) approximation is well justified for the levels of condensate typical for NSE turbulence.

We now compare the solution of the ODE model [5]-[6] with the results of numerical simulations of developed turbulence in nonlinear Schrödinger equation [1]. Our numerical simulations, based on the 4th order fully

\[ n_k \]  
\[ |\theta_k - \pi| \]

FIG. 2: (Color online) Spectrum of NSE turbulence \( n_k \) (left) and phase difference \( \theta_k \) (right), \( N = 3600 \).
dealiased split-step method \[14\] \[15\], are set up similarly to \[8\] and are described in detail in \[10\]. Simulation are done in periodic \(2\pi \times 2\pi\) domain. Extra terms are added to Eq. (1) to model forcing — the large scale multiplicative pumping and small scale damping. In simulations where thermal equilibrium is used as initial conditions, the pumping is needed to gradually raise the wave action and develop the condensate.

In such simulations, as the wave action increases with time, the system undergoes the series of phase transitions \[10\]. The "phases" are distinguished by different symmetries of the spectrum and by different spatial patterns in over-condensate fluctuations observed on small scales. The shape of the spectrum changes from radially symmetric to two-petal spectrum at \(N \sim 500\), to three-petal spectrum at \(N \sim 1000\), and to four-petal spectrum at \(N \sim 2000 - 4000\). Further transitions are possible. The transitions are related to the decrease of the angle of wave interactions, which causes the spectrum to break into a large number of narrow-angle bands. The typical four-petal spectrum, \(n_k\), is shown in Fig. 2 together with phase differences, \(\theta_k\). Even though the phases of different modes appear to be random, the modes with opposite wavenumbers are pairwise correlated by their interactions with the condensate.

Condensate oscillations at the frequencies of individual modes add up to the overall fluctuations around condensate’s average, shown in Fig. 3. The highest contribution to the signal is from the modes with lowest wave numbers. Our three-wave model predicts that the modes oscillate at twice the Bogolyubov frequency. Indeed, the NSE simulations show that the dominant frequency of condensate-turbulence oscillations, \(2\sqrt{N_0}\), is the frequency of \(|k| = 1\) modes, while the frequency of the second harmonic, \(4\sqrt{N_0}\), correspond to \(|k|^2 = 2\) modes. These frequencies (which are much smaller than the frequency of the phase rotation of the condensate \(N\)) essentially do not depend on the level of over-condensate fluctuations. They are clearly seen in the oscillations of the condensate amplitude and of the normal correlation functions.

Our three-wave model describes not only the frequencies but also the time dependence of the oscillations. The time dependence of the phase and the amplitude of an individual mode extracted from a NSE simulation is compared to the prediction of the model (5)-(6) in Fig. 4. Here, the \(n/N\) ratio is small, and the reduced model (7)-(9) works as good as the full one, as illustrated in phase portrait section of Fig. 4. As predicted by the model, the amplitude \(a_k\) has cusped shape, while the phase is localized around \(\theta = \pi\).

The biggest difference between the NSE modes and the model is the phase portrait. The model predicts monotonous increase of the phase from \((2j - 1)\pi\) to \((2j + 1)\pi\), while the phase of the NSE modes oscillates around \(\pm\pi\) in closed loops. Apparently, the interaction with other modes, unaccounted in our simple model, leads to phase locking.

The amplitude in the three-wave model is determined up to the constant of integration. In other word, the level of fluctuations must be obtained from the turbulence data. Our study of NSE turbulence at different levels of condensate indicates that the amplitude of lower modes remain roughly constant for the wide range of \(N\). This also means that \(n/N\) decreases as the condensate level increases, as shown in Fig. 5. (There is a possibility that \(n/N\) is somewhat larger near phase transition but this topic needs additional investigation.) In all cases considered, the ratio \(n/N\) is small, well within the applicability limits of the reduced model. At higher \(N\), the loops in the phase portrait are reconnecting tighter and closer to \(\theta = \pi\). It is also interesting that amplitudes of non-condensate modes are larger than fluctuations of the condensate.

Regarding the data presented in Fig. 5 we shall make the following technical comment, unaddressed in [10]. The states, similar to the state shown in Fig. 2, can be achieved in numerical simulation by using initial conditions where the condensate is superimposed on top of the thermal equilibrium spectrum. For such systems, it take only a few linear time units to relax to quasi-equilibrium with the appropriate symmetry, in contrast
FIG. 4: (Color online) First three panels: comparison of oscillations of $k = (1,1)$ mode extracted from simulation of turbulence at $N = 400$ with the prediction of three-wave model; from left to right, phase, amplitude, the phase portrait are shown. Fourth panel: comparison of phase portraits for $N = 3600$ and $k = (0,3)$). Here, as in Fig. 1 $a/A = \sqrt{n/N}$.

FIG. 5: (Color online) Amplitude of oscillations of low modes for different levels of condensate is averaged over time interval $[10,12]$ in simulations with preset condensate.

to thousands of time units of evolution needed for systems without preset condensate. Moreover, as shown in Fig. 3 the condensate-turbulence oscillations are very fast and can be studied in simulations without forcing (except for the small damping for smooth transition to de-aliased region).

To summarize, at the large level of condensate the three-wave model capture the following features of the turbulence-condensate oscillations: (i) the frequency of oscillations is twice the Bogolubov frequency, (ii) the system spends most of its time around $\theta = \pi$ state, and (iii) the amplitude as a function of time has a non-trivial, cusped shape. To describe the shape of the phase oscillations around $\pm \pi$, additional mechanisms need to be included.

This research was supported by the Kupcinet-Getz International Science School and by the grants of the BSF, ISF and the Minerva Foundation funded by the German Ministry for education and research. Work of N.V. was supported by NSF grants PHY 1004118 and PHY 1004110.

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