BPS solitons in Lifshitz field theories

Archil Kobakhidze, Jayne E. Thompson and Raymond R. Volkas

School of Physics, The University of Melbourne, Victoria 3010, Australia

Abstract

Lorentz-invariant scalar field theories in $d + 1$ dimensions with second-order derivative terms are unable to support static soliton solutions that are both finite in energy and stable for $d > 2$, a result known as Derrick’s theorem. Lifshitz theories, which introduce higher-order spatial derivatives, need not obey Derrick’s theorem. We construct stable, finite-energy, static soliton solutions in Lifshitz scalar field theories in $3 + 1$ dimensions with dynamical critical exponent $z = 2$. We exhibit three generic types: non-topological point defects, topological point defects, and topological strings. We focus mainly on Lifshitz theories that are defined through a superpotential and admit BPS solutions. These kinds of theories are the bosonic sectors of supersymmetric theories derived from the stochastic dynamics of a scalar field theory in one higher dimension. If nature obeys a Lifshitz field theory in the ultraviolet, then the novel topological defects discussed here may exist as relics from the early universe. Their discovery would prove that standard field theory breaks down at short distance scales.

*Electronic address: archilk@unimelb.edu.au
†Electronic address: j.thompson@pgrad.unimelb.edu.au
‡Electronic address: raymondv@unimelb.edu.au
I. INTRODUCTION

Hořava has suggested that nature may exhibit Lifshitz-like anisotropic scaling in the extreme ultraviolet (UV) regime in order to cure the divergence problem in quantum gravity [1, 2]. If gravity were to behave this way, then presumably all other interactions should also. In this paper, we shall show that Lifshitz scalar field theories can support stable, finite-energy soliton or defect solutions of a kind that are impossible in standard second-order Lorentz-invariant field theories. Specifically, we shall demonstrate that stable, static, finite-energy defects formed entirely from scalar fields can exist in three dimensions. We shall derive both point-like and string-like solitons, which we can call “Lifshitz poles (L-poles)” and “Lifshitz strings (L-strings)”, respectively. The non-existence of solutions having all of the above characteristics in standard field theories follows from Derrick’s theorem [3]. So, while point-like (“hedgehog”) and global string solutions do exist in standard theory, they have divergent energies or linear energy densities, as the case may be. The discovery of L-poles and/or L-strings as cosmological relics from the early universe would provide observational proof of the breakdown of standard field theory at short distances.

The basic idea of Lifshitz theories is to add high-order spatial derivative terms to the action while maintaining time derivatives at second order. This procedure preserves unitarity, while softening UV divergences through the presence of higher powers of 3-momentum in the denominators of propagators. This is an interesting idea, because it increases the range of power-counting renormalisable field theories. Hořava gravity [1] is the best-studied example; field-theory models involving extra dimensions of space is another [4]. The improved UV behaviour comes at the expense of Lorentz invariance, with the UV isotropic scaling of Lorentz-invariant field theories replaced by anisotropic scaling between space and time. This is obviously the most serious drawback of this approach, in both a phenomenological and an aesthetic sense. It is not known how to ensure that standard relativity emerges in the infrared, except by fine-tuning, though progress has recently been made on the gravity front [2]. More generally, it is neither clear how to ensure that all particle species acquire a common limiting speed, nor how to make the renormalisation group running of the limiting speed(s) phenomenologically acceptable [5]. So, there are arguments both for and against Lifshitz theories of particle physics and gravity. Experimental or observational evidence in favour of the breakdown of Lorentz invariance in the UV would lend support to the Lifshitz
approach. Such evidence can be searched for in very high energy cosmic ray events, but also through the possible existence of cosmological relics that would be inexplicable by standard field theory.

We shall examine Lifshitz theories having fourth-order spatial derivatives, which is the simplest possible choice from the point of view of the differential equations. In general, these theories are much more difficult to analyse than the usual second-order relativistic theories. However, a Lifshitz analogue of the Bogomolnyi-Prasad-Sommerfield (BPS) superpotential approach can be invoked to reduce the problem back to second-order (this is equivalent to the "detailed balance" condition [6]). We shall take this approach in this paper, partly for simplicity, and partly because the BPS system has intrinsic theoretical interest through a deep connection with supersymmetry and stochastic quantisation.

In the next section we review Lifshitz scalar field theory and anisotropic scaling. We then show why Derrick’s theorem is inapplicable. This is followed by the derivation of Lifshitz BPS soliton solutions, and then a brief section on the general fourth-order system. The final section is a conclusion.

II. LIFSHITZ SCALAR FIELD THEORY

We shall work exclusively in $3 + 1$ dimensions. The construction of a Lifshitz theory begins through the specification of the dynamical critical exponent $z$,

\[ \vec{x} \rightarrow \kappa \vec{x}, \quad t \rightarrow \kappa^z t, \quad (2.1) \]

which defines the anisotropic scaling invariance of the theory in the UV. Lorentz-invariant theories must have isotropic scaling, namely $z = 1$. Consider now the case of $z = 2$, adopted for definiteness and simplicity in our analysis. We shall construct the action step-by-step. One first chooses that the Lagrangian contains only the term $\dot{\phi}^2$ in time derivatives. This is to ensure the unitarity of the quantum theory. Anisotropic $z = 2$ scale invariance then follows with the observation that the action $\int d^3x \, dt \, \dot{\phi}^2$ is invariant under

\[ \vec{x} \rightarrow \kappa \vec{x}, \quad t \rightarrow \kappa^2 t, \quad \phi(\vec{x}, t) \rightarrow \kappa^{-1/2} \phi(\vec{x}, t), \quad (2.2) \]

\[ ^1 \text{We shall not consider the possibility that different dimensions of space may scale differently.} \]
or, equivalently, under the field substitution

$$\phi(\vec{x}, t) \rightarrow \kappa^{+1/2} \phi(\kappa \vec{x}, \kappa^2 t), \quad (2.3)$$

without the coordinates in the integration measure and derivatives participating in the transformation. We shall adopt the viewpoint of Eq. (2.2) for definiteness in this paper.

The $\dot{\phi}^2$ action is actually invariant for any $z$, with the field transforming as $\phi \rightarrow \kappa^{3/2} \phi$. The $z$ value is specified through the spatial-derivative terms we now introduce. The relativistic choice, $\int d^3x dt (\dot{\phi}^2 - c^2 \nabla \phi \cdot \nabla \phi)/2$, enforces $z = 1$. But we are interested in $z = 2$, which implies that

$$S \subset \int d^3x dt \frac{1}{2} \left[ \dot{\phi}^2 - (\nabla^2 \phi)^2 \right]. \quad (2.4)$$

The arbitrary constant in front of the second term has been absorbed into $\vec{x}$, and the minus sign is necessary to make the energy bounded from below.\(^2\) The higher spatial derivative implies that the propagator behaves like $1/[k_0^2 - (\vec{k} \cdot \vec{k})^2]$, so has improved UV behaviour compared to the relativistic case $1/(k_0^2 - \vec{k} \cdot \vec{k})$.

The most general $z = 2$ scale invariant action for a single real scalar field is then easily deduced to be\(^3\)

$$S = \int d^3x dt \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla^2 \phi)^2 - a \phi^5 \nabla^2 \phi - \lambda \phi^{10} \right], \quad (2.5)$$

where $a$ and $\lambda$ are coupling constants, and they are dimensionless with respect to the “anisotropic scaling units”,

$$[S] = 0, \quad \alpha = -1, \quad t = -2, \quad \phi = \frac{1}{2}, \quad (2.6)$$

that replace the usual “mass dimension units” of relativistic theories. The power-counting renormalisation properties follow from the adoption of these new units, with the usual rule that renormalisable terms have either dimensionless or positive-power coupling constants, corresponding to marginal and relevant operators, respectively. The scale-invariant action (2.5) contains only marginal operators. The most general power-counting renormalisable action contains in addition the terms,

$$\phi^n \nabla^2 \phi, \quad \text{with} \quad 0 \leq n \leq 4 \quad (2.7)$$

$$\phi^m, \quad \text{with} \quad 0 \leq m \leq 9 \quad (2.8)$$

---

\(^2\) Note that the constant must be reintroduced if one wishes to work in standard natural units.

\(^3\) Note that we are assuming that surface terms may be discarded, so that there is no need to include $\phi^4 \nabla \phi \cdot \nabla \phi$, $\nabla \phi \cdot \nabla(\nabla^2 \phi)$ and $\phi \nabla^2(\nabla^2 \phi)$ as independent terms.
These relevant operators explicitly, but softly, break the anisotropic scale invariance. The most general action may be compactly written as
\[ S = \int d^3x \, dt \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla^2 \phi)^2 - K(\phi) \nabla^2 \phi - V(\phi) \right], \]
where \( K(\phi) \) is quintic and the potential \( V(\phi) \) is tenth-order. Observe that the standard relativistic term \( \nabla \phi \cdot \nabla \phi \) is contained as the \( n = 1 \) term of \( K \), up to a discarded total derivative. The equation of motion that follows from Hamilton’s principle is
\[ \ddot{\phi} + \nabla \nabla (\nabla^2 \phi) - K_{\phi \phi} \nabla \phi \cdot \nabla \phi - 2K_{\phi} \nabla^2 \phi + V_{\phi} = 0, \]
where the subscripts \( \phi \) and \( \phi\phi \) denote first and second derivatives with respect to \( \phi \), respectively. The energy functional is
\[ E[\phi] = \int d^3x \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla^2 \phi)^2 + K(\phi) \nabla^2 \phi + V(\phi) \right]. \]

The generalisation of the above formalism to more than one scalar field is obvious.

**III. DERRICK’S THEOREM AND THE VIRIAL RELATION**

Derrick proved that stable, finite-energy, static solutions to second-order relativistic scalar field theories do not exist in greater than two spatial dimensions [3]. We now review his extremely simple argument for the case of three spatial dimensions. For static configurations, extremising the action functional is equivalent to extremising the energy functional,
\[ E[\phi] = \int d^3x \left[ \frac{1}{2} \nabla \phi \cdot \nabla \phi + V(\phi) \right]. \]
Consider the specific field variation given by
\[ \phi(\vec{x}) \to \phi(k\vec{x}), \]
under which the energy varies to
\[ E_k[\phi] = \int d^3x \left[ \frac{1}{2} \nabla \phi(k\vec{x}) \cdot \nabla \phi(k\vec{x}) + V(\phi(k\vec{x})) \right]. \]
A necessary condition for a solution is that the variation of the energy under this restricted field variation should vanish to first order, which is equivalent to the demand that
\[ \left. \frac{\partial E_k[\phi]}{\partial k} \right|_{k=1} = 0. \]
Now,

\[ E_k[\phi] = \int d^3 \xi \left[ k^{-1} \frac{1}{2} \nabla_\xi \phi(\tilde{\xi}) \cdot \nabla_\xi \phi(\tilde{\xi}) + k^{-3} V(\phi(\tilde{\xi})) \right], \]

\[ \equiv k^{-1} S_1 + k^{-3} S_3 \]  

(3.5)

where \( \tilde{\xi} \equiv k \tilde{x} \) and \( E \equiv S_1 + S_3 \). Imposing Eq. (3.4) we get the virial relation \(-S_1 = 3S_3\), which implies that \( E = 2S_1/3 \). A stable solution must have a positive second-order variation. Computing \( d^2 E_k/dk^2|_{k=1} \) one obtains \( 2S_1 + 12S_3 \), which equals \(-2S_1\) through the virial theorem. Since \( S_1 \) is positive-definite, no stable solutions can exist. It was implicit above that \( E[\phi] \) converged, so the theorem does not preclude infinite-energy, static, point-like solitons (the hedgehog being a well-known example).

Repeating this exercise for the Lifshitz energy functional (2.11) we obtain

\[ E_k[\phi] = kS_1 + k^{-1} S_2 + k^{-3} S_3, \]

(3.6)

where \( S_1 \equiv \int d^3 x dt (\nabla^2 \phi)^2/2 \) and \( S_2 \equiv \int d^3 x dt K(\phi) \nabla^2 \phi \), which implies the virial relation

\[ S_1 - S_2 - 3S_3 = 0. \]  

(3.7)

The second-order variation is now equal to \( 2S_2 + 12S_3 \), which in turn equals \( 4S_1 - 2S_2 \) through the virial relation. There is no necessity for this quantity to be negative, hence there is no generalisation of Derrick’s theorem to the \( z = 2 \) Lifshitz case. A simple example is the case \( K = 0 \) which implies \( S_2 = 0 \) and hence a positive second-order variation. Another is \( K = \phi^5 \), which sees \( S_2 \) turn into an integral of the negative-definite function \(-\phi^4 \nabla \phi \cdot \nabla \phi \) after integration by parts. This makes \( 4S_1 - 2S_2 \) necessarily positive. The addition of higher-derivative terms thus tends to stabilise scalar-field solitons, a phenomenon also observed in the (relativistic) Skyrme model [9]. However, evasion of Derrick’s theorem does not prove that stable, static, finite-energy, scalar-field solitons actually exist, because the variation (3.2) is not general. We therefore turn to explicit computations.

IV. THE LIFSHITZ BPS SOLITON SOLUTIONS

A technical problem with \( z = 2 \) Lifshitz theory is that the scalar field equations are fourth-order in spatial derivatives, as per Eq. (2.10). To get to concrete results in the simplest possible way, we therefore develop a Lifshitz generalisation of superpotential systems. This
will reduce the problem to second order, and as a bonus the solutions will be Lifshitz
generalisations of BPS states, which is interesting in itself because of the strong connection
with supersymmetry and stochastic quantisation. We shall derive three kinds of solutions:
a perturbatively-stable non-topological point-like soliton, a topological point-like soliton or
hedgehog, and a topological string. In the section after this, we shall also briefly discuss the
general fourth-order problem, and display a finite-energy solution by way of an existence
proof, but defer a systematic treatment to future work.

We first very briefly review how superpotential considerations lead to interesting simpli-
cfications in one-dimensional second-order systems, with

$$E[\phi] = \int dx \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + V(\phi) \right]. \quad (4.1)$$

For the special cases where

$$V(\phi) = \frac{1}{2} \left( \frac{dW(\phi)}{d\phi} \right)^2, \quad (4.2)$$

with $W$ being a superpotential, it is easy to see that

$$E[\phi] = \mp [W(\phi(x = +\infty)) - W(\phi(x = -\infty))] + \int_{-\infty}^{+\infty} dx \left( \frac{d\phi}{dx} \pm \frac{dW}{d\phi} \right)^2. \quad (4.3)$$

Since the integrand above is positive definite, solutions to the first-order equations

$$\frac{d\phi}{dx} \pm \frac{dW}{d\phi} = 0 \quad (4.4)$$

globally minimise the energy functional for given boundary conditions. They are stable BPS
solutions, and it is trivial to confirm that they satisfy the Euler-Lagrange equations.

We can achieve a situation that resembles the above if we restrict our Lifshitz action
functional to the special form

$$S = \int d^3x \, dt \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \left( \nabla^2 \phi + K(\phi) \right)^2 \right], \quad (4.5)$$

so that

$$V(\phi) = \frac{1}{2} K(\phi)^2. \quad (4.6)$$

A superpotential $W$ can then be defined as the negative indefinite integral of $K$ with respect
to $\phi$, so that $K = -W_\phi$ (the minus sign is just a convention). The convenience of the action
(4.5) is that the energy functional,

$$E[\phi] = \int d^3x \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left( \nabla^2 \phi + K(\phi) \right)^2 \right], \quad (4.7)$$
is positive definite. Configurations that obey

\[ \dot{\phi} = 0, \quad \nabla^2 \phi + K(\phi) = 0 \]  \hspace{1cm} (4.8)

minimise the energy, and in fact they have zero energy-density. It is straightforward to confirm that solutions of Eq. (4.8) are also solutions of the field equation (2.10). We shall call these “Lifshitz BPS” or LBPS solutions.

From Eq. (2.10), we see that static, spatially homogeneous solutions (vacua \( \phi = v \)) must obey \( V_\phi(v) = 0 \) as usual. For a Lifshitz superpotential system, this means that \( K(v)K_\phi(v) = 0 \), which implies that either \( K(v) = 0 \), or \( K_\phi(v) = 0 \) with \( K(v) \neq 0 \). The former type of vacuum obeys Eq. (4.8), while the latter does not. Suppose \( K(\phi) \neq 0 \ \forall \phi \). Then LBPS solutions must obey either \( \nabla^2 \phi > 0 \) everywhere, or \( \nabla^2 \phi < 0 \) everywhere. In either case, these solutions will be asymptotically unbounded, and thus forbidden once physically reasonable boundary conditions are imposed. We shall not consider such theories any further, by requiring that \( K \) vanishes for all vacuum configurations. This means the LBPS solutions for the allowed models are degenerate with the vacua, and hence at least perturbatively stable.

An interesting aspect of Lifshitz theories with \( z = 2 \) anisotropic scaling (4.5) is that they provide an equivalent description of certain stochastic systems \([10, 11]\) (for recent relevant works see \([12]\) and references therein). It is well known that the equilibrium limit — defined as the \( \tau \to \infty \) limit where \( \tau \) is a fictitious time (Markov parameter) — of the statistical correlators of a \((d + 1)\)-dimensional scalar field \( \phi_\eta(\tau, x^i) \) is equivalent to quantum correlators of the \( d \)-dimensional Euclidean scalar field \( \phi(x^i) \) with the classical action \( S_E[\phi] \):

\[
\lim_{\tau \to \infty} \langle \phi_\eta(\tau, x_1^i) \ldots \phi_\eta(\tau, x_n^i) \rangle_\eta = \langle \phi(x_1^i) \ldots \phi(x_n^i) \rangle. \hspace{1cm} (4.9)
\]

Here \( \phi_\eta(\tau, x^i) \) is a stochastic field, being a solution of the Langevin equation,

\[
\partial_\tau \phi(\tau, x^i) = -\frac{\delta S_E}{\delta \phi} \bigg|_{\phi(x^i)=\phi(\tau,x^i)} + \eta(\tau, x^i), \hspace{1cm} (4.10)
\]

where

\[
\frac{\delta S_E}{\delta \phi} \equiv \frac{\delta S_E}{\delta \phi(x^i)} \bigg|_{\phi(x^i)=\phi(\tau,x^i)}. \hspace{1cm} (4.11)
\]

\[\text{In what follows we explicitly discuss the d=3 case.}\]
and $\eta(\tau, x^i)$ is Gaussian white noise, i.e.,

$$\langle \eta(\tau', x^i) \eta(\tau, x^i) \rangle_\eta = \delta(\tau' - \tau) \delta^d(x^i - x^i),$$  \hspace{1cm} (4.12)$$

with all other correlators being zero. Note that in the large time limit, $\tau \to \infty$, the system approaches equilibrium with $\partial_\tau \phi(\tau, x^i) = \eta = 0$, and the Langevin equation (4.10), reduces to the equation of motion of the Euclidean scalar field $\phi(x^i)$ described by the action $S_E$.

The generating functional for the statistical correlators in Eq. (4.9) reads as:

$$Z[J] = \int D\eta \exp \left\{ \int d\tau d^3x \left[ -\frac{\eta^2}{2} + J_\eta \eta \right] \right\},$$  \hspace{1cm} (4.13)$$

and

$$\langle \phi_\eta(\tau, x^i_1) \ldots \phi_\eta(\tau, x^i_n) \rangle_\eta \equiv \left. \frac{\delta^n Z}{\delta J_\eta(\tau, x^i_1) \ldots \delta J_\eta(\tau, x^i_n)} \right|_{J=0}.$$  \hspace{1cm} (4.14)$$

Following the standard steps, we perform change of variables $\eta(\tau, x^i) \to \phi_\eta(\tau, x^i)$ using Eq. (4.12) and express the new functional measure through a path integral over the Grassmannian fields $\psi$ and $\bar{\psi}$:

$$D\eta \to D\phi \det \left( \frac{\delta \eta}{\delta \phi} \right) = D\phi \det \left( \partial_\tau + \frac{\delta^2 S_E}{\delta \phi^2} \right)$$

$$= D\phi \int D\bar{\psi} D\psi \exp \left\{ \int d\tau d^3x \bar{\psi} \left( \partial_\tau + \frac{\delta^2 S_E}{\delta \phi^2} \right) \psi \right\}.$$  \hspace{1cm} (4.15)$$

Plugging (4.15) into (4.13) and using the Langevin equation, we obtain the effective action

$$S_{\text{eff}} = \int d\tau d^3x \left[ \frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} \left( \frac{\partial S_E}{\partial \phi} \right)^2 - \bar{\psi} \left( \partial_\tau + \frac{\delta^2 S_E}{\delta \phi^2} \right) \psi \right],$$  \hspace{1cm} (4.16)$$

where we have dropped the full time derivative term $\partial_\tau \phi \frac{\delta S_E}{\delta \phi}$. Taking

$$S_E = \int d^3x \left[ \frac{1}{2} (\partial_\tau \phi)^2 + W(\phi) \right]$$  \hspace{1cm} (4.17)$$

in (4.16) and passing to the Lorentzian time $t = -i\tau$, we obtain a theory described by the action:

$$S_{\text{LSUSY}} = \int dt d^3x \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \left( \nabla^2 \phi - \frac{dW}{d\phi} \right)^2 + \bar{\psi} \left( -i\partial_t - \nabla^2 + \frac{d^2 W}{d\phi^2} \right) \psi \right].$$  \hspace{1cm} (4.18)$$

The bosonic part of the above action is precisely the action of the $z = 2$ Lifshitz theory (4.5) with $K(\phi) = -W_\phi$.  

9
The action (4.18) is invariant under rigid (quantum mechanical) supersymmetry transformations. To obtain transformations that are linear in the fields, we rewrite (4.18) in the equivalent form

\[ S_{\text{LSUSY}} = \int dt d^3x \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} D^2 - D \left( \nabla^2 \phi - \frac{dW}{d\phi} \right) + \bar{\psi} \left( -i \partial_t - \nabla^2 + \frac{d^2W}{d\phi^2} \right) \psi \right], \]  

(4.19)

where \( D(t, x^i) \) is an auxiliary field satisfying the non-dynamical equation of motion:

\[ D = \nabla^2 \phi - \frac{dW}{d\phi}. \]  

(4.20)

The infinitesimal supersymmetry transformations then read:

\[ \delta_\epsilon \phi = \bar{\epsilon} \psi - \bar{\psi} \epsilon, \quad \delta_\epsilon D = -i \partial_t (\bar{\epsilon} \psi + \bar{\psi} \epsilon), \]

\[ \delta_\epsilon \psi = \epsilon (D - i \dot{\phi}), \quad \delta_\epsilon \bar{\psi} = -\bar{\epsilon} (D + i \dot{\phi}). \]  

(4.21)

It is clear now that any solution of \( z = 2 \) Lifshitz theory that satisfies Eq. (4.8) gives vanishing D-term (4.20). Therefore, our LBPS solutions are supersymmetric and, hence, they are degenerate with the zero-energy ground state. Supersymmetry ensures the stability of the LBPS solitons.

From the perspective of stochastic dynamics (4.10), our supersymmetric LBPS solitons are relevant to the description of physics at equilibrium (\( \tau \to \infty \)), while any time-dependent solution describes physics in the non-equilibrium regime. A discussion of these interesting points is beyond the scope of the present paper.

**A. Non-topological BPS L-pole**

Our first explicit solution will be a non-topological BPS L-pole. Let \( \phi = \phi_0(r) \) be a radially symmetric solution of Eq. (4.8):

\[ \left( \frac{2}{r^2} \frac{d}{dr} + \frac{d^2}{dr^2} \right) \phi_0(r) = -K (\phi_0(r)). \]  

(4.22)

Impose the boundary condition that \( \phi_0 \) asymptotes to a finite value \( v \) as \( r \to \infty \). Regularity at the origin requires \( \phi_0 \sim r^2 \) as \( r \to 0 \). An obvious interpolation between these behaviours is provided by the function

\[ \phi_0(r) = \frac{vr^2}{R^2 + r^2}, \]  

(4.23)
where $R$ is a parameter that sets the characteristic size of the defect. It is easy to verify that this function solves Eq. (4.22) for

$$K(\phi) = \frac{2}{R^2 v^2}(4\phi - 3v)(\phi - v)^2,$$  \hspace{1cm} (4.24)

so that $v$ is a vacuum. This provides a simple analytical example of an L-pole. It is non-topological, because the asymptotic limit is always the same vacuum.

To get the simple analytical form of Eq. (4.23) we of course need a very specific function $K$. If we vary this function continuously from Eq. (4.24), then the solution will vary continuously away from our analytical example, with numerical computations being in general necessary.

We give another example of a non-topological BPS L-pole which corresponds to a simpler choice of $K$, namely

$$K(\phi) = \lambda \phi (\phi^2 - v^2)^2.$$  \hspace{1cm} (4.25)

The non-topological BPS L-pole corresponding to this choice of $K$ does not have an analytic solution, however a numerical solution is shown in Fig. 1.
B. Topological BPS L-pole or hedgehog

Our second example is an LBPS hedgehog. The goal is to produce a topologically non-trivial BPS L-pole. We need the vacuum manifold to be a 2-sphere, so we choose an $O(3)$ invariant theory with a triplet of scalar fields $\vec{\phi} = (\phi_i)$, $i = 1, 2, 3$, and work within the parameter regime that features $O(3) \rightarrow O(2)$ spontaneous symmetry breaking. The energy functional for static configurations is

$$E[\vec{\phi}] = \int d^3x \left[ \nabla^2 \phi_i + \phi_i F(\vec{\phi} \cdot \vec{\phi}) \right] \left[ \nabla^2 \phi_i + \phi_i F(\vec{\phi} \cdot \vec{\phi}) \right],$$

where there is a sum over $i$ and

$$F = \lambda (\vec{\phi} \cdot \vec{\phi} - v^2)^2.$$  

(4.27)

The hedgehog ansatz in spherical polar coordinates is

$$\phi_i(r, \theta, \varphi) = \hat{x}_i h(r)$$

(4.28)

where $\hat{x}_1, \hat{x}_2, \hat{x}_3$ is a unit vector in the $(\theta, \varphi)$ direction. The LBPS equations $\nabla^2 \phi_i + \phi_i F = 0$ reduce to

$$\frac{d^2 h}{dr^2} + \frac{2}{r} \frac{dh}{dr} - \frac{2h}{r} + \lambda h (h^2 - v^2)^2 = 0,$$

(4.29)
which is the same as the original hedgehog equation except that the non-derivative term is fifth-order rather than third-order. We demand that it asymptote to $v$ and be regular at the origin. A numerical solution is depicted in Fig. 2.

C. Topological BPS L-string

The third solution is a topological BPS L-string. The theory has a complex scalar field $\Phi$ and a spontaneously broken $\Phi \rightarrow e^{i\alpha} \Phi$ global $U(1)$ symmetry, with energy functional:

$$E[\Phi] = \int d^3x \left[ \nabla^2 \Phi^* + \lambda \Phi^* (\Phi^* \Phi - v^2)^2 \right] \left[ \nabla^2 \Phi + \lambda \Phi (\Phi^* \Phi - v^2)^2 \right].$$ (4.30)

The topological global string ansatz is

$$\Phi(\rho, \theta) = e^{i\theta} f(\rho)$$ (4.31)

where $(\rho, \theta, z)$ are the usual cylindrical coordinates, and the string defines the $z$-axis. The BPS equation of motion $\nabla^2 \Phi + \lambda \Phi (\Phi^* \Phi - v^2)^2 = 0$ reduces to

$$\frac{d^2 f}{d\rho^2} + \frac{1}{\rho} \frac{df}{d\rho} - \frac{f}{\rho^2} + \lambda f (f^2 - v^2)^2 = 0.$$ (4.32)

The asymptotic boundary condition is $f \rightarrow v$ as $\rho \rightarrow \infty$, and we demand regularity at the origin. A numerical solution to Eq. (4.32) is given in Fig. 3.

![FIG. 3: Numerical solution to Eq. (4.32) with $\lambda = v = 1$.](image-url)
One may wonder if the existence of the finite-energy, static solutions displayed above relies on the special superpotential form of the action. The answer is no, which we shall establish by displaying a numerical solution for a non-LBPS hedgehog. A full analysis of non-LBPS solutions is beyond the scope of this paper, in part because the stability of non-topological soliton solutions requires the development of a fourth-order linear stability formalism.

We shall exhibit a non-LBPS hedgehog solution for an $O(3)$ triplet model with $K = 0$ and

$$V = \frac{\lambda}{10} \left( \vec{\phi} \cdot \vec{\phi} - v^2 \right)^5,$$

where, using the ansatz of Eq. (4.28), the equation of motion reduces to

$$\frac{d^4 h}{dr^4} + \frac{4}{r} \frac{dh}{dr^3} - \frac{4}{r^2} \frac{d^2 h}{dr^2} + \lambda h (h^2 - v^2)^4 = 0.$$  \hspace{1cm} (5.2)

A numerical solution for $h(r)$ is given in Fig. (4). This configuration is topologically stable.

FIG. 4: Numerical solution to Eq. (5.2) with $\lambda = v = 1$.  

14
VI. CONCLUSIONS

Lifshitz scalar field theories permit the existence of finite-energy, static soliton solutions that have only infinite-energy analogues in standard second-order relativistic theories (such as the global string and the hedgehog). We have displayed several such solutions, mostly within a superpotential-based subclass of $z = 2$ Lifshitz scalar theories that admit stable zero energy density BPS solutions. These systems constitute the bosonic sector of supersymmetric theories that are themselves related to a stochastic field theory in one higher dimension. If the world is described by a Lifshitz theory, as suggested recently by Hořava as a way to improve ultraviolet behaviour in a ghost-free manner, then the existence of new classes of finite-energy defects may provide a way to look for observational evidence in support of this hypothesis. To really make this connection, one would need to extend our analysis from $z = 2$ (which was chosen for simplicity), to at least $z = 3$. The gravitational properties of Lifshitz solitons could then also be studied.

Acknowledgements

We thank Damien George for useful comments. This work was supported in part by the Australian Research Council and in part by the Puzey bequest to the University of Melbourne. RRV thanks O. Yasuda and H. Minakata for their kind hospitality at Tokyo Metropolitan University where this work was completed.

[1] P. Hořava, Phys. Rev. D79, 084008 (2009) [arXiv:0901.3775]; JHEP 03, 020 (2009) [arXiv:0901.3775]; Phys. Rev. Lett. 102, 161301 (2009) [arXiv:0902.3657].
[2] P. Hořava and C. M. Melby-Thompson, [arXiv:1007.3657].
[3] G. H. Derrick, J. Math. Phys. 5, 1252 (1964).
[4] J. E. Thompson and R. R. Volkas, [arXiv:1008.2054].
[5] R. Iengo, J. G. Russo and M. Serone, JHEP 0911, 020 (2009) [arXiv:0906.3477]; R. Iengo, and M. Serone, Phys. Rev. D81, 125005 (2010) [arXiv:1003.4430]; D. Anselmi, Annals Phys. 324, 874 (2009) [arXiv:0808.3470]; ibid. 324, 1058 (2009) [arXiv:0808.3474]; Phys. Rev. D79,
025017 (2009) \texttt{arXiv:0808.3475}; B. Chen and Q. G. Huang, Phys. Lett. B683, 108 (2010) \texttt{arXiv:0904.4565}.

[6] E. B. Bogomol’nyi, Sov. J. Nucl. Phys. 24, 4 (1976); M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[7] P. Hořava, Phys. Lett. B \textbf{694}, 172 (2010) \texttt{arXiv:0811.2217}.

[8] A. M. Polyakov, JETP Lett. 20, 194 (1974).

[9] T. H. R. Skyrme, Nucl. Phys. 31, 556 (1962).

[10] G. Parisi and Y.-S. Wu, Sci. Sin. 24, 483 (1981).

[11] G. Parisi and N. Sourlas, Nucl. Phys. B206, 321 (1982).

[12] R. Dijkgraaf, D. Orlando and S. Reffert, Nucl. Phys. B824, 365 (2010) \texttt{arXiv:0903.0732}; D. Orlando and S. Reffert, Class. Quant. Grav. 26, 155021 (2009) \texttt{arXiv:0905.0301}; D. Orlando and S. Reffert, Phys. Lett. B 683, 62 (2010) \texttt{arXiv:0908.4429}; I. Bakas, \texttt{arXiv:1009.6173}. 