Abstract

Using Mazur’s theorem on torsions of elliptic curves, an upper bound 24 for the order of the finite Galois group $H$ associated with weighted walks in the quarter plane $\mathbb{Z}_2^+$ is obtained. The explicit criterion for $H$ to have order 4 or 6 is rederived by simple geometric argument. Using division polynomials, a recursive criterion for $H$ having order $4m$ or $4m+2$ is also obtained. As a corollary, explicit criterion for $H$ to have order 8 is given and is much simpler than the existing method.

1 Introduction

Counting lattice walks is a classic problem in combinatorics. A combinatoric walk with nearest-neighbour step length can be seen as a weighted walk with weight 1 for the allowed directions and weight 0 for the forbidden directions. If a multiple step length requirement is allowed, a combinatoric walk can be seen as a weighted walk with integer weights. Without loss of generality, for a weighted walk, we may assume that the weights sum to 1 by normalization. If we allow the weights of a walk to take arbitrary non-negative real values that sum to 1, then we arrive at the realm of probabilistic walks in the quarter plane. So a weighted walk is the same thing as a probabilistic walk and a weighted walk with rational weights is the same thing as a combinatoric walks with different step lengths in different directions.

In the probabilistic scenario, an approach called the ”kernel method” has been well developed and summarized in the book [4]. In the kernel method, Malyshev [8] defined a group $H$, called the Galois group associated with any walk in $\mathbb{Z}_2^+$. The finiteness of $H$ turns out to be important. Here are some applications of $H$

1. For the 2-demands queueing system, Flato and Hahn [6, 5] exploited the finiteness of $H$ to obtain an exact formula for the stationary distribution.
2. The generating function of the walk satisfies some differential equation if and only if $H$ has finite order. Moreover the generating function is algebraic if and only if the orbit sum is zero. See Theorem 42 in [2].

Bousquet-Mélou [1] showed that for combinatoric walks with nearest-neighbour step length in the quarter plane, $H$ can have order 4, 6, or 8, if $H$ has finite order. For a weighted walk, Kauers and Yatchak found three walks with order 10 [7].

In this paper, we only consider the generic case when the kernel of the walk determines genus 1 surfaces. We give 24 as an upper bound on the finite order of $H$ when the weights of the walk are rationals. In particular, this result says that if the order of $H$ is finite, then it cannot be arbitrarily large. The following list summarizes different objects considered in the paper and also serves as an outline of the proof.

1. A biquadratic polynomial $Q(x, y)$ defines a connected real curve $Q \subset \mathbb{R}^2$. The composition of the horizontal and the vertical switches is called a QRT map $\delta$ on $Q$.

2. By going to complex numbers, $Q(x, y)$ defines a Riemann surface, also called $Q \subset \mathbb{C}^2$. The Abel-Jacobi map $J$ determines a lattice $\Lambda$ generated by $\omega_1, \omega_2 \in \mathbb{C}$, unique up to the modular group $\text{PSL}(2, \mathbb{Z})$ action, such that $Q \cong \mathbb{C}/\Lambda$.

3. The Weierstrass function $\wp$ and its derivative $\wp'$ can be used to construct a map $J'^{-1}$, an "inverse" of the Abel-Jacobi map $J$. It is not an actual inverse because the image of $J'^{-1}$ is not $Q$ but an elliptic curve $E$ in the Weierstrass normal form.

4. Both $J$ and $J'^{-1}$ are defined analytically. However, the composition of them turns out to be a polynomial map. So if we start with a $Q(x, y)$ with rational coefficients, we obtain an elliptic curve $E$ with rational coefficients.

5. Moreover, the QRT map $\delta$ induces an addition by a rational point on $E$.

6. So the Mazur’s theorem applies and the bound is obtained.

The organization of the paper is as follows: Section 2 covers 1-3 in the above list, Section 3 covers 4-5, Section 4 covers 6. Section 5 covers criteria for $H$ to have order $4m$ or $4m + 2$. Section 6 is discussion.

2 Preliminary

In this section, we provide the preliminaries that are needed for our main result.
2.1 The model

We shall consider walks in $\mathbb{Z}_2^2$ with step length limited to 1 (nearest-neighbor) and the walk is considered to be homogeneous, that is, the transition probabilities $p_{i,j}(-1 \leq i, j \leq 1)'s$ are independent of the current place. To determine the stationary distribution $\{\pi_{ij}, i, j \in \mathbb{N}\}$ of the walk, following [4], the generating function method is applied. The generating function

$$\pi(x, y) = \sum_{i,j \geq 1} \pi_{ij} x^{i-1} y^{j-1}$$

satisfies the following functional equation:

$$Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \tilde{q}(x, y)\tilde{\pi}(y) + \pi_0(x, y),$$

where

$$Q(x, y) = xy \left( \sum_{i,j} p_{i,j} x^i y^j - 1 \right).$$

Other terms reflects the boundary conditions on the random walk and do not enter our study.

$Q(x, y)$ is called the kernel of the random walk and is a biquadratic polyno-
mial, i.e. both quadratic in $x$ and quadratic in $y$:

$$Q(x,y) = (p_{1,1}x^2 + p_{0,1}x + p_{-1,1})y^2 + (p_{1,0}x^2 + (p_{0,0} - 1)x + p_{-1,0})y
+ p_{1,-1}x^2 + p_{0,-1}x + p_{-1,-1}
:= a(x)y^2 + b(x)y + c(x)
= (p_{1,1}y^2 + p_{0,0}y + p_{-1,1})x^2 + (p_{1,0}y^2 + (p_{0,0} - 1)y + p_{0,-1})x
+ p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}
:= \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y)$$

The partial discriminants of the kernel $Q$ is defined:

**Definition 2.1 (Partial discriminant).** The partial discriminants of $Q(x,y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y)$ are defined as

$$\Delta_1(y) := \tilde{b}^2(y) - 4\tilde{a}(y)\tilde{c}(y), \quad (4)$$

and

$$\Delta_2(x) := b^2(x) - 4a(x)c(x). \quad (5)$$

By using complex variable and compactification, the kernel $Q$ determines a compact Riemann surface $Q$. Since $Q(x,y), x,y \in \mathbb{C}$ is biquadratic, $Q$ double covers the Riemann sphere $\hat{\mathbb{C}}$ with 4 branching point if the partial discriminant $\Delta_1(y)$ or equivalently $\Delta_2(x)$ has no multiple zeros [4]. By Riemann-Hurwitz formula, the genus of $Q$ is

$$g(Q) = 2(g(\hat{\mathbb{C}}) - 1) + \frac{4}{2}(2 - 1) + 1 = 1 \quad (6)$$

We shall assume that $Q$ has genus 1. On $Q$, the following maps are defined:

**Definition 2.2 (Involutions and the QRT map).** The vertical switch $\xi$:

$$\xi(x, y) := \left( x, -\frac{b(x)}{a(x)} - y \right). \quad (7)$$

The horizontal switch $\eta$:

$$\eta(x, y) := \left( -\frac{\tilde{b}(y)}{\tilde{a}(y)} - x, y \right). \quad (8)$$

The QRT map:

$$\delta := \eta \circ \xi. \quad (9)$$

The QRT map generates a group, called the Galois group associated with the random walk:
Definition 2.3 (Galois group).
\[ H := \langle \xi, \eta \rangle \] (10)

Remark. The reason why \( H \) is coined as Galois is essentially that Malyshev adopted a field-theoretic definition of the Riemann surface \( Q \), where a point on \( Q \) is defined as a discrete valuation on the function field \( \mathbb{C}[x,y]/Q(x,y) \).

As involutions, both \( \xi \) and \( \eta \) have order 2. However, the subgroup \( H_0 := \langle \delta \rangle \) can have finite or infinite order. Lemma 2.4.3 of [4] says that \( H_0 \) is a normal subgroup of \( H \) and \( H/H_0 \) is a group of order 2. Obviously \( H_0 \) and \( H/H_0 \) have trivial intersection, hence
\[ H = H_0 \rtimes \mathbb{Z}_2. \] (11)

The following invariants are useful later:

Definition 2.4 (Eisenstein invariants). Let \( f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e \) be a quartic polynomial. The Eisenstein invariants of \( f \) are
\[ D(f) := ae + 3c^2 - 4bd, \] (12)
and
\[ E(f) := ad^2 + b^2e - ace - 2bcd + c^3. \] (13)

2.2 Abel-Jacobi map

Since the Riemann surface \( Q \) has genus 1, its topological structure is a torus. Hence \( Q \) can support a nowhere vanishing vector field. Indeed, we have the following nowhere vanishing

Definition 2.5 (Hamiltonian vector field). The Hamiltonian vector field \( v_H \) given by
\[ v_H := \frac{\partial Q}{\partial y} \frac{\partial}{\partial x} - \frac{\partial Q}{\partial x} \frac{\partial}{\partial y} \] (14)

In fact, this can be used as a more direct proof that \( Q \) has genus 1. Using Hamiltonian vector field, we can define a unique Abelian differential \( \omega_H \), such that \( (\omega_H, v_H) = 1 \), where the parentheses denote the canonical pairing between vector fields and differential forms. Since \( v_H \) is nowhere vanishing, so is \( \omega_H \).

The complex structure of \( Q \) is determined by a lattice \( \Lambda \):

Theorem 1 (Abel-Jacobi map). Let \( Q \) be the Riemann surface determined by \( Q(x,y) \) with genus 1. Let \( \omega \) be the Abelian differential determined by the Hamiltonian vector field \( v_H \). Define a lattice \( \Lambda = \langle \omega_1, \omega_2 \rangle \) by complete elliptic integrals:
\[ \begin{align*}
\omega_1 &= \int_{\gamma_1} \frac{dx}{\sqrt{\Delta_1(x)}} = \int_{\gamma_1} \frac{dy}{\sqrt{\Delta_1(y)}} \\
\omega_2 &= \int_{\gamma_2} \frac{dx}{\sqrt{\Delta_2(x)}} = \int_{\gamma_2} \frac{dy}{\sqrt{\Delta_2(y)}}
\end{align*} \] (15)
where \([\gamma_1]\) and \([\gamma_2]\) form a basis for \(H_1(Q, \mathbb{Z})\). Let \(p_0 \in Q\) be an arbitrary point. Then the Abel-Jacobi map

\[ J : Q \to \mathbb{C}/\Lambda \]  

given by incomplete elliptic integrals along a path from \(p_0\) to \(p\)

\[ p \mapsto \int_{p_0}^p \omega \pmod{\Lambda}, \]  

is well-defined and does not depend on the path.

Remark. Under a modular group \(PSL(2, \mathbb{Z})\) action, we may choose \(\Lambda = \mathbb{Z} + \tau \mathbb{Z}\), where \(\tau = \pm \omega_2 \omega_1\). The \(\pm\) sign here makes \(\text{Im}(\tau) > 0\).

Remark. \(J\) depends on the choice of \(p_0\) but in a trivial way: A different choice of \(p_0\) gives an integration constant and hence a translation on \(\mathbb{C}/\Lambda\).

The QRT map \(\delta\) induces an addition on \(\mathbb{C}/\Lambda\) via the Abel-Jacobi map:

**Proposition 1.** The following diagram is commutative

\[
\begin{array}{ccc}
Q & \xrightarrow{\delta} & Q \\
\downarrow J & & \downarrow J \\
\mathbb{C}/\Lambda & \xrightarrow{\delta^*} & \mathbb{C}/\Lambda
\end{array}
\]

where the map \(\delta^*\) is given by

\[ \delta^*(z) = z + \omega_3 \pmod{\Lambda} \]  

for \(z \in \mathbb{C}/\Lambda\).

### 2.3 Weierstrass normal form

Since the field \(K\), over which an elliptic curve \(E\) is defined, plays a role, we will write \(E(K)\) to mean that the polynomial defining \(E\) has coefficients over \(K\) and there exists a point on \(E\) with coordinates in \(K\).

Our goal is to transform a biquadratic curve \(Q\) to an elliptic curve \(E(\mathbb{C})\) in the Weierstrass normal form.

**Definition 2.6** (Weierstrass normal form). An elliptic curve \(E(\mathbb{C})\) is said to be in the Weierstrass normal form if \(E(\mathbb{C})\) is defined by the polynomial

\[ y^2 = 4x^3 - g_2x - g_3. \]  

An elliptic curve carries a natural Abelian group law \(+ : E \times E \to E\). In the Weierstrass normal form, the Abelian group law can be described by the usual chord-tangent construction with the identity element being the point at infinity.

We have transformed the biquadratic curve \(Q\) to the lattice \(\Lambda\) via the Abel-Jacobi map \(J\). We need Weierstrass \(\wp\) functions to transform \(\Lambda\) to \(E(\mathbb{C})\).
Definition 2.7 (Weierstrass function). The Weierstrass $\wp$ function for a lattice $\Lambda = \langle \omega_1, \omega_2 \rangle$ is

$$\wp(z) := \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}} \left\{ \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right\}$$  \hspace{0.5cm} (20)

with derivative being

$$\wp'(z) = -2 \sum_{m,n \in \mathbb{Z}} \frac{1}{(z - m\omega_1 - n\omega_2)^3}.$$  \hspace{0.5cm} (21)

Both series for $\wp$ and $\wp'$ converge locally uniformly in $\mathbb{C} - \Lambda$, hence they define holomorphic functions $\mathbb{C} - \Lambda$. $\wp$ and $\wp'$ are meromorphic in $\mathbb{C}$ and have pole of order 2 and 3 respectively on $\Lambda$.

Definition 2.8 (Modular invariants). The modular invariants for a lattice are defined by

$$g_2(\Lambda) := 60 \sum_{(m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^4}$$  \hspace{0.5cm} (22)

and

$$g_3(\Lambda) := 140 \sum_{(m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^6}.$$  \hspace{0.5cm} (23)

The following theorem transform a lattice to an elliptic curve in Weierstrass normal form

Theorem 2 (Inverse Abel-Jacobi map). The map

$$J'^{-1} : \mathbb{C}/\Lambda \rightarrow E(\mathbb{C})$$  \hspace{0.5cm} (24)

where $E(\mathbb{C})$ is an elliptic curve in the Weierstrass normal form given by

$$z \mapsto (\wp(z), \wp'(z))$$

is an isomorphism of analytic manifold and also a group isomorphism.

Remark. The map in the theorem is an inverse of the Abel-Jacobi map. First, we use $J$ to transform $Q$ to $\mathbb{C}/\Lambda$. Then we use $J'^{-1}$ to transform $\mathbb{C}/\Lambda$ to an elliptic curve in the Weierstrass normal form, which is isomorphic to $Q$ but in a different coordinate system. The composition $J'^{-1} \circ J$ has the effect of a change of coordinate.
\[ \begin{array}{c}
Q \xrightarrow{\delta} Q \\
\downarrow \mathcal{J} \quad \downarrow \mathcal{J} \\
\mathbb{C}/\Lambda \xrightarrow{\mathcal{J}^{-1}} \mathbb{C}/\Lambda \\
\downarrow \mathcal{J}^{-1} \quad \downarrow \mathcal{J}^{-1} \\
E(\mathbb{C}) \xrightarrow{\delta^{**}} E(\mathbb{C})
\end{array} \]

Denote \( \mathcal{J}^{-1}(\omega_3) \) by \( \Omega_3 \). Since \( \mathcal{J}^{-1} \) is a group isomorphism, we have

\[ \delta^{**}(P) = P + \Omega_3 \quad (25) \]

for \( P \in E(\mathbb{C}) \).

3 Complex to rational

For an elliptic curve over \( \mathbb{Q} \), following theorems hold.

**Theorem 3** (Mordell). The rational points on an elliptic curve form a finitely generated Abelian group.

**Theorem 4** (Mazur). For any \( E(\mathbb{Q}) \), the torsion subgroup \( T \) has only the following possibilities:

1. \( \mathbb{Z}/N\mathbb{Z} \), where \( 1 \leq N \leq 10 \) or \( N = 12 \),
2. \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} \), where \( 1 \leq N \leq 4 \).

In Section 2, we worked in \( \mathbb{C} \). To apply Mordell’s and Mazur’s theorems, we need to work in \( \mathbb{Q} \), that is, we need to work in \( E(\mathbb{Q}) \subset E(\mathbb{C}) \). Therefore, we need to show that the elliptic curve \( E = \mathcal{J}^{-1} \circ \mathcal{J}(Q) \) has rational coefficients and \( \Omega_3 = \mathcal{J}^{-1}(\omega_3) \in E(\mathbb{C}) \) is in fact in \( E(\mathbb{Q}) \).

**Lemma 5.** Let \( \Lambda \) be the lattice determined by \( Q \) as in Theorem 1. Then,

\[ g_2(\Lambda) = D(\Delta_1) = D(\Delta_2), \quad (26) \]

and

\[ g_3(\Lambda) = -E(\Delta_1) = -E(\Delta_2), \quad (27) \]

where \( E \) and \( D \) are Eisenstein invariants in Definition 2.4 and \( \Delta_1 \) and \( \Delta_2 \) are partial discriminants of \( Q \) in Definition 2.7

**Proof.** See Corollary 2.4.7 of [3].

**Remark.** The lemma says that although both the uniformization of \( Q \) by \( \Lambda \) and the Abel-Jacobi map of \( \mathbb{C}/\Lambda \) are analytic, their composition is completely given by polynomial functions.
The following lemma shows how the QRT map $\delta$ transforms to a polynomial map under $J$ and $J^{-1}$.

**Lemma 6.** The addition $\delta^*$ on $E(\mathbb{C})$ induced by the QRT map $\delta$ sends the point at infinity $O$ to $(X,Y)$, where

$$X = (p_{0,0}^2 - 4p_{0,-1}p_{1,0} - 4p_{-1,0}p_{1,0} + 8p_{-1,1}p_{1,-1} + 8p_{-1,-1}p_{1,1})/12$$

and

$$Y = -\det P,$$

where

$$P = \begin{pmatrix} p_{1,1} & p_{1,0} & p_{1,-1} \\ p_{0,1} & p_{0,0} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{pmatrix}$$

\begin{proof}
See Proposition 2.5.6 of [3].
\end{proof}

### 4 Main result

Gather around all information, we state the main result of this paper.

**Theorem 7.** A finite Galois group $\mathcal{H}$ of the weighted walk with rational coefficients can have order at most 24.

\begin{proof}
Since the kernel $Q(x,y)$ has rational coefficients, Lemma 5 says that the associated elliptic curve $E$ in the Weierstrass normal form

$$y^2 = 4x^3 - g_2x - g_3$$

also has rational coefficients, i.e. $g_2, g_3 \in \mathbb{Q}$. Lemma 6 says that $\Omega_3 = J^{-1}(\omega_3) \in E(\mathbb{Q})$. Then the group $\langle \Omega_3 \rangle$ generated by $\Omega_3$ is a subgroup of $E(\mathbb{Q})$. By Proposition 1 and Theorem 2

$$\mathcal{H}_0 \cong \langle \Omega_3 \rangle.$$

Hence $\mathcal{H}_0 \leq E(\mathbb{Q})$. Mordell’s theorem says that $E(\mathbb{Q})$ is finitely generated, hence by the fundamental theorem of finitely generated Abelian group, we have

$$\mathcal{H}_0 \leq \mathbb{Z}^r \oplus T,$$

where $r \in \mathbb{Z}_+$ and $T$ is the torsion subgroup. Since $\mathcal{H}$ is assumed to be finite, we have

$$\mathcal{H}_0 \leq T.$$

Mazur’s theorem says that $|T| \leq 12$, so $|\mathcal{H}_0| \leq 12$. By Equation (11), $|\mathcal{H}| \leq 24$.

We rederive two known criteria for the weighted walk having order 4 and 6 using geometric argument.
Theorem 8 (Criterion for \( \mathcal{H} \) of order 4). \( \mathcal{H} \) has order 4 if and only if \( \det P = 0 \).

**Proof.** \( \mathcal{H} \) has order 4 if and only if \( \Omega_3 \) is a torsion point of order 2 in \( E(\mathbb{C}) \). The result is obtained by the fact that a point in a Weierstrass normal curve has order 2 if and only if its \( Y \) coordinate is 0. \( \square \)

Theorem 9 (Criterion for \( \mathcal{H} \) of order 6). \( \mathcal{H} \) has order 6 if and only if

\[
\begin{vmatrix}
-12X & 0 & D \\
0 & 1 & Y \\
D & Y & DX + 3E
\end{vmatrix} = 0
\]

where \( \Omega_3 = (X,Y) \) is given by Lemma 6 and \( D := D(\Delta_1) = D(\Delta_2) \), \( E := E(\Delta_1) = E(\Delta_2) \) are Eisenstein invariants given by Lemma 5.

**Proof.** \( \mathcal{H} \) has order 6 \( \iff \) \( \Omega_3 = ((X,Y) \) is a torsion point of order 3 in \( E(\mathbb{C}) \) \( \iff \) \( \Omega_3 \) is a flex point \( \iff \) \( \det (\text{Hess}(f)) \) vanishes on \( (X,Y,1) \), where \( f(x,y,z) \) is the homogeneous polynomial

\[ f = y^2z - 4x^3 + g_2xz^2 + g_3z^3 \]

. The result is obtained by direct calculation. \( \square \)

**Remark.** The following result for \( \mathcal{H} \) to have order 6 is give by Proposition 4.1.8 in [4]:

\( \mathcal{H} \) has order 6 if and only if

\[
\begin{vmatrix}
\Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\
\Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\
\Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\
\Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33}
\end{vmatrix} = 0,
\]

where \( \Delta_{ij} \)'s are cofactors of the matrix \( P \).

5 Criterion for orders 4m and 4m+2

In this section, we give criteria for \( \mathcal{H} \) having orders \( 4m \) or \( 4m + 2 \) using division polynomials. The criteria given in [4] are abstract, requiring linear dependence of certain functions in some function field. Our result is completely given by polynomials.

**Definition 5.1** (Division polynomials). Let \( y^2 = x^3 + ax + b \) be an elliptic
The division polynomials are

\[
\begin{align*}
\psi_1 &= 1, \\
\psi_2 &= 2y, \\
\psi_3 &= 3x^4 + 6ax^2 + 12bx - a^2 \\
\psi_4 &= 4y(x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - 8b^2 - a^3), \\
\psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3, \quad m \geq 2, \\
\psi_{2m} &= \frac{1}{2y}\psi_m(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+2}^2), \quad m \geq 3.
\end{align*}
\]

**Remark.** These polynomials get the name because \(m|n \Rightarrow \psi_m|\psi_n\) in \(\mathbb{Z}[x,y,a,b]\) and \((x,y)\) is a torsion point of order dividing \(n\) if and only if \((x,y)\) is a zero of \(\psi_n\).

This is the definition commonly found in textbooks, for example page 105 of [9]. To use it in our case \(y^2 = 4x^3 - 2g_2x - g_3\), we need change variables. We believe these formula have been known for a long time but hard to find in literature.

**Theorem 10** (Division polynomials in \(g_2\) and \(g_3\)).

\[
\begin{align*}
\Psi_1 &= 1, \\
\Psi_2 &= y, \\
\Psi_3 &= 48x^4 - 24g_2x^2 - 48g_3x - g_3^2 \\
\Psi_4 &= y(64x^6 - 80g_2x^4 - 320g_3x^3 - 20g_2^2x^2 - 16g_2g_3x + g_3^3 - 32g_3^2), \\
\Psi_{2m+1} &= \Psi_{m+2}\Psi_m^3 - \Psi_{m-1}\Psi_{m+1}^3, \quad m \geq 2, \\
\Psi_{2m} &= \frac{1}{y}\Psi_m(\Psi_{m+2}\Psi_{m-1}^2 - \Psi_{m-2}\Psi_{m+2}^2), \quad m \geq 3.
\end{align*}
\]

**Proof.** Direct change of variables. \qed

Following corollaries give criteria for \(\mathcal{H}\) to have various orders. Their proofs are obvious.

**Corollary 10.1** (Criterion for \(\mathcal{H}\) of order 8). \(\mathcal{H}\) has order 8 if and only if \(Y \neq 0\) and

\[
64X^6 - 80DX^4 - 320EX^3 - 20D^2X^2 - 16DEX + D^3 - 32E^2 = 0.
\]

where \(\Omega_3 = (X,Y)\) and \(D, E\) are Eisenstein invariants.

**Remark.** The following result for \(\mathcal{H}\) to have order 8 is give by Proposition 4.1.11 in [4]:

\(\mathcal{H}\) has order 8 if and only if

\[
\begin{vmatrix}
A & B & C \\
D & E & F \\
G & H & I
\end{vmatrix} = 0,
\]

11
where $A = 2\Delta_{22}\Delta_{32} - (\Delta_{21}\Delta_{33} + \Delta_{31}\Delta_{23})$, $B = 2(\Delta_{22}^2 - \Delta_{12}\Delta_{31} + \Delta_{21}\Delta_{23}) + \Delta_{11}\Delta_{33} + \Delta_{31}\Delta_{13}$, $C = 2\Delta_{12}\Delta_{22} - (\Delta_{11}\Delta_{23} + \Delta_{21})$, $D = \Delta_{22}^2 - \Delta_{31}\Delta_{33}$, $E = -2\Delta_{32}\Delta_{22} + \Delta_{31}\Delta_{23} + \Delta_{21}\Delta_{33}$, $F = \Delta_{22}^2 - \Delta_{21}\Delta_{23}$, $G = \Delta_{22}^2 - \Delta_{21}\Delta_{23}$, $H = -2\Delta_{22}\Delta_{12} + \Delta_{11}\Delta_{23} + \Delta_{13}\Delta_{21}$, and $I = \Delta_{12}^2 - \Delta_{11}\Delta_{13}$, and $\Delta_{ij}$'s are cofactors of the matrix $\mathbb{P}$.

Our result is much simpler.

**Corollary 10.2** (Criterion for $\mathcal{H}$ of order $4m$). $\mathcal{H}$ has order $4m$ if and only if $\Psi_n(X,Y) \neq 0$ for all $n|2m$, $n \neq 2m$ and $\Psi_{2m}(X,Y) = 0$.

**Corollary 10.3** (Criterion for $\mathcal{H}$ of order $4m + 2$). $\mathcal{H}$ has order $4m + 2$ if and only if $\Psi_n(X,Y) \neq 0$ for all $n|(2m + 1)$, $n \neq 2m + 1$ and $\Psi_{2m+1}(X,Y) = 0$.

### 6 Discussion

We found that the finite $\mathcal{H}$ can have order at most 24 for weighted walks with rational weights. Geometric proofs of the criterion for $\mathcal{H}$ to have order 4 and 6 are given. In particular for the case of order 6, the result is simpler than Proposition 4.1.8 of [4]. Using division polynomial, a recursive criterion for $\mathcal{H}$ to have order $4m$ or $4m + 2$ is also obtained and explicit criterion for $\mathcal{H}$ to have order 8 is given almost with no computation, much simpler than Proposition 4.1.11 of [4].

Since 24 is only an upper bound, further work on finding possible realizations of different group orders is needed.

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