Random periodic solutions of nonautonomous stochastic feedback systems with multiplicative noise

Zhao Dong\textsuperscript{1,2}, Weili Zhang\textsuperscript{1,2,*} and Zuohuan Zheng \textsuperscript{3,1,2}

\textsuperscript{1} Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
\textsuperscript{2} School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, China
\textsuperscript{3} College of Mathematics and Statistics, Hainan Normal University, Haikou, Hainan 571158, China
dzhao@amt.ac.cn, zhangweili@amss.ac.cn, zhzheng@amt.ac.cn

Abstract: We investigate the dynamical behavior of pull-back trajectories for nonautonomous stochastic feedback systems with multiplicative noise. We proved that there exists a random periodic solution of this system and all pull-back trajectories converge to this random periodic solution as time goes to infinitely almost surely. Our results can be applied to nonautonomous stochastic Goodwin negative feedback system, nonautonomous stochastic Othmer-Tyson positive feedback system and nonautonomous stochastic competitive systems etc.

MSC: 93E03; 93E15; 93C10; 37H05; 60H10

Keywords: Stochastic feedback system, stochastic flow, random dynamical system, random periodic solution.

1 Introduction

For deterministic systems, there is a well-developed and constructive theory of systems interconnections and feedback, such as the very successful and widely applied backstepping approach \cite{20,16} and stability analysis \cite{4,11,13,14,15,32,33}. Feedback loops play important roles in many biochemical control systems, which often occur in the study of the reaction process in

* Corresponding author
cellular signalling, such as [3, 27, 28]. It is natural to attempt to extend such work to stochastic systems considering the real world phenomena. As a matter of fact, much excellent research has been done pursuing such extensions, notably studies on stochastic stability [6, 23, 25, 26, 35]. Recently, Marcondes de Freitas and Sontag have initiated a different approach based upon random dynamical systems to investigated the stability of feedback systems involving real noise perturbation in [10]. Motivated by them, Jiang and Lv considered the global stability of nonlinear stochastic feedback systems driven by additive and multiplicative white noise respectively in [17, 18].

It is natural to attempt to extend nonlinear output (feedback) function to time-periodic feedback function, that is, consider non-autonomous stochastic systems. Indeed, we have considered the stable random periodic solution of non-autonomous stochastic feedback systems driven by additive white noise in [5]. Our goal in this paper is to prove that there exists a random periodic solutions of non-autonomous stochastic feedback systems driven by multiplicative white noise and all pull-back trajectories converge to this random periodic solution as time tends to infinitely almost surely. In this paper, we will make full use of the theory of random dynamical systems established by L. Arnold [1] and stochastic flows established by H. Kunita [21, 22]. And the powerful theory of monotone random dynamical systems [2] can be applied to investigate the global stability of stochastic flows while the stochastic system admits the stochastic comparison principle, i.e., the system is cooperative or monotone. There are several literatures which use random dynamical systems to investigate the existence of random periodic solution, such as [36, 7, 8, 9]. Compared with the existing ones, we also consider the global stability.

As a motivation, we first look at a simple biochemical circuit. This biochemical circuit contains three chemical species $X_1$, $X_2$, $X_3$ that interact with one another as shown in Figure 1. Systems of this type are routinely studied as molecular biology, biochemical reaction systems. Furthermore, the strength of the interactions between the species, may depend on environmental factors such as temperature and the concentrations of other biochemical compounds not explicitly modeled. This dependence may be periodicity and randomness intrinsically. If this is the case, then a more realistic mathematical model would be a non-autonomous stochastic feedback system of the following form:

$$dx_i = (\alpha_i x_i + h_i(t, x_{i-1}))dt + \sigma_i x_i dW^i_t, \quad i = 1, 2, 3, \quad (1.1)$$

which indices taken modulo three, so $x_0 = x_3$. Here, $\alpha_i$, $i = 1, 2, 3$ are negative constants, $h_i(\cdot, x_{i-1}), i = 1, 2, 3$ are nonincreasing functions in $x_{i-1}$, $h_i(t + T, x_{i-1}) = h_i(t, x_{i-1}), i = 1, 2, 3$, $T$ is a positive constant, and $W_i(\omega) = (W^1_i(\omega), W^2_i(\omega), W^3_i(\omega))$ is a three dimensional standard Brownian motion with $W^i_0(\omega) = 0, i = 1, 2, 3, \omega \in \Omega$. 

![Figure 1: Biochemical circuit. The symbol “$X \to Y$” means that species $X$ represses the production of species $Y$.](image-url)
Figure 2: Decomposition of the biochemical circuit from Figure 1 into input-output modules. In each partition, $v_i$ indicates the input into the element $X_i$ and $h_i(\cdot, x_{i-1})$ indicates the subsequent output—feedback of the current state.

The nonlinearity of $h_i$ makes the system difficult to study directly. To overcome this difficulty, we use the decomposition motivated by the work of [10, 17, 18]. This idea is to look at (1.1) as a network of smaller input-output modules as shown in Figure 2, and then we can derive the closed system’s properties from these smaller modules’ emerging properties. The first step is to open up the feedback loop, rewriting the model as a stochastic system with inputs

$$dx_i = (\alpha_i x_i + v_i(t))dt + \sigma_i x_i dW_t^i, \quad i = 1, 2, 3,$$

(1.2)

together with a set of outputs

$$y_i(t) = v_i(t) = h_i(t, x_{i-1}(t)), \quad i = 1, 2, 3.$$  

(1.3)

Observe that (1.2) is much easier to study. In fact, in this particular example, we can show that (1.2) has a unique, globally attracting random periodic solution $K(v)$ for each random periodic input $v$. We call $K$ defined in this way the input-to-state characteristic of the system.

The closed-loop system satisfies certain conditions, the next step is to look at the gain of the system. The output function is read at $K(v)$ for each random periodic input $v$, and an operator $K^h$ is so defined on the space of random periodic inputs. If $K^h$ has a unique, globally attracting random periodic solution, then the input-output system (1.2) and (1.3) is said to satisfy the small-gain condition. It is natural to believe that the closed-loop system should have random periodic solution under such circumstances. Periodicity and monotonicity assumptions ensures the system has a unique, globally attracting random periodic solution.

The goal of this paper may be described as to give a rigorous treatment of this example and its generalizations to more general situations. Considering the following $T$-periodic stochastic feedback system with multiplicative linear noise in $\mathbb{R}^d_+$:

$$dX_t = (AX_t + h(t, X_t))dt + \sum_{k=1}^d \sigma_k X_t dW_t^k,$$

(1.4)

where $W_t = (W_t^1, ..., W_t^d)$ is a two-side time Wiener process with values in $\mathbb{R}^d$ on the canonical Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$, i.e., $\mathcal{F}$ is the Borel $\sigma$-algebra of $\Omega = C_0(\mathbb{R}, \mathbb{R}^d) = \{\omega : \omega(t) \text{ continuous, } \omega(0) = 0, t \in \mathbb{R}\}; \mathcal{F}_s^d$ is the least complete $\sigma$-field for which all $W_u - W_v, s \leq v \leq u \leq t$ are measurable and $\mathcal{F}^t = \mathcal{F}^t_{\infty} = \bigvee_{s \leq t} \mathcal{F}_s^d; \mathbb{P}$ is the Wiener measure. $A = (a_{ij})_{d \times d}$ is a $(d \times d)$-dimensional matrix. $h : \mathbb{R} \times \mathbb{R}^d_+ \rightarrow \mathbb{R}^d_+, h(t + T, x) = h(t, x)$ for any $t \in \mathbb{R}, x \in \mathbb{R}^d_+, T > 0$ is a constant. $\sigma_k, k = 1, \cdots, d$ are $(d \times d)$-dimensional matrices.

This paper is organized as follows. In section 2, we review some preliminary concepts and definitions, present the assumptions for the stochastic differential equation (1.4), and define the
input-to-state characteristic operator of the system via the pull-back of the discretised stochastic differential equation. In section 3, we describes the asymptotic behavior of stochastic solution, give some auxiliary lemmas, present the definition of gain operator and its properties. In section 4, the main theorem is proved and the global convergence to a unique random periodic solution is presented. In section 5, we present some examples.

Convention: Throughout this paper, without loss of generality we always denote a universal set of full $\mathbb{P}$-measure by $\Omega$.

2 Preliminaries

For the convenience of readers, we recall some definitions and basic facts about random dynamical systems and stochastic flows, see [1, 2, 21, 22] for more details. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X$ be a Polish space and $\mathcal{B}(X)$ be its Borel $\sigma$-algebra. Denote $\Delta := \{(t, s) \in \mathbb{R}^2, s \leq t\}$.

**Definition 2.1 ([1])** A family of mapping on the sample space $\Omega$, $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$ is called a measurable dynamical system if the following conditions are satisfied

(i) Identity property: $\theta_0$ is the identity on $\Omega;

(ii) Flow property: $\theta_{t+s} = \theta_t \circ \theta_s$, where $\circ$ means composition of mappings;

(iii) Measurability: $(\omega, t) \mapsto \theta_t \omega$ is measurable.

It is called a measure-preserving or metric dynamical system, if furthermore

(iv) Measure-preserving property: $\mathbb{P}(\theta_t(A)) = \mathbb{P}(A)$, for every $A \in \mathcal{F}$ and $t \in \mathbb{R}$.

In this case, $\mathbb{P}$ is called an invariant measure with respect to the dynamical system $\theta_t$.

**Definition 2.2 ([1])** A (continuous) random dynamical system (RDS) on the Polish space $X$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with time $\mathbb{R}_+$ is a mapping $\Phi: \mathbb{R}_+ \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \Phi(t, \omega, x)$ which is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies the following properties:

(i) Continuity: $\Phi(\cdot, \omega, \cdot): \mathbb{R}_+ \times X \to X$, $(t, x) \mapsto \Phi(t, \omega, x)$ is continuous for all $\omega \in \Omega$.

(ii) Cocycle property: The mappings $\Phi(t, \omega) := \Phi(t, \omega, \cdot): X \to X$ form a cocycle over $\theta(\cdot)$, i.e. they satisfy for all $\omega \in \Omega$,

$$
\Phi(0, \omega) \text{ is the identity on } X,
\Phi(t + s, \omega) = \Phi(t, \theta_s \omega) \circ \Phi(s, \omega), \text{ for all } s, t \in \mathbb{R}_+.
$$
Definition 2.3 \([11]\) A random variable \(R : \Omega \to \mathbb{R}_+\) is called tempered with respect to the dynamical system \(\theta\) if
\[
\lim_{t \to \pm \infty} \frac{\log R(\theta_t \omega)}{|t|} = 0.
\]
This condition is equivalent to the subexponential growth of \(t \mapsto R(\theta_t \omega)\),
\[
\lim_{t \to \pm \infty} \left\{ e^{-\gamma |t|} R(\theta_t \omega) \right\} = 0, \text{ for any } \gamma > 0, \omega \in \Omega,
\]
which implies that, for any \(\gamma > 0, \omega \in \Omega\)
\[
\sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t|} R(\theta_t \omega) \right\} < \infty.
\]

Definition 2.4 \([22]\) A map \(\varphi : \Delta \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d\), \((t,s,x,\omega) \mapsto \varphi(t,s,x,\omega)\) is called a forward stochastic flow if for almost all \(\omega \in \Omega\), it satisfies the following properties:
\begin{itemize}
  \item[(i)] \(\varphi(t,s,x,\cdot)\) is continuous with respect to \((s,t,x)\).
  \item[(ii)] \(\varphi(u,s,\omega) = \varphi(u,t,\omega) \circ \varphi(t,s,\omega)\) holds for all \(s \leq t \leq u\).
  \item[(iii)] \(\varphi(s,s,\omega)\) is the identity map on \(X\) for all \(s\).
\end{itemize}

Definition 2.5 \([9]\) A random periodic solution of period \(T > 0\) for the forward stochastic flow \(\varphi : \Delta \times \Omega \times X \to X\) is an \(\mathcal{F}\)-measurable map \(Y : \mathbb{R} \times \Omega \to X\) such that for almost all \(\omega \in \Omega\),
\[
\varphi(t,s,\omega)Y(s,\omega) = Y(t,\omega), \ Y(r + T,\omega) = Y(r,\theta_T \omega), \text{ for any } (t,s) \in \Delta, \ r \in \mathbb{R}. \quad (2.1)
\]
Firstly, we consider the corresponding linear homogeneous Itô stochastic differential equations:
\[
\frac{dX_t}{dt} = AX_t dt + \sum_{k=1}^{d} \sigma_k X_t dW^k_t.
\]
Without loss of generality, we assume that \(\sigma_k, k = 1, \cdots, d\) has the following form
\[
\sigma_k = \begin{pmatrix}
  \sigma^1_k \\
  \cdots \\
  \sigma^d_k
\end{pmatrix}, \quad \sigma^i_k \in \mathbb{R}, \ k, i = 1, \cdots, d.
\]
A more general situation can be reduced to this one by a diagonalizing linear transformation. According to Definition 3.3.13 in \([19]\), it is equivalent to the following Stratonovich stochastic differential equations
\[
\frac{dX_t}{dt} = (A - \frac{1}{2}B)X_t dt + \sum_{k=1}^{d} \sigma_k X_t \circ dW^k_t, \quad (2.2)
\]
where
\[
B = \begin{pmatrix}
\sum_{k=1}^d (\sigma_k^1)^2 \\
\vdots \\
\sum_{k=1}^d (\sigma_k^d)^2
\end{pmatrix}.
\]

Next, we introduce assumptions that guarantee that the random dynamical system generated by (2.2) in \( R^d_+ \) is order-preserving and the existence and uniqueness of solutions for stochastic differential equations (1.4). In order to make use of the technique for monotone systems, we make the following standing assumption on \( A \).

\((A)\) \( A \) is cooperative, i.e., \( a_{ij} \geq 0 \) for all \( i, j \in \{1, \ldots, d\} \) and \( i \neq j \).

Throughout this paper, we will use the norm \(|x| := \max\{|x_i| : i = 1, \ldots, d\}\), \(|x|_2 := (\sum_{i=1}^d |x_i|^2)^{1/2}\), \(x \in \mathbb{R}^d\) and \(|M|_2 := (\sum_{i,j=1}^d |M_{ij}|^2)^{1/2}\), where \( M \) is a \((d \times d)\)-dimensional matrix.

Let \( \Phi_j(t) = (\Phi_{1j}(t), \ldots, \Phi_{dj}(t))^T \) be the solution of equation (2.2) with initial value \( X(0) = e_j, j = 1, \ldots, d \). Define the \( d \times d \) matrix
\[
\Phi(t) = (\Phi_1(t), \ldots, \Phi_d(t)) = (\Phi_{ij}(t))_{d \times d}.
\]
Then \( \Phi(t) \) is the fundamental matrix of equation (2.2). It is useful to note that \( \Phi(0) \) is the \( d \times d \) identity matrix and
\[
d\Phi(t) = A\Phi(t)dt + \sum_{k=1}^d \sigma_k \Phi(t)dW_t^k.
\]
By Proposition 6.2.2 in [2], it is clear that (2.2) generates a order-preserving random dynamical system \((\theta, \Phi)\) in \( R^d_+ \) and \( \Phi(t, \omega)(\mathbb{R}^d_+ \setminus \{0\}) \subset \mathbb{R}^d_+ \setminus \{0\} \) for any \( t \geq 0, \omega \in \Omega \), where \( \theta \) is the time shift on \( \Omega \), i.e.,
\[
\theta \omega(\cdot) := \omega(t + \cdot) - \omega(t), \quad t \in \mathbb{R}.
\]
\( \Phi \) satisfies the cocycle property: \( \Phi(t+s, \omega) = \Phi(t, \theta_s \omega) \circ \Phi(s, \omega) \) for all \( t, s \in \mathbb{R}_+, \omega \in \Omega \), and \( \Phi(t, \omega)x \geq_{R^d_+} \Phi(t, \omega)y \) for all \( x, y \in \mathbb{R}^d_+ \) such that \( x \geq_{R^d_+} y \), where \( x \geq_{R^d_+} y \) means that \( x - y \in \mathbb{R}^d_+ \).

Now we give the assumption of \((\theta, \Phi)\) which will be needed in what follows.

\((L)\) The top Lyapunov exponent for the linear RDS \((\theta, \Phi)\) is a negative real number, i.e., there exist a constant \( \lambda > 0 \) and a stochastic process \( R(t, \omega) > 0 \) which satisfies that for any \( \gamma > 0 \),
\[
\sup_{t \in \mathbb{R}} \{ e^{-\gamma t} \sup_{s \in \mathbb{R}_+} R(s, \theta_t \omega) \} < \infty \quad \text{such that for all } t \geq 0, \omega \in \Omega,
\]
\[
\| \Phi(t, \omega) \| := \max\{|\Phi_{ij}(t, \omega)| : i, j = 1, \ldots, d\} \leq R(t, \omega)e^{-\lambda t}.
\]
In the remainder of this section, we discuss the questions of existence and uniqueness of solutions for stochastic differential equation (1.4), as well as its pull-back trajectories. Let us start with the following assumptions on $h$.

\[ (H) \quad h \in C^1_b(\mathbb{R} \times \mathbb{R}^d_+, \mathbb{R}^d_+ \setminus \{0\}) , \text{i.e., the function } h \text{ and its derivatives are both bounded. And } h \text{ is order-preserving in } \mathbb{R}^d_+, \text{i.e., for any } t \in \mathbb{R} \]

\[ h(t, x_1) \leq_R^d h(t, x_2) \text{ whenever } x_1, x_2 \in \mathbb{R}^d_+ \text{ such that } x_1 \leq_R^d x_2, \]

or anti-order-preserving in $\mathbb{R}^d_+$, i.e.,

\[ h(t, x_1) \geq_R^d h(t, x_2) \text{ whenever } x_1, x_2 \in \mathbb{R}^d_+ \text{ such that } x_1 \leq_R^d x_2. \]

By $(H)$, it is easy to check that (1.4) satisfies the conditions of the global Lipschitz and linear growth in $\mathbb{R}^d_+$, since $h$ and its derivatives are both bounded in $\mathbb{R}^d_+$. Let $\tilde{h}$ be an extension from $\mathbb{R} \times \mathbb{R}^d_+$ to $\mathbb{R} \times \mathbb{R}^d$ such that $\tilde{h}(t, x) = h(t, |x^1|, \cdots, |x^d|), x = (x^1, \cdots, x^d) \in \mathbb{R}^d$. It is clearly that $\tilde{h}$ satisfies the conditions of global Lipschitz and linear growth in $\mathbb{R}^d$, we thus have the existence and uniqueness of global solutions for

\[ dX_t = [AX_t + \tilde{h}(t, X_t)]dt + \sum_{k=1}^d \sigma_k X_t dW^k_t \]

in $\mathbb{R}^d$, which is a forward stochastic flow $\varphi(t, s, \omega) : \mathbb{R}^d \to \mathbb{R}^d$ (cf. [19, 22, 26, 29]). From the form of $\tilde{h}$, we can deduce that the set $\mathbb{R}^d_+$ is forward invariant under the forward stochastic flow, i.e., $\varphi(t, s, \omega) \mathbb{R}^d_+ \subset \mathbb{R}^d_+$ for $(t, s) \in \Delta$, $\omega \in \Omega$ and $\varphi(t, s, \omega)x = X(t, s, x, \omega), x \in \mathbb{R}^d_+$ is a unique solution of (1.4). Furthermore, it can be written as

\[ \varphi(t, s, \omega)x = \Phi(t - s, \theta_s \omega)x + \int_s^t \Phi(t - r, \theta_r \omega)h(r, \varphi(r, s, \omega)x)dr, \ (t, s) \in \Delta, \ x \in \mathbb{R}^d_+. \tag{2.5} \]

**Proposition 2.6** \( \varphi(t, s, \omega)x \) in (2.5) has the following properties:

(i) For all $(t, s) \in \Delta$ and $x \in \mathbb{R}^d_+$, $\varphi(t, s, \cdot)x$ is $\mathcal{F}^t_{s-}$-measurable.

(ii) For all $\omega \in \Omega$, $\varphi(t, s, \omega)x$ is continuous in $(t, s, x)$ and satisfies $\lim_{t \downarrow s} \varphi(t, s, x, \omega) = x$.

(iii) For all $\omega \in \Omega$,

\[ \varphi(t, s, \omega) = \varphi(t, r, \omega) \circ \varphi(r, s, \omega), \text{ for all } s \leq r \leq t, \ s, r, t \in \mathbb{R}. \]

(iv) For all $(t, s) \in \Delta, \omega \in \Omega$,

\[ \varphi(t + T, s + T, \omega) = \varphi(t, s, \theta_T \omega). \tag{2.6} \]
Proof. The properties (i), (ii), (iii) are obvious. Next we prove (iv). For \((t, s) \in \Delta\), by the periodicity of \(h(\cdot, x)\) we have
\[
\varphi(t + T, s + T, \omega)x = \Phi(t - s, \theta_{s+T}\omega)x + \int_{s+T}^{t+T} \Phi(t + T - r, \theta_{r+T}\omega)h(r, \varphi(r, s + T, \omega)x)dr
\]
\[
= \Phi(t - s, \theta_{s+T}\omega)x + \int_s^t \Phi(t - r, \theta_{r+T}\omega)h(r, \varphi(r, s + T, \omega)x)dr
\]
\[
= \Phi(t - s, \theta_{s+T}\omega)x + \int_s^t \Phi(t - r, \theta_{r+T}\omega)h(r, \varphi(r + T, s + T, \omega)x)dr
\]
We thus have found that the function
\[
\psi(r, s, \theta_{T}\omega)x = \varphi(r + T, s + T, \omega)x, \ s \leq r
\]
satisfies
\[
\psi(t, s, \theta_{T}\omega)x = \Phi(t - s, \theta_{s+T}\omega)x + \int_t^s \Phi(t - r, \theta_{r+T}\omega)h(r, \psi(r, s, \theta_{T}\omega)x)dr.
\]
By the uniqueness of the solution
\[
\varphi(t, s, \theta_{T}\omega)x = \psi(t, s, \theta_{T}\omega)x = \varphi(t + T, s + T, \omega)x.
\]
which implies that \((2.6)\) holds.

For any \(n \in \mathbb{N}_+, t \geq -nT\), let \(\varphi(t, -nT, \omega)x\) be the solution with the initial value \(X(-nT) = x\), by \((2.5)\) we have
\[
\varphi(t, -nT, \omega)x = \Phi(t + nT, \theta_{-nT}\omega)x + \int_{-nT}^t \Phi(t - s, \theta_{s}\omega)h(s, \varphi(s, -nT, \omega)x)ds.
\]
By \((L)\), it is evident that for all \(x \in \mathbb{R}_+^d\), \(\omega \in \Omega\), \(\lim_{t \to \infty} \Phi(t, \theta_{-t}\omega)x = 0\). Regarding the feedback function \(h\) as an input term, we can define the input-to-state characteristic operator \(K\) associated with given inputs in \(\mathbb{R}_+^d\) as follows:
\[
K(v)(t, \omega) = \int_{-\infty}^t \Phi(t - s, \theta_{s}\omega)v(s, \omega)ds, \ t \in \mathbb{R}, \ \omega \in \Omega, \tag{2.7}
\]
where the stochastic process \(v : \mathbb{R} \times \Omega \to \mathbb{R}_+^d\) is bounded. It is evident that \(K\) is well defined by condition \((L)\). In fact, by \((2.4)\) we have \(\|\Phi(t, \omega)\|_2 \leq d\|\Phi(t, \omega)\| \leq dR(t, \omega)e^{-\lambda t}, \ \lambda > 0\), and so for any \(t \in \mathbb{R}, \ \omega \in \Omega\),
\[
\int_{-\infty}^t |\Phi(t - s, \theta_{s}\omega)v(s, \omega)|_2ds \\
\leq d \int_{-\infty}^t R(t - s, \theta_{s}\omega)e^{-\lambda(t-s)}|v(s + t, \omega)|_2ds \\
\leq Cd \sup_{s \in \mathbb{R}} \{e^{-\lambda|s|} \sup_{r \in \mathbb{R}_+} R(r, \theta_{s}\omega)\} \int_{-\infty}^t e^{-\lambda(t-s)+\lambda|s|}ds \\
< \infty.
\]
3 Measurability and asymptotic behavior

In this section, we give some lemmas to describe the measurability and the dynamical behavior of the pull-back trajectory which will be used in the proof of our main result. Throughout this paper, we set

\[ N = (N_1, ..., N_d), \quad N_i = \sup_{t \in \mathbb{R}, x \in \mathbb{R}^d_+} |h_i(t, x)|, \quad i = 1, ..., d. \]

Lemma 3.1 For each \( n \in \mathbb{N}^+, \ x \in \mathbb{R}_+^d, \ \alpha \in [0, T), \) let

\[ a^h_n(t, \omega) = \inf \{ h(t, \varphi(t, t - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+ \}, \ t \in \mathbb{R}, \ \omega \in \Omega, \]

and

\[ b^h_n(t, \omega) = \sup \{ h(t, \varphi(t, t - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+ \}, \ t \in \mathbb{R}, \ \omega \in \Omega, \]

where \( \inf \) and \( \sup \) mean the greatest lower bound and the least upper bound, respectively. Then \( a^h_n(t, \omega) \) and \( b^h_n(t, \omega) \) are progressively measurable with respect to \( \{ \mathcal{F}^t \} \).

Proof. We only prove the case of \( b^h_n(t, \omega) \) for the sake of convenience and the case of \( a^h_n(t, \omega) \) can be proved analogously. Define

\[ B^h_n(t, \omega) := \{ h(t, \varphi(t, t - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+ \}. \]

First we show that \( b^h_n(t, \omega) \) are well defined. For each \( n \in \mathbb{N}^+, \ x \in \mathbb{R}_+^d, \) by the boundedness of \( h \) we know that \( B^h_n(t, \omega) \) is a bounded set for any \( t \in \mathbb{R}, \ \omega \in \Omega, \) which implies that \( B^h_n(t, \omega) \) is order-bounded. And since \( \mathbb{R}_+^d \) is strongly minihedral (see [2], Definition 3.1.7), \( b^h_n(t, \omega) \) exists.

Next, we prove that \( b^h_n(t, \omega) \) are progressively measurable with respect to \( \{ \mathcal{F}^t \} \). Define

\[ \beta^h_{n,M}(t, \omega) = \sup \{ h(t, \varphi(t, t - \alpha - mT, \omega)x) : n \leq m \leq M, m \in \mathbb{N}_+ \}. \]

By the continuity of \( h \), the properties of \( \varphi \) and Corollary 3.1.1(ii) in [2], \( \beta^h_{n,M}(t, \omega) \) is progressively measurable with respect to \( \{ \mathcal{F}^t \} \) for every \( M = 1, 2, \cdots \). It is clear that

\[ \beta^h_{n,1}(t, \omega) \leq \beta^h_{n,2}(t, \omega) \leq \cdots \leq \beta^h_{n,M}(t, \omega) \leq \cdots . \]

Moreover, by the boundedness of \( h \) in \( \mathbb{R}_+^d \), \( b^h_n(t, \omega) = \lim_{M \to \infty} \beta^h_{n,M}(t, \omega) \) is a progressively measurable with respect to \( \{ \mathcal{F}^t \} \). \( \square \)

Lemma 3.2 Assume that conditions (A) (L) and (H) hold. Then we have

\[ \mathcal{K}(\lim h(\cdot, \varphi)) \leq \lim_{n \to \infty} \varphi \leq \liminf_{n \to \infty} \varphi \leq \mathcal{K}(\lim h(\cdot, \varphi)), \quad (3.1) \]

where

\[ \liminf_{n \to \infty} \varphi(t, \omega) := \lim_{n \to \infty} \inf \{ \varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+ \}, \ t \in \mathbb{R}, \ \omega \in \Omega, \]
\[
\lim \varphi(t, \omega) := \lim_{n \to \infty} \sup \{ \varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+ \}, \quad t \in \mathbb{R}, \omega \in \Omega,
\]
\[
\lim h(\cdot, \varphi)(t, \omega) := \lim_{n \to \infty} a_n^h(t, \omega), \quad t \in \mathbb{R}, \omega \in \Omega,
\]
\[
\lim h(\cdot, \varphi)(t, \omega) := \lim_{n \to \infty} b_n^h(t, \omega), \quad t \in \mathbb{R}, \omega \in \Omega,
\]
for each \( x \in \mathbb{R}^d, \alpha \in [0, T) \).

**Proof.** By the definition of \( \lim \varphi \) and \( \lim \varphi \) we have \( \lim \varphi \leq \lim \varphi \), the second inequality in (3.1) is proved. For the sake of convenience we only prove the first inequality in (3.1) and the third inequality can be proved analogously. Similar to Lemma 3.1, we can easily get that \( \lim \varphi(t, \omega) \) and \( \lim h(\cdot, \varphi)(t, \omega) \) exist, which are also progressively measurable with respect to \( \{F_t\} \). Then by the boundedness of \( h \), (2.7) and the Fubini Theorem, \( K(\lim h(\cdot, \varphi)) \) is well defined, progressively measurable with respect to \( \{F_t\} \). By (2.7), the definition of \( \lim h(\cdot, \varphi) \) and Lebesgue’s dominated convergence theorem, we have
\[
K(\lim h(\cdot, \varphi)) = \lim_{n \to \infty} K(a_n^h(t, \omega)).
\]
For fixed \( n \in \mathbb{N}_+ \), it is enough to prove that
\[
K(a_n^h)(t, \omega) = \int_{t-nT}^{t} \Phi(t-s, \theta_s \omega) \inf \{h(s, \varphi(s, s - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+ \} ds
\]
\[
= \lim_{\tilde{m} \to \infty} \left\{ \Phi(t + \tilde{m}T, \theta_{-\tilde{m}T} \omega)x + \int_{nT - \tilde{m}T}^{t} \Phi(t-s, \theta_s \omega) \inf \{h(s, \varphi(s, s - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+ \} ds \right\}
\]
\[
= \lim_{\tilde{m} \to \infty} \inf \left\{ \Phi(t + \tilde{m}T, \theta_{-\tilde{m}T} \omega)x + \int_{nT - \tilde{m}T}^{t} \Phi(t-s, \theta_s \omega) h(s, \varphi(s, s - \alpha - \tilde{m}T, \omega)x) ds : \tilde{m} \geq \tilde{n}, \tilde{m} \in \mathbb{N}_+ \right\}
\]
where the third equality has used Lemma A.2 in [10], while the second-to-last inequality has applied the order-preserving property of \( \Phi \) and the positivity of \( h \).
Lemma 3.3 Assume that conditions $(A)$, $(L)$ and $(H)$ hold. Then we have the following:

(i) if $h$ is order-preserving in $\mathbb{R}^d_+$, then

$$h(\cdot, \lim \varphi) \leq \lim h(\cdot, \varphi) \leq \overline{\lim} h(\cdot, \varphi) \leq h(\cdot, \overline{\lim} \varphi);$$  \hspace{1cm} (3.2)

(ii) if $h$ is anti-order-preserving in $\mathbb{R}^d_+$, then

$$h(\cdot, \overline{\lim} \varphi) \leq \lim h(\cdot, \varphi) \leq \overline{\lim} h(\cdot, \varphi) \leq h(\cdot, \lim \varphi).$$  \hspace{1cm} (3.3)

Proof. Indeed, the proof of the first inequality in (3.2) is adequate and the rest of the results of this lemma can be obtained analogously. Observe that $h$ is order-preserving in $\mathbb{R}^d_+$, then for fixed $n, k \in \mathbb{N}_+, k \geq n$,

$$h(t, \inf \{\varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+\}) \leq h(t, \varphi(t, t - \alpha - kT, \omega)x),$$

thus we have

$$h(t, \inf \{\varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+\}) \leq \inf \{h(t, \varphi(t, t - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+\}. \hspace{1cm} (3.4)$$

By the definition of $\overline{\lim} \varphi$, the continuity of $h$, (3.4) and the definition of $\lim h(\cdot, \varphi)$, we have

$$h(\cdot, \overline{\lim} \varphi)(t, \omega) = h(t, \lim_{n \to \infty} \inf \{\varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+\})$$

$$= \lim_{n \to \infty} h(t, \inf \{\varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+\})$$

$$\leq \lim_{n \to \infty} \inf \{h(t, \varphi(t, t - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+\}$$

$$= \lim h(\cdot, \varphi)(t, \omega).$$

The proof is complete. \hfill \Box

Lemma 3.4 Assume that conditions $(A)$, $(L)$ and $(H)$ hold. We have

$$\mathcal{K}(a^h_n) \leq \lim \varphi \leq \lim \varphi \leq \mathcal{K}(b^h_n), \hspace{1cm} n \in \mathbb{N}_+, \hspace{1cm} (3.5)$$

where $a^h_n$ and $b^h_n$ are as defined in Lemma 3.1. Furthermore, we define the gain operator

$$\mathcal{K}^h(u)(t, \omega) = h(t, \mathcal{K}(u)(t, \omega)), \hspace{1cm} t \in \mathbb{R}, \omega \in \Omega,$$

and we have the following:

(i) if $h$ is order-preserving in $\mathbb{R}^d_+$, then for fixed $n \in \mathbb{N}_+$,

$$(\mathcal{K}^h)^k(a^h_n) \leq \lim h(\cdot, \varphi) \leq \lim h(\cdot, \varphi) \leq (\mathcal{K}^h)^k(b^h_n), \hspace{1cm} k \in \mathbb{N}_+, \hspace{1cm} (3.6)$$

(ii) if $h$ is anti-order-preserving in $\mathbb{R}^d_+$, then for fixed $n \in \mathbb{N}_+$,

$$(\mathcal{K}^h)^{2k}(a^h_n) \leq \lim h(\cdot, \varphi) \leq \lim h(\cdot, \varphi) \leq (\mathcal{K}^h)^{2k}(b^h_n), \hspace{1cm} k \in \mathbb{N}_+. \hspace{1cm} (3.7)$$
Proof. By the definition of $a_n^h$ and $b_n^h$, it is evident that
$$a_n^h \leq \lim h(\cdot, \varphi) \leq \overline{\lim h(\cdot, \varphi)} \leq b_n^h, \quad n \in \mathbb{N}_+.$$ By the order-preserving property of $\Phi$, $K(u)$ is monotone with respect to $u$, and consequently
$$K(a_n^h) \leq K(\lim h(\cdot, \varphi)) \leq K(\overline{\lim h(\cdot, \varphi)}) \leq K(b_n^h), \quad n \in \mathbb{N}_+.$$ By (3.1), we have
$$K(a_n^h) \leq \lim \varphi \leq \overline{\lim \varphi} \leq K(b_n^h), \quad n \in \mathbb{N}_+.$$ which implies that (3.5) holds.

In what follows, we claim that (3.6) and (3.7) hold.

If $h$ is order-preserving in $\mathbb{R}_+^d$, then it deduces that $h$ preserves the inequalities in (3.5):
$$K^h(a_n^h) \leq h(\cdot, \lim \varphi) \leq h(\cdot, \overline{\lim \varphi}) \leq K^h(b_n^h), \quad n \in \mathbb{N}_+.$$ which together with (3.2) implies
$$K^h(a_n^h) \leq \lim h(\cdot, \varphi) \leq \overline{\lim h(\cdot, \varphi)} \leq K^h(b_n^h), \quad n \in \mathbb{N}_+.$$ This proves (3.6) for $k = 1$. Next we assume that, for some $k \in \mathbb{N}$,
$$(K^h)^k(a_n^h) \leq \lim h(\cdot, \varphi) \leq \overline{\lim h(\cdot, \varphi)} \leq (K^h)^k(b_n^h), \quad n \in \mathbb{N}_+,$$ holds. From the monotonicity of $K$ and (3.1), we have:
$$K ((K^h)^k(a_n^h)) \leq K(\lim h(\cdot, \varphi)) \leq \lim \varphi \leq \overline{\lim \varphi} \leq K(\overline{\lim h(\cdot, \varphi)}) \leq K ((K^h)^k(b_n^h)).$$ By the monotonicity of $h$ in $\mathbb{R}_+^d$ and (3.2), we get that
$$(K^h)^{k+1}(a_n^h) \leq \lim h(\cdot, \varphi) \leq \overline{\lim h(\cdot, \varphi)} \leq (K^h)^{k+1}(b_n^h), \quad n \in \mathbb{N}_+.$$ Therefore, we conclude that (3.6) holds by mathematical induction.

If $h$ is anti-order-preserving in $\mathbb{R}_+^d$, similar to $h$ is order-preserving in $\mathbb{R}_+^d$, we deduce that
$$K^h(b_n^h) \leq h(\cdot, \lim \varphi) \leq h(\cdot, \overline{\lim \varphi}) \leq K^h(a_n^h), \quad n \in \mathbb{N}_+.$$ by (3.3), we have
$$K^h(b_n^h) \leq \lim h(\cdot, \varphi) \leq \overline{\lim h(\cdot, \varphi)} \leq K^h(a_n^h), \quad n \in \mathbb{N}_+.$$ Combining the monotonicity of $K$ and (3.1), it shows that
$$K (K^h(b_n^h)) \leq \lim \varphi \leq \overline{\lim \varphi} \leq K (K^h(a_n^h)), \quad n \in \mathbb{N}_+,$$ which together with the anti-monotonicity of $h$ in $\mathbb{R}_+^d$ and (3.3) implies
$$(K^h)^2(a_n^h) \leq \lim h(\cdot, \varphi) \leq \overline{\lim h(\cdot, \varphi)} \leq (K^h)^2(b_n^h), \quad n \in \mathbb{N}_+.$$ The rest of the proof of (3.7) can be obtained analogously to $h$ is order-preserving in $\mathbb{R}_+^d$ by the mathematical induction. \qed
4 Main results

In this section, we state our main result on the stable random periodic solution of nonautonomous stochastic feedback system (1.4) and prove them.

Let \( \mathcal{M} \) be the space of all progressively measurable (with respect to \( \{\mathcal{F}_t\} \)) processes \( f : \mathbb{R} \times \Omega \to [0, N] \), and

\[
f(t + T, \omega) = f(t, \theta_T \omega), \quad \text{for any } t \in \mathbb{R}, \ \omega \in \Omega.
\]

Here the metric on \( \mathcal{M} \) is given as follows:

\[
\rho(f_1, f_2) := \sup_{t \in \mathbb{R}} \mathbb{E}|f_1(t, \omega) - f_2(t, \omega)|, \quad \text{for all } f_1, f_2 \in \mathcal{M}.
\]

Lemma 4.1 \((\mathcal{M}, \rho)\) is a complete metric space.

Proof. It is clear that \((\mathcal{M}, \rho)\) is a metric space. To prove completeness assume that \( \{f_n, n \in \mathbb{N}\} \) is a Cauchy sequence in \((\mathcal{M}, \rho)\), i.e.

\[
\sup_{t \in \mathbb{R}} \mathbb{E}|f_n(t, \omega) - f_m(t, \omega)| \to 0, \quad \text{as } m, n \to \infty.
\]

Then one can find a subsequence \( \{f_{n_k}\} \) such that

\[
\sup_{t \in \mathbb{R}} \mathbb{E}|f_{n_{k+1}}(t, \omega) - f_{n_k}(t, \omega)| \leq 2^{-k}.
\]

Thus

\[
\sup_{t \in \mathbb{R}} (\sum_{k=1}^{\infty} |f_{n_{k+1}}(t, \omega) - f_{n_k}(t, \omega)| + \mathbb{E}|f_1(t, \omega)|) \\
\leq \sum_{k=1}^{\infty} \sup_{t \in \mathbb{R}} \mathbb{E}|f_{n_{k+1}}(t, \omega) - f_{n_k}(t, \omega)| + \sup_{t \in \mathbb{R}} \mathbb{E}|f_1(t, \omega)| < \infty,
\]

which implies that for any \( t \in \mathbb{R} \),

\[
\mathcal{N}_t := \left\{ \omega \in \Omega \mid \sum_{k=1}^{\infty} |f_{n_{k+1}}(t, \omega) - f_{n_k}(t, \omega)| + |f_1(t, \omega)| = +\infty \right\}
\]

is a null set and thus

\[
\sum_{k=1}^{\infty} |f_{n_{k+1}}(t, \omega) - f_{n_k}(t, \omega)| + |f_1(t, \omega)| < \infty, \quad t \in \mathbb{R}, \ \omega \in \mathcal{N}_t^c.
\]

So for any \( t \in \mathbb{R}, \omega \in \mathcal{N}_t^c \), \( \lim_{k \to \infty} f_{n_k}(t, \omega) \) exists and

\[
\lim_{k \to \infty} f_{n_k}(t, \omega) = \sum_{k=1}^{\infty} (f_{n_{k+1}}(t, \omega) - f_{n_k}(t, \omega)) + f_1(t, \omega).
\]
For any \( t \in \mathbb{R}, \omega \in \Omega \), set
\[
f^j(t, \omega) = \liminf_{k \to \infty} f^j_{n_k}(t, \omega), \quad j = 1, \ldots, d
\]
and
\[
f(t, \omega) = (f^1(t, \omega), \ldots, f^d(t, \omega)).
\]
It is clear that \( f \) is well defined, progressively measurable (with respect to \( \{ F^t \} \)), and \( f \in [0, N] \).

Since
\[
f_{n_k}(t + T, \omega) = f_{n_k}(t, \theta_T \omega), \quad \text{for any} \ t \in \mathbb{R}, \ \omega \in \Omega.
\]
We have
\[
f^j(t + T, \omega) = \liminf_{k \to \infty} f^j_{n_k}(t + T, \omega) = \liminf_{k \to \infty} f^j_{n_k}(t, \theta_T \omega) = f^j(t, \theta_T \omega),
\]
which implies that
\[
f(t + T, \omega) = f(t, \theta_T \omega), \quad \text{for any} \ t \in \mathbb{R}, \ \omega \in \Omega.
\]
Therefore, \( f \in \mathcal{M} \).

In fact, \( f(t, \omega) = \sum_{l=k}^{\infty} (f_{n_{l+1}}(t, \omega) - f_{n_l}(t, \omega)) + f_{n_k}(t, \omega) \) for any \( t \in \mathbb{R}, \ \omega \in \mathcal{N}_t^c \). And thus
\[
\sup_{t \in \mathbb{R}} \mathbb{E}|f_{n_k} - f| = \sup_{t \in \mathbb{R}} \mathbb{E} \left| \sum_{l=k}^{\infty} (f_{n_{l+1}} - f_{n_l}) \right| \leq \sum_{l=k}^{\infty} 2^{-l} = 2^{1-k},
\]
which implies that \( d(f, f_{n_k}) \to 0 \) as \( k \to \infty \). Hence \( d(f, f_n) \to 0 \) as \( k \to \infty \) since \( \{ f_n \} \) is a Cauchy sequence. Thus \( (\mathcal{M}, \rho) \) is a complete metric space. \( \square \)

**Lemma 4.2** Assume that conditions \((A) (L) (H)\) hold. Assume additionally that the following conditions on \( R \) and \( h \) are satisfied:

\((R)\) Let \( R(t-s, \theta_s \omega) \), \( (t, s) \in \Delta \) be \( F_t^s \)-measurable and \( \sup_{t \in \mathbb{R}^+} \mathbb{E}(R(t, \omega)) < \infty \).

\((H_1)\) Let \( L := \max \{ \sup_{t \in \mathbb{R}, x \in [0, T]} |\partial_{x_j} \Phi(t, x)|, i, j = 1, \ldots, d \} \) such that \( \sup_{t \in \mathbb{R}^+} \mathbb{E}(R(t, \omega)) < \frac{\lambda}{L_d} \).

Then the gain operator \( \mathcal{K}^h : \mathcal{M} \to \mathcal{M} \), \( f \mapsto \mathcal{K}^h(f) \) is a contractive mapping, where \( \mathcal{K}^h(f)(t, \omega) = h(t, [\mathcal{K}(f)](t, \omega)) \).

**Proof.** First, we prove that \( \mathcal{K}^h : \mathcal{M} \to \mathcal{M} \) is well defined. For any \( f \in (\mathcal{M}, \rho) \), by the definition of \( \mathcal{K}^h \), \( \mathcal{K}^h(f) \in [0, N] \). And by \( (2.7) \), the order-preserving of \( \Phi \) and the positivity of \( f \), for any \( t \in \mathbb{R}, \ \omega \in \Omega \), we have
\[
[\mathcal{K}(f)](t, \theta_T \omega) = \int_{-\infty}^{t} \Phi(t - r, \theta_{r+T} \omega) f(r, \theta_T \omega) dr
\]
\[
= \int_{-\infty}^{t+T} \Phi(t + T - r, \theta_\omega) f(r - T, \theta_T \omega) dr
\]
\[
= \int_{-\infty}^{t+T} \Phi(t + T - r, \theta_\omega) f(r, \omega) dr
\]
\[
= [\mathcal{K}(f)](t + T, \omega).
\]
By the fact that \( h(t + T, x) = h(t, x), t \in \mathbb{R}, x \in \mathbb{R}^d \), we have

\[
\mathcal{K}^h(f)(t + T, \omega) = h(t + T, \mathcal{K}(f)(t + T, \omega)) = h(t, \mathcal{K}(f)(t, \theta_T \omega)) = \mathcal{K}^h(f)(t, \theta_T \omega).
\]

By (H) the measurability of \( \Phi \), and the Fubini theorem, it is evident that \( \mathcal{K}^h(f) \) is progressively measurable with respect to \( \{\mathcal{F}^t\} \). So \( \mathcal{K}^h : \mathcal{M} \to \mathcal{M} \) is well defined.

Next we prove that \( \mathcal{K}^h \) is a contractive mapping. By (H1) we have

\[
\rho(\mathcal{K}^h(f_1), \mathcal{K}^h(f_2)) = \sup_{t \in \mathbb{R}} \mathbb{E} \left| \mathcal{K}^h(f_1) - \mathcal{K}^h(f_2) \right| = \sup_{t \in \mathbb{R}} \mathbb{E} \left| h(\cdot, \mathcal{K}(f_1)) - h(\cdot, \mathcal{K}(f_2)) \right|
\]

\[
= \sup_{t \in \mathbb{R}} \mathbb{E} \left| \int_0^t \nabla_x h(\cdot, \mathcal{K}(f_1)) + r(\mathcal{K}(f_1) - \mathcal{K}(f_2)) \right| \cdot \mathbb{E} \left| \mathcal{K}(f_1) - \mathcal{K}(f_2) \right|
\]

\[
\leq d \sup_{t \in \mathbb{R}, x \in \mathbb{R}^d} \left| \nabla_x (h(t, x)) \right| \cdot \sup_{t \in \mathbb{R}} \mathbb{E} \left| \mathcal{K}(f_1) - \mathcal{K}(f_2) \right|
\]

\[
\leq L d \sup_{t \in \mathbb{R}} \mathbb{E} \left| \int_{-\infty}^t \Phi(t - s, \theta_s \omega) f_1(s, \omega) ds - \int_{-\infty}^t \Phi(t - s, \theta_s \omega) f_2(s, \omega) ds \right|
\]

\[
\leq L d^2 \sup_{t \in \mathbb{R}} \mathbb{E} \left| \int_{-\infty}^t \Phi(t - s, \theta_s \omega) ||f_1(s, \omega) - f_2(s, \omega)|| ds \right|
\]

\[
\leq L d^2 \sup_{t \in \mathbb{R}} \mathbb{E} \left| \int_{-\infty}^t e^{-\lambda(t-s)} R(t - s, \theta_s \omega) \cdot |f_1(s, \omega) - f_2(s, \omega)| ds \right|
\]

\[
= L d^2 \sup_{t \in \mathbb{R}} \mathbb{E} \left( \int_{-\infty}^t e^{-\lambda(t-s)} R(t - s, \theta_s \omega) \cdot |f_1(s, \omega) - f_2(s, \omega)| P(d\omega) ds \right)
\]

\[
= L d^2 \sup_{t \in \mathbb{R}} \mathbb{E} \left( \int_{-\infty}^t e^{-\lambda(t-s)} \mathbb{E}(R(t - s, \theta_s \omega)) |f_1(s, \omega) - f_2(s, \omega)| ds \right)
\]

\[
= L d^2 \sup_{t \in \mathbb{R}} \mathbb{E} \left( \int_{-\infty}^t e^{-\lambda(t-s)} \mathbb{E}(R(t - s, \omega)) |f_1(s, \omega) - f_2(s, \omega)| ds \right)
\]

\[
\leq L d^2 \sup_{t \in \mathbb{R}} \mathbb{E}(R(t, \omega)) \sup_{t \in \mathbb{R}} |f_1(t, \omega) - f_2(t, \omega)| \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\lambda(t-s)} ds
\]

\[
= \frac{L d^2}{\lambda} \mathbb{E}(R(t, \omega)) \rho(f_1, f_2)
\]

\[
< \rho(f_1, f_2),
\]

15
where the third-to-last equality holds because of \( \BR \) and the independence of \( R(t - s, \theta s \omega) \) and \( f_1(s, \omega) - f_2(s, \omega) \), the third-to-last inequality holds because of the \( \mathbb{P} \)-measure preserving property of \( \theta \).

\[ \square \]

**Theorem 4.3** Assume that conditions \((A) \) \((L) \) \((H) \) \((R) \) and \((H_1) \) hold. Then there exist a unique nonnegative fixed point \( u \in (\mathcal{M}, \rho) \) for the gain operator \( K^h \) such that for any \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} \varphi(t, -nT, \cdot) x = K^h(t, \cdot), \quad \forall x \in \mathbb{R}_+^d, \quad \mathbb{P} - a.s.
\]

Furthermore, for almost all \( \omega \in \Omega \),

\[
\varphi(t, s, \omega) K^h(u)(s, \omega) = K^h(u)(t, \omega), \quad \forall (t, s) \in \Delta, \quad r \in \mathbb{R}.
\]

i.e., \( K^h(u) \) is a random periodic solution of the forward stochastic flow generated by \((T, A)\) in \( \mathbb{R}_+^d \) and for any \( t \in \mathbb{R} \), all pull-back trajectories originating from nonnegative orthant converge to this positive random periodic solution almost surely.

**Proof.** By (3.6), (3.7), regardless of the monotonicity or anti-monotonicity for \( h \), for fixed \( n \in \mathbb{N}_+ \), we have

\[
(K^h)^{2k}(a_n^h) \leq \lim h(\cdot, \varphi) \leq \lim h(\cdot, \varphi) \leq (K^h)^{2k}(b_n^h), \quad k \in \mathbb{N}_+
\]

where \( a_n^h \) and \( b_n^h \) are as defined in Lemma 3.1. By Lemma 3.1 \( a_n^h \) and \( b_n^h \) are bounded progressively measurable processes with respect to \( \{\mathcal{F}^t\} \). By the definition of \( a_n^h \), \( h(t + T, x) = h(t, x), t \in \mathbb{R}, x \in \mathbb{R}_+^d \) and (2.6), for any \( t \in \mathbb{R}, \omega \in \Omega \), we have

\[
a_n^h(t + T, \omega) = \inf \{ h(t + T, \varphi(t + T, t + T - \alpha - mT, \omega)x) : m \geq n, m \in \mathbb{N}_+ \} \\
= \inf \{ h(t, \varphi(t, \alpha - mT, \theta T \omega)x) : m \geq n, m \in \mathbb{N}_+ \} \\
= a_n^h(t, \theta T \omega).
\]

Similarly, we have \( b_n^h(t + T, \omega) = b_n^h(t, \theta T \omega) \) for any \( t \in \mathbb{R}, \omega \in \Omega \). And it is clear that \( a_n^h, b_n^h \in [0, N] \). Therefore, \( a_n^h \) and \( b_n^h \) are both in \( (\mathcal{M}, \rho) \).

Since \( K^h \) is a contracting mapping on the complete metric space \( (\mathcal{M}, \rho) \), by the Banach fixed point theorem (3.4), there exists a unique \( u \in \mathcal{M} \) such that for any \( t \in \mathbb{R} \),

\[
[K^h(u)](t, \omega) = u(t, \omega), \quad \mathbb{P} - a.s.
\]

and

\[
\lim_{k \to \infty} \sup_{t \in \mathbb{R}} \mathbb{E}[|(K^h)^{2k}(a_n^h) - u|] = \lim_{k \to \infty} \sup_{t \in \mathbb{R}} \mathbb{E}[|(K^h)^{2k}(b_n^h) - u|] = 0,
\]

which implies that \( [(K^h)^{2k}(a_n^h)](t, \cdot) \overset{\mathbb{P}}{\rightarrow} u(t, \cdot) \) and \( [(K^h)^{2k}(b_n^h)](t, \cdot) \overset{\mathbb{P}}{\rightarrow} u(t, \cdot) \) for all \( t \in \mathbb{R} \). Therefore, there exists a subsequence \( \{k_j\}_{j \in \mathbb{N}} \) such that for any \( t \in \mathbb{R} \),

\[
\lim_{j \to \infty} [(K^h)^{2k_j}(a_n^h)](t, \cdot) = u(t, \cdot) = \lim_{j \to \infty} [(K^h)^{2k_j}(b_n^h)](t, \cdot), \quad \mathbb{P} - a.s. \quad (4.3)
\]
Combining (4.2) and (4.3), we obtain that for any \( t \in \mathbb{R} \),
\[
[\lim h(\cdot, \varphi)](t, \cdot) = [\lim h(\cdot, \varphi)](t, \cdot) = u(t, \cdot) \quad \mathbb{P} - a.s.
\]
which together with (3.1) implies that for any \( t \in \mathbb{R} \),
\[
[\lim \varphi](t, \cdot) = [\lim \varphi](t, \cdot) = \mathcal{K}(u)(t, \cdot) \quad \mathbb{P} - a.s.
\] (4.4)

By the definition of infimum and supremum, it is clear that
\[
\inf \{ \varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+ \} \\
\leq \varphi(t, t - \alpha - nT, \omega)x \\
\leq \sup \{ \varphi(t, t - \alpha - mT, \omega)x : m \geq n, m \in \mathbb{N}_+ \}, \ x \in \mathbb{R}^d.
\]

Let \( n \to \infty \) in the above inequality, by (4.4), for any \( t \in \mathbb{R} \), we have
\[
[\lim \varphi](t, \omega) = [\lim \varphi](t, \omega) = \lim_{n \to \infty} \varphi(t, t - \alpha - nT, \omega)x, \quad \mathbb{P} - a.s.
\] (4.5)

Combining (4.4) and (4.5), we obtain that for any \( t \in \mathbb{R}, \ t = m_0T + \alpha_0, m_0 \in \mathbb{Z}, \alpha_0 \in [0, T) \),
\[
\lim_{n \to \infty} \varphi(t, -nT, \cdot)x = \lim_{n \to \infty} \varphi(t, t - \alpha_0 - nT, \cdot)x = \mathcal{K}(u)(t, \cdot), \ x \in \mathbb{R}^d_{+}, \ \mathbb{P} - a.s.
\]

which proves that for any \( t \in \mathbb{R} \),
\[
\lim_{n \to \infty} \varphi(t, -nT, \cdot)x = \mathcal{K}(u)(t, \cdot), \ x \in \mathbb{R}^d_{+}, \ \mathbb{P} - a.s.
\] (4.6)

By (2.7), the progressively measurability with respect to \( \{F^t\} \) of \( u \) and the Fubini Theorem, \( \mathcal{K}(u)(t, \omega) \) is continuous and progressively measurable with respect to \( \{F^t\} \).

By the continuity of \( \varphi \) in \( \mathbb{R}^d_{+} \) and (4.6), we can show that for any fixed \( (t, s) \in \Delta \),
\[
\varphi(t, s, \omega)\mathcal{K}(u)(s, \omega) = \varphi(t, s, \omega) \lim_{n \to \infty} \varphi(s, -nT, \omega)x \\
= \lim_{n \to \infty} \varphi(t, s, \omega)\varphi(s, -nT, \omega)x \\
= \lim_{n \to \infty} \varphi(t, -nT, \omega)x \\
= \mathcal{K}(u)(t, \omega), \ \mathbb{P} - a.s.
\] (4.7)

Next, we claim that for almost all \( \omega \in \Omega \),
\[
\varphi(t, s, \omega)\mathcal{K}(u)(s, \omega) = \mathcal{K}(u)(t, \omega), \ (t, s) \in \Delta.
\]

By the continuity of \( \varphi(t, s, \omega, x) \) in \( (t, s, x) \) for all \( \omega \in \Omega \) and the continuity of \( \mathcal{K}(u)(t, \omega) \) in \( t \)
for all \( \omega \in \Omega \), we have

\[
\mathcal{N}_{s,t} := \{ \omega \in \Omega : \varphi(t, s, \omega)\mathcal{K}(u)(s, \omega) \neq \mathcal{K}(u)(t, \omega) \} 
\]

\[
= \bigcup_{k=1}^{\infty} \left\{ \omega \in \Omega : |\varphi(t, s, \omega)\mathcal{K}(u)(s, \omega) - \mathcal{K}(u)(t, \omega)| > \frac{1}{k} \right\}
\]

\[
\subset \bigcup_{k=1}^{\infty} \bigcup_{(q,p)\in Q^2\cap \Delta} \left\{ \omega \in \Omega : |\varphi(q, p, \omega)\mathcal{K}(u)(p, \omega) - \mathcal{K}(u)(q, \omega)| > \frac{1}{2k} \right\}
\]

\[
= \bigcup_{(q,p)\in Q^2\cap \Delta} \bigcup_{k=1}^{\infty} \left\{ \omega \in \Omega : |\varphi(q, p, \omega)\mathcal{K}(u)(p, \omega) - \mathcal{K}(u)(q, \omega)| > \frac{1}{2k} \right\}
\]

Thus we obtain \( \mathbb{P}(\bigcup_{(t,s)\in \Delta} \mathcal{N}_{s,t}) \leq \mathbb{P}(\bigcup_{(q,p)\in Q^2\cap \Delta} \mathcal{N}_{p,q}) = 0 \), since \( \mathbb{P}(\mathcal{N}_{s,t}) = 0, (t,s) \in \Delta \) (by (4.7)).

Furthermore, \( u(t + T, \omega) = u(t, \theta_T \omega) \), for any \( t \in \mathbb{R}, \omega \in \Omega \) since \( u \in \mathcal{M} \). Thus by (2.7) we have

\[
\mathcal{K}(u)(t, \theta_T \omega) = \int_{-\infty}^{t} \Phi(t - r, \theta_T \omega)u(r, \theta_T \omega)dr
\]

\[
= \int_{-\infty}^{t+T} \Phi(t + T - r, \theta_T \omega)u(r - T, \theta_T \omega)dr
\]

\[
= \int_{-\infty}^{t+T} \Phi(t + T - r, \theta_T \omega)u(r, \omega)dr
\]

\[
= \mathcal{K}(u)(t + T, \omega), \quad t \in \mathbb{R}, \quad \omega \in \Omega.
\]

So for almost all \( \omega \in \Omega \),

\[
\varphi(t, s, \omega)\mathcal{K}(u)(s, \omega) = \mathcal{K}(u)(t, \omega), \quad \mathcal{K}(u)(r + T, \omega) = \mathcal{K}(u)(r, \theta_T \omega), \quad \text{for any} \ (t, s) \in \Delta, \ r \in \mathbb{R}.
\]

\[\square\]

5 Examples

In this section, we show the efficiency of our result. Our main results Theorem 4.3 works for the \( T \)-periodic stochastic Goodwin negative feedback system, \( T \)-periodic stochastic Othmer-Tyson positive feedback system and \( T \)-periodic stochastic competitive systems. Our main task is to check the condition (L), (R) and (H₁) in order to use Theorem 4.3. That is to say, we need to choose a suitable \( \lambda > 0 \) and random process \( R \).
Now we consider nonautonomous stochastic single loop feedback system

\[
\begin{cases}
\frac{dx_1}{dt} = (-\alpha_1 x_1 + f(t, x_n)) dt + \sigma_1 x_1 dW_1^t, \\
\frac{dx_i}{dt} = (x_{i-1} - \alpha_i x_i) dt + \sigma_i x_i dW_i^t, \quad 2 \leq i \leq n,
\end{cases}
\]

(5.1)

where \(\alpha_i > 0, \sigma_i > 0\) for \(i = 1, \cdots, n\) and \(f \in C_b^1(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)\), i.e., \(f\) and its derivatives are both bounded. Moreover, we assume that \(f(t + T, x) = f(t, x), t \in \mathbb{R}, x \in \mathbb{R}^+\), and \(f(t, \cdot), t \in \mathbb{R}\) is increasing or decreasing in \(\mathbb{R}^+\).

The corresponding linear homogeneous stochastic Itô type differential equations is

\[
\begin{cases}
\frac{dx_1}{dt} = -\alpha_1 x_1 dt + \sigma_1 x_1 dW_1^t, \\
\frac{dx_i}{dt} = (x_{i-1} - \alpha_i x_i) dt + \sigma_i x_i dW_i^t, \quad 2 \leq i \leq n.
\end{cases}
\]

(5.2)

By the variation-of-constants formula, we can easily calculate the fundamental matrix \(\Phi(t, \omega)\) of (5.2) as follows:

\[
\Phi(t, \omega) = \begin{pmatrix}
\Phi_{11}(t, \omega) & 0 & \cdots & 0 \\
\Phi_{21}(t, \omega) & \Phi_{22}(t, \omega) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{n1}(t, \omega) & \Phi_{n2}(t, \omega) & \cdots & \Phi_{nn}(t, \omega)
\end{pmatrix}
\]

for all \(t \geq 0\) and \(\omega \in \Omega\), where

\[
\Phi_{ij}(t, \omega) = \begin{cases}
\int_0^t \Phi_{ii}(t-s, \theta_s \omega) \Phi_{i-1,j}(s, \omega) ds, & 1 \leq j \leq i - 1 \\
e^{-(\alpha_i + \frac{1}{2} \sigma_i^2) t + \sigma_i W_i^t(\omega)}, & j = i \\
0, & i + 1 \leq j \leq n
\end{cases}
\]

(5.3)

for all \(i = 1, \cdots, n\). Let \(\lambda = \frac{1}{n+1} \min\{\alpha_1, \cdots, \alpha_n\}\), then it is easy to check that

\[
\Phi_{ii}(t, \omega) \leq e^{-[(n+1)\lambda + \frac{1}{2} \sigma_i^2] t + \sigma_i W_i^t(\omega)}
\]

\[
e^{-((i+1)\lambda + \frac{1}{2} \sigma_i^2) t + \sigma_i W_i^t(\omega)} e^{-(n+1-i)\lambda t}
\]

\[
= R_{ii}(t, \omega) e^{-(n+1-i)\lambda t},
\]

(5.4)

for all \(t \geq 0\) and \(\omega \in \Omega\), where

\[
R_{ii}(t, \omega) = e^{-((i+1)\lambda + \frac{1}{2} \sigma_i^2) t + \sigma_i W_i^t(\omega)}, \quad i = 1, \cdots, n.
\]

(5.5)

We prove that \(\sup_{s \in \mathbb{R}^+} R_{ii}(s, \omega)\) is a tempered random variable, for any \(\gamma > 0\), by (5.5) we have
the following estimate

\[
\sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \sup_{s \in \mathbb{R}^+} R_{3i}(s, \theta_i \omega) \right\} 
= \sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \sup_{s \in \mathbb{R}^+} \left\{ e^{-\left((i \lambda + \frac{1}{2} \sigma^2\right)s + \sigma_1 W^i_s(\omega)} \right\} \right\} 
= \sup_{s \in \mathbb{R}^+, t \in \mathbb{R}} \left\{ e^{-\gamma |t| - (i \lambda + \frac{1}{2} \sigma^2)s + \sigma_1 \left(W^i_{s+t}(\omega) - W^i_t(\omega)\right)} \right\} 
\leq \sup_{s \in \mathbb{R}^+, t \in \mathbb{R}} \left\{ e^{-\left(\frac{\gamma}{2} (i \lambda + \frac{1}{2} \sigma^2)\right) |t| + \sigma_1 W^i_t(\omega)} e^{-\gamma |t| - \sigma_1 W^i_t(\omega)} \right\} 
\leq \sup_{t \in \mathbb{R}} \left\{ e^{-\left(\frac{\gamma}{2} (i \lambda + \frac{1}{2} \sigma^2)\right) |t| + \sigma_1 W^i_t(\omega)} \right\} \sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t| - \sigma_1 W^i_t(\omega)} \right\} 
< \infty,
\]

where the last inequality holds because of the law of the iterated logarithm of Brownian motions.

Next, we claim that

\[
\Phi_{ij}(t, \omega) \leq e^{-(n+1-i)\lambda t} R_{ij}(t, \omega), \quad 1 \leq j \leq i - 1 \tag{5.6}
\]

where

\[
R_{ij}(t, \omega) = \int_0^t e^{-\lambda s} R_{ii}(t - s, \theta_i \omega) R_{i-1,j}(s, \omega) ds, \quad 1 \leq j \leq i - 1. \tag{5.7}
\]

In order to check (5.6), we only present the proof of \(\Phi_{21}(t, \omega)\) and \(\Phi_{31}(t, \omega)\), the rest can be analogously completed by induction. Combining (5.3), (5.4), we have

\[
\Phi_{21}(t, \omega) = \int_0^t \Phi_{22}(t - s, \theta_i \omega) \Phi_{11}(s, \omega) ds 
\leq \int_0^t e^{-(n-1)\lambda(t-s)} R_{22}(t - s, \theta_i \omega) e^{-n\lambda s} R_{11}(s, \omega) ds 
= e^{-(n-1)\lambda t} \int_0^t e^{-\lambda s} R_{22}(t - s, \theta_i \omega) R_{11}(s, \omega) ds 
= e^{-(n-1)\lambda t} R_{21}(t, \omega)
\]

and

\[
\Phi_{31}(t, \omega) = \int_0^t \Phi_{33}(t - s, \theta_i \omega) \Phi_{21}(s, \omega) ds 
\leq \int_0^t e^{-(n-2)\lambda(t-s)} R_{33}(t - s, \theta_i \omega) e^{-(n-1)\lambda s} R_{21}(s, \omega) ds 
= e^{-(n-2)\lambda t} \int_0^t e^{-\lambda s} R_{33}(t - s, \theta_i \omega) R_{21}(s, \omega) ds 
= e^{-(n-2)\lambda t} R_{31}(t, \omega)
\]

for all \(t \geq 0\) and \(\omega \in \Omega\).
Furthermore, we only prove that \( \sup_{s \in \mathbb{R}^+} R_{21}(s, \omega) \) and \( \sup_{s \in \mathbb{R}^+} R_{31}(s, \omega) \) are tempered random variables, the rest can be analogously completed by induction. For any \( \gamma > 0 \), by (5.7) we have the following estimate

\[
\begin{align*}
\sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \sup_{s \in \mathbb{R}^+} R_{21}(s, \theta_t \omega) \right\} \\
= \sup_{s \in \mathbb{R}^+, t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \int_0^s e^{-\lambda r} R_{22}(s - r, \theta_{r+t} \omega) R_{11}(r, \theta_t \omega) dr \right\} \\
\leq \sup_{s \in \mathbb{R}^+, t \in \mathbb{R}} \left\{ \int_0^s e^{-\frac{1}{2} \lambda r} e^{-\frac{R(\lambda \gamma)}{2}(|t| + r)} R_{22}(s - r, \theta_{r+t} \omega) e^{-\frac{\gamma |t|}{2}} R_{11}(r, \theta_t \omega) dr \right\} \\
\leq \sup_{t \in \mathbb{R}} \left\{ e^{-\frac{(\lambda \gamma + |t|)}{2}} \sup_{s \in \mathbb{R}^+} R_{22}(s, \theta_t \omega) \sup_{s \in \mathbb{R}^+} R_{11}(s, \theta_t \omega) \right\} \sup_{t \in \mathbb{R}} \left\{ e^{-\frac{\gamma |t|}{2}} \sup_{s \in \mathbb{R}^+} R_{11}(s, \theta_t \omega) \right\} \int_0^\infty e^{-\frac{1}{2} \lambda r} dr \\
< \infty,
\end{align*}
\]

and

\[
\begin{align*}
\sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \sup_{s \in \mathbb{R}^+} R_{31}(s, \theta_t \omega) \right\} \\
= \sup_{s \in \mathbb{R}^+, t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \int_0^s e^{-\lambda r} R_{33}(s - r, \theta_{r+t} \omega) R_{21}(r, \theta_t \omega) dr \right\} \\
\leq \sup_{s \in \mathbb{R}^+, t \in \mathbb{R}} \left\{ \int_0^s e^{-\frac{1}{2} \lambda r} e^{-\frac{R(\lambda \gamma)}{2}(|t| + r)} R_{33}(s - r, \theta_{r+t} \omega) e^{-\frac{\gamma |t|}{2}} R_{21}(r, \theta_t \omega) dr \right\} \\
\leq \sup_{t \in \mathbb{R}} \left\{ e^{-\frac{(\lambda \gamma + |t|)}{2}} \sup_{s \in \mathbb{R}^+} R_{33}(s, \theta_t \omega) \right\} \sup_{t \in \mathbb{R}} \left\{ e^{-\frac{\gamma |t|}{2}} \sup_{s \in \mathbb{R}^+} R_{21}(s, \theta_t \omega) \right\} \int_0^\infty e^{-\frac{1}{2} \lambda r} dr \\
< \infty.
\end{align*}
\]

Let

\[
R(t, \omega) := \max_{i,j=1,\ldots,n} R_{ij}(t, \omega). \tag{5.8}
\]

Then \( \sup_{t \in \mathbb{R}^+} R(t, \omega) \) is a tempered random variable and

\[
\| \Phi(t, \omega) \| := \max\{|\Phi_{ij}(t, \omega)| : i, j = 1, \ldots, n\} \leq R(t, \omega) e^{-\lambda t}, \quad t \geq 0, \ \omega \in \Omega.
\]

It follows from (2.3), (5.5) and (5.7) that \( R_{ij}(t - s, \theta_s \omega) \), \( i, j = 1, \ldots, n \) are \( F_{s}^{t} \)-measurable, and thus \( R(t - s, \theta_s \omega) \) is \( F_{s}^{t} \)-measurable. By (5.5) and the maximal inequality of geometric Brownian motion (see [12], p.858 and [31], p.1639), we obtain that

\[
\mathbb{E}(\sup_{t \in \mathbb{R}^+} R_{ii}(t, \omega)) \leq 1 + \frac{\sigma_i^2}{2t \lambda}, \quad i = 1, \ldots, n. \tag{5.9}
\]

Combining (5.7), (5.9), the \( \mathbb{P} \)-measure preserving property of \( \theta \), Fubini theorem and the fact
that an $n$-dimensional Brownian motion has $n$ independent components, we obtain that

$$
\sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{ij}(t, \omega)) \\
= \sup_{t \in \mathbb{R}^+} \mathbb{E}\left( \int_0^t e^{-\lambda s} R_{ii}(t-s, \theta_s \omega) R_{i-1,j}(s, \omega) ds \right) \\
= \sup_{t \in \mathbb{R}^+} \int_0^t e^{-\lambda s} \mathbb{E}(R_{ii}(t-s, \theta_s \omega) R_{i-1,j}(s, \omega)) ds \\
= \sup_{t \in \mathbb{R}^+} \int_0^t e^{-\lambda s} \mathbb{E}(R_{ii}(t-s, \theta_s \omega)) \mathbb{E}(R_{i-1,j}(s, \omega)) ds \\
\leq \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{i-1,j}(t, \omega)) \sup_{t \in \mathbb{R}^+} \int_0^t e^{-\lambda s} \mathbb{E}(R_{ii}(t-s, \omega)) ds \\
\leq \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{ii}(t, \omega)) \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{i-1,j}(t, \omega)) \int_0^\infty e^{-\lambda s} ds \\
\leq \frac{1}{\lambda} \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{ii}(t, \omega)) \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{i-1,j}(t, \omega)), \ 1 \leq j \leq i - 1.
$$

By induction, we conclude that

$$
\sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{ij}(t, \omega)) \leq \frac{1}{\lambda^{i-j}} \prod_{k=j}^i \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{kk}(t, \omega)), \ 1 \leq j \leq i.
$$

So

$$
\sup_{t \in \mathbb{R}^+} \mathbb{E}(R(t, \omega)) \leq \sup_{t \in \mathbb{R}^+} \sum_{i=1}^n \sum_{j=1}^i \mathbb{E}(R_{ij}(t, \omega)) \\
\leq \sum_{i=1}^n \sum_{j=1}^i \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{ij}(t, \omega)) \\
\leq \sum_{i=1}^n \sum_{j=1}^i \frac{1}{\lambda^{i-j}} \prod_{k=j}^i \sup_{t \in \mathbb{R}^+} \mathbb{E}(R_{kk}(t, \omega)) \\
\leq \sum_{i=1}^n \sum_{j=1}^i \frac{1}{\lambda^{i-j}} \prod_{k=j}^i \mathbb{E}\left( \sup_{t \in \mathbb{R}^+} R_{kk}(t, \omega) \right) \\
\leq \sum_{i=1}^n \sum_{j=1}^i \frac{1}{\lambda^{i-j}} \prod_{k=j}^i \left( 1 + \frac{\sigma_k^2}{2k\lambda} \right).
$$

Let $h(t, x) = (f(t, x_n), 0, \cdots, 0)^T$, $x \in \mathbb{R}^n$, $N_1 = \sup_{t \in \mathbb{R}, x_n \in \mathbb{R}^+_n} \left| f(t, x_n) \right|$, $N_i = 0$ for all $2 \leq i \leq n$ and $L = \sup_{t \in \mathbb{R}, x_n \in \mathbb{R}^+_n} \left| \frac{df(t, x_n)}{dx_n} \right|$. Then from Theorem 4.3 we conclude the following.
Proposition 5.1 Let $\alpha_i > 0$ for $i = 1, \cdots, n$ and $f \in C^1_b(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+)$. Assume that $f(t + T, x) = f(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}_+$ and $f(t, \cdot)$, $t \in \mathbb{R}$ is increasing or decreasing in $\mathbb{R}_+$. If

$$\frac{\ln^2 \sup_{t \in \mathbb{R}_+} \mathbb{E}(R(t, \omega))}{\lambda} \leq \frac{\ln^2}{\lambda} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{\lambda^{i-j}} \prod_{k=j}^{i} \left(1 + \frac{\sigma_k^2}{2k\lambda}\right) < 1$$

holds, where $R(t, \omega)$ is defined in (5.8) and $L = \sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}_+} \left|\frac{d(t, x)}{dx}\right|$. Then equation (5.1) has a unique random periodic solution of periodic $T$ in $\mathbb{R}_+^n$.

Example 5.2 (Time-periodic stochastic Goodwin system). Consider $n$-dimensional stochastic differential equation

\begin{equation}
\begin{cases}
dx_1 = (-\alpha_1 x_1 + \frac{V}{K + \sin t + x_{n}^m})dt + \sigma_1 x_1 dW_t^1, \\
dx_i = (x_{i-1} - \alpha_i x_i)dt + \sigma_i x_i dW_t^i, 
\end{cases}
\tag{5.10}
\end{equation}

where $m > 1$, $K > 2$, $V > 0$ and $\alpha_i > 0$ for $i = 1, \cdots, n$. It is clear that (5.10) is a non-monotone system, which can be regarded as the stochastic Goodwin model with time-periodic coefficient; see [31, 32]. By the direct calculation, it is obvious that

$$L = \sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}_+} \left|\frac{mV x_{n}^{m-1}}{(K + \sin t + x_{n}^m)^2}\right| \leq \sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}_+} \left|\frac{mV(1 + x_{n}^m)}{(K + \sin t + x_{n}^m)^2}\right| \leq \frac{mV}{K - 1}.$$ 

Applying Proposition 5.1, we get that if

$$\frac{mn^2V}{\lambda(K - 1)} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{\lambda^{i-j}} \prod_{k=j}^{i} \left(1 + \frac{\sigma_k^2}{2k\lambda}\right) < 1 \tag{5.11}$$

is satisfied, then (5.10) has a unique globally stable random periodic solution of periodic $2\pi$ in $\mathbb{R}_+^n$. Here, (5.11) holds for $V$ sufficiently small or $K$ sufficiently large.

Example 5.3 (Time-periodic stochastic Othmer-Tyson system). Consider the following $n$-dimensional nonautonomous stochastic Othmer-Tyson positive feedback system:

\begin{equation}
\begin{cases}
dx_1 = (-\alpha_1 x_1 + \frac{k_0(1 + x_{n}^m)}{K + \sin t + x_{n}^m})dt + \sigma_1 x_1 dW_t^1, \\
dx_i = (x_{i-1} - \alpha_i x_i)dt + \sigma_i x_i dW_t^i, 
\end{cases}
\tag{5.12}
\end{equation}

where $m > 1$, $K > 2$, $k_0 > 0$ and $\alpha_i > 0$ for $i = 1, \cdots, n$. This is a stochastic cooperative system; see [31, 32]. By the direct calculation, it is obvious that

$$L = \sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}_+} \left|\frac{mk_0(K + \sin t - 1)x_{n}^{m-1}}{(K + \sin t + x_{n}^m)^2}\right| \leq \sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}_+} \left|\frac{mk_0(K + \sin t - 1)(1 + x_{n}^m)}{(K + \sin t + x_{n}^m)^2}\right| \leq \frac{mk_0}{K - 1}.$$ 

Applying Proposition 5.1, we get that if

$$\frac{mk_0n^2K}{\lambda(K - 1)} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{\lambda^{i-j}} \prod_{k=j}^{i} \left(1 + \frac{\sigma_k^2}{2k\lambda}\right) < 1 \tag{5.13}$$

is satisfied, then (5.12) has a unique globally stable random periodic solution of periodic $2\pi$ in $\mathbb{R}_+^n$. Here, (5.13) holds for $k_0$ sufficiently small.
Example 5.4 Consider an $n$-dimensional stochastic competitive system with time-periodic coefficient

$$dx_i = [-\alpha_ix_i + h_i(t,x)]dt + \sigma_ix_idW^i_t,$$

(5.14)

where $\alpha_i > 0$ for all $i = 1, \cdots, n$ and

$$h_i(t,x) := |\sin t| + \frac{1}{K_i + x_1^m + \cdots + x^n_m}, x \in \mathbb{R}^n_+, i = 1, \cdots, n,$$

where $m > 1$ and $K_i > 1$ for all $i = 1, \cdots, n$. Then $h(t,\cdot)$ is a decreasing function from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+ \setminus 0$. The fundamental matrix of the following corresponding linear homogeneous stochastic Itô type differential equations

$$dx_i = -\alpha_ix_idt + \sigma_ix_idW^i_t,$$

is

$$\Phi(t,\omega) = \begin{pmatrix}
\Phi_{11}(t,\omega) & 0 & \cdots & 0 \\
0 & \Phi_{22}(t,\omega) & \cdots & 0 \\
0 & 0 & \cdots & \Phi_{nn}(t,\omega)
\end{pmatrix}$$

for all $t \geq 0$ and $\omega \in \Omega$, where

$$\Phi_{ij}(t,\omega) = e^{-(\alpha_i + \frac{1}{2}\sigma^2_i)t + \sigma_i W^i_t(\omega)}.$$

It is clear that

$$\|\Phi(t,\omega)\| := \max\{|\Phi_{ij}(t,\omega)| : i, j = 1, \ldots, n\} \leq R(t,\omega)e^{-\lambda t}, \quad t \geq 0, \quad \omega \in \Omega,$$

where $\lambda = \frac{1}{2} \min\{\alpha_1, \ldots, \alpha_n\}$ and

$$R(t,\omega) = \bigvee_{i=1}^n e^{-(\lambda + \frac{1}{2}\sigma^2_i)t + \sigma_i W^i_t(\omega)}.$$

Similar to the analysis for system (5.1), it is easy to check that $R(t-s,\theta_s\omega)$ is $\mathcal{F}^s_t$-measurable and for any $\gamma > 0$,

$$\sup_{t \in \mathbb{R}} \{e^{-\gamma|t|} \sup_{s \in \mathbb{R}^+} R(s,\theta_t\omega)\} < \infty.$$

By the maximal inequality of geometric Brownian motion, we can obtain that

$$\sup_{t \in \mathbb{R}^+} \mathbb{E}(R(t,\omega)) = \sup_{t \in \mathbb{R}^+} \mathbb{E}\left( \bigvee_{i=1}^n e^{-(\lambda + \frac{1}{2}\sigma^2_i)t + \sigma_i W^i_t(\omega)} \right) \leq \sum_{i=1}^n \sup_{t \in \mathbb{R}^+} \mathbb{E}e^{-(\lambda + \frac{1}{2}\sigma^2_i)t + \sigma_i W^i_t(\omega)}$$

$$\leq \sum_{i=1}^n \mathbb{E}(\sup_{t \in \mathbb{R}^+} e^{-(\lambda + \frac{1}{2}\sigma^2_i)t + \sigma_i W^i_t(\omega)}) \leq \sum_{i=1}^n \left(1 + \frac{\sigma^2_i}{2\lambda}\right).$$
By the direct calculation, it is obvious that

\[ L = \max \left\{ \sup_{t \in \mathbb{R}, x \in \mathbb{R}^d_+} \left| \frac{\partial h_i(t, x)}{\partial x_j} \right|, i, j = 1, \ldots, n \right\} \]

\[ = \max \left\{ \sup_{x \in \mathbb{R}^d_+} \left| \frac{m x_j^{m-1}}{(K_1 + x_1^m + \cdots + x_n^m)^2} \right|, i, j = 1, \ldots, n \right\} \]

\[ \leq \frac{m}{K} \leq \frac{m}{K}, \]

where \( K = \min\{K_i, i = 1, \ldots, n\} \). Therefore, if

\[ \frac{mn^2}{\lambda K} \sum_{i=1}^{n} \left( 1 + \frac{\sigma_i^2}{2\lambda} \right) < 1 \] (5.15)

is satisfied, then (5.14) has a unique globally stable random periodic solution of periodic \( \pi \) in \( \mathbb{R}^n_+ \). Here, (5.15) holds when \( \lambda \) or \( K \) is large enough.

As a further specific example, we give the following example.

**Example 5.5** Consider the following 3-dimensional stochastic Othmer-Tyson positive feedback system with time-periodic coefficient:

\[
\begin{align*}
\frac{dx_1}{dt} &= (-8x_1 + \frac{1}{12} \cdot \frac{1+x^3}{3+\sin(t+x^3)})dt + \frac{1}{2}x_1dW_1^1, \\
\frac{dx_2}{dt} &= (x_1 - 9x_2)dt + \frac{1}{4}x_2dW_2^2, \\
\frac{dx_3}{dt} &= (x_2 - 10x_3)dt + \frac{1}{3}x_3dW_3^3.
\end{align*}
\] (5.16)

By the direct calculation, it is obvious that

\[ L = \sup_{t \in \mathbb{R}, x_3 \in \mathbb{R}_+} \left| \frac{(2 + \sin t)x^3_3}{4(3 + \sin t + x^3_3)^2} \right| \]

\[ = \sup_{t \in \mathbb{R}} \frac{(2 + \sin t)x^3_3}{4(3 + \sin t + x^3_3)^2} \cdot \frac{3+\sin t}{x_3=\frac{3+\sin t}{2}} \]

\[ = \sup_{t \in \mathbb{R}} \frac{2 + \sin t}{36(3+\sin t)^\frac{3}{2}} \cdot \frac{3+\sin t}{x_3=\frac{3+\sin t}{2}} \left| \sin t=1 \right| \]

\[ = \frac{1}{24 \cdot 2^\frac{3}{2}}. \]

The corresponding linear homogeneous stochastic Itô type differential equations is

\[
\begin{align*}
\frac{dx_1}{dt} &= -8x_1dt + \frac{1}{2}x_1dW_t^1, \\
\frac{dx_2}{dt} &= (x_1 - 9x_2)dt + \frac{1}{4}x_2dW_t^2, \\
\frac{dx_3}{dt} &= (x_2 - 10x_3)dt + \frac{1}{3}x_3dW_t^3.
\end{align*}
\] (5.17)
By the variation-of-constants formula, we can easily calculate the fundamental matrix $\Phi(t, \omega)$ of (5.17) as follows:

\[
\Phi(t, \omega) = \begin{pmatrix}
\Phi_{11}(t, \omega) & 0 & 0 \\
\Phi_{21}(t, \omega) & \Phi_{22}(t, \omega) & 0 \\
\Phi_{31}(t, \omega) & \Phi_{32}(t, \omega) & \Phi_{33}(t, \omega)
\end{pmatrix}
\]

for all $t \geq 0$ and $\omega \in \Omega$, where

\[
\Phi_{11}(t, \omega) = e^{-(8+\frac{1}{8})t + \frac{1}{8}W_1^2(\omega)},
\]

\[
\Phi_{22}(t, \omega) = e^{-(9+\frac{1}{32})t + \frac{1}{8}W_2^2(\omega)},
\]

\[
\Phi_{33}(t, \omega) = e^{-(10+\frac{1}{18})t + \frac{1}{8}W_3^2(\omega)}.
\]

and

\[
\Phi_{ij}(t, \omega) = \begin{cases}
\int_0^t \Phi_{ii}(t - s, \theta_s \omega)\Phi_{i-1,j}(s, \omega))ds, & 1 \leq j \leq i - 1 \\
0, & i + 1 \leq j \leq 3.
\end{cases}
\]

Hence, it is easy to check that

\[
\Phi_{ii}(t, \omega) = R_{ii}(t, \omega)e^{-(4-i)t}, \quad i = 1, 2, 3
\]

for all $t \geq 0$ and $\omega \in \Omega$, where

\[
R_{11}(t, \omega) = e^{-(5+\frac{1}{8})t + \frac{1}{8}W_1^2(\omega)},
\]

\[
R_{22}(t, \omega) = e^{-(7+\frac{1}{32})t + \frac{1}{8}W_2^2(\omega)},
\]

\[
R_{33}(t, \omega) = e^{-(9+\frac{1}{18})t + \frac{1}{8}W_3^2(\omega)}.
\]

and thus

\[
\Phi_{21}(t, \omega) = \int_0^t \Phi_{22}(t - s, \theta_s \omega)\Phi_{11}(s, \omega))ds
\]

\[
= \int_0^t e^{-2(t-s)}R_{22}(t - s, \theta_s \omega)e^{-3s}R_{11}(s, \omega))ds
\]

\[
= e^{-2t}R_{21}(t, \omega),
\]

where $R_{21}(t, \omega) = \int_0^t e^{-s}R_{22}(t - s, \theta_s \omega)R_{11}(s, \omega))ds$.

\[
\Phi_{31}(t, \omega) = \int_0^t \Phi_{33}(t - s, \theta_s \omega)\Phi_{21}(s, \omega))ds
\]

\[
= \int_0^t e^{-(t-s)}R_{33}(t - s, \theta_s \omega)e^{-2s}R_{21}(s, \omega))ds
\]

\[
= e^{-t}R_{31}(t, \omega),
\]
where \( R_{31}(t, \omega) = \int_0^t e^{-s} R_{33}(t - s, \theta_s \omega) R_{21}(s, \omega) \) ds,

\[
\Phi_{32}(t, \omega) = \int_0^t \Phi_{33}(t - s, \theta_s \omega) \Phi_{22}(s, \omega) ds
= \int_0^t e^{-(t-s)} R_{33}(t - s, \theta_s \omega) e^{-2s} R_{22}(s, \omega) ds
= e^{-t} R_{32}(t, \omega),
\]

where \( R_{32}(t, \omega) = \int_0^t e^{-s} R_{33}(t - s, \theta_s \omega) R_{22}(s, \omega) \) ds. In order to verify conditions in Theorem 4.3, we choose \( \lambda = 1 \) and

\[
R(t, \omega) = \max_{i,j=1,2,3} R_{ij}(t, \omega).
\]

Similar to the analysis for system (5.1), we can prove that \( \sup_{t \in \mathbb{R}_+} R(t, \omega) \) is a tempered random variable and

\[
\| \Phi(t, \omega) \| := \max_i \{|\Phi_{ij}(t, \omega)| : i, j = 1, 2, 3\} \leq R(t, \omega)e^{-t}, \quad t \geq 0, \ \omega \in \Omega.
\]

It is clear that \( R_{ij}(t - s, \theta_s \omega), i, j = 1, \cdots, n \) is \( \mathcal{F}_s^t \)-measurable, and thus \( R(t - s, \theta_s \omega) \) is \( \mathcal{F}_s^t \)-measurable. By the maximal inequality of geometric Brownian motion, i.e.,

\[
\mathbb{E}\left( \sup_{t \in \mathbb{R}_+} e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t(\omega)} \right) = 1 - \frac{\sigma^2}{2\mu}
\]

for \( \mu < 0 \) and \( \sigma > 0 \), we obtain that

\[
\sup_{t \in \mathbb{R}_+} \mathbb{E}(R(t, \omega)) = \sup_{t \in \mathbb{R}_+} \mathbb{E}(\max_{i,j=1,2,3} R_{ij}(t, \omega))
\leq \mathbb{E}(\sup_{t \in \mathbb{R}_+} R_{31}(t, \omega)) + \mathbb{E}(\sup_{t \in \mathbb{R}_+} R_{32}(t, \omega)) + \mathbb{E}(\sup_{t \in \mathbb{R}_+} R_{33}(t, \omega))
\leq \frac{41}{40} \times \frac{225}{224} \times \frac{163}{162} + \frac{225}{224} \times \frac{163}{162} + \frac{163}{162}
\leq 3.0528.
\]

Therefore, we have that

\[
\frac{Ld^2 \sup_{t \in \mathbb{R}_+} \mathbb{E}(R(t, \omega))}{\lambda} \leq \frac{1}{24 \times 2^t} \times 9 \times 3.0528 < 1.
\]

Then by Theorem 4.3, we conclude that (5.10) has a unique globally stable random periodic solution of period \( 2\pi \) in \( \mathbb{R}_+^3 \).

**Acknowledgements** This work was supported in part by National Key R&D Program of China (No. 2020YFA0712700), National Natural Science Foundation of China (Nos. 11931004, 12031020, 12090014), CAS Key Project of Frontier Sciences (No. QYZDJ-SSW-JSC003), the Key Laboratory of Random Complex Structures and Data Sciences, CAS(No. 2008DP173182).
References

[1] L. Arnold, Random Dynamical Systems, Springer Monogr. Math., Springer-Verlag, Berlin, 1998.

[2] I. Chueshov, Monotone Random Systems Theory and Applications, Lecture Notes in Math. 1799, Springer-Verlag, Berlin, 2002.

[3] O. Cinquin and J. Demongeot, Positive and negative feedback: Striking a balance between necessary antagonists, J. Theoret. Biol., 216 (2002), 229-241.

[4] C. A. Desoer and M. Vidyasagar, Feedback Systems: Input-Output Properties, Academic Press, New York, 1975.

[5] Z. Dong, W. L. Zhang and Z. H. Zheng, Random periodic solutions of non-autonomous stochastic differential equations, 2021, arXiv:2104.01423.

[6] V. Dragan, T. Morozan, and A. Stoica, Mathematical Methods in Robust Control of Linear Stochastic Systems, Vol. 50, 2nd ed., Springer, Berlin, 2013.

[7] C. R. Feng, Y. Wu and H. Z. Zhao, Anticipating random periodic solutions-I. SDEs with multiplicative linear noise, J. Funct. Anal., 271 (2016), 365-417.

[8] C. R. Feng and H. Z. Zhao, Random periodic processes, periodic measures and ergodicity, J. Differ. Equ., 269(2020), 7382-7414.

[9] C. R. Feng, H. Z. Zhao and B. Zhou, Pathwise random periodic solutions of stochastic differential equations, J. Differ. Equ., 251(2011), 119-149.

[10] M. Marcondes de Freitas and E. D. Sontag, A small-gain theorem for random dynamical systems with inputs and outputs, SIAM J. Control Optim., 53 (2015), 2657-2695.

[11] B. C. Goodwin, Oscillatory behavior in enzymatic control processes, Adv. Enzyme. Regul., 3 (1965), 425-438.

[12] S. E. Graversen and G. Peskir, Optimal stopping and maximal inequalities for geometric Brownian motion, J. Appl. Probab., 35 (1998), 856-872.

[13] J. S. Griffith, Mathematics of cellular control processes II. Positive feedback to one gene, J. Theoret. Biol., 20 (1968), 209–216.

[14] S. Hastings, J. Tyson and D. Webster, Existence of periodic solutions for negative feedback cellular control systems, J. Differ. Equ., 25 (1977), 39-64.

[15] M. W. Hirsch, Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere, SIAM J. Math. Anal., 16 (1985), 423-439.

[16] A. Isidori, Nonlinear Control Systems. II, Comm. Control Engrg. Ser., Springer-Verlag, Berlin, 1999.
[17] J. F. Jiang and X. Lv, A small-gain theorem for nonlinear stochastic systems with inputs and outputs I: Additive white noise, SIAM J. Control Optim., 54(2016), 2383-2402.

[18] J. F. Jiang and X. Lv, Global stability of feedback systems with multiplicative noise on the nonnegative orthant, SIAM J. Control Optim., 56(2018), 2218-2247.

[19] I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus, Grad. Texts in Math. 113, Springer, New York, 1988.

[20] M. Krstic, I. Kanellakopoulos and P. V. Kokotovic, Nonlinear and Adaptive Control Design, Wiley-Interscience, Hoboken, NJ, 1995.

[21] H. Kunita, 1984, Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms, Ecole d’été de Probabilités de Saint-Flour 12, 1982 Lect.Notes Math.1097, 143-303.

[22] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge University Press, Cambridge, 1990.

[23] S. J. Liu, Z. P. Jiang, and J. F. Zhang, Global output-feedback stabilization for a class of stochastic non-minimum-phase nonlinear systems, Automatica, 44 (2008), 1944–1957.

[24] R. C. Liu and K. N. Lu, Statistical properties of 2d stochastic Navier-Stokes equations with time-periodic forcing and degenerate stochastic forcing, 2021, [arXiv:2105.00598].

[25] S. J. Liu, J. F. Zhang, and Z. P. Jiang, Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems, Automatica, 43 (2007), 238–251.

[26] X. R. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, UK, 1997.

[27] J. Monod, J. Changeaux, and F. Jacob, Allosteric proteins and cellular control systems, J.Mol. Biol., 6 (1963), 306-329.

[28] M. Nomura, R. Course, and G. Baughman, Regulation of the synthesis of ribosomes and ribosomal components, Annu. Rev. Biochem., 53 (1984), 75-117.

[29] B.Øksendal, Stochastic Differential Equations: An Introduction with Applications, 5th ed., Springer-Verlag, Berlin, 1998.

[30] H. G. Othmer, The qualitative dynamics of a class of biochemical control circuits, J. Math. Biol., 3 (1976), 53-78.

[31] G. Peskir, Optimal stopping of the maximum process: The maximality principle, Ann. Probab., 26 (1998), 1614-1640.

[32] J. F. Selgrade, Asymptotic behavior of solutions to single loop positive feedback systems, J. Differential Equations, 38 (1980), 80–103.

[33] J. J. Tyson and H. G. Othmer, The dynamics of feedback control circuits in biochemical pathways, Progr. Theor. Biol., 5 (1978), 1-62.
[34] K. Yoshida, Functional Analysis, 6th ed., Springer, New York, 1980.

[35] X. Yu, X. J. Xie and N. Duan, Small-gain control method for stochastic nonlinear systems with stochastic iISS inverse dynamics, Automatica, 46 (2010), 1790–1798.

[36] H. Z. Zhao and Z. H. Zheng, Random periodic solutions of random dynamical systems, J. Differ. Equ., 246(2009), 2020-2038.