BOX-COUNTING MEASURE OF METRIC SPACES

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Abstract. In this paper, we introduce a new notion called the box-counting measure of a metric space. We show that for a doubling metric space, an Ahlfors regular measure is always a box-counting measure; consequently, if $E$ is a self-similar set satisfying the open set condition, then the Hausdorff measure restricted to $E$ is a box-counting measure. We show two classes of self-affine sets, the generalized Lalley-Gatzouras type self-affine sponges and Barański carpets, always admit box-counting measures; this also provides a very simple method to calculate the box-dimension of these fractals. Moreover, among others, we show that if two doubling metric spaces admit box-counting measures, then the multi-fractal spectra of the box-counting measures coincide provided the two spaces are Lipschitz equivalent.

1. Introduction

Let $X$ and $Y$ be two metric spaces with common Hausdorff dimension $s$. Let $\mu$ and $\nu$ be the restrictions of the $s$-dimensional Hausdorff measure of $X$ and $Y$, respectively. It is well known that if $\mu$ and $\nu$ are finite measures and $f : X \to Y$ is a bi-Lipschitz map, then

$$f^* \mu \sim \nu,$$

where $f^* \mu(A) := \mu(f^{-1}(A))$; see for instance, Falconer [6]. (Recall that two measures $\mu$ and $\nu$ on $X$ are said to be equivalent, and denoted by $\mu \sim \nu$, if there exists $\zeta > 0$ such that $\zeta^{-1} \mu(\cdot) \leq \nu(\cdot) \leq \zeta \mu(\cdot)$.)

Recall that a measure $\mu$ on a metric space $X$ is said to be Ahlfors regular with index $s$ if there is a constant $C > 0$ such that $C^{-1} r^s \leq \mu(B(x,r)) \leq C r^s$ holds for any ball $B(x,r)$ with center $x \in X$ and radius $r \in (0,1)$. If $\mu$ and $\nu$ are Ahlfors regular measures of $X$ and $Y$ with index $s$, respectively, then relation (1.1) still holds. Indeed, in this case $\mu \sim \mathcal{H}^s|_X$ and $\nu \sim \mathcal{H}^s|_Y$. We ask the question that do there exist other metric spaces and measures such that (1.1) holds?

Recently, Rao et al. [27] found that (1.1) holds for uniform Bernoulli measures on Bedford-McMullen carpets.

**Proposition 1.1** ([27]). Let $X$ and $Y$ be two totally disconnected Bedford-McMullen carpets, let $\mu$ and $\nu$ be the uniform Bernoulli measures of $X$ and $Y$ respectively. If $f : X \to Y$ is bi-Lipschitz, then $f^* \mu \sim \nu$. 

Date: November 30, 2022.

The work is supported by NSFS No. 11971195.

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2020 Mathematics Subject Classification: 26A16, 28A12, 28A80.

Key words and phrases: box-counting measure, self-affine sponge, Lipschitz invariant.
Consequently, $\mu$ and $\nu$ have the same multi-fractal spectrum, and $\mu$ is doubling if and only if $\nu$ is doubling. Hence the multi-fractal spectrum and the doubling property can serve as new Lipschitz invariants. Thanks to these new Lipschitz invariants, Yang and Zhang [31] gave a complete Lipschitz classification of totally disconnected Bedford-McMullen carpets whose Hausdorff dimension and box dimension coincide.

Recently, Falconer, Fraser and Kempton [7] introduced a new notion called intermediate dimension. Banaji and Kolossváry [1] calculated the intermediate dimensions of Bedford-McMullen carpets, and proved that two Bedford-McMullen carpets have the same intermediate dimensions if and only if they have the same multi-fractal spectrum. Therefore, they obtain that the following result without assuming the totally disconnected condition.

**Proposition 1.2** ([1]). Let $X$ and $Y$ be two Bedford-McMullen carpets, let $\mu$ and $\nu$ be the uniform Bernoulli measures of $X$ and $Y$ respectively. If $f : X \to Y$ is bi-Lipschitz, then $\mu$ and $\nu$ have the same multi-fractal spectrum.

In the present paper, we introduce a new notion called the box-counting measure, and we will show that (1.1) holds for very general settings.

Let $(X,d_X)$ be a compact metric space and let $A \subset X$. A family of balls with radius $\delta$ is called a $\delta$-ball-packing of $A$, if they are disjoint and their centers are located in $A$. For any $\delta > 0$, we define

$$N_\delta(A) := \max \{\#\mathcal{P}; \mathcal{P} \text{ is a } \delta\text{-ball-packing of } A\},$$

where $\#\mathcal{P}$ denotes the cardinality of $\mathcal{P}$. The upper and lower box dimensions of $X$ are defined by $\overline{\dim}_B X = \lim_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}$ and $\underline{\dim}_B X = \lim_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}$. If the two values coincide, then the common value is called the box dimension of $X$ and denoted by $\dim_B X$.

First, we define a new type covering of a metric space.

**Definition 1.3** (Compact Vitalii-type covering). Let $X$ be a compact metric space. For each $k \geq 1$, let $\mathcal{F}_k$ be a covering of $X$. We call $\mathcal{F} = \{\mathcal{F}_k\}_{k \geq 1}$ a compact Vitalii-type covering of $X$, if

(i) every element in $\mathcal{F}_k$ is compact;

(ii) there is an integer $\lambda$ such that for all $k \geq 1$, every point $x \in X$ is covered by at most $\lambda$ elements of $\mathcal{F}_k$;

(iii) $\max\{\text{diam } A; A \in \mathcal{F}_k\} \to 0$ as $k \to \infty$.

Now we define the box-counting measure as follows.

**Definition 1.4** (Box-counting measure). Let $X$ be a compact metric space such that $\beta = \dim_B X$ exists. Let $\mu$ be a finite Borel measure on $X$. Let $\mathcal{F} = \{\mathcal{F}_k\}_{k \geq 1}$ be a compact Vitalii-type covering of $X$. We call $\mu$ an $\mathcal{F}$-box-counting measure, if there is a constant $M > 0$ such that for any $R \in \bigcup_{k \geq 1} \mathcal{F}_k$,

$$M^{-1} \mu(R) \leq N_\delta(R) \delta^\beta \leq M \mu(R) \quad (1.2)$$

holds for $\delta$ small enough (i.e., for $0 < \delta \leq \delta_0(R)$); in this case, we call the triple $(X, \mu, \mathcal{F})$ a box-counting space.
Example 1.5. Let $\epsilon > 0$. Two points $x, y \in X$ are said to be $\epsilon$-equivalent if there exists a sequence \( \{x_1 = x, x_2, \ldots, x_{k-1}, x_k = y\} \subset X \) such that \( d_X(x_i, x_{i+1}) \leq \epsilon \) for \( 1 \leq i \leq k - 1 \). An $\epsilon$-equivalent class is called an $\epsilon$-connected component of $X$. Let $C(X, \epsilon)$ be the collection of all $\epsilon$-connected components of $X$.

If $X$ is a totally disconnected compact metric space, then $\{F_k = C(X, 1/2^k)\}_{k \geq 1}$ is a compact Vitalli-type covering.

For a large class of IFS’s, their attractors admit a natural compact Vitalli-type covering. An iterated function system (IFS) is a family of contractions $\Phi = \{\varphi_j\}_{j=1}^N$ on a compact set $X \subset \mathbb{R}^d$. In this paper, we will always assume that all $\varphi_j$’s are injections. The attractor of the IFS is the unique nonempty compact set $E$ satisfying $E = \bigcup_{j=1}^N \varphi_j(E)$; especially, it is called a self-similar set if all $\varphi_j$’s are similitudes. An IFS $\{\varphi_j\}_{j=1}^N$ is said to satisfy the open set condition (OSC), if there is a bounded nonempty open set $U \subset \mathbb{R}^d$ such that for all $1 \leq i \leq N$, $\varphi_i(U) \subset U$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. See [6, 13].

Denote $\Sigma = \{1, \ldots, N\}$ and $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$. For $I = i_1 \ldots i_n \in \Sigma^*$, we call $E_I = \varphi_I(E) = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(E)$ an $n$-th cylinder of $E$. Let $x \in E$, we call $J = (j_k)_{k \geq 1} \in \Sigma^\infty$ a coding of $x$ if $x \in \varphi_{j_1 \ldots j_n}(E)$ for all $n \geq 1$.

Definition 1.6. Let $F_k$ be the collection of $k$-th cylinders of an IFS $\Phi$. Clearly $\{F_k\}_{k \geq 1}$ is a compact Vitalli-type covering of the attractor $E$ if and only if the number of codings of points in $E$ is uniformly bounded. Let $\mu$ be a finite Borel measure on $E$. We call $\mu$ a cylinder box-counting measure if $(E, \mu, F = \{F_k\}_{k \geq 1})$ is a box-counting space.

Figure 1. Let $F$ be the attractor of the above self-affine IFS, then $F$ satisfies the OSC, but the left-bottom point has infinitely many codings.

A metric space $X$ is said to be a doubling space, if there is a constant $C > 0$ such that for any $x \in X$ and $r > 0$, $B(x, 2r)$ can be covered by $C$ numbers of balls of radius $r$ (see [11]). For a doubling space, the box-counting measure is a generalization of Ahlfors regular measure.

Theorem 1.1. Let $X$ be a compact doubling space. Then every Ahlfors regular measure of $X$ is a box-counting measure.
Example 1.7. Let \( E \subset \mathbb{R}^d \) be a self-similar set satisfying the OSC. Let \( s = \dim_H E \). Denote \( \mathcal{H}^s \) the \( s \)-dimensional Hausdorff measure. It is well-known that \( \mathcal{H}^s|_E \) is an Ahlfors measure, and hence it is also a cylinder box-counting measure. A direct proof is given in Theorem 2.1.

It is well known that on a metric space, any two Ahlfors regular measures are equivalent. Similarly, we have

Theorem 1.2. Let \((X, \mu, \mathcal{F})\) and \((X, \mu', \mathcal{F}')\) be two box-counting spaces. Then \( \mu \sim \mu' \).

Next, we investigate Lipschitz invariants related to box-counting measures.

Theorem 1.3. Let \( X \) and \( Y \) be two compact doubling spaces. Suppose \((X, \mu, \mathcal{F})\) is a box-counting space, and \( f : X \to Y \) is bi-Lipschitz. Then \((Y, f^* \mu, f(\mathcal{F}))\) is a box-counting space.

As a corollary of Theorem 1.2 and 1.3, we have

Corollary 1.8. Let \( X \) and \( Y \) be two compact doubling spaces. Suppose \( \mu \) and \( \nu \) are box-counting measures of \( X \) and \( Y \), respectively. If \( f : X \to Y \) is bi-Lipschitz, then \( f^* \mu \sim \nu \).

Consequently, \( \mu \) and \( \nu \) have the same multi-fractal spectrum, and \( \mu \) is doubling if and only if \( \nu \) is doubling.

Proof. Since both \((Y, f^* \mu, f(\mathcal{F}))\) and \((Y, \nu, \mathcal{F}')\) are box-counting spaces, by Theorem 1.2, we have \( f^* \mu \sim \nu \). Therefore, \( \mu, f^* \mu \) and \( \nu \) have the same multi-fractal spectrum and the same doubling property. \( \square \)

We remark that the above corollary generalizes the corresponding results in Rao, Yang and Zhang \[27\] and Banaji and Kolossváry \[1\].

Finally, we show that several class of diagonal self-affine sets admits cylinder box-counting measures.

Definition 1.9. We call \( f : \mathbb{R}^d \to \mathbb{R}^d \), \( f(x) = Tx + b \) a diagonal self-affine mapping if \( T \) is a \( d \times d \) diagonal matrix such that all the diagonal entries are positive numbers. An IFS \( \Phi = \{\phi_j(x)\}_{j=1}^m \) is called a diagonal self-affine IFS if all the maps \( \phi_j(x) \) are diagonal self-affine contractions; the attractor is called a diagonal self-affine sponge, which we denote by \( \Lambda_\Phi \). Without loss of generality, we will always assume that \( \Lambda_\Phi \subset [0,1]^d \).

Das and Simmons \[4\] gave an equivalent definition as following. For each \( i \in \{1, \ldots, d\} \), let \( A_i \) be a finite index set, and let \( \Phi_i = \{\phi_{a,i} \}_{a \in A_i} \) be a collection of contracting similarities of \([0,1] \) coming from the \( i \)-th components of maps in \( \Phi \), called the base IFS in coordinate \( i \). Let \( A = A_1 \times \cdots \times A_d \), and for each \( a = (a_1, \ldots, a_d) \in A \), define the contracting affine map \( \phi_a : [0,1]^d \to [0,1]^d \) by

\[
\phi_a(x_1, \ldots, x_d) = (\phi_{a,1}(x_1), \ldots, \phi_{a,d}(x_d)),
\]

where \( \phi_{a,i} \) is shorthand for \( \phi_{a,i} \) in the formula above. Given \( D \subset A \), then the collection \( \Phi = \{\phi_a\}_{a \in D} \) is a diagonal self-affine IFS.
Remark 1.10. Recently, there are a lot of works on diagonal self-affine sponges, Feng and Wang [9], Barański [2], Mackay [22], Fraser [10], Das and Simmons [4], Banaji and Kolossváry [1] on dimensions; King [16], Jordan and Rams [14], Olsen [25], Reeve [29] on multi-fractal formalism; Li, Li and Miao [21], Miao, Xi and Xiong [23], Liang, Miao and Ruan [20], Rao, Yang and Zhang [27] on metric and topology classifications.

Now let $\Lambda_\Phi$ be a diagonal self-affine sponge. Given a permutation $\tau$ of the coordinate set $\{1, \ldots, d\}$, a sequence $\{\beta_j\}_{j=1}^d \subset [0, 1]$ can be defined inductively by (4.2); see Section 4 for details. For $a \in D$, we define

\begin{equation}
(1.3)
p_a = \prod_{j=1}^d (\phi'_{a,j})^{\beta_j},
\end{equation}

which is a probability weight. Denote $\beta(\tau) = \sum_{j=1}^d \beta_j$ and let $\mu_\tau$ be the Bernoulli measure determined by the above probability weight. This construction has been given in [2] and [18] for calculating the box dimensions of self-affine sets.

A diagonal self-affine sponge $\Lambda_\Phi$ is said to satisfy the coordinate ordering condition if there is a permutation $\tau$ such that $\{\phi'_{a,j}\}_{j=1}^d$ is a strictly decreasing sequence for each $a \in D$; is said to satisfy the weak coordinate ordering condition if $\{\phi'_{a,j}\}_{j=1}^d$ is non-increasing.

A diagonal self-affine sponge is said to be of the generalized Lalley-Gatzouras type (resp. Lalley-Gatzouras type), if it satisfies the weak coordinate ordering condition (resp. coordinate ordering condition) as well as the neat projection condition. (See Section 4 for details).

Theorem 1.4. Let $\Lambda_\Phi$ be a diagonal self-affine sponge of generalized Lalley-Gatzouras type. Let $\tau$ be the permutation such that $\{\phi'_{a,j}\}_{j=1}^d$ is a non-increasing sequence for each $a \in D$. Then $\dim_B \Lambda_\Phi = \beta(\tau)$, and the Bernoulli measure $\mu_\tau$ defined by the weight in (1.3) is a cylinder box-counting measure.

Remark 1.11. Actually, we will show that for a cylinder $R = \Lambda_I$, $\delta_0(R)$ can be chosen to be the length of the shortest side of $\phi_I([0, 1]^d)$. 
Let $\Lambda$ be a Barański carpet (see Section 5 for details). Let $\tau_1$ be the identity map and $\tau_2$ be the permutation $1 \mapsto 2, 2 \mapsto 1$. Following [2], denote $\alpha = \beta_\tau_1, \mu_A = \mu_\tau_1$ and $\beta = \beta_\tau_2, \mu_B = \mu_\tau_2$.

Barański [2] proved that $\dim_B \Lambda = \max\{\alpha, \beta\}$.

**Theorem 1.5.** Let $\Lambda$ be a Barański carpet. Then

$$
\mu = \begin{cases} 
\mu_A, & \text{if } \alpha > \beta; \\
\mu_B, & \text{if } \alpha < \beta; \\
\mu_A + \mu_B, & \text{if } \alpha = \beta 
\end{cases}
$$

is a cylinder box-counting measure.

**Remark 1.12.** In the proof of Theorem 1.4 and 1.5, as a by-product, we obtain the box dimensions of the generalized Lalley-Gatzouras type sponges and the Barański carpets, respectively. The first one is also calculated in Lalley and Gatzouras [18] and Kolossváry [17]. Due to the notion of box-counting measure, our proofs are considerably simpler than [18], [17] and Barański [2].

**Remark 1.13.** The box dimension is closely related to Minkowski content (see Section 7 for precise definition). For applications of Minkowski content in analysis, we refer to Lapidus and Pomerance [19], Falconer [5] and the references therein. We show that if $(X, \mu, F)$ is a box-counting space, denote $\beta = \dim_B X$, then both the lower and upper $\beta$-dimensional Minkowski contents are finite (Lemma 7.1).

**Example 1.14** (Self-similar sets admitting no box-counting measures). Let $F_\lambda$ be the self-similar set generated by the IFS

$$
\left\{ f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x+1}{3}, \quad f_3(x) = \frac{x+\lambda}{3} \right\}.
$$

When $\lambda$ is an irrational number, it was shown by Kenyon [15] that the Lebesgue measure of $F_\lambda$ is zero, and it was proved by Hochman [12] that $\dim_H F_\lambda = \dim_B F_\lambda = 1$. Therefore the Minkowski content of $F_\lambda$ is zero, and $F_\lambda$ does not admit any box-counting measure by Lemma 7.1.

**Remark 1.15.** Another motivation of this paper is to investigate the locally measure preserving property of bi-Lipschitz maps. Cooper and Pignataro [3], Xi and Ruan [30] proved that if $E$ and $F$ are two self-similar sets satisfying the strong separation condition, and $f : E \to F$ is bi-Lipschitz, then there is a cylinder $U$ of $E$ such that $f : U \to f(U)$ preserves the Hausdorff measure in dimension $\dim_H E$. This property plays an important rôle in the Lipschitz classification of self-similar sets, see [8, 26, 28], etc.

In a sequential paper, Yang and Zhang [32] proved that if $(E, \mu)$ and $(F, \nu)$ are two box-counting spaces, then the above measure preserving property is valid provided that (i) both $E$ and $F$ are perfectly disconnected; (ii) both $\mu$ and $\nu$ are arithmetically doubling. For precise definition of these terminologies, we refer to [32].
Motivated by Banaji and Kolossváry [1], we ask the following question.

**Open problem 1.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be two self-affine sponges of generalized Lalley-Gatzouras type (or Barański carpets) and let \( \mu_1 \) and \( \mu_2 \) be their box-counting measures, respectively. Is it true that \( \Lambda_i \ (i = 1, 2) \) have the same intermediate dimensions if and only if \( \mu_i \ (i = 1, 2) \) have the same multi-fractal spectrum? (It is well-known that the calculation of intermediate dimensions and multi-fractal spectra are both tedious for self-affine sets.)

The paper is organized as follows. In Section 2, we prove Theorem 1.1. Theorem 1.2 and Theorem 1.3 are proved in Section 3. In Section 4, we study the box-counting measure of self-affine sponges of generalized Lalley-Gatzouras type. In Section 5, we prove Theorem 1.5. In Section 6, we discuss the box-counting measures of some symbolic spaces. The relation between box-counting measure and Minkowski content is discussed in Section 7.

### 2. Ahlfors regular measures are box-counting measures

Let \( X \) be a metric space admitting an Ahlfors regular measure \( \mu \). It is well-known that \( \dim_H X = \dim_B X = s \). Clearly, if \( \mu' \) is another Ahlfors regular measure of \( X \), then \( \mu \sim \mu' \).

For a set \( A \subset X \), let
\[
[A]_\delta := \{ x \in X; \quad d_X(x, a) \leq \delta \text{ for some } a \in A \},
\]
and we call it the \( \delta \)-**neighbourhood** of \( A \). Let \( \mathcal{L} \) be the collection of all compact subsets of \( X \). If \( A, B \in \mathcal{L} \), the Hausdorff metric \( d_H \) is defined as follows,
\[
d_H(A, B) = \inf \{ \delta; \ A \subset [B]_\delta \text{ and } B \subset [A]_\delta \}.
\]

It is well-known that the space \( \mathcal{L} \) is compact in the Hausdorff metric \( d_H \) (see for instance, [24, Sec. 45]).

Through the whole paper, when we write \( f(x) \asymp g(x), \ x \in X \), we mean that there is a constant \( C \) independent of \( x \) such that \( C^{-1}f(x) \leq g(x) \leq Cf(x) \) for all \( x \in X \). In this case, we say that \( f(x) \) and \( g(x) \) are **comparable** for \( x \in X \).

**Proof of Theorem 1.1.** Let \( X \) be a compact doubling space with \( \beta = \dim_B X \), and let \( \mu \) be an Ahlfors regular measure on \( X \).

First, we construct a compact Vitalli-type covering of \( X \). That \( X \) is compact implies \( X \) is totally bounded. Let \( \{ r_k \}_{k \geq 1} \) be a sequence of reals decreasing to 0. For each integer \( k \geq 1 \), there exists a finite set \( A'_k \subset X \) such that
\[
\mathcal{F}'_k = \{ B(a, r_k); \ a \in A'_k \}
\]
is a covering of \( X \). By the 5r-covering lemma, we can choose \( A_k \subset A'_k \) such that the balls \( B(a, r_k), a \in A_k \), are disjoint and
\[
\mathcal{F}_k = \{ B(a, 5r_k); \ a \in A_k \}
\]
is a covering of \( X \).

Pick \( x \in X \). Suppose \( x \) is covered by \( p \) numbers of balls in \( \mathcal{F}_k \). Then the ball \( B(x, 10r_k) \) contains at least \( p \) numbers of balls in \( \mathcal{F}'_k \). Since \( X \) is a doubling space,
we conclude that \( p \leq \lambda = C^4 \) where \( C \) is the constant in the definition of doubling space. This proves that \( \{ F_k \}_{k \geq 1} \) is a compact Vitalli-type covering.

Next, let \( \mathcal{N}_\delta(R) \) be the minimal cardinality of \( \delta \)-ball-coverings of \( R \). By the doubling property of \( X \), we have that for any \( R \subset X \),

\[
\mathcal{N}_\delta(R) \geq \mathcal{N}_{2\delta}(R) \geq C^{-1}\mathcal{N}_\delta(R).
\]

Finally, we show that for any \( R \in \bigcup_{k=1}^\infty \mathcal{F}_k \), \( \mu(R) \) and \( \mathcal{N}_\delta(R)\delta^\beta \) are comparable for \( \delta \) small enough. Since \( \mu \) is an Ahlfors regular measure, for any \( x \in X \) and \( \delta \in (0, 1) \),

\[
C_1^{-1}\delta^\beta \leq \mu(B(x, \delta)) \leq C_1\delta^\beta
\]

for some \( C_1 > 0 \). On one hand, let \( \{ B_i \}_{i=1}^k \) be a minimal \( \delta \)-ball-covering of \( R \), we deduce that

\[
\mu(R) \leq \sum_{i=1}^k \mu(B_i) \leq \mathcal{N}_\delta(R)C_1\delta^\beta \leq CC_1\mathcal{N}_\delta(R)\delta^\beta.
\]

On the other hand, let \( \{ B_j' \}_{j=1}^\ell \) be a maximal \( \delta \)-ball-packing of \( R \), we have

\[
\mu([R_\delta]) \geq \sum_{j=1}^\ell \mu(B_j') \geq \mathcal{N}_\delta(R)C_1^{-1}\delta^\beta.
\]

Note that \( 2\mu(R) \geq \mu([R_\delta]) \) holds for \( \delta \) small, we obtain the other side estimation and finish the proof. \( \square \)

For a self-similar set \( E \) satisfying the OSC, the Hausdorff dimension \( s \) is given by Moran’s formula \( \sum_{j=1}^N r_j^s = 1 \), where \( r_j \)'s are contraction ratios of maps in the IFS. We call the Bernoulli measure with probability weight \( (r_j^s)_j \) the canonical Bernoulli measure of \( E \).

**Theorem 2.1.** Let \( E \subset \mathbb{R}^d \) be a self-similar set satisfying the OSC. Let \( \mu \) be the canonical Bernoulli measure of \( E \). Then there is a constant \( M > 0 \) such that for any cylinder \( E_I \) and \( \delta \leq \text{diam}(E_I) \), it holds that

\[
M^{-1}\mu(E_I)\delta^{-\beta} \leq \mathcal{N}_\delta(E_I) \leq M\mu(E_I)\delta^{-\beta}.
\]

**Proof.** Let \( \{ \varphi_j \}_{j=1}^N \) be an IFS generating \( E \). Denote \( \beta = \dim_H E \). For \( J \in \{1, \ldots, N\}^* \), let \( c_J \) be the contraction ratio of \( \varphi_J \). Let \( \delta \in (0, \text{diam } E] \). We set

\[
\Omega_\delta = \{ J = j_1 \ldots j_k; c_J \leq \delta < c_{j_1 \ldots j_{k-1}} \}.
\]

It is well known that (see, for instance, Falconer [6])

\[
\mathcal{N}_\delta(E) \asymp \#\Omega_\delta \asymp \delta^{-\beta}.
\]

Therefore, since \( \varphi_I \) is a similarity, we obtain that for \( \delta \in (0, \text{diam } E_I] \),

\[
\mathcal{N}_\delta(E_I) = \mathcal{N}_{\delta/c_I}(E) \asymp (\delta/c_I)^{-\beta} = \mu(E_I)\delta^{-\beta}.
\]

The theorem is proved. \( \square \)

The above theorem shows that \( \delta_0(R) \) in Definition 1.4 can be chosen to be \( \text{diam}(R) \).
3. Proofs of Theorem 1.2 and Theorem 1.3

In this section, we investigate the Lipschitz invariants related to box-counting spaces.

3.1. Equivalence of box-counting measures.

**Proof of Theorem 1.2.** Let \( F = \{F_k\}_{k \geq 1} \) and \( F' = \{F'_k\}_{k \geq 1} \) be the compact Vitalli-type coverings in the theorem. Let \( \lambda \) be the maximal covering multiplicity of \( F_k \) as well as \( F'_k \) for all \( k \geq 1 \). For \( R \in \bigcup_{k=1}^{\infty} F_k \), denote \( R_k = \bigcup_{A' \in F'_k \text{ and } A' \cap R \neq \emptyset} A' \).

First, since the maximal diameter of elements of \( F'_k \) tends to 0 as \( k \to \infty \), we conclude that \( d_H(R_k, R) \to 0 \) as \( k \to \infty \). Secondly, since \( R \) is compact, we have that \( R_k \) decreases to \( R \) and it follows that \( \lim_{k \to \infty} \mu'(R_k) = \mu'(R) \).

Since \((X, \mu, F)\) and \((X, \mu', F')\) are box-counting spaces, there are positive constants \( M \) and \( M' \) such that
\[
M^{-1} \mu(A) \leq N_\delta(A) \delta^\beta \leq M^1 \mu(A) \quad \text{for any } A \in F_k, k \geq 1; \\
M'^{-1} \mu(A') \leq N_\delta(A') \delta^\beta \leq M'^1 \mu(A') \quad \text{for any } A' \in F'_k, k \geq 1.
\]

Choose \( k \) large enough so that \( \mu'(R_k) < 2 \mu'(R) \), then we have
\[
N_\delta(R) \leq N_\delta(R_k) \leq \sum_{A' \subset R_k} N_\delta(A') \leq \sum_{A' \subset R_k} M' \mu'(A') \delta^{-\beta} \leq \lambda M' \mu'(R_k) \delta^{-\beta} \leq 2 \lambda M' \mu'(R) \delta^{-\beta}.
\]

From the above relations we conclude that \( \mu(R) \leq 2 \lambda M M' \mu'(R) \).

Since \( \bigcup_{k=1}^{\infty} F_k \) generates the Borel algebra of \( X \), we conclude that
\[
\mu(A) \leq 2 \lambda M M' \mu'(A)
\]
for any Borel set \( A \subset X \). By symmetry, we also have \( \mu'(A) \leq 2 \lambda M M' \mu(A) \). Hence \( \mu \sim \mu' \). The theorem is proved.

3.2. Bi-Lipschitz maps between doubling spaces.

Two metric spaces \((X, d_X)\) and \((Y, d_Y)\) are said to be Lipschitz equivalent, and denoted by \((X, d_X) \sim (Y, d_Y)\), if there exist a bijection \( f : X \to Y \) and a constant \( C > 0 \) such that
\[
C^{-1} d_X(x, y) \leq d_Y(f(x), f(y)) \leq C d_X(x, y), \quad \text{for all } x, y \in X;
\]
in this case, we call \( f \) a bi-Lipschitz map.

**Lemma 3.1.** Let \( X \) and \( Y \) be two doubling spaces, and let \( f : X \to Y \) be a bi-Lipschitz map. Then there exists a constant \( C > 0 \) such that for any \( \delta > 0 \),
\[
C^{-1} N_\delta(X) \leq N_\delta(Y) \leq CN_\delta(X).
\]
Proof. Let $C_1 > 0$ be a Lipschitz constant of $f$. For $r > 0$ and $x \in X$, we have
\[ B(f(x), C_1^{-1}r) \subset f(B(x, r)), \]
it follows that $\mathcal{N}_\delta(X) \leq \mathcal{N}_{C_1^{-1}\delta}(Y)$.

Since $Y$ is a doubling space, there exists a constant $C_2 > 0$ such that for any $y \in Y$ and $r > 0$, $B(y, r)$ contains at most $C_2$ disjoint balls of radius $C_1^{-1}r/3$. Let $\mathcal{P}_1 = \{B(x_1, C_1^{-1}\delta), \ldots, B(x_k, C_1^{-1}\delta)\}$ be a $(C_1^{-1}\delta)$-ball-packing of $Y$ such that $k = \mathcal{N}_{C_1^{-1}\delta}(Y)$, let $\mathcal{P}_2 = \{B(y_1, \delta), \ldots, B(y_\ell, \delta)\}$ be a $\delta$-ball-packing of $Y$ such that $\ell = \mathcal{N}_\delta(Y)$. Then for each $x_i (1 \leq i \leq k)$, there exists $y_j (1 \leq j \leq \ell)$ such that $d_Y(x_i, y_j) \leq 2\delta$. Let $\mathcal{P}_3 = \{B(y_1, 3\delta), \ldots, B(y_\ell, 3\delta)\}$, we see that $\bigcup \mathcal{P}_1 \subset \bigcup \mathcal{P}_3$. Therefore,
\[ \mathcal{N}_{C_1^{-1}\delta}(Y) = k \leq C_2\ell = C_2\mathcal{N}_\delta(Y). \]
This proves the first inequality of (3.1). By symmetry, we have the second inequality. \hfill $\square$

Proof of Theorem 1.3. First, we claim $f(\mathcal{F})$ is a compact Vitalli-type covering of $Y$. Clearly, $f(\mathcal{F}_k)$ is a covering of $Y$ preserving the maximal covering multiplicity $\lambda$, the maximal diameter (up to the bi-Lipschitz constant $C$), and the compactness. Our claim is proved.

Let $\beta$ be the common value of the box dimension of $X$ and $Y$. Let $R \in \bigcup_{k=1}^{\infty} \mathcal{F}_k$. By Lemma 3.1 we have
\[ (3.2) \quad \mathcal{N}_\delta(f(R)) \asymp \mathcal{N}_\delta(R). \]
That $\mu$ is a box-counting measure of $X$ implies for $\delta \leq \delta_0(R)$,
\[ (3.3) \quad \mathcal{N}_\delta(R) \asymp \mu(R)\delta^{-\beta} = (f^*\mu)(f(R))\delta^{-\beta}. \]
Combining (3.2) and (3.3), we obtain that
\[ \mathcal{N}_\delta(f(R)) \asymp (f^*\mu)(f(R))\delta^{-\beta}, \quad \text{for } \delta \leq \delta_0(R). \]
Hence $(Y, f^*\mu, f(\mathcal{F}))$ is a box-counting measure space. The theorem is proved. \hfill $\square$

4. Box-counting measures of self-affine sponges of generalized Lalley-Gatzouras type

Let $\Phi = \Phi_\mathcal{D}$ be a diagonal self-affine IFS on $\mathbb{R}^d$ and let $\Lambda_\Phi$ be the attractor of $\Phi$. That $\Phi$ is said to satisfy the weak coordinate ordering condition if there exists a permutation $\sigma$ of $\{1, 2, \ldots, d\}$ such that
\[ \phi^{\prime}_{a, \sigma(1)} \geq \cdots \geq \phi^{\prime}_{a, \sigma(d)}; \quad \text{for all } a \in \mathcal{D}. \]
Without loss of generality, we always assume that
\[ (4.1) \quad \phi^{\prime}_{a, 1} \geq \cdots \geq \phi^{\prime}_{a, d}; \quad \text{for all } a \in \mathcal{D}. \]
Let $\pi_j : \mathbb{R}^d \to \mathbb{R}^j$ be the projection $\pi_j(x_1, \ldots, x_d) = (x_1, \ldots, x_j)$. We define the $j$-th projection IFS of $\Phi$ to be
\[ \Phi_{\{1, \ldots, j\}} = \{(\phi_{d, 1}, \ldots, \phi_{d, j})\}_{d \in \pi_j(\mathcal{D})}, \]
which is a diagonal self-affine IFS on $\mathbb{R}^j$. (Here we emphasize that each map occurs at most once in the above IFS.) Clearly, 

$$\Lambda_j := \pi_j(\Lambda_\Phi)$$

is the attractor of the IFS $\Phi_{\{1, \ldots, j\}}$.

**Definition 4.1** ([4]). Let $\Lambda_\Phi$ be a diagonal self-affine sponge satisfying (4.1). We say $\Phi$ satisfies the *neat projection condition*, if for each $j \in \{1, \ldots, d\}$, the IFS $\Phi_{\{1, \ldots, j\}}$ satisfies the OSC with the open set $\mathbb{D}^j = (0, 1)^j$, that is,

$$\{\phi_{\mathbb{D},\{1, \ldots, j\}}(\mathbb{D}^j)\}_{d \in \pi_j(D)}$$

are disjoint.

In the rest of this section, we always assume that $\Lambda_\Phi$ satisfies the weak coordinate ordering condition as well as the neat projection condition, in other words, $\Lambda_\Phi$ is of *generalized Lalley-Gatzouras type*.

Now we define a sequence of Bernoulli measures related to $\Lambda_j, j = 1, \ldots, d$.

First, we define a sequence $\{\beta_j\}_{j=1}^d$ related to $\Lambda_{\Phi}$. Let $\beta_1 > 0$ be the unique real number satisfying

$$\sum_{f_1 \in \Phi_{\{1\}}} (f_1')^{\beta_1} = 1.$$ 

If $\beta_1, \ldots, \beta_{j-1}$ are defined, we define $\beta_j > 0$ to be the unique real number such that

$$\sum_{(f_1, \ldots, f_j) \in \Phi_{\{1, \ldots, j\}}} \prod_{k=1}^j (f_k')^{\beta_k} = 1.$$ 

Next, for $f = (f_1, \ldots, f_j) \in \Phi_{\{1, \ldots, j\}}$, define

$$p_f = \prod_{k=1}^j (f_k')^{\beta_k}.$$ 

Let $\mu_j$ be the Bernoulli measure on $\Lambda_j$ defined by the weight $(p_r)_{r \in \Phi_{\{1, \ldots, j\}}}$. Especially, $\mu_d$ is the measure in Theorem 1.4.

For $I = i_1 \ldots i_n \in \mathcal{D}^n$, we call $\phi_I(\Lambda_\Phi)$ an $n$-th cylinder of $\Lambda_\Phi$, and call $\phi_{i_1 \ldots i_{n-1}}(\Lambda_\Phi)$ the ancestor of $\phi_I(\Lambda_\Phi)$. For a cylinder $R = \phi_I(\Lambda_\Phi)$, let $S(R) = \prod_{k=1}^n f_{i_k,d}'$ be the ‘shortest side’ of it.

**Remark 4.2.** Let $R$ and $R'$ be two $n$-th cylinders of $\Lambda_\Phi$, and $\nu$ be a Bernoulli measure of $\Lambda_\Phi$. Then $\nu(R \cap R') = 0$ is always true. See for instance [27].

**Proof of Theorem 1.4.** For $1 \leq j \leq d$, define

$$\alpha_j := \sum_{k=1}^j \beta_k.$$ 

We shall prove by induction on $j$ that for any cylinder $R$ of $\Lambda_j$ and any $\delta \leq S(R)$,

$$N_\delta(R) \asymp \mu_j(R) \delta^{-\alpha_j}.$$ 

If \( j = 1 \), \( \Lambda_1 \) is a one-dimensional self-similar set satisfying the OSC, hence \( \alpha_1 = \beta_1 = \dim_H \Lambda_1 = \dim_B \Lambda_1 \). By Theorem 2.1, \( \mu_1 \) is a box-counting measure of \( \Lambda_1 \) and (4.4) holds.

Now suppose (4.4) holds for \( j = d - 1 \). Let \( R = \phi_{i_1, \ldots, i_n}(\Lambda_\Phi) \) be a cylinder of \( \Lambda_d = \Lambda_\Phi \). Let us denote
\[
\rho := \min\{\phi'_{a,d} : a \in D\}.
\]

We start with the special case that \( S(R)\rho < \delta \leq S(R) \). A crucial observation is that
\[
N_\delta(R) \asymp N_\delta(\pi_{d-1}(R)).
\]

Denote \( \tilde{R} = \pi_{d-1}(R) \), then \( \tilde{R} \) is a cylinder of \( \Lambda_{d-1} \). By induction hypothesis, we have
\[
N_\delta(\tilde{R}) \asymp \mu_{d-1}(\tilde{R})\delta^{-\alpha_{d-1}}.
\]

For simplicity, let us denote \( \phi_{i_k} = (f_{k,1}, \ldots, f_{k,d}) \). Then
\[
(4.5) \quad \mu_d(R) = \prod_{k=1}^{n} \prod_{j=1}^{d} (f'_{k,j})^{\beta_k}
\]

and
\[
(4.6) \quad \delta \leq S(R) = \prod_{k=1}^{n} f'_{k,d} < \delta/\rho.
\]

It follows that
\[
(4.7) \quad N_\delta(R) \asymp \mu_{d-1}(\tilde{R})\delta^{-\alpha_{d-1}} = \left(\prod_{k=1}^{n} \prod_{j=1}^{d-1} (f'_{k,j})^{\beta_k}\right) \delta^{-\alpha_{d-1}} \\
\leq \left(\prod_{k=1}^{n} \prod_{j=1}^{d} (f'_{k,j})^{\beta_k}\right) \left(\prod_{k=1}^{n} (f'_{k,d})^{-\beta_d} \right) \delta^{-\alpha_{d-1}} \\
\asymp \left(\prod_{k=1}^{n} \prod_{j=1}^{d} (f'_{k,j})^{\beta_k}\right) \delta^{-\beta_d} \delta^{-\alpha_{d-1}} \quad \text{(by (4.6))} \\
= \mu_d(R)\delta^{-\alpha_d}, \quad \text{(by (4.3) and (4.5))}
\]

which verifies (4.4).

Now we consider the general case that \( \delta \leq S(R) \). Let \( \mathcal{V}_\delta \) be the collection of cylinders \( H \) such that \( S(H) \leq \delta < S(\hat{H}) \), where \( \hat{H} \) is the ancestor of \( H \); then
\[
\delta \rho < S(H) \leq \delta.
\]

First, by (4.7) we have
\[
(4.8) \quad N_\delta(H) \asymp \mu_d(H)\delta^{-\alpha_d}
\]

Clearly, \( \mathcal{V}_\delta \) is a covering of \( \Lambda_d \), and the elements in \( \mathcal{V}_\delta \) have disjoint interiors. Moreover, \( R \) is a union of some elements of \( \mathcal{V}_\delta \). These properties guarantee that
\[
N_\delta(R) \asymp \sum_{H \in \mathcal{V}_\delta \text{ and } H \subset R} N_\delta(H),
\]

where
\[
N_\delta(R) \asymp \mu_d(R)\delta^{-\alpha_d}
\]
which together with (4.8) imply that
\[ \mathcal{N}_\delta(R) \leq \sum_{H \in \mathcal{V}_3 \text{ and } H \subseteq R} \mu_d(H) \delta^{-\alpha_d} = \mu_d(R) \delta^{-\alpha_d}. \]
So (4.4) holds. Consequently, \( \dim_B \Lambda_d = \alpha_d \). The theorem is proved. \( \square \)

5. Box-counting measures of Barański carpets

Let \( r, s \geq 2 \) be two integers. Let \( \{\Phi_{i,1}\}_{i=1}^r \) and \( \{\Phi_{j,2}\}_{j=1}^s \) be the base IFS’s such that \( \{\Phi_{k,1}[0,1); 1 \leq k \leq r \} \) and \( \{\Phi_{k,2}[0,1); 1 \leq k \leq s \} \) are two disjoint families. Let \( \mathcal{D} \subset \{1, \ldots, r\} \times \{1, \ldots, s\} \). For \( \mathbf{d} = (i, j) \), let \( \Phi_{\mathbf{d}} = (\Phi_{i,1}, \Phi_{j,2}) \). The attractor \( \Lambda \) of the diagonal self-affine IFS
\[
\{\Phi_{\mathbf{d}}\}_{\mathbf{d} \in \mathcal{D}}
\]
is called a Barański carpet (Barański [2]).

Write \( a_i = \Phi_{i,1}' \) and \( b_j = \Phi_{j,2}' \). For \( \mathbf{d} = (i, j) \), we denote \( a_{\mathbf{d}} = a_i \) and \( b_{\mathbf{d}} = b_j \); in other words, \( \Phi_{\mathbf{d}}' = (a_{\mathbf{d}}, b_{\mathbf{d}}) \). Let \( \pi_1(x,y) = x \) and \( \pi_2(x,y) = y \) be the canonical projections. For \( I = i_1 \ldots i_k \in \mathcal{D}^k \), we denote
\[
a_I = a_{i_1} \cdots a_{i_k}, \quad b_I = b_{i_1} \cdots b_{i_k}, \quad \text{and} \quad s_I = \min\{a_I, b_I\}.
\]

Let \( \mathcal{D}^* = \bigcup_{k \geq 0} \mathcal{D}^k \) with \( \mathcal{D}^0 = \emptyset \). Set
\[
\mathcal{V}^A = \{I \in \mathcal{D}^*; a_I \geq b_I\}, \quad \mathcal{V}^B = \{I \in \mathcal{D}^*; a_I < b_I\},
\]
and for \( 0 < \delta < 1 \), set
\[
\mathcal{V}_\delta = \{I = i_1 \ldots i_k \in \mathcal{D}^*; s_I \leq 1 < s_{i_1 \ldots i_{k-1}}\}
\]
be the collection of cylinders whose shorter sides are of ‘approximately equal size’. Moreover, we define
\[
\mathcal{V}_\delta^A = \{I \in \mathcal{V}_\delta; a_I \geq b_I\}, \quad \mathcal{V}^B = \mathcal{V}_\delta \setminus \mathcal{V}_\delta^A.
\]

Let \( \alpha_1 \) and \( \alpha_2 \) be the positive real numbers satisfying
\[
\sum_{i \in \pi_1(\mathcal{D})} a_i^{\alpha_1} = 1, \quad \sum_{\mathbf{d} \in \mathcal{D}} a_{\mathbf{d}}^{\alpha_1} b_{\mathbf{d}}^{\alpha_2} = 1.
\]
Similarly, let \( \beta_1 \) and \( \beta_2 \) be the positive real numbers satisfying
\[
\sum_{j \in \pi_2(\mathcal{D})} b_j^{\beta_1} = 1, \quad \sum_{\mathbf{d} \in \mathcal{D}} b_{\mathbf{d}}^{\beta_1} a_{\mathbf{d}}^{\beta_2} = 1.
\]
Set \( \alpha = \alpha_1 + \alpha_2 \) and \( \beta = \beta_1 + \beta_2 \). (Barański [2] proved that \( \dim_B \Lambda = \max\{\alpha, \beta\} \).)

Let \( \nu_A \) be the Bernoulli measure on \( \pi_1(\Lambda) \) with probability weight \( (a_i^{\alpha_1})_{i \in \pi_1(\mathcal{D})} \), and \( \nu_B \) on \( \pi_2(\Lambda) \) with probability weight \( (b_j^{\beta_1})_{j \in \pi_2(\mathcal{D})} \). Then \( \nu_A \) and \( \nu_B \) are box-counting measures since they are Hausdorff measures of self-similar sets (Theorem 2.1).

Let \( \mu_A \) and \( \mu_B \) be the Bernoulli measures on \( \Lambda \) with probability weights
\[
(a_d^{\alpha_1} b_d^{\alpha_2})_{\mathbf{d} \in \mathcal{D}} \quad \text{and} \quad (b_d^{\beta_1} a_d^{\beta_2})_{\mathbf{d} \in \mathcal{D}}
\]
respectively.
Lemma 5.1. Let $\Lambda$ be a Barański carpet. Then

\begin{equation}
N_{b_I}(\Lambda_I) \asymp \mu_A(\Lambda_I)b_I^{-\alpha} \quad \text{for } I \in \mathcal{V}^A,
\end{equation}

and

\begin{equation}
N_{a_I}(\Lambda_I) \asymp \mu_B(\Lambda_I)a_I^{-\beta} \quad \text{for } I \in \mathcal{V}^B.
\end{equation}

Proof. If $I \in \mathcal{V}^A$, then $a_I \geq b_I$. The crucial observation is that

$N_{b_I}(\Lambda_I) \asymp N_{b_I}(\pi_1(\Lambda_I))$.

Notice that $\pi_1(\Lambda)$ is a self-similar set satisfying the OSC, $\nu_A$ is the canonical Bernoulli measure on $\pi_1(\Lambda)$, and $\dim_B \pi_1(\Lambda) = \alpha_1$. By Theorem 2.1, for $\delta \leq a_I$ we have

$N_\delta(\pi_1(\Lambda_I)) \asymp \frac{\nu_A(\pi_1(\Lambda_I))}{\delta^{\alpha_1}} = \frac{a_I^{\alpha_1}}{\delta^{\alpha_1}}$.

Set $\delta = b_I$ in the above formula, we obtain (5.1). Formula (5.2) can be obtained in the same manner. □

Lemma 5.2 (Barański [2]). Let $\Lambda$ be a Barański carpet. Then

$\max\{\alpha, \beta\} \leq \alpha_1 + \beta_1$.

Proof. Note that

$\sum_{d \in \mathcal{D}} a_d^{\alpha_1} b_d^{\beta_1} \leq \left( \sum_{i \in \pi_1(\mathcal{D})} a_i^{\alpha_1} \right) \left( \sum_{j \in \pi_2(\mathcal{D})} b_j^{\beta_1} \right) = 1$.

So we have $\beta_1 \geq \alpha_2 = \alpha - \alpha_1$ and $\alpha_1 \geq \beta_2 = \beta - \beta_1$, which imply the lemma. □

Lemma 5.3. Let $\Lambda$ be a Barański carpet. Then

$\mu_A(\Lambda_I)b_I^{-\alpha} \geq \mu_B(\Lambda_I)b_I^{-\beta}$, if $I \in \mathcal{V}^A$;

$\mu_A(\Lambda_I)a_I^{-\alpha} \leq \mu_B(\Lambda_I)a_I^{-\beta}$, if $I \in \mathcal{V}^B$.

Proof. Let $I \in \mathcal{V}^A$, then $a_I \geq b_I$. By Lemma 5.2, we have $\alpha_1 \geq \beta_2$, so

$\frac{\mu_A(\Lambda_I)b_I^{-\alpha}}{\mu_B(\Lambda_I)b_I^{-\beta}} = \left( \frac{a_I}{b_I} \right)^{\alpha_1 - \beta_2} \geq 1$,

which implies the first inequality of the lemma. The second inequality can be obtained in the same manner. □

For a cylinder $R = \Lambda_I$, we denote $S(R) = s_I$ to be the length of its shorter side.

**Proof of Theorem 1.5.** Without loss of generality, we assume that $\alpha \geq \beta$. We will show that for any cylinder $R$,

$N_\delta(R) \asymp (\mu_A(R) + \mu_B(R))\delta^{-\alpha}$, for $\delta \leq S(R)$,

in case of $\alpha = \beta$, and

$N_\delta(R) \asymp \mu_A(R)\delta^{-\alpha}$

holds for $\delta \leq \delta_0(R)$ where

\begin{equation}
\delta_0(R) = \min\{S(R), (\mu_A(R)/\mu_B(R))^{\frac{1}{\alpha - \beta}}\}
\end{equation}
in case of $\alpha > \beta$.

Let $\delta \leq S(R)$. Note that $R$ can be written as a finite disjoint union of cylinders in $V_\delta$, that is, $R = \bigcup_{I \in V_\delta} \Lambda_I$. Notice that for each $I \in V_\delta$, it holds that

$$r_\ast \delta < \min \{a_I, b_I\} \leq \delta,$$

where $r_\ast := \min(\{a_d; d \in D\} \cup \{b_d; d \in D\})$. By Lemma 5.1, we have

\begin{equation}
N_\delta(R) \asymp \sum_{I \in V_\delta} N_A(\Lambda_I) = \sum_{I \in V_\delta} \mu_A(\Lambda_I) \delta^{-\alpha} + \sum_{I \in V_\delta} \mu_B(\Lambda_I) \delta^{-\beta}.
\end{equation}

So there exists $M_1 > 0$ such that

\begin{equation}
N_\delta(R) \leq M_1 (\mu_A(R) \delta^{-\alpha} + \mu_B(R) \delta^{-\beta}).
\end{equation}

According to Lemma 5.3, we have

$$\sum_{I \in V_\delta} \mu_B(\Lambda_I) \delta^{-\beta} \geq \sum_{I \in V_\delta} \mu_A(\Lambda_I) \delta^{-\alpha},$$

so by (5.4) we obtain that there is a universal constant $M_2 > 0$ such that

$$N_\delta(R) \geq M_2^{-1} \mu_A(R) \delta^{-\alpha}.$$

By symmetry, we can show that $N_\delta(R) \geq M_3^{-1} \mu_B(R) \delta^{-\beta}$ for a universal constant $M_3 > 0$. It follows that

$$N_\delta(R) \geq \frac{\mu_A(R) \delta^{-\alpha} + \mu_B(R) \delta^{-\beta}}{M_2 + M_3}.$$

This together with (5.5) implies that

$$N_\delta(R) \asymp \mu_A(R) \delta^{-\alpha} + \mu_B(R) \delta^{-\beta}.$$

In case of $\alpha = \beta$, this already gives us the desired relations.

Now suppose that $\alpha > \beta$. Choose $\delta_0$ as in (5.3), then for $\delta \leq \delta_0$, we have $\mu_A(R) \delta^{-\alpha} \geq \mu_B(R) \delta^{-\beta}$, so

$$N_\delta(R) \asymp \mu_A(R) \delta^{-\alpha}.$$

This completes the proof. As a by-product, we obtain that $\dim_B \Lambda = \alpha$.  

\section{Box-counting measures of symbolic spaces}

In this section, we consider the box-counting measures of several symbolic spaces related to self-affine sponges.

Let $0 < \xi < 1$. Let $D_\xi$ be the metric on $\mathbb{Z}^\infty$ defined by

\begin{equation}
D_\xi(I, J) = \xi^{|I \wedge J|}, \quad I, J \in \mathbb{Z}^\infty,
\end{equation}

where $I \wedge J$ is the maximal common prefix of $I$ and $J$, and $|W|$ denotes the length of a finite word $W$. 

\[15\]
6.1. The first metric ($\lambda$-metric).

Let $1 > \xi_1 > \xi_2 > \cdots > \xi_d > 0$ be a sequence of real numbers. The metric $\lambda$ on $(\mathbb{Z}^d)^\infty$ is defined as the product metric

$$(\mathbb{Z}^d)^\infty, \lambda) = (\mathbb{Z}^\infty, D_{\xi_1}) \times \cdots \times (\mathbb{Z}^\infty, D_{\xi_d})$$

with $\lambda = \max\{D_{\xi_1}, \ldots, D_{\xi_d}\}$.

Let $D \in \mathbb{Z}^d$ be a finite set. Then $(D^\infty, \lambda)$ is a totally disconnected metric space. This symbolic space has been used in many works related to Bedford-McMullen carpets, for example, [14, 16] on multi-fractal analysis, and [31, 32] on Lipschitz classification.

For $I \in D^k$, recall that $[I]$ is a $k$-th cylinder of $D^\infty$. Denote $N := \#D$. Let $\mu$ be the measure on $D^\infty$ such that for any cylinder $[I]$ of rank $k$, $\mu([I]) = 1/N^k$, and we call it the uniform Bernoulli measure on $D^\infty$.

**Theorem 6.1.** The uniform Bernoulli measure is a box-counting measure of the space $(D^\infty, \lambda)$.

**Proof.** Similar to Section 4, we define $\beta_1, \ldots, \beta_d$ inductively as following. Let $\beta_1 > 0$ be the unique real number satisfying $\#(\pi_1(D)) (\xi_1)^{\beta_1} = 1$. If $\beta_1, \ldots, \beta_{j-1}$ are defined, we set $\beta_j > 0$ to be the unique real number satisfying

$$\#(\pi_j(D)) \prod_{k=1}^j (\xi_k)^{\beta_k} = 1.$$ 

Then by the same argument as the proof of Theorem 1.4, one can show that $\beta = \sum_{j=1}^d \beta_j$ is the box dimension of $(D^\infty, \lambda)$, and the uniform Bernoulli measure is a box-counting measure. \[ \square \]

6.2. The second metric ($\rho$-metric).

To study the Lipschitz classification of Bedford-McMullen carpets, [31] introduced a quasi-metric $\rho$ on $D^\infty$ as following. Let $2 \leq m < n$ be two integers. Assume that

$$D \subset \{0, 1, \ldots, m-1\} \times \mathbb{Z}.$$ 

For $I = (i, j), I' = (i', j') \in D^\infty$, set

$$\rho(I, I') = \max\{r_{1/m}(i, i'), D_{1/n}(j, j')\},$$

where $D_{1/n}$ is defined by (6.1) and

$$r_{1/m}(i, i') = \left| \sum_{k=1}^\infty \frac{i_k}{m^k} - \sum_{k=1}^\infty \frac{i'_k}{m^k} \right|.$$ 

If for any two distinct points $I = (i, j), I' = (i', j') \in D^\infty$, it holds that $\rho(I, I') \neq 0$, then we say $D$ satisfies the non-overlapping condition; in this case $(D^\infty, \rho)$ is a metric space.

**Theorem 6.2.** Suppose $D$ satisfies the non-overlapping condition. Then the uniform Bernoulli measure is a box-counting measure of the space $(D^\infty, \rho)$. 

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Proof. Let $s = \#(\pi_1(D))$, $\beta_1 = \log s / \log m$, and $\beta_2 = \log(N/s) / \log n$ where $N = \#D$. By the same argument as the proof of Theorem 1.4, one can show that $\beta = \beta_1 + \beta_2$ is the box dimension of $(D^\infty, \rho)$, and the uniform Bernoulli measure is a box-counting measure. □

7. Relation to Minkowski content

For $A \subset \mathbb{R}^d$, and $0 \leq \beta \leq d$, the $\beta$-dimensional upper Minkowski content is

$$M^*\beta(A) = \limsup_{\delta \to 0} \frac{\mathcal{L}^d([A]_\delta)}{\delta^{d-\beta}}$$

where $\mathcal{L}^d$ is the $d$-dimensional Lebesgue measure. By taking lower limit instead of upper limit, we define the $\beta$-dimensional lower Minkowski content $M_\beta^*(A)$. (See for instance, §3.1 of [6].)

The following lemma provides a necessary condition for the existence of the box-counting measure.

**Lemma 7.1.** Let $X$ be a compact subset of $\mathbb{R}^d$ with $\beta = \dim_B X$. If $\mu$ is a box-counting measure of $X$, then there exists a constant $M > 0$ such that for any $\delta$-connected component $R$ of $X$, it holds that

$$M^{-1} \mu(R) \leq M_\beta^*(R) \leq M^*\beta(R) \leq M \mu(R).$$

**Proof.** For $A \subset X$, define

$$g_\delta(A) = \frac{\mathcal{L}^d([A]_\delta)}{\delta^d}.$$ 

It is easy to show that $g_\delta(A) \asymp N_\delta(A)$. Hence $g_\delta(R) \asymp \mu(R)\delta^{-\beta}$, which proves the lemma. □

**References**

[1] A. Banaji and I. Kolossváry, *Intermediate dimensions of Bedford-McMullen carpets with applications to Lipschitz equivalence*, 2021, Preprint (arXiv:2111.05625 [math.DS]).

[2] K. Barański, *Hausdorff dimension of the limit sets of some planar geometric constructions*, Advances in Mathematics, 2007, 210(1): 215-245.

[3] D. Cooper, T. Pignataro, *On the shape of Cantor sets*, J. Differential Geom., 1988, 28: 203-221.

[4] T. Das, D. Simmons, *The Hausdorff and dynamical dimensions of self-affine sponges: a dimension gap result*, Inventiones Mathematicae, 2016, 2: 1-50.

[5] K.J. Falconer, *On the Minkowski measurability of fractals*, Proc. Amer. Math. Soc, 1995, 123: 1115-1124.

[6] K.J. Falconer, *Fractal geometry: mathematical foundations and applications*, John Wiley & Sons, 1990.

[7] K.J. Falconer, J.M. Fraser, T. Kempton, *Intermediate dimensions*, Math. Z., 2020, 296(1-2): 813-830.

[8] K.J. Falconer, D.T. Marsh, *On the Lipschitz equivalence of Cantor sets*, Mathematika, 1992, 39: 223-233.

[9] D.J. Feng, Y. Wang, *A class of self-affine sets and self-affine measures*, J. Fourier Anal. Appl., 2005, 11(1): 107-124.

[10] J.M. Fraser, *On the packing dimension of box-like self-affine sets in the plane*, Nonlinearity, 2012, 25(7): 2075-2092.
[11] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, MR1800917, 2001.
[12] M. Hochman, *On self-similar sets with overlaps and inverse theorems for entropy*, Annals of Mathematics, 2012, 183(2): 493-537.
[13] J.E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., 1981, 30: 713-747.
[14] T. Jordan, M. Rams, *Multifractal analysis for Bedford-McMullen carpets*, Math. Proc. Cambridge Philos. Soc., 2011, 150: 147-156.
[15] R. Kenyon, *Projecting the one-dimensional Sierpiński gasket*, Israel Journal of Mathematics, 1997, 97(1): 221-238.
[16] J.F. King, *The singularity spectrum for general Sierpiński carpets*, Advances in Mathematics, 1995, 116(1): 1-11.
[17] I. Kolossváry, *Calculating box dimension with the method of types*, 2021, Preprint (arXiv:2102.11049 [math.MG]).
[18] S.P. Lalley, D. Gatzouras, *Hausdorff and box dimensions of certain self-affine fractals*, Indiana Univ. Math. J., 1992, 41(2): 533-568.
[19] M.L. Lapidus, C. Pomerance, *The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums*, Proc. Lond. Math. Soc, 1993, 66: 41-69.
[20] Z. Liang, J.J. Miao, H. J. Ruan, *Gap sequences and topological properties of Bedford-McMullen sets*, Nonlinearity, 2022, 35(8): 4043-4063.
[21] B.M. Li, W.X. Li, J.J. Miao, *Lipschitz equivalence of McMullen sets*, Fractals, 2013, 21(3 & 4), 1350022, 11 pages.
[22] J.M. Mackay, *Assouad dimension of self-affine carpets*, Conform. Geom. Dyn., 2011, 15(12): 177-187.
[23] J.J. Miao, L.F. Xi, Y. Xiong, *Gap sequences of McMullen sets*, Proc. Amer. Math. Soc., 2017, 145: 1629-1637.
[24] J.R. Munkres, *Topology (second edition)*, Prentice Hall, Upper Saddle River, 2000.
[25] L. Olsen, *Symbolic and geometrical local dimensions of self-affine multifractal Sierpiński sponges in $\mathbb{R}^d$*, Stochastics and Dynamics, 2007, 7(01): 37-51.
[26] H. Rao, H.J. Ruan, Y. Wang, *Lipschitz equivalence of Cantor sets and algebraic properties of contraction ratios*, Trans. Amer. Math. Soc., 2012, 364: 1109-1126.
[27] H. Rao, Y.M. Yang and Y. Zhang, *Invariance of multifractal spectrum of uniform self-affine measures and its applications*, 2021, Preprint (arXiv:2005.07451 [math.DS]).
[28] H. Rao, Y. Zhang, *Higher dimensional Frobenius problem and Lipschitz equivalence of Cantor sets*, J. Math. Pures Appl., 2015, 104: 868-881.
[29] H.W. Reeve, *Multifractal analysis for Birkhoff averages on Lalley-Gatzouras repellers*, Fundamenta Mathematicae, 2010, 212 (1).
[30] L.F. Xi, H.J. Ruan, *Lipschitz equivalence of self-similar sets satisfying strong separation condition*, Acta Mathematica Sinica, 2008, 51(3): 493-500.
[31] Y.M. Yang, Y. Zhang, *Lipschitz classification of Bedford-McMullen carpets with uniform horizontal fibers*, J. Math. Anal. Appl., 2020, 495(2): 124742.
[32] Y.M. Yang, Y. Zhang, *Locally measure preserving property of bi-Lipschitz maps*, 2022, Preprint.
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