Curvature constraints in A-twisted heterotic Landau-Ginzburg models

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Abstract: In this paper, we study a class of A-twisted heterotic Landau-Ginzburg models. We show that the action can be written as a sum of BRST-exact and non-exact terms. The non-exact terms involve the pullback of the complexified Kähler form to the worldsheet and terms arising from the superpotential, which is a Grassmann-odd holomorphic function of the superfields. We then demonstrate that the action is invariant on-shell under supersymmetry transformations up to a total derivative. Finally, we extend the analysis to the case in which the superpotential is not holomorphic. In this case, we find that supersymmetry imposes a constraint which relates the nonholomorphic parameters of the superpotential to the Hermitian curvature. Various special cases of this constraint have previously been used to establish properties of Mathai-Quillen form analogues which arise in the corresponding heterotic Landau-Ginzburg models. There, it was claimed that supersymmetry imposes those constraints. Our goal in this paper is to support that claim.
1 Introduction

A Landau-Ginzburg model is a nonlinear sigma model with a superpotential. For a heterotic Landau-Ginzburg model [1–7], the nonlinear sigma model possesses only $(0,2)$ supersymmetry and the superpotential is a Grassmann-odd function of the superfields which may or may not be holomorphic. These heterotic models have field content consisting of $(0,2)$ bosonic chiral superfields $\Phi_i = (\phi_i, \psi_i)$, and $(0,2)$ fermionic chiral superfields $\Lambda^a = (\lambda^a, H^a, E^a)$, along with their conjugate antichiral superfields $\Phi_ı = (\phi_ı, \psi_ı)$, and $\Lambda^a = (\lambda^a, H^a, E^a)$. The $\phi_i$ are local complex coordinates on a Kähler manifold $X$. The $E^a$ are local smooth sections of a Hermitian vector bundle $E$ over $X$, i.e. $E^a \in \Gamma(X, E)$. The $H^a$ are nonpropagating auxiliary fields. The fermions couple to bundles as follows:

$$
\psi^i_+ \in \Gamma \left( K^1/2_\Sigma \otimes \Phi^* (T^{1,0} X) \right), \quad \lambda^a_- \in \Gamma \left( K^1/2_\Sigma \otimes (\Phi^* E)^\vee \right),
$$

$$
\psi^a_- \in \Gamma \left( K^1/2_\Sigma \otimes (\Phi^* (T^{1,0} X))^\vee \right), \quad \lambda^a_+ \in \Gamma \left( K^1/2_\Sigma \otimes \Phi^* E \right),
$$

where $\Phi : \Sigma \rightarrow X$ and $K_\Sigma$ is the canonical bundle on the worldsheet $\Sigma$. In [5], an A-twisted version of the above with superpotential of the form

$$
W = \Lambda^a F_a ,
$$

was considered. In this paper, we will study supersymmetry in these A-twisted heterotic Landau-Ginzburg models with $E^a = 0$. Such models yield the A-twisted $(2,2)$ Landau-Ginzburg models of [8] when $E = TX$ and $\Lambda^i F_i = \Lambda^i \partial_i W^{(2,2)}$, where $W^{(2,2)}$ is the $(2,2)$ superpotential. It was claimed in [7] that, when the superpotential is not holomorphic, supersymmetry imposes a constraint which relates the nonholomorphic parameters of the superpotential to the Hermitian curvature. Our goal in this paper is to support that claim.

This paper is organized as follows: In section 2, we will write down the action for the class of A-twisted heterotic Landau-Ginzburg models that we are considering. In section 3,
for the case of a holomorphic superpotential, we will show that the action can be written as a sum of BRST-exact and non-exact terms. We will then demonstrate that the action is invariant on-shell under supersymmetry transformations up to a total derivative. Finally, in section 4, we will extend the analysis to the case in which the superpotential is not holomorphic. In this case, we will show that supersymmetry imposes a constraint which relates the nonholomorphic parameters of the superpotential to the Hermitian curvature.

2 Action

Let \( X \) be a Kähler manifold with metric \( g \), antisymmetric tensor \( B \), local real coordinates \( \phi^\mu \), and local complex coordinates \( \phi^i \) with complex conjugates \( \phi^\bar{i} \). Furthermore, let \( E \) be a vector bundle over \( X \) with Hermitian fiber metric \( h \). We consider the action [5] of an A-twisted heterotic Landau-Ginzburg model on \( X \) with gauge bundle \( E \):

\[
S = 2t \int_{\Sigma} d^2z \left[ \frac{1}{2} (g_{\mu\nu} + iB_{\mu\nu}) \partial_{\mu} \phi^\mu \partial_{\nu} \phi^\nu + ig_{\mu\nu} \psi_+^i \partial_{\pi} \psi_+^i + ih_{\alpha\beta} \lambda^\alpha_+ \partial_\lambda \lambda^-_+ \right.
\]

\[
+ F_{i\alpha\pi} \psi_+^i \psi_+^\alpha \lambda^-_+ + h_{\alpha\beta} F_a \partial_\alpha F_a + \psi_+^i \lambda^-_+ D_i F_a + \psi_+^\alpha \lambda^-_+ \partial_\alpha F_a \right]
\]

(2.1)

Here, \( t \) is a coupling constant, \( \Sigma \) is a Riemann surface, \( F_a \in \Gamma(X, E^\vee) \), and

\[
D_\pi \psi_+^i = \partial_\pi \psi_+^i + \partial_{\mu} \phi^\mu \Gamma_{jk}^i \psi_+^k,
\]

\[
D_i F_a = \partial_i F_a - A^b_{ia} F_b,
\]

\[
A^b_{ia} = h^{ab} h_{a\alpha} \lambda^-_+,
\]

\[
\Gamma_{jk}^i = g^{\alpha\beta} g_{\alpha\gamma} \lambda^-_+.
\]

The fermions couple to bundles as follows:

\[
\psi_+^i \in \Gamma \left( \Phi^* (T^{-1,0} X) \right), \quad \lambda^\alpha_+ \in \Gamma \left( \Phi^* (K_{\Sigma} \otimes \Phi^* E^\vee) \right),
\]

\[
\psi_+^\alpha \in \Gamma \left( K_{\Sigma} \otimes (\Phi^* (T^{-1,0} X))^\vee \right), \quad \lambda^-_+ \in \Gamma \left( \Phi^* E \right),
\]

where \( \Phi : \Sigma \to X \) and \( K_{\Sigma} \) is the canonical bundle on \( \Sigma \).

3 Supersymmetry invariance for holomorphic superpotential

In this section, we will show that, when the superpotential is holomorphic, the action is invariant on-shell under the supersymmetry transformations

\[
\delta \phi^i = i\alpha_- \psi_+^i, \quad \delta \phi^\bar{i} = 0,
\]

\[
\delta \psi_+^i = 0, \quad \delta \psi_+^\alpha = -\alpha_- \partial_\pi \phi^\alpha,
\]

\[
\delta \lambda^\alpha_+ = -i\alpha_- A^b_{ib} \lambda^-_+ + i\alpha_- h^{\alpha\pi} F_\alpha, \quad \delta \lambda^-_+ = 0
\]

(3.1)

up to a total derivative. To this end, using the \( \lambda^\alpha_+ \) equation of motion, we will show in section 3.1 that the action (2.1) can be written

\[
S = it \int_{\Sigma} d^2z \{Q, V\} + t \int_{\Sigma} \Phi^* (K) + 2t \int_{\Sigma} d^2z \left( \psi_+^i \lambda^-_+ \partial_\pi \lambda^-_+ \psi_+^i - \psi_+^i \lambda^\alpha_+ D_a F_a \right)
\]

(3.2)
where $Q$ is the BRST operator,

$$V = 2\left(g_{\alpha i}\psi_+^i \partial_\alpha \phi^i + i\lambda_\alpha F_a\right), \quad (3.3)$$

and

$$\int \Phi^*(K) = \int \Sigma d^2 z \left(g_{\alpha i} + iB_{\alpha i}\right) \left(\partial_\alpha \phi^i \partial_\alpha \phi^j - \partial_\alpha \phi^j \partial_\alpha \phi^i\right) \quad (3.4)$$

is the integral over the worldsheet $\Sigma$ of the pullback to $\Sigma$ of the complexified Kähler form $K = -(g_{\alpha i} + iB_{\alpha i}) dz^i d\bar{z}^j$. Since $\delta f = -i\alpha \left\{Q, f\right\}$, where $f$ is any field, the $Q$-exact part of (3.2) is $\delta$-exact and hence $\delta$-closed. In section 3.2, we will complete our argument by establishing that the remaining terms are $\delta$-closed on shell up to a total derivative.

### 3.1 BRST-exact and non-exact terms

Let us now derive (3.2). The BRST transformations are

$$\begin{align*}
\{Q, \phi^i\} &= -\psi_+^i, \\
\{Q, \psi_+^i\} &= 0, \\
\{Q, \lambda_\alpha\} &= i\alpha \partial_\alpha \phi^i - h^{\alpha \alpha} F_\pi, \\
\{Q, \lambda_\alpha\} &= 0.
\end{align*} \quad (3.5)$$

Now, we compute

$$\frac{\{Q, V\}}{2} = \left\{Q, g_{\alpha i}\psi_+^i \partial_\alpha \phi^i + i\lambda_\alpha F_a\right\}$$

$$= \left\{Q, g_{\alpha i}\psi_+^i \partial_\alpha \phi^i + g_{\alpha i}\{Q, \psi_+^i\} \partial_\alpha \phi^i - g_{\alpha i}\psi_+^i \partial_\alpha \{Q, \phi^i\} + i\{Q, \lambda_\alpha\} F_a - i\lambda_\alpha \{Q, F_a\}\right\}$$

$$= \left(\gamma_{\alpha i}\left\{Q, \phi^k\right\}\right) \psi_+^i \partial_\alpha \phi^j + g_{\alpha i}\left(-i\partial_\alpha \phi^i \partial_\alpha \phi^j - g_{\alpha i}\psi_+^i \partial_\alpha \{Q, \phi^i\}\right)$$

$$+ \left(i\gamma_{\alpha i} A_{jk}^b \lambda_\alpha - h^{\alpha \alpha} F_\pi\right) F_a - i\lambda_\alpha \left\{Q, \phi^k\right\} + \left(\gamma_{\alpha i} A_{jk}^b F_a - i\lambda_\alpha \left\{Q, \phi^k\right\}\right)$$

$$= -i\gamma_{\alpha i} \partial_\alpha \phi^i \partial_\alpha \phi^j + g_{\alpha i}\psi_+^i \left(\partial_\alpha \phi^j + \partial_\alpha \phi^j \Gamma_{jk}^i \psi_+^k\right)$$

$$- i\lambda_\alpha \left\{Q, F_a - A_{ia}^b F_b\right\}$$

$$= -i\gamma_{\alpha i} \partial_\alpha \phi^i \partial_\alpha \phi^j + g_{\alpha i}\psi_+^i \partial_\alpha \phi^j - h^{\alpha \alpha} F_\pi F_a - i\lambda_\alpha D_i F_a,$$

where we have used $g_{\alpha i,k} = \gamma_{\alpha i,k}$ in the fourth step. It follows that

$$\int d^2 z \{Q, V\} = 2t \int d^2 z \left(g_{\alpha i} \partial_\alpha \phi^i \partial_\alpha \phi^j + g_{\alpha i}\psi_+^i \partial_\alpha \phi^j + h^{\alpha \alpha} F_\pi F_a + i\lambda_\alpha D_i F_a\right).$$
Using the identity
\[ \int \Sigma d^2 z \, g_{\alpha} \partial_z \phi^i \partial \pi^j = \int \Sigma d^2 z \, \left( \frac{1}{2} (g_{\mu \nu} + iB_{\mu \nu}) \partial_z \phi^\mu \partial \phi^\nu - \frac{1}{2} \int \Phi^*(K) \right), \tag{3.6} \]
we obtain
\[ \int \Sigma d^2 z \{Q, V\} \]
\[ = 2t \int \Sigma d^2 z \left[ \frac{1}{2} (g_{\mu \nu} + iB_{\mu \nu}) \partial_z \phi^\mu \partial \pi^\nu + ig_{\alpha} \psi^{i}_{+} \partial \pi \psi^{i}_{+} + h^{\alpha \pi} F_\alpha F_\pi + \psi^{i}_{+} \lambda^\alpha D_i F_\alpha \right] \]
\[ - t \int \Sigma \Phi^*(K) \]
\[ = S - t \int \Sigma \Phi^*(K) - 2t \int \Sigma d^2 z \left( ih_{\alpha \pi} \lambda^\alpha D_\pi \lambda_\pi + F_{\alpha \pi} \psi^{i}_{+} \lambda^\alpha \lambda_\pi + \psi^{i}_{+} \lambda^\alpha D_\pi F_\alpha \right), \]
and hence
\[ S = it \int \Sigma d^2 z \{Q, V\} + t \int \Sigma \Phi^*(K) \]
\[ + 2t \int \Sigma d^2 z \left( ih_{\alpha \pi} \lambda^\alpha D_\pi \lambda_\pi + F_{\alpha \pi} \psi^{i}_{+} \lambda^\alpha \lambda_\pi + \psi^{i}_{+} \lambda^\alpha D_\pi F_\alpha \right). \]

An analogous result was found in [4] for the case in which the gauge fields are absent and \( B = 0 \). Finally, using the \( \lambda_\pi^\alpha \) equation of motion
\[ \lambda_\pi^\alpha : \quad ih_{\alpha \pi} \lambda_\pi^\alpha D_\pi \lambda_\pi - \psi^{i}_{+} \lambda_\pi^\alpha D_i F_\alpha = 0, \tag{3.7} \]
we obtain
\[ ih_{\alpha \pi} \lambda_\pi^\alpha D_\pi \lambda_\pi + F_{\alpha \pi} \psi^{i}_{+} \lambda_\pi^\alpha \lambda_\pi = - \psi^{i}_{+} \lambda_\pi^\alpha D_i F_\alpha \]
and hence
\[ S = it \int \Sigma d^2 z \{Q, V\} + t \int \Sigma \Phi^*(K) + 2t \int \Sigma d^2 z \left( \psi^{i}_{+} \lambda_\pi D_\pi F_\pi - \psi^{i}_{+} \lambda_\pi D_i F_\alpha \right), \]
which is (3.2).

### 3.2 Supersymmetry invariance of non-exact terms

Let us now complete our argument that the action (3.2) is \( \delta \)-closed on shell up to a total derivative. As we previously noted, the \( Q \)-exact part of (3.2) is \( \delta \)-exact and hence \( \delta \)-closed. The following computation establishes that the non-exact term of (3.2) involving \( \Phi^*(K) \) is
\[ \delta \left[ (g_{\alpha \pi} + i B_{\alpha \pi}) \left( \partial_\alpha \phi^i \overline{\partial}_{\overline{\pi}} \phi^j - \overline{\partial}_{\overline{\pi}} \phi^i \partial_\alpha \phi^j \right) \right] \\
\quad = (g_{\alpha \pi, k} + i B_{\alpha \pi, k}) \delta \phi^k \left( \partial_\alpha \phi^i \overline{\partial}_{\overline{\pi}} \phi^j - \overline{\partial}_{\overline{\pi}} \phi^i \partial_\alpha \phi^j \right) \\
\quad \quad + (g_{\alpha \pi} + i B_{\alpha \pi}) \left[ \left( \partial_\alpha \delta \phi^i \right) \overline{\partial}_{\overline{\pi}} \phi^j + \partial_\alpha \phi^i \left( \overline{\partial}_{\overline{\pi}} \delta \phi^j \right) - \left( \overline{\partial}_{\overline{\pi}} \delta \phi^i \right) \partial_\alpha \phi^j - \overline{\partial}_{\overline{\pi}} \phi^i \left( \partial_\alpha \phi^j \right) \right] \\
\quad = (g_{\alpha \pi, k} + i B_{\alpha \pi, k}) \left( i \alpha_{- \psi_+^k} \right) \left( \partial_\alpha \phi^i \overline{\partial}_{\overline{\pi}} \phi^j - \overline{\partial}_{\overline{\pi}} \phi^i \partial_\alpha \phi^j \right) \\
\quad \quad + (g_{\alpha \pi} + i B_{\alpha \pi}) \left[ \partial_\alpha \left( i \alpha_{- \psi_+^i} \right) \overline{\partial}_{\overline{\pi}} \phi^j - \overline{\partial}_{\overline{\pi}} \left( i \alpha_{- \psi_+^i} \right) \partial_\alpha \phi^j \right] \\
\quad = - (g_{\alpha \pi} + i B_{\alpha \pi}) \partial_\alpha \left( i \alpha_{- \psi_+^i} \right) \left( \partial_\alpha \phi^i \overline{\partial}_{\overline{\pi}} \phi^j - \overline{\partial}_{\overline{\pi}} \phi^i \partial_\alpha \phi^j \right) \\
\quad \quad - \partial_\alpha \left[ (g_{\alpha \pi} + i B_{\alpha \pi}) \overline{\partial}_{\overline{\pi}} \phi^j \right] \left( i \alpha_{- \psi_+^i} \right) + \overline{\partial}_{\overline{\pi}} \left[ (g_{\alpha \pi} + i B_{\alpha \pi}) \partial_\alpha \phi^j \right] \left( i \alpha_{- \psi_+^i} \right) \\
\quad = - (g_{\alpha \pi} + i B_{\alpha \pi}) \left( i \alpha_{- \psi_+^k} \right) \left[ \left( \partial_\alpha \delta_\alpha^k \right) \overline{\partial}_{\overline{\pi}} \phi^j - \overline{\partial}_{\overline{\pi}} \phi^i \left( \partial_\alpha \delta_\alpha^k \right) \right] \\
\quad \quad - \left[ (g_{\alpha \pi, k} + i B_{\alpha \pi, k}) \partial_\alpha \phi^k + \left( g_{\alpha \pi, k} + i B_{\alpha \pi, k} \right) \partial_\alpha \phi^j \right] \overline{\partial}_{\overline{\pi}} \phi^j \left( i \alpha_{- \psi_+^i} \right) \\
\quad \quad - \left[ (g_{\alpha \pi} + i B_{\alpha \pi}) \partial_\alpha \overline{\partial}_{\overline{\pi}} \phi^j \right] \left( i \alpha_{- \psi_+^i} \right) \\
\quad \quad + \left[ g_{\alpha \pi, k} + i B_{\alpha \pi, k} \right] \overline{\partial}_{\overline{\pi}} \phi^k \left( i \alpha_{- \psi_+^i} \right) \\
\quad \quad + \left[ g_{\alpha \pi} + i B_{\alpha \pi} \right] \overline{\partial}_{\overline{\pi}} \partial_\alpha \phi^j \left( i \alpha_{- \psi_+^i} \right) \right] = 0, \quad (3.8) \]

where we have integrated by parts in the third and fifth steps and assumed that the boundary terms vanish. It remains to consider the non-exact expression of (3.2) involving \( 2 \left( \psi_+^k, \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} - \psi_+^k \lambda_\pi^\alpha D_\pi F_{\alpha} \right) \). First, we compute

\[ \delta \left( \psi_+^k, \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} \right) = \left( \delta \psi_+^k \right) \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} + \psi_+^k \left( \delta \lambda_\pi^\alpha \right) \overline{D}_\pi F_{\alpha} + \psi_+^k \lambda_\pi^\alpha \left[ \delta \left( \overline{D}_\pi F_{\alpha} \right) \right] \\
\quad = \left( - \alpha_{- \partial_\alpha \phi^k} \right) \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} + \psi_+^k \lambda_\pi^\alpha \left[ \delta \left( \overline{D}_\pi F_{\alpha} \right) \right] \\
\quad \quad + \left( - \alpha_{- \partial_\alpha \phi^k} \right) \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} + \psi_+^k \lambda_\pi^\alpha \left[ \delta \left( \overline{D}_\pi F_{\alpha} \right) \right] - \left( \delta A_{\alpha \pi, k}^\beta \right) F_{\beta} - A_{\alpha \pi, k}^\beta \left( \delta F_{\beta} \right) \\
\quad = \left( \delta \psi_+^k, \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} \right) + \psi_+^k \lambda_\pi^\alpha \left[ \partial_\alpha \overline{D}_\pi F_{\alpha} \right] \left( \delta \phi^k \right) + \psi_+^k \lambda_\pi^\alpha \left( \delta \phi^k \right) \left[ \overline{D}_\pi F_{\alpha} \right] - \left( \delta A_{\alpha \pi, k}^\beta \right) F_{\beta} - A_{\alpha \pi, k}^\beta \left( \delta F_{\beta} \right) \right] = \left( - \alpha_{- \partial_\alpha \phi^k} \right) \lambda_\pi^\alpha \overline{D}_\pi F_{\alpha} - \psi_+^k \lambda_\pi^\alpha \lambda_{\alpha \pi, k}^\beta \left( i \alpha_{- \psi_+^k} \right) F_{\beta}, \quad (3.9) \]
where we have used $F_{\pi,k} = 0$ in the last step. Now, we compute

\[
\delta (-\psi^i_+ \lambda^a D_i F_a) = - \left( \delta \psi^i_+ \lambda^a D_i F_a - \psi^i_+ (\delta \lambda^a_+) D_a F_a - \psi^i_+ \lambda^a_- [\delta (D_i F_a)] \right) \\
= - \psi^i_+ \left( -i\alpha_- \psi^i_+ A^b_+ \lambda^b_- + i\alpha_- h^{ab} F_{\pi} \right) D_i F_a - \psi^i_+ \lambda^a_- \left[ \partial_i \left( \delta F_a \right) - \left( \delta A^b_{ia} \right) F_b - A^b_{ia} \left( \delta F_b \right) \right] \\
= - \psi^i_+ \left( i\alpha_- h^{ab} F_{\pi} \right) D_i F_a \\
- \psi^i_+ \lambda^a_- \left\{ \partial_i \left[ F_{a,k} \left( \delta \phi^k \right) \right] - \left[ A^b_{ia,k} \left( \delta \phi^k \right) \right] F_b - A^b_{ia} \left( \delta \phi^k \right) \right\} \\
= - \psi^i_+ \left( i\alpha_- h^{ab} F_{\pi} \right) D_i F_a + \left( \text{total } \phi^k \text{ derivative} \right) \\
= i\alpha_- h^{ab} F_{\pi} D_z \lambda^\pi_- + F_{\pi a} \psi^i_+ \psi^i_+ \left( i\alpha_- h^{ab} F_{\pi} \right) \lambda^\pi_- \\
+ \left( \text{total } \phi^k \text{ derivative} \right), \\
\tag{3.10}
\]

where we have used the $\lambda^a_-$ equation of motion (3.7) in the last step. Note that the first term on the right-hand side of (3.10) cancels the first term on the right-hand side of (3.9):

\[
\begin{align*}
\alpha_- F_{\pi} D_z \lambda^\pi_- &= - \alpha_- F^\pi D_z \lambda^\pi_- \\
&= - \alpha_- F_{\pi} \left( \partial_z \lambda^\pi_- + \partial_\phi^\pi A^\pi_{\tau b} \lambda^\pi_- \right) \\
&= \alpha_- \left( F_{\pi,k} \partial_\phi^k + F_{\pi,k}^\pi \partial_\phi^\pi \right) - \alpha_- F_{\pi} \partial_\phi^\pi A^\pi_{\tau b} \lambda^\pi_- \\
&= \left( \alpha_- \partial_\phi^\pi \right) \lambda^\pi_- \left( \partial_{\tau} F_{\pi} - A_{\tau b}^\pi \right) \\
&= \left( \alpha_- \partial_\phi^\pi \right) \lambda^\pi_- \partial_{\tau} F_{\pi}, \\
\tag{3.11}
\end{align*}
\]

where we have integrated by parts in the third step, assumed that the boundary term vanishes, and used $F_{\pi,k} = 0$ in the fourth step. Furthermore, the second term on the right-hand side of (3.10) cancels the second term on the right-hand side of (3.9):

\[
F_{\pi a} \psi^i_+ \psi^i_+ \left( i\alpha_- h^{ab} F_{\pi} \right) \lambda^\pi_- = h_{a\pi} A_{\tau \pi, i}^\pi \psi^i_+ \psi^i_+ \left( i\alpha_- h^{ab} F_{\pi} \right) \lambda^\pi_- \\
= A_{\tau \pi,k}^\pi \psi^i_+ \psi^i_+ \left( i\alpha_- F_{\pi} \right) \lambda^\pi_- \\
= \psi^i_+ \lambda^\pi_- A_{\tau \pi,k}^\pi \left( i\alpha_- \psi^i_+ \right) F_{\pi}.
\tag{3.12}
\]

It follows that (3.10) cancels (3.9) up to a total derivative, i.e.

\[
\delta (-\psi^i_+ \lambda^a_- D_i F_a) = - \delta \left( \psi^i_+ \lambda^\pi_- \partial_{\tau} F_{\pi} + \left( \text{total } \phi^k \text{ derivative} \right) \right).
\tag{3.13}
\]

This completes our argument.
4 Supersymmetry invariance for nonholomorphic superpotential

In this section, we will extend the analysis of section 3 to the case in which the superpotential is not holomorphic. Most of the analysis in section 3 still applies, so let us focus on what changes. In (3.11), the first term of the third line no longer vanishes outright, but now yields zero after integrating by parts, assuming that the boundary term vanishes:

$$\alpha \bar{F}_{\pi,k} \partial_z \phi^k = -\alpha \bar{F}_{\pi} \partial_z \phi^k = 0. \tag{4.1}$$

Furthermore, in the next to last line of (3.9), we now have

$$\psi^\tau_+ \lambda^\pi_- \left\{ \overline{\partial}_\tau \left[ \bar{F}_{\pi,k} \left( \delta \phi^k \right) \right] - A^\pi_- F_{\bar{h},k} \left( \delta \phi^k \right) \right\}$$

$$= \psi^\tau_+ \lambda^\pi_- \left\{ \overline{\partial}_\tau \left[ \bar{F}_{\pi,k} \left( i\alpha_+ \psi^k_+ \right) \right] - A^\pi_- F_{\bar{h},k} \left( i\alpha_+ \psi^k_+ \right) \right\}$$

$$= \psi^\tau_+ \lambda^\pi_- \left( \overline{\partial}_\tau \bar{F}_{\pi,i} - A^\pi_- F_{\bar{h},i} \right) \left( i\alpha_+ \psi^i_+ \right)$$

$$= \psi^\tau_+ \lambda^\pi_- \left( \overline{\partial}_\tau \bar{F}_{\pi,i} + A^\pi_- F_{\bar{h},i} \right) \left( i\alpha_+ \psi^i_+ \right)$$

$$= \psi^\tau_+ \lambda^\pi_- \left( \overline{\partial}_\tau \bar{F}_{\pi,i} + h^a_{\alpha a} A^\pi_- F_{\bar{h},i} \right) \left( i\alpha_+ \psi^i_+ \right)$$

$$= \psi^\tau_+ \lambda^\pi_- \left( \overline{\partial}_\tau \bar{F}_{\pi,i} + F_{\pi a \bar{a}} h^a_{\alpha a} \bar{F}_{\bar{b}} \right) \left( i\alpha_+ \psi^i_+ \right), \tag{4.2}$$

where we have integrated by parts in the third step and assumed that the boundary term vanishes. It follows that supersymmetry imposes the constraint

$$\overline{\partial}_\tau \bar{F}_{\pi,i} + F_{\pi a \bar{a}} h^a_{\alpha a} \bar{F}_{\bar{b}} = 0. \tag{4.3}$$

Various special cases of this constraint were used in [7] to establish properties of Mathai-Quillen form analogues which arise in the corresponding heterotic Landau-Ginzburg models. In that paper, it was claimed that supersymmetry imposes those constraints. In this paper, we have worked out the details supporting that claim.

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