Periodic Maxwell–Chern–Simons vortices with concentrating property

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Abstract
In order to study electrically and magnetically charged vortices in fractional quantum Hall effect and anyonic superconductivity, the Maxwell–Chern–Simons (MCS) model was introduced by Lee et al. (Phys Lett B 252:79–83, 1990) as a unified system of the classical Abelian–Higgs model (AH) and the Chern–Simons (CS) model. In this article, the first goal is to obtain the uniform (CS) limit result of (MCS) model with respect to the Chern–Simons parameter, without any restriction on either a particular class of solutions or the number of vortex points, as the Chern–Simons mass scale tends to infinity. The most important step for this purpose is to derive the relation between the Higgs field and the neutral scalar field. Our (CS) limit result also provides the critical clue to answer the open problems raised by Ricciardi and Tarantello (Comm Pure Appl Math 53:811–851, 2000) and Tarantello (Milan J Math 72:29–80, 2004), and we succeed to establish the existence of periodic Maxwell–Chern–Simons vortices satisfying the concentrating property of the density of superconductive electron pairs. Furthermore, we expect that the (CS) limit analysis in this paper would help to study the stability, multiplicity, and bubbling phenomena for solutions of the (MCS) model.

Mathematics Subject Classification 35B40 · 35J20

1 Introduction
As the pioneering work by Ginzburg and Landau, the classical Abelian-Higgs (AH) model (or, Maxwell–Higgs) was proposed in order to describe the superconductivity phenomena at low temperature (see [4,36,40,50]). This model has been studied in [6,36,59,62] for various domains. However, (AH) model can only describes electrically neutral vortices, which are static solutions of the corresponding Euler–Lagrange

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equation. In order to study the fractional quantum Hall effect and high temperature superconductivity, we should investigate electrically and magnetically “charged” vortices. For this purpose, one might attempt to include Chern–Simons (CS) term into (AH) model. However, just adding (CS) term into (AH) model loses the self-dual structure, which is characterized by a special class of static solution corresponding to a constrained energy minimizer. The self-dual equation has a benefit in the gauge field theory since it is a reduced first-order equation, so called “Bogomol’nyi equation”, for the more complicated second order equation of motion (see [5,46]). In order to obtain a self-dual Chern–Simons theory, Hong–Kim–Pac [34] and Jackiw–Weinberg [35] independently proposed a model for charged vortices with electrodynamics governed only by the (CS) term without Maxwell term, which was included in the (AH) model. This pure (CS) model was suggested from the observation such that the (CS) term is dominant over the Maxwell term in the large scale. During the last few decades, the (CS) model has been extensively studied in [13,14,18,54,55,61,63] for entire solutions on a full space, in [8,18,19,21–24,28–30,49,56,57] for the periodic case, and in [33] for bounded domains (see also [9,10,17,31,37–39,53]).

As stated above, a naive inclusion of both (AH) term and (CS) term in the Lagrangian fails to make the system self-dual. However, in [41], Lee, Lee, and Min succeeded in restoring the self-duality in Maxwell–Chern–Simons (MCS) model as a unified self-dual system of (AH) and (CS), by introducing a neutral scalar field. Moreover, the authors in [41] showed formally that the self-dual equation of (MCS) owns both (AH) model and (CS) model as limiting problems according to the limit behavior of the electric charge and the Chern–Simons mass scale (see also [25]). This formal argument in [25,41] could be supported with mathematically rigorous proof in [11,12,51,52]. In [11], Chae and Kim established the existence of topological multivortex solution for (MCS) model in a full space $\mathbb{R}^2$. Here, the topological entire solution in $\mathbb{R}^2$ satisfies the specific boundary condition such that its first component vanishes at infinity. Moreover, the authors in [11] showed the convergence of topological multivortex solutions to the (CS) model and (AH) model. The convergence depends on the asymptotic behavior of the electric charge and the Chern–Simons mass scale. In [12], they also obtained the corresponding result for topological solutions on a flat two torus (see (1.7) for the definition of topological solution on a flat two torus). In [52], Ricciardi and Tarantello showed that there exist at least two gauge distinct periodic multivortices (topological solution and mountain pass solution), and analyzed their asymptotic behavior in terms of the (CS) limit and the (AH) limit. Moreover, Ricciardi in [51] obtained the stronger convergence result for an arbitrary sequence of periodic multivortices while the Chern–Simons parameter, which is the ratio between electric charge and the Chern–Simons mass scale, is fixed.

In this article, one of main goals is to derive the uniform (CS) limit result for (MCS) model with respect to the Chern–Simons parameter, without any restriction on either a particular class of solutions or the number of vortex points, as the Chern–Simons mass scale tends to infinity. Moreover, in view of our first result, we could also obtain the affirmative answers for the open problems raised by Ricciardi and Tarantello in [52], and Tarantello in [58].

In order to introduce our results more precisely, let us recall the Lagrangian density $\mathcal{L}_{MCS}$ for the (MCS) model, which is defined in the $(2 + 1)$-dimensional Minkowski
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space $\mathbb{R}^{2,1}$ with the metric $\text{diag}(1, -1, -1)$:

$$L^{\text{MCS}}(A, \phi, n) = -\frac{1}{4q^2} F_{\alpha\beta} F^{\alpha\beta} - \frac{\mu}{4q^2} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + D_\alpha \phi (D^\alpha \phi) + \frac{1}{2q^2} \partial_\alpha n \partial^\alpha n$$

$$- |\phi|^2 \left( n - \frac{q^2}{\mu} \right)^2 - \frac{q^2}{2} \left( |\phi|^2 - \frac{\mu}{q^2} n \right)^2,$$

(1.1)

where the metric is used to raise or lower indices, all the Greek indices run over $0, 1, 2$, and $\epsilon^{\alpha\beta\gamma}$ is the totally skew-symmetric tensor fixed so that $\epsilon^{012} = 1$. Here, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex valued Higgs field, $n : \mathbb{R}^{1+2} \to \mathbb{R}$ is the neutral scalar field, $A_\alpha : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_\alpha = \partial_\alpha - iq A_\alpha$ is the gauge covariant derivative associated with $A_\alpha$ where $i = \sqrt{-1}$, and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength. The constant $q > 0$ denotes the electric charge and $\mu > 0$ is the Chern–Simons mass scale. The gauge potential field $A$ with a 1-form (connection) is identified as $A = -i A_\alpha dx^\alpha$, and the Maxwell gauge field $F_A$ is expressed by $F_A = dA = -\frac{i}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ is expressed by the 2-form (curvature). Let us denote the self-dual potential by

$$V(|\phi|, n) = |\phi|^2 \left( n - \frac{q^2}{\mu} \right)^2 + \frac{q^2}{2} \left( |\phi|^2 - \frac{\mu}{q^2} n \right)^2.$$

Note that in $L^{\text{MCS}}$, the Maxwell term for $A$ is denoted by $F_{\alpha\beta} F^{\alpha\beta}$ and the Chern–Simons term is represented by the quantity $\frac{\mu}{4q^2} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma}$. Indeed, the Lagrangian of the (AH) model and the (CS) model are given by

$$L^{\text{AH}}(A, \phi) = -\frac{1}{4q^2} F_{\alpha\beta} F^{\alpha\beta} + D_\alpha \phi (D^\alpha \phi) - \frac{q^4}{2} \left( |\phi|^2 - 1 \right)^2,$$

and

$$L^{\text{CS}}(A, \phi) = -\frac{\mu}{4q^2} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + D_\alpha \phi (D^\alpha \phi) - \frac{q^4}{\mu^2} |\phi|^2 \left( |\phi|^2 - 1 \right)^2,$$

respectively. If we fix $q$, and assume the identity $n = \frac{q^2}{\mu}$ in (1.1), then as $\mu \to 0$, a limiting Lagrangian for $L^{\text{MCS}}$ formally would be $L^{\text{AH}}$. On the other hand, if we fix $\frac{q^2}{\mu}$, and insert the identity $n = \frac{q^2}{\mu} |\phi|^2$ into the potential of $L^{\text{MCS}}$, then as $\mu \to \infty$, a limiting Lagrangian for $L^{\text{MCS}}$ formally would be $L^{\text{CS}}$.

The periodic patterns of vortex configurations have been predicted and founded in the experiment for the study of superconductivity (see [1]). Periodic vortices (or condensates) relative to (1.1) are defined as the static solutions, which is independent of the $x^0$-variable, for the following Euler–Lagrangian equations subject to the ’t Hooft...
type periodic boundary conditions (see [60]):

\[
\begin{cases}
  D_\alpha D^\alpha \phi = -\frac{\partial V}{\partial \phi}, \\
  \frac{1}{q^2} \partial_\alpha \partial^\alpha n = -\frac{\partial V}{\partial n}, \\
  \frac{1}{q^2} \partial_\alpha F^{\alpha\beta} + \frac{\mu}{2q^2} e^{\alpha\beta\gamma} F_{\beta\gamma} = J^\alpha,
\end{cases}
\]

(1.2)

where \( J^\alpha = i (\bar{\phi} D^\alpha \phi - \phi (D^\alpha \bar{\phi})) \) is the conserved current for the system. We say that \((\phi, A_\alpha, n)\) is gauge equivalent to \((\psi, B_\alpha, n)\), if there exists a smooth function \( \omega \) satisfying

\[
(\psi, B_\alpha, n) = \left( e^{i\omega} \phi, A_\alpha + \partial_\alpha \omega, n_1 \right), \quad \alpha = 0, 1, 2.
\]

The 't Hooft type periodic boundary conditions are required for the invariance of (1.2) with respect to the gauge transformation. More precisely, the periodic cell domain is given by

\[
\Omega = \left\{ x \in \mathbb{R}^2 \mid x = t_1 a_1 + t_2 a_2, \ t_1, t_2 \in (0, 1) \right\},
\]

where \( a_1 \) and \( a_2 \) are linearly independent vectors in \( \mathbb{R}^2 \). Let \( \Gamma_k = \{ x \in \mathbb{R}^2 \mid x = t_k a_k, \ t_k \in (0, 1) \}, k = 1, 2 \), be a part of the boundary of \( \Omega \). We assume that \((A, \phi, n)\) is a static (that is, independent of the \( x^0 \)-variable) solution of (1.2), and there exist smooth functions \( \omega_k, (k = 1, 2) \) in a neighborhood of \( \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k \), satisfying

\[
\begin{cases}
  A_j (x + a_k) = A_j (x) + \partial_j \omega_k (x), \quad j, k = 1, 2, \\
  A_0 (x + a_k) = A_0 (x), \\
  \phi (x + a_k) = e^{-i\omega_k (x) \frac{\phi}{\mu}} (x), \\
  n (x + a_k) = n (x), \quad k = 1, 2,
\end{cases}
\]

(1.3)

for \( x \in \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k, k = 1, 2 \). We set \( \omega_k (s^1, s^2) = \omega_k (s^1 a_1, s^2 a_2), k = 1, 2 \) so that \( \phi \) is single-valued in \( \Omega \). In view of the compatibility condition, we have

\[
\omega_1 (0, 0^+) - \omega_1 (0, 1^-) + \omega_2 (1^-, 0) - \omega_2 (0^+, 0) = 2\pi \mathcal{M},
\]

(1.4)

where \( \mathcal{M} \in \mathbb{Z}_+ \) is called the vortex number and coincides with the total number of zeroes of \( \phi \) in \( \Omega \) counted according to their multiplicities.

Since the Euler–Lagrangian equation (1.2) is very complicated to study even for stationary solution, we restrict to consider energy minimizers only. It is well known from the arguments in [5] that a global minimizer of static energy on suitable function spaces is achieved by the following self-dual equations:

\[
\begin{cases}
  (D_1 + i D_2) \phi = 0 \\
  F_{12} = q^2 |\phi|^2 - \mu n \\
  -A_0 = n - \frac{q^2}{\mu} \\
  -\Delta A_0 + \mu F_{12} = -2q^2 A_0 |\phi|^2
\end{cases}
\]

(1.5)
together with the boundary conditions (1.3). Due to Jaffe-Taubes argument in [36,59],
the self-dual equation (1.5) is reduced to the following elliptic system (see [11,25,32,
52,57] for the detail):

\[
\begin{align*}
\Delta u &= \lambda \mu e^u - \mu N + 4\pi \sum_{i=1}^{n} m_i \delta_{p_i}, \\
\Delta N &= \mu(\mu + \lambda e^u) N - \lambda \mu (\mu + \lambda)e^u, \quad \text{in } \Omega.
\end{align*}
\tag{1.6}
\]

where \( u = \ln |\phi|^2, \lambda = \frac{2q^2}{\mu}, \) and \( N = 2n. \) Here, \( \delta_{p_i} \in \Omega \) stands for the Dirac measure
concentrated at \( p_i, \) and \( p_i \neq p_j \) if \( i \neq j. \) Each \( p_i \) is called a vortex point and \( m_i \in \mathbb{N} \) is the multiplicity of \( p_i. \)

In view of Remark 3 below, the Eq. (1.6) has two different kinds of periodic solutions
satisfying one of the following asymptotic behaviors:

\[
\begin{align*}
(u_{\lambda,\mu}, \frac{N_{\lambda,\mu}}{\lambda}) \to (0, 1) \quad \text{a.e. on } \Omega \quad \text{as } \lambda \to \infty, \mu \gg \lambda, \quad \text{(topological solution)}
\end{align*}
\]

\[
\begin{align*}
(u_{\lambda,\mu}, \frac{N_{\lambda,\mu}}{\lambda}) \to (-\infty, 0) \quad \text{a.e. on } \Omega \quad \text{as } \lambda \to \infty, \mu \gg \lambda, \quad \text{(nontopological solution)}
\end{align*}
\tag{1.7}
\]

Among the results obtained in [12,51,52] for (MCS) model, let us review the (CS)
limit results for (1.6) on a flat two torus \( \Omega. \) In [12], Chae and Kim showed the existence
of topological solution for (1.6), and its (CS) convergence whenever \( \mu \to \infty \) and \( \lambda \) is
fixed (see [11] for the study in \( \mathbb{R}^2). \) In [52], Ricciardi and Tarantello extended the (CS)
limit to other class of solutions. They showed that there exists \( \lambda_0 > 0 \) sufficiently large
such that for any \( \lambda > \lambda_0, \) there is \( \mu_\lambda > 0 \) satisfying that if \( \mu > \mu_\lambda, \) then (1.6)
has at least two distinct solutions, topological solution and mountain pass solution, which
converge to (CS) multivortices as \( \mu \to \infty. \) Moreover, they derived the asymptotic
behavior of these (CS) multivortices for not only topological solution but also mountain
pass solution provided \( \mathfrak{M} = \sum_{i=1}^{n} m_i = 1 \) as \( \lambda \to \infty. \) In [51], Ricciardi improved
the results [12,52] by obtaining the (CS) limit for arbitrary sequence of solutions in
\( C^q \) norm for any \( q \geq 0 \) whenever \( \lambda = 1. \)

For given arbitrary configuration of vortex points, our first goal is to obtain the
uniform (CS) limit result of (MCS) model with respect to \( \lambda > 0 \) and any class of
solutions for (1.6) as \( \mu \to \infty, \) and derive the following Brezis–Merle type alternatives
for (MCS) model.

**Theorem 1.1** Let \( Z \equiv \cup_i \{p_i\}. \) We assume that \( \{(u_{\lambda,\mu}, N_{\lambda,\mu})\} \) is a sequence of solutions of (1.6). Then

\[
\lim_{\lambda,\mu \to \infty, \frac{\lambda}{\mu} \to 0} \left\| e^{u_{\lambda,\mu}} - \frac{N_{\lambda,\mu}}{\lambda} \right\|_{L^\infty(\Omega)} = 0.
\tag{1.8}
\]

Moreover, as \( \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0, \) up to subsequences, one of the following holds:

(i) \( u_{\lambda,\mu} \to 0 \) uniformly on any compact subset of \( \Omega \setminus Z; \)
(ii) \( u_{\lambda,\mu} + 2 \ln \lambda - u_0 \to \hat{\omega} \) in \( C^1_{\text{loc}}(\Omega), \) where \( \hat{\omega} \) satisfies \( \Delta \hat{\omega} + e^{\hat{\omega} + u_0} = 4\pi \mathfrak{M}; \)
(iii) there exists a nonempty finite set $B = \{ \hat{q}_1, \ldots, \hat{q}_k \} \subset \Omega$ and $k$-number of sequences of points $q_{\lambda, \mu}^j \in \Omega$ such that $\lim_{\lambda, \mu \to \infty} \frac{1}{\mu} q_{\lambda, \mu}^j = \hat{q}_j$,

$$(u_{\lambda, \mu} + 2 \ln \lambda)(q_{\lambda, \mu}^j) \to +\infty, \text{ and } u_{\lambda, \mu} + 2 \ln \lambda \to -\infty \text{ uniformly on any compact subset of } \Omega \setminus B.$$ 

Moreover,

$$\lambda^2 e^{u_{\lambda, \mu}} \left(1 - \frac{N_{\lambda, \mu}}{\lambda}\right) \to \sum_j \alpha_j \delta_{\hat{q}_j}, \quad \alpha_j \geq 8\pi,$$

in the sense of measure.

The most important step in the proof for Theorem 1.1 is to derive the relation (1.8) between $u_{\lambda, \mu}$ and $N_{\lambda, \mu}$. In order to achieve this purpose, we apply the Green’s representation formula for the gradient estimation of $u_{\lambda, \mu}$, and use the nondegeneracy of the operator $-\Delta_1 + \frac{1}{\mu}$ after a suitable scaling.

We note that the elliptic system (1.6) is equivalent to

$$\begin{cases}
\Delta \left( u + \frac{N}{\mu} \right) = -\lambda^2 e^u \left(1 - \frac{N}{\lambda}\right) + 4\pi \sum_{i=1}^n m_i \delta_{p_i}, \\
\Delta N = \mu^2 \left(1 + \frac{\lambda}{\mu} e^u\right) N - \lambda \mu^2 \left(1 + \frac{\lambda}{\mu}\right) e^u
\end{cases} \text{ in } \Omega. \quad (1.9)$$

To the best of our knowledge, the estimation (1.8) in Theorem 1.1 has been known for a fixed constant $\lambda > 0$ as $\mu \to \infty$. We improve this result holds uniformly for large $\lambda > 0$ satisfying $\lambda \ll \mu$. Due to the estimation (1.8), (1.9) would be regarded as a perturbation of the following equation arising from (CS) model:

$$\Delta u = -\lambda^2 e^u \left(1 - e^u\right) + 4\pi \sum_{i=1}^n m_i \delta_{p_i} \quad \text{in } \Omega. \quad (1.10)$$

The corresponding result (i)–(iii) in Theorem 1.1 for (CS) equation (1.10) has been proved in [20] based on the arguments for Brezis–Merle type alternatives (see [2,3,7,20,48,49]). However, since our case is the coupled system problem, a major obstacle arises from the interaction between two components $u_{\lambda, \mu}$ and $N_{\lambda, \mu}$. In order to overcome this difficulty, we should carry out a careful estimation for the gradient of $N_{\lambda, \mu}$ in the Pohozaev identity. This is one of different features from the scalar equation (1.10).

In [52], the authors made a conjecture such that the density of superconducting particles $e^{u_{\lambda, \mu}}$ of (1.6) converges to $e^{u_{\lambda}}$ of (1.10) as $\mu \to \infty$ without the restriction $\mathfrak{M} = 1$, and it was proved in [51] for fixed $\lambda = 1$. This result would be valid even uniformly for $\lambda > 0$ since (1.9) and (1.10) share the similar asymptotic behavior in (i)–(iii) of Theorem 1.1 for any sequence of solutions to (1.6) including even mountain pass solution and for any $\mathfrak{M}_0 > 0$. Moreover, we can improve the (CS) convergence for blow up solutions, which are constructed below, in terms of not only $e^{u_{\lambda, \mu}}$ but also $u_{\lambda, \mu}$. We will continue to discuss the detail of uniform (CS) convergence for arbitrary solutions in forthcoming paper.
Now we consider the asymptotic behavior (iii) in Theorem 1.1. The case (iii) is called blow up phenomena. More precisely, we define the blow up solutions as follows:

**Definition 1.1** Let $B = \{ \hat{q}_j \}_{j=1}^k \subset \Omega$ be a set of finite points. If $\{(u_{\lambda,\mu}, N_{\lambda,\mu})\}$ is a family of solutions of (1.6) and there exist $k$-number of sequence of points $q^j_{\lambda,\mu}$, $j = 1, \ldots, k$, satisfying

(i) $\lim_{\lambda,\mu \to \infty, \frac{\lambda}{\mu} \to 0} (u_{\lambda,\mu} + 2 \ln \lambda) (q^j_{\lambda,\mu}) = +\infty$, and

(ii) $\lim_{\lambda,\mu \to \infty, \frac{\lambda}{\mu} \to 0} q^j_{\lambda,\mu} = \hat{q}_j$,

then $B$ is called a blow-up set and $\{(u_{\lambda,\mu}, N_{\lambda,\mu})\}$ is called a family of bubbling solutions (or blow up solutions) of (1.6) at $B$.

In view of Theorem 1.1, we note that the blow up phenomena implies the concentration of density for the nonlinear terms in the first equation in (1.9). We emphasize that this observation provides the affirmative answer for the open problem raised in [58]. In other words, we would like to show the existence of blow up solutions with the concentrating property at the vortex points. It turns out that the construction of solutions blowing up outside vortex points, that is, at the regular points, is more difficult than at the vortex points since the limit problem for the first one has nontrivial kernel. We first construct solutions blowing up at a regular point, and continue to study solutions blowing up at a vortex point.

**Theorem 1.2** Assume $\mathfrak{M} > 2$. Let $\hat{q}$ be a non-degenerate critical point of $u_0$ defined in (3.1). Assume that $\lambda, \mu$ are large enough and satisfy $(\ln \lambda) \lambda^2 \ll \mu$. Then (1.6) has a solution $(u_{\lambda,\mu}, N_{\lambda,\mu})$ satisfying

(i) $\lambda^2 e^{u_{\lambda,\mu}} \left(1 - \frac{N_{\lambda,\mu}}{\lambda}\right) \to 4\pi \mathfrak{M} \delta_{\hat{q}}$ in the sense of measure as $\lambda, \mu \to \infty$, $\frac{(\ln \lambda) \lambda^2}{\mu} \to 0$,

(ii) $\max_{y \in \Omega} u_{\lambda,\mu}(y) \geq c$ for some constant $c \in \mathbb{R}$, and

(iii) $\frac{N_{\lambda,\mu}}{\lambda} \to 0$ uniformly on any compact subset of $\Omega \setminus \{\hat{q}\}$ as $\lambda, \mu \to \infty$, $\frac{(\ln \lambda) \lambda^2}{\mu} \to 0$.

**Remark 1** By integrating the first equation of (1.9), we have

$$\int_{\Omega} \lambda^2 e^{u} (1 - \frac{N}{\lambda}) = 4\pi \mathfrak{M}.$$ 

Moreover, in view of Lemma 2.1 below, one knows that the local mass of the Chern–Simons equation without vortex points is strictly greater than $8\pi$. So necessarily one has $4\pi \mathfrak{M} > 8\pi$, that is, $\mathfrak{M} > 2$. This implies that when there is only one vortex point with multiplicity one, there should be no such kind of bubbling solutions considered in Theorem 1.2.

Motivated by Theorems 1.1 and 1.2, we also could solve the open problem raised in [58], and show the existence of blow up solutions with the concentrating property at the vortex point.
Theorem 1.3 Assume $\mathfrak{N} > 4$, $p_1 \neq p_j$, $j = 2, \ldots, n$, and $1 \ll (\ln \lambda)^5 \lambda^5 \ll \mu$. Then (1.6) has a solution $(u_{\lambda, \mu}, N_{\lambda, \mu})$ satisfying

(i) $\lambda^2 e^{u_{\lambda, \mu}} \left(1 - \frac{N_{\lambda, \mu}}{\lambda}\right) \to 4\pi \mathfrak{N} \delta_{p_1}$ in the sense of measure as $\lambda, \mu \to \infty$, $\frac{(\ln \lambda)^5 \lambda^5}{\mu} \to 0$,
(ii) $\max_{y \in \Omega} u_{\lambda, \mu}(y) \geq c$ for some constant $c \in \mathbb{R}$,
(iii) $\frac{N_{\lambda, \mu}}{\lambda} \to 0$ uniformly on any compact subset of $\Omega \setminus \{p_1\}$ as $\lambda, \mu \to \infty$, $\frac{(\ln \lambda)^5 \lambda^5}{\mu} \to 0$.

Remark 2 If we consider the blow up solutions at the vortex point with the multiplicity one, and assume that the maximum of the first component has a finite lower bound, then the limit equation becomes the Chern–Simons equation containing the vortex point with the multiplicity one. In this case, the local mass should be greater than $16\pi$, necessarily $4\pi \mathfrak{N} > 16\pi$, and thus we need the condition $\mathfrak{N} > 4$ in Theorem 1.3.

We note that the conditions for $\lambda, \mu > 0$ in Theorems 1.2–1.3 is stronger than the condition $\lambda \ll \mu$ in Theorem 1.1 because of technical reason, which occurs from the lower bound of $u_{\lambda, \mu}$. The maximum of the first component for solutions in Theorem 1.2 and Theorem 1.3 has a finite lower bound since the profile of approximate solutions comes from the entire solution of (CS) model. In forthcoming paper, we will study the blow up solutions whose first component has no lower bound for the maximum value such that the limiting profile will be the Liouville equation.

The paper is organized as follows. In Sect. 2, we review some preliminaries in the gauge theory. In Sect. 3, we analyze the asymptotic behavior of solutions and prove Theorem 1.1. In Sects. 4 and 5, we study the existence of blow up solutions.

2 Preliminaries

In this section, we review some known results in the gauge theory. Firstly, we consider the following problem

$$\Delta w + e^w (1 - e^w) = 4\pi m \delta_0 \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

We recall the following results.

Lemma 2.1 [7,15] [20, Lemma 3.2] Let $m$ be a nonnegative integer, and $w$ be a solution of (2.1). If $e^w (1 - e^w) \in L^1(\mathbb{R}^2)$, then either

(i) $w(x) \to 0$ as $|x| \to \infty$, or
(ii) $w(x) = -\beta \ln |x| + O(1)$ near $\infty$, where $\beta = -2m + \frac{1}{2\pi} \int_{\mathbb{R}^2} e^w (1 - e^w) dx$.

Assume that $w$ satisfies the boundary condition (ii). Then we have

$$\int_{\mathbb{R}^2} e^{2w} dx = \pi (\beta^2 - 4\beta - 4m^2 - 8m), \quad \text{and} \quad \int_{\mathbb{R}^2} e^w dx = \pi (\beta^2 - 2\beta - 4m^2 - 4m).$$

In particular, $\int_{\mathbb{R}^2} e^w (1 - e^w) dx > 8\pi (1 + m)$. 

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Next we introduce the following result, which will help us to study the asymptotic behavior of solutions in $\Omega$.

**Lemma 2.2** [14, Theorem 2.1] [16, Theorem 3.2] [54, Theorem 2.2] Let $m = 0$, and $w$ be a solution of (2.1) with $e^w(1 - e^w) \in L^1(\mathbb{R}^2)$. Then, $w(x)$ is smooth, radially symmetric with respect to some point $x_0$ in $\mathbb{R}^2$, and strictly decreasing function of $r = |x - x_0|$.

Assume $w(r; s)$ be the radially symmetric solution with respect to $0$ of (2.1) such that

$$\lim_{r \to 0} w(r; s) = s, \quad \text{and} \quad \lim_{r \to 0} w'(r; s) = 0,$$

where $w'$ denotes $\frac{dw}{dr}(r; s)$, and let us set

$$\beta(s) \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{w(r; s)}(1 - e^{w(r; s)}) dx = \int_0^\infty e^{w(r; s)}(1 - e^{w(r; s)}) r dr. \quad (2.2)$$

Then one has

(i) $\beta(0) = 0$ and $w(\cdot; 0) \equiv 0$;

(ii) $\beta : (-\infty, 0) \to (4, +\infty)$ is strictly increasing, bijective, and

$$\lim_{s \to -\infty} \beta(s) = 4, \quad \text{and} \quad \lim_{s \to 0} \beta(s) = +\infty.$$

**Lemma 2.3** (Lemma 2.1, [52]) Let $(u_{\lambda, \mu}, N_{\lambda, \mu})$ be solutions of (1.6) over $\Omega$. Then

$$u_{\lambda, \mu}(x) < 0, \quad 0 < N_{\lambda, \mu}(x) < \lambda \quad \text{for any} \quad x \in \Omega.$$

In view of Lemma 2.3, we can show that the nonlinear term of the first equation in (1.9) is uniformly bounded in $L^1(\Omega)$ with respect to $\lambda, \mu > 0$ as in the following corollary.

**Corollary 2.1** Let $(u_{\lambda, \mu}, N_{\lambda, \mu})$ satisfy (1.6) over $\Omega$. Then we have

$$\int_{\Omega} \lambda^2 e^{u_{\lambda, \mu}} \left(1 - \frac{N_{\lambda, \mu}}{\lambda}\right) dx = \int_{\Omega} \lambda^2 e^{u_{\lambda, \mu}} \left|1 - \frac{N_{\lambda, \mu}}{\lambda}\right| dx = 4\pi \mathcal{M}.$$

**Proof** By integrating (1.9) over $\Omega$ and using Lemma 2.3, we can obtain Corollary 2.1.

**Remark 3** In view of Corollary 2.1, we obtain

$$\lim_{\lambda \to \infty} \left[\int_{\Omega} e^{u_{\lambda, \mu}} \left|1 - \frac{N_{\lambda, \mu}}{\lambda}\right| dx\right] = 0,$$

which implies

$$\text{either} \quad u_{\lambda, \mu} \to -\infty \quad \text{or} \quad \frac{N_{\lambda, \mu}}{\lambda} \to 1 \quad \text{a.e. in} \quad \Omega \quad \text{as} \quad \lambda \to \infty. \quad (2.3)$$
Moreover, by integrating the second equation of (1.6) on $\Omega$, we also see that

$$\int_{\Omega} \left( 1 + \frac{\lambda}{\mu} e^{\mu_{\lambda,\mu}} \right) \frac{N_{\lambda,\mu}}{\lambda} dx = \int_{\Omega} \left( 1 + \frac{\lambda}{\mu} \right) e^{\mu_{\lambda,\mu}} dx. \quad (2.4)$$

If $\mu > \lambda$, then it is reasonable to consider the class of solutions satisfying the asymptotic behavior in (1.7).

Let us also recall the following form of the Harnack inequality.

**Lemma 2.4** ([3,27]) Let $D \subseteq \mathbb{R}^2$ be a smooth bounded domain and $v$ satisfy:

$$-\Delta v = f \text{ in } D,$$

with $f \in L^p(D)$, $p > 1$. For any subdomain $D' \subset D$, there exist two positive constants $\sigma \in (0, 1)$ and $\tau > 0$, depending on $D'$ only such that:

1. If $\sup_{\partial D} v \leq C$, then $\sup_{D'} v \leq \sigma \inf_{D'} v + (1 + \sigma)\tau \|f\|_{L^p} + (1 - \sigma)C$,

2. If $\inf_{\partial D} v \geq -C$, then $\sigma \sup_{D'} v \leq \inf_{D'} v + (1 + \sigma)\tau \|f\|_{L^p} + (1 - \sigma)C$.

### 3 Asymptotic behavior of solutions

In this section, we will study the asymptotic behavior of solutions to (1.9) and prove Theorem 1.1. We firstly introduce some notations. Let $G(x, y)$ be the Green’s function satisfying

$$-\Delta_x G(x, y) = \delta_y - \frac{1}{|\Omega|}, \quad \int_{\Omega} G(x, y) dy = 0,$$

where $|\Omega|$ is the measure of $\Omega$, and we denote the regular part of $G(x, y)$ by

$$\gamma(x, y) = G(x, y) + \frac{1}{2\pi} \ln |x - y|.$$ 

Let $M = \sum_{i=1}^n m_i$, and

$$u_0(x) = -4\pi \sum_{i=1}^n m_i G(x, p_i). \quad (3.1)$$
We set $u = v + u_0$, and assume $|\Omega| = 1$. Then (1.9) is equivalent to

$$
\begin{align*}
\Delta \left( v + \frac{N}{\mu} \right) &= -\lambda^2 e^{v + u_0} \left( 1 - \frac{N}{\lambda} \right) + 4\pi M, \\
\Delta N &= \mu (\mu + \lambda e^{v + u_0}) N - \lambda \mu (\lambda + \mu) e^{v + u_0} \\
\end{align*}
$$

in $\Omega$. \hfill (3.2)

**Lemma 3.1** Let $(u_{\lambda, \mu}, N_{\lambda, \mu})$ satisfy (1.9) over $\Omega$. Then there exists a constant $C > 0$, independent of $\lambda > 0$ and $\mu > 0$, such that

$$
\left\| \nabla \left( u_{\lambda, \mu} - u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) \right\|_{L^\infty(\Omega)} \leq C\lambda.
$$

**Proof** By applying the Green’s representation formula for a solution $(u_{\lambda, \mu}, N_{\lambda, \mu})$ of (1.9), we see

$$
\begin{align*}
\left( u_{\lambda, \mu} - u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) (x) - \int_{\Omega} \left( u_{\lambda, \mu} - u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) dy \\
&= \int_{\Omega} \lambda^2 e^{u_{\lambda, \mu}(y)} \left( 1 - \frac{N_{\lambda, \mu}(y)}{\lambda} \right) G(x, y) dy.
\end{align*}
$$

Together with Lemma 2.3 and Corollary 2.1, we can obtain

$$
\begin{align*}
\left| \nabla_x \left( u_{\lambda, \mu} - u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) (x) \right| \\
&\leq \left| \int_{B_d(x)} \lambda^2 e^{u_{\lambda, \mu}(y)} \left( 1 - \frac{N_{\lambda, \mu}(y)}{\lambda} \right) \left( -\frac{x - y}{2\pi |x - y|^2} + \nabla \gamma(x, y) \right) dy \right| + c_0 \\
&\leq \frac{\lambda^2}{2\pi} \left\| e^{u_{\lambda, \mu}} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) \right\|_{L^\infty(\Omega)} \left[ \int_{|x - y| \leq \frac{1}{\lambda}} \frac{1}{|x - y|^2} \right] \\
&\quad + \int_{\frac{1}{\lambda} \leq |x - y| \leq d} \frac{\lambda^2 e^{u_{\lambda, \mu}} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right)}{2\pi |x - y|} dy + c_1 \leq C\lambda,
\end{align*}
$$

where $c_0, c_1, C > 0$ are constants, independent of $\lambda, \mu > 0$. \hfill \Box

Next we will have the key estimate which will reduce (1.9) to an almost decoupled system whose first equation is a perturbation of a single Chern–Simons equation.

**Lemma 3.2** Let $(u_{\lambda, \mu}, N_{\lambda, \mu})$ satisfy (1.9) over $\Omega$. Then

$$
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left\| e^{u_{\lambda, \mu}} - \frac{N_{\lambda, \mu}}{\lambda} \right\|_{L^\infty(\Omega)} = 0.
$$
Proof Let \( v_{\lambda, \mu} = u_{\lambda, \mu} - u_0. \) Then \( v_{\lambda, \mu} \) satisfies (3.2). We argue by contradiction and suppose that there exists \( x_{\lambda, \mu} \in \Omega \) such that

\[
\left| e^{v_{\lambda, \mu}(x_{\lambda, \mu}) + u_0(x_{\lambda, \mu})} \frac{N_{\lambda, \mu}(x_{\lambda, \mu})}{\lambda} \right| \geq c > 0. \quad (3.4)
\]

Let

\[
y = \mu^{-1} x + x_{\lambda, \mu}, \quad \text{and} \quad \tilde{N}_{\lambda, \mu}(x) = \frac{N_{\lambda, \mu}(\mu^{-1} x + x_{\lambda, \mu})}{\lambda} = \frac{N_{\lambda, \mu}(y)}{\lambda}. \quad (3.5)
\]

Then we see that

\[
\Delta_x \tilde{N}_{\lambda, \mu}(x) - (1 + \lambda e^{v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu}}) \tilde{N}_{\lambda, \mu}(x) = -\left(1 + \frac{\lambda}{\mu}\right) e^{v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu}} \frac{N_{\lambda, \mu}(\mu^{-1} x + x_{\lambda, \mu})}{\mu} = -\left(1 + O(1)\right) e^{v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu}} \frac{N_{\lambda, \mu}(\mu^{-1} x + x_{\lambda, \mu})}{\mu} \quad \text{as} \quad \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0. \quad (3.6)
\]

Here, the last equality is obtained from Lemma 2.3.

Fix a constant \( R > 0, \) independent of \( \lambda, \mu > 0. \) The mean value theorem and

\[
\left\| \nabla \left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right) \right\|_{L^\infty(\Omega)} = O(\lambda) \quad \text{in Lemma 3.1 yield some} \quad \theta \in [0, 1] \quad \text{satisfying}
\]

\[
\left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right)(\mu^{-1} x + x_{\lambda, \mu}) = (v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu}) + \nabla_z \left( v_{\lambda, \mu}(z) + \frac{N_{\lambda, \mu}(z)}{\mu} \right)|_{z = \mu^{-1} x + x_{\lambda, \mu}} \cdot (\mu^{-1} x)
\]

\[
= (v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu}) + O \left( \frac{\lambda}{\mu} |x| \right)
\]

\[
= (v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu}) + o(1) \quad \text{for} \quad |x| \leq R \quad \text{as} \quad \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0. \quad (3.7)
\]

We are going to consider the following cases according to the location of limit point for \( x_{\lambda, \mu}, \) up to subsequence.

Case 1 \( \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} x_{\lambda, \mu} = x_0 \notin \bigcup_{i=1}^n \{ p_i \}. \)
Since $B_d(x_0) \cap \bigcup_{i=1}^{n} \{ p_i \} = \emptyset$ for a sufficiently small constant $d > 0$, $u_0$ is smooth in $B_d(x_0)$. Together with (3.7), we see that

$$
\left( v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) (\mu^{-1} x + x_{\lambda, \mu}) = \left( v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) (x_{\lambda, \mu}) + o(1) \quad \text{for } |x| \leq R,
$$

here we used that if $|x| \leq R$, then $\mu^{-1} x + x_{\lambda, \mu} \in B_d(x_0)$.

In view of Lemma 2.3, we have $|\tilde{N}_{\lambda, \mu}| \leq 1$, and thus there exists a function $N_0$ satisfying $\tilde{N}_{\lambda, \mu} \to N_0$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ as $\lambda, \mu \to \infty$, $\frac{\lambda}{\mu} \to 0$, where $N_0$ is a solution of

$$
\Delta N_0 - N_0 = -c_0 \quad \text{in } \mathbb{R}^2, \quad \text{and } \|N_0\|_{L^\infty(\mathbb{R}^2)} \leq 1,
$$

and $c_0 = \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} e^{(v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu})}$. Then we have $N_0 \equiv c_0$ in $\mathbb{R}^2$ (for example, see [26, Proposition 2.3]). We also note that

$$
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \frac{N_{\lambda, \mu}(x_{\lambda, \mu})}{\lambda} = \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \tilde{N}_{\lambda, \mu}(0) = N_0(0) = c_0. \quad (3.8)
$$

and

$$
c_0 = \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} e^{(v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu})} = \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} e^{(v_{\lambda, \mu} + u_0)(x_{\lambda, \mu})} \left( 1 + O \left( \frac{\|N_{\lambda, \mu}\|_{L^\infty(\Omega)}}{\mu} \right) \right) = \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} e^{(v_{\lambda, \mu} + u_0)(x_{\lambda, \mu})} (1 + o(1)) \quad (3.9)
$$

here, we used Lemma 2.3 and the assumption $1 \ll \lambda \ll \mu$ in the third equality. However, (3.8) and (3.9) contradict the assumption (3.4).

**Case 2** $x_{\lambda, \mu} \to x_0 = p_i$ for some $i$.

Define $\hat{u}_0(x) \in C^\infty(B_d(p_i))$ such that

$$
\hat{u}_0(x) = u_0(x) - 2m_i \ln |x - p_i|. \quad (3.10)
$$
Together with (3.7), we have

\[(v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})(\mu^{-1}x + x_{\lambda, \mu}) \]

\[= (v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu}) + o(1), \quad \text{if } |x| \leq R.\]

There are two cases according to the behavior of \(|x_{\lambda, \mu} - p_i|\).

**Case 2-(1)** \(\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} |x_{\lambda, \mu} - p_i|\mu = \infty.\)

In this case, the Eqs. (3.6) and (3.10) imply that if \(|x| \leq R\), then

\[
\begin{align*}
\Delta x \tilde{N}_{\lambda, \mu}(x) &= -(1 + o(1))|\mu|^{-1}x + x_{\lambda, \mu} - p_i|2mi e^{(v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})(\mu^{-1}x + x_{\lambda, \mu})} \\
\Delta x \tilde{N}_{\lambda, \mu}(x) &= -(1 + o(1))\left|\frac{x}{\mu|x_{\lambda, \mu} - p_i|} + \frac{x_{\lambda, \mu} - p_i}{|x_{\lambda, \mu} - p_i|} \right|^{2mi} |x_{\lambda, \mu} \\
\Delta x \tilde{N}_{\lambda, \mu}(x) &= -(1 + o(1))e^{(v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})(x_{\lambda, \mu})}. \tag{3.11}
\end{align*}
\]

Then the same arguments in Case 1 implies a contradiction again.

**Case 2-(2)** \(\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} |x_{\lambda, \mu} - p_i|\mu \leq c \) for some constant \(c > 0.\)

In view of (3.2), Lemma 2.3, and the condition \(\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0\), we see that

\[
\begin{align*}
\Delta \left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right) &= -\lambda^2 e^{(v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})} e^{-\frac{N_{\lambda, \mu}}{\mu}} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) + 4\pi \mathbb{M} \\
\Delta \left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right) &= -\lambda^2 e^{(v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})} \left( 1 + \frac{O(||N_{\lambda, \mu}||_{L^\infty(Q)})}{\mu} \right) \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) + 4\pi \mathbb{M} \\
\Delta \left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right) &= -(1 + o(1))\lambda^2 |x - p_i|2mi e^{(v_{\lambda, \mu} + \hat{u}_0 + \frac{N_{\lambda, \mu}}{\mu})} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) + 4\pi \mathbb{M}. \tag{3.12}
\end{align*}
\]

Let

\[
\hat{v}_{\lambda, \mu}(x) = \left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right)(\mu^{-1}x + p_i) - 2mi \ln \mu
\]

and \(z = \mu^{-1}x + p_i.\)

Then, we have
\[
\Delta_x \hat{v}_{\lambda, \mu}(x) + \lambda^2 \mu^{-2} |x|^{2m_i} e^{\hat{v}_{\lambda, \mu}(x) + \hat{u}_0(\mu^{-1}x + p_i)} (1 + o(1)) \left( 1 - \frac{N_{\lambda, \mu}(\mu^{-1}x + p_i)}{\lambda} \right)
= 4\pi \mathcal{M} \mu^{-2} = o(1) \text{ as } \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0.
\] (3.13)

In view of Lemma 2.3 and \(\hat{u}_0 \in C^\infty(B_d(p_i))\), we see that
\[
\hat{v}_{\lambda, \mu}(x) + 2m_i \ln |x| = \left( v_{\lambda, \mu} + u_0 + \frac{N_{\lambda, \mu}}{\mu} \right) (\mu^{-1}x + p_i) - \hat{u}_0(\mu^{-1}x + p_i) \\
\leq \left\| \frac{N_{\lambda, \mu}}{\mu} \right\|_{L^\infty(\Omega)} + \|\hat{u}_0\|_{L^\infty(B_d(p_i))} \leq c_0 \text{ in } B_{d\mu}(0),
\] (3.14)

for some constant \(c_0 > 0\), independent of \(\lambda, \mu > 0\).

By Lemma 3.1, we also see that
\[
|\nabla x \hat{v}_{\lambda, \mu}(x)| = \mu^{-1} \left| \nabla_z \left( v_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} \right)(z) \right|_{z=\mu^{-1}x+p_i} \\
= O \left( \frac{\lambda}{\mu} \right) = o(1) \text{ in } B_{d\mu}(0) \text{ as } \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0.
\] (3.15)

From (3.14) and (3.15), we note that there are two possibilities as follows:

(i) \(\sup_{\partial B_1(0)} |\hat{v}_{\lambda, \mu}| \leq C\).

In this case, (3.15) implies that \(|\hat{v}_{\lambda, \mu}|\) is uniformly bounded in \(C^{0}_{\text{loc}}(B_{d\mu}(0))\) for \(\lambda, \mu > 0\). Then there exists a function \(v_0\) such that \(\hat{v}_{\lambda, \mu} \to v_0\) in \(C^{1}_{\text{loc}}(B_{d\mu}(0))\) and \(\nabla v_0 \equiv 0\) in \(\mathbb{R}^2\). It implies that \(v_0 \equiv c\) for some constant \(c \in \mathbb{R}\) and \(\Delta v_0 = 0\) in \(\mathbb{R}^2\). From the mean value theorem for harmonic function and (3.14), we see that for any constant \(R > 0\),
\[
-\infty < c = v_0(0) = \frac{1}{|\partial B_R(0)|} \int_{\partial B_R(0)} v_0(y) dS_y \\
\leq \frac{1}{|\partial B_R(0)|} \int_{\partial B_R(0)} (c_0 - 2m_i \ln |y|) dS_y = c_0 - 2m_i \ln R.
\] (3.16)

We get a contradiction as \(R \to \infty\) in (3.16).

(ii) \(\sup_{\partial B_1(0)} \hat{v}_{\lambda, \mu} \to -\infty\).

In this case, (3.15) implies that \(\hat{v}_{\lambda, \mu} \to -\infty\) is uniformly in \(C^{0}_{\text{loc}}(B_{d\mu}(0))\) for \(\lambda, \mu > 0\). By (3.6), we have
\[
\Delta_x \tilde{N}_{\lambda,\mu}(x) - \left( 1 + \frac{\lambda}{\mu} e^{(v_{\lambda,\mu} + u_0)(\mu^{-1}x + x_{\lambda,\mu})} \right) \tilde{N}_{\lambda,\mu}(x)
= - \left( 1 + \frac{\lambda}{\mu} \right) e^{(v_{\lambda,\mu} + u_0)(\mu^{-1}x + x_{\lambda,\mu})}
= -(1 + o(1)) \left| x + \mu(x_{\lambda,\mu} - p_i) \right|^{2m_i} e^{\hat{v}_{\lambda,\mu}(x + \mu(x_{\lambda,\mu} - p_i) + \tilde{u}_0(\mu^{-1}x + x_{\lambda,\mu}))} = o(1).
\]

(3.17)

Since \(|\tilde{N}_{\lambda,\mu}(x)| \leq 1\) for all \(x \in B_{d\mu}(0)\), there is a function \(\tilde{N}_0\) satisfying \(\tilde{N}_{\lambda,\mu} \to \tilde{N}_0\) in \(C^1_{\text{loc}}(\mathbb{R}^2)\), and

\[
\Delta \tilde{N}_0 - \tilde{N}_0 = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \|\tilde{N}_0\|_{L^\infty(\mathbb{R}^2)} \leq 1,
\]

which implies \(\tilde{N}_0 \equiv 0\) in \(\mathbb{R}^2\) (for example, see [26, Proposition 2.3]). We note that

\[
\lim_{\lambda,\mu \to \infty, \mu \to 0} \left( \frac{N_{\lambda,\mu}(x_{\lambda,\mu})}{\lambda} \right) = \lim_{\lambda,\mu \to \infty, \mu \to 0} \tilde{N}_{\lambda,\mu}(0) = \tilde{N}_0(0) = 0, \quad (3.18)
\]

and

\[
\lim_{\lambda,\mu \to \infty, \mu \to 0} e^{(v_{\lambda,\mu} + u_0)(x_{\lambda,\mu})} = \lim_{\lambda,\mu \to \infty, \mu \to 0} \left( 1 + o(1) \left| x_{\lambda,\mu} - p_i \right| \right)^{2m_i} e^{\hat{v}_{\lambda,\mu}(\mu(x_{\lambda,\mu} - p_i) + \tilde{u}_0(x_{\lambda,\mu}))} = 0.
\]

(3.19)

However, (3.18) and (3.19) contradict the assumption (3.4).

\[\square\]

In view of Lemma 3.2, the first equation of (1.9) can be regarded as a perturbation of a single Chern–Simons equation (1.10). By applying the arguments in [20, Lemma 4.1], we can obtain the following result.

**Lemma 3.3** Suppose that there exists a sequence of solutions \((u_{\lambda,\mu}, N_{\lambda,\mu})\) of (1.9) such that

\[
\lim_{\lambda,\mu \to \infty, \mu \to 0} \left( \inf_{\Omega} |u_{\lambda,\mu}| \right) = 0.
\]

Then, we have

\[
\lim_{\lambda,\mu \to \infty, \mu \to 0} \|u_{\lambda,\mu}\|_{L^\infty(K)} = 0 \quad \text{for any compact set} \quad K \subset \Omega \setminus Z. \quad (3.20)
\]
**Proof** Choose a sequence of points \( \{x_{\lambda, \mu}\} \subseteq \Omega \) such that

\[
|u_{\lambda, \mu}(x_{\lambda, \mu})| = \inf_{\Omega} |u_{\lambda, \mu}|, \quad \text{and} \quad \lim_{\lambda, \mu \to \infty, \frac{x}{\mu} \to 0} u_{\lambda, \mu}(x_{\lambda, \mu}) = 0. \tag{3.21}
\]

Passing to a subsequence (still denoted by \( u_{\lambda, \mu} \)), we may assume that

\[
\lim_{\lambda, \mu \to \infty, \frac{x}{\mu} \to 0} x_{\lambda, \mu} = x_0 \in \Omega. \quad \text{We consider the following two cases according to the location of } x_0.
\]

**Case 1** \( x_0 \notin Z \).

Let \( d > 0 \) be a small constant satisfying \( B_d(x_0) \cap Z = \emptyset \). We argue by contradiction and suppose that there exist a compact set \( K \subset \Omega \setminus Z \), a positive constant \( c_K > 0 \), and a sequence \( \{z_{\lambda, \mu}\} \subset K \) such that \( \sup_K |u_{\lambda, \mu}| = |u_{\lambda, \mu}(z_{\lambda, \mu})| \geq c_K > 0 \) for large \( \lambda, \mu \). We choose a connected compact set \( K_1 \subset \Omega \setminus Z \) satisfying \( B_d(x_0) \cup K \subset K_1 \).

Since \( u_{\lambda, \mu}(z_{\lambda, \mu}) \leq -c_K < 0 \), Lemma 2.2 implies there is a constant \( s_1 < 0 \) such that

\[
\beta(s_1) > 4\mathfrak{M} \quad \text{and} \quad -c_K < s_1 < 0.
\]

We can also choose \( y_{\lambda, \mu} \in K_1 \) such that \( u_{\lambda, \mu}(y_{\lambda, \mu}) = s_1 \) by the intermediate value theorem.

Let \( \bar{u}_{\lambda, \mu}(x) = (u_{\lambda, \mu} + N_{\lambda, \mu}) (\lambda^{-1}x + y_{\lambda, \mu}) \) for \( x \in \Omega_{y_{\lambda, \mu}} \equiv \{ x \in \mathbb{R}^2 \mid \lambda^{-1}x + y_{\lambda, \mu} \in K_1 \} \).

By Corollary 2.1 and \( u_{\lambda, \mu}(y_{\lambda, \mu}) = s_1 \), we see that \( \bar{u}_{\lambda, \mu} \) satisfies

\[
\begin{align*}
\Delta \bar{u}_{\lambda, \mu} + \bar{u}_{\lambda, \mu} (\lambda^{-1}x + y_{\lambda, \mu}) & = \frac{N_\lambda}{\mu} \left( \frac{\lambda}{\lambda} - \frac{N_\lambda}{\mu} (\lambda^{-1}x + y_{\lambda, \mu}) \right) \left( 1 - \frac{N_\lambda (\lambda^{-1}x + y_{\lambda, \mu})}{\lambda} \right) = 0 \quad \text{in } \Omega_{y_{\lambda, \mu}}, \\
\bar{u}_{\lambda, \mu}(0) & = s_1 + \frac{N_\lambda}{\mu} (y_{\lambda, \mu}), \\
\int_{\Omega_{y_{\lambda, \mu}}} \left| \bar{u}_{\lambda, \mu}(x) - \frac{N_\lambda}{\mu} (\lambda^{-1}x + y_{\lambda, \mu}) \left( 1 - \frac{N_\lambda (\lambda^{-1}x + y_{\lambda, \mu})}{\lambda} \right) \right| dx & \leq 4\pi \mathfrak{M}. \tag{3.22}
\end{align*}
\]

By using Lemma 3.1 and \( W^{2,p} \) estimation, we see that \( \bar{u}_{\lambda, \mu} \) is bounded in \( C^{1,\sigma}_{\text{loc}}(\Omega_{y_{\lambda, \mu}}) \) for some \( \sigma \in (0, 1) \). In view of Lemmas 2.3 and 3.2, we see that if \( 0 < \lambda \ll \mu \), then

\[
eu_{\lambda, \mu}(x) - \frac{N_\lambda}{\mu} (\lambda^{-1}x + y_{\lambda, \mu}) \left( 1 - \frac{N_\lambda (\lambda^{-1}x + y_{\lambda, \mu})}{\lambda} \right) \left( 1 - \frac{N_\lambda (\lambda^{-1}x + y_{\lambda, \mu})}{\lambda} \right) \left( 1 - \frac{N_\lambda (\lambda^{-1}x + y_{\lambda, \mu})}{\lambda} \right)
\]

and \( \bar{u}_{\lambda, \mu}(0) = s_1 + o(1) \). Passing to a subsequence, \( \bar{u}_{\lambda, \mu} \) converges in \( C_{\text{loc}}^1(\mathbb{R}^2) \) to a function \( u_* \), which is a solution of

\[
\begin{align*}
\Delta u_* + e^{u_*} (1 - e^{u_*}) & = 0 \quad \text{in } \mathbb{R}^2, \\
u_*(0) & = s_1, \\
\int_{\mathbb{R}^2} |e^{u_*} (1 - e^{u_*})| dx & \leq 4\pi \mathfrak{M}. \tag{3.24}
\end{align*}
\]
By using Lemma 2.2, we see that $u_*$ is radially symmetric with respect to a point $\bar{p}$ in $\mathbb{R}^2$.

In view of Lemma 2.2, we have

$$4\pi \mathcal{M} \geq \left| \int_{\mathbb{R}^2} e^{u_*} (1 - e^{u_*}) \, dx \right| = 2\pi |\beta(u_*(\bar{p}))| \geq 2\pi |\beta(s_1)| > 8\pi \mathcal{M}, \quad (3.25)$$

which implies a contradiction. Thus (3.20) holds true in Case 1.

**Case 2** $x_0 = p_i \in Z$ for some $i$.

Fix a small constant $r_0 > 0$ such that $B_{r_0}(x_0) \cap Z = \{x_0\}$. For simplicity, we assume that $x_0 = 0$. We are going to show that

$$\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \int_{|x| = r_0} |u_{\lambda, \mu}(x)| \right) = 0. \quad (3.26)$$

Once we have (3.26), the argument in Case 1 implies (3.20). In order to prove (3.26), we argue by contradiction again and suppose that, up to a subsequence,

$$\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \int_{|x| = r_0} |u_{\lambda, \mu}(x)| \right) \geq \tau_0$$

for some constant $\tau_0 > 0$. Since $u_{\lambda, \mu} < 0$,

$$\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \sup_{|x| = r_0} u_{\lambda, \mu}(x) < -\tau_0. \quad (3.27)$$

We divide our discussion into the following two cases.

(i). $\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \lambda |x_{\lambda, \mu}| \right) < +\infty$.

Note that $u_{\lambda, \mu}(x) = 2m_i \ln |x| + v_{\lambda, \mu}(x)$ near $x = 0$ for some $1 \leq i \leq n$, where $v_{\lambda, \mu}$ is a smooth function in $B_d(0)$. Let

$$\hat{v}_{\lambda, \mu}(x) = v_{\lambda, \mu}(|x_{\lambda, \mu}| x) + 2m_i \ln |x_{\lambda, \mu}| + \frac{N_{\lambda, \mu}(|x_{\lambda, \mu}|)}{\mu} \quad \text{for} \quad |x| \leq \frac{r_0}{|x_{\lambda, \mu}|}. \quad (3.28)$$

Then $\hat{v}_{\lambda, \mu}$ satisfies

$$\int_{B_{r_0}(0)} \lambda^2 |x_{\lambda, \mu}|^2 |x| e^{\hat{v}_{\lambda, \mu}(x)} \left( 1 - \frac{N_{\lambda, \mu}(|x_{\lambda, \mu}|)}{\lambda} \right) \, dx \leq 4\pi \mathcal{M} \quad \text{in} \quad B_{r_0}(0). \quad (3.29)$$

By (3.21), we note that

$$\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \hat{v}_{\lambda, \mu}(\frac{x_{\lambda, \mu}}{|x_{\lambda, \mu}|}) = \lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( u_{\lambda, \mu}(x_{\lambda, \mu}) + \frac{N_{\lambda, \mu}(x_{\lambda, \mu})}{\mu} \right) = 0. \quad (3.29)$$
Together with Lemma 3.1 and the assumption \( \liminf_{\lambda, \mu \to \infty} \frac{N_{\lambda, \mu}}{\lambda} \to 0 (\lambda |x\lambda, \mu|) < +\infty \), we see that \( \hat{v}_{\lambda, \mu} \) is bounded in \( C^0_{\text{loc}}(B_{R_1}(0)) \). Passing to a subsequence, we may assume that

\[
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{{\mu}} \to 0} \frac{x_{\lambda, \mu}}{|x_{\lambda, \mu}|} = y_0 \in \mathbb{S}^1, \quad \lim_{\lambda, \mu \to \infty, \frac{\lambda}{{\mu}} \to 0} (\lambda |x_{\lambda, \mu}|) = c_0 \geq 0, \quad \text{and } \hat{v}_{\lambda, \mu} \to \hat{v} \text{ in } C^1_{\text{loc}}(B_{R_1}(0)),
\]

for some function \( \hat{v} \). By using Lemmas 2.3 and 3.2 as in (3.23), we see that \( \hat{u}(x) = \hat{v}(x) + 2m_i \ln |x| \) satisfies

\[
\Delta \hat{u} + c_0^2 e^{\hat{u}} (1 - e^{\hat{u}}) = 4\pi m_i \delta_0 \text{ in } \mathbb{R}^2.
\]

In view of Lemma 2.3, we have \( \int_0^{N_{\lambda, \mu}(\lambda x_{\lambda, \mu}|x|)} \lambda \leq 1 \). Moreover, Lemma 3.2 implies

\[
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{{\mu}} \to 0} \frac{N_{\lambda, \mu}(\lambda x_{\lambda, \mu}|x|)}{\lambda} = e^{\hat{u}(x)} \in [0, 1]. \text{ Since } \hat{u} \leq 0 \text{ in } \mathbb{R}^2 \text{ and } \hat{u}(y_0) = 0, \text{ we have } \hat{u} \equiv 0 \text{ by Hopf Lemma, which implies a contradiction.}
\]

(ii). \( \lim_{\lambda, \mu \to \infty, \frac{\lambda}{{\mu}} \to 0} (\lambda |x_{\lambda, \mu}|) = +\infty \).

Lemma 2.2 implies there is a constant \( s_2 < 0 \) such that

\[
\beta(s_2) > 4\mathcal{M} \text{ and } -\tau_0 < s_2 < 0,
\]

where \( \tau_0 \) is the constant in (3.27). We can also choose \( \hat{y}_{\lambda, \mu} \) on the line segment joining \( x_{\lambda, \mu} \) and \( \frac{\tau_0 x_{\lambda, \mu}}{|x_{\lambda, \mu}|} \) such that \( u_{\lambda, \mu}(\hat{y}_{\lambda, \mu}) = s_2 \) and \( |y_{\lambda, \mu}| \geq |x_{\lambda, \mu}| \) by the intermediate value theorem. Let \( \hat{u}_{\lambda, \mu}(x) = (u_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu}) (\lambda^{-1} x + \hat{y}_{\lambda, \mu}) \) \( (\lambda^{-1} x + \hat{y}_{\lambda, \mu}) \in B_{\frac{2}{\mu}}(\hat{y}_{\lambda, \mu}) \). Here we note that \( 0 \notin B_{\frac{2}{\mu}}(\hat{y}_{\lambda, \mu}) \). Then \( \hat{u}_{\lambda, \mu} \) satisfies

\[
\begin{cases}
\Delta \hat{u}_{\lambda, \mu} + e^{\hat{u}_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu} (\lambda^{-1} x + \hat{y}_{\lambda, \mu})} (1 - \frac{N_{\lambda, \mu}}{\mu} (\lambda^{-1} x + \hat{y}_{\lambda, \mu})) = 0 \text{ in } \hat{\Omega}_{\lambda, \mu}, \\
\hat{u}_{\lambda, \mu}(0) = s_2 + \frac{N_{\lambda, \mu}(\hat{y}_{\lambda, \mu})}{\mu}, \\
\int_{\hat{\Omega}_{\lambda, \mu}} |e^{\hat{u}_{\lambda, \mu} - e^{\hat{u}_{\lambda, \mu}}} |dx \leq 4\pi \mathcal{M}.
\end{cases}
\] (3.30)

Using the same argument as in Case 1, we get a contradiction by comparing the upper/lower bound of \( L^1 \) norm of the nonlinear term in (3.30). Thus the claim (3.26) holds true. Then we can again apply the arguments in Case 1 and prove (3.20) holds true.

As a corollary of Lemma 3.3, we get the following proposition.

**Proposition 3.1** Let \( (u_{\lambda, \mu}, N_{\lambda, \mu}) \) be a sequence of solutions of (1.9). Then, up to subsequences, one of the following holds true:

(i) \( u_{\lambda, \mu} \to 0 \) uniformly on any compact subset of \( \Omega \setminus Z \) as \( \lambda, \mu \to \infty, \frac{\lambda}{{\mu}} \to 0 \), or
(ii) there exists a constant \( \nu_0 > 0 \) such that \( \sup_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \sup_{\Omega} u_{\lambda, \mu} \right) \leq -\nu_0 \).

In order to complete the proof of Theorem 1.1, we note that the analysis of the gradient of \( N_{\lambda, \mu} \) would play an important role since our main problem (1.9) is the coupled system problem.

**Completion of the proof of Theorem 1.1** Note that Proposition 3.1-(i) corresponds to Theorem 1.1-(i). In order to complete the proof of Theorem 1.1, from now on, we will study the asymptotic behavior for the solution \((u_{\lambda, \mu}, N_{\lambda, \mu})\) of (1.9) satisfying Proposition 3.1-(ii). Let us denote

\[
w_{\lambda, \mu} = u_{\lambda, \mu} + \frac{N_{\lambda, \mu}}{\mu} + 2 \ln \lambda \quad \text{in } \Omega.
\]

Clearly \( w_{\lambda, \mu} \) satisfies

\[
\begin{align*}
\Delta w_{\lambda, \mu} + e^{w_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) &= 4\pi \sum_{i=1}^{n} m_i \delta_{p_i} \quad \text{in } \Omega, \\
\sup_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \sup_{\Omega} \left( w_{\lambda, \mu} - 2 \ln \lambda - \frac{N_{\lambda, \mu}}{\mu} \right) &\leq -\nu_0 < 0, \\
\int_{\Omega} e^{w_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) |dx| = \int_{\Omega} e^{w_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}} \left( 1 - \frac{N_{\lambda, \mu}}{\lambda} \right) dx &= 4\pi M.
\end{align*}
\]  

By Lemma 2.3 and Lemma 3.2, as \( \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0 \), we have (for example, see (3.23))

\[
\begin{align*}
\Delta w_{\lambda, \mu} + e^{w_{\lambda, \mu}} (1 + o(1)) \left( 1 - \lambda^{-2} e^{w_{\lambda, \mu}} + o(1) \right) &= 4\pi \sum_{i=1}^{n} m_i \delta_{p_i} \quad \text{in } \Omega, \\
\int_{\Omega} e^{w_{\lambda, \mu}} (1 + o(1)) \left( 1 - \lambda^{-2} e^{w_{\lambda, \mu}} + o(1) \right) |dx| &= 4\pi M.
\end{align*}
\]

Then we have

\[
0 < 2\pi M \leq \|e^{w_{\lambda, \mu}}\|_{L^1(\Omega)} \leq \frac{8\pi M}{1 - e^{-\nu_0}}. \tag{3.33}
\]

We consider the following two cases:

**Case 1** \( \sup_{\Omega} w_{\lambda, \mu} \leq C \) for some constant \( C > 0 \).

In this case, we note that the Harnack inequality (i.e. Lemma 2.4) and (3.33) imply \( w_{\lambda, \mu} - u_0 \) is uniformly bounded in \( L^\infty(\Omega) \). Moreover, in view of \( W^{2,2} \) estimation, \( w_{\lambda, \mu} - u_0 \) is uniformly bounded in \( C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \). Then \( w_{\lambda, \mu} - u_0 \to \hat{w} \) in \( C^{1}_{\text{loc}}(\Omega) \), where \( \hat{w} \) satisfies

\[
\Delta \hat{w} + e^{\hat{w} + u_0} = 4\pi M.
\]
We note that Case 1 implies Theorem 1.1-(ii).

**Case 2** \(\lim_{\lambda,\mu \to \infty, \frac{\lambda}{\mu} \to 0} \sup_{\Omega} w_{\lambda,\mu} = +\infty\).

Following [7], we say that a point \(q \in \Omega\) is a blow-up point for \(\{w_{\lambda,\mu}\}\) if there exists a sequence \(\{x_{\lambda,\mu,q}\}\) such that

\[
x_{\lambda,\mu,q} \to q \quad \text{and} \quad w_{\lambda,\mu}(x_{\lambda,\mu,q}) \to \infty \quad \text{as} \quad \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0.
\]

The set \(S \subset \Omega\), which consists of blow-up points for \(\{w_{\lambda,\mu}\}\), is called the blow-up set for \(\{w_{\lambda,\mu}\}\).

**Step 1** Let \(p \in \Omega\) be a blow-up point for \(\{w_{\lambda,\mu}\}\). Then we have the following “minimal mass” result.

\[
\lim_{\lambda,\mu \to \infty, \frac{\lambda}{\mu} \to 0} \int_{B_d(p)} e^{w_{\lambda,\mu} - \frac{N_{\lambda,\mu}}{\lambda}} \left(1 - \frac{N_{\lambda,\mu}}{\lambda}\right) dx \geq 8\pi \quad \text{for any} \quad d > 0. \tag{3.34}
\]

Indeed, we note that the equation (3.32) is a perturbation of

\[
\Delta w_{\lambda} + e^{w_{\lambda}} \left(1 - \lambda^{-2} e^{w_{\lambda}}\right) = 4\pi \sum_{i=1}^{n} m_i \delta_{p_i} \quad \text{in} \quad \Omega, \tag{3.35}
\]

and the minimal mass result for (3.35) was obtained in [20, Lemma 4.2]. By the similar arguments in [20], we can also get (3.34) for the solution \(w_{\lambda,\mu}\) of (3.32). Here we skip the detail and refer to [20].

The estimation (3.34) shows that if Case 2 happens, then \(\{w_{\lambda,\mu}\}\) has a nonempty finite blow-up set \(B \subset \Omega\), and \(|B| \leq \frac{4\pi \mathcal{M}}{8\pi} = \frac{\mathcal{M}}{2}\). We also see that for any compact set \(K \subset \Omega \setminus B\), there exists a constant \(C_K > 0\) such that

\[
\sup_{K} w_{\lambda,\mu} \leq C_K. \tag{3.36}
\]

**Step 2** In this step, we are going to prove that the blow up phenomena implies the concentration of mass as in [3,7,20]. However, our case is a coupled system problem, we should carry out a delicate analysis in order to prove the concentration of mass. Firstly, we claim that

\[
w_{\lambda,\mu} - u_0 \to -\infty \quad \text{uniformly on any compact set} \quad K \subset \Omega \setminus B. \tag{3.37}
\]

Choose a small constant \(d > 0\) satisfying for any \(q \in B, B_{2d}(q) \cap [B \cup Z] = \{q\}\). For each \(q \in B\), we let \(\{x_{\lambda,\mu,q}\}\) be a sequence of points such that

\[
x_{\lambda,\mu,q} \to q \in B \quad \text{and} \quad w_{\lambda,\mu}(x_{\lambda,\mu,q}) = \sup_{B_d(q)} w_{\lambda,\mu} \to \infty \quad \text{as} \quad \lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0.
\]
We shall prove that
\[
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \inf_{\partial B_r(q)} (w_{\lambda, \mu} - u_0) \right) = -\infty
\] (3.38)
for any \( r \in (0, d] \) and all \( q \in B \). Then by (3.36) and Harnack’s inequality, we get that
\[
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \sup_{\Omega \setminus \bigcup_{j \in B} B_r(q_j)} (w_{\lambda, \mu} - u_0) \right) = -\infty \quad \text{for any } r \in (0, d].
\]
To prove (3.38), we argue by contradiction and suppose that there exist \( r \in (0, d] \) and \( q \in B \) such that
\[
\lim_{\lambda, \mu \to \infty, \frac{\lambda}{\mu} \to 0} \left( \inf_{\partial B_r(q)} (w_{\lambda, \mu} - u_0) \right) \geq c,
\]
for some constant \( c \in \mathbb{R} \). For simplicity, we assume that \( q = 0 \). By using (3.36) and Harnack’s inequality, we can verify that \( \{w_{\lambda, \mu} - u_0\} \) is bounded in \( C^0_0(\Omega) \). Moreover, Green’s representation formula implies that
\[
\xi(x) = -\frac{\alpha_q}{2\pi} \ln |x| + \phi(x) + \eta(x),
\]
where \( \eta \in C^1(B_r(0)) \) for every \( r \in (0, d) \), and we let
\[
\phi(x) = \frac{1}{2\pi} \int_{B_d(0)} \ln \left( \frac{1}{|x - y|} \right) e^{(\xi + u_0)(y)} dy.
\] (3.39)
We note that
\[
\phi \geq \frac{1}{2\pi} \ln \left( \frac{1}{2d} \right) \|e^{\xi + u_0}\|_{L^1(B_d(0))} \quad \text{on } B_d(0).
\]
Then we see that $e^{\xi(x)} = |x|^{-\alpha_q/2\pi} e^{\phi + \eta} \geq c|x|^{-\alpha_q/2\pi}$ for $0 < |x| \leq d$ and some constant $c > 0$. Then the integrability of $e^{\xi + u_0}$ implies that

$$4\pi (1 + m) > \alpha_q,$$

(3.40)

where $m = m_i$ if $q = p_i \in B \cap Z$, and $m = 0$ if $q \in B \setminus Z$.

Let $\phi_{\lambda, \mu}(x) = w_{\lambda, \mu}(x) - 2m \ln |x|$. Then $\phi_{\lambda, \mu}$ satisfies

$$\Delta \phi_{\lambda, \mu} + |x|^{2m} e^{\phi_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}} \left(1 - \frac{N_{\lambda, \mu}}{\lambda}\right) = 0 \text{ in } B_d(0).$$

(3.41)

Multiplying (3.41) by $x \cdot \nabla \phi_{\lambda, \mu}$ and integrating over $B_r(0)$ for $r \in (0, d)$, we get that

$$\int_{\partial B_r(0)} \left[ \frac{(x \cdot \nabla \phi_{\lambda, \mu})^2}{|x|} - |x||\nabla \phi_{\lambda, \mu}|^2 \frac{2}{|x|^{2m+1}} e^{\phi_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}} \left(1 - \frac{N_{\lambda, \mu}}{\lambda}\right) \right] d\sigma$$

$$= \int_{B_r(0)} \left(2 + 2m\right) e^{\phi_{\lambda, \mu} + 2m \ln |x| - \frac{N_{\lambda, \mu}}{\mu}} \left(1 - \frac{N_{\lambda, \mu}}{\lambda}\right) dx$$

$$+ \int_{B_r(0)} e^{\phi_{\lambda, \mu} + 2m \ln |x| - \frac{N_{\lambda, \mu}}{\mu}} \left[ \frac{\nabla N_{\lambda, \mu} \cdot x - \nabla N_{\lambda, \mu} \cdot x}{\lambda - 1} \right] dx.$$

(3.42)

We recall the second equation in (1.9), which can be written into

$$\Delta N_{\lambda, \mu} = \mu^2 \left(1 + \frac{|x|^{2m} e^{\phi_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}}}{\lambda - 1} \right) N_{\lambda, \mu} - \frac{\mu}{\lambda} (\lambda + \mu) |x|^{2m} e^{\phi_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}} \text{ in } \Omega.$$

(3.43)

Multiplying (3.43) by $x \cdot \nabla N_{\lambda, \mu}$ and integrating over $B_r(0)$ for $r \in (0, d)$, we have

$$\int_{B_r(0)} e^{\phi_{\lambda, \mu} + 2m \ln |x| - \frac{N_{\lambda, \mu}}{\mu}} \left[ \left(\frac{\nabla N_{\lambda, \mu}}{\mu} - 1\right) \frac{\nabla N_{\lambda, \mu}}{\lambda} \cdot x - \frac{\nabla N_{\lambda, \mu}}{\lambda} \cdot x \right] dx$$

$$= \frac{1}{\mu^2} \int_{\partial B_r(0)} \left[ \frac{(x \cdot \nabla N_{\lambda, \mu})^2}{|x|} - |x||\nabla N_{\lambda, \mu}|^2 \frac{2}{|x|^{2m+1}} \right] d\sigma - \int_{\partial B_r(0)} \frac{N_{\lambda, \mu}^2 |x|}{2} d\sigma$$

$$+ \int_{B_r(0)} N_{\lambda, \mu}^2 dx$$

$$\geq \frac{1}{\mu^2} \int_{\partial B_r(0)} \left[ \frac{\partial N_{\lambda, \mu}}{\partial r} \right]^2 - \frac{\partial^2 N_{\lambda, \mu}}{2} - \frac{r \mu^2 N_{\lambda, \mu}^2}{2} \right] d\sigma$$

$$- \frac{1}{\mu^2} \int_{\partial B_r(0)} r \left[ \frac{\partial^2 N_{\lambda, \mu}}{2r^2} \right] d\sigma,$$

(3.44)
here we used $\nabla N_{\lambda, \mu} \cdot x = \frac{\partial N_{\lambda, \mu}}{\partial r} r$ and $|\nabla N_{\lambda, \mu}|^2 = \left(\frac{\partial N_{\lambda, \mu}}{\partial r}\right)^2 + \left(\frac{\partial N_{\lambda, \mu}}{\partial \theta}\right)^2$.

We claim that there is a sequence $\{r_l\}_{l \in \mathbb{N}}$ satisfying

$$\lim_{l \to \infty} r_l = 0 \quad \text{and} \quad \lim_{l \to \infty} \left( \frac{1}{\mu^2} \int_{\partial B_{r_l}(0)} r_l \left[ \left( \frac{\partial N_{\lambda, \mu}}{\partial r} \right)^2 + \mu^2 N_{\lambda, \mu}^2 \right] d\sigma \right) = 0.$$  

(3.45)

In order to prove the claim (3.45), we multiply the second equation in (1.9) by $N_{\lambda, \mu}$ and integrate over $\Omega$. Then we have

$$\int_{\Omega} |\nabla N_{\lambda, \mu}|^2 + \mu^2 \left( 1 + \frac{\lambda}{\mu} e^{\mu N_{\lambda, \mu}} \right) N_{\lambda, \mu}^2 dx = \int_{\Omega} \lambda \mu^2 \left( 1 + \frac{\lambda}{\mu} \right) e^{\mu N_{\lambda, \mu}} N_{\lambda, \mu} dx.$$

Together with Lemma 2.3 and (3.33), we see that there is a constant $C > 0$ such that

$$\frac{1}{\mu^2} \int_0^d \int_{\partial B_r(0)} \left[ \left( \frac{\partial N_{\lambda, \mu}}{\partial \theta} \right)^2 + \mu^2 N_{\lambda, \mu}^2 \right] d\sigma dr \leq \frac{1}{\mu^2} \left( \int_{\Omega} |\nabla N_{\lambda, \mu}|^2 + \mu^2 \left( 1 + \frac{\lambda}{\mu} e^{\mu N_{\lambda, \mu}} \right) N_{\lambda, \mu}^2 dx \right) = \int_{\Omega} \lambda \left( 1 + \frac{\lambda}{\mu} \right) e^{\mu N_{\lambda, \mu}} N_{\lambda, \mu} dx \leq \int_{\Omega} \lambda^2 \left( 1 + \frac{\lambda}{\mu} \right) e^{\mu N_{\lambda, \mu}} dx \leq C.$$

(3.46)

In view of (3.46), we note that there exists a sequence $r_l$ satisfying (3.45). Otherwise, there exist constants $\varepsilon > 0$ and $\bar{r} > 0$ satisfying

$$\frac{1}{\mu^2} \int_{\partial B_{r_l}(0)} r_l \left[ \left( \frac{\partial N_{\lambda, \mu}}{\partial \theta} \right)^2 + \mu^2 N_{\lambda, \mu}^2 \right] d\sigma \geq \varepsilon \quad \text{for any} \quad r \in (0, \bar{r}).$$

From (3.46), we have

$$C \geq \frac{1}{\mu^2} \int_0^{\bar{r}} \int_{\partial B_r(0)} \left[ \left( \frac{\partial N_{\lambda, \mu}}{\partial \theta} \right)^2 + \mu^2 N_{\lambda, \mu}^2 \right] d\sigma dr \geq \int_0^{\bar{r}} \frac{\varepsilon}{r} dr = +\infty,$$

which is a contradiction.
At this point, in view of (3.42), (3.44), and (3.45), we see that for any $\varepsilon > 0$, there is $l_\varepsilon$ such that if $l \geq l_\varepsilon$, then

$$
\begin{align*}
\int_{\partial B_{r_1}(0)} \left[ \frac{(x \cdot \nabla \phi_{\lambda,\mu})^2}{|x|} - \frac{|x||\nabla \phi_{\lambda,\mu}|^2}{2} + |x|^{2m+1} e^{\phi_{\lambda,\mu} - \frac{N_{\lambda,\mu}}{\mu}} \left( 1 - \frac{N_{\lambda,\mu}}{\lambda} \right) \right] d\sigma \\
= \int_{B_{r_1}(0)} (2 + 2m) e^{\phi_{\lambda,\mu} + 2m \ln |x| - \frac{N_{\lambda,\mu}}{\mu}} \left( 1 - \frac{N_{\lambda,\mu}}{\lambda} \right) dx \\
+ \int_{B_{r_1}(0)} e^{\phi_{\lambda,\mu} + 2m \ln |x| - \frac{N_{\lambda,\mu}}{\mu}} \left[ \left( \frac{N_{\lambda,\mu}}{\mu} - 1 \right) \nabla \phi_{\lambda,\mu} \cdot x - \nabla N_{\lambda,\mu} \cdot x \right] dx \\
\geq \int_{B_{r_1}(0)} (2 + 2m) e^{\phi_{\lambda,\mu} + 2m \ln |x| - \frac{N_{\lambda,\mu}}{\mu}} \left( 1 - \frac{N_{\lambda,\mu}}{\lambda} \right) dx \\
- \frac{1}{\mu^2} \int_{\partial B_{r_1}(0)} r_1 \left[ \left( \frac{\partial N_{\lambda,\mu}}{\partial \theta} \right)^2 + \frac{\mu^2 N_{\lambda,\mu}^2}{2} \right] d\sigma \\
\geq \int_{B_{r_1}(0)} (2 + 2m) e^{\phi_{\lambda,\mu} + 2m \ln |x| - \frac{N_{\lambda,\mu}}{\mu}} \left( 1 - \frac{N_{\lambda,\mu}}{\lambda} \right) dx - \varepsilon. \quad (3.47)
\end{align*}
$$

Let $\varphi_0(x) = (\xi + u_0)(x) - 2m \ln |x|$. Letting $\lambda, \mu \to \infty$, $\frac{\lambda}{\mu} \to 0$ in (3.47), Lemmas 2.3 and 3.2 as in (3.23) imply

$$
\begin{align*}
\int_{\partial B_{r_1}(0)} \left[ \frac{(x \cdot \nabla \varphi_0)^2}{|x|} - \frac{|x||\nabla \varphi_0|^2}{2} + |x|^{2m+1} e^{\varphi_0} \right] d\sigma \\
\geq (2m + 2) \left( \alpha_\varphi + \int_{B_{r_1}(0)} e^{\xi + u_0} dx \right) - \varepsilon. \quad (3.48)
\end{align*}
$$

There exists a constant $c > 0$ such that $|x|^{2m} e^{\varphi_0} = e^{\xi + u_0} \leq c|x|^{-\tau} e^\phi$ in $B_r(0)$ for any $r \in (0, \delta)$, where $\tau \equiv \max\{0, \frac{2\varphi_0}{2\pi} - 2m\}$. We note that $\tau \in [0, 2)$ from (3.40). In view of (3.39) and Corollary 1 in [7], we see that $e^{\phi} \in L^k_{loc}(B_d(0))$ for any $k \in [1, \infty)$. Since $\xi \in C^2_{loc}(B_{2\delta}(0)\setminus\{0\})$, we have $|x|^{2m} e^{\varphi_0} \in L^t(B_d(0))$ for any $t \in (1, \frac{2}{\tau})$. Then Hölder’s inequality implies that $\phi \in L^\infty(B_d(0))$ and

$$
|x|^{2m} e^{\varphi_0} = e^{\xi + u_0} \leq C|x|^{-\tau} \quad \text{for some constant} \ C > 0. \quad (3.49)
$$

We note that for $|x| = r < \delta$,

$$
|\nabla \phi(x)| \leq \frac{1}{2\pi} \int_{B_d(0)} \frac{1}{|x - y|} e^{(\xi + u_0)(y)} dy \\
= \frac{1}{2\pi} \left[ \int_{B_d(0) \setminus B_{r/2}(x)} \frac{e^{(\xi + u_0)(y)} |x - y|}{|x - y|} dy + \int_{B_d(0) \cap B_{r/2}(x)} e^{(\xi + u_0)(y)} |x - y| dy \right].
$$
Fix $t \in (1, \frac{2}{\tau})$ and choose a constant $a \in (0, \min\{1, 2 - \tau\})$ such that $\frac{at}{t+1} < 2$. Hölder’s inequality implies that

$$\int_{B_d(0) \setminus B_{r/2}(x)} \frac{e^{(\xi+u_0)(y)}}{|x-y|} dy \leq \int_{B_d(0) \setminus B_{r/2}(x)} \frac{Cr^a-1}{|y-x|^a} e^{(\xi+u_0)(y)} dy \leq Cr^a-1.$$  

Since $|x| = r$, we have $B_{r/2}(x) \subseteq \Omega \setminus B_{r/2}(0)$. It follows from (3.49) that

$$\int_{B_d(0) \cap B_{r/2}(x)} \frac{e^{(\xi+u_0)(y)}}{|x-y|} dy \leq \int_{|y-x| \leq r/2} \frac{Cr^{-\tau}}{|y-x|} dy = O(r^{1-\tau}).$$

Since $a \in (0, 2-\tau)$, we see that $|\nabla \phi(x)| = O(|x|^{a-1} + 1)$ as $|x| \to 0$. Consequently $\nabla \phi_0(x) = -\frac{\alpha_q x}{2\pi |x|^2} + \nabla h(x)$ with $|\nabla h(x)| = O(|x|^{a-1} + 1)$ as $|x| \to 0$. Letting $\varepsilon \to 0$ and $\rho_l \to 0$ in (3.48), we obtain that $(2m+2)\alpha_q \leq \frac{\alpha_q}{\pi}$, which contradicts (3.40).

Therefore, it follows from Harnack’s inequality that $w_{\lambda, \mu} - u_0 \to -\infty$ and $w_{\lambda, \mu} \to -\infty$ uniformly on any compact subset of $\Omega \setminus B$.

**Step 3** In view of Lemma 2.3 and Corollary 2.1, along a subsequence, $e^{w_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}(1 - \frac{N_{\lambda, \mu}}{\lambda})}$ converges to a nonnegative measure. However, this measure must be supported in $B$ since $w_{\lambda, \mu} \to -\infty$ uniformly on any compact set $K \subset \Omega \setminus B$. Then we see that as $\lambda, \mu \to \infty$, $\frac{\lambda}{\mu} \to 0$,

$$e^{w_{\lambda, \mu} - \frac{N_{\lambda, \mu}}{\mu}(1 - \frac{N_{\lambda, \mu}}{\lambda})} \to \sum_{q \in B} \alpha_q \delta_q \quad (\alpha_q \geq 8\pi)$$

in the sense of measure. In view of the above arguments, we conclude that Case 2 implies Theorem 1.1-(iii). Now we complete the proof of Theorem 1.1. \(\square\)

**4 Proof of Theorem 1.2**

In this section, we are going to construct blow up solutions of (3.2) such that

$$\sup u_{\lambda, \mu} \geq -c_0 > -\infty.$$  

Based on Theorem 1.1 above, our construction in this section was inspired by the construction in [44] where the authors constructed blow up solutions for the $SU(3)$ Chern–Simons system on torus using an entire regular solution for the single Chern–Simons equation as the building blocks.
4.1 The approximate solution and the reduction

Without loss of generality, we assume $|\Omega| = 1$. We recall the equation (3.2) as follows:

\[
\begin{aligned}
\Delta (u + \frac{N}{\mu}) &= -\lambda^2 e^{u^* + u_0^*} (1 - \frac{N}{\lambda}) + 4\pi \mathfrak{M}, \\
\Delta \frac{N}{\lambda} &= \mu (\mu + \lambda e^{u^* + u_0^*}) \frac{N}{\lambda} - \mu (\lambda + \mu) e^{u^* + u_0^*} \text{ in } \Omega.
\end{aligned}
\] (4.1)

We are going to define the approximate solutions for (3.2). Let $w$ be the radially symmetric solution of

\[
\begin{aligned}
\Delta w + e^w (1 - e^w) &= 0 \text{ in } \mathbb{R}^2, \\
w'(|x|) &\to -\frac{2\mathfrak{M}}{|x|} + \frac{a_1 (2\mathfrak{M} - 2)}{|x|^{2\mathfrak{M} - 1}} + O\left(\frac{1}{|x|^{2\mathfrak{M} - 1}}\right), \quad |x| \gg 1, \\
w(|x|) &= -2\mathfrak{M} \ln |x| + I_1 - \frac{a_1}{|x|^{2\mathfrak{M} - 2}} + O\left(\frac{1}{|x|^{2\mathfrak{M} - 2}}\right), \quad |x| \gg 1,
\end{aligned}
\] (4.2)

where $a_1$ and $I_1$ are constants (see [14, Theorem 2.1, Lemma 2.6] for the existence of $w$ satisfying (4.2)). Motivated by [44,45], we set

\[
\begin{aligned}
u_{1,\lambda,q}(y) &= w(\lambda |y - q|) - u_0(q) + 4\pi \mathfrak{M}(\gamma (y, q) - \gamma(q, q)), \\
u_{2,\lambda,q}(y) &= w(d) - u_0(q) + 4\pi \mathfrak{M}(G(y, q) - \gamma(q, q) + \frac{1}{2\pi} \ln d), \\
u_{\lambda,q}(y) &= u_{1,\lambda,q}(y) \chi(|y - q|) + u_{2,\lambda,q}(y) (1 - \chi(|y - q|)),
\end{aligned}
\] (4.3)

where $q \in \Omega \setminus \cup_i \{p_i\}$, and $\chi(y) = \chi(|y|)$ is a smooth cut off function such that

\[\chi = 1 \text{ in } B_d(0), \quad \chi = 0 \text{ in } \mathbb{R}^2 \setminus B_{2d}(0), \quad 0 \leq \chi \leq 1.\]

We note that $\nu_{\lambda,q} \in C^\infty(\Omega)$. We would find a solution of (3.2) with the following form:

\[
v + \frac{N}{\mu} = \nu_{\lambda,q} + \varphi \quad \text{and} \quad \frac{N}{\lambda} = e^{\nu_{\lambda,q} + u_0}(1 + \varphi) + S,
\] (4.4)

here $(\varphi, S)$ would be regard as an error term. For the convenience, we also denote

\[
\begin{aligned}
w_{\lambda,q}(y) &= w(\lambda |y - q|), \\
h(\varphi, S) &= e^{\nu_{\lambda,q} + u_0}(1 + \varphi) + S, \\
F(t) &= e^t (1 - e^t) \quad \text{and} \quad f(t) = F'(t) = e^t (1 - 2e^t).
\end{aligned}
\] (4.5)

Then the Eq. (3.2) is reduced to a system for $(\varphi, S)$:

\[
\begin{aligned}
\Delta \varphi + \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi &= g_{1,\lambda,\mu}(\varphi, S), \\
\Delta S - \mu^2 S &= g_{2,\lambda,\mu}(\varphi, S),
\end{aligned}
\] (4.6)
where
\[ g_{1,\lambda,\mu}(\varphi, S) := -\Delta U_{\lambda,q} + \lambda^2 f(w_{\lambda,q}(y))\chi(|y - q|)\varphi - \lambda^2 F(U_{\lambda,q} + u_0) + 4\pi\mathcal{M} \]
\[-\lambda^2 f(U_{\lambda,q} + u_0)\varphi + \lambda^2 F(U_{\lambda,q} + u_0)(1 + \varphi - e^{\varphi - \frac{1}{\mu} h(\varphi, S)}) + \lambda^2 e^{U_{\lambda,q} + u_0} S + \lambda^2 e^{U_{\lambda,q} + u_0}(e^{\varphi - \frac{1}{\mu} h(\varphi, S)} - 1)(e^{U_{\lambda,q} + u_0} \varphi + S), \tag{4.7} \]
\[ g_{2,\lambda,\mu}(\varphi, S) := -\Delta \left\{ e^{U_{\lambda,q} + u_0}(1 + \varphi) \right\} + \mu^2 e^{U_{\lambda,q} + u_0} \left\{ 1 + \varphi - e^{\varphi - \frac{1}{\mu} h(\varphi, S)} \right\} + \lambda \mu \left\{ e^{2U_{\lambda,q} + 2u_0 + \varphi - \frac{1}{\mu} h(\varphi, S)} (1 + \varphi) + (S - 1)e^{U_{\lambda,q} + u_0 + \varphi - \frac{1}{\mu} h(\varphi, S)} \right\}. \]

The following result would be useful to analyze the nonlinear parts in (4.6).

**Lemma 4.1** There is a constant \( C > 0 \) satisfying
\[
\|u_{1,\lambda,q} - u_{2,\lambda,q}\|_{C^1(B_{2d}(q) \setminus B_d(q))} + \|\Delta(u_{1,\lambda,q} - u_{2,\lambda,q})\|_{L^\infty(B_{2d}(q) \setminus B_d(q))} \leq C\lambda^{-2\mathcal{M} + 2}. \tag{4.8} 
\]

**Proof** We note that
\[
u_{1,\lambda,q}(y) - u_{2,\lambda,q}(y) = w(\lambda|y - q|) - w(d\lambda) + 4\pi\mathcal{M} \left( y(q, q) - G(y, q) - \frac{1}{2\pi} \ln d \right). 
\]
In view of (4.2), we see that if \( y \in B_{2d}(q) \setminus B_d(q) \), then
\[
u_{1,\lambda,q}(y) - u_{2,\lambda,q}(y) = -\frac{a_1}{(\lambda|y - q|)^{2\mathcal{M} - 2}} + \frac{a_1}{(\lambda d)^{2\mathcal{M} - 2}} + O(\lambda^{-2\mathcal{M}}) = O(\lambda^{-2\mathcal{M} + 2}), 
\]
\[
\nabla_y \left( u_{1,\lambda,q}(y) - u_{2,\lambda,q}(y) \right) = w'(\lambda|y - q|) \frac{(y - q)}{|y - q|} \lambda + 2\mathcal{M} \frac{(y - q)}{|y - q|^2} = \frac{a_1(2\mathcal{M} - 2)\lambda(y - q)}{(\lambda|y - q|)^{2\mathcal{M} - 1}} + O(\lambda^{-2\mathcal{M}}) = O(\lambda^{-2\mathcal{M} + 2}), \]

and
\[
\Delta_y \left( u_{1,\lambda,q}(y) - u_{2,\lambda,q}(y) \right) = \Delta_y \left( w(\lambda|y - q|) + 2\mathcal{M} \ln |y - q| \right) = -\exp(w(\lambda|y - q|))(1 - \exp(w(\lambda|y - q|)))\lambda^2 = O(\lambda^{-2\mathcal{M} + 2}). 
\]

The above estimations imply that Lemma 4.1 holds. \( \square \)
4.2 The linear and nonlinear problem

In this subsection, we are going to study the linear and nonlinear problem. First, let us introduce the space we are going to work in. Fix a small constant $0 < \alpha < \frac{1}{2}$. Let us introduce two function spaces $X_{\alpha,q}$ and $Y_{\alpha,q}$. Define

$$\rho(z) = (1 + |z|)^{1+\frac{\alpha}{2}}, \quad \text{and} \quad \tilde{\rho}(z) = \frac{1}{(1 + |z|)(\ln(2 + |z|))^{1+\frac{\alpha}{2}}}.$$  

We say that $\psi \in X_{\alpha,q}$ if

$$\|\psi\|^2_{X_{\alpha,q}} = \|\Delta \tilde{\psi}\rho\|^2_{L^2(B_{2d\lambda}(0))} + \tilde{\psi}\rho\|^2_{L^2(B_{2d\lambda}(0))} + \|\Delta \psi\|^2 + \psi^2 \|_{L^1(\Omega/B_d(q))} < +\infty$$

We say that $\psi \in Y_{\alpha,q}$ if

$$\|\psi\|^2_{Y_{\alpha,q}} = \frac{1}{\lambda^4} \|\tilde{\psi}\rho\|^2_{L^2(B_{2d\lambda}(0))} + \|\tilde{\psi}\|^2_{L^2(\Omega/B_d(q))} < +\infty.$$  

Let

$$W_{q,j}(y) = \chi(y - q) \frac{\partial w(\lambda|y - q|)}{\partial q_j}, \quad j = 1, 2,$$

and

$$Z_{q,j}(y) = -\Delta W_{q,j} + \lambda^2 e^{w(\lambda|y - q|)} W_{q,j}, \quad j = 1, 2.$$  

We define two subspace of $X_{\alpha,q}$ and $Y_{\alpha,q} as

$$E_q = \left\{ u : u \in X_{\alpha,q}, \int_\Omega Z_{q,j}u = 0, \quad j = 1, 2 \right\},$$

and

$$F_q = \left\{ u : u \in Y_{\alpha,q}, \int_\Omega W_{q,j}u = 0, \quad j = 1, 2 \right\}.$$  

Moreover, we define the projection operator to $F_q$ as

$$Q_q u = u - \sum_{j=1}^2 c_j Z_{q,j}$$

where $c_j$ are chosen so that $Q_q u \in F_q$. We have the following estimates:
Lemma 4.2 There holds \( \|Q_qu\|_{Y_{a,q}} \leq C\|u\|_{Y_{a,q}} \) for some positive constant \( C \) independent of \( q \).

First we need the preliminary results for the linear operators \( L_{1,q}, L_2 \) in [26,44], where
\[
\begin{aligned}
L_{1,q}(\varphi) &:= \Delta \varphi + \lambda^2 f(w_{\lambda,q}(y)) \chi(y-q)\varphi, \\
L_2(S) &:= \Delta S - \mu^2 S.
\end{aligned}
\]  \( (4.16) \)

Theorem 4.1 The operator \( Q_q L_{1,q} \) is an isomorphism from \( E_q \) to \( F_q \). Moreover if \( \varphi \in E_q \) and \( g \in F_q \) satisfies \( Q_q L_{1,q}(\varphi) = g \), then there exists a constant \( C > 0 \), independent of \( \lambda > 0 \) and \( q \), such that
\[
\|\varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{X_{a,q}} \leq C \ln \|g\|_{Y_{a,q}}.
\]  \( (4.17) \)

Proof In [44, Theorem B.3], Theorem 4.1 was proved for the linearized operator \( \tilde{L}_{1,q}(\varphi) := \Delta \varphi + \lambda^2 f(w_{\lambda,q}(y)) 1_{B_{2d}(q)} \varphi \), where \( 1_{B_{2d}(q)}(x) = 1 \) if \( x \in B_{2d}(q) \), and \( 1_{B_{2d}(q)}(x) = 0 \) if \( x \notin B_{2d}(q) \). Since \( L_{1,q} \) is a small perturbation from \( \tilde{L}_{1,q} \), Theorem 4.1 also holds for our linearized operator \( L_{1,q} \). In this paper, we introduce a new simpler proof for (4.17), which unifies the approaches in [44] and [42]. We put the detail of the proof in the Sect. 6 below. \( \square \)

Theorem 4.2 The operator
\[
L_2 : W^{2,2}(\Omega) \rightarrow L^2(\Omega)
\]
is an isomorphism. Moreover, for any \( S \in W^{2,2}(\Omega) \) and \( g_2 \in L^2(\Omega) \) satisfies \( L_2(S) = g_2 \), there exists a positive constant \( C > 0 \), independent of \( \mu > 0 \), such that
\[
\begin{aligned}
\mu^2 \|S\|_{L^2(\Omega)} + \mu \|\nabla S\|_{L^\infty(\Omega)} + \mu \|\nabla S\|_{L^2(\Omega)} + \|\partial_i \partial_j S\|_{L^2(\Omega)} &\leq C \|g_2\|_{L^2(\Omega)}, \\mu^2 \|S\|_{L^\infty(\Omega)} &\leq C \|g_2\|_{L^\infty(\Omega)} \quad \text{if} \quad g_2 \in L^\infty(\Omega).
\end{aligned}
\]  \( (4.18) \)

Proof It has been shown in [26, Theorem 2.4] that \( L_2 \) is an isomorphism, \( \mu \|S\|_{L^\infty(\Omega)} \leq C \|g_2\|_{L^2(\Omega)} \), and \( \mu^2 \|S\|_{L^\infty(\Omega)} \leq C \|g_2\|_{L^\infty(\Omega)} \) if \( g_2 \in L^\infty(\Omega) \). In order to complete the proof of Theorem 4.1, if is enough to prove
\[
\mu^2 \|S\|_{L^2(\Omega)} + \mu \|\nabla S\|_{L^2(\Omega)} + \|\partial_i \partial_j S\|_{L^2(\Omega)} \leq C \|g_2\|_{L^2(\Omega)}.
\]

Here we prove the estimate (4.18).

Multiply the equation \( L_2(S) = g_2 \) by \( S \) and integrate over \( \Omega \), one has
\[
\int_{\Omega} |\nabla S|^2 + \mu^2 \int_{\Omega} S^2 = - \int_{\Omega} g_2 S.
\]  \( (4.19) \)

By Hölder inequality,
\[
\int_{\Omega} |\nabla S|^2 + \mu^2 \int_{\Omega} S^2 \leq \mu^{-2} \int_{\Omega} g_2^2 + \frac{\mu^2}{4} \int_{\Omega} S^2,
\]
which implies that
\[ \mu^2 \|S\|_{L^2(\Omega)} + \mu \|\nabla S\|_{L^2(\Omega)} \leq C \|g_2\|_{L^2(\Omega)}. \]

Since \( \Delta S - \mu^2 S = g_2 \), one can get that
\[ \|\Delta S\|_{L^2(\Omega)} \leq \mu^2 \|S\|_{L^2(\Omega)} + \|g_2\|_{L^2(\Omega)}, \]
so
\[ \mu^2 \|S\|_{L^2(\Omega)} + \mu \|\nabla S\|_{L^2(\Omega)} + \|\partial_i \partial_j S\|_{L^2(\Omega)} \leq C \|g_2\|_{L^2(\Omega)}. \]

Now, we complete the proof of Theorem 4.2. \qed

Next, let us consider the corresponding nonlinear problem. We define an operator \( \Psi \) by
\[
\Psi(\varphi, S) = \left( (Q_q L_{1,q})^{-1} (Q_q g_{1,\lambda,\mu}(\varphi, \hat{S})), \hat{S} \right),
\]
where \( \hat{S} = L_2^{-1}(g_{2,\lambda,\mu}(\varphi, S)) \), and a subset \( M_{\lambda,\mu} \) of \( E_q \times W^{2,2}(\Omega) \) by
\[
M_{\lambda,\mu} = \left\{ (\varphi, S) \in E_q \times W^{2,2}(\Omega) \mid \| (\varphi, S) \| \leq \frac{\left( \ln \lambda \right)^2}{\lambda} \right\}.
\]

We note that if \( (\varphi, S) \in M_{\lambda,\mu} \), then
\[
\| \varphi \|_{L^\infty(\Omega)} + \| \varphi \|_{X_{a,q}} \leq \frac{(\ln \lambda)^2}{\lambda}, \quad \text{and} \quad \mu^2 \|S\|_{L^2(\Omega)} + \mu \|S\|_{L^\infty(\Omega)} + \|S\|_{W^{2,2}(\Omega)} \leq \frac{\mu^2}{\lambda}.
\]

In order to apply contraction mapping theorem, we need some estimations for the right hand side of (4.6).

**Lemma 4.3** There exists a constant \( C \) such that
\[ \| \Delta e^{U_{\lambda,q}+u_0} (1 + \varphi) \|_{L^2(\Omega)} \leq C \lambda (1 + \| \varphi \|_{L^\infty(\Omega)} + \| \varphi \|_{X_{a,q}}). \]
for any \( (\varphi, S) \in M_{\lambda,\mu} \).
Proof  We have

\[
\begin{align*}
\Delta [e^{U_{\lambda,q} + u_0} (1 + \varphi)]
&= e^{U_{\lambda,q} + u_0} (1 + \varphi) \Delta (U_{\lambda,q} + u_0) + e^{U_{\lambda,q} + u_0} \Delta \varphi \\
&\quad + 2e^{U_{\lambda,q} + u_0} \nabla (U_{\lambda,q} + u_0) \cdot \nabla \varphi + e^{U_{\lambda,q} + u_0} (1 + \varphi) |\nabla (U_{\lambda,q} + u_0)|^2.
\end{align*}
\]  
(4.21)

First, we consider the \(L^2\) norm of \(\Delta [e^{U_{\lambda,q} + u_0} (1 + \varphi)]\) in \(B_d(q)\).
In \(B_d(q)\), we get that

\[
\Delta U_{\lambda,q} = \Delta w_{\lambda,q} (y) + 4\pi \mathfrak{M} \Delta \gamma (y, q) = -\lambda^2 F (w_{\lambda,q} (y)) + 4\pi \mathfrak{M}
\]  
(4.22)

and

\[
\Delta u_0 = -4\pi \mathfrak{M}.
\]

Moreover, since \(\gamma (y, q)\) and \(u_0\) are smooth functions in \(B_d(q)\), we get that

\[
\frac{\partial U_{\lambda,q}}{\partial y_j} = \lambda w'_{\lambda,q} (y) \frac{y_j - q_j}{|y - q|} + 4\pi \mathfrak{M} \frac{\partial \gamma (y, q)}{\partial y_j}, \quad j = 1, 2,
\]

and

\[
\frac{\partial u_0}{\partial y_j} = O(1), \quad j = 1, 2.
\]

By the definitions of \(U_{\lambda,q}\) and \(u_0\), we obtain that

\[
e^{U_{\lambda,q}} = e^{w_{\lambda,q} (y) + O(1)} \quad \text{and} \quad e^{u_0} = O(1) \quad \text{in} \ B_d(q).
\]

Then,

\[
\|\Delta [e^{U_{\lambda,q} + u_0} (1 + \varphi)]\|_{L^2(B_d(q))} \\
\leq C \{\|e^{w_{\lambda,q} (y)} (1 + \varphi) \lambda^2 F (w_{\lambda,q} (y))\|_{L^2(B_d(q))} + \|e^{w_{\lambda,q} (y)} \Delta \varphi\|_{L^2(B_d(q))} \\
+ \|e^{w_{\lambda,q} (y)} (\lambda w'_{\lambda,q} (y) + 1) |\nabla \varphi|\|_{L^2(B_d(q))} \\
+ \|e^{w_{\lambda,q} (y)} (1 + \varphi) (\lambda^2 (w'_{\lambda,q})^2 + 1)\|_{L^2(B_d(q))}\} \\
=: C \{I + II + III + IV\}.
\]  
(4.23)

Let

\[
z := \lambda (y - q) \quad \text{and} \quad \bar{u}(z) := u (\lambda^{-1} z + q)
\]  
(4.24)
for any function \( u \). Then,
\[
I = \|e^{u_{\lambda,q}}(y)(1 + \varphi)\lambda^2 F(w_{\lambda,q}(y))\|_{L^2(B_d(q))}
\]
\[
= \lambda^{-1}\|e^{w(z)}(1 + \tilde{\varphi})\lambda^2 F(w(z))\|_{L^2(B_{d,q}(0))}
\]
\[
\leq C\lambda\|e^{w(z)}(1 + \tilde{\varphi})\|_{L^2(B_{d,q}(0))} \leq C\lambda(1 + \|\varphi\|_{L^\infty(\Omega)}), \tag{4.25}
\]
where \( \tilde{\varphi}(z) := \varphi(\lambda^{-1}z + q) \) as defined in (4.24). In the last inequality in (4.25), we used the decay rate of \( w(z) \) in (4.2). From (4.2), we also have
\[
II = \|e^{u_{\lambda,q}}(y)\Delta\varphi\|_{L^2(B_d(q))} \leq \lambda\|e^{w(z)}\Delta\tilde{\varphi}(z)\|_{L^2(B_{d,q}(0))} \leq C\lambda\|\Delta\tilde{\varphi}\|_{L^2(B_{d,q}(0))}.
\]
We can rewrite III by
\[
III = \|e^{u_{\lambda,q}}(y)(\lambda w_{\lambda,q} + 1)|\nabla\varphi|\|_{L^2(B_d(q))}
\]
\[
\leq \lambda\|e^{w(z)}w'(z)|\nabla\tilde{\varphi}|\|_{L^2(B_{d,q}(0))} + \|e^{w(z)}|\nabla\tilde{\varphi}|\|_{L^2(B_{d,q}(0))}.
\tag{4.26}
\]
Since \( |e^{w(z)}w'(z)| \leq C(1 + |z|)^{-2\eta t - 1}, \)
\[
\lambda\|e^{w(z)}w'(z)|\nabla\tilde{\varphi}|\|_{L^2(B_{d,q}(0))} \leq C\lambda(1 + |z|)^{-2\eta t - 1}|\nabla\tilde{\varphi}|\|_{L^2(B_{d,q}(0))}.
\]
There exist finite number of points \( \{z_i\} \) such that \( B_{d,q}(0) \subseteq \bigcup_i B_1(z_i) \subseteq \bigcup_i B_2(z_i) \subseteq B_{d,q}(0) \) and for any index \( i \), \( |\{z_j \mid B_1(z_i) \cap B_1(z_j) \neq \emptyset\}| \leq c_1 \), where a positive constant \( c_1 \) is independent of \( \lambda \). Then, we have
\[
\lambda\|\Delta\tilde{\varphi}\|_{L^2(B_{d,q}(0))} \leq \lambda 2^{2\eta t + 1}\|\Delta\tilde{\varphi}\|_{L^2(B_{d,q}(0))}.
\tag{4.27}
\]
Moreover, there exists some constant \( c_2 \) from \( W^{2,2} \) estimate satisfying
\[
|\nabla\tilde{\varphi}|\|_{L^2(B_{d,q}(0))} \leq \sum_i \|\Delta\tilde{\varphi}\|_{L^2(B_{d,q}(0))} + \|\tilde{\varphi}|\|_{L^2(B_{d,q}(0))}
\]
\[
\leq 3^{1 + \eta t}c_2\left( (1 + |z_i|)^{-1 + \eta t} \|\rho\Delta\tilde{\varphi}\|_{L^2(B_{d,q}(0))} \right.
\]
\[
+ (1 + |z_i|)(\ln(2 + |z_i|))^{1 + \eta t} \|\tilde{\varphi}\|_{L^2(B_{d,q}(0))}\right).
\tag{4.28}
\]
Therefore, it follows that
\[
\|e^{w_{\lambda,q}}(y)\lambda w_{\lambda,q}'|\nabla\varphi|\|_{L^2(B_d(q))}
\]
\[
\leq 2^{2\eta t + 1}\|\Delta\tilde{\varphi}|\|_{L^2(B_{d,q}(0))} + \|\tilde{\varphi}|\|_{L^2(B_{d,q}(0))}
\]
\[
\leq 2^{2\eta t + 1}\|\Delta\tilde{\varphi}|\|_{L^2(B_{d,q}(0))} + \|\tilde{\varphi}|\|_{L^2(B_{d,q}(0))}
\leq C\lambda\|\varphi\|_{X_{a,q}}.
\tag{4.29}
\]
Similarly, we have

\[ \| e^{w(z)} |\nabla_z \tilde{\varphi}| \|_{L^2(B_d(0))} \leq C \| \varphi \|_{X_{\alpha,q}}. \]

From the decay rate of \( e^{w(z)} w'(z) \), we have

\[
IV = \| e^{w_{\lambda,q}(y)} (1 + \varphi)(\lambda^2 (w'_{\lambda,q})^2 + 1) \|_{L^2(B_0(y))} \\
\leq \lambda \| e^{w(z)} (1 + \tilde{\varphi})(w'(z))^2 \|_{L^2(B_{d\lambda}(0))} + \lambda^{-1} \| e^{w(z)} (1 + \tilde{\varphi}) \|_{L^2(B_{d\lambda}(0))} \\
\leq C \lambda(1 + \| \varphi \|_{L^\infty(B_d(0))}). \tag{4.30}
\]

Next, we consider \( \| \Delta [e^{U_{\lambda,q} + u_0}(1 + \varphi)] \|_{L^2(\Omega \setminus B_d(q))} \). By \( U_{\lambda,q} = u_{2,\lambda,q} + (u_{1,\lambda,q} - u_{2,\lambda,q}) \chi(|y - q|) \) and Lemma 4.1, we have

\[
\Delta U_{\lambda,q} = 4\pi \mathcal{M} + O(\lambda^{-2m}\n), \quad \Delta u_0 = -4\pi \mathcal{M} + 4\pi \sum_{i=1}^n m_i \delta_{p_i} \text{ in } \Omega \setminus B_d(q),
\]

and

\[
\nabla U_{\lambda,q} = 4\pi \mathcal{M} \nabla G(y, q) + O(\lambda^{-2m}\n), \\
\nabla u_0 = -4\pi \sum_{i=1}^n m_i \nabla G(y, p_i) \text{ in } \Omega \setminus B_d(q).
\]

We see that \( G(y, q) \) is a smooth function in \( \Omega \setminus B_d(q) \) and thus

\[
|\Delta U_{\lambda,q}| + |\nabla U_{\lambda,q}| + |e^{U_{\lambda,q}}| \\
= |4\pi \mathcal{M}| + |4\pi \mathcal{M} \nabla G(y, q)| + e^{U_{\lambda,q}} + O(\lambda^{-2m}\n) = O(1) \text{ in } \Omega \setminus B_d(q). \tag{4.31}
\]

We also see that

\[
u_0(y) = 2m_i \ln |y - p_i| - 4\pi m_i \gamma(y, p_i) - 4\pi \sum_{i \neq j} m_j G(y, p_j), \text{ in } B_d(p_i),
\]

where \( m_i \) is the multiplicity of \( p_i \). Since \( \gamma(y, p_i) \) and \( G(y, p_j) \), \( j \neq i \), are smooth functions in \( B_d(p_i) \), we have \( e^{u_0(y)} = O(|y - p_i|^{2m_i}) \in C^2(B_d(p_i)) \). Obviously, we also have \( e^{u_0} \in C^2(\Omega \setminus \cup_{i=1}^n B_d(p_i)) \). From this observation, we see that

\[
|\Delta e^{u_0}| + |\nabla e^{u_0}| + |e^{u_0}| = O(1) \text{ in } \Omega \setminus B_d(q). \tag{4.32}
\]
We also see that

\[
\Delta \{e^{U_{\lambda,q} + u_0} (1 + \varphi)\} \\
= \left\{ e^{U_{\lambda,q}} \left( \Delta U_{\lambda,q} + |\nabla U_{\lambda,q}|^2 \right) e^{u_0} + 2e^{U_{\lambda,q}} \nabla U_{\lambda,q} \cdot \nabla e^{u_0} + e^{U_{\lambda,q}} \Delta e^{u_0} \right\} (1 + \varphi) \\
+ 2 \left( e^{U_{\lambda,q} + u_0} \nabla U_{\lambda,q} + e^{U_{\lambda,q}} \nabla e^{u_0} \right) \cdot \nabla \varphi + e^{U_{\lambda,q} + u_0} \Delta \varphi.
\] (4.33)

We are going to estimate \( \|\Delta \{e^{U_{\lambda,q} + u_0} (1 + \varphi)\}\|_{L^2(\Omega \setminus B_d(q))} \) by dividing the region \( \Omega \setminus B_d(q) \) into \( \Omega \setminus B_{\frac{3}{2}d}(q) \) and \( B_{\frac{3}{2}d}(q) \setminus B_d(q) \).

Firstly, the estimations (4.31)–(4.33) again imply

\[
\|\Delta \{e^{U_{\lambda,q} + u_0} (1 + \varphi)\}\|_{L^2(\Omega \setminus B_{\frac{3}{2}d}(q))} \\
\leq C (1 + \|\varphi\|_{L^2(\Omega \setminus B_{\frac{3}{2}d}(q))} + \|\nabla \varphi\|_{L^2(\Omega \setminus B_{\frac{3}{2}d}(q))} + \|\Delta \varphi\|_{L^2(\Omega \setminus B_{\frac{3}{2}d}(q)))} \\
\leq C (1 + \|\varphi\|_{L^2(\Omega \setminus B_d(q))} + \|\Delta \varphi\|_{L^2(\Omega \setminus B_d(q)))} \leq C (1 + \|\varphi\|_{X_{\alpha,q}}) \). (4.34)

Secondly, in \( B_{\frac{3}{2}d}(q) \setminus B_d(q) \), the estimations (4.31)–(4.33) again imply

\[
\|\Delta \{e^{U_{\lambda,q} + u_0} (1 + \varphi)\}\|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_d(q))} \\
\leq \|\left( e^{U_{\lambda,q}} \left( \Delta U_{\lambda,q} + |\nabla U_{\lambda,q}|^2 \right) e^{u_0} + 2e^{U_{\lambda,q}} \nabla U_{\lambda,q} \cdot \nabla e^{u_0} + e^{U_{\lambda,q}} \Delta e^{u_0} \right) (1 \\
+ \varphi)\|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_d(q))} \\
+ \|2 \left( e^{U_{\lambda,q} + u_0} \nabla U_{\lambda,q} + e^{U_{\lambda,q}} \nabla e^{u_0} \right) \cdot \nabla \varphi\|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_d(q))} \\
+ \|e^{U_{\lambda,q} + u_0} \Delta \varphi\|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_d(q))} \\
\leq C (1 + \|\varphi\|_{L^2(\Omega \setminus B_d(q))} + \|\Delta \varphi\|_{L^2(\Omega \setminus B_d(q)))} \\
+ 2\|e^{U_{\lambda,q} + u_0} \nabla \left( U_{\lambda,q} + u_0 \right) \cdot \nabla \varphi\|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_d(q))}.
\]

In order to estimate \( \|e^{U_{\lambda,q} + u_0} \nabla \left( U_{\lambda,q} + u_0 \right) \cdot \nabla \varphi\|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_d(q))} \), we note that the estimation (4.2) yields

\[
e^{U_{\lambda,q} + u_0} \leq C \lambda^{-2M}, \quad \text{in } B_{\frac{3}{2}d}(q) \setminus B_d(q).
\]
From the definition of $U_{\lambda, q}$ and $u_0$, it is clear that $\nabla \left( U_{\lambda, q} + u_0 \right)$ are uniformly bounded in $B_{\frac{3}{2}d}(q) \setminus B_{d}(q)$. Therefore, we obtain that

$$ \| e^{U_{\lambda, q} + u_0} \nabla (U_{\lambda, q} + u_0) \cdot \nabla \varphi \|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_{d}(q))} \leq C \lambda^{-2\eta} \| \nabla \varphi \|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_{d}(q))} \leq C \lambda^{-2\eta} \| \nabla \tilde{\varphi} \|_{L^2(B_{2}(z_i))} \leq C \lambda^{-2\eta} \sum_i \| \nabla \tilde{\varphi} \|_{L^2(B_{2}(z_i))} \leq C \lambda^{-2\eta} \sum_i (\| \Delta \tilde{\varphi} \|_{L^2(B_{2}(z_i))} + \| \tilde{\varphi} \|_{L^2(B_{2}(z_i))}) $$

(4.35)

where $\{ B_1(z_i) \}$ is finite covering of $B_{\frac{3}{2}d}(0) \setminus B_{d}(0)$ as in the calculus in III. Since

$$ \| \Delta \tilde{\varphi} \|_{L^2(B_2(z_i))} \leq (1 + |z_i|)^{-\frac{q}{2}} \| \Delta \tilde{\varphi} \rho \|_{L^2(B_2(z_i))} \leq C \lambda^{-1-\frac{q}{2}} \| \Delta \tilde{\varphi} \rho \|_{L^2(B_2(z_i))} $$

and

$$ \| \tilde{\varphi} \|_{L^2(B_2(z_i))} \leq (1 + |z_i|) \{ \ln(2 + |z_i|) \}^{1+\frac{q}{2}} \| \tilde{\varphi} \rho \|_{L^2(B_2(z_i))} \leq C \lambda^{1+\frac{q}{2}} \| \tilde{\varphi} \rho \|_{L^2(B_2(z_i))} $$

we obtain

$$ \| e^{U_{\lambda, q} + u_0} \nabla (U_{\lambda, q} + u_0) \cdot \nabla \varphi \|_{L^2(B_{\frac{3}{2}d}(q) \setminus B_{d}(q))} \leq C \lambda^{-2\eta} \| \Delta \tilde{\varphi} \rho \|_{L^2(B_2(z_i), 0)} + \| \tilde{\varphi} \rho \|_{L^2(B_2(z_i), 0))} \leq C \lambda^{-2\eta} + \frac{q}{2} \| \varphi \|_{X_{\alpha, q}}. $$

(4.36)

Therefore, the proof is complete.

□

**Proposition 4.1** If $1 \ll (\ln \lambda) \lambda^2 \ll \mu$, then there exists a fixed point $(\varphi_q, S_q) \in M_{\lambda, \mu}$ of the operator $\Psi$.

**Proof** In order to prove Proposition 4.1, it is enough to show that $\Psi$ is a contraction map from $M_{\lambda, \mu}$ to $M_{\lambda, \mu}$ due to the contraction mapping theorem. We are going to prove that $\Psi$ is a contraction map from $M_{\lambda, \mu}$ to $M_{\lambda, \mu}$ with the following two steps.

**Step 1** We claim that $\Psi(\varphi, S) \in M_{\lambda, \mu}$ for any $(\varphi, S) \in M_{\lambda, \mu}$. First, we consider $\| g_{2, \lambda, \mu}(\varphi, S) \|_{L^2(\Omega)}$.

From the definition of (4.5), we note that

$$ \| \frac{\lambda}{\mu} h(\varphi, S) \|_{L^\infty(\Omega)} \leq \frac{\lambda}{\mu} \| e^{U_{\lambda, q} + u_0} (1 + \varphi) \|_{L^\infty(\Omega)} + \frac{\lambda}{\mu} \| S \|_{L^\infty(\Omega)} $$

$$ \leq C \lambda \mu (1 + \| \varphi \|_{L^\infty(\Omega)}) + \frac{\lambda}{\mu} \| S \|_{L^\infty(\Omega)}. $$

(4.37)

This implies that for any $(\varphi, S) \in M_{\lambda, \mu}$, $\| \frac{\lambda}{\mu} h(\varphi, S) \|_{L^\infty(\Omega)} = O(1)$. 

\( \Box \) Springer
From the definition of $U_{\lambda,q}$ and $u_0$, $e^{U_{\lambda,q}} = e^{u_{\lambda,q}(y)} + O(1)$ and $u_0 = O(1)$ in $B_d(q)$. This implies that

$$
\| e^{U_{\lambda,q} + u_0} \|_{L^2(B_d(q))} = C\lambda^{-1} \| e^{u(0)} \|_{L^2(B_d(0))} \leq C\lambda^{-1}.
$$

In $\Omega \setminus B_d(q)$, we have $e^{U_{\lambda,q}} = O(\lambda^{-2M})$ and $e^{u_0} = O(1)$. This implies that

$$
\| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega \setminus B_d(q))} = O(\lambda^{-2M}).
$$

It follows that $\| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)} \leq C\lambda^{-1}$ and $\| e^{U_{\lambda,q} + u_0} \|_{L^\infty(\Omega)} = O(1)$.

From (4.7), we get that

$$
\| g_{2,\lambda,\mu} \|_{L^2(\Omega)} \leq \| \Delta \{ e^{U_{\lambda,q} + u_0}(1 + \varphi) \} \|_{L^2(\Omega)} + \| \mu^2 e^{U_{\lambda,q} + u_0}(1 + \varphi - e^{\varphi - \frac{\lambda}{\mu} h(\varphi,S)}) \|_{L^2(\Omega)}
\begin{align*}
&+ \lambda \mu \| e^{U_{\lambda,q} + u_0 + \varphi - \frac{\lambda}{\mu} h(\varphi,S)} (e^{U_{\lambda,q} + u_0}(1 + \varphi) + S - 1) \|_{L^2(\Omega)}. \\
&= \lambda \mu \| S \|_{L^2(\Omega)} + \lambda \mu \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}(1 + \| \varphi \|_{L^\infty(\Omega)})
\begin{align*}
&+ C \mu^2 \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}(\varphi - \frac{\lambda}{\mu} h(\varphi,S))^2
&\leq \lambda \mu \| S \|_{L^2(\Omega)}^2 + 2 \mu \| \varphi \|_{L^\infty(\Omega)} + \lambda \mu \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}^2 + C \mu^2 \| \varphi \|_{L^\infty(\Omega)}^2 \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}
\begin{align*}
&+ C \lambda^2 \| S \|_{L^2(\Omega)}^2 + \mu (1 + \| \varphi \|_{L^\infty(\Omega)}).
\end{align*}
\end{align*}
$$

Using Taylor’s Theorem, we see that for some $0 \leq \sigma \leq 1$,

$$
\| \mu^2 e^{U_{\lambda,q} + u_0}(1 + \varphi - e^{\varphi - \frac{\lambda}{\mu} h(\varphi,S)}) \|_{L^2(\Omega)} \leq \mu^2 \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}(1 + \varphi)
\begin{align*}
&- \frac{1}{2} \mu^2 e^{\sigma \varphi - \frac{\lambda}{\mu} h(\varphi,S)} (\varphi - \frac{\lambda}{\mu} h(\varphi,S))^2 \|_{L^2(\Omega)}
&\leq \lambda \mu \| S \|_{L^2(\Omega)} + \lambda \mu \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}(1 + \| \varphi \|_{L^\infty(\Omega)})
\begin{align*}
&+ C \mu^2 \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}(\varphi - \frac{\lambda}{\mu} h(\varphi,S))^2
&\leq \lambda \mu \| S \|_{L^2(\Omega)} + \mu \| \varphi \|_{L^\infty(\Omega)} + C \mu^2 \| \varphi \|_{L^\infty(\Omega)}^2 \| e^{U_{\lambda,q} + u_0} \|_{L^2(\Omega)}
\begin{align*}
&+ C \lambda^2 \| S \|_{L^2(\Omega)}^2 + \mu (1 + \| \varphi \|_{L^\infty(\Omega)}).
\end{align*}
\end{align*}
$$

We also obtain

$$
\begin{align*}
&\leq \lambda \mu \| S \|_{L^2(\Omega)} + \mu (1 + \| \varphi \|_{L^\infty(\Omega)}).
\end{align*}
$$

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From Lemma 4.3, we obtain that
\[
\|g_{2,\lambda,\mu}\|_{L^2(\Omega)} \leq \mu \left( 1 + \|\varphi\|_{L^\infty(\Omega)} \right) + \lambda \|\varphi\|_{X_{\alpha,q}} + \frac{\mu^2}{\lambda} \|\varphi\|_{L^2(\Omega)}^2 + \lambda \|S\|_{L^2(\Omega)} (\mu + \lambda \|S\|_{L^\infty(\Omega)}).
\] (4.41)

Therefore,
\[
\|g_{2,\lambda,\mu}(\varphi, S)\|_{L^2(\Omega)} \leq C \mu \left( 1 + \frac{\mu (\ln \lambda)^4}{\lambda^3} \right), \quad \text{for any } (\varphi, S) \in M_{\lambda,\mu}.
\] (4.42)

Next, we consider \(\|g_{1,\lambda,\mu}(\varphi, \hat{S})\|_{Y_{\alpha,q}}\), where \(\hat{S} = L_2^{-1}(g_{2,\lambda,\mu}(\varphi, S))\).

Since \(U_{\lambda,q} = u_{1,\lambda,q} + (u_{2,\lambda,q} - u_{1,\lambda,q}) (1 - \chi(|y - q|))\), we can rewrite (4.7) in \(B_{2d}(q)\) as
\[
g_{1,\lambda,\mu}(\varphi, \hat{S}) = \left[ \lambda^2 F(w_{\lambda,q}(y)) - \lambda^2 F(U_{\lambda,q} + u_0) + \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi - \lambda^2 f(U_{\lambda,q} + u_0) \varphi \right]
\]
\[
+ \left[ \lambda^2 F(U_{\lambda,q} + u_0) (1 + \varphi - e^{\varphi - \frac{\lambda}{2} h(\varphi, \hat{S})}) \right]
\]
\[
- \Delta \left[ (u_{2,\lambda,q} - u_{1,\lambda,q}) (1 - \chi(|y - q|)) \right]
\]
\[
+ \left[ \lambda^2 e^{U_{\lambda,q} + u_0 \hat{S}} + \lambda^2 e^{U_{\lambda,q} + u_0} (e^{\varphi - \frac{\lambda}{2} h(\varphi, \hat{S})} - 1) (e^{U_{\lambda,q} + u_0} \varphi + \hat{S}) \right] =: I + III + III + IV
\] (4.43)

From the decay of \(w\) and mean value theorem, there exists some \(0 \leq \sigma, \sigma' \leq 1\) such that
\[
I = \lambda^2 F(w_{\lambda,q}(y)) - \lambda^2 F(U_{\lambda,q} + u_0) + \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi
\]
\[
- \lambda^2 f(U_{\lambda,q} + u_0) \varphi
\]
\[
= -\lambda^2 f \left( \sigma w_{\lambda,q} + (1 - \sigma)(U_{\lambda,q} + u_0) \right) (u_0(y) - u_0(q))
\]
\[
+ 4\pi \mathfrak{M}(\gamma(y, q) - \gamma(q, q))
\]
\[
- \lambda^2 f' \left( \sigma' w_{\lambda,q} + (1 - \sigma')(U_{\lambda,q} + u_0) \right) (u_0(y) - u_0(q) + 4\pi \mathfrak{M}(\gamma(y, q)
\]
\[
- \gamma(q, q)) \varphi \chi(|y - q|) + O(\lambda^{-2\mathfrak{M}+2}).
\] (4.44)

This implies that
\[
\| \tilde{I} (1 + |z|)^{1+\sigma/2} \|_{L^2(B_{2d}(0))}
\]
\[
\leq C \lambda^2 \left( 1 + \|\varphi\|_{L^\infty(\Omega)} \right) \|e^{w(z)} (1 + |z|)^{1+\sigma/2} (u_0(\lambda^{-1} z + q) - u_0(q)) \|_{L^2(B_{2d}(0))}
\]
\[
+ C \lambda^2 \left( 1 + \|\varphi\|_{L^\infty(\Omega)} \right) \|e^{w(z)} (1 + |z|)^{1+\sigma/2} (\gamma(\lambda^{-1} z + q, q)
\]
\[
- \gamma(q, q)) \|_{L^2(B_{2d}(0))} + O(\lambda^{-2\mathfrak{M}+4+\sigma}).
\] (4.45)
We again apply mean value theorem to \( u_0(\lambda^{-1}z + q) - u_0(q) \) and \( \gamma(\lambda^{-1}z + q, q) - \gamma(q, q) \), then for some \( 0 \leq \sigma, \sigma' \leq 1 \) we get that

\[
|u_0(\lambda^{-1}z + q) - u_0(q)| = \lambda^{-1}|\nabla_y u_0(y)|_{y = \sigma \lambda^{-1}z + q} |z|
\]

and

\[
|\gamma(\lambda^{-1}z + q, q) - \gamma(q, q)| = \lambda^{-1}|\nabla_y \gamma(y)|_{y = \sigma \lambda^{-1}z + q} |z|.
\]

Since \( u_0 \) and \( \gamma \) are regular and \( e^{w(z)} \leq C(1 + |z|)^{-2M} \) in \( B_{2d\lambda}(0) \), we get that

\[
\| \hat{I}(1 + |z|)^{1+\alpha/2} \|_{L^2(B_{2d\lambda}(0))} \leq C\lambda.
\]

From Taylor’s Theorem, there exists some \( 0 \leq \sigma \leq 1 \) such that

\[
II = \lambda^2 F(U_{\lambda, q} + u_0) \left( 1 + \varphi - 1 - \varphi + \frac{\lambda}{\mu} h(\varphi, \hat{S}) \right) - \frac{1}{2} e^{\sigma(\varphi - \frac{\lambda}{\mu} h(\varphi, \hat{S}))} \left( \varphi - \frac{\lambda}{\mu} h(\varphi, \hat{S}) \right)^2 \]

\[
= \lambda^2 F(U_{\lambda, q} + u_0) \left( \frac{\lambda}{\mu} h(\varphi, \hat{S}) - \frac{1}{2} e^{\sigma(\varphi - \frac{\lambda}{\mu} h(\varphi, \hat{S}))} \left( \varphi - \frac{\lambda}{\mu} h(\varphi, \hat{S}) \right)^2 \right). \tag{4.46}
\]

We have

\[
\|e^{w(z)}(1 + |z|)^{1+\alpha/2} \hat{S}(\lambda^{-1}z + q) \|_{L^2(\Omega_1)}
\]

\[
= \left( \int_{B_{2d\lambda}(0)} e^{2w(z)} (1 + |z|)^{2+\alpha} \hat{S}^2 (\lambda^{-1}z + q) dz \right)^{1/2}
\]

\[
\leq C \left( \lambda^2 \int_{B_{2d\lambda}(0)} \hat{S}^2 (\lambda^{-1}z + q) \lambda^{-2} dz \right)^{1/2}
\]

\[
= C\lambda \left( \int_{B_{2d\lambda}} \hat{S}^2 (y) dy \right)^{1/2} = C\lambda \| \hat{S} \|_{L^2(B_{2d\lambda}(q))}, \tag{4.47}
\]

where \( y := \lambda^{-1}z + q \). From Theorem 4.2 and \( \|g_{2, \lambda, \mu}(\varphi, \hat{S}) \|_{L^2(\Omega)} \), we note that

\[
\| \frac{\lambda}{\mu} h(\varphi, \hat{S}) \|_{L^\infty(\Omega)} \leq \frac{\lambda}{\mu} \|e^{U_{\lambda, q} + u_0}(1 + \varphi) \|_{L^\infty(\Omega)} + \frac{\lambda}{\mu} \| \hat{S} \|_{L^\infty(\Omega)}
\]

\[
\leq C\frac{\lambda}{\mu} (1 + \| \varphi \|_{L^\infty(\Omega)}) + \frac{\lambda}{\mu} \| \hat{S} \|_{L^\infty(\Omega)} \leq 1. \tag{4.48}
\]
This implies that

\[
\lambda^{-2} \| \tilde{I} \tilde{I} (1 + |z|)^{1+\frac{\alpha}{2}} \|_{L^2(B_{2d\lambda}(0))} = \| F(\tilde{U}_q + \tilde{u}_0) \left( \frac{\lambda}{\mu} h(\tilde{\varphi}, \tilde{S}) - \frac{1}{\mu} e^{\sigma(\tilde{\varphi} - \frac{\lambda}{\mu} h(\tilde{\varphi}, \tilde{S}))} \right) (\tilde{\varphi} + \frac{\lambda}{\mu} h(\tilde{\varphi}, \tilde{S}))^2 + |z|^{1+\frac{\alpha}{2}} \|_{L^2(B_{2d\lambda}(0))} \leq \frac{C\lambda}{\mu} \| e^w(z) (1 + |z|)^{1+\alpha/2} (1 + \tilde{\varphi}) \|_{L^2(B_{2d\lambda}(0))} + \frac{C\lambda}{\mu} \| e^w(z) (1 + |z|)^{1+\alpha/2} \|_{L^2(B_{2d\lambda}(0))} + C \| e^w(z) (1 + |z|)^{1+\alpha/2} (\tilde{\varphi}^2 + \frac{\lambda^2}{\mu^2} (1 + \tilde{\varphi})^2 + \frac{\lambda^2}{\mu^2} \tilde{z}^2) \|_{L^2(B_{2d\lambda}(0))} \leq \frac{C\lambda}{\mu} (1 + \| \varphi \|_{L^\infty(\Omega)} + C \| \varphi \|_{L^\infty(\Omega)}^2 + \frac{C\lambda^2}{\mu} \| \tilde{S} \|_{L^2(B_{2d\lambda}(q))} \left( 1 + \frac{\lambda}{\mu} \| \tilde{S} \|_{L^\infty(\Omega)} \right).
\]

(4.49)

From Lemma 4.1, it follows that

\[
\| \tilde{I} \tilde{I} (1 + |z|)^{1+\frac{\alpha}{2}} \|_{L^2(B_{2d\lambda}(0))} \leq C \lambda^{-2\theta + 2} (1 + |z|)^{1+\frac{\alpha}{2}} \|_{L^2(B_{2d\lambda}(0))} \leq C \lambda^{-2\theta + 4 + \frac{\alpha}{2}}.
\]

(4.50)

Finally,

\[
\lambda^{-2} \| \tilde{I} \tilde{V} (1 + |z|)^{1+\frac{\alpha}{2}} \|_{L^2(B_{2d\lambda}(0))} \leq C \| e^w(z) (1 + |z|)^{1+\frac{\alpha}{2}} \varphi(e^{\psi - \frac{\lambda}{\mu} h} - 1) \|_{L^2(B_{2d\lambda}(0))} + C \| e^w(z) (1 + |z|)^{1+\frac{\alpha}{2}} \tilde{S}(\lambda^{-1} z + q) \|_{L^2(B_{2d\lambda}(0))}.
\]

(4.51)

Since

\[
|e^{\psi - \frac{\lambda}{\mu} h} - 1| \leq \frac{C\lambda}{\mu} + C |\varphi| + \frac{C\lambda}{\mu} |\tilde{S}|,
\]

this implies that

\[
\lambda^{-2} \| \tilde{I} \tilde{V} (1 + |z|)^{1+\frac{\alpha}{2}} \|_{L^2(B_{2d\lambda}(0))} \leq \frac{C\lambda}{\mu} \| \varphi \|_{L^\infty(\Omega)} + C \| \varphi \|_{L^\infty(\Omega)}^2 + C \lambda \| \tilde{S} \|_{L^2(B_{2d\lambda}(q))}.
\]

(4.52)
By $U_{\lambda,q} = u_{2,\lambda,q} + (u_{1,\lambda,q} - u_{2,\lambda,q}) \chi(|y - q|)$, we have in $\Omega \backslash B_d(q)$,

$$
g_{1,\lambda,\mu}(\varphi, \hat{S}) = \lambda^2 F(U_{\lambda,q} + u_0)(\varphi - e^{\varphi - \frac{1}{\mu} h(\varphi, \hat{S})}) + \lambda^2 f(U_{\lambda,q} + u_0)\varphi - \Delta [(u_{1,\lambda,q} - u_{2,\lambda,q}) \chi(|y - q|)] - \lambda^2 f(U_{\lambda,q} + u_0)\varphi + \lambda^2 e^{U_{\lambda,q} + u_0} \left(e^{U_{\lambda,q} + u_0 + \varphi - \frac{1}{\mu} h(\varphi, \hat{S})} \varphi - e^{U_{\lambda,q} + u_0} \varphi + e^{\varphi - \frac{1}{\mu} h(\varphi, \hat{S})} \hat{S}\right). \quad (4.53)
$$

Since $e^{U_{\lambda,q}} \leq c\lambda^{-2\eta}$ in $\Omega \backslash B_d(q)$, it follows that

$$
\|g_{1,\lambda,\mu}(\varphi, \hat{S})\|_{L^2(\Omega \backslash B_d(q))} \leq C\lambda^{-2\eta} + (1 + \|\varphi\|_{L^\infty(\Omega)} + \|\hat{S}\|_{L^2(\Omega \backslash B_d(q))}).
$$

Therefore, Theorem 4.2 and the assumption $\lambda^2 \ln \lambda < \mu$ yield for any $(\varphi, S) \in M_{\lambda,\mu}$

$$
\|g_{1,\lambda,\mu}(\varphi, \hat{S})\|_{\gamma_{\lambda,q}} \leq C \frac{1}{\lambda} + C \left(\frac{\lambda}{\mu} + \lambda^{-2\eta} + \frac{2}{\lambda} + \lambda^2 \|\hat{S}\|_{L^2(\Omega)} \right)(1 + \frac{\lambda^2}{\mu^2} \|\hat{S}\|_{L^\infty(\Omega)})
$$

(4.54)

From Theorems 4.1 and 4.2, the inequalities (4.42) and (4.54) yield that $\Psi(\varphi, S) \in M_{\lambda,\mu}$ for any $(\varphi, S) \in M_{\lambda,\mu}$.

**Step 2** We claim that for any $(\varphi_1, S_1)$ and $(\varphi_2, S_2)$ in $M_{\lambda,\mu}$, there exists some constant $0 < \tau < 1$ such that

$$
\|\Psi(\varphi_1, S_1) - \Psi(\varphi_2, S_2)\| < \tau \|(\varphi_1, S_1) - (\varphi_2, S_2)\|. \quad (4.55)
$$

Firstly, we see that

$$
\|g_{2,\lambda,\mu}(\varphi_1, S_1) - g_{2,\lambda,\mu}(\varphi_2, S_2)\|_{L^2(\Omega)} \
\leq \|\Delta \{e^{U_{\lambda,q} + u_0}(\varphi_1 - \varphi_2)\}\|_{L^2(\Omega)} + \|\mu^2 e^{U_{\lambda,q} + u_0}(\varphi_1 - \varphi_2 + e^{\varphi - \frac{1}{\mu} h(\varphi, S_2)})\|_{L^2(\Omega)} \
- e^{\varphi_1 - \frac{1}{\mu} h(\varphi_1, S_1)}\|_{L^2(\Omega)} \
+ \lambda \mu \|e^{U_{\lambda,q} + u_0 + \varphi_1 - \frac{1}{\mu} h(\varphi_1, S_1)}\{e^{U_{\lambda,q} + u_0}(1 + \varphi_1) + S_1 - 1\} \
- e^{U_{\lambda,q} + u_0 + \varphi_2 - \frac{1}{\mu} h(\varphi_2, S_2)}\{e^{U_{\lambda,q} + u_0}(1 + \varphi_2) + S_2 - 1\}\|_{L^2(\Omega)}
$$
By the similar way in Step 1, we can get that
\[
\|g_{2,\lambda,\mu}(\varphi_1, S_1) - g_{2,\lambda,\mu}(\varphi_2, S_2)\|_{L^2(\Omega)} = O\left(\mu \left(1 + \frac{\mu(\ln \lambda)^2}{\lambda^2}\right) \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} + \lambda \|\varphi_1 - \varphi_2\|_{X_{\alpha,q}} + \lambda \mu \|S_1 - S_2\|_{L^2(\Omega)}\right).
\]
(4.56)

Next, we consider \(\|g_{1,\lambda,\mu}(\varphi_1, \hat{S}_1) - g_{1,\lambda,\mu}(\varphi_2, \hat{S}_2)\|_{Y_{\alpha,q}}\), where \(\hat{S}_i = L_2^{-1}(g_{2,\lambda,\mu}(\varphi_i, S_i))\), \(i = 1, 2\). We see that
\[
g_{1,\lambda,\mu}(\varphi_1, \hat{S}_1) - g_{1,\lambda,\mu}(\varphi_2, \hat{S}_2) = [\lambda^2 f(u_{\lambda,q}(y)) \chi(\|y - q\|) - \lambda^2 f(U_{\lambda,q} + u_0)](\varphi_1 - \varphi_2)
+ [\lambda^2 F(U_{\lambda,q} + u_0)(\varphi_1 - \varphi_2) + e^{\frac{1}{2}h(\varphi_2, \hat{S}_2)} - e^{\frac{1}{2}h(\varphi_1, \hat{S}_1)}] + \lambda^2 e^{U_{\lambda,q} + u_0} \sum_{j=1}^N c_{q,j} Z_{q,j},
\]
(4.57)

By the similar way in Step 1, we can get that
\[
\|g_{1,\lambda,\mu}(\varphi_1, \hat{S}_1) - g_{1,\lambda,\mu}(\varphi_2, \hat{S}_2)\|_{Y_{\alpha,q}} = O\left(\frac{(\ln \lambda)^2}{\lambda} \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} + \lambda \|\hat{S}_1 - \hat{S}_2\|_{L^2(\Omega)}\right).
\]
(4.58)

In view of (4.56)–(4.58), Theorem 4.1, and Theorem 4.2, we can prove the claim (4.55) if \(1 \ll (\ln \lambda)\lambda^2 \ll \mu\).

\[\square\]

**Completion of the proof of Theorem 1.2** By Proposition 4.1, we get that for any large \(\lambda, \mu > 0\) and any \(q\) close to \(\hat{q}\), where \(\hat{q}\) is a non-degenerate critical point of \(u_0\), there are \((\varphi_q, S_q) \in M_{\lambda,\mu}\) and constants \(c_{q,j}\) such that
\[
\left\{ \begin{array}{l}
\Delta \varphi_q + \lambda^2 f(w(\lambda|y - q|)) \chi(|y - q|) \varphi_q = g_{1,\lambda,\mu}(\varphi_q, S_q) + \sum_{j=1}^N c_{q,j} Z_{q,j}, \\
\Delta S_q - \mu^2 S_q = g_{2,\lambda,\mu}(\varphi_q, S_q).
\end{array} \right.
\]
(4.59)

In the following, we will choose \(q\) suitably (depending on \(\lambda, \mu > 0\)) such that the corresponding constants \(c_{q,j}\) are zero and thus \((u_{\lambda,\mu}, N_{\lambda,\mu})\) is a solution to (1.9), where
\[
u_{\lambda,\mu} + \frac{N_{\lambda,\mu}}{\mu} = U_{\lambda,q} + \varphi_q \quad \text{and} \quad \frac{N_{\lambda,\mu}}{\lambda} = e^{U_{\lambda,q} + u_0}(1 + \varphi_q) + S_q.
\]
It is standard to prove the following.
Lemma 4.4  If

\[ \int_{\Omega} \left( \Delta \varphi_q + \lambda^2 f(w(\lambda |y - q|)) \chi(|y - q|) \varphi_q - g_{1,\lambda,\mu}(\varphi_q, S_q) \right) W_{q,j} \, dx = 0, \quad j = 1, 2, \]

(4.60)

then \( c_{q,j} = 0 \) for \( j = 1, 2 \).

Next we have the following reduced problem:

Lemma 4.5

\[ \int_{\Omega} \left( \Delta \varphi_q + \lambda^2 f(w(\lambda |y - q|)) \chi(|y - q|) \varphi_q - g_{1,\lambda,\mu}(\varphi_q, S_q) \right) W_{q,j} \, dy \\
= a_0 D_j u_0(q) + o(1) \quad \text{as} \quad \lambda, \mu \to \infty, \quad \frac{(\ln \lambda)\lambda^2}{\mu} \to 0, \quad j = 1, 2, \]

(4.61)

for some \( a_0 \neq 0 \).

Proof Since \( W_{q,j} = \chi(y - q) \frac{\partial w(\lambda |y - q|)}{\partial d_j} \), we see that

\[
\int_{\Omega} \left( \Delta \varphi_q + \lambda^2 f(w(\lambda |y - q|)) \chi(|y - q|) \varphi_q - g_{1,\lambda,\mu}(\varphi_q, S_q) \right) W_{q,j} \, dy \\
= -\lambda \int_{B_{2d_j}(0)} \left( \Delta \tilde{\varphi}_q + f(w(z)) \chi(\lambda^{-1}z) \tilde{\varphi}_q \right) \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} \, dz \\
+ \lambda \int_{B_{2d_j}(0)} [F(w(z)) - F((U_{\lambda,q} + u_0)(\lambda^{-1}z + q))] \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} \, dz \\
+ \lambda \int_{B_{2d_j}(0)} [f(w(z)) \chi(\lambda^{-1}z) - f((U_{\lambda,q} + u_0)(\lambda^{-1}z + q))] \tilde{\varphi}_q(z) \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} \, dz \\
+ \lambda \int_{B_{2d_j}(0)} [F((U_{\lambda,q} + u_0)(\lambda^{-1}z + q))(1 + \tilde{\varphi}_q - e^{\tilde{\varphi}_q - \frac{\mu}{\lambda} h(\tilde{\varphi}_q, \tilde{S}_q)})] \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} \, dz \\
+ \lambda \int_{B_{2d_j}(0)} e^{(U_{\lambda,q} + u_0)(\lambda^{-1}z + q)} [e^{(U_{\lambda,q} + u_0)(\lambda^{-1}z + q)}(e^{\tilde{\varphi}_q - \frac{\mu}{\lambda} h(\tilde{\varphi}_q, \tilde{S}_q)} - 1)\tilde{\varphi}_q \\
+ e^{\tilde{\varphi}_q - \frac{\mu}{\lambda} h(\tilde{\varphi}_q, \tilde{S}_q)} \tilde{S}_q] \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} \, dz \\
- \lambda \int_{B_{2d_j}(0)} \Delta_{\chi} [(\tilde{u}_{\lambda,q} - \tilde{u}_{1,\lambda,q})(1 - \chi(\lambda^{-1}z)) \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} \, dz \\
=: I + II + III + IV + V + VI. \]

(4.62)

We will estimate the above term by term.

Step 1 We claim that \( I = o(1) \).

Note that

\[ \Delta \left( \frac{\partial w}{\partial z_j} \right) + f(w) \frac{\partial w}{\partial z_j} = 0 \quad \text{in} \quad \mathbb{R}^2. \]

(4.63)
Together with the integration by parts, we have

\[
\lambda \int_{B_{2d}(0)} \left( \Delta_z \tilde{\varphi}_q + f(w(z)) \chi(\lambda^{-1}z) \partial w(z) \right) \chi(\lambda^{-1}z) \partial z_j dz
\]

\[
= \lambda \int_{B_{2d}(0)} \left( \Delta_z \chi(\lambda^{-1}z) \partial w(z) + f(w(z)) \chi^2(\lambda^{-1}z) \partial w(z) \right) \tilde{\varphi}_q dz
\]

\[
= \lambda \int_{B_{2d}(0)} \left( \Delta_z \chi(\lambda^{-1}z) \partial w(z) + 2 \nabla_z (\chi(\lambda^{-1}z)) \cdot \nabla_z \left( \chi(\lambda^{-1}z) \tilde{\varphi}_q \right) \right) dz
\]

\[
+ \lambda \int_{B_{2d}(0)} f(w(z)) \chi(\lambda^{-1}z) \partial w(z) \chi(\lambda^{-1}z) \tilde{\varphi}_q dz
\]

\[
= \lambda \int_{B_{2d}(0)} f(w(z)) \left( \chi(\lambda^{-1}z) - 1 \right) \partial w(z) \chi(\lambda^{-1}z) \tilde{\varphi}_q dz
\]

\[
= O(\|\varphi_q\|_{L^\infty(\Omega)}) = o(1). \tag{4.64}
\]

**Step 2** We claim that \( II = a_0 D_j u_0(q) + o(1) \) for some \( a_0 \neq 0 \).

Recall the definition of \( U_{\lambda,q} \) from (4.3), and let \( \Gamma(y) = u_0(y) + 4\pi \mathcal{M}_1 \cdot \gamma(y, q) \). From the radial symmetry and decay rate of \( w(z) \), we see that for some \( 0 \leq \tau \leq 1 \)

\[
\lambda \int_{B_{2d}(0)} \left[ F(w(z)) - F((U_{\lambda,q} + u_0)(\lambda^{-1}z + q)) \right] \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} dz
\]

\[
= \lambda \int_{B_{2d}(0)} \left[ F(w(z)) - F(w(z) + \Gamma(\lambda^{-1}z + q) - \Gamma(q)) \right] \frac{\partial w(z)}{\partial z_j} dz
\]

\[
= \chi(\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} + O(\lambda^{-2\mathcal{M}+2})
\]

\[
= \lambda \int_{B_{2d}(0)} f(w(z)) \left( \Gamma(q) - \Gamma(\lambda^{-1}z + q) \right) \frac{\partial w(z)}{\partial z_j} dz
\]

\[
= \lambda \int_{B_{2d}(0)} f(w(z)) \left( \Gamma(q) - \Gamma(\lambda^{-1}z + q) \right) \frac{\partial w(z)}{\partial z_j} dz
\]

\[
= \left( \Gamma(q) - \Gamma(\lambda^{-1}z + q) \right) \frac{\partial w(z)}{\partial z_j} \frac{\partial w(z)}{\partial z_j} + O(\lambda^{-2\mathcal{M}+2})
\]

\[
= - \left( \int_{B_{2d}(0)} f(w(z)) \nabla(\gamma) \cdot \frac{\partial w(z)}{\partial z_j} dz \right) + O(\lambda^{-1})
\]

\[
= - \left( \int_{B_{2d}(0)} f(w(z)) D_j \Gamma(q) w'(z) \frac{\partial w(z)}{\partial z_j} dz \right) + O(\lambda^{-1})
\]

\[
= - \pi D_j \Gamma(q) \left( \int_0^\infty f(w(r)) \frac{dr}{\partial r} r^2 dr \right) + O(\lambda^{-1})
\]

\[
= - \pi D_j \Gamma(q) \left( F(w(r)) r^2 \right) \int_0^\infty - 2 \int_0^\infty F(w(r)) r dr + O(\lambda^{-1})
\]

\[
= D_j \Gamma(q) \int_{\mathbb{R}^2} F(w(x)) dx + O(\lambda^{-1}). \tag{4.65}
\]

It has been known that \( D_j \Gamma(q, q) = 0 \) for any \( q \in \Omega \) and \( j = 1, 2 \), which implies \( D_j \Gamma(q) = D_j u_0(q) \). Together with \( F(w(x)) > 0 \) for all \( x \in \mathbb{R}^2 \), we prove the claim.
**Step 3** We claim that $III + IV + V + VI = o(1)$. For some $0 \leq \tau \leq 1$ we note that

$$
\left|III\right| \leq \lambda \int_{B_{2d,\lambda}(0)} |f'(\tau w + (1 - \tau)(U_{\lambda, q} + u_0))| + \lambda \|U_{\lambda, q}(\lambda^{-1}z + q) + u_0(\lambda^{-1}z + q) - w(z)\| \bar{\varphi}_q(z) (\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} dz
$$

$$
+ \lambda \int_{B_{2d,\lambda}(0)} f(w(z)) \left(\chi(\lambda^{-1}z) - 1\right) \bar{\varphi}_q(z) (\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} dz
$$

$$
\leq C \lambda \int_{B_{d,\lambda}(0)} e^{w(z)} |u_0(\lambda^{-1}z + q)) - u_0(q)| + 4\pi \mathcal{M}(\gamma(\lambda^{-1}z + q, q) - \gamma(q, q)) \bar{\varphi}_q(z) (\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} dz + o(1)
$$

$$
\leq C \lambda \int_{B_{d,\lambda}(0)} e^{w(z)} \lambda^{-1}z \bar{\varphi}_q(z) (\lambda^{-1}z) \frac{\partial w(z)}{\partial z_j} dz + o(1)
$$

$$
= O\left(\|\varphi_q\|_{L^\infty(\Omega)}\right) = o(1).
$$

(4.66)

Moreover, since $(\varphi_q, S_q)$ is a fixed point of $\Psi$, we have

$$
S_q = L_2^{-1}(g_{2, \lambda, \mu}(\varphi_q, S_q)).
$$

(4.67)

From the proof of Proposition 4.1, we know that $\|g_{2, \lambda, \mu}(\varphi_q, S_q)\|_{L^2(\Omega)} = O(\mu + \frac{\mu^2(\ln \lambda)^4}{\lambda^2})$ since $(\varphi_q, S_q) \in \mathcal{M}_{\lambda, \mu}$. From Theorem 4.2, it follows $\|S_q\|_{L^\infty(\Omega)} + \frac{1}{\mu} \|S_q\|_{L^2(\Omega)} \leq O\left(\frac{1}{\mu} + \frac{(\ln \lambda)^4}{\lambda^2}\right)$. Then, by the assumption $\lambda^2 \ln \lambda < \mu$ and the similar way in (4.66), we get that

$$
|IV| = O\left(\frac{\lambda^2}{\mu} (1 + \|\varphi_q\|_{L^\infty(\Omega)} + \|S_q\|_{L^2(\Omega)}) + \lambda \|\varphi_q\|^2_{L^\infty(\Omega)} + \frac{\lambda^4}{\mu^2} \|S_q\|_{L^\infty(\Omega)} \|S_q\|_{L^2(\Omega)}\right) = o(1).
$$

We recall $h(\varphi_q, S_q) = e^{U_{\lambda, q} + u_0}(1 + \varphi_q) + S_q$. In the estimation in (4.51), the assumption $\lambda^2 \ln \lambda < \mu$ yields that

$$
|V| = O(\lambda \|\varphi_q\|_{L^\infty(\Omega)} \left(\frac{\lambda}{\mu} + \|\varphi_q\|_{L^\infty(\Omega)} + \lambda^2 \|S_q\|_{L^2(\Omega)}\right) = o(1).
$$

(4.68)

By Lemma 4.1, we also have $|VI| = o(1)$. From the above estimates, we can derive (4.61). 

From Lemma 4.5, we can derive (4.61). Since we assume that $Du_0(\hat{q}) = 0$ and $D^2(u_0)(\hat{q})$ is nondegenerate, from Lemma 4.5, we can find a point $q$ near $\hat{q}$ such that the right hand side of (4.61) is equal to zero. Together with Lemma 4.4, we can find $q$.
satisfying $c_{q,j} = 0$ for $j = 1, 2$. At this point, we complete the proof of Theorem 1.2.

\[ \square \]

5 Proof of Theorem 1.3

In this section, we are going to construct blow up solutions of (3.2) at the vortex point satisfying $\sup u_{\lambda,\mu} \geq -c_0 > -\infty$. Based on Theorem 1.1, our construction in this section was inspired by the arguments in Sect. 4 and the construction in [43] where the authors construct blow up solutions at the vortex point using an entire solution for the Chern–Simons equation, which has a singularity, as the building blocks.

We recall the equation (3.2) as follows:

\[
\begin{aligned}
\Delta (u + \frac{N}{\mu}) &= -\lambda^2 e^{u+u_0} \left(1 - \frac{N}{\lambda}\right) + 4\pi \mathcal{M}, \\
\Delta \frac{N}{\lambda} &= \mu (\mu + \lambda e^{u+u_0}) \frac{N}{\lambda} - \mu (\lambda + \mu) e^{u+u_0} \quad \text{in } \Omega.
\end{aligned}
\]

Throughout this section, we assume that $\mathcal{M} > 4$. First of all, we are going to define the approximate solutions for (3.2). Let $V$ be the radially symmetric solution of

\[
\begin{aligned}
\Delta V + |x|^2 e^V (1 - |x|^2 e^V) &= 0 \quad \text{in } \mathbb{R}^2, \\
V'(|x|) &\to -\frac{22N}{|x|} + a_1 \frac{22N - 4}{|x|^{22N - 3}} + O \left( \frac{1}{|x|^{22N - 4}} \right), \; |x| \gg 1, \\
V(|x|) &= -22\ln |x| + I_1 - \frac{a_1}{|x|^{22N - 3}} + O \left( \frac{1}{|x|^{22N - 4}} \right), \; |x| \gg 1,
\end{aligned}
\]

(5.1)

where $a_1$ and $I_1$ are constants (see [14, Theorem 2.1, Lemma 2.6] for the existence of $V$ satisfying (5.1)). Motivated by [43,45], we set

\[
\begin{aligned}
u_{1,\lambda}(y) &= V(\lambda |y - p_1|) + 4\pi \mathcal{M} (\gamma(y, p_1) - \gamma(p_1, p_1)) + C_\lambda, \\
u_{2,\lambda}(y) &= V(d\lambda) + 4\pi \mathcal{M} (G(y, p_1) - \gamma(p_1, p_1)) + \frac{1}{2\pi} \ln d) + C_\lambda, \\
U_\lambda(y) &= u_{1,\lambda}(y) \chi(|y - p_1|) + u_{2,\lambda}(y) (1 - \chi(|y - p_1|)),
\end{aligned}
\]

(5.2)

where $C_\lambda = 2 \ln \lambda + 4\pi \left( \gamma(p_1, p_1) + \sum_{j=2}^N m_j G(p_1, p_j) \right)$. We note that $U_\lambda \in C^\infty(\Omega)$.

We would find a solution of (3.2) with the following form:

\[
\begin{aligned}
u + \frac{N}{\mu} = U_\lambda + \varphi \quad \text{and} \quad \frac{N}{\lambda} = e^{U_\lambda + u_0 + \varphi} + S,
\end{aligned}
\]

(5.3)

here $(\varphi, S)$ would be regard as an error term.

We note that $e^{U_\lambda} = O(\lambda^2)$ in (5.2), but $e^{U_{\lambda,q}} = O(1)$ in the Sect. 4. In order to control the difficulties arising from the error parts related to $\varphi^2$ term we need to make the difference between (4.4) and (5.3).

For the convenience, we also denote

\[
\begin{aligned}
V_\lambda(y) &= V(\lambda |y - p_1|), \quad h(\varphi, S) = e^{U_\lambda + u_0 + \varphi} + S, \\
F(t) &= t^2 e^{V(t)} (1 - t^2 e^{V(t)}) \quad \text{and} \quad f(t) = t^2 e^{V(t)} (1 - 2t^2 e^{V(t)}).
\end{aligned}
\]

(5.4)
The Eq. (3.2) is reduced to a system for \((\varphi, S)\):
\[
\begin{align*}
\Delta \varphi + \lambda^2 f(\lambda |x - p_1|) \chi(|x - p_1|) \varphi &= h_{1,\lambda,\mu}(\varphi, S), \\
\Delta S - \mu^2 S &= h_{2,\lambda,\mu}(\varphi, S),
\end{align*}
\] (5.5)

where
\[
\begin{align*}
h_{1,\lambda,\mu}(\varphi, S) := -\Delta U_\lambda + \lambda^2 f(\lambda |x - p_1|) \chi(|x - p_1|) \varphi + 4\pi M \\
    - \lambda^2 e^{U_\lambda + u_0 + \varphi - \frac{\lambda}{\mu} h(\varphi, S)} (1 - e^{U_\lambda + u_0 + \varphi - S}), \\
h_{2,\lambda,\mu}(\varphi, S) := -\Delta e^{U_\lambda + u_0 + \varphi} + \mu^2 \left( 1 + \frac{\lambda}{\mu} e^{U_\lambda + u_0 + \varphi - \frac{1}{\mu} h(\varphi, S)} \right) \\
    - \mu^2 \left( 1 + \frac{\lambda}{\mu} \right) e^{U_\lambda + u_0 + \varphi - \frac{1}{\mu} h(\varphi, S)}. \quad (5.6)
\end{align*}
\]

For a small constant \(0 < \alpha < \frac{1}{2}\), recall that
\[
\rho(z) = (1 + |z|)^{1+\frac{\alpha}{2}}, \quad \text{and} \quad \tilde{\rho}(z) = \frac{1}{(1 + |z|)(\ln(2 + |z|))^{1+\frac{\alpha}{2}}}. \quad (5.7)
\]

We say that \(\psi \in X_\alpha\) if
\[
\|\psi\|^2_{X_\alpha} = \|(\Delta \tilde{\psi})\rho\|^2_{L^2(B_{2d}(0))} + \|\tilde{\psi} \tilde{\rho}\|^2_{L^2(B_{2d}(0))} \\
+ \|\Delta \tilde{\psi}\|^2 + \psi^2\|_{L^1(\Omega \setminus B_d(p_1))} < +\infty \quad (5.8)
\]

where \(\tilde{\psi}(z) = \psi(\lambda^{-1}z + p_1)\), and that \(\psi \in Y_\alpha\) if
\[
\|\psi\|^2_{Y_\alpha} = \frac{1}{\lambda^4} \|\tilde{\psi} \rho\|^2_{L^2(B_{2d}(0))} + \|\psi\|^2_{L^2(\Omega \setminus B_d(p_1))} < +\infty.
\]

We note that \(\| \cdot \|_{X_\alpha}\) and \(\| \cdot \|_{Y_\alpha}\) are similar to the norms \(\| \cdot \|_{X_{\alpha,q}}\) and \(\| \cdot \|_{Y_{\alpha,q}}\) in Sect. 4, but the scaled area is different. We consider the following linearized operator
\[
L_1(\varphi) := \Delta \varphi + \lambda^2 f(\lambda |y - p_1|) \chi(|y - p_1|) \varphi.
\]

**Theorem 5.1** \(L_1\) is an isomorphism from \(X_\alpha\) to \(Y_\alpha\). Moreover, if \(w \in X_\alpha\) and \(h \in Y_\alpha\) satisfy \(L_1(w) = h\), then there is a constant \(C > 0\), independent of \(\lambda > 0\), such that
\[
\|w\|_{L^\infty(\Omega)} + \|w\|_{X_\alpha} \leq C (\ln \lambda) \|h\|_{Y_\alpha}. \quad (5.9)
\]

**Proof** In [43, Theorem B.1], Theorem 5.1 was proved for the linearized operator \(\tilde{L}_1(\varphi) := \Delta \varphi + \lambda^2 f(\lambda |y - p_1|) 1_{B_{2d}(p_1)} \varphi\). Since \(L_1\) is a small perturbation from \(\tilde{L}_1\), Theorem 5.1 also holds for our linearized operator \(L_1\). By using a similar argument for
the proof of Theorem 4.1 in the Sect. 6, we can prove Theorem 5.1. In fact, since we
do not need to consider the kernel parts for $L_\lambda$, the proof is simpler than Theorem 4.1,
and thus we skip the detail. □

Next, let us consider the corresponding nonlinear problem. We define an operator
$\Psi$ by

$$
\Psi(\varphi, S) = \left( L_1^{-1}(h_{1,\lambda,\mu}(\varphi, \hat{S})), \hat{S} \right),
$$

where $\hat{S} = L_2^{-1}(h_{2,\lambda,\mu}(\varphi, S))$, and a subset $C_{\lambda,\mu}$ of $X_\alpha \times W^{2,2}(\Omega)$ by

$$
C_{\lambda,\mu} = \left\{ (\varphi, S) \in X_\alpha \times W^{2,2}(\Omega) \mid \| (\varphi, S) \| \leq (\ln \lambda)^{-3} \right\}.
$$

where

$$
\| (\varphi, S) \| = \| \varphi \|_{L^\infty(\Omega)} + \| \varphi \|_{X_\alpha} + \frac{\lambda}{\mu^2 (\ln \lambda)^3} (\mu^2 \| S \|_{L^2(\Omega)} + \mu \| S \|_{L^\infty(\Omega)} + \| S \|_{W^{2,2}(\Omega)}).
$$

We note that if $(\varphi, S) \in C_{\lambda,\mu}$, then

$$
\| \varphi \|_{L^\infty(\Omega)} + \| \varphi \|_{X_\alpha} \leq (\ln \lambda)^{-3}, \quad \text{and} \quad \mu^2 \| S \|_{L^2(\Omega)} + \mu \| S \|_{L^\infty(\Omega)} + \| S \|_{W^{2,2}(\Omega)} \leq \frac{\mu^2}{\lambda}.
$$

The following estimation would be important for the contraction argument.

**Lemma 5.1** There exists a constant $C$ such that

$$
\| \Delta (e^{U_{\lambda,q} + u_0 + \varphi}) \|_{L^2(\Omega)} \leq C\lambda^3 (1 + \| \varphi \|_{L^\infty(\Omega)} + \| \varphi \|_{X_\alpha}).
$$

for any $(\varphi, S) \in C_{\lambda,\mu}$.

**Proof** Although we have $e^{U_{\lambda,q}} = O(1)$ from (4.3) in the Sect. 4, we note that $e^{U_{\lambda}} = O(\lambda^2)$ from (5.2). Except this observation, we can follow the arguments in the proof of Lemma 4.3, and obtain Lemma 5.1. We skip the detail. □

**Completion of the proof of Theorem 1.3** First of all, we claim that there exists a fixed point $(\hat{\varphi}, \hat{S}) \in C_{\lambda,\mu}$ of the operator $\Psi$.

As in the proof of Proposition 4.1, Lemma 5.1 and (5.2) imply that there is a constant $C > 0$ satisfying

$$
\| h_{2,\lambda,\mu} \|_{L^2(\Omega)} \leq C \left\{ \mu \lambda^4 (1 + \| \varphi \|_{L^\infty(\Omega)}) + \lambda^3 \| \varphi \|_{X_\alpha} + \lambda^3 \| S \|_{L^2(\Omega)} (\mu + \lambda \| S \|_{L^\infty(\Omega)}) \right\},
$$

(5.10)
We remark that the difference between (4.42)–(4.54) and (5.10)–(5.11) comes from the setting of solution in (4.4) and (5.3) in addition to Lemma 4.3 and Lemma 5.1. From Theorem 5.1 and 4.2, the inequalities (5.10)–(5.11) and the assumption \(1 \ll (\ln \lambda)^5 \lambda^5 \ll \mu\) yield that \(\Psi(\varphi, S) \in C_{\lambda, \mu}\) for any \((\varphi, S) \in C_{\lambda, \mu}\).

Similarly, we can also get that if \(1 \ll (\ln \lambda)^5 \lambda^5 \ll \mu\) and \((\varphi_1, S_1), (\varphi_2, S_2) \in C_{\lambda, \mu}\), then

\[
\|h_{2, \lambda, \mu}(\varphi_1, S_1) - h_{2, \lambda, \mu}(\varphi_2, S_2)\|_{L^2(\Omega)} = O(\mu\|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} + \lambda^3\|\varphi_1 - \varphi_2\|_{X_\alpha} + \lambda\mu\|S_1 - S_2\|_{L^2(\Omega)}),
\]

and

\[
\|h_{1, \lambda, \mu}(\varphi_1, S_1) - h_{1, \lambda, \mu}(\varphi_2, S_2)\|_{Y_\alpha} = O((\ln \lambda)^{-3}\|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} + \lambda\|S_1 - S_2\|_{L^2(\Omega)}).
\]

The estimations (5.12)-(5.13), Theorem 5.1, and Theorem 4.2 imply that if \((\varphi_1, S_1), (\varphi_2, S_2) \in C_{\lambda, \mu}\), then there exists a constant \(0 < \tau < 1\) satisfying

\[
\|\Psi(\varphi_1, S_1) - \Psi(\varphi_2, S_2)\|_* \leq \tau\|((\varphi_1, S_1) - (\varphi_2, S_2))\|_*.
\]

In view of the contraction mapping theorem, we get that if \(1 \ll (\ln \lambda)^5 \lambda^5 \ll \mu\), there exists \((\bar{\varphi}, \bar{S}) \in C_{\lambda, \mu}\) satisfying

\[
\begin{align*}
\Delta \bar{\varphi} + \lambda^2 f(\lambda|y - p_1|)\chi(|y - p_1|)\bar{\varphi} &= h_{1, \lambda, \mu}(\bar{\varphi}, \bar{S}), \\
\Delta \bar{S} - \mu^2 \bar{S} &= h_{2, \lambda, \mu}(\bar{\varphi}, \bar{S}).
\end{align*}
\]

We note that \((u_{\lambda, \mu}, N_{\lambda, \mu}) := \left(U_{\lambda} + \bar{\varphi} - \frac{\lambda}{\mu} (e^{U_{\lambda} + u_0 + \bar{\varphi}} + \bar{S}), \lambda (e^{U_{\lambda} + u_0 + \bar{\varphi}} + \bar{S})\right)\) satisfies the system (3.2), and thus complete the proof of Theorem 1.3. \(\square\)

6 Proof of Theorem 4.1

Once we get the estimation (4.17), then a standard argument with Fredholm alternative implies that \(Q_q L_{1,q}\) is isomorphism from \(E_q\) to \(F_q\) (for example, see the Appendix B in [44]). Therefore, for the proof of Theorem 4.1, it is enough to derive the estimation (4.17). In order to prove (4.17), we argue by contradiction, and suppose that there are \(\varphi_\lambda \in E_q, g_\lambda \in F_q\), and constants \(c_{\lambda, 1}, c_{\lambda, 2}\) satisfying
Here $q = (q_1, q_2) \in \Omega \setminus \bigcup_i \{p_i\}$.

We will derive a contradiction from (6.1) by unifying the approaches based on [44] and [42].

**Lemma 6.1** Let $\tilde{\varphi}_\lambda(y) = \varphi_\lambda(\lambda^{-1}y + q)$. Then $\tilde{\varphi}_\lambda \to 0$ in $C_{loc}(\mathbb{R}^2)$.

**Proof** We will apply the arguments in [44], and prove Lemma 6.1 with the following two steps.

**Step 1** In this step, we will prove that there is a constant $C_0 > 0$ satisfying

$$\sum_{j=1}^{2} |c_{\lambda,j}| \leq C_0 \left( \frac{\|\varphi_\lambda\|_{L^\infty(B_{2d}(q))}}{\lambda^2} \right).$$

(6.2)

We recall $W_{q,j}(x) = \chi(x - q) \frac{\partial w(\lambda|x-q|)}{\partial q_j}$, and $Z_{q,j}(x) = -\Delta W_{q,j} + \lambda^2 e^{w(\lambda|x-q|)} W_{q,j}$, $j = 1, 2$. Since $w$ and $\chi$ are radially symmetric functions, it is easy to see that if $i \neq j$, then

$$\int_{\Omega} Z_{q,j} W_{q,i} dx = 0.$$  

(6.3)

Moreover, by using the integration by parts, we also have a constant $c > 0$ satisfying

$$\int_{\Omega} Z_{q,j} W_{q,j} dx = \lambda^2 \int_{B_{2d}(0)} \left( \chi(\lambda^{-1}|z|) w'(|z|) \frac{z_j}{|z|} \right)^2 dz + e^{w(z)} \left( \chi(\lambda^{-1}|z|) w'(|z|) \frac{z_j}{|z|} \right)^2 d^2z$$

$$= \lambda^2 \left( c + o(1) \right) \text{ as } \lambda \to \infty.$$  

(6.4)
Now we multiply (6.1) by $W_{q,j}(x)$ and integrate the equation. By using (6.3), $g_{\lambda} \in F_{q}$, and the integration by parts, we see that

\begin{align*}
&c_{\lambda,j} \int_{\Omega} Z_{q,j} W_{q,j} \, dx \\
&= \int_{\Omega} (L_{1,q} \varphi_{\lambda} - g_{\lambda}) W_{q,j} \, dx = \int_{\Omega} (L_{1,q} \varphi_{\lambda}) W_{q,j} \, dx \\
&= \int_{\Omega} \varphi_{\lambda} L_{1,q} \left( \chi(x - q) \frac{\partial w}{\partial q_j} (\lambda|x - q|) \right) \, dx \\
&= \lambda \int_{B_{2d}(0)} \varphi_{\lambda}(\lambda^{-1}z + q) \left[ \frac{\partial w}{\partial z_j} \Delta[\chi(\lambda^{-1}z)] + 2[\nabla \chi(\lambda^{-1}z)] \cdot \nabla \frac{\partial w}{\partial z_j} \\
&+ \chi(\lambda^{-1}z) \left( \Delta \frac{\partial w}{\partial z_j} + F'(w) \frac{\partial w}{\partial z_j} \right) \right] \\
&= \lambda \int_{B_{2d}(0)} \varphi_{\lambda}(\lambda^{-1}z + q) \left[ \frac{\partial w}{\partial z_j} \Delta[\chi(\lambda^{-1}z)] + 2[\nabla \chi(\lambda^{-1}z)] \cdot \nabla \frac{\partial w}{\partial z_j} \right] \\
&= O \left( \max_{x \in B_{2d}(q)} |\varphi_{\lambda}(x)| \right), \\
&\text{(6.5)}
\end{align*}

here, we used $\Delta \frac{\partial w}{\partial z_j} + F'(w) \frac{\partial w}{\partial z_j} = 0$ and the property of $w'$ in (4.2). Together with (6.4), we obtain (6.2).

Step 2 We note that

\begin{align*}
\Delta \tilde{\varphi}_{\lambda} + f(w(z)) \chi(\lambda^{-1}z) \tilde{\varphi}_{\lambda} &= \lambda^{-2} g_{\lambda}(\lambda^{-1}z + q) + \lambda^{-2} \sum_{j=1}^{2} c_{\lambda,j} Z_{q,j}(\lambda^{-1}z + q). \\
&\text{(6.6)}
\end{align*}

For any $p > 1$ but close to 1, Hölder inequality and (6.1) imply that

\begin{align*}
\lambda^{-2} \|g_{\lambda}(\lambda^{-1}z + q)\|_{L^p(B_{2d}(0))} \\
&\leq \lambda^{-2} \left( \int_{B_{2d}(0)} |g_{\lambda}(\lambda^{-1}z + q)|^2 \rho^2 \right)^{1/2} (1 + |z|)^{(1+2/p)} \|L^{2p} \|_{L^{2p}(\mathbb{R}^2)} \\
&\leq C \|g_{\lambda}\|_{Y_{a,q}} = o(1) \quad \text{as} \quad \lambda \to \infty.
\end{align*}

For any $R > 0$ and $p > 1$, we see from (6.2) and $\|\varphi_{\lambda}\|_{L^\infty(\Omega)} \leq 1$ that

\begin{align*}
\lambda^{-2} \sum_{j=1}^{2} \|c_{\lambda,j} Z_{q,j}(\lambda^{-1}z + q)\|_{L^p(\Omega)} \\
&\leq C \lambda \sum_{j=1}^{2} |c_{\lambda,j}| = O \left( \frac{\|\varphi_{\lambda}\|_{L^\infty(B_{2d}(q))}}{\lambda} \right) = o(1) \quad \text{as} \quad \lambda \to \infty.
\end{align*}
In view of the $L^p$ estimate, we see that a function $\varphi$ satisfying $\tilde{\varphi}_\lambda \to \varphi$ in $C_{\text{loc}}(\mathbb{R}^2)$, and

$$\Delta \varphi + f(w)\varphi = 0 \quad \text{in} \quad \mathbb{R}^2.$$ 

In [44, Lemma B.2], it has been known that $\varphi = a_1 \frac{\partial w}{\partial z_1} + a_2 \frac{\partial w}{\partial z_2}$. Since $\varphi_\lambda \in E_q$, we can deduce

$$\int_{\mathbb{R}^2} \varphi \left( \Delta \frac{\partial w}{\partial z_j} + e^w \frac{\partial w}{\partial z_j} \right) = a_j \int_{\mathbb{R}^2} \left( \left| \nabla w \right|^2 + e^w \left| \frac{\partial w}{\partial z_j} \right|^2 \right) = 0, \quad j = 1, 2,$$ 

and thus $\varphi = 0$. Therefore, we can conclude that $\tilde{\varphi}_\lambda \to 0$ in $C_{\text{loc}}(\mathbb{R}^2)$. \hfill \qed

For a function $g$ satisfying $g(z)(1 + |z|)^{1 + \frac{\alpha}{2}} \in L^2(\mathbb{R}^2)$, we recall the following result.

**Lemma 6.2** [14] There is a constant $c > 0$, independent of $x \in \mathbb{R}^2 \setminus B_2(0)$ and $g$, such that

$$\left| \int_{\mathbb{R}^2} (\ln |x - z| - \ln |x|)g(z)dz \right| \leq C|x|^{-\frac{\alpha}{2}}(\ln |x| + 1)\|g(z)(1 + |z|)^{1 + \frac{\alpha}{2}}\|_{L^2(\mathbb{R}^2)}.$$ 

Motivated by [42], we will apply the Green representation formula and Lemma 6.2 in order to estimate the difference of the values for $\varphi_\lambda$.

**Lemma 6.3** If $2\lambda^{-1} \leq |x - q| \leq |x' - q| \leq d$, then

$$|\varphi_\lambda(x) - \varphi_\lambda(x')| \leq C_0 \left( (\lambda|x - q|)^{-\frac{\alpha}{2}}(\ln |\lambda|x - q|| + 1)\|\varphi_\lambda\|_{X_{x,q}} + |x - x'|\|\varphi_\lambda\|_{X_{x,q}} \right),$$

where $C_0 > 0$ is a constant, independent of $\lambda > 0$ and $x, x' \in B_d(q) \setminus B_{2\lambda^{-1}}(q)$.

**Proof** We will prove Lemma 6.3 with the following two steps.

**Step 1** By the Green representation formula, we have for any $x, x' \in B_d(q) \setminus B_{2\lambda^{-1}}(q)$,

$$\varphi_\lambda(x) - \varphi_\lambda(x') = \int_{\Omega} (G(x', \zeta) - G(x, \zeta)) \Delta \varphi_\lambda(\zeta) d\zeta. \quad (6.7)$$

By Hölder inequality, we see that

$$\int_{\Omega \setminus B_{2d}(q)} (G(x, \zeta) - G(x', \zeta)) \Delta \varphi_\lambda(\zeta) d\zeta + \int_{B_{2d}(q)} (R(x, \zeta) - R(x', \zeta)) \Delta \varphi_\lambda(\zeta) d\zeta \leq O(|x - x'|\|\varphi_\lambda\|_{X_{x,q}}). \quad (6.8)$$
Step 2 By the change of variables $\zeta = \lambda^{-1} z + q$, we see that

$$
\int_{B_{2d}(q)} \ln \left( \frac{|x' - \zeta|}{|x - \zeta|} \right) \Delta \varphi_{\lambda} d\zeta = \int_{B_{2d}(0)} \ln \left( \frac{|x' - q - \lambda^{-1} z|}{|x - q - \lambda^{-1} z|} \right) \Delta \tilde{\varphi}_{\lambda} dz
$$

$$
= \int_{B_{2d}(0)} \ln \left( \frac{\lambda(x' - q) - z}{\lambda(x - q) - z} \right) \Delta \tilde{\varphi}_{\lambda} dz. \quad (6.9)
$$

Let

$$
3_{\lambda, q} = \ln \left( \frac{|\lambda(x' - q)|}{|\lambda(x - q)|} \right) \int_{B_{2d}(0)} \Delta \tilde{\varphi}_{\lambda} dz = \ln \left( \frac{|x' - q|}{|x - q|} \right) \int_{B_{2d}(0)} \Delta \tilde{\varphi}_{\lambda} dz.
$$

By adding and substituting the same constant $3_{\lambda, q}$ in the last line of (6.9), we get

$$
\int_{B_{2d}(q)} \ln \left( \frac{|x' - \zeta|}{|x - \zeta|} \right) \Delta \varphi_{\lambda} d\zeta
$$

$$
= 3_{\lambda, q} + \int_{B_{2d}(0)} \left\{ \ln \left( \frac{|\lambda(x' - q) - z|}{|\lambda(x' - q)|} \right) - \ln \left( \frac{|\lambda(x - q) - z|}{|\lambda(x - q)|} \right) \right\} \Delta \tilde{\varphi}_{\lambda} dz.
$$

(6.10)

In view of $2 \leq |\lambda(x - q)| \leq |\lambda(x' - q)|$, and Lemma 6.2, we see that

$$
\int_{B_{2d}(q)} \ln \left( \frac{|x' - \zeta|}{|x - \zeta|} \right) \Delta \varphi_{\lambda} d\zeta
$$

$$
= 3_{\lambda, q} + O \left( (\lambda|x - q|)^{-\frac{\alpha}{2}} (\ln |\lambda|x - q|| + 1) \|\varphi_{\lambda}\|_{X_{\alpha, q}} \right). \quad (6.11)
$$

By (6.7)–(6.11), we prove Lemma 6.3. \qed

By Lemma 6.1 and the behavior of $w$ in (4.2), we can get

$$
\lim_{\lambda \to \infty} \int_{B_{2d}(0)} f(w(z)) \chi(\lambda^{-1} z) \tilde{\varphi}_{\lambda}(z) dz = 0. \quad (6.12)
$$

Now we will improve the estimation (6.12) as follows.

**Lemma 6.4**

(i) $\int_{B_{2d}(0)} f(w(z)) \chi(\lambda^{-1} z) \tilde{\varphi}_{\lambda}(z) dz = o(|\ln \lambda|^{-1})$.

(ii) $\int_{B_{2d}(0)} (-\Delta \tilde{\varphi}_{\lambda}(z)) dz = o(|\ln \lambda|^{-1})$.

**Proof**

(i) By integrating (6.1) over $\Omega$, we have

$$
\int_{\Omega} \lambda^2 f(w_{\lambda, q}(y)) \chi(|y - q|) \varphi_{\lambda} dy = \int_{\Omega} \left( g_{\lambda} + c_{\lambda, 1} Z_{q, 1} + c_{\lambda, 2} Z_{q, 2} \right) dy. \quad (6.13)
$$
Since \( w \) and \( \chi \) are radially symmetric functions, it is easy to see that \( \int_{\Omega} Z_{q,i} \, dy = 0 \), \( i = 1, 2 \). Moreover, in view of Hölder inequality, we also see that

\[
\left| \int_{\Omega} g_{\lambda} \, dy \right| \leq O \left( \| g_{\lambda} \|_{L^2(\Omega \setminus B_d(q))} + \lambda^{-2} \| g_{\lambda} (\lambda^{-1} z + q) \rho \|_{L^2(B_d(0))} \right) \\
\leq O \left( \| g_{\lambda} \|_{Y_{a,q}} \right).
\]

(6.14)

Since \( \int_{\Omega} \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi_{\lambda} \, dy = \int_{B_{2d\lambda}(0)} f(w(z)) \chi(\lambda^{-1} z) \widetilde{\varphi}_{\lambda}(z) \, dz \) and \( \| g_{\lambda} \|_{Y_{a,q}} = o(|\ln \lambda|^{-1}) \), we can get Lemma 6.4-(i).

(ii) Let us integrate (6.6) over \( B_{2d\lambda}(0) \). By the radial symmetric property of \( w \) and \( \chi \), and Hölder inequality, we see that

\[
\int_{B_{2d\lambda}(0)} (-\Delta \widetilde{\varphi}_{\lambda}(z)) \, dz = \int_{B_{2d\lambda}(0)} f(w(z)) \chi(\lambda^{-1} z) \widetilde{\varphi}_{\lambda}(z) \, dz + O(1)(\| g_{\lambda} \|_{Y_{a,q}}).
\]

(6.15)

By Lemma 6.4-(i) and \( \| g_{\lambda} \|_{Y_{a,q}} = o(|\ln \lambda|^{-1}) \), we get the Lemma 6.4-(ii). \( \square \)

Now we are ready to prove (4.17).

Completion of the proof for the estimation (4.17) We recall the first equation of (6.1) as follows:

\[
L_{1,q} \varphi_{\lambda} = \Delta \varphi_{\lambda} + \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi_{\lambda} = g_{\lambda} + c_{\lambda,1} Z_{q,1} + c_{\lambda,2} Z_{q,2}. \quad (6.16)
\]

Fix a small constant \( r > 0 \). By using the property of \( w \) in (4.2), we get that if \( 1 < p < 2 \), then

\[
\| \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi_{\lambda} \|_{L^p(\Omega \setminus B_r(q))} = O(\lambda^{-2+2\bar{m}}),
\]

and

\[
\| Z_{q,i} \|_{L^p(\Omega \setminus B_r(q))} = O(1).
\]

In view of Hölder inequality, we see that if \( 1 < p < 2 \), then

\[
\| \lambda^2 f(w_{\lambda,q}(y)) \chi(|y - q|) \varphi_{\lambda} \|_{L^p(\Omega \setminus B_r(q))} + \| g_{\lambda} \|_{L^p(\Omega \setminus B_r(q))} + \sum_{i=1}^{2} |c_{\lambda,i}| \| Z_{q,i} \|_{L^p(\Omega \setminus B_r(q))}
\]

\[
= O(\lambda^{-2+2\bar{m}}) + O(\| g_{\lambda} \|_{L^2(\Omega \setminus B_d(q))})
\]

\[
+ O(\| \lambda^{-2} g_{\lambda} (\lambda^{-1} z + q) \rho \|_{L^2(B_d(0) \setminus B_{\lambda}(0))}) + O \left( \sum_{i=1}^{2} |c_{\lambda,i}| \right)
\]

\[
= o(1) \text{ as } \lambda \to \infty,
\]

(6.17)

here, we used \( \bar{m} > 2 \), \( \| g_{\lambda} \|_{Y_{a,q}} = o(|\ln \lambda|^{-1}) \), and (6.2). In view of \( \| \varphi_{\lambda} \|_{L^\infty(\Omega)} \leq 1 \), (6.1), and the Sobolev imbedding theorem, we see that there is a function \( \varphi_0 \) satisfying \( \varphi_{\lambda} \to \varphi_0 \) in \( C^0_{\text{loc}}(\Omega \setminus \{q\}) \) for some \( \beta \in (0, 1) \), and

\[
\Delta \varphi_0 = 0 \quad \text{in } \Omega \setminus \{q\}.
\]

(6.18)
Since $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$, $\varphi_0$ is smooth near 0, then we can extend the equation (6.18) to $\Omega$. Then $\varphi_0 \equiv c_0$, where $c_0$ is a constant. Together with Lemma 6.1, we see that for a large constant $R > 0$ and a small constant $r > 0$, if $R\lambda^{-1} = |x - q|$ and $|x' - q| = r$, then we have $|\varphi_\lambda(x) - \varphi_\lambda(x')| = |c_0| + o(1)$ as $\lambda \to \infty$. In view of Lemma 6.3 and Lemma 6.4, we also see that if $2\lambda^{-1} \leq R\lambda^{-1} = |x - q| \leq |x' - q| = r \leq d$, then

$$|c_0| + o(1) = |\varphi_\lambda(x) - \varphi_\lambda(x')|$$

$$\leq C_0 \left( \ln \frac{|x' - q|}{|x - q|} \int_{B_{2d_\beta}(0)} \Delta \tilde{\varphi}_\lambda(z) \, dz \right) + (\lambda |x - q|)^{-\frac{a}{2}} \ln(\lambda |x - q| + 1) \|\varphi_\lambda\|_{X_{a,q}}$$

$$+ |x - x'||\varphi_\lambda\|_{X_{a,q}}$$

$$= o(1) + O(R^{-\frac{a}{2}} \ln R) + O(r).$$

(6.19)

where $o(1) \to 0$ as $\lambda \to \infty$. Since $R, r > 0$ are arbitrary constants, we get that $c_0 = 0$, and thus $\varphi_\lambda \to 0$ in $C^{0,\beta}_{loc}(\Omega \setminus \{q\})$. As in (6.19), we can see that if $2\lambda^{-1} \leq R\lambda^{-1} = |x - q| \leq |x' - q| \leq r \leq d$, then

$$o(1) + |\varphi_\lambda(x')| = |\varphi_\lambda(x) - \varphi_\lambda(x')|$$

$$\leq C_0 \left( \ln \frac{|x' - q|}{|x - q|} \int_{B_{2d_\beta}(0)} \Delta \tilde{\varphi}_\lambda(z) \, dz \right) + (\lambda |x - q|)^{-\frac{a}{2}} \ln(\lambda |x - q| + 1) \|\varphi_\lambda\|_{X_{a,q}}$$

$$+ |x - x'||\varphi_\lambda\|_{X_{a,q}}$$

$$= o(1) + O(R^{-\frac{a}{2}} \ln R) + O(r).$$

Together with Lemma 6.1, we can conclude that $\lim_{\lambda \to \infty} \|\varphi_\lambda\|_{L^\infty(\Omega)} = 0$. The definition of the norm $\|\cdot\|_{X_{a,q}}$ and the equation (6.1) imply $\lim_{\lambda \to \infty} \left( \|\varphi_\lambda\|_{L^\infty(\Omega)} + \|\varphi_\lambda\|_{X_{a,q}} \right) = 0$, which contradicts (6.1). Now we prove the estimation (4.17). \hfill \Box

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