Birkhoff’s theorem in higher derivative theories of gravity

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Abstract
In this paper, we present a class of higher derivative theories of gravity which admit Birkhoff’s theorem. In particular, we explicitly show that in this class of theories, although generically the field equations are of fourth order, under spherical (plane or hyperbolic) symmetry, all the field equations reduce to second order and have exactly the same or similar structure to those of Lovelock theories, depending on the spacetime dimensions and the order of the Lagrangian.

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1. Introduction

In general relativity, Birkhoff’s theorem states that the spherically symmetric solutions of Einstein’s equations in the vacuum are locally isometric to the Schwarzschild solution. This theorem has recently been generalized for a class of higher curvature theories namely the Lovelock theories [1]. These theories are natural generalizations of Einstein’s theory in higher dimensions. While the field equations of a general higher curvature theory involve second derivatives of the Riemann tensor, Lovelock theories share the property of Einstein gravity that no derivative of the curvature tensor arises, and hence the field equations are second order in the derivative of the metric [2]. In \(D\)-spacetime dimensions, the general Lovelock Lagrangian consists of an arbitrary linear combination of all the \(2k\)-dimensional Euler densities, where \(2k < D\). Birkhoff’s theorem in Lovelock gravity states that spherically (plane or hyperbolic) symmetric solutions of the Lovelock field equations are isometric to the corresponding static Lovelock black hole solution. The proof relies on the second-order nature of the Lovelock field equations. The admittance of Birkhoff’s theorem is related to the lack of spin-0 mode excitations in the linearized field equations. It is of fundamental importance in proving uniqueness theorems in general relativity.
Lately, there has been a great interest in other higher curvature theories of gravity coming from high-energy physics, cosmology and astrophysics. These theories generically have higher order field equations in the metric formalism. However, as of yet, none of these theories are known to admit Birkhoff’s theorem. In this paper, we present a class of higher derivative theories which includes the Lovelock theories as a subclass and admits Birkhoff’s theorem. These theories are characterized by the Lagrangian densities which give second-order traced field equations and further all the field equations reduce to second order for spherically (plane or hyperbolic) symmetric ansatz.

2. Construction of Lagrangian densities

It is well known \[3\] that in dimensions greater than or equal to 4, the most general Lagrangian density which is quadratic in the curvature and gives second-order traced field equations can be expressed as an arbitrary linear combination of the Gauss–Bonnet density and the quadratic conformal density \[3\]. The Gauss–Bonnet density being the quadratic Lovelock Lagrangian gives second-order field equations for generic metrics. The field equations obtained from the quadratic conformal density on the other hand are generically of fourth order. However, the trace of the field equations is proportional to the invariant itself, the proportionality factor being \((4 - D)/2\). This is not very surprising if one considers the variation of the action under infinitesimal conformal rescalings of the metric \(\delta g_{ab} = \Omega g_{ab}\) which gives

\[
\delta I = \int \sqrt{-g} \delta g_{ab} \delta g^{ab} = \int \delta (\sqrt{-g} \mathcal{L}) = \int \Omega \left(2 - \frac{D}{2}\right) \sqrt{-g} \mathcal{L},
\]

where \(\mathcal{L} = C_{ab}^{cd} C_{cd}^{ab}\). It turns out that in dimensions greater than or equal to 4, the Gauss–Bonnet density and the quadratic conformal density are the only two linearly independent invariants which generically give second-order traced field equations. Thus, there is a two-dimensional space of quadratic invariants with this special property in dimensions greater than or equal to 4.

Now let us consider the same question in three dimensions. One might naively be tempted to think that since in three dimensions, the conformal tensor identically vanishes and so does the Gauss–Bonnet invariant, there are no quadratic invariants which give second-order traced field equations. However, this assertion can explicitly be shown to be incorrect. One may easily check that the density

\[
\sqrt{-g} \left[R_{ab} R_{ab} - \frac{1}{8} R^2\right]
\]

does give field equations whose trace is of second order and is proportional to the density itself\[4\]. In fact this is the only quadratic invariant which has this special property in three dimensions. Let us now ask ourselves the same question for cubic densities. First let us try to guess all the linearly independent cubic invariants which have this property. Obviously, first there is the six-dimensional Euler density all of whose field equations are of second order. Next one can repeat the same argument for the quadratic conformal density to the cubic conformal densities. Note that in dimensions greater than 5, there are two independent ways of contracting three conformal tensors. Hence, in dimensions greater than or equal to 5, there are at least three independent invariants which give second-order traced field equations, namely the

\[3\] Hereafter, by a conformal density, we mean a scalar invariant constructed by contracting all the indices of a certain number of conformal tensors among themselves times \(\sqrt{-g}\).

\[4\] This density supplemented by an Einstein–Hilbert term and a cosmological constant has lately gained a lot of attention in the community as a toy model for quantum gravity in three dimensions and is known as the theory of new massive gravity \([4]\).
six-dimensional Euler density and the two independent cubic conformal densities. In [5], we
had explicitly shown that these are the only invariants which possess this property. However,
in lower dimensions, the Euler density identically vanishes and the two conformal densities
are not linearly independent. In fact, in four dimensions, the only cubic invariant which has
the property is given by the conformal density. On the other hand, in five dimensions, in
addition to the cubic conformal density, there is another independent density which shares the
property. It is given by

\[ \sqrt{-g} \left[ 24 R_{ab}^{cd} R_{cd}^{ab} R_{e}^{a} + 21 \frac{1}{4} R_{ab}^{cd} R_{cd}^{ab} R + 40 R_{ab}^{cd} R_{e}^{a} R_{e}^{c} + 320 \frac{1}{9} R_{a}^{b} R_{b}^{a} R_{c}^{e} - \frac{67}{3} R_{a}^{b} R_{b}^{a} R \right] + \frac{21}{2} R^3. \]  

(3)

This invariant is the cubic counterpart of the special quadratic density (2) in three dimensions.
Now, to generalize the case for arbitrary higher order, one needs to classify or understand the
nature of these special invariants.

Let us scrutinize the special quadratic invariant more carefully. First realize that in
dimensions 4 and higher, there are three linearly independent Riemannian invariants namely
\( R^{abcd} R_{abcd}, R_{ab} R_{ab}, \) and \( R^2. \) However, in three dimensions, only two of them are independent.
In other words, any one of the three invariants can be expressed in terms of the other two.
This identity is responsible for vanishing of both the Gauss–Bonnet density and the conformal
density\(^5\). Analogously, there are eight linearly independent cubic Riemannian invariants in
dimensions greater than or equal to 6. However, in five (and lower) dimensions, the six-
dimensional Euler density vanishes identically. In fact, this is the only independent identity
in five dimensions among the cubic scalar invariants. So, this identity is also responsible for
the linear dependence of the two conformal invariants. The same occurs for the invariants of
arbitrary higher order \( k \) in dimensions \( D = 2k - 1 \) because of the following relation:

\[ \delta_{c_{d_1} \cdots c_{d_k}} a_{b_1} \cdots a_{b_k} R_{c_{d_1}}^{a_{b_1}} \cdots R_{c_{d_k}}^{a_{b_k}} = \delta_{a_{b_1} \cdots a_{b_k}} c_{d_1} \cdots C_{c_{d_k}}^{a_{b_1}} \cdots C_{a_{b_k}}^{c_{d_1}} = 0, \]

(4)

where \( \delta \) \(^7\) is the generalized Kronecker delta. Now, in the case of quadratic invariants, when one
takes a particular linear combination of the Gauss–Bonnet density and the conformal density
in arbitrary dimensions and re-expresses the conformal invariant in terms of the Riemannian
invariants, it factorizes by \( (D - 3) \). Explicitly,

\[ R^{abcd} R_{abcd} = 4 R_{ab} R_{ab} + R^2 - C^{abcd} C_{abcd} = -\frac{D - 3}{D - 2} \left( 4 R_{ab} R_{ab} - \frac{D}{D - 1} R^2 \right). \]

(5)

Obviously, since the left-hand side is a linear combination of the Gauss–Bonnet density and the
conformal density, it gives second-order traced field equations in arbitrary dimensions. This
implies that the invariant inside the parenthesis on the right-hand side also gives second-order
traced field equations in arbitrary dimensions. However, the left-hand side vanishes in three
dimensions, whereas the term inside the parenthesis on the right-hand side does not. In fact,
in three dimensions, it gives the special invariant mentioned earlier. Similar situation arises
in the cubic case where a particular linear combination of the six-dimensional Euler density
and the two conformal invariants can be factorized by \( D - 5 \) and the remaining invariant
does not vanish in five dimensions identically. This gives the special cubic invariant in five
dimensions (3). Interestingly, this can be generalized to arbitrary higher orders when one
realizes that there is always a particular combination of the \( 2k \)-dimensional Euler density and the
\( k \)th-order conformal invariants which can be expressed as

\[ \delta_{a_{b_1} \cdots a_{b_k}} c_{d_1} \cdots C_{c_{d_k}}^{a_{b_1}} \cdots C_{a_{b_k}}^{c_{d_1}} = \]

(6)

\(^5\) Of course, this can be equivalently understood in terms of vanishing of the conformal tensor. However, here we
are concerned with scalar identities.
which when expanded (re-expressed) in terms of Riemannian invariants is factorized by 
\((D - 2k + 1)\). The remaining invariant is non-vanishing in dimensions \(D = 2k - 1\) but vanishes identically in lower dimensions. Based on this observation, we had proposed a conjecture in [5] that any \(k\)th-order Riemannian invariant which gives second-order traced field equations can be expressed as \(\sqrt{-g}\) times a linear combination of the special invariant, all the linearly independent conformal invariants and a divergence term\(^6\). Explicitly, if in 
\(D\) dimensions, there are \(N^{(k)}_{D}\) linearly independent \(k(\geq 2)\)th-order conformal invariants \(W^{(k)}_{i}\) for \(\{i = 1, \ldots, N^{(k)}_{D}\}\), then the most general action which gives second-order traced field equations is given by

\[
I^{(k)} = \int \sqrt{-g} \left( a^{(k)}_0 \mathcal{N}^{(k)} + \sum_{i=1}^{N^{(k)}_{D}} a^{(k)}_i W^{(k)}_i + \text{a divergence term} \right),
\]

where

\[
\mathcal{N}^{(k)} = \frac{1}{2k} \left( \frac{D - 2}{D - 2k + 1} \right) \xi^{c_1 \cdots c_k}_{a_1 a_2 \cdots a_k} \left\{ R^{a_1 b_1 \cdots a_k b_k} - C^{a_1 b_1 \cdots a_k b_k} \right\}.
\]

Note that in dimensions \(D \geq 2k\), the special invariant \(\mathcal{N}^{(k)}\) can be expressed as a linear combination of the \(2k\)-dimensional Euler density and all the conformal invariants, and in dimensions \(D < 2k - 1\), it vanishes identically.

Now we show that a subclass of these theories generically gives fourth-order field equations, but for spherically (plane or hyperbolic) symmetric spacetimes all the field equations reduce to second order. It is in this class where Birkhoff’s theorem can be shown to hold.

### 3. Field equations for spherically (plane or hyperbolic) symmetric spacetimes

Consider the general spherically (plane or hyperbolic) symmetric spacetimes given by the following line element:

\[
d\mathbb{x}^2 = \tilde{g}_{ij}(x) dx^i dx^j + e^{2\lambda(x)} d\Sigma^2_{\gamma}.
\]

where \(d\Sigma^2_{\gamma} = \tilde{g}_{\alpha\beta}(y) dy^\alpha dy^\beta\) is the line element of a \((D - 2)\)-dimensional space of constant curvature \(\gamma\). Let \(\tilde{\nabla}\) be the Lévi-Civitá connection on the two-dimensional space orthogonal to the constant curvature space and \(\tilde{R}\) be the corresponding scalar curvature. Then the nontrivial components of the Riemann curvature tensor and the conformal tensor are given by

\[
R^{ik}_{jl} = \frac{1}{2} \tilde{R} \delta^{ik}_{jl}, \quad R^\mu_{ij} = \tilde{B} \delta^\mu_{ij}, \quad R^{ij}_\nu = -\tilde{A}^{ij}_\nu,
\]

\[
C^{ik}_{jl} = \frac{(D - 3) \tilde{S}}{2(D - 1)} \delta^{ik}_{jl}, \quad C^\mu_{ij} = \frac{\tilde{S}}{(D - 1)(D - 2)} \delta^\mu_{ij}, \quad C^\mu_{j\nu} = -\frac{(D - 3) \tilde{S}}{2(D - 1)(D - 2)} \delta^\mu_{j\nu},
\]

where

\[
\tilde{B} = \gamma e^{-2\lambda} - (\tilde{\nabla}_m \lambda)(\tilde{\nabla}^m \lambda),
\]

\[
\tilde{A}^j = \tilde{\nabla}^j \lambda + (\tilde{\nabla} \lambda)(\tilde{\nabla}^j \lambda),
\]

\(^6\) We have checked this conjecture up to the fourth order.
Note that since all the components of the conformal tensor are a mere multiple of the function \( \tilde{S} \), each of the conformal densities \( W_m^{(k)} \)'s evaluated on the metric (9) are proportional to \( \tilde{S}^k \). Let \( W_m^{(k)} = \omega_m(D, k) \tilde{S}^k \). Then the field equations for action (7) evaluated on the metric ansatz (9) are given by

\[
G_m^{(k)j} = \frac{(D-2)!}{2(D-2k+1)!} [(D-2k+1)\tilde{B}^j B] \\
+ k \sum_{m=1}^{N_D^{(k)}} \alpha_m^{(k)} \omega_m(D, k) \frac{(D-2)\alpha_0^{(k)} \omega_0(D, k)}{2^k (D-2k+1)} \tilde{P}_j^{(k)\tilde{S}^{k-1}} \tag{15}
\]

\[
G_m^{(k)\rho} = - \frac{(D-2)!}{(D-2k-1)!} \frac{(D-2k-2)\tilde{B}^2}{(D-2k+1)!} \\
+ k (\tilde{R} - 2(D-2k-1)\tilde{A}_i^j) \tilde{B}^2 + 2k (k-1) \delta^r_{ij} \tilde{A}_i^j \tilde{A}_j^r] \\
+ k \sum_{m=1}^{N_D^{(k)}} \alpha_m^{(k)} \omega_m(D, k) \frac{(D-2)\alpha_0^{(k)} \omega_0(D, k)}{2^k (D-2k+1)} \delta^r_{ij} \tilde{Q}^{(k)\tilde{S}^{k-1}} \tag{16}
\]

\[
G_m^{(k)i} = G_m^{(k)i} = 0, \tag{17}
\]

where \( \tilde{P}_j \) and \( \tilde{Q} \) are two (related) second-order linear differential operators defined on the two-dimensional space orthogonal to the constant curvature base manifold and \( \omega_0(D, k) \) is a positive number (see the appendix). Note that the fourth derivative terms arise when the operators \( \tilde{P}_j \) and \( \tilde{Q} \) act on the function \( S^{k-1} \). However, in all the field equations, these terms are multiplied by a numerical factor which depends on the coupling constants \( \alpha_i^{(k)} \), for \( \{i = 0, \ldots, N_D^{(k)}\} \), the dimensions \( D \) and the order \( k \). Hence, if one chooses the coupling constants in such a way that this factor vanishes, then the field equations reduce to second order. In particular, choosing \( \{\alpha_0^{(k)} \neq 0, \alpha_i^{(k)}\} \) such that

\[
\sum_{m=1}^{N_D^{(k)}} \alpha_m^{(k)} \omega_m(D, k) = \frac{(D-2)\alpha_0^{(k)} \omega_0(D, k)}{2^k (D-2k+1)} \tag{18}
\]

give the same equations as in pure Lovelock gravity theories in dimensions \( D > 2k \). Note that for any dimension \( D \geq 2k \) for a given \( k \), \( N_D^{(k)} = N_2^{(k)} \) whereas \( N_2^{(k)} = N_2^{(k-1)} + 1 \) as explained previously due to identity (4). Therefore, the number of independent densities of order \( k \geq 2 \) satisfying (18) in dimensions \( D > 2k \) is \( N_2^{(k)} \). Also, note that in \( D = 2k \) or \( D < 2k - 1 \), all the field equations corresponding to the densities satisfying (18) vanish identically when evaluated on metric (9).

4. Classification of theories admitting Birkhoff’s theorem

Let us now discuss the different cases up to the first few orders. Since there are no conformal invariants of order 1, the only non-trivial action, in any dimension \( D > 2 \), is given by the Einstein–Hilbert term. Next, there is only one conformal invariant of order \( k = 2 \) in dimensions \( D > 4 \). This indicates that there is only a one-parameter family of densities in
$D > 4$ which admit Birkhoff’s theorem. So, this must be the Gauss–Bonnet density. One can explicitly check that this is indeed the case by calculating $\omega_0(D, 2)$ and $\omega_1(D, 2)$. In $D = 4$, the Gauss–Bonnet density is a topological term, and in $D < 4$, it vanishes identically and hence there are no densities quadratic in curvature which admit Birkhoff’s theorem in dimensions $D \leq 4$. Now, let us consider the cubic invariants. It turns out that there are two independent conformal invariants in dimensions $D \geq 6$. Therefore, there must be two linearly independent densities which admit Birkhoff’s theorem in $D > 6$. Obviously, a particular linear combination of these two densities gives the third-order Euler density, which in turn has second-order field equations for any metric and is already known to admit Birkhoff’s theorem. This implies that any other linearly independent combination of the two densities represents a four-derivative theory, whose field equations when evaluated on metric (9) are either the same as those of cubic Lovelock theory or are identically vanishing. On the other hand, in $D = 6$, since the third-order Euler density is a topological term, there is only one linearly independent non-trivial density which represents a four-derivative theory. However, when evaluated on metric (9), all the corresponding field equations vanish identically. Now, in five dimensions, as explained previously, there is only one independent cubic conformal invariant. Also, the third-order Euler density vanishes identically in $D = 5$. So, there is a unique cubic theory in $D = 5$, which is a four-derivative theory and admits Birkhoff’s theorem. Even though there is no cubic Lovelock theory in five dimensions, the field equations of this theory when evaluated on metric (9) have a similar structure as that of the cubic Lovelock theory in higher dimensions. This theory was first presented in [7] and Birkhoff’s theorem was proven explicitly.

So for an arbitrary order $k \geq 2$, in dimensions

- $D > 2k$: there are $N(k)_{2k}$ independent densities whose field equations are generically of fourth order but when evaluated on metric (9), either give the same field equations as those of the corresponding $k$th-order pure Lovelock theory or vanish identically;
- $D = 2k$: there are $N(k)_{2k}$ independent densities out of which one particular linear combination gives the $2k$-dimensional Euler density which is a topological term. The field equations from any other linearly independent combination are generically of fourth order but when evaluated on metric (9) vanish identically;
- $D < 2k - 1$: there are $N(k)_{2k - 1}$ independent densities whose field equations are generically of fourth order but when evaluated on metric (9) vanish identically. This is because both the special invariant (8) and the right-hand side of equation (18) vanish.
- $D = 2k - 1$: there are $N(k)_{2k - 1} = N(k)_{2k} - 1$ independent densities whose field equations are generically of fourth order but when evaluated on metric (9), either vanish identically or reduce to second order and have a similar structural form as that of pure $k$th-order Lovelock theory. In the later case, Birkhoff’s theorem can be proved in the following way.

Let us begin by fixing the coordinates in the two-dimensional spacetime with metric $\tilde{g}_{ij}$, and the gauge freedom by choosing $\tilde{\nabla} e^{(x)}$ to be a spacelike vector on $\tilde{g}_{ij}$:

$$d\tilde{s}^2 = -f(t, r) \, dt^2 + \frac{dr^2}{g(t, r)} + r^2 \, d\Sigma_\gamma.$$  \hspace{1cm} (19)

The equation $G^{(k)\gamma}_{\gamma\gamma} = 0$ implies

$$\left(\gamma - g\right)^{k-1} \left(2(\gamma - g) + krg'\right) = 0.$$ \hspace{1cm} (20)

7 Interestingly, a weaker version of Birkhoff’s theorem has been shown to hold for conformal gravity in four dimensions [6]. There it has been proved that the most general spherically symmetric solution of conformal gravity is static up to a conformal gauge transformation.

8 It was simultaneously found in [8] where it was named quasi-topological gravity.

9 The proof of Birkhoff’s theorem for timelike and null cases follows along the same lines.
while $G^{(k)\nu}_{\nu} = 0$ reduces to
\[ (\gamma - g)^{k-1} (2(\gamma - g) f + krgf') = 0, \]  
where the prime on $g$ and $f$ denotes a partial differentiation with respect to $r$. Focussing first on the non-degenerate case, i.e. $g(t,r) \neq \gamma$, from equation (20) we obtain
\[ g(t,r) = F_1(t) r^{2/k} + \gamma, \]  
where $F_1(t)$ is an arbitrary function of $t$. Replacing this in equation (21), we obtain
\[ f(t,r) = F_2(t) g(t,r). \]  
The arbitrary function $F_2(t)$ can be reabsorbed by a coordinate transformation without any loss of generality. Consequently,
\[ f(t,r) = g(t,r) = F_1(t) r^{2/k} + \gamma. \]  
Equation $G^{(k)\nu}_{\nu} = 0$ then implies $dF_1/dt = 0$, which in turn implies that $F_1$ is a constant $c$. Then the equations along the base manifold $G^{(k)\beta}_{\beta} = 0$ are satisfied without any further restriction, and we obtain the following metric:
\[ ds^2 = -(cr^{2/k} + \gamma) dt^2 + \frac{dr^2}{cr^{2/k} + \gamma} + r^2 d\Sigma_1^2, \]  
where $c$ is an integration constant. The spacetime is asymptotically locally flat. In the hyperbolic case ($\gamma = -1$), the metric describes a black hole, provided $c$ is positive. This completes the proof of Birkhoff’s theorem for the non-degenerate case.

In the degenerate case, where $g(t,r) = \gamma$, all the equations are trivially satisfied, and the metric function $f(t,r)$ is left undetermined. Birkhoff’s theorem does not hold in this case.

One can also consider actions which are non-homogenous in the order $k$. In such a case, unless there is a non-vanishing contribution to the action from invariant (8) of order $k = \frac{D+1}{2}$, the spherically (plane or hyperbolic) symmetric solutions are given by the corresponding solutions of general Lovelock theory of order $k = \left[ \frac{D+1}{2} \right]$. However, if there is such a contribution (necessarily in odd dimensions), then the general spherically (plane or hyperbolic) symmetric solution is again static but does not belong to the family of solutions of general Lovelock theory. Nevertheless, the general solution has the same structural form as that in general Lovelock theories of order $k = \frac{D+1}{2}$ and can implicitly be written in terms of an algebraic equation for the metric function $g_{tt}$. Generically, the solution is of the form
\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_1^2, \]  
where the function $f(r)$ solves the following polynomial equation:
\[ \sum_{k=0}^{[\frac{D+1}{2}]} \tilde{\alpha}^{(k)}_0 (D - 2k) r^0 - 2k - 1 (\gamma - f(r))^k = M, \]  
where $M$ is an integration constant and
\[ \tilde{\alpha}^{(k)}_0 = \frac{(D - 2)(D - 2)!}{(D - 2k + 1)!} \alpha^{(k)}_0. \]  
Therefore, in any dimension $D$, in addition to the cosmological constant, there is a $p$-parameter family of (non-trivial) Lagrangian densities which generically admit Birkhoff’s
theorem, where

\[ p = 1 + \left( \sum_{1<k<D/2} N^{(k)}_{2k} \right) + \left( \sum_{k>D/2} \left( N^{(k)}_D - 1 \right) \right) \text{ for even } D \]

\[ 1 + \left( \sum_{1<k\leq D+1} N^{(k)}_{2k} - 1 \right) + \left( \sum_{k>D+1} \left( N^{(k)}_D - 1 \right) \right) \text{ for odd } D. \]

Here, for both even and odd \( D \), the unity on the right-hand side corresponds to the Einstein–Hilbert term, whereas the second and the third term correspond to the terms in the action which generically gives fourth-order field equations but when evaluated on the metric ansatz (9) respectively reduces to second order and vanish entirely.

5. Conclusions

Our findings raise several natural questions. Firstly, is there an even wider class of theories which admit Birkhoff’s theorem? Secondly, we have seen that the solutions of the theories discussed here all have the same structural form namely those of Lovelock gravity theories. Is there any theory which admits Birkhoff’s theorem but the corresponding solution does not have the same structure? It seems to us that, if it is necessary for Birkhoff’s theorem to hold, that the field equations reduce to second order under spherical (plane or hyperbolic) symmetry, then the corresponding solutions will have the same structure.

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Appendix

The operators \( \tilde{P}_j^i \) and \( \tilde{Q} \) in (15) and (16) are second-order linear differential operators defined on the two-dimensional space orthogonal to the constant curvature base manifold and are given by

\[ \tilde{P}_j^i = \left[ \delta_j^i \left( \frac{\tilde{R}}{2} + (D-1)\tilde{\nabla}_k \tilde{\nabla}^k \lambda + (D-2)(D-1)\tilde{\nabla}_i \lambda \tilde{\nabla}^k \lambda + \tilde{\nabla}_k \tilde{\nabla}^k + (2D-3)\tilde{\nabla}_k \lambda \tilde{\nabla}^k - \frac{\tilde{S}}{2k} \right) 
- (D-2)(\tilde{\nabla}^i \tilde{\nabla}_j \lambda + D\tilde{\nabla}_i \lambda \tilde{\nabla}_j \lambda) - \tilde{\nabla}^i \tilde{\nabla}_j - (D-1)(\tilde{\nabla}^i \lambda \tilde{\nabla}_j + \tilde{\nabla}_j \lambda \tilde{\nabla}^i) \right]. \]  

(A.1)

\[ \tilde{Q} = -\frac{1}{D-2} \left[ \tilde{P}_j^i - \tilde{S} \left( 1 - \frac{D}{2k} \right) \right]. \]  

(A.2)
The numerical factor $\omega_0(D, k)$ is given by
\[
\omega_0(D, k) = \frac{(D - 2)!2^k}{(D - 2k)![(D - 2k)(D - 2k - 1) + k(k - 2)]D(D - 3)}.
\]

(A.3)

References

[1] Zegers R 2005 J. Math. Phys. 46 072502
Deser S and Franklin J 2005 Class. Quantum Grav. 22 L103
[2] Lovelock D 1971 J. Math. Phys. 12 498
[3] Farhoudi M 2009 Gen. Rel. Grav. 41 117
[4] Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. Lett. 102 201301
Bergshoeff E A, Hohm O and Townsend P K 2009 Phys. Rev. D 79 124042
[5] Oliva J and Ray S 2010 Phys. Rev. D 82 124030
[6] Riegert R J 1984 Phys. Rev. Lett. 53 315
[7] Oliva J and Ray S 2010 Class. Quantum Grav. 27 225002
[8] Myers R C and Robinson B 2010 J. High Energy Phys. JHEP08(2010)067