One-dimension cubic–quintic Gross–Pitaevskii equation in Bose-Einstein condensates
in a trap potential

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By means of new general variational method we report a direct solution for the quintic self-focusing
nonlinearity and cubic-quintic 1D Gross Pitaevskii equation (GPE) in a harmonic confined potential.

I. INTRODUCTION

Nowadays one and quasi-one dimensional Bose-Einstein condensates (BEC) are common experimental
procedures. The transition from 3D to 1D system was invoked long time ago. In general, the 3D
Gross-Pitaevskii equation (GPE) cannot be factorized into transverse and longitudinal motions, nevertheless,
under certain parameter regions we can assert that the BEC follows a 1D behavior (for a detailed discussion see
Ref. \textsuperscript{3}). In the case of the harmonic trapping potential and considering that the atoms are tightly confined in
two transverse directions, a transition to the quasi-1D description is possible. Starting with the standard 3D GPE,
employing the adiabatic approximation and using the anzat wavefunction
\[ \Psi(x, r; t) = \exp(-i\mu_0 t/\hbar)\Phi(x)\chi(r; t), \]
we can derive an effective 1D GPE, which describes the physical characteristics of the cigar-like shape condensate
\textsuperscript{4,5}

\[ \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \Phi + \frac{1}{2} m\omega^2 x^2 \Phi + g_{1D} |\Phi|^2 \Phi - g |\Phi|^4 \Phi = \mu \Phi, \tag{1} \]

where \( \mu_0 \) is the chemical potential, \( m \) is the atomic mass, \( \omega \) is the longitudinal harmonic oscillator
frequencies, and \( g_{1D} \in \mathbb{R}, g \in \mathbb{R} \) are the effective 1D nonlinear self-interaction coefficients. These
two coefficients depend on the total number \( N \) of particles in the condensate, the transverse harmonic oscillator frequency \( \omega \)
and the scattering length \( a_s \) (\( a_s > 0 \) or \( a_s < 0 \) for attractive or repulsive interatomic interaction, respectively)
by the relations \( g_{1D} = 2a_s N\hbar \omega \) and \( g = 6 \ln(4/3)g_{1D}^2 / \hbar \omega^2 \), where we have chosen for the stationary state \( \Phi \) the normalization condition
\( \int dx |\Phi|^2 = 1 \).

Equation (1) is a cubic-quintic nonlinear Schrödinger equation (NLSE) with real coefficients. The presence
of the \( -g |\Phi|^4 \) term in (1) is due to the deviation from one dimension on the longitudinal condensate dynamics,
\i.e. a residual three dimensionality on an effective one-dimensional GPE. In the case of a homogeneous medium,
\i.e. assuming that \( \omega \) is zero, the cubic-quintic NLSE was widely used to describe the physical process of an optical
medium with a nonlinear polarization including susceptibilities up to fifth order. Also, the soliton solutions
have been extensively studied in Refs. \textsuperscript{5,6}.

As we have mentioned above, Eq. (1) is the cigar-like shape approach from the 3D GPE. An important issue is
the range of validity of (1), which is directly linked to the existence and stability of set of ground states solutions of
3D NLSE. It is well known that for any value of \( a_s > 0 \) the 3D GPE does not collapse. However, for attractive
interatomic interactions, the solution is dynamically stable if and only if \( a_s \) is within the range \textsuperscript{6}

\[ \frac{N|a_s|}{a_\perp} < 0.627, \tag{2} \]

with \( a_\perp = \sqrt{\hbar/m\omega} \). Hence, the validity of the cigar-like shape approach represented by Eq. (1) is also restricted
to the constrain (2).

Rescaling to dimensionless variables
\[ \xi = x/l_o, \lambda = 2g_{1D} / (l_o \hbar \omega), \mu = 2\mu_0 / (l_o \hbar), \psi(\xi) = \Phi(\xi) / \sqrt{l_o}, \epsilon = 3 \ln(4/3) / \omega / \omega_r, \]
Eq. (1) can be cast as

\[ -\frac{d^2 \psi}{d\xi^2} + \xi^2 \psi + \lambda |\psi|^2 \psi - \epsilon \lambda^2 |\psi|^4 \psi = \mu \psi. \tag{3} \]

The main task of this paper is the implementation of a more general variational mathematical approach to solve
Eq. (3). Based on this result, we provide approximate solutions for the order parameter, the chemical potential
and minimal energy for the quintic and cubic-quintic GPEs. The paper is organized as follows. First, in Sec.
II we present the bases of our formalism, i.e., we present exact formulae for the energy and the chemical potential
as functions of relevant parameters of (3). By considering a trial function for the ground state, we derive in Sec.
III a representation for the energy, \( E_{\text{app}}(\lambda) \), and chemical potential, \( \mu_{\text{app}}(\lambda) \). Section IV is devoted to the application
of our results to get explicit approximate solutions for the quintic and also cubic-quintic NLSEs. An estimation
of the error due to the influence of the interaction

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between the axial and radial degrees of freedom on the 1D cigar-shape model is presented both graphically and analytically as function of the self-interaction parameter \( \lambda \) and the coefficient \( \varepsilon \) leading the quintic nonlinear term.

II. CUBIC-QUINTIC NONLINEAR GROSS-PITAEVSKII EQUATION

In the following we will consider a more general nonlinear Gross-Pitaevskii equation

\[
- \frac{d^2 \psi}{d \xi^2} + \varepsilon^2 \psi + a \lambda |\psi|^2 \psi - b \lambda^2 |\psi|^4 \psi = \mu \psi, \quad \xi \in \mathbb{R}, \quad (4)
\]

where \( a \geq 0 \) and \( b \) are real constants.

Let \( \mathbf{V} = \left\{ \psi \in H^1(\mathbb{R}) \mid \int \varepsilon^2 |\psi(\xi)|^2 d\xi < +\infty \right\} \) be the Hilbert space endowed \( \mathbf{H}_10 \) with the norm

\[
\| \psi \|_{\mathbf{V}} = \left[ \int \varepsilon^2 |\psi(\xi)|^2 d\xi + \int \varepsilon^2 |\psi(\xi)|^2 d\xi \right]^{1/2}
\]

and the corresponding inner product

\[
(\phi, \psi)_{\mathbf{V}} = \int \left( \frac{d\phi}{d\xi} \frac{d\psi}{d\xi} + \varepsilon^2 \phi(\xi)\psi(\xi) \right) d\xi.
\]

In \( \mathbf{V} \) we define the energy functional

\[
E_{\lambda}[\psi] = \int \left( \frac{d\psi}{d\xi} \right)^2 d\xi + \int \varepsilon^2 |\psi(\xi)|^2 d\xi + \frac{a \lambda}{2} \int |\psi(\xi)|^4 d\xi - \frac{b \lambda^2}{3} \int |\psi(\xi)|^6 d\xi. \quad (5)
\]

We denote by \( \mathcal{G}_{\lambda} \) the set of ground states of Eq. (4), i.e., the set of functions of \( \mathbf{V} \) that minimize the energy functional \( E_{\lambda}[\psi] \) under the condition

\[
Q[\psi] = \int |\psi(\xi)|^2 d\xi = 1. \quad (6)
\]

Notice that in the case of attractive interaction where \( b > 0 \), it is possible to show (by applying the Gagliardo-Nirenberg inequalities) that the set of ground states \( \mathcal{G}_{\lambda} \) is nonempty if the condition \( |\lambda| < 2/\sqrt{|b|} \) is satisfied. Hence, for \( \psi_\lambda \in \mathcal{G}_{\lambda} \) we obtain

\[
\frac{dE_{\lambda}[\psi_\lambda]}{d\lambda} = \left\langle \frac{\delta E_{\lambda}[\psi_\lambda]}{\delta \psi_\lambda} , \frac{d\psi_\lambda}{d\lambda} \right\rangle + \frac{a}{2} \|\psi_\lambda\|_4^4 - \frac{2b \lambda^2}{3} \|\psi_\lambda\|_6^6.
\]

Hence, from Eqs. (7) and (9) we obtain the useful formula

\[
\frac{dE_{\lambda}[\psi_\lambda]}{d\lambda} = \frac{a}{2} \|\psi_\lambda\|_4^4 - \frac{2b \lambda^2}{3} \|\psi_\lambda\|_6^6.
\]

Thus, the minimum energy is given by

\[
E_{\min}(\lambda) = 1 + \frac{a}{2} \int_0^\lambda \|\psi_\lambda\|_4^4 ds - \frac{2b \lambda^2}{3} \int_0^\lambda s \|\psi_\lambda\|_6^6 ds. \quad (10)
\]

Moreover, from Eq. (10) is straightforward that the chemical potential can be written as

\[
\mu_{\min}(\lambda) = E_{\min}(\lambda) + \frac{a \lambda}{2} \|\psi_\lambda\|_4^4 - \frac{2b \lambda^2}{3} \|\psi_\lambda\|_6^6.
\]

or equivalently

\[
\mu_{\min}(\lambda) = 1 + \frac{a}{2} \left[ \lambda \|\psi_\lambda\|_4^4 + \int_0^\lambda \|\psi_\lambda\|_4^4 ds \right] - \frac{2b \lambda^2}{3} \left[ \lambda^2 \|\psi_\lambda\|_6^6 + \int_0^\lambda s \|\psi_\lambda\|_6^6 ds \right]. \quad (11)
\]

It is important to remark that Eqs. (10) and (11) are exact under the condition of knowing the ground state \( \psi_\lambda \in \mathcal{G}_{\lambda} \) and therefore, independent of the method or approach we employ to get the solution of the Eq. (3).

III. APPROXIMATE FORMULAE

It is possible to show \( \delta \) that any solution of Eq. (3) belonging to the Hilbert space \( \mathbf{V} \) has the asymptotic behavior \( \exp(-\tau \xi^2) \), with \( \tau > 0 \), as \( |\xi| \to \infty \). So, to evaluate the minimal energy \( E_{\min}(\lambda) \), we can consider for the ground state the trial function
\[ \psi_\tau(\xi) = \left( \frac{2\tau}{\pi} \right)^{1/4} \exp(-\tau \xi^2). \]  \hspace{1cm} (12)

Using the function (12) and evaluating the energy functional (5), we obtain the algebraic expression

\[ E_\lambda[\psi_\tau] = (1 - \epsilon\lambda^2)\sigma^2 + \frac{1}{4\sigma^2} + \frac{a\lambda}{2\sqrt{\pi}} \]  \hspace{1cm} (13)

with \( \epsilon = 2b/(3\sqrt{3}\pi) \) and \( \sigma = 1/\sqrt{\pi} \). In the case \( \epsilon \leq 0 \) the Eq. (13) presents a global minimum in \( (0, +\infty) \) for any \( a \) and \( \lambda \in \mathbb{R} \), but for \( \epsilon > 0 \) a global minimum is guaranteed if \( |\lambda| < 1/\sqrt{\epsilon} = \sqrt{3\pi/3\sqrt{2b}}. \)

Let \( \sigma(\lambda) = \sqrt{\sigma(\lambda)} \) be the minimizer of Eq. (13), in this way the ground state solution in \( \mathcal{G}_\lambda \) can be searched considering the function \( \varphi_\lambda(\xi) := \psi_\tau(\lambda)(\xi) \). By using Eqs. (10) and (11) we obtain the approximate energy, \( E_{\text{app}}(\lambda) \), and chemical potential, \( \mu_{\text{app}}(\lambda) \), namely

\[ E_{\text{app}}(\lambda) = 1 + \frac{a}{2\sqrt{\pi}} \int_0^\lambda \sigma(s)ds - 2\epsilon \int_0^\lambda \sigma^2(s)ds \]  \hspace{1cm} (14)

and

\[ \mu_{\text{app}}(\lambda) = 1 + \frac{a}{2\sqrt{\pi}} \left[ \lambda\sigma(\lambda) + \int_0^\lambda \sigma(s)ds \right] - 2\epsilon \left[ \lambda^2\sigma^2(\lambda) + \int_0^\lambda \sigma^2(s)ds \right]. \]  \hspace{1cm} (15)

It becomes clear that \(|\lambda| < 2/\sqrt{b} < \lambda_S = \sqrt{3\pi/3\sqrt{2b}}\), ensuring that the set \( \mathcal{G}_\lambda \) is nonempty and that the functions \( E_{\text{app}}(\lambda) \) and \( \mu_{\text{app}}(\lambda) \) can be good approximations of \( E_{\text{min}}(\lambda) \) and \( \mu_{\text{min}}(\lambda) \), respectively.

IV. APPLICATIONS AND DISCUSSION OF THE RESULTS

Explicit formulae (10), (11) and the approximate relationships (14), (15) are among the main results of our work. Nevertheless, more explicit expressions for the order parameter \( \varphi_\lambda \), the energy and the chemical potential as a function of the atom-atom interaction term would be desirable. To do so, let \( \sigma_{\text{min}} = \sigma_{\text{min}}(a, \epsilon; \lambda) \) be the global minimizer of Eq. (13). Thus, Eqs. (14) and (15) with \( \sigma = \sigma_{\text{min}} \) allow to obtain the dependence \( E_{\text{app}} \) and \( \mu_{\text{app}} \) on the relevant physical parameters \( a, \epsilon \) and \( \lambda \). Depending on the values of \( \lambda \) and the sign of the parameter \( \epsilon \), we can distinguish several phases linked to both type of interaction strengths, i.e. i) pure attractive \((\lambda < 0, \epsilon > 0)\), ii) pure repulsive \((\lambda > 0, \epsilon < 0)\), and iii) a mixture \((\lambda < 0, \epsilon < 0 \text{ or } \lambda > 0, \epsilon < 0)\).

More precisely, the value of \( \tau_{\text{min}} \) \( (\sigma_{\text{min}}(\lambda) := \sqrt{\tau_{\text{min}}}) \) that minimizes the function \( \tau \mapsto E_\lambda[\psi_\tau] \) fulfill the equation

\[ (1 - \epsilon^2\lambda^2)s^4 + \frac{a\lambda}{4\sqrt{\pi}} = \frac{1}{4}. \]  \hspace{1cm} (16)

Notice that, for \( \epsilon \leq 0 \) or \( \epsilon > 0 \) under the condition \( |\lambda| < 1/\sqrt{\epsilon} \), the function (13) is strictly convex and coercive on the interval \((0, +\infty)\), and in consequence Eq. (16) has a unique solution, while for \(|\lambda| > 1/\sqrt{\epsilon} \) the function (13) is not bounded from below.

A. Quintic NLSE

Firstly and for sake of comparison, we will consider the quintic NLSE in a harmonic potential. Choosing \( a = 0 \), from (10) we have

\[ \sigma_{\varphi_5}^2 = \frac{1}{2} (1 - \epsilon^2\lambda^2)^{1/2}. \]  \hspace{1cm} (17)

Inserting (17) into Eqs. (14) and (15) with \( a = 0 \), we obtain for the approximate energy \( E_{\varphi_5} \) and chemical potential \( \mu_{\varphi_5} \) the expressions

\[ E_{\varphi_5} = (1 - \epsilon^2\lambda^2)^{1/2} \]  \hspace{1cm} (18)

and

\[ \mu_{\varphi_5} = (1 - \epsilon^2\lambda^2)^{1/2} - \frac{\epsilon\lambda^2}{(1 - \epsilon^2\lambda^2)^{1/2}}. \]  \hspace{1cm} (19)

FIG. 1: (Color online) \( \mu_{\varphi_5} - \lambda \) map for the parameter values \( \epsilon = \pm 0.06 \) and \( \pm 0.12 \). According to the signs of \( \lambda \) and \( \epsilon \) the character of the \( \mu_{\varphi_5} \), as approximated solution of Eq. (4) with \( a = 0 \), can be mapped into four zones: \( \lambda > 0, \epsilon < 0 \) -pure repulsive interaction; \( \lambda < 0, \epsilon > 0 \) -pure attractive interaction; \( \lambda > 0, \epsilon > 0 \) and \( \lambda < 0, \epsilon < 0 \) -mixture region.
Figure 2 displays the $\mu_{\varphi^5} - \lambda$ map diagram for several values of the parameter $\epsilon$. It can be seen that the reduced chemical potential $\mu_{\varphi^5} = 2\mu_0/(\hbar \omega)$ shows a strong dispersion as a function of $\lambda$, moreover and following the symmetry properties of Eq. (19), we observe that $\mu_{\varphi^5}$ increases (decreases) for the pure repulsive phase, $\lambda > 0$, $\epsilon < 0$ (pure attractive phase, $\lambda < 0$, $\epsilon > 0$), while in the mixture region an opposite behavior is reached with respect to the pair of values $(\lambda, \epsilon)$.

Following Eqs. (12) and (17) we obtain for the wave-function $\varphi_{\varphi^5, \lambda}$ the expression

$$\varphi_{\varphi^5, \lambda}(\xi) = \frac{1}{\pi (1 - \epsilon \lambda^2)^{1/2}} \exp \left[ -\frac{\xi^2}{2 (1 - \epsilon \lambda^2)^{1/2}} \right]$$

valid for $1 > \lambda^2 \epsilon$. The above obtained wave function exhibits different behavior depending on the sign of $\epsilon$ and independent of the type interaction (attractive with $\lambda < 0$ or repulsive for $\lambda > 0$). The function becomes effectively less confined for $\epsilon < 0$, i.e., $\varphi_{\varphi^5, \lambda}$ is delocalized and its maximum decreases, while for $\epsilon > 0$ the function $\varphi_{\varphi^5, \lambda}(\xi)$ gets more localized and the maximum increases as the nonlinear potential $\lambda^2 \epsilon$ increases.

B. Cubic-quintic NLSE

Although one can solve Eq. (10) numerically and to obtain from Eqs. (14) and (15) the energy $E_{\varphi^3 - \varphi^5}(\lambda, b)$ and the chemical potential $\mu_{\varphi^3 - \varphi^5}(\lambda, b)$, it will be very useful to report explicit compact approximate solution of the cubic-quintic nonlinear 1D GPE (13). Searching the solution of Eq. (10) with $a = 1$ and $b = \epsilon > 0$ as a Taylor series on $\lambda$, $\sigma_{\varphi^3 - \varphi^5} = \sum_{n=0}^{\infty} \sigma_n \lambda^n$, we get

$$\sigma_{\varphi^3 - \varphi^5} = \frac{1}{\sqrt{2}} - \frac{\lambda}{16 \sqrt{\pi}} + \frac{\sqrt{2}}{2 \pi} \left( \frac{3}{256} + \frac{\epsilon}{6 \sqrt{3}} \right) \lambda^2 - \frac{1}{2 \pi \sqrt{\pi}} \left( \frac{1}{512} + \frac{\epsilon}{12 \sqrt{3}} \right) \lambda^3 + \frac{\sqrt{2}}{3 \pi^2 \sqrt{2}} \left( \frac{45}{128} + \frac{42 \epsilon}{\sqrt{3}} \right) \lambda^4 + \ldots$$

(21)

Under the condition $|\epsilon| << 1$ and by substituting Eq. (21) in (14) and (15) we have, for $\lambda$ small enough

$$E_{\varphi^3 - \varphi^5} = E_{\varphi^3}(\lambda) + \Delta E_{\varphi^3 - \varphi^5}(\lambda, \epsilon), \quad (22)$$

$$\mu_{\varphi^3 - \varphi^5} = \mu_{\varphi^3}(\lambda) + \Delta \mu_{\varphi^3 - \varphi^5}(\lambda, \epsilon), \quad (23)$$

where $E_{\varphi^3}(\lambda)$ and $\mu_{\varphi^3}(\lambda)$ correspond to the energy and the chemical potential, respectively, for the cubic NLSE and are given, up to the 5th order, by (10).

$$E_{\varphi^3}(\lambda) = 1 + \frac{\sqrt{2}}{4 \sqrt{\pi}} - \frac{1}{\pi 64} \lambda^2 + \frac{\sqrt{2}}{12 \pi \sqrt{\pi} 256} \lambda^3 - \frac{1}{32 \pi^2 256} \lambda^4, \quad (24)$$

$$\mu_{\varphi^3}(\lambda) = 1 + \frac{\sqrt{2}}{2 \sqrt{\pi}} - \frac{1}{\pi 64} \lambda^2 + \frac{\sqrt{2}}{2 \sqrt{\pi} 256} \lambda^3 - \frac{5}{32 \pi^2 256} \lambda^4 + \frac{3 \sqrt{2}}{2 \pi^2 \sqrt{\pi} 256} \lambda^5. \quad (25)$$

The terms $\Delta E_{\varphi^3 - \varphi^5}(\lambda, \epsilon)$ and $\Delta \mu_{\varphi^3 - \varphi^5}(\lambda, \epsilon)$ tell us how much error is made by neglecting the potential $-\epsilon \lambda^2 |\psi|^4$ in the 1D cigar-like shape model. These quantities are expressed by

$$\Delta E_{\varphi^3 - \varphi^5}(\lambda, \epsilon) = \frac{\sqrt{3}}{9 \pi} \left[ -\lambda + \frac{\sqrt{2}}{8 \sqrt{\pi}} \lambda^2 - \frac{1}{32 \lambda^3} + \frac{7 \sqrt{2}}{21 \pi \sqrt{\pi} 2 \lambda^4} \epsilon \lambda \right], \quad (26)$$

and
\[ \Delta \mu_{\varphi^3 - \varphi^5}(\lambda, \varepsilon) = \frac{\sqrt{3}}{3\pi} \left[ -\lambda + \frac{\sqrt{2}}{6\sqrt{3}} \lambda^2 - \frac{5}{96\pi} \lambda^3 + \frac{7\sqrt{2}}{2^{10/3}\sqrt{3}} \lambda^4 \right] \varepsilon \lambda. \]  

Figure (2) is devoted to the calculated chemical potential \( \mu_{\varphi^3 - \varphi^5} \) using Eqs. (23), (25), and (27) as a function of \( \lambda \) for \( \varepsilon = 0, 0.05, \) and 0.1. First, by comparing Figs. (1) and (2) we see the strong qualitative difference between the two types equations here considered, quintic and cubic-quintic NLSEs. Notice that this difference is remarkable even at \( \lambda \approx 0 \). From the Fig. (2) we can assert that the cigar-like shape approximation retaining term up to \( |\varphi|^4 \) is a good approach solely for the repulsive case. The chemical potential (and also, the energy) for \( \lambda > 0 \) and \( \omega << \omega_r \) is almost independent of \( \varepsilon = 3\ln(4/3)\omega/\omega_r \), while a non-negligible contribution is reached to \( \mu_{\varphi^3 - \varphi^5} \) if the atom-atom interaction is attractive, even for very small value of \( \varepsilon \). In the former case the term \(-\varepsilon\lambda^2|\varphi|^4\) is responsible for the strong obtained dispersion compared to the \( \varepsilon = 0 \) limit. In order to understand from the physical point of view the behavior of the chemical potential on \( \lambda \), we define the effective potential

\[ U_{\text{eff}} = \xi^2 + \lambda |\varphi_\lambda|^2 - \varepsilon \lambda^2 |\varphi_\lambda|^4, \]  

where the order parameter \( \varphi_\lambda(\xi) := \psi_\lambda(\lambda)(\xi) \) has been substituted by the trial function \( \psi_\lambda(\lambda)(\xi) \) with \( \varphi_{\varphi^3 - \varphi^5}(\lambda; \varepsilon) := \sqrt{\varphi_{\varphi^3 - \varphi^5}} \). Figure (2) shows the potential \( U_{\text{eff}} \) for both considered cases, attractive (\( \lambda = -5 \)) and repulsive interatomic (\( \lambda = 5 \)) interactions. Also, in the figure is represented the values of \( \mu_{\varphi^3 - \varphi^5} \) for \( \varepsilon = 0, 0.05, \) and 0.1. It becomes clear that for a given \( \lambda > 0 \), the variation of the function \( U_{\text{eff}}(\xi; \lambda, \varepsilon) \) with respect to the parameter \( \varepsilon \) is negligible. Hence, the corresponding chemical potentials \( \mu_{\varphi^3 - \varphi^5}(\lambda, \varepsilon) \approx \mu_{\varphi^3 - \varphi^5}(\lambda, 0) \). Thus, the nonlinear potential \( \lambda |\psi|^2 \) for \( \lambda > 0 \) becomes a very good description to study the physical properties of a 1D cigar-shape BEC under the condition that the transverse harmonic oscillator frequency \( \omega_r \) is much larger than the perpendicular frequency trap \( \omega \). This result is in agreement with the conclusions of Ref. 12. As one can see from Fig. (2) the error \( \Delta \mu_{\varphi^3 - \varphi^5}(\lambda > 0, \varepsilon) \) ranges between 2-7 \% with respect to the value of \( \mu_{\varphi^3}(\lambda) \). Now, if we consider the opposite case, i.e. an attractive interaction, the behavior of \( U_{\text{eff}}(\xi; \lambda, \varepsilon) \) presents a strong localized potential and, in correspondence, the chemical potential will change drastically as the parameter \( \varepsilon \) increases. This characteristic of the effective potential \( \mu_{\varphi^3 - \varphi^5} \) for \( \lambda < 0 \) determines clearly that the term \( \lambda |\psi|^2 \) is not enough for a correct description of 1D cigar-like shape BECs. In this case, the residual three dimensionality term \(-\varepsilon\lambda^2|\varphi|^4\), for the effective 1D GPE under a harmonic trap, strongly modify the corresponding chemical potential. Small variation of the strength \( \varepsilon\lambda^2 \) leads to strong change of \( \mu_{\varphi^3 - \varphi^5} \) value and hence the ground state energy \( E_{\varphi^3 - \varphi^5} \) as well.

This peculiar behavior is related to the orbital stability of the nonlinear Schrödinger equation \( 1 \). As mentioned above, the 3D GPE presents a set of ground states which is orbitally stable for any value of the self-repulsive interaction, while for the attractive interparticle interaction regime \( (\alpha_s < 0) \) the solution does not collapse if and only if the condition \( N\alpha_s > -a_\perp 0.627 \) is fulfilled. The former criterion can be rewritten as \(-\lambda < \lambda_{SD} = 2.33/\sqrt{\varepsilon} \) and it should be compared with condition of existence and validity, for \(-\lambda < \lambda_S = 2.857/\sqrt{\varepsilon} \), of the obtained solutions Eqs. (24-27). In the inset Fig. (2) for the attractive interaction, the intersection of these two sets, \( \lambda_{SD}(\varepsilon) \cap \lambda_S(\varepsilon) \), is represented by a shaded region in the \( \lambda - \varepsilon \) diagram.

In conclusion, a new variational approach is presented, which allows to construct for the cubic-quintic GPE closed analytical expressions for the order parameter, the minimal energy, and the corresponding chemical. Focusing on the compact analytical expressions, we report the contribution of the quintic term and the systematic error of the residual 3D contribution to the 1D cigar-shape model for both attractive and repulsive interaction. By the calculations the obtained chemical potential solution highlight the strong dependence on the sign of strength \( \lambda \) and the values of quintic self-interaction parameter \( \varepsilon \).

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