The Necessary and Sufficient Conditions for Wavelet Frames in Sobolev Space Over Local Fields

Ashish Pathak*, Dileep Kumar, Guru P Singh

ABSTRACT: In this paper we construct wavelet frame on Sobolev space. A necessary condition and sufficient conditions for wavelet frames in Sobolev space are given.

Key Words: Wavelet, Wavelet frame, Local fields, Sobolev space, Fourier transform.

Contents

1 Introduction .................................................................................................................. 81
   1.1 Distributions over local fields ........................................................................... 82

2 A necessary condition of wavelet frame for $H^s(F)$ .................................................. 83

3 Sufficient conditions of wavelet frame for $H^s(F)$ ..................................................... 86

1. Introduction

Let $F$ be an algebraic field and topological space with the topological properties of non-discrete, complete, locally compact and totally disconnected. Let $F^*$ and $F^+$ are the multiplicative and additive groups of $F$ respectively. Now we define Haar measure $d\xi$ for $F^+$. Then for $\beta \neq 0 (\beta \in F)$, $d(\beta \xi)$ is also a Haar measure. Let $d(\beta \xi) = |\beta| d\xi$ and we say $|\beta|$ is the absolute value or valuation of $\beta$. Let $|0| = 0$. The valuation or absolute value has following properties:

(a) $|\xi| \geq 0$ and $|\xi| = 0$ if and only if $\xi = 0$;
(b) $|\xi \eta| = |\xi| |\eta| ;$
(c) $|\xi + \eta| \leq max(|\xi|, |\eta|)$.

The last property is called ultrametric inequality. The set $\mathcal{D} = \{\xi \in F : |\xi| \leq 1\}$ is the ring of integers in $F$ and is the unique maximal compact subring of $F$. Define $\mathfrak{P} = \{\xi \in F : |\xi| < 1\}$ The set $\mathfrak{P}$ is called the prime ideal in $F$. The prime ideal in $F$ is the unique maximal ideal in $\mathcal{D}$. Then set $\mathfrak{P}$ is principal and prime. Let $A$ be a measurable subset of $F$ and $|A| = \int_{F} \zeta_A(\xi) d\xi$, where $\zeta_A$ is the characteristic function of $A$ and $d\xi$ is the Haar measure of $F$ normalized so that $|\mathcal{D}| = 1$. Then we observe that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{P}| = q^{-1}$. Therefore for $\xi \neq 0 (\xi \in F)$, $|\xi| = q^k$ for some $k \in \mathbb{Z}$.

Define $\mathfrak{P}^k = p^k \mathcal{D} = \{\xi \in F : |\xi| \leq q^{-k}, k \in \mathbb{Z}\}$ . These are called fractional ideals. Each $\mathfrak{P}^k$ is a subgroup of $F^+$. It is to see that $\mathfrak{P}^k$ is open as well as compact. If $F$ is a local field, then there is a nontrivial, unitary, continuous character $\chi$ on $F^+$.
It can be proved that \( \mathbb{F}^+ \) is self-dual.

Let \( \chi \) be a fixed character on \( \mathbb{F}^+ \) that is trivial on \( \mathcal{D} \) but is nontrivial on \( \mathcal{P}^{-1} \).

We will define fixed character \( \chi \) for a local field of positive characteristic by
\[
\chi_n(\xi) = \chi(\eta\xi), \quad \text{for} \quad \xi, \eta \in \mathbb{F}.
\]

**Definition 1.1.** If \( g \in L^1(\mathbb{F}) \), then the Fourier transform of \( g \) is the function \( \hat{g} \) defined by
\[
\hat{g}(\eta) = \int_{\mathbb{F}} g(\xi)\overline{\chi(\xi)}d\xi = \int_{\mathbb{F}} g(\xi)\chi(-\eta\xi)d\xi.
\]

The Fourier transform in \( L^p(\mathbb{F}) \), \( 1 < p \leq 2 \), can be defined similarly as in \( L^p(\mathbb{R}) \).

The inner product is defined by
\[
\langle g, f \rangle = \int_{\mathbb{F}} g(\xi)\overline{f(\xi)}d\xi \quad \text{for} \quad f, g \in L^2(\mathbb{F}).
\]

The “natural” order on the sequence \( \{v(n) \in \mathbb{F} \}_{n=0}^{\infty} \) is described as follows.

Recall that \( \mathcal{P} \) is the prime ideal in \( \mathcal{D} \), \( \mathcal{D}/\mathcal{P} \cong GF(q) = \tau, q = p^c, p \) is a prime, \( c \) a positive integer and \( \Omega : \mathcal{D} \to \tau \) the canonical homomorphism of \( \mathcal{D} \) on to \( \tau \). Note that \( \tau = GF(q) \) is a \( c \)-dimensional vector space over \( GF(p) \subset \tau \). We choose a set \( \{1 = \epsilon_0, \epsilon_1, ..., \epsilon_{c-1}\} \subset \mathcal{D}^* = \mathcal{D}\setminus \mathcal{P} \) such that \( \{\Omega(\epsilon_k)\}_{k=0}^{c-1} \) is a basis of \( GF(q) \) over \( GF(p) \).

**Definition 1.2.** For \( k, 0 \leq k < q, k = a_0 + a_1p + ... + a_{c-1}p^{c-1}, 0 \leq a_i < p, i = 0, 1, ..., c-1, \) we define
\[
v(k) = (a_0 + a_1\epsilon_1 + ... + a_{c-1}\epsilon_{c-1})p^{-1} \quad (0 \leq k < q).
\]

For \( k = b_0 + b_1q + ... + b_sq^s, 0 \leq b_i < q, \) \( k \geq 0, \) we set
\[
v(k) = v(b_0) + p^{-1}v(b_1) + ... + p^{-s}v(b_s).
\]

Note that for \( k, l \geq 0, v(k + l) \neq v(k) + v(l) \). However, it is true that for all \( r, s \geq 0, v(rq^s) = p^{-s}v(r), \) and for \( r, s \geq 0, 0 \leq t < q^s, v(rq^s + t) = v(rq^s) + v(t) = p^{-s}v(r) + v(t) \).

We will denote \( \chi_{v(n)} \) by \( \chi_n(n \geq 0) \) and use the notation \( \mathbb{N}_0 = \{0, 1, 2, ...\} \) and \( \mathbb{N} = \{1, 2, 3, ...\} \) throughout this paper.

**1.1. Distributions over local fields**

We denote \( \mathcal{S}(\mathbb{F}) \) the spaces of all finite linear combinations of characteristics functions of ball of \( \mathbb{F} \). The Fourier transform is homeomorphism of \( \mathcal{S}(\mathbb{F}) \) onto \( \mathcal{S}(\mathbb{F}) \). The distribution space of \( \mathcal{S}(\mathbb{F}) \) is denoted by \( \mathcal{S}'(\mathbb{F}) \). The Fourier transform of \( g \in \mathcal{S}(\mathbb{F}) \) is denoted by \( \hat{g}(\omega) \) and defined by
\[
\hat{g}(\omega) = \int_{\mathbb{F}} g(\xi)\overline{\chi(\xi)}d\xi = \int_{\mathbb{F}} g(\xi)\chi(-\omega\xi)d\xi, \quad \omega \in \mathbb{F}, \quad (1.1)
\]
and the inverse Fourier transform defined by,

$$g(\xi) = \int_{\mathbb{F}} \hat{g}(\omega) \chi_{\xi}(\omega) d\omega, \quad \xi \in \mathbb{F}. \tag{1.2}$$

The Fourier transform and inverse Fourier transforms of a distributions $g \in \mathcal{S}'(\mathbb{F})$ is defined by

$$\langle \hat{g}, \varphi \rangle = \langle g, \hat{\varphi} \rangle, \quad \langle g^\nu, \varphi \rangle = \langle g, \varphi^\nu \rangle, \text{ for all } \varphi \in \mathcal{S}(\mathbb{F}). \tag{1.3}$$

**Definition 1.3. Sobolev space over local fields.**

Let $s \in \mathbb{R}$. Sobolev space over local fields denote by $H^s(\mathbb{F})$, defined by the space of all $g \in \mathcal{S}'(\mathbb{F})$ such that

$$\hat{\nu}_s^{\hat{\omega}}(\omega) \hat{g}(\omega) \in L^2(\mathbb{F}), \text{ where } \hat{\nu}^\nu(\omega) = (\text{Max}(1, |\omega|))^s.$$

We equip $H^s(\mathbb{F})$ with the inner product

$$\langle g, h \rangle_{H^s(\mathbb{F})} = \int_{\mathbb{F}} \hat{\nu}^{\hat{\omega}}(\omega) \hat{g}(\omega) \hat{h}(\omega) d\omega,$$

which induces the norm

$$||g||_{H^s(\mathbb{F})}^2 = \int_{\mathbb{F}} \hat{\nu}^{\hat{\omega}}(\omega) |\hat{g}(\omega)|^2 d\omega.$$

**Theorem 1.4.** The space $\mathcal{S}(\mathbb{F})$ is dense in $H^s(\mathbb{F})$.

*Proof.* See [15].

2. A necessary condition of wavelet frame for $H^s(\mathbb{F})$

Let $\psi \in H^s(\mathbb{F})$, $\psi_{j,k}(\xi) = q^{j/2} \psi(p^{-j}\xi - v(k))$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$. The function system $\{\psi_{j,k}(\xi)\}_{(j,k) \in \mathbb{Z} \times \mathbb{N}_0}$ a wavelet frame for $H^s(\mathbb{F})$, if there are two constants $C, D \geq 0$ such that

$$C||g||_{H^s(\mathbb{F})}^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} |\langle g, \psi_{j,k} \rangle|^2 \leq D||g||_{H^s(\mathbb{F})}^2 \tag{2.1}$$

satisfies for all $g \in H^s(\mathbb{F})$.

**Theorem 2.1.** If $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ is a wavelet frame for $H^s(\mathbb{F})$ with bounds $C$ and $D$ then

$$C \leq \hat{\nu}^\nu(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p^j \omega)|^2 \leq D \text{ a.e. } \omega \in \mathbb{F}.$$
Proof. For \( g \in S(F) \) and \( \psi \in H^s(F) \), we have
\[
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k} \rangle_{H^s(F)}|^2 = \sum_{k=0}^{\infty} \int_{F} \hat{v}^*(\omega) \hat{g}(\omega) q^\frac{j}{2} \psi(p^{-j} \omega - v(k)) d\omega|^2
\]
\[
= \sum_{k=0}^{\infty} \int_{F} \hat{v}^*(\omega) \hat{g}(\omega) q^{-\frac{j}{2}} \psi(p^{-j} \omega) \chi_k(p^{-j} \omega) d\omega|^2
\]
\[
= \sum_{k=0}^{\infty} q^{-j} \int_{F} \hat{v}^*(\omega) \hat{g}(\omega) \chi_k(p^{-j} \omega) d\omega
\]
\[
\times \left\{ \int_{F} \hat{v}^*(\omega) \hat{g}(\omega) \psi(p^{-j} \omega) \chi_k(p^{-j} \omega) d\omega \right\}
\]
\[
= \sum_{k=0}^{\infty} q^{-j} \int_{F} \int_{\mathbb{D}} \hat{v}^*(p^{-j} \omega) \hat{g}(p^{-j} \omega) \chi_k(\omega) d\omega
\]
\[
\times \left\{ \hat{v}^*(p^{-j} \omega) \hat{g}(p^{-j} \omega) \psi(\omega) \chi_k(\omega) \right\} d\omega.
\]
Since \( g \in S(F) \) so the \( \sum_{l=0}^{\infty} \) contains only finite non-zero terms and \( \chi_k(v(l)) = 1 \)
for all \( k, l \in \mathbb{N}_0 \), then we get
\[
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k} \rangle_{H^s(F)}|^2 = \sum_{k=0}^{\infty} q^{-j} \int_{F} \left( \sum_{l=0}^{\infty} \hat{v}^*(p^{-j} \omega) \hat{g}(p^{-j} \omega) \chi_k(\omega) \right)
\times \left\{ \hat{v}^*(p^{-j} \omega) \hat{g}(p^{-j} \omega) \psi(\omega) \chi_k(\omega) \right\} d\omega.
\]
\[
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k} \rangle_{H^s(F)}|^2 = \int_{F} \hat{v}^*(\omega) \hat{g}(\omega) \psi(p^j \omega) \left( \sum_{k=0}^{\infty} v^*(\omega + p^{-j} v(k)) \chi_k(\omega) \right)
\times \left\{ \hat{v}^*(p^j \omega) \hat{g}(p^j \omega) \psi(p^j \omega) \chi_k(\omega) \right\} d\omega.
\]
By the convergence theorem of Fourier Series on \( \mathcal{D} \), we get
\[
\sum_{k=0}^{\infty} |\langle g, \psi_{j,k} \rangle_{H^s(F)}|^2 = \int_{F} \hat{v}^*(\omega) \hat{g}(\omega) \psi(p^j \omega) \left( \sum_{k=0}^{\infty} v^*(\omega + p^{-j} v(k)) \chi_k(\omega) \right)
\times \left\{ \hat{v}^*(p^j \omega) \hat{g}(p^j \omega) \psi(p^j \omega) \chi_k(\omega) \right\} d\omega.
\]
Let \( A_j \) be the set of regular point of \( \hat{v}^*(\omega) \hat{p}(p^j \omega) \), so for all \( \omega \in A_j \)
\[
q^{-j} \int_{\omega = \omega_0}^{\infty} \hat{v}^*(\omega) |\hat{p}(p^j \omega)|^2 d\omega \rightarrow \hat{v}^*(\omega_0) |\hat{p}(p^j \omega_0)|^2, \quad \text{as} \ l \rightarrow +\infty.
\]
If \( A = \bigcup_{j \in \mathbb{Z}} A_j \), then \( |A| = 0 \).
Suppose that \( \omega_0 \in F - A \). So for each fixed positive integer \( M \), set
\[ \hat{g}(\omega) = \frac{q^k \varphi_l(\omega - \omega_0)}{\hat{\psi}(\omega)} \quad \text{for all } l \geq M, \]

where \( \varphi_l \) is the characteristic function of \( \omega_0 + \mathbb{P}^l \). Then for \( l \in \mathbb{N} \) and \( j \geq -M \), \( \hat{g}(\omega) \hat{g}(\omega + p^{-j}v(l)) = 0 \). Since \( \omega \) and \( \omega_0 + \mathbb{P}^m \) cannot be in \( \omega_0 + \mathbb{P}^m \) simultaneously. Now, we have

\[ \sum_{j \geq -M} \sum_{k=0}^{\infty} |(g, \psi_{j,k})|^2 = \sum_{j \geq -M} \int_{\mathbb{P}^l} q^j \hat{v}^s(\omega) |\hat{\psi}(p^j \omega)|^2 d\omega \leq D. \quad (2.3) \]

Let \( l \to +\infty \) and \( M \to +\infty \), we have

\[ \sum_{j \in \mathbb{Z}} q^j \hat{v}^s(\omega_0) |\hat{\psi}(p^j \omega_0)|^2 \leq D. \quad (2.4) \]

To prove the left hand inequality,

\[ \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} |(g, \psi_{j,k})|^2 = T_1 + T_2, \quad (2.5) \]

where

\[ T_1 = \sum_{j \geq -M} \sum_{k \in \mathbb{N}_0} |(g, \psi_{j,k})|^2 \quad \text{and} \quad T_2 = \sum_{j \leq -M} \sum_{k \in \mathbb{N}_0} |(g, \psi_{j,k})|^2. \]

By condition of frame, \( T_1 \geq C - T_2 \). Since we have already show that

\[ T_1 = \sum_{j \geq -M} \hat{v}^s(\omega_0) |\hat{\psi}(p^{-j} \omega_0)|^2. \]

So we only need to show that \( T_2 \to 0 \) as \( M \to \infty \). Now, using the fact \( \mathscr{F}(\mathbb{F}) \) is dense in \( H^s(\mathbb{F}) \) in (2.2) and Schwarz’s inequality, we have

\[ T_2 \leq \sum_{j \geq -M} \sum_{k=0}^{\infty} \left\{ \int_{\mathbb{P}} \hat{v}^s(\omega) |\hat{f}(\omega)|^2 |\hat{\psi}(p^j \omega)|^2 d\omega \right\}^{\frac{1}{2}} \times \left\{ \int_{\mathbb{P}} \hat{v}^s(\omega + p^{-j}v(k)) |\hat{f}(\omega + p^{-j}v(k))|^2 |\hat{\psi}(p^j \omega + v(k))|^2 d\omega \right\}^{\frac{1}{2}}. \]

where \( \hat{g} = (\hat{v}^{-s} \hat{f}) \) and \( \hat{f} \in \mathscr{F}(\mathbb{F}) \).

Since \( \hat{f} \in \mathscr{F}(\mathbb{F}) \), so there exists a characteristic function \( \varphi_r(\omega - \omega_0) \) of the set \( \omega_0 + \mathbb{P}^r \), where \( r \) is some integers. Now \( \hat{f} \) can be written as \( \hat{f}(\omega) = q^{s-r} \varphi_r(\omega - \omega_0) \).

If \( \omega + p^{-j}v(k) \in \omega_0 + \mathbb{P}^m \), then \( |p^{-j}v(k)| \leq q^{-r} \), hence \( |v(k)| \leq q^{-r-j}. \) Then summation index \( k \) is bounded by \( q^{-r-j}. \) So using this, we get

\[ T_2 \leq q^{-r} \int_{p^{-j} \omega_0 + \mathbb{P}^{-j+r}} \hat{v}^s(p^{-j} \omega) |\hat{\psi}(\omega)|^2 d\omega. \]
Suppose that $\omega_0 \neq 0$. For any $\epsilon > 0$, choose $J < 0$ enough small satisfies the following two inequalities: $q^J < |\omega_0| = q^\rho$ such that $J + \rho < 0$ and $\int_{\mathbb{R}} |\hat{\varphi}^{(p^{-J}\omega)}| |\hat{\psi}(\omega)|^2 d\omega < \epsilon$.

We have

$$p^{-j}\omega_0 + \mathfrak{p}^{-j+\rho} \subset \mathfrak{p}^{-J-\rho} \text{ for all } j \leq J.$$  (2.6)

Since $|p^{-j}\omega_0| = q^j q^\rho \leq q^J q^\rho$ and $\mathfrak{p}^{-j+\rho} \subset \mathfrak{p}^{-J-\rho}$.

Hence, $T_2 \to 0$ as $j \to -\infty$. Therefore there exists $j$ such that $T_2 < \epsilon$.

Hence we obtain required result. $\Box$

3. Sufficient conditions of wavelet frame for $H^s(\mathbb{F})$

To find the sufficient conditions of wavelet frame for $H^s(\mathbb{F})$, we need the following Lemma

Lemma 3.1. Let $g$ be in $\mathcal{S}(\mathbb{F})$ and $\psi \in H^s(\mathbb{F})$. If $\sup \{\hat{\nu}^s(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p_j^l \omega)|^2 : \omega \in \mathfrak{p}^{-1}\setminus \mathfrak{D} \} < +\infty$, then

$$\sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} |\langle g, \psi_{j,k} \rangle_{H^s(\mathbb{F})}|^2 = \int_{\mathbb{F}} \hat{\nu}^s(\omega) |\hat{g}(\omega)|^2 |\hat{\nu}^s(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p_j^l \omega)|^2 d\omega + T_2,$$  (3.1)

where

$$T_2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) |\hat{g}(\omega)|^2 |\hat{\nu}^s(\omega + p^{-J} v(l)) \hat{\psi}(p_j^l \omega + v(l))| \hat{\psi}(p_j^l \omega + v(l)) d\omega.$$  (3.2)

Then iterated series in (3.2) is absolutely convergent.

Proof. Since $g \in \mathcal{S}(\mathbb{F})$ so the $\sum_{l=0}^{\infty}$ in (3.2) contains only finite non-zero terms.

Hence,

$$\sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) |\hat{g}(\omega)|^2 |\hat{\nu}^s(\omega + p^{-J} v(l)) \hat{\psi}(p_j^l \omega + v(l))| \hat{\psi}(p_j^l \omega + v(l)) d\omega$$

$$= \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) |\hat{g}(\omega)|^2 |\hat{\nu}^s(\omega + p^{-J} v(l)) \hat{\psi}(p_j^l \omega + v(l))| \hat{\psi}(p_j^l \omega + v(l)) d\omega.$$  (3.3)
The Necessary and Sufficient Conditions for Wavelet Frames in $H^s(\mathbb{F})$

We claim that,

$$\sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} |(g, \psi_{j,k})_{H^s(\mathbb{F})}|^2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{F}} |\hat{g}(\omega)|^2 \hat{\nu}^s(\omega) |\hat{\nu}^s(\omega)|^2 d\omega + T_2 \quad (3.4)$$

holds for all $g \in \mathcal{S}(\mathbb{F})$. We have

$$\sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} |(g, \psi_{j,k})_{H^s(\mathbb{F})}|^2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{F}} \hat{\nu}^{2s}(\omega) |\hat{g}(\omega)|^2 |\hat{\nu}^s(\omega)|^2 d\omega + T_2, \quad (3.5)$$

where

$$T_2 = \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) \hat{\nu}(\omega) \hat{\nu}^s(\omega + p^{-j}v(l)) \hat{\nu}^s(\omega + p^{-j}v(l)) d\omega \quad (3.6)$$

By using the condition $\sup \{\hat{\nu}^s(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p^j \omega)|^2 : \omega \in \mathbb{P}^{-1} \setminus \mathcal{D} \} < +\infty$ and Levi’s Lemma for integral, we get

$$\sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} |(g, \psi_{j,k})_{H^s(\mathbb{F})}|^2 = \int_{\mathbb{F}} \hat{\nu}^s(\omega) |\hat{g}(\omega)|^2 \hat{\nu}^s(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p^j \omega)|^2 d\omega + T_2. \quad (3.7)$$

Now, we show that series (3.6) is absolutely convergent.

$$|T_2| \leq \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) \hat{\nu}(\omega) \hat{\nu}^s(\omega + p^{-j}v(l)) \hat{\nu}(\omega + p^{-j}v(l)) \hat{\nu}^s(\omega + p^{-j}v(l)) \hat{\psi}(p^j \omega + v(l)) d\omega$$

$$\leq \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) |\hat{g}(\omega)| |\hat{\nu}^s(\omega + p^{-j}v(l))| \hat{\nu}(\omega + p^{-j}v(l)) |\hat{\nu}^s(\omega + p^{-j}v(l))|^2$$

$$\times |\hat{\psi}(p^j \omega + v(l))|^2 d\omega. \quad (3.8)$$

Since $g \in \mathcal{S}(\mathbb{F})$, there exist a constant $J > 0$ such that for all $|j| > J$

$$\hat{g}(p^{-j} \omega) \hat{g}(p^{-j} \omega + p^{-j}v(l)) = 0. \quad (3.9)$$

On the other hand, for each $|j| > J$, there exist a constant $L$ such that for all $l \geq L$

$$\hat{g}(p^{-j} \omega + p^{-j}v(l)) = 0. \quad (3.10)$$

Therefore only finite number of terms of the iterated series in (3.8) are nonzero.

$$|T_2| \leq C \|\hat{\nu}^s(\cdot)\|_{L^2}^2 \|\hat{\psi}\|_{H^s(\mathbb{F})}. \quad (3.11)$$
Hence the $T_2$ is absolutely convergent. The proof is complete. □

Now using above lemma, we establish sufficient condition of frame for $H^s(\mathbb{F})$. Let

$$
\Delta_1 = \text{ess sup}\{\hat{\nu}^s(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p^j \omega)|^2 : \omega \in \mathcal{P}^{-1} \setminus \mathcal{D}\}, \quad (3.12)
$$

and

$$
\Delta_2 = \text{ess inf}\{\hat{\nu}^s(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p^j \omega)|^2 : \omega \in \mathcal{P}^{-1} \setminus \mathcal{D}\}. \quad (3.13)
$$

We set

$$
\beta_\psi(v(l)) = \text{Sup}\{\sum_{j \in \mathbb{Z}} |h_\psi(v(l), p^j \omega)| : \omega \in \mathcal{P}^{-1} \setminus \mathcal{D}\}, \quad (3.14)
$$

where

$$
h_\psi(v(l), \omega) = \sum_{l \in \mathbb{N}_0} \hat{\nu}^s(\omega) \hat{\psi}(p^{-j} \omega) \overline{\hat{\psi}(p^{-j} \omega + v(l))}. \quad (3.15)
$$

Suppose that $Q = \{1, 2, 3, 4, \ldots, q - 1\}$ and $q\mathbb{N}_0 = \{qk : k = 0, 1, 2, 3, \ldots\}$.

**Theorem 3.1.** Suppose $\psi \in H^s(\mathbb{F})$ such that

$$
\rho_1(\psi) = \Delta_2 - \sum_{m \in q\mathbb{N}_0 + Q} [\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}} > 0,
$$

$$
\rho_2(\psi) = \Delta_1 + \sum_{m \in q\mathbb{N}_0 + Q} [\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}} < +\infty.
$$

Then $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ is wavelet frame for $H^s(\mathbb{F})$ with bounds $\rho_1(\psi)$ and $\rho_2(\psi)$.

**Proposition 3.2.** For a given $l \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and unique $m \in q\mathbb{N}_0 + Q$ such that $l = q^k m$. Thus we have $\{v(l)\}_{l \in \mathbb{N}} = \{p^{-k}v(m)\}_{(k,m) \in \mathbb{N}_0 \times (q\mathbb{N}_0 + Q)}$. Since the last series in equation (3.2) is absolutely convergent. Therefore equation (3.2) become

$$
T_2 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{F}} \hat{\omega}^s(\omega) \tilde{g}(\omega) \hat{\psi}(p^j \omega) \sum_{l \in \mathbb{N}} \hat{\omega}^s(\omega + p^{-j}v(l)) \tilde{g}(\omega + p^{-j}v(l)) \times \hat{\psi}(p^j \omega + v(l)) d\omega.
$$
We derive further that

\[
T = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega) \int_{\mathbb{R}} \hat{\psi}(p^{j-k} \omega) \hat{\gamma}(\omega + p^{-j-k} v(m)) d\omega
\]

\[
= \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega) \sum_{k \in \mathbb{N}_0} \sum_{m \in \mathbb{Q}_0 + Q} \hat{\psi}(p^{j-k} \omega) \hat{\gamma}(\omega + p^{-j-k} v(m)) d\omega
\]

\[
= \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega) \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Q}_0 + Q} \hat{\psi}(p^{j-k} \omega) \hat{\gamma}(\omega + p^{-j-k} v(m)) d\omega
\]

\[
= \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Q}_0 + Q} \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega) |\hat{\gamma}(\omega + p^{-j} v(m))| d\omega \right] \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega + p^{-j} v(m)) d\omega
\]

\[
\leq \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Q}_0 + Q} \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) |\hat{\gamma}(\omega)|^2 \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega + p^{-j} v(m)) d\omega \right]^\frac{1}{2}
\]

\[
\times \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega + p^{-j} v(m)) |h_\psi(v(m), p^j \omega)| d\omega \right]^\frac{1}{2}
\]

\[
\leq \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Q}_0 + Q} \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) |\hat{\gamma}(\omega)|^2 \int_{\mathbb{R}} \hat{\psi}(p^j \omega) \hat{\gamma}(\omega - v(m), p^j \omega) d\omega \right]^\frac{1}{2}
\]

\[
\times \left[ \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \hat{\psi}(p^j \omega) |\hat{\gamma}(\omega)|^2 h_\psi(-v(m), p^j \omega) d\omega \right]^\frac{1}{2}
\]

\[
\leq \sum_{m \in \mathbb{Q}_0 + Q} \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) |\hat{\gamma}(\omega)|^2 |h_\psi(v(m), p^j \omega)| d\omega \right]^\frac{1}{2}
\]

\[
\times \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) |\hat{\gamma}(\omega)|^2 \beta_\psi(v(m)) d\omega \right]^\frac{1}{2}
\]

\[
\times \left[ \int_{\mathbb{R}} \hat{\psi}(p^j \omega) |\hat{\gamma}(\omega)|^2 \beta_\psi(-v(m)) d\omega \right]^\frac{1}{2}
\]
\[ = \int_{\mathbb{F}} \hat{\nu}^{*}(\omega) |\hat{g}(\omega)|^{2} d\omega \sum_{m \in \mathbb{Q}} [\beta_{\psi}(v(m))\beta_{\psi}(-v(m))]^{\frac{1}{2}}. \]

Now it follows from equation (3.1) in Lemma 3.1 that
\[
\int_{\mathbb{F}} \hat{\nu}^{*}(\omega) |\hat{g}(\omega)|^{2} \left\{ \sum_{j \in \mathbb{Z}} \hat{\nu}^{*}(\omega) |\hat{\psi}(p_{j}\omega)|^{2} - \sum_{m \in \mathbb{Q}} [\beta_{\psi}(v(m))\beta_{\psi}(-v(m))]^{\frac{1}{2}} \right\} d\omega \\
\leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_{0}} |\langle g, \psi_{j,k} \rangle|_{H^{s}(\mathbb{F})}^{2}, \quad (3.16)
\]
and
\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_{0}} |\langle g, \psi_{j,k} \rangle|_{H^{s}(\mathbb{F})}^{2} \leq \int_{\mathbb{F}} \hat{\nu}^{*}(\omega) |\hat{g}(\omega)|^{2} \left( \sum_{j \in \mathbb{Z}} \hat{\nu}^{*}(\omega) |\hat{\psi}(p_{j}\omega)|^{2} + \sum_{m \in \mathbb{Q}} [\beta_{\psi}(v(m))\beta_{\psi}(-v(m))]^{\frac{1}{2}} \right) d\omega. \quad (3.17)
\]

Taking infimum and suprimum in above two inequality respectively, we get
\[
\rho_{2}(\psi)\|g\|_{H^{s}(\mathbb{F})} \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_{0}} |\langle g, \psi_{j,k} \rangle|^{2} \leq \rho_{1}(\psi)\|g\|_{H^{s}(\mathbb{F})}. \quad (3.18)
\]

The proof of theorem 3.1 is complete.

**Theorem 3.3.** Suppose \( \psi \in H^{s}(\mathbb{F}) \) such that
\[
\Delta_{3}(\psi) = \text{ess inf}_{\omega \in \mathbb{P}^{-1} \setminus \mathbb{Z}} \left\{ \hat{\nu}^{*}(\omega) \sum_{j \in \mathbb{Z}} |\hat{\psi}(p_{j}\omega)|^{2} - \hat{\nu}^{*}(\omega) \sum_{j \in \mathbb{Z}} \hat{\psi}(p_{j}\omega)\psi(p_{j}\omega + v(l)) \right\} > 0, \quad (3.19)
\]
\[
\Delta_{4}(\psi) = \text{ess sup}_{\omega \in \mathbb{P}^{-1} \setminus \mathbb{Z}} \left\{ \hat{\nu}^{*}(\omega) \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_{0}} |\hat{\psi}(p_{j}\omega)\overline{\psi}(p_{j}\omega + v(l))| \right\} < +\infty. \quad (3.20)
\]

Then \( \{ \psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_{0} \} \) is a wavelet frame for \( H^{s}(\mathbb{F}) \) with bounds \( \Delta_{3}(\psi) \) and \( \Delta_{4}(\psi) \).

**Proof.** We use Lemma 3.1 to calculate \( T_{2} \) in (3.2) for \( g \in \mathbb{S}(\mathbb{F}) \) with another way. We first deduce that
\[ |T_2| = \left| \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} \hat{\nu}^s(\omega) \overline{\hat{\nu}(p^j \omega)} \hat{\nu}(\omega + p^{-j} v(l)) \hat{g}(\omega + p^{-j} v(l)) \times \overline{\hat{\nu}(p^j \omega + v(l))} \right| \]
\[ \leq \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \left\{ \int_{\mathbb{F}} |\hat{\nu}(\omega)|^2 |\hat{\nu}^s(\omega)| \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega + v(l))} \right| d\omega \right\}^{\frac{1}{2}} \]
\[ \times \left\{ \int_{\mathbb{F}} |\hat{\nu}(\omega + p^{-j} v(l))|^2 |\hat{\nu}^s(\omega)| \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega + v(l))} \right| d\omega \right\}^{\frac{1}{2}} \]
\[ = \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \left\{ \int_{\mathbb{F}} |\hat{\nu}(\omega)|^2 |\hat{\nu}^s(\omega)| \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega + v(l))} \right| d\omega \right\}^{\frac{1}{2}} \]
\[ \times \left\{ \int_{\mathbb{F}} |\hat{\nu}(\omega + p^{-j} v(l))|^2 |\hat{\nu}^s(\omega)| \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega - v(l))} \right| d\omega \right\}^{\frac{1}{2}}. \]

Since \( \{v(k) : k \in \mathbb{N}\} = \{-v(k) : k \in \mathbb{N}_0\} \), we have

\[ |T_2| \leq \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \int_{\mathbb{F}} |\hat{\nu}(\omega)|^2 |\hat{\nu}^s(\omega)| \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega - v(l))} \right| d\omega. \tag{3.21} \]

By Levi Lemma we obtain,

\[ |T_2| \leq \int_{\mathbb{F}} |\hat{\nu}(\omega)|^2 |\hat{\nu}^s(\omega)| \left\{ \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega - v(l))} \right| \right\} d\omega. \tag{3.22} \]

Using equation (3.1), we get

\[ \int_{\mathbb{F}} |\hat{\nu}(\omega)|^2 |\hat{\nu}^s(\omega)| \left\{ \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega - v(l))} \right| \right\} d\omega \]
\[ \leq \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} |\langle g, \psi_{j,k} \rangle|^2, \tag{3.23} \]

and

\[ \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} |\langle g, \psi_{j,k} \rangle|^2 \leq \int_{\mathbb{F}} |\hat{\nu}(\omega)|^2 |\hat{\nu}^s(\omega)| \left\{ \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \left| \hat{\nu}(p^j \omega) \overline{\hat{\nu}(p^j \omega + v(l))} \right| \right\} d\omega. \tag{3.24} \]

Taking infimum in (3.23) and supremum in (3.24), we obtain that

\[ \Delta_4(\psi) \|g\|_{H^s(\mathbb{F})}^2 \leq \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} |\langle g, \psi_{j,k} \rangle|^2 \leq \Delta_4(\psi) \|g\|_{H^s(\mathbb{F})}^2. \tag{3.25} \]
hold for all \( g \in \mathcal{H}(F) \). The proof of theorem 3.3 is complete. □

Acknowledgments

The authors are thankful to the referee for his thorough review and appreciate suggestions and the comments to various improvements in the manuscript. The work of third author is supported by the CSIR grant no : 09/013(0647)/2016 - EMR - 1, New Delhi.

References

1. C. K. Chui, *An Introduction to Wavelets, Wavelet Analysis and Its Applications*, Academic Press, Boston, MA, 1, (1992).
2. S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of \( L^2(\mathbb{R}) \)*, Trans. Amer. Math. Soc. 315, 69-87, (1989).
3. Y. Meyer, *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992.
4. S. Dahlke, *Multiresolution analysis and wavelets on locally compact abelian groups. in Wavelets, images, and surface fitting*, A K Peters, 141-156,(1994).
5. J. J. Benedetto and R. L. Benedetto, *A wavelet theory for local fields and related groups*, J. Geom. Anal. 14 , 423-456,(2004).
6. R. L. Benedetto, *Examples of wavelets for local fields*, in: *Wavelets, Frames and Operator Theory*, Contemporary Mathematics, American Mathematical Society, Providence, RI, 345, 27-47,(2004).
7. S. Albeverio and S. Kozyrev, *Multidimensional basis of p-adic wavelets and representation theory*, P-Adic Numbers Ultrametrie, Anal. Appl. 1 , 181-189,(2009).
8. A.Y. Khrennikov, V. M. Shikovich and M. Skopina, *p-adic refinable functions and MRA-based wavelets*, J. Approx. Theory 161, 226-236 (2009).
9. S. Kozyrev, *Wavelet theory as p-adic spectral analysis* Izv. Ross. Akad. Nauk Ser. Mat. 66, 149-158, (2002).
10. H. Jiang, D. Li and N. Jin, *Multiresolution analysis on local fields*, J. Math. Anal. Appl. 294, 523-532, (2004).
11. D. Ramakrishnan and R. J. Valenza, *Fourier Analysis on Number Fields*, Graduate Texts in Mathematics , Springer-Verlag, New York,186 (1999).
12. M. H. Taibleson, *Fourier Analysis on Local Fields*, Mathematical Notes , Princeton University Press, Princeton, NJ,15 (1975).
13. Biswaranjan Behera and Qaiser Jahan, *Multiresolution analysis on local fields and characterization of scaling functions*, Adv. Pure Appl. Math. 3 , 181-202,(2012).
14. F. Bastin and P.Laubin, *Regular Compactly Supported Wavelets in Sobolev spaces*, Duke Mathematical Journal, 87 (3), 481-508,(1997).
15. Ashish Pathak and Guru P. Singh, *Wavelet in Sobolev space over local fields of positive characteristic*, Int. J. of Wavelets Multiresolut. Inf. Process, 16(3),16 pp, (2018).
16. Ashish Pathak, Dileep Kumar and Guru P. Singh *Multiresolution Analysis on Sobolev space over local fields of positive characteristic and Characterization of scaling function* (preprint).