Three-dimensional Ricci solitons which project to surfaces

Paul Baird and Laurent Danielo

Département de Mathématiques, Université de Bretagne Occidentale
6 Avenue Le Gorgeu, B.P. 452, 29285 Brest Cedex, France
E-mail: Paul.Baird@univ-brest.fr  Laurent.Danielo@univ-brest.fr

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Abstract

We study 3-dimensional Ricci solitons which project via a semi-conformal mapping to a surface. We reformulate the equations in terms of parameters of the map; this enables us to give an ansatz for constructing solitons in terms of data on the surface. A complete description of the soliton structures on all the 3-dimensional geometries is given, in particular, non-gradient solitons are found on Nil and Sol.

1 Introduction

Fixed points of the Ricci flow:

\[ \frac{\partial g}{\partial t} = -2\text{Ricci}(g) \]

on a manifold \((M, g)\) are called Ricci solitons. It is usual to look for fixed points up to diffeomorphism of \(M\) and scaling of \(g\), whence the equations for a Ricci soliton become:

\[ -2\text{Ricci}(g) = \mathcal{L}_E g + 2Ag \]

where \(E\) is a vector field on \(M\), \(\mathcal{L}_E g\) denotes Lie derivation of \(g\) with respect to \(E\) and \(A\) is a constant. The soliton is called shrinking, stationary or expanding according as \(A < 0, = 0, > 0\), respectively. It is of gradient type if \(E = \text{grad} F\) for some function \(F\), in which case \(\mathcal{L}_E g = 2\nabla dF\).

Apart from their interest as fixed points of the Ricci flow, see [13], and [16] for an overview, they are interesting geometric objects in their own right. Warped product
solutions have been constructed by Bryant and Ivey [15]. Kähler solitons in even
dimension are studied by Koiso [17], Cao [6, 7], Feldman, Ilmanen and Knopf [9] and
Bryant [5]. A construction using a doubly warped product metric in higher dimensions
is given by Gastel and Krong [11]. We note that all of the above constructions are of
gradient type. By a result of Ivey [14], in dimension 3 the only compact examples are
of constant curvature. Our aim in this article is to construct 3-dimensional solitons
which admit a semi-conformal projection onto a surface. Some of our examples are
not of gradient type. The existence of such solitons has important consequences for the
stability of the Ricci flow about non-Ricci flat metrics, which is studied in the Ricci-flat
case in [12, 22].

A Lipschitz map \( \varphi : (M^m, g) \rightarrow (N^n, h) \) between Riemannian manifolds is said to be
semi-conformal if, at each point \( x \in M \) where \( \varphi \) is differentiable (dense by Radmacher’s
Theorem), the derivative \( d\varphi_x : T_x M \rightarrow T_{\varphi(x)} N \) is either the zero map or is conformal
and surjective on the compliment of \( \ker d\varphi_x \) (called the horizontal distribution). Thus,
there exists a number \( \lambda(x) \) (defined almost everywhere), called the dilation, such that
\( \lambda(x)^2 g(X,Y) = \varphi^* h(X,Y) \), for all \( X,Y \in (\ker d\varphi_x)^\bot \). If \( \varphi \) is of class \( C^1 \), then we have
a useful characterisation in local coordinates, given by
\[
g^{ij} \varphi^i_\alpha \varphi^j_\beta = \lambda^2 h^{\alpha\beta},
\]
where \( (x^i), (y^\alpha) \) are coordinates on \( M, N \), respectively and \( \varphi^i_\alpha = \partial(y^\alpha \circ \varphi)/\partial x^i \) (here
and throughout we sum over repeated indices). The fibres of a smooth submersive semi-
conformal map determine a conformal foliation, see [23] and conversely, with respect to
a local foliated chart, we may put a conformal structure on the leaf space with respect
to which the projection is a semi-conformal map. We then have the identity:
\[
(L_U g)(X,Y) = -2U(\ln \lambda) g(X,Y),
\]
for \( U \) tangent and \( X,Y \) orthogonal to the foliation.

The question of which 3-manifolds admit a conformal foliation (equivalently, local
semi-conformal maps to surfaces) remains a delicate one. It is equivalent, locally, to
finding a function which admits a ‘conjugate’; in [2] it is shown that such functions are
characterized as satisfying a 2nd order differential inequality and a 3rd order differential
equation. We shall see that we can calculate the Ricci curvature in terms of geometric
quantities associated to the projection, for example the mean curvature of the fibres,
the integrability tensor of the orthogonal distribution and the dilation.
In general, the equations are still difficult to study and in this article we concentrate on the case when objects are basic, that is they are defined in terms of data on the surface. This leads to the following ansatz for the construction of solitons.

**Theorem 1.1** Let \((N, h)\) be a Riemannian surface. Let \(\overline{\psi}, \overline{\lambda}, \overline{\rho}, \overline{\nu} : N \to \mathbb{R}\) be functions with \(\overline{\psi} \geq 0\), satisfying the system of equations

\[
\begin{align*}
(i) & \quad K^N + \frac{1}{2} \left( \Delta^N \ln(\overline{\lambda}^2 \overline{\nu}) - |\text{grad} \ln \overline{\rho}|^2 \right) + \frac{\overline{\lambda} - \overline{\psi}}{\overline{\lambda}} = 0 \\
(ii)(a) & \quad \Delta^N \ln(\overline{\rho}^2 \overline{\psi}^{1/2}) + |\text{grad} \ln \overline{\rho}|^2 - \frac{1}{2} |\text{grad} \ln \overline{\psi}|^2 \\
& \quad + \frac{1}{2} h(\text{grad} \ln \overline{\psi}, \text{grad} \ln \overline{\rho}) + \frac{\overline{\nu} + \Delta}{\overline{\lambda}} = 0 \\
(ii)(b) & \quad \nabla \text{d} \ln \overline{\nu} + 2 \text{d} \ln \overline{\lambda} \otimes \text{d} \ln \overline{\nu} - (\text{d} \ln \overline{\rho})^2 = a h \quad \text{(some } a : N \to \mathbb{R})
\end{align*}
\]

(2)

where we take (ii)(a) to be vacuous whenever \(\overline{\psi} \equiv 0\) and where \(K^N\) is the Gaussian curvature of \(N\). Now set \(M = N \times (-\delta, \delta)\) for some \(\delta > 0\) and let \(\varphi : M \to N\) be the canonical projection. Let \(\overline{\sigma} = \sqrt{2\overline{\psi}/\overline{\lambda}^2}\) and set \(\overline{\Omega} = \overline{\varphi}^* \overline{\theta}\), \(\rho = \overline{\rho} \circ \varphi\) and \(\lambda = \overline{\lambda} \circ \varphi\). Let the 1-form \(\theta\) be a solution to the exterior differential equation

\[
d\theta + d \ln \rho \wedge \theta = \overline{\Omega}
\]

(3)

which is everywhere non-vanishing on \(\ker d\varphi\). Write \(g = \frac{\varphi^* h}{\overline{\lambda}^2} + \theta^2\). Then \((M^3, g)\) is a Ricci soliton.

The function \(\overline{\psi}\) measures the non-integrability of the orthogonal complement of \(\ker d\varphi\), in particular if \(\overline{\psi} \equiv 0\), then this distribution is integrable. The function \(\ln \rho\) is the potential for the mean curvature \(\mu\) of the fibres of \(\varphi\); that is, \(\mu = \text{grad} \ln \rho\). A special case of the theorem is when \(\overline{\rho}\) is constant, so the fibres are minimal. In this case we can give a complete description of the solutions of (2) in terms of holomorphic data on the surface \(N\), which leads to the following conclusion.

**Corollary 1.2** Let \((M^3, g)\) be a soliton derived from the ansatz given by Theorem 1.1 with \(\rho\) constant, then \(\overline{\psi} = C\) is constant and either

(i) \(C = 0\), in which case \(M\) is locally isometric to a Riemannian product \(N^2 \times \mathbb{R}\), where \(N^2\) is 2-dimensional gradient Ricci soliton, or

(ii) \(C \neq 0\), in which case \(M\) either has constant curvature or is locally isometric to the geometry \(\text{Nil}\).

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We will give an explicit description of the soliton structure on Nil, which exists and is unique by [18]. On the other hand, to identify the geometry Sol as a soliton (Example 4.11) we will require the more general set-up of Theorem 1.1. This is because Sol, even locally, does not admit a semi-conformal map to a surface with geodesic fibres [3]. For both of these geometries the soliton structure is not of gradient type. Finally, we show that $SL_2(R)$ does not admit any soliton structure whatsoever. To complete the picture, we show uniqueness of the soliton structures on the other 3-dimensional geometries, except for $R^3$ where the Gaussian soliton appears.

**Remarks:**

1. The constant $A$ is the same constant that occurs in (1) and so its sign determines whether the soliton is shrinking, stationary or expanding. To be consistent with notation, functions, forms and vector fields on the codomain $N$ will have a ‘bar’ over them.

2. The system (2) depends only on $\text{grad} \ln \nu$ and $\text{grad} \ln \rho$. In its most general form we may replace $\text{grad} \ln \nu$ by $Y$, where $Y$ is a vector field on $N$. For example, $\Delta^N \ln \nu$ would then become $\text{div} Y^b$, where $Y^b = h(Y, \cdot)$ is the dual of $Y$. Our hypothesis is not the same as supposing the soliton is of gradient type (see also Section 4).

3. We will refer to the condition $(u)$ in the above theorem and corollary as the ‘umbilicity condition’; the reason for this will become apparent in Section 4.

4. The system (2) is invariant under conformal changes in the metric $h$. This is a natural consequence of our construction. Indeed, if we replace $h$ by $e^{2u}h$, then $\lambda$ is replaced by $e^u \nabla, K^N$ by $e^{-2u}(-\Delta^N u + K^N)$ and the Laplacian $\Delta^N$ by $e^{-2u} \Delta^N$.

5. Locally, on a topologically trivial domain, it is always possible to solve (3). In fact we can write it in the equivalent form: $d(\rho \theta) = \rho \tilde{\Omega}$. In order to solve this for $\theta$ we require that $d(\rho \tilde{\Omega}) = 0$. But $\rho \tilde{\Omega} = \varphi^*(\tilde{\Omega})$, which is clearly closed.

6. The condition that a soliton given by Theorem 1.1 has constant curvature is given in Proposition 4.9.

7. A semi-conformal map which is harmonic is called a harmonic morphism (see [10, 3]). If the mapping takes values in a surface then this is equivalent to the fibres being minimal. The second named author has constructed Einstein metrics in dimension 4 from harmonic morphisms with 1-dimensional fibres [8]. On the other hand, any harmonic morphism with 1-dimensional fibres from an Einstein manifold of dimension $\geq 5$ is of warped product type or of Killing type (the fibres are the integral curves of a Killing vector field). This was shown for manifolds of constant curvature by Bryant.
and for more general Einstein manifolds by Pantilie and Wood [21]. In dimension 4 one other type can occur [20].

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2 The fundamental equations

Let $\varphi : (M^3, g) \to (N^2, h)$ be a $(C^\infty)$ semi-conformal submersion with dilation $\lambda$. We will use the following notation: Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal distributions, respectively. We will use the same letters to denote orthogonal projection onto the respective distributions. Let $U$ denote the unit vertical vector field, i.e. $d\varphi(U) = 0$ and $g(U, U) = 1$. If $M$ is oriented, there are two choices for $U$; we will select one of these - the equations are invariant of this choice. If $M$ is not oriented, then we work locally with a choice of $U$. Let $\theta = U^b = g(U, \cdot) = g|U$ denote its dual and write $\Omega = d\theta$; then the integrability tensor $I(X,Y) = \mathcal{V}[X,Y], X,Y \in \mathcal{H}$ is related by the formula $\Omega(X,Y) = -g(I(X,Y), U)$. We will use the norm: $||\Omega||^2 = \sum_{a,b} |\Omega(e_a, e_b)|^2$, where $\{e_a\}$ is an orthonormal frame, similarly for $||I||^2$. Write $d^V f$ for the vertical component of the exterior differential of a function $f$, i.e. $d^V f = U(f)\theta$. Set $\omega \otimes \eta = \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega)$ for the symmetric product of two 1-forms. Finally, our sign convention for the Laplacian is such that $\Delta f = f''$ for a function of a single variable.

Proposition 2.1 Let $\varphi : (M^3, g) \to (N^2, h)$ be a $(C^\infty)$ semi-conformal submersion with dilation $\lambda$. Then, in the notations defined above, the Ricci tensor of $(M, g)$ is given by the formula:

$$
\text{Ricci} (g) = \left\{ \lambda^2 K^N + \Delta \ln \lambda + \mu(\ln \lambda) \right\} (g - \theta^2) - \frac{1}{4} ||I||^2 g + \frac{1}{2} L_{\mu} g - (\mu^b + d^V \ln \lambda)^2 - (d^V \ln \lambda)^2 + 2d(U(\ln \lambda)) \otimes \theta + d^* \Omega \otimes \theta,
$$

where $\mu$ denotes the mean-curvature vector field of the fibres of $\varphi$ and $K^N$ is the Gauss curvature of $N$.

The above formula will be deduced by evaluating the Ricci tensor on different combinations of horizontal and vertical vectors. In what follows, we let $\{e_i\} = \{e_a, U\}$
denote a local orthonormal frame field, where the index \( i \) ranges over 1, 2, 3 and the index \( a \) over 1, 2. The following lemma is a straightforward calculation.

**Lemma 2.2** Let \( \tilde{\Omega} \) be the 2-form defined by \( \tilde{\Omega}(E,F) = -g(U,[\mathcal{H}E,\mathcal{H}F]) \), for all vectors \( E,F \). Then

\[
\Omega = \tilde{\Omega} - \mu^3 \wedge \theta .
\]

We will call \( \tilde{\Omega} \) the integrability 2-form associated to \( \varphi \). The following lemma is useful and is to be found in [1].

**Lemma 2.3** The mean curvature \( \frac{1}{2}g(U,\nabla e_a e_a) \) of the horizontal distribution \( \mathcal{H} \) is given by \( U(\ln \lambda) \).

**Proof**: We note that the quantity \( g(U,\nabla e_a e_a) \) is independent of the horizontal frame \( \{ e_a \} \). Suppose that, for each \( a = 1, 2 \), the vector field \( e_a \) is the normalised lift of a unit vector field \( \overline{\nabla}_a \) on a domain of \( N \): \( d\varphi(e_a) = \lambda \overline{\nabla}_a \circ \varphi \). By the symmetry of the second fundamental form \( \nabla d\varphi \), we have

\[
-d\varphi(\nabla e_a U) = -d\varphi(\nabla_U e_a) + \nabla_U e_a - T d\varphi(e_a),
\]

so that \( \mathcal{H}[U,e_a] = U(\ln \lambda)e_a \). Thus

\[
g(U,\nabla e_a e_a) = g([U,e_a],e_a) = g(U(\ln \lambda)e_a,e_a) = 2U(\ln \lambda) .
\]

q.e.d.

For horizontal vectors \( X,Y \), let \( A_X Y = \nabla \nabla_X Y = \frac{1}{2} \nabla[X,Y] + g(X,Y)\nabla \ln \lambda \) be one of the fundamental tensors associated to a submersion. The second fundamental tensor \( B \) is defined by \( B_U V = \mathcal{H} \nabla_U V \) for vertical vectors \( U,V \) [3, 19]. We write \( A^*, B^* \) for their respective adjoints.

The formula for the Ricci tensor evaluated on vertical vectors is as follows.

**Lemma 2.4** Ricci \( \text{Ricci}(U,U) = 2U(U(\ln \lambda)) - 2U(\ln \lambda)^2 - \frac{1}{4} ||I||^2 + \frac{1}{2}(L_{\mu}g)(U,U) + d^*\Omega(U) \).

**Proof**: First note that

\[
\text{Ricci}(U,U) = \sum_a g(R(U,e_a)e_a,U)) = \sum_a K(e_a \wedge U),
\]

where \( K(e_a \wedge U) \) is the sectional curvature of the plane spanned by \( e_a \) and \( U \). By Proposition 11.2.2 of [3], this latter term equals:

\[
\sum_a \{ \nabla d\ln \lambda(U,U) + d\ln \lambda(B_U U) - 2U\ln \lambda^2 + |A^*_e a U|^2 + g((\nabla e_a B^*_U e_a, U) - |B^*_a e_a|^2) .
\]
Then we have
\[
\sum_a |A^*_a U|^2 = \frac{1}{4} ||I||^2 + 2U(\ln \lambda)^2, \\
g((\nabla e_a B^* U)e_a, U) = g(e_a, \nabla e_a \mu), \\
\sum_a |B^*_U e_a|^2 = \sum_a g(e_a, \nabla_U)^2 = |\mu|^2.
\]

On noting that \(d^* \mu^\flat = -g(e_a, \nabla e_a \mu) + |\mu|^2\), \(d^* \Omega(U) = -d^* \mu^\flat + |\mu|^2 + \frac{1}{2} ||I||^2\) and \((\mathcal{L}_\mu g)(U, U) = 2g(\nabla_U \mu, U) = -2|\mu|^2\), the formula follows. \(\text{q.e.d.}\)

The Ricci tensor evaluated on a horizontal and vertical vector is given by the following expression.

**Lemma 2.5**

\[
\text{Ricci } (X, U) = X(U(\ln \lambda)) + \frac{1}{2} d^* \Omega(X) - U(\ln \lambda) g(X, \mu) + \frac{1}{2} (\mathcal{L}_\mu g)(X, U).
\]

**Proof:** By Theorem 11.2.1 of [3],

\[
\text{Ricci } (X, U) = g(R(X, e_i)e_i, U) = g(R(X, e_a)e_a, U) = g((\nabla X A)e_a e_a - (\nabla e_a A)x e_a, U) + g(B^*_U e_a, I(X, e_a)).
\]

On calculating, we obtain

\[
g((\nabla X A)e_a e_a, U) - g((\nabla e_a A)x e_a, U) = X(U(\ln \lambda)) + \frac{1}{2} d^* \Omega(X) - U(\ln \lambda) g(X, \mu) - \frac{1}{2} g(X, \mathcal{L}_\mu U) + \frac{1}{2} g(U, \mathcal{L}_\mu X).
\]

Also \(g(B^*_U e_a, I(X, e_a)) = g(U, [X, \mu])\) and the formula follows. \(\text{q.e.d.}\)

In order to calculate the Ricci tensor on horizontal vectors, we first establish a useful lemma which we will need later on.

**Lemma 2.6** Suppose that, for each \(a = 1, 2\), the vector field \(X_a\) is the horizontal lift of a unit vector field \(\bar{X}_a\) on a domain of \(N\): \(d\varphi(X_a) = \bar{X}_a \circ \varphi\). Then

(i) \(\mathcal{H}[U, X_a] = 0\);

(ii) \(U\{g(U, [X_1, X_2])\} = -d\mu^\flat(X_1, X_2)\).

**Proof:** On the one hand \(\nabla d\varphi(X_a, U) = -d\varphi(\nabla X_a U)\). Whereas

\[
\nabla d\varphi(U, X_a) = -d\varphi(\nabla_U X_a) + \nabla_U (\bar{X}_a \circ \varphi) = -d\varphi(\nabla_U X_a).
\]
But by the symmetry of the second fundamental form, $\nabla d\varphi(X_a, U) = \nabla d\varphi(U, X_a)$ and (i) follows.

Now $g(\nabla_U [X_1, X_2], U) = g([U, [X_1, X_2]] + \nabla_{[X_1, X_2]} U, U) = g([U, [X_1, X_2]], U)$. Furthermore, from the Jacobi identity: $[U, [X_1, X_2]] = [[U, X_1], X_2] + [[X_2, U], X_1]$. From (i), $\mathcal{H}[U, X_a] = 0$; also $\mathcal{V}[U, X_a] = -g(X_a, \mu)U$. Therefore

$$[U, [X_1, X_2]] = [-g(X_1, \mu)U, X_2] - [-g(X_2, \mu)U, X_1]$$

$$= -g(X_1, \mu)[U, X_2] + g(X_2, \mu)[U, X_1] + X_2 \{g(X_1, \mu)\}U - X_1 \{g(X_2, \mu)\}U$$

$$= X_2 \{g(X_1, \mu)\}U - X_1 \{g(X_2, \mu)\}U.$$ 

But $d\mu^\lambda(X_1, X_2) = X_1 g(X_2, \mu) - X_2 g(X_1, \mu) - g([X_1, X_2], \mu)$. The formula follows.

q.e.d.

**Lemma 2.7** For horizontal vectors $X, Y$,

$$\text{Ricci}(X, Y) = \left\{ \lambda^2 K^N + \Delta \ln \lambda + \mu(\ln \lambda) - \frac{1}{4}|I|^2 \right\} g(X, Y)$$

$$+ \frac{1}{2} \mathcal{L}_\mu g(X, Y) - g(X, \mu)g(Y, \mu).$$

*Proof:* Since both sides are tensorial in $X$ and $Y$, it suffices to verify the formula on a basis. Let $e_a = \lambda X_a$, where $X_a$ is the basis taken in Lemma 2.6; thus $\{e_1, e_2\}$ is an orthonormal frame for the horizontal space. We first of all compute $\text{Ricci}(e_1, e_2) = g(R(U, e_1)e_2, U)$. By Theorem 11.2.1 of [3], we have

$$g(R(U, e_1)e_2, U) = g((\nabla_U A)_{e_1} e_2, U) + g(A^*_{e_1} U, A^*_{e_2} U)$$

$$+ g((\nabla_{e_1} B^*) U e_2, U) - g(B^*_{U e_2}, B^*_{U e_1}) - 2U(\ln \lambda)g(A_{e_1} e_2, U).$$

We compute the individual terms in this expression:

$$g((\nabla_U A)_{e_1} e_2, U) = \frac{1}{2} U \{g([e_1, e_2], U)\}.$$

It is easily checked that $g(A^*_{e_1} U, A^*_{e_2} U)$ vanishes, whereas

$$g((\nabla_{e_1} B^*) U e_2, U) = g(e_2, \nabla_{e_1} \mu).$$

We also have $g(B^*_{U e_2}, B^*_{U e_1}) = g(e_2, \mu)g(e_1, \mu)$ and $g(A_{e_1} e_2, U) = \frac{1}{7} g([e_1, e_2], U)$. Combining and applying Lemma 2.6, gives the required formula with $X = e_1$ and $Y = e_2$.

On the other hand

$$\text{Ricci}(e_1, e_1) = g(R(U, e_1)e_1, U) + (R(e_2, e_1)e_1, e_2)$$

$$= K(e_1 \wedge U) + K(e_2 \wedge e_1),$$

$$g(A_{e_1} e_2, U) = \frac{1}{7} g([e_1, e_2], U).$$
We now apply Proposition 11.2.2 of [3], which expresses the sectional curvatures of a semi-conformal submersion, to give the required formula with $X = Y = e_1$. We omit the details. q.e.d.

**Proof of Proposition 2.1:** The formula of the proposition follows on combining Lemmata 2.4, 2.5 and 2.7. q.e.d.

We will now equate the expression for the Ricci curvature with the right hand side of (1). However, it is useful to decompose the vector field $E$ into its different components. It is no loss of generality to set $E = -\mu + X + fU$ for some function $f$ and horizontal vector field $X$, in order to cancel the term $\frac{1}{2}L_\mu g$ in (4).

**Lemma 2.8**

$$L_{(fU)}g = -2fU(\ln \lambda)(g - \theta^2) + 2df \odot \theta + 2f \mu^\flat \odot \theta. \quad (5)$$

**Proof:** It suffices to evaluate $L_{(fU)}g$ on different combinations of vectors, using the fact that $fU$ is tangent to a conformal foliation, so that $(L_{(fU)}g)(X,Y) = -2fU(\ln \lambda)g(X,Y)$ for all $X,Y \in \mathcal{H}$. q.e.d.

We can now write out the fundamental equations for a soliton derived from a semi-conformal map.

**Proposition 2.9** Let $\varphi : (M^3,g) \to (N^2,h)$ be a $(C^\infty)$ submersive semi-conformal map. Then $(M^3,g)$ is a Ricci soliton if and only if there is a function $f : M^3 \to \mathbb{R}$, a horizontal vector field $X$ and a constant $A$, such that, in the notations defined above,

$$0 = \left\{ \lambda^2 K^N + \Delta \ln \lambda + \mu(\ln \lambda) - fU(\ln \lambda) \right\} (g - \theta^2) - \frac{1}{4}|I|^2g + \frac{1}{2}L_X g + Ag - (\mu^\flat + dV \ln \lambda)^2 - (dV \ln \lambda)^2 + \left\{ df + f \mu^\flat + 2d(U(\ln \lambda)) + d^*\Omega \right\} \odot \theta. \quad (6)$$

**Example 2.10** (Warped product solutions) Locally, a warped product is of the form $M^3 = N^2 \times J$ with metric $g = (h/\lambda^2) + dt^2$, where $J$ is an open interval in $\mathbb{R}$ with its coordinate $t$ and where $\lambda = \lambda(t)$. We take $\varphi$ to be the canonical projection $\varphi : M \to N$. Then the fibres are geodesic, so that, in addition to being semi-conformal, $\varphi$ now has the further property of being a harmonic morphism and our expression (4) for the Ricci curvature reduces to a well-known, much simpler case of our formula, see [3]. Note also that the horizontal distribution is integrable so that $I$ and $\Omega$ vanish. We will...
suppose that the horizontal component $X$ of the vector field $E$ vanishes and that $K^N$ is constant. Then the system of equations (6) is equivalent to the pair of equations

\[
\begin{cases}
0 &= \lambda^2 K^N + \Delta \ln \lambda - f U(\ln \lambda) + A \\
0 &= U(f) + 2U(U(\ln \lambda)) - 2U(\ln \lambda)^2 + A
\end{cases}
\] (7)

Let $t$ denote a unit speed parameter along the fibres, so that $t = \text{const.}$ gives the (integrable) horizontal spaces, $U = \partial/\partial t$ and $\theta = dt$. As a consequence of the equations, we also have $f = f(t)$. We note the special solutions given by $\lambda \equiv \text{const}$. Without loss of generality, we may suppose that $\lambda \equiv 1$. If now $K^N = 1$, then $A = -1$ and $f' = 1$. This gives the soliton $S^2 \times \mathbb{R}$ with $E = t(\partial/\partial t)$. Similarly, if $K^N = -1$, then $A = 1$ and $f' = -1$ leading to the soliton $H^2 \times \mathbb{R}$ with $E = -t(\partial/\partial t)$.

We will now suppose that $\lambda$ is non-constant and work on a neighbourhood where $\lambda' \neq 0$. Then, letting $\{e_a\}$ denote a local orthonormal frame, by Lemma 2.3,

\[
\Delta \ln \lambda = \text{Tr} \nabla d \ln \lambda = d \ln \langle \nabla_{e_a} e_a \rangle + U(U(\ln \lambda)) = -2\{\ln \lambda\}'^2 + (\ln \lambda)''.
\]

The system (7) now becomes the pair of ordinary differential equations:

\[
\begin{cases}
0 &= \lambda^2 K^N + (\ln \lambda)'' - 2\{\ln \lambda\}'^2 - f(\ln \lambda)' + A \\
0 &= f' + 2(\ln \lambda)'' - 2\{\ln \lambda\}'^2 + A
\end{cases}
\] (8)

We can solve these to express $f$ in terms of $\lambda$:

\[
f = \frac{\lambda'' + A\lambda + K^N \lambda^3}{\lambda} - \frac{3\lambda'}{\lambda}.
\] (9)

Substituting back, we obtain the following 3rd order ordinary differential equation:

\[
\lambda''' \lambda' \lambda^2 - \lambda \lambda'' (\lambda'^2 + \lambda \lambda'') + A \lambda^2 (2\lambda'^2 - \lambda \lambda'') + K^N \lambda^4 (3\lambda'^2 - \lambda \lambda'') - \lambda'^4 = 0.
\] (10)

Thus (9) and (10) are equivalent to (8). It is easily checked that the solution has constant curvature if and only if $(\ln \lambda)'' - \lambda^2 K^N = 0$.

If we look for solutions of the form

\[
(\lambda')^2 = 2F(\lambda)
\] (11)

for some function $F$, then (10) becomes

\[
2\lambda^2 F'' - \lambda F'(2F + \lambda F') + A \lambda^2 (4F - \lambda F') + K^N \lambda^4 (6F - \lambda F') - 4F^2 = 0.
\]

It is easy to see that the only polynomial solutions are given by

\[
F = \frac{K^N \lambda^4}{2} + \frac{A \lambda^2}{4}.
\]
Then (11) can be integrated explicitly, however all these solutions have constant curvature. On the other hand, the exceptional solutions when $A = K^N = 0$, given by $F(\lambda) = B\lambda^k$ with $k = 2(1 \pm \sqrt{2})$ are of non-constant curvature. Choosing $B = 2$, this gives $\lambda(t) = t^{\pm 1/\sqrt{2}}$. We may express the corresponding (incomplete) metric in the form $g = t^{\pm \sqrt{2}}(dx^2 + dy^2) + dt^2$.

3 Prescribing the mean-curvature of the fibres

We wish to characterize those semi-conformal maps $\varphi : M^3 \to N^2$ which can be recovered from data on the surface $N^2$. Consider the 2-form $\Omega = d\theta$. Then by Lemma 2.2,

$$\Omega = \tilde{\Omega} - \mu^\flat \wedge \theta,$$

where $\tilde{\Omega} = \Omega \circ \mathcal{H} = -g(U, I)$, with $I$ the integrability tensor $I(X, Y) = \mathcal{V}[X, Y]$. Say that a 1-form $\omega$ is closed relative to a distribution $D$ if and only if $d\omega(X, Y) = 0$ for all $X, Y \in D$.

**Lemma 3.1** The 2-form $\tilde{\Omega}$ is basic if and only if $\mu^\flat$ is closed relative to the horizontal distribution, i.e. $d\mu^\flat(X, Y) = 0$ for all $X, Y \in \mathcal{H}$. In particular, this is the case whenever $\mu$ is the gradient of a function.

**Proof**: If $X, Y$ are basic horizontal vector fields, then, as in Lemma 2.6(i), $\mathcal{H}L_U X = \mathcal{H}L_U Y = 0$ and

$$(L_U \tilde{\Omega})(X, Y) = U \left( \tilde{\Omega}(X, Y) \right) - \tilde{\Omega}(L_U X, Y) - \tilde{\Omega}(X, L_U Y)
= U(\tilde{\Omega}(X, Y)) = U(\Omega(X, Y)).$$

It follows that $\tilde{\Omega}$ is basic if and only if $U\{\Omega(X, Y)\} = 0$ for all basic horizontal vector fields $X, Y$. But by Lemma 2.6(ii), $U\{\tilde{\Omega}(X, Y)\} = d\mu^\flat(X, Y)$. The result follows. q.e.d.

**Corollary 3.2** If $\mu = \mathcal{H}\text{grad } \ln \rho$ for some function $\rho$. Then $\mu^\flat$ is closed relative to the horizontal distribution if and only if either $\tilde{\Omega} \equiv 0$ (the horizontal distribution is integrable), or $U(\ln \rho) = 0$, i.e. $\rho$ is a basic function.

**Proof**: Let $X, Y$ be horizontal vector fields. Then

$$d\mu^\flat(X, Y) = X(Y(\ln \rho)) - Y(X(\ln \rho)) - g(\mathcal{H}\text{grad } \ln \rho, [X, Y])
= g(\mathcal{V}\text{grad } \ln \rho, [X, Y]) = -U(\ln \rho)\tilde{\Omega}(X, Y).$$
The result follows.

Suppose that $\varphi : (M^3, g) \to (N^2, h)$ is a semi-conformal map with dilation $\lambda$ and with associated integrability 2-form $\tilde{\Omega}$ basic. We will suppose further that $M^3$ is a domain whose 2-cohomology vanishes and on which $\varphi$ is submersive with connected fibres. This is no loss of generality, since otherwise we work locally on an open set with these properties. Then we may write $\tilde{\Omega} = \varphi^*\Omega$ for some 2-form $\Omega$ on $N$. Clearly $\tilde{\Omega}$ is closed, since $d\tilde{\Omega} = d\varphi^*\Omega = \varphi^*d\Omega$. Since $H^2(M, \mathbb{R}) = 0$, we have $\tilde{\Omega} = d\tilde{\theta}$ for some 1-form $\tilde{\theta}$ (defined up to addition of a derivative). Set

$$\tilde{g} = \frac{\varphi^*h}{\lambda^2} + \tilde{\theta}^2.$$  

**Proposition 3.3** The map $\varphi : (M^3, \tilde{g}) \to (N^2, h)$ is both semi-conformal and harmonic, equivalently, $\varphi$ is a harmonic morphism with respect to the metric $\tilde{g}$.

**Proof:** For a semi-conformal map onto a surface, harmonicity is equivalent to the fibres being minimal ([1]). Let $\tilde{U}$ be a unit vertical vector field with respect to $\tilde{g}$. Then it is easily checked that the fibres are geodesic if and only if $L_{\tilde{U}}\tilde{\theta} = 0$ (see [3]). But

$$L_{\tilde{U}}\tilde{\theta} = d(\tilde{\theta}|\tilde{U}) + d\tilde{\theta}|\tilde{U} = \tilde{\Omega}|\tilde{U} = 0.$$

q.e.d.

We therefore see how, to a semi-conformal map $\varphi : (M^3, g) \to (N^2, h)$ with $\mu^h$ closed relative to the horizontal distribution, we can associate a harmonic morphism $\varphi : (M^3, \tilde{g}) \to (N^2, h)$. We now consider the converse problem: given a harmonic morphism $\varphi : (M^3, \tilde{g}) \to (N^2, h)$, construct a semi-conformal map $\varphi : (M^3, g) \to (N^2, h)$ with fibres having prescribed mean curvature $\mu$.

**Proposition 3.4** Let $\varphi : (M^3, \tilde{g}) \to (N^2, h)$ be a submersive harmonic morphism with dilation $\lambda$ and integrability 2-form $\tilde{\Omega}$. Let $\eta$ be a given 1-form and let $\theta$ be a 1-form satisfying the exterior differential equation

$$d\theta + \eta \wedge \theta = \tilde{\Omega}. \quad (12)$$

Suppose that $\theta(\tilde{U}) \neq 0$ at every point, where $\tilde{U}$ is a unit vertical vector with respect to $\tilde{g}$. Let $g = \tilde{g} + \theta^2$. Then $\varphi : (M^3, g) \to (N^2, h)$ is semi-conformal with fibres having mean curvature $\mathcal{H}(\eta^\sharp)$. Furthermore, if $\eta$ is the derivative of a basic function, then we can always solve (12) on any domain $M^3$ with $H^2(M^3, \mathbb{R}) = 0$.  

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Proof: Clearly \( \varphi : (M^3, g) \to (N^2, h) \) is semi-conformal. Set \( \Omega = d\theta \). Then by Lemma 2.2, \( \Omega(X, U) = -\mu^h(X) = -g(\mu, X) \), where \( X \) is horizontal (with respect to \( g \)) and \( \mu \) denotes the mean-curvature of the fibres. But since \( U \) and \( \tilde{U} \) are colinear (both being in the kernel of \( d\varphi \)), \( \tilde{\Omega} \) is \( U \). Then by (12), \( \Omega(X, U) = -\eta(X) \), i.e. \( \eta(X) = g(\mu, X) \) as required.

For the last part of the proposition, write \( \eta = d\ln \rho \), where \( \rho = \varphi \circ \varphi \). Then equation (12) has the form

\[
d(\rho \theta) = \rho \tilde{\Omega}.
\]

Then we can solve the equation \( d\omega = \rho \tilde{\Omega} \) on a domain with vanishing 2-cohomology if and only if \( d(\rho \tilde{\Omega}) = 0 \). But \( \rho \tilde{\Omega} = \varphi^* (\rho \Omega) \), which is clearly closed.

In order to construct a semi-conformal map from data on \( N^2 \) we proceed as follows. Let \( (N^2, h) \) be a Riemannian surface with \( H^2(N^2, \mathbb{R}) = 0 \). Let \( \lambda, \rho : N^2 \to \mathbb{R} \) be smooth positive functions and let \( \Omega \) be a smooth 2-form on \( N^2 \) (equivalently, we may let \( \sigma : N^2 \to \mathbb{R} \) be a smooth function and write \( \Omega = \sigma \mu^N \), where \( \mu^N \) is the volume form on \( N^2 \)). Set \( M^3 = N^2 \times (-\delta, \delta) \) for some \( \delta > 0 \) and let \( \varphi : M^3 \to N^2 \) be the projection. Define \( \tilde{\Omega} = \varphi^* \Omega \). Then \( d\tilde{\Omega} = 0 \) so that by vanishing 2-cohomology, we have \( \tilde{\Omega} = d\tilde{\theta} \) for some 1-form \( \tilde{\theta} \). On writing \( t \) for the coordinate of \( (-\delta, \delta) \), by replacing \( \tilde{\theta} \) with \( \tilde{\theta} + Rd\alpha \) if necessary (\( R \) a sufficiently large constant) we may suppose that \( \tilde{\theta}(\partial/\partial t) \neq 0 \) at every point. Set \( \lambda = \tilde{\lambda} \circ \varphi, \rho = \varphi \circ \rho \) and let \( \bar{g} = \frac{\varphi^2 h}{\lambda^2} + \tilde{\theta}^2 \). Then \( \varphi : (M^3, \bar{g}) \to (N^2, h) \) is a harmonic morphism. In fact, since the gradient of the dilatation is horizontal, it is a harmonic morphism of \( \text{Killing type} \), that is the fibres are the integral curves of a Killing vector field (see [4]).

Now let \( \theta \) solve the exterior differential equation

\[
d\theta + d\ln \rho \wedge \theta = \tilde{\Omega}.
\]

By Proposition 3.4, if we set \( g = \frac{\varphi^2 h}{\lambda^2} + \theta^2 \), then provided \( \theta(\partial/\partial t) \neq 0 \), the metric \( g \) is positive definite and \( \varphi : (M^3, g) \to (N^2, h) \) is a semi-conformal map with fibres having mean curvature \( \text{grad} \ln \rho \).

In the next section, we will establish conditions on the functions \( \lambda, \rho, \sigma \) defined above, which are equivalent to the property that \( (M^3, g) \) be a Ricci soliton.

## 4 Solitons constructed from data on a surface

Our aim in this section is to establish Theorem 1.1. The ansatz is derived from the construction of a semi-conformal mapping whose dilation and mean curvature of its
fibres are both basic.

Let \( \varphi : M^3 \to N^2 \) be a semi-conformal submersion with connected fibres, basic
dilation \( \lambda = \overline{\lambda} \circ \varphi \) and whose fibres have mean curvature which is the gradient of a
basic function: \( \mu = \text{grad} \ln \rho \) (\( \rho = \overline{\rho} \circ \varphi \)). A special type of horizontal vector field \( X \) is
given by the gradient of a basic function, that is \( X = \text{grad} \ln \nu \), where \( \nu = \overline{\nu} \circ \varphi \) with \( \overline{\varphi} : N \to \mathbb{R} \). We will now make this assumption on \( X \); in particular this implies
that \( \mathcal{L}_X g = 2\nabla d \ln \nu \). This need not imply that a corresponding soliton is of gradient
type; for this we require that \( E \) be the gradient of a function, which depends essentially
on the function \( f \). Since \( d\varphi(\text{grad} \ln \nu) = \lambda^2 \text{grad} \ln \nu \circ \varphi \), we have the correspondence
\( \lambda X = \lambda^2 \text{grad} \ln \nu \). Thus with reference to Remark 2 of the Introduction, we need to
make the substitution \( Y = X/\lambda^2 \).

Equation (6) for a soliton now has the form:

\[
\left\{ \lambda^2 K^N + \Delta^M \ln \lambda + \mu(\ln \lambda) \right\} g|_{H \times H} - \frac{1}{4} ||I||^2 g + 4g + \nabla d \ln \nu - (\mu^b)^2 + \{df + f u^b + \Omega^* \} \circ \theta = 0. \tag{13}
\]

An immediate necessary condition for this to be satisfied is the requirement that
\( \nabla d \ln \nu - (\mu^b)^2 \) be umbilic on \( H \), i.e. \( \nabla d \ln \nu|_{H \times H} - (\mu^b)^2 = \alpha g|_{H \times H} \) for some function \( \alpha : M \to \mathbb{R} \). We will first of all investigate this condition more fully. For a
covariant tensor \( S \), we will write \( S \circ X \) for its contraction with a vector \( X \), so that
\( (S \circ X)(Y_1, \ldots, Y_k) = S(X, Y_1, \ldots, Y_k) \).

**Lemma 4.1** Let \( F = \overline{F} \circ \varphi : M \to \mathbb{R} \) be any smooth basic function, then

\[
\nabla d F = \varphi^* \left\{ \nabla d \overline{F} + 2d \ln \overline{\lambda} \circ d \overline{F} - h(\text{grad} \ln \overline{\lambda}, \text{grad} \overline{F}) h \right\} + (\Omega|\text{grad} F) \circ \theta, \tag{14}
\]

with \( \Omega|\text{grad} F = \varphi^*(\overline{\lambda}^2 \overline{\Omega}|\text{grad} \overline{F}) \).

**Proof:** First let \( X, Y \) be horizontal vectors and set \( \overline{X} = d\varphi(X) \) etc.. Let \( \{\overline{X}_a\}_{a=1,2} \)
be an orthonormal frame on a domain of \( N \) and write \( X_a \) for the horizontal lift of \( \overline{X}_a \). Note that \( \{\lambda X_a\}_{a=1,2} \) is an orthonormal frame for \( H \). Then

\[
\nabla d F(X, Y) = -d F(\nabla^M Y) + X(Y(F)) = -d \overline{F}(d \varphi(\nabla^N Y)) + \overline{X}(\overline{Y}(\overline{F})) \circ \varphi.
\]

But

\[
d \varphi(\nabla^M Y) = d \varphi(g(\nabla^M Y, \lambda X_a) \lambda X_a)
= \frac{\lambda^2}{2} \left( X g(Y, X_a) + Y g(X, X_a) - X_a g(X, Y) \right)
\]
Let \( \varphi \) be any smooth basic function, then

\[
\Delta^M F + \mu(F) = \lambda^2 \Delta^N \varphi.
\]

Proof: We have \( \Delta^M F = \text{Tr} \nabla dF = \text{Tr}_H \nabla dF + \nabla dF(U, U) \). But \( \nabla dF(U, U) = -dF(\nabla_U U) = -\mu(F) \). On taking the trace over the horizontal space in (14), the formula follows. q.e.d.

We can now characterize the umbilicity condition discussed above.

Corollary 4.3 \( \nabla d \ln \nu \mid_\mathcal{H} \otimes \mathcal{H} - (\mu^b)^2 \) is umbilic on \( \mathcal{H} \) if and only if

\[
\nabla d \ln \nu + 2 d \ln \lambda \otimes d \ln \nu - (d \ln \nu)^2 = \alpha h,
\]

for some function \( \alpha : N \to \mathbb{R} \).
Proof: For the mean curvature, we have \( \mu^b = d \ln \rho = \varphi^*(d \ln \rho) \), so that from (14),

\[
\nabla d \ln \nu|_{\mathcal{H} \times \mathcal{H}} - (\mu^b)^2 = \varphi^* \left\{ \nabla d \ln \nu + 2d \ln \lambda \otimes d \ln \nu - h(\text{grad} \ln \lambda, \text{grad} \ln \nu) h - (d \ln \rho)^2 \right\}.
\]

We require the bracket to be proportional to \( h \), which is the assertion of the corollary.

q.e.d.

By taking traces, we have the following consequence.

**Corollary 4.4** If \( \nabla d \ln \nu|_{\mathcal{H} \times \mathcal{H}} - (\mu^b)^2 \) is umbilic on \( \mathcal{H} \times \mathcal{H} \), then

\[
\nabla d \ln \nu|_{\mathcal{H} \times \mathcal{H}} - (\mu^b)^2 = \frac{1}{2} \left( \Delta^M \ln \nu + \mu(\ln \nu) - |\mu|^2 \right) g|_{\mathcal{H} \times \mathcal{H}}
\]

\[
= \frac{1}{2} \varphi^* \left\{ (\Delta^N \ln \nu - |\text{grad} \ln \rho|^2) h \right\}.
\]

**Lemma 4.5** Let \( \tilde{\Omega} \) be the integrability 2-form associated to \( \varphi \) and let \( \tilde{g} \) be the corresponding metric (cf. Section 3). Then \( ||\tilde{\Omega}||^2_{\tilde{g}} = ||I||^2_g \). In particular, \( ||I||^2_g \) is a basic function.

Proof: Let \( Y \) be horizontal and basic with respect to \( g \): \( d \varphi(Y) = \tilde{Y}, g(Y, U) = 0 \). Then, recalling that \( \tilde{g} = \varphi^* h + \bar{g}^2 \), it follows that \( \tilde{Y} = Y - \frac{\tilde{g}(Y)}{\tilde{g}(U)} U \) is horizontal and basic with respect to \( \tilde{g} \). With the same notations, let \( \{ \lambda X_a \} \) be an orthonormal basis on \( N \), so that \( \{ \lambda X_a \} \) is an orthonormal basis for \( \mathcal{H} \) on \( (M, g) \) and \( \{ \lambda \tilde{X}_a \} \) is an orthonormal basis for \( \tilde{H} \) on \( (M, \tilde{g}) \). But \( \tilde{\Omega}|U = 0 \), so that

\[
||\tilde{\Omega}||^2_{\tilde{g}} = 2\lambda^4 \tilde{\Omega}(\lambda \tilde{X}_1, \lambda \tilde{X}_2)^2 = 2\lambda^4 \tilde{\Omega}(X_1, X_2)^2
\]

\[
= 2\lambda^4 \Omega(X_1, X_2)^2 = 2\Omega(\lambda X_1, \lambda X_2)^2 = ||I||^2_g.
\]

For the last part:

\[
||\tilde{\Omega}||^2_{\tilde{g}} = 2\lambda^4 \tilde{\Omega}(\lambda \tilde{X}_1, \lambda \tilde{X}_2)^2 = 2\lambda^4 \varphi^* \Omega(\lambda \tilde{X}_1, \lambda \tilde{X}_2)^2 = 2\lambda^4 \Omega(X_1, X_2)^2 \circ \varphi = \lambda^4 ||\Omega||^2_h.
\]

But since \( \lambda = \lambda \circ \varphi \) is basic, so is \( ||I||^2_g \).

q.e.d.

It will be convenient in what follows to write \( \frac{1}{4} ||I||^2 = \psi = \varphi^* \varphi \). Let us review the soliton equation (13) under the assumption that the umbilicity condition (2)(u) is satisfied. By the above calculations it takes the form:

\[
0 = \left\{ \nabla^\mathcal{H} K^N + \nabla^\mathcal{H} \Delta^N \ln \lambda + \frac{\lambda^2}{2} \left( \Delta^N \ln \nu - |\text{grad} \ln \rho|^2 \right) - \nabla \psi + A \right\} g|_{\mathcal{H} \times \mathcal{H}}
\]

\[
+ \left\{ df + f \mu^b + \Omega | \text{grad} \ln \nu + d^* \Omega - \psi \theta + A \theta \right\} \otimes \theta,
\]
with its component parts:

\[
\begin{cases}
(i) & K^N + \Delta^N \ln \lambda + \frac{1}{2} \left( \Delta^N \ln \sigma - |\text{grad} \ln \rho|^2 \right) - \frac{\sigma - A}{\lambda} = 0 \\
(ii) & df + f \mu^b + \Omega |\text{grad} \ln \nu + d^* \Omega - (\psi - A) \theta = 0.
\end{cases}
\] (18)

We will now study (18)(ii). Recall that \( \Omega = \overline{\Omega} - \mu^b \wedge \theta = \varphi^* \overline{\Omega} - d \ln \rho \wedge \theta \). The following lemma is easily established.

**Lemma 4.6** For two 1-forms \( \omega, \eta \), we have

\[
d^*(\omega \wedge \eta) = d^*\omega \cdot \eta - d^*\eta \cdot \omega - [\omega^b, \eta^b].
\]

In particular

\[
d^*(\mu^b \wedge \theta) = (d^* \mu^b - |\mu|^2) \theta = -(\Delta^M \rho - |\mu|^2) \theta = -\lambda^2 (\Delta^N \ln \sigma) \circ \varphi \cdot \theta.
\]

**Lemma 4.7**

\[
d^* \overline{\Omega} = \lambda^2 \varphi^* \left\{ d^* \overline{\Omega} + \overline{\Omega} |\text{grad} \ln (\rho \lambda^{-2}) \right\} + 2 \psi \theta.
\]

**Proof:** Let \( \{X_a\} \) be an orthonormal frame on a domain of \( N \) and let \( X_a \) be the horizontal lift of \( X_a \) (\( a = 1, 2 \)). Set \( e_a = \lambda X_a \) and let \( e_3 = U \), so that \( \{e_1, e_2, e_3\} \) is an orthonormal frame on a domain of \( M \). Then

\[
d^* \overline{\Omega}(X_b) = -(\nabla_{e_i} \overline{\Omega})(e_i, X_b) = -\lambda^2 (\nabla_{X_a} \overline{\Omega})(X_a, X_b) - (\nabla_U \overline{\Omega})(U, X_b)
\]

\[
= -\lambda^2 X_a \{ \overline{\Omega}(X_a, X_b) \} + \overline{\Omega}(\nabla_{X_a} X_a, X_b) + \overline{\Omega} U (\nabla_U U, X_b)
\]

\[
= -\lambda^2 X_a \{ \overline{\Omega}(X_a, X_b) \} \circ \varphi \} + \lambda^2 \overline{\Omega}(d\varphi(\nabla_{X_a} X_a), X_b)
\]

\[
+ \lambda^2 \overline{\Omega}(X_a, d\varphi(\nabla_{X_a} X_b)) + \overline{\Omega}(d\varphi(\mu), X_b).
\]

We now employ the expression (15) for \( d\varphi(\nabla_{X_a} X_b) \), to give

\[
d^* \overline{\Omega}(X_b) = \lambda^2 \left\{ d^* \overline{\Omega}(X_b) - 2 \overline{\Omega}(\text{grad} \ln \lambda, X_b) + \overline{\Omega}(\text{grad} \ln \sigma, X_b) \right\} \circ \varphi. \quad (19)
\]

On the other hand

\[
d^* \overline{\Omega}(U) = -(\nabla_{e_i} \overline{\Omega})(e_i, U) = -\lambda^2 (\nabla_{X_a} \overline{\Omega})(X_a, U)
\]

\[
= \lambda^2 \overline{\Omega}(X_a, \nabla_{X_a} U) = \lambda^2 \overline{\Omega}(X_a, g(\nabla_{X_a} U, e_b)e_b)
\]

\[
= -g(U, \nabla_{e_a} e_b) \overline{\Omega}(e_a, e_b) = -\frac{1}{2} g(e_a, e_b) \overline{\Omega}(e_a, e_b) = \frac{1}{2} ||I||^2.
\]

q.e.d.
We can now express equation (18)(ii) in the form:

\[
d(f \rho) = -\rho \lambda^2 \varphi^* \left\{ d^* \Omega + \Omega \right\} \nabla \ln(\rho \nu \lambda^{-2}) + \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \theta.
\]

(20)

Written in this way, we see that, on a simply connected domain, we can find a function \( f \) such that (18)(ii) is satisfied if and only if the derivative of the right-hand side vanishes, that is, if and only if

\[
d \ln(\lambda^2 \rho) \wedge \varphi^* \left\{ d^* \Omega + \Omega \right\} \nabla \ln(\rho \nu \lambda^{-2}) + d \left\{ \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \right\} \circ \varphi \wedge \theta
\]

\[
+ d \left\{ \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \right\} \circ \varphi 
\]

\[
+ d \left\{ \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \right\} \circ \varphi \cdot \varphi^* \Omega = 0.
\]

Here, \( \Delta \Omega \) is the Laplacian on forms: \( \Delta = d d^* + d^* d \). On simplifying and separating into components, we have equivalence with the following pair of equations on \( N \):

(a) \[ d \ln(\lambda^2 \rho) \wedge \left\{ d^* \Omega + \Omega \right\} \nabla \ln(\rho \nu \lambda^{-2}) + d \left\{ \Omega \right\} \nabla \ln(\rho \nu \lambda^{-2}) + \Delta \Omega \]

\[ + \left\{ \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \right\} \circ \varphi \cdot \varphi^* \Omega = 0.
\]

(b) \[ \left\{ \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \right\} \wedge d \nabla^2
\]

\[ + d \left\{ \Delta^N \ln \rho - h(\nabla \ln \rho, \nabla \ln \nu) + \frac{A + \psi}{\lambda^2} \right\} = 0.
\]

In order to simplify these, it is useful to write \( \Omega = \sigma \mu^N \), where \( \mu^N \) is the volume form on \( (N, h) \). Then it is easily checked that the following relation holds:

\[ 2\psi = \lambda^4 \sigma^2. \]

(21)

The following identities are useful.

**Lemma 4.8** Let \( \alpha, \beta \) be arbitrary smooth functions on \( N \), then

(i) \( d \alpha \wedge (\mu^N \nabla \beta) = h(\nabla \alpha, \nabla \beta) \mu^N \);

(ii) \( d(\mu^N \nabla \alpha) = \Delta^N \alpha \cdot \mu^N \);

(iii) \( d^*(\alpha \mu^N) = -\mu^N \nabla \alpha \).
On applying these to equation (a) above, we obtain
\[
\left\{ \Delta^N \ln(\mathcal{F}^2) \mathcal{F}^{-2} \right\} + |\text{grad} \ln \mathcal{F}|^2 - |\text{grad} \ln(\mathcal{X}^2)|^2 \\
+ h(\text{grad} \ln(\mathcal{X}^2), \text{grad} \ln \mathcal{F}) + \frac{A + \psi}{\lambda^2} \right\} \Omega = 0.
\]

From (21) we can eliminate \( \sigma \). Then, either \( \Omega = 0 \), i.e. \( \psi = 0 \) and the horizontal distribution is integrable, or equation (ii)(a) of Theorem 1.1 holds. Equation (ii)(b) of the theorem is immediate from (b) above. This establishes Theorem 1.1.

We now isolate the condition of constant curvature.

**Proposition 4.9** Let \((M^3, g)\) be a Ricci soliton as in Theorem 1.1. Then \((M^3, g)\) has constant curvature if and only if both

(i) either \( \bar{\psi} = 0 \) (\( \mathcal{H} \) integrable) or \( \mathcal{F}^2/\bar{\psi}^{-1/2} = \text{const.} \); and

(ii) \( \{ K^N + \Delta^N \ln(\mathcal{X}^2)^{-1} \} + |\text{grad} \ln \mathcal{F}|^2 - h(\text{grad} \ln \mathcal{X}, \text{grad} \ln \mathcal{F}) - \frac{2\bar{\psi}}{\lambda^2} \} h \\
+ \nabla d \ln \mathcal{F} + 2d \ln \mathcal{X} \odot d \ln \mathcal{F} - (d \ln \mathcal{F})^2 = 0.
\]

are satisfied.

**Proof**: The manifold \( M^3 \) has constant curvature if and only if, for all unit horizontal vectors \( Y \), (i) Ricci \( (Y, U) = 0 \) and (ii) Ricci \( (Y, Y) = \text{Ricci} (U, U) \). We now employ the expressions for the Ricci curvature given by Lemmata 2.4, 2.5 and 2.7, as well as the formula for the divergence \( d^* \Omega \) given by Lemmata 4.6 and 4.7. We omit the details.

q.e.d.

The following example is instructive, even though the soliton behind it is the trivial one \((M^3, g) = \mathbb{R}^3\). It is based on an example in [2].

**Example 4.10** (Helix example) Choose cylindrical coordinates for \( \mathbb{R}^3 \): \((r, \alpha, z)\), where \( r^2 = x_1^2 + x_2^2 \), \( \tan \alpha = x_2/x_1 \) and \( z = x_3 \). Define \( \varphi : \mathbb{R}^3 \to \mathbb{R}^2 \) by

\[
\varphi(r, \alpha, z) = \left( \ln \left\{ \frac{1 + cr^2}{\sqrt{c} r} - 1 \right\} + \sqrt{1 + cr^2}, -\alpha + \sqrt{c} z \right),
\]

where \( c \) is an arbitrary positive constant. Then \( \varphi \) is semi-conformal with fibres helices which wind around the concentric cylinders \( r = \text{constant} \). Its gradient is given by

\[
\text{grad} \varphi = \left( \frac{\sqrt{1 + cr^2}}{r} \frac{\partial}{\partial r}, -\frac{1}{r^2} \frac{\partial}{\partial \alpha} + \sqrt{c} \frac{\partial}{\partial z} \right),
\]
and its dilation by $\lambda^2 = (1 + cr^2)/r^2$. Let $(u, v)$ denote coordinates on the codomain $\mathbb{R}^2$; then $u = u(r)$ with $\frac{du}{dr} = \frac{\sqrt{1 + cr^2}}{r}$, whereas $v = v(\alpha, z)$. The objects associated to our construction of solitons are given as follows:

$$\begin{aligned}
\theta &= -\frac{\sqrt{cr^2}}{\sqrt{1 + cr^2}} d\alpha - \frac{1}{\sqrt{1 + cr^2}} dz \\
\Omega &= -\frac{\sqrt{cr}(2 + cr^2)}{(1 + cr^2)^{3/2}} d\alpha \wedge d\psi + \frac{cr}{(1 + cr^2)^{3/2}} dr \wedge dz \\
\mu^\flat &= d \ln \left( \frac{1}{\sqrt{1 + cr^2}} \right) \\
\tilde{\Omega} &= \frac{2\sqrt{cr^2}}{(1 + cr^2)^{3/2}} du \wedge dv.
\end{aligned}$$

Note that the functions $\lambda$ and $\rho$ are basic, so we are in the situation of Theorem 1.1. It is routine to check that the equations of the Theorem are satisfied.

**Example 4.11** In this example we generalize a construction of Ivey [15] to include examples of non-gradient type. In particular we will see the geometry Sol arising in this way. Ivey considers doubly warped product metrics of the form $g = dx_1^2 + a(x_1)^2 dx_2^2 + b(x_1)^2 dx_3^2$ (but where $dx_2^2$ and $dx_3^2$ are to be considered as metrics on a sphere and an Einstein manifold, respectively). Let $N^2 = \mathbb{R}^2$ with metric $h = dx_1^2 + a(x_1)^2 dx_2^2$ and let $\varphi : (\mathbb{R}^3, g) \to (\mathbb{R}^2, h)$ be the projection $\varphi(x_1, x_2, x_3) = (x_1, x_2)$. Then $\varphi$ is semi-conformal with dilation $\lambda \equiv 1$ and with corresponding form dual to its kernel given by $\theta = b(x_1) dx_3$. The dual of the mean curvature of the fibres is given by the form $\mu^\flat = d \ln b^{-1}$ so that $\rho = b^{-1}$. Then $\tilde{\Omega} = d\theta + d\ln \rho \wedge \theta = 0$ and $\mathcal{H}$ is integrable. In fact Ivey takes the vector field $E$ in (1) to be the gradient of a function which depends on $x_1$ only. This would correspond to the vanishing of the function $f$ in the decomposition $E = X + fU$. From (20), we see that this corresponds to the vanishing of the constant on the right-hand side of equation (ii)(b) of Theorem 1.1. We will suppose that the function $\overline{\nabla} = \overline{\nabla}(x_1)$, but we could conceivably have $f$ non-zero, leading to a more general situation. On noting that $K^N = -a''/a$ and writing $\beta = \ln \overline{\nabla}$, the equations for a soliton become:

\[
\begin{cases}
\text{(i)} & -\frac{a''}{a} + \frac{1}{2} \left( \beta'' + \frac{a'}{a} \beta' - \left( \frac{\beta'}{\beta} \right)^2 \right) + A = 0 \\
\text{(ii)(b)} & \left( \frac{\beta'}{\beta} \right)^2 - \frac{\beta''}{\beta} - \frac{a' \beta'}{a} + \frac{1}{\beta} \beta' = \text{const.} \\
\text{(u)} & \beta'' - \left( \frac{\beta'}{\beta} \right)^2 = \frac{a'}{a} \beta'.
\end{cases}
\]

On setting $a = b$, we retrieve the solitons of Example 2.10. Another particular solution is given by $a = b^{-1}$ with $a = e^{x_1}$, $A = 2$ and the constant on the right-hand
side of (ii)(b) equal to 2. This gives the metric \( g = dx_1^2 + e^{2x_1} dx_2^2 + e^{-2x_1} dx_3^2 \), which corresponds to the 3-dimensional geometry Sol. We can be more explicit in describing the vector field \( E \) and seeing if the soliton is of gradient type. For this we apply equation (18)(ii), which takes the form

\[
\begin{align*}
df + f dx_1 + 4e^{-x_1} dx_3 &= 0.
\end{align*}
\]

This is solved by

\[
f = -4x_3 e^{-x_1}.
\]

Now \( E = -\mu + fU + \text{grad } \ln \nu \), which is (locally) a gradient if and only if \( dE^\flat = 0 \), i.e. if and only if \( df \wedge \theta + f\Omega = 0 \). But it is easily checked that \( df \wedge \theta + f\Omega = 8x_3e^{-2x_1} dx_1 \wedge dx_3 \), which is clearly non-vanishing, so Sol, viewed as a soliton in this way, is not of gradient type. In fact we can exhibit its soliton flow:

\[
E = -4x_3 \frac{\partial}{\partial x_3} - 2 \frac{\partial}{\partial x_1}.
\]

## 5 The case of minimal fibres

When the potential function \( \rho \) is constant, the fibres of \( \varphi \) are minimal and the equations for a soliton become more accessible; in this case, we are able to give a complete local description of the solutions as specified by Corollary 1.2. Suppose then that \( \rho \) is constant. From (2)(ii)(b), we see that \( \overline{\varphi} \) is also a constant, \( C \) say. The equations (2) now become

\[
\begin{align*}
&\left\{ \begin{array}{l}
(i) \quad \Delta^N \ln \overline{\lambda} + K^N + \frac{1}{2} \Delta^N \ln \overline{\nu} + \frac{(A-C)}{\lambda} = 0 \\
(ii) \quad \Delta^N \ln \overline{\nu} + \frac{(C+A)}{\lambda} = 0 \\
(u) \quad \nabla d \ln \overline{\nu} + 2d \ln \overline{\lambda} \odot d \ln \overline{\nu} = \alpha h,
\end{array} \right.
\end{align*}
\]

for some function \( \alpha : N \to \mathbb{R} \), with (ii) vacuous whenever \( H \) is integrable, i.e. \( C = 0 \).

**Proposition 5.1** Let \( \overline{\lambda}, \overline{\nu} \) satisfy the system (23) with \( C \neq 0 \). Then either \( C + A = 0 \) and the corresponding soliton has constant curvature, or \( 3C - A = 0 \) and, in terms of a local isothermal coordinate \( z = x + iy \) on a simply connected domain of \( N^2 \), we have

\[
\overline{\lambda}(z) = \frac{B}{|v(z)|}, \quad \frac{\partial \ln \overline{\nu}}{\partial z} = \frac{Cv(z)}{B^2} \int \overline{v(z)} d\overline{\nu}, \tag{24}
\]

where \( v(z) \) is a non-vanishing holomorphic function and \( B \) is a positive constant.
Remark: In fact equations (2) and (23) depend only on the gradient of \( \gamma = \ln \nu \), so we only expect to be able to explicitly express \( \gamma_x - i \gamma_y = 2 \frac{\partial \gamma}{\partial z} \), as in the proposition above.

Proof: First note that, on combining Proposition 4.9 with equation (23)(i), we see that a solution has constant curvature if and only if \( C + A = 0 \). In what follows, write \( \beta = \ln \lambda, \gamma = \ln \nu \). Since solutions to the system (23) are invariant under conformal changes \( h \to \tilde{h} = e^{2u}h \), and since we can always choose a local isothermal coordinate \( z = x + iy \) with respect to which \( h = \delta(z)^2(dx^2 + dy^2) \), it is no loss of generality to work on a simply connected domain \( U \subset \mathbb{R}^2 \) with metric \( h = dx^2 + dy^2 \). Write \( \gamma_z = \frac{\partial \gamma}{\partial z} = \frac{1}{2} \left( \frac{\partial \gamma}{\partial x} - i \frac{\partial \gamma}{\partial y} \right) \), etc.. The system (23) now takes the form

\[
\begin{align*}
(\text{i}) & \quad 4 \beta_{z\bar{\tau}} - \left( \frac{3C-A}{2} \right) e^{-2\beta} = 0 \\
(\text{ii}) & \quad 4 \gamma_{z\bar{\tau}} + (C + A)e^{-2\beta} = 0 \\
(\text{u}) & \quad \gamma_{zz} + 2 \gamma z \beta_z = 0.
\end{align*}
\]

On differentiating (u) with respect to \( \tau \), (ii) with respect to \( z \) and combining, we obtain

\[ \gamma_z \beta_{z\tau} = 0. \]

Thus either \( \gamma_z \equiv 0 \) on some open set, in which case from (ii) we have \( C + A = 0 \), so that the corresponding soliton has constant curvature, or \( \beta_{z\tau} \equiv 0 \) on some open set, which from (i) implies that \( 3C - A = 0 \). Henceforth we will suppose that we are in the latter situation, so that \( \beta_{z\tau} \equiv 0 \) and \( A = 3C \). Note that since \( C \geq 0 \), this implies that \( A \geq 0 \) and any corresponding soliton is expanding. The system (25) now takes the form

\[
\begin{align*}
(\text{i}) & \quad 4 \beta_{z\tau} = 0 \\
(\text{ii}) & \quad \gamma_{z\tau} = Ce^{-2\beta} \\
(\text{u}) & \quad \gamma_{zz} + 2 \gamma z \beta_z = 0.
\end{align*}
\]

If \( \gamma_z \equiv 0 \), then \( C = 0 \), which in turn implies that \( A = 0 \) and we are in the constant curvature case again. So we will work on a domain where \( \gamma_z \neq 0 \). Then

\[
\frac{\gamma_{zz}}{\gamma_z} = -2 \beta_z \quad \Rightarrow \quad \frac{\partial}{\partial z} \ln \gamma_z = -2 \beta_z \quad \Rightarrow \quad \gamma_z = \mu(z, \bar{z})e^{-2\beta},
\]

where \( \frac{\partial \mu}{\partial z} = 0 \), i.e. where \( \mu = \mu(z) \) is antiholomorphic in \( z \). Then

\[ \gamma_{z\tau} = (\mu' - 2\mu \beta_{z\tau}) e^{-2\beta}, \]

and from (26) we require that \( C = \mu' - 2\mu \beta_{z\tau} \). Thus

\[ \beta_{z\tau} = \frac{\mu' - C}{2\mu}, \]
with $\mu = \mu(\overline{z})$ antiholomorphic. Since the right-hand side of this expression is antiholomorphic, we can integrate along any path in a simply connected domain. To write this integral more conveniently, we let $p(\overline{z})$ be an antiholomorphic function which is a primitive of $1/\mu$, i.e. $p'(\overline{z}) = 1/\mu(\overline{z})$. Then

$$\beta = \int \frac{\mu' - C}{2\mu} \cdot d\overline{z} = -\frac{1}{2} \ln p'(\overline{z}) - \frac{C}{2} p(\overline{z}) + q(z),$$

with $q(z)$ holomorphic in $z$ ($\frac{\partial q}{\partial \overline{z}} = 0$).

Let us change notation once more, by first setting $r(z) = e^{p(z)}$, $s(z) = e^{q(z)}$. Then

$$\beta = -\frac{1}{2} \ln \left\{ \left( \frac{r^C}{C} \right)' s^{-2} \right\}.$$

Finally, write $u(\overline{z}) = \left( \frac{r^C}{C} \right)'$ and $v(z) = s(z)^{-2}$, to obtain

$$\beta = -\frac{1}{2} \ln (u(\overline{z})v(z)), $$

where we take the principal branch of $\ln$ and require that $u(\overline{z})v(z)$ be real and positive for all $z$. But this is easily seen to be equivalent to the condition

$$u(\overline{z}) = av(z),$$

with $a$ a real positive constant. We now have

$$\beta = -\frac{1}{2} \ln \left( a |v(z)|^2 \right),$$

where $v(z)$ is a holomorphic function, which we require to be non-vanishing to avoid singular points. In particular, this gives

$$\lambda = \frac{B}{|v(z)|},$$

with $B$ a positive constant.

Returning to $\gamma_z$; from (27), this is given by

$$\gamma_z = \mu e^{-2\beta} = \frac{1}{p} a |v(z)|^2.$$

But $p' = \frac{\mu'}{r} = \frac{r^C}{r^C}$, with $r^C = \int C u(\overline{z}) d\overline{z}$. On noting that $a^{-1/2} = B$, we obtain

$$\gamma_z = \frac{C v(z)}{B^2} \int \overline{v(z)} d\overline{z},$$

as required.

q.e.d.
Corollary 5.2  Any solution to the equations (24) determines a Riemannian 3-manifold which is locally isometric to the geometry Nil.

Proof: We apply the ansatz of Theorem 1.1. Without loss of generality, we may take the constants $B = 1$ and $C = \frac{1}{2}$. Then

$$\Omega = \frac{i}{2} v(z) \bar{v}(z) \, dz \wedge d\bar{z} = d \left( \frac{i}{2} \int v(z) \, dz \, \bar{v}(z) \, d\bar{z} \right).$$

Let $u(z)$ be a primitive for $v(z): u'(z) = v(z)$. Then

$$\Omega = d \left( \text{Re} \left\{ \frac{i}{2} u \, d\bar{u} \right\} + dt \right),$$

to give the metric $g$ in the form

$$g = du \, d\bar{u} + \left( \text{Re} \left\{ \frac{i}{2} u \, d\bar{u} \right\} + dt \right)^2.$$  

We claim this is the metric for Nil. Indeed, if we take $u = y_1 + iy_2$ as a local coordinate, then

$$g = dy_1^2 + dy_2^2 + (y_1 dy_2 + dy_3)^2$$  

(28)

where we have set $y_3 = t - \frac{y_1 y_2}{2}$. q.e.d.

The geometry Nil as a soliton: On expressing the metric $g$ for Nil as in (28), we can solve (18)(ii) for $f$ to obtain the soliton flow $E$ explicitly. Indeed (18)(ii) becomes

$$df + y_1 dy_2 + y_2 dy_1 + 2 dy_3 = 0,$$

which has solution $f = -y_1 y_2 - 2 y_3$. Then the soliton is of gradient type if and only if

$$dE^\flat = df \wedge \theta + f \Omega = 0.$$  

But

$$df \wedge \theta + f \Omega = -2(y_1 y_2 + y_3) dy_1 \wedge dy_2 - y_2 dy_1 \wedge dy_3 + y_1 dy_2 \wedge dy_3,$$

which is non-vanishing and so Nil, viewed as a soliton in this way, is not of gradient type. The soliton flow $E$ is given explicitly by

$$E = -y_1 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_2} - 2y_3 \frac{\partial}{\partial y_3}$$  

(29)

It remains to consider the case when $C = 0$, so that the horizontal distribution is integrable and equations (23) become the system:

$$\begin{align*}
(i) \quad & \Delta^N \ln \lambda + K^N + \frac{1}{2} \Delta^N \ln \nu + \frac{1}{\lambda} = 0 \\
(ii) \quad & \nabla d \ln \nu + 2d \ln \lambda \odot d \ln \nu = \alpha h,
\end{align*}$$
On replacing $h$ by the conformally related metric $h/\lambda^2$, we may suppose that $\lambda \equiv 1$. But then the system becomes

$$
\begin{align*}
(i) \quad & K^N + \frac{1}{2} \Delta N \ln \nu + A = 0 \\
(ii) \quad & \nabla d \ln \nu = \alpha h.
\end{align*}
$$

But this is precisely the equation for a 2-dimensional gradient soliton: $-K^N h = \nabla d \ln \nu + Ah$. This proves Corollary 1.2.

**The case of $SL_2(\mathbb{R})$:** The geometry $SL_2(\mathbb{R})$ naturally admits a semi-conformal map with minimal fibres $\varphi : SL_2(\mathbb{R}) \to H^2$ onto the hyperbolic plane. Specifically, if we write its metric in the form

$$g = \frac{dx_1^2 + dx_2^2}{x_2^2} + \left(\frac{dx_1}{x_2} + dx_3\right)^2,$$

then the projection is given by $\varphi(x_1, x_2, x_3) = (x_1, x_2)$, where we take the metric on the codomain to be $h = \frac{dx_1^2 + dx_2^2}{x_2^2}$. This fact enables us to establish the following theorem.

**Theorem 5.3** The geometry $SL_2(\mathbb{R})$ admits no soliton structure.

**Proof**: We apply equation (6) directly. We have $\theta = \frac{dx_1}{x_2} + dx_3$, so that $\Omega = d\theta = (dx_1 \wedge dx_2)/x_2^2$ and $\psi = \frac{1}{4}||I||^2 = \frac{1}{4}||\Omega||^2 = \frac{1}{2}$. From Lemma 4.7, $d^* \Omega = 2\psi \theta = \theta$, so that equation (6) becomes

$$
\left(A - \frac{3}{2}\right) g + \frac{1}{2} \mathcal{L}X g + df \cup \theta + 2\theta^2 = 0. \tag{30}
$$

Thus $SL_2(\mathbb{R})$ admits a soliton structure if and only if this has a solution for $f$, $X$ and $A$.

Let $Y_1 = x_2 \partial_1 - \partial_3$, $Y_2 = x_2 \partial_2$ be an orthonormal frame for $\mathcal{H}$, and let $X = \alpha Y_1 + \beta Y_2$. Then a routine calculation gives

$$
\frac{1}{2} \mathcal{L}X g = \frac{1}{x_2} (\alpha dx_2 - \beta dx_1) \left(\frac{2dx_1}{x_2} + dx_3\right) + \frac{\alpha dx_1}{x_2} + \frac{\beta dx_2}{x_2}.
$$

On substituting into (30) and equating the various coefficients of $dx^i dx^j$ to zero, we are led to the following system of equations:

$$
\begin{align*}
(i) \quad & 2A - 1 - 2\beta + x_2 \partial_1 \alpha + x_2 \partial_1 f = 0 \\
(ii) \quad & A - \frac{3}{2} + x_2 \partial_2 \beta = 0 \\
(iii) \quad & A + \frac{1}{2} + \partial_3 f = 0 \\
(iv) \quad & \frac{2\alpha}{x_2} + \partial_2 \alpha + \partial_1 \beta + \partial_2 f = 0 \\
(v) \quad & 2(A + \frac{1}{2}) - \beta + \partial_3 \alpha + \partial_3 f + x_2 \partial_1 f = 0 \\
(vi) \quad & \alpha + \partial_3 \beta + x_2 \partial_2 f = 0
\end{align*} \tag{31}
$$

25
We claim this has no solution in \( \alpha, \beta, f \), whatever the value of the constant \( A \).

It is convenient to write \( a = A + \frac{1}{2} \). Then (iii) and (ii) imply

\[
\begin{align*}
  f &= -ax_3 + p(x_1, x_2) \\
  \beta &= -(a - 2) \ln x_2 + q(x_1, x_3)
\end{align*}
\]

for functions \( p = p(x_1, x_2), q = q(x_1, x_3) \). From (vi), \( \alpha = -\partial_3 q - x_2 \partial_2 p \), and we obtain the following system in \( p \) and \( q \):

\[
\begin{align*}
  (\text{vii}) \quad 2(a - 1) + 2(a - 2) \ln x_2 - 2q - x_2 \partial_13q - x_2^2 \partial_21p + x_2 \partial_1p &= 0 \\
  (\text{viii}) \quad -\frac{2}{x_2^2} \partial_3 q - 2\partial_2 p - x_2 \partial_22p + \partial_1 q &= 0 \\
  (\text{ix}) \quad a + (a - 2) \ln x_2 - q - \partial_33 q + x_2 \partial_1p &= 0
\end{align*}
\]

On differentiating (ix) with respect to \( x_2 \), substituting into (vii) and using (ix) once more, we obtain the equation \( a - 4 - x_2 \partial_13q + 2\partial_33q = 0 \), which, on integrating with respect to \( x_3 \), implies that

\[
(a - 4)x_3 - x_2 \partial_1q + 2\partial_3q = \tilde{r}(x_1, x_2),
\]

for some function \( \tilde{r} = \tilde{r}(x_1, x_2) \). But since \( q \) is independent of \( x_2 \), we must have \( -\partial_1q = \partial_2\tilde{r} \), which implies that \( \tilde{r} = -x_2 \partial_1q + r(x_1) \), for a function \( r = r(x_1) \). Therefore \( (a - 4)x_3 + 2\partial_3q = r(x_1) \), which, on integrating with respect to \( x_3 \), gives

\[
q = \frac{1}{2} \left\{ x_3 r(x_1) + s(x_1) - (a - 4) \frac{x_3^2}{2} \right\},
\]

for some function \( s = s(x_1) \). But the fact that \( \tilde{r} \) is independent of \( x_3 \) shows that \( r' \) vanishes and so \( r \) is constant. We can now substitute back into the system (32) to obtain:

\[
\begin{align*}
  (\text{x}) \quad 2(a - 1) + 2(a - 2) \ln x_2 - x_3 r - s + \frac{1}{2}(a - 4)x_3^2 - x_2^2 \partial_12p + x_2 \partial_1p &= 0 \\
  (\text{xi}) \quad -\frac{1}{x_2} (r - (a - 4)x_3) - 2\partial_2 p - x_2 \partial_22p + \frac{1}{2}s' &= 0 \\
  (\text{xii}) \quad a + (a - 2) \ln x_2 - \frac{1}{2} \left( r x_3 + s - \frac{a - 4}{2} x_3 \right) + \frac{1}{2}(a - 4) + x_2 \partial_1p &= 0
\end{align*}
\]

Then (xi) implies that \( a = 4 \), which, on substituting into (xii) gives \( r = 0 \). From (xi), we have \( 2\partial_2 p + x_2 \partial_22p = \frac{1}{2} s' \), which implies that \( x_2 \partial_2 p + p = \frac{x_2}{2} s' + t(x_1) \), for some function \( t = t(x_1) \). But then \( x_2 \partial_12p + \partial_1p = \frac{x_2}{2} s'' + t' \). Equations (x) and (xii) now quickly lead to a contradiction.

q.e.d.

The other geometries: We can apply the same techniques as above to the other geometries (and in principle to any 3-manifold admitting a semi-conformal map to a surface),
to obtain a system of equations similar to (31). The calculations are long and tedious so we omit the details, but a complete set of solutions can be obtained. We summarize the conclusions as follows: up to addition of a Killing vector field, the soliton flows $E$ on $\text{Nil}$ given by (29) and on $\text{Sol}$ given by (22), are unique; the soliton flows on $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ given in Example 2.10 are unique; the only soliton flows on $S^3$ and $H^3$ are given by Killing vector fields; the only non-Killing solitons on $\mathbb{R}^3$ are the well-known Gaussian solitons.

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