A TVERBERG TYPE THEOREM
FOR COLLECTIVELY UNAVOIDABLE COMPLEXES

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ABSTRACT
We prove that the symmetrized deleted join $\text{SymmDelJoin}(K)$ of a “balanced family” $K = \langle K_i \rangle_{i=1}^r$ of collectively $r$-unavoidable subcomplexes of $2^{[m]}$ is $(m-r-1)$-connected. As a consequence we obtain a Tverberg–Van Kampen–Flores type result which is more conceptual and more general than previously known results. Already the case $r = 2$ of this result seems to be new as an extension of the classical Van Kampen–Flores theorem. The main tool used in the paper is R. Forman’s discrete Morse theory.
1. Introduction

Tverberg–Van Kampen–Flores type results have been for many years one of the central research themes in topological combinatorics. The last decade has been particularly fruitful with some longstanding conjectures resolved, as summarized by several review papers [1, 12, 16, 19], covering different aspects of the theory.

Certainly the most striking among the new results is the resolution (in the negative!) of the general “Topological Tverberg Problem” [13, 6, 3]. On the positive side is the proof [11, Theorem 1.2] of the “Balanced Van Kampen–Flores theorem” indicating in which direction one can expect new positive results.

In this paper we prove a result (Theorem 3.2) which we see as a candidate for the currently most general and far reaching result of Van Kampen–Flores type. Indeed, this result contains the “Balanced Van Kampen–Flores theorem” as a special case (Corollary 3.4), as well as other results of this type. Note that already the case \( r = 2 \) of the theorem (see Section 3.1), which extends the classical Van Kampen–Flores theorem, doesn’t seem to have been recorded before.

Surprisingly enough, Theorem 3.2 is not only more general but it also provides a more conceptual and possibly more elegant and transparent approach. The new approach relies on the concepts of “collectively unavoidable” (Definition 2.4) and “balanced” (Definition 2.1) \( r \)-tuples \( \mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle \) of simplicial complexes. Recall that collectively unavoidable complexes were originally introduced and studied in [9] as a common generalization of pairs \( \langle K, K^\circ \rangle \) of Alexander dual complexes (Alexander 2-tuples) and \( r \)-unavoidable complexes of [2] and [8].

Here is a brief outline of the paper. In Section 2 we prove the first main result of the paper, Theorem 2.5, which estimates the connectivity of the symmetrized deleted join of an \( r \)-tuple of complexes, under the assumption that the \( r \)-tuple is both balanced and unavoidable. The main tool in the proof is R. Forman’s discrete Morse theory. The construction of the discrete Morse function is particularly well adapted for applications to deleted joins and symmetrized deleted joins of complexes. The new Van Kampen–Flores type result (Theorem 3.2) is obtained in Section 3 as a corollary of Theorem 2.5. The proof uses the usual Configuration Space/Test Map Scheme, see [11, 18, 19], and relies on Volovikov’s version of the Borsuk–Ulam theorem [19]. In Section 4 we
discuss criteria for an \( r \)-tuple of simplicial complexes to be both balanced and collectively unavoidable. Finally, for the reader’s convenience, we outline in the Appendix (Section A) the fundamental principles of discrete Morse theory \([4, 5]\), including the definitions of some of the most basic terms (migrating element, matching step, splitting step, etc.), used in the paper.

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2. Connectivity of the symmetrized deleted join

2.1. Preliminary definitions.

Definition 2.1: We say that a simplicial complex \( K \subseteq 2^{[m]} \) is \((m, k)\)-balanced if it is positioned between two consecutive skeleta of a simplex on \( m \) vertices,

\[
\binom{[m]}{\leq k} \subseteq K \subseteq \binom{[m]}{\leq k+1}.
\]

By definition \( \binom{[m]}{\leq k} \subseteq 2^{[m]} \) (the \((k-1)\)-dimensional skeleton of \( 2^{[m]} \)) is the collection of all subsets of \([m]\) of cardinality at most \( k \).

Definition 2.2: The deleted join ([10], [14, Section 6]) of a family

\[
\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle
\]

of subcomplexes of \( 2^{[m]} \) is the complex

\[
\mathcal{K}^*_\Delta = K_1 *_{\Delta} \cdots *_{\Delta} K_r \subseteq (2^{[m]})^*^r
\]

where \( A = A_1 \sqcup \cdots \sqcup A_r \in \mathcal{K}^*_\Delta \) if and only if \( A_j \) are pairwise disjoint and \( A_i \in K_i \) for each \( i = 1, \ldots, r \). In the case \( K_1 = \cdots = K_r = K \) this reduces to the definition of an \( r \)-fold deleted join \( \mathcal{K}^*_{\Delta r} \); see [14].
The \textit{symmetrized deleted join} of $\mathcal{K}$ is defined as
\[
\text{SymmDelJoin}(\mathcal{K}) := \bigcup_{\pi \in S_r} K_{\pi(1)} *_{\Delta} \cdots *_{\Delta} K_{\pi(r)} \subseteq (2^{[m]})^{*r}_{\Delta},
\]
where the union is over the set of all permutations of $r$ elements and
\[
(2^{[m]})^{*r}_{\Delta} \cong [r]^*m
\]
is the $r$-fold deleted join of a simplex with $m$ vertices.

An element $A_1 \sqcup \cdots \sqcup A_r \in (2^{[m]})^{*r}_{\Delta}$ is from here on recorded as
\[
(A_1, A_2, \ldots, A_r; B)
\]
where $B$ is the complement of $\bigcup_{i=1}^r A_i$, so in particular $A_1 \sqcup \cdots \sqcup A_r \sqcup B = [m]$ is a partition of $[m]$ such that $A_i \neq \emptyset$ for some $i \in [r]$.

\text{Lemma 2.3: The dimension of the simplex can be read off from $|B|$ as}
\[
\dim(A_1, \ldots, A_r; B) = m - |B| - 1.
\]

Collectively unavoidable $r$-tuples of complexes are introduced in [9]. They were originally studied as a common generalization of pairs of Alexander dual complexes, Tverberg unavoidable complexes of [2] and $r$-unavoidable complexes from [8].

\text{Definition 2.4: An ordered $r$-tuple $\mathcal{K} = (K_1, \ldots, K_r)$ of subcomplexes of $2^{[m]}$ is \textit{collectively $r$-unavoidable} if for each ordered collection $(A_1, \ldots, A_r)$ of disjoint sets in $[m]$ there exists $i$ such that $A_i \in K_i$.}

\section{2.2. The main theorem.}

\text{Theorem 2.5: Suppose that $\mathcal{K} = (K_i)_{i=1}^r = (K_1, \ldots, K_r)$ is a collectively $r$-unavoidable family of subcomplexes of $2^{[m]}$. Moreover, we assume that there exists $k \geq 1$ such that $K_i$ is $(m, k)$-balanced for each $i = 1, \ldots, r$. Then the associated symmetrized deleted join}
\[
\text{SymmDelJoin}(\mathcal{K}) = \text{SymmDelJoin}(K_1, \ldots, K_r)
\]
is $(m - r - 1)$-connected.
Proof outline. In Section 2.3 we construct a discrete Morse function on the symmetrized deleted join \( \text{SymmDelJoin}(K) \). In other words we describe an acyclic matching of simplices in \( \text{SymmDelJoin}(K) \) (see the Appendix for a brief description of this technique). The proof of the acyclicity is given in Section 2.4. Following one of the central principles of Discrete Morse Theory, the complex \( \text{SymmDelJoin}(K) \) is homotopy equivalent to a complex built from critical simplices. So the proof is concluded (Section 2.5) by showing that the dimension of all critical simplices is at least \((m-r)\) (with the exception of the unique simplex of dimension 0).

2.3. CONSTRUCTION OF A DISCRETE MORSE FUNCTION. Assume that \( A_1 \sqcup \cdots \sqcup A_r \sqcup B = [m] \) is an ordered partition, interpreted as a simplex \((A_1, A_2, \ldots, A_r; B) \in (2^{[m]})^*_{\Delta} \). A simplex labeled by \((A_1, \ldots, A_r; B)\) is called large if \(|B| \leq r - 1\). By Lemma 2.3, the dimension of a large simplex is at least \(m - r\).

Our aim is to construct a discrete Morse function (DMF) such that all simplices that are not large are matched (with one 0-dimensional exception). This is precisely the condition needed for the \((m-r-1)\)-connectivity; see the Appendix.

**Step 1.** Assume that \( B \cup A_1 \neq \emptyset \). Set

\[ a_1 := \min(B \cup A_1) \]

and match the simplices \((A_1, \ldots, A_r; Ba_1)\) and \((A_1a_1, \ldots, A_r; B)^1\) whenever both of them are elements of the complex \( \text{SymmDelJoin}(K) \).

(An alert reader may have noticed a slight abuse of notation in the statement above. A more precise statement is the following: If \( a_1 \in A_1 \) we match the simplices \((A_1, \ldots, A_r; B)\) and \((A_1 \setminus \{a_1\}, \ldots, A_r; Ba_1)\) and if \( a_1 \in B \) the simplex \((A_1, \ldots, A_r; B)\) is matched with \((A_1a_1, \ldots, A_r; B \setminus \{a_1\})\). In the sequel, for brevity and clarity of the argument, we keep the simplified notation.)

Let us analyze simplices which remain unmatched after Step 1.

1. Take an arbitrary simplex \( \sigma \in \text{SymmDelJoin}(K) \) with \( B \cup A_1 \neq \emptyset \). Assume that \( a_1 \) is not in \( B \), which means that the simplex has the form

\[ \sigma = (A_1a_1, \ldots, A_r; B) \in \text{SymmDelJoin}(K). \]

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1 Here and in the sequel we abbreviate \( B \cup \{a\} \) as \( Ba \) and \( A_1 \cup \{a\} \) as \( A_1a \). Moreover, by writing \( Ba \) we indicate that \( a \notin B \).
Then it must be matched, since the simplex \((A_1, \ldots, A_r; Ba_1)\), as a facet of \(\sigma\), belongs to \(\text{SymmDelJoin}(K)\). The only exception appears when all the \(A_i\) are empty, \(a_1 = 1\), and \((A_1, \ldots, A_r; Ba_1)\) is not a simplex. Hence there is exactly one 0-dimensional unmatched simplex with \(a_1 \in A_1\), namely the ‘exceptional simplex’

\[((\{1\}, \emptyset, \ldots, \emptyset; [m] \setminus \{1\})).\]

We conclude that unmatched simplices are some of the simplices of the form \((A_1, \ldots, A_r; Ba_1)\). They are characterized by the property

\((A_1a_1, \ldots, A_r; B) \not\in \text{SymmDelJoin}(K)\).

For bookkeeping purposes these simplices are recorded as **Step 1-Type 1** unmatched simplices.

(2) Assume now that for a simplex \((A_1, \ldots, A_r; B)\), the set \(B \cup A_1\) is empty. Such a simplex is not matched. We say that this is a **Step 1-Type 3** simplex. All Type 3 simplices are large.

**Step 2.** Assuming that \((B \cup A_2) \setminus [1, a_1]\) is non-empty, set

\[a_2 := \min((B \cup A_2) \setminus [1, a_1])\]

and match the simplices

\((A_1, \ldots, A_r; Ba_2)\) and \((A_1, A_2a_2, \ldots, A_r; B)\)

whenever

1. both of them belong to \(\text{SymmDelJoin}(K)\),
2. both of them were not matched before (that is, at Step 1).

Let us analyze the simplices which remain unmatched after Step 2. In particular we consider below only those simplices that are unmatched at Step 1, ignoring the exceptional zero-dimensional simplex.

(1) Assume that \((B \cup A_2) \setminus [1, a_1]\) is non-empty. Then \(a_2\) is well-defined.

(a) Consider a simplex of type \((A_1, A_2, \ldots, A_r; Ba_2)\) which is not matched at Step 1. If \((A_1, A_2a_2, \ldots, A_r; B) \in \text{SymmDelJoin}(K)\), then the simplex \((A_1, A_2a_2, \ldots, A_r; B)\) is also not matched at Step 1, and according to our rule, these two simplices get matched at Step 2.

We conclude that \((A_1, \ldots, A_r; Ba_2)\) is not matched at Step 2 iff \((A_1, A_2a_2, \ldots, A_r; B) \not\in \text{SymmDelJoin}(K)\). In this case we say that \((A_1, \ldots, A_r; Ba_2)\) is a **Step 2-Type 1** simplex.
(b) Now take a simplex of type

\((A_1, A_2a_2, \ldots, A_r; B) \in \text{SymmDelJoin}(\mathcal{K}).\)

Since

\((A_1, \ldots, A_r; Ba_2) \in \text{SymmDelJoin}(\mathcal{K}),\)

the simplex \((A_1, A_2a_2, \ldots, A_r; B)\) is non-matched at step 2 iff \((A_1, \ldots, A_r; Ba_2)\) is matched at Step 1.

We say that \((A_1, A_2a_2, \ldots, A_r; B)\) is a \textbf{Step 2-Type 2} simplex.

(2) Now assume that for a simplex \((A_1, \ldots, A_r; B)\) the set \((B \cup A_2) \setminus [1, a_1]\) is empty. In particular, this means that \(|B| \leq 1\). Then the simplex is unmatched. We say that this is a \textbf{Step 2-Type 3} simplex. All the simplices of Type 3 are large by Lemma 2.3.

Other steps are similar; the Step \(s\) is described as follows:

**Step \(s\).** Set

\[a_s := \min((B \cup A_s) \setminus [1, a_{s-1}])\]

and match the simplices

\((A_1, \ldots, A_r; Ba_s)\) and \((A_1, \ldots, A_s a_s, \ldots, A_r; B)\)

whenever

1. both of them belong to \(\text{SymmDelJoin}(\mathcal{K})\),
2. both of them were not matched before (that is, at Step \(\leq s - 1\)).

As before the unmatched simplices fall into three types:

1. \((A_1, \ldots, A_r; Ba_s)\). We say that this is a \textbf{Step \(s\)-Type 1} simplex. Here we necessarily have

   \((A_1, \ldots, A_s a_s, \ldots, A_r; B) \notin \text{SymmDelJoin}(\mathcal{K}).\)

2. \((A_1, \ldots, A_s a_s, \ldots, A_r; B)\). We say that this is a \textbf{Step \(s\)-Type 2} simplex. Here \((A_1, \ldots, A_r; Ba_s)\) is matched at an earlier step (by the migration of some \(a_j\), for \(j < s\)).

3. Those with \((B \cup A_s) \setminus [1, a_{s-1}] = \emptyset\). In particular, this means that \(|B| \leq s - 1\). We say that this is a \textbf{Step \(s\)-Type 3} simplex. All the simplices of type 3 are large.

We proceed analogously for \(s = 1, \ldots, r\).
Lemma 2.6: The matching described above is a discrete vector field on the symmetrized deleted join $\text{SymmDelJoin}(\mathcal{K})$.

Proof. Each simplex of the complex is matched at most once. Indeed, at each step $s$ there is an explicit condition that the simplices matched at this step have not been matched before. In each matched pair, one of the simplices is a facet of the other. Indeed, the simplex $(A_1, \ldots, A_r; Ba_s)$ is a face of $(A_1, \ldots, A_s a_s, \ldots, A_r; B)$, and the dimensions of the simplices are respectively $m - |B| - 2$ and $m - |B| - 1$.

It remains to be shown that this discrete vector field is acyclic.

2.4. The acyclicity of the discrete vector field.

Definition 2.7: Given a simplex $\sigma = (A_1, \ldots, A_r; B) \in \text{SymmDelJoin}(\mathcal{K})$, its passport $p(\sigma) = (a_1, \ldots, a_r)$ is inductively defined as

$$a_i := \min((A_i \cup B) \setminus [1, a_{i-1}]),$$

provided the indicated set is non-empty. Otherwise we set $a_i := \infty$. Note that in this case $a_j = \infty$ for each $j > i$.

We assume that the passports are linearly ordered by the lexicographic ordering.

Claim 1: The passport does not increase along a gradient path.

Proof. In a gradient path, a matching step does not change the value of the passport $p$. Indeed, in this case one of the minima $a_i$ migrates from $B$ to $A_i$ and the set $B \cup A_i$ remains unchanged. A splitting step can only decrease (lexicographically) the value of $p$. Indeed, in this case some element $b$ migrates from $A_i$ to $B$ and the sets $B \cup A_j$ can only get larger for $j \leq i$. More explicitly, the new $a_1 := \min(A_1 \cup B)$ cannot increase. If $a_1$ gets smaller, we are done. If it persists, we conclude that $b > a_1$. Now we observe that

$$a_2 := \min((A_2 \cup B) \setminus [1, a_1])$$

either persists or gets smaller, since the interval $[1, a_1]$ persists. If $a_2$ gets smaller, we are done. If $a_2$ persists, we pass to $a_3$, etc.

Corollary: If the constructed discrete vector field is not acyclic, i.e. if there is a closed gradient path, then the passport must be constant along this path.
For the next claim recall that a **migrating element** (see the Appendix or [9]), corresponding to the “splitting step”

\[ \beta^{p+1}_i \searrow \alpha^{p+1}_{i+1} \]

in a gradient path, is the vertex \( v \in \beta^{p+1}_i \setminus \alpha^p_{i+1} \). Similarly in the “matching step”

\[ \alpha^p_i \nearrow \beta^{p+1}_i \]

it is the element \( v \in \beta^{p+1}_i \setminus \alpha^p_i \). For illustration, the matching step

\[(A_1, \ldots, A_r; Ba_s) \nearrow (A_1, \ldots, A_s a_s, \ldots, A_r; B)\]

can be described as a migration of \( a_s \) from \( B' = Ba_s \) to \( A'_s = A_s a_s \). Similarly, in the splitting step

\[(A_1, \ldots, A_s \nu, \ldots, A_r; B) \searrow (A_1, \ldots, A_s, \ldots, A_r; B \nu)\]

the element \( \nu \) migrates from \( A'_s = A_s \nu \) to \( B \nu \).

**Claim 2:** Assuming that there is a closed path, the migrating elements can come only from the set \( \{a_i\}_{i=1}^r \).

**Proof.** Indeed, assume that in a closed gradient path, some element \( b_i \in A_i \) migrates from \( A_i \) to \( B \). Since the path is closed, at some other step \( b_i \) should migrate back to \( A_i \) at a matching step. By construction, this happens only if \( b_i = a_i \). \( \blacksquare \)

Summarizing, we conclude that a closed path, if it exists, is uniquely determined by the sequence of indices of migrating elements.

For instance, a fragment of a closed path, producing indices

\[ i_1 = 3, i_2 = 4, i_3 = 2, \]

looks exactly as follows:

\[
\begin{align*}
(A_1, A_2, A_3, A_4 a_4; Ba_1 a_2 a_3) \\
\downarrow 3 \\
(A_1, A_2, A_3 a_3, A_4 a_4; Ba_1 a_2) \\
\downarrow 4 \\
(A_1, A_2, A_3 a_3, A_4; Ba_1 a_2 a_4) \\
\downarrow 2 \\
(A_1, A_2 a_2, A_3 a_3, A_4; Ba_1 a_4)
\end{align*}
\]

Note that each of the migrating elements \( a_i \) participates in an equal number of matching and splitting steps.
Assuming that the indexing of steps in the closed path is chosen so that the even steps correspond to the “matching steps” (and the odd steps are the “splitting steps”), then the indices satisfy the following relation:

\[(\forall j) \ i_j \neq i_{j+1},\]
\[(\forall j) \ i_{2j+1} > i_{2j+2}.\]

Indeed, if \(k = i_{2j+1} < i_{2j+2} = l\) then \(a_k < a_l = \min(B \cup A_l)\), but this is not possible since \(a_k\) migrated to \(B\) in the previous step.

Finally, by taking the minimal migrating index \(i\) we obtain a contradiction with the second inequality in (2). This observation completes the proof of the acyclicity of the constructed discrete vector field.

2.5. CRITICAL SIMPLICES ARE LARGE.

**Lemma 2.8:** Assume that \((A_1, \ldots, A_r; Bb) \in \text{SymmDelJoin}(K)\). Assume that for some \(i\), \(|A_i| < k\). Then

\((A_1, \ldots, A_i b, \ldots, A_r; B) \in \text{SymmDelJoin}(K)\).

**Proof.** By the condition, there exists a permutation \(\sigma\) such that each \(A_j\) belongs to \(K_{\sigma(j)}\). The same permutation serves for \((A_1, \ldots, A_i b, \ldots, A_r; B)\), since whatever \(K_{\sigma(i)}\) is, \(A_i b\) belongs to \(K_{\sigma(i)}\).

Let \(\Phi(A_1, \ldots, A_r; B)\) be the set of all permutations \(\phi \in S_r\) such that \(A_i \in K_{\phi(i)}\) for each \(i = 1, \ldots, r\). Clearly, \((A_1, A_2, \ldots, A_r; B)\) belongs to \(\text{SymmDelJoin}(K)\) iff \(\Phi(A_1, \ldots, A_r; B)\) is non-empty.

Let us look at the non-matched simplices after the final step (the step \(r\)). We need to show that each non-matched simplex is large (except for the exceptional 0-dimensional simplex). Let us assume that the simplex \((A_1, \ldots, A_r; B)\) is unmatched. If it was of Type 3 on some step, then it is large. So assume that at each step, \((A_1, \ldots, A_r; B)\) is either Type 1 or Type 2.

**Lemma 2.9:** Suppose that \(\tau = (A_1, \ldots, A_i, \ldots, A_r; B)\) is Type 2 at step \(i\), meaning that \(A_i = A'_i a_i\) and \(\tau' = (A_1, \ldots, A'_i, \ldots, A_r; B a_i)\) was matched on some earlier step \(j < i\). Then

\(|A_i| = k + 1\).
Proof. By assumption \( \tau \) is not matched at step \( j \), but \( \tau' \) was matched. There are two cases:

(1) \( B = B' a_j \), and \( \tau' = (A_1, \ldots, A_j, \ldots, A'_i, \ldots, A_r; B' a_i) \) is matched to \((A_1, \ldots, A_j a_j, \ldots, A'_i, \ldots, A_r; B' a_i)\). Then

\[
\tau = (A_1, \ldots, A_j, \ldots, A'_i a_i, \ldots, A_r; B' a_j)
\]

is matched to

\[
(A_1, \ldots, A_j a_j, \ldots, A'_i a_i, \ldots, A_r; B')
\]
at step \( j \).

The latter simplex belongs to \( \text{SymmDelJoin}(K) \) since \( |A_i| \leq k \) by Lemma 2.8, which leads to a contradiction.

(2) \( A_j = A'_j a_j \), and \( \tau' = (A_1, \ldots, A'_j a_j, \ldots, A'_i, \ldots, A_r; B a_i) \) is matched to \((A_1, \ldots, A'_j a_j, \ldots, A'_i, \ldots, A_r; B a_j a_i)\) at step \( j \). In this case the simplex

\[
\tau = (A_1, \ldots, A'_j a_j, \ldots, A'_i a_i, \ldots, A_r; B)
\]
is matched to

\[
(A_1, \ldots, A'_j a_j, \ldots, A'_i a_i, \ldots, A_r; B a_j)
\]
at step \( j < i \), which again leads to a contradiction. \( \blacksquare \)

Let \( I \subset [r] \) be the set of all indices such that

\[
i \in I \iff (A_1, \ldots, A_r; B) \text{ is Type } 1 \text{ at step } i.
\]

Then \( s \notin I \) implies that the simplex \((A_1, \ldots, A_r; B)\) is Type 2 at step \( s \).

Assume now that the simplex is not large, that is, \( |B| \geq r \). Choose a permutation \( \phi \in \Phi(A_1, \ldots, A_r; B) \). If \( s \in I \) then \( a_s \in B \), and \( A_s a_s \notin K_{\phi(s)} \). If \( s \notin I \) then, by Lemma 2.9, \( |A_s| = k + 1 \), so \( A_s \) plus any other element \( b \) is no longer in \( K_{\phi(s)} \).

Fix an injection \( \pi : [r] \to B \) with the property \( \pi(i) = a_i \) if \( i \in I \). Set

\[
A'_i := A_i \cup \{ \pi(i) \}.
\]

By construction \( A'_i \notin K_{\pi(i)} \) and, since \( A'_i \) are pairwise disjoint, we arrive at a contradiction with the collective unavoidability of the family

\[
\mathcal{K} = (K_i)_{i=1}^r.
\]
3. A general Tverberg–Van Kampen–Flores theorem for balanced complexes

The following theorem of Tverberg–Van Kampen–Flores type is the main result of [11]. It is very likely the most general known result that evolved from the classical Van Kampen–Flores theorem [19, Section 22.4.3]. For example it extends and contains as a special case the ‘Generalized Van Kampen–Flores Theorem’ of Sarkaria [15], Volovikov [17], and Blagojević, Frick and Ziegler [2].

**Theorem 3.1** ([11, Theorem 1.2]): Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq (r - 1)(d + 2)$, and $rk + s \geq (r - 1)d$ for integers $k \geq 0$ and $0 \leq s < r$. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$, with $\dim \sigma_i \leq k + 1$ for $1 \leq i \leq s$ and $\dim \sigma_i \leq k$ for $s < i \leq r$.

Theorem 3.1 confirmed the conjecture of Blagojević, Frick and Ziegler about the existence of ‘balanced Tverberg partitions’ (Conjecture 6.6 in [2]). Among the consequences of this theorem is a positive answer (see [11, Theorem 7.2]) to the ‘balanced case’ of the problem whether each admissible $r$-tuple is Tverberg prescribable, [2, Question 6.9].

The term ‘balanced partitions’ in both results refers to the constraint that a Tverberg $r$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_r)$ is sought in the symmetric deleted join

$$\text{SymmDelJoin}(K_1, \ldots, K_r)$$

of adjacent skeleta of the simplex $\Delta_N = 2^{[N+1]}$,

$$K_1 = \cdots = K_s = \begin{pmatrix} [N+1] \\ \leq k + 2 \end{pmatrix}, \quad K_{s+1} = \cdots = K_r = \begin{pmatrix} [N+1] \\ \leq k + 1 \end{pmatrix}.$$  

It is known [9] that the collection of subcomplexes of $2^m$,

$$\left( \begin{pmatrix} m \\ \leq m_1 \end{pmatrix}, \ldots, \begin{pmatrix} m \\ \leq m_r \end{pmatrix} \right),$$

is always a collectively $r$-unavoidable, provided $m = \sum_{i=1}^r m_i + r - 1$ and in particular (4) is such a collection if $N + 1 = s(k + 2) + (r - s)(k + 1) + r - 1$.

Conditions (4) and (5) indicate that collectively $r$-unavoidable complexes behave very well if in addition we assume that they are balanced. This is precisely the content of Theorem 2.5. From here it is not difficult to derive a general theorem of Van Kampen–Flores type which includes Theorem 3.1 as a special case and which seems to be new already in the case $r = 2$ (Theorem 3.7).
Theorem 3.2: Suppose that $\mathcal{K} = \langle K_i \rangle_{i=1}^{r} = \langle K_1, \ldots, K_r \rangle$ is a collectively $r$-unavoidable family of subcomplexes of $2^{|m|}$, where $r = p^\nu$ is a power of a prime number. Let $k \geq 1$ and assume that $K_i$ is $(m, k)$-balanced for each $i = 1, \ldots, r$. Suppose that $N \geq (r - 1)(d + 2)$, where $N = m - 1$. Then for each continuous map $f : \Delta_N \to \mathbb{R}^d$, from an $N$-dimensional simplex into a $d$-dimensional euclidean space, there exist vertex-disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$$

and

$$\sigma_1 \in K_1, \sigma_2 \in K_2, \ldots, \sigma_r \in K_r.$$ 

Remark 3.3: We are primarily interested in the “optimal” case

$$k = \left\lfloor \frac{m - r + 1}{r} \right\rfloor,$$

which is the smallest number for which unavoidable $(m, k)$-balanced $r$-tuples exist (see Proposition 4.2). Let us also mention that in the optimal case, $k - 1$ is the minimal allowed dimension that appears in “Tverberg prescribable $r$-tuples”; see [7, Theorem 2.8].

Proof. Let $V = ((\mathbb{R}^d)^\ast r) \cong \mathbb{R}^{rd + r - 1}$ be the $r$-fold join of $\mathbb{R}^d$ and let $D \cong \mathbb{R}^d$ be the diagonal subspace in $V$. A map $f : \Delta_N \to \mathbb{R}^d$ induces an $S_r$-equivariant map

$$F : \text{SymmDelJoin}(\mathcal{K}) \to V/D \cong \mathbb{R}^{(r-1)(d+1)}$$

and the theorem follows from the observation that this map must have a zero. If not, there arises an equivariant map

$$F' : \text{SymmDelJoin}(\mathcal{K}) \to S(V/D) \cong S^{(r-1)(d+1)-1}$$

which contradicts Volovikov’s theorem [19], since the space SymmDelJoin($\mathcal{K}$) is $(m - r - 1)$-connected (Theorem 2.5) and $m - r - 1 \geq (r - 1)(d + 1) - 1$ is equivalent to $N \geq (r - 1)(d + 2)$. 

Corollary 3.4: Theorem 3.1 is a special case of Theorem 3.2.

Proof. It can be easily shown that Theorem 3.1 is reduced to the case when

$$N = (r - 1)(d + 2) \quad \text{and} \quad rk + s = (r - 1)d.$$

Indeed, for given $r$ and $d$ one is interested in the smallest $k$ and $N$ for which the theorem is still valid.
All that remains to be checked is that the collection $K$, described by (4), is collectively $r$-unavoidable. However, knowing that (5) is collectively $r$-unavoidable if \( m = \sum_{i=1}^{r} m_i + r - 1 \), it is sufficient to check the condition

\[
m = N + 1 = s(k + 2) + (r - s)(k + 1) + r - 1.
\]

Since $rk + s = (r - 1)d$ this is equivalent to $N = (r - 1)(d + 2)$.

Remark 3.5: The conditions in Theorem 3.2 may appear at first sight somewhat artificial or even superfluous. As we now know both conditions are quite natural and in some sense necessary. The prime power condition $r = p^\nu$ appears in all Borsuk–Ulam type theorems for the $p$-toral group of symmetries $G = (\mathbb{Z}_p)^\nu$, as illustrated by Volovikov’s theorem [17]. More importantly, by the far reaching results of Mabillard and Wagner [13], this condition is necessary for general theorems of Van Kampen–Flores type. The condition that the complexes are balanced is also quite natural in light of the fact that “prescribable dimensions” of intersecting simplices should be roughly the same, see [7].

3.1. Example. Let $m = 2k + 2$ and assume that $K$ is a $(m, k)$-balanced simplicial subcomplex of $2^{[m]}$. Let $K^\circ$ be the Alexander dual of $K$. Then the pair $(K, K^\circ)$ of simplicial complexes is collectively 2-unavoidable (by the definition of Alexander duality, see also [9]). Moreover, $K^\circ$ is also $(m, k)$-balanced. By Theorem 2.5 the symmetric deleted join

\[
\text{SymmDelJoin}(K, K^\circ)
\]

is an $(m - 3)$-connected, $(m - 2)$-dimensional simplicial complex.

Remark 3.6: The symmetric deleted join (9) is a union of two overlapping Bier spheres of dimension $(m - 2)$. It follows (essentially from the Mayer–Vietoris exact sequence) that $(K *_{\Delta} K^\circ) \cup (K^\circ *_{\Delta} K)$ is $(m - 3)$-connected if and only if $(K *_{\Delta} K^\circ) \cap (K^\circ *_{\Delta} K)$ is $(m - 4)$-connected. This holds for balanced complexes and can be established by a direct argument.

As a consequence we obtain a result which (in a single statement) includes and extends both the classical Van Kampen–Flores theorem and the “sharpened Van Kampen–Flores theorem” (Theorem 6.8 from [2]).
Theorem 3.7: Let \( K \subset 2^{[m]} \) be a simplicial complex and let \( K^\circ \) be its Alexander dual. Let \( k \) be an integer such that both \( K \) and \( K^\circ \) are \((m,k)\)-balanced. Then for each continuous map \( f: \Delta_{m-1} \to \mathbb{R}^{m-3} \) there exist faces \( F_1 \in K \) and \( F_2 \in K^\circ \) such that \( f(F_1) \cap f(F_2) \neq \emptyset \).

Moreover, the condition that both \( K \) and \( K^\circ \) are \((m,k)\)-balanced is satisfied precisely in one of the following two cases:

1. If \( m \) is even then \( m = 2k + 2 \), and if \( K \subset 2^{[m]} \) is an arbitrary \((m,k)\)-balanced complex then \( K^\circ \) is \((m,k)\)-balanced as well.
2. If \( m = 2n + 1 \) is odd then \( k = n \), and the only pair \( \{K, K^\circ\} \) of \((m,k)\)-balanced simplicial complexes is \( K = K^\circ = ([m]_{\leq n}) \).

Proof. The first part of the theorem is precisely the case \( r = 2 \) of Theorem 3.2.

For the second part of the theorem let us find all pairs \((m,k)\) of integers such that there exist complexes \( K \) and \( K^\circ \) which are both \((m,k)\)-balanced. By the definition of Alexander duality there is an equivalence

\[
\binom{[m]}{\leq k} \subseteq K \subseteq \binom{[m]}{\leq k+1} \iff \binom{[m]}{\leq k'} \subseteq K^\circ \subseteq \binom{[m]}{\leq k'+1}
\]

where \( k + k' = m - 2 \). If both \( K \) and \( K^\circ \) are \((m,k)\)-balanced then, as a consequence of (10), we obtain the inequalities \( k \leq k' + 1, k' \leq k + 1 \) which imply the inequality \( 2k + 1 \leq m \leq 2k + 3 \).

Conversely, let us assume that \( 2k+1 \leq m \leq 2k+3 \) is satisfied and let \( K \subset 2^{[m]} \) be an arbitrary \((m,k)\)-balanced complex. If \( m = 2k + 2 \) then \( k' = k \) and \( K^\circ \) is automatically \((m,k)\)-balanced.

If \( m = 2k + 1 \), then \( k' = k - 1 \) which forces \( K^\circ \) to be the complex \( K^\circ = ([m]_{\leq k}) \).

It follows that the original complex is \( K = K^\circ = ([m]_{\leq k}) \). If \( m = 2k + 3 \), then \( k' = k + 1 \) which forces \( K^\circ \) to be the complex \( K^\circ = ([m]_{\leq k+1}) \). In this case the original complex is

\[
K = K^\circ = \binom{[m]}{\leq k+1}.
\]
Let $m = 9$, $k = 2$. Let $K_1 = \binom{[9]}{\leq 2} \cup \left( \binom{9}{\leq 3} \setminus \{A\} \right)$ be the complex on 9 vertices 1, ..., 9 containing all 2-element subsets and all 3-element sets except $A = \{7, 8, 9\}$. Let $\Delta$ be an Alexander self-dual complex on the vertices 1, ..., 6, for example the minimal, 6-vertex triangulation of the real projective plane. Let $K_2 = K_3$ be the complex on 9 vertices 1, ..., 9 containing all 2-element subsets together with all 3-element subsets that belong to $\Delta$.

**Lemma 3.8:** Both the triple $K = (K_1, K_2, K_3)$ and the triple of skeleta $\mathcal{L} = (L_1, L_2, L_3) = \left( \binom{9}{\leq 3}, \binom{9}{\leq 2}, \binom{9}{\leq 2} \right)$ are collectively unavoidable.

Let us observe that the symmetrized deleted join $\text{SymmDelJoin}(\mathcal{L})$ is not contained in $\text{SymmDelJoin}(K)$. For example,

$$\left( \{7, 8, 9\}, \{3, 4\}, \{1, 2\}, \{5, 6\} \right) \in \text{SymmDelJoin}(\mathcal{L}) \setminus \text{SymmDelJoin}(K).$$

This fact indicates that there does not exist an obvious $S_3$-equivariant map $f : \text{SymmDelJoin}(\mathcal{L}) \to \text{SymmDelJoin}(K)$ and therefore this argument cannot be applied to deduce Theorem 3.2 from the “Balanced Van Kampen–Flores theorem” (Theorem 3.1).

4. **Collective unavoidability of balanced $r$-tuples**

Let $K_1, \ldots, K_r$ be a collection of $(m, k)$-balanced complexes. Each $K_i$ can be represented as

$$K_i = \left( \binom{[m]}{\leq k+1} \right) \setminus A_i^j = \left( \binom{[m]}{\leq k+1} \right) \setminus \{A_i^1, \ldots, A_i^k\}$$

where $|A_i^j| = k + 1$ for each $i$ and $j$.

Let us define an $r$-partite graph $\Gamma = \Gamma(K_1, \ldots, K_r)$ whose vertices are labeled by pairs $(i, j)$ where $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$ for each $i$. Two vertices $(i, j)$ and $(i', j')$ share an edge if $i \neq i'$ and $A_i^j \cap A_{i'}^{j'} = \emptyset$.

**Remark 4.1:** In agreement with the standard definition of the Kneser graph $KG(\mathcal{F})$ of a family of sets (see [14]) the graph $\Gamma$ can be also described as the $r$-partite Kneser graph $KG(\mathcal{A})$ of the disjoint union $\mathcal{A} = \mathcal{A}^1 \sqcup \cdots \sqcup \mathcal{A}^r$. 
Proposition 4.2: Assume that $D = r(k + 2) - m$.
Suppose that $K_1, \ldots, K_r$ is a collection of $(m,k)$-balanced subcomplexes of $2^{[m]}$ and let $\Gamma = \Gamma(K_1, \ldots, K_r)$ be the associated $r$-partite Kneser graph.

1. If $D > r$, then the collection $(K_1, \ldots, K_r)$ is always collectively unavoidable.
2. If $D < 1$, then the collection $(K_1, \ldots, K_r)$ is not collectively unavoidable.
3. If $D = 1$, the only collectively unavoidable $r$-tuple is $K_1 = K_2 = \cdots = K_r = \left( \left\lvert \frac{m}{k+1} \right\rvert \right)$.
4. If $1 < D \leq r$, then the collection $(K_1, \ldots, K_r)$ is collectively unavoidable iff the $r$-partite associated Kneser graph $\Gamma = \Gamma(K_1, \ldots, K_r)$ contains no $D$-clique.

Proof. (1) In this case $D > r \iff r(k + 2) - m > r \iff r(k + 1) > m$. For each partition $[m] = B_1 \sqcup \cdots \sqcup B_r$ there exists $i$ such that $|B_i| \leq k$, hence $B_i \in K_i$.

(2) If $D < 1$ then $m > r(k + 2)$. In this case there exists a partition $[m] = B_1 \sqcup \cdots \sqcup B_r$ where $|B_i| \geq k + 2$ for each $i$, proving that $(K_1, \ldots, K_r)$ is not collectively unavoidable.

(3) In this case $m = r(k + 2) - 1$. Let $B \in \left( \left\lvert \frac{m}{k+1} \right\rvert \right)$ be an arbitrary set. For a given $i$ let $[m] = B_1 \sqcup \cdots \sqcup B_r$ be a partition where $B_i = B$ and $|B_j| = k + 2$ for $j \neq i$. If the $r$-tuple $(K_1, \ldots, K_r)$ is collectively unavoidable, then $B = B_i \in K_i$ which implies that $K_i = \left( \left\lvert \frac{m}{k+1} \right\rvert \right)$.

(4) Assume that there is a $D$-clique in the graph $\Gamma = \Gamma(K_1, \ldots, K_r)$, associated to a collectively $r$-unavoidable $r$-tuple $K_1, \ldots, K_r$. Assume that the vertices of the $D$-clique are $A_1^1, A_1^2, \ldots, A_1^D$. Let us choose a partition $[m] = A_1^1 \sqcup A_1^2 \sqcup \cdots \sqcup A_1^D \sqcup B_1 \sqcup \cdots \sqcup B_{r-D}$ such that $|B_i| = k + 2$ for each $i$. This is possible since $D(k+1) + (r-D)(k+2)$ adds up to $m$. This is in contradiction with the collective unavoidability of the collection $\langle K_i \rangle_{i=1}^r$ since $A_1^1 \notin K_1, A_1^2 \notin K_2, \ldots, A_1^D \notin K_D, B_1 \notin K_{D+1}, \ldots, B_{r-D} \notin K_r$.

For the opposite direction, assume that there exists a partition $[m] = B_1 \sqcup \cdots \sqcup B_r$.
such that $B_i \notin K_i$ for each $i = 1, \ldots, r$. It follows that $|B_i| \geq k + 1$ for each $i$ and, since $m = r(k + 2) - D$, at least $D$ of them have exactly $k + 1$ elements. This is a clique with desired properties.

Appendix A. Discrete Morse theory

By definition, a discrete Morse function (a DMF, for short) is an acyclic matching on the Hasse diagram of a simplicial complex.

Suppose that $K$ is a simplicial complex. By $\alpha^p$, $\beta^p$ we denote its $p$-dimensional simplices, or $p$-simplices for short. A discrete vector field is a matching

$$(\alpha^p, \beta^{p+1})$$

such that:

1. each simplex of the complex is matched at most once, and
2. in each matched pair, the simplex $\alpha^p$ is a facet of $\beta^{p+1}$.

Given a discrete vector field, a gradient path is a sequence of simplices $\alpha_0^p, \beta_0^{p+1}, \alpha_1^p, \beta_1^{p+1}, \alpha_2^p, \beta_2^{p+1}, \ldots, \alpha_k^p, \beta_k^{p+1}, \alpha_{k+1}^p$, which satisfies the conditions:

1. $\alpha_i^p$ and $\beta_i^{p+1}$ are matched.
2. Whenever $\alpha$ and $\beta$ are neighbors in the path, $\alpha$ is a facet of $\beta$.
3. $\alpha_i \neq \alpha_{i+1}$.

A path is closed if $\alpha_{k+1}^p = \alpha_0^p$.

Along a gradient path we distinguish the “matching steps” $\alpha_i^p \nearrow \beta_i^{p+1}$ from the “splitting steps” $\beta_i^{p+1} \searrow \alpha_i^p$. Moreover, the elements

$$\{u\} = \beta_i^{p+1} \setminus \alpha_i^p$$

(respectively $\{v\} = \beta_i^{p+1} \setminus \alpha_{i+1}^p$) are referred to as migrating elements (they “migrate” from one simplex to another).

A discrete Morse function on a simplicial complex is a discrete vector field without closed paths. Assuming that a discrete Morse function is fixed, the critical simplices are the non-matched simplices.

A DMF provides a blueprint for contracting all the simplices of the complex that are matched: if a simplex $\beta$ is matched with its facet $\alpha$, then these two can be contracted by pushing $\alpha$ inside $\beta$. Acyclicity guarantees that if we have
many matchings at a time, one can consecutively perform the contractions. The order of contractions does not matter, and eventually one arrives at a complex homotopy equivalent to the initial one.

We use in the paper the following fundamental fact, which can be easily deduced from the analysis outlined above; see [4, 5] for details.

**Proposition A.1:** If a simplicial complex has a single zero-dimensional critical simplex, and the dimensions of the other critical simplices are at least $N \geq 1$, then the complex is $(N - 1)$-connected.

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