On Integer-Forcing Precoding for the Gaussian MIMO Broadcast Channel

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Abstract

Integer-forcing (IF) precoding, a generalization of so-called reverse compute-and-forward, is a promising new approach for communication over multiple-input multiple-output (MIMO) broadcast channels (BC), also known as multiuser MIMO downlink. Inspired by the integer-forcing framework for multiple-access channels, it generalizes linear precoding by inducing an effective channel matrix that is approximately integer, rather than approximately identity. Combined with lattice encoding and a pre-inversion of the channel matrix at the transmitter, the scheme has the potential to outperform any linear precoding scheme, despite enjoying similar low complexity.

In this paper, a specific IF precoding scheme, called diagonally-scaled exact IF (DIF), is proposed and shown to achieve maximum spatial multiplexing gain. For the special case of two receivers, in the high SNR regime, an optimal choice of parameters is derived analytically, leading to an almost closed-form expression for the achievable sum rate. In particular, it is shown that the gap to the sum capacity is upper bounded by 0.27 bits for any channel realization. For general SNR, a regularized version of DIF (RDIF) is proposed. Numerical results for two receivers under Rayleigh fading show that RDIF can achieve performance superior to optimal linear precoding and very close to zero-forcing dirty-paper coding.
Index Terms

Multiuser MIMO, broadcast channel, linear precoding, beamforming, compute-and-forward, integer-forcing.

I. INTRODUCTION

Multiple-input multiple-output (MIMO) broadcast channels (BC) have received significant attention in recent years. One reason is that, provided that channel state information is available at the transmitter (CSIT), the sum rate of the system (also called throughput) scales linearly with the minimum of the number of transmit antennas and the aggregate number of receive antennas [1]. Thus, given enough receivers, the throughput can be increased simply by increasing the number of transmit antennas, even if each receiver has only a single antenna [2].

While the capacity region of the Gaussian MIMO BC is well known, it has only been achieved so far using the so-called dirty-paper coding (DPC) technique [2]–[6]. However, DPC is widely regarded as being mostly of theoretical value, due to its significantly high implementation complexity [2], [7]. As a consequence, suboptimal, lower-complexity schemes have been intensively investigated, with the goal of enabling a practical implementation that achieves good performance.

The simplest and yet very rich class of alternative methods is that of linear precoding (or beamforming), which consists of multiplying the (complex-valued) signals to be transmitted by a well-chosen matrix [7], [8], aiming at producing parallel channels to each receiver, while balancing residual interference and noise. Popular methods in this class include zero-forcing (ZF) precoding and regularized zero-forcing (RZF) precoding.

While these methods are simple and achieve the same multiplexing gain as DPC, they suffer a significant penalty when the aggregate number of receive antennas is equal or slightly less than the number of transmit antennas, even for asymptotically high SNR [9]. However, even optimal linear precoding [8] (which is currently infeasible to implement in practice) falls short of achieving the DPC performance.

A. Integer-Forcing Precoding

Recently, an integer-forcing (IF) approach to the problem has been proposed that appears to provide promising gains without a significant increase in complexity [10]–[13]. The approach
is inspired by the compute-and-forward (CF) framework for the multiple-access channel (MAC) [14] and can be understood as the dual of the integer-forcing approach for the MIMO MAC [15]. Fundamentally, it is a nonlinear technique that nevertheless enjoys many of the properties of traditional linear schemes.

From a high level, the IF approach to precoding can be described as a concatenation of three layers. In the inner layer (closest to the channel), the transmitter applies linear beamforming so that the precoded channel matrix becomes approximately an integer matrix $A$; this is referred to as precoding at the signal level, or signal precoding. In the outer layer, the transmitter precodes the original messages into auxiliary messages, pre-inverting the integer channel matrix $A$ that will be seen by the receivers. This is referred to precoding at the message level, or message precoding, and is entirely performed using finite-field operations. In the middle layer, the transmitter channel encodes the auxiliary messages using nested lattice codes, forming the codewords to which signal precoding is applied. Each receiver then attempts to decode an integer linear combination of these codewords, where the coefficients of the linear combination correspond to the rows of the integer matrix $A$. Due to the message precoding, each correctly decoded message is exactly the original message intended to that receiver. Moreover, assuming that lattice encoding and decoding can be done efficiently, the overall complexity is only slightly higher than that of classical linear precoding, essentially due to the message precoding stage.

Originally, an integer-forcing scheme without beamforming was proposed in [10], [11], where it was named reverse compute-and-forward (RCF). A more general version allowing beamforming was also proposed in [10]; however, a specific choice of the beamforming matrix was used, which enforced the precoded channel matrix to be exactly integer. The most general form of IF, allowing arbitrary linear beamforming, as well as potentially different shaping lattices for each user, was later proposed in [12] in the context of an uplink-downlink duality result for IF. However, a disadvantage of such a high degree of generality is that finding an optimal choice of beamforming and integer matrices is a very hard and currently still an open problem [12], [13].

B. Outline of This Work

In this work, we restrict our attention to the special case of IF with a single shaping lattice for all users and focus on the optimization of the signal precoding and integer matrices.
We propose a specific structure for the precoding matrix called \textit{diagonally-scaled exact integer-forcing} (DIF). This scheme is a generalization of the exact IF scheme in \cite{12} by allowing arbitrary scaling by a diagonal matrix, akin to the power scaling in conventional beamforming. As we shall see, such a \textit{moderate} degree of generalization is key to obtaining a scheme that is flexible yet amenable to optimization.

We analyze the performance of DIF for high SNR and show that it achieves maximum spatial multiplexing gain. For the specific case of two receivers and high SNR, we derive the optimal solution for both the precoding matrix and the integer matrix. In a surprising contrast to IF for the MAC case—which typically requires solving a lattice basis reduction problem in order to find an optimal integer matrix—we show that both matrices can be found analytically and therefore very efficiently. This result in turn provides an expression for the achievable sum rate of DIF in almost closed form. In particular, we show that the gap between the sum rate achievable by DIF and that of DPC is upper bounded by 0.27 bits for any channel matrix. For general SNR, we propose a regularized version of DIF (RDIF) and show numerically that the method achieves a sum-rate performance very close to DPC.

The remainder of this paper is organized as follows. The Gaussian MIMO BC model and the necessary background on IF are reviewed in Section \textbf{II} along with the problem statement. In Section \textbf{IV}, we describe our proposed scheme and analyze its performance in the high SNR regime. In Section \textbf{V}, we focus on the two receiver case and derive the optimal selection of the precoding and integer matrices in high SNR regime. Moreover, we show in this section that the gap for the sum capacity in this scenario is bounded. Section \textbf{VI} extends the scheme for the general SNR case. Finally, in Section \textbf{VII} we present numerical results on the average sum rate and the average gap to capacity of the proposed schemes under Rayleigh fading.

\section{Preliminaries}

\subsection{Notation}

For any $x > 0$, let $\log_{10}^{+}(x) = \max\{0, \log_{10}(x)\}$. Vectors and matrices are denoted by boldface $\mathbf{a}$ and $\mathbf{A}$, respectively, and by default vectors are treated as row vectors. For any vector $\mathbf{a}$, $\|\mathbf{a}\|$ denotes its Euclidean norm. For any matrix $\mathbf{A}$, $\mathbf{A}^H$ denotes its conjugate transpose and $\text{tr}(\mathbf{A})$ denotes its trace. The identify matrix is denoted by $\mathbf{I}$. 
Let \( \mathbb{Z}[j] = \mathbb{Z} + j\mathbb{Z} \) denote the ring of Gaussian integers. For any prime \( p \in \mathbb{Z} \), let \( \mathbb{Z}_p \) denote the ring of integers modulo \( p \) and let \( \mathbb{Z}_p[j] = \mathbb{Z}_p + j\mathbb{Z}_p \cong \mathbb{Z}[j]/p\mathbb{Z}[j] \) denote the ring of Gaussian integers modulo \( p \). For \( x \in \mathbb{Z}[j] \), we use \( x \mod p \) to denote the modulo-\( p \) reduction of both the real and imaginary parts of \( x \), i.e., \( x \mod p = \{a + jb : a, b \in \{0, 1, \ldots, p - 1\}\} \).

Whenever scalar functions are applied to vectors and matrices, it should be understood that the functions are applied element-wise.

### B. Lattices

We review a few basic concepts on lattices, which can be found in more detail in, e.g., [16].

A lattice \( \Lambda \in \mathbb{R}^n \) is a discrete \( \mathbb{Z} \)-submodule of \( \mathbb{R}^n \), i.e., it is closed to integer linear combinations [16]. More generally, a \( \mathbb{Z}[j] \)-lattice \( \Lambda \in \mathbb{C}^n \) is a discrete \( \mathbb{Z}[j] \)-submodule of \( \mathbb{C}^n \), i.e., it is closed to Gaussian integer linear combinations. A full-rank \( \mathbb{Z}[j] \)-lattice may be specified by a full-rank generator matrix \( G \in \mathbb{C}^{n \times n} \) such that \( \Lambda = \{uG : u \in \mathbb{Z}[j]^n\} \).

The nearest neighbor quantizer \( Q_\Lambda : \mathbb{C}^n \rightarrow \Lambda \) is defined by \( x \mapsto \arg \min_{\lambda \in \Lambda} \|x - \lambda\| \), with ties broken in a systematic manner. The Voronoi region of \( \Lambda \) is defined as \( V_\Lambda \triangleq \{x \in \mathbb{C}^n : Q_\Lambda(x) = 0\} \) and the modulo-\( \Lambda \) operation is defined as \( x \mod \Lambda \triangleq x - Q_\Lambda(x) \in V_\Lambda \). Note that \( x \mod \Lambda = y \) implies that \( x - y \in \Lambda \). The second moment (per dimension) of \( \Lambda \) is defined as \( P_\Lambda = \frac{1}{n} \mathbb{E}[\|x\|^2] \), where \( x \) is a random vector uniformly distributed over \( V_\Lambda \).

If \( \Lambda \) and \( \Lambda_s \subseteq \Lambda \) are lattices, then \( \mathcal{C} = \Lambda \cap V_{\Lambda_s} \) is said to be a nested lattice code [book]. Note that \( \mathcal{C} = \Lambda \mod \Lambda_s \).

### C. System Model

Consider the discrete-time complex baseband model of a Gaussian MIMO BC with one transmitter and \( K \) receivers, where the transmitter has \( M \geq K \) antennas and each receiver has a single antenna.

Let \( x_j \in \mathbb{C}^n \) be the vector sent by the transmitter on its \( j \)-th antenna, \( j = 1, \ldots, M \), over \( n \) channel uses. For \( i = 1, \ldots, K \), the vector received by the \( i \)-th receiver is given by

\[
y_i = \sum_{j=1}^{M} h_{ij} x'_j + z_i
\]

(1)

where \( h_i = [h_{i1} \ \cdots \ h_{iM}] \in \mathbb{C}^{1 \times M} \) is the vector of channel coefficients and \( z_i \) is a circularly symmetric jointly-Gaussian complex random vector with i.i.d. components \( \sim \mathcal{CN}(0, 1) \).
Equivalently, we can write
\[ Y = HX' + Z \]  
where \( Y \in \mathbb{C}^{K \times n} \), \( H \in \mathbb{C}^{K \times M} \), \( X' \in \mathbb{C}^{M \times n} \), and \( Z \in \mathbb{C}^{K \times n} \) are matrices having the vectors \( y_i \), \( h_i \), \( x'_j \) and \( z_i \), respectively, as rows.

The transmit signals must satisfy an average total power constraint
\[
\frac{1}{n} \mathbb{E}\left[ \text{tr}(X'X'^H) \right] = \frac{1}{n} \sum_{j=1}^{M} \mathbb{E}\left[ \|x'_j\|^2 \right] \leq \text{SNR}
\]  
which is denoted by SNR since the noise is assumed to have unit variance. We assume that the transmitter and each receiver have perfect knowledge of their respective channel coefficients. We also assume that \( H \) is full-rank.

We consider the problem where, for \( i = 1, \ldots, K \), an independent message \( w_i \in \mathcal{W}_i \), of rate \( R_i = \frac{1}{n} \log_2 |\mathcal{W}_i| \), taken from a message space \( \mathcal{W}_i \), is to be transmitted to the \( i \)th receiver. The sum rate of the scheme is given by \( R_{\text{sum}} = R_1 + \cdots + R_K \).

A sum rate \( R \) is said to be achievable if, for any \( \epsilon > 0 \) and sufficiently large \( n \), there exists a coding scheme with sum rate \( R \) that allows each receiver to recover its intended message with probability of error smaller than \( \epsilon \). The sum capacity is the supremum of all achievable sum rates. It is well-known that the sum capacity of the MIMO BC is given by \([3]–[6]\)
\[
C_{\text{sum}} = \sup_{Q : \text{tr}(Q) \leq 1} \log_2 \det \left( I + \text{SNR} H^HQ \right)
\]  
where \( Q \) is a \( K \times K \) diagonal matrix.

### D. Integer-Forcing Scheme

We now review the integer-forcing approach to the MIMO BC (also called downlink integer-forcing), originally proposed in \([10]\) and extended in \([12]\). The scheme consists of the steps of message precoding, nested lattice encoding, and linear beamforming, performed at the transmitter, together with compute-and-forward (lattice) decoding at each receiver.

We restrict our attention to the case of IF with symmetric coding power allocation, where a single shaping lattice is used for all receivers, which is enough for the purposes of this paper. This is a special case of the scheme in \([12]\), which allows multiple shaping lattices with unequal second moments. Such a special case, while potentially inferior to \([12]\) in terms of achievable rates,
has the advantage of being easier to optimize and requiring lower implementation complexity, which is relevant if IF is to be competitive against conventional low-complexity beamforming schemes.

1) Construction: Let $p \in \mathbb{Z}$ be a prime satisfying $p - 3 \in 4\mathbb{Z}$, so that $\mathbb{Z}_p[j]$ becomes a finite field \[17\].

For $i = 1, \ldots, K$, let the message space for the $i$th user be given by $\mathcal{W}_i = \mathbb{Z}_p[j]^{k_i} \times \{0\}^{n-k_i} \subseteq \mathcal{W}$, where $\mathcal{W} = \mathbb{Z}_p[j]^n$ is the ambient space. In other words, the elements of $\mathcal{W}_i$ are length-$n$ (row) vectors over $\mathbb{Z}_p[j]$ whose last $n - k_i$ entries are zeros. Let $\Lambda \subseteq \mathbb{C}^n$ be a $\mathbb{Z}[j]$-lattice, referred to as the fine or coding lattice, and let $\Lambda_s = p\Lambda$, referred to as the coarse or shaping lattice. The scaling for $\Lambda$ is chosen so that $P_{\Lambda_s} = \text{SNR}$. Let $\mathcal{C} = \Lambda \cap \mathcal{V}_{\Lambda_s}$ be a nested lattice code, which is to be used as a “mother” code for the transmitter. Let $\varphi : \Lambda \to \mathcal{W}$ be a surjective $\mathbb{Z}[j]$-linear map\[18\] with kernel $\Lambda_s$ (referred to as a linear labeling in \[18\]) and let $\tilde{\varphi} : \mathcal{W} \to \mathcal{C}$ be a bijective encoding function such that $\varphi(\tilde{\varphi}(w)) = w$, for all $w \in \mathcal{W}$. For all $i$, define the lattice code $\mathcal{C}_i = \tilde{\varphi}(\mathcal{W}_i) \subseteq \mathcal{C}$ as the image of the encoder restricted to $\mathcal{W}_i$, which naturally results in a subcode of $\mathcal{C}$.

Each $i$th receiver is assumed to have a lattice decoder for the corresponding $\mathcal{C}_i$, while the transmitter is assumed to have an encoder for $\mathcal{C}$.

2) Encoding: Let $\mathbf{W} \in \mathcal{W}^K = \mathbb{Z}_p[j]^{K \times n}$ be a matrix whose rows are the messages $\mathbf{w}_1, \ldots, \mathbf{w}_K$. Let $\mathbf{A} \in \mathbb{Z}[j]^{K \times K}$ be an integer matrix that is invertible modulo $p$, i.e., for which
\[\det(\mathbf{A}) \mod p \neq 0\] (5)
and let $\tilde{\mathbf{A}} \in \mathbb{Z}[j]^{K \times K}$ be any matrix satisfying
\[\mathbf{A}\tilde{\mathbf{A}} \mod p = \mathbf{I}.\] (6)

First, the original messages are linearly precoded with $\tilde{\mathbf{A}}$, resulting in the precoded messages $\mathbf{w}_1', \ldots, \mathbf{w}_K' \in \mathcal{W}$ given as the rows of the matrix
\[\mathbf{W}' = \tilde{\mathbf{A}}\mathbf{W}.\] (7)

Then, each precoded message $\mathbf{w}_i'$ is encoded with the lattice code $\mathcal{C}$, yielding a codeword $\mathbf{c}_i' = \tilde{\varphi}(\mathbf{w}_i') \in \mathcal{C}$. Next, for each $i$, a vector
\[\mathbf{x}_i = \mathbf{c}_i' + \mathbf{d}_i \mod \Lambda_s\] (8)

\[\text{More precisely, a } \mathbb{Z}[j]-\text{module homomorphism.}\]
is computed, where $d_i \in \mathbb{C}^n$ is a dither vector, selected so as to ensure that $\frac{1}{n} \mathbb{E} [\| x_i \|^2] \leq P_{\Lambda_s} = \text{SNR}$.

Finally, linear beamforming—also referred to here as signal precoding—with a matrix $T \in \mathbb{C}^{M \times K}$ is performed, producing the transmission matrix

$$X' = TX$$

(9)

where $X \in \mathbb{C}^{K \times n}$ is the matrix whose rows are $x_1, \ldots, x_K$. Note that the beamforming matrix must satisfy the constraint

$$\text{tr}(T^H T) \leq 1$$

(10)
in order to guarantee that the power constraint (3) is respected.

3) Decoding: Let $C', D \in \mathbb{C}^{K \times n}$ be matrices whose rows are, respectively, the codewords $c'_1, \ldots, c'_K$ and the dither vectors $d_1, \ldots, d_K$, and let $a_1, \ldots, a_K \in \mathbb{Z}[j]^K$ denote the rows of $A$.

Recall that the observation of the $i$th receiver is given by

$$y_i = h'_i X' + z_i = h'_i X + z_i$$

(11)

where $h'_i = h_i T$. The receiver selects a scalar $\alpha_i \in \mathbb{C}$ and computes

$$y_{\text{eff},i} = \alpha_i y_i - a_i D \mod \Lambda_s$$

(12)

$$= \alpha_i(h'_i X + z_i) - a_i D \mod \Lambda_s$$

(13)

$$= a_i(X - D) + \underbrace{(\alpha_i h'_i - a_i)X + \alpha_i z_i}_{z_{\text{eff},i}} \mod \Lambda_s$$

(14)

$$= a_i C' + z_{\text{eff},i} \mod \Lambda_s$$

(15)

$$= c_i + z_{\text{eff},i} \mod \Lambda_s$$

(16)

where $c_i = a_i C' \mod \Lambda_s$ and

$$z_{\text{eff},i} = (\alpha_i h'_i - a_i)X + \alpha_i z_i$$

(17)

is the effective noise.

\(^2\)For instance, $d_i$ may be chosen to be uniformly distributed over $\mathbb{V}_{\Lambda_s}$, but fixed dithers are also possible \(^3\).

\(^3\)The expressions signal precoding and beamforming are used interchangeably in this paper.
Since $\phi$ is $\mathbb{Z}[j]$-linear with kernel $\Lambda_s$, we have that
\[
\phi(a_iC' \mod \Lambda_s) = a_i\phi(C') = a_iW' = a_i\tilde{A}W = w_i
\]
which implies that $c_i = \tilde{\phi}(w_i) \in C_i$, i.e., $c_i$ corresponds exactly to the encoding of the original message $w_i$, even though it is never explicitly computed at the transmitter.

It follows that, if the receiver can correctly decode $c_i$ from $y_{\text{eff},i}$, rejecting the effective noise $z_{\text{eff},i}$, then the message $w_i$ can be easily recovered by inverting $\tilde{\phi}$.

4) Achievable rates: According to (16), the $i$th receiver effectively sees an independent modulo-$\Lambda_s$ channel [20], free of interference from other messages (except as already incorporated in $z_{\text{eff},i}$). With an appropriate lattice construction, it can be shown [14] that, provided that $k_1, \ldots, k_K$ and $p$ are allowed to grow with $n$, the following rate tuple is achievable:
\[
R_i = \log_2 \left( \frac{\text{SNR}}{\sigma_{\text{eff},i}^2} \right), \quad i = 1, \ldots, K
\]
where
\[
\sigma_{\text{eff},i}^2 = \frac{1}{n} \mathbb{E} \left[ \|z_{\text{eff},i}\|^2 \right]
\]
is the per-component variance of the effective noise.

After optimizing each scalar $\alpha_i$, this achievable rate tuple can be expressed more simply as
\[
R_i = R_{\text{comp}}(h_i, a_i), \quad i = 1, \ldots, K
\]
where
\[
R_{\text{comp}}(h_i, a_i) \triangleq \log_2 \left( \frac{1}{a_i \left( I - \frac{\text{SNR}}{\|h_i\|^2 + 1} h_i^H h_i \right) a_i^H} \right)
\]
is known as a computation rate in the compute-and-forward framework [14].

It follows that the sum rate achievable by the IF scheme is given by
\[
R_{\text{IF}}(A, T) \triangleq \sum_{i=1}^K R_{\text{comp}}(h_i T, a_i)
\]
where, for conciseness, we omit the dependence on $H$ and $\text{SNR}$. 
III. Problem Statement

In this paper, we focus on the problem of choosing $A$ and $T$, based on $H$ and SNR, in order to maximize the achievable sum rate $R_{IF}(A, T)$. Note that this problem has to be solved at the transmitter for each channel realization. Additionally, in practice, the selected matrix $A$ would have to be communicated to the receivers, possibly in a preamble section, prior to the main transmission.

For any fixed integer matrix $A$, we can optimize over $T$ to obtain an achievable sum rate

$$R_{IF}(A) \triangleq \max_{T : \text{tr}(T^H T) \leq 1} R_{IF}(A, T).$$

(23)

Then, by optimizing $R_{IF}(A)$ over $A$, the maximum achievable sum rate for the scheme is obtained as

$$R_{IF} \triangleq \max_{A : \text{rank} A = K} R_{IF}(A).$$

(24)

Note that, in the above expression, we require only that $A$ be invertible over $\mathbb{C}$, rather than modulo $p$. This is due to the fact that, without loss of optimality, we may safely restrict the search to matrices $A$ of bounded determinant, while, as far as we are concerned with achievable rates, the modulus $p$ has to grow without bound, implying that any optimal matrix for (24) will also satisfy (5). In other words, the modulo $p$ issue is irrelevant for achievable rates, although it may be important in a practical implementation.

A. Related Schemes

When $M = K$ and $T = I$, the scheme reduces to the original RCF [11]. In this case, a natural strategy for optimizing $A$ is to choose each coefficient vector $a_i$ independently for each receiver as the one which maximizes (21) for its corresponding channel gain vector $h_i$. This approach is optimal (for this specific $T$) if the resulting matrix $A$ turns out to be full rank, but this condition is not guaranteed to occur. This problem is avoided in [11] by restricting the transmission to a subset of receivers (and antennas) for which the full rank condition is obtained. However, this solution is not applicable when the number of receivers is fixed, which is the scenario considered here.

This follows provided that $\|h_i\|^2$ is bounded, which implies that both $\|h_i'\|^2$ and $\|a_i\|^2$ are bounded as well. See [14, Remark 10] for details.
When \( M = K \) and \( T = cH^{-1}A \), where \( c > 0 \) is chosen to satisfy a per-antenna power-constraint, then the scheme reduces to RCF with “integer-forcing beamforming” proposed in [10], which produces an exactly integer effective channel matrix. In this case it is no longer possible to optimize each \( a_i \) individually, since \( T \) (and thus \( h_i' = h_iT \)) depends on \( A \). Therefore, the optimization of \( A \) must be based on the complete sum rate \( R_{IF}(A, T) \). This is a hard problem in general, suggesting the use of suboptimal methods, such as the one proposed in [10].

However, as we shall see, having more flexibility on the choice of \( T \) actually makes the problem more structured and potentially easier to solve.

It is also worth mentioning that if we take \( A = I \), then (23) becomes equivalent to optimal linear precoding [8] (with the sum rate as objective). This is due to the fact that, in this case, the computation rate \( R_{comp}(h_iT, a_i) \) becomes equal to the capacity of the channel to the \( i \)th receiver when treating interference from all other users as noise. In particular, if we take \( A = I \) and \( T = cH^H(HH^H)^{-1}D \), where \( D \) is a diagonal matrix, then we recover ZF precoding [8].

### IV. Proposed Scheme

In this section we propose and analyze a signal precoding structure that is the main focus of this paper. Our definition is motivated by the following result.

**Theorem 1:** For any full-rank \( A \), integer-forcing precoding with a fixed \( T \) achieves maximum spatial multiplexing gain if and only if

\[
HT = DA
\]

for some diagonal matrix \( D = \text{diag}(d_1, \ldots, d_K) \) with nonzero diagonal entries. In this case,

\[
R_{IF}(A, T) = \sum_{i=1}^{K} \log_2^+ \left( \frac{1}{\|a_i\|^2} + |d_i|^2\text{SNR} \right)
\]

and

\[
\lim_{\text{SNR} \to \infty} \frac{R_{IF}(A, T)}{\log_2(\text{SNR})} = K.
\]
Proof: The computation rate (21) for the $i$th receiver can be rewritten as

$$R_{\text{comp}}(h'_i, a_i) = \log_2 \left( \frac{1}{a_i \left( I - \frac{\text{SNR}}{\|h'_i\|^2 + 1} h'^H_i h'_i \right) a^H_i} \right)$$

$$= \log_2 \left( \frac{1 + \|h'_i\|^2 \text{SNR}}{a_i \left[ (1 + \|h'_i\|^2 \text{SNR}) I - \text{SNR} h'^H_i h'_i \right] a^H_i} \right)$$

$$= \log_2 \left( \frac{1 + \|h'_i\|^2 \text{SNR}}{\|a_i\|^2 + (\|a_i\|^2 \|h'_i\|^2 - |h'_i a^H_i|^2) \text{SNR}} \right).$$  (28)

From the Cauchy-Schwarz inequality, we know that the term

$$\|a_i\|^2 \|h'_i\|^2 - |h'_i a^H_i|^2$$  (29)

is always non-negative, and it is equal to zero if and only if $h'_i = d_i a_i$, for some $d_i \in \mathbb{C}$. Thus, for fixed $h'_i, a_i$, the computation rate grows with SNR if and only if $h'_i \neq 0$ and $h'_i = d_i a_i$. In this case, we obtain

$$R_{\text{comp}}(h'_i, a_i) = \log_2 \left( \frac{1}{\|a_i\|^2 + |d_i|^2 \text{SNR}} \right).$$  (30)

The remaining statements follow directly.

Theorem[1] shows that the optimal IF precoder structure for high SNR is such that the precoded channel matrix becomes exactly an integer matrix, up to scaling for each user. We call any such scheme \textit{diagonally-scaled exact integer-forcing} (DIF) precoding. Clearly, DIF generalizes ZF, which corresponds to the special case $A = I$. It also generalizes the “exact IF” scheme in [10], which corresponds to the special case $D = c I$.

For any $A$ and $D = \text{diag}(d_1, \ldots, d_K)$, let

$$R_{\text{DIF}}(A, D) \triangleq \sum_{i=1}^{K} \log_2 \left( \frac{1}{\|a_i\|^2 + |d_i|^2 \text{SNR}} \right)$$  (31)

$$R_{\text{DIF}}(A) \triangleq \max_{D,T} R_{\text{DIF}}(A, D)$$

s.t. $H T = D A$

$$\text{tr}(T T^H) \leq 1$$  (32)

and

$$R_{\text{DIF}} \triangleq \max_{A: \text{rank} A = K} R_{\text{DIF}}(A).$$  (33)
The question of optimally choosing $T$ given $A$ and $D$ under a power constraint is addressed in the following theorem.

**Theorem 2:** For any full-rank $A$,

$$R_{\text{DIF}}(A) = \max_D R_{\text{DIF}}(A, D)$$

subject to $\text{tr}(TT^H) \leq 1$

with $T$ given by

$$T = H^H(HH^H)^{-1}DA. \quad (34)$$

**Proof:** The proof is a direct generalization of that of [21, Theorem 1] and is therefore omitted. \hfill \blacksquare

We are particularly interested in analyzing the DIF scheme in the high SNR regime. For any $A$ and diagonal $D$, let

$$R_{\text{DIF}}^{\text{HI}}(A, D) \triangleq \log_2 \left( |\det D|^2 \text{SNR}^K \right) \quad (35)$$

and

$$R_{\text{DIF}}^{\text{HI}}(A) \triangleq \max_D R_{\text{DIF}}^{\text{HI}}(A, D)$$

subject to the same constraint as in Theorem 2 and let

$$R_{\text{DIF}}^{\text{HI}} \triangleq \max_A R_{\text{DIF}}^{\text{HI}}(A).$$

It is easy to see that $R_{\text{DIF}} \geq R_{\text{DIF}}^{\text{HI}}$ and

$$\lim_{\text{SNR}\to\infty} R_{\text{DIF}} - R_{\text{DIF}}^{\text{HI}} = 0.$$ 

Thus, for high SNR, $R_{\text{DIF}}^{\text{HI}}(A, D)$ can serve as an equivalent objective function that is easier to analyze than $R_{\text{DIF}}(A, D)$. We will focus on this function for the remainder of the paper.

**Theorem 3:** For any full-rank $A$,

$$R_{\text{DIF}}^{\text{HI}}(A) = \max_{D} \max_{|\det D| = 1} K \log_2 \left( \frac{\text{SNR}}{\text{tr}(T_0T_0^H)} \right) \quad (36)$$

where $D_0$ is a diagonal matrix and

$$T_0 = H^H(HH^H)^{-1}D_0A. \quad (37)$$
This value is achievable by choosing \( D = cD_0 \), with
\[
c = \frac{1}{\sqrt{\text{tr}(T_0 T_0^H)}}. \tag{38}
\]

**Proof:** Without loss of generality, we can express \( D \) as
\[
D = cD_0 \tag{39}
\]
where \( c > 0 \) and \( D_0 = \text{diag}(\bar{d}_1, \ldots, \bar{d}_K) \) satisfies\(^5\)
\[
|\det D_0| = 1. \tag{40}
\]
This implies that
\[
R_{\text{DIF}}^H(A, D) = \log_2 \left( c^{2K} \text{SNR}^K \right). \tag{41}
\]

Note also that this choice results in \( T = cT_0 \), with \( T_0 \) given in (37). Now, since \( 1 \geq \text{tr}(TT^H) = c^2 \text{tr}(T_0 T_0^H) \), it is easy to see that the rate is maximized under the power constraint by choosing \( c \) according to (38). Replacing this value in (41) gives the desired result. \( \blacksquare \)

It follows from Theorem 3 that, for high SNR, optimizing the DIF scheme amounts to minimizing \( \text{tr}(T_0 T_0^H) \) under a constraint on \( D_0 \).

**V. THE TWO-USER CASE FOR HIGH SNR**

In this section, we focus on the the special case of \( K = 2 \) receivers in the high SNR regime, assuming that DIF is used.

**A. Optimal Signal Precoding**

We start by optimizing the matrix \( D_0 \) in (36) and finding the resulting achievable rate.

Let
\[
\rho(H) \triangleq \frac{|h_1 h_2^H|}{\|h_1\|\|h_2\|} \tag{42}
\]
denote the normalized inner product between the rows of \( H \), and define
\[
f(A, \rho) \triangleq \|a_1\|\|a_2\| - \rho|a_2 a_1^H|. \tag{43}
\]

\(^5\)Specifically, take \( c = \left( \prod_{i=1}^{K} |d_i| \right)^{1/K} \) and \( \bar{d}_i = d_i/c, i = 1, \ldots, K \).
Theorem 4: For any full-rank $A \in \mathbb{Z}[i]^{2 \times 2}$, 
\[
R_{\text{DIF}}^H(A) = 2 \log_2 \left( \frac{\det(HH^H) \cdot \text{SNR}}{2\|h_1\|\|h_2\| \cdot f(A, \rho(H))} \right)
\] (44)
achievable with
\[
D_0 = \begin{bmatrix}
\sqrt{\|a_2\|\|h_1\|} & 0 \\
0 & \sqrt{\|a_1\|\|h_2\|} e^{-j\angle(a_2a_1^Hh_1h_2^H)}
\end{bmatrix}.
\] (45)

Proof: Recall that $D_0 = \text{diag}(\bar{d}_1, \bar{d}_2)$. Without loss of generality, let $\bar{d}_1 = e^{\beta+j\theta_1}$ and $\bar{d}_2 = e^{-\beta+j\theta_2}$, where $\beta \in \mathbb{R}$. Note that $\det(D_0) = |\bar{d}_1||\bar{d}_2| = 1$. In order to simplify notation, let
\[
M = (HH^H)^{-1} = \frac{1}{\det(HH^H)} \begin{bmatrix}
\|h_2\|^2 & -h_1h_2^H \\
-h_2h_1^H & \|h_1\|^2
\end{bmatrix}
\] (46)
with entries $M_{ij}$. Since $(AA^H)_{ij} = a_i^Ha_j^H$,
\[
(D_0^HMD_0)_{ij} = \bar{d}_i^*M_{ij}\bar{d}_j
\] (47)
and $M^H = M$, we have that
\[
\text{tr}(T_0^HT_0) = \text{tr} \left( A^HD_0^HMD_0A \right)
= \text{tr} \left( AA^H \right) = \|a_1\|^2M_{11}e^{2\beta} + \|a_2\|^2M_{22}e^{-2\beta} + 2\Re\{a_2a_1^HM_{12}e^{\Delta\theta}\}
\] (48)
where $\Delta \theta = \theta_2 - \theta_1$.

We wish to minimize (48) by the choice of $\beta$ and $\Delta \theta$. Clearly, the optimal choice of $\Delta \theta$ is
\[
\Delta \theta = -\angle(-a_2a_1^HM_{12}) = -\angle(a_2a_1^Hh_1h_2^H)
\] (49)
which gives
\[
\text{tr}(T_0^HT_0) = \|a_1\|^2M_{11}e^{2\beta} + \|a_2\|^2M_{22}e^{-2\beta} - 2|a_2a_1^HM_{12}|.
\] (50)

Solving for the optimal $\beta$, we have
\[
0 = \frac{\beta}{\partial \beta} \text{tr}(T_0^HT_0)
= 2\|a_1\|^2M_{11}e^{2\beta} - 2\|a_2\|^2M_{22}e^{-2\beta}
\] (51)
which gives

\[ e^{2\beta} = \frac{\|a_2\| \sqrt{M_{22}}}{\|a_1\| \sqrt{M_{11}}} = \frac{\|a_2\| \|h_1\|}{\|a_1\| \|h_2\|}. \]  

(52)

Substituting this optimal choice of \( \beta \), we have

\[ \text{tr}(T_0^H T_0) = 2\|a_1\|\|a_2\| \sqrt{M_{11}M_{22}} - 2|a_2 a_1^H M_{12}| \]

\[ = 2\sqrt{M_{11}M_{22}}(\|a_1\|\|a_2\| - \rho |a_2 a_1^H|) \]

\[ = 2\sqrt{M_{11}M_{22}} f(A, \rho) \]

\[ = 2 \frac{\|h_1\|\|h_2\|}{\det(HH^H)} f(A, \rho) \]  

(53)

where

\[ \rho = \rho(H) = \frac{|M_{12}|}{\sqrt{M_{11}M_{22}}}. \]  

(54)

The result now follows by replacing (53) in (56) and setting \( \theta_1 = 0 \).

It follows from Theorem 4 that

\[ R_{\text{DIF}}^{HH} = 2 \log_2 \left( \frac{\det(HH^H) \cdot \text{SNR}}{2\|h_1\|\|h_2\| \cdot f(\rho(H))} \right) \]  

(55)

where

\[ f(\rho) \triangleq \min_{A \in \mathbb{Z}[j]^{2 \times 2} : \text{rank} A = 2} f(A, \rho). \]  

(56)

**B. Optimal Message Precoding**

We now address the optimal choice of the message precoding matrix \( A \) in (56).

First, we need a few definitions. Let

\[ N_2 = \{ |a|^2 : a \in \mathbb{Z}[j] \} = \{ a^2 + b^2 : a, b \in \mathbb{Z} \} \subseteq \mathbb{Z} \]  

(57)

be the set of all integers that are sums of two squares. For any \( x \in \mathbb{R} \), let \( \lfloor x \rfloor_{N_2} \) denote the largest element of \( N_2 \) smaller than or equal to \( x \). Similarly, let \( \lceil x \rceil_{N_2} \) denote the smallest element of \( N_2 \) greater than or equal to \( x \).

**Theorem 5:** We have

\[ f(\rho) = \min_{N \in N_2} \sqrt{N + 1} - \rho \sqrt{N}. \]  

(58)

This value is achievable by any full-rank \( A \) satisfying

\[ \|a_1\|^2\|a_2\|^2 - 1 = |a_1 a_2^H|^2 = N \]
in particular, by

\[ A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{21} & 1 \end{bmatrix} \]  \hspace{1cm} (59)

where \( a_{21} \in \mathbb{Z}[j] \) is such that \( |a_{21}|^2 = N \).

**Proof:** Recall that \( f(A, \rho) = \|a_1\|\|a_2\| - \rho|a_2a_1^H| \). Our approach to solve (56) is to find a pair of linearly independent vectors \( a_1, a_2 \in \mathbb{Z}[j] \) that minimizes \( \|a_1\|\|a_2\| \), for every possible value of \( |a_1a_2^H|^2 \in \mathcal{N}_2 \).

For all \( N \in \mathcal{N}_2 \), let

\[ \pi_N \triangleq \min_{a_1, a_2 : |a_1a_2^H|^2 = N} \|a_1\|\|a_2\| \]  \hspace{1cm} (60)

where the minimization is restricted to \( a_1, a_2 \in \mathbb{Z}[j] \) that are linearly independent. It follows that \( f(\rho) \) is equal to the minimum value of \( \pi_N - \rho\sqrt{N} \) over all \( N \in \mathcal{N}_2 \).

From the Cauchy-Schwarz inequality, we know that \( \|a_1\|^2\|a_2\|^2 \geq |a_1a_2^H|^2 \), achievable if and only if \( a_2 \) is a multiple of \( a_1 \). However, this condition violates the requirement of linear independence. Thus, since both \( \|a_1\|^2\|a_2\|^2 \) and \( |a_1a_2^H|^2 \) must be integers, we must have

\[ \|a_1\|^2\|a_2\|^2 \geq |a_1a_2^H|^2 + 1. \]  \hspace{1cm} (61)

Since equality is always achievable, e.g., by (59), we have that \( \pi_N^2 = N + 1 \), for all \( N \in \mathcal{N}_2 \).

Table I lists all possible non-equivalent solutions of (56) for \( N \leq 20 \). Equivalent solutions can be found by permuting rows and/or columns and by multiplying rows and/or columns by \(-1, j, \) or \(-j\). Note that there can be multiple non-equivalent solutions for certain values of \( N \).

The next theorem shows that the optimal value of \( N \) can be found in an almost closed form.

**Theorem 6:** The optimal value of \( N \) in (58) satisfies

\[ N \in \left\{ \left[ \frac{\rho^2}{1 - \rho^2} \right]_{\mathcal{N}_2}, \left[ \frac{\rho^2}{1 - \rho^2} \right]_{\mathcal{N}_2} \right\} . \]  \hspace{1cm} (62)

Moreover, each value of \( N \in \mathcal{N}_2 \) is an optimal solution for \( \rho_N \leq \rho \leq \rho_{N^+} \), where \( \rho_0 \triangleq 0 \),

\[ \rho_N \triangleq \frac{\sqrt{N+1} - \sqrt{N^-+1}}{\sqrt{N} - \sqrt{N^-}}, \quad N \geq 1, \]  \hspace{1cm} (63)

\( N^- = \lfloor N - 1 \rfloor_{\mathcal{N}_2} \) and \( N^+ = \lceil N + 1 \rceil_{\mathcal{N}_2} \).
TABLE I

| $\mathbf{a}_1$ | $\mathbf{a}_2$ | $||\mathbf{a}_1||^2 ||\mathbf{a}_2||^2$ | $N = ||\mathbf{a}_1\mathbf{a}_2^H||^2$ | $\rho_N$ |
|-------------|-------------|-------------------------------|--------------------------|--------|
| (1, 0)      | (0, 1)      | 1                             | 0                        | 0      |
| (1, 0)      | (1, 1)      | 2                             | 1                        | 0.4142 |
| (1, 0)      | (1 + j, 1)  | 3                             | 2                        | 0.7673 |
| (1, 0)      | (2, 1)      | 5                             | 4                        | 0.8604 |
| (1, 0)      | (2 + j, 1)  | 6                             | 5                        | 0.9041 |
| (1, 1)      | (1 + j, 1)  |                               |                          |        |
| (1, 0)      | (2 + 2j, 1) | 9                             | 8                        | 0.9294 |
| (1, 0)      | (3, 1)      | 10                            | 9                        | 0.9458 |
| (1, 1)      | (2, 1)      |                               |                          |        |
| (1, 0)      | (3 + j, 1)  | 11                            | 10                       | 0.9511 |
| (1, 0)      | (3 + 2j, 1) | 14                            | 13                       | 0.9588 |
| (1, 0)      | (4, 1)      | 17                            | 16                       | 0.9670 |
| (1, 0)      | (4 + j, 1)  | 18                            | 17                       | 0.9710 |
| (1 + j, 1)  | (2 + j, 1)  |                               |                          |        |
| (1, 0)      | (3 + 3j, 1) | 19                            | 18                       | 0.9726 |
| (1, 0)      | (4 + 2j, 1) | 21                            | 20                       | 0.9746 |

Proof: Let

$$f(N, \rho) = \sqrt{N+1} - \rho \sqrt{N}. \quad (64)$$

We have that $f(N, \rho) \leq f(N^-, \rho)$ if and only if

$$\sqrt{N+1} - \rho \sqrt{N} \leq \sqrt{N^- + 1} - \rho \sqrt{N^-} \quad (65)$$

i.e., if and only if $\rho \geq \rho_N$. It follows that $N$ is optimal for all $\rho$ satisfying $\rho_N \leq \rho \leq \rho_{N^+}$.

Now, suppose $\rho$ is fixed and consider the relaxed function $g(x) = \sqrt{x+1} - \rho \sqrt{x}$, where $x \in \mathbb{R}$. This function has a single critical point which is a global minimum, given by

$$x^* = \frac{\rho^2}{1 - \rho^2}. \quad (66)$$

It follows that the optimal value of $N$ is either the floor or ceiling of $x^*$ in $\mathbb{N}_2$, i.e., $N$ must satisfy (62).

It is worth treating the special case where $\mathbf{A}$ is constrained to be a real integer matrix, $\mathbf{A} \in \mathbb{Z}^{2 \times 2}$, in which case $N$ must be of the form $N = k^2$, where $k \in \mathbb{Z}$. Although suboptimal, this choice gives an upper bound on $f(\rho)$ which is easier to analyze.
Theorem 7: We have
\[ f(\rho) \leq \min_{k \in \mathbb{Z}, k \geq 0} \sqrt{k^2 + 1} - \rho k \]
achievable by some \( k \in \mathbb{Z} \) satisfying
\[ k \in \{\lceil u \rceil, \lfloor u \rfloor\} \quad (67) \]
where \( u = \rho / \sqrt{1 - \rho^2} \). Moreover, each value of \( k \geq 0 \) is an optimal solution for \( u_k \leq u \leq u_{k+1} \), where \( u_0 = 0 \) and, for \( k \geq 1 \), \( u_k \) is defined by
\[ \frac{1}{\sqrt{1/u_k^2 + 1}} = \sqrt{k^2 + 1} - \sqrt{(k-1)^2 + 1}. \quad (68) \]

Proof: Except for a change of variables, the proof is very similar to that of Theorem 6 and is therefore omitted. \( \blacksquare \)

Corollary 8: For all \( k \geq 0 \), \( \lceil u_k \rceil = \lfloor u_{k+1} \rfloor = k \).

Proof: The statement is true for \( k = 0 \), as can be easily checked. For \( k \geq 1 \), assume the statement is true for \( k-1 \), i.e., \( \lceil u_{k-1} \rceil = \lfloor u_k \rfloor = k - 1 \). From Theorem 7 we know that
\[ k \in \{\lceil u_k \rceil, \lfloor u_k \rfloor\} \cap \{\lceil u_{k+1} \rceil, \lfloor u_{k+1} \rfloor\}. \]
Since \( u_k \notin \mathbb{Z} \) and \( u_k < u_{k+1} \), this implies that either
\[ \lceil u_k \rceil = \lfloor u_{k+1} \rfloor < \lceil u_k \rceil = \lfloor u_{k+1} \rfloor \quad (69) \]
or
\[ \lceil u_k \rceil = \lfloor u_{k+1} \rfloor = k. \quad (70) \]
By the induction hypothesis, \( (69) \) implies that \( \lceil u_{k+1} \rceil = k - 1 \) and \( \lfloor u_{k+1} \rfloor = k \), which is a contradiction since, from Theorem 7 we must have \( k + 1 \in \{\lceil u_{k+1} \rceil, \lfloor u_{k+1} \rfloor\} \). Thus, \( (70) \) must be true, proving the statement. \( \blacksquare \)

C. Gap to Sum Capacity

We now investigate the asymptotic gap of DIF to the sum capacity, \( C_{\text{sum}} \).

Theorem 9: Let \( \rho = \rho(\mathbf{H}) \). We have
\[
\lim_{\text{SNR} \to \infty} C_{\text{sum}} - R_{\text{DIF}} = 2 \log_2 \left( \frac{f(\rho)}{\sqrt{1 - \rho^2}} \right) \\
\leq \log_2 \left( \frac{1 + \sqrt{2}}{2} \right).
\]
Proof: It is known from [9] that, for high SNR,
\[
\lim_{\text{SNR} \to \infty} C_{\text{sum}} - C_{\text{sum}}^{\text{HI}} = 0
\] (71)
where
\[
C_{\text{sum}}^{\text{HI}} = K \log_2(SNR/K) + \log_2 \det(\mathbf{HH}^H)
\] (72)
with $K = 2$ in the present case.

Let $\delta = C_{\text{sum}}^{\text{HI}} - P_{\text{DIF}}^{\text{HI}}$ and $\Delta = 2^{\delta/2}$. We have that
\[
\Delta = \frac{1}{2} \text{SNR} \sqrt{\frac{\det(\mathbf{HH}^H)}{\frac{1}{2} f(\rho)}}
\]
\[
= \frac{\|\mathbf{h}_1\| \|\mathbf{h}_2\|}{\sqrt{\det(\mathbf{HH}^H)}} f(\rho)
\]
\[
= \frac{f(\rho)}{\sqrt{1 - \rho^2}}
\]
where the last equality follows since
\[
\frac{\det(\mathbf{HH}^H)}{\|\mathbf{h}_1\|^2 \|\mathbf{h}_2\|^2} = 1 - \rho(\mathbf{H})^2.
\]

We now proceed to give an upper bound on $\Delta$. First, note that, from Theorem [7]
\[
\Delta \leq \Delta_k^*(u)
\]
where $u = \rho/\sqrt{1 - \rho^2}$.
\[
\Delta_k(u) = \sqrt{u^2 + 1} \sqrt{k^2 + 1} - uk
\]
and $k^* \geq 0$ is such that $u_{k^*} \leq u \leq u_{k^* + 1}$.

Now, $\Delta_k(u)$ is a convex function of $u$, which implies that
\[
\Delta_k^*(u) \leq \max\{\Delta_k^*(u_{k^*}), \Delta_k^*(u_{k^* + 1})\}
\]
\[
\leq \max_{k \geq 0} \Delta_k(u_k)
\]
where in the last equation we have used the fact that $\Delta_k(u_k) = \Delta_k(u_{k+1})$, for all $k$.

Numerical evaluation reveals that $\Delta_0(u_0) = 1$, $\Delta_1(u_1) = \sqrt{\frac{1 + \sqrt{2}}{2}}$. 
and \( \Delta_1(u_1) \geq \Delta_2(u_2) \geq \Delta_3(u_3) \). For larger \( k \), note that

\[
\Delta_k(u_k) = \sqrt{u_k^2 k^2 + u_k^2 + k^2 + 1 - u_k k}
\]

\[
= u_k k \sqrt{1 + \frac{u_k^2 + k^2 + 1}{u_k^2 k^2}} - u_k k
\]

\[
\leq u_k k \left( 1 + \frac{u_k^2 + k^2 + 1}{2u_k^2 k^2} \right) - u_k k
\]

\[
= \frac{u_k^2 + k^2 + 1}{2u_k k}
\]

\[
= \frac{(k - u_k)^2 + 2u_k k + 1}{2u_k k}
\]

\[
= 1 + \frac{(k - u_k)^2 + 1}{2u_k k}
\]

\[
\leq 1 + \frac{1}{k^2}
\]

where the last inequality follows since \( 0 \leq k - u_k < 1 \). It can be easily checked that \( \Delta_k(u_k) \leq \Delta_1(u_1) \) for \( k \geq 4 \), completing the proof.  

Theorem 9 shows that, for high SNR, DIF achieves a small gap to sum capacity, upper bounded by about 0.27 bits, or approximately 0.4 dB. This maximum value occurs for \( \rho = \sqrt{2} - 1 \) in the transition from \( N = 0 \) to \( N = 1 \) in Table I.

The gap to sum capacity as a function of \( \rho \) is illustrated in Fig. 1 for both the complex case \( A \in \mathbb{Z}[i]^{2\times 2} \) (Theorem 6) and the real case \( A \in \mathbb{Z}^{2\times 2} \) (Theorem 7). Peaks occur at values \( \rho = \rho_N \) given in Table II for the complex case, and at \( \rho \) equal to the right hand side of (68) for the real case. Interestingly, the gap vanishes both as \( \rho \to 0 \) (orthogonal channels) and as \( \rho \to 1 \).

VI. EXTENSION TO GENERAL SNR

We have shown in Section IV that, among all integer-forcing precoding schemes, DIF is optimal for high SNR. For general SNR, however, designing an optimal integer-forcing precoder is much more challenging, since we have much more freedom in the choice of the precoding matrix \( T \) (besides the choice of \( A \)). Indeed, a special case of this problem (for \( A = I \)) is that of designing an optimal linear precoder, which is still an open problem [8].

One way to approach this problem is to keep the general structure of DIF and make certain adaptations to improve its performance under finite SNR, as we describe next. Note that the discussion below is valid for any \( K \).
A. Regularized DIF

Recall that DIF is a generalization of ZF precoding; namely, the latter corresponds to the special case $A = I$. It is well-known that, for finite SNR, the performance of ZF can be improved by regularization, i.e., by choosing a precoding matrix of the form

$$T = H^H \left( \frac{K}{SNR} I + HH^H \right)^{-1} c D_0$$

(73)

where $c$ is chosen to satisfy (with equality) the power constraint. In this case, while the performance for high SNR is unchanged, the performance for low SNR improves considerably.

We can apply a similar principle to the proposed DIF method. Namely, we choose

$$T = c T_0$$

(74)

$$T_0 = H^H \left( \frac{K}{SNR} I + HH^H \right)^{-1} D_0 A$$

(75)

where $c > 0$ is chosen as in (38) in order to satisfy the power constraint. We call this method regularized DIF (RDIF). Note that $A$ and $D_0$ may still be designed based on the high SNR lower bound; only $T$ needs to be adapted. Note also that (75) tends to (37) as SNR grows. Thus, RDIF reduces to DIF in the high SNR scenario.
B. Justification via Uplink-Downlink Duality

We can give a more formal justification for RDIF based on the uplink-downlink duality for integer forcing proved in [12].

It is shown in [12] that any rate tuple achievable in the downlink with a certain power vector can also be achieved in the uplink with a certain (possibly different) power vector, and the same result holds with the roles of downlink and uplink reversed. Each power coefficient refers to the second moment of the shaping lattice for the corresponding user (in the case where multiple shaping lattices are allowed). In addition, the uplink and downlink channels are subject to the same total power constraint and are related by a transpose (or Hermitian, in the complex case) of all corresponding matrices, similarly to the general uplink-downlink duality for the MAC and the BC.

By starting with the downlink problem with a rate tuple that achieves a certain sum rate $R$, we can create a virtual dual uplink problem where the same rate tuple is achievable. For that problem, with all other parameters fixed, the optimal beamforming matrix can be obtained in closed form, which can only possibly improve the sum rate. Next, going back to original downlink problem, we have a solution that achieves a sum rate equal or higher than $R$. This is the basis for an iterative optimization algorithm proposed in [13]. Here, however, we note that if $R$ is already optimal, then these steps essentially provide and optimal solution for the downlink with a specific, known form of the beamforming matrix.

Thus, as a consequence of [13, (27)–(29)], we can claim that every optimal sum rate $R$ for the downlink can be achieved with a beamforming matrix of the form

$$
T = (I + HH^TQ)^{-1}H^TQDA
$$

(76)

$$
= H^T(I + QHH^T)^{-1}QDA
$$

(77)

$$
= H^T(Q^{-1} + HH^T)^{-1}DA
$$

(78)

for some diagonal matrices $Q, D \in \mathbb{C}^{K \times K}$ and some nonnegative diagonal power matrix $P_d \in \mathbb{R}$ satisfying $\text{tr}(Q) = \text{SNR}$ and $\text{tr}(T^H T P_d) = \text{SNR}$. In other words, it is safe to restrict attention to $T$ of this form, at least when multiple power levels are allowed.

---

6In the notation of [13], $D = C_d^{-1}, Q = C_u P_{c,u} C_u^H$ and $C_d = C_u^H$, all of which are diagonal matrices.
Our choice differs from the optimal in [12] in two ways. First, in our problem only a single shaping lattice is allowed, which corresponds to the choice \( P_d = \text{SNR} \cdot I \). Second, we choose \( Q = qI \), which gives \( q = \text{SNR}/K \). Note that the latter choice is analogous to the heuristic derivation of RZF as a suboptimal solution to the problem of optimal linear precoding [8].

It is important to emphasize that, even with these simplifications, the optimization problem for IF is still seemingly hard, due to the appearance of \( A \) in (75), making it difficult to optimize \( D_0 \) as well as \( A \). Thus, we do not attempt to obtain an optimal \( D_0 \) based on \( A \) for general SNR. Rather, we find optimal \( A \) and \( D_0 \) for high SNR and simply replace them in (75) for finite values of SNR.

VII. Numerical Results

In this section, numerical results on the sum-rate performance of the proposed schemes are presented for the case \( M = K = 2 \). These results correspond to an average over a total of 1000 channel realizations for each value of SNR. Independent Rayleigh fading is assumed, where \( H \) has i.i.d entries \( \sim \mathcal{CN}(0,1) \).

Fig. 2 shows the average sum rate of DIF and RDIF over a large range of SNR values. For comparison, the sum capacity (4) obtained by DPC [2] is also shown. It can be seen that RDIF improves on the performance of DIF for low/medium SNR, while both schemes tend to the optimal performance for high SNR.

Fig. 3 shows a close up of Fig. 2 on a range of medium SNR values. For comparison, the performances of several other precoders are also included, namely: the well-known linear precoders ZF and RZF, the optimal linear solution given in [8], and the nonlinear zero-forcing DPC (ZF-DP) [2]. As can be seen, for an average sum rate of 6 bits/channel use, RDIF is less than 0.33 dB away from the sum capacity.

In order to clearly quantify the loss in performance compared to the optimal solution, Fig. 4 shows the average gap to sum capacity of all suboptimal schemes. As can be seen, for SNR \( > 7 \) dB, RDIF outperforms all linear schemes. Moreover, RDIF performs very close to ZF-DP for all SNR. Both DIF and RDIF (as well as ZF-DP) have its worst performance for low to medium SNR. As predicted in Section V-C, we can see that the two proposed schemes obey the gap of \( \delta < 0.27 \) bits for high SNR, although the actual gap is much smaller on average.
Fig. 2. Average sum rate for Rayleigh fading with $M = 2$ transmit antennas and $K = 2$ single-antenna receivers.

Fig. 3. Average sum rate of Fig. 2 for the medium SNR region.

VIII. CONCLUSION

In this paper, we propose two integer-forcing precoding schemes for the Gaussian MIMO BC, called diagonally-scaled exact integer-forcing (DIF) and regularized DIF (RDIF). These precoders generalize ZF and RZF, respectively. Essentially, DIF creates an effective channel matrix that is exactly integer up to diagonal scaling, while RDIF modifies DIF with a certain
matrix regularization in order to improve its performance for low to medium SNR.

We show that the DIF structure is optimal for high SNR, in the sense that it achieves maximum spatial multiplexing gain. Moreover, in the special case of two receivers, we have shown that an analytical solution to the problem of optimal design of IF in the high SNR regime is possible. In this case, we show that the gap to sum capacity of the Gaussian MIMO BC is upper bounded by 0.27 bits.

For general SNR, still in the two-receiver case, we have presented numerical results showing that RDIF, when properly optimized, achieves performance superior to optimal linear precoding and very close to the non-linear ZF-DP precoder from [2].

Besides the results presented in this paper, we also have preliminary results indicating that, for $K = 3$, the performance of RDIF is also very close to the sum capacity, and it is possible to design such a scheme very efficiently. We hope to extend our results for general $K > 2$ in a forthcoming paper.

Other potential extensions of this work include studying other objective functions such as weighted sum rate, other practical constraints such as a per-antenna power constraint, as well as the impact of imperfect CSIT.

Fig. 4. Average gap to sum rate for Rayleigh fading with $M = 2$ transmit antennas and $K = 2$ single-antenna receivers.
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