SHARP WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR 
THE FOURTH ORDER NONLINEAR SCHRÖDINGER 
EQUATION

YUANYUAN REN AND YONGSHENG LI
School of Mathematics, South China University of Technology
Guangzhou, Guangdong 510640, China

WEI YAN
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan 453007, China

(Communicated by Alain Miranville)

ABSTRACT. In this paper, we investigate the Cauchy problem for the fourth 
order nonlinear Schrödinger equation

$$i\partial_t u + \partial^4_x u = u^2, \quad (t, x) \in [0, T] \times \mathbb{R}.$$ 

Zheng (Adv. Differential Equations, 16(2011), 467-486.) has proved that the 
problem is locally well-posed in \(H^s(\mathbb{R})\) with \(-\frac{7}{4} < s \leq 0\). In this paper, we 
am at extending Zheng’s work to a lower regularity index. We prove that 
the equation is locally well-posed in \(H^s(\mathbb{R})\) when \(s \geq -2\) and ill-posed when 
\(s < -2\) in the sense that the solution map is discontinuous for \(s < -2\). The 
key ingredient used in this paper is Besov-type space introduced by Bejenaru 
and Tao (Journal of Functional Analysis, 233(2006), 228-259.).

1. Introduction. In this paper, we consider the Cauchy problem of the following 
quadratic fourth order nonlinear Schrödinger equation

$$i\partial_t u + \partial^4_x u = u^2, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad (1.1)$$

$$u(0, x) = \varphi(x), \quad x \in \mathbb{R}, \quad (1.2)$$

where \(u = u(t, x)\) is a complex-valued unknown function, and \(\varphi(x)\) is prescribed 
complex-valued function.

Several authors have investigated the Cauchy problem for

$$iu_t + \lambda \Delta u + \mu \Delta^2 u = F(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad (1.3)$$

where \(\lambda \in \mathbb{R}, \mu\) is a nonzero real number, and \(n \in \mathbb{N}_+\), see, for instance [4, 10, 
12, 13, 14, 15, 16, 18].

2000 Mathematics Subject Classification. Primary: 35Q53; Secondary: 35G25, 46E35.

Key words and phrases. Fourth order Schrödinger equation, Cauchy problem, sharp well-
posedness, Besov-type space.

This work is supported by NSFC under grant numbers 11571118, 11771127 and 11401180, and 
also by the Fundamental Research Funds for the Central Universities of China under the grant 
number 2017ZD094.
When $\lambda = 0$, $\mu \neq 0$ and $F(u) = |u|^{2\sigma}u$ with $\sigma > 0$, (1.3) be reduced to the biharmonic nonlinear Schrödinger equation
\begin{equation}
 i\partial_t u + \mu \Delta^2 u = |u|^{2\sigma} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n,
\end{equation}
which was studied by Ivanov and Kosevich [9] and Turitsyn [20] in the context of stability of solutions in magnetic materials where the effective quasi-particle mass becomes infinite. Pecher and von Wahl in [17] proved the existence of classical global solutions of (1.4) for the space dimensions $n \leq 14$. Guo and Wang [4] considered the existence and scattering theory for this equation. They proved the local well-posedness for any initial data and the global well-posedness for small initial data in $H^s(\mathbb{R}^n)$.

In this paper, we mainly study the Cauchy problem of the fourth order nonlinear Schrödinger equation (1.1). Inspired by [1, 8], we get a lower regularity index of the equation is locally well-posed in $H^s(\mathbb{R})$ when $s \geq -1$ and ill-posed when $s < -2$ in the sense that the solution map is discontinuous for $s < -2$.

We give some notations before presenting the main results. For any $k \in \mathbb{N}_+$, let $\eta_k \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function such that $\eta_k(t) \equiv 1$ for $|t| \leq k$ and $\eta_k(t) \equiv 0$ for $|t| \geq 2k$, and let $a(t) := \frac{1}{2} \text{sgn}(t)\eta_k(t)$. Note that, the identity
\begin{equation}
 1_{[0,1]}(t') = a(t') + a(t-t')
\end{equation}
is valid for $t \in (0,2)$ and $t' \in (-3,3)$, then we have
\begin{equation}
  \int_0^t g(t')dt' = \int_\mathbb{R}a(t')g(t')dt + \int_\mathbb{R}a(t-t')g(t')dt'
\end{equation}
for all $0 \leq t \leq 1$ and any $g : \mathbb{R} \rightarrow \mathbb{R}$ with supp$g \subset [-2,2]$ (see [1, 2, 3]). From now on, for any $1 \leq p, q \leq \infty$, $L^p L^q$ always denotes the mixed Lebesgue space of time-space variable $(\tau, \xi)$:
\begin{equation}
  L^p L^q := L^p_\tau L^q_\xi(\mathbb{R}^2) \quad \text{and} \quad \|f\|_{L^p L^q} := \|f(\tau, \xi)\|_{L^p_\tau L^q_\xi(\mathbb{R})}.
\end{equation}
For any domain $\Omega \subset \mathbb{R}^2$, we also use the restricted norm,
\begin{equation}
  \|f\|_{L^p L^q(\Omega)} := \|f \cdot 1_\Omega\|_{L^p L^q},
\end{equation}
where $1_\Omega$ denotes the characteristic function of $\Omega$. We call a function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ reasonable if $f \in L^\infty(\mathbb{R}^2)$ and supp$f$ is compact. For any given $s, b \in \mathbb{R}$, we define $X^{s,b}$ as the completion of the reasonable functions with the following norm
\begin{equation}
  \|f\|_{X^{s,b}} := \|\langle \xi \rangle^s (\tau - \xi^b)^b f\|_{L^2_\tau L^2_\xi}.
\end{equation}
where \( \langle \cdot \rangle := (1 + |\cdot|^2)^{1/2} \). Denote by
\[
\hat{u}(\tau, \xi) = \int_{\mathbb{R}^2} e^{-i(\tau x + \xi \xi_2)} u(t, x) dt dx
\]
the space-time Fourier transform \( \hat{u} \) for \((\tau, \xi) \in \mathbb{R}^2 \). For \((\tau, \xi) \in \mathbb{R}^2 \), we define
\[
f * g(\tau, \xi) = \int_{\mathbb{R}^2} f(\tau_1, \xi_1) g(\tau - \tau_1, \xi - \xi_1) d\tau_1 d\xi_1.
\]
For any \( j \geq 0 \) and \( d \geq 0 \), we define
\[
A_j := \{(\tau, \xi) \in \mathbb{R}^2 : 2^j \leq \langle \xi \rangle < 2^{j+1}\},
\]
\[
B_d := \{(\tau, \xi) \in \mathbb{R}^2 : 2^d \leq \langle \tau - \xi^4 \rangle < 2^{d+1}\}.
\]
Thus the sets \( A_j \cap B_d \) for \( j, d \geq 0 \) partition the frequency space. By the definition of the space \( \dot{X}^{s, b} \), we have
\[
\|f\|_{\dot{X}^{s, b}} \sim \left( \sum_{j \geq 0} 2^{2sj} \sum_{d \geq 0} 2^{2bd} \|f\|_{L^2(\cap A_j \cap B_d)}^2 \right)^{1/2}. \tag{1.6}
\]
For convenience, we define the following notations
\[
A_{< j} := \bigcup_{j' < j} A_{j'}, \quad B_{[d_1, d_2]} := \bigcup_{d \in [d_1, d_2]} B_d
\]
and similarly define \( A_{> j}, B_{< d}, B_{> d} \), etc. Also, the first (or the second) subscript of a function means the restriction to \( \{A_j\} \) (or \( \{B_d\} \)) for example
\[
f_j := f \cdot 1_{A_j}, \quad f_{\geq j, [d_1, d_2]} := f \cdot 1_{A_{\geq j} \cap B_{[d_1, d_2]}}
\]
and other cases can be similarly defined.

For \( s, b \in \mathbb{R} \), we define \( \dot{X}^{-2, \frac{1}{2}, 1} \) as follows:
\[
\|f\|_{\dot{X}^{-2, \frac{1}{2}, 1}} := \left( \sum_{j \geq 0} 2^{-4j} \left( \sum_{d \geq 0} 2^{d/2} \|f\|_{L^2(\cap A_j \cap B_d)} \right)^2 \right)^{1/2}. \tag{1.7}
\]
Throughout this paper, \( C \) is a positive constant which may vary from line to line. We use the notation \( A \lesssim B \) if there exists a constant \( C > 0 \) independent of any variable appearing in the estimate, such that \( A \leq CB \), while \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \). In addition, \( A \gtrsim B \) equals to \( B \lesssim A \). Let \( a \lor b = \max\{a, b\} \) and \( a \land b = \min\{a, b\} \).

For a Banach space \( (H, \|\cdot\|_H) \), we use \( B_H(r) := \{f \in H : \|\cdot\|_H \leq r\} \) to denote the usual open ball of radius \( r \) around the origin.

The Cauchy problem (1.1)-(1.2) can be rewritten as follows:
\[
u(t) = S(t) \phi + \int_0^t S(t - t') u(t') dt', \tag{1.8}
\]
where \( S(t) := \exp(it\partial_\xi^2) \).

By identity (1.5), we can replace (1.8) on the time interval \( 0 \leq t \leq 1 \) by the following equation
\[
u(t) = \eta_1(t) S(t) \phi + \eta_1(t) S(t) \int_\mathbb{R} a(t') S(-t')(\eta_1 u^2)(t') dt'
+ \int_\mathbb{R} a(t - t') S(t - t')(\eta_1 u^2)(t') dt' \tag{1.9}
\]
For simplicity, we rewrite (1.9) as follows:
\[ u(t) = L(\varphi) + N_2(u, u), \]  
where \( L \) is the linear operator
\[ L(\varphi)(t) := \eta_1(t)S(t)\varphi \]  
and \( N_2 \) is the bilinear operator
\[ N_2(u, v)(t) := \eta_1(t)S(t) \int_{\mathbb{R}} a(t')S(-t')(\eta_1 uv)(t')dt' \]
\[ + \int_{\mathbb{R}} a(t - t')S(t - t')(\eta_1 uv)(t')dt'. \]

Now we state our main results as follows.

**Theorem 1.1.** Let \( s \geq -2 \). Then, for given \( \varphi \in H^s(\mathbb{R}) \), there exist a time \( T = T(\|\varphi\|_{H^{-2}(\mathbb{R})}) > 0 \) and a unique solution \( u \in C([0, T]; H^s(\mathbb{R})) \) to (1.1)-(1.2).

Theorem 1.1 is sharp in the following sense.

**Theorem 1.2.** Let \( B_r := B_{H^{-2}(\mathbb{R})}(r) \) for any \( r > 0 \). Let \( T \) be as in Theorem 1.1. Then the solution map \( \varphi \mapsto u[\varphi] \) is discontinuous on \( B_r \) with the \( H^s(\mathbb{R}) \)-topology to \( C([0, T]; H^s(\mathbb{R})) \) for any \( s < -2 \) and \( s' \in \mathbb{R} \).

It is easy to know that (1.1) is invariant under the scaling transformation
\[ u(t, x) \mapsto u_\gamma(t, x) := \gamma^4 u(\gamma^4 t, \gamma x) \quad \text{for any } \gamma > 0. \]
If we define
\[ \varphi^{(\lambda)}(x) := \lambda^{-4} \varphi \left( \frac{x}{\lambda} \right), \]
a direct computation yields
\[ \|\varphi^{(\lambda)}\|_{H^{-2}(\mathbb{R})} \lesssim \lambda^{-3/2}\|\varphi\|_{H^{-2}(\mathbb{R})} \]
for \( \lambda > 1 \). When \( \lambda \) is sufficiently large, \( \|\varphi^{(\lambda)}\|_{H^{-2}(\mathbb{R})} \) becomes small. Thus, it suffices to prove Theorem 1.1 when \( T = 1 \) and \( \|\varphi\|_{H^{-2}(\mathbb{R})} \) is sufficiently small.

The rest of the paper is arranged as follows. In Section 2, we construct a new weighted space. In Section 3, we establish some important bilinear estimates and complete the proof of Proposition 2.4. In Section 4, we give the proof of Theorem 1.1. In Section 5, we prove Theorem 1.2.

2. **Construction of the work space \( W \).** We want to study the optical regularity index of the solution to the fourth order nonlinear Schrödinger equation, as the classical Bourgain space \( X^{s,b} \) is not so perfect to our situation, then we will construct a weaker space. In this section, motivated by [1], we modify the working space based on the special property of the nonlinearity \( u^2 \). We will construct a new space \( W \) to prove the main theorems.

We need to introduce the spaces \( Y \) and \( Z \). \( Y \) is defined by the following norm
\[ \|f\|_Y := \|\xi\|^{-2}\|f\|_{L^1_tL^1_x} + \|f\|_{L^2_tL^2_x}, \]
and \( Z \) is the sum space of \( \hat{X}^{-2,\frac{1}{2},1} \) and \( Y \), whose norm is defined as
\[ \|f\|_Z := \inf \{ \|f_1\|_{\hat{X}^{-2,\frac{1}{2},1}} + \|f_2\|_Y : f_1 \in \hat{X}^{-2,\frac{1}{2},1}, f_2 \in Y, f = f_1 + f_2 \}. \]
It is easy to deduce that \( Z \) has an equivalent norm: \( \|f\|_Z \sim (\sum_{j \geq 0} \|1_j f\|_{L^2}^2)^{1/2} \),
\[ \|f\|_Z \leq \|f\|_{\hat{X}^{-2,\frac{1}{2},1}} \quad \text{and} \quad \|f\|_Z \leq \|f\|_Y. \]
On the other hand, for a Banach space $Z$ and a linear operator $T$, if there is a $M > 0$ such that $\|Tf\|_Z \leq M\|f\|_{\tilde{X}^{-2, \frac{1}{4}, 1}}$ and $\|Tf\|_Z \leq M\|f\|_{Y}$, then it is easy to know that $\|Tf\|_Z \leq M\|f\|_Z$.

Now we introduce a weighted space $W$ defined by

$$\|f\|_W := \|\omega f\|_Z,$$

where

$$\omega(\tau, \xi) = \omega(\tau) := \min(-1, \tau)^{10}.$$  

It’s easy to check that $\omega \geq 1$ and $\omega(\tau, \xi) \leq (\tau - \xi^4)^{10}$.

Firstly, for the relationship between $\tilde{X}^{-2, \frac{1}{4}, 1}$ and $\tilde{X}^{-2, b}$, we have the following lemma.

**Lemma 2.1.** Let $	ext{supp } f \subset B_{2^d}$ for some $d \geq 0$. Then we have

$$\|f\|_{\tilde{X}^{-2, b}} \lesssim 2^{-(\frac{1}{2} - b)d}\|f\|_{\tilde{X}^{-2, \frac{1}{4}, 1}}, \text{ if } b < \frac{1}{2};$$

and

$$\|f\|_{\tilde{X}^{-2, \frac{1}{4}, 1}} \lesssim 2^{-(b - \frac{1}{2})d}\|f\|_{\tilde{X}^{-2, b}}, \text{ if } b > \frac{1}{2}.$$  

**Proof.** When $b < \frac{1}{2}$, from the definition of the spaces $\tilde{X}^{-2, b}$ and $\tilde{X}^{-2, \frac{1}{4}, 1}$ and $l^1 \hookrightarrow l^2$

$$\|f1_{A_j}\|_{\tilde{X}^{-2, b}} \sim 2^{-2j} \left( \sum_{d' \geq d} 2^{bd'}\|f\|_{L^2(A_j \cap B_{d'})}^2 \right)^{\frac{1}{2}}$$

$$\lesssim 2^{-2j} \sum_{d' \geq d} 2^{bd'}\|f\|_{L^2(A_j \cap B_{d'})}$$

$$= 2^{-2j} \sum_{d' \geq d} 2^{-(\frac{1}{2} - b)d'} 2^{d'/2}\|f\|_{L^2(A_j \cap B_{d'})}$$

$$\leq 2^{-(\frac{1}{2} - b)d - 2j} \sum_{d' \geq d} 2^{d'/2}\|f\|_{L^2(A_j \cap B_{d'})}$$

$$= 2^{-(\frac{1}{2} - b)d}\|f1_{A_j}\|_{\tilde{X}^{-2, \frac{1}{4}, 1}},$$

then square-summing in $j$ yields the claim.

When $b > \frac{1}{2}$, by using Hölder’s inequality, we have

$$\|f1_{A_j}\|_{\tilde{X}^{-2, \frac{1}{4}, 1}} \sim 2^{-2j} \sum_{d' \geq d} 2^{d'/2}\|f\|_{L^2(A_j \cap B_{d'})}$$

$$= 2^{-2j} \sum_{d' \geq d} 2^{-(b - \frac{1}{2})d'} 2^{bd'}\|f\|_{L^2(A_j \cap B_{d'})}$$

$$\leq 2^{-(b - \frac{1}{2})d}\|f1_{A_j}\|_{\tilde{X}^{-2, b}},$$

then square-summing in $j$ yields the claim. \hfill \Box

Furthermore, motivated by [1], we have the following basic estimates.

**Proposition 2.2.** For any reasonable function $f$, we have

$$\|(\xi)^{-2}f\|_{L^2 L^1} \lesssim \|f\|_Z.$$  

(2.15)
Furthermore, if \( \text{supp} f \subset A_j \cap B_{\geq d} \) for some \( j, d \geq 0 \), then we have
\[
\|f\|_{L^1} \lesssim 2^{5j/2} \|f\|_Z, \quad (2.16)
\]
\[
\|f\|_{L^2} \lesssim (1 + 2^{2j} 2^{-d/2}) \|f\|_Y, \quad (2.17)
\]
\[
\|f\|_{L^1} \lesssim 21/2 (1 + 2^{2j} 2^{-d/2}) \|f\|_Z, \quad (2.18)
\]
\[
\|f\|_{L^2} \lesssim 2^{2j} \|f\|_Z. \quad (2.19)
\]

Proof. \((2.15)-(2.18)\) can be proved similarly to \((32)-(35)\) of \([1]\). By using the definition of the space \( Z \) and \((2.15)\), we obtain that \((2.19)\) is valid. This completes the proof of the proposition.

By using Proposition 2.2, we easy get the following lemma.

**Lemma 2.3.** Let \( f \) be a reasonable function and \( j \geq 0 \). If \( \text{supp} f \subset \cup_j (A_j \cap B_{\geq4j-100}) \), then we have
\[
\|f\|_Y \lesssim \|f\|_Z.
\]
If \( \text{supp} f \subset \cup_j (A_j \cap B_{\leq4j+100}) \), then we have
\[
\|f\|_{\tilde{X}^{-2,\frac{1}{2}}} \lesssim \|f\|_Z.
\]

Proof. For the proof, we refer the readers to Lemma 2 of \([1]\). \( \square \)

Now, we give the following proposition which plays an important role in the proof of main theorems.

**Proposition 2.4.** For any reasonable functions \( f \) and \( g \) satisfy the following properties.

(i) If \( |f| \leq |g| \) pointwise, then we have \( \|f\|_W \leq \|g\|_W \).

(ii) We derive
\[
\|\langle \xi \rangle^{-2} f \|_{L^2} \lesssim \|f\|_W. \quad (2.20)
\]

(iii) We derive
\[
\|f\|_W \lesssim \|f\|_{\tilde{X}^{-2,100}}. \quad (2.21)
\]

(iv) We derive
\[
\|f\|_W \lesssim \|f\|_{\tilde{X}^{-2,100}} \lesssim \|f\|_{\tilde{X}^{-2,100}} \lesssim \|f\|_{\tilde{X}^{-2,100}} \lesssim \|f\|_{\tilde{X}^{-2,100}}. \quad (2.22)
\]

Proof. Obviously, (i) is true. We now prove \((2.20)\). Combining \((2.15)\) with \( \omega \geq 1 \), we have
\[
\|\langle \xi \rangle^{-2} f \|_{L^2} \lesssim \|f\|_Z \leq \|\omega f\|_Z = \|f\|_W,
\]
thus \((2.20)\) is proved.

Now we prove \((2.21)\). By using Lemma 2.1, we obtain
\[
\|f\|_W = \|\omega f\|_Z \leq \|\omega f\|_{\tilde{X}^{-2,\frac{1}{2}}} \lesssim \|\omega f\|_{\tilde{X}^{-2,100}} \lesssim \|f\|_{\tilde{X}^{-2,100}},
\]
where the last inequality, we have used the fact that the estimate \( \omega(\tau, \xi) \lesssim (\tau - \xi^4)^{10} \).

Thus \((2.21)\) is proved.

Finally, to prove \((2.22)\), by \((2.14)\) and (i), it suffices to show
\[
\left\| \frac{\omega}{(\tau - \xi^4)} \left( \frac{f}{\omega} * \frac{g}{\omega} \right) \right\|_Z \lesssim \|f\|_Z \|g\|_Z, \quad (2.23)
\]
for any reasonable functions \( f \) and \( g \).
From now on, we always use variables \((\tau_1, \xi_1)\) for \(f\), \((\tau_2, \xi_2)\) for \(g\) and \((\tau, \xi)\) for \(f * g\) under the convention
\[
(\tau, \xi) = (\tau_1, \xi_1) + (\tau_2, \xi_2).
\]

Next, we split \(f = \sum j_1 \geq 0 f_{j_1}\) and \(g = \sum j_2 \geq 0 g_{j_2}\), thus
\[
\left\| \frac{\omega}{\langle \tau - \xi^4 \rangle} \left( \frac{f * g}{\omega} \right) \right\|_Z = \left\| \sum A_j \frac{\omega}{\langle \tau - \xi^4 \rangle} \left( \frac{f_{j_1}}{\omega} * \frac{g_{j_2}}{\omega} \right) \right\|_Z,
\]

where \((\tau, \xi) \in A_j, (\tau_1, \xi_1) \in A_{j_1}, (\tau_2, \xi_2) \in A_{j_2}\) obey (2.24).

We only need consider the following two cases:

- **High-low interaction:** \(|\xi| \sim |\xi_1| \gtrsim |\xi_2|\).
- **High-high interaction:** \(|\xi| \ll |\xi_1| \sim |\xi_2|\).

The rest of possible cases can be treated similarly.

For (2.25), in order for the inner summand to be non-zero, we have the following two cases:

- \(|j_1 - j| \leq 10\) (which implies \(j_2 \leq j_1 + 11\)) \text{ (High-low interaction)},
- \(|j_1 - j_2| \leq 1\) (which implies \(j < j_1 - 10, j_2 - 10\)) \text{ (High-high interaction)}.

Thus, to prove (2.23), it suffices to show
\[
\left\| \sum_{|j_1 - j| \leq 10, j_2 \leq j_1 + 11} A_j \frac{\omega}{\langle \tau - \xi^4 \rangle} \left( \frac{f_{j_1}}{\omega} * \frac{g_{j_2}}{\omega} \right) \right\|_Z \lesssim \|f\|_Z \|g\|_Z,
\]
\[
\left\| \sum_{|j_1 - j| \leq 1, j_2 < j_1 - 10} A_j \frac{\omega}{\langle \tau - \xi^4 \rangle} \left( \frac{f_{j_1}}{\omega} * \frac{g_{j_2}}{\omega} \right) \right\|_Z \lesssim \|f\|_Z \|g\|_Z.
\]

Noting that under (2.24) that the weight \(\omega\) satisfies
\[
\omega(\tau, \xi) \lesssim \omega(\tau_1, \xi_1) \omega(\tau_2, \xi_2),
\]
so we have the estimate
\[
\left\| \frac{\omega}{\langle \tau - \xi^4 \rangle} \left( \frac{f * g}{\omega} \right) \right\|_Z \lesssim \langle \tau - \xi^4 \rangle^{-1} \|f * g\|_Z.
\]

Combining Schur’s lemma with the above inequality, to obtain the bilinear estimate (2.23), it suffices to show that the following lemmas are valid.

**Lemma 2.5.** (High-low interaction estimate) If non-negative integers \(j, j_1, j_2\) satisfy \(|j_1 - j| \leq 10\) and \(j_2 \leq j_1 + 11\), then
\[
\|A_j \langle \tau - \xi^4 \rangle^{-1} f_{j_1} * g_{j_2}\|_Z \lesssim (2^{-j_2/6} + 2^{-(j_2 + j_1)/6}) \|f_{j_1}\|_Z \|g_{j_2}\|_Z
\]
holds.

**Lemma 2.6.** (High-high interaction estimate) If non-negative integers \(j_1, j_2\) satisfy \(|j_1 - j_2| \leq 1\), then
\[
\left\| A_{<j_1 - 10} \frac{\omega}{\langle \tau - \xi^4 \rangle} \left( \frac{f_{j_1}}{\omega} * \frac{g_{j_2}}{\omega} \right) \right\|_Z \lesssim \|f_{j_1}\|_Z \|g_{j_2}\|_Z
\]
holds.
In fact, if we assume that Lemma 2.5 is valid, then for each \( j \), by (2.26) and the Cauchy-Schwarz inequality, we have
\[
\left\| 1_{A_j} \frac{\omega}{(\tau - \xi^4)} \left( \frac{f_{j-10,j+10}}{\omega} \ast g \right) \right\|_Z \\
\leq \sum_{j=10}^{j+10} \sum_{j_2 \leq j} \left\| 1_{A_j} (\tau - \xi^4)^{-1} f_{j1} \ast g_{j_2} \right\|_Z \\
\lesssim \sum_{j_1=10}^{j+10} \sum_{j_2 \leq j} (2^{-j_2/6} + 2^{-(j-j_2)/6}) \|f_{j_1}\|_Z \|g_{j_2}\|_Z \\
\lesssim \sum_{j_1=10}^{j+10} \|f_{j_1}\|_Z \left( \sum_{j_2 \geq 0} 2^{-j_2/3} + \sum_{j_2:j_2-j_2 \geq -11} 2^{-(j-j_2)/3} \right)^{1/2} \left( \sum_{j_2 \geq 0} \|g_{j_2}\|_Z^2 \right)^{1/2} \\
\lesssim \sum_{j_1=10}^{j+10} \|f_{j_1}\|_Z \|g\|_Z.
\]
Then square-summing over \( j \) and by the triangle inequality, we obtain (2.23) in the High-low interaction case.

Similarly, if we assume Lemma 2.6 is valid, then for each \( j_1 \), we have
\[
\left\| 1_{A_{j_1-10}} \frac{\omega}{(\tau - \xi^4)} \left( \frac{f_{j_1}}{\omega} \ast \frac{g_{j_1-1,j_1+1}}{\omega} \right) \right\|_Z \\
\leq \sum_{j_1=11}^{j_1+1} \left\| 1_{A_{j_1-10}} \frac{\omega}{(\tau - \xi^4)} \left( \frac{f_{j_1}}{\omega} \ast \frac{g_{j_1-10}}{\omega} \right) \right\|_Z \\
\lesssim \sum_{j_1=11}^{j_1+1} \|f_{j_1}\|_Z \|g_{j_1-10}\|_Z.
\]
Then square-summing over \( j_1 \) and by the triangle inequality and the Cauchy-Schwarz inequality, we obtain (2.23) in the High-high interaction case.

From the discussion above, in order to complete the proof of (2.22), it suffices to prove Lemmas 2.5 and 2.6. Thus, we will give the proofs of Lemmas 2.5 and 2.6 in the next section.

3. The proofs of Lemmas 2.5 and 2.6. In this section, motivated by [1, 8], we first show two bilinear estimates.

**Proposition 3.1.** Suppose that \( \text{supp} f \subset A_{j_1} \), \( \text{supp} g \subset A_{j_2} \) for some \( j_1, j_2 \geq 0 \). Then
\[
\|f \ast g\|_{L^2L^2} \lesssim 2^{(j_1+j_2)} \|f\|_{X^{-2, \frac{1}{2}, 1}} \|g\|_{X^{-2, \frac{1}{2}, 1}}.
\]

**Proof.** Without loss of generality, we may assume \( f \) and \( g \) to be non-negative. Let \( \text{supp} f_{d_1} \subset B_{d_1} \) and \( \text{supp} g_{d_2} \subset B_{d_2} \). By the definition of the space \( X^{-2, \frac{1}{2}, 1} \), we have
\[
\|f\|_{X^{-2, \frac{1}{2}, 1}} = 2^{-2j_1} \sum_{d_1 \geq 0} 2^{d_1/2} \|f_{d_1}\|_{L^2L^2},
\]
\[
\|g\|_{X^{-2, \frac{1}{2}, 1}} = 2^{-2j_2} \sum_{d_2 \geq 0} 2^{d_2/2} \|g_{d_2}\|_{L^2L^2},
\]
using the triangle inequality, it suffices to prove that, for any \( d_1, d_2 \geq 0 \),
\[
\|f_{d_1} \ast g_{d_2}\|_{L^2L^2} \lesssim 2^{(d_1+d_2)/2} \|f_{d_1}\|_{L^2L^2} \|g_{d_2}\|_{L^2L^2}.
\]
We may assume $d_1 \geq d_2$ by symmetry. Applying Schwarz’s inequality, we have
\[
\|f_1 \ast g_2\|_{L^2 L^2} = \left\| \int_{B_{d_1} \cap ((\tau, \xi) - B_{d_2})} f_1(\tau_1, \xi_1)g_2(\tau - \tau_1, \xi - \xi_1) \, d\tau_1 d\xi_1 \right\|_{L^2 L^2} \\
\leq \left\| m_1(\tau, \xi)^{1/2} \left( \int_{\mathbb{R}^2} f_1(\tau_1, \xi_1)g_2(\tau - \tau_1, \xi - \xi_1) \, d\tau_1 d\xi_1 \right)^{1/2} \right\|_{L^2 L^2} \\
\leq \left[ \sup_{\tau, \xi} m_1(\tau, \xi) \right]^{1/2} \|f_1\|_{L^2 L^2} \|g_2\|_{L^2 L^2},
\]
where $m_1(\tau, \xi)$ is the measure of the set
\[ \{(\tau_1, \xi_1) \in B_{d_1} \cap ((\tau, \xi) - B_{d_2})\} \].
Thus, it remains to be proved
\[
m_1(\tau, \xi) \leq 2^{d_1+d_2}. \quad (3.29)
\]
Note that if $(\tau_1, \xi_1) \in B_{d_1} \cap ((\tau, \xi) - B_{d_2})$, then $\tau_1 = \xi_1^3 + O(2^{d_1})$, $\tau_2 = \xi_2^4 + O(2^{d_2})$, and thus $\tau = \xi_1^3 + \xi_2^4 + O(2^{d_1})$. From the following identity
\[ ((\xi_1 - \xi_2)^2 + 3\xi^2)^2 = 8(\xi_1^3 + \xi_2^4 + \xi^4), \]
we deduce that
\[
((\xi_1 - \xi_2)^2 + 3\xi^2)^2 = 8(\tau + \xi^4) + O(2^{d_1}).
\]
Then elementary algebra shows that $((\xi_1 - \xi_2)^2 + 3\xi^2$ belongs to at most two intervals of size $O(2^{d_1/2})$. This in turn implies that $\xi_1 - \xi_2$ belongs to at most four intervals of size $O(2^{d_1/4})$, which means that the variation of $\xi_1$ is estimated by $2^{d_1/4}$ for fixed $(\tau, \xi)$.

If we also fix $\xi_1$, then the estimates
\[ |\tau_1 - \xi_1^3| \leq 2^{d_1}, \quad |\tau_1 - (\xi - \xi_1)^4| \leq 2^{d_2} \]
imply that the variation of $\tau_1$ is bounded by $2^{d_2}$. Thus we obtain (3.29).
This completes the proof of Proposition 3.1.

Proposition 3.2. Suppose that $\text{supp} f \subset A_{j_1}$ for some $j_1 \geq 0$ and $\text{supp} g \subset A_{j_2}$ for some $j_2 > 0$. Then, for any $d > 0$,
\[
\|f \ast g\|_{L^2 L^2(B_d)} \lesssim 2^{d/2} 2^{j_1} 2^{-j_2/2} (2^{2j_1} \lor 2^{2j_2})^{-1/2} \|f\|_{X_{-d, \frac{d}{2}}} \|g\|_{L^2 L^2}.
\]

Proof. We may assume $f$ and $g$ to be non-negative without loss of generality. For any non-negative test function $h \in L^2 L^2$ restricted to $B_d$, by Fubini’s theorem and Schwarz’s inequality, we have
\[
\left| \int_{\mathbb{R}^2} (f \ast g)(\tau, \xi)h(\tau, \xi) \, d\tau d\xi \right| = \left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} f(\tau_1, \xi_1)g(\tau_2, \xi_2) \, d\tau_1 d\xi_1 \right) h(\tau, \xi) \, d\tau d\xi \right| \\
= \left| \int_{\mathbb{R}^2} (f_\ast h)(\tau_2, \xi_2)g(\tau_2, \xi_2) \, d\tau_2 d\xi_2 \right| \\
\leq \|f_\ast h\|_{L^2 L^2(A_{j_2})} \|g\|_{L^2 L^2},
\]
where $f_\ast(\tau, \xi) := f(-\tau, -\xi)$.

We decompose $f = \sum_{d_i} f_{d_i}$ where each $f_{d_i}$ is supported on $B_{d_i}$, thus $f = \sum_{d_1} (f_{d_1})_\ast$. By duality and the triangle inequality it clearly suffices to show that
\[
\|(f_{d_1})_\ast h\|_{L^2 L^2(A_{j_2})} \lesssim 2^{-j_2/2} (2^{2j_1} \lor 2^{2j_2})^{-1/2} 2^{(d_1+d_2)/2} \|f_{d_1}\|_{L^2 L^2} \|h\|_{L^2 L^2},
\]
for each \(d_1 \geq 0\).

Using Cauchy-Schwarz’s inequality and Fubini’s theorem, we get
\[
\| (f_{d_1} \ast h) \|_{L^2 \cdot L^2(A_{j_2})} \\
= \left\| \int_{\mathbb{R}^2} f_{d_1}(\tau_1, \xi_1) h(\tau_1 + \tau_2, \xi_1 + \xi_2) \, d\tau_1 d\xi_1 \right\|_{L^2 \cdot L^2(A_{j_2})} \\
\leq \left\| m_2(\tau_2, \xi_2)^{1/2} \left( \int_{\mathbb{R}^2} |f_{d_1}(\tau_1, \xi_1) h(\tau_2, \xi_2)|^2 \, d\tau_1 d\xi_1 \right)^{1/2} \right\|_{L^2 \cdot L^2(A_{j_2})} \\
\leq \left[ \sup_{(\tau_2, \xi_2) \in A_{j_2}} m_2(\tau_2, \xi_2) \right]^{1/2} \| f_{d_1} \|_{L^2} \| h \|_{L^2},
\]
where \(m_2(\tau_2, \xi_2)\) is the measure of the set
\[
\{(\tau_1, \xi_1) \in A_{j_1} \cap B_{d_1} : (\tau_1 + \tau_2, \xi_1 + \xi_2) \in B_d\}.
\]
Then, it suffices to show that
\[
m_2(\tau_2, \xi_2) \lesssim 2^{d_1 + d} 2^{-j_2} (2^{2j_1} \lor 2^{j_2})^{-1}
\]
for \((\tau_2, \xi_2) \in A_{j_2}.

Now assume \((\tau_1, \xi_1) \in A_{j_1} \cap B_{d_1}, (\tau_2, \xi_2) \in A_{j_2}\) and \((\tau_1 + \tau_2, \xi_1 + \xi_2) \in B_d\). It is easy to know that
\[
\tau - \xi^4 - (\tau_1 - \xi_1^4) - (\tau_2 - \xi_2^4) = -2\xi_1\xi_2(2\xi_1^2 + 3\xi_1 \xi_2 + 2\xi_2^2).
\]
On the one hand, set \(M = \tau_2 - \xi_2^4\), we have
\[
|2\xi_1\xi_2(2\xi_1^2 + 3\xi_1 \xi_2 + 2\xi_2^2)| \lesssim 2^{d_1} + 2^d + |M|,
\]
by simple calculations imply
\[
|\xi_i| \lesssim 2^{-j_2} (2^{2j_1} \lor 2^{j_2})^{-1} (2^{d_1} \lor 2^d),
\]
On the other hand, if we also fix \(\xi_1\), then the estimates
\[
|\tau_1 - \xi_1^4| \lesssim 2^{d_1}, \quad |\tau_1 + \tau_2 - (\xi_1 + \xi_2)^4| \lesssim 2^d
\]
imply that the variation of \(\tau_1\) is bounded by \(2^{d_1} \lor 2^d\). Then we obtain
\[
m_2(\tau_2, \xi_2) \lesssim 2^{-j_2} (2^{2j_1} \lor 2^{j_2})^{-1} (2^{d_1} \lor 2^d) (2^{d_1} \lor 2^d)
= 2^{-j_2} (2^{2j_1} \lor 2^{j_2})^{-1} 2^{d_1 + d}.
\]
This completes the proof of the proposition. \(\square\)

We also have the following estimate with respect to time variable.

**Lemma 3.3.** For any \(\psi_1, \psi_2 \in L^2(\mathbb{R})\), we have
\[
\left\| \frac{\omega}{\langle \tau \rangle} \left( \frac{\psi_1}{\omega} + \frac{\psi_2}{\omega} \right) \right\|_{L^1(\mathbb{R})} \lesssim \|\psi_1\|_{L^2(\mathbb{R})} \|\psi_2\|_{L^2(\mathbb{R})},
\]
where \(\omega(\tau) = \left[ \min(-1, \tau) \right]^{10}\).

**Proof.** See Lemma Ex in [8]. \(\square\)

Now, we give the proof of Lemma 2.5.
Proof. Note that, for this case, we have a priori assumptions \(|j_1 - j| \leq 10\) and \(j_2 \leq j_1 + 11\).

We firstly show that the simple case \(j_2 = 0\). By Lemma 2.1, Young’s inequality and Proposition 2.2, we have

\[
\| A_j (\tau - \xi^4)^{-1} f_{j_1} \ast g_{j_2} \|_{L^2 L^1} \lesssim \| A_j (\tau - \xi^4)^{-1} f_{j_1} \ast g_0 \|_{L^2 L^1} \lesssim \| f_{j_1} \|_{L^2 L^2} \| g_0 \|_{L^1 L^1} \lesssim \| f_{j_1} \|_Z \| g_0 \|_Z,
\]

then (2.27) holds when \(j_2 = 0\).

From now on, we may assume \(j_2 > 0\). Then the assumption \(|j_1 - j| \leq 10\) and the following relation

\[
|\tau - \xi^4 - (\tau_1 - \xi_1^4) - (\tau_2 - \xi_2^4)| = 2|\xi_1\xi_2|(2\xi_1^2 + 3\xi_1\xi_2 + 2\xi_2^2) \geq 14\xi_1^2\xi_2^2,
\]

imply the resonance estimate

\[
\max(|\tau - \xi_1^4, |\tau_1 - \xi_1^4, |\tau_2 - \xi_2^4|) \geq 2^{2(j+j_2-20)}.
\]

Therefore, we split the L.H.S. of (2.27) into four parts,

\[
I = \| A_j (\tau - \xi^4)^{-1} f_{j_1} \ast g_{j_2} \|_Z
\]

Then from Proposition 3.1 and Lemma 2.1, we have

\[
I \lesssim 2^{-2(j+j_2)} \| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^1} + \| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^2} \| g_{j_2} \|_{L^1 L^1}.
\]

Applying Young’s inequality and Proposition 2.2, we have

\[
\| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^1} \leq \| f_{j_1, \geq 2(j+j_2-20)} \|_{L^2 L^1} \| g_{j_2} \|_{L^1 L^1} \lesssim 2^{2j_1} 2^{3j_2/2} \| f_{j_1} \|_Z \| g_{j_2} \|_Z,
\]

and

\[
\| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^2} \leq \| f_{j_1, \geq 2(j+j_2-20)} \|_{L^2 L^2} \| g_{j_2} \|_{L^1 L^1} \lesssim 2^{2j_1} 2^{3j_2/2} \| f_{j_1} \|_Z \| g_{j_2} \|_Z.
\]

Putting these estimates together, we get

\[
I \lesssim 2^{-3j_2/2} \| f_{j_1} \|_Z \| g_{j_2} \|_Z.
\]

Next we consider \(\mathcal{I}\). By using \(2.13\), we have

\[
\mathcal{I} \lesssim 2^{-2(j+j_2)} \| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^1} + \| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^2} \| g_{j_2} \|_{L^1 L^1}.
\]

On the one hand, if we measure \(g_{j_2}\) in \(\hat{X}_{-2,1}^{s,b}\), then from Proposition 3.1 and Lemma 2.3, we have

\[
\| f_{j_1, \geq 2(j+j_2-20)} \ast g_{j_2} \|_{L^2 L^2} \lesssim 2^{2j_1+j_2} \| f_{j_1, \geq 2(j+j_2-20)} \|_{\hat{X}_{-2,1}^{s,b}} \| g_{j_2} \|_{\hat{X}_{-2,1}^{s,b}} \lesssim 2^{2j_1+j_2} \| f_{j_1} \|_Z \| g_{j_2} \|_{\hat{X}_{-2,1}^{s,b}}.
\]
On the other hand, if we measure $g_{j_2}$ in $Y$, then from Young’s inequality, Proposition 2.2 we have
\[
\|f_{j_1, < 2(j + j_2 - 20)} * g_{j_2}\|_{L^2L^2} \leq \|f_{j_1}\|_{L^2L^1} \|g_{j_2}\|_{L^1L^2} \lesssim 2^{2j_1} 2^{j_2/2} \|f_{j_1}\|_Z \|g_{j_2}\|_Y.
\]
Therefore, we have
\[
\|f_{j_1, < 2(j + j_2 - 20)} * g_{j_2}\|_{L^2L^2} \lesssim 2^{2(j_1 + j_2)} \|f_{j_1}\|_Z \|g_{j_2}\|_Z.
\]
Inserting this last inequality into (3.30) yields
\[
\|II\| \lesssim 2^{-(j-j_2)} \|f_{j_1}\|_Z \|g_{j_2}\|_Z.
\]
Next we consider III. On the one hand, according to Propositions 3.2 and 2.2, we get
\[
III \sim 2^{-2j} \sum_{d < 2(j + j_2 - 20)} 2^{-d/2} \|f_{j_1, \geq 2(j + j_2 - 20)} * g_{j_2}\|_{L^2L^2(A_j \cap B_d)} \lesssim 2^{-2j} \sum_{d < 2(j + j_2 - 20)} 2^{2j_2} 2^{-j_2/2} (2^{2j_1} + 2^{2j_2})^{-1/2} \|f_{j_1, \geq 2(j + j_2 - 20)}\|_{L^2L^2} \|g_{j_2}\|_{X^{-j_2/2} L^1} \lesssim (j + j_2) 2^{-2j} 2^{2j_2} 2^{-j_2/2} \|f_{j_1}\|_Z \|g_{j_2}\|_{X^{-j_2/2} L^1} \lesssim 2^{-(j-j_2)} \|f_{j_1}\|_Z \|g_{j_2}\|_Z.
\]
On the other hand, let $b \in \left(\frac{1}{2}, \frac{3}{4}\right)$, applying Lemma 2.1, Hölder’s inequality and Young’s inequality we deduce that
\[
III \lesssim \|1_{A_j} (\tau - \xi^4)^{-1} f_{j_1, \geq 2(j + j_2 - 20)} * g_{j_2}\|_{X^{-j_2/2} L^1} - 2^{-2j} \|1_{A_j} (\tau - \xi^4)^{b-1} f_{j_1, \geq 2(j + j_2 - 20)} - 2^{-2j} \|1_{A_j} (\tau - \xi^4)^{b-1} f_{j_1, \geq 2(j + j_2 - 20)} * g_{j_2}\|_{L^2L^2} \lesssim 2^{-2j} \|f_{j_1, \geq 2(j + j_2 - 20)}\|_{L^2L^2} \|g_{j_2}\|_{X^{-j_2/2} L^1} \lesssim 2^{-(j-j_2)} \|f_{j_1}\|_Z \|g_{j_2}\|_Y.
\]
Combining these two estimates, we have
\[
III \lesssim 2^{-(j-j_2)} \|f_{j_1}\|_Z \|g_{j_2}\|_Z.
\]
Finally, we estimate IV. By using Proposition 3.2, Lemma 2.3 and Proposition 2.2, we obtain
\[
IV \sim 2^{-2j} \sum_{d < 2(j + j_2 - 20)} 2^{-d/2} \|f_{j_1, < 2(j + j_2 - 20)} * g_{j_2, \geq 2(j + j_2 - 20)}\|_{L^2L^2(A_j \cap B_d)} \lesssim (j + j_2) 2^{-2j} 2^{j_2/2} 2^{-j_2/2} \|f_{j_1}\|_{X^{-j_2/2} L^1} \|g_{j_2, \geq 2(j + j_2 - 20)}\|_{L^2L^2} \lesssim 2^{-j_2/2} \|f_{j_1}\|_Z \|g_{j_2}\|_Z.
\]
From the above estimates, we obtain
\[
I + II + III + IV \lesssim (2^{-j_2/2} + 2^{-(j-j_2)} \|f_{j_1}\|_Z \|g_{j_2}\|_Z),
\]
we get the desired inequality. This completes the proof of Lemma 2.5. \qed

Next, we give the proof of Lemma 2.6.
Proof. Note that, in this case, we have a priori assumption $|j_1 - j_2| \leq 1$. Meanwhile, we may also assume $\tilde{j}_1 \geq 10$.

Now, from Hölder’s inequality, Young’s inequality and Proposition 2.2 we deduce that

$$
\|\langle \xi \rangle^{-2} f_{j_1} * g_{j_2} \|_{L^2 L^1} \lesssim \|f_{j_1} * g_{j_2}\|_{L^\infty L^1} \lesssim \|f_{j_1}\|_{L^2 L^1} \|g_{j_2}\|_{L^2 L^1}
\lesssim 2^{j_1} 2^{j_2} \|f_{j_1}\|_Y \|g_{j_2}\|_Z.
$$

(3.31)

According to Proposition 3.1, we have

$$
\|f_{j_1} * g_{j_2}\|_{L^2 L^2} \lesssim 2^{(j_1 + j_2)} \|f_{j_1}\|_{X^{-2, \frac{4}{1}}} \|g_{j_2}\|_{X^{-2, \frac{4}{1}}}.
$$

On the other hand, by using Young’s inequality and Proposition 2.2, we have

$$
\|f_{j_1} * g_{j_2}\|_{L^2 L^2} \leq \|f_{j_1}\|_{L^2 L^2} \|g_{j_2}\|_{L^1 L^1} \lesssim 2^{j_3/2} \|f_{j_1}\|_Y \|g_{j_2}\|_Z
\leq 2^{j_3/2} \|f_{j_1}\|_Y \|g_{j_2}\|_{X^{-2, \frac{4}{1}}},
$$

and similarly, we obtain

$$
\|f_{j_1} * g_{j_2}\|_{L^2 L^2} \lesssim 2^{j_1/2} \|f_{j_1}\|_{X^{-2, \frac{4}{1}}} \|g_{j_2}\|_Y.
$$

Putting all these estimates together, we obtain

$$
\|f_{j_1} * g_{j_2}\|_{L^2 L^2} \lesssim 2^{j_1} \|f_{j_1}\|_Y \|g_{j_2}\|_Z.
$$

(3.32)

Inserting (3.31) and the above inequality into (2.13), we get

$$
\|f_{j_1} * g_{j_2}\|_Y \lesssim 2^{j_1} \|f_{j_1}\|_Y \|g_{j_2}\|_Z.
$$

We now return to (2.28). Let us first restrict $A_{< j_1 - 10}$ to the region $A_{< j_1 - 10} \cap B_{\geq 4j_1 - 10}$. In this case, from (2.26) and (3.32), we deduce

$$
\left\| 1_{A_{< j_1 - 10} \cap B_{\geq 4j_1 - 10}} \frac{\omega}{\langle \tau - \xi \rangle^{1}} \left( \frac{f_{j_1}}{\omega} * \frac{g_{j_2}}{\omega} \right) \right\|_Y \lesssim 2^{-4j_1} \|f_{j_1} * g_{j_2}\|_Y \lesssim \|f_{j_1}\|_Z \|g_{j_2}\|_Z;
$$

thus, (2.28) holds in this case.

Next, we consider the domain $A_{< j_1 - 10} \cap B_{< 4j_1 - 10}$. We discuss in two different cases:

- $\text{supp} f_{j_1} \subset A_{j_1} \cap B_{< 4j_1 - 100}$ or $\text{supp} g_{j_2} \subset A_{j_2} \cap B_{< 4j_2 - 100}$;
- $\text{supp} f_{j_1} \subset A_{j_1} \cap B_{\geq 4j_1 - 100}$ and $\text{supp} g_{j_2} \subset A_{j_2} \cap B_{\geq 4j_2 - 100}$.

For the first case, by the inequality $|j_1 - j_2| \leq 1$ it suffices to assume that $\text{supp} f_{j_1} \subset A_{j_1} \cap B_{< 4j_1 - 100}$, the other case can be proved in the similar way.

A direct calculation leads to $\tau_1 \geq 2^{4j_1}/10$ and $|\tau| \leq 2^{4j_1}/200$, then by using identity (2.24) we have $\tau_2 \leq -2^{4j_1}/20$. Thus we have $\omega(\tau_2, \xi_2) \gtrsim 2^{40j_1}$.

On the other hand, it is easy to know that for $(\tau, \xi) \in A_{< j_1 - 10} \cap B_{< 4j_1 - 10}$

$$
\frac{\omega(\tau, \xi)}{\langle \tau - \xi \rangle^{1}} \lesssim \langle \tau - \xi \rangle^{9} \lesssim 2^{36j_1},
$$

where we use the estimate $\omega(\tau, \xi) \lesssim \langle \tau - \xi \rangle^{10}$. Therefore, we have

$$
\left\| 1_{A_{< j_1 - 10} \cap B_{< 4j_1 - 10}} \frac{\omega}{\langle \tau - \xi \rangle^{1}} \left( \frac{f_{j_1}}{\omega} * \frac{g_{j_2}}{\omega} \right) \right\|_Y \lesssim 2^{-4j_1} \|f_{j_1} * g_{j_2}\|_Y.
$$

Inserting the above inequality into (3.32) yields (2.28).
Proposition 2.2. Applying (2.26), we get the proof of Proposition 2.4.

For any \( \langle \tau - \xi^4 \rangle^{-1} \), we have (2.28). This completes the proof of Lemma 2.6.

From what has been discussed above, we complete the proof of (2.22) and thus the proof of Proposition 2.4.
4. The proof of Theorem 1.1. In this section, we will give the proof of the first main theorem. First, we take \( S^s \) and \( N_s \) to be the closure of the Schwartz functions under the norms

\[
\|u\|_{S^s} := \|\xi^{s+2}\hat{u}\|_W, \quad \|F\|_{N_s} := \|\xi^{s+2}(\tau - \xi^4)^{-1}\hat{F}\|_W.
\]

Then we have the following proposition.

**Proposition 4.1.** For any \( s \in \mathbb{R} \), Banach spaces \( S^s(\mathbb{R}^2) \) and \( N_s(\mathbb{R}^2) \) have the following properties.

(i) The Schwartz functions on \( \mathbb{R}^2 \) are dense in \( S^s(\mathbb{R}^2) \) and in \( N_s(\mathbb{R}^2) \).

(ii) If \( s \leq s' \) and \( u \in S^s(\mathbb{R}^2) \), then

\[
\|u\|_{S^{s'}(\mathbb{R}^2)} \leq \|u\|_{S^{s'}(\mathbb{R}^2)}.
\]

Similarly, let \( F \in N_s(\mathbb{R}^2) \), then

\[
\|F\|_{N_{s'}(\mathbb{R}^2)} \leq \|F\|_{N_{s'}(\mathbb{R}^2)}.
\]

(iii) Let \( u \in S^s(\mathbb{R}^2) \), then

\[
\|u\|_{C(\mathbb{R}, H^s(\mathbb{R}))} \leq \|u\|_{S^s(\mathbb{R}^2)}.
\]

(iv) Let \( u_0 \in H^s(\mathbb{R}) \), \( u(t) = S(t)u_0 \), and \( \eta(t) \) is a smooth bump function, then \( \eta u \in S^s(\mathbb{R}^2) \) and

\[
\|\eta u\|_{S^s(\mathbb{R}^2)} \leq \|u_0\|_{H^s(\mathbb{R})}.
\]

(v) Let \( F \in N_s(\mathbb{R}^2) \), and \( \eta(t) \) is a smooth bump function, then

\[
\left\| \int_{\mathbb{R}} \text{sgn}(t)\eta(s-t)F(t)dt \right\|_{H^s(\mathbb{R})} \leq \|F\|_{N_s(\mathbb{R}^2)}.
\]

(vi) Let \( F \in N_s(\mathbb{R}^2) \), and \( \eta(t) \) is a smooth bump function, then

\[
\left\| \int_{\mathbb{R}} \text{sgn}(t-t')\eta(s-t')F(t')dt' \right\|_{S^s(\mathbb{R}^2)} \leq \|F\|_{N_s(\mathbb{R}^2)}.
\]

(vii) For any \( s \geq -2 \), if \( u, v \in S^s(\mathbb{R}^2) \), then

\[
\|uv\|_{S^s(\mathbb{R}^2)} \leq \|u\|_{S^s(\mathbb{R}^2)}\|v\|_{H^{-2}(\mathbb{R}^2)} + \|u\|_{S^{-2}(\mathbb{R}^2)}\|v\|_{S^s(\mathbb{R}^2)}.
\]

(viii) Let \( u \in S^s(\mathbb{R}^2) \), and \( \eta(t) \) is a smooth bump function, then

\[
\|\eta u\|_{S^s(\mathbb{R}^2)} \leq \|u\|_{S^s(\mathbb{R}^2)}.
\]

**Proof.** By Proposition 2.4, we can easily give the proof of (i)-(vii) where completely similar to the proof of Proposition 2 in [1]. So we omit it here.

Now, we only prove (viii). By the definition of the space \( S^s \), it suffices to show

\[
\|\xi^{s+2}\omega\hat{u}\|_{L^2} \lesssim \|\xi^{s+2}\omega\hat{u}\|_{L^2}.
\]

It is not difficult to show that \( \omega(\tau, \xi) \lesssim \omega(\tau_1, \xi)\omega(\tau_2, \xi) \), where \( \tau = \tau_1 + \tau_2 \). On the one hand, by using Young’s inequality, we have

\[
\|\xi^{s+2}\omega\hat{u}\|_{L^2} \lesssim \|\omega\hat{u}\|_{L^2} \lesssim \|\omega\hat{u}\|_{L^2} \lesssim \|\xi^{s+2}\omega\hat{u}\|_{L^2}.
\]

where \( \hat{\cdot} \) denotes the time Fourier transform

\[
\hat{\eta}(\tau) = \int_{\mathbb{R}} e^{-it\tau}\eta(t)dt
\]
and

\[ \| \xi^{-2}(\xi)^{s+2}\omega \tilde{\eta}u \|_{L^2 L^1} \lesssim \| (\omega \tilde{\eta}) \ast_\tau ( (\xi)^{-2}(\xi)^{s+2}\omega \tilde{u} ) \|_{L^2 L^1} \]
\[ \leq \| (\omega \tilde{\eta}) \|_{L^1} \| ( (\xi)^{-2}(\xi)^{s+2}\omega \tilde{u} ) \|_{L^2 L^1} \]
\[ \lesssim \| \xi^{s+2}\omega \tilde{u} \|_Y. \]  

(4.35)

Combining (4.34) with (4.35), we deduce from the definition of spaces \( Z \) and \( Y \) that

\[ \| \xi^{s+2}\omega \tilde{\eta}u \|_Z \leq \| \xi^{s+2}\omega \tilde{\eta}u \|_Y \lesssim \| \xi^{s+2}\omega \tilde{u} \|_Y. \]  

(4.36)

On the other hand, by the definition of the space \( \tilde{X}^{-2,1,1} \) and (4.34), we obtain

\[ \| \xi^{s+2}\omega \tilde{\eta}u \|_{\tilde{X}^{-2,1,1}} = \left( \sum_{j \geq 0} 2^{-2j} \left( \sum_{d \geq 0} 2^{d/2} \| \xi^{s+2}\omega \tilde{\eta}u \|_{L^2 L^2(A_j \cap B_d)} \right)^2 \right)^{1/2} \]
\[ \leq \left( \sum_{j \geq 0} 2^{-4j} \left( \sum_{d \geq 0} 2^{d/2} \| \xi^{s+2}\omega \tilde{u} \|_{L^2 L^2(A_j \cap B_d)} \right)^2 \right)^{1/2} \]
\[ = \| \xi^{s+2}\omega \tilde{u} \|_{\tilde{X}^{-2,1,1}}. \]

So, the definition of the space \( Z \) and the above inequality imply that

\[ \| \xi^{s+2}\omega \tilde{\eta}u \|_Z \leq \| \xi^{s+2}\omega \tilde{\eta}u \|_{\tilde{X}^{-2,1,1}} \lesssim \| \xi^{s+2}\omega \tilde{u} \|_{\tilde{X}^{-2,1,1}}. \]  

(4.37)

Combining (4.36)-(4.37) gives (4.33). Thus, (viii) is proved.

We are now in a position to give the proof of Theorem 1.1.

Proof. We use the above proposition to study the local well-posedness to the Cauchy problem (1.1)-(1.2) by a contraction argument. To be more specific, local solution of (1.1)-(1.2) is constructed as a fixed point \( u \) of contraction mapping \( u \mapsto \Gamma u \), where \( \Gamma u(t) := L(\varphi) + N_2(u, u) \), on a suitable complete metric space of functions.

Applying (iv)-(viii) of Proposition 4.1, we deduce from (1.11) and (1.12) that

\[ \| L(\varphi) \|_{S^{-2}(R^2)} \leq C \| \varphi \|_{H^{-2}(R)}, \]  

(4.38)

\[ \| N_2(u, u) \|_{S^{-2}(R^2)} \leq C \| \eta_1 u^2 \|_{N^{-2}(R^2)} \leq C \| \eta_1 u \|_{S^{-2}(R^2)}^2 \leq C \| u \|_{S^{-2}(R^2)}^2, \]  

(4.39)

for some constant \( C > 0 \).

Thus, for \( \varphi \in B_{H^{-2}(R)}(r) \), we choose \( r = \frac{1}{8C^2} \), by a simple computation, it is easy to know that \( \Gamma \) is a strict contraction on \( B_{S^{-2}}(\frac{1}{16C^2}) \), and thus has a unique fixed point \( u \in B_{S^{-2}}(\frac{1}{16C^2}) \). In fact, for \( u, v \in B_{S^{-2}}(\frac{1}{16C^2}) \), by using (4.38)-(4.39), we have

\[ \| \Gamma u \|_{S^{-2}(R^2)} \leq C \| \varphi \|_{H^{-2}(R)} + C \| u \|_{S^{-2}(R^2)}^2 \leq C \left( \frac{1}{8C^2} + \frac{1}{16C^2} \right) \leq \frac{1}{4C}, \]

and

\[ \| \Gamma u - \Gamma v \|_{S^{-2}(R^2)} \leq C \| \eta_1 u^2 - \eta_1 v^2 \|_{N^{-2}(R^2)} \]
\[ \leq C(\| u \|_{S^{-2}(R^2)} + \| v \|_{S^{-2}(R^2)}) \| u - v \|_{S^{-2}(R^2)}, \]
\[ \leq C \left( \frac{1}{4C} + \frac{1}{4C} \right) \| u - v \|_{S^{-2}(R^2)} \]
\[ = \frac{1}{2} \| u - v \|_{S^{-2}(R^2)}. \]
It is easy to see by the property (iii) that the space $S^{-2}(\mathbb{R}^2)$ is continuously embedded into $C(\mathbb{R}; H^s(\mathbb{R}))$. Thus, we see that (1.10) is well-posed in the spaces $C(\mathbb{R}; H^s(\mathbb{R}))$.

Similarly, for any $s \geq -2$, by using (4.38)-(4.39), we have
\[
\|L(\varphi)\|_{S^{s}(\mathbb{R}^2)} \leq C\|\varphi\|_{H^s(\mathbb{R})},
\]
\[
\|N_2(u, u)\|_{S^{s}(\mathbb{R}^2)} \leq C\|\eta_1 u^2\|_{L^s(\mathbb{R}^2)} \leq C\|u\|_{S^{-2}(\mathbb{R}^2)}\|u\|_{S^{s}(\mathbb{R}^2)},
\]
by Theorem 4 in [1], we see that (1.10) has persistence of regularity for the spaces $H^s(\mathbb{R})$ and $S^s(\mathbb{R}^2)$ for any $s \geq -2$. Finally, combining with (ii) and (iii) of Proposition 4.1, we complete the proof of Theorem 1.1.

5. The proof of Theorem 1.2. In this section, we prove Theorem 1.2.

*Proof.* Let $s < -2$ and $s' \in \mathbb{R}$. Without loss of generality, we may rescale $T$ to equal 1. We assume that, on the contrary, the solution map $\varphi \mapsto u[\varphi]$ is continuous on $B_r$ (with the $H^s(\mathbb{R})$-topology) to $C([0, T]; H^{-2}(\mathbb{R}))$ (with the $C([0, T]; H^s(\mathbb{R}))$ topology). By Proposition 1 in [1], we obtain that the map $\varphi \mapsto N_2(L(\varphi), L(\varphi))$ is continuous from $H^s(\mathbb{R})$ to $C([0, 1]; H^s(\mathbb{R}))$. More precisely, we have
\[
\sup_{0 \leq t \leq 1} \|N_2(L(\varphi), L(\varphi))\|_{H^{s'}(\mathbb{R})} \lesssim \|\varphi\|_{H^s(\mathbb{R})}.
\]
From (1.11) and (1.12), we have
\[
\sup_{0 \leq t \leq 1} \|N_2(L\varphi, L\varphi)\|_{H^{s'}(\mathbb{R})} = \sup_{0 \leq t \leq 1} \left\| \int_0^t S(t - t')(S(t')\varphi)^2 dt' \right\|_{H^{s'}(\mathbb{R})}
\]
\[
\leq \sup_{0 \leq t \leq 1} \left\| \int_0^t \exp(it(t - t')\xi^4) \exp(it'(\xi_1^4 + (\xi - \xi_1)^4)) \right. \times \langle \xi \rangle^{s'} \hat{\varphi}(\xi_1) \hat{\varphi}(\xi - \xi_1) d\xi_1 dt'
\]
\[
\left. \left\| \right\|_{L^2_\xi(\mathbb{R})}.
\]
Let $N$ be sufficiently large and take
\[
\hat{\varphi}(\xi) := N^{-s}1_{[-10, 10]}(|\xi| - N).
\]
It is easy to check that $\|\hat{\varphi}\|_{H^s(\mathbb{R})} \sim 1$, thus we get
\[
\|\langle \xi \rangle^{s'} \int_0^t \exp(it(t' - t')\xi^4) \exp(it'(\xi_1^4 + (\xi - \xi_1)^4)) \hat{\varphi}(\xi_1) \hat{\varphi}(\xi - \xi_1) d\xi_1 dt' \|_{L^2_\xi(\mathbb{R})} \lesssim 1
\]
for all $0 \leq t \leq 1$.

We take $t := 1/(1000N^4)$ and localize to the region where $-1 \leq \xi \leq 1$. Straightforward calculations yields
\[
\Re\left[ \exp(it(t - t')\xi^4) \exp(it'(\xi_1^4 + (\xi - \xi_1)^4)) \right] > \frac{1}{2},
\]
where $0 \leq t' \leq t$ and $\xi_1$ belongs to supp$\hat{\varphi}$, then, we get
\[
\|\langle \xi \rangle^{s'} \int_0^t \exp(it(t - t')\xi^4) \exp(it'(\xi_1^4 + (\xi - \xi_1)^4)) \hat{\varphi}(\xi_1) \hat{\varphi}(\xi - \xi_1) d\xi_1 dt' \|_{L^2_\xi(\mathbb{R})}
\]
\[
\gtrsim N^{-2s-4},
\]
which leads to a contradiction to (5.40) for sufficiently large $N$ (since $s < -2$).
This completes the proof of Theorem 1.2.

Acknowledgments. The authors would like to express their great gratitude to the referees for their valuable suggestions, which lead to improvements of the paper.

REFERENCES

[1] I. Bejenaru and T. Tao, Sharp well-posedness and ill-posedness results for a quadratic nonlinear Schrödinger equation, *J. Funct. Anal.*, 233 (2006), 228–259.

[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I. Schrödinger equations, *Geom. Funct. Anal.*, 3 (1993), 107–156.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness for KdV and modified KdV on R and T, *J. Amer. Math. Soc.*, 16 (2003), 705–749.

[4] B. L. Guo and B. X. Wang, The global Cauchy problem and scattering of solutions for nonlinear Schrödinger equations in $H^s$, *Diff. Int. Eqns.*, 15 (2002), 1073–1083.

[5] C. Hao, L. Hsiao and B. X. Wang, Well-posedness for the fourth-order Schrödinger equations, *J. Math. Anal. Appl.*, 320 (2006), 246–265.

[6] C. Hao, L. Hsiao and B. X. Wang, Well-posedness of the Cauchy problem for the fourth-order Schrödinger equations in high dimensions, *J. Math. Anal. Appl.*, 328 (2007), 58–83.

[7] V. I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: Fourth order nonlinear Schrödinger-type equations, *Phys. Rev. E*, 53 (1996), 1336–1339.

[8] N. Kishimoto, Remark on the paper “Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation” by I. Bejenaru and T. Tao, *Atl. Electron. J. Math.*, 4 (2011), 35–48.

[9] B. A. Ivanov and A. M. Kosevich, Stable three-dimensional small-amplitude soliton in magnetic materials, *Sov. J. Low Temp. Phys.*, 9 (1983), 439–442.

[10] C. Hao, G. X. Xu and L. F. Zhao, Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case, *J. Diff. Eqns.*, 246 (2009), 3715–3749.

[11] C. X. Miao and J. Q. Zheng, Scattering theory for the defocusing fourth-order Schrödinger equation, *Nonlinearity*, 29 (2016), 692–736.

[12] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case, *Dyn. Partial Diff. Eqns.*, 4 (2007), 197–225.

[13] B. Pausader, The cubic fourth-order Schrödinger equation, *J. Funct. Anal.*, 256 (2009), 2473–2517.

[14] B. Pausader, The focusing energy-critical fourth-order Schrödinger equation with radial data, *Discrete Contin. Dyn. Syst.*, 24 (2009), 1275–1292.

[15] B. Pausader and S. L. Shao, The mass-critical fourth-order Schrödinger equation in high dimensions, *J. Hyperbolic Diff. Eqns.*, 7 (2010), 651–705.

[16] B. Pausader and S. X. Xia, Scattering theory for the fourth-order Schrödinger equation in low dimensions, *Nonlinearity*, 26 (2013), 2175–2191.

[17] H. Pecher and W. von Wahl, Time dependent nonlinear Schrödinger equations, *Manuscripta Math.*, 27 (1979), 125–157.

[18] J. Segata, Modified wave operators for the fourth-order nonlinear Schrödinger-type equation with cubic non-linearity, *Math. Methods. Appl. Sci.*, 26 (2006), 1785–1800.

[19] T. Tao, Multilinear weighted convolution of $L^2$ functions, and applications to nonlinear dispersive equations, *Amer. J. Math.*, 123 (2001), 839–908.

[20] S. K. Turitsyn, Three-dimensional dispersion of nonlinearity and stability of multidimensional solitons, *Teoret. Mat. Fiz.*, 64 (1985), 226–232 (Russian).

[21] J. Q. Zheng, Well-posedness for the fourth-order Schrödinger equations with quadratic non-linearity, *Adv. Diff. Eqns.*, 16 (2011), 467–486.