PARTIAL TENSOR-PRODUCT FUNCTORS AND CROSSED-PRODUCT FUNCTORS

JULIAN KRANZ AND TIMO SIEBENAND

Abstract. For a given discrete group $G$, we apply results of Kirchberg on exact and injective tensor products of $C^*$-algebras to give an explicit description of the minimal exact correspondence crossed-product functor and the maximal injective crossed-product functor for $G$ in the sense of Buss, Echterhoff and Willett. In particular, we show that the former functor dominates the latter.

1. Introduction

A fruitful approach to construct examples of $C^*$-algebras is to complete $*$-algebras with respect to certain $C^*$-norms. For instance, if $G \curvearrowright A$ is an action of a discrete group on a $C^*$-algebra, one can complete the algebraic crossed product $A \rtimes_{\text{alg}} G$ to get the maximal crossed product $A \rtimes G$ or the reduced crossed product $A \rtimes_r G$.

In the last decade, there has been an increasing interest in exotic completions of $A \rtimes_{\text{alg}} G$, i.e. completions which strictly lie between the maximal and reduced completion. One important motivation comes from the Baum–Connes conjecture with coefficients [BCH94] which predicts that the Baum–Connes assembly map

$$\mu: K_*^G(EG, A) \to K_*(A \rtimes_r G)$$

is an isomorphism. Counterexamples to the conjecture were constructed in [HLS02] by exploiting non-exactness of the functor $- \rtimes_r G$ for certain groups $G$. Later, in [BGW16] it was suggested to modify the conjecture by replacing the reduced crossed product with the minimal exact Morita compatible crossed product. This modification strictly enlarges the class of actions $G \curvearrowright A$ for which the conjecture is known to hold and does not change the statement of the conjecture for exact groups. Other motivations to study exotic crossed-product functors come from a-T-menability and property $(T)$ [BG13] or from non-commutative duality [KLQ13, KLQ18, BE14].

General exotic crossed–product functors and their properties were studied systematically by Buss, Echterhoff and Willett [Bew17, Bew18a, Bew18b, Bew20a] Bew20b. They introduced the minimal exact crossed-product functor $- \rtimes_{\varepsilon} G$, the minimal exact correspondence crossed-product functor $- \rtimes_{\varepsilon, \text{c}} G$ (which agrees with the minimal exact Morita compatible crossed-product functor of [BGW16] for

2010 Mathematics Subject Classification. 46L55 (Primary) 46M15; 46L80 (Secondary).
Key words and phrases. $C^*$-algebras, exotic crossed products, tensor products.

Both authors were funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 427320536 - SFB 1442, as well as by Germany’s Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics-Geometry-Structure.
separable $G$-$C^*$-algebras [BEW18a Cor. 8.13]) and the maximal injective crossed-product functor $\otimes i, G$. All these functors agree with the reduced crossed product for exact groups, but their interrelations for non-exact groups are still unclear. In particular, it is unclear whether or not $\otimes \ell_e G$ and $\otimes \ell_{err} G$ agree. A positive answer to this question would imply that the “new” Baum-Connes conjecture of [BGW16] agrees with the old conjecture of [BCH94] for complex coefficients $A = \mathbb{C}$. The aim of this article is to provide an explicit description of $\otimes \ell_{inj} G$ and $\otimes \ell_{err} G$. We hope that the interplay of the universal properties and the explicit descriptions of these functors turn out useful in the future. Our main ingredient is the following construction by Kirchberg:

**Theorem A** ([Kir95]). There is a tensor-product functor $\otimes i, G$ satisfying the following properties:

1. For every $C^*$-algebra $A$, $A \otimes i, G$ is the minimal exact partial tensor-product functor for $A$.
2. For every $C^*$-algebra $B$, $\otimes i, B$ is the maximal injective partial tensor-product functor for $B$.

In particular, $\otimes i, G$ is the unique tensor-product functor which is injective in the first variable and exact in the second variable. Furthermore, $\otimes i, G$ is functorial for completely positive maps in both variables.

In terms of Kirchberg’s tensor product, we can describe $\otimes \ell_{inj} G$ and $\otimes \ell_{err} G$ as follows:

**Theorem B** (Theorem 4.3). Let $G$ be a discrete group and let $A$ be a $G$-$C^*$-algebra. Then there are injective $*$-homomorphisms

1. $A \otimes \ell_{inj} G \hookrightarrow (A \otimes r G) \otimes i, G^*(G)$
2. $A \otimes \ell_{err} G \hookrightarrow G^*(G) \otimes i, (A \otimes G)$

given by $a \delta_g \mapsto a \delta_g \otimes \delta_g$ and $a \delta_g \mapsto \delta_g \otimes a \delta_g$ respectively.

We obtain an even more concrete picture using $G$-injective $G$-$C^*$-algebras (see p.4 for the definition). Note that $G$-injective $G$-$C^*$-algebras are always unital.

**Proposition C** (Proposition 4.5). Let $G$ be a discrete group, let $A$ be a $G$-$C^*$-algebra and let $I$ be a $G$-injective $G$-$C^*$-algebra (e.g. $I = \ell^\infty(G)$). Then the canonical embedding $A \hookrightarrow A \otimes I, a \mapsto a \otimes 1$ induces an injective $*$-homomorphism $A \otimes \ell_{err} I \otimes G \hookrightarrow (A \otimes I, G) \otimes I, G$.

Note that for $I = \ell^\infty(G)$, this provides a positive solution to a question asked in [BEW18a Question 9.4] and [BGW16 8.2]. As an application, we are able to compare $\otimes \ell_{inj} G$ and $\otimes \ell_{err} G$:

**Corollary D** (Corollary 4.6). For any discrete group $G$, we have $\otimes \ell_{inj} G \leq \otimes \ell_{err} G$ and $C^*_\inj(G) = C^*_\ell_{err}(G)$.

Thus, in order to prove that $\otimes \ell_{inj} G$ and $\otimes \ell_{err} G$ coincide, it would suffice to construct a crossed-product functor which is both exact and injective.

**Acknowledgements.** The authors would like to thank Siegfried Echterhoff for helpful discussions and comments and the anonymous referee for pointing out an error in a previous version of this article.
2. Preliminaries

In this section we fix some terminology regarding crossed-product and tensor-product functors. For definitions and basic properties of crossed products and tensor products we refer to [BO08, Wil07].

Let \( * \text{Alg} \) denote the category of \(*\)-algebras with \(*\)-homomorphisms as morphisms and let \( C^* \text{Alg} \) denote the full subcategory of \( C^*\)-algebras. For a discrete group \( G \), we denote by \( C^* \text{Alg}_G \) the category of \( G\)-\( C^*\)-algebras with \( G\)-equivariant \(*\)-homomorphisms.

Let \( C \) be a category. A functor \( F^* : C \to C^* \text{Alg} \) is a \( C^*\)-completion of a functor \( F : C \to \text{Alg} \), if for every object \( X \) in \( C \), \( F^*(X) \) is a \( C^*\)-completion of \( F(X) \) and if for every morphism \( f \) in \( C \), \( F^*(f) \) is an extension of \( F(f) \). We define a partial order on the class of \( C^*\)-completions of a given functor \( F \) by declaring \( F^* \geq F^\nu \) if for every object \( X \) in \( C \), the identity on \( F(X) \) extends to a \(*\)-homomorphism \( F^*(X) \to F^\nu(X) \).

For two \( C^*\)-algebras \( A \) and \( B \), we denote by \( A \odot B \) the algebraic tensor product, by \( A \otimes_{\text{max}} B \) the maximal tensor product and by \( A \otimes B \) the minimal tensor product. A tensor-product functor \(- \odot - : C^* \text{Alg} \times C^* \text{Alg} \to \text{Alg}\).

A partial tensor-product functor \(- \odot_\alpha B \) for \( B \) is a \( C^*\)-completion of the functor \( - \odot : C^* \text{Alg} \to \text{Alg} \).

A partial tensor-product functor \(- \odot_\alpha B \) is

1. called exact if it maps exact sequences to exact sequences;
2. called injective if it maps injective \(*\)-homomorphisms to injective \(*\)-homomorphisms;
3. said to have the cp-map property if for each completely positive map \( \varphi : A \to C \), the induced map \( \varphi \odot \text{id}_B : A \odot B \to C \odot B \) extends to a completely positive map \( \varphi \otimes_\alpha \text{id}_B : A \otimes_\alpha B \to C \otimes_\alpha B \).

Every (partial) tensor-product functor is dominated by the maximal tensor-product functor and dominates the minimal tensor-product functor. For a fixed \( C^*\)-algebra \( B \), the functor \(- \otimes_{\text{max}} B \) is exact [BO08 Prop. 3.7.1] whereas the functor \(- \odot B \) is injective. Both functors have the cp-map property [BO08 Thm. 3.5.3].

For a discrete group \( G \) and a \( G\)-\( C^*\)-algebra \( A \), we denote by \( A \ltimes_{\text{alg}} G = A[G] \) the algebraic crossed product, by \( A \rtimes G \) the maximal crossed product and by \( A \rtimes_r G \) the reduced crossed product. A crossed-product functor \(- \rtimes_\mu G \) is a \( C^*\)-completion of the algebraic crossed-product functor

\[- \ltimes_{\text{alg}} G : C^* \text{Alg}_G \to \text{Alg}\]

which dominates the reduced crossed product. We write \( C^*_{\mu}(G) := \mathbb{C} \rtimes_\mu G \). A crossed-product functor \(- \rtimes_\mu G \) is

1. called exact if it maps exact sequences to exact sequences;
2. called injective if it maps injective \( G\)-equivariant \(*\)-homomorphisms to injective \(*\)-homomorphisms;
3. said to have the cp-map property if for each \( G\)-equivariant completely positive map \( \varphi : A \to B \), the induced map \( \varphi \ltimes_{\text{alg}} G : A \ltimes_{\text{alg}} G \to B \ltimes_{\text{alg}} G \) extends to a completely positive map \( \varphi \rtimes_\mu G : A \rtimes_\mu G \to B \rtimes_\mu G \).
Every crossed-product functor is dominated by the maximal crossed product and dominates the reduced crossed product. The maximal crossed product \(-\rtimes G\) is exact \([Ech17,\text{Prop. 4.8}]\) and the reduced crossed product \(-\rtimes_r G\) is injective \([EKR06,\text{Lem. A.16}]\). Both functors have the cp-map property \([BEW20a,\text{Lem. 4.8}]\). Every injective crossed-product functor has the cp-map property \([BEW18a,\text{Thm. 4.9}]\). Moreover there is a maximal injective crossed-product functor \(-\rtimes_{\text{inj}} G\) \([BEW20b,\text{Prop. 3.5}]\) and a minimal exact crossed-product functor with the cp-map property \(-\rtimes_{\text{ex}} G\) \([BEW18a,\text{Cor. 8.8}]\).

Remark 2.1. It was shown in \([BEW18a,\text{Thm. 4.9}]\) that a crossed-product functor has the cp-map property if and only if it extends to a functor on the \(G\)-equivariant correspondence category \(\text{Corr}(G)\) as defined in \([BEW18a,\text{Def. 4.4}]\). Therefore crossed-product functors with the cp-map property are called correspondence crossed-product functors \([BEW18a]\) and \(-\rtimes_{\text{ex}} G\) is called the minimal exact correspondence crossed-product functor. One can prove a similar characterization for partial tensor-product functors.

A \(G\)-\(C^*\)-algebra \(I\) is called \(G\)-injective if for every injective \(G\)-equivariant \(*\)-homomorphism \(\iota: A \hookrightarrow B\) and every \(G\)-equivariant completely positive contractive map \(\varphi: B \to I\) such that \(\varphi \circ \iota = \varphi\). We say that \(\varphi\) extends \(\varphi\) along \(\iota\). In this case \(I\) is unital since there exists a conditional expectation from the unitization \(\tilde{I}\) onto \(I\).

3. Exact and injective tensor-product functors

In this section we give a detailed proof of a theorem that was stated in \([Kir95]\) for convenience of the reader. We need a folklore lemma.

Lemma 3.1. Let
\[
\begin{array}{cccc}
0 & \to & I & \xrightarrow{\iota} & A & \xrightarrow{q} & B & \xrightarrow{\varphi} & 0 \\
0 & \to & I' & \xrightarrow{i'} & A' & \xrightarrow{q'} & B' & \to & 0
\end{array}
\]
be a commutative diagram of \(C^*\)-algebras and \(*\)-homomorphisms. Assume that \(i\) is an ideal inclusion, that the lower row is exact, and that the vertical maps are non-degenerate inclusions. Then we have \(\ker q \subseteq \text{im}(i)\).

Proof. Let \(x \in \ker q\). By exactness, we find \(y \in I'\) such that \(i'(y) = \varphi_A(x)\). Let \((e_\lambda)\) be an approximate unit for \(I\). Since \(\varphi_I\) is non-degenerate, \((\varphi_I(e_\lambda))\) is an approximate unit for \(I'\) and thus \(\|\varphi_I(e_\lambda) y - y\| \to 0\). We obtain \(\|\varphi_A(i(e_\lambda)x - x)\| = \|i'(\varphi_I(e_\lambda) y - y)\| \to 0\). This implies \(\|i(e_\lambda)x - x\| \to 0\) because \(\varphi_A\) is isometric and therefore \(x \in \text{im}(i)\) since \(i\) is an ideal inclusion. 

Theorem 3.2 \([Kir95]\). There is a tensor-product functor \(-\otimes_{i,\varepsilon} -\) satisfying the following properties:

1. For every \(C^*\)-algebra \(A\), \(A \otimes_{i,\varepsilon} -\) is the minimal exact partial tensor-product functor for \(A\).
2. For every \(C^*\)-algebra \(B\), \(-\otimes_{i,\varepsilon} B\) is the maximal injective partial tensor-product functor for \(B\).
In particular, \( - \otimes_{i,e} - \) is the unique tensor-product functor which is injective in the first variable and exact in the second variable. Furthermore, \( - \otimes_{i,e} - \) has the cp-property in both variables.

**Proof.** Let \( A \) and \( B \) be \( C^* \)-algebras and let \( \iota \colon A \hookrightarrow \mathcal{B}(H) \) be an embedding into the bounded operators on a Hilbert space. We define

\[
A \otimes_{i,e} B := \iota \otimes \text{id}_B(A \otimes_{\max} B) \subseteq \mathcal{B}(H) \otimes_{\max} B.
\]

To show that \( - \otimes_{i,e} - \) has the desired properties, we verify the following claims:

**Claim 1.** Up to canonical isomorphism, the definition of \( A \otimes_{i,e} B \) is independent of \( \iota \).

Let \( \iota' \colon A \hookrightarrow \mathcal{B}(H') \) be another embedding. Then by Arveson’s extension theorem there exist completely positive contractive maps \( \Psi \colon \mathcal{B}(H) \to \mathcal{B}(H') \) extending \( \iota' \) along \( \iota \) and \( \Phi \colon \mathcal{B}(H') \to \mathcal{B}(H) \) extending \( \iota \) along \( \iota' \). Then \( \Psi \otimes_{\max} \text{id}_B \) and \( \Phi \otimes_{\max} \text{id}_B \) restrict to mutually inverse \(*\)-isomorphisms

\[
\iota \otimes \text{id}_B(A \otimes_{\max} B) \cong \iota' \otimes \text{id}_B(A \otimes_{\max} B).
\]

**Claim 2.** \( A \otimes_{i,e} B \) is functorial for completely positive maps in both variables.

Functoriality for completely positive maps in \( B \) follows immediately from the definition. To see functoriality in \( A \), let \( \varphi \colon A_1 \to A_2 \) be a completely positive map and let \( \iota_j \colon A_j \hookrightarrow \mathcal{B}(H_j), j = 1, 2 \) be embeddings. Let \( \Psi \colon \mathcal{B}(H_1) \to \mathcal{B}(H_2) \) be a completely positive map extending \( \iota_2 \circ \varphi \) along \( \iota_1 \). Then \( \Psi \otimes_{\max} \text{id}_B \) restricts to a completely positive map \( A_1 \otimes_{i,e} B \to A_2 \otimes_{i,e} B \) extending the canonical map \( \varphi \otimes \text{id}_B \colon A_1 \otimes B \to A_2 \otimes B \).

**Claim 3.** The functor \( - \otimes_{i,e} - \) is the maximal injective partial tensor-product functor for \( B \).

Let \( \varphi \colon A_1 \hookrightarrow A_2 \) be an injective \(*\)-homomorphism and let \( \iota \colon A_2 \hookrightarrow \mathcal{B}(H) \) be an embedding. Then \( \varphi \circ \iota \colon A_1 \hookrightarrow \mathcal{B}(H) \) is an embedding too. Inserting this embedding into (1) shows that \( \varphi \otimes \text{id}_B \colon A_1 \otimes_{i,e} B \to A_2 \otimes_{i,e} B \) is isometric and therefore injective. Now let \( - \otimes_{\alpha} B \) be another injective partial tensor-product functor for \( B \) and let \( A \hookrightarrow \mathcal{B}(H) \) be an embedding. Then the canonical quotient map \( \mathcal{B}(H) \otimes_{\max} B \to \mathcal{B}(H) \otimes_{\alpha} B \) restricts to a quotient map \( A \otimes_{i,e} B \to A \otimes_{\alpha} B \). Thus, \( - \otimes_{i,e} B \) is maximal.

**Claim 4.** The functor \( A \otimes_{i,e} - \) is exact.

Let \( 0 \to I \to B \to Q \to 0 \) be an exact sequence of \( C^* \)-algebras. Assume first that \( A \) is unital and choose a unital embedding \( A \hookrightarrow \mathcal{B}(H) \). Then the upper row of the diagram

\[
\begin{array}{cccccc}
0 & \to & A \otimes_{i,e} I & \to & A \otimes_{i,e} B & \to & A \otimes_{i,e} Q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B(H) \otimes_{\max} I & \to & B(H) \otimes_{\max} B & \to & B(H) \otimes_{\max} Q & \to & 0
\end{array}
\]
is exact by Lemma 3.1. Now assume that $A$ is not unital and denote by $\tilde{A}$ its unitization. By the above, the middle and lower row of the diagram

$$
0 \longrightarrow A \otimes_{i,\varepsilon} I \longrightarrow A \otimes_{i,\varepsilon} B \longrightarrow A \otimes_{i,\varepsilon} Q \longrightarrow 0
$$

are exact. Since the extension $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$ splits, the columns of (2) are exact as well. Now exactness of the upper row of (2) follows from the $3 \times 3$-Lemma.

**Claim 5.** The functor $A \otimes_{i,\varepsilon} -$ is the minimal exact partial tensor-product functor.

Let $A \otimes_{\alpha} -$ be another exact partial tensor-product functor and fix a $C^*$-algebra $B$. Assume first that $B$ is unital and pick a surjective $\ast$-homomorphism $C^*(F_X) \to B$ where $F_X$ denotes the free group on a set $X$ of unitaries generating $B$. Denote by $I$ the kernel of $C^*(F_X) \to B$ and choose an embedding $\iota: A \hookrightarrow B(H)$. By Kirchberg’s tensor-product functor $- \otimes B$ where $B$ is a non-unital $C^*$-algebra, we can fill the following diagram with the dashed $\ast$-homomorphism $\psi$:

$$
0 \longrightarrow A \otimes_{\alpha} I \longrightarrow A \otimes_{\alpha} C^*(F_X) \longrightarrow A \otimes_{\alpha} B \longrightarrow 0
$$

By definition, we have $\psi(A \otimes_{\alpha} B) = A \otimes_{i,\varepsilon} B$. If $B$ is a non-unital $C^*$-algebra, we can apply the same argument to its unitization and use exactness to produce a canonical quotient map $A \otimes_{\alpha} B \to A \otimes_{i,\varepsilon} B$. This proves maximality.

**Remark 3.3.** Let $F$ be a non-amenable free group and $H$ an infinite-dimensional Hilbert space. Then the flip isomorphism $B(H) \otimes C_r^*(F) \cong C_r^*(F) \otimes B(H)$ does not extend to an isomorphism $B(H) \otimes_{i,\varepsilon} C_r^*(F) \otimes_{i,\varepsilon} B(H)$. Therefore, Kirchberg’s tensor-product functor $- \otimes_{i,\varepsilon} -$ is not symmetric. Indeed, we have $C_r^*(F) \otimes_{i,\varepsilon} B(H) = C_r^*(F) \otimes B(H)$ since $C_r^*(F)$ is exact and $B(H) \otimes_{i,\varepsilon} C_r^*(F) = B(H) \otimes_{\max} C_r^*(F)$ by construction. But the identity map on $B(H) \otimes C_r^*(F)$ does not extend to an isomorphism $B(H) \otimes_{\max} C_r^*(F) \cong B(H) \otimes C_r^*(F)$ since $C_r^*(F)$ does not have the local lifting property [BO08] Cor. 3.7.12, Thm. 13.1.6, Cor. 13.2.5].
4. APPLICATION TO CROSSED PRODUCTS

Throughout this section, let $G$ be a discrete group. We recall a version of Fell’s absorption principle from \[ABES21\].

Proposition 4.1 (\[ABES21\] Prop. 2.8]). Let $- \rtimes \mu G$ be a crossed-product functor with the cp-map property and let $A$ be a $C^*$-algebra equipped with the trivial $G$-action. Then the canonical map $A \otimes C^*_\mu(G) \to A \rtimes \mu G$ is injective. In particular, $A \mapsto A \otimes \mu C^*_\mu(G) := A \rtimes \mu G$ is a partial tensor-product functor for $C^*_\mu(G)$.

Although only stated for $\rho = \max$ in \[ABES21\], the proof of the following lemma works verbatim for every crossed-product functor $- \rtimes \rho G$:

Lemma 4.2 (\[ABES21\] Lem. 2.10]). Let $- \rtimes \mu G$ be a crossed-product functor with the cp-map property and let $- \rtimes \rho G$ be any crossed-product functor. Then for every $G$-$C^*$-algebra $A$, there is an injective *-homomorphism

$$A \rtimes \mu G \hookrightarrow (A \rtimes \rho G) \otimes \mu C^*_\mu(G)$$

given by $a\delta_g \mapsto a\delta_g \otimes \delta_g$ for $a \in A, g \in G$.

Theorem 4.3. For every $G$-$C^*$-algebra $A$, there are injective *-homomorphisms

1. $A \rtimes_{\text{inj}} G \hookrightarrow (A \rtimes \gamma) \otimes_{i,\varepsilon} C^*(G)$, $a\delta_g \mapsto a\delta_g \otimes \delta_g$.
2. $A \rtimes_{\varepsilon_{\text{err}} G} \hookrightarrow C^*_{\varepsilon_{\text{err}}}(G) \otimes_{i,\varepsilon} (A \rtimes G)$, $a\delta_g \mapsto \gamma_g \otimes \delta_g$.

Proof. We first prove the statement for $- \rtimes_{\text{inj}} G$. Denote by $A \rtimes \gamma G$ the image of $A \rtimes G$ in $(A \rtimes \gamma) \otimes_{i,\varepsilon} C^*(G)$ under the map $a\delta_g \mapsto a\delta_g \otimes \delta_g$. Then $- \rtimes \gamma G$ is an injective crossed-product functor and therefore $- \rtimes \gamma G \leq - \rtimes_{\text{inj}} G$. On the other hand, Lemma 4.2 gives us an embedding

$$A \rtimes_{\text{inj}} G \hookrightarrow (A \rtimes \gamma G) \otimes \gamma C^*_\gamma(G), \quad a\delta_g \mapsto a\delta_g \otimes \delta_g.$$
Proposition 4.5. Let $I$ be a $G$-injective $G$-$C^*$-algebra and let $A$ be any $G$-$C^*$-algebra. Then the canonical embedding $A \hookrightarrow A \otimes_{\max} I, a \mapsto a \otimes 1$ induces an embedding

$$A \times_{\varepsilon_{\text{extr}}} G \hookrightarrow (A \otimes_{\max} I) \times G.$$  

Proof. Denote by $A \times_{\alpha} G$ the image of $A \times G$ in $(A \otimes_{\max} I) \times G$ under the map $a \delta_g \mapsto (a \otimes 1) \delta_g$. Then $- \times_{\alpha} G$ is an exact crossed-product functor with the cp-map property by Lemma 3.1 and therefore $- \times_{\alpha} G \geq - \times_{\varepsilon_{\text{extr}}} G$. It remains to prove the converse inequality. Consider $B(\ell^2(G))$ as a $G$-$C^*$-algebra equipped with the conjugation action of the left regular representation $\lambda: G \to U(\ell^2(G))$. By $G$-injectivity, there is a $G$-equivariant unital completely positive map $\varphi: B(\ell^2(G)) \to I$. Consider the “untwisting isomorphism”

$$\Psi: B(\ell^2(G)) \otimes_{\max} (A \times G) \cong (B(\ell^2(G)) \otimes_{\max} A) \times G, \quad T \otimes a \delta_g \mapsto T\lambda_{g^{-1}} \otimes a \delta_g$$

denote by $\kappa$ the following composition of contractive maps.

$$A \times_{\varepsilon_{\text{extr}}} G \xrightarrow{\text{Thm} \ref{Bl}} C^*_r(G) \otimes_{i,\varepsilon} (A \times G) \xrightarrow{\lambda \otimes \text{id}} B(\ell^2(G)) \otimes_{\max} (A \times G) \xrightarrow{\Psi} (I \otimes_{\max} A) \times G \xleftarrow{(\varphi \otimes \text{id}) \times G} (B(\ell^2(G)) \otimes_{\max} A) \times G$$

A straightforward computation shows that $\kappa(a \delta_g) = (a \otimes 1) \delta_g$ for $a \in A$ and $g \in G$. Thus, we have $\kappa(A \times_{\varepsilon_{\text{extr}}} G) = A \times_{\alpha} G$ and therefore $- \times_{\varepsilon_{\text{extr}}} G \geq - \times_{\alpha} G$. \hfill $\square$

Corollary 4.6. For any discrete group $G$, we have $- \times_{\text{inj}} G \leq - \times_{\varepsilon_{\text{extr}}} G$ and $C^*_\text{inj}(G) = C^*_\varepsilon_{\text{extr}}(G)$.

Proof. Let $A, I$ be $G$-$C^*$-algebras where $I$ is $G$-injective. The embedding $A \hookrightarrow A \otimes_{\max} I$ induces an embedding $A \times_{\text{inj}} G \hookrightarrow (A \otimes_{\max} I) \times_{\text{inj}} G$. The first statement now follows from Proposition 4.5. The second statement follows from the same argument and the fact that $I \times G = I \times_{\text{inj}} G$ \cite{BEW20b} [Cor. 3.3]. \hfill $\square$

References

[ABES21] Paolo Antonini, Alcides Buss, Alexander Engel, and Timo Siebenand. Strong Novikov conjecture for low degree cohomology and exotic group $C^*$-algebras. Trans. Amer. Math. Soc., 374(7):5071–5093, 2021.

[BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group $C^*$-algebras. In $C^*$-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of Contemp. Math., pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[BE14] Alcides Buss and Siegfried Echterhoff. Universal and exotic generalized fixed-point algebras for weakly proper actions and duality. Indiana Univ. Math. J., 63(6):1659–1701, 2014.

[BEW17] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. Exotic crossed products. In Operator algebras and applications—the Abel Symposium 2015, volume 12 of Abel Symp., pages 67–114. Springer, [Cham], 2017.

[BEW18a] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. Exotic crossed products and the Baum-Connes conjecture. J. Reine Angew. Math., 740:111–159, 2018.

[BEW18b] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. The minimal exact crossed product. Doc. Math., 23:2043–2077, 2018.

[BEW20a] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. Injectivity, crossed products, and amenable group actions. In K-theory in algebra, analysis and topology, volume 749 of Contemp. Math., pages 105–137. Amer. Math. Soc., [Providence], RI, 2020.
[BEW20b] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. The maximal injective crossed product. *Ergodic Theory Dynam. Systems*, 40(11):2995–3014, 2020.

[BG13] Nathaniel P. Brown and Erik P. Guentner. New C*-completions of discrete groups and related spaces. *Bull. Lond. Math. Soc.*, 45(6):1181–1193, 2013.

[BGW16] Paul Baum, Erik Guentner, and Rufus Willett. Expanders, exact crossed products, and the Baum-Connes conjecture. *Ann. K-Theory*, 1(2):155–208, 2016.

[BO08] Nathaniel P. Brown and Narutaka Ozawa. $C^*$-algebras and finite-dimensional approximations, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Ech17] Siegfried Echterhoff. Crossed products and the Mackey–Rieffel–Green machine. In *K-Theory for Group C*-Algebras and Semigroup C*-Algebras*, pages 5–79. Springer, 2017.

[EKQR06] Siegfried Echterhoff, S. Kaliszewski, John Quigg, and Iain Raeburn. A categorical approach to imprimitivity theorems for $C^*$-dynamical systems. *Mem. Amer. Math. Soc.*, 180(850):viii+169, 2006.

[HLS02] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[Kir94] Eberhard Kirchberg. Commutants of unitaries in UHF algebras and functorial properties of exactness. *J. Reine Angew. Math.*, 452:39–77, 1994.

[Kir95] Eberhard Kirchberg. Exact $C^*$-algebras, tensor products, and the classification of purely infinite algebras. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 943–954. Birkhäuser, Basel, 1995.

[KLQ13] S. Kaliszewski, Magnus B. Landstad, and John Quigg. Exotic group $C^*$-algebras in noncommutative duality. *New York J. Math.*, 19:689–711, 2013.

[KLQ16] S. Kaliszewski, Magnus B. Landstad, and John Quigg. Coaction functors. *Pacific J. Math.*, 284(1):147–190, 2016.

[KLQ18] S. Kaliszewski, Magnus B. Landstad, and John Quigg. Coaction functors, II. *Pacific J. Math.*, 293(2):301–339, 2018.

[Pis96] Gilles Pisier. A simple proof of a theorem of Kirchberg and related results on $C^*$-norms. *J. Operator Theory*, 35(2):317–335, 1996.

[Wil07] Dana P. Williams. *Crossed products of $C^*$-algebras*, volume 134 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.