Flexible Multibody System Linear Modeling for Control using Component Modes Synthesis and Double-Port Approach

Jose Alvaro Perez  
PhD Student  
Department of Flight Dynamics and Control  
ONERA Toulouse, France  
Email: jose-alvaro.perez_gonzalez@onera.fr

Daniel Alazard  
Professor  
System Dynamics and Control  
ISAE-SUPAERO Toulouse, France  
Email: daniel.alazard@isae.fr

Thomas Loquen  
Research Engineer  
Department of Flight Dynamics and Control  
ONERA Toulouse, France  
Email: thomas.loquen@onera.fr

Christelle Pittet  
Control Engineer  
Department of AOCS  
CNES Toulouse, France  
Email: christelle.pittet@cnes.fr

Christelle Cumer  
Research Engineer  
Department of Flight Dynamics and Control  
ONERA Toulouse, France  
Email: christelle.cumer@onera.fr

ABSTRACT

The main objective of this study is to propose a methodology to build a parametric linear model of Flexible Multibody Systems for control design. This approach uses a combined Finite Element - State Space Approach based on component modes synthesis and double-port approach. The proposed scheme offers the advantage of automatic assembly of substructures, preserving the elastic dynamical behavior of the whole system. Substructures are connected following the double-port approach; i.e., through the transfer of accelerations and loads at the connection points, which take into account the dynamic coupling among them. In addition, parametric variations can be included in the model for accomplishing integrated control/structure design purposes. The method can be applied to combinations of chain-like or star-like flexible systems, and it is validated through its comparison with the assumed modes method for the case of a rotatory spacecraft.

1 Introduction

Modeling of flexible structures has been a major concern in control applications during the last thirty years due to the tendency of developing larger and lighter structures [1]. Large size and reduced mass imply lower natural frequencies which are typically closely spaced [2]. This can be problematic since natural frequencies may interfere with controller’s bandwidth, affecting the whole dynamical behavior of the system. Simple and accurate models for large flexible structures which predict such characteristics are indispensable to design, optimize and control engineering systems.

A strategy for modeling large flexible structures is considering them as a Flexible Multibody System (FMS), a group of interconnected rigid and deformable components, each of which undergoing translational and rotational motions [3]. Then, Flexible Multibody Dynamics (FMD) are used for analyzing the dynamic response of FMS due to external conditions. FMD has a significant importance for the design, optimization and control of many practical systems such as space vehicles [3, 4].
or robot manipulators [5,6]. This obliges control modeling techniques to deal with FMS substructures and thus to include the necessary control inputs/outputs to model the varying dynamical behavior according to varying boundary conditions. Indeed, these boundary conditions depend on the controller and are unknown during the open-loop modeling phase.

Modeling techniques for control purposes include the use of the Finite Element (FE) models as input data due to their powerfullness and their widely usage for structural analysis [7-9]. However, when using FE techniques for the study of FMD the complexity increases. In order to reduce the size of the matrices, substructuring techniques (ST) have been proposed, consisting in a “macrodiscretization” of the large system into a set of subsystems known as substructures, which in turn are discretized using a FE method [10]. Among these techniques, two methods have drawn researchers’ attention for decades: the “Finite Element Transfer Matrix” (FE-TM) method and the “Component Modes Synthesis (CMS) Method”. The FE-TM method has not enjoyed the same degree of application as the CMS method, but it has been proved a powerful tool for the analysis of structures. Leckie [11] stated the fundamentals of the transfer matrix method, which can be seen as a continuity function for a flexible system with transferable boundaries among the substructures. Later, Dokainish [12] merged the FE method with the transfer matrix method for the dynamic analysis of plates and presented it as the FE-TM method. Since then, many researchers extended the technique [13-15] for several boundaries and different applications, and modified it for control purposes like in Yousuff [16] and Tan [17]. This method continues being improved by the works of authors such as Rong [8] and Rui [18] and enriched with other control applications as in Krauss [19], where a non-collocated feedback is applied using the transfer matrix method.

The CMS method has received significant attention in the aerospace industry since its idea of matrix condensation lends itself particularly well to the concept of substructuring and its facility to be translated into a state-space representation. It is an extension of modal analysis that is particularly applicable to large modular systems [20]. First, it uses modal analysis to obtain a proper model of each subsystem. Then it assembles these subsystem models conforming the whole structure. Among the available dynamic substructuring methods, CMS stands the most systematic and efficient procedure for developing a satisfactory decomposed model [21-22-20]. Hurty [23], Craig [24], Hintz [25] and Macneal [26] have developed extensive studies in this area and the main aspects of the theory will be resumed in Section 2.

Many works have indirectly studied how to obtain proper representations of FE models using CMS technique for control purposes. One of the first authors who merged FE modeling of FMS, state-space modeling and control was Young [27], who substructured the complete FE model of a two-dimensional truss in order to synthesize a distributed control law. Although the used substructuring technique was similar in many aspects to CMS, it was mainly based on the overlapping among the inertia and stiffness loading provided by the FE model of the substructures, which requires deep knowledge of the FE model. Shortly after, Sunar [28] started studying FE models substructuring at a more general level applying Guyan static condensation, a specific type of CMS. Su [29] proposed a method to decompose a structure into a collection of substructures in order to design a controller for each substructure, and called it Substructural Controller Synthesis. This method is closed to CMS but the overlapping between systems is still done at the matrix level. Butler [30] studied integrated structure and control design using substructure decomposition based on CMS as well. With a clear idea of the need of converting FE models into state-space representation, Bokhari [9] developed an algorithm which converts a FE model into a state space model of an interconnected system and apply it to a simple cantilevered beam.

Guy [4], Alazard [31] and Cumer [32] also came up with a state-space representation from FE model in order to represent the dynamics of a spacecraft with flexible appendages in star structure. Although not directly mentioned, a particular application of CMS was used in order to transfer the influence of flexible appendages to the spacecraft main hub. The main advantage of this method relied on the overlapping between substructures, expressed as an acceleration-force transfer through the common boundaries, as explained in the double-port approach of Alazard [33]. Perez [34] generalized the method for chain-like type of substructures and successfully applied it to a modular spacecraft [35], based on the Craig-Bampton decomposition [24,36,37], a particular form of CMS.

This article proposes a general methodology for linear modeling of FMS, called Two-Input Two-Output Port (TITOP) model, with special emphasis on the modularity, chain-like type of structures, without restricting the substructures to be of the same nature. A special feature, described here, is the treatment given to the inputs/outputs of the substructure model obtained through CMS transformation, which are reduced to a transfer of loads-acceleration in what is called double-port approach. Each substructure is described by a TITOP block diagram allowing arbitrary boundary conditions to be taken into account without any modification of the substructure block-diagram or overlapping manipulations in the mass and stiffness matrices. Another contribution is the added possibility of interacting with several structural parameters inside the model. In Section 2 the highlights of CMS are explained. In Section 3 CMS is applied to develop linear models in state-space form in order to model Flexible Multibody Systems. One connection point and two connection points are developed, as well as the assembly technique and parametrization possibilities. Finally, the modeling technique is applied to a rotary spacecraft and results are compared with a largely accepted method, the assumed modes method (AMM).
2 Introduction to Component Modes Synthesis

When a finite element (FE) modeling technique is applied to a given substructure, equations of motion, written in terms of generalized coordinates, have the following matrix form:

\[
[M] \{ \ddot{u} \} + [D] \{ \dot{u} \} + [K] \{ u \} = \{ F \}
\]

(1)

It is generally assumed that the existence of damping does not cause coupling of the undamped natural modes of vibration [23, 38]. Therefore, the following undamped equation can be used in order to determine the substructure’s natural modes:

\[
[M] \{ \ddot{u} \} + [K] \{ u \} = \{ F \}
\]

(2)

The aim is to obtain suitable models for control theory application from Eqn. (2). This implies a formulation of the equations of motion which allows to establish the correct relation between applied forces and accelerations to the substructure under study in a linearized manner. In this work the re-formulation of the equations of motion is accomplished through component-mode transformation [23, 22, 25]. This method allows to separate substructure displacement sources into three categories: rigid-body displacements, redundant boundary displacements and natural vibration displacements.

As it can be seen in Fig. 1, the most general type of substructure presents three displacement categories [23, 22]: rigid displacements (Fig. 1a), redundant boundary displacements (Fig. 1b) and natural vibration displacements (Fig. 1c). If the substructure is not constrained, six independent rigid-body displacements modes exist, corresponding to three translations and three rotations with respect to a set of fixed orthogonal coordinate axes (the set \( \mathcal{R} = \{ r_i \} \)). The modes produced in this way are called rigid-body modes. Fewer than six rigid modes may exist if the substructure is partially or totally constrained. The constraint system is statically indeterminate with the redundant constraints (denoted by the set \( \mathcal{C} = \{ c_i \} \)). These constraints are the cause of the attachment to other substructures of the system, and they produce the called constraint modes. Finally, the displacements of other points of the structure relative to the constraints are given by a set of independent modes in which all constraints are fixed, called fixed-constraint natural modes of vibration of the structure (set \( \mathcal{I} = \{ i_i \} \)). Therefore, an arbitrary displacement of the constraints can be divided into rigid-body, constrained and fixed-constraint displacements. Generally speaking, the displacement of any point \( P(x,y,z) \) is given by the the superposition of these three displacements:

\[
\vec{u}(x,y,z) = \vec{u}_R(x,y,z) + \vec{u}_C(x,y,z) + \vec{u}_N(x,y,z)
\]

(3)

When the equations of motion are obtained with FE analysis, the substructure is discretized so that the displacements are defined at only a set of points. In this case the displacement at each point can be written as a component of a column vector, and Eqn. (3) becomes:

\[
\{ u \} = \{ u_r \} + \{ u_c \} + \{ u_n \}
\]

(4)
The number of coordinates in these sets are \( N_r \) for rigid-body modes, \( N_c \) for redundant constraint modes and \( N_n \) for natural vibration modes, respectively, with \( N = N_r + N_c + N_n \). In component-mode synthesis, each of these displacements is expressed as a superposition of discretized mode functions in the form of modal matrices \( \tilde{\phi} \) and a set of generalized coordinates \( \eta \). Thus, the term \( \tilde{\phi}_{ij} \) represents the displacement at point \( i \) in the \( j \)th mode. Consequently the three types of displacements take the following matrix form:

\[
\begin{align*}
\{u_r\} &= [\tilde{\phi}^r] \{\eta_r\} \\
\{u_c\} &= [\tilde{\phi}^c] \{\eta_c\} \\
\{u_n\} &= [\tilde{\phi}^n] \{\eta_n\} 
\end{align*}
\]  

Substituting Eqn. (5) into Eqn. (4) the total displacement may be written as

\[
\{\bar{u}\} = [\tilde{\phi}] \{\eta\}
\]  

where the complete transformation matrix reads as follows:

\[
[\tilde{\phi}] = [\tilde{\phi}^r \mid \tilde{\phi}^c \mid \tilde{\phi}^n]
\]  

The computation of submatrices in Eqn. (7) is explained in Appendix A. As a consequence of classifying the modes in three categories, namely, rigid-body modes, constraint modes, and normal modes, the Eqn. (2) can be partitioned as follows:

\[
\begin{bmatrix} M_{nn} & M_{nc} & M_{nr} \\ M_{cn} & M_{cc} & M_{cr} \\ M_{rn} & M_{rc} & M_{rr} \end{bmatrix} \begin{bmatrix} \bar{u}_n \\ \bar{u}_c \\ \bar{u}_r \end{bmatrix} + \begin{bmatrix} K_{nn} & K_{nc} & K_{nr} \\ K_{cn} & K_{cc} & K_{cr} \\ K_{rn} & K_{rc} & K_{rr} \end{bmatrix} \begin{bmatrix} \eta_n \\ \eta_c \\ \eta_r \end{bmatrix} = \begin{bmatrix} F_n \\ F_c + \bar{F}_c \\ F_r + \bar{F}_r \end{bmatrix}
\]  

where \( [M], [K], \{u\} \) and \( \{F\} \) are the substructure’s mass matrix, stiffness matrix, displacement vector and vector of externally applied forces, respectively. The “tilde” load term denotes the force resulting from the connection to adjacent structures at the boundary points [22]. The following partitioned form of Eqn. (1) is useful in the derivation of component modes:

Applying the modal transformation given in Eqn. (56) in Appendix A and pre-multiplying by \([\tilde{\phi}^T]\), and considering that neither interior forces nor external forces apply \((F_n = F_c = F_r = 0)\) Eqn. (8) yields:

\[
\begin{bmatrix} \hat{M}_{nn} & \hat{M}_{nc} & \hat{M}_{nr} \\ \hat{M}_{cn} & \hat{M}_{cc} & \hat{M}_{cr} \\ \hat{M}_{rn} & \hat{M}_{rc} & \hat{M}_{rr} \end{bmatrix} \begin{bmatrix} \hat{\eta}_n \\ \hat{\eta}_c \\ \hat{\eta}_r \end{bmatrix} + \begin{bmatrix} \hat{K}_{nn} & 0 & 0 \\ 0 & \hat{K}_{cc} & 0 \\ 0 & 0 & \hat{K}_{rr} \end{bmatrix} \begin{bmatrix} \eta_n \\ \eta_c \\ \eta_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \hat{F}_c \\ F_r + \hat{\phi}_c F_c \end{bmatrix}
\]  

Equation (9) is the partitioned transformed form of the equations of motion. It should be noted that for the transformed stiffness matrix several submatrices are null matrices. The submatrix \([\hat{K}_{rr}]\) is null since the work done by a self-equilibrating force system on a rigid-body displacement is zero [23]. The same occurs to the submatrix \([\hat{K}_{cn}]\) since the work done by the constraint forces on a normal mode displacement is zero because in normal mode the constraints are fixed. In the same way, submatrix \([\hat{K}_{cr}]\) is a null matrix.

In consequence of the foregoing results, the partitioned transformed equation of motion takes on a simpler form:

\[
\begin{bmatrix} \hat{M}_{nn} & \hat{M}_{nc} & \hat{M}_{nr} \\ \hat{M}_{cn} & \hat{M}_{cc} & \hat{M}_{cr} \\ \hat{M}_{rn} & \hat{M}_{rc} & \hat{M}_{rr} \end{bmatrix} \begin{bmatrix} \hat{\eta}_n \\ \hat{\eta}_c \\ \hat{\eta}_r \end{bmatrix} + \begin{bmatrix} \hat{K}_{nn} & 0 & 0 \\ 0 & \hat{K}_{cc} & 0 \\ 0 & 0 & \hat{K}_{rr} \end{bmatrix} \begin{bmatrix} \eta_n \\ \eta_c \\ \eta_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \hat{F}_c \\ \hat{F}_r + \hat{\phi}_c \hat{F}_c \end{bmatrix}
\]  

Equation (10) presents then submatrices which are more attractive for modeling purposes. Physical interpretations can be extrated from several submatrices. The square submatrix \([\hat{K}_{nn}]\) is a diagonal matrix containing the fixed-constraint natural vibration modes, and related with \([\hat{M}_{nn}]\) by the relationship of Eqn. (57) in Appendix A. The square submatrix \([\hat{K}_{cc}]\) is the stiffness matrix associated with the redundant constraints, and its order is equal to the number of redundant constraints. The square matrix \([\hat{M}_{rr}]\) is the rigid body matrix; i.e, the mass matrix if the substructure is considered as rigid. It contains
the whole mass of the system, gravity center position and rotatory inertia with respect to the rigid body boundaries. The submatrices \([\hat{M}_{mn}]\) and \([\hat{M}_{nr}]\) are the modal participation matrices of the natural modes and constraint boundaries on the rigid-body motion; i.e., how the natural modes and constraint boundaries affect the rigid dynamics.

If damping is taken into account, the damping matrix \([\hat{D}]\) may be partitioned in the same way as the mass and stiffness matrices:

\[
[\hat{D}] = \begin{bmatrix}
\hat{D}_{nn} & \hat{D}_{nc} & \hat{D}_{nr} \\
\hat{D}_{cn} & \hat{D}_{cc} & \hat{D}_{cr} \\
\hat{D}_{rn} & \hat{D}_{rc} & \hat{D}_{rr}
\end{bmatrix}
\] (11)

In general, all of the submatrices are not null as in the case of the mass matrix. However, if all damping forces are internal, then rigid body motions are not damped and in this case the third row and the third column of Eqn. (11) are null matrices [23]. In this case, Eqn. (10) is written with viscous damping as:

\[
\begin{bmatrix}
\hat{M}_{nn} & \hat{M}_{nc} & \hat{M}_{nr} \\
\hat{M}_{cn} & \hat{M}_{cc} & \hat{M}_{cr} \\
\hat{M}_{rn} & \hat{M}_{rc} & \hat{M}_{rr}
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_n \\
\ddot{\eta}_c \\
\ddot{\eta}_r
\end{bmatrix}
+ \begin{bmatrix}
\hat{D}_{mn} & \hat{D}_{mc} & \hat{D}_{mr} \\
\hat{D}_{cn} & \hat{D}_{cc} & \hat{D}_{cr} \\
\hat{D}_{rn} & \hat{D}_{rc} & \hat{D}_{rr}
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_n \\
\dot{\eta}_c \\
\dot{\eta}_r
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\ddot{\eta}_n \\
\ddot{\eta}_c \\
\ddot{\eta}_r
\end{bmatrix}
+ \begin{bmatrix}
\hat{K}_{nn} & 0 & 0 \\
0 & \hat{K}_{cc} & 0 \\
0 & 0 & \hat{K}_{rr}
\end{bmatrix}
\begin{bmatrix}
\eta_n \\
\eta_c \\
\eta_r
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\phi_r^T F_r
\end{bmatrix}
\]

\] (12)

3 Implementation of CMS for flexible structures linear modeling

The properties of the partitioned equations of motion in Eqn. (12) obtained in Section 2 can be used for simple, accurate and intuitive modeling of FMS. The advantages of this transformation are maximized when they are applied to one connection point or two connection points. More connection points are possible to model as well, but in the control domain this is not really advantageous. Beyond that, connection complexity obliges to manipulate the FE model itself and the problem becomes rather a structural problem than a control modeling problem.

Therefore, two uses of CMS equations are studied in this section. Firstly, the case of one connection point is explained. Secondly, the case of two connection points is addressed. Next, the modeling case of a revolute joint is described. Finally, the assembly technique with both models is explained, and some guidelines for parametrization with TITOP and superelement techniques are described.

3.1 One connection point

In this section modeling of a flexible substructure connected with another structure through one connection point is explained. As shown in Fig. 2 a flexible body (substructure) \(A\) linked to the parent structure \(P\) at the point \(P\). It is assumed that the only external loads applied to \(A\) are the interactions with \(P\) at point \(P\).

The problem is thus how to consider the coupling between \(P\) and \(A\). As explained in the review, several authors opted for overlapping stiffness and mass matrices at a matrix level (Young [27], Su [29] and [30]) or transferring boundary conditions with the transfer matrix method (Rong [8] and Mucino [10]). However, other approaches (Alazard [31, 33], Guy [4] and
Perez [34] took advantage of other particular transformations (cantilevered appendage and Craig-Bampton decomposition) and expressed the coupling as a transfer between loads and accelerations through the connection points; i.e., the overlapping between substructures is expressed as an acceleration-load transfer through the common boundaries. In this study, a further generalization is presented for the CMS method.

Therefore, the coupling transfer between \( \mathcal{P} \) and \( \mathcal{A} \) is expressed as an acceleration-load transfer through the connection point \( P \). The advantage of Eqn. (12) is that it offers a straightforward means of casting the FE Model of substructure \( \mathcal{A} \) in the state-space representation using as inputs/outputs accelerations and loads through the boundaries. This is possible thanks to the decoupling of the stiffness matrix when performing CMS transformation. In the case of one connection point, there are no redundant constraint displacements besides the rigid-body displacements. This implies that the rigid body displacements, sharing the rigid-body motions of the ensemble. As there are not redundant constraint displacements, second row and second column of Eqn. (12) can be removed leading to:

\[
\begin{bmatrix}
\dot{M}_{nn} & \dot{M}_{nr} \\
\dot{M}_{rn} & \dot{M}_{rr}
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_n \\
\dot{\eta}_r
\end{bmatrix}
+ 
\begin{bmatrix}
\dot{D}_{nn} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\eta}_n \\
\dot{\eta}_r
\end{bmatrix}
+ 
\begin{bmatrix}
\dot{K}_{nn} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\eta_n \\
\eta_r
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
F_r
\end{bmatrix}
\tag{13}
\]

The coupling is established as an exchange acceleration-load through the connection point:

\[
\{ F_{\mathcal{A}/\mathcal{P},P} \} = \big[ G_p^{\mathcal{A}}(s) \big] \{ \ddot{u}_P \} 
\tag{14}
\]

where \( F_{\mathcal{A}/\mathcal{P},P} \) is the load transmitted to the structure \( \mathcal{P} \) by the appendage \( \mathcal{A} \), \( G_p^{\mathcal{A}}(s) \) is the linear model of the appendage \( \mathcal{A} \) when connected at point \( P \), and \( \{ \ddot{u}_P \} \) the acceleration of the displacements at point \( P \). In the 3D case, where 6 degrees of freedom are needed to describe rigid body motion, \( G_p^{\mathcal{A}}(s) \) is a \( 6 \times 6 \) transfer matrix (i.e. \( r = 6 \)). It is trivial that the loads experienced by \( \mathcal{A} \) due to adjacent connections, \( F_r \), are in the opposite direction of the loads experienced by \( \mathcal{P} \), \( F_{\mathcal{A}/\mathcal{P},P} \). Identifying terms with Eqn. (13):

\[
F_r = -F_{\mathcal{A}/\mathcal{P},P}; \quad \ddot{\eta}_r = \ddot{u}_P; \quad \dot{M}_{nn} = L_p^T; \quad \dot{M}_{rr} = J_p^3
\tag{15}
\]

In the case of one connection point, normalized rigid-body accelerations are equal to the acceleration at point \( P \). The matrix \( L_p \) is the modal participation matrix of natural modes at point \( P \); i.e., it expresses how the motion of \( P \) is affected by the natural modes of vibration and viceversa. The square matrix \( J_p^3 \) is the direct dynamic model, at point \( P \), of the substructure \( \mathcal{A} \) assumed rigid [33] takes the following form for \( r = 6 \):

\[
J_p^3 = \tau_{AP}^T \begin{bmatrix}
m^3I_3 & 0_3 \\
0_3 & I_3^A
\end{bmatrix} \tau_{AP}
\tag{16}
\]

where \( \tau_{AP} \) is the kinematic model between the mass center of substructure \( \mathcal{A} \), \( A \), and the connection point \( P \), written as:

\[
\tau_{AP} = \begin{bmatrix}
I_3 & (\times AP) \\
0_3 & I_3
\end{bmatrix}
\tag{17}
\]

with \((\times AP)\) being the skew-symmetric matrix associated to the vector \( AP \). Considering that the natural vibration modes are normalized with respect to the mass matrix, the submatrix \( [M_{mn}] \) becomes the identity matrix, \([K_{nn}]\) is a diagonal matrix containing the natural modes (fixed-constraint natural modes, \( \omega_n^2 \)) and \([D_{mn}]\) a diagonal matrix expressed with a damping ratio \( \xi_n \). Consequently, the linear model of the appendage \([G_p^{\mathcal{A}}(s)]\) reads:
3.2 Two connection points

interactions with the parent matrix to the next substructure in the chain (the fixed-boundary natural modes (the modes obtained when clamping the appendage at point P to each structure, is explained. As shown in Fig. 4, this time the flexible body (substructure) at the opposite end, since there is no information about its displacement. In Section 3.2 an expansion of rigid-body displacement of appendage A is linked to the parent structure Q at the point P and to a child substructure Q at the point Q. It is assumed that the only external loads applied to A are the interactions with Q at point Q and with Q at point Q.

As seen in Section 3.1 the main problem is how to consider the coupling between P, A and Q. Again, the overlapping between substructures is expressed as an acceleration-load transfer through the common boundaries. A generalization of the double-port approach, proposed by Alazard [33] and Perez [34], is presented in this study for the general CMS transformation. In this case, both points, P and Q, suffer an acceleration-load transfer, in such a way that the acceleration is transferred to the next substructure in the chain (Q in this case) and the load is transmitted to the previous substructure in the chain (the parent P structure). Therefore the objective is to build a double-port model of the substructure A such that:

\[
\begin{bmatrix}
\dot{u}_Q \\
\dot{F}_{A/P,P}
\end{bmatrix} = \left[ G^A_{P,Q}(s) \right] \begin{bmatrix}
F_{A/P,P} \\
\dot{u}_P
\end{bmatrix}
\] (19)
As there are only two connection points, the assignment of degrees of freedom is simple: rigid-body displacements to connection point \( P \) and the redundant constraint displacements to connection point \( Q \). Thus the accelerations read:

\[
\ddot{u}_P = \ddot{\eta}_r \\
\ddot{u}_Q = \ddot{\eta}_c + \phi_r \ddot{\eta}_r
\]  

(20)

where \( [\phi_r] \) is described in Appendix A. Equation (20) implies that the rigid motion is supported by point \( P \) and the constrained motion of connection point \( Q \) is a result of the rigid body motion in \( P \) transported to point \( Q \) \( (\phi_r \ddot{\eta}_r) \) plus the constrained motion due to flexibility \( (\ddot{\eta}_c) \). In the same way, loads are received and transmitted by appendage \( A \) with the following directions:

\[
F_{A/P,P} = -F_r \\
F_{Q/A,Q} = F_c
\]  

(21)

Using the relations given in Eqn. (20) and (21) in combination with Eqn. (12), a state-space representation can be obtained for the substructure \( A \):

\[
G_{P,Q}'(s) \begin{bmatrix} \ddot{\eta}_r \\ \dot{\eta}_c \\ \ddot{\eta}_r \\ \dot{\eta}_c \end{bmatrix} = A \begin{bmatrix} \eta_r \\ \dot{\eta}_c \\ \ddot{\eta}_r \\ \dot{\eta}_c \end{bmatrix} + B \begin{bmatrix} \dot{F}_{Q/A,Q} \\ \ddot{u}_P \end{bmatrix}
\]

\[
G_{P,Q}'(s) \begin{bmatrix} \dot{\eta}_c \\ \ddot{u}_Q \end{bmatrix} = C \begin{bmatrix} \eta_r \\ \dot{\eta}_c \\ \ddot{\eta}_r \\ \dot{\eta}_c \end{bmatrix} + (D + D_\delta) \begin{bmatrix} \dot{F}_{Q/A,Q} \\ \ddot{u}_P \end{bmatrix}
\]  

(22)

where \( A, B, C, D \) and \( D_\delta \) are the short hand notation of the following state-space matrices:

\[
A = \begin{bmatrix} 0_{n+c} & I_{n+c} \\ -\hat{M}_Q^{-1}\hat{K}_Q & -\hat{M}_Q^{-1}\hat{D}_Q \end{bmatrix}
\]  

(23)

\[
B = \begin{bmatrix} 0_{n+c+c+r} \\ \hat{M}_Q^{-1} \begin{bmatrix} 0_{nc} & -\hat{M}_{nr} \\ I_{cc} & -\hat{M}_{cr} \end{bmatrix} \end{bmatrix}
\]  

(24)

\[
C = \begin{bmatrix} [0_{cn} I_{cc}] \begin{bmatrix} -\hat{M}_Q^{-1}\hat{K}_Q & -\hat{M}_Q^{-1}\hat{D}_Q \end{bmatrix} \\ [\hat{M}_rn \hat{M}_{rc}] \begin{bmatrix} -\hat{M}_Q^{-1}\hat{K}_Q & -\hat{M}_Q^{-1}\hat{D}_Q \end{bmatrix} \end{bmatrix}
\]  

(25)

\[
D = \begin{bmatrix} [0_{cn} I_{cc}] \hat{M}_Q^{-1} \begin{bmatrix} 0_{nc} & -\hat{M}_{nr} \\ I_{cc} & -\hat{M}_{cr} \end{bmatrix} \\ -[\hat{M}_rn \hat{M}_{rc}] \hat{M}_Q^{-1} \begin{bmatrix} 0_{nc} & -\hat{M}_{nr} \\ I_{cc} & -\hat{M}_{cr} \end{bmatrix} \end{bmatrix}
\]  

(26)
Another advantage of this kind of approach is its versatility. By setting inputs to 0, \([G_{PQ}^A(s)]\) represents the clamped (at \(P\))-free (at \(Q\)) model of \(A\). In the same way, \([G_{PQ}^A(s)]^{-1}\) represents the free (at \(P\)) - clamped (at \(Q\)) model of \(A\). Both “channels” can be inverted and thus the models

\[
\begin{bmatrix}
G_{PQ}^A(s)
\end{bmatrix}^{-1} \begin{bmatrix}
F_{Q/A,Q} \\
F_{A/P,P}
\end{bmatrix} = \begin{bmatrix}
G_{PQ}^A(s)
\end{bmatrix}^{-1} \begin{bmatrix}
\ddot{u}_A \\
\ddot{u}_P
\end{bmatrix}
\]

(28)

\[
\begin{bmatrix}
G_{PQ}^A(s)
\end{bmatrix}^{-1} \begin{bmatrix}
\ddot{u}_P \\
\ddot{u}_Q
\end{bmatrix} = \begin{bmatrix}
G_{PQ}^A(s)
\end{bmatrix}^{-1} \begin{bmatrix}
F_{Q/A,Q} \\
F_{A/P,P}
\end{bmatrix}
\]

(29)
can be used to take into account boundary conditions at $P$ or $Q$. Indexes $u$ and $l$ are used to describe the “upper” channel and “lower” channel respectively. It should be noted that removing connection point $Q$ the same model as for the one connection point case is found.

3.3 Revolute joint

The double-port approach allows taking into account constraints at the level of the connection points by simply restricting or releasing degrees of freedom. This study shows that a revolute joint at the connection point $P$ between the bodies $\mathcal{A}$ and $\mathcal{Q}$, as depicted in Fig. 6, can be modeled as well for the two connection points.

Augmenting the double port model $G^3_{\mathcal{A}Q}(s)_{\mathcal{A}_0}$ of the body $\mathcal{A}$ projected in the frame $\mathcal{A}_0$: let $\{e_a\} = \{x_{e_a} y_{e_a} z_{e_a}\}_{\mathcal{A}_0}^T$ be a unit vector along the revolute joint axis, then:

$$\begin{bmatrix} \ddot{u}_Q \\ F_{\mathcal{A}Q/P,P} \\ t_{rj,P} \end{bmatrix}_{\mathcal{A}_0} = \begin{bmatrix} I_{r+c} \\ E_a \end{bmatrix}_{\mathcal{A}_0} G^3_{\mathcal{A}Q}(s)_{\mathcal{A}_0} \begin{bmatrix} I_{r+c} \\ E_a \end{bmatrix}_{\mathcal{A}_0} \begin{bmatrix} \ddot{u}_Q \\ \dot{u}_p \\ \dot{\alpha} \end{bmatrix}_{\mathcal{A}_0} \begin{bmatrix} F_{\mathcal{A}Q/\mathcal{A}Q} \\ \dot{F}_{\mathcal{A}Q/\mathcal{A}Q} \\ \dot{t}_{rj,P} \end{bmatrix}_{\mathcal{A}_0}$$

(30)

with the selection matrix:

$$E_a = \begin{bmatrix} 0_{1 \times (r+c/2)} & x_{e_a} & y_{e_a} & z_{e_a} \end{bmatrix}$$

(31)

where $H^3_{\mathcal{A}Q}(s)_{\mathcal{A}_0}$ is the double port model augmented with a 13th input: $\dot{\alpha}$, the angular acceleration inside the revolute joint and a 13th output: $t_{rj,P}$ the torque applied by an actuator located inside the revolute joint.

This augmentation allows to release the desired degrees of freedom (setting $t_{rj,P} = 0$) or to take into account the model $K(s)$ of a local mechanism inside an actuated revolute joint as it can be seen in Fig. 7. In this case, the system $H^3_{\mathcal{A}Q}(s)_{\mathcal{A}_0}$ has to be inverted between its 13th input and its 13th output, defining a new inversion operation: $[H^3_{\mathcal{A}Q}(s)]^{-1}_{\mathcal{A}_0}$, the operation corresponding to the inversion of the 13-th input output channel of $H^3_{\mathcal{A}Q}(s)_{\mathcal{A}_0}$ such that:

$$\begin{bmatrix} \ddot{u}_Q \\ F_{\mathcal{A}Q/P,P} \\ \dot{\alpha} \end{bmatrix}_{\mathcal{A}_0} = [H^3_{\mathcal{A}Q}(s)]^{-1}_{\mathcal{A}_0} \begin{bmatrix} \ddot{u}_Q \\ \dot{u}_p \\ t_{rj,P} \end{bmatrix}_{\mathcal{A}_0}$$

(32)

The effect of $K(s)$ on the boundary condition at point $P$ is strictly taken into account.

3.4 Modeling Flexible Multibody Systems

The state-space realizations found for FE models transformed with CMS decomposition and double-port approach serve as elemental bricks for building FMS with small deflections. Indeed, one-connection-point TITOP model can be used to
model flexible systems in star-like structures or to end chain-like structures. Two-connection-point TITOP model can be used to connect every type of chain-like structures between them, taking into account its flexibility.

For instance, the FMS shown in Fig. 8 can be modeled as different TITOP models interacting among them as depicted in Fig. 9. The flexible multibody spacecraft is composed of a rigid main body or hub in which other appendages are attached such as an antenna, masts and solar panels. For control purposes, it is useful to choose as inputs the loads applied to the hub, $F_G$ and as outputs the induced accelerations at the hub, $\ddot{u}_G$. These accelerations are transported to the connection point $P_i$ ($P_1$ for the antenna connection point, $P_2$ for one of the masts) through the kinematic model $\tau_{P_iG}$ [31], where they are transmitted to the TITOP models of the flexible appendages. Eventually, rotation matrices can be included in the diagram in order to change from the hub’s frame to the appendage’s frame. These models transmit what can be called “disturbance” loads at the level of the hub, thus taken into account their flexibility. A more illustrative example of FMS modeling is explained in Section 4.

Therefore, the TITOP model allows synthetic, simple and intuitive modeling schemes for control purposes. Given its simplicity and the easy access to some measurements and inputs such as external forces, accelerations in different parts of the FMS, this kind of modeling approach has been used by authors such as Alazard [39] and [40] for integrated control/structure design.

### 3.5 Parametrization

This section underlines another utility of the TITOP model. It can be used for methods of integrated structure/control design since variations of several parameters can be taken into account. For a structure with varying configuration or varying mass and stiffness properties, like some space structures, the TITOP modeling technique may be specially efficient since it is able to reflect those changes. That is the case for studies such in Alazard [39] and [40].

Physical parameters are accessible in the TITOP model through the rigid-body matrix, denoted as $[\hat{M}_r]$ or $[J_p^3]$ in Eqn. (17). In fact, total system mass or geometric parameters can be parametrized by accessing to this matrix. Natural modes can be parametrized by accessing to matrix $[\hat{K}_m]$ or $[I_n\omega_n]$ in the one connection point case. Matrix $[\phi_{cr}]$ reflects geometrical properties of the appendage since it transports the kinematics from point $P$ to $Q$, and its modifications can also be taken into account within the model, affecting its dynamics. Consequently, parameters variation inside the state-space representations in Eqn. (14) and (22) lead to an uncertain Linear Fractional Representation (LFR) of the substructure in which the $\Delta$ block...
contains the effects in the substructure dynamics of these variations, as depicted in Fig. 10. Such a formalism of parametric variation is commonly used for sensitivity analysis [41].

For elementary substructures like mast or boom, the analytical TITOP model proposed in Murali [42], recalled in Section 3.5.1 can be used to obtain a fully parametrized model with length, cross-section inertia, young modulus, etc.

### 3.5.1 Superelements

When a substructure linking two other substructures has a beam-like shape, a technique called “superelement” modeling can be used. This technique provides a full parametric representation for the length, section surface, section inertia and material properties of the beam-like structure. Main aspects of this method are recalled thereafter. The interested reader can refer to [42] for a complete description.

The superelement technique uses a FE approach that exploits the uniformity of the beam, using a polynomial function of higher order than the typical finite element for beams. Conventional elements have 3rd order while superelement is 5th order, leading to more accurate mode shapes for a single superelement than for two conventional elements sequence. With this polynomial approximation, the mass and stiffness matrices of the beam superelement in planar deflection read:

\[
\tilde{M} = \frac{\rho s l}{55440} \begin{bmatrix}
55440 & 27720l & 462l^2 & 27720 & -5544l & 462l^2 \\
27720l & 18480l^2 & 198l^3 & 19800l & -3432l^2 & 264l^3 \\
462l^2 & 198l^3 & 64l^4 & 181l^2 & -52l^3 & 51l^4 \\
27720l & 19800l & 181l^2 & 21720 & -3732l & 281l^2 \\
-5544l & -3432l^2 & -52l^3 & -3732l & 832l^2 & -69l^3 \\
462l^2 & 264l^3 & 5l^4 & 281l^2 & -69l^3 & 6l^4 
\end{bmatrix}
\]  

(33)
of the double-port model

\[ u_3(t) = \frac{T_{yP}(t)}{EI_x} \]

\[ u_4(t) = y_Q(t) \]

\[ u_5(t) = \theta_Q(t) \]

being the time-dependent kinematic vector of the superelement (see Fig. 11):

\[ \vec{u}(t) = \begin{bmatrix} y_P(t) \\ \theta_P(t) \\ T_{yP}(t)/(EI_x) \\ y_Q(t) - y_P(t) - l\theta_P(t) \\ \theta_Q(t) - \theta_P(t) \\ T_{yQ}(t)/(EI_x) \end{bmatrix} \]  

which is the same one as a standard beam element but adding the curvature of the deflected geometry, \( T_{yP}(t)/(EI_x) \), and the relative motion between the two beam tips.

It should be noted that vertical displacement, rotation, force and torque at points \( P \) and \( Q \), respectively \( y_P/y_Q \), \( \theta_P/\theta_Q \), \( f_P/f_Q \) and \( t_P/t_Q \) are the following projections in the deflection plane \( \pi(\vec{x}_a, \vec{y}_a) \):

\[
\begin{align*}
y_P &= \vec{y}_a \cdot \{q_P\} \\
\theta_P &= \vec{z}_a \cdot \{\theta_P\} \\
f_P &= \vec{y}_a \cdot \{f_{3x} \} \\
t_P &= \vec{z}_a \cdot \{t_{3x} \}
\end{align*}
\]  

(36)

Murali [42] uses Eqn. (33) and the expressions in Eqn. (36) to arrange the following state-space representation of the double-port model \( T_yR_z(s) \) of the beam restricted to bending in the plane \( \pi(\vec{x}_a, \vec{y}_a) \):

\[
\begin{bmatrix}
\dot{\vec{u}}_{(3,6)} \\
\vec{u}_{(3,6)}
\end{bmatrix}
= 
\begin{bmatrix}
A_{Ty,R_z} & B_{Ty,R_z} \\
C_{Ty,R_z} & D_{Ty,R_z}
\end{bmatrix}
\begin{bmatrix}
\{q_P\} \\
\{f_P\} \\
\{t_P\}
\end{bmatrix}
\]  

(37)

with:

\[
A_{Ty,R_z} = 
\begin{bmatrix}
0_{4 \times 4} & I_4 \\
-I_{(3,6,3,6)}^{-1} & 0_{4 \times 4}
\end{bmatrix}
\]  

(38)
are beam-like structures) and compared it with the AMM approach. Two additional input torques, rigid hub, and it is denoted by a right-handed triad \( \vec{a} \). Planar rotational maneuvers about the inertially fixed axis \( \vec{z} \) are shown in Fig. 12. The configuration parameters are provided in Table 1. Under normal operation, the spacecraft undergoes other models based on AMM such as the one developed in Junkins \[45\] and Turner \[46\].

Developing the equations of the hub-beam system using the AMM and Lagrangian principle. This section provides a generic approach for modeling such a system and which can be expanded for every kind of FMS. Results will be compared with a widely-used approach, the Assumed Modes Method (AMM). A rotating flexible spacecraft is actuated by three different torques. The main torque, tip, and results are compared with a widely-used approach, the Assumed Modes Method (AMM). A rotating flexible spacecraft is considered and results are compared with a widely-used approach, the Assumed Modes Method (AMM). A rotating flexible spacecraft is often modeled as a coupled rigid hub and flexible beam-like structures with tip masses at their ends. Rotating thin flexible beams with tip masses are a well-known modeling problem, studied by authors such as Choura \[43\] and Zhu \[44\], but always and presents a double-port model superelement \( S_{P,Q}(s) \) of the 6 d.o.f. beam is (in projection in the frame \( R_a = (P, x_a, y_a, z_a) \)):

\[
\begin{bmatrix}
\{ \ddot{u}_Q \}_{R_a} \\
\{ F_{R/P,P} \}_{R_a}
\end{bmatrix} =
\begin{bmatrix}
S_{P,Q}(s) \\
D_{P,Q}(s)
\end{bmatrix}
\begin{bmatrix}
\{ \ddot{F}_{Q,P} \}_{R_a} \\
\{ \ddot{u}_P \}_{R_a}
\end{bmatrix}
\]

where:

\[
S_{P,Q}(s)_{R_a} = T^T \begin{bmatrix}
T_i R_z(s) \\
T_i R_y(s) \\
R_z(s) \\
R_y(s)
\end{bmatrix} T
\]

with \( T \) being a permutation matrix of the inputs/outputs computed in Eqn. \[64\] described in Appendix B.

As it can be appreciated, the parametrization is more thoroughful since length, section area, cross inertia or density appear at all the levels of the system, having a more accurate influence on the dynamics.

## 4 TITOP Modeling of a Rotating Flexible Spacecraft

To demonstrate the validity of the TITOP modeling method for FMS, a maneuvering flexible spacecraft is considered and results are compared with a widely-used approach, the Assumed Modes Method (AMM). A rotating flexible spacecraft is often modeled as a coupled rigid hub and flexible beam-like structures with tip masses at their ends. Rotating thin flexible beams with tip masses are a well-known modeling problem, studied by authors such as Choura \[43\] and Zhu \[44\], but always and presents a double-port model superelement \( S_{P,Q}(s) \) of the 6 d.o.f. beam is (in projection in the frame \( R_a = (P, x_a, y_a, z_a) \)):

\[
\begin{bmatrix}
\{ \ddot{u}_Q \}_{R_a} \\
\{ F_{R/P,P} \}_{R_a}
\end{bmatrix} =
\begin{bmatrix}
S_{P,Q}(s) \\
D_{P,Q}(s)
\end{bmatrix}
\begin{bmatrix}
\{ \ddot{F}_{Q,P} \}_{R_a} \\
\{ \ddot{u}_P \}_{R_a}
\end{bmatrix}
\]

where:

\[
S_{P,Q}(s)_{R_a} = T^T \begin{bmatrix}
T_i R_z(s) \\
T_i R_y(s) \\
R_z(s) \\
R_y(s)
\end{bmatrix} T
\]

with \( T \) being a permutation matrix of the inputs/outputs computed in Eqn. \[64\] described in Appendix B.

As it can be appreciated, the parametrization is more thoroughful since length, section area, cross inertia or density appear at all the levels of the system, having a more accurate influence on the dynamics.

## 4.1 System Description

The system is composed of a rigid main hub with four identical cantilevered flexible appendages and tip masses as shown in Fig. 12. The configuration parameters are provided in Table 1. Under normal operation, the spacecraft undergoes planar rotational maneuvers about the inertially fixed axis \( \vec{z} \). The spacecraft body frame is attached to the mass center of the rigid hub, and it is denoted by a right-handed triad \( \vec{x}, \vec{y} \) and \( \vec{z} \). The rotation about the axis \( \vec{z} \) is denoted by the angle \( \theta \) and the translational deformation of each tip by \( w_{tip} \), with superscript \( i \) denoting the beam number.

The system is actuated by three different torques. The main torque, \( T_{hub} \) is provided by the main hub about the axis \( \vec{z} \). Two additional input torques, \( T_{tip,1} \) and \( T_{tip,2} \), are applied at the tip masses 1-3 and 2-4 respectively. These torques can be applied purposely for control reasons or can be the result of environment disturbances.

The purpose is to model this particular FMS using the TITOP method (superelement method applies as well since there are beam-like structures) and compared it with the AMM approach.
4.2 System Modeling
The modeling problem of a single axis rotating flexible spacecraft is addressed extensively using two modeling schemes: TITOP method and AMM method.

4.2.1 TITOP Approach
The TITOP approach needs two sets of data. The first one corresponds to structural data: rigid body matrices of the hub and tip masses, and a FE model for the beams. The second one corresponds to connection matrices: for each appendage, the kinetic transportation matrix $\tau_{GP}$ and the rotation frame matrix $R$ must be provided.

The rigid body matrices are straightforward for the planar case:

$$J_{Hub}^G = \begin{bmatrix} m_h & 0 & 0 \\ 0 & m_h & 0 \\ 0 & 0 & J_h \end{bmatrix} \quad J_{Tip}^Q = \begin{bmatrix} m_t & 0 & 0 \\ 0 & m_t & 0 \\ 0 & 0 & J_t \end{bmatrix}$$

The kinematic models between points $G$ and $P_i$, being $i$ the appendage number $i$, are in the planar case:
\( \tau_{P_i G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \)

\( \tau_{P_2 G} = \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

\( \tau_{P_3 G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{bmatrix} \)

\( \tau_{P_4 G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

(45)

and the rotation matrices can be written as follows:

\[
R_i = \begin{bmatrix}
\cos \beta_i & -\sin \beta_i & 0 \\
\sin \beta_i & \cos \beta_i & 0 \\
0 & 0 & 1
\end{bmatrix}_{\text{App} \rightarrow \text{Hub}}
\]

(46)

where \( \beta_i \) is the angle of the \( i \)th appendage with \( \vec{x} \). Beam’s FE model is obtained with classical FE discretization, and mass and stiffness matrices are transformed as explained in Section 3.2 to get the double-port TITOP model of the beam.

As the tip mass is considered as rigid, there is no need of applying CMS to this substructure. Eqn. (44) is used for such a purpose.

The assembly for each appendage is the one shown in Fig. 13. Accelerations at the hub are transmitted to the attachment point \( P_i \) through the kinematic model \( \tau_{P_i G} \), and then changed to the appendage frame through \( R_i \). The acceleration of the hub, together with the load exerted by the tip mass at the opposite end, are the inputs of the beam TITOP model, which delivers the acceleration transmitted to the tip mass and the load transmitted to the hub, which has to be transported to the hub and change its frame.

Being the TITOP approach a generic approximation, all the planar degrees of freedom are taken into account. Thus, it should be noted that the kinematic vector transmitted through the appendages is a vector containing the following accelerations:

\[
\ddot{u}_G = \begin{bmatrix}
\ddot{q}_G \\
\ddot{\theta}_G
\end{bmatrix}_{\text{Hub}} ;
\ddot{u}_{P_i} = \begin{bmatrix}
\ddot{q}_{P_i} \\
\ddot{\theta}_{P_i}
\end{bmatrix}_{\text{App}}
\]

(47)

and the loads vector:

\[
F_{\text{ext}} = \begin{bmatrix}
f_{xG} \\
f_{yG} \\
t_{\text{hub}}
\end{bmatrix} ;
F_{A_i/h,P_i} = \begin{bmatrix}
f_{xP_i} \\
f_{yP_i} \\
t_z
\end{bmatrix}
\]

(48)

The same process is performed to the four appendages, obtaining the final assembly shown in Fig. 14. It can be observed that the resulting system has the applied torques as inputs when the following inputs are assigned the following values: \( f_{\text{ext}} = f_{xG} = f_{yG} = 0, t_{\text{ext}} = t_{\text{hub}} \), and the hub accelerations as outputs \( \theta_G = \theta \). Tip acceleration can be observed through the signal transmitted from the beam to the tip.

4.2.2 AMM Approach

For the comparison objective the classical assumed modes (AM) solution is utilized. Although AMM can be applied in many different ways, the most general case is deriving the hub-beam-tip equations.

The AMM assumes a decoupled spatial and time deformation approximated by the series:

\[
w(x,t) = \sum_{i=1}^{\text{sum}} \phi_i(x) u_i(t)
\]

(49)
where $\phi_i(x)$ denotes the assumed mode shape, $u_i(t)$ denotes the $i$-th generalized coordinate, $asm$ denotes the number of terms retained in the approximation and $x$ the distance from the considered point in the beam to the attachment point.

Then, the kinetic and potential energy of the spacecraft, containing space and time partial derivatives of $w(x,t)$, are derived using the approximation in Eqn. (49) and performing the integration with respect to $x$, writing the kinetic energy and potential energy in the quadratic forms:

$$T(t) = \frac{1}{2} \sum_{i=1}^{asm} \sum_{j=1}^{asm} M_{ij} \dot{u}_i(t) \dot{u}_j(t) = \frac{1}{2} \{ \dot{u}(t) \}^T [M] \{ \dot{u}(t) \}$$

(50)

$$V(t) = \frac{1}{2} \sum_{i=1}^{asm} \sum_{j=1}^{asm} K_{ij} u_i(t) u_j(t) = \frac{1}{2} \{ u(t) \}^T [K] \{ u(t) \}$$

(51)

where $M_{ij}$ denotes the $(i,j)$-th element of the symmetric mass matrix $[M]$ (respectively for the stiffness matrix $[K]$).

The equations of motion follow on introducing $T$ and $V$ into Lagrange’s equations:

$$\frac{d}{dt} \left( \frac{\delta T}{\delta u_r} \right) - \frac{\delta T}{\delta u_r} + \frac{\delta V}{\delta u_r} = Q_r, \hspace{1em} r = 1, \ldots, asm$$

(52)

where $Q_r$ denotes the generalized non-conservative forces, the applied torques. The following equations of motion are obtained:
which written in matrix compact form gives:

$$\sum_{j=1}^{asm} M_{rj} \ddot{u}_j(t) + \sum_{j=1}^{asm} K_{rj} u_j(t) = Q_r, \ r = 1, \ldots, asm \quad (53)$$

The analytical formulation of mass and stiffness submatrices in Eqn. (54) for the rotatory spacecraft can be found in Junkins [45], which have been developed with the following admissible functions satisfying the boundary conditions for clamped-free appendages:

$$\phi_i(x) = 1 - \cos \left( \frac{j \pi x}{L} \right) + \frac{1}{2} (-1)^{j+1} \left( \frac{j \pi x}{L} \right)^2 \quad (55)$$

Equation (54) provides thus the desired equations of motion in which the time-varying amplitudes are generalized coordinates. Given the instantaneous vector \( \{u(t)\} \), the instantaneous deformation of the structure is approximated by the assumed modes expansion.

### 4.3 Compared Numerical Results

A comparison between the TITOP modeling and the numerical assumed modes method, assuming 13 modes, is presented in this Section. In Fig. 15 (frequency response) one of the main interest of control modeling is presented: the effects of flexible appendages on the main hub motion, \( \ddot{\theta}_{hub} \). It can be seen that the results obtained by both methods are in complete agreement with each other both for frequency response and step response. The superelement modeling technique has been compared as well, giving excellent results. Hereinafter superelement is not compared since it provides exactly the same results as the TITOP model.

Figure 16 shows that the dynamical response of the tip masses for both methods is in perfect agreement. The TITOP model frequency response in Fig. 16 fits perfectly the lowest frequency resonances and then shifts with respect to the AMM from the first antiresonance. It should be noted that the measured tip acceleration is composed of the hub acceleration and the tip displacement around the neutral position.

In Fig. 17 the frequency response of the hub acceleration rate due to tip torque input, \( t_{tip,1} \), is compared for both methods. Curves are perfectly in agreement as well, with a slight shift with frequency increase, mainly due to the choice of mode shape functions.
A comparison for all methods (AMM, TITOP and superelement) among the first six flexible modes is shown in Table 2. The computed frequencies converge accurately for the superelement and 5-nodes TITOP solution, whereas the AMM solutions are not accurate for modes 4-6. The Relative Mean Square (RMS) error of these values is shown in Fig. 18, showing that for the same number of degrees of freedom the TITOP modeling technique is slightly more accurate. Therefore, TITOP modeling is able to provide accurate models with less degrees of freedom and achieving more accurate results than the AMM.

4.4 System Parametrization

The TITOP modeling technique allows taking into account the variations of certain structural parameters inside the model, since they can be easily found inside the state-space representation of the substructures. In this Section parametric variations are performed to the rotatory spacecraft, including variations on beams’ lengths and tip masses.

Being the appendage a beam, its length variations are introduced through the superelement technique (Section 3.5.1), which is more accurate. Tip mass variations are introduced through the rigid-body matrix of the tip mass, Eqn. (44). After assembly, the system appears like a model as the one shown in Fig. 10 with variations in tip mass and beam’s length included in the Δ-block.

Using this approach, interesting features can be discovered which can help for future integrated control/structure spacecraft. As it can be seen in Fig. 19 the first natural frequency of the system decreases either increasing length or increasing tip mass in all appendages in the same manner. This is because an increment in beam’s length decreases beam’s stiffness, reducing the needed effort to bend the beam, and thus the frequency needed to excite this mode is lower. In the same way, when incrementing the mass at the tip the inertia at the tip is higher, increasing oscillations amplitude at the tip for equal effort, and thus lowering the frequency needed to excite the first mode.
Table 2: Table showing the natural frequencies ($\omega$, rad/s$^2$) corresponding to the rigid body mode and the first six flexible modes for each modeling method. $N_1$ denotes the number of assumed modes retained in the approximation and $N_2$ the number of finite elements used for the beam model in the TITOP method. Superelement is denoted as SP and the exact frequency is obtained with a FE model of 50 nodes.

| Mode | AMM | TITOP | SP | Exact |
|------|-----|-------|----|-------|
| No   | $N_1 = 5$ | $N_1 = 10$ | $N_1 = 15$ | $N_2 = 5$ | $N_2 = 10$ | $N_2 = 15$ | - | 50 |
| 0    | 0   | 0     | 0  | 0     | 0     | 0     | 0 | 0 |
| 1    | 4.3731 | 4.3723 | 4.3722 | 4.3722 | 4.3722 | 4.3722 | 4.3722 | 4.3722 |
| 2    | 7.9084 | 7.9070 | 7.9067 | 7.9066 | 7.9066 | 7.9066 | 7.9066 | 7.9066 |
| 3    | 51.7234 | 51.4518 | 51.4259 | 51.4286 | 51.3999 | 51.3989 | 51.4538 | 51.3987 |
| 4    | 53.0829 | 52.8066 | 52.7797 | 52.7819 | 52.7525 | 52.7515 | 52.8106 | 52.7513 |
| 5    | 160.2661 | 157.5591 | 156.7351 | 156.4701 | 155.7516 | 155.7257 | 157.7792 | 155.7203 |
| 6    | 161.0962 | 158.3683 | 157.5382 | 157.2609 | 156.5407 | 156.5147 | 158.6570 | 156.5094 |

Nevertheless, the most interesting remark can be done when only one appendage varies its beam’s length and tip mass. As it can be appreciated in Fig. 20, the first frequency mode is much lower than the one where all appendages varied their length and tip mass simultaneously. This happens even when tip mass or beam’s length are reduced, which theoretically should raise the first mode frequency. The first mode frequency of the whole system lowers because there is an asymmetric distribution of the appendages. Since one appendage has varied its dimensions, no matter in what direction, efforts are no longer compensated by the opposite appendage, and then it is easier to destabilize the spacecraft.

As it has been demonstrated, parametrization can be easily taken into account with the TITOP method. AMM approach made by Junkins [45] for this problem considers several simplifications for the model, such as symmetric displacements between appendages. If such kind of variations were done within AMM approach, the equations, integrals and other assumptions would have been changed, reinitialising the modeling process. On the contrary, the TITOP model does not need re-formulating the problem since parameters are easily found inside the state-space structures, simplifying the modeling process.

5 Conclusions

In this study, a new flexible multibody system linear modeling approach more appropriate for control purposes is proposed, the TITOP model. The method is based on many aspects of component-mode synthesis, which has been a widely accepted tool for the analysis of complex structures. Connections among elastic substructures are established using the double-port approach, which uses exchanges of accelerations-loads at the connection points to express the overlapping
among the components.

Chain-like and/or star-structures can be handled by this technique using an intuitive assembly process which eases the access to certain measurements needed for control purposes, such as accelerations or applied loads at the connection points. In addition, several parametric variations, such as changes in mass or geometry, can be taken into account for effective integrated control/structure design.

The modeling and design variations of a rotatory flexible spacecraft demonstrate the feasibility, accuracy and effectiveness of the approach compared with other accepted methods. Dynamic responses, both in the frequency domain and time domain, are in complete agreement with the largely accepted Assumed Modes Method. Continued research in extending the proposed method will be in the directions of examining alternative structure/control configurations, performing integrated structure/control design with structured $H_\infty$ techniques and developing control approaches for complex space structures.

Acknowledgements
The research efforts were supported by a grant from the AOCS Department of CNES and from the Flight Dynamics and Control System Department of ONERA Toulouse.

References
[1] Wasfy, T., and Noor, A., 2003. “Computational strategies for flexible multibody systems”. ASME Journal of Applied Mechanics, 56(6), November.
[2] Schoen, M. P., Hoover, R. C., Chinvorarat, S., and Schoen, G. M., 2009. “System identification and robust controller design using genetic algorithms for flexible space structures”. ASME Journal of Dynamic Systems, Measurement, and Control, 131(3), p. 031003.

[3] Masoudi, R., and Mahzoon, M., 2011. “Maneuvering and vibrations control of a free-floating space robot with flexible arms”. ASME Journal of Dynamic Systems, Measurement, and Control, 133(5), p. 051001.

[4] Guy, N., Alazard, D., Cumer, C., and Charbonnel, C., 2014. “Dynamic Modeling and Analysis of Spacecraft With Variable Tilt of Flexible Appendages”. ASME Journal of Dynamic Systems, Measurement and Control, 136.

[5] Chatlatanagulchawi, W., and Meckl, P. H., 2009. “Model-independent control of a flexible-joint robot manipulator”. ASME Journal of Dynamic Systems, Measurement, and Control, 131(4), p. 041003.

[6] Boscariol, P., Gasparetto, A., and Zanotto, V., 2010. “Active position and vibration control of a flexible links mechanism using model-based predictive control”. ASME Journal of Dynamic Systems, Measurement, and Control, 132(1), p. 014506.

[7] Usoro, P., Nadira, R., and Mahil, S., 1986. “A finite element/lagrange approach to modeling lightweight flexible manipulators”. ASME Journal of Dynamic Systems, Measurement, and Control, 108(3), pp. 198–205.

[8] Rong, B., Rui, X., and Wang, G., 2011. “Modified Finite Element - Transfer Matrix Method for Eigenvalue Problem of Flexible Structures”. ASME Journal of Applied Mechanics, 78.

[9] Bokhari, S. F. R. A., Chughtai, S. S., and Werner, H., 2008. “A tool for converting fem models into representations suitable for control synthesis”. In World Congress, Vol. 17, pp. 6066–6071.

[10] Mucino, V. H., and Pavelic, V., 1981. “An Exact Condensation Procedure for Chain-Like Structures Using a Finite Element - Transfer Matrix Approach”. ASME Journal of Mechanical Design, 103, pp. 295–303.

[11] Leckie, F., and Pestel, E., 1960. “Transfer Matrix fundamentals”. International Journal of Mechanical Sciences, 2, pp. 137–167.

[12] Dokainish, M., 1972. “A new approach for plate vibrations: combination of transfer matrix and finite element technique”. Transactions of the American Society of Mechanical Engineers, Journal of Engineering for Industry, 94, pp. 526–530.

[13] McDaniel, T. J., and Eversole, K., 1977. “A combined Finite Element - Transfer Matrix structural analysis method”. Journal of Sound and Vibration, 51(2), pp. 157–169.

[14] Sankar, S., and Hoa, S. V., 1980. “An extended Transfer Matrix - Finite Element Method for free vibration of plates”. Journal of Sound and Vibration, 70(2), pp. 205–211.

[15] Ohga, M., Shigematsu, T., and Hara, T., 1983. “Structural Analysis by a combined Finite Element - Transfer Matrix Method”. Computers & Structures, 17(3), pp. 321–326.

[16] Yousuff, A., 1986. Controller Design via Structural Modeling by FETM. Tech. rep., Drexel University, Dep. of Mechanics, Philadelphia, Pennsylvania 19104.

[17] Tan, T., Yousuff, A., Bahar, L., and Konstantinidis, M., 1990. “A modified Finite Element - Transfer Matrix for control design of space structures”. Computers & Structures, 36(1), pp. 47–55.

[18] Rui, X., Wang, G., Lu, Y., and Yun, L., 2008. “Transfer matrix method for linear multibody system”. Multibody System Dynamics, 19, pp. 179–207.

[19] Krauss, R., and Book, W. J., 2010. “Transfer Matrix Modelling of Systems with Noncollocated Feedback”. ASME Journal of Dynamic Systems, Measurement and Control, 132.

[20] T.Ersal, Fathy, H., Rideout, D., Louca, L., and Stein, J., 2008. “A Review of Proper Modeling Techniques”. ASME Journal of Dynamic Systems, Measurement and Control, 130, November.

[21] Dhingra, A., and Lee, B., 1994. “Optimal placement of actuators in actively controlled structures”. Engineering Optimization, 23, pp. 99–118.

[22] Craig Jr, R. R., 2000. “A brief tutorial on substructure analysis and testing”. In Proceedings of the International Modal Analysis Conference-IMAC, Vol. 1, pp. 899–908.

[23] Hurty, W., 1965. “Dynamic Analysis of Structural Systems Using Component Modes”. AIAA Journal, 3(4), pp. 678–685.

[24] Craig, R., and Bampton, M., 1968. “Coupling of substructures for dynamic analysis”. AIAA Journal, 6(7).

[25] Hintz, R. M., 1975. “Analytical methods in component modal synthesis”. AIAA Journal, 13(8), pp. 1007–1016.

[26] MacNeal, R. H., 1971. “A hybrid method of component mode synthesis”. Computers & Structures, 1(4), pp. 581–601.

[27] Young, K. D., 1990. “Distribute Finite - Element Modeling and Control Approach for Large Flexible Structures”. Journal of Guidance, 13(4), July-August, pp. 703–713.

[28] Sunar, M., and Rao, S., 1992. “A substructure-decomposition method for the control design of large flexible structures”. AIAA Journal, 30(10), pp. 2573–2575.

[29] Su, T., Babuska, V., and Craig, R., 1995. “Substructure-Based Controller Design Method for Flexible Structures”. Journal of Guidance, Control and Dynamics, 18(5), September-October.

[30] Butler, S., and Dhingra, A., 2003. “Integrated structure and control design of actively controlled structures using substructure decomposition”. Journal of Engineering Optimization, 35(4), pp. 325–340.
[31] Alazard, D., Cumer, C., and Tantawi, K., 2008. “Linear dynamic modeling of spacecraft with various flexible appendages and on-board angular momentums”. In 7th ESA Guidance, Navigation and Control Conference.
[32] Manceaux-Cumer, C., and Chretien, J., 2001. “Minimal lift form of a spacecraft built up from two bodies”. In AIAA Guidance, Navigation and Control Conference., Montreal, Canada.
[33] Alazard, D., Perez, J. A., Loquen, T., and Cumer, C., 2015. “Two-Input Two-Output Port Model for Mechanical Systems”. In AIAA Science and Technology Forum and Exposition.
[34] Perez, J., Alazard, D., Loquen, T., Cumer, C., and Pittet, C., 2015. “Linear dynamic modeling of spacecraft with open-chain assembly of flexible bodies for ACS/Structure Co-design”. In EURO Guidance, Navigation and Control Conference.
[35] Perez, J., Pittet, C., Alazard, D., Loquen, T., and Cumer, C., 2015. “A Flexible Appendage Model for Use in Integrated Control/Structure spacecraft Design”. In IFAC Workshop on Advanced Control and Navigation for Autonomous Aerospace Vehicles.
[36] Young, J., 2000. Primer on Craig-Bampton CMS procedure method: an introduction to boundary node functions, base shake analyses, load transformation matrices, modal synthesis and much more. Tech. rep., NASA, October.
[37] Rixen, D. J., 2004. “A dual Craig-bampton method for dynamic substructuring”. Journal of Computational and Applied Mathematics, 168, pp. 383–391.
[38] Kraker, B., 1993. Generalization of the Craig-Bampton CMS procedure for general damping. Tech. rep., Technische Universität Eindhoven, Eindhoven.
[39] Alazard, D., Loquen, T., de Plinval, H., Cumer, C., Toglia, C., and Pavia, P., 2013. “Optimal Co-Design for Earth Observation Satellites with Flexible Appendages”. In AIAA Guidance, Navigation, and Control (GNC) Conference.
[40] Alazard, D., Loquen, T., de Plinval, H., and Cumer, C., 2013. “Avionics/Control co-design for large flexible space structures”. In AIAA Guidance, Navigation, and Control (GNC) Conference.
[41] Ferreres, G., 1999. A practical approach to robustness analysis with aeronautical applications. Springer Science & Business Media.
[42] Murali, H., Alazard, D., Massotti, L., Ankersen, F., and Toglia, C., 2015. “Mechanical-attitude controller co-design of large flexible space structures”. In EURO Guidance, Navigation and Control Conference.
[43] Choura, S., Jayasuriya, S., and Medick, M. A., 1991. “On the modeling, and open-loop control of a rotating thin flexible beam”. ASME Journal of Dynamic Systems, Measurement, and Control, 113(1), pp. 26–33.
[44] Zhu, W., and Mote, C., 1997. “Dynamic modeling and optimal control of rotating euler-bernoulli beams”. ASME Journal of Dynamic Systems, Measurement, and Control, 119(4), pp. 802–808.
[45] Jenkins, J. L., and Kim, Y., 1993. Introduction to dynamics and control of flexible structures. AIAA.
[46] Turner, J., and Chun, H., 1984. “Optimal Distributed Control of a Flexible Spacecraft During a Large-Angle Maneuver”. Journal of Guidance, 7(3), May - June.

Nomenclature

Generalized displacements in equations and figures are expressed as follows:
\{u\} column matrix of generalized displacements.
\{\dot{u}\} column matrix of generalized velocities.
\{\ddot{u}\} column matrix of generalized accelerations.

The vector \{u\} of generalized displacements is often decomposed as follows (identically for generalized velocities and generalized accelerations):
\{q\} generalized translations.
\{\theta\} generalized rotations.

Generalized coordinates can be projected in the following directions:
\hat{x} unit vector along x axis.
\hat{y} unit vector along y axis.
\hat{z} unit vector along z axis.

The equations of motion and figures may include one of the following notations:
\[K\] square matrix of system generalized stiffness.
\[D\] square matrix of system generalized damping coefficients.
\[M\] square matrix of system generalized masses.
\{F\} column matrix of generalized loads.
\[\phi^R\] matrix of rigid-body modes.
\[\phi^C\] matrix of constraint modes.
\[\phi^N\] matrix of natural modes.
\[F_c\] vector of externally applied forces at the constraint degrees of freedom.
\( \vec{F}_c \) vector of loads acting on a substructure as a result of its connection to adjacent substructure at the constraint degrees of freedom.

\( F_r \) vector of externally applied forces at the rigid-body degrees of freedom.

\( \vec{F}_r \) vector of loads acting on a substructure rigid body degrees of freedom as a result of its connection to adjacent substructure.

\( \{ \eta \} \) column matrix of normalized displacements.

\( N_r \) or subscript \( r \) dimension of rigid body modes.

\( N_c \) or subscript \( c \) dimension of redundant constraint modes.

\( N_n \) or subscript \( n \) dimension of fixed-constraint modes.

\( N \) dimension of total degrees of freedom.

The TITOP model uses the following notations:

\( F_{A/P} \) vector of loads exerted by a substructure \( A \) to a substructure \( P \) at point \( P \).

\( f_{A/P} \) force exerted by a substructure \( A \) to a substructure \( P \) at point \( P \).

\( t_{A/P} \) torque exerted by a substructure \( A \) to a substructure \( P \) at point \( P \).

\( [G^A_P(s)] \) one-connection-point TITOP model of substructure \( A \) at point \( P \).

\( [G^A_{P,Q}(s)] \) two-connection-point TITOP model of substructure \( A \) at points \( P \) and \( Q \).

\( [J^A_P] \) rigid body matrix or direct dynamic model of substructure \( A \) at point \( P \).

\( [L^P] \) modal participation matrix of natural modes at point \( P \).

\( [H^A_{P,Q}(s)] \) two-connection-point TITOP model of substructure \( A \) at points \( P \) and \( Q \) with a revolute joint at point \( P \).

\( \tau_{PG} \) kinematic model between points \( P \) and \( G \).

In Section 3.5.1 the superelement technique is described, using the following notation:

\( \rho \) beam volumetric density.

\( s \) beam section surface.

\( l \) beam length.

\( E \) beam Young’s modulus.

\( I_z \) beam cross section inertia.

\( \gamma_P(t) \) beam displacement at point \( P \) in the deflection plane.

\( \theta_P(t) \) beam rotation at point \( P \) in the deflection plane.

\( [S^A_{P,Q}(s)] \) superelement model of a beam-like structure between points \( P \) and \( Q \).

Appendix A: Component Modes Obtention

In component-mode synthesis, the substructure’s physical displacements can be expressed in terms of substructure generalized coordinates \( \{ \eta \} \) by the Rayleigh-Ritz coordinate transformation:

\[
\{ u \} = \phi \{ \eta \}
\]  

(56)

where the component-mode matrix \( \phi \) is a matrix of preselected component modes including: fixed-constraint modes, constraint modes and rigid-body modes. Then the matrix \( \phi \) is obtained as follows:

a) Using a set of \( N_n \) substructure fixed-constraint normal modes, \( \phi^N \), obtained from the solution of the eigenproblem:

\[
[\mathbf{K}_{nn} - \omega^2 \mathbf{M}_{nn}] \{ \phi_n \}_j = [0], \quad j = 1, 2, \ldots, N_n
\]

(57)

\[
\phi^N = \begin{bmatrix} \phi_{n1} \\ 0_{(N_n - 1) \times N_n} \end{bmatrix}
\]  

(58)

b) Using a set of redundant constraint modes, defined relative to the redundant boundary coordinate set:

\[
\phi^c = \begin{bmatrix} \phi_{nc} \\ I_{cc} \end{bmatrix} = \begin{bmatrix} -K_{nn}^{-1}K_{nc} \\ I_{cc} \end{bmatrix}
\]

(59)
c) Using a set of rigid-body modes, obtained by solving the equation resulting from restraining the rigid-body motion of the substructure:

\[
\phi^R_{N \times N_r} = \begin{bmatrix} \phi_{nr} \\ \phi_{cr} \\ I_{rr} \end{bmatrix} = \begin{bmatrix} -[K_{nn} K_{nc}]^{-1} [K_{nr}] \\ [K_{cn} K_{cc}] \\ I_{rr} \end{bmatrix}
\] (60)

The set of \(N_n\) fixed - interface normal modes can be reduced to a smaller set of kept normal modes, denoted as \(\phi_k\). The combined set \([\phi^R, \phi^C]\) spans the static response of the substructure to interface loading and allows for arbitrary interface displacements \(u_b\). These interface displacements can be accompanied by the displacements of the interior of the substructure as shown in Fig. P

Appendix B: Short hand notations for Superelement Modeling

In Section 3.5.1 the following matrices are used:

\[
D^{11}_{T_y, R z} = \Phi \tilde{M}^{-1}_{(3:6,3:6)} \Phi^T
\] (61)

\[
D^{11}_{T_y, R z} = \left( \tau - \Phi \tilde{M}^{-1}_{(3:6,3:6)} \tilde{M}_{(3:6,1:2)} \right)
\] (62)

\[
D^{22}_{T_y, R z} = -\tilde{M}_{(1:2,1:2)} + \tilde{M}^T_{(3:6,1:2)} \tilde{M}^{-1}_{(3:6,3:6)} \tilde{M}_{(3:6,1:2)}
\] (63)

\(T\) is a permutation matrix defined by:

\[
T = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\] (64)
\[
\{\ddot{u}_Q\} = \begin{bmatrix} \ddot{d}_Q \end{bmatrix} \quad \dot{\omega} \quad F_{C/A,C} = \begin{bmatrix} \ddot{F}_{C/A} \end{bmatrix}
\]

\[
\ddot{x}_P = \begin{bmatrix} \ddot{d}_P \end{bmatrix} \quad \dot{\omega}_P \quad F_{A/P,P} = \begin{bmatrix} \ddot{F}_{A/P} \end{bmatrix}
\]
Bode response $t_{tip,1} \rightarrow \ddot{w}_{tip}$