Any flat connection on a principal fibre bundle comes from a linear representation of the fundamental group. The noncommutative analog of this fact is discussed here.

1 Motivation. Preliminaries

1.1 Coverings

Definition 1.1. [10] Let $\tilde{\pi} : \tilde{X} \to X$ be a continuous map. An open subset $U \subset X$ is said to be evenly covered by $\tilde{\pi}$ if $\tilde{\pi}^{-1}(U)$ is the disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto $U$ by $\tilde{\pi}$. A continuous map $\tilde{\pi} : \tilde{X} \to X$ is called a covering projection if each point $x \in X$ has an open neighborhood evenly covered by $\tilde{\pi}$. $\tilde{X}$ is called the covering space and $X$ the base space of the covering.

Definition 1.2. [10] Let $p : \bar{X} \to X$ be a covering. A self-equivalence is a homeomorphism $f : \bar{X} \to \bar{X}$ such that $p \circ f = p$. This group of such homeomorphisms is said to be the group of covering transformations of $p$ or the covering group. Denote by $G \left( \bar{X} \mid X \right)$ this group.

Remark 1.3. Above results are copied from [10]. Below the covering projection word is replaced with covering.
1.2 Flat connections in the differential geometry

Here I follow to [8]. Let $M$ be a manifold and $G$ a Lie group. A *(differentiable) principal bundle over $M$ with group $G$* consists of a manifold $P$ and an action of $G$ on $P$ satisfying the following conditions:

(a) $G$ acts freely on $P$ on the right: $(u,a) \in P \times G \mapsto ua = R_au \in P$;

(b) $M$ is the quotient space of $P$ by the equivalence relation induced by $G$, i.e. $M = P / G$, and the canonical projection $\pi : P \to M$ is differentiable;

(c) $P$ is locally trivial, that is, every point $x$ of $M$ has an open neighborhood $U$ such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$ where $\varphi$ is a mapping of $\pi^{-1}(U)$ into $G$ satisfying $\psi(ua) = (\psi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M, G, \pi)$, $P(M, G)$ or simply $P$.

Let $P(M, G)$ be a principal fibre bundle over a manifold with group $G$. For each $u \in P$ let $T_u(P)$ be a tangent space of $P$ at $u$ and $G_u$ the subspace of $T_u(P)$ consisting of vectors tangent to the fibre through $u$. A *connection* $\Gamma$ in $P$ is an assignment of a subspace $Q_u$ of $T_u(P)$ to each $u \in P$ such that

(a) $T_u(P) = G_u \oplus Q_u$ (direct sum);

(b) $Q_u = (R_a)\ast Q_u$ for every $u \in P$ and $a \in G$, where $R_a$ is a transformation of $P$ induced by $a \in G$, $R_au = ua$.

Let $P = M \times G$ be a trivial principal bundle. For each $a \in G$, the set $M \times \{a\}$ is a submanifold of $P$. The canonical flat connection in $P$ is defined by taking the tangent space to $M \times \{a\}$ at $u = (x,a)$ as the horizontal tangent subspace at $u$. A connection in any principal bundle is called *flat* if every point has a neighborhood such that the induced connection in $P|_U = \pi^{-1}(U)$ is isomorphic with the canonical flat connection.

**Corollary 1.4.** *(Corollary II 9.2 [8]*) Let $\Gamma$ be a connection in $P(M, G)$ such that the curvature vanishes identically. If $M$ is paracompact and simply connected, then $P$ is isomorphic to the trivial bundle and $\Gamma$ is isomorphic to the canonical flat connection in $M \times G$.

If $\tilde{\pi} : \tilde{M} \to M$ is a covering then the $\tilde{\pi}$-*lift* of $P$ is a principal $\tilde{P}(\tilde{M}, G)$ bundle, given by

$$\tilde{P} = \left\{ (u, \tilde{x}) \in P \times \tilde{M} \mid \pi(u) = \tilde{\pi}(\tilde{x}) \right\}.$$  

If $\Gamma$ is a connection on $P(M, G)$ and $\tilde{M} \to M$ is a covering then is a canonical connection $\tilde{\Gamma}$ on $\tilde{P}(\tilde{M}, G)$ which is the *lift* of $\Gamma$, that is, for any $\tilde{u} \in \tilde{P}$ the horizontal space $\tilde{Q}_{\tilde{u}}$ is isomorphically mapped onto the horizontal space $Q_{\tilde{\pi}(\tilde{u})}$ associated with the connection.
\[ \Gamma. \] If \( \Gamma \) is flat then from the Proposition (II 9.3 [8]) it turns out that there is a covering \( \widetilde{M} \to M \) such that \( \check{P} (\widetilde{M}, G) \) (which is the lift of \( P (M, G) \)) is a trivial bundle, so the lift \( \check{\Gamma} \) of \( \Gamma \) is a canonical flat connection (cf. Corollary [1.4]). From the the Proposition (II 9.3 [8]) it follows that for any flat connection \( \Gamma \) on \( P (M, G) \) there is a group homomorphism \[ \varphi : G (\widetilde{M} \mid M) \to G \] such that

(a) There is an action \( G (\widetilde{M} \mid M) \times \check{P} \to \check{P} \approx \widetilde{M} \times G \) given by
\[ g (\check{x}, a) = (g \check{x}, \varphi (g) a); \forall \check{x} \in \widetilde{M}, a \in G, \]

(b) There is the canonical diffeomorphism \( P = \check{P} / G (\widetilde{M} \mid M) \),

(c) The lift \( \check{\Gamma} \) of \( \Gamma \) is a canonical flat connection.

**Definition 1.5.** In the above situation we say that the flat connection \( \Gamma \) is induced by the covering \( \widetilde{M} \to M \) and the homomorphism \( G (\widetilde{M} \mid M) \to G \), or we say that \( \Gamma \) comes from \( G (\widetilde{M} \mid M) \to G \).

**Remark 1.6.** The Proposition (II 9.3 [8]) assumes that \( \widetilde{M} \to M \) is the universal covering however it is not always necessary requirement.

**Remark 1.7.** If \( \pi_1 (M, x_0) \) is the fundamental group [10] then there is the canonical surjective homomorphism \( \pi_1 (M, x_0) \to G (\widetilde{M} \mid M) \). So there exist the composition \( \pi_1 (M, x_0) \to G (\widetilde{M} \mid M) \to G \). It follows that any flat connection comes from the homomorphisms \( \pi_1 (M, x_0) \to G \).

Suppose that there is the right action of \( G \) on \( P \) and suppose that \( F \) is a manifold with the left action of \( G \). There is an action of \( G \) on \( P \times F \) given by \( a (u, \xi) = (ua, a^{-1} \xi) \) for any \( a \in G \) and \( (u, \xi) \in P \times F \). The quotient space \( P / G = (P \times F) / G \) has the natural structure of a manifold and if \( E = P \times_G F \) then \( E (M, F, G, P) \) is said to be the fibre bundle over the base \( M \), with (standard) fibre \( F \), and (structure) group \( G \) which is associated with the principal bundle \( P \) (cf. [8]). If \( P = M \times G \) is the trivial bundle then \( E \) is also trivial, that is, \( E = M \times F \). If \( F = \mathbb{C}^n \) is a vector space and the action of \( G \) on \( \mathbb{C}^n \) is a linear representation of the group then \( E \) is the linear bundle. Denote by \( T (M) \) (resp. \( T^* (M) \)) the tangent (resp. contangent) bundle, and denote by \( \Gamma (E) \), \( \Gamma (T (M)) \), \( \Gamma (T^* (M)) \) the spaces of sections of \( E \), \( T (M) \), \( T^* (M) \) respectively. Any connection \( \Gamma \) on \( P \) gives a covariant derivative on \( E \), that is, for any section \( X \in \Gamma (T (M)) \) and any section \( \xi \in \Gamma (E) \) there is the derivative given by
\[ \nabla_X (\xi) \in \Gamma (E). \]

If \( E = M \times \mathbb{C}^n \), \( \Gamma \) is the canonical flat connection and \( \xi \) is a trivial section, that is, \( \xi = M \times \{ x \} \) then
\[ \nabla_X \xi = 0, \quad \forall X \in T (M). \quad (1.1) \]
For any connection there is the unique map
\[ \nabla : \Gamma(E) \to \Gamma(E \otimes T^* (M)) \tag{1.2} \]
such that
\[ \nabla_X \xi = (\nabla \xi, X) \]
where the pairing \((\cdot, \cdot) : \Gamma(E \otimes T^* (M)) \times \Gamma(T (M)) \to \Gamma(E)\) is induced by the pairing \(\Gamma(T^* (M)) \times \Gamma(T (M)) \to \mathcal{C}^\infty (M)\).

### 1.3 Noncommutative generalization of connections

The noncommutative analog of manifold is a spectral triple and there is the noncommutative analog of connections.

#### 1.3.1 Connection and curvature

**Definition 1.8.**

(a) A cycle of dimension \(n\) is a triple \((\Omega, d, \int)\) where \(\Omega = \bigoplus_{j=0}^{n} \Omega^j\) is a graded algebra over \(\mathbb{C}\), \(d\) is a graded derivation of degree 1 such that \(d^2 = 0\), and \(\int : \Omega^n \to \mathbb{C}\) is a closed graded trace on \(\Omega\),

(b) Let \(\mathcal{A}\) be an algebra over \(\mathbb{C}\). Then a cycle over \(\mathcal{A}\) is given by a cycle \((\Omega, d, \int)\) and a homomorphism \(\mathcal{A} \to \Omega^0\).

**Definition 1.9.** Let \(\mathcal{A} \xrightarrow{\rho} \Omega\) be a cycle over \(\mathcal{A}\), and \(\mathcal{E}\) a finite projective module over \(\mathcal{A}\). Then a connection \(\nabla\) on \(\mathcal{E}\) is a linear map \(\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\) such that
\[ \nabla(\xi x) = \nabla(\xi) x = \xi \otimes d_{\mathcal{A}}(x) ; \forall \xi \in \mathcal{E}, \forall x \in \mathcal{A}. \tag{1.3} \]

Here \(\mathcal{E}\) is a right module over \(\mathcal{A}\) and \(\Omega^1\) is considered as a bimodule over \(\mathcal{A}\).

**Remark 1.10.** The map \(\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\) is an algebraic analog of the map \(\nabla : \Gamma(E) \to \Gamma(E \otimes T^* (M))\) given by (1.2).

**Proposition 1.11.**

Following conditions hold:

(a) Let \(e \in \mathrm{End}_\mathcal{A} (\mathcal{E})\) be an idempotent and \(\nabla\) is a connection on \(\mathcal{E}\); then
\[ \xi \mapsto (e \otimes 1) \nabla \xi \tag{1.4} \]
is a connection on \(e\mathcal{E}\),

(b) Any finite projective module \(\mathcal{E}\) admits a connection,

(c) The space of connections is an affine space over the vector space \(\mathrm{Hom}_\mathcal{A} (\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1)\),

\[4\]
(d) Any connection $\nabla$ extends uniquely up to a linear map of $\tilde{E} = E \otimes_A \Omega$ into itself such that
$$\nabla (\xi \otimes \omega) = \nabla (\xi) \omega + \xi \otimes d\omega; \quad \forall \xi \in E, \ \omega \in \Omega. \quad (1.5)$$

A curvature of a connection $\nabla$ is a (right $A$-linear) map
$$F_\nabla : E \to E \otimes_A \Omega^2 \quad (1.6)$$
defined as a restriction of $\nabla \circ \nabla$ to $E$, that is, $F_\nabla = \nabla \circ \nabla|_E$. A connection is said to be flat if its curvature is identically equal to 0 (cf. \[1\]).

**Remark 1.12.** Above algebraic notions of curvature and flat connection are generalizations of corresponding geometrical notions explained in \[8\] and the Section 1.2.

For any projective $A$ module $E$ there is a trivial connection $\nabla : E \to E \otimes_A \Omega^1$
$$\nabla = \text{Id}_E \otimes d.$$
From $d^2 = d \circ d = 0$ it follows that $(\text{Id}_E \otimes d) \circ (\text{Id}_E \otimes d) = 0$, i.e. any trivial connection is flat.

**Lemma 1.13.** If $\nabla : E \to E \otimes_A \Omega^1$ is a trivial connection and $e \in \text{End}_A (E)$ is an idempotent then the given by (1.4)
$$\xi \mapsto (e \otimes 1) \nabla \xi$$
connection $\nabla_e : eE \to eE \otimes \Omega^1$ on $eE$ is flat.

**Proof.** From
$$(e \otimes 1) (\text{Id}_E \otimes d) \circ (e \otimes 1) (\text{Id}_E \otimes d) = e \otimes d^2 = 0$$
it turns out that $\nabla_e \circ \nabla_e = 0$, i.e. $\nabla_e$ is flat. \hfill \Box

**Remark 1.14.** The notion of the trivial connection is an algebraic version of geometrical canonical connection explained in the Section 1.2.

### 1.3.2 Spectral triples

This section contains citations of \[7\].

**Definition of spectral triples**

**Definition 1.15.** \[7\] A (unital) spectral triple $(A, \mathcal{H}, D)$ consists of:
- a pre-$C^*$-algebra $A$ with an involution $a \mapsto a^*$, equipped with a faithful representation on:
  - a Hilbert space $\mathcal{H}$; and also
  - a selfadjoint operator $D$ on $\mathcal{H}$, with dense domain $\text{Dom} D \subset \mathcal{H}$, such that $a(\text{Dom} D) \subseteq \text{Dom} D$ for all $a \in A$.

There is a set of axioms for spectral triples described in \[7, 11\].
Noncommutative differential forms

Any spectral triple naturally defines a cycle \( \rho : \mathcal{A} \to \Omega_D \) (cf. Definition 1.9). In particular for any spectral triple there is an \( \mathcal{A} \)-module \( \Omega^1_D \subset B(\mathcal{H}) \) of order-one differential forms which is a linear span of operators given by

\[
a [D, b]; \ a, b \in \mathcal{A}.
\]

(1.7)

There is the differential map

\[
d : \mathcal{A} \to \Omega^1_D,
\]

\[
a \mapsto [D, a].
\]

(1.8)

2 Noncommutative finite-fold coverings

2.1 Coverings of \( \mathcal{C}^* \)-algebras

Definition 2.1. If \( \mathcal{A} \) is a \( \mathcal{C}^* \)-algebra then an action of a group \( G \) is said to be involutive if \( ga^* = (ga)^* \) for any \( a \in \mathcal{A} \) and \( g \in G \). The action is said to be non-degenerated if for any nontrivial \( g \in G \) there is \( a \in \mathcal{A} \) such that \( ga \neq a \).

Definition 2.2. Let \( \mathcal{A} \hookrightarrow \tilde{\mathcal{A}} \) be an injective *-homomorphism of unital \( \mathcal{C}^* \)-algebras. Suppose that there is a non-degenerated involutive action \( G \times \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \) of a finite group \( G \), such that \( \mathcal{A} = \tilde{\mathcal{A}}^G \) def \( \{ a \in \tilde{\mathcal{A}} \mid a = ga; \forall g \in G \} \). There is an \( \mathcal{A} \)-valued product on \( \tilde{\mathcal{A}} \) given by

\[
\langle a, b \rangle_{\tilde{\mathcal{A}}} = \sum_{g \in G} g (a^*b)
\]

(2.1)

and \( \tilde{\mathcal{A}} \) is an \( \mathcal{A} \)-Hilbert module. We say that a triple \((A, \tilde{A}, G)\) is an unital noncommutative finite-fold covering if \( \tilde{A} \) is a finitely generated projective \( \mathcal{A} \)-Hilbert module.

Remark 2.3. Above definition is motivated by the Theorem 2.4.

Theorem 2.4. Suppose \( X \) and \( Y \) are compact Hausdorff connected spaces and \( p : Y \to X \) is a continuous surjection. If \( C(Y) \) is a projective finitely generated Hilbert module over \( C(X) \) with respect to the action

\[
(f\xi)(y) = f(y)\xi(p(y)), \ f \in C(Y), \ \xi \in C(X),
\]

then \( p \) is a finite-fold covering.

2.2 Coverings of spectral triples

Definition 2.5. Let \((A, \mathcal{H}, D)\) be a spectral triple, and let \( A \) be the \( \mathcal{C}^* \)-norm completion of \( \mathcal{A} \). Let \((A, \tilde{A}, G)\) be an unital noncommutative finite-fold covering such that there is
then

\[ \tilde{H} = \tilde{A} \otimes \mathcal{H} \] is a Hilbert space such that the Hilbert product \((\cdot, \cdot)_{\tilde{H}}\) is given by

\[ (a \otimes \xi, b \otimes \eta)_{\tilde{H}} = \frac{1}{|G|} \left( \xi, \sum_{g \in G} g \left( \tilde{a} \tilde{b} \right) \eta \right)_{\tilde{H}} \quad \forall \tilde{a}, \tilde{b} \in \tilde{A}, \, \xi, \eta \in \mathcal{H} \]

where \((\cdot, \cdot)_{\tilde{H}}\) is the Hilbert product on \(\mathcal{H}\). There is the natural representation \(\tilde{A} \to B \left( \tilde{H} \right)\).

A spectral triple \(\left( \tilde{A}, \tilde{H}, \tilde{D} \right)\) is said to be a \(\left( A, \tilde{A}, \tilde{G} \right)\)-lift of \((A, \mathcal{H}, D)\) if following conditions hold:

(a) \(\tilde{A}\) is a \(C^*\)-norm completion of \(\tilde{A}\),

(b) \(\tilde{D} \left( 1_{\tilde{A}} \otimes A \xi \right) = 1_{\tilde{A}} \otimes A \tilde{D} \xi; \ \forall \xi \in \text{Dom} \tilde{D}\),

(c) \(\tilde{D} \left( g \tilde{\xi} \right) = g \left( \tilde{D} \tilde{\xi} \right)\) for any \(\tilde{\xi} \in \text{Dom} \tilde{D}, \ g \in G\).

**Remark 2.6.** It is proven in [5] that for any spectral triple \((A, \mathcal{H}, D)\) and any unital noncommutative finite-fold covering \(\left( A, \tilde{A}, \tilde{G} \right)\) there is the unique \(\left( A, \tilde{A}, \tilde{G} \right)\)-lift \(\left( \tilde{A}, \tilde{H}, \tilde{D} \right)\) of \((A, \mathcal{H}, D)\).

**Remark 2.7.** It is known that if \(M\) is a Riemannian manifold and \(\tilde{M} \to M\) is a covering, then \(\tilde{M}\) has the natural structure of Riemannian manifold (cf. [8]). The existence of lifts of spectral triples is a noncommutative generalization of this fact (cf. [5]).

### 3 Construction of noncommutative flat coverings

Let \((A, \mathcal{H}, D)\) be a spectral triple, let \((\tilde{A}, \tilde{H}, \tilde{D})\) is the \(\left( A, \tilde{A}, \tilde{G} \right)\)-lift of \((A, \mathcal{H}, D)\). Let \(V = \mathbb{C}^n\) and with left action of \(G\), i.e. there is a linear representation \(\rho : G \to GL(\mathbb{C}, n)\).

Let \(\tilde{E} = A \otimes \mathbb{C}^n \approx \tilde{A}^n\) be a free module over \(\tilde{A}\), so \(\tilde{E}\) is a projective finitely generated \(A\)-module (because \(\tilde{A}\) is a finitely generated projective \(A\)-module). Let \(\tilde{\nabla} : \tilde{E} \to \tilde{E} \otimes \tilde{A} \Omega^1_D\) be the trivial flat connection. In [5] it is proven that \(\Omega^1_D = \tilde{A} \otimes A \Omega^1_D\) it follows that the connection \(\tilde{\nabla} : \tilde{E} \to \tilde{E} \otimes \tilde{A} \Omega^1_D\) can be regarded as a map \(\nabla' : \tilde{E} \to \tilde{E} \otimes \tilde{A} \tilde{H} \otimes A \Omega^1_D = \tilde{E} \otimes A \Omega^1_D\), i.e. one has a connection

\[ \nabla' : \tilde{E} \to \tilde{E} \otimes A \Omega^1_D. \]

From \(\nabla' \circ \nabla'|_{\tilde{E}} = 0\) it turns out that \(\nabla' \circ \nabla'|_{\tilde{E}} = 0\), i.e. \(\nabla'\) is flat. There is the action of \(G\) on \(\tilde{E} = \tilde{A} \otimes \mathbb{C}^n\) given by

\[ g \left( \tilde{a} \otimes x \right) = g \tilde{a} \otimes gx; \ \forall g \in G, \ \tilde{a} \in \tilde{A}, \ x \in \mathbb{C}^n. \] (3.1)

Denote by

\[ \mathcal{E} = \tilde{E}^G = \left\{ \tilde{\xi} \in \tilde{E} \mid g \tilde{\xi} = \tilde{\xi} \right\} \] (3.2)
Clearly $\mathcal{E}$ is an $A$-$A$-bimodule. For any $\tilde{\xi} \in \tilde{\mathcal{E}}$ there is the unique decomposition
\[
\tilde{\xi} = \xi + \xi \perp,
\]
\[
\xi = \frac{1}{|G|} \sum_{g \in G} g \tilde{\xi},
\]
\[
\xi \perp = \tilde{\xi} - \xi.
\]

From the above decomposition it turns out the direct sum $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}^G \oplus \tilde{\mathcal{E}} \perp$ of $A$-modules. So $\mathcal{E} = \tilde{\mathcal{E}}^G$ is a projective finitely generated $A$-module, it follows that there is an idempotent $e \in \text{End}_A \tilde{\mathcal{E}}$ such that $\mathcal{E} = e\tilde{\mathcal{E}}$. The Proposition 1.11 gives the canonical connection
\[
\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_D
\]
which is defined by the connection $\nabla' : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes_A \Omega^1_D$ and the idempotent $e$. From the Lemma 1.13 it turns out that $\nabla$ is flat.

**Definition 3.1.** We say that $\nabla$ is a flat connection induced by noncommutative covering $\left(\mathcal{E} = \tilde{\mathcal{E}}^G \oplus \tilde{\mathcal{E}} \perp\right)$ and the linear representation $\rho : G \to GL(C, n)$, or we say the $\nabla$ comes from the representation $\rho : G \to GL(C, n)$.
4 Mapping between geometric and algebraic constructions

The geometric (resp. algebraic) construction of flat connection is explained in the Section 1.2 (resp. 3). Following table gives a mapping between these constructions.

| Geometry | Algebra |
|----------|---------|
| 1 Riemannian manifold $M$. | Spectral triple $(C^\infty (M), L^2 (M, \mathcal{S}), \mathcal{D})$. |
| 2 Topological covering $\tilde{M} \to M$. | Noncommutative covering, $(C (M), C (\tilde{M}), G (\tilde{M} | M))$, given by the Theorem 2.4. |
| 3 Natural structure of Riemannian manifold on the covering space $\tilde{M}$. | Triple $(C^\infty (\tilde{M}), L^2 (\tilde{M}, \tilde{\mathcal{S}}), \mathcal{D})$ is the $(C (M), C (\tilde{M}), G (\tilde{M} | M))$-lift of $(C^\infty (M), L^2 (M, \mathcal{S}), \mathcal{D})$. |
| 4 Group homomorphism $G (\tilde{M} | M) \to GL (n, \mathbb{C})$. | Action $G (\tilde{M} | M) \times \mathbb{C}^n \to \mathbb{C}^n$. |
| 5 Trivial bundle $\tilde{M} \times \mathbb{C}^n$. | Free module $C^\infty (\tilde{M}) \otimes \mathbb{C}^n$. |
| 6 Canonical flat connection on $\tilde{M} \times \mathbb{C}^n$. | Trivial flat connection on $C^\infty (\tilde{M}) \otimes \mathbb{C}^n$. |
| 7 Action of $G (\tilde{M} | M)$ on $\tilde{M} \times \mathbb{C}^n$. | Action of $G (\tilde{M} | M)$ on $C^\infty (\tilde{M}) \otimes \mathbb{C}^n$. |
| 8 Quotient space $P = (\tilde{M} \times \mathbb{C}^n) / G (\tilde{M} | M)$. | Invariant module $\mathcal{E} = (C^\infty (\tilde{M}) \otimes \mathbb{C}^n)^{G(\tilde{M} | M)}$. |
| 9 Geometric flat connection on $P$ | Algebraic flat connection on $\mathcal{E}$. |

5 Noncommutative examples

5.1 Noncommutative tori

Following text is the citation of [5]. If $\Theta$ be a real skew-symmetric $n \times n$ matrix. There is a $C^*$-algebra $C (T^n_\Theta)$ which is said to be the noncommutative torus (cf. [5]). There is a pre-$C^*$-algebra $C^\infty (T^n_\Theta)$ and the spectral triple $(C^\infty (T^n_\Theta), \mathcal{H}, \mathcal{D})$ such that it is the dense...
inclusion $C^\infty(T^n_\Theta) \hookrightarrow C(T^n_\Theta)$. If $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{N}^n$ and

$$\bar{\Theta} = \begin{pmatrix} 0 & \bar{\theta}_{12} & \ldots & \bar{\theta}_{1n} \\ \bar{\theta}_{21} & 0 & \ldots & \bar{\theta}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\theta}_{n1} & \bar{\theta}_{n2} & \ldots & 0 \end{pmatrix}$$

is a skew-symmetric matrix such that

$$e^{-2\pi i \theta_{kn}} = e^{-2\pi i \bar{\theta}_{kn} k_n}$$

then one has a following theorem.

**Theorem 5.1.** [4] The triple $(C(T^n_\Theta), C(T^n_\Theta), Z_{k_1} \times ... \times Z_{k_n})$ is an unital noncommutative finite-fold covering.

There is $(C(T^n_\Theta), C(T^n_\Theta), Z_{k_1} \times ... \times Z_{k_n})$-lift $(C^\infty(T^n_\Theta), \bar{\mathcal{H}}, \bar{\mathcal{D}})$ of $(C^\infty(T^n_\Theta), \mathcal{H}, D)$. From the construction of the Section 3 it follows that for any representation $\rho: Z_{k_1} \times ... \times Z_{k_n} \to GL(N, \mathbb{C})$ there is a finitely generated $C^\infty(T^n_\Theta)$-module $\mathcal{E}$ and a flat connection

$$\mathcal{E} \to \mathcal{E} \otimes_{C^\infty(T^n_\Theta)} \Omega^1_D$$

which comes from $\rho$.

### 5.2 Isospectral deformations

A very general construction of isospectral deformations of noncommutative geometries is described in [4]. The construction implies in particular that any compact Spin-manifold $M$ whose isometry group has rank $\geq 2$ admits a natural one-parameter isospectral deformation to noncommutative geometries $M_{\mathfrak{g}}$. We let $(C^\infty(M), L^2(M, S), \mathcal{D})$ be the canonical spectral triple associated with a compact spin-manifold $M$. We recall that $C^\infty(M)$ is the algebra of smooth functions on $M$, $S$ is the spinor bundle and $\mathcal{D}$ is the Dirac operator. Let us assume that the group Isom($M$) of isometries of $M$ has rank $r \geq 2$. Then, we have an inclusion

$$T^2 \subset Isom(M),$$

with $T^2 = \mathbb{R}^2/2\pi \mathbb{Z}^2$ the usual torus, and we let $U(s), s \in T^2$, be the corresponding unitary operators in $\mathcal{H} = L^2(M, S)$ so that by construction

$$U(s) \mathcal{D} = \mathcal{D} U(s).$$

Also,

$$U(s) a U(s)^{-1} = a_s(a), \quad \forall a \in \mathcal{A}, \quad \text{5.1}$$

where $\alpha \in \text{Aut}(\mathcal{A})$ is the action by isometries on the algebra of functions on $M$. In [4] is constructed a spectral triple $(\mathcal{C}^\infty(M), L^2(M, S), \mathcal{D})$ such that $\mathcal{C}^\infty(M)$ is a noncommutative algebra which is said to be an isospectral deformation of $C^\infty(M)$. For any
finite-fold topological covering $\tilde{M} \to M$ there is the finite-fold noncommutative covering $\left( lC \left( \tilde{M} \right), l \left( M \right), G \left( \tilde{M} | M \right) \right)$ (cf. [6]). So there is the $\left( lC \left( \tilde{M} \right), l \left( M \right), G \left( M \right) \right)$-lift

$$\left( lC^{\infty} \left( \tilde{M} \right), L^2 \left( \tilde{M}, \tilde{S} \right), D \right)$$

of $\left( L^\infty \left( M \right), L^2 \left( M, S \right), D \right)$. From the construction of the Section 3 it follows that for any representation $\rho : G \left( \tilde{M} | M \right) \to GL \left( N, \mathbb{C} \right)$ there is a finitely generated $lC^\infty \left( M \right)$-module $E$ and a flat connection

$$E \to E \otimes_{lC^\infty \left( M \right)} \Omega^1_D$$

which comes from $\rho$.

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