I compare collinear and $k_T$ factorization theorems in perturbative QCD, and discuss their application to exclusive $B$ meson decays. Especially, I concentrate on the recently measured time-dependent CP asymmetry of the $B_0^0 \rightarrow \pi^+ \pi^-$ modes, from which the unitarity angle can be extracted.

1 Introduction

Both collinear factorization theorem and $k_T$ factorization theorem are the fundamental tools of perturbative QCD (PQCD), where $k_T$ denotes parton transverse momenta. In these theorems the amplitude for an exclusive process can be calculated as an expansion of $\alpha_s(Q)$ and $\Lambda/Q$ with $Q$ being a large momentum transfer and $\Lambda$ a small hadronic scale. Similarly, in the heavy quark limit $m_b \rightarrow \infty$, $m_b$ being the $b$ quark mass, a $B$ meson decay amplitude can be calculated as an expansion of $\alpha_s(m_b)$ and $\Lambda/m_b$. A study of $B$ meson decays is important for determining Standard Model parameters, such as the unitarity angles.

2 Collinear Factorization vs. $k_T$ Factorization

We explain how to derive collinear and $k_T$ factorization theorems for the pion form factor involved in the scattering process $\pi(P_1)\gamma^\ast(q) \rightarrow \pi(P_2)$. The momenta are chosen in the light-cone coordinates as $P_1 = (P_1^+, 0, 0_T)$, $P_2 = (0, P_2^−, 0_T)$, and $Q^2 = −q^2$. At leading order, $O(\alpha_s)$, shown in Fig. 1(a), the hard kernel is proportional to $H^{(0)}(x_1, x_2) \propto −1/(x_1 P_1 − x_2 P_2)^2 = 1/(x_1 x_2 Q^2)$. Here $x_1$ and $x_2$ are the parton momentum fractions carried by the lower quarks in the incoming and outgoing pions, respectively. At next-to-leading order, $O(\alpha_s^2)$, collinear divergences are generated in loop integrals, and need to be factorized into the pion wave function. In the collinear region with the loop momentum $l$ parallel to $P_1$, we have an on-shell gluon $l^2 \sim P_1^2 \sim O(\Lambda^2)$ with the hierarchy of the components, $l^+ \sim P_1^+ \gg l_T \sim \Lambda \gg l^- \sim \Lambda^2/P_1^+$. 
Fig. 1: (a) Lowest-order diagram for $F_\pi$. (b) Radiative correction to (a).

An example of next-to-leading-order diagrams is shown in Fig. 1(b). The factorization of Fig. 1(b) is trivial: one performs the Fierz transformation to separate the fermion flows, so that the right-hand side of the cut corresponds to the lowest-order hard kernel $H^{(0)}$. Since the loop momentum $l$ flows into the hard gluon, we have the gluon momentum $x_1 P_1 - x_2 P_2 + l$ and

$$H^{(0)} \propto \frac{-1}{(x_1 P_1 - x_2 P_2)^2 + 2 x_1 P_1^+ l - 2 x_2 P_2^- l^+ + 2 l^- - l_T^2}. \quad (1)$$

Dropping $l^-$ and $l_T$ as a collinear approximation, the above expression reduces to

$$H^{(0)}(\xi_1, x_2) \propto \frac{1}{2 x_1 x_2 P_1^+ P_2^- + 2 x_2 P_2^- l^+} \equiv \frac{1}{\xi_1 x_2 Q^2}, \quad (2)$$

where $\xi_1 = x_1 + l^+/P_1^+$ is the parton momentum fraction modified by the collinear gluon exchange. The left-hand side of the cut then contributes to the $O(\alpha_s)$ distribution amplitude $\phi^{(1)}_\pi(\xi_1)$, which contains the integration over $l^-$ and $l_T$. Therefore, factorization to all orders gives a convolution only in the longitudinal components of parton momentum,

$$F_\pi = \int d\xi_1 d\xi_2 ^2 \phi_\pi(\xi_1) H(\xi_1, \xi_2) \phi_\pi(\xi_2). \quad (3)$$

In the region with small parton momentum fractions, the hard scale $x_1 x_2 Q^2$ is not large. In this case one may drop only $l^-$, and keep $l_T$ in $H^{(0)}$. This weaker approximation gives

$$H^{(0)}(\xi_1, x_2, l_T) \propto \frac{1}{2(\xi_1 x_2 + l^+/P_1^+) x_2 P_2^- l_T^+} \equiv \frac{1}{\xi_1 x_2 Q^2 + l_T^2}, \quad (4)$$

which acquires a dependence on a transverse momentum. We factorize the left-hand side of the cut in Fig. 1(b) into the $O(\alpha_s)$ wave function $\phi^{(1)}_\pi(\xi_1, l_T)$, which involves the integration over $l_T$. It is understood that the collinear gluon exchange not only modifies the momentum fraction, but introduces the transverse momentum dependence of the pion wave function. Extending the above procedure to all orders, we derive the $k_T$ factorization,

$$F_\pi = \int d\xi_1 d\xi_2 d^2 k_{1T} d^2 k_{2T} \phi_\pi(\xi_1, k_{1T}) H(\xi_1, k_{1T}, k_{2T}) \phi_\pi(\xi_2, k_{2T}). \quad (5)$$

### 3 Semileptonic Decays

Collinear factorization theorem for the semileptonic decay $B(P_1) \to \pi(P_2) l \nu(q)$ can be constructed in a similar way with the $B$ meson momentum $P_1 = (P_1^+, P_1^-, 0_T)$. Hence, the involved $B \to \pi$ transition form factor is written as $F_{B\pi} = \int dx_1 dx_2 \phi_B(x_1) H(x_1, x_2) \phi_\pi(x_2)$ with the lowest-order hard kernel $H^{(0)} \propto 1/(x_1 x_2^2)$. The parton momentum fractions $x_1$ and $x_2$ are defined via the spectator quark momenta $k_1 = (x_1 P_1^+, 0, 0_T)$ and $k_2 = x_2 P_2$ on the $B$ meson and pion sides, respectively. Obviously, the above integral is logarithmically divergent for the asymptotic model $\phi_\pi \propto x(1-x)$.
There are two options to handle the above end-point singularity:

1. An end-point singularity in collinear factorization implies that exclusive $B$ meson decays are dominated by soft dynamics. Therefore, a heavy-to-light form factor is not calculable\cite{footnote}, and $F_{B\pi}$ should be treated as a soft object, like $\phi_\pi$.

2. An end-point singularity in collinear factorization implies its breakdown. $k_T$ factorization theorem then becomes a more appropriate framework, in which $F_{B\pi}$ is calculable by means of the convolution $F_{B\pi} = \int dx_1 dx_2 d^2k_1 d^2k_2 \phi_B(x_1,k_{1T})H(x_1,x_2,k_{1T},k_{2T})\phi_\pi(x_2,k_{2T})$\cite{footnote}, with the lowest-order hard kernel $H^{(0)} \propto 1/\left[ (x_1 m_0 B)^2 + (k_{1T} - k_{2T})^2 \right]$. It is easy to observe that the above formula does not develop an end-point singularity.

We emphasize that there is no preference between options 1 and 2 for semileptonic $B$ meson decays, since the unknowns $F_{B\pi}^0$ and the meson wave functions are more or less equivalent: from experimental data one either determines $F_{B\pi}$ in option 1 or $\phi_B$ in option 2. However, when extending the two options to two-body nonleptonic $B$ meson decays, predictions are very different. Options 1 and 2 lead to the so-called QCD-improved factorization (QCDF)\cite{footnote} and perturbative QCD (PQCD)\cite{footnote} approaches, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Sources of the strong phase.}
\end{figure}

4 Nonleptonic Decays

Nonleptonic $B$ meson decays occur through a weak effective Hamiltonian $H_{\text{eff}}$. The decay amplitudes involve several topologies: factorizable emission, nonfactorizable emission, factorizable annihilation, and nonfactorizable annihilation. A goal of QCD approaches is to estimate all the topologies of amplitudes.

There are many important quantities involved in two-body nonleptonic $B$ meson decays. Here we shall discuss only the time-dependent CP asymmetry of the $B \rightarrow \pi\pi$ modes,

\begin{equation}
A_{CP}(t) \equiv \frac{B(B^0(t) \rightarrow \pi^+\pi^-) - B(B^0(t) \rightarrow \pi^-\pi^+)}{B(B^0(t) \rightarrow \pi^+\pi^-) + B(B^0(t) \rightarrow \pi^-\pi^+)} = S_{\pi\pi} \sin(\Delta m t) - C_{\pi\pi} \cos(\Delta m t),
\end{equation}

with $\Delta m$ being the mixing frequency. The coefficients $S_{\pi\pi}$ and $C_{\pi\pi}$ depend on the weak angle $\phi_2$ and the tree-over-penguin ratio. We discuss the calculation of the strong phase associated with the tree-over-penguin ratio, and the predictions for $S_{\pi\pi}$ and $C_{\pi\pi}$ in collinear factorization (QCDF) and in $k_T$ factorization (PQCD).

Sources of the strong phase in QCDF are displayed in Fig. 2. Figure 2(a) represents the leading $O(\alpha_s^0)$ soft contribution involving the real $B \rightarrow \pi$ form factor. Figure 2(b) is the imaginary annihilation amplitude from one gluon exchange. Since annihilation occurs through the scalar penguin operator, it is of $O(\alpha_s r_\chi)$, $r_\chi = 2m_0/m_B$ being the chiral enhancing factor with $m_0 \sim 1.4$ GeV. Figure 2(c) denotes the imaginary $O(\alpha_s)$ vertex correction to the four-fermion operators. Because of $\alpha_s r_\chi < \alpha_s$ ($r_\chi$ is of order unity), Fig. 2(c) is the most important source of the strong phase $\delta$. Compared to the leading contribution of $O(\alpha_s^0)$, $\delta$ derived from QCDF is expected to be small (and positive). This is the reason QCDF predicts a smaller and positive $C_{\pi\pi}$ as shown in the left-hand plot of Fig. 3\cite{footnote}.
In $k_T$ factorization power counting for the topologies of amplitudes changes. Figure 2(a) represents the $O(\alpha_s)$ calculable $B \rightarrow \pi$ form factor, which contains one hard gluon exchange. Figure 2(b) has the same power counting as in QCDF. Figure 2(c) becomes higher-power, i.e., $O(\alpha_s^2)$. Because of $\alpha_s r_\chi \gg \alpha_s^2$, Fig. 2(b) is the most important source of the strong phase $\delta$. It is smaller but comparable to the leading contribution of $O(\alpha_s)$. Therefore, $\delta$ derived from PQCD is expected to be large (and negative). This is the reason PQCD predicts a larger and negative $C_{\pi\pi} \sim -30\%$ as shown in the right-hand plot of Fig. 3. The central value of $S_{\pi\pi} \sim 0$ measured by BaBar then corresponds to $\phi_2 \sim 80^0$. The boundary of the square represents $1\sigma$ uncertainty, from which the range of $\phi_2$, $60^0 < \phi_2 < 100^0$ is extracted.

5 Summary

In collinear factorization a heavy-to-light transition form factor exhibits an end-point singularity, while in $k_T$ factorization it is infrared-finite. Hence, soft dominance is postulated and the form factor is parametrized as a nonperturbative input in the former. Hard dominance is postulated and the form factor can be calculated as a convolution of a hard kernel with meson wave functions in the latter. Extending the above theorems to two-body nonleptonic $B$ meson decays, QCDF (collinear factorization) prefers a small positive $C_{\pi\pi}$, direct CP asymmetry in the $B \rightarrow \pi\pi$ decays, while PQCD ($k_T$ factorization) prefers a large negative $C_{\pi\pi}$. More precise experimental data can soon discriminate the two approaches.

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