Local unitary invariants for multipartite quantum systems

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Abstract
A method is presented to obtain local unitary invariants for multipartite quantum systems consisting of fermions or distinguishable particles. The invariants are organized into infinite families, in particular, the generalization to higher dimensional single-particle Hilbert spaces is straightforward. Many well-known invariants and their generalizations are also included.

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1. Introduction
The possibility of entanglement between subsystems is a purely quantum-mechanical phenomenon, related to the nonlocal nature of the fundamental laws governing our world. In order to understand quantum entanglement, one constructs functions on the space of quantum states invariant under groups modelling local operations. The two main approaches consider local unitary (LU) operations which describe local transformations that can be applied with probability 1, or the stochastic local operations and classical communication (SLOCC) group corresponding to transformations that can be done with nonzero probability [1].

Various results exist for both groups acting on composite quantum systems with distinguishable constituents [2–4], but much less is known about the entanglement of indistinguishable particles. Recent work revealed that an understanding of fermionic entanglement can also provide us with information about the entanglement of distinguishable subsystems [5, 6]. Motivated by this, in this paper, we present a way to construct fermionic entanglement measures for pure states.

Generally, one looks for invariant functions that are polynomials in the coefficients and their conjugates of the pure state with respect to a fixed orthonormal basis. More abstractly (and independently of basis choices), one would like to find the subrepresentations of \( S(V \oplus V^*) \) isomorphic to the trivial one or the one-dimensional subrepresentations of \( S(V) \), where \( V \) is a representation of some group \( G \) (either LU or SLOCC) and \( S(\cdot) \) denotes the symmetric algebra on a vector space.
Here, we take a slightly different approach, and use the projections to every subrepresentation in $S(V)$ and associate invariants with them. Here, $V$ is the state space of a $k$-fermion system, and the group considered is the LU group, i.e. the unitary group acting on the one-particle Hilbert space. This approach has the advantage that the resulting formulae are independent of the dimension of the one-particle state space.

The outline of the paper is as follows. In section 2, we recall how each graded part of the symmetric algebra of a Hilbert space comes equipped with an invariant inner product. In section 3, this inner product is utilized in order to associate LU invariants with every isotypic subspace of the space of degree $m$ polynomials in the coefficients of a multi-fermion state. In section 4, a special case is considered, namely the invariant associated with the subrepresentation containing the weight space with highest weight. The method to obtain explicit formulae is also presented here. In section 5, some examples are worked out illustrating various features of our approach. In section 6, the relationship between LU invariants generated this way and SLOCC invariants is highlighted. In section 7, it is briefly mentioned that how the fermionic invariants obtained can be used to construct LU invariants for quantum systems with distinguishable constituents. For the readers’ convenience, a summary of some concepts from the representation theory of the unitary groups can be found in the appendix.

2. The symmetric algebra of a Hilbert space

Throughout this section, $\mathcal{H}$ denotes a finite-dimensional complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. We regard $\mathcal{H}$ as a representation of $U(\mathcal{H}) = \{\varphi : \mathcal{H} \to \mathcal{H} | \forall v \in \mathcal{H} : \|\varphi v\| = \|v\|\}$.

Let $S(\mathcal{H})$ denote the symmetric algebra of $\mathcal{H}$, that is, the algebra of polynomials in vectors of $\mathcal{H}$. $S(\mathcal{H})$ has the structure of a graded algebra; its degree $m$ homogenous subspace will be denoted by $S^m(\mathcal{H})$. As $S^1(\mathcal{H}) = \mathcal{H}$, and this subspace generates $S(\mathcal{H})$ as a unital commutative algebra, we have that $U(\mathcal{H})$ acts on $S(\mathcal{H})$ with algebra automorphisms.

The inner product on $\mathcal{H}$ induces one on $S^m(\mathcal{H})$ by the following requirement: for $u, v \in \mathcal{H}$, let $\langle u^m, v^m \rangle = \langle u, v \rangle^m$. This turns out to be equivalent to saying that for a unit vector $u \in \mathcal{H}$, $\|u^m\| = 1$. Clearly, this inner product will be preserved by the action of $U(\mathcal{H})$ on $S(\mathcal{H})$, restricted to each homogenous subspace. It is known from the representation theory of the unitary groups that in this way each $S^m(\mathcal{H})$ becomes an irreducible unitary representation of $U(\mathcal{H})$, and hence the induced inner product is essentially the only one invariant under this group action.

To be more explicit, if we fix an orthonormal basis $\{e_1, \ldots, e_d\}$ in $\mathcal{H}$, then $S^m(\mathcal{H})$ is the space of degree $m$ homogenous polynomials in the basis elements, and the degree $m$ monomials with coefficient 1 form a basis. These monomials are mutually orthogonal, but they are not unit vectors. If $v = \sum_{i=1}^d \alpha_i e_i$, then

$$v^m = \sum_{k_1, \ldots, k_d=0}^m \sum_{\alpha_1, \ldots, \alpha_d} (\prod_{i=1}^d \alpha_i^{k_i}) e_1^{k_1} e_2^{k_2} \cdots e_d^{k_d},$$

(1)

where $\prod_{i=1}^d \alpha_i^{k_i}$ is the multinomial coefficient and hence

$$\|v^m\|^2 = \sum_{k_1, \ldots, k_d=0}^m \left(\prod_{i=1}^d \alpha_i^{k_i}\right) \|e_1^{k_1} e_2^{k_2} \cdots e_d^{k_d}\|^2.$$

(2)
Comparing this with
\[ (\|v\|^2)^m = \left( \sum_{i=1}^{d} |\alpha_i|^2 \right)^m = \sum_{k_1, \ldots, k_d \geq 0, k_1 + \ldots + k_d = m} \left( k_1 \alpha_1^k \alpha_2^k \ldots \alpha_d^k \right)^2 \]
we conclude that
\[ \|e_1^{k_1} e_2^{k_2} \cdots e_d^{k_d}\| = \left( \sum_{k_1, k_2, \ldots, k_d} \left( k_1 \alpha_1^k \alpha_2^k \ldots \right)^{-1/2} \right) \]

3. Invariants for multi-fermion systems

In this section, \( \mathcal{H} \) will be a finite-dimensional complex Hilbert space, playing the role of the single-particle state space of a fermionic quantum system of \( k \) particles. If \( n = \dim \mathcal{H} \), then the \( k \)-particle Hilbert space is isomorphic to
\[ \bigwedge^k \mathcal{H} \cong \bigwedge^k \mathbb{C}^n \]
and hence its dimension is \( \binom{n}{k} \). This space also comes equipped with an inner product induced from that of \( \mathcal{H} \), and a unitary action of \( U(\mathcal{H}) \) which models LU transformations of the \( k \)-particle states.

Now let us look at the symmetric algebra of the \( k \)-fermion state space. On its homogenous subspaces \( S^m(\bigwedge^k \mathcal{H}) \), we have an action of \( U(\mathcal{H}) \) which factors through \( U(\bigwedge^k \mathcal{H}) \) and an inner product which is invariant under \( U(\bigwedge^k \mathcal{H}) \) and hence also invariant under \( U(\mathcal{H}) \). This time the representation of \( U(\mathcal{H}) \) is not irreducible, and \( S^m(\bigwedge^k \mathcal{H}) \) can be split into the orthogonal sum of \( U(\mathcal{H}) \)-invariant subspaces in a non-trivial way:
\[ S^m(\bigwedge^k \mathcal{H}) = \bigoplus_{\lambda} V_{\lambda} \]
where \( \lambda \) ranges over the partitions of \( km \), and \( V_{\lambda} \) is the corresponding isotypic component of the representation. Interestingly, this decomposition is independent of \( n \) (apart from the vanishing of the subrepresentations associated with partitions involving more than \( n \) parts, but for \( n \geq km \) this certainly cannot happen). This is essentially due to the fact that a degree \( j \) symmetric polynomial in \( n \) variables can be reconstructed even if we only know its restriction to a subspace in which only \( j \) variables take nonzero values.

This decomposition allows us to introduce unitary invariants, one for each isotypic subspace. Let \( \psi \in \bigwedge^k \mathcal{H} \) be a \( k \)-fermion state vector, and \( \psi^m \) its \( m \)th power which is an element of \( S^m(\bigwedge^k \mathcal{H}) \). Let \( P_{\lambda} : S^m(\bigwedge^k \mathcal{H}) \to V_{\lambda} \) denote the orthogonal projection. This commutes with the representation of \( U(\mathcal{H}) \); therefore, the value of \( I_{\lambda}(\psi) := \langle \psi^m, P_{\lambda} \psi^m \rangle = \| P_{\lambda} \psi^m \|^2 \) is invariant:
\[ \forall g \in U(\mathcal{H}) : \langle (g \cdot \psi)^m, P_{\lambda} (g \cdot \psi)^m \rangle = \langle g \cdot (\psi^m), g \cdot (P_{\lambda} \psi^m) \rangle = \langle \psi^m, P_{\lambda} \psi^m \rangle. \]
Note that the number of linearly independent invariants is 1 less than the number of non-vanishing isotypic components because
\[ \sum_{\lambda} \langle \psi^m, P_{\lambda} \psi^m \rangle = \langle \psi^m, \left( \sum_{\lambda} P_{\lambda} \right) \psi^m \rangle = \langle \psi^m, \psi^m \rangle = 1. \]
4. Invariant subspaces with maximal highest weight

Let us now fix an ordered orthonormal basis \((e_1, \ldots, e_n)\) in \(\mathcal{H}\). This also gives the isomorphisms \(\mathcal{H} \cong \mathbb{C}^n\), and \(U(\mathcal{H}) \cong U(n, \mathbb{C})\). The maximal torus \(T\) which acts diagonally in this basis is then identified with the subgroup of diagonal unitary matrices. The set of one-dimensional representations \(T \rightarrow \mathbb{C}^\times\) is a commutative group isomorphic to \(\mathbb{Z}^n\). We will use the following identification:

\[
(r_1, r_2, \ldots, r_n) : T \rightarrow \mathbb{C}^\times
\]

\[
(r_1, r_2, \ldots, r_n)(\text{diag}(\lambda_1, \ldots, \lambda_n)) = \prod_{i=1}^{n} \lambda_i^{r_i}.
\]  

On the set of \(n\)-tuples of integers, we have the usual partial ordering: \((r_1, r_2, \ldots, r_n)\) is called positive iff \(r_1 + \cdots + r_n = 0\) and \(r_1, r_1 + r_2, \ldots, r_1 + r_2 + \cdots + r_{n-1}\) are nonnegative, and \(\lambda \geq \mu\) iff \(\lambda - \mu\) is positive. A finite-dimensional representation of \(U(\mathcal{H})\), when restricted to \(T\), splits into one-dimensional subrepresentations. The representations with nonzero multiplicity are called weights, and a vector whose orbit under \(T\) spans a one-dimensional subspace is called a weight vector. The isomorphism class of an irreducible representation of \(U(\mathcal{H})\) is determined by its highest weight.

For \(I = \{i_1, i_2, \ldots, i_k\}\), where \(1 \leq i_1 < i_2 < \cdots < i_k \leq n\), let us introduce the following notation:

\[
e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} = \frac{1}{\sqrt{k!}} \sum_{\pi \in S_k} \sigma(\pi) e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(k)}},
\]

where \(S_k\) is the symmetric group on \(k\) elements, and \(\sigma : S_k \rightarrow \{1, -1\}\) denotes the alternating representation. The set \(\{e_{[i_1, \ldots, i_k]}\} | 1 \leq i_1 < i_2 < \cdots < i_k \leq n\) forms an orthonormal basis of \(\bigwedge^k \mathcal{H}\), and therefore every \(k\)-fermion pure state can be expressed uniquely as a linear combination of these vectors:

\[
\psi = \sum_{I \in \binom{[n]}{k}} \psi_I e_I, \quad \text{where} \quad \sum_{I \in \binom{[n]}{k}} |\psi_I|^2 = 1.
\]  

(Here, we have used the short notation \([n] = \{1, 2, \ldots, n\}\) and \(\binom{[n]}{k}\) denotes the set of \(k\)-element subsets of \([n]\).) For each \(m \in \mathbb{N}\), the \(m\)th power of \(\psi\) is a vector in \(S^m(\bigwedge^k \mathcal{H})\):

\[
\psi^m = \sum_{I_{1}, \ldots, I_{m}} \psi_{I_{1}} \psi_{I_{2}} \cdots \psi_{I_{m}} e_{I_{1}} e_{I_{2}} \cdots e_{I_{m}}.
\]

We find a vector in \(S^m(\bigwedge^k \mathcal{H})\) which generates an irreducible \(U(\mathcal{H})\)-representation. In general, we cannot say much about all the irreducible subrepresentations, but we always have one weight vector, \(e_{[m, \ldots, m]}^{e_{[m, \ldots, m]}, \ldots, e_{[m, \ldots, m]}}\) corresponding to the highest weight, which is easily calculated to be \((m, m, \ldots, m, 0, \ldots, 0)\) with \(k\) nonzero entries. We now have that \(U(\mathcal{H})e_{[m, \ldots, m]}^{e_{[m, \ldots, m]}, \ldots, e_{[m, \ldots, m]}} := W\) is irreducible. The next step will be to find an orthonormal basis for \(W\).

Our first goal will be to find a generating set for \(W\) as a linear space, and then we can orthogonalize it to yield an orthonormal basis. To this end, we will use the fact that \(W\) is also an irreducible representation of \(GL(n, \mathbb{C})\) whose action on \(S^m(\bigwedge^k \mathcal{H})\) is defined in the same way as that of \(U(\mathcal{H})\).

In order to find a generating set which is easy to handle, we will look for one that is the union of orbits under \(S_n \leq GL(n, \mathbb{C})\) (possibly up to a nonzero multiple) which permutes the basis elements of \(\mathcal{H}\). It turns out that we can also require that the generating set consists of weight vectors. We will call sets with these properties good.
Definition. Let \( S \subseteq \mathbb{S}^n(\bigwedge^k \mathcal{H}) \) be a subset, \( e_1, \ldots, e_n \) an orthonormal basis in \( \mathcal{H} \) and \( S_\pi \subseteq U(\mathcal{H}) \) the subgroup which permutes these basis elements. The subset \( S \) will be called good (with respect to this basis) if it has the following two properties.

1. The subset
   \[
   C S := \bigcup_{w \in S} C w \subseteq S^n(\bigwedge \mathcal{H})
   \]
   is fixed under the action of \( S_\pi \).
2. If \( v \) is an element of \( S \), then if we write \( v \) as a polynomial in the vectors \( \{e_I\}_{I \in \mathcal{V}} \) then every index \( i \in [n] \) appears the same number of times in every term. Or equivalently, \( v \) is a weight vector for the maximal torus fixing the given orthonormal basis.

We can immediately see that \( \{e_I^m\}_{I \in \mathcal{V}} \) is the smallest good subset containing \( e_{1,2,3,\ldots,k}^m \).

To reach every element in \( W \), we will use the fact that \( GL(n, \mathbb{C}) \) is generated by matrices of the form \( u_{ij}(s) = id + se_{ij} \) where \( E_{ij} \) is a matrix with 1 at the intersection of the \( i \)th row and the \( j \)th column, and zeros everywhere else. We need to know how these matrices act on the basis elements of \( \bigwedge^k(\mathcal{H}) \). One can calculate using equation (10) that

\[
\begin{align*}
   u_{ij}(s) \cdot e_I &= \begin{cases} 
   e_I & j \notin I \\
   e_I + (-1)^{(\gamma(I,j))} s e_{I \cup \{j\}\backslash \{i\}} & j \in I, \ i \notin I \\
   e_I & i, j \in I.
   \end{cases}
\end{align*}
\]

The first and last cases are not interesting, but the second one allows us to build a generating set step by step starting from the above-mentioned elements. Keeping track of the appearing sign could cause some difficulty, but we can overcome this by letting \( \varepsilon_{abc} = -\varepsilon_{bac} \) etc and simply substituting \( j \) with \( i \) without reordering the indices.

Observe that when \( u_{ij}(s) \) acts on a degree \( m \) polynomial in the \( e_I \)-s, we get a polynomial in \( s \) with coefficients in \( S^n(\bigwedge \mathcal{H}) \). Since \( W \) contains this polynomial for any \( s \in \mathbb{C} \), and it is a linear subspace, \( W \) must also contain the coefficient of \( s' \) for each \( 0 \leq l \leq m \) (because of non-vanishing of a Vandermonde determinant). Using this method, one can calculate in a few steps a generating set for the isotypic (in fact, irreducible) subspace corresponding to the highest weight. The following lemma shows which terms should one concentrate on:

Lemma. Let \( W \subseteq S^n(\bigwedge^k \mathcal{H}) \) be an invariant subspace and \( S \subseteq W \) a good subset.

Suppose that \( w \in S \) and \( i \neq j \) are indices such that \( i \) does not appear in \( w \) when written in the monomial basis as above. Then,

(a) The coefficients of every power of \( s \) in \( u_{ij}(s) \cdot w \) as a polynomial in \( s \) are weight vectors.
(b) If the degree of this polynomial is \( d \), then the one-dimensional subspaces spanned by the coefficients of \( s' \) and \( s^{d-r} \) are in the same \( S_\pi \)-orbit.
(c) The coefficient of the constant and the leading terms is contained in \( CS \).
(d) If \( Cw = \mathbb{C} \pi \cdot w' \) for some \( \pi \in S_n \), then the minimal good subsets containing \( S \) and each coefficient in the polynomial \( u_{ij}(s) \cdot w \) or \( u_{\sigma^{-1}(i)\pi^{-1}(j)}(s) \cdot w' \) generate the same subspace.

Proof.

(a) \( u_{ij}(s) \cdot e_{i_1} e_{i_2} \cdots e_{i_m} = (e_{I_1} + se_{I_1})(e_{I_2} + se_{I_2}) \cdots (e_{I_m} + se_{I_m}) \) where \( I'_l \) is obtained from \( I_l \) by replacing \( j \) with \( i \) if \( I_l \) contains \( j \) and \( e_{I_l} = 0 \) else. The coefficient of \( s' \) contains exactly those terms in the expansion in which the number of replaced \( j \)-indices is \( l \).

(b) \( d \) is the (common) number of occurrences of the index \( j \) in each term of \( w \). The coefficient of the \( s^{d-r} \) term is therefore proportional to the coefficient of \( s' \) under the transposition swapping \( e_i \) and \( e_j \).
(c) The constant term is \( w \).

(d) Let \( \pi \in S_r \subseteq U(\mathcal{H}) \) be an element such that \( \mathbb{C}w = \mathbb{C}\pi \cdot w' \). Then,

\[
\mathbb{C}u_{ij}(s) \cdot w = u_{ij}(s)\mathbb{C}w
\]

\[
= u_{ij}(s)\mathbb{C}\pi \cdot w'
\]

\[
= \mathbb{C}u_{ij}(s)\pi \cdot w'
\]

\[
= \mathbb{C}\pi u_{s^{-1}(j)s^{-1}(j)}(s) \cdot w'.
\]  
(15)

**Corollary.** If \( S \subseteq S^m(\wedge^k \mathcal{H}) \) is a good subset and \( w \in S \) such that in each term of \( w \) the index \( j \) appears exactly once and \( w \) does not contain the index \( i \), then \( u_{ij}(s) \cdot w \in \langle S \rangle \) for all \( s \in \mathbb{C} \).

**Proof.** In this case, \( u_{ij}(s) \cdot w \) is a degree 1 polynomial in \( s \); therefore, by the lemma above, both of its terms are in \( \mathbb{C}S \), and hence their sum is in \( \langle S \rangle \).

To sum up, we begin with the vector \( e_{12}^m \), then act on it and the distinct types of obtained coefficients of \( s \) successively with the matrices \( u_{ij}(s) \), as long as we get new types of vectors. Finally, we take union of the \( S_r \)-orbits of the vectors we have obtained. This will result in a generating set of \( W \).

Once we have a generating set, we orthogonalize it, and for each vector \( w \) in the orthogonal set, we calculate the value of \( |\langle w, \psi^m \rangle|^2 \|w\|^{-2} \). Finally, the sum of these numbers is the value of the invariant evaluated on the state \( \psi \). Explicitly, suppose that \( \psi = \sum_i \psi_i e_i \), and \( w = \sum_{k_1, \ldots, k_d} \beta_{k_1, \ldots, k_d} e_{k_1}^{l_1} \cdots e_{k_d}^{l_d} \), where \( d = \binom{d}{i} \) and \( I_1, \ldots, I_d \) are the possible \( k \)-element subsets of \([n]\), and \( k_1, \ldots, k_d \) run over nonnegative integers such that their sum equals \( m \). Then, by equation (4)

\[
\langle w, \psi^m \rangle = \left( \sum_{k_1, \ldots, k_d} \beta_{k_1, \ldots, k_d} e_{k_1}^{l_1} \cdots e_{k_d}^{l_d} \right) \left( \sum_{k_1', \ldots, k_d'} \psi_{l_1}^{k_1'} \cdots \psi_{l_d}^{k_d'} \right) \left( \sum_{k_1, \ldots, k_d} e_{k_1}^{l_1} \cdots e_{k_d}^{l_d} \right)
\]

\[
= \sum_{k_1, \ldots, k_d} \beta_{k_1, \ldots, k_d} \psi_{l_1}^{k_1} \cdots \psi_{l_d}^{k_d} \left( \sum_{k_1, \ldots, k_d} \psi_{l_1}^{k_1} \cdots \psi_{l_d}^{k_d} \right) \|e_{k_1}^{l_1} \cdots e_{k_d}^{l_d}\|^2.
\]  
(16)

We remark that if we are to use these invariants as measures of entanglement, then, taking into account constraint (8) and the fact that the \( m \)th power of a decomposable state is always in the irreducible subspace generated by \( e_{12}^m \), we should use \( 1 - \langle \psi^m, P_W \psi^m \rangle = \langle \psi^m, P_W \psi \rangle \), or the invariants associated with the subspaces other than \( W \).

If we wanted to calculate the projectors of the other isotypic subspaces, then we simply needed to take the orthogonal complement of \( W \), and find the weight vectors corresponding to the highest weight, and proceed with it the same way as we did with \( e_{12}^m \).

**5. Examples**

**5.1. \( k = m = 2 \) case**

The first non-trivial case is the space of quadratic polynomials in vectors of the space of two fermions. As we have seen, a weight vector with maximal weight is \( e_{12}^2 \); therefore, \( W := GL(n, \mathbb{C}) e_{12}^2 \) contains \( e_{ij}^2 \) for \( 1 \leq i < j \leq n \). In the next step, we let \( u_{ij}(s) \) act on an element:

\[
u_{ij}(s)(e_{ij}^2) = (e_{ij} + se_{ik})^2 = e_{ij}^2 + 2se_{ij}e_{ik} + s^2e_{ik}^2.
\]  
(17)
This shows that we must add $e_{ij}e_{ik}$ for each triple $i, j, k$, where the appearing indices are distinct. Now as

$$u_{ij}(s)(e_{ij}e_{ik}) = (e_{ij} + s e_{ij})(e_{ij} + s e_{ik}) = e_{ij}e_{ik} + s(e_{ij}e_{ik} + e_{ij}e_{ik}) + e_{ij}e_{ik}$$

we also have to add $e_{ij}e_{ik} + e_{ik}e_{ij}$ for each combination of indices.

By the corollary after the lemma, we are ready, but it is instructive to verify the dimension of the generated subspace. Clearly, \$\{e_{ij}\}_{1 \leq i < j \leq n} \cup \{e_{ij}e_{ik}\}_{1 \leq i < j < k \leq n}$ consists of pairwise orthogonal elements. The third type in the generating set is $\{e_{ij}e_{ik} + e_{ik}e_{ij}\}$ which is seen to generate a two-dimensional space for each set of four indices, and these subspaces are pairwise orthogonal and also orthogonal to the other elements. Therefore,

$$\dim W = \left(\begin{array}{c} n \\ 2 \end{array}\right) + n\left(\begin{array}{c} n-1 \\ 2 \end{array}\right) + 2\left(\begin{array}{c} n \\ 4 \end{array}\right) = \frac{n^2(n^2-1)}{12}$$

which is exactly the dimension of the irreducible representation of $GL(n, \mathbb{C})$ corresponding to the partition $(2, 2)$.

Orthogonalization needs to be performed only within the two-dimensional subspaces, and this leads to the vectors $e_{ij}e_{kl} + e_{kl}e_{ij}$ and $e_{ij}e_{ik} + 2e_{il}e_{jk} - e_{ik}e_{jl}$ for $1 \leq i < j < k < l \leq n$. The expression for the invariant corresponding to $W$ is therefore (using equation (16))

$$I_{(2,2)}(\psi) = \langle \psi, P_W \psi \rangle$$

$$= \sum_{1 \leq i < j \leq n} \left| \psi_{ij} \right|^2 + \sum_{i=1}^n \sum_{1 \leq j < k \leq n, j \neq k} 2\left| \psi_{ij} \psi_{ik} \right|^2$$

$$\quad + \sum_{1 \leq i < j < k < l \leq n} \left( \left| \psi_{ij} \psi_{kl} + \psi_{ik} \psi_{jl} \right|^2 + \frac{1}{3} \left| \psi_{ij} \psi_{ik} + 2\psi_{il} \psi_{jk} - \psi_{ik} \psi_{jl} \right|^2 \right).$$

In this case, we can also show that $W^\perp$ is irreducible. To this end, let us recall that for $n = 4$, there exists a degree 2 $SL(4, \mathbb{C})$-invariant over $\wedge^2 \mathbb{C}^4$, namely the polynomial in the Plücker relation which is known to be a sufficient and necessary condition of separability. The subrepresentation generated by this polynomial is the representation indexed by the partition $(1, 1, 1, 1)$; therefore, this one must appear also in the $n \neq 4$ case. As the dimension of this is $\binom{n}{2}$, and

$$\dim W + \binom{n}{4} = \frac{n(n-1)(n^2-n+2)}{8} = \dim S^2 \left( \bigwedge^2 \mathbb{C}^n \right)$$

therefore $W^\perp$ is irreducible, and the unitary invariant associated with it gives a generalization of the Plücker relation. The explicit formula turns out to be simpler than the previous one:

$$I_{(1,1,1,1)}(\psi) = \langle \psi, P_{W^\perp} \psi \rangle = \sum_{1 \leq i < j < k < l \leq n} \frac{2}{3} \left| \psi_{ij} \psi_{kl} + \psi_{ik} \psi_{jl} + \psi_{il} \psi_{jk} \right|^2.$$

5.2. $k = 2, m = 3$ case

In this case, a weight vector for the highest weight is $e_{12}^3$. Again, $W := \langle GL(n, \mathbb{C})e_{12}^3 \rangle$. We are looking for a generating set of $W$. We extend $e_{12}^3$ into a good set $\{e_{ij}\}_{1 \leq i < j \leq n}$. Now we need to add the coefficient of $s$ in

$$u_{32}(s)(e_{12}^3) = (e_{12} + s e_{13})^3 = e_{12} + 3s e_{12} e_{13} + 3s^2 e_{12} e_{13} + s^3 e_{13}$$

(23)
and one vector from each element of the orbit of the subspace generated by it: \( \{ e_{ij}^2 e_{ik} \}_{i,j,k \in [n]} \).

The next steps are

\[
\begin{align*}
\text{u}_{43}(s)(e_{12}^2 e_{13}^2) &= e_{12}(e_{13} + s e_{14})^2 = \cdots + 2 s e_{12} e_{13} e_{14} + s^2 (\cdots) \\
\text{u}_{m1}(s)(e_{12}^2 e_{13}^2) &= (e_{12} + s e_{m2})(e_{13} + s e_{m3})^2 \\
&= \cdots + s (2 e_{12} e_{13} e_{m3} + e_{m2} e_{14}^2) + s^2 (\cdots) + s^3 (\cdots).
\end{align*}
\]

(24)

Here, \( m = 2 \) is special, and in this case the second term in the coefficient of \( s \) vanishes; hence, we have to add \( \{ 2 e_{ij} e_{ik} e_{nk} + e_{mj} e_{mk}^2 \} \) for any ordered pair of disjoint pairs \((i, j, k, m)\). The remaining steps are

\[
\begin{align*}
\text{u}_{m1}(s)e_{12} e_{13} e_{14} &= (e_{12} + s e_{m2}) (e_{13} + s e_{m3}) (e_{14} + s e_{m4}) \\
&= \cdots + s (e_{m2} e_{13} e_{14} + e_{12} e_{m3} e_{14} + e_{12} e_{13} e_{m4}) + s^2 (\cdots) + s^3 (\cdots) \\
\text{u}_{m1}(s)(e_{52} e_{13} e_{14} + e_{12} e_{53} e_{14} + e_{12} e_{13} e_{54}) &= \cdots + s (e_{52} e_{m3} e_{14} + e_{52} e_{13} e_{m4} \\
&+ e_{m2} e_{53} e_{14} + e_{12} e_{53} e_{m4} + e_{m2} e_{13} e_{54} + e_{12} e_{m3} e_{54}) + s^2 (\cdots).
\end{align*}
\]

(25)

(26)

Here, \( m \leq 5 \) does not lead to a new subspace.

It turns out that the vectors obtained so far are enough to generate \( W \). In this case, orthogonalization turns out to be a bit lengthy, especially in the case of the six-term vectors like in equation (27). These span a five-dimensional subspace for each six-element set of indices \( i_1, \ldots, i_6 \). For these, the coefficients of the monomials are given as a matrix:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 3 & 0 \\
0 & 0 & 2 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & 1 & 0 \\
4 & 2 & 2 & 2 & 1 & 1
\end{pmatrix}
\]

(28)

The orders of the monomials are \((12)(34)(56), (12)(35)(46), (12)(36)(45), (13)(24)(56), (13)(25)(46), (13)(26)(45), (14)(23)(56), (14)(25)(36), (14)(26)(35), (15)(23)(46), (15)(24)(36), (15)(26)(34), (16)(23)(45), (16)(24)(35), (16)(25)(34), \) where \((ab)(cd)(ef)\) is a short notation for \( e_{i_1,i_2} e_{i_3,i_4} e_{i_5,i_6} \). The norms inverse squared of these vectors are

\[
1, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5},
\]

(29)

respectively. The orthogonal generators coming from the remaining vectors are given in table 1.

Using these data, the value of the invariant \( I_{(3,3)} \), can be calculated in a straightforward way, but the full formula is too long to be presented explicitly.

The orthogonal complement of \( W \) clearly has a highest weight of \((2,2,1,1)\), and we could find a generator of the unique one-dimensional weight space corresponding to it, and calculate the projector of its invariant subspace. Instead of this, we follow another approach. According to the plethysm

\[
s_{(3)}[s_{(1,1,1)}] = s_{(3,3)} + s_{(2,2,1,1)} + s_{(1,1,1,1,1)}
\]

(30)

for \( n = 6 \), an \( SL(6, \mathbb{C}) \)-invariant polynomial appears. It is easy to guess how this should look like: for a state \( \psi \in \wedge^2 \mathbb{C}^6 \), we can construct \( \psi \wedge \psi \wedge \psi \) which is an element of \( \wedge^6 \mathbb{C}^6 \), a one-dimensional vector space on which \( GL(6, \mathbb{C}) \) acts by multiplication with the determinant. Therefore, this element remains unchanged under \( SL(6, \mathbb{C}) \), and its norm squared is an \( U(6, \mathbb{C}) \)-invariant polynomial in the coefficients of \( \psi \) and their conjugates. Our
invariant corresponding to the subrepresentation indexed by the partition \((1, 1, 1, 1, 1, 1)\) must be proportional to it. Explicitly, it equals

\[
\frac{1}{11520} \sum_{\pi \in S_6} \sigma(\pi) \prod_{j=1}^{6} \psi_{\pi(1), \pi(2)} \psi_{\pi(3), \pi(4)} \psi_{\pi(5), \pi(6)} |^{2}.
\] (31)

Here, the sum is over all the permutations, but actually there are 15 different terms, each counted \(48 = 3! \cdot 2^3\) times. Alternatively, we could sum over the partitions of \([n]\) into three two-element sets.

The \(n \geq 6\) case can be obtained similarly to the previous section. Taking all the six-element subsets of \([n]\) polynomials like this span an \(6^n\)-dimensional subspace which is also the dimension of the invariant subspace we are looking for. Therefore, in the general case the invariant is

\[
I_{(1,1,1,1,1,1)}(\psi) = \frac{1}{11520} \sum_{\pi \in S_6} \sum_{\pi \in S_6} \sigma(\pi) \prod_{j=1}^{6} \psi_{\pi(1), \pi(2)} \psi_{\pi(3), \pi(4)} \psi_{\pi(5), \pi(6)} |^{2},
\] (32)

where \(I = \{i_1, \ldots, i_6\}\).

These two invariants are linearly independent, and they sum to 1 with the one associated with the third irreducible subspace.

5.3. \(k = 3, m = 2\) case

Now we turn to the first case with more than two particles. In \(S^2(\wedge^3 \mathfrak{h})\), the vector with highest weight is \(e_{123}^2\). We proceed in a similar way as before:

\[
u_{a3}(x)(e_{123}^2) = e_{123}^2 + 2s e_{123} e_{12n} + s^2 e_{12n}^2
\] (33)

\[
u_{a2}(x)(e_{12} e_{12n}) = \cdots + s(e_{123} e_{12n} + e_{12n} e_{124}) + s^2(\cdots)
\] (34)

\[
u_{a1}(x)(e_{124} e_{154} + e_{153} e_{124}) = \cdots + s(e_{123} e_{154} + e_{123} e_{154} + e_{153} e_{124} + e_{153} e_{124}) + s^2(\cdots).
\] (35)

These vectors already form a generating set; we only need to orthogonalize this set. For a fixed subset of six indices, the vectors of the form like in (35) span a five-dimensional subspace.
Table 2. Orthogonalized generators for the subspace generated by the highest weight vector. Indices shown in one set are indistinguishable for counting purposes.

| Form | Indices | Dimension | \|·\|^{-2} |
|------|---------|-----------|----------|
| \(e_{ijk}^{l}\) | \([i, j, k]\) | \(\binom{0}{3}\) | 1 |
| \(e_{ijkl}^{l}\) | \([i, j], [k, l]\) | \(\binom{0}{3}(\binom{0}{2})\) | 2 |
| \(e_{ijml}^{l} e_{ijkl} + e_{ijkl} e_{iml}\) | \([i], [j, k, l, m]\) | \(2n(\binom{1}{4})\) | 1 |
| \(e_{ijml}^{l} e_{ijkl} + 2e_{ijkl} e_{iml} - e_{ijkl} e_{iml}\) | | | \(\frac{1}{2}\) |

Orthogonal generators for this are again given with the coefficients of the monomials as a matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 2 & 1 & 0 & 0 & 2 & 1 \\
1 & 4 & 2 & -1 & 2 & 1 & -2 & 0 & -2 & -1 \\
1 & 1 & -1 & -1 & 1 & 3 & 1 & -1 & -1 \\
\end{pmatrix}
\] (36)

The orders of monomials are \((123)(456), (124)(356), (125)(346), (126)(345), (134)(256), (135)(246), (136)(245), (145)(236), (146)(235), (156)(234)\), where \((abc)(def)\) is a shorthand notation for the vector \(e_{ia,ib,ic} e_{ia,ib,ic}\). The inverse squared norms of these vectors are

\[
\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{18}, \frac{1}{5}.
\] (37)

respectively. The orthogonal generators coming from the remaining vectors are given in table 2.

The value of \(I_{(2, 2, 2)}\) can now be calculated. This time the orthocomplement is also irreducible, so we get one independent invariant in this case.

6. SLOCC invariants and local unitary invariants

In the examples, we have seen LU invariants with a special property: for a particular value of \(n\), the corresponding irreducible subspace becomes one dimensional, and the subspace is pointwise fixed under the action of \(SL(n, \mathbb{C})\), that is, the LU invariant turns out to be a SLOCC invariant. Let us examine this case in more detail.

The irreducible polynomial representation of \(SL(n, \mathbb{C})\) indexed by the partition \(\lambda\) is one dimensional precisely when \(\lambda\) consists of equal parts. In this case, \(\lambda = (r, r, \ldots, r)\) is a partition of \(nr\); hence, a necessary condition for it to occur as a subrepresentation of \(S^{m}(\wedge^{k} \mathbb{C}^{n})\) is that \(mk = nr\), and in this case \(GL(n, \mathbb{C})\) acts on it by multiplication with the \(r\)th power of the determinant. The norm squared is therefore invariant under \(U(n, \mathbb{C})\).

In our notations, this subspace is spanned by a polynomial \(w\) in the basis vectors \(e_{1}, \ldots, e_{n}\). \(w\) is a weight vector with weight \((r, r, \ldots, r)\), and it generates a one-dimensional \(U(n, \mathbb{C})\)-invariant subspace. The crucial thing is that when we increase the dimension \(n\) of the single-particle state space to \(n'\), \(w\) remains a weight vector that generates an irreducible \(U(n', \mathbb{C})\)-invariant subspace, but it is no longer one dimensional. Therefore, the invariant corresponding to this subspace will be a generalization of the SLOCC invariant we have begun with, but is now only a unitary invariant.

The explicit form of the resulting invariant can be obtained in general using the method outlined above: we must act on it with \(u_{ij}(x) - x\) and elements of \(S_{w}\). A particularly simple
special case is when \( r = 1 \). In this case, the dimension of the representation corresponding to \( \lambda \) is \( \binom{n}{r} \), and an orthonormal basis can be obtained by acting on \( w \) by the elements of \( S_r \). Therefore, the invariant can be obtained by calculating the value of the SLOCC invariant with the initial index set \([n]\) replaced by every element of \( \binom{[n]}{r} \), and summing their absolute values squared.

### 7. Distinguishable particles

We can also obtain many (but not all) LU invariants for distinguishable subsystems from our fermionic ones \([5–7]\). Suppose that we would like to find LU invariants for a quantum system containing \( k \) subsystems with Hilbert space dimensions \( n_1, \ldots, n_k \), and let \( n = n_1 + \cdots + n_k \). Then, using the branching rule

\[
U(n, \mathbb{C}) \cong U(n_1, \mathbb{C}) \times \cdots \times U(n_k, \mathbb{C})
\]

we can identify the full state space with a subspace of a fermionic Hilbert space with \( n \)-dimensional single-particle state space. Then, we can pull back the fermionic invariants constructed by the above method. For example, for \( k = 2 \), the restriction of \( I_{1(1,1,1)} \) is proportional to the square of the norm of the concurrence vector \([3]\).

Observe, that these special LU invariants have a larger symmetry group, in particular, if \( n_1 = \cdots = n_k \), then they are also permutation invariant.

As an other example, let us consider the quantum system of three qubits. Using the scheme outlined above, its Hilbert space can be thought of as a subspace of the Hilbert space of a three-fermion system with six single-particle states. In the previous section (see equation (36) and table 2), we have derived a formula for a LU invariant which can be restricted to this subspace. The restriction turns out to be the following:

\[
I_{(2,2,2)}(\psi) = \sum_{i,j,k=0} \left| \psi_{ijk} \right|^2 + 2 \sum_{i,j=0} \left( |\psi_{i0j0}\psi_{1j1}|^2 + |\psi_{i0j1}\psi_{1j1}|^2 + |\psi_{i0j1}\psi_{1j1}|^2 \right)
+ \sum_{i=0} \left( |\psi_{i01}\psi_{1i0}|^2 + \frac{1}{3}|\psi_{i01}\psi_{1i0} + 2\psi_{0i0}\psi_{111}|^2 + |\psi_{i01}\psi_{1i0}|^2 \right)
+ \frac{1}{3}|\psi_{0i1}\psi_{1i0} + 2\psi_{0i0}\psi_{111}|^2 + |\psi_{0i1}\psi_{1i0}|^2 + \frac{1}{3}|\psi_{0i1}\psi_{1i0} + 2\psi_{0i0}\psi_{111}|^2
+ \frac{1}{2}|\psi_{0i0}\psi_{1i0}|^2 + \frac{1}{6}|\psi_{0i0}\psi_{111} + 2\psi_{0i1}\psi_{1i0}|^2 + \frac{1}{6}|\psi_{0i0}\psi_{111} + 2\psi_{0i1}\psi_{1i0}|^2
+ \frac{1}{18}|\psi_{0i0}\psi_{111} - 2\psi_{0i1}\psi_{1i0} - 2\psi_{0i1}\psi_{110}|^2
+ \frac{1}{9}|\psi_{0i0}\psi_{111} + 3\psi_{0i1}\psi_{1i0} + 3\psi_{0i1}\psi_{100}|^2,
\]

where \( \psi = \sum_{i,j,k=0} \psi_{ijk} e_i \otimes e_j \otimes e_k \) is a three-qubit state. This quantity is invariant under LU transformations and permutations of the three subsystems. Plugging in the coefficients of certain states into this formula reveals that for a separable state the value is 1 as expected, for the GHZ state it is 3/4, for the W-state it is 7/9, and for a biseparable state with maximal entanglement shared by two qubits the value is 5/6. This behaviour is in contrast with the well-known three-qubit invariant, Cayley’s hyperdeterminant, which is only sensitive to GHZ-type entanglement.

\[
E_{111}(\psi) = \begin{vmatrix} \psi_{000} & \psi_{001} & \psi_{010} & \psi_{011} \end{vmatrix}^2
\]

and

\[
E_{111}(\psi) = \begin{vmatrix} \psi_{000} & \psi_{001} & \psi_{010} & \psi_{011} \end{vmatrix}^2
\]
Note that these values are 1 minus the quarter of the values of the entanglement measure proposed in [8] for three qubits, so one is tempted to conjecture that the restriction of $4-4I(2,2,2)$ equals their measure. In their paper, this measure appears as a member of an infinite family of multiqubit entanglement measures. It would be interesting to see whether every member can be found using our method. If this is the case, then we could generalize them to arbitrary dimensional single-particle states and also to fermionic systems.

A more detailed treatment of this method of constructing entanglement measures for distinguishable subsystems from fermionic measures along with more examples can be found in [6].

8. Conclusion

In this paper, a certain class of entanglement measures for fermionic quantum systems has been introduced and studied. A way to obtain their explicit form is presented, and it was pointed out that this form is independent of the dimension of the single-particle state space. Some examples are discussed in detail. The connection to SLOCC invariants and the case of distinguishable subsystems are also mentioned.

At this point a number of questions arise. Further study is needed to explore the behaviour of these LU invariants. For instance: are there any entanglement monotones among them, and is there a way to characterize these? Could we find the convex roof extension of any of them? Is there a physical meaning for these quantities, and in what way do they measure entanglement?

Also, a similar method could be applied directly to quantum systems with distinguishable particles, or to mixed states. It would be interesting to see if one could obtain entanglement measures this way which can be useful in practice.

Appendix. Representation theory of the unitary group

In this appendix, the basic aspects of the representation theory of the unitary groups is summarized. These and many more can be found in many textbooks, see e.g. [9].

The unitary group $U(n, \mathbb{C})$ consists of $n \times n$ complex matrices satisfying $A^* = A^{-1}$. Let $T \leq U(n, \mathbb{C})$ be the subgroup of diagonal matrices, and $T$ is a maximal torus in $U(n, \mathbb{C})$. We are interested solely in the finite-dimensional polynomial representations of $U(n, \mathbb{C})$ which are determined by their characters which are in turn uniquely encoded in the restriction of the characters to $T$. The characters themselves are symmetric polynomials in the eigenvalues (the diagonal elements, in the case of $T$).

For example, the action of $U(n, \mathbb{C})$ on the $n$-dimensional vector space $\mathbb{C}^n$ of column vectors by left multiplication is called the standard representation. Its character is represented by the symmetric polynomial $x_1 + x_2 + \cdots + x_n$.

A representation $V$ of $U(n, \mathbb{C})$ is also a representation of $T$ whose irreducible representations are onedimensional, and hence can be considered to be homomorphisms $T \to \mathbb{C}^\times$. An irreducible representation of $T$ with nonzero multiplicity in $V$ is called a weight. The weight space corresponding to weight is the union of the subrepresentations in $V$ isomorphic to a given weight.

Continuing the previous example, the weights of the standard representation are the representations $\rho_k : \text{diag}(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_k$, and the weight space of $\rho_k$ is spanned by the $k$th standard basis element. From now on, the product $\rho_1^{r_1} \otimes \cdots \otimes \rho_n^{r_n}$ will be simply denoted by $(r_1, \ldots, r_n)$. 

There is the usual partial ordering on the set of weights which we identify with \( \mathbb{Z}^n \): \((r_1, r_2, \ldots, r_n)\) being positive iff \( r_1 + \cdots + r_n = 0 \) and \( r_1, r_1 + r_2, \ldots, r_1 + r_2 + \cdots + r_{n-1} \) are nonnegative, and \( \lambda \succeq \mu \) iff \( \lambda - \mu \) is positive. The isomorphism class of an irreducible representation of \( U(n, \mathbb{C}) \) is determined by its highest weight and the weight space for the highest weight is one dimensional. Using this fact, we can decompose any finite-dimensional representation of \( U(n, \mathbb{C}) \) into the direct sum of irreducible representations.

The dimension of an irreducible representation of \( U(n, \mathbb{C}) \) with the highest weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is

\[
\dim S(\lambda_1, \ldots, \lambda_n)V = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]

(A.1)

Note that if we start to increase \( n \) and pad \( \lambda \) with zeros on the right, then the dimension in the function of \( n \) turns out to be a polynomial of degree \( \lambda_1 + \cdots + \lambda_n \). In fact, the entries of the representing matrices are polynomials in the entries of the represented matrix with this same degree. The symmetric polynomial giving the character of this representation is the Schur polynomial.

Calculating symmetric and exterior powers, tensor products, decomposition into irreducible subrepresentations and many more can be done working only with symmetric polynomials. This is a crucial fact for our purposes, as a symmetric polynomial of degree \( d \) is determined by its terms containing unknowns only from a fixed set of \( d \) variables. Indeed, no term can contain more than \( d \) variables, so the missing ones are obtained by permutations of the variables. This fact implies that the decomposition of \( S^d(\mathbb{C}^n) \) into irreducible \( U(n, \mathbb{C}) \) representations is independent of \( n \) when \( n \leq km \).

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