Stability of leaderless multi-agent systems.
Extension of a result by Moreau

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Abstract. The paper presents a result which relates connectedness of the interaction graphs in a multi-agent systems with the capability for global convergence to a common equilibrium of the system. In particular we extend a previously known result by Moreau by including the possibility of arbitrary bounded time-delays in the communication channels and relaxing the convexity of the allowed regions for the state transition map of each agent.

1 Introduction

Recent years have witnessed a growing interest in the study of the dynamical behaviour of the so called multi-agent systems. Roughly speaking these can be thought of as complex dynamical systems composed by a high number of simpler units, the agents. Each of them updates its state according to some rule, whose Input-Output dynamics are typically much simpler and much better understood, and on the basis of the available information coming from the other agents. All of them, though not necessarily identical, share in fact some common feature of interest (say for instance a given output variable) and are coupled together by communication channels. The focus of the current research is precisely on how the global behaviour of the system, (for instance questions concerning the global stability or the overall synchronization) is influenced by the topology of the coupling on one hand (this is an analysis problem in many respects) or the dual question of how to induce a certain desired property of the ensemble based on some form of local coupling for the agents. Problems of this nature arise in many different fields, such as in the theory of coupled oscillators [5, 11], in neural networks [3], in economics or in the manouversing of groups of vehicles [6]. For instance in [7] the so called rendezvous problem is considered, namely how to design a local updating rule, based on nearest neighbor interactions, which would ensure convergence of all of the agents to an unspecified common meeting point. Emergence of a global behaviour is therefore a consequence of the local updating rule, without the need for a leader nor of centralized directions being broadcasted.

Despite the common traits, the most powerful results are obtained when specializing to systems of a given simple form. Hereby we take a slightly different approach. The emphasis is on how the topology of interconnections between agents (possibly time-varying) affects the convergence of all agents to a common equilibrium. This analysis will be carried out in the presence of limited transmission speed of the information between the agents. In particular, we propose an extension of the contributions by Moreau [8, 9], mainly in two directions:

- The new setting allows the presence of arbitrary bounded communication delays.
- A central assumption in the results [8, 9], namely that the future evolution of the studied system is constrained to occur in the convex hull of the agents states, is removed.

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The first aspect comes as a very natural question both from a practical and a theoretical point of view. Communication delays are in fact ubiquitous in the “real” world and it is well-known their potential destabilizing effect in conjunction with feedback loops, here induced by the graph topology of the communication channels which need not be of a hierarchical type. It is therefore remarkable to see how, at least in the specific set-up we are considering, this destabilizing effect does not take place and the same global behaviour of the multi-agent system in terms of convergence to a common equilibrium follows also in the extended set-up.

The second extension deals with convexity issues; one of the technical tools used in order to enforce a common behaviour in systems whose state takes value in Euclidean space, is to have local evolutions point always inside the convex hull of all variables. This makes life easier in a certain respect but it is an unnatural assumption in more general contexts, for instance when oscillators networks are considered (these are typically modeled as systems evolving on a torus) or systems evolving in partially obstructed Euclidean spaces (for instance on a plane minus a circle). Relaxing convexity is meant as a first step in the quest for stability conditions which can work in more general spaces.

Before going on further, we present the main elements of this construction, developed below. The multi-agent system under study will be described by a time-dependent graph $A(t)$, describing the transfer of information between the agents at time $t$, and a set of rules according to which each agent updates its state at time $t + 1$. The definition of the latter is done by the introduction of two types of objects, which we present now (complete definitions are to be found in Section 2 below).

- A set-valued map $\sigma$, is defined, which associates to the set of present and past states of the agents a compact set in the state space common to all the agents. This map will play the central role of a set-valued Lyapunov function for the system.

- It is then necessary to define the rules according to which the agents update their state, given the (possibly delayed) information on the position of the other agents they received. For this, each agent $k$ is attributed a set-valued map $\sigma_k$ which, given the communication graph $A(t)$, defines the set of allowed positions $\sigma_k(A(t))$. An important point here is that, whatever the information received by each agent, the new positions cannot induce an increase of the set-valued Lyapunov function along the trajectories.

The definition of the new class of multi-agent systems studied here is done and commented in Section 2. The stability is studied and the main results are given in Section 3.
Finally, for the comfort of the reader we indicate that the Theorems 1, 2, 3 and 5 in reference [8] are numbered respectively 4, 1, 2 and 5 in [10].

2 A class of multi-agent dynamical systems

This section is devoted to the presentation of the dynamical system under study. We study here a special class of nonlinear difference inclusions with delay, that we write:

\[ x_k(t + 1) \in e_k(\mathcal{A}(t))(\tilde{x}(t)) . \] (1)

Recall that \( x_k(t) \) represents the “position” at time \( t \) of the agent \( k \). The evolution of the latter depends upon the complete system state \( \tilde{x}(t) \) (including delayed components), through the time-varying map \( e_k(\mathcal{A}(t)) \). For a trajectory of (1), we call decision set of agent \( k \) at time \( t \) the value taken by \( e_k(\mathcal{A}(t))(\tilde{x}(t)) \). The specificity of the problem lies in these maps: they depend upon the topology of the inter-agent communications, modeled by the graph \( \mathcal{A}(t) \).

The modeling of the communication network is presented below in Section 2.2. Last, we provide some examples in Section 2.3.

2.1 Inter-agent communications modeling

The first ingredient of the construction is the family of continuous set-valued maps \( e_k(\mathcal{A}) : X^h \ni X \) taking on compact values, and defined for \( k \in \mathcal{N} \) and any directed graph \( \mathcal{A} \). The latter will define, according to the position of the other agents, in which subset of \( X \) agent \( k \) is allowed to choose its future state.

Here, we are concerned by information transfer from the past to the present. In other words, we need to consider graphs in \( X^h \) linking some past and/or present values \( x_k(t - j) \) of the states of an agent \( k \) to another agent \( l \). This motivates the following adaptations of some notions of the graph theory. Recall that \( \mathcal{N} = \{1, \ldots, n\}, \mathcal{H} = \{0, \ldots, h-1\} \), where \( n \) is the number of agents and \( h-1 \) the larger transmission delay.

**Definition 1.** We call directed graph with delays any subset \( \mathcal{A} \) of \( (\mathcal{N} \times \mathcal{H}) \times \mathcal{N} \) satisfying \( ((k,0),k) \notin \mathcal{A} \) for all \( k \in \mathcal{N} \). The elements of \( \mathcal{N} \) are called nodes and an element \( ((k,j),l) \) of \( \mathcal{A} \) is referred to as an arc from \( k \) to \( l \). A node \( k \in \mathcal{N} \) is said connected to a node \( l \in \mathcal{N} \) if there is a path from \( k \) to \( l \) in the graph with delays which respects the orientation of the arcs. Given a sequence of directed graphs with delays \( \mathcal{A}(t) \), a node \( k \in \mathcal{N} \) is said connected to a node \( l \in \mathcal{N} \) across an interval \( I \subseteq \mathbb{N} \) if \( k \) is connected to \( l \) for the graph with delays \( \bigcup_{t \in I} \mathcal{A}(t) \).

Figure 1 provides an example of graph with delays. For the graph represented therein, agents 1 and 2 are mutually connected and agent 3 is connected to 1 and 2, but neither 1 nor 2 is connected to 3. Notice that generally speaking there may exist more than one arc between two distinct nodes, and that a node may be connected to itself (via delayed values).

**Definition 2.** Consider a directed graph with delays \( \mathcal{A} \) and a nonempty subset \( \mathcal{L} \subseteq \mathcal{N} \). The set Neighbors(\( \mathcal{L}, \mathcal{A} \)) is the set of those nodes \( k \in \mathcal{N} \setminus \mathcal{L} \) for which there is \( l \in \mathcal{L} \) such that (at least) one arc from \( k \) to \( l \) exists. When \( \mathcal{L} \) is a singleton \( \{l\} \), the notation Neighbors(\( l, \mathcal{A} \)) is used instead of Neighbors(\( \{l\}, \mathcal{A} \)).

We impose to the maps \( e_k \) the following assumption.

**Assumption A.** For all \( k \in \mathcal{N} \) and all directed graph with delays \( \mathcal{A} \), the set-valued map \( e_k \) is continuous and takes on compact values. Moreover,

- \( e_k(\mathcal{A})(\tilde{x}) = \{x_k\} \) if \( \{x_{i,j} : ((i,j),k) \in \mathcal{A}\} = \{x_k\} \);
- \( e_k(\mathcal{A})(\tilde{x}) \subset \sigma(\{x_k\} \cup \{x_{i,j} : ((i,j),k) \in \mathcal{A} \}) \) otherwise.

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The exact meaning and the properties of the set-valued \( \sigma \) are the subject of Section 2.2. However, we may already make some remarks on the form of the right-hand side of the problem. Clearly, Assumption A implies that the evolution of each agent depends only upon the possibly delayed information received from its neighbors. The case where \( \{x_{i,j} : ((i,j), k) \in A\} = \{x_k\} \) is realized when either the agent \( k \) has no neighbor and the set involved in the formula is empty, or all the (possibly delayed) positions received from the neighboring agents are also equal to the present position \( x_k \) of agent \( k \); in this case, no motion is allowed. We shall see below that in the present framework the use by each agent of the present value of its own position is mandatory for stability, see counterexample in Example 6.

### 2.2 Construction of the decision sets

The second ingredient necessary for the construction of the dynamical system under study is a set-valued map \( \sigma : 2^X \Rightarrow X \), taking on compact values. It has a central role in the definition of the dynamics, and it will be shown afterwards (cf. in particular the proof of Theorem 2) that it plays the role of a “set-valued Lyapunov function” for the studied system.

In order to state the properties that \( \sigma \) should fulfil, we have to introduce beforehand some notions. First of all, define \( S \), a set of subsets of \( X \) in which \( \sigma \) will be compelled to take on its values, as:

\[
S = \{S \subset X : S \text{ compact and } \exists \varphi : X \to X, \varphi \text{ bijective, } \varphi, \varphi^{-1} \text{ Lipschitz and } \varphi(S) \text{ convex}\}. \tag{2}
\]

Important consequences will proceed from the fact that \( \sigma \) takes on values in \( S \), inherited from properties of \( S \) summarized in the following result (see proof of Lemma 1 in Appendix).

**Lemma 1.** Let \( S \) be defined by (2).

1. for any \( S \in \mathcal{S} \), the function \( d_S(x^0, x^1) : S \times S \to [0, +\infty) \) defined as

\[
d_S(x^0, x^1) = \inf \{\text{length}(\psi) : \psi : [0, 1] \to S, \psi(0) = x^0, \psi(1) = x^1, \text{Lipschitz}\}
\]

is well-defined and continuous. Define \( \mu : \mathcal{S} \to \mathbb{R}^+ \) by:

\[
\mu(S) = \max_{x^0, x^1 \in S} d_S(x^0, x^1). \tag{3}
\]

Then, for all \( S \in \mathcal{S} \),
• \( \mu(S) < +\infty \).
• \( \mu(S) = 0 \) if and only if \( S \) is a singleton.
• \( \mu(S) \) is at least equal to the (euclidian) diameter of \( S \), and equal to this value if \( S \) is convex.
• \( \mu \) is lower semicontinuous in \( S \), but nowhere continuous.

2. for any \( S \in \mathcal{S} \), let \( \varphi \) be as in (2) and

\[
\text{ri}(S) = \varphi^{-1}(\text{ri}(\varphi(S)))
\]

where \( \text{ri}(\varphi(S)) \) designates the relative interior of the convex set \( \varphi(S) \), i.e. its interior when regarded as a topological subspace of its affine hull. Then, for all \( S \in \mathcal{S} \),

• \( \text{ri}(S) \) is independent of the choice of \( \varphi \).
• \( \text{ri}(S) = \emptyset \) if and only if \( S \) is a singleton.
• \( \text{int} S \subseteq \text{ri} S \subset \mathcal{S} \).
• \( \text{ri}(S) \) is the relative interior of \( S \) if \( S \) is convex.

Lemma 1 permits to measure the distance between points of a set \( S \in \mathcal{S} \) “along the arcs”. It permits to define extended notions of diameter and of relative interior, which coincide with the usual ones for convex subsets of \( X \). By definition, we call “relative boundary” of sets \( S \) in \( \mathcal{S} \) the following set:

\[
\partial^r(S) = S \setminus \text{ri}(S).
\]

Also, according to the definition of \( d_S \) in Lemma 1, we define, for any subsets \( S' \), \( S'' \) of a set \( S \) in \( \mathcal{S} \) the \( S \)-distance from \( S' \) to \( S'' \) as:

\[
d_S(S',S'') = \inf_{x^0 \in S', x^1 \in S''} d_S(x^0, x^1).
\]

We now gather the properties that \( \sigma \) must fulfil, and afterwards comment on their meaning and consequences.

**Assumption B.** The set-valued map \( \sigma : 2^X \to \mathcal{S} \) is continuous with respect to the topology induced by Hausdorff metric. Moreover, the following should hold:

1. \( S \subseteq \sigma(S) \) with equality if \( S \) is a singleton.
2. \( \sigma(S) = \sigma \circ \sigma(S) \) for all \( S \in 2^X \).
3. \( S' \subseteq S \Rightarrow \sigma(S') \subseteq \sigma(S) \) for all \( S, S' \in 2^X \).
4. If \( S \) is not a singleton, for all \( x \in S \), there exists \( \Sigma_x \subseteq \partial^r\sigma(S) \) such that \( \Sigma_x \cap S \neq \emptyset \) and \( x \notin \Sigma_x \). Moreover, if \( S' \subseteq \sigma(S) \):
   (a) if \( \text{ri} \sigma(S') \cap \Sigma_x \neq \emptyset \), then \( S' \subseteq \Sigma_x \) (and in particular, \( x \notin S' \)).
   (b) if \( d_S(S',\Sigma_x) > 0 \), then \( \mu(\sigma(S')) < \mu(\sigma(S)) \).
5. \( \mu \circ \sigma \) is continuous.

Remark that at this point, the problem under study is fully understandable: our goal is to find stability conditions for systems defined by (1), where the maps \( e_k \) verify Assumption A for a given map \( \sigma \) fulfilling Assumption B and where the meaning of the relative interior \( \text{ri} \) has been defined previously by Lemma 1.

Important consequences of Assumptions B.1 to B.5 are now stated. We shall see further (see Theorem 1) that Assumptions B.1 B.3 are indeed sufficient to forbid increase along time of the natural set-valued Lyapunov function of the system. The additional Assumptions B.4 B.5 induce the strict decrease of the set-valued Lyapunov function (see Theorem 2).

We provide in the following lemma a direct consequence of Assumption B.1.
Lemma 3. Assume Assumptions B.1-B.4 be fulfilled. Then, for any $S \subset X$ we have: $\text{card}(S) > 1 \Rightarrow \mathfrak{r}(\sigma(S)) \neq \emptyset$ and $\mu(\sigma(S)) > 0$.

Proof. By Assumption B.1 it follows from $\text{card}(S) > 1$ that $\sigma(S) > 1$. Since $\sigma(S)$ is not a singleton, it is homeomorphic to a sphere of non-zero dimension. Hence, $\mathfrak{r}(\sigma(S)) \neq \emptyset$. Moreover, $\mu(\sigma(S)) \geq \text{diam}(\sigma(S)) > 0$, since the euclidean diameter of a set vanishes if and only if the set is a singleton.

We now come to the central hypothesis, stated in Assumption B.4. This Assumption applies to arbitrary (but non trivial) groups of agents $S$, which may comprise indifferently true agents or “virtual” agents, viz. informations relative to the position of a true agent at previous sampling times. More closely, for each agent $x$, there exists a portion of the boundary of $\sigma(S)$, denoted by $\Sigma_x$, whose elements are irreversibly attracted outside of it when using information received from any agent not in $\Sigma_x$ (such as $x$ itself) according to the rule edited in Assumption B.1. The second part of Assumption B.4 imposes that such an irreversible escape from $\Sigma_x$ comes with a strict decrease of the diameter of the set-valued Lyapunov function of the system (for convex sets $S$, $S' \subseteq X$, $S' \subseteq S$ implies $\mu(S') \leq \mu(S)$, but this is not true for general sets in $S$ defined by B.1).

Generally speaking, the set $\Sigma_x$ looks like as an union of “faces” of $x\partial \sigma(x)$ containing an extremity of each geodesic of maximal length in $\sigma(S)$ (that is of length equal to $\mu(\sigma(S))$), except the point $x$ itself if it belongs to $x\partial \sigma(x)$. Remark that sets $\Sigma_x$, $\Sigma_y$ associated to different points $x, y$ in $S$ may be equal, and that the union of the sets $\Sigma_x$ over $x \in S$ does not have to cover $x \partial \sigma(S)$.

Similarly to what happens within Moreau’s setting, one has the following result.

Lemma 2. Assume Assumptions B.1-B.4 are fulfilled. Then, $\text{card}(S) > 1 \Rightarrow \text{card}(S \cap x\partial \sigma(S)) \geq 2$.

Proof. Let $x \in S$. Since $S$ is not a singleton, by Assumption B.4 there exists $\Sigma_x \subset x\partial \sigma(S)$ so that $\Sigma_x \cap S \neq \emptyset$. Let $y \in \Sigma_x \cap S$. Since $y \in S$ we can apply the same property to $y$ in order to conclude that there exists $\Sigma_y \subset x\partial \sigma(S)$ and such that $\Sigma_y \cap S \neq \emptyset$. Let $z \in \Sigma_y \cap S$. Since by assumption $y \notin \Sigma_y$ we have $y \neq z$. Therefore $y$ and $z$ both belong to $S \cap x\partial \sigma(S)$ and are different from each other. This completes the proof of the lemma.

Last, notice that Lemma 1 and the continuity assumption on $\sigma$ implies that the map $\mu \circ \hat{V}$ is already lower semicontinuous on $X^{hn}$. Assumption B.4 thus represents a slightly stronger regularity assumption.

2.3 Examples

We present here different examples and counter-examples of maps $\sigma$ fulfilling the properties previously defined.

Example 1 (convex hull). In Moreau’s work, $\sigma(S)$ is taken to be the convex hull of $S$, see Figure 2. One may check easily that Assumptions B.1 to B.4 are all fulfilled. Here, the sets $\Sigma_x$ involved in Assumption B.4 can be defined as follows:

$$\Sigma_x = \bigcup_{c \in TC_{\sigma(S)}(x), |c|=1} x + \max\{t : x + ct \in \sigma(S)\}c,$$

where $TC_{\sigma(S)}(x)$ denotes the Bouligand contingent cone to the set $\sigma(S)$ at $x$ (otherwise called tangent cone, as $\sigma(S)$ is convex here; see [1] pp. 176–177 and 219 for details).

Example 2 (a different convex example). For a given basis $e_j$, $j = 1, \ldots, p$ of $X$, take

$$\sigma(S) = \left[ \min_{x \in S} e_j^T x, \max_{x \in S} e_j^T x \right] \times \cdots \times \left[ \min_{x \in S} e_p^T x, \max_{x \in S} e_p^T x \right].$$

In this example, the convex hull is applied “componentwise”, see Figure 3. Remark that $\text{conv}(S) \subseteq \sigma(S)$ for this case, but this relation is not mandatory, see Example 4 below.

In the example depicted on Figure 3 one may check that the choice consisting in taking for $\Sigma_x = \bigcup_{c \in TC_{\sigma(S)}(x), |c|=1} x + \max\{t : x + ct \in \sigma(S)\}c$, fulfills the Assumptions.
Figure 2: The convex-hull, Moreau’s set-valued Lyapunov function.

Figure 3: Illustration of Example 2.
Example 3 (other convex examples). One may also define $\sigma(S)$ as the smaller set containing $S$ and with boundary parallel to given $p + 1$ non-parallel hyperplans (where $X = \mathbb{R}^p$), see Figure 4. More precisely, let $\Sigma = \text{conv}(S)$ and $e_1, \ldots, e_{p+1}$ be $(p+1)$ vectors in $X$ such that for some positive $\lambda \in \mathbb{R}^{p+1}$ we have $\sum_j \lambda_j e_j = 0$. The set $\sigma(S)$ is a polytope defined as: $\{ x \in X : e_j^T x \leq \max_{x' \in \Sigma} e_j^T x' \, , \, j = 1, \ldots, p+1 \}$, containing the points $x_1, \ldots, x_n$. Symmetrically we may define $\sigma(x) = \{ x \in X : e_j^T x \geq \min_{x' \in \Sigma} e_j^T x' \, , \, j = 1, \ldots, p+1 \}$. Similarly to what occurs in Example 2 one may take for $\Sigma_x$ the portion of the boundary obtained by following the vectors coming out from the tangent cone at $x$ all the way to their extreme intersection point with the boundary of $\sigma(S)$, and the Assumptions B.1–B.5 are fulfilled.

Remark that the smallest ball or the smallest hypercube containing $S$ does not fulfill the requested properties. For instance the smallest circle containing a triangle never contains the smallest circle containing the shortest of its edges, which violates monotonicity of the map $\sigma$.

Example 4 (nonconvex examples). For any bijective transformation $\varphi : X \rightarrow X$ which is Lipschitz together with its inverse, one may take $\sigma_\varphi(S) = \varphi^{-1}(\sigma((\varphi(S)))$, where $\sigma$ fulfils all the Assumptions. In general $\sigma_\varphi(S) \not\subseteq \text{conv}(S)$ and is not convex: indeed, this latter property is not essential. Such an example of nonconvex sets is given in Figure 5 obtained for $X = \mathbb{R}^2$, $x_1 = \left[ \begin{array}{c} 2 \\ 0 \end{array} \right]$, $x_2 = \left[ \begin{array}{c} 1 \\ 5 \end{array} \right]$, $x_3 = \left[ \begin{array}{c} 0 \\ -1 \end{array} \right]$, $\varphi(x) = \left( \begin{array}{cc} \cos \alpha \|x\|_2^2 & \sin \alpha \|x\|_2^2 \\ -\sin \alpha \|x\|_2 & \cos \alpha \|x\|_2 \end{array} \right) x$, $\alpha = 0.04$, and $\sigma(S) = \text{conv}(S)$.

Notice that, generally speaking, the systems generated along this principle are such that the map $\varphi$ in (2) is identical for all the sets $\sigma(S)$. The sets $\Sigma_x$ may be obtained as for Example 1 up to transformation by $\varphi$.

Example 5 (intersection of decision sets). When $\sigma, \sigma'$ fulfil the properties stated above, an interesting issue is to see whether $\sigma \cap \sigma'$ do. One verifies easily that Assumptions B.1–B.5 are automatically fulfilled. The
Figure 5: An example of map $\sigma$ giving rise to nonconvex sets. Notice that $\text{conv}(S) \not\subseteq \sigma(S)$, and that $\mu(\sigma(S))$ is larger than the diameter of $\text{conv}(S)$. 
validity of $\s_\s \cap \tilde{\s}$ depends upon the configuration of the sets $\s_x, \s'_x$ corresponding to $\s$ and $\s'$. In Figure 6 is presented an example where the resulting map fulfills all the properties.

3 Results

Before stating the results of this paper, we recall the notions under discussion below, see [8, 10]. As in Moreau’s papers, we call equilibrium point any element of the state space which is the constant value of an equilibrium solution.

**Definition 3.** Let $\s X$ be a finite-dimensional Euclidean space and consider a continuous set-valued map $e : \s N \times \s X \Rightarrow \s X$ taking on closed values, giving rise to the difference inclusion

$$x(t + 1) \in e(t, x(t)).$$

Consider a collection of equilibrium solutions of this equation and denote the corresponding set of equilibrium points by $\Phi$. By definition, $\varphi \in \Phi$ if and only if $\varphi \in e(t, \varphi)$ for all $t \in \s N$.

With respect to the considered collection of equilibrium solutions, the dynamical system is called

1. stable if for each $\varphi \in \Phi$, for all $c_2 > 0$ and for all $t_0 \in \s N$, there is $c_1 > 0$ such that every solution $\zeta$ of $\text{(5)}$ satisfies: if $|\zeta(t_0) - \varphi| < c_1$ then $|\zeta(t) - \varphi| < c_2$ for all $t \geq t_0$.

2. bounded if for each $\varphi \in \Phi$, for all $c_1 > 0$ and for all $t_0 \in \s N$, there is $c_2 > 0$ such that every solution $\zeta$ of $\text{(5)}$ satisfies: if $|\zeta(t_0) - \varphi| < c_1$ then $|\zeta(t) - \varphi| < c_2$ for all $t \geq t_0$.

3. globally attractive if for each $\varphi_1 \in \Phi$, for all $c_1, c_2 > 0$ and for all $t_0 \in \s N$, there is $T \geq 0$ such that every solution $\zeta$ of $\text{(5)}$ satisfies: if $|\zeta(t_0) - \varphi_1| < c_1$ then there is $\varphi_2 \in \Phi$ such that $|\zeta(t) - \varphi_2| < c_2$ for all $t \geq t_0 + T$. 

Figure 6: Map obtained by intersection of the maps from Figures 3 and 4.
4. globally asymptotically stable if it is stable, bounded and globally attractive.

If \( c_1 \) (respectively \( c_2 \) and \( T \)) may be chosen independently of \( t_0 \) in Item 1 (respectively Items 2 and 3) then the dynamical system is called uniformly stable (respectively uniformly bounded and uniformly globally attractive) with respect to the considered collection of equilibrium solutions.

Notice that the above notions are uniform with respect to all trajectories of [5].

We now state a first result on boundedness and (simple) stability, analogous to [8, Theorem 2].

**Theorem 1.** Assume that Assumptions [A] and [B1]-[B3] are fulfilled. Then the discrete-time system (1) is uniformly globally bounded and uniformly globally stable with respect to the collection of equilibrium solutions 

\[ x_1(t) = \cdots = x_n(t) = \text{constant} \]

**Proof.** The proof of Theorem 1 is based on the evolution of the following set-valued function \( \tilde{V} : X^h \Rightarrow X \),

\[
\tilde{V}(\tilde{x}) = \sigma(\pi(\tilde{x}))
\]

along the solutions of (1). The fact that \( t \mapsto \tilde{V}(\tilde{x}(t)) \) is non-increasing is stated in the following result.

**Lemma 4.** Let \( x \) be a solution of equation (1). Then, for all \( t \in \mathbb{N} \),

\[
\tilde{V}(\tilde{x}(t+1)) \subseteq \tilde{V}(\tilde{x}(t)).
\]

Let us first prove Lemma 4. For any \( k \in \mathbb{N} \), for any \( t \in \mathbb{N} \),

\[
x_k(t+1) \in \sigma(\{x_k(t)\} \cup \{x_i(t-j) : ((i,j),k) \in A(t)\}) \subseteq \tilde{V}(\tilde{x}(t))
\]

successively by Assumption [A] and Assumption [B3] and one concludes the demonstration of Lemma 4 by use of Assumptions [B3] and [B2]. The proof of Theorem 1 is then obtained as a direct consequence.

In view of Lemma 4 one may now have a clearer understanding of the fact that the map \( \sigma \) has a double role: it is necessary to define the flow, but also serves as a set-valued Lyapunov function of the systems. Indeed, Assumption [A] states that each agent has to remain in the set \( \tilde{V}(\tilde{x}(t)) \), of which it has only an imperfect knowledge, and does its best to come closer from the other agents it has detected (this is the meaning of the use of the relative interior). In particular, when no new information is received, the only possible choice is to stay at the same place.

As detailed in Section 2.2 contrary to \( \sigma \), the map \( \text{ri} \sigma \) is not monotone: violation of this rule may occur when \( S' \subset S \) and the \( \sigma \)-hulls \( \sigma(S) \), \( \sigma(S') \) have different topological dimensions as spheres. Up to this subtlety, a consequence of Assumption [A] is that, in general, the larger the quantity of information received by agent \( k \) from its neighborhood, the largest the set of possible updates it may choose (see the monotony property in Assumption [B3]). Although this may sound paradoxical at first glance, this increase of the decision possibilities is quite natural: it means that supplementary information either leads to make a choice which could have been done otherwise (it is ignored or makes more valuable the decision) or allows to adopt choices which would not have been done otherwise. The “subtlety” comes from the fact that, when the information available to an agent is poor, some decisions are taken which would not have been possible with richer data. For example, the possibility of staying in the same place, which occurs when an agent, say agent 1, is isolated from the other world, disappears when the position of another agent located elsewhere, agent 2, is received. However, the unique choice \( \sigma(\{x_1\}) = \{x_1\} \) is then located “on the boundary” of the decision set \( \text{ri} \sigma(\{x_1, x_2\}) \), see Lemma [3].

The key result of the paper is now stated. It provides a necessary and sufficient stability condition for system (1), which extends [8, Theorem 3].

**Theorem 2.** Assume that Assumptions [A] and [B2] are fulfilled. Then the discrete-time system (1) is uniformly globally attractive with respect to the collection of equilibrium solutions \( x_1(t) = \cdots = x_n(t) = \text{constant} \) if and only if there exists \( T \geq 0 \) such that for all \( t_0 \in \mathbb{N} \) there is a node connected to all other nodes across \([t_0,t_0+T]\).
The uniformity which is meant in the statement of Theorems \(1\) and \(2\) is with respect to time. One may check from the proofs that it is also valid with respect to the different trajectories of \(1\).

**Proof of Theorem \(2\)** (Only if.) The proof consists in an adaptation of the contraposition argument developed by Moreau [8, Proof of Theorem 3]. Assume that for every \(T \geq 0\) there is \(t_0 \in \mathbb{N}\) such that the sequence of graphs with delays has no node connected to each other across the interval \([t_0, t_0 + T]\). This implies that for every \(T \geq 0\) there is \(t_0 \in \mathbb{N}\) and nonempty, disjoint subsets \(L_1, L_2 \subset \mathcal{N}\) such that \(\text{Neighbors}(L_1, \mathcal{A}(t)) = \text{Neighbors}(L_2, \mathcal{A}(t)) = \emptyset\) for all \(t \in [t_0, t_0 + T]\). The proof of this fact consists in checking that the proof of [8, Theorem 5] holds also in the case of graph with delays as defined above in Definition \(1\).

Let \(y, \bar{y} \in X\) and consider any solution \(\zeta\) of \(1\) departing from initial data defined by:

\[
\zeta_k(t_0 - j) = \begin{cases}
 y & \forall (k, j) \in L_1 \times \mathcal{H}, \\
 \bar{y} & \forall (k, j) \in L_2 \times \mathcal{H}, \\
 \in \sigma(\{y, \bar{y}\}) & \forall (k, j) \in (\mathcal{N} \setminus (L_1 \cup L_2)) \times \mathcal{H}.
\end{cases}
\]

As in the proof of [8, Theorem 5], we still have the same relation at time \(t_0 + T + 1\), since \(\text{Neighbors}(L_1, \mathcal{A}(t)) = \text{Neighbors}(L_2, \mathcal{A}(t)) = \emptyset\) for all \(t \in [t_0, t_0 + T]\). As the time \(T\) may be chosen arbitrarily large, this contradicts uniform global attractivity of \(1\) with respect to the equilibrium solutions \(x_k(t) = \cdots = x_n(t) = \text{constant}\).

(If.) As in [8, Theorem 3], the proof is based on a (strict) decrease property of the set-valued function \(\bar{V}\) introduced in \(0\).

Let \(T \geq 0\) chosen as in the statement of Theorem \(2\) \(x\) an arbitrary solution of \(1\), and \(t_0 \in \mathbb{N}\) for which the values of \(x_k(t - j), k \in \mathcal{N}, j \in \mathcal{H}\) are not all equal. For all \(k \in \mathcal{N}\), define the integer-valued function \(\alpha_k(t), t \geq t_0\), by:

\[
\alpha_k(t) = \text{card} \{j \in \mathcal{N} : x_j(t) \in \Sigma_{x_k(t_0)}\}.
\]

where \(\Sigma_x\) is meant relative to the set \(S = \pi(\bar{x}(t_0))\). In words, this is the number of agents which at time \(t \geq t_0\) are still belonging to the critical portion of the boundary \(\Sigma_{x_k(t_0)}\). Assume by contradiction that at time \(t_1 > t_0\) a new agent \(x_\alpha\) enters \(\Sigma_{x_k(t_0)}\) which was not there at time \(t_1 - 1\) \((x_\alpha(t_1-1) \notin \Sigma_{x_k(t_0)})\). Let \(S'\) denote the set of points in \(X\) used by \(x_\alpha\) at time \(t_1 - 1\) in order to update its state. Of course, by Assumption \(1\), \(S'\) comprises \(x_\alpha(t_1 - 1)\) itself. Moreover, by monotonicity of the set-valued Lyapunov function \(\bar{V}, S' \subset \sigma(S)\).

Now, by the updating rule \(x_\alpha(t_1) \in \Sigma_{x_k(t_0)}\) is only possible provided that \(\text{ri}(\sigma(S')) \cap \Sigma_{x_k(t_0)} \neq \emptyset\), and therefore, application of Assumption \(3\) implies that \(\text{ri}(\sigma(S')) \cap \Sigma_{x_k(t_0)} \neq \emptyset\), which contradicts what was previously stated. Hence, we can conclude that agents can only leave \(\Sigma_{x_k(t_0)}\), but never get back in. In particular then, the functions \(\alpha_k\) satisfy the inequality:

\[
\forall t \in [t_0, +\infty), \forall k \in \mathcal{N}, \alpha_k(t + 1) \leq \alpha_k(t)
\]

and

\[
\alpha_k(t_0) = \alpha_k(t_1) \Rightarrow \{j \in \mathcal{N} : x_j(t_0) \in \Sigma_{x_k(t_0)}\} = \{j \in \mathcal{N} : x_j(t) \in \Sigma_{x_k(t_0)}\} \quad \forall t \in \{t_0, \ldots, t_1\}.
\]

(7)

On the other hand, if \(t \geq t_0\) is such that \(\alpha_k(t) = 0\) for a certain \(k \in \mathcal{N}\), then Assumption \(3\) implies that \(\mu(\sigma(\bar{x}(t))) < \mu(\sigma(\bar{x}(t_0)))\). An important step consists in showing that:

\[
t > t_0 + T' \Rightarrow \exists k \in \mathcal{N}, \alpha_k(t) < \alpha_k(t_0), \quad T' = h + T.
\]

(8)

There are at most \(n\) different sets \(\Sigma_k\) in \(\sigma(\bar{x}(t_0))\), and \(\alpha_k(t_0) \leq n - 1\). Consequently, the repetition of the argument used to get implication \(3\) (if allowed) will yield:

\[
t > t_0 + (n - 1)^2T' \Rightarrow \exists k \in \mathcal{N}, \alpha_k(t) = 0.
\]

As a consequence, the estimate:

\[
t > t_0 + T'' \Rightarrow \mu(\bar{V}(\bar{x}(t))) < \mu(\bar{V}(\bar{x}(t_0))), \quad T'' = (n - 1)^2T',
\]

(9)
will be deduced from Assumption [44], because \( \pi(\tilde{x}(t)) \) being a finite set of points located in \( \sigma(\pi(\tilde{x}(t_0))) \setminus \Sigma_{x_k(t_0)} \), is thus at a nonzero distance (more precisely a \( \sigma(\pi(\tilde{x}(t_0))) \)-distance, see [4]) from \( \Sigma_{x_k(t_0)} \). In order to get [4], let us now prove [6].

1. Inequality [6] is fulfilled if it holds for \( t < t_0 + h + T \).

2. Otherwise, one has \( \alpha_k(t_0 + h + T) = \alpha_k(t_0) \) for all \( k \) in \( N \) and, by virtue of [7], the set \( L_k = \{ j < N : x_j(t) \in \Sigma_{x_k(t)} \} \) has not changed for \( t \in \{ t_0, \ldots, t_0 + h + T \} \).

Using the hypothesis in the statement of Theorem [2], there exists an agent, numbered \( k \), connected to all others across the interval \( [t_0 + h, t_0 + h + T] \). By definition, the set \( \Sigma_{x_k(t_0)} \) does not contain \( x_k(t_0) \), and, since \( L_k \) has not varied in time, then also \( x_k(t) \not\in \Sigma_{x_k(t)} \) for \( t = t_0, \ldots, t_0 + h + T \). Moreover, because of the connectivity property of the graph put in the statement, \( \text{Neighbors}(L_k, \cup_{t \in [t_0 + h, t_0 + h + T]} A(t)) \neq \emptyset \).

Let \( i \in L_k \) be such that \( \text{Neighbors}(i, \cup_{t \in [t_0 + h, t_0 + h + T]} A(t)) \setminus L_k \neq \emptyset \), viz. an agent which over the time-interval \( [t_0 + h, t_0 + h + T] \) is receiving information from outside \( L_k \). Clearly \( x_i(t_0 + h - 1) \in \Sigma_{x_k(t_0)} \), but \( x_i(t_0 + h + T) \not\in \Sigma_{x_k(t_0)} \), as Assumption [44] implies that elements in \( \Sigma_x \) are attracted outside of it, as soon as they receive information from agents sitting outside \( \Sigma_x \). This yields finally: \( \alpha_k(t_0 + h + T) < \alpha_k(t_0) \) for the value of \( k \) previously exhibited. Inequality [6] is thus proved. Of course this is true only as long as \( \Sigma_{x_k(t_0)} \) is non-empty to start with.

One verifies easily that the same argument may be used recursively, because the sets \( \Sigma_{x_k(t_0)} \) may be kept unchanged as long as \( \alpha_k(t) > 0 \) for all \( k \in N \). Thus, [7] is proved.

Considering now \( \tilde{x}(t_0) \in X^{hn} \) as a variable, let

\[
\beta(\tilde{x}(t_0)) = \inf \mu \left( \tilde{V}(\zeta(0)) - \mu \left( \tilde{V}(\zeta(T'')) \right) \right),
\]

where the infimum is taken over all sequences \( \zeta(1), \ldots, \zeta(T'') \) in \( X^{hn} \) such that \( \zeta(0) = \tilde{x}(t_0) \) and, for all \( t = 1, \ldots, T'' \), for all \( k \in N \),

\[
\zeta_{k,0}(t) \in e_k(A(t_0 + t)) (\zeta(t - 1)) \text{ and } \zeta_{k,j}(t) = \zeta_{k,j-1}(t - 1) \text{ for } j \in H \setminus \{0\}.
\]

The meaning of the previous line is precisely that the infimum is computed over all possible trajectories of the difference inclusion [11]. Now, the collection of \( \zeta(t) \in X^{hn}, t = 1, \ldots, T'' \) satisfying the previous condition is nonempty and compact for all initial value \( \tilde{x}(t_0) \in X^{hn} \). Indeed, by Assumption [11] the set-valued functions \( e_k(A) \) are continuous and take compact values.

The quantity to be minimized is strictly positive when the \( hn \) components of \( \zeta(0) \) are not all equal, due to the strict decrease property of \( \tilde{V} \) established above and Assumption [44]. Also, the expression to be minimized is lower semicontinuous with respect to \( \zeta(1), \ldots, \zeta(T'') \), as \( \tilde{V} \) is continuous (by Assumption [4]) and \( \mu \) is lower semicontinuous (by Lemma [11]). Thus, \( \beta(\tilde{x}(t_0)) > 0 \), except if all the components of \( \tilde{x}(t_0) \) are all equal. In other words, \( \beta \) is definite positive with respect to \( \{ \tilde{x} \in X^{hn} : x_1 = \cdots = x_{hn} \} \).

By Assumption [44] the map \( X^{hn} \to \mathbb{R}^+, \tilde{x} \mapsto \mu(\tilde{V}(\tilde{x})) \) is continuous. The proof of Theorem [2] is then achieved as for [8, Theorem 3], by use of a result on set-valued Lyapunov functions. The latter, Theorem [8] is an extension of [8, Theorem 1] to differential inclusions, given in Appendix.

**Example 6.** The necessity for each agent to take into account the present values of its own position may be seen by the following counter-example. We take \( n = 3 \) and \( h = 2 \). Let

\[
A(2t) = (((2, 1, 1), (1, 0, 2)), A(2t + 1) = (((2, 1, 3), ((3, 0, 2)).
\]

In other words, agent 2 sends alternatively to agent 1 and 3 the value of its position at the previous instant, and receives in the same time the present value of their position, see Figure [7]. Assume the agents use at time \( t \) the value of their position at time \( t - 1 \) to elaborate the update applied at time \( t + 1 \). Clearly, for the corresponding graph with delays, the agent 2 is connected to all other agents across any interval \([t, t + 1]\). However, one sees easily that provided that the agents 1 and 3 are located initially at different positions, the positions of agent 2 at even and odd times tend in general toward two different values. As indicated by the existence of periodic motions, the strict decrease of the map \( \tilde{t} \mapsto \mu(\tilde{V}(\tilde{x}(\tilde{t}))) \) may fail.
Figure 7: Graph with delays representing the information flow for Example 8 even (dots) and odd (dash) times.

Appendix A – Proof of Lemma 1

1. We first recall that the length of a Lipschitz arc $\psi(\lambda)$ defined on $[0,1]$ is equal to

$$\text{length}(\psi) \doteq \int_0^1 \|\frac{d\psi}{d\lambda}\| \cdot d\lambda.$$ 

Notice that, by Rademacher’s theorem, $\psi$ Lipschitz implies differentiability almost everywhere, and therefore the previous integral is well defined. Let $x^0, x^1 \in S$. Taking $\varphi : X \to X$ as in (2), define the map $\psi : [0,1] \to X$

$$\psi(\lambda) = \varphi^{-1} \left((1 - \lambda)\varphi(x^0) + \lambda\varphi(x^1)\right). \quad (10)$$

As $\varphi(S)$ is convex, $\psi$ maps $[0,1]$ in $S$. Moreover, due to the regularity assumption on $\varphi$, it is a Lipschitz arc, and $\psi(0) = x^0, \psi(1) = x^1$. Thus the set $\left\{\text{length}(\psi) : \psi : [0,1] \to S, \psi(0) = x^0, \psi(1) = x^1\right\}$ is non-void, and the definition of the map $d_S(x^0, x^1)$ given in the statement is meaningful.

Let us show its continuity with respect to $x^0, x^1 \in S$. Let $(x^0, x^1) \in S \times S$. Consider a sequence $(x^s)_{s > 0}$ of elements of $S$ tending towards $x^0$. Let $\psi^s$ be a fixed Lipschitz arc linking $x^s$ to $x^0$. For any piecewise Lipschitz arc $\psi^s$ linking $x^0$ to $x^1$, one may construct by concatenation of $\psi^s$ and another Lipschitz arc $\psi^s$ linking $x^0$ to $x^1$. One has

$$\text{length}(\psi^s) = \text{length}(\psi^s) + \text{length}(\psi^s),$$

so

$$d_S(x^s, x^1) \leq d_S(x^s, x^0) + d_S(x^0, x^1).$$

Arguing similarly, one gets that $|d_S(x^s, x^1) - d_S(x^0, x^1)| \leq d_S(x^0, x^0)$.

On the other hand, one may take $\psi$ as in (10), in such a way that

$$\inf \left\{\text{length}(\psi) : \psi : [0,1] \to S, \psi(0) = x^s, \psi(1) = x^0\right\} \leq \left\|\frac{d\psi}{d\lambda}\right\|_{L^\infty} \|x^s - x^0\|,$$

which shows the desired continuity property.

Defining $\mu(S)$ as in (3), one has $\mu(S) < +\infty$, because the image of a compact set by a continuous function is bounded. Moreover, if $\mu(S) = 0$, then, for any $(x^0, x^1) \in S \times S$, the length defined by the map $\psi$ in (10) is zero, that is $x^0 = x^1$ and $S$ is a singleton. Conversely, if $S$ is a singleton, then $\mu(S) = 0$. Last, for any Lipschitz arc $\psi$ linking $x^0$ to $x^1$ and defined as in (10),

$$\text{length}(\psi) = \int_0^1 \left\|\frac{d\psi}{d\lambda}\right\| \cdot d\lambda \geq \int_0^1 \left\|\frac{d\psi}{d\lambda}\right\| \cdot d\lambda = \|x^0 - x^1\|,$$
and this shows that $\mu(S)$ is at least equal to the maximal euclidian distance between two points of $S$, that is its diameter. The equality when $S$ is convex is straightforward, taking the identity for $\varphi$ in (10).

We now prove the lower semicontinuity of $\mu$. Let $S \subseteq S$, and a sequence of sets $S_n \in S$ tending towards $S$ for the topology induced by Hausdorff distance. Our goal is to prove that:

$$\liminf_{n \to +\infty} \mu(S_n) \geq \mu(S).$$

(11)

Obviously, in order to establish inequality (11), it is sufficient to consider only sets $S_n$ such that $\mu(S_n)$ is bounded from above by a given constant, say by twice the value of $\mu(S)$. Let $\varepsilon > 0$, and consider two arbitrary sequences $x_n^0, x_n^1 \in S_n$ and a sequence of Lipschitz arcs $\psi_n$ linking $x_n^0$ to $x_n^1$ in $S_n$ and of length at most equal to $d_{S_n}(x_n^0, x_n^1) + \varepsilon$. We thus have:

$$\liminf_{n \to +\infty} d_{S_n}(x_n^0, x_n^1) + \varepsilon \geq \liminf_{n \to +\infty} \text{length}(\psi_n) \geq \liminf_{n \to +\infty} d_{S_n}(x_n^0, x_n^1).$$

(12)

However, due to the previous remark on the boundedness of the sequence $\mu(S_n)$, one may assume without loss of generality that the arcs $\psi_n$ are covered with a rate of variation $\|\frac{d\psi_n}{dt}\|_{L^\infty}$ uniformly bounded. Indeed, if this is not the case, replace $\psi_n$ by the map $\tilde{\psi}_n$ defined by

$$\tilde{\psi}_n \doteq \psi_n \circ \theta^{-1}, \quad \theta(t) \doteq \frac{\int_0^t \left\| \frac{d\psi_n}{dt} \right\| \cdot d\lambda}{\int_0^1 \left\| \frac{d\psi_n}{dt} \right\| \cdot d\lambda}.$$

The map $\tilde{\psi}_n$ has the same image and length than $\psi_n$, but the norm of its derivative is equal almost everywhere on $[0, 1]$ to the constant $\int_0^1 \left\| \frac{d\psi_n}{dt} \right\| \cdot d\lambda$. In particular,

$$\liminf_{n \to +\infty} \text{length}(\tilde{\psi}_n) = \liminf_{n \to +\infty} \text{length}(\psi_n).$$

(13)

The arcs considered at this stage are thus equicontinuous. By compactness, one deduces that there exist subsequences (denoted similarly $x_n^0, x_n^1$ and $\tilde{\psi}_n$) such that

$$x_n^0 \to x^0 \in S, \quad x_n^1 \to x^1 \in S, \quad \tilde{\psi}_n \to \tilde{\psi} \in \text{Lipschitz}([0, 1]; S).$$

In particular, since by Arzela-Ascoli $\tilde{\psi}_n \to \tilde{\psi}$ uniformly over compact sets, we also have $\text{length}(\tilde{\psi}_n) \to \text{length}(\tilde{\psi})$. Thus, since $\mu(S_n) \geq d_{S_n}(x_n^0, x_n^1)$, we have:

$$\liminf_{n \to +\infty} \mu(S_n) + \varepsilon \geq \liminf_{n \to +\infty} d_{S_n}(x_n^0, x_n^1) + \varepsilon \geq \liminf_{n \to +\infty} \text{length}(\tilde{\psi}_n) = \text{length}(\tilde{\psi}) \geq d_S(x^0, x^1).$$

(14)

By arbitrariness of $\varepsilon$:

$$\liminf_{n \to +\infty} \mu(S_n) \geq d_S(x^0, x^1).$$

We finally use arbitrariness of $x^0, x^1$ in $S$ (arbitrary converging sequences in $S_n$ yield arbitrary limit points in $S$ since $S_n \to S$) to conclude $\liminf_{n \to +\infty} \mu(S_n) \geq \mu(S)$. The lower semicontinuity of $\mu$ is demonstrated.

2. To prove that the definition of $\text{ri}(S)$ is independent of $\varphi$ amounts to show that: if the image of a convex set $S$ by a bijective bi-Lipschitz map $f : X \to X$ is a convex set, then the image of the relative interior $\text{ri}(S)$ is the relative interior of the image of $S$. Let $x \in \text{ri}(S)$. Consider the restriction of $f$ to the affine hull $\text{ah}(S)$ of $S$. The set $\text{ri}(S)$ is convex, thus there exists a convex neighborhood $V \subseteq S$ of $x$ in $\text{ah}(S)$. By continuity of $f^{-1}$, the image of $V$ through $f$ is a neighborhood of $f(x)$ in $f(\text{ah}(S))$. By hypothesis, $f(S)$ is convex, thus there exists a convex neighborhood $W \subseteq f(S)$ of $f(x)$ in $\text{ah}(f(S))$. Now, the intersection $f(V) \cap W \subseteq f(S)$
is a neighborhood of \( f(x) \) in \( ah(f(S)) \), so \( f(x) \in ri(f(S)) \). We thus get \( ri(S) \subseteq f^{-1}(ri(f(S))) \), and one shows similarly the converse inclusion.

Let \( S \in S \). If \( ri(S) = \emptyset \), then \( ri(\varphi(S)) = \emptyset \) and \( S \) is a singleton. Conversely, if \( S \) is a singleton, \( \varphi(S) \) is also a singleton and \( ri(\varphi(S)) = \emptyset \).

It is clear that \( ri(S) \subseteq S \). Also, the fact that \( int (\varphi(S)) \subseteq ri (\varphi(S)) \subset \varphi(S) \) implies that

\[
\int S = \int \varphi^{-1}(\varphi(S)) \subseteq \varphi^{-1}(\int (\varphi(S))) \quad \text{as } \varphi^{-1} \text{ is continuous}
\]

\[
\subseteq \varphi^{-1}(ri (\varphi(S))) = ri (S) \quad \text{by definition of } ri (S).
\]

This ends the proof of Lemma \[1\]

**Appendix B – Stability based on set-valued Lyapunov functions**

The following result is an adaptation of \([3\) Theorem 1\] to difference inclusions. For sake of completeness, a proof is provided, intimately linked to the proof of Moreau’s result.

**Theorem 3.** Let \( \mathcal{X} \) be a finite-dimensional Euclidean space and consider a continuous set-valued map \( e : \mathbb{N} \times \mathcal{X} \rightarrow \mathcal{X} \) taking on closed values, giving rise to the difference inclusion \([\\[3\]. Let \( \Xi \) be a collection of equilibrium solutions and denote the corresponding set of equilibrium points by \( \Phi \). Consider an upper semicontinuous \([1\) p. 41\] set-valued function \( V : \mathcal{X} \rightrightarrows \mathcal{X} \) satisfying

1. \( x \in V(x), \forall x \in \mathcal{X} \);
2. \( \bigcup_{y \in e(t,x)} V(y) \subseteq V(x), \forall t \in \mathbb{N}, \forall x \in \mathcal{X} \).

If \( V(\phi) = \{ \phi \} \) for all \( \phi \in \Phi \) then the dynamical system is uniformly stable with respect to \( \Phi \). If \( V(x) \) is bounded for all \( x \in \mathcal{X} \) then the dynamical system is uniformly bounded with respect to \( \Phi \).

Consider in addition a function \( \mu : \text{Image}(V) \rightarrow \mathbb{R}_{\geq 0} \) and a lower semicontinuous function \( \beta : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0} \) satisfying

3. \( \mu \circ V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0} \) is bounded uniformly with respect to \( x \) in bounded subsets of \( \mathcal{X} \);
4. \( \beta \) is positive definite with respect to \( \Phi \) that is, \( \beta(\phi) = 0 \) for all \( \phi \in \Phi \) and \( \beta(x) > 0 \) for all \( x \in \mathcal{X} \setminus \Phi \);
5. \( \sup_{y \in e(t,x)} \mu(V(y)) - \mu(V(x)) \leq -\beta(x), \forall t \in \mathbb{N}, \forall x \in \mathcal{X} \).

If \( V(\phi) = \{ \phi \} \) for all \( \phi \in \Phi \) and \( V(x) \) is bounded for all \( x \in \mathcal{X} \) then the dynamical system is uniformly globally asymptotically stable with respect to \( \Phi \).

The results stated above remain true if, for a fixed \( \tau \in \mathbb{N} \), the decrease relations in 2. and 5. occur between \( V(y_{t+1}) \) and \( V(y_t) \), \( i = 0, \ldots, \tau - 1, y_0 = x, y_\tau = y \), instead of \( V(y) \) and \( V(x) \).

**Proof.** (Uniform stability.) Consider arbitrary \( \varphi \in \Phi \) and \( c_2 > 0 \). If \( V(\varphi) = \{ \varphi \} \) then, by upper semicontinuity of \( V \), there is \( c_1 > 0 \) such that \( V(x) \subseteq B(\varphi, c_2) \) for all \( x \in B(\varphi, c_1) \). Consider arbitrary \( t_0 \in \mathbb{N} \) and \( x_0 \in B(\varphi, c_1) \) and let \( \zeta \) denote any solution of inclusion \([3\] with \( \zeta(t_0) = x_0 \). Conditions 1 and 2 of Theorem \[3\] imply that, for all \( t \geq t_0 \),

\[
\zeta(t) \in \bigcup_{y \in e(t,x_0)} V(y) \subseteq V(x_0) \subseteq B(\varphi, c_2) \quad .
\]

(Uniform boundedness.) Consider arbitrary \( \varphi \in \Phi \) and \( c_1 > 0 \). If \( V(x) \) is bounded for all \( x \in \mathcal{X} \) then, by upper semicontinuity of \( V \), there is \( c_2 > 0 \) such that \( V(x) \subseteq B(\varphi, c_2) \) for all \( x \in B(\varphi, c_1) \). Consider arbitrary
$t_0 \in \mathbb{N}$ and $x_0 \in B(\varphi, c_1)$ and let $\zeta$ be any solution of (5) with $\zeta(t_0) = x_0$. Conditions 1 and 2 of Theorem 3 imply that for all $t \geq t_0$,

$$
\zeta(t) \in \bigcup_{y \in \{t, x_0\}} V(y) \subseteq V(x_0) \subset B(\varphi, c_2).
$$

(Uniform global asymptotic stability.) It remains only to prove uniform global attractivity with respect to $\Xi$.

Consider arbitrary $\varphi_1 \in \Phi$ and $c_1 > 0$. If $V(x)$ is bounded for all $x \in \mathcal{X}$ then, by upper semicontinuity of $V$, there is a compact set $K \subset \mathcal{X}$ such that $V(x) \subseteq K$ for all $x \in B(\varphi_1, c_1)$. Similarly as above, Conditions 1 and 2 of Theorem 3 imply that every solution of (5) initiated in $B(\varphi_1, c_1)$ remains in $K$.

Consider in addition arbitrary $c_2 > 0$. If $V(\varphi) = \{\varphi\}$ for all $\varphi \in \Phi$ then, by upper semicontinuity of $V$, there is $c_3 > 0$ such that for all $x \in B(\Phi \cap K, c_3)$ there is $\varphi_2 \in \Phi$ such that $V(x) \subset B(\varphi_2, c_2)$. Similarly as above, Conditions 1 and 2 of Theorem 3 imply that every solution of (5) entering $B(\Phi \cap K, c_3)$ remains in a $c_2$-ball around some equilibrium point $\varphi_2 \in \Phi$.

It remains to prove the existence of $T \geq 0$ such that every solution of (5) starting in $B(\varphi_1, c_1)$ cannot remain longer than $T$ subsequent times in $K$ without entering $B(\Phi \cap K, c_3)$. In agreement with Conditions 3 and 4 of Theorem 3 and the lower semicontinuity of $\beta$, we introduce two real numbers:

$$
M \doteq \sup_{x \in B(\varphi_1, c_1)} \mu(V(x)) < \infty \quad \text{and} \quad \Delta \doteq \min_{x \in K \setminus B(\Phi, c_3)} \beta(x) > 0.
$$

Let $T > 0$ be such that $T \Delta > M$. Consider arbitrary $t_0 \in \mathbb{N}$ and $x_0 \in B(\varphi_1, c_1)$ and let $\zeta$ denote any solution of (5) with $\zeta(t_0) = x_0$. Then Condition 5 of Theorem 3 implies that for some $t_1 \in [t_0, t_0 + T]$, $\zeta(t_1) \in B(\Phi \cap K, c_3)$, since otherwise $\zeta(t) \in K \setminus B(\Phi, c_3)$ for all $t \in [t_0, t_0 + T]$ and

$$
\mu(V(\zeta(t_0 + T))) \leq \mu(V(\zeta(t_0))) - T \min_{x \in K \setminus B(\Phi, c_3)} \beta(x) \leq M - T \Delta < 0,
$$

contradicting that $\mu$ takes only non-negative values. Putting everything together, we conclude that for some $\varphi_2 \in \Phi$ and for all $t \geq t_0 + T$,

$$
\zeta(t) \in V(\zeta(t)) \subseteq V(\zeta(t_1)) \subset B(\varphi_2, c_2).
$$

This achieves the proof of Theorem 3. \hfill \Box

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