p-Laplacian Fractional Sturm-Liouville Problem for Diffusion Operator via Impulsive Conditions

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Abstract. In this study, the existence results of solution is given for fractional p-Laplacian Sturm-Liouville problem for diffusion operator of order with impulsive conditions. The derivatives are described in Riemann-Liouville and Caputo sense. The Riemann-Liouville integral operator is used to acquire the integral representation of solution. The existence of solution is demonstrate via Schaefer fixed point theorem.

Keywords. Sturm-Liouville Problem, Fractional, Impulsive Condition, Schaefer Fixed Point, p-Laplacian

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1. Introduction

The last half of the past century has witnessed to both intensive improvement of the theory of differential equations involving derivatives fractional order and the applications such as physics, control systems, polymer rheology, aerodynamics and other areas. Fractional differential equations have been constantly drawing interest of many autors. The attention in the study of fractional differential equations is based upon the fact that fractional calculus service as an great tool in common usage for the applications of such constructions in various sciences, the description of properties of diverse materials, processes and important part of the physical mathematics and also a large part of the literature is related to fractional differential equations. In consequence, the fractional order models is more factual and useful than the integer order models. For more information and applications about fractional calculus, see [1-10,14-21] and references therein. At the same time, the fractional p-Laplacian operator appear naturally in the applied sciences and is extensively used in the mathematical modeling of physical and natural phenomena, blood flow problems, turbulent filtration in porous media, rheology modeling of viscoplasticity, mechanics, material science and many other related fields. Therefore, a continuous increasing attention has been shown towards problems involving the fractional p-Laplacian operator and on the existence of solutions for this problem. But there is no known study about the existence of solutions for fractional p-Laplacian Sturm-Liouville problem thus this paper is a main study on literature. For examples and details, see [22-26]

Impulsive differential equations have arisen as a significant area for applied sciences in recent years. Impulsive differential equations are accepted as significant mathematical devices to make many real world problems plausible in applied sciences. There is a great deal of study for boundary value problems of impulsive differential equations of integer order in the literature. On the other hand, there is very little known about impulsive boundary value problems for fractional order and many aspect of these problems are yet to be discovered.

Recently, there has been too much attention on the existence of solutions for impulsive boundary-value problems for fractional differential equations by means of techniques (fixed point theorems, Banach contraction mapping principle, etc.). This subject has been studied in the various papers [27–29,32-37]. For example, in [32], Yuansheng Tian and Zhanbing Bai discussed the existence results for the three-point impulsive boundary value problem involving fractional differential equations given by

$$\begin{align*}
{\mathcal{C}D}^\alpha u(t) &= f(t, u(t)), \quad 0 < t < 1, \quad t \neq t_k, \quad k = 1, 2, \ldots, p,

\Delta u|_{t=t_k} &= I_k(u(t_k)), \quad \Delta u'|_{t=t_k} = \tilde{I}_k(u(t_k)), \quad k = 1, 2, \ldots, p,

u(0) + u'(0) &= 0, \quad u(1) + u' (\xi) = 0,

\end{align*}$$

where $\mathcal{C}D^\alpha$ is the Caputo fractional derivative, \( q \in R, 1 < q \leq 2, \ f : [0, 1] \times R \to R \) is a continuous function $I_k, \tilde{I}_k : R \to R, \xi \in (0,1), \xi \neq t_k, k = 1, 2, \ldots, p \) and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) , \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-) \) $u(t_k^+)$ and $u(t_k^-)$ shows the right and left-hand limit of the function $u(t)$ at $t = t_k$, and the sequences $\{ t_k \}$ satisfy that $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = 1, p \in \mathbb{N}$. 


Ravi P. Agarwal, Mouffak Benchohra and Boualem Attou Slimani [34] investigated the existence and uniqueness of solutions for the initial value problems for fractional order differential equations as the following form

\[ C D^\alpha y(t) = f(t, y), \quad \text{for each } t \in J = [0, T], \]

\[ t = t_k, \; k = 1, 2, ..., m, \quad 1 < \alpha \leq 2 \]

\[ \Delta y|_{t=t_k} = I_k (y(t_k^-)), \quad \Delta y'|_{t=t_k} = \bar{I}_k (y(t_k^-)), \; k = 1, 2, ..., m, \]

where \( C D^\alpha \) is the Caputo fractional derivative, \( f : J \times R \rightarrow R \) is a continuous function, \( I_k, \bar{I}_k : R \rightarrow R \) are real-valued continuous functions, \( \Delta \) is a constant such that \( h \rightarrow 0^+ \) \( y(t_k + h) \) and \( y(t_k^-) \) at \( t = t_k, \; k = 1, 2, ..., m \). Moreover, Jie Zhou and Meiqiang Feng [37] study fractional Sturm-Liouville problem with impulsive conditions.

The object of this study is to develop main parts of Sturm-Liouville theory for the \( p \)-Laplacian and in order to continue this study by giving several existence results for fractional \( p \)-Laplacian Sturm-Liouville problem having diffusion operator with impulsive conditions and is to prosper the theoretical knowledge of the above.

Therefore we analyze the following fractional \( p \)-Laplacian Sturm-Liouville problem having diffusion operator with impulsive conditions

\[ -D_{0+}^\alpha \phi_p C D_{0+}^\alpha y(t) + (2\lambda p(t) + q(t)) y(t) = 0, \]

\[ \Delta y|_{t=t_k} = I_k (y(t_k^-)), \quad \Delta y'|_{t=t_k} = \bar{I}_k (y(t_k^-)), \quad k = 1, 2, ..., n, \]

\[ y(0) + y'(0) = 0, \quad y(\pi) + y'(\pi) = 0, \]

where \( D_{0+}^\alpha \) is the Riemann-Liouville fractional derivative, \( C D_{0+}^\alpha \) is the Caputo fractional derivative, \( p \in W, q \in L^2 [0, \pi], I_k, \bar{I}_k : R \rightarrow R \) are real-valued continuous functions, \( \Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-), \quad \Delta y'|_{t=t_k} = \lim_{h \rightarrow 0^+} y(t_k + h) - y(t_k^-), \quad k = 1, 2, ..., n \).

\[ \phi_p(s) = |s|^{p-2} s, \quad p > 1. \]

Obviously, \( \phi_p \) is invertible and its inverse operator is \( \phi_q \), where \( q > \frac{1}{p-1} \) is a constant such that \( \frac{1}{p} + \frac{1}{q} = 1 \).

Some necessary notations, definitions and lemmas are given in Section 2. We establish a theorem on existence of solution for (1)-(3) problem by using Schaefer’s fixed point theorem for \( p \)-Laplacian fractional Sturm-Liouville problem via diffusion operator in Section 3.

2. Preliminaries

We give some material related to fractional calculus theory. For more details about this field, see [4,5,8].

Considering the following space

\[ PC(J, R) = \{ y : J \rightarrow R : y \in C((t_k, t_{k+1}], R), k = 0, ..., n + 1 \text{ and there exist } y(t_k^+), \; y(t_k^-), \quad k = 1, 2, ..., n, \; y(t_n^+) = y(t_0^-) \} \]

where \( J = [0, \pi] \). \( PC(J, R) \) is a Banach space with the norm

\[ \|y\|_{PC} = \sup_{t \in J}|y(t)|. \]

**Definition 1.** The left and right-sided Riemann-Liouville integrals of order \( \alpha \) are given by [4].

**Definition 2.** [4] The left and right-sided Riemann-Liouville derivatives are defined as respectively, \( 0 < \alpha < 1\),

\[ (D_{0+}^\alpha f)(r) = D (I_{0+}^{1-\alpha} f)(r), \quad r > a, \]

\[ (D_{0-}^\alpha f)(r) = -D (I_{0-}^{1-\alpha} f)(r), \quad r < b. \]
Similar formulas give the left and right-sided Caputo derivatives of order $\alpha$:

$$
(CD_{a+}^{\alpha} f)(r) = (I_{a+}^{1-\alpha} Df)(r) \quad r > a
$$

$$
(CD_{b-}^{\alpha} f)(r) = (I_{b-}^{1-\alpha} (-D)f)(r) \quad r < b
$$

**Definition 3.** [30] If $K$ is a compact metric space then a subset $F \subset C(K)$ of the space of continuous functions on $K$ equipped with the uniform distance, is compact if and only if it is closed, bounded and equicontinuous.

**Lemma 4.** [31] Let $\alpha > 0$. Then the differential equation

$$
CD^{\alpha} h(t) = 0,
$$

has solution $h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_n t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n, n = [\alpha] + 1$.

**Lemma 5.** [31] Let $\alpha > 0$. Then

$$
I^\alpha CD^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_n t^{n-1},
$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n, n = [\alpha] + 1$.

**Lemma 6.** [4] $\text{Re}(\alpha) > 0$, $n = \text{Re}(\alpha) + 1$ and let $f_{n-\alpha}(x) = (I^{n-\alpha}_b f)(x)$ be the fractional integral of order $n - \alpha$.

a) If $1 \leq p \leq \infty$ and $f(x) \in I^\alpha_b (L_p)$, then

$$
(I^{n-\alpha}_b CD^\alpha_{b-} f)(x) = f(x)
$$

where

$$
I^\alpha_b (L_p) = \{ f : f = I^\alpha_b \varphi, \varphi \in L_p(a, b) \}.
$$

b) If $f(x) \in L_1 (a, b)$ and $f_{n-\alpha, \alpha} (x) \in AC^n [a, b]$, then the equality

$$
(I^{n-\alpha}_b CD^\alpha_{b-} f)(x) = f(x) - \sum_{j=1}^n \frac{f^{(n-j)}_{n-\alpha}}{\Gamma(\alpha - j + 1)} (x - a)^{\alpha - j},
$$

holds almost everywhere on $[a, b]$.

**Lemma 7.** [30] (Schaefer’s fixed point theorem) Let $X$ be a Banach space and $T : X \to X$ be a continuous and compact mapping. If the set

$$
\{ x \in X : x = \lambda T(x) \text{ for some } \lambda \in [0, 1] \}
$$

is bounded, then $T$ has a fixed point.

**3. Existence Result**

Recently, problems involving the fractional $p$-Laplacian operator have been of great interest and this subject is studied by many mathematician. Taiyong Chen, Wenbin Liu discuss the existence of solutions for the anti-periodic boundary value problem of a fractional $p$-Laplacian equation given by

$$
D_{0+}^\beta \phi_p \left(D_{0+}^\alpha x(t)\right) = f(t, x(t)), \quad t \in [0, 1],
$$

$$
x(0) = -x(1), \quad D_{0+}^\alpha x(0) = -D_{0+}^\alpha x(1),
$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $D_{0+}^\alpha$ is a Caputo fractional derivative. We investigate fractional $p$-Laplacian Sturm-Liouville problem having diffusion operator with impulsive conditions and establish a theorem on existence of solution the problem. We use Riemann-Liouville and Caputo fractional derivatives and potential function.
Theorem 8. For a given \( y \in PC(J, R) \). A function \( y \) is a solution of fractional Sturm-Liouville problem

\[
-D_{0+}^\beta \phi_p \, D_{0+}^\alpha y(t) + (2\lambda p(t) + q(t)) y(t) = 0,
\]

\[
\Delta y|_{t=t_k} = I_k^*(y(t_k)), \, \Delta y'|_{t=t_k} = I_k^*(y(t_k)), \, t_k \in (0, \pi), \, k = 1, 2, \ldots, n,
\]

\[
y(0) + y'(0) = 0, \, y(\pi) + y'(\pi) = 0,
\]

if and only if \( y \) is a solution of the fractional integral equation

\[
y(t) = \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left( \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ \left[ \frac{1-t}{\pi} \int_{t_k}^t \frac{(\pi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds \right]
\]

\[
+ \frac{(1-t)}{\pi} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left( \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ \left[ \frac{\pi+1-t}{\pi} \int_{t_k}^t y(t_i) + \frac{(\pi+1-t)}{\pi} \sum_{i=1}^n I_i^* (y(t_i)) (1-t_i) \right].
\]

Proof. Assuming \( y \) satisfies (1) – (3). Using Lemma 6 and Lemma 5, for some constants \( b_0, b_1 \in \mathbb{R}, \, t \in (0, t_1) \), we have

\[
y(t) = I_{0+}^\alpha \phi_q I_{0+}^\beta (2\lambda p(t) + q(t)) y(t) + b_0 + b_1 t,
\]

\[
= \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds + b_0 + b_1 t,
\]

(5)

It follows from (5) that

\[
y'(t) = \int_{0}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds + b_1,
\]

if \( t \in (t_1, t_2) \) and \( c_0, c_1 \in \mathbb{R} \) are arbitrary constants then we have

\[
y(t) = \int_{t_1}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds + c_0 + c_1 (t-t_1),
\]

\[
y'(t) = \int_{t_1}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds + c_1,
\]
using the impulse conditions (2)

\[
c_0 = \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds + b_0 + b_1 t_1 + I_1 (y(t_1)),
\]

\[
c_1 = \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds + b_1 + I_1^* (y(t_1)),
\]

thus,

\[
y(t) = \int_{t_k}^t \left[ \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds
\]

\[
+ \int_{t_{i-1}}^{t_i} \left[ \frac{(t - s)(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} + \frac{(t_i - s)^{\alpha - 1}}{\Gamma(\alpha)} \right] \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ b_0 + b_1 t + \sum_{i=1}^{n} I_1 (y(t_i)) + \sum_{i=1}^{n} I_1^* (y(t_i)) (t - t_i),
\]

repeating the process in this way, for \( t \in (t_k, t_{k+1}] \), we have

\[
y(t) = \int_{t_k}^t \left[ \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds
\]

\[
+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left[ \frac{(t - s)(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} + \frac{(t_i - s)^{\alpha - 1}}{\Gamma(\alpha)} \right] \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ b_0 + b_1 t + \sum_{i=1}^{n} I_1 (y(t_i)) + \sum_{i=1}^{n} I_1^* (y(t_i)) (t - t_i),
\]

and

\[
y'(t) = \left[ \int_{t_k}^t \frac{(t - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds \right]
\]

\[
+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 2}}{\Gamma(\alpha - 1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ b_1 + \sum_{i=1}^{n} I_1^* (y(t_i)),
\]

applying the boundary condition \( y(0) + y'(0) = 0, \ y(\pi) + y'(\pi) = 0 \), we find that
Theorem 9. Presume that

\[(H_1) \text{ There exist constants } N, R, M > 0 \text{ such that} \]
\[|\lambda| \leq N, \quad |p(t)| \leq R, \quad |q(t)| \leq M \text{ for each } t \in J.\]

\[(H_2) \text{ The functions } I_k, I_k^* : \mathbb{R} \rightarrow \mathbb{R} \text{ are continuous and there exist constant } r_1, r_2 > 0 \text{ such that} \]
\[|I_k (y)| < r_1, \quad |I_k^* (y)| < r_1, \quad |I_k (y)| < r_2, \quad |I_k^* (y)| < r_2.\]

then the (1) – (3) problem has at least one solution on $J$. 

\[b_1 = \frac{1}{\pi} \int_{t_i}^{\pi} \frac{(\pi - s)^{\alpha - 1}}{\Gamma (\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) ds \]
\[- \left\{ \frac{1}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{((\pi - t_i) (t_i - s)^{\alpha - 2}}{\Gamma (\alpha - 1)} + \frac{(t_i - s)^{\alpha - 1}}{\Gamma (\alpha)} \right\} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) ds \]
\[- \left\{ \frac{1}{\pi} \sum_{i=1}^{n} I_i (y(t_i)) - \frac{1}{\pi} \sum_{i=1}^{n} I_i^* (y(t_i)) (\pi - t_i) \right\} \]
\[- \left\{ \frac{1}{\pi} \int_{t_i}^{\pi} \frac{(\pi - s)^{\alpha - 2}}{\Gamma (\alpha - 1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) ds \right\} \]
\[- \left\{ \frac{1}{\pi} \sum_{i=1}^{n} I_i (y(t_i)) + \frac{1}{\pi} \sum_{i=1}^{n} I_i^* (y(t_i)) (\pi - t_i) \right\} \]
\[- \frac{1}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha - 2}}{\Gamma (\alpha - 1)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) ds \]
\[- \frac{1}{\pi} \sum_{i=1}^{n} I_i^* (y(t_i)) \right\}, \quad (8)\]

substituting (7), (8) into (6), we obtain (4). The proof completes.
Proof. Define the operator \( T : PC(J, R) \to PC(J, R) \) as

\[
T(y(t)) = \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds
\]

\[
+ \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds \right]
\]

\[
+ \left[ \frac{(1-t)}{\pi} \int_{t_k}^{\pi} \frac{(\pi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds \right]
\]

\[
+ \left[ \frac{(1-t)}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q I_{0+}^\beta (2\lambda p(s) + q(s)) y(s) \, ds \right]
\]

Now, to prove that \( T \) has a fixed point, we use Schaefer fixed point theorem and it will be proven in four steps.

Step 1: \( T \) is continuous.

Let \( \{y_n\} \) be a sequence such that \( y_n \to y \) in \( PC(J, R) \). Then for each \( t \in J \)

\[
|T(y_n)(t) - T(y)(t)| \leq \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left| I_{0+}^\beta (2\lambda p(s) + q(s)) (y_n(s) - y(s)) \right| \, ds
\]

\[
+ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q \left| I_{0+}^\beta (2\lambda p(s) + q(s)) (y_n(s) - y(s)) \right| \, ds
\]

\[
+ \frac{(1-t)}{\pi} \int_{t_k}^{\pi} \left( \frac{(\pi-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\pi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \phi_q \left| I_{0+}^\beta (2\lambda p(s) + q(s)) (y_n(s) - y(s)) \right| \, ds
\]

\[
+ \frac{(1-t)}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_q \left| I_{0+}^\beta (2\lambda p(s) + q(s)) (y_n(s) - y(s)) \right| \, ds
\]

\[
+ \frac{(\pi+1-t)}{\pi} \sum_{i=1}^{n} |I_i(y_n(t_i)) - I_i(y(t_i))| + \frac{(\pi+1-t)}{\pi} \sum_{i=1}^{n} |I_i^*(y_n(t_i)) - I_i^*(y(t_i))| (1-t_i)
\]

Since \( I_k, I_k^*, k = 1, ..., n \), are continuous functions, we have

\[
\|T(y_n) - T(y)\|_\infty \to 0 \text{ as } n \to \infty.
\]

Step 2: \( T \) operator bounded on bounded sets of \( PC(J, R) \).
In fact, it is enough to show that for any $\nu > 0$ there exists a positive constant $\delta$ such that for each $y \in B = \{y \in PC(J, \mathbb{R}) : \|y\|_{\infty} < \nu\}$ we have $\|T(y)\|_{\infty} \leq \delta$. There exists constant $K > 0$ such that $\left|I_{0,+}^{\beta} (2\lambda p(s) + q(s)) y(s)\right| \leq K$. By (H$_1$) and (H$_2$), we have for each $t \in J$

$$|Ty(t)| \leq \int_{t_k}^{t} \left[ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}I_{0,+}^{\beta} (2\lambda p(s) + q(s)) y(s) \right] + \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t} \left( \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_{q}I_{0,+}^{\beta} (2\lambda p(s) + q(s)) y(s) ds \right]$$

$$+ \left[ \frac{(1-t)}{\pi} \int_{t_n}^{\pi} \left( \frac{(\pi-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\pi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \phi_{q}I_{0,+}^{\beta} (2\lambda p(s) + q(s)) y(s) ds \right]$$

$$+ \left[ \frac{(1-t)}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t} \left( \frac{(\pi-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi_{q}I_{0,+}^{\beta} (2\lambda p(s) + q(s)) y(s) ds \right]$$

$$+ \left[ \frac{(1-t)}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t} (t_i-s)^{\alpha-2} \phi_{q}I_{0,+}^{\beta} (2\lambda p(s) + q(s)) y(s) ds \right]$$

$$+ \left[ \sum_{i=1}^{n} I_{i} (y(t_i)) \right] + \left[ \frac{\pi+1-t}{\pi} \sum_{i=1}^{n} I_{i} (y(t_i)) (1-t_i) \right]$$

$$\leq \left[ K^{-1}q \int_{t}^{t} (t-s)^{\alpha-1} ds + K^{-1}q \sum_{i=1}^{n} \int_{t_{i-1}}^{t} \left( \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) ds \right]$$

$$+ \left[ \frac{1}{\pi} K^{-1}q \int_{t_n}^{\pi} \left( \frac{(\pi-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(\pi-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) ds \right]$$

$$+ \frac{\pi}{K^{-1}q} \sum_{i=1}^{n} \int_{t_{i-1}}^{t} \left( \frac{(\pi-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \right) ds$$

$$+ \frac{\pi}{K^{-1}q} \sum_{i=1}^{n} \int_{t_{i-1}}^{t} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \frac{(1+\pi) nr_1}{\pi} + \frac{(1+\pi) nr_2}{\pi}$$

$$\leq \frac{K^{-1}q \alpha}{\Gamma(\alpha+1)} + \frac{n K^{-1}q \alpha \pi^{\alpha-1}}{\Gamma(\alpha)} + \frac{n K^{-1}q \alpha \pi^{\alpha-1}}{\Gamma(\alpha)} + \frac{K^{-1}q \alpha}{\Gamma(\alpha+1) \pi}$$

$$+ \frac{K^{-1}q \alpha \pi^{\alpha-1}}{\Gamma(\alpha) \pi} + \frac{n K^{-1}q \alpha \pi^{\alpha-1}}{\Gamma(\alpha+1) \pi} + \frac{n K^{-1}q \alpha \pi^{\alpha-1}}{\Gamma(\alpha) \pi}$$

$$+ \frac{(1+\pi) nr_1}{\pi} + \frac{(1+\pi) nr_2}{\pi}$$

$$\leq \frac{K^{-1}q \alpha}{\pi} \frac{(n+1)(\pi+1)}{\Gamma(\alpha+1)} + \frac{n (\pi+1)}{\pi \Gamma(\alpha)} + \frac{K^{-1}q \alpha (n+1)}{\Gamma(\alpha) \pi}$$

$$+ \frac{n (\pi+1)(r_1+r_2)}{\pi}$$
Thus

\[ ||T(y)||_{\infty} \leq K^{\alpha-1} \pi^{\alpha} \left[ \frac{(n+1)(\pi+1)}{\pi} + \frac{n(\pi+1)}{\pi} \right] + K^{\alpha-1} \frac{(n+1)}{\Gamma(\alpha) \pi} + \frac{n(\pi+1)(r_1+r_2)}{\pi} = \delta. \]

**Step 3** T operator bounded equicontinuous on sets of $PC(J,R)$.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, $B$ be a bounded set of $PC(J,R)$ as in Step 2, and let $y \in B$. Then

\[ |T(y)(\tau_2) - T(y)(\tau_1)| \leq \left[ \int_{\tau_1}^{\tau_2} \frac{(T_\alpha-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]

\[ + \left[ \frac{(\tau_2-\tau_1)}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(T_\alpha-t_i(s))^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]

\[ + \left[ \frac{(\tau_2-\tau_1)}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(T_\alpha-t_i(s))^{\alpha-2}}{\Gamma(\alpha-1)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]

\[ + \left[ \frac{(\tau_2-\tau_1)}{\pi} \sum_{i=1}^{n} |I_i(y(t_i))| + \frac{(\tau_2-\tau_1)}{\pi} \sum_{i=1}^{n} |I_i^*(y(t_i))| (1-t_i) \right], \]

As a result of Step 1 to Step 3 by Arzela-Ascoli theorem, we can deduce that $T : PC(J,R) \rightarrow PC(J,R)$ is continuous and completely continuous. Furthermore, as $\tau_2 \rightarrow \tau_1$, T operator is equicontinuous.

**Step 4:** Now, let’s show that the set

\[ L = \{ y \in PC[J,R] : y = \theta T(y), \ 0 < \theta < 1 \}, \]

is bounded.

Let $y \in L$. Then $y = \theta T(y)$, for some $0 < \theta < 1$. Thus for each $t \in J$, we have

\[ y(t) = \theta \int_{t_n}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \]

\[ + \left[ \theta \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]

\[ + \left[ \frac{(1-t)\theta}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]

\[ + \left[ \frac{(1-t)\theta}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]

\[ + \left[ \frac{(1-t)\theta}{\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{(t-t_i)(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left[ I_{\alpha,0}^\beta (2\lambda p(s) + q(s)) y(s) \right] ds \right] \]
This implies by \((H_1)\) and \((H_2)\) (as in Step 2) that for each \(t \in J\) we have

\[
|y(t)| \leq K^{q-1} \pi^\alpha \left[ \frac{(n+1)(\pi+1)}{\pi \Gamma(\alpha+1)} + \frac{n(\pi+1)}{\pi \Gamma(\alpha)} \right] + K^{q-1} \pi^{\alpha-1} \frac{(n+1)}{\Gamma(\alpha) \pi} + \delta,
\]

Furthermore, the set \(L\) is bounded. We conclude that \(T\) has a fixed point in the solution of the problem (1)–(3), according to Schaefer’s

**Conclusions**

In this paper, we investigate fractional \(p\)-Laplacian Sturm-Liouville problem having diffusion operator with impulsive conditions at \(\alpha \in (1, 2]\). The derivatives are described in the Riemann-Liouville and Caputo sense. The fractional impulsive differential equation and boundary value problem involving fractional \(p\)-Laplacian is analyzed for the case of our fractional Sturm-Liouville problem. This paper is dealt with Sturm-Liouville problem involving impulsive differential equation of fractional order. We show an explicit representation of solution of the problem. By using Schaefer fixed point theorem we proved existence of solution for fractional \(p\)-Laplacian Sturm-Liouville problem having diffusion operator with impulsive conditions. We hope that our study will make a new research in the area of fractional Sturm-Liouville problems begin with different boundary condition and many of its variations.

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