On the Asymptotic Behavior of Ring $Q$-homeomorphisms with Respect to $P$-modulus

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Abstract. We study the behavior at infinity of ring $Q$-homeomorphisms with respect to $p$-modulus for $p > n$.

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1. Introduction

Let us recall some definitions, see [1]. Let $\Gamma$ be a family of curves $\gamma$ in $\mathbb{R}^n$, $n \geq 2$. A Borel measurable function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, (abbr. $\rho \in \text{adm } \Gamma$), if

$$\int_{\gamma} \rho(x) \, ds \geq 1$$

for any curve $\gamma \in \Gamma$. Let $p \in (1, \infty)$.

The quantity

$$M_p(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \rho^p(x) \, dm(x)$$

is called $p$-modulus of the family $\Gamma$.

For arbitrary sets $E$, $F$ and $G$ of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all continuous curves $\gamma : [a, b] \to \mathbb{R}^n$ that connect $E$ and $F$ in $G$, i.e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and $d_0 = \text{dist}(x_0, \partial D)$. Set

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

$$S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}, \quad i = 1, 2.$$
Let a function \( Q : D \to [0, \infty] \) be Lebesgue measurable. We say that a homeomorphism \( f : D \to \mathbb{R}^n \) is ring \( Q \)-homeomorphism with respect to \( p \)-modulus at \( x_0 \in D \) if the relation

\[
M_p(\Delta(fS_1, fS_2, fD)) \leq \int_A Q(x) \eta^p(|x - x_0|) \, dm(x)
\]

holds for any ring \( A = A(x_0, r_1, r_2) \), \( 0 < r_1 < r_2 < d_0 \), \( d_0 = \text{dist}(x_0, \partial D) \), and for any measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr = 1.
\]

The theory of \( Q \)-homeomorphisms for \( p = n \) was studied in works [2]–[6], for \( 1 < p < n \) in works [7]–[14] and for \( p > n \) in works [15]–[19].

Denote by \( \omega_{n-1} \) the area of the unit sphere \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) in \( \mathbb{R}^n \) and by \( q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0, r)} Q(x) \, dA \) the integral mean over the sphere \( S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \} \), here \( dA \) is the element of the surface area.

Now we formulate a criterion which guarantees for a homeomorphism to be the ring \( Q \)-homeomorphisms with respect to \( p \)-modulus for \( p > 1 \) in \( \mathbb{R}^n, n \geq 2 \).

**Proposition 1.** Let \( D \) be a domain in \( \mathbb{R}^n, n \geq 2 \), and let \( Q : D \to [0, \infty] \) be a Lebesgue measurable function such that \( q_{x_0}(r) \neq \infty \) for a.e. \( r \in (0, d_0) \), \( d_0 = \text{dist}(x_0, \partial D) \). A homeomorphism \( f : D \to \mathbb{R}^n \) is ring \( Q \)-homeomorphism with respect to \( p \)-modulus at a point \( x_0 \in D \) if and only if the quantity

\[
M_p(\Delta(fS_1, fS_2, fA)) \leq \frac{\omega_{n-1}}{\left( \int_{r_1}^{r_2} \frac{dr}{\omega_{n-1} r^{n-1} q_{x_0}^{p-1}(r)} \right)^{p-1}}
\]

holds for any \( 0 < r_1 < r_2 < d_0 \) (see [12], Theorem 2.3).

Following the paper [21], a pair \( \mathcal{E} = (A, C) \) where \( A \subset \mathbb{R}^n \) is an open set and \( C \) is a nonempty compact set contained in \( A \), is called condenser. We say that a condenser \( \mathcal{E} = (A, C) \) lies in a domain \( D \) if \( A \subset D \). Clearly, if \( f : D \to \mathbb{R}^n \) is a homeomorphism and \( \mathcal{E} = (A, C) \) is a condenser in \( D \) then \((fA, fC)\) is also condenser in \( fD \). Further, we denote \( f\mathcal{E} = (fA, fC) \).

Let \( \mathcal{E} = (A, C) \) be a condenser. Denote by \( C_0(A) \) a set of continuous functions \( u : A \to \mathbb{R}^1 \) with compact support. Let \( W_0(\mathcal{E}) = W_0(A, C) \) be a family of nonnegative functions \( u : A \to \mathbb{R}^1 \) such that 1) \( u \in C_0(A) \), 2) \( u(x) \geq 1 \) for \( x \in C \) and 3) \( u \) belongs to the class ACL and
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\[ |\nabla u| = \left( \sum_{i=1}^{n} \left( \frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}. \]

For \( p \geq 1 \) the quantity

\[ \text{cap}_p \mathcal{E} = \text{cap}_p (A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^p \, dm(x) \]

is called \( p \)-capacity of the condenser \( \mathcal{E} \). It is known that for \( p > 1 \)

\[ \text{cap}_p \mathcal{E} = M_p(\Delta(\partial A, \partial C; A \setminus C)), \]

see in ([22], Theorem 1). For \( p > n \) the inequality

\[ \text{cap}_p (A, C) \geq n \Omega_n^\frac{p}{p-n} \left( \frac{p-n}{p-1} \right)^{p-1} \left[ m^{\frac{p-n}{p-1}}(A) - m^{\frac{p-n}{p-1}}(C) \right]^{1-p} \]  

(1.2)

holds where \( \Omega_n \) is a volume of the unit ball in \( \mathbb{R}^n \) (see, e.g., the inequality 8.7 in [23]).

2. Main results

Now we consider the main result of our paper on the behavior at infinity of ring Q-homeomorphisms with respect to \( p \)-modulus for \( p > n \). The case \( p = n \) was studied in the work [20]. Let

\[ L(x_0, f, R) = \sup_{|x-x_0| \leq R} |f(x) - f(x_0)|. \]

Theorem 2.1 (Main Theorem). Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a ring Q-homeomorphism with respect to \( p \)-modulus at a point \( x_0 \) with \( p > n \) where \( x_0 \) is some point in \( \mathbb{R}^n \) and for some numbers \( r_0 > 0, K > 0 \) the condition

\[ q_{x_0}(t) \leq K t^\alpha \]  

(2.1)

holds for a.e. \( t \in [r_0, +\infty) \). If \( \alpha \in [0, p-n) \) then

\[ \lim_{R \to \infty} \frac{L(x_0, f, R)}{R^{\frac{n-n-\alpha}{p-n}}} \geq K^{\frac{1}{n-p}} \left( \frac{p-n}{p-n-\alpha} \right)^{\frac{p-1}{p-n}} > 0. \]

If \( \alpha = p-n \) then

\[ \lim_{R \to \infty} \frac{L(x_0, f, R)}{(\ln R)^{\frac{p-1}{p-n}}} \geq K^{\frac{1}{n-p}} \left( \frac{p-n}{p-1} \right)^{\frac{p-1}{p-n}} > 0. \]
Proof. Consider a condenser $\mathcal{E} = (A, C)$ in $\mathbb{R}^n$, where $A = \{ x \in \mathbb{R}^n : |x - x_0| < R \}$, $C = \{ x \in \mathbb{R}^n : |x - x_0| \leq r_0 \}$, $0 < R < r_0 < \infty$. Then $f\mathcal{E} = (fA, fC)$ is a ringlike condenser in $\mathbb{R}^n$ and by (1.1) we have equality

$$\text{cap}_p f\mathcal{E} = M_p (\Delta (\partial fA, \partial fC; f(A \setminus C))).$$

Due to the inequality (1.2)

$$\text{cap}_p (fA, fC) \geq n \Omega_n \left( \frac{p - n}{p - 1} \right)^{p-1} \left[ m^{\frac{p-n}{p-1}}(fA) - m^{\frac{p-n}{p-1}}(fC) \right]^{1-p},$$

we obtain

$$\text{cap}_p (fA, fC) \geq n \Omega_n \left( \frac{p - n}{p - 1} \right)^{p-1} \left[ m(fA) \frac{n-p}{n} \right]. \quad (2.2)$$

On the other hand, by Proposition 1, one gets

$$\text{cap}_p (fA, fC) \leq \omega_{n-1} \left( \int_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q^{\frac{n}{p-1}}(t)} \right)^{p-1}. \quad (2.3)$$

Combining the inequalities (2.2) and (2.3), we obtain

$$n \Omega_n \left( \frac{p - n}{p - 1} \right)^{p-1} \left[ m(fA) \frac{n-p}{n} \right] \leq \omega_{n-1} \left( \int_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q^{\frac{n}{p-1}}(t)} \right)^{p-1}. \quad (2.4)$$

Due to $\omega_{n-1} = n \Omega_n$, the last inequality can be rewritten as

$$\Omega_n^{\frac{p}{n-1}} \left( \frac{p - n}{p - 1} \right)^{p-1} \left[ m(fA) \frac{n-p}{n} \right] \leq \left( \int_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q^{\frac{n}{p-1}}(t)} \right)^{1-p}. \quad (2.4)$$

Consider a case when $\alpha \in [0, p - n)$. Then from the condition (2.1) the estimate

$$\Omega_n^{\frac{p}{n-1}} \left( \frac{p - n}{p - 1} \right)^{p-1} \left[ m(fA) \frac{n-p}{n} \right] \leq K \left( \frac{p - n - \alpha}{p - 1} \right)^{p-1} \left( R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}} \right)^{1-p}$$

holds. Therefore

$$m(fB(x_0, R)) \geq \Omega_n K^{\frac{n}{n-p}} \left( \frac{p - n}{p - n - \alpha} \right)^{\frac{(p-1)n}{p-n}} \left( R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}} \right)^{\frac{n(p-1)}{p-n}}. \quad (2.5)$$

Due to

$$m(fB(x_0, R)) \leq \Omega_n L^n(x_0, f, R), \quad (2.6)$$

from the inequality (2.5) we have

$$\Omega_n \left[ \frac{(p-n-\alpha)^{n(p-1)}}{(p-n)^{n(p-1)}} \right]^{\frac{n}{n-p}} \left( R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}} \right)^{1-p} \leq K \left( \frac{p - n - \alpha}{p - 1} \right)^{p-1} \left( R^{\frac{p-n-\alpha}{p-1}} - r_0^{\frac{p-n-\alpha}{p-1}} \right)^{1-p}.$$
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\[ L(x_0, f, R) \geq K^{\frac{1}{n-p}} \left( \frac{p-n}{p-n-\alpha} \right)^{\frac{p-1}{p-n}} \left( \frac{R^{\frac{n-\alpha}{p-n}}}{p-n-\alpha} - r_0^{\frac{n-\alpha}{p-n}} \right)^{\frac{p-1}{p-n}}. \]

Dividing the last inequality by \( R^{\frac{n-\alpha}{p-n}} \) and taking the lower limit for \( R \to \infty \), we conclude

\[ \lim_{R \to \infty} \frac{L(x_0, f, R)}{R^{\frac{n-\alpha}{p-n}}} \geq K^{\frac{1}{n-p}} \left( \frac{p-n}{p-n-\alpha} \right)^{\frac{p-1}{p-n}}. \]

Now we consider a case when \( \alpha = p-n \). Then from (2.4) we get

\[ \Omega_n^{\frac{n}{p-1}} \left( \frac{p-n}{p-1} \right)^{p-1} [m(fA)]^{\frac{n}{p-n}} \leq K \left( \frac{\ln R}{r_0} \right)^{1-p}. \]

Therefore

\[ m(fB(x_0, R)) \geq \Omega_n K^{\frac{n}{n-p}} \left( \frac{p-n}{p-1} \right)^{\frac{n(p-1)}{p-n}} \left( \frac{\ln R}{r_0} \right)^{\frac{n(p-1)}{p-n}}. \]

Due to the estimate (2.6) we obtain

\[ L(x_0, f, R) \geq K^{\frac{1}{n-p}} \left( \frac{p-n}{p-1} \right)^{\frac{p-1}{p-n}} \left( \frac{\ln R}{r_0} \right)^{\frac{p-1}{p-n}}. \]

Finally, dividing the last inequality by \( (\ln R)^{\frac{p-1}{p-n}} \) and taking the lower limit for \( R \to \infty \), we conclude

\[ \lim_{R \to \infty} \frac{L(x_0, f, R)}{(\ln R)^{\frac{p-1}{p-n}}} \geq K^{\frac{1}{n-p}} \left( \frac{p-n}{p-1} \right)^{\frac{p-1}{p-n}}. \]

This completes the proof of Main Theorem. \( \square \)

Let us consider some examples.

**Example 2.1.** Let \( f_1 : \mathbb{R}^n \to \mathbb{R}^n \), where

\[ f_1(x) = \begin{cases} K^{\frac{1}{n-p}} \left( \frac{p-n}{p-n-\alpha} \right)^{\frac{p-1}{p-n}} \frac{x}{|x|} & , x \neq 0 \\ 0 , & x = 0 \end{cases} \]

It can be easily seen that \[ \lim_{x \to \infty} \frac{|f(x)|}{|x|^{\frac{p-n}{p-n-\alpha}}} = K^{\frac{1}{n-p}} \left( \frac{p-n}{p-n-\alpha} \right)^{\frac{p-1}{p-n}}. \]

Let us show that the mapping \( f_1 \) is a ring \( Q \)-homeomorphism with respect to \( p \)-modulus with the function \( Q(x) = K |x|^\alpha \) at the point \( x_0 = 0 \). Clearly, \( q_{x_0}(t) = K t^\alpha \).

Consider a ring \( A(0, r_1, r_2), 0 < r_1 < r_2 < \infty \). Note that the mapping \( f_1 \) maps the ring \( A(0, r_1, r_2) \) onto the ring \( A(0, \tilde{r}_1, \tilde{r}_2) \), where

\[ \tilde{r}_i = K^{\frac{1}{n-p}} \left( \frac{p-n}{p-n-\alpha} \right)^{\frac{p-1}{p-n}} r_i^{\frac{n-\alpha}{p-n}}, \quad i = 1, 2. \]
Denote by $\Gamma$ a set of all curves that join the spheres $S(0, r_1)$ and $S(0, r_2)$ in the ring $A(0, r_1, r_2)$. Then one can calculate $p$-modulus of the family of curves $f_1\Gamma$ in implicit form:

$$M_p(f_1\Gamma) = \omega_{n-1} \left( \frac{p-n}{p-1} \right)^{p-1} \left( \frac{p-n}{\tilde{r}_2^{p-1} - \tilde{r}_1^{p-1}} \right)^{1-p}$$

(see, e.g., the relation (2) in [24]). Substituting in the above equality the values $\tilde{r}_1$ and $\tilde{r}_2$, defined above, one gets

$$M_p(f_1\Gamma) = \omega_{n-1} K \left( \frac{p-n-\alpha}{p-1} \right)^{p-1} \left( \frac{p-n-\alpha}{r_2^{p-1} - r_1^{p-1}} \right)^{1-p}.$$ 

Note that the last equality can be written by

$$M_p(f_1\Gamma) = \omega_{n-1} \left( \frac{\tilde{r}_2}{\int_{r_1}^{r_2} dt} \right)^{p-1},$$

where $q_{x_0}(t) = K t^\alpha$.

Hence, by Proposition 1, the homeomorphism $f_1$ is a ring $Q$-homeomorphism with respect to $p$-modulus for $p > n$ with the function $Q(x) = K |x|^\alpha$ at the point $x_0 = 0$.

**Example 2.2.** Let $\alpha = p - n$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^n$, where

$$f_2(x) = \begin{cases} 
K^{\frac{1}{p-1}} \left( \frac{p-n}{p-1} \right)^{\frac{p-1}{p-n}} (\ln |x|)^{\frac{p-1}{p-n}} \frac{x}{|x|}, & x \neq 0 \\
0, & x = 0.
\end{cases}$$

It can be easily seen that $\lim_{x \to \infty} \frac{|f(x)|}{(\ln |x|)^{\frac{p-n}{p-1}}} = K^{\frac{1}{p-1}} \left( \frac{p-n}{p-1} \right)^{\frac{p-1}{p-n}}$. By analogy with Example 2.1, we can show that the mapping $f_2$ is a ring $Q$-homeomorphism with respect to $p$-modulus with the function $Q(x) = K |x|^{p-n}$.

**Remark 2.1.** Examples 2.1 and 2.2 show that the estimates in Main Theorem are exact, i.e. are attained on the above mappings.

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