Quasi-energies of coupled qubits: Magnus-Floquet states and their probing by weak signal

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Abstract. The behaviour of two coupled qubits in a periodic electromagnetic field of arbitrary amplitude was investigated. Quasienergy levels of the system as a function of the driving field parameters and the qubit coupling energy were found on the basis of the Floquet theory and the Magnus approximation. It is shown that the quasienergy levels may intersect depending on the control parameters. This fact can manifest itself as appearance of the novel resonances and interesting nonlinear phenomena.

1. Introduction

Recently, the influence of the alternating field on quantum systems (atoms, molecules) has been actively studied due to the creation of high-power lasers and the development of spectroscopy of fast processes. This made it possible to detect some interesting nonlinear effects: (i) dynamic Stark effect (modification of the spectrum in the monochromatic driving field); (ii) the Autler-Townes effect (the splitting of the frequency into a periodic driving field); (iii) induced transparency effect; (iv) laser cooling, etc. (see articles [1-3] for more details). Similar effects can also be observed when the microwave driving field is applied to manipulate the states of artificial atoms – superconducting qubits. Moreover, since the qubits have micron dimensions, the observation of these nonlinear effects may be realized at lower amplitudes of the alternating field [4].

This work is devoted to the development of a technique for calculating quasi-energy levels of two coupled qubits in a strong alternating field. Our approach is based on approximate calculations of the evolution operator (propagator) for the quantum system on a field period by using of the Magnus method [5-7] and Floquet theory (outside the perturbation theory) [8-11] (see for review [12, 13]). The resulting expression for the propagator is further used to calculate the qubit quasienergies depending on the system parameters: the amplitude of the driving field and the qubit coupling parameter. Note that the obtained dependences are important for development of the microwave spectroscopy of qubits. The technique developed in the present work allows us to consider the scenario when the main signal (pump) may be send on the strip line simultaneously with a weak test signal (probe), which allows to probe the quasienergy levels.
2. Quasienergy

Let us remind briefly the main ideas underlying the concept of quasi-energy [8-10]. If the Hamiltonian of a multilevel quantum system is periodic in time \( H(t) = H(t + T) \), then the solution of the Schrödinger equation \( i \hbar \frac{\partial \psi}{\partial t} = H(t) \psi(t) \) according to Floquet's theorem can be decomposed into a complete set of Floquet functions \( |\Phi_k(t)\rangle \)

\[
|\psi_k(t)\rangle = |\Phi_k(t)\rangle e^{-iQ_k\hbar/T},
\]

(1)

where \( Q_k \) are time-independent quasienergies [11]. They can be found from the solution of the equation:

\[
U(T)|\Phi_k(0)\rangle = e^{-iQ_k\hbar/T}|\Phi_k(0)\rangle.
\]

(2)

where \( U(T) = \hat{P} e^{\int_0^T H(t) \, dt} \) is an evolution operator on a period, \( \hat{P} \) is the chronological ordering. Since quasienergies are not uniquely defined, \( Q_k = Q_0 + n\hbar\omega \), we will translate them in the first “Brillouin zone” ( \( 0 \leq Q_k \leq \hbar\omega \)). The evolution operator is determined by solving the equation:

\[
i\hbar \frac{\partial U}{\partial t} = H(t)U.
\]

(3)

In the theory of magnetic resonance (when describing the spin motion in an alternating high-frequency field), an approximate solution of Eq. (3) and the expression for quasi-energies have been obtained in the framework of the resonance perturbation theory (Rabi approximation) [14]. In this case, they are weakly dependent on the field amplitude. It is also known that the Rabi approximation is not applicable in the case of periodic signals of large amplitude, when the level shifts in the field are comparable to the distances between the levels. In this case, the influence of the driving field cannot be considered in the framework of standard perturbation theory and should be applied an approach beyond of the Rabi approximation.

In this work, to calculate the evolution operator on the period \( U(T) \), the Magnus expansion will be used as an approximation. We consider the propagator on the field period in the form:

\[
U(T) = e^{-iM(T)},
\]

(4)

where \( M \) is a hermitian operator, \( M^* = M \). According to [5-7], the operator \( M \) is generally defined by an infinite series

\[
M(T) = \sum_{j=1}^{\infty} M_j(T),
\]

(5)

where the terms of the Magnus operator are proportional to the corresponding powers of the Hamilton operator \( H(t) \). The advantage of the Magnus approximation method is that the solution of the evolutionary problem “step by step” on the field period is not required to find quasienergies \( Q_k \). We need to find and diagonalize the evolution operator Eq. (4) only at the moment of time \( t = T \). Since the \( M \) operator commutes with the Hamiltonian, the problem reduces to the diagonalization of the Magnus operator.

Note that an important issue for the use of the Magnus approximation approach is the convergence of the series Eq. (5). We will not deal with mathematical subtleties in this paper, and we only will refer to the discussion of this question in terms of the eigenvalues of the operator \( M \) in [15]. In practice, this criterion is reduced to the evaluation of the norm of the Hamiltonian of the system according to:
\[ \int_0^\pi \|H(t)\| \, dt < \pi , \]  

where \( \| \cdot \| \) is the Euclidean norm of the Hamiltonian \( H(t) \).

3. Quasienergies of two interacting qubits in a strong field

The main features of the behavior of two superconducting flux coupled qubits studied in experiments [16, 17] can be described by the Hamiltonian:

\[
H(t) = -\frac{1}{2} \begin{pmatrix}
\varepsilon_1(t) + \varepsilon_2(t) + J & \Delta_2 & \Delta_4 & 0 \\
\Delta_2 & \varepsilon_1(t) - \varepsilon_2(t) - J & 0 & \Delta_4 \\
\Delta_4 & 0 & \varepsilon_1(t) - \varepsilon_2(t) + J & \Delta_2 \\
0 & \Delta_4 & \Delta_2 & -\varepsilon_1(t) + \varepsilon_2(t) + J
\end{pmatrix},
\]

where \( \varepsilon_i(t) \) is the control parameter of qubit \( i \) \( (i = 1, 2) \), \( \Delta_i \) is the corresponding tunneling matrix element, and the parameter \( J \) quantifies the strength of the interaction between the qubits. For simplicity, we will discuss the case when the qubits are affected by the monochromatic driving field \( (T = 2\pi/\omega) \), which will be described by functions:

\[
\varepsilon_{1,2}(t) = \varepsilon_{1,2}^\omega + \varepsilon_{1,2}^d \cos(\omega t),
\]

where \( \varepsilon_{1,2}^\omega \) and \( \varepsilon_{1,2}^d \) are the amplitude of the dc- and ac-fields. It is assumed that a sequence of synchronized pulses of a driving field acts on the qubit system. The pulse duration is much longer than the field period. In the absence of the ac field (\( \varepsilon_{1,2}^d = 0 \)) dispersion curves depending on the constant field are found from the solution of the equation \( H \mid \psi_{1,2} \rangle = E_{1,2} \mid \psi_{1,2} \rangle \) which is shown in the figure 1 (a).

We investigate the behavior of quasienergy levels at arbitrary amplitude of the external driving field on the basis of direct numerical solution of the Eqs. (3) and (2) of the Floquet theory, as well as on the basis of the Magnus approximation from Eq. (4). For two coupled qubits we take into account three terms in the expansion (5). For the case \( \varepsilon_{1,2}^d = \varepsilon_{1,2}^\omega = k \varepsilon_{1,2}^\omega \), \( \omega = 1 \ \text{GHz} \), the Magnus operator Eq. (5) on the field period may be written as:

\[
M(T) = \begin{pmatrix}
\frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J - k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J + k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J - k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J - k^2 \right) \\
\frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J - k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J + k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J - k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J + k^2 \right) \\
\frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J + k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J - k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J + k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J + k^2 \right) \\
\frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J + k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J + k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 - 4\varepsilon^\omega k J - k^2 \right) & \frac{\varepsilon_{12}^\omega}{4} \left(-4(\varepsilon^\omega)^2 k^2 + 4\varepsilon^\omega k J - k^2 \right)
\end{pmatrix}
\]

We present the behavior of the dispersion curves \( E_{1,2}(\varepsilon^\omega) \) in figure 1 (a) for two coupled qubits. Let us mentally draw on dependencies \( E_{1,2}(\varepsilon^\omega) \) a set of lines parallel to the vertical axis at distances \( n\omega \) from each other and then move the fragments of dispersion curves from each line to the first Brillouin zone \(-\hbar \omega / 2 \leq Q_i \leq \hbar \omega / 2\) (dashed gray lines in figure 1 (a)). The obtained picture will approximately correspond to the pictures shown in figure 1 (b), which were calculated using the numerical solution of Eq. (2) or the approximation decomposition of Magnus Eq. (4). As shown in figure 1 (b) the dependence of quasienergies on the parameter \( \varepsilon^\omega \) is very simple at \( \varepsilon^\omega \gg \Delta_i \): the quasienergies behave in accordance with the almost-linear laws of dispersion of the uncoupled qubits. Note that when \( \varepsilon^\omega \sim \Delta_i \), the curvature of the qubit dispersion plays an important role in the formation of the quasienergy.
Figure 1. (a) Dispersion curves $E_k$ of two coupled qubits as a function of the amplitude dc-field $\varepsilon^{dc}$. Quasienergies $Q_k$ (b, c, d) as a function of the amplitude of the amplitude dc-field $\varepsilon^{dc}$ (b), ac-field $\varepsilon^{ac}$ (c) and the interaction parameter of qubits $J$ (d). The solid line represents the numerical calculation, the dots show the solution based on the expansion of Magnus. The following parameters of the qubit and fields were used: $\omega = 1$ GHz, $\Delta_1 = 2\Delta_0 = 0.02$ GHz, $k = 1.2$, $\varepsilon^{ac} = 1$ GHz (c), $\varepsilon^{dc} = 0.4$ GHz (d), $\varepsilon^{ac} = 1.2$ GHz (b, d), $J = 0.1$ GHz (d, c). The colors on the figures establish the correspondence between the levels (a) and the quasi-levels (b, c, and d).

Figure 1 (b, c, d) demonstrates the quasienergy behavior of two coupled qubits when the field parameters $\varepsilon^{ac}, \varepsilon^{dc}$ and the interaction parameter of the qubits $J$ are changed. It is found experimentally that the qubits coupling parameter can be varied in a wide range, for example, by means of an additional superconducting circuit placed between the qubits [18]. In addition, the so-called programmable qubits are being actively developed, in which changing the parameters of the system it is possible to organize ferromagnetic ($J > 0$) or antiferromagnetic ($J < 0$) interaction in pairs between different qubits [19].

The results of the Magnus approximation (denoted by points in figure 1 (b, c, d)) describes the behavior of quasi-levels calculated numerically on the basis of Floquet theory (solid lines in figure 1 (b, c, d)) with high accuracy. Application of Magnus approximation for calculation $\Delta T \ll 1$ that corresponds to the chosen parameters of modeling and according to the analysis of experimental works on research of superconducting flux qubits. Note that in some cases tunnel level splitting $\Delta_i (i = 1, 2)$ is small in comparison with other energy parameters of the system [16, 17].

4. Conclusion
For the observation of atomic spectra it is necessary to excite atoms by a weak variable field and observe the emitted radiation by changing the frequency of the probe signal. For example, we can use a weak magnetic field for scanning when a system is irradiated by a monochromatic field. According
to the Bohr's postulates transitions between levels take place when the distance between them is compared with the energy of the electromagnetic field quantum.

In the case of a driving field of large amplitude, we can only talk about the transitions between the quasilevels of the qubits, which should be accompanied by the appearance of resonance of the response function. If we add a weak probe signal to the Hamiltonian of qubits Eq. (7), it is easy to obtain an analogue of the formula for the transition probability with the usual perturbation theory [20]. In this case, peaks of response function appear if the differences of quasienergies coincide with the energy of the absorbed quantum. Note that the above ambiguity of quasienergies (\( \epsilon = Q_n + n\hbar \omega \)) is insignificant when considering (neglecting the transfer processes), if the splitting of levels is small compared to \( \hbar \omega \).

Thus, the behavior of qubits in a strong driving field demonstrates new possibilities of nonlinear spectroscopy, which carries additional information about the qubit parameters.

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