HOMOGENIZATION OF PERIODIC PARABOLIC SYSTEMS
IN THE $L_2(\mathbb{R}^d)$-NORM
WITH THE CORRECTOR TAKEN INTO ACCOUNT

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ABSTRACT. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider a self-adjoint matrix second order elliptic differential operator $B_\varepsilon$, $0 < \varepsilon \ll 1$. The principal part of the operator is given in a factorised form, the operator contains first and zero order terms. The operator $B_\varepsilon$ is positive definite, its coefficients are periodic and depend on $x/\varepsilon$. We study the behaviour in the small period limit of the operator exponential $e^{-B_\varepsilon t}$, $t \geq 0$. The approximation in the ($L_2 \to L_2$)-operator norm with the error estimate of the order $O(\varepsilon^2)$ is obtained. The corrector is taken into account in this approximation. The results are applied to homogenization of the solutions for the Cauchy problem for parabolic systems.

INTRODUCTION

The research is devoted to homogenization of periodic differential operators (DO’s). A broad literature is devoted to homogenization problems; see, for example, books [BaPa, BeLP, ZhiKO, Sa]. We rely on the spectral approach to homogenization problems based on the Floquet-Bloch theory and the analytic perturbation theory. This approach was developed in the series of papers [BSu1, BSu2, BSu3, BSu4, BSu5] by M. Sh. Birman and T. A. Suslina. The great deal of considerations are carried out in abstract operator-theoretic terms.

0.1. Problem setting. We study the behavior in the small period limit of the solution for parabolic system

\[
\begin{cases}
G(x/\varepsilon)\partial_s u_\varepsilon(x,s) = -B_\varepsilon u_\varepsilon(x,s), & x \in \mathbb{R}^d, \ s > 0; \\
G(x/\varepsilon)u_\varepsilon(x,0) = \phi(x), & x \in \mathbb{R}^d.
\end{cases}
\]

(0.1)

Here $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, $B_\varepsilon$ is a matrix elliptic second order DO acting in $L_2(\mathbb{R}^d; C^n)$. The measurable $(n \times n)$-matrix-valued function $G(x)$ is assumed to be bounded, uniformly positive definite, and periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. The coefficients of the operator $B_\varepsilon$ are $\Gamma$-periodic functions depending on $x/\varepsilon$. For any measurable $\Gamma$-periodic function $\varphi(x)$, $x \in \mathbb{R}^d$, we use the notation $\varphi^{\varepsilon}(x) = \varphi(x/\varepsilon)$.

The principal part $A_\varepsilon$ of the operator $B_\varepsilon$ is given in a factorized form

$A_\varepsilon = b(D)^*g(\varepsilon)b(D),$

(0.2)

where $b(D)$ is a matrix homogeneous first order DO, $g(\varepsilon)$ is $\Gamma$-periodic bounded and positive definite matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on the coefficients of the operator $B_\varepsilon$ are given below in (4)). The operator $B_\varepsilon$ contains first and zero order terms

$B_\varepsilon u = A_\varepsilon u + \sum_{j=1}^d (a_j^*(x)D_j u + D_j(a_j^*(x))^*u) + Q^\varepsilon(x)u + \lambda u.$

(0.3)

Here $a_j(x)$, $j = 1, \ldots, d$, are $\Gamma$-periodic $(n \times n)$-matrix-valued functions belonging to a suitable $L_2^\varepsilon$-space on the cell $\Omega$ of the lattice $\Gamma$. In general, the potential $Q^\varepsilon(x)$ is a distribution generated by a rapidly oscillating measure with values in the Hermitian matrices. (Since the coefficients are not assumed to be bounded, the precise definition of the operator (0.3) is given via the quadratic form.) The constant $\lambda$ is chosen such that the operator $B_\varepsilon$ oscillate rapidly as $\varepsilon \to 0$.

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The principal term of approximation for the solution of the problem (0.1) for small $\varepsilon$ was obtained in [MI]:
\[
\|u_{\varepsilon}(\cdot, s) - u_0(\cdot, s)\|_{L_2(\mathbb{R}^d)} \leq C_1 \varepsilon(s + \varepsilon^2)^{-1/2} e^{-C_2 s} \|\phi\|_{L_2(\mathbb{R}^d)}, \quad s \geq 0.
\]  
(0.4)
Here $u_0$ is the solution of the “homogenized” problem
\[
\begin{align*}
(\overline{G}\partial_s u_0)(x, s) &= -B^0 u_0(x, s), \quad x \in \mathbb{R}^d, \quad s > 0; \\
(\overline{G} u_0)(x, 0) &= \phi(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]
where $B^0$ is the effective operator with constant coefficients, $\overline{G}$ is the mean value of the matrix-value function $G$ over the cell of periodicity: $\overline{G} = |\Omega|^{-1} \int_{\Omega} G(x) \, dx$. In the case when $G(x) = 1_n$, estimate (0.3) admits an obvious formulation in operator terms
\[
\|e^{-\overline{B} s} - e^{-B^0 s}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_1 \varepsilon(s + \varepsilon^2)^{-1/2} e^{-C_2 s}.
\]
Thus, results of such type are called operator error estimates in homogenization theory. (In the general case, we deal with the “bordered” exponential $e^{-f(x)\ast B} e^{-f(x)\ast f\varepsilon} f\varepsilon$, where $G^{-1} = f f^\ast$ and the matrix-valued function $f$ is assumed to be $\Gamma$-periodic.)

Our aim is to refine approximation (0.1) by taking the corrector into account. In other words, for a fixed time $s > 0$, to approximate the solution $u_{\varepsilon}(\cdot, s)$ of problem (0.1) in the $L_2(\mathbb{R}^d; C^0)$-norm with the error estimate of the order $O(\varepsilon^2)$.

0.2. Main results. In Introduction, we discuss only the case when $G(x) = 1_n$. Our main result is the estimate
\[
\|e^{-\overline{B} s} - e^{-B^0 s} - \varepsilon \mathcal{K}(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_3 \varepsilon^2 (s + \varepsilon^2)^{-1} e^{-C_2 s}, \quad s > 0.
\]  
(0.5)
Here $\mathcal{K}(\varepsilon, s)$ is the corrector. It turns out that it is equal to sum of three terms and the two terms of the corrector are mutually conjugate and contain rapidly oscillating factors. The third summand has constant coefficients.

0.3. Survey on operator error estimates. An interest to operator error estimates was sparked by the paper [BSu1] of M. Sh. Birman and T. A. Suslina, where for the resolvent of the operator (0.2) it was obtained the estimate
\[
\|(A_{\varepsilon} + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon.
\]  
(0.6)
Here $A^0 = b(D)^* g_0^3 b(D)$ is the effective operator, $g_0^3$ is the constant effective matrix. Approximation for the resolvent $(A_{\varepsilon} + I)^{-1}$ with the corrector taken into account was obtained in [BSu4]:
\[
\|(A_{\varepsilon} + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon^2.
\]  
(0.7)
Approximation for the resolvent $(A_{\varepsilon} + I)^{-1}$ in the energy norm was achieved in [BSu5]:
\[
\|(A_{\varepsilon} + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon.
\]  
(0.8)
Here $K_1(\varepsilon)$ corresponds to the traditional corrector in homogenization theory. This operator contains rapidly oscillating factors, so $\|K_1(\varepsilon)\|_{L_2 \rightarrow H^1} = O(1)$. Note that the corrector in (0.7) has the form $K(\varepsilon) = K_1(\varepsilon) + K_1(\varepsilon)^\ast + K_3$, moreover $K_3$ does not depend on $\varepsilon$. Later, estimates (0.6)–(0.8) were generalized by T. A. Suslina [Su4, Su7] to more wide class of the operators (0.3).

To parabolic systems the spectral approach was applied in the papers [Su1, Su2] by T. A. Suslina, where the principal term of approximation for the operator $e^{-A_{\varepsilon} s}$ was obtained, in the paper [V] by E. S. Vasilevskaya, where approximation in the $L_2(\mathbb{R}^d; C^0)$-operator norm with the corrector taken into account was achieved, and also in [Su3], where the approximation for the operator exponential in the energy norm was obtained:
\[
\|e^{-A_{\varepsilon} s} - e^{-A^0 s}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon(s + \varepsilon^2)^{-1/2}, \quad s \geq 0;
\]  
(0.9)
\[
\|e^{-A_{\varepsilon} s} - e^{-A^0 s} - \varepsilon K_1(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon^2(s + \varepsilon^2)^{-1}, \quad s \geq 0;
\]  
(0.10)
\[
\|e^{-A_{\varepsilon} s} - e^{-A^0 s} - \varepsilon K_1(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon(s^{-1/2} + s^{-1}), \quad s \geq \varepsilon^2.
\]  
(0.11)
There are no exponentially decaying factors in these estimates since the point zero is the lower edge of spectra for the operators $A_{\varepsilon}$ and $A^0$. Generalization of estimates (0.9) and (0.11) to the

\[\text{end of text} \]
operator (0.3) was achieved in the paper [M] by the author. Generalization of estimate (0.10) is the main result of the present paper.

Another approach to proving operator error estimates in homogenization theory was suggested by V. V. Zhikov [Zh] and developed by him together with S. E. Pastukhova [ZhPas1, ZhPas2]. The authors of the method called it the “shift method” or the “modified method of the first approximation.” The method is based on introduction of an additional parameter that is a shift on a vector from \( \mathbb{R}^d \) to the problem, on a careful analysis of the first order approximation to the solution, and subsequent integration over the shift parameter. The important role is played by the Steklov smoothing. In papers [Zh, ZhPas1], the analogues of estimates (0.6) and (0.8) were proven for the operators of acoustics and elasticity theory. To parabolic problems the shift method was applied in [ZhPas2], where the analogues of inequalities (0.9) and (0.11) were obtained. A summary of further results of V. V. Zhikov, S. E. Pastukhova, and their students can be found in the review [ZhPas3].

So far, we discussed operator error estimates for the operators acting in the whole space \( \mathbb{R}^d \). For completeness, note that it is more traditional for homogenization theory to study operators acting in a bounded domain \( \Omega \subset \mathbb{R}^d \). Operator error estimates for such problems were studied by many authors. We highlight, in particular, the works [MoVo, Gr1, Gr2, ZhPas1, KeLiS, Su6, Xu, GeS]. (A more detailed survey of results on operator error estimates can be found in introduction to the paper [MSu].)

0.4. The method of investigation. Consider the case \( G = 1_n \). By using the scaling transformation, we reduce the problem of estimate (0.5) to obtaining the analogues estimate for the operator exponential \( e^{-B(\varepsilon)x^{-2}\tau} \), where \( B(\varepsilon) \) is the operator acting in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) and given by the differential expression

\[
B(\varepsilon) = b(D)^*g(x)b(D) + \varepsilon \sum_{j=1}^d (a_j(x)D_j + D_ja_j(x)^*) + \varepsilon^2Q(x) + \varepsilon^2\lambda I.
\]

Thus, from study of the exponential of the operator with rapidly oscillating coefficients we switch to study the large time behavior of the exponential of the operator with coefficients depending on \( x \) (not on \( x/\varepsilon \)).

According to the Floquet-Bloch theory, the operator \( B(\varepsilon) \) decomposes into the direct integral of operators \( B(k, \varepsilon) \), acting in \( L_2(\Omega; \mathbb{C}^n) \) and depending on the parameter \( k \in \mathbb{R}^d \) called the quasi-momentum. The operator \( B(k, \varepsilon) \) is formally given by the expression

\[
B(k, \varepsilon) = b(D + k)^*g(x)b(D + k) + \varepsilon \sum_{j=1}^d (a_j(x)(D_j + k_j) + (D_j + k_j)a_j(x)^*)
\]

\[+ \varepsilon^2Q(x) + \varepsilon^2\lambda I\]

with periodic boundary conditions. The spectrum of the operator \( B(k, \varepsilon) \) is discrete. In accordance with [Su4, Su7], we distinguish the one-dimensional parameter \( \tau = (|k|^2 + \varepsilon^2)^{1/2} \) and study the family \( B(k, \varepsilon) \) by using the methods of analytic perturbation theory with respect to \( \tau \).

0.5. The structure of the paper. The paper consists of Introduction and three chapters. In Chapter 1 (§§4–12), the abstract operator-theoretic scheme is expounded. In Chapter 2 (§§13–19), periodic differential operators are studied. The approximation for the “bordered” operator exponential is obtained (see [13]). Chapter 3 (§§16–12) is devoted to homogenization for parabolic systems. In §10, we derive the main result of the paper that is estimate (0.3). In §11, results in operator terms are applied to homogenization of solutions for parabolic systems. In §12, we consider the scalar elliptic operator as an example.

0.6. Notation. Let \( \mathcal{H}, \mathcal{H}_n \) be complex separable Hilbert spaces. The symbols \((\cdot, \cdot)_0\) and \(\| \cdot \|_0\) denote the scalar product and the norm in \( \mathcal{H} \); the symbol \(\| \cdot \|_{\mathcal{H} \rightarrow \mathcal{H}_n}\) denotes the norm of a linear continuous operator from \( \mathcal{H} \) to \( \mathcal{H}_n \).

By \((\cdot, \cdot)_0\) and \(\| \cdot \|\) we denote the scalar product and the norm in \( \mathbb{C}^n \), respectively, by \(1_n\) we denote the unit \((n \times n)\)-matrix. If \(a\) is an \((m \times n)\)-matrix, then the symbol \(|a|\) denotes the norm
of the matrix $a$ as an operator form $\mathbb{C}^n$ to $\mathbb{C}^m$. We use the notation $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1, \ldots, d$, $D = -i\nabla = (D_1, \ldots, D_d)$. The class $L_p$ of vector-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ with values in $\mathbb{C}^n$ is denoted by $L_p(\mathcal{O}; \mathbb{C}^n)$, $1 \leq p \leq \infty$. The Sobolev class of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ is denoted by $H^s(\mathcal{O}; \mathbb{C}^n)$. For $n = 1$, we simply write $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, and so on, but (if this does not lead to confusion) we use such short notation also for the spaces of vector-valued or matrix-valued functions. The symbol $L_p((0, T); \mathcal{H})$, $1 \leq p \leq \infty$, means the $L_p$-space of $\mathcal{H}$-valued functions on the interval $(0, T)$.

The different constants in estimates are denoted by $c, C, \mathcal{C}$ (possibly, with indices and marks).

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Chapter 1. Abstract scheme

1. Preliminaries

The content of Sections 1.1–1.9 is borrowed from [Su4, Su5].

1.1. Operators $X(t)$ and $A(t)$. Let $\mathcal{H}$ and $\mathcal{H}_s$ be complex separable Hilbert spaces. Let $X_0: \mathcal{H} \to \mathcal{H}_s$ be a densely defined closed operator, and let the operator $X_1: \mathcal{H} \to \mathcal{H}_s$ be bounded. On the domain $\text{Dom} \ X(t) = \text{Dom} \ X_0$, we define the operator

$$X(t) := X_0 + tX_1: \mathcal{H} \to \mathcal{H}_s, \quad t \in \mathbb{R}. \quad (1.1)$$

In $\mathcal{H}$, consider a family of self-adjoint non-negative operators

$$A(t) := X(t)^*X(t), \quad t \in \mathbb{R}, \quad (1.2)$$

corresponding to closed in $\mathcal{H}$ quadratic forms

$$\|X(t)u\|_{\mathcal{H}_s}^2, \quad u \in \text{Dom} \ X_0.$$

Denote $A(0) = X_0^*X_0 =: A_0$. Set

$$\mathcal{N} := \text{Ker} \ A_0 = \text{Ker} \ X_0, \quad \mathcal{N}_s := \text{Ker} \ X_0^*.$$

Let $P$ be the orthogonal projection of the space $\mathcal{H}$ onto $\mathcal{N}$, and let $P_s$ be the orthogonal projection of $\mathcal{H}_s$ onto $\mathcal{N}_s$.

Impose the following condition.

Condition 1.1. The point $\lambda_0 = 0$ is isolated in the spectrum of the operator $A_0$, and

$$0 < n := \dim \mathcal{N} < \infty, \quad n \leq n_s := \dim \mathcal{N}_s \leq \infty.$$

By $d^0$ we denote the distance between the point zero and the rest of the spectrum of the operator $A_0$.

1.2. The operators $Y(t)$ and $Y_2$. Let $\widetilde{\mathcal{H}}$ be yet another separable Hilbert space. Let $Y_0: \mathcal{H} \to \widetilde{\mathcal{H}}$ be a densely defined linear operator such that $\text{Dom} \ X_0 \subset \text{Dom} \ Y_0$, and let $Y_1: \mathcal{H} \to \widetilde{\mathcal{H}}$ be a bounded linear operator. Denote $Y(t) := Y_0 + iT_1$, $\text{Dom} \ Y(t) = \text{Dom} \ Y_0$. The following condition is assumed to be satisfied.

Condition 1.2. There exists a constant $c_1 > 0$ such that

$$\|Y(t)u\|_{\widetilde{\mathcal{H}}} \leq c_1\|X(t)u\|_{\mathcal{H}_s}, \quad u \in \text{Dom} \ X_0, \quad t \in \mathbb{R}. \quad (1.3)$$

From estimate (1.3) with $t = 0$ it follows that $\text{Ker} \ X_0 \subset \text{Ker} \ Y_0$, i. e., $Y_0^*P = 0$.

Let $Y_2: \mathcal{H} \to \widetilde{\mathcal{H}}$ be a densely defined linear operator, and $\text{Dom} \ X_0 \subset \text{Dom} \ Y_2$. The following condition is assumed to be satisfied.

Condition 1.3. For any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$\|Y_2u\|_{\widetilde{\mathcal{H}}}^2 \leq \nu\|X(t)u\|_{\mathcal{H}_s}^2 + C(\nu)\|u\|_{\mathcal{H}_s}^2, \quad u \in \text{Dom} \ X_0, \quad t \in \mathbb{R}.$$
1.3. The form $q$. In the space $\mathcal{H}$, let $q[u,v]$ be a densely defined Hermitian sesquilinear form, and $\operatorname{Dom} X_0 \subset \operatorname{Dom} q$. We impose the following condition.

**Condition 1.4.** 1°. There exist constants $c_2 \geq 0$ and $c_3 \geq 0$ such that

$$|q[u,v]| \leq (c_2\|X(t)u\|_{\mathcal{H}}^2 + c_3\|u\|_{\mathcal{H}}^2)^{1/2} (c_2\|X(t)v\|_{\mathcal{H}}^2 + c_3\|v\|_{\mathcal{H}}^2)^{1/2},$$

$u,v \in \operatorname{Dom} X_0, \ t \in \mathbb{R}.$

2°. There exist constants $0 < \kappa \leq 1$ and $c_0 \in \mathbb{R}$ such that

$$q[u,u] \geq -(1-\kappa)\|X(t)u\|_{\mathcal{H}}^2 - c_0\|u\|_{\mathcal{H}}^2, \ u \in \operatorname{Dom} X_0, \ t \in \mathbb{R}.$$

1.4. The operator $B(t,\varepsilon)$. Let $\varepsilon \in (0,1]$. In $\mathcal{H}$, we consider a Hermitian sesquilinear form

$$b(t,\varepsilon)[u,v] = (X(t)u, X(t)v)_{\mathcal{H}} + \varepsilon \left((Y(t)u, Y(t)v)_{\mathcal{H}} + (Y(t)Y^*(t))_{\mathcal{H}}\right)$$

$$+ \varepsilon^2 q[u,v], \ u,v \in \operatorname{Dom} X_0.$$

By using Conditions 1.2, 1.3 and 1.4, it is easy to see that

$$b(t,\varepsilon)[u,u] \leq (2 + c_2^2 + c_2^2)\|X(t)u\|_{\mathcal{H}}^2 + (C(1) + c_3)\varepsilon^2\|u\|_{\mathcal{H}}^2, \ u \in \operatorname{Dom} X_0,$$

$$b(t,\varepsilon)[u,u] \geq \frac{\kappa}{2}\|X(t)u\|_{\mathcal{H}}^2 - (c_0 + c_4)\varepsilon^2\|u\|_{\mathcal{H}}^2, \ u \in \operatorname{Dom} X_0.$$ (1.5)

Here

$$c_4 = 4\kappa^{-1}c^2_1C(\nu) \quad \text{for} \quad \nu = \kappa^2(16c_2^2)^{-1}. \quad \text{(1.6)}$$

A detailed proof of these inequalities can be found in [Su1, Subsec. 1.4]. Thus, the form $b(t,\varepsilon)$ is closed and lower semi-bounded.

The self-adjoint operator acting in the space $\mathcal{H}$ and corresponding to the form (1.4) is denoted by $\mathfrak{B}(t,\varepsilon)$. Formally,

$$\mathfrak{B}(t,\varepsilon) = A(t) + \varepsilon(Y^*_2Y(t) + Y(t)^*Y_2) + \varepsilon^2 Q.$$ (1.7)

Here $Q$ is the formal object that corresponds to the form $q$.

Let $Q_0: \mathcal{H} \to \mathcal{H}$ be bounded positive definite operator. We add the constant $\lambda Q_0$ to $Q$ in such a way that the operator

$$B(t,\varepsilon) := \mathfrak{B}(t,\varepsilon) + \lambda \varepsilon^2 Q_0 = A(t) + \varepsilon(Y^*_2Y(t) + Y(t)^*Y_2) + \varepsilon^2 (Q + \lambda Q_0)$$

corresponding to the form

$$b(t,\varepsilon)[u,v] := b(t,\varepsilon)[u,v] + \lambda \varepsilon^2 (Q_0u,v), \ u,v \in \operatorname{Dom} X_0,$$ (1.8)

is positive definite. To guarantee this, we impose on $\lambda$ the following restriction

$$\lambda > \|Q_0^{-1}\|(c_0 + c_4), \quad \text{if} \ \lambda \geq 0,$$

$$\lambda > \|Q_0\|^{-1}(c_0 + c_4), \quad \text{if} \ \lambda < 0 \quad \text{(and} \ c_0 + c_4 < 0).$$ (1.9)

Condition (1.9) leads to the inequality

$$\lambda(Q_0u,u)_{\mathcal{H}} \geq (c_0 + c_4 + \beta)\|u\|_{\mathcal{H}}^2, \ u \in \mathcal{H},$$

where $\beta > 0$ is defined in terms of $\lambda$ by the rule

$$\beta = \lambda \|Q_0^{-1}\|^{-1} - c_0 - c_4, \quad \text{if} \ \lambda \geq 0,$$

$$\beta = \lambda \|Q_0\| - c_0 - c_4, \quad \text{if} \ \lambda < 0 \quad \text{(and} \ c_0 + c_4 < 0).$$ (1.11)

From (1.5) and (1.10) it follows the lower semi-bound for the form (1.8):

$$b(t,\varepsilon)[u,u] \geq \frac{\kappa}{2}\|X(t)u\|_{\mathcal{H}}^2 + \beta \varepsilon^2\|u\|_{\mathcal{H}}^2, \ u \in \operatorname{Dom} X_0.$$ (1.12)

Thus, $B(t,\varepsilon) > 0$. The operator $B(t,\varepsilon)$ is the main object of investigation in Chapter 1.
1.5. **Passage to the parameters** $\tau, \vartheta$. The operator family $B(t, \varepsilon)$ is an analytic with respect to the parameters $t$ and $\varepsilon$. If $t = \varepsilon = 0$, then the operator $B(0, 0)$ matches with $A_0$ and, in accordance with Condition [1.1], has an isolated eigenvalue $\lambda_0 = 0$ of multiplicity $n$. It seems natural to apply the analytic perturbation theory. But, for $n > 1$, we deal with a multiple eigenvalue and a two-dimensional parameter. Analytical perturbation theory is not directly applicable. Thus, we distinguish a one-dimensional parameter $\tau = (t^2 + \varepsilon^2)^{1/2}$ and trace the dependence on the additional parameters $\vartheta_1 = t\tau^{-1}$ and $\vartheta_2 = \varepsilon \tau^{-1}$. Note that the vector $\vartheta = (\vartheta_1, \vartheta_2)$ belongs to the unit circle. In that follows, the operator $B(t, \varepsilon)$ is denoted by $B(\tau; \vartheta)$ and the corresponding form $b(t, \varepsilon)$ is denoted by $b(\tau; \vartheta)$. The form can be written as

$$b(\tau; \vartheta) [u, v] = (X_0 u, X_0 v)_{\mathcal{H}_\tau^*} + \tau \vartheta_1 ((X_0 u, X_1 v)_{\mathcal{H}_\tau^*} + (X_1 u, X_0 v)_{\mathcal{H}_\tau^*}) + \tau^2 \vartheta_1^2 ((X_1 u, X_1 v)_{\mathcal{H}_\tau^*} + (X_1 u, X_1 v)_{\mathcal{H}_\tau^*}) + \tau^2 \vartheta_2 (Y_0 u, Y_0 v)_{\mathcal{H}_\tau^*} + (Y_2 u, Y_2 v)_{\mathcal{H}_\tau^*}) + \tau^2 \vartheta_2^2 (q[u, v] + \lambda (Q_0 u, v)_{\mathcal{H}_\tau^*}), \quad u, v \in \text{Dom} \ X_0.$$  

Formally,  

$$B(\tau; \vartheta) = X_0 X_0 + \tau \vartheta_1 (X_0 X_1 + X_1 X_0) + \tau^2 \vartheta_1^2 X_1 X_1 + \tau \vartheta_2 (Y_0 Y_0 + Y_0 X_2) + \tau^2 \vartheta_2 (Y_2 Y_2 + Y_2 Y_2) + \tau^2 \vartheta_2^2 (Q + \lambda Q_0).$$

Let $F(\tau; \vartheta; \mu)$ be the spectral projector of the operator $B(\tau; \vartheta)$ corresponding to the interval $[0, \mu]$. Fix a constant $\delta \in (0, n\kappa^0/13)$ and choose a number $\tau_0 > 0$ such that

$$\tau_0 \leq \delta^{1/2} \left( (2 + c_1^2 + c_2) ||X_1||^2 + C(1) + c_3 + |\lambda||Q_0|| \right)^{-1/2}.$$

As was shown in [Su4, Proposition 1.5], for $|\tau| \leq \tau_0$, we have

$$F(\tau; \vartheta; \delta) = F(\tau; \vartheta; 3\delta), \quad \text{rank} F(\tau; \vartheta; \delta) = n, \quad |\tau| \leq \tau_0.$$  

In that follows we often write $F(\tau; \vartheta)$ instead of $F(\tau; \vartheta; \delta)$.

In Chapter 1, we trace the dependence of constants in estimates on the following “data”:

$$\kappa^{1/2}, \kappa^{-1/2}, \delta, \delta^{-1/2}, \tau_0, c_1, c_2, c_3, C(1)^{1/2}, |\lambda|, ||X_1||, ||Y_1||, ||Q_0||$$

(1.14)

(and also on constant $\tilde{c}_s^{-1}$ introduced in Subsection [2.1] below). It is important for application of the results of the present chapter to differential operators that the constants in estimates (possibly after completion) depend polynomially on these quantities, and the coefficients of polynomials are positive numbers.

1.6. **The operators** $Z$ and $\bar{Z}$. In the present and next subsections, we define operators appearing in considerations inspired by the analytic perturbation theory.

Set $\mathcal{D} := \text{Dom} X_0 \cap \mathfrak{N}^1$. Since the point $\lambda_0 = 0$ is isolated in the spectrum of the operator $A_0$, the form $(X_0 \omega, X_0 \omega)_{\mathcal{H}_\tau^*}, \phi, \psi \in \mathcal{D}$ defines a scalar product in $\mathcal{D}$, so $\mathcal{D}$ becomes a Hilbert space. Let $\omega \in \mathfrak{N}$. Consider an equation for the element $\phi \in \mathcal{D}$ (cf. [BSu2, Chapter 1, (1.7)]):

$$X_0 (X_0 \phi + X_1 \omega) = 0,$$

(1.15)

which is understood in a weak sense. In other words, $\phi \in \mathcal{D}$ satisfies the identity

$$(X_0 \phi, X_0 \zeta)_{\mathcal{H}_\tau^*} = -(X_1 \omega, X_0 \zeta)_{\mathcal{H}_\tau^*}, \quad \forall \zeta \in \mathcal{D}.$$  

(1.16)

Thus the right-hand side in (1.16) is an anti-linear continuous functional on $\zeta \in \mathcal{D}$, from the Riesz theorem it follows that equation (1.16) has a unique solution $\phi(\omega)$. Obviously, $||X_0 \phi||_{\mathcal{H}_\tau^*} \leq ||X_1 \omega||_{\mathcal{H}_\tau^*}$. Define a bounded operator $Z : \mathcal{H} \to \mathcal{H}$ by identities

$$Z \omega = \phi(\omega), \quad \omega \in \mathfrak{N}; \quad Z x = 0, \quad x \in \mathfrak{N}^1.$$  

Obviously,

$$Z P = Z, \quad P Z = 0, \quad P Z^* = Z^*, \quad Z^* P = 0.$$  

(1.17)

It is easy to check that (see [Su3], (1.21))

$$||Z||_{\mathcal{H} \to \mathcal{H}} \leq \kappa^{1/2} (13\delta)^{-1/2} ||X_1||.$$  

(1.18)
Similarly, for given \( \omega \in \mathfrak{N} \) consider equation
\[
X_0^*X_0\psi + Y_0^*Y_0\omega = 0
\]  
(1.19)
for the element \( \psi \in \mathcal{D} \). This equation is understood as identity
\[
(X_0\psi, X_0\zeta)_{\mathcal{H}} = -(Y_0\omega, Y_0\zeta)_{\mathcal{H}}, \quad \zeta \in \mathcal{D}.
\]
By Condition \( \mathbf{1.2} \) the right-hand side is anti-linear continuous functional on \( \zeta \in \mathcal{D} \). By the Riesz theorem, there exists a unique solution \( \psi(\omega) \). Define bounded operator \( \tilde{Z} : \mathfrak{H} \to \mathfrak{H} \) by identities
\[
\tilde{Z}\omega = \psi(\omega), \quad \omega \in \mathfrak{N}; \quad \tilde{Z}x = 0, \quad x \in \mathfrak{N}^\perp.
\]
Note that \( \tilde{Z} \) maps \( \mathfrak{N} \) onto \( \mathfrak{N}^\perp \), and \( \mathfrak{N}^\perp \) onto \( \{0\} \). So,
\[
\tilde{Z}P = \tilde{Z}, \quad P\tilde{Z} = 0, \quad P\tilde{Z}^* = \tilde{Z}^*, \quad \tilde{Z}^*P = 0.
\]
(1.20)
We need the estimate (see \[\text{Su5}\] (1.25)):
\[
\|\tilde{Z}\|_{\mathcal{B} \to \mathcal{B}} \leq c_1K^{1/2}C(1)^{1/2}(13\delta)^{-1/2}.
\]
(1.21)

1.7. The operators \( R \) and \( S \). Define the operator \( R : \mathfrak{N} \to \mathfrak{N}^* \) (see \[\text{BSu2}\] Chapter 1, Subsec. 1.2] as follows \( R\omega = X_0\psi(\omega) + X_1\omega \in \mathfrak{N}^* \). Another definition of the operator \( R \) is given by the formula \( R = P_*X_1|_{\mathfrak{N}} \).

By the spectral germ of the operator family (\[\text{1.2} \]) for \( t = 0 \) we call (see \[\text{BSu2}\] Chapter 1, Subsection 1.3) the self-adjoint operator \( S = R^*R : \mathfrak{N} \to \mathfrak{N} \), which also satisfies the identity \( S = PX_1^*P_*X_1|_{\mathfrak{N}} \).

1.8. The analytic branches of eigenvalues and eigenvectors of the operator \( B(\tau; \omega) \). According to analytic perturbation theory, (see \[]Ka\]), for \( |\tau| \leq \tau_0 \) there exist real-analytic functions \( \lambda_l(\tau; \omega) \) and real-analytic (in \( \tau \)) \( \mathfrak{H} \)-valued functions \( \phi_l(\tau; \omega) \) such that
\[
B(\tau; \omega)\phi_l(\tau; \omega) = \lambda_l(\tau; \omega)\phi_l(\tau; \omega), \quad l = 1, \ldots, n, \quad |\tau| \leq \tau_0,
\]
(1.22)
moreover, \( \phi_l(\tau; \omega) \), \( l = 1, \ldots, n \), form an orthogonal basis in the eigenspace \( F(\tau; \omega)\mathfrak{H} \). For sufficiently small \( \tau_*(\omega) \) \((\leq \tau_0)\) and \( |\tau| \leq \tau_*(\omega) \) we have the following convergent power series expansions
\[
\lambda_l(\tau; \omega) = \gamma_l(\omega)\tau^2 + \mu_l(\omega)\tau^3 + \ldots, \quad \gamma_l(\omega) \geq 0, \quad l = 1, \ldots, n;
\]
(1.23)
\[
\phi_l(\tau; \omega) = \omega_l(\omega) + \tau\phi_l^1(\omega) + \tau^2\phi_l^2(\omega) + \ldots, \quad l = 1, \ldots, n.
\]
The elements \( \omega_l(\omega) \), \( l = 1, \ldots, n \), form orthonormal basis in \( \mathfrak{N} \).

In \[\text{Su4}\] (1.32), (1.33)), it was obtained that the elements
\[
\tilde{\omega}_l(\omega) := \phi_l^1(\omega) - \theta_1 Z\omega_l(\omega) - \theta_2 Z\omega_l(\omega), \quad l = 1, \ldots, n,
\]
satisfy identities
\[
(\tilde{\omega}_k(\omega), \omega_l(\omega))_{\mathfrak{H}} + (\omega_k(\omega), \tilde{\omega}_l(\omega))_{\mathfrak{H}} = 0, \quad k, l = 1, \ldots, n.
\]
(1.24)

In accordance with \[\text{Su4}\] Subsec. 1.8], by the spectral germ of the operator family \( B(\tau; \omega) \) for \( \tau = 0 \) we call the operator acting in \( \mathfrak{N} \) and given by the expression
\[
S(\omega) = \theta_1^2S + \theta_1\theta_2(-X_0Z^*X_0\tilde{Z} - (X_0\tilde{Z})^*X_0\tilde{Z} + P(Y_2^*Y_1^* + Y_1^*Y_2^*))|_{\mathfrak{N}}
\]
\[
+ \theta_2^2\left( -(X_0\tilde{Z})^*X_0\tilde{Z}|_{\mathfrak{N}} + Q_{\mathfrak{N}} + \lambda Q_{\mathfrak{N}} \right).
\]
(1.25)
Here \( Q_{\mathfrak{N}} \) is the self-adjoint operator in \( \mathfrak{N} \), generated by the form \( q[\omega, \omega] \) on \( \omega \in \mathfrak{N} \), and \( Q_{\mathfrak{N}} := PQ_{\mathfrak{N}}|_{\mathfrak{N}} \).

In \[\text{Su4}\] Proposition 1.6], it was obtained that the numbers \( \gamma_l(\omega) \) and elements \( \omega_l(\omega) \), \( l = 1, \ldots, n \), are eigen-ones for the operator \( S(\omega) \):
\[
S(\omega)\omega_l(\omega) = \gamma_l(\omega)\omega_l(\omega), \quad l = 1, \ldots, n.
\]
(1.26)
The quantities \( \gamma_l(\omega) \) and \( \omega_l(\omega) \), \( l = 1, \ldots, n \), are called the threshold characteristics at the bottom of the spectrum for the operator family \( B(\tau; \omega) \).
1.9. **Threshold approximations.** As was shown in [Su5, Theorem 3.1],

\[
F(\tau; \vartheta) - P = \Phi(\tau; \vartheta), \quad \|\Phi(\tau; \vartheta)\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_1|\tau|, \quad |\tau| \leq \tau_0. \tag{1.27}
\]

Besides (1.27), we need a more accurate approximation for the spectral projection obtained in [Su5, Theorem 3.1]:

\[
F(\tau; \vartheta) = P + \tau F_1(\vartheta) + F_2(\tau; \vartheta), \quad \|F_2(\tau; \vartheta)\| \leq C_2 \tau^2, \quad |\tau| \leq \tau_0. \tag{1.28}
\]

According to [Su5, (1.41)], the operator \(F_1(\vartheta)\) has the form

\[
F_1(\vartheta) = \vartheta_1(Z + Z^*) + \vartheta_2(\tilde{Z} + \tilde{Z}^*). \tag{1.29}
\]

From (1.17), (1.20), and (1.29) it follows that

\[
F_1(\vartheta) = \vartheta_1 Z + \vartheta_2 \tilde{Z}, \quad PF_1(\vartheta) = \vartheta_1 Z^* + \vartheta_2 \tilde{Z}^*. \tag{1.30}
\]

From (1.18), (1.21), and (1.29), we derive the estimate

\[
\|F_1(\vartheta)\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_{F_1} = 2\kappa^{1/2}(13\delta)^{-1/2}(\|X_1\| + c_1C(1)^{1/2}). \tag{1.31}
\]

We need an approximation obtained in [Su4, Theorem 2.2]:

\[
B(\tau; \vartheta)F(\tau; \vartheta) - \tau^2 S(\vartheta)P = \Phi_1(\tau; \vartheta), \quad \|\Phi_1(\tau; \vartheta)\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_3|\tau|^3, \quad |\tau| \leq \tau_0. \tag{1.32}
\]

Refinement of approximation (1.32) was achieved in [Su5, Theorem 3.3]:

\[
B(\tau; \vartheta)F(\tau; \vartheta) = \tau^2 S(\vartheta)P + \tau^{-3}K(\vartheta) + \Phi_2(\tau; \vartheta), \quad \|\Phi_2(\tau; \vartheta)\| \leq C_4 \tau^4, \quad |\tau| \leq \tau_0. \tag{1.33}
\]

According to [Su5, (3.18)–(3.20)], we have \(K(\vartheta) = K_0(\vartheta) + N(\vartheta)\), where

\[
K_0(\vartheta) = \sum_{l=1}^{n_1} \gamma_l(\vartheta) \left( \gamma_l(\vartheta) + \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta) \right),
\]

\[
N(\vartheta) = N_0(\vartheta) + N_*(\vartheta), \tag{1.35}
\]

\[
N_0(\vartheta) = \sum_{l=1}^{n} \mu_l(\vartheta) \left( \gamma_l(\vartheta) + \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta) \right),
\]

\[
N_*(\vartheta) = \sum_{l=1}^{n} \gamma_l(\vartheta) \left( \gamma_l(\vartheta) + \vartheta_1 Z \omega_l(\vartheta) + \vartheta_2 \tilde{Z} \omega_l(\vartheta) \right). \tag{1.36}
\]

In the basis \(\{\omega_l(\vartheta)\}_{l=1}^{n}\), the operators \(N(\vartheta)\), \(N_0(\vartheta)\), and \(N_*(\vartheta)\) (after restriction on \(\mathcal{M}\)) are represented by the \((n \times n)\)-matrices. The operator \(N_0(\vartheta)\) is diagonal: \((N_0(\vartheta)\omega_j(\vartheta) , \omega_l(\vartheta)) = \mu_j(\vartheta)\delta_{jl}, \; j, l = 1, \ldots, n\). According to (1.21), the matrix elements of the operator \(N_*(\vartheta)\) has the form

\[
(N_*(\vartheta)\omega_j(\vartheta) , \omega_l(\vartheta)) = \gamma_j(\vartheta) (\omega_j(\vartheta), \tilde{\omega}_l(\vartheta)) + \gamma_l(\vartheta) (\tilde{\omega}_j(\vartheta), \omega_l(\vartheta)) = (\gamma_j(\vartheta) - \gamma_l(\vartheta)) (\omega_j(\vartheta), \omega_l(\vartheta)). \tag{1.36}
\]

Thus, the diagonal elements \(N_*(\vartheta)\) are equal to zero. Moreover, \((N_*(\vartheta)\omega_j(\vartheta) , \omega_l(\vartheta)) = 0\), if \(\gamma_j(\vartheta) = \gamma_l(\vartheta)\). In the case when \(n = 1\), we have \(N_*(\vartheta) = 0\), i. e., \(N(\vartheta) = N_0(\vartheta)\).

An invariant representations for the operators \(K_0(\vartheta)\) and \(N(\vartheta)\) were found in [Su5, (3.21), (3.30)–(3.34)]:

\[
K_0(\vartheta) = \vartheta_1 (ZS(\vartheta)P + S(\vartheta)PZ^*) + \vartheta_2 \left( \tilde{Z}S(\vartheta)P + S(\vartheta)PZ^* \right), \tag{1.37}
\]

\[
N(\vartheta) = \vartheta_1^2 N_{11} + \vartheta_2^2 N_{12} + \vartheta_1 \vartheta_2 N_{21} + \vartheta_1 \vartheta_2^2 N_{22}. \tag{1.38}
\]
Here

\[ N_{11} = (X_1Z)^*RP + (RP)^*X_1Z, \]
\[ N_{12} = (X_1\tilde{Z})^*RP + (RP)^*X_1\tilde{Z} + (X_1Z)^*X_0\tilde{Z} \]
\[ + (X_0\tilde{Z})^*X_1Z + (Y_0\tilde{Z})^*Y_0Z + (Y_0Z)^*Y_2Z \]
\[ + (Y_2Z)^*Y_1P + (Y_1P)^*Y_2Z + (Y_2P)^*Y_1Z + (Y_1Z)^*Y_2P, \]
\[ N_{21} = (X_0\tilde{Z})^*X_1\tilde{Z} + (X_1\tilde{Z})^*X_0\tilde{Z} + (Y_0\tilde{Z})^*Y_0\tilde{Z} \]
\[ + (Y_0\tilde{Z})^*Y_2Z + (Y_2\tilde{Z})^*Y_0Z + (Y_0\tilde{Z})^*Y_2\tilde{Z} + (Y_2\tilde{Z})^*Y_1P \]
\[ + (Y_1P)^*Y_2\tilde{Z} + (Y_1\tilde{Z})^*Y_2P + (Y_2P)^*Y_1\tilde{Z} \]
\[ + Z^*Q P + (PQ Z^*)^* + \lambda(Z^*Q_0 P + PQ_0 Z), \]
\[ N_{22} = (Y_0\tilde{Z})^*Y_2\tilde{Z} + (Y_2\tilde{Z})^*Y_0\tilde{Z} + Z^*Q P + PQ Z^* \]
\[ + \lambda(\tilde{Z}^*Q_0 P + PQ_0 \tilde{Z}). \]

Clarity that, in (1.41), by the formal notion \( Z^*Q P + PQ Z \) we mean the bounded self-adjoint operator acting in \( \mathfrak{H} \) corresponding to the form \( q[Pu, Zu] + q[Zu, Pu], u \in \mathfrak{H} \). Similarly, in (1.42), \( \tilde{Z}^*Q P + PQ \tilde{Z} \) is understood as the bounded self-adjoint operator in \( \mathfrak{H} \) corresponding to the form \( q[Pu, \tilde{Z} u] + q[\tilde{Z} u, Pu], u \in \mathfrak{H} \).

By (1.17), (1.20), and (1.37)–(1.42), for the operator \( K(\vartheta) = K_0(\vartheta) + N(\vartheta) \) we have identity

\[ PK(\vartheta)P = PN(\vartheta)P = N(\vartheta). \]  

We need the following estimates obtained in Su5, (3.35), (3.36), (3.44)–(3.49)):

\[ \|N(\vartheta)\|_{\mathfrak{B} \to \mathfrak{H}} \leq C_N, \]  
\[ \|K(\vartheta)\|_{\mathfrak{B} \to \mathfrak{H}} \leq C_K. \]

**Remark 1.5.** The constants \( C_1, C_2, C_3, C_4, \) and \( C_F \), introduced above depend on data polynomials. The corresponding polynomials have numeric positive coefficients. The constants \( C_N \) and \( C_K \) can be estimated in terms of polynomials with numeric coefficients and variables set (1.14). The dependence of constants \( C_1, C_2, C_3, C_4, C_N, \) and \( C_K \) on the data (1.14) was traced in Su4, Su5.

2. **Approximation of the operator** \( e^{-B(t, \varepsilon)s} \) **with the corrector taken into account**

2.1. **Approximation of the operator** \( e^{-B(\tau; \vartheta)s} \). Assume that for some \( c_\ast > 0 \) we have

\[ A(t) \geq c_\ast t^2 I, \quad c_\ast > 0, \quad |t| \leq \tau_0. \]  

Then, by (1.12),

\[ B(\tau; \vartheta) \geq \hat{c}_\ast \tau^2 I, \quad |\tau| \leq \tau_0; \quad \hat{c}_\ast = \frac{1}{2} \min\{\kappa c_\ast; 2\beta\}. \]  

Together with (1.22) this implies that the eigenvalues \( \lambda_l(\tau; \vartheta) \) of the operator \( B(\tau; \vartheta) \) satisfy estimates

\[ \lambda_l(\tau; \vartheta) \geq \hat{c}_\ast \tau^2, \quad l = 1, \ldots, n, \quad |\tau| \leq \tau_0. \]  

Bringing together (1.23) and (2.3), we see that \( \gamma_l(\vartheta) \geq \hat{c}_\ast, l = 1, \ldots, n \), and so (see (1.20))

\[ S(\vartheta) \geq \hat{c}_\ast I_{\mathfrak{H}}. \]  

Let \( |\tau| \leq \tau_0 \). Rewrite the operator \( e^{-B(\tau; \vartheta)s}, s \geq 0 \), as

\[ e^{-B(\tau; \vartheta)s} = e^{-B(\tau; \vartheta)s} F(\tau; \vartheta)^{\perp} + e^{-B(\tau; \vartheta)s} F(\tau; \vartheta). \]  

Let \( s > 0 \). By (2.2), the first summand in the right-hand side (2.5) satisfies the estimate

\[ \|e^{-B(\tau; \vartheta)s} F(\tau; \vartheta)^{\perp}\|_{\mathfrak{B} \to \mathfrak{H}} \leq e^{-\delta s/2} e^{-\hat{c}_\ast \tau^2 s/2} \leq 2(\delta s)^{-1} e^{-\hat{c}_\ast \tau^2 s/2}. \]  

The second summand can be represented as

\[ e^{-B(\tau; \vartheta)s} F(\tau; \vartheta) = P e^{-B(\tau; \vartheta)s} F(\tau; \vartheta) + P^\perp e^{-B(\tau; \vartheta)s} F(\tau; \vartheta). \]  

(2.7)
By \((1.28)\),
\[
P^{\perp} e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) = (I - P) e^{-B(\tau;\vartheta)s} F(\tau; \vartheta)
\]
\[
= (\tau F_1(\vartheta) + F_2(\tau; \vartheta)) e^{-B(\tau;\vartheta)s} F(\tau; \vartheta)
\]
\[
= (\tau F_1(\vartheta) + F_2(\tau; \vartheta)) \left( e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) - e^{-\tau^2 S(\vartheta)s} P + e^{-\tau^2 S(\vartheta)s} P \right).
\]
Set
\[
\Pi(\tau; \vartheta; s) := e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) - e^{-\tau^2 S(\vartheta)s} P.
\]
Then
\[
P^{\perp} e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) = \tau F_1(\vartheta) e^{-\tau^2 S(\vartheta)s} P + \tau F_1(\vartheta) \Pi(\tau; \vartheta; s)
\]
\[
+ F_2(\tau; \vartheta) e^{-B(\tau;\vartheta)s} F(\tau; \vartheta).
\]
We need the estimate obtained in \(M\) Subsec. 2.2:
\[
\|\Pi(\tau; \vartheta; s)\|_{\mathcal{B}} \lesssim (2C_1|\tau| + C_3|\tau|^3) e^{-\varepsilon_s \tau^2 s}.
\]
Together with \((1.31)\) this implies
\[
\|\tau F_1(\vartheta) \Pi(\tau; \vartheta; s)\|_{\mathcal{B}} \lesssim C_F(2C_1 \tau^2 + C_3 \tau^4) e^{-\varepsilon_s \tau^2 s}.
\]
Introducing parameter \(\alpha = \tau^2 s\), we arrive at the inequality
\[
\|\tau F_1(\vartheta) \Pi(\tau; \vartheta; s)\|_{\mathcal{B}} \lesssim C_F(2C_1 \alpha + C_3 \alpha^2) e^{-\varepsilon_s \alpha s^{-1}}.
\]
The third summand in the right-hand side of \((2.9)\) we estimate with the help of \((1.28)\) and \((2.2)\):
\[
\|F_2(\tau; \vartheta) e^{-B(\tau;\vartheta)s} F(\tau; \vartheta)\|_{\mathcal{B}} \lesssim C_2 \tau^2 e^{-\varepsilon_s \tau^2 s} = C_2 e^{-\varepsilon_s \alpha s^{-1}}.
\]
Bringing together \((2.9)\), \((2.11)\), and \((2.12)\), we obtain
\[
\|P^{\perp} e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) - \tau F_1(\vartheta) e^{-\tau^2 S(\vartheta)s} P\|_{\mathcal{B}} \lesssim \phi(\alpha) s^{-1}, \quad s > 0,
\]
where
\[
\phi(\alpha) = C_F(2C_1 \alpha + C_3 \alpha^2) e^{-\varepsilon_s \alpha} + C_2 \alpha e^{-\varepsilon_s \alpha}.
\]
Now, consider the first summand in the right-hand side of \((2.7)\):
\[
P e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) = e^{-\tau^2 S(\vartheta)s} P + \Upsilon(\tau; \vartheta; s),
\]
where \(\Upsilon(\tau; \vartheta; s) := P e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) - e^{-\tau^2 S(\vartheta)s} P\). In \(M\) Subsec. 2.2, it was obtained the representation
\[
\Upsilon(\tau; \vartheta; s) = e^{-\tau^2 S(\vartheta)s} P \Phi(\tau; \vartheta) - \mathcal{J}(\tau; \vartheta; s),
\]
\[
\mathcal{J}(\tau; \vartheta; s) := \int_0^1 e^{-\tau^2 S(\vartheta)\alpha} P \Phi(\tau; \vartheta) e^{-B(\tau;\vartheta)s} F(\tau; \vartheta) \, d\alpha.
\]
Here the operators \(\Phi(\tau; \vartheta)\) and \(\Phi_1(\tau; \vartheta)\) are defined in \((1.24)\) and \((1.32)\). Taking \((1.27)\) and \((1.28)\) into account, we represent the operator \(\Upsilon(\tau; \vartheta; s)\) as
\[
\Upsilon(\tau; \vartheta; s) = e^{-\tau^2 S(\vartheta)s} P \Phi(\tau; \vartheta) + e^{-\tau^2 S(\vartheta)s} P \Phi_1(\tau; \vartheta) - \mathcal{J}(\tau; \vartheta; s).
\]
The second summand in the right-hand side of \((2.16)\) can be estimated with the help of \((1.28)\) and \((2.4)\):
\[
\|e^{-\tau^2 S(\vartheta)s} P \Phi_1(\tau; \vartheta)\|_{\mathcal{B}} \lesssim C_2 \tau^2 e^{-\varepsilon_s \tau^2 s} = C_2 e^{-\varepsilon_s \alpha} s^{-1}.
\]
By (1.32), (1.33), (1.43), and (2.8), the third summand in the right-hand side of (2.16) can be rewritten as

\[ J(\tau; \vartheta; s) \]

\[ = \int_{0}^{s} e^{-\tau^2 S(\vartheta)P(s-\tilde{s})} P(\vartheta)^3 K(\vartheta) + \Phi_2(\tau; \vartheta)) e^{-B(\tau;\vartheta)\tilde{s}} F(\tau; \vartheta) d\tilde{s} \]

(2.18)

Let us estimate the term \( J_2(\tau; \vartheta; s) \) with the help of (1.45), (2.4), and (2.10):

\[ \| J_2(\tau; \vartheta; s) \|_{\delta \to \delta} \leq |\tau|^3 \int_{0}^{s} e^{-\epsilon_s^s (s-\tilde{s})} C_K (2C_1|\tau| + C_3|\tau|^3 \tilde{s}) e^{-\epsilon_s^s \tilde{s}} d\tilde{s} \]

\[ = |\tau|^3 e^{-\epsilon_s^s \tilde{s}} C_K \int_{0}^{s} (2C_1|\tau| + C_3|\tau|^3 \tilde{s}) d\tilde{s} \]

\[ = e^{-\epsilon_s^s \tilde{s}} C_K (2C_1 \tau^4 s + 2^{-1} C_3^3 s^2) \]

\[ = C_K e^{-\epsilon_s^s \tilde{s}} (2C_1 \tau^4 s + 2^{-1} C_3^3 s^2) \]

(2.19)

Using (1.32), (2.2), and (2.4), we estimate \( J_3(\tau; \vartheta; s) \):

\[ \| J_3(\tau; \vartheta; s) \|_{\delta \to \delta} \leq \int_{0}^{s} e^{-\epsilon_s^s (s-\tilde{s})} C_4 \tau^4 e^{-\epsilon_s^s \tilde{s}} d\tilde{s} \]

\[ = e^{-\epsilon_s^s \tilde{s}} C_4 \tau^4 s = C_4 e^{-\epsilon_s^s \tilde{s}} \]

(2.20)

Let us summarise the results. By (2.15)–(2.20),

\[ \left\| P e^{-B(\tau;\vartheta)\tilde{s}} F(\tau; \vartheta) - e^{-\tau^2 S(\vartheta)\tilde{s}} P - \tau e^{-\tau^2 S(\vartheta)\tilde{s}} P F_1(\vartheta) + \tau^3 M(\tau; \vartheta; s) \right\|_{\delta \to \delta} \]

\[ \leq (C_2 a + 2C_1 C_K \alpha^2 + 2^{-1} C_3 C_K^3 \alpha^3 + C_4^3) e^{-\epsilon_s^s \tilde{s}} \]

(2.21)

where

\[ M(\tau; \vartheta; s) := \int_{0}^{s} e^{-\tau^2 S(\vartheta)P(s-\tilde{s})} P N(\vartheta) P e^{-\tau^2 S(\vartheta)\tilde{s}} d\tilde{s} \]

(2.22)

Now from (1.30), (2.5), (2.7), (2.13), (2.14), (2.21), and (2.22), we derive identity

\[ e^{-B(\tau;\vartheta)s} = e^{-\tau^2 S(\vartheta)\tilde{s}} P + \tau(\vartheta Z + \vartheta_2 Z) e^{-\tau^2 S(\vartheta)\tilde{s}} P \]

\[ + e^{-\tau^2 S(\vartheta)\tilde{s}} P \tau(\vartheta Z + \vartheta_2 Z) - \tau^3 M(\tau; \vartheta; s) + \mathcal{R}(\tau; \vartheta; s) \]

(2.23)
where the remainder term \( R(\tau; \vartheta; s) \) is subject to the estimate
\[
\| R(\tau; \vartheta; s) \|_{\vartheta \to \vartheta_0} \leq C_5 s^{-1} e^{-\tilde{c}_s \vartheta^2 s/2}, \quad s > 0;
\]
\[
C_5 = 2\delta^{-1} + \max_{\alpha > 0} \tilde{\phi}(\alpha) e^{-\tilde{c}_s \alpha/2}.
\]
(2.24)

Here we introduced the notation
\[ \tilde{\phi}(\alpha) = 2(C_1 C_{F_1} + C_2)\alpha + (C_3 C_{F_1} + 2C_1 C_{K} + C_4)\alpha^2 + 2 - 1 C_3 C_K \alpha^3. \]

Changing variable \( \tilde{c}_s \alpha/2 =: \alpha \), we can show that the constant \( C_5 \) is controlled by the polynomial of \( \tilde{c}_s^{-1} \) and the data (1.11).

Obviously, for \( s \geq 1 \) one has \( s^{-1} \leq 2(s + 1)^{-1} \), and, consequently, \( \| R(\tau; \vartheta; s) \| \leq 2C_5 (s + 1)^{-1} e^{-\tilde{c}_s \vartheta^2 s/2} \).

For \( 0 < s < 1 \), from (1.11), (2.22) it follows that for \( |\tau| \leq \tau_0 \) we have the inequality \( |\tau| \| M(\tau; \vartheta; s) \| \leq \tau_0^3 C_N e^{-\tilde{c}_s \vartheta^2 s/2} \).

Theorem 2.1. For \( |\tau| \leq \tau_0, s \geq 0 \), identity (2.23) makes sense, where the operator \( M(\tau; \vartheta; s) \) is defined in (2.22), and the operator \( R(\tau; \vartheta; s) \) is subject to estimates
\[
\| R(\tau; \vartheta; s) \| \leq C_6 (1 + s)^{-1} e^{-\tilde{c}_s \vartheta^2 s/2}, \quad 0 < s < 1,
\]
where \( C_6 = 4 + 4\kappa^{-1/2} (13\delta)^{-1/2} \tau_0 \| X_1 \| + c_3 C(1)^{1/2} + 2\tau_0^3 C_N \).

Taking Remark 1.3 into account, we arrive at the following result.

2.2. Enumeration of the matrix elements of the operator \( M(\tau; \vartheta; s) \). Let us look at the operator (2.22) in more detail. By (1.35),
\[
M(\tau; \vartheta; s) = M_0(\tau; \vartheta; s) + M_*(\tau; \vartheta; s),
\]
(2.25)

where
\[
M_0(\tau; \vartheta; s) = \int_0^s e^{-\tau^2 S(\vartheta) P(s-\vartheta)} P N_0(\vartheta) e^{-\tau^2 S(\vartheta) P^{\tau^2} s} d\tilde{s},
\]
\[
M_*(\tau; \vartheta; s) = \int_0^s e^{-\tau^2 S(\vartheta) P(s-\vartheta)} P N_*(\vartheta) e^{-\tau^2 S(\vartheta) P^{\tau^2} s} d\tilde{s}.
\]
The operators \( S(\vartheta) \) and \( N_0(\vartheta) \) are diagonal in the basis \( \{ \omega_l(\vartheta) \}_{l=1}^p \), thus \( N_0(\vartheta) S(\vartheta) = S(\vartheta) N_0(\vartheta) \) and
\[
M_0(\tau; \vartheta; s) = N_0(\vartheta)e^{-\tau^2 S(\vartheta) P^{\tau^2}_s} P_s.
\]
(2.26)

Let us calculate the matrix elements of the operator \( M_*(\tau; \vartheta; s) \) in the basis \( \{ \omega_l(\vartheta) \}_{l=1}^p \). According to (1.26) we have \( e^{-\tau^2 S(\vartheta) P^{\tau^2}_s} \omega_j(\vartheta) = e^{-\tau^2 \gamma_j(\vartheta) s} \omega_j(\vartheta) \). Thus, the matrix elements of the operator \( M_*(\tau; \vartheta; s) \) has the form
\[
\left( M_*(\tau; \vartheta; s) \omega_j(\vartheta), \omega_k(\vartheta) \right) = \int_0^s \left( e^{-\tau^2 S(\vartheta) P(s-\vartheta)} N_*(\vartheta) e^{-\tau^2 S(\vartheta) P^{\tau^2}_s} \omega_j(\vartheta), \omega_k(\vartheta) \right) d\tilde{s}
\]
\[
= \left( N_*(\vartheta) \omega_j(\vartheta), \omega_k(\vartheta) \right) e^{-\tau^2 \gamma_j(\vartheta) s} \int_0^s e^{-\tau^2 (\gamma_j(\vartheta) - \gamma_k(\vartheta)) s} d\tilde{s}.
\]

Together with (1.36) this implies that \( \gamma_j(\vartheta) = \gamma_k(\vartheta) \) satisfy
\[
\left( M_*(\tau; \vartheta; s) \omega_j(\vartheta), \omega_k(\vartheta) \right) = 0.
\]
Now, let $\gamma_j(\vartheta) \neq \gamma_k(\vartheta)$. By \((1.24)\) and \((1.36)\),
\[
\left( M_\ast(\tau; \vartheta; s)\omega_j(\vartheta), \omega_k(\vartheta) \right) = (N_\ast(\vartheta)\omega_j(\vartheta), \omega_k(\vartheta)) e^{-\tau^2\gamma_j(\vartheta)s} e^{-\tau^2(\gamma_j(\vartheta)-\gamma_k(\vartheta))s} \frac{1}{(\gamma_k(\vartheta)-\gamma_j(\vartheta))^2}
\]
\[
= \frac{e^{-\tau^2\gamma_j(\vartheta)s} - e^{-\tau^2\gamma_k(\vartheta)s}}{\tau^2} (\omega_j(\vartheta), \omega_k(\vartheta))
\]
\[
= \frac{e^{-\tau^2\gamma_j(\vartheta)s}}{\tau^2} (\omega_j(\vartheta), \omega_k(\vartheta)) + \frac{e^{-\tau^2\gamma_k(\vartheta)s}}{\tau^2} (\omega_j(\vartheta), \omega_k(\vartheta)).
\]

Thus,
\[
\left( M_\ast(\tau; \vartheta; s)\omega_j(\vartheta), \omega_k(\vartheta) \right) = \tau^{-2} \left( (e^{-\tau^2\gamma_j(\vartheta)s} \omega_j(\vartheta), \omega_k(\vartheta)) + (\omega_j(\vartheta), e^{-\tau^2\gamma_k(\vartheta)s} \omega_k(\vartheta)) \right).
\]

According to \((1.24)\), for $\gamma_j(\vartheta) = \gamma_k(\vartheta)$ the right-hand side in \((2.27)\) is equal to zero, so equality \((2.27)\) holds true in this case. From \((2.27)\) with the help of identities \((1.24)\) we get the representation
\[
M_\ast(\tau; \vartheta; s) = -\tau^{-2} \sum_{l=1}^{n} e^{-\tau^2\gamma_l(\vartheta)s} \left( (\cdot, \tilde{\omega}_l(\vartheta)) \omega_l(\vartheta) + (\cdot, \omega_l(\vartheta)) \tilde{\omega}_l(\vartheta) \right).
\]

Combining \((2.25)\), \((2.26)\), and \((2.28)\), we find
\[
M(\tau; \vartheta; s) = N_0(\vartheta) e^{-\tau^2 S(\vartheta)P_s} P S - \tau^{-2} \sum_{l=1}^{n} e^{-\tau^2\gamma_l(\vartheta)s} \left( (\cdot, \tilde{\omega}_l(\vartheta)) \omega_l(\vartheta) + (\cdot, \omega_l(\vartheta)) \tilde{\omega}_l(\vartheta) \right).
\]

Finally, note that formula \((2.28)\) simplifies if $N(\vartheta) = 0$ or $N_\ast(\vartheta) = 0$.

**Proposition 2.2.** Let $N(\vartheta) = 0$. Then, under conditions of Theorem \(\text{2.1}\) we have
\[
e^{-B(\tau; \vartheta)s} = e^{-\tau^2 S(\vartheta)s} P + \tau(\vartheta_1 Z + \vartheta_2 \tilde{Z}) e^{-\tau^2 S(\vartheta)s} P + e^{-\tau^2 S(\vartheta)s} P \tau(\vartheta_1 Z^* + \vartheta_2 \tilde{Z}^*) + R(\tau; \vartheta; s).
\]

**Proposition 2.3.** Under conditions of Theorem \(\text{2.1}\) let $N_\ast(\vartheta) = 0$. Thus
\[
e^{-B(\tau; \vartheta)s} = e^{-\tau^2 S(\vartheta)s} P + \tau(\vartheta_1 Z + \vartheta_2 \tilde{Z}) e^{-\tau^2 S(\vartheta)s} P + e^{-\tau^2 S(\vartheta)s} P \tau(\vartheta_1 Z^* + \vartheta_2 \tilde{Z}^*) - \tau^3 N_0(\vartheta)e^{-\tau^2 S(\vartheta)s} P S + R(\tau; \vartheta; s).
\]

2.3. **Returning to parameters $t$ and $\varepsilon$.** Let us return back to the original parameters $t, \varepsilon$, remaining that $t = \tau \vartheta_1$, $\varepsilon = \tau \vartheta_2$. By \((1.23)\), the operator $\tau^2 S(\vartheta) =: L(t, \varepsilon)$ is given by the expression
\[
L(t, \varepsilon) = t^2 S + t \varepsilon \left( - (X_0 Z)^* X_0 \tilde{Z} - (X_0 \tilde{Z})^* X_0 Z + P(Y_2 Y_1 + Y_1 Y_2) \right)_{\mathbb{R}} + \varepsilon^2 \left( - (X_0 \tilde{Z})^* X_0 \tilde{Z} |_{\mathbb{R}} + Q_{\mathbb{R}} + \lambda Q_{\mathbb{R}} \right).
\]

Note the estimate follows from \((2.4)\):
\[
L(t, \varepsilon) \geq \tilde{c}_s (t^2 + \varepsilon^2) I_{\mathbb{R}}, \quad t \in \mathbb{R}, \quad 0 \leq \varepsilon \leq 1.
\]

By \((1.35)\), the operator $\tau^3 N(\vartheta) =: N(t, \varepsilon)$ has the form
\[
N(t, \varepsilon) = t^3 N_{11} + t^2 \varepsilon N_{12} + t \varepsilon^2 N_{21} + \varepsilon^3 N_{22}.
\]

According to \((1.44)\),
\[
\|N(t, \varepsilon)\| \leq C N(t^2 + \varepsilon^2)^{3/2}, \quad t \in \mathbb{R}, \quad 0 \leq \varepsilon \leq 1.
\]
Rewriting (2.23) in terms of original parameters, we give an equivalent formulation of Theorem 2.1 that is convenient for further applications to differential operators. By the corrector we call the operator

\[ K(t, \varepsilon, s) := (tZ + \varepsilon \bar{Z})e^{-L(t, \varepsilon)s}P + e^{-L(t, \varepsilon)s}P(tZ^* + \varepsilon \bar{Z}^*) \]

\[ - \int_0^s e^{-L(t, \varepsilon)(s-\tau)}PN(t, \varepsilon)e^{-L(t, \varepsilon)s}P d\tau. \]  

(2.33)

By (2.20) and (2.22),

\[ \|K(t, \varepsilon, s)\| \leq 2 \max\{\|Z\|; \|\bar{Z}\|\}(t + \varepsilon)e^{-\varepsilon^2(t^2 + \varepsilon^2)s/2} + C_N s(t^2 + \varepsilon^2)^{3/2}e^{-\varepsilon^2(t^2 + \varepsilon^2)s}. \]

(2.34)

Combining (1.11), (1.21), and (2.33) and using elementary inequalities \(e^{-\alpha} \leq \alpha^{-1}e^{-\alpha/2}\) and \(e^{-\alpha} \leq 3\alpha^{-2}e^{-\alpha/2}\), \(\alpha > 0\), we get

\[ \|K(t, \varepsilon, s)\| \leq C_8 s^{-1}(t^2 + \varepsilon^2)^{-1/2}e^{-\varepsilon^2(t^2 + \varepsilon^2)s/2}, \quad s > 0; \]

\[ C_8 = 2^{3/2}e^{-\varepsilon^2}(13\delta)^{-1/2} \max\{\|X_1\|; c_1 C(1)^{1/2}\}. \]

(2.35)

For \(s \geq 0\), the estimate is more bulky: now for exponentials we use inequalities of the form \(e^{-\alpha} \leq 2(1 + \alpha) - e^{-\alpha/2}\) and \(e^{-\alpha} \leq 4(1 + \alpha) - e^{-\alpha/2}\), where \(\alpha > 0\). As result, we obtain

\[ \|K(t, \varepsilon, s)\| \leq C_9 (t^2 + \varepsilon^2)^{-1/2}(1 + \varepsilon(1 + \varepsilon^2)s)^{-1}e^{-\varepsilon^2(t^2 + \varepsilon^2)s/2}, \quad s \geq 0; \]

\[ C_9 = 2^{5/2}e^{-\varepsilon^2}(13\delta)^{-1/2} \max\{\|X_1\|; c_1 C(1)^{1/2}\}. \]

(2.36)

Theorem 2.4. We have

\[ e^{-B(t, \varepsilon)s} = e^{-L(t, \varepsilon)s}P + K(t, \varepsilon, s) + R(t, \varepsilon, s), \]

(2.37)

where the operators \(B(t, \varepsilon), L(t, \varepsilon), \) and \(K(t, \varepsilon, s)\) are defined in (1.1), (2.22), and (2.33), respectively. For the operator \(K(t, \varepsilon, s)\) we have estimates (2.35) and (2.36). The operator \(R(t, \varepsilon, s) := R(\tau; \varepsilon, s)\) for \(t^2 + \varepsilon^2 \leq \tau_0^2\) is subject to inequalities:

\[ \|R(t, \varepsilon, s)\| \leq C_7(s + 1)^{-1}e^{-\varepsilon^2(t^2 + \varepsilon^2)s/2}, \quad s \geq 0; \]

\[ \|R(t, \varepsilon, s)\| \leq C_5 s^{-1}e^{-\varepsilon^2(t^2 + \varepsilon^2)s/2}, \quad s > 0. \]

(2.38)

(2.39)

The constants \(C_5, C_7, C_8, \) and \(C_9\) are controlled in terms of polynomials whose coefficients are positive numbers and whose variables are \(\varepsilon^{-1}\) and data set (1.14).

3. Approximation for the bordered operator exponential

3.1. The operator family \(A(t) = M^* \hat{A}(t)M\). Let \(\hat{\mathcal{H}}\) be yet another Hilbert space, and let \(\hat{X}(t) = \hat{X}_0(t) + t\hat{X}_1(t)\) be the family of the form (1.1) satisfying conditions of Subsec. 1.1. We emphasize that the space \(\mathcal{H}\) is the same as before. All the objects corresponding to \(\hat{X}(t)\) we denote by the mark \(\hat{\cdot}\). Let \(M : \hat{\mathcal{H}} \to \hat{\mathcal{H}}\) be an isomorphism

\[ M\text{Dom} X_0 = \text{Dom} \hat{X}_0, \]

(3.1)

then

\[ X(t) = \hat{X}(t)M : \mathcal{H} \to \mathcal{H}_s; \quad X_0 = \hat{X}_0M, \quad X_1 = \hat{X}_1M. \]

Then \(A(t) = M^* \hat{A}(t)M\), where \(\hat{A}(t) = \hat{X}(t)^* \hat{X}(t)\). Note that \(\hat{\mathcal{R}} = M\mathcal{R}, \hat{n} = n, \) and \(\hat{\mathcal{H}}_s = \mathcal{H}_s, \hat{n}_s = n_s, \hat{P}_s = P_s\). Set

\[ G = (MM^*)^{-1} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}. \]

(3.2)

Let \(G_{\hat{\mathcal{R}}}\) be the block of the operator \(G\) in the subspace \(\hat{\mathcal{R}}: G_{\hat{\mathcal{R}}} = \hat{P}G_{\mid \hat{\mathcal{R}}}; \hat{\mathcal{R}} \to \hat{\mathcal{R}}.\) Obviously, \(G_{\hat{\mathcal{R}}}\) is an isomorphism in \(\hat{\mathcal{R}}.\)

Let \(\tilde{\mathcal{F}} = \hat{\mathcal{R}} \hat{R}: \hat{\mathcal{H}} \to \hat{\mathcal{R}}\) be the spectral germ of the operator family \(\hat{A}(t)\) with \(t = 0.\) According to [BSH2] Chapter 1, Subsec. 1.5], we have

\[ R = \hat{R}M_{\mid \hat{\mathcal{R}}}, \quad \text{rank } R = \text{rank } \hat{R}, \]

(3.3)

and \(S = PM^* \hat{SM}_{\mid \hat{\mathcal{R}}}.\)
3.2. The operator family $B(t, \varepsilon) = M^* \hat{B}(t, \varepsilon) M$. Assume that the operator $\hat{Y}_0: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ satisfies conditions of Subsec. 1.2. Note that the space $\tilde{\mathcal{H}}$ does not change. Set $Y_0 := \hat{Y}_0 M$, $M \text{Dom} Y_0 = \text{Dom} \hat{Y}_0$. By (3.11) and condition $\text{Dom} \hat{X}_0 \subset \text{Dom} \hat{Y}_0$, the inclusion $\text{Dom} X_0 \subset \text{Dom} Y_0$ makes sense. Let $\hat{Y}_1: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be a bounded operator and let $Y_1 = \hat{Y}_1 M: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$. Set $\tilde{Y}(t) := \hat{Y}_0 + t \hat{Y}_1: \bar{\mathcal{H}} \to \tilde{\mathcal{H}}$, $\text{Dom} \tilde{Y}(t) = \text{Dom} \hat{Y}_0$. Let $Y(t) = \tilde{Y}(t) M = Y_0 + t Y_1: \bar{\mathcal{H}} \to \bar{\mathcal{H}}$, $\text{Dom} Y(t) = \text{Dom} Y_0$. Assume that the operators $\tilde{X}(t)$ and $\tilde{Y}(t)$ are subject to Condition 1.2 with some constant $\tilde{c}_1$. Then we automatically have that $\|Y(t)u\|_{\tilde{\mathcal{H}}} \leq c_1 \|X(t)u\|_{\mathcal{H}}$, $u \in \text{Dom} X_0$, where $c_1 = \tilde{c}_1$.

Let the operator $\hat{Y}_2: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ satisfy conditions of Subsec. 1.2. Put $Y_2 := \hat{Y}_2 M: \bar{\mathcal{H}} \to \bar{\mathcal{H}}$, $M \text{Dom} Y_2 = \text{Dom} \hat{Y}_2$. Since $M$ is an isomorphism, and the operator $\hat{Y}_2$ is densely defined, the operator $Y_2$ is also densely defined. By (3.11), the inclusion $\text{Dom} X_0 \subset \text{Dom} Y_2$ holds true. Assume that the operators $\tilde{X}(t)$ and $\tilde{Y}_2$ are subject to Condition 1.3 with some constants $\tilde{C}(\nu) > 0$. Then, automatically, for any $\nu > 0$ where exists a constant $C(\nu) = \tilde{C}(\nu)||M||^2 > 0$ such that for $u \in \text{Dom} X_0$, $t \in \mathbb{R}$, we have $\|Y_2 u\|^2_{\tilde{\mathcal{H}}} \leq \nu \|X(t)u\|^2_{\mathcal{H}} + C(\nu)||u||^2_{\mathcal{H}}$.

Put $Q_0 := M^* X_0 M$. Then $Q_0$ is a bounded positive definite operator in $\bar{\mathcal{H}}$. (The role of $\hat{Q}_0$ is played by the identity operator in $\bar{\mathcal{H}}$.)

In the space $\tilde{\mathcal{H}}$, consider the form $\hat{q}$ satisfying conditions of Subsec. 1.3. Define the form $q$ acting by the rule $q[u, v] = \hat{q}[Mu, Mv]$, $u, v \in \text{Dom} q$. $M \text{Dom} q = \text{Dom} \hat{q}$. Formally, $Q = M^* Q M$. Assume that the operator $\hat{X}(t)$ and the form $\hat{q}$ satisfy Condition 1.4 with some constants $\kappa, \tilde{c}_0, \tilde{c}_2$, and $\tilde{c}_3$. Taking (3.11) into account, one can check that the operator $X(t) = \hat{X}(t) M$ and the form $q$ also satisfy Condition 1.4 with the constants

$$c_0 = \|M\|^2 \tilde{c}_0, \quad c_0 = \|M^{-1}\|^{-2} \tilde{c}_0, \quad c_0 < 0,$$

$$c_1 = \|M\|^2 \hat{c}_0, \quad c_2 = \tilde{c}_0, \quad c_3 = \|M\|^2 \hat{c}_3, \quad \text{and the same constant } \kappa.$$

From (1.16) it follows that the constants $c_4$ and $\tilde{c}_4 = 4 \kappa^{-1} \hat{c}_0^2 \hat{C}(\nu)$ for $\nu = \kappa^2 (16 \hat{c}_0^2)^{-1}$ are related by the identity

$$c_4 = \|M\|^2 \hat{c}_4. \tag{3.5}$$

Under the above assumptions, the operator pencil

$$\hat{B}(t, \varepsilon) = \hat{A}(t) + \varepsilon (\hat{Y}_2^* \hat{Y}(t) + \hat{Y}(t) \hat{Y}_2) + \varepsilon^2 \hat{Q} + \lambda \varepsilon^2 I \tag{3.6}$$

is related to the pencil (1.7) by the identity

$$B(t, \varepsilon) = M^* \hat{B}(t, \varepsilon) M. \tag{3.7}$$

The constant $\lambda$ is chosen from Condition 1.9 for the operator (1.7). By using relations (3.4), (3.5), and the equality $Q_0 = M^* M$, we find that, under our choice of $\lambda$, Condition 1.9 for the operator (3.6) is also satisfied.

Note that for the operator (3.6) both variants of relation (1.11) has the form $\hat{\beta} = \lambda - c_0 - c_4$. Together with (1.11), (3.4), and (3.5), this implies that

$$\beta \leq \|M^{-1}\|^{-2} \hat{\beta}. \tag{3.8}$$

3.3. The relation between the spectral germs $S(\vartheta)$ and $\hat{S}(\vartheta)$. In expressions for the operators $B(t, \varepsilon)$ and $\hat{B}(t, \varepsilon)$ we now switch to the parameters $\tau, \vartheta$. Consider the spectral germ (1.25) and the similar germ for the family (3.6):

$$\hat{S}(\vartheta) = \vartheta_1^2 \hat{S} - \vartheta_1 \vartheta_2 (\hat{X}_0 \hat{Z})^* (\hat{X}_0 \hat{Z})|_{\mathcal{H}} - \vartheta_1 \vartheta_2 (\hat{X}_0 \hat{Z})^* (\hat{X}_0 \hat{Z})|_{\mathcal{H}}$$

$$- \vartheta_2^2 (\hat{X}_0 \hat{Z})^* (\hat{X}_0 \hat{Z})|_{\mathcal{H}} + \vartheta_1 \vartheta_2 \hat{P} (\hat{Y}_1^* \hat{Y}_2 + \hat{Y}_1^* \hat{Y}_2)|_{\mathcal{H}} + \vartheta_2^2 (\hat{Q}_\vartheta + \lambda \mathcal{L}_\vartheta). \tag{3.9}$$

In [M] Proposition 1.8, the relation between operators (1.25) and (3.9) was proven:

$$S(\vartheta) = PM^* \hat{S}(\vartheta) M|_{\mathcal{H}}. \tag{3.10}$$
Let’s go back to the parameters \( t \) and \( \varepsilon \). The operator \( \tau^2 \tilde{S}(\vartheta) =: \hat{L}(t, \varepsilon) \) is given by the expression
\[
\hat{L}(t, \varepsilon) = t^2 \hat{S} + t\varepsilon \left( - \langle \hat{X}_0 \hat{Z} \rangle^* \hat{X}_0 \hat{Z} - \langle \hat{X}_0 \hat{Z} \rangle^* \hat{X}_0 \hat{Z} + \hat{P}(\hat{Y}_2^* \hat{Y}_1 + \hat{Y}_1^* \hat{Y}_2) \right) |_{\hat{R}}^* + \varepsilon^2 \left( - \langle \hat{X}_0 \hat{Z} \rangle^* \hat{X}_0 \hat{Z} |_{\hat{R}}^* + \hat{Q}_R + \lambda |_{\hat{R}}^* \right).
\] (3.11)
By (3.10), \( L(t, \varepsilon) = PM^* \hat{L}(t, \varepsilon)M |_{\hat{R}} \). Here \( L(t, \varepsilon) \) is the operator (2.29).

3.4. The operators \( \hat{Z}_G \) and \( \hat{\tilde{Z}}_G \). Let \( \hat{Z}_G \) be the operator in \( \hat{\mathfrak{H}} \) mapping an element \( \hat{u} \in \hat{\mathfrak{H}} \) into the (unique) solution \( \hat{\varphi}_G \) of the equation
\[
\hat{X}_0^* (\hat{X}_0 \hat{\varphi}_G + \hat{X}_1 \hat{\varphi}) = 0, \quad G \hat{\varphi}_G \perp \hat{R},
\]
where \( \hat{\varphi} = \hat{P} \hat{u} \). The equation is understood in a weak sense (cf. (1.16)). Thus, as was shown in [BSn3] Lemma 6.1,
\[
\hat{Z}_G = MZM^{-1} \hat{P}.
\] (3.12)
Similarly, let \( \hat{\tilde{Z}}_G \) be the operator in \( \hat{\mathfrak{H}} \) mapping an element \( \hat{u} \in \hat{\mathfrak{H}} \) into the (unique) solution \( \hat{\psi}_G \) of the equation
\[
\hat{X}_0^* \hat{X}_0 \hat{\psi}_G + \hat{Y}_0 \hat{Y}_2 \hat{\psi} = 0, \quad G \hat{\psi}_G \perp \hat{R},
\]
where \( \hat{\psi} = \hat{P} \hat{u} \). The equation is understood in a weak sense. By recalculating in equation (1.19) and identities \( MM = R, (3.1) \), and (3.2), we get
\[
\hat{\tilde{Z}}_G = M \hat{\tilde{Z}} \hat{M}^{-1} \hat{P}.
\] (3.13)

3.5. The operator \( \hat{N}_G(t, \varepsilon) \). Using relation between operators \( X_0, X_1, Y_0, Y_1, Y_2, \) and \( Q \) and the corresponding operators marked by the “hat”, applying identities \( \hat{R} = M \hat{R}, Q_0 = M^*M, \) (1.39), (1.42), (2.31), (3.3), (3.12), and (3.13), we conclude that the operator
\[
\hat{N}_G(t, \varepsilon) := \hat{P}(M^*)^{-1}N(t, \varepsilon)M^{-1} \hat{P}
\] (3.14)
has the form
\[
\hat{N}_G(t, \varepsilon) = t^3 \hat{N}_{G,11} + t^2 \varepsilon \hat{N}_{G,12} + t \varepsilon^2 \hat{N}_{G,21} + \varepsilon^3 \hat{N}_{G,22},
\] (3.15)
where
\[
\hat{N}_{G,11} = (\hat{X}_1 \hat{Z}_G)^* \hat{R} \hat{P} + (\hat{R} \hat{P})^* \hat{X}_1 \hat{Z}_G,
\] (3.16)
\[
\hat{N}_{G,12} = (\hat{X}_1 \hat{Z}_G)^* \hat{R} \hat{P} + (\hat{R} \hat{P})^* \hat{X}_1 \hat{Z}_G + (\hat{X}_1 \hat{Z}_G)^* \hat{X}_0 \hat{Z}_G
\]
\[
+ (\hat{X}_0 \hat{Z}_G)^* \hat{X}_1 \hat{Z}_G + (\hat{Y}_0 \hat{Z}_G)^* \hat{Y}_0 \hat{Z}_G + (\hat{Y}_0 \hat{Z}_G)^* \hat{Y}_2 \hat{Z}_G
\]
\[
+ (\hat{Y}_2 \hat{Z}_G)^* \hat{Y}_1 \hat{P} + (\hat{Y}_1 \hat{P})^* \hat{Y}_2 \hat{Z}_G + (\hat{Y}_2 \hat{P})^* \hat{Y}_1 \hat{Z}_G + (\hat{Y}_1 \hat{Z}_G)^* \hat{Y}_2 \hat{P},
\] (3.17)
\[
\hat{N}_{G,21} = (\hat{X}_0 \hat{Z}_G)^* \hat{X}_1 \hat{Z}_G + (\hat{X}_1 \hat{Z}_G)^* \hat{X}_0 \hat{Z}_G + (\hat{Y}_2 \hat{Z}_G)^* \hat{Y}_0 \hat{Z}_G
\]
\[
+ (\hat{Y}_0 \hat{Z}_G)^* \hat{Y}_2 \hat{Z}_G + (\hat{Y}_2 \hat{Z}_G)^* \hat{Y}_0 \hat{Z}_G + (\hat{Y}_0 \hat{Z}_G)^* \hat{Y}_2 \hat{Z}_G + (\hat{Y}_2 \hat{Z}_G)^* \hat{Y}_1 \hat{P}
\]
\[
+ (\hat{Y}_1 \hat{P})^* \hat{Y}_2 \hat{Z}_G + (\hat{Y}_2 \hat{P})^* \hat{Y}_1 \hat{Z}_G
\]
\[
+ \hat{Z}_G \hat{Q} \hat{P} + \hat{P} \hat{Q} \hat{Z}_G + \lambda (\hat{Z}_G \hat{P} + \hat{P} \hat{Z}_G),
\]
\[
\hat{N}_{G,22} = (\hat{Y}_0 \hat{Z}_G)^* \hat{Y}_2 \hat{Z}_G + (\hat{Y}_2 \hat{Z}_G)^* \hat{Y}_0 \hat{Z}_G + (\hat{Y}_2 \hat{Z}_G)^* \hat{Q} \hat{P} + \hat{P}\hat{Q} \hat{Z}_G
\]
\[
+ \lambda \left( (\hat{Z}_G)^* \hat{P} + \hat{P} \hat{Z}_G \right).
\] (3.19)

Note the estimate that follows from (2.32) and (3.14):
\[
\| \hat{N}_G(t, \varepsilon) \| \leq C_G (t^2 + \varepsilon^2)^{3/2}; \quad C_G = C_N \| M^{-1} \|^2.
\] (3.20)

According to Remark 1.5, the following observation holds true.

**Remark 3.1.** The constant \( C_G \) is controlled in terms of a polynomial with positive numeric coefficients and variables set consisting of \( \| M^{-1} \| \) and (1.14).
3.6. Approximation for the bordered operator exponential. In the present subsection, our goal is to approximate the operator exponential $e^{-B(t, \varepsilon)s}$ generated by the operator \((3.7)\) in terms of the isomorphism $M$ and the threshold characteristics of the operator $B(t, \varepsilon)$.

Assume that the operator $A(t)$ is subject to inequality \((2.1)\) with some constant $c_\ast > 0$: $A(t) \geq c_\ast t I$, $|t| \leq \tau_0$. Here $\tau_0$ satisfies condition \((1.13)\) for the operator $B(t, \varepsilon)$. Then for $B(t, \varepsilon)$ estimate \((2.2)\) holds.

Therefore, the operator $B(t, \varepsilon)$ of the form \((3.7)\) satisfies all the assumptions of Theorem \(2.4\). Applying theorem and multiplying equality \((2.37)\) on the operator $M$ from the left and on $M^*$ from the right, we obtain

$$Me^{-B(t, \varepsilon)s}M^* = Me^{-L(t, \varepsilon)s}PM^* + MK(t, \varepsilon, s)M^* + M\mathcal{R}(t, \varepsilon, s)M^*. \quad (3.21)$$

By \((3.3)\),

$$MK(t, \varepsilon, s)M^* = M(tZ + \varepsilon \tilde{Z})e^{-L(t, \varepsilon)s}PM^* + Me^{-L(t, \varepsilon)s}P(tZ^* + \varepsilon \tilde{Z}^*)M^* \quad (3.22)$$

$$- \int_0^s Me^{-L(t, \varepsilon)(s-\tau)}PNe^{-L(t, \varepsilon)s}PM^* d\tau. \quad (3.22)$$

Now we take in mind the identity obtained in \([M, \text{Proposition 3.1}]\):

$$Me^{-L(t, \varepsilon)s}PM^* = M_0e^{-M_0\tilde{L}(t, \varepsilon)\hat{M}\hat{M}^*}M_0\hat{P}. \quad (3.23)$$

Here $L(t, \varepsilon)$ and $\tilde{L}(t, \varepsilon)$ are the operators \((2.29)\) and \((3.11)\), respectively, and

$$M_0 := (G_{\hat{G}})^{-1/2}. \quad (3.24)$$

Denote $K_G(t, \varepsilon, s) := MK(t, \varepsilon, s)M^*$. We note at once the estimates following from \((2.35)\) and \((2.36)\):

$$\|K_G(t, \varepsilon, s)\| \leq C_8\|M\|^2 s^{-1}(t^2 + \varepsilon^2)^{-1/2} e^{-c_\ast(t^2 + \varepsilon^2)s/2}, \quad s > 0; \quad (3.25)$$

$$\|K_G(t, \varepsilon, s)\| \leq C_9\|M\|^2 (t^2 + \varepsilon^2)^{1/2}(1+c_\ast(t^2 + \varepsilon^2)s)^{-1} e^{-c_\ast(t^2 + \varepsilon^2)s/2}, \quad s \geq 0. \quad (3.26)$$

By \((3.12)\), \((3.14)\), \((3.22)\), and \((3.23)\), the operator $K_G(t, \varepsilon, s)$ represents as

$$K_G(t, \varepsilon, s) = (t\tilde{Z}_G + \varepsilon \tilde{Z}_G)M_0e^{-M_0\tilde{L}(t, \varepsilon)\hat{M}\hat{M}^*}M_0\hat{P}$$

$$+ M_0e^{-M_0\tilde{L}(t, \varepsilon)\hat{M}\hat{M}^*}M_0\hat{P}(t\tilde{Z}_G^* + \varepsilon (\tilde{Z}_G)^*)$$

$$- \int_0^s M_0e^{-M_0\tilde{L}(t, \varepsilon)\hat{M}\hat{M}^*}M_0\hat{N}_G(t, \varepsilon)M_0e^{-M_0\tilde{L}(t, \varepsilon)\hat{M}\hat{M}^*}M_0\hat{P} d\tau. \quad (3.27)$$

Combining \((3.28)\), \((3.29)\), \((3.21)\), \((3.23)\), and \((3.27)\), we arrive at the following result.

**Theorem 3.2.** Let the assumptions of subsections \(3.1\) and \(3.2\) be satisfied. Suppose that the operator $A(t)$ is subject to inequality \((2.1)\). Let the operator $\tilde{L}(t, \varepsilon)$ be defined in \((3.11)\). Let $M_0$ be the operator \((3.24)\). Then

$$Me^{-B(t, \varepsilon)s}M^* = Me^{-M_0\tilde{L}(t, \varepsilon)\hat{M}\hat{M}^*}M_0\hat{P} + K_G(t, \varepsilon, s) + M\mathcal{R}(t, \varepsilon, s)M^*. \quad (3.27)$$

Here $K_G(t, \varepsilon, s)$ is the operator \((3.27)\) that satisfies estimates \((3.25)\) and \((3.26)\), and the remainder is subject to inequalities

$$\|M\mathcal{R}(t, \varepsilon, s)M^*\| \leq C_7\|M\|^2(s + 1)^{-1} e^{-c_\ast(t^2 + \varepsilon^2)s/2}, \quad s \geq 0; \quad (3.27)$$

$$\|M\mathcal{R}(t, \varepsilon, s)M^*\| \leq C_8\|M\|^2 s^{-1} e^{-c_\ast(t^2 + \varepsilon^2)s/2}, \quad s > 0. \quad (3.27)$$

The constants $C_5$, $C_7$, $C_8$, and $C_9$ are controlled in terms of polynomials with positive numeric coefficients and variables $\hat{c}_\ast^{-1}$ and \((1.14)\).
Chapter 2. Periodic differential operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

4. Preliminaries

4.1. The lattices $\Gamma$ and $\tilde{\Gamma}$. Let $\Gamma$ be a lattice in $\mathbb{R}^d$ generated by a basis $a_1, \ldots, a_d$: $\Gamma = \{a \in \mathbb{R}^d : a = \sum_{j=1}^d n^j a_j, n^j \in \mathbb{Z}\}$. By $\Omega$ we denote the elementary cell of the lattice $\Gamma$: $\Omega = \{x \in \mathbb{R}^d : x = \sum_{j=1}^d \xi^j a_j, 0 < \xi^j < 1\}$. The basis $b_1^1, \ldots, b_1^d$ dual to $a_1, \ldots, a_d$, is defined by the relations $(b_1^j, a_j) = 2\pi \delta_1^j$. This basis generates a lattice $\tilde{\Gamma}$ dual to the lattice $\Gamma$. By $\tilde{\Omega}$ we denote the Brillouin zone of the lattice $\tilde{\Gamma}$:

$$\tilde{\Omega} = \{k \in \mathbb{R}^d : |k| < |k-b|, 0 \neq b \in \tilde{\Gamma}\}.$$  

The domain $\tilde{\Omega}$ is fundamental for $\tilde{\Gamma}$. We use the notation $|\Omega| = \text{meas}\,\Omega$, $|\tilde{\Omega}| = \text{meas}\,\tilde{\Omega}$. Let $r_0$ be the radius of the ball enclosed to clos $\tilde{\Omega}$ and let $2r_1 = \text{diam}\,\tilde{\Omega}$.

If $\Phi(x)$ is a $\Gamma$-periodic matrix-valued function in $\mathbb{R}^d$, denote

$$\overline{\Phi} := |\Omega|^{-1} \int_{\Omega} \Phi(x) \, dx, \quad \Phi := \left(|\Omega|^{-1} \int_{\Omega} \Phi(x)^{-1} \, dx\right)^{-1}. \quad (4.1)$$

Here in definition of $\overline{\Phi}$ it is assumed that $\Phi \in L_{1, \text{loc}}(\mathbb{R}^d)$ and in definition $\Phi$ it is assumed that $\Phi$ is squared non-degenerate matrix-valued function and $\Phi^{-1} \in L_{1, \text{loc}}(\mathbb{R}^d)$.

4.2. The factorized second-order operator families. (See [Bšnt2].) Let $b(D) = \sum_{l=1}^d b_l D_l : L_2(\mathbb{R}^d; \mathbb{C}^m) \to L_2(\mathbb{R}^d; \mathbb{C}^m)$ be a first order DO. Here $b_l$ are $(m \times n)$-matrices with constant entries. Suppose that $m \geq n$. The symbol $b(\xi) = \sum_{l=1}^d b_l \xi_l$ is assumed to satisfy rank $b(\xi) = n$, $0 \neq \xi \in \mathbb{R}^d$. Then for some $\alpha_0, \alpha_1 > 0$ we have

$$\alpha_0 1_n \leq b(\theta)^* b(\theta) \leq \alpha_1 1_n, \quad \theta \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty. \quad (4.2)$$

Let $(n \times n)$-matrix-valued function $f(x)$ and $(m \times m)$-matrix-valued function $h(x)$, $x \in \mathbb{R}^d$, are bounded together with the inverse:

$$f, \ f^{-1} \in L_\infty(\mathbb{R}^d); \quad h, \ h^{-1} \in L_\infty(\mathbb{R}^d). \quad (4.3)$$

The functions $f$ and $h$ are $\Gamma$-periodic. Consider DO

$$X := h b(D)f : L_2(\mathbb{R}^d; \mathbb{C}^m) \to L_2(\mathbb{R}^d; \mathbb{C}^m), \quad (4.4)$$

$$\text{Dom} X := \{u \in L_2(\mathbb{R}^d; \mathbb{C}^m) : f u \in H^1(\mathbb{R}^d; \mathbb{C}^m)\}. \quad (4.5)$$

The operator $[L.4]$ is closed on the domain $[L.5]$. In $L_2(\mathbb{R}^d; \mathbb{C}^m)$, consider a self-adjoint operator $A := X^* X$ corresponding to the quadratic form

$$a[u, u] := \|X u\|^2_{L_2(\mathbb{R}^d)}, \quad u \in \text{Dom} X. \quad (4.6)$$

Formally, $A = f^* b(D)^* g b(D) f$, where $g = h^* h$. By using the Fourier transform and ([L.2], and [L.3]), it is easily seen that for $u \in \text{Dom} X$ we have

$$\alpha_0 g^{-1} \|L_2^{-1} D(f u)\|_{L_2(\mathbb{R}^d)}^2 \leq a[u, u] \leq \alpha_1 \|g\|_{L_\infty} \|D(f u)\|_{L_2(\mathbb{R}^d)}^2. \quad (4.7)$$

4.3. The operators $\mathcal{Y}$ and $\mathcal{Y}_2$. Now we introduce lower order terms. Let the operator $\mathcal{Y} : L_2(\mathbb{R}^d; \mathbb{C}^m) \to L_2(\mathbb{R}^d; \mathbb{C}^{dn})$ acts by the rule

$$\mathcal{Y} u = D(f u) = \text{col} \{D_1(f u), \ldots, D_d(f u)\}, \quad \text{Dom} \mathcal{Y} = \text{Dom} X. \quad (4.8)$$

The lower estimate $[L.7]$ means that

$$\|\mathcal{Y} u\|_{L_2(\mathbb{R}^d)} \leq c_1 \|X u\|_{L_2(\mathbb{R}^d)}, \quad u \in \text{Dom} X: \quad c_1 = \alpha_0^{-1/2} \|g^{-1}\|_{L_\infty}^{1/2}. \quad (4.9)$$

In $\mathbb{R}^d$, let $\Gamma$-periodic $(n \times n)$-matrix-valued functions $a_j(x)$, $j = 1, \ldots, d$, be such that

$$a_j \in L_\varrho(\Omega), \quad \varrho = 2 \text{ for } d = 1, \quad \varrho > d \text{ for } d \geq 2; \quad j = 1, \ldots, d. \quad (4.10)$$

Let the operator $\mathcal{Y}_2 : L_2(\mathbb{R}^d; \mathbb{C}^m) \to L_2(\mathbb{R}^d; \mathbb{C}^{dn})$ acts on the domain $\text{Dom} \mathcal{Y}_2 = \text{Dom} X$ by the rule $\mathcal{Y}_2 u = \text{col} \{a_1^* f u, \ldots, a_d^* f u\}$. Formally, $\mathcal{Y}_2^* \mathcal{Y} + \mathcal{Y}^* \mathcal{Y}_2 u = \sum_{j=1}^d \left(f^* a_j D_j(f u) + f^* D_j(a_j^* f u)\right)$.
Using the Hölder inequality, conditions 13, 14, and the compactness of the embedding $H^1(\Omega) \hookrightarrow L_p(\Omega)$ for $p = 2g/(g-2)$, one can show (cf. [Su4 Subsec. 5.2]) for any $\nu > 0$ there exist a constant $C(\nu) > 0$ such that

$$
\| \mathcal{Y}_2 u \|_{L^2(\mathbb{R}^d)}^2 \leq \nu \| \mathcal{X} u \|_{L^2(\mathbb{R}^d)}^2 + C(\nu) \| u \|_{L^2(\mathbb{R}^d)}^2, \quad u \in \text{Dom} \mathcal{X}.
$$

(4.11)

For $\nu$ fixed, the constant $C(\nu)$ depends on the norms $\| a_j \|_{L^p(\Omega)}$, $j = 1, \ldots, d$, on $\| f \|_{L^\infty}$, $\| g^{-1} \|_{L^\infty}$, $a_0$, $d$, $g$, and on the parameters of the lattice $\Gamma$.

From (4.9) and (4.11) one can derive inequality

$$
2\varepsilon \text{Re}(\mathcal{Y} u, \mathcal{Y}_2 u)_{L^2(\mathbb{R}^d)} \leq \frac{K}{2} \| \mathcal{X} u \|_{L^2(\mathbb{R}^d)}^2 + c_4 \varepsilon^2 \| u \|_{L^2(\mathbb{R}^d)}^2,
$$

(4.12)

$u \in \text{Dom} \mathcal{X}$; $c_4 = 4\kappa^{-1}c_1^2C(\nu)$ for $\nu = \kappa^2(16\epsilon_1)^{-1}$.

### 4.4. The operator $Q_0$, the form $q[u, u]$

Let $Q_0$ be the operator acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ as multiplication by a $\Gamma$-periodic positive definite bounded matrix-valued function $Q_0(x) := f(x)^* f(x)$.

Suppose that $d\mu(x) = \{d\mu_{jl}(x)\}$, $j, l = 1, \ldots, n$ is a $\Gamma$-periodic Borel $\sigma$-finite measure in $\mathbb{R}^d$ with values in the class of Hermitian $(n \times n)$-matrices. In other words, $d\mu_{jl}(x)$ is a complex $\Gamma$-periodic measure in $\mathbb{R}^d$ and $d\mu_{jl} = d\mu_{lj}^*$.

Assume that the measure $d\mu$ is such that the function $|\nu(x)|^2$ is summable on each measure $d\mu_{jl}$ for any function $\nu \in H^1(\mathbb{R}^d)$.

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the form

$$
q[u, u] = \int_{\mathbb{R}^d} \langle d\mu(x) f u, f u \rangle, \quad u \in \text{Dom} \mathcal{X}.
$$

Assume that the measure $d\mu$ is subject to the following condition.

**Condition 4.1.** 1°. There exist constants $\hat{c}_2 \geq 0$ and $\hat{c}_3 \geq 0$ such that for any $u, v \in H^1(\Omega; \mathbb{C}^n)$ we have

$$
\left| \int_{\Omega} \langle d\mu(x) u, v \rangle \right| \leq \left( \hat{c}_2 \| Du \|_{L^2(\Omega)}^2 + \hat{c}_3 \| u \|_{L^2(\Omega)}^2 \right)^{1/2}
$$

$$
\times \left( \hat{c}_2 \| Dv \|_{L^2(\Omega)}^2 + \hat{c}_3 \| v \|_{L^2(\Omega)}^2 \right)^{1/2}.
$$

2°. We have

$$
\int_{\Omega} \langle d\mu(x) u, u \rangle \geq -\hat{c} \| Du \|_{L^2(\Omega)}^2 - \hat{c}_0 \| u \|_{L^2(\Omega)}^2, \quad u \in H^1(\Omega; \mathbb{C}^n),
$$

with some constants $\hat{c}_0 \in \mathbb{R}$ and $\hat{c}$ such that $0 < \hat{c} < \alpha_0\| g^{-1} \|_{L^\infty}^{-1}$.

Note that Condition 4.1 imply such that $c_0 = \hat{c}_0\| f \|_{L^\infty}^2$, $\hat{c}_0 = \hat{c}_0\| f^{-1} \|_{L^\infty}^{-2}$ if $\hat{c}_0 > 0$ and $c_0 = \hat{c}_0\| f^{-1} \|_{L^\infty}^{-2}$ if $\hat{c}_0 < 0$; $c_3 = \| f \|_{L^\infty}^2 \hat{c}_3$. 


For \( \mathbf{u}, \mathbf{v} \in \text{Dom}\, \mathcal{X} \), we write inequalities (4.13), (4.14) over shifted cells \( \Omega + \mathbf{a} \), \( \mathbf{a} \in \Gamma \), and sum up. We get

\[
|q(\mathbf{u}, \mathbf{v})| \leq \left( \tilde{c}_2 \|D(\mathbf{u})\|_{L_2(\mathbb{R}^d)}^2 + c_3 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \right)^{1/2} \times \left( \tilde{c}_2 \|D(\mathbf{v})\|_{L_2(\mathbb{R}^d)}^2 + c_3 \|\mathbf{v}\|_{L_2(\mathbb{R}^d)}^2 \right)^{1/2},
\]

\[
q(\mathbf{u}, \mathbf{u}) \geq -\tilde{c}_2 \|D(\mathbf{u})\|_{L_2(\mathbb{R}^d)}^2 - c_0 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2.
\]

By (4.7), this implies

\[
|q(\mathbf{u}, \mathbf{v})| \leq \left( c_2 \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 + c_3 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 \right)^{1/2} \times \left( c_2 \|\mathcal{X}\mathbf{v}\|_{L_2(\mathbb{R}^d)}^2 + c_3 \|\mathbf{v}\|_{L_2(\mathbb{R}^d)}^2 \right)^{1/2},
\]

\[
q(\mathbf{u}, \mathbf{u}) \geq -(1 - \kappa) \|\mathcal{X}\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2 - c_0 \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2.
\]

\( \mathbf{u}, \mathbf{v} \in \text{Dom}\, \mathcal{X} \). Here \( \tilde{c}_2 = \tilde{c}_2 \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \), \( \kappa = 1 - \tilde{c}_1 \alpha_0^{-1} \|g^{-1}\|_{L_\infty} \), \( 0 < \kappa \leq 1 \).

Examples of measures \( d\mu \) satisfying Condition (4.1) are given in [Su4, Subsec. 5.5]. Here we write down only the main example.

**Example 4.2.** Let the measure \( d\mu \) be absolute continuous with respect to the Lebesgue measure: \( d\mu(\mathbf{x}) = Q(\mathbf{x}) \, d\mathbf{x} \), where \( Q(\mathbf{x}) \) is a \( \Gamma \)-periodic Hermitian \((n \times n)\)-matrix-valued function in \( \mathbb{R}^d \), and

\[
Q \in L_\sigma(\Omega), \quad \sigma = 1 \text{ for } d = 1; \quad \sigma > d/2 \text{ for } d \geq 2.
\]

(4.17)

Then

\[
q(\mathbf{u}, \mathbf{u}) = \int_{\mathbb{R}^d} \langle Q(\mathbf{x}) f(\mathbf{x}) \mathbf{u}(\mathbf{x}), f(\mathbf{x}) \mathbf{u}(\mathbf{x}) \rangle \, d\mathbf{x}, \quad \mathbf{u} \in \text{Dom}\, \mathcal{X}.
\]

By the embedding theorem, under condition (4.17) for any \( \nu > 0 \) there exists a positive constant \( C_Q(\nu) \) such that

\[
\int_{\Omega} |Q(\mathbf{x})||\mathbf{v}|^2 \, d\mathbf{x} \leq \nu \int_{\Omega} |D\mathbf{v}|^2 \, d\mathbf{x} + C_Q(\nu) \int_{\Omega} |\mathbf{v}|^2 \, d\mathbf{x}, \quad \mathbf{v} \in H^1(\Omega; \mathbb{C}^n).
\]

For \( \nu \) fixed, the constant \( C_Q(\nu) \) is controlled by \( d, \sigma \), the norm \( \|Q\|_{L_\sigma(\Omega)} \) and the parameters of the lattice \( \Gamma \). So, Condition (4.7) is satisfied and the constants can be chosen in the following way: \( \tilde{c}_2 = 1, \tilde{c}_3 = C_Q(1), \tilde{c} = \nu, \) and \( \tilde{c}_0 = C_Q(\nu) \) for \( 2\nu = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \).

4.5. **The Operator** \( B(\varepsilon) \). In \( L_2(\mathbb{R}^d; \mathbb{C}^n) \), we consider a quadratic form

\[
b(\varepsilon)[\mathbf{u}, \mathbf{u}] = q[\mathbf{u}, \mathbf{u}] + 2\varepsilon \mathcal{R}e \langle \mathcal{Y}_1 \mathbf{u}, \mathcal{Y}_2 \mathbf{u} \rangle_{L_2(\mathbb{R}^d)} + \varepsilon^2 q[\mathbf{u}, \mathbf{u}],
\]

\[
+ \lambda \varepsilon^2 (Q_0 \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in \text{Dom}\, \mathcal{X},
\]

where \( 0 < \varepsilon \leq 1 \), and the parameter \( \lambda \in \mathbb{R} \) satisfies the following condition:

\[
\lambda > \|Q_0^{-1}\|_{L_\infty}^{-1}(c_0 + c_4), \text{ if } \lambda \geq 0,
\]

\[
\lambda > \|Q_0^{-1}\|_{L_\infty}^{-1}(c_0 + c_4), \text{ if } \lambda < 0 \text{ (and } c_0 + c_4 < 0 \).
\]

(4.19)

Now we estimate the form (4.18) from below. Let \( \beta > 0 \) be defined by the equality

\[
\beta = \lambda \|Q_0^{-1}\|_{L_\infty}^{-1} - c_0 - c_4, \text{ if } \lambda \geq 0,
\]

\[
\beta = \lambda \|Q_0\|_{L_\infty} - c_0 - c_4, \text{ if } \lambda < 0 \text{ (and } c_0 + c_4 < 0 \).
\]

(4.20)

Combining (4.12), (4.16), (4.19), and (4.20), we arrive at

\[
b(\varepsilon)[\mathbf{u}, \mathbf{u}] \geq \frac{\varepsilon}{2} q[\mathbf{u}, \mathbf{u}] + \varepsilon^2 \beta \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in \text{Dom}\, \mathcal{X}, \quad 0 < \varepsilon \leq 1.
\]

(4.21)

Thus, the form \( b(\varepsilon) \) is positive definite. Bringing together (4.9), (4.11) for \( \nu = 1 \), and estimate (4.15) for the quadratic form \( q[\mathbf{u}, \mathbf{u}] \), we obtain

\[
b(\varepsilon)[\mathbf{u}, \mathbf{u}] \leq (2 + c_1^2 + c_2) q[\mathbf{u}, \mathbf{u}]
\]

\[
+ \varepsilon^2 (C(1) + c_3 + |\lambda||Q_0\|_{L_\infty}) \|\mathbf{u}\|_{L_2(\mathbb{R}^d)}^2, \quad \mathbf{u} \in \text{Dom}\, \mathcal{X}.
\]

(4.22)
Inequalities (1.21) and (1.22) imply that the form \( b(\varepsilon) \) is closed. The corresponding positive definite operator acting in the space \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) we denote by \( \mathcal{B}(\varepsilon) \). Formally,
\[
\mathcal{B}(\varepsilon) = \mathcal{A} + \varepsilon (X_0^2 + \mathcal{Y}_0^2) + \frac{\varepsilon^2}{2} f^* \mathcal{Q} f + \varepsilon^2 \lambda \mathcal{Q}_0
\]
\[
= f^* b(D)^* g b(D) f + \varepsilon^2 \sum_{j=1}^d f^* (a_j D_j + D_j a_j^*) f + \varepsilon^2 f^* \mathcal{Q} f + \varepsilon^2 \lambda \mathcal{Q}_0,
\]
where \( \mathcal{Q} \) should be interpreted as a generalized matrix potential generated by the measure \( d \mu \).

For convenience of the further references, by the „initial data“ we call the following set of parameters:
\[
d, m, n, q; \alpha_0, \alpha_1, \parallel g \parallel_{L_\infty}, \parallel g^{-1} \parallel_{L_\infty}, \parallel f \parallel_{L_\infty}, \parallel f^{-1} \parallel_{L_\infty}, \parallel a_j \parallel_{L_p(\Omega)},
\]
\[
j = 1, \ldots, d; \tilde{c}, \tilde{c}_{0}, \tilde{c}_{2}, \tilde{c}_{3} \text{ from Condition 4.1}; \lambda; \text{ the parameters of the lattice } \Gamma.
\]

We will trace the dependence of constants in estimates on these data. The values of \( c_1 \), \( C(1) \), \( \kappa \), \( c_2 \), \( c_3 \), \( c_4 \), \( c_0 \), and \( \beta \) are completely determined by initial data (1.21).

5. DIRECT INTEGRAL DECOMPOSITION FOR THE OPERATOR \( \mathcal{B}(\varepsilon) \)

5.1. The Gelfand transformation. The Gelfand transformation \( \mathcal{U} \) is initially defined on functions from the Schwartz class \( \mathcal{V} \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}^n) \) by the rule
\[
\tilde{v}(k, x) = (Uv)(k, x) = (\tilde{\Omega})^{-1/2} \sum_{\lambda \in \Gamma} \exp(-i\lambda(k, x + a))v(x + a), \quad x \in \Omega, \ k \in \tilde{\Omega}.
\]
Moreover, \( \int_{\tilde{\Omega}} \int_\Omega |\tilde{v}(k, x)|^2 \, dx \, dk = \int_{\mathbb{R}^d} |v(x)|^2 \, dx \), so \( \mathcal{U} \) can be extended by continuity to the unitary transformation
\[
\mathcal{U} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to \int_\Omega + L_2(\Omega; \mathbb{C}^n) \, dk =: \mathcal{H}.
\]

By \( \tilde{H}^1(\Omega; \mathbb{C}^n) \) we denote the subspace of functions from \( H^1(\Omega; \mathbb{C}^n) \) those \( \Gamma \)-periodic extension on \( \mathbb{R}^d \) belongs to the class \( H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^n) \).

The inclusion \( v \in H^1(\mathbb{R}^d; \mathbb{C}^n) \) is equivalent to
\[
\tilde{v}(k, \cdot) \in \tilde{H}^1(\Omega; \mathbb{C}^n)
\]
for a.e. \( k \in \tilde{\Omega} \) and
\[
\int_{\tilde{\Omega}} \int_\Omega \left( \parallel (D + k)\tilde{v}(k, x) \parallel^2 + \parallel \tilde{v}(k, x) \parallel^2 \right) \, dx \, dk < \infty.
\]

Under the transformation \( \mathcal{U} \), the operator of multiplication by a bounded periodic matrix-valued function in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) turns into the operator of multiplication on the same function and fibres of the direct integral \( \mathcal{H} \). The action of the operator \( b(D) \) on the function \( v \in H^1(\mathbb{R}^d; \mathbb{C}^n) \) turns into the fibre-wise action of the operator \( b(D + k) \) on \( \tilde{v}(k, \cdot) \in \tilde{H}^1(\Omega; \mathbb{C}^n) \).

5.2. The operators \( \mathcal{A}(k) \). (See [BSu2, Subsec. 2.2.1.].) We set
\[
\mathcal{J}_1 = L_2(\Omega; \mathbb{C}^n), \quad \mathcal{J}_2 = L_2(\Omega; \mathbb{C}^m), \quad \mathcal{J}_3 = L_2(\Omega; \mathbb{C}^{dm})
\]
and consider the closed operator \( \mathcal{X}(k) : \mathcal{J}_1 \to \mathcal{J}_2 \), \( k \in \mathbb{R}^d \), defined by the relations
\[
\mathcal{X}(k) = bb(D + k)f, \quad k \in \mathbb{R}^d,
\]
\[
\mathcal{J} := \text{Dom} \mathcal{X}(k) = \{ u \in \mathcal{J}_1 : f u \in \tilde{H}^1(\Omega; \mathbb{C}^n) \}.
\]

The self-adjoint operator \( \mathcal{A}(k) := \mathcal{X}(k)^* \mathcal{X}(k) : \mathcal{J}_1 \to \mathcal{J}_1 \), \( k \in \mathbb{R}^d \), is generated by the closed quadratic form \( a(k)[u, u] := \parallel \mathcal{X}(k)u \parallel_{\mathcal{J}_2}^2 \), \( u \in \mathcal{J}_1, k \in \mathbb{R}^d \). By (1.3) and (1.3),
\[
a_0 \parallel g^{-1} \parallel_{L_\infty} \parallel (D + k)\tilde{v} \parallel_{L_2(\Omega)}^2 \leq a(k)[u, u] \leq a_1 \parallel g \parallel_{L_\infty} \parallel (D + k)\tilde{v} \parallel_{L_2(\Omega)}^2,
\]
\[
v = f u \in \tilde{H}^1(\Omega; \mathbb{C}^n).
\]
From (5.4), and the compactness of the embedding of $\tilde{H}^1(\Omega; \mathbb{C}^n)$ into $\mathcal{H}$ it follows that the spectrum of the operator $A(k)$ is discrete. Put $\mathcal{R} := \text{Ker} A(0) = \text{Ker} \mathcal{X}(0)$. Inequalities (5.4) with $k = 0$ imply that

$$\mathcal{R} = \text{Ker} A(0) = \{u \in L_2(\Omega; \mathbb{C}^n) : f u = c \in \mathbb{C}^n\}, \quad \dim \mathcal{R} = n. \quad (5.5)$$

As was shown in [BSu2] (2.2.11), (2.2.12),

$$A(k) \geq c|k|^2 I, \quad k \in \text{clos} \tilde{\Omega}; \quad c_0 = \alpha_0 \| f^{-1} \|_{L_\infty}^2 \| g^{-1} \|_{L_\infty}^2. \quad (5.6)$$

According to [BSu2] (2.2.14), the distance $d_0$ from the point $\lambda_0 = 0$ to the rest of the spectrum of the operator $A(0)$ satisfy the estimate

$$d_0 \geq 4c_0r_0^2. \quad (5.7)$$

5.3. The operators $\mathcal{Y}(k)$ and $Y_2$. Consider the operator $\mathcal{Y}(k) : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ acting on the domain $\text{Dom} \mathcal{Y}(k) = \mathcal{H}$ by the rule

$$\mathcal{Y}(k)u = (D + k)f u = \text{col} \{(D_1 + k_1)f u, \ldots, (D_d + k_d)f u\}, \quad u \in \mathcal{H}. \quad (5.8)$$

From the lower estimate (5.4) it follows that

$$\|\mathcal{Y}(k)u\|_{\mathcal{H}} \leq c_1 \|A(k)u\|_{\mathcal{H}}, \quad u \in \mathcal{H}, \quad (5.9)$$

where the constant $c_1$ is defined in (4.9).

Consider the operator $Y_2 : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ defined by the relation

$$Y_2u = \text{col} \{a_1u, \ldots, a_d u\}, \quad \text{Dom} Y_2 = \mathcal{H}. \quad (5.10)$$

As was shown in [Su4] Subsec. 5.7, for any $\nu > 0$ there exist constants $C_j(\nu) > 0$, $j = 1, \ldots, d$, such that for $k \in \mathbb{R}^d$ we have

$$\|a_j^* v\|_{L_2(\Omega)}^2 \leq \nu \|\mathcal{Y}(k)v\|_{L_2(\Omega)}^2 + C_j(\nu)\|v\|_{L_2(\Omega)}^2, \quad v \in \tilde{H}^1(\Omega; \mathbb{C}^n), \quad j = 1, \ldots, d. \quad (5.11)$$

5.4. The operator $Q_0$ and the form $q_0[u, u]$. Let $Q_0$ be the bounded operator in $\mathcal{H}$, acting as the multiplication by the matrix-valued function $Q_0(x) = f(x)^* f(x)$.

In $L_2(\Omega; \mathbb{C}^n)$, consider the form

$$q_0[u, u] = \int_\Omega (d\mu(x) f u, f u), \quad u \in \mathcal{H}. \quad (5.12)$$

In (4.13) and (4.14), we replace the function $f(x)u(x)$ by $f(x)u(x) \exp(i(k, x))$ (these functions simultaneously belong to the space $H^1(\Omega; \mathbb{C}^n)$) and $f(x)v(x)$ by $f(x)v(x) \exp(i(k, x))$. By using (5.4), we get

$$|q_0[u, v]| \leq (c_2 \|A(k)u\|_{\mathcal{H}}^2 + c_3 \|u\|_{\mathcal{H}}^2)^{1/2} (c_2 \|A(k)v\|_{\mathcal{H}}^2 + c_3 \|v\|_{\mathcal{H}}^2)^{1/2}, \quad (5.13)$$

$$q_0[u, u] \geq -(1 - \kappa) \|A(k)u\|_{\mathcal{H}}^2 - c_0 \|u\|_{\mathcal{H}}^2, \quad u, v \in \mathcal{H}, \quad k \in \mathbb{R}^d. \quad (5.14)$$

Here the constants $\kappa$, $c_0$, $c_2$, and $c_3$ are the same as in (4.15) and (4.16).

5.5. The operator pencil $\mathcal{B}(k, \varepsilon)$. In $\mathcal{H}$, consider the quadratic form

$$b(k, \varepsilon)[u, u] = a(k)[u, u] + 2\varepsilon \text{Re} \langle \mathcal{Y}(k)u, Y_2u \rangle_{\mathcal{H}}$$

$$+ \varepsilon^2 q_0[u, u] + \lambda \varepsilon^2 (Q_0u, u)_{\mathcal{H}}, \quad u \in \mathcal{H}. \quad (5.15)$$

From (4.19), (4.20), (5.9), (5.11), and (5.13) it follows that

$$b(k, \varepsilon)[u, u] \geq \frac{\kappa}{2} a(k)[u, u] + \beta \varepsilon^2 \|u\|_{\mathcal{H}}^2, \quad u \in \mathcal{H}. \quad (5.16)$$
Next, from (5.9), (5.11) for \( \nu = 1 \), and (5.12) we derive that
\[
\mathbf{b}(\mathbf{k}, \varepsilon)[\mathbf{u}, \mathbf{u}] \leq (2 + c_1^2 + c_2)\mathbf{a}(\mathbf{k})[\mathbf{u}, \mathbf{u}]
+ (C(1) + c_3 + |\lambda||\mathbf{Q}_0|_{L_\infty})\varepsilon^2||\mathbf{u}||^2_{S_\mathbf{D}}, \quad \mathbf{u} \in \mathcal{D}.
\] (5.15)

Inequalities (5.14) and (5.15) show that the form \( \mathbf{b}(\mathbf{k}, \varepsilon) \) is positive definite and closed on the domain (5.3). The self-adjoint positive-definite operator in \( \mathcal{H} \) corresponding to this form is denoted by \( \mathcal{B}(\mathbf{k}, \varepsilon) \). Formally we can write
\[
\mathcal{B}(\mathbf{k}, \varepsilon) = \mathcal{A}(\mathbf{k}) + \varepsilon(\mathcal{Y}(\mathbf{k}) + \mathcal{Y}(\mathbf{k})^*)\mathcal{Y}_2 + \varepsilon^2 f^* \mathcal{Q} f + \lambda \varepsilon^2 \mathbf{Q}_0
\]
\[
= f^*(\mathbf{D} + \mathbf{k})^* g\mathbf{b}(\mathbf{D} + \mathbf{k}) f + \varepsilon \sum_{j=1}^d f^*(a_j(D_j + k_j) + (D_j + k_j)a_j^*) f
+ \varepsilon^2 f^* \mathcal{Q} f + \lambda \varepsilon^2 f^* f.
\] (5.16)

5.6. The direct integral decomposition for the operator \( \mathcal{B}(\varepsilon) \). Under the Gelfand transformation \( \mathcal{U} \), the operator (4.23) acting in the space \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) is decomposed into the direct integral of operators (5.15) acting in \( L_2(\Omega; \mathbb{C}^n) \):
\[
\mathcal{U}\mathcal{B}(\varepsilon)\mathcal{U}^{-1} = \int_\Omega \mathcal{B}(\mathbf{k}, \varepsilon) d\mathbf{k}.
\]

This means the following. Let \( \mathbf{u} = \mathcal{U} \mathbf{u} \), where \( \mathbf{u} \in \text{Dom} \mathbf{b}(\varepsilon) = \text{Dom} \mathcal{X} \). Then
\[
\mathbf{u}(\mathbf{k}, \cdot) \in \mathcal{D} \text{ for a. e. } \mathbf{k} \in \Omega,
\] (5.17)
\[
\mathbf{b}(\varepsilon)[\mathbf{u}, \mathbf{u}] = \int_\Omega \mathbf{b}(\mathbf{k}, \varepsilon)[\mathbf{u}(\mathbf{k}, \cdot), \mathbf{u}(\mathbf{k}, \cdot)] d\mathbf{k}.
\] (5.18)

Conversely, if for \( \mathbf{u} \in \mathcal{H} \) one has (5.17) and the integral in (5.18) is finite, then \( \mathbf{u} \in \text{Dom} \mathbf{b}(\varepsilon) = \text{Dom} \mathcal{X} \) and (5.18).

6. Incorporation of the operators \( \mathcal{B}(\mathbf{k}, \varepsilon) \) into the abstract scheme

The content of Subsections 6.1–7.5 is borrowed from [Su7].

6.1. For \( d > 1 \), the operator \( \mathcal{B}(\mathbf{k}, \varepsilon) \) depends on the multi-dimensional parameter \( \mathbf{k} \). According to [BSu2, Chapter 2], we distinct a one-dimensional parameter \( t \), setting \( \mathbf{k} = t\theta, \ t = ||\mathbf{k}||, \ \theta \in \mathbb{S}^{d-1} \). We apply the scheme of Chapter 1. In this case, the resulting objects depend on an additional parameter \( \theta \) and we must make our considerations and estimates uniform in \( \theta \). The spaces \( \mathfrak{N}, \mathfrak{N}_n, \) and \( \mathfrak{S} \) are defined in (5.11). Put \( X(t) = X(t; \theta) := \mathcal{X}(t\theta) \). According to (5.2), we have \( X(t; \theta) = X_0 + tX_1(\theta) \), where
\[
X_0 = \mathcal{X}(0) = hh(D)f, \quad \text{Dom} \ X_0 = \mathcal{D}; \quad X_1(\theta) = hh(\theta)f.
\] (6.1)

Next, we set \( A(t) = A(t; \theta) := \mathcal{A}(t\theta) \). In accordance with (5.5), the kernel \( \mathfrak{N} := \text{Ker} X_0 = \text{Ker} \mathcal{A}(0) \) in \( n \)-dimensional. Condition (1.1) is satisfied. The quantity \( d^\theta \) admits estimate (5.7).

As was shown in [BSu2, Chapter 2, §3], the condition \( n \leq n_+ = \dim \text{Ker} X_0^* \) is also satisfied. Estimate (5.6) corresponds to inequality (2.1).

Next, the role of \( Y(t) \) is played by the operator \( Y(t; \theta) := \mathcal{Y}(t\theta) \). According to (5.8), we have the inequality
\[
Y_0(\mathbf{u}) = D(f\mathbf{u}) = \text{col} \{D_1f\mathbf{u}, \ldots, D_df\mathbf{u}\}, \quad \text{Dom} \ Y_0 = \mathcal{D};
\]
\[
Y_1(\theta)\mathbf{u} = \text{col} \{\theta_1f\mathbf{u}, \ldots, \theta_d f\mathbf{u}\}.
\] (6.2)

Condition (1.2) holds true because of estimate (5.9). The operator \( Y_2 \) is defined in (5.10). Condition (1.3) is satisfied due to (5.11). The role of the form \( q \) from Subsec. (1.3) is played by the form \( q_1 \). Condition (1.4) is valid due to (5.12), (5.13). The role of the operator \( Q_0 \) from Subsec. (1.4) is played by the operator of multiplication by the matrix-valued function \( Q_0(x) \). The restriction (1.9) on the parameter \( \lambda \) holds true according to (4.19).
Finally, the role of the operator pencil \( B(t, \varepsilon) \) (see Subsec. 6.4) is played by the operator family \( (5.16) \): \( B(t, \varepsilon; \theta) := \hat{B}(t, \varepsilon; \theta) \).

Thus, all the assumptions of the abstract scheme are fulfilled.

6.2. According to Subsec. 6.5 we should fix a positive number \( \delta \) such that \( \delta < \kappa \nu_0 / 13 \). Taking \( (5.6) \) and \( (5.7) \) into account, we put \( \delta = \frac{1}{4} \kappa c_s r_0^2 = \frac{1}{4} \kappa a_0 \| f \|_{L_\infty}^{-2} \| g \|_{L_\infty}^{-1} \| f \|_{L_\infty}^{-1} r_0^2 \).

Note that (4.2), (4.3), (6.1), and (6.2) imply relations
\[
\| X_1(\theta) \| \leq \alpha_1^{1/2} \| g \|_{L_\infty}^{1/2} \| f \|_{L_\infty}, \quad \| V_1(\theta) \| = \| f \|_{L_\infty}, \quad \theta \in \mathbb{S}^{d-1}, \tag{6.3}
\]
and depends on \( \theta \). We choose the following value for the constant \( \tau_0 \) suitable for all \( \theta \in \mathbb{S}^{d-1} \):
\[
\tau_0 = \delta^{1/2} (2 + c_1^2 + c_2) \alpha_1 \| g \|_{L_\infty} \| f \|_{L_\infty}^2 + C(1) + c_3 + |\lambda| \| f \|_{L_\infty}^2)^{-1/2}. \tag{6.4}
\]
By (5.6) and (5.14), the operator \( B(t, \varepsilon; \theta) \) satisfies condition of the form (2.2):
\[
\hat{B}(k, \varepsilon) = B(t, \varepsilon; \theta) \geq c_s (t^2 + \varepsilon^2) I, \quad k = t\theta \in \hat{\Omega}, \quad 0 < \varepsilon \leq 1;
\]
\[
c_s = \frac{1}{2} \min \{ \kappa c_s ; 2 \beta \}. \tag{6.6}
\]

6.3. The case when \( f = 1_n \). In that follows, all the objects corresponding to the case \( f = 1_n \), are marked by the upper sing "~". We have \( \hat{\mathcal{F}} = \tilde{\mathcal{F}} = L_2(\Omega; \mathbb{C}^n) \). According to Subsec. 6.1
\[
\hat{X}(t; \theta) = \hat{X}_0 + t\hat{X}_1(\theta), \tag{6.7}
\]
and \( \hat{X}_1(\theta) \) is a bounded operator of multiplication by the matrix \( h(x)b(\theta) \):
\[
\hat{X}_1(\theta) = h b(\theta). \tag{6.8}
\]
Formally, \( \hat{A}(t; \theta) = \hat{X}(t; \theta)^* \hat{X}(t; \theta) \). In the case when \( f = 1_n \), the kernel \( (5.5) \) matches with the subspace of constants \( \hat{\mathcal{R}} = \{ u \in \hat{\mathcal{F}} : u = c \in \mathbb{C}^n \} \). The orthogonal projection \( \tilde{P} \) of the space \( \hat{\mathcal{F}} = L_2(\Omega; \mathbb{C}^n) \) onto the subspace \( \hat{\mathcal{R}} = \mathbb{C}^n \) is the operator of averaging over the cell \( \Omega \):
\[
\tilde{P} u = |\Omega|^{-1} \int_{\Omega} u(x) \, dx. \tag{6.9}
\]

Next, \( \hat{Y}(t; \theta) = \hat{Y}_0 + t\hat{Y}_1(\theta) : \hat{\mathcal{F}} \rightarrow \hat{\mathcal{F}} \), where
\[
\hat{Y}_0 u = D u = \text{col} \{ D_1 u, \ldots, D_d u \}, \quad \text{Dom} \hat{Y}_0 = \tilde{H}^1(\Omega; \mathbb{C}^n), \tag{6.10}
\]
\[
\hat{Y}_1(\theta) u = \text{col} \{ \theta_1 u, \ldots, \theta_d u \}. \tag{6.11}
\]
The operator \( \hat{Y}_2 : \hat{\mathcal{F}} \rightarrow \hat{\mathcal{F}} \) acts by the rule
\[
\hat{Y}_2 u = \text{col} \{ a_1^* u, \ldots, a_d^* u \}, \quad \text{Dom} \hat{Y}_2 = \tilde{H}^1(\Omega; \mathbb{C}^n). \tag{6.12}
\]
The role of the form \( \hat{q}(u, u) \) is played by the form \( \int_{\Omega} (d\mu(x) u, u) \), \( u \in \tilde{H}^1(\Omega; \mathbb{C}^n) \). The role of the operator \( \hat{Q}_0 \) is played by the identity operator \( I \).

The operator pencil \( \hat{B}(t, \varepsilon; \theta) \) is formally given by the expression
\[
\hat{B}(t, \varepsilon; \theta) = \hat{A}(t; \theta) + \varepsilon (\hat{Y}_2 \hat{Y}(t; \theta) + \hat{Y}(t; \theta)^* \hat{Y}_2) + \varepsilon^2 Q + \lambda \varepsilon^2 I.
\]
(We emphasise that the formal object \( Q \) does not changed.)
6.4. The case \( f \neq 1_n \). The implementation of the assumptions of \([33]\) Now we return to the consideration of the operators \( B(\varepsilon) \) of the general form \([1.23]\) and the corresponding families \( B(t, \varepsilon; \theta) \) defined in Subsec. \([6.1]\). We held the upper mark \( \sim \) to indicate the objects corresponding to the case where \( f = 1_n \) and \( b(D), g, a_j, j = 1, \ldots, d, \lambda, \) and \( d\mu \) are the same as before.

Let us show that the operator families \( B(t, \varepsilon; \theta) \) and \( \hat{B}(t, \varepsilon; \theta) \) satisfy the assumptions from \([33]\) of the abstract scheme. Indeed, identity \([3.7]\) corresponds to the obvious equality \( B(t, \varepsilon; \theta) = f^*\hat{B}(t, \varepsilon; \theta)f \). The role of the isomorphism \( M \) is now played by the operator of multiplication by the matrix-valued function \( f(x) \). The the role of the operator \( G \) (see \([6.2]\)) is played by the operator of multiplication by the matrix-valued function \((f(x)f(x)^*)^{-1}\). The block of \( G \) in the kernel \( \hat{N} = \mathbb{C}^n \) is the operator of multiplication by the constant matrix \( \overline{G} = |\Omega|^{-1} \int_{\Omega}(f(x)f(x)^*)^{-1} \, dx \).

The role of the operator \( M_0 \) (see \([3.21]\)) is played by the operator of multiplication by the matrix

\[
 f_0 := (\overline{G})^{-1/2} = (f^*)^{-1/2}, \tag{6.13}
\]

Note that

\[
 |f_0| \leq \|f\|_{L_\infty}, \quad |f_0^{-1}| \leq \|f^{-1}\|_{L_\infty}. \tag{6.14}
\]

6.5. On the constants in estimates. We are aimed to apply the results of Chapter 1 to the operator \( B(t, \varepsilon; \theta) \) depending on the additional parameter \( \theta \). To realise the abstract scheme, we need to make all the estimates from Chapter 1 for this operator uniform in \( \theta \), i.e. to choose for each constant the suitable for all \( \theta \in \mathbb{S}^{d-1} \) value. In Chapter 1, it was traced that (see Remarks \([1.3]\) and \([6.1]\) and Theorem \([3.2]\) the values of the constants \( C_N, C_5, C_7, C_8, \) and \( C_9 \) are controlled in terms of polynomials with positive numeric coefficients and the variables \([1.1]\) and \([3.2]\), and the constant \( C_G \) is controlled in terms of the same variables and on \( |M^{-1}| \). In the case under consideration, \( |M^{-1}| = \|Q_0\|_{L_\infty}^{-1/2} \), the data \([1.1]\) matches with the set of data \( \delta, \delta^{-1/2}, \tau_0, \kappa^{1/2},\kappa^{-1/2}, c_1, c_2^{1/2}, c_3^{1/2}, C(1)^{1/2}, |\lambda|, \|X_1(\theta)\|, \|Y_1(\theta)\|, \) and \( \|Q_0\|_{L_\infty} \). The quantities \( \delta, \delta^{-1/2}, \kappa^{1/2},\kappa^{-1/2}, c_1, c_2^{1/2}, c_3^{1/2}, C(1)^{1/2} \) do not depend on \( \theta \) and can be controlled in terms of the initial data \([3.23]\); the number \( \tau_0 \) is already chosen independent on \( \theta \) (see \([6.3]\)). By using \([6.3]\), instead of \( \|X_1(\theta)\| \) we can take \( a_1^{1/2} \|g\|_{L_\infty}^{1/2} \|f\|_{L_\infty} \), and instead of \( \|Y_1(\theta)\| \) we can take the norm \( \|f\|_{L_\infty} \).

Thus, the following observation holds true.

Remark 6.1. The values of constants \( C_G, C_5, C_7, C_8, \) and \( C_9 \) for the operator \( B(t, \varepsilon; \theta) \) can be chosen independent on \( \theta \in \mathbb{S}^{d-1} \).

7. The effective characteristics

7.1. The operators \( \hat{Z}(\theta), \tilde{Z}, \) and \( \hat{R}(\theta) \). The operator \( \hat{Z} \) defined in Subsec. \([6.6]\) now depends on \( \theta \). We define a \( \Gamma \)-periodic \((n \times m)\)-matrix-valued function \( \Lambda(x) \) as the weak solution of the equation

\[
 b(D)^*g(x)(b(D)\Lambda(x) + 1_m) = 0, \quad \int_{\Omega} \Lambda(x) \, dx = 0. \tag{7.1}
\]

As was shown in [Su4 Subsec. 6.3], the operator \( \hat{Z}(\theta) : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}} \) is defined by the equality

\[
 \hat{Z}(\theta) = \Lambda b(\theta) \hat{P}, \tag{7.2}
\]

where \( \hat{P} \) is the projector \([6.9]\).

According to [Su4 Subsec. 6.3]

\[
 \hat{Z} = \tilde{\Lambda} \hat{P}, \tag{7.3}
\]

where \( \tilde{\Lambda}(x) \) is a \( \Gamma \)-periodic \((n \times n)\)-matrix-valued solution of the problem

\[
 b(D)^*g(x)b(D)\tilde{\Lambda}(x) + \sum_{j=1}^d D_j a_j(x)^* = 0, \quad \int_{\Omega} \tilde{\Lambda}(x) \, dx = 0. \tag{7.4}
\]

The operator \( \hat{R}(\theta) : \hat{\mathcal{N}} \rightarrow \hat{\mathcal{N}}_* \) acts as multiplication by the matrix-valued function

\[
 \hat{R}(\theta) = h(b(D)\Lambda + 1_m)b(\theta). \tag{7.5}
\]
7.2. The operator $\hat{\mathcal{S}}(\theta)$. The effective matrix. The spectral germ $\hat{\mathcal{S}}$ introduced in Subsec. 1.7 now depends on $\theta$. According to [BSu2], Chapter 3, §1, the operator $\hat{\mathcal{S}}(\theta) : \hat{\mathcal{N}} \to \hat{\mathcal{N}}$ acts as the operator of multiplication by the matrix $b(\theta)^* g^0 b(\theta)$, $\theta \in S^{d-1}$. Here $g^0$ is a constant $(m \times m)$-matrix called the effective matrix and defined by the expression

$$g^0 = |\Omega|^{-1} \int_{\Omega} \hat{g}(x) \, dx, \quad \hat{g}(x) := g(x)(b(D)\Lambda(x) + 1_m). \quad (7.6)$$

We need the following properties of the effective matrix, see [BSu2] Chapter 3, §1.

**Proposition 7.1.** The effective matrix $g^0$ is subject to estimates

$$g \leq g^0 \leq \overline{g}. \quad (Here \ g \ and \ \overline{g} \ is \ defined \ according \ to \ (4.1).) \ For \ m = n \ one \ has \ g^0 = g.$$  

**Proposition 7.2.** The equality $g^0 = \overline{g}$ is equivalent to the relations

$$b(D)^* g_k(x) = 0, \quad k = 1, \ldots, m, \quad (7.7)$$

for the columns $g_k(x)$ of the matrix $g(x)$.

**Proposition 7.3.** The identity $g^0 = \overline{g}$ is equivalent to the representations

$$l_k(x) = l_k^0 + b(D)w_k, \quad l_k^0 \in \mathbb{C}^m, \quad w_k \in \mathcal{H}^1(\Omega; \mathbb{C}^m), \quad k = 1, \ldots, m,$$

where $l_k(x)$, $k = 1, \ldots, m$, are the columns of the matrix $g(x)^{-1}$.

7.3. The operator $\hat{\mathcal{B}}^0(k, \varepsilon)$. The operator $\hat{\mathcal{L}}(t, \varepsilon)$ defined according to (2.29) and acting in the space $\hat{\mathcal{N}}$ now depends on $\theta$. It turns out (see [Su4], (7.2), (7.3), (7.8)) that

$$\hat{\mathcal{L}}(k, \varepsilon) = b(k)^* g^0 b(k) - \varepsilon(b(k)^* V + V^* b(k)) + \varepsilon \sum_{j=1}^d (a_j + a_j^*) k_j + \varepsilon^2 (-W + \overline{Q} + \lambda I),$$

where the constant matrices $(a_j + a_j^*)$ are defined in accordance with (4.1),

$$V := |\Omega|^{-1} \int_{\Omega} (b(D)\Lambda(x))^* g(x)b(D)\tilde{\Lambda}(x) \, dx, \quad (7.8)$$

$$W := |\Omega|^{-1} \int_{\Omega} (b(D)\tilde{\Lambda}(x))^* g(x)b(D)\tilde{\Lambda}(x) \, dx, \quad (7.9)$$

$$\overline{Q} := |\Omega|^{-1} \int d\mu(x). \quad (7.10)$$

In consistence with (5.6), $\hat{\Lambda}(k) \geq \hat{c}_* |k|^2 I$, $k \in \hat{\Omega}$, where $\hat{c}_* = \alpha_0 g^{-1} \|f^{-1}\|_{L^\infty}^{-1}$. Note that the constants $c_*$ and $\hat{c}_*$ are related by the identity $c_* = \|f^{-1}\|_{L^\infty}^{-2} \hat{c}_*$. In accordance with (4.3), $\beta \leq \|f^{-1}\|_{L^\infty}^{-2} \hat{c}_*$. By (6.1), $\hat{c}_* = \frac{1}{2} \min \{\kappa c_*; 2\beta\}$, $\hat{c}_* = \frac{1}{2} \min \{\kappa \hat{c}_*; 2\beta\}$. So, $\hat{c}_* \leq \|f^{-1}\|_{L^\infty}^{-2} \hat{c}_*$. According to (2.30),

$$\hat{\mathcal{L}}(k, \varepsilon) \geq \hat{c}_* (|k|^2 + \varepsilon^2) 1_n. \quad (7.11)$$

Together with (6.14) this implies the estimate

$$f_0 \hat{\mathcal{L}}(k, \varepsilon) f_0 \geq \hat{c}_* (|k|^2 + \varepsilon^2) 1_n, \quad k \in \mathbb{R}^d. \quad (7.12)$$

Set

$$\hat{\mathcal{A}}^0(k) = b(D + k)^* g^0 b(D + k), \quad \hat{\mathcal{A}}^0(k) = -b(D + k)^* V + \sum_{j=1}^d \sigma_j (D_j + k_j),$$

$$\hat{\mathcal{B}}^0(k, \varepsilon) = \hat{\mathcal{A}}^0(k) + \varepsilon(\hat{\mathcal{A}}^0(k) + \hat{\mathcal{A}}^0(k)^*) + \varepsilon^2 (\overline{Q} - W + \lambda I). \quad (7.13)$$

Then

$$\hat{\mathcal{L}}(k, \varepsilon) \hat{P} = \hat{\mathcal{B}}^0(k, \varepsilon) \hat{P}. \quad (7.14)$$
Denote
\[ B^0(k, \varepsilon) := f_0 \hat{B}^0(k, \varepsilon) f_0, \] (7.14)
where \( \hat{B}^0(k, \varepsilon) \) is the operator (7.12). Since the symbol of the operator \( B^0(k, \varepsilon) \) is subject to estimate (7.11), using the Fourier series expansion, one can show that
\[ B^0(k, \varepsilon) \geq \tilde{c}_0 (|k|^2 + \varepsilon^2) I, \quad k \in \tilde{\Omega}. \] (7.15)

7.4. The case when \( f \neq 1_n \). The operators \( \hat{Z}_G(\theta) \) and \( \hat{Z}_G \). Now we return back to analysis of the general case \( f \neq 1_n \). Let’s give a realisation for the operators from Subsec. 3.4. Define a \( \Gamma \)-periodic \((n \times m)\)-matrix-valued function \( \Lambda_G(x) \) as a (weak) solution of the problem
\[ b(D)^* g(x) (b(D) \Lambda_G(x) + 1_m) = 0, \quad \int_{\Omega} G(x) \Delta_G(x) \, dx = 0. \] (7.16)
Cf. [BSu4] §5. Obviously, \( \Lambda_G(x) \) differs from the solution \( \Lambda(x) \) of problem (7.1) by the constant summand:
\[ \Lambda_G(x) = \Lambda(x) + \Lambda_0, \quad \Lambda_0 = - (G)^{-1} (GA) . \] (7.17)
In [BSu4] Subsec. 7.3 it was obtained that
\[ |\Lambda_0| \leq \mathcal{C}_G = m^{1/2} (2r_0)^{-1} \alpha_0^{-1/2} \|g\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-1/2} . \] (7.18)
According to [BSu4] §5, the role of the operator \( \hat{Z}_G \) from Subsec. 3.4 is played by the operator
\[ \hat{Z}_G(\theta) = \Lambda_G b(\theta) \tilde{P}. \] (7.19)
By using \( b(D) \tilde{P} = 0 \), we get \( t \hat{Z}_G(\theta) = \Lambda_G b(D + k) \tilde{P}, \quad k \in \mathbb{R}^d \).

Next, similarly to (7.4), we consider a \( \Gamma \)-periodic \((n \times n)\)-matrix-valued solution \( \tilde{\Lambda}_G(x) \) of the problem
\[ b(D)^* g(x) b(D) \tilde{\Lambda}_G(x) + \sum_{j=1}^d D_j a_j(x)^* = 0, \quad \int_{\Omega} G(x) \tilde{\Lambda}_G(x) \, dx = 0. \] (7.20)
Note that
\[ \tilde{\Lambda}_G(x) = \tilde{\Lambda}(x) + \tilde{\Lambda}_0, \quad \tilde{\Lambda}_0 = - (G)^{-1} (GA) . \] (7.21)
In [M] (7.12)], it was obtained the estimate
\[ |\tilde{\Lambda}_0| \leq \tilde{\mathcal{C}}_G = (2r_0)^{-1} C_a n^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-1/2} |\Omega|^{-1/2} . \] (7.22)
Here \( C_a^2 = \sum_{j=1}^d \int_{\Omega} |a_j(x)|^2 \, dx \). By the definition of the operator \( \hat{Z}_G \) and the matrix-valued function \( \tilde{\Lambda}_G \) it follows the identity
\[ \hat{Z}_G = \tilde{\Lambda}_G \tilde{P}. \] (7.23)

7.5. The operator \( \hat{N}_G(k, \varepsilon) \). The operator \( \hat{N}_G(t, \varepsilon) \) defined according (3.15)–(3.19) now depends on \( \theta \) and represents as follows
\[ \hat{N}_G(t, \varepsilon; \theta) = t^3 \hat{N}_{G,11}(\theta) + t^2 \varepsilon \hat{N}_{G,12}(\theta) + t \varepsilon^2 \hat{N}_{G,21}(\theta) + \varepsilon^3 \hat{N}_{G,22}, \] (7.24)
where the operators \( \hat{N}_{G,jl}(\theta) \) are defined by identities (3.16)–(3.19) with the operators \( \hat{X}_1, \hat{Z}_G, \hat{R}, \) and \( \hat{Y}_1 \) depending on \( \theta \). (It is clear that \( \hat{N}_{G,22} \) does not depend on \( \theta \).)

By using equalities (6.5), (6.9), (7.5), and (7.19), one can show (cf. [BSu4] Subsec. 5.3)] that the first summand in the right-hand side of (7.24) has the form
\[ \hat{N}_{G,11}(k) := t^3 \hat{N}_{G,11}(\theta) = b(k)^* M_G(k) b(k) \tilde{P}, \]
where \( M_G(k) = \Lambda_G^\ast b(k)^* g + g^* b(k) \Lambda_G. \) Note that \( M_G(k) \) is a Hermitian \((m \times m)\)-matrix-valued function of \( k \) homogeneous of the degree one. Thus, \( M_G(k) \) is a symbol of a self-adjoint first order DO \( M_G(D) \) with constant coefficients.

Put \( \hat{N}_{G,12}(k) := t^2 \hat{N}_{G,12}(\theta) \). Using identities (6.7)–(6.12), (7.5), (7.19), and (7.23), by analogy with [Su7] Subsec. 5.2, one can check that
\[ \varepsilon \hat{N}_{G,12}(k) = \varepsilon (b(k)^* T_{G,0} b(k) + M_{G,1}(k) b(k) + b(k)^* M_{G,1}(k)^*) \tilde{P}, \]
where \( T_{G,0} = \tilde{\mathcal{C}}_G - \tilde{\mathcal{C}}_G \).
where
\[
M_{G,1}(k) = \Lambda_G^* b(k)^* \bar{g} + (b(D) \Lambda_G)^* g b(k) \Lambda_G + \sum_{j=1}^{d} (a_j + a_j^*) \Lambda_G k_j,
\]
\[
T_{G,0} = 2 \sum_{j=1}^{d} \text{Re} \Lambda_G^* a_j D_j \Lambda_G.
\]

Let \( \hat{N}_{G,21}(k) := t \hat{N}_{G,21}(\theta) \). By analogy with [Su7, Subsec. 5.2], one can show that
\[
\varepsilon^2 \hat{N}_{G,21}(k) = \varepsilon^2 (M_{G,2}(k) + M_{G,2}(k)^*) + T_G b(k) + b(k)^* T_G \hat{P}
\]
\[
+ 2 \varepsilon^2 \sum_{j=1}^{d} \text{Re} (a_j + a_j^*) \Lambda_G k_j \hat{P},
\]
where
\[
M_{G,2}(k) = (b(D) \Lambda_G)^* g b(k) \Lambda_G,
\]
\[
T_G = \sum_{j=1}^{d} \left( \Lambda_G^* a_j (D_j \Lambda_G) + (D_j \Lambda_G)^* a_j^* \Lambda_G \right) + \Lambda_G^* Q + \lambda \Lambda_G.
\]

Here \( \Lambda_G^* Q = |\Omega|^{-1} \int_{\Omega} \Lambda_G(x)^* d\mu(x) \).

Finally, similarly to [Su7, (5.30), (5.31)], \( \hat{N}_{G,22} = (\hat{T}_G + \hat{T}_G^*) \hat{P} \). Here
\[
\hat{T}_G = \sum_{j=1}^{d} \Lambda_G^* a_j (D_j \Lambda_G) + \Lambda_G^* Q + \lambda \Lambda_G,
\]
where \( \Lambda_G^* Q = |\Omega|^{-1} \int_{\Omega} \Lambda_G(x)^* d\mu(x) \).

As a result, the operator \( \hat{N}_G(k, \varepsilon) = \hat{N}_G(t, \varepsilon; \theta) \) defined in (7.21) is represented as
\[
\hat{N}_G(k, \varepsilon) = \hat{N}_{G,11}(k) + \varepsilon \hat{N}_{G,12}(k) + \varepsilon^2 \hat{N}_{G,21}(k) + \varepsilon^3 \hat{N}_{G,22}(k) + \varepsilon^2 \hat{N}_G.
\]

According to Remark 7.1, for the operator \( \hat{N}_G(k, \varepsilon) \), estimate of the form (3.20) holds true and the constant \( C_G \) can be chosen independent on \( \theta \). So, for \( k \in \mathbb{R}^d \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\| \hat{N}_G(k, \varepsilon) \|_{\mathcal{B}_1} = |\hat{N}_G(k, \varepsilon)| \leq C_G(|k|^2 + \varepsilon^2)^{3/2}.
\]
(We take into account that, for the operator of multiplication by the constant matrix, the operator norm in \( L_2(\Omega; \mathbb{C}^n) \) matches with the matrix norm.)

Since the projection \( \hat{P} \) stands in the expressions defining the operators \( \hat{N}_{G,11}(k), \hat{N}_{G,12}(k), \) and \( \hat{N}_{G,21}(k) \), we can replace \( k \) by \( D + k \) in these expressions: \( \hat{N}_G(k, \varepsilon) = \hat{N}(k, \varepsilon) \hat{P} \), where \( \hat{N}(k, \varepsilon) \) is a self-adjoint third order DO:
\[
\hat{N}(k, \varepsilon) = \hat{N}_{11}(D + k) + \varepsilon \hat{N}_{12}(D + k) + \varepsilon^2 \hat{N}_{21}(D + k) + \varepsilon^3 \hat{N}_{22}(k).
\]

Here the summands in the right-hand side are DO’s of the third, second, first, and zero order, respectively, given by the relations
\[
\hat{N}_{11}(D + k) = b(D + k)^* M_{G,1}(D + k) b(D + k),
\]
\[
\hat{N}_{12}(D + k) = b(D + k)^* T_{G,0} b(D + k) + M_{G,1}(D + k) b(D + k)
\]
\[
+ b(D + k)^* M_{G,1}(D + k)^*,
\]
\[
\hat{N}_{21}(D + k) = M_{G,2}(D + k) + M_{G,2}(D + k)^* + T_G b(D + k) + b(D + k)^* T_G
\]
\[
+ 2 \sum_{j=1}^{d} \text{Re} (a_j + a_j^*) \Lambda_G(D_j + k_j),
\]
\[
\hat{N}_{22}(k) = \hat{T}_G + \hat{T}_G^*.
\]

Remark 7.4. If the \( \Gamma \)-periodic solutions of problems (7.16) and (7.20) are equal to zero: \( \Lambda_G = 0, \) \( \tilde{\Lambda}_G = 0, \) then, by the construction, \( \hat{N}(k, \varepsilon) = 0. \)
8. Approximation for the operator \( f e^{-B(k, \varepsilon)} s f^* \)

8.1. Now we apply Theorem 3.2 to the operator \( B(k, \varepsilon) \). Taking equalities (7.2), (7.3), (7.13), and (7.14) into account, from Theorem 3.2 with \(|\tau| = (|k|^2 + \varepsilon^2)^{1/2} \leq \tau_0\) we deduce that

\[
f e^{-B(k, \varepsilon)} s f^* = f_0 e^{-B_0(k, \varepsilon)} s f_0 \hat{P} + K(k, \varepsilon, s) + f R(k, \varepsilon, s) f^*.
\] (8.1)

Here the operator \( B^0(k, \varepsilon) \) is defined in (7.14),

\[
K(k, \varepsilon, s) := \left( \Lambda_G b(D + k) + \varepsilon \tilde{\Lambda}_G \right) f_0 e^{-B^0(k, \varepsilon)} s f_0 \hat{P}
+ f_0 e^{-B^0(k, \varepsilon)} s f_0 \hat{P} \left( b(D + k)^* \Lambda_G^* + \varepsilon \tilde{\Lambda}_G^* \right)
- \int f_0 e^{-B^0(k, \varepsilon)} (s-s) f_0 N(k, \varepsilon) f_0 e^{-B^0(k, \varepsilon)} s f_0 \hat{P} d\sigma,
\] (8.2)

and the remainder term \( f R(k, \varepsilon, s) f^* \) for \((|k|^2 + \varepsilon^2)^{1/2} \leq \tau_0\) satisfies

\[
\|f R(k, \varepsilon, s) f^*\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_7 \|f\|_{L_{\infty}}^2 (s+1)^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}, \quad s \geq 0; \tag{8.3}
\]

\[
\|f R(k, \varepsilon, s) f^*\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_5 \|f\|_{L_{\infty}}^2 s^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}, \quad s > 0. \tag{8.4}
\]

8.2. Estimates for \(|k|^2 + \varepsilon^2 \geq \tau_0^2\). If \( k \in \tilde{\Omega}, 0 < \varepsilon \leq 1, \) and \(|k|^2 + \varepsilon^2 > \tau_0^2\), it suffices to estimate each operator in (8.1) separately. By (6.5) and the elementary inequalities \( e^{-\alpha} \leq \alpha^{-1} e^{-\alpha/2} \) for \( \alpha > 0 \) and \( e^{-\alpha} \leq 2(1+\alpha)^{-1} e^{-\alpha/2} \) for \( \alpha \geq 0 \), we obtain that for \( k \in \tilde{\Omega}, |k|^2 + \varepsilon^2 > \tau_0^2\) one has

\[
\|f e^{-B(k, \varepsilon)} s f^*\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \|f\|_{L_{\infty}}^2 e^{-c_1(|k|^2+\varepsilon^2)s} \leq \tau_0^{-2} c_1^{-1} s^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2} \|f\|_{L_{\infty}}^2, \quad s \geq 0; \tag{8.5}
\]

\[
\|f e^{-B(k, \varepsilon)} s f^*\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \max\{1; c_1^{-1} \tau_0^{-2}\} \times (1+s)^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2} \|f\|_{L_{\infty}}^2, \quad s \geq 0. \tag{8.6}
\]

Similarly, by (6.11) and (7.15) for \( k \in \tilde{\Omega}, |k|^2 + \varepsilon^2 > \tau_0^2\) we have

\[
\|f_0 e^{-B_0(k, \varepsilon)} f_0 \hat{P}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tau_0^{-2} c_1^{-1} s^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2} \|f_0\|_{L_{\infty}}^2, \quad s \geq 0; \tag{8.7}
\]

\[
\|f_0 e^{-B_0(k, \varepsilon)} f_0 \hat{P}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \max\{1; c_1^{-1} \tau_0^{-2}\} \times (1+s)^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2} \|f_0\|_{L_{\infty}}^2, \quad s \geq 0. \tag{8.8}
\]

By (8.25), (8.26) and Remark 6.1 for the correcor we have estimates with constants independent on \( \theta \):

\[
\|K(k, \varepsilon, s)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_8 \|f\|_{L_{\infty}}^2 s^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}
\leq C_8 \tau_0^{-1} \|f\|_{L_{\infty}}^2 s^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}, \quad s \geq 0, \quad 0 < \varepsilon \leq 1, \quad k \in \tilde{\Omega}, \quad |k|^2 + \varepsilon^2 > \tau_0^2; \tag{8.9}
\]

\[
\|K(k, \varepsilon, s)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_9 \|f\|_{L_{\infty}}^2 (\tau_0^2 + 1)^{1/2} (1 + c_1(|k|^2+\varepsilon^2)s)^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}
\leq C_9 \|f\|_{L_{\infty}}^2 \left( \tau_0^2 + 1 \right)^{1/2} \max\{1; c_1^{-1} \tau_0^{-2}\} \times (1+s)^{-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}, \tag{8.10}
\]

\[
\|f R(k, \varepsilon, s) f^*\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_{10} \max\{C_5; 2\tau_0^{-2} c_1^{-1} + C_8 \tau_0^{-1}\} \|f\|_{L_{\infty}}^2, \quad s \geq 0, \quad k \in \tilde{\Omega}. \tag{8.11}
\]

Combining (8.4), (8.5), (8.7), and (8.9), we conclude that for \( s > 0 \) and any \( k \in \tilde{\Omega}\) the representation (8.1) holds true with the error estimate

\[
\|f R(k, \varepsilon, s) f^*\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_{10}^{s-1} e^{-c_1(|k|^2+\varepsilon^2)s/2}, \quad s > 0, \quad k \in \tilde{\Omega}.
\]
By \( (8.3), (8.6), (8.8), \) and \( (8.10) \), for the remainder term in \( (8.1) \) for \( s \geq 0 \) and \( k \in \bar{\Omega} \) we have the estimate
\[
\| f R(k, \varepsilon, s)f^* \|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C_{11}(1+s)^{-1}(\| k \|^2 + \varepsilon^2)/2, \quad s \geq 0, \quad k \in \bar{\Omega},
\]
with the constant \( C_{11} = \max \{ C_7; (4 + C_9(1 + r_3^2))^{1/2} \} \max \{ 1; C_{11}^{-1} \} \) \( \| f \|_{L^2}^2 \).

We got the following result.

**Theorem 8.1.** Let \( B(k, \varepsilon) \) and \( B^0(k, \varepsilon) \) be the operators \( (5.16) \) and \( (7.14) \), respectively, and let \( K(k, \varepsilon, s) \) be the corrector \( (8.2) \). Then for \( s \geq 0, 0 < \varepsilon \leq 1, \) and \( k \in \Omega \) representation \( (8.1) \) makes sense and the remainder term satisfies estimates \( (8.11) \) and \( (8.12) \), where the constants \( C_{10} \) and \( C_{11} \) are controlled in terms of the initial data \( (4.24) \).

### 9. Approximation for the Operator \( f e^{-B(\varepsilon)s}f^* \)

9.1. Now we return to analysis of the operator \( B(\varepsilon) \) acting in \( L^2(\mathbb{R}^d; \mathbb{C}^n) \) and defined in Subsec. 4.3. Approximation for the bordered operator exponential \( f e^{-B(\varepsilon)s}f^* \) can be derived from Theorem 8.1 by using of the direct integral decomposition.

We introduce the effective operator with constant coefficients:
\[
B^0(\varepsilon) := f_0b(D)^*g^0b(D)f_0 + \varepsilon f_0 \left( -b(D)^*V - V^*b(D) + \sum_{j=1}^d (a_j + a_j^*) D_j \right) f_0 + \varepsilon^2 f_0(\mathbf{W} + \mathbf{Q} + \lambda I)f_0.
\]

The symbol of the operator \( (9.1) \) is the matrix \( \tilde{f}_0 \tilde{L}(k, \varepsilon)f_0 \) (see Subsec. 7.3).

Define the third order DO with constant coefficients:
\[
\mathcal{N}(\varepsilon) = N_{11}(D) + \varepsilon^2 N_{12}(D) + \varepsilon N_{21}(D) + \varepsilon^3 N_{22}, \quad (9.2)
\]
\[
N_{11}(D) = b(D)^*M_G(D)b(D),
\]
\[
N_{12}(D) = b(D)^*T_Gb(D) + M_{G,1}(D)b(D) + b(D)^*M_{G,1}(D)^*,
\]
\[
N_{21}(D) = M_{G,2}(D)^* + T_G^*b(D) + b(D)^*T_G + 2 \sum_{j=1}^d \text{Re} (a_j + a_j^*) \lambda_G D_j.
\]

(Recall that the matrix \( N_{22} \) is defined in \( (7.26) \).)

We use the direct integral decomposition for the operator \( B(\varepsilon) \), see Subsec. 5.6. Then for the operator exponential one has the representation
\[
e^{-B(\varepsilon)s} = U^{-1} \left( \int_{\bar{\Omega}} e^{-B(k, \varepsilon)s} dk \right) U.
\]

The similar identity holds true for \( e^{-B^0(\varepsilon)s} \). The operator
\[
\Lambda_G b(D) + \varepsilon \tilde{\Lambda}_G,
\]
decomposes into the direct integral with fibres \( \Lambda_G b(D + k) + \varepsilon \tilde{\Lambda}_G \). For the operator \( \mathcal{N}(\varepsilon) \), we have decomposition into the direct integral of the operators \( \mathcal{N}(k, \varepsilon) \). Thus, Remark 7.3 implies the following observation.

**Remark 9.1.** If \( \Lambda_G = 0 \) and \( \tilde{\Lambda}_G = 0 \), then \( \mathcal{N}(\varepsilon) = 0 \).

Define the bounded operator \( \Pi \) in \( L^2(\mathbb{R}^d; \mathbb{C}^n) \) by the relation
\[
\Pi = U^{-1} [\tilde{P}] U,
\]
where \( [\tilde{P}] \) is the operator acting in the space \( \mathcal{H} = \int_{\bar{\Omega}} L^2(\Omega; \mathbb{C}^n) dk \) fibre-wise as the operator \( \hat{P} \) of averaging over the cell. As was obtained in \( [\text{BSu4}] \) Subsec. 6.1, the operator \( \Pi \) can be
written as
\[(IIu)(x) = (2\pi)^{-d/2} \int e^{i(x,\xi)} (\mathcal{F}u)(\xi) \, d\xi, \quad (9.3)\]
where \((\mathcal{F}u)(\cdot)\) is the Fourier image of the function \(u\). Thus, \(II\) is PDO with the symbol \(\chi_{\tilde{\Omega}}(\xi)\), where \(\chi_{\tilde{\Omega}}\) is the characteristic function of the set \(\tilde{\Omega}\). This operator is the smoothing one. Note that \(II\) commutes with differential operators with constant coefficients.

Theorem 8.1 implies the identity
\[f e^{-B(\varepsilon)s} f^* = f_0 e^{-B(\varepsilon)s} f_0 II + K(\varepsilon, s) + R(\varepsilon, s). \quad (9.4)\]

Here the corrector \(K(\varepsilon, s) = \mathcal{U}^{-1} \left( \mathcal{F} + \mathcal{K}(k, \varepsilon, s) \right) \mathcal{U}\) has the form
\[K(\varepsilon, s) = \left( \Lambda_G b(D) + \varepsilon \Lambda_G \right) f_0 e^{-B(\varepsilon)s} f_0 II + f_0 e^{-B(\varepsilon)s} f_0 II \left( b(D)^* \Lambda_G^* + \varepsilon \Lambda_G^* \right) - \int_0^s f_0 e^{-B(\varepsilon)(s-t)} f_0 N(\varepsilon) f_0 e^{-B(\varepsilon)s} f_0 II ds, \quad (9.5)\]
and the operator \(R(\varepsilon, s) = \mathcal{U}^{-1} \left( \mathcal{F} + \mathcal{R}(k, \varepsilon, s) f^* \right) \mathcal{U}\) is subject to estimates
\[
\|R(\varepsilon, s)\|_{L_2(\mathbb{R}^d)} \leq \text{ess sup}_{k \in \Omega} \| f \mathcal{R}(k, \varepsilon, s) f^* \|_{L_2(\Omega)} \leq C_{10} s^{-1} e^{-\varepsilon^2 s^2/2}, \quad s > 0;
\]
\[
\|R(\varepsilon, s)\|_{L_2(\mathbb{R}^d)} \leq C_{11} (1 + s)^{-1} e^{-\varepsilon^2 s^2/2}, \quad s \geq 0.
\]

9.2. Elimination of the operator II. Now we discuss the opportunity to remove the smoothing operator from the corrector in the approximation (9.4). Put
\[\Xi(\varepsilon, s) := f_0 e^{-B(\varepsilon)s} f_0 (I - II). \quad (9.6)\]
Since the matrix \(f_0 \tilde{L}(\xi, \varepsilon) f_0\) is the symbol of the operator \(B(\varepsilon)\) and II is a PDO with the symbol \(\chi_{\tilde{\Omega}}(\xi)\), by (6.14), (7.11), and the elementary inequality \(e^{-\alpha} \leq 2(1 + \alpha)^{-1} e^{-\alpha/2}, \alpha > 0,\) for \(s \geq 0\) we have
\[
\|\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|f\|_{L_\infty}^2 \sup_{\xi \in \mathbb{R}^d} |e^{-f_0 \tilde{L}(\xi, \varepsilon) f_0}| |1 - \chi_{\tilde{\Omega}}(\xi)| \leq \|f\|_{L_\infty}^2 \sup_{\xi \in \mathbb{R}^d, |\xi| \geq 0} e^{-\varepsilon s^2/2} \leq 2\|f\|_{L_\infty}^2 \max\{1; \varepsilon^{-1} r_0^2\} (1 + s)^{-1} e^{-\varepsilon^2 s^2/2}.
\]
Thus, the smoothing operator II always can be removed from the principal term of approximation (9.4).

For \(s > 0\), we also can replace II by I in the third term of the corrector (9.5). Note that \(N(\varepsilon)\) is DO with the symbol \(\tilde{N}_G(\xi, \varepsilon)\) (see Subsec. 7.5) satisfying the estimate (7.25). From this and (6.14) and estimate (7.11) for the symbol of the operator \(B(\varepsilon)\) we derive that
\[
\left\| \int_0^s f_0 e^{-B(\varepsilon)(s-t)} f_0 N(\varepsilon) f_0 e^{-B(\varepsilon)s} f_0 (I - II) ds \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|f\|_{L_\infty}^2 \sup_{\xi \in \mathbb{R}^d, |\xi| \geq 0} e^{-\varepsilon^2 s^2/2} C_G (|\xi|^2 + \varepsilon^2)^{3/2} \leq 3\varepsilon^{-2} r_0^{-1} C_G \|f\|_{L_\infty}^2 s^{-1} e^{-\varepsilon^2 s^2/2}, \quad s > 0.
\]
(We used the elementary inequality \(e^{-\alpha} \leq 3\alpha^{-2} e^{-\alpha/2}, \alpha > 0\).)
Discuss the opportunity of elimination of the operator $\Pi$ from the other terms of the corrector. Since the matrix-valued functions $\Lambda_0$ and $\Lambda$, $\tilde{\Lambda}$ differ to the constant summand (see (7.17) and (7.21)), we use the multiplier properties of matrix-valued functions $\Lambda$ and $\tilde{\Lambda}$ to eliminate the smoothing operator from the terms of the corrector containing $\Lambda_0$ and $\tilde{\Lambda}$. The following result was obtained in [BSu4, Proposition 6.8] for $d \leq 4$ and [V, (7.19) and (7.20)] for $d > 4$.

**Lemma 9.2.** Let $\Lambda$ be a $\Gamma$-periodic $(n \times m)$-matrix-valued solution of the problem (7.4). Let $l = 1$ for $d \leq 4$ and $l = d/2 - 1$ for $d > 4$. Then the operator $[\Lambda]$ of multiplication by the matrix-valued function $\Lambda$ continuously maps $H^1(\mathbb{R}^d; \mathbb{C}^m)$ into $L_2(\mathbb{R}^d; \mathbb{C}^n)$, and

$$\|[\Lambda]\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \mathcal{C}_\Lambda.$$  

The constant $\mathcal{C}_\Lambda$ is controlled in terms of $m$, $n$, $d$, $\alpha_0$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

For $d \leq 6$, we apply Proposition 6.9 from [Su7], and for $d > 6$ we rely on Lemma 6.5(1°) from [MSu].

**Lemma 9.3.** Let $\tilde{\Lambda}$ be a $\Gamma$-periodic $(n \times n)$-matrix-valued solution of the problem (7.4). Put $\sigma = 2$ for $d \leq 6$ and $\sigma = d/2 - 1$ for $d > 6$. Then the operator $[\tilde{\Lambda}]$ of multiplication by the matrix-valued function $\tilde{\Lambda}$ continuously maps $H^\sigma(\mathbb{R}^d; \mathbb{C}^n)$ into $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and

$$\|[\tilde{\Lambda}]\|_{H^\sigma(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \mathcal{C}_{\tilde{\Lambda}}.$$  

The constant $\mathcal{C}_{\tilde{\Lambda}}$ is controlled in terms of the initial data (4.24).

The following result was obtained in [M, Proposition 8.3].

**Proposition 9.4.** Let $\Xi(\varepsilon, s)$ be the operator (9.10). Then for $s > 0$ and all $l > 0$ the operators $b(D)\Xi(\varepsilon, s)$ and $\varepsilon\Xi(\varepsilon, s)$ are bounded from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ into $H^l(\mathbb{R}^d; \mathbb{C}^n)$ and

$$\|b(D)\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to H^l(\mathbb{R}^d)} \leq \alpha_1^{1/2} \mathcal{C}(l)s^{-(l+1)/2}e^{-\varepsilon\varepsilon^2/2},$$  

$$\varepsilon\|\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to H^l(\mathbb{R}^d)} \leq \mathcal{C}(l)s^{-(l+1)/2}e^{-\varepsilon\varepsilon^2/2}.$$  

The constant $\mathcal{C}(l)$ depends only on $l$ and on initial data (4.24).

Now we can consider the first summand in (9.5). (The second summand is adjoint to the first one does not require additional considerations.) By (7.17)

$$\Lambda_0 b(D)\Xi(\varepsilon, s) = \Lambda b(D)\Xi(\varepsilon, s) + \Lambda_0^0 b(D)\Xi(\varepsilon, s).$$  

Combining (4.2), (6.14), (7.11), (7.18), and (9.6), we get

$$\|\Lambda_0^0 b(D)\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \mathcal{C}_G \|f\|_{L_\infty}^2 \sup_{\xi \in \mathbb{R}^d} |b(\xi)| e^{-\varepsilon\varepsilon^2/2} (1 - \chi_0(\xi)) \leq \alpha_1^{1/2} \varepsilon^{-1} r_0^{-1} \mathcal{C}_G \|f\|_{L_\infty}^2 s^{1} e^{-\varepsilon\varepsilon^2/2},$$  

(9.10)  

In that follows we are only interested in large values of $s$, so now it suffices to consider $s \geq 1$. Lemma 9.2 and estimate (9.7) imply that

$$\|\Lambda b(D)\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{12} s e^{-\varepsilon\varepsilon^2/2}, \quad s \geq 1.$$  

(9.11)

Here $C_{12} = \mathcal{C}_A \alpha_1^{1/2} \mathcal{C}(l)$, where $l = 1$ for $d \leq 4$ and $l = d/2 - 1$ for $d > 4$. Combining (9.9)–(9.11), we get

$$\|\Lambda_0 b(D)\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{13} s e^{-\varepsilon\varepsilon^2/2}, \quad s \geq 1,$$

where $C_{13} = \alpha_1^{1/2} \varepsilon^{-1} r_0^{-1} \mathcal{C}_G \|f\|_{L_\infty}^2 + C_{12}$.

Similarly, using Lemma 9.3 and (7.21), (7.22), and (9.8), we arrive at the estimate

$$\|\varepsilon\tilde{\Lambda}_0\Xi(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{14} s e^{-\varepsilon\varepsilon^2/2}, \quad s \geq 1,$$
with the constant $C_{14} = \varepsilon_1^{-1}r_0^{-1}\|F\|_{L^2}^2 + c_3\varepsilon \mathcal{C} (\varepsilon\sigma)$, where $\sigma = 2$ for $d \leq 6$ and $\sigma = d/2 - 1$ for $d > 6$. Thus, we have shown that for $s \geq 1$ the smoothing operator $\Pi$ can be removed from the all terms of the corrector (15).

**Theorem 9.5.** Let $B(\varepsilon)$ and $B^0(\varepsilon)$ be the operators (4.23) and (9.1), respectively.

1°. Let $K(\varepsilon, s)$ be the operator (9.35). Then for $0 < \varepsilon \leq 1$ and $s \geq 0$ we have the estimate

$$
\|f e^{-B(\varepsilon)s} f^* - f_0 e^{-B^0(\varepsilon)s} f_0 - K(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_{15} (1 + s)^{-1} e^{-\hat{c}_0 \varepsilon^2 s/2}
$$

(9.12)

with the constant $C_{15} = C_{11} + 2 \|f\|_{L_\infty}^2 \max\{1, \hat{c}_0^{-1} r_0^{-2}\}$.

2°. Denote

$$
K^0(\varepsilon, s) := \left( \Lambda_G b(D) + \varepsilon \Lambda_G \right) f_0 e^{-B^0(\varepsilon)s} f_0
$$

$$
+ f_0 e^{-B^0(\varepsilon)s} f_0 \left( b(D)^* \Lambda_G^* + \varepsilon \Lambda_G^* \right)
$$

$$
- \int_0^s f_0 e^{-B^0(\varepsilon)(s-t)} f_0 N(\varepsilon) f_0 e^{-B^0(\varepsilon)t} f_0 d\tau.
$$

(9.13)

Then for $0 < \varepsilon \leq 1$ and $s \geq 1$ we have

$$
\|f e^{-B(\varepsilon)s} f^* - f_0 e^{-B^0(\varepsilon)s} f_0 - K^0(\varepsilon, s)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_{16} s^{-1} e^{-\hat{c}_0 \varepsilon^2 s/2},
$$

(9.14)

where $C_{16} = C_{10} + 2 \|f\|_{L_\infty}^2 \max\{1, \hat{c}_0^{-1} r_0^{-2}\} + 3 \hat{c}_0^{-2} r_0^{-1} C_G \|f\|_{L_\infty}^4 + 2 C_{13} + 2 C_{14}$.

**Chapter 3. Homogenization Problem for Parabolic Systems**

10. Homogenization of the Operator $f^* e^{-B^0(\varepsilon)s}(f^*)^*$

10.1. The problem setting. For any $\Gamma$-periodic function $\phi(x)$, $x \in \mathbb{R}^d$, we use the notation $\phi^\varepsilon(x) := \phi(x/\varepsilon), \varepsilon > 0$. In $L_2(\mathbb{R}^d, \mathbb{C}^n)$, we consider the operator

$$
A_\varepsilon = f^\varepsilon(x)b(D)^* g^\varepsilon(x)b(D)f^\varepsilon(x),
$$

(10.1)

corresponding to the quadratic form

$$
a_\varepsilon[u, u] = \int_{\mathbb{R}^d} \langle g^\varepsilon b(D)f^\varepsilon u, b(D)f^\varepsilon u \rangle dx
$$

(10.1)

on the domain $\mathcal{D}_\varepsilon = \{u \in L_2(\mathbb{R}^d, \mathbb{C}^n) : f^\varepsilon u \in H^1(\mathbb{R}^d, \mathbb{C}^n)\}$. The form (10.1) is subject to estimates that are analogous to inequalities (4.7):

$$
a_0 \|g^{-1/2}D(f^\varepsilon u)\|_{L_2(\mathbb{R}^d)}^2 \leq a_\varepsilon[u, u] \leq \alpha_1 \|g\|_{L_\infty} \|D(f^\varepsilon u)\|_{L_2(\mathbb{R}^d)}^2.
$$

(10.2)

Next, let $\mathcal{V}_\varepsilon : L_2(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d, \mathbb{C}^{dn})$ be the operator, acting by the rule

$$
\mathcal{V}_\varepsilon u = D(f^\varepsilon u) = \{D_1(f^\varepsilon u), \ldots, D_d(f^\varepsilon u)\},
$$

(10.2)

and let $\mathcal{V}_\varepsilon : L_2(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d, \mathbb{C}^{dn})$ be the operator of multiplication by the $(dn \times d)$-matrix-valued function consisting on the blocks $\alpha_j^\varepsilon(x) f^\varepsilon(x)$, $j = 1, \ldots, d$:

$$
\mathcal{V}_\varepsilon u = \{\alpha_1^\varepsilon f^\varepsilon u, \ldots, \alpha_d^\varepsilon f^\varepsilon u\},
$$

(10.2)

Let $d\mu$ a matrix-valued measure on $\mathbb{R}^d$ defined in Subsec. 4.4. By using it, we build the measure $d\mu^\varepsilon$ as follows. For any Borel set $\Delta \subset \mathbb{R}^d$, consider the set $\varepsilon^{-1}\Delta = \{y = \varepsilon^{-1}x : x \in \Delta\}$ and put $\mu^\varepsilon(\Delta) = \varepsilon^d \mu(\varepsilon^{-1}\Delta)$. Define the quadratic form $q_\varepsilon$ by the rule

$$
q_\varepsilon[u, u] = \int_{\mathbb{R}^d} \langle d\mu^\varepsilon(x)(f^\varepsilon u)(x), (f^\varepsilon u)(x) \rangle, \ u \in \mathcal{D}_\varepsilon.
$$

(10.2)

All the assumptions of Subsec. 4.2 are assumed to be satisfied. In the space $L_2(\mathbb{R}^d, \mathbb{C}^n)$, consider the quadratic form

$$
b_\varepsilon[u, u] = a_\varepsilon[u, u] + 2Re(\mathcal{V}_\varepsilon u, \mathcal{V}_\varepsilon u)_{L_2(\mathbb{R}^d)} + q_\varepsilon[u, u], \ u \in \mathcal{D}_\varepsilon,
$$

(10.3)

Let $T_\varepsilon$ be the unitary in $L_2(\mathbb{R}^d, \mathbb{C}^n)$ scaling transformation:

$$
(T_\varepsilon u)(x) = \varepsilon^{d/2}u(\varepsilon x), \ \varepsilon > 0.
$$

(10.4)
The forms (4.18) and (10.1), (10.3) satisfy the obvious identities
\[ a_0[u, u] = \varepsilon^{-2}a[T_\varepsilon u, T_\varepsilon u], \quad b_\varepsilon[u, u] = \varepsilon^{-2}b(\varepsilon)[T_\varepsilon u, T_\varepsilon u], \quad u \in \mathcal{D}. \]
Together with estimates (4.21) and (1.22), this implies that for \( u \in \mathcal{D} \) we have
\begin{align*}
&b_\varepsilon[u, u] \geq \frac{\kappa}{2} a_\varepsilon[u, u] + \beta \|u\|_{L^2(\mathbb{R}^d)}^2, \quad (10.5) \\
&b_\varepsilon[u, u] \leq (2 + \varepsilon^2 + c_2) a_\varepsilon[u, u] + (C(1) + c_3 + |\lambda|\|Q_0\|_{L^\infty}) \|u\|_{L^2(\mathbb{R}^d)}^2, \quad (10.6)
\end{align*}
By (10.2), (10.5), and (10.6), the form \( b_\varepsilon \) is closed and positive definite. By \( \mathcal{B}_\varepsilon \) we denote the corresponding self-adjoint operator in the space \( L_2(\mathbb{R}^d; \mathbb{C}^n) \). Formally,
\[ \mathcal{B}_\varepsilon = A_\varepsilon + (\mathcal{A}_{2,\varepsilon}^\varepsilon \mathcal{Y}_\varepsilon + \mathcal{A}_{2,\varepsilon}^\varepsilon \mathcal{Y}_\varepsilon) + (f^\varepsilon)^* Q_\varepsilon f^\varepsilon + \lambda Q_0^0 \]
\[ = (f^\varepsilon)^* b(D)^* g^\varepsilon b(D) f^\varepsilon + \sum_{j=1}^d (f^\varepsilon)^* (\alpha_j^\varepsilon D_j + D_j(\alpha_j^\varepsilon)^*) f^\varepsilon \]
\[ + (f^\varepsilon)^* Q_\varepsilon f^\varepsilon + \lambda Q_0^0, \quad (10.7) \]
where \( Q_\varepsilon \) should be interpreted as a generalised matrix-valued potential generated by the rapidly oscillating measure \( d\mu^\varepsilon \).

10.2. The principal term of approximation for \( f^\varepsilon e^{-B_\varepsilon^s (f^\varepsilon)^*} \). The principal term of approximation was found at [M, Subsec. 9.3]. To formulate the result, we need to write down the effective operator:
\[ \mathcal{B}_0 = f_0 b(D)^* g^0 b(D) f_0 + f_0 \left( -b(D)^* V - V^* b(D) + \sum_{j=1}^d (a_j + a_j^\varepsilon) D_j \right) f_0 + f_0 (W + \mathcal{Q} + \lambda I) f_0. \]
Here the constant matrices \( f_0, g^0, V, W, (a_j + a_j^\varepsilon) \), and \( \mathcal{Q} \) are defined according to (6.13), (7.6), (7.8), (7.9), (4.1), and (7.10). The symbol of the operator (10.8) is the matrix \( f_0 \tilde{L}(|\xi|, 1) f_0 \) (see Subsec. 7.3). Thus, estimate (7.11) implies that the operator \( \mathcal{B}_0 \) is positive definite: \( \mathcal{B}_0 \geq \tilde{c}_s I \).

In [M, Theorem 9.1], the following result was obtained.

**Theorem 10.1.** Let the assumptions of Subsec. 4.2 and 4.4 be satisfied. Let \( \mathcal{B}_\varepsilon \) be the operator (10.7) and let \( \mathcal{B}_0^s \) be the effective operator (10.8). Then for \( s \geq 0 \) and \( 0 < \varepsilon \leq 1 \) we have
\[ \| f^\varepsilon e^{-B_\varepsilon^s (f^\varepsilon)^*} - f_0^s e^{-B_0^s f_0^s} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_{17} \varepsilon (s + \varepsilon^2)^{-1/2} e^{-\tilde{c}_s s^2/2}. \]
The constant \( C_{17} \) depends only on the initial data (19.2).

10.3. Approximation for the operator \( f^\varepsilon e^{-B_\varepsilon^s (f^\varepsilon)^*} \) with the corrector taken into account. Now we are going to derive a more accurate approximation for the operator exponential from the results of [9] with the help of the scaling transformation (10.4). The operators (10.7) and (10.8) and (9.1) are related by the identities \( \mathcal{B}_\varepsilon = \varepsilon^{-2} T_\varepsilon^* \mathcal{B}(\varepsilon) T_\varepsilon \) and \( \mathcal{B}_0^s = \varepsilon^{-2} T_\varepsilon^* \mathcal{B}^0(\varepsilon) T_\varepsilon \).

Thus,
\[ f^\varepsilon e^{-B_\varepsilon^s (f^\varepsilon)^*} = T_\varepsilon^* f e^{-B(\varepsilon)^{-2} s f^\varepsilon T_\varepsilon}, \quad (10.9) \]
\[ f_0 e^{-B_0^s f_0^s} = T_\varepsilon^* f_0 e^{-B^0(\varepsilon)^{-2} s f_0 T_\varepsilon}. \quad (10.10) \]

Next, by \( \Pi_\varepsilon \) we define a PDO acting in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \), which symbol is a characteristic function \( \chi_{\tilde{\Omega}}(\xi) \) of the set \( \tilde{\Omega}/\varepsilon \):
\[ (\Pi_\varepsilon f)(\mathbf{x}) = (2\pi)^{-d/2} \int_{\tilde{\Omega}/\varepsilon} e^{i \langle \mathbf{x}, \xi \rangle} (\hat{f})(\xi) \, d\xi. \quad (10.11) \]
The operators (9.3) and (10.11) are related via the equality
\[ \Pi_\varepsilon = T_\varepsilon^* \Pi T_\varepsilon. \quad (10.12) \]
Denote 
\[ N := N_{11}(D) + N_{12}(D) + N_{21}(D) + N_{22}. \]

Then 
\[ N = \varepsilon^{-3}T_\varepsilon^* N(\varepsilon) T_\varepsilon, \]  
(10.13)

where the operator \( N(\varepsilon) \) is defined in (9.2).

We point out the identities
\[ [\Lambda^*_\varepsilon] b(D) = \varepsilon^{-1}T_\varepsilon^* [\Lambda_G] b(D) T_\varepsilon, \quad [\Lambda_{\varepsilon}^* \tilde{\varepsilon}] = T_\varepsilon^* [\Lambda_G]\tilde{T}_\varepsilon. \]  
(10.14)

Introduce the correctors:
\[ K_\varepsilon(s) := \left( \Lambda^*_\varepsilon b(D) + \Lambda_{\varepsilon}^* \tilde{\varepsilon} \right) f_0 e^{-B^{0s}} f_0 \Pi \varepsilon \]
\[ + f_0 e^{-B^{0s}} f_0 \Pi \varepsilon \left( b(D)^* (\Lambda^*_\varepsilon)^* + (\Lambda_{\varepsilon}^* \tilde{\varepsilon})^* \right) \]
\[ - \int_0^s f_0 e^{-B^{(s-\tau)}} f_0 \Pi \varepsilon e^{-B^{0\tilde{\varepsilon}}} f_0 \Pi \varepsilon d\tau, \]
(10.15)

\[ K^0_\varepsilon(s) := \left( \Lambda^*_\varepsilon b(D) + \Lambda_{\varepsilon}^* \tilde{\varepsilon} \right) f_0 e^{-B^{0s}} f_0 \]
\[ + f_0 e^{-B^{0s}} f_0 \left( b(D)^* (\Lambda^*_\varepsilon)^* + (\Lambda_{\varepsilon}^* \tilde{\varepsilon})^* \right) \]
\[ - \int_0^s f_0 e^{-B^{(s-\tau)}} f_0 \Pi \varepsilon e^{-B^{0\tilde{\varepsilon}}} f_0 d\tau. \]
(10.16)

Combining (10.9), (10.10), and (10.12)–(10.14), we conclude that the operators (9.3) and (10.15) are related by the identity
\[ \varepsilon K_\varepsilon(s) = T_\varepsilon^* K(\varepsilon, \varepsilon^{-2}s) T_\varepsilon. \]  
(10.17)

For the operators (9.13) and (10.16), the similar identity holds true. Now (10.12), (10.14), (10.9), (10.10), and (10.17) imply the following result.

**Theorem 10.2.** Let the assumptions of Theorem 10.1 be satisfied. Let \( K_\varepsilon(s) \) and \( K^0_\varepsilon(s) \) be the operators (10.15) and (10.16), respectively. Then for \( 0 < \varepsilon \leq 1 \) and \( s \geq 0 \) we have the approximation
\[ \| f^\varepsilon e^{-B_{\varepsilon s}(f^\varepsilon)^*} - f_0 e^{-B^{0s}} f_0 - \varepsilon K_\varepsilon(s) \|_{L_2(\mathbb{R}^d)} \leq C_{15} \varepsilon^2 (s + \varepsilon^2)^{-1} e^{-\tilde{c} s/2}. \]  
(10.18)

For \( 0 < \varepsilon \leq 1 \) and \( s \geq \varepsilon^2 \) we have
\[ \| f^\varepsilon e^{-B_{\varepsilon s}(f^\varepsilon)^*} - f_0 e^{-B^{0s}} f_0 - \varepsilon K^0_\varepsilon(s) \|_{L_2(\mathbb{R}^d)} \leq C_{16} \varepsilon^2 s^{-1} e^{-\tilde{c} s/2}. \]

The constants \( C_{15}, C_{16}, \) and \( \tilde{c} \) depend only on the initial data (4.24).

### 10.4. The case when the corrector is equal to zero.

Assume that \( g^0 = \mathbf{7} \), i.e., relations (7.7) hold true. Then the \( \Gamma \)-periodic solutions of problems (7.1), (7.11) are equal to zero: \( \Lambda(x) = \Lambda_G(x) = 0 \). In addition, we assume that \( \sum_{j=1}^d D_j a_j(x)^* = 0 \). Then the \( \Gamma \)-periodic solutions of problems (7.3), (7.21) are also equal to zero: \( \tilde{\Lambda}(x) = \tilde{\Lambda}_G(x) = 0 \). Thus, (7.8) and (7.9) imply that \( V = 0 \) and \( W = 0 \). The effective operator (10.8) takes the form
\[ B^0 = f_0 b(D)^* g^0 b(D) f_0 + f_0 \sum_{j=1}^d (a_j + a_j^*) D_j f_0 + f_0 (\mathcal{Q} + \lambda I) f_0. \]  
(10.19)

Using (10.13) and Remark 9.1, we conclude that \( N = 0 \) in the case under consideration. Thus, all the corrector terms in (10.15) are equal to zero and (10.18) implies the following result.
Proposition 10.3. Under the assumptions of Theorem 10.2 let relations (7.7) and \( \sum_{j=1}^{d} D_j a_j(x)^\ast = 0 \) hold true. Then for \( 0 < \varepsilon \leq 1 \) and \( s \geq 0 \) we have
\[
\| f^\varepsilon e^{-b^0 \ast (f^\varepsilon)^\ast} - f^0 e^{-b^0 \ast f_0} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C 15 \varepsilon^2 (s + \varepsilon^2)^{-1} e^{-\varepsilon s/2}.
\]
Here \( B^0 \) is the operator (10.19).

11. Homogenization for solutions of the Cauchy problem for parabolic systems

11.1. Application to the Cauchy problem for a homogeneous equation. Let \( \hat{B}_\varepsilon \) be the operator of the form (10.7) with \( f = 1_n \):
\[
\hat{B}_\varepsilon = b(D)^\ast g^\varepsilon b(D) + \sum_{j=1}^{d} (a_j^\ast D_j + D_j (a_j^\ast)) + Q^\varepsilon + \lambda I.
\]

Consider the Cauchy problem
\[
G^\varepsilon(x) \frac{\partial u_\varepsilon(x, s)}{\partial s} = -\hat{B}_\varepsilon u_\varepsilon(x, s), \quad s > 0; \quad G^\varepsilon(x) u_\varepsilon(x, 0) = \phi(x), \quad (11.1)
\]
where \( \phi \in L_2(\mathbb{R}^d; C^n) \), and the \( \Gamma \)-periodic \((n \times n)\)-matrix-valued function \( G \) is assumed to be bounded and positive definite. We rewrite \( G(x) \) in a factorised form: \( G(x)^{-1} = f(x)^\ast f(x)^\ast \), where \( f(x) \) is a \( \Gamma \)-periodic matrix-valued function. Put \( v_\varepsilon = (f^\varepsilon)^\ast u_\varepsilon \). Then \( v_\varepsilon(x, s) \) is the solution of the problem
\[
\frac{\partial v_\varepsilon(x, s)}{\partial s} = -(f^\varepsilon)^\ast \hat{B}_\varepsilon f^\varepsilon v_\varepsilon(x, s), \quad s > 0; \quad v_\varepsilon(x, 0) = (f^\varepsilon)^\ast \phi(x).
\]
Let \( B_\varepsilon = (f^\varepsilon)^\ast \hat{B}_\varepsilon f^\varepsilon \). Then \( v_\varepsilon(\cdot, s) = e^{-B_\varepsilon \ast (f^\varepsilon)^\ast} \phi \). This implies that the solution \( u_\varepsilon = f^\varepsilon v_\varepsilon \) of the problem (11.1) admits a representation
\[
u_\varepsilon(\cdot, s) = f^\varepsilon e^{-B_\varepsilon \ast (f^\varepsilon)^\ast} \phi
\]
(11.2)
The corresponding effective problem has the form
\[
\frac{\partial u_0(x, s)}{\partial s} = -\hat{B}_0 u_0(x, s), \quad s > 0; \quad \hat{B}_0 u_0(x, 0) = \phi(x).
\]
(11.3)
Here
\[
\hat{B}_0 = b(D)^\ast g^0 b(D) - b(D)^\ast V - V^\ast b(D) + \sum_{j=1}^{d} (a_j^\ast + a_j^\ast) D_j - W + \hat{Q} + \lambda I
\]
Let \( f_0 = (\hat{G})^{-1/2} \) and \( B^0 = f_0 \hat{B}_0 f_0 \). Then
\[
u_0(\cdot, s) = f_0 e^{-B^0 \ast f_0} \phi
\]
(11.4)
From (11.2), (11.4), and Theorem 10.2 we derive the following result.

Theorem 11.1. Let \( u_\varepsilon \) and \( u_0 \) be the solutions of problems (11.1) and (11.3), respectively. Then for \( 0 < \varepsilon \leq 1 \) and \( s \geq e^2 \) we have the approximation
\[
\begin{align*}
\left\| u_\varepsilon(\cdot, s) - u_0(\cdot, s) - \varepsilon \left( \Lambda_\varepsilon \ast b(D) u_0(\cdot, s) + \tilde{\Lambda}_\varepsilon \ast u_0(\cdot, s) \right. \right. \\
\left. \left. + f_0 e^{-B^0 \ast f_0} \left( (\Lambda_\varepsilon \ast b(D))^\ast + (\tilde{\Lambda}_\varepsilon )^\ast \right) \phi \right) \\
- \int_0^s f_0 e^{-B^0(\cdot - \tilde{s})} f_0 \mathcal{N} u_0(\cdot, \tilde{s}) \, d\tilde{s} \right\|_{L_2(\mathbb{R}^d)} \leq C 16 \varepsilon^2 s^{-1} e^{-\varepsilon s/2} \| \phi \|_{L_2(\mathbb{R}^d)}.
\end{align*}
\]
(11.5)
Note that the third term of the corrector in (11.5), i.e., the function
\[
v^{(3)}(\cdot, s) := \int_0^s f_0 e^{-B^0(\cdot - \tilde{s})} f_0 \mathcal{N} u_0(\cdot, \tilde{s}) \, d\tilde{s},
\]
is the solution of the problem
\[
\overrightarrow{\mathbf{G}} \frac{\partial \mathbf{u}^{(3)}}{\partial s} = -\overrightarrow{\mathbf{B}}^{(3)} \mathbf{u}^{(3)}(\mathbf{x}, s) + \mathbf{N}u_0(\mathbf{x}, s), \quad s > 0; \quad \overrightarrow{\mathbf{G}} \mathbf{u}^{(3)}(\mathbf{x}, 0) = 0.
\]

11.2. The Cauchy problem for an inhomogeneous equation. Now, we consider a more general Cauchy problem
\[
G^e(\mathbf{x}) \frac{\partial \mathbf{u}_e(\mathbf{x}, s)}{\partial s} = -\overrightarrow{\mathbf{B}} \mathbf{u}_e(\mathbf{x}, s) + \mathbf{F}(\mathbf{x}, s), \quad s \in (0, T);
\]
\[
G^e(\mathbf{x}) \mathbf{u}_e(\mathbf{x}, 0) = \phi(\mathbf{x}),
\]
where \( \mathbf{x} \in \mathbb{R}^d, \ s \in (0, T), \ T > 0 \). Let
\[
\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n), \quad \mathbf{F} \in L_2((0, T); L_2(\mathbb{R}^d; \mathbb{C}^n)) =: \mathcal{H}_p
\]
for some \( 1 < p \leq \infty \). Let \( \mathbf{u}_0 \) be the solution of the “homogenized” problem
\[
\overrightarrow{\mathbf{G}} \frac{\partial \mathbf{u}_0(\mathbf{x}, s)}{\partial s} = -\overrightarrow{\mathbf{B}}^0 \mathbf{u}_0(\mathbf{x}, s) + \mathbf{F}(\mathbf{x}, s), \quad s \in (0, T); \quad \overrightarrow{\mathbf{G}} \mathbf{u}_0(\mathbf{x}, 0) = \phi(\mathbf{x}).
\]
By analogy with the proof of representation (11.2), one can show that
\[
\mathbf{u}_e(\cdot, s) = f^e e^{-B^e s} (f^e)^* \phi + \int_0^s f^e e^{-B^e(s-s')} (f^e)^* \mathbf{F}(\cdot, s') d\bar{s},
\]
\[
\mathbf{u}_0(\cdot, s) = f_0 e^{-B^0 s} f_0 \phi + \int_0^s f_0 e^{-B^0(s-s')} f_0 \mathbf{F}(\cdot, s') d\bar{s}.
\]
The principal term of approximation for the solution \( \mathbf{u}_e \) was obtained in [M] Theorem 10.1:
\[
\| \mathbf{u}_e(\cdot, s) - \mathbf{u}_0(\cdot, s) \|_{L_2(\mathbb{R}^d)} \leq C_1 \varepsilon (s + \varepsilon^2)^{-1/2} e^{-\varepsilon s/2} \| \phi \|_{L_2(\mathbb{R}^d)} + C_{18} \theta_1(\varepsilon, p) \| \mathbf{F} \|_{\mathcal{H}_p},
\]
where the constant \( C_{18} \) depends only on \( p \) and the problem data (11.24),
\[
\theta_1(\varepsilon, p) = \begin{cases} 
\varepsilon^{2-2/p}, & 1 < p < 2, \\
\varepsilon (1 + |\ln \varepsilon|)^{1/2}, & p = 2, \\
\varepsilon, & 2 < p \leq \infty.
\end{cases}
\]

Repeating considerations of [V] Subsec. 10.1, from Theorem (10.2) and representations (11.8), (11.9) we derive the following result.

Theorem 11.2. Let \( \mathbf{u}_e \) and \( \mathbf{u}_0 \) be the solutions of problems (11.6) and (11.9), respectively, where \( \phi \in L_2(\mathbb{R}^d; \mathbb{C}^n) \) and \( \mathbf{F} \in \mathcal{H}_p \). Then for \( 0 < s < T \), \( 0 < \varepsilon \leq 1 \) and \( 2 < p \leq \infty \) we have
\[
\mathbf{u}_e(\cdot, s) = \mathbf{u}_0(\cdot, s) + \varepsilon \left( \Lambda^e_G b(D) \Pi e \mathbf{u}_0(\cdot, s) + \tilde{\Lambda}^e_G \Pi e \mathbf{u}_0(\cdot, s) \right.
\]
\[
+ f_0 e^{-B^0 s} f_0 ( (\Lambda^e_G b(D) \Pi e)^* + (\tilde{\Lambda}^e_G \Pi e)^*) \phi 
\]
\[
- \varepsilon \int_0^s f_0 e^{-B^0(s-s')} f_0 \mathbf{N} \Pi e f_0 e^{-B^0 s} f_0 \phi d\bar{s}
\]
\[
+ \varepsilon \int_0^s f_0 e^{-B^0(s-s')} f_0 ( (\Lambda^e_G b(D) \Pi e)^* + (\tilde{\Lambda}^e_G \Pi e)^*) \mathbf{F}(\cdot, s') d\bar{s}
\]
\[
- \varepsilon \int_{s-s'}^{s-s''} ds' f_0 e^{-B^0(s-s'-s'')} f_0 \mathbf{N} \Pi e f_0 e^{-B^0 s'} f_0 \mathbf{F}(\cdot, s) + \tilde{u}_e(\cdot, s''),
\]
where the remainder term \( \tilde{u}_e(\cdot, s) \) admits the estimate
\[
\| \tilde{u}_e(\cdot, s) \|_{L_2(\mathbb{R}^d)} \leq C_{15} \varepsilon^2 (s + \varepsilon^2)^{-1} e^{-\varepsilon s/2} \| \phi \|_{L_2(\mathbb{R}^d)} + C_{19} \varepsilon^{2/p} \theta_2(\varepsilon, p) \| \mathbf{F} \|_{\mathcal{H}_p},
\]
$p^{-1} + (p')^{-1} = 1$. Here the constant $C_{10}$ depends only on $p$ and the initial data, $\theta_2(\varepsilon, p) = 1$ for $p < \infty$ and $\theta_2(\varepsilon, p) = 1 + |\ln \varepsilon|$ for $p = \infty$.

For $0 < s < T$, $0 < \varepsilon \leq 1$, and $1 < p < 2$ we have the representation

$$u_\varepsilon(\cdot, s) = u_0(\cdot, s) + \varepsilon \left( A_0 b(D) \Pi_x f_0 e^{-B_0 s} f_0 \phi + \tilde{A}_0^\ast \Pi_x f_0 \phi \right)$$

$$+ f_0 e^{-B_0 s} f_0 \left( (A_0^\ast b(D) \Pi_x)^\ast + (\tilde{A}_0^\ast \Pi_x)^\ast \right) \phi$$

$$- \varepsilon \int_0^s f_0 e^{-B_0 (s-\tilde{s})} f_0 \Pi_x e^{-B_0 \tilde{s}} f_0 \phi d\tilde{s} + \tilde{u}_\varepsilon(\cdot, s),$$

and

$$||\tilde{u}_\varepsilon(\cdot, s)||_{L_2(\mathbb{R}^d)} \leq C_{15} \varepsilon^2 (s + \varepsilon^2)^{-1} e^{-\varepsilon s^{1/2}} ||\phi||_{L_2(\mathbb{R}^d)} + C_{18} \theta_1(\varepsilon, p)||F||_{H_p}.$$ 

Here $\theta_1(\varepsilon, p)$ is quantity $\|1.11\|$. 

Remark 11.3. The different approximations of the solution $u_\varepsilon$ for different conditions on $p$ in Theorem 11.2 are caused by the fact that the corrector term taken into account at the first summand in the right-hand side of (11.8) makes the approximation more precise compared to (11.10) for any $1 < p \leq \infty$, while the corrector taken into account at the second summand does the same only for $2 < p \leq \infty$.

12. Example of application of the general results: 

The scalar elliptic operator 

In the present section, we consider one of examples of application of the general method. For elliptic problems, this example was earlier studied in [Su4, Su7]. Other examples also can be found there.

12.1. The scalar elliptic operator. Consider the case, when $n = 1$, $m = d$, $b(D) = D$, and $g(x)$ is a $\Gamma$-periodic symmetric $(d \times d)$-matrix with the real entries, that is bounded and positive definite. Then the operator $A_\varepsilon$ has the form

$$A_\varepsilon = D^g(x)D = -\text{div} g^\varepsilon(x) \nabla.$$

Obviously, in the case under consideration, $\alpha_0 = \alpha_1 = 1$; see (11.2).

Next, let $A(x) = \text{col} \{ A_1(x), \ldots, A_d(x) \}$, where $A_j(x)$ are $\Gamma$-periodic real-valued functions, and

$$A_j \in L_2(\Omega), \quad q = 2 \text{ for } d = 1, \quad q > d \text{ for } d \geq 2; \quad j = 1, \ldots, d.$$ 

Let $v(x)$ and $\mathcal{V}(x)$ be real-valued $\Gamma$-periodic functions such that

$$v, \mathcal{V} \in L_1(\Omega), \quad \sigma = 1 \text{ for } d = 1, \quad \sigma > \frac{d}{2} \text{ for } d \geq 2; \quad \int_\Omega v(x) \, dx = 0.$$ 

In $L_2(\mathbb{R}^d)$, we consider the operator $\mathfrak{B}_\varepsilon$, formally given by the differential expression

$$\mathfrak{B}_\varepsilon = \langle D - A^\varepsilon(x) \rangle^\ast g^\varepsilon(x) \langle D - A^\varepsilon(x) \rangle + \varepsilon^{-1} v^\varepsilon(x) + \mathcal{V}^\varepsilon(x).$$ 

(12.1)

The precise definition of the operator $\mathfrak{B}_\varepsilon$ is given via the quadratic form

$$b_\varepsilon[u, u] = \int_{\mathbb{R}^d} \left( \langle g^\varepsilon(D - A^\varepsilon)u, (D - A^\varepsilon)u \rangle + \varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon \right) |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^d).$$

The operator (12.1) can be treated as a periodic Schrödinger operator with the metric $g^\varepsilon$, the magnetic potential $A^\varepsilon$, and the electric potential $\varepsilon^{-1} v^\varepsilon + \mathcal{V}^\varepsilon$ containing a “singular” first summand.

In [Su4, Subsec. 13.1], it was obtained that the operator (12.1) can be written in a required form

$$\mathfrak{B}_\varepsilon = D^g(x)D + \sum_{j=1}^d \langle a_j^\varepsilon(x)D_j + D_j(a_j^\varepsilon(x))^\ast \rangle + \mathcal{Q}(x).$$

The real-valued function $\mathcal{Q}(x)$ is defined as

$$\mathcal{Q}(x) = \mathcal{V}(x) + \langle g(x)A(x), A(x) \rangle.$$ 

(12.2)
The complex-valued functions $a_j(x)$ are given by
\[ a_j(x) = -\eta_j(x) + i\zeta_j(x), \] (12.3)
where $\eta_j(x)$ are the components of the vector-valued function $\eta(x) = g(x)A(x)$ and the functions $\zeta_j(x)$ are defined in terms of the $\Gamma$-periodic solution of the equation $\Delta \Phi(x) = v(x)$ by the relation $\zeta_j(x) = -\partial_j \Phi(x)$. Moreover,
\[ v(x) = -\sum_{j=1}^{d} \partial_j \zeta_j(x). \] (12.4)
It is easily seen that the functions (12.3) satisfy the assumption (4.10) for a suitable exponent $\tilde{\eta}$ (depending on $\varepsilon$ and $\sigma$); herewith, the norms $\|a_j\|_{L^p(\Omega)}$ are controlled in terms $\|g\|_{L^p(\Omega)}$, $\|A\|_{L^p(\Omega)}$, $\|v\|_{L^p(\Omega)}$, and the parameters of the lattice $\Gamma$. Function (12.2) is subject to condition (4.19) with a suitable factor $\tilde{\sigma} = \min\{\sigma; \varepsilon/2\}$. Thus, now Example 1.2 is implemented.

Let the parameter $\lambda$ be chosen in accordance with condition (1.19) with $c_0$ and $c_4$ corresponding to the operator (12.1). Denote $B_\varepsilon := B_\varepsilon + \lambda I$. We are interested in approximation of the exponential for this operator. In the case under consideration, initial data (12.1) match with the data:
\[ d, \sigma; \|g\|_{L^\infty(\Omega)}, \|g^{-1}\|_{L^\infty(\Omega)}, \|A\|_{L^p(\Omega)}, \|v\|_{L^p(\Omega)}, \|V\|_{L^p(\Omega)}, \lambda; \] the parameters of the lattice $\Gamma$. (12.5)

12.2. **The effective operator.** Write down the effective operator. In our case, the $\Gamma$-periodic solution of problem (7.1) is a row-matrix: $\Lambda(x) = i\Psi(x)$, $\Psi(x) = (\psi_1(x), \ldots, \psi_d(x))$, where $\psi_j \in \tilde{H}^1(\Omega)$ is the solution of the problem
\[ \text{div} \, g(x)(\nabla \psi_j(x) + e_j) = 0, \quad \int_{\Omega} \psi_j(x) \, dx = 0. \]
Here $e_j$, $j = 1, \ldots, d$, form the standard basis in $\mathbb{R}^d$. It is clear that the functions $\psi_j(x)$ are real-valued and the elements of the row-matrix $\Lambda(x)$ are purely imaginary. According to (7.6), the columns of the $(d \times d)$-matrix-valued function $\tilde{g}(x)$ are the vector-valued functions $g(x)(\nabla \psi_j(x) + e_j)$, $j = 1, \ldots, d$. The effective matrix is defined according to (7.6):
\[ g_0 = |\Omega|^{-1} \int_{\Omega} \tilde{g}(x) \, dx. \]
It is clear that $\tilde{g}(x)$ and $g_0$ have real entries.

According to (12.3) and (12.4), the periodic solution of problem (12.4) can be represented as $\tilde{\Lambda}(x) = \tilde{\Lambda}_1(x) + i\tilde{\Lambda}_2(x)$, where the real-valued $\Gamma$-periodic functions $\Lambda_1(x)$ and $\Lambda_2(x)$ are the solutions of the problems
\[ -\text{div} \, g(x)\nabla \Lambda_1(x) + v(x) = 0, \quad \int_{\Omega} \Lambda_1(x) \, dx = 0; \]
\[ -\text{div} \, g(x)\nabla \Lambda_2(x) + \text{div} \, g(x)A(x) = 0, \quad \int_{\Omega} \Lambda_2(x) \, dx = 0. \]
The column $V$ (see (7.8)) has the form $V = V_1 + iV_2$, where $V_1$, $V_2$ are the columns with real entries defined by
\[ V_1 = |\Omega|^{-1} \int_{\Omega} (\nabla \Psi(x))^t g(x) \nabla \tilde{\Lambda}_2(x) \, dx, \]
\[ V_2 = -|\Omega|^{-1} \int_{\Omega} (\nabla \Psi(x))^t g(x) \nabla \tilde{\Lambda}_1(x) \, dx. \]
According to (7.9), the constant $W$ can be written as
\[ W = |\Omega|^{-1} \int_{\Omega} \left( g(x)\nabla \Lambda_1(x), \nabla \Lambda_1(x) \right) + \left( g(x)\nabla \Lambda_2(x), \nabla \Lambda_2(x) \right) dx. \]
The effective operator for $B_\varepsilon$ acts by the rule
\[ B_\varepsilon^0 u = -\text{div} \, g^0 \nabla u + 2i(\nabla u, V_1 + \mathbf{\Phi}) + (-W + Q + \lambda) u, \quad u \in H^2(\mathbb{R}^d). \]
The corresponding differential expression can be written as follows
\[ B^0 = (D - A^0)^* g^0 (D - A^0) + V^0 + \lambda, \]
where
\[ A^0 = (g^0)^{-1} (V_1 + gA), \quad V^0 = V + (gA, A) - \langle g^0 A^0, A^0 \rangle - W. \]

12.3. The operator \( \mathcal{N} \). In the case under consideration, the structure of the operator \( \mathcal{N} \) was found in \( \text{[Bu7]} \) (9.12)–(9.15). We write down the result:
\[ \mathcal{N} = \sum_{k,l=1}^d N_{12,kl} D_k D_l + \sum_{k=1}^d N_{21,k} D_k + N_{22}, \]
where
\[ N_{12,kl} = 2 \Lambda_1 \tilde{g}_{kl} + \bar{v} \psi_k \psi_l - \sum_{j=1}^d (g_{jk} \psi_k + g_{jk} \psi_l) \partial_j \Lambda_1, \]
\[ N_{21,k} = 2 \sum_{j=1}^d g_{jk} (\tilde{A}_1 \partial_j \tilde{A}_2 - \tilde{A}_2 \partial_j \tilde{A}_1) \]
\[ + 2 \bar{v} \psi_k \langle \eta, \nabla \tilde{A}_1 \rangle - 2 \Lambda_1 \langle \eta, \nabla \psi_k \rangle + 2 \bar{v} \psi_k - 4 \eta_k \tilde{A}_1, \]
\[ N_{22} = 2 \Lambda_2 \langle \eta, \nabla \tilde{A}_1 \rangle - 2 \Lambda_1 \langle \eta, \nabla \tilde{A}_2 \rangle + \bar{v} \left( \tilde{A}_1^2 + \tilde{A}_2^2 \right) + 2 \Lambda_1 (Q + \lambda). \]

12.4. Approximation for the exponential. According to the equality \( \Lambda(x) = i \Psi(x) \), the operator \( \text{[10.16]} \) can be written as
\[ \mathcal{K}_0^0(s) = (\Psi^* \nabla + \tilde{\Lambda}^\epsilon) e^{-B^0 s} \epsilon^{B^0 s} (\Psi^* \nabla + \tilde{\Lambda}^\epsilon)^s - \int_0^s e^{-B^0 (s-\tilde{s})} \mathcal{N} e^{-B^0 \tilde{s}} \, d\tilde{s} \]
\[ = (\Psi^* \nabla + \tilde{\Lambda}^\epsilon) e^{-B^0 s} + e^{-B^0 s} (\Psi^* \nabla + \tilde{\Lambda}^\epsilon)^s - s \mathcal{N} e^{-B^0 s}. \]

We take into account that, in the our case, the scalar differential operator \( \mathcal{N} \) commutes with the exponential generated by the effective operator \( B^0 \) with constant coefficients.

From Theorem \( \text{[10.2]} \) we derive the following result.

**Theorem 12.1.** Let the operators \( \mathcal{B}_x \) and \( B^0 \) be defined in Subsec. \( \text{[12.1]} \) and \( \text{[12.2]} \), respectively, and let \( \mathcal{K}_0^0(s) \) be the corrector \( \text{[12.6]} \). Then for \( 0 < \epsilon \leq 1 \) and \( s > 0 \) we have the approximation
\[ \| e^{-B^0 s} - e^{-B^0 s} - \epsilon \mathcal{K}_0^0(s) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1 \epsilon \gamma s^{1 - \frac{1}{2}}. \]
The constants \( C_1 \) and \( C_\gamma \) depend only on the initial data \( \text{[12.5]} \).

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