QUANTUM SOLITON MASS CORRECTIONS IN SL(N) AFFINE TODA FIELD THEORY

TIMOTHY HOLLOWOOD*

Theoretical Physics, 1 Keble Road
Oxford, OX1 3NP, U.K.

ABSTRACT

The first quantum mass corrections for the solitons of complex $sl(n)$ affine Toda field theory are calculated. The corrections are real and preserve the classical mass ratios. The formalism also proves that the solitons are classically stable.

* Email: holl%dionysos.thphys@prg.oxford.ac.uk
Address after 1st October 1992: Theory Division, CERN, 1211 Geneva 23, Switzerland
1. Introduction

The equation of motion for the complex $sl(n)^{(1)}$ Toda field theory can be written

$$\Box \phi = -\frac{m^2}{i\beta} \sum_{j=1}^{n} \alpha_j \exp i\beta \alpha_j \cdot \phi. \quad (1.1)$$

The field $\phi(x, t)$ is an $n-1$-dimensional vector: an element of the Cartan subalgebra of the finite Lie algebra $sl(n)$. The inner products are taken with respect to the Killing form of $sl(n)$ restricted to the Cartan subalgebra. The $\alpha_j$’s, for $j = 1, 2, \ldots, n-1$ are the simple roots of $sl(n)$; $\alpha_0$ is the extended root (minus the highest root of $sl(n)$). The fact that the extended root is included in the sum, distinguishes the affine theories from the non-affine ones. The $\alpha_j$’s are linearly dependent:

$$\sum_{j=1}^{n} \alpha_j = 0. \quad (1.2)$$

The constant $m$ sets an arbitrary mass scale and $\beta$ is a coupling constant. In the complex theories $\beta$ is real, whereas, in the real theories $\beta$ is purely imaginary.

The real theories, when $\tilde{\beta} = i\beta \in \mathbb{R}$, are well understood, both in the classical and quantum regime (see for example [1]). The spectrum consists of $n-1$ particles of mass

$$m_a = 2m \sin \frac{\pi a}{n}, \quad a = 1, 2 \ldots, n-1. \quad (1.3)$$

In the quantum theory the spectrum is preserved, except for an overall renormalization of the mass scale $m$. It is known from a Feynman diagram calculation to one-loop [1], that

$$\hat{m}_a = m_a \left[ 1 - \tilde{\beta}^2 \frac{2}{4n} \cot \frac{\pi}{n} + O(\tilde{\beta}^4) \right]. \quad (1.4)$$

(In the following we shall use “hats” to denote exact quantum masses.) The complex theories, when $\beta \in \mathbb{R}$, have a much more complicated spectrum, since they admits kink or soliton solutions. This is well-known when $n = 2$, for which the real case is the sinh-Gordon theory and the complex case is the sine-Gordon theory. Soliton solutions were first written down in ref. [2] for the affine $sl(n)$ and $d_4$ theories; recently, soliton solutions have been found for all the affine Toda theories [3].
For the $sl(n)$ theories the general $N$-soliton solution is constructed in the following way. To each soliton one associates the data $\{\sigma, \lambda, a, \xi\}$, where $\sigma$ and $\lambda$ are real parameters satisfying

$$F(\sigma, \lambda, a) = 0,$$  \hspace{1cm} (1.5)

where the characteristic polynomial is

$$F(\sigma, \lambda, a) = \sigma^2 - \lambda^2 - 4m^2 \sin^2 \frac{\pi a}{n},$$  \hspace{1cm} (1.6)

$a$ is an integer in the set $\{1, 2, \ldots, n-1\}$ and $\xi$ is, for the moment, an arbitrary complex parameter. To each soliton, say the $p$th, one associates the $n$ functions

$$\Phi_j^{(p)}(x, t) = \sigma_p x - \lambda_p t + \frac{2\pi i}{n} a_p j + \xi_p, \hspace{1cm} j = 1, 2, \ldots, n,$$  \hspace{1cm} (1.7)

and to each soliton pair, say the $p$th and $q$th, one associates the "interaction function"

$$\exp \gamma^{(pq)} = -\frac{F(\sigma_p - \sigma_q, \lambda_p - \lambda_q, a_p - a_q)}{F(\sigma_p + \sigma_q, \lambda_p + \lambda_q, a_p + a_q)}.$$  \hspace{1cm} (1.8)

The general $N$-soliton solution to the equations of motion is

$$\phi = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_j \log \left[ \prod_{\mu_1=0}^{1} \cdots \prod_{\mu_N=0}^{1} \exp \left( \sum_{p=1}^{N} \mu_p \Phi_j^{(p)} + \sum_{1 \leq p < q \leq N} \mu_p \mu_q \gamma^{(pq)} \right) \right].$$  \hspace{1cm} (1.9)

So the one-soliton solution is

$$\phi(x, t) = -\frac{1}{i\beta} \sum_{j=1}^{n} \alpha_j \log \left[ 1 + \exp \left( \sigma x - \lambda t + \xi + \frac{2\pi i a}{n} j \right) \right],$$  \hspace{1cm} (1.10)

with

$$\sigma^2 - \lambda^2 = 4m^2 \sin^2 \frac{\pi a}{n}.$$  \hspace{1cm} (1.11)

The soliton is a kink, whose centre-of-mass is at $\sigma^{-1}[\lambda t - \text{Re} \xi]$, moving with velocity $\lambda/\sigma$ and having characteristic size $\sigma^{-1}$. The parameters $a$ and $\text{Im} \xi$ determine the topological charge of the soliton. The topological charge is defined to be

$$t = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial \phi}{\partial x},$$  \hspace{1cm} (1.12)

from which one readily verifies that (1.10) has a topological charge which a weight of the $a$th fundamental representation of $sl(n)$, where the exact weight is determined by $\text{Im} \xi$.  


The classical masses of the solitons are easily determined by explicit calculation from the Hamiltonian:

\[ H = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{m^2}{\beta^2} \sum_{j=1}^{n} \left[ \exp(i\beta \alpha_j \cdot \phi) - 1 \right]. \quad (1.13) \]

One finds

\[ M_a = \frac{4mn}{\beta^2} \sin \frac{\pi a}{n}, \quad a = 1, 2, \ldots, n - 1. \quad (1.14) \]

Notice that these masses are real, despite the complex form for the Hamiltonian. More particularly, notice that they are proportional to the fundamental masses of (1.3): \( M_a \propto m_a \). Both these facts were explained in [2].

In ref. [4] the quantum theory of the solitons was investigated. A form for the soliton-soliton S-matrix was proposed, which agrees with the WKB quantization of the classical scattering theory, discussed above, in the semi-classical limit. In general, it was shown that the S-matrix represents a non-unitary quantum field theory, but that there was a unitary region depending on the coupling constant, for which, in addition, the bootstrap closes on a finite set of states corresponding to solitons transforming in the fundamental representations of \( sl(n) \). Central to the proposal in ref. [4] for the quantum theory of the solitons, is the idea that the ratios of the soliton masses should not be altered in the quantum theory: \( \hat{M}_a \propto m_a \). It is clearly of interest to calculate the quantum corrections to the soliton masses directly, in order to verify the truth of this fact.

There exists a standard method for computing the first quantum corrections to soliton masses in a 1 + 1-dimensional field theory: one basically sums the zero-point energies of the fluctuations around the soliton solution. Obviously, such a sum is divergent, but these divergences may be removed, in the standard way, by renormalization. Moreover, there is a piece of folklore for 1+1-integrable theories that says that the masses of solitons obtained in this way are exact and there are no higher-order corrections. This is appears to be true for the sine-Gordon theory (the case for \( g = sl(2) \)), where the exact soliton mass is

\[ \hat{M} = \frac{8m}{\beta^2} - \frac{2m}{\pi}, \quad (1.15) \]

in our conventions — the first term being the classical contribution from (1.14), and the second term being the first quantum correction. This piece of folklore can, to some extent, be justified by noting that at some level the theory can be cast, via a transformation to action-angle variables, into a set of harmonic oscillators — albeit of infinite number — and
the saddle-point calculation of the soliton masses should, therefore, be exact. Although this argument has a certain verisimilitude, it is not clear how general the principle is; in particular, does it apply to Toda theories, beyond the sine-Gordon theory?

As a by-product of our calculation of the quantum corrections to the soliton masses, we are able to prove that the soliton solutions are classically stable; an issue that is important given the fact that the Hamiltonian (1.13) is complex.

2. Classical Stability

In this section, we consider the question of the classical stability of the one-soliton solutions. This question is closely related to the calculation of the quantum corrections to the soliton masses, as we shall see.

The standard way to approach the question of stability, is to consider the effect of adding a small perturbation $\eta$ to the one-soliton solution (1.10), which we denote $\bar{\phi}$. It is convenient to take the soliton to be stationary with the centre-of-mass at the origin: $\lambda = 0$, $\sigma = m_a$ and $\text{Re} \, \xi = 0$. To first order in $\eta$ the equation of motion (1.1) becomes

$$\Box \eta + m^2 \sum_{j=1}^{n} \alpha_j (\alpha_j \cdot \eta) \exp i \beta \alpha_j \cdot \bar{\phi} = 0.$$  \hfill (2.1)

Now consider a perturbation with a time-dependence of the form $\eta(x, t) = \tilde{\eta}(x) \exp i \nu t$, so that

$$D \tilde{\eta} = \nu^2 \tilde{\eta},$$  \hfill (2.2)

where we have defined the following differential operator:

$$D = -\frac{\partial^2}{\partial x^2} + m^2 \sum_{j=1}^{n} \alpha_j \otimes \alpha_j \exp i \beta \alpha_j \cdot \bar{\phi}(x).$$  \hfill (2.3)

The question of stability now boils down to showing that the spectrum of $D$ — for bounded eigenfunctions — is real and positive; hence, the frequencies $\nu$ are real and small perturbations to $\bar{\phi}$ do not diverge. Our strategy for proving this will be to find the exact spectrum of $D$.

First of all, the spectrum of $D$ is real, since it is actually a hermitian operator, with respect to some inner product. To show this, we first notice that the one-soliton solutions
— for some particular choice of \( \text{Im} \xi \) — actually satisfy a reality condition of the form

\[
\bar{\phi}^*(x) = -M \bar{\phi}(x),
\]  

(2.4)

where \( M \) acts as a \( \mathbb{Z}_2 \) symmetry of the roots \( \alpha_j \):

\[
M \alpha_j = \alpha_{m(j)}, \quad m^2(j) = j.
\]  

(2.5)

This is explained in detail in refs. [2,5]. From this it follows that

\[
D^\dagger = MD M.
\]  

(2.6)

Hence, \( D \) is hermitian with respect to the inner product

\[
(f, g) = \int_{-\infty}^{\infty} dx \, f^\dagger \cdot Mg,
\]  

(2.7)

for functions \( f(x) \) and \( g(x) \) in the Cartan subalgebra of \( sl(n) \).

We now turn to the determination of the spectrum of bounded eigenfunctions of \( D \). The eigenvalue equation (2.2) can be viewed as a multi-component Schrödinger problem for the potential

\[
V(x) = \sum_{j=1}^{n} \alpha_j \otimes \alpha_j \exp i\beta \alpha_j \cdot \bar{\phi}.
\]  

(2.8)

The spectrum has contributions from two sources: the bound states and the scattering solutions. The former have a discrete spectrum and the latter a continuous spectrum. There are also eigenfunctions which are not bounded, but these have no relevance for the discussion of stability and for the quantum mass corrections.

In general the scattering solutions have the following asymptotic behaviour

\[
\lim_{x \to -\infty} \tilde{\eta}(x) = A \exp i k x + B(k) \exp -i k x
\]

\[
\lim_{x \to \infty} \tilde{\eta}(x) = C(k) \exp i k x,
\]  

(2.9)

where the incoming, reflection and transmission coefficients, \( A, B(k) \) and \( C(k) \), are Cartan subalgebra-valued. Remarkably, the potential in (2.8) is “reflection-less”, so that \( B(k) = 0 \). This can be shown directly using the following argument. In the introduction, we wrote down an expression for the \( N \)-soliton solution. Consider a two-soliton solution where the first soliton is \( \bar{\phi} \), the static one-soliton solution that we started with, and the second soliton — the “probe” — will act as a small perturbation of the first. To this end, we take \( \sigma_2 = ik \),
\(a_2 = b, \lambda_2 = -i\nu\) and treat \(\exp \xi_2\) as a small parameter; in which case to first order in \(\exp \xi_2\) we deduce

\[
\tilde{\eta}_b(k; x) = \sum_{j=1}^{n} \alpha_j \left[ \frac{1 + \omega^{aj} \exp(m_a x + \xi)}{1 + \omega^{aj} \exp(m_a x + \xi)} \right] \omega^{bj} \exp ikx, \tag{2.10}
\]

where we have introduced \(\omega\), the primitive \(n^{th}\) root of unity. In the above, \(\gamma\) is the interaction parameter (1.8) which gives in this case

\[
\exp \gamma(k) = -\frac{m_a^2 + m_b^2 - m_{a-b}^2 - 2im_ak}{m_a^2 + m_b^2 - m_{a+b}^2 + 2imak}. \tag{2.11}
\]

The expression in (2.10) is the exact scattering solution to the Schrödinger problem (2.2) with eigenvalue \(\nu^2 = k^2 + m_b^2\) and

\[
B(k) = 0, \quad A = \sum_{j=1}^{n} \omega^{bj} \alpha_j, \quad C(k) = \exp \gamma(k) \sum_{j=1}^{n} \omega^{bj} \alpha_j. \tag{2.12}
\]

Moreover, the set of solutions \(\tilde{\eta}_b(k; x)\), for \(b = 1, 2, \ldots, n - 1\) and \(k \in \mathbb{R}\), forms a complete set of scattering solutions, since the vectors \(\sum_{j=1}^{n} \omega^{bj} \alpha_j\), for \(b = 1, 2, \ldots, n - 1\), span the Cartan subalgebra. Hence, (2.8) is a reflection-less potential.

The beauty of dealing with a reflection-less potential is that it allows for a simple determination of the bound-states as well. These solutions occur for values of \(k\) for which the transmission coefficient has a zero, and which are, in addition, normalizable. It is easy to see, from the explicit expression for \(\exp \gamma(k)\) in (2.11), that \(C(k)\) has a zero when

\[
k = \frac{m_a^2 + m_b^2 - m_{a-b}^2 - 2im_ak}{2im_a} = -im_b \cos \frac{\pi(a-b)}{n}. \tag{2.13}
\]

The exact solution for the bound-states follows from (2.10) and the particular value for \(k\) in (2.13):

\[
\tilde{\eta}_b(x) = \sum_{j=1}^{n} \alpha_j \omega^{bj} \exp \left[ m_b x \cos \frac{\pi(a-b)}{n} \right], \tag{2.14}
\]

which has a frequency

\[
\nu = m_b \left| \sin \frac{\pi(a-b)}{n} \right|. \tag{2.15}
\]

The asymptotic limits of the above solution are

\[
\lim_{x \to -\infty} \tilde{\eta}(x) = \sum_{j=1}^{n} \alpha_j \omega^{bj} \exp \left[ 2m_x \sin \frac{\pi b}{n} \cos \frac{\pi(a-b)}{n} \right]
\]

\[
\lim_{x \to \infty} \tilde{\eta}(x) = \sum_{j=1}^{n} \alpha_j \omega^{(b-a)j} \exp \left[ 2m_x \sin \frac{\pi(a-b)}{n} \cos \frac{\pi b}{n} - \xi \right]. \tag{2.16}
\]
Of course, not all these solutions have the right asymptotics to be \textit{bona-fide} bound-states of the Schrödinger problem. The \textit{bona-fide} bound-states must be normalizable which means

$$\lim_{x \to \pm \infty} \tilde{\eta}(x) \sim \exp \mp |\kappa^\pm| x,$$

(2.17)

for some real non-zero constants \(\kappa^\pm \neq 0\). The values of \(b\) which lead to \textit{bona-fide} bound-states are

$$1 \leq b < a \quad \text{or} \quad \frac{n}{2} < b < \frac{n}{2} + a, \quad \text{for } a \leq \frac{n}{2}$$

$$a - \frac{n}{2} < b < \frac{n}{2} \quad \text{or} \quad a < b \leq n - 1, \quad \text{for } a \geq \frac{n}{2}. \quad (2.18)$$

The solution with \(b = a\) has not been included since this corresponds to the zero-mode of \(D\) given by

$$\tilde{\eta}_a(x) = \sum_{j=1}^{n} \alpha_j \frac{\omega^a_j \exp m_a x}{1 + \omega^a_j \exp (m_a x + \xi)}. \quad (2.19)$$

The appearance of this zero-mode was only to be expected since it is proportional to \(\partial \tilde{\phi} / \partial x\), and is due to the freedom to shift the centre-of-mass of the soliton.

In addition to these bound-states there are bounded solutions which must also be considered. Such eigenfunctions have a constant asymptote, and hence are not normalizable. They appear only when \(n\) is even and are given by (2.14) when \(b = \frac{n}{2}\) or \(b = a + \frac{n}{2}\), if \(a < \frac{n}{2}\), and \(b = a - \frac{n}{2}\) or \(b = \frac{n}{2}\), if \(a > \frac{n}{2}\).

From the preceding analysis, we conclude that the spectrum of bounded eigenfunctions of \(D\) is real and positive, except for the zero-mode which reflects the freedom to move the centre-of-mass of the soliton. Hence the one-soliton solutions are classically stable to small perturbations.

### 3. Quantum Corrections to the Masses

In this section we calculate the first quantum corrections to the soliton masses. We employ the semi-classical WKB method described in [6] for the kinks, or solitons, of the \(\phi^4\) and sine-Gordon theories — see also the review in [7]. The method is completely standard; however, we feel that the Toda theories are sufficiently more complicated than these other examples that it is worthwhile including all the calculational details. In particular, the question of boundary conditions is not completely standard.
The dimensionless expansion parameter in our theory is $\hbar \beta^2 / m^2$, and so the expansion in $\hbar$ coincides with the weak-coupling expansion. From now on we set $\hbar = 1$. The WKB method of [6] is straightforward to apply in the present situation since the one-soliton solution is, in its rest-frame, time-independent. The idea is to compute the zero-point energy of the small oscillations around the classical solution. The sum over all the modes can be done when an infra-red regulator is introduced; this is most easily achieved by putting the theory in a box with rigid boundary conditions. The zero-point energy of the vacuum must then be subtracted. The resulting expression is independent of the size of the box, $L$, as $L \to \infty$; however, the final expression has an ultra-violet divergence. This divergence is not a problem of the soliton solution per se, rather, it is just a manifestation of the fact that the bare-mass needs to be renormalized. This is achieved simply by normal-ordering the Hamiltonian. The extra correction removes the divergence and the resulting finite residue which remains is then the mass correction.

The small fluctuations which contribute to the mass shift come from either the bound-states or the scattering solutions of the linearized equation of motion around the soliton solution: the Schrödinger problem (2.2).

3.1 Contributions to the Mass from the Bound-States

Each of the bound-states $\tilde{\eta}_b(x)$, with $b$ in the range (2.18), contributes an amount $\frac{1}{2} \nu$ to the mass shift, where $\nu$ is given by (2.15). In addition to this, the bounded solutions — which only occur for $n$ even — also contribute to the mass shift. However, we shall not include the states (2.14) with $b = a + \frac{n}{2}$, for $a < \frac{n}{2}$, and $b = a - \frac{n}{2}$, for $a > \frac{n}{2}$, which correspond to the $k = 0$ solutions of (2.10), and whose contribution to the mass shift will be included in the contribution from the scattering states discussed in the next subsection. So the contributing modes of the form (2.14) are

\[
1 \leq b < a \quad \text{or} \quad \frac{n}{2} \leq b < \frac{n}{2} + a, \quad \text{for} \ a < \frac{n}{2}
\]

\[
a - \frac{n}{2} < b \leq \frac{n}{2} \quad \text{or} \quad a < b \leq n - 1, \quad \text{for} \ a > \frac{n}{2}
\]

\[
1 \leq b < \frac{n}{2} \quad \text{or} \quad \frac{n}{2} < b \leq n - 1, \quad \text{for} \ a = \frac{n}{2}.
\]

The contribution to the mass shift from these states, for $a < \frac{n}{2}$, is

\[
\Delta M_a(1) = m \left[ \sum_{b=1}^{a-1} \sin \frac{\pi b}{n} \sin \frac{\pi (a - b)}{n} - \sum_{b \geq \frac{n}{2}}^{\frac{n}{2} + a} \sin \frac{\pi b}{n} \sin \frac{\pi (a - b)}{n} \right]. \tag{3.2}
\]
(Notice, that the expression is also valid for \( a = \frac{n}{2} \), since in that case the term with \( b = \frac{n}{2} \) does not contribute.) It is straightforward to evaluate the sums involved, giving

\[
\Delta M_a(1) = \begin{cases} 
\frac{1}{2} m_a \left( \cot \frac{\pi}{n} + \cosec \frac{\pi}{n} \right) & n \in \text{Odd} \\
\frac{1}{2} m_a \cot \frac{\pi}{n} & n \in \text{Even}.
\end{cases} \tag{3.3}
\]

It is not difficult to show that the cases when \( a > \frac{n}{2} \) give the same results as (3.3), as a function of \( a \).

### 3.2 Contributions to the Mass from the Scattering States

In this subsection, we calculate the contribution of the scattering solutions of (2.2) to the mass correction. The idea is to sum the zero-point energies of all the scattering modes. Such a sum will, of course, be infra-red divergent, so first of all it is necessary to introduce some finite boundary conditions so that the modes become discrete and therefore enumerable. Usually, periodic boundary conditions are chosen; however, they are not appropriate here because \( \gamma(k) \) in (2.11) is not purely imaginary. It turns out that the appropriate boundary conditions are \( \tilde{\eta}(x=0) = \tilde{\eta}(x=L) = 0 \), where the centre-of-mass of the soliton lies somewhere inside the box, and \( L \) is much larger than the size of the soliton: \( L \gg m_a^{-1} \). The solutions satisfying these boundary conditions are

\[
\tilde{\eta}_b(k_p; x) - \tilde{\eta}_b(-k_p; x),
\]

where \( \tilde{\eta}_b(k; x) \) is the solution in (2.10), with

\[
k_p L + \rho_b(k_p) = \pi p, \quad k_p \geq 0, \quad p \in \mathbb{Z}.
\]

(3.5)

where \( \rho_b(k) = \text{Im} \gamma(k) \) and \( \gamma(k) \) is defined in (2.11).

The quantum correction to the soliton mass is then obtained by taking the zero-point energy of the modes around the soliton solution and subtracting the zero-point energy of modes around the vacuum, which satisfy (3.5) with \( \rho = 0 \). That is

\[
\Delta M_a(2) = \frac{1}{2} \sum_{b=1}^{n-1} \sum_{k_p \geq 0} \left\{ \sqrt{k_p^2 + m_b^2} - \sqrt{[k_p + \rho_b(k_p)/L]^2 + m_b^2} \right\},
\]

(3.6)

where the sum over \( k_p \) is the sum over distinct solutions of (3.5). Since \( L \) will eventually be taken to infinity, we can expand in \( L^{-1} \), the leading term in the sum being

\[
-L^{-1} \frac{k \rho_b(k)}{\sqrt{k^2 + m_b^2}} = -L^{-1} \rho_b(k) \frac{d \rho_b(k)}{dk},
\]

(3.7)
with \( \epsilon_b(k) = \sqrt{k^2 + m_b^2} \). Now, we take \( L \to \infty \) in which case we can replace the sum over \( k_p \) by an integral over \( k \):

\[
\sum_{p \geq 0} = L \left[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} + O(L^{-1}) \right],
\]

where we have used the fact that the integrand is an even function of \( k \). Therefore, the mass correction is

\[
\Delta M_a = -\sum_{b=1}^{n-1} \int_{-\infty}^{\infty} \frac{dk}{4\pi} \rho_b(k) \frac{d\epsilon_b(k)}{dk} = -\frac{1}{4\pi} \sum_{b=1}^{n-1} \left[ \rho_b(k)\epsilon_b(k) \bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dk \epsilon_b(k) \frac{d\rho_b(k)}{dk} \right].
\]

(3.9)

Now as \( k \to \pm\infty \), \( \epsilon_b(k) \to |k| \) and

\[
\rho_b(k) \to \frac{1}{2m_ak} \left( 2m_a^2 + 2m_b^2 - m_{a-b}^2 - m_{a+b}^2 \right) + \pi.
\]

(3.10)

and thus the first contribution from (3.9) is

\[
-\frac{1}{4\pi} \sum_{b=1}^{n-1} \rho_b(k)\epsilon_b(k) \bigg|_{-\infty}^{\infty} = -\frac{1}{4\pi m_a} \sum_{b=1}^{n-1} \left( 2m_a^2 + 2m_b^2 - m_{a-b}^2 - m_{a+b}^2 \right).
\]

(3.11)

The sum is straightforward to perform, the result being a contribution

\[
-\frac{m_an}{2\pi},
\]

(3.12)

to the mass. As it stands, the second contribution in (3.9) is not well defined because the integral has a logarithmic divergence. The problem is not due to any special nature of the soliton solution, indeed it is to be expected, being due to the fact that we have not renormalized the theory. We now pause to consider this aspect in more detail.

Mercifully, the renormalization process in a two-dimensional field theory is straightforward. The divergences can simply be removed by normal-ordering the Hamiltonian. Working to lowest order in \( \beta^2 \) and introducing an ultra-violet cut-off \( \Lambda \)

\[
: \exp i\beta \alpha_j \cdot \phi := \exp i\beta \alpha_j \cdot \phi \left[ 1 + \frac{1}{2} \beta^2 \sum_{a=1}^{n-1} (\zeta_a \cdot \alpha_j)^2 \Delta_a + O(\beta^4) \right],
\]

(3.13)

where

\[
\Delta_a = \int_{-\Lambda}^{\Lambda} \frac{dk}{4\pi} \frac{1}{\sqrt{k^2 + m_a^2}}.
\]

(3.14)
and $\zeta_a$ is the $a^{th}$ eigenvector of $\sum_{j=1}^{n} \alpha_j \otimes \alpha_j$ of eigenvalue $(m_a/m)^2$. Hence, to lowest order in $\beta^2$, the renormalized Hamiltonian is equal to

$$H_{\text{ren}} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{\beta^2} \sum_{j=1}^{n} (m^2 + \partial m_j^2) \left[ \exp (i\beta \alpha_j \cdot \phi) - 1 \right], \quad (3.15)$$

with

$$\partial m_j^2 = \frac{1}{2} (m\beta)^2 \sum_{a=1}^{n-1} (\zeta_a \cdot \alpha_j)^2 \Delta_a. \quad (3.16)$$

The presence of the additional terms clearly changes the energy of the soliton solution. These changes must be added to the quantum correction to the soliton mass. It might be thought that we should now consider the one-soliton solution for the renormalized equations of motion; fortunately, this is unnecessary since the correction due to change in the form of the solution does not occur at the lowest order in $\beta^2$, precisely because the soliton solution satisfies the “bare” equations of motion (1.1).

Taking into account the counter-term from the renormalization, the final expression for the mass correction for the $a^{th}$ soliton from the continuum solutions is

$$\Delta M_a(2) = - \frac{m_a n}{2\pi} + \sum_{b=1}^{n-1} \int_{-\Lambda}^{\Lambda} \frac{dk}{4\pi} \epsilon_b(k) \frac{d\rho_b(k)}{dk}$$

$$- \frac{1}{\beta^2} \sum_{j=1}^{n} \int_{-\infty}^{\infty} dx \left[ \exp (i\beta \alpha_j \cdot \bar{\phi}) - 1 \right] \partial m_j^2. \quad (3.17)$$

We now pause to calculate the integral

$$\int_{-\infty}^{\infty} dx \left[ \exp (i\beta \alpha_j \cdot \bar{\phi}) - 1 \right]. \quad (3.18)$$

Using the explicit form for the stationary one-soliton solution, one finds

$$\exp (i\beta \alpha_j \cdot \bar{\phi}) - 1 = - \left( \frac{m_a}{m} \right)^2 \frac{\omega^{a_j} \exp (m_a x + \xi)}{[1 + \omega^{a_j} \exp (m_a x + \xi)]^2}. \quad (3.19)$$

The integral (3.18) may now be calculated explicitly yielding

$$\left( \frac{m_a}{m} \right)^2 \frac{1}{m_a [1 + \omega^{a_j} \exp (m_a x + \xi)]} \bigg|_{-\infty}^{\infty} = - \frac{m_a}{m^2}. \quad (3.20)$$
The important point about the resulting expression for the integral is that it is independent of \( j \). The contribution to the mass correction (3.17) from the renormalization counter-term can now be simplified:

\[
-\frac{1}{\beta^2} \sum_{j=1}^{n} \int_{-\infty}^{\infty} dx \left[ \exp(i\beta \alpha_{j} \cdot \vec{\phi}) - 1 \right] \partial m_{j}^2
\]

\[
= \frac{1}{2} m_{a} \sum_{b=1}^{n-1} \sum_{j=1}^{n} (\zeta_{b} \cdot \alpha_{j})^2 \Delta_{b}
\]

\[
= \frac{1}{2} m_{a} \sum_{b=1}^{n-1} \left( \frac{m_{b}}{m} \right)^2 \int_{-\Lambda}^{\Lambda} \frac{dk}{4\pi} \frac{1}{\sqrt{k^2 + m_{b}^2}},
\]

where we used the fact that \( \zeta_{b} \) is an eigenvector of \( \sum_{j=1}^{n} \alpha_{j} \otimes \alpha_{j} \) of eigenvalue \( (m_{b}/m)^2 \).

The expression for the mass correction is now

\[
\Delta M_{a}(2) = -\frac{m_{a} n}{2\pi} + m_{a} \sum_{b=1}^{n-1} \int_{-\Lambda}^{\Lambda} \frac{dk}{4\pi} \left\{ \frac{1}{2} \left( \frac{m_{b}}{m} \right)^2 \frac{1}{\sqrt{k^2 + m_{b}^2}} - 4\sqrt{k^2 + m_{b}^2} \frac{m_{a}^2 + m_{b}^2 - m_{a+b}^2}{(m_{a}^2 + m_{b}^2 - m_{a+b}^2)^2 + 4m_{a}^2 k^2} \right\}.
\]

Using the fact that

\[
\sum_{b=1}^{n-1} \left( \frac{m_{b}}{m} \right)^2 = \text{Tr} \left( \sum_{j=1}^{n} \alpha_{j} \otimes \alpha_{j} \right) = \sum_{j=1}^{n} \alpha_{j} \cdot \alpha_{j} = 2n,
\]

one can easily verify that for large \( |k| \) the integrand behaves as \( k^{-2} \) and so, as required, the logarithmic divergence is cancelled by the renormalization counter-term. We can now let the ultra-violet cut-off \( \Lambda \) tend to infinity.

The remaining contribution is given in terms of a convergent integral:

\[
\Delta M_{a}(2) = m_{a} \left[ -\frac{n}{2\pi} + I(a, n) \right],
\]

where

\[
I(a, n) = \int_{-\infty}^{\infty} \frac{dk}{4\pi} \sum_{b=1}^{n-1} \left\{ -4\sqrt{k^2 + m_{b}^2} \frac{m_{a}^2 + m_{b}^2 - m_{a+b}^2}{(m_{a}^2 + m_{b}^2 - m_{a+b}^2)^2 + 4m_{a}^2 k^2} + \frac{1}{2} \left( \frac{m_{b}}{m} \right)^2 \frac{1}{\sqrt{k^2 + m_{b}^2}} \right\}.
\]
Unfortunately, we have not managed to evaluate the integral \( I(a, n) \) analytically; except when \( n = 2 \) in which case the integrand is zero. However, by evaluating the integral numerically for a range of \( a \) and \( n \) we find, to a very high degree of accuracy, that, firstly, the integral is independent of \( a \), and secondly it is equal to the following functional form [8]:

\[
I(a, n) = \begin{cases} 
-\frac{1}{2} \csc \frac{n \pi}{n} & n \in \text{Odd} \\
-\frac{1}{2} \cot \frac{n \pi}{n} & n \in \text{Even}.
\end{cases}
\] (3.27)

We can put the two contributions from the bound-states and the continuum together to arrive at our main result:

\[
\Delta M_a = \Delta M_a(1) + \Delta M_a(2) = m_a \left( -\frac{n}{2\pi} + \frac{1}{2} \cot \frac{\pi}{n} \right).
\] (3.28)

Notice that the final result is valid for \( n \) even or odd: the asymmetry has cancelled out on adding the two contributions.

4. Discussion

We have succeeded in calculating the first quantum corrections to the solitons masses in complex \( sl(n) \) affine Toda field theories; and as a by-product proving that the solitons are classically stable. Although the calculations were somewhat lengthy the result is simple:

\[
\hat{M}_a = 2nm_a \left[ \frac{1}{\beta^2} - \frac{1}{4\pi} + \frac{1}{4n} \cot \frac{\pi}{n} + O(\beta^2) \right],
\] (4.1)

where the first term is the classical mass. For the sine-Gordon theory it is thought that the result is exact and there are no higher order corrections; this example is recovered by setting \( n = 2 \), giving

\[
\hat{M} = \frac{8m}{\beta'^2},
\] (4.2)

where \( \beta' \) is the characteristic “renormalized” coupling

\[
\beta'^2 = \frac{\beta^2}{1 - \beta^2/4\pi}.
\] (4.3)

In the sine-Gordon theory it is well-known that the fundamental particle is, in the quantum theory, a soliton anti-soliton bound-state. However, it has been conjectured in ref. [4] that the same is true in the more general theories. The soliton-soliton \( S \)-matrix written down
in ref. [4] has poles corresponding to soliton bound states; in particular there are \( n - 1 \) distinct sets of scalar states, carrying zero topological charge. The ground states of these sets, have mass

\[
\hat{m}_a = 2\hat{M}_a \sin \frac{\pi}{n\lambda}, \quad a = 1, 2, \ldots, n - 1,
\]

(4.4)

where \( \hat{M}_a \) is the soliton mass and \( \lambda \) is a coupling constant which is known to be related to \( \beta \) as [4]

\[
\lambda = \frac{4\pi}{\beta^2} + \tilde{\lambda},
\]

(4.5)

where \( \tilde{\lambda} \) is \( O(1) \), but was not determined in ref. [4]. By using (4.1) and (4.5) we can expand (4.4) to first order:

\[
\hat{m}_a = m_a \left[ 1 + \left( \frac{1}{4n} \cot \frac{\pi}{n} - \frac{1}{4\pi} - \frac{\tilde{\lambda}}{4\pi} \right) \beta^2 + O(\beta^4) \right].
\]

(4.6)

But (4.6) should be compared with the one-loop formula for the masses of the fundamental particles in (1.4) (with \( \tilde{\beta} = i\beta \)). They are consistent if \( \tilde{\lambda} = -1 \) so

\[
\lambda = \frac{4\pi}{\beta'^2},
\]

(4.7)

to this order in \( \beta \), where \( \beta' \) is the shifted coupling appearing in the sine-Gordon theory (4.3).

It would be interesting to investigate the question as to whether the masses in (4.1) are actually exact. One way to do that would be to repeat the analysis of ref. [6], and calculate the corrections to the masses of the breathers.

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