Generalized CCR Flows

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Abstract: We introduce a new construction of $E_0$-semigroups, called generalized CCR flows, with two kinds of descriptions: those arising from sum systems and those arising from pairs of $C_0$-semigroups. We get a new necessary and sufficient condition for them to be of type III, when the associated sum system is of finite index. Using this criterion, we construct examples of type III $E_0$-semigroups, which cannot be distinguished from $E_0$-semigroups of type I by the invariants introduced by Boris Tsirelson. Finally, by considering the local von Neumann algebras, and by associating a type III factor to a given type III $E_0$-semigroup, we show that there exist uncountably many type III $E_0$-semigroups in this family, which are mutually non-cocycle conjugate.

1. Introduction

An $E_0$-semigroup is a weakly continuous semigroup of unital $*$-endomorphisms on $B(H)$, the algebra of all bounded operators on a separable Hilbert space $H$. $E_0$-semigroups are classified into three broad categories, namely type I, II, and III, depending upon the existence of intertwining semigroups called units. William Arveson completely classified the $E_0$-semigroups of type I, by showing that the CCR flows exhaust all the type I $E_0$-semigroups, up to the identification of cocycle conjugacy ([5]). But the theory of $E_0$-semigroups belonging to type II and type III remained mysterious for quite some time. There is no hope of completely classifying the whole class of $E_0$-semigroups even now, basically due to the presence of type II and type III examples.

For quite some time there were essentially only one example for each type II and type III $E_0$-semigroups, due to R. T. Powers ([14,15]). In this context Boris Tsirelson produced uncountable families of both type II and type III $E_0$-semigroups by using measure type spaces arising from several models in probability theory ([18]). It is equivalent to

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study the product systems of Hilbert spaces, a complete invariant introduced by Arveson, in order to understand the associated $E_0$-semigroups. Tsirelson basically produced uncountable families of both type II and type III product systems of Hilbert spaces.

Tsirelson’s construction of type III product systems uses off white noises, which are Gaussian generalized (i.e. distribution valued) processes with a slight correlation between “past and future”. After discussing Tsirelson’s results, Arveson concludes his book ([5]) by saying, ‘It is clear that we have not achieved a satisfactory understanding of the existence of ‘logarithms’ in the category of product systems.’ This was clarified for Tsirelson’s type III examples in [6] from the viewpoint of operator algebras. Inspired by the results of Tsirelson, a purely operator algebraic construction of Tsirelson’s type III examples was provided in [6]. They were called ‘product systems arising from sum systems’. In particular, a dichotomy result about types was proved in [6], namely, it was proved that the product system arising from a divisible sum system is either of type I or of type III. A sum system is said to be divisible if it has sufficiently many real and imaginary addits (called additive units in [6]).

On the other hand, motivated by Tsirelson’s construction of type III examples, a class of $C_0$-semigroups acting on $L^2(0, \infty)$, which are Hilbert-Schmidt perturbations of the unilateral shift semigroup, was investigated in [10]. Description of such semigroups in terms of analytic functions on the right-half plane was given and several examples were constructed.

In this paper we discuss the consequences of these developments. First we describe the $E_0$-semigroups associated with the above mentioned product systems arising from sum systems. This would generalize the simplest kind of $E_0$-semigroups called CCR flows. These generalized CCR flows are given by a pair of $C_0$-semigroups. By studying the product system we get a new necessary and sufficient condition for the $E_0$-semigroup to be of type III, when the associated sum system is of finite index. This criterion is much more powerful than the sufficient condition already proved in [6].

Using the results proved in [10], we compute the additive cocycles for the pairs of the shift semigroup and its perturbations, and show that the sum systems associated with them are always divisible. This class of sum systems include those coming from off white noises of Tsirelson. Then we concentrate on a special subclass of $C_0$-semigroups, which give rise to new type III $E_0$-semigroups. These new examples cannot be distinguished from type I $E_0$-semigroups by the invariants introduced by Tsirelson, and later discussed in [6]. Let us give an intuitive explanation of this phenomenon in terms of Tsirelson’s off white noise picture here. Although our examples also come from Tsirelson’s off white noises, spectral density functions for them tend to 1 at infinity. This means that our off white noises are so close to white noise that Tsirelson’s invariant can not work for them.

Finally, we associate a type III factor as an invariant to each of these type III $E_0$-semigroups, and using that we prove that there are uncountably many examples in this family which are not cocycle conjugate to each other. Toeplitz operator plays an essential role throughout these discussions.

We end this section by reviewing some of the very basic definitions about $E_0$-semigroups. For the definitions of notions related to $E_0$-semigroups, such as cocycle conjugacy, index etc., we refer to [5]. A unit for an $E_0$-semigroup $\{\alpha_t\}$ acting on $\mathbb{B}(H)$ is a strongly continuous semigroup of bounded operators $\{T_t\}$, which intertwines $\alpha$ and the identity, that is

$$\alpha_t(X)T_t = T_t X, \quad \forall \ A \in \mathbb{B}(H), \ t \geq 0.$$