Abstract

In [14], the author derived an exact rotation-strain model in two dimensions for the motion of incompressible viscoelastic materials via the polar decomposition of the deformation tensor. Based on the rotation-strain model, the author constructed a family of large global classical solutions for the 2D incompressible viscoelasticity. To get such a global well-posedness result, the equation for the rotation angle was essential to explore the underlying weak dissipative structure of the whole viscoelastic system even though the momentum equation for the velocity field and the transport equation for the strain tensor have already formed a closed subsystem. In this paper, we revisit such a result without making use of the equation of the rotation angle. The proof relies on a new identity satisfied by the strain matrix. The smallness assumptions are only imposed on the $H^2$ norm of initial velocity field and the initial strain matrix, which implies that the deformation tensor is allowed being away from the equilibrium of 2 in the maximum norm.

Keyword: Rotation-strain model, viscoelastic fluids, partial dissipation, weak dissipative mechanism.

1 Introduction

In this paper, we revisit the two dimensional rotation-strain model for the incompressible viscoelastic fluid flows:

$$
\begin{align*}
\nabla \cdot u &= 0, \\
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \mu \Delta u + \nabla \cdot (VV^T) + 2 \nabla \cdot V, \\
\frac{\partial V}{\partial t} + u \cdot \nabla V &= S(u) + \frac{1}{2}(\nabla u V + V \nabla u^T) + \frac{1}{2} \omega_{12}(u)(VA - AV) - \frac{1}{2} \gamma (VA - AV), \\
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta &= -\omega_{12}(u) + \gamma.
\end{align*}
$$

(1.1)
Here $u(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity field, $p(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is the scalar pressure, $V$ and $\theta$ are the strain tensor and the rotation angle of the deformation tensor, $\mu > 0$ is the viscosity coefficient and $A$ is a constant matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

Besides, $\omega(u)$ denotes the vorticity tensor:

$$\omega(u) = \frac{1}{2}(\nabla u - \nabla u^T), \quad \omega_{12}(u) = \frac{1}{2}(\partial_2 u_1 - \partial_1 u_2),$$

$S(u)$ denotes the symmetric part of the gradient of velocity

$$S(u) = \frac{1}{2}(\nabla u + \nabla u^T),$$

and $\gamma$ is given by

$$\gamma = \frac{1}{2 + \text{tr}V}[\text{tr}V\omega_{12}(u) - (\nabla_k u_1 V_k - \nabla_k u_2 V_k)]. \tag{1.3}$$

System (1.1) was derived in [14] from one of the most basic macroscopic models for viscoelastic flows

$$\begin{cases} \nabla \cdot u = 0, \\
u_t + u \cdot \nabla u + \nabla p = \mu \Delta u + \nabla \cdot (F F^T), \\
F_t + u \cdot \nabla F = \nabla u F, \end{cases} \tag{1.4}$$

via the rotation-strain decomposition of the deformation tensor

$$F = (I + V) R, \tag{1.5}$$

where $R$ is orthogonal

$$RR^T = I$$

and $I + V$ is positive definite and symmetric

$$V = V^T.$$

Physically, this means that the deformation is decomposed into stretching and rotation. Following a suggestion by K. O. Friedrichs [6], the tensor $I + V$ is called the left stretch tensor, $V$ the strain matrix and $R$ the rotation matrix. We refer to [7] [23] for detailed discussions on the relevant physical background of system (1.4) and [14] for the derivation of (1.1).
Note that the momentum equation for the velocity field \( u \) and the transport equation for the strain tensor \( V \) in (1.1) have already formed a closed subsystem:

\[
\begin{aligned}
\begin{cases}
    u_t + u \cdot \nabla u + \nabla p = \Delta u + \nabla \cdot (VV^T) + 2\nabla \cdot V, \\
    V_t + u \cdot \nabla V = S(u) + \frac{1}{2}(\nabla u V + V \nabla u^T) \\
    \quad + \frac{1}{2} \omega_{12}(u)(VA - AV) - \frac{1}{2} \gamma(VA - AV), \\
    \nabla \cdot u = 0.
\end{cases}
\end{aligned}
\]  

We have set the viscosity \( \mu \) to be one in the above system since we are not concerned the limit \( \mu \to 0^+ \) here. For the initial data

\[
u(0, x) = u_0(x), \quad V(0, x) = V_0(x), \quad x \in \mathbb{R}^2,
\]

we impose the following constraints

\[
\begin{aligned}
\begin{cases}
    \nabla \cdot u_0 = 0, \\
    \det(I + V_0) = 1, \\
    \nabla \cdot V_0 = A(I + V_0)\nabla \theta_0.
\end{cases}
\end{aligned}
\]  

The main result of this paper is the following theorem:

**Theorem 1.1.** There exist two positive absolute constants \( C > 1 \) and \( \epsilon_0 < 1 \) such that there exists a unique global classical solution \( (u, V) \) to the Cauchy problem for the rotation-strain viscoelastic model (1.6) and (1.7) with the intrinsic physical constraints (1.8) on the initial data provided that \( u_0, V_0 \in H^2 \) and

\[\|u_0\|_{H^2}^2 + \|V_0\|_{H^2}^2 < \epsilon_0.\]

Moreover, the solution \( (u, V) \) satisfies the following a priori estimate:

\[
\sup_{t \geq 0} \left( \|u(t, \cdot)\|_{H^2}^2 + \|V(t, \cdot)\|_{H^2}^2 \right) + \int_0^\infty \left( \|\nabla u\|_{H^2}^2 + \|\nabla V\|_{H^1}^2 \right) dt \leq C(\|u_0\|_{H^2}^2 + \|V_0\|_{H^2}^2).
\]

**Remark 1.2.** 1). The constraints on the initial data in (1.8) are all the consequences of the incompressibility. See [14] for more details. 2). The result is also true in the two-dimensional torus case and bounded domain case. The periodic domain case can be treated similarly as in this paper. The bounded domain case is more involved. One can follow, for instance, the method in [21]. 3). Even though we don’t utilize the equation for the rotation angle \( \theta \), there is still a smallness constraint on the norm \( \|\nabla \theta_0\|_{L^2} \). However, the smallness of \( \|\nabla \theta_0\|_{L^2} \) does not imply that \( \theta_0 \) is a small perturbation from a constant angle, and thus the deformation tensor \( F \) can be away from the identity at the distance of 2 in \( L^\infty \):

\[
\|F(t, \cdot) - I\|_{L^\infty} = \|I + V(t, \cdot) - R^T\|_{L^\infty} \leq 2.
\]
This constraint on $\theta$ (see the last equation in (1.1)) is due to the inherent incompressible constraints (1.8). Besides, the dynamics of the rotation angle $\theta$ is still inherently determined by the dynamics of velocity field and the strain tensor. It is expected to study this problem in critical Besov space, but we do not pursue such a result in this paper. Theorem 1.1 can be extended to the case of general energy function $W = W(F)$ by the same argument in this paper.

We emphasize that similar result in Theorem 1.1 has been proved in [14] by exploring the weak dissipative mechanics of the whole system (1.1), where the proof essentially relies on the use of the equation for angle. We avoid using the equation for angle in this paper by deriving the following new structural identity (see Lemma 2.1)

$$\nabla \cdot \nabla \cdot V = -\nabla \cdot [AV(I + V)^{-1}A\nabla \cdot V].$$

This new identity together with the following Hodge decomposition (see step 4 of section 2 for its derivation)

$$\Delta V = \nabla \nabla \cdot V + (\nabla^{-1})^2 \text{tr} V - \nabla^{-1}(A\nabla \cdot V)$$

is sufficient to prove that both $u$ and $u + 2\mu^{-1}\Delta^{-1}\nabla \cdot V$ are dissipative using the similar arguments as in [14].

Let us end this introduction by mentioning some related works on incompressible viscoelastic system of Oldroyd-B type, which has attracted numerous attentions in the past and recent years. The study of the different contributions of strain and rotation can be traced back to Friedrich [6] where he observed that the smallness of the strain in nonlinear elasticity can be realized through the polar decomposition of the deformation tensor. John [11, 12] showed that no pointwise estimate for rotations in terms of strains can exist even in the case of small strain. In the work of Friedrich and John, no PDE is involved. From the PDE point of view, Liu and Walkington [23] considered some approximating systems resulting from the special linearization of the original system with respect to the strain. Lei, Liu and Zhou [16] constructed a 2D rotation-strain viscoelastic model and proved the global existence of classical solutions with small strain, but the dynamics of strain and rotation are not equivalent to that of the deformation tensor. The exact rotation-strain model was then derived in [16].

The global existence of classical solutions to (1.4) near equilibrium was established by Lin, Liu and Zhang [20] in the two-dimensional case (see also Lei and Zhou [19] via the incompressible limit method). The three dimensional case was then proved by Lei, Liu and Zhou [17] (see also Chen and Zhang [3]). Very recently, an improved result was obtained by Kessenich [13] by removing the dependence of the smallness of the initial data on the viscosity via the hyperbolic energy method. We also mention the works on elastodynamics [1, 29, 30, 31, 32] and on viscoelastic Oldroyd-B model [2, 4, 15, 22, 5, 8, 9, 10, 18, 24, 25, 26, 27, 28, 33, 34, 35].

The paper is simply organized as follows. In section 2, after some preparations, we will present the proof of Theorem, which will be divided into four steps.
2 Proof of the Theorem

Before the proof of Theorem [1.1], let us give a brief explanation of the constraints on the initial data in (1.8). Due to the incompressibility, one has \( \det F = 1 \). Since the deformation tensor \( F \) is decomposed into stretching and rotation \( F = (I + V)R \), one has \( \det(I + V) = \det F = 1 \), which is the second equality in (1.8). The third equality in (1.8) can be derived from the identity \( \nabla \cdot F^T = 0 \) together with the polar decomposition of the deformation tensor. All of these constraints in (1.8) are intrinsic for the viscoelastic Oldroyd model (1.4). In fact, these identities in (1.8) are preserved in time (see [14]):

\[
\begin{aligned}
\det(I + V) &= 1, \\
\nabla \cdot V &= A(I + V)\nabla \theta.
\end{aligned}
\]  

(2.1)

A direct consequence of the above intrinsic properties is that the linear terms and nonlinear terms appearing in the equations are not transparent any more, which is revealed in the following lemma:

Lemma 2.1. Let \( F = (I + V)R \) be the polar decomposition and \( \det F = 1 \). Then there hold

\[
\text{tr} V = - \det V
\]

(2.2)

and

\[
\nabla \cdot \nabla \cdot V = - \nabla \cdot [AV(I + V)^{-1}A\nabla \cdot V],
\]

(2.3)

where \( A \) is given in (1.2).

The proof of Lemma 2.1 is straightforward. In fact, (2.2) is equivalent to the first identity in (2.1). To see this, by the second identity in (2.1), one has that

\[
\nabla \theta = -(I + V)^{-1}A\nabla \cdot V
\]

(2.4)

and

\[
\nabla \cdot V = A\nabla \theta + AV\nabla \theta.
\]

(2.5)

Apply the divergence operator to (2.5), one has

\[
\nabla \cdot \nabla \cdot V = \nabla \cdot (AV\nabla \theta).
\]

Then (2.3) follows by plugging (2.4) into the above equality.

The two identities (2.2) and (2.3) make system (1.6) fully dissipative. It is not necessary to use the equation for angle to complete the proof any more once one has those identities which are inherent in the fluid motion. Even though the proof of Theorem [1.1] is similar to that in [14], but the treatment of the term involving pressure in (2.15) is different. Besides, we need the space-time estimate not only for \( \nabla \Delta U \), but also for \( \Delta U \). Here \( U \) is an auxiliary function

\[
U = u + 2\mu^{-1}\Delta^{-1}\nabla \cdot V.
\]
For a self-contained presentation, we will still present the whole proofs, but we will omit some of the similar parts. The proofs will be divided into four steps. In the first step we will show the basic energy law (2.9). However, the basic energy law is not enough to give an estimate of the low frequency part of the whole strain matrix, which will be complemented by an $L^2$ estimate. In the second step we will do higher order energy estimate for $u$ and $V$. We then estimate the space-time $L^2$ norms of $\Delta U$ and $\nabla \Delta U$ in the third step. In the last step we apply Hodge’s decomposition (2.28) and the structural identity (2.3) to close the argument.

**Step 1. Basic Energy Law and an $L^2$ Energy Estimate.**

Let us first take the $L^2$ inner product of $u$-equation with $u$ and $V$-equation in (1.6) with $V$, and then add up the resulting equations to get

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + |V|^2) \, dx + \| \nabla u \|^2_{L^2} = - \int_{\mathbb{R}^2} u \cdot \nabla \left( \frac{|u|^2 + |V|^2}{2} + p \right) \, dx
$$

$$
+ \int_{\mathbb{R}^2} < V, \frac{1}{2} (\nabla uV + V \nabla u^T) + \frac{1}{2} \omega_{12}(u)(VA - AV) > \, dx
$$

$$
+ \int_{\mathbb{R}^2} < u, 2 \nabla \cdot V > + < V, S(u) > \, dx
$$

$$
+ \int_{\mathbb{R}^2} < u, \nabla \cdot (VV^T) > \, dx - \int_{\mathbb{R}^2} < V, \frac{1}{2} \gamma (VA - AV) > \, dx.
$$

On the other hand, the $V$-equation also gives that

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} 2 \text{tr} V \, dx = - \int_{\mathbb{R}^2} u \cdot \nabla \text{tr} V \, dx + \int_{\mathbb{R}^2} \text{tr} S(u) \, dx
$$

$$
+ \int_{\mathbb{R}^2} \text{tr} (\nabla uV + \frac{1}{2} (VA - AV)(\omega_{12}(u) - \gamma)) \, dx.
$$

Adding up the above two equations and using the divergence-free property of $u$, we obtain that

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + |V|^2 + 2 \text{tr} V) \, dx + \| \nabla u \|^2_{L^2} = \int_{\mathbb{R}^2} < V, \frac{1}{2} (\nabla uV + V \nabla u^T) + \frac{1}{2} \omega_{12}(u)(VA - AV) > \, dx
$$

$$
+ \int_{\mathbb{R}^2} < u, 2 \nabla \cdot V > + < V, S(u) > \, dx + \int_{\mathbb{R}^2} \text{tr} (\nabla uV) \, dx
$$

$$
+ \int_{\mathbb{R}^2} < u, \nabla \cdot (VV^T) > \, dx - \int_{\mathbb{R}^2} < V, \frac{1}{2} \gamma (VA - AV) > \, dx
$$

$$
+ \int_{\mathbb{R}^2} \text{tr} \left( \frac{1}{2} (VA - AV)(\omega_{12}(u) - \gamma) \right) \, dx.
$$
Let us compute the terms on the right hand side of (2.6). First of all, by noting that $V$ is symmetric, it is rather easy to see that

$$
\int_{\mathbb{R}^2} < u, 2\nabla \cdot V > + < V, S(u) > \, dx + \int_{\mathbb{R}^2} \text{tr}(\nabla u V) \, dx = 0. \tag{2.7}
$$

On the other hand, using the definitions in (1.2) and (1.3), we have

$$
\int_{\mathbb{R}^2} < V, \frac{1}{2}(\nabla u V + V \nabla u^T) > + \frac{1}{2} \omega_{12}(u)(VA - AV) > \, dx \tag{2.8}
$$

$$
+ \int_{\mathbb{R}^2} < u, \nabla \cdot (V V^T) > \, dx - \int_{\mathbb{R}^2} < V, \frac{1}{2} \gamma(VA - AV) > \, dx
$$

$$
+ \int_{\mathbb{R}^2} \frac{1}{2} ((VA - AV)(\omega_{12}(u) - \gamma)) \, dx
$$

$$
= \int_{\mathbb{R}^2} < V, \nabla u V > + < u, \nabla \cdot (V V^T) > \, dx
$$

$$
+ \int_{\mathbb{R}^2} \frac{1}{2}(\omega_{12}(u) - \gamma)(\text{tr}(VA - AV) + < V, VA - AV >) \, dx
$$

$$
= 0.
$$

Then, combining (2.6), (2.7) with (2.8), we have the basic energy law

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + |V|^2 + 2\text{tr} V) \, dx + \|\nabla u\|^2_{L^2} = 0. \tag{2.9}
$$

However, noting (2.3), one has

$$
|V|^2 + 2\text{tr} V = |V_{11} - V_{22}|^2 + |V_{12} + V_{21}|^2.
$$

Thus, the basic energy law does not give an $L^2$ estimate of $V$. On the meantime, we have

$$
\int_0^\infty \|\nabla u\|^2_{L^2} \, ds \leq \frac{1}{2} (\|u\|^2_{L^2} + \|V\|^2_{L^2}). \tag{2.10}
$$

This estimate will be crucial to derive the space-time $L^2$ estimate of $\Delta U$ and $\nabla \Delta U$ in step 3.

Next, let us derive the $L^2$ energy estimate. By taking the $L^2$ inner product of the first and second equations in (1.1) with $u$ and $2V$, and then adding up the resulting equations and using integration by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|u|^2 + 2|V|^2)dx + \|\nabla u\|^2_{L^2} \tag{2.11}
$$

$$
\lesssim \|\nabla u\|_{L^2}\|V\|^2_{L^4}
$$

$$
\lesssim \|V\|_{L^2}\|\nabla u\|_{L^2}\|\nabla V\|_{L^2},
$$

provided that

$$
\|\text{tr} V\|_{L^\infty} \leq 1,
$$

7
which is guaranteed by (2.2) and the smallness of \( \| V \|_{H^2} \):

\[
\| \text{tr} \ V \|_{L^\infty} = \| \text{det} \ V \|_{L^\infty} \lesssim \| V \|_{H^2}^2.
\]

**Step 2. Higher Order Energy Estimates.**

In order to exploit the higher order energy estimates, we apply \( \Delta \) to equations for \( u \) and \( V \) of system (1.1), and then take \( L^2 \) inner product of the resulting equations with \( \Delta u \) and \( 2\Delta V \), respectively, to yield

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left( |\Delta u|^2 + 2|\Delta V|^2 \right) dx + (\Delta u, \nabla \Delta p)
\]

\[
+ \int_{\mathbb{R}^2} u \cdot \nabla \left( \frac{|\Delta u|^2}{2} + |\Delta V|^2 \right) dx
\]

\[
= \left( \Delta V, \Delta \left[ \nabla uV + V \nabla u^T + \omega_{12}(u)(VA - AV) \right] \right)
\]

\[
- \gamma(VA - AV) \right) + (\Delta u, \Delta \nabla \cdot (VV^T)) + (\Delta u, 2\Delta \nabla \cdot V)
\]

\[
+ (2\Delta V, \Delta S(u)) + (\Delta u, \Delta \Delta u) - \left( \Delta u, [\Delta (u \cdot \nabla u) - u \cdot \nabla \Delta u] \right)
\]

\[
- \left( 2\Delta V, [\Delta (u \cdot \nabla V) - u \cdot \nabla \Delta V] \right).
\]

By careful calculations, we have (for the details, see [14])

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|\Delta u|^2 + 2|\Delta V|^2) dx + \| \nabla \Delta u \|_{L^2}^2 \geq (\| u \|_{H^2} + \| V \|_{H^2}) (\| \nabla u \|_{L^2}^2 + \| \nabla \Delta u \|_{L^2}^2 + \| \Delta V \|_{L^2}^2).
\]

**Step 3. Weak Dissipation for the Strain Matrix \( V \).**

Let us recall the definition of the auxiliary function \( U \):

\[
U = u + 2\Delta^{-1} \nabla \cdot V.
\]

Below we will estimate the space-time \( L^2 \) norm of both \( \Delta U \) and \( \nabla \Delta U \). The part for \( \int_0^t \| \nabla \Delta U \|_{L^2}^2 ds \) has been done in [14]. However, to get the fully dissipative nature of the whole system (1.1), the argument in [14] involves another auxiliary function \( \Theta = \Delta u + 2\nabla^\perp \theta \). We are going to avoid using \( \Theta \) below, and thus provide the global well-posedness of solutions to the subsystem (1.6).

The equation for \( U \) is as follows:

\[
\Delta U_t + u \cdot \nabla \Delta U + \nabla \Delta p = \Delta^2 U + \Delta u + f,
\]
where

\[
\begin{split}
f &= -[\Delta(u \cdot \nabla u) - u \cdot \nabla u] - [\nabla \cdot (u \cdot \nabla V) \\
& \quad - u \cdot \nabla (\nabla \cdot V)] + \Delta \nabla \cdot (VV^T) \quad \text{and} \quad [\nabla uV + V \nabla u^T] \\
& \quad + \omega_{12}(u)(VA - AV) - \gamma(VA - AV).
\end{split}
\]

Take the inner product of (2.14) with \(\Delta U\) in \(L^2(\mathbb{R}^2)\), and use the similar arguments as in (2.7), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\Delta U|^2 dx + \|\nabla \Delta U\|_{L^2}^2 = (\Delta p, \nabla \cdot \Delta U) + (\Delta u, \Delta U) + (f, \Delta U). \quad (2.15)
\]

In [14], the following estimate has been obtained:

\[
|(f, \Delta U)| \lesssim (\|u\|_{H^2} + \|V\|_{H^2}) \left(\|\nabla u\|_{H^2}^2 + \|\nabla \Delta U\|_{L^2}^2 + \|\Delta V\|_{L^2}^2\right), \quad (2.16)
\]

\[
|\Delta u, \Delta U| \leq \frac{1}{4} \|\nabla \Delta U\|_{L^2}^2 + \|\nabla u\|_{L^2}^2. \quad (2.17)
\]

Here we treat the pressure term by making use of the structural identity (2.3) in Lemma 2.1. We first apply the divergence operator to the momentum equation of system (1.6) to get

\[
\Delta p = -\text{tr}(\nabla u \nabla u) + \nabla \cdot [\nabla \cdot (VV^T)] - 2 \nabla \cdot [AV(I + V)^{-1}A \nabla \cdot V]. \quad (2.18)
\]

Plugging (2.3) into the above inequality, one has

\[
|\Delta p, \nabla \cdot \Delta U| \lesssim (\|u\|_{H^2} + \|V\|_{H^2}) \left(\|\nabla u\|_{H^2}^2 + \|\Delta u\|_{L^2}^2 + \|\Delta V\|_{L^2}^2\right). \quad (2.19)
\]

The combination of (2.15), (2.16), (2.17) and (2.19) gives

\[
\frac{d}{dt} \|\Delta U\|_{L^2}^2 + \|\nabla \Delta U\|_{L^2}^2 \lesssim \|\nabla u\|_{L^2}^2 \quad (2.20)
\]

\[
+ \left(\|u\|_{H^2} + \|V\|_{H^2}\right) \left(\|\nabla u\|_{H^2}^2 + \|\nabla \Delta U\|_{L^2}^2 + \|\Delta V\|_{L^2}^2\right).
\]

Next, let us derive a space-time \(L^2\) estimate for \(\Delta U\). For this purpose, we first take the \(L^2\) inner product of \(u\)-equation in (1.6) with \(\Delta U\) to get

\[
\|\Delta U\|_{L^2}^2 - \int_{\mathbb{R}^2} \Delta U \cdot u_t dx = \int_{\mathbb{R}^2} \Delta U \cdot [u \cdot \nabla u + \nabla p - \nabla \cdot (VV^T)] dx. \quad (2.21)
\]

First of all, one has

\[
- \int_{\mathbb{R}^2} \Delta U \cdot u_t dx = \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 - 2 \int_{\mathbb{R}^2} (\nabla \cdot V) \cdot u_t dx. \quad (2.22)
\]
Secondly, by using (2.3), it is easy to see that
\[
\int_{\mathbb{R}^2} \Delta U \cdot \nabla p \, dx = -2 \int_{\mathbb{R}^2} p \nabla \cdot \nabla V \, dx \quad (2.23)
\]
\[
= 2 \int_{\mathbb{R}^2} p \nabla \cdot [AV(I + V)^{-1}A\nabla V] \, dx
\]
\[
= -2 \int_{\mathbb{R}^2} \nabla p \cdot [AV(I + V)^{-1}A\nabla V] \, dx.
\]
Consequently, by (2.23) and (2.18), we have
\[
\left| \int_{\mathbb{R}^2} \Delta U \cdot [u \cdot \nabla u + \nabla p - \nabla \cdot (VV^T)] \, dx \right|
\leq \|\Delta U\|_{L^2} \|u \cdot \nabla u - \nabla \cdot (VV^T)\|_{L^2} + \|\nabla p\|_{L^2} \|AV(I + V)^{-1}A\nabla V\|_{L^2}
\leq \frac{1}{2} \|\Delta U\|_{L^2}^2 + \|u \cdot \nabla u - \nabla \cdot (VV^T)\|_{L^2}^2 + \|AV(I + V)^{-1}A\nabla V\|_{L^2}^2
\leq \frac{1}{2} \|\Delta U\|_{L^2}^2 + (\|u\|_{H^2}^2 + \|V\|_{H^2}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla V\|_{L^2}^2).
\]
Now plugging (2.22) and (2.24) into (2.21), we get
\[
\|\Delta U\|_{L^2}^2 + \frac{d}{dt} \|\nabla u\|_{L^2}^2 - 4 \int_{\mathbb{R}^2} (\nabla \cdot V) \cdot u_t \, dx \quad (2.25)
\leq (\|u\|_{H^2}^2 + \|V\|_{H^2}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla V\|_{L^2}^2).
\]
Taking the $L^2$ inner product of $V$-equation in (1.6) with $4\nabla u$, one has
\[
-4 \int_{\mathbb{R}^2} u \nabla \cdot V_t \, dx = 4 \int_{\mathbb{R}^2} u \nabla \left[ -u \cdot \nabla V + S(u) + \frac{1}{2} (\nabla u V + V \nabla u^T) \right. \quad (2.26)
\left. + \frac{1}{2} \omega_{12}(u)(VA - AV) - \frac{1}{2} \gamma(VA - AV) \right] \, dx
\leq \|\nabla u\|_{L^2}^2 + (\|u\|_{H^2}^2 + \|V\|_{H^2}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla V\|_{L^2}^2).
\]
Adding up (2.25) and (2.26), one gets the following:
\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \quad (2.27)
\leq (\|u\|_{H^2}^2 + \|V\|_{H^2}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla V\|_{L^2}^2) + \|\nabla u\|_{L^2}^2.
\]

**Step 4. Weakly Dissipative Mechanics of (1.6).**

This step ties everything together. Firstly, by Hodge’s decomposition, we derive from straightforward calculations that
\[
\Delta V = \nabla \nabla \cdot V - \nabla \times \nabla \times V \quad (2.28)
\]
\[
= \nabla \nabla \cdot V - \left( \begin{array}{c}
\nabla_2(\nabla_1 V_{12} - \nabla_2 V_{11}) \\
\nabla_2(\nabla_1 V_{22} - \nabla_2 V_{21})
\end{array} \right)
\]
\[
= \nabla \nabla \cdot V + (\nabla^\perp)^2 \text{tr} V - \nabla^\perp (A \nabla \cdot V).
\]
Thus, by Lemma 2.1, we obtain
\[ \| \nabla V \|_{H^1} \lesssim \| \nabla \cdot V \|_{H^1} + \| \nabla \det V \|_{L^2}, \]
which gives that
\[ \| \nabla V \|_{H^1} \lesssim \| \nabla \cdot V \|_{H^1}, \tag{2.29} \]
provided that \( \| V \|_{H^2} \) is sufficiently small. Using (2.13) and (2.29), we have
\[ \| \nabla V \|_{H^1} \lesssim \| \Delta u \|_{H^1} + \| \Delta U \|_{H^1}. \tag{2.30} \]

Plug (2.30) into (2.11), (2.12), (2.20) and (2.27), and then combine them to derive that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \left( \| u \|_{H^2}^2 + \| V \|_{H^2}^2 + \| \Delta U \|_{L^2}^2 \right) dx + \left( \| \nabla u \|_{H^2}^2 + \| \Delta U \|_{H^1}^2 \right) \leq \left( \| u \|_{H^2} + \| V \|_{H^2} \right) \left( \| \nabla u \|_{H^2}^2 + \| \Delta u \|_{H^1}^2 \right) + \| \nabla u \|_{L^2}^2. \tag{2.31} \]
Noting (2.10), one can easily conclude from (2.31) that the following a priori estimate
\[
\sup_{t \geq 0} \left( \| u(t, \cdot) \|_{H^2}^2 + \| V(t, \cdot) \|_{H^2}^2 \right) + \int_0^\infty \left[ \| \nabla u \|_{H^2}^2 \\
+ \| (\Delta u + 2 \nabla \cdot V) \|_{H^1}^2 \right] dt \leq C \left( \| u_0 \|_{H^2}^2 + \| V_0 \|_{H^2}^2 \right). \]
is true if the \( H^2 \) norms of \( u_0 \) and \( V_0 \) are sufficiently small. This further gives that
\[
2 \int_0^\infty \| \nabla \cdot V \|_{H^1}^2 dt \leq \int_0^\infty \left[ \| (\Delta u + 2 \nabla \cdot V) \|_{H^1}^2 \\
+ \| \Delta u \|_{H^1}^2 \right] dt \leq C \left( \| u_0 \|_{H^2}^2 + \| V_0 \|_{H^2}^2 \right). \]
Then the a priori estimate in Theorem 1.1 follows by the above inequality and (2.29). The proof of Theorem 1.1 is completed.

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