Black-bounce to traversable wormhole

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Abstract. So-called “regular black holes” are a topic currently of considerable interest in the general relativity and astrophysics communities. Herein we investigate a particularly interesting regular black hole spacetime described by the line element

\[ ds^2 = -\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\sqrt{r^2 + a^2}}} + \left(r^2 + a^2\right)\left(d\theta^2 + \sin^2\theta\,d\phi^2\right). \]

This spacetime neatly interpolates between the standard Schwarzschild black hole and the Morris-Thorne traversable wormhole; at intermediate stages passing through a black-bounce (into a future incarnation of the universe), an extremal null-bounce (into a future incarnation of the universe), and a traversable wormhole. As long as the parameter $a$ is non-zero the geometry is everywhere regular, so one has a somewhat unusual form of “regular black hole”, where the “origin” $r = 0$ can be either spacelike, null, or timelike. Thus this spacetime generalizes and broadens the class of “regular black holes” beyond those usually considered.

Keywords: GR black holes, gravity, Wormholes

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1 Introduction

Ever since Bardeen initially proposed the concept of a regular black hole in 1968 [1], see also the more recent references [2–13], the notion has been intuitively attractive due to its nonsingular nature. When exploring various candidates for regular black hole geometries within the framework of general relativity, it pays to compile examples of various metrics of interest, and provide thorough analyses of their phenomenological properties [12, 13]. As such, we propose the following candidate regular black hole specified by the spacetime metric:

\[
ds^2 = -\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\sqrt{r^2 + a^2}}} + (r^2 + a^2)\left(d\theta^2 + \sin^2 \theta d\phi^2\right).
\]  

This spacetime is carefully designed to be a minimalist modification of the ordinary Schwarzschild spacetime. Adjusting the parameter \(a\), this metric represents either:

1. The ordinary Schwarzschild spacetime;
2. A regular black hole geometry with a one-way spacelike throat;
3. A one-way wormhole geometry with an extremal null throat;
   (compare particularly with reference [10]); or
4. A canonical traversable wormhole geometry, (in the Morris-Thorne sense [14–29]), with a two-way timelike throat.

In the region where the geometry represents a regular black hole the geometry is unusual in that it describes a bounce into a future incarnation of the universe, rather than a bounce back into our own universe [30–42]. Conducting a standard analysis of this metric within the context of general relativity we find the locations of photon spheres and ISCOs for each case, calculate Regge-Wheeler potentials, and draw conclusions concerning the nature of the curvature of this spacetime.
2 Metric analysis and Carter-Penrose diagrams

Consider the metric:

\[
ds^2 = -\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\sqrt{r^2 + a^2}}} + (r^2 + a^2)\left(d\theta^2 + \sin^2\theta d\phi^2\right).
\] (2.1)

Note that if \(a = 0\) then this is simply the Schwarzschild solution, so enforcing \(a \neq 0\) is a sensible starting condition if we are to conduct an analysis concerning either regular black holes or traversable wormholes (trivially, the Schwarzschild solution models a geometry which is neither). Furthermore, this spacetime geometry is manifestly static and spherically symmetric. That is, it admits a global, non-vanishing, timelike Killing vector field that is hypersurface orthogonal, and there are no off-diagonal components of the matrix representation of the metric tensor; fixed \(r\) coordinate locations in the spacetime correspond to spherical surfaces. This metric does not correspond to a traditional regular black hole such as the Bardeen, Bergmann-Roman, Frolov, or Hayward geometries [1–11]. Instead, depending on the value of the parameter \(a\), it is either a regular black hole (bouncing into a future incarnation of the universe) or a traversable wormhole.

Before proceeding any further, note that the coordinates have natural domains:

\[
\begin{align*}
    r &\in (-\infty, +\infty); & t &\in (-\infty, +\infty); & \theta &\in [0, \pi]; & \phi &\in (-\pi, \pi].
\end{align*}
\] (2.2)

Analysis of the radial null curves in this metric yields (setting \(ds^2 = 0, d\theta = d\phi = 0\)):

\[
\frac{dr}{dt} = \pm \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right),
\] (2.3)

It is worth noting that this defines a “coordinate speed of light” for the metric (2.1),

\[
c(r) = \left|\frac{dr}{dt}\right| = \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right),
\] (2.4)

and hence an effective refractive index of:

\[
n(r) = \frac{1}{\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)}.
\] (2.5)

Let us now examine the coordinate location(s) of horizon(s) in this geometry:

- If \(a > 2m\), then \(\forall r \in (-\infty, +\infty)\) we have \(\frac{dr}{dt} \neq 0\), so this geometry is in fact a (two-way) traversable wormhole [14–29].

- If \(a = 2m\), then as \(r \to 0\) from either above or below, we have \(\frac{dr}{dt} \to 0\). Hence we have a horizon at coordinate location \(r = 0\). However, this geometry is not a black hole. Rather, it is a one-way wormhole with an extremal null throat at \(r = 0\).

- If \(a < 2m\), then consider the two locations \(r_{\pm} = \pm\sqrt{(2m)^2 - a^2}\); this happens when \(\sqrt{r_{\pm}^2 + a^2} = 2m\). Thence

\[
\exists r_{\pm} \in \mathbb{R} : \quad \frac{dr}{dt} = 0.
\] (2.6)

That is, when \(a < 2m\) there will be symmetrically placed \(r\)-coordinate values \(r_{\pm} = \pm| r_{\pm} |\) which correspond to a pair of horizons.
The coordinate location \( r = 0 \) maximises both the non-zero curvature tensor components (see section 3) and the curvature invariants (see section 4). We may therefore conclude that in the case where \( a > 2m \) the traversable wormhole throat is a timelike hypersurface located at \( r = 0 \), and negative \( r \)-values correspond to the universe on the other side of the geometry from the perspective of an observer in our own universe.

We then have the standard Carter-Penrose diagram for traversable wormholes as presented in figure 1. Similarly for the null case \( a = 2m \) the null throat is located at the horizon \( r = 0 \). Note that in this instance the wormhole geometry is only one-way traversable. The Carter-Penrose diagram for the maximally extended spacetime in this case is given in figure 2.

As an alternative construction we can identify the past null bounce at \( r = 0 \) with the future null bounce at \( r = 0 \) yielding the ‘looped’ Carter-Penrose diagram of figure 3.

For regular black holes, we can restrict our attention to the interval \( a \in (0, 2m) \). Then the hypersurface \( r = 0 \) is a spacelike spherical surface which marks the boundary between our universe and a bounce into a separate copy of our own universe. For negative values of \( r \) we have ‘bounced’ into another universe. See figure 4 for the relevant Carter-Penrose diagram. (Contrast these Carter-Penrose diagrams with the standard one for the maximally extended Kruskal-Szekeres version of Schwarzschild — see for instance references [43–45] — the major difference is that the singularity has been replaced by a spacelike hypersurface representing a “bounce”.)

Another possibility of interest for when \( a \in (0, 2m) \) arises when the \( r = 0 \) coordinate for the ‘future bounce’ is identified with the \( r = 0 \) coordinate for the ‘past bounce’. That is, there is still a distinct time orientation but we impose periodic boundary conditions on the time coordinate such that time is cyclical. This case yields the Carter-Penrose diagram of figure 5. (Note that the global causal structure is much milder than that for the so-called “twisted” black holes [46].)
Figure 2. Carter-Penrose diagram for the maximally extended spacetime in the case when $a = 2m$. In this example we have a one-way wormhole geometry with a null throat.

3 Curvature tensors

Next it is prudent to check that there are no singularities in the geometry, otherwise we do not satisfy the requirements for the regularity of our black hole. In view of the diagonal metric environment of (2.1) we can clearly see that the chosen coordinate basis is orthogonal though not orthonormal, and it therefore follows that the (mixed) non-zero components of the Riemann tensor shall be the same with respect to this basis as to any orthonormal tetrad, ensuring that the appearance (or lack thereof) of any singularities is not simply a coordinate artefact.

With this in mind, for simplicity we first consider the non-zero components of the Weyl tensor (these are equivalent to orthonormal components):

$$
C^{\theta \theta}_{\theta \theta} = C^{\phi \phi}_{\phi \phi} = C^{r \theta}_{r \theta} = C^{r \phi}_{r \phi} = -\frac{1}{2} C^{\theta r}_{\theta r} = -\frac{1}{2} C^{\phi r}_{\phi r} = \frac{6r^2m + a^2 \left( 2\sqrt{r^2 + a^2} - 3m \right)}{6 \left( r^2 + a^2 \right)^2}.
$$

(3.1)

Note that as $r \to 0$ these Weyl tensor components approach the finite value $\frac{2a - 3m}{6a^2}$.
Figure 3. Carter-Penrose diagram for the case when $a = 2m$ where we have identified the future null bounce at $r = 0$ with the past null bounce at $r = 0$.

For the Riemann tensor the non-zero components are a little more complicated:

\[
R^{tr}_{tr} = \frac{m(2r^2 - a^2)}{(r^2 + a^2)^\frac{3}{2}}; \\
R^{t\theta}_{t\theta} = R^{\phi t \phi} = \frac{-r^2 m}{(r^2 + a^2)^\frac{3}{2}}; \\
R^{r\theta}_{r\theta} = R^{r\phi r \phi} = \frac{m(2a^2 - r^2) - a^2 \sqrt{r^2 + a^2}}{(r^2 + a^2)^\frac{3}{2}}; \\
R^{\theta \phi}_{\theta \phi} = \frac{2r^2 m + a^2 \sqrt{r^2 + a^2}}{(r^2 + a^2)^\frac{3}{2}}. \tag{3.2}
\]

Provided $a \neq 0$, as $|r| \to 0$ all of these Riemann tensor components approach finite limits:

\[
R^{tr}_{tr} \to \frac{-m}{a^3}; \\
R^{t\theta}_{t\theta} = R^{\phi t \phi} \to 0; \\
R^{r\theta}_{r\theta} = R^{r\phi r \phi} \to \frac{2m - a}{a^3}; \\
R^{\theta \phi}_{\theta \phi} \to \frac{1}{a^2}. \tag{3.3}
\]

As $|r|$ increases, with $m$ and $a$ held fixed, all components asymptote to multiples of $m/r^3$, hence as $|r| \to +\infty$, all components tend to 0 (this is synonymous with the fact that for large $|r|$ this geometry models weak field general relativity). Hence $\forall r \in (-\infty, +\infty)$ the components of the Riemann tensor are strictly finite. We may conclude that on the interval
Figure 4. Carter-Penrose diagram for the maximally extended spacetime when $a \in (0, 2m)$. In this example the time coordinate runs up the page, ‘bouncing’ through the $r = 0$ hypersurface in each black hole region into a future copy of our own universe \textit{ad infinitum}.

Figure 5. Periodic boundary conditions in time when $a \in (0, 2m)$. In this example we impose periodic boundary conditions on the time coordinate such that the future bounce is identified with the past bounce.
\[ a \in (0, 2m) \] there is a horizon, but no singularity, and the metric really does represent the geometry of a regular black hole. In the case when \( a > 2m \) and we have a traversable wormhole, trivially there are also no singularities.

The Ricci tensor has non-zero (mixed) components:

\[ -2 R^t_t = R^\theta_\theta = R^\phi_\phi = \frac{2a^2 m}{(r^2 + a^2)^2}, \quad R^r_r = \frac{a^2 (3m - 2\sqrt{r^2 + a^2})}{(r^2 + a^2)^5}. \quad (3.4) \]

The Einstein tensor has non-zero (mixed) components:

\[
G^t_t = \frac{a^2 \left( \sqrt{r^2 + a^2} - 4m \right)}{(r^2 + a^2)^{\frac{5}{2}}}, \quad
G^r_r = \frac{-a^2}{(r^2 + a^2)^2},
\]

\[
G^\theta_\theta = G^\phi_\phi = \frac{a^2 \left( \sqrt{r^2 + a^2} - m \right)}{(r^2 + a^2)^{\frac{5}{2}}}. \quad (3.5)
\]

4 Curvature invariants

The Ricci scalar is:

\[ R = \frac{2a^2 \left( 3m - \sqrt{r^2 + a^2} \right)}{(r^2 + a^2)^{\frac{5}{2}}}. \quad (4.1) \]

The Ricci contraction \( R_{ab} R^{ab} \) is:

\[ R_{ab} R^{ab} = \frac{a^4 \left( 4 \left( \sqrt{r^2 + a^2} - \frac{3}{2}m \right)^2 + (3m)^2 \right)}{(r^2 + a^2)^5}. \quad (4.2) \]

Note that this is a sum of squares and so automatically non-negative (and finite).

The Weyl contraction \( C_{abcd} C^{abcd} \):

\[ C_{abcd} C^{abcd} = \frac{4}{3 (r^2 + a^2)^5} \left( 3m \left( 2r^2 - a^2 \right) + 2a^2 \sqrt{r^2 + a^2} \right)^2. \quad (4.3) \]

That this is a perfect square and so is automatically non-negative (and finite).

The Kretschmann scalar is:

\[ R_{abcd} R^{abcd} = C_{abcd} C^{abcd} + 2 R_{ab} R^{ab} - \frac{1}{3} R^2, \quad (4.4) \]

and so (in view of the above) is guaranteed finite without further calculation. Explicitly

\[
R_{abcd} R^{abcd} = \frac{4}{(r^2 + a^2)^5} \left\{ \sqrt{r^2 + a^2} \left[ 8a^2 m \left( r^2 - a^2 \right) \right] + 3a^4 \left( r^2 + a^2 \right) + 3m^2 \left( 3a^4 - 4a^2 r^2 + 4r^4 \right) \right\}. \quad (4.5)
\]
5 Stress-energy tensor and energy conditions

Let us examine the Einstein field equations for this spacetime. We first note that for \( \sqrt{r^2 + a^2} > 2m \), that is, outside any horizon that may potentially be present, one has \( \rho = -T^t_t \) while \( p_\parallel = T_r^r \) and \( p_\perp = T_\theta^\theta = T_\phi^\phi \). Using the mixed components \( G^\mu_\nu = 8\pi G_N T^\mu_\nu \), this yields the following form of the stress-energy-momentum tensor:

\[
\rho = -\frac{a^2 \left( \sqrt{r^2 + a^2} - 4m \right)}{8\pi G_N (r^2 + a^2)^{\frac{3}{2}}},
\]

\[
p_\parallel = \frac{a^2}{8\pi G_N (r^2 + a^2)^{\frac{3}{2}}},
\]

\[
p_\perp = \frac{a^2 \left( \sqrt{r^2 + a^2} - m \right)}{8\pi G_N (r^2 + a^2)^{\frac{3}{2}}}.
\]

Now a necessary condition for the NEC (null energy condition) to hold is that both \( \rho + p_\parallel \geq 0 \) and \( \rho + p_\perp \geq 0 \) for all \( r, a, m \). It is sufficient to consider

\[
\rho + p_\parallel = \frac{1}{8\pi G_N} \left\{ -\frac{a^2 \left( \sqrt{r^2 + a^2} - 4m \right)}{(r^2 + a^2)^{\frac{3}{2}}} - \frac{a^2}{(r^2 + a^2)^{\frac{3}{2}}} \right\}
\]

\[
= -\frac{a^2 (\sqrt{r^2 + a^2} - 2m)}{4\pi G_N (r^2 + a^2)^{\frac{3}{2}}},
\]

(5.1)

Assuming \( \sqrt{r^2 + a^2} > 2m \), this is manifestly negative for all values of \( a \) and \( m \) in our domain, and the NEC is clearly violated.

Note that for \( \sqrt{r^2 + a^2} < 2m \), that is, inside any horizon that may potentially be present, the \( t \) and \( r \) coordinates swap their timelike/spacelike characters and one has \( \rho = -T^r_r \) while \( p_\parallel = T_t^t \) and \( p_\perp = T_\theta^\theta = T_\phi^\phi \). So inside the horizon

\[
\rho = \frac{a^2}{8\pi G_N (r^2 + a^2)^{\frac{3}{2}}};
\]

\[
p_\parallel = \frac{a^2 \left( \sqrt{r^2 + a^2} - 4m \right)}{8\pi G_N (r^2 + a^2)^{\frac{3}{2}}},
\]

(5.3)

and

\[
\rho + p_\parallel = \frac{a^2 (\sqrt{r^2 + a^2} - 2m)}{4\pi G_N (r^2 + a^2)^{\frac{3}{2}}},
\]

(5.4)

But since we are now working in the region \( \sqrt{r^2 + a^2} < 2m \) this is again negative, and the NEC is again violated.

We can summarize this by stating

\[
\rho + p_\parallel = -\frac{a^2 |\sqrt{r^2 + a^2} - 2m|}{4\pi G_N (r^2 + a^2)^{\frac{3}{2}}},
\]

(5.5)

which now holds for all values of \( r \) and is negative everywhere except on any horizon that may potentially be present.
Demonstrating that the NEC is violated is sufficient to conclude that the weak, strong, and dominant energy conditions shall also be violated [18]. We therefore have a spacetime geometry that accurately models that of a regular black hole or a traversable wormhole depending on the value of $a$, but clearly violates all the classical energy conditions associated with the stress-energy-momentum tensor [47–62] .

### 6 Surface gravity and Hawking temperature

Let’s calculate the surface gravity at the event horizon for the regular black hole case when $a \in (0, 2m]$. The Killing vector which is null at the event horizon is $\xi^\mu = \partial_t$. This yields the following norm:

$$\xi^\mu \xi_\mu = g_{\mu\nu} \xi^\mu \xi^\nu = g_{tt} = -\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right). \quad (6.1)$$

Then we have the following relation for the surface gravity $\kappa$ (see for instance [43–45]):

$$\nabla_\nu (-\xi^\mu \xi_\mu) = 2\kappa \xi_\nu. \quad (6.2)$$

That is:

$$\nabla_\nu \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) = 2\kappa \xi_\nu; \quad (6.3)$$

Keeping in mind that the event horizon is located at radial coordinate $r = \sqrt{(2m)^2 - a^2}$ we see:

$$\kappa = \frac{1}{2} \left. \partial_r \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \right|_{r=\sqrt{(2m)^2 - a^2}} = \frac{\sqrt{(2m)^2 - a^2}}{8m^2} = \kappa_{\text{Sch}} \sqrt{1 - \frac{a^2}{(2m)^2}}. \quad (6.4)$$

As a consistency check it is easily observed that for the Schwarzschild case when $a = 0$, we have $\kappa = \frac{1}{4m}$, which is the expected surface gravity for the Schwarzschild black hole. For $a = 2m$ the null horizon (one-way throat) is seen to be extremal. It now follows that the temperature of Hawking radiation for our regular black hole is as follows (see for instance [43–45]):

$$T_H = \frac{\hbar \kappa}{2\pi k_B} = \frac{\hbar \sqrt{(2m)^2 - a^2}}{16\pi k_B m^2} = T_{H,\text{Sch}} \sqrt{1 - \frac{a^2}{(2m)^2}}. \quad (6.5)$$

### 7 ISCO analysis

Let us now find the location of both the photon sphere for massless particles [63–68] and the ISCO for massive particles as functions of $m$ and $a$. Consider the tangent vector to the worldline of a massive or massless particle, parameterized by some arbitrary affine parameter, $\lambda$:

$$g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = -g_{tt} \left(\frac{dt}{d\lambda}\right)^2 + g_{rr} \left(\frac{dr}{d\lambda}\right)^2 + (r^2 + a^2) \left\{ \left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\lambda}\right)^2 \right\}. \quad (7.1)$$

We may, without loss of generality, separate the two physically interesting cases (timelike and null) by defining

$$\epsilon = \begin{cases} -1 & \text{massive particle, i.e. timelike worldline} \\ 0 & \text{massless particle, i.e. null worldline} \end{cases}. \quad (7.2)$$
That is, \( \frac{ds^2}{d\lambda^2} = \epsilon \). Due to the metric being spherically symmetric we may fix \( \theta = \frac{\pi}{2} \) arbitrarily and view the reduced equatorial problem:

\[
g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = -g_{tt} \left( \frac{dt}{d\lambda} \right)^2 + g_{rr} \left( \frac{dr}{d\lambda} \right)^2 + (r^2 + a^2) \left( \frac{d\phi}{d\lambda} \right)^2 = \epsilon. \tag{7.3}
\]

The Killing symmetries yield the following expressions for the conserved energy \( E \) and angular momentum \( L \) per unit mass (see for instance \([43–45]\)):

\[
\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \left( \frac{dt}{d\lambda} \right) = E; \quad (r^2 + a^2) \left( \frac{d\phi}{d\lambda} \right) = L. \tag{7.4}
\]

Hence:

\[
\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)^{-1} \left\{ -E^2 + \left( \frac{dr}{d\lambda} \right)^2 \right\} + \frac{L^2}{r^2 + a^2} = \epsilon; \tag{7.5}
\]

implying

\[
\left( \frac{dr}{d\lambda} \right)^2 = E^2 + \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \left\{ \epsilon - \frac{L^2}{r^2 + a^2} \right\}. \tag{7.6}
\]

This gives “effective potentials” for geodesic orbits as follows:

\[
V_0(r) = \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \left\{ -\epsilon + \frac{L^2}{r^2 + a^2} \right\}. \tag{7.7}
\]

- For a photon orbit we have the massless particle case \( \epsilon = 0 \). Since we are in a spherically symmetric environment, solving for the locations of such orbits amounts to finding the coordinate location of the “photon sphere”. That is, the value of the \( r \)-coordinate sufficiently close to our central mass such that photons are forced to propagate along circular geodesic orbits. These circular orbits occur at \( V_0'(r) = 0 \). That is

\[
V_0(r) = \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \left( \frac{L^2}{r^2 + a^2} \right), \tag{7.8}
\]

leading to:

\[
V_0'(r) = \frac{2rL^2}{(r^2 + a^2)^{\frac{3}{2}}} \left\{ 3m - \sqrt{r^2 + a^2} \right\}. \tag{7.9}
\]

When \( V_0'(r) = 0 \), if we discount the solution \( r = 0 \) (as this spherical surface is clearly invalid for the location of the photon sphere), this gives the location of these circular orbits as \( r = \pm \sqrt{(3m)^2 - a^2} \). Firstly note that if \( a \in (0, 2m] \), \((3m)^2 > a^2 \forall a \), hence this solution does in fact correspond to a real-valued \( r \)-coordinate within our domain.

Hence the photon sphere in our universe (i.e. taking positive solution) for the case when the geometry is a regular black hole has coordinate location \( r = \sqrt{(3m)^2 - a^2} \). It also follows that in the case when \( a > 2m \) and we have a traversable wormhole, since we have strictly defined our \( r \)-coordinate to take on real values, there exists a photon sphere location in our universe only for the case when \( 2m < a < 3m \). When \( a > 3m \) we have no photon sphere. To verify stability, check the sign of \( V_0''(r) \):

\[
V_0''(r) = \frac{2L^2}{(r^2 + a^2)^{\frac{3}{2}}} \left\{ \sqrt{r^2 + a^2} (3r^2 - a^2) - 3m (4r^2 - a^2) \right\}. \tag{7.10}
\]
For ease of notation let us first establish that when \( r = \sqrt{(3m)^2 - a^2} \), then \( r^2 + a^2 = (3m)^2 \), hence it can be shown that:

\[
V_0'' \left( r = \sqrt{(3m)^2 - a^2} \right) = \frac{-2L^2}{(3m)^6} \left( (3m)^2 - a^2 \right) < 0. \tag{7.11}
\]

Now \( V_0'' < 0 \) implies instability, hence there is an unstable photon sphere at \( r = \sqrt{(3m)^2 - a^2} \) as presumed. For the Schwarzschild solution the location of the unstable photon sphere is at \( r = 3m \); which provides a useful consistency check.

- For massive particles the geodesic orbit corresponds to a timelike worldline and we have the case that \( \epsilon = -1 \). Therefore:

\[
V_{-1}(r) = \left( 1 - \frac{2m}{\sqrt{r^2 + a^2}} \right) \left( 1 + \frac{L^2}{r^2 + a^2} \right), \tag{7.12}
\]

and it is easily verified that this leads to:

\[
V_{-1}'(r) = \frac{2r}{(r^2 + a^2)^{3/2}} \left\{ L^2 \left( 3m - \sqrt{r^2 + a^2} \right) + m \left( r^2 + a^2 \right) \right\}. \tag{7.13}
\]

Equating this to zero and rearranging for \( r \) gives a messy solution for \( r \) as a function of \( L, m \) and \( a \). Instead it is preferable to assume a fixed circular orbit at some \( r = r_c \), and rearrange the required angular momentum \( L_c \) to be a function of \( r_c, m \), and \( a \). It then follows that the innermost circular orbit shall be the value of \( r_c \) for which \( L_c \) is minimised. Hence if \( V_{-1}'(r_c) = 0 \), we have:

\[
L_c^2 \left( 3m - \sqrt{r_c^2 + a^2} \right) + m \left( r_c^2 + a^2 \right) = 0, \tag{7.14}
\]

implying

\[
L_c(r_c, m, a) = \sqrt{\frac{m (r_c^2 + a^2)}{\sqrt{r_c^2 + a^2} - 3m}}, \tag{7.15}
\]

As a consistency check, for large \( r_c \) (i.e. \( r_c \gg a \)) we observe that \( L_c \approx \sqrt{mr_c} \), which is consistent with the expected value when considering circular orbits in weak-field GR. Note that in classical physics the angular momentum per unit mass for a particle with angular velocity \( \omega \) is \( L_c \sim \omega r_c \). Kepler’s third law of planetary motion implies that \( \omega^2 \sim G_N m / r_c \). (Here \( m \) is the mass of the central object, as above.) It therefore follows that \( L_c \sim \sqrt{G_N m / r_c} \). That is \( L_c \sim \sqrt{mr_c} \), as above.

It is then easily obtained that:

\[
\frac{\partial L_c}{\partial r_c} = \left( \frac{\sqrt{mr_c}}{2 \sqrt{r_c^2 + a^2} - 3m} \right) \left( \frac{2}{\sqrt{r_c^2 + a^2}} - \frac{1}{\sqrt{r_c^2 + a^2} - 3m} \right). \tag{7.16}
\]

Solving for stationary points, and excluding \( r_c = 0 \) (as this lies within the photon sphere, which is clearly an invalid solution for the ISCO of a massive particle):

\[
\sqrt{r_c^2 + a^2} - 6m = 0; \quad \Rightarrow \quad r_c = \sqrt{(6m)^2 - a^2}, \tag{7.17}
\]
(once again, discounting the negative solution for \( r_c \) in the interests of remaining in our own universe). We therefore have a coordinate ISCO location at \( r_c = \sqrt{(6m)^2 - a^2} \). This is consistent with the expected value (\( r = 6m \)) for Schwarzschild, when \( a = 0 \). For our traversable wormhole geometry, provided \( 2m < a < 6m \) we will have a valid ISCO location in our coordinate domain. When \( a > 6m \), we have a traversable wormhole with no ISCO.

Denoting \( r_H \) as the location of the horizon, \( r_{\text{Photon}} \) as the location of the photon sphere, and \( r_{\text{ISCO}} \) as the location of the ISCO, we have the following summary:

- \( r_H = \sqrt{(2m)^2 - a^2} \);
- \( r_{\text{Photon}} = \sqrt{(3m)^2 - a^2} \);
- \( r_{\text{ISCO}} = \sqrt{(6m)^2 - a^2} \).

### 8 Regge-Wheeler analysis

Considering the Regge-Wheeler Equation in view of the formalism developed in [69], (see also reference [28]), we may explicitly evaluate the Regge-Wheeler potentials for particles of spin \( S \in \{0, 1\} \) in our spacetime. Firstly define a tortoise coordinate as follows

\[
dr_* = \frac{dr}{\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)},
\]

which gives the following expression for the metric (2.1):

\[
ds^2 = \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \left\{ -dt^2 + dr_*^2 \right\} + (r^2 + a^2) \left( d\theta^2 + \sin^2 \theta \ d\phi^2 \right). \tag{8.2}
\]

It is convenient to write this as

\[
ds^2 = A(r_*)^2 \left\{ -dt^2 + dr_*^2 \right\} + B(r_*)^2 \left( d\theta^2 + \sin^2 \theta \ d\phi^2 \right). \tag{8.3}
\]

The Regge-Wheeler equation is [69]:

\[
\partial_{r_*}^2 \phi + \{\omega^2 - V_S\} \phi = 0, \tag{8.4}
\]

where \( \phi \) is the scalar or vector field, \( V \) is the spin-dependent Regge-Wheeler potential for our particle, and \( \omega \) is some temporal frequency component in the Fourier domain. For a scalar field (\( S = 0 \)) examination of the d’Alembertian equation quickly yields

\[
V_{S=0} = \left\{ \frac{A^2}{B} \right\} \ell(\ell + 1) + \frac{\partial_{r_*}^2 B}{B}. \tag{8.5}
\]

For a vector field (\( S = 1 \)) conformal invariance in 3+1 dimensions guarantees that the Regge-Wheeler potential can depend only on the ratio \( A/B \), whence normalizing to known results implies

\[
V_{S=1} = \left\{ \frac{A^2}{B} \right\} \ell(\ell + 1). \tag{8.6}
\]
Collecting results, for $S \in \{0,1\}$ we have

$$V_S = \left\{ \frac{A^2}{B^2} \right\} \ell (\ell + 1) + (1 - S) \frac{\partial_r^2 B}{B} . \quad (8.7)$$

The spin 2 axial mode is somewhat messier, and not of immediate interest.

Noting that for our metric $\partial_r = \left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right) \partial_r$ and $B = \sqrt{r^2 + a^2}$ we have:

$$\frac{\partial_r^2 B}{B} = \left\{1 - \frac{2m}{\sqrt{r^2 + a^2}} \right\} \left( \frac{2m (r^2 - a^2) + a^2 \sqrt{r^2 + a^2}}{(r^2 + a^2)^{3/2}} \right) . \quad (8.8)$$

Therefore:

$$V_{S \in \{0,1\}} = \left\{1 - \frac{2m}{\sqrt{r^2 + a^2}} \right\} \left\{ \ell (\ell + 1) \frac{1}{r^2 + a^2} + (1 - S) \frac{2m (r^2 - a^2) + a^2 \sqrt{r^2 + a^2}}{(r^2 + a^2)^{5/2}} \right\} . \quad (8.9)$$

This has the correct behaviour as $a \to 0$. Note that this Regge-Wheeler potential is symmetric about $r = 0$. For $a < 2m$ the situation is qualitatively similar to the usual Schwarzschild case (the tortoise coordinate diverges at either horizon, and $V_{S \in \{0,1\}} \to 0$ at either horizon). For $a = 2m$ the tortoise coordinate diverges at the extremal horizon (one-way null throat), while we still have $V_{S \in \{0,1\}} \to 0$. For $a > 2m$ the tortoise coordinate converges at the wormhole throat, while we now have $V_{S \in \{0,1\}}$ nonzero and positive at the throat:

$$V_{S \in \{0,1\}} \to \left(1 - \frac{2m}{a} \right) \left\{ \ell (\ell + 1) \frac{1}{a^2} + (1 - S) \left( \frac{a - 2m}{a^3} \right) \right\} . \quad (8.10)$$

9 Discussion

The regular black hole presented above in some sense represents minimal violence to the standard Schwarzschild solution. Indeed for $a = 0$ it is the standard Schwarzschild solution. For $a \in (0, 2m)$ the Carter-Penrose diagram is in some sense “as close as possible” to that for the maximally extended Kruskal-Szekeres version of Schwarzschild, except that the singularity is converted into a spacelike hypersurface representing a “bounce” into a future incarnation of the universe. This is qualitatively different from the picture where the collapsing regular black hole “bounces” back into our own universe [30–42], and is a scenario that deserves some attention in its own right. The specific model introduced above also has the very nice feature that it analytically interpolates between black holes and traversable wormholes in a particularly clear and tractable manner. The “one-way” wormhole at $a = 2m$, where the throat becomes null and extremal, is particularly interesting and novel.

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