Quantile and Copula Spectrum: A New Approach to Investigate Cyclical Dependence in Economic Time Series

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1 Introduction

Why using quantile spectrum. The purpose of traditional spectrum analysis is to analyze periodic behavior of a stationary time series. Let $X_t$, $t \in \mathbb{Z}$ be a single time series and $\gamma_X(j)$ be its autocovariance function at lag $j$, where $\gamma_X(j) = E(X_0X_j) - \{E(X_0)\}^2$. To this end, the spectrum or spectral density at frequency $\lambda$ is estimated as

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \gamma_X(j) \cos(j\lambda), \quad \lambda \in (-\pi, \pi]. \quad (1)$$

To ensure this function continuous and symmetric about 0, the assumption that the autocovariance function $\gamma_X(j)$ is absolutely summable is imposed on this process. In other words, $X_t$ is assumed to have a finite second moment.

When $X_t$ is covariance stationary, its first and second moments exist and depend not on time $t$ but on its difference. The Wold decomposition theorem shows that a certain covariance stationary series, e.g., $X_t$, can be decomposed into deterministic and stochastic components, e.g., $X_t = \mu + \sum_{j=1}^{\infty} \phi_j \epsilon_{t-j}$, where $\mu$ is a constant mean and $\epsilon_t$ is an i.i.d. random variable and $\sum_{j=1}^{\infty} |\phi_j| < \infty$. This also implies the linearity of the time series. Therefore, even if we transform a time series in the time domain
to the frequency domain, the existence of the second moment and the linearity of the series are implicitly assumed.

In practice, we often encounter the situation where financial time series and macroeconomic time series are not properly shown by ordinary spectrums. These time series are often considered to have heavy tail distributions, infinite moments, nonlinearity, or non-stationarity.

For example, Fig. 1 shows the density functions of three stock returns of the Dow Jones industrial average index, CAC (Cotation Assistée en Continu) 40 index and Nikkei 225 average stock price index, which are solid lines. As a reference, the standard normal density function is also shown as a dotted line. Although the return of the Dow Jones average index more densely distributes around its mean value compared to the standard normal distribution, those of the CAC and Nikkei stock price indices apparently have thicker or heavier tails of their distributions. Like this case, when we estimate single spectrums or cross spectrums of several time series that have peculiarities shown in Fig. 1, ordinary procedures of a spectrum density estimation may not be suited to precisely capture cyclical data variation or data dependent structures due to the restriction of normality. To deal with such cases, we will introduce some novel techniques in the frequency domain in the subsequent sections.

![Fig. 1 Distribution of the returns](image)

Note: The number of observations is 2,516 and the Epachenikov kernel is used for estimation.
Periodogram analyses for understanding and interpreting the properties of macroeconomic and financial series have received a renewed interest in recent years. This interest can be explained by the development of new tools allowing to capture a variety of phenomena: regime changes, structural breaks, volatility giving rise to clusters, non-stationary and nonlinear dynamics and time-irreversibility. The classical spectral analysis, based on Fourier transforms of autocovariance and autocorrelation functions, cannot capture such phenomena, because it is based on some assumptions that are too restrictive to provide a full characterization of the data. For a significant number of economic series, the Gaussianity hypothesis does not fully explain the dependencies in the series when their distributions exhibit extreme events, strong asymmetries and regime-switching dynamics.

To remedy this drawback, several alternative approaches to the classical spectral analysis have been proposed in the literature. The literature is vast (periodograms for locally stationary processes, polyspectrum analysis, evolutionary spectral methods, time-varying second-order spectra). In this chapter, we focus on a thread of the literature which has proposed new types of frequency domain concepts to analyze joint distribution functions at different lags. New concepts such as Laplace, quantile, copula spectrum are increasingly used to detect periodicities in time series by considering their entire distribution rather than focusing on the first two moments.

The survey proposed in this chapter uses some materials provided by researchers during the last decade. Just to mention a few, Hong (1999, 2000), Li (2008, 2012), Hagemann (2013) propose a concept of Laplace spectrum and Laplace periodogram, which are median-related spectral analysis tools. This was extended to cross-spectrum by Hagemann (2013) to detect cycles at different points of a distribution without imposing assumptions on the existence of the moments. Lim and Oh (2015) generalize this approach by introducing a concept of composite quantile spectrum (weighted linear combinations of quantile spectra). Lee and Subba Rao (2012), Dette et al. (2015), Kley et al. (2016) have proposed the concept of copula-based spectrum which does not require any distributional assumption such as the existence of finite moments. Local copula spectral density functions are defined as the Fourier transform of copula cross-covariance kernels. Baruník and Kley (2019) propose an extension to the multivariate case. They introduce quantile cross-spectral quantities (quantile coherency) to capture different dependence structures between time series across quantile.

2 Harmonic Regression Models and Laplace Periodograms

For a given time series \( \{Y_t\} \) of length \( n \) and its frequency \( \omega \in (0, \pi) \), the ordinary periodogram is defined as

\[
G_n(\omega) := \frac{1}{n} \left| \sum_{t=1}^{n} Y_t \exp(-it\omega) \right|^2.
\]
In the above equation, if $\omega = 2\pi k/n$, where $k$ is a certain integer, it can also be expressed as

$$G_n(\omega) = \frac{1}{4} n \|\widehat{\beta}_n(\omega)\|^2 = \frac{1}{4} n \widehat{\beta}'_n(\omega)\widehat{\beta}_n(\omega),$$

(3)

where $\|\cdot\|$ denotes the Euclidean norm, and $\widehat{\beta}_n(\omega)$ denotes the least squares estimator in the linear model with regressors $x_t(\omega) = [\cos(\omega t), \sin(\omega t)]'$, corresponding to an $L_2$-projection of the observed series onto the harmonic basis, which are obtained as the solution of the following equation.

$$\{\lambda_n(\omega), \beta_n(\omega)\} := \arg\min_{\lambda \in \mathbb{R}, \beta \in \mathbb{R}^2} \sum_{t=1}^{n} (Y_t - \lambda - x_t'(\omega)\beta)^2.$$  

(4)

When the OLS criterion is replaced by the least absolute deviation (LAD) criterion in the harmonic regression, the LAD coefficient $\tilde{\beta}_n(\omega)$ is obtained as follows:

$$\{\tilde{\lambda}_n(\omega), \tilde{\beta}_n(\omega)\} := \arg\min_{\lambda \in \mathbb{R}, \beta \in \mathbb{R}^2} \sum_{t=1}^{n} |Y_t - \lambda - x_t'(\omega)\beta|.$$  

(5)

By using $\tilde{\beta}_n(\omega)$, Li (2008) has defined the Laplace periodogram as

$$L_n(\omega) := \frac{1}{4} n \|\tilde{\beta}_n(\omega)\|^2.$$  

(6)

Therefore, both $G_n(\omega)$ and $L_n(\omega)$ are obtained by the squared norm (or sum of squares) of harmonic regression coefficients multiplied by some constant terms. In particular, the Laplace periodogram inherits the robustness properties of linear LAD regression. Just as the OLS estimator is used to characterize the sample mean, the LAD estimator applied captures the behavior of the observation around the median (0.5 quantile).

Li (2008) has derived the asymptotic normality and useful related theorems of the Laplace periodogram, which are very useful to consider asymptotic behaviors of several periodograms. His results are based on the concept of zero-crossings.

**Definition (Stationarity in zero-crossings)** The lagged zero-crossing rate of a random process $\{\varepsilon_t\}$ between $t$ and $s$ is defined as $\gamma_{ts} := P(\varepsilon_t\varepsilon_s < 0)$, and $\{\varepsilon_t\}$ is called to be stationary in zero-crossings if and only if $\gamma_{ts}$ depends only on $t - s$, that is, $\gamma_{ts} = \gamma_{t-s}$ for all $t$ and $s$. $\gamma_t$ is called as the lag-zero-crossing rate of $\{\varepsilon_t\}$ and $S(\omega) := \sum_{t=-\infty}^{\infty} (1 - 2\gamma_t) \cos(\omega t)$ is called as the zero-crossing spectrum of $\{\varepsilon_t\}$.

Using the definition described above, a strictly stationary process is also stationary in zero-crossings.

**Theorem (Asymptotic normality of the Laplace periodogram)** Let $\tilde{\beta}_n(\omega)$ and $L_n(\omega)$ be defined by Eqs. (5) and (6) with $Y_t = \varepsilon_t (t = 1, \ldots, n)$, where $\{\varepsilon_t\}$ is a
random process with a common marginal distribution function \( F(x) \) and density \( f(x) \) such that \( F(0) = 1/2 \) and \( f(0) > 0 \) and such that the assumption of continuous differentiability in a neighborhood of \( x = 0 \) is satisfied. We also assume that (i) \( \{\varepsilon_t\} \) is either an \( m \)-dependent process stationary in zero-crossings or a linear process of the form \( \varepsilon_t = \sum_{j=-\infty}^{\infty} \phi_j \varepsilon_{t-j} \), where \( \{\varepsilon_t\} \) is an i.i.d. random sequence with \( E(|\varepsilon_t|) < \infty \) and \( \{\phi_j\} \) is an absolutely summable deterministic sequence satisfying \( \sum_{|j|>m} |\phi_j| = O(n^{-1}) \) for some \( m = O(n^\delta) \) and \( \delta \in [0, 1/4) \), and (ii) in either case its zero-crossing rates \( \{\gamma_t\} \) satisfy \( \sum_{t=0}^{\infty} |1 - 2\gamma_t| < \infty \).

Let \( S(\omega) \) be the zero-crossing spectrum of \( \{\varepsilon_t\} \), and let \( \{\omega_1, \ldots, \omega_q\} \) be a set of distinct values in \((0, \pi)\) that may depend on \( n \) but that satisfy the condition (iii)

\[
D_{jkn} := n^{-1} \sum_{i=1}^{n} x_i(\omega) x_i'(\omega_k) = \frac{1}{2} \delta_{j-k} I + O(1).
\]

Assume further that \( S(\omega_j) > 0 \) for all \( j \). Then, as \( n \rightarrow \infty \),

\[
n^\frac{1}{2} \text{vec}\{\tilde{\beta}_n(\omega_j)\}_{j=1}^{q} \overset{d}{\to} N(0, 2\eta^2 S) \quad \text{and} \quad \{L_n(\omega_j)\}_{j=1}^{q} \overset{d}{\to} \left\{ \frac{1}{2} \eta^2 S(\omega_j) Z_j \right\},
\]

where \( \overset{d}{\to} \) denotes convergence in distribution, \( \eta^2 := 1/(4f^2(0)) \).

\[
S := \text{diag}\{S(\omega_1), S(\omega_1), \ldots, S(\omega_q), S(\omega_q)\}, \quad \text{and} \quad Z_j \sim \text{i.i.d.} \chi^2(2) \quad (j = 1, \ldots, q).
\]

**Proof** See Li (2008).

This theorem implies that the asymptotic mean of \( L_n(\omega) \) is equal to \( L(\omega) = \eta^2 S(\omega) \), which is the Laplace spectrum of \( \{\varepsilon_t\} \). When \( \{\varepsilon_t\} \) is stationary in second moments, \( G_n(\omega) \overset{d}{\to} \frac{1}{2} \sigma^2 R(\omega) \chi^2(2) \) (Brockwell and Davis 1991), where the autocorrelation spectrum \( R(\omega) := \sum_{\tau=-\infty}^{\infty} \rho_\tau \cos(\omega \tau) = 2 \sum_{\tau=0}^{\infty} \rho_\tau \cos(\omega \tau) - 1 \). In addition, the asymptotic mean of \( G_n(\omega) \), which is called the power spectrum, is \( G(\omega) = \sigma^2 R(\omega) \). The Laplace spectrum \( L(\omega) \) is the counterpart of the power spectrum \( G(\omega) \). The former is proportional to the zero-crossing spectrum \( S(\omega) \), while the latter is proportional to the autocorrelation spectrum \( R(\omega) \). The zero-crossing spectrum is obtained as a Fourier transform of the zero-crossing rates, and the autocorrelation spectrum is obtained as a Fourier transform of the autocorrelation coefficients.

More importantly, \( \eta^2 \) is distribution-dependent, but in general a finite \( \eta^2 \) does not require a finite variance of the asymptotic distribution. This is a theoretical advantage of the Laplace periodogram because it does not require the existence of any moments to have a well-defined asymptotic distribution.\(^1\) This implies that in practice the Laplace periodogram is expected to be more robust to high volatilities of financial data.

\(^1\)When \( \{\varepsilon_t\} \) is a white noise process with a finite variance \( \sigma^2 \), \( L(\omega) = \eta^2 (= 1/(4f^2(0))) \) and \( G(\omega) = \sigma^2 \). Obviously, to obtain the spectrum as the mean of the asymptotic distribution, the ordinary periodogram needs the existence of a finite variance, while the Laplace periodogram needs only the condition of \( f(0) > 0 \).
The median-based approach can be generalized to any quantile regression. Li (2012) has extended the approach to arbitrary quantiles with \(0 < \tau_1 = \tau_2 < 1\). A quantile regression estimator \(\hat{\beta}_{n,\tau}(\omega)\) is obtained as the following solution:

\[
\left\{ \hat{\lambda}_{n,\tau}(\omega), \hat{\beta}_{n,\tau}(\omega) \right\} := \text{argmin}_{\lambda \in \mathbb{R}, \beta \in \mathbb{R}^2} \frac{1}{n} \sum_{t=1}^{n} \rho_{\tau}(Y_t - \lambda - x'_t(\omega)\beta),
\]

(7)

where the check function \(\rho_{\tau}(u) := u(\tau - I(u \leq 0)) = (1 - \tau)|u|I(u \leq 0) + \tau|u|I(u > 0)\) for \(\tau \in (0, 1)\) (see Koenker 2005). Li (2012) has defined the quantile periodogram (of the first kind) at quantile level \(\tau\) as

\[
Q^I_{n,\tau}(\omega) := \frac{1}{4}n \left\| \hat{\beta}_{n,\tau}(\omega) \right\|^2.
\]

(8)

This is a scaled version of the squared norm of the quantile regression coefficients corresponding to the trigonometric regressors. With the special choice of \(\tau = 1/2\) (median or LAD regression), the quantile periodograms are reduced to the Laplace periodograms.

An alternative quantile periodogram (of the second kind) at quantile level \(\tau\) is defined as

\[
Q^{II}_{n,\tau}(\omega) := \sum_{i=1}^{n} \rho_{\tau}\left(Y_t - \hat{\lambda}_{n,\tau}\right) - \sum_{i=1}^{n} \rho_{\tau}\left(Y_t - \hat{\lambda}_{n,\tau}(\omega) - x'_t(\omega)\hat{\beta}_{n,\tau}\right)
\]

(9)

where \(\hat{\lambda}_{n,\tau}\) is the sample \(\tau\)th quantile given by \(\hat{\lambda}_{n,\tau} := \text{argmin}_{\lambda \in \mathbb{R}} \sum_{t=1}^{n} \rho_{\tau}(Y_t - \lambda)\).

The main difference with Eq. (8) is that, in Eq. (9), we consider the net effect of the trigonometric regressors. It can be shown that

\[
\left\{ Q^{I}_{n,\tau}(\omega) \right\} \overset{d}{\to} \left\{ \frac{1}{2} \eta^2_I S(\omega_j) Z^1_j \right\}, \quad Z^1_j \sim \text{i.i.d.} \chi^2(2), \quad j = 1, \ldots, q,
\]

(10a)

\[
\left\{ Q^{II}_{n,\tau}(\omega) \right\} \overset{d}{\to} \left\{ \frac{1}{2} \eta^2_{II} S(\omega_j) Z^2_j \right\}, \quad Z^2_j \sim \text{i.i.d.} \chi^2(2), \quad j = 1, \ldots, q,
\]

(10b)

where \(\eta^2_I = \tau(1 - \tau)/\kappa^2, \eta^2_{II} = \tau(1 - \tau)/\kappa\), are scaling constants.

**Proof** See Li (2012).

\(Q^I_{n,\tau}(\omega)\) and \(Q^{II}_{n,\tau}(\omega)\) provide information about the cyclical behavior of the time series \(\{Y_t\}\) around its \(\tau\)th quantile level. They both have asymptotic exponential distributions with mean \(\eta^2_I S(\omega_j)\) and \(\eta^2_{II} S(\omega_j)\) which are called quantile spectra. For fixed values of \(\tau\), \(S(\omega_j)\) is the Fourier transform of the autocorrelation function of \(\{\text{sgn}(Y_t - \hat{\lambda}_{n,\tau})\}\), or the spectral cumulative representation of the serial dependence of \(\{Y_t\}\) in terms of the bivariate cumulative probabilities.
\begin{equation}
F_{\tau}(\lambda, \lambda) = P(Y_t \leq \lambda, Y_{t+\tau} \leq \lambda). \tag{11}
\end{equation}

Dette et al. (2015) have generalized the quantile periodogram for arbitrary \((\tau_1, \tau_2) \in (0, 1)^2\) (not necessarily equal), which is defined as

\begin{equation}
L_{n,\tau_1,\tau_2}(\omega) := \frac{1}{4} \hat{n}^{\tau_1,\tau_2}(\omega) \left( \begin{array}{c}
i \\
-1 \\
i \\
1 \end{array} \right) \hat{\beta}_{n,\tau_2}(\omega). \tag{12}
\end{equation}

This can be regarded as an extended version of the Laplace periodogram. The asymptotic properties of \(L_{n,\tau_1,\tau_2}(\omega)\) can be established under two assumptions.

**Notation**
Consider the sequence \(\{Y_t\}_t\) where each \(Y_t\) is a measurable function defined by the probability space \((\Omega, \mathcal{F}, P)\). We define \(Y^j_i := \{Y_t\}_{j=i}^\infty\), where \(i\) and \(j\) are integers. The \(\sigma\)-field generated by \(X^j_i\) is \(\sigma^j_i\) and \(P^j_i\) the joint distribution of \(Y^j_i\).

Moreover, define \(q_\tau = P^{-1}(\tau), \tau \in [0, 1]\) the \(\tau\)th quantile of the distribution \(P\).

**Assumption 1** \(\{Y_t\}\) is strictly stationary and \(\beta\)-mixing. The stationarity hypothesis implies that, for all \(t\) and \(i, j\) (non-negative integers), the vectors \(Y_{t+i}\) and \(Y_{t+j}\) have the same distribution. \(\beta\)-mixing can be defined as follows (there are several equivalent definitions in the literature). For each positive integer \(n\), we have:

\begin{equation}
\lim_{k \to \infty} \beta(k) := \sup \| P^1_i \otimes P^{n+k}_i - P_{t+k} \|_{TV} = O(k^{-\delta}), \quad \delta > 1 \tag{13}
\end{equation}

where \(\| \cdot \|_{TV}\) is the total variation mean, \(P_{t,a}\) is the joint distribution of \((X^1_i, P^{n}_i)\), \(\beta(k)\) is the distance between the joint distribution of random variables separated by \(k\) time units and a distribution under which these random variables are independent. Many time series satisfy Assumption 1 (for instance ARMA processes and stochastic volatility models like GARCH).

**Assumption 2** The distribution function \(P_t\) of \(Y_t\) and the joint distribution function \(P_t^{n+k}\) of \((Y_t, Y_{t+k})\) are twice differentiable with uniformly bounded derivatives.

**Theorem** Asymptotic normality of \(\hat{\beta}_{n,\tau}(\omega)

Under Assumptions 1 and 2, we have

\begin{equation}
n^{\frac{3}{2}} \text{vec} \left( \hat{\beta}_{n,\tau}(\omega) \right)^j \overset{d}{\to} N(O, M) \tag{14}
\end{equation}

where \(M := M_{\tau_1,\tau_2}(\omega_1, \omega_2)\) is the covariance matrix defined by

\(M_{\tau_1,\tau_2}(\omega_1, \omega_2) :=
\begin{cases}
\pi \left( \hat{\mathbb{R}} \hat{f}_{\tau_1,\tau_2}(\omega) \hat{\mathbb{S}} \hat{f}_{\tau_1,\tau_2}(\omega) \right), & \text{if } \omega_1 = \omega_2 = \omega \\
\left( 
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array} \right), & \text{if } \omega_1 \neq \omega_2.
\end{cases}
\)

\(\hat{f}_{\tau_1,\tau_2}(\omega)\) is the scaled version of the spectral density function:
\[ \hat{f}_{\tau_1, \tau_2}(\omega) = \frac{f_{\lambda_1, \lambda_2}(\omega)}{(f_{\lambda_1}(\omega)f_{\lambda_2}(\omega))}, \lambda_1 = P^{-1}(\tau_1), \lambda_2 = P^{-1}(\tau_2), \]

and \( f_{\lambda_1, \lambda_2}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \gamma_k(\tau_1, \tau_2)e^{-ik\omega}, ~ (\tau_1, \tau_2) \in (0, 1)^2. \)

\( \gamma_k(\tau_1, \tau_2) \) is the Laplace cross-covariance kernel of lag \( k \) defined by

\[ \gamma_k(\tau_1, \tau_2) := \text{COV}[I(Y_t \leq \tau_1), I(Y_{t+k} \leq \tau_2)]. \]

\( \Re \hat{f}_{\tau_1, \tau_2}(\omega) \) and \( \Im \hat{f}_{\tau_1, \tau_2}(\omega) \) are the real and imaginary parts of \( \hat{f}_{\tau_1, \tau_2}(\omega) \).

**Proof** See Dette et al. (2015).

**Theorem** Asymptotic normality of \( L_{n, \tau_1, \tau_2}(\omega) \)

Under Assumptions 1 and 2, we have

\[ L_{n, \tau_1, \tau_2}(\omega_1, \omega_2) \left\{ \begin{array}{l}
\sim \pi \hat{f}_{\tau_1, \tau_2}(\omega)Z, \quad Z \sim \text{i.i.d.} \chi^2(2), \text{ if } \tau_1 = \tau_2 \\
\Rightarrow \frac{1}{4}(Z_{11}, Z_{12}) \left( \begin{array}{cc}
1 & i \\
-i & 1 \\
\end{array} \right) (Z_{21}, Z_{22}), \quad \text{if } \tau_1 \neq \tau_2.
\end{array} \right. \quad (15) \]

where \( (Z_{11}, Z_{12}, Z_{21}, Z_{22}) \) is a Gaussian vector with mean \( \theta \) and covariance matrix

\[ \Sigma(\omega) = \left( \begin{array}{cccc}
\hat{f}_{\tau_1, \tau_2}(\omega) & 0 & \Re \hat{f}_{\tau_1, \tau_2}(\omega) & \Im \hat{f}_{\tau_1, \tau_2}(\omega) \\
0 & \hat{f}_{\tau_1, \tau_2}(\omega) & -\Im \hat{f}_{\tau_1, \tau_2}(\omega) & \Re \hat{f}_{\tau_1, \tau_2}(\omega) \\
\Re \hat{f}_{\tau_1, \tau_2}(\omega) & -\Im \hat{f}_{\tau_1, \tau_2}(\omega) & \hat{f}_{\tau_1, \tau_2}(\omega) & 0 \\
\Im \hat{f}_{\tau_1, \tau_2}(\omega) & \Re \hat{f}_{\tau_1, \tau_2}(\omega) & 0 & \hat{f}_{\tau_1, \tau_2}(\omega)
\end{array} \right) \]

**Proof** See Dette et al. (2015).

### 3 Sample and Smoothed Laplace Periodogram

Define the following new variable of interest called a *quantile crossing indicator*:

\[ V_t(\tau, q(\tau)) = \tau - I\{Y_t < q(\tau)\}. \quad (16) \]

If the distribution function of \( Y_t \) is continuous and increasing at

\[ q(\tau) := \inf\{y : P(Y \leq y)\}, \quad (17) \]

\( V_t(\tau) \) is bounded, stationary and mean zero random variable. Using Koenker and Basset’s approach, we define an estimate of \( V_t(\tau) \) as follows:

\[ \hat{V}_t(\tau) = V_t(\tau, \hat{q}_n(\tau)) \quad (18) \]
where \( \hat{q}_n(\tau) = \arg\min_{q \in \mathbb{R}} \sum_{t=1}^{n} \rho_{\tau}(Y_t - y), \rho_{\tau}(x) = x[\tau - I(x < 0)] \). \( \hat{q}_n(\tau) \) is the estimate of the \( \tau \)th quantile. The \( \tau \)th quantile periodogram is given by

\[
Q_{n,\tau}(\omega) := \frac{1}{2\pi} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \hat{V}_t(\tau) e^{-it\omega} \right| = \frac{1}{2\pi} \sum_{|j| < n} \hat{r}_{n,\tau}(j) \cos(\omega j),
\]

where \( i^2 = -1 \) and \( \hat{r}_{n,\tau}(j) = \frac{1}{n} \sum_{t=|j|+1}^{n} \hat{V}_t(\tau) \hat{V}_{t-|j|}(\tau), \quad |j| < n \). \( Q_{n,\tau}(\omega) \) is an unbiased estimate of the \( \tau \)th spectral density, but is not consistent. A consistent estimator is obtained by smoothing the periodogram using kernel functions (all the results below are taken from Hagemann 2013).

We obtained a smoothed \( \tau \)th quantile periodogram as

\[
\hat{Q}_{n,\tau}(\omega) = \frac{1}{2\pi} \sum_{|j| < n} \lambda(j/b_n) \hat{r}_{n,\tau}(j) \cos(\omega j).
\]

\( \lambda(j/b_n) \) is a lag window and \( b_n \) is a bandwidth parameter. It is known from the literature on spectral analysis that an optimal lag window leading a non-negative periodogram is the so-called quadratic spectral window defined as

\[
\lambda_{QS}(x) = \frac{25}{12\pi^2 x^2} \left\{ \sin\left(\frac{6\pi x}{5}\right) - \cos\left(\frac{6\pi x}{5}\right) \right\}.
\]

The following results hold (for the proofs, see Hagemann 2013).

**Result 1: Confidence interval**

Define

\[
\overline{Q}_{n,\tau}(\omega, k) = \frac{1}{2k+1} \sum_{|j| < k} \hat{Q}_{n,\tau}\left(\omega + \frac{2\pi j}{n}\right), \quad k \in \mathbb{Z}.
\]

Then, a confidence interval of the smoothed periodogram with a probability of \( (1 - \alpha)\% \) is

\[
\left[ \frac{(4k + 2) \overline{Q}_{n,\tau}(\omega, k)}{Z_1}, \frac{(4k + 2) \overline{Q}_{n,\tau}(\omega, k)}{Z_2} \right],
\]

\( Z_1 \sim \text{i.i.d.} \chi^2_{(4k + 2)}, \quad Z_2 \sim \text{i.i.d.} \chi^2_{(4k + 2)} \)

**Result 2: Asymptotic normality**

Assume that
(i) \( F_Y(y) := P(Y \leq y) \) is Lipschitz continuous in a neighborhood of \( q(\tau) := \inf\{y : P(Y \leq y) \geq \tau\} \) and has a continuous density at \( q(\tau) \).

(ii) There is some \( n^* \) such that for \( n > n^* \), \( \hat{F_Y}(y) := P(\hat{Y} \leq y) \) is Lipschitz continuous in a neighborhood of \( q(\tau) \) and \( E|Y_n - \hat{Y}_n| = O(\rho^n) \), for some \( \rho \in (0, 1) \) (geometric moment contracting, a property that is satisfied for many linear and nonlinear stochastic models).

(iii) \( \lambda \) is even and Lipschitz continuous with support \([-1, 1] \), \( \lambda(0) = 1 \),

\[
\lim_{x \to 0} [1 - \lambda(x)]|x|^3 < \infty, \quad b_n \to \infty, \quad b_n = O\left(n^{\frac{1}{2}}\right), \quad n = o\left(b_n^7\right).
\]

Then,

\[
\sqrt{\frac{\text{int}\left(\frac{n}{2}\right)}{b_n}} \left[ \hat{Q}_{n,\tau}(\omega) - Q_{n,\tau}(\omega) \right] \sim N(0, \sigma^2(\eta)), \quad (24)
\]

where \( \sigma^2(\eta) = [1 + h(2\eta)]Q_{n,\tau}(\omega) \int_{-1}^{1} \lambda(x)dx, \quad h(\eta) = \begin{cases} 1, & \text{if } \omega = 2\pi k, \quad k \in \mathbb{Z} \\ 0, & \text{Otherwise.} \end{cases} \)

\[
\hat{Q}_{n,\tau}(\omega) = \frac{1}{2\pi \text{int}\left(\frac{n}{2}\right)} \sum_{|j| < \text{int}\left(\frac{n}{2}\right)} \lambda(j/4b_n) \hat{r}_{n,\tau}(j) \cos(\omega j) \quad (25)
\]

\( \hat{r}_{n,\tau}(j) \) can be used to test whether the spectrum is informative at a given \( \tau \)th quantile. We present here two approaches proposed by Hagemann (2013) to test the flatness of the quantile spectrum based on two Cramer-von Mises tests.

The null hypothesis to test is

\[
H_0: r_{n,\tau}(j) = 0, \quad \forall j > 0 \text{ against } H_1: r_{n,\tau}(j) \neq 0, \quad \text{for some } j > 0. \quad (26)
\]

Under \( H_0 \) the flatness of the quantile spectrum implies that \( Q_{n,\tau}(\omega) \equiv \frac{\tau(1-\tau)}{2\pi} \).

The Cramer-von Mises statistic is defined by:

\[
\text{CM}_{n,\tau}(j) = \frac{n}{j} \sum_{j=1}^{n-1} \left( \hat{r}_{n,\tau}(j) / j \right)^2. \quad (27)
\]

The following quantity can be used to realize the test:

\[
\hat{\text{CM}}_{n,\tau} = \frac{1}{2\pi n} \sum_{j=1}^{n-1} j^{-2} \left( \sum_{t=j+1}^{n-1} (\tau - J_t)(\tau - J_t) \right)^2, \quad (28a)
\]
where \( J_1, J_2, \ldots, J_n \) are independent Bernoulli \((\tau)\). Under the assumption that \( Y_t \) is i.i.d., the statistics in (28a) has the same distribution as

\[
\widetilde{\text{CM}}_{n, \tau} = \frac{1}{2\pi n} \sum_{j=1}^{n-1} j^{-2} \left( \sum_{t=j+1}^{n-1} \widehat{V}_t(\tau) \widehat{V}_{t-j}(\tau) \right)^2.
\] (28b)

In large samples, \( V_t(\tau, q(\tau)) \) is close to \( \tau - I\{Y_t < q(\tau)\} \) in probability. However, in small samples, \( \{Y_t < q(\tau)\} \) is a Bernoulli random variable with a success probability \( \tau \).

The test is realized through a Monte Carlo procedure which consists of the following steps:

- **Step 1.** Draw \( n \) i.i.d. copies of \( J_1, J_2, \ldots, J_n \) of Bernoulli \((\tau)\) random variables.
- **Step 2.** Compute (28a) with the variables drawn at step 1.
- **Step 3.** Repeat steps 1 and 2, \( R \) times with \( R \) large.
- **Step 4.** Define \( c_{n, \tau}(1 - \alpha) \) as the \((1 - \alpha)\) empirical quantile of the \( R \) realizations of (28a).

\[
c_{n, \tau}(1 - \alpha) := \inf \{ z \in \mathbb{R} : P(\text{CM}_{n, \tau} > z) \leq \alpha \}.
\]

The null hypothesis is rejected if \( \text{CM}_{n, \tau} \) is larger than \( c_{n, \tau}(1 - \alpha) \).

4 Copula-Based Periodogram and Rank-Based Laplace Periodogram

Laplace periodograms can be used to estimate copula spectra density kernels. We briefly present the methodology here since copula models have become widely used in economics and finance (see Patton 2012 for a review of theory and empirical estimation). One important advantage of copulas is that they do not require any distributional assumption, such as for instance the existence of finite moments.

Let us consider again a strictly stationary time series \( \{Y_t\}_{t \in \mathbb{Z}} \) and its marginal distribution function \( F \). In the traditional approach, the spectral density kernels are defined associated with autocovariance kernels of the series. To capture more general features of pairs of \( Y_t \) and \( Y_{t-k} \), the clipped processes \( (I\{Y_t \leq q\})_{t \in \mathbb{Z}} \) and \( (I\{U_t \leq \tau\})_{t \in \mathbb{Z}} \), where \( U_t := F(Y_t) \) are introduced; then, the spectral density kernels are defined associated with covariance kernels of these clipped processes, which are shown below.

\[
\gamma_k(q_1, q_2) := \text{Cov}(I\{Y_t \leq q_1\}, I\{Y_{t-k} \leq q_2\}), \quad q_1, q_2 \in \mathbb{R}, k \in \mathbb{Z},
\] (29)
where $I \{ \cdot \}$ denotes the indicator function and $\mathbb{R} := \mathbb{R} \cup \{-\infty, \infty\}$ the extended real line. The definition described above is the Laplace cross-covariance. The copula cross-covariance is

$$\gamma^U_k(\tau_1, \tau_2) := \text{Cov}(I\{U_t \leq \tau_1\}, I\{U_{t-k} \leq \tau_2\}), \quad \tau_1, \tau_2 \in [0, 1], k \in \mathbb{Z}. \quad (30)$$

By using the Laplace cross-covariance and the copula cross-covariance, researchers can consider more general dependence structures of $Y_t$ and $Y_{t-k}$ that traditional covariance-based methods unable to deal with, such as time-irreversibility, tail dependence, varying conditional skewness or kurtosis, and so on, though various extensions and revisions have been proposed in the $L_2$-periodograms (Kleiner et al. 1979; Klüppelberg and Mikosch 1994; Mikosch 1998; Katkovnik 1998; Hong 1999; Hill and McCloskey 2014).

Under summability conditions on $\gamma_k$ and $\gamma^U_k$, the population spectral densities are defined as follows.

$$f_{q_1, q_2}(\omega) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k(q_1, q_2)e^{-ik\omega}, \quad q_1, q_2 \in \mathbb{R}, \omega \in \mathbb{R}, \quad (31)$$

$$f_{q_{\tau_1}, q_{\tau_2}}(\omega) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma^U_k(\tau_1, \tau_2)e^{-ik\omega}, \quad \tau_1, \tau_2 \in [0, 1], \omega \in \mathbb{R}, \quad (32)$$

where $q_{\tau_i} := F^{-1}(\tau_i) (i = 1, 2)$. The former equation is called the Laplace spectral density, and the latter one is called the copula spectral density. These formulas are Fourier transforms of $\{\gamma_k\}$ and $\{\gamma^U_k\}$; therefore, based on the Fourier theorem, the inverse Fourier transform provides

$$\gamma_k(q_1, q_2) = \int_{-\infty}^{\infty} e^{i k \omega} f_{q_1, q_2}(\omega) d\omega, \quad (33)$$

$$\gamma^U_k(\tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{i k \omega} f_{q_{\tau_1}, q_{\tau_2}}(\omega) d\omega. \quad (34)$$

Since the clipped processes which take binary values satisfy the strict stationarity assumption, the approach based on the Laplace- and Copula spectral densities is still applicable to detect various statistical relationships in more appropriate manners, which are beyond the scope of the traditional spectral approach. In this aspect, the vast amount of theoretical work has been conducted by Li (2008, 2012, 2013, 2014), Hagemann (2013), Lee and Subba Rao (2012), Dette et al. (2015), and Kley et al. (2016).

An estimate of copula spectrum can be obtained by computing the copula periodogram $L^U_{n, \tau_1, \tau_2}(\omega)$, associated with the series $U_1, \ldots, U_n$, where $U_t := F(Y_t)$, is
defined. \( L^U_{n,\tau_1,\tau_2}(\omega) \) is obtained in Eq. (12) by replacing \( \hat{\beta}_{n,\tau} \) by the \( \hat{\beta}^U_{n,\tau} \), which is obtained below.

\[
\left\{ \lambda_{n,\tau}(\omega), \hat{\beta}^U_{n,\tau}(\omega) \right\} := \arg\min_{\lambda \in \mathbb{R}, \beta \in \mathbb{R}^2} \sum_{t=1}^{n} \rho_\tau(U_t - \lambda - x'_t(\omega)\beta).
\]

Since the distribution function \( F(Y_t) \) is unknown, it is replaced by the rank of \( Y_t \). In this case, the periodogram is called the empirical or rank-based Laplace periodogram, which is defined associated with the normalized rank series \( n^{-1}R^{(n)}_1, \ldots, n^{-1}R^{(n)}_n \), where \( R^{(n)}_t \) is the rank of \( Y_t \) among \( Y_1, \ldots, Y_n \). The rank-based Laplace periodogram \( L^R_{n,\tau_1,\tau_2}(\omega) \) is obtained in Eq. (12) by replacing \( \hat{\beta}_{n,\tau} \) by \( \hat{\beta}^R_{n,\tau} \). We have:

\[
\left\{ \hat{\lambda}_{n,\tau}(\omega), \hat{\beta}^R_{n,\tau}(\omega) \right\} := \arg\min_{\lambda \in \mathbb{R}, \beta \in \mathbb{R}^2} \sum_{t=1}^{n} \rho_\tau(\tau[n^{-1}R^{(n)}_t - \lambda - x'_t(\omega)\beta]).
\]

5 The Multivariate Case

Now we consider the multivariate case of quantile spectral densities and periodograms based on the copula- and Laplace-related concepts, which have been already introduced in the univariate case in the previous sections. Let \( \{X_t\}_{t \in \mathbb{Z}} \) be a \( d \)-variate strictly stationary process, with components \( X_{t,l} \), \( l = 1, \ldots, d \); i.e., \( X_t = (X_{t,1}, \ldots, X_{t,d})' \), \( X_{t,l} \) has its marginal distribution function \( F_l(q) \) and inverse function \( q_l(\tau) := F_l^{-1}(\tau) := \inf\{q \in \mathbb{R}: \tau \leq F_l(q)\} \), where \( \tau \in [0, 1] \). The matrix of quantile cross-covariance, \( \Gamma(\tau_1, \tau_2) := \left( \gamma_{l_1l_2}^{(\tau_1, \tau_2)} \right)_{l_1, l_2 = 1, \ldots, d} \), where \( \gamma_{l_1l_2}^{(\tau_1, \tau_2)} \) is the cross-covariance of a pair of \( (X_{t,l_1}, X_{t,l_2}) \), which is as follows.

\[
\gamma_{l_1l_2}^{(\tau_1, \tau_2)} := \text{Cov}\left( I\{X_{t,l_1} \leq q_{l_1}(\tau_1)\}, I\{X_{t-k,l_2} \leq q_{l_2}(\tau_2)\}\right),
\]

where \( l_1, l_2 \in \{1, \ldots, d\}, k \in \mathbb{Z}, \) and \( \tau_1, \tau_2 \in [0, 1] \). The quantile-based quantities are functions of \( \tau_1 \) and \( \tau_2 \), which are quantiles of a quantile regression. In the frequency domain, under approximate mixing conditions, the quantile cross-spectral densities are

\[
f_{q_1, q_2}(\omega) := \left( f_{q_1, q_2}^{l_1l_2}(\omega) \right)_{l_1, l_2 = 1, \ldots, d},
\]

where
\[ f_{q_1, q_2}^{l_1, l_2} (\omega) := \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{k}^{l_1, l_2} (\tau_1, \tau_2) e^{-ik\omega}, \]  

(39)

\[ l_1, l_2 \in \{1, \ldots, d\}, \omega \in \mathbb{R}, \text{ and } \tau_1, \tau_2 \in [0, 1]. \] Each quantile cross-spectral density, i.e., \( f_{q_1, q_2}^{l_1, l_2} (\omega) \), is a complex-valued function. As considered in traditional spectral analysis, its real and imaginary parts are referred to as the quantile cospectrum and quantile quadrature spectrum.

To measure dynamic dependence structure of the two processes \( \{X_{t, l}\}_{t \in \mathbb{Z}} \) and \( \{X_{t, l}\}_{t \in \mathbb{Z}} \), the quantile coherency is defined as follows.

\[ R_{q_1, q_2}^{l_1, l_2} (\omega) := \frac{f_{q_1, q_2}^{l_1, l_2} (\omega)}{\left( f_{q_1, q_1}^{l_1, l_1} (\omega) f_{q_2, q_2}^{l_2, l_2} (\omega) \right)^{1/2}}, \]  

(40)

where \((\tau_1, \tau_2) \in [0, 1]^2\). The modulus squared of this quantile coherency, e.g., \( |R_{q_1, q_2}^{l_1, l_2} (\omega)|^2 \), is referred to as the quantile coherence.

When we use the empirical distribution function of \( X_{t, l} \), i.e., \( \hat{F}_{n,l}(x) := n^{-1} \sum_{i=0}^{n-1} I \{ X_{t, l} \leq x \} \), the rank-based copula cross-periodograms (CCR periodograms) are defined as

\[ I_{n,R}^{l_1, l_2} (\omega; \tau_1, \tau_2) := \frac{1}{2\pi n} d_{n,R}^{l_1}(\omega; \tau_1) d_{n,R}^{l_2}(\omega; \tau_2), \]  

(41)

where \( l_1, l_2 \in \{1, \ldots, d\}, \omega \in \mathbb{R}, (\tau_1, \tau_2) \in [0, 1]^2 \), and

\[ d_{n,R}^{l}(\omega; \tau) := \sum_{i=0}^{n-1} I \left\{ \hat{F}_{n,l}(X_{t,l}) \leq \tau \right\} e^{-i\omega t} = \sum_{i=0}^{n-1} I \left\{ R_{t,l}^{(n)} \leq n\tau \right\} e^{-i\omega t}. \]  

(42)

where \( l = 1, \ldots, d, \omega \in \mathbb{R}, \tau \in [0, 1] \), and \( R_{t,l}^{(n)} \) is also the rank of \( X_{t,l} \) among \( X_{0,l}, \ldots, X_{t-1,l} \).

Kley et al. (2016) have shown that the CCR periodogram does not have the consistency when it is used to estimate the quantile cross-spectral density \( f_{q_1, q_2}^{l_1, l_2} (\omega) \). On the other hand, its smoothed versions, i.e., smoothed CCR periodogram shown below gains the consistency by correcting biases (see Kley et al. 2016, Theorem 3.5).

\[ \hat{G}_{n,R}^{l_1, l_2} (\omega; \tau_1, \tau_2) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) I_{n,R}^{l_1, l_2} \left( \frac{2\pi s}{n}; \tau_1, \tau_2 \right). \]  

(43)

where \( W_n \) denotes a sequence of weight functions. The smoothed CCR periodogram also maintains the asymptotic normality. In addition, note that fixing \( l_1, l_2 \) and
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\( \tau_1, \tau_2 \), the smoothed CCR periodogram is asymptotically equivalent to traditional smoothed periodogram determined from the unobservable bivariate time series \( \{ I \{ F_{l1}(X_{t,l1}) \leq \tau_1 \}, I \{ F_{l2}(X_{t,l2}) \leq \tau_2 \} \} \) \( (t = 0, \ldots, n - 1) \). By using this asymptotic normality property, the pointwise asymptotic confidence intervals for the real and imaginary parts of the spectrum can be computed for a pair of \( (\tau_1, \tau_2) \).

The consistent estimators of quantile coherency are also defined as

\[
\hat{\mathcal{R}}_{n,R,q_{\tau_1},q_{\tau_2}}^{l_1 l_2}(\omega) := \frac{\hat{G}_{n,R}(\omega; \tau_1, \tau_2)}{\left( \hat{G}_{n,R}(\omega; \tau_1, \tau_1) \hat{G}_{n,R}(\omega; \tau_2, \tau_2) \right)^{1/2}}.
\]

(44)

The difference between this coherency and \( \mathcal{R}_{q_{\tau_1},q_{\tau_2}}^{l_1 l_2}(\omega) \), with bias correction terms, asymptotically converges to a normal distribution, which implies the asymptotic consistency (see Barunik and Kley 2019, Theorem 4.1).

6 Empirical Example

This section shows an example of the quantile-based spectral analysis for the stock returns by using the R package “QUANTSPEC version 1.2.1”. The following returns of daily stock average indexes (Dow Jones Industrial Average, CAC 40, and Nikkei 225) were taken from “Factiva.com” during the post-period of global financial crisis from July 27th 2009 to March 27th 2020 (2516 observations).

We first plot the following three types of data for each stock index: (1) \( Y_t \): returns (2) \( \text{Cov}(Y_{t+k}, Y_t) \): autocovariances of the returns, and (3) \( \text{Cov}(Y_{t+k}^2, Y_t^2) \): autocovariances of the squared returns. Figure 2 shows the stock prices and their returns of three stock average indexes, DJ (Dow Jones Industrial Average in the United States), CAC (CAC 40 in France), and Nikkei (Nikkei 225 in Japan). Each return seems to have zero-mean with some outliers.

Their highly volatile periods correspond to “Flash crash” in May 2010, “Black Monday” in August 2011, “China shock” in August 2015, “Brexit” in June 2016, and “VIX shock” in February 2018, and “Coronavirus shock” in March 2020. Additionally, the highly volatile period, especially limited to Nikkei (Japanese market), corresponds to the “East Japan great earthquake” in March 2011.

Figure 3 shows their autocovariances with lag \( k \). DJ has significantly negative serial correlations (Lag = 1, 3, 5, 8, or 19) and positive correlations (Lag = 2, 9, or 11). CAC has a significantly negative serial correlation (Lag = 5) and a positive correlation (Lag = 6). Nikkei seems to have no serial correlation. Thus, only Japanese stock market appears to be uncorrelated. This is a typical characteristic of many financial returns, as long as we use a linear measure of dependence.

Figure 4 shows the autocovariances of the squared returns, i.e., their volatilities. In the series of all volatilities, we can find significant and persistent autocovariances. These squared returns are clearly correlated. However, all autocovariances persist
until at least lag 14 (more than 2 weeks). The persistency of their volatilities suggests that an ARCH or GARCH model will be required if we focus on the traditional approach of financial analyses. In this section, we focus on another approach, i.e., quantile-based spectral analysis.
Before going on quantile-based spectral analysis, we estimate the traditional smoothed periodograms of the returns in Figure 5 (Non-parametric model modified by Daniell smoothing). All the periodograms show some peaks, but do not near zero frequency. CAC and Nikkei have peaks around 0.15 and 0.35 frequencies, which means highly volatile stock returns in 42 and 18 periods \((2\pi/0.15\) and \(2\pi/0.35\)). DJ also fluctuates but its amplitude is much smaller than the other two. The feature of less volatile US market has already been observed in Fig. 1.

Figure 6a–c show the copula rank periodograms \(I_{n,R}^{\tau_1,\tau_2}(\omega)\) for \(\tau_1, \tau_2 \in \{0.1, 0.5, 0.9\}\), all Fourier frequencies \(\omega \in (0, \pi)\), 200 moving blocks bootstrap replications with block length 40 (about 2 months). In Figure 7a–c, we draw the smoothed copula rank periodograms \(\hat{G}_{n,R}(\omega; \tau_1, \tau_2)\) by using the computed quantile periodogram with the Epanechnikov kernel and bandwidth 0.07. In addition, pointwise confidence intervals are obtained by a normal approximation to the distribution of the estimator. The periodograms suggest serial dependent structure of each stock return below the 10, 50, and 90% quantiles.

The figures show that the copula spectra of three stock returns are not flat, showing clear evidence of serial dependency. This result turns out to be different from their

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\[^2\] \(I_{n,R}^{\tau_1,\tau_2}(\omega)\) is one-variate case of \(I_{n,R}^{l_1l_2}(\omega; \tau_1, \tau_2)\) for \(X_t = X_{t,1}\) in Eq. (41), which is called the CR periodogram. Its smoothed version \(\hat{G}_{n,R}(\omega; \tau_1, \tau_2)\) is also a special case of \(\hat{G}_{n,R}^{l_1l_2}(\omega; \tau_1, \tau_2)\) in Eq. (43).
slight evidence of small autocovariances in Fig. 3. The top left and bottom right figures show the serial dependency structure in negative and positive extreme events. The copula rank periodograms in Fig. 6a–c show that high serial dependency is observed near 0.0 for some cases. The smoothed copula rank periodograms in Fig. 7a–c also suggest that both negative and positive extreme events have some power to change the serial dependency structures in all stock markets. In addition, the copula rank periodograms (or its smoothed versions) indeed have apparent peaks in some frequencies and fluctuate over frequencies as well.

We also show the rank-based Laplace periodograms and its smoothed versions in Fig. 8. Figure 8a–c correspond to the rank-based Laplace periodograms $L_{n,\tau_1,\tau_2}^R(\omega)$. Figure 8d–f correspond to the smoothed rank-based Laplace periodograms $\tilde{f}_{n,R}(\omega; \tau_1, \tau_2)$.\(^3\) Obviously, compared to the (smoothed) traditional $L_2$-periodograms shown in Fig. 5, the rank-based Laplace periodograms or its smoothed rank-based Laplace periodograms seem to better capture the correlation structures in the stock return series and provide much richer views.

As is evident in these figures, one important aspect is that the copula rank periodograms and the rank-based Laplace periodograms, including their smoothed

\(^3\)The smoothed rank-based Laplace periodogram is defined as $\tilde{f}_{n,R}(\omega; \tau_1, \tau_2) := \frac{2\pi}{\pi} \sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) L_{n,\tau_1,\tau_2}^R(\frac{2\pi s}{\pi})$, where $W_n$ denotes a sequence of weight functions.
versions, are quantile dependent, i.e., their values change across quantile \( \tau \). The periodograms estimated at the tails of the distribution (\( \tau_1 = \tau_2 = 0.1 \) and \( \tau_1 = \tau_2 = 0.9 \)) suggest strong correlations of the returns. We indeed have a significant frequency near zero with a spectrum which decreases across frequencies. This is a typical pattern of a long-memory process.

Next, to grasp the sources of systemic risk, we refer to the behavior of joint quantiles in stock return distributions. The volatility of stock market returns is regarded as time-varying. Thus, common volatility can be obtained as a result of the dependence of concerned two stock markets. Figure 9 shows the rank-based copula cross-periodograms \( \mathbf{P}_{n,R}^{1/2}(\omega; \tau_1, \tau_2) \) [Eq. (41)], and the smoothed rank-based copula cross-periodograms \( \tilde{G}_{n,R}^{1/2}(\omega; \tau_1, \tau_2) \) [Eq. (43)].

Figure 9 shows the frequency dynamics in quantiles of the joint distribution of the returns. The copula rank cross-periodograms (or its smoothed version) estimated at the tails of the distribution (\( \tau_1 = \tau_2 = 0.1 \) and \( \tau_1 = \tau_2 = 0.9 \)) suggest a persistent and strong dependences of the returns for the concerned two markets. Moreover, during the normal period (i.e., the 0.5|0.5 combinations of quantile levels of the joint distribution), US-France, US-Japan, and France-Japan markets have two similar peaks around 0.15 and 0.35 (cycles per day) (0.9 and 2.2 frequencies).
Fig. 6  

a Copula rank periodogram: Dow Jones.  
b Copula rank periodogram: CAC40.  
c Copula rank periodogram: Nikkei 225
Fig. 6  (continued)

Fig. 7  a Smoothed Copula rank periodogram: Dow Jones. b Smoothed Copula rank periodogram: CAC 40. c Smoothed Copula rank periodogram: Nikkei 225
Fig. 7 (continued)
Fig. 8 Rank-based Laplace periodograms $L^R_{n,\tau_1,\tau_2}(\omega)$ and smoothed rank-based Laplace periodograms $\hat{f}^R_{n,R}(\omega; \tau_1, \tau_2)$
e  Nikkei 225: $I^R_{n,t_1,t_2}(\omega)$

![Graphs showing time-frequency analysis for Nikkei 225](image)

\[\text{\omega}/2\pi\]

\[\tau_0.1\]
\[\tau_0.5\]
\[\tau_0.9\]

\[\text{Dow Jones: } f_{n,R}(\omega; \tau_1, \tau_2)\]

![Graphs showing time-frequency analysis for Dow Jones](image)

\[\text{\omega}/2\pi\]
\[\tau_0.1\]
\[\tau_0.5\]
\[\tau_0.9\]

**Fig. 8** (continued)
e  CAC 40: $\hat{f}_{n,R}(\omega; \tau_1, \tau_2)$

f  Nikkei 225: $\hat{f}_{n,R}(\omega; \tau_1, \tau_2)$

Fig. 8  (continued)
Fig. 9  Rank-based copula cross-periodograms $I_{n,R}^{i_1i_2}(\omega; \tau_1, \tau_2)$ [Eq. (41)], and the smoothed rank-based copula cross-periodograms $\hat{G}_{n,R}^{i_1i_2}(\omega; \tau_1, \tau_2)$ [Eq. (43)]
c  CAC and Nikkei $\hat{I}_{n,R}^{112}(\omega; \tau_1, \tau_2)$

d  DJ and CAC $\tilde{C}_{n,R}^{112}(\omega; \tau_1, \tau_2)$

Fig. 9  (continued)
e DJ and Nikkei $\bar{G}_{n, R}^{l_1 l_2}(\omega; \tau_1, \tau_2)$

f CAC and Nikkei $\bar{G}_{n, R}^{l_1 l_2}(\omega; \tau_1, \tau_2)$

Fig. 9 (continued)
Finally, we refer to the quantile coherency. The left panels of Fig. 10a–c show the quantile coherency estimates for the \(0.1|0.1, 0.5|0.5, 0.9|0.9\) combinations of quantile levels of the joint distribution for (a) Dow Jones and CAC 40, (b) Dow Jones and Nikkei 225, and (c) CAC 40 and Nikkei 225. The right panels of Fig. 10a–c focus only on the \(0.1|0.9\) combination of quantile levels. Figure 10 also plots

![Fig. 10](image)

**Fig. 10**  
(a) Quantile coherency for Dow Jones and CAC 40.  
(b) Quantile coherency for Dow Jones and Nikkei 225.  
(c) Quantile coherency for CAC 40 and Nikkei 225
the x-axis in daily cycles and shows the frequencies that correspond to yearly (Y), monthly (M), and weekly (W) periods for the purpose of confirming how weekly, monthly, or yearly cycles are connected across quantiles of the joint distribution. Figure 11 shows the quantile coherency estimates for three fixed yearly, monthly,
and weekly periods which correspond to $\omega \in 2\pi \{\frac{1}{250}, \frac{1}{22}, \frac{1}{5}\}$ at all quantile levels $\tau_1 = \tau_2 \in \{0.05, 0.1, \ldots, 0.95\}$.

Figure 10 shows that the cycles at the lower quantiles relatively have more strong dependence than those at the upper quantiles in all combinations of the markets. The lower quantiles are more strongly related in periods longer than at least one month. The same result can be obtained in Fig. 11. In other words, large negative returns brought by extreme events seem to explain mainly the monthly or yearly common cycles of the concerned markets.

7 Conclusion

This chapter has presented some recent techniques which are becoming more and more influential in the analysis of time series. Quantile spectrum is a concept that can be used when one suspects high nonlinear and non-stationary interactions between financial and economic time series. There are several advantages compared to the standard spectrum. First, one does not need to make any distributional assumptions, especially by imposing the existence of finite moments. Secondly, we can grasp the correlations at different locations of joint distribution of series, by considering their cyclical dynamics. Thirdly, we avoid biased projections of the correlations in time series.

We voluntarily focus on techniques and methodologies that are easy to apply. One approach is based on harmonic regressions and leads to a new family of periodograms and spectra, i.e., Laplace periodograms and spectra. A second approach is based on the Fourier transform of copulas which are widely used in the analysis of the dependence of series. Empirical applications are made easy by the existence of software proposing such analyses (see Kley 2016).

Quantile spectrum analysis opens new areas of research. First, these approaches could be generalized to polyspectrum, when the frequency analysis applies to moments higher than 2. Secondly, quantile-based periodograms can also be used to display correlations in time series with non-Gaussian distributions (Student-t, Weibull, Gumbel, Fréchet, etc.). Recent applications of quantile spectrum for nonlinear and GARCH-type models are studied in Li (2019).

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