Computational simulation and the search for a quantitative description of simple reinforcement schedules

Paulo Sergio Panse Silveira$^{1,2}$, Jose de Oliveira Siqueira$^2$, João Lucas Bernardy$^{3,4}$, Jessica Santiago$^3$, Thiago Cersosimo Meneses$^3$, Bianca Sanches Portela$^3$, and Marcelo Frota Benvenuti$^{3,4}$

$^1$Department of Pathology, Medical School at the University of Sao Paulo

$^2$Department of Legal Medicine, Medical Ethics, Work and Social Medicine, Medical School at the University of Sao Paulo

$^3$Department of Experimental Psychology, Institute of Psychology at the University of Sao Paulo

$^4$National Institute of Science and Technology: Behavior, Cognition and Teaching (INCT-ECCE)

* corresponding author

marcelobenva@gmail.com

Department of Experimental Psychology
Av. Professor Mello Moraes, 1721
05508-030, São Paulo, SP, Brazil
Phone: +55 11 30914178
ORCID

Paulo S. P. Silveira: 0000-0003-4110-1038
Jose O. Siqueira: 0000-0002-3357-8939
João L. Bernardy: 0000-0002-3805-7366
Jessica Santiago: 0000-0002-7788-5455
Thiago C. Meneses: 0000-0003-3473-5841
Bianca S. Portela: 0000-0002-1351-652X
Marcelo F. Benvenuti: 0000-0002-9397-3033

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https://github.com/jlbernardy/simpleschedules.
Abstract

We aim to discuss schedules of reinforcement in its theoretical and practical terms pointing to practical limitations on implementing those schedules while discussing the advantages of computational simulation. In this paper, we present a R script named Beak, built to simulate rates of behavior interacting with schedules of reinforcement. Using Beak, we’ve simulated data that allows an assessment of different reinforcement feedback functions (RFF). This was made with unparalleled precision, since simulations provide huge samples of data and, more importantly, simulated behavior isn’t changed by the reinforcement it produces. Therefore, we can vary it systematically. We’ve compared different RFF for RI schedules, using as criteria: meaning, precision, parsimony and generality. Our results indicate that the best feedback function for the RI schedule was published by Baum (1981). We also propose that the model used by Killeen (1975) is a viable feedback function for the RDRL schedule. We argue that Beak paves the way for greater understanding of schedules of reinforcement, addressing still open questions about quantitative features of schedules. Also, they could guide future experiments that use schedules as theoretical and methodological tools.
Schedules of reinforcement are core concepts for the experimental analysis of behavior. The algorithms and rules that define schedules, however, are usually taken for granted, except for initial works (e.g., Catania & Reynolds, 1968; Ferster & Skinner, 1957; Fleshler & Hoffman, 1962; Millenson, 1963). The absence of schedule appraisal in the current literature is a potential problem. Pioneering methods to study schedule parameters were restricted to the limits of state-of-the-art technology at that time. In fact, current operant chambers, although controlled by modern computers, are still based on algorithms strongly tied to the primordial electromechanical devices, which can be regarded as a waste of resources. More recent technologies pave the way for a precise quantitative description of schedules.

Such a quantitative analysis would directly address some old yet still pending questions about schedules of reinforcement (e.g., Baum, 1973, 1993; Catania & Reynolds, 1968; Rachlin, 1978; Killeen, 1975) and guide future research that uses schedules as a methodological tool. In this work, we aim to resume the long-dormant discussion about quantitative features of simple schedules. For this purpose, we present a computational routine called Beak, built to simulate rates of behavior interacting with schedules of reinforcement. Our major contribution is that this software allows us to test insurmountable possibilities of rates of responses without having to rely on extensive experimentation with actual subjects.

The absence of discussions addressing the schedule’s algorithms used along many experiments suggests an apparent, but false, consensus. There are several critical aspects to defining and implementing schedules of reinforcement, which were already recognized by Ferster & Skinner in their seminal work. According to these authors, every schedule of reinforcement could be “represented by a certain arrangement of timers, counters and relay circuits” (Ferster & Skinner, 1957, p. 10). Still, most textbooks and technical papers omit relevant details about schedule algorithms
and emphasize the behavioral patterns associated with each simple schedule (e.g., Catania & Reynolds, 1968; Mazur, 2016; Pierce & Cheney, 2017).

This discussion, however, is not confined to solely theoretical matters. Schedules of reinforcement are held as crucial methodological tools for behavioral scientists to analyze many experimental results. The correct interpretation of these results relies on clarity of schedule definitions when applied to problems, such as discrimination learning by the use of multiple schedules (Ferster & Skinner, 1957; Weiss & Ost, 1974), observing behavior and conditioned reinforcement (Wyckoff, 1969), choice (Herrnstein, 1961, 1970) by the use of concurrent schedules, self-control (Rachlin & Green, 1972) by the use of concurrent chained schedules, behavior pharmacology (Dews, 1962; Reilly, 2003), decision making and bias (Fantino, 1998; Goodie & Fantino, 1995).

It is important to emphasize that these simulations do not replace the study of animal behavior. Simulations are concerned with mapping of an entire schedule, going through a large range of possible response rates and exhaustively repeating these conditions. In this sense, Beak can provide orientation for a researcher in creating an experimental scenario to which a biological being can be purposefully subjected. Since this biological being will behave with certain response rate, its confrontation with the simulation predictions may clarify biases and constraints of actual behavior. In other words, simulations map the normative rules of schedules while experiments map effective behaviors of organisms.

On Simple Schedules

A schedule of reinforcement is a set of rules that describe how behavior can produce reinforcers (Ferster & Skinner, 1957). Although the literature on the topic presents a myriad of schedule designations, all of them derive from the criteria used to define the so-called simple schedules. Fundamentally, reinforcers can be a function of a number
of responses, of the passage of time, or some combination of both.

Schedules that depend only on behavior are ratio schedules (R), while those that depend solely on time are time schedules (T). These requirements can be combined in several ways. If reinforcers become available after some time, but still require at least one response, we have an interval schedule (I). If reinforcers depend on a minimum time between responses (in other words, limits the rate of responses in a period), we have a differential reinforcement of low rates of responses (DRL). Instead, if reinforcers depend on several responses within a time limit, we have a differential reinforcement of high rates of responses (DRH).

All these schedule requirements can be either fixed (F) or variable (V). On fixed schedules, the criterion to be met (schedule size) is constant between reinforcers. For instance, an interval schedule in which the reinforcer is available every 2 seconds is a FI 2s. On variable schedules, this criterion is an average of a set of values. The same schedule with an average of 2 seconds is denoted as VI 2s. Back in the late fifties, implementing a variable schedule could be a challenge. Ferster & Skinner (1957) did so, selecting a series of values with an intended mean and “scrambling” them. However simple, this handmade solution raises some important questions. How many values should one use? How should the relative frequency of such intervals be distributed? Does scrambling mean randomness?

Intuitively, one should build a schedule with as many values as possible in order to diminish predictability. Yet, back in the day, researchers implemented schedules using a punched tape, in which the distances between holes corresponded to the values that originated the variable schedule. Therefore, this method imposed a practical limitation, because too many values meant very long tapes, which could lead to more technical difficulties (Catania & Reynolds, 1968). The electromechanical apparatus also constrained choices regarding the distribution of frequency of interval
values. Since it limited the number of values, distributions were always discrete. Even though these limitations are long gone, many works still implement variable schedules through scrambled values (e.g., Katz & Lattal, 2020; Fisher et al., 2018; Li et al., 2018). Instead of variable schedules and with the advent of modern computers, implementations can easily apply random procedures with intervals distributed according to continuous density probability functions; as such, an interval schedule with an average of 2 seconds to present a reinforcer is indicated by RI 2s.

**On Feedback Functions**

The general definition of operant behavior implies that behavior controls environmental changes. Ferster & Skinner (1957) emphasized how these changes shaped different patterns of behavior. For them, behavior was the dependent variable. On the other hand, reinforcement feedback functions (RFF) expand this analysis, treating environmental changes as the dependent variable and rates of behavior as the independent variable (Baum, 1973; Rachlin, 1978). Therefore, RFF clarifies how rates of reinforcement are constrained by basic schedules in a molar level of analysis.

The general shape of some RFF is well known. Time schedules do not depend on behavior. Therefore, the RFF is a horizontal line with an intercept equal to the rate of reinforcement deduced from the schedule’s size. In ratio schedules, rates of behavior and reinforcement have a linear relation, with an intercept equal to zero and a slope that is the reciprocal of the ratio size. In interval schedules, reinforcement rate is further constrained by a temporal criterion, altering the prior linear function. In such cases, rates of response control increasing rates of reinforcement only until an asymptotic level. Figure 1 depicts schematic RFF, based on Rachlin (1989) analysis.
Figure 1. Schematic feedback function for three fundamental variable schedules: VT, VI and VR. VT does not depend on animal behavior for reinforcers are provided at average time intervals. VR completely depends on animal’s behavior since reinforcers are provided after a given number of responses. VI is a middle ground, in which the reinforcer becomes available at intervals, but only received after an animal response.

This quantitative signature of schedules precedes the empirical pattern associated with each schedule and the ensuing controversy on differences among species, related repertoires and stability criterion (e.g., Galizio & Buskist, 1988; Stoddard et al., 1988). RFFs allows us to discover optimal relations between behavior and reinforcement for each schedule, and so pose a way to propose normative rules for what to expect from actual (optimal) behavior. That’s why RFFs are a research topic in their own right. Still, the precise quantitative description for many schedules remains an open subject.

As Baum (1992) pointed out, a viable RFF should fit the experimental data. But one cannot directly manipulate rates of behavior in the animal laboratory without changing critical aspects of the environment. That’s an important limitation, since RFFs models the environment as a function of behavior and the rate of response is the independent variable, but one that we cannot manipulate systematically. Therefore,
even with large samples of behavior, experiments rarely cover a sufficiently wide range of response rates (Baum, 1992), while Beak allows the experimenter to explore and predict what optimal performances would look like for a wide range of environmental conditions. For that reason, we argue that the ideal conditions for investigation of RFFs are only achievable through computer simulation, because we can prevent simulated behavior from changing as a function of rates of outcomes. Furthermore, we can analyze with unprecedented precision the quantitative features of the relation between rates of reinforcement and rates of responses and build normative rules for different contingencies. Another consequence of this perception is that experiments with humans or laboratory animals do not seem to be the best choice to define a RFF, which was the historical attempt; their utility is the discovery of what strategies among many and under which circumstances an organism can adopt, given the purified normative rules predicted by simulations (literally providing a map), thus opening a whole new string of research.

In the present paper, we will use Beak to discuss the curve fit presented by Baum (1981); Rachlin (1978); Prelec (1982) for the RI schedule. We also show that a curve from Killeen (1975), which was originally proposed in a different theoretical context, is a viable RFF for RI schedule. More interestingly, this function is also a suitable RFF for the RDRL schedule.

*Implementing simple schedules of reinforcement*

Here we describe how we’ve implemented simple schedules and responses on Beak. For the sake of parsimony, we’ll describe the random interval (RI) and random differential reinforcement of low rates (RDRL). The other two basic schedules are simpler and do not pose any fitting challenge: RT is a horizontal line at the schedule size and RR is a simple straight line with slope reciprocal of the ratio size. Our implementa-
tions of simple schedules are mainly based on initial work by Millenson (1963) and Ambler (1973). We consider their implementation ideal, because they’re continuous versions of the discrete (and more widely used) algorithms (e.g., Fleshler & Hoffman, 1962). Our implementation of responses is like the one by Green et al. (1983). Distinctly, here, \( p \) stands simply for response probability, while \( 1-p \) stands for a probability of no response at all. Also, trials can happen every fraction of a second, depending on the response rates we want to investigate.

**Random Interval (RI).** Back in 1963, Millenson proposed the random interval (RI) schedule as a random version of VI schedules (Millenson, 1963). Millenson’s RI is a function of the parameters \( T \) and \( p \), where \( T \) stands for the duration of a cycle in any unit of time, at the end of which there is a probability \( p \) of reinforcement assignment. The inter-assignment time (IAT) is the number of cycles with duration \( T \) until reinforcement assignment.

For every specific RI size, there are infinite combinations of \( T \) and \( p \). However, not every combination is eligible for our purposes: behavioral researchers should find values of \( T \) and \( p \) that will produce an IAT with geometric distribution with mean equal to:

\[
\mu_{RI} = \frac{T}{p} \tag{1}
\]

In order to achieve a geometric distribution, we must meet two requirements. First, the distribution’s average (Equation 1) must be equal to the standard deviation of the distributions (\( \sigma \)), given by:

\[
\sigma_{RI} = \frac{T}{p} \sqrt{1-p} \tag{2}
\]
Second, the geometric distribution of IAT will approach an exponential distribution as $T$ approaches zero. The exponential distribution is desirable because it has the inherent property of lack of memory (Feller, 1968), videlicet, its past behavior bears no information about the future behavior of IAT distribution. This property is key for a more refined implementation of variable schedules because it ensures minimum predictability, as Fleshler & Hoffman (1962) intended. Also, the exponential distribution conveniently portrays the continuity of time. This can be especially useful when using computational simulations, since we have means to investigate exhaustively long sessions with this method.

On the other hand, Millenson (1963) pointed out that $T$ should also be greater than the average time of reinforcer consumption. For studies with approximately zero consumption time, we argue that $T \leq 1$ second is a convenient heuristic for $T$ to meet both requirements simultaneously.

Given that the implemented schedule is a function of $T$ and $p$, it is unlikely that the average and standard deviation will be identical to the planned value. Therefore, we suggest a 1% margin of tolerance. If $x$ is the planned schedule average and standard deviation, this margin of tolerance for the mean can be described as:

$$\left| \frac{x - T}{p} \right| \leq 0.01$$

Applying the same margin of tolerance to the standard deviation:

$$1 \geq \frac{T}{x} \sqrt{1 - p} \geq 0.99$$

In other words, values of $T$ and $p$ which meet the requirements expressed in Equations 3 and 4 will produce an RI with exponential distribution of inter-assignment intervals that is sufficiently close to a RI$x$ (of same size) as planned beforehand. A
small R script to determine adequate combinations of \( T \) and \( p \) is available as supplemental material.

After choosing appropriate values for \( T \) and \( p \), the simulation starts running. A given interval will elapse until the first reinforcer is assigned. After every reinforced response, the chronometer restarts. That poses the interval schedule’s criterion for reinforcement presentation based on the time period between two consecutive reinforcers (reinforcement as a function of both responding and passage of time). Using such an implementation, based on Millenson (1963), we’ll discuss the shape of the RFF RI produced using Beak.

As a result of this implementation, the rate of reinforcement in RI schedules is monotonically increased and negatively accelerated, as shown in Figure 2. Each point represents the average of 500 sessions, each one lasting one hour and vertical bars represent the 95% high density intervals.

**Random Differential Reinforcement of Low Rates (RDRL).** In the well-known DRL schedule (differential reinforcement of low rates of behavior), a minimum inter-response time (IRT) must precede rewarded responses (Ferster & Skinner, 1957). Using Beak, we were able to implement the variable differential reinforcement of low rates - the RDRL schedule (Ambler, 1973; Logan, 1967). In a RDRL schedule, the required IRT varies randomly. Such variation is a function of parameters like those used to implement the RI schedule (Millenson, 1963).

Just like the previously defined RI, a reinforcement is assigned with probability equal to \( p \) every \( T \) seconds. The difference relies on the fact that, in the RI schedule, the parameter \( T \) is not affected by the organism’s behavior, whereas the same parameter, in the RDRL, is directly affected by the organism’s IRTs. This happens because the chronometer that registers the passage for each cycle resets after every response.
Figure 2. Simulated data relating mean reinforcement per minute with responses per minute in three RI schedules applying the best approximation of RI 5s, RI 15s and RI 60s with IAT geometrical distribution as function of $T$ and $p$ (see text). Average is represented by points and 95% high density intervals by vertical bars from 500 repetitions of simulated sessions of 1 hour.

emitted, which causes a cycle of time $T$ to only be fully completed if no responses are emitted in the meantime. Such a condition makes $p$ conditional to the organism’s IRTs, so in order to obtain a mean value for the probability of reinforcement in the session one must consider the minimum IRT the schedule requires (the size of the RDRL).

In other words, while the relation between $T$ and $p$ defines the average IAT of an RI, the same relation defines the average IRT which the organism is required to comply with in order to produce reinforcers in a RDRL. Therefore, substituting $T$ for $T'$, in order to emphasize such a difference, the mean RDRL size is given by:
The parameter $T'$ is the minimum IRT required by the schedule for reinforcement assignment and $p$ is the probability that a reinforcer is actually assigned by the end of $T'$. Here we’ll use Beak to draw the RFF RDRL and discuss a convenient curve fit. Even though the VDRL was implemented in animal laboratory (Aasvee et al., 2015)Logan1967, to the best of our knowledge, no further studies have been published about the RFF RDRL. Therefore, we’ll discuss this matter in the section in which we cover the advances we were able to make using Beak.

**Simulating Responses.** Here we will present the assumptions of Beak regarding the implementation of responses to study schedules of reinforcement using computational simulation. Beak produces instantaneous responses programmed as a Bernoulli process, where a success corresponds to the emission of a response. The simulation explores a range of response rates ($B$), being $B$ constant along each session. The probability of response emission at each instant of time ($p_b$) for each session is given by:

$$p_b = \frac{B}{60t}$$

The simulation evolves in discrete steps. Each second is fractioned according to $t$ (the minimum possible IRT). The mean rate of responses, $B$, is provided in minutes (the correspondence from minutes to seconds is represented by the constant 1/60 in Equation 6). For instance, a response rate of 100 per minute and a second partitioned in intervals of 5/1000 of a second, would result in $p_b \approx 0.0083$ (the probability of response in each iteration step). To investigate higher values of $B$, $t$ needs to be
smaller, making the simulation finer with greater computational cost. Additionally, as this rate of trials increases, the Bernoulli process approaches a time continuity, as in a Poisson process.

The researcher also determines session duration and the number of session repetitions. Beak stores the reinforcement rate (reinforcers per minute) of each repetition and computes the 95% high density interval [Hyndman, 1996]. For this work we computed 500 repetitions of one-hour sessions, therefore, each point of our simulations corresponding to a given $B$ was the result of 500 sessions, each one depending on 720,000 iterations ($3600 \cdot \frac{1}{0.005}$), totaling $3.6 \cdot 10^8$ trials. Since $B$ ranged from 0 to 200 (integer values), the definition of each RFF depicted below was obtained by $7.236 \cdot 10^{10}$ events. With such a number of trials, the obtained average rate of responses draws itself nearer to the nominal rate of responses determined by the experimenter.

Discussion

As it was shown, simulations executed using Beak were able to reproduce the general shape of known RFF. Moreover, it has also enabled the investigation of how precisely and parsimoniously the RFF curves proposed can describe simulated or experimental data, which gives us grounds to point out advantages and disadvantages of different ways to implement each RFF, while also suggesting possible new curves in addition to existing ones, as we do by testing the curve from Killeen (1975). We will use our simulations to compare different RFF proposed and explore the precision, meaning, parsimony and generality of each one. This provides experimenters with better ways to describe the relationship between behavior and environmental constraints. We will also propose an RFF for the RDRL schedule, which is novel to the literature, and extend the discussion on curve fitting to it.
Table 1: Equations explored herein investigating best RI RFF fit ($V$ provided in seconds and scaled by 60 for conversion in minutes).

| Reference       | RFF RI                                      |
|-----------------|---------------------------------------------|
| Baum (1981)     | $R = \frac{1}{(V/60) + x}$                 |
| Prelec (1982)   | $R = B \left(1 - \exp\left(-\frac{1}{(V/60)B}\right)\right)$ |
| Rachlin (1978)  | $R = \frac{1}{(V/60)} \left(\frac{B}{B_{max}}\right)^m$ |
| Killeen (1975)  | $R = \frac{1}{(V/60)} \left(1 - \exp\left(-\frac{B}{c}\right)\right)$ |

RI curve fit

Deciding between curve fits is no simple matter, given that there are no definitive criteria. Therefore, we’ll address the issue systematically, highlighting the pros and cons of each one of the four curve fits - Baum (1981); Prelec (1982); Rachlin (1978); Killeen (1975) - compared to data obtained through our simulation. The RFF are summarized in Table 1.

In all RFF, $B$ stands for the response rate, $R$ stands for reinforcement rate and $V$ stands for the size schedule in minutes, while $c$ and $m$ are free parameters that are estimated $a$ posteri ori. Ahead, we’ll compare these functions regarding meaning, precision, parsimony and generality.

Meaning. To describe the relationship between $B$ and $R$, Baum’s (1981) and Prelec’s (1982) RFF rely only on $V$, a single parameter which has a built-in meaning and is supposed to correspond to the schedule’s size determined from experimental planning. That is convenient, especially because, at least in theory, they do not require estimation methods. On the other hand, Rachlin’s (1978) depends on $m$, and Killeen’s (1975) depends on $c$. As far as we know, these parameters have no empirical
Table 2: Parameter estimates for each RFF RI using Beak

| RFF Parameter | RI5  | RI15 | RI60  |
|---------------|------|------|-------|
| Baum (1981) V | 4.91 | 14.88| 59.56 |
| Prelec (1982) V | 5.25 | 15.32| 60.17 |
| Rachlin (1978) V | 4.90 | 14.52| 58.60 |
| Rachlin (1978) m | 0.210| 0.111| 0.043 |
| Killeen (1975) V | 5.41 | 15.59| 60.58 |
| Killeen (1975) c | 18.474| 7.383| 2.107 |

meaning.

To fit the curves presented to the data we obtained through simulation, we allowed all parameters of the equations to vary (except $B$ and $R$ which are the variables we want to describe) in order to best fit the curve to the data according to a nonlinear least squares method. From that ensues that we have estimated parameters, which are not exactly those obtained empirically but are a good approximation. For instance, we have an estimated $V$ that approaches the schedule’s size that was defined by the experimenter but renders a more accurate description of the data than that one defined *a priori*. We do that because we want to know how faithfully one can assume that this parameter actually approaches the schedule’s size. Therefore, we investigated how $V$, $m$ and $c$ vary across RI. These results are summarized in Table 2. Our estimations of $V$ are all fairly close to the schedule sizes.

Rachlin (1978, 1989) showed that $m$ always falls between zero and one for any interval schedule and suggested that $m = 0.1$ (Rachlin, 1989) or $m = 0.2$ (Rachlin, 1978) across schedules of different sizes. In fact, if $m$ approaches zero, the interval schedule approaches a RT; if $m$ approaches one, it approaches a RR. However we did not find a constant value for $m$, which varies with RI size.
Killeen's (1975) $c$ seems to be a positive value with no upper limit. Like Rachlin's (1978) $m$, it seems to have an inverse relation with the schedule's size. Still, there's no obvious way to derive them a priori. Therefore, we argue that Baum's (1981) and Prelec's (1982) feedback functions have a didactic advantage, since they rely on a single and interpretable parameter $V$.

**Precision and Parsimony.** As previously mentioned, an appropriate feedback function should fit the data (Baum, 1992). In order to compare fit qualities, one possible criteria is the goodness of fit measure, $R^2$, for what we suggest the cutoffs 0.9 and 0.95 for good and excellent fit. Notwithstanding, using $R^2$ as the only criterion could be misleading, since it usually favors more complex RFF. Hence, we’ll use a Bayesian information criterion (BIC) to compare models with different numbers of parameters (Schwarz, 1978). Table 3 summarizes the $R^2$ and BIC estimated for each RFF.

Our results favor Baum's (1981) RFF regarding both excellent precision (highest $R^2$) and parsimony (lowest BIC). While the other RFF seem to struggle with larger RI, Baum's (1981) RFF is fairly stable. Adding the fact that it has a single meaningful parameter, $V$, it seems to be the best available RFF RI.

Baum (1992) stated that Rachlin's (1978) RFF has many fallouts. First, it doesn’t have a horizontal asymptote, an important feature of interval schedules. Second, and more importantly, it doesn’t fit the data properly (see Table 3 and Figure 3).

Even though Prelec's (1982) and Killeen's (1975) do better than Rachlin's (1978), their $R^2$ also drops significantly for the larger RI. We cannot state that this represents a tendency for even larger RI, but that’s further evidence that Baum's (1981) is the best among them. Figure 3 brings graphical representation for the RI 5s, 15s, and 60 seconds and helps to understand where some of this function fails to fit our simulated data.
Figure 3. Curve fit and $R^2$ for each feedback function using simulated data of RI 5s, 15s, and 60s.
Table 3: Fit precision and parsimony for each feedback function using simulated data from three RI schedules

|                | RI | $R^2$ | BIC  |
|----------------|----|-------|------|
| Baum (1981)    | 5s | 0.999 | -1070|
| Prelec (1982)  |    | 0.933 | 304  |
| Rachlin (1978) |    | 0.888 | 414  |
| Killeen (1975) |    | 0.963 | 189  |
| Baum (1981)    | 15s| 0.999 | -1220|
| Prelec (1982)  |    | 0.917 | -255 |
| Rachlin (1978) |    | 0.770 | -45  |
| Killeen (1975) |    | 0.922 | -263 |
| Baum (1981)    | 60s| 0.988 | -1485|
| Prelec (1982)  |    | 0.864 | -1006|
| Rachlin (1978) |    | 0.589 | -780 |
| Killeen (1975) |    | 0.835 | -963 |

Generality. Following the above criteria, one should readily decide in favor of Baum’s (1981) RFF. However, Baum’s function, like Prelec’s, applies only to interval schedules.

Conversely, Rachlin’s (1978) and Killeen’s (1975) functions also describe other simple schedules (see also Killeen & Sitomer, 2003). Rachlin’s (1978) exponential function describes time, interval, and ratio schedules, but in this case such generality doesn’t compensate for the fallouts we already discussed. Similarly, Killeen’s (1975) functions have generality as the primary advantage, since it models not only interval schedules but also ratio schedules and the still undocumented RDRL feedback function, as shown in the next topic.
RDRL Feedback Function

Logan (1967) exposed rats to a variable DRL with only two equally likely IRT requirements. Here, we’ve implemented a RDRL, a continuous version of the somewhat minimalist Logan’s VDRL. Even though Logan (1967) described his results in terms of proportion of IRTs, the data allows us to conjecture about the RFF VDRL shape.

Logan found that the most likely IRT “approximated an optimal strategy for maximizing reward” (Logan, 1967, p. 393). This meant that the subjects’ first response after reinforcement occurred with an IRT slightly longer than the smaller VDRL interval out of the two programmed, and further responses happened with IRT around the other VDRL interval, which was longer. Therefore, he found two peaks of likely IRT that matched the VDRL intervals used.

Considering that behavior rate equals the reciprocal of IRT, Logan’s results allow us to sense what a RFF RDRL should look like. Reinforcers per minute should increase along with response rate until a certain maximum. However, if the response rate increases beyond this optimal point, reinforcers income would decrease asymptotically. Since Logan (1967) built his VDRL out of two intervals, optimal rates of response were predictable. In fact, rats that served as subjects learned how to maximize reinforcers responding after the shorter interval and then waiting for the longer one.

A RDRL could reinforce any IRT with a certain probability. Therefore, we expect an optimal rate greater than the size of RDRL and, as a result, a maximum of reinforcers per minute falls short of the theoretical asymptote deduced from the size of schedule. All these features are shown in Figure 4, which depicts the points resulting from our simulation of four RDRL.
Our simulations are well described by the equation:

\[ R = \frac{1}{(V/60)} \left( \exp \left( -\frac{B}{b} \right) - \exp \left( -\frac{B}{c} \right) \right) \]  

(7)

As for the RI schedule, \( R \) and \( B \) stand for rates of reinforcement and responses, respectively. Also, \( V \) still stands for the schedule size in minutes. The parameter \( \frac{1}{(V/60)} = \frac{60}{V} \) is a theoretical asymptote of reinforcers per minute.

Killeen (1975) used Equation 7 to model two competing processes controlled by parameters \( b \) (concurrent) and \( c \) (inhibitory). Killeen (1975) interpreted the former as a measure of an increasing competition among different activities, and the latter as post-reinforcement inhibition. Despite the main concern in Killeen (1975) analysis is the probability of behavior in inter-reinforcement intervals, we found an analogous
conflict in our analysis. The RDRL poses a similar competition between contingency and postponement of reinforcers: on one hand, we have the negative punishment imposed by the schedule to rates above the schedule’s criterion (controlled by $b$), on the other, we have the direct relation between rates of response and reinforcement (controlled by $c$). This kind of analysis shows the potential of our findings for further discussion about the theory of schedules of reinforcement.

Using an iterative least squares algorithm, we have estimated the parameters $b$ and $c$ for all RDRLs in Figure 4. Table 4 summarizes these estimations, as well as $R^2$.

The $R^2$ values summarized in Table 4 show that Equation 7 is a proper RFF for the RDRL schedule. In the present paper, we dismiss a Bayesian Information Criterion analysis, simply because we do not know any viable alternative to model the RDRL. Table 4 also shows our estimations for $b$ and $c$. For further discussion of these parameters, Figure 5 exemplifies this relation, comparing 2, 4, 8 and 16 seconds RDRLs.

The decreasing dashed lines for each RDRL in Figure 5 are controlled by the parameter $b$, while the rising dashed lines are controlled by $c$. Greater values of $b$ mean a slower decay in reinforcer income at higher rates of response. Greater values

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**Table 4: Parameter estimates and $R^2$ for simulated data of four RDRL schedules**

| RDRL | V/60 | $b$  | $c$  | $R^2$ |
|------|------|------|------|-------|
| 2s   | 30.00| 212.55| 74.94| 0.994 |
| 4s   | 15.00| 99.68 | 35.46| 0.986 |
| 8s   | 7.50 | 48.25 | 17.14| 0.994 |
| 16s  | 3.75 | 23.94 | 8.51 | 0.999 |
Figure 5. Curve fit for simulated data of RDRL 2s, 4s, 8s, and 16s. Horizontal dashed line is the maximum nominal reinforcements that could be obtained from a RT of the size $V$, decreasing dashed lines are adjusted by $\frac{1}{(V/60)} \cdot \exp(-B/b)$, rising dashed lines are adjusted by $\frac{1}{(V/60)} \cdot (1 - \exp(-B/c))$, empty circles are simulated data, and solid tick line is RDRL fit by $\frac{1}{(V/60)} \cdot (\exp(-B/b) - \exp(-B/c))$ where $B$ is the response rate.
of $c$ mean a slower increase in reinforcement at low rates of response.

As shown in Table 4, we’ve found greater values of $b$ and $c$ for smaller (richer) RDRL. That matches our interpretation of the model, because smaller RDRL are more demanding and permissive. They are demanding because they require greater response rates in order to reach maximum reinforcement income, and they are permissive because they allow greater response rates to go unpunished (see Figure 5).

An interesting result we observed across many simulations is that the maximum reinforcement rate is a constant $0.364 \cdot \frac{1}{(V/60)}$. This normative rule not only confers a priori the maximum reinforcement an animal can obtain given the schedule size but also the prediction of the response rate at which this maximum will be achieved.

These findings are important because they successfully add complexity in our basic description of simple schedules of reinforcement. This complexity may be viewed as consistent with other schedule parameters, showing the capacity of our computational model to compare different quantitative models of simple schedules of reinforcement as a starting point to analysis of other sources of control, including conflict between excitatory and inhibitory control (Staddon, 1977) and temporal control (e.g., Machado, 1997).

Conclusion

The main objective of the present paper was to implement and discuss simple schedules of reinforcement from a quantitative perspective. In Beak, we’ve implemented random interval (RI) and random DRL (RDRL) schedules. Also, we’ve implemented a random distribution of responses that allow us to investigate wide ranges of response rate and their effects on reinforcers per minute (i.e., RFF).

Our results demonstrate the power of our computational simulation to discuss basic schedules of reinforcement and refine ways to implement them. Based on the
results, we revised RFF RI and proposed a RFF RDRL.

Regarding schedules in which reinforcement may depend both on the passage of time and the occurrence of responses, the RDRL is a way to further constrain reinforcement in comparison to the RI schedule. The RFF RDRL is like RFF RI, in a sense that in both cases the rate of reinforcement depends on the response rate. Therefore, we’ve found increasing functions at low rates of response. However, these functions are also negatively accelerated functions. This represents the restriction imposed by time, which is present in both schedules.

The RFF of both schedules differ in the extent to which the RDRL schedules further constraints reinforcement. In the interval schedule the rate of responses has a positive monotonic relation with the ever-increasing rates of reinforcement. That is not the case in the RDRL. In the RDRL schedule, high rates of response are negatively punished by the postponement of reinforcement. In fact, this feedback system is well described by two competing processes (Killeen, 1975).

The new implementation methods presented paves way for a richer study of schedules of reinforcement and their normative maximization rules, serving also as a guide towards promising questions which future experiments may want to explore.
References

Aasvee, K., Rasmussen, M., Kelly, C., Kurvinen, E., Giacchi, M. V., & Ahluwalia, N. (2015). Validity of self-reported height and weight for estimating prevalence of overweight among estonian adolescents: The health behaviour in school-aged children study. *BMC Research Notes, 8*. doi: 10.1186/s13104-015-1587-9

Ambler, S. (1973). A mathematical model of learning under schedules of interresponse time reinforcement. *Journal of Mathematical Psychology, 10*. doi: 10.1016/0022-2496(73)90023-0

Baum, W. M. (1973). The correlation-based law of effect. *Journal of the Experimental Analysis of Behavior, 20*. doi: 10.1901/jeab.1973.20-137

Baum, W. M. (1981). Optimization and the matching law as accounts of instrumental behavior. *Journal of the Experimental Analysis of Behavior, 36*. doi: 10.1901/jeab.1981.36-387

Baum, W. M. (1992). In search of the feedback function for variable-interval schedules. *Journal of the Experimental Analysis of Behavior, 57*. doi: 10.1901/jeab.1992.57-365

Baum, W. M. (1993). Performances on ratio and interval schedules of reinforcement: data and theory. *Journal of the Experimental Analysis of Behavior, 59*. doi: 10.1901/jeab.1993.59-245

Catania, A. C., & Reynolds, G. S. (1968). A quantitative analysis of the responding maintained by interval schedules of reinforcement. *Journal of the Experimental Analysis of Behavior, 11*. doi: 10.1901/jeab.1968.11-s327

Dews, P. B. (1962). *Psychopharmacology* (4th ed.). Basic Books, Inc.

Fantino, E. (1998). Behavior analysis and decision making. *Journal of the Experimental Analysis of Behavior, 69*. doi: 10.1901/jeab.1998.69-355
Feller, W. (1968). *An introduction to probability theory and its applications, vol. 1, 3rd edition*. Wiley.

Ferster, C. B., & Skinner, B. F. F. (1957). *Schedules of reinforcement*. Appleton.

Fisher, W. W., Greer, B. D., Fuhrman, A. M., Saini, V., & Simmons, C. A. (2018). Minimizing resurgence of destructive behavior using behavioral momentum theory. *Journal of Applied Behavior Analysis, 51*. doi: 10.1002/jaba.499

Fleshler, M., & Hoffman, H. S. (1962). A progression for generating variable-interval schedules. *Journal of the Experimental Analysis of Behavior, 5*. doi: 10.1901/jeab.1962 .5-529

Galizio, M., & Buskist, W. (1988). Laboratory lore and research practices in the experimental analysis of human behavior: Selecting reinforcers and arranging contingencies. *The Behavior Analyst, 11*. doi: 10.1007/bf03392457

Goodie, A. S., & Fantino, E. (1995). An experientially derived base-rate error in humans. *Psychological Science, 6*. doi: 10.1111/j.1467-9280.1995.tb00314.x

Green, L., Rachlin, H., & Hanson, J. (1983). Matching and maximizing with concurrent ratio-interval schedules. *Journal of the Experimental Analysis of Behavior, 40*. doi: 10.1901/jeab.1983.40-217

Herrnstein, R. J. (1961). Relative and absolute strength of response as a function of frequency of reinforcement. *Journal of the Experimental Analysis of Behavior, 4*. doi: 10.1901/jeab.1961.4-267

Herrnstein, R. J. (1970). On the law of effect. *Journal of the Experimental Analysis of Behavior, 13*. doi: 10.1901/jeab.1970.13-243

Hyndman, R. J. (1996). Computing and graphing highest density regions. *American Statistician, 50*. doi: 10.1080/00031305.1996.10474359
Katz, B., & Lattal, K. A. (2020). An experimental analysis of the extinction-induced response burst. *Journal of the Experimental Analysis of Behavior, 114*. doi: 10.1002/jeab.611

Killeen, P. R. (1975). On the temporal control of behavior. *Psychological Review, 82*. doi: 10.1037/h0076820

Killeen, P. R., & Sitomer, M. T. (2003). Mpr. *Behavioural Processes, 62*, 49-64. Retrieved from [https://www.sciencedirect.com/science/article/pii/S0376635703000172](https://www.sciencedirect.com/science/article/pii/S0376635703000172) (Theories in Progress: Proceedings of the Meeting of the Society for the Quantitative Analyses of Behaviour) doi: [https://doi.org/10.1016/S0376-6357(03)00017-2](https://doi.org/10.1016/S0376-6357(03)00017-2)

Li, D., Hautus, M. J., & Elliffe, D. (2018). The natural mathematics of behavior analysis. *Journal of the Experimental Analysis of Behavior, 109*. doi: 10.1002/jeab.330

Logan, F. A. (1967). Variable drl. *Psychonomic Science, 9*. doi: 10.3758/BF03330862

Machado, A. (1997). Learning the temporal dynamics of behavior. *Psychological Review, 104*. doi: 10.1037/0033-295X.104.2.241

Mazur, J. E. (2016). *Learning and behavior*. Routledge.

Millenson, J. R. (1963). Random interval schedules of reinforcement. *Journal of the Experimental Analysis of Behavior, 6*. doi: 10.1901/jeab.1963.6-437

Pierce, W. D., & Cheney, C. D. (2017). *Behavior analysis and learning: a biobehavioral approach*. Routledge.

Prelec, D. (1982). Matching, maximizing, and the hyperbolic reinforcement feedback function. *Psychological Review, 89*. doi: 10.1037/0033-295X.89.3.189

Rachlin, H. (1978). A molar theory of reinforcement schedules. *Journal of the Experimental Analysis of Behavior, 30*. doi: 10.1901/jeab.1978.30-345
Rachlin, H. (1989). *Judgment, decision, and choice: a cognitive/behavioral synthesis*. W.H. Freeman.

Rachlin, H., & Green, L. (1972). Commitment, choice and self-control 1. *Journal of the Experimental Analysis of Behavior, 17*. doi: 10.1901/jeab.1972.17-15

Reilly, M. P. (2003). Extending mathematical principles of reinforcement into the domain of behavioral pharmacology. *Behavioural Processes, 62*. doi: 10.1016/S0376-6357(03)00027-5

Schwarz, G. (1978). "estimating the dimension of a model.". *The Annals of Statistics, 6*. doi: 10.2307/2958889

Staddon, J. E. R. (1977). *Handbook of operant behavior*. Prentice-Hall.

Stoddard, L. T., Sidman, M., & Brady, J. V. (1988). Fixed-interval and fixed-ratio reinforcement schedules with human subjects. *The Analysis of Verbal Behavior, 6*. doi: 10.1007/bf03392827

Weiss, S. J., & Ost, S. L. V. (1974). Response discriminative and reinforcement factors in stimulus control of performance on multiple and chained schedules of reinforcement. *Learning and Motivation, 5*. doi: 10.1016/0023-9690(74)90004-6

Wyckoff, L. B. J. (1969). *The role of observing responses in discrimination learning: Part ii*. The Dorsey Press.