Large deviations for functionals of Gaussian processes

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Abstract

We prove large deviation principles for $\int_0^t \gamma(X_s)\,ds$, where $X$ is a $d$-dimensional Gaussian process and $\gamma(x)$ takes the form of the Dirac delta function $\delta(x)$, $|x|^{-\beta}$ with $\beta \in (0, d)$, or $\prod_{i=1}^d |x_i|^{-\beta_i}$ with $\beta_i \in (0, 1)$. In particular, large deviations are obtained for the functionals of $d$-dimensional fractional Brownian motion, sub-fractional Brownian motion and bi-fractional Brownian motion. As an application, the critical exponential integrability of the functionals is discussed.

Keywords: Gaussian process, fractional Brownian motion, sub-fractional Brownian motion, bi-fractional Brownian motion, reproducing kernel Hilbert space, local time, large deviation.

Subject Classification: Primary 60G15, 60G18, 60G22, 60J55, 60F10.

1 Introduction

The large deviations for functionals of symmetric Lévy stable processes such as the (intersection) local time and Riesz potentials of additive processes were studied in [8, 3], where the properties of symmetric Lévy stable process such as self-similarity and independent increment property play a crucial role in the analysis. Later, exact forms of large deviations for (intersection) local times of fractional Brownian motion (fBm for short) and the Riemann-Liouville process were obtained in [10]. The results in [10] are surprising and are not a “natural” extension of [8, 3], in the sense that fBm and the Riemann-Liouville process are not Markovian and the techniques for Lévy processes do not apply.

Let $\gamma(x)$ be one of the following functions: the Dirac delta function $\delta(x)$, $|x|^{-\beta}$ with $\beta \in (0, d)$, and $\prod_{i=1}^d |x_i|^{-\beta_i}$ with $\beta_i \in (0, 1)$. Throughout the article, we use the
convention that $\beta = d$ if $\gamma(x) = \delta(x)$ and that $\beta = \sum_{i=1}^{d} \beta_i$ if $\gamma(x) = \prod_{i=1}^{d} |x_i|^{-\beta_i}$.

Under this convention, the functional $\gamma$ has homogeneity, i.e., $\gamma(ax) = a^{-\beta} \gamma(x)$ for $a > 0$ and $x \in \mathbb{R}^d$.

This article concerns the large deviations for $\int_0^t \gamma(X_s)ds$, where $X$ is a $d$-dimensional self-similar Gaussian process satisfying some conditions. In particular, the large deviations for the functionals of fractional Brownian motion $B^H$ (fBm for short) with $H \in (0, 1)$ and $H\beta < 1$, sub-fractional Brownian motion $S^H$ (sub-fBm for short) with $H < \frac{1}{2}$ and $H\beta < 1$, and bi-fractional Brownian motion $Z^{H,K}$ (bi-fBm for short) with $H \in (0, 1)$, $K \in (0, 1]$ and $HK\beta < 1$.

Instead of carrying out a direct analysis for the functionals of fBm, sub-fBm and bi-fBm, we first obtain the large deviations for the Riemann-Liouville process $\int_0^t (t-s)^{H-\frac{1}{2}}dW_s$, where $W$ is $d$-dimensional standard Brownian motion. In light of Lemma 3.4, the large deviation principle for the functional of the Riemann-Liouville process is reduced to proving the existence of the limit for the log moments of the functional, for which it suffices to show the sub-additivity (see Propositions 3.1 and 3.2). After we obtain the results for the functionals of the Riemann-Liouville process, we study the large deviations for the functionals of fBm, sub-fBm and bi-fBm by comparing them with the functionals of the Riemann-Liouville process. The comparison strategy was initially developed in [10], and we briefly interpret the two key ingredients of the idea below.

Firstly, we observe that, for general Gaussian processes $X$ and $Y$ which both possess certain self-similarity, if $Y = X + \eta$ such that $X$ and $\eta$ are independent, and $\eta$ belongs to the Cameron-Martin space of $X$ almost surely, then the large deviations of $X$ and $Y$, if one of them exists, both exist and coincide with each other (see Proposition 2.13 for details). The crucial condition here is that $\eta$ belongs to the Cameron-Martin space of $X$, which yields that, conditioned on $\eta$, the distributions of $X$ and $Y$ are equivalent. An heuristic explanation for the coincidence of the large deviations for the functionals of $X$ and $Y$ is that, $\eta$ is “regular” enough in comparison with $X$, and thus the perturbation of $\eta$ is just negligible.

The second key ingredient in the comparison strategy is to show that the decompositions for fBm, sub-fBm ([20]) and bi-fBm ([15]) satisfy the conditions in Proposition 2.13, for which one needs to characterize the Cameron-Martin spaces for fBm, sub-fBm and bi-fBm (see Section 4.1).

This article is organized as follows. In Section 2, the comparison principle for the large deviations of functionals of Gaussian processes is developed in a general context. Large deviations for the functionals of the Riemann-Liouville process are
obtained in section 3. Finally, section 4 is devoted to the study of large deviations for functionals of fBm, sub-fBm and bi-fBm.

2 Large deviations by comparison

Suppose that $X$ and $Y$ are two $d$-dimensional self-similar Gaussian processes such that $\{X_{at}, t \geq 0\} \overset{d}{=} \{a^\alpha X_t, t \geq 0\}$ and $\{Y_{at}, t \geq 0\} \overset{d}{=} \{a^\alpha Y_t, t \geq 0\}$ with $\alpha > 0$ and $a > 0$, and that $Y \overset{d}{=} X + \eta$ where $\eta$ is a Gaussian process which is independent of $X$ and belongs to the reproducing kernel Hilbert space of $X$ almost surely. Let $\gamma(x)$ take the forms of the Dirac delta function $\delta(x)$ with $\alpha d < 1$, $\prod_{i=1}^d |x_i|^{-\beta_i}$ with $\beta_i \in (0, 1)$ and $\alpha \sum_{i=1}^d \beta_i < 1$, or $|x|^{-\beta}$ with $\beta \in (0, d)$ and $\alpha \beta < 1$.

The major goal of this section is to prove the following equality under some conditions,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(X_s) ds \right) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(Y_s) ds \right).$$

This result is useful, for instance, to derive the large deviations for fBm, sub-fBm and bi-fBm (see Section 4).

2.1 Reproducing kernel Hilbert spaces

In this subsection, we summarize some preliminaries on reproducing kernel Hilbert spaces associated with Gaussian processes. We refer readers to [13] for more details.

In a probability space $(\Omega, \mathcal{F}, P)$, consider a one-dimensional centered Gaussian process $X = \{X_t, 0 \leq t \leq T\}$ with covariance function

$$R(s, t) = \mathbb{E}[X_s X_t], \quad 0 \leq s \leq t \leq T.$$ 

The reproducing kernel Hilbert space (RKHS) associated with the Gaussian process $X$, denoted by $\mathbb{H}(X)$, is the completion of the linear span of the functions $\sum_{i=1}^n a_i R(s_i, \cdot)$ with $n \in \mathbb{N}^+$, $a_i \in \mathbb{R}$, $s_i \in [0, T]$, $i = 1, \ldots, n$, under the norm induced by the inner product

$$\langle R(s, \cdot), R(t, \cdot) \rangle_{\mathbb{H}(X)} = R(s, t).$$

Note that the RKHS is also referred to as the Cameron-Martin space ([13] Theorem 8.15]), and in this article we do not distinguish these two terminologies.
As a comparison, we also recall the space of integrands for Wiener integrals with respect to $X$, denoted by $\mathcal{H}(X)$, which is defined as the completion of the linear span of the simple functions $\sum_{i=1}^{n} a_i 1_{[s_i, t_i]}$ under the norm induced by the inner product
\[
\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}(X)} = R(s,t).
\]

Denote by $X(\hat{f})$ the Wiener integral for $\hat{f} \in \mathcal{H}(X)$. The collection of these Wiener integrals is the first Wiener chaos $\mathcal{H}_1$ of $X$ (see, e.g., [19]). Then $E[X(\hat{f})X(\hat{g})] = \langle \hat{f}, \hat{g} \rangle_{\mathcal{H}(X)}$, for $\hat{f}, \hat{g} \in \mathcal{H}(X)$. Furthermore, setting $f(t) = E[X(\hat{f})X(t)]$, we have $f \in \mathbb{H}(X)$, and $\langle f, g \rangle_{\mathbb{H}(X)} = \langle \hat{f}, \hat{g} \rangle_{\mathcal{H}(X)}$. Therefore, the RKHS $\mathbb{H}(X)$, the space $\mathcal{H}(X)$ of integrands of Wiener integrals, and the first Wiener chaos $\mathcal{H}_1$ of $X$ are isometric to each other. For example, when $X = W$ is a Brownian motion, $\mathcal{H}(W) = L^2[0,T]$, $\mathbb{H}(W) = \{ \int_0^T f(s)ds : \int_0^T |f(s)|^2ds < \infty \}$ and $\mathcal{H}_1 = \{ W(h) = \int_0^T h(s)dW_s, h \in L^2[0,T] \}$.

One important feature of the RKHS is the following. For a function $h : [0,T] \to \mathbb{R}$, the laws of $X + h$ and $X$ are mutually absolutely continuous (resp. mutually singular) if $h \in \mathbb{H}(X)$ (resp. if $h \notin \mathbb{H}(X)$), see, e.g., [13, Theorem 14.17]. Moreover, for $h \in \mathbb{H}(X)$, by the Cameron-Martin theorem, the measure $\tilde{P}$ defined by
\[
\frac{d\tilde{P}}{dP} = \exp \left( -X(h) - \frac{1}{2} \|h\|^2_{\mathbb{H}(X)} \right)
\]
is a probability measure, under which $X + h$ has the same distribution as $X$ under $P$.

## 2.2 Preliminaries on large deviation principles

In this subsection, let $L = \{L_t, t \geq 0\}$ be a stochastic process with non-negative values. We will recall some results on the large deviation principle for the process $L$.

**Definition 2.1** A function $I : \mathbb{R}^+ \to [0, \infty]$ is called a rate function on $\mathbb{R}^+$, if for each $M < \infty$ the level set $\{ x \in \mathbb{R}^+ : I(x) \leq M \}$ is a closed subset of $\mathbb{R}^+$. If the level set $\{ x \in \mathbb{R}^+ : I(x) \leq M \}$ is compact for any $M < \infty$, then $I(\cdot)$ is said to be a good rate function. For any $A \in \mathcal{B}(\mathbb{R}^+)$, we define $I(A) = \inf_{x \in A} I(x)$.

**Definition 2.2** Let $I(\cdot)$ be a rate function on $\mathbb{R}^+$, and let $\{ b(t), t \geq 0 \}$ be a sequence of positive real numbers such that $\lim_{t \to \infty} b(t) = \infty$. The stochastic process $L$ is said
to satisfy the large deviation principle with speed \( \{b(t)\} \) and rate function \( I(\cdot) \) if the following two conditions hold:

\[
\limsup_{t \to \infty} \frac{1}{b(t)} \log P(L_t \in F) \leq -\inf_{\lambda \in F} I(\lambda), \quad \forall \text{ closed set } F \subseteq \mathbb{R}^+,
\]

and

\[
\liminf_{t \to \infty} \frac{1}{b(t)} \log P(L_t \in G) \geq -\inf_{\lambda \in G} I(\lambda), \quad \forall \text{ open set } G \subseteq \mathbb{R}^+.
\]

The following result shows that under some mild conditions on the rate function \( I(\cdot) \), the large deviation principle defined above is equivalent to the asymptotic behavior of tail properties (see [9, Theorem 1.2.1]).

**Theorem 2.3** Suppose that the rate function \( I(\cdot) \) is strictly increasing and continuous on \( \mathbb{R}^+ \). The following two statements are equivalent:

(a) The large deviation principle given in Definition 2.2 holds.

(b) For any \( \lambda > 0 \),

\[
\lim_{t \to \infty} \frac{1}{b(t)} \log P(L_t \geq \lambda) = -I(\lambda).
\]

**Definition 2.4** A convex function \( \Lambda(\theta) : \mathbb{R}^+ \to [0, \infty] \) is said to be essentially smooth on \( \mathbb{R}^+ \), if

1. there is a \( \theta_0 > 0 \) such that \( \Lambda(\theta) < \infty \) for every \( \theta \in [0, \theta_0] \).
2. the function \( \Lambda(\cdot) \) is differentiable in the interior \( D_\Lambda^o = (0, a) \) \( (0 < a \leq \infty) \) of the domain \( D_\Lambda = \{\theta \in \mathbb{R}^+ : \Lambda(\theta) < \infty\} \).
3. the function \( \Lambda(\cdot) \) is steep at the right end of the domain and is flat at the left end of the domain, i.e.,

\[
\lim_{\theta \to a^-} \Lambda'(\theta) = \infty \quad \text{and} \quad \Lambda'(0^+) = \lim_{\theta \to 0^+} \frac{\Lambda(\theta) - \Lambda(0)}{\theta} = 0.
\]

The following result appeared in [9, Theorem 1.2.4] is a version of the Ga\text{"a}rtner-Ellis large deviation.
Theorem 2.5 Assume that for all \( \theta \geq 0 \), the limit
\[
\Lambda(\theta) = \lim_{t \to \infty} \frac{1}{b(t)} \log \mathbb{E}\{\theta b(t) L_t\}
\]
exists as an extended real number, and that the function \( \Lambda(\cdot) \) is essentially smooth on \( \mathbb{R}^+ \). Then, the function
\[
I(\lambda) = \sup_{\theta > 0} \{\theta \lambda - \Lambda(\theta)\}, \quad \lambda \geq 0
\]
is strictly increasing and continuous on \( \mathbb{R}^+ \). Moreover, the large deviation principle in Definition 2.2 and equation (2.1) hold and they are equivalent.

As the converse of the Gärtner-Ellis theorem, we have the following Varadhan’s integral lemma (see [9, Theorem 1.1.6]).

Lemma 2.6 (Varadhan’s integral lemma) Assume that the stochastic process \( L \) satisfy the large deviation principle with speed \( \{b(t), t \geq 0\} \) and a good rate function \( I(\cdot) \). Let \( \phi: \mathbb{R}^+ \to \mathbb{R} \) be any continuous function. Suppose that, for some \( \rho > 1 \), the following condition holds
\[
\limsup_{t \to \infty} \frac{1}{b(t)} \log \mathbb{E}\exp\{\rho b(t) \phi(L_t)\} < \infty,
\]
then we have
\[
\lim_{t \to \infty} \frac{1}{b(t)} \log \mathbb{E}\exp\{b(t) \phi(L_t)\} = \sup_{\lambda \in \mathbb{R}^+} \{\phi(\lambda) - I(\lambda)\}.
\]

2.3 Comparison strategy

We first validate the definition of \( \int_0^T \delta(X_s)ds \) for a class of Gaussian process. Denote the heat kernel on \( \mathbb{R}^d \) by \( p_\varepsilon(x) = (2\pi\varepsilon)^{-d/2}e^{-|x|^2/2\varepsilon} \).

Proposition 2.7 Let \( \{X_t = (X^1_t, \ldots, X^d_t), 0 \leq t \leq T\} \) be a centered Gaussian process, the components of which are independent and have the same distribution. If there exist constants \( C_T > 0 \) and \( 0 < \alpha < 1/d \), such that, for \( 0 \leq r \leq s \leq T \),
\[
\text{Var}(X^1_r) \geq C_T r^{2\alpha} \quad \text{and} \quad \text{Var}(X^1_s|X^1_r) = \mathbb{E}\left\{ \left[ X^1_s - \mathbb{E}\left( X^1_s | X^1_r \right) \right]^2 | X^1_r \right\} \geq C_T (s-r)^{2\alpha},
\]
then \( \int_0^T p_\varepsilon(X_t)dt \) converges in \( L^2 \) as \( \varepsilon \) goes to 0. The limit is denoted by \( L_T(X) := \int_0^T \delta(X_s)ds \) and called the local time of \( X \).
Proof. It suffices to show that the sequence \( \mathbb{E} \left[ \int_0^T p_\varepsilon(X_r)dr \int_0^T p_\delta(X_s)ds \right] \) converges to the same limit as \( \varepsilon \) and \( \delta \) go to zero. Note that the Fourier transform of \( p_\varepsilon(x) \) is
\[
\hat{p}(\xi) = \int_{\mathbb{R}^d} e^{-i\varepsilon \cdot \xi} p_\varepsilon(x) dx = e^{-\varepsilon|\xi|^2/2},
\]
and also note that the inverse Fourier transform implies
\[
p_\varepsilon(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\varepsilon \cdot \xi} e^{-\varepsilon|\xi|^2/2} d\xi,
\]
where \( dx = dx_1 \ldots dx_d \) and \( d\xi = d\xi_1 \ldots d\xi_d \).

Then for fixed \( r, s \in [0, T] \), we have
\[
\mathbb{E} [p_\varepsilon(X_r)p_\delta(X_s)] = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \exp \left( -\frac{1}{2} (\varepsilon |\xi|^2 + \delta |\eta|^2) \right) \mathbb{E} \exp \left( i (X_r \cdot \xi + X_s \cdot \eta) \right) d\xi d\eta \\
= \left( 2\pi \right)^{-1} \int_{\mathbb{R}^2} \exp \left( -\frac{1}{2} (\varepsilon \xi^2 + \delta \eta^2) \right) \mathbb{E} \exp \left( i (X_r^1 \xi + X_s^1 \eta) \right) d\xi d\eta \\
= \left( 2\pi \right)^{-1} \int_{\mathbb{R}^2} \exp \left( -\frac{1}{2} (\varepsilon \xi^2 + \delta \eta^2) \right) \exp \left( -\frac{1}{2} \text{Var}(X_r^1 \xi + X_s^1 \eta) \right) d\xi d\eta \\
= \left( 2\pi \right)^{-1} \int_{\mathbb{R}^2} \exp \left( -\frac{1}{2} (\varepsilon \xi^2 + \delta \eta^2) \right) \exp \left( -\frac{1}{2} (\xi, \eta) \text{Cov}(r, s)(\xi, \eta)^T \right) d\xi d\eta,
\]
where \( Q(r, s) \) is the covariance matrix of \( (X_r^1, X_s^1) \). It is well known (see, e.g., [4] or [10, Lemma 3.8]) that \( \det Q(r, s) = \text{Var}(X_r^1) \text{Var}(X_s^1 | X_r^1) = \text{Var}(X_s^1) \text{Var}(X_r^1 | X_s^1) \), and hence, by \( (2.2) \),
\[
\det Q(r, s) \geq C_T^2 (r \wedge s)^{2\alpha} |r - s|^{2\alpha}.
\]

Therefore, by the dominated convergence theorem, we can get
\[
\lim_{\varepsilon, \delta \to 0} \mathbb{E} [p_\varepsilon(X_r)p_\delta(X_s)] = (\det Q(r, s))^{-d/2} \leq C_T^{-d} (r \wedge s)^{-\alpha d} |r - s|^{-\alpha d}.
\]

Since \( \alpha d < 1 \), we obtain \( \int_0^T \int_0^T (r \wedge s)^{-\alpha d} |r - s|^{-\alpha d} dr ds < \infty \). Then one can apply the dominated convergence theorem to deduce
\[
\lim_{\varepsilon, \delta \to 0} \mathbb{E} \left[ \int_0^T p_\varepsilon(X_r)dr \int_0^T p_\delta(X_s)ds \right] = \int_0^T \int_0^T (\det Q(r, s))^{-d/2} dr ds.
\]
The proof is concluded. \[\square\]
Remark 2.8 If the conditions in (2.2) are satisfied, we say that the Gaussian process \( X \) has local nondeterminism. In particular, when \( HD < 1 \), \( HKd < 1 \), by Proposition 2.7 and the local nondeterminism of \( fBm B^H \), sub-fBm \( S^H \) and bi-fBm \( Z^{H,K} \) (see [5, 17, 23]), the local times \( L_l(B^H) \), \( L_l(S^H) \) and \( L_l(Z^{H,K}) \) exist.

Remark 2.9 Let \( X^\alpha_t = \int_0^t (t - s)^{\alpha - \frac{1}{2}} dW_s \) be the 1-dimensional Riemann-Liouville process, where \( W \) is a standard Brownian motion. Then, we can show that

(i) \( \text{Var}(X^\alpha_t) = \int_0^t (t - s)^{2\alpha - 1} ds = \frac{t^{2\alpha}}{2\alpha} \);

(ii) for any \( 0 \leq r < t < \infty \),

\[
\mathbb{E} \left( X^\alpha_t \mid F^W_r \right) = \mathbb{E} \left( \int_0^t (t - s)^{\alpha - \frac{1}{2}} dW_s \bigg| F^W_r \right) = \int_0^r (t - s)^{\alpha - \frac{1}{2}} dW_s,
\]

and

\[
\text{Var}(X^\alpha_t \mid X^\alpha_r) \geq \mathbb{E} \left( \text{Var}(X^\alpha_t \mid F^W_r) \mid X^\alpha_r \right) = \mathbb{E} \left( \mathbb{E} \left( \left[ \int_r^t (t - s)^{\alpha - \frac{1}{2}} dW_s \right]^2 \bigg| F^W_r \right) \mid X^\alpha_r \right) = \frac{1}{2\alpha} (t - r)^{2\alpha};
\]

(iii) for any \( 0 \leq r < t < \infty \), by some changes of variables

\[
\text{Var}(X^\alpha_t - X^\alpha_r) = \mathbb{E} \left( \int_0^r [ (t - s)^{\alpha - \frac{1}{2}} - (r - s)^{\alpha - \frac{1}{2}} ] dW_s \right)^2 + \mathbb{E} \left( \int_r^t (t - s)^{\alpha - \frac{1}{2}} dW_s \right)^2
\]

\[
= \int_0^r [ (t - s)^{\alpha - \frac{1}{2}} - (r - s)^{\alpha - \frac{1}{2}} ]^2 ds + \int_r^t (t - s)^{2\alpha - 1} ds
\]

\[
= (t - r)^{2\alpha} \int_0^{r/(t-r)} \left[ (1 + u)^{\alpha - \frac{1}{2}} - u^{\alpha - \frac{1}{2}} \right]^2 du + \frac{1}{2\alpha} (t - r)^{2\alpha}
\]

\[
\leq (t - r)^{2\alpha} \int_0^{\infty} \left[ (1 + u)^{\alpha - \frac{1}{2}} - u^{\alpha - \frac{1}{2}} \right]^2 du + \frac{1}{2\alpha} (t - r)^{2\alpha}
\]

\[
= C_\alpha (t - r)^{2\alpha},
\]

where \( C_\alpha = \int_0^{\infty} \left[ (1 + u)^{\alpha - \frac{1}{2}} - u^{\alpha - \frac{1}{2}} \right]^2 du + \frac{1}{2\alpha} \).
From (i) and (ii), we see that the Riemann-Liouville process has local nondeterminism. An estimate for the variance of the increment of this process is given in (iii).

In the following sections, we always assume that the process \( \{X_t, 0 \leq t \leq 1\} \) can be viewed as a Gaussian random vector in a separable Banach space. The result below is an important property of Gaussian measure (see e.g. [10, Lemma 3.7]).

**Lemma 2.10** Suppose \( \mu \) is a centered Gaussian measure on a separable Banach space \( B \). Let \( \mathbb{H}_\mu \) denote the RKHS of \( \mu \), and let \( h : B \mapsto \mathbb{R}^+ \) be a symmetric measurable function (\( h(-x) = h(x) \) for any \( x \in B \)). Then, for every \( y \) in \( \mathbb{H}_\mu \), we have

\[
\int_B h(x+y)\mu(dx) \geq \exp \left( -\frac{1}{2}\|y\|_{\mathbb{H}_\mu}^2 \right) \int_B h(x)\mu(dx),
\]

where \( \|y\|_{\mathbb{H}_\mu} \) is the norm of \( y \) in \( \mathbb{H}_\mu \).

The inequalities stated in the following will be used in the proof of Proposition 2.13 and the sub-additive property for the Riemann-Liouville process.

**Lemma 2.11** Let \( \gamma \) be a tempered distribution on \( \mathbb{R}^d \) with its Fourier transform \( \nu(dx) \) being a non-negative measure on \( \mathbb{R}^d \), i.e., \( \gamma \) is a non-negative definite distribution. Then for any centered Gaussian random vector \( X \sim N(0, \Sigma) \), where \( \Sigma \) is a positive definite matrix in \( \mathbb{R}^{d \times d} \), we have

\[
E[\gamma(X + a)] \leq E[\gamma(X)], \text{ for all } a \in \mathbb{R}^d.
\]

**Proof** Denote the probability density function of the Gaussian random vector \( X \) by \( p_\Sigma(x) \). Note that \( p_\Sigma(x) \) belongs to the Schwartz space \( S(\mathbb{R}^d) \) and its Fourier transform is \( \hat{p}_\Sigma(\xi) = \exp\{-\frac{\xi^T \Sigma \xi}{2}\} \). Then

\[
E[\gamma(X + a)] = \int_{\mathbb{R}^d} \gamma(x + a)p_\Sigma(x)dx
= \int_{\mathbb{R}^d} \hat{p}_\Sigma(\xi)e^{-ia\cdot\xi}\nu(d\xi) \leq \int_{\mathbb{R}^d} \hat{p}_\Sigma(\xi)\nu(d\xi) = E[\gamma(X)].
\]

We complete the proof. \( \blacksquare \)
Remark 2.12 Similarly, one can show that for a centered Gaussian vector \((X_1, \ldots, X_n)\) with \(X_i\) being a \(d\)-dimensional Gaussian vector, we have for any \(a = (a_1, \ldots, a_n) \in \mathbb{R}^{d \times n}\)

\[
\mathbb{E} \left[ \prod_{i=1}^{n} \gamma(X_i + a_i) \right] \leq \mathbb{E} \left[ \prod_{i=1}^{n} \gamma(X_i) \right].
\]

When \(\gamma\) is a measurable function which is also symmetric \((\gamma(-x) = \gamma(x))\), and the result can also be obtained by [10, Lemma 3.7 (i)].

The following proposition is the main result in this subsection.

Proposition 2.13 Let \(\{X_t, t \geq 0\}, \{Y_t, t \geq 0\} \) and \(\{\eta_t, t \geq 0\}\) be \(d\)-dimensional centered Gaussian processes, such that

(i) \(\{X_{at}, t \geq 0\} \overset{d}{=} a^\alpha \{X_t, t \geq 0\}\) and \(\{Y_{at}, t \geq 0\} \overset{d}{=} a^\alpha \{Y_t, t \geq 0\}\) for some \(\alpha > 0\);
(ii) \(Y = X + \eta\);
(iii) \(X\) and \(\eta\) are independent;
(iv) For any \(\varepsilon \in (0, 1)\), there exists a process \(\eta^\varepsilon\) such that \(\eta^\varepsilon_t = \eta_t\) for \(t \geq \varepsilon\), and \(\{\eta^\varepsilon_t, t \in [0, 1]\}\) belongs to the RKHS of \(\{X_t, t \in [0, 1]\}\) almost surely.

If either \(\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(X_s) ds \right)\) or \(\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(Y_s) ds \right)\) exists as a finite number, then both limits exist and are equal to each other.

Proof Without loss of generality, we assume \(\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(X_s) ds \right)\) exists.

It follows from Remark 2.12 that

\[
\mathbb{E} \left( \int_0^t \gamma(Y_s) ds \right)^n \leq \mathbb{E} \left( \int_0^t \gamma(X_s) ds \right)^n,
\]

and hence

\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(Y_s) ds \right) \leq \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(X_s) ds \right).
\]

To get the desired result, we shall prove the opposite direction of the above inequality with \(\limsup\) replaced by \(\liminf\). Fixing an arbitrary \(\varepsilon \in (0, 1)\), and
denoting $Y^\varepsilon = X + \eta^\varepsilon$, by Lemma 2.10 Hölder’s inequality and the scaling property
\[ \int_0^a \gamma(X_s) ds \overset{d}{=} a^{1-\alpha\beta} \int_0^1 \gamma(X_s) ds, \]
we have
\[ \mathbb{E} \left( \int_0^1 \gamma(Y_s) ds \right)^n \geq \mathbb{E} \left( \int_\varepsilon^1 \gamma(Y_s^\varepsilon) ds \right)^n \]
\[ \overset{\text{by Lemma 2.10}}{=} \mathbb{E} \exp \left( - \frac{1}{2} \| \eta^\varepsilon \|^2_{H(X)} \right) \mathbb{E} \left( \int_\varepsilon^1 \gamma(X_s) ds \right)^n \]
\[ = \mathbb{E} \exp \left( - \frac{1}{2} \| \eta^\varepsilon \|^2_{H(X)} \right) \mathbb{E} \left( \int_0^1 \gamma(X_s) ds - \int_0^\varepsilon \gamma(X_s) ds \right)^n \]
\[ \geq A_\varepsilon \left( \left( \mathbb{E} \left( \int_0^1 \gamma(X_s) ds \right)^n \right)^{1/n} - \left( \mathbb{E} \left( \int_0^\varepsilon \gamma(X_s) ds \right)^n \right)^{1/n} \right)^n \]
\[ = A_\varepsilon \left( 1 - \varepsilon^{1-\alpha\beta} \right)^n \mathbb{E} \left( \int_0^1 \gamma(X_s) ds \right)^n, \quad (2.3) \]

where $\| \eta^\varepsilon \|_{H(X)} < \infty$ a.s. is the norm endowed in the RKHS of \{X, s \in [0, 1]\}, $A_\varepsilon = \mathbb{E} \exp \left( - \frac{1}{2} \| \eta^\varepsilon \|^2_{H(X)} \right) \in (0, 1]$.

Thus, by the scaling property for the functional of $Y$ and (2.3), we obtain
\[ \mathbb{E} \exp \left( \int_0^T \gamma(Y_s) ds \right) = \sum_{n \geq 0} \frac{1}{n!} t^{n(1-\alpha\beta)} \mathbb{E} \left( \int_0^1 \gamma(Y_s) ds \right)^n \]
\[ \overset{\text{by Lemma 2.10}}{=} A_\varepsilon \sum_{n \geq 0} \frac{1}{n!} (t^{1-\alpha\beta}(1 - \varepsilon^{1-\alpha\beta}))^n \mathbb{E} \left( \int_0^1 \gamma(X_s) ds \right)^n \]
\[ = A_\varepsilon \mathbb{E} \exp \left( t^{1-\alpha\beta}(1 - \varepsilon^{1-\alpha\beta}) \int_0^1 \gamma(X_s) ds \right). \quad (2.4) \]

For random variables $F, G$ with $\mathbb{E} e^{\theta G} < \infty$ and $\mathbb{E} e^{\theta F} < \infty$ for all $\theta > 0$, Hölder’s inequality yields
\[ \log \mathbb{E} e^{F-G} \geq p \log \mathbb{E} e^F - \frac{p}{q} \log \mathbb{E} e^{\frac{q}{p} G}, \]
where $p, q > 1$ and $1/p + 1/q = 1$. Applying this inequality to the right-hand side of
and using the scaling property $\int_0^a \gamma(X_s)ds \overset{d}{=} a^{1-\alpha} \int_0^1 \gamma(X_s)ds$, we get

\[
\log \mathbb{E} \exp \left( \int_0^t \gamma(Y_s)ds \right)
\geq \log A_\varepsilon + p \log \mathbb{E} \exp \left( p^{-1} \int_0^1 \gamma(X_s)ds \right) - \frac{p}{q} \log \mathbb{E} \exp \left( p^{-1} q \varepsilon t \int_0^1 \gamma(X_s)ds \right)
\geq \log A_\varepsilon + p \log \mathbb{E} \left( \int_0^{tp^{-1}(1-\alpha\beta)^{-1}} \gamma(X_s)ds \right) - \frac{p}{q} \log \mathbb{E} \left( \int_0^{tp^{-1}(1-\alpha\beta)^{-1}} \gamma(X_s)ds \right),
\]

and hence

\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(Y_s)ds \right)
\geq \lim_{t \to \infty} \frac{p}{t} \log \mathbb{E} \left( \int_0^{tp^{-1}(1-\alpha\beta)^{-1}} \gamma(X_s)ds \right) - \lim_{t \to \infty} \frac{p}{qt} \log \mathbb{E} \left( \int_0^{tp^{-1}(1-\alpha\beta)^{-1}} \gamma(X_s)ds \right)
= \left( p^{1-(1-\alpha\beta)^{-1}} - \varepsilon (pq^{-1})^{1-(1-\alpha\beta)^{-1}} \right) \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(X_s)ds \right).
\]

Since $\varepsilon > 0$ can be arbitrarily small and $p$ can be arbitrarily close to 1, we obtain

\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(Y_s)ds \right) \geq \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t \gamma(X_s)ds \right).
\]

The proof is completed.

**Remark 2.14** It is obvious that if condition (iv) is replaced by

(iv') $\{\eta_t, t \in [0, 1]\}$ belongs to the RKHS of $\{X_t, t \in [0, 1]\}$ almost surely,

the result of Proposition 2.13 still holds.

### 3 Large deviations for the functionals of Riemann-Liouville process

In this section, we let $X^\alpha = \{X^\alpha_t, t \geq 0\}$ be the $d$-dimensional Riemann-Liouville process with parameter $\alpha \in (0, 1)$, i.e., $X^\alpha_t = \int_0^t (t - s)^{\alpha - \frac{1}{2}} dW_s$, where $\{W_t, t \geq 0\}$ is a $d$-dimensional Brownian motion. This section is devoted to deriving the large deviations for $\int_0^t \gamma(X^\alpha_s)ds$, where $\gamma$ is the functional given in Section 2.
Proposition 3.1 Suppose \( X_t = \int_0^t K(t-s)dW_s \), where \( K(s) : \mathbb{R}^+ \to \mathbb{R}^d \) is a measurable function such that \( \int_0^T |K(s)|^2ds \) for all \( T > 0 \). Let \( \gamma \) be a tempered distribution on \( \mathbb{R}^d \) with its Fourier transform \( \nu(dx) \) being a non-negative measure on \( \mathbb{R}^d \), i.e., \( \gamma \) is a non-negative definite distribution. Then \( \log (\gamma(X_s)) \) is sub-additive in \( m \), where \( \tau \) is an exponential time with parameter 1 independent of \( X \).

Proof Denote \([0,t]_m^m = [0 < s_1 < s_2 < \cdots < s_m < t] \) and \( \mathbb{P}_+^{m,+} = [0 < s_1 < s_2 < \cdots < s_m < \infty] \). Notice that

\[
\frac{1}{m!} \mathbb{E} \left( \int_0^\tau \gamma(X_s)ds \right)^m = \frac{1}{m!} \int_0^\infty e^{-s} \mathbb{E} \left( \int_0^s \gamma(X_s)ds \right)^m ds \\
= \int_0^\infty e^{-s} \int_{[0,s]_m^{m}} \mathbb{E} \left( \prod_{k=1}^m \gamma(X_{s_k}) \right) ds_1 \cdots ds_m ds \\
= \int_{\mathbb{P}_+^{m,+}} e^{-s m} \mathbb{E} \left( \prod_{k=1}^m \gamma(X_{s_k}) \right) ds_1 \cdots ds_m.
\]

Therefore,

\[
\frac{1}{(m+n)!} \mathbb{E} \left( \int_0^\tau \gamma(X_s)ds \right)^{m+n} \\
= \int_{\mathbb{P}_+^{m+n}} e^{-s (m+n)} \mathbb{E} \left( \prod_{k=1}^{m+n} \gamma(X_{s_k}) \right) ds_1 \cdots ds_{m+n} \\
= \int_{\mathbb{P}_+^{m+n}} e^{-s m} e^{-s (m+n)} \mathbb{E} \left( \prod_{k=1}^{m+n} \gamma(X_{s_k}) \mathbb{E} \left( \prod_{k=m+1}^{m+n} \gamma(X_{s_k}) \mathbb{F}_{s_m} \right) \right) ds_1 \cdots ds_{m+n}. \tag{3.1}
\]

For \( k = m+1, \ldots, m+n \), let \( X_{s_k} = \int_0^{s_k} K(s_k - s)dW_s = A_{s_m,s_k} + Y_{s_m,s_k} \), where \( A_{s_m,s_k} = \int_0^{s_k} K(s_k - s)dW_s \) and \( Y_{s_m,s_k} = \int_{s_m}^{s_k} K(s_k - s)dW_s \). Furthermore, note that

\[
(Y_{s_m,s_{m+1}}, \ldots, Y_{s_m,s_{m+n}}) \overset{d}{=} (X_{s_{m+1}-s_m}, \ldots, X_{s_{m+n}-s_m}). \tag{3.2}
\]

Hence, by the fact that \( A_{s_m,s_k} \in \mathcal{F}_{s_m} \), (3.2) and Remark 2.12 we have

\[
\mathbb{E} \left( \prod_{k=m+1}^{m+n} \gamma(X_{s_k}) | \mathcal{F}_{s_m} \right) \leq \mathbb{E} \left( \prod_{k=m+1}^{m+n} \gamma(X_{s_k} - s_m) \right).
\]
Thus, it follows from (3.1) and a change of variables,

\[
\frac{1}{(m+n)!} E \left( \int_0^\tau \gamma(X_s) ds \right)^{m+n} 
\leq \int_{\mathbb{R}_{+}} e^{-s_m} e^{-(s_{m+n}+s_m)} \mathbb{E} \left[ \prod_{k=1}^m \gamma(X_{s_k}) \right] \mathbb{E} \left[ \prod_{k=m+1}^{m+n} \gamma(X_{s_k-s_m}) \right] ds_1 \ldots ds_{m+n} 
\leq \int_{\mathbb{R}_{+}} e^{-s_m} \mathbb{E} \left[ \prod_{k=1}^m \gamma(X_{s_k}) \right] ds_1 \ldots ds_m \int_{\mathbb{R}_{+}} e^{-s_n} \mathbb{E} \left[ \prod_{k=1}^n \gamma(X_{s_k}) \right] ds_1 \ldots ds_n 
= \frac{1}{m!} \mathbb{E} \left( \int_0^\tau \gamma(X_s) ds \right)^m \frac{1}{n!} \mathbb{E} \left( \int_0^\tau \gamma(X_s) ds \right)^n.
\]

The proof is completed. \(\blacksquare\)

**Proposition 3.2** Let \(\gamma(x)\) be given in Section 3. Then for all \(\theta > 0\),

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \int_0^t \gamma(X_s^\alpha) ds \right) = \mathcal{E}(\gamma, \alpha, d) \theta^{1-\alpha \beta},
\]

where \(\mathcal{E}(\gamma, \alpha, d)\) is a positive constant depending on \((\gamma, \alpha, d)\).

**Proof** Let \(\tau\) be an exponential time with parameter 1 which is independent of \(X\). By Proposition 3.1 and Fekete’s lemma, we know that there exists an extended number \(A \in [-\infty, \infty)\), such that

\[
A := \lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{m!} \mathbb{E} \left( \int_0^\tau \gamma(X_s^\alpha) ds \right)^m \right) = \inf_m \left\{ \frac{1}{m} \log \left( \frac{1}{m!} \mathbb{E} \left( \int_0^\tau \gamma(X_s^\alpha) ds \right)^m \right) \right\}. \tag{3.3}
\]

By the scaling property, we have

\[
\frac{1}{m} \log \left( \frac{1}{m!} \mathbb{E} \left( \int_0^\tau \gamma(X_s^\alpha) ds \right)^m \right) = \frac{1}{m} \log \left( \mathbb{E}[\tau^{(1-\alpha)\beta}m!] \frac{1}{m!} \mathbb{E} \left( \int_0^\tau \gamma(X_s^\alpha) ds \right)^m \right). \tag{3.4}
\]

First we show that \(A\) is a real number. Noting that for \(x \in \mathbb{R}^d\), \(|x|^{-(\beta_1 + \cdots + \beta_d)} \leq |x_1|^{-\beta_1} \cdots |x_d|^{-\beta_d}\), we only need to show \(A > -\infty\) for the cases \(\gamma(x) = \delta(x)\) and \(\gamma(x) = |x|^{-\beta}\).
For the case $\gamma(x) = \delta(x)$, we have

$$\frac{1}{m!} \mathbb{E} \left( \int_0^1 \gamma(X_s^\alpha) ds \right)^m$$

$$= \int_{[0,1]^m} \mathbb{E} \left[ \prod_{i=1}^m \gamma(X_{s_i}^\alpha) \right] ds_1 \ldots ds_m$$

$$= \int_{[0,1]^m} \int_{\mathbb{R}^m} \prod_{i=1}^m \hat{\gamma}(\xi_i) \exp \left( -\frac{1}{2} \text{Var} \left( \sum_{i=1}^m \xi_i \cdot X_{s_i}^\alpha \right) \right) d\xi_1 \ldots d\xi_m ds_1 \ldots ds_m.$$  \hspace{1cm} (3.5)

Note (see, e.g., [4] or [10, Lemma 3.8]) that, for $0 < s_1 < s_2 < \ldots < s_n < 1$,

$$\det \left[ \text{Cov}(X_{s_1}^{\alpha,1}, \ldots, X_{s_n}^{\alpha,1}) \right]$$

$$= \text{Var}(X_{s_1}^{\alpha,1}) \text{Var}(X_{s_2}^{\alpha,1} | X_{s_1}^{\alpha,1}) \ldots \text{Var}(X_{s_n}^{\alpha,1} | X_{s_1}^{\alpha,1}, \ldots, X_{s_{n-1}}^{\alpha,1})$$

$$= \text{Var}(X_{s_1}^{\alpha,1}) \text{Var}(X_{s_2}^{\alpha,1} - X_{s_1}^{\alpha,1} | X_{s_1}^{\alpha,1}) \ldots \text{Var}(X_{s_n}^{\alpha,1} - X_{s_1}^{\alpha,1} | X_{s_1}^{\alpha,1}, \ldots, X_{s_{n-1}}^{\alpha,1})$$

$$\leq \text{Var}(X_{s_1}^{\alpha,1}) \text{Var}(X_{s_2}^{\alpha,1} - X_{s_1}^{\alpha,1}) \ldots \text{Var}(X_{s_n}^{\alpha,1} - X_{s_{n-1}}^{\alpha,1})$$

$$\leq C^m s_1^{2\alpha} (s_2 - s_1)^{2\alpha} \ldots (s_m - s_{m-1})^{2\alpha}.$$  \hspace{1cm} (3.6)

where and in the following $C$ denotes a generic positive constant independent of $m$ which may vary from line to line.

When $\gamma(x) = \delta(x), \hat{\gamma}(\xi) = 1$, and together with (3.6), the equation (3.5) equals

$$\int_{[0,1]^m} \int_{\mathbb{R}^m} \exp \left( -\frac{1}{2} \text{Var} \left( \sum_{i=1}^m \xi_i \cdot X_{s_i}^\alpha \right) \right) d\xi_1 \ldots d\xi_m ds_1 \ldots ds_m$$

$$= \int_{[0,1]^m} \left( 2\pi \det \left[ \text{Cov}(X_{s_1}^{\alpha,1}, \ldots, X_{s_m}^{\alpha,1}) \right] \right)^{-d/2} ds_1 \ldots ds_m$$

$$\geq C^m \int_{[0,1]^m} s_1^{-\alpha d} (s_2 - s_1)^{-\alpha d} \ldots (s_m - s_{m-1})^{-\alpha d} ds_1 \ldots ds_m$$

$$= C^m \frac{\Gamma^m(1 - \alpha d)}{\Gamma(1 + (1 - \alpha d)m)}.$$

The above inequality, the fact $\mathbb{E}[\tau^{(1-\alpha d)m}] = \Gamma(1 + (1 - \alpha d) m)$ with $\Gamma(\cdot)$ being the Gamma function, (3.3) and (3.4) imply that $A \geq \log \Gamma(1 - \alpha d) + \log C > -\infty$.

Now we show that $A > -\infty$ when $\gamma(x) = |x|^{-\beta}$. Assume instead that $A = -\infty$, then by (3.4), the fact $\mathbb{E}[\tau^{(1-\alpha \beta)m}] = \Gamma(1 + (1 - \alpha \beta)m)$, and the Stirling formulas
\[ \Gamma(1 + x) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \] and \( m! \sim \sqrt{2\pi m} \left( \frac{m}{e} \right)^m \), we have
\[
\lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{(m!)^{\alpha \beta}} \mathbb{E} \left( \int_0^1 \gamma(X_s^\alpha) ds \right)^m \right) = -\infty. \tag{3.7}
\]

Thus Lemma 3.4 (i) implies that
\[
\limsup_{t \to \infty} \frac{1}{t^{1/\alpha \beta}} \log \mathbb{P} \left( \int_0^1 \gamma(X_s^\alpha) ds \geq t \right) = -\infty,
\]
which is equivalent to, by scaling property \( f^t \gamma(X_s^\alpha) ds = t^{1-\alpha \beta} \int_0^1 \gamma(X_s^\alpha) ds \),
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{1}{t} \int_0^t \gamma(X_s^\alpha) ds \geq \lambda \right) = -\infty, \quad \text{for all } \lambda > 0. \tag{3.8}
\]

Note that the (3.7) implies
\[
\mathbb{E} \exp \left( \theta \int_0^1 \gamma(X_s^\alpha) ds \right) < \infty, \quad \text{for all } \theta > 0,
\]
and moreover,
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \int_0^t \gamma(X_s^\alpha) ds \right) < \infty, \quad \text{for all } \theta > 0. \tag{3.9}
\]

Then, it follows from (3.8), (3.9), Theorem 1.2.9 and Lemma 1.2.10 in [9] that
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left( \int_0^t \gamma(X_s^\alpha) ds \right) = 0,
\]
which contradicts Lemma 3.3. Therefore, \( A > -\infty \) when \( \gamma(x) = |x|^{-\beta} \).

Since \( A \) in (3.3) is a real number, by (3.4) and the Stirling formula, there exists \( a \in (-\infty, 0) \) (depending on the function \( \gamma(x) \)) such that
\[
\lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{(m!)^{\alpha \beta}} \mathbb{E} \left( \int_0^1 \gamma(X_s^\alpha) ds \right)^m \right) = a. \tag{3.10}
\]
By the scaling property and Lemma 3.4 (ii), we have for all \( \lambda > 0 \),
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left( \frac{1}{t} \int_0^t \gamma(X_s^\alpha) ds \geq \lambda \right) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left( \frac{1}{t} \cdot t^{1-\alpha \beta} \int_0^1 \gamma(X_s^\alpha) ds \geq \lambda \right)
\]
\[
= \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left( \int_0^1 \gamma(X_s^\alpha) ds \geq \lambda t^{\alpha \beta} \right)
\]
\[
= \lambda^{1/\alpha \beta} \lim_{u \to \infty} \frac{1}{u^{1/\alpha \beta}} \log \mathbb{P}\left( \int_0^1 \gamma(X_s^\alpha) ds \geq u \right)
\]
\[
= -\alpha \beta e^{-a/\alpha \beta} \lambda^{1/\alpha \beta}.
\]

Notice that (3.10) implies that
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \int_0^t \gamma(X_s^\alpha) ds \right) < \infty, \quad \text{for all } \theta > 0.
\]
Hence, by Theorem 2.3 and Varadhan’s integral lemma (see Lemma 2.6), we have for \( \theta > 0 \),
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \int_0^t \gamma(X_s^\alpha) ds \right) = \mathcal{E}(\gamma, \alpha, d) \theta^{1/\alpha \beta}.
\]
The proof is concluded. 

The remaining part of this section consists of lemmas that were used in the previous proof.

**Lemma 3.3** For the Riemann-Liouville process \( X^\alpha \) we have
\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t |X_s^\alpha|^{-\beta} ds \right) > 0.
\]

**Proof** For any \( \varepsilon > 0 \), by the small ball probability result provided in [16] Theorem 4.1], there exists a constant \( c_0 \in (0, \infty) \) such that
\[
\mathbb{P}\left( \sup_{0 \leq s \leq 1} |X_s^\alpha,1| \leq \varepsilon \right) \geq \exp(-c_0 \varepsilon^{-1/\alpha}). \tag{3.11}
\]
Denote \( S_\varepsilon = \left\{ \sup_{j \in \{1, \ldots, d\}} \sup_{0 \leq s \leq 1} |X_s^{\alpha,j}| \leq \varepsilon \right\} \). By (3.11), we have
\[
P(S_\varepsilon) = \left( \mathbb{P}\left( \sup_{0 \leq s \leq 1} |X_s^{\alpha,1}| \leq \varepsilon \right) \right)^d \geq \exp(-c_0 d \varepsilon^{-1/\alpha}).
\]
Then,
\[
\mathbb{E} \exp \left( \int_0^t |X_s^\alpha|^{-\beta} \, ds \right) \geq \mathbb{E} \left[ \exp \left( t^{1-\alpha\beta} \int_0^1 |X_s^\alpha|^{-\beta} \, ds \right) 1_{S_\varepsilon} \right] \\
\geq \exp \left( c_d t^{1-\alpha\beta} \varepsilon^{-\beta} \right) \mathbb{P}(S_\varepsilon) \\
\geq \exp(c_d t^{1-\alpha\beta} \varepsilon^{-\beta} - c_0 d \varepsilon^{-1/\alpha}),
\]
where \(c_d\) is a positive constant depending on \(d\).

Now, choose \(\varepsilon = \left( \frac{2c_0 d}{c_d} \right)^{\alpha/(1-\alpha\beta)} t^{-\alpha}\) such that
\[
c_d t^{1-\alpha\beta} \varepsilon^{-\beta} - c_0 d \varepsilon^{-1/\alpha} = c_0 d \varepsilon^{-1/\alpha} = Ct,
\]
where \(C = c_0 d \left( \frac{c_d}{2c_0 d} \right)^{1/(1-\alpha\beta)}\). Consequently, we can show
\[
\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \int_0^t |X_s^\alpha|^{-\beta} \, ds \right) \geq C > 0.
\]

We prove the desired result. \(\blacksquare\)

The following lemma (see [14, Lemma 2.3] or [9, Theorem 1.2.8]) connects the moments and large deviations.

**Lemma 3.4** Let \(F \geq 0\) be a random variable and let \(p > 0\). Then, for any \(a \in \mathbb{R}\), the following results hold.

(i) If
\[
\limsup_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{(m!)^p} \mathbb{E} F^m \right) \leq a,
\]
for some \(a \in \mathbb{R}\), then
\[
\limsup_{x \to \infty} \frac{1}{x^{1/p}} \log \mathbb{P}(F \geq x) \leq -pe^{-a/p}.
\]

(ii) If
\[
\lim_{m \to \infty} \frac{1}{m} \log \left( \frac{1}{(m!)^p} \mathbb{E} F^m \right) = a,
\]
for some \(a \in \mathbb{R}\), then
\[
\lim_{x \to \infty} \frac{1}{x^{1/p}} \log \mathbb{P}(F \geq x) = -pe^{-a/p}.
\]
4 Large deviations for the functionals of fBm, sub-fBm and bi-fBm

4.1 Some preliminaries on fBm, sub-fBm and bi-fBm

In this subsection, we will first recall some preliminaries on fBm, sub-fBm and bi-fBm, and then we will provide some detailed results on the RKHSs associated to some of them.

**Definition 4.1** A centered 1-dimensional Gaussian process \( \{B^H_t, t \geq 0\} \) is called a fBm with Hurst parameter \( H \in (0, 1) \), if the covariance function is given by

\[
R(t, s) = \mathbb{E}[B^H_t B^H_s] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}). \tag{4.1}
\]

It immediately implies from the above covariance function that fBm has self-similarity: \( \{B^H_{at}, t \geq 0\} \overset{d}{=} a^H \{B^H_t, t \geq 0\} \), for any \( a > 0 \).

When \( H = \frac{1}{2} \), the process \( B^\frac{1}{2} \) is a standard Brownian motion. As an extension of classical Brownian motion, fBm is essentially different from Brownian motion in the sense that fBm is not a semi-martingale nor a Markov process when \( H \neq \frac{1}{2} \). We refer readers to [6, 19] and the references therein for more details on the analysis of fBm.

In this subsection, let \( W = \{W_t, t \in \mathbb{R}\} \) be a Brownian motion on \( \mathbb{R} \). Then, fBm \( B^H \) has the following representation (see [18] and [22]):

\[
B^H_t = \alpha_H \int_{-\infty}^{t} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right] dW_s
= \alpha_H X^H_t + \alpha_H \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right] dW_s
= : \alpha_H X^H_t + \eta^H_t,
\]

where \( \alpha_H = \left( \int_0^{\infty} \left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right]^2 ds + \frac{1}{2H} \right)^{\frac{1}{2}} \), \( X^H \) is the Riemann-Liouville process with parameter \( H \), and the process \( \eta^H_t = \alpha_H \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right] dW_s \) is independent of \( X^H \).
The left-sided fractional Riemann-Liouville integrals of \( f \in L^2[0, 1] \) of order \( \alpha > 0 \) are defined for almost all \( t \in [0, 1] \) by

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds.
\]

Let \( I_0^\alpha(L^2[0, 1]) \) denote the image of \( L^2[0, 1] \) under \( I_0^\alpha \).

Lemma 10.2 in [24] provides the following result on the RKHS of the Riemann-Liouville process \( X^H \):

\[
\mathbb{H}(X^H) = I^{H+\frac{1}{2}}(L^2[0, 1]).
\]

Regarding the decomposition (1.2) of fBm \( B^H \), we summarize Propositions 3.3 and 3.5 in [10] as follows.

**Proposition 4.2** For any \( 0 < \varepsilon < 1 \), the process \( \eta^H_t, t \geq \varepsilon \) has \( C^\infty \)-sample paths a.s., and there is a Gaussian process \( \{ \eta^{\varepsilon,H}_t, t \geq 0 \} \) such that

(a) \( \eta^{\varepsilon,H}_t = \eta^H_t \) for all \( t \geq \varepsilon \);

(b) \( \mathbb{P}\left( \{ \eta^{\varepsilon,H}_t, 0 \leq t \leq 1 \} \in \mathbb{H}(X^H) \right) = 1 \).

Next, we discuss the RKHS of fBm \( B^H \). The covariance function of fBm can be expressed as (see [11])

\[
R(t, s) = \mathbb{E}[B^H_t B^H_s] = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr,
\]

where

\[
K_H(t, s) = c_H(t - s)^{H-\frac{1}{2}} F \left( H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s} \right) 1_{[0,t]}(s),
\]

with \( c_H = \left[ \frac{2^{2H}(\frac{3}{2} - H)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})} \right]^{1/2} \) and \( F(a, b, c, z) \) being the Gauss hypergeometric function.

Note that for any fixed \( t \in [0, 1] \), \( K_H(t, \cdot) \in L^2[0, 1] \). Consider the integral transform \( K_H \) defined on \( L^2[0, T] \) by

\[
(K_H f)(t) := \int_0^t K_H(t, s) f(s) ds,
\]

for any \( f \in L^2[0, 1] \).
We can easily see from (4.3) that $R(s, \cdot) = K_H(K_H(s, \cdot))$. In fact, the RKHS of $B^H$ is $\mathbb{H}(B^H) = I_0^{H+\frac{1}{2}}(L^2[0, 1])$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}(B^H)}$. The integral transform $K_H$ is an isomorphism from $L^2[0, 1]$ onto $I_0^{H+\frac{1}{2}}(L^2[0, 1])$ (see Theorems 2.1 and 3.3 and Remark 3.1).

**Definition 4.3** Let $\{S_t^H, t \geq 0\}$ denote a 1-dimensional sub-fBm with index $H \in (0, 1)$ introduced in [7], which is a centered Gaussian process with covariance function

$$
\mathbb{E}[S_t^H S_s^H] = t^{2H} + s^{2H} - \frac{1}{2} \left( (t + s)^{2H} - |t - s|^{2H} \right).
$$

(4.4)

Clearly when $H = \frac{1}{2}$, $S^\frac{1}{2}$ is a standard Brownian motion. Note that the process $S^H$ has self-similarity $\{S_{a^t}^H, t \geq 0\} \overset{d}{=} a^H \{S_t^H, t \geq 0\}$, for any $a > 0$.

Define, for $\alpha \in (0, \frac{1}{2})$,

$$
Y_t^\alpha = \int_0^\infty (1 - e^{-rt}) r^{-\alpha - \frac{1}{2}} dW_r.
$$

(4.5)

By some basic calculation, we get

$$
\mathbb{E}(Y_t^\alpha Y_s^\alpha) = \frac{\Gamma(1 - 2\alpha)}{2\alpha} \left( (4\alpha + s^{2\alpha} - (t + s)^{2\alpha}) \right).
$$

(4.6)

Assume that $W$ in (4.5) is independent of $B^H$, part (a) of [20, Theorem 3.5] provides the following decomposition result for sub-fBm.

**Proposition 4.4** For $0 < H < \frac{1}{2}$, $\{B_t^H + \sqrt{\frac{H(1 - 2H)}{1(2 - 2H)}} Y_t^H, t \geq 0\}$ has the same law as $\{S_t^H, t \geq 0\}$.

**Definition 4.5** The bi-fBm $\{Z_t^{H,K}, t \geq 0\}$ with parameters $H \in (0, 1)$ and $K \in (0, 1]$ is a generalization of fBm, defined as a centered 1-dimensional Gaussian process with covariance function

$$
\mathbb{E}[Z_t^{H,K} Z_s^{H,K}] = \frac{1}{2K} \left( (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right).
$$

(4.7)
It is obvious that the bi-fBm $Z_{H,1}^{H,1}$ degenerates to a fBm when $K = 1$ and that $Z_{H,K}^{H,K}$ has self-similarity $\{Z_{at}^{H,K}, t \geq 0\} \overset{d}{=} a^{HK} \{Z_{t}^{H,K}, t \geq 0\}$, for any $a > 0$.

Now assume that $W$ in (4.5) is independent of $Z_{H,K}^{H,K}$. The following decomposition result for bi-fBm is obtained in [15].

**Proposition 4.6** For $H, K \in (0, 1)$, \(\left\{Z_{t}^{H,K} + \sqrt{\frac{K}{2\pi \Gamma(1-K)}} Y_{t^{2H}}^{K}, t \geq 0\right\}\) has the same law as \(\left\{2^{1-K} B_{HK}^{H,K}, t \geq 0\right\}\), where $B_{HK}^{H,K}$ is a fBm with Hurst parameter $HK$.

Denote the RKHSs of $\{B_{H}^{H}, 0 \leq t \leq 1\}$ with parameter $H$ and $\{Z_{t}^{H,K}, 0 \leq t \leq 1\}$ with parameters $H$ and $K$ by $\mathbb{H}(B_{H}^{H})$ and $\mathbb{H}(Z_{H,K}^{H,K})$ respectively. For bi-fBm $Z_{H,K}^{H,K}$, unlike fBm, we don’t have an explicit representation for its RKHS when $K \in (0, 1)$. However, we have the following property.

**Proposition 4.7** For $H, K \in (0, 1)$, $\mathbb{H}(B_{H}^{H}) \subseteq \mathbb{H}(Z_{H,K}^{H,K}) \subseteq \mathbb{H}(B_{HK}^{H,K})$. In particular, $I_{0+}^{3}(L^{2}[0,1]) \subseteq \mathbb{H}(B_{H}^{H}) \subseteq \mathbb{H}(Z_{H,K}^{H,K})$.

**Proof** The fact $\mathbb{H}(B_{H}^{H}) \subseteq \mathbb{H}(Z_{H,K}^{H,K})$ follows from [1] Theorem 5.2.

Next, we will show $\mathbb{H}(Z_{H,K}^{H,K}) \subseteq \mathbb{H}(B_{HK}^{H,K})$. Note from (4.1), (4.6) and (4.7) that

\[
E\left(B_{t}^{2HK}B_{s}^{2HK}\right) = \frac{2K}{2} E\left(Z_{t}^{H,K}Z_{s}^{H,K}\right) + \frac{K}{2\pi \Gamma(1-K)} E\left(Y_{t^{2H}}^{K}Y_{s^{2H}}^{K}\right).
\]

It follows from Theorem I in [2], p. 354] that the $\mathbb{H}\left(\sqrt{\frac{2}{2}} Z_{H,K}\right) = \mathbb{H}(Z_{H,K}^{H,K}) \subseteq \mathbb{H}(B_{HK}^{H,K})$.

For any $0 \leq \alpha \leq \beta$, we have $I_{0+}^{\beta}(L^{2}[0,1]) \subseteq I_{0+}^{\alpha}(L^{2}[0,1])$. Indeed, for any $f \in I_{0+}^{\beta}(L^{2}[0,1])$, there exists $g \in L^{2}[0,1]$ such that $f = I_{0+}^{\beta}(g)$. Then by [21] Theorem 2.5, we get $f = I_{0+}^{\beta}(I_{0+}^{\beta-\alpha}(g))$. In addition, $I_{0+}^{\beta-\alpha}(g) \in L^{2}[0,1]$ by [21] Theorem 2.6, we can show that $f = I_{0+}^{\beta}(g) = I_{0+}^{\alpha}(I_{0+}^{\beta-\alpha}(g))$ belongs to $I_{0+}^{\alpha}(L^{2}[0,1])$. Consequently, the relationship $I_{0+}^{\beta}(L^{2}[0,1]) \subseteq \mathbb{H}(B_{H}^{H})$ holds by noting $\mathbb{H}(B_{H}^{H}) = I_{0+}^{H+\frac{1}{2}}(L^{2}[0,1])$.
4.2 Large deviation results

Throughout this subsection, let $B^H$, $S^H$ and $Z^{H,K}$ be $d$-dimensional processes, and for a general $d$-dimensional Gaussian process $X$, denote

$$\mathcal{L}_t^\gamma(X) = \int_0^t \gamma(X_s)ds,$$

where $\gamma(x)$ is given in Section 2. In particular, when $\gamma = \delta$, $Hd < 1$ and $HKd < 1$, from Remarks 2.8 and 2.9, the local times $L_t(B^H) = \mathcal{L}_t^\delta(B^H)$, $L_t(S^H) = \mathcal{L}_t^\delta(S^H)$ and $L_t(Z^{H,K}) = \mathcal{L}_t^\delta(Z^{H,K})$ exist.

Due to the self-similarity possessed by $B^H$, $S^H$, $Z^{H,K}$ and the homogeneity $\gamma(ax) = a^{-\beta}\gamma(x)$ for $a > 0$, the following scaling property holds: for any $a > 0$,

$$\mathcal{L}_t^\gamma(B^H) \overset{d}{=} a^{1-H\beta}\mathcal{L}_t^\gamma(B^H),$$

(4.8)

$$\mathcal{L}_t^\gamma(S^H) \overset{d}{=} a^{1-H\beta}\mathcal{L}_t^\gamma(S^H),$$

(4.9)

$$\mathcal{L}_t^\gamma(Z^{H,K}) \overset{d}{=} a^{1-HK\beta}\mathcal{L}_t^\gamma(Z^{H,K}).$$

(4.10)

We first provide the following large deviation result for $\mathcal{L}_t^\gamma(B^H)$, which will be used later to obtain the large deviation for $\mathcal{L}_t^\gamma(S^H)$ with $H \in (0, \frac{1}{2})$ and $Hd < 1$, and $\mathcal{L}_t^\gamma(Z^{H,K})$ with $H \in (0, 1)$, $K \in (0, 1)$ and $HKd < 1$.

**Theorem 4.8** Assume $H \in (0, 1)$ and $H\beta < 1$, then

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \mathcal{L}_t^\gamma(B^H) \right) = \mathcal{E}(\gamma, H, d) \alpha_H^{-\frac{d}{2-H}} \theta^{1-H\beta} =: \mathcal{E}_1(\gamma, H, d) \theta^{1-H\beta},$$

(4.11)

and consequently,

$$\lim_{x \to \infty} \frac{1}{x^{\frac{d}{2-H}} \theta} \log \mathbb{P}(\mathcal{L}_1^\gamma(B^H) \geq x) = -C(\gamma, H, d),$$

(4.12)

where $\mathcal{E}(\gamma, H, d)$ is a positive constant given in Proposition 3.2, the constant $\alpha_H$ is given in (4.2), $\mathcal{E}_1(\gamma, H, d) = \mathcal{E}(\gamma, H, d) \alpha_H^{-\frac{d}{2-H}}$, and

$$C(\gamma, \alpha, d) = \mathcal{E}_1(\gamma, \alpha, d)^{1-\frac{d}{2-H}}(1-\alpha\beta)^{-\frac{1}{2-H}} \alpha \beta.$$

(4.13)
Proof By the representation (4.2) and Proposition 4.2, for any \( \varepsilon \in (0, 1) \), we may construct \( \eta^\varepsilon \) such that \( \eta^\varepsilon_t = \eta_t \) for \( t \geq \varepsilon \), and \( \{ \eta^\varepsilon_t, t \in [0, 1] \} \) belongs to the RKHS \((I^{H+1/2}(L^2[0, 1]))^d\) of \( \{ \alpha_H X^t_H, t \in [0, 1] \} \) almost surely. Thus, (4.11) follows from Proposition 2.13 and Proposition 3.2.

By the Gärtner-Ellis theorem (see Theorem 2.5), we have for \( \lambda > 0 \)

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{1}{t} \mathcal{L}^\gamma_t(B^H) \geq \lambda \right) = \sup_{\theta > 0} \left\{ \theta \lambda - \mathcal{E}_1(\gamma, H, d) \theta \right\} = C(\gamma, H, d) \lambda^{\frac{1}{H-d}},
\]

where \( C(\gamma, H, d) \) is given in (4.13). By scaling property (4.8) and a change of variables, we have

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{1}{t} \mathcal{L}^\gamma_t(B^H) \geq \lambda \right) = \lim_{x \to \infty} \frac{1}{x} \log \mathbb{P} \left( \mathcal{L}_{1}^\gamma(B^H) \geq x \right).
\]

Hence, equation (4.12) follows from (4.14) and (4.15). \( \blacksquare \)

Remark 4.9 When \( \gamma(x) = \delta(x) \), we can prove Theorem 4.8 directly based on the result in [10, Theorem 2.1]. Moreover, we have

\[
\mathcal{E}_1(\delta, H, d) = C(H, d)^{-\frac{Hd}{1-H-d}} \left[ (Hd)^{\frac{Hd}{1-H-d}} - (Hd)^{-\frac{1}{H-d}} \right],
\]

where \( C(H, d) \) is given by (4.6) in [10] and satisfies

\[
\left( \frac{\pi c_H^2}{H} \right)^{\frac{1}{H}} \varphi(Hd) \leq C(H, d) \leq (2\pi)^{\frac{1}{H}} \varphi(Hd),
\]

with \( c_H = \frac{\sqrt{\pi}H^{H/2}}{\sqrt{B(1-H,H+1/2)}} \), \( B(\cdot, \cdot) \) being the beta function and

\[
\varphi(x) = x(1-x)^{\frac{1-x}{x}} \frac{1}{\Gamma(1-x)^{\frac{1}{x}}}. \]

Proof It follows from Theorem 2.1 in [10] that

\[
\lim_{x \to \infty} \frac{1}{x^{1-H-d}} \log \mathbb{P}(L_1(B^H) \geq x) = -C(H, d),
\]

for any \( \varepsilon \in (0, 1) \), we may construct \( \eta^\varepsilon \) such that \( \eta^\varepsilon_t = \eta_t \) for \( t \geq \varepsilon \), and \( \{ \eta^\varepsilon_t, t \in [0, 1] \} \) belongs to the RKHS \((I^{H+1/2}(L^2[0, 1]))^d\) of \( \{ \alpha_H X^t_H, t \in [0, 1] \} \) almost surely. Thus, (4.11) follows from Proposition 2.13 and Proposition 3.2.

By the Gärtner-Ellis theorem (see Theorem 2.5), we have for \( \lambda > 0 \)

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{1}{t} \mathcal{L}^\gamma_t(B^H) \geq \lambda \right) = \sup_{\theta > 0} \left\{ \theta \lambda - \mathcal{E}_1(\gamma, H, d) \theta \right\} = C(\gamma, H, d) \lambda^{\frac{1}{H-d}},
\]

where \( C(\gamma, H, d) \) is given in (4.13). By scaling property (4.8) and a change of variables, we have

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P} \left( \frac{1}{t} \mathcal{L}^\gamma_t(B^H) \geq \lambda \right) = \lim_{x \to \infty} \frac{1}{x} \log \mathbb{P} \left( \mathcal{L}_{1}^\gamma(B^H) \geq x \right).
\]

Hence, equation (4.12) follows from (4.14) and (4.15). \( \blacksquare \)

Remark 4.9 When \( \gamma(x) = \delta(x) \), we can prove Theorem 4.8 directly based on the result in [10, Theorem 2.1]. Moreover, we have

\[
\mathcal{E}_1(\delta, H, d) = C(H, d)^{-\frac{Hd}{1-H-d}} \left[ (Hd)^{\frac{Hd}{1-H-d}} - (Hd)^{-\frac{1}{H-d}} \right],
\]

where \( C(H, d) \) is given by (4.6) in [10] and satisfies

\[
\left( \frac{\pi c_H^2}{H} \right)^{\frac{1}{H}} \varphi(Hd) \leq C(H, d) \leq (2\pi)^{\frac{1}{H}} \varphi(Hd),
\]

with \( c_H = \frac{\sqrt{\pi}H^{H/2}}{\sqrt{B(1-H,H+1/2)}} \), \( B(\cdot, \cdot) \) being the beta function and

\[
\varphi(x) = x(1-x)^{\frac{1-x}{x}} \frac{1}{\Gamma(1-x)^{\frac{1}{x}}}. \]

Proof It follows from Theorem 2.1 in [10] that

\[
\lim_{x \to \infty} \frac{1}{x^{1-H-d}} \log \mathbb{P}(L_1(B^H) \geq x) = -C(H, d),
\]
and $C(H, d)$ satisfies (4.16).

Fixing a constant $\lambda > 0$ and letting $x = u\lambda$, we have

$$\lim_{u \to \infty} \frac{1}{u^{\frac{1}{Hd}}} \log \mathbb{P} \left( \frac{1}{u} L_1(B^H) \geq \lambda \right) = -C(H, d)\lambda^{\frac{1}{Hd}}.$$

Now, by Varadhan’s integral lemma (see Lemma 2.6), we get, for all $\theta > 0$,

$$\lim_{u \to \infty} \frac{1}{u^{\frac{1}{Hd}}} \log \mathbb{E} \exp \left( \theta \frac{1}{u^{\frac{1}{Hd}}} L_1(B^H) \right) = \sup_{\lambda > 0} \{ \lambda \theta - C(H, d)\lambda^{\frac{1}{Hd}} \} = C(H, d)^{-\frac{Hd}{1-Hd}} \left[ (Hd)^{\frac{Hd}{1-Hd}} - (Hd)^{\frac{1}{Hd}} \right] \theta^{\frac{1}{Hd}}.$$

On the other hand, a change of variables and the scaling property (4.8) yield

$$\lim_{u \to \infty} \frac{1}{u^{\frac{1}{Hd}}} \log \mathbb{E} \exp \left( \theta \frac{1}{u^{\frac{1}{Hd}}} L_1(B^H) \right) = \lim_{t \to \infty} \frac{1}{t^{\frac{1}{Hd}}} \log \mathbb{E} \exp \left( \theta t^{1-Hd} L_1(B^H) \right) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta L_t(B^H) \right).$$

Therefore, the desired result follows from the above equalities and Theorem 4.8.

The following result is the large deviations for sub-fBm and bi-fBm.

**Theorem 4.10**

(i) Assume $H \in (0, \frac{1}{2})$ and $H\beta < 1$, then

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \mathcal{L}^\gamma_t(S^H) \right) = \mathcal{E}_1(\gamma, H, d) \theta^{\frac{1}{1-Hd}},$$

and consequently,

$$\lim_{x \to \infty} \frac{1}{x^{\frac{1}{Hd}}} \log \mathbb{P}(\mathcal{L}^\gamma_1(S^H) \geq x) = -\mathcal{E}(\gamma, H, d),$$

where the constants $\mathcal{E}_1(\gamma, \alpha, d)$ and $\mathcal{E}(\gamma, H, d)$ are given in Theorem 4.8.

(ii) Assume $H, K \in (0, 1)$ and $HK\beta < 1$, then

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left( \theta \mathcal{L}^\gamma_t(Z^{H,K}) \right) = 2 \mathcal{E}_1(\gamma, H, K, d) \theta^{\frac{1}{1-HKd}},$$

and consequently,

$$\lim_{x \to \infty} \frac{1}{x^{\frac{1}{HKd}}} \log \mathbb{P}(\mathcal{L}^\gamma_1(Z^{H,K}) \geq x) = -2 \mathcal{E}_1(\gamma, H, K, d).$$
Proof Let us first prove (4.18). By Propositions 4.4 and 2.13 and together with the fact \( \left( \tilde{I}_{0+}^{3/2}(L^2[0,1]) \right)^d \subseteq \left( I_{0+}^{H+\frac{1}{2}}(L^2[0,1]) \right)^d \), it suffices to show that for any \( 0 < \varepsilon < 1 \), there exists a process \( \eta^\varepsilon \) such that such that \( \eta^\varepsilon_t = Y^H_t \) for \( t \geq \varepsilon \), and \( \{ \eta^\varepsilon_t, 0 \leq t \leq 1 \} \) belongs to \( \left( \tilde{I}_{0+}^{3/2}(L^2[0,1]) \right)^d \) almost surely.

Note that \( Y^H_t = \int_0^t M^H_s ds \), where \( M^H_s = \int_0^\infty u^{\frac{1}{2} - H} e^{-us} dW_u \) and \( M^H \) has a version denoted again by \( M^H \) that is infinitely differentiable on \( (0, \infty) \) (see [15, Theorem 1]). We construct \( \eta^\varepsilon \) by following the proof of [10, Proposition 3.5] as follows:

\[
\eta^\varepsilon_t = \begin{cases} 
  a_1 t^2 + a_2 t^3, & 0 \leq t \leq \varepsilon, \\
  Y^H_t, & t > \varepsilon,
\end{cases}
\]

where \( a_1 = 3\varepsilon^{-2}Y^H_\varepsilon - \varepsilon^{-1}M^H_\varepsilon \) and \( a_2 = -2\varepsilon^{-3}Y^H_\varepsilon + \varepsilon^{-2}M^H_\varepsilon \). It is easy to verify that \( \eta^\varepsilon \) and \((\eta^\varepsilon)’\) exist as continuous functions on \([0,1]\) with \( \eta^\varepsilon_0 = (\eta^\varepsilon)'_0 = 0 \), and \((\eta^\varepsilon)'' \in L^2[0,1] \). Then, by [10, Proposition 3.4], we obtain \( \{ \eta^\varepsilon_t, t \in [0,1] \} \in \left( \tilde{I}_{0+}^{3/2}(L^2[0,1]) \right)^d \) a.s. Noting that \( \left( \tilde{I}_{0+}^{3/2}(L^2[0,1]) \right)^d \subseteq \left( I_{0+}^{H+\frac{1}{2}}(L^2[0,1]) \right)^d = \mathbb{H}(B^H) \), we have \( \{ \eta^\varepsilon_t, t \in [0,1] \} \in \mathbb{H}(B^H) \) a.s. Hence, equation (4.18) follows from Theorem 4.8 and Propositions 2.13 and 4.4.

Next, we will prove (4.20) by an analogue to the proof of (4.18). Since \( \{ Y^{K/2}_t = \int_0^t M^K_{s/2} ds, t \geq 0 \} \) is infinitely differentiable on \( (0, \infty) \) (see [15, Theorem 1]), we obtain that

\[
\tilde{Y}^{H,K}_t := Y^{K/2}_{t/2} = \int_0^t 2Hs^{2H - 1} M^{K/2}_s ds =: \int_0^t \tilde{M}^{H,K}_s ds
\]

is infinitely differentiable on \( (0, \infty) \). Thus we can construct \( \tilde{\eta}^\varepsilon \) in the same way as for \( \eta^\varepsilon \) in (4.22) by replacing \( Y^H_t \) with \( \tilde{Y}^{H,K}_t \) and replacing \( M^H \) with \( \tilde{M}^{H,K} \). For the process \( \tilde{\eta}^\varepsilon \) we have \( \tilde{\eta}^\varepsilon_t = \tilde{Y}^{H,K}_t \) for \( t \geq \varepsilon \), and almost surely \( \{ \tilde{\eta}^\varepsilon_t, t \in [0,1] \} \in \left( \tilde{I}_{0+}^{3/2}(L^2[0,1]) \right)^d \) which belongs to \( \mathbb{H}(Z^{H,K}) \) by Proposition 4.7. Therefore, we can obtain (4.20) follows by Theorem 4.8 and Propositions 4.6 and 2.13 and the homogeneity of \( \gamma(\cdot) \).

Using the same argument as in the proof of equation (4.12), we can prove (4.19) and (4.21) respectively.

Remark 4.11 There are other examples of self-similar Gaussian processes \( X \) for which \( \mathcal{L}_t^\gamma(X) \) has large deviations. For instance, the Gaussian process \( X \) with parameters \( H \in (0,1) \) and \( \lambda \in (0, H) \) introduced in [12] can be decomposed in law as...
the sum of a fBm (up to a factor of a constant) and the Gaussian process \( Y \) defined in (4.5) (up to a factor of a constant) independent of the fBm. Hence, using the same technique in dealing with sub-fBm, we can show the large deviations for \( \mathcal{L}_1^\gamma(X) \). In the case \( H \in (\frac{1}{2}, 1) \), the Gaussian process \( X \) has the same law (up to a constant) as the process \( u(t,0) \), where \( u(t,x) \) is the solution to the following stochastic heat equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \dot{W}(t,x), \ t \geq 0, \ x \in \mathbb{R}^d, \\
u(0,x) = 0,
\end{cases}
\]

with \( \dot{W} \) being a zero mean Gaussian field with a covariance of the form

\[\mathbb{E} \left( \dot{W}(t,x) \dot{W}(s,y) \right) = |s-t|^{2H-2} |x-y|^{-\beta},\]

where \( 0 < \beta < \min\{d, 2\} \).

As an application of the results in Theorem 4.10, we can show the following critical exponential integrability.

**Corollary 4.12**  (i) Assume \( H \in (0, 1) \) and \( H\beta < 1 \). We have

(a) when \( p < \frac{1}{H\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(B^H) \right)^p \right) < \infty \) for all \( \lambda > 0 \);

(b) when \( p > \frac{1}{H\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(B^H) \right)^p \right) = \infty \) for all \( \lambda > 0 \);

(c) when \( p = \frac{1}{H\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(B^H) \right)^p \right) < \infty \) for \( \lambda < C(\gamma, H, d) \),

and \( \mathbb{E} \exp \left( \lambda \left( L_1^\gamma(B^H) \right)^p \right) = \infty \) for \( \lambda > C(\gamma, H, d) \),

where \( C(\gamma, H, d) \) is given in (4.13).

(ii) Assume \( H \in (0, \frac{1}{2}) \) and \( H\beta < 1 \). We have

(a) when \( p < \frac{1}{H\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(S^H) \right)^p \right) < \infty \) for all \( \lambda > 0 \);

(b) when \( p > \frac{1}{H\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(S^H) \right)^p \right) = \infty \) for all \( \lambda > 0 \);

(c) when \( p = \frac{1}{H\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(S^H) \right)^p \right) < \infty \) for \( \lambda < C(\gamma, H, d) \),

and \( \mathbb{E} \exp \left( \lambda \left( L_1^\gamma(S^H) \right)^p \right) = \infty \) for \( \lambda > C(\gamma, H, d) \).

(iii) Assume \( H, K \in (0, 1) \) and \( HK\beta < 1 \). We have

(a) when \( p < \frac{1}{HK\beta} \), \( \mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma(Z^{H,K}) \right)^p \right) < \infty \) for all \( \lambda > 0 \);
when $p > \frac{1}{HK\beta}$, $\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (Z^{H,K}) \right)^p \right) = \infty$ for all $\lambda > 0$;

(c) when $p = \frac{1}{HK\beta}$, $\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (Z^{H,K}) \right)^p \right) < \infty$ for $\lambda < 2^{(1-K)\beta}C(\gamma, HK, d)$, and $\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (Z^{H,K}) \right)^p \right) = \infty$ for $\lambda > 2^{(1-K)\beta}C(\gamma, HK, d)$.

Proof. By Fubini’s theorem, we have

$$\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (B^H) \right)^p \right) - 1 = \int_0^\infty \mathbb{P} \left( \mathcal{L}_1^\gamma (B^H) \geq \left( \lambda^{-1} y \right)^{1/p} \right) e^y dy.$$ 

Then, the desired result for $\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (B^H) \right)^p \right)$ follows from (4.12).

Similarly, one can show the result for $\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (S^H) \right)^p \right)$ and $\mathbb{E} \exp \left( \lambda \left( \mathcal{L}_1^\gamma (Z^{H,K}) \right)^p \right)$.

We complete the proof. \hfill \blacksquare

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