The structure of the trace anomaly of higher spin conformal currents in the bulk of $AdS_4$

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Abstract

The two-point function of the conserved traceless spin-$\ell$ currents which are constructed from the scalar field $\sigma(z)$ is evaluated and renormalized by a dimensional regularization procedure. The anomaly is managed to arise only in the trace part. To isolate this trace anomaly it is sufficient to analyze only the maximum singular part of the two-point function and its trace terms to leading order. The corresponding part of the effective action which is quadratic in the trace of the higher spin field is explicitly given. For the spin-2 field which is identical with the gravitational field the results known from the literature are reproduced.
1 Introduction

The AdS$_4$/CFT$_3$ correspondence of the critical $O(N)$ sigma model and four dimensional higher spin gauge theory in anti de Sitter space\cite{1} increased interest in the complicated problem of quantization and interaction of the higher spin gauge theories in AdS space\cite{2,3}. In this case the well known boundary theory has been used for the reconstruction of the unknown interacting field theory in the bulk of AdS$_4$\cite{4,6,5,7,8}. This AdS$_4$ case is interesting also in view of the presence of the conformal anomaly in even dimensional space-time\cite{9}. The anomaly should arise during the one-loop calculation and can be used for checking of the correctness of the linearized interaction between higher spin gauge and scalar fields\cite{10} and always should reproduce for $\ell = 2$ the well known results of the trace anomaly of the scalar mode in curved background\cite{11}. Investigation of this problem could also be important for a deeper understanding of the geometrical and topological structure of the linearized interaction of the higher spin gauge fields.

In the previous article\cite{12} we investigated the anomalous behaviour of the renormalized one-loop effective action for the conformal scalar mode in the external higher spin gauge field

$$W(h^{(\ell)}) = \frac{1}{2\ell^2} \int_{z_1} \int_{z_2} h^{(\ell)\mu_1...\mu_\ell}(z_1) J^{(\ell)\mu_1...\mu_\ell}(z_2) J^{(\ell)\nu_1...\nu_\ell}(z_2), \quad (1)$$

$$\int_z := \int d^4z \sqrt{g(z)} \quad (2)$$

Here $h^{(\ell)\mu_1...\mu_\ell}(z)$ is the spin $\ell$ gauge field and $J^{(\ell)\mu_1...\mu_\ell}(z)$ the conserved traceless current constructed from the conformally coupled scalar $\sigma(z)$\cite{10} in $D = d + 1$ dimensional AdS space*. The additional factor $1/z$ comes from the following normalization of the linearized interaction

$$S^{(\ell)}_{int} = \frac{1}{\ell} \int d^4z \sqrt{g(z)} h^{(\ell)\mu_1...\mu_\ell}(z) J^{(\ell)\mu_1...\mu_\ell}(z) \quad (3)$$

*We will use Euclidian AdS$_{d+1}$ with conformal flat metric, curvature and covariant derivatives satisfying

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{L^2}{(z^0)^2}\delta_{\mu\nu}dz^\mu dz^\nu, \quad \sqrt{g} = \frac{L^{d+1}}{(z^0)^{d+1}},$$

$$[\nabla_\mu, \nabla_\nu] V_\lambda^\rho = R_{\mu\nu\lambda}^\sigma V_\sigma^\rho - R_{\mu\nu\sigma}^\rho V_\lambda^\sigma,$$

$$R_{\mu\nu\lambda}^\rho = -\frac{1}{(z^0)^2} (\delta_{\mu\lambda}\delta^\rho_\nu - \delta_{\nu\lambda}\delta^\rho_\mu) = -\frac{1}{L^2} (g_{\mu\lambda}(z)\delta^\rho_\nu - g_{\nu\lambda}(z)\delta^\rho_\mu),$$

$$R_{\mu\nu} = -\frac{d}{(z^0)^2}\delta_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu}(z), \quad R = -\frac{d(d+1)}{L^2}.$$
leading to the following rule of calculation of the correlation functions

\[ < J_{\mu_1 \ldots \mu_\ell}^\ell (z) \cdot \cdot \cdot > := \frac{\ell}{\sqrt{g(z)}} \delta h_{\mu_1 \ldots \mu_\ell}^\ell (z) \cdot \cdot \cdot W(h_{\mu \nu})^\ell. \] (4)

This is in agreement with the usual normalization of the energy-momentum tensor (\( \ell = 2 \) case) for the conformally coupled scalar matter field [11]

\[ < T_{\mu \nu} (z) > := \frac{2}{\sqrt{g(z)}} \delta h_{\mu \nu} (z) W(g_{\mu \nu}). \] (5)

In the future for shortening the notation and calculation we contract all rank \( \ell \) symmetric tensor fields at \( z \) with the \( \ell \)-fold tensor product of a vector \( a^\mu \) from the tangential space at \( z \). In this notation the symmetric tensor, it’s trace, symmetrized gradient and divergence can be written as

\[ J^{(\ell)} (a; z) = J^{(\ell)}_{\mu_1 \ldots \mu_\ell} a^{\mu_1} \ldots a^{\mu_\ell}, \] (6)

\[ Tr J^{(\ell)} (a; z) = \frac{1}{\ell (\ell - 1)} \nabla_a J^{(\ell)} (a; z), \quad \nabla_a = g^{\mu \nu} (z) \frac{\partial^2}{\partial a^\mu \partial a^\nu}; \] (7)

\[ (a^\mu \nabla_\mu) J^{(\ell)} (a; z), \quad \nabla \cdot J^{(\ell)} (a; z) = \frac{1}{\ell} \nabla^\mu \frac{\partial}{\partial a^\mu} J^{(\ell)} (a; z). \] (8)

We introduce also the following notations for contractions in \( z \) and \( a \) spaces

\[ \int d^4 z \sqrt{g(z)} J^{(\ell)}_{\mu_1 \ldots \mu_\ell} (z) h^{(\ell)}_{\mu_1 \ldots \mu_\ell} (z) = J^{(\ell)} (a; z) \ast, \quad h^{(\ell)} (a; z) \] ,

\[ J^{(\ell)}_{\mu_1 \ldots \mu_\ell} h^{(\ell)}_{\mu_1 \ldots \mu_\ell} = J^{(\ell)} (a) \ast, \quad h^{(\ell)} (a) \] (9)

In [12] we considered the extraction of singularities and anomalous behaviour of the two point function \(< J^{(\ell)} (z_1; a) J^{(\ell)} (z_2; c) > \) due to the ”trace terms”. It means that we performed all our calculations with the exception of \( O(a^2) \) and \( O(c^2) \) terms. This was enough for the investigation of the renormalized Ward identities and the observation of the violation of the tracelessness condition under the stipulation that the renormalized two-point function satisfies the Ward identities following from the conservation condition (gauge invariance). In this article we enlarge our consideration and include the first \( O(a^2) \) and \( O(c^2) \) terms in order to understand the exact structure and anomaly coefficients of the conformal scalar mode in the external higher spin field. To control our calculation we will compare the particular case \( \ell = 2 \) with the known results for the trace anomaly in an external gravitational field [11] and the connection of the anomalous coefficients with the correlation functions in the flat [13] and the constant curvature backgrounds [14].
2 Loop function and Ward identity with trace terms

In the previous article we calculated the one-loop two-point function

\[ \Pi^{(\ell)}(z_1; a|z_2; c) := \langle J^{(\ell)}(z_1; a) J^{(\ell)}(z_2; c) \rangle \]  

(11)

applying just Wick’s theorem to the leading part of the conserved and traceless currents avoiding \(O(a^2)\) and \(O(c^2)\) contributions from so-called ”trace terms” [12]. Calculations were done using the propagator of the scalar field in \(AdS_4\) quantized with a boundary condition corresponding to the free conformal point of the boundary \(O(N)\) model

\[ < \sigma(z_1) \sigma(z_2) > = \frac{1}{8\pi^2} \left( \frac{1}{u} + \frac{1}{u+2} \right), \]  

(12)

where

\[ u = \zeta - 1 = \frac{(z_1 - z_2)^2}{2z_1^0 z_2^0} \]  

(13)

is the invariant chordal distance in \(AdS\).

The general rule for working with such objects is analyzed in detail in the same article. The main point is the following. The tensorial structure of any two-point function in \(AdS\) space can be described using a general basis of the independent bitensors [15, 16, 17, 18, 19, 20]

\[ I_1(a, c) := (a\partial)_{1}(c\partial)_{2}u(z_1, z_2), \]  

(14)

\[ I_2(a, c) := (a\partial)_{1}u(z_1, z_2)(c\partial)_{2}u(z_1, z_2), \]  

(15)

\[ I_3(a, c) := a_1^2 I_{2a} + c_2^2 I_{1a}, \]  

(16)

\[ I_4 := a_1^2 c_2^2, \]  

(17)

\[ I_{1a} := (a\partial)_{1}u(z_1, z_2), \quad I_{2a} := (c\partial)_{2}u(z_1, z_2), \]  

(18)

\[ (a\partial)_{1} = a^\mu \frac{\partial}{\partial z_1^\mu}, \quad (c\partial)_{2} = c^\mu \frac{\partial}{\partial z_2^\mu}, \]  

(19)

\[ a_1^2 = g_{\mu\nu}(z_1)a^\mu a^\nu, \quad c_2^2 = g_{\mu\nu}(z_2)c^\mu c^\nu. \]  

(20)

In this case this basis should appear automatically after contractions of scalars and action of the vertex derivatives. In general we have to get an expansion with all four basis elements

\[ \Pi^{(\ell)}(z_1; a|z_2; c) = \Psi^{(\ell)}[F] + \sum_{n,m; 0<2(n+m)<\ell} I_3^n I_4^m \Psi^{(\ell-2(n+m))[G^{(n,m)}]}. \]  

(21)

Here we introduce a special map from the set \(\{F_k(u)\}_{k=0}^{\ell}\) of the \(\ell + 1\) functions of \(u\) to the space of \(\ell \times \ell\) bitensors

\[ \Psi^{(\ell)}[F] = \sum_{k=0}^{\ell} I_1^{\ell-k}(a, c) I_2^k(a, c) F_k(u). \]
But in [12] we restricted our consideration on the first part of (21) connected with the $I_1, I_2$ bitensors only calling all monomials corresponding to $I_3$ and $I_4$ in the above sum and the corresponding sets of functions $\{G_k^{(n,m)}\}_{k=0}^{\ell-2(n+m)}$ the "trace terms"

$$
\Pi^\ell(z_1; a|z_2; c) = \Psi^\ell[F] + \text{trace terms.} \quad (22)
$$

Here for the investigation of the exact structure (not only existence) of the trace anomaly we take the first $O(a^2)$ and $O(c^2)$ terms into account. It means that we will use instead of (22) the following ansatz

$$
\Pi^\ell(z_1; a|z_2; c) = n(\ell)K^\ell(F, G, H) + \text{next trace terms,} \quad (23)
$$

$$
K^\ell(F, G, H) = \Psi^\ell[F] + I_3\Psi^{\ell-2}[G] + I_4\Psi^{\ell-2}[H]. \quad (24)
$$

The normalization factor $n(\ell)$ will be fixed below. This ansatz must satisfy the following naive Ward identities † following from the conservation and tracelessness conditions of our currents with contribution of the corresponding partner trace terms described by the expansion in the other two bitensors $I_3, I_4$

$$
\Box_a K^\ell(F, G, H) = \frac{\partial^2}{\partial a_\mu \partial a^\mu} K^\ell(F, G, H) = 0, \quad (25)
$$

$$
(\nabla \cdot \partial_a)K^\ell(F, G, H) = \nabla^\mu \frac{\partial}{\partial a^\mu} K^\ell(F, G, H) = 0. \quad (26)
$$

All important relations for the calculations including $O(a^2)$ and $O(c^2)$ terms to be performed below can be found in Appendix A of this article. In the main text we will only present the Ward identities (25), (26) in the language of the sets of functions $\{F_k(u)\}_{k=0}^{\ell-2}, \{G_k(u)\}_{k=0}^{\ell-2}$ and $\{H_k(u)\}_{k=0}^{\ell-2}$. For that purpose we apply (A.12)-(A.26) to our ansatz (24) and keep only the leading trace terms. We get

1) the Divergence map

$$
\nabla^\mu \frac{\partial}{\partial a^\mu} K^\ell(F, G, H) = I_2c\Psi^{\ell-1}[D_1^\ell(F, G)] + c_2^2 I_4a \Psi^{\ell-2}[D_1^2(F, G, H)], \quad (27)
$$

$$
D_1^\ell(F, G) = (\text{Div}_F)k + 2(k + 2)G_k + 2G'_{k-1}, \quad (28)
$$

$$
(\text{Div}_F)k = (\ell - k)(u + 1)F_k + (k + 1)u(u + 2)F_k \quad + (\ell - k)(\ell + d + k)F_k + (k + 1)(\ell + d + k + 1)uF_{k+1}, \quad (29)
$$

$$
D_2^\ell(F, G, H) = (k + 1)(\ell - k - 1)F_{k+1} + u(u + 2)(k + 2)G_k \quad + (u + 1)(\ell - k - 1)G_{k-1} + (u + 1)(k + 2)(d + \ell + k)G_k \quad + (\ell - k - 1)(\ell + d + k - 1)G_{k-1} + 2H_k + 2(k + 1)H_{k+1}, \quad (30)
$$

where $F_k := \frac{\partial}{\partial u} F_k(u)$ and so on;

†For the purpose of derivation with respect to $z_1$ or $z_2$ we understand the invariant functions $F, G, H$ as analytic functions on the half plane $\Re u > 0$. Distributions on the positive real $u$ axis are obtained as boundary values of these later on.
2) the Trace map

\[ \Box_a K^\ell(F, G, H) = I_2^2 \Psi^\ell-2[T_1^\ell(F, G)] + c_2^2 \Psi^\ell-2[T_2^\ell(F, G, H)] \],

(31)

\[ T_1^\ell(F, G) = (Tr_\ell F)_k + 2(d + 2\ell - 3)G_k, \]

(32)

\[ (Tr_\ell F)_k = (\ell - k)(\ell - k - 1)F_k + 2(k + 1)(\ell - k - 1)(u + 1)F_{k+1} \\
+ (k + 2)(k + 1)u(u + 2)F_{k+2}, \]

(33)

\[ T_2^\ell(F, G, H) = (\ell - k)(\ell - k - 1)F_k + (k + 1)(k + 2)u(u + 2)G_k \\
+ 2(k + 1)(\ell - k - 1)(u + 1)G_{k-1} + (\ell - k)(\ell - k - 1)G_{k-2} \\
+ 2(d + 2\ell - 3)H_k. \]

(34)

Then we obtain two relations from the tracelessness condition

\[ T_1^\ell(F, G) = 0, \]

(35)

\[ T_2^\ell(F, G, H) = 0, \]

(36)

and further two relations from the conservation condition

\[ D_1^\ell(F, G) = 0, \]

(37)

\[ D_2^\ell(F, G, H) = 0. \]

(38)

In [12] we discovered that essential for the renormalization procedure of (23) described by the function set \( \{F_k(u)\}_{k=0}^{\ell} \) is the main singular part of the form

\[ \Pi_B^\ell(z_1; a|z_2; c) = \frac{(-1)^\ell(2\ell)!}{2^\ell \pi^4} \sum_{k=0}^{\ell} I_1^{\ell-k} I_2^k F_B^k(u), \]

(39)

\[ F_B^k(u) = (-1)^k \binom{\ell}{k} \frac{1}{u^{\ell+k+2}}. \]

(40)

This was proven by direct calculation of the loop diagram after extraction of the gauge-gradient and regular terms. The next important point is that this main singular part satisfies the following simple Ward identities without entanglement of the sets \( \{G_k(u)\}_{k=0}^{\ell-2} \) and \( \{H_k(u)\}_{k=0}^{\ell-2} \)

\[ (Div_\ell F_B)_{d=3}^k = 0, \quad (Tr_\ell F_B)_{d=3}^k = 0. \]

(41)

As a result we could, using only the \( F \)-set, calculate the corresponding singular part of the effective action (see Eq.(80) of [12]) and prove that after renormalization we observe a trace anomaly.

For this consideration this result will simplify our tasks in the following two points:

- We can now fix the normalization coefficient \( n(\ell) \) in (21)

\[ n(\ell) = \frac{(-1)^\ell(2\ell)!}{2^\ell \pi^4}, \]

(42)
• We can obtain the main singular $G^B$- and $H^B$-dependent “bare” part of $K^\ell(F^B, G^B, H^B)$ without loop calculation but just using the Ward identities \((35)-(38)\) (for \(d=3\)) and expression \((40)\).

Following this we obtain immediately from \((35)-(38)\) the answer for the main singular parts of $G^B_k$ and $H^B_k$

\[
G^B_k = 0, \quad H^B_k = -\frac{(\ell - k)(\ell - k - 1)}{(4\ell)} F_k^B = (-1)^{k+1} \left(\frac{\ell - 2}{k}\right) \frac{\ell - 1}{4u\ell^{k+2}}. \quad (43)
\]

Checking consistency of the solutions we can insert these in \((37)-(38)\) and see that they are satisfied automatically. So we see that for the main singular term we can put $G^B_k = 0$ which will dramatically simplify the naive Ward identities.

### 3 Renormalized trace

Now we introduce a regularization for the remaining set of functions $F^R_k$ and $H^R_k$. Following the prescription of \([12]\) we see that we can continue our $u^{-n}$ distributions analytically away from integer $d$ and observe that regularized distributions

\[
F^R_k(u) = (-1)^k \left(\frac{\ell}{k}\right) \frac{1}{u^{\ell+k+d-1}},
\]

\[
H^R_k = -\frac{(\ell - k)(\ell - k - 1)}{(4\ell)} F_k^B = (-1)^{k+1} \left(\frac{\ell - 2}{k}\right) \frac{\ell - 1}{4u\ell^{k+2}} \quad (46)
\]

satisfy the regularized Ward identities (away from $d = 3$)

\[
D^1_\ell(F^R, 0) = (\ell - k)(u + 1)\partial_u F^R_k + (k + 1)u(u + 2)\partial_u F^R_{k+1} + (\ell - k)(\ell + d + k)F^R_k + (k + 1)(\ell + d + k + 1)uF^R_{k+1} = 0, \quad (47)
\]

\[
D^2_\ell(F^R, 0, H^R) = (k + 1)(\ell - k - 1)F^R_{k+1} + 2\partial_u H^R_k + 2(k + 1)H^R_{k+1} = 0,\quad (48)
\]

\[
T^1_\ell(F^R, 0) = (\ell - k)(\ell - k - 1)F^R_k + 2(k + 1)(\ell - k - 1)(u + 1)F^R_{k+1} + (k + 2)(k + 1)u(u + 2)F^R_{k+2} = 0, \quad (49)
\]

\[
T^2_\ell(F^R, 0, H^R) = (\ell - k)(\ell - k - 1)F^R_k + 2(d + 2\ell - 3)H^R_k = 0. \quad (50)
\]

It is the usual picture for dimensional regularization \([21]\) when the regularized effective action is conformal invariant and a non zero trace arises only after subtraction of the singularities. The same behaviour we expect to get here.

Indeed we can just put in \((47), (50)\) $d = 3 - \epsilon$ and declare these to be the regularized Ward identities. Then using the standard relation

\[
\left[\frac{1}{u^{n-\epsilon}}\right]_{sing} = \frac{1}{\epsilon (n - 1)!} \delta^{(n-1)}(u). \quad (51)
\]
we can split (45), (46) for \( d = 3 - \epsilon \) in singular and renormalized parts

\[
F^R_k(u) = \left( \frac{\ell}{k} \right) \left( \frac{1}{\epsilon} \frac{(-1)^{\ell+1}}{(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) + f_k(u) \right) + F^{\text{Ren}}_k(u),
\]

\[
H^R_k(u) = \frac{\ell - 1}{4} \left( \frac{\ell - 2}{k} \right) \left( \frac{1}{\epsilon} \frac{(-1)^{\ell}}{(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) + h_k(u) \right) + H^{\text{Ren}}_k(u).
\]

Here we introduced also sets of finite distributions \( f_k(u), h_k(u) \) (without \( \epsilon \) pole) to describe the finite renormalization freedom. These singular parts correspond to the local counterterms of the effective action (they are proportional to \( \delta^{(n)}(u) \)). As usual we will concentrate on the subtraction parts

\[
F^S_k(u) = \left( \frac{\ell}{k} \right) \left( \frac{1}{\epsilon} \frac{(-1)^{\ell+1}}{(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) + f_k(u) \right),
\]

\[
H^S_k(u) = \frac{\ell - 1}{4} \left( \frac{\ell - 2}{k} \right) \left( \frac{1}{\epsilon} \frac{(-1)^{\ell}}{(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) + h_k(u) \right),
\]

understanding that the renormalized correlation function formed by \( F^{\text{Ren}}_k(u), H^{\text{Ren}}_k(u) \) will on the quantum level get the same trace as a subtracted singular part but with opposite sign because the regularized expression is traceless and conserved.

The general solution of our problem is the following. We have to switch on finite (without \( \epsilon \) pole) distributions \( G^S_k(u) \) and solve the conservation Ward identities

\[
D^1_\ell(F^S, G^S)_{d=3-\epsilon} = 0,
\]

\[
D^2_\ell(F^S, G^S, H^S)_{d=3} = 0,
\]

using as general ansatz short distance expansions

\[
f_k(u) = \sum_{p=0}^{\ell} g^p_k \frac{\delta^{(\ell+k+1-p)}(u)}{\ell + k + 1 - p}!,
\]

\[
h_k(u) = \sum_{p=0}^{\ell} b^p_k \frac{\delta^{(\ell+k+1-p)}(u)}{\ell + k + 1 - p}!,
\]

\[
G^S_k(u) = \frac{\ell - 1}{4} \left( \frac{\ell - 2}{k} \right) \sum_{p=0}^{\ell-2} d^p_k \frac{\delta^{(\ell+k+1-p)}(u)}{\ell + k + 1 - p}!.
\]

This fixes \( G^S_k(u), h_k(u) \) completely and a part of the freedom in \( f_k(u) \). Then we put this solution in the expression for the traces \( T^1_\ell(F^S, G^S)_{d=3}, T^2_\ell(F^S, G^S, H^S)_{d=3-\epsilon} \) and removing the remaining freedom in the coefficients \( g^p_k \) obtain the anomalous trace

\[
T^1_\ell(F^S, G^S)_{d=3} = (-1)^{\ell+1} \ell(\ell - 1) \frac{\ell - 2}{k} \frac{2\delta^{(\ell+k+2)}(u)}{(\ell + k + 3)!},
\]

\[
T^2_\ell(F^S, G^S, H^S)_{d=3-\epsilon} = (-1)^{\ell+1} \ell(\ell - 1) \frac{\ell - 2}{k} \frac{\delta^{(\ell+k+1)}(u)}{\ell(\ell + k + 1)!}.
\]
The general complicated proof of this universal result will be sketched in Appendix B of this article.

In the main text we will only watch the highest delta-function’s derivative terms of the Ward identities. The reason for this is the following: Anomaly is located only in the highest derivative part of Ward identities and described by $g^0_k$ and $b^0_k$ coefficients. The finite distributions $G^S_k(u)$ do have no direct influence in the expressions for anomalous traces and are necessary only for the cancellation of the lower delta-function’s derivatives in the Ward identities and in the rigorous proof of the existence of the solutions for (56), (57), (61) and (62). This is easy to see inserting the ansatz (58)-(60) in the Ward identities and using the relation $u\delta^{(n)}(u) = -n\delta^{(n-1)}(u)$. As a result we can now put in as first approximation

$$G^S_k = 0$$  (63)

$$f_k(u) = g^0_k \frac{\delta^{(\ell+k+1)}(u)}{\ell + k + 1}!$$  (64)

$$h_k(u) = b^0_k \frac{\delta^{(\ell+k+1)}(u)}{\ell + k + 1}!$$  (65)

and solve

$$D^1_\ell(F^S, 0)_{d=3-\epsilon} = 0,$$  (66)

$$D^2_\ell(F^S, 0, H^S)_{d=3} = 0,$$  (67)

watching only the highest delta-function’s derivative terms. From (66) and in agreement with [12] we obtain

$$g^0_k - g^0_{k+1} = \frac{1}{\ell + k + 2}, \quad k = 0, \ldots, \ell - 1.$$  (68)

From (67) follows the recursion

$$2\ell g^0_{k+1} - (\ell - k - 2)b^0_{k+1} - (\ell + k + 2)b^0_k = 0, \quad k = 0, \ldots, \ell - 3$$  (69)

with the boundary condition

$$b^0_{\ell - 2} = g^0_{\ell - 1},$$  (70)

which implies the unique solution for (69)

$$b^0_k = g^0_k - \frac{1}{2\ell}, \quad k = 0, \ldots, \ell - 2.$$  (71)

Inserting this solution in the expression for the trace we obtain up to lower order derivative terms again

$$T^1_\ell(F^S, 0)_{d=3-\epsilon} = (-1)^{\ell+1}\ell(\ell - 1)\left(\ell - 2\right) \frac{2\delta^{(\ell+k+2)}(u)}{k} \frac{\delta^{(\ell+k+1)}(u)}{\ell + k + 3}!,$$  (72)

$$T^2_\ell(F^S, 0, H^S)_{d=3-\epsilon} = (-1)^{\ell+1}\ell(\ell - 1)\left(\ell - 2\right) \frac{\delta^{(\ell+k+1)}(u)}{k} \frac{\delta^{(\ell+k+1)}(u)}{\ell(\ell + k + 1)}!.$$  (73)

‡We also performed exact calculations with a computer program (Mathematica 5) for the cases $\ell = 2$ and $\ell = 4$ with the results in agreement with [31, 62].
Finally we can derive from (31) and (61), (62) the following expression (\( \ell \) is even)

\[
Tr K^\ell(F^S, G^S, H^S) = \frac{1}{\ell(\ell - 1)} \Box a K^\ell(F^S, G^S, H^S)
= -\int_2 \Psi^{-2} \left[ \left( \ell - 2 \right) \frac{2\delta^{(\ell+k+2)}(u)}{k} \right] - c_2^2 \Psi^{-2} \left[ \left( \ell - 2 \right) \frac{\delta^{(\ell+k+1)}(u)}{k} \ell(\ell + k + 1)! \right]
\]

(74)

4 Effective action and trace anomaly

First we return to the singular part of the two-point function without trace terms already calculated in [12]

\[
\Pi^\ell_{\text{sing}}(z_1; a|z_2; c) = n(\ell) K^\ell_{\text{sing}}(F^S, 0, 0) + \text{trace terms},
\]

(75)

\[
K^\ell_{\text{sing}}(F^S, 0, 0) = \Psi^\ell \left[ \left( \ell \right) \frac{1}{k} \frac{(-1)^{\ell+1}}{\ell(\ell + k + 1)!} \delta^{(\ell+k+1)}(u) \right].
\]

(76)

Inserting this in (11) for a transversal and traceless higher spin field \( h^{(\ell)} \) and using our technique for integration and transformation of \( u \)-derivatives of the delta functions to the covariant derivatives of the four-dimensional delta function in the general coordinate system (see [12] for details), we arrive at the same expression as in the previous article

\[
W_{\text{Sing}}^{(\ell)}(h^{(\ell)}) = \frac{1}{\epsilon} Z^\ell \Omega_3 \int \sqrt{g} d^4 z \nu^{(\ell)}_{\mu_1 \ldots \mu_\ell} K^\ell(\Box) h^{(\ell)\mu_1 \ldots \mu_\ell},
\]

(77)

\[
K^\ell(\Box) = \left\{ 2 \hat{D}_\ell + (\ell + 2) \right\} \prod_{m=1}^{\ell-1} \left[ \hat{D}_m + \frac{\ell}{2m} \right],
\]

(78)

\[
\hat{D}_n = \frac{1}{2n} \left[ \Box + 2 - n(n+1) \right], \quad Z^\ell = \frac{n(\ell)}{2\ell^2(2\ell + 1)!}.
\]

(79)

In the case of \( \ell = 2 \) this integral should be proportional to the integrated square of the gravitational Weyl tensor linearized in the \( AdS_4 \) background (see [11], [13], [14] and ref. there)

\[
C^{\mu\nu}_{\lambda\rho}(G)C^{\lambda\rho}_{\mu\nu}(G) = R^{\mu\nu}_{\lambda\rho}(G)R^{\lambda\rho}_{\mu\nu}(G) - 2R^{\mu\nu}(G)R_{\mu\nu}(G) + \frac{1}{3} R(G) R(G),
\]

(80)

\[
G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}^{(2)}, \quad \nabla^\mu h_{\mu\nu}^{(2)} = h_{\mu\nu}^{(2)} = 0.
\]

(81)

For traceless and transversal \( h_{\mu\nu}^{(2)} \) in an \( AdS_4 \) background (we put as before \( L=1 \) we have

\[
R^{\mu\nu}_{\lambda\rho}(G) = R^{\mu\nu}_{\lambda\rho}(h^{(2)}) = 2
\n\left. \nabla^\mu \nabla_{[\lambda} h_{\rho]}^{(2)\nu] \right) - 2\delta^{[\mu}_{[\lambda} h_{\rho]}^{(2)\nu]},
\]

(82)

\[
R^\mu_{\lambda}(h^{(2)}) = \frac{1}{2} \left\{ \nabla^\mu \nabla_{\lambda} h_{\nu}^{(2)\nu} + \square h_{\lambda}^{(2)\mu} - 2\nabla^\nu (\nabla \cdot h_{\lambda}^{(2)})_{\lambda\nu} \right\} + h_{\lambda}^{(2)\mu} - g_{\lambda}^{\mu} h_{\nu}^{(2)\nu}
\]

\[
= \frac{1}{2} \square h_{\lambda}^{(2)\mu} + h_{\lambda}^{(2)\mu},
\]

(83)

\[
R(h^{(2)}) = \square h_{\mu\nu}^{(2)} - \nabla^\mu \nabla^\nu h_{\mu\nu}^{(2)} - 3h_{\mu}^{(2)\mu} = 0,
\]

(84)
and straightforward calculations lead to
\[
\int \sqrt{g} d^4 z C^{\mu\nu}_{\lambda\rho}(h^{(2)}(\square^2 + 6\square + 8)) = \frac{1}{2} \int \sqrt{g} d^4 z h^{(2)}_{\mu\nu} \left[ \square^2 + 6\square + 8 \right] h^{(2)\mu\nu}. \tag{85}
\]
Then we can evaluate (78) for \( \ell = 2 \) and obtain
\[
K^2(\Box) = \frac{1}{4} \left[ \square^2 + 6\square + 8 \right]. \tag{86}
\]
So we see that
\[
W^{(2)}_{\text{Sing}}(h^{(\ell)}) = \frac{1}{\epsilon} \frac{Z^2\Omega_3}{2} \int \sqrt{g} d^4 z C^{\mu\nu}_{\lambda\rho}(h^{(2)}) C^{\lambda\rho}_{\mu\nu}(h^{(2)}). \tag{87}
\]
Next we compare the coefficient \( \frac{Z^2\Omega_3}{2} \) with the textbook result (see formula (6.102) in [11]). For doing this carefully we note that our current \( J^{(2)}_{\mu\nu} \) defined according to formulas (1)-(5) of [12] as
\[
J^{(2)}_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta h^{\mu\nu}} S^{(2)\text{conf}} = \sigma \nabla_\mu \partial_\nu \sigma - 2\partial_\mu \sigma \partial_\nu \sigma + \ldots \tag{88}
\]
and the energy-momentum tensor of the conformal scalar in \textit{AdS}_4
\[
T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta h^{\mu\nu}} W_{\text{cl}}(G = g + h) = \frac{1}{3} \sigma \nabla_\mu \partial_\nu \sigma - \frac{2}{3} \partial_\mu \sigma \partial_\nu \sigma + \ldots, \tag{89}
\]
\[
W_{\text{cl}}(G = g + h) = \frac{1}{2} \int \sqrt{G} d^4 z \left( G^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{6} R(G) \sigma^2 \right) \tag{90}
\]
are related in the following way
\[
T_{\mu\nu} = \frac{1}{3} J^{(2)}_{\mu\nu}. \tag{91}
\]
So we see that we can compare our spin-two effective action with the gravitational one after taking into account the additional factor \( \frac{1}{3} \). This leads to
\[
\frac{1}{9} \frac{Z^2\Omega_3}{2} = \frac{1}{2} \frac{1}{16\pi^2} \frac{1}{120}, \tag{92}
\]
which exactly coincides with the coefficient in front of the \( C^2 \) term of the singular part of the effective action in [11] and produces the first part of the common expression for the anomalous trace of the energy-momentum tensor
\[
< T^{\mu\nu}_{\text{Sing}} > = \frac{2}{\sqrt{G}} G^{\mu\nu} \frac{\delta W_{\text{Sing}}(G)}{\delta G^{\mu\nu}} = \frac{1}{16\pi^2} \left[ \alpha C^{\mu\nu}_{\lambda\rho}(G) C^{\lambda\rho}_{\mu\nu}(G) - \beta E_4 \right], \tag{93}
\]
\[
\alpha = \frac{1}{120}, \quad \beta = \frac{1}{360}. \tag{94}
\]
\[^{\$}\text{Restoring } L \text{ dependence } K^2(\Box) = \frac{1}{4} \left[ \square^2 + 6\frac{1}{L^2} \Box + 8\frac{1}{L^4} \right], \text{ and therefore } K^2(\Box) \to \frac{1}{4} \square^2 \text{ when } L \to \infty. \text{ It means that this anomaly contribution can be determined from the divergent part of the two point function in both the flat and the constant curvature background [13, 14].} \]
\[^{\dagger}\text{In the literature instead of } \alpha \text{ and } \beta \text{ one can also find other notations (e.g. } c \text{ and } a, \text{ respectively) for the coefficients of the trace anomaly [13].} \]
Note that in the conformal flat $AdS$ background $C_{\lambda\rho}^{\mu\nu}(G = g + h) = C_{\lambda\rho}^{\mu\nu}(h)$ because $C_{\lambda\rho}^{\mu\nu}(g) = 0$.

It is important that we got from a perturbative calculation only a $C^2$ type pole term of the effective action but not a topological Euler density term. This means that a perturbative calculation with dimensional regularization of the correlation function does not feel this kind of pole terms with $\frac{0}{0}$ behaviour when $\epsilon$ goes to 0 (see the corresponding discussion in [21]).

Nevertheless we can extract information about this topological term from our calculation of the non-zero finite trace of the two-point function. First of all we construct the effective action for a transversal field $h^{(\ell)}$ with nonvanishing trace but vanishing double trace

$$< Tr J^\ell > = Tr \frac{\ell}{\sqrt{g}} \frac{\delta W_{\text{sing}}^\ell}{\delta h^{(\ell)}} = \ell n(\ell) Tr K^\ell *_{c_2} h^{(\ell)}(c; z_2)$$

(94)

We will therefore decompose $Tr K^\ell$ into three parts

$$Tr K^\ell = A + Bc_\mu \nabla_2^\mu + C(c_2^2)^2$$

(95)

so that B and C drop out after insertion into (94). Thus A represents a restclass and our issue is to find an optimal representative for the rest class of $Tr K^\ell$. We denote rest class equivalence by $\equiv$.

The reduction of $Tr K^\ell$ to this optimal form is achieved by a series of partial integrations based on

$$I_{2c} \delta^{(m)}(u) = c_\mu \nabla_2^\mu \delta^{(m-1)}(u)$$

(96)

Any partial integration is performed employing the formulae of Appendix A. First we transform the first part of (74) proportional to $I_{2c}^2$ and add to the result the second part proportional to $c_2^2$. After $\ell - 2$ times repeated partial integrations we can render $Tr K^\ell$ equivalent to the following intermediate form

$$Tr K^\ell \equiv -c_2^2(\ell - 2)! \left\{ \Psi^{\ell-2} \left[ \sum_{p=0}^{\ell-2} M_k^p(u) \right] \right. + \Psi^{\ell-2} \left[ \frac{\delta^{(\ell+k+1)}}{k!(\ell - 2 - k)!(\ell + k + 1)!} \right] \left. \right\}$$

(97)

$$M_k^p(u) = \frac{(-1)^{p+1}(k + 1)(u + 1)\delta^{(\ell+k+1)}(u)}{k!(\ell - 2 - k - p)!(\ell + k + 3 + p)!} + \frac{(-1)^{p+1}(\ell - k - 1)\delta^{(\ell+k)}(u)}{(k - 1)!(\ell - 1 - k - p)!(\ell + k + 2 + p)!}$$

(98)
Then using \( u\delta^{(\ell+k+1)}(u) = -(\ell+k+1)\delta^{(\ell+k)}(u) \) and the summation rules
\[
\sum_{p=0}^{m} \frac{(-1)^p}{(m-p)!(r+p)!} = \frac{1}{(m+r)(m+r-1)!},
\]
\[
\sum_{p=0}^{m} \frac{(-1)^{p+1}}{p!(m-p)!(r+p)!} = \frac{1}{(m+r-1)(m+r)(m+r-1)!}.
\]
we obtain
\[
\sum_{p=0}^{\ell-2} M_p^k(u) = \frac{(\ell+1)\delta^{(\ell+k)}(u) - (k+1)\delta^{(\ell+k+1)}(u)}{(2\ell+1)k!(\ell-2-k)!(\ell+k+2)!}.
\]

Inserting this in (97) we obtain an expression proportional to \( c_2^2 \) which produces the trace of \( h^{(\ell)} \). Next we apply the same algorithm of partial integration to \( \Psi^{\ell-2} \)
\[
\sum_{k=0}^{\ell-2} \phi_k\delta^{(m+k)}(u)I_1^{\ell-2-k}I_2^k \equiv I_1^{\ell-2}\delta^{(m)}(u)\sum_{k=0}^{\ell-2} (-1)^k\phi_k k!
\]
and again taking into account (99), (100) we obtain finally
\[
TrK^{\ell}(u) \equiv -\frac{c_2^2I_1^{\ell-2}}{\ell!(2\ell+1)} \left[ \frac{2\delta^{(\ell+1)}(u)}{2\ell-1} + \frac{\delta^{(\ell)}(u)}{\ell} \right].
\]

In the next step we have to express the derivatives of the delta function of the chordal distance by polynomials of the Laplacian applied to the standard delta function. This problem was solved by the authors in general in [12]
\[
\delta^{(n)}(u) = (-1)^n\Omega_3 \left\{ 2\hat{D}_{n-1} + n \right\} \left\{ \prod_{m=1}^{n-2} \hat{D}_m \right\} \frac{\delta(z-z_{\text{pole}})}{\sqrt{g(z)}},
\]
with \( \hat{D}_m \) as in [19]. Moreover we need the commutation relation
\[
\square I_1^{\ell-2} \hat{\phi}_c h^{(\ell)}(c; z_2) = I_1^{\ell-2} \hat{\phi}_c \{ \square + \ell + 2 \} h^{(\ell)}(c; z_2).
\]
Using these relations and the fact that
\[
\lim_{u \to 0} I_1(a, c; u) = a^\mu c_\mu
\]
we obtain the final formula for the trace anomaly
\[
<T r J^{(\ell)}> = \frac{\Omega_3 n(\ell)}{\ell^2(\ell-1)!4(4\ell^2-1)} T^{\ell}(\square) Tr h^{(\ell)}(c; z_2),
\]
\[
T^{\ell}(\square) = \left[ \square^2 - (\ell^2 - \ell - 1)\square - \ell(\ell^2 - 1) \right] \prod_{m=1}^{\ell-2} \hat{D}_m + \frac{\ell - 2}{2m}.
\]
Now we compare this expression with the known answer for the $\ell = 2$ case with the second part of (93). First of all the Euler density has the linear term in the linearized AdS background

$$E_4 = C_{\lambda \rho}^{\mu \nu} (G) C_{\mu \nu}^{\lambda \rho} (G) - 2 \left( R^{\mu \nu} (G) R_{\mu \nu} (G) - \frac{1}{3} R (G) R (G) \right)$$

$$G_{\mu \nu} = g_{\mu \nu} + h_{\mu \nu}^{(2)}, \quad \nabla^\mu h_{\mu \nu}^{(2)} = 0. \quad (109)$$

Note that in the flat background $E_4$ starts from the quadratic terms in $h^{(2)}$ and could be discovered only from the three-point function $[13]$. So we should compare our expression with the following one

$$< T_{\mu}^{\mu} > = -\frac{\beta}{16 \pi^2} 4 (\Box - 3) Tr h^{(2)}, \quad \beta = \frac{1}{360}. \quad (111)$$

Indeed inserting $\ell = 2$ in (108) we obtain

$$T^2 (\Box) Tr h^{(2)} = (\Box^2 - 6) Tr h^{(2)} = \Box R + 2 R. \quad (112)$$

But the first term $\Box R$ corresponds to the regularization scheme dependent contribution in the trace anomaly (so-called "trivial anomaly") and can be absorbed by the appropriate choice of a finite local counterterm $R^2 [11].$

So finally we obtain the following result

$$< Tr J^{(2)} >= -\frac{1}{40 \cdot 16 \pi^2} E_4. \quad (113)$$

Then remembering the different normalization of the currents $[11]$ leading to

$$< TT >= \frac{1}{3^2} < J^{(2)} J^{(2)} >$$

we get

$$< T_{\mu}^{\mu} >= -\frac{1}{360 \cdot 16 \pi^2} E_4, \quad (114)$$

which is in full agreement with $[11, 11]$ and $[11].$

Now we return to the general $\ell$ case. We can rewrite our general answer (107), (108) using the gauge invariant object – the so-called "Fronsdal” operator $[2]$ (see for details $[7, 8]$)

$$F(h^{(\ell)} (z; a)) = \Box h^{(\ell)} (z; a) - (a \nabla)^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} (z; a) + \frac{1}{2} (a \nabla)^2 \Box_a h^{(\ell)} (z; a)$$

$$- \left( \ell^2 + \ell (d - 5) - 2 (d - 2) \right) h^{(\ell)} (z; a) - a^2 \Box_a h^{(\ell)} (z; a). \quad (115)$$

\footnote{In other words restoring $L$ dependence we see that the linearized Euler density is $E_4 = -4 (\frac{1}{4} \Box - \frac{1}{2} R) Tr h^{(2)}$ and therefore $E_4 \to 0$ when $L \to \infty$}
Actually we need only the trace of this object which looks very simple for a transversal gauge field \( \nabla^\mu \partial_\mu h^{(\ell)} = 0 \)

\[
Tr F(h^{(\ell)}(z; a)) = 2 \left[ \Box - (\ell^2 - 1) \right] Tr h^{(\ell)}(z; a). \tag{116}
\]

Introducing the new notation

\[
\hat{R}^{(\ell)} Tr h^{(\ell)} = \left[ \Box - (\ell^2 - 1) \right] Tr h^{(\ell)}(z; a), \tag{117}
\]

and expanding \( T^{\ell}(\square) \) in powers of \( \hat{R}^{(\ell)} \) we obtain

\[
T^{\ell}(\square) = \frac{\hat{R}^{(\ell)}}{2\ell^2(\ell - 2)!} \prod_{m=0}^{\ell-2} \left[ \hat{R}^{(\ell)} + (\ell^2 + \ell - 1) - m(m + 1) \right]. \tag{118}
\]

This is a polynomial of \( \ell \)'th order in \( \hat{R}^{(\ell)} \) without a constant term. Factorizing

\[
m(m + 1) - (\ell^2 + \ell - 1) = (m - a_+)(m - a_-), \tag{119}
\]

\[
a_\pm = -\frac{1}{2} \pm \sqrt{\lambda + \frac{1}{4}}, \quad \lambda = \ell^2 + \ell - 1 \tag{120}
\]

we obtain the coefficient of the term linear in \( \hat{R}^{(\ell)} \)

\[
\gamma_0 = \frac{(-a_+)_{\ell-1}(-a_-)_{\ell-1}}{2\ell^2(\ell - 2)!}. \tag{121}
\]

and as a coefficient of \( (\hat{R}^{(\ell)})^{n+1} \)

\[
\gamma_n = \gamma_0 \sum_{0 \leq m_1 < m_2 < \cdots < m_n \leq \ell - 2} \prod_{i=1}^{n} \frac{1}{\ell^2 + \ell - 1 - m_i(m_i + 1)}. \tag{122}
\]

Another elegant representation for \( \gamma_n \) we can obtain directly from (115) using Taylor’s expansion and replacing the \( n \)'th derivative \( \left( \frac{d}{d\lambda} \right)^n (\ldots)_{\hat{R}=0} \) with \( \left( \frac{d}{d\lambda} \right)^n (\ldots)_{\hat{R}=0} \)

\[
\gamma_n = \frac{1}{n!} \left( \frac{d}{d\lambda} \right)^n \gamma_0(\lambda), \tag{123}
\]

\[
\gamma_0(\lambda) = \frac{1}{2\ell^2(\ell - 2)!} \prod_{m=0}^{\ell-2} [\lambda - m(m + 1)], \tag{124}
\]

where after differentiation we have to put \( \lambda = \ell^2 + \ell - 1 \).

In a similar way for the operator \( K^{\ell}(\Box) \) in [115] we can consider [115] for transversal and traceless \( h^{(\ell)} \)

\[
\mathcal{F}(h^{(\ell)}(z; a)) = (\Box - \ell^2 + 2\ell + 2) h^{(\ell)}(z; a) = \hat{\mathcal{F}}^{(\ell)} h^{(\ell)}(z; a), \tag{125}
\]
and obtain the following representation for (78)

$$K^\ell(\Box) = \frac{\hat{F}^{(\ell)}}{\ell!2^\ell-1} \prod_{m=0}^{\ell-2} \left[ \hat{F}^{(\ell)} + \lambda - m(m+1) \right], \quad \lambda = \ell^2 - \ell. \quad (126)$$

Then expanding

$$\prod_{m=0}^{\ell-2} \left[ \hat{F}^{(\ell)} + \lambda - m(m+1) \right] = \sum_{n=0}^{\ell-2} \hat{\gamma}_n(\lambda) \left( \hat{F}^{(\ell)} \right)^n, \quad (127)$$

we get again

$$\hat{\gamma}_n = \frac{1}{n!} \left( \frac{d}{d\lambda} \right)^n \hat{\gamma}_0(\lambda), \quad (128)$$

$$\hat{\gamma}_0(\lambda) = \prod_{m=0}^{\ell-2} [\lambda - m(m+1)], \quad (129)$$

but with the simpler expression for $\hat{\gamma}_0(\lambda)$ evaluated at $\lambda = \ell^2 - \ell$

$$\hat{\gamma}_0(\lambda)_{\lambda=\ell^2-\ell} = (2\ell - 2)!. \quad (130)$$

**Conclusions**

In this article we considered the two-point correlation function for traceless conserved higher spin currents in $AdS_4$ including the first trace terms. We have shown that extracting the delta function singularities we can observe the trace anomaly in the external higher spin gauge fields. In the particular $\ell = 2$ case our result is in full agreement with the answer for the conformal anomaly in an external gravitational field and produces the right anomaly numbers for both the Weyl invariant and the topological part of anomaly. The important point here is that in the flat background the topological part of the anomaly can not be determined from the two point function. This two-point function acts on transversal external gauge fields which have either vanishing trace (case 1) or vanishing doubletrace (case 2). In case 1 it has $\ell + 1$ components corresponding to the parts $I_1^{\ell-k}I_2^k$, $0 \leq k \leq \ell$. In case 2 there exist $2(\ell - 1)$ terms corresponding to $I_3A I_1^{\ell-2-k}I_2^k$, $0 \leq k \leq \ell - 2$. For $\ell = 2$ the first three terms reduce to one term, the square of the Weyl tensor with coefficient $\alpha$. The two terms of case 2 reduce to the single topological (Euler density) term with coefficient $\beta$. The reduction is due to equations of motion, vanishing of traces and skipping trivial anomaly terms.

For the general spin-$\ell$ case we present the anomaly as a polynomial of a gauge covariant differential operator which contains information about a trivial and the
topological part of the trace anomaly. Unfortunately at the moment we have no means to classify the possible finite local gauge invariant counterterms constructed from the higher spin field and can neither extract the trivial contribution in the anomaly nor define the closed form of the topological part. But at least we know that the term linear in $\hat{R}^{(\ell)}$ of (118) should contribute to the topological part and the corresponding numerical coefficient of this term is presented in closed form as a function of $\ell$.

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References

[1] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical O(N) vector model,” Phys. Lett. B 550 (2002) 213, [arXiv:hep-th/0210114].

[2] C. Fronsdal, “Singletons And Massless, Integral Spin Fields On De Sitter Space (Elementary Particles In A Curved Space Vii),” Phys. Rev. D 20, (1979)848; “Massless Fields With Integer Spin,” Phys. Rev. D 18 (1978) 3624.

[3] E. S. Fradkin and M. A. Vasiliev, “On The Gravitational Interaction Of Massless Higher Spin Fields,” Phys. Lett. B 189 (1987) 89; E. S. Fradkin and M. A. Vasiliev, “Cubic Interaction In Extended Theories Of Massless Higher Spin Fields,” Nucl. Phys. B 291 (1987) 141; M. A. Vasiliev, “Higher-spin gauge theories in four, three and two dimensions,” Int. J. Mod. Phys. D 5 (1996) 763 [arXiv:hep-th/9611024]; M. A. Vasiliev, “Nonlinear equations for symmetric massless higher spin fields in (A)dS(d),” [arXiv:hep-th/0304049].

[4] W. Rühl, “Lifting a conformal field theory from $d$-dimensional flat space to $(d + 1)$-dimensional $AdS$ space” Nucl. Phys. B 705 (2005) 437, [arXiv:hep-th/0403114].

[5] A. C. Petkou, ”Evaluating the AdS dual of the critical O(N) sigma model”, JHEP 0303 (2003) 049, [arXiv:hep-th/0302063].

[6] W. Rühl, “The masses of gauge fields in higher spin field theory on AdS(4),” Phys.Lett. B 605 (2005) 413, [arXiv:hep-th/0409252].

[7] R. Manvelyan and W. Rühl, “The masses of gauge fields in higher spin field theory on the bulk of AdS(4)”, Phys. Lett. B 613 (2004) 197, [arXiv:hep-th/0412252].
[8] R. Manvelyan and W. Rühl, “The off-shell behaviour of propagators and the Goldstone field in higher spin gauge theory on $AdS_{d+1}$ space,” Nucl. Phys. B 717 (2005) 3, [arXiv:hep-th/0502123v.3 28 Feb.2005].

[9] M. J. Duff, “Observations on conformal anomalies,” Nucl. Phys. B 125 (1977) 334; “Twenty years of the Weyl anomaly,” Class. Quant. Grav. 11 (1994) 1387, [hep-th/9308075]; S. Deser, M. J. Duff and C. J. Isham, “Non-local conformal anomalies,” Nucl. Phys. B 111 (1976) 45; S. Deser and A. Schwimmer, “Geometric classification of conformal anomalies in arbitrary dimensions, Phys. Lett. B 309 (1993) 279, [hep-th/9302047].

[10] R. Manvelyan and W. Rühl, “Conformal coupling of higher spin gauge fields to a scalar field in $AdS(4)$ and generalized Weyl invariance,” Phys. Lett. B 593 (2004) 253, [arXiv:hep-th/0403241].

[11] N. D. Birrell and P. C. Davies, “Quantum fields in curved space”, Cambridge University Press, 1982.

[12] R. Manvelyan and W. Rühl, “The quantum one loop trace anomaly of the higher spin conformal conserved currents in the bulk of $AdS(4)$,” Nucl. Phys. B 733 (2006) 104, [arXiv:hep-th/0506185].

[13] H. Osborn and A. C. Petkou, “Implications of conformal invariance in field theories for general dimensions,” Annals Phys. 231 (1994) 311 [arXiv:hep-th/9307010].

[14] H. Osborn and G. M. Shore, “Correlation functions of the energy momentum tensor on spaces of constant curvature,” Nucl. Phys. B 571 (2000) 287 [arXiv:hep-th/9909043].

[15] B. Allen and T. Jacobson, “Vector Two Point Functions In Maximally Symmetric Spaces,” Commun. Math. Phys. 103 (1986) 669.

[16] B. Allen and M. Turyn, “An Evaluation Of The Graviton Propagator In De Sitter Space,” Nucl. Phys. B 292 (1987) 813.

[17] M. Turyn, “The Graviton Propagator In Maximally Symmetric Spaces,” J. Math. Phys. 31 (1990) 669.

[18] T. Leonhardt, R. Manvelyan and W. Rühl, “The group approach to $AdS$ space propagators,” Nucl. Phys. B 667 (2003) 413, [arXiv:hep-th/0305235].

[19] T. Leonhardt, W. Rühl and R. Manvelyan, “The group approach to $AdS$ space propagators: A fast algorithm,” J. Phys. A 37 (2004) 7051, [arXiv:hep-th/0310063].
Appendix A

The Euclidian $AdS_{d+1}$ metric

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{1}{(z^0)^2} \delta_{\mu\nu}dz^\mu dz^\nu$$ (A.1)

can be realized as an induced metric for the hypersphere defined by the following embedding procedure in $d+2$ dimensional Minkowski space

$$X^A X^B \eta_{AB} = -X_{-1}^2 + X_0^2 + \sum_{i=1}^{d} X_i^2 = -1,$$ (A.2)

$$X_{-1}(z) = \frac{1}{2} \left( \frac{1}{z_0} + \frac{z_0^2 + \sum_{i=1}^{d} z_i^2}{z_0} \right),$$ (A.3)

$$X_0(z) = \frac{1}{2} \left( \frac{1}{z_0} - \frac{z_0^2 + \sum_{i=1}^{d} z_i^2}{z_0} \right),$$ (A.4)

$$X_i(z) = \frac{z_i}{z_0}.$$ (A.5)

Using this embedding rules we can realize that the chordal distance $\zeta(z, w)$ is just an $SO(1, d + 1)$ invariant scalar product

$$-X^A(z)Y^B(w)\eta_{AB} = \frac{1}{2z_0w_0} \left( 2z_0w_0 + \sum_{\mu=0}^{d} (z-w)^\mu \right) = \zeta = u + 1,$$ (A.6)

and therefore can be realized by a hyperbolic angle. Indeed we can introduce another embedding

$$X_{-1}(\Theta, \omega_\mu) = \cosh \Theta,$$ (A.7)

$$X_\mu(\Theta, \omega_\mu) = \sinh \Theta \omega_\mu \quad \sum_{\mu=0}^{d} \omega_\mu^2 = 1,$$ (A.8)

$$ds^2 = d\Theta^2 + \sinh^2 \Theta d\Omega_d.$$ (A.9)
In these coordinates the chordal distance between an arbitrary point $X^A(\Theta, \Omega_\mu)$ and the pole of the hypersphere $Y^A(\Theta = 0, \omega_\mu)$ is simply

$$\zeta = -X^A Y^B \eta_{AB} = \cosh \Theta.$$  \hfill (A.10)

Therefore the invariant measure is expressed as

$$\sqrt{g} d\Theta d\Omega_d = (\sinh \Theta)^d d\Theta d\Omega_d = u(u + 2)^{d+1} du d\Omega_d. \hfill (A.11)$$

So we see that the integration measure for $d = 3$ ($D = d + 1 = 4$) will cancel one order of $u^{-n}$ in short distance singularities and we have to count the singularities starting from $u^{-2}$.

In this article we use the following rules and relations for $u(z, z')$, $I_{1a}$, $I_{2c}$ and the bitensorial basis $\{I_i\}_{i=1}^4$

\[
\Box u = (d + 1)(u + 1), \quad \nabla_\mu \partial_\nu u = g_{\mu\nu}(u + 1), \quad g^{\mu\nu} \partial_\mu \partial_\nu u = u(u + 2), \quad (A.12)
\]
\[
\partial_\mu \partial_\nu u \nabla^{\mu} u = u \partial_\nu u, \quad \partial_\mu \partial_\nu u \nabla^{\nu} \partial_\rho u = g_{\mu\rho} \partial_\nu u + \partial_\nu u \partial_\mu u, \quad (A.13)
\]
\[
\nabla_\mu \partial_\nu \partial_\rho u \nabla^{\rho} u = \partial_\nu u \partial_\rho u, \quad \nabla_\mu \partial_\nu \partial_\rho u = g_{\mu\rho} \partial_\nu u, \quad (A.14)
\]
\[
\frac{\partial}{\partial a_\mu} I_{1a} \frac{\partial}{\partial a_\mu} I_{1a} = u(u + 2), \quad \frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_{1a} = \zeta I_{2c}, \quad (A.15)
\]
\[
\frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = c_2^2 + I_{2c}^2, \quad \frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_2 = (u + 1)I_{2c}, \quad \Box a I_4 = 2(d + 1)c_2^2, \quad (A.16)
\]
\[
\frac{\partial}{\partial a_\mu} I_2 \frac{\partial}{\partial a_\mu} I_2 = u(u + 2)^2 I_{2c}, \quad \Box a I_3 = 2(d + 1)I_{2c}^2 + 2c_2^2 u(u + 2), \quad (A.17)
\]
\[
\nabla^\mu \frac{\partial}{\partial a_\mu} I_1 = (d + 1)I_{2c}, \quad \nabla^\mu \frac{\partial}{\partial a_\mu} I_2 = (d + 2)(u + 1)I_{2c}, \quad \nabla^\mu I_1 \partial_\mu u = I_2, \quad (A.18)
\]
\[
\nabla^\mu \frac{\partial}{\partial a_\mu} I_3 = 4I_1I_2 + 2(d + 2)(u + 1)c_2^2 I_{1a}, \quad \nabla^\mu I_2 \partial_\mu u = 2(u + 1)I_2, \quad (A.19)
\]
\[
\frac{\partial}{\partial a_\mu} I_1 \partial_\mu u = (u + 1)I_2c, \quad \frac{\partial}{\partial a_\mu} I_2 \partial_\mu u = u(u + 2)I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_1 = I_1I_{2c}, \quad (A.20)
\]
\[
\frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_2 = I_{2c}((u + 1)I_1 + I_2) + c_2^2 I_{1a}, \quad \frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_1 = I_{2c}I_2, \quad (A.21)
\]
\[
\frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_2 = 2(u + 1)I_2cI_2, \quad \nabla^\mu I_1 \nabla_\mu I_1 = a_1^2 I_{2c}^2, \quad \Box I_1 = I_1, \quad (A.22)
\]
\[
\nabla^\mu I_1 \nabla_\mu I_2 = I_2 I_1 + a_1^2 (u + 1)I_{2c}, \quad \Box I_2 = (d + 2)I_2 + 2(u + 1)I_1, \quad (A.23)
\]
\[
\nabla^\mu I_2 \nabla_\mu I_2 = I_2^2 + 2(u + 1)I_1I_2 + a_1^2 I_{2c}(u + 1)^2 + c_2^2 I_{1a}, \quad (A.24)
\]
\[
a_1^2 \nabla_\mu I_{1a} = a_1^2 (u + 1), \quad a_1^2 \nabla_\mu I_{2c} = I_1, \quad a_1^2 \nabla_\mu I_3 = a_1^2 I_{2c}, \quad (A.25)
\]
\[
a_1^2 \nabla_\mu I_2 = a_1^2 (u + 1)I_2c + I_{1a}I_1, \quad (A.26)
\]

**Appendix B**

We derive an algorithm to determine the finite renormalization functions $g_k^p$ and through these the functions $h_k^p, d_k^p$ from the current conservation constraints $D_1^1 = \ldots$
0 and $D^2_{\ell} = 0$ (37), (38) and the tracelessness conditions $T^1_{\ell} = 0$, $T^2_{\ell} = 0$ modulo anomalous terms already considered in the main text. The idea is to replace $G_k$ and $H_k$ in $D^1_{\ell} = 0$ and $D^2_{\ell} = 0$ by eliminating them with the help of the two $T^1_{\ell} = 0$, $T^2_{\ell} = 0$ equations

In $D^1_{\ell} = 0$ with

$$D^{(n)} := \frac{1}{n!}\delta^{(n)}(u), \quad (B.1)$$

and

$$\Delta f_k := f_k - f_{k+1}, \quad (B.2)$$

$$\Delta^2 f_k = \Delta f_k - \Delta f_{k+1}, \quad (B.3)$$

we get

$$\sum_p D^{(\ell+k+1-p)} \{2\ell [(p+1)\Delta g^p_k + (\ell+k+4)\Delta g^{p+1}_k - (p+3)g^{p+1}_k + 2(p+2)g^{p+1}_{k+1}]$$

$$-(k+2)(\ell-k-1) \left[2\Delta g^{p+1}_{k+1} + \Delta^2 g^p_k \right] - k(\ell+k+1-p) \left[2\Delta g^{p+1}_k + \Delta^2 g^p_{k-1} \right] \} = 0. \quad (B.4)$$

To solve these equations we use as ansatz

$$g^\ell_{\ell-q} = g^\ell_{\ell-q} + \sum_{n=1}^{q} \left( \sum_{r=1}^{n} Q^{(q)}(n, r, \ell) k^r \right) g^\ell_{\ell-q+n}, \quad 0 \leq q \leq \ell - 1, \quad (B.5)$$

and

$$g^0_0 = g^0_0 + \sum_{n=0}^{k} \frac{1}{\ell + n + 1}. \quad (B.6)$$

The functions $Q^{(q)}(n, r, \ell)$ are rational and can be determined by inserting (B.5) in (B.4) recursively. Explicit results are

$$Q^{(1)}_{1, 1}(\ell) = 1, \quad Q^{(2)}_{2, 2}(\ell) = \frac{\ell - 1}{2\ell - 1}, \quad Q^{(2)}_{2, 1}(\ell) = \frac{\ell^2 - 5\ell + 1}{(\ell - 1)(2\ell - 1)}. \quad (B.7)$$

The equation $D^1_{\ell} = 0$ is then completely exploited. The remaining variables

$$\left\{ \frac{g^\ell_{\ell-q}}{g^0_0} \right\}_{0 \leq q \leq \ell}$$

enter the system of equations derived from $D^2_{\ell} = 0$:

$$\sum_{q=0}^{r} M_{r,q} g^\ell_{\ell-q} = A\delta_{r,\ell}, \quad (0 \leq r \leq \ell). \quad (B.8)$$
In order that this system is consistent we need

\[ M_{0,0} = 0. \] (B.9)

In fact, the contribution to \( M_{0,0} \) is

1. from the \( F_{k+1} \) term
   \[ + \ell(\ell - 1) \left( \binom{\ell - 2}{k} \right); \]
2. from the \( H'_{k} + (k + 1)H_{k+1} \) term
   \[ - \ell(\ell - 1) \left( \binom{\ell - 2}{k} \right) \left[ \frac{k+2}{2\ell-1} + \frac{\ell-2-k}{2\ell-1} \right]; \]
3. from the sum over \( G'_{k}, G'_{k-1}, G_{k}, G_{k-1} \) terms after many cancellations
   \[ - \ell(\ell - 1) \left( \binom{\ell - 2}{k} \right) \frac{\ell-1}{2\ell-1}. \]

So (1), (2), (3) cancel each other. The explicit forms of \( Q_{1,1}^{(1)} \) and \( Q_{2,2}^{(2)} \) have been used in these derivations.

There remain in this way \( \ell \) equations for \( \ell + 1 \) unknowns. We choose \( g_{0}^{0} \) as free parameter and identified this one as renormalization parameter \( \mu \) in [12].