Guide for the establishment of the decomposition model of the probability density function based on the hyphal extension rate of fungi

Shangyang Shi1; Senxin Guo1*; Yaotian Zhang1
1 College of Applied Science and technology, Hainan University, Danzhou, Hainan, 571737, China
*Corresponding author’s e-mail: 20177402320013@hainanu.edu.cn

Abstract. The main purpose of this paper is to provide readers with a guide to mathematical modeling. The idea of establishing a model is roughly as follows: extract the probability factors of the research object, analyze the research object itself, observe the changes of the research object, and observe the effects of environmental changes on the research object. In this article, taking the 2021 American Mathematical Competition in Modeling (MCM) as an example, using the proposed modeling guidelines, a mathematical model was established to describe the decomposition of woody fiber by fungi, and the independent variable was set to hyphal extension rate, temperature conditions, climate conditions, etc. Set the dependent variable to decomposition rate, and fit a series of logarithmic expression and exponential expression according to the data provided by references [1][2]. Establish a community evolution model to realize the dynamic evolution of the interaction between different types of fungi, fungi with a higher competitive ranking will have a higher hyphal extension rate than the original without the interaction. As a result, fungi with a higher competitive ranking is in a dominant position in the colony, and fungi with a lower competitive ranking is in a subordinate or inferior position in the colony. In order to use the relative advantages and disadvantages of various fungi in different environments to predict the exact composition of fungi in the colony, considering the influence of initial climatic conditions on the evolution of the community, and bring about relative changes in the competitive ranking of different types of fungi. The model example in this article reflects the correctness of the modeling guidelines.

1. Introduction
In recent years, using mathematical models to solve biological problems has the following advantages: it can specifically and intuitively reflect the growth process and changes of organisms, and can predict the results of biological growth and evolution, simple and easy to implement, fast to solve, and so on. This article will take 2021 MCM Problem A as an example for a practice. The following is a simple restatement of Problem A, establish a mathematical model to describe the decomposition of woody fiber in the presence of multiple fungi. The interactions of different types of fungi with different growth rates and different moisture resistance are combined, and the impact of environmental and climate changes on the decomposition rate of fungi is considered.

Start from reality and look for probabilistic factors. Because there is no woody fiber and water in the air from the very beginning, the fungi are in a dormant state, so the initial biomass of each fungus is the same, so it is assumed that each fungus in the air has the same biomass. For fungi scattered in the air, the air is mobile, so the process of fungi attaching to woody fibers is a random process. Then, we analyze
the research object--fungi. Under our research theme, we can ignore the difference in their names of each fungus. The main focus is on five indicators (hyphal extension rate, decomposition rate, moisture trade-off, competitive ranking and moisture niche width), these five indicators are our main basis for distinguishing different types of fungi. After that, focus on observing the changing factors of the research object. Considering that the fungi will proliferate, the proportion of the biomass of the fungi will change. What is the relationship between the five indicators and the biomass of the fungi? The answer is in the main text. Finally, observe the impact of environmental changes on the research object, such as the decomposition rate and growth rate of fungi in tropical and temperate climates.

In this process, it is necessary to pay attention to the establishment of the model, from simple to complex, from static to dynamic, and to prepare for the transition from considering a single factor to considering multiple factors. This process is "static model → dynamic model → modified model". People's cognition of complex things often transits from simple to complex, and from static to dynamic, which also reflects the rationality of this mathematical model process.

First, extract the probability factors of the research object, secondly, analyze the research object itself, then observe the changes of the research object, and finally observe the impact of environmental changes on the research object. Figure 1 shows the process of this modeling process.

2. Static model of fungus decomposition

2.1. Determine 5 relevant indicators of various fungus in the colony

In actual situations, the hyphal extension rate of a fungus isolated from a colony is distributed according to a probability density function \( f(\nu_{HE}) \). According to the data in Figure 1C in [1], the probability function of the hyphal extension rate of the colony is obtained

\[
\frac{dN}{N} = f(\nu_{HE}) d\nu_{HE}
\]

\[
f(\nu_{HE}) = 0.1705 \cdot x^2 - 2.7591 \cdot x + 11.5278 \text{ (Not normalized)}
\]

\( N = 34 \) is the number of fungal species in the colonies studied by Nicky Lustenhouver et al.

![Probability density function](image)

Figure 1. Probability density function \( f(\nu_{HE}) \) of hyphal extension rate.

Using probability density function \( f(\nu_{HE}) \), the specific value of hyphal extension rate of each fungus in the colony can be directly confirmed through the random process of MATLAB programming.

According to the data in Figure 1C in [1], the logarithmic growth relationship between geometric mean of decomposition rate of various fungi and geometric mean of hyphal extension rate at different temperatures (10°C, 16°C, 22°C) is obtained

\[
\nu_D = 3.9865 \cdot \ln(8.0459 \cdot \nu_{HE}) - 0.4057
\]
According to the data in Figure 4A in [1], the geometrical average decomposition rate of various fungi at different temperatures (10°C, 16°C, 22°C) and the exponential growth of moisture trade-off is obtained

$$v_D = 7.4182 \cdot e^{1.0969 \cdot MT} + 0.1197$$  \hspace{1cm} (3)

According to the relevant data in article [2], the exponential growth relationship between the competition ranking of various fungi and their moisture trade-off was obtained

$$CR = 0.2727 \cdot e^{4.848 \cdot MT} - 0.0837$$  \hspace{1cm} (4)

The values of moisture niche width $MNW$ of various fungi were obtained. After summarizing the above steps, we obtained the graph of 5 related indicators for a particular fungus.
2.2. Introducing interactions between various fungus (competition, cooperation and non-interference) into the colony

For colonies, it is assumed that

1. Under the initial conditions, the biomass \( m_i \) of all kinds of fungi was equal, and the number of fungi was \( n \).
2. There will be interaction between two fungi \((i \text{ and } j)\). The effect of interaction is that the original hyphal extension rate of each fungus will fluctuate. According to the formula (2), the decomposition rate of each fungus will be affected
   1. In the competition relationship, the fungi with high ranking in competition were less hurt.
   2. In the cooperation relationship, the fungi with low competition rank get more income.
   3. If the relationship between them is non-interference, the decomposition rate and hyphal extension rate of each fungus will not change.
3. The probability of competition, cooperation and non-interference between two fungi is 12%, 12% and 76%, respectively.

According to our hypothesis on the colony and the formula (2), the expression of describing the interaction relationship between the two types of fungi can be obtained

\[
\begin{aligned}
    v_{HE,i,j}^- &= v_{HE,i} \cdot (1 - \eta_{j-i}) \\
    v_{D,i,j}^- &\propto \ln v_{HE,j-i} \\
    v_{HE,i,j}^+ &= v_{HE,j} \cdot (1 - \eta_{i-j})' \quad j \neq i \\
    v_{D,i,j}^+ &\propto \ln v_{HE,i-j} 
\end{aligned}
\]  

(6)

For the volatility coefficient \( \eta_{j-i} \) and \( \eta_{i-j} \), we discuss as follows

1. If two fungi are in competition, and \( CR_i > CR_j \), then
   \[
   \begin{aligned}
   \eta_{j-i} &= \frac{1}{20} |CR_i - CR_j| \\
   \eta_{i-j} &= \frac{1}{10} |CR_i - CR_j| 
\end{aligned}
\]  

(7)

2. If two fungi are cooperative, and \( CR_i > CR_j \), then
   \[
   \begin{aligned}
   \eta_{j-i} &= -\frac{1}{20} |CR_i - CR_j| \\
   \eta_{i-j} &= -\frac{1}{10} |CR_i - CR_j| 
\end{aligned}
\]  

(8)

3. If two fungi do not interfere with each other, then
   \[
   \eta_{j-i} = \eta_{i-j} = 0
\]  

(9)

In summary, and according to formula (6), for a specific fungus \( i \), the modified decomposition rate \( v_{D,i}^- \) and hyphal extension rate \( v_{HE,i}^+ \) are
\[
\begin{align*}
\nu'_{HEi} &= \frac{1}{n-1} \sum_{j \neq i} \nu'_{HE,j-i} \\
\nu'_{Di} &= \frac{1}{n-1} \sum_{j \neq i} \nu'_{D,j-i}
\end{align*}
\]

(10)

For the whole colony, the weighted sum of its decomposition rate is

\[
\nu_{WSD} = \sum_{i=1}^{n} WC_i \cdot \nu'_{Di}
\]

(11)

Similarly, the weighted sum of hyphal extension rate of the whole colony can be obtained as follows

\[
\nu_{WSHE} = \sum_{i=1}^{n} WC_i \cdot \nu'_{HEi}
\]

(12)

The calculation method of weight coefficient \(WC_i\) is as follows

\[
WC_i = \frac{m_i}{M}, \quad M = \sum_{i=1}^{n} m_i
\]

(13)

Among them, the total biomass of colonies is \(M\), and the biomass of individual fungus is \(m_i\). In summary, we established a static model of fungal decomposition without considering the growth of bacterial colonies and the effects of environment and climate.

3. Dynamic model of fungal decomposition

Define the time step as \(\lambda = 1d\), and the sequence number of the time grid as \(k = 0,1,2,\ldots\). We believe that for a specific fungus \(i\), the biomass \(m_{i,k+1}\) under a specific time grid \(k\) can be calculated as follows

\[
m_{i,k+1} = m_{i,k} + \nu_{HE,i,k} \cdot \Delta t
\]

(14)

According to the formula (13), given the biomass \(m_{i,k+1}\) of various fungi \(i\), we can calculate the weight coefficient \(WC_{i,k+1}\) in the corresponding time grid \((k + 1)\). Furthermore, according to formula (11) and formula (12), the weighted sum of the decomposition rate \(\nu_{WSD,k+1}\) of the whole colony and the weighted sum of hyphal extension rate \(\nu_{WSHE,k+1}\) of the whole colony is calculated under the corresponding time grid \((k + 1)\).

According to the content in Section 2, we can define \(\nu'_{WSD,k+1}\) and \(\nu'_{WSHE,k+1}\) represent the dynamic value of the weighted sum of the decomposition rate of the whole colony and the weighted sum of hyphal extension rate after the interaction between various fungi is introduced into the colony in the corresponding time grid \((k + 1)\).

4. Fungal decomposition model (Overview)

4.1. Define and Initialize

(1) The fungal grid is defined as \(i\) and \(j\), \(n\) is the number of fungal species in the colony. Define time step \(\lambda\) and time grid \(k\), \(der\) is the number of iterations of colony over time \(t\)

\[
\begin{align*}
i &= 1,2,3,\ldots,n \\
j &= 1,2,3,\ldots,n \\
k &= 0,1,2,3,\ldots,der
\end{align*}
\]

(2) Let \(k = 0\), define \(\nu_{HE,i}\), \(\nu_{Di}\), \(MT_i\), \(CR_i\) and \(MNW_i\) are the five indexes describing specific species of fungi in the colony respectively
\[ \frac{dN}{N} = f(v_{HE,i})dv_{HE,i} \]
\[ v_{D,i} = 3.9865 \ln(8.0459 \cdot v_{HE,i}) - 0.4057 \]
\[ v_{D,i} = 7.4182 \cdot e^{1.0969 \cdot MT_i} + 0.1197 \]
\[ CR_i = 0.2727 \cdot e^{1.4848 \cdot MT_i} - 0.0837 \]
\[ MT_i = CR_i - MNW_i \]

4.2. Initialization correction
(1) Make: \( k = 0 \), define the climatic conditions \( MT^{new} \), and get the modified value of one index of fungi \( i \)

\[ CR_i^{new} = MT^{new} + MNW_i \]

(2) Let \( k = 0 \), define \( \eta_{j \rightarrow i} \) and \( \eta_{i \rightarrow j} \) are the interaction coefficient of fungi, and obtain the first revision value of the two indicators of fungi \( i \)

\[ v'_{H,i} = \frac{1}{n-1} \sum_{j \neq i} v'_{H,j \rightarrow i} \]
\[ v'_{D,i} = \frac{1}{n-1} \sum_{j \neq i} v'_{D,j \rightarrow i} \]

4.3. Iterative correction
(1) Define the temperature conditions \( T_k \) corresponding to the time grid \( k \), and get the second modification value of the two indexes of fungi \( i \)

\[ v''_{H,i,k} = v'_{H,i,k} \cdot \left[ 1 + \frac{T_k - 16^\circ C}{16^\circ C} \right] \]
\[ v''_{D,i,k} = v'_{D,i,k} \cdot \left[ 1 + \frac{T_k - 16^\circ C}{16^\circ C} \right] \]

(2) The biomass \( m_{i,k} \) and weight coefficient \( WC_{i,k} \) of a certain fungus defined as a colony corresponding to the fungus grid \( i \) and the corresponding time grid \( k \)

\[ m_{i,k+1} = m_{i,k} + v_{H,i,k} \cdot \Delta t = m_{i,k} + v_{HE,i,k} \cdot \lambda \]
\[ WC_{i,k} = \frac{m_{i,k}}{M_k}, \quad M_k = \sum_{i=1}^{n} m_{i,k} \]

4.4. Results
Corresponding to the time grid \( k \), 2 indicators of the entire colony were defined

\[ v_{WSHE,k} = \sum_{i=1}^{n} WC_{i,k} \cdot v''_{H,i,k} \]
\[ v_{WSD,k} = \sum_{i=1}^{n} WC_{i,k} \cdot v''_{D,i,k} \]

5. Conclusion
This article provides readers with a guide to mathematical modeling, and takes the 2021 American Mathematical in Contest Modeling Question A as an example to establish a fungal decomposition model. In the model, there are detailed data and formulas for readers to learn and refer to.

Here are the advantages of the mathematical model established follow the guide, clearly and accurately reflect the interaction between difference organisms, intuitively observe and predict the growth trend of organisms, specify the environmental changes to facilitate observation of their impact
on organisms (such as the impact of environment on the five major indicators in this article), ability to reveal the abstract concept of a problem, which can reveal the essence of the problem more concisely. It is convenient to use the computer to process the main variables and factors of a model, and it is easy to understand the influence of one variable on other variables.

We summarize the guide in four steps

First, extract the probability factors of the research object, secondly, analyze the research object itself, then observe the changes of the research object itself, and finally observe the impact of environmental changes on the research object. Grasp these four steps, the mathematical model will be established.

References
[1] Nicky Lustenhouwer, Daniel S. Maynard, Mark A. Bradford, Daniel L. Lindner, Brad Oberle, Amy E. Zanne, and Thomas W. Crowther, "A trait-based understanding of wood decomposition by fungi," Proceedings of the National Academy of Sciences of the United States, May 13, 2020.
[2] Daniel S. Maynard, Mark A. Bradford, Kristofer R. Covey, et al. Consistent trade-offs in fungal trait expression across broad spatial scales. 2019, 4(5):846-853.
[3] T. K. A. Kumar et al., An ontology of fungal subcellular traits. Am. J. Bot. 98, 1504–1510 (2011).
[4] P. Baldrian, Forest microbiome: Diversity, complexity and dynamics. FEMS Microbiol. Rev. 41, 109–130 (2017).