The group theory of oxidation

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Abstract

Dimensional reduction of (super-)gravity theories to 3 dimensions results in
sigma models on coset spaces $G/H$, such as the $E_8/\text{SO}(16)$ coset in the bosonic
sector of 3 dimensional maximal supergravity. The reverse process, oxidation, is the
reconstruction of a higher dimensional gravity theory from a coset sigma model. Us-
ing the group $G$ as starting point, the higher dimensional models follow essentially
from decomposition into subgroups. All equations of motion and Bianchi identities
can be directly reconstructed from the group lattice, Kaluza-Klein modifications and
Chern-Simons terms are encoded in the group structure. Manipulations of extended
Dynkin diagrams encode matter content, and (string) dualities. The reflection sym-
metry of the “magic triangle” for $E_n$ gravities, and approximate reflection symmetry
of the older “magic triangle” of supergravities in 4 dimensions, are easily understood
in this framework.

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1 Introduction

It has been appreciated for a long time that supergravity theories, or more generally, any theory of which general relativity is a subsector, are rich in algebraic structures. The best established of these are the coset symmetries that occur in theories that are related to higher dimensional theories by toroidal compactification, and subsequently truncating to the massless sector\textsuperscript{2}. The cosets are of the form $G/H$, where $G$ is a non-compact Lie group, and $H$ its maximal compact subgroup. For the maximal supergravity theories, the groups $G$ are members of the exceptional $E_{n(n)}$-series as found by Cremmer and Julia \cite{2}, while for theories with 16 supersymmetries the groups $SO(8, 8 + r)$ are found.

Other papers have found suggestive relations with, and/or made conjectures involving affine, hyperbolic or still more general Kac-Moody algebra’s (an incomplete list being \cite{3, 4, 5, 6, 7, 8}), deformed superalgebra’s \cite{9} and Borcherds superalgebra’s \cite{10}. At the moment, these structures are only partly understood.

The supergravities turn up in the low energy description of string theories, and their hypothetical non-perturbative ancestor, M-theory. An exciting conjecture suggests that discrete subgroups of the Cremmer-Julia groups are realized as exact symmetries in toroidally compactified M-theory \cite{11, 12} (for a review see \cite{13}). Even bolder conjectures have been put forward pointing out that these discrete subgroups might fit in a bigger structure, related to the infinite-dimensional algebra $E_{11}$ and suggesting that some form of $E_{11}$ is realized as a symmetry in an appropriate, to be discovered, formulation of M-theory \cite{8}.

A lot of the structure is not specific for supersymmetric theories, and we will not restrict to these. In this paper we study the well established coset theories. Our starting point will be 3 dimensional theories, where the groups involved are finite-dimensional. As demonstrated in \cite{14, 15}, it is possible to interpret these theories as dimensional reductions of higher dimensional theories. Reconstructing the higher dimensional theory from the lower dimensional one is called “oxidation”. An interesting aspect of oxidation is that, in contrast to dimensional reduction, it is not unique: there can be different “branches” leading to the same lower dimensional theory. In relevant cases, this nicely coincides with T-duality symmetries from string theory.

This paper describes some group theoretical aspects of the oxidation procedure. Not only does this give a unified point of view on the case-by-case studies of \cite{14, 15}, it also provides insights in the way in which certain symmetries leave their mark in higher dimensions. Our analysis puts some “observations” in the literature on firmer ground, and, under some reasonable assumptions, it is possible to enumerate all the possible theories that can be oxidized from a certain coset theory. As a byproduct, the discrete symmetries relating different gravity theories, grouped in so-called “magic triangles” can be explained.

\textsuperscript{2}In this paper, we will refer to this procedure as “dimensional reduction”, even though one can of course reduce over other manifolds than tori.
In this paper we will mostly restrict ourselves (as reference [15] also did) to the maximally non-compact, or split forms of simple Lie groups. In a follow-up paper [16] (see also [17]) the formalism will be extended to cover the other non-compact forms of the various simple Lie-groups.

The outline of this paper will be as follows. In section 2 we will describe general ideas. As an immediate corollary, we find a first criterion on the subgroups we are looking for. In section 3 we review dimensional reduction. In section 4 we develop a group theoretical recipe for oxidation, compare it to what is obtained by dimensional reduction, and comment on equations of motion, Bianchi identities, Kaluza-Klein modifications and Chern-Simons terms, which all follow from group theory. Section 5 is devoted to the graphical language provided by extended Dynkin diagrams: These encode the matter content of the theory, and various dualities. Section 6 explains the symmetries of two “magic” triangles found in the literature. Section 7 summarizes, and comments on some previous developments in the understanding of symmetries of (compactified) theories of gravity. Our conventions on Lie algebras are summarized in appendix A. Appendix B lists relevant subgroup decompositions for the split forms of the simple finite dimensional Lie groups. With these, and the formalism developed in the text, the reader can rederive all the results of [15], and will be able to fill in some small omissions in that paper (that were mostly filled in in [10]). Appendix C provides more relevant group theoretic information, included for easy reference.

We have listed the most relevant papers in the reference section. Many older papers can be found in [18], while reviews such as [13] and [19] list more recent references.

We conclude the introduction with a caveat (and an apology) to the mathematically inclined reader. As common in the physics literature, we will mention various groups while completely neglecting their global structure. As an example, we will not distinguish between $SO(5)$ and $Sp(2)$, but rather use the first form when the group acts on a “real” structure, and the second if it acts on a “symplectic” structure. Appendix C includes a list of groups, isomorphic up to a quotient by a subgroup of the center.

## 2 Intuitive idea

### 2.1 Symmetries in reduced gravity theories

This section reviews various, sometimes old, ideas on the coset theories resulting from compactifying gravity. Much of the discussion here can be found in [20], whereas the discussion on the representation of various forms follows [21] to some extent.

In gravity- and coset theories the concept of a vielbein, and its generalizations play an important role. A vielbein can be regarded as a local frame. A priori, this associates an
element of $GL(D, \mathbb{R})$ to each point. Vielbeins have two indices, a “curved” and a “flat” one. The curved one transforms under general coordinate transformations. An element of $GL(D, \mathbb{R})$ can be written as the product of a matrix proportional to the identity, times an element of $SL(D, \mathbb{R})$ (modulo a $\mathbb{Z}_2$ ambiguity when $n$ is even). In the following, this diagonal part decouples from the discussion, and we will further ignore it.

The flat index transforms under the transformations that takes orthonormal frames to orthonormal frames, hence $SO(D-1, 1)$. We do not want to distinguish frames related by orthogonal transformations, so we divide by these. The vielbein therefore describes the coset $SL(D, \mathbb{R})/SO(D-1, 1)$, which can be parametrized by

$$
(D^2 - 1) - \frac{D(D-1)}{2} = \frac{D(D+1)}{2} - 1
$$

parameters, which is the number of components of a symmetric tensor minus its trace: the metric. In general relativity, the metric becomes dynamical, describing the dynamics of a massless field, and the degrees of freedom for a massless field should organize in representations of $SO(D-2)$. Therefore, the cosets $SL(D-2)/SO(D-2)$ intuitively seem to represent the vielbein formalism restricted to the degrees of freedom. Moreover, upon dimensional reduction, the number of degrees of freedom stays constant, and the coset also seems to be the relevant structure for the lower dimensional theories \cite{20}.

The group $SL(n, \mathbb{R})$ is easily constructed from $SU(n)$. $SU(n)$ has a $\mathbb{Z}_2$ automorphism (inner for $n = 2$, outer for $n > 2$), that acts as complex conjugation. Of the antihermitean generators of $SU(n)$, the imaginary ones change sign, whereas the real ones are invariant. Multiplying the imaginary generators with $i$, they turn into real (but hermitian, and therefore symmetric) generators, and the resulting set of generators generates $SL(n, \mathbb{R})$.

On this real form the previous outer automorphism now acts on the algebra as transposing plus multiplication with $-1$, which leaves the anti-symmetric generators (generating the compact subgroup) invariant.

There may be other massless fields in the theory, which are classified in representations of $SL(n, \mathbb{R})$. Applying an element of $SL(n, \mathbb{R})/SO(n)$ allows one to switch between $SL(n, \mathbb{R})$ and $SO(n)$. However, after conversion to $SL(n, \mathbb{R})$, fields related by Poincaré duality transform differently. To be precise, Poincare duality precisely corresponds to the automorphism that inverts the generators of the non-compact part of the group.

The completely antisymmetric $k$-tensor of $SL(n, \mathbb{R})$ has $\binom{n}{k}$ components. Poincaré duality relates it to the antisymmetric $(n-k)$ tensor, which has $\binom{n}{n-k} = \binom{n}{k}$ components. The two have equal dimension, but transform oppositely under the non-compact part of $SL(n, \mathbb{R})$.

The various representations of $SL(n, \mathbb{R})$ will be denoted by their dimension, with a bar over the number whenever the irrep transforms oppositely under the compact generators. This conforms with notation used commonly for representations of $SU(n)$.

Non-trivial representations of $SL(n, \mathbb{R})$ are not invariant under its $\mathbb{Z}_2$ automorphism. There are two possibilities: the automorphism maps the representation to an equivalent
one, or to a different representation. An example of the first kind is the adjoint representation, \( n^2 - 1 \). Its invariant part corresponds to the adjoint of \( SO(n) \), and its non-invariant part is a real symmetric traceless matrix, identified with the degrees of freedom of the metric. All other massless bosonic fields are forms, completely antisymmetric tensors. The \( \mathbb{Z}_2 \) automorphism maps the \( k \)-form \( \binom{n}{k} \) degrees of freedom to the \( (n - k) \)-tensor \( \binom{n}{k} \), which is identified with its Poincaré dual.

The form and its dual can only represent the same degrees of freedom after projection on the \( SO \) subgroup of \( SL(n, \mathbb{R}) \). It is useful to think of the tensor and its dual as a priori independent. Projecting to \( SO(n, \mathbb{R}) \) one has to make a “choice of gauge”, and choose either the tensor or its dual to represent the relevant degrees of freedom. The other tensor is expressed in terms of the first, by means of the antisymmetric Levi-Civita tensor. In [9] the treatment of tensor and dual as independent is taken quite far; their “twisted self-duality constraint” is analogous to what we have called a choice of gauge. Later in this paper, we will associate a Bianchi identity to the tensor field, and the equation of motion to its dual. That there is no elementary distinction between the elementary and the dual form is our version of “twisted self-duality” (sometimes called a “silver rule” [21]).

Reducing a \( \tilde{D} \)-dimensional theory to 3 dimensions, we expect at least the \( SL(\tilde{D} - 3) \) symmetry of the \( \tilde{D} - 3 \)-torus. In fact the symmetry is always larger. Vectors are dual to scalars in 3 dimensions, and one finds a coset model with the full \( SL(\tilde{D} - 2) \) symmetry[9] from the transverse degrees of freedom. The symmetry can however still be larger.

In a 3-dimensional theory based on a coset \( G/H \), we expect to find an \( SL(D-2) \) subgroup in\(^3 G \). Decomposing with respect to this subgroup, the adjoint representation of \( G \) decomposes into the adjoint of \( SL(D-2) \), plus a number of other representations. If \( SL(D-2) \) is interpreted as the symmetry of the transverse degrees of freedom, the representations should be interpreted as matter fields. There are a few noteworthy points:

- The centralizer of \( SL(D-2) \) in \( G \) acts as a symmetry group on the \( SL(D-2) \) representations. We will call the centralizer the \( U \)-duality group, even though most cosets do not have a direct relation with string theory (generically they do not even allow a supersymmetric extension);

- Concentrating on the massless sectors, the irreducible representations (irreps) of \( SL(D-2) \) in the decomposition besides the adjoint should allow an interpretation as form fields. This leads to a constraint that is explained in the next subsection;

- In decomposing the adjoint representation, one finds self-conjugate, or pairs of conjugate irreps. As conjugation corresponds to Poincaré duality, a pair of conjugate representations is interpreted as a field and its dual;

\(^3\)Although in this paper we will restrict to \( G \)'s that are split, this statement and the following ones are key ingredients for the general case [16].
• Representations that are selfconjugate under $SL(\tilde{D}−2)$ require some care. There are two cases: In each there are only half the number of degrees of freedom. For $\tilde{D}−2 = 4k$ the antisymmetric ($\tilde{D}/2 −1$)-tensors can be (anti-)self-dual, and hence are interpreted as (anti-)selfdual form fields; for $\tilde{D} = 4k$, half of the fields are interpreted as the duals of the other half, but the forms are not identified with their duals\(^4\) (see also [21]). In the latter case only forms plus their duals fill complete U-duality representations (as an example: The 28 vector fields in maximal supergravity in 4 dimensions [2] fill half of the 56 of $E_{7(7)}$; their duals fill the other half).

Summarizing, the claim is that higher dimensional theories can be recovered from 3 dimensions by decomposing the 3 dimensional U-duality group into an $SL(D−2)$ group encoding gravity, times a group which is the relevant U-duality group in the higher dimension. The irreducible representations of $SL(D−2)$ are interpreted as scalars (singlets), forms (anti-symmetric tensors) and the graviton (the adjoint irrep). The rest of this paper will be devoted to making this idea more precise.

2.2 Regular subgroups of long roots

There is an immediate consequence of the previous ideas. To make sense of the recipe, we demand that the only irreps of $SL(D−2)$ that are found in the decomposition can be interpreted as one (and exactly one) graviton, a number of antisymmetric tensors (form fields), and singlets (scalars). This can be translated to a constraint on the relevant subgroups. In this section we will make use of some results in group theory, which can mostly be found in a classical paper by Dynkin [25].

Let $L_G$ be the complexification of a simple Lie algebra, and $L_{\tilde{G}}$ a subalgebra of $L_G$. If $L_G$ would be the complexification of $SL(2)$, $A_1$, the discussion below would trivialize, so we assume that this is not the case. The Cartan subalgebra of $L_{\tilde{G}}$ can be chosen such that it is embedded in the Cartan subalgebra\(^5\) of the algebra $L_G$. An immediate corollary is that the roots of $L_{\tilde{G}}$ can be expressed as linear combinations of simple roots of $L_G$. Moreover, the coefficients appearing in the expansion of the roots of $L_{\tilde{G}}$ in simple roots of $L_G$ are integers.

The Killing form provides a symmetric bilinear two-form on the root space of $L_G$. It is unique up to normalization; we normalize it such that the long roots of the algebra have length $\sqrt{2}$, and denote the resulting form as $\langle \cdot, \cdot \rangle_G$. Completely analogously there

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\(^4\)If we had discussed $SU(\tilde{D}−2)$ instead of $SL(\tilde{D}−2)$, the corresponding representations would be called real (for $\tilde{D}−2 = 4k$) and pseudoreal (for $\tilde{D} = 4k$) respectively.

\(^5\)This is true for compact forms, and for complexified algebra’s, but not in general. The discussion on the representations in this subsection does however not at all rely on the particular real form of the algebra. Hence, to avoid unnecessary complications, we have turned to the complexification. I thank B. Julia for pointing out that an earlier version of this subsection was inaccurate at this point.
exists a bilinear form on the root space of $L_\tilde{G}$. We normalize this such that the long roots of $L_\tilde{G}$ have length $\sqrt{2}$, and denote the resulting form as $\langle \cdot, \cdot \rangle_\tilde{G}$. Then on $L_\tilde{G}$ we have defined two forms, and uniqueness up to normalization implies

$$\kappa \langle \cdot, \cdot \rangle_G = \langle \cdot, \cdot \rangle_\tilde{G}$$

(1)

where $\kappa$ is a constant, that is actually a positive integer [25], and called the index of the subgroup. Obviously, we can then extend $\langle \cdot, \cdot \rangle_\tilde{G}$ to $L_G$.

Consider a root $\alpha$ of $L_\tilde{G}$. Via standard arguments, $\alpha$, $e_\alpha$ and $e_{-\alpha}$ generate an $A_1$ subalgebra of $L_\tilde{G}$. Decomposing $L_G$ with respect to this subalgebra, we need to find all weights. If $\beta$ is a root of $L_G$, then the corresponding weight of the $A_1$ subalgebra is $\langle \alpha, \beta \rangle_\tilde{G}$.

Now take $L_\tilde{G}$ to be a $A_{D-3}$ algebra (the complexification of the algebra of $SL(D-2)$). It is clear that if $\beta \neq \pm \alpha$, but instead corresponds to a weight of the adjoint or an antisymmetric tensor representation, that $\beta$ must correspond to either a zero-weight, or a weight of the 2 dimensional irrep of $A_1$. Hence we should require that

$$|\langle \alpha, \beta \rangle_\tilde{G}| = \kappa |\langle \alpha, \beta \rangle_G| < 2$$

(2)

We have chosen $\beta$ to lie on the root lattice of $L_G$, and according to [25], $\alpha$ lies also on the root lattice. As we assumed $L_G \neq A_1$ and $L_G$ simple, one can always find a $\beta$ for which $\langle \alpha, \beta \rangle_G \neq 0$. If we also require $\beta \neq \pm \alpha$, and the inequality (2), $\kappa$ must equal one.

Index 1 subgroups are special. They must be regular, which means that the root lattice of the subgroup can be chosen to be a sublattice of the lattice of the original group. All regular subgroups of a given group can be found by a procedure described by Dynkin [25], which we will explain in section 5.

If the group is simply laced, all regular embeddings have index 1. If the group is non-simply laced, then regular subgroups involving only short roots are possible. Such subgroups have an index bigger than one, and hence are excluded.

The conclusion is that regular $SL$ subgroups of long roots are the appropriate mathematical structure for encoding the graviton in the theories under study. Regularity of subgroups is a criterion that has been observed before [20], and is implicit in many discussions on algebraic aspects of compactified gravity. Here we have justified regularity as a consequence of reasonable assumptions, instead of observing it to be obeyed by the available data.

The requirement of long roots is correlated to the observation in the literature that what is referred to as group disintegration has to start at one end of the Dynkin diagram [3]. We postpone a more detailed discussion to section 5.
3 Decreasing the number of dimensions: Reduction

The claim, to be made precise in the next 2 sections, is that the oxidation process is completely determined by group theory. We will now review a systematic procedure for dimensional reduction developed in [22, 23, 15], and emphasize elements that are significant to our discussion. In the next section we will develop our recipe for oxidation, and demonstrate that it is precisely inverse to dimensional reduction.

3.1 Dimensional reduction

In [22, 23, 15] dimensional reduction is developed as an inductive scheme, by first reducing over 1 dimension, then over a second etc. We will use the possibility of rotations on the scalar manifold to simplify some expressions. We will not use some symbols defined in [22, 23, 15], and on the other hand introduce symbols that do have a direct significance to our subsequent discussion.

We reduce a theory in $\tilde{D}$ dimensions, over $n$ toroidal dimensions, to a $D$ dimensional theory. Of course $\tilde{D} = n + D$, but we will keep all symbols to simplify expressions. First we introduce some definitions. Consider the vector space $\mathbb{R}^n$, with a basis of unit vectors $e_i$. For explicit computations one can choose $(e_i)^k = \delta^k_i$. First define

$$ S = \sum_{i=1}^{n} e_i $$

In the dimensional reduction procedure, we will extensively need the vectors

$$ s = \frac{S}{\sqrt{(\tilde{D} - 2)(D - 2)}} \quad f_i = \sqrt{\frac{D - 2}{\tilde{D} - 2}} \left( \frac{S}{n} \right) + f_i^\perp $$

where in turn

$$ f_i^\perp = e_i - \frac{S}{n} \quad \rightarrow \quad \langle s, f_i^\perp \rangle = 0 $$

These vectors obey

$$ \langle s, s \rangle = \frac{n}{(\tilde{D} - 2)(D - 2)} \quad \langle s, f_i \rangle = \frac{1}{D - 2} \quad \langle f_i, f_j \rangle = \delta_{ij} + \frac{1}{D - 2}. $$

Up to a factor of 2, these relations are also satisfied by the vectors $s$ and $f_i$ defined in [15], so the vectors appearing here may be identified with the vectors in [15] (up to rescaling with $\sqrt{2}$ which we absorb in the dilaton), as they can be rotated into each other by an $O(n)$ rotation.
The reason for these conventions is that the groups \( SL(n) \) (their discrete versions \( SL(n, \mathbb{Z}) \) describing the symmetry group of the compactification torus) play a prominent role. The \( f_j^\perp \) form a non-orthogonal basis of the \((n-1)\) dimensional subspace of \( \mathbb{R}^n \) orthogonal to \( s \). There are \( n \) vectors \( f_j^\perp \), so this basis is overcomplete. The combinations

\[
    f_i - f_j = f_i^\perp - f_j^\perp = e_i - e_j
\]

span an \((n-1)\)-dimensional lattice; this is a well-known representation of the root lattice of \( SL(n) \) (\cite{15} defines the Dynkin diagram with the above vectors). The \( f_j^\perp \) are now easily interpreted: From the innerproducts of the root vectors of \( SL(n) \) with the \( f_j^\perp \) one finds that they span the weight lattice of \( SL(n) \). The \( f_j^\perp \) form the weights of the \( n \)-dimensional, fundamental representation. The weights encode the possible eigenvalues of the generators, and the relation \( \sum_j f_j^\perp = 0 \) expresses tracelessness of the generators of the algebra of the \( n \)-dimensional representation.

The Kaluza-Klein metric ansatz is

\[
    ds_D^2 = e^{(s,\phi)} ds_D^2 + e^{-\frac{D-2}{n}(s,\phi)} \sum_{i=1}^n e^{-\langle f_i^\perp,\phi \rangle} (h_i)^2
\]

with

\[
    h_j = \tilde{\gamma}^i_j (d z_i + \hat{A}_i), \quad \tilde{\gamma}^i_j = \delta^i_j + A_i^{(0)j}
\]

The matrix \( A_i^{(0)j} \) has non-zero entries only for \( i > j \). In spite of its different appearance the metric eq. (8) is nothing but a rewriting of eq. (A.2) in \cite{15}, to exhibit the connection with group theory. Having claimed that the metric degrees of freedom take values in the vielbein \( SL(D-2)/SO(D-2) \), we expect that reducing the vielbein, we have to decompose \((\mathbb{R} \cong SO(1,1) \) is the unique 1-dimensional non-compact group, see appendix \cite{C} \)

\[
    SL(D-2) \rightarrow SL(D-2) \times SL(n) \times \mathbb{R}.
\]

Identifying \( SL(D-2) \) with the lower dimensional vielbein, \( SL(n) \) is represented in eq. (8) by the appearance of the weights \( f_i^\perp \). Factors corresponding to the remaining \( \mathbb{R} \) factor have been split off. Commuting with \( SL(n,\mathbb{R}) \) means that its representation vector is orthogonal to the \( f_i^\perp \), hence proportional to \( s \). Tracelessness of the \( \mathbb{R} \) generator fixes the ratio between the exponent of the factor in front of the \( D \) dimensional metric, and the one before the compact metric, to \(- (D-2)/n \). The overall values are fixed by the normalization of the dilaton kinetic terms in the Lagrangian.

With respect to the \( \tilde{\gamma}^i_j \) we note that, after having set the vectors \( \hat{A}_i = 0 \), they are part of a vielbein on the compact space. The metric on the compact space is given by

\[
    ds_n^2 = \exp(-\langle f_i^\perp,\phi \rangle) \delta_{ij} \tilde{\gamma}^i_k \tilde{\gamma}^j_l \, dz^k \otimes dz^l.
\]

As in \cite{22,23,15} we also define the inverse of \( \gamma \) by \( \gamma = \tilde{\gamma}^{-1} \).
The Lagrangian in $\tilde{D}$ dimensions is
\[ L_{\tilde{D}} = R \ast 1 - \frac{1}{2} F_{(p)} \wedge F_{(p)} \] (10)

After reduction on the $n$-torus, the Lagrangian in $D$ dimensions becomes
\[ L_D = R \ast 1 - \frac{1}{2} \left( \ast d\phi \right) - \frac{1}{2} e^{(a_1, \phi)} \ast F_{(p)} \wedge F_{(p)} \\
- \frac{1}{2} \sum_i e^{(a_i, \phi)} F_{(p-1)i} \wedge F_{(p-1)i} - \frac{1}{2} \sum_{i,j} e^{(a_{ij}, \phi)} F_{(p-2)ij} \wedge F_{(p-2)ij} - \ldots \] (11)

We have followed Cremmer et al. [15] and defined the following dilaton vectors in (11)
\[ a = - (p - 1)s \quad a_{i_1 \ldots i_p} = \sum_{k=1}^p f_{i_k} - (p - 1)s \quad b_i = - f_i \quad b_{ij} = f_j - f_i \] (12)

In case one already has a dilaton and dilaton couplings in the top dimension, one simply forms direct sums of the dilaton vector before reduction, with the above dilaton vectors, and couples these to $\tilde{D}$-dimensional dilatons and metric scalars in the obvious way.

Noting that
\[ \sum_{k=1}^n f_k = (\tilde{D} - 2)s \] (13)
we see
\[ a = - \left( \sum_{k=1}^n f_k - (\tilde{D} - p - 1)s \right); \quad a_i = - \left( \sum_{k \neq i} f_k - (\tilde{D} - p - 1)s \right); \quad \text{etc.} \]

These identities are a consequence of Poincaré duality in dimensionally reduced theories: Reducing a form, its dilaton prefactor is the inverse of the one found by reducing the Poincaré dual form precisely over those dimensions that the form was not reduced over [24]. Kaluza-Klein vectors coming from the metric obviously do not follow this pattern.

The field strengths appearing in (11) are defined as follows:
\[ F_{(q)i_1 \ldots i_{p-q}} = \gamma^1_{i_1} \ldots \gamma^{p-q}_{i_{p-q}} \hat{F}_{(q)i_1 \ldots i_{p-q}} \quad F_{(2)}^i = \gamma^i_{j} \hat{F}_{(2)}^j \quad F_{(1)}^{ij} = \gamma^k_{j} \hat{F}_{(1)}^{ik} \] (14)
with the hatted field strengths defined as
\[ \hat{F}_{(p)} = dA_{(p-1)} - dA_{(p-2)i} \hat{A}_{(1)}^i + \frac{1}{2} dA_{(p-3)ij} \hat{A}_{(1)}^i \hat{A}_{(1)}^j - \frac{1}{6} dA_{(p-4)ijk} \hat{A}_{(1)}^i \hat{A}_{(1)}^j \hat{A}_{(1)}^k \ldots \]
\[
\hat{F}_{(p-1)i} = dA_{(p-2)i} + dA_{(p-3)ij} \hat{A}_{(1)j} + \frac{1}{2} dA_{(p-4)ijk} \hat{A}_{(1)k} \\
+ \frac{1}{6} dA_{(p-5)ijkl} \hat{A}_{(1)j} \hat{A}_{(1)k} \hat{A}_{(1)l} \ldots \\
\vdots \\
\hat{F}_{(1)i_1 \ldots i_{p-1}} = dA_{(0)i_1 \ldots i_{p-1}} \\
\hat{F}_{(2)} = d\hat{A}_{(1)} \\
\hat{F}_{(1)j} = d\hat{A}_{(0)j}
\] 

(15)

The hatted field strengths are solutions to the identity

\[
d\hat{F}_{(p-q)i_1 \ldots i_q} = (-)^p \sum_j \hat{F}_{(p-q-1)i_1 \ldots i_q j} \hat{F}_j^{(2)}
\] 

(16)

Using \(d\hat{\gamma}^i_j = \hat{F}_{(1)j}^i\) and \(d\gamma^i_j = -\mathcal{F}^k_{(1)j} \gamma^i_k\) one finds:

\[
dF_{(p-q)i_1 \ldots i_q} = - \sum_k \mathcal{F}^k_{(1)ij} \wedge F_{(p-q)ij \ldots i(q)} + (-)^p \sum_j F_{(p-q-1)i_1 \ldots i_q j} \wedge \mathcal{F}_j^{(2)}; \\
d\mathcal{F}_{(2)} = \mathcal{F}_{(1)j}^{i} \mathcal{F}_{(2)}^{ij}; \\
d\mathcal{F}_{(1)j} = \mathcal{F}_{(1)k}^{i} \mathcal{F}_{(1)j}^{k}
\] 

(17) (18)

These important identities are the Bianchi identities for the dimensionally reduced theory. They form a vital clue to our discussion in section 4. There we will use a different bookkeeping for the indices.

With the reduced Lagrangian given we may dualize a certain number of fields, to reach the formulation of the theory we desire. Note that once a form is fully reduced, the length of its dilaton vector takes a value independent of \(D\) (but not of \(\tilde{D}\))

\[
| \sum_{k=1}^p f_{ik} - (p - 1)s |^2 = \frac{(p - 1)(\tilde{D} - p - 1)}{\tilde{D} - 2}
\] 

(19)

Notice the symmetry under \(p \leftrightarrow \tilde{D} - p\). This result hints at the significance of 3-forms in 11-dimensions, and 4-forms in 10 dimensions, as then a fully reduced form can play the role of a root of a group lattice (having length squared 2). Other significant combinations are 2 forms in 6 dimensions, and 1-forms in 5 dimensions, possibly giving rise to short roots of groups. Indeed, all these cases are realized in the \(E_8\), \(E_7\), \(B_3\) and \(G_2\) models, and give rise to a Freudenthal-like construction of the groups as the composition of an \(SL\) group with some of its representations. All have supersymmetric extensions, except for the \(E_7\) theory. The latter can however be viewed as a truncation of IIB supergravity.

### 3.2 Projection to sublattices

Reducing gravity from \(\tilde{D}\) to 3 dimensions gives a \(SL(\tilde{D} - 2)/SO(\tilde{D} - 2)\) coset. So suppose that we are given a coset \(G/H\) parametrizing a scalar theory coupled to gravity.
in 3 dimensions. If this can be oxidized, part of the coset must be a reduced gravity theory, and hence we expect $SL(\tilde{D} - 2)$ to be contained in $G$, for some value(s) of $\tilde{D}$. The first step in oxidation consist of identifying the $SL(\tilde{D} - 2)$ sublattice (recall that we are dealing with a regular subgroup) in the lattice spanned by the dilaton vectors. Having found such a sublattice, we select a sublattice of this sublattice corresponding to $SL(\tilde{D} - 3)$. As, in a sense, a $SL(\tilde{D} - 3)$ group is geometrical, coming from the symmetry of the torus, we can regard this as recovering the geometry. There are $\tilde{D} - 3$ positive roots of $SL(\tilde{D} - 2)$ that are not roots of $SL(\tilde{D} - 3)$. We label them $g_i$, $i = 1, \ldots, \tilde{D} - 3$, and order them, $g_1$ being the highest root in the set, $g_2$ the highest except for $g_1$ etc. From the $g_i$ we can build all the positive roots of $SL(\tilde{D} - 3)$: they have the form $g_i - g_j$, with $i < j$. Hence, at least in 3 dimensions, we can identify $g_i = f_i$. We can then also find a vector $t$, defined by $\sum g_i = (\tilde{D} - 2)t$. Of course, in 3 dimensions $t = s$.

The vectors $g_i$ and $(g_i - g_j)$ form the positive roots of the $SL(\tilde{D} - 2)$ lattice. We now wish to oxidize to $D$ dimensions, and according to our earlier story, we have to decompose $G$ in an $SL(D - 2)$ group plus complement. Without loss of generality, we can assume that $SL(D - 2)$ has as simple roots $g_{D-3}$, and if $D > 4$ the roots $g_{D-3-k} - g_{D-2-k}$, with $1 \leq k \leq D - 4$. The orthogonal complement to span$(g_{D-3}, g_{D-4}, \ldots)$ then gives the U-duality group in $D$ dimensions. We will give the results of the projection.

The projection of $g_i$ (where we assume that $i$ is not in the set $\tilde{D} - 3, \tilde{D} - 4, \tilde{D} - 5, \ldots$) we denote by $g_{P,i}$, the projection of $t$ by $t_P$. They are given by

$$g_{P,i} = g_i - \frac{1}{D - 2} \sum_{k=0}^{D-4} g_{D-3-k} t_P = t_P - \frac{1}{D - 2} \sum_{k=0}^{D-4} g_{D-3-k}$$ (20)

One easily verifies that

$$\langle g_{P,i}, g_{D-3-k} \rangle = 0 \quad \langle t_P, g_{D-3-k} \rangle = 0 \quad 0 \leq k \leq D - 4$$ (21)

More crucial is that these vectors obey

$$\langle t_P, t_P \rangle = \frac{n}{(D - 2)(D - 2)} \quad \langle t_P, g_{P,i} \rangle = \frac{1}{D - 2} \quad \langle g_{P,i}, g_{P,j} \rangle = \delta_{ij} + \frac{1}{D - 2}$$ (22)

Comparing these equations with (13) we see that the $g_{P,i}$ form a set of $n$ independent vectors, and having precisely the same innerproducts as the $f_i$ appropriate for $D$ dimensions. Hence the $n$ dimensional subspace spanned by the $g_{P,i}$ and the $\mathbb{R}^n$ spanned by the $f_i$ can be rotated into each other, such that the $g_{P,i}$ and $f_i$, and $t_P$ and $s$, coincide. We can therefore switch to previous notation, and denote the vectors again by $f_i$ and $s$.

This demonstrates how to recover the lattice of dilaton vectors for the higher dimensional theory from the coset in 3 dimensions. In the next section, we will recover the full theory.
4 Increasing the number of dimensions: Oxidation

With the formalism of \([22, 23, 15]\), reviewed in the previous subsection, one can construct, starting in the maximal dimension, the lower dimensional reduced theories. This section starts at the other end: Suppose we do have a certain scalar coset in 3 dimensions, how to reconstruct the higher dimensional theories?

4.1 Coset sigma models

To come to grips with the full structure, we first study the relevant coset sigma models. Given a group \(G\) with maximal compact subgroup \(H\), there are 2 convenient forms for an action. Consider a group element \(V \in G\). We form the invariant “metric” \(\mathcal{M} = V^\# V\).

The sigma model Lagrangian can be written as:

\[
L_{G/H} = \frac{1}{4} e \text{tr}(\partial \mathcal{M}^{-1} \partial \mathcal{M}) \tag{23}
\]

\[
= -e \text{tr} \left( (\partial V) V^{-1} \frac{1}{2} (1 + T)(\partial V) V^{-1} \right) \tag{24}
\]

where we have introduced the operator \(T\), acting as \(T(a) = a^\#\). We use the superscript \(^\#\) for “generalized transpose”, as in [23]. We have included a discussion in our appendix A. Because \((a^\#)^\# = a\), the expression \(\frac{1}{2} (1 + T)\) represents a projection operator. The first form of the action (23) features the metric \(\mathcal{M}\), obviously constant under the action of \(H\), while covariant under \(G\). We will focus on the second form of the action (24), featuring \((\partial V) V^{-1}\) which is invariant under global \(G\) transformations, and transforms as a connection 1-form under local \(H\) transformations.

The 1-form \((\partial V) V^{-1}\) is a tangent form on group space, and hence it can be expressed in the group generators \((r\) denoting the (real) rank)

\[
(\partial V) V^{-1} = \frac{1}{2} \sum_{i=1}^{r} \text{d}\phi^i H^i + \sum_{\alpha \in \Delta^+} e^{\frac{1}{2}(\alpha, \phi)} F_{(1)\alpha} E_\alpha \tag{25}
\]

The second sum runs only over the set of positive roots \(\Delta^+\), because by virtue of the Iwasawa decomposition, we can always use an \(H\) transformation to set terms with negative roots to zero. Henceforth, we will work in this positive root gauge. One can then decompose \(V = V_H V_E\), with \(V_H = \exp(\frac{1}{2} \phi \cdot H)\) an exponentiated element of the Cartan subalgebra, and \(V_E\) generated by the ladder operators \(E_\alpha\) associated to the positive roots. Then

\[
(\partial V) V^{-1} = d(V_H) V_H^{-1} + V_H d(V_E) V_E^{-1} V_H^{-1}
\]

which is the reason for the appearance of the exponential prefactors in (25).
Inserting \(25\) in the action \(24\), we obtain (with the wedge product understood in the first term)

\[
L_{G/H} = -\langle *d\phi, d\phi \rangle - \frac{1}{2} \sum_{\alpha \in \Delta^+} e^{(\alpha,\phi)} * F_{(1)\alpha} \wedge F_{(1)\alpha}
\]  

(26)

The \(F_{(1)\alpha}\) are one forms. To get more detail on these, we take the derivative

\[
d((dV)V^{-1}) = -(dV) \wedge (dV^{-1}) = ((dV)V^{-1}) \wedge ((dV)V^{-1})
\]

Substituting, one finds

\[
\sum_{\alpha \in \Delta^+} d(e^{\frac{1}{2}(\alpha,\phi)} F_{(1)\alpha}) E_{\alpha} = \frac{1}{8} \sum_{i,j} d\phi_i \wedge d\phi_j [H_i, H_j] + \frac{1}{2} \sum_{\alpha \in \Delta^+} e^{\frac{1}{2}(\alpha,\phi)} d\phi_i \wedge F_{(1)\alpha} [H_i, E_{\alpha}]
\]

\[
+ \frac{1}{2} \sum_{\beta,\gamma \in \Delta^+} e^{\frac{1}{2}(\beta+\gamma,\phi)} F_{(1)\beta} \wedge F_{(1)\gamma} [E_{\beta}, E_{\gamma}]
\]  

(27)

Substituting the commutators, working out the differentials, and checking per component (by multiplying each side with \(E_{-\alpha}\) for some \(\alpha\), and taking the trace) one finds

\[
dF_{(1)\gamma} = \frac{1}{2} \sum_{\alpha,\beta} N_{\alpha,\beta} F_{(1)\alpha} \wedge F_{(1)\beta} * = \begin{cases} \alpha, \beta, \gamma \in \Delta^+ \\ \alpha + \beta = \gamma \end{cases}
\]  

(28)

The right hand side of this equation is symmetric in \(\alpha\) and \(\beta\): interchanging the two gives a minus sign from the forms, and another from \(N_{\alpha,\beta}\). Because of the symmetry, each term occurs 2 times; the factor of \(\frac{1}{2}\) cancels this.

These Bianchi identities are an important clue. We want to arrive at equations that do not explicitly mention \(V\) (which is not an invariant object). The Bianchi identities are an alternative way of expressing that the right hand side of \(25\) can be written as its left hand side, that is, as a tangent form on \(G/H\); they form a set of integrability conditions that ensure this. As such, they encode group theoretic information, previously stored in \(V\). There is a field \(F_{(1)\alpha}\) for every positive root \(\alpha\), the structure of \(25\) is determined from the geometry of the root lattice of \(G\), and the structure constants \(N_{\alpha,\beta}\) appear. Note that the positive roots of lowest height, the simple roots, satisfy standard Bianchi identities.

The structure becomes even nicer when also considering the equations of motion for the scalar coset. There are two ways of obtaining these from the action \(26\). The standard way would be to regard the equations \(28\) as Bianchi identities, solve for \(F_{(1)\alpha}\) in terms of potentials from these, and then derive the equations of motion from \(26\) with the solutions for the \(F_{(1)\alpha}\) inserted. This is fairly tedious if \(\text{dim } G\) is large, and hides some of the nice covariance properties that will show up in the following.
Instead we regard the $F_{(1)\alpha}$ as independent fields, and enforce (28) by Lagrange multipliers (which will be $(D-2)$-forms). Therefore we add to the Lagrangian

$$L_{\text{Bianchi}} = \sum_{\alpha \in \Delta^+} \left( dF_{(1)\alpha} - \sum_{\alpha = \beta + \gamma} N_{\beta,\gamma} F_{(1)\beta} \wedge F_{(1)\gamma} \right) \wedge A_{(D-2)-\alpha}$$  \hspace{1cm} (29)$$

The labels $-\alpha$ appearing on the $(D-2)$-forms will turn out meaningful.

Varying with respect to $A_{(D-2)-\alpha}$ will return the Bianchi identities. Varying with respect to $F_{(1)\gamma}$ we find the equations

$$F_{(D-1)-\gamma} \equiv e^{(\gamma,\phi)} \ast F_{(1)\gamma} = dA_{(D-2)-\gamma} - \sum_{\beta - \alpha = -\gamma} N_{\beta,-\alpha} F_{(1)\beta} \wedge A_{(D-2)-\alpha}$$  \hspace{1cm} (30)$$

We defined $F_{(D-1)-\alpha}$, and used the identity (30) for the structure constants. In the end, we are not interested in the dual potential $A_{(D-2)-\alpha}$, but in the field strengths $F_{(D-1)-\alpha}$. Taking the derivatives of these (note that in general $dF_{(1)\beta} \neq 0$), one finds (see appendix D for details of the derivation)

$$dF_{(D-1)-\gamma} = \sum_{\ast} N_{\alpha,-\beta} F_{(1)\alpha} \wedge F_{(D-1)-\beta} \ast = \begin{cases} \alpha, \beta, \gamma \in \Delta^+ \\ \alpha - \beta = -\gamma \end{cases}$$  \hspace{1cm} (31)$$

This should be compared to (28). Notice the absence of a factor $\frac{1}{2}$.

In terms of $F_{(D-1)-\alpha}$, the Lagrangian can be rewritten to

$$L_{G/H} = -\langle \ast d\phi, d\phi \rangle - \frac{1}{2} \sum_{\alpha \in \Delta^+} F_{(D-1)-\alpha} \wedge F_{(1)\alpha},$$  \hspace{1cm} (32)$$

from which one finds the equation of motion for $\phi$

$$2d(\ast d\phi^i) = \sum_{\alpha \in \Delta^+} \alpha^i F_{(D-1)-\alpha} \wedge F_{(1)\alpha}$$  \hspace{1cm} (33)$$

Note that with (33), (28) and (31) we have established a one-to-one relation between the conventional basis of the Lie-algebra of $G$, and the equations relevant to the coset sigma model. For every $E_\alpha$ where $\alpha$ is a positive root, we have a Bianchi identity from (28), when $\alpha$ is a negative root we have an equation of motion from (31), while the Cartan subalgebra determines the equations for $\phi$. In particular, the Cartan subalgebra only gives us rank $G$ equations, because the “Bianchi identity” for the potential for $\phi$, $d^2\phi = 0$ is trivial. This is a marked difference between our philosophy, and the one from [9]. For the special case of the coset manifolds found from compactifying 11-d supergravity the equations for the “double” of the fields as found in [9] coincide with our field equations.
**4.2 Coupling to other fields**

Coupling to other forms \( F_{(n)} \) is most easily done by adding quadratic terms to the action. If the \( F_{(n)} \) form a non-trivial representation of U-duality, we need to contract on the internal metric to render the action U-duality invariant. This takes the form

\[
L_m = \frac{1}{2} * F_{(n)} \wedge \mathcal{M} F_{(n)}
\]  

(34)

We note that \( \mathcal{M} \) (or rather \( \mathcal{V} \)) should be in an appropriate representation. We will not discuss the possibility of Chern-Simons modifications in this subsection, as these will appear naturally in our discussion later.

The equation of motion for \( F_{(n)} \) then becomes

\[
d(\mathcal{M} * F_{(n)}) = 0
\]  

(35)

We find it more convenient to substitute \( M = \mathcal{V}^\# \mathcal{V} \) in this equation, and rewrite it to

\[
(d + (d \mathcal{V} \mathcal{V}^{-1})^\#) \mathcal{V} * F_{(n)} = 0
\]  

(36)

The expression between the brackets is a covariant derivative, ensuring that the equation of motion is covariant under the compact part of the U-duality group, which is a local symmetry. We rewrite the standard Bianchi identity \( dF_{(n)} = 0 \) to

\[
(d - (d \mathcal{V} \mathcal{V}^{-1})) \mathcal{V} F_{(n)} = 0
\]  

(37)

If the fields \( *F_{(n)} \) and \( F_{(n)} \) represent the same degrees of freedom, we cannot allow them to transform differently. Hence, only local transformations \( O \) with \( O^\#O = 1 \) are allowed, and this is the restriction to the compact subgroup that was imposed already.

Both \( \mathcal{V}F_{(n)} \) and \( \mathcal{V}*F_{(n)} \) represent a full \( G \) multiplet of fields. The components of a multiplet can be labelled by their weights, hence we decompose into components by writing

\[
\mathcal{V}F_{(n)} \equiv \sum_{\lambda \in \Lambda} e^{\frac{1}{2} \langle \lambda, \phi \rangle} F_{(n)\lambda},
\]  

(38)

where the sum runs over the weights \( \lambda \) on the weight lattice \( \Lambda \) of the representation.

One can work out the Bianchi identity (37) by inserting \( (d \mathcal{V}) \mathcal{V}^{-1} \) from (25), to obtain

\[
dF_{(n)\lambda'} = \sum_{\ast} N_{\alpha, \lambda} F_{(1)\alpha} \wedge F_{(n)\lambda} \ast = \left\{ \begin{array}{ll}
\lambda, \lambda' \in \Lambda \\
\alpha \in \Delta^+ \\
\alpha + \lambda = \lambda'
\end{array} \right.
\]  

(39)

The constants \( N_{\alpha, \lambda} \) can be computed. We will not need them explicitly; when finding expressions like (39) in the future, we will find that the constants are already determined in the derivation.
The equation of motion (36) can be rewritten similarly. In this case, the covariant derivative acts on the dual form, which belongs to a different, conjugate $G$ representation. The weights for the conjugate representation are the negatives of the weights of the representation, so we write

$$V \ast F_{(n)} = \sum_{-\lambda \in \Lambda} e^{-\frac{1}{2} \langle \lambda, \phi \rangle} F_{(D-n)-\lambda},$$  \hspace{1cm} (40)$$

where we denote the weight lattice of the conjugate representation by $\Lambda$. Again the reader should note that the forms of degree $D-n$ are not the duals to $F_{(n)}$. Rather,

$$e^{-\frac{1}{2} \langle \lambda, \phi \rangle} \ast F_{(D-n)-\lambda} = e^{\frac{1}{2} \langle \lambda, \phi \rangle} F_{(n)}\lambda,$$  \hspace{1cm} (41)$$

The equation of motion becomes

$$dF_{(D-n)-\lambda'} = \sum_* N_{\alpha,-\lambda} F_{(1)\alpha} \wedge F_{(D-n)-\lambda} \ast = \begin{cases} \lambda, \lambda' \in \Lambda \\ \alpha \in \Delta^+ \\ \alpha - \lambda = -\lambda' \end{cases}$$  \hspace{1cm} (42)$$

It can happen that form and dual form transform in a self-conjugate representation; in theories with self-dual tensors they must be in such a representation. In that case the equation of motion (42) and Bianchi identity (39) are essentially the same equation, and we can consistently impose self-duality.

The Lagrangian for coupled matter becomes

$$L_m = -\frac{1}{2} \sum_{\lambda \in \Lambda} F_{(D-n)-\lambda} \wedge F_{(n)\lambda}$$  \hspace{1cm} (43)$$

The sum over $\lambda$ indicates a sum over the weights of the representation. Again, we get one Bianchi identity, and one equation of motion, labelled by $\lambda$ resp. $-\lambda$. For self-dual representations constraint equation and Bianchi identity imply each other, but since then $\lambda$ and $-\lambda$ belong to the same representation we precisely get as many equations as weights.

With extra matter, the equation of motion for the dilatonic scalars (33) is modified to

$$2d(\ast d\phi^i) = \sum_{\alpha \in \Delta^+} \alpha^i F_{(D-1)-\alpha} \wedge F_{(1)\alpha} + \sum_{\lambda \in \Lambda} \lambda^i F_{(D-n)-\lambda} \wedge F_{(n)\lambda},$$  \hspace{1cm} (44)$$

while (31) is not modified. Note that singlet representations of U-duality do not couple to $\phi$ (as for these, $\lambda = 0$).

For the non-dilatonic fields we have used roots and weights as labels. In the action, combinations of forms always occur such that: the degrees sum up to $D$; their labels sum up to zero. In equations we find also that various terms have to have same degrees, but also that their labels sum up to the same vector, which is either a root or a weight.
These properties are easily traced back to symmetries. The rule on addition of forms is implied by Lorentz symmetry in the non-compact directions. The rule for addition of the vector and weight labels follows from the U-duality group. The theory is invariant under $\mathcal{V} \rightarrow \mathcal{V}U$, with $U$ a constant element of $G$. With the expansion of $d\mathcal{V}V^{-1}$, most of these symmetries became implicit. An exception is when $U$ is an element obtained by exponentiating an element of the Cartan sub-algebra, then we have

$$\phi \rightarrow \phi + \zeta; \quad F_{(n)\xi} \rightarrow e^{-\frac{1}{2} \langle \xi, \zeta \rangle} F_{(n)\xi},$$

regardless of whether $\xi$ is a weight or a root. Covariance of the equations of motion and Bianchi identities therefore requires all the labels of various terms to add up on both sides. The exponential factors included in the definitions of fields labelled by negative roots and conjugate weights ensure the transformation behavior implied by their labels.

4.3 Oxidizing

We now know the relevant equations for a $G/H$ scalar coset theory coupled to gravity in 3 dimensions. We shall not consider the addition of other scalar fields. The equations are then given by the Einstein equation, and the equations and Bianchi identities (28), (31) and (33).

We will now (re)-construct the higher dimensional theories. In section 3.2 we recovered the lattice of the higher dimensional theory, by choosing an $SL(D-2)$-sublattice, and decomposing with respect to this lattice. The group $G$ decomposes into $SL(D-2) \times U_D$. The representations in the decomposition are at least the adjoints of $SL(D-2)$ and of $U_D$, but in general there are more irreps. Because $SL(D-2)$ is a level 1 subgroup, there is only one adjoint irrep of $SL(D-2)$ and the other irreps are antisymmetric tensors.

We choose a basis of $D-3$ simple roots $\alpha_i$ for the $SL(D-2)$-lattice (which is a sublattice of the $G$-lattice). We order the indices of the $\alpha_i$ along a direction of the Dynkin diagram of $SL(D-2)$ (which is a single chain of nodes). We then have:

$$\langle \alpha_i, \alpha_i \rangle = 2, \quad \langle \alpha_i, \alpha_{i+1} \rangle = \langle \alpha_{i+1}, \alpha_i \rangle = -1, \quad \langle \alpha_i, \alpha_j \rangle = 0 \text{ otherwise.}$$

The $D-3$ fundamental weights $\lambda_j$ are defined by restricting to the space spanned by the $SL(D-2)$ root lattice, and demanding

$$2 \frac{\langle \alpha_i, \lambda_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \langle \alpha_i, \lambda_j \rangle = \delta_{ij},$$

where we used that $SL(D-2)$ is simply laced. The $\lambda_j$ are highest weights for the antisymmetric $j$-tensor representation of $SL(D-2)$. The statement that the irreps of $SL(D-2)$

\footnote{But note that this is not an irrelevant possibility: Reduction of a generic theory with gravity and matter results in a $SL(n)/SO(n)$ theory with additional scalars in representations of $SL(n)$. The semi-simple Lie groups discussed here require fine tuning of dilaton factors and Chern-Simons terms.}
other than the adjoint are forms, means that among their weights there is a highest
weight, which is either 0 for a singlet, or \( \lambda_j \) for some value of \( j \). We associate a form
with degree \( j + 1 \) to the representation with highest weight \( \lambda_j \). The singlets transform in
the adjoint of \( U_D \). For the semi-simple part of \( U_D \) we choose a basis of positive roots, for
Abelian factors a positive direction. To the positive roots of the semi-simple part of \( U_D \)
we associate forms of degree 1, to the negative roots forms of degree \( D - 1 \) (the Cartan
generators, and non semi-simple parts correspond to the dilatons).

The equations for the oxidized theory are found as follows. The roots of the \( SL(D - 2) \)
subgroup represent the graviton; in the oxidized theory, it is encoded in the Einstein
equation. For the other roots \( \alpha \), we associate to \( \alpha \) the field \( F_{(n)\alpha'} \), where \( n \) corresponds to
the degree defined previously, and \( \alpha' \) is the projection of the root \( \alpha \) to the plane orthogonal
to the \( SL(D - 2) \) lattice (it defines a “representation vector” for \( U_D \); if \( U_D \) is semisimple
it is a weight). There will be \( \binom{D - 2}{n - 1} \) fields \( F_{(n)\alpha'} \), the dimension of the antisymmetric
\( n - 1 \)-tensor irrep of \( SL(D - 2) \). If \( n \neq 1, D - 1 \), this number is bigger than 1, and these
multiple fields give equivalent descriptions of the same field. In the oxidized theories, they
correspond to multiple degrees of freedom grouped in a single \( SL(D - 2) \) tensor.

The equations for the forms, and the non-dilatonic scalars then become

\[
\frac{dF_{(n)\alpha'}}{2} = \sum \eta_{l,\beta;m,\gamma} N_{\beta,\gamma} F_{(l)\beta'} \wedge F_{(m)\gamma'} \quad * = \left\{ \begin{array}{c} l + m = n + 1 \\ \alpha' + \beta' = \gamma' \end{array} \right.,
\]

(47)

We sum over all combinations such that the sum of the degrees and the labels of the right
hand side match with degree and label on the left hand side. The constants \( \eta_{l,\beta;m,\gamma} \) are
sign factors: \( N_{\alpha,\beta} \) is antisymmetric, but the combination of the 2 forms is not necessarily,
and the insertion of \( \eta_{l,\beta;m,\gamma} \) will prevent such terms from vanishing pairwise. We explain
how to compute \( \eta_{l,\beta;m,\gamma} \) in the next subsection.

The multiple representatives of \( F_{(n)\alpha'} \) in the end all give the same equation, as a conse-
quence of group theory. The dilatonic equation of motion is (44), as usual. The above
recipe can be trivially applied to 3-dimensional theories themselves, \( SL(1) \) being the
trivial group, leading to singlets only.

Note that, whereas the right hand sides of equations (28), (31), (39) and (42) always
involve at least a one form appearing once, we have made no such restriction in (47).
These additional terms clearly go beyond the coset structure defined by (28), (31), (39)
and (42), and are known to appear in the higher dimensional theories we wish to construct.

Interpreting equation (47) as a Bianchi identity, one reason for the appearance of terms
with exclusively higher forms is due to the dimensional reduction procedure: Not only
KK-scalars, but also KK-vectors coming from the metric couple to the various fields
(compare with (17)). Another possibility is that a higher dimensional field may have
a definition that includes a “Chern-Simons” term modifying its Bianchi identity. Such
modifications are often found in supergravity theories. In oxidation, one can recover such
terms as hallmarks of symmetries that are only manifest in lower dimensions.

If (47) is interpreted as an equation of motion, then an action for this equation of motion must have terms beyond those considered in (26) and (34), of the Chern-Simons type. Varying a Chern-Simons term consisting of one bare potential and two field strengths, bilinear terms with field strengths of degree higher than one will appear in the equations of motion. Of course Chern-Simons terms are common in supergravity theories; from the point of view of oxidation they are again signs of lower dimensional symmetries. The oxidation procedure fixes the coefficient of the Chern-Simons term to a structure constant of the algebra; this is to be compared with the old observation that its value is crucial to get the right symmetry in lower dimensions.

4.4 Sign factors

For completeness, the sign factors $\eta_{\alpha,\beta;m,\gamma}$, introduced in (47), have to be specified. Their computation is rather technical; however, for many purposes it is not necessary to know the explicit signs.

From our analysis of coset sigma models we easily deduce

$$
\eta_{1,\beta;1,\gamma} = 1 \quad \eta_{1,\beta;D-1,\gamma} = 1 \quad \eta_{D-1,\beta;1,\gamma} = (-1)^D \quad (48)
$$

To find the other sign factors, we have to deal with a complication. To oxidize, we claim that one has to decompose $G$ into $SL(D-2) \times U_D$. This can be done at the level of the lattices. We decompose the $G$ lattice into an $SL(D-2)$ and a $U_D$ lattice; for the 3 dimensional theory we need a positive root decomposition. Either decomposition is only specified up to lattice automorphisms. A way to ensure that one theory is directly related to the other without invoking lattice automorphisms, proceeds as follows. First, choose a basis of positive roots of $G$, and draw the corresponding Dynkin diagram. Then extend the diagram with the lowest root of the $G$ lattice (the next section will feature more on extended Dynkin diagrams). Now choose an $SL(D-2)$ subdiagram of the extended $G$ diagram, such that one of the ends of the $SL(D-2)$ chain coincides with the extended node; this guarantees that the positive roots of $U_D$ are all made up of positive roots of $G$. To relate the theories by dimensional reduction, one must choose the right orientation of the $SL(D-2)$ diagram; this is done by attaching the highest index to the extended node, and labelling downwards along the chain.

Consider a positive root $\alpha$ of $G$. To find its representative in the oxidized theory, decompose with respect to the $SL(D-2)$ lattice. Let $\beta$ be the component parallel to the space spanned by the $SL(D-2)$ lattice; $\beta$ must be a weight of an antisymmetric tensor representation of $SL(D-2)$, say the $k$-form. Calling the complement of $\beta \alpha'$, the oxidation recipe tells us that the fields in the 3-dimensional and the $D$-dimensional theory are

$$
\alpha = \alpha' + \beta \quad \Rightarrow \quad F_{(1)\alpha} \rightarrow F_{(k+1)\alpha'}
$$
One can do the same for the negative root $-\alpha$; decomposing the weight $-\beta$ of the $D-2-k$ antisymmetric tensor irrep is found. This leads to the representatives

$$-\alpha = -\alpha' - \beta \quad \Rightarrow \quad F_{(2)-\alpha} \rightarrow F_{(D-k-1)-\alpha'}$$

Now suppose we have a term in the equations for the oxidized theory:

$$dF_{(n)\alpha'} = \ldots + \eta_{l,\beta;m,\gamma} N_{\beta,\gamma} F_{(l)\beta'} \wedge F_{(m)\gamma'} + \ldots \quad (49)$$

We reduce straightforwardly on a rectangular torus (a non-rectangular torus would only result in cluttering the equations with extra terms due to KK-fields from the metric). To do so, we replace each field $F_{(n)\alpha'}$ by either $F_{(1)\alpha} dz_{i_{n-1}} \ldots dz_{i_1}$ or $F_{(2)\alpha} dz_{i_{n-2}} \ldots dz_{i_1}$, whichever one is appropriate. The addition of differential forms $dz_i$ leads to sign ambiguities in the reduced forms. As a matter of fact, we can choose any convention as long as we do so consistently. We will choose to order the indices on the $dz_{i_j}$ in decreasing order, i.e. $i_j > i_k$ if $j > k$. Of course the products of $dz_{i_j}$’s have to match on both sides.

The above example becomes then ($l', m', n'$ are 1 or 2, whichever one is appropriate)

$$dF_{(n')\alpha} dz_{n-n'} \ldots dz_1 = \ldots + \eta_{l,\beta;m,\gamma} N_{\beta,\gamma} F_{(l')\beta'} dz_{n-n'} \ldots dz_{m-m'+1} \wedge F_{(m')\gamma'} dz_{m-m'} \ldots dz_1 + \ldots \quad (50)$$

Reordering, and comparing with (31) or (28), one finds (using that $l'$ is either 1 or 2)

$$\eta_{l,\beta;m,\gamma} = (-1)^{(l-l')m'+l'+1} \quad (51)$$

Note that running the same argument with $F_{(l)\beta'}$ and $F_{(m)\gamma'}$ interchanged leads to

$$\eta_{m,\gamma;l,\beta} = (-1)^{(m-m')l'+m'+1} \quad (52)$$

which is not the same as (51) with $l$ and $m$, $l'$ and $m'$ interchanged. Using that $l'm'+l'+m'$ is odd (at most one of $l'$ or $m'$ is even), one easily shows that

$$\eta_{l,\beta;m,\gamma} = (-1)^{m+1} \eta_{m,\gamma;l,\beta} \quad (53)$$

which is a necessary condition to prevent terms from (47) from vanishing pairwise.

### 4.5 Comparison to dimensional reduction

The above oxidation recipe is simple and elegant, but still has to be compared to what is obtained by dimensional reduction. To do so, we construct theories from a $G$ lattice in $D+1$ and in $D$ dimensions, and compare them with each other. Applying dimensional reduction to the $D+1$ dimensional theory should give the $D$ dimensional theory. To
prove full equivalence, one can then pick any theory in the chain of theories related by reduction and oxidation, and appeal to induction.

According to the recipe, we have to decompose $G$ into $SL(D-1)$ and $SL(D-2)$ subgroups respectively. To relate the two, we embed $SL(D-2)$ in $SL(D-1)$, and compare their representations. The adjoint of $SL(D-1)$ corresponds to the $(D+1)$ dimensional graviton; it decomposes into adjoint, fundamental, anti-fundamental and a singlet of $SL(D-2)$. The adjoint is the $D$-dimensional graviton; the fundamental and anti-fundamental representations give equation of motion and Bianchi identity for a single KK-vector; and the singlet is a dilatonic scalar, corresponding to 1 equation. An $n$-form for $SL(D-1)$ decomposes into an $n$-form and an $n-1$ form of $SL(D-2)$. These decompositions are clearly compatible with the dimensional reduction procedure.

Next we compute the relation between the $(D+1)$ dimensional dilaton factors and the $D$ dimensional ones (this partly doubles the discussion in section 3.2, but we think this is instructive). We will use that the lengths of the weights for the antisymmetric $k$-tensor representation of $SL(m)$ are given by

$$\sqrt{\frac{(m-k)k}{m}}.$$  \hspace{1cm} (54)

The $D$-dimensional KK-vector coming from the $(D+1)$-dimensional metric is represented by roots of $SL(D-1)$ that are no longer a root of $SL(D-2)$. Rather, they are represented by the fundamental irrep of $SL(D-2)$. Knowing the lengths of the roots and weights involved, the dilaton coefficient can be computed by using Pythagoras’ theorem:

$$\left(2 - \frac{D-3}{D-2}\right)^{\frac{1}{2}} = \frac{D-1}{\sqrt{(D-1)(D-2)}} = x_2$$  \hspace{1cm} (55)

The computation for the forms proceeds similarly. For the $n$-form of $SL(D)$, one finds

$$\left(\frac{n(D-1-n)}{D-1} - \frac{n(D-2-n)}{D-2}\right)^{\frac{1}{2}} = \frac{n}{\sqrt{(D-1)(D-2)}} = x_{n+1}$$  \hspace{1cm} (56)

The $(n-1)$ form of $SL(D-2)$ gives

$$\left(\frac{n(D-1-n)}{D-1} - \frac{(n-1)(D-1-n)}{D-2}\right)^{\frac{1}{2}} = \frac{D-n-1}{\sqrt{(D-1)(D-2)}} = x_n$$  \hspace{1cm} (57)

That $x_{n+1} + x_n = x_2$ is no coincidence, but a simple consequence of the relations between the relevant weight lattices. These are absolute values. Tracelessness of the algebra generators of the $n$-tensor representation of $SL(D-1)$ requires the coefficients $x_n$ and $x_{n+1}$ to have opposite signs. An explicit computation reveals that $x_2$ has the same sign as $x_{n+1}$. The overall sign can be absorbed in a redefinition of the dilatons, so we choose a (positive) sign for $x_n$. 

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In summary, the argument for the dilaton factor may be defined recursively as

\[
\alpha_D = (\alpha_{D+1}, x) \quad x = \begin{cases} 
-x_{n+1} & \text{for } F_{(n+1)} \to F_{(n+1)} \\
x_n & \text{for } F_{(n+1)} \to F_{(n)} \\
-x_2 & \text{for } F_{(2)}
\end{cases},
\]

(58)
demonstrating complete agreement with \[22\] (up to a factor of \(\sqrt{2}\) that we absorbed in \(\phi\)). Hence, upon dimensional reduction, the labels referring to the U-duality grow as claimed.

This is also sufficient to guarantee that the action has the right form, with the right coefficients, and that the Einstein equation and the equation for the dilatons take the right form. Complete agreement requires that the field strengths for the forms and the axions are defined in the same way: This is checked by comparing the Bianchi identities in both formalisms.

We first check that oxidation gives the right answer for the fields coming from the reduction of the gravity sector. Pure gravity is obtained by oxidizing the \(SL(D-2)/SO(D-2)\)-coset \[9\]. Assuming we are not in the maximal dimension, we need the following decomposition (see appendix \[13\] \(d + d' = n\))

\[
SL(n, \mathbb{R}) \to SL(d, \mathbb{R}) \times SL(d', \mathbb{R}) \times \mathbb{R} \\
(n^2 - 1) \to (d^2 - 1, 1)^0 \oplus (1, d^2 - 1)^0 \oplus (1, 1)^0 \oplus (d, d')^n \oplus (d', d')^{-n}
\]

Setting \(d\) to \(D - 2\), we find the fields for the \(SL(d')/SO(d') \times \mathbb{R}\) coset, and \(d'\) vectors transforming in the fundamental irrep of \(SL(d')\). The \(SL(d')\) adjoint gives us \(d'-1\) dilatons we group in \(\phi\), and there is an additional \(\phi_R\) for the \(\mathbb{R}\) factor. The \(\frac{1}{2}d'(d' - 1)\) positive roots of \(SL(d')\) give rise to axion field strengths \(F_{(1)\alpha,0}\), and the same number of negative roots result in forms \(F_{(D-1)\alpha,0}\). The label \(\pm \alpha\) corresponds to a positive/negative root of \(SL(d')\), while we have added a third label for the charge under \(\mathbb{R}\). The fundamental irrep \(d'\) gives rise to \(d'\) vector field strengths \(F_{(2)\alpha,0}\), with \(\lambda\) a weight of the irrep, while there are \(d'\) forms \(F_{(D-2)\alpha,0}\) from the \(\overline{d'}\) irrep.

Our recipe leads to the following Bianchi identities for the axions and the form fields

\[
dF_{(1)\gamma,0} = \frac{1}{2} \sum_{\alpha, \beta} N_{\alpha, \beta} F_{(1)\alpha,0} \wedge F_{(1)\beta,0} \quad \ast = \begin{cases} 
\alpha, \beta, \gamma \in \Delta^+ \\
\alpha + \beta = \gamma \\
\alpha \in \Delta^+ \\
\alpha + \lambda = \lambda'
\end{cases}
\]

(60)

Setting \(N_{\alpha, \lambda} = 1\), the equations \[60\] agree with \[18\], in any dimension.

For the form fields we explicitly compare the \(D+1\) dimensional theory with the \(D\) dimensional one. An \(n\)-form of \(SL(D-1)\) decomposes to an \(n\) form and an \(n - 1\) form of \(SL(D-2)\), both charged under the dilaton appearing when reducing from \(D+1\) to \(D\)-dimensions. Their charges have opposite sign, and tracelessness of the \(n\) tensor representation of \(SL(D-1)\) means that the charges of the \(n\)-tensor resp. the \(n - 1\) tensor
of $SL(D-2)$ are $-n$ and $D-1-n$. We also need that the axions are not charged under the new dilaton. Then, for the $n$-tensor of $SL(D-2)$, our formalism predicts:

$$dF_{(n+1)\lambda_n,-n} = \sum N(\alpha,0,\lambda_n,-n)F_{(1)\alpha,0} \wedge F_{(n+1),\lambda_n,-n} + \sum N(\lambda_n,D-1-n),(0,D-1)F_{(n),\lambda_n,D-1-n} \wedge F_{(2)0,D-1}$$

We denoted the label of the $n$-tensor of $SL(D-1)$ by $\lambda_n$ and added the charges as a second label. The structure of (61) is identical to (17); closer inspection shows that detailed matching can be done. The reader can convince himself that addition of Chern-Simons terms modifies the equations in the way our formalism predicts, in the formalism of [22] these have to be added by hand.

5 Extended Dynkin diagrams

It has been known for long that aspects of the supergravity theory can be encoded in Dynkin diagrams. Extended Dynkin diagrams\textsuperscript{7} encode even more information about the theory. This can be traced back to the original reason for their introduction, in [25], as a tool for the classification of regular subalgebra’s of a given semi-simple Lie algebra. These diagrams have appeared previously in the supergravity literature (e.g. in [20, 3]); here we focus on applications to theories with dimension $\geq 3$. Appendix B, figure 3 lists all extended Dynkin diagrams.

5.1 Representation content of a theory

We first recall some definitions. Given the root lattice for a simple Lie group, one can choose a basis of linearly independent simple roots, the simple roots being defined by the property that the difference of two simple roots is not a root. The geometrical relations between these basisvectors can be conveniently encoded in a Dynkin diagram.

Dropping the requirement of linear independence, the basis can be extended by adding the lowest root of the root lattice to the basis. Though no longer independent, all elements of such a basis still have the property that the difference of two of them is not a root. As a consequence, any proper subset of the vectors of the extended basis defines a set of simple roots for some lattice, and the lattice in question is a sublattice of the original group lattice. The extended basis can also be encoded in a diagram, the extended Dynkin diagram, and the process of dropping some basis elements corresponds to erasing some nodes of the diagram. Repetition of the procedure (extension of the diagram, choosing a subdiagram) allows one to find all the regular sublattices [25].

\textsuperscript{7}These are also known as the Dynkin diagrams of the affine untwisted Kac-Moody algebra’s. They are however useful in many contexts where a relation with Kac-Moody algebra’s is not obvious.
Applied to our problem, the sublattices we are interested in are the $SL(n)$ sublattices, realized as chains in the extended Dynkin diagram. In the generic case, we obtain a disconnected diagram when erasing nodes such that we obtain a $SL$-chain. The remaining diagram, with the chain taken out, corresponds to the lattice of a (regular) subgroup of the U-duality group in that dimension. If the extended node is part of the $SL(n)$-chain, the complementary diagram precisely gives the lattice of the semi-simple part of the U-duality group. If the final (not extended) diagram has less nodes than the original (not extended) diagram, the difference in nodes corresponds to $\mathbb{R}$ factors in the U-duality group. It has been known for a long time that "group disintegration" should start at the end of the Dynkin diagram where one attaches the affine vertex (In [15] this is verified for all split simple Lie-groups, for the $E_8(8)$-series of maximal supergravity the observation is much older [3]). The above argument demonstrates why this observation is correct.

As an immediate corollary, we notice that for every coset theory $G/H$ in 3 dimensions with split $G$, the maximal oxidation dimension can be found by taking the extended diagram for $G$, looking for the largest chain of nodes representing long roots, and counting the number of nodes of this chain. Adding 3 to this number, we find the maximal oxidation dimension. Comparing with the list in the summary section of [15], we see that this simple rule covers all cases.

Another important point is that if there are multiple ways of realizing the diagram of a sublattice, the various possibilities correspond to subgroups that are conjugate to each other, except for a finite number of exceptions, listed in Dynkins paper [25]. These play a central role in the next subsection.

Before arriving at the final diagram, we have to erase a certain node. We know the geometrical relations of this node to the final sublattice, they were encoded in the diagram before erasing the node. This gives us information on the representation content of the theory. The Dynkin diagram encodes the entries of the Cartan matrix, which is given by

$$N_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

(62)

We are interested in $SL$ groups of long roots, for which we have $\langle \alpha_i, \alpha_i \rangle = 2$. Hence $N_{ij} = \langle \alpha_i, \alpha_j \rangle = -1$, if the $j^{th}$ node was linked to the $i^{th}$ one. This implies that the node that was erased corresponds to a lowest weight for a representation, and the node it was connected to reveals which representation: If the erased node connected to the $k^{th}$ node of the $SL(n)$ chain (which has $n-1$ nodes), the relevant representation is the $(n-k)$-form (see also [21]). Of course one can enumerate the nodes in the $SL(n)$ chain in 2 ways, and one finds also a $k$-form (provided $2k \neq n$). But this should be expected: The adjoint is a self-conjugate irrep, and should decompose in selfconjugate representations. Hence for each $k$-form one should also find the conjugate $(n-k)$-form, the exception being $n/2$-forms for $n$ even, which are selfconjugate. In principle also the representation of the U-duality group can be read off this way, but here we can also have $N_{ij} = -2, -3$.
for non-simply laced groups. Also, because the U-duality group is not an $SL$-group in general, additional knowledge of the representations of the U-duality group is required.

![Figure 1: Manipulations of Dynkin diagrams](image)

As an example we depicted in figure 1 the manipulations that lead to the $SL(6) \times SL(2) \times SL(3)$ subgroup of $E_8$. This is relevant for the toroidal compactification of maximal supergravity to $6 + 2 = 8$ dimensions, where indeed the U-duality group is $SL(2) \times SL(3)$. The relevant Dynkin diagram can be obtained in various ways. The simplest is by erasing the black node in the diagram denoted by $a$. This node is connected to the ends of the $A_1$, $A_2$ and $A_5$ diagrams, hence we expect vectors (from the end of $A_5$), transforming as a doublet under $SL(2)$ and a triplet under $SL(3)$. Another way of arriving at the same diagram, depicted in $b_1$, $b_2$, proceeds by first erasing a node to obtain $A_1 \oplus E_7$, then extending $E_7$ and erasing the black node in $b_2$. This node is connected to the second node of the $A_5$ chain, and hence corresponds to 2-tensors. They are singlets under $SL(2)$ (not connected), and form a triplet under $SL(3)$. The third way of getting to the final diagram proceeds via $A_2 \oplus E_6$, and from this we learn that there are 3-tensors, in a doublet under $SL(2)$ and singlet under $SL(3)$.

Instead of giving more examples, we encourage the reader to play with the extended Dynkin diagrams to recover the matter content of the relevant theories. For comparison, the relevant representations are listed in appendix B.

### 5.2 Dualities

A interesting phenomenon is the appearance of diagrams with vertices. At such a vertex, an embedded chain, corresponding to an $SL(n)$ group can turn in different ways. The corresponding $SL(n)$ groups are non-conjugate: They are listed in [25].
A very prominent example can be found inside the $E_8$ diagram. There are two different, and inequivalent ways of embedding an $SL(8)$ chain in this diagram. In one case we obtain a 10 dimensional theory with U-duality group $\mathbb{R}$. The various ways of embedding the chain inside the “long end” of the $E_8$ diagram reveal a vector, a 2-tensor, and a 3-tensor, and hence this theory should be identified with IIA supergravity. A second possibility is to insert the $A_7$ chain in the “short” end. Its complement is then an $SL(2)$, and the erased node reveals a doublet of 2-tensors. A second way to reach $SL(8) \times SL(2)$ proceeds by first breaking $E_8$ to $E_7 \times SL(2)$, and subsequently breaking $E_7$ to $SL(8)$. This path to the duality group reveals a 4-tensor, which is a singlet under $SL(2)$. Of course this second theory is IIB supergravity. The $SL(8)$ of IIB can not be embedded in the $SL(9)$ of 11 dimensional supergravity, but smaller $SL(n)$ subgroups can be embedded in either the $SL(8)$ of IIB-theory, or the $SL(9)$ of 11 dimensional supergravity.

The other examples of diagrams with vertices are the $B_n$ and $D_n$ series, and $E_6$ and $E_7$. The $D_n$-diagrams with their two vertices are particularly interesting. The $D_8$ diagram corresponds to type I supergravity, that also occurs in the heterotic string. Because of the forks there are two different ways of embedding an $A_7$ chain. Comparing the two options, one sees that the two different ways correspond to exchanging moduli coming from the metric with moduli that come from a 2-tensor: Again this represents T-duality in this context. Also interesting are the two inequivalent embeddings\(^8\) of $SL(4)$. One of them corresponds to compactifying the bosonic sector of 10 dimensional type I supergravity to 6 dimensions, the other is a theory of gravity with only self-dual and anti-selfdual

\(^8\)In Dynkins paper \cite{25} the inequivalent $SL(4)$’s are denoted as $A_3$ and $D_3$, respectively.
2-tensors, and corresponds to the bosonic sector of $(2,0)$ gravity in 6 dimensions with tensor multiplets. The theories can be related in 5 dimensions, as they can be regarded as truncations of the compactification of IIA and IIB string theory on $K3$, and adding an additional circle makes T-duality possible. The reader should compare the number of vectors found by group theory with the discussion in [12].

Intriguingly, these well known dualities seem to be present in all the $D_n$ theories (at least as far as the massless field are concerned), in particular in the $D_{24}$ theory, which corresponds to the massless sector of the bosonic string. Also this gravity theory has a branch in 6-d, with 21 self-dual and 21 anti-selfdual tensors transforming in a single vector irrep of $SO(21,21)$. Though it is not obvious that this is possible, it would be interesting to attempt to construct a string theory realizing these massless fields. Doing so might provide clues on issues such as little string theories, and a possible role for the bosonic string in string dualities. Let us also point out the $D_4$ extended Dynkin diagram, the only diagram with a four-vertex, yielding a highly symmetric spectrum upon compactification. It gives rise to a 6 dimensional theory that, upon compactification on a circle exhibits a “T-triality” rather than T-duality.

The $B_n$ chains have similar symmetries as the $D_n$-series. Again there is the possibility of different ways of embedding a “long” chain, and an additional 6 dimensional branch. The difference between the $D_n$ and $B_n$ model in their maximal dimension is the presence of an extra vector. Apart from the modification of the action, this leads to a modification of the equation for the 2-form field strength: It becomes

$$dF_3 = -\frac{1}{2}F_2 \wedge F_2$$

(63)

Note that for $n > 4$ ($D > 6$) this is usually interpreted as a Bianchi identity, while for $n < 4$ ($D < 6$) it is an equation of motion. This can be seen by using group theory. Let $e_i$ be unit vectors on the $\mathbb{R}^n$, then one can pick the following positive roots for $B_n$: $e_i - e_j$, these will be the $SL(n)$ in the maximal dimension; $e_i + e_j$; these correspond to the antisymmetric tensor in the top dimension; and $e_i$; these form the vector in the top dimension. One then immediately sees that one has to find an equation of the form of (63). The identity (63) is well known in string theory in a version where the r.h.s. is replaced by $\Tr(F_2 \wedge F_2)$; it plays a crucial role in anomaly cancellation [26] (A second first Pontryagin class associated to the Riemann tensor does not turn up at the level of classical supergravity, but is a higher derivative effect). Our derivation here has nothing to do with supersymmetry or string theory, but produces the term from the group structure. A similar identity occurs when enlarging the $D_n = SO(n,n)$ or $B_n = SO(n+1,n)$ groups to $SO(n + r, n)$; this also implies vectors in the maximal dimension, and modification of the 2-form Bianchi identity, as we will explain in [16, 17].

Although it is a bit outside the scope of the rest of this subsection, that focuses on T-duality, note that the fork in $B_n$ and $D_n$ diagrams with $n \geq 2$ leads to a separate $SL(2,\mathbb{R})$
factor in 4 dimensions, commonly associated to S-duality of gauge theories. Also note that all of the $B_n$ and $D_n$ theories oxidize in their maximal dimension to a theory with antisymmetric 2-tensors, and hence, strings. Together with the fact that all these theories have a 6-dimensional branch, this stresses once more various known relations between strings and gauge theories.

The $E_7$ theory is most easily interpreted as a truncation of IIB-theory, by decomposing $E_8 \rightarrow E_7 \times SL(2)$, and truncating to the sector that contains only the singlets under $SL(2)$. The two inequivalent ways of embedding an $A_6$ chain lead to a branch in $d = 8$. This can also be seen from the viewpoint of $E_7$ theory as a truncation of a maximally supersymmetric theory. This would have U-duality group $SL(2) \times SL(3)$, with the original IIB $SL(2)$ embedded in $SL(3)$. Of course it is also possible to truncate by the other $SL(2)$, that arises because the 3-form of 11 dimensional supergravity reduced over a 3-torus forms a complex combination with a dilaton, which transform under $SL(2)$.

The extended diagram of $E_6$ is highly symmetric. Viewing $E_6$ theory as a truncation of $E_8$ theory, comparison of the branches in the diagram shows a relation with “M-theory T-duality”; a version of T-duality acting on 2 directions of the compactification torus simultaneously.

### 6 Discrete symmetries: Magic triangles

As a direct application of our results, we point out that the discrete symmetries pictured in so-called magic triangles, can be easily understood in the developed formalism.

#### 6.1 Magic triangle

In [15], a “magic triangle” appears. This is a table of the U-duality groups appearing in the oxidation of $E_n$ cosets from 3 dimensions. Its content is given in table [11].

This table of groups appearing in oxidation has a reflection symmetry along the diagonal. This structure is easy to understand in the present formalism\(^9\). Starting with a U-duality group $E_{n(n)}$ in 3-dimensions, to find the U-duality group in $D$ dimensions, we decompose

\[ E_{n(n)} \rightarrow SL(D - 2, \mathbb{R}) \times U_{n,D} \quad (64) \]

On the other hand, the $E$-series can be defined as the groups appearing in the following decomposition of regular subgroups

\[ E_{8(8)} \rightarrow SL(9 - n) \times E_{n(n)} \quad (65) \]

\(^9\)The proof of the symmetry given here is the one alluded to in a footnote in [11]. It becomes identical to the proof given in [10], upon invoking one of the well known $A - D - E$ correspondences, between Kleinian singularities and the classification of simply laced Lie groups.
\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
10 & $\mathbb{R}$ & $A_1$ & $\{e\}$ &  \\
\hline
9 & $\mathbb{R} \times A_1$ & $\mathbb{R}$ &  \\
\hline
8 & $A_1 \times A_2$ & $\mathbb{R} \times A_2$ & $A_1$ &  \\
\hline
7 & $D_4$ & $A_4$ & $\mathbb{R} \times A_2$ & $\mathbb{R} \times A_1$ & $A_1$ & $\{e\}$ \cellcolor{gray!25}  \\
\hline
6 & $E_6$ & $D_5$ & $A_1 \times A_3$ & $\mathbb{R} \times A_2^2$ & $\mathbb{R}$ &  \\
\hline
5 & $E_7$ & $E_6$ & $A_5$ & $\mathbb{R} \times A_1^2$ & $\mathbb{R} \times A_1$ & $A_1$ & $\{e\}$ \cellcolor{gray!25}  \\
\hline
3 & $E_8$ & $E_7$ & $E_6$ & $D_5$ & $A_4$ & $A_2 \times A_1$ & $\mathbb{R} \times A_1$ & $\mathbb{R}$ & $\{e\}$ \cellcolor{gray!25}  \\
\hline
D & $\mathbb{n} = 8$ & $\mathbb{n} = 7$ & $\mathbb{n} = 6$ & $\mathbb{n} = 5$ & $\mathbb{n} = 4$ & $\mathbb{n} = 3$ & $\mathbb{n} = 2$ & $\mathbb{n} = 1$ & $\mathbb{n} = 0$ \\
\hline
\end{tabular}
\end{table}

Table 1: The $E_8$ triangle

i.e. the $E_{n(n)}$ lattice is the orthogonal complement to the $SL(9 - n)$ lattice in $E_{8(8)}$.

Combination of the 2 equations reveals that the groups $U_{n,D}$ follow from the decomposition

$$E_8 \rightarrow SL(9 - n, \mathbb{R}) \times SL(D - 2, \mathbb{R}) \times U_{n,D}$$

(66)

There is an obvious symmetry, which is the symmetry of the magic triangle:

$$U_{n,D} = U_{11-D,11-n}.$$  

(67)

It is possible to define a triangle for every simply laced group $G$, by decomposing

$$G \rightarrow SL(n' - n, \mathbb{R}) \times SL(D - 2, \mathbb{R}) \times U_{n,D},$$

(68)

where $n'$ is defined by the largest $SL(n', \mathbb{R})$ factor one can find (it is the length of the largest chain in the extended Dynkin diagram, plus 1). Such a triangle has the symmetry

$$U_{n,D} = U_{n' + 2 - D, n' + 2 - n}.$$  

(69)

If we do not impose maximality on $n'$ the triangle will not be symmetric, unless we impose a similar restriction on $D$. We have renamed the original magic triangle to “$E_8$”-triangle, to distinguish it from other $A - D - E$-triangles.

For non-simply laced groups the story is more complicated. The reason is that we required the $SL(D-2)$ of gravity to be made up from long roots. To recover a symmetric triangle,
we should demand the same for the U-duality group. As illustration of the procedure, consider the “triangle” that we would obtain for $F_4$, depicted in table 2.

The groups inside the boxes in the triangle are the ones one obtains by restriction to level 1 subgroups. This looks odd, since $A_2$ and $A_1$ appear in the $D = 3$ row, and one surely would expect to be able to decompactify from these. Hence we have added the groups that would result from oxidation, but then evidently, the symmetry is gone.

Summarizing, for the non-simply laced case the triangle either loses its meaning as a table of groups encountered in oxidation, or it is not symmetric.

### 6.2 The supergravity triangle

There is a second “magic triangle”. Again we put maximal supergravity in the lower left corner, while going up represents oxidation. To the right we list the theories obtained by truncating the amount of 4-dimensional supersymmetries. The symmetries of the theories obtained this way are displayed in the tables 4 and 5 which contain the information from a table in [3] (an earlier version in [20] has some small errors). Generically, the groups in this triangle are non-split. We have listed along a vertical bar, two characteristics of non-compact groups, being their signature (the difference between the number of non-compact and the number of compact generators, upper right corner), and their real rank (lower right corner). In [16, 17] cosets of non-split groups will be discussed in more detail. Table 5 lists the explicit forms appearing in table 4.

This triangle has an intriguing approximate reflection symmetry in the diagonal. The entries that exhibit the symmetry have groups that have the same complexifications (but different real forms). With our theory we can explain not only the approximate symmetry, but also why and when it breaks down.

We need however a new ingredient, supersymmetry. It will be sufficient to use some elements of the theory of spinors. Supersymmetric theories enjoy (at the classical level) R-symmetries, mixing the various supercharges. For supergravity theories, these R-
Symmetries are (necessarily) local, and since they are compact, they must be embedded inside maximal compact subgroups.

The cosets reflect transverse degrees of freedom \[20\]. Hence we are led to study spinors in \(D-2\) Euclidean dimensions. Spinors are real, complex or quaternionic; they are naturally acted upon by orthogonal, unitary, and symplectic groups. We have listed the Clifford algebra’s relevant for \(D-2\) dimensions in table 3 (see e.g. \[27\] for a recent review). We have also listed the symmetry groups acting on \(N\) supercharges, with the number \(N\) referring to the number of 4-dimensional supersymmetries. One of the things to note is that the series of groups mentioned for \(N=8\) are precisely the compact subgroups for the \(E_{n(n)}\) series; in this sense these groups are “tailor-made” for maximal supergravity.

| \(D\) | Clifford | \(N=8\) | \(1 \leq N \leq 6\) |
|-------|----------|---------|---------------------|
| 11    | \(\mathbb{R}^{16}\) | \{e\}   | \{e\}               |
| 10    | \(\mathbb{R}^8 \oplus \mathbb{R}^8\) | \(O(1)_{SO(2)}\) | \{e\}               |
| 9     | \(\mathbb{R}^8\) | \(O(2)\) | \(O(N/4)\)         |
| 8     | \(\mathbb{C}^4\) | \(U(2)\) | \(U(N/4)\)         |
| 7     | \(\mathbb{H}^2\) | \(Sp(2)\) | \(Sp(N/4)\)        |
| 6     | \(\mathbb{H} \oplus \mathbb{H}\) | \(Sp(2) \times Sp(2)\) | \(Sp(n) \times Sp(m)\) when \(n + m = N/2\), \(n, m \leq 2\) |
| 5     | \(\mathbb{H}\) | \(Sp(4)\) | \(Sp(N/2)\)        |
| 4     | \(\mathbb{C}\) | \(SU(8)\) | \(U(N)\)           |
| 3     | \(\mathbb{R}\) | \(SO(16)\) | \(SO(2N)\)         |

Table 3: Clifford algebra’s, and R-symmetries

The supergravity triangle \[4\] is based on the number of supersymmetries in 4 dimensions. In 4 dimensions, the helicity group is \(O(2)\). There are two transverse dimensions, and spinors in two dimensions are complex. Hence the R-symmetry group for \(N\) supersymmetries must contain \(U(N)\) with \(N\) the number of supersymmetries \[28\]. Exception to the rule is \(N=7\). As is well known, building a supergravity multiplet with 7 supersymmetries, one automatically finds an eighth supersymmetry, because a realistic theory has to contain the CPT-conjugate, and the \(N=7\) multiplet and its conjugate automatically fill a single \(N=8\) multiplet. Then for \(N=8\), the supergravity multiplet is its own conjugate (in 4-d). The \(U(1)\) factor in \(U(N)\) multiplies a multiplet with a phase, and the CPT-conjugate with the opposite phase. If a multiplet is its own CPT-conjugate, then \(U(1)\) must act trivially on the multiplet, and since there is no other multiplet for \(N=8\) than the supergravity multiplet, the R-symmetry is truncated to \(SU(8)\) (see also \[19\]).

Now suppose we truncate to fewer supersymmetries, say \(n\). We decompose the \(N=8\) multiplet, by separating the supercharges in a group of \(n\), and one of \(8-n\) supercharges. The \(8-n\) supercharges have \(SU(8-n)\) symmetry, and the remaining ones \(U(n)\). We
### Table 4: The supergravity triangle

| $N$ | $D$ | group | real form | compact |
|-----|-----|-------|-----------|---------|
| 6   | 6   | $A_1 \times A_3 | A_1^8$ | $Sp(1) \times SU^*(4)$ | $Sp(1) \times Sp(2)$ |
| 6   | 5   | $A_5 | A_2^7$ | $SU^*(6)$ | $Sp(3)$ |
| 6   | 4   | $D_6 | D_6^{-6}$ | $SO^*(12)$ | $U(6)$ |
| 6   | 3   | $E_7 | E_7^{-14}$ | $E_7(-5)$ | $SU(2) \times SO(12)$ |
| 5   | 4   | $A_5 | A_5^{-14}$ | $SU(5, 1)$ | $U(5)$ |
| 5   | 3   | $E_6 | E_6^{-14}$ | $U(1) \times SO(10)$ |
| 4   | 4   | $A_1 \times A_3 | A_1^9$ | $SL(2) \times SU(4)$ | $U(4)$ |
| 4   | 3   | $D_5 | D_5^{-13}$ | $SO(8, 2)$ | $U(1) \times SO(8)$ |
| 3   | 3   | $A_1 | A_1^{-14}$ | $SU(4, 1)$ | $U(1) \times SO(6)$ |
| 2   | 3   | $A_2 \times A_1 | A_2^{-3}$ | $SU(2, 1) \times SU(2)$ | $U(1) \times SO(4)$ |
| 1   | 3   | $U(1) \times A_1 | U(1) \times SL(2, \mathbb{R})$ | $U(1) \times SO(2)$ |
| 0   | 3   | $A_1 | SL(2, \mathbb{R})$ | $SO(2)$ |

### Table 5: Addendum to the supergravity triangle

| $N$ | $D$ | group | real form | compact |
|-----|-----|-------|-----------|---------|
| 6   | 6   | $A_1 \times A_3 | A_1^8$ | $Sp(1) \times SU^*(4)$ | $Sp(1) \times Sp(2)$ |
| 6   | 5   | $A_5 | A_5^{-7}$ | $SU^*(6)$ | $Sp(3)$ |
| 6   | 4   | $D_6 | D_6^{-6}$ | $SO^*(12)$ | $U(6)$ |
| 6   | 3   | $E_7 | E_7^{-14}$ | $E_7(-5)$ | $SU(2) \times SO(12)$ |
| 5   | 4   | $A_5 | A_5^{-14}$ | $SU(5, 1)$ | $U(5)$ |
| 5   | 3   | $E_6 | E_6^{-14}$ | $U(1) \times SO(10)$ |
| 4   | 4   | $A_1 \times A_3 | A_1^9$ | $SL(2) \times SU(4)$ | $U(4)$ |
| 4   | 3   | $D_5 | D_5^{-13}$ | $SO(8, 2)$ | $U(1) \times SO(8)$ |
| 3   | 3   | $A_1 | A_1^{-14}$ | $SU(4, 1)$ | $U(1) \times SO(6)$ |
| 2   | 3   | $A_2 \times A_1 | A_2^{-3}$ | $SU(2, 1) \times SU(2)$ | $U(1) \times SO(4)$ |
| 1   | 3   | $U(1) \times A_1 | U(1) \times SL(2, \mathbb{R})$ | $U(1) \times SO(2)$ |
| 0   | 3   | $A_1 | SL(2, \mathbb{R})$ | $SO(2)$ |
should therefore decompose via

\[ SU(8) \to U(n) \times SU(8-n) \]  

(70)

In 4-d maximal supergravity, the U-duality group is \( E_{7(7)} \), and its maximal compact subgroup is \( SU(8) \). Truncating to fewer supersymmetries means truncating the R-symmetry group by an \( SU(8-n) \) factor, and because \( SU(8) \) is the maximal level 1 compact subgroup, it means truncating \( E_{7(7)} \) by a regular \( SU(8-n) \) factor.

From the previous section we already knew that \( E_{7(7)} \) itself is obtained by decomposing \( E_{8(8)} \) with an \( SL(2) \) factor. We can now move in two directions: Fewer (four-dimensional) supersymmetries; and, more dimensions. For the U-duality group \( U_{N,D} \) of a theory in \( D \) dimensions, with \( N \) supersymmetries as counted in 4 dimensions, we decompose

\[ E_{8(8)} \to SU(8-N) \times SL(D-2, \mathbb{R}) \times U_{N,D}, \]

(71)

where \( SU(8-N) \) and \( SL(D-2) \) are regular (level 1) subgroups. Note that for \( N = 7 \) we get the groups of \( N = 8 \) supersymmetry; the formula reflects that we get an extra supersymmetry for free.

To exhibit the symmetry, we replace the groups of (71) by their complexifications:

\[ E_S \to A_{7-N} \times A_{D-3} \times \tilde{U}_{N,D} \]

(72)

It is clear that \( \tilde{U}_{N,D} = \tilde{U}_{10-D,10-N} \). This is responsible for the approximate symmetry of the supergravity triangle. The symmetry would be exact, if the symmetry extended from \( \tilde{U}_{N,D} \) to \( U_{N,D} \), but it is easy to see that it does not. For example, the maximal \( SL(D-2) \) subgroup in \( E_{8(8)} \) is \( SL(9, \mathbb{R}) \) giving 11-d supergravity, while the maximal \( SU(8-N) \) group is \( SU(8) \) (sitting inside \( SO(16) \)), resulting in no supersymmetry.

The last example leads to the decomposition

\[ \frac{E_{8(8)}}{SO(16)} \to \frac{E_{7(7)}}{SU(8)} \times \frac{SL(2)}{SO(2)} \]

(73)

It is amusing to see that in this way, the maximal supersymmetric theory delivers us the correct Ehlers symmetry of 4-d non-supersymmetric gravity, compactified to 3-d. It appears next to the U-duality group of maximal supergravity in 4-d.

7 Discussion and conclusions

The central theme of this paper is that the inverse process to dimensional reduction, oxidation, is completely governed by group theory. We have restricted to oxidation starting
from coset theories in 3 dimensions, where the groups involved are split. In follow-up papers [16, 17] the formalism will be extended to all non-compact forms, and results will be presented for the oxidation chains for these groups, in particular their maximal dimension. The analysis will require a few extra technicalities, but still be completely based on group theory.

The final oxidation recipe is very simple: It can be found in section 4.3 and described in a few lines. Note that it defines the oxidized theory in terms of equations of motion and Bianchi identities; in particular, there is no difficulty describing theories with self-dual forms. Our formalism is democratic in its equations; it does not prescribe which one is supposed to be a dynamical equation, and which one a constraint equation. As such, we are also insensitive to the modifications of the symmetries resulting from dualizing various forms [29, 23]; these affect the actions, and therefore the interpretation of the equations, but not the equations themselves.

We started with 3 dimensional theories. This has a number of profound consequences: The groups involved are finite dimensional, and the relevant representation theory is relatively simple. We believe that there are no fundamental obstacles in relating the theories in this paper to 2- and lower dimensional theories, but evidently, this should involve the representation theory of infinite dimensional groups, and therefore be more involved.

There are a number of previous developments that deserve some comments.

A paper that develops a number of directions similar to the ones expressed in this paper, yet differs profoundly in its philosophy, is [9]. In this paper, the fields of maximal supergravities are supplemented with their “double” (essentially the Poincaré dual). This leads to doubled Lagrangians, with twice the amount of fields, from which the original formulation can be rederived by imposing a “twisted self-duality” equation. For axions and form fields, the “doubles” of [9] obey Bianchi identities that are equivalent to our equations of motion. An important difference between our work and [9], is that in the latter paper also the dilatons are doubled, a procedure that has no analogue in our paper. This seems to be the source of another key difference: [9] finds a particular kind of supergroups, while we have claimed that “ordinary” Lie groups can account for all the structure. Note also an important difference in the treatment of Chern-Simons terms: In [9] these are an ingredient in deforming the superalgebra, whereas in our work, we have ignored them for a long while, finding in the end that they come out automatically. In view of the flexibility and content of both formalisms, it seems that there should be some well-defined map between them. The supergroups formed one of the ingredients for the development in [10].

Another intriguing paper [30] suggests relations between the classification of Del Pezzo surfaces, and maximal supergravities in various dimensions. The work [10] elaborates on the theme, and relates it to the supergroups of [9]. Many of the relations found are linked to the fact that the second homology lattice of the Del Pezzo’s are the root lattices of the
$E_n$ groups. Our work emphasizes the physical information that can be extracted from the root lattices. It seems fair to say that the Del Pezzo-supergravity correspondence requires more motivation from the physical side, as there are clear links between supergravity and $E_n$ groups, as well as between Del Pezzo’s and $E_n$ groups, but it is not clear to what extent there is a third link of a potential triangle, between Del Pezzo’s and supergravities. Much of the information uncovered in [30, 10] might actually be recoverable from group theory (certainly the BPS-states should have some place; we have all the bosonic equations of the theory), and a direct physical motivation for the involvement of Del Pezzo’s seems to be still lacking (apart from some remarks in [30]). On the other hand, if there is a deeper reason for a Del Pezzo-gravity correspondence, the present formalism might provide valuable tools for exploring it.

An old idea is that the symmetries that (maximal) supergravities exhibit upon compactification, are in some way already realized in the higher dimensional theories [20]. This conjecture in its most advanced form is at the heart of the $E_{11}$-program of [8]. Our paper fits into the strategy of identifying substructures of $E_{11}$: We have demonstrated how to reconstruct the bosonic sector of 11 dimensional supergravity (and its compactifications) from the finite dimensional algebra $E_8$. We believe that its very plausible that an extension of our methods to compactifications lower than 3 dimensions exists (the recent paper [31] has partial results on an $E_{10}$ coset which seem reminiscent to ours for finite dimensional cosets) and that such an extension might lead to support, and perhaps even a proof of the conjecture of [8].

Some papers are hinting at evidence for a “12th dimension” for maximal supergravities (see [13, 9, 32] for some motivations). We have nothing to say in support of this conjecture, but do note that restrictions and assumptions that formed the basis of this paper made it unlikely that we would be able to probe such a dimension. The restrictions imposed by supersymmetry require that a 12th dimension be very different in nature from the other 11 space-time dimensions, which in turn seems incompatible with our methods, which are (indirectly) rooted in Lorentz symmetries.

Similarly, although our methods are easily applicable to the massless sector of the bosonic string, (governed by the split form of $D_{24}$, $SO(24, 24$)), we find no hints for “bosonic M-theory” [33]. Again it can be argued that our assumptions are not suitable for dealing with this issue, but it is less clear to us why (from a physical perspective) this hypothetical theory should not fit somewhere in our framework (it cannot fit anywhere by virtue of the classification of simple Lie algebra’s).

After these negative results we would like to stress once more a remarkable positive result. We have found novel branches for the $B_n$ and $D_n$ series in 6 dimensions. This is remarkable, because these theories oxidize in their maximal dimension to a theory containing gravity, an antisymmetric 2-tensor, and a dilaton scalar (the $B_n$ series, and the non-split forms [16, 17] have extra vectors); at the classical level, they allow elementary string solutions, and hence are string theories. In 6 dimensions, we find gravity, scalars,
and multiple antisymmetric tensors (for the $B_n$ series and the non-split forms [16] [17], the number of self-dual tensors is not equal to the number of antisymmetric tensors, and the theories are chiral, even when fermions are absent). The only known theory in this category is a branch of the $D_8$ chain, giving an extension of the bosonic sector of $(2,0)$ gravity with tensor multiplets (which can be obtained by compactifying $IIB$ gravity on a $K3$, and truncating; $IIB$ on $K3$ itself gives rise to a non-split form of $D_{16}$). The other ones, in particular the theory arising as a branch on the $D_{24}$ oxidation chain of the massless sector of the bosonic string, represent new challenges to those who believe that algebraic structure must lead to meaningful physics.

Apart from the above problems there are many other perspectives for future work. We will extend our analysis to cosets of non-split groups [16] [17]. Some fundamental directions for future research are the relation with supersymmetry, and the extension to theories that have less than 3 non-compact dimensions. We also believe that there is more to say on the microscopic realization of the symmetries. On the other hand, we hope that this work may also lead to applications. Possible directions are the investigation of gaugings of (super)gravities, and the study of solutions of the theories.

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## A Conventions for Lie algebra’s

We deal exclusively with real Lie algebra’s. We follow to some extent (but not completely), the conventions of [34]. This reference also discusses some alternative conventions.

Every real Lie algebra $L$ has an adjoint representation. With a basis $b_i$ for the Lie algebra, the matrices of the adjoint representation are defined by

$$\{\text{ad}(a)\}_{ij} b_j = [a, b_i] \quad a, b, \in L$$  \hspace{1cm} (74)

The Killing form, which we denote by $\langle . , . \rangle$, is defined by

$$\langle a, b \rangle = \text{tr}(\text{ad}(a)\text{ad}(b)) \quad a, b \in L$$  \hspace{1cm} (75)

This fixes many properties of the Killing form, but not its normalization. In our paper
the Killing form defines the inner product on the scalar manifold spanned by the dilatons, and hence its normalization corresponds to the normalization of the dilatons.

The Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ is a maximal Abelian subalgebra, with as additional property that $\text{ad}\mathcal{H}$ is completely reducible. The dimension $r$ of the Cartan subalgebra is called the rank of the algebra. We choose an orthonormal basis $H_1, \ldots, H_r$ of $\mathcal{H}$:

$$\langle H_i, H_j \rangle = \delta_{ij} \quad (76)$$

As the $\text{ad}(H_i)$ are mutually commuting, they can be simultaneously diagonalized. The simultaneous eigenvectors for the $\text{ad}(H_i)$, corresponding to non-zero eigenvalues are denoted by $E_\alpha$. The commutation relations following from these definitions are

$$[H_i, H_j] = 0 \quad [H_i, E_\alpha] = \alpha_i E_\alpha \quad (77)$$

It is common to regard the $\alpha_i$ as components of a vector. Contracting them with the basis elements of $\mathcal{H}$, we define $\alpha = \alpha_i H_i$. We have $\langle \alpha, \beta \rangle = \alpha_i \beta_i$, like a normal inner product. The $\alpha$'s appearing as label on $E_\alpha$ are called roots. Inspection of the simple Lie algebra's shows that the norm of the roots can take at most two values, hence one speaks of short and long roots. We normalize, as conventional, the long roots to have length $\sqrt{2}$, but note that this leads to an unconventional dilaton kinetic term in the matter actions in this paper. The normalization of $\langle E_\alpha, E_{-\alpha} \rangle$ is not yet determined, and we set

$$\langle E_\alpha, E_{-\alpha} \rangle = 1 \quad (78)$$

That this norm is independent of $\alpha$ is important for the normalization of the axion kinetic terms in the paper. Combinations of two generators inserted into the Killing form that cannot be brought to the form (76) or (78) are zero.

The remaining commutation relations are

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i \quad [E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha+\beta}, \quad \alpha + \beta \neq 0 \quad (79)$$

In the last of these commutation relations, the structure constants $N_{\alpha, \beta}$ appear, that are non-zero whenever $\alpha + \beta$ is a root. They are antisymmetric $N_{\beta, \alpha} = -N_{\alpha, \beta}$, and may be chosen such that $N_{\alpha, \beta} = -N_{-\alpha, -\beta}$. Further properties are

$$N_{\alpha, \beta} = N_{\beta, \gamma} = N_{\gamma, \alpha} \quad \text{if } \alpha + \beta + \gamma = 0 \quad (80)$$

$$N_{\alpha, \beta} N_{\gamma, \delta} + N_{\beta, \gamma} N_{\alpha, \delta} + N_{\gamma, \alpha} N_{\beta, \delta} = 0 \quad \text{if } \alpha + \beta + \gamma + \delta = 0 \quad (81)$$

For completeness (though we hardly will use it) we mention that

$$\{N_{\alpha, \beta}\}^2 = \frac{1}{2} \langle \alpha, \alpha \rangle q(p + 1) \quad (82)$$

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with \( p \) and \( q \) positive integers such that the \( \alpha \)-string containing \( \beta \) is \( \beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha \). These conventions do not fix all signs, the remaining ones may be chosen freely.

The groups in this paper are in split form. We define the split form as the form that is generated by combinations of the \( H_j \) and \( E_\alpha \) with real coefficients (while for example the compact form has generators like \( iH_j, i(E_\alpha + E_{-\alpha}) \)). The maximal compact subgroup of the split form is generated by generators of the form \( E_\alpha - E_{-\alpha} \). The compact subgroup can be used to generate Weyl reflections on the root space, so, as in the case of compact groups, different labellings of the roots related by Weyl reflections are equivalent.

The set of roots is denoted by \( \Delta \). One can choose a basis for the root lattice consisting of simple roots. The simple roots \( \alpha_i \) are characterized by the property, that if \( \alpha_i \) and \( \alpha_j \) are simple roots, then \( \alpha_i - \alpha_j \notin \Delta \). The roots can be expanded in the simple roots, as

\[
\alpha = \sum_i p_i \alpha_i
\]

The coefficients \( p_i \) are integers, that are either all non-negative or all non-positive; consequently the non-zero roots can be divided in positive and negative roots. The set of positive roots is denoted by \( \Delta^+ \). We also define the height \( h \) of the root \( \alpha \) by, \( h = \sum_i p_i \); for a non-zero root this a non-zero number.

The fundamental weights are defined by

\[
2\frac{\langle \alpha_i, \lambda_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}
\]  

The weight lattice is spanned by the fundamental weights. Also to the weights one can attach a height, by expanding them in the simple roots, and summing up the coefficients in the expansion (that need not be integer). A standard result is that each irreducible representation is characterized by a unique highest weight.

The algebra defined by the \( H_i \) and \( E_\alpha \) allows the Chevalley involution \( \omega \), with action

\[
\omega : H_i \rightarrow -H_i \hspace{1cm} E_\alpha \rightarrow -E_{-\alpha}
\]  

The significance here of this involution lies in the fact that the generators invariant under the involution are of the form \( E_\alpha - E_{-\alpha} \), and generate the compact subgroup. Specifically, for the algebra of \( SL(n, \mathbb{R}) \), the invariant generators are the generators of \( SO(n) \). Given an algebra element \( T \), we define \( T^\# = -\omega(T) \). For the fundamental representation of \( SL(n, \mathbb{R}) \) the generators can be chosen such that \( T^\# \) acts as matrix transpose. The generators invariant under \( \omega \) satisfy \( T^\# = -T \), and hence are antisymmetric. This is the root of the name “generalized transpose” used in \([23]\) and the present paper. It is however possible to choose the matrix representation of the adjoint representation such that the “generalized transpose” is actually the transpose (be it in a much bigger representation). It is therefore possible to extend the action of the generalized transpose to the group, by setting \( (\exp A)^\# = \exp(A^\#) \), a property obviously satisfied by the “normal” transpose. The invariant subgroup is then defined by \( O^\#O = 1 \).
B Oxidation for cosets of split forms of Lie groups

In this appendix we list the decompositions of split Lie groups into the $SL(D-2) \times U_D$ groups that are central to this paper. The table has been compiled with the help of [35]. We also list all extended Dynkin diagrams, in figure 3.

![Extended Dynkin diagrams](image)

Figure 3: Extended Dynkin diagrams: The marked dot is the extended node. Erasing this node and all its connections leaves the standard Dynkin diagram.

We use the notations $\{e\}$ and $SL(1, \mathbb{R})$ for the trivial group, signifying triviality of the U-duality group, or that we are dealing with the 3-dimensional theory, respectively. A number in boldface denotes the dimension of the representation; the conjugate representation is denoted by putting a bar over the number. The representations of the Abelian groups ($\mathbb{R}$ factors) are labeled by their charges. They are normalized such that charge 1 would correspond to the minimal charge in the compact, $U(1)$-case. The listed charges therefore reflect the ratios rather than absolute charges. At the level of the equations of motion and Bianchi identities, the unit of charge can be absorbed in a redefinition of the corresponding dilaton. In the action however this may lead to an unconventional kinetic term for the dilaton.

With the contents of this appendix and the formalism developed in the text, the reader may rederive the results of [15], and fill in some small omissions in that paper (already mostly filled in in [10]). The list of representations in this appendix is complete in the sense that all relevant level 1 embeddings are given here, and hence, under the assumptions that lead us to restrict to level 1 subgroups, this appendix lists all the possible theories that can be oxidized from a given split simple Lie group. In [16] we will complete this project by setting up the theory for the non-split Lie groups.

The paper [15] discusses in detail the theories appearing here, in 3 dimensions and in their maximal dimension, and sometimes in 4 dimensions. Other references are given in the relevant subsections.
B.1 $A_n$

The split real form of $A_n$ is $SL(n+1, \mathbb{R})$. These theories oxidize to pure gravity in $n + 3$ dimensions. The reduction of these theories is discussed in detail in [9].

The table of decompositions is $(d + d' = n)$:

$$SL(n, \mathbb{R}) \times \{e\} : (n^2 - 1)$$

$$SL(d, \mathbb{R}) \times SL(d', \mathbb{R}) \times \mathbb{R} : (d^2 - 1, 1)^{0} \oplus (1, d^2 - 1)^{0} \oplus (1, 1)^{0} \oplus (d, d')^{n} \oplus (d, d')^{-n}$$

$$SL(1, \mathbb{R}) \times SL(n, \mathbb{R}) : (n^2 - 1)$$

B.2 $B_n$

The split real form of $B_n$ is $SO(n+1, n)$. Generically, these oxidize to $n + 2$ dimensions, and then contain gravity, an antisymmetric 2-tensor, a dilaton, and a vector. This matter content is typical for string theories; these theories are indeed included in the $SO(d+r, d)$ theories following from general considerations on toroidally compactified strings [36].

The $B_8$ case allows a supersymmetric extension. The $B_2$ case is interesting, because it is the minimal simple group that leads to (classical) S-duality in 4 dimensions. The $B_3$ case oxidizes to gravity in 6 dimensions coupled to a single self dual tensor. All $B_n$-theories with $n > 3$ have a separate branch in 6 dimensions, and all theories with $n \geq 2$ have a separate $SL(2, \mathbb{R})$ factor in 4 dimensions, signifying S-duality.

The table of decompositions is $(d + d' = n)$:

$$SL(n, \mathbb{R}) \times \mathbb{R} : (n^2 - 1)^{0} \oplus 1^{0} \oplus n^{1} \oplus \frac{n(n-1)}{2}^{2} \oplus \frac{n(n-1)}{2}^{-2} \oplus n^{-1}$$

$$SL(d, \mathbb{R}) \times SO(d', d' + 1) \times \mathbb{R} : (d^2 - 1, 1)^{0} \oplus (1, d'(2d' + 1))^{0} \oplus (1, 1)^{0} \oplus (d, 2d' + 1)^{1} \oplus (d, 2d' + 1)^{-1} \oplus (\frac{d(d-1)}{2}, 1)^{2} \oplus (\frac{d(d-1)}{2}, 1)^{-2}$$

$$SL(4, \mathbb{R}) \times SO(n-3, n-4) \times \mathbb{R} : (15, 1)^{0} \oplus (1, (n - 4)(2n - 7))^{0} \oplus (1, 1)^{0} \oplus (4, 2n - 7)^{1} \oplus (4, 2n - 7)^{-1} \oplus (6, 1)^{2} \oplus (6, 1)^{-2}$$

$$SL(3, \mathbb{R}) \times SO(n-2, n-3) \times \mathbb{R} : (8, 1)^{0} \oplus (1, (n - 3)(2n - 5))^{0} \oplus (1, 1)^{0} \oplus (3, 2n - 5)^{1} \oplus (3, 2n - 5)^{-1} \oplus (3, 1)^{-2} \oplus (3, 1)^{2}$$

$$SL(2, \mathbb{R}) \times SO(n-1, n-2) \times SL(2, \mathbb{R}) : (3, 1, 1)^{0} \oplus (1, (n - 2)(2n - 3), 1)^{0} \oplus (1, 1, 3)^{0} \oplus (2, 2n - 3, 2)$$

$$SL(1, \mathbb{R}) \times SO(n+1, n) : n(2n + 1)$$
There exists an alternative decomposition with a maximal $SL(4, \mathbb{R})$ factor

$$SL(4, \mathbb{R}) \times SO(n-2, n-3): \ (15, 1) \oplus (1, (n-3)(2n-5)) \oplus (6, 2n-5)$$

The table also gives the correct results for $4 \geq n > 1$, provided one restricts to the lines where the number $p$ in the always present $SO(p+1, p)$ factor is bigger or equal than 0 ($SO(1, 0)$ is the trivial group), and omits representations that would be 0-dimensional according to the above formulas.

We have $SO(2, 1) \cong SL(2, \mathbb{R})$ and this theory can be found in the previous paragraph. For $n = 3$ the “extra” line is possible, while the one in the table for $SL(4, \mathbb{R})$ is not.

### B.3 $C_n$

The split real form of $C_n$ is $Sp(n, \mathbb{R})$. These theories all oxidize to 4 dimensions, no matter what the value of $n$ is. The $C_2 \cong B_2$ theory has S-duality in 4 dimensions. The $C_n$ theories with $n > 2$ have an extended version of S-duality, involving a higher dimensional $Sp(n-1, \mathbb{R})$ group.

The table of decompositions is:

$$SL(2, \mathbb{R}) \times Sp(n-1, \mathbb{R}): \ (3, 1) \oplus (1, (n-1)(2n-1)) \oplus (2, 2n-2)$$

$$SL(1, \mathbb{R}) \times Sp(n, \mathbb{R}) : \ n(2n+1)$$

Note that for $n = 2$ we have $Sp(2, \mathbb{R}) \cong SO(3, 2)$ which confirms the previous analysis, while for $n = 1$ we have $Sp(1, \mathbb{R}) \cong SL(2, \mathbb{R})$.

### B.4 $D_n$

The split real form of $D_n$ is $SO(n, n)$. These theories generically oxidize to $n + 2$ dimensions, and contain gravity, an antisymmetric 2-tensor, a dilaton, again typical for string theories. Indeed, they are included in the $SO(d+r, d)$ theories following from general considerations [36]. The $D_8$ case allows a supersymmetric extension, and corresponds then to pure type I supergravity in 10 dimensions [37]. The $D_3$ case oxidizes to pure gravity in 6 dimensions. All $D_n$-theories with $n > 3$ have a separate branch in 6 dimensions, and all theories with $n \geq 2$ have a separate $SL(2, \mathbb{R})$ factor in 4 dimensions, signifying S-duality. The $D_4$ theory can be oxidized to 6 dimensions; compactifying this theory on a circle it displays a triality rather than a duality.
The table of decompositions is:

\[
\begin{align*}
SL(n, \mathbb{R}) \times \mathbb{R} & : (n^2-1)^0 \oplus 1^0 \oplus \frac{n(n-1)}{2}^2 \oplus \frac{n(n-1)}{2}^{-2} \\
\vdots & : \vdots \\
SL(d, \mathbb{R}) \times SO(d', d') \times \mathbb{R} & : (d^2-1, 1)^0 \oplus (1, d'(2d'-1))^0 \oplus (1, 1)^0 \oplus (d, 2d')^1 \oplus (d', 2d')^{-1} \oplus (\frac{d(d-1)}{2}, 1)^2 \oplus (\frac{d(d-1)}{2}, 1)^{-2} \\
\vdots & : \vdots \\
SL(4, \mathbb{R}) \times SO(n-4, n-4) \times \mathbb{R} & : (15, 1)^0 \oplus (1, (n-4)(2n-9))^0 \oplus (1, 1)^0 \oplus (4, 2n-8)^1 \oplus (3, 2n-8)^{-1} \oplus (6, 1)^2 \oplus (6, 1)^{-2} \\
SL(3, \mathbb{R}) \times SO(n-3, n-3) \times \mathbb{R} & : (8, 1)^0 \oplus (1, (n-3)(2n-7))^0 \oplus (1, 1)^0 \oplus (3, 2n-6)^1 \oplus (3, 2n-6)^{-1} \oplus (3, 1)^2 \oplus (3, 1)^{-2} \\
SL(2, \mathbb{R}) \times SO(n-2, n-2) \times SL(2, \mathbb{R}) : (3, 1, 1) \oplus (1, (n-2)(2n-5), 1) \oplus (1, 1, 3) \oplus (2, 2n-4, 2) \\
SL(1, \mathbb{R}) \times SO(n, n) & : n(2n-1) \\
\end{align*}
\]

There exists an alternative decomposition with a maximal $SL(4, \mathbb{R})$ factor

\[
SL(4, \mathbb{R}) \times SO(n-3, n-3) : (15, 1) \oplus (1, (n-3)(2n-7)) \oplus (6, 2n-6)
\]

The table gives the correct results for $4 \geq n > 1$, provided one restricts to the lines where the number $p$ in the always present $SO(p, p)$ factor is bigger or equal than 0 ($SO(0, 0)$ is the trivial group), and omit representations that would be 0-dimensional according to the above formula’s.

Notice that $SO(2, 2) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is not simple. This theory allows an outer automorphism (T-duality) in 3 dimensions, interchanging the $SL(2, \mathbb{R})$ of gravity with the $SL(2, \mathbb{R})$ of S-duality in 4 dimensions. Finally, $SO(1, 1) \cong \mathbb{R}$; this theory cannot be oxidized, because we require a semi-simple factor.

Note that again for $n = 3$ the “extra” line is possible, while the one in the table for $SL(4, \mathbb{R})$ is not. This is due to the isomorphism $SO(3, 3) \cong SL(4, \mathbb{R})$. Notice that this theory gives pure gravity, as it should.

### B.5 $E_6$

The split form of $E_6$ is $E_{6(6)}$. This theory oxidizes to 8 dimensions. Compactifying it on a 2-torus it allows a version of T-duality acting on two dimensions simultaneously (“M-theory T-duality”)
The relevant decompositions are:

\[
\begin{align*}
SL(6, \mathbb{R}) \times SL(2, \mathbb{R}) & : (35, 1) \oplus (1, 3) \oplus (20, 2) \\
SL(5, \mathbb{R}) \times SL(2, \mathbb{R}) \times \mathbb{R} & : (24, 1)^0 \oplus (1, 3)^0 \oplus (1, 1)^0 \oplus (5, 1)^0 \oplus (10, 2)^3 \oplus (\overline{10}, 2)^{-3} \\
SL(4, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times \mathbb{R} & : (15, 1, 1)^0 \oplus (1, 3, 1)^0 \oplus (1, 1, 3)^0 \oplus (1, 1, 1)^0 \oplus (4, 1, 2)^3 \oplus (4, 2, 1)^{-3} \oplus (\overline{4}, 1, 2)^{-3} \oplus (\overline{4}, 2, 1)^3 \oplus (6, 2, 2)^0 \\
SL(3, \mathbb{R}) \times SL(3, \mathbb{R}) \times SL(3, \mathbb{R}) & : (8, 1, 1) \oplus (1, 8, 1) \oplus (1, 1, 8) \oplus (3, 3, 3) \oplus (\overline{3}, \overline{3}, \overline{3}) \\
SL(2, \mathbb{R}) \times SL(6, \mathbb{R}) & : (3, 1) \oplus (1, 35) \oplus (2, 20) \\
SL(1, \mathbb{R}) \times E_6(6) & : 78
\end{align*}
\]

B.6 \( E_7 \)

The split form of \( E_7 \) is \( E_{7(7)} \). The theory oxidizes to 10 dimensions, where it represents a truncated version of IIB gravity, but also has a separate branch in 8 dimensions.

The relevant decompositions are:

\[
\begin{align*}
SL(8, \mathbb{R}) \times \{e\} & : 63 \oplus 70 \\
SL(7, \mathbb{R}) \times \mathbb{R} & : 48^0 \oplus 1^0 \oplus 7^4 \oplus 7^{-4} \oplus 35^2 \oplus 35^{-2} \\
SL(6, \mathbb{R}) \times SL(2, \mathbb{R}) \times \mathbb{R} & : (35, 1)^0 \oplus (1, 3)^0 \oplus (1, 1)^0 \oplus (6, 2)^2 \oplus (\overline{6}, 2)^{-2} \oplus (15, 1)^{-2} \oplus (\overline{15}, 1)^2 \oplus (20, 2)^0 \\
SL(5, \mathbb{R}) \times SL(3, \mathbb{R}) \times \mathbb{R} & : (24, 1)^0 \oplus (1, 8)^0 \oplus (1, 1)^0 \oplus (5, 1)^{-6} \oplus (\overline{5}, 1)^6 \oplus (5, 3)^4 \oplus (\overline{5}, \overline{3})^{-4} \oplus (10, 3)^{-2} \oplus (\overline{10}, 3)^2 \\
SL(4, \mathbb{R}) \times SL(4, \mathbb{R}) \times SL(2, \mathbb{R}) & : (15, 1, 1) \oplus (1, 15, 1) \oplus (1, 1, 3) \oplus (4, 4, 2) \oplus (\overline{4}, \overline{4}, 2) \oplus (6, 6, 1) \\
SL(3, \mathbb{R}) \times SL(6, \mathbb{R}) & : (8, 1) \oplus (1, 35) \oplus (3, 15) \oplus (\overline{3}, 15) \\
SL(2, \mathbb{R}) \times SO(6, 6) & : (3, 1) \oplus (1, 66) \oplus (2, 32) \\
SL(1, \mathbb{R}) \times E_7(7) & : 133
\end{align*}
\]

There exists an alternative decomposition with a maximal \( SL(6, \mathbb{R}) \) factor

\[
SL(6, \mathbb{R}) \times SL(3, \mathbb{R}) : (35, 1) \oplus (1, 8) \oplus (15, 3) \oplus (\overline{15}, \overline{3})
\]

B.7 \( E_8 \)

The split form of \( E_8 \) is \( E_{8(8)} \). The 3 dimensional theory was constructed in [39]. This theory oxidizes to the bosonic sector of 11 dimensional supergravity [1]. A separate branch in 10 dimensions oxidizes to the bosonic sector of IIB gravity [38]. This chain includes all maximal supergravity theories in dimensions \( \geq 3 \) [2](see e.g. [13]).
The relevant decompositions are:

\[
\begin{align*}
SL(9, \mathbb{R}) \times \{e\} &: 80 \oplus 84 \oplus 84 \\
SL(8, \mathbb{R}) \times \mathbb{R} &: 63^9 \oplus 1^1 \oplus 8^{-1} \oplus 8^1 \oplus 28^2 \oplus 28^{-2} \oplus 56^{-3} \oplus 56^3 \\
SL(7, \mathbb{R}) \times SL(2, \mathbb{R}) \times \mathbb{R} &: (48, 1)^0 \oplus (1, 3)^0 \oplus (1, 1)^0 \oplus (7, 1)^4 \oplus (7, 1)^{-4} \oplus (7, 2)^{-3} \oplus (7, 2)^3 \oplus (21, 2)^1 \oplus (21, 2)^{-1} \oplus (35, 1)^{-2} \oplus (35, 1)^2 \\
SL(6, \mathbb{R}) \times SL(3, \mathbb{R}) \times SL(2, \mathbb{R}) &: (35, 1, 1) \oplus (1, 8, 1) \oplus (1, 1, 3) \oplus (6, 3, 2) \oplus (6, \overline{3}, 2) \oplus (15, \overline{3}, 1) \oplus (15, 3, 1) \oplus (20, 1, 2) \\
SL(5, \mathbb{R}) \times SL(5, \mathbb{R}) &: (24, 1) \oplus (1, 24) \oplus (5, 10) \oplus (10, 5) \oplus (10, 5) \oplus (5, 10) \\
SL(4, \mathbb{R}) \times SO(5, 5) &: (15, 1) \oplus (1, 45) \oplus (4, 16) \oplus (4, \overline{16}) \oplus (6, 10) \\
SL(3, \mathbb{R}) \times E_6(6) &: (8, 1) \oplus (1, 78) \oplus (3, 27) \oplus (3, \overline{27}) \\
SL(2, \mathbb{R}) \times E_7(7) &: (3, 1) \oplus (1, 133) \oplus (2, 56) \\
SL(1, \mathbb{R}) \times E_8(8) &: 248
\end{align*}
\]

There exist an alternative decomposition with a maximal \(SL(8, \mathbb{R})\) factor.

\[
SL(8, \mathbb{R}) \times SL(2, \mathbb{R}) : (63, 1) \oplus (1, 3) \oplus (28, 2) \oplus (\overline{28}, 2) \oplus (70, 1)
\]

**B.8 \( F_4 \)**

The split form of \(F_4\) is \(F_{4(4)}\). This gives an intriguing theory in 6 dimensions, discussed in detail in an appendix of [15]. The lower dimensional theories can be found in [40]. All allow supersymmetric extensions.

The relevant decompositions are:

\[
\begin{align*}
SL(4, \mathbb{R}) \times SL(2, \mathbb{R}) &: (15, 1) \oplus (1, 3) \oplus (4, 2) \oplus (\overline{4}, 2) \oplus (6, 3) \\
SL(3, \mathbb{R}) \times SL(3, \mathbb{R}) &: (8, 1) \oplus (1, 8) \oplus (3, \overline{6}) \oplus (\overline{3}, 6) \\
SL(2, \mathbb{R}) \times Sp(3, \mathbb{R}) &: (3, 1) \oplus (1, 21) \oplus (2, 14) \\
SL(1, \mathbb{R}) \times F_{4(4)} &: 52
\end{align*}
\]

**B.9 \( G_2 \)**

The split form of \(G_2\) is \(G_{2(2)}\). This theory oxidizes to the bosonic sector of simple supergravity in 5 dimensions. Details can be found in [11] (see also [12]).

The relevant decompositions are:

\[
\begin{align*}
SL(3, \mathbb{R}) \times \{e\} &: 8 \oplus 3 \oplus 3 \\
SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) &: (3, 1) \oplus (2, 4) \oplus (1, 3) \\
SL(1, \mathbb{R}) \times G_{2(2)} &: 14
\end{align*}
\]
C Other relevant group theoretic information

C.1 Maximal compact subgroups for split forms

The maximal compact subgroups for the split forms are compiled in the following table:

|   |   |   |
|---|---|---|
| $A_n$ | $SL(n, \mathbb{R})$ | $SO(n)$ |
| $B_n$ | $SO(n+1, n)$ | $SO(n+1) \times SO(n)$ |
| $C_n$ | $Sp(n, \mathbb{R})$ | $U(n)$ |
| $D_n$ | $SO(n, n)$ | $SO(n) \times SO(n)$ |
| $E_6$ | $E_6(6)$ | $Sp(4)$ |
| $E_7$ | $E_7(7)$ | $SU(8)$ |
| $E_8$ | $E_8(8)$ | $SO(16)$ |
| $F_4$ | $F_4(4)$ | $Sp(3) \times SU(2)$ |
| $G_2$ | $G_2(2)$ | $SO(4)$ |

C.2 Isomorphism of groups

In this section we have collected several groups with isomorphic algebra’s. Although their global topology may be different, we have adapted to the common practice in the physics literature, and disregarded global differences.

Equivalences for compact groups:

$U(1) \cong SO(2)$
$SU(2) \cong SO(3) \cong Sp(1)$
$SU(4) \cong SO(6)$
$Sp(2) \cong SO(5)$
$SO(4) \cong SU(2) \times SU(2)$

Equivalences for non-compact groups:

$\mathbb{R} \cong SO(1, 1)$
$SU(1, 1) \cong SL(2, \mathbb{R}) \cong SO(1, 2) \cong Sp(1, \mathbb{R})$
$SL(4, \mathbb{R}) \cong SO(3, 3)$
$SU(2, 2) \cong SO(4, 2)$
$SU^*(4) \cong SO(5, 1)$
$SU(1, 3) \cong SO^*(6)$
$Sp(1, 1) \cong SO(4, 1)$
$Sp(2, \mathbb{R}) \cong SO(3, 2)$
$SO(2, 2) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$
$SO(1, 3) \cong SL(2, \mathbb{C})$
$SO(6, 2) \cong SO^*(8)$
D  The sigma model equations of motion

The derivation of the equations and Bianchi identities (28), (39) and (42) is straightforward. Proving that the sigma model equations of motion are given by (31) however requires a few subtleties.

Taking the derivative of (30), and using the Bianchi identity (28) and again (30), one arrives at (all Greek symbols stand for positive roots, wedges are omitted to save space)

\[
dF_{(D-1)-\gamma} = \sum_{\alpha-\beta=-\gamma} N_{\alpha,\beta} F_{(1)\alpha} F_{(D-2)-\beta} + \\
\sum_{\alpha-\beta=-\gamma, \delta-\epsilon=-\beta} N_{\alpha,\beta} N_{\delta,\epsilon} F_{(1)\alpha} F_{(1)\delta} A_{(D-2)-\epsilon} - \frac{1}{2} \sum_{\alpha-\beta=-\gamma, \delta+\epsilon=\alpha} N_{\alpha,\beta} N_{\delta,\epsilon} F_{(1)\delta} F_{(1)\epsilon} A_{(D-2)-\beta} \tag{85}
\]

The first line of eq. (85) is eq. (31), so we have to demonstrate that the second line vanishes. Note that it must vanish, since terms with bare \( A_{(D-2)-\beta} \) cannot be gauge invariant.

We proceed with the second line of eq. (85). We eliminate \( \beta \) from the first sum, and rename dummy indices in the second sum:

\[
\sum_{\alpha+\delta-\epsilon=-\gamma} N_{\alpha,\delta} N_{\delta,\epsilon} F_{(1)\alpha} F_{(1)\delta} A_{(D-2)-\epsilon} - \frac{1}{2} \sum_{\alpha+\delta-\gamma} N_{\beta,\alpha} N_{\alpha,\delta} F_{(1)\alpha} F_{(1)\delta} A_{(D-2)-\epsilon} \tag{86}
\]

To eliminate \( \beta \) from the second term, we use (80). In the first term, we antisymmetrize the combination of the two structure constants in \( \alpha \) and \( \delta \), and obtain

\[
\frac{1}{2} \sum_{\alpha+\delta-\epsilon=-\gamma} (N_{\alpha,\delta-\epsilon} N_{\delta,\epsilon} - N_{\delta,\alpha-\epsilon} N_{\alpha,\epsilon} - N_{\epsilon,\gamma} N_{\alpha,\delta}) F_{(1)\alpha} F_{(1)\delta} A_{(D-2)-\epsilon} \tag{87}
\]

We now study the terms of (87), for fixed \( \alpha, \delta, \epsilon \). Of course one has \( \alpha \neq \delta \), and also \( \alpha \neq -\epsilon \), \( \delta \neq -\epsilon \). A possibility is that \( \alpha = \gamma \). In that case one has \( \delta - \epsilon = -2\alpha \) and \( N_{\alpha,\delta-\epsilon} \) vanishes. One can use the antisymmetry of the structure constants to show that the remaining two terms (with the same \( \alpha, \delta, \epsilon \)) cancel against each other. A similar argument applies if \( \delta = \gamma \). The last possibility is that \( \alpha, \gamma, \delta \) and \( -\epsilon \) are all distinct. Then we use antisymmetry of the structure constants, and (80) to rewrite the relevant terms to

\[
(N_{\epsilon,\delta} N_{\alpha,\gamma} + N_{\alpha,-\epsilon} N_{\delta,\gamma} + N_{\delta,\alpha} N_{\epsilon,\gamma}) F_{(1)\alpha} F_{(1)\delta} A_{(D-2)-\epsilon} \tag{88} \quad (\alpha + \delta - \epsilon + \gamma = 0),
\]

which vanishes by virtue of eq. (81). This completes the derivation of eq. (31).
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