Privacy Amplification via Random Participation in Federated Learning

Burak Hasırcıoğlu and Deniz Gündüz

Information Processing and Communications Lab, Imperial College London, UK

E-mail: {b.hasircio6lu18, d.gunduz}@imperial.ac.uk

Abstract

Running a randomized algorithm on a subsampled dataset instead of the entire dataset amplifies differential privacy guarantees. In this work, in a federated setting, we consider random participation of the clients in addition to subsampling their local datasets. Since such random participation of the clients creates correlation among the samples of the same client in their subsampling, we analyze the corresponding privacy amplification via non-uniform subsampling. We show that when the size of the local datasets is small, the privacy guarantees via random participation is close to those of the centralized setting, in which the entire dataset is located in a single host and subsampled. On the other hand, when the local datasets are large, observing the output of the algorithm may disclose the identities of the sampled clients with high confidence. Our analysis reveals that, even in this case, privacy guarantees via random participation outperform those via only local subsampling.

1 Introduction

Federated learning (FL) framework allows several clients to collaboratively and iteratively learn from each other’s data with the coordination of a parameter server (PS) [McMahan et al., 2017]. In a typical scenario, the PS averages the model updates received from clients based on their local data. Then, it updates the global model and broadcasts the updated global model back to clients for the next iteration. In this learning model, since the data itself never leaves the participating clients, in terms of privacy, the FL framework is usually perceived as superior to centralized training, in which entire data is offloaded to a central server. Although this is true to some extent, many recent works [Melis et al., 2019, Zhu et al., 2019, Geiping et al., 2020, Carlini et al., 2019, Salem et al., 2020] have shown that the model updates from the clients as well as the final deployed model can leak many features of the clients’ local datasets, including reconstruction of some data samples used for training. Therefore, additional mechanisms with formal and quantifiable privacy guarantees need to be employed in FL.

Differential privacy (DP) [Dwork et al., 2014] is the gold standard quantifying the privacy leakage in privacy-preserving data analysis tasks. It is a measure of the indistinguishability...
between two outputs of an algorithm when two neighboring datasets are fed into the algorithm as inputs. A deterministic algorithm can be made differentially private by randomizing its output as long as the introduced randomness is independent from the input and secret to the adversary. Output perturbation via an additive noise such as Gaussian or Laplace noise [Dwork et al., 2006] is a common example of such randomization techniques. However, there exist other sources of randomness that do not guarantee indistinguishability alone, but they can amplify the final DP guarantees when they are cascaded with techniques already guaranteeing DP. Subsampling [Chaudhuri and Mishra, 2006, Balle et al., 2018, Wang et al., 2019, Mironov et al., 2019] and shuffling [Erlingsson et al., 2019, Feldman et al., 2020] are among the most prominent of such tools. In subsampling, a random subset from the original dataset is sampled to be used as the input to the privacy-preserving mechanism. In shuffling, on the other hand, the outputs resulting from different inputs are randomly shuffled so that mapping between the inputs and the outputs are masked, resulting in anonymized outputs. Such privacy amplification techniques further confuse an adversary when used in conjunction with output perturbation techniques, and the amplified privacy guarantees are achieved without sacrificing the utility.

In this work, we consider a FL setting with distributed stochastic gradient descent (DSGD) using a trusted PS, in which formal DP guarantees are achieved for all intermediate models including the final deployed one. For this, we update the global model by the average of the gradients collected from participating clients, which is perturbed via an additive Gaussian noise, following the seminal work of Abadi et al. [Abadi et al., 2016]. To further amplify the privacy guarantee, we consider two types of sampling: (1) client sampling, and (2) local dataset sampling. In each iteration, first, available clients during that iteration randomly decide whether to participate or not, independently from the decisions of other clients. Then, each client that decides to participate samples a subset from its local dataset such that each element is sampled independently from the other elements in the local dataset. That is, in both sampling phases, we employ Poisson sampling separately. Analysing the privacy amplification guarantee of this sampling technique in a federated setting is challenging. More specifically, random participation of the clients introduces correlation between the elements located in a single client in their sampling. For example, given an element is sampled from a specific client, the probability of another element being sampled from the same client is larger than the probability of another element from another client being sampled. Therefore, employing random participation of clients and local dataset sampling together poses a non-uniform sampling problem, and in this work, we analyse the central DP guarantees of such a setting.

1.1 Related Work and Motivation

For FL scenarios with many participants, sampling the participating clients is essential for efficient use of the power and the communication resources even when privacy is not a constraint. Consider a learning problem with millions of mobile devices. Thousands of iterations are carried out in a typical learning task, and if each mobile device participates in every
iteration, the power spent on the computations and the extra communication to transmit the results to the PS will result in a very large overall energy cost. In addition, communication overhead of such a dense participation may cause congestion in the communication networks; and thus, is not scalable. Moreover, for the learning tasks employing mini-batch optimization techniques, limited number of samples per iteration is preferable. Hence, client sampling can help to reach the targeted batch size in addition to the local dataset sampling. All these aforementioned benefits make client sampling one of the indispensable components of FL, whose effect on the convergence and final performance of FL algorithms have been widely studied [Chen et al., 2020, Ruan et al., 2021, Cho et al., 2020]. In addition to these benefits, since it brings additional randomness to the training procedure, it is expected to bring some additional amplification to the DP guarantees. Hence, rather than proposing the inclusion of a new mechanism to amplify DP guarantees, in this work, our goal is to utilize an already-employed mechanism to improve the privacy guarantees of the task.

Differentially private deep learning in a centralized setting is studied in [Abadi et al., 2016]. The authors employ Poisson subsampling of the dataset and the gradient of every sample is clipped and Gaussian noise is added to it. They further introduce the moments accountant technique to calculate the composition of the total privacy leakage throughout the iterations. Similar techniques to those in [Abadi et al., 2016] are extended to the federated setting in [Geyer et al., 2017, McMahan et al., 2018, Seif et al., 2021]. In these works, similarly to our approach, client sampling is considered, but the achieved DP guarantees are at the client-level, which means the participation of a client is indistinguishable from its non-participation. Although client-level DP guarantees are meaningful when the local dataset is composed of the elements from the same source, e.g., personal photos stored in a mobile phone, in some cases, it might be too conservative. To achieve DP guarantees protecting all the elements in a client, a larger amount of noise must be introduced compared to sample-level guarantees, and hence, client-level guarantees result in a larger loss in utility. Moreover, in some cases, the datasets stored by the clients may comprise of samples from different individuals, hence, it may not be necessary to protect client-level privacy. For instance, while learning from medical data, each client may represent a hospital and sample-level DP guarantees would suffice [Malekzadeh et al., 2021].

Sample-level DP in the federated setting with client sampling is studied in [Balle et al., 2020, Girgis et al., 2021a, Girgis et al., 2021b] together with shuffling. In the analysis carried out in [Balle et al., 2020], each client is assumed to store only one sample, in which case the client-level guarantees are equivalent to sample-level guarantees. While each client is allowed to store more than one sample in [Girgis et al., 2021a, Girgis et al., 2021b], the number of sampled elements is determined in advance. In [Girgis et al., 2021b], only one element is sampled from the local dataset at each iteration, while more than one but a constant number of elements are sampled in [Girgis et al., 2021a]. It is also worth noting that different from [Balle et al., 2020] and the current paper, in [Girgis et al., 2021a, Girgis et al., 2021b], communication efficiency is studied together with privacy, and a random compression mechanism is used as the randomizer. In all of these works [Balle et al., 2020, Girgis et al., 2021a, Girgis et al., 2021b], the clients are assumed to employ a local randomizer that satisfies pure DP, which we will
formally define in Section 2, and the privacy analysis is based on shuffling the local responses. Such an assumption of pure DP for the local randomizers turns out to be useful while jointly analysing client sampling and shuffling. However, if the local randomizers provide approximate DP guarantees instead of pure DP, which is the case when Gaussian noise is used as the randomizer, central privacy guarantees may be substantially degraded. Therefore, novel methods to jointly analyze privacy amplification via client sampling and local dataset sampling that do not rely on shuffling are needed, and this constitutes the main motivation and contribution of our work.

1.2 Our Contributions

The major contributions of our work and its novelty with respect to the current literature can be summarized as follows:

- We propose a client sampling algorithm for privacy amplification in FL. Each available client randomly decides to participate or not at each iteration. Having decided to participate, each client then samples its local dataset randomly.

- Since each client decides to participate or not independently from any other external factor, it can account for its own privacy loss after each iteration without requiring any other iteration-specific information about the system, such as the number of available clients, the number of sampled clients, or the number of sampled elements by the participating clients. This local sampling and privacy accounting mechanism makes the implementation of our algorithm feasible in practice.

- We provide a theoretical analysis of the central privacy guarantees of the proposed algorithm. Since random client participation leads to a non-uniform sampling among the data points stored at different clients, the analysis is non-trivial and cannot be directly obtained from the standard privacy amplification results via subsampling. Unlike the literature, our analysis does not rely on shuffling, and we do not need local randomizers with pure DP guarantees. This is especially useful when Gaussian noise is used as a randomizer.

2 Preliminaries

In this section, we review some preliminary notions about DP and subsampling, which will be used in the discussion of our algorithm and its privacy analysis.

**Definition 2.1** (Differential Privacy). A randomized mechanism $M : D \rightarrow S$ is $(\varepsilon, \delta)$ differentially private (DP) if

$$\Pr[M(D) \in S] \leq e^\varepsilon \Pr[M(D') \in S] + \delta,$$

for all neighboring datasets $D$ and $D'$, i.e., sets differing in only one element, and $\forall S \subset S$, where $\varepsilon > 0$ and $\delta \in [0, 1)$. If $\delta = 0$, then $M$ is called $\varepsilon$ pure differentially private, or $\varepsilon$-DP.
The neighboring relation between $D$ and $D'$ depends on the context. In our work, we assume $D$ can be generated from $D'$ by either removing or adding one element. $\delta$ in the definition above quantifies the failure probability of the relation $\Pr[\mathcal{M}(D) \in \mathcal{S}] \leq e^\varepsilon \Pr[\mathcal{M}(D') \in \mathcal{S}]$. Thus, $(\varepsilon, \delta)$-DP is, in fact, a relaxation of $\varepsilon$-DP.

Alternative to the above definition, $(\varepsilon, \delta)$-DP can be equivalently expressed in terms of hockey stick divergence, which we introduce next.

**Definition 2.2** ([Sason and Verdú, 2016]). We define the **hockey stick divergence** between two probability measures $\mu$ and $\mu'$ as

$$D_\alpha(\mu \| \mu') \triangleq \int_Z [d\mu(z) - \alpha d\mu'(z)]_+ d(z)$$

where $[\cdot]_+ \triangleq \max\{0, \cdot\}$.

Based on hockey stick divergence, $(\varepsilon, \delta)$-DP can be expressed as follows.

**Theorem 2.3** ([Barthe and Olmedo, 2013], Theorem 1 in [Balle et al., 2018]). A mechanism $\mathcal{M}$ is $(\varepsilon, \delta)$-DP if and only if $\sup_{D, D'} D_\alpha(\mathcal{M}(D) \| \mathcal{M}(D')) \leq \delta$, where $D$ and $D'$ are two neighboring datasets and $\alpha = e^\varepsilon$.

From Theorem 2.3, we can conclude that there is a trade-off between $\varepsilon$ and $\delta$ and we can perceive $\delta$ as a function of $\varepsilon$.

Gaussian mechanism is commonly used to achieve differential privacy guarantees for machine learning tasks [Abadi et al., 2016]. The following theorem formally characterizes its guarantees.

**Theorem 2.4** (Theorem 8 in [Balle and Wang, 2018]). Let $f : \mathcal{D} \to \mathbb{R}^m$ be a function with $\|f(D) - f(D')\|_2 \leq C$, where $D$ and $D'$ are neighboring datasets and $\| \cdot \|_2$ denotes the $L_2$ norm. A mechanism $\mathcal{M}(D) = f(D) + \mathcal{N}(0, \sigma^2)$ is $(\varepsilon, \delta)$-DP if and only if

$$\Phi\left(\frac{C}{2\sigma} - \frac{\varepsilon\sigma}{C}\right) - e^\varepsilon \Phi\left(-\frac{C}{2\sigma} - \frac{\varepsilon\sigma}{C}\right) \leq \delta,$$

where $\Phi$ is the cumulative distribution function (CDF) of the standard normal distribution.

Next, we present another result from [Balle et al., 2018], which characterizes the privacy amplification via subsampling.

**Theorem 2.5** (Theorem 8 in [Balle et al., 2018]). Let $\mathcal{M}$ be a randomized mechanism satisfying $(\varepsilon, \delta_M(\varepsilon))$-DP. We define $\mathcal{M}'$ as the randomized mechanism when the input is first sampled from a larger dataset with Poisson sampling with probability $q$, then $\mathcal{M}$ is applied on the sampled subset. Then, we have $\delta_{\mathcal{M}'}(\varepsilon') \leq q \delta_M(\varepsilon)$, where $\varepsilon' = \log(1 + q(e^\varepsilon - 1))$.

Note that this result is only valid for uniform sampling, i.e., the cases in which a subset is sampled from a larger dataset such that each element is sampled independently and identically distributed (i.i.d.) with probability $q$. Hence, it does not consider any correlation between the sampled elements. However, it serves as a baseline to evaluate the performance of our scheme.
3 Problem Setting

In our setting, we consider a FL scenario with $N$ clients and a PS. Client $i \in [N]$ has a dataset $D_i$ such that $D = \bigcup_{i \in [N]} D_i$. For simplicity, we assume that each client holds the same number of data samples, i.e., $|D_i| = d$, $\forall i \in [N]$, but our analysis holds for different $|D_i|$ values as well. We allow one of the clients to violate this assumption to satisfy the neighboring relation, i.e., at most one of the clients can store a dataset of size $d + 1$. We consider differentially private distributed stochastic gradient descent (DP-DSGD) in a FL setting. Namely, the clients and the PS collectively learn a model $w \in \mathbb{R}^m$ through $T \in \mathbb{Z}^+$ iterations by minimizing the loss function

$$
\hat{\ell}(w, D) = \frac{1}{N} \sum_{i \in [N]} \frac{1}{|D_i|} \sum_{s \in D_i} \ell(w, s),
$$

where $\ell$ is a general loss function depending on the nature of the learning problem. In general, we allow it to be non-convex; hence, our results are applicable to deep neural networks. Since clients can go offline for several reasons in FL, in each iteration $t$, we assume $N_t \leq N$ clients are available and $N_t$ is known to the PS but not necessarily to the clients.

3.1 Threat Model

Following [Balle et al., 2020, Girgis et al., 2021a, Girgis et al., 2021b], we assume a trusted PS and honest but curious clients, i.e., they do not deviate from the protocol but they can try to learn about the local datasets of other clients. We further assume the presence of secure communication channel between each client and the PS such that neither the presence of communication nor the content of the transmitted messages is disclosed to third parties. Therefore, in the case of random participation, for an adversary, it is not possible to infer the participating clients by tapping into the channels.

Trusted PS assumption may sound strong, but due to the nature of the client sampling problem, it is essential. Keeping the identities of the participating clients secret from the PS is difficult without employing a trusted third party. Hence, as done in similar works [Balle et al., 2020, Girgis et al., 2021a, Girgis et al., 2021b], we make this assumption. This assumption is valid for the scenarios in which the training is done with the help of a trusted PS, and the model is publicly deployed after training. If the PS is not trusted, client sampling can still be employed for a price of increased communication cost, and our analysis still holds. We elaborate on this in Section 5.5.

4 Main Results

In this section, we first present the details of the proposed algorithm and then we state its privacy guarantees.

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$^1$We define $[N] \triangleq \{1, 2, \ldots, N\}$. 
4.1 Algorithm Description

At the beginning of each iteration $t \in [T]$, independently from other clients, each client randomly decides whether to participate or not in that iteration, with probability $p$. If a client decides to participate, then it further samples a subset from its local dataset such that each element is sampled i.i.d. with probability $q$. Let the set of sampled clients in iteration $t$ be $P_t$ and the set of sampled elements in client $i \in P_t$ be $S_{i,t}$. After sampling, client $i \in P_t$ computes the gradients $\nabla \ell(w, s)$ for all $s \in S_{i,t}$. If the $L_2$ norm of the gradient of any sample is greater than a predetermined constant $C$, then it is scaled down to $C$ to guarantee $\|\nabla \ell(w, s)\|_2 \leq C$. Then, the client $i$ aggregates all the sample gradients, and sends the sum to the PS. The PS further aggregates all the summations from the participating clients and adds a Gaussian noise $N(0, \sigma^2 I_m)$ to the sum, where $I_m$ is the identity matrix with dimension $m$. Then, it scales the noisy sum by $\frac{1}{pN_t q d}$ to have an unbiased estimate of the gradient. The final expression after these operations at the PS is

$$\hat{g}_t = \frac{1}{pN_t q d} \left( \sum_{i \in P_t} \sum_{s \in S_{i,t}} \nabla \ell(w, s) + N(0, \sigma^2 I_m) \right).$$

Finally, the PS updates the model as $w_{t+1} = w_t - \eta_t \hat{g}_t$, where $\eta_t$ is the learning rate for the iteration $t$. The pseudo-code for our procedure is given in Algorithm 1.

4.2 Privacy Guarantees

In the following theorem, we present the privacy guarantees of DP-DSGD with random participation.

**Theorem 4.1.** Each iteration of DP-DSGD with random participation algorithm is $(\varepsilon, \delta)$-DP, where, for any $\varepsilon' > 0$,

$$\varepsilon = \log \left( 1 + pq \left( e^{\varepsilon'} - 1 \right) \right),$$

$$\delta = pq \left( 1 - \alpha' \bar{c}_2 + \alpha' \bar{c}_1 \left( \Phi_{0,\sigma}(z^*) - 1 \right) + \sum_{i=0}^{d} \binom{d}{i} q^i (1-q)^{d-i} \left( \alpha' \bar{c}_2 \Phi_{C,\sigma}(z^*) - \Phi_{(i+1)C,\sigma}(z^*) \right) \right),$$

where

$$\bar{c}_1 \triangleq (1 - e^{\varepsilon - \varepsilon'}) \frac{1-p}{1-pq}, \quad \bar{c}_2 \triangleq \frac{p(1-q)}{1-pq} \left( 1 - e^{\varepsilon - \varepsilon'} \right) + e^{\varepsilon - \varepsilon'}, \quad \Phi_{C,\sigma}(z^*)$$

is the CDF of the Gaussian distribution with mean $C$ and standard deviation $\sigma$, and $z^*$ is the solution, for $z$, of the following equation:

$$\sum_{i=0}^{d} \binom{d}{i} q^i (1-q)^{d-i} \left( N(z, (i+1)C, \sigma^2) - \alpha' \bar{c}_2 N(z, iC, \sigma^2) \right) - \alpha' \bar{c}_1 N(z, 0, \sigma^2) = 0. \quad (7)$$

Note that although the analytical solution of Equation (7) is not tractable, it can be efficiently solved numerically. We provide a more detailed discussion on this and the proof of Theorem 4.1 in Appendix A.
Algorithm 1 DP-DSGD with random participation

Protocol in client $i$:

for $t \in [T]$ do
    Sample $B \sim \text{Bern}(p)$
    if $B = 1$ then
        Inform the PS that $B = 1$
        Receive $w_t$ from the PS
        Initialize $g_i = 0$
        Sample $S_i$ from $D_i$, w.p. $q$, i.i.d. for each sample
        for $s \in S_i$ do
            Increment $g_i$ by $\nabla \ell(w_t, s) / \max\left\{1, \frac{\|\nabla \ell(w_t, s)\|_2}{C}\right\}$
        end for
        Send $g_i$ to PS
    end if
end for

Protocol in the PS:

for $t \in [T]$ do
    Learn $P_t$ from the clients
    Broadcast $w_t$ to $\forall i \in P_t$
    Receive $g_i$ from $\forall i \in P_t$
    Aggregate, randomize and scale:
    $\hat{g}_t = \frac{1}{pNq} \left( \sum_{i \in P_t} g_i + \mathcal{N}(0, \sigma^2 I_m) \right)$
    Update $w_{t+1} = w_t - \eta \hat{g}_t$
end for
5 Discussion and Numerical Results

First, we would like to emphasize that the number of available clients $N_t$ and the sizes of the datasets $d$ at the clients are needed only at the PS to correctly scale the average of the gradients. They are not needed to compute the privacy loss, and hence, each client can keep track of its own privacy loss without needing any iteration-specific information about the system.

In our algorithm, the noise is added by the PS centrally, and the variance of the noise does not depend on any iteration-specific information. Since the number of sampled clients in each iteration can vary, central noise addition makes each client keep track of its own privacy loss much more easier in the FL setting without the need to know the number of sampled clients in each iteration. Central noise addition might be seen as a disadvantage of our scheme since there is no local randomizers at the clients, but in case of a trusted PS, our privacy guarantees are central and this does not constitute a limitation.

Next, we introduce three baselines and then, compare their privacy guarantees with those of the proposed scheme. Note that the previous works on random participation provide client-level pure DP guarantees. Hence, comparing these schemes with our work would not be fair. Nevertheless, in Section 5.3, we provide some comparisons to emphasize what improvement our scheme brings upon previous work.

5.1 Baselines

5.1.1 Only Local Sampling (OLS)

The first setting we consider is the one in which all the clients participate in each iteration, i.e., $p = 1$, and each client samples from its local dataset such that each element is sampled with probability $q$ in an i.i.d. fashion. We consider OLS to demonstrate the benefits of random participation. The privacy guarantees of local subsampling in OLS is given by the following theorem.

Theorem 5.1. Each iteration of OLS scheme is $(\varepsilon, \delta)$-DP, where, for any $\varepsilon' > 0$,

$$\varepsilon = \log \left(1 + q \left(e^{\varepsilon' - 1}\right)\right), \quad (8)$$

$$\delta = q \left(\Phi \left(\frac{C}{2\sigma} - \frac{\sigma\varepsilon'}{C}\right) - e^{\varepsilon'} \Phi \left(-\frac{C}{2\sigma} - \frac{\sigma\varepsilon'}{C}\right)\right). \quad (9)$$

The proof of Theorem 5.1 is based on Theorem 2.5 and given in Appendix A.

5.1.2 Weak Client Sampling (WCS)

In this setting, our sampling procedure is exactly the same as in Algorithm 1 but we assume that when the clients randomly decide whether to participate or not, the identities of the participating clients are disclosed. Hence, in WCS, random participation of clients helps amplifying DP guarantees only since when some clients are not sampled, no information is leaked
from them. This obviously results in weaker privacy guarantees than those of Algorithm 1, and we consider this setting as an upper bound. To elaborate more, let us assume \( D' \) has exactly the same elements as \( D \) except an additional element \( x' \). If the client storing \( x' \) is sampled, then the privacy leakage is the same as OLS. On the other hand, when this specific client is not sampled, no information is leaked at all about the existence of \( x' \). This implies that the privacy guarantees of WCS should still be better than OLS. In the following theorem, we present the formal privacy guarantees of WCS.

**Theorem 5.2.** Each iteration of WCS scheme is \((\varepsilon, \delta)\)-DP, where, for any \( \varepsilon' > 0 \)

\[
\varepsilon = \log \left(1 + pq \left( e^{\varepsilon'} - 1 \right) \right),
\]

\[
\delta = pq \left( \Phi \left( \frac{C}{2\sigma} - \frac{\sigma \varepsilon''}{C} \right) - e^{\varepsilon''} \Phi \left( -\frac{C}{2\sigma} - \frac{\sigma \varepsilon''}{C} \right) \right),
\]

where \( \varepsilon'' = \varepsilon' + \log \left( e^{\varepsilon - \varepsilon'} + (1 - e^{\varepsilon - \varepsilon'}) \frac{(1-q)}{1-pq} \right) \).

The proof of Theorem 5.2 is given in Appendix A.

5.1.3 Centralized Shuffling (CS)

In CS, we have the same client sampling and dataset sampling procedure with probabilities \( p \) and \( q \), respectively, but at the beginning of each iteration, the elements stored in all the clients are shuffled centrally and uniformly. This is obviously too costly and far from practice, but we consider this case as a lower bound to measure the performance of our scheme. The privacy guarantees of CS are derived in the same way as in Theorem 5.1. Notice that the case in Theorem 5.1 and CS are exactly the same except for the probability of an element being sampled, which is \( pq \) in CS and \( q \) in OLS. Hence, the next corollary follows.

**Corollary 5.3.** Each iteration of the CS scheme is \((\varepsilon, \delta)\)-DP where, for any \( \varepsilon' > 0 \)

\[
\varepsilon = \log \left(1 + q \left( e^{\varepsilon'} - 1 \right) \right),
\]

\[
\delta = q \left( \Phi \left( \frac{C}{2\sigma} - \frac{\sigma \varepsilon'}{C} \right) - e^{\varepsilon'} \Phi \left( -\frac{C}{2\sigma} - \frac{\sigma \varepsilon'}{C} \right) \right).
\]

5.2 Comparison to Baselines

For a single iteration of the learning task, we present the trade-offs \( \varepsilon \) vs. \( \delta \), \( \varepsilon \) vs. \( \sigma \) and \( \delta \) vs. \( \sigma \), in Figure 1, Figure 2 and Figure 3, respectively, for the proposed scheme and the baselines we consider. While generating the plots, we assume \( C = 1 \) and set \( \sigma = 1 \), \( p = q = 0.1 \), unless otherwise stated. We observe that the proposed scheme’s privacy guarantees lie in between CS and WCS. Inferring whether a client is sampled or not by observing the corresponding model update is easier if the participation of one client changes the model update significantly, i.e., large sensitivity to addition or removal of one client. Since the sensitivity of the gradient sum received by the PS with respect to one element is bounded by \( C \), the number of sampled
elements from a client $i$, i.e., $|S_{i,t}|$, determines how easy it is to make an inference about the participation of that client. That is because the sensitivity of the gradient sum with respect to the participation of client $i$ becomes $|S_{i,t}|C$. Since smaller $d\cdot q$ values imply smaller expected values of $|S_{i,t}|C$, in the figures, we observe that smaller $d\cdot q$ values result in stronger privacy guarantees, and our scheme performs quite close to the CS when $d\cdot q$ is small. For example, for $d=1$, CS and our scheme are almost identical since it is quite hard to distinguish if the sole element of each client is sampled or not and hence, it is hard to make inference about the participation of any client. Increasing $d$ gives more information to an adversary about the identities of the sampled clients, but for $d=10$ the privacy guarantees if our scheme and still very close to CS. As the value of $d\cdot q$ increases, so does the privacy leakage of our scheme as we observe in all three figures, and the privacy guarantees of the proposed scheme converge to those of WCS as $d \to \infty$. For the considered set of parameters, since there are no significant differences between the privacy guarantees of WCS and the proposed scheme for values $d\cdot q > 10$, for clarity, we only show $d=100$, which is almost the same curve as WCS. Such a convergence of our scheme to WCS is intuitive since when $d$ is large, an adversary can gain a significant amount of information about the number of sampled clients and their identities. Since the assumption in WCS is that the identities of the sampled clients are known, our scheme becomes very close to the WCS, when $d$ is large.

Note that in many practical cases, the dataset sizes may be much larger than those considered in this section. This does not necessarily mean that for the cases $d>100$, the proposed scheme will be always equivalent to WCS. If the local dataset sizes of the clients are large, then much smaller $q$ values can be employed. In fact, smaller $d\cdot q$ values result in better privacy guarantees in general. This is the essence of what we have observed in this section. We consider an example with larger datasets in Section 6.2.
5.3 Comparison to Prior Work

Previous works on random participation provide client-level guarantees and assumes pure $\varepsilon$-DP guarantees. Hence, we think that the comparison of these schemes with our work would not be fair for these works since the conversion from $\varepsilon$-DP guarantees to $(\varepsilon, \delta)$-DP considerably degrades the privacy guarantees. Further, these guarantees are client-level, so they may not be directly comparable with our results. Nevertheless, here we provide the comparison of our results with those of [Balle et al., 2020] in Figure 4 for a region that we were able to get meaningful privacy guarantees for [Balle et al., 2020]. We used sampling probabilities $p = q = 0.1$, the sensitivity per gradient $C = 1.0$, noise std $\sigma = 3.0$. As we observe, in terms of approximate DP guarantees, the proposed scheme significantly improves upon [Balle et al., 2020].

5.4 Local Sampling vs. Client Sampling

Due to Theorem 4.1, when we decrease $p$ and $q$, the DP guarantees of our scheme improves. However, we may have some constraints that limit the minimum values $p$ and $q$ can take. For example, batch size is a crucial hyperparameter affecting the accuracy of learning algorithms, and in most cases, choosing the batch size too small results in too noisy updates while choosing it too large may forbid learning details in the dataset. Hence, the value of $p \cdot q$ should be chosen to attain the required batch size. In this section, we investigate how the DP guarantees of the proposed scheme and our baselines are affected by the choice of the individual values of $p$ and $q$ given that $pq$ is fixed.

In Figure 5, given $pq = 10^{-4}$, we plot the $\varepsilon$ vs. $q$ trade-off, when $C = 1$ and $\sigma = 1$. We observe that, while we increase $q$, which corresponds to decreasing $p$ at the same rate, the DP guarantees of all the schemes degrade. When all clients participate, i.e., $p = 1.0$, all the schemes become equivalent and $\varepsilon$ is at its best value. Although this implies that we should
In accordance with the observations in Section 5.2, we further observe that as \( q \) increases, smaller dataset sizes at the clients, i.e., smaller \( d \), become more favorable since the proposed scheme converges to WCS at larger values of \( q \).

5.5 Modification of the proposed scheme when the PS is not trusted

When the PS is not trusted, client sampling can still be applied, and our theoretical analysis is still valid. In Algorithm 1, since Gaussian noise is added by the PS, the same noise protects all the clients’ updates at the same time. In case of an untrusted PS, we cannot rely on the noise addition by the PS; and hence, each client must add Gaussian noise to protect its own
privacy. This causes a higher variance of the effective noise added to the total update received by the PS; and hence, reduces the accuracy compared to the trusted PS case.

On the other hand, the sampling can still be done in the following manner. First, each client decides to participate or not independently with probability \( p \), and if a client decides not to participate, since we do not want the PS to learn if a client is sampled or not, it sends pure noise to the PS without processing its local dataset. If it decides to participate, then it samples its local dataset, calculates the gradients of the sampled elements, and sends the average of the gradients by adding Gaussian noise. This way, the PS will always receive a message from all the available clients, and our analysis on client sampling is still valid. One disadvantage of this modified protocol is that an available client sends a message to the PS at each iteration, even when it is not participating, and hence, the communication efficiency of the scheme degrades, which is one of the main motivations of the client sampling. Note that the reduction in the computational costs due to client sampling is still valid.

If the communication costs are scarce in a setting, alternatively, we may prefer a client communicates via the PS only if it decides to participate in a round. In this case, the PS would clearly learn if a client is participating or not, and our privacy guarantees reduce to the one provided by WCS. Although it provides weaker guarantees than the proposed method, we have seen in Section 5 and Section 6 that WCS considerably improves upon OLS, and can still provide meaningful privacy guarantees.

6 Empirical Evaluation

In this section, we consider two different experimental settings to empirically show the improvement on the accuracy when the variance of the Gaussian noise added for DP guarantees is determined according to Theorem 4.1. In both settings, we use EMNIST dataset [Cohen et al., 2017], which is an extension of the MNIST dataset [LeCun et al., 1998], where,
in addition to the handwritten digits, uppercase and lowercase letters are also included. Although there are different possible splits in this dataset, we use the split based on digit and letter classification, referred to as ByClass in [Cohen et al., 2017]. Hence, we have 62 possible classes. This dataset includes 814255 data samples in total, and we use 697932 of them as the training set and the rest as the test set, as implemented in PyTorch [Paszke et al., 2019]. In both scenarios, we use a simple deep neural network, which is composed of two layers of CNNs followed by two fully connected layers. We give further details on the experimental setting in Appendix B. We also provide the code for the experiments in [github link].

6.1 Many Clients and Small Datasets Scenario

In the first scenario, we assume there are large number of users while each user has only a few number of samples. Motivated by the findings in Figure 1, we choose that each client has \( d = 30 \) samples and we equally split the training set across 23264 clients. Since we have relatively large number of clients, we choose the client sampling probability as \( p = 0.001 \), and each client that decides to participate samples its local dataset with probability \( q = 0.1 \). These sampling rates correspond to an average batch size of 70, which we verify to provide a good accuracy via hyperparameter search. For each iteration of Algorithm 1, we require a fixed privacy guarantee such that \( \varepsilon = 0.015 \) and \( \delta = 10^{-6} \). These values may seem too conservative, but note that training deep learning models is an iterative procedure, and at the end of training, the total privacy leakage should be reasonably low. To attain these guarantees, for the proposed method, the server should add Gaussian noise with a standard deviation of \( \sigma = 1.065 \) according to Theorem 4.1. Similarly, for WCS, according to Theorem 5.2, we should have \( \sigma = 7.65 \), and for OLS, according Theorem 5.1, we should have \( \sigma = 22.4 \). Moreover, to provide a baseline, we also train our model ignoring privacy constraints, i.e., \( \sigma = 0 \). For the proposed method, WCS and OLS, we use a gradient clipping value of \( C = 1 \), while no such restriction is imposed in the non-private case.

In Figure 6, we present the test accuracies of the proposed scheme and all the baselines with respect to the number of iterations. We observe that the proposed method attains much higher accuracy than the other analysis methods. We further observe that, although it constitutes an upper bound on \((\varepsilon, \delta)\), WCS still brings a non-trivial improvement over OLS. In fact, this shows that even if the identities of the sampled clients can be inferred with high confidence from the gradients, analysis provided in Theorem 5.2 for client sampling still provides useful privacy guarantees. Finally, we observe that the analysis based on only local sampling, i.e., OLS, has quite a poor performance due to the very high noise addition in this scheme, in order to attain \( \varepsilon = 0.015 \).

6.2 Few Clients and Large Datasets Scenario

Next, we consider the scenario with fewer clients, but each client has a larger dataset. We assume each client stores \( d = 1000 \) data points and there are 697 clients. Each client is independently sampled with a probability of \( p = 0.1 \) and each participating client samples
its data points independently each with probability of $q = 0.001$. Similar to the scenario in Section 6.1, we have an average batch size of 70. Again, we fix per-iteration privacy guarantees as $\varepsilon = 0.015$ and $\delta = 10^{-6}$. To attain them, according to Theorem 4.1, the server should add Gaussian noise with $\sigma = 0.646$ for the proposed scheme. For WCS and OLS, we have $\sigma = 0.873$ and $\sigma = 1.103$, according to Theorem 5.2 and Theorem 5.1, respectively. While we use $C = 1$ for the gradient clipping parameter in the private cases, we do not clip the gradients or add any noise in the non-private case.

In Figure 7, we present our experiment results. Again, we observe that the accuracy of each scheme is inversely proportional to the amount of added noise; and hence, the proposed scheme has the best accuracy. However, compared to the setting in Section 6.1, in this setting,
the performance of the schemes are closer to each other. As discussed in Section 5.4, that is because we have a smaller value of $q$ although $pq$ value is the same. Hence, smaller amount of noise is enough to attain $\varepsilon = 0.015$.

7 Conclusion

Client sampling is widely accepted as a core aspect of FL to reduce the energy consumption of clients and to limit the communication load. In this paper, we have analyzed the privacy amplification provided by client sampling in FL to amplify sample-level DP guarantees. While the privacy amplification effect of local data sampling has been well studied, to the best of our knowledge, this is the first work to study the approximate DP implications of client sampling in FL. Previous works employing client sampling either provide user-level DP guarantees, or their analyses rely on shuffling. Although shuffling provides strong central privacy guarantees when local randomizers with pure DP guarantees are available, when the local randomizers satisfy approximate DP, the central privacy guarantees degrade considerably. This poses a challenge when Gaussian noise is used as a randomizer in FL settings. Instead, we have provided an analysis that does not rely on shuffling and allows the use of Gaussian noise without degrading the privacy guarantees. We have also shown that when the number of samples available at each client is small, we obtain privacy guarantees close to those of centralized training, in which uniform sampling is possible. Moreover, even if the local dataset sizes are very large, we have shown that client sampling is still quite beneficial compared to only local dataset sampling. Moreover, since available clients locally decide to participate or not, independently from each other and from any other system parameter, they can keep track of their own privacy losses; and therefore, our scheme trivially scales to large systems, and is feasible to apply in practice.
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A Proofs

A.1 Proof of Theorem 4.1

Without loss of generality, we assume that $D'$ has one more element than $D$ and this extra element is stored by the first client, i.e., $D'_1 = D_1 \cup \{x'\}$. Thus, the distribution of the model update in a single iteration when user 1 stores $D'_1$ is

$$\xi(z) = (1-p)\xi_0(z) + p\left((1-q)\xi_1(z) + q\xi_2(z)\right)$$  \hspace{1cm} (14)

and the distribution when user 1 stores $D_1$ is

$$\xi'(z) = (1-p)\xi_0(z) + p\xi_1(z)$$  \hspace{1cm} (15)

where $\xi_0$ is the distribution when user 1 is not sampled, $\xi_1$ is the distribution when user 1 is sampled but $x'$ is not sampled, $\xi_2$ is the distribution when both user 1 and $x'$ are sampled and $z$ is the observed model update, and hence, $z \in \mathbb{R}^m$. In the proposed algorithm, all these distributions are multivariate Gaussian mixture distributions of dimension $m$. For clearer notation, we omit $z$ in the rest of the section.

Observe that the event $x'$ being sampled has a probability of $pq$. Thus, we can write Equation (14) and Equation (15) as

$$\xi = (1-pq)(c_1\xi_0 + c_2\xi_1) + pq\xi_2,$$  \hspace{1cm} (16)

and

$$\xi' = (1-pq)(c_1\xi_0 + c_2\xi_1) + pq\xi_1,$$  \hspace{1cm} (17)

where $c_1 = \frac{(1-p)}{(1-pq)}$ and $c_2 = \frac{p(1-q)}{(1-pq)}$.

When the output distribution of a mechanism is a mixture distribution, advanced joint convexity of hockey stick divergence [Balle et al., 2018], which we introduce next, helps tightly upper bounding it.

**Theorem A.1** (Theorem 2 in [Balle et al., 2018]). Let $\mu$ and $\mu'$ be measures composed of mixtures of other measures $\mu_0, \mu_1$ and $\mu'_1$ such that $\mu = (1-\gamma)\mu_0 + \gamma\mu_1$ and $\mu' = (1-\gamma)\mu_0 + \gamma\mu'_1$. For $\alpha \geq 1$, we have

$$D_\alpha'(\mu||\mu') = \gamma D_\alpha(\mu_1||\mu_0 + (1-\beta)\mu_1)$$  \hspace{1cm} (18)

where $\alpha' = 1 + \gamma(\alpha - 1)$ and $\beta = \alpha'/\alpha$. The relation in Equation (18) is referred to as advanced joint convexity of $D_\alpha$ in [Balle et al., 2018].

Using Theorem A.1, we can write the divergence between these two measures as follows

$$D_\alpha(\xi||\xi') = pq D_{\alpha'}(\xi_2||\mu_0 + (1-\beta)\mu_0 + \beta\mu_1).$$  \hspace{1cm} (19)

where $\alpha = e^\varepsilon$, $\alpha' = e^{\varepsilon'}$, $\beta = e^{\varepsilon-\varepsilon'}$ and $\varepsilon = \log\left(1 + pq\left(e^{\varepsilon'} - 1\right)\right)$. Since hockey stick divergence is also jointly convex in its arguments, we can bound Equation (19) as

$$D_\alpha(\xi||\xi') \leq pq(1-\beta)c_1 D_{\alpha'}(\xi_2||\xi_0) + pq((1-\beta)c_2 + \beta) D_{\alpha'}(\xi_2||\xi_1).$$  \hspace{1cm} (20)
However, $D_{\alpha'}(\xi_2||\xi_0)$ can be quite large compared to $D_{\alpha'}(\xi_2||\xi_1)$ since $\xi_2$ has the effect of all the sampled data points by user 1 while $\xi_0$ does not. Thus, Equation (20) is not a tight bound. Instead, we try to bound Equation (19) in a more involved way.

Observe that there is a coupling between $\xi_0, \xi_1$ and $\xi_2$ such that except sampling of user 1 and $x'$, the other sampled users and the sampled elements are the same. Therefore, $\xi_1$ has a set of elements sampled from user 1 except $x'$ and the same sampled users and elements sampled in $\xi_0$. Similarly, $\xi_2$ has $x'$ sampled as well as all the users and the elements sampled in $\xi_1$. As we stated previously, $\xi_0, \xi_1$ and $\xi_2$ are multivariate Gaussian mixture distributions. Next, let us define $\mathcal{P}$ be the set of all possible samplings of clients given client 1 is not sampled. Similarly, let us define $\mathcal{S}_I$ as the set of all possible samplings of elements from the clients in the set $I$ given these clients are sampled. Moreover, let $\mathcal{S}_1$ be the set of all possible samplings of the elements stored in client 1 given $x'$ is not sampled.

Thus, given $P \in \mathcal{P}_I, S \in \mathcal{S}_P$ and $S_1 \in \mathcal{S}_1$ are actual samplings of clients and the elements, respectively, we can write $\xi_1(P, S, S_1) \sim \mathcal{N}(\mu(P \cup \{1\}, S \cup S_1), \sigma^2 I_m), \xi_2(P, S, S_1) \sim \mathcal{N}(\mu(P \cup \{1\}, S \cup S_1 \cup \{x'\}), \sigma^2 I_m)$ and $\xi_0(P, S) \sim \mathcal{N}(\mu(P, S), \sigma^2 I_m)$ where $\mu(P, S)$ is the observed model update when the set of sampled services is $P$ and the set of sampled elements is $S$.

As a result, we can write Equation (19) as

$$D_{\alpha'}(\xi||\xi') = pq D_{\alpha'}\left( \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}_P} \sum_{S_1 \in \mathcal{S}_1} \Pr(P, S) \Pr(S_1) \xi_2(P, S, S_1) || \bar{c}_1 \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}_P} \Pr(P, S) \xi_0(P, S) \\
+ \bar{c}_2 \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}_P} \sum_{S_1 \in \mathcal{S}_1} \Pr(P, S) \Pr(S_1) \xi_1(P, S, S_1) \right).$$  \hspace{1cm} (21)

where $\bar{c}_1 = (1 - \beta)c_1$ and $\bar{c}_2 = c_2 + \beta(1 - c_2)$. Since hockey stick divergence is jointly convex, we can upper bound Equation (21) as

$$D_{\alpha'}(\xi||\xi') \leq \sum_{P \in \mathcal{P}} \sum_{S \in \mathcal{S}_P} \Pr(P, S) pq D_{\alpha'}\left( \sum_{S_1 \in \mathcal{S}_1} \Pr(S_1) \xi_2(P, S, S_1) || \bar{c}_1 \xi_0(P, S) + \bar{c}_2 \sum_{S_1 \in \mathcal{S}_1} \Pr(S_1) \xi_1(P, S, S_1) \right)$$

$$\leq \max_{P \in \mathcal{P}, S \in \mathcal{S}_P} pq D_{\alpha'}\left( \sum_{S_1 \in \mathcal{S}_1} \Pr(S_1) \xi_2(P, S, S_1) || \bar{c}_1 \xi_0(P, S) + \bar{c}_2 \sum_{S_1 \in \mathcal{S}_1} \Pr(S_1) \xi_1(P, S, S_1) \right).$$  \hspace{1cm} (22)

Since the hockey stick divergence is shift-invariant, we can take a reference distribution and then define all the other distributions accordingly. Without loss of generality, we take $\xi_0$ as the reference, and assume its mean $\mu = 0$. Then, Equation (22) can be written as

$$D_{\alpha'}(\xi||\xi') \leq \max_{P \in \mathcal{P}, S \in \mathcal{S}_P} pq D_{\alpha'}\left( \sum_{S_1 \in \mathcal{S}_1} \Pr(S_1) \mathcal{N}(\bar{\mu}_2(S_1), \sigma^2 I_m) || \bar{c}_1 \mathcal{N}(0, \sigma^2 I_m) \\
+ \bar{c}_2 \sum_{S_1 \in \mathcal{S}_1} \Pr(S_1) \mathcal{N}(\bar{\mu}_1(S_1), \sigma^2 I_m) \right).$$  \hspace{1cm} (23)

where

$$\bar{\mu}_1(S_1) = \mathcal{N}(\mu(P \cup \{1\}, S \cup S_1), \sigma^2 I_m) - \mathcal{N}(\mu(P, S), \sigma^2 I_m),$$  \hspace{1cm} (24)

and

$$\bar{\mu}_2(S_2) = \mathcal{N}(\mu(P \cup \{1\}, S \cup S_1 \cup \{x'\}), \sigma^2 I_m) - \mathcal{N}(\mu(P, S), \sigma^2 I_m).$$  \hspace{1cm} (25)
Observe that we have \( \| \tilde{\mu}_1(S_1) \|_2 \leq iC \) and \( \| \tilde{\mu}_2(S_1) \|_2 \leq (i+1)C \) if there are exactly \( i \) elements sampled from client 1, i.e., \( |S_1| = i \), where \( C \) is the sensitivity of the noiseless output to one sample.

Remember from Definition 2.2 that the hockey-stick divergence between two distributions is written as

\[
D_\alpha(\xi || \xi') \triangleq \int_Z [\xi(z) - \alpha \xi'(z)]_+ d(z). \tag{26}
\]

Since all the Gaussian distributions have a diagonal covariance matrices in Equation (23), and \( \Pr(S_1) = q^i(1 - q)^{d-i} \) when \( |S_1| = i \), we can write it as a hockey stick divergence of univariate Gaussian mixture distributions as follows

\[
D_\alpha (\xi || \xi') \leq pqD_\alpha' \left( \sum_{i=0}^{d} \binom{d}{i} q^i (1 - q)^{d-i} \mathcal{N}((i+1)C, \sigma^2) \right) - \alpha \mathcal{N}^{(0, \sigma^2)} + \tilde{c}_2 \sum_{i=0}^{d} \binom{d}{i} q^i (1 - q)^{d-i} \mathcal{N}(iC, \sigma^2), \tag{27}
\]

where \( d \) is the number of elements stored by user 1 excluding \( x' \).

Let us define \( \mathcal{N}(z, \mu, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \) and substitute Equation (27) into Equation (26), which results in

\[
D_\alpha (\xi || \xi') \leq pq \int \left[ \sum_{i=0}^{d} \binom{d}{i} q^i (1 - q)^{d-i} \left( \mathcal{N}(z, (i+1)C, \sigma^2) - \alpha \tilde{c}_2 \mathcal{N}(z, iC, \sigma^2) \right) - \alpha \tilde{c}_1 \mathcal{N}(z, 0, \sigma^2) \right]_+ dz. \tag{28}
\]

In Equation (28), the expression inside \([ \cdot ]_+ \) goes to zero when \( z \to \infty \) and \( z \to -\infty \). Other than these, the expression has only one zero-crossing \( z^* \) and for all finite \( z > z^* \), it is positive. Therefore, taking the integral \( z \geq z^* \) is enough. Moreover, since there is only one zero-crossing for the expression, it is easy to numerically solve

\[
\sum_{i=0}^{d} \binom{d}{i} q^i (1 - q)^{d-i} \left( \mathcal{N}(z, (i+1)C, \sigma^2) - \alpha \tilde{c}_2 \mathcal{N}(z, iC, \sigma^2) \right) - \alpha \tilde{c}_1 \mathcal{N}(z, 0, \sigma^2) = 0 \tag{29}
\]

for \( z \), where \( z^* \) is the solution. Therefore, we have

\[
D_\alpha (\xi || \xi') \leq pq \sum_{i=0}^{d} \binom{d}{i} q^i (1 - q)^{d-i} \left( 1 - \Phi_{(i+1)C, \sigma}(z^*) - \alpha \tilde{c}_2(1 - \Phi_{iC, \sigma}(z^*)) \right) - pq\alpha \tilde{c}_1(1 - \Phi_{0, \sigma}(z^*))
\]

\[
= pq \left( 1 - \alpha \tilde{c}_2 + \alpha \tilde{c}_1 \Phi_{0, \sigma}(z^*) - 1 \right) + \sum_{i=0}^{d} \binom{d}{i} q^i (1 - q)^{d-i} \left( \alpha \tilde{c}_2 \Phi_{iC, \sigma}(z^*) - \Phi_{(i+1)C, \sigma}(z^*) \right), \tag{30}
\]

where \( \Phi_{\mu, \sigma} \) is the cumulative distribution function of a Gaussian random variable with mean \( \mu \) and standard deviation \( \sigma \).

Together with Theorem 2.3, Equation (30) proves the claim.
A.2 Proof of Theorem 5.1

According to Theorem 2.5, if a mechanism $M$ satisfying $(\varepsilon, \delta)$-DP takes as the input a subset sampled from the entire dataset with Poisson sampling, we have $\delta_{M'}(\varepsilon') \leq q\delta_M(\varepsilon)$ and $\varepsilon' = \log(1 + q(e^\varepsilon - 1))$, where $M'$ is the mechanism $M$ cascaded with sampling. Moreover, from Theorem 2.4, we know $\varepsilon$ vs $\delta$ trade-off for Gaussian mechanism. The claim follows from combining Theorem 2.5 and Theorem 2.4. □

A.3 Proof of Theorem 5.2

The proof can be obtained by following the same steps as in the proof of Theorem 4.1 until Equation (27). In Equation (27) if we ignore $\bar{c}_1\mathcal{N}(0, \sigma^2)$, then we obtain a looser upper bound according to Equation (26). Moreover, ignoring this expression is equivalent to ignoring the contribution of the client sampling when client 1 is sampled. This results in

$$D_\alpha(\xi||\xi') \leq pqD_{\alpha'} \left( \sum_{S_1 \in S_1} \Pr(S_1)\mathcal{N}((i + 1)C, \sigma^2)||\bar{c}_2 \sum_{S_1 \in S_1} \Pr(S_1)\mathcal{N}(iC, \sigma^2) \right) \quad (31)$$

$$\leq \max_{S_1 \in S_1} pqD_{\alpha'} \left( \mathcal{N}((i + 1)C, \sigma^2)||\bar{c}_2 \mathcal{N}(iC, \sigma^2) \right), \quad (32)$$

where last inequality is due to the joint convexity of $D_{\alpha'}$. Since it is also shift-invariant, we have

$$D_\alpha(\xi||\xi') \leq pq \int_Z \left[ \mathcal{N}(C, \sigma^2) - \alpha' \bar{c}_2 \mathcal{N}(0, \sigma^2) \right]_+ d(z). \quad (33)$$

Since $\varepsilon'' = \varepsilon + \log(\bar{c}_2)$, the claim follows from Theorem 2.4 and Equation (33). □

B Details of the Experimental Setting

B.1 Architecture

For the experiments in Section 6.1 and Section 6.2 we use a deep neural network with two convolutional layers followed by two fully connected layers. In Figure 8, we give the visualization of the network with all necessary details.

![Network Architecture](image)

Figure 8: Network Architecture Used in the Experiments

B.2 Hyperparameters

In Section 6.1, we use SGD optimizer with a momentum 0.9. Learning rate for the non-private case and the proposed method is 0.002. For WCS and OLS, we observed the learning rate
should be smaller due to the high noise addition. Hence, the learning rate of 0.0001 is used for these schemes.

In Section 6.2, we used SGD optimizer with a momentum of 0.9. The learning rate is 0.002, for all schemes.

In all experiments, we used an average batch size of 70.