A note on a generalisation of a definite integral involving the Bessel function of the first kind

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Abstract

We consider a generalisation of a definite integral involving the Bessel function of the first kind. It is shown that this integral can be expressed in terms of the Fox-Wright function of one variable. Some consequences of this representation are explored by suitable choice of parameters. In addition, two closed-form evaluations of infinite series of the Fox-Wright function are deduced.

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\section{1. Introduction and Preliminaries}

The Fox-Wright function \( _p\Psi_q(z) \) of one variable \([2, 3]\) is given by

\[
_p\Psi_q\left[ \begin{array}{l} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{array} \right| z \right] = _p\Psi_q\left[ \begin{array}{l} (\alpha_j, A_j)_{1,p} \\ (\beta_j, B_j)_{1,q} \end{array} \right| z \right]
= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + kA_1) \ldots \Gamma(\alpha_p + kA_p)}{\Gamma(\beta_1 + kB_1) \ldots \Gamma(\beta_q + kB_q)} \frac{z^k}{k!},
\]

where \( z \in \mathbb{C}, \alpha_j, \beta_j \in \mathbb{C} \) and the coefficients \( A_j \geq 0, B_j \geq 0 \), it being supposed throughout that the \( \alpha_j, \beta_j \) and the \( A_j, B_j \) are such that the gamma functions are well defined. With

\[
\Delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \quad \delta = \prod_{j=1}^{p} A_j^{-A_j} \prod_{j=1}^{q} B_j^{B_j}, \quad \mu^* = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j + \frac{1}{2}(p - q),
\]
the series in (1.1) converges for \(|z| < \infty\) when \(\Delta > -1\), for \(|z| < \delta\) when \(\Delta = -1\) and for \(|z| = \delta\) if, in addition, \(\Re(\mu) > \frac{1}{2}\).

Particular cases of (1.1) that we shall employ are the Wright function [3, p. 438(1.2)] defined by
\[
\Phi(\alpha, \beta, z) = 0 \Psi_1 \left( \frac{z}{(\beta, \alpha)} \right) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\beta + \alpha k)}, \tag{1.2}
\]
where \(z, \beta \in \mathbb{C}\) and \(\alpha > 0\) and the Wright generalised-Bessel function [3, p. 438(1.3)] defined by
\[
J^\mu(\nu, z) = \Phi(\mu, \nu, 1, -z) = 0 \Psi_1 \left( \frac{-z}{(\nu + 1, \mu)} \right) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(\nu + 1 + \mu k)}, \tag{1.3}
\]
where \(z, \nu \in \mathbb{C}\) and \(\mu > 0\). The Mittag-Leffler function [3, p. 450(6.1)] is given by
\[
E_{\alpha, \beta}(z) = 1 \Psi_1 \left( \frac{(1, 1)}{(\beta, \alpha)} \right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \tag{1.4}
\]
where \(z, \beta \in \mathbb{C}\) and \(\alpha > 0\). The Bessel function of the first kind is defined by (see [5, p. 217])
\[
J_\nu(z) = (\frac{z}{2})^\nu \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{2k}}{k! \Gamma(\nu + 1 + k)}.
\]
Of special interest in this paper are the elementary functions corresponding to \(\nu = \pm \frac{1}{2}\), namely
\[
J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z. \tag{1.5}
\]

The infinite Fourier sine and cosine transforms of \(g(x)\) over the interval \([0, \infty)\) are defined by [1, 4]
\[
F_{S,C}(g(x); b) = \int_0^{\infty} g(x) \sin_{\cos} (bx) \, dx = G_{S,C}(b) \quad (b > 0). \tag{1.6}
\]
If \(\Re(z) > 0\) and \(\Re(b) > 0\), the Mellin transforms of \(\cos(bx)/(e^{ax} - 1)\) and \(\sin(bx)/(e^{ax} - 1)\) are given by [1, p. 15, 1.4(7)]
\[
\int_0^{\infty} \frac{x^{\mu-1} \cos(bx)}{e^{ax} - 1} \, dx = \frac{\Gamma(\mu)}{2a^{\mu}} \left( \zeta(\mu, 1 + ib/a) + \zeta(\mu, 1 - ib/a) \right),
\]
and
\[
\int_0^{\infty} \frac{x^{\mu-1} \sin(bx)}{e^{ax} - 1} \, dx = \frac{i \Gamma(\mu)}{2a^{\mu}} \left( \zeta(\mu, 1 + ib/a) - \zeta(\mu, 1 - ib/a) \right),
\]
where \(\zeta(a, z) = \sum_{k \geq 0} (k + z)^{-a}\) is the Hurwitz zeta function. In the special case \(\mu = 2\) and \(a = 2\pi\), \(b = \pi n\), we have
\[
\int_0^{\infty} \frac{x \cos(\pi n x)}{e^{2\pi x} - 1} \, dx = \frac{1}{2\pi^2 n^2} + \frac{1}{4(1 - \cosh \pi n)} \tag{1.7}
\]
and [5, (5.15.1)]
\[
\int_0^{\infty} \frac{x \sin(\pi n x)}{e^{2\pi x} - 1} \, dx = \frac{i}{8\pi^2} \left( \psi'(1 + \frac{1}{2}in) - \psi'(1 - \frac{1}{2}in) \right), \tag{1.8}
\]
where $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$ and the prime denotes differentiation.

A natural generalisation of the Gauss hypergeometric function $\, _2F_1(z)$ is the generalised hypergeometric function $\, _pF_q(z)$, with $p$ numerator parameters $\alpha_1, \ldots, \alpha_p$ and $q$ denominator parameters $\beta_1, \ldots, \beta_q$, defined by

$$
\, _pF_q\left(\frac{\alpha_1, \ldots, \alpha_p}{\beta_1, \ldots, \beta_q} ; z\right) = \, _pF_q\left(\frac{(\alpha_j)_{1,p}}{(\beta_j)_{1,q}} ; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k} \ldots (\alpha_p)_{k}}{(\beta_1)_{k} \ldots (\beta_q)_{k}} \frac{z^k}{k!},
$$

where $\alpha_j \in \mathbb{C}$ ($j = 1, \ldots, p$) and $\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ ($j = 1, \ldots, q$), $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$ and $(a)_k = \Gamma(a + k)/\Gamma(a)$ is the Pochhammer symbol. The series in (1.9) is convergent for $|z| < \infty$ if $p \leq q$ and for $|z| < 1$ if $p = q + 1$. If we define the parametric excess by $\omega = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j$, then it is known that the $\, _pF_q(z)$ series with $p = q + 1$ is (i) absolutely convergent for $|z| = 1$ if $\Re(\omega) > 0$, (ii) is conditionally convergent for $|z| = 1$ if $-1 < \Re(\omega) \leq 0$ and (iii) is divergent for $|z| = 1$ if $\Re(\omega) \leq -1$.

The central aim in this paper is to give a generalisation and find some consequences of a definite integral involving the Bessel function of the first kind which we express in terms of the Fox-Wright $Ψ$ function. The work is motivated by the papers by one of the present authors in [6, 7, 8]. In order to generalise the definite integrals introduced by Ramanujan $\phi_{S,C}(m, n)$ defined by

$$
\phi_{S,C}(m, n) = \int_0^{\infty} \frac{x^m}{e^{2\pi x} - 1} \sin (\pi nx) \, dx, \quad (1.10)
$$

we introduce the following integrals:

$$
F_1(\mu, \xi, a, \nu, y) = \int_0^{\infty} x^\mu e^{-ax\xi} \sqrt{xy} J_\nu(xy) \, dx, \quad (1.11)
$$

$$
F_2(\mu, \xi, b, c, \nu, y) = \sum_{k=0}^{\infty} \frac{\Theta(k)}{k!} \int_0^{\infty} x^\mu e^{-(b+ck)x} \sqrt{xy} J_\nu(xy) \, dx, \quad (1.12)
$$

$$
F_3(\mu, \xi, b, c, \nu, y) = \int_0^{\infty} x^\mu e^{-bx\xi} \Psi_s \left[ \frac{(\alpha_j, A_j)_{1,r}}{(\beta_j, B_j)_{1,s}} \right] e^{-cx\xi} \sqrt{xy} J_\nu(xy) \, dx \quad (1.13)
$$

and

$$
F_4(\mu, \xi, b, c, \nu, y) = \int_0^{\infty} x^\mu e^{-bx\xi} F_s \left( \frac{(\alpha_j, A_j)_{1,r}}{(\beta_j, B_j)_{1,s}} ; e^{-cx\xi} \right) \sqrt{xy} J_\nu(xy) \, dx. \quad (1.14)
$$

Here $\xi > 0$ and $\{\Theta(k)\}_{k=0}^{\infty}$ is a bounded sequence of real or complex quantities. We show how the main general theorem given in Section 3 is applicable for obtaining new and interesting results by suitable adjustment of the parameters and variables.

2. Evaluation of the definite integral $F_1(\mu, \xi, a, \nu, y)$

In this section, we evaluate the integral in (1.11) involving the Bessel function of the first kind in terms of the Fox-Wright function. We suppose throughout that the parameter $\xi > 0$. 

Theorem 1. With \( \sigma := \frac{(2\mu + 2\nu + 3)}{(2\xi)} \), we have

\[
F_1(\mu, \xi, a, \nu, y) = \int_0^\infty x^\mu e^{-ax} \sqrt{xy} J_\nu(xy) \, dx = \frac{y^{\nu+1/2}}{2^\nu \xi a^\sigma} 1 \Psi_1 \left[ \frac{(\sigma, 2/\xi)}{(\nu + 1, 1)} \left| -\frac{y^2}{4a^2/\xi} \right. \right],
\]

where \( a > 0, \ y > 0 \) and \( \Re(\mu + \nu) > -\frac{1}{2} \).

Proof: Expanding the Bessel function in its series form, followed by reversal of the order of summation and integration, we find

\[
F_1(\mu, \xi, a, \nu, y) = \int_0^\infty x^\mu e^{-ax} \sqrt{xy} J_\nu(xy) \, dx = \sum_{k=0}^{\infty} (-1)^k (y^2/4)^k \int_0^\infty t^{\sigma+2k/\xi} e^{-at} dt
\]

upon evaluation of the integral as a gamma function, where \( \sigma = \frac{(2\mu + 2\nu + 3)}{(2\xi)} \). If we now employ the definition of the Fox-Wright function in (1.1) in the above series, we obtain the right-hand side of (2.1).

If we set \( \xi = 1 \) and \( \nu = \pm \frac{1}{2} \) in (2.1), we obtain the Fourier sine and cosine transforms of \( x^{\eta-1}e^{-ax} \) in the form

\[
\int_0^\infty x^{\eta-1}e^{-ax} \cos(xy) \, dx = \frac{\Gamma(\eta)}{a^\eta} \sum_{k=0}^{\infty} \frac{(\eta)_{2k}}{(\frac{1}{2})_{k}k!} \left( -\frac{y^2}{4a^2} \right)^k
\]

and

\[
\int_0^\infty x^{\eta-1}e^{-ax} \sin(xy) \, dx = \frac{y\Gamma(\eta + 1)}{a^{\eta+1}} \sum_{k=0}^{\infty} \frac{(\eta + 1)_{2k}}{(\frac{3}{2})_{k}k!} \left( -\frac{y^2}{4a^2} \right)^k
\]

Use of the standard evaluations of the hypergeometric function given in [5, (15.4.8), (15.4.10)] then yields the results stated in [4]

\[
\int_0^\infty x^{\eta-1}e^{-ax} \sin \frac{\cos(xy) \, dx}{\cos} = \Gamma(\eta) (a^2 + y^2)^{-\eta/2} \sin \frac{\cos(\eta \arctan(y/a))}{},
\]

where \( \Re(a) > 0, \ y > 0 \) and \( \Re(\eta) > -1 \) and \( \Re(\eta) > 0 \) for the sine and cosine integral, respectively.

3. Evaluation of the definite integrals \( F_j(\mu, \xi, a, \nu, y), \ j = 2, 3, 4 \)

Here we let \( \sigma = \frac{(2\mu + 2\nu + 3)}{(2\xi)} \) throughout this section.
Theorem 2. Let \( \{ \Theta(k) \}_{k=0}^{\infty} \) be a bounded sequence of arbitrary real or complex numbers. Then when \( \xi > 0 \) and \( \Re(b) > 0 \) we have

\[
F_2(\mu, \xi, b, c, \nu, y) = \sum_{k=0}^{\infty} \frac{\Theta(k)}{k!} \int_0^{\infty} x^\mu e^{-(b+ck)x^\xi} \sqrt{xy} J_\nu(xy) \, dx
\]

\[
= \frac{y^{\nu+1/2}}{2^\nu b^\xi} \sum_{k=0}^{\infty} \frac{\Theta(k)}{k!} \left( \frac{(b/c)_k}{(1+b/c)_k} \right)^{\sigma} \Psi_1 \left[ \frac{(\sigma, 2/\xi)}{\Re(\nu + 1, 1)} - \frac{y^2}{4(b+c)^{2/\xi}} \right].
\]  

(3.1)

Proof: The proof follows the same procedure as in Theorem 1 by expressing the Bessel function in series form and integrating term by term. We find

\[
F_2(\mu, \xi, b, c, \nu, y) = \frac{y^{\nu+1/2}}{2^\nu b^\xi} \sum_{k=0}^{\infty} \frac{\Theta(k)}{k!} \frac{\Gamma(\sigma + 2\ell/\xi)}{\ell!\Gamma(\nu + 1 + \ell)} \left( \frac{-y^2}{4(b+c)^{2/\xi}} \right)^\ell.
\]

Employing the definition of the Fox-Wright function function in (1.1) then yields the stated result in (3.1).

Corollary 1. For \( \xi > 0, \Re(b) > 0 \) and \( \Re(c) > 0 \), we have

\[
F_3(\mu, \xi, b, c, \nu, y) = \int_0^{\infty} x^\mu e^{-bx^\xi} r_{\nu, s} \left[ \frac{(\alpha_j, A_j, B_j)}{\nu, 1, s} \right] e^{-cx^\xi} \sqrt{xy} J_\nu(xy) \, dx
\]

\[
= \frac{y^{\nu+1/2}}{2^\nu b^\xi} \sum_{k=0}^{\infty} \prod_{j=1}^{r} \frac{\Gamma(\alpha_j + kA_j)}{k! \prod_{j=1}^{r} \Gamma(\beta_j + kB_j)} \left( \frac{(b/c)_k}{(1+b/c)_k} \right)^{\sigma} \Psi_1 \left[ \frac{(\sigma, 2/\xi)}{\Re(\nu + 1, 1)} - \frac{y^2}{4(b+c)^{2/\xi}} \right],
\]  

(3.2)

where the parameters \( \alpha_j, \beta_j \in \mathbb{C} \) and \( A_j, B_j \geq 0 \). This result follows immediately from (3.1) by substituting

\[
\Theta(k) = \frac{\Gamma(\alpha_1 + kA_1) \ldots \Gamma(\alpha_r + kA_r)}{\Gamma(\beta_1 + kB_1) \ldots \Gamma(\beta_s + kB_s)} \quad (k = 0, 1, 2, \ldots).
\]

Corollary 2. For \( \xi > 0, \Re(b) > 0 \) and \( \Re(c) > 0 \), we have

\[
F_4(\mu, \xi, b, c, \nu, y) = \int_0^{\infty} x^\mu e^{-bx^\xi} r_{\nu, s} \left[ \frac{(\alpha_j, A_j, B_j)}{\nu, 1, s} \right] e^{-cx^\xi} \sqrt{xy} J_\nu(xy) \, dx
\]

\[
= \frac{y^{\nu+1/2}}{2^\nu b^\xi} \sum_{k=0}^{\infty} \prod_{j=1}^{r} \frac{\Gamma(\alpha_j)}{k! \prod_{j=1}^{r} \Gamma(\beta_j)} \left( \frac{(b/c)_k}{(1+b/c)_k} \right)^{\sigma} \Psi_1 \left[ \frac{(\sigma, 2/\xi)}{\Re(\nu + 1, 1)} - \frac{y^2}{4(b+c)^{2/\xi}} \right],
\]  

(3.3)

where the parameters \( \alpha_j \in \mathbb{C}, \beta_j \in \mathbb{C}\setminus\mathbb{Z}_0 \) and \( r \leq s + 1 \). This result follows by setting \( A_1 = \ldots = A_r = 1 \) and \( B_1 = \ldots = B_s = 1 \) in (3.2).

3.1 Special cases of the integral (3.2)

Special cases of the integral in (3.2) are given by the following. When \( r = 0, s = 1 \) in (3.2), we obtain

\[
\int_0^{\infty} x^\mu e^{-bx^\xi} \Phi(B_1, \beta_1, e^{-cx^\xi}) \sqrt{xy} J_\nu(xy) \, dx
\]

\[
= \frac{y^{\nu+1/2}}{2^\nu b^\xi} \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\beta_1 + kB_1)} \left( \frac{(b/c)_k}{(1+b/c)_k} \right)^{\sigma} \Psi_1 \left[ \frac{(\sigma, 2/\xi)}{\Re(\nu + 1, 1)} - \frac{y^2}{4(b+c)^{2/\xi}} \right],
\]  

(3.4)
where $\beta_1 \in \mathbb{C}, B_1 > 0$ and $\Phi(B_1, \beta_1, z)$ is the Wright function defined in (1.2).

When $r = 0, s = 1, B_1 = \mu, \beta_1 = \gamma + 1$ and $e^{-cx}$ is replaced by $-e^{-cx}$ in (3.2), we obtain

$$
\int_0^\infty x^\mu e^{-bx} \frac{1}{2^\nu b^{\sigma}} \sum_{k=0}^\infty \frac{1}{k! \Gamma(\gamma + \mu k)} \left(\frac{(b/c)_k}{(1 + (b/c)_k)}\right) \frac{\Gamma(\nu + 1)}{\Gamma(\nu)} \Psi_1 \left[ \frac{\nu, (\xi, 2)}{(\sigma, 2/\xi)} \right] - \frac{y^2}{4(b + c)^{2/\xi}}
$$

where $\gamma \in \mathbb{C}, \mu > 0$ and $J_{\mu}^n(z)$ is the Wright generalised Bessel function defined in (1.3).

When $r = s = 1$ and $\alpha_1 = A_1 = 1$ in (3.2), we obtain

$$
\int_0^\infty x^\mu e^{-bx} \frac{1}{2^\nu b^{\sigma}} \sum_{k=0}^\infty \frac{1}{k! \Gamma(\beta_1 + kB_1)} \left(\frac{(b/c)_k}{(1 + (b/c)_k)}\right) \frac{\Gamma(\nu + 1)}{\Gamma(\nu)} \Psi_1 \left[ \frac{\nu, (\xi, 2)}{(\sigma, 2/\xi)} \right] - \frac{y^2}{4(b + c)^{2/\xi}}
$$

where $\beta_1 \in \mathbb{C}, B_1 > 0$ and $E_{B_1, \beta_1}(z)$ is the Mittag-Leffler function defined in (1.4).

4. Expressions for Fourier cosine and sine transforms

The expressions for the Fourier cosine and sine transforms of $x^{\eta-1}e^{-(b+cx)}$ in terms of infinite series of the Fox-Wright function $1 \Psi_1(*)$ will hold true for $y > 0, \Re(\eta) > 0, \Re(b) > 0$ and $\Re(c) > 0$. If we let $\mu = \eta - 1, \xi = 1$ and $\nu = \pm \frac{1}{2}$ in the main theorem in (3.1), we obtain after some simplification the following expressions:

$$
F_2(\eta - 1, 1, b, c, -\frac{1}{2}, y) \equiv F_C^{(1)}(\eta, b, c, y) = \sqrt{2} \sum_{k=0}^\infty \frac{\Theta(k)}{k!} \int_0^\infty x^{\eta-1}e^{-(b+ck)x} \cos(xy) dx
$$

$$
= \sqrt{2} \sum_{k=0}^\infty \frac{\Theta(k)}{k!} \frac{1}{(b + ck)^{\eta}} \Psi_1 \left[ \frac{\eta, 2}{(\sigma, 2/\xi)} \right] - \frac{y^2}{4(b + c)^{2/\xi}}
$$

(4.1)

$$
F_2(\eta - 1, 1, b, c, \frac{1}{2}, y) \equiv F_S^{(1)}(\eta, b, c, y) = \sqrt{2} \sum_{k=0}^\infty \frac{\Theta(k)}{k!} \int_0^\infty x^{\eta-1}e^{-(b+ck)x} \sin(xy) dx
$$

$$
= \frac{y}{\sqrt{2}} \sum_{k=0}^\infty \frac{\Theta(k)}{k!} \frac{1}{(b + ck)^{\eta+1}} \Psi_1 \left[ \frac{\eta + 1, 2}{(\sigma, 2/\xi)} \right] - \frac{y^2}{4(b + c)^{2/\xi}}
$$

(4.2)

$$
F_3(\eta - 1, 1, b, c, -\frac{1}{2}, y) \equiv F_C^{(2)}(\eta, b, c, y)
$$

$$
= \sqrt{2} \sum_{k=0}^\infty \frac{1}{k! \Gamma(\beta_j + kB_j)} \frac{\Gamma(\alpha_j + kA_j)}{(\sigma, 2/\xi)} \cos(xy) dx
$$

$$
= \sqrt{2} \sum_{k=0}^\infty \frac{\Gamma(\alpha_j + kA_j)}{k! \Gamma(\beta_j + kB_j)} \frac{1}{(b + ck)^{\eta+1}} \Psi_1 \left[ \frac{\eta, 2}{(\sigma, 2/\xi)} \right] - \frac{y^2}{4(b + c)^{2/\xi}}
$$

(4.3)

$$
F_3(\eta - 1, 1, b, c, \frac{1}{2}, y) \equiv F_S^{(2)}(\eta, b, c, y)
$$
If we choose $\Theta(k) = k!$, $\eta = m + 1$, $y = \pi n$ and $b = c = 2\pi$ in (4.1)–(4.6), we find the following representations involving infinite sums of the Fox-Wright function:

$$
\int_0^\infty \frac{x^m \cos(\pi nx)}{e^{2\pi x} - 1} \, dx = \sum_{k=0}^\infty \int_0^\infty x^m e^{-2\pi x(1+k)} \cos(\pi nx) \, dx
$$

$$
= \frac{\sqrt{\pi}}{(2\pi)^{m+1}} \sum_{k=0}^\infty \frac{1}{(k+1)^{m+1}} \Psi_1 \left[ \frac{(m+1,2)}{1} \mid -\frac{n^2}{16(1+k)^2} \right], \quad (5.1)
$$

$$
\int_0^\infty \frac{x^m \sin(\pi nx)}{e^{2\pi x} - 1} \, dx = \sum_{k=0}^\infty \int_0^\infty x^m e^{-2\pi x(1+k)} \sin(\pi nx) \, dx
$$

$$
= \frac{\sqrt{\pi} n}{4(2\pi)^{m+1}} \sum_{k=0}^\infty \frac{1}{(k+1)^{m+2}} \Psi_1 \left[ \frac{(m+2,2)}{1} \mid -\frac{n^2}{16(1+k)^2} \right], \quad (5.2)
$$

$$
\int_0^\infty x^m e^{-2\pi x} \Psi_s \left[ \frac{(\alpha_j, A_j)_{1,r}}{(\beta_j, B_j)_{1,s}} \mid -e^{-2\pi x} \right] \cos(\pi nx) \, dx
$$

$$
= \frac{\sqrt{\pi}}{(2\pi)^{m+1}} \sum_{k=0}^\infty \frac{\prod_{j=1}^r \Gamma(\alpha_j + kA_j)}{\prod_{j=1}^s \Gamma(\beta_j + kB_j)} \frac{1}{(k+1)^{m+1}} \Psi_1 \left[ \frac{(m+1,2)}{1} \mid -\frac{n^2}{16(1+k)^2} \right], \quad (5.3)
$$

$$
\int_0^\infty x^m e^{-2\pi x} \Psi_s \left[ \frac{(\alpha_j, A_j)_{1,r}}{(\beta_j, B_j)_{1,s}} \mid -e^{-2\pi x} \right] \sin(\pi nx) \, dx
$$

5. Closed-form infinite summation formulas
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\[ \frac{\sqrt{\pi} n}{4(2\pi)^{m+1}} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(\alpha_j + kA_j)}{\prod_{j=1}^{s} \Gamma(\beta_j + kB_j)} \frac{1}{(k+1)^{m+2}} 1\Psi_1 \left[ \frac{m+2, 2}{\frac{3}{2}, 1} \left| -\frac{n^2}{16(1+k)^2} \right. \right], \quad (5.4) \]

\[ \int_0^{\infty} x^m e^{-2\pi x} \mathcal{F}_s \left[ \frac{(\alpha_j)_{1,r}}{(\beta_j)_{1,s}} e^{-2\pi x} \right] \cos(\pi n x) \, dx \]

\[ = \frac{\sqrt{\pi}}{(2\pi)^{m+1}} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(\alpha_j)_k}{\prod_{j=1}^{s} \Gamma(\beta_j)_k} \frac{1}{(k+1)^{m+1}} 1\Psi_1 \left[ \frac{m+1, 2}{\frac{1}{2}, 1} \left| -\frac{n^2}{16(1+k)^2} \right. \right], \quad (5.5) \]

\[ \int_0^{\infty} x^m e^{-2\pi x} \mathcal{F}_s \left[ \frac{(\alpha_j, A_j)_{1,r}}{(\beta_j, B_j)_{1,s}} e^{-2\pi x} \right] \sin(\pi n x) \, dx \]

\[ = \frac{\sqrt{\pi} n}{4(2\pi)^{m+1}} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(\alpha_j)_k}{\prod_{j=1}^{s} \Gamma(\beta_j)_k} \frac{1}{(k+1)^{m+2}} 1\Psi_1 \left[ \frac{m+2, 2}{\frac{3}{2}, 1} \left| -\frac{n^2}{16(1+k)^2} \right. \right]. \quad (5.6) \]

Finally, from (1.7), (1.8), (5.1) and (5.2), we obtain the summation formulas

\[ \sum_{k=0}^{\infty} \frac{1}{(1+k)^2} 1\Psi_1 \left[ \frac{(2, 2)}{\frac{1}{2}, 1} \left| -\frac{n^2}{16(1+k)^2} \right. \right] = \sqrt{\pi} \left( \frac{2}{\pi n^2} + \frac{\pi}{1 - \cosh \pi n} \right), \quad (5.7) \]

and

\[ \sum_{k=0}^{\infty} \frac{1}{(1+k)^3} 1\Psi_1 \left[ \frac{(3, 2)}{\frac{3}{2}, 1} \left| -\frac{n^2}{16(1+k)^2} \right. \right] = \frac{2i}{\sqrt{\pi} n} \left\{ \psi'(1 + \frac{1}{2} in) - \psi'(1 - \frac{1}{2} in) \right\}. \quad (5.8) \]

where \( n \) is free to be chosen. These sums have been verified numerically using Mathematica.

6. Conclusion

In this note we have shown that a certain integral involving the Bessel function of the first kind can be expressed in terms of the Fox-Wright hypergeometric function of a single variable. Special cases of this integral lead to similar representations given in (5.1) and (5.2) for an integral considered by Ramanujan. Two infinite sums involving the Fox-Wright function are evaluated in closed form.

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