Elastic scattering of Dirac fermions on Schwarzschild black holes

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Abstract

Approximative analytic solutions of the Dirac equation in the Schwarzschild geometry are used for building the partial wave analysis of the Dirac fermions scattered by black holes. The analytic expressions of the differential cross section and induced polarization degree are derived in terms of scattering angle, mass of the black-hole, energy and mass of the fermion. We perform a graphical study of differential cross section analysing the forward/backward scattering (known also as glory scattering) and the polarization degree as functions of scattering angle. The graphical analysis shows the presence of oscillations in scattering intensity around forward/backward directions, phenomena known as spiral scattering. In addition, we find that the scattering probability increases significantly for fermions with large angular momentum. The energy dependence of the differential cross section is also established by using analytical and graphical methods.
I. INTRODUCTION

The scattering of quantum particles on black holes was studied using mainly massless scalar and electromagnetic fields [5]-[20],[35]-[37] since the equations of massive fields cannot be solved analytically in the Schwarzschild geometry. This is one of the reasons why the scattering of massive Dirac fermions on black holes, was studied in particular cases [1], using combined analytical and numerical methods [2]-[4]. In Ref. [4] such methods were used for obtaining solutions of the Dirac equation on this background and investigating the glory and spiral scattering as well as the degree of polarization of these scattering processes. Moreover, it is a difficult task to study the interaction of a quantum particle with a black hole since there one has to combine quantities at quantum scale (mass, energy of the fermion) with quantities at galactic scale (mass of the black hole).

In this paper we would like to continue this study exploiting, in addition, the analytical properties of the approximative asymptotic solutions of the Dirac equation we have found some time ago [21]. These were obtained in the chart with Schwarzschild coordinates where we considered the Cartesian gauge that preserves the global central symmetry of the field equations [22, 23]. Under such circumstances, the separation of the spherical variables can be done as in central problems of special relativity [24]. After the separation of angular variables we remain with a pair of simple radial equations depending only on the gauge field components. These equations can be approximatively solved in the case of the Schwarzschild geometry by using the Novikov radial coordinate [25]. We obtained thus the asymptotic radial solutions we intend to use here for performing the partial wave analysis of the elastic scattering of fermions on black holes. The elastic channel will be selected by a suitable boundary condition on the black hole horizon which has to prevent the absorption of fermions by black hole. The method consists in identifying the scattering phase shifts after imposing the boundary condition as in the case of the Dirac-Coulomb scattering in Minkowski spacetime. Thus we obtain the analytic expressions of phase shifts, amplitudes, differential and total cross sections and polarization degree.

The paper is organized as follows. In the second section we briefly present our method of separating variables for the Dirac equation on central backgrounds. In the next section we present the mentioned approximative solutions of the Dirac equation in the black hole gravitational field, corresponding to the continuous energy spectrum. The fourth section is
devoted to our partial wave analysis obtaining the analytical expressions of the amplitudes whose principal properties have to be outlined before deriving the differential cross section and polarization degree. The fifth section is dedicated to the graphical analysis and discussion of the physical consequences of our results. Our conclusions are summarized in the last section.

II. THE DIRAC EQUATION IN CENTRAL BACKGROUNDS

The Dirac equation in curved spacetimes is defined in the frames \{x; e\} formed by a local chart of coordinates \(x^\mu\), labelled by natural indices, \(\alpha, \ldots, \mu, \nu, \ldots = 0, 1, 2, 3\), and the orthogonal local frames and coframes defined by the gauge fields (or tetrads), \(e_\alpha\) and respectively \(\hat{e}_\alpha\), labelled by the local indices \(\hat{\alpha}, \ldots, \hat{\mu}, \ldots\) with the same range. In local-Minkowskian manifolds \((M, g)\), having as flat model the Minkowski spacetime \((M^0, \eta)\) of metric \(\eta = \text{diag}(1, -1, -1, -1)\), the gauge fields satisfy the usual duality conditions, \(\hat{e}_\alpha e_\beta = \delta_\alpha^\beta\) and the orthogonality ones, \(e_\mu \cdot e_\nu = 0\), \(\hat{e}_\mu \cdot \hat{e}_\nu = 0\). The gauge fields define the local derivatives \(\hat{\partial}_\mu = e_\mu \partial_\mu\) and the 1-forms \(\omega_\alpha^\beta = e_\alpha \omega_\beta^\gamma dx_\gamma\) giving the line element \(ds^2 = \eta_{\alpha\beta} \omega_\alpha^\beta\). Moreover, these fields allow one to write down the connection components in the local frames as \(\hat{\Gamma}_\gamma^\alpha_{\hat{\beta} \hat{\gamma}} = e_\gamma^\alpha \omega_{\gamma \hat{\alpha} \hat{\beta}}\) while the notation \(\Gamma^{\gamma}_{\alpha \beta}\) stands for the usual Christoffel symbols.

In the frame \(\{x; e\}\), the Dirac equation of a free spinor field \(\psi\) of mass \(m\) has the form

\[
    i\gamma^\hat{\alpha} D_\hat{\alpha} \psi - m \psi = 0 ,
\]

where \(\gamma^\hat{\alpha}\) are the point-independent Dirac matrices that satisfy \(\{\gamma^\hat{\alpha}, \gamma^\hat{\beta}\} = 2\eta^{\hat{\alpha}\hat{\beta}}\) and define the generators of the spinor representation of the \(SL(2, C)\) group, \(S^{\hat{\alpha}\hat{\beta}} = \frac{i}{4}[\gamma^\hat{\alpha}, \gamma^\hat{\beta}]\), which give the spin connections of the covariant derivatives.

\[
    D_\hat{\alpha} = e_\mu^\hat{\alpha} D_\mu = \partial_\hat{\alpha} + \frac{i}{2} S_{\hat{\gamma} \hat{\beta}}^{\hat{\alpha}} \hat{\Gamma}_{\hat{\gamma} \hat{\beta}} .
\]

Thus the Dirac equation \([1]\) takes the explicite form

\[
    i\gamma^\hat{\alpha} e_\mu^\alpha \partial_\mu \psi - m \psi + \frac{i}{2\sqrt{-g}} \partial_\mu (\sqrt{-g} e_\mu^\alpha) \gamma^\hat{\alpha} \psi - \frac{1}{4} \{\gamma^\hat{\alpha}, S_{\hat{\gamma} \hat{\beta}}^{\hat{\alpha}}\} \hat{\Gamma}_{\hat{\gamma} \hat{\beta}} \psi = 0 ,
\]

where \(g = \text{det}(g_{\mu\nu})\). Moreover, from the conservation of the electric charge, one deduces that the time-independent relativistic scalar product of two spinors is given by the integral
\[(\psi, \psi') = \int_D d^3x \sqrt{-g(x)} e^0_\mu(x) \bar{\psi}(x) \gamma^\mu \psi'(x), \] (4)

over the space domain \(D\) of the the local chart under consideration.

In general, a manifold with central symmetry has a central static chart with spherical coordinates \((t, r, \theta, \phi)\), covering the space domain \(D = D_r \times S^2\), i.e. \(r \in D_r\) while \(\theta\) and \(\phi\) cover the sphere \(S^2\). The local frames for which the central symmetry behaves as a global one are given by the so called Cartesian gauge,

\[
\omega^0 = w(r) dt,
\]

\[
\omega^1 = \frac{w(r)}{u(r)} \sin \theta \cos \phi \, dr + \frac{r w(r)}{v(r)} \cos \theta \cos \phi \, d\theta - \frac{r w(r)}{v(r)} \sin \theta \sin \phi \, d\phi,
\]

\[
\omega^2 = \frac{w(r)}{u(r)} \sin \theta \sin \phi \, dr + \frac{r w(r)}{v(r)} \cos \theta \sin \phi \, d\theta + \frac{r w(r)}{v(r)} \sin \theta \cos \phi \, d\phi,
\]

\[
\omega^3 = \frac{w(r)}{u(r)} \cos \theta \, dr - \frac{r w(r)}{v(r)} \sin \theta \, d\theta,
\]

we defined in Refs. [22, 23] in terms of three arbitrary functions of \(r\), denoted by \(u, v\) and \(w\), which allow us to write the line element as

\[
ds^2 = \eta_{\alpha\beta} \omega^\alpha \omega^\beta = w(r)^2 \left[ dt^2 - \frac{dr^2}{u(r)^2} - \frac{r^2}{v(r)^2} (d\theta^2 + \sin^2 \theta d\phi^2) \right].\] (9)

We have shown that in this gauge the last term of Eq. (3) does not contribute and, moreover, there is a simple transformation, \(\psi \rightarrow vw^{-\frac{3}{2}} \psi\), able to eliminate the terms containing the derivatives of the functions \(u, v\) and \(w\) leading thus to a simpler reduced Dirac equation [22]. The separation of the spherical variables of this reduced equation can be done as in the case of the central problems in Minkowski flat spacetime such that the Dirac field can be written as a linear combination of particular solutions of given energy, \(E\). Those of positive frequency,

\[U_{E,\kappa,m_j}(x) = U_{E,\kappa,m_j}(t, r, \theta, \phi)\]

\[= \frac{v(r)}{rw(r)^{3/2}} [f^+_x(r)\Phi^+_x(m_j,\kappa)(\theta, \phi) + f^-_x(r)\Phi^-_x(m_j,\kappa)(\theta, \phi)] e^{-iEt},\]

are particle-like energy eigenspinors expressed in terms of radial wave functions \(f^\pm_x\) and usual four-component angular spinors \(\Phi^\pm_{m_j,\kappa}\) [24]. It is known that these spinors are orthogonal to each other being labelled by the angular quantum numbers \(m_j\) and

\[
\kappa = \begin{cases} 
  j + \frac{1}{2} = l & \text{for } j = l - \frac{1}{2} \\
  -(j + \frac{1}{2}) = -l - 1 & \text{for } j = l + \frac{1}{2}
\end{cases}
\]

(11)
which encapsulates the information about the quantum numbers $l$ and $j = l \pm \frac{1}{2}$ as defined in Refs. [24, 27] (while in Ref. [4] $\kappa$ is of opposite sign). The spherical spinors are normalized to unity with respect to their own angular scalar product. We note that the antiparticle-like energy eigenspinors can be obtained directly using the charge conjugation as in the flat case [26].

Thus the problem of the angular motion is completely solved for any central background. We remain with a pair of radial wave functions, $f^\pm$ (denoted from now without indices) which satisfy two radial equations that can be written in compact form as the eigenvalue problem $H_r \mathbf{F} = E \mathbf{F}$ of the radial Hamiltonian [22],

$$H_r = \begin{pmatrix} m \, w(r) & -u(r) \frac{d}{dr} + \kappa \frac{v(r)}{r} \\ u(r) \frac{d}{dr} + \kappa \frac{v(r)}{r} & -m \, w(r) \end{pmatrix},$$

in the space of two-component vectors (or doublets), $\mathbf{F} = (f^+, f^-)^T$, equipped with the radial scalar product [22]

$$(\mathcal{F}_1, \mathcal{F}_2) = \int_{D_r} \frac{dr}{u(r)} \mathcal{F}_1^\dagger \mathcal{F}_2,$$

resulted from the general formula (4). This selects the 'good' radial wave functions, i.e. square integrable functions or tempered distributions, which enter in the structure of the particle-like energy eigenspinors (10).

III. APPROXIMATING THE DIRAC SPINORS IN SCHWARZSCHILD’S GEOMETRY

Let us assume that a Dirac particle of mass $m$ is moving freely (as a perturbation) in the central gravitational field of a black hole of mass $M$ with the Schwarzschild line element

$$d\sigma^2 = \left(1 - \frac{r_0}{r}\right) dt^2 - \frac{dr^2}{1 - r_0/r} - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2),$$

defined on the radial domain $D_r = (r_0, \infty)$, where $r_0 = 2MG$. Hereby we identify the functions

$$u(r) = 1 - \frac{r_0}{r}, \quad v(r) = w(r) = \sqrt{1 - \frac{r_0}{r}},$$

that give the radial Hamiltonian (12). The resulting radial problem cannot be solved analytically as it stays forcing one to resort to numerical methods [4] or to some approximations.
In Ref. [21] we proposed an effective method of approximating this radial problem expanding the radial equations in terms of the Novikov dimensionless coordinate, \( x = \sqrt{\frac{r}{r_0}} - 1 \in (0, \infty) \). (16)

Using this new variable and introducing the notations
\[
\mu = r_0 m, \quad \epsilon = r_0 E,
\]
we rewrite the exact radial problem as
\[
\begin{pmatrix}
\mu \sqrt{1 + x^2} - \epsilon \left(x + \frac{1}{x}\right) & -\frac{1}{2} \frac{d}{dx} + \frac{\kappa}{\sqrt{1 + x^2}} \\
\frac{1}{2} \frac{d}{dx} + \frac{\kappa}{\sqrt{1 + x^2}} & -\mu \sqrt{1 + x^2} - \epsilon \left(x + \frac{1}{x}\right)
\end{pmatrix}
\begin{pmatrix}
f^+(x) \\
f^-(x)
\end{pmatrix} = 0. \quad (18)
\]

Moreover, from Eq.(13) we find that the radial scalar product takes now the form
\[
(F_1, F_2) = 2r_0 \int_0^\infty dx \left(x + \frac{1}{x}\right) F_1^\dagger F_2. \quad (19)
\]

For very large values of \( x \), we can use the approximation \( \sqrt{1 + x^2} \sim x \) that leads to the asymptotic radial problem [21]
\[
\begin{pmatrix}
\frac{1}{2} \frac{d}{dx} + \frac{\kappa}{x} & -\mu \left(x + \frac{\delta}{x}\right) - \epsilon \left(x + \frac{1}{x}\right) \\
-\mu \left(x + \frac{\delta}{x}\right) + \epsilon \left(x + \frac{1}{x}\right) & \frac{1}{2} \frac{d}{dx} - \frac{\kappa}{x}
\end{pmatrix}
\begin{pmatrix}
f^+(x) \\
f^-(x)
\end{pmatrix} = 0. \quad (20)
\]

The new parameter \( \delta \in [0, 1) \) could be seen as a fit parameter but if we consider rigorously the approximation of the order \( O(1/x) \) then we must take \( \delta = \frac{1}{2} \). These equations can be solved analytically for any values of \( \epsilon \). In Ref. [21] we studied the spinors of the discrete spectrum, in the domain \( \epsilon < \mu \), and we outlined the scattering modes corresponding to the continuous spectrum \( \epsilon \in [\mu, \infty) \) but without investigating scattering effects.

Here we would like to derive the fermion scattering amplitudes starting with the solutions of Ref. [21] we briefly present below. For finding them, we need to diagonalize the term proportional to \( x \) of the operator (20) using the transformation matrix
\[
T = \begin{vmatrix}
-i\sqrt{\mu + \epsilon} & i\sqrt{\mu + \epsilon} \\
\sqrt{\epsilon - \mu} & \sqrt{\epsilon - \mu}
\end{vmatrix}. \quad (21)
\]
After the transformation $\mathcal{F} \rightarrow \hat{\mathcal{F}} = T^{-1} \mathcal{F} = (\hat{f}^+, \hat{f}^-)^T$ we obtain the new system of radial equations
\[
\left[\frac{1}{2} x \frac{d}{dx} \pm i \left(\frac{\delta \mu^2 - \epsilon^2}{\nu} - \nu x^2\right)\right] \hat{f}^\pm = \left(\kappa \pm \frac{i \epsilon \mu (\delta - 1)}{\nu}\right) \hat{f}^\mp ,
\] (22)
where $\nu = \sqrt{\epsilon^2 - \mu^2}$. These equations can be analytically solved for the continuous energy spectrum, $\epsilon > \mu$, in terms of Whittaker functions as [21]
\[
\hat{f}^+(x) = \frac{C_1^+}{x} M_{r+,s}(2i\nu x^2) + \frac{C_2^+}{x} W_{r+,s}(2i\nu x^2) \quad \hat{f}^-(x) = \frac{C_1^-}{x} M_{r-,s}(2i\nu x^2) + \frac{C_2^-}{x} W_{r-,s}(2i\nu x^2) ,
\quad (23)
\quad (24)
\]
where we denote
\[
s = \sqrt{\kappa^2 + \mu^2 \delta^2 - \epsilon^2} , \quad r_\pm = \mp \frac{1}{2} - iq , \quad q = \nu + \frac{\mu^2 (1 - \delta)}{\nu} ,
\quad (25)
\]
while the normalization constants must satisfy
\[
\frac{C_1^-}{C_1^+} = \frac{s - iq}{\kappa - i\lambda} , \quad \frac{C_2^-}{C_2^+} = -\frac{1}{\kappa - i\lambda} , \quad \lambda = \frac{\epsilon \mu}{\nu}(1 - \delta) .
\quad (26)
\]
We observe that these solutions are similar to those of the relativistic Dirac-Coulomb problem. The functions $M_{r\pm,s}(2i\nu x^2) = (2i\nu x^2)^{s\pm \frac{1}{2}}[1 + O(x^2)]$ are regular in $x = 0$, where the functions $W_{r\pm,s}(2i\nu x^2)$ diverge as $x^{1-2s}$ if $s > \frac{1}{2}$. These solutions will help us to find the scattering amplitudes of the Dirac particles on black holes.

IV. PARTIAL WAVE ANALYSIS

We consider now the elastic scattering of the Dirac fermions on a black hole. This is described by the energy eigenspinor $U$ whose asymptotic form,
\[
U \rightarrow U_{\text{plane}}(\vec{p}) + A(\vec{p}, \vec{n}) U_{\text{sph}} ,
\quad (27)
\]
for $r \rightarrow \infty$ (where the gravitational field vanishes) is given by the plane wave spinor of momentum $\vec{p}$ and the free spherical spinors of the flat case behaving as
\[
U_{\text{sph}} \propto \frac{1}{r} e^{ipr - iEt} , \quad p = \sqrt{E^2 - m^2} = \frac{\nu}{r_0} ,
\quad (28)
\]
Here we fix the geometry such that $\vec{p} = p \vec{e}_3$ while the direction of the scattered fermion is given by the scattering angles $\theta$ and $\phi$ which are just the spheric angles of the versor $\vec{n}$. Then the scattering amplitude
\[
A(\vec{p}, \vec{n}) = f(\theta) + \frac{1}{p} \vec{p} \wedge \vec{n} g(\theta)
\quad (29)
\]
depends only on two scalar functions on $\theta$ that can be determined by using the partial wave analysis.

### A. Phase shifts

This method exploits the asymptotic form of the analytical solutions,

$$\mathcal{F} \propto \frac{\sqrt{E + m} \sin \left( pl - \frac{\pi l}{2} + \delta_\kappa \right)}{\sqrt{E - m} \cos},$$

(30)

defining the scattering phase shifts, $\delta_\kappa$, which allows one to write down the amplitudes in terms of Lagrange polynomials of $\cos \theta$.

In our case the key of this analysis is the observation that for real values of $s$ we have $|C_1^+| = |C_1^-|$ since $s^2 + q^2 = \kappa^2 + \lambda^2$. This suggests that the suitable boundary condition is $C_2^+ = C_2^- = 0$ since this selects the regular spinors in $x = 0$ eliminating simultaneously the terms which may have an unwanted asymptotic behaviour. In other words, this condition selects the *elastic channel* of the fermion scattering on black-hole. We recover thus a well-known conjecture, similar to that of the Dirac-Coulomb scattering in special relativity. Therefore, it is convenient to chose

$$C_1^+ = \frac{1}{\sqrt{2\nu}} Ce^{i\xi - i\zeta}, \quad C_1^- = \frac{1}{\sqrt{2\nu}} Ce^{-i\xi - i\zeta}$$

(31)

where $C \in \mathbb{R}$ is an arbitrary constant, $\zeta = \frac{q}{2}(s + \frac{1}{2})$ and

$$e^{-2i\xi} = \frac{C_1^-}{C_1^+} = \frac{s - iq}{\kappa - i\lambda}.$$ 

(32)

Then by using Eq. (A1) we can rewrite the solutions as

$$\hat{f}^+(x) = Ce^{i\xi}(2\nu x^2)^{s-\frac{1}{2}}e^{-i\nu^2x^2} F_1(1 + s + iq, 2s + 1, 2i\nu x^2),$$

(33)

$$\hat{f}^-(x) = Ce^{-i\xi}(2\nu x^2)^{s-\frac{1}{2}}e^{-i\nu^2x^2} F_1(s + iq, 2s + 1, 2i\nu x^2).$$

(34)

According to Eq. (A2) these have the remarkable property $\left[\hat{f}^+(x)\right]^* = \hat{f}^-(x)$ which is the starting point of the partial wave analysis since the functions of the doublet $\mathcal{F} = T\hat{\mathcal{F}}$ can be written now as

$$f^+ = i\sqrt{\nu + \mu}(\hat{f}^- - \hat{f}^+) = \sqrt{\nu + \mu} 2\Re(\hat{f}^+)$$

(35)

$$f^- = \sqrt{\nu - \mu}(\hat{f}^+ + \hat{f}^-) = \sqrt{\nu - \mu} 2\Im(\hat{f}^+)$$

(36)
For $x \to \infty$ we obtain the asymptotic form

$$f^+(x) \to Ce^{-\frac{1}{2}\pi q} \frac{\Gamma(2s + 1)}{\Gamma(1 + s + iq)} e^{i(\xi + \nu x^2 - \frac{1}{2}\pi s + q \ln(2\nu x^2))}$$  \hspace{1cm} (37)$$

resulted from Eqs. (A3) and (A4). Then, bearing in mind that according to Eqs. (16) and (28b) we have $\nu x^2 = p(r - r_0)$, we obtain the asymptotic form,

$$\mathcal{F} \propto \frac{\sqrt{E + m} \sin \left( pr - \frac{\pi l}{2} + \delta + \vartheta(r) \right)}{\sqrt{E - m} \cos \left( pr - \frac{\pi l}{2} + \delta + \vartheta(r) \right)},$$  \hspace{1cm} (38)$$

defining the constants phase shifts $\delta$ that satisfy

$$e^{2i\delta} = \left( \frac{\kappa - i\lambda}{s - iq} \right) \frac{\Gamma(1 + s - iq)}{\Gamma(1 + s + iq)} e^{i\pi(l - s)}$$  \hspace{1cm} (39)$$

where the values of $\kappa$ and $l$ are related as in Eq. (11), i. e. $l = |\kappa| - \frac{1}{2}(1 - \text{sign}\,\kappa)$. The remaining point-dependent phase,

$$\vartheta(r) = -pr_0 + q \ln[2p(r - r_0)],$$  \hspace{1cm} (40)$$

which does not depend on angular quantum numbers, may be ignored as in the Dirac-Coulomb case [4, 27].

We arrived thus at the final result (39) depending on the parameters introduced above that can be expressed in terms of physical constants by using Eq. (28b) and fixing $\delta = \frac{1}{2}$. Then we obtain

$$s = \left[ \kappa^2 - G^2M^2(4p^2 + 3m^2) \right]^{\frac{1}{2}} = |\kappa| - \frac{G^2M^2}{2|\kappa|}(4p^2 + 3m^2) + O(G^4),$$  \hspace{1cm} (41)$$

$$q = \frac{GM}{p} (2p^2 + m^2),$$  \hspace{1cm} (42)$$

$$\lambda = \frac{GM}{p} m\sqrt{m^2 + p^2}.$$  \hspace{1cm} (43)$$

We remind the reader that this result holds only for real values of $s$ such that the condition

$$|\kappa| \geq GM \sqrt{4p^2 + 3m^2} \geq \sqrt{3m}GM,$$  \hspace{1cm} (44)$$

becomes now mandatory.

This is the basic framework of the relativistic partial wave analysis of the Dirac fermions scattered by Schwarzschild black holes. Our result is in accordance with the Newtonian approach in exact $\frac{1}{r}$ potential where the phase shifts are given by Eq. (B2) [11]. Notice that
this happens only when we choose $\delta = \frac{1}{2}$ since otherwise $q$ does not take the suitable form \((42)\). Therefore, we must keep this value as being the only correct one.

On the other hand, we must point out that our phase shifts are given by a formula which has the same form as that deduced for the Dirac-Coulomb scattering. Indeed, by replacing in Eq. \((39)\) the parameters

$$s \to \sqrt{\kappa^2 - Z^2\alpha^2}, \quad q \to \frac{Z\alpha}{p} E, \quad \lambda \to \frac{Z\alpha}{p} m.$$ \((45)\)

we recover the phase shifts of the Dirac-Coulomb scattering. However, there are significant differences between these two parametrization, given by Eqs. \((41)-(43)\) and respectively Eqs. \((45)\), showing that these two systems are of different natures.

**B. Partial amplitudes and cross sections**

The amplitude \((29)\) can be written now in terms of phase shifts deduced above and Lagrange polynomials as,

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos \theta) \quad (46)$$

$$g(\theta) = \sum_{l=1}^{\infty} (2l+1) g_l P_l(\cos \theta) \quad (47)$$

where we consider the partial amplitudes \([4, 27]\)

$$f_l = \frac{1}{2i\alpha} \frac{1}{2l+1} \left[ (l+1)(e^{2i\delta_{l-1}} - 1) + l(e^{2i\delta_l} - 1) \right] \quad (48)$$

$$g_l = \frac{1}{2i\alpha} \frac{1}{2l+1} \left( e^{2i\delta_{l-1}} - e^{2i\delta_l} \right) \quad (49)$$

Hereby one can deduce the scattering intensity or differential cross section,

$$\mathcal{I}(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2; \quad (50)$$

and the total cross section by integrating after the angles $\theta$ and $\phi$ in \((50)\).

The total cross section

$$\sigma = \sum_{l=0}^{\infty} \sigma_l = 2\pi \int_{-1}^{1} d \cos \theta \left[ |f(\theta)|^2 + |g(\theta)|^2 \right] \quad (51)$$

calculated according to the normalization integral

$$\int_{-1}^{1} dx P_l^m(x) P_l^m(x) = \frac{2\delta_{lm}}{2l+1} \frac{(l+m)!}{(l-m)!}.$$ \((52)\)
yields the partial cross sections
\[ \sigma_l = 4\pi (2l + 1) \left[ |f_l|^2 + l(l + 1) |g_l|^2 \right] \]
\[ = \frac{4\pi}{p^2} [(l + 1) \sin^2(\delta_{-l-1}) + l \sin^2(\delta_l)]. \] (53)

Another important quantity which allows us to study the physical consequences of our calculations is the polarization degree. This quantity refers to the scattering of fermions which have mass and represents the induced polarization for an unpolarized initial beam:
\[ \mathcal{P}(\theta) = -i \frac{f(\theta)^* g(\theta) - f(\theta) g(\theta)^*}{|f(\theta)|^2 + |g(\theta)|^2}. \] (54)

Now we have all the elements for calculating the analytic expressions of the partial amplitudes and cross sections in terms of the phase shifts defined by Eq. (39). However, in general, these have very complicated formulas that cannot be written down but can be manipulated under algebraic codes on computers. Nevertheless, in some particular cases we obtain even simple formulas. For example, we find that the phase shifts \( \delta_l \) and \( \delta_{-l} \) are related as
\[ e^{2i(\delta_l - \delta_{-l})} = \frac{pl - iG M m \sqrt{p^2 + m^2}}{pl + iG M m \sqrt{p^2 + m^2}} \] (55)
which means that for massless Dirac fermions we have \( \delta_l = \delta_{-l} \).

In weak gravitational fields with small \( GM \) we can expand,
\[ f_l = GM f_l^{(1)} + G^2 M^2 f_l^{(2)} + \cdots \quad l = 0, 1, 2, \ldots, \] (56)
\[ g_l = GM g_l^{(1)} + G^2 M^2 g_l^{(2)} + \cdots \quad l = 1, 2, \ldots. \] (57)

Our algebraic codes under Maple indicate that we must consider separately the cases \( l = 0 \) and \( l > 0 \) since it seems that the expansion of \( f_l \) does not commute with its limit for \( l \to 0 \). However, this is not surprising since a similar phenomenon can be met in the Dirac-Coulomb problem. Then, according to Eqs. (48), (49) and (39), we obtain first for \( l = 0 \),
\[ f_0^{(1)} = \frac{\gamma^2 2p^2 + m^2}{p^2} - \frac{2p^2 + m^2 - m \sqrt{p^2 + m^2}}{2p^2}, \] (58)
\[ f_0^{(2)} = i \gamma \frac{2p^2 + m^2}{p^3} - i \gamma \frac{2p^2 + m^2}{p^3} (2p^2 + m^2 - m \sqrt{p^2 + m^2}) \]
\[ - \frac{m(2p^2 + m^2)}{2p^3} \sqrt{p^2 + m^2} + i \frac{4p^4 + 5p^2 m^2 + 2m^4}{4p^3} + \pi \frac{4p^2 + 3m^2}{4p}. \] (59)
where $\gamma$ is the Euler constant. Furthermore, we find for any $l > 0$ the terms of first order,

\[
f^{(1)}_l = -\psi(l + 1) \frac{2p^2 + m^2}{p^2}, \quad \text{and} \quad g^{(1)}_l = -\frac{1}{2l(l + 1)} \frac{2p^2 + m^2 - m\sqrt{p^2 + m^2}}{p^2},
\]

where $\psi$ is the digamma function, as well as the more complicated ones of second order

\[
f^{(2)}_l = i\psi(l + 1) \frac{(2p^2 + m^2)^2}{p^3} - \frac{i}{2l(l + 1)} \frac{m(2p^2 + m^2)}{p^3} \frac{\sqrt{p^2 + m^2}}{p} + \frac{i}{4l(l + 1)} \frac{2m^4 + 5p^2m^2 + 4p^4}{p^3} + \frac{\pi}{2(2l + 1)} \frac{4p^2 + 3m^2}{p},
\]

\[
g^{(2)}_l = i\psi(l + 1) \frac{2p^2 + m^2}{l(l + 1)} \frac{(2p^2 + m^2 - m\sqrt{p^2 + m^2})}{p^3} + \frac{i}{l^2(l + 1)^2} \frac{m(2p^2 + m^2)}{p^3} \frac{\sqrt{p^2 + m^2}}{p} - \frac{i}{4l^2(l + 1)^2} \frac{4p^4 + 5m^2p^2 + 2m^4}{p^3} - \frac{\pi}{4l(l + 1)(2l + 1)} \frac{4p^2 + 3m^2}{p} ,
\]

which lay out the dependence on $l$ and $p$. Then, from Eq. (B3) we observe that $f^{(1)}_l$ and the first term of $f^{(2)}_l$ are of Newtonian form, increasing with $l$ because of the digamma function. The other terms decrease when $l$ is increasing so that we can say that our partial wave analysis has a correct Newtonian limit for large $l$, as it happens in the well-known case of the scalar particles [1]. In other respects, we see that the partial amplitudes of the massless fermions are regular in $p = 0$ while those of the massive particles diverge in any order. Thus the problem of removing the infrared catastrophe in the massive case seems to remain a serious challenge.

V. NUMERICAL EXAMPLES

Our purpose now is to use the graphical analysis for understanding the physical consequences of our analytical results encapsulated in quite complicated formulas and infinite series. First of all we must truncate the series (46) and (47) ignoring the terms with $l > L$ where the integer $L$ is a now a new parameter. We obtain thus the truncated amplitudes $f_L(\theta)$ and $g_L(\theta)$ giving the corresponding truncated scattering intensity $I_L(\theta)$ and polarization degree $P_L(\theta)$ as defined by Eqs. (50) and respectively (54). In what follows we focus on these functions of $\theta$ bearing in mind that these depend, in addition, on four free parameters: the fermion mass $m$ and momentum $p$, the black hole mass $M$ and the truncation parameter.
We remind the reader that we work in the asymptotic zone where the fermion energy is \( E = \sqrt{m^2 + p^2} \) such that we can use the fermion speed \( v = p/E \) as a relevant parameter.

We must stress that our analytical result for the scattering intensity holds for any values of the scattering angle such that we are able to study for the first time how this quantity behaves for small angles, near \( \theta \sim 0 \), returning then to the case when \( \theta \) is close to \( \pi \). Moreover, after this investigation, we continue to study the behaviour of the differential and total cross sections as function of energy.

We remind the reader that one can construct two dimensionless quantities relevant for scattering problems by using the constants \( M \) (black hole mass), \( E \) (fermion energy or frequency of the field) and \( m \) (fermion mass) \([4]\). The first one is given by the product \( GmM/(\hbar c) \) and can be seen as (proportional to) the ratio of the black hole horizon to the fermion associated Compton wavelength. The second relevant quantity is \( GME/(hc^3) \). Because we are using units in which \( c = \hbar = G = 1 \) in all our analytical formulas will appear products like \( mM \) and \( ME \), which together with \( L \) are used to label our graphs.

A. Forward and backward scattering

We begin the graphical analysis by plotting the differential cross section given by Eq. \([50]\), as a function of angle \( \theta \) for fixed values of \( mM = 1.2 \) and \( ME = 2 \) and summing after the angular momentum quantum number \( l \) up to \( L = 50 \). The graphs from Figs. \([12]\) shows how the scattering intensity depends on scattering angle for different angular momenta and small speeds of the fermion while the graphs from Figs. \([34]\) shows the same dependence at relativistic speeds. The axis of \( \theta \) is shorten in order to observe the oscillations of the scattering intensity around \( \theta = 0 \) for the forward scattering.
FIG. 1: The function $I_L(\theta)$ for different values of the fermion speed, $L = 10$, $mM = 1.2$ and $ME = 2$.

FIG. 2: The function $I_L(\theta)$ for different values of the fermion speed, $L = 30$, $mM = 1.2$ and $ME = 2$.

Our graphs from Figs. (1-4) show that the scattering intensity is maximal in the forward direction ($\theta = 0$) and that there are small oscillations for small values of $\theta$, that are known as orbiting scattering [39]. From our graphs we observe that the scattering intensity is increasing when we sum up to larger values of $L$ in Eq. (50). Few comments are needed here. From the classical point of view the forward scattering is in fact a diffraction on the black hole horizon. This could be possible only in the case when the wavelength of the incident particle is comparable with the size of the event horizon. Another consequence
that can be seen from Figs. (1-2) is that the scattering intensity in the forward direction is larger in the case of small speeds comparatively with the case of relativistic speeds given in Figs. (3-4). This happens since the wavelength of a quantum particle (given by $\lambda = \frac{h}{p}$) is very large for small momenta of the incident fermion such that the probability of the forward scattering increases. In other words, the slow fermions have higher probabilities to be diffracted by the black hole.

Let us now address the opposite situation when the scattering takes place in the backward
direction. Plotting the scattering intensity \( I(\theta) \) as function of \( \theta \) around \( \theta \sim \pi \) we observe two interesting phenomena. The first one is associated to the presence of a maximum of scattering intensity in the backward direction which is known as the glory scattering \[39\], while the second one is referring to the presence of oscillations in the scattering intensity at intermediate scattering angles around \( \theta = \pi \), phenomenon which is known as orbiting scattering \[39\]. Our results for values of \( \theta \) around \( \pi \) are presented in Figs. (5-8):

**FIG. 5:** The function \( I_L(\theta) \) for different values of the fermion speed, \( L = 20, \ mM = 1.2 \) and \( ME = 2 \).

**FIG. 6:** The function \( I_L(\theta) \) for different values of the fermion speed, \( L = 30, \ mM = 1.2 \) and \( ME = 2 \).
FIG. 7: The function $I_L(\theta)$ for different values of the fermion speed, $L = 20$, $mM = 1.2$ and $ME = 2$.

FIG. 8: The function $I_L(\theta)$ for different values of the fermion speed, $L = 30$, $mM = 1.2$ and $ME = 2$.

Hereby we observe that the oscillations in the scattered intensity around $\pi$ corresponds to the orbiting or spiral scattering [39]. Moreover, it is worth pointing out the presence of the glory scattering [39] in Figs. (5-8) thanks to the central maximum at $\theta = \pi$. On the other hand, we observe that the scattering intensity increases considerably when we sum up to larger values of $L$ like in the previous case of forward scattering. In both these cases the scattering intensity $I_L(\theta)$ presents a series of oscillations around forward and backward directions which show that the fermions could be scattered by black hole at intermediate
angles. The main difference between the two types of oscillations is that in the case of the forward scattering these are very small comparatively with the central maximum in $\theta = 0$.

Another interesting aspect related to our result is that the oscillations around $\theta = 0$ and $\theta = \pi$, are important when we sum after the angular quantum number up to $L = 30$ in Eq. (50). After this value (around $L = 32$) the scattering intensity will be concentrated mainly around backward and forward directions. So one could ask what are the conditions in which forward/backward scattering becomes dominant. In order to find an answer and understanding the difference between the two types of scattering we plot the scattering intensity as function of $\theta$ but this time we allow the values of the scattering angle to run from 0 to values larger than $\pi$. These results are presented in Figs. (9-13).

![FIG. 9: The function $I_L(\theta)$ for different values of the fermion speed, $L = 10$, $mM = 1.2$ and $ME = 2$.](image)

We observe from Figs. (9-13) that the forward scattering seems to be dominant as long as we take the sum up to $L = 30$ in Eq. (50). After this value the backward scattering begins important and when we sum up to $L = 34$ we obtain the first value that produces a significant backward intensity. The conclusion is that the forward scattering is dominant for small values of the fermion angular momentum. This result is in accordance with our observation that the forward scattering is more probable for fermions with small momenta, which means large wavelength. So the scattering intensity will be mainly concentrated in the forward direction for a beam of fermions which have small angular momenta.

The situation changes drastically if we take large values for $L$ in the sum (50). We can
FIG. 10: The function $I_L(\theta)$ for different values of the fermion speed, $L = 34$, $mM = 1.2$ and $ME = 2$.

FIG. 11: The function $I_L(\theta)$ for different values of the fermion speed, $L = 49$, $mM = 1.2$ and $ME = 2$.

see in Figs. (11-13), that for large values of angular momentum the backward scattering becomes important. Our result prove that the forward scattering and backward scattering have close probabilities when we sum for large values of $L$. To conclude, we can say that if an incident beam of fermions (in which the particles have different values for angular momenta) is scattered by a black hole then the scattered intensities will be mainly concentrated in the forward and backward directions.

From our graphs Figs. (9-13) we observe that the scattering intensities in the forward
and backward direction increases as the angular momentum becomes larger or equivalently if we sum more terms in Eq. (50). This suggest that the scattering probability increases with the angular momentum. Let us comment this result from the classical point of view. One knows that the differential cross section represents the area that the incident particle must cross in the target zone in order to be detected in the solid angle $d\Omega$. In the case of a scattering process between two quantum objects this quantity is very small. So in our case it is not so surprising that the differential cross section becomes very large, since the
target is of the size of the black hole event horizon. This can be understood better if we recall the result from classical physics according to which the scattering probability for a classical particle that is moving on spiral trajectory is larger than the scattering probability of a particle moving in a straight line. A classical particle with large angular momentum will always cross a larger area in the target zone, so that a large angular momentum means a large target area. The situation is the same in the case of a quantum particle scattered by a black hole but with the observation that in this case the notion of trajectory is not well-defined. Taking into consideration that the minimum area of the target is of the size of the event horizon we see that the area crossed by the fermion to be detected in a solid angle could be very large. To conclude we can say that the fermions with large angular momentum have a larger probability to be scattered by the black hole.

B. Dependence on energy

Furthermore we plot the scattering intensity \([50]\) as function of energy for different speeds and scattering angles using a logarithmic scale for the scattering intensity. We mention that for obtaining these plots as function of energy we use fermions with a Planck size mass since we can perform useful plots only for such very heavy fermions. However, our analytical formulas can be, in principle, plotted for any fermion mass as long as one has sufficiently computing power.
FIG. 14: $I_L(\theta)$ as function of energy for different values of the fermion speed, $\theta = 0$, $mM = 1.2$ and $L = 10$.

For the backward scattering, we see in Fig. (15) that the energy dependence of scattering intensity has a more pronounced oscillatory behavior compared with the forward scattering in Fig. (14). Moreover, this oscillatory behavior is decreasing as the energy increases. On the other hand, we observe in Figs. (14-15) that the scattering intensity is decreasing too when the energy increases. The shape of these graphs are the result of the fact that our scattering intensity is proportional with the usual factor $1/E^2$ and the oscillatory effect is given by the more complicated dependence of energy from the phase shift given in Eq. (39). Another important observation is that the differential cross section seems to tend to a constant value for very high energies, as it was expected.
FIG. 15: $I_L(\theta)$ as function of energy for different values of the fermion speed, $\theta = \pi$, $mM = 1.2$ and $L = 10$.

Now we plot the total cross section $[51]$ as function of energy. The shape of the curves remain basically the same as in the case of scattering intensity but the oscillatory behavior is less pronounced, as we observe from Fig. (16).
FIG. 16: Total cross as function of energy for $mM = 1.2$ and different fermion speeds. $L = 10$ in the first graph and $L = 15$ in the second one.

C. Polarization degree

Here we consider that the fermions of the incident beam are not polarized but the scattered beam could become partially polarized. It is interesting to study this effect by plotting the degree of polarization (54) as a function of scattering angle for $mM = 1.2$, $ME = 2$ and different speeds of the fermion.

As we observe in Figs. (17-18) the polarization is a very oscillatory function in terms of scattering angle. These oscillations are due to the forward and backward scattering as well as to the spiral scattering. These three types of scattering induce the oscillatory behaviour of polarization, since the scattering intensity oscillates with the scattering angle.
Let us study now the case of $ME = 2$ and fixed fermion speeds, but this time we chose different values of $mM$. Plotting the polarization as function of scattering angle we obtain the results given in Figs. (19-20). In Fig. (19) consider a small fermion speed while in Fig. (20) we take relativistic values.

Here one can observe that the oscillatory behaviour of the polarization, Figs. (19-20) is modified as we change the parameter $mM$. If the mass of the fermion $m$ is fixed, then we
can draw the conclusion that the oscillatory behavior of polarization depends on the black hole mass $M$. 

FIG. 19: The function $\mathcal{P}(\theta)$ for different $mM$, fermion speed $v = 0.2$ and $ME = 2$. $L = 10$ in the first graph and $L = 20$ in the second one.
FIG. 20: The function $P(\theta)$ for different $mM$, fermion speed $v = 0.8$ and $ME = 2$. $L = 10$ in the first graph and $L = 20$ in the second one.

To see how the spin of the fermion is aligned with a given direction after scattering on a black hole, we present the polar plot for the degree of polarization for different speeds.
FIG. 21: Polar plot for $\mathcal{P}(\theta)$ for different fermion speeds, $ME = 2, mM = 1.2$ and $L = 10$. In the third graph the fermion speed is $v = 0.25$ while in the forth graph this is $v = 0.75$.

It is interesting to point out that from our Figs. 21, 22 we observe that the scattered wave can be partially polarized in the direction orthogonal with the scattering plane. This phenomenon is similar with the Mott polarization [40], which appear in the electromagnetic scattering. This conclusion was also underlined in Ref. [4].
VI. CONCLUDING REMARKS

In this paper we performed the partial wave analysis of the Dirac fermions scattered on black-holes by using an approximative analytical solutions of the Dirac equation in Schwarzschild geometry. Thus we obtained the analytical expression for the phase shifts computing the differential cross section and the degree of polarization.

From our analytical and graphical results we established that the quantum wave can be scattered in forward and backward directions and that the spiral scattering is present in both these cases. Moreover, we established what are the values of physical parameters
such that the forward/backward scattering could become dominant or the situation when
the two types of scattering could have almost the same probabilities to occur. The forward
scattering seems to be dominant if we sum in the scattering intensity only over small values
of the angular quantum number \( l \). For larger values of angular momentum the forward and
backward scattering have close probabilities. We also found that the scattering intensity in
both forward and backward directions is increasing with the angular momentum.

The polarization degree has an oscillatory behaviour in terms of scattering angle and de-
PENDS on the black hole mass. Our polar plots for polarizations show what are the directions
in which the spin is aligned after the interaction. Our graphical analysis can be performed
for different values of the parameters \( mM, ME \) and \( v \) leading to similar conclusions.

For further study it will be interesting to complete this program with an analytical study
of the scattering of the electromagnetic field in this geometry. The absorption and emis-
sion by black holes may be also investigated using our method of approximative analytical
solutions of different field equations in the black hole gravitational field.

Appendix A: Confluent hypergeometric functions

The Wittaker functions \( M \) are related to the confluent hypergeometric ones as \[28\]
\[
M_{\kappa,\mu}(z) = e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu} \, _1F_1(\frac{1}{2} + \mu - \kappa, 2\mu + 1, z). \tag{A1}
\]

These have the property
\[
_1F_1(a, b, -z) = e^{-z} \, _1F_1(b - a, b, z), \tag{A2}
\]

and the asymptotic representation,
\[
_1F_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(b - a)}(-z)^{-a} G(a, a - b + 1, -z) + \frac{\Gamma(b)}{\Gamma(a)}(z)^{a-b} e^{z} G(b - a, 1 - a, z) \tag{A3}
\]

where
\[
G(a, b, z) = \sum_{a=0}^{\infty} \left(1 + \frac{ab}{z} + \frac{a(a + 1)b(b + 1)}{2!} \frac{1}{z^2} + \cdots \right). \tag{A4}
\]
Appendix B: Newtonian scattering

In the non-relativistic theory with exact potential $\frac{1}{r}$ the Newtonian scattering amplitude \[ f^N(\theta) = \sum_{l=0}^{\infty} (2l + 1) f^N_l P_l(\cos\theta), \quad f^N_l = \frac{1}{2i\mu} \left( e^{2i\delta^N_l} - 1 \right), \] (B1)
is determined by the phase shifts $\delta^N_l$ that in our notations satisfy \[ e^{2i\delta^N_l} = \frac{\Gamma(1 + l - iq)}{\Gamma(1 + l + iq)}, \] (B2)
where $q$ is given by Eq. (42). Then the partial amplitudes can be expanded as \[ f^N_l = -GM \frac{\beta}{p^2} + iG^2 M^2 \frac{\beta^2}{p^3} + O(G^3), \quad \beta = (2p^2 + m^2)\psi(l + 1). \] (B3)

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