ON THE HADAMARD’S TYPE INEQUALITIES FOR
L-LIPSCHITZIAN MAPPING

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ABSTRACT. In this paper, we establish some new inequalities of Hadamard’s
type for L-Lipschitzian mapping in two variables.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval $I$ of real
numbers and $a, b \in I$, with $a < b$, the following double inequality is well known in
the literature as the Hermite-Hadamard inequality:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$  \hspace{1cm} (1.1)

Let us now consider a bidimensional interval $\Delta = : [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$
and $c < d$. A mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on $\Delta$ if the following
inequality:

$$f(tx + (1 - t)z, ty + (1 - t)w) \leq tf(x, y) + (1 - t)f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be
on the co-ordinates on $\Delta$ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(x, y)$
and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and
$y \in [c, d]$ (see [3]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. A function $f : \Delta \rightarrow \mathbb{R}$ will be called co-ordinated convex on $\Delta$, for
all $t, s \in [0, 1]$ and $(x, y), (u, w) \in \Delta$, if the following inequality holds:

$$f(tx + (1 - t)y, su + (1 - s)w) \leq tsf(x, u) + s(1 - t)f(y, u) + t(1 - s)f(x, w) + (1 - t)(1 - s)f(y, w).$$  \hspace{1cm} (1.1)

Clearly, every convex function is co-ordinated convex. Furthermore, there exist
co-ordinated convex function which is not convex, (see, [3]). For several recent
results concerning Hermite-Hadamard’s inequality for some convex function on the
co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1]-[3], [5],
[6], [8], [9] and [11]).

Recently, in [3], Dragomir establish the following similar inequality of Hadamard’s
type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

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inequality and L-Lipschitzian.}
Theorem 1. Suppose that \( f : \Delta \rightarrow \mathbb{R} \) is co-ordinated convex on \( \Delta \). Then one has the inequalities:

\[
\begin{align*}
&f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \\
\leq & \frac{1}{2} \left[ \frac{1}{b-a} \int_{a}^{b} f \left( x, \frac{c+d}{2} \right) \, dx + \frac{1}{d-c} \int_{c}^{d} f \left( \frac{a+b}{2}, y \right) \, dy \right] \\
\leq & \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \\
\leq & \frac{1}{4} \left[ \frac{1}{b-a} \int_{a}^{b} f(x, c) \, dx + \frac{1}{b-a} \int_{a}^{b} f(x, d) \, dx \\
&+ \frac{1}{d-c} \int_{c}^{d} f(a, y) \, dy + \frac{1}{d-c} \int_{c}^{d} f(b, y) \, dy \right] \\
\leq & \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{align*}
\]

The above inequalities are sharp.

Definition 2. Consider a function \( f : V \rightarrow \mathbb{R} \) defined on a subset \( V \) of \( \mathbb{R}^n \), \( n \in \mathbb{N} \). Let \( L = (L_1, L_2, ..., L_n) \) where \( L_i \geq 0 \), \( i = 1, 2, ..., n \). We say that \( f \) is \( L \)-Lipschitzian function if

\[
|f(x) - f(y)| \leq \sum_{i=1}^{n} L_i |x_i - y_i|
\]

for all \( x, y \in V \).

The authors in [4] and [10] have proved the following inequalities of Hadamard’s type for Lipschitzian mapping.

Theorem 2. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be \( L \)-Lipschitzian on \( I \) and \( a, b \in I \) with \( a < b \). Then, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{L(b-a)}{3}
\]

and

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{L(b-a)}{4}.
\]

For several recent results concerning Hadamard’s type inequality for some \( L \)-Lipschitzian function, we refer the reader to ([4], [7], [10]).

The main purpose of this paper is to establish some Hadamard’s type inequalities for \( L \)-Lipschitzian mapping in two variables.
2. Hadamard’s Type Inequalities

Firstly, we will start the proof of the Theorem 1 by using the definition of the co-ordinated convex functions as follows:

**Theorem 3.** Suppose that \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \). Then one has the inequalities:

\[
f\left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx
\]

(2.1)

\[
\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\]

**Proof.** According to (1.1) with \( x = t_1 a + (1 - t_1)b, \ y = (1 - t_1)a + t_1 b, \ u = s_1 c + (1 - s_1)d, \ w = (1 - s_1)c + s_1 d \) and \( t = s = \frac{1}{2} \), we find that

\[
f\left( \frac{a+b}{2}, \frac{c+d}{2} \right)
\]

\[
\leq \frac{1}{4} \left[ f(t_1a + (1 - t_1)b, s_1c + (1 - s_1)d) + f((1 - t_1)a + t_1 b, s_1 c + (1 - s_1)d)
\]

\[
+ f(t_1a + (1 - t_1)b, (1 - s_1)c + s_1 d) + f((1 - t_1)a + (1 - t_1)b, (1 - s_1)c + s_1 d) \right].
\]

Thus, by integrating with respect to \( t_1, s_1 \) on \( [0, 1] \times [0, 1] \), we obtain

\[
f\left( \frac{a+b}{2}, \frac{c+d}{2} \right)
\]

\[
\leq \frac{1}{4} \left[ \int_{0}^{1} \int_{0}^{1} \left[ f(ta + (1 - t)b, sc + (1 - s)d) + f(ta + (1 - t)b, (1 - s)c + sd) \right] \, ds \, dt
\]

\[
+ \int_{0}^{1} \int_{0}^{1} \left[ f((1 - t)a + tb, sc + (1 - s)d) + f((1 - t)a + tb, (1 - s)c + sd) \right] \, ds \, dt \right].
\]

Using the change of the variable, we get

\[
f\left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx,
\]

(2.2)

which the first inequality is proved. The proof of the second inequality follows by using (1.1) with \( x = a, \ y = b, \ u = c \) and \( w = d \), and integrating with respect to
Here, using the change of the variable \( x = ta + (1-t)b \) and \( y = sc + (1-s)d \) for \( s, t \in [0,1] \), we have

\[
\int_{0}^{1} \int_{0}^{1} \left[ ts f(a, c) + s(1-t)f(b, c) + t(1-s)f(a, d) + (1-t)(1-s)f(b, d) \right] dsdt = f(a, c) + f(a, d) + f(b, c) + f(b, d).
\]

We get the inequality (2.1) from (2.2) and (2.3). The proof is complete. \( \square \)

**Theorem 4.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) satisfy \( L \)-Lipschitzian conditions. That is, for \((t_1, s_1)\) and \((t_2, s_2)\), \( |f(t_1, s_1) - f(t_2, s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2| \)

where \( L_1 \) and \( L_2 \) are positive constants. Then, we have the following inequalities:

\[
\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \right| \leq \frac{1}{16} (M_1 |b-a| + M_2 |d-c|)
\]

\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \right| \leq \frac{1}{12} (M_1 |b-a| + M_2 |d-c|)
\]

where \( M_1 = [L_1 + L_3 + L_5 + L_7] \) and \( M_2 = [L_2 + L_4 + L_6 + L_8] \).
Proof. Let \( t, s \in [0,1] \). Since \( ts + s(1-t) + t(1-s) + (1-t) (1-s) = 1 \), then we have

\[
|tsf(a,c) + s(1-t)f(b,c) + t(1-s)f(a,d) + (1-t)(1-s)f(b,d) - f(ta + (1-t)b, sc + (1-s)d)|
\]

\[= |ts[f(a,c) - f(ta + (1-t)b, sc + (1-s)d)] + s(1-t)[f(b,c) - f(ta + (1-t)b, sc + (1-s)d)] + t(1-s)[f(a,d) - f(ta + (1-t)b, sc + (1-s)d)] + (1-t)(1-s)[f(b,d) - f(ta + (1-t)b, sc + (1-s)d)]|
\]

\leq ts[(1-t)L_1|b-a| + (1-s)L_2|d-c|] + s(1-t)[tL_3|b-a| + (1-s)L_4|d-c|] + t(1-s)[(1-t)L_5|b-a| + sL_6|d-c|] + (1-t)(1-s)[tL_7|b-a| + sL_8|d-c|]

\[= (ts(1-t)[L_1 + L_3] + t(1-s)(1-t)[L_5 + L_7])|b-a| + (ts(1-s)[L_2 + L_6] + s(1-s)(1-t)[L_4 + L_8])|d-c|.
\]

If we choose \( t = s = \frac{1}{2} \) in (2.7), we get

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|
\]

\[\leq \frac{1}{8} ([L_1 + L_3 + L_5 + L_7]|b-a| + [L_2 + L_6 + L_4 + L_8]|d-c|).
\]

Thus, if we put \( ta + (1-t)b \) instead of \( a \), \( (1-t)a + tb \) instead of \( b \), \( sc + (1-s)d \) instead of \( c \) and \( (1-s)c + sd \) instead of \( d \) in (2.7), respectively, then it follows that

\[
\left| \frac{f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd)}{4} + \frac{f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)}{4} - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|
\]

\[\leq \frac{1}{8} ([L_1 + L_3 + L_5 + L_7]|1-2t| |b-a| + [L_2 + L_6 + L_4 + L_8]|1-2s| |d-c|)
\]

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for all \( t, s \in [0, 1] \). If we integrate the inequality (2.8) with respect to \( s, t \) on \([0, 1] \times [0, 1]\)

\[
\begin{align*}
&\int_0^1 \int_0^1 [f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd)] \, ds \, dt \\
&+ \int_0^1 \int_0^1 [f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)] \, ds \, dt \\
&- f\left(\frac{a + b + c + d}{2}\right)
\end{align*}
\]

\[
\leq \frac{1}{8} \left\{ [L_1 + L_3 + L_5 + L_7] |b - a| \int_0^1 \int_0^1 |1 - 2t| \, ds \, dt \\
+ [L_2 + L_6 + L_4 + L_8] |d - c| \int_0^1 \int_0^1 |1 - 2s| \, ds \, dt \right\}.
\]

Thus, using the change of the variable \( x = ta + (1-t)b, \ y = (1-t)a + tb, \ u = sc + (1-s)d \) and \( w = (1-s)c + sd \) for \( t, s \in [0, 1] \), and

\[
\int_0^1 \int_0^1 |1 - 2t| \, ds \, dt = \int_0^1 \int_0^1 |1 - 2s| \, ds \, dt = \frac{1}{2}
\]

we obtain the inequality (2.4).

Note that, by the inequality (2.6), we write

\[
|tsf(a, c) + s(1-t)f(b, c) + t(1-s)f(a, d) + (1-t)(1-s)f(b, d) \\
- f(ta + (1-t)b, sc + (1-s)d)|
\]

(2.9)

\[
\leq (ts(1-t) [L_1 + L_3] + s(1-s)(1-t) [L_5 + L_7]) |b - a| \\
+ (ts(1-s) [L_2 + L_6] + s(1-s)(1-t) [L_4 + L_8]) |d - c|.
\]

for all \( t, s \in [0, 1] \). If we integrate the inequality (2.9) with respect to \( s, t \) on \([0, 1] \times [0, 1]\), we have

\[
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \right|
\]

\[
\leq \frac{1}{12} ([L_1 + L_3 + L_5 + L_7] |b - a| + [L_2 + L_6 + L_4 + L_8] |d - c|)
\]
and so we have the inequality (2.5), where we use the fact that

\[
\int_0^1 \int_0^1 st(1-t)dsdt = \int_0^1 \int_0^1 s(1-s)(1-t)dsdt = \frac{1}{12}.
\]

This completes the proof. \(\square\)

3. The Mapping \(H\)

For a \(L\)-Lipschitzian function \(f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}\), we can define a mapping \(H : [0, 1] \times [0, 1] \to \mathbb{R}\) by

\[
H(t, s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( tx + (1-t) \frac{a+b}{2}, sy + (1-s) \frac{c+d}{2} \right) dydx.
\]

Now, we give some properties of this mapping as follows:

**Theorem 5.** Suppose that \(f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}\) be \(L\)-Lipschitzian on \(\Delta := [a, b] \times [c, d]\). Then:

(i) The mapping \(H\) is \(L\)-Lipschitzian on \([0, 1] \times [0, 1]\).

(ii) We have the following inequalities

\[
(3.1) \quad \left| H(t, s) - f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right| \leq \frac{L_1 t}{4} (b-a) + \frac{L_2 s}{4} (d-c)
\]

\[
(3.2) \quad \left| H(t, s) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right| \leq \frac{L_1 (1-t)}{4} (b-a) + \frac{L_2 (1-s)}{4} (d-c).
\]
Proof. (i) Let \( t_1, t_2, s_1, s_2 \in [0, 1] \). Then, we have
\[
|H(t_2, s_2) - H(t_1, s_1)|
\]
\[
= \frac{1}{(b - a)(d - c)} \left| \int_a^b \int_c^d f \left( t_2 x + (1 - t_2) \frac{a + b}{2}, s_2 y + (1 - s_2) \frac{c + d}{2} \right) dy dx \right. \\
\left. - \int_a^b \int_c^d f \left( t_1 x + (1 - t_1) \frac{a + b}{2}, s_1 y + (1 - s_1) \frac{c + d}{2} \right) dy dx \right|
\]
\[
\leq \frac{1}{(b - a)(d - c)} \left| \int_a^b \int_c^d \left[ L_1 |t_2 - t_1| \left| x - \frac{a + b}{2} \right| + L_2 |s_2 - s_1| \left| y - \frac{c + d}{2} \right| \right] dy dx \right|
\]
\[
= \frac{L_1 (b - a)}{4} |t_2 - t_1| + \frac{L_2 (d - c)}{4} |s_2 - s_1| ,
\]
i.e., for all \( t_1, t_2, s_1, s_2 \in [0, 1] \),
\[
|H(t_2, s_2) - H(t_1, s_1)| \leq \frac{L_1 (b - a)}{4} |t_2 - t_1| + \frac{L_2 (d - c)}{4} |s_2 - s_1| ,
\]
which yields that the mapping \( H \) is \( L \)-Lipschitzian on \( [0, 1] \times [0, 1] \).

(ii) The inequalities (3.1) and (3.2) follow from (3.3) by choosing \( t_1 = 0, t_2 = t, s_1 = 0, s_2 = s \) and \( t_1 = 1, t_2 = t, s_1 = 1, s_2 = s \), respectively. \( \square \)

Another result which is connected in a sense with the inequality (2.5) is also given in the following:

**Theorem 6.** Under the assumptions Theorem 5, then we get the following inequality
\[
\left| f \left( at + (1 - t) \frac{a + b}{2}, cs + (1 - s) \frac{c + d}{2} \right) + f \left( at + (1 - t) \frac{a + b}{2}, ds + (1 - s) \frac{c + d}{2} \right) \right|
\]
\[
+ \left| f \left( bt + (1 - t) \frac{a + b}{2}, cs + (1 - s) \frac{c + d}{2} \right) + f \left( bt + (1 - t) \frac{a + b}{2}, ds + (1 - s) \frac{c + d}{2} \right) \right|
\]
\[
= \frac{1}{(n_2 - n_1)(m_2 - m_1)} \int_{n_1}^{m_1} \int_{n_1}^{m_1} f(u, w) dw du
\]
\[
\leq \frac{1}{12} (M_1 |n_2 - n_1| t + M_2 |m_2 - m_1| s)
\]
where \( M_1 = [L_1 + L_3 + L_5 + L_7] \) and \( M_2 = [L_2 + L_4 + L_6 + L_8] \).
Proof. If we denote \( n_1 = at + (1-t)\frac{a+b}{2} \), \( n_2 = bt + (1-t)\frac{a+b}{2} \), \( m_1 = cs + (1-s)\frac{c+d}{2} \) and \( m_2 = ds + (1-s)\frac{c+d}{2} \), then, we have

\[
H(t, s) = \frac{1}{(n_2 - n_1)(m_2 - m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u, w) \, dw \, du.
\]

Now, using the inequality (2.5) applied for \( n_1, n_2, m_1 \) and \( m_2 \), we have

\[
\left| f(n_1, m_1) + f(n_1, m_2) + f(n_2, m_1) + f(n_2, m_2) \right| - \frac{1}{(n_2 - n_1)(m_2 - m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u, w) \, dw \, du \leq \frac{1}{12} (M_1 |n_2 - n_1| + M_2 |m_2 - m_1|)
\]

from which we have the inequality (3.4). This completes the proof. \( \square \)

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