TOPICAL REVIEW

Discrete quantum mechanics

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Abstract

A comprehensive review of the discrete quantum mechanics with the pure imaginary shifts and the real shifts is presented in parallel with the corresponding results in the ordinary quantum mechanics. The main subjects to be covered are the factorized Hamiltonians, the general structure of the solution spaces of the Schrödinger equation (Crum’s theorem and its modification), the shape invariance, the exact solvability in the Schrödinger picture as well as in the Heisenberg picture, the creation/annihilation operators and the dynamical symmetry algebras, the unified theory of exact and quasi-exact solvability based on the sinusoidal coordinates, and the infinite families of new orthogonal (the exceptional) polynomials. Two new infinite families of orthogonal polynomials, the $X\ell$ Meixner–Pollaczek and the $X\ell$ Meixner polynomials, are reported.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the discrete quantum mechanics (dQM), developed by the present authors, the time-independent Schrödinger equation is a second order difference equation instead of a second order differential equation in the ordinary quantum mechanics (oQM). The basic framework of quantum mechanics (QM), such as the probability interpretation, the Hilbert space structure, the continuous real time, etc, is unchanged and the only difference is the discrete forms of various Hamiltonians. The dQM can be considered as a generalization (deformation) of the oQM in the sense that a differential equation can be obtained from a difference equation in an appropriate limit. The generalization is also consistent with Heisenberg’s idea that a certain fundamental length would appear in the theory at some stage of descending the microscopic ladder. This generalization has turned out to be extremely fruitful. Most of the concepts and methods of oQM can be easily transplanted to dQM, offering a unified platform for the systematic understanding of various orthogonal polynomials satisfying second order
difference equations with the pure imaginary shifts (idQM) as well as the real shifts (rdQM). Sometimes these polynomials are said to have the bi-spectral property [1]. They satisfy two types of spectral conditions, the difference eigenvalue (Schrödinger) equation (2.23) and the three-term recurrence relations (2.139). Those polynomials satisfying the difference equations with the real shifts are also called the orthogonal polynomials of a discrete variable [2]. They contain most of the $(q)$-hypergeometric orthogonal polynomials belonging to the Askey scheme [2–6]. The power of the discrete QM is demonstrated most eloquently in the discovery of various infinite families of new orthogonal (exceptional) polynomials. In less than 2 years after the discovery of the infinite family of the exceptional Laguerre and Jacobi polynomials [7–9], the generic counterparts in dQM, the exceptional Wilson, Askey–Wilson, Racah and $q$-Racah polynomials are constructed by the present authors [10, 11]. These new orthogonal polynomials are expected to play a significant role in mathematics, mathematical physics and related disciplines. The exceptional Jacobi polynomials provide infinitely many global solutions of Fuchsian differential equations with more than three and arbitrarily many regular singularities [9], although the locations of the extra singularities are rigidly specified.

This review of dQM provides a comprehensive overview of the subjects related to the exactly and quasi-exactly solvable quantum particle dynamics [12–26]. Due to the length constraints, we have to concentrate on the systems of single degree of freedom, which is the most basic and best established part of the theory. The subjects to be covered in this review are the adaptation to dQM of the methods and concepts of oQM accumulated over 80 years after the birth of QM. After the general setting of the dQM with the pure imaginary shifts (idQM) as well as the real shifts (rdQM), we start with the factorized Hamiltonians and the Schrödinger equations [12, 20, 21]. The general structure of the solution spaces is explored by the intertwining relations and Crum’s theorem together with its modification [27–29, 24–26]. The exact solvability in the Schrödinger picture is explained by the shape invariance [30, 31, 12, 20, 21]. The generic eigenvalue formula, unified Rodrigues formulas and the forward-/backward-shift operators are deduced. The solvability in the Heisenberg picture is derived based on the closure relation between the sinusoidal coordinate and the Hamiltonian [14, 15]. The creation/annihilation operators are introduced and their connection with the three-term recurrence relations of the orthogonal polynomials is emphasized. The dynamical symmetry algebra generated by the Hamiltonian and the creation/annihilation operators are also established for all the solvable systems in the Heisenberg picture [21, 23]. This includes the dynamical realization of the $q$-oscillator algebras [19]. For the rdQM, the dual pair of orthogonal polynomials (the Leonard pair [32, 33]) and the related Askey–Wilson algebras [34] are explored. The unified theory of exactly and quasi-exactly solvable dQM is another major theme of this review. Based on the fundamental properties of various sinusoidal coordinates, a simple recipe to construct a solvable dQM Hamiltonian is provided [23]. It generates all known exactly solvable theories and many new ones. As a byproduct, it also gives a recipe of constructing quasi-exactly solvable dQM Hamiltonians. The final major subject of the review is the new (exceptional) orthogonal polynomials [7–11, 35–38].

Throughout the review, we stress on the similarity and differences among oQM, idQM and rdQM. As far as possible, we adopt the same notation and present one all-inclusive formula for the three categories, oQM, idQM and rdQM. The differences are exhibited by explicit examples. We usually adopt three examples from each group denoted by the name of the corresponding eigenpolynomials with an abbreviation. For the oQM, they are the Hermite (H), the Laguerre (L) and the Jacobi (J) polynomials. For the idQM, we choose the special case of the Meixner–Pollaczek (MP) (the parameter $\phi$ is fixed to $\phi = \frac{\pi}{2}$), the Wilson (W) and the Askey–Wilson (AW) polynomials. For the rdQM, the Meixner (M), the Racah (R) and
the q-Racah (qR) polynomials are selected. In some cases, we present only one representative from each group. They are usually the most generic ones J, AW and qR or the simplest ones H, MP and M. For the full lineups, the original references should be consulted [20, 21, 23]. We will provide cross references to the original articles with the section or the equation numbers.

This review is organized as follows. In section 2, various concepts and methods in dQM are explained, which are as follows. (a) The factorized Hamiltonians (2.5)–(2.12) and the ‘Schrödinger equations’ for the eigenpolynomials (2.23)–(2.27) together with the groundstate eigenfunctions (2.52), (2.55) as the orthogonality weight functions (2.49)–(2.50), (b) The intertwining relations connecting various Hamiltonians (2.58), (2.65)–(2.66), and eigenfunctions (2.61), (2.68) and (2.73). This is a brief summary of Crum’s theory in our language. Its deformation in the manner of Krein–Adler is also mentioned. (c) Shape invariance (2.105) and the full energy spectrum (2.106) and the unified Rodrigues type formula (2.107) for all the eigenfunctions. (d) Solvability in the Heisenberg picture (2.144) and the closure relation (2.142). The formulas for the creation/annihilation operators (2.147)–(2.148) are given. (e) Dual closure relation (2.182) together with its connection with the Askey–Wilson algebra. In section 3, we provide the essence of the unified theory of exact and quasi-exact solvability in dQM. In section 4, the new (exceptional) orthogonal polynomials satisfying second order differential (difference) equations are explored. The exceptional Meixner–Pollaczek and the exceptional Meixner polynomials are new results. The final section is for a summary and comments. Basic symbols and definitions are listed in the appendix. A substantially shorter version of the present review is published recently [39]. We apologize to all the authors whose good works could not be referred to in this review due to lack of space.

2. Discrete quantum mechanics

The dynamical variables of one-dimensional QM are the coordinate $x$ and its conjugate momentum $p$, which is realized as a differential operator $p = -i\hbar \frac{d}{dx} = -i\hbar \partial_x$. Hereafter, we adopt the convention $\hbar = 1$. The dQM is a generalization of oQM in which the Schrödinger equation is a difference equation instead of differential in the oQM [12–14, 20, 21]. In other words, the Hamiltonian contains the momentum operator in exponentiated forms $e^{\pm \beta \hat{p}}$ which works as shift operators on the wavefunction

$$e^{\pm \beta \hat{p}} \psi(x) = \psi(x \mp i\beta). \tag{2.1}$$

For the two choices of the parameter $\beta$, either real or pure imaginary, we have two types of dQM with (i) pure imaginary shifts $\beta = \gamma \in \mathbb{R}\backslash\{0\}$ (idQM), or (ii) real shifts $\beta = i$ (rdQM), respectively. In the case of idQM, $\psi(x \mp iy)$, we require the wavefunction and potential functions, etc to be analytic functions of $x$ with their domains including the real axis or a part of it on which the dynamical variable $x$ is defined. In contrast, in the rdQM, the difference equation gives constraints on wavefunctions only on equally spaced lattice points. Then, we choose, after proper rescaling, the variable $x$ to take the value of non-negative integers, with the total number either finite ($N + 1$) or infinite ($x_{\text{max}} = N$ or $\infty$). To sum up, the dynamical variable $x$ of the one-dimensional dQM takes continuous or discrete values:

$$\text{idQM} : \ x \in \mathbb{R}, \quad \text{rdQM} : \ x \in \mathbb{Z}_{\geq 0}, \quad x \in [0, x_{\text{max}}]. \tag{2.2}$$

Here $x_1, x_2$ may be finite, $-\infty$ or $+\infty$. Correspondingly, the inner product of the wavefunctions has the following form:

$$\text{idQM} : \ (f, g) = \int_{x_1}^{x_2} f(x)^*g(x) \, dx; \quad \text{rdQM} : \ (f, g) = \sum_{x=0}^{x_{\text{max}}} f(x)^*g(x), \tag{2.3}$$

and the norm of $f(x)$ is $\|f\| = \sqrt{(f, f)}$. 

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We will consider the Hamiltonians having a finite (rdQM with finite \(N\)) or semi-infinite number of discrete energy levels only:
\[
0 = \mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \cdots .
\] (2.4)

The additive constant of the Hamiltonian is so chosen that the ground state energy vanishes. That is, the Hamiltonian is positive semi-definite. It is a well-known theorem in linear algebra that any positive semi-definite Hermitian matrix can be factorized as a product of a certain matrix, say \(A\), and its Hermitian conjugate \(A^\dagger\). As we will see shortly, the Hamiltonians we consider always have factorized forms in one dimension as well as in higher dimensions.

2.1. Factorized Hamiltonian

The Hamiltonian we consider has a simple factorized form \([40]\):
\[
\mathcal{H} \overset{\text{def}}{=} A^\dagger A \quad \text{or} \quad \mathcal{H} \overset{\text{def}}{=} A^\dagger_j A_j \quad \text{in } D \text{ dimensions.}
\] (2.5)

The operators \(A\) and \(A^\dagger\) in one-dimensional QM are
\[
\text{oQM : } \begin{align*}
A & \overset{\text{def}}{=} \frac{d}{dx} - \frac{dw(x)}{dx}, \\
A^\dagger & \overset{\text{def}}{=} -\frac{d}{dx} - \frac{dw(x)}{dx}, \\
\mathcal{H} & = p^2 + U(x), \\
\phi_0(x) & = e^{w(x)},
\end{align*}
\] (2.6)

\[
\text{idQM : } \begin{align*}
A & \overset{\text{def}}{=} i(e^{\frac{i}{2}\gamma} \sqrt{V^*(x)} - e^{-\frac{i}{2}\gamma} \sqrt{V(x)}), \\
A^\dagger & \overset{\text{def}}{=} -i(\sqrt{V(x)} e^{\frac{i}{2}\gamma} - \sqrt{V^*(x)} e^{-\frac{i}{2}\gamma}), \\
\mathcal{H} & = \sqrt{V(x)} e^{iy} \sqrt{V^*(x)} + \sqrt{V^*(x)} e^{-iy} \sqrt{V(x)} - V(x) - V^*(x),
\end{align*}
\] (2.8)

\[
\text{rdQM : } \begin{align*}
A & \overset{\text{def}}{=} \sqrt{B(x) - \epsilon^2 D(x)}, \\
A^\dagger & \overset{\text{def}}{=} \sqrt{B(x) - \epsilon^2 D(x)} e^{-\epsilon^2}, \\
\mathcal{H} & = -\sqrt{B(x)} e^{\epsilon^2} \sqrt{D(x)} - \sqrt{D(x)} e^{-\epsilon^2} \sqrt{B(x)} + B(x) + D(x).
\end{align*}
\] (2.10)

Here \(w(x)\) is called the \textit{prepotential}. The function \(V^*(x)\) in idQM is an analytic function of \(x\) obtained from \(V(x)\) by the *-operation, which is defined as follows. If \(f(x) = \sum_n a_n x^n\), \(a_n \in \mathbb{C}\), then \(f^*(x) \overset{\text{def}}{=} \sum_n a^*_n x^n\), in which \(a^*_n\) is the complex conjugation of \(a_n\). Obviously \(f^{**}(x) = f(x)\) and \(f(x)^* = f^*(x^*)\). If a function satisfies \(f^* = f\), then it takes real values on the real line. To the best of our knowledge, the factorized Hamiltonian approach to dQM is our invention. The existing factorization methods for dQM are not for the Hamiltonian \(\mathcal{H}\), but for Hamiltonian minus an eigenvalue \(\mathcal{H} - \lambda(n)\) \([41, 42]\). The condition \(D(0) = 0\) in rdQM (2.11) is necessary for the term \(\psi(-1)\) not to appear in \(\mathcal{H}\psi(0)\). Likewise, \(B(N) = 0\) is necessary for
the finite case. Roughly speaking these correspond to the regular singularities in oQM. The Hamiltonians of idQM and rdQM can be written in a unified notation:

\[ H = \epsilon \left( \sqrt{V(x)} \frac{e^{\beta p}}{\sqrt{V(x)}} + \sqrt{V(x)} \frac{e^{-\beta p}}{\sqrt{V(x)}} - V(x) - V(x) \right), \]  

(2.13)

idQM : \( \beta = \gamma, \quad \epsilon = 1, \quad V_+(x) = V(x), \quad V_-(x) = V^*(x) \),

(2.14)

rdQM : \( \beta = i, \quad \epsilon = -1, \quad V_+(x) = B(x), \quad V_-(x) = D(x) \).

Multi-particle exactly solvable systems can be constructed in a similar way. The prepotential approach is also useful in Calogero–Moser systems [43] in oQM [44, 45]:

\[ A^0_j = \frac{\partial}{\partial x_j} - \frac{\partial w(x)}{\partial x_j}, \quad A_j^0 = -\frac{\partial}{\partial x_j} - \frac{\partial w(x)}{\partial x_j} \quad (j = 1, \ldots, D), \quad \phi_0(x) = e^{w(x)}. \]  

(2.15)

We also adopt the factorized Hamiltonian for the discrete analogues of the Calogero–Moser systems:

\[ A_j^\text{idQM} = i \left( e^{i \gamma p_j} \sqrt{V_j^*(x)} - e^{-i \gamma p_j} \sqrt{V_j(x)} \right), \quad V_j(x), V_j^*(x) \in \mathbb{C}, \]

(2.16)

\[ A_j^\text{rdQM} = -i \left( \sqrt{V_j^*(x)} e^{i \gamma p_j} - \sqrt{V_j(x)} e^{-i \gamma p_j} \right) \quad (j = 1, \ldots, D), \]

which is markedly different from the approach of Ruijsenaars et al [46]. The multi-particle version of rdQM is now being developed.

The Schrödinger equation

\[ \mathcal{H}_0(x) = \mathcal{E}(n) \phi_0(x) \quad (n = 0, 1, 2, \ldots), \quad 0 = \mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \cdots, \]

(2.17)

is a second order differential (oQM) or difference equation (dQM) and the ground state wavefunction \( \phi_0(x) \) is determined as a zero mode of the operator \( \mathcal{A} (\mathcal{A}_j) \) which is a first order equation:

\[ \mathcal{A}_j \phi_0(x) = 0 \quad (\mathcal{A}_j \phi_0(x) = 0, \quad j = 1, \ldots, D) \quad \Rightarrow \mathcal{H}_0(x) = 0. \]

(2.18)

By definition all the eigenfunctions are required to have a finite positive norm:

\[ 0 < (\phi_n, \phi_n) < \infty. \]

(2.19)

Throughout this review, we look for the eigenfunctions in a factorized form:

\[ \phi_n(x) = \phi_0(x) P_n(\eta(x)), \]

(2.20)

in which \( P_n(\eta(x)) \) is a degree \( n \) (except for those discussed in section 4) polynomial in the sinusoidal coordinate \( \eta(x) \). However, some basic results, such as the (modified) Crum’s theorem in sections 2.2–2.3 and the shape invariance in section 2.4, hold without this assumption (2.20). The explicit forms of the eigenfunctions will be given for various examples (2.30–2.39). Here we will briefly introduce their general property for dQM. The sinusoidal coordinate \( \eta(x) \) in dQM is a monotone increasing function of \( x \) with the initial (boundary) condition

\[ \eta(0) = 0. \]

(2.21)

With this, we also normalize the eigenpolynomials \( \{ P_n(\eta) \} \) in rdQM as

\[ \eta(0) = 1 \quad (n = 0, 1, \ldots). \]

(2.22)

In the multi-particle case, \( n \) will be a multi-index. Here, we require \( \phi_0(x) \) to be chosen real and positive for the physical values of \( x \), which is always guaranteed in oQM. One
important distinction between a differential and a difference equation is the uniqueness of the solution. For a linear differential equation, the solution is unique when the initial conditions are specified. In contrast, a solution of a difference Schrödinger equation (2.17) multiplied by any periodic function of a period $i\beta$ ($i\gamma$ for idQM, 1 for rdQM) is another solution. In the case of idQM, this non-uniqueness is removed by requiring the finite norm condition for the ground-state wavefunction $\phi_0$, $(\phi_0, \phi_0) < \infty$ and the Hermiticity (self-adjointness) condition of the Hamiltonian. For details, see [21], appendix A. For the rdQM, the periodic ambiguity of period 1 is harmless since the values of eigenfunctions on the integer lattice points only count for the inner products, etc. It is also important to realize that the periodic ambiguity does not appear in the polynomial solutions of (2.23).

It is important to stress that the second order equation for $P_n(\eta(x))$ is square root free:

$$\tilde{H} P_n(\eta(x)) = E(n) P_n(\eta(x)).$$  \hspace{0.5cm} (2.23)

In other words, the similarity-transformed Hamiltonian $\tilde{H}$ in terms of the ground-state wavefunction $\phi_0(x)$ has a much simpler form than the original Hamiltonian $H$:

$$\tilde{H} \overset{\text{def}}{=} \phi_0(x)^{-1} \circ H \circ \phi_0(x).$$  \hspace{0.5cm} (2.24)

\begin{align*}
oQM : \quad \tilde{H} &= -\frac{d^2}{dx^2} - 2\frac{dw(x)}{dx}, \\
dQM : \quad \tilde{H} &= \epsilon(V_+(x)(e^{\beta p} - 1) + V_-(x)(e^{-\beta p} - 1)) = \begin{cases} V(x)(e^{\gamma p} - 1) + V^*(x)(e^{-\gamma p} - 1): & \text{idQM} \\ B(x)(1 - e^{\gamma}) + D(x)(1 - e^{-\gamma}): & \text{rdQM}. \end{cases}
\end{align*}

For all the examples discussed in this section, $\tilde{H}$ is lower triangular

$$\tilde{H} \eta(x)^n = E(n) \eta(x)^n + \text{lower orders in } \eta(x),$$  \hspace{0.5cm} (2.28)

in the special basis

\begin{align*}
1, \eta(x), \eta(x)^2, \ldots, \eta(x)^n, \\
\text{spanned by the sinusoidal coordinate } \eta(x). \hspace{0.5cm} \text{This situation is expressed as}
\end{align*}

$$\tilde{H} \mathcal{V}_n \subseteq \mathcal{V}_n, \quad \mathcal{V}_n \overset{\text{def}}{=} \text{Span}[1, \eta(x), \ldots, \eta(x)^n].$$  \hspace{0.5cm} (2.29)

Here are some explicit examples. For the oQM, the prepotential $w(x)$ determines the potential $U(x)$ of the Hamiltonian $H = p^2 + U(x)$, $U(x) \overset{\text{def}}{=} (\partial_x w(x))^2 + \partial_x^2 w(x)$:

\begin{align*}
\text{H} : \quad w(x) &= \frac{1}{2} x^2, \quad -\infty < x < \infty, \\
U(x) &= x^2 - 1, \quad \eta(x) = x.
\end{align*}

\begin{align*}
\text{L} : \quad w(x) &= -\frac{1}{2} x^2 + g \log x, \quad g > 0, \quad 0 < x < \infty, \\
U(x) &= x^2 + \frac{g(g-1)}{x^2} - 1 - 2g, \quad \eta(x) = x^2.
\end{align*}

\begin{align*}
\text{J} : \quad w(x) &= g \log \sin x + h \log \cos x, \quad g > 0, \quad h > 0, \quad 0 < x < \frac{\pi}{2}, \\
U(x) &= \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2, \quad \eta(x) = \cos 2x.
\end{align*}

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Let us note that \( x = 0 \) for L and \( x = 0, \frac{q}{g} \) for J are the \textit{regular singular points} of the Fuchsian differential equations. The monodromy at the regular singular point is determined by the \textit{characteristic exponent} \( \rho \):

\[ M_\rho = e^{2\pi i \rho}. \]  

The corresponding exponents are \( \rho = g, 1 - g \) for L and \( \rho = g, 1 - g \) and \( \rho = h, 1 - h \) for J.

For the \textit{rdQM} \((0 < q < 1)\):

\[ V(x) = a + ix, \quad a > 0, \quad -\infty < x < \infty, \quad \gamma = 1, \quad \eta(x) = x, \]  

\[ (2.34) \]

For the \textit{idQM} \((0 < q < 1)\):

\[ V(x) = \prod_{j=1}^{q} (1 - a_j e^{i\alpha_j}), \quad [a_j] < 1, \quad 0 < x < \pi, \quad \gamma = \log q, 
\]

\[ \eta(x) = 1 - \cos x, \quad [a_1', a_2', a_3', a_4'] = [a_1, a_2, a_3, a_4] \]  

\[ (2.35) \]

For the \textit{rdQM} \((0 < q < 1)\):

\[ B(x) = \frac{c}{1-c}(x + \beta), \quad D(x) = \frac{1}{1-c}x, \quad \beta > 0, \quad 0 < c < 1, \]

\[ \eta(x) = x, \quad x_{\text{max}} = \infty, \]  

\[ (2.37) \]

The \textit{R} and \textit{qR} cases admit four different parametrizations specified by \( \epsilon, \epsilon' = \pm 1 \), see [20]. Here we present the \( \epsilon = \epsilon' = 1 \) case only. It should be emphasized that the sinusoidal coordinates \( \eta(x) \) in \textit{rdQM} in general depend on the parameters in contradistinction to those of the \textit{oQM} or \textit{idQM}.

The eigenfunctions of these examples have the form \( (2.20) \). The eigenpolynomials \( P_n(\eta(x)) \) are as follows (for notation, see [6]):

\[ \text{oQM} : \]

\[ H : \quad P_n(\eta(x)) = H_n(x) \overset{\text{def}}{=} (2x)^{\alpha} F_0 \left( \begin{array}{c} \frac{n}{2}, \frac{n-1}{2} \end{array} \bigg| \frac{1}{x^2} \right). \]  

\[ (2.40) \]
Let us emphasize that the weight function, or \( \phi_0(x) \), is determined as a solution of a first order differential (difference) equation (2.18), without recourse to a moment problem. This situation becomes crucially important when various deformations of orthogonal polynomials
are considered. For the oQM, the weight function is simply given by the prepotential \( \phi_0(x)^2 = e^{2ix} \) and the normalization constants are

\[
    h_n = \begin{cases} 
      2^n \sqrt{\pi} & : H \\
      \frac{1}{2n!} \Gamma \left( n + g + \frac{1}{2} \right) & : L \\
      \Gamma \left( n + g + \frac{1}{2} \right) \Gamma \left( n + h + \frac{1}{2} \right) & : J, \\
      \end{cases}
\]  

The explicit forms of the squared ground-state wavefunction (weight function) \( \phi_0(x)^2 \) and the normalization constants \( h_n \) for the above examples in pure imaginary shifts dQM are

\[
    \phi_0(x)^2 = \begin{cases} 
      \Gamma(a + ix) \Gamma(a - ix) & : MP \\
      \left( \Gamma(2ix) \Gamma(-2ix) \right)^{-1} \prod_{j=1}^{4} \Gamma(a_j + ix) \Gamma(a_j - ix) & : W \\
      (e^{2ix}; q)_{\infty} (e^{-2ix}; q)_{\infty} \prod_{j=1}^{4} \left( (a_j e^{ix}; q)_{\infty} (a_j e^{-ix}; q)_{\infty} \right)^{-1} & : AW, \\
    \end{cases}
\]

\[
    h_n = \begin{cases} 
      2\pi (2^{2n}n!)^{-1} \Gamma(n + 2a) & : MP \\
      2\pi n! (n + b_1 - 1)_n \prod_{1 \leq i < j \leq 4} \Gamma(n + a_i + a_j) \cdot \Gamma(2n + b_1)^{-1} & : W \\
      2\pi (p q^{p-1}; q)_{\infty} (b_1 q^{2n}, q)_{\infty} (q^{n+1}; q)_{\infty}^{-1} \prod_{1 \leq i < j \leq 4} (a_i a_j q^p; q)_{\infty}^{-1} & : AW. \\
    \end{cases}
\]

For the rdQM, the zero-mode equation \( A \phi_0(x) = 0 \) is a two-term recurrence relation, which can be solved elementarily by using the boundary condition (2.11):

\[
    \phi_0(x)^2 = \prod_{j=0}^{x-1} \frac{B(y)}{D(y + 1)}, \quad \phi_0(0) = 1.
\]

The explicit forms of \( \phi_0(x)^2 \) and \( d_n^2 \), which are related by duality, are:

\[
    \phi_0(x)^2 = \frac{(\beta)_x c^x}{x!} \frac{(a, b, c, d)_x}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_x} \frac{d}{d} \frac{2 + d}{d} \frac{(a, b, c, d; q)_x}{(1 - dq^2d, 1 - dq^2d)} : M
\]

\[
    d_n^2 = \frac{(\beta)_x c^x \times (1 - c)^d}{n!} \frac{(a, b, c, d)_n}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_n} \frac{d}{d} \frac{2 + d}{d} \frac{(a, b, c, d; q)_n}{(1 - dq^2d, 1 - dq^2d)} : M
\]

\[
    d_n^2 = \left( \frac{(\beta)_n c^n}{n!} \times (1 - c)^d \right) \left( \frac{(a, b, c, d)_n}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_n} \frac{d}{d} \frac{2 + d}{d} \frac{(a, b, c, d; q)_n}{(1 - dq^2d, 1 - dq^2d)} \right) : M
\]

\[
    d_n^2 = \frac{(\beta)_n c^n}{n!} \frac{(a, b, c, d)_n}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_n} \frac{d}{d} \frac{2 + d}{d} \frac{(a, b, c, d; q)_n}{(1 - dq^2d, 1 - dq^2d)} : M
\]

\[
    d_n^2 = \frac{(\beta)_n c^n}{n!} \frac{(a, b, c, d)_n}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_n} \frac{d}{d} \frac{2 + d}{d} \frac{(a, b, c, d; q)_n}{(1 - dq^2d, 1 - dq^2d)} : M
\]
It should be stressed that $\phi_0(x)^2$ for the rdQM (2.54), when continued analytically in $x$, vanish on the integer points outside the defining domain, i.e., $x \in \mathbb{Z}_{<0}$ and $x \in \mathbb{Z}_{>0}$.

To the best of our knowledge, the first example of a factorized idQM Hamiltonian was mentioned in 2001 (equation (4.28) of [47]) for the Meixner–Pollaczek polynomial. But it was not considered as a general principle.

2.2. Intertwining relations: Crum’s theorem

The general structure of the intertwining relations works equally well for the oQM as well as the dQM. Let us denote by $\mathcal{H}^{[0]}$ the original factorized Hamiltonian and by $\mathcal{H}^{[1]}$ its partner (associated) Hamiltonian obtained by changing the order of $\mathcal{A}$ and $\mathcal{A}^\dagger$:

$$\mathcal{H}^{[0]} \overset{\text{def}}{=} \mathcal{A}^\dagger \mathcal{A}, \quad \mathcal{H}^{[1]} \overset{\text{def}}{=} \mathcal{A} \mathcal{A}^\dagger. \quad (2.57)$$

One simple and most important consequence of the factorized Hamiltonian (2.5) is the intertwining relations

$$\mathcal{A} \mathcal{H}^{[0]} = \mathcal{A} \mathcal{A}^\dagger \mathcal{A} = \mathcal{H}^{[1]}; \quad \mathcal{A}^\dagger \mathcal{H}^{[1]} = \mathcal{A}^\dagger \mathcal{A} \mathcal{A}^\dagger = \mathcal{H}^{[0]} \mathcal{A}^\dagger, \quad (2.58)$$

which are equally valid in the oQM and the dQM. The pair of Hamiltonians $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$ are essentially iso-spectral and their eigenfunctions $\phi^{[0]}_n(x)$ and $\phi^{[1]}_n(x)$ are related by the Darboux–Crum transformations [48, 27]:

$$\mathcal{H}^{[0]} \phi^{[0]}_n(x) = \mathcal{E}(n) \phi^{[0]}_n(x) \quad (n = 0, 1, \ldots); \quad \mathcal{A} \phi^{[0]}_0(x) = 0, \quad (2.59)$$

$$\mathcal{H}^{[1]} \phi^{[1]}_n(x) = \mathcal{E}(n) \phi^{[1]}_n(x) \quad (n = 1, 2, \ldots), \quad (2.60)$$

$$\phi^{[1]}_n(x) = \mathcal{A} \phi^{[0]}_n(x), \quad \phi^{[0]}_n(x) = \frac{\mathcal{A}^\dagger}{\mathcal{E}(n)} \phi^{[1]}_n(x) \quad (n = 1, 2, \ldots), \quad (2.61)$$

$$\left(\phi^{[1]}_n, \phi^{[1]}_m\right) = \mathcal{E}(n) (\phi_n, \phi_m) \quad (n, m = 1, 2, \ldots). \quad (2.62)$$

The partner Hamiltonian $\mathcal{H}^{[1]}$ has the lowest eigenvalue $\mathcal{E}(1)$. If the ground-state energy $\mathcal{E}(1)$ is subtracted from the partner Hamiltonian $\mathcal{H}^{[1]}$, it is again positive semi-definite and can be factorized in terms of new operators $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[1]}\dagger$: \n
$$\mathcal{H}^{[1]} = \mathcal{A}^{[1]} \mathcal{A}^{[1]}\dagger + \mathcal{E}(1), \quad \mathcal{A}^{[1]} \phi^{[1]}_1(x) = 0. \quad (2.63)$$

It should be stressed that in rdQM with a finite $N$, the size of the Hamiltonian decreases by 1, since the lowest eigenstate is removed, see [26].

By changing the orders of $\mathcal{A}^{[1]}\dagger$ and $\mathcal{A}^{[1]}$, a new Hamiltonian $\mathcal{H}^{[2]}$ is defined:

$$\mathcal{H}^{[2]} \overset{\text{def}}{=} \mathcal{A}^{[1]} \mathcal{A}^{[1]}\dagger + \mathcal{E}(1). \quad (2.64)$$

These two Hamiltonians are intertwined by $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[1]}\dagger$:

$$\mathcal{A}^{[1]} (\mathcal{H}^{[1]} - \mathcal{E}(1)) = \mathcal{A}^{[1]} \mathcal{A}^{[1]}\dagger \mathcal{A}^{[1]} = (\mathcal{H}^{[2]} - \mathcal{E}(1)) \mathcal{A}^{[1]}, \quad (2.65)$$

$$\mathcal{A}^{[1]}\dagger (\mathcal{H}^{[2]} - \mathcal{E}(1)) = \mathcal{A}^{[1]}\dagger \mathcal{A}^{[1]}\dagger \mathcal{A}^{[1]} = (\mathcal{H}^{[1]} - \mathcal{E}(1)) \mathcal{A}^{[1]}\dagger. \quad (2.66)$$

The iso-spectrality of the two Hamiltonians $\mathcal{H}^{[1]}$ and $\mathcal{H}^{[2]}$ and the relationship among their eigenfunctions follow as before:

$$\mathcal{H}^{[2]} \phi^{[2]}_n(x) = \mathcal{E}(n) \phi^{[2]}_n(x) \quad (n = 2, 3, \ldots), \quad (2.67)$$
This process can go on indefinitely by successively deleting the lowest lying energy level:

\[ \phi^{[n]}_n(x) = \mathcal{A}^{[1]} \phi^{[1]}_n(x), \quad \phi^{[1]}_n(x) = \frac{\mathcal{A}^{[1]} \phi^{[2]}_n(x)}{\mathcal{E}(n) - \mathcal{E}(1)} \phi^{[2]}_n(x) \quad (n = 2, 3, \ldots). \quad (2.68) \]

(\phi^{[n]}_n, \phi^{[m]}_m) = (\mathcal{E}(n) - \mathcal{E}(1)) (\phi^{[1]}_n, \phi^{[1]}_m) \quad (n, m = 2, 3, \ldots). \quad (2.69) \]

\[ \mathcal{H}^{[2]} = \mathcal{A}^{[2]} \mathcal{A}^{[1]} + \mathcal{E}(2), \quad \mathcal{A}^{[2]} \phi^{[2]}_2(x) = 0. \quad (2.70) \]

This process can go on indefinitely by successively deleting the lowest lying energy level:

\[ \mathcal{H}^{[s]} \equiv \mathcal{A}^{[s-1]} \mathcal{A}^{[s-1]^\dagger} + \mathcal{E}(s - 1) = \mathcal{A}^{[s]} \mathcal{A}^{[s]^\dagger} + \mathcal{E}(s), \quad (2.71) \]

\[ \mathcal{H}^{[s]} \phi^{[s]}_n(x) = \mathcal{E}(n) \phi^{[s]}_n(x) \quad (n = s, s + 1, \ldots), \quad \mathcal{A}^{[s]} \phi^{[s]}_s(x) = 0, \quad (2.72) \]

\[ \phi^{[s]}_n(x) = \mathcal{A}^{[s-1]} \phi^{[s-1]}_n(x), \quad \phi^{[s-1]}_n(x) = \frac{\mathcal{A}^{[s-1]^\dagger}}{\mathcal{E}(n) - \mathcal{E}(s - 1)} \phi^{[s]}_n(x) \quad (n = s, s + 1, \ldots). \quad (2.73) \]

(\phi^{[s]}_n, \phi^{[m]}_m) = (\mathcal{E}(n) - \mathcal{E}(s - 1)) (\phi^{[s-1]}_n, \phi^{[s-1]}_m) \quad (n, m = s, s + 1, \ldots). \quad (2.74) \]

The quantities in the \( s \)th step are defined by those in the \((s - 1)\)st step: \( s \geq 1 \)

oQM : \( u^{(s)}(x) \equiv \log |\phi^{(s)}_s(x)| \).

\[ \mathcal{A}^{[s]} \equiv \partial_x - \partial_x u^{(s)}(x), \quad \mathcal{A}^{[s]^\dagger} = -\partial_x - \partial_x u^{(s)}(x), \quad (2.75) \]

idQM : \( V^{(s)}(x) \equiv \sqrt{V^{[s-1]}(x - i \frac{\gamma}{2}) V^{[s-1]^\dagger}(x - i \frac{\gamma}{2}) \frac{\phi^{[s]}_n(x - i \gamma)}{\phi^{[s]}_n(x)}} \).

\[ \mathcal{A}^{[s]} \equiv i(\gamma \tau^1 \sqrt{V^{[s]}(x)} - e^{-\gamma \tau^1} \sqrt{V^{[s]}(x)}), \quad \mathcal{A}^{[s]^\dagger} = -i(\sqrt{V^{[s]}(x)} e^{\gamma \tau^1} - \sqrt{V^{[s]}(x)} e^{-\gamma \tau^1}), \quad (2.77) \]

rdQM : \( B^{(s)}(x) \equiv \sqrt{B^{[s-1]}(x + 1) D^{[s-1]}(x + 1) \frac{\phi^{[s]}_n(x + 1)}{\phi^{[s]}_n(x)}} \).

\[ D^{[s]}(x) \equiv \sqrt{B^{[s-1]}(x) D^{[s-1]}(x) \frac{\phi^{[s]}_n(x - 1)}{\phi^{[s]}_n(x)}}, \quad (2.79) \]

\[ \mathcal{A}^{[s]} \equiv \sqrt{B^{[s]}(x) - e^\alpha \sqrt{D^{[s]}(x)}}, \quad \mathcal{A}^{[s]^\dagger} = \sqrt{B^{[s]}(x) - \sqrt{D^{[s]}(x)}} e^{-\alpha}. \quad (2.81) \]

The eigenfunctions at the \( s \)th step have succinct determinant forms in terms of the Wronskian (oQM) and the Casoratian (dQM) \[ [27, 49, 24, 26] \]: \( n \geq s \geq 0 \)

oQM : \( W[f_1, \ldots, f_m](x) \equiv \det \left( \frac{d^{j-1} f_k(x)}{dx^{j-1}} \right)_{1 \leq j, k \leq m} \) (Wronskian). \( (2.82) \)
\[ \phi_n^{(j)}(x) = \frac{W[\phi_0, \phi_1, \ldots, \phi_{j-1}, \phi_j](x)}{W[\phi_0, \phi_1, \ldots, \phi_{j-1}](x)}, \quad (2.83) \]

\[ \text{idQM} : \quad W_C[f_1, \ldots, f_m](x) \overset{\text{def}}{=} \left( \frac{\gamma}{i} \right)^{m(m-1)} \det \left( f_i \left( x + \frac{m + 1 - 2j}{2} \right) \right)_{1 \leq i, k \leq m} \quad \text{(Casoratian)}, \quad (2.84) \]

\[ \phi_n^{(j)}(x) = \prod_{j=0}^{s-1} \sqrt{B^{(j)}(x)} \cdot \frac{W_C[\phi_0, \phi_1, \ldots, \phi_{j-1}, \phi_j](x)}{W_C[\phi_0, \phi_1, \ldots, \phi_{j-1}](x+1)} \quad (2.87) \]

\[ = (-1)^s \prod_{k=0}^{s-1} \sqrt{D^{(k)}(x + s - k)} \cdot \frac{W_C[\phi_0, \phi_1, \ldots, \phi_{j-1}, \phi_j](x)}{W_C[\phi_0, \phi_1, \ldots, \phi_{j-1}](x)} \quad (2.88) \]

The norm of the \( s \)th step eigenfunctions has a simple uniform expression

\[ \left( \phi_n^{(j)}, \phi_m^{(j)} \right) = \prod_{j=0}^{s-1} (E(n) - E(j)) \cdot (\phi_n, \phi_m). \quad (2.89) \]

This situation of Crum’s theorem is illustrated in figure 1.

A quantum mechanical system with a factorized Hamiltonian \( \mathcal{H} = \mathcal{A}^\dagger \mathcal{A} \) together with the associated one \( \mathcal{H}^{(1)} = \mathcal{A} \mathcal{A}^\dagger \) is sometimes called a ‘supersymmetric’ QM [50, 51]. This appears rather a misnomer, since as we have shown that the factorized form is generic and it implies no extra symmetry. The iso-spectrality is shared by all the associated Hamiltonians, not merely by the first two. In this connection, the transformation of mapping the \( s \)th to the \((s+1)\)st associated Hamiltonian is sometimes called susy transformation. It is also known as the Darboux or Darboux–Crum transformation. Those covering multi-steps are sometimes called ‘higher derivative’ or ‘nonlinear’ or ‘\( \mathcal{N} \)-fold’ susy transformations [52–54].

2.3. Modified Crum’s theorem

Crum’s theorem provides a new iso-spectral Hamiltonian system by deleting successively the lowest lying levels from the original Hamiltonian systems \( \mathcal{H} \) and \( \{\phi_n(x)\} \). The modification of Crum’s theorem by Krein–Adler [28, 29] is achieved by deleting a finite number of eigenstates indexed by a set of non-negative distinct integers\(^3\) \( \mathcal{D} \overset{\text{def}}{=} \{d_1, d_2, \ldots, d_l\} \subset \mathbb{Z}_{\geq 0} \), satisfying certain conditions to be specified later. After the deletion, the new groundstate has the label \( \mu \) given by

\[ \mu \overset{\text{def}}{=} \min \{n \mid n \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D} \}. \quad (2.90) \]

\(^3\) Although this notation \( d_i \) conflicts with the notation of the normalization constant \( d_n \) in (2.50), we think this does not cause any confusion because the latter appears as \( \frac{1}{d_n} \delta_{nn}. \)
Corresponding to (2.75)–(2.88), the new iso-spectral Hamiltonian system is [25, 26]

\[ \tilde{H} \defeq \tilde{A}^{\dagger} \tilde{A} + \mathcal{E}(\mu), \quad \tilde{A}_\phi(x) = 0, \tag{2.91} \]

\[ \tilde{H}\phi_n(x) = \mathcal{E}(n)\tilde{\phi}_n(x) \quad (n \in \mathbb{Z}_{\geq 0}\setminus \mathcal{D}), \tag{2.92} \]

\( \text{oQM : } \tilde{H} = p^2 + \tilde{U}(x), \quad \tilde{U}(x) = U(x) - 2\partial_x^{\ell}(\log W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_L}](x)). \tag{2.93} \]

\[ \tilde{A} \defeq \partial_x - \partial_x \tilde{w}(x), \quad \tilde{A}^{\dagger} = -\partial_x - \partial_x \tilde{w}(x), \quad \tilde{w}(x) \defeq \log |\tilde{\phi}_\mu(x)|, \tag{2.94} \]

\[ \tilde{\phi}_n(x) \defeq \frac{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_L}, \phi_n](x)}{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_L}](x)}, \tag{2.95} \]

\( \text{idQM : } \tilde{V}(x) \defeq \sqrt{V \left( x - i \frac{\ell}{2} \gamma \right)} \tilde{V}^* \left( x - i \frac{\ell + 2}{2} \gamma \right) \times \frac{W_{\gamma}[\phi_{d_1}, \ldots, \phi_{d_L}](x + i \frac{\ell}{2} \gamma)}{W_{\gamma}[\phi_{d_1}, \ldots, \phi_{d_L}](x - i \frac{\ell}{2} \gamma)} \frac{W_{\gamma}[\phi_{d_1}, \ldots, \phi_{d_L}, \phi_n](x - i \gamma)}{W_{\gamma}[\phi_{d_1}, \ldots, \phi_{d_L}, \phi_n](x)}, \tag{2.96} \)
\[ \mathcal{H} \triangleq i(e^{i\rho} \sqrt{\mathcal{V}^*}(x) - e^{-i\rho} \sqrt{\mathcal{V}}(x)), \]
\[ \mathcal{H}^\dagger = -i(\sqrt{\mathcal{V}}(x) e^{i\rho} - \sqrt{\mathcal{V}^*}(x) e^{-i\rho}), \] (2.97)

\[ F(x) \triangleq \sqrt{\prod_{j=0}^{\ell} W_j[\phi_{d_1}, \ldots, \phi_{d_\ell}](x - i \frac{\ell - 2j}{2} \gamma)} W_j[\phi_{d_1}, \ldots, \phi_{d_\ell}](x), \]
\[ \phi_n(x) \triangleq \sqrt{F(x) W_j[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_n](x)}, \] (2.98)

\[ \mathcal{B}(x) = \sqrt{\mathcal{B}(x + \ell) D(x + \ell + 1)} W_C[\phi_{d_1}, \ldots, \phi_{d_\ell}](x) \]
\[ \times W_C[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_n](x + 1), \]
\[ \mathcal{D}(x) = \sqrt{\mathcal{D}(x + 1) D(x)} W_C[\phi_{d_1}, \ldots, \phi_{d_\ell}](x + 1) W_C[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_n](x), \] (2.100)

\[ \mathcal{A} \triangleq \sqrt{\mathcal{A}(x) - e^{-\beta} \sqrt{\mathcal{D}(x)}}, \]
\[ \mathcal{A}^\dagger = \sqrt{\mathcal{A}(x) - \sqrt{\mathcal{D}(x)} e^{-\beta}}, \] (2.101)

\[ F(x) \triangleq \sqrt{\prod_{j=0}^{\ell} \mathcal{B}(x + k - 1) D(x + k)} \]
\[ \phi_n(x) = (-1)^j \sqrt{F(x) W_C[\phi_{d_1}, \ldots, \phi_{d_\ell}, \phi_n](x)}, \] (2.102)

and

\[ (\phi_n, \phi_m) = \prod_{j=1}^{\ell} (\mathcal{E}(n) - \mathcal{E}(d_j)) \cdot (\phi_n, \phi_m) \quad (n, m \in \mathbb{Z}_{\geq 0}) \] (2.103)

It should be emphasized that the Hamiltonian \( \mathcal{H} \) and the eigenfunction \( \{\phi_n(x)\} \) are symmetric with respect to \( d_1, \ldots, d_\ell \), and thus they are independent of the order of \( \{d_j\} \). In order to guarantee the positivity of the norm (2.103) of all the eigenfunctions \( \{\phi_n(x)\} \) of the modified Hamiltonian, the set of deleted energy levels \( \mathcal{D} = \{d_1, \ldots, d_\ell\} \) must satisfy the necessary and sufficient conditions \([28, 29]\)

\[ \sum_{j=1}^{\ell}(m - d_j) \geq 0, \quad m \in \mathbb{Z}_{\geq 0}. \] (2.104)

And the Hamiltonian \( \mathcal{H} \) is non-singular under this condition. Crum’s theorem in section 2.2 corresponds to the choice \( \{d_1, d_2, \ldots, d_\ell\} = \{0, 1, \ldots, \ell - 1\} \).

Starting from an exactly solvable Hamiltonian, one can construct infinitely many variants of exactly solvable Hamiltonians and their eigenfunctions by Adler and García et al’s methods [25]. The resulting systems are, however, not shape invariant, even if the starting system is. For the dQM with real shifts, see the recent work [26] and a related work [55].
2.4. Shape invariance

Shape invariance [30] is a sufficient condition for the exact solvability in the Schrödinger picture. Combined with Crum’s theorem [27], or the factorization method [40] or the so-called supersymmetric QM [31, 51], the totality of the discrete eigenvalues and the corresponding eigenfunctions can be easily obtained. It was shown by the present authors that the concept of shape invariance worked equally well in the dQM with the pure imaginary shifts [12, 21] as well as the real shifts [20], providing quantum mechanical explanation of the solvability of the Askey scheme of hypergeometric orthogonal polynomials in general. Throughout this review, we concentrate on the simple situations that the entire spectra are discrete. When continuous spectra exist, however, shape invariance and Crum’s theorem fail to provide the full set of eigenvalues and eigenfunctions.

In many cases, the Hamiltonian contains some parameter(s), $\lambda = (\lambda_1, \lambda_2, \ldots)$. Here we write the parameter dependence explicitly, $\mathcal{H}(\lambda), \mathcal{A}(\lambda), \mathcal{E}(n; \lambda), \phi_n(x; \lambda), P_n(\eta(x; \lambda); \lambda)$, etc, since it is the central issue. The shape invariance condition with a suitable choice of parameters is

$$\mathcal{A}(\lambda) \mathcal{A}(\lambda)^\dagger = \kappa \mathcal{A}(\lambda + \delta)^\dagger \mathcal{A}(\lambda + \delta) + \mathcal{E}(1; \lambda), \tag{2.105}$$

where $\kappa$ is a real positive constant and $\delta$ is the shift of the parameters. In other words, $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$ have the same shape, only the parameters are shifted by $\delta$. The $n$th step Hamiltonian $\mathcal{H}^{[n]}$ in section 2.2 is $\mathcal{H}^{[n]}(\lambda) = \kappa^n \mathcal{H}(\lambda + n\delta) + \mathcal{E}(x; \lambda)$. The energy spectrum and the excited state wavefunction are determined by the data of the ground-state wavefunction $\phi_0(x; \lambda)$ and the energy of the first excited state $E(1; \lambda)$ as follows [31, 12, 20, 21]:

$$\mathcal{E}(n; \lambda) = \sum_{s=0}^{n-1} \kappa^s \mathcal{E}(1; \lambda)^{[s]}, \quad \lambda^{[s]} \overset{\text{def}}{=} \lambda + s\delta, \tag{2.106}$$

$$\phi_n(x; \lambda) \propto \mathcal{A}(\lambda)^{[0]} \mathcal{A}(\lambda)^{[1]} \mathcal{A}(\lambda)^{[2]} \cdots \mathcal{A}(\lambda)^{[n-1]} \phi_0(x; \lambda^{[n]}). \tag{2.107}$$

The above formula for the eigenfunctions $\phi_n(x; \lambda)$ can be considered as the universal Rodrigues formula for the Askey scheme of hypergeometric polynomials and their $q$-analogues. For the explicit forms of the Rodrigues-type formula for each polynomial, one only has to substitute the explicit forms of the operator $\mathcal{A}(\lambda)$ and the ground-state wavefunction $\phi_0(x; \lambda)$. For the nine explicit examples given in (2.30)–(2.39), it is straightforward to verify the shape invariance conditions (2.105) and the energy (2.106) and the eigenfunction (2.107) formulas.

In the case of a finite number of bound states, e.g. the Morse potential, the eigenvalue has a maximum at a certain level $n, \mathcal{E}(n; \lambda)$. Beyond that level, the formula (2.106) ceases to work and the Rodrigues formula (2.107) does not provide square integrable eigenfunctions, although $\phi_m (m > n)$ continues to satisfy the Schrödinger equation with $\mathcal{E}(m; \lambda)$.

The above shape invariance condition (2.105) is equivalent to the following conditions:

**oQM:** \[(\partial_x w(x; \lambda))^2 - \partial_x^2 w(x; \lambda) = (\partial_x w(x; \lambda + \delta))^2 + \partial_x^2 w(x; \lambda + \delta) + \mathcal{E}(1; \lambda), \quad \kappa = 1, \tag{2.108}\]

**idQM:** \[V \left( x - i \frac{V}{2}; \lambda \right) V^* \left( x - i \frac{V}{2}; \lambda \right) = \kappa^2 V \left( x; \lambda + \delta \right) V^*(x - iy; \lambda + \delta), \tag{2.109}\]

\[V \left( x + i \frac{V}{2}; \lambda \right) + V^* \left( x - i \frac{V}{2}; \lambda \right) = \kappa (V(x; \lambda + \delta) + V^*(x; \lambda + \delta)) - \mathcal{E}(1; \lambda). \tag{2.110}\]
rdQM:  \[ B(x + 1; \lambda) D(x + 1; \lambda) = \kappa^2 B(x; \lambda + \delta) D(x + 1; \lambda + \delta), \]  \[ B(x; \lambda) + D(x + 1; \lambda) = \kappa (B(x; \lambda + \delta) + D(x; \lambda + \delta)) + \mathcal{E}(1; \lambda). \]  \( (2.111) \)

For the idQM, the first condition \((2.109)\) is multiplicative. If \(V_1, V'_1, \kappa_1\) and \(V_2, V'_2, \kappa_2\) satisfy the condition independently, then \(V = V_1 V_2, V' = V'_1 V'_2, \kappa = \kappa_1 \kappa_2\) also satisfies it. The second condition \((2.110)\) provides a substantial constraint. The situation is the same in rdQM.

It is straightforward to verify the shape invariance for the nine examples \((2.30)–(2.39)\) in section 2.1 with the following data:

\[ \Lambda: \phi = \phi \text{ (null)}, \quad \delta = \phi, \quad \kappa = 1, \quad \mathcal{E}(n; \lambda) = 2n, \]  \( (2.113) \)

\[ \Lambda: \lambda = g, \quad \delta = 1, \quad \kappa = 1, \quad \mathcal{E}(n; \lambda) = 4n, \]  \( (2.114) \)

\[ \Lambda: \lambda = (g, h), \quad \delta = (1, 1), \quad \kappa = 1, \quad \mathcal{E}(n; \lambda) = 4n(n + g + h). \]  \( (2.115) \)

It should be stressed that the above shape invariant transformation \(\lambda \to \lambda + \delta, \mathcal{H}^{l+1} \to \mathcal{H}^{l+1}\) for L and J, that is, \(g \to g + 1, h \to h + 1,\) preserves the monodromy \((2.33)\) at the regular singularities:

\[ \Lambda: \lambda = a, \quad \delta = \frac{1}{2}, \quad \kappa = 1, \quad \mathcal{E}(n; \lambda) = 2n, \]  \( (2.116) \)

\[ \Lambda: \lambda = (a_1, a_2, a_3, a_4), \quad \delta = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \kappa = 1, \quad \mathcal{E}(n; \lambda) = 4n(n + b_1 - 1), \]  \( (2.117) \)

\[ \Lambda: \lambda = q^k, \quad \delta = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \kappa = q^{-1}, \quad \mathcal{E}(n; \lambda) = (q^{-n} - 1)(1 - b_4 q^{n-1}), \]  \( (2.118) \)

where \(q^{(k_1, k_2, \ldots)} = (q^{k_1}, q^{k_2}, \ldots).\) For more complete lists of exactly solvable oQM, see [40, 51] and for idQM and rdQM, see [21] and [20]. It is well known that the solutions of Schrödinger equations provide those of the corresponding Fokker–Planck equations [56], which describe the time evolution of probability density functions. This connection can be readily generalized to discrete Schrödinger equations, which would correspond to, for example, discretized stochastic processes, the Markov chains. The birth and death processes [5, 57] are
the best-known examples. It is interesting to point out that all the exactly solvable rdQM examples in [20] and their modifications [26, 11] also provide exactly solvable birth and death processes [58].

The simplest example of rdQM, the Charlier polynomial, has no shiftable parameter. Its shape invariance relation $A^\dagger A - A^\dagger - A = 1$ gives another realization of the oscillator algebra. Likewise, the simplest example of idQM, the $q$-Hermite ($qH$) polynomial, has no parameter other than $q$. It is obtained as a special case of the Askey–Wilson (AW) (2.36) by setting $a_j = 0$, $j = 1, \ldots, 4$. Like the harmonic oscillator (Hermite) case, it is shape invariant and the shape invariance relation ($\kappa = q^{-1}$) (2.105) becomes the $q$-oscillator algebra $A^\dagger q^{-1}A^\dagger - A^\dagger A = q^{-1} - 1$ itself [19].

The shape invariance and Crum’s theorem imply that $\phi_n(x; \lambda)$ and $\phi_{n-1}(x; \lambda + \delta)$ are mapped to each other by the operators $A(\lambda)$ and $A(\lambda)^\dagger$:

\begin{equation}
A(\lambda)\phi_n(x; \lambda) = f_n(\lambda)\phi_{n-1}(x; \lambda + \delta) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B(0; \lambda)}} & : \text{rdQM}, \end{cases}
\end{equation}

(2.122)

\begin{equation}
A(\lambda)^\dagger\phi_{n-1}(x; \lambda + \delta) = b_{n-1}(\lambda)\phi_n(x; \lambda) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B(0; \lambda)}} & : \text{rdQM}. \end{cases}
\end{equation}

(2.123)

Here the constants $f_n(\lambda)$ and $b_{n-1}(\lambda)$ depend on the normalization of $\phi_n(x; \lambda)$ but their product does not. It gives the energy eigenvalue

\begin{equation}
\mathcal{E}(n; \lambda) = f_n(\lambda)b_{n-1}(\lambda).
\end{equation}

(2.124)

The factor $\sqrt{B(0; \lambda)}$ for rdQM is introduced for later convenience. For our choice of $\phi_n(x; \lambda)$ and $P_n(\eta(x; \lambda); \lambda)$, the data for $f_n(\lambda)$ and $b_{n-1}(\lambda)$ are

\begin{align}
o\text{QM} : & \quad f_n(\lambda) = \begin{cases} 2n & : \text{H} \\ -2 & : \text{L}, \end{cases} \quad b_{n-1}(\lambda) = \begin{cases} 1 & : \text{H} \\ -2n & : \text{L, J}. \end{cases} \\
\text{idQM} : & \quad f_n(\lambda) = \begin{cases} \frac{n}{q^n} & : \text{MP} \\ -n(n+b_1-1) & : \text{W}, \end{cases} \quad b_{n-1}(\lambda) = \begin{cases} n & : \text{MP} \\ -1 & : \text{W} \\ q^{\frac{n}{2}}(q^n-1)(1-b_4q^{n-1}) & : \text{AW}. \end{cases}
\end{align}

(2.125)

(2.126)

rdQM : \quad f_n(\lambda) = \mathcal{E}(n; \lambda), \quad b_{n-1}(\lambda) = 1.

(2.127)

By removing the ground-state contributions, the forward- and backward-shift operators acting on the polynomial eigenfunctions, $\mathcal{F}(\lambda)$ and $B(\lambda)$, are introduced:

\begin{equation}
\mathcal{F}(\lambda) \overset{\text{def}}{=} \phi_0(x; \lambda + \delta)^{-1} \circ A(\lambda) \circ \phi_0(x; \lambda) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{B(0; \lambda)} & : \text{rdQM} \end{cases}
\end{equation}

(2.128)

\begin{equation}
\begin{aligned}
\mathcal{F}(\lambda) &= \frac{d}{d\eta} \begin{cases} c_{\eta} & : \text{oQM} \\ i\varphi(x)^{-1}(e^{\frac{i\eta}{2}} - e^{-\frac{i\eta}{2}}) & : \text{idQM} \\ B(0; \lambda)e^{\lambda_1}(1 - e^{\lambda_1}) & : \text{rdQM}, \end{cases} \\
B(\lambda) &= \phi_0(x; \lambda)^{-1} \circ A(\lambda)^\dagger \circ \phi_0(x; \lambda + \delta) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \sqrt{B(0; \lambda)} & : \text{rdQM} \end{cases}
\end{aligned}
\end{equation}

(2.129)

(2.130)
\[
\begin{align*}
\begin{cases}
-4c_F^{-1}c_2(\eta) \left( \frac{d}{d\eta} + \frac{c_1(\eta, \lambda)}{c_2(\eta)} \right) & : \text{oQM} \\
-i(V(x; \lambda) e^{i\hat{p}} - V^*(x; \lambda) e^{-i\hat{p}})\psi(x) & : \text{idQM} \\
\frac{1}{B(0; \lambda)} (B(x; \lambda) - D(x; \lambda) e^{-\hat{p}})\psi(x; \lambda) & : \text{rdQM},
\end{cases}
\end{align*}
\]

(2.131)

where \(c_F, c_1(\eta, \lambda)\) and \(c_2(\eta)\) are

\[
\begin{align*}
c_F \equiv & \begin{cases}
1 & : H \\
2 & : L \\
-4 & : J
\end{cases}, \\
c_1(\eta, \lambda) \equiv & \begin{cases}
-\frac{1}{2} & : H \\
g + \frac{1}{2} - \eta & : L \\
h - g - (g + h + 1)\eta & : J
\end{cases}, \\
c_2(\eta) \equiv & \begin{cases}
\frac{1}{2} & : H \\
\eta & : L \\
1 - \eta^2 & : J,
\end{cases}
\end{align*}
\]

(2.132)

and the auxiliary functions \(\psi(x)\) are

\[
\begin{align*}
\text{idQM} : \quad \psi(x) & = \begin{cases}
1 & : \text{MP} \\
2x & : \text{W} \\
2\sin x & : \text{AW}
\end{cases}, \\
\text{rdQM} : \quad \psi(x; \lambda) & = \begin{cases}
1 & : \text{M} \\
\frac{2x + d + 1}{d + 1} & : \text{R} \\
\frac{q^{-s} - dq^{s+1}}{1 - dq} & : q\text{R}.
\end{cases}
\end{align*}
\]

(2.133)

Then, the above relations (2.122)–(2.123) become

\[
\begin{align*}
\mathcal{F}(\lambda) \hat{P}_n(x; \lambda) & = f_n(\lambda) \hat{P}_{n-1}(x; \lambda + \delta), \\
B(\lambda) \hat{P}_n(x; \lambda + \delta) & = b_{n-1}(\lambda) \hat{P}_n(x; \lambda),
\end{align*}
\]

(2.134, 2.135)

where we have used the notation

\[
\hat{P}_n(x; \lambda) \equiv P_n(\eta(x; \lambda); \lambda).
\]

(2.136)

Corresponding to (2.5), the forward- and backward-shift operators give a factorization of the similarity transformed Hamiltonian (2.24):

\[
\hat{\mathcal{H}}(\lambda) = B(\lambda)\mathcal{F}(\lambda).
\]

(2.137)

2.5. Solvability in the Heisenberg picture

As is well known, the Heisenberg operator formulation is central to quantum field theory. The creation/annihilation operators of the harmonic oscillators are the cornerstones of modern quantum physics. However, until recently, it had been generally conceived that the Heisenberg operator solutions are intractable. Here, we show that most of the shape invariant dQM Hamiltonian systems are exactly solvable in the Heisenberg picture too [14, 15]. To be more precise, the Heisenberg operator of the sinusoidal coordinate \(\eta(x)\)

\[
e^{i\hat{r}}\eta(x) e^{-i\hat{r}}
\]

(2.138)

can be evaluated in a closed form. It is well known that any orthogonal polynomials satisfy the three-term recurrence relations [3, 5]

\[
\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta) \quad (n \geq 0),
\]

(2.139)
with $P_1(\eta) = 0$. Here the coefficients $A_0$, $B_i$, and $C_p$ are real and $A_{p-1}C_p > 0$ ($n \geq 1$). Conversely all the polynomials starting with degree 0 and satisfying the above three-term recurrence relations are orthogonal (Favard’s theorem [59]). These relations can also be considered as an eigenvalue equation, in which $\eta$ is the eigenvalue. For the polynomials in rdQM with the universal normalization $P_n(0) = 1$ (2.22), the coefficients of the three-term recurrence relations are restricted by the condition

$$
\text{rdQM} : \quad B_n = -(A_n + C_n) \quad (n = 0, 1, \ldots).
$$

(2.140)

For the factorized quantum mechanical eigenfunctions (2.20), these relations mean

$$
\eta(x)\phi_n(x) = A_n\phi_{n+1}(x) + B_n\phi_n(x) + C_n\phi_{n-1}(x) \quad (n \geq 0).
$$

(2.141)

In other words, the operator $\eta(x)$ acts like a creation operator which sends the eigenstate $n$ to $n + 1$ as well as like an annihilation operator which maps an eigenstate $n$ to $n - 1$. This fact combined with the well-known result that the annihilation/creation operators of the harmonic oscillator are the positive/negative frequency part of the Heisenberg operator solution for the coordinate $x$ is the starting point of this subsection. As will be shown below, the sinusoidal coordinate $\eta(x)$ undergoes sinusoidal motion (2.144), whose frequencies depend on the energy. Thus, it is not harmonic in general. To the best of our knowledge, the sinusoidal coordinate was first introduced in a rather broad sense for general (not necessarily solvable) potentials as a useful means for coherent state research by Nieto and Simmons [60].

The sufficient condition for the closed form expression of the Heisenberg operator (2.138) is the closure relation

$$
[\mathcal{H}, [\mathcal{H}, \eta(x)]] = \eta(x)R_0(\mathcal{H}) + [\mathcal{H}, \eta(x)]R_1(\mathcal{H}) + R_{-1}(\mathcal{H}).
$$

(2.142)

Here the coefficients $R_i(\gamma)$ are polynomials in $\gamma$. It is easy to see that the cubic commutator $[\mathcal{H}, [\mathcal{H}, \eta(x)]] \equiv (\text{ad}\mathcal{H})^3\eta(x)$ is reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with $\mathcal{H}$ depending coefficients:

$$(\text{ad}\mathcal{H})^3\eta(x) = [\mathcal{H}, \eta(x)]R_0(\mathcal{H}) + [\mathcal{H}, [\mathcal{H}, \eta(x)]]R_1(\mathcal{H})$$

$$= \eta(x)R_0(\mathcal{H})R_1(\mathcal{H}) + [\mathcal{H}, \eta(x)](R_1(\mathcal{H})^2 + R_0(\mathcal{H}^2) + R_{-1}(\mathcal{H})R_1(\mathcal{H})),$n

(2.143)

in which the definition $\text{(ad}\mathcal{H})X \equiv [\mathcal{H}, X]$ is used. It is trivial to see that all the higher commutators $(\text{ad}\mathcal{H})^n\eta(x)$ can also be reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with $\mathcal{H}$ depending coefficients. The second order closure (2.142) simply reflects the Schrödinger equation, which is a second order differential or difference equation. Thus, we arrive at

$$
e^{i\mathcal{H}t}\eta(x)e^{-i\mathcal{H}t} = \sum_{n=0}^{\infty} \frac{(i\mathcal{H})^n}{n!}(\text{ad}\mathcal{H})^n\eta(x)$$

$$= [\mathcal{H}, \eta(x)]\frac{e^{i\mathcal{H}t}\eta(x) - e^{-i\mathcal{H}t}\eta(x)}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}$$

$$+ (\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1})\frac{-\alpha_-(\mathcal{H})e^{i\mathcal{H}t}\eta(x) + \alpha_+(\mathcal{H})e^{-i\mathcal{H}t}\eta(x)}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})}.$$n

(2.144)

This simply means that $\eta(x)$ oscillates sinusoidally with two energy-dependent ‘frequencies’ $\alpha_\pm(\mathcal{H})$ given by

$$\alpha_\pm(\mathcal{H}) = \frac{1}{2}(R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2 + 4R_0(\mathcal{H})}).$$

(2.145)

$$\alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H}) = R_1(\mathcal{H}), \quad \alpha_+(\mathcal{H})\alpha_-(\mathcal{H}) = -R_0(\mathcal{H}).$$

(2.146)

The energy spectrum is determined by the over-determined recursion relations $\mathcal{E}(n + 1) = \mathcal{E}(n) + \alpha_+(\mathcal{E}(n))$ and $\mathcal{E}(n - 1) = \mathcal{E}(n) + \alpha_-(\mathcal{E}(n))$ with $\mathcal{E}(0) = 0$. It should be stressed...
that for the known spectra \([\mathcal{E}(n)]\) determined by the shape invariance, the quantity inside the square root in the definition of \(\alpha_\pm(\mathcal{H})\) (2.145) for each \(n\)

\[ R_1(\mathcal{E}(n))^2 + 4R_0(\mathcal{E}(n)) \]

becomes a complete square and the above two conditions are consistent. For oQM, the Hamiltonian and the sinusoidal coordinate satisfying the closure relation (2.142) are classified and then the eigenfunctions have the factorized form (2.20) [14]. For dQM, we assume (2.20). The *annihilation* and *creation* operators \(a^{(+)}\) are extracted from this exact Heisenberg operator solution:

\[
e^{i\mathcal{H}t}q(x) e^{-i\mathcal{H}t} = a^{(+)} e^{i\omega_+(\mathcal{H})t} + a^{(-)} e^{i\omega_-(\mathcal{H})t} - R_{-1}(\mathcal{H}) R_0(\mathcal{H})^{-1}, \tag{2.147}\]

\[
a^{(\pm)} \overset{\text{def}}{=} \pm(\mathcal{H}, \eta(x)) - (\eta(x) + R_{-1}(\mathcal{H}) R_0(\mathcal{H})^{-1} \alpha_\pm(\mathcal{H}) (\alpha_+ - \alpha_- (\mathcal{H})))^{-1}
= \pm(\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H}))^{-1} (\mathcal{H}, \eta(x)) + \alpha_\pm(\mathcal{H}) (\eta(x) + R_{-1}(\mathcal{H}) R_0(\mathcal{H})^{-1})), \tag{2.148}\]

\[
a^{(+)} = a^{-1}, \quad a^{+} \phi_n(x) = A_n \phi_{n+1}(x), \quad a^{(-)} \phi_n(x) = C_n \phi_{n-1}(x). \tag{2.149}\]

It should be stressed that the annihilation operator and creation operators are Hermitian conjugate of each other and they act on the eigenstate (2.149). Simple commutation relations

\[
[\mathcal{H}, a^{(\pm)}] = a^{(\pm)} \alpha_\pm(\mathcal{H}) \tag{2.150}\]

follow from (2.148) and (2.142). Commutation relations of \(a^{(\pm)}\) are expressed in terms of the coefficients of the three-term recurrence relation by (2.149):

\[
a^{(-)} a^{(+)} \phi_n = A_n C_{n+1} \phi_n, \quad a^{(+) a^{(-)}} \phi_n = C_n A_{n-1} \phi_n,
\Rightarrow [a^{(-)}, a^{(+) a^{(-)}}] \phi_n = (A_n C_{n+1} - A_{n-1} C_n) \phi_n. \tag{2.151}\]

These relations simply mean the operator relations

\[
a^{(-)} a^{(+)} = f(\mathcal{H}), \quad a^{(+) a^{(-)}} = g(\mathcal{H}), \tag{2.152}\]

in which \(f\) and \(g\) are analytic functions of \(\mathcal{H}\) explicitly given for each example. In other words, \(\mathcal{H}\) and \(a^{(\pm)}\) form a so-called quasi-linear algebra [61]. Various *dynamical symmetry algebras* associated with exactly solvable QM, including the \(q\)-oscillator algebra, were explicitly identified in [20, 21]. It should be stressed that the situation is quite different from those of the wide variety of proposed annihilation/creation operators for various quantum systems [62], most of which were introduced within the framework of ‘algebraic theory of coherent states’, without exact solvability. In all such cases, there is no guarantee for symmetry relations like (2.152). The explicit form of the annihilation operator (2.148) allows us to define the *coherent state* as its eigenvector, \(a^{(-)} \psi(\alpha, x) = \alpha \psi(\alpha, x), \alpha \in \mathbb{C}\). See [14, 21] for various coherent states.

The excited state wavefunctions \(\{\phi_n(x)\}\) are obtained by the successive action of the creation operator \(a^{(+)}\) on the ground-state wavefunction \(\phi_0(x)\). This is the exact solvability in the Heisenberg picture.

The data for the three examples in oQM (2.30)–(2.32) are

\[
H:\quad R_1(y) = 0, \quad R_0(y) = 4, \quad R_{-1}(y) = 0, \tag{2.153}\]

\[
L:\quad R_1(y) = 0, \quad R_0(y) = 16, \quad R_{-1}(y) = -8(y + 2g + 1), \tag{2.154}\]

\[
J:\quad R_1(y) = 8, \quad R_0(y) = 16(y + (g + h)^2 - 1), \quad R_{-1}(y) = 16(g - h)(g + h - 1). \tag{2.155}\]
The data for the three examples in idQM (2.34)–(2.36) are

\[ \text{MP} : \quad R_1(y) = 0, \quad R_0(y) = 4, \quad R_{-1}(y) = 0, \]  
\[ \text{(2.156)} \]

\[ \text{W} : \quad R_1(y) = 2, \quad R_0(y) = 4y + b_1(b_1 - 2), \quad R_{-1}(y) = -2y^2 + (b_1 - 2b_2)y + (2 - b_1)b_3, \]
\[ b_2 \overset{\text{def}}{=} \sum_{1 \leq j < k \leq 4} a_j a_k, \quad b_3 \overset{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_j a_k a_l, \]  
\[ \text{(2.157)} \]

\[ \text{AW} : \quad R_1(y) = (q^{-1} - q^1)^2 y', \quad R_0(y) = (q^{-1} - q^1)^2 (y^2 - (1 + q^{-1})^2 b_2), \]
\[ R_{-1}(y) = \frac{1}{2} (q^{-1} - q^1)^2 ((b_1 + q^{-1}b_2)y' - (1 + q^{-1})(b_3 + q^{-1}b_1b_4)) - R_0(y), \]
\[ y' \overset{\text{def}}{=} y + 1 + q^{-1}b_4, \quad b_1 \overset{\text{def}}{=} \sum_{j=1}^4 a_j, \quad b_3 \overset{\text{def}}{=} \sum_{1 \leq j < k < l \leq 4} a_j a_k a_l, \]  
\[ \text{(2.158)} \]

The closure relation and the creation/annihilation operators of the MP polynomials were obtained in 2001 in [47]. The data for the three examples in rdQM (2.37)–(2.39) are

\[ \text{M} : \quad R_1(y) = 0, \quad R_0(y) = 1, \quad R_{-1}(y) = -\frac{1+c}{1-c} y - \frac{\beta c}{1-c}, \]  
\[ \text{(2.159)} \]

\[ \text{R} : \quad R_1(y) = 2, \quad R_0(y) = 4y + \tilde{d}^2 - 1, \]
\[ R_{-1}(y) = 2y^2 + (2(ab + bc + ca) - (1 + d)(1 + \tilde{d}))y + abc(\tilde{d} - 1), \]  
\[ \text{(2.160)} \]

\[ q\text{R} : \quad R_1(y) = (q^{-1} - q^1)^2 y', \quad R_0(y) = (q^{-1} - q^1)^2 (y'^2 - (1 + q^{-1})^2 \tilde{d}), \quad y' \overset{\text{def}}{=} y + 1 + \tilde{d}, \]
\[ R_{-1}(y) = (q^{-1} - q^1)^2 ((1 + \tilde{d})y'^2 - (a + b + c + d + \tilde{d} + (ab + bc + ca)q^{-1})y' \]
\[ + (1-a)(1-b)(1-c)(1-\tilde{d}q^{-1}) \]
\[ + (a + b + c - 1 - \tilde{d} \tilde{d} + (ab + bc + ca)q^{-1})(1 + \tilde{d})). \]  
\[ \text{(2.161)} \]

For oQM, the necessary and sufficient condition for the existence of the sinusoidal coordinate satisfying the closure relation (2.142) is analysed in appendix A of [14]. It was shown that such systems constitute a sub-group of the shape invariant oQM. We also mention that exact Heisenberg operator solutions for independent sinusoidal coordinates, as many as the degree of freedom, were derived for the Calogero systems based on any root system [16]. These are novel examples of infinitely many multi-particle Heisenberg operator solutions.

There were attempts to relate q-oscillator algebras to the difference equation of the q-Hermite polynomials [63]. None of them is based on a Hamiltonian; thus, Hermiticity is not manifest and the logic for factorization is unclear. There are several derivations of annihilation/creation operators together with the dynamical symmetry algebras (including the q-oscillator) acting on the polynomials.

2.6. Dual polynomials in rdQM

The dual polynomials are an important concept in the theory of orthogonal polynomials of a discrete variable [2], that is, the orthogonal polynomials appearing in rdQM. We show that the dual polynomials arise naturally as the solutions of the original eigenvalue problem, the
Schrödinger equation (2.17) or (2.23), obtained in an alternative way [20]. We do believe that this derivation of the duality is more intuitive than the existing ones [32, 33]. The Hamiltonian $\mathcal{H}$ (2.12) for rQM is a real symmetric tri-diagonal (Jacobi) matrix of a finite (\(x_{\text{max}} = N\)) or infinite (\(x_{\text{max}} = \infty\)) dimension:

$$\mathcal{H} = (\mathcal{H}_{x,y})_{0 \leq x, y \leq x_{\text{max}}}, \quad \mathcal{H}_{x,y} = \mathcal{H}_{y,x}. \quad (2.162)$$

$$\mathcal{H}_{x,y} = -\sqrt{B(x)}D(x+1)\delta_{x+1,y} - \sqrt{B(x-1)}D(x)\delta_{x-1,y} + (B(x) + D(x))\delta_{x,y}. \quad (2.163)$$

It is well known that the spectrum of a Jacobi matrix is simple (2.4), that is, no degeneracy. The factor $\mathcal{A}^T \mathcal{A}$ (2.10) is a lower triangular matrix with the diagonal and sub-diagonal entries only, and $\mathcal{A}$ is an upper triangular matrix having the diagonal and super-diagonal entries only. Throughout this review, we adopt the (standard) convention that the (0, 0) element of the matrix is at the upper-left corner. The similarity-transformed Hamiltonian $\tilde{\mathcal{H}}$ (2.27) is again tri-diagonal:

$$\tilde{\mathcal{H}} = (\tilde{\mathcal{H}}_{x,y})_{0 \leq x, y \leq x_{\text{max}}}, \quad \tilde{\mathcal{H}}_{x,y} = B(x)(\delta_{x,y} - \delta_{x+1,y}) + D(x)(\delta_{x,y} - \delta_{x-1,y}). \quad (2.164)$$

The similarity-transformed eigenvalue problem $\tilde{\mathcal{H}} \psi(x) = \mathcal{E} \psi(x)$ (2.23) can be rewritten into an explicit matrix form with the change of the notation $\psi(x) \rightarrow \psi(Q_0, Q_1, \ldots, Q_x, \ldots)$:

$$\sum_{y=0}^{x_{\text{max}}} \tilde{\mathcal{H}}_{x,y} \psi_y = \mathcal{E} \psi_x \quad (x = 0, 1, \ldots, x_{\text{max}}). \quad (2.165)$$

Because of the tri-diagonality of $\tilde{\mathcal{H}}$, the above eigenvalue equations are in fact the three-term recurrence relations for $\{Q_x\}$ as polynomials in $\mathcal{E}$:

$$\mathcal{E} Q_0(\mathcal{E}) = B(x)(Q_x(\mathcal{E}) - Q_{x+1}(\mathcal{E})) + D(x)(Q_x(\mathcal{E}) - Q_{x-1}(\mathcal{E})) \quad (x = 0, 1, \ldots, x_{\text{max}}). \quad (2.166)$$

Starting with the boundary (initial) condition $Q_0 = 1$, $Q_x(\mathcal{E})$ is determined as a degree $x$ polynomial in $\mathcal{E}$. It is easy to see that

$$Q_x(0) = 1 \quad (x = 0, 1, \ldots, x_{\text{max}}). \quad (2.167)$$

When $\mathcal{E}$ is replaced by the actual value of the $n$th eigenvalue $\mathcal{E}(n)$ (2.28) in $Q_x(\mathcal{E})$, we obtain the explicit form of the eigenvector

$$\sum_{y=0}^{x_{\text{max}}} \tilde{\mathcal{H}}_{x,y} Q_y(\mathcal{E}(n)) = \mathcal{E}(n) Q_x(\mathcal{E}(n)) \quad (x = 0, 1, \ldots, x_{\text{max}}). \quad (2.168)$$

In the finite-dimensional case, \{\(Q_0(\mathcal{E}), \ldots, Q_N(\mathcal{E})\)\} are determined by (2.166) for $x = 0, \ldots, N - 1$. The last equation

$$\mathcal{E} Q_N(\mathcal{E}) = D(N)(Q_N(\mathcal{E}) - Q_{N-1}(\mathcal{E})) \quad (2.169)$$

is the degree $N + 1$ algebraic equation (characteristic equation) for the determination of all the eigenvalues $\{\mathcal{E}(n)\}$.

We now have two expressions (polynomials) for the eigenvectors of problem (2.23) belonging to the eigenvalue $\mathcal{E}(n)$: $P_n(\eta(x))$ and $Q_n(\mathcal{E}(n))$. Due to the simplicity of the spectrum of the Jacobi matrix, they must be equal up to a multiplicative factor $\alpha_n$:

$$P_n(\eta(x)) = \alpha_n Q_n(\mathcal{E}(n)) \quad (x = 0, 1, \ldots, x_{\text{max}}),$$

which turns out to be unity because of the boundary (initial) condition at $x = 0$ (2.21), (2.22), (2.167) (or at $n = 0$ (2.18), (2.22), (2.167)): ($\eta_{\text{max}} \overset{\text{def}}{=} x_{\text{max}}$)

$$P_n(\eta(0)) = P_n(0) = 1 = Q_0(\mathcal{E}(n)) \quad (n = 0, 1, \ldots, n_{\text{max}}). \quad (2.170)$$
There is an important relation among the four important quantities: the polynomial $\eta(x)$, the functions $Q_n(\mathcal{E}(n))$ $(n = 0, 1, \ldots, \eta_{\text{max}}; x = 0, 1, \ldots, x_{\text{max}})$.

We have established that two polynomials, $\{P_n(\eta)\}$ and its dual polynomial $\{Q_n(\mathcal{E})\}$, coincide at the integer lattice points:

$$P_n(\eta(x)) = Q_n(\mathcal{E}(n)) \quad (n = 0, 1, \ldots, n_{\text{max}}; x = 0, 1, \ldots, x_{\text{max}}).$$

The completeness relation, which is dual to the orthogonality relation (2.50),

$$\sum_{n=0}^{n_{\text{max}}} d_n^2 P_n(\eta(x)) P_n(\eta(y)) = \sum_{n=0}^{n_{\text{max}}} d_n^2 Q_n(\mathcal{E}(n)) Q_n(\mathcal{E}(n)) = \frac{1}{\phi_0(x)^2} \delta_{xy}$$

is now understood as the orthogonality relation of the dual polynomials $Q_n(\mathcal{E})$, and the previous normalization constant $d_n^2$ is now the orthogonality measure.

The real symmetric (Hermitian) matrix $\mathcal{H}_{x,y}$ (2.163) can be expressed in terms of the complete set of the eigenvalues and the corresponding normalized eigenvectors

$$\mathcal{H}_{x,y} = \sum_{n=0}^{n_{\text{max}}} \mathcal{E}(n) \hat{\phi}_n(x) \hat{\phi}_n(y),$$

$$\hat{\phi}_n(x) = d_n \phi_0(x) P_n(\eta(x)) = d_n \phi_0(x) Q_n(\mathcal{E}(n)).$$

The very fact that it is tri-diagonal can be easily verified by using the difference equation for the polynomial $P_n(\eta(x))$ or the three-term recurrence relations for $Q_n(\mathcal{E}(n))$.

Here is a list of the dual correspondence:

$$x \leftrightarrow n, \quad \eta(x) \leftrightarrow \mathcal{E}(n), \quad \eta(0) = 0 \leftrightarrow \mathcal{E}(0) = 0,$$

$$B(x) \leftrightarrow -A_n, \quad D(x) \leftrightarrow -C_n, \quad \frac{\phi_0(x)}{\phi_0(0)} \leftrightarrow d_0.$$

The functions $B(x)$ and $D(x)$ govern the difference equation for the polynomials $P_n(\eta)$, the solution of which requires the knowledge of the sinusoidal coordinate $\eta(x)$. The same quantities $B(x)$ and $D(x)$ specify the three-term recurrence of the dual polynomials $Q_n(\mathcal{E})$ without the knowledge of the spectrum $\mathcal{E}(n)$. It is required for them to be the eigenvectors of the eigenvalue problem (2.23). Likewise, $A_n$ and $C_n$ in (2.139) specify the polynomials $P_n(\eta)$ without the knowledge of the sinusoidal coordinate. As for the dual polynomial $Q_n(\mathcal{E}(n))$, $A_n$ and $C_n$ provide the difference equation (in $n$), the solution of which needs the explicit form of $\mathcal{E}(n)$. Let us stress that it is the eigenvalue problem (2.23) with the specific Hamiltonian (2.12) that determines the polynomials $P_n(\eta)$ and their dual $Q_n(\mathcal{E})$, the spectrum $\mathcal{E}(n)$, the sinusoidal coordinate $\eta(x)$ and the orthogonality measures $\phi_0(x)^2$ and $d_n^2$.

By comparing the explicit Heisenberg operator solution for $\eta(x)$ with the three-term recurrence relations (2.139), the coefficients $A_n$ and $C_n$ are determined:

$$A_0 = R_{-1}(0)R_0(0)^{-1}, \quad C_0 = 0,$$

$$A_n = \frac{R_{-1}(\mathcal{E}(n)) + \eta(1)(\mathcal{E}(n) - B(0))\alpha_+(\mathcal{E}(n))}{\alpha_+(\mathcal{E}(n))(\alpha_+(\mathcal{E}(n)) - \alpha_-(\mathcal{E}(n)))} \quad (n \geq 1),$$

$$C_n = \frac{R_{-1}(\mathcal{E}(n)) + \eta(1)(\mathcal{E}(n) - B(0))\alpha_-(\mathcal{E}(n))}{\alpha_-(\mathcal{E}(n))(\alpha_-(\mathcal{E}(n)) - \alpha_+(\mathcal{E}(n)))} \quad (n \geq 1).$$

There is an important relation among the four important quantities:

$$A_0 \mathcal{E}(1) + B(0)\eta(1) = 0.$$ 

For the details of the derivation, see (4.45)–(4.53) of [20].
2.7. Dual closure relation in dQM

The dual closure relation has the same form as the closure relation (2.142) with the roles of the Hamiltonian $\hat{\mathcal{H}}$ and the sinusoidal coordinate $\eta(x)$ interchanged [20]:

$$[\eta, [\eta, \hat{\mathcal{H}}]] = \hat{\mathcal{H}}R^\text{dual}_0(\eta) + [\eta, \hat{\mathcal{H}}]R^\text{dual}_1(\eta) + R^\text{dual}_{-1}(\eta),$$  \hspace{1cm} (2.182)

in which

$$R^\text{dual}_1(\eta(x)) = (\eta(x - i\beta) - \eta(x)) + (\eta(x + i\beta) - \eta(x)),$$  \hspace{1cm} (2.183)

$$R^\text{dual}_0(\eta(x)) = -(\eta(x - i\beta) - \eta(x))(\eta(x + i\beta) - \eta(x)),$$  \hspace{1cm} (2.184)

$$R^\text{dual}_{-1}(\eta(x)) = \varepsilon(V_+(x) + V_-(x))R^\text{dual}_0(\eta(x)).$$  \hspace{1cm} (2.185)

The dual closure relation is the characteristic feature shared by all the ‘Hamiltonians’ $\hat{\mathcal{H}}$ which map a polynomial in $\eta(x)$ onto another. Therefore, its dynamical contents are not so constraining as the closure relation, except for the rdQM exactly solvable case, where the closure relation and the dual closure relations are on the same footing and they form a dynamical symmetry algebra which is sometimes called the Askey–Wilson algebra [34, 64, 33, 20].

2.8. Bochner’s theorem

In 1884 [65], Routh showed that polynomials satisfying the three-term recurrence relations and a second order differential equation were one of the classical polynomials, the Hermite, Laguerre, Jacobi and Bessel. Later in 1929 [66], Bochner classified all polynomial solutions to second order Sturm–Liouville operators with polynomial coefficients and arrived at the same conclusions. See section 20.1 of [5] for more details. This was a kind of No–Go theorem in oQM, since it declared that no essentially new exactly solvable oQM could be achieved as the solutions of the ordinary Schrödinger equation. Thus, avoiding Bochner’s theorem was one of the strongest motivations for the introduction of the dQM, whose difference Schrödinger equations were not constrained by the theorem and provided various exactly solvable examples [20, 21]. As shown in section 4, the new orthogonal (exceptional ($X_\ell$) Laguerre and Jacobi) polynomials [67, 68, 7, 8] were discovered in an attempt to evade the restrictions of the theorem by allowing the polynomials to start at degree $\ell \geq 1$. The new orthogonal polynomials in dQM, the exceptional ($X_\ell$) Wilson, Askey–Wilson, Racah and $q$-Racah polynomials satisfying second order difference equations, were constructed by the present authors in less than 2 years.

Here we present Bochner’s theorem for dQM. As in the original Bochner’s theorem, the conditions for the lowest three degrees $n = 0, 1$ and 2 are essential to determine the constraints. Reformulation of Bochner’s theorem for second order difference equations was pursued by several authors. In [69], it was shown (in our language) that if orthogonal polynomials in $\eta(x) = \cos x$ satisfy $q$-difference equations (2.23), then the conditions characterize the Askey–Wilson polynomials. In [70], it was shown that if the $q$-difference equations (2.23) have polynomial solutions, then the sinusoidal coordinate $\eta(x)$ is at most $q$-quadratic and that the polynomials are at most Askey–Wilson polynomials. In these papers, the distinction between the pure imaginary and the real shifts is blurred.

The starting point is the difference equation for the polynomials (2.23) with (2.26):

$$\varepsilon V_+(x)(P_n(\eta(x - i\beta)) - P_n(\eta(x))) + \varepsilon V_-(x)(P_n(\eta(x + i\beta)) - P_n(\eta(x)))$$

$$= \mathcal{E}(n)P_n(\eta(x)) \quad (n = 0, 1, \ldots),$$  \hspace{1cm} (2.186)
and the three-term recurrence relations (2.139). The above equation is trivially satisfied for \( n = 0 \), since \( \mathcal{E}(0) = 0 \). Two relations for \( n = 1 \) and \( n = 2 \) are linear equations:

\[
\begin{pmatrix}
P_1(\eta(x - i\beta)) - P_1(\eta(x)) & P_1(\eta(x + i\beta)) - P_1(\eta(x)) \\
\mathcal{E}(2)P_2(\eta(x - i\beta)) - P_2(\eta(x)) & \mathcal{E}(2)P_2(\eta(x + i\beta)) - P_2(\eta(x))
\end{pmatrix}
\begin{pmatrix}
V_+(x) \\
V_-(x)
\end{pmatrix}
= \begin{pmatrix}
\mathcal{E}^{-1}(\eta(x)) \\
\mathcal{E}(\eta(x))
\end{pmatrix},
\]

which determine the potential functions \( V_+(x) \) and \( V_-(x) \) uniquely. By using the explicit forms of \( P_1 \) and \( P_2 \),

\[ A_0P_1(\eta) = \eta - B_0, \quad A_0A_1P_2(\eta) = (\eta - B_0)(\eta - B_1) - A_0C_1, \quad (2.187) \]
derived from the three-term recurrence relations (2.139), the above linear equation gives

\[ V_\pm(x) = \frac{S_2 \pm S_1\eta(x \pm i\beta)}{(\eta(x \mp i\beta) - \eta(x))(\eta(x \pm i\beta) - \eta(x \mp i\beta))}, \quad (2.188) \]
in which \( S_1 \) and \( S_2 \) are a linear and a quadratic polynomial in \( \eta(x) \), respectively:

\[ S_1 = -\mathcal{E}^{-1}(\eta(x)), \quad S_2 = \mathcal{E}(\eta(x))A_0 P_2(\eta(x)) - \mathcal{E}(\eta(x))A_0 P_1(\eta(x)) - B_1). \quad (2.189) \]

These \( V_\pm \) have essentially the same forms as those which will be presented in section 3. If the symmetric shift-addition property of the sinusoidal coordinate \( \eta(x) \) (3.4) is satisfied, the above expressions for \( V_\pm(x) \) (2.188) are equivalent to \( V_\pm(x) \) (3.2)–(3.3) with \( L = 2 \). Moreover, when the symmetric shift-multiplication property (3.5) is satisfied, these potential functions give the exactly solvable models. For rdQM, the sinusoidal coordinates satisfying (3.4) can be classified into five types (3.15)–(3.19) [20] and they also satisfy (3.5).

### 3. Unified theory of exactly solvable dQM

In this section, we present a simple theory of constructing exactly solvable ‘Hamiltonians’ in dQM based on the two required properties of the sinusoidal coordinates (3.4)–(3.5). The general strategy is to construct the similarity transformed ‘Hamiltonian’ \( \tilde{\mathcal{H}} \) (2.26) in such a way that it maps a polynomial in \( \eta(x) \) into another:

\[ \tilde{\mathcal{H}} \mathcal{V}_n \subseteq \mathcal{V}_{n+L-2} \subset \mathcal{V}_\infty \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.1) \]

where \( L \) is a fixed positive integer. Here, \( \mathcal{V}_n \) is defined by (2.29) and \( \mathcal{V}_\infty \) \( \overset{\text{def}}{=} \lim_{n \to \infty} \mathcal{V}_n \). When \( L = 2 \), the above relation (3.1) is simply the lower triangularity of the ‘Hamiltonian’ \( \tilde{\mathcal{H}} \), leading to exact solvability.

In the following, we will take the similarity-transformed Hamiltonian \( \tilde{\mathcal{H}} \) (2.24) instead of \( \mathcal{H} \) as the starting point. That is, we reverse the argument and construct directly the ‘Hamiltonian’ \( \tilde{\mathcal{H}} \) (2.26) based on the sinusoidal coordinate \( \eta(x) \). This section is a brief review of [23].

#### 3.1. Potential functions

The general form of the ‘Hamiltonian’ \( \tilde{\mathcal{H}} \) mapping a polynomial in \( \eta(x) \) into another is achieved by the following form of the potential functions \( \tilde{V}_\pm(x) \):

\[ V_\pm(x) = \frac{\tilde{V}_\pm(x)}{(\eta(x \mp i\beta) - \eta(x))(\eta(x \pm i\beta) - \eta(x \mp i\beta))}. \quad (3.2) \]

\[ \tilde{V}_\pm(x) = \sum_{k,l \geq 0} v_{k,l} \eta(x)^l \eta(x \pm i\beta)^k, \quad (3.3) \]
where \( L \) is a natural number indicating the degree of \( \eta(x) \) in \( \hat{V}_k(x) \) and \( v_{k,i} \) are real constants, with the constraint \( \sum_{k=1}^{L} v_{k,i}^2 \neq 0 \). It is important that the same \( v_{k,i} \) appears in both \( \hat{V}_k(x) \).

The ‘Hamiltonian’ \( \hat{H} \) with the above \( \hat{V}_\pm(x) \) maps a degree \( n \) polynomial in \( \eta(x) \) to a degree \( n + L - 2 \) polynomial. This can be shown elementarily based on the two basic properties of the sinusoidal coordinates called the symmetric shift-addition property:

\[
\eta(x - i\beta) + \eta(x + i\beta) = (2 + r_{1}^{(1)})\eta(x) + r_{-1}^{(2)},
\]

and the symmetric shift-multiplication property:

\[
\eta(x - i\beta)\eta(x + i\beta) = (\eta(x) - \eta(-i\beta))(\eta(x) - \eta(i\beta)),
\]

together with \( \eta(x) \neq \eta(x - i\beta) \neq \eta(x + i\beta) \neq \eta(x) \). Here \( r_{1}^{(1)} \) and \( r_{-1}^{(2)} \) are real parameters.

In fact, these parameters also appear in the three functions \( R_0, R_1 \) and \( R_{-1} \) in the closure relation (2.142):

\[
R_1(y) = r_1^{(1)}y + r_1^{(0)}, \quad R_0(y) = r_0^{(2)}y^2 + r_0^{(1)}y + r_0^{(0)}, \quad R_{-1}(y) = r_{-1}^{(2)}y^2 + r_{-1}^{(1)}y + r_{-1}^{(0)}.
\]

Here are the lists of the known sinusoidal coordinates satisfying the above two conditions (3.4)–(3.5). There are eight sinusoidal coordinates for the idQM:

\[
\begin{align*}
\text{(i):} & \quad \eta(x) = x, \quad -\infty < x < \infty, \quad \gamma = 1, \\
\text{(ii):} & \quad \eta(x) = x^2, \quad 0 < x < \infty, \quad \gamma = 1, \\
\text{(iii):} & \quad \eta(x) = 1 - \cos x, \quad 0 < x < \pi, \quad \gamma \in \mathbb{R}_{\neq 0}, \\
\text{(iv):} & \quad \eta(x) = \sin x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \gamma \in \mathbb{R}_{\neq 0}, \\
\text{(v):} & \quad \eta(x) = 1 - e^{-x}, \quad -\infty < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \\
\text{(vi):} & \quad \eta(x) = e^{x} - 1, \quad -\infty < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \\
\text{(vii):} & \quad \eta(x) = \cosh x - 1, \quad 0 < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}, \\
\text{(viii):} & \quad \eta(x) = \sinh x, \quad -\infty < x < \infty, \quad \gamma \in \mathbb{R}_{\neq 0}.
\end{align*}
\]

and five sinusoidal coordinates for the rdQM: \( 0 < q < 1 \)

\[
\begin{align*}
\text{(i)':} & \quad \eta(x) = x, \\
\text{(ii)':} & \quad \eta(x) = e^{x}(x + d), \quad e' = \begin{cases} 1 & \text{for } d > -1 \\ -1 & \text{for } d < -N' \end{cases} \\
\text{(iii)':} & \quad \eta(x) = 1 - q', \\
\text{(iv)':} & \quad \eta(x) = q^{-x} - 1, \\
\text{(v)':} & \quad \eta(x) = e^{x}(q^{-x} - 1)(1 - dq'), \quad e' = \begin{cases} 1 & \text{for } d < q^{-1} \\ -1 & \text{for } d > q^{-N'}. \end{cases}
\end{align*}
\]

As shown in detail in section 4C of [20], the above five sinusoidal coordinates for rdQM (3.15)–(3.19) exhaust all the solutions of (3.4)–(3.5) up to a multiplicative factor. On the other hand, those for idQM (i)–(viii) (3.7)–(3.14) are merely typical examples satisfying all the postulates for the sinusoidal coordinate (3.4)–(3.5) and the extra one used for the shape invariance, (3.50) of [23].
The essential part of formula (3.2) is the denominators. They have the same form as the generic formula for the coefficients of the three-term recurrence relations of the orthogonal polynomials, (2.179) and (2.180) (see (4.52) and (4.53) in [20]). The translation rules are the duality correspondence itself, (2.176)–(2.177) (see also (3.14)–(3.18) in [20]):

\[ E(n) \rightarrow \eta(x), \quad -A_n \rightarrow V_+(x), \quad -C_n \rightarrow V_-(x), \]

\[ \alpha_r(E(n)) \rightarrow \eta(x - i\beta) - \eta(x), \quad \alpha_-(E(n)) \rightarrow \eta(x + i\beta) - \eta(x). \]

(3.20)

Some of the parameters \( v_{k,l} \) in (3.3) are redundant. It is sufficient to keep \( v_{k,l} \) with \( l = 0, 1 \). The remaining \( 2L + 1 \) parameters \( v_{k,l} \) (\( k + l \leq L, l = 0, 1 \)) are independent, one of which corresponds to the overall normalization of the Hamiltonian. See section II of [23] for the actual proof of the property (3.1).

The \( L = 2 \) case is exactly solvable. This corresponds to the general hypergeometric equations having at most degree 2 polynomial coefficients. Since the Hamiltonian of the polynomial space \( \mathcal{H} \) is expressed as an upper triangular matrix, its eigenvalues and eigenvectors are easily obtained explicitly. For the solutions of a full quantum mechanical problem, however, one needs the square-integrable ground-state wavefunction \( \varphi \) are easily obtained explicitly. For the solutions of a full quantum mechanical problem, however, one needs the square-integrable ground-state wavefunction \( \varphi \).

3.2. Askey–Wilson algebra

Here we briefly comment on the Askey–Wilson algebra, which are generated by the closure plus the dual closure relations. By simply expanding the double commutators in the closure (2.142) and the dual closure (2.182) relations, we obtain two cubic relations generated by the two operators \( \mathcal{H} \) and \( \eta \):

\[ \mathcal{H}^2 \eta - (2 + r^{(1)}_1)\mathcal{H}\eta \mathcal{H} \eta + \eta \mathcal{H}^2 - r^{(0)}_1(\mathcal{H}\eta + \eta \mathcal{H}) - r^{(0)}_0 \eta = r^{(2)}_2 \mathcal{H}^2 + r^{(1)}_1 \mathcal{H} + r^{(0)}_0, \]

\[ \eta^2 \mathcal{H} - (2 + r^{(1)}_1)\eta \mathcal{H}\eta + \mathcal{H}\eta^2 - r^{(2)}_1(\eta \mathcal{H} + \mathcal{H}\eta) + \eta(-i\beta)\eta(i\beta) \mathcal{H} = r^{(0)}_1 \eta^2 + r^{(1)}_1 \eta + \epsilon v_{0,0}. \]

(3.21)

From its structure, the closure relation is at most linear in \( \eta \) and at most quadratic in \( \mathcal{H} \). So the lhs of (3.21) has terms containing one factor of \( \eta \) and the rhs, none. It is simply \( R_{-1}(\mathcal{H}) \).

Likewise, the rhs of (3.22) has terms containing one factor of \( \mathcal{H} \) and the rhs, none. It is simply \( R_{0,0}^{\text{real}}(\eta) \). In (3.22), \( \eta(-i\beta)\eta(i\beta) \) is just a real number, not an operator. These have the same form as the so-called Askey–Wilson algebra, which has many different expressions [34, 64, 61, 33]. While the Askey–Wilson algebra has no inherent structure, the closure relation (2.142) has the right structure to lead to the Heisenberg operator solution for \( \eta(x) \), whose positive and negative frequency parts are the annihilation–creation operators [14, 20, 21]. It is the Hamiltonian and the annihilation–creation operators that form the dynamical symmetry algebra of the system [20, 21], not the closure or dual-closure relations, nor the Askey–Wilson algebra relations.
3.3. Quasi-exact solvability in dQM

The unified theory is general enough to generate quasi-exactly solvable Hamiltonians in the same manner. The quasi-exact solvability means, in contrast to the exact solvability, that only a finite number of energy eigenvalues and the corresponding eigenfunctions can be obtained exactly. Many examples are known in the oQM [71–73], but only a few are known in dQM in spite of the proposal that the $sl(2, R)$ algebra characterization of quasi-exact solvability could be extended to difference Schrödinger equations [72]. The unified theory also incorporates the known examples of quasi-exactly solvable Hamiltonians in dQM [17, 18, 22]. A new type of quasi-exactly solvable Hamiltonians is constructed. This approach reveals the common structure underlying the exactly and quasi-exactly solvable theories, in particular, the important roles played by the sinusoidal coordinates.

The higher $L \geq 3$ cases are obviously non-solvable. Among them, the tame non-solvability of $L = 3$ and 4 can be made quasi-exactly solvable (QES) by adding suitable compensation terms. This is a simple generalization of the method of Sasaki and Takasaki [74] for multi-particle QES in the oQM. For a given positive integer $M$, let us try to find a QES ‘Hamiltonian’ $\tilde{H}$, or more precisely its modification $\tilde{H}'$, having an invariant polynomial subspace $V_M$:

$$\tilde{H}' V_M \subseteq V_M.$$  \hspace{1cm} (3.23)

For $L = 3$, $\tilde{H}'$ is defined by adding one single compensation term of degree 1

$$\tilde{H}' \overset{\text{def}}{=} \tilde{H} - e_0(M)\eta(x),$$  \hspace{1cm} (3.24)

and we have achieved the quasi-exact solvability $\tilde{H}' V_M \subseteq V_M$. Known discrete QES examples belong to this class [18, 22].

For $L = 4$ case, $\tilde{H}'$ is defined by adding a linear and a quadratic in $\eta(x)$ compensation terms to the Hamiltonian $\tilde{H}$:

$$\tilde{H}' \overset{\text{def}}{=} \tilde{H} - e_0(M)\eta(x)^2 - e_1(M)\eta(x),$$  \hspace{1cm} (3.25)

and one condition between $v_{3,1}$ and $v_{4,0}$ is imposed. Then, we have $\tilde{H}' V_M \subseteq V_M$. This type of QES theory is new. For explicit forms of $e_0(M)$ and $e_1(M)$, see [23].

4. New orthogonal polynomials

In this section, we present a brief review of the recent hot topic, the discovery of infinitely many new orthogonal polynomials satisfying second order differential or difference equations. They are obtained as the main parts of eigenfunctions of exactly solvable QM in all the three categories, oQM, idQM and rdQM. They are the exceptional ($X_\ell$) Laguerre (XL) and Jacobi (XJ) polynomials in oQM, the exceptional Meixner–Pollaczek (XMP), Wilson (XW) and Askey–Wilson (XAW) polynomials in idQM and the exceptional Meixner (XM), Racah (XR) and $q$-Racah (XqR) polynomials in rdQM. The XMP and XM are new results. For any positive integer $\ell$, these $X_\ell$ polynomials have degrees $\ell + n$, $n = 0, 1, 2, \ldots$, and are orthogonal to each other in (4.43), (4.50), (4.55) with respect to explicitly given positive definite weight functions. Thus, they do not satisfy the three-term recurrence relations (2.139) and therefore the restrictions of Bochner’s theorem and its discrete counterparts in section 2.8 do not apply. As will be explained later, the quantum mechanical Hamiltonians for these exceptional polynomials are deformations of those for the original polynomials in terms of a degree $\ell$ eigenpolynomial with twisted parameters. The deformation terms satisfy the same type of equations as the original polynomials, but with the twisted parameters. The parameters are so chosen that the deformed system retains the shape invariance, too. The general scheme of the
deformation is common for various exceptional polynomials as shown in detail below. The original system corresponds to \( \ell = 0 \).

It should be stressed that the \( X_\ell \) Jacobi polynomials provide infinitely many global solutions of Fuchsian differential equations having \( 3 + \ell \) regular singularities [9].

### 4.1. Deforming polynomials

In order to introduce new orthogonal polynomials, we prepare the deforming polynomials. The deforming polynomials \( \xi_\ell (\eta) \) are eigenpolynomials \( P_\ell (\eta) \) with twisted parameters (or twisted coordinates) and they are crucial for the construction of new orthogonal polynomials. Two identities relating \( \xi_\ell \) with \( \lambda + \delta \), (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26), will play important roles in the derivation of various results.

For oQM, there are two types of the exceptional Jacobi polynomials, called \( X_{J1} \) and \( X_{J2} \). Their confluent limits give the exceptional Laguerre polynomials, \( X_{L1} \) and \( X_{L2} \). The deforming polynomials are

\[
\xi_\ell (\eta; \lambda) = \begin{cases}
P_\ell (-\eta; \lambda + (\ell - 1)\delta) & : \text{XL1} \\
P_\ell (\eta; t(\lambda + (\ell - 1)\delta)), & : \text{XL2} \\
t(\lambda) & : \text{JXJ}
\end{cases}
\tag{4.1}
\]

Explicitly they are

\[
\xi_\ell (\eta; \lambda) = \begin{cases}
\begin{align*}
L_{\ell - \frac{\ell - 1}{2}}^{(\ell + \frac{1}{2}, -\frac{1}{2})}(-\eta) & : \text{XL1} \\
L_{\ell - \frac{\ell - 1}{2}}^{(-\frac{1}{2}, -\frac{1}{2})}(\eta) & : \text{XL2}
\end{align*}
\end{cases}
\quad \xi_\ell (\eta; \lambda) = \begin{cases}
\begin{align*}
p_\ell^{(\ell + \frac{1}{2}, -\frac{1}{2})}(-\eta) & : \text{JX1} \\
p_\ell^{(-\frac{1}{2}, -\frac{1}{2})}(\eta) & : \text{JX2}
\end{align*}
\end{cases}
\tag{4.2}
\]

The two identities relating \( \xi_\ell (\eta; \lambda) \) and \( \xi_\ell (\eta; \lambda + \delta) \) are

\[
d_1 (\lambda + \ell \delta) \xi_\ell (\eta; \lambda) + d_2 (\eta) \partial_\eta \xi_\ell (\eta; \lambda) = d_3 (\lambda) \xi_\ell (\eta; \lambda + \delta),
\tag{4.3}
\]

\[
d_3 (\lambda, \ell) \xi_\ell (\eta; \lambda + \delta) + \frac{c_2(\eta)}{d_2(\eta)} \partial_\eta \xi_\ell (\eta; \lambda + \delta) = d_3 (\lambda + \ell \delta, \ell) \xi_\ell (\eta; \lambda),
\tag{4.4}
\]

where \( d_1 (\lambda), d_2 (\eta) \) and \( d_3 (\lambda, \ell) \) are given by

\[
d_1 (\lambda) \overset{\text{def}}{=} \begin{cases}
1 & : \text{XL1} \\
g + \frac{\ell}{2} & : \text{XL2, XJ2} \\
h + \frac{\ell}{2} & : \text{JXJ}
\end{cases}
\quad d_2 (\eta) \overset{\text{def}}{=} \begin{cases}
1 & : \text{XL1} \\
-\eta & : \text{XL2} \\
(1 \pm \eta) & : \text{JX1/JX2}
\end{cases}
\tag{4.5}
\]

\[
d_3 (\lambda, \ell) \overset{\text{def}}{=} \begin{cases}
1 & : \text{XL1} \\
0 & : \text{XL2} \\
0 & : \text{JX1}
\end{cases}
\tag{4.6}
\]

Note that these two identities, (4.3) and (4.4), imply the differential equation for the deforming polynomial

\[
c_2(\eta) \partial_\eta^2 \xi_\ell (\eta; \lambda) + c_1(\eta, \lambda + (\ell - 1)\delta) \partial_\eta \xi_\ell (\eta; \lambda) = -\frac{1}{2} \tilde{E}(\ell; \lambda) \xi_\ell (\eta; \lambda),
\tag{4.7}
\]

\[
\tilde{c} (\eta, \lambda, \ell) \overset{\text{def}}{=} \begin{cases}
\begin{align*}
c_1 (-\eta, \lambda + (\ell - 1)\delta) & : \text{XL1} \\
c_1 (\eta, t(\lambda + (\ell - 1)\delta)) & : \text{XL2, XJ1, XJ2}
\end{align*}
\end{cases}
\tag{4.8}
\]

\[
\tilde{E}(\ell; \lambda) \overset{\text{def}}{=} \begin{cases}
\begin{align*}
-\tilde{E}(\ell; \lambda) & : \text{XL1} \\
\tilde{E}(\ell; t(\lambda)) & : \text{XL2, XJ1, XJ2}
\end{align*}
\end{cases}
\tag{4.9}
\]
which is to be compared with the differential equation for the eigenpolynomial (2.23):
\[ c_2(\eta) \Delta_\lambda^2 P_\lambda(\eta; \lambda) + c_1(\eta, \lambda) \lambda \partial_\eta P_\lambda(\eta; \lambda) = -1/2 \mathcal{E}(\eta; \lambda) P_\lambda(\eta; \lambda). \] (4.10)

Note that \( \mathcal{E}(\ell; \lambda) = \mathcal{E}(\ell; \lambda + (\ell - 1)\delta) \). The deforming polynomial \( \xi_\ell(\eta; \lambda) \) has the same sign in the orthogonality domain, that is, \((0, \infty)\) for \( L \) and \((-1, 1)\) for \( J \), see (2.39), (2.40) of [35] and (3.2) of [38].

For idQM, we restrict the parameters:

**XMP**: \( \ell \) even,

\[ a_1, a_2 \in \mathbb{R}, \quad \{a_1^*, a_2^*\} = \{a_3, a_4\} \text{ (as a set)}, \quad 0 < a_j < \text{Re} \ a_k \quad (j = 1, 2; k = 3, 4), \] (4.11)

**XW**: \( a_1, a_2 \in \mathbb{R}, \quad \{a_1^*, a_2^*\} = \{a_3, a_4\} \text{ (as a set)}, \quad 1 > a_j > |a_k| \quad (j = 1, 2; k = 3, 4). \] (4.12)

**XAW**: \( a_1, a_2 \in \mathbb{R}, \quad \{a_1^*, a_2^*\} = \{a_3, a_4\} \text{ (as a set)}, \quad 1 > a_j > |a_k| \quad (j = 1, 2; k = 3, 4). \] (4.13)

The deforming polynomials for XMP, XW and XAW are
\[ \xi_\ell(\eta; \lambda) \overset{\text{def}}{=} P_\lambda(\eta; \lambda + (\ell - 1)\delta), \quad t(\lambda) \overset{\text{def}}{=} \begin{cases} -\lambda_1 & \text{: XMP} \\ (-\lambda_1, -\lambda_2, \lambda_3, \lambda_4) & \text{: XW, XAW}. \end{cases} \] (4.14)

The two identities relating \( \xi_\ell(\eta(x); \lambda) \) and \( \xi_\ell(\eta(x); \lambda + \delta) \) are [37]
\[ \frac{i}{\varphi(x)}(v_1(x; \lambda + \ell \delta) e^{\tilde{x}^\rho} - v_1(x; \lambda + \ell \delta) e^{\tilde{x}^\rho}) \xi_\ell(\eta(x); \lambda) = \hat{f}_{\ell,0}(\lambda) \xi_\ell(\eta(x); \lambda + \delta), \] (4.15)
\[ \frac{-i}{\varphi(x)}(v_2(x; \lambda + (\ell - 1)\delta) e^{\tilde{x}^\rho} - v_2(x; \lambda + (\ell - 1)\delta) e^{\tilde{x}^\rho}) \xi_\ell(\eta(x); \lambda + \delta) = \hat{b}_{\ell,0}(\lambda) \xi_\ell(\eta(x); \lambda), \] (4.16)
where \( v_1(x; \lambda), v_2(x; \lambda) \) are the factors of the potential function \( V(x; \lambda) \):
\[ V(x; \lambda) = -\sqrt{-1} \frac{\varphi(x)}{|\varphi(x)|} \frac{v_1(x; \lambda)v_2(x; \lambda)}{(x - i\frac{\lambda}{2})}, \] (4.17)
\[ v_1(x; \lambda) \overset{\text{def}}{=} \prod_{j=1}^{2} (a_j + ix), \quad v_2(x; \lambda) \overset{\text{def}}{=} \prod_{j=3}^{4} (a_j + ix). \] (4.18)

The constants \( \hat{f}_{\ell,n}(\lambda) \) and \( \hat{b}_{\ell,n}(\lambda) \) are given by
\[ \hat{f}_{\ell,n}(\lambda) \overset{\text{def}}{=} \begin{cases} 2a + n & \text{: XMP} \\ a_1 + a_2 + n & \text{: XW} \\ -q^{-\frac{n}{2}} (1 - a_1 a_2 q^n) & \text{: XAW} \end{cases}, \] (4.19)
\[ \hat{b}_{\ell,n}(\lambda) \overset{\text{def}}{=} \begin{cases} 2 & \text{: XMP} \\ a_1 + a_4 + n + 2\ell - 1 & \text{: XW} \\ -q^{-\frac{n}{2}} (1 - a_1 a_4 q^n 2\ell - 1) & \text{: XAW}. \end{cases} \] (4.19)

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These two identities (4.15) and (4.16) imply the difference equation for the deforming polynomial

\[
(V(x; t(\lambda + (\ell - 1)\delta))(e^{\gamma p} - 1) + V^*(x; t(\lambda + (\ell - 1)\delta))(e^{-\gamma p} - 1))\xi_{\ell}(\eta(x); \lambda) = E(\ell; t(\lambda))\xi_{\ell}(\eta(x); \lambda),
\]

(4.20)

which should be compared with the difference equation for the eigenpolynomial

\[
(V(x; \lambda)(e^{\gamma p} - 1) + V^*(x; \lambda)(e^{-\gamma p} - 1))P_n(\eta(x); \lambda) = E(n; \lambda)P_n(\eta(x); \lambda).
\]

(4.21)

Note that \(E(\ell; t(\lambda)) = E(\ell; t(\lambda + (\ell - 1)\delta))\). For the appropriate parameter ranges, the deforming polynomial \(\xi_{\ell}(\eta(x); \lambda)\) has no zero in the rectangular domain \(x_1 \leqslant \text{Re}x \leqslant x_2, |\text{Im}x| \leqslant |\gamma|\), which is necessary for the Hermiticity of the Hamiltonian.

For rQdM, the deforming polynomials for XM, XR and XqR are

\[
\tilde{\xi}_\ell(x; \lambda) \equiv \xi_{\ell}(\eta(x; \lambda + (\ell - 1)\delta); \lambda), \quad \tilde{P}_n(x; \lambda) \equiv P_n(\eta(x; \lambda); \lambda),
\]

(4.22)

\[
\tilde{\xi}_\ell(x; \lambda) \equiv \begin{cases} e^\ell \tilde{P}_\ell(-x + \beta + \ell - 1; \lambda + (\ell - 1)\delta) : \text{XM} \\ \tilde{P}_\ell(t(\lambda + (\ell - 1)\delta)), t(\lambda) \equiv (\lambda_1 - \lambda_1, \lambda_4 - \lambda_2, \lambda_3, \lambda_4) : \text{XR, XqR,} \end{cases}
\]

(4.23)

which satisfies the normalization

\[
\xi_{\ell}(0; \lambda) = 1.
\]

(4.24)

The two identities relating \(\tilde{\xi}_\ell(x; \lambda)\) and \(\tilde{\xi}_\ell(x; \lambda + \delta)\) are [11]

\[
\frac{1}{\varphi(x; \lambda + (\ell - 1)\delta + \delta)}(v_1^B(x; \lambda + \ell\delta) - v_1^D(x; \lambda + \ell\delta) e^{\delta})\tilde{\xi}_\ell(x; \lambda + \delta) = f_{\ell,0}(\lambda)\tilde{\xi}_\ell(x; \lambda + \delta),
\]

(4.25)

\[
\frac{1}{\varphi(x; \lambda + (\ell - 1)\delta + \delta)}(v_2^B(x; \lambda + (\ell - 1)\delta) - v_2^D(x; \lambda + (\ell - 1)\delta) e^{\delta})\tilde{\xi}_\ell(x; \lambda + \delta)
\]

\[
= \tilde{b}_{\ell,0}(\lambda)\tilde{\xi}_\ell(x; \lambda).
\]

(4.26)

Here \(v_1^B(x; \lambda), v_2^B(x; \lambda), v_1^D(x; \lambda), v_2^D(x; \lambda)\) are the factors of the potential functions \(B(x; \lambda)\) and \(D(x; \lambda):\)

\[
B(x; \lambda) = -\sqrt{\lambda} \frac{v_1^B(x; \lambda)v_1^B(x; \lambda)}{\varphi(x; \lambda + \delta)\varphi(x + \frac{1}{2}; \lambda + \delta)},
\]

(4.27)

\[
D(x; \lambda) = -\sqrt{\lambda} \frac{v_1^D(x; \lambda)v_1^D(x; \lambda)}{\varphi(x; \lambda + \delta)\varphi(x - \frac{1}{2}; \lambda + \delta)},
\]

(4.28)

\[
v_1^B(x; \lambda) \equiv \begin{cases} -e^x : \text{XM} \\ d^{-1}(x + a)(x + b) : \text{XR} \\ g^{-x}(1 - aq^p)(1 - bq^p) : \text{XqR,} \end{cases}
\]

(4.29)

\[
v_2^B(x; \lambda) \equiv \begin{cases} e^{-\frac{1}{2}}B(x; \lambda) : \text{XM} \\ d^{-1}(x + c)(x + d) : \text{XR} \\ g^{-x}(1 - cq^p)(1 - dq^p) : \text{XqR,} \end{cases}
\]

(4.30)
The deforming polynomial 

\[ \tilde{v}_1^D(x; \lambda) \text{ def } \left\{ \begin{array}{ll}
-c^{-1} & : \text{XM} \\
d^{-1}(x + d - a)(x + d - b) & : \text{XR} \\
n^{-x}a^{1-d} \left(1 - a^{-1} d\right) \left(1 - b^{-1} d\right) & : \text{XqR},
\end{array} \right. \tag{4.31}
\]

\[ \tilde{v}_2^D(x; \lambda) \text{ def } \left\{ \begin{array}{ll}
c^2D(x; \lambda) & : \text{XM} \\
d^{-1}(x + d - c)x & : \text{XR} \\
n^{-x}c \left(1 - c^{-1} d\right) \left(1 - q^c\right) & : \text{XqR},
\end{array} \right. \tag{4.32}
\]

where \( \hat{\delta} \) is

\[ \hat{\delta} \text{ def } \left\{ \begin{array}{ll}
(-1, 0) & : \text{XM} \\
(0, 0, -1, -1) & : \text{XR, XqR}.
\end{array} \right. \tag{4.33}
\]

The constants \( \hat{f}_{\ell,n}(\lambda) \) and \( \hat{b}_{\ell,n}(\lambda) \) are given by

\[ \hat{f}_{\ell,n}(\lambda) \text{ def } \left\{ \begin{array}{ll}
\frac{1 - c \beta + n + 2 \ell - 1}{\sqrt{c}} & : \text{XM} \\
(a + b - d + n) \frac{c + 2 \ell + n - 1}{c + \ell - 1} & : \text{XR} \\
n^{-x} \left(1 - abd^{-1} q^c\right) \frac{1 - c q^{2+\ell-1}}{1 - c q^{\ell-1}} & : \text{XqR},
\end{array} \right. \tag{4.34}
\]

\[ \hat{b}_{\ell,n}(\lambda) \text{ def } \left\{ \begin{array}{ll}
\frac{\sqrt{c}}{1 - c} (\beta - \ell - 1) & : \text{XM} \\
c + \ell - 1 & : \text{XR} \\
1 - c q^{\ell-1} & : \text{XqR}.
\end{array} \right. \tag{4.35}
\]

Again these two identities (4.25) and (4.26) imply the difference equation for the deforming polynomial

\[ (\tilde{B}(x; \lambda) (1 - e^\lambda) + \tilde{D}(x; \lambda) (1 - e^{-\lambda})) \tilde{\xi}_\ell(x; \lambda) = \tilde{\mathcal{E}}(\ell; \lambda) \tilde{\xi}_\ell(x; \lambda), \tag{4.36} \]

\[ \tilde{B}(x; \lambda) \text{ def } \left\{ \begin{array}{ll}
-D(-x + \beta + \ell - 1; \lambda + (\ell - 1)\delta) & : \text{XM} \\
B(x; t(\lambda + (\ell - 1)\delta)) & : \text{XR, XqR},
\end{array} \right. \tag{4.37}
\]

\[ \tilde{D}(x; \lambda) \text{ def } \left\{ \begin{array}{ll}
-B(-x + \beta + \ell - 1; \lambda + (\ell - 1)\delta) & : \text{XM} \\
D(x; t(\lambda + (\ell - 1)\delta)) & : \text{XR, XqR},
\end{array} \right. \tag{4.38}
\]

\[ \tilde{\mathcal{E}}(\ell; \lambda) \text{ def } \left\{ \begin{array}{ll}
-\mathcal{E}(\ell; \lambda) & : \text{XM} \\
\mathcal{E}(\ell; t(\lambda)) & : \text{XR, XqR},
\end{array} \right. \tag{4.39}
\]

to be compared with the difference equation for the eigenpolynomial

\[ (\mathcal{B}(x; \lambda) (1 - e^\lambda) + \mathcal{D}(x; \lambda) (1 - e^{-\lambda})) \mathcal{P}_\ell(x; \lambda) = \mathcal{E}(\ell; \lambda) \mathcal{P}_\ell(x; \lambda). \tag{4.40} \]

Note that \( \mathcal{E}(\ell; t(\lambda)) = \mathcal{E}(\ell; t(\lambda + (\ell - 1)\delta)). \) For the appropriate parameter ranges, the deforming polynomial \( \tilde{\xi}_\ell(x; \lambda) \) is positive at integer points \( x = 0, 1, \ldots, x^\text{max} + 1 \) \((x^\text{max} \text{ def } x_{\text{max}} - \ell). \)

Note that the range of parameters can be enlarged. For XJ, see [75].
4.2. New orthogonal polynomials

In this subsection, we present the explicit forms of the exceptional orthogonal polynomials and the orthogonality relations. They are bilinear in the deforming polynomial $\xi_\ell$ (with $\lambda$ and $\lambda + \delta$) and the eigenpolynomial $P_\ell$ (with $\lambda + \ell \delta + \tilde{\delta}$).

For oQM, the exceptional polynomials are

$$P_{\ell,n}(\eta; \lambda) \equiv \frac{2}{f_{\ell,n}(\lambda)} \left( (d_\ell(\eta))\xi_\ell(\eta; \lambda) \partial_\eta P_\ell(\eta; \lambda + \ell \delta + \tilde{\delta}) - d_\ell(\lambda)\xi_\ell(\eta; \lambda + \delta) P_\ell(\eta; \lambda + \ell \delta + \tilde{\delta}) \right) \times \begin{cases} 1 & : XJ1/XJ2 \ \text{or} \\ \pm 1 & : XL1/XL2 \end{cases}$$

(4.41)

where $\tilde{\delta}$ is

$$\tilde{\delta} \equiv \begin{cases} 1 & : XJ1/XJ2 \\ \pm (1,-1) & : XL1/XL2 \end{cases}$$

(4.42)

and $f_{\ell,n}(\lambda)$ will be given in (4.45). They satisfy the orthogonality relation

$$\int_{x_1}^{x_2} \psi_\ell(x; \lambda)^2 P_{\ell,n}(\eta(x); \lambda) P_{\ell,m}(\eta(x); \lambda) \ dx = h_{\ell,n}(\lambda) \delta_{nm},$$

(4.43)

where the weight function $\psi_\ell(x; \lambda)^2$ will be given in (4.63). The normalization constant $h_{\ell,n}(\lambda)$ is related to $h_\lambda(\lambda)$ (2.51) as

$$h_{\ell,n}(\lambda) = \frac{b_{\ell,n}(\lambda)}{f_{\ell,n}(\lambda)} h_\lambda(\lambda + \ell \delta + \tilde{\delta}) = \frac{\tilde{b}_{\ell,n}(\lambda)}{f_{\ell,n}(\lambda)} \frac{\tilde{f}_{\ell,n}(\lambda + \ell \delta)}{b_{\ell,n}(\lambda + \ell \delta)} h_\lambda(\lambda + \ell \delta),$$

(4.44)

where $\tilde{f}_{\ell,n}(\lambda)$ and $\tilde{b}_{\ell,n}(\lambda)$ are given by

$$\tilde{f}_{\ell,n}(\lambda) \equiv 2 \begin{cases} -1 & : XL1 \\ n + g + \frac{1}{2} & : XL2 \\ -(n + h + \frac{1}{2}) & : XJ1 \end{cases}, \quad \tilde{b}_{\ell,n}(\lambda) \equiv 2 \times \begin{cases} 1 & : XL2 \\ -(n + g + 2\ell - \frac{1}{2}) & : XJ1 \\ n + h + 2\ell - \frac{1}{2} & : XJ2 \end{cases}$$

(4.45)

The deforming polynomial $\xi_\ell(\eta; \lambda)$ has the same sign in the orthogonality domain, that is, $(0, \infty)$ for XL and $(-1, 1)$ for XJ.

The two types of the XJ polynomials are the mirror images of each other

$$P_{\ell,n}(x; g, h) = (-1)^{\ell+n} P_{\ell,n}^{XJ}(x; h, g),$$

(4.46)

due to the parity property of the Jacobi polynomial $P_\ell^{(\alpha, \beta)}(-x) = (-1)^\alpha P_\ell^{(\alpha, \beta)}(x)$, see (50)–(54) of [8]. The Laguerre polynomials are known to be obtained from the Jacobi polynomials in a certain limit:

$$\lim_{\beta \to \infty} P_\ell^{(\alpha, \pm \beta)}(1 - 2x \beta^{-1}) = L_\ell^{(\alpha)}(\pm x).$$

(4.47)

This connects the XJ polynomials to the XL polynomials as shown in (43) and (58) of [8].

For idQM, the exceptional polynomials are [37]

$$P_{\ell,n}(\eta(x); \lambda) \equiv \frac{1 - i}{f_{\ell,n}(\lambda) \psi(x)} \left( v_1(x; \lambda + \ell \delta) \xi_\ell \left( \eta \left( x + i \frac{\gamma}{2} \right); \lambda \right) P_\ell \left( \eta \left( x - i \frac{\gamma}{2} \right); \lambda + \ell \delta + \tilde{\delta} \right) - v_1(x; \lambda + \ell \delta) \xi_\ell \left( \eta \left( x - i \frac{\gamma}{2} \right); \lambda \right) P_\ell \left( \eta \left( x + i \frac{\gamma}{2} \right); \lambda + \ell \delta + \tilde{\delta} \right) \right),$$

(4.48)

where $\tilde{\delta}$ is

$$\tilde{\delta} \equiv \begin{cases} \frac{1}{2} & : XMP \\ \left( \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) & : XW, XAW. \end{cases}$$

(4.49)
They satisfy the orthogonality relation

\[ \int_{x_{\min}}^{x_{\max}} \psi_\ell(x; \lambda)^2 P_{\ell,n}(\eta(x); \lambda) P_{\ell,m}(\eta(x); \lambda) \, dx = h_{\ell,n}(\lambda) \delta_{nm}, \tag{4.50} \]

where the weight function \( \psi_\ell(x; \lambda)^2 \) will be given in (4.67). The normalization constant \( h_{\ell,n}(\lambda) \) is related to \( h_0(\lambda) \) (2.53) as

\[ h_{\ell,n}(\lambda) = \frac{\hat{b}_{\ell,n}(\lambda)}{f_{\ell,n}(\lambda)} h_0(\lambda + \ell \delta + \tilde{\delta}) = \frac{\hat{b}_{\ell,n}(\lambda)}{f_{\ell,n}(\lambda)} \frac{\hat{f}_{0,n}(\lambda + \ell \delta)}{b_{0,n}(\lambda + \ell \delta)}. \tag{4.51} \]

We have verified by numerical calculation for the parameter ranges in (4.11)–(4.13) that the deforming polynomial \( \xi_\ell(\eta(x); \lambda) \) has the same sign in the orthogonality domain, that is, \( x \in (-\infty, \infty) \) for XM, \( x \in (0, \infty) \) for XW and \( x \in (0, \pi) \) for XAW.

For rQD, the exceptional polynomials are [11]

\[ \tilde{P}_{\ell,n}(x; \lambda) \equiv P_{\ell,n}(\eta(x; \lambda + \ell \delta); \lambda) \]

\[ = \frac{1}{f_{\ell,n}(\lambda)} \psi(x; \lambda + \ell \delta + \tilde{\delta}) \left( v_\ell^0(x; \lambda + \ell \delta) \tilde{\xi}_\ell(x; \lambda) \tilde{P}_{\ell}(x + 1; \lambda + \ell \delta + \tilde{\delta}) \right. \]

\[ \left. - v_\ell^0(x; \lambda + \ell \delta) \tilde{\xi}_\ell(x + 1; \lambda) \tilde{P}_{\ell}(x; \lambda + \ell \delta + \tilde{\delta}) \right), \tag{4.52} \]

which satisfy the normalization

\[ P_{\ell,n}(0; \lambda) = 1. \tag{4.53} \]

Here, \( \tilde{\delta} \) is

\[ \tilde{\delta} \equiv \begin{cases} (-1, 0) & : \text{XM} \\ (0, 0, -1, -1) & : \text{XR}, \text{XqR}. \end{cases} \tag{4.54} \]

The orthogonality relation reads

\[ \sum_{x=0}^{x_{\max}} \frac{\psi_\ell(x; \lambda)^2}{\tilde{\xi}_\ell(1; \lambda)} \tilde{P}_{\ell,n}(x; \lambda) \tilde{P}_{\ell,m}(x; \lambda) = \frac{\delta_{nm} d_{\ell,n}(\lambda)^2}{\delta_{\ell,m}(\lambda)^2} \tag{4.55} \]

where the weight function \( \psi_\ell(x; \lambda)^2 \) will be given in (4.71). The normalization constant \( d_{\ell,n}(\lambda)^2 \) is related to \( d_0(\lambda)^2 \) (2.56) as

\[ d_{\ell,n}(\lambda)^2 = d_0(\lambda + \ell \delta + \tilde{\delta})^2 \frac{\hat{f}_{\ell,n}(\lambda)}{b_{\ell,n}(\lambda)} \frac{1}{s_1(\lambda)} \]

\[ = d_0(\lambda + \ell \delta + \tilde{\delta})^2 \frac{\hat{f}_{\ell,n}(\lambda)}{b_{\ell,n}(\lambda)} \frac{\hat{f}_{0,n}(\lambda + \ell \delta)}{b_{0,n}(\lambda + \ell \delta)} s_0(\lambda + \ell \delta). \tag{4.56} \]

where \( s_1(\lambda) \) is

\[ s_1(\lambda) \equiv \begin{cases} \frac{1 - c}{c} & : \text{XM} \\ \frac{1 - b + \ell - 1}{(d - a)(d - b)} & : \text{XR} \\ \frac{1 - abd^{-1} q^{1 - (1 - a^{-1}d)(1 - b^{-1}d)}}{(1 - c q^{1 - (1 - a^{-1}d)(1 - b^{-1}d)})} & : \text{XqR}. \end{cases} \tag{4.57} \]

We have verified by numerical calculation for the parameter ranges in (2.37)–(2.39) that the deforming polynomial \( \xi_\ell(x; \lambda) \) has the same sign in the orthogonality domain, that is, \( x = 0, 1, \ldots, x_{\max} \).
The exceptional polynomial $P_{\ell,n}(\eta; \lambda)$ is a degree $\ell + n$ polynomial in $\eta$ but has only $n$ zeros in $(0, \infty)$ for XL, $(-1, 1)$ for XJ, $(\eta(x_1), \eta(x_2))$ for idQM and $(0, \eta(x_{\text{max}}))$ for rdQM. For $\ell = 0$, the expressions of $P_{\ell,n}(\eta; \lambda)$ reduce to those of the original polynomials.

These new orthogonal polynomials do not satisfy the ordinary three-term recurrence relations (2.139). For the XL and XJ polynomials, a new type of bi-spectrality is demonstrated in section 4 of [36]. The bi-spectrality of the other new orthogonal polynomials, XW, XAW, XR, XJ polynomials is reported in section 10 of [9]. The generating functions $\sum_{n=0}^{\infty} \xi^P P_{\ell,n}(\eta; \lambda)$ for the XL and XJ polynomials are reported in section 9 of [9]. For the XL1 and XL2 polynomials, the closed form expression of the double generating functions

$$G(s, t; \eta; \lambda) \equiv \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} s^\ell t^n P_{\ell,n}(\eta; \lambda)$$

are given there too. The zeros of orthogonal polynomials have always attracted the interest of researchers. In [76], the behaviours of the zeros of the XL1, XL2 and XJ2 polynomials $P_{\ell,n}(\eta; \lambda)$ are explored as the parameters change. The structure of the extra zeros of the other new orthogonal polynomials, XW, XAW, XR and XqR, are also an interesting problem.

4.3. Deformed Hamiltonians

The new orthogonal polynomials are discovered as the main part of the eigenfunctions of quantum mechanical systems in which the original Hamiltonians are deformed in terms of the deforming polynomial $\xi_\ell$. The parameters are so chosen that the deformed systems retain shape invariance [7, 10, 35, 8, 37, 36, 11].

The deformed Hamiltonian and their eigenfunctions have the following forms:

$$\mathcal{H}_\ell(\lambda) \equiv \mathcal{A}_\ell(\lambda)^{\dagger} \mathcal{A}_\ell(\lambda), \quad (4.58)$$

$$\mathcal{H}_\ell(\lambda) \phi_{\ell,n}(x; \lambda) = \mathcal{E}_{\ell,n}(\lambda) \phi_{\ell,n}(x; \lambda), \quad \mathcal{E}_{\ell,n}(\lambda) = \mathcal{E}(n; \lambda + \ell \delta), \quad (4.59)$$

$$\phi_{\ell,n}(x; \lambda) = \psi_{\ell}(x; \lambda) P_{\ell,n}(\eta(x; \lambda + \ell \delta); \lambda). \quad (4.60)$$

The deformation is achieved additively at the level of the prepotential for oQM and multiplicatively at the level of the potential functions $V(x)$, $V^*(x)$, $B(x)$ and $D(x)$ for dQM. For oQM, operators $\mathcal{A}_\ell(\lambda)$, $\mathcal{A}_\ell(\lambda)^{\dagger}$ and $\psi_{\ell}(x; \lambda)$ are [7, 8]

$$\mathcal{A}_\ell(\lambda) \equiv \frac{d}{dx} - \delta \xi \psi_{\ell}(x; \lambda), \quad \mathcal{A}_\ell(\lambda)^{\dagger} = - \frac{d}{dx} - \delta \xi \psi_{\ell}(x; \lambda), \quad (4.61)$$

$$\psi_{\ell}(x; \lambda) \equiv \frac{B(x; \lambda + \ell \delta)}{\xi_\ell(\eta(x); \lambda)}, \quad \phi_{\ell,0}(x; \lambda) = \psi_{\ell}(x; \lambda) \xi_\ell(\eta(x); \lambda + \delta). \quad (4.62)$$

For idQM, they are [10]

$$\mathcal{A}_\ell(\lambda) \equiv i(e^{i\delta \nu} \sqrt{V^*_\ell(x; \lambda) - e^{-i\delta \nu} \sqrt{V^*_\ell(x; \lambda)}}, \quad (4.63)$$

$$\mathcal{A}_\ell(\lambda)^{\dagger} = -i(\sqrt{V^*_\ell(x; \lambda)} e^{i\delta \nu} - \sqrt{V^*_\ell(x; \lambda)} e^{-i\delta \nu} \), \quad (4.64)$$

$$V_\ell(x; \lambda) \equiv V(x; \lambda + \ell \delta) \xi_\ell(\eta(x + \nu); \lambda) \xi_\ell(\eta(x - \nu); \lambda + \delta) \xi_\ell(\eta(x); \lambda + \delta). \quad (4.65)$$
\[ V_\ell^* (x; \lambda) = V^* (x; \lambda + \ell \delta) \frac{\xi_\ell (\eta (x - i \frac{\lambda}{2}); \lambda)}{\xi_\ell (\eta (x + i \frac{\lambda}{2}); \lambda)} \frac{\xi_\ell (\eta (x + i \frac{\lambda}{2}); \lambda)}{\xi_\ell (\eta (x + i \frac{\lambda}{2}); \lambda)} \xi_\ell (\eta (x); \lambda + \delta). \quad (4.66) \]

\[ \psi_\ell (x; \lambda) \equiv \frac{\phi_0 (x; \lambda + \ell \delta)}{\sqrt{\xi_\ell (\eta (x + i \frac{\lambda}{2}); \lambda) \xi_\ell (\eta (x - i \frac{\lambda}{2}); \lambda)}}. \quad (4.67) \]

For rDQM, they are [11]
\[ A_\ell (\lambda) \equiv \sqrt{B_\ell (x; \lambda)} - e^\delta \sqrt{D_\ell (x; \lambda)}, \quad A_\ell (\lambda)^\dagger = \sqrt{B_\ell (x; \lambda)} - \sqrt{D_\ell (x; \lambda)} e^{-\delta}, \quad (4.68) \]

\[ B_\ell (x; \lambda) \equiv B (x; \lambda + \ell \delta) \frac{\xi_\ell (\eta (x); \lambda)}{\xi_\ell (\eta (x + 1; \lambda); \lambda)} \frac{\xi_\ell (\eta (x + 1; \lambda); \lambda)}{\xi_\ell (\eta (x + 1; \lambda); \lambda)} \xi_\ell (\eta (x); \lambda + \delta), \quad (4.69) \]

\[ D_\ell (x; \lambda) \equiv D (x; \lambda + \ell \delta) \frac{\xi_\ell (\eta (x + 1; \lambda); \lambda)}{\xi_\ell (\eta (x + 1; \lambda); \lambda)} \frac{\xi_\ell (\eta (x + 1; \lambda); \lambda)}{\xi_\ell (\eta (x + 1; \lambda); \lambda)} \xi_\ell (\eta (x); \lambda + \delta), \quad (4.70) \]

\[ \psi_\ell (x; \lambda) \equiv \frac{\phi_0 (x; \lambda + \ell \delta)}{\sqrt{\xi_\ell (\eta (x); \lambda)} \xi_\ell (\eta (x); \lambda + \delta)} \frac{\xi_\ell (1; \lambda)}{\xi_\ell (\eta (x); \lambda) \xi_\ell (\eta (x); \lambda + \delta)} \xi_\ell (\eta (x); \lambda + \delta). \quad (4.71) \]

These deformed systems are shape invariant:
\[ A_\ell (\lambda) A_\ell (\lambda)^\dagger = \kappa A_\ell (\lambda + \delta)^\dagger A_\ell (\lambda + \delta) + E_{\ell, 1} (\lambda), \quad (4.72) \]

or equivalently (2.108)–(2.112) with replacements \( (w, V, B, D) \rightarrow (w_\ell, V_\ell, B_\ell, D_\ell) \). The shape invariance of these deformed systems is shown analytically by making use of the two identities of the deforming polynomials (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26) [35, 37, 11].

The action of the operators \( A_\ell (\lambda) \) and \( A_\ell (\lambda)^\dagger \) on the eigenfunctions is
\[ A_\ell (\lambda) \phi_{\ell,n} (x; \lambda) = f_{\ell,n} (\lambda) \phi_{\ell,n-1} (x; \lambda + \delta) \times \begin{cases} 1 \\ \frac{1}{\sqrt{B_\ell (0; \lambda)}} \end{cases} : \text{oQM, idQM} \]
\[ A_\ell (\lambda)^\dagger \phi_{\ell,n-1} (x; \lambda + \delta) = b_{\ell,n-1} (\lambda) \phi_{\ell,n} (x; \lambda) \times \begin{cases} 1 \\ \frac{1}{\sqrt{B_\ell (0; \lambda)}} \end{cases} : \text{oQM, idQM} \]

where \( f_{\ell,n} (\lambda) \) and \( b_{\ell,n-1} (\lambda) \) are
\[ f_{\ell,n} (\lambda) = f_n (\lambda + \ell \delta) \times \begin{cases} \frac{c}{1 - c} (\beta + \ell) \\ 1 \end{cases} : \text{XM} \]
\[ b_{\ell,n-1} (\lambda) = b_{n-1} (\lambda + \ell \delta) \times \begin{cases} \left( \frac{1}{1 - c} (\beta + \ell) \right)^{-1} \\ 1 \end{cases} : \text{XM} \]

The forward-shift operator \( F_\ell (\lambda) \) and the backward-shift operator \( B_\ell (\lambda) \) are defined by
\[ F_\ell (\lambda) \equiv \psi_\ell (x; \lambda + \delta)^{-1} \circ A_\ell (\lambda) \circ \psi_\ell (x; \lambda) \times \begin{cases} 1 \\ \frac{1}{\sqrt{B_\ell (0; \lambda)}} \end{cases} : \text{oQM, idQM} \]
\[ (4.77) \]
The explicit forms of $\text{rQM}$:

\[
\begin{align*}
\mathcal{F}_\ell(x) &\equiv \psi_\ell(x; \lambda)^{-1} \circ \mathcal{A}_\ell(\lambda) \circ \psi_\ell(x; \lambda + \delta) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B_\ell(0; \lambda)}} & : \text{rdQM}, \end{cases} \\
\mathcal{B}_\ell(\lambda) &\equiv \psi_\ell(x; \lambda)^{-1} \circ \mathcal{A}_\ell(\lambda) \circ \psi_\ell(x; \lambda + \delta) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{B_\ell(0; \lambda)}} & : \text{rdQM}, \end{cases}
\end{align*}
\]  

and their action on the polynomials is $(\tilde{P}_{\ell,n}(x; \lambda)) \overset{\text{def}}{=} \tilde{P}_{\ell,n}(\eta(x; \lambda))$ for oQM and idQM

\[
\begin{align*}
\mathcal{F}_\ell(x) \tilde{P}_{\ell,n}(x; \lambda) &= f_{\ell,n}(\lambda) \tilde{P}_{\ell,n-1}(x; \lambda + \delta), \\
B_\ell(\lambda) \tilde{P}_{\ell,n-1}(x; \lambda + \delta) &= b_{\ell,n-1}(\lambda) \tilde{P}_{\ell,n}(x; \lambda).
\end{align*}
\]  

The explicit forms of $\mathcal{F}_\ell(\lambda)$ and $\mathcal{B}_\ell(\lambda)$ are as follows:

**oQM:**

\[
\begin{align*}
\mathcal{F}_\ell(\lambda) &= \frac{i}{\varphi(x)\xi_\ell(\eta(x); \lambda)} \left( \frac{d}{d\eta} \log \xi_\ell(\eta; \lambda + \delta) \right), \\
\mathcal{B}_\ell(\lambda) &= -4e^{-x}c_2(\eta) \frac{\xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda + \delta)} \left( \frac{d}{d\eta} + \frac{c_1(\eta, \lambda + \delta)}{c_2(\eta)} - \partial_\eta \log \xi_\ell(\eta; \lambda) \right).
\end{align*}
\]

**idQM:**

\[
\begin{align*}
\mathcal{F}_\ell(\lambda) &= \frac{i}{\varphi(x)\xi_\ell(\eta(x); \lambda)} \left( \xi_\ell(\eta(x) + \frac{i}{2}) : \lambda + \delta \right) e^{\xi_\ell} - \xi_\ell(\eta(x) - \frac{i}{2}) : \lambda + \delta \right) e^{-\xi_\ell}, \\
\mathcal{B}_\ell(\lambda) &= \frac{\ell}{\xi_\ell(\eta(x); \lambda + \delta)} \left( \varphi(x) : \lambda + \delta \right) \left( \xi_\ell(\eta(x) + \frac{i}{2}) : \lambda + \delta \right) e^{\xi_\ell} - \xi_\ell(\eta(x) - \frac{i}{2}) : \lambda + \delta \right) e^{-\xi_\ell}.
\end{align*}
\]

**rdQM:**

\[
\begin{align*}
\mathcal{F}_\ell(\lambda) &= \frac{B(0, \lambda + \ell\delta)}{\varphi(x; \lambda + \ell\delta)\xi_\ell(x + 1; \lambda)} (\tilde{\xi}_\ell(x + 1 + \lambda + \delta) - \tilde{\xi}_\ell(x; \lambda + \delta) e^3), \\
\mathcal{B}_\ell(\lambda) &= \frac{1}{B(0; \lambda + \ell\delta)} \frac{1}{\xi_\ell(x; \lambda + \delta)} \left( B(x; \lambda + \ell\delta)\tilde{\xi}_\ell(x; \lambda) \\
& \quad - D(x; \lambda + \ell\delta)\tilde{\xi}_\ell(x + 1; \lambda) e^{-3} \right) \varphi(x; \lambda + \ell\delta).
\end{align*}
\]  

### 4.4. Second order equations for the new polynomials

The new orthogonal polynomials $P_{\ell,n}(\eta; \lambda)$ satisfy second order differential or difference equations with rational coefficients. Here we list the explicit forms.

The similarity-transformed Hamiltonian $\tilde{\mathcal{H}}_\ell(\lambda)$ is defined by

\[
\tilde{\mathcal{H}}_\ell(\lambda) \overset{\text{def}}{=} \psi_\ell(x; \lambda)^{-1} \circ \mathcal{H}_\ell(\lambda) \circ \psi_\ell(x; \lambda) = B_\ell(\lambda)\mathcal{F}_\ell(\lambda),
\]

and its action on the polynomials is

\[
\tilde{\mathcal{H}}_\ell(\lambda) \tilde{P}_{\ell,n}(x; \lambda) = E_{\ell,n}(\lambda) \tilde{P}_{\ell,n}(x; \lambda).
\]

From (4.79)–(4.80) and (4.75)–(4.76), we have

\[
E_{\ell,n}(\lambda) = f_{\ell,n}(\lambda) b_{\ell,n-1}(\lambda) = f_{\ell}(\lambda + \ell\delta) b_{\ell,n-1}(\lambda + \ell\delta) = E(n; \lambda + \ell\delta).
\]

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The explicit forms of $\tilde{\mathcal{H}}_\ell(\lambda)$ are as follows:

**oQM:**

$$
\tilde{\mathcal{H}}_\ell(\lambda) = -4 \left( c_2(\eta) \frac{d^2}{d\eta^2} + c_1(\eta, \lambda + \ell \delta) - 2c_2(\eta)\partial_\eta \log \xi_\ell(\eta; \lambda) \right) \frac{d}{d\eta} \\
- 2d_2(\eta)d_3(\lambda, \ell)\partial_\eta \log \xi_\ell(\eta; \lambda) - \frac{1}{4} \xi_\ell(\lambda),
$$

(4.90)

**idQM:**

$$
\tilde{\mathcal{H}}_\ell(\lambda) = V(x; \lambda + \ell \delta) \frac{\xi_\ell(\eta(x + i\frac{\lambda}{2}); \lambda)}{\xi_\ell(\eta(x - i\frac{\lambda}{2}); \lambda)} \left( e^{\gamma p} - \frac{\xi_\ell(\eta(x + iy); \lambda + \delta)}{\xi_\ell(\eta(x); \lambda + \delta)} \right) \\
+ V^*(x; \lambda + \ell \delta) \frac{\xi_\ell(\eta(x - i\frac{\lambda}{2}); \lambda)}{\xi_\ell(\eta(x + i\frac{\lambda}{2}); \lambda)} \left( e^{-\gamma p} - \frac{\xi_\ell(\eta(x + iy); \lambda + \delta)}{\xi_\ell(\eta(x); \lambda + \delta)} \right),
$$

(4.91)

**rdQM:**

$$
\tilde{\mathcal{H}}_\ell(\lambda) = B(x; \lambda + \ell \delta) \frac{\xi_\ell(x; \lambda)}{\xi_\ell(x + 1; \lambda)} \left( \xi_\ell(x + 1; \lambda + \delta) - e^\delta \right) \\
+ D(x; \lambda + \ell \delta) \frac{\xi_\ell(x + 1; \lambda)}{\xi_\ell(x; \lambda + \delta)} \left( \xi_\ell(x + 1; \lambda + \delta) - e^\delta \right).
$$

(4.92)

For the XL and XJ polynomials in oQM we write concretely [9]

**XL1:** \( \eta^2 \partial_\eta^2 P_{\ell,n}(\eta; \lambda) + \left( g + \ell + \frac{1}{2} - \eta - 2 \eta \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} \right) \partial_\eta P_{\ell,n}(\eta; \lambda) \)

\[+ \left( -2 \eta \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} + n - \ell \right) P_{\ell,n}(\eta; \lambda) = 0, \]

(4.93)

**XL2:** \( \eta^2 \partial_\eta^2 P_{\ell,n}(\eta; \lambda) + \left( g + \ell + \frac{1}{2} - \eta - 2 \eta \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} \right) \partial_\eta P_{\ell,n}(\eta; \lambda) \)

\[+ \left( -2 \left( g + \frac{1}{2} \right) \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} + n + \ell \right) P_{\ell,n}(\eta; \lambda) = 0, \]

(4.94)

**XJ1:** \((1 - \eta^2) \partial_\eta^2 P_{\ell,n}(\eta; \lambda) \)

\[+ \left( h - g - (g + h + 2\ell + 1)\eta - 2 (1 - \eta^2) \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} \right) \partial_\eta P_{\ell,n}(\eta; \lambda) \]

\[+ \left( -2 \left( h + \frac{1}{2} \right) (1 - \eta^2) \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} \right) \xi_\ell(\eta; \lambda) \]

\[+ \ell (\ell + g - h - 1) + n(n + g + h + 2\ell) P_{\ell,n}(\eta; \lambda) = 0, \]

(4.95)

**XJ2:** \((1 - \eta^2) \partial_\eta^2 P_{\ell,n}(\eta; \lambda) + \left( h - g - (g + h + 2\ell + 1)\eta \right. \)

\[+ \left. - 2 (1 - \eta^2) \frac{\partial_\eta \xi_\ell(\eta; \lambda)}{\xi_\ell(\eta; \lambda)} \right) \partial_\eta P_{\ell,n}(\eta; \lambda) + \left( 2 \left( g + \frac{1}{2} \right) (1 + \eta) \frac{\partial_\eta \xi_\ell(\eta; \lambda + \delta)}{\xi_\ell(\eta; \lambda)} \right) \xi_\ell(\eta; \lambda) \]

\[+ \ell (\ell + h - g - 1) + n(n + g + h + 2\ell) P_{\ell,n}(\eta; \lambda) = 0. \]

(4.96)
The zeros of the deforming polynomial $\xi_\ell(\eta; \lambda) = 0 \ (j = 1, \ldots, \ell)$ are the extra regular singularities on top of those for the original Laguerre and Jacobi polynomials. In all the four cases, the characteristic exponents at these extra $\ell$ zeros are the same:

$$\rho = 0, \ 3.$$  \hfill (4.97)

The new polynomial solutions have $\rho = 0$ at all the extra singularities. The XJ polynomials provide an infinite family ($n = 0, 1, 2, \ldots$) of global solutions of the Fuchsian differential equations (4.95)–(4.96) with $3 + \ell$ regular singularities. To the best of our knowledge, global solutions of a Fuchsian differential equation with more than four singularities had been utterly unknown, although in this case, the locations of the singularities are very special.

### 4.5. Intertwining the original and the deformed systems

Here we present the transformation intertwining the original and the deformed Hamiltonian systems. It provides a simple derivation of the bi-linear expression of the new orthogonal polynomials in the original and the deforming polynomials. It also yields another proof of the shape invariance of the deformed systems. This type of transformations are sometimes called Darboux–Crum transformations.

The general setting of the transformation theory is common to oQM, idQM and rdQM. Let us define a pair of Hamiltonians $\hat{H}_\ell^{(+)}(\lambda)$ and $\hat{H}_\ell^{(-)}(\lambda)$ in terms of the operators $\hat{A}_\ell(\lambda)$ and $\hat{A}_\ell(\lambda)^\dagger$:

$$\hat{H}_\ell^{(+)}(\lambda) \overset{\text{def}}{=} \hat{A}_\ell(\lambda)^\dagger \hat{A}_\ell(\lambda), \quad \hat{H}_\ell^{(-)}(\lambda) \overset{\text{def}}{=} \hat{A}_\ell(\lambda) \hat{A}_\ell(\lambda)^\dagger,$$  \hfill (4.98)

and consider their Schrödinger equations

$$\hat{H}_\ell^{(+)}(\lambda) \hat{\phi}_{\ell,n}^{(+)}(x; \lambda) = \hat{\xi}_{\ell,n}(\lambda) \hat{\phi}_{\ell,n}^{(+)}(x; \lambda), \quad \hat{H}_\ell^{(-)}(\lambda) \hat{\phi}_{\ell,n}^{(-)}(x; \lambda) = \hat{\xi}_{\ell,n}(\lambda) \hat{\phi}_{\ell,n}^{(-)}(x; \lambda) \quad (n = 0, 1, 2, \ldots).$$  \hfill (4.99)

By definition, all the eigenfunctions must be square integrable (square summable). The pair of Hamiltonians are intertwined:

$$\hat{H}_\ell^{(+)}(\lambda) \hat{A}_\ell(\lambda)^\dagger = \hat{A}_\ell(\lambda)^\dagger \hat{H}_\ell^{(-)}(\lambda) = \hat{A}_\ell(\lambda)^\dagger \hat{H}_\ell^{(-)}(\lambda),$$

$$\hat{A}_\ell(\lambda) \hat{H}_\ell^{(+)}(\lambda) = \hat{A}_\ell(\lambda) \hat{H}_\ell^{(-)}(\lambda) = \hat{H}_\ell^{(+)}(\lambda) \hat{A}_\ell(\lambda).$$  \hfill (4.100)

In all the cases to be discussed here, we have $\hat{A}_\ell(\lambda) \hat{\phi}_{\ell,n}^{(+)}(x; \lambda) \neq 0$ and $\hat{A}_\ell(\lambda)^\dagger \hat{\phi}_{\ell,n}^{(-)}(x; \lambda) \neq 0$. This situation is called the ‘broken susy’ case in the supersymmetric quantum mechanics. This means that the two systems are exactly iso-spectral and there is one-to-one correspondence between the eigenfunctions:

$$\hat{\xi}_{\ell,n}^{(+)}(\lambda) = \hat{\xi}_{\ell,n}^{(-)}(\lambda),$$

$$\hat{\phi}_{\ell,n}^{(+)}(x; \lambda) \propto \hat{A}_\ell(\lambda) \hat{\phi}_{\ell,n}^{(+)}(x; \lambda), \quad \hat{\phi}_{\ell,n}^{(-)}(x; \lambda) \propto \hat{A}_\ell(\lambda)^\dagger \hat{\phi}_{\ell,n}^{(-)}(x; \lambda).$$  \hfill (4.102)

(4.103)

The strategy is to find the operators $\hat{A}_\ell$ and $\hat{A}_\ell^\dagger$ in such a way that $\hat{H}_\ell^{(+)}$ is the original Hamiltonian (up to an additive constant) shown in section 2.1 and $\hat{H}_\ell^{(-)}$ becomes the deformed Hamiltonian (up to an additive constant) demonstrated in section 4.3. Then the above formula (4.103) gives the expression of the new orthogonal polynomials in terms of the original ones.

We define operators $\hat{A}_\ell(\lambda)$ and $\hat{A}_\ell^\dagger(\lambda)$ as follows:

$$\hat{A}_\ell(\lambda) \overset{\text{def}}{=} \frac{d}{dx} - \partial_x \hat{u}_\ell(x; \lambda), \quad \hat{A}_\ell(\lambda)^\dagger = -\frac{d}{dx} - \partial_x \hat{u}_\ell(x; \lambda),$$(4.104)

$$\hat{u}_\ell(x; \lambda) \overset{\text{def}}{=} \log \xi_\ell(\eta(x); \lambda) + \begin{cases} \frac{1}{2} x^2 + (g + \ell - 1) \log x \quad &\text{XL1} \\ w(x; t(\lambda + (\ell - 1)\delta)) \quad &\text{XL2, XJ1, XJ2}. \end{cases}$$  \hfill (4.105)
(Note that $\frac{1}{2}x^2 + (g + \ell - 1) \log x = u(ix; \lambda + (\ell - 1)\delta) + \text{const. for XL1.})$

\[ \hat{A}_\ell(\lambda) \overset{\text{def}}{=} i(e^{\tilde{\zeta}p} \sqrt{\hat{V}_\ell^*(x; \lambda)} - e^{-\tilde{\zeta}p} \sqrt{\hat{V}_\ell(x; \lambda)}), \]
\[ \hat{A}_\ell(\lambda) = -i(\sqrt{\hat{V}_\ell(x; \lambda)} e^{\tilde{\zeta}p} - \sqrt{\hat{V}_\ell^*(x; \lambda)} e^{-\tilde{\zeta}p}), \quad (4.106) \]
\[ \hat{V}_\ell(x; \lambda) \overset{\text{def}}{=} V(x; t(\lambda + (\ell - 1)\delta)) \frac{\hat{\xi}_\ell(\eta(x + iy); \lambda)}{\hat{\xi}_\ell(\eta(x); \lambda)} , \quad (4.107) \]
\[ \hat{V}_\ell^*(x; \lambda) = V^*(x; t(\lambda + (\ell - 1)\delta)) \frac{\hat{\xi}_\ell(\eta(x + iy); \lambda)}{\hat{\xi}_\ell(\eta(x); \lambda)} , \quad (4.108) \]

\[ \hat{A}_\ell(\lambda) \overset{\text{def}}{=} \sqrt{\hat{B}_\ell(x; \lambda)} - e^{\xi} \sqrt{\hat{D}_\ell(x; \lambda)}, \quad \hat{A}_\ell(\lambda) = \sqrt{\hat{B}_\ell(x; \lambda)} - \sqrt{\hat{D}_\ell(x; \lambda)} e^{-\xi} , \quad (4.109) \]

\[ \hat{B}_\ell(x; \lambda) \overset{\text{def}}{=} \frac{\hat{\xi}_\ell(x + 1; \lambda)}{\hat{\xi}_\ell(x; \lambda)} \times \begin{cases} -D(-(x + \beta + \ell - 1); \lambda + (\ell - 1)\delta) : \text{XM} \\ B(x; t(\lambda + (\ell - 1)\delta)) : \text{XR, XqR}, \end{cases} \quad (4.110) \]
\[ \hat{D}_\ell(x; \lambda) \overset{\text{def}}{=} \frac{\hat{\xi}_\ell(x - 1; \lambda)}{\hat{\xi}_\ell(x; \lambda)} \times \begin{cases} -B(-(x + \beta + \ell - 1); \lambda + (\ell - 1)\delta) : \text{XM} \\ D(x; t(\lambda + (\ell - 1)\delta)) : \text{XR, XqR}. \end{cases} \quad (4.111) \]

Then, by using the two identities of the deforming polynomials (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26), we can show that [36, 37, 38, 39, 41]

\[ \hat{H}_\ell^{(+)}(\lambda) = \hat{k}_\ell(\lambda)(\hat{H}(\lambda + \ell\delta + \tilde{\delta}) + \hat{f}_{\ell,0}(\lambda)\hat{b}_{\ell,0}(\lambda)) , \quad (4.112) \]
\[ \hat{H}_\ell^{(-)}(\lambda) = \hat{k}_\ell(\lambda)(\hat{H}(\lambda) + \hat{f}_{\ell,0}(\lambda)\hat{b}_{\ell,0}(\lambda)) , \quad (4.113) \]

where $\hat{k}_\ell(\lambda)$ is

\[ \hat{k}_\ell(\lambda) \overset{\text{def}}{=} 1 : \text{oQM,} \quad \hat{k}_\ell(\lambda) \overset{\text{def}}{=} \frac{1}{(a_1a_2q')^{-1}} : \text{XMP, XW} \overset{\text{def}}{=} \frac{1}{(a_1a_2q')^{-1}} : \text{XAW,} \]
\[ \hat{k}_\ell(\lambda) \overset{\text{def}}{=} \frac{1}{(abd^{-1}q')^{-1}} : \text{XM, XR} \overset{\text{def}}{=} \frac{1}{(abd^{-1}q')^{-1}} : \text{XqR}. \quad (4.114) \]

Therefore, the original system with the shifted parameters $(\hat{H}(\lambda + \ell\delta + \tilde{\delta}))$ and the deformed system $(\hat{H}_\ell(\lambda))$ are exactly isospectral.

Based on the results (4.112)–(4.113), we have

\[ \hat{\phi}_{\ell,n}^{(+)}(x; \lambda) = \phi_\ell(x; \lambda + \ell\delta + \tilde{\delta}), \quad \hat{\phi}_{\ell,n}^{(-)}(x; \lambda) = \phi_{\ell,n}(x; \lambda) , \quad (4.115) \]
\[ \hat{\xi}_{\ell,n}^{(+)}(\lambda) = \hat{k}_\ell(\lambda)(\hat{\xi}_\ell(\lambda + \ell\delta + \tilde{\delta}) + \hat{f}_{\ell,0}(\lambda)\hat{b}_{\ell,0}(\lambda)) = \hat{k}_\ell(\lambda)(\hat{\xi}_{\ell,n}(\lambda) + \hat{f}_{\ell,0}(\lambda)\hat{b}_{\ell,0}(\lambda)) , \quad (4.116) \]

The correspondence of the pair of eigenfunctions $\hat{\phi}_{\ell,n}^{(+)}(x)$ with their own normalization specified in the preceding sections is related by

\[ \hat{\phi}_{\ell,n}^{(-)}(x; \lambda) = \frac{\hat{A}_\ell(\lambda)\hat{\phi}_{\ell,n}^{(+)}(x; \lambda)}{\sqrt{\hat{k}_\ell(\lambda)f_{\ell,n}(\lambda)}} \times \frac{1}{\sqrt{\hat{\xi}_\ell(1; \lambda)s_\ell(\lambda)}} : \text{oQM, idQM} \]
\[ \frac{\hat{A}_\ell(\lambda)\hat{\phi}_{\ell,n}^{(+)}(x; \lambda)}{\sqrt{\hat{k}_\ell(\lambda)f_{\ell,n}(\lambda)}} \times \frac{1}{\sqrt{\hat{\xi}_\ell(1; \lambda)s_\ell(\lambda)}} : \text{rdQM} . \quad (4.117) \]
By removing the effects of the orthogonality weight functions, we define the operators \( \hat{F}_x(\lambda) \) and \( \hat{B}_x(\lambda) \):

\[
\hat{F}_x(\lambda) \overset{\text{def}}{=} \psi_x(\lambda; \lambda)^{-1} \circ \frac{\hat{A}_x(\lambda)}{\sqrt{\kappa_x(\lambda)}} \circ \phi_x(\lambda; \lambda + \ell \delta + \delta) \times \begin{cases} 1 & : \text{oQM, idQM} \\ \frac{1}{\sqrt{\xi_x(\lambda)}} & : \text{rdQM}. \end{cases}
\]

Their explicit forms are as follows:

**oQM:**

\[
\hat{F}_x(\lambda) = 2 \left( d_x(\eta) \xi_x(\eta; \lambda) \frac{d}{d\eta} - d_x(\lambda) \xi_x(\eta; \lambda + \delta) \right) \times \begin{cases} \pm 1 & : \text{XL1/XL2} \\ \pm 1 & : \text{XL1/XL2}. \end{cases}
\]

**idQM:**

\[
\hat{F}_x(\lambda) = -2 \xi_x(\eta; \lambda) \left( \frac{c_x(\eta)}{d_x(\eta)} \frac{d}{d\eta} + d_x(\lambda; \ell) \right) \times \begin{cases} \pm 1 & : \text{XL1/XL2} \\ \pm 1 & : \text{XL1/XL2}. \end{cases}
\]

**rdQM:**

\[
\hat{F}_x(\lambda) = \frac{1}{\xi_x(\lambda; \lambda)} \left( v_2(x; \lambda + (\ell - 1)\delta) e^{\tilde{\tau}_p} - v_2^*(x; \lambda + (\ell - 1)\delta) e^{-\tilde{\tau}_p} \right),
\]

The operators \( \hat{F}_x(\lambda) \) and \( \hat{B}_x(\lambda) \) act as the forward- and backward-shift operators connecting the original polynomials \( P_n \) and the exceptional polynomials \( P_{e,n} \):

\[
\hat{F}_x(\lambda) \hat{P}_n(x; \lambda + \ell \delta + \delta) = \hat{F}_{e,n}(\lambda) \hat{P}_{e,n}(x; \lambda),
\]

\[
\hat{B}_x(\lambda) \hat{P}_{e,n}(x; \lambda + \ell \delta + \delta) = \hat{B}_{e,n}(\lambda) \hat{P}_n(x; \lambda + \ell \delta + \delta).
\]
The former relation (4.127) with the explicit forms of \( \hat{\mathcal{F}}_\ell (\lambda) \) (4.121), (4.123) and (4.125) provides the explicit expressions (4.41), (4.48) and (4.52) of the exceptional orthogonal polynomials. In terms of \( \hat{\mathcal{F}}_\ell (\lambda) \) and \( \hat{\mathcal{B}}_\ell (\lambda) \), relations (4.112)–(4.113) become
\[
\hat{B}_\ell (\lambda) \hat{\mathcal{F}}_\ell (\lambda) = \tilde{\mathcal{H}}_\ell (\lambda + \ell \delta + \tilde{\delta}) + \hat{f}_{\ell,0}(\lambda) \hat{b}_{\ell,0}(\lambda), \\
(4.129)
\]
\[
\hat{\mathcal{F}}_\ell (\lambda) \hat{B}_\ell (\lambda) = \tilde{\mathcal{H}}_\ell (\lambda) + \hat{f}_{\ell,0}(\lambda) \hat{b}_{\ell,0}(\lambda). \\
(4.130)
\]

The other simple consequences of these relations are
\[
\hat{\xi}^{(\pm)}(\lambda) = \hat{s}_\ell (\lambda) \hat{\mathcal{F}}_\ell (\lambda) \hat{b}_{\ell,n}(\lambda), \\
\hat{\xi}^{(\pm)}(\lambda) = \hat{s}_\ell (\lambda) \hat{\mathcal{F}}_\ell (\lambda) \hat{b}_{\ell,n}(\lambda) - \hat{f}_{\ell,0}(\lambda) \hat{b}_{\ell,0}(\lambda). \\
(4.131)
\]

Orthogonality relations (4.43)–(4.44), (4.50)–(4.51) and (4.55)–(4.56) can be shown by using these intertwining relations [38, 37, 11].

It is interesting to note that the operator \( \hat{\mathcal{A}}_\ell (\lambda) \) intertwines those of the original and deformed systems \( \mathcal{A}(\lambda) \) and \( \hat{\mathcal{A}}_\ell (\lambda) \):
\[
\hat{\mathcal{A}}_\ell (\lambda + \delta) \mathcal{A}(\lambda + \ell \delta + \tilde{\delta}) = \mathcal{A}(\lambda) \hat{\mathcal{A}}_\ell (\lambda), \\
(4.132)
\]
\[
\hat{\mathcal{A}}_\ell (\lambda + \delta) \mathcal{A}(\lambda + \ell \delta + \tilde{\delta})^\dagger = \mathcal{A}(\lambda)^\dagger \hat{\mathcal{A}}_\ell (\lambda + \delta). \\
(4.133)
\]

In terms of the definitions of the forward-shift operators \( \mathcal{F}(\lambda) \) (2.128), \( \mathcal{F}_\ell (\lambda) \) (4.77), \( \hat{\mathcal{F}}_\ell (\lambda) \) (4.119), and \( \mathcal{B}(\lambda) \) (2.130) and \( \mathcal{B}_\ell (\lambda) \) (4.78), the above relations are rewritten as
\[
\hat{s}_\ell (\lambda + \delta) \hat{\mathcal{F}}_\ell (\lambda + \ell \delta) \mathcal{F}(\lambda + \ell \delta + \tilde{\delta}) = \hat{s}_\ell (\lambda) \mathcal{F}_\ell (\lambda) \hat{\mathcal{F}}_\ell (\lambda), \\
(4.134)
\]
\[
\hat{s}_\ell (\lambda + \delta) \hat{\mathcal{F}}_\ell (\lambda) \mathcal{B}(\lambda + \ell \delta + \tilde{\delta}) = \hat{s}_\ell (\lambda + \delta) \mathcal{B}_\ell (\lambda) \mathcal{F}_\ell (\lambda), \\
(4.135)
\]

where \( \hat{s}_\ell (\lambda) \) is
\[
\hat{s}_\ell (\lambda) \overset{\text{def}}{=} 1 : \text{oQM}, \quad \hat{s}_\ell (\lambda) \overset{\text{def}}{=} \sqrt{\kappa_\ell (\lambda)} : \text{idQM}, \\
\hat{s}_\ell (\lambda) \overset{\text{def}}{=} \tilde{\kappa}_\ell (\lambda) \times \begin{cases} 
\beta + \ell - 1 : \text{XM} \\
c + \ell - 1 : \text{XR} \\
1 - cn^{\ell - 1} : \text{XqR}.
\end{cases} \\
(4.136)
\]

These relations can be proven by explicit calculation with the help of the two identities of the deforming polynomial (4.3)–(4.4), (4.15)–(4.16) and (4.25)–(4.26), and provide a proof of (4.73)–(4.74) and (4.79)–(4.80) without recourse to the shape invariance of the deformed system, see [38, 37, 11].

The intertwining relation offers another proof of the shape invariance of the \( \ell \)th new orthogonal polynomials through the established shape invariance of the original polynomials as depicted in the following commutative diagram:

Two ways of proving shape invariance of the new orthogonal polynomials system.
Historically, the first members of the $X_l$ Laguerre polynomials were discussed in the framework of ‘conditionally exactly solvable problems’ [77] in 1997. About 10 years later, the first members of the $X_l$ Laguerre and Jacobi polynomials were constructed by Gómez–Ullate et al [67] in 2008 in the framework of the Sturm–Liouville theory. They were rederived as the main part of the eigenfunctions of shape invariant quantum mechanical Hamiltonians by Quesne and collaborators [68]. In 2009, the present authors derived the infinitely many $X_l$ Laguerre and Jacobi polynomials by deforming the Hamiltonian systems of the radial oscillator and the Pöschl–Teller potential in terms of the eigenpolynomials of degree $\ell$ [7]. The main idea was very simple. The $X_l$ Jacobi Hamiltonian of Quesne [68] in our notation reads

$$\mathcal{H}_l(g, h) = -\frac{d^2}{dx^2} + \frac{g(g+1)}{\sin^2 x} + \frac{h(h+1)}{\cos^2 x} - (2 + g + h)^2$$

$$\frac{8(g + h + 1)}{1 + g + h + (g - h) \cos 2x} - \frac{8(2g + 1)(2h + 1)}{(1 + g + h + (g - h) \cos 2x)^2}.$$ (4.137)

Its groundstate wavefunction was

$$\phi_{l,0}(x; g, h) = (\sin x)^{g+1}(\cos x)^{h+1}\frac{3 + g + h + (g - h) \cos 2x}{1 + g + h + (g - h) \cos 2x}$$

$$= (\sin x)^{g+1}(\cos x)^{h+1}\frac{P_{\ell}^{(g+2-\frac{1}{2},-h-\frac{1}{2})}(\cos 2x)}{P_{\ell}^{(g+1-\frac{1}{2},-h-\frac{1}{2})}(\cos 2x)}.$$ (4.138)

From this, it did not take long to guess the general $\ell$ form:

$$\phi_{l,0}(x; g, h) = (\sin x)^{g+\ell}(\cos x)^{h+\ell}\frac{P_{\ell}^{(g+\ell+1-\frac{1}{2},-h-\ell-1-\frac{1}{2})}(\cos 2x)}{P_{\ell}^{(g+\ell-\frac{1}{2},-h-\ell-\frac{1}{2})}(\cos 2x)},$$ (4.139)

which was $e^{w(x, \lambda)}$ in (4.63) for XJ1. After the discovery of the second lowest members ($\ell = 2$) of the XL2 family [78], the entire XJ2 and XL2 families were constructed in [8]. Then, the construction of the exceptional Wilson and Askey–Wilson polynomials [10] and the exceptional Racah and $q$-Racah polynomials [11] followed by taking shape invariance as a guiding principle. The Fuchsian properties and many other aspects of XL and XJ polynomials were demonstrated in [9]. The intertwining relations were developed in [36, 37, 11]. The general knowledge of the solution spaces of exactly solvable (discrete) quantum mechanical systems governed by Crum’s theorem [27] and its modifications [29, 24–26] had been very helpful for the discovery of various exceptional orthogonal polynomials. Slightly different formulation of the Darboux–Crum transformation for the XL polynomials were reported in [79, 80].

Naively one expects that most exactly solvable QM Hamiltonian systems would admit similar shape invariant deformations. Since the ‘twisting’ of the parameters is essential, the systems corresponding to the Hermite and the Gegenbauer ($g = h$ case of the Jacobi) polynomials do not admit such a deformation. So far, in addition to the examples presented in this review, the exceptional polynomials for the continuous Hahn [37], dual $(q)$-Hahn and little $q$-Jacobi [11] are explicitly known. Various exceptional orthogonal polynomials can be obtained from XAW and XqR polynomials by taking appropriate limits. It is a big challenge to construct various new orthogonal polynomials, for example, those corresponding to the Morse and Scarf potentials and the $(q)$-Hahn polynomials, and to study them in detail.
5. Summary and comments

Discrete quantum mechanics, as formulated and developed by the present authors during the last decade, is briefly reviewed following the logical structure rather than the historical developments. The parallelism and contrast among the oQM and the dQM are emphasized. As far as possible, one universal formula valid for oQM and dQM with the pure imaginary shifts (idQM) and the real shifts (rdQM) is presented first and various ramifications follow. In one aspect, this is a reformulation of the theory of bi-spectral orthogonal polynomials in the language and logic of QM and every quantity is given explicitly without recourse to the moment problem as exemplified in section 2. While it offers structural understanding of various specific properties of the bi-spectral polynomials, a new unified theory is established to generate all the known bi-spectral polynomials from the first principle, as reviewed in section 3. As evidenced by the discovery of various kinds of infinite families of new orthogonal polynomials reviewed in section 4, this reformulation has been extremely successful.

Here are a few words on how our project of dQM started. The classical and quantum integrability were very closely related in the multi-particle dynamics of Calogero–Moser systems [43]. Reflecting the quantized eigenvalues, etc, the corresponding classical quantities showed Diophantine properties at various levels [81]. In particular, the classical equilibrium points of the Calogero–Moser systems were described by the zeros of the classical polynomials, the Hermite, Laguerre and Jacobi [82, 83, 84], which in turn constituted the eigenpolynomials of single-particle oQM. The classical equilibrium point, namely the minimum of the classical potential, could be better rephrased by the maximum of the groundstate eigenfunction \( \phi_0(x) = e^{w(x)} \), or the maximum of the prepotential \( w(x) \). Stieltjes investigated the equilibrium problems of multi-particle systems having ‘logarithmic potential’ which was nothing but the corresponding prepotential. Investigation of the discrete counterparts of the Calogero–Moser systems, that is, the Ruijsenaars–Schneider–van-Diejen [46] systems revealed similar Diophantine properties [85]. Later it was discovered that the classical equilibrium points of the Ruijsenaars–Schneider–van-Diejen systems were described by the zeros of the Wilson and Askey–Wilson polynomials [86, 87]. Then it was quite natural to seek for a single-particle quantum mechanical formulation in which the Wilson and Askey–Wilson polynomials were the eigenpolynomials. Thus, the first dQM was born [12]. The factorized Hamiltonian was the guiding principle.

The dQM is still in its infancy. Although the single-particle dynamics has been understood rather well by now, the multi-particle dynamics is virtually untouched. A lot of interesting problems, for example, the construction of the multi-particle version of various results in this review, are waiting to be clarified.

Due to the length limit, many interesting and important topics could not be covered in this review, for example, the limits from dQM to oQM and those of the corresponding polynomials, continuous \( \ell \) version of XL and XJ, etc. We refer to our papers [7–38, 86] and others.

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5. Appendix. Symbols and definitions

Here are several symbols and definitions related to the \((q)\)-hypergeometric functions.

- **Shifted factorial (Pochhammer symbol)\((a)_n\):**
  \[
  (a)_n \stackrel{\text{def}}{=} \prod_{k=1}^{n} (a + k - 1) = a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}, \tag{A.1}
  \]

- **\(q\)-shifted factorial (\((q)\)-Pochhammer symbol)\((a ; q)_n\):**
  \[
  (a ; q)_n \stackrel{\text{def}}{=} \prod_{k=1}^{n} (1 - a q^{k-1}) = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}). \tag{A.2}
  \]

- **Hypergeometric series \(\,_{r}F_{s}\):**
  \[
  \,_{r}F_{s} \left( \begin{array}{c} a_{1}, \ldots, a_{r} \\ b_{1}, \ldots, b_{s} \end{array} \right) \left( z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_{1}, \ldots, a_{r})_{n}}{(b_{1}, \ldots, b_{s})_{n}} \frac{z^{n}}{n!}, \tag{A.3}
  \]
  where \((a_{1}, \ldots, a_{r})_{n} \stackrel{\text{def}}{=} \prod_{j=1}^{r} (a_{j})_{n} = (a_{1})_{n} \cdots (a_{r})_{n}.

- **\(q\)-Hypergeometric series (the basic hypergeometric series) \(\,_{r}\phi_{s}\):**
  \[
  \,_{r}\phi_{s} \left( \begin{array}{c} a_{1}, \ldots, a_{r} \\ b_{1}, \ldots, b_{s} \end{array} \right) \left( q ; z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_{1}, \ldots, a_{r} ; q)_{n}}{(b_{1}, \ldots, b_{s} ; q)_{n}} \frac{(-1)^{(1+z-r)n} q^{(1+z-r)n(n-1)/2}}{(q ; q)_{n}} \frac{z^{n}}{n!}, \tag{A.4}
  \]
  where \((a_{1}, \ldots, a_{r} ; q)_{n} \stackrel{\text{def}}{=} \prod_{j=1}^{r} (a_{j} ; q)_{n} = (a_{1} ; q)_{n} \cdots (a_{r} ; q)_{n}.

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