A symplectic analytical singular element for the inclined crack terminating at the material interface of composite structures

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Abstract. The inclined bimaterial interface crack problem is studied from theoretical and computational aspects. The original problem is transformed into an eigen value problem by using the symplectic approach, and the eigen solution are obtained. A Symplectic Analytical Singular Element (SASE), of which the interior fields are described by the obtained eigen solutions, is developed. The stress intensity factors (SIFs) can be calculated directly without any post-processing. Numerical example on a square bimaterial plate containing an inclined crack is provided to validate the proposed method. The comparison of the predicted results with the existing result indicates that the proposed method is accurate for the modelling of the inclined bimaterial interface crack.

1. Introduction

With the increasing of the demand from modern manufacturing, more and more composite materials are used in engineering applications. However, the strength of composite material is still facing with challenges where extensive material interfaces exist. Cracks may initiate from, propagate along or terminate at these interfaces[1, 2, 3, 4]. This study investigates a typical case where crack is terminating at the material interface, to provide an effective tool for numerical modelling. Finite element method (FEM) is popular in engineering to calculate the stress intensity factors (SIFs) to estimate the state of cracked structure. However, extensive mesh refinement around crack tip is required to ensure the accuracy of simulation if conventional elements are used[5]. Bouhala et al.[6] studied the inclined interface crack problem by using the extended finite element method (XFEM), the stress and displacement enrichment functions were determined by considering plane elasticity solution based on Airy functions. Natarajan et al. applied the extended scaled boundary finite element method (XSBFEM) to the inclined interface crack problem, and both SIFs and T stresses were solved numerically without any necessary of path independent integral[7]. Nasri et al. investigated the behaviour of the crack terminating at the zinc/steel bimaterial interface with different orientations by using the commercial FE software package ABAQUS[8]. Recently, cohesive crack model was applied to the modeling of the bimaterial interface cracks. Adams modelled the crack which was perpendicular to the bimaterial interface and found that the critical value for crack propagation depended upon the maximum stress of the cohesive crack model, as well as the Dundures parameters[9]. Mehidi et al. investigated the crack normal to the metal/alumina interface by determining the $J$ integral and the plastic zone at the crack tip using the three-dimensional finite element methods[10].
In this study, a new symplectic analytical singular element (SASE) is developed for numerical study of inclined interface crack. By using the proposed method, mesh refinement around the crack tip is not required, and hence the solving efficiency is improved. The SIFs can be calculated directly without any post-processing. Verification of the proposed method by comparing with the benchmark has indicated that the developed method is very accurate and efficient.

2. Theoretical study

![Figure 1](image)

**Figure 1.** The configuration of inclined crack terminating at bimaterial interface

Considering an inclined interface crack terminating at the bimaterial interface as shown in Figure 1, the origin of the polar coordinate system is located at the crack tip. In the figure, $\theta = \pi - \omega$ is used to denote the crack orientation while $\pi \geq \omega \geq 0$. The discussed domain around the crack tip is divided into three regions by the crack and the material interface. The subscript “$i$” ($i = 1, 2, 3$) is employed to represent each region. The Young’s modulus and Poisson’s ratio of the material in the $i$th region are $E_i$, $\nu_i$. We have $E_1 = E_3$ and $\nu_1 = \nu_3$ since the regions #1 and #3 are occupied by the material 1. The fundamental equations can be derived from the following variational principle:

$$\delta \sum_{i=1}^{3} \int_{\varrho_i}^{\varrho_i} \left[ \sigma_{r.i} \frac{\partial u_{r.i}}{\partial r} + \sigma_{\theta.i} \left( \frac{\partial u_{\theta.i}}{\partial \theta} + u_{r.i} \right) + r_{\theta,i} \left( \frac{\partial u_{\theta.i}}{\partial r} - \frac{u_{\theta.i}}{r} + \frac{1}{r} \frac{\partial u_{r.i}}{\partial \theta} \right) \right] r dr d\theta = 0$$

(1)

where $\vartheta_1 = -\pi - \omega$, $\vartheta_2 = -\pi$, $\vartheta_3 = 0$ and $\vartheta_4 = \pi - \omega$. Assume that the two materials are perfectly bounded and the crack surfaces are free of external traction, so the compatibility condition as well as the boundary condition can be obtained easily. In the following discussion, the subscript “$i$” ($i = 1, 2, 3$) is omitted in the equations except where it may cause confusion. In order to solve this problem, the symplectic approach is used. By introducing the generalized coordinate

$$\xi = \ln r$$

(2)

and the generalized stress components

$$S_r = r \sigma_r, \quad S_{\theta} = r \sigma_{\theta}, \quad S_{r\theta} = r \tau_{r\theta}$$

(3)

the variational principle expressed in Eq.(1) can be further simplified into
\[
\delta \sum_{i=1}^{3} \int_{\theta_i}^{\theta_1} \int_{0}^{\infty} \left\{ \frac{E_i}{2} \left( u_{r,i} + \frac{\partial u_{r,i}}{\partial \theta} \right)^2 - \frac{1}{2E_i} \left[ (1 - \nu_i^2)S_{r,i}^2 + 2(1 + \nu_i)S_{\theta,i}^2 \right] \right\} \, d\xi \, d\theta = 0
\]

Making variation with respect to the independent variables gives the symplectic dual equation,

\[
\frac{\partial Z}{\partial \xi} = HZ
\]

where \( Z = [u_r, u_\theta, S_r, S_\theta]^T \) is the vector of displacements and generalized stresses. The Hamiltonian operator matrix \( H \) is presented as follows,

\[
H = \begin{bmatrix}
-\nu & -\nu \frac{\partial}{\partial \theta} & \frac{1 - \nu^2}{E} & 0 \\
-\frac{\partial}{\partial \theta} & 1 & 0 & \frac{2(1 + \nu)}{E} \\
E & E \frac{\partial}{\partial \theta} & \nu & -\frac{\partial}{\partial \theta} \\
-E \frac{\partial}{\partial \theta} & -E \frac{\partial^2}{\partial \theta^2} & -\nu \frac{\partial}{\partial \theta} & -1
\end{bmatrix}
\]

The symplectic equation can be solved by using the method of separation of variables, let

\[
Z(\xi, \theta) = e^{\mu \theta} \psi(\theta)
\]

where \( \mu \) is the symplectic eigenvalue and \( \psi(\theta) = [\psi_u, \psi_v, \psi_r, \psi_{r\theta}]^T \) is the corresponding symplectic eigenvector. \( \psi_u, \psi_r, \psi_v, \psi_{r\theta} \) are variables which are, respectively, separated from the original variables \( u_r, u_\theta, S_r \) and \( S_{r\theta} \). The eigenvalue and the corresponding eigenvector can be solved according to the standard solving procedure of symplectic approach which has been discussed in Refs.[11, 12]. It is found that \( \mu^{(1)} = \mu^{(2)} = 0 \) and generally \( \mu^{(3)} \) is the first eigenvalue which brings stress singularity. After obtaining the eigen solution, the solution of the original problem can be expressed in the form of

\[
Z = \sum_{i=1}^{\infty} a^{(i)} e^{\mu^{(i)} \theta} \psi^{(i)}
\]

where \( a^{(i)} \) is the unknown eigen expanding coefficient.
3. Symplectic analytical singular element (SASE)

Solving SIFs by using the conventional finite elements is still unsatisfactory even though refined mesh around the crack tip is used. The eigen solution obtained above should be used in the numerical method. In this study, the eigen solution is used to define a novel SASE which occupies the area around the crack tip while the other area of the structure is meshed by using conventional elements. The radius of the SASE is denoted by $\rho$. The node indexes are arranged from 1 to $N$. Assume that the unknown fields of the SASE is expressed by Eq. (8) with the first $2N$ expanding terms. Substituting Eq. (8) back into the variational principle (4) gives the deformation energy of the SASE which is specified by

$$\Pi = a^T R a$$

(9)

Meanwhile, the explicit form of $R$ can be obtained. Noted that the nodal displacements of the SASE can be expressed as

$$d = [u_r^{(1)}, u_\theta^{(1)}, \ldots, u_\theta^{(2N)}]^T = T_a a$$

(10)

where $d = [u_r^{(1)}, u_\theta^{(1)}, \ldots, u_\theta^{(2N)}]^T$ is the vector of the nodal displacement and $a = [a^{(1)}, a^{(2)}, \ldots, a^{(2N)}]^T$. So the deformation energy of the SASE is transformed into

$$\Pi = d^T K d$$

(11)

where $K = T_a^T R T_a^{-1}$. It is clear that $K$ is the stiffness matrix of the SASE. Assemble the stiffness matrices of the SASE and conventional elements, the FE system can be solved directly. Then, the expanding coefficients can be solved directly by Eq. (10).

The definition of the SIFs for inclined bimaterial interface crack in Ref.[5] is employed in this study, and the relationship between the eigen expanding coefficient and the SIF can be determined and specified as follows

$$K_1 = 2\sqrt{2\pi} \text{Re}[a^{(3)}(L)^{-1}\text{Im}(\rho^3)^{-1} \psi_\theta^{(3)}(-\omega)]$$

and

$$K_\| = 2\sqrt{2\pi} \text{Re}[a^{(3)}(L)^{-1}\text{Im}(\rho^{3n})^{-1} \psi_{\theta}^{(3)}(-\omega)]$$

(12)

So, once the expanding coefficients are solved, the SIFs can be solved directly without any post-processing.
4. Verification

Figure 3, The configuration of a bimaterial plate containing an interior crack terminating at the interface and the FE mesh

Consider the $4\text{mm} \times 4\text{mm}$ bimaterial square plate containing a crack terminating at the interface as shown in Figure 3, the material properties are $E_1 = 73\text{GPa}$, $\nu_1 = 0.17$, $E_2 = 206\text{GPa}$ and $\nu_1 = 0.3$. The tensile loading is $\sigma_y = 1\text{MPa}$. The FE mesh is also shown in Figure 3. The numerical results are listed in Table 1. The results obtained in Ref.[5] by using extrapolation method on the FE results with extremely refined mesh are listed as a reference. It is shown that the present results agree very well with the reference results. Moreover, the computing cost of the present method is much more cheaper because mesh refinement is not required.

Table. 1 Numerical results of the non-dimensional SIFs of the inclined bimaterial interface crack

| $\omega$ | $\mu_i$ | $K_i$ | $K_{II}$ |
|---------|---------|-------|---------|
| 15      | 0.5303±0.0746i | 6.3740 | 2.7036 |
| 30      | 0.5648±0.0667i | 7.5944 | 2.6679 |
| 45°     | 0.5996±0.0448i | 8.9794 | 2.2099 |
| 45° [5] | 0.5996±0.0448i | 8.9890 | 2.0978 |

5. Conclusion
The inclined interface crack problem in composite has been investigated systematically in this study. The eigen solution for the inclined interface crack were derived by using the symplectic approach. Based on the derived eigen solution, a new Symplectic Analytical Singular Element (SASE) has been developed to solve the SIFs. The comparison between the present results and the benchmarks has indicated that the developed model is very accurate in capturing the stress state around the inclined interface crack and mesh refinement around crack tip is not required.

6. Acknowledgments
The supports of the National Natural Science Foundation of China (No.11502045, and No. 11372065).
7. References

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