Gravitational Waves from Incomplete Inflationary Phase Transitions

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We study the observable implications of an incomplete first order phase transition during inflation. In such a phase transition, the nucleated bubbles do not percolate and instead are continuously produced until the onset of reheating. The process creates an inhomogeneity with a distinct power spectrum that depends on both the physics of the phase transition and the inflationary dynamics. Upon horizon re-entry, this spectrum generates gravitational waves through non-linear effects. This stochastic gravitational wave background is predicted to have unique signatures that may be detectable by future experiments spanning a wide frequency range. The discovery of such a gravitational wave signal would shed light on the detailed dynamics of inflation.

I. INTRODUCTION

The detection of gravitational waves (GWs) by LIGO and VIRGO [1] has opened a new window into our universe. Upcoming and far future experiments are expected to cover a wide range of frequencies and also improve current sensitivities [2–12], making the future detection of a stochastic gravitational wave background possible. Intriguingly, the NANOGRAV collaboration has reported some hints for the existence of a stochastic gravitational wave background at low frequencies of order 1 yr$^{-1}$ [13], although it is not clear if the origin is cosmological. Since gravitational waves propagated freely through the universe even while it was opaque to light, a gravitational wave background of primordial origin, if it exists, can hold crucial information about the very early history of our universe.

Inflation is the leading paradigm for these first moments [14, 15]. A period of rapid, exponential expansion explains why the observable universe is flat, homogeneous and isotropic to a very good accuracy. In the simplest scenario, the expansion can be driven by a single scalar field, called the inflaton. The initially small inhomogeneities of the universe originated in inflationary vacuum fluctuations, which grew to cosmological scales. These inhomogeneities are observed through the cosmic microwave background (CMB), giving strong constraints on inflationary models [16–20]. For a review of inflation, see [21].

GWs provide a promising tool to further explore the inflationary epoch. Although nothing is known about the pre-inflationary epoch, it is commonly assumed that the universe has been in a high-energy state. During inflation, the universe has rapidly cooled and it is natural to expect the system to be away from its global minimum, which may be eventually reached through one or more phase transitions (PTs). It is conceivable that some of these PTs are first order and proceed through bubble nucleation. Fast enough first order PTs proceed through a percolation stage which produce (possibly observable) GWs [22, 24] (for GWs from PTs after inflation, see, e.g. [25, 38]). Another possibility, however, which is at the focus of this letter, is slow PTs, which do not complete during inflation. In that case, bubbles nucleate too far from each other and never go through the percolation stage, due to the shrinking Hubble sphere. One may therefore wonder whether a GW signal still forms and if so, how would it be distinguished from the previously studied scenarios?

As we show below, while bubbles do not collide and percolate, their presence serves as a new source of inhomogeneities on small length scales. As a consequence of the PT remaining incomplete during inflation, the bubbles are produced continuously, resulting in a broad and flat inhomogeneity spectrum, spanning across a large range of modes. After inflation, once those modes enter the horizon, the inhomogeneities induce GWs through secondary effects [39–43] with a similarly broad and rather unique spectrum which could be measured by multiple upcoming and future experiments. The spectrum depends not only on the sector which goes through the PT but also on the details of inflation.

To demonstrate the above, we study a simplified single-field model which captures the relevant features of various PTs, including the well-known Coleman-de Luccia (CdL) bubble nucleation [44] and the Hawking-Moss (HM) instanton [45]. We then calculate the expected anisotropies and resulting GW background, concluding that a new and promising signal may appear in future GW observatories, shedding light on hidden sectors as well as on the physics of inflation.

II. INFLATION AND PHASE TRANSITIONS

A first order PT in the early universe takes place via bubble nucleation. The bubbles may or may not collide depending on the competition between their expansion and nucleation rates with the expansion rate of the universe. In this sense, such phase transitions exhibit two distinct regimes. The phenomenology of PTs with bubble collisions has been thoroughly studied [22, 38], and
here we focus on the signatures of PTs where the bubbles can’t meet.

During inflation, any slow enough first order PT does not complete \[46, 47\]. Schematically, if the bubble nucleation rate per unit volume is smaller than the Hubble expansion rate, i.e.,

\[
\Gamma / V \lesssim H^4, \tag{1}
\]

the mean distance between two neighboring bubbles is larger than the cosmological horizon (which shrinks in comoving coordinates). Hence bubbles cannot meet and percolate, leaving most of space in the false vacuum and the transition incomplete for as long as the universe inflates. While signals, such as GWs, are typically known to be produced during the percolation period, in this paper we show that stochastic GW signals are also predicted in slow PTs that can’t percolate during inflation, and the resulting signal records the entire duration of the PT.

To be concrete, consider first the CdL tunneling process \[44\] which describes the quantum process of vacuum tunneling in a gravitational background, and the rate of which is calculated with the instanton method based on the saddle-point approximation. In the semiclassical calculation, the bubbles are produced at rest with their radius equal to the critical radius - the minimal radius for an expanding bubble. Once formed, such bubbles expand classically, quickly approaching the speed of light. As soon as the physical radius of the bubble becomes larger than the Hubble radius, \(H^{-1}\), the surface velocity becomes negligible and the Hubble drift dominates the bubble evolution. At this point the bubble is “frozen”, i.e. it does not expand with respect to the comoving frame. As a consequence, a single bubble can never overtake the entire universe and for low enough nucleation rate, too few bubbles can form to complete the PT.

The Hubble radius represents the region contained inside a cosmological horizon created by the expanding universe. Therefore, in order to maintain causality, a CdL bubble must form with a smaller radius and if the critical bubble radius is larger than \(H^{-1}\), the CdL instanton does not exist. Instead, tunneling is still possible through the HM solution \[45\]. In the HM case, an entire Hubble patch tunnels simultaneously to the top of the potential barrier. This phenomenon is best understood through the formalism of stochastic inflation \[48\] where the inflationary horizon gives rise to a temperature, analogous to the Hawking temperature of a black hole. The thermal fluctuations then allow a trapped scalar field to diffuse, eventually reaching the top of a potential barrier. Once the barrier is crossed, the field may classically roll to the true minimum. As with the CdL PT, here too the Hubble-size bubble remains frozen and a slow nucleation rate implies that the PT is never complete. We stress, however, that as opposed to the CdL case, the stochastic formalism shows that the HM instanton calculation only holds in the limit of a very slow transition \[49\], and thus by construction can only describe an incomplete PT during inflation.

### III. A MODEL

The necessary details needed to study an incomplete PT can be described by a simple toy model. The PT is driven by the field \(\chi\), acting as a spectator during inflation. The inflationary dynamics are dominated by the inflaton \(\phi\), for which we assume the slow-roll conditions to hold, but whose detailed potential we otherwise remain agnostic to. While interactions between \(\chi\) and \(\phi\) may exist, their presence do not significantly affect our conclusions and we ignore them here. The potential is thus

\[
V = V_{PT}(\chi) + V_{inf}(\phi), \tag{2}
\]

where,

\[
V_{PT}(\chi) \ll V_{inf}(\phi). \tag{3}
\]

The PT potential, \(V_{PT}(\chi)\), is illustrated in Fig. 1. We take \(\chi\) to be initially on the left side of the potential and away from the false vacuum, classically slow-rolling down. To evade CMB constraints \[16, 20\] we assume that \(\chi\) settles at the unstable minimum sufficiently late in the inflationary epoch, so that bubble nucleation takes place only after the CMB modes had exited the horizon. Once at the local minimum, \(\chi\) can tunnel over the potential
barrier and roll to the global minimum where $V_{PT} = 0$, nucleating true vacuum bubbles. For simplicity, we assume this classical rolling after barrier crossing to be instantaneous.

The bubble nucleation rate during the PT is directly dictated by the potential parameters. We thus choose $V_{PT}$ such that Eq. 1 is fulfilled, ensuring a slow PT and implying that the physical volume of space where $\chi$ is “stuck” in the unstable minimum increases with time. Once the inflaton decays, regions of false vacuum may dominate the energy density and lead to an unwanted eternal inflation within our Hubble patch, driven by $\chi$. To evade such a catastrophe, one may either assume that the reheating temperature is larger than the energy density in the false vacuum and its effect drives to destabilize it, or ever simpler, that the nucleation rate is larger than the value of Hubble in the false vacuum so that rapid nucleation and percolation becomes possible after the inflaton decays\(^1\). With this, the PT suddenly and instantaneously completes everywhere and inflation truly ends at least within our visible universe.

The process of horizon exit and re-entry is illustrated in Fig. 2. The bubbles are either created small and rapidly expand to horizon size, as in the case of CdL, or created exactly at horizon size, as in the case of HM. Either way, once at the horizon, the comoving radius is completely frozen. After the end of inflation, the phase transition completes everywhere but the imprint on the curvature power spectrum remains. Upon horizon re-entry, the inhomogeneities produce GWs from secondary effects. Independent of the fine details of the model, the dynamics are governed by merely three parameters: the tunneling rate per unit volume $\Gamma/V$, the vacuum energy difference $\Delta V_{PT}$ between the false and true vacuum, and the time $t_0$ at which the transition commences (shortly after $\chi$ reaches the false vacuum). With this simplified description we now turn to calculate the GW spectrum produced by such inflationary incomplete PTs and arrive at predictions for future experiments.

IV. SCALAR CURVATURE SPECTRUM

We now move to calculate the scalar spectrum. To this end, we first find the energy-momentum tensor $T_{\mu \nu}$, neglecting the energy density in the bubble wall. This is justified because the ratio of energy in the interior volume over the wall energy scales as $a$ (the metric scale factor), and grows as the universe rapidly expands. Given the above, we omit the spatial derivatives of $\chi$ and $\phi$, which are localized in the bubble walls, and write the energy density and pressure after the PT starts, at $t \geq t_0$,

$$\rho(t, \vec{x}) = \frac{1}{2} \dot{\phi}^2 + V_{inf}(\phi(t)) + [1 - \theta(t - t_\inf)]\Delta V_{PT}, \quad (4)$$

$$p(t, \vec{x}) = \frac{1}{2} \dot{\phi}^2 - V_{inf}(\phi(t)) - [1 - \theta(t - t_\inf)]\Delta V_{PT}. \quad (5)$$

Here $t_\inf$ is the time when the transition occurred at point $\vec{x}$, and $\theta$ is the Heaviside step function. Using a step function is justified under the assumption of a rapid roll to the true vacuum once $\chi$ tunnels out of the false minimum. Furthermore, the kinetic energy stored in $\chi$ around the true minimum is quickly dissipated and is therefore neglected. All other components of the energy-momentum tensor can be neglected.

The main effect of the PT on the curvature spectrum is through the change in the Hubble constant due to the shift in the vacuum energy. We use the linearized Einstein equations in Newtonian gauge to calculate this induced curvature perturbation to first order. To this end, we need to find the inhomogeneous part of $T_{\mu \nu}$:

$$\rho(t, \vec{x}) = \bar{\rho}(t) + \delta \rho(t, \vec{x}), \quad \delta \rho \ll \rho, \quad (6)$$

and similarly for $p$. The homogeneous background is taken to be

$$\bar{\rho}(t) = \frac{1}{2} \dot{\phi}^2 + V_{inf}(\phi(t)) + [1 - \theta(t - \langle t_\inf \rangle)]\Delta V_{PT}, \quad (7)$$

$$\bar{p}(t) = \frac{1}{2} \dot{\phi}^2 - V_{inf}(\phi(t)) - [1 - \theta(t - \langle t_\inf \rangle)]\Delta V_{PT}, \quad (8)$$

while the perturbations are given by

$$\delta \rho(t, \vec{x}) = \Delta V_{PT}\theta(t - \langle t_\inf \rangle) - \theta(t - t_\inf), \quad (9)$$

\(^1\) We will ignore the GW from this final stage of the PT, as these occur not far from the reheating time, and the frequency range is likely beyond any near-future experiment.
and $\delta p = -\delta \rho$. The (scalar) perturbed metric in Newtonian gauge (for a review see, e.g. [21]) is,

$$ds^2 = -(1+2\Phi)dt^2+a^2(t)(1-2\Psi)\left(dx^2+dy^2+dz^2\right), \quad (10)$$

and the gauge invariant comoving curvature perturbation is defined as

$$\mathcal{R} = \Psi - \frac{H}{\dot{\rho} + \ddot{\rho}} \delta q , \quad (11)$$

where $\delta q$ is the scalar momentum perturbation. When specifying the energy momentum tensor in Eqs. (7), (8), and (9), we have neglected the wall energy, which is equivalent to setting $\delta q = 0$ and $\mathcal{R} = \Psi$.

The only Einstein equation we will need for calculating $\mathcal{R}$ is

$$\ddot{\mathcal{R}} + H\mathcal{R} = 0. \quad (12)$$

Using the continuity equation

$$\dot{\rho} + (\dot{\rho} + \ddot{\rho})\Phi = 0 , \quad (13)$$

we extract $\Phi$ and plug the result into Eq. (12),

$$\ddot{\mathcal{R}} = H \frac{\dot{\rho}}{\dot{\rho} + \ddot{\rho}} = -H \frac{\dot{\rho}}{\dot{\rho}^2} . \quad (14)$$

Equation (3) shows that $\mathcal{R}$ is constant for $t < t_1 = \min(t_{\mathcal{R}}, t_{\delta q})$ and for $t > t_2 = \max(t_{\mathcal{R}}, t_{\delta q})$. With this, and using the slow-roll approximation, we take $\dot{\phi}$ and $H$ to be constant and assume an initial flat background. We will later relax these assumptions in order to demonstrate the sensitivity of the predicted spectrum to the inflationary dynamics. The integrated Eq. (14) then gives

$$\mathcal{R}(\bar{x}) = -\frac{H \Delta V_{\mathcal{PT}}}{\dot{\phi}^2} (t_x - \langle t_{\mathcal{R}} \rangle) \equiv -\frac{H \Delta V_{\mathcal{PT}}}{\dot{\phi}^2} \delta t_x . \quad (15)$$

We move to calculate the scalar power spectrum $P_R(k)$, defined by

$$\langle k R_k \rangle = \frac{\delta (k + k')}{k^2} P_R(k) . \quad (16)$$

Equation (15) implies that one has to calculate the correlation between the tunneling times at different points in space, $\langle \delta t_x \delta t_x \rangle$. The details of this calculation are given in Appendix A, where we assume spherical bubbles and a constant tunneling rate $\Gamma/V$. The calculation further assumes that bubbles nucleate frozen at horizon size, a valid assumption in the HM case, and a reasonable approximation in the CdL case.

The scalar spectrum is shown in Fig. 3 for three different choices of the relevant parameters. The PT commences at $t_0$ and carries on until the end of inflation at $t_{\text{reheating}}$, continuously producing bubbles. The effect on the primordial power spectrum therefore spans the range of momentum modes that exit the horizon at this period of time, i.e. from $k_0 \equiv H_0(t_0)$ to $k_{\text{B}} \equiv H_0(t_{\text{reheating}})$. Since bubbles are created at a fixed rate in a universe with a shrinking co-moving Hubble sphere, the spectrum is expected to be approximately flat (varying only logarithmically). For concreteness, throughout this work we fix $k_0 = 4 \cdot 10^2 \text{Mpc}^{-1}$. Under our assumptions above of inflation with fixed slow-roll parameters, the shape and position of the peak are determined by the two $k_0$’s, while the amplitude further depends linearly on the dimensionless parameter $\Gamma_{\mathcal{PT}} \equiv \frac{\Gamma}{V} \frac{\Delta V_{\mathcal{PT}}}{\dot{\phi}^2}$ (see Eqs. (15) and (16) as well as App. A for details). We find that the maximal value of the power spectrum, $P_{R_{\text{max}}}$, only weakly depends on $k_0$ and is roughly $P_{R_{\text{max}}} / \gamma_{\mathcal{PT}} \approx O(10^3)$.
We emphasize that the power spectrum is sensitive not only to the spectator field which drives the PT, but also to the inflationary dynamics themselves, and its shape records the entire inflationary history from the beginning of the PT to the onset of reheating. Consequently, a measurement of the spectral shape can reveal detailed information about the dynamics of inflation. So far, we have assumed that the slow-roll parameter and scale of inflation are constant (see Eqs. (14) and (15)). Relaxing this assumption strongly affects the spectrum as is demonstrated by the blue and orange dashed-dotted lines in Fig. 3, which assume a sudden change in the value of $H/\dot{\phi}$ occurring at time $t_{\text{drop}}$. For a detailed derivation of $\mathcal{R}(\vec{x})$ in this case, we refer the reader to App. B.

Figure 3 further shows constraints from the overclosure due to primordial black hole (PBH) abundance (red line), and distortions to the CMB black-body spectrum (red region). We note that the PBH constraint, taken from [19], and the primordial black hole abundance (red line), are connected” part of the four-point correlation function does not induce GWs. This point is further explained and proven in App. C. In the following we briefly review the relevant result of [43], before applying it to the spectrum derived in Sec. IV. Since the scalar spectrum is almost scale-invariant over a wide range of momenta, the induced gravitational wave energy density is given by

\[
\frac{d\Omega_{\text{GW}}}{d\log k}(\eta, k) \equiv \frac{1}{\rho_{\text{tot}}(\eta)} \frac{d\rho_{\text{GW}}}{d\log k} = \frac{1}{24} \left( \frac{k}{a(\eta)H(\eta)} \right)^2 \mathcal{P}_h(\eta, k),
\]

where $\eta$ is the conformal time and $\mathcal{P}_h(\eta, k)$ is the time averaged tensor spectrum, given by

\[
\mathcal{P}_h(\eta, k) = 4 \int_0^\infty dv \int_{1-v}^{1+v} du \left( \frac{4v^2 - (1 + u^2 - a^2)^2}{4vu} \right)^2 \mathcal{I}^2(v, u, k\eta)\mathcal{P}_R(kv)\mathcal{P}_R(ku). \tag{18}
\]

FIG. 4: The GW energy density induced by the scalar spectrum shown in Fig. 3 multiplied by the square of the scaling factor of the Hubble expansion rate, $\mathcal{R}$. The amplitude $\mathcal{R}$ is quadratically dependent on the dimensionless parameter $\gamma_T \equiv \frac{1}{\mathcal{R}} \left( \frac{\Delta x_{\text{PT}}}{\sigma^2} \right)^2$, which is assumed to be constant. The momentum scale of the horizon at the beginning of the phase transition, $k_0 \equiv H\alpha(t_0)$, acts as a minimal scale, below which the spectrum is strongly suppressed. The scale of reheating $k_{\text{reh}} \equiv H(t_{\text{reh}})$ was fixed at $4 \cdot 10^{22} \text{Mpc}^{-1}$. The blue, dash-dotted, dashed, and dotted lines correspond to the parameter choices $k_0 = 2 \cdot 10^4, 8 \cdot 10^9, 2 \cdot 10^{14} \text{Mpc}^{-1}$ and $\gamma_T = 5 \cdot 10^{-7}, 10^{-6}, 10^{-4}$, respectively. The dashed-dotted blue and orange lines show the spectrum for an alternative scenario where the value of $\frac{H}{\phi}$ changes by a factor of $1/10$ at time $t_{\text{drop}}$. We define $k_{\text{drop}} \equiv H(t_{\text{drop}})$ and take its value to be $k_{\text{drop}} \equiv 10^9, 10^{13} \text{Mpc}^{-1}$ for the blue and orange lines respectively. We further choose $k_0 = 2 \cdot 10^4$ and $\gamma_T = 7 \cdot 10^{-6}, 2 \cdot 10^{-6}$ to ensure that the peaks align, thereby demonstrating the effect of the drop in $\frac{H}{\phi}$ on the spectral shape. Current constraints and future detector sensitivity regions are shown with solid, and semi-transparent colored regions respectively. The detector sensitivity curves for SKA [51], LISA [3], [52], TianQin [4], Ligo [2], and Ligo A+, are taken from [54]. The green violin plots represent the free-spectrum fit to the NANOGrav data [19][45]. The red region and pale red line are CMB distortion and primordial black hole constraints derived from those in Fig. 5.

Here the quantity $\mathcal{I}^2(v, u, k\eta)$ is defined in Eq. (C4). $d\Omega_{\text{GW}}/d\log k$ approaches a constant value during radiation domination because the GWs redshift like radiation. The density during radiation domination can thus be related to the density today through

\[
\frac{d\Omega_{\text{GW}}(\eta_0, k)}{d\log k} = \Omega_r(\eta_0) \frac{\Omega_{\text{GW}}(\eta_c, k)}{d\log k}, \tag{19}
\]

where $d\Omega_{\text{GW}}(\eta_c, k)/d\log k$ is the constant value reached during the radiation dominated era, and $\Omega_r(\eta_0) \approx 10^{-4}$ is the energy fraction of radiation today.

Integrating Eq. (18) numerically, we obtain the GW spectrum shown in Fig. 4 for the parameters discussed in Sec. IV.
abundance of GWs can be approximated by the analytical result given in [13] for a scale invariant case, $d\Omega_{GW}(\eta, k)/d\log k \approx 0.87P_{R}^2$. Applying this approximation along with Eq. (19), we find that much like the power spectrum, the peak of the GW energy density today can be estimated directly from the model parameters,

$$\frac{d\Omega_{GW}(\eta_0, k)}{d\log k}_{\text{peak}} \sim \mathcal{O}(100)\gamma_{\widetilde{T}}^2.$$ (20)

In addition to the predicted lines, Fig. 4 shows the corresponding constraints from Fig. 3, as well as constraints from LIGO [2] and expected sensitivity from various future GW detectors. See caption for details.

VI. DISCUSSION

An incomplete phase transition occurring in the late stages of inflation may generate observable gravitational waves, summarized in Fig. 4. Since the transition occurs over a long period of time, the resulting spectrum is very wide. The unique shape allows it to be detected by future experiments and to be distinguished from other scenarios, which typically predict a specific peak frequency. Furthermore, as we showed, the resulting spectrum depends on the details of the inflationary dynamics and a corresponding measurement would probe the slow-roll parameters at small scales.

The model studied in this paper may produce a significant amount of primordial black holes. Calculating the primordial black hole abundance requires a detailed study of the gravitational collapse which we leave for future work.

Acknowledgments. We thank Nadav Outmezguine for useful discussions. The work of TV is supported by the Israel Science Foundation (grant No. 1862/21), by the Binational Science Foundation (grant No. 2020220) and by the European Research Council (ERC) under the EU Horizon 2020 Programme (ERC-CoG-2015 - Proposal n. 682676 LDMThExp). MG is supported in part by Israel Science Foundation under Grant No. 1302/19. MG is also supported in part by the US-Israeli BSF grant 2018236

Appendix A: The 2-point correlation function of the tunneling time

Here we calculate the correlation function of $\delta t_{\bar{x}}$ defined in Eq. (15).

$$\langle \delta t_{\bar{x}} \delta t_{\bar{x}} \rangle = \langle t_{\bar{x}} t_{\bar{x}} \rangle - \langle t_{\bar{x}} \rangle^2$$ (A1)

as a function of $r = |\bar{x} - \bar{x}'|$. For simplicity, we will ignore the end of inflation for now, and calculate the correlation as if inflation goes on forever. In the next section, we will show how the calculation has to be modified in order to account for the end of inflation.

After deriving the detailed correlation function, we find a simple approximate expression valid under the assumption of slow tunneling rate, $\Gamma/V \ll H^4$. This simplification will come handy in the next section where we compute the effect of non-Gaussianity.

1. The correlation without ending inflation

To compute the correlation function, we first need to find the probability distribution for the decay of the false vacuum. For a general phase transition, the probability to find a given point in space in the false vacuum by the time $t$ is given by [47]

$$p(t_{\bar{x}} > t) = e^{-I(t)}$$ (A2)

with

$$I(t) = \frac{4\pi}{3} \int_{t_0}^{t} dt' \frac{\Gamma(t')V}{V} a^3(t')r^3(t, t').$$ (A3)

Here $I(t)$ is the ratio of the space volume inside the bubbles over the entire volume at time $t$, without accounting for the overlaps. The latter are automatically taken care of by the exponent in Eq. (A2). Note that by definition, Eq. (A2) is the complementary cumulative probability. In Eq. (A3), $r(t, t')$ is the comoving radius of a bubble that was created at time $t'$ and measured at time $t$, $\frac{\Gamma}{V}(t')$ is the tunneling rate per unit volume at time $t'$, and $t_0$ the time when the phase transition commences. We will take $t_0 = 0$ to simplify the expressions.

For the toy models studied in this work, $\Gamma/V$ is constant. We further assume that at formation, the bubble radius coincides with the Hubble radius and thus,

$$r(t, t') = (a_0 e^{Ht'} H)^{-1} \equiv r_H(t'),$$ (A4)

where $a_0$ is the scale factor at $t = 0$. We stress that this expression is exact for the case of the HM transition, but is only approximate for the Cink transition. In the latter case, we neglect the time it takes the bubble to grow to horizon-size. Plugging the radius back into Eq. (A3) gives the exponential decay

$$p(t_{\bar{x}} > t) = e^{-t/\tau},$$ (A5)

with $\tau$ being the mean lifetime of the false vacuum at any given point

$$\tau = \left[ \frac{4\pi}{3} \left( \frac{1}{H^3} \right) \left( \frac{\Gamma}{V} \right) \right]^{-1}.$$ (A6)

which agrees with a more direct calculation.

Now the second term in Eq. (A1) can be extracted by taking the mean of the probability at Eq. (A5):

$$\langle t_{\bar{x}} \rangle = \tau.$$ (A7)
The remaining part of this section is focused on computing the covariance $\langle t_{x} t_{x'} \rangle$. We use the law of total expectation

$$\langle t_{x} t_{x'} \rangle = \langle t_{x} E(t_{x'} | t_{x}) \rangle,$$

where $E(\cdot)$ is the conditional expectation value. Applying this to the exponential probability distribution described by Eq. (A7), gives

$$\langle t_{x} t_{x'} \rangle = \frac{1}{\tau} \int_0^\infty dt_x t_x e^{-t_x/\tau} E(t_{x'} | t_{x}) \, . \quad (A9)$$

With the above, we are left with computing the conditional expectation value, $E(t_{x'} | t_{x})$. This describes the expectation of $t_{x'}$ assuming we know $t_{x}$. Given the cumulative probability distribution in the domain of $(0, +\infty)$, the expectation is given by

$$E(t_{x'} | t_{x}) = \int_0^\infty dt \, p(t_{x'} > t | t_{x}) \, . \quad (A10)$$

To compute Eq. (A10), we separate the integral over $t_{x'}$ into two intervals, $t_{x'} \in [0, t_s)$ and $t_{x'} \in [t_s, \infty)$, where $t_s$ is the separation time, defined to be the moment when the distance between $x$ and $x'$ becomes larger than twice the Hubble radius, i.e. $r = 2r_H(t_s)$. One finds,

$$t_s = \frac{1}{H} \log \frac{2}{a_0 H r} \, . \quad (A11)$$

When the points are more than two radii apart, they can no longer be contained inside any single bubble, and thus $t_s$ represents the moment at which these two points became independent.

**a. first interval: $0 < t_x < t_s$**

Let us first consider the case where the false vacuum at $x$ decays before the two points are causally disconnected. In principle a bubble that contains $x'$ may form at any time $t$. To evaluate the integral in Eq. (A10), we further break it into two intervals, $0 \leq t < t_x$ and $t \geq t_x$. We begin with the first interval.

In the case where $t < t_x$, the bubble cannot contain $x$. We therefore have to modify Eq. (A3) in order exclude from $I$ the contribution of bubbles which form at $t < t_x$ and include $x$. The centers of all possible bubbles that could form at $t$ containing $x'$ form a ball of radius $r_H(t)$ centered at $x'$. Similarly, the centers of bubbles containing $x$ form a Hubble ball of the same radius centered at $x$. Therefore, if we randomly draw a bubble that contains $x'$, the probability this bubble doesn’t contain $x$ is given by the fraction of the Hubble ball centered at $x'$ that doesn’t overlap with the ball centered at $x$: $f_V(t) = \frac{V_H(t) - V_O(t)}{V_H(t)} = \frac{3r}{4r_H(t)} - \frac{r^3}{16r_H^3(t)} \, . \quad (A12)$

where the Hubble volume is $V_H = 4\pi r_H^3(t)/3$ and the overlapping volume is $V_O = \pi (4r_H(t) + r) (2r_H(t) - r)^2/12$. Now, excluding from $[A3]$ the bubbles that include $x$ means replacing $I(t)$ with

$$J(t) = \frac{4\pi}{3} \int_0^t dt' \frac{\Gamma}{V} a^3(t') r_H^3(t') f_V(t') = \frac{1}{\tau H} \left( \frac{3r}{4r_H(t')} - \frac{r^3}{48r_H^3(t')} \right)_{t'=0} \, . \quad (A13)$$

For $t < t_x$, the probability is thus given by,

$$p(t_{x'} > t | t_{x}) = e^{-J(t_x)} \, . \quad (A14)$$

We move on to the interval $t \geq t_x$. At $t = t_x$, we know a bubble forms around $x$. To compute the probability this bubble does not include $x'$ let us first map out the allowed region for the bubble center. To contain $x'$ the bubble center can be placed anywhere in a sphere that is centered at $x$ with a radius of $r_H(t_x)$. To avoid containing $x'$, the center of the bubble cannot be anywhere inside a sphere centered at $x'$ with radius $r_H(t_{x'})$. Therefore, if we randomly place a bubble such that it contains $x$, the probability it does not contain $x'$ is given by the non-overlapping fraction $f_V(t_{x'})$. As a result, the probability that $x'$ will remain in the false vacuum when the bubble around $x$ is formed, is given by

$$p(t_{x'} > t | t_{x}) = e^{-J(t_{x'})} f_V(t_{x'}) \, . \quad (A15)$$

We note that the discontinuity of $p(t_{x'} > t | t_{x})$ at $t = t_x$ (see Eqs. (A14) and (A15)), is due to the instantaneous formation of a bubble at $x$.

The known value of $t_x$ does not pose any constraints on bubble formation after $t_x$, so for $t > t_x$ the probability will decay exponentially with the same rate as in Eq. (A5):

$$p(t_{x'} > t | t_{x}) = e^{-J(t_x)} f_V(t_{x}) e^{-(t - t_x)/\tau} \, . \quad (A16)$$

Finally, putting Eq. (A14) and Eq. (A16) into Eq. (A10) we obtain the conditional expectation

$$E(t_{x'} | t_{x}) = \int_0^{t_x} e^{-J(t)} dt + \tau e^{-J(t_x)} f_V(t_{x}) \quad (A17)$$

**b. second interval: $t_x \geq t_s$**

This corresponds to the scenario where the false vacuum at $x$ decays after the two bubbles are causally disconnected. For $t < t_s$ the probability is given by Eq. (A14) as before. At $t_s$ the two points become separated and the decay becomes independent. Therefore, $t_x$ is irrelevant to the conditional probability. The cumulative distribution function is given by

$$p(t_{x'} > t | t_{x}) = \begin{cases} e^{-J(t_x)}, & t < t_s \\ e^{-J(t_x)} e^{-(t - t_x)/\tau}, & t \geq t_s \end{cases} \, . \quad (A18)$$
which means the conditional expectation is
\[ E(t' | t_e) = \int_0^{t_e} e^{-J(t)} \, dt + \tau e^{-J(t_e)}. \] (A19)

Since Eq. (A19) is independent of \( t_e \), it can be integrated analytically when plugged back into Eq. (A9).

Putting both intervals back into Eq. (A9) we reach the final expression for the two-point expectation value:
\[
\langle t_xt_x' \rangle = \frac{1}{\tau} \int_0^{t_e} dt \, t_x \, e^{-t_e/\tau} E(t_x | t_e) + \left[ \int_0^{t_e} e^{-J(t)} \, dt + e^{-J(t_e)} \right] e^{-t_e/\tau} (t_e + \tau)
\] (A20)

where in the first term we plug in Eq. (A17) and integrate numerically. This result is only valid for \( t_e > 0 \). If \( t_e \leq 0 \), the points were separated before the phase transition started, so they decay independently and \( \langle \delta t_x \delta t_x' \rangle = 0 \).

### 2. Adding the end time of inflation

Now, we have to repeat the above calculation, taking into account the end of inflation at \( t_e = t_{\text{reheating}} \), when reheating starts. As discussed in the main text, we assume that all of the volume of space which remained in the false vacuum will move to the true vacuum immediately at \( t_e \), which means the probability distribution Eq. (A5) has to be replaced with
\[
p(t_x > t) = \begin{cases} e^{-t/\tau}, & t < t_e \\ 0, & t \geq t_e \end{cases}
\] (A21)

and the corresponding expectation value Eq. (A7) is replaced with
\[
\langle t_x \rangle = \tau \left( 1 - e^{-t_e/\tau} \right).
\] (A22)

Note that having an end of inflation is similar to putting a regulator to \( \langle t_x \rangle \). Without \( t_e \) the expected transition time \( \tau \) diverges in the slow tunneling limit \( \Gamma/(VH^3) \rightarrow 0 \).

With the new probability distribution given in Eq. (A21), the expectation approaches \( t_e \) in the limit of slow tunneling.

The law of total expectation can be applied in the same manner as above, but now the integral in Eq. (A9) ends at \( t_e \):
\[
\langle t_xt_x' \rangle = \frac{1}{\tau} \int_0^{t_e} dt \, t_x \, e^{-t_e/\tau} E(t_x | t_e) + t_e \, e^{-t_e/\tau} E(t_e | t_e)
\] (A23)

where the second term accounts for the finite probability that \( \vec{x} \) does not tunnel until the end of inflation, \( P(t_x = t_e) = e^{-t_e/\tau} \).

The derivation of the conditional expectation is the same as in the previous section, except for the fact that the integrals end at \( t_e \) instead of \( \infty \). We are only interested in scales which exited the horizon before the end of inflation, which means \( t_e < t_e \). Under this assumption, we split the conditional expectation to two cases as above, \( t_e < t_s \) and \( t_e < t_e \leq t_e \). In the first case, Eq. (A14) and Eq. (A16) remain unchanged, but the integral over \( t \) ends at \( t_e \), which means Eq. (A17) has to be replaced with
\[
E(t_x | t_e) = \int_0^{t_e} e^{-J(t)} \, dt + \frac{1}{\tau} \left[ \int_0^{t_e} e^{-J(t)} \, dt + e^{-J(t_e)} \right] e^{-t_e/\tau} \left( 1 - e^{-t_e} / \tau \right).
\] (A24)

Similarly, Eq. (A19) has to be replaced with
\[
E(t_x | t_e) = \int_0^{t_e} e^{-J(t)} \, dt + \frac{1}{\tau} \left[ \int_0^{t_e} e^{-J(t)} \, dt + e^{-J(t_e)} \right] e^{-t_e/\tau} \left( 1 - e^{-t_e} / \tau \right).
\] (A25)

Eq. (A24) and Eq. (A25) can be plugged into Eq. (A23) and integrated numerically to get the required correlation \( \langle t_xt_x' \rangle \).

### 3. The small \( \Gamma/V \) limit

The above derivation allows the 2-point correlation to be calculated without assuming anything about \( \Gamma \). Here, we derive a simpler closed-form formula by assuming \( \Gamma/(VH^3) \ll t_e^{-1} \). This assumption makes the resulting power spectrum linear in \( \Gamma \), and the approximation will be used in the next section to show that non-Gaussianity can be ignored when calculating the GW spectrum.

Once again, we denote by \( t_s \) the moment at which the points \( \vec{x} \) and \( \vec{x}' \) become causally separated, and consider the case in which the two points we are looking at were separated before the end of inflation, \( t_e < t_s \). Since the rate of bubble nucleation is very small, the probability that more than a single bubble formed around any of the two points before the separation time \( t_s \), can be neglected. The correlation of \( \delta t_x \) can therefore be written as
\[
\langle \delta t_x \delta t_x' \rangle = E(\delta t_x \delta t_x' | \text{points before } t_s) P(\text{points before } t_s)
\]
\[
+ E(\delta t_x \delta t_x' | \text{bubble only } \vec{x} \text{ before } t_s) P(\text{bubble only } \vec{x} \text{ before } t_s)
\]
\[
+ E(\delta t_x \delta t_x' | \text{no bubble} \text{ before } t_s) P(\text{no bubble} \text{ before } t_s)
\] (A26)

We now show that in the \( \Gamma/V \ll H^4 \) limit, the first term in Eq. (A26) becomes linear in \( \Gamma/V \), while the other three terms are of order \( (\Gamma/V)^2 \).
Consider first the last term. Since the points become independent after \(t_s\), the expectation value in the that term can be written as

\[
E(\delta t_\bar{x} | \text{no bubble before } t_s) = E(\delta t_{\bar{x}} | t_s > t_s)^2. \tag{A27}
\]

This expectation can be calculated by integrating the probability distribution given in Eq. (A21) from \(t_s\) to \(t_e\),

\[
E(\delta t_{\bar{x}} | t_s > t_s) = t_s + \tau \, e^{-t_s/\tau} \left(1 - e^{t_s/\tau}\right)
= t_s \left(1 - \frac{t_s}{\tau} + O\left((t_s/\tau)^2\right)\right). \tag{A28}
\]

Together with Eq. (A6), this result shows the last term in Eq. (A20) is at least of order \((\Gamma/V)^2\). Similarly, the expectation value in the second and third terms can be factorized, e.g.,

\[
E(\delta t_{\bar{x}} | \text{bubble with } \bar{x} \text{ only before } t_s) = E(\delta t_{\bar{x}} | \text{bubble with } \bar{x} \text{ only before } t_s)E(\delta t_{\bar{x}} | t_s > t_s),
\]

and Eq. (A28) implies that the second factor is linear in \(\Gamma/V\). Since the first factor is regular in \(\Gamma/V\), the whole expression is at least linear in \(\Gamma/V\). Given that the probabilities of forming a bubble are also linear in \(\Gamma/V\), the second and third terms of Eq. (A20) must be at least of order \((\Gamma/V)^2\).

Finally, we are ready to calculate the dominant term: the contribution to the correlation from a single bubble forming around both points. This contribution can be rewritten as

\[
\int_{t_0}^{t_s} E(\delta t_{\bar{x}} | \text{bubble with } \bar{x} \text{ only before } t_s) dP(\text{bubble with } \bar{x} \text{ only at } t_s)\tag{A29}
= \int_{t_0}^{t_s} (t - \langle t_{\bar{x}} \rangle)^2 \frac{\Gamma}{V} V_{\text{overlap}}(t) dt
\]

where \(dP \propto \Gamma dt\) is the probability of forming a bubble in the infinitesimal time interval \(dt\), and \(V_{\text{overlap}}(t)\) is the physical volume of overlap between two Hubble spheres, each centered at \(\bar{x}\) and \(\bar{x}'\). The expectation value \(\langle t_{\bar{x}} \rangle\) is given by Eq. (A22), but since we are calculating only the first order in \(\Gamma\) we can take \(\langle t_{\bar{x}} \rangle \simeq t_s\). From this point, it is straightforward to write \(V_{\text{overlap}}\) explicitly and integrate Eq. (A30) directly. Instead, we introduce a trick that will be useful later. The overlap volume between two spheres can be written as an integral over a \(\delta\) function:

\[
V_{\text{overlap}}(t) = a^3(t) \int_{r_H} d^3 y_1 \int_{r_H} d^3 y_2 \delta(\bar{y}_1 - \bar{y}_2 + \bar{r})
\]

where both integrals are over a sphere of radius \(r_H\) centered at the origin, and \(\bar{r} = \bar{x}_1 - \bar{x}_2\) is the separation between the centers of the overlapping spheres. This expression automatically vanishes for \( t > t_s \), so the upper limit of the integral in Eq. (A30) can be replaced with \(\infty\). After doing so, the only dependence of Eq. (A30) on \(\bar{x}_1\) and \(\bar{x}_2\) is through \(\bar{r}\) in the \(\delta\) function. That makes taking the Fourier transform trivial,

\[
\langle \delta t_{E_1} \delta t_{E_2} \rangle = \delta(\bar{k}_1 + \bar{k}_2) \left(\frac{12\pi}{k^2}\right) \times \frac{1}{\tau H} \int_0^\infty dt \frac{\phi}{a(t)} (t - \langle t_{\bar{x}} \rangle)^2 j_2^2(kr_H)
\]

where \(j_1(x) = \frac{1}{2} \left(\sin x - x \cos x\right)\) is the first spherical Bessel function. Using \(a(t) = a_0 e^{H t}\) and \(r_H(t) = (H a(t))^{-1}\) the integral can be evaluated numerically, and by comparing with Eq. (15) and Eq. (16) we can extract the spectrum \(P_R\). This result is in agreement with the full calculation of the previous section in the \(\Gamma/V \to 0\) limit.

**Appendix B: Alleviating the assumption of constant \(\phi\) and \(H\)**

In deriving Eq. (15) from Eq. (14), we assumed \(H/\phi^2\) to be constant, which resulted with \(\mathcal{R}\) being a linear function of \(t_{\bar{x}}\). This result meant we only have to calculate correlations of the tunneling times \(t_{\bar{x}}\), and then convert the final results to correlations of \(\mathcal{R}\) with the use of Eq. (15). We will now discuss how the spectrum can be approximated without this simplification.

\(\mathcal{R}\) is obtained by integrating Eq. (14) over \(t\), with an initial condition of \(\mathcal{R} = 0\). The theta functions in Eq. (9) ensure the integrand vanishes unless \(t\) is between \(\langle t_{\bar{x}} \rangle\) and \(t_{\bar{x}}\), allowing us to write the integral as

\[
\mathcal{R}(t_{\bar{x}}) = -\Delta V_{\text{PT}} \int_{\langle t_{\bar{x}} \rangle}^{t_{\bar{x}}} \mathcal{R}(t) \frac{H}{\phi^2} dt. \tag{B1}
\]

Eq. (15) can be recovered from this more general result by assuming the integrand is constant.

As we have shown in Section A 3, in the limit of small \(\Gamma/V\) we only have to consider the contribution of a single bubble to the correlation function, given by Eq. (A30). Assuming \(\mathcal{R}(t_{\bar{x}})\) is a well-behaved function, the approximation is still valid, but we have to modify Eq. (A30) to calculate the correlation of \(\mathcal{R}\) directly instead of using \(\delta t_{\bar{x}}\):

\[
\int_0^{t_s} E(\mathcal{R}_{\bar{x}} | \text{bubble with both points at } t_s) dP(\text{bubble with both points at } t_s) \tag{B2}
= \int_0^{t_s} \mathcal{R}(t) \frac{\Gamma}{V} V_{\text{overlap}}(t) dt.
\]

This result can be used to calculate the scalar spectrum for a general inflationary background, but only in the limit of small \(\Gamma/V\).

Let us now consider a concrete example to demonstrate how the spectral shape can be affected by the time dependence of \(H/\phi^2\). In this example, \(H/\phi^2\) starts at some value \(H/\phi^0\), and remains constant until \(t_{\text{drop}}\), when it instantly...
changes to a new value smaller from the original one by a factor of 10. In that scenario, the integrand of Eq. \[B1\] can be written as
\[
\frac{H}{\dot{\phi}^2} = \frac{H}{\dot{\phi}_0^2} \left[ \theta(t_{\text{drop}} - t) + \frac{1}{10} \theta(t - t_{\text{drop}}) \right], \tag{B3}
\]
and after integrating it, we get
\[
\mathcal{R}(t_x) = -\frac{H \Delta V_{\text{PT}}}{\dot{\phi}_0^2} \left\{ \begin{array}{ll}
t_x - t_{\text{drop}} + \frac{1}{10}(t_{\text{drop}} - \langle t_x \rangle), & t_x < t_{\text{drop}} \\
\frac{1}{10}(t_x - \langle t_x \rangle), & t_x \geq t_{\text{drop}} \end{array} \right. \tag{B4}
\]
The colored lines in Fig. \[3\] were calculated by plugging this result into \[B2\].

**Appendix C: Non-Gaussianity and the induced gravitational waves**

Here, we very briefly review the equations necessary for calculating the secondary gravitational waves, taken from \[43\] and \[55\]. We then use the methods of the previous section to calculate the four-point correlation function and show that Eq. \[15\] can be applied to our model, ignoring non-Gaussianity.

1. **The induced gravitational waves**

To calculate the gravitational wave spectrum, we need the four-point correlation function of \(\mathcal{R}\). In the general, non-Gaussian case the correlation can be split into disconnected and connected components \[55\]:
\[
\left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_4} \right\rangle = \left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_4} \right\rangle_d + \left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_4} \right\rangle_c. \tag{C1}
\]
The disconnected part satisfies Wick’s theorem,
\[
\left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_4} \right\rangle_d = \left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_2} \right\rangle \left\langle \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_4} \right\rangle + \left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_4} \right\rangle + \left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_4} \right\rangle, \tag{C2}
\]
and the connected part defines the connected trispectrum \(\mathcal{T}\),
\[
\left\langle \mathcal{R}_{\vec{k}_1} \mathcal{R}_{\vec{k}_2} \mathcal{R}_{\vec{k}_3} \mathcal{R}_{\vec{k}_4} \right\rangle_c = \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \mathcal{T}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4). \tag{C3}
\]
In the Gaussian case, the connected part vanishes because of Wick’s theorem and the four-point correlation function is fully described by the scalar spectrum defined by Eq. \[16\].

The GW spectrum induced by the disconnected part is given by Eq. \[18\], where \(I(v, u, k\eta)\) is a Green’s function integral that was calculated analytically in \[43\]. Since we are only interested in the energy density of gravitational waves today, we only need the late-time oscillation average of \(\mathcal{I}\), which, during radiation domination, is given by
\[
\mathcal{I}(v, u, k\eta \rightarrow \infty) = \frac{1}{2} \left( \frac{3(u^2 + v^2 - 3)}{4u^3v^3k^3} \right)^2 \\
\left[ (4uv + (u^2 + v^2 - 3) \log \frac{3 - (u + v)^2}{3 - (u - v)^2} \right]^2 \\
+ \pi^2(u^2 + v^2 - 3)^2 \Theta(v + u - \sqrt{3}) \right]. \tag{C4}
\]
The connected contribution to the GW spectrum is\(^2\)
\[
\mathcal{P}_{\lambda} (k) = \frac{k^3}{\pi^3} \int d^3q_1 d^3q_2 Q_\lambda (\vec{k}, \vec{q}_1) I \left( |\vec{k} - \vec{q}_1|, q_1, \eta \right) I \left( |\vec{k} - \vec{q}_2|, q_2, \eta \right) \times \mathcal{T} \left( \vec{q}_1, \vec{k} - \vec{q}_1, -\vec{q}_2, \vec{q}_2 - \vec{k} \right), \tag{C5}
\]
where the \(Q_\lambda\) are polarization factors, given by
\[
Q_\lambda (\vec{k}, \vec{q}) \equiv c_{ij}^\lambda (\vec{k}) q_i q_j, \tag{C6}
\]
where \(c_{ij}^\lambda\) with \(\lambda = +, \times\) is a basis of traceless transverse polarization tensors. \(I\) in Eq. \[C5\] is related to \(\mathcal{I}\) used above through a change of variables, \(I(v, u, x) \equiv k^2 I(vk, uk, x/k)\). Taking \(\vec{k}\) to be in the \(z\) direction and writing \(\vec{q}\) in spherical coordinates, \((q, \theta, \phi)\), the polarization factors can be written as
\[
Q_\lambda (\vec{k}, \vec{q}) = \frac{q^2 \sin^2(\theta)}{\sqrt{2}} \times \left\{ \begin{array}{l}
\cos(2\phi) \quad \lambda = + \\
\sin(2\phi) \quad \lambda = \times \end{array} \right. \tag{C7}
\]
Because of the polarization factors, the integral in Eq. \[C5\] does not vanish only when the connected trispectrum has a non-trivial dependence on the azimuthal angles of \(\vec{q}_1\) and \(\vec{q}_2\). In the next section, we will show the connected trispectrum in our model does not depend on these angles. This is why the known result, Eq. \[18\], can be used on our model as if it was Gaussian.

2. **The four-point correlation function**

We use the same method shown in App. \[A3\] above for the two point correlation, and write an expression analogous to Eq. \[A20\] for the 4-point correlation, assuming

\(^2\)Eq. \[C5\] has a different coefficient compared to equation (2.29) in \[55\] because here we use the dimensionless spectrum.
at most a single bubble (which is the leading contribution when expanding in small $\Gamma/V$),

$$\langle \delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l \rangle \leq 1 \text{--bubble} =$$

$$E(\delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l) \text{ bubble with all pts before } t_s) P(\text{bubble with all pts before } t_s) +$$

$$\sum E(\delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l) P(\text{bubble with only 3 pts before } t_s) P(\text{bubble with only 3 pts before } t_s) +$$

$$\sum E(\delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l) P(\text{bubble with only 2 pts before } t_s) P(\text{bubble with only 2 pts before } t_s) +$$

$$\sum E(\delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l) P(\text{bubble with only 1 pt before } t_s) P(\text{bubble with only 1 pt before } t_s) +$$

$$E(\delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l) \text{ bubble with no bubble before } t_s) P(\text{no bubble before } t_s),$$

(C8)

where $t_s$ is now defined to be the moment after which no pair of two points is contained in a common Hubble sphere. The sums are over all possible choices of different points to be included in the bubble. As before, in the $\Gamma/V \to 0$ limit the first term is linear in $\Gamma/V$, while the others are of order $(\Gamma/V)^2$ and above. This is because the probability of forming a single bubble is linear in $\Gamma/V$, and the expectation values that multiply them have a factor of $\Gamma/V$ for every $\delta \vec{x}$ that doesn’t tunnel before $t_s$.

The first term is given by an expression very similar to Eq. (A30),

$$\int_0^{t_s} (\langle \delta \vec{x} \rangle) \frac{\Gamma}{V} V_{\text{overlap}}(t) dt, \quad \text{(C9)}$$

but this time $V_{\text{overlap}}$ is the overlap volume between four Hubble spheres:

$$V_{\text{overlap}}(t) = \alpha^3(t) \int_{r_H} d^3 y_1 \int_{r_H} d^3 y_2 \int_{r_H} d^3 y_3 \int_{r_H} d^3 y_4 \delta(\vec{y}_1 - \vec{y}_2 + \vec{r}_1) \delta(\vec{y}_1 - \vec{y}_3 + \vec{r}_2) \delta(\vec{y}_2 - \vec{y}_3 + \vec{r}_3) \delta(\vec{y}_3 - \vec{y}_4 + \vec{r}_4) \quad \text{(C10)}$$

where $\vec{r}_i = \vec{x}_i - \vec{x}_1$ are the three independent separations between the $\vec{x}_i$'s. In a very similar manner to the above, we Fourier transform the volume to get the contribution to the trispectrum, analogous to Eq. (A32). Omitting numerical coefficients, the result is

$$T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \propto \int_0^{t_s} dt (\langle \delta \vec{x} \rangle)^2 \alpha^3(t) \int_{r_H} d^3 y_1 e^{i \vec{k}_1 \cdot \vec{y}_1} \int_{r_H} d^3 y_2 e^{i \vec{k}_2 \cdot \vec{y}_2} \int_{r_H} d^3 y_3 e^{i \vec{k}_3 \cdot \vec{y}_3} \int_{r_H} d^3 y_4 e^{i \vec{k}_4 \cdot \vec{y}_4}. \quad \text{(C11)}$$

This expression depends only on the magnitudes of the $\vec{k}$'s, not their directions: $T(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = T(k_1, k_2, k_3, k_4)$. As mentioned above, this trispectrum gives zero when plugged into Eq. (C5), because of the integral over the azimuthal angle in the polarization factors. This result has a physical interpretation: the leading contribution we have calculated corresponds to the inhomogeneity created by the presence of a single spherical bubble. A spherically symmetric inhomogeneity cannot emit gravitational waves. Contributions to the tunneling due to the formation of non-spherical bubbles may change this conclusion and consequently strengthen the predicted signal. The study of such effects goes beyond the scope of this paper and is left for future work.

Since the leading term of order $\Gamma/V$ in Eq. (C8) does not contribute, we have to calculate the $(\Gamma/V)^2$ terms. In Eq. (C8), only the second term is of that order. However, taking the $(\Gamma/V)^2$ order means we have to add terms with the probabilities that two bubbles formed before $t_s$, which were not present in Eq. (C8). Since we are only interested in terms of order $(\Gamma/V)^2$, the two bubbles have to cover all four points. We split this scenario into three cases:

1. One of the bubbles includes all 4 points.
2. One bubble includes a single point, and the other bubble includes the remaining three points.
3. Each bubble contains two points.

We neglect the chance of two bubbles forming with a distance smaller than the Hubble radius, $H^{-1}$, which means the three cases above are distinct. This is justified since the mean distance between bubbles is of order $(\Gamma/V)^{-1/4} \gg H^{-1}$ for an incomplete PT.

The first case has the same symmetry as in the single-bubble calculation above, and therefore gravitational waves are not produced. In the second case, after adding the second term from Eq. (C8), one of the points is independent of the other three, and since $\langle \delta \vec{x} \rangle = 0$, the contribution to the correlation function is zero. We are therefore only left with the last case, where each of the two bubbles contains two points. The corresponding contribution to the four-point correlation function is then given by a sum over the three possible ways of distributing the four points into two bubbles,

$$\langle \delta \vec{x}_i, \delta \vec{x}_j, \delta \vec{x}_k, \delta \vec{x}_l \rangle =$$

$$\left( \frac{\Gamma}{V} \right)^2 \int_0^{t_s} (\langle \delta \vec{x} \rangle)^2 V_{1,2}(t) dt \int_0^{t_s} (\langle \delta \vec{x} \rangle)^2 V_{3,4}(t) dt +$$

$$\left( \frac{\Gamma}{V} \right)^2 \int_0^{t_s} (\langle \delta \vec{x} \rangle)^2 V_{1,3}(t) dt \int_0^{t_s} (\langle \delta \vec{x} \rangle)^2 V_{2,4}(t) dt +$$

$$\left( \frac{\Gamma}{V} \right)^2 \int_0^{t_s} (\langle \delta \vec{x} \rangle)^2 V_{1,4}(t) dt \int_0^{t_s} (\langle \delta \vec{x} \rangle)^2 V_{2,3}(t) dt \quad \text{(C12)}$$

where $V_{i,j}$ is the overlap physical volume of two Hubble spheres centered around $\vec{x}_i$ and $\vec{x}_j$. Each term is in fact a product of two 2-point correlation functions as given by Eq. (A30), so the above correlation satisfies Eq. (C2). This is the promised result: the leading non-vanishing contribution has only a disconnected component, so we can use Eq. (C8) safely.
