Fractal AC circuits and propagating waves on fractals

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Abstract

We extend Feynman’s analysis of the infinite ladder AC circuit to fractal AC circuits. We show that the characteristic impedances can have positive real part even though all the individual impedances inside the circuit are purely imaginary. This provides a physical setting for analyzing wave propagation of signals on fractals, by analogy with the Telegrapher’s Equation, and generalizes the real resistance metric on a fractal, which provides a measure of distance on a fractal, to complex impedances.

1 Introduction

It is well known that fractal-like structures have novel spectral and response properties, both classically and quantum mechanically. However, the concept of wave propagation on fractals has been a long-standing puzzle, since even the notion of velocity is unclear on such singular geometrical structures. Here we propose a new approach to this problem, by considering a fractal circuit comprised of inductors and capacitors. Part of our motivation comes from the familiar textbook example of the infinite ladder circuit, described for example in Feynman’s lectures [1]: see Figure 1. This is an important example relating AC circuits to wave propagation, as these ladders are directly related to the wave equation known as the Telegrapher’s Equation (see, for example, [2]). In more general terms, the introduction of complex impedances in fractal circuits leads to a physical picture of energy flow in a fractal. In fractal circuits consisting only of resistors, the effective resistance, derived from Kirchhoff’s laws, gives a concrete physical realization of the static distribution of energy in the circuit. This idea is a key element in the theory of energy forms on fractals, and provides a definition of a ‘metric’ on a fractal, leading also to a bridge between spectral properties and

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geometry. Algebraically, this analysis can be extended to include inductance and capacitance, using again the Kirchhoff rules combined with the same geometric properties of the fractal. Surprisingly, the results are quite different, and describe situations in which signals propagate through the fractal in a manner very different from conventional circuits. This simple idea should have profound consequences for the description of time-dependent phenomena on fractals, as well as leading to AC circuits with novel physical transport properties.

![Figure 1: The Feynman infinite ladder LC circuit](image)

The conventional infinite ladder circuit is shown in Figure 1. It is made of infinite sequences of inductances $L$ and capacitors $C$, and iterating the circuit elements leads to the characteristic complex impedance at frequency $\omega$:

\[
Z = \frac{i}{2C} \left( \omega LC + \sqrt{\omega^2 L^2 C^2 - 4LC} \right)
\]

if $\omega^2 LC > 4$, but

\[
Z = \frac{1}{2C} \left( i\omega LC + \sqrt{4LC - \omega^2 L^2 C^2} \right)
\]

if $\omega^2 LC < 4$. Note that in the latter case the characteristic impedance has a positive real part even though all elements in the circuit have purely imaginary impedances. This circuit behaves as a low pass filter because the propagation factor $\alpha$ has $|\alpha| < 1$ for $\omega^2 LC > 4$, and $|\alpha| = 1$ for $\omega^2 LC < 4$. In this Letter we show how to extend Feynman’s analysis to fractal networks (see [3, 4] for some technical details concerning taking the corresponding limits and choosing the branch of the complex square root).

Our analysis, although simple in nature, relies on the ideas of some recent papers on physics on fractals [5, 6, 7, 8, 9] which are motivated by mesoscopic physics, [10, 11], some recent connections between gravity and fractals [12, 13], and early physics papers on fractals [14, 15, 16, 17, 18] which highlighted the unusual physical properties of fractal structures. For physics applications it is important to note that even though some of these results have been proven mathematically for infinitely-iterated mathematical fractals, many of the unusual physical features are visible even for physical finitely-iterated structures which technically speaking are not mathematical fractals. Energy measures in related real circuits are analyzed in [19, 20]. Our analysis of the Hanoi-type graphs (discussed below) is motivated by related analytic [21, 22] and algebraic [23, 24] features.

Furthermore, the Hanoi-type graphs play an important role in [25, 26], which is closely related to the physical renormalization group discussion in [27]. In more general terms, our work relies on the understanding of energy in fractal resistance networks, as developed in [28, 29, 30, 31, 32, 33, 34, 35, 36, 37] and related spectral analysis [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50].

The mathematical literature implies (see [4]) that any physically relevant self-similar AC circuit on the Sierpinski gasket is a constant complex multiple of a purely real self-similar circuit. This simple fact has been an obstacle to progress in this direction. Therefore one needs to modify the standard construction in order to obtain more interesting properties. We present the most representative examples of such modifications in the following sections.
2 Fractal modified Sierpinski circuits

Our first non-trivial fractal AC circuit is constructed by a substitution procedure, the first two steps of which are shown in Figure 2. In each case the upward-pointing triangles with dark outlines are replaced by the circuit in the central diagram. This takes the triangle at the left to the central circuit, and the central circuit to that at the right. The substitutions are repeated infinitely many times to obtain the final circuit, which is made of purely imaginary impedances \(i \omega L\) and \(1/(i \omega C)\). Our main idea is that the modified Sierpinski gasket (SG) circuit contains 3 copies of itself, which allows for the fractal version of Feynman’s analysis.

![Figure 2: A fractal SG-type circuit construction.](image)

If we denote the impedance of all sides of the original triangular circuit at the left in Figure 2 by \(Z\) and set it equal to the corresponding impedance of the limiting circuit then we find

\[
Z = \frac{1}{10 \omega C} \left( 2i \omega^2 LC + 9i + \sqrt{144 \omega^2 LC - 4(\omega^2 LC)^2 - 81} \right)
\] (2.1)

provided

\[
9(4 - \sqrt{15}) < 2 \omega^2 LC < 9(4 + \sqrt{15}).
\] (2.2)

In particular, this circuit is a filter for this frequency range. Outside the range (2.2) we have

\[
Z = \frac{i}{10 \omega C} \left( 2 \omega^2 LC + 9 - \sqrt{4(\omega^2 LC)^2 + 81 - 144 \omega^2 LC} \right) \quad \text{if} \quad 9(4 - \sqrt{15}) > 2 \omega^2 LC
\] (2.3)

\[
Z = \frac{i}{10 \omega C} \left( 2 \omega^2 LC + 9 + \sqrt{4(\omega^2 LC)^2 + 81 - 144 \omega^2 LC} \right) \quad \text{if} \quad 9(4 + \sqrt{15}) < 2 \omega^2 LC
\] (2.4)

and so \(Z\) is a purely imaginary impedance under these conditions. Note that the signs in front of the square root can be justified by considering the cases \(\omega \to 0\) and \(\omega \to \infty\).

To obtain these results, observe that the Sierpinski gasket has the resistance scaling factor 5/3. This remains true for complex impedances because it depends only on the symmetries in the geometric structure of the gasket. Therefore the three solid triangles of impedances \(Z\) are equivalent to one triangle with side impedance \(5Z/3\). We can then do a delta-Y transform to obtain an Y-shaped circuit where each leg has impedance \(5Z/9 + 1/(i \omega C)\), followed by a Y-delta transform to obtain

\[
\frac{1}{Z} = \frac{1}{i \omega L} + \frac{1}{5Z/3 + 3/(i \omega C)}
\]

which implies (2.1).

An interesting additional fact proved in [4] is that the characteristic impedances of the sequence of finite approximations of this infinite Sierpinski circuit fail to converge to
the impedance $Z$ in [2.1] but that introducing a small positive resistance $\epsilon$ in series with each of the capacitors and inductors gives a sequence of scale $N$ approximating circuits for which the impedances $Z_{n,\epsilon}$ converge as $N \to \infty$. Moreover for the regularized limit $\lim_{\epsilon \to 0^+} \lim_{N \to \infty} Z_{N,\epsilon} = Z$.

Figure 3: The infinite ladder fractafold, [51, 42, 43].

For a physical configuration analogous to a two-sided AC ladder circuit, we can consider the infinite ladder fractafold shown in Figure 3, the (static) spectral properties of which have been studied in [51, 42, 43]. Replacing the Sierpinski elements by the AC fractal SG-type circuits constructed by the process in Figure 2, we can construct an infinite fractafold ladder with transmission properties very different from those of the ladders in Figure 1.

3 Fractal Hanoi circuits

Another class of fractal AC circuits can be constructed as follows. We begin with a bilaterally symmetric Y-shaped circuit, defined by the vertical impedance $Z_v$, and left and right impedances $Z_{\ell} = Z_r$, to obtain a weakly self-similar AC fractal circuit such that this original circuit is equivalent to the Hanoi-type circuit shown in Figure 4. The idea is, again, that the Hanoi circuit contains rescaled copies on the original Y circuit, which allows for the fractal analysis to proceed. The main difference is that when we go from scale to scale we reduce all impedances by a factor $r$. Thus at the $n$th iteration of the construction, each of the $3^n$ Y-shaped pieces have the vertical impedance $r^n Z_v$ and left and right impedances $r^n Z_{\ell}$. The inductances and capacitors are $r^{k-1} L$ and $r^{1-k} C$, where $k$ is the scale when this particular element first appeared in the construction. We can compute effective impedance top-to-bottom and left-to-right to obtain the following two equations:

\[
Z_v + Z_{\ell}/2 = rZ_v + (rZ_v + 2rZ_{\ell} + 1/(i\omega C))/2
\]  

(3.1)
and
\[ 2Z_\ell = 2rZ_\ell + \left( \frac{1}{2rZ_\ell + i\omega L} + \frac{1}{2rZ_v + 2rZ_\ell + 2/(i\omega C)} \right)^{-1}. \] (3.2)

From (3.1) we obtain
\[ (2 - 3r)Z_v + (1 - 2r)Z_\ell = \frac{1}{i\omega C} \] (3.3)
which implies the following special case:

if \( r = \frac{1}{2} \) then \( Z_v = \frac{2}{i\omega C} \) and \( Z_\ell = Z_n = 2\sqrt{L/C}. \) (3.4)

In this case (3.2) can be simplified to \( \frac{1}{Z_\ell} = \frac{1}{Z_\ell + i\omega L} + \frac{1}{Z_\ell + 4/(i\omega C)}. \) Equation 3.4 is remarkable not only because it has a purely real impedance for an infinite circuit of imaginary impedances, but also this real impedance is independent of \( \omega. \) Similar behavior in a different simpler circuit was obtained numerically in [52].

In general, we have characteristic impedances with positive real part when \( 0 < r < \frac{3}{5} \) and

\[ \gamma(r) - \sqrt{[\gamma(r)]^2 - 1} < 2LC\omega^2 < \gamma(r) + \sqrt{[\gamma(r)]^2 - 1}. \] (3.5)

where
\[ \gamma(r) = 1 + \frac{r(3 - 5r)}{(2r - 1)^2}. \] (3.6)

Further details are given in [4], including the determination of a filter condition from the quadratic equation

\[ r(5r - 3)i\omega C Z_\ell^2 + (2r - 1)(2 - \omega^2 LC)Z_\ell + i\omega L = 0. \] (3.7)

In particular it follows that if \( r = \frac{3}{5} \) then there are formal but unphysical solutions \( Z_v = \frac{10}{i\omega C(2 - \omega^2 LC)}, \) \( Z_\ell = Z_n = \frac{-5i\omega L}{2 - \omega^2 LC}. \)

## 4 Conclusions

We have shown how to construct nontrivial AC circuits with fractal geometrical structure which have novel transmission properties. This provides a novel experimental and theoretical framework for investigating the propagation of waves on fractal structures. It also generalizes the concept of a “spatial” resistance-metric to a “space-time” setting.

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