In non-attractor single-field inflation models producing a scale-invariant power spectrum, the curvature perturbation on super-horizon scales grows as $R \propto a^3$. This is so far the only known class of self-consistent single-field models with a Bunch-Davies initial state that can produce a large squeezed-limit bispectrum violating Maldacena’s consistency relation. Given the importance of this result, we calculate the bispectrum with three different methods: using quantum field theory calculations in two different gauges, and classical calculations (the $\delta N$ formalism). All the results agree, giving the local-form bispectrum parameter of $f^\text{local}_{NL} = 5(1 + c_s^2)/(4c_s^2)$. This result is valid for arbitrary values of the speed of sound parameter, $c_s$, for a particular non-attractor model we consider in this paper.

PACS numbers:

I. INTRODUCTION

Recent high-precision measurements of fluctuations in the cosmic microwave background (CMB) from WMAP \cite{1, 2} and Planck \cite{3, 4} strongly support inflation \cite{5, 4} as the leading theory of the early universe and structure formation. The simplest models of inflation are based on a single scalar field slowly rolling down on a flat potential. These models predict adiabatic, almost scale-invariant, and almost Gaussian primordial fluctuations in the CMB \cite{10, 14}, in agreement with all the data we have today.

Non-Gaussianity has emerged as a powerful observational tool to discriminate between different inflationary models \cite{13, 18} for reviews. While the Planck data show no evidence for non-Gaussianity, the current limits \cite{10} are yet to reach the levels expected from the simple single-field models. \cite{1} Therefore, there is still significant room for non-Gaussianity to be found, with the amplitude much greater than that predicted by the simple models.

In particular, the so-called squeezed-limit of the bispectrum of primordial curvature perturbations, $R$, plays a special role. The bispectrum is defined by $\langle R_{k_1} R_{k_2} R_{k_3} \rangle = (2\pi)^3 \delta^3(\sum k_i) B_R(k_1, k_2, k_3)$, and the squeezed limit is the limit in which one of the wavenumbers is much smaller than the other two, i.e., $k_3 \ll k_1 \approx k_2$. One can show that all of the simple single-field models \cite{1} satisfy Maldacena’s consistency relation \cite{21} given by

$$B_R(k_1, k_2, k_3) \rightarrow (1 - n_s) \frac{(2\pi)^4}{4k_1^3 k_3^3} \delta^3(k_1) \delta^3(k_2) \delta^3(k_3),$$

for $k_3 \ll k_1 \approx k_2$. Here, the curvature-perturbation power spectrum per logarithmic intervals in momentum space, $P_R(k) \propto k^{n_s - 1}$, is defined by $\langle R_k R_{k'} \rangle = (2\pi)^3 \delta^3(k + k') \frac{2\pi^2}{k^3} P_R(k)$. A convenient quantity to express the magnitude of the bispectrum in the squeezed limit is the (local form) $f^\text{local}_{NL}$ parameter defined by \cite{20}

$$\frac{6}{5} f^\text{local}_{NL} = \frac{B_R(k_1, k_2, k_3)}{\delta^3(k_1) \delta^3(k_2) \delta^3(k_3) + 2 \text{ perm.}},$$

which approaches $\frac{6}{5} f^\text{local}_{NL} \rightarrow \frac{1}{2} (1 - n_s)$ for $k_3 \ll k_1 \approx k_2$.

---

\footnote{1 By “simple single-field models,” we refer to single-field models with the canonical kinetic term and a Bunch-Davies initial state, which have approached attractor solutions. Namely, one of the two solutions of the curvature perturbation on super-horizon scales is a constant, and the other is a decaying solution. In these conditions, the condition of the canonical kinetic term can be generalized to non-canonical terms without changing the consistency condition.}
Usually, single-field inflationary models predict an almost scale-invariant spectrum, i.e., $1 - n_s = \mathcal{O}(\epsilon, \eta)$, where $\epsilon$ and $\eta$ are the slow-roll parameters and they are of order $\mathcal{O}(10^{-2})$ or smaller. As a result, Maldacena’s analysis [21] shows that all of the simple single-field models give $f_{NL}^{\text{local}} = \frac{5}{12}(1 - n_s) = \mathcal{O}(10^{-2})$. An intuitive way to understand Maldacena’s consistency relation is to note that the large-scale mode, $k_3$, leaving the horizon long before the small-scale modes, $k_1$ and $k_2$, provides a constant re-scaling of the background scale factor (hence the comoving coordinates) for the small-scale modes $k_2$. 

Until recently, the only known class of single-field inflation models which violate Maldacena’s consistency relation (for finite values of $k_3$) were the models with non-Bunch-Davies initial states [17, 24–31] (also see [32–34] for earlier work studying the effects of non-Bunch-Davies initial states).

However, Refs. [35–39] find that single-field inflation models containing a non-attractor phase at the initial stage of inflation can yield $f_{NL}^{\text{local}}$ that violates the consistency relation, without invoking non-Bunch-Davies initial states. (Also see [10] earlier work on non-attractor inflation models.) In conventional models of single-field inflation, one of the two solutions of the curvature perturbation on super-horizon scales remains constant, and the other solution decays. On the other hand, in non-attractor models of inflation, what-would-be a decaying mode of the curvature perturbation in conventional models of single-field inflation grows and dominates over the constant mode during the non-attractor phase of inflation. This time evolution of the curvature perturbation violates the consistency relation, as one can not simply absorb the effects of the long-wavelength modes $k_{\lambda}$, into a constant re-scaling of the background scale factor for the short-wavelength modes. This property thus calls for explicit calculations.

In the non-attractor models explored so far, one of the slow-roll parameters decays as $\epsilon = \dot{\phi}^2/(2H^2) \propto 1/a^6$ and the curvature perturbation on super-horizon scales grows as $\mathcal{R} \propto a^3$ during the non-attractor phase. The simplest example is given by a scalar field with the canonical kinetic term rolling on a constant potential [32]. The kinetic energy of the scalar field is thus given by the initial velocity, which decays as $\dot{\phi}^2 \propto a^{-6}$. This gives $\epsilon \propto 1/a^6$, and $\mathcal{R} = -H/\sqrt{2\epsilon} \propto a^3$. Using both the quantum field theory calculation in the comoving gauge [21] and the $\delta N$ formalism [14, 42, 46]. Ref. [35] finds $f_{NL}^{\text{local}} = 5/2 \gg 1 - n_s$, violating the consistency relation (also see [37]). The second example has a scalar field with a non-canonical kinetic term, yielding the speed of sound of $c_s \ll 1$ [38] (see Section II for the details of this set up). The non-attractor inflation is still driven by a constant potential, leading to $\epsilon \propto 1/a^6$ and $\mathcal{R} \propto a^3$. Using both the quantum field theory calculation in the comoving gauge as well as in the flat gauge [17]. Ref. [37] finds $f_{NL}^{\text{local}} \simeq 5/(4c_s^2) \gg 1 - n_s$ (for $c_s \ll 1$), again violating the consistency relation (also see [38]).

In this paper, we first provide more detailed derivations of the quantum field theory calculations used by Ref. [36]. We first show that the in-in formalism in the comoving gauge yields $f_{NL}^{\text{local}} = 5(1 + c_s^2)/(4c_s^2)$ for arbitrary values of $c_s$. We then present a new derivation of the same result using the $\delta N$ formalism. This is a non-trivial task: the usual application of the $\delta N$ formalism is limited to the case in which attractor solutions have been reached. Then, one needs to consider derivatives of the number of e-folds of inflation, $N$, only with respect to the value of a scalar field, $\phi$, on the initial flat hypersurface. However, one must take into account the full phase space, i.e., the values of both $\phi$ and $\dot{\phi}$, when attractor solutions have not yet been reached [33].

One may worry about validity of the classical calculation such as the $\delta N$ formalism when $c_s \ll 1$, as the $\delta N$ formalism is based on the gradient expansion [11] and thus ignores non-Gaussian contributions from modes at the horizon crossing [13, 48]. However, as we shall show below, the dynamics responsible for the interactions between the modes and the generation of the local-type non-Gaussianity in our models happens on super-horizon scales. The $\delta N$ formalism thus gives accurate results, which are insensitive to intrinsic non-Gaussianities generated at the horizon crossing.

The rest of the paper is organized as follows. In Section II we present our set up. In Section III we present the linear cosmological perturbation for our model and calculate the power spectrum. In Section IV we calculate the bispectrum using the in-in formalism in both the comoving and flat gauges. In Section V we calculate the bispectrum using the $\delta N$ formalism. We conclude in Section VI. In the Appendix we show that the actions in the flat and comoving gauges are equivalent to each other at the leading order in $c_s^{-2} \gg 1$.

II. NON-ATTRACTOR BACKGROUND WITH LARGE SELF-INTERACTIONS

Here we present our set up. The model is the same as that studied in Ref. [36] with the following action for a scalar field with a non-standard kinetic energy such as in models of k-inflation [49]:

$$S = \int dt \, d^3 x \, P(X, \phi) ,$$  

(3)
where \( X \equiv -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \), and

\[
P(X, \phi) = X + \frac{X^{\alpha}}{M^{4 \alpha - 4}} - V(\phi), \quad V(\phi) = V_{0} + v \left( \frac{\phi}{M_{P}} \right)^{\beta},
\]

with \( M, \alpha, v, V_{0}, \) and \( \beta \) being free constant parameters. As studied in Ref. [36], inflation has two phases: the non-attractor phase, followed by the attractor phase.

As we shall show below, the second term of the potential is negative, \( v < 0 \). Inflation during the non-attractor phase is driven by a constant term in the potential, \( V_{0} \). The initial velocity of \( \phi \) is arranged such that the field climbs up the potential initially, and the first term of \( P(X, \phi) \), i.e., the term linear in \( X \), is sub-dominant during the non-attractor phase. Towards the end of the non-attractor phase (at which the kinetic energy decays sufficiently), the term linear in \( X \) dominates and the second phase of inflation starts. We assume that the second phase is a usual slow-roll inflation in which the super-horizon modes are frozen. However, the crucial point is that the curvature perturbation is not conserved on super-horizon scales during the early non-attractor phase. We find a large non-Gaussianity when the CMB modes leave the horizon during the non-attractor phase.

The background equations of motion are

\[
3M_{P}^{2} H^{2} = 2XP_{X} - P, \tag{5}
\]
\[
M_{P}^{2} \ddot{H} = -XP_{X}, \tag{6}
\]

and

\[
(P_{X} + 2XP_{XX}) \ddot{\phi} + 3H \dot{\phi} P_{X} + 2XP_{X\phi} - P_{,\phi} = 0, \tag{7}
\]

where a dot indicates the derivative with respect to the cosmic time, \( t \), and \( H \equiv \dot{a}/a \) is the Hubble constant during inflation. As usual we use the convention that \( P_{X} \equiv \partial P/\partial X \) and so on.

Here we define the sound speed, \( c_{s} \), and slow-roll parameters, \( \epsilon \) and \( \eta \), by

\[
c_{s}^{2} \equiv \frac{P_{X}}{P_{X} + 2XP_{XX}}, \tag{8}
\]

\[
\epsilon \equiv -\frac{\dot{H}}{H^{2}} = \frac{XP_{X}}{M_{P}^{2}H^{2}}, \tag{9}
\]

\[
\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = \left( 1 + \frac{1}{c_{s}^{2}} \right) + \frac{\dot{\phi} P_{X\phi}}{H P_{X}} + 2\epsilon. \tag{10}
\]

For future references, we also define the following variables:

\[
\Sigma \equiv XP_{X} + 2X^{2}P_{XXX} = \frac{H^{2}M_{P}^{2}\epsilon}{c_{s}^{2}}, \tag{11}
\]

\[
\lambda \equiv X^{2}P_{XX} + \frac{2}{3}X^{3}P_{XXX}. \tag{12}
\]

To support inflation we assume that the constant potential term dominates in the total energy density such that

\[
3M_{P}^{2} H^{2} \simeq V_{0}. \tag{13}
\]

Let us first consider the non-attractor phase in which the term linear in \( X \) can be neglected. To be able to do analytic calculation, we require \( c_{s} \) and \( \eta \) to be nearly constant. The latter requirement implies \( \epsilon \propto a^{\eta} \). We will check below that our Lagrangian can satisfy these requirements. The sound speed during the non-attractor phase is given by

\[
c_{s}^{2} \simeq \frac{1}{2\alpha - 1}. \tag{14}
\]

Since \( P_{X\phi} = 0 \), the Klein-Gordon equation Eq. (7) can be rewritten as

\[
\frac{P_{X}}{c_{s}^{2}} \ddot{X} + 6HX P_{X} - P_{,\phi} = 0. \tag{15}
\]
Finding analytical solutions of the above equation is not easy. Instead, we propose the following ansatz:

$$\phi(t) = \text{const} \cdot a^\kappa, \quad (16)$$

where $\kappa$ is a constant and should be determined by the consistency of the equations. By this ansatz, and noting that $H$ is nearly constant, we have

$$\dot{\phi} \simeq H \kappa \phi, \quad \ddot{\phi} \simeq H^2 \kappa^2 \phi, \quad X \simeq \frac{1}{2} H^2 \kappa^2 \phi^2. \quad (17)$$

Using the above relations we obtain two equations to solve Eq. (15), one for the cancellation of powers of $\phi$ and the other for the cancellation of constant pre-factors:

$$\beta = 2\alpha = \frac{1}{c_s^2} + 1, \quad (18)$$

$$v = -\frac{M^4}{c_s^2} \left( \frac{V_0 \kappa^2}{6M^4} \right)^\alpha \left( 1 + \frac{3c_s^2}{\kappa} \right). \quad (19)$$

In addition, we also have

$$\epsilon = \frac{XP_X}{M_p^2 H^2} \propto a^{2\alpha \kappa}. \quad (20)$$

Recalling $\epsilon \propto a^\eta$, the parameter $\kappa$ being a constant is consistent with the parameter $\eta$ being a constant, as desired. We find

$$\kappa \simeq \frac{\eta}{2\alpha}. \quad (21)$$

We have five free parameters in our action. In addition, $\kappa$ is another parameter obtained from the solution. Among all, two of them are determined by requiring the ansatz given in (16) to be a consistent solution, and two others are fixed for a given value of $\eta$ and $c_s$. In the end, two parameters remain undetermined. As we shall show in Section III a scale-invariant power spectrum requires $\eta \simeq -6$, and a large non-Gaussianity requires $c_s \ll 1$.

Using Eq. (17) and Eq. (18), one can easily check that

$$\frac{X^\alpha / M^{4\alpha - 4}}{v (\phi / M_p)^2} \simeq -c_s^2 \left( 1 + \frac{3c_s^2}{\kappa} \right)^{-1}. \quad (22)$$

As a result, for $c_s \ll 1$, the kinetic term is always sub-dominant in comparison to the potential term.

In the above calculations, we have assumed that the term linear in $X$ is negligible, i.e., $X \ll X^\alpha / M^{4\alpha - 4}$, or equivalently $(X/M^4)^{\alpha - 1} \gg 1$. For $\alpha \gg 1$ (i.e., $c_s \ll 1$), this implies $X/M^4 > 1$. Using the ansatz given in (16), this condition translates to

$$\sqrt{\frac{V_0}{6M_p^2 M^2}} |\kappa| \phi > 1, \quad (23)$$

and the condition breaks down at $\phi = \phi_*$ or $t = t_*$ defined by

$$\frac{\phi_*}{M_p} \simeq \sqrt{\frac{6}{V_0}} \frac{M^2}{|\kappa|}. \quad (24)$$

After $\phi_*$, we enter the slow-roll inflation phase for a relatively large range of initial conditions. If this does not happen, we lose our analytic control on the solution and the curvature perturbation may not be conserved in the second phase. Therefore, in what follows, we shall assume that the second inflationary phase does occur and is in the slow-roll regime. Furthermore, as $v < 0$ for $\eta \simeq -6$, we need another phase to have a graceful exit from inflation before the negative potential dominates. This can be achieved by coupling $\phi$ to another heavy (waterfall) field, as in hybrid inflation models [50]. Note that the waterfall field is needed only for ending inflation and does not contribute to super-horizon fluctuations, and thus our model remains a single-field model.

In this set up of the model, the field climbs up the potential during the first phase of inflation. This is why we have a non-attractor background initially. Note that the ansatz given in Eq. (16) and the fine-tuning between parameters given in Eq. (18) and Eq. (21) are not necessary conditions for obtaining the non-attractor behavior. We require
FIG. 1: Evolution of $\phi(n)$ in the undershoot situation as a function of the number of $e$-folds, $n$, counted from the beginning of inflation. The inflaton field climbs up the potential, stops somewhere before the top of the potential ($\phi = 0$), turns around and goes back to plus infinity. The dashed red curve is the analytic ansatz for the non-attractor phase, while the solid blue curve is the full numerical solution. The transition to a slow-roll inflation phase is sharp, and an extended slow-roll phase follows afterward. The parameters, consistent with Eqs. (18) and (21), in units of $M_P$ are $V_0 \approx 6.25 \times 10^{-6}$, $M = 5 \times 10^{-5}$, $\alpha = 10$, and $\eta = -6$.

these specific values of the parameters in order to be able to do analytic calculations of the bispectrum. In fact, we find a non-attractor phase for a half of the ranges of possible initial conditions.

Depending on initial conditions, the solution for $\phi$ shows three different behaviors: the undershoot, the critical or the overshoot. In the undershoot case, the inflaton field climbs up the potential, stops somewhere before crossing the origin (the top of the potential), turns around and rolls down on the same side of the potential. In this case $\phi$ always has a unique sign while $\dot{\phi}$ changes the sign (see Fig. 1). In the overshoot case, the inflaton field climbs up the potential with a large enough initial velocity, so that it goes over the top of the potential, and rolls down on the other side of the potential. In this case $\phi$ changes the sign while $\dot{\phi}$ always has a unique sign (see Fig. 2). The critical limit occurs when the initial conditions are such that it takes infinite amount of time for the inflaton field to reach the top of the potential.

These different behaviors of the inflaton-field evolution in phase space are shown in Fig. 3. The critical limit (the black solid line in Fig. 3) separates the overshoot and undershoot solutions. The early-time behavior of this curve for large $|\phi|$ and $|\dot{\phi}|$ (for which the power-law term, $X^\alpha$, dominates) is asymptotically the same as the ansatz we obtained above. On the other hand, the linear term in $X$ dominates near the origin, and one should solve the equation of motion for a canonically normalized field, i.e.,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{v\beta}{M_P} \left( \frac{\phi}{M_P} \right)^{\beta-1} = 0.$$  \hspace{1cm} (25)

For $\beta \gg 1$, the last term proportional to $(\phi/M_P)^{\beta-1}$ is small relative to the first two terms near the origin, $\phi/M_P \approx 0$. Thus, the slow-roll condition no longer holds, and we have $\ddot{\phi} + 3H\dot{\phi} \approx 0$. This is similar to the scenario of a constant potential studied in Ref. [35]. The solution is $\phi \propto a^{-3}$ and, as a result, $d\phi/dn \approx -3\phi$, where $n$ is the number of $e$-folds counted from the beginning of inflation. This asymptotic solution is in agreement with the numerical one for $\phi/M_P \approx 0$ (see the purple line in Fig. 3).
FIG. 2: Same as Fig. but for the overshoot situation. The inflaton field climbs up the potential, goes over the top of the potential, and rolls down on the other side of the potential.

FIG. 3: Phase-space diagram of the model given by Eq. (4). The black line separates two different trajectories: the undershoot (dashed blue lines) and overshoot (dot-dashed red lines) trajectories. The purple dashed line near the origin shows the asymptotic solution, $\phi \propto a^{-3}$. In the undershoot case, the inflaton field climbs up the potential, stops somewhere before reaching the top of the potential, and returns back. In the overshoot case, the field climbs up the potential, crosses the top of the potential, and rolls down on the other side of the potential. The symmetry in the plot reflects the fact that our Lagrangian given in Eq. (4) is symmetric under the transformation $\phi \rightarrow -\phi$ and $\dot{\phi} \rightarrow -\dot{\phi}$. 
III. POWER SPECTRUM

In this section, we calculate the power spectrum of curvature perturbations generated during the non-attractor phase and obtain the condition for a scale-invariant power spectrum. As usual, we have the following quadratic action for curvature perturbation [51–53]:

$$S = \frac{1}{2} \int d^3x \, d\tau \, z^2 \left[ R'^2 - c_s^2 (\nabla R)^2 \right], \quad (26)$$

where

$$z^2 \equiv \frac{2\epsilon a^2}{c_s^2} M_p^2. \quad (27)$$

Recalling $\epsilon \propto a^\eta$ and assuming a Bunch-Davies initial state deep inside the horizon, the solution for the mode function is given by

$$R_k = C x^\nu H^{(1)}_\nu(x), \quad (28)$$

where we have defined

$$x \equiv -c_s k \tau, \quad \nu \equiv \frac{3 + \eta}{2}, \quad (29)$$

$\tau$ is the conformal time defined by $d\tau \equiv dt/a(t)$, and

$$|C|^2 \equiv \frac{\pi c_s}{8k_c a_i^2 M_p^2} x^{1-2\nu}. \quad (30)$$

The subscripts $i$ denote the corresponding values at the start of inflation.

The power spectrum of curvature perturbations at the end of the non-attractor phase is given by

$$P_R = \frac{k^3}{2\pi^2} |R_k|^2. \quad (31)$$

Note that this power spectrum will be the observed power spectrum, as $R$ is conserved outside the horizon after the end of the non-attractor phase.

Using Eq. (30) we write the power spectrum in terms of the parameters at the end of the non-attractor phase as

$$P_R \simeq \frac{\Gamma(\nu)^2}{\pi^3 2^{2\nu+1}} \left( \frac{H_*}{M_p} \right)^2 \frac{1}{c_s^2 \epsilon_*} \left( \frac{c_s k}{H_* a_*} \right)^{3+2\nu}, \quad (32)$$

where we have assumed $\nu = (3 + \eta)/2 < 0$, so that we can expand the Hankel function for a small argument, $x \ll 1$.

During the non-attractor phase, the fast decay of $\epsilon$ makes the curvature perturbation grow very rapidly on superhorizon scales. This growth continues until the end of the non-attractor phase. The subsequent slow-roll phase begins at $t = t_* \text{ or } \phi = \phi_*$ given in Eq. (24). The curvature perturbation is conserved during the slow-roll phase.

The spectral index is given by

$$n_s - 1 \simeq 3 + 2\nu = 6 + \eta. \quad (33)$$

Therefore, $\eta = -6$ is required for a scale-invariant power spectrum. A slightly red-tilted power spectrum, $n_s = 0.96$ [2, 4], can be easily obtained by choosing $\eta = -6.04$.

IV. BISPECTRUM: IN-IN FORMALISM

In this section, we calculate the bispectrum of curvature perturbations generated during the non-attractor phase using the in-in formalism in two different gauges. We provide the detailed derivations of the results presented earlier in Ref. [36].

The first gauge is the comoving gauge, which enables the most complicated but rigorous calculations. The bispectrum we obtain in the comoving gauge is valid for arbitrary values of $c_s$, including $c_s = 1$. The second gauge is the flat gauge, and we use this gauge to show that a large bispectrum comes from the matter sector. As we use the decoupling-limit approximation when computing the bispectrum in the flat gauge, the bispectrum in this gauge is valid only for $c_s \ll 1$. We also explicitly show that the results from two gauges are equivalent to each other in the small sound-speed limit.
A. Comoving gauge

The cubic action in the comoving gauge, in which the scalar field is unperturbed, is given by 32, 47

\[
S_3 = \int dt d^3x \left\{ -a^3 \left[ \Sigma \left( 1 - \frac{1}{c_s^2} \right) + 2\lambda \right] \frac{\dot{R}^3}{H^3} + \frac{a^3 \epsilon}{c_s^4} (\epsilon - 3 + 3c_s^2) R \dot{R}^2 \\
+ \frac{a \epsilon}{c_s^2} (\epsilon - 2s + 1 - c_s^2) \dot{R} (\partial \dot{R}) \right\} , \\
\]

where

\[
\dot{R}^2 \equiv a^2 \epsilon \dot{R} , \\
\]

\[
\frac{\delta L}{\delta \dot{R}} \bigg|_{1} \equiv a \left( \frac{d \delta^2 \chi}{dt} + H \delta \chi - \epsilon \delta^2 \dot{R} \right) , \\
\]

\[
f(\dot{R}) \equiv \frac{\eta}{4c_s^2} R^2 + \frac{1}{c_s^2 H} R \dot{R} + \frac{1}{4a^2 H^2} \left[ (\partial \dot{R}) (\partial \dot{R}) + \partial^{-2} (\partial_i \partial_j (\partial_i \partial_j R) \right) \\
+ \frac{1}{2a^2 H^2} \left[ (\partial \dot{R}) (\partial \dot{R}) - \partial^{-2} (\partial_i \partial_j (\partial_i \partial_j \dot{R})) \right] .
\]

The terms in the last line in Eq. (34) are higher-order in \( \epsilon \) and can be ignored. In the usual attractor inflation models, for which \( \dot{R} \approx 0 \) on super-horizon scales and thus the contributions to the integral come from the horizon-crossing epoch, \( kc_s \approx aH \), all of the terms in the first two lines in Eq. (34), which are proportional to \( \dot{R}^3, \dot{R} \dot{R}^2, (\partial \dot{R})^2 \), yield the equilateral bispectrum 32, 47. However, in this non-attractor model, for which \( \dot{R} = 3H \dot{R} \) on super-horizon scales, the integral receives dominant contributions after the horizon exit. As a result, the terms in the first line, \( \dot{R}^3 \) and \( \dot{R} \dot{R}^2 \), are proportional to \( \dot{R}^3 \) on super-horizon scales, yielding the local-form bispectrum. Also in this non-attractor case, all the terms with spatial derivatives are suppressed by factors of the scale factor at the end of the non-attractor inflationary phase, \( k/(a_{\text{end}} H) \), and negligible.

Therefore, we have a rather different situation here: in the usual attractor case, all the terms in the first three lines in Eq. (34) must be included for consistent computation of the bispectrum up to \( f_{NL} \sim O(\epsilon) \), whereas in the non-attractor case only the terms in the first line are necessary. Note that this statement is independent of the value of \( c_s \), and thus the results given in this section are valid for arbitrary values of \( c_s \), including \( c_s = 1 \).

In the usual attractor case, the terms in the first two lines in Eq. (34) give the equilateral bispectrum with \( f_{NL}^{\text{equil}} = O(1/c_s^2) \) for \( c_s \ll 1 \). In the non-attractor case, the terms in the first line give a large local bispectrum with \( f_{NL}^{\text{local}} = O(1/c_s^2) \) for \( c_s \ll 1 \) as we shall show below, whereas the other terms are negligible. Therefore, the non-attractor model yields un-observable signals in the equilateral bispectrum.

How about the fourth line in Eq. (34), which can be removed by a field redefinition? Again, we only need to keep the terms that do not have extra spatial derivatives. Ignoring the terms suppressed by spatial derivatives in Eq. (37), we redefine the curvature perturbation as

\[
\mathcal{R} \rightarrow \mathcal{R}_n + \frac{\eta}{4c_s^2} \mathcal{R}_n^2 + \frac{1}{c_s^2 H} \mathcal{R}_n \dot{\mathcal{R}}_n , \\
\]

where \( \mathcal{R}_n \) is the redefined field. After the horizon crossing, when the argument of the Hankel function with rank \( \nu < 0 \) is small, we have

\[
\mathcal{R}' = -c_s k x^\nu H_{\nu-1}^{(1)}(x) \simeq -c_s k^2 \frac{2\nu}{x} \mathcal{R} .
\]
As a result, the above field redefinition becomes
\[ R \rightarrow R_n + \left( \frac{\eta}{4c_s^2} - \frac{2\nu}{c_s^2} \right) R_n^2. \] (40)

The quadratic terms in Eq. (40) give the following contribution to the local-form \( f_{NL} \) parameter (denoted as “\( f_{NL}^{FR} \)):
\[ \frac{3}{5} f_{NL}^{FR} = \frac{1}{4c_s^2} (\eta - 8\nu) = -\frac{1}{4c_s^2} (12 + 3\eta). \] (41)

The first term in Eq. (34) gives the bispectrum of
\[ \langle R_{k_1} R_{k_2} R_{k_3} \rangle = 6 \times 2M_P^6 \Im \left[ R_{k_1}(\tau) R_{k_2}(\tau) R_{k_3}(\tau) \int_{-\infty}^{\tau_s} d\tau \left[ \Sigma(1 - 1/c_s^2) + 2\lambda \right] R_{k_1}^* (\tau) R_{k_2}^* (\tau) R_{k_3}^* (\tau) \right], \] (42)
where \( \tau_s \) is the conformal time at the end of the non-attractor phase. As the kinetic term is dominated by \( X^\alpha \) during the non-attractor phase, \( \lambda \) is given by
\[ \lambda = \frac{\Sigma}{6} \left( \frac{1}{c_s^2} - 1 \right), \] (43)
where we have used Eq. (14). Recall that \( \Sigma = H^2 M_P^2 \epsilon/c_s^2. \) Ignoring a small tilt and setting \( \eta = -6 \) and \( \nu = -3/2, \) the mode function simplifies to
\[ R_k = C_k \sqrt{\frac{2}{\pi}} e^{-i c_s k \tau} (-1 - i c_s k \tau). \] (44)

As a result, the first term in Eq. (34) gives the local-form bispectrum parameter of
\[ \frac{3}{5} f_{NL}^{R^3} = -\frac{3}{2c_s^2} (1 - c_s^2). \] (45)

With a similar procedure, the second term gives the local-form bispectrum parameter of
\[ \frac{3}{5} f_{NL}^{R^2} = \frac{3}{4c_s^2} (1 - c_s^2). \] (46)

The total local-form bispectrum parameter, \( f_{NL}^{local} \), is given by the sum of the above contributions:
\[ \frac{3}{5} f_{NL}^{local} = \frac{3}{5} \left( f_{NL}^{R^3} + f_{NL}^{R^2} + f_{NL}^{FR} \right) = \frac{3}{4c_s^2} (1 + c_s^2). \] (47)

Once again, this result is valid for arbitrary values of \( c_s \), including \( c_s = 1 \). As emphasized in Ref. \[36\], Eq. (47) shows that the presence of a large primordial \( f_{NL}^{local} \) would not rule out all single-field models in full generality. Rather, it would rule out all single-field models which have reached the attractor solution and with a Bunch-Davies initial state.\(^2\)

As we have shown above, this model gives a local-form bispectrum because \( R^3 \) and \( R R^2 \) become proportional to \( R^3 \) on super-horizon scales. This implies that we can obtain the same result using classical calculations such as the \( \delta N \) formalism, which uses gradient expansion. We shall confirm this in Section \[V\].

The subsequent slow-roll phase of inflation after the first non-attractor phase cannot change the value of \( f_{NL}^{local} \), as the super-horizon curvature perturbation remains constant during the slow-roll phase. However, one may wonder what would happen to \( f_{NL}^{FR} \), i.e., the contribution from the field-redefinition terms given in Eq. (38), which are suppressed during the slow-roll phase by \( \eta \ll 1 \) and \( \mathcal{R} \approx 0 \). While it is true that the field-redefinition terms become negligible during the slow-roll phase, we find that, in the comoving gauge, a boundary term in the cubic action at the end of the non-attractor phase replaces the contributions from the field-redefinition terms.

\[^2\] Also see the workshop summary of “Critical Tests of Inflation Using Non-Gaussianity” in \[http://www.mpa-garching.mpg.de/~komatsu/meeting/ng2012/\]
To show this explicitly, let us model the evolution of $\eta$ such that it is equal to $\eta = \eta_0 = -6$ during the non-attractor phase and vanishes during the slow-roll phase. Specifically,

$$\eta = \eta_0 (1 - \theta(t - t_\ast)),$$

where $t_\ast$ is the transition time at which $\phi(t_\ast) = \phi_\ast$ given in Eq. (24). This step function becomes a delta function upon a time derivative with respect to $t$. As a result, this gives a boundary term in the cubic action which has a non-negligible contribution:

$$S_3 \ni \int d\tau d^3x \frac{a^2 \epsilon}{2c_s^2} \frac{d}{dt}\left( \frac{\eta}{c_s^2} \right) R^2 R' \simeq \int d^3x \left[ \frac{a^2 \epsilon}{2c_s^2} \eta_0 R^2 R' \right]_*. \quad (49)$$

One can check that $f_{N,L}^{\text{local}}$ from this term is equal to $f_{N,L}^{\text{flat}}$ given by Eq. (41). Therefore, this term replaces $f_{N,L}^{\text{flat}}$ after the end of the non-attractor phase, and the total $f_{N,L}^{\text{local}}$ remains equal to that given by Eq. (17).

B. Flat gauge

In the small sound-speed limit, the bispectrum is sourced primarily by interactions in the scalar-field sector, and the interactions involving gravity become negligible. In such cases, it is known that the computation of the bispectrum can be made simpler by using the so-called “inflaton approximation” or the “decoupling limit” [54–59]. In this approximation, we ignore metric perturbations entirely, and consider only the scalar-field perturbation:

$$\phi(x, t) = \phi_0(t) + \delta \phi(x, t). \quad (50)$$

To derive the cubic action in $\delta \phi$, we simply perturb $P(X, \phi)$ with respect to $X$, and obtain

$$\mathcal{L}_3 = a^3 \frac{2\lambda}{\phi_0^2} \delta \phi^3 - a \frac{\Sigma(1 - c_s^2)}{\phi_0^2} \delta \phi (\partial \delta \phi)^2. \quad (51)$$

The perturbations of $P(X, \phi)$ with respect to $\phi$ can be ignored, as they are not enhanced by $c_s^{-2}$ or $\lambda/\Sigma$. While we shall loosely call this action the “cubic action in the flat gauge” in this paper, this action is not the full cubic action in the flat gauge, as we have ignored terms coming from the lapse function and the shift vector via the constraint equations (Lagrange multipliers). Once again, ignoring these terms and working only with the above two terms is justified only in the decoupling limit, in which the scalar-field interactions overwhelm the gravitational ones.

Using the relation $R = -H \delta \phi / \phi$, we rewrite this action as

$$\mathcal{L}_3 = -2a^3 \frac{\lambda}{H^2} \dot{R}^3 + a \frac{\Sigma(1 - c_s^2)}{H^3} \dot{R} (\partial \dot{R})^2. \quad (52)$$

In Appendix A, we show that the action given in Eq. (52) is equivalent to that in the comoving gauge in the leading order of $\lambda/\Sigma$ and $c_s^{-2}$, and for $H$, $\dot{\phi}$, $\eta$, $c_s \sim \text{const.}$, including the field-redefinition terms.

In our model, the field interactions build up on super-horizon scales. The second term in Eq. (51) is thus subdominant due to the spatial derivative, and we only need to compute the first term. We obtain

$$\frac{3}{5} f_{N,L}^{\text{local}} \simeq \frac{3}{4} \left( \frac{1}{c_s^2} - 1 \right). \quad (53)$$

As expected, for $c_s \ll 1$, this simple method reproduces the leading order result of the previous section, i.e., Eq. (17).

V. $\delta N$ FORMALISM

In this section we shall use the $\delta N$ formalism [12, 42, 44] to calculate $f_{N,L}^{\text{local}}$. We shall show that $f_{N,L}^{\text{local}}$ we have calculated using the in-in formalism in Section IV agrees precisely with that we find from the $\delta N$ formalism in this section. As we have shown already in Section IV, this is because the intrinsic bispectrum of the quantum fluctuations present at the time of the horizon crossing is sub-dominant, and the dominant contribution comes from the interactions of the scalar field on super-horizon scales. Fortunately this is all one needs in using the $\delta N$ formalism based on a separate universe assumption [45] (see [48] for more precise conditions under which the $\delta N$ formalism is valid).
Nevertheless, extra cares must be taken when we use the $\delta N$ formalism in non-attractor backgrounds. Once the solution reaches the attractor solution, we need to consider only the perturbations of the scalar-field trajectories with respect to the field value at the initial hypersurface, $\phi$, as the velocity, $\dot{\phi}$, is uniquely determined by $\phi$. However, in the non-attractor case, the scalar-field trajectories are not uniquely determined by the field value $\phi$ alone. We also need the information of $\dot{\phi}$ to determine the trajectory [33].

In order to find the scalar-field trajectories, we need to solve the equation of motion of the scalar field, which is a second-order differential equation. We thus need to provide two initial conditions ($\phi$ and $\dot{\phi}$) on the initial hypersurface. We can then integrate the equation of motion to the final time, $t = t_*$. We assume that the universe has already arrived at the attractor phase (often called the adiabatic limit) by this epoch, or a phase transition to an attractor phase occurs at $t = t_*$. More specifically, we assume that the evolution of the universe is unique after the value of the scalar field has arrived at $\phi = \phi_*$, irrespective of the value of its velocity $\dot{\phi}_*$. In other words, at and after $t = t_*$, the scalar field plays the role of a clock. We note that this is a necessary condition for the validity of the $\delta N$ formalism, since only in this case $\delta N$ is equal to the final value of the comoving curvature perturbation $\mathcal{R}$ which is conserved at $t \geq t_*$. Thus the number of $e$-folds $N$ counted backward from the epoch when $\phi = \phi_*$ to an earlier epoch is a function of $\phi$ and $\dot{\phi}$, $N = N(\phi, \dot{\phi}; \phi_*)$.

With this in mind we apply the $\delta N$ formalism. Our program is as follows. In order to find the background scalar-field trajectories, we need to solve the equation of motion of the scalar field perturbatively by expanding it around a particular trajectory given by $\phi \propto e^{cHt}$. We then use these background solutions for the field trajectories to compute the perturbations of the number of $e$-folds with respect to the initial field value and its time derivative.

### A. The case with $c_s = 1$

To familiarize ourselves with the $\delta N$ calculation in the non-attractor background, let us first work out the simplest case with the canonical kinetic term, $c_s = 1$. During the non-attractor phase whose potential is dominated by a constant term, the background Klein-Gordon equation is given by

$$\ddot{\phi} + 3H \dot{\phi} = 0,$$

which has the following solution

$$\phi = \lambda + \mu e^{-3Ht},$$

where $\lambda$ and $\mu$ are constants of integration. Without loss of generality, we assume $\dot{\phi} > 0$. We set $\phi(t_*) = \phi_*$ at which the non-attractor phase ends.

The number of $e$-folds counted backward in time from $t = t_*$ is

$$N = \int_t^{t_*} H dt = H(t_* - t) = -H t_*,$$

where we have set $t_* = 0$ without loss of generality. With this definition of time, the above solution becomes

$$\phi = \lambda + \mu e^{3N} = \lambda + (\phi_* - \lambda)e^{3N}.$$  

This gives

$$\dot{\phi} = -3H \mu e^{3N} = 3H(\lambda - \phi_*)e^{3N}.$$  

As clear from the above, the different trajectories in the phase space $(\phi, \dot{\phi})$ are parameterized by $\lambda$ with $N$ being the parameter along each trajectory. That is,

$$\phi = \phi(N, \lambda); \quad \dot{\phi} = \dot{\phi}(N, \lambda).$$

In other words, the variables $(N, \lambda)$ may be regarded as another set of coordinates in the phase space. Thus one can invert the above to obtain $N$ and $\lambda$ as functions of $(\phi, \dot{\phi})$. Specifically we obtain

$$N = N(\phi, \dot{\phi}) = \frac{1}{3} \ln \left( \frac{\dot{\phi}}{\phi + 3H(\phi - \phi_*)} \right),$$

$$\lambda = \lambda(\phi, \dot{\phi}) = \phi + \frac{\dot{\phi}}{3H}.$$  

---

3 We change the notation. Henceforth, $\lambda$ is not given by Eq. [2], but is a constant of integration.
In the present case of \( c_s = 1 \), Eq. (60) for \( N \) in terms of \( \phi \) and \( \dot{\phi} \) is sufficient to derive the \( \delta N \) formula. Nevertheless, for the sake of the discussion in the next subsection in which the case with \( c_s \neq 1 \) is considered, we insert an intermediate step for the derivation of the \( \delta N \) formula as follows.

In place of \((\phi, \dot{\phi})\), we may introduce a yet another set of coordinates in the phase space. Here we choose \((\phi, \lambda)\). A special feature of this choice is that one of the coordinates \( \lambda \) is a constant of integration along each trajectory. Therefore, in particular, its perturbation \( \delta \lambda \) can be evaluated at any point along the trajectory. With this choice, we have \( N = N(\phi, \lambda) \). This expression can be immediately obtained by inverting the solution of \( \dot{\phi} \) given by Eq. (57),

\[
N = \frac{1}{3} \ln \left( \frac{\phi - \lambda}{\phi_\ast - \lambda} \right).
\]

Then one may expand this by setting \( \phi \to \phi + \delta \phi \) and \( \lambda \to \lambda + \delta \lambda \). Up to the second order, we have

\[
\delta N = \frac{\partial N}{\partial \phi} \delta \phi + \frac{\partial N}{\partial \lambda} \delta \lambda + \frac{1}{2} \frac{\partial^2 N}{\partial \phi^2} \delta \phi^2 + \frac{1}{2} \frac{\partial^2 N}{\partial \lambda^2} \delta \lambda^2 + \frac{\partial^2 N}{\partial \phi \partial \lambda} \delta \phi \delta \lambda.
\]

Now we identify the perturbations \( \delta \phi \) and \( \delta \lambda \) with those evaluated on the flat hypersurface at or after which the scale of interest has crossed out of horizon. For \( \delta \phi \), this is immediate. As for \( \delta \lambda \), however, we need its relation to \( \delta \phi \) and \( \dot{\delta} \phi \). In the present case, we can readily find this from Eq. (61).

\[
\delta \lambda = \delta \phi + \frac{\delta \dot{\phi}}{3H}.
\]

If we recall that the quantum fluctuations are dominated by the constant mode \( \delta \phi = \text{const.} \) on superhorizon scales, we immediately obtain \( \delta \dot{\phi} = 0 \), and hence \( \delta \lambda = \delta \phi \). Inserting this to Eq. (62), we finally obtain

\[
\delta N = \frac{\delta \phi}{3(\phi_\ast - \lambda)} + \frac{\delta \phi^2}{6(\phi_\ast - \lambda)^2}.
\]

This \( \delta N \) yields the following \( f_{NL}^{\text{local}} \):

\[
f_{NL}^{\text{local}} = \frac{5}{2}.
\]

This is of course in agreement with the result obtained by differentiating \( N \) directly with respect to \( \phi \) and \( \dot{\phi} \).

As mentioned in the above, for this particular setup, not only \( \delta \lambda \) which is a constant of motion by definition but also \( \delta \phi \) is conserved on superhorizon scales, and \( \delta \lambda = \delta \phi \). This implies that we may choose the initial hypersurface to be infinitesimally close to the end of the non-attractor phase, i.e., \( \phi \to \phi_\ast \). In other words, \( \delta N \) is simply given by the difference in the number of \( e \)-folds between the flat and comoving slices at \( t = t_\ast \),

\[
\delta N = \frac{\partial N}{\partial \phi} \big|_{t_\ast} \delta \phi_\ast + \frac{\partial N}{\partial \lambda} \big|_{t_\ast} \delta \lambda + \frac{1}{2} \frac{\partial^2 N}{\partial \phi^2} \big|_{t_\ast} \delta \phi^2 + \frac{1}{2} \frac{\partial^2 N}{\partial \lambda^2} \big|_{t_\ast} \delta \lambda^2 + \frac{\partial^2 N}{\partial \phi \partial \lambda} \big|_{t_\ast} \delta \phi_\ast \delta \lambda,
\]

where \( \delta \phi_\ast \) is the fluctuation evaluated on the flat slicing at \( t = t_\ast \). We find that two of these derivatives, \( \partial N/\partial \lambda \big|_{t_\ast} \) and \( \partial^2 N/\partial \lambda^2 \big|_{t_\ast} \), vanish, and thus we need to evaluate only the other three terms. The result is

\[
\delta N = \frac{\delta \phi_\ast}{3(\phi_\ast - \lambda)} + \frac{\delta \phi^2_\ast}{6(\phi_\ast - \lambda)^2}.
\]

Recalling \( \delta \phi_\ast = \delta \phi = \text{const.} \), this again gives \( f_{NL}^{\text{local}} = 5/2 \).

In general, provided that we know how \( \delta \phi \) evolves on superhorizon scales, we can obtain \( \delta N \) by evaluating the fluctuations at \( t = t_\ast \). In this case, since we only need to know the dependence of the derivatives of \( N \) on \( \phi \) and a constant of integration \( \lambda \) at \( t = t_\ast \), the evaluation procedure can be simplified considerably. We shall exploit this simplification in the next section where we deal with the case with \( c_s \neq 1 \).

**B. The case with \( c_s \neq 1 \)**

Having familiarized ourselves with the new \( \delta N \) calculation for the simplest case, let us now move onto the case with \( c_s \neq 1 \). In what follows, we shall again neglect the canonical kinetic term during the non-attractor phase for simplicity. The background equation of motion is

\[
\ddot{\phi} + 3c_s^2 H \dot{\phi} - F = 0,
\]

(69)
where

\[ F \equiv c_s^2 \frac{P}{P_{\delta X}}. \]  

(70)

Finding a general analytical solution to this equation is not easy. We thus first consider a particular solution, \( \phi = \phi_0 \propto e^{\kappa H t} \) (i.e., \( \phi = \phi_0(N) = \phi_0 e^{-\kappa N} \)), and then obtain a more general solution for the background up to the second order in perturbations around this particular solution. Here, as before, we assume that the non-attractor phase ends when \( \phi = \phi_* \). Using \( \phi_0 \propto e^{\kappa H t} \) in \( F \) yields

\[ F = c_s^2 \frac{P}{P_{\delta X}} = F_0 \left( \frac{\phi}{\phi_0} \right)^{2\alpha-1} \left( \frac{\dot{\phi}}{\phi_0} \right)^{2-2\alpha}, \]  

(71)

where

\[ F_0 \equiv \frac{\dot{\phi}_0}{\phi_0} + 3c_s^2 H \phi_0 = \kappa (\kappa + 3c_s^2) H^2 \phi_0. \]  

(72)

Let us expand \( F \) around \( \phi = \phi_0 \) to the second order. Defining

\[ \chi \equiv \phi - \phi_0, \]

for notational simplicity,\(^4\) the result is

\[ F = F_0 \left[ 1 + (2 - 2\alpha) \frac{\dot{\chi}}{\phi_0} + (2\alpha - 1) \frac{\chi}{\phi_0} \right. \]

\[ + \frac{(2 - 2\alpha)(1 - 2\alpha)}{2} \left( \frac{\dot{\chi}}{\phi_0} \right)^2 + (2 - 2\alpha)(2\alpha - 1) \frac{\dot{\chi} \chi}{\phi_0 \phi_0} \]

\[ + \frac{(2 - 2\alpha)(1 - 2\alpha)}{2\kappa^2} \left( \frac{\dot{\chi}}{H \phi_0} \right)^2 + \frac{(2 - 2\alpha)(2\alpha - 1)}{\kappa} \frac{\dot{\chi} \chi}{H \phi_0^2} \]

\[ + \frac{(2 - 2\alpha)(1 - 2\alpha)}{2\kappa^2} \left( \frac{\dot{\chi}}{H \phi_0} \right)^2 + \frac{(2 - 2\alpha)(2\alpha - 1)}{\kappa} \frac{\dot{\chi} \chi}{H \phi_0^2} \]

(73)

Having obtained \( F \) in Eq. (69) to the linear and quadratic orders in \( \chi \), as given in Eq. (73), we are ready to solve Eq. (69) perturbatively.

\[ \chi \equiv \phi - \phi_0, \]

for notational simplicity,\(^4\) the result is

\[ F = F_0 \left[ 1 + (2 - 2\alpha) \frac{\dot{\chi}}{\phi_0} + (2\alpha - 1) \frac{\chi}{\phi_0} \right. \]

\[ + \frac{(2 - 2\alpha)(1 - 2\alpha)}{2} \left( \frac{\dot{\chi}}{\phi_0} \right)^2 + (2 - 2\alpha)(2\alpha - 1) \frac{\dot{\chi} \chi}{\phi_0 \phi_0} \]

\[ + \frac{(2 - 2\alpha)(1 - 2\alpha)}{2\kappa^2} \left( \frac{\dot{\chi}}{H \phi_0} \right)^2 + \frac{(2 - 2\alpha)(2\alpha - 1)}{\kappa} \frac{\dot{\chi} \chi}{H \phi_0^2} \]

(73)

1. Linear perturbation

Let us consider the linear perturbation, \( \chi_1 \). The equation of motion is

\[ 0 = \ddot{\chi} + 3c_s^2 H \dot{\chi} - F_0 \left[ \frac{(2 - 2\alpha)}{\kappa} \frac{\ddot{\chi}}{H \phi_0} + (2\alpha - 1) \frac{\dot{\chi}}{\phi_0} \right] \]

\[ = \ddot{\chi} + \left[ 3c_s^2 + (2\alpha - 2)(\kappa + 3c_s^2) \right] H \dot{\chi} - (2\alpha - 1) \kappa (\kappa + 3c_s^2) H^2 \chi \]

\[ = \ddot{\chi} + \left[ 3 + (2\alpha - 1) \kappa - \kappa \right] H \dot{\chi} - [3 + (2\alpha - 1) \kappa] H^2 \chi. \]  

(74)

The general solution is given by

\[ \chi = \chi_1 \propto \begin{cases} \exp[\kappa H t], \\ \exp[-(3 + \eta - \kappa) H t], \end{cases} \]

(75)

where we have set \( 2\alpha \kappa = \eta \), and \( \eta = \dot{\epsilon}/H \kappa \). A scale-invariant spectrum requires \( \eta \approx -6 \); thus, the second solution, \( \propto \exp[-(3 + \eta - \kappa) H t] \), will eventually dominate.

\[ \text{4 We change the notation. Henceforth, } \chi \text{ is not given by in Eq. (35), but is the difference between the true background solution and the reference solution, } \chi \equiv \phi - \phi_0. \]
2. Second-order perturbation

Next we consider the second-order perturbation, $\chi_2$. The equation of motion is

$$\dot{\chi}_2 + \left[ 3 + (2\alpha - 1)\kappa - \kappa \right] H\dot{\chi}_2 - \kappa \left[ 3 + (2\alpha - 1)\kappa \right] H^2 \chi_2 = S,$$

where the source term, $S$, is given by

$$S = F_0 \left[ \frac{(2 - 2\alpha)(1 - 2\alpha)}{2\kappa^2} \left( \frac{\dot{\chi}_1}{H\phi_0} \right)^2 + \frac{(2 - 2\alpha)(2\alpha - 1)}{\kappa} \chi_1 \frac{\dot{\chi}_1}{H\phi_0^2} + \frac{(2\alpha - 1)(2\alpha - 2)}{2} \left( \frac{\chi_1}{\phi_0} \right)^2 \right].$$

Now we assume that the second solution of Eq. (75), $\chi_2 \propto e^{\mu H t}$, is a solution to Eq. (76), with $\mu$ determined by the time-dependence of $S$. We thus obtain

$$\phi = \phi_0 + \chi_1 + \left( \frac{g}{\phi_0} \right) \chi_1^2,$$

with

$$g = \left( \frac{3e^2 + \kappa}{4\kappa} \right) (2 - 2\alpha)(1 - 2\alpha),$$

and

$$\mu = -2(3 + \eta) + \kappa.$$

3. Calculating $\delta N$

We are ready to compute the perturbations of the number of $e$-folds, $\delta N$. The background solution of $\phi$ (computed up to the second-order perturbations around the reference trajectory, $\phi_0 \propto e^{-\kappa N}$) in terms of $N$ is

$$\phi = \phi_0 \left( e^{-\kappa N} + \lambda e^{(3 + \eta - \kappa)N} + g\lambda^2 e^{(2(3 + \eta - \kappa))N} \right)$$

$$= \frac{\phi_0}{1 + \lambda + g\lambda^2} \left( e^{-\kappa N} + \lambda e^{(3 + \eta - \kappa)N} + g\lambda^2 e^{(2(3 + \eta - \kappa))N} \right),$$

where $\lambda$ is an integration constant that parameterizes different trajectories, and we have set $\phi(0, \lambda) = \phi_*$ for any value of $\lambda$ in accordance with the assumption that the end of the non-attractor phase is determined only by the value of the scalar field, $\phi = \phi_*$. Inverting Eq. (81) for a fixed $\lambda$, we would obtain $N$ as a function of $\phi$ and $\lambda$. Then the $\delta N$ formula can be obtained by

$$\delta N = N(\phi + \delta \phi, \lambda) - N(\phi, 0) = \sum_{n,m} \frac{1}{n!m!} \frac{\partial^{n+m} N(\phi, 0)}{\partial \phi^n \partial \lambda^m} \delta \phi^n \lambda^m.$$ 

In practice, the explicit inversion of Eq. (81) is neither easy nor necessary. We may just assume $N$ in the right-hand side of it as a function of $\phi$ and $\lambda$, $N = N(\phi, \lambda)$. Then we may set $\phi \to \phi + \delta \phi$ and $N \to N + \delta N$ on both sides of Eq. (81) and solve for $\delta N$ iteratively.

So far, we have obtained approximate, perturbative solutions of the scalar-field trajectories around the particular reference solution, $\phi_0 = \phi_* e^{e^{Ht}}$. These solutions are valid only when the perturbed trajectories are not far away from the reference solution, i.e., $|\chi_1 + \chi_2|/\phi_0 \ll 1$. However, the dominant linear solution, $\chi_1/\phi_0 = \lambda e^{-(3 + \eta)Ht} \sim e^\kappa H t$, quickly diverges as a function of time. Furthermore, as we can see from the time-dependence of the second order solution, $\chi_2/\phi_0^2$ diverges even faster, $\chi_2/\phi_0^2 \sim e^{2Ht}$. From this, one suspects that the approximate solutions can be trusted only for a short lapse of time. In addition, since we have neglected the subdominant solution, $\chi_1 \propto e^{e^{Ht}}$, our approximation is valid only at sufficiently late times. These considerations suggest that we should choose the initial time as close as possible to the final time, $N \lesssim 1$. Then the simplest choice is to take the initial time to be infinitesimally close to $t = t_*$. 

Perturbing $N = N(\phi, \lambda)$ up to the second order at $t = t_*$, we have

$$\delta N = \frac{\partial N}{\partial \phi} \bigg|_* \delta \phi_* + \frac{\partial N}{\partial \lambda} \bigg|_* \lambda + \frac{1}{2} \frac{\partial^2 N}{\partial \phi^2} \bigg|_* \delta \phi_*^2 + \frac{1}{2} \frac{\partial^2 N}{\partial \lambda^2} \bigg|_* \lambda^2 + \frac{\partial^2 N}{\partial \phi \partial \lambda} \bigg|_* \delta \phi_* \lambda. \quad (83)$$

Using Eq. (81), we can easily evaluate the derivatives at the final hypersurface for a fixed $\phi_*$. In particular, the $\lambda$-independence of $N$ at $N = 0$ implies $\partial N / \partial \lambda|_* = \partial^2 N / \partial \lambda^2|_* = 0$. Thus we need to evaluate only the other three terms, just as in Section V A. This means that, in evaluating $\delta N$, we only need the linear terms in $\lambda$, while we have to take into account the $\phi$ dependence of $N$ up to the second order. That is, we can obtain $\delta N$, up to the second order, by using

$$\phi = \frac{\phi_*}{1 + \lambda} \left( e^{-\kappa N} + \lambda e^{(3 + \eta - \kappa)N} \right). \quad (84)$$

By taking the derivatives of both sides of Eq. (84) and setting $N = \lambda = 0$ in the end, the necessary derivatives are easily computed as

$$\frac{\partial N}{\partial \phi} \bigg|_* = \frac{1}{-\kappa \phi_*}, \quad \frac{\partial^2 N}{\partial \phi^2} \bigg|_* = \frac{3 + \eta}{\kappa^2 \phi_*}, \quad \frac{1}{2} \frac{\partial^2 N}{\partial \phi \partial \lambda} \bigg|_* = \frac{1}{2 \kappa^2 \phi_*^2}. \quad (85)$$

Now we are to identify $\delta \phi_*$ and $\lambda$ with those generated from quantum fluctuations on flat slicing, $\delta \phi$. To do so let us consider the evolution of $\delta \phi$ on superhorizon scales. To the leading order in the slow-roll parameter $\epsilon$, which is an extremely good approximation in the present case, $\delta \phi$ on flat slicing satisfies exactly the same equation as for the background, perturbatively given by Eqs. (74) and (76). Naturally $\delta \phi_*$ contains both growing and decaying modes initially, where the subscript 1 denotes it is of linear order. From Eq. (75), we may set

$$\delta \phi_1(N) = C e^{(3 + \eta - \kappa)(N - N_b)} + D e^{-\kappa(N - N_b)}, \quad (86)$$

where $N_b$ is the number of $e$-folds at horizon crossing and one expects $|C| \sim |D|$. As the background trajectory is given by $\phi_0 \propto e^{-\kappa N}$, it follows that the $D$-term, which has the same time-dependence as $\phi_0$, corresponds to the adiabatic perturbation along the background trajectory. Thus the $C$-term corresponds to the perturbation of the background trajectory. Since the $D$-term is completely negligible at the end of the non-attractor phase, $N = 0$, we conclude

$$\delta \phi_1(0) = \delta \phi_{1*} = \lambda \phi_* \quad (87)$$

hence to the second order,

$$\delta \phi_* = \delta \phi_{1*} + \frac{g}{\phi_*} \delta \phi_{1*}^2, \quad \lambda = \frac{\delta \phi_{1*}}{\phi_*}. \quad (88)$$

Combining Eqs. (85) and (88), we obtain $\delta N$ as

$$\delta N = -\frac{\delta \phi_{1*}}{\kappa \phi_*} + \left[ -\kappa \left( g - \frac{1}{2} \right) - (3 + \eta) \right] \frac{\delta \phi_{1*}^2}{\kappa^2 \phi_*^2}, \quad (89)$$

from which we find $f_{NL}^{local}$ as

$$\frac{3}{5} f_{NL}^{local} = -\kappa g + \frac{\kappa}{2} - (3 + \eta) = \frac{(3 + \eta + 3c_s^2)}{4(1 + c_s^2)} \left( 1 - \frac{1}{c_s^2} \right) - (3 + \eta) + \frac{\eta c_s^2}{2(1 + c_s^2)} = -\frac{1}{4 c_s^2} \left( 9 c_s^2 + 2 \eta c_s^2 + 3 + \eta \right). \quad (90)$$

With $\eta = -6$ to obtain a scale-invariant power spectrum,

$$f_{NL}^{local} = \frac{5}{4c_s^2(1 + c_s^2)}. \quad (91)$$

This result is valid for any values of $c_s$, as we have not assumed $c_s \ll 1$, and agrees exactly with the result obtained from the in-in formalism (Eq. (47)). Furthermore, we obtain $f_{NL}^{local} = 5/2$ for $c_s = 1$, in agreement with the result we find in Section V A.
VI. CONCLUSION

The non-attractor inflation models giving $R \propto a^3$ on super-horizon scales are so far the only examples of self-consistent single-field inflation models based on a Bunch-Davies initial state that give a scale-invariant power spectrum and a large squeezed-limit bispectrum, violating Maldacena’s consistency relation \cite{35,38}. The previous work \cite{35,38} shows that the local-form bispectrum parameters from these models are $f_{NL}^{local} = 5/2$ and $3/(4c_s^2)$ for $c_s = 1$ and $c_s \ll 1$, respectively. Therefore, detection of a large local-form bispectrum violating Maldacena’s consistency relation would not rule out all single-field inflation models in full generality; rather, it would rule out all single-field inflation models which are based on a Bunch-Davies initial state and have reached the attractor solutions.

Given the importance of this statement, in this paper we have provided more detailed derivation. We find that two completely different methods, the quantum field theory calculation using the in-in formalism in the comoving gauge and the classical calculation using the $\delta N$ formalism, give the same result, $f_{NL}^{local} = 5(1 + c_s^2)/(4c_s^2)$, which is valid for arbitrary values of $c_s$. This is because the non-attractor model generates non-Gaussianity on super-horizon scales, as the interactions such as $\dot{R}^3$ and $R \dot{R}^2$, which yield the equilateral bispectrum in the usual attractor case (for which $\dot{R} \approx 0$ on super-horizon scales), become proportional to $R^3$ on super-horizon scales via $\dot{R} = 3H \dot{R}$. We also find that the third method using the in-in formalism in the flat gauge and decoupling limit ($c_s \ll 1$) gives the same answer in the appropriate limit. In contrast to the usual attractor single field case, this model does not predict observable equilateral bispectrum.

While (a shorter version of) the derivation of $f_{NL}^{local} = 3/(4c_s^2)$ for $c_s < 1$ using the in-in formalism has already been presented in Ref. \cite{35}, the full derivation for arbitrary $c_s$ and the derivation using the $\delta N$ formalism are new. As the scalar field trajectory is determined by two parameters, we usually specify the scalar field value and its derivative at some epoch, $\phi_i$ and $\dot{\phi}_i$, when applying the $\delta N$ formalism \cite{35}. If the solution has reached the attractor solution, $\phi_i$ is uniquely determined by $\dot{\phi}_i$, and thus we need to differentiate the number of $c$-folds with respect to $\phi_i$ only; otherwise, we must differentiate the number of $c$-folds with respect to both $\phi_i$ and $\dot{\phi}_i$. In this paper, we find it more convenient to use two parameters naturally characterizing the scalar field trajectories, which are not necessarily $\dot{\phi}_i$ or $\phi_i$. We have used this methodology to derive both $f_{NL}^{local} = 5/2$ for $c_s = 1$ (which was previously derived by Ref. \cite{35} using $\phi_i$ and $\dot{\phi}_i$) and $f_{NL}^{local} = 5(1 + c_s^2)/(4c_s^2)$ for $c_s \neq 1$ (which had not previously been derived).

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Appendix A: Equivalence of Lagrangians in the comoving and flat gauges

In this Appendix, we prove that two Lagrangians in the comoving (Section IV A) and flat (Section IV B) gauges are equivalent to each other at the leading order in $c_s^2$ and $\lambda/\Sigma$. For the comoving gauge, the leading terms include the first three terms in Eq. (44) and the field-redefinition terms in Eq. (45); for the flat gauge, they include both terms in Eq. (52). We assume that $H$, $\dot{\phi}$, $\eta$, and $c_s^2$ are approximately constant, and $c_s^2$ and $\lambda/\Sigma$ are much bigger than unity. The difference between the first three terms in Eq. (44) and both terms in Eq. (52) is

$$\mathcal{L}_3^R - \mathcal{L}_3^\phi = \frac{H}{c_s^2} - \frac{3 \epsilon}{c_s^2} \dot{R} R^2 \frac{a \mathcal{R} R}{H} \frac{\partial \mathcal{R}}{\partial \mathcal{R}} - \frac{3 \epsilon}{c_s^2} a \mathcal{R} (\partial \mathcal{R})^2 - \frac{\epsilon}{c_s^2} a \mathcal{R} (\partial \mathcal{R})^2 \frac{\partial \mathcal{R}}{\partial \mathcal{R}} \frac{\partial \mathcal{R}}{\partial \mathcal{R}}.$$

Defining

$$- \frac{1}{2} \frac{\delta \mathcal{L}_2}{a^2} \frac{\delta \mathcal{R}}{H} \equiv \frac{\epsilon}{c_s^2} \mathcal{R} \frac{\partial \mathcal{R}}{\partial \mathcal{R}} + (3 + \eta) H \frac{\epsilon}{c_s^2} a \mathcal{R} \frac{\partial \mathcal{R}}{\partial \mathcal{R}} - \frac{\epsilon}{a^2} \mathcal{R} \frac{\partial \mathcal{R}}{\partial \mathcal{R}},$$

and integrating by parts the first term in Eq. (A1), Eq. (A1) becomes

$$\frac{1}{H c_s^2} \left( \eta H a^3 R^2 \frac{\partial \mathcal{R}}{\partial \mathcal{R}} + 2 a c_s^2 R \mathcal{R} \frac{\partial \mathcal{R}}{\partial \mathcal{R}} + \epsilon c_s^2 a H \mathcal{R} (\partial \mathcal{R})^2 + c_s^2 a H \mathcal{R} (\partial \mathcal{R})^2 \right) + \frac{1}{H c_s^2} \mathcal{R} \frac{\delta \mathcal{L}_2}{\delta \mathcal{R}},$$

where we have dropped a temporal total derivative proportional to $\frac{d}{dt}(a^3 \mathcal{R}^2 \mathcal{R})$. 
Using the following integration by parts (ignoring spatial total derivatives),
\[
aR \dot{R} \partial^2 \! R \to \frac{1}{4} \frac{d}{dt} (a R^2 \partial^2 \! R) + \frac{a H}{2} R (\partial R)^2 - \frac{a}{2} \dot{R} (\partial R)^2,
\]
the difference between Lagrangians in two gauges given in Eq. (A1) becomes
\[
\frac{1}{H c_s^2} \left( \eta c_s a R^3 \dot{R} + \frac{1}{2} \eta c_s^2 a \dot{R} \partial^2 \! R \right) + \frac{1}{H c_s^2} R \dot{R} \frac{\delta L_2}{\delta R},
\]
where we have dropped a temporal total derivative proportional to \( \frac{d}{dt} (\dot{\eta} a R^2 \partial^2 \! R) \). Integrating by parts the first term in Eq. (A5) and ignoring \( \dot{\eta} \), this term gives
\[
- \frac{1}{2} \eta c_s a R^3 \ddot{R} - \frac{1}{2} c_s (\eta + 3) H^2 a^3 \dot{R}^2,
\]
where we have dropped a temporal total derivative proportional to \( \frac{d}{dt} (c_s \dot{R} R^2) \).

Inserting Eq. (A6) into Eq. (A5), the difference finally becomes
\[
\left( \frac{\eta}{4 c_s^2} R^2 + \frac{1}{c_s^2 H} R \dot{R} \right) \frac{\delta L_2}{\delta R},
\]
which is equivalent to the field-redefinition terms given in Eq. (38). The three temporal total-derivative terms we have dropped give no contribution to the three-point function, as the contraction of two \( R \)’s at the equal time vanishes. This ends the proof.

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