Control Under Action-Dependent Markov Packet Drops: An Event-Triggered Approach

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Abstract—In this paper, we consider the problem of second moment stabilization of a scalar linear plant with process noise. We assume that the sensor must communicate with the controller over an unreliable channel, whose state evolves according to a Markov chain, with the transition matrix on a timestep depending on whether there is a transmission or not on that timestep. Under such a setting, we propose an event-triggered transmission policy which meets the objective of exponential convergence of the second moment of the plant state to an ultimate bound. Furthermore, we provide upper bounds on the transmission fraction of the proposed policy. The guarantees on performance and transmission fraction are verified using simulations.

Index Terms—networked control systems, control under communication constraints, action-dependent channel, event-triggered control, second moment stability

I. INTRODUCTION

In networked control systems (NCS) feedback occurs over a communication channel that may introduce a number of effects such as sampling, packet drops and time delays. The resulting limitations on the communication resources necessitates the design of parsimonious, system aware communication and control. In this paper, we study the problem of controlling a scalar linear system, using an event-triggered approach, over an unreliable action-dependent Markov channel.

Literature Review: The last two decades have seen extensive research on NCS or control over networks [2]–[4]. Stochastic NCS over lossy or unreliable communication channels is also an extensively studied area [4]–[6]. A common assumption in this literature is that the packet drops are independent and identically distributed (i.i.d.) on every timestep. However, some works consider packet drop probabilities which evolve in time as a Markov process. For example, [7], [8] consider the problem of estimation and [9] considers the problem of stabilization of nonlinear systems, each under Markov packet drops. References [10], [11] explore the problem of finite data rate control under Markovian packet losses. On the other hand, [12] considers the problem of control over a channel that supports a finite data rate, with the data rate evolving according to a Markov chain. Finally, [13] is concerned with mean-square stabilization of a linear system under Markov losses and Gaussian transmission noise, while [14] considers mean-square stabilization over an AWGN channel with fading subject to a Markov chain.

In the literature on communication systems, Markov models for channels have a long history, starting with the work of Gilbert [15] and Elliott [16]. The paper [17] is a relatively recent survey on Markov modeling of fading channels. Channels whose properties depend on past actions have also been explored, including as models for other applications. The reference [18] is a recent survey on models and research work on systems whose operation depends on a “utilization dependent component” such as queuing in action dependent servers, iterative learning algorithms and systems with energy harvesting components, among other problems.

In this paper, the actions we seek to design are the transmission times from a sensor to a controller. In particular, we take an event-triggered approach, which in the last decade has been a very popular method for parsimonious transmissions in NCS [19]–[21]. However, the volume of work on event-triggered control in a stochastic setting is still not as considerable as in the deterministic setting. Some papers that consider random packet drops are [22]–[26], while [27], [28] study stochastic stability with event-triggered control. This paper builds upon our previous work on event-triggered control under Bernoulli packet drops [29] and event-triggered control over Markov packet drops [1].

Contributions: In this paper, we study the problem of second-moment stabilization of a scalar linear plant with process noise over an unreliable channel/network. In particular, the channel state determines the packet drop probability and evolves according to a finite state space Markov process that also depends on the past transmission actions.

Our first contribution is modeling an NCS with an action-dependent Markov channel and design of a transmission policy over such a channel. To the best of our knowledge, such channels have not been considered before in the context of NCS. Our second contribution is a two-step design of event-triggered transmission policy. This design approach is in a spirit similar to our earlier work [1], [29]. We provide a necessary condition on the plant dynamics and the channel parameters for our transmission policy to work. This necessary condition is similar to the conditions often found in the data rate limited control [30] and NCS in general. Our third contribution is analysis of the proposed event-triggered transmission policy and guarantee of second moment stability with an exponential convergence to a desired ultimate bound. The fourth contribution is an upper bound on the transmission fraction (the fraction of timesteps, in a time duration, on which a transmission occurs) resulting from the event-triggered policy. We provide upper bounds on the asymptotic transmission fraction as well as for the 'transient' transmission fraction.
Notation: We let \( \mathbb{R} \), \( \mathbb{Z} \), \( \mathbb{N} \), and \( \mathbb{N}_0 \) denote the sets of real numbers, integers, natural numbers and non-negative integers, respectively. We use the standard font for scalar quantities while boldface for vectors and matrices. The notations \( 1 \), \( \delta \), and \( I \) denote the vector with all 1s, the vector whose \( i \)th takes the value 1 and 0 everywhere else, and the identity matrix, respectively, of appropriate dimensions. We use \( \rho (\mathbf{A}) \) to denote the spectral radius of a real square matrix \( \mathbf{A} \). We denote the space of probability vectors (i.e. vectors with non-negative entries that sum to 1) of \( n \) dimensions as \( \mathbb{P}^n \). The notation \( \text{Pr}_i \) denotes the probability of an event. We denote a generic transmission policy using \( T \), and \( \mathbb{E}_T \) denotes expectation of a random variable under a given transmission policy \( T \). We denote the cardinality of a finite set \( S \) as \( |S| \). For integers \( a \) and \( b \), we let \( [a, b]_{\mathbb{Z}} \), \( (a, b]_{\mathbb{Z}} \), and \( (a, b)_{\mathbb{Z}} \) represent the finite sets \( [a, b] \cap \mathbb{Z} \), \( (a, b] \cap \mathbb{Z} \), and \( (a, b) \cap \mathbb{Z} \), respectively. For random variables \( X \), \( Y \), and \( Z \), the tower property of conditional expectation is
\[
\mathbb{E} \left[ \mathbb{E} [X \mid Y, Z] \mid Y \right] = \mathbb{E} [X \mid Y].
\]

II. System Description

In this section, we describe the model of the plant, channel, controller and the control objective.

A. Plant and Controller Model

Consider a scalar linear plant with process noise
\[
x_{k+1} = ax_k + u_k + v_k, \quad x_k, u_k, v_k \in \mathbb{R}, \quad \forall k \in \mathbb{N}_0.
\]
The parameter \( a \) is the inherent plant gain, which we assume is unstable, i.e. \( |a| > 1 \). The variables \( x_k \), \( u_k \) and \( v_k \) are the plant state, the control input and the process noise, respectively at \( k \). Apart from the plant state \( x_k \), we let \( [a, b]_{\mathbb{Z}}, (a, b]_{\mathbb{Z}} \) and \( (a, b)_{\mathbb{Z}} \) represent the finite sets \( [a, b] \cap \mathbb{Z}, (a, b] \cap \mathbb{Z} \), and \( (a, b) \cap \mathbb{Z} \), respectively. For random variables \( X \), \( Y \), \( Z \), the tower property of conditional expectation is
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We assume that, at each timestep, a sensor perfectly measures the plant state and can decide on whether to transmit a packet with the plant state to the controller. We denote the sensor’s transmission decision on timestep \( k \) by \( t_k \) and we let
\[
t_k := \begin{cases} 
1, & \text{if sensor transmits at } k \\
0, & \text{if sensor does not transmit at } k.
\end{cases}
\]
The sensor determines \( t_k \) at each timestep \( k \) according to an event-triggered transmission policy on the basis of plant state and all the information available to it on timestep \( k \). Even if the sensor transmits a packet at timestep \( k \) (\( t_k = 1 \)), the packet may be dropped by the communication channel according to a packet drop model which we describe in Section II-B. We let \( r_k \) be the reception indicator, which takes values as follows
\[
r_k := \begin{cases} 
1, & \text{if } t_k = 1 \text{ and packet received} \\
0, & \text{if } t_k = 1 \text{ and packet dropped} \\
0, & \text{if } t_k = 0.
\end{cases}
\]
The controller maintains a controller state, \( \hat{x}_k^+ \), which it uses to generate the input \( u_k := L \hat{x}_k^+ \), where \( L \) is a constant such that \( \bar{a} := (a + L) \in (-1, 1) \). The controller state evolves as
\[
\hat{x}_k^+ = \begin{cases}
x_k, & \text{if } r_k = 1 \\
\hat{x}_k, & \text{if } r_k = 0,
\end{cases}
\] where \( \hat{x}_k := \hat{x}_{k-1}^+ \) is the estimate of the plant state given past data. Corresponding to the controller state and plant state estimate, we define the estimation error \( z_k \) and controller state error \( \hat{z}_k^+ \) as follows.
\[
\hat{z}_k := x_k - \hat{x}_k, \quad \hat{z}_k^+ := x_k - \hat{x}_k^+.
\]

The two quantities differ only on successful reception times. It is possible to write the plant state evolution equation in terms of these errors as follows.
\[
x_{k+1} = ax_k + L\hat{x}_k^+ + v_k = \hat{a}x_k - L\hat{x}_k^+ + v_k
\]
\[
\hat{x}_{k+1} = \hat{a}\hat{x}_k^+.
\]

Equations (2)-(4) compositely describe the evolution of the plant state, controller state and the estimate of plant state.

B. Channel Model

We let the communication channel be an action-dependent finite state space Markov channel (FSSMC). We denote the channel state at timestep \( k \) by \( \gamma_k \in \{1, \ldots, n\} \), with \( n \) a finite positive integer. We assume that the probability distribution of \( \gamma_{k+1} \) depends on \( \gamma_k \) and \( t_k \), the transmission decision on timestep \( k \). Thus, the evolution of the channel is an action-dependent Markov process. We let \( p_{ij}^{(0)} \) and \( p_{ij}^{(1)} \) denote the probabilities of the channel state transitioning from \( j \) to \( i \) given \( t_k \), respectively.
\[
\begin{align*}
p_{ij}^{(0)} &= \text{Pr}[\gamma_{k+1} = i | \gamma_k = j, t_k = 0] \\
p_{ij}^{(1)} &= \text{Pr}[\gamma_{k+1} = i | \gamma_k = j, t_k = 1].
\end{align*}
\]

We let \( \mathbf{P}_0 \) and \( \mathbf{P}_1 \) be column-stochastic matrices, whose \((i,j)\)th elements are \( p_{ij}^{(0)} \) and \( p_{ij}^{(1)} \), respectively.

We model the unreliability of the channel through a packet drop probability \( e_i \) for each element \( i \) of the channel state space. Thus, if on timestep \( k \) the channel state \( \gamma_k = i \) and if the sensor transmits a packet then the channel drops the packet with probability \( e_i \) and communicates the packet successfully to the controller with probability \( 1 - e_i \), i.e.,
\[
r_k := \begin{cases} 
1, & \text{w.p. } (1 - e_i) \text{ if } t_k = 1 \\
0, & \text{w.p. } e_i \text{ if } t_k = 1 \\
0, & \text{otherwise}
\end{cases}
\]
where “w.p.” stands for “with probability”. Thus, the packet drops on each timestep is Bernoulli, though not i.i.d. We collect the packet drop probabilities across all possible channel states in the vector \( \mathbf{e} := [e_1, e_2, \ldots, e_n]^T \in [0, 1]^n \). Correspondingly, we define the transmission success probability vector \( \mathbf{d} \) as \( \mathbf{d} := 1 - \mathbf{e} \).

C. Sensor’s Information Pattern

Next, we describe the information available to the sensor to make the transmissions decisions \( t_k \). Apart from the plant state \( x_k \) that the sensor can measure perfectly on each timestep \( k \), we assume that if a successful reception occurs on timestep \( k \), then the controller acknowledges it by relaying the reception indicator variable \( r_k \) and the channel state \( \gamma_k \) over an error-free feedback channel, which the sensor may use the information only on subsequent timesteps.

To describe all the information available to the sensor on timestep \( k \) more formally, we first introduce the variables \( R_k \)
and \( R_k^+ \) to track the latest reception time before and latest reception time until timestep \( k \), respectively. Thus,
\[
R_k := \max \{ i < k : r_i = 1 \}, \quad R_k^+ := \max \{ i \leq k : r_i = 1 \}.
\]
The variable \( R_k \) is useful for the sensor’s decision making while \( R_k^+ \) is helpful in the analysis. Further, we let \( S_j \) for \( j \in \mathbb{N}_0 \) be the \( j^\text{th} \) successful random reception time, that is,
\[
S_0 = 0, \quad S_{j+1} := \min \{ k > S_j : r_k = 1 \}, \quad \forall j \in \mathbb{N},
\]
where without loss of generality, we have assumed that the zeroth successful reception occurs on time 0.

From the controller feedback, the sensor knows \( R_k \) and \( \gamma_{R_k} \) before deciding \( t_k \), from which the sensor can utilize the channel evolution model to obtain the probability distribution of the channel state \( p_k \in \mathbb{P}^n \) given \( R_k, \gamma_{R_k} \) and all the transmission decisions from \( R_k \) to \( k \)-1, that is,
\[
p_k(i) := \Pr \left[ \gamma = i \mid R_k, \gamma_{R_k}, \{ t_w \}_{R_k}^{k-1} \right],
\]
where \( p_k(i) \) is the \( i^\text{th} \) element of the vector \( p_k \). Thus, we can obtain \( p_k \) recursively as
\[
p_{k+1} = \begin{cases} p_k \delta_{\gamma_k}, & \text{if } t_k = 1 \text{ and } r_k = 1 \\ p_k \rho_k, & \text{if } t_k = 0 \text{ and } r_k = 0 \\ p_k \theta_k, & \text{if } t_k = 1 \text{ and } r_k = 0. \end{cases}
\]
We also let
\[
p_k^+ := \begin{cases} \delta_{\gamma_k}, & \text{if } r_k = 1 \\ \rho_k, & \text{if } r_k = 0. \end{cases}
\]

We represent by \( I_k \) the information available to the sensor about the controller’s knowledge of plant state before transmission while we use \( I_k^+ \) to denote the information available to the sensor after channel state feedback (if any). Thus, \( I_k := I_k \) when \( r_k = 0 \), and \( I_k^+ \) contains \( r_k \) and \( \gamma_k \) over \( I_k \) when \( r_k = 1 \). Noting the same, we define \( I_k \) and \( I_k^+ \) as
\[
I_k := \{ k, x_k, z_k, R_k, x_{R_k}, p_k, t_{k-1}, r_{k-1}, \gamma_{k-1} \},
\]
\[
I_k^+ := \{ k, x_k, z_k^+, R_k^+, x_{R_k}^+, p_k^+, t_k, r_k, \gamma_k \}. \tag{6a}
\]
Note that the channel state feedback by the controller is represented as \( r_{k-1} \gamma_{k-1} \) and \( r_k \gamma_k \) in \( I_k \) and \( I_k^+ \), respectively. If \( r_k = 1 \) then \( r_k \gamma_k = \gamma_k \), and if \( r_k = 0 \) then \( r_k \gamma_k = 0 \) and thus no channel state feedback is available. Note that \( \{ I_k \}_{k \in \mathbb{N}_0} \) and \( \{ I_k^+ \}_{k \in \mathbb{N}_0} \) are action-dependent Markov processes. In particular, the probability distribution of \( I_k \) conditioned on \( \{ I_s, t_s \}_{s=0}^{k-1} \) can be shown to be the same as the one conditioned on \( \{ I_{k-1}, t_{k-1} \} \). Similarly, \( \{ I_k^+ \} \) is “sufficient information” to determine the distribution of \( I_{k+1} \) given all the past information

### D. Control Objective

The control objective is exponential second-moment stabilization of the plant state to an ultimate bound. Given the plant and the controller models in Section II-A, the only decision making left to be designed is the sensor’s transmission policy \( \mathcal{T} \), which determines \( t_k \) for each timestep \( k \). In particular, we seek to design a feedback transmission policy using the available information \( I_k \) on timestep \( k \). The **offline control objective** that we seek to guarantee is
\[
E_{\mathcal{T}} \left[ x_k^2 \mid I_0^+ \right] \leq \max \{ c^2 x_0^2, B \}, \quad \forall k \in \mathbb{N}_0, \quad \tag{7}
\]
which is to have the second moment of the plant state decay exponentially at least at a rate of \( c^2 \) until it settles to the ultimate bound \( B \). We assume that the convergence rate parameter \( c^2 \in (n^2, 1) \). Note that (7) prescribes the restriction on the plant state evolution in an offline fashion, in terms of only the initial information. However, a recursive formulation of the control objective is more conducive to designing a feedback transmission policy.

To design a feedback transmission policy, we need to define an online version of the control objective. First, we define the **performance function** \( h_k \) for every timestep \( k \) as follows
\[
h_k := x_k^2 - \max \{ c^{2(k-k)}x_k^2, B \}.
\]
Then, the **online objective** is to ensure
\[
E_{\mathcal{T}} \left[ h_k^2 \mid I_0^+ \right] \leq 0, \quad \forall k \in \mathbb{N}_0. \quad \tag{8}
\]
We borrow from Lemma III.1 from [29], which demonstrates that any transmission policy that satisfies the online objective also satisfies the offline objective.

**Lemma II.1** ( Sufficiency of the online objective [29]). If a transmission policy \( \mathcal{T} \) satisfies the online objective (8) then it also satisfies the offline objective (7).

Note that in the control objective (7), the sources of randomness that determine the expectation are the transmission policy \( \mathcal{T} \), the random channel behavior, and the process noise. The transmission policy and the random channel behavior determine the successful reception times while the process noise affects the evolution of the performance function during the inter-reception times. As the online objective (8) is essentially a condition on the evolution of the performance function during the inter-reception times, Lemma II.1 continues to hold in the setting of this paper.

### III. Two-Step Design of Transmission Policy

Designing a transmission policy so that the described system meets the control objective (7) or even the stricter online objective (8) poses many challenges. The main challenge stems from the random packet drops, which makes the necessity of a transmission on timestep \( k \) depend on future transmission decisions. Furthermore, the evolution of the channel state depends on all the past and current transmission decisions. Thus, the transmission decisions \( t_k \) cannot be made in a myopic manner and instead must be made by evaluating their impact on the channel and the control objective over a sufficiently long time frame. To tackle this problem, we adopt a two-step design procedure. This general design principle is the same as in [29], wherein the reader can find a more detailed discussion about this procedure as well as its merits. We now describe the two steps of the design procedure.

In the first step, for each timestep \( k \), we consider a family of nominal policies with look-ahead parameter \( D \in \mathbb{N} \). A nominal policy with parameter \( D \) involves a ‘hold-off’ period of \( D \) timesteps from \( k \) to \( k + D - 1 \) during which \( t_k = 0 \), and then there is perpetual transmission, that is \( t_k = 1 \) for all timesteps after \( k + D - 1 \). Thus, letting \( \mathcal{T}_k^D \) be the nominal policy with parameter \( D \), we can formally express it as
\[
\mathcal{T}_k^D : t_i = \begin{cases} 0, & \text{if } i \in \{ k, k + 1, \cdots, k + D - 1 \} \\ 1, & \text{for } i \geq k + D. \end{cases} \tag{9}
\]
In the second step of the design procedure, we construct the event-triggered policy, \( \mathcal{T}_{ct}^D \), using the nominal policies as building blocks. Given (9), one can reason that if the nominal policy with parameter \( D \in \mathbb{N} \) satisfies the online objective from the current timestep \( k \), then a transmission on the current timestep is not necessary to meet the online objective. Further, if the online objective cannot be met from timestep \( k \) using the nominal policy \( \mathcal{T}_k^D \) then it may be necessary to transmit on timestep \( k \). This forms the basis for the construction of the event-triggered policy, which we detail next.

First, we need a method to check if the nominal policy \( \mathcal{T}_k^D \) satisfies the online objective from timestep \( k \). For this, we define the look-ahead function, \( G_D^p \), as the expected value of the performance function \( h_k \) at the next successful reception timestep \( k = S_{j+1} \) under the nominal policy, that is,

\[
G_D^p := \mathbb{E}_{\mathcal{T}_D^p} [h_{S_{j+1}} \mid I_k, S_j = R_k].
\]  

We can evaluate \( G_D^p \) as a total expectation, over all possible values of \( S_{j+1} \), as

\[
G_D^p = \sum_{w=D}^{\infty} \mathbb{E}_{\mathcal{T}_D^p} [h_{S_{j+1}} \mid I_k, S_j = R_k, S_{j+1} = k + w] \Omega_D(w, p_k),
\]  

where \( \Omega_D(w, p) \) is the probability of the event that the first successful reception after timestep \( k \) is at timestep \( k + w \) under the nominal policy \( \mathcal{T}_D^p \) and given \( p \), the probability distribution of the channel state at time \( k \), conditioned on the information at time \( R_k \). Formally,

\[
\Omega_D(w, p) := \mathbb{P}[S_{j+1} = k + w \mid \mathcal{T} = \mathcal{T}_D, p_k = p, S_j = R_k].
\]  

The closed form of \( \Omega_D(w, p) \) is given as follows.

\[
\Omega_D(w, p) = d^T (\mathbf{P}_1 \mathbf{E})^{(w-D)} \mathbf{P}_0^{(D)} p.
\]  

A. The Event-Triggered Policy

We now describe the event-triggered policy. The main idea behind the proposed event-triggered policy is the following. A negative sign of the look-ahead function \( G_D^p \) indicates that it is not “necessary” to transmit on timestep \( k \) as there exists a transmission sequence (given by the nominal policy) that meets the objective at least on the next random reception timestep. However, if the sign of \( G_D^p \) is non-negative, it means that the sensor cannot afford to hold off transmission for \( D \) timesteps from the current timestep \( k \), and still ensure that the online objective is not violated on some future timestep. In the proposed event-triggered transmission policy, the sensor evaluates \( G_D^p \) at every timestep \( k \), and when it turns non-negative the sensor keeps transmitting on every timestep until a successful reception occurs, and then the sensor again waits for \( G_D^p \) to turn non-negative. The event-triggered transmission policy may be described formally as follows.

\[
\mathcal{T}_{ct}^D : t_k = \begin{cases} 0, & \text{if } k \in \{R_k + 1, \ldots, \tau_k - 1\} \\ 1, & \text{if } k \in \{\tau_k, \ldots, Z_k\} \end{cases}
\]  

where \( \tau_k \) is the first timestep after \( R_k \) when \( G_D^p \geq 0 \) and \( Z_k \) is the first timestep, after \( R_k \), on which there is a successful reception. Thus, formally,

\[
\tau_k := \min \{m > R_k : G_D^m \geq 0\},
\]

\[
Z_k := \min \{m > R_k : R_k^* = m\}.
\]

Note that the event-triggered policy is described recursively in terms of \( R_k \), the latest reception time before \( k \), and the look-ahead function \( G_D^p \). As a result, the policy in (14) is valid over the entire infinite time horizon. In the analysis of the policy (14) in the sequel, it is useful to refer to the \( j \)th reception time, denoted by \( S_j \). Similarly, we let

\[
T_j := \min \{m > S_j : G_D^m \geq 0\}.
\]

So, if \( S_j = R_k \) then \( T_j = \tau_k \) and \( S_{j+1} = Z_k \).

IV. IMPLEMENTATION AND PERFORMANCE GUARANTEES

In this section, we describe the implementation details of the proposed event-triggered policy, and analyze the system under this policy through several intermediate results. At the end of the section, we provide sufficient conditions on the ultimate bound \( B \) and the look-ahead parameter \( D \) such that the system meets the online objective (and hence the offline objective) under the event-triggered policy.

A. Closed Form Expression of the Look Ahead Criterion

For implementation of the event-triggered policy (14), we need an easy method to compute the look-ahead function \( G_D^p \). In particular, we provide here a closed form expression of the look-ahead function. We begin by expanding the expectation term in (11) as follows [31]

\[
\mathbb{E} [h_{S_{j+1}} \mid I_k, S_j = R_k, S_{j+1} = k + w] = a^2 w^2 x_k^2 + 2a^2 w (a^w - a^w) x_k z_k + (a^2 w^2 - 2a^w a^w + a^2 w) z_k^2 + M (a^2 w - 1) - \max \{c^2 w^2 (k - R_k), B\}.
\]  

From (11) and (15), it is evident that convergence of \( G_D^p \) requires the convergence of infinite series of the form

\[
g_D(b, p) := \sum_{w=D}^{\infty} b^w \Omega_D(w, p)
\]

\[
= b^D \sum_{w=D}^{\infty} b^{w-D} d^T (\mathbf{P}_1 \mathbf{E})^{(w-D)} \mathbf{P}_0^{(D)} p,
\]  

with \( p \in \mathbb{P}^n \), and \( D \in \mathbb{N} \) and for values of \( b \) equal to \( a^2 \), \( c^2 \), \( a^2 \), \( \bar{a} \bar{a} \) and 1, which satisfy

\[
0 < a^2 < c^2 < 1 < a^2, \ |\bar{a}\bar{a}| < a^2.
\]  

Each of the terms \( g_D(b, p) \) is an infinite matrix geometric series. The criteria for convergence and the closed form of \( g_D(b, p) \) for these values of \( b \) would allow us to determine the
same for $G_k^D$. Thus, our first aim is to examine the condition for convergence of $g_D(b, p)$. The following lemma establishes the necessary and sufficient condition for a matrix geometric series to converge. Although this is a well-known result, we could not find a concise proof in the literature and hence we present a proof ourselves in Appendix A.

**Lemma IV.1** (Convergence of Matrix Geometric Series). Consider a square matrix $K$. The matrix geometric series $\sum_{w=0}^{\infty} K^w$ converges if and only if $\rho(K) < 1$. Further, if $\rho(K) < 1$ then the series converges to $(I - K)^{-1}$.

We can obtain a closed form expression of $g_D(b, p)$ defined in (16) by first expressing it as

$$g_D(b, p) = b^D d^T \left[ \sum_{w=0}^{\infty} (bp_1E)^w \right] P_0^{(D)} p.$$ 

and then applying Lemma IV.1. In particular if $\rho(bp_1E) < 1$ then we obtain

$$g_D(b, p) = b^D d^T (I - bp_1E)^{-1} P_0^{(D)} p.$$ 

In the following result, we apply Lemma IV.1 to provide a necessary and sufficient condition for $G_k^D$ to be well-defined. Its proof appears in Appendix A.

**Lemma IV.2** (Necessary and sufficient condition for the existence of $G_k^D$). $G_k^D$ converges for all values of the probability distribution vector $p_k$ if and only if $a^2 \rho (p_1E) < 1$.

We now proceed to give a closed form expression of the look-ahead function $G_k^D$ in the following lemma. Its proof is presented in Appendix A.

**Lemma IV.3** (Closed form of the look-ahead function). Suppose that $a^2 \rho (p_1E) < 1$. The following is a closed-form expression of the look-ahead function $G_k^D$.

$$G_k^D = g_D(a^2, p_k) x_k^2 + 2 \left( g_D(a, p_k) - g_D(a^2, p_k) \right) x_k z_k + \left( g_D(a^2, p_k) + g_D(a^2, p_k) - 2 g_D(a, p_k) \right) z_k^2 + M \left( g_D(a, p_k) - g_D(1, p_k) \right) - B f_D(1, p_k) + N_k \left[ g_D(c^2, p_k) - f_D(c^2, p_k) \right]$$

where $M := M(a^2 - 1)^{-1}, N_k := c^{2(k-R_k)} x_k^2$, the closed form of the function $g_D(b, p)$ is given in (18), while $f_D(b, p)$ is given by

$$f_D(b, p) := b^D d^T (p_1E)^{(\mu - D)} (I - bp_1E)^{-1} P_0^{(D)} p.$$ 

Finally, $\mu$ is defined as follows

$$\mu := \max \left\{ D, \left[ \log(x_k^2/R_k)/B \right]/(\log(1/c^2)) \right\}.$$ 

Note that the closed form of $G_k^D$ is a third-degree polynomial of the plant state $x_k$, error $z_k$, and individual elements of $p_k$, and is amenable for online computation. Furthermore, note that the look-ahead function $G_k^D$ possesses a mathematical structure comprising of a linear operator with unit dimensional rowspace acting on the stochastic vector $p_k$.

**B. Necessary Condition on the Ultimate Bound B**

We now seek a necessary condition on the ultimate bound $B$ for there to exist a transmission policy that satisfies the online objective. To this end, we introduce the open loop performance function, $H(w, y)$, which we define as the expectation of the performance function $h_{S_j+1}$ conditioned upon $I_{S_j}^+$, and the event that $S_j+1 = S_j + w$ and $x_{S_j}^2 = y$, that is,

$$H(w, y) := \mathbb{E} \left[ h_{S_j+1} \mid I_{S_j}^+, x_{S_j}^2 = y, S_j+1 = S_j + w \right].$$ 

Note that $H(w, x_{S_j}^2)$ is very similar to (15) except that $H$ is conditioned upon $I_{S_j}^+$ and defined for the special case of $k = S_j$. Thus, the closed form of $H(w, x_{S_j}^2)$ may be obtained from (15) by replacing $k$ with $S_j$, $x_k$ with $x_{S_j}$ and $z_k$ with $z_{S_j}^2 = 0$ and $R_k$ with $R_{S_j} = S_j$. Hence we have

$$H(w, x_{S_j}^2) = \tilde{a}^2 w x_{S_j}^2 + \tilde{M}(a^2w - 1) - \max\{c^2 w x_{S_j}^2, B\}.$$ 

Note that $H(w, x_{S_j}^2) < 0$ indicates that given the information $I_{S_j}^+$, the online objective is expected to be satisfied on timestep $S_j + w$. Conversely, a positive sign implies that the online objective is expected to be violated on timestep $S_j + w$. Using this observation, we demonstrate in the following proposition that for $B$ less than a critical $B_0$, there exists no transmission policy that can satisfy the online objective. We provide its proof in Appendix A.

**Proposition IV.4** (Necessary condition on the ultimate bound for meeting the online objective). If $B < B_0 := \tilde{M} \log(a^2)/\log(c^2/a^2)$ then no transmission policy satisfies the online objective.

Proposition IV.4 demonstrates that $B > B_0$ is a necessary condition on $B$ for a transmission policy to satisfy the online objective. In the following subsection, we further analyse the open-loop performance function $H(w, y)$ to find a sufficient criterion to check whether a given $B$ and $D$ ensure that the online objective is met under the event-triggered policy.

**C. The Performance-Evaluation Function, $J_{S_j}^D$**

For the purpose of analysing system performance between any two successive reception times $S_j$ and $S_{j+1}$, we define the performance-evaluation function, $J_{S_j}^D$. It is defined similarly as $G_k^D$ in (10), though only for $k = S_j$ (successful reception times) and conditioned upon the information set $I_{S_j}^+$ instead of $I_{S_j}$. In particular, we let

$$J_{S_j}^D := \mathbb{E}_{I_{S_j}^{D+1}} \left[ h_{S_j+1} \mid I_{S_j}^+ \right] = \sum_{w=D}^{\infty} H(w, x_{S_j}^2) \tilde{N}_D(w, \gamma_{S_j}).$$

Here, $\tilde{N}_D(w, \gamma)$ denotes the probability of getting a successful reception $w$ timesteps after $S_j$ starting with channel state $\gamma$ on $S_j$ under the nominal policy $\mathcal{T}_{S_j+1}$. The purpose of the function $\tilde{N}_D(w, \gamma)$ is analogous to that of $\Omega_D(w, p)$ in $G_k^D$, and is formally defined as

$$\tilde{N}_D(w, \gamma) := \mathbb{P}_{\tilde{S}_{j+1} = S_j + w \mid \mathcal{T} = \mathcal{T}_{S_j+1}^+, \gamma_{S_j} = \gamma}.$$ 

Note that there are two differences between the closed forms of $\Omega_D(w, p)$ and $\tilde{N}_D(w, \gamma)$. First, we use channel state $\gamma$ instead probability vector $p$ in the definition of $\Omega_D(w, \gamma)$, since channel state distribution $p_{S_j} = \delta_{\gamma_{S_j}}$ can be inferred from $I_{S_j}^+$ using (5). Second, the expectation in (22) is conditioned upon nominal policy $\mathcal{T}_{S_j+1}^+$ as opposed to the policy $\mathcal{T}_{S_j}^D$ in the definition of $G_k^D$ in (11). This is because $G_k^D$ is defined
for the purpose of deciding $t_k$, while $\mathcal{J}_D^D$ is defined post-transmission on timestep $S_i$ for the purpose of convergence analysis. Therefore $\mathcal{G}^D_k$ is calculated for a timestep with an underlying nominal policy in which $t_i = 0$ for $i \in \{k, k+1, \ldots, k+D-1\}$, while the definition of $\mathcal{J}_D^D$ is already conditioned upon the fact that $t_{S_j} = 1$ for all $j$. The closed form of $\Omega_D(w; \gamma)$ can be obtained similarly to the closed form of $\Omega_D(w, y; p)$, and is given as

$$
\tilde{\Omega}_D(w, \gamma) = d^T(P_1)_{(w-D)}P_0^{(D-1)}P_1\delta_\gamma. \quad (24)
$$

For a well-chosen value of $B$, it can be shown that the open loop performance function possesses the property of sign monotonicity. This property is an important characteristic of $H(w, y)$ and will prove useful in later results.

**Theorem IV.5** (Sign behaviour of the open-loop performance function, Proposition IV.6, [29]). There exists a $B^* \geq B_0$ with $B_0$ defined in Proposition IV.4 such that if $B > B^*$, then $H(w, y) > 0$ implies $H(s, y) > 0$ for all $s \geq w$. \hfill $\square$

The value of $B^*$ defined in Theorem IV.5 can be numerically computed using the procedure in Appendix B, which is based on the proof of Lemma IV.13 in [29]. We now provide a closed form expression of the performance evaluation function $\mathcal{J}_s^D$, similar to the closed form of $\mathcal{G}^D_k$ in Lemma IV.3. The proof appears in Appendix A.

**Lemma IV.6** (Closed form of performance-evaluation function). Suppose that $a^2 \rho(P_1, E) < 1$. A closed form of the performance-evaluation function $\mathcal{J}_s^D$ is given as

$$
\mathcal{J}_s^D := \tilde{g}_D(\theta^2, \gamma_0) + \bar{M} \left[ \tilde{g}_D(\theta^2, \gamma_0) - \tilde{g}_D(1, \gamma_0) \right] - B \tilde{f}_D(1, \gamma_0) + x \tilde{f}_D(\theta, \gamma_0),
$$

where

$$
\tilde{f}_D(b, \gamma) := b^\nu d^T(P_1)_{(\nu-D)}(1 - bP_1)^{-1}P_0^{(D-1)}P_1\delta_\gamma,
$$

and finally, $\nu$ is defined as

$$
\nu := \max \left\{ D, \frac{\log(x^2, \gamma_0)}{\log(1/c^2)} \right\}.
$$

The next result is concerned with the expected value of $\mathcal{G}^D_k$ after no transmission or after successful reception and the channel state feedback on timestep $k$. Note that this result is valid for any transmission policy $T$.

**Theorem IV.7** (Expected value of look-ahead function on next timestep). Let $T$ be any transmission policy. Then, the following hold.

(a) $\mathbb{E}_T[\mathcal{G}^D_k + 1 \mid I_k, t_k = 0] = \mathcal{G}^D_{k+1}.$

(b) $\mathbb{E}_T[\mathcal{G}^D_k + 1 \mid I_k, t_k = 1, \gamma_k] = \mathcal{J}_D^D + 1.$

**Proof.** (a): Note that

$$
\mathbb{E}_T[\mathcal{G}^D_k + 1 \mid I_k, t_k = 0] = \left[ \begin{array}{c}
\mathbb{E}_T[\mathbb{E}_T[d_0^T \mid h_{S+1} \mid I_k, t_k = 0] \mid I_k = I_k, t_k = 0] \\[= \mathbb{E}_T[d_0^T \mid h_{S+1} \mid I_k+1, S_j = R_k+1] \mid I_k, t_k = 0] \end{array} \right],
$$

where [r1] follows from (10), while in [r2] we can replace the policy $T$ with $T_{k+1}^{D+1}$ because the event $t_k = 0$ is consistent with the policy $T_k^{D+1}$ on time step $k$ and once $t_k = 0$ is fixed the expected value of $\mathcal{G}^D_k$ is independent of the transmission policy used on subsequent timesteps. In [r3], we also use the fact that if $t_k = 0$ then $R_{k+1} = R_k$. Finally, [r3] uses the fact that $\{ I_k, t_k \}$ is sufficient information and then the tower property.

(b): For proving this part, we observe that $I_k$ and the additional information that $r_k = 1$ and $\gamma_k$ implies the knowledge of $I_k^+$. Considering this fact and proceeding with a similar methodology as the proof of claim (a), we observe that

$$
\mathbb{E}_T[\mathcal{G}^D_k + 1 \mid I_k, t_k = 1, \gamma_k] = \mathbb{E}_T[\mathbb{E}_T[d_0^T \mid h_{S+1} \mid I_k+1, S_j = R_k+1] \mid I_k^+, t_k = 1],
$$

where $I_k^+$ is calculated for a subsequent policy used on subsequent timesteps. In [r2], we also use the fact that if $t_k = 0$ then $R_{k+1} = R_k$. Finally, [r3] uses the fact that $\{ I_k, t_k \}$ is sufficient information and then the tower property.
of \( Q(\theta) \) is monotonically increasing in \( \theta \), and thus, \( Q(D) < 0 \) ensures \( Q(\theta) < 0 \) for \( \theta \in \{1, \ldots, D\} \). The first and the second derivatives of \( Q(\theta) \) with respect to \( \theta \) are

\[
\frac{dQ(\theta)}{d\theta} = B \frac{2^a}{C^2} \log \left( \frac{a^2}{C^2} \right) Z_0(a^2) + M \log(a^2) Z_0(a^2)
\]

\[
\frac{d^2 Q(\theta)}{d\theta^2} = B \frac{2^a}{C^2} \log^2 \left( \frac{a^2}{C^2} \right) Z_0(a^2) + M \log^2(a^2) Z_0(a^2).
\]

Note that each element of the second derivative is strictly positive. Thus, each element of \( Q(\theta) \) is strictly convex in \( \theta \). Also, note that the first derivative of \( Q(\theta) \) at \( \theta = 0 \) is

\[
\frac{dQ(\theta)}{d\theta} \big|_{\theta=0} > B \log \left( \frac{a^2}{C^2} \right) [Z_0(a^2) - Z_0(a^2)] > 0,
\]

where [r1] follows from the fact that \( B \geq B_0 \). Since each element of \( Q(\theta) \) is strictly convex for \( \theta \in \mathbb{R} \) and increasing at \( \theta = 0 \), it follows that each element of \( Q(\theta) \) is monotonically increasing for \( \theta > 0 \). Thus, \( Q(D) < 0 \) implies \( Q(\theta) < 0 \), and thereby \( \mathcal{J}_S^D < 0 \) for all \( \theta \in \{1, \ldots, D\} \).

We consolidate the results so far to provide a theoretical performance guarantee that the event-triggered policy satisfies the online objective (8).

**Theorem IV.10** (Performance guarantee of the event-triggered policy). If \( B > B^* \) (see Appendix B) and the lookahead parameter \( D \) satisfies the condition \( Q(D) < 0 \) then the event-triggered policy (14) guarantees that the online objective (8), and therefore the original offline objective (7), are met.

**Proof.** Given Lemma II.1, it suffices to show that the online objective (8) is met by the event-triggered policy. We center the proof around the following two claims.

**Claim (a):** For any \( j \in \mathbb{N}_0 \), \( \mathbb{E}_{\mathcal{T}_D}[h_{S_j+1} \mid I_{S_j}^+] \leq 0 \) implies

\[
\mathbb{E}_{\mathcal{T}_D}[h_k \mid I_0^+] \leq 0 \quad \text{forall } k \in [S_j, S_j+1].
\]

**Claim (b):** For any \( j \in \mathbb{N}_0 \), \( \mathbb{E}_{\mathcal{T}_D}[h_{S_j+1} \mid I_{S_j}^+] < 0 \).

Together these two claims guarantee that the online objective is met, because

\[
\mathbb{E}_{\mathcal{T}_D}[h_k \mid I_0^+] = \mathbb{E}_{\mathcal{T}_D}\left[ \cdots \mathbb{E}_{\mathcal{T}_D}\left[ h_k \mid I_{S_j}^+ \right] \mid I_{S_j-1}^+ \right] \cdots \mid I_0^+,
\]

where \( \{S_j\} \) are the random reception times and \( S_j = R_k^+ \).

To prove Claim (a), we note that by the definition of open-loop performance function \( H(w, y) \) in (20), we have

\[
\mathbb{E}_{\mathcal{T}_D}[h_k \mid I_0^+] = H(k - S_j, x_{S_j}^2), \quad \forall k \in [S_j, S_j+1].
\]

If \( \mathbb{E}_{\mathcal{T}_D}[h_{S_j+1} \mid I_{S_j}^+] = H(S_j+1 - S_j, x_{S_j}^2) < 0 \), then the sign monotonicity property of the open-loop performance function (Theorem IV.5) implies \( H(k - S_j, x_{S_j}^2) \leq 0 \) for all \( k \in [S_j, S_j+1] \), which proves Claim (a).

We now prove Claim (b). It can be seen from Theorem IV.7 that for all \( k \in (S_j, T_j) \),

\[
\mathbb{E}_{\mathcal{T}_D}[G_{k+1}^D \mid k \in (S_j, T_j) \zeta \mid I_{S_j}^+] \leq \mathbb{E}_{\mathcal{T}_D}[G_{k+1}^D \mid I_{k, t_k} = 0] \mid I_{S_j}^+] \leq \mathbb{E}_{\mathcal{T}_D}[G_{k+1}^D \mid I_{S_j}^+] \tag{26}
\]

where [r1] is obtained by using the tower property and the fact that \( t_k = 0 \) for \( k \in (S_j, T_j) \), while [r2] is obtained from Theorem IV.7. Furthermore, Theorem IV.7 (b) implies that

\[
\mathbb{E}_{\mathcal{T}_D}[G_{S_j+1}^D \mid I_{S_j}^+] = \mathbb{E}_{\mathcal{T}_D}[G_{S_j+1}^D \mid I_{S_j}^+, r_{S_j} = 1, \gamma_{S_j}^+] = \mathcal{J}_S^{D+1}.
\]

Next, we condition the expected value of \( h_{S_j+1} \) over information from timestep \( T_j \) as well as timestep \( S_j \) and using the tower property of conditional expectations, we obtain

\[
\mathbb{E}_{\mathcal{T}_D}[h_{S_j+1} \mid I_{S_j}^+] \leq \mathbb{E}_{\mathcal{T}_D}[h_{S_j+1} \mid I_{T_j}, \gamma_{S_j}^+] \mid I_{S_j}^+] = \mathbb{E}_{\mathcal{T}_D}[G_{I_{S_j}}^D \mid I_{S_j}^+] \tag{28}
\]

where the inner expectation in [r3] is conditioned under the nominal policy \( T_j \). Since for all times \( k \in [T_j, S_j+1] \), we have transmissions \( (t_k = 1) \). We consider two cases: \( T_j \leq S_j + D \) and \( T_j > S_j + D \). In the first case, since \( t_k = 0 \) for \( k \in (S_j, T_j) \), we use (26) and (27) to write (28) as

\[
\mathbb{E}_{\mathcal{T}_D}[G_{I_{S_j}}^D \mid I_{S_j}^+] = \mathbb{E}_{\mathcal{T}_D}[G_{S_j+1}^D \mid I_{S_j}^+] = \mathcal{J}_{S_j}^{D+1},
\]

where Theorem IV.9 ensures that if \( T_j - S_j \leq D \) then \( \mathcal{J}_{S_j}^{T_j-S_j} < 0 \). We now consider the second case in which \( T_j > S_j + D \). Since we have \( t_k = 0 \) for \( k \in (S_j, T_j) \), we use (26) to write (28) as

\[
\mathbb{E}_{\mathcal{T}_D}[G_{I_{S_j}}^D \mid I_{S_j}^+] = \mathbb{E}_{\mathcal{T}_D}[G_{S_j+1}^D \mid I_{S_j}^+] < 0,
\]

since \( G_{I_{S_j}}^D \) is negative, by definition, for \( k \in (S_j, T_j) \). This proves Claim (b), and hence also the result.

**V. Transmission Fraction**

This section analyzes the efficiency of the proposed event-triggered transmission policy in terms of the fraction of times the sensor transmits \( (t_k = 1) \) over a given time horizon. First, we introduce the transmission fraction up to timestep \( K \) as

\[
\mathcal{F}^K := \frac{\mathbb{E}_{\mathcal{T}_D}\left[ \sum_{i=1}^K t_i \mid I_0^+ \right]}{\mathbb{E}_{\mathcal{T}_D}[K \mid I_0^+]},
\]

wherein the stopping timestep \( K \) could itself be a random variable. We call the limit of \( \mathcal{F}^K \) when \( K \to \infty \) as the asymptotic transmission fraction, denoted by \( \mathcal{F}^\infty \).

We also consider another type of transmission fraction which we call the transmission fraction up to state \( \mathcal{X} \), and denote it with \( \mathcal{F}_{\mathcal{X}} \). It is defined as the transmission fraction up to the first reception timestep such that the squared plant state is lesser than \( \mathcal{X} \). That is,

\[
\mathcal{F}_{\mathcal{X}} := \frac{\mathbb{E}_{\mathcal{T}_D}\left[ \sum_{i=1}^{S_{\mathcal{X}}} t_i \mid I_0^+, \{x_{S_i}^2\}_{i=0}^{S_{\mathcal{X}}-1} \geq \mathcal{X}, x_{S_i}^2 < \mathcal{X} \right]}{\mathbb{E}_{\mathcal{T}_D}[S_{\mathcal{X}} \mid I_0^+, \{x_{S_i}^2\}_{i=0}^{S_{\mathcal{X}}-1} \geq \mathcal{X}, x_{S_i}^2 < \mathcal{X}]}.
\]

In Theorem V.1, we provide an upper bound on \( \mathcal{F}_{\mathcal{X}} \) which only involves plant and channel parameters, and \( \mathcal{X} \). From this result, we derive an upper bound on the asymptotic transmission fraction \( \mathcal{F}^\infty \) as a corollary. Together, these results form a figure-of-merit to determine channel utilization for different values of plant and channel parameters, as well as the operational value \( D \) of the look-ahead parameter.
Theorem V.1 (Upper bound on $F_X$). Suppose $Q(D) < 0$ for a given value of $D$. The transmission fraction up to state $X$ is upper bounded by

$$F_X \leq \frac{C^{(1)}}{C^{(0)} + C^{(1)}},$$

where

$$C^{(0)} := \arg\max_{B \in \mathbb{N}_0} \{Q_X(D + B) < 0\}$$

and

$$Q_X(\theta) := [Z_\theta(a^2) - Z_\theta(c^2)] \max\{X, Bc^{-20}\} + M[Z_\theta(a^2) - Z_\theta(1)],$$

with $Z_\theta(b)$ as defined in Theorem IV.9, while $C^{(1)}$ is given by

$$C^{(1)} = \max_{i \in \{1, \ldots, n\}} \{d^T(P_1E)(I - P_1E)^{-1} \delta_i\}.$$

Proof. We find an upper bound on $F_X$ by first considering the time horizon between two successive reception times, and then extending the analysis to an arbitrary number of interception cycles. For $j \in \mathbb{N}_0$, we let $\Delta_j$ be the time horizon $(S_j, S_{j+1})$. Further, throughout this proof, we use the shorthand $\Pi_\theta(\gamma_{S_j}) := P_0(\theta)^{(\theta-1)}P_1\delta_{S_j}$ for notational convenience.

Using the structure of the event-triggered policy, we split $\Delta_j$ into two parts as $\Delta^{(0)}_j := (S_j, T_j)$ and $\Delta^{(1)}_j := (T_j, S_{j+1})$. Hence, for $k \in \Delta^{(0)}_j$, no transmission occurs ($t_k = 0$) while for each $k \in \Delta^{(1)}_j$, a transmission occurs ($t_k = 1$). Now, consider the following two claims.

Claim (a): $\mathbb{E}_{T^D}[\Delta^{(0)}_j | I^+_j, x^2_j > \chi] \geq C^{(0)}$.

Claim (b): $\mathbb{E}_{T^D}[\Delta^{(1)}_j | I^+_j] \leq C^{(1)}$, for all $x_j \in \mathbb{R}$.

Supposing the two claims are true, consider the transmission fraction during the $j$th horizon, $\Delta_j$, conditioned on $I^+_j$.

$$\mathbb{E}_{T^D}[\Delta^{(0)}_j | I^+_j, x^2_j > \chi] \geq C^{(0)}$$

since the transmission fraction is increasing in the term $\mathbb{E}_{T^D}[\Delta^{(0)}_j | I^+_j]$, and decreasing in the term $\mathbb{E}_{T^D}[\Delta^{(1)}_j | I^+_j]$. Now, as this upper bound is independent of the state of the system as long as $x^2_j > \chi$, we obtain the upper bound on $F_X$, stated in the result. Thus all that remains now is to prove claims (a) and (b).

To prove Claim (a), we start by demonstrating that, for a given value of $\theta \in \mathbb{N}$ and under the assumption that $x^2_j \geq \chi$, $J^D_j \leq Q_X(\theta)\Pi_\theta(\gamma_{S_j})$. To this end, we consider two cases, $\chi \in \Lambda_1 = [0, Bc^{-20}]$ and $\chi \in \Lambda_2 = [Bc^{-20}, \infty)$ respectively. If $\chi \in \Lambda_1$, then we have

$$J^D_j \leq R_j(\theta) = Q(\theta)(\Pi_\theta(\gamma_{S_j})) = Q_X(\theta)\Pi_\theta(\gamma_{S_j}),$$

where the inequality is from Proposition IV.8, the first equality from (25) and the second equality from the fact that $X \in \Lambda_1$. Now, consider the case of $x^2_j \geq \chi \in \Lambda_2$. Recall from the proof of Proposition IV.8 that

$$J^D_j \leq [\hat{g}_0(\hat{a}^2, \gamma_{S_j}) - \hat{g}_0(c^2, \gamma_{S_j})] x^2_j + M[\hat{g}_0(a^2, \gamma_{S_j}) - \hat{g}_0(1, \gamma_{S_j})]$$

where $[r_1] = (Z_\theta(a^2) - Z_\theta(c^2)) \max\{X, Bc^{-20}\} + M (Z_\theta(\bar{a}^2) - Z_\theta(\bar{a}))\Pi_\theta(\gamma_{S_j})$, for any given $X \in \Lambda_2$.

Hence, from the design of the event-triggered policy (14), it follows that $T_j > S_j + B$, or in other words, no transmission takes place at least $B$ timesteps from $S_j$, in expectation. Thus,

$$\mathbb{E}_{T^D}[\Delta^{(0)}_j | I^+_j, x^2_j > \chi] \geq C^{(0)}.$$

We now consider Claim (b). Note that $t_k = 1$ for all $k \in \Delta^{(1)}_j$, and by the structure of the event-triggered policy, $\mathbb{E}_{T^D}[\Delta^{(1)}_j]$ is simply the expected number of timesteps for reception under a string of continuous transmission attempts, starting from timestep $T_j$ and channel state $\gamma_{T_j}$. To capture the same, we define the constant $C^{(1)}_i$ for $i \in \{1, \ldots, n\}$ as

$$C^{(1)}_i := \mathbb{E}[w - T_j | w \geq T_j : r_w = 1, t_w = 1, r_w = 0, t_w = 1 | \gamma_{T_j} = i]$$

$$= d^T \sum_{s=0}^{\infty} s (P_1E)^s \delta_i = d^T (P_1E)(I - P_1E)^{-1} \delta_i.$$

We bound $|\Delta^{(1)}_j|$ by simply choosing the highest value of $C^{(1)}_i$ among $i \in \{1, \ldots, n\}$, thereby showing that $C^{(1)}$ is indeed an upper bound on $|\Delta^{(1)}_j|$, and proving Claim (b) and hence also the result.

Note that the term $C^{(0)}$ in the upper bound on $F_X$ is basically the $B$-maximizer of $Q_X(D + B)$ under the constraint that $Q_X(D + B) < 0$. This fact illuminates the trade-off between control performance and transmission fraction, which we highlight in the following remark.

Remark V.2 (Tradeoff between control performance and transmission fraction). Suppose for a given value of $X$ and some $\psi \in \mathbb{N}$, we have $Q_X(\psi) < 0$ but $Q_X(\psi + 1) \delta_i \geq 0$ for at least one $i \in \{1, \ldots, n\}$. Then if the operational value of the look-ahead parameter is $D$, we note that $D + B = \psi$. The system designer can either choose a high value of $D$ (conservative control) but this results in a lower value of $B$, and thus a larger upper bound on $F_X$. Conversely, a lower value of $D$ (aggressive control) leads to a higher $B$, and thus a smaller upper bound on $F_X$.

We show in the following result that an upper bound on the asymptotic transmission fraction, $F^\infty$ can be obtained by setting $X = Bc^{-20}$ in the upper bound of $F_X$ provided in Theorem V.1. We present its proof in Appendix A.

Corollary V.3 (Upper bound on asymptotic transmission fraction). The asymptotic transmission fraction $F^\infty$ is upper
bounded by
\[ F^\infty \leq \frac{C^{(1)}}{C^{(0)}_\infty + C^{(1)}} \],
where
\[ C^{(0)}_\infty := \arg\max_{B \in \mathbb{N}_0} \{ Q \{ D + B \} < 0 \} \]
and \( C^{(1)} \) as defined in Theorem VI.

VI. SIMULATIONS

In this section, we present simulation results to validate the event-triggered policy. We choose a scalar plant with \( \alpha = 1.10 \), with desired convergence rate \( c = 0.98 \) and \( \hat{a} = 0.95c \). We choose the following \( P_0 \) and \( P_1 \) matrices.
\[
P_0 = \begin{bmatrix} 0.5 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.3 \\ 0.0 & 0.1 & 0.2 & 0.1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.1 & 0.0 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.6 & 0.4 & 0.3 \end{bmatrix}.
\]
The matrices \( P_0 \) and \( P_1 \) are chosen to have the characteristic that the probability of going to a higher numbered state is lower in \( P_0 \), and higher in \( P_1 \), as compared to staying in the same or going to a lower numbered state. Correspondingly, we choose the following values of packet drop probability for channel states 1 through 4:
\[
e = [0.1, 0.2, 0.3, 0.4]^T.
\]

We choose the process noise \( v_k \) randomly with independent and identical normal distribution having mean 0 and variance \( M = 1 \). For the chosen parameters, the procedure in Appendix B gives \( B^* = 8.8483 \). We choose \( B = 18 > B^* \) and pick the initial plant state as \( x_0 = 15.5B \).

Simulation Results

We simulated the above system, using MATLAB, for various values of \( D \). We obtained 10,000 empirical iterations for each value of \( D \). Figure 1a demonstrates the effect of the choice of the look-ahead parameter \( D \) on the closed-loop plant state evolution. For a higher value of \( D \), the policy \( T^D \) is more conservative in the sense that the value of squared plant state is in general lesser and farther away from the desired envelope than for a higher value of \( D \) (refer to Remark V.2). Note that in Figure 1a, the squared plant state in general has a lower value with \( D = 2 \) than with \( D = 1 \).

Figure 1b demonstrates the empirical value of \( F^k \) for two values of \( D \) over a horizon of 5,000 timesteps. Note that very large \( k \), the transmission fraction reaches an asymptotic value. Higher values of \( D \) lead to a greater steady state value of the transmission fraction. Moreover, we also calculate and present the theoretical upper bound on transmission fraction for both \( D = 1 \) and \( D = 2 \), using Corollary V.3. Figure 1c depicts empirically calculated \( F_X \), transmission fraction up to state \( X' \), for \( D = 1, D = 2, \) and \( D = 3 \). To generate the empirical value of \( F_X \) the \( X' \)-axis was bucketed with buckets of exponentially increasing size, since finding a continuous empirical relation between \( X' \) and \( F_X \) would be impossible within a finite number of simulated trajectories. We again see that higher values of \( D \) result in higher transmission fractions.

Figure 1d is a plot of the theoretical upper bounds on both \( F^\infty \) and \( F_X \), the asymptotic transmission fraction and the transmission fraction up to state \( X' \), respectively, for various values of the look-ahead parameter \( D \). From this figure, it can be visually verified that the upper bound on \( F^\infty \) given by Corollary V.3 is the same as the upper bound on \( F_{Bc^{-2D}} \) given by Theorem VI, for different values of \( D \).

VII. CONCLUSION

In this paper, we have considered a networked control system consisting of a scalar linear plant with process noise and non-collocated sensor and controller. Further, the sensor communicates over a time-varying channel whose state evolves according to an action-dependent Markov process. The state of the channel determines the probability with which a packet transmitted by the sensor is dropped. In this setting, we have designed an event-triggered transmission policy that guarantees second moment stabilization of the plant state at a desired rate of convergence to an ultimate bound. We also derived upper bounds on the transient and the asymptotic transmission fraction, the fraction of timesteps on which the sensor transmits. We have verified and illustrated our analysis and theoretical guarantees through simulations. Future work in this direction includes incorporation of imperfect measurement of plant and channel state, application of the proposed action-dependent Markov channel framework to specific scenarios such as control over a shared channel and control with energy-harvesting components.

REFERENCES

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APPENDIX A
PROOFS OF AUXILIARY RESULTS

Proof of Lemma IV.1. First we assume that ρ(K) < 1. We let Sη := Σw=0η−1 Kw. Observe that (I − K)Sη = I − Kη. The assumption ρ(K) < 1 implies (I − K) is invertible and hence the following holds Sη = (I − K)−1(I − Kη).

Again since ρ(K) < 1, Kη converges to zero as η → ∞ and thus Sη converges to (I − K)−1 asymptotically.

Now we assume that ρ(K) ≥ 1. Let λ be an eigenvalue of K with |λ| ≥ 1. Consider a corresponding eigenvector v (with possibly complex entries).

Under the assumption ρ(K) ≥ 1, the right hand side of the above equation diverges and therefore the matrix-geometric series (∑w=0∞ Kw) also diverges.

Proof of Lemma IV.2. From (11)-(12) and (15)-(16), we see that an expansion of G^D_k involves terms such as gD(b,p) with b equal to a^2, a^2, a^2, and c^2. Using (17) and noting that ρ(bP_kE) > ρ(bP_kE) when |b_1| > |b_2|, we can state that ρ(a^2P_kE) > ρ(bP_kE) for b assuming values a^2, a^2, and c^2. Thus by Lemma IV.1, ρ(a^2P_kE) < 1 is a necessary and sufficient condition for convergence of G^D_k.

Proof of Lemma IV.3. Most terms in the closed form of G^D_k follow directly from (11), the series expansion of G^D_k, the closed form of ΩD(w,p) in (13), the expansion of the expectation term (15), the definition (16) and the closed form of gD(b,p). We only need to find the closed form of

\[ ∑_{w=D}^{∞} \max\{2w^22(k−R_w)x^2_{R_w}, B\} \Omega_D(b, p_k). \]

We split this summation into two parts based on if c^2wN_k is larger or smaller than B. Observe that μ, defined in (19), is the smallest integer w ≥ D such that B ≥ c^2wN_k. Then, for w ≥ μ

\[ ∑_{w=μ}^{∞} \max\{2w^22(k−R_w)x^2_{R_w}, B\} \Omega_D(w, p_k). \]

This completes the proof.

Proof of Theorem 4.1. The event-triggered rule (1) is

\[ τ_k = inf \left\{ t \geq t_k : y(t) \notin \mathcal{G}_{t_k} \right\} \]

with \( \mathcal{G}_{t_k} \) being the set of triggering conditions defined in (23). The closed form of G^D_k follows from (11) and the expansion of gD(b,p) in (13). The closed form of ΩD(w,p) in (15) is given by

\[ ΩD(w,p) = \sum_{j=0}^{∞} \frac{(−1)^j}{j!} (w^2)^j. \]

This completes the proof.
\[ B f_D(1, p_k) + N_k \left[ g_D(c^2, p_k) - f_D(c^2, p_k) \right], \]
where we obtain \([r1]\) by observing that
\[
\sum_{w=\mu}^{\infty} b^w \Omega_D(w, p) = \sum_{w=\mu}^{\infty} b^w d^T(P_1 E)(w-D) P_0^D p = b^w d^T(P_1 E)(w-D) (b, p),
\]
assuming \(\rho(b P_1 E) < 1\) (see Lemma IV.1). With this we obtain the closed form expression of the look-ahead function \(G^D_k\) provided in Lemma IV.3.

Proof of Proposition IV.4. The proof relies on demonstrating that \(H(w, y) > 0\) for all \(w \in \mathbb{N}\) and for all \(y \in (B, B_0)\). This implies that if \(x_{S_j}^2 \in (B, B_0)\), then the system would violate the online objective on the very next timestep. From (21), note that for a fixed \(y\), the function \(H(w, y)\) can be written as
\[
H(w, y) = \begin{cases} 
1, & \text{if } w \leq w_*(y), \\
2, & \text{if } w > w_*(y),
\end{cases}
\]
with
\[
l_1(w, y) := \bar{a}^2 y + \bar{M}(a^2 w - 1) - c^2 y \\
l_2(w, y) := \bar{a}^2 y + \bar{M}(a^2 w - 1) - B,
\]
where \(w_*(y) := \log(y/B) \log(1/c^2)\) is such that, now it suffices to prove the following two claims.

Claim (a): \(l_1(0, y) > 0\) for all \(w \in \mathbb{N}\) for \(y \in (B, B_0)\).

Claim (b): \(l_2(w, y) > 0\) for all \(w \in \mathbb{N}\) for \(y \in (B, B_0)\).

For proving Claim (a), we first recall the term \(\nu\) in the closed form of \(f_\theta(b, \gamma)\) from Lemma IV.3 and note that \(\nu = 0\) when \(x_{S_j}^2 < Bc^{-2\theta}\). Thus, \(g_\theta(b, \gamma) = 0\) when \(x_{S_j}^2 \in \Lambda_1\).

Proof of Lemma IV.6. Recall the infinite series expansion of \(J^D_{S_j}\) in (22). To evaluate it, we substitute \(H(w, x_{S_j}^2)\) with its closed form from (21) and that of \(\Omega_D(w, \gamma_{S_j})\) from (24). Correspondingly, we get an expression that is the sum of multiple infinite series, as in the derivation of \(G^D_k\) in Lemma IV.3. To evaluate said terms, we define the summation functions \(f_\theta(b, \gamma)\) and \(g_\theta(b, \gamma)\) given in the statement of the lemma and which are analogous to \(f_\theta(b, p)\) and \(g_\theta(b, p)\), respectively and used for obtaining the expression for \(G^D_k\). Proceeding exactly like in Lemma IV.3, we obtain the expression for \(J^D_{S_j}\).

Proof of Proposition IV.8. We partition the possible values of \(x_{S_j}^2\) into two sets,
\[
\Lambda_1 := [0, Bc^{-2\theta}], \quad \Lambda_2 := [Bc^{-2\theta}, \infty),
\]
and demonstrate that \(J^D_{S_j} < R_j(\theta)\) in each case. The proof is centered around the following two claims, which establish bounds on some important terms of the closed form of \(J^D_{S_j}\) from Lemma IV.3.

Claim (a): If \(x_{S_j}^2 \in \Lambda_1\), then
\[
B f_\theta(1, \gamma_{S_j}) \geq \frac{B}{c^{2\theta}} g_\theta(c^2, \gamma_{S_j}).
\]

Claim (b): If \(x_{S_j}^2 \in \Lambda_2\), then
\[
B f_\theta(1, \gamma_{S_j}) \geq \frac{B}{c^{2\theta}} f_\theta(c^2, \gamma_{S_j}).
\]

Now, we recall the closed form of \(f_\theta(b, \gamma)\) from Lemma IV.3 and note that \(\nu = 0\) when \(x_{S_j}^2 < Bc^{-2\theta}\). Thus, \(\tilde{g}_\theta(b, \gamma_{S_j}) = 0\) when \(x_{S_j}^2 \in \Lambda_1\). We now observe that
\[
Bc^{-2\theta} \tilde{g}_\theta(c^2, \gamma_{S_j}) = \frac{[r1]}{c^{2\theta}} \frac{B}{c^{2\theta}} d^T(I - c^2 P_1 E)^{-1} P_0^{(\theta-1)} P_1 \delta_{S_j} \leq \frac{B}{c^{2\theta}} d^T(I - P_1 E)^{-1} P_0^{(\theta-1)} P_1 \delta_{S_j} = B f_\theta(1, \gamma_{S_j}),
\]
where \([r1]\) uses the definition of \(\tilde{g}_\theta(b, \gamma_{S_j})\), \([r2]\) follows from the fact that the matrix \((I - c^2 P_1 E)^{-1}\) is element-wise smaller than \((I - P_1 E)^{-1}\) because \(c^2 < 1\) and all elements of \(P_1 E\) are non-negative, along with the fact that \((I - K)^{-1} = \sum_{w=0}^{\infty} K^w\). This completes the proof of Claim (a).

To prove Claim (b), we establish an upper bound on \(c^{2\nu}\) under the assumption that \(x_{S_j}^2 \in \Lambda_2\). Note that
\[
c^{2\nu} = c^{\max\left\{\theta \left| \frac{\log(x_{S_j}^2/c^2)}{\log(1/c^2)} \right| \right\}} \leq c^{\frac{\log(x_{S_j}^2/c^2)}{\log(1/c^2)}} \leq \frac{B}{x_{S_j}^2},
\]
where we have again used the fact that \(c^2 < 1\). From this bound, one can upper bound \(x_{S_j}^2 f_\theta(c^2, \gamma_{S_j})\) as
\[
x_{S_j}^2 f_\theta(c^2, \gamma_{S_j}) \leq B d^T(P_1 E)^{-(\theta-1)} P_0^{(\theta-1)} P_1 \delta_{S_j} \leq B d^T(P_1 E)^{-(\theta-1)} P_0^{(\theta-1)} P_1 \delta_{S_j} = B f_\theta(1, \gamma_{S_j}).
\]
This concludes the proof of Claim (b).

Now, we recall the closed form of \(J^D_{S_j}\). If \(x_{S_j}^2 \in \Lambda_1\), we have \(f_\theta(c^2, \gamma_{S_j}) \geq 0\) and \(x_{S_j}^2 < Bc^{-2\theta}\), while \(\tilde{g}_\theta(c^2, \gamma_{S_j}) \geq 0\). These facts along with Claim (a) imply that
conclude that

\[ \mathcal{J}_{S_j}^0 \leq R_j(\theta) \] when \( x_{S_j}^2 \in \Lambda_1 \). In the case that \( x_{S_j}^2 \in \Lambda_2 \), we rearrange the closed form of \( \mathcal{J}_{S_j}^0 \) as

\[
\mathcal{J}_{S_j}^0 = [\tilde{g}_0(\tilde{a}^2, \gamma_{S_j}) - \tilde{g}(c^2, \gamma_{S_j})]x_{S_j}^2 + M[\tilde{g}_0(\tilde{a}^2, \gamma_{S_j}) - \tilde{g}_0(1, \gamma_{S_j}) - x_{S_j}^2 \tilde{f}_0(c^2, \gamma_{S_j})].
\]

Then using Claim (b), the fact that \( \tilde{g}_0(\tilde{a}^2, \gamma_{S_j}) < \tilde{g}_0(c^2, \gamma_{S_j}) \) (since \( \tilde{a}^2 < c^2 \)) and lastly the fact that \( x_{S_j}^2 \geq Bc^{-2\theta} \), we conclude that \( \mathcal{J}_{S_j}^0 \leq R_j(\theta) \) when \( x_{S_j}^2 \in \Lambda_2 \). Thus, \( R_j(\theta) \) uniformly upper bounds \( \mathcal{J}_{S_j}^0 \) for all \( x_{S_j}^2 \in [0, \infty) \).

\[ \square \]

**Proof of Corollary V.3.** The proof is similar to that of Theorem V.1 except for one key difference. We note that in Theorem V.1, \( C(0) \) was obtained as the \( B \)-maximizer of \( Q_X(D + B) \) under the constraint that \( Q_X(D + B) < 0 \). This ensured that the transmission fraction over the horizon \( (S_j, S_j + 1)_j \) is upper bounded by \( C(1)(C(0) + C(1))^{-1} \), under the assumption that \( x_{S_j}^2 \geq X \). In case of asymptotic transmission fraction, we know that said upper bound on transmission fraction over the horizon \( (S_j, S_j + 1)_j \) has to hold for all \( j \in \mathbb{N}_0 \), and equivalently for all \( x_{S_j}^2 > 0 \). Thus we derive the term \( C(0) \) by first maximizing \( Q_X(D + B) \) over all possible values of \( \lambda \) and then choosing the largest value of \( B \) such that \( Q_X(D + B) < 0 \) and setting \( C(0) \) equal to said value.

The former maximization is carried out because \( Q_X(D + B) \mid_{D + B(\gamma_{S_j})} \) acts as an upper bound on \( \mathcal{J}_{S_j}^{D + B} \), which we want to be negative so that (29) is valid. Thus, we let

\[
C(0) := \arg\max_{B \in \mathbb{N}_0} \left\{ \max_{\lambda \in \mathbb{R}, \lambda \geq 0} Q_X(D + B) \right\} < 0
\]

\[
= \arg\max_{B \in \mathbb{N}_0} \{Q(D + B) < 0\},
\]

which follows from the fact that \( c^2 > \tilde{a}^2 \) and the definitions of \( Q_X(\theta) \) and \( Q(\theta) \). The rest of the proof follows along similar lines as that of Theorem V.1.

\[ \square \]

**APPENDIX B**

**PROCEDURE TO COMPUTE A SUFFICIENT LOWER BOUND \( B^* \) ON THE ULTIMATE BOUND \( B \)**

Here, we provide a procedure to compute the lower bound \( B^* \) on \( B \), referred to in in Theorem IV.5. This procedure is based on the proof of Lemma IV.13 in [29] and we present it here for completeness. First, we define the following constants

\[
P_1 := \log(\tilde{a}^2 / \tilde{a}^2), \quad P_2 := \log(\tilde{a}^2 c^2 / \tilde{a}^2),
\]

\[
P_3 := \log(1 / c^2), \quad P_4 := \log \left( \frac{\log(1 / \tilde{a}^2)}{M \log(\tilde{a}^2)} \right).
\]

Then, consider the following functions of \( B \)

\[
U(B) := e^{(P_1 P_2 / P_3)} B^{P_1 / P_3}, \quad w_*(U(B)) := \frac{\log(B)}{P_2} + \frac{P_2}{P_4},
\]

\[
Y(B) := \tilde{a}^{2w_*(U(B))} U(B) + \tilde{M} \tilde{a}^{2w_*(U(B))}, \quad F_*(U(B)) := Y(B) - \tilde{M} - B.
\]

The function \( F_*(U(B)) \) is strictly concave in \( B \) (Lemma IV.13, [29]). Thus, it has at most two zeroes, one of which is

\[ B_0 = \frac{\tilde{M} \log(\tilde{a}^2)}{\log(c^2 / \tilde{a}^2)}. \]

There is another zero \( B_2 > B_0 \) of \( F_*(U(B)) \) only if \( F_*(U(B)) \) is non-increasing at \( B = B_0 \), with \( B_2 > B_0 \), and lastly the fact that \( x_{S_j}^2 \geq Bc^{-2\theta} \), we conclude that \( \mathcal{J}_{S_j}^0 \leq R_j(\theta) \) when \( x_{S_j}^2 \in \Lambda_2 \). Thus, \( R_j(\theta) \) uniformly upper bounds \( \mathcal{J}_{S_j}^0 \) for all \( x_{S_j}^2 \in [0, \infty) \).

\[ \square \]

**Procedure to Compute a Sufficient Lower Bound \( B^* \) on the Ultimate Bound \( B \)**

Here, we provide a procedure to compute the lower bound \( B^* \) on \( B \), referred to in in Theorem IV.5. This procedure is based on the proof of Lemma IV.13 in [29] and we present it here for completeness. First, we define the following constants

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\]

\[
P_3 := \log(1 / c^2), \quad P_4 := \log \left( \frac{\log(1 / \tilde{a}^2)}{M \log(\tilde{a}^2)} \right).
\]

Then, consider the following functions of \( B \)

\[
U(B) := e^{(P_1 P_2 / P_3)} B^{P_1 / P_3}, \quad w_*(U(B)) := \frac{\log(B)}{P_2} + \frac{P_2}{P_4},
\]

\[
Y(B) := \tilde{a}^{2w_*(U(B))} U(B) + \tilde{M} \tilde{a}^{2w_*(U(B))}, \quad F_*(U(B)) := Y(B) - \tilde{M} - B.
\]

The function \( F_*(U(B)) \) is strictly concave in \( B \) (Lemma IV.13, [29]). Thus, it has at most two zeroes, one of which is

\[ B_0 = \frac{\tilde{M} \log(\tilde{a}^2)}{\log(c^2 / \tilde{a}^2)}. \]