ON THE HOMOLOGY OF INDEPENDENCE COMPLEXES

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Abstract. The independence complex $I(G)$ of a graph $G$ is the simplicial complex formed by the independent sets in $G$. This article introduces a deformation of its simplicial boundary map that gives rise to a double complex with trivial homology. Filtering this double complex in the right direction induces a spectral sequence that converges to zero and contains on its first page the homology of the independence complexes of $G$ and of some special subgraphs of $G$, obtained from it by deleting independent sets and their neighborhoods. It is shown that this spectral sequence may be used to extract information about the homology of $I(G)$. Furthermore, a careful study of the sequences’ first page gives rise to a conjecture relating the cardinality of maximal independent sets in $G$ to the vanishing of certain homology groups of the independence complexes of various subgraphs of $G$.

1. Introduction

An independent set in a graph $G = (V, E)$ is a subset of its vertices $I \subset V$ such that no two elements in $I$ are adjacent. More generally, a subset $I \subset V$ is $r$-independent if every connected component of the induced subgraph $G[I] := (I, E')$ with $E' = \{ e \in E \mid e \in I \times I \}$ has at most $r$ vertices. Since the property of being $r$-independent is closed under taking subsets, the set of all $r$-independent sets of $G$ forms a simplicial complex, the $r$-independence complex $I_r(G)$ of $G$; the vertex set of $I_r(G)$ is $V$ and $I \subset V$ forms a simplex if and only if $I$ is $r$-independent in $G$. In the following we write $I(G)$ for $I_1(G)$. See Figure 1 for an example.

These complexes are a special instance of a great variety of simplicial complexes associated to graphs [Jon08], some of them related to each other. For example, the independence complex of a graph is the matching complex of its line graph and the clique complex of its dual graph.

The topology of such complexes is a well-studied topic. For independence complexes, most work has been done on connectivity and homotopy-type, cf. [Koz90, EH06, ST06, Eng08, Eng09, Cso09, Ada15], with applications, for example, to the study of graph colorings [BK04, Bar13]. For the case of higher independence complexes, see [ST06] and [DS20], as well as references therein.

The homology groups of independence complexes have also been investigated, see [Mes03, Jon10, DS13]. Here, applications reach from statistical physics, where the Euler characteristic of $I(L)$ for $L$ a periodic lattice is referred to as its Witten index [BMLN07, HS09, Ada11], to group theory, where certain local homology groups of the classical braid groups are related to the homology of certain (higher) independence complexes [Sal15, PS17].

The main result of this paper is a computational recipe for the homology groups of $I(G)$ (and $I_r(G)$ to some extent, see the discussion below). It is implied by Corollary 5.1 of Theorem 4.1 and strengthened by the properties established in Theorem 5.3.

**Theorem 1.1.** Let $G$ be a finite simple graph. There exists a spectral sequence whose $E^1$-page contains a copy of the homology of $I(G)$ and its other entries are given by homology groups of independence complexes of graphs obtained from $G$ by deleting independent sets and their neighborhoods. The sequence collapses to $E^\infty$ which has one entry isomorphic to $\mathbb{Z}$ and all other vanish. Moreover, the differential $d^1 : E^1 \to E^1$ is explicitly given and easy to compute.

![Figure 1. A graph, its independence complex and its 2-independence complex](image-url)
This means that we can use this spectral sequence to study the homology of \( I(G) \) by monitoring the typical dramatic action associated with these objects.

Moreover, the proof of this theorem and all “empirical data” hint at a rather peculiar property of independence complexes:

**Conjecture 1.2.** If \( G \) has no maximal independent set of cardinality \( p \), then

\[
\tilde{H}_{p-q-1}(I(G_U)) \cong 0
\]

holds for all \( q > 0 \) and all independent sets \( U \subset V \) with \( |U| = q \). Here \( G_U \) is the subgraph obtained from \( G \) by deleting all vertices in \( U \) together with their neighbors.

This effectively reduces the computation of the homology of \( I(G) \) to the problem of determining the homology of independence complexes of the “building blocks” \( G_U \) and inspecting the first page(s) of the above mentioned spectral sequence. In good cases this allows to determine all homology groups of \( I(G) \). In general one gets at least some relations between them (and the homology of the building blocks).

The basic idea is to consider a deformation of the simplicial boundary map of \( I(G) \) which gives rise to the above mentioned spectral sequence. For this we model the simplicial chain complex of \( I(G) \) by decorations on the vertices on \( G \) so that the boundary map \( d \) is given by removing these decorations, hereafter called *markings*. We then enhance this picture by introducing a second type of marking and a differential \( \delta \) that changes the first into the second type. These two differentials commute, so that we may form a double complex of markings on \( G \), graded by the number of markings of the first and second type, called 1- and 2-markings, respectively. Setting \( D := d + \delta \) defines a differential on the total complex

\[
T(G) := \bigoplus_n T_u(G), \quad T_u(G) := \bigoplus_{i-j=n} T_{i,j}(G),
\]

where \( T_{i,j}(G) \) denotes the markings with \( i \) marked vertices and \( j \) 2-marked vertices.

It turns out that the total complex \((T(G), D)\) is acyclic. Filtering it by the number of 2-markings induces a spectral sequence with first page

\[
E^1_{p,q} = H_p(T_{•,q}(G), d),
\]

so it contains the homology of \( I(G) \) in the row \( q = 0 \). Since the spectral sequence converges to zero, this allows then to apply standard techniques from homological algebra to study \( H(I(G)) \).

The whole construction is motivated by the article [KSvS13] by Kreimer, Sars and van Suijlekom where two similar complexes were used to encode consistency conditions in the perturbative quantization of gauge theories. The masters thesis [Kni17] by Knispel studied the cohomology of these complexes in detail. He showed that every variant of marking can be pulled back to the case of marking vertices in an associated simple graph \( G \), allowing to compute the cohomology for all cases at once – a streamlined version, using the above introduced spectral sequence, in a slightly different guise, can be found in [BK20]. He continued to study the non-trivial part \( d \) of the differential \( D = d + \delta \), relating it to the notion of independent sets and cliques in \( G \). However, we will see below that this relation can in fact be pushed much further. Firstly, the map \( d \) really is the boundary map of the independence complex of \( G \), and secondly, the total complex \((T(G), D)\) may be used to study the homology of \( I(G) \).

The exposition is organized as follows. In Section 2 we define the notion of markings to model independent sets in a graph \( G \). We then introduce two differentials \( d \) and \( \delta \) to set up the double complex \((T(G), D)\) that contains a copy of the simplicial chain complex of \( I(G) \).

The next two sections recite the results of [BK20] (Sections 3.1 and 3.2 therein). In Section 3 the vertical differential \( \delta \) of \((T(G), D)\) is studied and its homology is shown to be trivial, except in bidegree \((0, 0)\) where it is isomorphic to \( \mathbb{Z} \). We use this in Section 4 to compute the homology of the total complex \((T(G), D)\), showing that it is acyclic as well.

Section 5 contains the heart of the paper. We introduce the spectral sequence that allows to study the homology of \( I(G) \) and derives some crucial properties of it. The section finishes with a conjecture on the vanishing of certain homology groups of the independence complexes of various subgraphs of \( G \).

The last section contains some elaborated examples.

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1Citing J.F. Adams from [McC00] “... the behavior of this spectral sequence ... is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage strewn with corpses and only one actor left alive (namely the one who has to speak the last few lines).”
2. Independent sets and markings

Let $G = (V, E)$ be a finite, simple graph. We start by introducing a model for independent sets $I \subset V$ of $G$. For this we simply label the vertices in $I$, and call such a labeling a marking of $G$. The raison d'être is that this point of view allows to

1. model the simplicial boundary map on $I(G)$ as a map on $G$ that removes labels on vertices,
2. introduce a second kind of label which gives rise to a deformation of the simplicial boundary map.

In what follows everything will depend on the chosen graph $G$, but whenever there is no risk of confusion, this dependence is dropped from notation. Throughout this paper $\hat{H}$ and $\check{H}$ denote homology and reduced homology, respectively, with integer coefficients.

**Definition 2.1.** Let $G$ be a graph. A marking of $G$ is a map $m : V \to \{0, 1, 2\}$ such that $V_m := m^{-1}(\{1, 2\})$ is an independent set in $G$. For $i = 1, 2$ we refer to the elements of $V_i := m^{-1}(i)$ as $i$-marked and to the elements of $V_0 := m^{-1}(0)$ as unmarked.

**Definition 2.2.** Choose an order on $V$ such that $V = \{v_1, \ldots, v_n\}$ with $v_i < v_j$ if and only if $i < j$. Let $T_{i,j} = T_{i,j}(G)$ be the free abelian group generated by all markings of $G$ with $i$ marked and $j$ 2-marked vertices,

$$
T_{i,j} := \mathbb{Z}\langle m : V \to \{0, 1, 2\} \mid |V_m| = i, |V_2| = j \rangle.
$$

Define linear maps $d : T_{i,j} \to T_{i-1,j}$ and $\delta : T_{i,j} \to T_{i,j+1}$ by

$$
dm := \sum_{v \in V_i} (-1)^{\#\{w \in V_1 \mid w < v\}} m_{v \to 0},
$$

$$
dm := \sum_{v \in V_i} (-1)^{\#\{w \in V_1 \mid w < v\}} m_{v \to 2},
$$

where $m_{v \to i}(x) := \begin{cases} m(x) & \text{if } x \neq v \\ i & \text{if } x = v \end{cases}$, changes the marking $m$ by relabeling the vertex $v$ with $i$.

**Example 2.3.** Let $G = P_3 = \bullet \ldots \bullet$ with $V = \{v_1, v_2, v_3\}$ ordered from left to right. Let us denote 1-marked and 2-marked vertices by orange and white filled circles. For instance, the marking

$$
m : v_1 \mapsto 1, v_2 \mapsto 0, v_3 \mapsto 1
$$

is represented by

$$
m = \bullet \ldots \bullet.
$$

Then

$$
d \bullet \ldots \bullet = \bullet \ldots \bullet \ldots \bullet
$$

$$
\delta \bullet \ldots \bullet = \bigcirc \ldots \bigcirc
$$

$$
\delta d \bullet \ldots \bullet = \bullet \ldots \bigcirc \ldots \bigcirc, \quad \delta \delta \bullet \ldots \bullet = \bigcirc \ldots \bigcirc.
$$

If on the other hand $m = \bullet \ldots \bigcirc$, then $dm = \delta m = 0$.

**Proposition 2.4.** $d^2 = \delta^2 = 0$ and $d\delta = \delta d$.

**Proof.** The first statement follows by a standard computation,

$$
ddm = d \sum_{v \in V_1} (-1)^{\#\{w \in V_1 \mid w < v\}} m_{v \to 0}
$$

$$
= \sum_{v \in V_1} (-1)^{\#\{w \in V_1 \mid w < v\}} \sum_{v' \in V_1 \{v\}} (-1)^{\#\{w' \in V_1 \mid w' < v'\}} m_{v, v' \to 0}
$$

$$
= \sum_{v, v' \in V_1, v < v'} (-1)^{\#\{w \in V_1 \mid v < v'\} + \#\{w' \in V_1 \mid v' < v'\}} m_{v, v' \to 0}
$$

$$
= \sum_{v, v' \in V_1, v < v'} (-1)^{\#\{w \in V_1 \mid v < v'\} + 1} m_{v, v' \to 0}
$$

$$
+ \sum_{v, v' \in V_1, v < v'} (-1)^{\#\{w \in V_1 \mid v < u < v'\}} m_{v, v' \to 0} = 0,
$$

3
and similarly for $\delta$. To see that $d$ and $\delta$ commute we calculate
\[
dm = d \sum_{v \in V} (-1)^{\# \{w \in V \mid w < v\}} m_{v \to 2}
= \sum_{v \in V} \sum_{v' \in V \setminus \{v\}} (-1)^{\# \{w \in V \mid w < v\} + \# \{w' \in V \setminus \{v\} \mid w' < v'\}} m_{v \to 2, v' \to 0}
\]
and
\[
\delta m = \delta \sum_{v \in V} (-1)^{\# \{w \in V \mid w < v\}} m_{v \to 0}
= \sum_{v \in V} \sum_{v' \in V \setminus \{v\}} (-1)^{\# \{w \in V \mid w < v\} + \# \{w' \in V \setminus \{v\} \mid w' < v'\}} m_{v \to 0, v' \to 2} = d\delta m.
\]

Note that the complex $(T_{\bullet,j}, d)$ models the simplicial chain complex of $\mathcal{I}$. More precisely, for any choice of order on $V$ and orientation of $\mathcal{I}$ there exists a unique isomorphism of chain complexes
\[(T_{\bullet,j}, d) \cong (C_{\bullet-1}(\mathcal{I}), \partial)\]
where $(C_{\bullet}(\mathcal{I}), \partial)$ denotes the augmented simplicial chain complex of $\mathcal{I}$. On the level of chain groups this isomorphism is given by simply mapping every independent set $I$ in $G$ to the marking $m_I$ that marks the vertices in $I$ by 1 and every other vertex by 0. Since an orientation of $\mathcal{I}$ is the same as a linear order on its vertex set, which is equal to $V$, this correspondence defines a chain map. Thus, $H_n(T_{\bullet,j}, d)$ is isomorphic to the reduced simplicial homology $\tilde{H}_{n-1}(\mathcal{I})$ of the independence complex of $G$.

What about the complexes with 2-marked vertices, i.e. the case $j \neq 0$? In this case we can relate the complex $(T_{\bullet,j}, d)$ to independence complexes of graphs obtained from $G$ by removing $j$ vertices together with their neighborhoods.

**Definition 2.5.** The *neighborhood* of a vertex $v$ in a graph $G$ is $N(v) := \{ w \in V \mid \{v, w\} \in E \}$, the set of vertices of $G$ adjacent to $v$. The set $B(v) := \{v\} \cup N(v)$ is called the *ball* around $v$. Likewise, for a subset $U \subset V$ we define $B(U) := \bigcup_{v \in U} B(v)$.

We write $G_U$ for the graph obtained from $G$ by deleting all elements in $B(U)$, i.e. for the graph $G_U = (V', E')$ with
\[
V' = V \setminus B(U), \quad E' = \{c = \{v, w\} \in E \mid v, w \in V'\}.
\]

**Proposition 2.6.** For each $j \geq 0$ there is an isomorphism of chain complexes,
\[(T_{\bullet,j}(G), d) \cong \bigoplus_{U \subset V \text{ independent } |U|=j} (C_{\bullet-1}(G_U), \partial),\]

$(C_{\bullet-1}(G_U), \partial)$ denoting the augmented (and degree shifted) simplicial chain complex of $\mathcal{I}(G_U)$ with the conventions
\[
G_0 := G \text{ and } (C_{\bullet}(\emptyset), \partial) := 0 \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\partial_0} 0.
\]

**Proof.** The case $j = 0$ is clear, so let $j > 0$. By definition every set of 2-marked vertices forms an independent set in $G$. Conversely, every independent set of size $j$ can be modeled by an appropriate 2-marking. Since $d$ acts only on 1-marked vertices, the complex $(T_{\bullet,j}, d)$ splits into a direct sum of complexes with one summand for each independent/2-marked set of size $j$.

Given such an independent set $U$, the remaining vertices that can be marked are precisely the non-neighbors of vertices in $U$, i.e. the vertices of the graph $G_U$. If $G_U$ is not empty, then \[(2.2)\] follows from the interpretation \[(2.1)\].

In the special case that $U$ is a maximal independent set there is no possibility to put any 1-markings on the remaining vertices, so $G_U = \emptyset$. The corresponding chain complex has only one nontrivial chain group in degree zero, generated by a single element, the marking that marks every vertex in $U$ by 2.

In terms of chain complexes of markings $(T(\cdot), d)$ we may rephrase the previous proposition. It states that
\[(T_{\bullet,j}(G), d) \cong \bigoplus_{U \subset V \text{ independent } |U|=j} (T_{\bullet-j,0}(G_U), d),\]
This leads to two important observations. Firstly, even in the presence of 2-marked vertices the complexes \((T_{*,j}, d)\) can be interpreted as chain complexes of independence complexes of graphs (more precisely, of subgraphs of \(G\)). Secondly, as the number \(j\) of 2-markings increases, the complexes \((T_{*,j}, d)\) split into simpler and simpler building blocks. We will see that these observations, together with the simplicity of the second differential \(\delta\), allow to set up a spectral sequence to study \(H(I)\) as the homology of \((T_{*,0}, d)\).

**Remark 2.7.** The whole construction outlined in this paper works also in the case of higher independence complexes (as well as for more general markings where a given set of subgraphs is allowed to be marked, cf. [BK20]). One simply requires in Definition 2.1 the set \(V_m\) of marked elements to be \(r\)-independent. Then everything works exactly as presented here for the case \(r = 1\), except for one crucial difference. The splitting of \((T_{*,j}, d)\) in (2.3) or (2.4) becomes more complicated: The direct sum runs now over \(r\)-independent sets in \(G\) and the appropriate replacements of the graphs \(G_U\) are not necessarily subgraphs of \(G\) anymore due to the varying cardinality of \(r\)-independent sets. Only for \(r = 2\) a similar looking formula can be recovered where the summands in (2.3) have to be replaced by 1-independence complexes of appropriately associated graphs. See Example 6.4 and 6.5.

### 3. The second differential \(\delta\)

While \(d\) models the boundary map of independence complexes, the map \(\delta : T_{i,j} \to T_{i,j+1}\) acts by relabeling already marked vertices. Thus, it is effectively independent of the topology of \(G\). However, this differential may also be interpreted as the (co-)boundary map of a simplicial complex, albeit of a very simple one, the standard simplex \(\Delta^{i-1}\) on \(i\) vertices.

**Remark 3.1.** Note that \(\delta\) has bidegree \((0,1)\), going in the “wrong” direction. Nevertheless, we will use homological terminology for both maps, \(d\) and \(\delta\). This avoids awkwardly changing between homology and cohomology. For the purists this choice of convention may be justified by flipping the sign in the second part of the grading of \(T_{*,j}\), i.e. by defining

\[
T_{i,j} := \mathbb{Z}\langle m : V \to \{0,1,2\} \mid |V_m| = i, |V_2| = -j \rangle.
\]

**Proposition 3.2.** All homology groups \(H_n(T_{*,\delta})\) are trivial unless \(i = 0\) and \(n = 0\). In this case \(H_0(T_{*,\delta}) \cong \mathbb{Z}\), generated by the trivial marking

\[
m_0 : V \to \{0,1,2\}, m_0(v) = 0 \text{ for all } v \in V.
\]

The proof of this proposition is not needed in the sequel and may be omitted on a first read; the following example should give an intuitive idea why the statement holds.

**Example 3.3.** Let \(G = P_3\) as in Example 2.3. For \(i = 2\) we have

\[
\delta \circ \circ \circ = \circ \circ \circ - \circ \circ \circ = 0, \\
\delta \circ \circ \circ = \circ \circ \circ - \circ \circ \circ = \delta \circ \circ \circ = 0.
\]

A formal proof of Proposition 3.2 relies on the following two lemmata.

**Lemma 3.4.** For \(k > 1\) let \((C_{\bullet}(k), \partial)\) denote the augmented and degree shifted simplicial chain complex of the standard simplex on \(k\) vertices, \((C_{\bullet+k-1}(\Delta^{k-1}), \partial)\). Then for \(i > 0\)

\[
(T_{*,\delta}) \cong \bigoplus_{U \subset V \text{ independent}} (C_{\bullet}(i), \delta).
\]

**Proof.** Fix \(i > 0\). The map \(\delta\) neither creates any new marked vertices nor does it see adjacency relations in \(G\), it simply operates on the set of marked vertices, regardless of their distribution in \(G\). Therefore, the complex \((T_{*,\delta})\) splits into a direct sum

\[
(T_{*,\delta}) \cong \bigoplus_{U \subset V \text{ independent}} (T^f(U), \delta),
\]

where

\[
T^f(U) := \mathbb{Z}\langle m : U \to \{1,2\} \rangle
\]

denotes the free abelian group generated by all “full” markings of the graph \(U\) on \(i = |U|\) disjoint vertices, graded by the number of 1-marked elements.
Identifying the vertices in $U$ with the vertices of $\Delta^{i-1}$, there is a unique orientation preserving bijection between the elements $m \in T^f(U)$ and the simplices in $\Delta^{i-1}$, sending $m$ to the (oriented) simplex $m^{-1}(1) \subset U$. This is clearly a chain map (after shifting the degree by one), so that

$$(T^f(U)_\bullet, \delta) \cong (C_{i-1}(\Delta^{i-1}), \partial)$$

and the claim follows. □

**Lemma 3.5.** For each $i > 0$ and all $n \in \mathbb{N}$ the groups $H_n(C_\bullet(i), \partial)$ are trivial.

**Proof.** A simplex is contractible, so its reduced homology vanishes. □

**Proof of Proposition 3.2.** Combining the two lemmata shows the first assertion in the proposition, the second one follows by direct computation: Clearly, $m_0 \in \ker \delta$, and since $\delta$ keeps the number of marked elements constant, $m_0$ cannot be an element of $\text{im} \, \delta$. □

Our next task is to study the differential $d + \delta$, viewed as a deformation of $d$, and to use it to extract information about the differential $d$, especially when restricted to the subcomplex of markings with no 2-marked vertices.

4. The double complex and its associated spectral sequence

To study the homology of $(T_{\bullet,0}, d)$ we first consider the total complex associated to $T_{i,j}$ with $d$ as horizontal and $\delta$ as vertical differential. For this let

$$T := \bigoplus_n T_n, \quad T_n := \bigoplus_{i-j=n} T_{i,j},$$

and define a differential

$$D_n : T_n \to T_{n-1}, \quad D_n := d + (-1)^n \delta.$$

Proposition 2.4 implies that $(T, D)$ is a chain complex. Its homology is given by

**Theorem 4.1.** The complex $(T, D)$ is acyclic,

$$H_n(T, D) \cong \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

**Proof.** Consider a descending exhaustive filtration

$$T = F_0T \supset \ldots \supset F_{p-1}T \supset F_pT \supset \ldots,$$

defined by

$$(4.1) \quad F_pT_n := \bigoplus_{i-j=n, j \geq p} T_{i,j}.$$

It induces an associated spectral sequence which starts with

$$E^0_{p,q} := F_p T_{p-q} / F_{p-1} T_{p-q} = \bigoplus_{i-j=p-q, i=p} T_{i,j} = T_{p,q},$$

$$d^0_{p,q} : E^0_{p,q} \to E^0_{p,q+1} = \delta : T_{p,q} \to T_{p,q+1}.$$

On its first page we have $E^1_{p,q} = H_p(T_{\bullet,0}, \delta)$ which by Proposition 3.2 vanishes for $(p, q) \neq (0, 0)$, while for $p = q = 0$ we computed $H_0(T_{\bullet,0}, \delta) \cong \mathbb{Z}$. It follows that all the differentials

$$d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$$

are trivial and the sequence collapses with $E^\infty = E^1$. By standard spectral sequence arguments we obtain from $E^\infty$ the (associated graded pieces of the) homology of $(T, D)$. In the present case this is very simple,

$$H_0(T, D) \cong E^1_{0,0} \cong \mathbb{Z}, \quad H_n(T, D) \cong 0 \text{ for all } n > 0.$$

\[2\] We follow the conventions in [GM99]. For nice introductions see [Cho06] as well as [DR17], and [McC00] for a concise treatment of the subject.
From the theory of spectral sequences associated to double complexes \cite{GM99}, it follows that if we consider the other canonical filtration of $T$, i.e. filtering $T$ in the horizontal direction, the associated spectral sequence converges to the same limit. Furthermore, on its first page we have $E^1_{p,q} = H_p(T_{\bullet,q},d)$. In particular, for $q = 0$ we find the homology of $\mathcal{I}$, while for $q > 0$ the entries of $E^1$ are given by the homology of the independence complexes of the graphs $G_U$ for $U \subset V$ independent.

We will see below how this allows to extract information about the homology of the independence complex of $G$.

5. The homology of $\mathcal{I}(G)$

We now turn our attention to the homology groups $H_n(T_{\bullet,0},d) \cong \tilde{H}_{n-1}(\mathcal{I})$. The proof of Theorem 4.1 implies the following

**Corollary 5.1.** There is a spectral sequence converging to $H_n(T,D)$ with its first page containing a copy of the (reduced) homology of the independence complex of $G$.

**Proof.** Consider the spectral sequence associated to a filtration of $T$, “opposite” to the one in (4.1), defined by

$$0 = F_{-1}T \subset F_0T \subset \ldots \subset F_pT \subset F_{p+1}T \subset \ldots$$

with

$$F_pT_n := \bigoplus_{i+j=p} T_{i,j}.$$  

Since there are only finitely many nonvanishing $T_{i,j}$, the associated spectral sequence converges to the same limit as the one in the proof of Theorem 4.1 (Proposition 3.5.1 in \cite{GM99}). Its zeroth page is given by

$$E^0_{p,q} = F_pT_{p-q}/F_{p+1}T_{p-q} = T_{p,q},$$

$$d^0_{p,q} : E^0_{p,q} \to E^0_{p-1,q} = d : T_{p,q} \to T_{p-1,q},$$

so that on the next page we find

$$E^1_{p,q} = H_p(T_{\bullet,q},d),$$

and the maps $d^1_{p,q} : H_p(T_{\bullet,q},d) \to H_p(T_{\bullet,q-1},d)$ induced by $\delta$.

Applying Proposition 2.6 identifies the $q = 0$ column with $\tilde{H}_{p-1}(\mathcal{I})$. \hfill \Box

The usefulness of this corollary lies in the simplicity of the spectral sequence’s limit $E^\infty$. If we know the homology of the complexes $(T_{\bullet,j},d)$ for $j > 0$, we can deduce information about $H(\mathcal{I})$ from studying the $E^1$ (or $E^2$-) page of this spectral sequence. Since we know that it eventually collapses, every entry except for a single copy of $\mathbb{Z}$ must disappear at some stage. In more dramatic words, as the spectral sequences progresses further, all but one entries of $E^1$ are eventually paired together, both partners doomed to killing each other.

There will be only one survivor, a lucky representative from the group of maximum independent sets of $G$.

**Example 5.2.** Let us look at a very simple example, $G = K_n$ the complete graph on $n$ vertices. The $E^1$ page of the spectral sequence from Corollary 5.1 has two nontrivial rows, $E^1_{p,0} = H_p(T_{\bullet,0},d)$ containing the homology of $\mathcal{I}(K_n)$ and

$$E^1_{p,1} = H_p(T_{\bullet,1},d) \cong \bigoplus_{U \subset V \text{ independent} \atop |U| = 1} H_{p-1}(T_{\bullet,0}(G_U),d).$$

Here $G_U = \emptyset$ because every vertex of $K_n$ is already a maximal (and maximum) independent set. By Proposition 2.6 the row $E^1_{p,1}$ is thus given by $E^1_{n,0} \cong \mathbb{Z}^n$ and $E^1_{p,0} \cong 0$ for $p \neq 1$.

The only way for this spectral sequence to exhibit the expected convergence is to have the differential

$$d^1_{1,0} : E^1_{1,0} \to E^1_{1,1}$$

$$\text{im} d^1_{1,0} \cong \mathbb{Z}^{n-1} \quad \text{and} \quad \ker d^1_{1,0} \cong 0.$$  

This implies $E^1_{1,0} \cong \tilde{H}_0(\mathcal{I}(K_n)) \cong \mathbb{Z}^{n-1}$ and $E^1_{p,0} \cong \tilde{H}_{p-1}(\mathcal{I}(K_n)) \cong 0$ for $p \neq 1$.

In the general case there is a similar behavior with respect to maximum (and maximal) independent sets.
Theorem 5.3. Let \( \alpha(G) \) denote the independence number of \( G \), i.e. the cardinality of a maximum independent set in \( G \). Then we have for the spectral sequence from Corollary 5.7:

1. \( E^\infty_{p,q} \cong \begin{cases} \mathbb{Z} & p = q = \alpha(G) \\ 0 & \text{else} \end{cases} \)
2. \( E^1_{p,p} \cong \mathbb{Z}^n \) where \( n_p \) is the number of maximal independent sets of size \( p \) in \( G \)
3. all diagonal entries with \( p < \alpha(G) \) vanish already on the \( E^2 \) page, \( E^2_{p,p} = H_p(E^1_{p,p}(d^1)) \cong 0 \)

Proof. Assertion (1) holds by construction because the spectral sequence converges to the associated graded piece of the homology of \( (T,D) \). It follows however also from (3) by Theorem 4.4 which assures that the surviving entry must lie on the diagonal; these entries represent the total degree 0 in which the only nontrivial homology of \( (T,D) \) is concentrated.

To prove (2) we consider the diagonal entries \( E^1_{p,p} = H_p(T_{p,p},d) \). Proposition 2.6 shows that this group is nonzero if and only if \( G \) has a maximal independent set of cardinality \( p \) (because every nonempty graph has \( H_0(T_{p,0},d) \cong 0 \)). Since \( d \) does not act on 2-marked vertices, the complex \( (T_{p,p},d) \) splits into a direct sum of complexes, one for each 2-marking of size \( p \). Thus, we find one generator for each maximal independent set of cardinality \( p \). In particular, if \( G \) does not have such maximal independent sets, then \( E^1_{p,p} \cong 0 \).

For (3) let now \( p < \alpha(G) \). The map \( d^1 \) on \( E^1 \) is given by the restriction of \( \delta \) to the homology classes of \( (T,d) \). Our goal is therefore to find for each generator of \( H_p(T_{p,p},d) \) a homology class in \( H_p(T_{p,p-1},d) \) that gets mapped to it by \( \delta \). Note that, if it were not for the restriction to homology classes of \( d \), this task would be trivial as the homology with respect to \( \delta \) vanishes (Proposition 3.2).

By (2) the generators of \( H_p(T_{p,p},d) \) are represented by maximal independents set \( I \) with \( |I| = p \). For each such \( I \) there exists a vertex \( z \in I \) such that the graph \( G_{I \setminus \{z\}} \) has at least two vertices that can be simultaneously marked – otherwise \( I \) would be not only maximal, but also maximum independent.

Choose two such vertices, denote them by \( x \) and \( y \), and consider the markings

\[
m_{0}: v \mapsto \begin{cases} 2 & \text{if } v \in I, \\ 0 & \text{else,} \end{cases} \quad m_{z}: v \mapsto \begin{cases} 1 & \text{if } v = z, \\ 2 & \text{if } v \in I \setminus \{z\}, \\ 0 & \text{else,} \end{cases} \quad m_{x}: v \mapsto \begin{cases} 1 & \text{if } v = x, \\ 2 & \text{if } v \in I \setminus \{x,z\}, \\ 0 & \text{else,} \end{cases} \quad m_{x,y}: v \mapsto \begin{cases} 1 & \text{if } v = y, \\ 2 & \text{if } v \in I \setminus \{y,z\}, \\ 0 & \text{else.} \end{cases}
\]

See Figure 3 for an example.

We have \( d(m_{z} - m_{x}) = 0 \) and, since \( \{v,z\} \in E \) holds for every vertex \( v \) of \( G_{I \setminus \{z\}} \), the element \( m_{z} - m_{x} \) cannot lie in the image of \( d \). Hence,

\[
0 \neq [m_{z} - m_{x}] \in H_p(T_{p,p-1},d).
\]

Moreover, since \( \delta m_{z} = dm_{x,y} \), we have found a preimage of \([m_{0}]\),

\[
[\delta(m_{z} - m_{x})] = [m_{0} - \delta m_{z}] = [m_{0}] \in H_p(T_{p,p},d).
\]

The preceding theorem implies that in “good” cases the spectral sequence contains already enough information to fully determine the homology of \( I \) (or at least to find relations between the groups in different dimensions). In “not so good” cases one has to examine the differential \( d^1 \) or throw in some additional information. Fortunately, \( d^1 \) is induced by the map \( \delta \) and therefore rather simple.
Here the term “good” essentially means that we know or are able to compute the homology of the independence complexes of the graphs $G_U$ for $U \subset V$ independent. For instance, if $G$ has many vertices of high valence or is a very symmetric graph, then the graphs $G_U$ become very simple as the cardinality of $U$ grows. This is demonstrated by the examples in the next section.

Last, but not least, there is one peculiar property of the spectral sequence which has so far withstood all attempts of proving or falsifying it.

**Conjecture 5.4.** If $E^1_{p,p} \cong 0$, i.e. $G$ has no maximal independent set of cardinality $p$, then all entries of the upper column $E^1_{p,q}$, $q > 0$, vanish.

This is quite remarkable, because by

$$E^1_{p,q} = H_p(T_d, d) \cong \bigoplus_{U \subset V \text{ independent}} H_{p-q}(T_{d}(G_U), d)$$

this would allow to deduce the vanishing of the rank $p - q - 1$ homology groups of the independence complex of every subgraph of $G$ that can be obtained by deleting $q$ independent vertices and their neighborhoods in $G$.

6. Examples

Our first goal is to compute the homology of $I(P)$ for $P$ the Petersen graph. For this we need a preparatory calculation. Throughout this section let $H_n$ denote $H_n(T_{d}, d)$.

**Example 6.1 ($C_6$, the cyclic graph on six vertices).** To set up the spectral sequence for $H(I(C_6))$ we need to calculate the homology of (2.3) for $i = 1, 2, 3$.

Removing any vertex with its neighbors from $C_6$ gives a path $P_3$ on three vertices, so

$$(T_{d}, d) \cong \bigoplus_{k=1}^{6} (T_{d} - 1(P_3), d).$$

The homology of $I(P_3)$ may be calculated directly, or simply read off from Figure 1. We find

$$H_n(T_{d}, d) \cong \begin{cases} \mathbb{Z} & n = 1, \\ 0 & n \neq 1. \end{cases}$$

Removing two independent vertices together with their neighborhoods produces either an empty graph or a single vertex. The latter is trivial on homology while the former adds a copy of $\mathbb{Z}$ in degree zero. There are three different maximum independent sets of size two, so we get

$$H_n(T_{d}, d) \cong \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Finally, $C_6$ has two maximal independent sets, so

$$H_n(T_{d}, d) \cong \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

It seems that a proof along the lines of (3) in Theorem 5.3 should work, but it is very hard to keep track of all possible cases. Therefore, a more elegant argument is likely to be required. On the other hand, if it is not true in general, then an interesting question would be for which classes of graphs it fails/holds.

In a way reflecting the ancient calculus tables where all kinds of functions were differentiated to find solutions of integrals.
Figure 4. The Petersen graph $P$. The ball $B(v_1)$ around $v_1$ is depicted in red, its complement graph $P_{v_1}$ in blue.

Filling out the first page of the associated spectral sequence

\[
\begin{array}{cccc}
Z^2 \\
Z^3 & 0 \\
0 & Z^6 & 0 \\
0 & H_1 & H_2 & H_3
\end{array}
\]

we immediately deduce that $H_3$ must vanish and $H_2 \cong \text{im} d_{1,0}^1 \leq Z^6$. The next page looks like

\[
\begin{array}{cccc}
Z^2 \\
E_2^2 & 0 \\
0 & E_2^2 & 0 \\
0 & H_1 & 0 & 0
\end{array}
\]

with $E_{2,1}^2 \cong \ker d_{1,1}^3 / H_2$ and $E_{2,2}^2 \cong Z^3 / \text{im} d_{1,1}^1$. From Theorem 5.3 we know that $E_{2,2}^3 \cong 0$. Hence, $\text{im} d_{1,1}^1 \cong Z^3$ and $\ker d_{1,1}^1 \cong Z^3$.

Since this page’s differential $d^2$ goes one step to the right and two steps up, we must have $E_{2,1}^3 \cong Z$ and $H_1 \cong E_{2,2}^3 \cong 0$ for the spectral sequence to exhibit its expected convergence behavior. Putting everything together we conclude

$$H_1 \cong 0, \ H_2 \cong Z^2 \quad \Longrightarrow \quad H_n(I(C_6)) \cong \begin{cases} Z^2 & \text{if } n = 1, \\ 0 & \text{else.} \end{cases}$$

**Example 6.2** (The Petersen graph $P$). Removing the ball around each of the vertices of $P$ produces a cyclic graph $C_6$ on six vertices. See Figure 4 for the case of an “interior” vertex and note that we get an isomorphic graph if we do the same with one of the $w_i$, $i = 1, \ldots, 5$.

Using the previous example we have thus

$$H_n(T_{\bullet,1}, d) \cong \begin{cases} Z^{20} & n = 2, \\ 0 & n \neq 2. \end{cases}$$

Deleting another vertex and its neighbors in the remaining graph produces $P_3$, a path on three vertices. There are twenty different pairs, so

$$H_n(T_{\bullet,2}, d) \cong \begin{cases} Z^{30} & n = 1, \\ 0 & n \neq 1. \end{cases}$$
Next, if we remove the ball around the "middle" vertex of \( P_3 \), there is nothing left. This can be done in ten different ways, hence

\[
H_n(T_{\bullet,3}, d) \cong \begin{cases} 
Z^{10} & n = 0, \\
0 & n \neq 0.
\end{cases}
\]

Finally, the Petersen graph has five maximum independent sets, so that

\[
H_n(T_{\bullet,4}, d) \cong \begin{cases} 
Z^5 & n = 0, \\
0 & n \neq 0.
\end{cases}
\]

The first page of the associated spectral sequence is then given by

\[
\begin{array}{cccccc}
\mathbb{Z}^5 \\
\mathbb{Z}^{10} & 0 \\
0 & \mathbb{Z}^{30} & 0 \\
0 & 0 & \mathbb{Z}^{20} & 0 \\
0 & H_1 & H_2 & H_3 & H_4
\end{array}
\]

We deduce \( H_3 \cong \text{im} \, d^1_{3,0} \) and \( H_4 \cong 0 \). Furthermore, it must hold that \( \text{im} \, d^1_{3,3} \cong \mathbb{Z}^{10} \) because \( E^2_{3,3} \cong 0 \) by Theorem 5.3. This implies \( \ker \, d^1_{3,3} \cong \mathbb{Z}^{20} \).

Therefore, we find on the \( E_2 \)-page

\[
\begin{array}{cccccc}
\mathbb{Z}^5 \\
0 & 0 \\
0 & \mathbb{Z}^{20} / Y & 0 \\
0 & 0 & X / H_3 & 0 \\
0 & H_1 & H_2 & 0 & 0
\end{array}
\]

for \( X := \ker d^1_{3,1} \) and \( Y := \text{im} \, d^1_{3,1} \), satisfying \( X \oplus Y \cong \mathbb{Z}^{20} \). The only nontrivial differentials are

\[
d^2_{1,0} : H_1 \to 0, \quad d^2_{2,0} : H_2 \to \mathbb{Z}^{20} / Y, \quad d^2_{3,2} : \mathbb{Z}^{20} / Y \to \mathbb{Z}^5, \quad d^2_{3,1} : X / H_3 \to 0.
\]

Convergence of the spectral sequence implies

\[
H_1 \cong 0, \quad \text{im} \, d^2_{3,2} \cong \mathbb{Z}^4, \quad \ker \, d^2_{3,2} \cong H_2, \quad X \cong H_3,
\]

and, using \( X \oplus Y \cong \mathbb{Z}^{20} \), this is equivalent to \( H_3 \cong \mathbb{Z}^4 \oplus H_2 \).

We see that the spectral sequence does not always solve the problem completely; additional calculations may be necessary. In the present case one finds \( H_2 \cong 0 \), so that

\[
\tilde{H}_n(I(P)) \cong \begin{cases} 
\mathbb{Z}^4 & n = 2, \\
0 & n \neq 2.
\end{cases}
\]
Example 6.3 \((K,\text{the }1\text{-skeleton of a three dimensional cube})\). Here the \(E^1\)-page of the associated spectral sequence looks like

\[
\begin{array}{cccc}
\mathbb{Z}^2 & 0 & 0 & \\
\mathbb{Z}^4 & 0 & 0 & \\
0 & \mathbb{Z}^8 & 0 & 0 \\
0 & H_1 & H_2 & H_3 & H_4
\end{array}
\]

from which it follows that

\[
\tilde{H}_n(I(K)) \cong \begin{cases} 
\mathbb{Z}^3 & n = 1, \\
0 & n \neq 1.
\end{cases}
\]

The details of the computation are left to the interested reader.

We finish with two examples on the homology of 2-independence complexes. Note that Proposition 2.6 still applies, but in (2.2) we have to change the definition of the graphs \(G_v\).

In the following let \(T_{i,j}\) be given as in Definition 2.2 except that markings \(m : V \to \{0,1,2\}\) are now defined by requiring that \(V_m = m^{-1}(|1,2|)\) is a 2-independent set in \(G\).

Example 6.4 \((C_4,\text{the cyclic graph on four vertices})\). We consider the 2-independence complex \(I_2(C_4)\) whose geometric realization is homeomorphic to the 1-skeleton of a tetrahedron \(\Delta^3\).

To find the first page of the associated spectral sequence we need to know the homology of the complexes \((T_{i,j},d)\) for \(j = 1,2\).

Now let \(j = 2\), i.e. two vertices be 2-marked. Every such 2-independent set is already maximum, so

\[
(T_{\bullet,2},d) \cong \bigoplus_{U \subset V \text{ maximum } \text{2-independent}} (C_{\bullet-1}(U\emptyset),\partial).
\]

The homology groups of the latter two complexes are easy to compute: For the first one observe that \(I(\mathbb{Z}K_3)\) consists of three disjoint vertices, for the second one note that there are six maximum 2-independent sets in \(C_4\). Thus,

\[
H_n(T_{\bullet,1},d) \cong \begin{cases} 
(\mathbb{Z}^2)^4 & n = 1, \\
0 & n \neq 1
\end{cases}, \\
H_n(T_{\bullet,2},d) \cong \begin{cases} 
\mathbb{Z}^6 & n = 0, \\
0 & n \neq 0
\end{cases}.
\]

The first page of the associated spectral sequence is then

\[
\begin{array}{cccc}
0 & \mathbb{Z}^6 & 0 & \\
0 & \mathbb{Z}^8 & 0 & \\
0 & H_1 & H_2 & 
\end{array}
\]

which implies \(H_1 \cong 0\) and \(H_2 \cong \mathbb{Z}^3\) (the other possible solution, \(H_1 \cong \mathbb{Z}^5\) and \(H_2 \cong \mathbb{Z}^8\), cannot be true), so that \(\tilde{H}_n(I_2(G)) \cong \begin{cases} 
\mathbb{Z}^3 & \text{if } n = 1, \\
0 & \text{else.}
\end{cases}\)
Example 6.5 (\(C_5\), the cyclic graph on five vertices). \(C_5\) admits 2-independent sets of cardinality up to three, so we need to know the homology of the complexes \((T_{\bullet,j},d)\) for \(j = 1, 2, 3\).

Let a single vertex of \(C_5\) be 2-marked, say \(v_1\). Ordering the vertices of \(C_5\) cyclically, its maximal 2-independent sets containing \(v_1\) are

\[
\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}.
\]

We see that, to model the allowed 1-markings if \(v_1\) is 2-marked, we have to replace in the decomposition formula (2.2) the summand corresponding to \(C_{5v_1}\) by a path graph \(P_4\) on 4 vertices with edge set

\[
E(P_4) = \{(v_3, v_2), \{v_2, v_5\}, \{v_5, v_4\}\}.
\]

Thus, using that \(I(P_4)\) is contractible,

\[
(T_{\bullet,1},d) \cong \bigoplus_{k=1}^5 (C_{k-1}(P_4), \partial) \implies H_n(T_{\bullet,1},d) \cong 0 \text{ for all } n \in \mathbb{N}.
\]

Now let \(j = 2\), i.e. two vertices be 2-marked. The graphs encoding the remaining possible markings consist of either a single vertex or a \(K_2\), two vertices connected by an edge (if we start with \(v_1\) these cases correspond to 2-marking the sets \(\{v_1, v_2\}, \{v_1, v_3\}\) or \(\{v_1, v_4\}\), respectively). The first case has trivial (reduced) homology, the latter contributes a copy of \(\mathbb{Z}\) in degree one,

\[
(T_{\bullet,2},d) \cong \left( \bigoplus_{\{v,v_j\} \in E} (C_{v,v_j}(P_4), \partial) \right) \oplus \left( \bigoplus_{\{v,v_j\} \in Y \atop \{v,v_j\} \subseteq \{v_1, v_2, v_3\}} (C_{v,v_j}(K_2), \partial) \right) \implies H_n(T_{\bullet,2},d) \cong \begin{cases} \mathbb{Z}^5 & n = 1, \\ 0 & n \neq 1. \end{cases}
\]

Finally, there are five maximal 2-independent sets of size three, so \(H_n(T_{\bullet,3},d) \cong \begin{cases} 0 & n = 0, \\ \mathbb{Z}^5 & n = 1, \\ 0 & n \neq 0. \end{cases}\)

Filling out the first page of the associated spectral sequence gives

\[
\begin{array}{cccc}
\mathbb{Z}^5 & 0 & \mathbb{Z}^5 \\
0 & 0 & 0 & 0 \\
0 & H_1 & H_2 & H_3
\end{array}
\]

so that \(H_3 \cong 0\). The next page \(E^2\) reads

\[
\begin{array}{cccc}
\mathbb{Z}^5/Y & 0 & X \\
0 & 0 & 0 & 0 \\
0 & H_1 & H_2 & 0
\end{array}
\]

with \(X := \ker d^1_{1,2} \text{ and } Y := \im d^1_{1,2}, X \oplus Y \cong \mathbb{Z}^5\).

For the spectral sequence to converge accordingly we must have \(H_2 \cong X \cong 0\) (if \(\ker d^1_{1,2} \cong 0\), then \(Y \cong 0\)) and \((\mathbb{Z}^5/Y)/H_1 \cong \mathbb{Z}\). This shows \(H_2 \cong H_1 \oplus \mathbb{Z}\). Now, either by inspecting the differential \(d^1_{1,2}\) more closely, or by simply noting that \(I_d(C_5)\) is connected, we find \(H_1 \cong 0\) and therefore

\[
H_n(I_d(C_5)) \cong \begin{cases} \mathbb{Z} & n = 1, \\ 0 & n \neq 1. \end{cases}
\]
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