A GRAPHICAL APPROACH TO THE DRINFELD CENTRE

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Abstract. Let $C$ be a spherical fusion category. The goal of this article is to present
the tube category of $C$, denoted $\mathcal{T}C$, as giving an alternative graphical perspective on
the Drinfeld centre of $C$, denoted $Z(C)$. We then exploit this perspective to obtain an
alternative proof of the equivalence between $Z(C)$ and $C \boxtimes C$ when $C$ is modular. To
aid accessibility, the preliminaries are introduced gently through Sections 2-6. Readers
already familiar with graphical calculus in fusion categories should start at Section 7.

1. Introduction

Let $C$ be a spherical fusion category. The idea that morphisms in the Drinfeld centre
of $C$, denoted $Z(C)$, should correspond to graphical calculus in $C$ drawn on a cylinder (or
tube) goes back to Ocneanu’s definition of the tube algebra [Ocn94]. Although originally
defined in an operator algebra context, the concept generalises to an arbitrary spherical
fusion category and a result of Popa, Shlyakhtenko and Vaes [PSV18, Proposition 3.14]
proves that the category of representations of the tube algebra is equivalent to $Z(C)$.
Independently, Kirillov Jr. worked on the string-net spaces introduced in the papers of
Levin and Wen [Lev06]. These spaces may be thought of as the Hom-spaces in categories
whose graphical calculus is drawn on an arbitrary surface. In particular, [Jr11, Theorem
6.4] proves that, when the surface is a cylinder, one recovers a category whose idempotent
completion is $Z(C)$.

The purpose of this article is to provide a ‘from the ground up’ exposition of this
perspective by analysing the so called tube category, first introduced in [HK19]. This
construction may be viewed as a middle ground between the tube algebra and string-
net approaches. Although the resulting object is a category rather than an algebra, the
definition more closely resembles that of the tube algebra. In particular, considering
the endomorphism algebra of a generating object in the tube category directly recovers
Ocneanu’s tube algebra. Working with a category, however, allows us to describe the
idempotents which appear in Kirillov’s work more easily.

The first six sections provide a gentle introduction to all the required pre-requisites.
The tube category is then defined and it is shown that its category of representations (or
equivalently, its idempotent completion) is equivalent to $Z(C)$. This equivalence stems
from the fact that, when $C$ is a spherical fusion category, the Yoneda embedding is an
equivalence and that the data required to extend the image of $X$ under the Yoneda embedding
to $\mathcal{T}C$ corresponds to a half braiding on $X$. To present an alternative perspective
on this equivalence, for an object $(X, \tau)$ in $Z(C)$ we describe an idempotent

$$\epsilon_\tau = \frac{1}{d(C)} \bigoplus_S d(S)$$

$$\in \text{End}_{\mathcal{T}C}(X)$$
which yields the corresponding representation under the Yoneda embedding. Finally, the \( \mathcal{T} \mathcal{C} \) framework is exploited to provide an alternative proof of the equivalence between \( Z(\mathcal{C}) \) and \( \mathcal{C} \boxtimes \overline{\mathcal{C}} \) when \( \mathcal{C} \) is a modular tensor category (this result is originally due to M"uger [Mu"g03]). This is achieved by applying graphical calculus to the idempotent

\[
\varepsilon_X = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S) \begin{tikzpicture}[-latex,scale=0.5, every node/.style={transform shape}] 
  \node (X) at (0,0) {X}; 
  \node (Y) at (0,1) {Y}; 
  \node (S) at (1,0) {S}; 
  \draw (X) -- (Y) -- (S) -- (X); 
\end{tikzpicture} \in \text{End}_{\mathcal{T} \mathcal{C}}(XY)
\]

which, under the Yoneda embedding, corresponds to the image of \( X \boxtimes Y \) under the canonical functor \( \Phi: \mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C}) = \mathcal{R} \mathcal{T} \mathcal{C} \). In particular it is shown that the complete set of simples \( \{I \boxtimes J\}_{I,J \in \text{Irr}(\mathcal{C})} \) in \( \mathcal{C} \boxtimes \overline{\mathcal{C}} \) maps to a complete set of simples in \( \mathcal{R} \mathcal{T} \mathcal{C} = Z(\mathcal{C}) \).

Acknowledgements. The author thanks Alastair King for his guidance during the period this work was carried out. He is also grateful to Ingo Runkel for multiple helpful conversations.

2. \( \mathbb{K} \)-linear Categories

Let \( \mathbb{K} \) be a field. From now on all categories are assumed to be linear categories over \( \mathbb{K} \), i.e. the Hom-spaces are finite dimensional vector spaces over \( \mathbb{K} \) and composition is bilinear. For example, the category of finite dimensional vector spaces \( \text{Vect} \) is a linear category as \( \text{Hom}(V, W) \) is a \( \dim(V) \times \dim(W) \)-dimensional vector space. Furthermore all functors are linear functors, i.e. functors between two linear categories such that the corresponding maps between Hom-spaces are linear.

Definition 2.1. Let \( X \) and \( Y \) be objects in a category \( \mathcal{C} \). A direct sum of \( X \) and \( Y \) is an object \( Z \) in \( \mathcal{C} \) such that \( \text{Hom}_\mathcal{C}(Z, A) \) is naturally identified with \( \text{Hom}_\mathcal{C}(X, A) \oplus \text{Hom}_\mathcal{C}(Y, A) \) and \( \text{Hom}_\mathcal{C}(A, Z) \) is naturally identified with \( \text{Hom}_\mathcal{C}(A, X) \oplus \text{Hom}_\mathcal{C}(A, Y) \).

Definition 2.2. Let \( X \) be in \( \mathcal{C} \) and let \( V \) be in \( \text{Vect} \). The product of \( V \) with \( X \) is an object \( Z \) in \( \mathcal{C} \) such that \( \text{Hom}_\mathcal{C}(Z, Y) \) is naturally identified with \( V^* \otimes \text{Hom}_\mathcal{C}(X, Y) \) and \( \text{Hom}_\mathcal{C}(Y, Z) \) is naturally identified with \( V \otimes \text{Hom}_\mathcal{C}(Y, X) \).

Remark 2.3. For fixed \( X, Y \) in \( \mathcal{C} \) and \( V \) in \( \text{Vect} \), one can check that both the direct sum of \( X \) and \( Y \) and the product of \( V \) with \( X \) are unique. This can also be seen as a consequence of the Yoneda Lemma. They are denoted \( X \oplus Y \) and \( V \cdot X \) respectively.

In practice we never consider the question of whether or not direct sums or products exist. This is due to the fact that, if they do not exist, they may be formally added unambiguously. The following lemma captures the relationship between products and direct sums.

Lemma 2.4. Let \( V \) in \( \text{Vect} \), let \( b \) be a basis of \( V \) and let \( X \) be in \( \mathcal{C} \). Then

\[
V \cdot X = \bigoplus_b X.
\]

Proof. Let \( b^* \) be the dual basis to \( b \). The maps \( \bigoplus_b b \otimes \text{id}_X \in \text{Hom}_\mathcal{C}(\bigoplus_b X, V \cdot X) \) and \( \bigoplus_b b^* \otimes \text{id}_X \in \text{Hom}_\mathcal{C}(V \cdot X, \bigoplus_b X) \) are inverse to one another. \( \square \)

We recall that an object \( X \) in \( \mathcal{C} \) is called simple if \( X \) has no proper subobjects. Schur’s Lemma implies that, for a simple object \( S \) in \( \mathcal{C} \), \( \text{End}_\mathcal{C}(S) \) is a division algebra over \( \mathbb{K} \).
We call an object $X$ Schurian if $\text{End}_C(X) = \mathbb{K}$. We call a category Schurian if all of its simple objects are Schurian. In particular if $\mathbb{K}$ is algebraically closed then $C$ is Schurian.

We recall that a category $C$ is called semisimple if every object in $C$ is a direct sum of finitely many simple objects.

**Definition 2.5.** For a category $C$ a complete set of simples $\text{Irr}(C)$ is a set such that for all $I$ in $\text{Irr}(C)$, $I$ is a simple object in $C$ and for all simple object $S$ in $C$ there exists a unique $I \in \text{Irr}(C)$ such that $\text{Hom}_C(S, I) \neq 0$.

The following canonical decomposition of an object in a semisimple category will be of great importance for the remainder of this article.

**Proposition 2.6.** Let $C$ be a semisimple, Schurian category, let $\text{Irr}(C)$ be a complete set of simples and let $X$ be an object in $C$. Then

$$X = \bigoplus_{S \in \text{Irr}(C)} \text{Hom}_C(S, X) \cdot S.$$ 

**Proof.** As $X$ is semisimple we have

$$X = \bigoplus_{i \in I} X_i$$

where the $X_i$ are simple objects and $I$ is an indexing set. We therefore have

$$\bigoplus_{S \in \text{Irr}(C)} \text{Hom}_C(S, X) \cdot S = \bigoplus_{i \in I} \text{Hom}_C(S, X_i) \cdot S = \bigoplus_{i \in I_S} \text{Hom}_C(S, X_i) \cdot S$$

where $I_S = \{i \in I \mid X_i \cong S\}$. As $C$ is Schurian $\text{Hom}_C(S, X_i)$ is one-dimensional and the canonical morphism

$$\text{id} \in \text{End}(\text{Hom}_C(S, X_i)) = \text{Hom}_C(S, X_i)^* \otimes \text{Hom}_C(S, X_i) = \text{Hom}_C(\text{Hom}_C(S, X_i) \cdot S, X_i)$$

is an isomorphism by Schur’s Lemma. Therefore

$$\bigoplus_{i \in I_S} \text{Hom}_C(S, X_i) \cdot S = \bigoplus_{i \in I_S} X_i = X.$$ 

3. Idempotent Completions

**Definition 3.1.** Let $X$ be an object in a category. A morphism $\varepsilon \in \text{End}_C(X)$ such that $\varepsilon \circ \varepsilon = \varepsilon$ is called an idempotent.

Let $V$ be in $\text{Vect}$ and suppose $\varepsilon \in \text{End}(V)$ is an idempotent. In this case $V_\varepsilon = \{v \in V \mid \varepsilon(v) = v\}$ forms a subspace of $V$ and $\varepsilon$ may be thought of as a projection from $V$ onto $V_\varepsilon$ composed with an inclusion of $V_\varepsilon$ back into $V$. Furthermore this correspondence between idempotents on $V$ and split subspaces of $V$ defines a bijection. To say this more generally we made the following definition,

**Definition 3.2.** Let $\varepsilon \in \text{End}_C(X)$ be an idempotent in a category $C$. An image object for $\varepsilon$ is a object $X_\varepsilon$ in $C$ together with morphisms $\pi: X \to X_\varepsilon$ and $i: X_\varepsilon \to X$ such that $i \circ \pi = \varepsilon$ and $\pi \circ i = \text{id}_{X_\varepsilon}$. 
Remark 3.3. Suppose \((X_1, \pi_1, i_1)\) and \((X_2, \pi_2, i_2)\) are two image objects for an idempotent \(\varepsilon \in \text{End}_C(X)\). Then \(\pi_2 \circ i_1\) and \(\pi_1 \circ i_2\) give inverse isomorphisms between \(X_1\) and \(X_2\). Furthermore, as the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & X_1 \\
\downarrow{\pi_2} & & \downarrow{\pi_1 \circ i_1} \\
X & \xrightarrow{\pi_2 \circ i_1} & X_2
\end{array}
\]

commutes, image objects of \(\varepsilon\) are unique as summands of \(X\).

Now let \(C\) be any (linear) category. If there exists an image object for every idempotent in \(C\) we say that \(C\) is idempotent complete. As this property fails for many categories it can be desirable to fully embed \(C\) into another category \(\overline{C}\) that is idempotent complete.

Definition 3.4. An idempotent completion of a category \(C\) is a category \(\overline{C}\) together with a covariant functor \(\Psi: C \rightarrow \overline{C}\) such that

- \(\Psi\) is fully faithful.
- \(\overline{C}\) is idempotent complete.
- For every object \(X\) in \(\overline{C}\) there exists an idempotent \(\varepsilon\) in \(C\) such that \(X\) is an image object for \(\Psi(\varepsilon)\).

Remark 3.5. Idempotent completions are unique up to equivalence of categories [Lur09, Section 5.1.4].

We shall now describe a realization of an idempotent completion for any category \(C\). Let \(RC\) denote the category of contravariant functors from \(C\) into \(\text{Vect}\). We consider the functor

\[Y: C \rightarrow RC\]

\[X \mapsto X^{\sharp}\]

where \(X^{\sharp} = \text{Hom}_{C}(-, X)\).

It is a well known corollary of the Yoneda Lemma that \(Y\) is fully faithful and is therefore referred to as the Yoneda embedding. Furthermore for any idempotent \(\varepsilon \in \text{End}_C(X)\), there is, in \(RC\), a subfunctor \((X, \varepsilon)^{\sharp} \leq X^{\sharp}\) given by

\[(X, \varepsilon)^{\sharp}: C \rightarrow \text{Vect}\]

\[Y \mapsto \text{Hom}_{C}(Y, \varepsilon) := \{f \in \text{Hom}_{C}(Y, X) \mid \varepsilon \circ f = f\}\]

\[(\alpha: Y \rightarrow Z) \mapsto (f \mapsto \alpha \circ f)\]

which is an image object for \(Y(\varepsilon)\) in \(RC\). Indeed, \((X, \varepsilon)^{\sharp}\) is a summand of \(X^{\sharp}\). This image object exists because \(RC\) is an abelian category, so idempotent complete, even if \(C\) may not be. Concretely, \((X, \varepsilon)^{\sharp}(Y)\) is the image of \(\varepsilon_Y^\sharp = \varepsilon_*\), which is an idempotent endomorphism of \(X^\sharp(Y) = \text{Hom}_{C}(Y, X)\). The naturality of \(\varepsilon^{\sharp}\), i.e. the fact that \(\varepsilon_*\) commutes with \(\phi^\sharp\) for any \(\phi: Z \rightarrow Y\), makes \((X, \varepsilon)^{\sharp}\) a functor.

Let \(\overline{C}_{Y}\) be the full subcategory of \(RC\) spanned by object of the form \((X, \varepsilon)^{\sharp}\). We note that \(Y\) factors through \(\overline{C}_{Y}\) as \((X, \text{id}_X)^{\sharp} = \Psi(X)\) for all \(X\) in \(C\). We therefore obtain the following result.

Proposition 3.6. The category \(\overline{C}_{Y}\) together with the Yoneda embedding is an idempotent completion of \(C\).
Proposition 3.7. Let $\mathcal{C}$ be a semisimple Schurian category with finitely many isomorphism classes of simple objects. Then the Yoneda embedding
\[
\mathcal{C} \to \mathcal{R}\mathcal{C} \\
X \mapsto X^{\#}
\]
is an equivalence.

Proof. As the Yoneda embedding is fully faithful we only have to show that it is essentially surjective. For $F$ in $\mathcal{R}\mathcal{C}$ and $X$ in $\mathcal{C}$, we have
\[
F(X) = \bigoplus_{S \in \text{Irr}(\mathcal{C})} F(S) \otimes \text{Hom}_\mathcal{C}(S, X)^* \\
= \bigoplus_{S \in \text{Irr}(\mathcal{C})} F(S) \otimes \text{Hom}_\mathcal{C}(X, S) \\
= \bigoplus_{S \in \text{Irr}(\mathcal{C})} F(S) \otimes S^{\sharp}(X)
\]
where the first equality uses the semisimplicity of $\mathcal{C}$ and the contravariance of $F$ and the second equality uses the fact $S$ is Schurian. \qed

Let $\mathcal{C}$ be a semisimple category together with a complete set of simples $\text{Irr}(\mathcal{C})$. Suppose we choose an element $I \in \text{Irr}(\mathcal{C})$ and consider the full subcategory of $\mathcal{C}$ whose objects are non-isomorphic to $I$. Clearly this new category fails to be semisimple, however the missing simple objects may still be detected by considering the idempotent endomorphisms of any object that has a proper summand isomorphic to $I$. There is, therefore, a notion analogous to a complete set of simples for idempotents.

Definition 3.8. A set of primitive orthogonal idempotents in a linear category $\mathcal{C}$ is a set of idempotents $I$ in $\mathcal{C}$ such that
\[
\text{Hom}_\mathcal{C}(\varepsilon, \varepsilon') = \begin{cases} 
K & \text{if } \varepsilon = \varepsilon' \\
0 & \text{else.}
\end{cases}
\]
A set of primitive orthogonal idempotents is called complete if we have
\[
\bigoplus_{\varepsilon \in I} \text{Hom}_\mathcal{C}(X, \varepsilon) \otimes \text{Hom}_\mathcal{C}(\varepsilon, Y) = \text{Hom}_\mathcal{C}(X, Y)
\]
for all $X, Y$ in $\mathcal{C}$.

The proof Proposition 3.7 shows that the Yoneda embedding maps a complete set of Schurian simples in $\mathcal{C}$ to a complete set of Schurian simples in $\mathcal{R}\mathcal{C}$. The corresponding claim for a complete set of primitive orthogonal idempotents also holds.

Proposition 3.9. Let $\mathcal{C}$ be a linear category with a complete set of primitive orthogonal idempotents $I$. Then $\mathcal{R}\mathcal{C}$ is a semisimple Schurian category and $\{(X_{\varepsilon}, \varepsilon)^{\sharp}\}_{\varepsilon \in I}$ forms a complete set of simples in $\mathcal{R}\mathcal{C}$ (where $\varepsilon \in \text{End}_\mathcal{C}(X_{\varepsilon})$).

Proof. It is straightforward to check that the set $\{(X_{\varepsilon}, \varepsilon)^{\sharp}\}_{\varepsilon \in I}$ contains distinct simple Schurian objects in $\mathcal{R}\mathcal{C}$. The condition that $I$ is a complete set of orthogonal idempotents may be rephrased as
\[
Y^{\sharp} = \bigoplus_{\varepsilon \in I} \text{Hom}_\mathcal{C}(\varepsilon, Y) \cdot (X_{\varepsilon}, \varepsilon)^{\sharp}
\]
for all \( Y \) in \( C \). Then, using a similar argument to the proof of Proposition 3.7, we take \( F \) in \( RC \), \( Y \) in \( C \) and compute,

\[
F(Y) = \text{Hom}_{RC}(Y^2, F)
= \bigoplus_{\varepsilon \in I} \text{Hom}_C(\varepsilon, Y)^* \otimes \text{Hom}_{RC}((X_{\varepsilon}, \varepsilon)^2, F)
= \bigoplus_{\varepsilon \in I} \text{Hom}_C(Y, \varepsilon) \otimes \text{Hom}_{RC}((X_{\varepsilon}, \varepsilon)^2, F)
= \bigoplus_{\varepsilon \in I} \text{Hom}_{RC}((X_{\varepsilon}, \varepsilon)^2, F) \otimes (X_{\varepsilon}, \varepsilon)^2(Y)
\]
as desired.  

\[\square\]

4. Monoidal Categories

**Definition 4.1.** A monoidal category is a category \( C \) together with a tensor product bifunctor \( \otimes: C \times C \to C \) with natural associativity isomorphisms \( a_{X,Y,Z}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \) and a tensor identity \( 1 \) in \( C \) with natural unit isomorphisms \( r_x: 1 \otimes X \to X \) and \( l_x: X \otimes 1 \to X \) such that

- The diagram

\[
\begin{array}{ccc}
(X \otimes (Y \otimes Z)) \otimes W & \xrightarrow{a_{X,Y,W} \otimes \text{id}} & (X \otimes Y) \otimes (Z \otimes W) \\
\downarrow a_{X,Y,Z} & & \downarrow a_{X,Y,Z} \\
X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{\text{id} \otimes a_{Y,Z,W}} & X \otimes (Y \otimes (Z \otimes W))
\end{array}
\]

commutes.

- The diagram

\[
\begin{array}{ccc}
(X \otimes 1) \otimes Y & \xrightarrow{a_{X,1,Y}} & X \otimes (1 \otimes Y) \\
\downarrow r_X \otimes \text{id} & & \downarrow \text{id} \otimes l_Y \\
X \otimes Y & & 
\end{array}
\]

commutes.

**Remark 4.2.** By interpreting objects as 1-morphisms and \( \otimes \) as composition of 1-morphisms we see that the notion of a monoidal category is equivalent to the notion of a 2-category with exactly one object.

**Remark 4.3.** For the remainder of this article we will suppress the associativity and unit isomorphisms.

**Definition 4.4.** Let \( C \) be a monoidal category and let \( X \) be an object in \( C \). An object \( Y \) in \( C \) together with morphisms

\[
cr: 1 \to X \otimes Y \quad \text{and} \quad an: Y \otimes X \to 1
\]
such that

\[
(id_X \otimes an) \circ (cr \otimes id_X) = id_X \quad (1)
\]
and
\[(\text{an} \otimes \text{id}_Y) \circ (\text{id}_Y \otimes \text{cr}) = \text{id}_Y\] (2)
is called a right dual to \(X\). An object \(Z\) in \(\mathcal{C}\) together with morphisms
\[
\text{cr}: 1 \to Z \otimes X \quad \text{and} \quad \text{an}: X \otimes Z \to 1
\]
is called a left dual to \(X\). The maps \(\text{cr}\) and \(\text{an}\) are called the creation and annihilation morphisms respectively.

**Remark 4.5.** If \(X\) in \(\mathcal{C}\) admits a right (resp. left) dual, then the dual is unique [EGNO15, Proposition 2.10.5].

**Definition 4.6.** A monoidal category \(\mathcal{C}\) is called rigid if every object in \(\mathcal{C}\) admits a left and a right dual.

For an object \(X\) in \(\mathcal{C}\) we use \(X^\vee\) to denote a right dual to \(X\) and \(\vee X\) to denote a left dual to \(X\). The corresponding creation and annihilation morphisms are denoted \(\text{cr}_X\) and \(\text{an}_X\) respectively.

**Lemma 4.7.**
\[
(X \otimes Y)^\vee = Y^\vee \otimes X^\vee \quad \text{and} \quad \vee(X \otimes Y) = \vee Y \otimes \vee X.
\]

**Proof.** The isomorphism
\[
(X \otimes Y)^\vee \xrightarrow{\text{id} \otimes \text{cr}_X} (X \otimes Y)^\vee \otimes X \otimes X^\vee \xrightarrow{\text{id} \otimes \text{cr}_Y \otimes \text{id}} (X \otimes Y)^\vee \otimes X \otimes Y^\vee \otimes X^\vee \xrightarrow{\text{an}_X \otimes \text{id}} Y^\vee \otimes X^\vee
\]
gives the first identify, the second may be proved analogously. \(\Box\)

**Lemma 4.8.**
\[
\text{Hom}_{\mathcal{C}}(X \otimes Y, Z) = \text{Hom}_{\mathcal{C}}(X, Z \otimes Y^\vee), \quad \text{Hom}_{\mathcal{C}}(X, Y \otimes Z) = \text{Hom}_{\mathcal{C}}(Y^\vee \otimes X, Z)\]
\[
\text{Hom}_{\mathcal{C}}(X \otimes \vee Y, Z) = \text{Hom}_{\mathcal{C}}(X, Z \otimes Y), \quad \text{Hom}_{\mathcal{C}}(Y \otimes X, Z) = \text{Hom}_{\mathcal{C}}(X, \vee Y \otimes Z).
\]

**Proof.** We consider the following canonical map
\[
\text{Hom}_{\mathcal{C}}(X \otimes Y, Z) \to \text{Hom}_{\mathcal{C}}(X, Z \otimes Y^\vee) \quad g \mapsto (g \otimes \text{id}_{Y^\vee}) \circ (\text{id}_X \otimes \text{cr}_Y)
\]
it has inverse
\[
\text{Hom}_{\mathcal{C}}(X, Z \otimes Y^\vee) \to \text{Hom}_{\mathcal{C}}(X \otimes Y, Z)) \quad g \mapsto (\text{id}_Z \otimes \text{an}_Y) \circ (g \otimes \text{id}_Y)
\]
and is therefore a canonical isomorphism. The proofs of the other equalities are analogous. \(\Box\)

**Corollary 4.9.**
\[
\text{Hom}_{\mathcal{C}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y^\vee, X^\vee) = \text{Hom}_{\mathcal{C}}(1, Y \otimes X^\vee).
\]

Let \(\mathcal{C}\) be a rigid category. Lemma 4.5 and Corollary 4.9 give us contravariant endo-functors \(-^\vee\) and \(\vee -\) on \(\mathcal{C}\). The following rephrasing of Lemma 4.8 in terms of the Yoneda embedding will prove useful.

**Lemma 4.10.** Let \(\mathcal{C}\) be a rigid monoidal category. Then we have
\[
Y^\sharp \circ (X^\vee \otimes \text{--}) = (X \otimes Y)^\sharp = X^\sharp \circ (\text{--} \otimes Y^\vee).
\]
Proof. The natural isomorphisms from the proof of Lemma 4.8 give the desired result. □

We now introduce a graphical notation, due to Roger Penrose [Pen71], that shall be used throughout the remainder of this article. To represent a morphism \( \alpha \in \text{Hom}_C(X, Y) \) we draw a strand labelled \( X \), a strand labelled \( Y \) and a connection between them labelled \( \alpha \) as follows

\[
\begin{array}{c}
\text{X} \\
\text{α} \\
\text{Y}
\end{array}
\]

Composition is then depicted by vertical juxtaposition and the monoidal product by horizontal juxtaposition:

\[
\beta \circ \alpha = \begin{array}{c}
\text{α} \\
\text{β}
\end{array}, \quad \alpha \otimes \beta = \begin{array}{c}
\text{α} \\
\text{β}
\end{array}.
\]

As the monoidal product is depicted by horizontal juxtaposition the associativity maps are implicitly used but not depicted. Similarly any strand labelled by the tensor identity is not drawn, therefore the unit isomorphisms are also implicitly used but not depicted. The maps \( \text{cr}_X \) and \( \text{an}_X \) are drawn as a cap and cup respectively:

\[
\text{cr}_X = \begin{array}{c}
\text{X} \\
\text{X} \vee
\end{array}, \quad \text{an}_X = \begin{array}{c}
\text{X} \\
\text{X} \vee
\end{array}.
\]

Note that this is consistent with horizontal juxtaposition depicting the monoidal product by Lemma 4.7. With graphical notation in hand we can rewrite conditions (1) and (2) as

\[
\begin{array}{c}
\text{X} \\
= \\
\text{X} \\
\end{array} = \begin{array}{c}
\text{X}
\end{array}.
\]

As all three diagrams are isotopic the graphical intuition behind dual objects is made clear. In general, when working with rigid categories it can be important to distinguish between left and right duals. However, to establish the graphical calculus we are interested in we need to identify them.

Definition 4.11. Let \( C \) be a rigid category. A pivotal structure on \( C \) is a choice of natural isomorphism

\[
\delta_X : \vee X \rightarrow X \vee
\]

such that (under the identification of Lemma 4.7)

\[
\delta_{X \otimes Y} = \delta_Y \otimes \delta_X.
\]

A rigid category equipped with a pivotal structure is called a pivotal category.

The map \( \delta_X \) is suppressed from graphical notation; condition (4) guarantees that this doesn’t cause inconsistencies. As a pivotal structure identifies left and right duals, we use \( X \vee \) to denote both. For example

\[
\begin{array}{c}
\text{X} \\
\text{X} \vee
\end{array}
\]

(5)
is valid graphical notation for an element in $\text{End}_C(1)$ if $C$ is pivotal but not if $C$ is merely rigid. However, even in a pivotal category, (5) is not necessarily equal to

$$X^\vee \bigotimes X.$$ (6)

**Definition 4.12.** A pivotal category is called spherical if (3) and (4) define the same element of $\text{End}_C(1)$ for all $X$ in $C$. In this case said element is called the dimension of $X$ and is denoted $d(X)$.

**Proposition 4.13.** Suppose $C$ is a semisimple spherical category, $\text{Irr}(C)$ is a complete set of simples in $C$ and $S \in \text{Irr}(C)$ is Schurian. Then $d(S)$ is an automorphism of $1$.

**Proof.** By Proposition 2.6 we have

$$S \otimes S' = \text{Hom}_C(1, S \otimes S') \cdot 1 \oplus \bigoplus_{T \in \text{Irr}(C)} \text{Hom}_C(T, S \otimes S') \cdot T$$

Let $\pi$ be projection onto the $\text{Hom}_C(1, S \otimes S') \cdot 1$ summand and let $i$ be the corresponding inclusion. As, by (2), $\text{cr}_S$ and $\text{an}_S$ are non zero and

$$\text{Hom}_C(1, S \otimes S') = \text{Hom}_C(S \otimes S', 1) = \text{End}_C(S) = \mathbb{K}$$

$\pi \circ \text{cr}_S$ and $\text{an}_S \circ i$ are isomorphisms. As $d(S) = \text{an}_S \circ \text{cr}_S = \text{an}_S \circ i \circ \pi \circ \text{cr}_S$ this concludes the proof. □

Combining the categorical properties discussed in the previous section with a rigid monoidal structure yields the following crucial definition.

**Definition 4.14.** A fusion category is a Schurian, semisimple, rigid, monoidal category $C$ such that there is a complete set of simples $\text{Irr}(C)$ satisfying $|\text{Irr}(C)| < \infty$ and $1 \in \text{Irr}(C)$.

**Definition 4.15.** The dimension of a spherical fusion category $C$ is given by

$$d(C) := \sum_{S \in \text{Irr}(C)} d(S)^2.$$ 

For the remainder of this article any fusion category is assumed to have non-zero dimension. This holds automatically if the underlying field is algebraically closed, see

5. **Braided Categories**

**Definition 5.1.** Let $C$ be a monoidal category. A braiding on $C$ is a collection of natural isomorphisms $\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$, such that the diagrams

$$X \otimes Y \otimes Z \xrightarrow{\sigma_{X,Y} \otimes 1} Y \otimes Z \otimes X \xrightarrow{1 \otimes \sigma_{X,Z}} Y \otimes X \otimes Z$$

and

$$X \otimes Y \otimes Z \xrightarrow{1 \otimes \sigma_{Y,Z}} Y \otimes Z \otimes X \xrightarrow{\sigma_{X,Z} \otimes 1} Y \otimes X \otimes Z$$

commute. These conditions are often referred to as the hexagon identities (our diagrams are triangular as we have suppressed the associativity isomorphisms).
A monoidal category equipped with a braiding is said to be braided. In graphical notation this braiding is depicted by the over-crossing,

\[
\begin{array}{c}
X \\
\downarrow \alpha \\
Y \\
\end{array}
\quad \quad \quad \quad
\begin{array}{c}
X \\
\downarrow \beta \\
Y \\
\end{array}
\]

The hexagon identities guarantee that this notation is consistent with horizontal juxta-position depicting the monoidal product. Naturality of the braiding allows morphisms to pass over and under strands, i.e.

\[
\begin{array}{c}
X \\
\downarrow \alpha \\
Y \\
\end{array}
\quad \quad \quad \quad
\begin{array}{c}
X \\
\downarrow \beta \\
Y \\
\end{array}
\]

If \( \sigma \) is a braiding on a monoidal category \( \mathcal{C} \) then \( \overline{\sigma} \), given by \( \overline{\sigma}_{X,Y} := (\sigma_{Y,X}^{-1}) \), also defines a braiding on \( \mathcal{C} \) called the opposite braiding to \( \sigma \). In graphical notation the opposite braiding is depicted by the under-crossing,

\[
\begin{array}{c}
X \\
\downarrow \alpha \\
Y \\
\end{array}
\quad \quad \quad \quad
\begin{array}{c}
X \\
\downarrow \beta \\
Y \\
\end{array}
\]

We therefore have the “Reidemeister II” rule

\[
\begin{array}{c}
X \\
\downarrow \beta \\
Y \\
\end{array}
\quad \quad \quad \quad
\begin{array}{c}
X \\
\downarrow \alpha \\
Y \\
\end{array}
\]

This, together with \( \Box \) (which resembles a “Reidemeister 0” rule), makes one wonder whether, for a spherical braided category, graphical notation is well defined up to tangle isotopy. For this to be possible we have to assume a certain compatibility between the braiding and the pivotal structure which may be described graphically as

\[
\begin{array}{c}
X \\
\downarrow \beta \\
Y \\
\end{array}
\quad \quad \quad \quad
\begin{array}{c}
X \\
\downarrow \beta \\
Y \\
\end{array}
\]

A braiding which satisfies this condition is called balanced; for the remainder of this article we suppose that all braidings on spherical categories are balanced. Even when the braiding is balanced, however, graphical notation is not well defined up to tangle isotopy
as, in general,

$$X$$

does not define \( \text{id}_X \in \text{End}_C(X) \) and so there is no “Reidemeister I” rule. Dropping the “Reidemeister I” rule gives us the notion of ribbon tangle isotopy and it is true that graphical notation is well defined up to ribbon tangle isotopy [RT90]. This motivates the following terminology.

**Definition 5.2.** A (balanced) braided spherical category is called a **ribbon** category.

However, we are interested in the more specific case when the category is also assumed to be fusion.

**Definition 5.3.** A **pre-modular tensor category** (PTC) is a (balanced) braided spherical fusion category.

This is clearly loaded terminology. What additional property must \( C \) satisfy to achieve the full status of modular tensor category? The answers lie in something called the **modular data** of \( C \). As \( C \) is fusion it admits a finite complete set of simples \( \text{Irr}(C) \). We consider the following \( \text{Irr}(C) \times \text{Irr}(C) \) matrices,

$$\mathcal{T}_{I,J} := \delta_{I,J}, \quad \mathcal{S}_{I,J} := \delta_{I,J}.$$  \hspace{1cm} (7)

**Remark 5.4.** If we replace the \( I \) strand in the definition of \( \mathcal{T}_{I,J} \) with an \( I^\vee \) strand we obtain an equivalent definition [EGNO15 Section 8.10.]. Similarly if we replace the \( I \) and \( J \) strands in the definition of \( \mathcal{S}_{I,J} \) with \( I^\vee \) and \( J^\vee \) strands respectively we also obtain an equivalent definition [EGNO15 Remark 8.1.12.3].

These matrices are called the T-matrix and the S-matrix respectively. Collectively they are know are the modular data of \( C \).

**Definition 5.5.** A **modular tensor category** (MTC) is a pre-modular tensor category such that the above defined \( S \) and \( T \) matrices are invertible, i.e. the modular data is non-singular.

**Remark 5.6.** Modular tensor categories are so called as their modular data satisfies the following equations,

$$(ST)^3 = \lambda S^2, \quad S^2 = d(C) C, \quad TC = CT$$

where \( \lambda \in \mathbb{K} \) and \( C := (\delta_{I^\vee,J})_{I,J \in \text{Irr}(C)} \) is the charge conjugation matrix [EGNO15 Section 8.16.]. As \( C^2 = \text{id} \) these equations imply that the modular data gives a projective representation of \( \text{SL}_2(\mathbb{Z}) \) a.k.a. the **modular group**.

**Remark 5.7.** From Remark 5.4 we see that \( S \) and \( T \) commute with the charge conjugation matrix even when \( C \) is only assumed to be pre-modular.
6. Further Developing Graphical Calculus

The goal of this section is to gather together certain graphical results which will be useful throughout the remainder of this article. For the sake of expediency many of the proofs are omitted, however references are always provided. The first of these lemmas applies to semisimple Schurian categories and describes how one may decompose an arbitrary strand into a sum of strands labelled by simple objects.

**Lemma 6.1.** Let $\mathcal{C}$ be a semisimple Schurian category and let $X$ be in $\mathcal{C}$. We have

$$X = \sum_{R,b} X \bigg\| \begin{array}{c} \overline{b^*} \\ b \\ R \\ b \\ X \end{array}$$

where $R$ ranges over a basis of $\text{Irr}(\mathcal{C})$, $\{b\}$ is a basis of $\text{Hom}_\mathcal{C}(R, X)$ and $\{b^*\}$ is the corresponding dual basis with respect to the perfect pairing given by

$$\text{Hom}_\mathcal{C}(R, X) \otimes \text{Hom}_\mathcal{C}(X, R) \rightarrow \text{End}_\mathcal{C}(R)$$

$$f \otimes g \mapsto g \circ f.$$ 

*Proof.* A proof is provided by Lemma 3.3 in [HK19].

As decompositions of the form (8) come up fairly frequently, we now make the following conventions: unless otherwise specified, a sum over a variable object in $\mathcal{C}$ ranges over $\text{Irr}(\mathcal{C})$ and a sum over a variable morphism in $\mathcal{C}$ ranges over a basis of the appropriate Hom-space.

A standard technique when using graphical calculus in a spherical fusion category is to decompose a strand, using Lemma 6.1, rearrange the diagram (up to ribbon isotopy), and then reapply Lemma 6.1 to pack the simple strands back into an object. During the rearranging step of this procedure, the strands attached to certain morphisms may get pulled up or down. This can affect the final re-packaging step, as described by the following lemma.

**Lemma 6.2.** Let $\mathcal{C}$ be a spherical fusion category. Let $X$ be in $\mathcal{C}$ and $S$ be in $\text{Irr}(\mathcal{C})$. We have

$$\sum_{T,b} d(T) X \bigg\| \begin{array}{c} S \\ b \\ T \\ b^* \\ S \\ X \end{array} = d(S) X \bigg\| S$$

and

$$\sum_{T,b} d(T) S \bigg\| \begin{array}{c} X \\ b \\ T \\ b^* \\ S \\ X \end{array} = d(S) S \bigg\| X.$$

*Proof.* A proof is provided by Lemma 5.1 in [Kon08], or alternatively, Lemma 3.11 in [HK19].
An interesting consequence of this lemma is the following relationship between the dimension of simples objects and the dimension of the corresponding Hom-spaces. We consider the expression

\[ \sum_{S,T,b} d(S)d(T) \]

As, by definition, \( b \) and \( b^* \) compose to the identity this expression is equal to the scalar \( \sum_{S,T} \text{hom}_C(R, ST)d(S)d(T) \) in \( \text{End}_C(R) = \mathbb{K} \), where \( \text{hom}_C(R, ST) \) denotes the dimension of \( \text{Hom}_C(R, ST) \). However, by rearranging the diagram as follows:

\[ \sum_{S,T,b} d(S)d(T) \]

we may apply Lemma 6.2 and evaluate the expression to be \( d(R)d(C) \) giving the following lemma.

**Lemma 6.3.** Let \( R \) be in \( \text{Irr}(\mathcal{C}) \). Then

\[ \sum_{S,T} \text{hom}_C(R, ST)d(S)d(T) = d(R)d(C). \]

We now equip \( \mathcal{C} \) with a braiding and furthermore assume that \( \mathcal{C} \) is a modular tensor category. The main difference between graphical calculus in modular tensor categories (as opposed to pre-modular tensor categories) is the so called ‘killing ring’ lemma. It goes as follows.

**Lemma 6.4.** Let \( \mathcal{C} \) be an MTC and let \( R \) be in \( \text{Irr}(\mathcal{C}) \). Then

\[ \sum_s d(S) \sum_{b,c} \delta_{R,1} = \delta_{R,1}d(C) \]

where \( 1 \) is the tensor identity.

**Proof.** A proof is provided by Corollary 3.1.11 in [BK01].

Heuristically speaking, the ring labelled \( S \) (as weighted by \( d(S) \)) may be though of as ‘killing’ any strand that passes through it. This allows the ring to ‘slice horizontally’ through a diagram, as illustrated by the following corollary.

**Corollary 6.5.** Let \( \mathcal{C} \) be an MTC and let \( X \) and \( Y \) be in \( \mathcal{C} \). Then

\[ \sum_s d(S) \]
Proof. A straightforward generalisation of the proof of Corollary 3.13 in [HK19] proves this result. □

Although it requires a bit more work to see, the killing ring may also be used to slice a diagram vertically.

**Proposition 6.6.** Let $\mathcal{C}$ be an MTC. We consider $X, Y, A, B$ in $\mathcal{C}$, and $\alpha \in \text{Hom}_\mathcal{C}(IJ, XY)$. Then

\[
\sum_s d(S) \cdot S^\vee \cdot \alpha \cdot X \cdot Y = d(C) \cdot \sum_{T, b, c} \frac{1}{d(T)} \cdot T \cdot \alpha \cdot b^* \cdot c^* \cdot X \cdot Y.
\]

Proof. A straightforward generalisation of the proof of Proposition 3.14 in [HK19] proves this result. □

7. INTRODUCING THE TUBE CATEGORY

Let $\mathcal{C}$ be a spherical fusion category, see Definition 4.14 and 4.12. The *tube category*, denoted $\mathcal{T}\mathcal{C}$, shares the same objects as $\mathcal{C}$ but has more morphisms i.e. $\text{Hom}_\mathcal{C}(X, Y) \leq \text{Hom}_{\mathcal{T}\mathcal{C}}(X, Y)$. The intuition is that whereas morphisms in $\mathcal{C}$ may be represented graphically as diagrams drawn on a bounded region of the plane, morphisms in $\mathcal{T}\mathcal{C}$ are given by diagrams drawn on a *cylinder*. For example, for any $f \in \text{Hom}_\mathcal{C}(X, Y)$ diagrammatically represented by

\[
\begin{array}{c}
\text{X} \\
\text{f} \\
\text{Y}
\end{array}
\]

there will be a morphism in $\mathcal{T}\mathcal{C}$ diagrammatically represented by

\[
\begin{array}{c}
\text{X} \\
\text{Y}
\end{array}
\]

We capture such morphisms by drawing diagrams in a diamond and glueing the upper left and lower right edges. For example morphism (10) is represented by

\[
\begin{array}{c}
\text{Y} \\
\text{Y}^\vee
\end{array}
\]

\[
\begin{array}{c}
\text{X} \\
\text{Y}
\end{array}
\]

\[
\begin{array}{c}
\text{Y} \\
\text{Y}^\vee
\end{array}
\]
We note that this diagram may also be read vertically and interpreted as an element in $\text{Hom}_\mathcal{C}(Y^\vee X, Y Y^\vee)$. We also note that due to Lemma 6.1 we may restrict ourselves to only gluing \textit{simple} strands. In this way morphism (10) would be represented as

$$\sum_{R,b} R X \rightarrow Y \rightarrow R$$

where $b$ ranges over a basis of $\text{Hom}_\mathcal{C}(R, Y^\vee)$. We note that each diagram may now be read vertically as an element in $\text{Hom}_\mathcal{C}(RX, Y R)$. With this motivation in mind we may proceed with the definition of $\mathcal{T}C$.

**Definition 7.1.** Let $\mathcal{C}$ be a spherical fusion category. The associated \textit{tube category}, denoted $\mathcal{T}C$, is defined as the following category,

1. $\text{Obj}(\mathcal{T}C) := \text{Obj}(\mathcal{C})$
2. $\text{Hom}_{\mathcal{T}C}(X, Y) := \bigoplus_R \text{Hom}_\mathcal{C}(RX, Y R)$
3. Let $f$ be in $\text{Hom}_{\mathcal{T}C}(X, Y)$ and let $g$ be in $\text{Hom}_{\mathcal{T}C}(Y, Z)$. We define $g \circ f$ as follows (using the diagrams explained above)

$$g \circ f := \bigoplus_T \sum_{S, R, b} S R T X \rightarrow Z$$

where $f_R$ and $g_S$ are the $\text{Hom}_\mathcal{C}(RX, Y R)$ and $\text{Hom}_\mathcal{C}(SY, Z S)$ components of $f$ and $g$ respectively and $b$ ranges over a basis of $\text{Hom}_\mathcal{C}(T, SR)$. We note that $g \circ f \in \bigoplus_T \text{Hom}_\mathcal{C}(TX, ZT) = \text{Hom}_{\mathcal{T}C}(X, Z)$ as desired.

From Lemma 6.1 we see that this definition agrees with the intuition that composition corresponds to vertically stacking the cylinders upon which the diagrams are drawn. This intuition, together with the associativity of the tensor product, makes it clear that composition in $\mathcal{T}C$ is associative.

**Remark 7.2.** At this point, a careful reader might protest that the tensor product is merely weakly associative and yet composition in a category must be strongly associative. However, this is not an issue as the associator isomorphisms will simply modify the basis appearing in (11) leaving the composition unchanged.

**Remark 7.3.** The summand indexed by 1 in $\text{Hom}_{\mathcal{T}C}(X, Y)$ is $\text{Hom}_\mathcal{C}(X, Y)$. This gives a map $\text{Hom}_\mathcal{C}(X, Y) \hookrightarrow \text{Hom}_{\mathcal{T}C}(X, Y)$ such that
commutes. In other words $\mathcal{C}$ is a subcategory of $\mathcal{T C}$. In particular the identity in $\text{End}_{\mathcal{T C}}(X)$ is given by the image of $\text{id}_X \in \text{End}_\mathcal{C}(X)$ under this embedding.

**Remark 7.4.** If we consider the algebra

$$\mathcal{T A} := \text{End}_{\mathcal{T C}} \left( \bigoplus_S S \right)$$

we recover Ocneanu’s tube algebra $[\text{Ocn94}]$. As $\bigoplus S$ is a projective generator in $\mathcal{T C}$ the functor

$$\mathcal{R TC} \to \text{Mod-} \mathcal{T A}$$

$$F \mapsto \text{Hom}_{\mathcal{R TC}} \left( F, \bigoplus_S S \right)$$

gives an equivalence, i.e. $\mathcal{T C}$ is Morita equivalent to $\mathcal{T A}$.

**Remark 7.5.** The definition of Hom-spaces in $\mathcal{T C}$ has the following interesting consequence. Let $\mathcal{K}(\mathcal{C})$ denote the Grothendieck ring of $\mathcal{C}$ and let $\mathcal{K}_\mathbb{K}(\mathcal{C})$ denote $\mathcal{K}(\mathcal{C}) \otimes \mathbb{K}$. Then $\text{End}_{\mathcal{T C}}(\mathbf{1})$ and $\mathcal{K}_\mathbb{K}(\mathcal{C})$ are canonically isomorphic algebras. Indeed, $\text{End}_{\mathcal{T C}}(\mathbf{1}) = \bigoplus_S \text{End}(S) = \bigoplus_S \mathbb{K}$ is precisely the underlying vector space of $\mathcal{K}_\mathbb{K}(\mathcal{C})$. Furthermore, composition in $\text{End}_{\mathcal{T C}}(\mathbf{1})$ corresponds to the tensor product in $\mathcal{K}_\mathbb{K}(\mathcal{C})$ by Lemma 6.1.

For $X, Y, G$ in $\mathcal{C}$ and $\alpha \in \text{Hom}_\mathcal{C}(GX, Y G)$ we use

$$\alpha_G = \begin{array}{ccc}
\alpha & G \\
X & \downarrow & Y \\
G & \leftarrow & \\
\end{array}$$

as shorthand for

$$\bigoplus_s \sum_b \begin{array}{ccc}
S & \alpha & X \\
Y & \downarrow & S \\
S & \leftarrow & \\
\end{array} \in \bigoplus_S \text{Hom}_\mathcal{C}(SX, Y S) = \text{Hom}_{\mathcal{T C}}(X, Y).$$

**Remark 7.6.** This new notation may potentially create confusion with the pre-existing convention that, for $f \in \text{Hom}_{\mathcal{T C}}(X, Y)$ and $R \in \text{Irr}(\mathcal{C})$, $f_R$ denotes the $\text{Hom}_\mathcal{C}(RX, Y R)$ competent of $f$. To avoid such confusion we will restrict our new notation to the letters $G$ and $H$. 
We note that, using this new notation, we have

\[ \alpha X G_1 G_1 G_2 g = \alpha X G_2 G_2 G_1 g. \]  

Indeed the \( S \)-summand of the left hand side of (12) is

\[ \sum_b \alpha b g \alpha^* S X Y \]  

and similarly the right hand side of (12) is

\[ \sum_b \alpha \bar{b} g \alpha^* S X Y. \]

This serves as an effective reality-check that Definition 7.1 captures our original motivation of constructing an annular analogue of \( C \).

8. REPRESENTATIONS OF THE TUBE CATEGORY

Let \( C \) be a spherical fusion category let \( \mathcal{RT}C \) be the category of (contravariant) functors from \( \mathcal{T}C \) to \( \text{Vect} \). As \( C \) is a subcategory of \( \mathcal{T}C \) we have a canonical (covariant) functor

\[ \mathcal{RT}C \to \mathcal{RC} \]

\[ F \mapsto \bar{F} \]

that simply restricts \( F \) to morphisms in \( C \). A natural question now arises: for a given object \( \bar{F} \) in \( \mathcal{RC} \) what additional data could be provided to specify a unique extension to an object \( F \) in \( \mathcal{RT}C \)? To answer this question we consider the following morphisms in \( \mathcal{T}C \):

\[ c_{G,X} = \]

where \( G \) and \( X \) are in \( C \).
For $f$ and $g$ in $\text{Hom}_C(X, Y)$ and $\text{Hom}_C(G_1, G_2)$ respectively, we have

\begin{equation}
(g \otimes f) \circ c_{G_1,X} = \begin{array}{c}
\begin{array}{c}
G_1 \\
G_2
\end{array} \\
\begin{array}{c}
X \\
Y
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
G_2 \\
G_1
\end{array} \\
\begin{array}{c}
X \\
Y
\end{array}
\end{array} = c_{G_2,Y} \circ (f \otimes g).
\end{equation}

Furthermore, $c_{G,X}$ is an isomorphism and satisfies $c_{G,HX} \circ c_{H,XG} = c_{GH,X}$. The principal aim of this section is to prove that specifying how to evaluate $F$ on $c_{G,X}$ precisely captures how to extend $\bar{F}$ to $F$ uniquely and therefore provides an answer to the question mentioned above. Applying $F$ to $c_{G,X}$ gives a collection of maps

\[ \kappa_{G,X} : \bar{F}(GX) \to \bar{F}(XG). \]

By (13) $\kappa$ is natural in both $X$ and $G$. The additional properties of $c_{G,X}$ listed above then imply that $\kappa_{G,X}$ is an isomorphism and satisfies $\kappa_{H,XG} \circ \kappa_{G,HX} = \kappa_{GH,X}$. Suppose we start with an arbitrary object $\bar{F}$ in $\mathcal{R}C$ and isomorphisms $\kappa_{G,X}$ that satisfy naturality in $G$ and $X$ and $\kappa_{H,XG} \circ \kappa_{G,HX} = \kappa_{GH,X}$. We shall prove that there is a unique functor $F$ in $\mathcal{R}TC$ such that

(i) $F(X) = \bar{F}(X)$ for all $X$ in $C$
(ii) $F(\alpha) = \bar{F}(\alpha)$ for all $\alpha \in \text{Hom}_C(X, Y)$
(iii) $F(c_{G,X}) = \kappa_{G,X}$ for all $G, X$ in $C$.

If such a functor exists it is certainly unique as these conditions determine the functor on any morphism in $\mathcal{T}C$. Indeed for any $\alpha_G \in \text{Hom}_\mathcal{T}C(Y, X)$ we have

\begin{equation}
\alpha_G = \begin{array}{c}
\begin{array}{c}
G \\
Y
\end{array} \\
\begin{array}{c}
G^\vee \\
X
\end{array}
\end{array} \circ \begin{array}{c}
\begin{array}{c}
G \\
Y
\end{array} \\
\begin{array}{c}
G^\vee \\
X
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
Y \\
G^\vee
\end{array} \\
\begin{array}{c}
G
\end{array}
\end{array} \circ \begin{array}{c}
\begin{array}{c}
Y \\
G^\vee
\end{array} \\
\begin{array}{c}
G
\end{array}
\end{array}.
\end{equation}

The first and last terms are morphisms in $C$ and therefore determined by Condition (iii). The middle term is $c_{G,YG^\vee}$ and is therefore determined by Condition (iii). The following proposition establishes existence.

**Proposition 8.1.** There exists a unique object $F$ in $\mathcal{R}TC$ that satisfies Conditions (i), (ii) and (iii).

**Proof.** We first check that Conditions (ii) and (iii) don’t contradict one another. The only case when $c_{G,X}$ is a morphism in $C$ is when $G = 1$ and $c_{G,X} = \text{id}_X$. As we have

\[ \kappa_{1,X} \circ \kappa_{1,X} = \kappa_{1,X}, \]

and $\kappa_{1,X}$ is an isomorphism this implies $\kappa_{1,X} = \text{id}_X$. Therefore Conditions (ii) and (iii) are equivalent in this case.
To aid legibility when the domain of a map of the form $\kappa_{G,X}$ is clear from the context we will suppress the second argument and simply write $\kappa_G$. As discussed before the proof, any $F$ that satisfies Conditions (i), (ii) and (iii) also satisfies

$$F(\alpha_G) = \bar{F} \left( \begin{array}{c} Y \\ G \end{array} \right) \circ \kappa_G \circ \bar{F} \left( \begin{array}{c} G \\ Y \end{array} \right) \circ \bar{F} \left( \begin{array}{c} \alpha \\ G \end{array} \right) \circ \bar{F} \left( \begin{array}{c} X \end{array} \right)$$

for any $\alpha_G \in \text{Hom}_\mathcal{TC}(Y, X)$. We therefore only have to check that the equation does indeed define a functor. Let $\beta_H$ be in $\text{Hom}_\mathcal{TC}(Z, Y)$. We have

$$F(\beta_H) \circ F(\alpha_G) = \bar{F} \left( \begin{array}{c} H \\ Z \end{array} \right) \circ \kappa_H \circ \bar{F} \left( \begin{array}{c} \beta \\ Y \end{array} \right) \circ \kappa_G \circ \bar{F} \left( \begin{array}{c} G \\ Y \end{array} \right) \circ \bar{F} \left( \begin{array}{c} X \end{array} \right).$$

By the naturality of $\kappa_H$ and $\kappa_G$ this equation may be rearranged. The creation morphism of $G$ in the middle term may be moved to after $\kappa_H$, giving

$$\bar{F} \left( \begin{array}{c} H \\ Z \end{array} \right) \circ \kappa_H \circ \bar{F} \left( \begin{array}{c} \beta \\ Y \end{array} \right) \circ \bar{F} \left( \begin{array}{c} G \end{array} \right) \circ \kappa_G \circ \bar{F} \left( \begin{array}{c} X \end{array} \right).$$

Then $\beta$ together with the annihilation morphism of $H$ may be moved to before $\kappa_G$. This yields

$$\bar{F} \left( \begin{array}{c} H \\ Z \end{array} \right) \circ \kappa_H \circ \bar{F} \left( \begin{array}{c} \beta \\ Y \end{array} \right) \circ \bar{F} \left( \begin{array}{c} G \end{array} \right) \circ \kappa_G \circ \bar{F} \left( \begin{array}{c} X \end{array} \right).$$

Definition 8.2. We call the functor constructed in Proposition 8.1 the extension of $\bar{F}$ by $\kappa$ and denote it $(\bar{F}, \kappa)$.

This new description of objects in $\mathcal{RTC}$ then also yields a new description of morphisms as follows.

Proposition 8.3. We consider $F = (\bar{F}, \kappa)$ and $F' = (\bar{F}', \kappa')$ in $\mathcal{RTC}$ then we have

$$\text{Hom}_{\mathcal{RTC}}(F, F') = \{ \alpha \in \text{Hom}_{\mathcal{RC}}(\bar{F}, \bar{F}') \mid \alpha_{XG} \circ \kappa_{G,X} = \kappa'_{G,X} \circ \alpha_{GX} \}.$$
Proof. Let $\alpha \in \text{Hom}_{\mathcal{RC}}(\bar{F}, \bar{F}')$ be such that $\alpha_{XG} \circ \kappa_{G,X} = \kappa'_{G,X} \circ \alpha_{GX}$. As $\alpha$ is in $\text{Hom}_{\mathcal{RC}}(\bar{F}, \bar{F}')$, $\alpha$ is natural with respect to all morphism in $\mathcal{C}$. Furthermore, the additional condition on $\alpha$ implies that it is also natural with respect to all morphisms of the form $c_{G,X}$. From (14) we see that any morphism in $\mathcal{T} \mathcal{C}$ may be written as a composition of morphisms in $\mathcal{C}$ and morphisms of the form $c_{G,X}$. Therefore $\alpha$ is natural with respect to all morphisms in $\mathcal{T} \mathcal{C}$. \hfill \square

9. Equivalence with the Drinfeld Centre

Definition 9.1. Let $\mathcal{C}$ be a monoidal category and let $X$ be an object in $\mathcal{C}$. A half-braiding on $X$ is a collection of natural isomorphisms

$$\tau_G : G \otimes X \to X \otimes G$$

such that

$$\tau_{GH} = (\tau_G \otimes \text{id}_H) \circ (\text{id}_G \otimes \tau_H)$$

for all $G, H$ in $\mathcal{C}$. From a graphical perspective the condition that $\tau_G$ is natural allows us to ‘push’ morphisms through $\tau$:

$$\begin{array}{c}
 G & X \\
 \alpha \\
 X & H \\
 \tau_H & \\
 H & G \\
 G & X \rightleftharpoons \tau_G \\
 X & H \\
 \end{array}$$

this motivates the name ‘half-braiding’. Pushing the graphical perspective further, we may rewrite (16) as

$$\begin{array}{c}
 G & H & X \\
 \tau_{GH} \\
 X & G & H \\
 \alpha \\
 \end{array} = \begin{array}{c}
 G & H & X \\
 \tau_H \\
 X & H & G \\
 \tau_G \\
 \end{array}$$

The (Drinfeld) centre of $\mathcal{C}$, denoted $Z(\mathcal{C})$, is a category with objects of the form $(X, \tau)$ where $X$ is in $\mathcal{C}$ and $\tau$ is a half braiding on $X$. Hom$_{Z(\mathcal{C})}((X, \tau), (Y, \gamma))$ is then given by the subspace of Hom$_{\mathcal{C}}(X, Y)$ defined by the condition that $f \in \text{Hom}_\mathcal{C}(X, Y)$ satisfies

$$(f \otimes \text{id}_G) \circ \tau_G = \gamma_G \circ (\text{id}_G \otimes f)$$

for all $G$ in $\mathcal{C}$. This category is monoidal [EGNO15 Section 7.13] with tensor product $(X, \tau) \otimes (Y, \gamma) = (X \otimes Y, \iota)$ where $\iota_G = (\text{id}_X \otimes \gamma_G) \circ (\tau_G \otimes \text{id}_Y)$. In fact $Z(\mathcal{C})$ also admits a natural braiding [Kas98 Theorem XIII.4.2.] given by

$$\sigma_{(X, \tau), (Y, \gamma)} = \gamma_X \cdot$$

Now, for the remainder of this section, let $\mathcal{C}$ be a spherical fusion category. Then $\mathcal{C}$ satisfies the conditions of Proposition 5.7 and so the Yoneda embedding gives an equivalence between $\mathcal{C}$ and $\mathcal{RC}$. This induces an equivalence

$$Z(\mathcal{C}) \to Z(\mathcal{RC})$$

$$\begin{array}{c}
 (X, \tau) \\
 \mapsto \end{array} \begin{array}{c}
 (X^\sharp, \tau^\sharp) \\
 \end{array}.$$
Here $\tau^z$ is a natural isomorphism from $(- \otimes X)^z$ to $(X \otimes -)^z$ such that

$$\tau^z_{GH} = (\tau_G \otimes \text{id}_H)^z \circ (\text{id}_G \otimes \tau_H)^z.$$ 

By Lemma 4.10 this is equivalent to an isomorphism

$$\kappa_{G^\vee, Z} : X^z(G^\vee Z) \to X^z(ZG^\vee)$$

that is natural in both $Z$ and $G^\vee$ and satisfies

$$\kappa_{(HG)^\vee, Z} = \kappa_{H^\vee, ZG^\vee} \circ \kappa_{G^\vee, H^\vee Z}$$

or, more simply,

$$\kappa_{GH, Z} = \kappa_{H, ZG} \circ \kappa_{G, HZ}.$$ 

Therefore an object in $Z(R\mathcal{C})$ may simply be thought of as an object in $R\mathcal{C}$ together with isomorphisms $\kappa_{G, Z}$ as above, i.e. an object in $R\mathcal{T}\mathcal{C}$. Furthermore, we recall that a morphism in $Z(\mathcal{C})$ between $(X, \tau)$ and $(Y, \tau')$ is a map $f \in \text{Hom}_\mathcal{C}(X, Y)$ such that

$$(f \otimes \text{id}_G) \circ \tau_G = \tau'_G \circ (\text{id}_G \otimes f).$$

Applying the Yoneda embedding we get

$$\left((f \otimes \text{id}_G)^z \circ \tau^z_G\right)_Z = \left((\tau'_G)^z \circ (\text{id}_G \otimes f)^z\right)_Z$$

which is equivalent to

$$f^z_{ZG^\vee} \circ \kappa_{G^\vee, Z} = \kappa'_{G^\vee, Z} \circ f^z_{G^\vee Z}$$

which is precisely the condition that $f^z$ is a morphism from $(X^z, \kappa_{G^\vee, Z})$ to $(Y^z, \kappa'_{G^\vee, Z})$ in $R\mathcal{T}\mathcal{C}$. Therefore $R\mathcal{T}\mathcal{C}$ and $Z(\mathcal{C})$ are equivalent. To see this equivalence more explicitly let $(X, \tau)$ be in $Z(\mathcal{C})$. The corresponding object $F$ in $R\mathcal{T}\mathcal{C}$ is given on objects by

$$F(Y) = \text{Hom}_\mathcal{C}(Y, X)$$

and on morphisms by

$$F(\alpha_G) : \text{Hom}_\mathcal{C}(Y, X) \to \text{Hom}_\mathcal{C}(Z, X)$$

where $\alpha_G \in \text{Hom}_{\mathcal{T}\mathcal{C}}(Z, Y)$. This functor may also be understood in terms of an idempotent in $\mathcal{T}\mathcal{C}$. Indeed, for $(X, \tau)$ in $Z(\mathcal{C})$ we may consider the following endomorphism in $\mathcal{T}\mathcal{C}$,

$$\epsilon_\tau = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S) \tau_{TS}.$$
Proposition 9.2. For \( Y \) in \( \mathcal{C} \) and \( \alpha_G \in \text{Hom}_{\mathcal{T}\mathcal{C}}(Y, X) \) we have

\[
\epsilon_{\tau} \circ \alpha_G = \frac{1}{d(C)} \bigoplus_R d(R)
\]

and, for \( \beta_G \in \text{Hom}_{\mathcal{T}\mathcal{C}}(X, Y) \), we have

\[
\beta_G \circ \epsilon_{\tau} = \frac{1}{d(C)} \bigoplus_R d(R)
\]

Proof. By the definition of composition in \( \mathcal{T}\mathcal{C} \), we have

\[
\epsilon_{\tau} \circ \alpha_G = \frac{1}{d(C)} \bigoplus_R \sum_{s,b} d(S)
\]

\[
= \frac{1}{d(C)} \bigoplus_R \sum_{s,b} d(S)
\]

\[
= \frac{1}{d(C)} \bigoplus_R d(R)
\]
where the first equality is achieved by pushing $b$ though $\tau$ and the second by applying Proposition 6.2. This proves the first equality, the second is proved analogously. □

**Corollary 9.3.** Let $(X, \tau)$ be in $\mathcal{Z}(\mathcal{C})$. Then $\epsilon_\tau$ is an idempotent.

**Proof.** By Proposition 9.2 we have

\[
\epsilon_\tau \circ \epsilon_\tau = \frac{1}{d(C)^2} \bigoplus_R \sum_S d(R)d(S) = \epsilon_\tau.
\]

□

**Proposition 9.4.** Let $(X, \tau)$ be in $\mathcal{Z}(\mathcal{C})$, let $F$ in $\mathcal{R_TC}$ be given by (18) and let $(X, \epsilon_\tau)\sharp$ be as defined in Section 3. Then $F \cong (X, \epsilon_\tau)\sharp$.

**Proof.** We consider the following two linear maps,

\[
\Xi_Y: \text{Hom}_C(Y, X) \to \text{Hom}_{T\mathcal{C}}(Y, \epsilon_\tau) \quad \quad \alpha \mapsto \epsilon_\tau \circ \alpha
\]

and

\[
\Psi_Y: \text{Hom}_{T\mathcal{C}}(Y, \epsilon_\tau) \to \text{Hom}_C(Y, X) \quad \quad \beta \mapsto G^\beta \circ G^\tau_\alpha.
\]

For $\alpha \in \text{Hom}_C(Y, X)$, we have

\[
\Psi \circ \Xi(\alpha) = \frac{1}{d(C)} \sum_S d(S) \tau^S_\alpha = \frac{1}{d(C)} \sum_S d(S) \tau^S_\alpha = \alpha.
\]
and, for $\beta_G \in \text{Hom}_{\tau C}(Y, \epsilon_{\tau})$, we have

$$\beta_G = \epsilon_{\tau} \circ \beta_G = \frac{1}{d(C)} \bigoplus_s d(S) = \Xi \circ \Psi(\beta_G).$$

As $\Xi_Y$ and $\Psi_Y$ are inverse we only have to check naturality for one of them. For $\alpha \in \text{Hom}_C(Z, X)$ and $\beta_G \in \text{Hom}_{\tau C}(Y, Z)$, we have

$$\chi^t(\beta_G) \circ \Xi_Z(\alpha) = \frac{1}{d(C)} \bigoplus_s d(S) = \Xi_Y \circ F(\beta_G)(\alpha).$$

This proposition has an interesting consequence. As $Z(C)$ and $\mathcal{R}T C$ are equivalent it also proves that every functor $F$ in $\mathcal{R}T C$ is represented by an idempotent (namely $\epsilon_{\tau}$ where $(X, \tau)$ is the corresponding object in $Z(C)$). In summary, we have the following.

**Corollary 9.5.** *The Yoneda embedding $\xi: \tau C \rightarrow \mathcal{R}T C$ is an idempotent completion.*

Informally, we may interpret this result as allowing us to study $\mathcal{R}T C$ (and therefore also the centre of $C$) by simply working with idempotents in $\tau C$. This idea is precisely what is meant by the terms ‘graphical approach’ which appear in the title of this article. To illustrate this approach we shall now describe an alternative proof of the equivalence between $Z(C)$ and $C \boxtimes \overline{C}$ when $C$ is modular.

### 10. Equivalence with $C \boxtimes \overline{C}$

We start by equipping $C$ with a (balanced) braiding $\sigma$; $C$ is now a pre-modular tensor category. As we have now chosen a braiding we get a (covariant) braided monoidal functor

$$\Phi: C \boxtimes \overline{C} \rightarrow Z(C)$$

$$X \boxtimes Y \mapsto (XY, (\text{id}_X \otimes \overline{\sigma}_Y) \circ (\sigma_X \otimes \text{id}_Y))$$

where $\boxtimes$ denotes the Deligne tensor product and $\overline{C}$ is obtained by equipping $C$ with the opposite braiding. It is also known that this functor is an equivalence if and only if $C$ is modular (see [Müg03] or [EGNO15, Proposition 8.20.12]). Our aim in this section is to provide an alternative proof of this result by exploiting graphical calculus in the tube category. First we note that, by the results of the previous section, we have $\Phi(X \boxtimes Y) = \Xi_Y \circ F(\beta_G)(\alpha)$.
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$(XY, \epsilon^Y_X)$ where

$$\epsilon^Y_X = \frac{1}{d(C)} \bigoplus_s d(S).$$

When we suppose $C$ is modular the killing ring lemma (Lemma 6.4) allows us to compute the Hom-spaces between these idempotents.

**Proposition 10.1.** Let $C$ be a modular tensor category and let $X, Y, A, B$ be in $C$. We have

$$\text{Hom}_{TC}(\epsilon^Y_X, \epsilon^B_A) = \text{Hom}_C(X, A) \otimes \text{Hom}_C(Y, B).$$

**Proof.** We consider the maps

$$\phi: \text{Hom}_C(X, A) \otimes \text{Hom}_C(Y, B) \to \text{Hom}_{TC}(\epsilon^Y_X, \epsilon^B_A)$$

$$f \otimes g \mapsto \epsilon^B_A \circ (f \otimes g) \circ \epsilon^Y_X$$

and

$$\varphi: \text{Hom}_{TC}(\epsilon^Y_X, \epsilon^B_A) \to \text{Hom}_C(X, A) \otimes \text{Hom}_C(Y, B)$$

$$\alpha_G \mapsto \sum_{T,b,c} \frac{1}{d(T)} \sum_{R,T} d(R) d(T) T.$$  

We have

$$\varphi \circ \phi(f \otimes g) = \varphi \left( \frac{1}{d(C)^2} \sum_{R,T} d(R) d(T) \right)$$

$$= \varphi \left( \frac{1}{d(C)} \sum_{R,T} d(R) \right)$$

$$= \sum_{T,b,c} b^*(g \circ c) f \otimes (b \circ e^*) = f \otimes g$$
and, for $\alpha_G \in \text{Hom}_{TC}(\epsilon_Y^X, \epsilon_A^B)$,

$$\alpha_G = \epsilon_A^B \circ \alpha_G \circ \epsilon_X^Y = \frac{1}{d(C)^2} \bigoplus_{R,b} d(R)d(T) \alpha_G = \phi \circ \varphi(\alpha_G)$$

where the penultimate equality uses Proposition 9.2 and the final equality uses Proposition 6.6.

**Corollary 10.2.** The set $\{\epsilon_I^J\}_{I,J \in \text{Irr}(C)}$ is a set of orthogonal primitive idempotents (see Definition 3.8).

**Proof.** Let $I, J, I', J'$ be in $\text{Irr}(C)$. By Proposition 10.1 we have

$$\text{Hom}(\epsilon_I^J, \epsilon_I'^{J'}) = \begin{cases} \mathbb{K} & \text{if } I = I' \text{ and } J = J' \\ 0 & \text{else} \end{cases}$$

which proves the claim. □

As the map from $\text{Hom}_{C}(X \boxtimes Y, A \boxtimes B) = \text{Hom}_{C}(X, A) \otimes \text{Hom}_{C}(Y, B)$ to $\text{Hom}_{TC}(\epsilon_Y^X, \epsilon_A^B)$ induced by $\Phi$ is precisely the map denoted $\phi$ in the proof of Proposition 10.1, we have already shown that $\Phi$ is fully faithful. It remains to be shown that $\Phi$ is essentially surjective, i.e. that the set $\{\epsilon_I^J\}_{I,J \in \text{Irr}(C)}$ forms a complete set of orthogonal primitive idempotents in $TC$. A straightforward consideration of the dimension of $\text{Hom}_{TC}(X, Y)$ achieves this.

**Theorem 10.3.** Let $C$ be an modular tensor category. We have

$$\text{Hom}_{TC}(X, Y) = \bigoplus_{I,J} \text{Hom}_{TC}(X, \epsilon_I^J) \otimes \text{Hom}_{TC}(\epsilon_I^J, Y),$$

in other words, the set $\{\epsilon_I^J\}_{I,J \in \text{Irr}(C)}$ forms a complete set of orthogonal primitive idempotents in $TC$.

**Proof.** Our aim is to show that the map giving by composition

$$\bigoplus_{I,J} \text{Hom}_{TC}(X, \epsilon_I^J) \otimes \text{Hom}_{TC}(\epsilon_I^J, Y) \to \text{Hom}_{TC}(X, Y). \quad (19)$$

is an isomorphism. As before, let $\text{hom}_{C}(X, Y)$ denote the dimension of $\text{Hom}_{C}(X, Y)$. By Corollary 10.2, (19) is injective and therefore

$$\sum_{I,J} \text{hom}_{TC}(X, \epsilon_I^J) \text{hom}_{TC}(\epsilon_I^J, Y) \leq \text{hom}_{TC}(X, Y) \quad (20)$$

with equality if and only if (19) is an isomorphism. Furthermore, by Proposition 9.4 we have

$$\text{hom}_{C}(X, IJ) = \text{hom}_{TC}(X, \epsilon_I^J) \quad \text{and} \quad \text{hom}_{C}(IJ, Y) = \text{hom}_{TC}(\epsilon_I^J, Y)$$
which allows us to compute
\[
\text{hom}_{\mathcal{T}C}(X, Y) = \sum_{I^\vee} \text{hom}_C(I^\vee X, Y I^\vee)
\]
\[
= \sum_{I^\vee, J} \text{hom}_C(I^\vee X, J) \text{hom}_C(J, Y I^\vee)
\]
\[
= \sum_{I, J} \text{hom}_C(X, I J) \text{hom}_C(I J, Y)
\]
\[
= \sum_{I J} \text{hom}_{\mathcal{T}C}(X, \epsilon^I_J) \text{hom}_{\mathcal{T}C}(\epsilon^I_J, Y),
\]
 implying that (19) is an isomorphism.

\[\square\]

**Corollary 10.4.** Let $\mathcal{C}$ be a modular tensor category. Then $\Phi: \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathbb{Z}(\mathcal{C})$ is an equivalence.

**Remark 10.5.** As mentioned at the beginning of this section, the converse statement is also true i.e. if $\mathcal{C}$ fails to be modular then $\Phi$ will also fail to be an equivalence. To prove this we require a converse of the killing ring lemma (Lemma 6.4), which is provided by Theorem 8.20.7 in [EGNO15]. In particular, this theorem implies that if the S-matrix is degenerate, then there exists an object $I \in \text{Irr}(\mathcal{C})$ such that
\[
\begin{array}{c}
I \\
\circlearrowleft
\end{array}
\quad X
\quad \begin{array}{c}
I \\
\circlearrowleft
\end{array}
\]
for all $X$ in $\mathcal{C}$. From there one may may check that
\[
\frac{1}{d(\mathcal{C})} \bigoplus_{S, d(S)} S
\]
\[
\begin{array}{c}
S \\
\circlearrowleft
\end{array}
\quad I
\quad \begin{array}{c}
\circlearrowleft
\end{array}
\]
\[
\quad S
\]
is a non-zero morphism between $\epsilon^I_{J^\vee}$ and $\epsilon^I_1$. As $I \boxtimes I^\vee$ and $1 \boxtimes 1$ are distinct simple objects in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$, $\Phi$ is not an equivalence.

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