PULLBACKS OF METRIC BUNDLES AND CANNON-THURSTON MAPS

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Abstract. Metric (graph) bundles were defined by Mj and Sardar in [MS12]. In this paper we introduce the notion of morphisms and pullbacks of metric (graph) bundles. Given a metric (graph) bundle $X$ over $B$ where $X$ and all the fibers are uniformly (Gromov) hyperbolic and nonelementary, and a Lipschitz qi embedding $i : A \rightarrow B$ we show that the pullback $i^*X$ is hyperbolic and the map $i^* : i^*X \rightarrow X$ admits a continuous boundary extension, i.e. a Cannon-Thurston (CT) map $\partial i^* : \partial(i^*X) \rightarrow \partial X$. As an application of our theorem we show that given a short exact sequence of nonelementary hyperbolic groups $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ and a finitely generated qi embedded subgroup $Q_1 < Q$, $G_1 := \pi^{-1}(Q_1)$ is hyperbolic and the inclusion $G_1 \rightarrow G$ admits a CT map $\partial G_1 \rightarrow \partial G$. We then derive several interesting properties of the CT map.

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1. Introduction

Given a hyperbolic group $G$ and a hyperbolic subgroup $H$ a natural question to ask is if the inclusion $H \to G$ extends continuously to $\partial H \to \partial G$ (see [Bos01]). Such a map is popularly known as a Cannon-Thurston map in Geometric Group Theory. More generally, one may ask the same question for a pair of (Gromov) hyperbolic metric spaces $Y \subset X$. This question of Mahan Mitra (MJ) has motivated numerous work. The reader is referred to [MJ14] for a detailed history of the problem. Although the general question for groups has been answered in the negative recently by Baker and Riley ([BR13]) there are many interesting questions to be answered in this context. In this paper we pick up the following.

**Question.** Suppose $1 \to N \to G \xrightarrow{\pi} Q \to 1$ is a short exact sequence of hyperbolic groups. Suppose $Q_1 < Q$ is qi embedded and $G_1 = \pi^{-1}(Q_1)$. Then does the inclusion $G_1 < G$ admit Cannon-Thurston map?

It follows by the results of [MS12] that $G_1$ is hyperbolic (see Remark 4.4, [MS12]) so that the question makes sense. In this paper we answer the above question affirmatively. However, we reformulate this question in terms of metric (graph) bundles as defined in [MS12] (see section 3 of this paper) and obtain the following more general result.

**Theorem 5.2.** Suppose $X$ is a metric (graph) bundle over $B$ such that $X$ is hyperbolic and all the fibers are uniformly hyperbolic and nonelementary. Suppose $i: A \to B$ is a Lipschitz, qi embedding and $Y$ is the pullback of $X$ under $i$ (see Definition 5.18). Then $i^*: Y \to X$ admits the CT map.

As an immediate application of this theorem we have the following theorem for groups.

**Theorem 6.1.** Suppose $1 \to N \to G \xrightarrow{\pi} Q \to 1$ is a short exact sequence of hyperbolic groups. Suppose $Q_1 < Q$ is qi embedded and $G_1 = \pi^{-1}(Q_1)$. Then the inclusion $G_1 < G$ admits Cannon-Thurston map.

We note that special cases of Theorem 5.2 and Theorem 6.1, namely when $A$ is a point and $Q_1 = (1)$ respectively, were already known. See Theorem 5.3 in [MS12] and Theorem 4.3 in [Mit98a]. Next we explore properties of the Cannon-Thurston map $\partial Y \to \partial X$. Suppose $F$ is a fiber of the bundle $Y$ over $A$. Then there is a CT map for the inclusions $i_{F,X}: F \to X$ and $i_{F,Y}: F \to Y$, and the map $i^*: Y \to X$. Since $\partial i_{F,X} = \partial i \circ \partial i_{F,Y}$ if $\alpha, \beta \in \partial F$ are identified under $\partial i_{F,X}$ then under $\partial i^*$ the points $\partial i_{F,Y}(\alpha)$ and $\partial i_{F,Y}(\beta)$ are identified too. It turns out that a sort of ‘converse’ of this is also true.

**Theorem 6.25.** Suppose we have the hypotheses of Theorem 5.2. Suppose $\gamma$ is a (quasi)geodesic line in $Y$ such that $\gamma(\infty)$ and $\gamma(-\infty)$ are identified by the CT map $\partial i^*: \partial Y \to \partial X$. Then $\pi_Y(\gamma)$ is bounded. In particular given any fiber $F$ of the metric bundle, $\gamma$ is at a finite Hausdorff distance from a quasigeodesic line of $F$.

On the other hand as an immediate application of Theorem 6.25 (in fact, see Corollary 6.26 and Proposition 6.3) we get the following:

**Theorem.** Suppose we have the hypotheses of Theorem 5.2. Let $F$ be the fiber over a point $b \in A$. Then the CT map $\partial i_{b,X} : \partial F \to \partial X$ is surjective if and only if the CT maps $\partial i_{b,Y}: \partial F \to \partial Y$ is surjective for all $\xi \in \partial B$ where $Y_\xi$ is the pullback of a (quasi)geodesic ray in $B$ asymptotic to $\xi$.

In particular $\partial i_{F,Y} : \partial F \to \partial Y$ is surjective if $\partial i_{F,X} : \partial F \to \partial X$ is surjective.
Following Mitra ([Mitr97]) we define the Cannon-Thurston lamination \( \Lambda' \) to be \( \{(z_1, z_2) \in \partial F \times \partial F : z_1 \neq z_2, \partial i_{F,X}(z_1) = \partial i_{F,X}(z_2)\} \) and following [Bow02] we defined for any point \( \xi \in \partial B \) a subset of this lamination denoted by \( \Lambda_{\xi}' \) or simply \( \Lambda_\xi' \) when \( X \) is understood, where \( (z_1, z_2) \in \Lambda_{\xi}' \) if and only if \( \partial i_{F,X}(z_1) = \partial i_{F,X}(z_2) = \tilde{\gamma}(\infty) \) where \( \tilde{\gamma} \) is a qi lift of a (quasi)geodesic ray \( \gamma \) in \( B \) converging to \( \xi \). If \( (z_1, z_2) \in \Lambda_{\xi}' \) and \( \alpha \) is a (quasi)geodesic line in \( F \) connecting \( z_1, z_2 \) then \( \alpha \) is referred to be a leaf of the lamination \( \Lambda_{\xi}' \). We have the following:

**Theorem.** (See Lemma 6.16 through Lemma 6.24) (Properties of \( \Lambda' \))

1. \( \Lambda' = \bigcup_{\xi \in \partial B} \Lambda_{\xi}' \).
2. \( \Lambda' \) and \( \Lambda_{\xi}' \) are all closed subsets of \( \partial(2)F \) where \( \partial(2)F = \{(z_1, z_2) \in \partial F \times \partial F : z_1 \neq z_2\} \).
3. \( \Lambda_{\xi_1}' \cap \Lambda_{\xi_2}' = \emptyset \) for all \( \xi_1 \neq \xi_2 \in \partial B \). Moreover, leaves of \( \Lambda_{\xi_1}' \), \( \Lambda_{\xi_2}' \) are coarsely transverse to each other for all \( \xi_1 \neq \xi_2 \in \partial B \):
   - Given \( \xi_1 \neq \xi_2 \in \partial B \) and \( D > 0 \) there exists \( R > 0 \) such that if \( \gamma_i \) is leaf of \( \Lambda_{\xi_i}' \), \( i = 1, 2 \) then \( \gamma_1 \cap N_D(\gamma_2) \) has diameter less than \( R \).
4. If \( \xi_n \to \xi \) in \( \partial B \) and \( \alpha_n \) is a leaf of \( \Lambda_{\xi_n}' \) for all \( n \in \mathbb{N} \) which converge to a geodesic line \( \alpha \) then \( \alpha \) is a leaf of \( \Lambda_{\xi}' \).
5. \( \Lambda_{X,\xi}' = \Lambda_{Y,\xi}' \) for all \( \xi \in \partial A \) if we have the hypothesis of Theorem A.

Finally we also prove the following interesting property of the CT lamination in the case of metric bundles coming from complexes of hyperbolic groups. (See Example 3)

**Theorem.** (Corollary 6.30) Suppose \( G \) is the fundamental group of a finite developable complexes of groups with nonelementary hyperbolic face groups where images of the homomorphisms between respective face groups are of finite index in the target groups. Suppose \( B \) is the universal cover of the complexes of groups and \( X \) is the metric bundle over \( B \) obtained from this data. Finally, suppose \( G \) is hyperbolic.

Then for all \( \xi \in \partial B \) we have \( \Lambda_{\xi}' \neq \emptyset \).

**Outline of the paper:** In section 2 we recall basic hyperbolic geometry, Cannon-Thurston maps etc. In section 3 we recall basics of metric (graph) bundles and we introduce morphisms of bundles, pullbacks. Here we prove existence of pullbacks under suitable assumptions. In section 4 we mainly recall the machinery of [MSTR] and we prove a few elementary results. Section 5 is devoted to the proof of the main theorem. In section 6 we derive applications of the main result and we mention some related results.

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2. Hyperbolic metric spaces

In this section we remark on the notation and convention to be followed in the rest of the paper and we put together basic definitions and results about hyperbolic metric spaces. We begin with some basic notions from large scale geometry. Most
of these are quite standard, e.g. see [Gro87], [Gd90]. We have used [MS12] where all the basic notions can be quickly found in one place.

**Notation, convention and some metric space notions.** One is referred to [BH99] Chapter I.1, for the definitions and basic facts about geodesic metric spaces, metric graphs and length spaces.

1. For any set $A$, $\text{Id}_A$ will denote the identity map $A \to A$. If $A \subset B$ then we denote by $i_{A,B} : A \to B$ the inclusion map of $A$ into $B$.
2. If $x \in X$ and $A \subset X$ then $d(x,A)$ will denote $\inf \{ d(x,y) : y \in A \}$ and will be referred to as the distance of $x$ from $A$. For $D \geq 0$ and $A \subset X$, $N_D(A) := \{ x \in X : d(x,a) \leq D \text{ for some } a \in A \}$ will be called the $D$-neighborhood of $A$ in $X$. For $A,B \subset X$ we shall denote by $d(A,B)$ the quantity $\inf \{ d(x,B) : x \in A \}$ and by $HD(A,B)$ the quantity $\inf \{ D > 0 : A \subset N_D(B), B \subset N_D(A) \}$ and will refer to it as the Hausdorff distance of $A,B$.
3. If $X$ is a length space we consider only subspaces $Y \subset X$ such that the induced length metric on $Y$ takes values in $[0,\infty)$, or equivalently for any pair of points in $Y$ there is a rectifiable path in $X$ joining them which is contained in $Y$. If $\gamma$ is a rectifiable path in $X$ then $l(\gamma)$ will denote the length of $\gamma$.
4. All graphs are connected for us. If $X$ is a metric graph then $\mathcal{V}(X)$ will denote the set of vertices of $X$. Generally we shall write $x \in X$ to mean $x \in \mathcal{V}(X)$. In metric graphs (see [BH99] Chapter I.1]) all the edges are assumed to have length 1.

In a graph $X$ the paths are assumed to be a sequence of vertices. In other words, these are maps $I \cap \mathbb{Z} \to X$ where $I$ is a closed interval in $\mathbb{R}$ with end points in $\mathbb{Z} \cup \{ \pm \infty \}$. We shall informally write this as $\alpha : I \to X$ and sometimes refer to it as a dotted path for emphasis. Length of such a path $\alpha : I \to X$ is defined to be $l(\alpha) = \sum d(\alpha(i),\alpha(i+1))$ where the sum is taken over all $i \in \mathbb{Z}$ such that $i, i+1 \in I$.
5. If $X$ is a geodesic metric space and $x,y \in X$ then we shall use $[x,y]_X$ or simply $[x,y]$ to denote a geodesic segment joining $x$ to $y$. This applies in particular to metric graphs. For $x,y,z \in X$ we shall denote by $\Delta xyz$ some geodesic triangle with vertices $x,y,z$.

(5) If $X$ is any metric space and then for all $A \subset X$, $diam(A)$ will denote the diameter of $A$.

2.1. **Basic notions from large scale geometry.** Suppose $X, Y$ are any two metric spaces and let $k \geq 1$, $\epsilon \geq 0, \epsilon' \geq 0$ are real numbers.

**Definition 2.1.** ([MS12] Definition 1.1.1])

1. A map $\phi : X \to Y$ is said to be metrically proper if there is an increasing function $f : [0,\infty) \to [0,\infty)$ such that for any $x,y \in X$ and $R \in [0,\infty)$, $d_Y(\phi(x),\phi(y)) \leq R$ implies $d_X(x,y) \leq f(R)$. In this case we say that $\phi$ is proper as measured by $f$.
2. A subset $A$ of a metric space $X$ is said to be $r$-dense in $X$ for some $r \geq 0$ if $N_r(A) = X$.
3. Suppose $A$ is a set. A map $\phi : A \to Y$ is said to be $\epsilon$-coarsely surjective if $\phi(A)$ is $\epsilon$-dense in $Y$. We will say that it is coarsely surjective if it is $\epsilon$-coarsely surjective for some $\epsilon \geq 0$. 

(4) A map \( \phi : X \to Y \) is said to be coarsely \((\epsilon, \epsilon')\)-Lipschitz if for every \( x_1, x_2 \in X \), we have \( d(\phi(x_1), \phi(x_2)) \leq \epsilon d(x_1, x_2) + \epsilon' \). A coarsely \((\epsilon, \epsilon)\)-Lipschitz map will be simply called a coarsely \( \epsilon \)-Lipschitz map. A map \( \phi \) is coarsely Lipschitz if it is coarsely \( \epsilon \)-Lipschitz for some \( \epsilon \geq 0 \).

(5) (i) A map \( \phi : X \to Y \) is said to be a \((k, \epsilon)\)-quasi-isometric embedding if for every \( x_1, x_2 \in X \), one has

\[
d(x_1, x_2)/k - \epsilon \leq d(\phi(x_1), \phi(x_2)) \leq k d(x_1, x_2) + \epsilon.
\]

A map \( \phi : X \to Y \) will simply be referred to as a quasi-isometric embedding if it is a \((k, \epsilon)\)-quasi-isometric embedding for some \( k \geq 1 \) and \( \epsilon \geq 0 \). A \((k, k)\)-quasi-isometric embedding will be referred to as a \( k \)-quasi-isometric embedding.

(ii) A map \( \phi : X \to Y \) is said to be a \((k, \epsilon)\)-quasi-isometry (resp. \( k \)-quasi-isometry) if it is a \((k, \epsilon)\)-quasi-isometric embedding (resp. \( k \)-quasi-isometric embedding) and moreover, it is \( D \)-coarsely surjective for some \( D \geq 0 \).

(iii) A \((k, \epsilon)\)-quasi-geodesic (resp. \( k \)-quasi-geodesic) in a metric space \( X \) is a \((k, \epsilon)\)-quasi-isometric embedding (resp. \( k \)-quasi-isometric embedding) \( \gamma : I \to X \), where \( I \subseteq \mathbb{R} \) is an interval.

We recall that a \((1, 0)\)-quasigeodesic is called a geodesic.

If \( I = [0, \infty) \), then \( \gamma \) will be called a quasigeodesic ray. If \( I = \mathbb{R} \), then we call it a quasigeodesic line. One similarly defines a geodesic ray and a geodesic line.

Quasigeodesics in a metric graph \( X \) will be maps \( I \cap \mathbb{Z} \to X \), informally written as \( I \to X \) where \( I \) is a closed interval in \( \mathbb{R} \).

(6) Suppose \( \phi, \phi' : X \to Y \) are two and \( \epsilon \geq 0 \).

(i) We define \( d(\phi, \phi') \) to be the quantity \( \sup \{ d_Y(\phi(x), \phi'(x)) : x \in X \} \)

provided the supremum exists in \( \mathbb{R} \); otherwise we write \( d(\phi, \phi') = \infty \).

(ii) A map \( \psi : Y \to X \) is called an \( \epsilon \)-coarse left (right) inverse of \( \phi \) if \( d(\psi \circ \phi, \text{Id}_X) \leq \epsilon \) (resp. \( d(\phi \circ \psi, \text{Id}_Y) \leq \epsilon \)).

If \( \psi \) is both an \( \epsilon \)-coarse left and right inverse then it is simply called an \( \epsilon \)-coarse inverse of \( \phi \).

(7) Suppose \( A \subseteq X \). The nearest point projection of \( X \) on \( A \) is a map \( P_A : X \to A \) such that \( d(x, P_A(x)) = \inf \{ d(x, y) : y \in A \} \) for all \( x \in X \).

Moreover, given \( r \geq 0 \), an \( r \)-approximate nearest point projection of \( X \) on \( A \) is a map \( X \to A \), still denoted by \( P_A \), such that \( d(x, P_A(x)) \leq r + \inf \{ d(x, y) : y \in A \} \) for all \( x \in X \setminus A \) and \( P_A(x) = x \) for all \( x \in A \).

We note that the nearest point projection map \( P_A \) need not be defined on the whole of \( X \) even when \( A \) is a closed subset of \( X \). However, for any \( \epsilon > 0 \), an \( \epsilon \)-approximate nearest point projection map always exists by axiom of choice.

**Remark on terminology:** (1) All the above definitions are about certain properties of maps and in each case some parameters are involved.

(i) When the parameters are not important or they are clear from the context then we say that the map has the particular property without explicit mention of the parameters, e.g. \( \phi : X \to Y \) is metrically proper’ if \( \phi \) is metrically proper as measured by some function.

(ii) When we have a set of pairs of metric spaces and a map between each pair possessing the same property with the same parameters then we say that the set of maps ‘uniformly’ have the property, e.g. uniformly metrically proper,
uniformly coarsely Lipschitz, uniform qi embeddings, uniform approximate nearest point projection etc.

(2) We will refer to quasiisometric embeddings as ‘qi embedding’ and quasiisometry as ‘qi’.

The following gives a characterization of quasiisometry to be used in the discussion on metric bundles.

Lemma 2.2. ([MS12] Lemma 1.1)

(1) For every $K_1, K_2 \geq 1$ and $D \geq 0$ there are $K_{2,1} = K_{2,1}(K_1, K_2, D)$, such that the following hold.

A $K_1$-coarsely Lipschitz map with a $K_2$-coarsely Lipschitz, $D$-coarse inverse is a $K_{2,1}$-quasi-isometry.

(2) Given $K \geq 1$, $\epsilon > 0$ and $R \geq 0$ there are constants $C_{2,2} = C_{2,2}(K, \epsilon, R)$ and $D_{2,2} = D_{2,2}(K, \epsilon, R)$ such that the following holds:

Suppose $X, Y$ are any two metric spaces and $f : X \to Y$ is a $(K, \epsilon)$-quasiisometry which is $R$-coarsely surjective. Then there is a $(K_{2,2}, C_{2,2})$-coarsely surjective $D_{2,2}$-coarse inverse of $f$.

The following lemma follows from a simple calculation.

Lemma 2.3. (1) Suppose we have a sequence of maps $f : X \to Y \to Z$ where $f, g$ are coarsely $L_1$-Lipschitz and $L_2$-Lipschitz respectively. Then $g \circ f$ is coarsely $(L_1 L_2, L_1 L_2 + L_2)$-Lipschitz.

(2) Suppose $f : X \to Y$ is a $(K_1, \epsilon_1)$-qi embedding and $g : Y \to Z$ is a $(K_2, \epsilon_2)$-qi embedding. Then $g \circ f : X \to Z$ is a $(K_1 K_2, K_2 \epsilon_1 + \epsilon_2)$-qi embedding.

Moreover, if $f$ is coarsely $D_1$-surjective and $g$ is coarsely $D_2$-surjective then $g \circ f$ is coarsely $(K_1 D_1 + \epsilon_2 + D_2)$-surjective.

In particular, composition of finitely many quasiisometries is a quasiisometry.

The following lemma appears in [KS]. We include a proof for the sake of completeness.

Lemma 2.4. Suppose $X$ is any metric space, $x, y \in X$, $\gamma$ is a (dotted) $k$-quasigeodesic joining $x, y$ and $\alpha : I \to X$ is a (dotted) coarsely $L$-Lipschitz path joining $x, y$. Suppose moreover, $\alpha$ is a proper embedding as measured by a function $f : [0, \infty) \to [0, \infty)$ and that $Hd(\alpha, \gamma) \leq D$ for some $D \geq 0$. Then $\alpha$ is (dotted) $K_{2,4} = K_{2,4}(k, f, D, L)$-quasigeodesic in $X$.

Proof. Suppose $\gamma$ is defined on an interval $J$. Let $a, b \in I$. Then we have (1) $d(\alpha(a), \alpha(b)) \leq L|a - b| + L$ since $\alpha$ is coarsely $L$-Lipschitz. Now let $a', b' \in J$ be such that $d(\alpha(a), \gamma(a')) \leq D$ and $d(\alpha(b), \gamma(b')) \leq D$. Let $R = d(\alpha(a), \alpha(b))$. Then by triangle inequality $d(\gamma(a'), \gamma(b')) \leq 2D + R$. Since $\gamma$ is a $k$-quasigeodesic we have $-k + |a' - b'| \leq d(\gamma(a'), \gamma(b')) \leq 2D + R$. Hence, $|a' - b'| \leq k(2D + R) + k^2$. Without loss of generality suppose $a' \leq b'$. Consider the sequence of points $a_0' = a'$, $a_1', \ldots, a_n' = b' \in J$ such that $a_{i+1}' = 1 + a_i'$ for $0 \leq i \leq n - 2$ and $a_n' - a_{n-1}' \leq 1$. We note that $n \leq 1 + k(2D + R) + k^2$. Let $a_i \in I$ be such that $d(\gamma(a_i'), \alpha(a_i)) \leq D$, $0 \leq i \leq n$ where $a_0 = a, a_n = b$. Once again by triangle inequality we have

$$d(\alpha(a_i), \alpha(a_{i+1})) \leq 2D + d(\gamma(a_i'), \gamma(a_{i+1}')) \leq 2D + 2k$$
Thus we have
\[ d_1 + (\alpha + 1) R \]
Hence, by (1) and (2) we can take
\[ \text{Lemma 2.5.} \]
this lemma being immediate we skip it.

Remark 1. We spend quite some time to restate some results proved in [MST12] in the generality of length spaces since the main result in our paper is about length spaces. For instance (1) the existence of pullback of metric bundles to be defined below is unclear within the category of geodesic metric spaces; and (2) we observe that for the definition of Cannon-Thurston maps the assumption of (Gromov) hyperbolic geodesic metric spaces is rather restrictive and unnecessary.

Lemma 2.6. Suppose \( X \) is a length space. (1) Given any \( \epsilon > 0 \), any pair of points of \( X \) can be joined by a continuous, rectifiable, arc length parameterized path which is a \((1, \epsilon)\)-quasigeodesic.

(2) Any pair of points of \( X \) can be joined by a dotted \( 1 \)-quasigeodesic.

Proof. (1) Let \( x, y \in X \). Given \( \epsilon > 0 \) there is a rectifiable arc-length parameterized path \( \gamma : [0, l] \to X \) such that \( \gamma(0) = x, \gamma(l) = y \) and \( l(\gamma) = l \) where \( l - \epsilon \leq d(x, y) \leq l \). We claim that it is a \((1, \epsilon)\)-quasigeodesic connecting \( x, y \). Given \( s \leq t \in [0, l] \) we have \( d(\gamma(s), \gamma(t)) \leq l(\gamma|[s,t]) - \epsilon \). We need to show that \( d(\gamma(s), \gamma(t)) \geq l(\gamma|[s,t]) - \epsilon \). However, if \( d(\gamma(s), \gamma(t)) < l(\gamma|[s,t]) - \epsilon \) then we can replace the portion of \( \gamma \) from \( \gamma(s) \) to \( \gamma(t) \) by another path, say \( \alpha \), whose length will be smaller than \( l(\gamma|[s,t]) - \epsilon \). This will mean that the length of the concatenation \( \gamma|[a,a] \ast \alpha \ast \gamma|[b,b] \) will be smaller than \( l(\gamma) - \epsilon \). This is impossible since \( d(x, y) \geq l(\gamma) - \epsilon \).

(2) Given \( x, y \in X \) by (1) there is a continuous \((1, 1)\)-quasigeodesic \( \gamma : [0, l] \to X \) joining \( x \) to \( y \). If \( l \in \mathbb{N} \) we can restrict the \( \gamma \) on \([0, l] \cap \mathbb{Z}\) to get a \((1, 1)\)-quasigeodesic. Suppose \( l \) is not an integer. Let \( n \) be the greatest integer less than \( l \). We then define \( \alpha : [0, n + 1] \to X \) by setting \( \alpha(i) = \gamma(i) \) for \( 0 \leq i \leq n \) and \( \alpha(n + 1) = \gamma(l) \). We claim that it is a dotted \((1, 1)\)-quasigeodesic.

Given \( i, j \in [0, n] \) we of course have \(-1 + |i - j| \leq d(\alpha(i), \alpha(j)) = d(\gamma(i), \gamma(j)) \leq 1 + |i - j| \). Suppose \( i \in [0, n] \). Then \(-1 + (l - i) \leq d(\gamma(i), \gamma(l)) = d(\alpha(i), \alpha(n + 1)) \leq 1 + (l - i) \). Since \( n < l < n + 1 \) we have \(-2 + (n + 1 - i) < d(\gamma(i), \gamma(l)) = d(\alpha(i), \alpha(n + 1)) < 1 + (n + 1 - i) \). The lemma follows from this.

Metric graph approximation to a length space
Let $X$ be any length space. We define a metric graph $Y$ as follows. We take the vertex set $V(Y) = X$. We join $x, y \in X$ by an edge (of length 1) if and only if $d_X(x, y) \leq 1$. We let $\psi_X : X \to V(Y) \subset Y$ be the identity map and let $\phi_X : Y \to X$ be defined as the inverse of $\psi_X$ on $V(Y)$ and any point in the interior of an edge is sent to one of the end points of the edge under $\phi_X$. The following lemma is taken from [KS]. We include a proof for the sake of completeness.

**Lemma 2.7.** (1) $Y$ is a (connected) metric graph. (2) The maps $\psi_X$ and $\phi_X|_{V(Y)}$ are coarsely 1-surjective, $(1, 1)$-quasigeodesics. (3) The map $\phi_X$ is a $(1, 3)$-quasimetric and it is a 1-coarse inverse of $\psi_X$.

**Proof.** We claim that it is a $(1, 1)$-quasimetric. Given $x, y, z \in X$, as in the proof of Lemma 2.8(1), we join them by an arclength parametrized path $γ : [0, l] \to X$ such that $l \leq d(x, y) + ϵ$ where $ϵ > 0$ is chosen in such a way that $d(x, y) < m + 1$ where $m$ is the nonnegative integer determined by $m \leq d(x, y) < m + 1$. Since $l(γ) < m + 1$ it follows that $d_Y(x, y) \leq m + 1 \leq d_X(x, y) + 1$. Suppose $x, y \in X$ such that $d_Y(x, y) = n$. Let $x = x_0, x_1, \ldots, x_n = y$ the consecutive vertices on a geodesic in $Y$ joining $x, y$. Then we know that $d_X(x_i, x_{i+1}) \leq 1$. Thus $d_Y(x, y) = \sum_{i=1}^{n} d_X(x_{i-1}, x_i) \leq n$. Thus we get $d_X(x, y) \leq d_Y(x, y) \leq d_X(x, y) + 1$. This proves the first statement of the lemma.

Finally, it is clear that $N_1(\psi_X(X)) = Y$ and hence $ψ_X$ is coarse 1-surjective. The remaining parts of the proof follows from a simple calculation and so we omit the proof.

**Remark 2.** We shall refer to the space $Y$ constructed in the proof of the above lemma as the (canonical) metric graph approximation to $X$. We also preserve the notations $ψ_X$ and $φ_X$ to be used in this context only.

**Definition 2.8.** Gromov inner product: Let $X$ be any metric space and let $p, x, y \in X$. Then the Gromov inner product of $x, y$ with respect to $p$ is defined to be the number $\frac{1}{2}(d(p, x) + d(p, y) - d(x, y))$. It is denoted by $(x, y)_p$.

**Lemma 2.9.** Let $X$ be any metric space and suppose $x, y, p, x', y', p' \in X$. The following holds.

1. $|\langle x, y \rangle_p - \langle x, y' \rangle_{p'}| \leq d(y, y')$.
2. $|\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| \leq d(x, x') + d(y, y')$.
3. $|\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| \leq d(x, x') + d(y, y')$.
4. $|\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| \leq |\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| + |\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| \leq d(x, x') + d(y, y')$.
5. Suppose $p, x, y$ are points on a $(1, C)$-quasigeodesic appearing in that order then $(x, y)_p \geq d(p, x) - 5C/2$.

**Proof.**
1. $|\langle x, y \rangle_p - \langle x, y' \rangle_{p'}| = \frac{1}{2}(d(p, y) - d(p, y') + d(x, y) - d(x, y')) \leq \frac{1}{2}(|d(p, y) - d(p, y')| + |d(x, y) - d(x, y')|) \leq d(y, y')$, using (1).
2. $|\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| \leq |\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| + |\langle x', y \rangle_{p'} - \langle x', y \rangle_{p'}| \leq d(x, x') + d(y, y')$.
3. $|\langle x, y \rangle_p - \langle x', y \rangle_{p'}| \leq \frac{1}{2}(d(x, p) - d(x, p')) + (d(y, p) - d(y, p')) \leq d(p, p')$.
4. $|\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| \leq |\langle x, y \rangle_p - \langle x' \cdot y \rangle_{p'}| + |\langle x', y \rangle_{p'} - \langle x', y \rangle_{p'}| \leq d(x, x') + d(y, y') + d(p, p')$, using (2) and (3).
5. Suppose $α : [0, l] \to X$ is a $(1, C)$-quasigeodesic, and $s \leq t \in [0, C]$ such that $α(0) = p, α(s) = x, α(t) = y$. Then $2\langle x, y \rangle_p = d(x, p) + d(y, p) - d(x, y) \geq s - C + t - C - (t - s + C) = 2s - 3C \geq 2d(p, x) - 5C$. Hence, $(x, y)_p \geq d(p, x) - 5C/2$. □
Lemma 2.10. Suppose $X$ is a length space and $x, y, p \in X$. Suppose $\gamma$ is a (dotted) $(1, \epsilon)$-quasigeodesic in $X$ joining $x, y$. Then for any $z \in \gamma$ we have $(x, y)_p \leq d(p, z) + \frac{1}{2} \epsilon$.

Proof. We have $d(x, y) \geq l(\gamma) - \epsilon = l(\gamma|_{[x, z]}) + l(\gamma|_{[z, y]}) - \epsilon \geq d(x, z) + d(z, y) - \epsilon$. Thus $(x, y)_p = \frac{1}{2}(d(p, x) + d(p, y) - d(x, y)) \leq \frac{1}{2}(d(p, x) + d(p, y) - d(x, z) - d(z, y) + \epsilon)$. Now, $d(p, z) - d(z, x) \leq d(p, z)$ and $d(p, y) - d(z, y) \leq d(p, z)$. Using these three inequalities we get $(x, y)_p \leq d(p, z) + \frac{1}{2} \epsilon$. 

Lemma 2.11. Suppose $X$ is a length space and $x_1, x_2, x_3 \in X$. Let $[x_i, x_j], i < j, 1 \leq i, j \leq 3$ denote $(1,1)$-quasigeodesics joining the respective pairs of points. Suppose there are points $w_1 \in [x_2, x_3], w_2 \in [x_1, x_3]$ and $w_3 \in [x_1, x_2]$ such that $d(w_1, w_3) \leq R$ for some $R \geq 0$. Then $|d(x_2, x_3)_{x_1} - d(x_1, w_1)| \leq 3 + 2R$.

Proof. We have $|d(x_2, w_1) - d(x_2, w_2)| \leq R, |d(x_3, w_1) - d(x_3, w_2)| \leq R, |d(x_1, w_1) - d(x_1, w_2)| \leq R, i = 2, 3$. Since all the three sides of the triangle are formed by $(1,1)$-quasigeodesics it is easy to see that $d(x_1, w_3) + d(w_3, x_2) \leq d(x_1, x_2) + 3, d(x_1, w_2) + d(w_2, x_3) \leq d(x_1, x_3) + 3$ and $d(x_2, w_1) + d(w_1, x_3) \leq d(x_2, x_3) + 3$. It then follows by a simple calculation that $2d(x_1, w_1) - 6 - 4R \leq d(x_1, x_2) + d(x_1, x_3) - d(x_2, x_3) \leq 2d(x_1, w_1) + 3 + 4R$.

Hence, we have $|d(x_2, x_3)_{x_1} - d(x_1, w_1)| \leq 3 + 2R$. 

Definition 2.12. (1) Suppose $X$ is a length space and $Y_1, Y_2, Z$ are nonempty subsets of $X$. We say that $Z$ coarsely disconnects $Y_1, Y_2$ in $X$ if (i) $Y_i \setminus Z \neq \emptyset$, $i = 1, 2$ and (ii) for all $K \geq 1$ there is $R \geq 0$ such that for all $y_i \in Y_i, i = 1, 2$ and all $K$-quasigeodesics $\gamma$ in $X$ joining $y_1, y_2$ we have $\gamma \cap N_R(Z) \neq \emptyset$.

(2) Suppose $Y, Z \subset X, Y_1, Y_2 \subset Y$. We say that $Z$ coarsely bisects $Y$ into $Y_1, Y_2$ in $X$ if $Y = Y_1 \cup Y_2$ and $Z$ coarsely disconnects $Y_1, Y_2$ in $X$.

(3) Suppose $\{X_i\}$ is a collection of length spaces and there are nonempty sets $Y_i, Z_i \subset X_i, Y_i^+, Y_i^- \subset Y_i$ such that $Y_i = Y_i^+ \cup Y_i^-$, $Y_i^+ \setminus Z_i \neq \emptyset$, and $Y_i^- \setminus Z_i \neq \emptyset$ for all $i$. We say that $Z_i$’s uniformly coarsely bisect $Y_i^+$’s into $Y_i^+$’s, and $Y_i^-$’s if for all $K \geq 1$ there is $R = R(K) \geq 0$ with the following property: For any $i$, and for any $x_i^+ \in Y_i^+, x_i^- \in Y_i^-$ and any $K$-quasigeodesic $\gamma_i \subset X_i$ joining $x_i^+$ we have $N_R(Z_i) \cap \gamma_i \neq \emptyset$.

We note that the first part of the above definition implies $Y_1 \cap Y_2 \subset N_R(1)(Z)$.

Moreover one would like to impose the condition that $Y_i \setminus Z$ are of infinite diameter.

Keeping the application we have in mind we do not assume that. The following lemma is immediate.

Lemma 2.13. Suppose $X$ is a length space, $Z \subset Y \subset X$ and $Z$ coarsely bisects $Y$ into $Y_1, Y_2$ in $X$. If $A \subset Y$ with $Z \subset A \subset Y$, and $(A \cap Y_i) \setminus Z \neq \emptyset$, $i = 1, 2$ then $Y$ coarsely bisects $A$ into $A \cap Y_1$ and $A \cap Y_2$ in $X$.

Definition 2.14. (Approximate nearest point projection) (1) Suppose $X$ is any metric space, $A \subset X$, and $x \in X$. Given $\epsilon \geq 0$ and $y \in A$ we say that $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $A$ if for all $z \in A$ we have $d(x, y) \leq d(x, z) + \epsilon$.

(2) Suppose $X$ is any metric space, $A \subset X$ and $\epsilon \geq 0$. An $\epsilon$-approximate nearest point projection map $f : X \to A$ is a map such that $f(a) = a$ for all $a \in A$ and $f(x)$ is an $\epsilon$-approximate nearest point projection of $x$ on $A$ for all $x \in X \setminus A$. 

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For $\epsilon = 0$ an $\epsilon$-approximate nearest point projection is simply referred to as a nearest point projection. A nearest point projection map from $X$ onto a subset $A$ will be denoted by $P_{A,X} : X \to A$ or simply $P_A : X \to A$ when there is no possibility of confusion.

We note that given a metric space $X$ and $A \subset X$ a nearest point projection map $X \to A$ may not be defined in general but an $\epsilon$-approximate nearest point projection map $X \to A$ exists by axiom of choice for all $\epsilon > 0$.

**Lemma 2.15.** Suppose $X$ is a metric space and $A \subset X$. Suppose $y \in A$ is an $\epsilon$-approximate nearest point projection of $x \in X$. Suppose $\alpha : I \to X$ is a $(1,1)$-quasigeodesic joining $x,y$. Then $y$ is $(\epsilon + 3)$-approximate nearest point of $A$ to $x'$ for all $x' \in \alpha$.

**Proof.** Suppose $z \in A$ is any point. Then we know that $d(x,y) \leq d(x,z) + \epsilon$. Since $\alpha$ is a $(1,1)$-quasigeodesic it is easy to see that $d(x,x') + d(x',y) \leq d(x,y) + 3$. Hence, $d(x,x') + d(x',y) \leq d(x,y) + 3 + \epsilon$ which in turn implies that $d(x',y) \leq d(x,z) - d(x,x') + 3 + \epsilon \leq d(x',z) + \epsilon + 3$. Hence, $y$ is an $(\epsilon + 3)$-approximate nearest point projection of $x'$ on $A$. \qed

**Corollary 2.16.** Suppose $X$ is any metric space and $x,y,z \in X$. Suppose $\alpha$, $\beta$ are $(1,1)$-quasigeodesics joining $x,y$ and $y,z$ respectively. If $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $\beta$ then $\alpha * \beta$ is $(3,3+\epsilon)$-quasigeodesic.

**Proof.** Let $x' \in \alpha$ and $y' \in \beta$. Let $\beta'$ denote the segment of $\beta$ from $y$ to $y'$. Then $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $\beta'$ too. Hence, by the previous lemma $y$ is an $(\epsilon + 3)$-approximate nearest point projection of $x'$ on $\beta'$. Without loss of generality, suppose $\alpha(a) = x'$, $\alpha(a + m) = y$, $\beta(0) = y$, and $\beta(n) = y'$. Now, $d(x',y) \leq d(x',y') + \epsilon + 3$. Hence $d(y,y') \leq d(x',y') + d(x',y) \leq 2d(x',y') + \epsilon + 3$. Since $\alpha$, $\beta$ are both $(1,1)$-quasigeodesics it follows that $m - 1 \leq d(x',y) \leq d(x',y') + \epsilon + 3$ and $n - 1 \leq d(y,y') \leq 2d(x',y') + \epsilon + 3$. Adding these we get $m + n - 2 \leq 3d(x',y') + 2\epsilon + 6$. On the other hand, $d(x',y') \leq d(x',y) + d(y,y') \leq m + n + 2$. Putting everything together we get

$$
\frac{1}{3}(m+n) - \frac{2\epsilon + 8}{3} \leq d(x',y') \leq (m+n) + 2
$$

from which the lemma follows immediately. \qed

2.2. **Rips hyperbolicity vs Gromov hyperbolicity.** This subsection gives a quick introduction to some basic notions and results about hyperbolic metric spaces. One is referred to [Gro87], [Gd90], [ABC+91] for more details.

**Definition 2.17.** (1) Suppose $\Delta x_1x_2x_3$ is a geodesic triangle in a metric space $X$ and $\delta \geq 0$, $K \geq 0$. We say that the triangle $\Delta x_1x_2x_3$ is $\delta$-slim if any side of the triangle is contained in the $\delta$-neighborhood of the union of the remaining two sides.

(2) Let $\delta \geq 0$ and $X$ be a geodesic metric space. We say that $X$ is a $\delta$-hyperbolic metric space if all geodesic triangles in $X$ are $\delta$-slim.

A geodesic metric space is said to be hyperbolic if it is $\delta$-hyperbolic for some $\delta \geq 0$.

This definition of hyperbolic metric space is due to E. Rips. However, as mentioned in the remark above in this paper we need to deal with length spaces a lot which a priori need not be geodesic. The following definition comes to use in that case.
Definition 2.18. (Gromov hyperbolicity) Suppose $X$ is any metric space, not necessarily geodesic and $\delta \geq 0$.

1. Let $p \in X$. We say that the Gromov inner product on $X$ with respect to $p$, i.e. the map $X \times X \to \mathbb{R}$ defined by $(x, y) \mapsto (x,y)_p$, is $\delta$-hyperbolic if

$$
(x,y)_p \geq \min\{(x,z)_p, (y,z)_p\} - \delta
$$

for all $x, y, z \in X$.

2. The metric space $X$ is called $\delta$-hyperbolic in the sense of Gromov if the Gromov inner product on $X$ is $\delta$-hyperbolic with respect to any point of $X$.

A metric space is called (Gromov) hyperbolic if it is $\delta$-hyperbolic in the sense of Gromov for some $\delta \geq 0$.

However, for geodesic metric spaces the two concepts are equivalent.

Lemma 2.19. [Gro87 Section 6.3C], [BH99 Proposition 1.22, Chapter III.H]) Suppose $X$ is a geodesic metric space. If it is $\delta$-hyperbolic in the sense of Rips then it is $\delta$-hyperbolic in the sense of Gromov.

Conversely, if $X$ is $\delta$-hyperbolic in the sense of Gromov then it is $\delta$-hyperbolic in the sense of Rips.

The above lemma is not true for general metric spaces (see [BH99 Exercise 1.23, Chapter III.H] although it is easy to see that it is true for length spaces. Since we could not find a proof of this fact in a reference we are providing a proof below (see Corollary 2.21 for the sake of completeness).

Lemma 2.20. Suppose $X$ is a metric space which is $\delta$-hyperbolic in the sense of Gromov. If $f : X \to Y$ is a coarsely $R$-surjective, $(1,C)$-quasiisometry then $Y$ is $D = D(\delta,R,C)$-hyperbolic in the sense of Gromov.

Proof. Fix an arbitrary point $p \in X$. Suppose $x, y \in X$. Then it follows from an easy calculation that $|(f(x),f(y))_p - (x,y)_p| \leq 3C/2$. Hence for all $x, y, z, p \in X$ we get

$$
(f(x),f(y))_p \geq (x,y)_p - 3C/2 \geq \min\{(x,z)_p, (y,z)_p\} - \delta - 3C/2
$$

$$
\geq \min\{(f(x),f(z))_p, (f(y),f(z))_p\} - \delta - 3C/2.
$$

Let $y_1, y_2, y_3 \in Y$. Then there are $x_1, x_2, x_3 \in X$ such that $d(y_i, f(x_i)) \leq R$. By Lemma 2.9(2) we have $|(y_i, y_j)_p - (f(x_i), f(x_j))_p| \leq 2R$. Thus it follows that

$$
(y_1, y_2)_p \geq (f(x_1), f(x_2))_p - 2R
$$

$$
\geq \min\{(f(x_1), f(x_3))_p, (f(x_2), f(x_3))_p\} - \delta - 3C - 2R
$$

$$
\geq \min\{(y_1, y_3)_p, (y_2, y_3)_p\} - \delta - 3C - 3R.
$$

Now let $y_1, y_2, y_3, y \in Y$ be arbitrary points. Let $x \in X$ be such that $d_Y(f(x), y) \leq R$. By Lemma 2.9(3) $|(y_1, y_j)_y - (y_i, y_j)_f(x)| \leq R$. It follows that

$$
(y_1, y_2)_y \geq \min\{(y_1, y_3)_y, (y_2, y_3)_y\} - \delta - 3C - 4R.
$$

□

Corollary 2.21. Gromov hyperbolicity is a $\text{qi}$ invariant among length spaces.

Proof. Suppose $X$ is a length space. Let $Y$ be the canonical metric graph approximation to $X$ as given by Lemma 2.7 and let $\psi_X : X \to Y$ and $d_X : Y \to X$ be respectively $(1,1)$-qi and $(1,3)$-qi which are 1-coarse inverse to each other. Hence, by Lemma 2.20 $X$ is Gromov hyperbolic if and only if $Y$ is Gromov hyperbolic.
Since $Y$ is a geodesic metric space it is Gromov hyperbolic if and only if it is Rips hyperbolic by Lemma 2.19. Hence, $X$ is Gromov hyperbolic if and only if $Y$ is hyperbolic in any of the two senses.

Now, suppose $X'$ is a length space and $f : X \to X'$ is a $\text{qi}$. Let $Y'$ be the canonical metric graph approximation to $X'$. Then $\psi_{X'} \circ f \circ \phi_X : Y \to Y'$ is a $\text{qi}$ by Lemma 2.3. We know that Rips hyperbolicity is a $\text{qi}$ invariant. This is a standard consequence of the stability of quasigeodesics which is the next lemma. Thus $Y'$ is hyperbolic in any sense if and only if so is $Y'$. Using this with the first part of the proof the lemma follows.

\begin{lemma}
(\text{Ga90}, \text{Stability of quasigeodesics in a Rips hyperbolic space}) For all $\delta \geq 0$ and $k \geq 1$, $\epsilon \geq 0$ there is a constant $D_{2.22} = D_{2.22}(\delta, k, \epsilon)$ such that the following holds:

Suppose $Y$ is a geodesic metric space $\delta$-hyperbolic in the sense of Rips. Then the Hausdorff distance between a geodesic and a $(k, \epsilon)$-quasi-geodesic joining the same pair of end points is less than or equal to $D_{2.22}$.

Using the metric graph approximation to a length space and Corollary 2.21 one easily obtains the following.

\begin{corollary}
\text{Stability of quasigeodesics in a Gromov hyperbolic space}:

Given $\delta \geq 0$, $k \geq 1$, $\epsilon \geq 0$ there is $D = D_{2.22}(\delta, k, \epsilon)$ such that the following holds.

Suppose $X$ is metric space which is $\delta$-hyperbolic in the sense of Gromov. Then given any $(k, \epsilon)$-quasigeodesics $\gamma_i$, $i = 1, 2$ with the same end points we have $Hd(\gamma_1, \gamma_2) \leq D$.

\begin{lemma}
Suppose $X$ is a length space. If $X$ is $\delta$-hyperbolic in the sense of Gromov then for all $K \geq 1$, $\epsilon \geq 0$ all $(K, \epsilon)$-quasigeodesic triangles in $X$ are $D_{2.24} = D_{2.24}(\delta, K, \epsilon)$-slim.

Conversely suppose all $(K, \epsilon)$-quasigeodesic triangles in $X$ are $R$-slim for some $R \geq 0$ and for some sufficiently large $K, \epsilon$ then $X$ is $D_{2.24} = D_{2.24}(\delta, K, \epsilon)$-hyperbolic in the sense of Gromov.

\end{lemma}

\begin{proof}
We briefly indicate a proof without explicit calculation of constants. Suppose $Y$ is the canonical metric graph approximation to $X$ and $\psi_X : X \to Y$ is the coarsely 1-surjective, $(1, 1)$-quasiisometry as constructed in Lemma 2.7. Note that there is a $(1, 3)$-qi 1-coarse inverse $\phi_X : Y \to X$ to $\psi_X$ too.

Suppose $X$ is $\delta$-hyperbolic in the sense of gromov. Then $X$ is $D_{2.24}(\delta, 1, 1)$-hyperbolic in the sense of Gromov and hence $\delta' = \delta'_{2.24}(D_{2.24}(\delta, 1, 1))$-hyperbolic in the sense of Rips by Lemma 2.19. By Lemma 2.3 the image under $\psi_X$ of any $(K, \epsilon)$-quasigeodesic in $X$ is a $(K, \epsilon + 1)$-quasigeodesic in $Y$. Then it follows from Lemma 2.24 that the image of any $(K, \epsilon)$-quasigeodesic triangle $\Delta$ in $X$ under $\psi_X$ is $(\delta' + D_{2.24}(\delta', K, \epsilon + 1))$-slim in $Y$. It follows that $\Delta$ is $D_{2.24} = (1 + \delta' + D_{2.24}(\delta', K, \epsilon + 1))$-slim.

Conversely suppose $(K, \epsilon)$-quasigeodesic triangles in $X$ are $R$-slim for some $R \geq 0$ and for sufficiently large $K \geq 1$, $\epsilon \geq 0$. Given a geodesic triangle $\Delta'$ in $Y$, $\phi_X(\Delta')$ is a $(1, 3)$-quasigeodesic triangle in $X$. Hence, if $\epsilon \geq 3$ then $\phi_X(\Delta')$ is $R$-slim. It follows that $\Delta'$ is $(R + 3)$-slim in $Y$. Thus $Y$ is $(R + 3)$-hyperbolic in the sense of Rips. Hence it is $\phi_X(\Delta') = (R + 3)$)-hyperbolic in the sense of Gromov. By Lemma 2.24 using $\phi_X$, $X$ is $D_{2.24}(\phi_X(\Delta'), R + 3), 1, 3)$-hyperbolic in the sense of Gromov.
\end{proof}
Corollary 2.25. Suppose that $X$ is a length space. If $X$ is $\delta$-hyperbolic in the sense of Gromov then for all $K \geq 1$, $\epsilon \geq 0$ all $(K, \epsilon)$-quasigeodesics $n$-gons in $X$ are $(n-2)D_{\delta,K,\epsilon} = (n-2)D_{\delta,K,\epsilon}$-slim.

Convention 2.26. For the rest of the paper when we refer to a space to be $\delta$-hyperbolic (or simply hyperbolic) we shall mean (1) $\delta$-hyperbolic (resp. hyperbolic) in the sense of Rips if $X$ is a geodesic metric space and (2) $\delta$-hyperbolic (resp. hyperbolic) in the sense of Gromov if $X$ is not a geodesic metric space.

2.3. Quasiconvex subspaces of hyperbolic spaces.

Definition 2.27. Let $X$ be a hyperbolic geodesic metric space and let $A \subseteq X$. For $K \geq 0$, we say that $A$ is $K$-quasiconvex in $X$ if any geodesic with end points in $A$ is contained in $N_K(A)$.

If $X$ is a Gromov hyperbolic length space and $A \subset X$ then we will say that $A$ is $K$-quasiconvex if any $(1,1)$-quasigeodesic joining a pair of points of $A$ is contained in $N_K(A)$.

A subset $A \subset X$ is said to be quasiconvex if it is $K$-quasiconvex for some $K \geq 0$.

Lemma 2.28. (Projection on quasiconvex set) Let $X$ be a $\delta$-hyperbolic metric space, $U \subset X$ be a $K$-quasiconvex set and $\epsilon \geq 0$. Suppose $y \in U$ is an $\epsilon$-approximate nearest point projection of a point $x \in X$ on $U$. Let $z \in U$. Suppose $\alpha$ is a (dotted) $k$-quasigeodesic joining $x$ to $y$ and $\beta$ is a (dotted) $k$-quasigeodesic joining $y$ to $z$. Then $\alpha \ast \beta$ is a (dotted) $K_{\delta,K,\epsilon}$-quasigeodesic joining $x$ to $z$.

In particular, if $y$ is $K$-quasigeodesic joining $x,z$ then $y$ is contained in the $D_{\delta,K,\epsilon}$-$\delta$-neighborhood of $\gamma$.

Proof. Without loss of generality we shall assume that $X$ is a $\delta$-hyperbolic length space. Suppose $\beta_1$ is a $(1,1)$-quasigeodesic in $X$ joining $y,z$. Since $U$ is $K$-quasiconvex it is clear that $y$ is an $(\epsilon+K)$-approximate nearest point projection of $x$ on $\beta_1$. Hence, if $\alpha_1$ is a $(1,1)$-quasigeodesic joining $x,y$ then $\alpha_1 \ast \beta_1$ is a $(3,3+\epsilon+K)$-quasigeodesic in $X$ by Corollary 2.14. By stability of quasigeodesics $Hd(\alpha_1,\alpha_1) \leq D_{\delta,K,\epsilon}(\delta,\epsilon)$, and $Hd(\beta_1,\beta_1) \leq D_{\delta,K,\epsilon}(\delta,\epsilon)$. Hence, $Hd(\alpha_1 \ast \beta_1,\alpha_1 \ast \beta_1) \leq D_{\delta,K,\epsilon}(\delta,\epsilon)$. By Lemma 2.4 it is enough to show now that $\gamma = \alpha \ast \beta$ is uniformly properly embedded. Let $\gamma_1 = \alpha_1 \ast \beta_1$ and $R = D_{\delta,K,\epsilon}$. Suppose $\alpha : [0,l] \to X$ with $\alpha(0) = x, \alpha(l) = y$ and $\beta : [0,m] \to X$ with $\beta(0) = y, \beta(m) = z$. Let $s \leq t \in [0,l+m]$ and $d(\gamma(s),\gamma(t)) \leq D$ for some $D \geq 0$. We need find a constant $D_1$ such that $t-s \leq D_1$ where $D_1$ depends on $\delta, k, K$ and $D$ only. However, if $s,t \in [0,l]$ or $s,t \in [l,l+m]$ then we have $-k + (t-s)/k \leq D$ since both $\alpha, \beta$ are $K$-quasigeodesics. Hence, in that case $t-s \leq k^2 + kD$. Suppose $s \in [0,l]$ and $t \in (l,m]$. In this case $\gamma(s) = \alpha(s),\gamma(t) = \beta(t-l)$. Let $x' \in \alpha_1, y' \in \beta_1$ be such that $d(x',\gamma(s)) \leq R$ and $d(y',\gamma(t)) \leq R$. Then $d(x',y') \leq 2R + D$. Suppose $\gamma_1(s') = x',\gamma_1(t') = y',\gamma_1(u) = y$ where $s' \leq u \leq t'$. Since $\gamma_1$ is a $(3,3+\epsilon+K)$-quasigeodesic we have $|s'-t'| \leq 3(3+\epsilon+K) + 3d(x',y') \leq 3(3+\epsilon+K) + 3(2R+D)$. It follows that $|s'-u|$ and $|u-t'|$ are both at most $3(3+\epsilon+K) + 3(2R+D) = 9+3\epsilon+3K+6R+3D$. Hence, $d(z,y),d(x,y)$ are both at most $3(9+3\epsilon+3K+6R+3D)+3+\epsilon+K = 30+10\epsilon+9K+18R+9D = D'$, say. Hence, $d(\gamma(s),y),d(y,\gamma(t))$ are both at most $R + D'$. Since $\alpha, \beta$ are $K$-quasigeodesics it follows that $l-s$ and $t-l$ are both at most $k^2 + k(R + D')$. Hence, $t-s \leq 2(k^2 + k(R + D'))$. Hence, we can take $D_1 = 2k^2 + 2kR + 2kD'$. This completes the proof.
Clearly one can take $D_{2.29}(\delta, K, k) = D_{2.23}(\delta, K, 2.29(\delta, K, k))$.

**Corollary 2.29.** Suppose $X$ is a $\delta$-hyperbolic metric space and $\alpha$ is a $k$-quasigeodesic in $X$ with an end point $y$. Suppose $x \in X$ and $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $\alpha$. Suppose $\beta$ is a $k$-quasigeodesic joining $x$ to $y$. Then $\beta \ast \alpha$ is a $K_{2.29}(\delta, k, \epsilon)$-quasigeodesic.

**Proof.** We briefly indicate the proof. One first notes by stability of quasigeodesics that images of uniform quasigeodesics are uniformly quasiconvex. Then one applies the preceding lemma.

The following corollary easily follows from Lemma 2.28 and Lemma 2.15. For instance the proof is similar to that of [MS12] Lemma 1.32.

**Corollary 2.30.** (Projection on nested quasiconvex sets) Suppose $X$ is a $\delta$-hyperbolic metric space and $V \subset U$ are two $K$-quasiconvex subsets of $X$. Suppose $x \in X$ and $x_1 \in U$, $x_2 \in V$ are $\epsilon$-approximate nearest point projection of $x$ on $U$ and $V$ respectively. Suppose $x_3$ is an $\epsilon$-approximate nearest point projection of $x_1$ on $V$. Then $d(x_2, x_3) \leq D_{2.30}(\delta, K, \epsilon)$.

In particular, for any two $\epsilon$-approximate nearest point projections $x_1, x_2$ of $x$ on $U$ we have $d(x_1, x_2) \leq D_{2.30}(\delta, K, \epsilon)$.

**Corollary 2.31.** Given $\delta \geq 0, K \geq 0, \epsilon \geq 0$ there are constants $L = 2.31(\delta, K, \epsilon), D = D_{2.31}(\delta, K, \epsilon)$ and $R = R_{2.31}(\delta, K, \epsilon)$ such that the following hold:

1. Suppose $X$ is a $\delta$-hyperbolic metric space and $U$ is a $K$-quasiconvex subset of $X$. Then for all $\epsilon \geq 0$ any $\epsilon$-approximate nearest point projection map $P : X \to U$ is coarsely $L$-Lipschitz.

2. Suppose $V$ is another $K$-quasiconvex subset of $X$ and $v_1, v_2 \in V$ and $u_i = P(v_i), i = 1, 2$. If $d(v_1, v_2) \geq D$ then $u_1, u_2 \in N_R(V)$.

In particular, if the diameter of $P(V)$ is at least $D$ then $d(U, V) \leq R$.

**Proof.** (1) Suppose $x, y \in X$ with $d(x, y) \leq 1$. Then $P(x)$ is an $(\epsilon + 1)$-approximate nearest point projection of $y$ on $U$. Hence, by Corollary 2.30 we have $d(P(x), P(y)) \leq D_{2.30}(\delta, K, \epsilon + 1)$. Hence, we may take $L_{2.31}(\delta, K, \epsilon) = D_{2.30}(\delta, K, \epsilon + 1)$ by Lemma 2.15.

(2) Consider the quadrilateral formed by $(1, 1)$-quasigeodesics joining the pairs $(u_1, u_2), (v_2, v_1)$ and $(v_1, u_1)$. This is $2D_{2.31}(\delta, 1, 1)$-slim by Corollary 2.28. Let $\delta' = 2D_{2.31}(\delta, 1, 1)$. Suppose no point of the side $v_1v_2$ is contained in a $\delta'$-neighborhood of the side $u_1u_2$. Then there are two points say $x_1, x_2 \in v_1v_2$ such that $x_i \in N_{\delta'}(u_1u_2), i = 1, 2$ and $d(x_1, x_2) \leq 2$. Hence there are points $y_i \in u_iu_1, i = 1, 2$ such that $d(y_1, y_2) \leq 2 + 2\delta'$. However, $u_i$ is an $(\epsilon + 3)$-approximate nearest point projection of $y_i$ on $U$ by Lemma 2.13. Hence, by the first of the Corollary 2.31 we have $d(u_1, u_2) \leq D_{2.31}(\delta, K, \epsilon + 3) + (2 + 2\delta')L_{2.31}(\delta, K, \epsilon + 3)$. Hence, if the diameter of $P(V)$ is bigger than $D = L_{2.31}(\delta, K, \epsilon + 3) + (2 + 2\delta')L_{2.31}(\delta, K, \epsilon + 3)$ then there is a point $x \in v_1v_2$ and $y \in u_1u_2$ such that $d(x, y) \leq \delta'$. Since $U$ is $K$-quasiconvex we have thus $x \in N_{K + \delta'}(U)$. Thus we may choose $R = K + \delta'$. □

The second part of the above corollary is implied in Lemma 1.35 of [MS12] too. This is also proved in [KS].

**Lemma 2.32.** Suppose $X$ is a $\delta$-hyperbolic metric graph and $Y \subset X$ is a connected subgraph such that the inclusion $(Y, d_Y) \to (X, d_X)$ is a $k$-qi embedding. Suppose $A \subset Y$ is $K$-quasiconvex in $Y$. Then the following holds.
(1) \(A\) is \(K, \beta, \Delta, \kappa\)-quasiconvex in \(X\).

(2) For any \(x \in Y\) if \(x_1, x_2 \in A\) are the nearest point projections of \(x\) on \(A\) in \(Y\) and \(X\) respectively then \(d_Y(x_1, x_2) \leq D, \beta, \Delta, \kappa\).

Proof. (1) Suppose \(x, y \in A\) and let \(\alpha, \beta\) be \(1\)-quasigeodesics joining \(x, y\) in \(Y\) and \(X\) respectively. Since, \(Y\) is \(k\)-qi embedded \(\alpha\) is a \((k, 2k)\)-quasigeodesic in \(X\) by Lemma 2.36. Hence, by stability of quasigeodesics projection of \(Y\) on \(X\) is \(\beta\)-quasigeodesic in \(X\).

However, \(A\) being \(K\)-quasiconvex in \(Y\), \(\alpha \subseteq N_K(A)\) in \(Y\) and hence in \(X\) as well. Thus \(\beta \subseteq N_K(D, \beta, \Delta, \kappa)(A)\) in \(X\). Hence, we can take \(K = K, \beta, \Delta, \kappa\).

(2) Suppose \(K_1 = K, \beta, \Delta, \kappa\). Then \(x_2 \in N_D([x, x_1]_X)\) in \(X\) where \(D = D, \beta, \Delta, \kappa\). We have \(Hd([x, x_1]_Y, [x, x_1]_X) \leq D, \beta, \Delta, \kappa\) by stability of quasigeodesics. Thus there is a point \(x'_2 \in [x, x_1]_Y\) such that \(d_X(x_2, x'_2) \leq D + D, \beta, \Delta, \kappa\). Hence, \(d_Y(x_1, x'_2) \leq k(D_1 + k)\) by triangle inequality. Thus we can take \(D, \beta, \Delta, \kappa\).

**Definition 2.33.** Suppose \(X\) is a \(\delta\)-hyperbolic metric space and \(A, B\) are two quasi-convex subsets. Let \(R > 0\). We say that \(A, B\) are mutually \(R\)-cobounded, or simply \(R\)-cobounded, if any \(1\)-approximate nearest point projection of \(A\) to \(B\) has diameter at most \(R\) and vice versa.

When the constant \(R\) is understood or is not important we just say that \(A, B\) are cobounded.

The following proposition is motivated by an analogous result due to Hamenstadt ([Ham05, Lemma 3.5]). See also [MS12, Corollary 1.52]. However, we have a weaker result here with stronger hypothesis and the proof is also different.

Suppose \(X\) is a \(\delta\)-hyperbolic metric graph and \(I\) is an interval in \(R\) whose both end points are in the set \(\{\infty, -\infty\}\). Suppose \(Y \subseteq X\) is a \(K\)-quasiconvex subset which admits a surjective map \(\Pi : Y \rightarrow I\). Let \(Y_i := \Pi^{-1}(i)\) for all \(i \in I \cap Z\) and \(Y_{ij} = \Pi^{-1}([i, j] \cap Z)\) for all \(i, j \in I \cap Z\) with \(i < j\). Suppose moreover that we have the following.

(1) All the sets \(Y_i, Y_{ij}\), \(i, j \in I\), \(i < j\) are \(K\)-quasiconvex in \(X\).

(2) \(Y_i\) uniformly coarsely bisects \(Y\) into \(Y_i^+ := (\infty, i] \cap I\) and \(Y_i^- := (\Pi^{-1}([i, \infty))\) for all \(i \in Z\) in the interior of \(I\).

(3) \(d(Y_{ii+1}, Y_{jj+1}) > 2K + 1\) for all \(i, j \in I\) if \(j + 1 \in I\) and \(i + 1 < j\).

**Proposition 2.34.** Given \(D \geq 0\), \(\lambda \geq 1\) and \(\epsilon \geq 1\) there are \(\lambda' \geq \lambda, K, D, \delta, \kappa\) such that the following holds.

Let \(m, n \in I \cap Z\), \(m < n\). Suppose the sets \(Y_i \subseteq Y_{ij}\) are \(D\)-cobounded in \(X\) for \(m + 1 \leq i < j \leq n + 1\) for some \(D\) independent of \(i, j\).

Let \(y \in Y_m, y' \in Y_n\) and let \(\{y_i\}, m \leq i \leq n\) be a finite sequence of points in \(Y\) defined as follows: \(y_m = y, y_{i+1}\) is an \(\epsilon\)-approximate nearest point projection of \(y_i\) on \(Y_{i+1}\) for \(m \leq i \leq n - 1\). Suppose \(\alpha_i \subseteq Y_{ii+1}\) is a \(\lambda\)-quasigeodesic in \(X\) joining \(y_i\) and \(y_{i+1}, m \leq i \leq n - 1\) and \(\beta\) is a \(\lambda\)-quasigeodesic joining \(y_n\) and \(y'\). Then the concatenation of the all the \(\alpha_i\)’s and \(\beta\), denoted by \(\alpha\), is a \(\lambda\)-quasigeodesic in \(X\) joining \(y, y'\).

Moreover, each \(y_i\) is an \(\epsilon\)-approximate nearest point projection of \(y\) on \(Y_i\) for \(m + 2 \leq i \leq n\).
Proof: The proof is broken into the following three claims.

Claim 1: Suppose \( x \in Y_i^- \) for some \( i \). Let \( \bar{x} \) be an \( \epsilon \)-approximate nearest point projection of \( x \) on \( Y_i^- \). Then \( \bar{x} \) is an \( \epsilon' \)-approximate nearest point projection of \( x \) on \( Y_i^+ \) where \( \epsilon' \) depends on \( \epsilon \) and the other hypotheses of the proposition.

Proof of Claim 1: Suppose \( x' \) is a 1-approximate nearest point projection of \( x \) on \( Y_i^+ \). Since \( Y_i^+ \) is \( K \)-quasiconvex \( [x, x'] \) is a \( K \)-quasigeodesic by Lemma 2.28. Let \( k_1 = K_2(\delta, K, 1, 1) \). Then by stability of quasigeodesics there is a point \( z \in \{x, \bar{x}\} \) such that \( d(x', z) \leq D(x, \bar{x}) \delta, k_1 = D_1 \), say. We claim that \( z \) is uniformly close to \( Y_i \). Since \( Y_i^- \) is \( K \)-quasiconvex there is a point \( w \in Y_i^- \) such that \( d(z, w) \leq K \). It follows that \( d(w, x') \leq D_1 + K \). Since \( Y_i \) coarsely uniformly bisects \( Y \) into \( Y_i^\pm \) there is a point \( z_1 \in \{w, x'\} \) such that \( d(z_1, Y_i) \leq D_1' \) for some uniform constant \( D'_1 \). Since, \( d(z_1, w) \leq d(w, x') \leq D_1 + K \) and \( d(w, z) \leq K \) it follows by triangle inequality that \( d(z, Y_i) \leq 2K + D_1 + D_1' \). Now, by Lemma 2.15 \( \bar{x} \) is an \((\epsilon + 3)\)-approximate nearest point projection of \( z \) on \( Y_i \). Hence, \( d(x', \bar{x}) \leq d(x', z) + d(z, \bar{x}) \leq D_1 + \epsilon + 3 + d(z, Y_i) \). It follows that \( \epsilon' = 3 + \epsilon + 2K + 2D_1 + D_1' \) works.

Claim 2. Next we claim that for all \( m + 2 \leq i \leq n - 1 \) there is uniformly bounded set \( A_i \subset Y_i \) such that \( \epsilon \)-nearest point projection of any point of \( Y_i^- \), \( j < i \) on \( Y_j \) is contained in \( A_i \).

Proof of Claim 2: Consider any \( Y_i \), \( m + 2 \leq i \leq n - 1 \). Let \( B_i \subset Y_i \) be the set of all 1-approximate nearest point projections of points of \( Y_i-1 \) on \( Y_i \) in \( X \). Then the diameter of \( B_i \) is at most \( D \). Suppose \( x \in Y_i^- \), \( j < i \). Let \( x_1, x_2 \) be respectively \( \epsilon \)-approximate nearest point projections of \( x \) on \( Y_i-1 \) and \( Y_i \) respectively. Let \( x_3 \) be an \( \epsilon \)-nearest point projection of \( x_1 \) on \( Y_i \). Now, by Step 1 \( x_1 \) is an \( \epsilon \)-approximate nearest point projection of \( x \) on \( Y_i^\pm \) and \( x_2, x_3 \) are \( \epsilon \)-approximate nearest point projection of \( x \) and \( x_1 \) respectively on \( Y_i^\pm \). Therefore, by the first part of Corollary 2.30 we have \( d(x_2, x_3) \leq D(x, \bar{x}) \). However, if \( x' \in B_i \) is a 1-approximate nearest point projection of \( x \) on \( Y_i \) then by the second part of the Corollary 2.30 we have \( d(x_3, B_i) \leq d(x_3, x') \leq D(x, \bar{x}) \delta, K, \epsilon \) since \( \epsilon \geq 1 \). Hence, \( d(x_2, B_i) \leq 2D(x, \bar{x}) \delta, K, \epsilon \). Therefore, we can take \( A_i = N_2(\bar{x}) \delta, K, \epsilon \) \( (B_i) \cap Y_i \).

Let \( r = \sup_{m + 2 \leq i \leq n - 1} \{ \text{diam}(A_i) \} \).

We note that \( r \leq D + 2D(x, \bar{x}) \delta, K, \epsilon \).

Claim 3. Finally we claim that (1) \( \alpha \) is contained in a uniformly small neighborhood of a geodesic joining \( y, y' \) and (2) \( \alpha \) is uniformly properly embedded in \( X \).

We note that the proposition follows from Claim 3 using Lemma 2.31.

Proof of Claim 3: Suppose \( x, x' \in \alpha \), \( \Pi(x) < \Pi(x') \). Choose smallest \( k, l \) such that \( x \in \alpha \cap Y_{kk+1}, x' \in \alpha \cap Y_{ll+1} \), where \( m \leq k \leq l \leq n \). Let \( \gamma \) be a geodesic in \( X \) joining \( x, x' \).

(1) It is enough to show that the segment of \( \alpha \) joining \( x \) to \( x' \) is contained in a uniformly small neighborhood of \( \gamma \). Hence, without loss of generality \( k < l \). Due to Corollary 2.25 it is enough to prove that the points \( y_i, k+1 \leq i \leq l-1 \) are contained in a uniformly small neighborhood of \( \gamma \) in order to show that the segment of \( \alpha \) joining \( x \) to \( x' \) is contained in a uniformly small neighborhood of \( \gamma \). (We note that the path \( \alpha_{n-1} * \beta \) is a \( D(x, \bar{x}) \delta, K, \lambda, \epsilon \)-quasigeodesic joining \( y_{n-1} \) and \( y' \).) For this first we note that \( x \) is on \( \alpha_k \). Let \( \gamma_k \) be a geodesic joining \( y_k, y_{k+1} \). Then by stability
of quasigeodesics there is a point \( x_1 \in \gamma_k \) such that \( d(x_1, x) \leq D_{\gamma,k,\epsilon}(\delta, \lambda, \lambda) \). Since \( y_{k+1} \) is an \( \epsilon \)-approximate nearest point projection of \( y_k \) on \( Y_{k+1} \), by Lemma 2.15 \( y_{k+1} \) is an \((\epsilon + 3 + D_{\gamma,k,\epsilon}(\delta, \lambda, \lambda))\)-approximate nearest point projection of \( x_1 \) on \( Y_{k+1} \). Hence, \( y_{k+1} \) is an \((\epsilon + 3 + D_{\gamma,k,\epsilon}(\delta, \lambda, \lambda))\)-approximate nearest point projection of \( x_1 \) on \( Y_{k+1} \). Let \( \epsilon_1 = \epsilon + 3 + D_{\gamma,k,\epsilon}(\delta, \lambda, \lambda) \). By Step 1 \( y_{k+1} \) is an \( \epsilon_1 \)-nearest point projection of \( x \) on \( Y_{k+1}^+ \) where \( \epsilon_1 = 3 + \epsilon_1 + 3D_1 + D_1' \). Now the concatenation of a geodesic joining \( y_{k+1} \) to \( x' \) with the segment of \( \alpha \) from \( x \) to \( y_{k+1} \) is a uniform quasigeodesic by Lemma 2.28. Thus by Corollary 2.23 \( y_{k+1} \) is uniformly close to \( \gamma \). On the other hand by Step 2 \( y_i \) is an \((\epsilon + r)\)-approximate nearest point projection of \( x \) on \( Y_i \) and hence an \((\epsilon + r')\)-approximate nearest point projection on \( Y_i^+ \) for all \( k + 2 \leq i \leq l - 1 \). Hence, again by Lemma 2.28 and Corollary 2.28 \( y_i \) is within a uniformly small neighborhood of \( \gamma \). This proves (1).

(2) Suppose \( L = \sup \{d(y_i, \gamma) : k + 1 \leq i \leq l - 1\} \). Suppose \( x, x' \in \alpha \) as above with \( d(x, x') \leq L \). Once again, without loss of generality \( k < l \). We claim that \( l \leq k + N \). To see this consider two adjacent vertices \( v_i, v_{i+1} \) on \( \gamma \). If \( v_i \in N_{Y}(y_{s+1}) \) and \( v_{i+1} \in N_{Y}(y_{t+1}) \) with \( s < t \) then by the condition (3) we have \( t = s + 1 \). The claim follows from this. Suppose \( \alpha(s_k) = x, \alpha(s_i) = y_i \) for \( k + 1 \leq i \leq l - 1 \) and \( \alpha(s_i) = x' \). We note that \( d(\alpha(s_i), \alpha(s_{i+1})) \leq N + 2L \) for \( k \leq i \leq l - 1 \). Since \( l - k \leq N \) and since the segments of \( \alpha \) joining \( \alpha(s_i), \alpha(s_{i+1}) \), \( k \leq i \leq l - 1 \) are uniform quasigeodesics we are done.

For the second part of the proposition we have already noticed that \( y_i \) is an \((\epsilon + r)\)-approximate nearest point projection of any point \( Y_j^{-} \), in particular of \( y \), on \( Y_i \) for all \( j \leq i \), \( m + 2 \leq i \leq n - 1 \). On the other hand, \( y_{n-1} \) is an \((\epsilon + r')\) \((\epsilon + r' + 3 + 3D_1 + D_1')\)-approximate nearest point projection of \( y \) on \( Y_{n-1}^+ \). Hence, by Corollary 2.30 if \( y_i' \) is a 1-approximate point projection of \( y \) on \( Y_n \subset Y_{n-1}^+ \) then \( d(y_i, y_{n-1}) \leq D(\gamma,k,\epsilon)(\delta, K, \epsilon + r') \). Thus \( y_n \) is an \((1 + D(\gamma,k,\epsilon)(\delta, K, \epsilon + r'))\)-approximate nearest point projection of \( y \) on \( Y_n \).

Lemma 2.35. Given \( \delta \geq 0, k \geq 1, \epsilon \geq 0 \) there is a constant \( D = D(\delta, k, \epsilon) \) such that the following is true.

Suppose \( X \) is a \( \delta \)-hyperbolic metric space. Suppose \( x_1, x_2, p \in X \) and \( \alpha \) is a \((k, \epsilon)\)-quasigeodesic in \( X \) joining \( x_1, x_2 \). Then \( d(x_1, x_2, p) - d(p, \alpha) \leq D \).

Proof. Without loss generality we shall assume that \( X \) is a length space \( \delta \)-hyperbolic in the sense of Gromov. Let \( w \in \alpha \) be a 1-approximate nearest point projection of \( p \) on \( \alpha \). Let \( \beta_1, \beta_2 \) be \((1, 1)\)-quasigeodesics joining the pairs of points \( (x_1, p), (x_2, p) \) respectively. Let \( \gamma \) be a \((1, 1)\)-quasigeodesic joining \( p, w \) and let \( \alpha' \) be a \((1, 1)\)-quasigeodesic joining \( x_1, x_2 \). Let \( C = D(\gamma,k,\epsilon)(\delta, k, \epsilon + 1) \). Now, by Corollary 2.23 \( H\mathrm{d}(\alpha, \alpha') \leq C \) and \( \alpha \) is \( C \)-quasiconvex. Let \( \alpha_1 \) be the portion of \( \alpha \) from \( x_1 \) to \( w \) and let \( \alpha_2 \) be the portion of \( \alpha \) from \( w \) to \( x_2 \). Then \( \alpha_1 \ast \gamma, \alpha_2 \ast \gamma \) are \( K = D(\gamma,k,\epsilon)(\delta, C, k, \epsilon + 1) \)-quasigeodesics. Hence by Corollary 2.23 \( H\mathrm{d}(\beta_1, \alpha_1 \ast \gamma) \leq D(\gamma,k,\epsilon)(\delta, K, K) \). Let \( w_1 \in \beta_1 \) be such that \( d(w, w_1) \leq D(\gamma,k,\epsilon)(\delta, K, K) \). Since \( H\mathrm{d}(\alpha, \alpha') \leq C \), there is a point \( w' \in \alpha' \) such that \( d(w, w') \leq C \). Hence, \( d(w', w_1) \leq C + D(\gamma,k,\epsilon)(\delta, K, K) = R \), say. Now by Lemma 2.11 \( d(x_1, x_2, p) - d(p, w) \leq 3 + 2R \). It follows that \( |d(x_1, x_2, p) - d(p, w)| \leq 3 + 2R + C \). Since \( w \) is a 1-approximate nearest point projection of \( p \) on \( \alpha \) we have for all \( z \in \alpha, d(p, w) \leq d(p, z) + 1 \). Thus \( |d(p, z) - d(p, w)| \leq 1 \). Hence, \( |d(x_1, x_2, p) - d(p, \alpha)| \leq 4 + 2R + C \).
Some of the results in this subsection are mentioned without proofs. One may refer to [BH99] and [ABC+91] for details.

**Definition 2.36.**

1. **Geodesic boundary.** Suppose $X$ is a (geodesic) hyperbolic metric space. Let $\mathcal{G}$ denote the set of all geodesic rays in $X$. The geodesic boundary $\partial X$ of $X$ is defined to be $\mathcal{G}/\sim$ where $\sim$ is the equivalence relation defined on $\mathcal{G}$ where $\alpha \sim \beta$ if $Hd(\alpha, \beta) < \infty$.

2. **Quasigeodesic boundary.** Suppose $X$ is a hyperbolic metric space and $\partial X$ is a proper geodesic hyperbolic metric space. Let $Q$ be the set of all quasigeodesic rays in $X$. Then the quasigeodesic boundary $\partial q X$ is defined to be $Q/\sim$ where $\sim$ is defined as above.

3. **Gromov boundary or sequential boundary.** Suppose $X$ is a hyperbolic metric space in the sense of Gromov and $p \in X$. Let $S$ be the set of all sequences $\{x_n\}$ in $X$ such that $\lim_{i,j \to \infty} (x_i, x_j)_p = \infty$. All such sequences are said to converge to infinity. On $S$ we define an equivalence relation where $\{x_n\} \sim \{y_n\}$ if and only if $\lim_{i,j \to \infty} (x_i, y_j)_p = \infty$ for some (any) base point $p \in X$. The Gromov boundary or the sequential boundary $\partial_S X$ of $X$, as a set, is defined to be $S/\sim$.

**Notation.**

1. The equivalence class of a geodesic ray or a quasigeodesic ray $\alpha$ in $\partial X$ or $\partial q X$ is denoted by $\alpha(\infty)$. It is customary to fix a base point and require that all the rays start from there to define $\partial X$ and $\partial q X$ but it is not essential.

2. If $\alpha$ is a (quasi)geodesic ray with $\alpha(0) = x$, $\alpha(\infty) = \xi$ then we say that $\alpha$ joins $x$ to $\xi$. We use $[x, \xi]$ to denote any (quasi)geodesic ray joining $x$ to $\xi$ when the parameter of the (quasi)geodesic ray is not important or is understood.

3. If $\alpha$ is a quasigeodesic line with $\alpha(\infty) = \xi_1, \alpha(-\infty) = \xi_2 \in \partial q X$ then we say that $\alpha$ joins $\xi_1, \xi_2$. We denote by $[\xi_1, \xi_2]$ any quasigeodesic line joining $\xi_1, \xi_2$ when the parameters of the quasigeodesic are understood.

4. If $\xi = \{[x_n]\} \in \partial S X$ then we write $x_n \to \xi$ or $\lim_{n \to \infty} x_n$ and say that the sequence $\{x_n\}$ converges to $\xi$.

5. We shall denote by $\hat{X}$ the set $X \cup \partial_s X$.

The following lemma and proposition summarizes all the basic properties of the boundary of hyperbolic spaces that we will need in this paper.

**Lemma 2.37.**

1. Given a $qi$ embedding $\phi : X \to Y$ we have an injective map $\partial \phi : \partial q X \to \partial q Y$.

2. (i) If $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ are $qi$ embeddings then $\partial(\psi \circ \phi) = \partial \psi \circ \partial \phi$

   (ii) $\partial(Id_X)$ is the identity map on $\partial q X$.

   (iii) A $qi$ induces a bijective boundary map.

The following proposition relates the three definitions of boundaries.

**Proposition 2.38.**

1. For any metric space $X$ the inclusion $\mathcal{G} \to Q$ induces an injective map $\partial X \to \partial q X$.

2. Given a quasigeodesic ray $\alpha$, $\lim_{n \to \infty} \alpha(n)$ is well defined and $\alpha \sim \beta$ implies $\lim_{n \to \infty} \alpha(n) = \lim_{n \to \infty} \beta(n)$. This induces an injective map $\partial q X \to \partial q X$.

3. If $X$ is a proper geodesic hyperbolic metric space then the map $\partial X \to \partial q X$ is a bijection.

4. The map $\partial q X \to \partial q X$ is a bijection for all Gromov hyperbolic length spaces.

   In fact, there is a constant $k_0 = k_0(\delta)$ such that given any $\delta$-hyperbolic length space $X$, any pair of points $x, y \in \hat{X}$ can be joined by a $k_0$-quasigeodesic ray or line.
Proof. (1), (2), (3) are standard. See [BH99] Chapter III.H for instance.

(4) is proved for geodesic metric spaces in the section 2 of [MS12]. See Lemma 2.4 there. The same result for a general length space then is a simple consequence of the existence of a metric graph approximation of a length space and the preceding lemma.

Lemma 2.39. (Ideal triangles are slim) Suppose $X$ is a $\delta$-hyperbolic metric space in the sense of Rips or Gromov. Suppose $x,y,z \in X$ and we have three $k$-quasigeodesics joining each pair of points from $\{x,y,z\}$. Then the triangle is $R = R_{2.39}(\delta, k)$-slim.

In particular, if $\gamma_1, \gamma_2$ are two $k$-quasigeodesic rays with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(\infty) = \gamma_2(\infty)$ then $H\delta(\gamma_1, \gamma_2) \leq R$.

The proof of this lemma is pretty standard and hence we omit it.

Corollary 2.40. (Ideal polygons are slim) Suppose $X$ is a $\delta$-hyperbolic metric space in the sense of Rips or Gromov. Suppose $x_1, x_2, \ldots, x_n \in X$ are $n$ points and we have $n$ $k$-quasigeodesics joining each pair of points from $\{x_1, x_2, \ldots, x_n\}$. Then this $n$-gon is $R = R_{2.40}(\delta, k, n)$-slim, i.e. every side is contained in $R$-neighborhood of the union of the remaining $n-1$ sides.

Lemma 2.41. Let $x \in X$ be any point. Suppose $\{x_n\}$ is any sequence of points in $X$ and $\beta_{m,n}$ is a $k$-quasigeodesic joining $x_m$ to $x_n$ for all $m, n \in \mathbb{N}$. Suppose $\alpha_n$ is a $k$-quasigeodesic joining $x$ to $x_n$. Then

1. $\{x_n\} \in S$ if and only if $\lim_{m,n \to \infty} d(x, \beta_{m,n}) = \infty$.
2. Suppose moreover $\xi \in \partial_s X$ and $\gamma_n$ is a $k$-quasigeodesic in $X$ joining $x_n$ to $\xi$ for all $n \in \mathbb{N}$ and $\alpha$ is a $k$-quasigeodesic joining $x$ to $\xi$.

Then $x_n \to \xi$ if and only if $d(x, \gamma_n) \to \infty$ if and only if $d(x, \gamma_n) \to \infty$.

We skip the proof of this lemma. In fact, the first statement of the lemma is an easy consequence of Lemma 2.35 and stability of quasigeodesics. The second statement is a simple consequence of Lemma 2.35 and stability of quasigeodesics and the Lemma 2.39

The following lemma is proved in section 2 of [MS12] (see Lemma 2.7 and Lemma 2.9 there) for hyperbolic geodesic metric spaces. The same statements are true for length spaces too. To prove it for length spaces one just takes a metric graph approximation. Since the proof is straightforward we omit it.

Lemma 2.42. (Barycenters of ideal triangles) There is a number $r_0$ such that given any length space $X$, any three distinct points $x, y, z \in X$ and any three $k_0$-quasigeodesics joining $x, y, z$ in pairs there is a point $x_0 \in X$ such that $N_{r_0}(x_0)$ intersects all the three quasigeodesics.

We refer to a point with this property to be a barycenter of $\Delta xyz$. There is a constant $L_0$ such that given any two barycenter $x_0, x_1$, $d(x_0, x_1) \leq L_0$.

Thus we have a coarsely well-defined map $\partial^2 X \to X$ which shall refer to as the barycenter map. It is a standard fact that for a nonelementary hyperbolic group $G$, if $X$ is a Cayley graph of $G$ then the barycenter map $\partial^2 X \to X$ is coarsely
surjective. In section 4 and 5 we deal with spaces with this properties. The following lemma is clear. For instance we can apply the proof of [MS12 Lemma 2.9].

**Lemma 2.43.** Barycenter maps being coarsely surjective is a qi invariant property among hyperbolic length spaces.

2.4.1. Topology on $\partial_s X$ and Cannon-Thurston maps.

**Definition 2.44.** (1) If $\{\xi_n\}$ is a sequence of points in $\partial_s X$, we say that $\{\xi_n\}$ converges to $\xi \in \partial_s X$ if the following holds: Suppose $\xi_n = \{x_n^i\}_k$ and $\xi = \{x_k\}_k$. Then $\lim_{n \to \infty} (\lim \inf_{i,j \to \infty} (x_i, x_j)_p) = \infty$.

(2) A subset $A \subset \partial_s X$ is said to be closed if for any sequence $\{\xi_n\}$ in $A$, $\xi_n \to \xi$ implies $\xi \in A$.

The definition of convergence that we have stated here is equivalent to the one stated in [ABC +91]. Moreover, that the convergence mentioned above is well-defined follows from [ABC +91] and hence we skip it. The following lemma gives a geometric meaning of the convergence.

**Lemma 2.45.** Given $k \geq 1$ and $\delta \geq 0$ there are constants $D = D(k, \delta)$, $L = L(k, \delta)$ and $r = r(k, \delta)$ with the following properties:

Suppose $\alpha, \beta$ are two $k$-quasigeodesic rays starting from a point $x \in X$ such that $\alpha(\infty) \neq \beta(\infty)$ and $\gamma$ is a $k$-quasigeodesic line joining $\alpha(\infty)$ and $\beta(\infty)$. Then (1) there exists $N \in \mathbb{N}$ such that $|\{(\alpha(m), \beta(n))_x - d(x, \gamma)| \leq D$ for all $m, n \geq N$.

In particular, $|\lim \inf_{m, n \to \infty} (\alpha(m), \beta(n))_x - d(x, \gamma)| \leq D$.

(2) Suppose $R = d(x, \gamma)$ then $\text{Hd}(\alpha \cap B(x; R - r), \beta \cap B(x; R - r)) \leq L$.

**Proof.** (1) Since $\alpha(\infty) \neq \beta(\infty)$ by Lemma 2.39 there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\alpha(m) \in N_{\frac{2.39}{2.39}}(\gamma)$ and $\beta(n) \in N_{\frac{2.39}{2.39}}(\gamma)$. Let $x_m, y_n \in \gamma$ be such that $d(x_m, \alpha(m)) \leq R\frac{2.39}{2.39}$ and $d(y_n, \beta(n)) \leq R\frac{2.39}{2.39}$. Then by joining $x_m, \alpha(m)$ and $y_n, \beta(n)$ and applying Corollary 2.23 we see that Hausdorff distance between any $(1,1)$-quasigeodesic joining $\alpha(m), \beta(n)$, say $c_{m,n}$ and the portion of $\gamma$ between $x_m, y_n$ is at most $R\frac{2.39}{2.39} + 2D\frac{2.24}{2.24}$. It is clear that for large enough $N$, $d(x, \gamma)$ is the same as the distance of the segment of $\gamma$ between $x_m, y_n$ if $m, n \geq N$. Thus for such $m, n$ we have $\{d(x, c_{m,n}) - d(x, \gamma)| \leq R\frac{2.39}{2.39} + 2D\frac{2.24}{2.24}$. But by Lemma 2.39 $\{(\alpha(m), \beta(n))_x - d(x, \gamma)| \leq R\frac{2.39}{2.39} + 2D\frac{2.24}{2.24}$ for all large $m, n$.

(2) To see this we take a 1-approximate nearest point projection of $x$ on each $\gamma$. Let $z$ be a 1-approximate nearest point projection. Let $x_z$ denote a 1-quasigeodesic joining $x, z$. Then by Corollary 2.20 concatenation of $xz$ and the portions of $\gamma$ joining $z$ to $\gamma(\pm \infty)$ respectively are both $R\frac{2.30}{2.30}, \delta, k, k)$-quasigeodesics. Call them $\alpha'$ and $\beta'$ respectively. Note that $\alpha(\infty) = \alpha'(\infty)$ and $\beta(\infty) = \beta'(\infty)$. Let $K = \max\{k, R\frac{2.30}{2.30}, \delta, k, \epsilon\}$. Then by the last part of Lemma 2.39 it follows that $z \in N_r(\alpha) \cap N_r(\beta)$ where $r = R\frac{2.30}{2.30}, \delta, K)$. Suppose $x' \in \alpha, y' \in \beta$ are such that $d(z, x') \leq r$ and $d(y', z) \leq r$. By Lemma 2.23 the Hausdorff distance between $xz$ and the portions of $\alpha$ from $x$ to $x'$ and the portion of $\beta$ from $x$ to $y'$ are each at most $D\frac{2.24}{2.24} \delta, k) + r$. Thus these segments of $\alpha$ and $\beta$ are at a Hausdorff distance at most $L = 2D\frac{2.24}{2.24} \delta, k) + 2r$ from each other. This completes the proof. □

**Lemma 2.46.** Let $x \in X$ be any point. Suppose $\{\xi_n\}$ is any sequence of points in $\partial_s X$. Suppose $\beta_m, n$ is a $k$-quasigeodesic line joining $\xi_m$ to $\xi_n$ for all $m, n \in \mathbb{N}$ and $\alpha_n$ is a $k$-quasigeodesic ray joining $x$ to $\xi_n$ for all $n \in \mathbb{N}$. Then
(1) \( \lim_{m,n \to \infty} d(x, \beta_{m,n}) = \infty \) iff there is a constant \( D = D(k, \delta) \) such that for all \( M > 0 \) there is \( N > 0 \) with \( \text{Hd} \( \alpha_m \cap B(x;M), \alpha_n \cap B(x;M) \) \leq D \) for all \( m, n \geq N \) and in this case \( \{ \xi_n \} \) converges to some point of \( \partial_s X \).

(2) Suppose moreover \( \xi \in \partial_s X \), \( \gamma_n \) is a \( k \)-quasigeodesic ray in \( X \) joining \( \xi_n \) to \( \xi \) for all \( n \), and \( \alpha \) is a \( k \)-quasigeodesic ray joining \( x \) to \( \xi \). Then \( \xi_n \to \xi \) iff \( d(x, \gamma_n) \to \infty \) iff there is constant \( D = D(k, \delta) \) such that for all \( M > 0 \) there is \( N > 0 \) with \( \text{Hd} \( \alpha \cap B(x;M), \alpha_n \cap B(x;M) \) \leq D \) for all \( n \geq N \). In this case \( \lim_{m,n \to \infty} d(x, \beta_{m,n}) = \infty \).

Proof. (1) The ‘iff’ part is an immediate consequence of Lemma 2.45. We prove the last part. Let \( n_i \) be an increasing sequence in \( \mathbb{N} \) such that for all \( m,n \geq n_i \) we have \( \text{Hd} \( \alpha_m \cap B(x;i), \alpha_n \cap B(x;i) \) \leq D \). Let \( y_i \) be a point of \( \alpha \cap B(x;i) \) such that \( d(x, y_i) + 1 \geq \sup \{ d(x, y) : x \in \alpha_m \cap B(x;i) \} \). We claim that \( y_i \) converges to a point of \( \partial_s X \). Clearly \( d(x, y_i) \to \infty \). Given \( i \leq j \in \mathbb{N} \) we have \( d(y_i, \alpha_n) \leq D \) and \( d(y_j, \alpha_n) \leq D \) for all \( n \geq n_j \). By slimness of polygons we see that any \( (1,1) \)-quasigeodesic joining \( y_i, y_j \) is uniformly close to \( \alpha_n \). It follows that \( \lim_{i,j \to \infty} (y_i, y_j) = \infty \). Let \( \xi = \{ \{ y_n \} \} \). It is clear that \( \xi_n \to \xi \).

(2) Both ‘iff’ statements are immediate from Lemma 2.45. The last part follows from slimness of ideal triangle since \( d(x, \gamma_n) \to \infty \).

Corollary 2.47. Suppose \( \{ x_n \} \) is a sequence of points in \( X \) such that \( \{ x_n \} \subset X \) or \( \{ x_n \} \subset \partial_s X \). Suppose \( x_n \to \xi \in \partial_s X \) and \( \gamma_n \) is a \( k \)-quasigeodesic joining \( x_n \) to \( \xi \) for each \( n \). Let \( y_n \in \gamma_n \) such that \( d(x, y_n) \to \infty \). Then \( \lim_{n \to \infty} y_n = \xi \).

Definition 2.48. (Cannon-Thurston map, [Mit98b]) If \( f : Y \to X \) is any map of hyperbolic metric spaces then we say that Cannon-Thurston (CT) map exists for \( f \) if \( f \) gives rise to a continuous map \( \partial f : \partial Y \to \partial X \) in the following sense:

- Given any \( \xi \in \partial Y \) and any sequence of points \( \{ y_n \} \) in \( Y \) converging to \( \xi \), the sequence \( \{ f(y_n) \} \) converges to a definite point of \( \partial X \) independent of the \( \{ y_n \} \) and the resulting map \( \partial f : \partial Y \to \partial X \) is continuous.

Generally, one assumes that the map \( f \) is a proper embedding but for the sake of the definition it is unnecessary.

Lemma 2.49. (Mitra’s criterion, [Mit98b], Lemma 2.1) Suppose \( X, Y \) are geodesic hyperbolic metric spaces and \( f : Y \to X \) is a proper embedding. Then \( f \) admits CT if the following holds:

(*) Let \( y_0 \in Y \). There exists a function \( \tau : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), with the property that \( \tau(n) \to \infty \) as \( n \to \infty \) such that for all geodesic segments \( [y_1, y_2]_Y \) in \( Y \) lying outside the \( n \)-ball around \( y_0 \in Y \), any geodesic segment \( [y_1, y_2]_X \) in \( X \) joining the same pair of points \( y_1, y_2 \) lies outside the \( \tau(n) \)-ball around \( f(y_0) \in X \).

Remark 3. (1) The main set of examples where the lemma applies comes from taking \( Y \) to be a subspace of a hyperbolic space \( X \) with induced length metric and the map \( f \) is assumed to be the inclusion map, or the orbit map from a hyperbolic group \( G \) acting properly by isometries on a hyperbolic metric space \( X \). In these examples the map \( f \) is coarsely Lipschitz as well as a proper embedding. The proof of the lemma by Mitra also assumes that \( X, Y \) are proper geodesic metric spaces and Mitra considered the geodesic boundaries. However, these conditions are not necessary as the following lemma and examples show.

(2) The proof of the above lemma by Mitra only checks that the map is a well-defined extension of \( f \) rather than it is continuous. However, with very little effort
the condition (*) can be shown to be sufficient for the well-definedness as well as
the continuity of the CT map.

(3) One can easily check that the condition (*) is also necessary provided $X, Y$
are proper hyperbolic spaces and $f$ is a coarsely Lipschitz proper embedding.

The following lemma is the main tool for the proof of our theorem of Cannon-
Thurston map. We shall refer to this as Mitra’s lemma.

**Lemma 2.50.** Suppose $X, Y$ are length spaces hyperbolic in the sense of Gromov
and $f : Y \to X$ is any map. Let $p \in Y$.

(**) Suppose for all $N > 0$ there is $M = M(N) > 0$ such that $N \to \infty$
implies $M \to \infty$ with the following property: For any $y_1, y_2 \in Y$, any $(1, 1)$-quasigeodesic
$\alpha$ in $Y$ joining $y_1, y_2$ and any $(1, 1)$-quasigeodesic $\beta$ in $X$ joining $f(y_1), f(y_2)$,
$B(p, N) \cap \alpha = \emptyset$ implies $B(f(p), M) \cap \beta = \emptyset$.

Then the CT map exists for $f : Y \to X$.

**Proof.** Suppose $\{y_n\}$ is any sequence in $Y$. Suppose $\alpha_{i,j}$ is a $(1, 1)$-quasigeodesic in
$Y$ joining $y_i, y_j$ and suppose $\gamma_{i,j}$ is a $(1, 1)$-quasigeodesic in $X$ joining $f(y_i), f(y_j)$. Then by Lemma 2.35 $\lim_{i,j \to \infty} (y_i, y_j)_p = \infty$ if and only if $\lim_{i,j \to \infty} d(p, \alpha_{i,j}) = \infty$ and $\lim_{i,j \to \infty} (f(y_i), f(y_j))_{f(p)} = \infty$ if and only if $\lim_{i,j \to \infty} d_X(f(p), \gamma_{i,j}) = \infty$. On
the other hand by (**), $\lim_{i,j \to \infty} d(p, \alpha_{i,j}) = \infty$ implies $\lim_{i,j \to \infty} d_X(f(p), \gamma_{i,j}) = \infty$. Thus $\{y_n\}$ converges to a point of $\partial_2 Y$ implies $\{f(y_n)\}$ converges to a point of $\partial_1 X$. The same argument shows that if $\{y_n\}$ and $\{z_n\}$ are two sequences in $Y$
representing the same point of $\partial_2 Y$ then $\{f(y_n)\} \cap \{f(z_n)\} \cap \partial_1 X$. Thus we have a well-defined map $\partial f : \partial_1 Y \to \partial_2 X$.

Now we prove the continuity of the map. We need to show that if $\xi_n \to \xi$ in $\partial_2 Y$
then $\partial f(\xi_n) \to \partial f(\xi)$. Suppose $\xi_n$ is represented by the class of $\{y^n_k\}$ and $\xi$ is the
equivalence class of $\{y_k\}$. Then

$$\lim_{n \to \infty} \lim_{i,j \to \infty} (y^n_i, y^n_j)_p = \infty.$$ 

By Lemma 2.35 then we have

$$\lim_{n \to \infty} \lim_{i,j \to \infty} d(p, \alpha^n_{i,j}) = \infty$$

for any $(1, \epsilon)$-quasigeodesic $\alpha^n_{i,j}$ in $Y$ joining $y^n_i$ and $y^n_j$. By (*) then we have

$$\lim_{n \to \infty} \lim_{i,j \to \infty} d(f(p), \gamma^n_{i,j}) = \infty$$

where $\gamma^n_{i,j}$ is any $(1, \epsilon)$-quasigeodesic in $X$ joining $f(y^n_i), f(y^n_j)$. This in turn implies that

$$\lim_{n \to \infty} \lim_{i,j \to \infty} (f(y^n_i), f(y^n_j)))_{f(p)} = \infty.$$ 

Therefore, $\partial f(\xi_n) \to \partial f(\xi)$ as was required. \qed

**Examples and remarks:**

(1) Suppose $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an exponential function. Then $f$ is not coarsely
Lipschitz but $f$ admits CT.

(2) One can easily cook up an example along the line of the above example
where properness is also violated but CT map exists like we see in the
example below. We will see another interesting example in Corollary 6.9.
(3) The condition (*) in the above lemma is also not necessary in general for the existence of CT map. Here is an example: Suppose $X$ is a tree built in two phases. First we have a star, i.e. a tree with one central vertex on which end points of finite intervals are glued. Assume the lengths of the intervals are unbounded. Then two distinct rays are glued to each integer points of the intervals. Suppose $Y$ is obtained by collapsing the central star in $X$ and $f$ is the identity map. The clearly CT exists but (*) is violated.

The following lemma is very standard and hence we skip mentioning its proof.

**Lemma 2.51.** (Functoriality of CT maps) (1) Suppose $X,Y,Z$ are hyperbolic metric spaces and $f:X \to Y$ and $g:Y \to Z$ admit CT maps. Then so does $g \circ f$ and $\partial (g \circ f) = \partial g \circ \partial f$.

(2) If $i:X \to X$ is the identity map then it admits a CT map $\partial i$ which is the identity map on $\partial_s X$.

(3) If two maps $f,h:X \to Y$ are at a finite distance admitting CT maps then they induce the same CT map.

(4) Suppose $f:X \to Y$ is a qi embedding of hyperbolic length spaces. There is a continuous injective CT map $\partial f: \partial_s X \to \partial_s Y$ which is a homeomorphism onto image.

If $f$ is a quasiconformal map then $\partial f$ is a homeomorphism. In particular, the action by left multiplication of a hyperbolic group $G$ on itself induces an action of $G$ on $\partial G$ by homeomorphisms.

### 2.4.2 Limit sets

**Definition 2.52.** Suppose $X$ is a hyperbolic metric space and $A \subset X$. Then the limit set of $A$ in $X$ is the set $\Lambda(A) = \{ \lim_{a_n \to \infty} a_n \in \partial_s X : \{a_n\} \subset A \}$.

In this subsection we collect some basic results on limit sets that we need in the section 6 of the paper. In each case we briefly indicate the proofs for the sake of completeness. The following is straightforward.

**Lemma 2.53.** Suppose $X$ is a hyperbolic metric space and $A,B \subset X$ with $Hd(A,B) < \infty$. Then $\Lambda(A) = \Lambda(B)$.

**Lemma 2.54.** Suppose $X$ is a hyperbolic metric space and $Y \subset X$. Suppose $Z \subset Y$ coarsely bisects $Y$ in $X$ into $Y_1,Y_2$. Then $\Lambda(Y_1) \cap \Lambda(Y_2) = \Lambda(Z)$.

**Proof.** This is a straightforward consequence of Lemma 2.35.

**Lemma 2.55.** Suppose $X$ is a $\delta$-hyperbolic metric space and $A \subset X$ is $\lambda$-quasiconvex. Suppose $\xi \in \Lambda(A)$ and $\gamma$ is a $K$-quasigeodesic ray converging to $\xi$. Then there is $D = D(\delta,\lambda,K) > 0$ such that $\gamma(n) \subset N_D(A)$ for all large enough $n$.

**Proof.** Rather than explicitly computing the constants we indicate how to obtain them. Suppose $x_n$ is a sequence in $A$ such that $x_n \to \xi$. Let $y_1 \in \gamma$ be a 1-approximate nearest point projection of $x_1$ on $\gamma$. Let $\alpha_1$ denote a $(1,1)$-quasigeodesic joining $x_1,y_1$. Then the concatenation, say $\gamma_1$, of $\alpha_1$ and the segment of $\gamma$ from $y_1$ to $\xi$ is a uniform quasigeodesic by Corollary 2.29. For all $m > 1$, let $y_m$ denote a 1-approximate nearest point projection of $x_m$ on $\gamma_1$. Then $y_m$ is contained in $\gamma_1$ for all large $m$. However, once again by Corollary 2.29 the concatenation of the portion of $\gamma_1$ between $x_1,y_m$ and a 1-quasigeodesic joining $x_m,y_m$ is a uniform quasigeodesic. Now it follows by stability of quasigeodesics that the segment of
Lemma 2.56. Suppose $X, Y$ are hyperbolic metric spaces and $f : Y \to X$ is any proper map. If $Y$ is a proper metric space and the CT map exists for $f$ then we have $\Lambda(f(Y)) = \partial f(\partial Y)$.

Proof. It is clear that $\partial f(\partial Y) \subset \Lambda(f(Y))$. Suppose $y_n$ is any sequence such that $f(y_n) \to \xi$ for some $\xi \in \partial_X$. Since $f$ is proper $\{y_n\}$ is an unbounded sequence. Since $Y$ is a proper length space it is a geodesic metric space by Hopf-Rinow theorem (see [BH99], Proposition 3.7, Chapter I.3). Now it is a standard fact that any unbounded sequence in a proper geodesic metric space has a subsequence converging to a point of the Gromov boundary of the space. Since $Y$ is proper, we have a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \eta$ for some $\eta \in \partial_Y$. It is clear that $\partial f(\eta) = \xi$. Hence $\Lambda(f(Y)) \subset \partial f(\partial Y)$.

3. Metric bundles

In this section we recall necessary definitions and some elementary properties of the primary objects of study in this paper namely, that of metric bundles and metric graph bundles from [MS12]. We make a minor modification to the definition of a metric bundle. See Definition 3.1 below. This is a slight generalization of the notion of metric bundles due to [MS12], but we use the same definition of metric graph bundles as in [MS12].

3.1. Basic definitions and properties.

Definition 3.1. (Metric bundles [MS12, Definition 1.2]) Suppose $(X, d)$ and $(B, d_B)$ are geodesic metric spaces; let $c \geq 1$ and let $\eta : [0, \infty) \to [0, \infty)$ be a function. We say that $X$ is an $(\eta, c)$-metric bundle over $B$ if there is a surjective $1$-Lipschitz map $\pi : X \to B$ such that the following conditions hold:

(1) For each point $z \in B$, $F_z := \pi^{-1}(z)$ is a geodesic metric space with respect to the path metric $d_z$ induced from $X$. The inclusion maps $i : (F_z, d_z) \to X$ are uniformly metrically proper as measured by $\eta$.

(2) Suppose $z_1, z_2 \in B$, $d_B(z_1, z_2) \leq 1$ and let $\gamma$ be a geodesic in $B$ joining them. Then for any point $z \in \gamma$ and $x \in F_z$ there is a path $\tilde{\gamma} : [0, 1] \to \pi^{-1}(\gamma) \subset X$ of length at most $c$ such that $\tilde{\gamma}(0) \in F_{z_1}$, $\tilde{\gamma}(1) \in F_{z_2}$ and $x \in \tilde{\gamma}$.

Given geodesic metric spaces $X$ and $B$ one says that $X$ is a metric bundle over $B$ if $X$ is an $(\eta, c)$-metric bundle over $B$ in the above sense for some function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ and some constant $c \geq 1$. If $X$ is a metric bundle over $B$ in the above sense then we shall refer to it as a geodesic metric bundle in this paper. However, the above definition seems a little restrictive. Therefore, we propose the following.

Definition 3.2. (Length metric bundles) Suppose $(X, d)$ and $(B, d_B)$ are length spaces, $c \geq 1$ and we have a function $\eta : [0, \infty) \to [0, \infty)$. We say that $X$ is an $(\eta, c)$-length metric bundle over $B$ if there is a surjective $1$-Lipschitz map $\pi : X \to B$ such that the following conditions hold:

(1) For each point $z \in B$, $F_z := \pi^{-1}(z)$ is a length space with respect to the path metric $d_z$ induced from $X$. The inclusion maps $i : (F_z, d_z) \to X$ are uniformly metrically proper as measured by $\eta$. 

(2) Suppose \( z_1, z_2 \in B \), and let \( \gamma \) be a path of length \( \leq 1 \) in \( B \) joining them. Then for any point \( z \in \gamma \) and \( x \in F_z \) there is a path \( \tilde{\gamma} : [0, 1] \to \pi^{-1}(\gamma) \subset X \) of length at most \( c \) such that \( \tilde{\gamma}(0) = F_{z_1}, \tilde{\gamma}(1) \in F_{z_2} \) and \( x \in \tilde{\gamma} \).

Given length spaces \( X \) and \( B \) we will say that \( X \) is a **length metric bundle** over \( B \) if \( X \) is an \((\eta, c)\)-length metric bundle over \( B \) in the above sense for some function \( \eta : \mathbb{R}^+ \to \mathbb{R}^+ \) and some constant \( c \geq 1 \).

**Convention 3.3.** From now on whenever we speak of a metric bundle we mean a length metric bundle.

**Definition 3.4.** (**Metric graph bundles** [MS12 Definition 1.5]) Suppose \( X \) and \( B \) are metric graphs. Let \( \eta : [0, \infty) \to [0, \infty) \) be a function. We say that \( X \) is an \( \eta \)-**metric graph bundle** over \( B \) if there exists a surjective simplicial map \( \pi : X \to B \) such that:

1. For each \( b \in \mathcal{V}(B) \), \( F_b := \pi^{-1}(b) \) is a connected subgraph of \( X \) and the inclusion maps \( i : F_b \to X \) are uniformly metrically proper as measured by \( \eta \) for the path metrics \( d_b \) induced on \( F_b \).
2. Suppose \( b_1, b_2 \in \mathcal{V}(B) \) are adjacent vertices. Then each vertex \( x_1 \) of \( F_{b_1} \) is connected by an edge with a vertex in \( F_{b_2} \).

**Remark 4.** Since the map \( \pi \) is simplicial it follows that it is \( 1 \)-Lipschitz.

For a metric (graph) bundle the spaces \((F_z, d_z)\), \( z \in B \) will be referred to as **fibers** and the distance between two points in \( F_z \) will be referred to as their **fiber distance**. A geodesic in \( F_z \) will be called a **fiber geodesic**. The spaces \( X \) and \( B \) will be referred to as the **total space** and the **base space** of the bundle respectively. By a statement of the form ‘\( X \) is a metric bundle (resp. metric graph bundle)’ we will mean that it is the total space of a metric bundle (resp. metric graph bundle).

Most of the results proved for geodesic metric bundles in [MS12] have their analogs for length metric bundles. We explicitly prove this phenomenon or provide sufficient arguments for all the results needed for our purpose.

**Convention 3.5.** Very often in a lemma, proposition, corollary or a theorem we shall omit explicit mention of some of the parameters on which a constant may depend if the parameters are understood.

**Definition 3.6.** Suppose \( \pi : X \to B \) is a metric (graph) bundle.

1. Suppose \( A \subset B \) and \( k \geq 1 \). A \( k \)-qi section over \( A \) is a \( k \)-qi embedding \( s : A \to X \) (resp. \( s : \mathcal{V}(A) \to X \)) such that \( \pi \circ s = Id_A \) (resp. \( \pi \circ s = Id_{\mathcal{V}(A)} \)) where \( A \) has the restricted metric from \( B \) and \( Id_A \) (resp. \( Id_{\mathcal{V}(A)} \)) denotes the identity map on \( A \to A \) (resp. \( \mathcal{V}(A) \to \mathcal{V}(A) \)).

2. More generally, given any metric space (resp. graph) \( Z \) and any qi embedding \( f : Z \to B \) (resp. \( f : \mathcal{V}(Z) \to \mathcal{V}(B) \)) a \( k \)-qi lift of \( f \) is a \( k \)-qi embedding \( \tilde{f} : Z \to X \) (resp. \( \tilde{f} : \mathcal{V}(Z) \to \mathcal{V}(X) \)) such that \( \pi \circ \tilde{f} = f \).

**Convention 3.7.** (1) Most of the time we shall refer to the image of a qi section or a qi lift to be the qi section (resp. qi lift).

(2) Suppose \( \gamma : I \to B \) is a (quasi)geodesic and \( \tilde{\gamma} \) is a qi lift of \( \gamma \). Let \( b = \gamma(t) \) for some \( t \in I \). Then we will denote \( \tilde{\gamma}(t) \) by \( \tilde{\gamma}(b) \) also.

The following lemma is immediate from the definition of a metric (graph) bundle.

**Lemma 3.8.** (Path lifting lemma) Suppose \( \pi : X \to B \) is an \((\eta, c)\)-metric bundle or an \( \eta \)-metric graph bundle.
(1) Suppose $b_1, b_2 \in B$. Suppose $\gamma : [0, L] \to B$ is a continuous, rectifiable, arc length parametrized path (resp. an edge path) in $B$ joining $b_1$ to $b_2$. Given any $x \in F_{b_1}$ there is a path $\tilde{\gamma}$ in $\pi^{-1}(\gamma)$ such that $l(\tilde{\gamma}) \leq (L + 1)c$ (resp $l(\tilde{\gamma}) = L$) joining $x$ to some point of $F_{b_2}$.

In particular, in case $X$ is a metric graph bundle over $B$ any geodesic $\gamma$ of $B$ can be lifted to a geodesic starting from any given point of $\pi^{-1}(\gamma)$.

(2) For any $k \geq 1$ and $\epsilon \geq 0$, any dotted $(k, \epsilon)$-quasigeodesic $\tilde{\beta} : [m, n] \to B$ has a lift $\tilde{\beta}$ starting from any point of $F_{\tilde{\beta}(m)}$ such that the following hold, where we assume $c = 1$ for metric graph bundles.

For all $i, j \in [m, n]$ we have

$$-\epsilon + \frac{1}{k}|i - j| \leq d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \leq c.(k + \epsilon + 1)|i - j|.$$  

In particular it is a $c.(k + \epsilon + 1)$-qi lift of $\beta$. Also we have

$$l(\tilde{\beta}) \leq ck(k + \epsilon + 1)(\epsilon + d_B(b_1, b_2)).$$

Proof. (1) We fix a sequence of points $0 = t_0, t_1, \ldots, t_n = L$ in $[0, L]$ such that the length of the portion of $\gamma$ joining $t_i, t_{i+1}$ is equal to 1 for $0 \leq i < n - 1$ and is less than or equal to 1 for $i = n - 1$ for the metric bundle case. For the metric graph bundle we have $t_i, 0 \leq i \leq L = n$. Now given any $x_i \in F_{t_i}$ we can find a path in $\pi^{-1}(\gamma[t_i, t_{i+1}])$ of length at most $c$ joining $x_i$ to some point of $F_{t_{i+1}}$ for $0 \leq i \leq n - 1$ using the definition of metric (graph) bundle where $c = 1$ for the metric graph bundle.

Hence, given any $x =: x_0 \in F_{t_0}$ we can inductively construct a sequence of points $x_i \in F_{t_i}, 0 \leq i \leq n$ and a sequence of paths $\alpha_i$ of length at most $c$ (resp. an edge) joining $x_i$ to $x_{i+1}$ for $0 \leq i \leq n - 1$. Concatenation of these paths gives a candidate for $\tilde{\gamma}$.

We also notice that in the case of a metric graph bundle $\tilde{\gamma}$ is a lift of $\gamma$. Moreover, if $\alpha$ is any edge path joining a point of $F_{b_1}$ to a point of $F_{b_2}$ then $d_B(b_1, b_2) = L \leq l(\pi \circ \alpha) = l(\alpha)$ since $\pi$ is 1-Lipschitz. Hence if $\gamma \subset B$ is a geodesic then $\tilde{\gamma}$ is a geodesic since $l(\gamma) = l(\tilde{\gamma})$.

(2) We construct a lift $\tilde{\beta}$ of $\beta$ starting from any point $x \in F_{\tilde{\beta}(m)}$ inductively as follows. We know that $d_B(\tilde{\beta}(i), \tilde{\beta}(i + 1)) \leq k + \epsilon$. Hence there is a path, say $\beta_i$ in $B$ joining $\beta(i)$ to $\beta(i + 1)$ which is of length at most $k + \epsilon + 1$ for $m \leq i \leq n - 1$. We can then find a path of length at most $(k + \epsilon + 1)c$ in $\pi^{-1}(\beta_i)$ (where $c = 1$ for metric graph bundle) joining any point of $F_{\beta(i)}$ to some point of $F_{\beta(i+1)}$. Hence, inductively we can construct a sequence of points $x_i \in F_{\beta(i)}$ for $m \leq i \leq n$ where $x_m = x$ and a sequence of paths $\tilde{\beta}_i$ in $\pi^{-1}(\beta_i)$ of length at most $(k + \epsilon + 1)c$, where $c = 1$ in the case of a metric graph bundle joining $x_i$ and $x_{i+1}$ for $m \leq i \leq n - 1$. Finally $\tilde{\beta}$ is defined by setting $\tilde{\beta}(i) = x_i, m \leq i \leq n$.

Clearly $d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \leq c.(k + \epsilon + 1)|i - j|$. Also, $d_B(\pi \circ \tilde{\beta}(i), \pi \circ \tilde{\beta}(j)) = d_B(\tilde{\beta}(i), \tilde{\beta}(j)) \leq d_X(\tilde{\beta}(i), \tilde{\beta}(j))$ since $\pi$ is 1-Lipschitz. Since $\beta$ is a dotted $(k, \epsilon)$ quasigeodesic, we have $-\epsilon + \frac{1}{k}|i - j| \leq d_B(\beta(i), \beta(j))$. This proves that

$$-\epsilon + \frac{1}{k}|i - j| \leq d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \leq c.(k + \epsilon + 1)|i - j|.$$
For the last part of (2) we see that
\[
\ell(\beta) = \sum_{i=m}^{n-1} d_X(\hat{\beta}(i), \hat{\beta}(i + 1)) \leq \sum_{i=m}^{n-1} c(k + \epsilon + 1) = (n - m)c(k + \epsilon + 1).
\]

On the other hand since \( \beta \) is a \((k, \epsilon)\)-quasigeodesic we have \(-\epsilon + \frac{1}{\epsilon}(n - m) \leq d_B(b_1, b_2)\). The conclusion immediately follows from these two inequalities. \( \square \)

The following corollary follows from the proof of Proposition 2.10 of [MST2]. We include it for the sake of completeness.

\begin{corollary}
Given any metric (graph) bundle \( \pi : X \to B \) and \( b_1, b_2 \in B \) we can define a map \( \phi : F_{b_1} \to F_{b_2} \) such that \( d_X(x, \phi(x)) \leq 3c + 3cd_B(b_1, b_2) \) (resp. \( d(x, \phi(x)) = d_B(b_1, b_2) \)) for all \( x \in F_{b_1} \).
\end{corollary}

\begin{proof}
The statement about the metric graph bundle is trivially true by Lemma 3.8(1). For the metric bundle case, fix a dotted 1-quigeodesic (resp. a geodesic) \( \gamma \) joining \( b_1 \) to \( b_2 \). Then for all \( x \in F_{b_1} \) fix for once and all a dotted lift (resp. an isometric lift) \( \tilde{\gamma} \) as constructed in the proof of the Lemma 3.8 which starts from \( x \) and set \( \phi(x) = \tilde{\gamma}(b_2) \). The statement then follows from Lemma 3.8(2). \( \square \)

\begin{remark}
For all \( b_1, b_2 \in B \) any map \( \phi : F_{b_1} \to F_{b_2} \) such that \( d_X(x, \phi(x)) \leq D \) for some constant \( D \) independent of \( x \) will be refereed to as a fiber identification map.
\end{remark}

The proof of the following lemma appears in the proof of Proposition 2.10 of [MST2]. We include a proof of this using the above lemma for the sake of completeness.

\begin{lemma}
Suppose \( \pi : X \to B \) is an \((\eta, c)\)-metric bundle or an \( \eta \)-metric graph bundle and \( R \geq 0 \). Suppose \( b_1, b_2 \in B \). The we have the following.
\begin{enumerate}
\item Suppose \( d_B(F_b_1, F_{b_2}) \leq 3c + 3cd_B(b_1, b_2) \) (resp. \( d_B(F_{b_1}, F_{b_2}) = d_B(b_1, b_2) \)).
\item Suppose \( \phi_{b_1 b_2} : F_{b_1} \to F_{b_2} \) is any map such that \( \forall x \in F_{b_1} \) and \( d(x, \phi_{b_1 b_2}(x)) \leq R \) for all \( x \in F_{b_1} \).
\end{enumerate}

Then \( \phi_{b_1 b_2} \) is a \((3, 10) \to K(3, 10) \)-quasiisometry which is \( F \)-surjective.
\end{lemma}

The proof of this lemma is similar to that of the above lemma. In particular, the maps \( \phi_{b_1 b_2} \) are coarsely unique.

In this lemma we deliberately suppress the dependence of \( K(3, 10) \) on the parameter(s) of the bundle.

\begin{proof}
(1) This clearly follows from Corollary 3.9.

(2) We first show that the map is coarsely Lipschitz. If \( x_1, x_2 \in F_{b_1}, d_{b_1}(x_1, x_2) \leq 1 \) then \( d(\phi_{b_1 b_2}(x_1), \phi_{b_1 b_2}(x_2)) \leq 2R + 1 \). Hence, \( d_{b_2}(\phi_{b_1 b_2}(x_1), \phi_{b_1 b_2}(x_2)) \leq \eta(2R + 1) \). This implies that \( \phi_{b_1 b_2} \) is \( \eta(2R + 1) \)-coarsely Lipschitz by Lemma 2.5.

Now, \( d_B(b_1, b_2) \leq d_X(x, \phi_{b_1 b_2}(x)) \leq R \). Hence by Corollary 3.9 we can define a map \( \phi_{b_2 b_1} : F_{b_2} \to F_{b_1} \) such that \( d_X(y, \phi_{b_2 b_1}(y)) \leq 3c + 3cR. \) (In the case of a metric graph bundle this is simply \( d_X(y, \phi_{b_2 b_1}(y)) \leq R \)). This by the first part of the proof then is \( \eta(6c + 6cR + 1) \)-coarsely Lipschitz for the metric bundles (resp. \( \eta(2R + 1) \)-coarsely Lipschitz for metric graph bundles). On the other hand, for all \( x \in F_{b_2} \), we have \( d(x, \phi_{b_2 b_1} \circ \phi_{b_1 b_2}(x)) \leq d(x, \phi_{b_2 b_1}(x)) + d(\phi_{b_2 b_1}(x), \phi_{b_2 b_1} \circ \phi_{b_1 b_2}(x)) \leq R + 3c + 3cR \) (resp. \( d(x, \phi_{b_2 b_1} \circ \phi_{b_1 b_2}(x)) \leq \eta(2R) \)). This implies that \( d_{b_1}(x, \phi_{b_2 b_1} \circ \phi_{b_1 b_2}(x)) \leq \eta(R + 3c + 3cR) \) (resp. \( d_{b_1}(x, \phi_{b_2 b_1} \circ \phi_{b_1 b_2}(x)) \leq \eta(2R) \)) for the case of metric graph bundles.
\end{proof}
bundles). In the same way one can show that \( d_b(y, \phi_{b_1} \circ \phi_{b_2}(y)) \) is bounded by the same quantity for all \( y \in F_{b_2} \). This means \( \phi_{b_1} \) and \( \phi_{b_2} \) are \( \eta(R + 3c + 3cR) \)-coarse inverse to each other for metric bundles and \( \eta(2R) \) for the metric graph bundle case. Therefore, we may take \( K_{\phi} \) for the same quantity for all \( \eta \), \( K \), \( D \), \( c \) for the metric bundles and \( K_{\phi} \) for the metric graph bundle case.

(3) This is immediate.

**Corollary 3.11.** Suppose \( \pi : X \to B \) is a metric (graph) bundle and \( b_1, b_2 \in B \) (resp. \( b_1, b_2 \in \mathcal{V}(B) \)) \( d_B(b_1, b_2) \leq R \). Suppose \( \phi_{b_1 b_2} : F_{b_1} \to F_{b_2} \) is a fiber identification map as constructed by Corollary 3.3. Then \( \phi_{b_1 b_2} \) is a \( K_{\phi} = K_{\phi} \)-quasimetric.

**Proof.** By Corollary 3.9 \( d_X(x, \phi_{b_1 b_2}(x)) \leq 3c + 3cd_B(b_1, b_2) \leq 3c + 3cR \) for all \( x \in F_{b_1} \) (resp. \( d_X(x, \phi_{b_1 b_2}(x)) = d_B(b_1, b_2) \leq R \) for all \( x \in \mathcal{V}(B) \)). Hence by Lemma 3.10 \( \phi_{b_1 b_2} \) is a \( K_{\phi} = K_{\phi} \)-quasimetric for the metric bundle and \( K_{\phi} = K_{\phi} \)-quasimetric for the metric graph bundle case.

The following corollary is proved as a simple consequence of Lemma 3.10 and Corollary 3.9 (See Corollary 1.14, and Corollary 1.16 of [MS12]). Therefore, we skip the proof of it.

**Corollary 3.12.** (Bounded flaring condition) For all \( k \in \mathbb{R}, k \geq 1 \) there is a function \( \mu_k : \mathbb{N} \to \mathbb{N} \) such that the following holds:

Suppose \( \pi : X \to B \) is an \( (\eta, c) \)-metric bundle or an \( \eta \)-metric graph bundle. Let \( \gamma \subset B \) be a dotted \( (1, 1) \)-quasigeodesic (resp. a geodesic) joining \( b_1, b_2 \in B \), and let \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) be two k-qi lifts of \( \gamma \) in \( X \). Suppose \( \tilde{\gamma}_i(b_1) = x_i \in F_{b_1} \) and \( \tilde{\gamma}_i(b_2) = y_i \in F_{b_2} \), \( i = 1, 2 \).

Then \( d_b(x_1, x_2) \leq \mu_k(N) \max\{d_b(x_1, x_2), 1\} \).

If \( d_B(b_1, b_2) \leq N \).

In the rest of the paper, we will summarize the conclusion of Corollary 3.12 by saying that a metric (graph) bundle satisfies the **bounded flaring condition**.

**Remark 6.** (Metric bundles in the literature) Metric (graph) bundles appear in a number of places in other people’s work. In [Bow02] Bowditch defines stacks of (hyperbolic) spaces which can easily be shown to be quasiisometric to metric graph bundles. Conversely a metric (graph) bundle is clearly a stack of spaces as per [Bow02]. In [Why10] Whyte defines coarse bundles which are also quasiisometric to metric graph bundles but with additional restrictions.

3.2. Some natural constructions and examples. In this section we discuss a number of natural examples and a few general constructions that produces metric (graph) bundles.

**Example 1.** Tangent bundle of a manifold. Suppose \( M \) is a (complete) Riemannian manifold. Consider the Sasaki metric on the tangent bundle \( TM \) of \( M \). We claim that \( (TM, M, \pi) \) is a metric bundle where \( \pi : TM \to M \) is the natural footpoint projection map. Given \( p \in M \) the fiber of \( \pi \) is the tangent space \( T_pM \). We know that the inclusion maps \( T_pM \to TM, p \in M \) are isometric embeddings in the
Riemannian sense and hence in our sense too. In particular the fibers of $\pi$ are uniformly properly embedded in $TM$. On the other hand given $p,q \in M$ $v \in T_pM$, and a piecewise smooth path $\gamma \subset M$ joining $p,q$ we can consider the parallel transport of $v$ along $\gamma$. This gives us a lift $\tilde{\gamma}(t) := (\gamma(t), v(t))$ of $\gamma$ in $TM$ joining $(p,v) \in T_pM$ to a point of $T_qM$. For the Sasaki metric $l(\tilde{\gamma}) = l(\gamma)$. This checks all the hypotheses of a metric bundle.

**Example 2.** Short exact sequence of groups. Given a short exact sequence of finitely generated groups $1 \to N \to G \to Q \to 1$ we have a naturally associated metric graph bundle. This is the main motivating example of metric graph bundles. One is referred to Example 1.8 of [MS12] for details. See also Example 4 below.

**Example 3.** Complexes of groups. For this example we refer to [Cor95]. Suppose $Y$ is a finite simplicial complex and $\mathbb{G}(Y)$ is a developable complex of groups defined over $Y$. Suppose $T$ is a maximal tree in the 1-skeleton of $Y$ and let $G = \pi_1(\mathbb{G}(Y),T)$ be the fundamental group of the complex of groups and let $\nu_T : \mathbb{G}(Y) \to G$ be the natural morphism. Let $B' = D(Y,\nu_T)$ be the development of $\mathbb{G}(Y)$ with respect to $\nu_T$. Now we assume the following properties:

1. All the face groups $G_\tau$ are finitely generated.
2. If $\sigma \subset \tau$ are faces then the corresponding homomorphism $G_\tau \to G_\sigma$ is a qi, i.e. it is an isomorphisms onto a finite index subgroup of $G_\sigma$.

Now we start with Eilenberg-McLane spaces of all the face groups which have finite 1-skeletons and build a complex of spaces $p : K \to Y$ after Corson (Theorem 2.5, [Cor95]) so that $G \simeq \pi_1(K)$ by Corollary 3 of [Cor95]. Now we take the universal cover $\pi_K : \tilde{K} \to K$ and collapse inverse image of each point of $\pi_K \circ p : \tilde{K} \to Y$ to get the universal complex $Y(p)$ and simplicial map $\tilde{p} : \tilde{K} \to Y(p)$ so that these maps fit into the commutative diagram:

$$
\begin{array}{ccc}
\tilde{K} & \xrightarrow{\tilde{p}} & Y(p) \\
\pi_K \downarrow & & \downarrow q \\
K & \xrightarrow{p} & Y
\end{array}
$$

**Figure 1.**

Next we restrict the map $\tilde{p}$ only to the 1-skeletons of the spaces $\tilde{K}$ and $Y(p)$. This gives us a metric graph bundle $\pi : \tilde{K}^1 \to Y(p)^1$ where for all vertex $v \in Y(p)^1$, $\pi^{-1}(v)$ is uniformly quasiisometric to the group $G_{q(v)}$.

**Definition 3.13.** (1) **Metric bundle morphisms** Suppose $(X_i, B_i, \pi_i)$, $i=1,2$ are metric bundles. A morphism from $(X_1, B_1, \pi_1)$ to $(X_2, B_2, \pi_2)$ (or simply from $X_1$ to $X_2$ when there is no possibility of confusion) consists of a pair of coarsely $L$-Lipschitz maps $f : X_1 \to X_2$ and $g : B_1 \to B_2$ for some $L \geq 0$ such that $\pi_2 \circ f = g \circ \pi_1$, i.e. the following diagram (Figure 2) is commutative.

(2) **Metric graph bundle morphisms** Suppose $(X_i, B_i, \pi_i)$, $i=1,2$ are metric graph bundles. A morphism from $(X_1, B_1, \pi_1)$ to $(X_2, B_2, \pi_2)$ (or simply from $X_1$ to $X_2$ when there is no possibility of confusion) consists of a pair of coarsely $L$-Lipschitz maps $f : \mathcal{V}(X_1) \to \mathcal{V}(X_2)$ and $g : \mathcal{V}(B_1) \to \mathcal{V}(B_2)$ for some $L \geq 0$ such that $\pi_2 \circ f = g \circ \pi_1$. 
We note that for any morphism \((f,g)\) from a metric (graph) bundle \((X_1,B_1,\pi_1)\) to a metric (graph) bundle \((X_2,B_2,\pi_2)\) we have \(f(\pi_1^{-1}(b)) \subset \pi_2^{-1}(g(b))\) for all \(b \in B_1\). We will denote by \(f_b : \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))\) the restriction of \(f\) to \(\pi_1^{-1}(b)\) for all \(b \in B_1\). We shall refer to these maps as the fiber maps of the morphisms. We also note that in the case of metric graph bundles coarse Lipschitzness is equivalent to Lipschitzness.

**Lemma 3.14.** Given \(k \geq 1, K \geq 1\) and \(L \geq 0\) there are constants \(L, K, K\) such that the following hold.

Suppose \((f,g)\) is a morphism of metric (graph) bundles as in the definition above. Then the following hold:

1. For all \(b \in B_1\) the map \(f_b : \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))\) is \(L\)-coarsely Lipschitz with respect to the induced length metrics on the fibers.
2. Suppose \(\gamma : I \to B_1\) is a dotted \((1,1)\)-quasigeodesic (or simply a geodesic in the case of a metric graph bundle) and suppose \(\tilde{\gamma}\) is a \(K\)-qi lift of \(\gamma\). If \(g\) is a \(K\)-qi-embedding then \(f \circ \tilde{\gamma}\) is a \(K\)-qi lift of \(g \circ \gamma\).

**Proof.** We shall check the lemma only for the metric bundle case because for the metric graph bundles the proofs are similar and in fact easier.

Suppose \(\pi_i : X_i \to B_i, i = 1, 2\) are \((\eta_i, c_i)\)-metric bundles.

1. Let \(b \in B_1\) and \(x, y \in \pi_1^{-1}(b)\) be such that \(d_\delta(x,y) \leq 1\). Since \(f\) is \(L\)-Lipschitz, \(d_{X_2}(f(x), f(y)) \leq L + Ld_{X_1}(x,y) \leq L + 2L \leq 2L\). Now, the fibers of \(\pi_2\) are uniformly properly embedded as measured by \(\eta_2\). Hence, \(d_\delta(b(f(x), f(y)) \leq \eta_2(2L)\). Therefore, by Lemma 2.3 the fiber map \(f_b : \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))\) is \(\eta_2(2L)\)-coarsely Lipschitz. Hence, \(L_k, L_k = \eta_2(2L)\) will do.

2. Let \(\gamma_2 = g \circ \gamma\) and \(\tilde{\gamma}_2 = f \circ \tilde{\gamma}\). Then clearly, \(\pi_2 \circ \tilde{\gamma}_2 = \gamma_2\) whence \(\tilde{\gamma}_2\) is a lift of \(\gamma_2\). By Lemma 2.3(1) \(\tilde{\gamma}_2 = f \circ \tilde{\gamma}\) is coarsely \((kL, kL)\)-Lipschitz. Hence, for all \(s, t \in I\) we have

\[
d_{X_2}(\tilde{\gamma}_2(s), \tilde{\gamma}_2(t)) \leq kL|s-t| + (kL + L).
\]

On the other hand, for \(s, t \in I\) we have

\[
d_{X_2}(\tilde{\gamma}_2(s), \tilde{\gamma}_2(t)) \geq d_{B_2}(\pi_2 \circ \tilde{\gamma}_2(s), \pi_2 \circ \tilde{\gamma}_2(t)) = d_{B_2}(\gamma_2(s), \gamma_2(t)).
\]

However, by Lemma 2.3(2) \(\gamma_2 = g \circ \gamma\) is a \((K, 2K)\)-qi embedding. Hence, we have

\[
d_{X_2}(\tilde{\gamma}_2(s), \tilde{\gamma}_2(t)) \geq d_{B_2}(\gamma_2(s), \gamma_2(t)) \geq -2K + \frac{1}{K}|s-t|.
\]
Therefore, it follows that $\tilde{\gamma}_2$ is a $K_{1.13} = \max\{2K_1kL + L\}$-qi lift of $\gamma_2$. □

The following theorem characterizes isomorphisms of metric (graph) bundles.

**Theorem 3.15.** If $(f, g)$ is an isomorphism of metric (graph) bundles as in the above definition then the maps $f, g$ are quasiisometries and all the fiber maps are uniform quasiisometries.

Conversely, if the map $g$ is a qi and the fiber maps are uniform qi then $(f, g)$ is an isomorphism.

**Proof.** We shall prove the theorem in the case of a metric bundle only. The proof in case of a metric graph bundle is very similar and hence we skip it.

If $(f, g)$ is an isomorphism then $f, g$ are qi by Lemma 2.21. We need to show that the fiber maps are quasiisometries.

Suppose $(f', g')$ is a coarse inverse of $(f, g)$ such that $d_{X_2}(f \circ f'(x_2), x_2) \leq R$ and $d_{X_1}(f \circ f(x_1), x_1) \leq R$ for all $x_1 \in X_1$ and $x_2 \in X_2$. It follows that for all $b_1 \in B_1, b_2 \in B_2$ we have $d_{b_1}(b_1, g(b_1)) \leq R$ and $d_{b_2}(b_2, g \circ g'(b_2)) \leq R$ since the maps $\pi_1, \pi_2$ are 1-Lipschitz. Suppose $f', g'$ are coarsely $L'$-Lipschitz. Let $L_1 = \eta_1(2L)$ and $L_2 = \eta_1(2L')$. Then for all $u \in B_1, f_u : \pi_1^{-1}(u) \to \pi_2^{-1}(g(u))$ is coarsely $L_1$-Lipschitz and for all $v \in B_2, f'_u : \pi_2^{-1}(v) \to \pi_1^{-1}(g'(v))$ is coarsely $L_2$-Lipschitz by Lemma 2.21.

Let $b \in B_1$. To show that $f_b : \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ is a uniform quasiisometry, it is enough by Lemma 2.21 to find a uniformly coarsely Lipschitz map $\pi_2^{-1}(g(b)) \to \pi_1^{-1}(b)$ which is uniform coarse inverse of $f_b$. We already know that $f'_g(b)$ is $L_2$-coarsely Lipschitz. Let $b_1 = g' \circ g(b)$. We also noted that $d_B(b, b_1) \leq R$. Hence, it follows by Corollary 3.10 and Corollary 3.11 that we have a $K_{b_1}(3R)$-qi $\phi_{b_1} : \pi_1^{-1}(b_1) \to \pi_1^{-1}(b)$ such that $d_{X_1}(x, \phi_{b_1}(x)) \leq 3c + 3cR$ for all $x \in \pi_1^{-1}(b_1)$. Let $h = \phi_{b_1} \circ f'_g(b)$. We claim that $h$ is a uniformly coarse Lipschitz, uniform coarse inverse of $f_b$. Since $f'_g(b)$ is $L_2$-coarsely Lipschitz and clearly $\phi_{b_1}$ is $K_{b_1}(3R)$-coarsely Lipschitz, it follows by Lemma 2.21 that $h$ is $(L_2K_{b_1}(3R) + K_{b_1}(3R))$-coarsely Lipschitz.

Moreover, for all $x \in \pi_1^{-1}(b)$ we have $d_{X_1}(x, h \circ f_b(x)) \leq d_{X_1}(x, f'_g(b) \circ f_b(x)) + d_{X_1}(f'_g(b) \circ f_b(x), h \circ f_b(x)) \leq R + 3c + 3cR$. Hence, $d_b(x, h \circ f_b(x)) \leq \eta_1(R + 3c + 3cR)$. Let $y \in \pi_2^{-1}(g(b))$. Then

$$d_{X_2}(y, f_b \circ h(y)) = d_{X_2}(y, f \circ \phi_{b_1} \circ f'(y)) \leq d_{X_2}(y, f \circ f'(y)) + d_{X_2}(f \circ f'(y), f \circ \phi_{b_1} \circ f'(y)) \leq R + L(3c + 3cR)$$

since $d_{X_2}(f'(y), \phi_{b_1} \circ f'(y)) \leq 3c + 3cR$. Hence, $d_{g(b)}(y, h \circ f(y)) \leq \eta_2(R + L(3c + 3cR))$. Hence by Lemma 2.21 $f_b$ is a uniform qi.

Conversely, suppose all the fiber maps of the morphism $(f, g)$ are $(\lambda, \epsilon)$-qi which are coarsely $R$-surjective and $g$ is a $(\lambda_1, \epsilon_1)$-qi which is $R_1$-surjective. Let $g'$ be a coarsely $(K, C)$-quasiisometric, $D$-coarse inverse of $g$ where $K = K_{2.2}(\lambda_1, \epsilon_1, R_1)$, $C = C_{2.2}(\lambda_1, \epsilon_1, R_1)$ and $D = D_{2.2}(\lambda_1, \epsilon_1, R_1)$. For all $u \in B_1$ let $f_u$ be a $D_1$-coarse inverse of $f_u : F_u \to F_{g(u)}$. We will define a map $f' : X_2 \to X_1$ such that $(f', g')$ is morphism from $X_2$ to $X_1$ and $f'$ is a coarse inverse of $f$ as follows.

For all $u \in B_2$ we define $f'_u : F_u \to F_{g'(u)}$ as the composition $f'_u = f_{g'(u)} \circ \phi_{u(g'(u))}$ where $\phi_{u(g'(u))}$ is a fiber identification map as constructed in the proof of Corollary 3.9. Collectively this defines $f'$. Now we shall check that $f'$ satisfies the desired properties.
(i) We first check that \((f', g')\) is a morphism. It is clear from the definition that \(\pi_1 \circ f' = g' \circ \pi_2\). Hence we will be done by showing that \(f'\) is coarsely Lipschitz. By Lemma 2.7 it is enough to show that for all \(u_2, v_2 \in B_2\) and \(x \in F_{u_2}, y \in F_{v_2}\) with \(d_{X_1}(x, y) \leq 1, d_{X_1}(f'(x), f'(y))\) is uniformly small. We note that \(d_{B_2}(u_2, v_2) \leq 1\). Let \(u_1 = g'(u_2)\) and \(v_1 = g'(v_2)\). Then \(\phi_{B_2}(u_1, v_1) \leq K + C, d_{B_2}(u_2, g(u_1)) \leq D\) and \(d_{B_2}(v_2, g(v_1)) \leq D\). This means \(d_{X_2}(x, \phi_{u_2g(u_1)(x)}) \leq 3Dc_2 + 3c_2\) and \(d_{X_2}(y, \phi_{u_2g(v_1)(y)}) \leq 3Dc_2 + 3c_2\) by Lemma 3.8 and Corollary 3.9.

Hence, \(d_{X_2}(\phi_{u_2g(u_1)(x)}, \phi_{u_2g(v_1)(y)}) \leq 1 + 6c_2 + 6Dc_2\). Let \(x_2 = \phi_{u_2g(u_1)(x)}, y_2 = \phi_{u_2g(v_1)(y)}, x_1 = f'(x_2) = \tilde{f}_g(u_1)(x_2)\) and \(y_1 = f'(y_2) = \tilde{f}_g(v_1)(y_2)\). Therefore, \(d_{X_2}(x_2, y_2) \leq 1 + 6c_2 + 6Dc_2 = R_2\), say and we want to show that \(d_{X_1}(x_1, y_1)\) is uniformly small. Let \(x'_2 = f(x_1) = f_{u_1}(x_1), y'_2 = f(y_1) = f_{v_1}(y_1)\). Then \(d_{X_2}(x_2, x'_2) \leq D_1\) and \(d_{X_2}(y_2, y'_2) \leq D_1\). Hence, \(d_{X_2}(x_2, y_2) \leq 2D_1\). Since \(d_{B_1}(u_1, v_1) \leq K + C\) there is a point \(y'_1 \in F_{u_1}\) such that \(d_{X_2}(x_1, y'_1) \leq (K + C)c_1 + c_1\). Hence, \(d_{X_2}(x_2, f(y'_1)) = ((K + C)c_1 + c_1)\). \(L + L\). Hence, \(d_{X_2}(f(y'_1), y'_2) \leq d_{X_2}(f(y'_1), x'_2) + d_{X_2}(x'_2, y'_2) \leq ((K + C)c_1 + c_1)\). \(L + L + 2D_1 + R_2\). This implies that \(d_{v_2}(f(y'_1), f(y'_2)) \leq \eta_2(((K + C)c_1 + c_1)\). \(L + 2D_1 + R_2\). Say. Since \(f_{v_1}\) is a \((\lambda, \epsilon)\)-qi we have \(-\epsilon + \tilde{h}_{v_1}(y_1, y'_1) \leq D_2\). Hence, \(d_{v_1}(y_1, y'_1) \leq (\epsilon + D_2)\). Then \(d_{v_1}(x_1, y_1) \leq (\epsilon + D_2)\). Thus, \(d_{X_1}(x_1, y_1) \leq (K + C)c_1 + c_1 + (\epsilon + D_2)\).

(ii) We already know that \(g'\) is a coarse inverse of \(g\). Hence \(d_{X_2}(x'_2, y'_2) = L(f \circ f')(x) \circ Id_{X_1} \neq \infty\) leaving the proof of \(d(f \circ f', Id_{X_1}) \neq \infty\) for the reader. Suppose \(b \in B_1\) and \(x \in \pi_1^{-1}(b)\). Then \(f'(f(x)) = \tilde{f}_g \circ \phi_{b(\phi_{g(b)} \circ g')(b)} \circ f_b(x)\). We want to show that \(d_{X_2}(x, f'(f(x)))\) is uniformly small. Let \(x = f'_g \circ f_{b(\phi_{g(b)} \circ g')(b)} \circ f_b(x) = f_{X_2}(f(x), \phi_{b(\phi_{g(b)} \circ g')(b)}(f_b(x))) + d_{X_2}(\phi_{b(\phi_{g(b)} \circ g')(b)}(f_b(x)), \phi_{b(\phi_{g(b)} \circ g')(b)}(f_b(x)))\). Now since \(d(y, \phi_{g'(b)} \circ f_{b(\phi_{g(b)} \circ g')(b)}(f_b(x))) \leq 3Dc_2 + 3c_2\). Let \(d_{B_2}(\phi_{g(b)} \circ g'(b), \phi_{b(\phi_{g(b)} \circ g')(b)}(f_b(x))) \leq D_1\). Thus \(d_{X_2}(f(x), f'(f(x))) \leq 3Dc_2 + 3c_2 + D_1\). Hence, it is enough to show that \(f\) is a proper embedding. Here is how this is proved. Suppose \(b, b' \in B, x \in \pi_1^{-1}(b)\) and \(x' \in \pi_1^{-1}(b')\). Suppose \(d_{X_2}(f(x), f(x')) \leq N\) for some \(N \geq 0\). This implies \(d_{B_2}(g(b), g(b')) = d_{B_1}(\pi_2 g f(x), \pi_2 g f(x')) \leq N\). Since \(g\) is a \((\lambda_1, \epsilon_1)\)-qi we have \(-\epsilon_1 + d_{B_2}(b, b')/\lambda_1 \leq N, i.e. d_{B_2}(b, b') \leq (N + \epsilon_1)\lambda_1 = N_1\). Say. Hence by Corollary 3.9 there is a point \(x'' \in \pi_1^{-1}(b')\) such that \(d_{X_2}(x, x'') \leq 3N_1c_1 + 3c_1\). Since \(f\) is coarsely \(L\)-Lipschitz we have \(d_{X_2}(f(x), f(x'')) \leq L(3N_1c_1 + 3c_1) + L\). It follows that \(d(f(x'), f(x'')) \leq d(f(x'), f(x)) + d(f(x), f(x'')) \leq N + L(N_1c_1 + c_1) = N_2\). Say. Hence, \(d_{B_2}(g(b'), g(b')) \leq \eta_2(N_2)\). Since \(f_b\) is a \((\lambda, \epsilon)\)-qi we have \(d_{X_2}(x, x'') \leq \lambda(\epsilon + \eta_2(N_2))\). Hence, \(d_{X_1}(x, x') \leq d_{X_2}(x, x') + d_{X_2}(x', x'') \leq 3N_1c_1 + 3c_1 + \lambda(\epsilon + \eta_2(N_2))\). This completes the proof. □

**Definition 3.16.** (Subbundle) Suppose \((X_i, B, \pi_i), i = 1, 2\) are metric (graph) bundles with the same base space \(B\). We say that \((X_1, B, \pi_1)\) is subbundle of \((X_2, B, \pi_2)\) or simply \(X_1\) is a subbundle of \(X_2\) if there is a metric (graph) bundle morphism \((f, g)\) from \((X_1, B, \pi_1)\) to \((X_2, B, \pi_2)\) such that all the fiber maps \(f_b, b \in B\) are uniform \(\epsilon\) embeddings and \(g\) is the identity map on \(B\) (resp. on \(V(B)\)).

The most important example of a subbundle that concerns us is that of ladders which we discuss in a later section. The following gives another way to construct metric (graph) bundle. We omit the proof since it is immediate.
**Lemma 3.17.** (Restriction bundle) Suppose $\pi : X \to B$ is a metric (graph) bundle and $A \subset B$ is a connected subset such that any pair of points in $A$ can be joined by a path of finite length in $A$ (resp. $A$ is a connected subgraph). Then the restriction of $\pi$ to $Y = \pi^{-1}(A)$ gives a metric (graph) bundle with the same parameters as that of $\pi : X \to B$ where $A$ and $Y$ are given the induced length metrics from $B$ and $X$ respectively.

Moreover, if $f : Y \to X$ and $g : A \to B$ are the inclusion maps then $(f, g) : (Y, A) \to (X, B)$ is a morphism of metric (graph) bundles.

**Definition 3.18.** (1) (Pullback of a metric bundle) Given a metric bundle $(X, B, \pi)$ and a coarsely Lipschitz map $g : B_1 \to B$ a pullback of $(X, B, \pi)$ under $g$ is a metric bundle $(X_1, B_1, \pi_1)$ together with a morphism $(f : X_1 \to X, g : B_1 \to B)$ such that the following universal property holds: Suppose $\pi_2 : Y \to B_1$ is another metric bundle and $(f', g)$ is a morphism from $Y$ to $X$. Then there is a coarsely unique morphism $(f', \text{Id}_{B_1})$ from $Y$ to $X_1$ making the following diagram commutative.

![Diagram](image)

**Figure 3.**

(2) (Pullback of a metric graph bundle) In the case of a metric graph bundle the diagram is replaced by one where we have the vertex sets instead of the whole spaces.

The following lemma follows by a standard argument.

**Lemma 3.19.** Suppose we have a metric bundle $(X, B, \pi)$ and a coarsely Lipschitz map $g : B_1 \to B$ for which there are two pullbacks i.e. metric bundles $(X_i, B_1, \pi_i)$ together with a morphisms $(f_i : X_i \to X, g : B_i \to B)$, $i = 1, 2$ satisfying the universal property of the Definition 3.18. Then there is a coarsely unique metric (graph) bundle isomorphism from $X_1$ to $X_2$.

With the above lemma in mind, in the context of Definition 3.18 we say that $X_1$ is the pullback of $X$ under $g$ or $f : X_1 \to X$ is the pullback of $X$ under $g : B_1 \to B$ when all the other maps are understood.

**Lemma 3.20.** Given $L \geq 0$ and functions $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ there is a function $\phi : [0, \infty) \to [0, \infty)$ such that the following hold:

Suppose we have the following commutative diagram of maps between metric spaces satisfying the properties (1)-(3) below.

1. All the maps (except possibly $f'$) are coarsely $L$-Lipschitz.
2. If $d_{B_1}(b, b') \leq N$ then $Hd(\pi_1^{-1}(b), \pi_1^{-1}(b')) \leq \phi_1(N)$ for all $b, b' \in B_1$ and $N \in [0, \infty)$. 


(3) The restrictions of $f$ on the fibers of $\pi_1$ are uniformly properly embedded as measured by $\phi_2$.

Then $d_Y(y, y') \leq R$ implies $d_X(f'(y), f'(y')) \leq \phi(R)$ for all $y, y' \in Y$ and $R \in [0, \infty)$. In particular, if $Y$ is a length space or the vertex set of a connected metric graph with restricted metric then $f'$ is coarsely $\phi(1)$-Lipschitz.

Moreover, $f'$ is coarsely unique, i.e. there is a constant $D > 0$ such that if $f'' : Y \to X_1$ is another map making the above diagram commutative then $d(f', f'') \leq D$.

Proof. Suppose $y, y' \in Y$ with $d_Y(y, y') \leq R$. Let $x = f'(y), x' = f'(y')$. Then

$$d_{B_1}(\pi_1(x), \pi_1(x')) = d_{B_1}(\pi_2(y), \pi_2(y')) \leq LR + L.$$ Let $b = \pi_2(y), b' = \pi_2(y')$.

Then $Hd(\pi_1^{-1}(b), \pi_1^{-1}(b')) \leq \phi_1(LR + L) = R_1$, say. Let $x'_1 \in \pi_1^{-1}(b')$ be such that $d_{X_1}(x, x'_1) \leq R_1$. Then $d_Y(f(x), f(x'_1)) \leq LR_1 + L$. On the other hand

$$d_X(f(x), f(x')) = d_X(f'(y), f'(y')) \leq LR + L.$$ By triangle inequality, we have

$$d_X(f(x'), f(x'_1)) \leq LR + L + LR_1 + L = 2L + RL + R_1L.$$ Hence, by the hypothesis (3) of the lemma $d_{X_1}(x', x'_1) \leq \phi_2(2L + RL + R_1L)$.

Thus $d_{X_1}(x, x') \leq d_{X_1}(x, x'_1) + d_{X_1}(x'_1, x'_1) \leq R_1 + \phi_2(2L + RL + R_1L)$. Hence, we may choose $\phi(t) = \phi_1(Lt + L) + \phi_2(2L + tL + L\phi_2(Lt + L))$.

In case $Y$ is a length space or the vertex set of a connected metric graph it follows by Lemma 2.3 $f'$ is coarsely $\phi(1)$-Lipschitz.

Lastly, suppose $f'' : Y \to X_1$ is another map making the diagram commutative. In particular we have $f'' = f \circ f' = f \circ f''$. Hence for all $y \in Y$ we have $f(f''(y)) = f(f'(y))$. Since $\pi_1(f'(y)) = \pi_1(f''(y)) = \pi_2(y)$ by the hypothesis (3) of the lemma it follows that $d_{X_1}(f'(y), f''(y)) \leq \phi_2(0)$. Hence $d(f', f'') \leq \phi_2(0)$.

□

Remark 7. We note that the condition (2) of the lemma above holds in case $\pi_1 : X_1 \to B_1$ is a metric (graph) bundle.

Proposition 3.21. (Pullbacks of metric bundles) Suppose $(X, B, \pi)$ is a metric bundle and $g : B_1 \to B$ is a Lipschitz map. Then there is a pullback.

More precisely the following hold: Suppose $X_1$ is the set theoretic pullback with the induced length metric from $X \times B_1$ and let $\pi_1 : X_1 \to B_1$ be the projection on the second coordinate and let $f : X_1 \to X$ be the projection on the first coordinate. Then (1) $\pi_1 : X_1 \to B_1$ is a metric bundle and $f$ is a coarsely Lipschitz map so that $(f, g)$ is a morphism from $X_1$ to $X$. (2) $f : X_1 \to X$ is the metric bundle pullback of $X$ under $g$. (3) All the fiber maps $f_b : \pi_1^{-1}(b) \to \pi^{-1}(g(b)), b \in B_1$ are isometries with respect to induced length metrics from $X_1$ and $X$ respectively.

Proof. By definition $X_1 = \{(x, t) \in X \times B_1 : g(t) = \pi(x)\}$. We put on it the induced length metric from $X \times B_1$. Let $\pi_1 : X_1 \to B_1$ be the restriction of the projection map $X \times B_1 \to B_1$ to $X_1$. We first show that $X_1$ is a length space. Suppose $g$ is $L$-Lipschitz. Let $(x, s), (y, t) \in X_1$. Let $\alpha$ be a rectifiable path joining $s, t$ in $B_1$. Then $g \circ \alpha$ is a rectifiable path in $B$ of length at most $l(\alpha)L$. By Lemma
Corollary 3.22. Suppose this path can be lifted to a rectifiable path in $X$ starting from $x$ and ending at some point say $z$ in $F_t$ such that the length of the path is at most $3c + 3cL(\alpha)$. By construction this lift is contained in $X_1$. Finally we can join $(y, t), (z, t)$ by a rectifiable path in $F_t$. This show that $(x, s)$ and $(y, t)$ can be joined in $X_1$ by a rectifiable path. This proves that $X_1$ is a length space. Now, since $\pi_1^{-1}(t) = \pi^{-1}(g(t))$ is uniformly properly embedded in $X$ for all $t \in B_1$ and $X$ is properly embedded in $X \times B_1$, $\pi_1^{-1}(t)$ is uniformly properly embedded in $X_1$ for all $t \in B_1$. The same argument also shows that any path in $B_1$ of length at most $cL$ can be lifted to a path of length at most $cL$ verifying the condition 2 of metric bundles.

Hence $(X_1, B_1, \pi_1)$ is a metric bundle. Let $f : X_1 \to X$ be the restriction of the projection map $X \times B_1 \to X$ to $X_1$. Clearly $f : X_1 \to X$ is a morphism of metric bundles. Finally, we check the universal property. If there is a metric bundle $\pi_2 : Y \to B_1$ and a morphism $(f^Y, g)$ from $Y$ to $X$ then there is a map $f' : Y \to X_1$ making the diagram \ref{eq:universal_property} commutative since we are working with the set theoretic pullback. That $f'$ is a coarsely unique, coarsely Lipschitz map now follows from Lemma \ref{lem:coarsely_unique_lipschitz}. In fact, condition (2) of that lemma follows from Lemma \ref{lem:coarsely_unique_lipschitz} since $\pi_1 : X_1 \to B_1$ is a metric bundle and (3) follows because fibers of metric bundles are uniformly properly embedded and in this case the restriction of $f, \pi_1^{-1}(b) \to \pi^{-1}(g(b)) \subset X$ is an isometry with respect to the induced path metric on $\pi_1^{-1}(b)$ and $\pi^{-1}(g(b))$ for all $b \in B_1$.

Corollary 3.22. Suppose $(X, B, \pi)$ is a metric bundle and $g : B_1 \to B$ is a Lipschitz map. Suppose $\pi_2 : X_2 \to B_1$ is an arbitrary metric bundle and $(f_2 : X_2 \to X, g)$ is a morphism of metric bundles. If $X_2$ is the pullback of $X$ under $g$ and $f_2 : X_2 \to X$ is the pullback map then for all $b \in B_1$ the fiber map $(f_2)_b : \pi_2^{-1}(b) \to \pi^{-1}(g(b))$ is a uniform quasiisometry with respect to the induced length metrics on the fibers of $\pi_2$ and $\pi$ respectively.

Proof. Suppose $X_1$ is the pullback of $X$ under $g$ as constructed in the proof of the proposition above. Then the fiber maps $f_b : \pi_1^{-1}(b) \to \pi^{-1}(g(b))$ are isometries with respect to the induced metrics on the fibers of $\pi_1$ and $\pi$ respectively. On the other hand by Lemma \ref{lem:coarsely_unique_lipschitz} there is a coarsely unique metric bundle isomorphism $(h, Id)$ from $X_2$ to $X_1$ making the diagram \ref{eq:universal_property} below commutative.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{}
\end{figure}

Now, by Theorem \ref{thm:coarsely_unique_lipschitz} the fiber maps $h_b : \pi_2^{-1}(b) \to \pi_1^{-1}(b)$ are uniform quasi-isometries with respect to the induced length metrics on the fibers of $\pi_2$ and $\pi_1$ respectively. Since $(f_2)_b = f_b \circ h_b$ for all $b \in B_1$ are done by Lemma \ref{lem:coarsely_unique_lipschitz}.

\end{proof}
Example 4. Suppose \((X, B, \pi)\) is a metric bundle and \(B_1 \subset B\) which is path connected and such that with respect to the path metric induced from \(B\), \(B_1\) is a length space. Let \(X_1 = \pi^{-1}(B_1)\) be endowed with the induced path metric from \(X\). Let \(\pi_1 : X_1 \rightarrow B_1\) be the restriction of \(\pi\) to \(X_1\). Let \(g : B_1 \rightarrow B\) and \(f : X_1 \rightarrow X\) be the inclusion maps. It is clear that \((X_1, B_1, \pi_1)\) is a metric bundle and also that \(X_1\) is the pullback of \(g\).

Remark 8. The notion of morphisms of metric bundles was implicit in the work of Whyte ([Why10]). Along the line of [Why10] one can define a more general notion of metric bundles by relaxing the hypothesis of length spaces. In that category of spaces pullbacks should exist under any coarsely Lipschitz maps. However, we do not delve into it here.

Proposition 3.23. (Pullbacks for metric graph bundles) Suppose \((X, B, \pi)\) is an \(\eta\)-metric graph bundle, \(B_1\) is a metric graph and \(g : \mathcal{V}(B_1) \rightarrow \mathcal{V}(B)\) is a coarsely \(L\)-Lipschitz map for some constant \(L \geq 1\). Then there is a pullback \(\pi_1 : X_1 \rightarrow B_1\) of \(g\) such that all the fiber maps \(f_s : \pi_1^{-1}(b) \rightarrow \pi^{-1}(g(b)), b \in \mathcal{V}(B_1)\) are isometries with respect to induced length metrics from \(X_1\) and \(X\) respectively.

Proof. The proof is a little long. Hence we break this into steps for the sake of clarity.

Step 1. Construction of \(X_1\) and \(\pi_1 : X_1 \rightarrow B_1\) and \(f : \mathcal{V}(X_1) \rightarrow \mathcal{V}(X)\). We first construct a metric graph \(X_1\), a candidate for the total space of the bundle. The vertex set of \(X_1\) is the disjoint union of the vertex sets of \(\pi^{-1}(g(b)), b \in \mathcal{V}(B_1)\). There are two types of edges. First of all for all \(b \in \mathcal{V}(B_1)\), we take all the edges appearing in \(\pi^{-1}(g(b))\). In other words, the full subgraph \(\pi^{-1}(g(b))\) is contained in \(X_1\). Let us denote that by \(F_b\). For all adjacent vertices \(s, t \in B_1\) we introduce some other edges with one end point in \(F_s\) and the other in \(F_t\). We note that \(F_s, F_t \subset X_1\) are identical copies of \(F_{g(s)}\) and \(F_{g(t)}\) respectively. Let \(f_s : F_s \rightarrow F_{g(s)}\) denote this identification. Let \(e\) be an edge joining \(s, t\) and let \(\alpha\) be a geodesic in \(B\) joining \(g(s), g(t)\). Now for each \(x \in F_s\) we lift the path \(\alpha\) starting from \(f_s(x)\) isometrically by Lemma 3.8(1) to say \(\tilde{\alpha}\). For each such lift we join \(x\) by an edge to \(y \in \mathcal{V}(F_t)\) iff \(f_s(y) = \tilde{\alpha}(g(t))\). This completes the construction of \(X_1\). We note that \(d_B(g(s), g(t)) \leq 2L\) and hence \(d(\tilde{\alpha}) \leq 2L\) too. Now we define \(f : \mathcal{V}(X_1) \rightarrow \mathcal{V}(X)\) by setting \(f(x) = f_{\pi_1(x)}(x)\) for all \(x \in \mathcal{V}(X_1)\). It is clear that this map is 2\(L\)-Lipschitz.

Step 2. \(\pi_1 : X_1 \rightarrow B_1\) is a metric graph bundle and \((f, g)\) is a morphism. We need to verify that the fibers are uniformly properly embedded in \(X_1\) so that \(X_1\) is a metric graph bundle. Suppose \(x, y \in F_s\) and \(d_{X_1}(x, y) \leq D\). Let \(\alpha\) be a (dotted) geodesic in \(X_1\) joining \(x, y\). Then \(f \circ \alpha\) is a (dotted) path of length at most \(2LD\). Thus \(d_X(f(x), f(y)) \leq 2LD\). Since \(X\) is an \(\eta\)-metric graph bundle the graph \(d_{g(s)}(f(x), f(y)) \leq \eta(2LD)\). Since \(f\) is an isometry when restricted to \(F_s\) we have \(d_s(x, y) \leq \eta(2LD)\). This proves that \(X_1\) is a metric graph bundle over \(B_1\).

On the other hand, \(f\) is 2\(L\)-Lipschitz by step 1 and \(g\) is coarsely \(L\)-Lipschitz by hypothesis. It is also clear that \(\pi \circ f = g \circ \pi_1\) by the definition of \(f\). Thus \((f, g)\) is a morphism of metric graph bundles from \(X_1\) to \(X\).

Step 3. \(X_1\) is a pullback. Now we check that \(X_1\) is a pullback of \(X\) under \(g\). Suppose \(\pi_2 : Y \rightarrow B_1\) is a metric graph bundle and \((f^Y, g)\) is a morphism of metric graph bundles from \(Y\) to \(X\) where \(f^Y\) is coarsely \(L_1\)-Lipschitz We need to find a coarsely unique, coarsely Lipschitz map \(f^Y : \mathcal{V}(Y) \rightarrow \mathcal{V}(X_1)\) such that
In this section we recall some results from [MS12] and also add a few of our own which are going to be useful for the proof of our main theorem in the next section.
 Especially some of the results which were stated for geodesic metric spaces in \[\text{MS12}\] but whose proofs require little to hold true for length spaces are mentioned here.

4.1. Metric graph bundles arising from metric bundles. An analogue of the following result is proved in \[\text{MS12}\] (see Lemma 1.17 through Lemma 1.21 in \[\text{MS12}\]). We give an independent and relatively simpler proof here. We also construct an approximating metric graph bundle morphism starting with a given metric bundle morphism. However, one disadvantage of our construction is that the metric graphs so obtained are never proper.

**Proposition 4.1.** Suppose \(\pi' : X' \to B'\) is an \((\eta, c)\)-metric bundle. Then there is a metric graph bundle \(\pi : X \to B\) along with quasiisometries \(\psi_B : B' \to B\) and \(\psi_X : X' \to X\) such that (1) \(\pi \circ \psi_X = \psi_B \circ \pi'\) and (2) for all \(b \in B'\) the map \(\psi_X\) restricted to \(\pi'^{-1}(b)\) is a uniform quasiisometry onto \(\pi^{-1}(\psi_B(b))\).

Moreover, the maps \(\psi_X, \psi_B\) have coarse inverses \(\phi_X, \phi_B\) respectively making the following diagram commutative:

\[
\begin{array}{ccc}
X' & \xrightarrow{\psi_X} & X \\
\pi' \downarrow & & \downarrow \pi \\
B' & \xleftarrow{\phi_X} & B
\end{array}
\]

**Figure 6.**

**Proof.** For the proof we use the construction of Lemma 2.7. We shall briefly recall the construction of the spaces. We define \(V(B) = B'\) and \(s, t \in V(B)\) are connected by an edge if and only if \(s \neq t\) and \(d_{B'}(s, t) \leq 1\). This defines the graph. We also have a natural map \(\psi_B : B' \to B\) which is just the inclusion map when \(B'\) is identified with the vertex set of \(B\). To define \(X\), we take \(V(X) = X'\). Edges are of two types.

**Type 1 edges:** For all \(s \in B', x, y \in \pi'^{-1}(s)\) are connected by an edge if and only if \(d_B(x, y) \leq 1\).

**Type 2 edges:** Then if \(s \neq t \in B', x \in \pi'^{-1}(s)\) and \(y \in \pi'^{-1}(t)\) then \(x, y\) are connected by an edge if and only if \(d_B'(s, t) \leq 1\) and \(d(x, y) \leq c\).

The map \(\psi_X : X' \to X\) is defined as before to be the inclusion map. By Lemma 2.7 \(\psi_B\) is a qi. We also note that \(\pi \circ \psi_X = \psi_B \circ \pi'\). We need to verify that \(\psi_X\) is a qi. For that it is enough to produce Lipschitz coarse inverses \(\phi_X, \phi_B\) as claimed in the second part of the proposition and then apply Lemma 2.2 since it is clear that \(\psi_X\) is 1-Lipschitz. We first choose a coarse inverse \(\phi_B\) of \(\psi_B\) as follows. On \(V(B)\) it is simply the identity map. The interior of each edge is then sent to one of its end points. The map \(\phi_X\) on \(V(X)\) is also defined as the identity map. The interior of a type 1 edge is sent to one of its end points. Then interior of each type 2 edge \(e = [x, y]\) is sent to one of the end points \(x\) or \(y\) according as the edge \(\pi(e)\) is mapped by \(\phi_B\) to \(\pi(x)\) or \(\pi(y)\) respectively. It follows that the diagram in Figure 6 commutes. We just need to check that \(\phi_X\) is coarsely Lipschitz, since \(\phi_B, \phi_X\) are inverses of \(\psi_B, \psi_X\) respectively on a 1-dense subset they will be coarse.
inverse automatically. However, by Lemma 2.5 it is enough to show that edges are mapped to small diameter sets. This is again clear. In fact, the image of an edge is diameter at most $c$. Let $s \in B'$ and $\psi_B(s) = t \in B$. By the construction of $X$, $\psi_X$ restricted to $\pi'^{-1}(s)$ is mapped to $\pi^{-1}(t)$ such that the vertex set of $\pi^{-1}(t)$ is $\pi^{-1}(s)$ and there exists an edge joining any pair of elements $x, y \in \pi^{-1}(t)$ if and only if $d_s(x, y) \leq 1$. Then, by the construction in Lemma 2.7, $\psi_X$ restricted to $\pi'^{-1}(s)$ is a $(1, 1)$-quasiisometry. Finally, we need to check that $(X, B, \pi)$ is a metric graph bundle. Let $s \in B$ and $x, y \in \pi^{-1}(s)$ such that $d_X(x, y) \leq M$ for some $M > 0$. Since $\phi_X$ is a quasiisometry, $d_{X'}(x, y) \leq M'$, where $M' > 0$ depends on $M$ and $\phi_X$. Since $\pi'^{-1}(\phi_B(s))$ is properly embedded in $X'$ as measured by $\eta$, we have $d_{\phi_B(s)}(x, y) \leq \eta(M')$. Now, using the above fact that $\pi'^{-1}(\phi_B(s))$ is $(1, 1)$-quasiisometric to $\pi^{-1}(s)$, we have $d_s(x, y) \leq \eta(M') + 1$. Hence, $\pi^{-1}(s)$ is uniformly properly embedded in $X$. Next we check the condition (2) of Definition 4.4. Suppose $s, t \in \mathcal{V}(B)$ are adjacent vertices. Then, $d_B(s, t) \leq 1$. Let $\alpha$ be a path in $B'$ joining $s, t$ with $l_B(\alpha) \leq 1$. Then, for any $x \in \pi'^{-1}(s), \alpha$ can be lifted to a path of length at most $c$, joining $x$ to some $y \in \pi'^{-1}(t)$. Then there exists an edge joining $x$ and $y$ in $X$, which is a lift of the edge joining $s$ and $t$ in $B$.

**Remark 9.** We shall refer to the metric graph bundle $X$ obtained from $X'$ as the canonical metric graph bundle associated to the bundle $X$. Since we are working with length metric spaces some of the machinery of [MS12] may not apply directly. The above proposition then comes to the rescue. We sometimes modify our definitions suitably to make things work. Consequently all the results proved for metric graph bundles have their close analogs in metric bundles. We shall make this precise for instance in Proposition 4.3 and Definition 4.4.

**Approximating a metric bundle morphism**

Suppose $\pi' : X' \to B'$ is a metric bundle and $g : A' \to B'$ is a Lipschitz map. Suppose $Y'$ is the pullback of the bundle under the map $g$ as constructed in the proof of Proposition 3.21 i.e. $Y'$ is also the set theoretic pullback. Let $g^*\pi' : Y \to A'$ is the corresponding bundle projection map and $f : Y' \to X'$ is the pullback map. Suppose we use the recipe of the above proposition to construct metric graph bundles $\pi_X : X \to B$, $\pi_Y : Y \to A$ with quasiisometries $\psi_A : A' \to A$, $\psi_B : B' \to B$, $\psi_Y : Y' \to Y$ and $\psi_X : X' \to X$ such that $\pi_Y \circ \psi_Y = \psi_A \circ g^*\pi'$ and $\pi_X \circ \psi_X = \psi_B \circ \pi'$.

Suppose $\phi_X, \phi_B, \phi_Y, \phi_A$ are the coarse inverses (as constructed in the proposition above) of $\psi_X, \psi_B, \psi_Y,$ and $\psi_A$ respectively. We then have a commutative diagram:

![Figure 7](image.png)

Let $\tilde{f}, \tilde{g}$ denote the restrictions of $\psi_X \circ f \circ \phi_Y$ and $\psi_B \circ g \circ \phi_A$ on the vertex sets of $Y$ and $A$ respectively.
**Proposition 4.2.** (1) The pair of maps $(\bar{f}, \bar{g})$ gives a morphism of metric graph bundles from $Y$ to $X$.

Moreover, if $Y'$ is the pullback of $X'$ under $g$ and $f$ is the pullback map then $Y$ is the pullback of $X$ under $\bar{g}$ and $\bar{f}$ is the pullback map.

(2) In case, $X', Y'$ are hyperbolic then $f$ admits a CT map if and only if so does $\bar{f}$.

**Proof.** (1) Since all the maps in consideration, i.e. $\psi_X, f, \phi_Y, \psi_B, g, \phi_A$ are coarsely Lipschitz the maps $\bar{f}, \bar{g}$ are also coarsely Lipschitz by Lemma 2.3(1). It also follows that $\pi_X \circ \bar{f} = \bar{g} \circ \pi_Y$. Thus $(\bar{f}, \bar{g})$ is a morphism.

![Figure 8.](image)

Suppose $Y'$ is a the pullback of $X'$ under $g$. To show that $Y$ is the pullback of $X$ we need to verify the universal property. Suppose $\pi_1 : Y_1 \to A$ is any metric bundle and $f_1 : V(Y_1) \to V(X)$ is a coarsely Lipschitz map such that the pair $(f_1, \bar{g})$ is a morphism of metric graph bundles from $Y_1$ to $X$. We note that $\pi' \circ (\phi_X \circ f_1) = g \circ (\phi_A \circ \pi_1)$. Since $Y'$ is a set theoretic pullback there is a unique map $f_2 : V(Y_1) \to Y'$ making the whole diagram below commutative.

Now, by Lemma 2.3(1) the maps $\phi_X \circ f_1$ and $\phi_A \circ \pi_1$ are coarsely Lipschitz. Hence, it follows by Lemma 3.20 and Remark 7 that the map $f_2$ is coarsely Lipschitz. Let $h = \psi_Y \circ f_2$. Then $h$ is coarsely Lipschitz by Lemma 2.3(1) and we have $f \circ h = f_1$ and $\pi' \circ h = \pi_1$. Hence, $(h, Id_A)$ is a morphism from $Y_1$ to $Y$. Finally coarse uniqueness of $h$ follows from Lemma 3.20.

(2) This is a simple application of Lemma 2.51.

**4.2. Metric bundles with hyperbolic fibers.** For the rest of this section we shall assume that all our metric (graph) bundles $\pi : X \to B$ have the following property:

(*) Each of the fibers $F_b, b \in B$ (resp. $b \in V(B)$) is a $\delta'$-hyperbolic metric space with respect to the path metric $d_b$ induced from $X$.

We will refer to this by saying that the metric (graph) bundle has uniformly hyperbolic fibers. Moreover, the following property is crucial for the existence of (global) qi sections.

(**) For all $b \in B$ the barycenter map $\phi_b : \partial^{\delta} F_b \to F_b$ is coarsely $N$-surjective for some constant $N \geq 0$ independent of $b$.

**Proposition 4.3.** ([MST12] Proposition 2.10, Proposition 2.12] Global qi sections for metric (graph) bundles: For all $\delta', c \geq 0, N \geq 0$ and $\eta : [0, \infty) \to [0, \infty)$ there exists $K_0 = K_0(c, \eta, \delta', N)$ such that the following holds.


Suppose \( p : X' \to B' \) is an \((\eta, c)\)-metric bundle or an \(\eta\)-metric graph bundle satisfying (*) and (**). Then there is a \(K_0\)-qi section over \(B'\) through each point of \(X'\) (where we assume \(c = 1\) for the metric graph bundle).

Proof. We shall briefly indicate a proof for the metric bundle case assuming the proposition for metric graph bundles. Suppose \(X'\) is a metric bundle over \(B'\) with the properties mentioned in the proposition and suppose \(X \to B\) is the canonical metric graph bundle associated to \(X'\). Since any length space is uniformly quasi-isometric to a metric graph by Lemma 2.7 and quasiisometries induce bijection of the boundaries of hyperbolic spaces, by Lemma 2.51(4), it follows the metric graph bundle satisfies the same properties (1) and (2), i.e. the fibers are uniformly hyperbolic and the barycenter maps are uniformly coarsely surjective. Hence by the existence of qi sections in a metric graph bundle through any point \(x \in X\) there is a uniform qi section \(\Sigma\) over \(B\). Now, clearly \(\phi_X(\Sigma)\) is a uniform qi section through \(x\) in \(X'\) where \(\phi_X : X \to X'\) is as in Proposition 4.1.

Convention 4.4. (1) Note that \(\phi_X(\Sigma) = \Sigma\) since \(\phi_X\) is the identity map when restricted to \(\mathcal{V}(X)\). We shall refer to it as a qi section of the metric graph bundle transported to the metric bundle.

(2) Whenever we talk about a \(K\)-qi section in a metric bundle we shall mean that it is the transport of a \(K\)-qi section contained in the associated canonical metric graph bundle.

Definition 4.5. (MS12, Definition 2.13) Suppose \(\Sigma_1\) and \(\Sigma_2\) are two \(K\)-qi sections of the metric graph bundle \(X\). For each \(b \in \mathcal{V}(B)\) we join the points \(\Sigma_1 \cap F_b, \Sigma_2 \cap F_b\) by a geodesic in \(F_b\). We denote the union of these geodesics by \(L(\Sigma_1, \Sigma_2)\), and call it a K-ladder (formed by the sections \(\Sigma_1\) and \(\Sigma_2\)).

For a metric bundle by a ladder we will mean one transported from its canonical metric graph bundle.

The following are the most crucial properties of a ladder summarized from MS12.

Proposition 4.6. Given \(K \geq 0, \delta \geq 0\) there are \(C = C(K) \geq 0, R = R(K, \delta, K) \geq 0\) such that the following holds:

Suppose \(\pi : X \to B\) is an \(\eta\)-metric graph bundle. Suppose \(\Sigma_1, \Sigma_2\) are two \(K\)-qi sections in \(X\) and \(L = L(\Sigma_1, \Sigma_2)\) is the ladder formed by them. Then the following hold.

(1) (Ladders are coarse Lipschitz retracts) There is a coarsely \(C\)-Lipschitz retraction \(\pi_L : X \to L\) defined as follows:

For all \(x \in X\) we define \(\pi_L(x)\) to be a nearest point projection of \(x\) in \(F_\pi(x)\) on \(L \cap F_\pi(x)\).

(2) Given a \(k\)-qi section \(\gamma\) in \(X\) over a geodesic in \(B\), \(\pi_L(\gamma)\) is a \((C + 2k)\)-qi section in \(X\) contained in \(L\) over the same geodesic in \(B\).

(3) (QI sections in ladders) If \(X\) satisfies (**) of the Proposition 4.3 then through any point of \(L\) there is \((1 + 2k)\)-qi section contained in \(L\).

(4) (Quasiconvexity of ladders) The \(R\)-neighborhood of \(L\) is (i) connected and (ii) uniformly qi embedded in \(X\).

In particular if \(X\) is \(\delta\)-hyperbolic then \(L\) is \(K(\delta, \delta)\)-quasiconvex in \(X\).

Proof. (1) is stated as Theorem 3.2 in MS12. (2), (3) are immediate from (1) or one can refer to Lemma 3.1 of MS12. (4) is proved in Lemma 3.6 in MS12.
assuming (**). However, we briefly indicate the argument here without assuming (**).

4(i) Suppose \( b, b' \in B \), \( d_B(b, b') = 1 \). Let \( x \in L \cap F_b \). Then there is a point \( x' \in F_{b'} \) such that \( d(x, x') = 1 \). Hence, \( d(\pi_L(x), \pi_L(x')) = d(x, \pi_L(x')) \leq 2C \). If we define \( R = 2C \) then clearly the \( R \)-neighborhood of \( L \) is connected.

4(ii) We first claim that the \( N_R(L) = Y \) say, is also properly embedded in \( X \). Suppose \( x', y' \in Y \) with \( d_X(x', y') \leq N \). Let \( x, y \in L \) be such that \( d(x, x') \leq R, d(y, y') \leq R \). Then \( d(x, y) \leq 2R + N \). Hence, \( d(\pi(x), \pi(y)) \leq 2R + N \). Let \( \alpha \) be a geodesic in \( B \) joining \( \pi(x), \pi(y) \). Then by Lemma 3.3 there is a geodesic lift \( \tilde{\alpha} \) of \( \alpha \) starting from \( x \). It follows that for all adjacent vertices \( b_1, b_2 \in \alpha \) we have \( d(\pi_L(\tilde{\alpha})(b_1), \pi_L(\tilde{\alpha})(b_2)) \leq 2C \). Hence, the length of \( \pi_L(\tilde{\alpha}) \) is at most \( 2C(2R + N) \). Hence, \( d(y, \tilde{\alpha}(\pi(y))) \leq d(x, y) + d(\pi_L(\tilde{\alpha})) \leq 2R + N + l(\pi_L(\tilde{\alpha})) \leq 2R + N + 2C(2R + N) \). Hence, \( d_{\pi}(y, \tilde{\alpha}(\pi(y))) \leq \eta(2R + N + 4CR + 2CN) \). Since \( \pi_L(\tilde{\alpha}) \in Y \), \( d_Y(x, y) \leq d_{\pi}(y, \tilde{\alpha}(\pi(y))) + l(\pi_L(\tilde{\alpha})) \leq \eta(2R + N + 4CR + 2CN) + 4CR + 2CN \). Hence, \( d_Y(x', y') \leq 4CR + 2CN + \eta(2R + N + 4CR + 2CN) \).

Finally we prove the qi embedding. Let \( f(N) = \eta(2R + N + 4CR + 2CN) + 4CR + 2CN \) for all \( N \in \mathbb{N} \). Given \( x, y \in L \), \( d_X(x, y) = n \) and a geodesic \( \gamma : [0, n] \to X \) joining them. By the proof of (4)(i) we have \( d_L(\pi_L(\gamma(i)), \pi_L(\gamma(i + 1))) \leq f(2C) \) for all \( 0 \leq i \leq n - 1 \) whence \( d_L(x, y) \leq nf(2C) = f(2C)d_X(x, y) \). Clearly \( d_X(x, y) \leq d_L(x, y) \). This proves the qi embedded part.

It follows that for all \( x, y \in L \) a geodesic joining \( x, y \) in \( Y \) is a \( (f(2C), 0) \)-quasigeodesic in \( X \). Since \( X \) is \( \delta \)-hyperbolic stability of quasigeodesics imply that \( L \) is uniformly quasiconvex. In fact we can take \( K_{10}(\delta, K) = R + L_{2.23} \delta, f(2C), 0) \).

\[ \textbf{Remark 10.} \] Part (3) and (4) are clearly true for metric bundles also which satisfy the corresponding properties (*) and (**).

The following corollary is immediate.

\[ \textbf{Corollary 4.7. (Ladders form subbundles)} \] Suppose \( \pi : X \to B \) is an \( \eta \)-metric graph bundle satisfying (*) and (**). Let \( C, R \) be as in the previous proposition. Suppose \( L = L(\Sigma_1, \Sigma_2) \) is a \( K \)-ladder. Consider the metric graph \( Z \) obtained from \( L \) by introducing some extra edges as follows: Suppose \( b, b' \in B \) are adjacent vertices then for all \( x \in L \cap F_b, x' \in L \cap F_{b'} \) we join \( x, x' \) by an edge iff \( d_X(x, x') \leq C + 2KC \). Let \( \pi_Z : Z \to B \) be the simplicial map such that \( \pi = \pi_Z \) on \( \mathcal{V}(Z) \) and the extra edges are mapped isometrically to edges of \( B \).

Then \( Z \) is a metric graph bundle and the natural map \( Z \to X \) gives a subbundle of \( X \) which is also a (uniform) qi onto \( N_R(L) \) and hence a (uniform) qi embedding in \( X \).

In the next section of the paper we will exclusively deal with bundles \( \pi : X \to B \) which are hyperbolic satisfying (*) and (**) and we will need to understand geodesics in \( X \). Since ladders are quasiconvex we look for quasigeodesics contained in ladders. The following lemma is the last technical piece of information needed for that purpose.

\[ \textbf{Definition 4.8.} \] Suppose \( X \) is a metric graph bundle over \( B \) and suppose \( \Sigma_1, \Sigma_2 \) are any two qi sections.

(1) \( \text{Neck of ladders (M02 Definition 2.16)}. \) Suppose \( R \geq 0 \). Then \( U_R(\Sigma_1, \Sigma_2) = \{ b \in B : d_b(\Sigma_1 \cap F_b, \Sigma_2 \cap F_b) \leq R \} \) is called the \( R \)-neck of the ladder \( L(\Sigma_1, \Sigma_2) \).
For a metric bundle $R$-neck of a ladder will be defined to be the one transported from its canonical metric graph bundle, i.e. the image under $\phi_B$.

(2) **Girth of ladders** ([MS12 Definition 2.15]). The quantity $\min\{d_b(\Sigma_1 \cap F_b, \Sigma_2 \cap F_b) : b \in B\}$ is called the girth of the ladder $L(\Sigma_1, \Sigma_2)$ and it will be denoted by $d_h(\Sigma_1, \Sigma_2)$.

The significance of necks of ladders is contained in the following lemma. However, we first need the following.

**Definition 4.9.** ([MS12 Definition 1.12]) Suppose $p : X \to B$ is a metric graph bundle. We say that it satisfies a **flaring condition** if for all $k \geq 1$, there exist $\nu_k > 1$ and $n_k, M_k \in \mathbb{N}$ such that the following holds:

Let $\gamma : [-n_k, n_k] \to B$ be a geodesic and let $\gamma_1$ and $\gamma_2$ be two $k$-qi lifts of $\gamma$ in $X$. If $d_{\gamma_{1(0)}}(\gamma_{1(0)}(0), \gamma_{2(0)}(0)) \geq M_k$, then we have

$$d_{\nu_k d_{\gamma_{1(0)}}}(\gamma_{1(0)}(0), \gamma_{2(0)}(0)) \leq \max\{d_{\gamma_{1(n_k)}}(\gamma_{1(n_k)}), \gamma_{2(n_k)}), d_{\gamma_{2(-n_k)}}(\gamma_{1(-n_k)}), \gamma_{2(-n_k)}\}.$$ 

**Lemma 4.10.** (Thick neck of a ladder is quasiconvex, [MS12 Lemma 2.18]) Let $X$ be an $\eta$-metric graph bundle over $B$ satisfying $(M_k, \nu_k, n_k)$-flaring condition for all $k \geq 1$. Then for all $c_1 \geq 1$ and $R > 1$ there are constants $D_{\ref{lem:thick-neck}}(c_1, R)$ and $D_{\ref{lem:thick-neck}}(c_1)$ such that the following holds:

Suppose $\Sigma_1, \Sigma_2$ are two $c_1$-qi sections of $B$ in $X$ and let $L \geq \max\{M_{\ref{lem:thick-neck}}, d_h(\Sigma_1, \Sigma_2)\}$.

1. Let $\gamma : [t_0, t_1] \to B$ be a geodesic, $t_0, t_1 \in \mathbb{Z}$, such that
   a) $d_{\gamma(t_0)}(\Sigma_1 \cap F_\gamma(t_0), \Sigma_2 \cap F_\gamma(t_0)) = LR$.
   b) $\gamma(t_1) \in U_L := U_{\Sigma_1}(\Sigma_1, \Sigma_2)$ but for all $t \in [t_0, t_1] \cap \mathbb{Z}$, $\gamma(t) \not\in U_L$.

   Then the length of $\gamma$ is at most $D_{\ref{lem:thick-neck}}(c_1, R)$.

2. $U_L$ is $D_{\ref{lem:thick-neck}}(c_1)$ quasi-convex in $B$.

3. If $d_h(\Sigma_1, \Sigma_2) \geq M_{\ref{lem:thick-neck}}$, then the diameter of the set $U_L$ is at most $D_{\ref{lem:thick-neck}}(c_1, L)$.

**Definition 4.11.** (Small girth ladders) Given two $K$-qi sections $\Sigma_1, \Sigma_2$ in a metric graph bundle satisfying the flaring condition the ladder $L(\Sigma_1, \Sigma_2)$ is called a small girth ladder if $U_L(\Sigma_1, \Sigma_2) \neq \emptyset$ where $L = M_K$.

**Remark 11.** Suppose $X' \to B'$ is a metric bundle and $X \to B$ is its canonical metric graph bundle approximation. Assume flaring condition holds for $X$. This will be the case for instance when $X$ or equivalently $X'$ is hyperbolic as will be discussed in the next section. In such a case, a small girth ladder in $X'$ is by definition the transport of a small girth ladder from $X$.

Next we find a relation between the girth of a ladder $L(\Sigma_1, \Sigma_2)$ and $d(\Sigma_1, \Sigma_2)$. Suppose $\pi : X \to B$ is an $\eta$-metric graph bundle satisfying (*).

**Lemma 4.12.** Given $D \geq 0, K \geq 1$ there is $R = R_{\ref{lem:small-girth}}(D, K)$ such that the following holds.

Suppose $\Sigma$ is a $K$-qi section in $X$ and $x \in X$. Let $b = \pi(x)$. Then $d(x, \Sigma) \geq D$ if $d_b(x, \Sigma \cap F_b) \geq R$.

**Proof.** Suppose $y \in \Sigma$ a nearest point from $x$. Let $\alpha \subset \Sigma$ be the lift of a geodesic $[b, \pi(y)]$ joining $b$ to $\pi(y)$ joining $y$ to $\Sigma \cap F_b$. We note that $d_B(b, \pi(y)) \leq d(x, y)$. Hence, $d(y, \alpha(b)) \leq Kd(x, y) + K$. Therefore, $d(x, \alpha(b)) \leq d(x, y) + d(y, \alpha(b)) \leq (K + 1)d(x, y) + K$. This implies $d(x, y) \geq \frac{1}{K+1}d(x, \alpha(b))$ since all distances are intergers in this case. Now fiber of $X$ are properly embedded as measured by
Now we suppose that \( \pi \) is a geodesic metric bundle or a metric graph bundle satisfying (H1, H2, H3, H3′). Suppose \( \Sigma_1, \Sigma_2 \) are two \( K\)-qi sections in \( X \). Then \( d(\Sigma_1, \Sigma_2) \geq D \) if \( U_R(\Sigma_1, \Sigma_2) = \emptyset \).

We have an immediate corollary.

**Corollary 4.13.** Given \( D \geq 0, K \geq 1 \) there is an \( R = R_{\mathbf{1.14}}(D, K) \) such that

Suppose \( \Sigma_1, \Sigma_2 \) are two \( K\)-qi sections in \( X \). Then \( d(\Sigma_1, \Sigma_2) \geq D \) if \( d_\mathcal{L}(\Sigma_1, \Sigma_2) \geq 1 \).

**Lemma 4.14.** Given \( K, D \) there is \( R = R_{\mathbf{1.14}}(K, D) \) such that the following holds.

Suppose \( \Sigma_1, \Sigma_2 \) are two \( K\)-qi sections in \( X \) and \( \Sigma = L(\Sigma_1, \Sigma_2) \). Suppose \( x \in X \) and \( \pi(x) = b \). Then \( d(x, \Sigma) \geq D \) if \( d_\mathcal{L}(x, \Sigma \cap F_b) \geq R \).

**Proof.** Suppose \( y \in \Sigma \) is a nearest point from \( x \). Let \( \alpha \) be a geodesic lift of any \( \eta \)-metric bundle satisfying \( H_1, H_2, H_3 \). Suppose \( \alpha \) joins \( y \) to \( F_b \). Now \( \pi_\mathcal{L}(\alpha) \) is a \( 2C\)-qi lift of \([b, \pi(y)]\) where \( C = C_{\mathbf{1.5}}(K) \). Thus \( d(y, \pi_\mathcal{L}(\alpha)(b)) \leq 2Cd_B(b, \pi(y) + 2C \leq 2Cd(x, y) + 2C \). Hence, \( d(x, \Sigma \cap F_b) \leq d(x, y) + d(y, \pi_\mathcal{L}(\alpha)(b)) \leq (2C + 1)d(x, y) + 2C \). Therefore, \( d(x, y) \geq \frac{1}{2C+1} d(x, \Sigma \cap F_b) \). Hence, we can take \( R = \eta((2C + 1)D) \).

5. Cannon-Thurston Maps for Pull-Back Bundles

In this section we prove the main result of the paper. Here is the set up. From now on we suppose that \( \pi : X \rightarrow B \) is an \((\eta, c)\)-metric bundle or an \( \eta \)-metric graph bundle satisfying the following hypotheses.

- **(H1)** \( B \) is a \( \delta_0 \)-hyperbolic metric space.
- **(H2)** Each of the fibers \( F_b, b \in B \) is a \( \delta_0 \)-hyperbolic metric space with respect to the path metric induced from \( X \).
- **(H3)** The barycenter maps \( \bar{\partial} F_b \rightarrow F_b, b \in B \) (resp. \( b \in V(B) \)) are coarsely \( N_0 \)-surjective for some constant \( N_0 \).
- **(H4)** The \((v_k, M_k, n_k)\)-flaring condition is satisfied for all \( k \geq 1 \).

The following theorem is the main result of [MS12]:

**Theorem 5.1.** ([MS12 Theorem 4.3 and Proposition 5.8]) If \( \pi : X \rightarrow B \) is a geodesic metric bundle or a metric graph bundle satisfying \( H1, H2, H3 \) then \( X \) is a hyperbolic metric space if and only if \( X \) satisfies a flaring condition.

**Remark 12.** The sole purpose of \( H3 \) is to have global uniform qi sections through every point of \( X \) which is guaranteed by Proposition 4.5. For the rest of this section we shall assume the following.

\((H3')\) Through any point of \( X \) there is a global \( K_0\)-qi section.

5.1. Proof of the main theorem. We are now ready to state and prove the main theorem of the paper.

**Theorem 5.2.** (Main Theorem) Suppose \( \pi : X \rightarrow B \) is a metric (graph) bundle satisfying the hypotheses \( H1, H2, H3, H3' \) and \( H4 \). Suppose \( g : A \rightarrow B \) is a Lipschitz \( k\)-qi embedding and suppose \( p : Y \rightarrow A \) is the pullback bundle. Let \( f : Y \rightarrow X \) be the pullback map.

Then \( Y \) is a hyperbolic metric space and the CT map exists for \( f : Y \rightarrow X \).

**Proof.** By Theorem 5.1 \( X \) is hyperbolic. We shall assume that \( X \) is \( \delta \)-hyperbolic. We first note two reductions. (1) It is enough to prove the theorem only for metric graph bundles: In fact if any metric bundle satisfies \( H1, H2, H3 \) and is hyperbolic then its canonical metric graph bundle approximation also has the same properties.
Then are we done by Proposition 4.2. Here, by hyperbolicity we will mean Rips hyperbolicity for the rest of the proof.

(2) We may moreover assume that $A$ is a connected subgraph and $g : A \to B$ is the inclusion map and $Y$ is the restriction bundle for that inclusion. In particular, $f : Y \to X$ is the inclusion map and $Y = \pi^{-1}(A)$:

Since $g : A \to B$ is a $k$-qi embedding and $B$ is $\delta_0$-hyperbolic, $g(A)$ is $D_{\delta_0,k,k}'(\delta_0, k, k)$-quasiconvex in $B$. Let $A'$ be the $D_{\delta_0,k,k}'(\delta_0, k, k)$-neighborhood of $g(A)$ in $B$. Then clearly $A'$ is connected subgraph of $B$ and $g : A \to A'$ is a quasiisometry with respect to the induced path metric on $A'$ from $B$. Clearly $A'$ is $(1, 4D_{\delta_0,k,k})$-qi embedded. Let $\pi' : X' = \pi^{-1}(A') \to A'$ be the restriction of $\pi$ on $X'$. Then $\pi' : X' \to A'$ is a metric graph bundle by Lemma 3.17. Also we note that $(f, g) : Y \to X'$ is a morphism of metric graph bundles. By Corollary 3.25, the fiber maps of the morphism $f : Y \to X'$ are uniform quasiisometries and hence by Theorem 3.15 we see that $f : Y \to X'$ is an isomorphism of metric graph bundles. Since (Rips) hyperbolicity of graphs is a qi invariant, we are reduced to proving hyperbolicity of $X'$ and also by Lemma 2.51(1) we are reduced to proving the existence of CT map for the inclusion $X' \to X$.

**Hyperbolicity of $Y$**

$Y$ is hyperbolic by Remark 4.4 of [MS12]. In fact, by Theorem 5.1 it is enough to check that flaring holds for the bundle $Y \to A$. This is a consequence of flaring of the bundle $\pi : X \to B$ and bounded flaring.

For simplicity we shall assume that $Y$ is $\delta$-hyperbolic too and that it therefore satisfies the same flaring condition for the rest of the proof.

**Existence of CT map**

**Outline of the proof:** To prove the existence of CT map we use Lemma 2.50. The different steps used in the proof are follows. (1) Given $y, y' \in Y$ first we define a uniform quasigeodesic $c(y, y')$ in $X$ joining $y, y'$. This is extracted from [MS12]. (2) In the next step we modify $c(y, y')$ to obtain a path $\bar{c}(y, y')$ in $Y$. (3) We then check that these paths are uniform quasigeodesics in $Y$. (4) Finally we verify the condition of Lemma 2.50 for the paths $c(y, y')$ and $\bar{c}(y, y')$. Since $X, Y$ are hyperbolic metric spaces, stability of quasigeodesics and Lemma 2.50 finishes the proof. To maintain modularity of the arguments we state intermediate observations as lemma, proposition etc.

**Remark 13.** Although we assumed that $y, y' \in Y$ as is necessary for our proof, $c(y, y')$ as defined below is a uniform quasigeodesic for all $y, y' \in X$ as it will follow from the proof.

However, we would like to note that description of uniform quasigeodesics in a metric graph bundle with the above properties H1-H4 is already contained in [MS12], e.g. see Proposition 3.4, and Proposition 3.14 of [MS12]. We make it more explicit with the help of Proposition 2.34.

**Step 1:** **Descriptions of the uniform quasigeodesic $c(y, y')$.**

The description of the paths and the proof that they are uniform quasigeodesics in $X$ is broken up into three further substeps.

**Step 1(a): Choosing a ladder containing $y, y'$.** We begin by choosing any two $K_0$-qi sections $\Sigma, \Sigma'$ in $X$ containing $y, y'$ respectively. Let $\mathbb{L}(\Sigma, \Sigma')$ be the ladder formed by them. Throughout the Step 1 we shall work with these qi sections.
and leader. The path \( c(y, y') \) that we shall construct in Step 1(c) will be contained in this ladder.

**Step 1(b): Decomposition of the ladder into small girth ladders.**

We next choose finitely many qi sections in \( L(\Sigma, \Sigma') \) after \([\text{MS12}, \text{Proposition 3.14}]\) in a way suitable for using Proposition 2.3. This requires a little preparation. We start with the following.

**Lemma 5.3.** For all \( K \geq 1 \) there is \( D_{5.3}(K) \) such that the following hold in \( X \) as well as in \( Y \).

Suppose \( \Sigma_1, \Sigma_2 \) are two \( K \)-qi sections and \( d_\ell(\Sigma_1, \Sigma_2) \geq M_K \). Then \( \Sigma_1, \Sigma_2 \) are \( D_{5.3}(K) \)-cobounded.

**Proof.** We note that \( \Sigma_1, \Sigma_2 \) are \( K' = D_{2.3}(\delta, K, K) \)-quasiconvex. Suppose \( P : X \to \Sigma_1 \) is an 1-approximate nearest point projection map and the diameter of \( P(\Sigma_2) \) is bigger than \( D = D_{2.3}(\delta, K', 1) \). Then \( d(\Sigma_1, \Sigma_2) \leq R = R_{2.3}(\delta, K', 1) \).

If \( x \in \Sigma_2 \) such that \( d(x, \Sigma_1) \leq R \) and \( b = \pi(x) \) then \( d_\ell(x, \Sigma_1 \cap F_b) \leq R_{2.3}(\delta, K', 1) \). Hence, \( \pi(P(\Sigma_2)) \subset U_{R_{2.3}(\delta, K', 1)}(\Sigma_2) \). However, by Lemma 4.10 the diameter of \( U_{R_{2.3}(\delta, K, 1)}(\Sigma_2) \) is at most \( D_{4.10}(K') \). It follows that the diameter of \( P(\Sigma_2) \) is at most \( K + DL_{1.11}(K', R) \). Hence we may choose \( D_{5.3}(K) = \max(D_{2.3}(\delta, K', 1), K + DL_{1.11}(K', R)) \).

**Lemma 5.4.** Suppose \( \Sigma_1, \Sigma_2 \) are two \( K \)-qi sections and \( \Sigma \subset L(\Sigma_1, \Sigma_2) \) is \( K \)-qi section. Then \( \Sigma \) coarsely uniformly bisects \( L(\Sigma_1, \Sigma_2) \) into the subladders \( L(\Sigma_1, \Sigma) \) and \( L(\Sigma, \Sigma_2) \).

**Proof.** First of any ladder formed by \( K \)-qi sections is \( K_{5.4}(\delta, K) \)-quasiconvex. Let \( K' = D_{1.11}(\delta, K) \). Let \( x_i \in \Sigma_i, i = 1, 2 \) be any points and let \( \gamma_{x_1 x_2} : I \to X \) be a \( K \)-quasigeodesics joining them where \( I \) is an interval. Then there are points \( t_1, t_2 \in I \) with \( |t_1 - t_2| \leq 1 \) such that \( \gamma_{x_1 x_2}(t_1) \in N_{K'}(L(\Sigma_1, \Sigma)) \) and \( \gamma_{x_1 x_2}(t_2) \in N_{K'}(L(\Sigma, \Sigma_2)) \). Let \( y_1 \in L(\Sigma_1, \Sigma) \) and \( y_2 \in L(\Sigma, \Sigma_2) \) be such that \( d(\gamma_{x_1 x_2}(t_1), y_1) \leq K' \) and \( d(\gamma_{x_1 x_2}(t_2), y_2) \leq 2K' \). Let \( b = \pi(y_1) \). Then \( d_\ell(y_1, L(\Sigma_1, \Sigma_2)) \leq D_{4.11}(\delta, K', 2K' + 2k) \). This implies \( d_\ell(y_1, L(\Sigma_1, \Sigma_2)) \leq D_{4.11}(\delta, K, 2K' + 2k) \). Thus \( d(\gamma_{x_1 x_2}(t_1), \Sigma) \leq K' + D_{4.11}(\delta, K, 2K' + 2k) \). This proves the lemma.

**Lemma 5.5.** If \( Q \) is a \( K \)-qi section in \( X \) then \( Q \cap Y \) is a \( D_{5.5}(K) \)-qi section of \( A \) in \( Y \).

**Proof.** Suppose \( s : B \to X \) is the \( K \)-qi embedding such that \( s(B) = Q \). Let \( s \) also denote the restriction on \( A \). Since the bundle map \( Y \to A \) is 1-Lipschitz we have \( d_A(u, v) \leq d_Y(s(u), s(v)) \) for all \( u, v \in A \). Thus it is enough to show that \( s : A \to Y \) is uniformly coarsely Lipschitz. Suppose \( u, v \in A \) are adjacent vertices. Then \( d_X(s(u), s(v)) \leq 2K \). Now, there is a vertex \( x \in F_u \) adjacent to \( s(u) \in F_u \). Hence, \( d_X(s(v), x) \leq 1 + 2K \). Therefore, \( d_(s(v), x) \leq \eta(1 + 2K) \).

Hence, \( d_Y(s(u), s(v)) \leq 1 + \eta(1 + 2K) \). It follows that for all \( u, v \in A \) we have \( d_Y(s(u), s(v)) \leq (1 + \eta(1 + 2K))d_A(u, v) \). Hence, we can take \( K_{5.5}(K) = 1 + \eta(1 + 2K) \).

The following corollary is proved exactly as Lemma 5.5. Hence we omit the proof.

**Corollary 5.6.** For all \( K \geq 1 \) there is \( D_{5.6}(K) \geq 0 \) such that the following holds.
Suppose \( \Sigma_1, \Sigma_2 \) are two \( K \)-qi sections in \( X \) and \( d_h(\Sigma_1, \Sigma_2) \geq M_K \). Then \( \Sigma_1 \cap Y, \Sigma_2 \cap Y \) are \( D_{K,0}(K) \)-cobounded in \( Y \).

Before describing the decomposition of ladders the following conclusions and notation on qi sections and ladders will be useful to record.

**Convention 5.7. (C0)** Clearly \( Y \) is an \( \eta \) metric graph bundle over \( A \) satisfying \( H2, H3 \). We shall assume that \( A \) is \( \delta_0 \)-hyperbolic. We shall also assume the bundle \( Y \) satisfies a \((\nu_k, M_k, n_k)\)-flaring condition for all \( k \geq 1 \). We recall that \( A \) is \( k \)-qi embedded in \( B \). We let \( k_0 = D_{K,0}(\delta_0, k, k) \) so that \( A \) is \( k_0 \)-quasiconvex in \( B \). Finally we assume that \( Y \) is \( \delta' \)-hyperbolic.

1. **(C1)** Let \( K_{i+1} = (1 + 2K_0)(\delta, K_i) \) for all \( i \in \mathbb{N} \) where \( K_0 \) is as in \( (H3') \). Therefore, through any point of a \( K_i \)-ladder in \( X \) there is a \( K_{i+1} \)-qi section contained in the ladder. Let \( K'_i = \delta(\delta, K_i) \).

2. **(C2)** We let \( \lambda_i = \max\{D_{\delta,0}(\delta, K_i, K_i), K_i^{[\delta,1]}(\delta, K_i), K_i^{[\delta',1]}(\delta', K_i), \delta_i^{[\delta',1]}(\delta', K'_i)\} \) so that any \( K_i \)-qi section \( \mathcal{Q} \) and any ladder \( L \) formed by two \( K_i \)-qi sections in \( X \) are \( \lambda_i \)-quasiconvex in \( X \) and moreover \( \mathcal{Q} \cap Y \) and \( L \cap Y \) are \( \lambda_i \)-quasiconvex in \( Y \).

3. **(C3)** If \( \Sigma_1, \Sigma_2 \) are two \( K_i \)-qi sections in \( X \) and \( d_h(\Sigma_1, \Sigma_2) \geq M_{K_i} \), then they are \( D_i \)-cobounded in \( X \), as are \( \Sigma_1 \cap Y, \Sigma_2 \cap Y \) in \( Y \) where \( D_i = \max\{D_{\delta,0}(K_i), D_{\delta',0}(K_i)\} \).

4. **(C4)** For each pair of \( K_i \)-qi sections \( \Sigma_1, \Sigma_2 \) in \( X \) with \( d_h(\Sigma_1, \Sigma_2) > r_i = \max\{D_{\delta,0}(2\delta_i + 1, K_i), D_{\delta',0}(2\delta_i + 1, K'_i)\} \) we have \( d_X(\Sigma_1, \Sigma_2) > 2\lambda_i + 1 \) and \( d_Y(\Sigma_1 \cap Y, \Sigma_2 \cap Y) > 2\lambda_i + 1 \).

The following proposition is extracted from Proposition 3.14 of [MS12]. The various parts of this proposition are contained in the different steps of the proof of [MS12 Proposition 3.14].

Let us fix a point \( b_0 \in A \) once and for all. Suppose \( \alpha : [0, l] \to F_{b_0} \cap L(\Sigma, \Sigma') \) is an isometry such that \( \alpha(0) = \Sigma \cap F_{b_0} \) and \( \Sigma' \cap F_{b_0} = \alpha(l) \).

**Proposition 5.8. (See [MS12 Corollary 3.13 and Proposition 3.14])** There is a constant \( L_0 \) such that for all \( L \geq L_0 \) there is a partition \( 0 = t_0 < t_1 < \cdots < t_n = l \) of \([0, l] \) and \( K_1 \)-qi sections \( \Sigma_i \) passing through \( \alpha(t_i) \), \( 0 \leq i \leq n \) inside \( L(\Sigma, \Sigma') \) such that the following holds.

1. \( \Sigma_0 = \Sigma, \Sigma_n = \Sigma' \).
2. For \( 0 \leq i \leq n - 2 \), \( \Sigma_{i+1} \subset L(\Sigma_i, \Sigma') \).
3. For \( 0 \leq i \leq n - 2 \) either (I) \( d_h(\Sigma_i, \Sigma_{i+1}) = L \), or (II) \( d_h(\Sigma_i, \Sigma_{i+1}) > L \) and there is a \( K_2 \)-qi section \( \Sigma_i' \) through \( \alpha(t_{i+1} - 1) \) inside \( L(\Sigma_i, \Sigma_{i+1}) \) such that \( d_h(\Sigma_i, \Sigma'_i) < C + CL \) where \( C = C_{K_1} \).
4. \( d_h(\Sigma_{n-1}, \Sigma_n) \leq L \).

However, we will need a slightly different decomposition of \( L(\Sigma, \Sigma') \) than what is described here. It is derived as the following corollary to the Proposition 5.8.

**Convention 5.9.** We shall fix \( L = L_0 + M_{K_1} + r_3 \) and denote it by \( R_0 \) for the rest of the paper. Also we shall define \( R_1 = C + CR_0 \) where \( C = C_{K_1} \). Thus we have the following.

**Corollary 5.10.** (Decomposition of \( L(\Sigma, \Sigma') \)) There is a partition \( 0 = t_0 < t_1 < \cdots < t_n = l \) of \([0, l] \) and \( K_1 \)-qi sections \( \Sigma_i \) passing through \( \alpha(t_i) \), \( 0 \leq i \leq n \) inside \( L(\Sigma, \Sigma') \) such that the following holds.

1. \( \Sigma_0 = \Sigma, \Sigma_n = \Sigma' \).
2. For \( 0 \leq i \leq n - 2 \), \( \Sigma_{i+1} \subset L(\Sigma_i, \Sigma') \).
(3) For $0 \leq i \leq n - 2$ either (I) $d_h(\Sigma_i, \Sigma_{i+1}) = R_0$, or (II) $d_h(\Sigma_i, \Sigma_{i+1}) > R_0$ and there is a $K_2$-qi section $\Sigma_i$ through $\alpha(t_{i+1} - 1)$ inside $L(\Sigma_i, \Sigma_{i+1})$ such that $d_h(\Sigma_i, \Sigma_{i+1}) < R_1$.

In either case $d(\Sigma_i, \Sigma_{i+1}) > 2\lambda_1 + 1$ and $\Sigma_i, \Sigma_{i+1}$ are $D_1$-cobounded in $X$.

(4) $d_h(\Sigma_{n-1}, \Sigma_n) \leq R_0$.

We note that the second part of (3) follows from (C1), (C2), (C3) above.

**Remark 14.** We shall use $\Sigma$ to mean qi sections in $L(\Sigma, \Sigma')$ exactly as in the corollary above for the rest of this section. Also we note that $\Sigma_n, \Sigma_{n-1}$ need not be cobounded in general.

**Lemma 5.11.** Let $\Pi : L(\Sigma, \Sigma') \to [0, n]$ be any map that sends $\Sigma_i$ to $i \in [0, n] \cap \mathbb{Z}$ and sends any point of $L(\Sigma_i, \Sigma_{i+1}) \setminus \{\Sigma_i \cup \Sigma_{i+1}\}$ to any point in $(i, i+1)$. We note that the hypotheses of Proposition [2.32] are verified for $\Pi$ and its restriction $L(\Sigma, \Sigma') \cap X \to [0, n]$.

**Proof.** (1) follows from (C2), (2) follows from Lemma [5.4], (3) follows from (C4). The coboundedness of $\Pi^{-1}(i) = \Sigma_i, \Pi^{-1}(j) = \Sigma_j, i < j < n$ follows from (C3). □

**Step 1(c): Joining $y, y'$ inside $L(\Sigma, \Sigma')$.** We now inductively define a finite sequence of points $y_i \in \Sigma_i, 0 \leq i \leq n + 1$ with $y_0 = y, y_{n+1} = y'$ such that each $y_i, 1 \leq i \leq n$, is a uniform approximate nearest point projection of $y_{i-1}$ on $\Sigma_i$ in $X$. We also define uniform quasigeodesics $\gamma_i$ joining $y_i, y_{i+1}$. The concatenation of these $\gamma_i$’s then forms a uniform quasigeodesic joining $y, y'$ by Proposition [2.34] and Lemma [5.11].

We define $\gamma_n$ to be the lift of $[\pi(y_n), \pi(y_{n+1})]$ in $\Sigma'$.

Suppose $y_0, \ldots, y_i$ and $\gamma_0, \ldots, \gamma_{i-1}$ are already constructed, $0 \leq i \leq n - 2$. We next explain how to define $y_{i+1}$ and $\gamma_i$.

**Case I.** Suppose $L_i = L(\Sigma_i, \Sigma_{i+1})$ is of type (I) or $i = n-1$. Then, $U_{R_0}(\Sigma_i, \Sigma_{i+1})$ is non-empty. Let $u_i$ be a nearest point projection of $\pi(y_i)$ on $U_{R_0}(\Sigma_i, \Sigma_{i+1})$. We define $y_{i+1} = \Sigma_{i+1} \cap F_{u_i}$. Let $\alpha_i$ be the lift of $[\pi(y_i), u_i]_i \in \Sigma_i$, let $\sigma_i$ be the subsegment of $F_{u_i} \cap L_i$ joining $\alpha_i(u_i)$ and $y_{i+1}$. We define $\gamma_i$ to be the concatenation of $\alpha_i$ and $\sigma_i$. Then clearly $\gamma_i$ is a $(K_1 + R_0)$-quasigeodesic in $X$. That $y_{i+1}$ is a uniform approximate nearest point projection of $y_i$ on $\Sigma_{i+1}$ follows from the following lemma.

**Lemma 5.12.** Given $K \geq 1$ and $R \geq M_K$ there are constants $6_{12}((K, R)$ and $5_{12}((K, R))$ such that the following holds.

Suppose $Q_1, Q_2$ are two $K$-qi sections and $d_h(Q_1, Q_2) \leq R$. Let $x \in Q_1$ and let $U = U_R(Q_1, Q_2)$. Suppose $b$ is a nearest point projection of $\pi(x)$ on $U$. Then $Q_2 \cap F_b$ is $5_{12}((K, R))-approximate nearest point projection of $x$ on $Q_2$.

If $d_h(Q_1, Q_2) \geq M_K$ then for any $b' \in U$ the point $Q_2 \cap F_{b'}$ is an $6_{12}((K, R))-approximate nearest point projection of any point of $Q_1$ on $Q_2$.

This lemma follows from Corollary 1.40 and Proposition 3.4 of [MSP12] given that ladders are quasiconvex. However, we give an independent proof using the hyperbolicity of $X$.

**Proof.** Suppose $\bar{x}$ is a nearest point projection of $x$ on $Q_2$ and let $x' = Q_2 \cap F_b$. Let $\gamma_{xx'}$ be the concatenation of the lift in $Q_1$ of any geodesic in $B$ joining $\pi(x)$ to $b$ and any geodesic in $F_b$ joining $Q_1 \cap F_b$ to $Q_2 \cap F_b$. Clearly it is a $(K+R)$-quasigeodesic in
Also by Lemma 2.28 the concatenation of any 1-quasigeodesics joining \( x, \bar{x} \) and \( \bar{x}, x' \) is a \( K, 2.28(\delta, K, 1, 0) \)-quasigeodesic. Hence, by stability of quasigeodesics we have \( \bar{z} \in N_D(\gamma_i) \) where \( D = D.2.28(\delta, K') \) and \( K' = \max(1 + 2K, 2.28(\delta, K, 1, 0)) \). This implies there is a point \( z \in \gamma_{xx'} \) such that \( d(z, \bar{x}) \leq D \). If \( z \in F_b \cap \gamma_{xx'} \) then \( d(\bar{x}, x') \leq D + R_0 \) and hence \( x' \) is a \((D + R_0)\)-approximate nearest point projection of \( x \) on \( Q_2 \).

Suppose \( z \in Q_1 \cap \gamma_{xx'} \). Then \( d_{\pi(z)}(z, Q_2 \cap F_{\pi(z)}) \leq R.4.11(12(D, K)) \). Hence, by Lemma 4.10 we have \( d_B(\pi(z), b) \leq R.4.11(K, K') \) where \( K = H.4.11(D, K)/R_0 \). Therefore, \( d(\bar{x}, x') \leq d(\bar{x}, z) + d(z, Q_1 \cap F_b) + d(Q_1 \cap F_b, x') \leq D + (K + K.4.11(K, K')) + R_0 \). Hence in this case \( x' \) is a \((D + K + K.4.11(K, K'), R_0)\)-approximate nearest point projection of \( x \) on \( Q_2 \). We may set \( \alpha.6.12(K, R) = D + K + K.4.11(K, K') + R_0 \).

For the last part, we note that the diameter of \( U \) is at most \( D.4.11(K, R) \). Thus clearly \( \alpha.6.12(K, R) = \alpha.6.12(K, R) + K + K.4.11(K, K') \) works. \( \square \)

**Case II.** Suppose \( L_i = L(\Sigma_i, \Sigma_{i+1}) \) is of type (II), i.e. \( d_H(\Sigma_i, \Sigma_{i+1}) > R_0 \). In this case there exists a \( K, 2\)-qi section \( \Sigma_i' \) inside \( \Sigma_i = L(\Sigma_i, \Sigma_{i+1}) \) passing through \( \alpha(t_{i+1} - 1) \) such that \( d_H(\Sigma_i, \Sigma_i') \leq R_1 \). We thus use Case (I) twice as follows. First we project \( y_i \) on \( \Sigma_i' \). Next we project \( y_i' \) on \( \Sigma_{i+1} \) and that is a uniform quasigeodesic follow immediately from Lemma 5.12 and the last part of Proposition 2.34.

**Remark 15.** We note that \( L(\Sigma, \Sigma') \cap Y \) is a ladder in \( Y \) formed by the qi sections \( \Sigma \cap Y \) and \( \Sigma' \cap Y \) defined over \( A \). However, in this case the subladders \( L(\Sigma_i, \Sigma_{i+1}) \cap Y \) may not be of type (I) or (II). Therefore, we cannot directly use the above procedure to construct a uniform quasigeodesic in \( Y \) joining \( y, y' \).

**Step 2: Modification of the path \( c(y, y') \)**

In this step we shall construct a path \( \tilde{c}(y, y') \) in \( Y \) joining \( y, y' \) by modifying \( c(y, y') \). For \( 0 \leq i \leq n \), let \( b_i \) be a nearest point projection of \( \pi(y_i) \) on \( A \) and let \( \bar{y}_i = F_{b_i} \cap \Sigma_i \). We define a path \( \bar{\gamma}_i \subset Y \) joining the points \( \bar{y}_i, \bar{y}_{i+1} \) for \( 0 \leq i \leq n \). Finally the path \( \tilde{c}(y, y') \) is defined to be the concatenation of these paths. The path \( \bar{\gamma}_n \) is the lift of \( [\pi(y_{n+1}), \pi(y_n)]_A \) in \( \Sigma' \cap Y \). The definition of \( \bar{\gamma}_i \), for \( 0 \leq i \leq n - 1 \), depends on the type of the subladder \( L_i = L(\Sigma_i, \Sigma_{i+1}) \) given by Corollary 5.12(4).

**Case 2(I):** Suppose \( L_i \) is of type (I) or \( i = n - 1 \). Let \( \bar{\alpha}_i \) denote the lift of \( [b_i, b_{i+1}]_A \) in \( \Sigma_i \) starting at \( \bar{y}_i \). The path \( \tilde{\gamma}_i \) is defined to be the concatenation of \( \bar{\alpha}_i \) and the fiber geodesic \( F_{b_{i+1}} \cap L(\Sigma_i, \Sigma_{i+1}) \).

**Case 2(II):** Suppose \( L_i \) is of type (II). In this case, we apply Case 2(I) to each of the subladders \( L(\Sigma_i, \Sigma_i') \) and \( L(\Sigma_i, \Sigma_{i+1}) \). Let \( y_i' \) be as defined in step 1(c). Let \( b'_i \in A \) be a nearest point projection \( \pi(y'_i) \) of \( A \) and let \( \bar{y}'_i = \pi^{-1}(b'_i) \). Next we connect \( \bar{y}_i, \bar{y}'_i \) and \( \bar{y}'_i, \bar{y}_{i+1} \) as in Case 2(I) inside the ladders \( L(\Sigma_i \cap Y, \Sigma_i' \cap Y) \) and \( L(\Sigma_i' \cap Y, \Sigma_{i+1} \cap Y) \) respectively. We shall denote by \( \bar{\alpha}_i \) and \( \bar{\beta}_i \) the lift of \( [b_i, b'_i]_A \) in \( \Sigma_i \cap Y \) and \( [b'_i, b_{i+1}]_A \) in \( \Sigma_i' \cap Y \) respectively. The concatenation of the paths \( \bar{\alpha}_i \),
Step 3: Proving that $\bar{c}(y,y')$ is a uniform quasigeodesic in $Y$. To show that $\bar{c}(y,y')$ is a quasigeodesic, by Proposition 2.3.2 it is enough to show that the paths $\bar{\gamma}_i$ are all uniform quasigeodesics in $Y$ and that for $0 \leq i \leq n-1$, $y_{i+1}$ is an approximate nearest point projection of $\bar{y}_i$ in $\Sigma_{i+1} \cap Y$. The proof of this is broken into three cases depending on the type of the ladder $L_i$. We start with the following lemma as a preparation for the proof.

The lemma below is true for any metric bundle that satisfies the hypotheses (H1)-(H4), (H3') although we are stating it for $X$ only. For instance it is true for $Y$ too.

**Lemma 5.13.** Suppose $b \in B$, $x,y \in F_b$. Suppose for all $K \geq K_0$ and $R \geq M_K$ there is a constant $D = D(K,R) \geq 0$ such that for all $x',y' \in [x,y]_b$ and any two $K$-qi sections $Q_1$ and $Q_2$ in $X$ passing through $x',y'$ respectively, either $U_R(Q_1, Q_2) = \emptyset$ or $d_{b,b}(U_R(Q_1, Q_2)) \leq D$. Then the following hold:

1. $[x,y]_b$ is a $D$-quasigeodesic in $X$ where $D$ depends on the function $D$ (and the parameters of the metric bundle).
2. If $Q$ and $Q'$ are two $K$-qi sections passing through $x,y$ respectively then $x$ is a uniform approximate nearest point projection of $y$ on $Q$ and $y$ is a uniform approximate nearest point projection of $x$ on $Q'$.

**Proof.** (1) Since the arc length parametrization of $[x,y]_b$ is a uniform proper embedding, by Lemma 2.3.2 it is enough to show that $[x,y]_b$ is uniformly close to a geodesic in $X$ joining $x,y$.

**Claim:** Suppose $\Sigma_x, \Sigma_y$ are two $K_0$-qi sections passing through $x,y$ respectively. Given any $z \in [x,y]_b$ and any $K_1$-qi section $\Sigma_z$ passing through $z$ contained in the ladder $L(\Sigma_x, \Sigma_y)$ the nearest point projection of $x$ on $\Sigma_z$ is uniformly close to $z$.

We note that once the claim is proved then applying Proposition 2.3.2 to the ladder $L(\Sigma_x, \Sigma_y) = L(\Sigma_x, \Sigma_z) \cup L(\Sigma_z, \Sigma_y)$ it follows that $z$ is uniformly close to a geodesic joining $x,y$. From this (1) follows immediately.

**Proof of the claim:** First suppose $U_{M_{K_1}}(\Sigma_x, \Sigma_z) \neq \emptyset$. Then we can find a uniform approximate nearest point projection of $x$ on $\Sigma_z$ using Step 1(c), Case I and Lemma 5.12 above which is uniformly close to $z$ by hypothesis.

Now suppose $U_{M_{K_1}}(\Sigma_x, \Sigma_z) = \emptyset$. Let $\alpha_{zx} : [0,l] \to F_b$ be the unit speed parametrization of the geodesic $L(\Sigma_x, \Sigma_z) \cap F_b$ joining $z$ to $x$. Applying Corollary 5.11 we can find a $K_2$-qi section $\Sigma_{z'}$ in the ladder $L(\Sigma_x, \Sigma_z)$ passing through $z' = \alpha_{zx}(t)$ for some $t \in [0,l]$ such that $L(\Sigma_x, \Sigma_{z'})$ is a $K_2$-ladder of type (I) or (II). Now we first take a nearest point projection say $x'$ of $x$ on $\Sigma_{z'}$. If we can define a uniform approximate nearest point projection of $x'$ on $\Sigma_z$ which is also uniformly close to $z$ then we will be done by applying the last part of Proposition 2.3.2 to $L(\Sigma_x, \Sigma_z)$. However, in this case $\Sigma_x, \Sigma_{z'}$ are $D_2$-cobounded. Hence it is enough to find uniform approximate nearest point projection of $z'$ on $\Sigma_z$ which is uniformly close to $z$. The proof of this in the two cases goes as follows.

(I) Suppose $d_{b,b}(\Sigma_x, \Sigma_{z'}) = R_0$. By the last part of Lemma 5.12 if $v \in U_{R_0}(\Sigma_z, \Sigma_{z'})$ then $F_v \cap \Sigma_z$ is a uniform approximate nearest point projection of any point of $\Sigma_z$. Since $d_{b,b}(b,v)$ is uniformly small by hypothesis $d(z,F_v \cap \Sigma_z)$ is also uniformly small.

(II) Suppose $d_{b,b}(\Sigma_z, \Sigma_{z'}) > R_0$. Then there is a $K_3$-qi section $\Sigma_{z''}$ in $L(\Sigma_z, \Sigma_{z'})$ passing through $z'' = \alpha_{zx}(t-1)$ such that $U_{R_0}(\Sigma_x, \Sigma_{z''}) \neq \emptyset$. Let $v'$ be a nearest
point projection of $b$ on $U_{R_b}(\Sigma_{x}, \Sigma_{x'})$. Then by hypothesis $d(b,v')$ is uniformly small and by Lemma 5.12 the point $\Sigma_{x} \cap F_{v'}$ is a uniform approximate nearest point projection of $z''$ on $\Sigma_{x}$. Since $d(z, z'') \leq 1$, $\Sigma_{x} \cap F_{v'}$ is a uniform approximate nearest point projection of $z'$. However, $\Sigma_{x}, \Sigma_{x'}$ are $D_2$-cobounded. Thus $\Sigma_{x} \cap F_{v'}$ is a uniform approximate nearest point projection of $z'$ on $\Sigma_{x}$.

(2) We shall prove only the first statement since the proof of the second would be an exact copy. Suppose $x_1 \in Q$ is a nearest point projection of $y$ on $Q$. Consider the $K$-qi section over $[b, \pi(x_1)]$ contained in $Q$. This is a $K$-quasigeodesic of $X$ joining $x, x_1$. Since $Q$ is a $K$-qi section it is $L_{\pi(x_1)}(\delta, K, K')$-quasiconvex in $X$. Hence by Lemma 2.28 the concatenation of this quasigeodesic with a geodesic in $X$ joining $y$ to $x_1$ is a $K_{\pi(x_1)}(\delta, K, K')$-quasigeodesic where $K = D_{\pi(x_1)}(\delta, K, K')$. Let $k' = \max\{K = D_{\pi(x_1)}(\delta, K, K'), K_{\pi(x_1)}(\delta, K, K')\}$. Since $[x, y)_b$ is a $K_{\pi(x_1)}(\delta, K, K')$-quasigeodesic we have by Lemma 2.22, $x_1 \in N_{D_{\pi(x_1)}(\delta, K, K')}$ where $D_{\pi(x_1)}(\delta, K, K')$. Suppose $z \in [x, y)_b$ be such that $d(x_1, z) \leq 2D'$. Then $d_{\pi(x_1)}(\pi(x_1), \pi(z)) = d_{\pi(x_1)}(\pi(x_1), b) \leq 2D'$. Hence, $d(x, x_1) \leq K + 2D'K$. Thus $x$ is a $(K + 2D'K)$-approximate nearest point projection of $y$ on $Q$.

\[ \square \]

Remark 16. The proof of the first part of the above lemma uses the hypothesis for $K \leq K_3$ only whereas the proof of the second part follows directly from the statement of the first part and is independent of the hypotheses of the lemma.

Lemma 5.14. Given $R \geq 0, K, K' \geq 1$ and $R' \geq M_K$, there is a constant $R_{\frac{5}{14}}(R, R', K, K')$ and $D_{\frac{5}{14}}(R, R', K, K')$ such that the following holds.

Suppose $u \in B$ and $P_A(u) = b$. Suppose $x, y \in F_b$ and let $\gamma_x, \gamma_y$ be two $K$-qi sections over $[u, b]$. Let $Q_1, Q_2$ be two $K$-qi sections over $A$ in $Y$ and $U = U_{R'}(Q_1, Q_2)$. If $d_b(\gamma_x(u), \gamma_y(u)) \leq R$ and $U \neq \emptyset$ then $d_{\pi(x_1)}(x, y) \leq R_{\frac{5}{14}}(R, R', K, K')$ and $d_{\pi(x_1)}(b, U) \leq D_{\frac{5}{14}}(R, R', K, K')$.

Proof. Suppose $U \neq \emptyset$ and $d_b(\gamma_x(u), \gamma_y(y)) \leq L$. Let $b' \in U_{M_K}(Q_1, Q_2)$ be any point and let $b'b'$ denote a geodesic in $A$ joining $b, b'$. This is the concatenation $[u, b] \times [b, b']$ is a $K_{\pi(x_1)}(\delta, K, 0)$-quasigeodesic in $B$ by Lemma 2.28. Since $A$ is $K$-qi embedded and $K_0$-quasiconvex. Concatenation of $\gamma_x, \gamma_y$ with the qi sections over $[b, b']$ are contained in $Q_1, Q_2$ respectively defines max$\{K, K'\}$-qi sections over $u \times b'$. Then by Lemma 2.3 these qi sections are $(k, k', k'' + k''')$-quasigeodesics in $X$. Since $X$ is $\delta$-hyperbolic and $d(\gamma_x(u), \gamma_y(u)) \leq R$ and $d(Q_1 \cap F_{b'}, Q_2 \cap F_{b'}) \leq R'$, by Corollary 2.23 $x$ is contained in the $D' = (R + R' + 2D_{\frac{5}{14}}(\delta, k, k', k'' + k'''))$-neighborhood of the qi section over $[u, b] \times [b, b']$ passing through $y$. Applying Lemma 4.12 to the restriction bundles over $[u, b] \times [b, b']$ we have $d_b(x, y) \leq R'_{\delta}$ where $R'_{\delta} = R_{\frac{5}{14}}(R, R', K, K').$ Hence, we can take $R_{\frac{5}{14}}(R, R', K, K') = R'_{\delta}$. Finally by Lemma 5.11, $d_{\pi(x_1)}(b, U) \leq D_{\frac{5}{14}}(K, R', R'/M_K).$ This completes the proof by taking $D_{\frac{5}{14}}(R, R', K, K') = D_{\frac{5}{14}}(K, R'/M_K).

\[ \square \]

Lemma 5.15. Given $K \geq K_0$ and $R \geq M_k$ there are constants $R_{\frac{5}{14}}(R, K, R)$ and $D_{\frac{5}{14}}(K, R)$ such that the following holds.

Suppose $Q, Q'$ are two $K$-qi sections in $X$ and $d_b(Q, Q') \leq R$ in $X$. Let $U = U_R(Q, Q')$. Suppose $d_b(Q \cap Y, Q' \cap Y) \geq R$ in $Y$. Then the following hold.

(1) The projection of $U$ on $A$ is of diameter $\leq D_{\frac{5}{14}}(K, R)$.

(2) For any $b \in P_A(U), F_b \cap L(Q, Q')$ is a $K_{\frac{5}{14}}(\delta, K, R)$-quasigeodesic in $Y$.

(3) $F_b \cap Q$ is an $\epsilon_{\alpha, \beta}$-approximate nearest point projection of any point of $Q'$ on $Q$ and vice versa.
Consider the restriction are such that the following holds. We know that by Lemma 5.13. Thus $Q ∩ Y, Q' ∩ Y$. Since $R_1 ≥ M_K$ by Lemma 5.10 we have $diam(U_{R_1}(Q ∩ Y, Q' ∩ Y)) ≤ D_{5.17}, k, R_1)$. This proves (1). In fact, we can take $D_{5.17} = max\{D_{5.17}(\delta, \lambda, 0), D_{5.17}(\lambda, 0, 0), D_{5.17}(\delta, \lambda, 0)\}$. We derive (2) and (3) from Lemma 5.13 as follows. Let $u ∈ U$ be such that $P_A(u) = b$ and let $x, y ∈ F_b \cap L(Q, Q')$. Suppose $Q_1, Q_1'$ are two $K'$-qi sections in $Y$ passing through $x, y$ respectively and $U' = U_{M_{K'}}(Q_1, Q_1')$. Suppose $U' ≠ \emptyset$. Consider the restriction $Z$ of the bundle $X$ on $[u, b] ⊂ B$. In this bundle $Q ∩ Z, Q' ∩ Z$ are $K$-qi sections. By Proposition 1.10(3) there are $(1 + 2K_0)\epsilon_r(K)$-qi sections over $u b$ contained in the ladder $L(Q ∩ Z, Q' ∩ Z)$ passing through $x, y$. Call them $\gamma_x, \gamma_y$ respectively. We note that $d(\gamma_x(u), \gamma_y(u)) ≤ R$. Now applying Lemma 5.13 we know that $d_M(b, U')$ is uniformly small. This verifies the hypothesis of Lemma 5.13. Thus $Q ∩ F_b$ is a uniform approximate nearest point projection of $Q' ∩ F_b$ on $Q$. Since $d_h(Q ∩ Y, Q' ∩ Y) ≥ R ≥ M_K$ the qi sections $Q ∩ Y, Q' ∩ Y$ are uniformly cobounded by Lemma 5.3. This shows that $Q ∩ F_b$ is a uniform approximate nearest point projection of any point of $Q'$ on $Q$. That $Q' ∩ F_b$ is a uniform approximate nearest point projection of any point of $Q$ on $Q'$ is similar and hence we skip it.

**Lemma 5.16.** Given $D ≥ 0, K ≥ K_0$ and $R ≥ M_K$ there are constants $K_{5.16} = K_{5.16}(D, K, R)$ such that the following holds.

Suppose $Q, Q'$ are two $K$-qi sections in $X$ and $d_h(Q, Q') ≤ R$ in $X$. Let $U = U_{R}(Q, Q')$. Suppose $U ≠ \emptyset$ and $diam(U) ≤ D$. Then the following holds.

1. $diam(P_A(U)) ≤ D_{5.16}.$
2. For any $b ∈ P_A(U)$, $F_b ∩ L(Q, Q')$ is a $K_{5.16}$-quasigeodesic in $Y$.
3. $F_b ∩ Q$ is an $\epsilon_{5.16}$-approximate nearest point projection of any point of $Q'$ on $Q$ and vice versa.

**Proof.** (1) Since $B$ is $δ_0$ is hyperbolic and $A$ is $k_0$-quasiconvex in $B$ any nearest point projection map $P_A : B → A$ is coarsely $L := L_{2.31}(\delta_0, k_0, 0)$-Lipschitz. Hence, $diam(P_A(U)) ≤ L + DL$.

We can derive (2), (3) from Lemma 5.13 and the hypotheses of Lemma 5.13 can be verified using Lemma 5.13. The proof is an exact copy of the proof of Lemma 5.13. Hence we omit it. The only part that requires explanation is why $Q ∩ Y, Q' ∩ Y$ are uniformly cobounded in $Y$. If $d_h(Q ∩ Y, Q' ∩ Y) > R$ then we are done by Lemma 5.3. Suppose this is not the case. Then by the hypothesis $diam(U_{R}(Q ∩ Y, Q' ∩ Y)) ≤ k(k + D)$ since $A$ is $k$-qi embedded in $B$. Then we are done by the first part of Lemma 5.12.

**Lemma 5.17.** Given $K ≥ K_0$ and $R ≥ M_K$ there is a constant $D_{5.17} = D_{5.17}(K, R)$ such that the following holds.

Suppose $Q, Q'$ are two $K$-qi sections in $X$ and $d_h(Q ∩ Y, Q' ∩ Y) ≤ R$. Let $U = U_{R}(Q, Q')$. Then the following holds. For any $b ∈ P_A(U)$, $d_h(Q ∩ F_b, Q' ∩ F_b) ≤ D_{5.17}$.
Proof. Suppose $P_A (u) = b$ where $u \in U$. If $u \in A$ then $b = u$ and $d_B (Q \cap F_b, Q' \cap F_b) \leq R$. Suppose $u \not\in A$. We note that $U (Q \cap Y, Q' \cap Y) \not= \emptyset$. Let $v \in U (Q \cap Y, Q' \cap Y)$. Then by Lemma 5.15 \[ u, b \in \cal X \ast [b, v] \] is a $K_1, 10$-quasigeodesic in $X$. Since $U$ is $K_1, 10$-quasiconvex in $X$. Let $k' = k_0, k_0, 1, 0)$. Hence, by Lemma 5.18 $b \in N_D (U)$ where $D = D_2, 28 (\delta_0, k', k)$. Finally by the bounded flaring $d_B (Q \cap F_b, Q' \cap F_b) \leq k_0 \in \{ 1, \mu_K (D) \}$. Hence we can take $D_2, 28 (\delta_0, k', 1, 0)$. Finally we are ready to finish the proof of step 3.

**Lemma 5.18.** For $0 \leq i \leq n - 1$ we have the following.

1. $\bar{y}_{i+1}$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma_{i+1} \cap Y$.
2. $\bar{\gamma}_i$ is a uniform quasigeodesic in $Y$.

**Proof.** The proof is broken into three cases depending on the type of $L_i$.

**Case 1:** $i \leq n - 2$ and $L_i$ is of type (I): By Corollary 4.10 $U_{R_0} (\Sigma_i, \Sigma_{i+1})$ has uniformly small diameter. Hence by Lemma 5.15 \[ \Sigma_i \cap F_{b_{i+1}}, \Sigma_{i+1} \cap F_{b_{i+1}} \] is a uniform quasigeodesic in $Y$. By the part (3) of the same lemma $\Sigma_{i+1} \cap F_{b_{i+1}}$ is a uniform approximate nearest point projection of $\Sigma_i \cap F_{b_{i+1}}$ on $\Sigma_{i+1} \cap Y$. Hence the second part of the lemma follows, in this case, by Lemma 5.18.

**Case 2:** $i \leq n - 2$ and $L_i$ is of type (II): Suppose $L_i$ is a ladder of type (II). In this case, by Proposition 2.33 it is enough to show the following two statements $(\gamma)$ and $(\gamma')$:

$(\gamma)$: $\bar{y}_i'$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma_i' \cap Y$ in $Y$ and the concatenation of $\bar{\alpha}_i$ and the fiber geodesic $[\Sigma_i' \cap F_{b_i'}, \Sigma_i' \cap F_{b_i'}]_{F_{b_i'}}$ is a uniform quasigeodesic in $Y$.

We know that $d_B (\Sigma_i, \Sigma_i') \leq R_1$. Depending on the nature of $d_B (\Sigma_i \cap Y, \Sigma_i' \cap Y)$ the proof of $(\gamma')$ is broken into the following two cases.

**Case (i):** Suppose $d_B (\Sigma_i \cap Y, \Sigma_i' \cap Y) \leq R_1$. In this case $d_B (\Sigma_i, \Sigma_i')$ is uniformly small by Lemma 5.17. By Lemma 5.12 if $b_i'$ is a nearest projection of $\pi (\bar{y}_i)$ on $U_{R_i} (\Sigma_i \cap Y, \Sigma_i' \cap Y)$ then $F_{b_i'} \cap \Sigma_i'$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma_i \cap Y$. Thus it is enough to show that $d_B (b_i', b_i')$ uniformly bounded to prove that $\bar{y}_i'$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma_i' \cap Y$. Then since $\Sigma_i \cap Y$ is $K_1$-quasi section in $Y$ and $d_B (\Sigma_i, \Sigma_i') \leq R_1$, $F_{b_i'} \cap \Sigma_i'$ is uniformly small it will follow that the concatenation of $\bar{\alpha}_i$ and the fiber geodesic $[\Sigma_i \cap F_{b_i'}, \Sigma_i' \cap F_{b_i'}]_{F_{b_i'}}$ is a uniform quasigeodesic in $Y$.

That $d_B (b_i', b_i')$ uniformly bounded is proved as follows. Let $U = U_{R_1} (\Sigma_i, \Sigma_i')$, $V = U \cap A = U_{R_1} (\Sigma_i \cap Y, \Sigma_i' \cap Y)$. Since $B$ is $\delta_0$-hyperbolic, $A$ is $k$-qi embedded in $B$ and $V$ is $\lambda_2$-quasiconvex in $A$, $V$ is $K_1, 10 (\delta_0, k, \lambda_2)$-quasiconvex in $B$. Let $k' = \max \{ \lambda_2, \delta_0, k, K_2 \}, K_2, 20 (\delta_0, k, \lambda_2)$. Then $A, U, V$ are all $k'$-quasiconvex in $B$. By the definitions of $y_i$, we know that $\pi (y_i)$ is the nearest point projection of $\pi (y_i)$ on $U$. Let $b_i'$ be a nearest point projection of $\pi (y_i)$ on $V$. Also $b_i' = \pi (\bar{y}_i)$ is the nearest point projection of $\pi (y_i)$ on $A$. On the other hand, $b_i = \pi (\bar{y}_i)$ is a nearest point projection of $\pi (y_i)$ on $A$ and $b_i$ is the nearest point projection of $b_i$ on $V$. Therefore, $d_B (b_i', b_i') \leq 2D_1 (\delta_0, k', 0)$ by Corollary 2.39.

Now, by Lemma 5.17 $d_B (\Sigma_i \cap F_{b_i'}, \Sigma_i' \cap F_{b_i'}) \leq D_1 (K_2, R_1)$. Hence, by Lemma 2.10 $d_A (b_i', V) \leq D_1 (K_2, R_1) = D_1$. Say, let $v$ be such that $d_A (b_i', v) \leq D_1$. Then $d_B (b_i', v) \leq kD_1 + k$.

Hence, $H_B (\pi (y_i), b_i', \pi (y_i), v) \leq \delta_0 + k + kD_1$. However, the concatenation $[\pi (y_i), b_i'] [b_i', v]_{B}$ is a $K_1, 10 (\delta_0, k', 1, 0)$-quasigeodesic. Hence, there is a point $w \in [\pi (y_i), v]_B$ such that $d_B (w, b_i') \leq \delta_0 + k + kD_1$.
Proof. Let \( \Sigma \) be \( K_0 \)-qi sections and \( \gamma_i \) be \( qi \) sections \( x \in \Sigma \). In the various \( qi \) sections \( \Sigma \), points \( z \in \gamma_i \) are joined by a geodesic quadrilateral. \( \sigma \) to \( \pi \).

Lemma 5.20. \( Y \) in \( \gamma_i \) is a uniform quasigeodesic joining \( \bar{y}_{i+1} \) to \( \bar{y}_i \) in \( Y \).

The conclusion of Lemma 5.16 is subsumed by Lemma 5.16 and Lemma 5.17. But we still keep Lemma 5.16 for the sake of ease of explanation.

Remark 17. The conclusion of Lemma 5.17 is subsumed by Lemma 5.16 and Lemma 5.17. But we still keep Lemma 5.16 for the sake of ease of explanation.

Thus by Lemma 5.11 and Lemma 5.18, we have proved the following.

Proposition 5.19. Let \( x, y \in Y \) and let \( \Sigma \) and \( \Sigma' \) be two \( K_0 \)-qi sections in \( X \) through \( x \) and \( y \) respectively. Let \( c(x, y) \) be a uniform quasigeodesic in \( X \) joining \( x \) and \( y \) which is contained in \( L(\Sigma, \Sigma') \) as constructed in step 1(c). Then the corresponding modified path \( c(x, y) \), as constructed in step 2, is a uniform quasigeodesic in \( Y \).

Step 4. Verification of the hypothesis of Lemma 2.50.

Lemma 5.20. Suppose \( u, v \in \Sigma \) and \( \bar{u}, \bar{v} \in A \) respectively \( u \) and \( v \) are their nearest point projections on \( A \). Suppose \( w \in [u, v] \) is such that \( d_B(w, A) \leq R \). Then \( d_B(w, [\bar{u}, \bar{v}]) \leq D(\Sigma, \Sigma', \Sigma) \).

Proof. Clearly geodesic quadrilaterals in \( B \) are \( 2\delta_0 \)-slim. Hence, there is \( w' \in [\bar{u}, \bar{v}] \) such that \( d_B(w, w') \leq 2\delta_0 \). If \( w' \in [\bar{u}, \bar{v}] \) then we are done. Suppose not. Without loss of generality let us assume that \( w' \in [u, v] \). Then \( d_B(w', A) \leq d_B(w, w') + d_B(w, A) \leq 2\delta_0 + R \). Since \( \bar{u} \) is a nearest point projection of \( u \) on \( A \) we have \( d_B(w, \bar{u}) \leq 2\delta_0 + R \). Hence, \( d_B(w, \bar{v}) \leq d_B(w, w') + d_B(w', \bar{u}) \leq 4\delta_0 + R \). Thus we may take \( 4\delta_0 + R \).

We recall that we fixed a vertex \( b_0 \in A \) to define the paths \( c(y, y') \) in the last step. Let \( y_0 \in F_{b_0} \).

Lemma 5.21. Given \( D > 0 \), there is \( D_1 > 0 \) such that the following holds.

If \( d_X(y_0, c(y, y')) \leq D \) then \( d_Y(y_0, c(y, y')) \leq D_1 \).

Proof. Let \( x \in c(y, y') \) be such that \( d_X(y_0, x) \leq D \). This implies that \( d_B(\pi(x), b_0) \leq D \). Suppose \( x \in \gamma_i \), \( 0 \leq i \leq n \). We claim that there is a point of \( \bar{\gamma}_i \) uniformly close to \( y_0 \).

We note that the path \( c(y, y') \) is a concatenation of a finite number of fiber geodesics each of which has length at most \( R_1 \) and some lifts of geodesic segments in the various \( qi \) sections \( \Sigma_{\gamma_i} \)’s and possibly \( \Sigma_{\gamma_{\gamma_i}} \)’s. Suppose \( Q \) is one of these \( qi \) sections and \( c(y, y') \) \( \cap Q \) joins the points \( z \in Q \) to \( w \in Q \) and that there is a fiber geodesic \( \sigma \in c(y, y') \) connecting \( Q \) to the next \( qi \) section \( Q' \). Then both the points \( z \) and \( Q' \cap Q \) belong to one of the \( y_i \)’s or \( y_{\gamma_i} \)’s. Let \( z' = Q' \cap Q \) and \( b' = \pi(Q' \cap \sigma) \). Let \( b \) be the nearest point projection of \( \pi(x) \) on \( A \). It follows that \( d_B(\pi(x), b) \leq D \).
Suppose $x \in \sigma$. Then $\pi(x) = b'$ and $d_\varphi(z', x) \leq R_1$ and therefore $d_X(z', y_0) \leq R_1 + D$. Also $d_{\varphi^*}(b', b_0) \leq d_X(x, y_0) \leq D$. However, by the definition of the modified paths $Q' \cap F_b \in \bar{c}(y, y)$. Hence, lifting $b', b_0$ in $Q'$ we find a $K_2$-quasigeodesic in $X$ joining $z'$ to $Q' \cap F_b$. Thus $d_X(z', Q' \cap F_b) \leq K_2d_B(b', b) + K_2 \leq K_2d_B(b', b_0) + K_2 \leq K_2 + DK_2$. Hence, $d(Q' \cap F_b, y_0) \leq R_1 + D + K_2 + DK_2$.

On the other hand suppose $x$ is contained in the lift of $[\pi(z), \pi(w)]_B$ in $Q$. We note that $\pi(x) \in [\pi(z), \pi(w)]_B$ and $d(\pi(x), A) \leq D$. Hence $d_B(\pi(x), [\pi(z), \pi(w)]_A) \leq D_{[5,20]}(D)$. Hence $d_X(x, Q \cap \bar{c}(y, y)) \leq K_2 + K_2D_{[5,20]}(D)$. Thus $dy([y_0, Q \cap \bar{c}(y, y)] \leq D + K_2 + K_2D_{[5,20]}(D)$.

5.2. An example. For convenience of the reader we briefly illustrate a special case of our main theorem where $B = \mathbb{R}, A = (-\infty, 0)$. This discussion will also be used in the proof of the last proposition of the next section. We shall assume $b_0 = 0$ here.

As in the proof of Lemma 5.24, suppose $Q, Q'$ are two qi sections among the various $\Sigma_i$, $\Sigma_j$'s and let $w' \in Q', z, w \in Q$ are points of $c(y, y')$ where $\pi(w') = \pi(w)$, $d_{\pi(w)}(w, w') \leq R_1$ and the concatenation of the lift say $\alpha$, of $[\pi(z), \pi(w)]$ in $Q$ and the vertical geodesic segment, say $\sigma$, in $F_{\pi(w)}$ is a part of $c(y, y')$. Following are the possibilities.

Case 1. If $w', z \in Y \cap c(y, y')$ then $\alpha * \sigma \subset Y$ and it is the corresponding part of $\bar{c}(y, y')$.

Case 2. $z \in Y, w' \not\in Y$. In this case the modified segment is formed as the concatenation of subsegment of $\alpha$ joining $z$ to $Q \cap F_0$ and the fiber geodesic $[Q \cap F_0, Q' \cap F_0]_0$.

![Figure 9. Case 2](image)

Case 3. $w' \in Y, z \not\in Y$. In this case the modified segment is the concatenation of the segment of $\alpha$ from $Q \cap F_0$ to $w$ and the fiber geodesic segment $\sigma$.

Case 4. $z, w' \not\in Y$. In this case the modified segment is the fiber geodesic $[Q \cap F_0, Q' \cap F_0]_0$.

Here, the dashed lines denote the portion of $c(y, y')$, the thick lines denote the portion of $\bar{c}(y, y')$ and dotted lines are portions of the qi sections $Q, Q'$.

6. Applications, Examples and Related Results

Given a short exact sequence of finitely generated groups there is a natural way to associate a metric graph bundle to it as was mentioned in Example 1.8 of MS12. See also Example 5. Having said that Theorem 5.2 gives the following as an immediate consequence.
Theorem 6.1. Suppose \( 1 \to N \to G \to Q \to 1 \) is a short exact sequence of hyperbolic groups where \( N \) is nonelementary hyperbolic. Suppose \( Q_1 \) is a finitely generated, qi embedded subgroup of \( Q \) and \( G_1 = \pi^{-1}(Q_1) \). Then the \( G_1 \) is hyperbolic and the inclusion \( G_1 \to G \) admits CT.

As we remarked in section 2 (Remark 3) proper embedding is not necessary for the existence of CT map. However, we have the following.

Lemma 6.2. (Proper embedding of the pullback \( Y \)) Suppose \( \pi : X \to B \) is a metric graph bundle, \( g : A \to B \) is a Lipschitz \( k \)-qi embedding and \( p : Y \to A \) is the pullback bundle satisfying the hypotheses of the main theorem in section 5. Then \( Y \) is metrically properly embedded in \( X \). In fact, the distortion function for \( Y \) is the composition of a linear function with \( \eta \)-the common distortion function for all the fibers of the bundle \( X \).

Proof. As was done in the proof of the main theorem, we shall assume that \( g \) is the inclusion map and \( Y = \pi^{-1}(A) \) and \( p \) is the restriction of \( \pi \). Let \( x, y \in Y \) such that \( d_X(x, y) \leq M \). Let \( \pi(x) = b_1 \) and \( \pi(y) = b_2 \). Then, \( d_B(b_1, b_2) \leq M \) and hence \( d_A(b_1, b_2) \leq k + kM \). Let \( [b_1, b_2]_A \) be a geodesic joining \( b_1 \) and \( b_2 \) in \( A \). This is a quasigeodesic in \( B \). By Lemma 5.8, there exists an isometric section \( \gamma \) over \( [b_1, b_2]_A \), through \( x \) in \( Y \). Clearly, \( \gamma \) is a qi lift in \( X \), say \( k'-qi \) lift. We have, \( l_X(\gamma) \leq k'(kM + k) + k' = D(M) \). The concatenation of \( \gamma \) and the fiber geodesic \( [\gamma \cap F_{b_2}, y]_{F_{b_2}} \) is a path, denoted by \( \alpha \), joining \( x \) and \( y \) in \( X \). So,

\[
d_X(\gamma \cap F_{b_2}, y) \leq d_X(\gamma \cap F_{b_2}, x) + d_X(x, y) \leq l_X(\gamma) + d_X(x, y) \leq D(M) + M.
\]
Now, since $F_{b_0}$ is uniformly properly embedded as measured by $\eta$, we have, $d_{b_0}(\gamma \cap F_{b_2}, y) \leq \eta(D(M) + M)$. Now, $\alpha$ lies in $Y$. So, $l_Y(\gamma) \leq kM + k$. As in the case of $X$, the concatenation of $\gamma$ and the fiber geodesic $[\gamma \cap F_{b_2}, y]_{F_{b_2}}$ is a path joining $x$ and $y$ in $Y$. Then,

$$d_Y(x, y) \leq l_Y(\alpha) \leq l_Y(\gamma) + d_Y(\gamma \cap F_{b_2}, y) \leq kM + k + d_{b_2}(\gamma \cap F_{b_2}, y).$$

Therefore, $d_Y(x, y) \leq kM + k + \eta(D(M) + M)$. Setting $\eta_0(M) := kM + k + \eta(D(M) + M)$, we have the following: For all $x, y \in Y$, $d(x, y) \leq M$ implies $d_Y(x, y) \leq \eta_0(M)$.

6.1. Some results on the boundary of a hyperbolic metric bundle. For this subsection we let $\pi : X \to B$ be a $\delta$-hyperbolic $(y, c)$-metric bundle or $\eta$-metric graph bundle over $B$ satisfying the hypothesis H1 and H2 of section 5. For simplicity paths in $B$ will be assumed to be continuous, arc length parametrized in the case of a metric bundle and dotted edge paths for the case of a metric graph bundle. We shall also assume that any $(1, 1)$-quasigeodesic in $B$ has a $K_0$-qi lift through any point of $X$, by dint of the path lifting lemma.

Lemma 6.3. Suppose $\alpha, \beta : [0, \infty) \to B$ are two $k$-quasigeodesic rays for some $k \geq 1$ with $\alpha(\infty) = \beta(\infty) = \xi$. Suppose $\tilde{\beta}$ is a $K$-qi lift of $\beta$ for some $K \geq 1$. Then there is a $K'$-qi lift $\tilde{\alpha}$ of $\alpha$ such that $\tilde{\alpha}(\infty) = \tilde{\beta}(\infty)$ where $K'$ depends on $k, K, d_B(\alpha(0), \beta(0))$ and the various parameters of the metric (graph) bundle.

Proof. Suppose $\alpha, \beta : [0, \infty) \to B$ are two $k$-quasigeodesic rays for some $k \geq 1$ with $\alpha(\infty) = \beta(\infty) = \xi$. This means $Hd(\alpha, \beta) < \infty$. Let $R = Hd(\alpha, \beta)$. Then for all $s \in [0, \infty)$ there is $t = t(s) \in [0, \infty)$ such that $d_B(\alpha(s), \beta(t)) \leq R$. Let $\phi_{ts} : F_{\beta(t)} \to F_{\alpha(s)}$ be fiber identification maps such that $d_X(x, \phi_{ts}(x)) \leq 3c + 3cR$ for all $x \in F_{\beta(t)}$, $t \in [0, \infty)$ where $c = 1$ for metric graph bundles. (See Lemma 3.10) Let $\tilde{\beta}$ be a $K$-qi lift of $\beta$. Now, for all $s \in [0, \infty)$ we define $\tilde{\alpha}(s) = \phi_{ts}(\tilde{\beta}(t))$. It is easy to verify that $\tilde{\alpha}$ thus defined is a uniform qi lift of $\alpha$. Also clearly $\tilde{\alpha} \subset N_{3c+3cR}(\tilde{\beta})$. It follows that $\tilde{\alpha}(\infty) = \tilde{\beta}(\infty)$. \hfill \Box

Corollary 6.4. Let $\xi \in \partial B$ and let $\alpha$ be a quasigeodesic ray in $B$ joining $b$ to $\xi$. Let $\partial^\xi X := \{\gamma(\infty) : \gamma$ is a qi lift of $\alpha\}$.

Then $\partial^\xi X$ is independent of $\alpha$; it is determined by $\xi$.

Due to the above corollary we shall use the notation $\partial^\xi X$ for all $\xi \in \partial B$ without further explanation. The following proposition is motivated by a similar result proved by Bowditch ([Bowd02 Proposition 8.2]).

Proposition 6.5. Let $b \in B$ be an arbitrary point and $F = F_b$. Then we have

$$\partial X = \Lambda(F) \cup (\cup_{\xi \in \partial B} \partial^\xi X).$$

Proof. We first fix a point $x \in F$. Let $\gamma$ be a quasigeodesic ray in $X$ starting from $x$. Let $b_n = \pi(\gamma(n))$. Let $\alpha_n$ be a $(1, 1)$-quasigeodesic in $B$ joining $b$ to $b_n$. Let $\tilde{\alpha}_n$ be a $K_0$-qi lift of $\alpha_n$ starting from $\gamma(n)$ and ending at $\tilde{\alpha}_n(b) = x_n \in F$. There are two possibilities.

Suppose $\{x_n\}$ is an unbounded sequence. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x) \to \infty$. Since $\tilde{\alpha}_{n_k}$’s are uniform quasigeodesics in $X$ whose distance from $x$ is going to infinity, by Lemma 2.35 $x_{n_k} \to \gamma(\infty)$ and thus $\gamma(\infty) \in \Lambda(F)$.

Otherwise suppose $\{x_n\}$ is a bounded sequence.
Claim: In this case \( \pi \circ \gamma \) is a quasigeodesic ray.

Proof of claim: We note that by stability of quasigeodesics (Corollary 2.23) and slimness of triangles (Lemma 2.24), \( \text{Hd}(\alpha_n, \gamma_{[0,n]}) \) is uniformly small for all \( n \). In particular \( d_X(\hat{\alpha}_m(m), \hat{\alpha}_n) = d_X(\gamma(m), \hat{\alpha}_n) \) is uniformly small for all \( n \geq m \). These imply that \( \text{Hd}(\alpha_n, (\pi \circ \gamma)_{[0,n]}) \) is uniformly small for all \( n \) and \( d_B(b_m, \alpha_n) \) is uniformly small for all \( n \geq m \). We note that \( d_B(b_m, \alpha_n) \rightarrow \infty \) for otherwise \( d(\gamma(n), x) \) will be bounded. It is also clear that \( \lim_{m,n \rightarrow \infty} (b_m, b_n) = \infty \). Let \( \xi = \lim_{m,n \rightarrow \infty} b_n \) and let \( \alpha \) be a \( \kappa_0 \)-quasigeodesic ray in \( B \) joining \( b \) to \( \xi \). Now, to show that \( \pi \circ \gamma \) is a quasigeodesic it is enough to show by Lemma 2.41 that \( \pi \circ \gamma \) is (1) uniformly close to \( \alpha \) and (2) properly embedded.

Fix an arbitrary \( m \in \mathbb{N} \) and consider all \( n \geq m \). Since \( \lim_{n \rightarrow \infty} b_n = \alpha(\infty) = \xi \), by Lemma 2.40 (2) for any \( \kappa_0 \)-quasigeodesic ray \( \beta_n \) joining \( b_n \) to \( \xi \) we have \( d(b_n, \beta_n) \rightarrow \infty \). Since the triangles with vertices \( b_n \), \( b \), \( \xi \) are uniformly slimmer by Lemma 2.40 and \( d_B(b_m, \alpha_n) \) are uniformly small it follows that \( b_m \) is uniformly close to \( \alpha \). This shows (1). Next suppose \( d_B(b_n, b_m) \leq D \) for some \( D \geq 0 \) and \( m, n \in \mathbb{N} \). We claim that \( d_X(\gamma(m), \gamma(n)) \) is uniformly small. Without loss of generality suppose \( n \geq m \). Suppose \( d_X(\gamma(i), \gamma(j)) \leq R \) for all \( i \leq j \) and some constant \( R \geq 0 \). Let \( y_{m,n} \in \hat{\alpha}_n \) be such that \( d_X(\gamma(m), y_{m,n}) \leq R \). Then \( d_B(\pi(y_{m,n}), b_m) \leq R + D \). But \( \hat{\alpha}_n \) is \( \kappa_0 \)-qi lift of \( \alpha_n \). It follows that \( d_X(y_{m,n}, \gamma(n)) \leq K_0(R + D) + K_0 \). Hence, \( d_X(\gamma(m), \gamma(n)) \leq R + K_0(R + D) + K_0 \). Since \( \gamma \) is quasigeodesic it follows that \( (n - m) \) is uniformly small. This proves (2) and along with this the claim.

Finally since \( \gamma \) is a lift of \( \pi \circ \gamma \), \( \gamma(\infty) \in \partial^F X \). \( \square \)

Corollary 6.6. Suppose \( F \) is a bounded metric space. Then \( \partial X = \cup_{\xi \in \partial B} \partial^F X \).

Next suppose \( \Sigma_1, \Sigma_2 \) are two qi sections and \( L = L(\Sigma_1, \Sigma_2) \) then by Corollary 4.7 there is a metric graph subbundle \( \pi_Z : Z \rightarrow B \) of \( X \) where the bundle map \( Z \rightarrow X \) is a qi embedding onto a finite neighborhood of \( L \). It follows that \( Z \) is hyperbolic and fibers are uniformly quasimetric to intervals. Therefore, the conclusion of Lemma 6.3 applies to the metric bundle \( Z \) too. Hence, informally speaking we have the following.

Corollary 6.7. For any ladder \( L = L(\Sigma_1, \Sigma_2) \) we have \( \partial L = \bigcup_{\xi \in \partial B} \partial^F L \).

Lemma 6.8. Suppose \( \alpha_n : [0, \infty) \rightarrow B \) is a sequence of uniform quasigeodesic rays starting from \( b \in B \) and \( \hat{\alpha}_n \) is a uniform qi lift of \( \alpha_n \) for all \( n \) such that the set \( \{\hat{\alpha}_n(0)\} \) has finite diameter. If \( \hat{\alpha}_n(\infty) \rightarrow z \in \partial X \) then \( \lim_{n \rightarrow \infty} \alpha_n(\infty) \) exists and if the limit is \( \xi \) and if \( \alpha : [0, \infty) \rightarrow B \) is a \( \kappa_0 \)-quasigeodesic ray joining \( b \) to \( \xi \) then there is a uniform qi lift \( \hat{\alpha} \) of \( \alpha \) such that \( \hat{\alpha}(\infty) = z \).

Proof. Since \( \hat{\alpha}_n(\infty) \rightarrow \xi \) there is a constant \( D \) such that for all \( M > 0 \) there is \( N > 0 \) with \( \text{Hd}(\hat{\alpha}_m_{[0,M]}, \hat{\alpha}_n_{[0,M]}) \leq D \) for all \( m, n \geq N \) by Lemma 2.40 (1). It follows that for all \( M > 0 \), \( \text{Hd}(\alpha_m_{[0,M]}, \alpha_n_{[0,M]}) \leq D \) for all \( m, n \geq N \). Hence, again by Lemma 2.40 (1) \( \alpha_n(\infty) \) converges to a point of \( \partial_B B \), say \( \xi \). Let \( \alpha \) be a \( \kappa_0 \)-quasigeodesic ray in \( B \) joining \( b \) to \( \xi \). We claim \( z \in \partial^F X \). Given any \( t \in [0, \infty) \) by Lemma 2.40 \( d(\alpha(t), \alpha_n) \leq D \) and \( d_X(\hat{\alpha}_m(t), \hat{\alpha}_n(t')) \leq D \) for some constant \( D \), \( t' \in [0, \infty) \) and for all \( m, n \geq N = N(t) \). Define \( \hat{\alpha}(t) = \phi_{uv}(\hat{\alpha}_{N(t)}(t)) \) where \( u = \alpha_{N(t)}(t), v = \alpha(t') \) and \( \phi_{uv} \) is a fiber identification map. It is now easy to check that this defines a qi section of \( \alpha \) and \( z = \hat{\alpha}(\infty) \).
Corollary 6.9. If \( X \) is a metric (graph) bundle over \( B \) where a fiber \( F \) has finite diameter then the map \( \partial X = \cup_{\xi \in \partial B} \partial \xi X \to \partial B \) defined by sending \( \partial \xi X \) to \( \xi \) for all \( \xi \in \partial B \) is continuous.

6.2. Cannon-Thurston lamination. Rest of the paper is devoted to properties of the boundary of metric (graph) bundles and Cannon-Thurston maps. We recall that \( \text{qi} \) sections, ladders etc for a metric bundle are defined as transport of the same from its canonical metric graph bundle. All the results in the rest of the section are meant for metric bundles as well as metric graph bundles. However, using the dictionary provided by Proposition 4.1 we shall prove the results only for the metric graph bundles.

Convention 6.10. (1) For the rest of the paper we shall assume that \( \pi: X \to B \) is a \( \delta \)-hyperbolic \((\eta,c)\)-metric bundle or \( \eta \)-metric graph bundle over \( B \) satisfying the hypothesis \( H1, H2, H3 \) and \( H4 \) of section 5. (2) By Proposition 2.38 any point of \( \partial B \) can be joined to any point of \( B \cup \partial B \) and any point of \( \partial X \) can be joined to \( X \cup \partial X \) by a uniform quasigeodesic ray or line. We shall assume that these are \( \kappa_0 \)-quasigeodesics. (3) We shall assume that any \( (1,1) \)-quasigeodesic in \( B \) has \( c \)-qi lift in \( X \) using the path lifting lemma for metric (graph) bundles.

Suppose \( b_0 \in B \) is an arbitrary point and \( F = F_{b_0} \). Then we know that the inclusion \( i = i_{F,X}: F \to X \) admits a CT map \( \partial i : \partial F \to \partial X \). Now, following Mitra ([Mit97]) we define the following.

Definition 6.11. (1) (Cannon-Thurston lamination) Let \( \Lambda_X^F(F) = \{(\alpha, \beta) \in \partial^{(2)}F : \partial i(\alpha) = \partial i(\beta)\} \).

(2) Suppose \( \xi \in \partial B \). Let \( \Lambda_X^\xi(F) = \{(\alpha, \beta) \in \partial^{(2)}F : \partial i(\alpha) = \partial i(\beta) \in \partial \xi X\} \).

We shall denote \( \Lambda_X^\xi(F) \) simply by \( \Lambda_X^\xi(F) \) when \( X \) is understood.

In this subsection we are going to discuss the various properites of the CT lamination. First we need some definitions. Suppose \( b \in B \) and \( z \in \partial F_b \). For all \( s \in B \) we have the fiber identification map \( \phi_{bs} : F_b \to F_s \) which is a uniform quasimorphism. This induces a bijection \( \partial \phi_{bs} : \partial F_b \to \partial F_s \). Let \( z_s = \partial \phi_{bs}(z) \). For the rest of the subsection by ‘quasigeodesic rays’ or ‘lines’ we shall always mean \( \kappa_0 \) quasigeodesic rays and lines in the fibers of a metric (graph) bundle.

Definition 6.12. (1) (Semi-infinite ladders) Suppose \( \Sigma_1 \) is a \( \text{qi} \) section over \( B \) in \( X \). For all \( s \in B \) let \( \gamma_s \subset F_s \) be a quasigeodesic ray joining \( \Sigma_1 \cap F_s \) to \( z_s \). The union of all the rays will be denoted by \( L(\Sigma_1; z) \).

This set is coarsely well-defined by Lemma 2.39. We shall refer to this as the semi-infinite ladder defined by \( \Sigma_1 \) and \( z \).

(2) (Bi-infinite ladders) Suppose \( b \in B \) and \( z, z' \in \partial F_b \). Now for all \( s \in B \) join \( z_s = \partial \phi_{bs}(z) \) to \( z'_s = \partial \phi_{bs}(z') \) by a quasigeodesic line in \( F_s \). The union of all these lines will be denote by \( L(z; z') \).

As before, this set is coarsely well-defined by Lemma 2.39. We shall refer to this as the bi-infinite ladder defined by \( z \) and \( z' \).

We shall refer to either of these ladders as an ‘infinite girth ladder’.

Lemma 6.13. (Properties of infinite girth ladders) Suppose \( L \) is an infinite girth ladder.
(1) (Coarse retract) There is a uniformly coarsely Lipschitz retraction \( \pi_L : X \to \mathbb{L} \) such that for all \( b \in B \) and \( x \in F_b \), \( \pi_L(x) \) is a (uniform approximate) nearest point projection of \( x \) in \( \mathbb{L} \cap F_b \).

Consequently, infinite girth ladders are uniformly quasiconvex.

(2) (QI section in ladders) Uniform QI sections exist through any point of \( L \) contained in \( L \).

(3) (QI sections coarsely bisects ladders) Any QI section in \( L \) coarsely bisects it into two subladders.

Proof. We shall briefly indicate the proofs comparing with the proof of the analogous result for finite girth ladders. (2) follows exactly as Lemma 4.5. (2) is immediate from (1). In fact given \( x \in \mathbb{L} \) one takes a \( K_0 \)-QI section \( \Sigma \) in \( X \) containing \( x \) and then \( \pi_L(\Sigma) \) is a required QI section. Therefore, we are left with proving (1). This is an exact analog of Proposition 4.6.1. The reader is referred to Mitra (Theorem 4.6) for supporting arguments.

\[ \square \]

Convection 6.14. All semi-infinite ladders \( \mathbb{L}(\Sigma; z) \) are formed by \( K_0 \)-QI section \( \Sigma \). We shall assume that through any point of a bi-infinite ladder there is a \( K_0 \)-QI section contained in the ladder. Also all infinite girth ladders are assumed to be \( \bar{\lambda}_0 \)-quasiconvex.

Following Mitra (Mit97) we have similar consequences of the coarse bisection of ladders by QI sections in this context. Let \( b_0 \in B \) and \( F = F_{b_0} \) as in Definition 6.2.

Suppose \( (z_1, z_2) \in \mathcal{N} = \Lambda_X^\prime(F) \) and \( \mathbb{L} = \mathbb{L}(z_1; z_2) \). Let \( i_{F,X} : F \to X \) denote the inclusion map and \( \partial i_{F,X} : \partial F \to \partial X \) denote the CT map.

Lemma 6.15. Suppose \( \Sigma \) is any QI section contained in \( L \). Then \( \partial i_{F,X}(z_i) \in \Lambda(\Sigma), i = 1, 2 \).

Proof. Let \( \Sigma \) be a QI section contained in \( \mathbb{L} \). Then \( \Sigma \) coarsely separates \( L \) in \( X \) into \( \mathbb{L}_1 = \mathbb{L}(\Sigma; z_1) \) and \( \mathbb{L}_2 = \mathbb{L}(\Sigma; z_2) \). We note that \( \partial i_{F,X}(z_1) = \partial i_{F,X}(z_2) \in \Lambda(\mathbb{L}_1) \cap \Lambda(\mathbb{L}_2) \). Hence we are done by Lemma 2.54. \[ \square \]

Lemma 6.16. \( \Lambda_X^\prime(F) = \bigcup_{\xi \in \partial B} \Lambda_X^{\prime}(F) \).

Proof. We need to show that \( \Lambda_X^\prime(F) \subset \bigcup_{\xi \in \partial B} \Lambda_X^{\prime}(F) \) since the reverse inclusion is automatic. Suppose \( (z_1, z_2) \in \mathcal{N} \). Let \( \mathbb{L} = \mathbb{L}(z; z') \). Let \( \sigma : B \to X \) be QI section with image \( \Sigma \) contained in \( \mathbb{L} \). By Lemma 6.15 \( \partial i_{F,X}(z_1) \in \Lambda(\Sigma) \). But \( \Lambda(\Sigma) = \partial \sigma(\partial B) \). Hence, there is a \( \kappa_0 \)-quasigeodesic ray \( \beta : [0, \infty) \to B \) such that \( \partial \sigma(\beta(\infty)) = \partial i_{F,X}(z_1) \). Let \( \xi = \beta(\infty) \). If \( \beta = \sigma \circ \beta \) then \( \tilde{\beta} \) is a QI lift of \( \beta \) and \( \partial i_{F,X}(z_1) = \beta(\infty) \in \Lambda_X^{\prime}(F) \). \[ \square \]

Remark 18. From the proof of Lemma 6.10 it follows that the point \( \xi \) is unique since \( \partial \sigma \) is injective; it is also independent of the \( \sigma \) chosen. In particular any two QI lifts of \( \beta \) contained in two QI sections in \( \mathbb{L} \) are asymptotic.

Let \( \beta : [0, \infty) \to B \) be a continuous, arc length parametrized \( \kappa_0 \)-quasigeodesic in \( B \) with \( \beta(0) = b_0 \) and \( \beta(\infty) = \xi \) as in the proof of Lemma 6.10. Let \( A = \beta([0, \infty)) \). Let \( Y = \pi^{-1}(A) \) be the restriction of the bundle \( X \) over \( A \). Let \( i_{Y,X} : Y \to X \), \( i_{Y,Y} : F \to Y \) inclusion maps.

Lemma 6.17. If \( (z_1, z_2) \in \Lambda_X^{\prime} \) then \( \partial i_{Y,Y}(z_1) = \partial i_{Y,Y}(z_2) \), i.e. \( (z_1, z_2) \in \Lambda_X^{\prime,Y}(F) \).
Proof. Let $\mathbb{L} = L(z_1; z_2)$ and let $\gamma : \mathbb{R} \to F$ be a $\kappa_0$-quasigeodesic line in $F$ joining $z_1$ to $z_2$ such that $\text{Im}(\gamma) = L \cap F$. Let $\Sigma_n$ be any qi section in $L$ passing through $\gamma(n)$, $n \in \mathbb{Z}$. Then by the remark above $\Sigma_m \cap Y$ and $\Sigma_n \cap Y$ are asymptotic for all $m, n \in \mathbb{Z}$ in $X$. Since $Y$ is properly embedded in $X$ by Lemma 6.2, they are still asymptotic in $Y$. Clearly $d_Y(\gamma(0), \Sigma_n \cap Y) \to \infty$ as $n \to \pm \infty$. Thus by Lemma 2.46(1) $\lim_{n \to \pm \infty} \gamma(n) = \hat{\beta}_0(\infty)$ in $Y$ where $\hat{\beta}_0$ is the lift of $\beta$ in $\Sigma_0$. This completes the proof.

Corollary 6.18. Since $\mathbb{L} \cap Y$ is qi embedded in $Y$ it follows that $z_1, z_2$ are identified under the CT map $\gamma \to \mathbb{L}$.

Corollary 6.19. Let $\hat{\beta}$ be any qi lift of $\beta$ in $\mathbb{L}$. Then $\hat{\beta}(\infty) = \partial_i F, Y(z_1)$. In particular any two qi lifts of $\beta$ in $\mathbb{L}$ are asymptotic.

Proof. We know that $\hat{\beta}$ coarsely separates $\mathbb{L} \cap Y$ into two semi-infinite ladders, $\mathbb{L}^+$ and $\mathbb{L}^-$ in $Y$. It follows that $\Lambda(\mathbb{L}^+) \cap \Lambda(\mathbb{L}^-) = \Lambda(\hat{\beta}) = \hat{\beta}(\infty)$. It then follows that the limit of $\gamma(n)$ in $\partial \mathbb{L}$ is $\hat{\beta}(\infty)$.

Corollary 6.20. $\partial(\mathbb{L} \cap Y)$ is a point. In particular, the limit set of $\mathbb{L} \cap Y$ in $Y$ and also in $X$ is a point.

Corollary 6.21. We have $\Lambda'(Y) = \Lambda'_{\xi,Y}(F) = \Lambda'_{\xi,X}(F)$.

In particular, each $\Lambda'_\xi$ is a closed subset of $\partial^{(2)} F$.

Proof. The first equality follows from Lemma 6.16 applied to the metric bundle $Y$ over $X$. We will now prove the second one. Since $\partial_i F, X = \partial_i Y, X \circ \partial F, Y$, clearly $\Lambda'_{\xi,Y}(F) \subset \Lambda'_{\xi,X}(F)$. The opposite inclusion is an immediate consequence of Lemma 6.17.

Since $\partial_i F, Y$ is continuous it follows that $\Lambda'_\xi$ is a closed subset of $\partial^{(2)} F$.

The following three results are motivated by similar results proved in [Mit97]. The proof ideas are very similar except that we got rid of the group actions and in our setting properness is never needed.

Lemma 6.22. Suppose $\xi_1 \neq \xi_2 \in \partial B$. If $(z_i, w_i) \in \Lambda'_{\xi_i}$, $i = 1, 2$ then $\{z_1, w_1\} \cap \{z_2, w_2\} = \emptyset$. In particular, $\Lambda'_{\xi_1} \cap \Lambda'_{\xi_2} = \emptyset$.

Proof. Suppose $\gamma_i$ is a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi_i$, $i = 1, 2$. Suppose $\gamma_i$ is a qi lift of $\gamma$ such that $\partial_i(z_i) = \gamma_i(\infty)$. Since $\gamma_1(\infty) = \gamma_2(\infty)$ if and only if $\gamma_i$’s are asymptotic in which case $\gamma_i$’s would also be asymptotic because $\pi : X \to B$ is 1-Lipschitz. This would be a contradiction since $\xi_1 \neq \xi_2$.

Lemma 6.23. Suppose $\xi_1 \neq \xi_2 \in \partial B$. Given $D > 0$ there exists $R = R_{\text{2.23}}(D) > 0$ such that the following holds:

Suppose $\gamma_1$ is leaf of $\Lambda'_{\xi_1}$ and $\gamma_2$ is a leaf of $\Lambda'_{\xi_2}$. Then $\gamma_1 \cap N_D(\gamma_2)$ has diameter less than $R$.

Proof. Let $\alpha$ be a $\kappa_0$-quasigeodesic line in $B$ joining $\xi_1, \xi_2$. Let $b_0' \in \alpha$ be a $1$-approximate nearest point projection of $b_0$ on $\alpha$. Let $\beta$ be a $1$-quasigeodesic in $B$ joining $b_0$ to $b_0'$. Let $\alpha_i$ be the concatenation of $\beta$ with the portion of $\alpha$ joining $b_0'$ to $\xi_i$, $i = 1, 2$. By stability of quasigeodesics $\kappa_0$-quasigeodesics in $B$ are $D_{\text{2.23}}(\delta_0, \kappa_0)$-quasiconvex. Let $K = D_{\text{2.23}}(\delta_0, \kappa_0)$. Hence, $\alpha_i$’s are $K_{\text{2.28}}(\delta_0, K, \kappa_0, 1)$-quasigeodesics by Lemma 2.28(2). Let $k = K_{\text{2.28}}(\delta_0, K, \kappa_0, 1)$. 


Next suppose \( x_i, x_i' \in \gamma_i \), \( i = 1, 2 \) are such that \( d_F(x_1, x_2) \leq D \) and \( d_F(x_1', x_2') \leq D \). Let \( \Sigma_i, \Sigma_i' \) be two qi sections in \( L = L(\gamma_i(\infty), \gamma_i(-\infty)) \) passing through \( x_i \) and \( x_i' \) respectively, \( i = 1, 2 \).

Let \( \partial \gamma \) be lifts of \( \gamma \) in \( X \). The first case is impossible to arise since \( q_i \) lifts in \( \partial \gamma \). Moreover, if \( \partial \gamma \) is qi and \( \gamma \) is qi in \( X \), then \( \partial \gamma \) is also qi in \( X \).

We will show that this case is also not possible. We fix a base point \( \Lambda \) in \( Y \). We now look at the quasigeodesic hexagon in \( X \) with vertices \( x_i, x_i', \xi_i \) where \( \xi_i \)'s and \( \partial \gamma \)'s form four sides and the other two sides are formed by 1-quasigeodesics joining \( x_1 \) to \( x_2 \) and \( x_1' \) to \( x_2' \) respectively. We note that the infinite sides of this polygon are all \((k \hat{K}_0 + k + \hat{K}_0)\)-quasigeodesics. Let \( k = k \hat{K}_0 + k + \hat{K}_0 \). Hence, such a hexagon is \( D_{\hat{K}_0}(\delta, \hat{k}) \)-slim by Corollary 2.46. Let \( R_1 = D_{\hat{K}_0}(\delta, \hat{k}) \). Let \( b_2 \) be a point on \( \alpha_2 \) such that \( d_B(b_2, \alpha_1) = 2D + R_1 + 1 = R \), say and let \( y_2 = \partial \gamma(b_2) \). Then \( y_2 \in N_R(\Sigma_2) \). In particular, \( y_2 \in N_R(\Sigma_2) \). Hence, by Lemma 4.12 \( d_{\Sigma_2} \Sigma_2 \cap F_{b_2}, \Sigma_2 \cap F_{b_2} \) \( \leq D_{\hat{K}_0}(\delta \hat{k}) \). It follows by bounded flaring that \( d_{\Sigma_2}(x_2, x_2') \leq \mu_{\Sigma_2}(D_{\hat{K}_0}(\delta \hat{k})) \).

**Lemma 6.24.** If \( \xi_n \to \xi \) in \( \partial B \), \( (z_n, w_n) \in \Lambda'_{\xi_n} \) and \( (z_n, w_n) \to (z, w) \in \partial(\gamma)X \). Then \( (z, w) \in \Lambda'_{\xi} \).

**Proof.** Since \( \partial i_{F,X}(z_n) = \partial i_{F,X}(w_n) \) for all \( n \) and \( \partial i_{F,X} \) is continuous it follows that \( \partial i_{F,X}(z) = \partial i_{F,X}(w) \) whence \( (z, w) \in \Lambda' \). Let \( [z_n, w_n], [z_n, z], [w_n, w] \) and \( [z, w] \) denote \( \kappa_0 \)-quasigeodesic lines in \( F \) joining these pairs of points. Let \( x \in [z, w] \cap F \). Since \( z_n \to z \) and \( w_n \to w \) by Lemma 2.46(1) \( d_{x}(x, [z_n, z]) \to \infty \) and \( d_{x}(x, [w_n, w]) \to \infty \). Hence, by Corollary 2.46 there is \( N \in \mathbb{N} \) such that \( d_{x}(x, [z_n, w_n]) \leq \mathcal{R} = \mathcal{R}(\delta_0, \kappa_0, 4) \) for all \( n \geq N \). Now, let \( x_n \in [z_n, w_n] \) such that \( d_{xy}(x, x_n) \leq \mathcal{R} \). Let \( \alpha_n \) be a \( \kappa_0 \)-quasigeodesic ray in \( B \) joining \( b \) to \( \xi_n \) and let \( \alpha \) be a \( \kappa_0 \)-quasigeodesic in \( B \) joining \( b \) to \( \xi \). Then we know that there is a uniform qi lift \( \alpha_n \) of each \( \alpha_n \), \( n \geq N \) such that \( \partial \gamma(b) = x_n \). The rest of the arguments then follows from Lemma 6.25.

The following result is motivated by a similar result proved in [KS] for trees of hyperbolic spaces which in turn was suggested by Mahan Mj. We gratefully acknowledge the same.

Suppose we have the hypothesis of Theorem 6.2. We identify \( Y \) as a subspace of \( X \). Similarly \( \partial A \) is identified as a subset of \( \partial B \). With that in mind we have the following:

**Theorem 6.25.** Suppose we have the hypotheses of the main theorem. Suppose \( \gamma \) is a quasigeodesic line in \( Y \) such that \( \gamma(\infty) \) and \( \gamma(-\infty) \) are identified by CT map \( \partial i_{Y,X} : \partial Y \to \partial X \). Then \( \pi(\gamma) \) is bounded.

In particular, given a fiber \( F_b = F \) of the bundle, \( \gamma \) is within a finite Hausdorff distance from a \( \kappa_0 \)-quasigeodesic line, say \( \beta \), of \( F \) such that \( \partial i_{F,Y}(\beta(\pm \infty)) = \gamma(\pm \infty) \). Moreover, \( (\beta(\infty), \beta(-\infty)) \in \Lambda'_{\xi} \) for some \( \xi \in \partial B \setminus \partial A \).

**Proof.** We fix a base point \( b \) in \( B \) and let \( F = F_b \) for the proof. Let us denote by \( \Lambda_Y(F) \) the limit set of \( F \) in \( \partial Y \) for the purpose of the proof. By Proposition 6.2 there are three possibilities for the points \( \gamma(\pm \infty) \).

**Case 1.** Suppose \( \gamma(\infty) \in \partial A'Y \) and \( \gamma(-\infty) \in \partial A'Y \) for some \( \xi_1, \xi_2 \in \partial A \). But this case is impossible to arise since qi lifts in \( Y \) of quasigeodesics in \( B \) are also qi lifts in \( X \) and since \( Y \) is properly embedded in \( X \) by Lemma 6.2 any two such lifts are asymptotic in \( X \) if and only if they are asymptotic in \( Y \).

**Case 2:** \( \gamma(\infty) \in \partial A'Y \) for some \( \xi \in \partial A \) and \( \gamma(-\infty) \in \Lambda_Y(F) \setminus \bigcup_{\xi \in \partial B} \partial A' \). We will show that this case is also not possible.
Let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $A$ joining $b$ to $\xi$ and let $\tilde{\alpha}$ be a $K_0$-qi lift of $\alpha$ such that $\tilde{\alpha}(\infty) = \gamma(\infty)$. Also let $\beta$ be a $\kappa_0$-quasigeodesic ray in $F$ such that $\partial F,Y(\beta(\infty)) = \gamma(\infty)$. Now, for all $n \in \mathbb{N}$ let $\Sigma_n$ be a $K_0$-qi section in $X$ passing through $\beta(n)$ and let $\mathbb{L}_n = \mathbb{L}(\Sigma_n, \beta(\infty))$. Then $\mathbb{L}_n$ is $\lambda_0$-quasiconvex in $X$. Clearly $\gamma(\infty) = \tilde{\alpha}(\infty) \in \Lambda(\mathbb{L}_n)$. Hence, by Lemma 2.59 $\tilde{\alpha}$ is asymptotic to $\mathbb{L}_n$. It follows by Proposition 4.10 and Lemma 4.14 that $\mathbb{L}_n(\tilde{\alpha})$ is a uniform qi lift of $\alpha$ and it is asymptotic to $\tilde{\alpha}$. Since $Y$ properly embedded in $X$ by Lemma 4.2 it follows that these qi lifts are asymptotic in $Y$ too. Now, it follows from Lemma 2.40 that $\lim_{n \to \infty} \beta(n) = \tilde{\alpha}(\infty)$. This gives a contradiction.

Therefore, the only possibility is the following.

**Case 3:** $\gamma(\pm \infty) \in \Lambda_Y(F) \setminus \cup_{\xi \in \partial F} \partial Y$.

Let $z, z' \in \partial F$ such that $\partial i_{F,Y}(z) = \gamma(\infty)$ and $\partial i_{F,Y}(z') = \gamma(-\infty)$. We have $(z, z') \in \Lambda'_Y(F)$ and hence $(z, z') \in \Lambda'_\xi,X(F)$ for some $\xi \in \partial B$ by Lemma 6.10.

By Corollary 6.21 we have $\xi \in \partial B \setminus \partial A$. Let $\mathbb{L} = \mathbb{L}(z; z')$ be the bi-infinite ladder in $X$ formed by $z, z'$. Let $\beta$ be an arc length parametrized $\kappa_0$-quasigeodesic line in $F$ joining $z, z'$ where $\beta = \mathbb{L} \cap F$. Let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi$.

Let $\Sigma_n$ be a $K_0$-qi section in $L$ passing through $\beta(n)$, $n \in \mathbb{N}$. By Corollary 6.19 qi lifts of $\alpha$ contained in these qi sections are asymptotic. Denote the qi section of $\alpha$ contained in $\Sigma_n$ by $\tilde{\alpha}_n$. We note that these are $k = (\bar{K}_0\kappa_0 + \bar{K}_0 + \kappa_0)$-quasigeodesics. Hence, by Lemma 2.39 given $m, n \in \mathbb{N}$ we have $\tilde{\alpha}_n(i) \in N_R(\tilde{\alpha}_m)$ and $\tilde{\alpha}_m(i) \in N_R(\tilde{\alpha}_n)$ where $R = D_2(\tilde{\alpha}_n, \tilde{\alpha}_m)$ as long as $\tilde{\alpha}(i)$ (resp. $\tilde{\alpha}_m(i)$) is not contained in the $R$-neighborhood of any 1-quasigeodesic joining $\beta(m), \beta(n)$. It follows that for such $i$ we have $\tilde{\alpha}_n(i) \in N_R(\tilde{\alpha}_m(i))$. Hence, by Lemma 4.12 we have $d_{\mathbb{L}}(\tilde{\alpha}_n(i), \tilde{\alpha}_m(i)) \leq R_1 = R_2(\bar{M}, \bar{M}, K_0)$. Let $R_2 = \max(R_1, 2M, K_0)$. Thus for all $n \in \mathbb{N}$, $U_n = U_{R_2}(\Sigma_n, \Sigma_n) \neq \emptyset$. Let $b_n \in U_n$ be a nearest point projection of $b_0$ on $U_n$ and let $\gamma'_n$ be a nearest point projection of $b_n$ on $A$. Then it follows from Lemma 6.18 and Lemma 2.28 that the concatenation of the segments of $\tilde{\alpha}_n, \tilde{\alpha}_m$ over the portion of $\alpha$ joining $b_0, b_n'$ and the fiber geodesic segment $L \cap F'_{\gamma'_n}$ is a uniform quasigeodesic in $Y$ joining $\beta(\pm n)$. Call it $\gamma'_n$. Since $\lim_{n \to \infty} \beta(n) \neq \lim_{n \to \infty} \beta(\pm n)$ in $Y$ there is a constant $D \geq 0$ such that $d_Y(\beta(0), \gamma'_n) \leq D$ by Lemma 2.39. We claim that this means $d_B(b_0, b'_n)$ is uniformly bounded. In fact $d_Y(\beta(0), \gamma'_n) \to \infty$ by Lemma 4.12. Thus for all large $n$ we have $d_Y(\beta(0), L \cap F'_{\gamma'_n}) \leq D$ whence $d_B(b_0, b'_n) \leq D$. It follows from Proposition 4.3 that the Hausdorff distance of $L \cap F'_{\gamma'_n}$ and the segment of $\beta$ between $\beta(n)$ and $\beta(-n)$ is at most $(1 + 2K_0)(\bar{M}, \bar{M}, K_0)$. Since $\beta$ is a proper embedding in $Y$ it follows by Lemma 2.43 that $\beta$ is a uniform quasigeodesic in $Y$. Suppose $\beta$ is a $K$-quasigeodesic in $Y$ and $\gamma$ is a $K'$-quasigeodesic. Since $Y$ is $\delta$-hyperbolic $Hd(\beta, \gamma) \leq R_2(\delta, K, K_2)$. Thus $diam(\pi(\gamma)) \leq R_{240}(\delta, K, K_2)$.}

**Corollary 6.26.** Suppose we have the hypothesis of the main theorem. Let $F$ be the fiber over a point $b \in A$. Suppose the CT map $\partial i_{F,X} : \partial F \to \partial X$ is surjective. Then the CT map $\partial i_{F,Y} : \partial F \to \partial Y$ is also surjective.

**Proof.** Let $\xi \in \partial Y$. Since $\partial i_{F,X} : \partial F \to \partial X$ is surjective there exists $z \in \partial F$ such that $\partial i_{F,X}(z) = \partial i_{Y,X}(\xi)$. Since $\partial i_{F,X} = \partial i_{Y,X} \circ \partial i_{F,Y}$, $\partial i_{F,Y}$ identifies the points $\partial i_{F,Y}(z)$ and $\xi$. By Theorem 6.25 we are now done. \[\square\]

A special case of the following corollary is proved by E. Field ([Fic, Theorem B]).
Corollary 6.27. Suppose $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ is a short exact sequence of infinite hyperbolic groups. Suppose $A \subset Q$ is qi embedded and $Y = \pi^{-1}(A)$. Then the CT map $\partial N \rightarrow \partial Y$ is surjective.

Proof. Since $N$ is a normal subgroup of the hyperbolic group $G$ it is a standard fact that $\Lambda(N) = \partial G$. Thus by Lemma 2.50 the CT map $\partial N \rightarrow \partial G$ is surjective. Now we are done by Corollary 6.26.

Theorem 6.28. Suppose $X$ is a metric (graph) bundle over $B$ satisfying hypotheses of section 5. Let $F = F_b$ where $b \in B$. Suppose $\partial F$ is not homeomorphic to a dendrite and also the CT map $\partial F \rightarrow \partial X$ is surjective. Finally suppose $X$ is a proper metric space.

Then for all $\xi \in \partial B$ we have $\Lambda_{\xi} \neq \emptyset$.

Proof. Suppose $\alpha$ is a 1-quasigeodesic ray in $B$ joining $b$ to $\xi$. Let $Y = \pi^{-1}(\alpha)$. Then the CT map $\partial i_{F,Y} : \partial F \rightarrow \partial Y$ is surjective by Corollary 6.26. Since $X$ is proper so is $F$ and so $\partial F$ is compact. Hence, $\partial i_{F,Y}$ is injective implies $\partial F$ is homeomorphic to $\partial Y$. Since $\partial F$ is not a dendrite this is impossible due to the following result of Bowditch.

Theorem 6.29. ([Bow02] Proposition 10.2) Suppose $X$ is hyperbolic metric (graph) bundle over $B = [0,\infty)$ satisfying the hypotheses H1-H4 of section 5. Suppose moreover that $X$ is a proper metric space. Then $\partial X$ is a dendrite.

Corollary 6.30. Suppose $G$ is the fundamental group of a finite developable complexes of groups with nonelementary hyperbolic face groups where images of the homomorphisms between respective face groups are of finite index in the target groups and $B$ is the universal cover of the complexes of groups. Suppose $X$ is the metric bundle over $B$ obtained from this data as constructed in Example 5. Suppose $G$ is hyperbolic. Then for all $\xi \in \partial B$, and any fiber $F$ of the bundle $\Lambda_{\xi,X}(F) \neq \emptyset$.

Proof. We need to check the hypotheses of Theorem 6.28. It is a standard fact that boundary of any hyperbolic group is not a dendrite. Since the fibers of the metric bundle under consideration are quasisymmetric to nonelementary hyperbolic groups $\partial F$ is not a dendrite for any fiber $F$. We also note that the metric bundle satisfies H1-H4 of section 5. Finally $G$ acts on $X$ and $B$ so that the map $\pi : X \rightarrow B$ is equivariant, on $X$ the action is proper and cocompact and on $B$ it is cocompact. Thus any orbit map $G \rightarrow X$ is a qi by Milnor-Svarc lemma and therefore induces a homeomorphism $\partial X \rightarrow \partial G$.

Now, given any fiber $F$ and $g \in G$, $gF$ is another fiber of the metric bundle. By Lemma 6.10 (1) $Hd(F, gF) < \infty$. Hence, by Lemma 2.53 $\Lambda(F) = \Lambda(gF) = g\Lambda(F)$. It is a standard fact that the action of a nonelementary hyperbolic group on its boundary is minimal, i.e. the only invariant closed subsets are the empty set and the whole set. Hence, it follows that $\Lambda(F) = \partial X$. By Lemma 2.56 we have $\Lambda(F) = \partial i_{F,X}(\partial F)$. Thus the CT map $\partial i_{F,X} : \partial F \rightarrow \partial X$ is surjective. Finally, clearly $X$ is a proper metric space. Hence, we are done by Theorem 6.28.

Definition 6.31. Suppose $Z$ is any hyperbolic metric space and $S \subset Z$. Then a point $z \in \Lambda(S) \subset \partial_s Z$ will be called a conical limit point of $S$ if for some (any) quasigeodesic $\gamma$ converging to $z$ in $Z$ there is a constant $D > 0$ such that $N_D(\gamma) \cap S$ is a subset of infinite diameter in $Z$. 

Proposition 6.32. Suppose we have hypothesis of Theorem 6.31. Let $\partial i_{Y,X} : \partial Y \to \partial X$ be the CT map. If $\xi \in \partial X$ is a conical limit point of $Y$, then $|\partial i_{Y,X}^{-1}(\xi)| = 1$.

Proof. Suppose $z \neq z' \in \partial Y$ such that $\partial i_{Y,X}(z) = \partial i_{Y,X}(z') = \xi$. Then by Theorem 6.23 there is $\xi_B \in \partial B \setminus \partial A$ and a qi lift of $\gamma$ of a quasigeodesic ray joining $b$ to $\xi_B$ such that $\xi = \gamma(\infty)$. Since $\xi_B \in \partial B \setminus \partial A$ and $A$ is quasiconvex $\xi_B$ is not a limit point of $A$ in $\partial B$. Thus it is clear that $\xi$ is not a conical limit point of $Y$. This gives a contradiction and proves the proposition.

The following result was pointed out to us by Misha Kapovich.

Lemma 6.33. Suppose $\pi : X \to \mathbb{R}$ is a metric (graph) bundle satisfying the hypotheses of section 5 and $X^\pm$ are the restrictions of it to $[0, \infty)$ and $(-\infty, 0]$ respectively. Then the diagonal embedding $f : F_0 \to X^+ \times X^-$ is a qi embedding where the latter is given the $l_2$ metric.

Proof. Without loss of generality, we assume $(X, d)$ is a metric graph bundle. Let $d_{\pm}$ be the induced length metric on $X^\pm$ respectively. Then the $l_2$ metric $d_Y$ on $Y := X^+ \times X^-$ is given by $d_Y((x_1, x_2), (y_1, y_2))^2 = d_+(x_1, y_1)^2 + d_-(x_2, y_2)^2$ for all $x_1, y_1 \in X^+$ and $x_2, y_2 \in X^-$. We note that the inclusion maps $F_0 \to X^\pm$ are $1$-Lipschitz.

Let $x, y \in F_0$. Then, $d_Y(f(x), f(y))^2 = d_Y((x, x), (y, y))^2 = d_+(x, y)^2 + d_-(x, y)^2 \leq d_0^2(x, y) + d_2^2(x, y) = 2d_0(x, y)^2$, which implies that $d_Y(f(x), f(y)) \leq \sqrt{2}d_0(x, y)$. A reverse inequality is obtained as follows.

Now, let $\Sigma, \Sigma'$ be a pair of $K_0$-qi sections in $X$ through $x, y$ respectively. Let $L = L(\Sigma, \Sigma')$ be the ladder formed by them. Let $\lambda = L \cap F_0$. This is a geodesic in $F_0$ joining $x, y$. Now, suppose $c(x, y)$ is a uniform quasigeodesic in $X$ joining $x, y$ constructed as in section 5 by decomposing $L$ into subladders using the the qi sections $\Sigma_i$'s and $\Sigma_i$'s. Let $\tilde{c}_+ := \tilde{c}_+(x, y), \tilde{c}_- := \tilde{c}_-(x, y)$ be the modified paths joining $x, y$ in $X^+, X^-$ respectively. By our main theorem in section 5, $\tilde{c}_+, \tilde{c}_-$ are uniform quasigeodesics in $X^+, X^-$ respectively. Suppose these are $K$-quasigeodesics. As in the discussion at the end of section 5, suppose $Q, Q'$ are consecutive qi sections in the decomposition of $L = L(\Sigma, \Sigma')$ and $z, w \in Q, w' \in Q'$ with $b' = \pi(w) = \pi(w')$ are such that $L(Q, Q') \cap c(y, y')$ is made of the fiber geodesic $[w, w']$ and the lift of $[\pi(z), \pi(w)]_B$ in $Q$. However, if $b' \in [0, \infty)$ then $\lambda \cap \mathbb{L}(Q, Q') \subseteq \tilde{c}_-$ and similarly if $b' \in (-\infty, 0]$ then $\lambda \cap \mathbb{L}(Q, Q') \subseteq \tilde{c}_+$. Thus $\lambda \subseteq \tilde{c}_+ \cup \tilde{c}_-$. Therefore we have,

$$d_0(x, y) \leq l_+(\tilde{c}_+) + l_-(\tilde{c}_+) \leq Kd_+(x, y) + K + Kd_-(x, y) + K = K(d_+(x, y) + d_-(x, y)) + 2K = 2Kd_Y((x, x), (y, y)) + 2K = 2Kd_Y(f(x), f(y)) + 2K.$$

Thus, $-1 + \frac{1}{2K}d_0(x, y) \leq d_Y(f(x), f(y)) \leq \sqrt{2}d_0(x, y)$. Hence, $f$ is $(2K, 1)$-qi embedding.

In the same way, we obtain the following.

Lemma 6.34. If $v_0$ is a cut point of $B$ and removing it produces two qc subsets $A_1, A_2$ and $Y_1, Y_2$ are the restrictions of the bundle to $A_1, A_2$ respectively then the diagonal map $F_{v_0} \to Y_1 \times Y_2$ is a qi embedding.

Corollary 6.35. If $v_0$ is a cut point of $B$ and removing it produces finitely many qc subsets $A_i, 1 \leq i \leq n$ and $Y_i$‘s are the restrictions of the bundle to $A_i$’s respectively then the diagonal map $F_{v_0} \to \Pi_i Y_i$ is a qi embedding.
Remark 19. In [Mit97] Mitra defined an ending lamination for an exact sequence of groups. Given any point \( \xi \in \partial Q \) he defined a lamination \( \Lambda_\xi \) and then showed that \( \Lambda_\xi = \Lambda'_\xi \). However, for formulating and proving this sorts of results one needs additional structure on the bundle, e.g. action of a group on the bundle through morphisms which has uniformly bounded quotients when restricted to the fibers. Results of this type are proved in [MR18, Section 3]; see also [Bow02 Section 17].

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