Spinning Q-balls in the complex signum-Gordon model

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Abstract

Rotational excitations of compact Q-balls in the complex signum-Gordon model in 2+1 dimensions are investigated. We find that almost all such spinning Q-balls have the form of a ring of strictly finite width. In the limit of large angular momentum $M_z$ their energy is proportional to $|M_z|^{1/5}$.

PACS: 11.27.+d, 98.80.Cq, 11.10.Lm

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1 Introduction

Q-balls belong to the most popular non topological solitons. Formed by a self interacting scalar field with a $U(1)$ global symmetry, they are a part of the still largely unexplored world of non-perturbative phenomena in field theory. As such, for years they have attracted well justified attention, see [1] for a review and references. Recent works have been focused mainly on gravitating Q-balls (also called boson stars) [2], interactions and stability [3], and rotational excitations (spinning Q-balls) [4].

The present paper is a sequel to [5], where non rotating Q-balls with astonishingly simple analytic form were found in the complex signum-Gordon model. The discussed below spinning Q-balls are their rotational excitations. Study of such excitations is a natural and desired step in the search for understanding the nonlinear dynamics of the scalar field.

Our main findings are as follows. First, we present detailed analytic description of the spinning Q-balls in the planar signum-Gordon model. It turns out, rather surprisingly, that all spinning axially symmetric Q-balls except the ones with the least non vanishing angular momentum ($|N| = 1$) have the form of a ring of strictly finite width. Outside the ring strip the scalar field has its exact vacuum value, and inside it the field is given by a quadratic combination of cylindrical Bessel functions. The inner and outer radii of the ring are determined from Eqs. (23) below which can be solved analytically in the limit of high angular momentum. It is quite remarkable that the signum-Gordon model allows for such a detailed analytic insight into the structure of the rotationally excited Q-balls.

In Section 2 below we present certain preliminary material. Section 3 is devoted to explicit solutions of the field equation. Basic physical characteristics of the spinning Q-balls are discussed in Section 4. Several remarks are collected in Section 5.

2 Preliminaries

The Lagrangian of the complex signum-Gordon model has the form

$$L = \partial_{\mu} \psi^{*} \partial^{\mu} \psi - \lambda |\psi|,$$  \hspace{1cm} (1)

where $\psi$ is a complex scalar field in $(2 + 1)$-dimensional Minkowski space-time, and $\lambda > 0$ is a coupling constant. The field $\psi$, the space-time coordinates $x^{\mu}$ and $\lambda$

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1 $^*$ denotes the complex conjugation, $|\psi|$ is the modulus of $\psi$
are dimensionless – in physical applications they have to be multiplied by certain dimensional constants. The self interaction term $\lambda|\psi|^2$ regarded as the function of $(\text{Re}\psi, \text{Im}\psi)$ has the shape of inverted cone with the tip at $\psi = 0$ (the vacuum field). It is an example of V-shaped field potential. Models with such potentials have several interesting features [6], such as compactness of Q-balls and of other solitonic objects, or a scale invariance of on-shell type.

Lagrangian (1) is invariant under the global $U(1)$ transformations $\psi(x) \rightarrow \exp(i\alpha)\psi(x)$ as well as under rotations, translations and Lorentz boosts in the $(x^1, x^2)$ plane. The conserved $U(1)$ charge $Q$, the angular momentum $M_z$ and the energy $E$ are given by the following formulas

$$Q = -\frac{i}{2} \int d^2x \left( \psi^* \partial_0 \psi - \partial_0 \psi^* \psi \right),$$  \hspace{1cm} (2)

$$M_z = -\frac{1}{2} \int d^2x \left( \partial_0 \psi^* \partial_\theta \psi + \partial_\theta \psi^* \partial_0 \psi \right),$$  \hspace{1cm} (3)

$$E = \int d^2x \left( \partial_0 \psi^* \partial_0 \psi + \partial_r \psi^* \partial_r \psi + r^{-2} \partial_\theta \psi^* \partial_\theta \psi + \lambda|\psi|^2 \right),$$  \hspace{1cm} (4)

where $\theta$ is the azimuthal angle and $r$ the radius in the $(x^1, x^2)$ plane.

The spinning Q-balls minimize the energy $E$ under the condition that $Q$ and $M_z$ have fixed values. Introducing Lagrange multipliers $\lambda_1, \lambda_2$ and the functional $F = E + \lambda_1 Q + \lambda_2 M_z$, the conditions necessary for the minimum have the form

$$\frac{\delta F}{\delta(\partial_0 \psi)} = 0 = \frac{\delta F}{\delta(\partial_0 \psi^*)}, \quad \frac{\delta F}{\delta \psi} = 0 = \frac{\delta F}{\delta \psi^*},$$

or explicitly

$$\partial_0 \psi + \frac{i}{2} \lambda_1 \psi - \frac{\lambda_2}{2} \partial_\theta \psi = 0,$$  \hspace{1cm} (5)

and

$$- \Delta \psi + \frac{\lambda}{2} \frac{\psi}{|\psi|} - \left( \frac{i}{2} \lambda_1 + \frac{\lambda_2}{2} \right) \partial_0 \psi = 0,$$  \hspace{1cm} (6)

where $\Delta = \partial_r^2 + r^{-1} \partial_r + r^{-2} \partial_\theta^2$. Note that conditions (5), (6) imply that $\psi$ obeys the Euler-Lagrange equation corresponding to (1):

$$\partial_0^2 \psi - \Delta \psi + \frac{\lambda}{2} \frac{\psi}{|\psi|} = 0,$$  \hspace{1cm} (7)

where by definition $\psi/|\psi| = 0$ if $\psi = 0$, see [5].
One can show that general solution of condition (5) has the form
\[ \psi = \exp(-i\lambda_1 x_0/2) \phi(r, \theta + \lambda_2 x_0/2), \] (8)
where \( \phi \) is an arbitrary (differentiable) function. Axially symmetric Q-balls obey the condition
\[ \psi(r, \theta + \theta_0, x_0) = \exp(iN\theta_0) \psi(r, \theta, x_0), \] (9)
where \( \theta_0 \in [0, 2\pi) \) is a rotation angle, \( N \) is an integer. The phase factor \( \exp(iN\theta_0) \) is allowed because of the global \( U(1) \) symmetry – the effect of rotation can be compensated by the \( U(1) \) transformation. The symmetry condition (9) together with formula (8) gives
\[ \psi = \exp(-i\omega x_0) \exp(iN\theta) \chi(r), \] (10)
where \( \omega = (\lambda_1 - N\lambda_2)/2 \). The unknown function \( \chi(r) \) obeys the following equation
\[ (\partial_r^2 + \frac{1}{r}\partial_r)\chi - \frac{N^2}{r^2} \chi - \frac{\lambda}{2} \frac{\chi}{|\chi|} = -\omega^2 \chi, \] (11)
obtained from Eq. (7) by inserting formula (10). In the case of rotating Q-balls \( N \neq 0 \) and therefore \( \chi(0) = 0 \), otherwise the function \( \psi \) would have a discontinuity at \( r = 0 \).

So far the function \( \chi(r) \) can have complex values. In the intervals of the \( r \) variable in which \( \chi \neq 0 \) we can uniquely split \( \chi(r) \) into the phase and the modulus, \( \chi(r) = \exp(iG(r)) \) \( F(r) \), which obey the following equations (obtained from (11)):
\[ \partial_r(F^2\partial_r G) = 0, \] (12)
\[ \partial_r^2 F - F(\partial_r G)^2 + \frac{1}{r}\partial_r F - \frac{N^2}{r^2} F - \frac{\lambda}{2} \text{sign} F = -\omega^2 F. \] (13)
The sign function has the values \( \pm 1 \) when \( F \neq 0 \) and \( \text{sign}(0) = 0 \). Equation (12) means that \( rF^2\partial_r G \) is constant; substituting \( r = 0 \) gives that constant equal to 0. Therefore, \( G(r) \) has a constant value (\( \partial_r G = 0 \)) in intervals in which \( rF(r) \neq 0 \). For simplicity, we assume that \( G(r) \) is constant in the whole range \([0, \infty)\) of the radial coordinate, hence it can be removed by the \( U(1) \) transformation. Thus, in the case of simplest axially symmetric Q-ball with minimal energy
\[ \psi = \exp(-i\omega x_0) \exp(iN\theta)F(r), \] (14)
where the nonnegative real function \( F(r) \) obeys the following equation
\[ \partial_r^2 F + \frac{1}{r}\partial_r F - \frac{N^2}{r^2} F - \frac{\lambda}{2} \text{sign} F = -\omega^2 F \] (15)
with the condition \( F(0) = 0 \).
3 Explicit form of the profile function $F(r)$

It is convenient to introduce a new variable $\rho$ and a new function $f(\rho)$,

$$\rho = |\omega|r, \quad f(\rho) = \frac{2\omega^2}{\lambda}F(r).$$ \hfill (16)

Then Eq. (15) acquires the parameter free form

$$\partial^2_\rho f + \frac{1}{\rho} \partial_\rho f + \left(1 - \frac{N^2}{\rho^2}\right)f = \text{sign}f. \hfill (17)$$

Let us first try the standard tool: a series expansion in a vicinity of $\rho = 0$,

$$f(\rho) = \rho k(a_0 + a_1\rho + \ldots), \quad a_0 \neq 0.$$ Because Eq. (17) is invariant under the reflection $f \rightarrow -f$ we may assume that $a_0 > 0$. Then $\text{sign}f = +1$ if we take not too large $\rho$. Eq. (17) considered in the leading order in $\rho$, represented by terms $\sim \rho^{k-2}$, implies that $k = |N|$. The next to leading term is then proportional to $\rho^{|N|-1}$. Therefore, when $|N| > 2$ we do not find a term $\sim \rho^0$ needed in order to cancel the term $\text{sign}f = +1$. For $|N| = 2$ it is the leading term which is $\sim \rho^0$, but in this case the series expansion also does not work: Eq. (17) in the order $\sim \rho^0$ gives $0 = 1$. We conclude that the assumed series form of the solution is applicable only when $|N| = 1$. In this case we find that $a_1 = 1/3$ while $a_0$ is a free parameter. Strangely enough, the case $|N| = 1$ is essentially different than $|N| > 1$.

Luckily, Eq. (17) with $\text{sign}f = +1$ coincides with inhomogeneous Bessel equation for which one can construct the general solution using a standard method \cite{7}. The method requires Wronskian $W$ of Bessel functions \cite{8},

$$W = J_{|N|}(\rho)Y_{|N|}'(\rho) - J'_{|N|}(\rho)Y_{|N|}(\rho) = 2/\pi\rho.$$ We denote the general solution by $f_+$ in order to emphasize the fact that it obeys Eq. (17) only in the intervals of $\rho$ in which $f_+ > 0$. It has the following form

$$f_+(\rho) = AJ_{|N|}(\rho) + BY_{|N|}(\rho)$$

$$+ \frac{2}{\pi} Y_{|N|}(\rho) \int_0^{\rho} d\rho' \rho'J_{|N|}(\rho') - \frac{2}{\pi} J_{|N|}(\rho') \int_0^{\rho} d\rho' \rho'Y_{|N|}(\rho').$$ \hfill (18)

Here $A, B, \rho_0$ are constants which are suitably adjusted in order to satisfy boundary or matching conditions. Further calculations are carried out separately for $|N| = 1$ and $|N| > 1$. 5
3.1 The $|N| = 1$ case

The condition $f_+(0) = 0$ gives $B = 0$ and $\rho_0 = 0$. Then, formula (18) gives $a_1 = 1/3$ as expected. It remains to find the constant $A$ and the interval of $\rho$ in which $f_+ > 0$. Motivated by the results of [5] we look for compact spinning Q-balls, for which there exists $\rho_1 > 0$ such that $f(\rho) \equiv 0$ if $\rho \geq \rho_1$ (note that $f \equiv 0$ is a solution of Eq. (17)). The matching conditions at $\rho = \rho_1$ have the form $f_+(\rho_1) = 0$, $f'_+(\rho_1) = 0$. The first condition is just the continuity of $f$ at $\rho_1$, the second one is obtained by integrating both sides of Eq. (17) in an infinitesimally small interval containing $\rho_1$. The matching conditions give

$$A = \frac{\pi}{2} \int_0^{\rho_1} d\rho \rho Y_1(\rho), \quad \int_0^{\rho_1} d\rho \rho J_1(\rho) = 0.$$ 

The latter condition determines $\rho_1$. Numerically, $\rho_1 = 5.8843...$ and $A = 2.6907...$.

To summarize, the full profile function in the case $|N| = 1$ has the form

$$f(\rho) = \begin{cases} 
    f_1(\rho) & \text{if } 0 \leq \rho \leq \rho_1, \\
    0 & \text{if } \rho \geq \rho_1, 
\end{cases}$$

where

$$f_1(\rho) = \frac{\pi}{2} Y_1(\rho) \int_0^\rho d\rho' \rho' J_1(\rho') + \frac{\pi}{2} J_1(\rho) \int_\rho^{\rho_1} d\rho' \rho' Y_1(\rho').$$

3.2 The $|N| > 1$ case

The puzzle with the lack of nontrivial series solution close to $\rho = 0$ is solved at once when we realize that we may take there the trivial solution $f \equiv 0$. Then the full solution has the form

$$f(\rho) = \begin{cases} 
    0 & \text{if } 0 \leq \rho \leq \rho_0, \\
    f_N(\rho) & \text{if } \rho_0 \leq \rho \leq \rho_1, \\
    0 & \text{if } \rho \geq \rho_1, 
\end{cases}$$

where $f_N(\rho)$ is obtained from the general solution (18) by imposing the matching conditions at $\rho_0$ and $\rho_1$:

$$f_+(\rho_0) = 0, \quad f'_+(\rho_0) = 0, \quad f_+(\rho_1) = 0, \quad f'_+(\rho_1) = 0.$$ 

The two conditions at $\rho_0$ imply that $A = B = 0$. Thus,

$$f_N(\rho) = \frac{\pi}{2} Y_{|N|}(\rho) \int_{\rho_0}^\rho d\rho' \rho' J_{|N|}(\rho') - \frac{\pi}{2} J_{|N|}(\rho) \int_{\rho_0}^\rho d\rho' \rho' Y_{|N|}(\rho').$$
The matching conditions at $\rho_1$ give the following equations for $\rho_0, \rho_1$:

$$\int_{\rho_0}^{\rho_1} \rho J_{|N|}(\rho) \, d\rho = 0, \quad \int_{\rho_0}^{\rho_1} \rho Y_{|N|}(\rho) \, d\rho = 0.$$  \hfill (22)

It is not difficult to determine the radii $\rho_0, \rho_1$ numerically, see Table 1.

| $|N|$ | 1   | 2   | 5   | 10  | 20  | 40  | 80  |
|------|-----|-----|-----|-----|-----|-----|-----|
| $\rho_0$ | 0.0 | 0.6654 | 6.4593 | 16.413 | 36.392 | 76.382 | 156.37 |
| $\rho_1$ | 5.8843 | 7.7942 | 13.699 | 23.664 | 43.646 | 83.637 | 163.63 |
| $A(|N|)$ | 22.082 | 39.996 | 97.238 | 193.72 | 387.06 | 773.95 | 1547.8 |

Table 1. Sample of numerical results for $\rho_0, \rho_1, A(|N|)$. $\rho_0, \rho_1$ have been found directly from purely numerical solutions of Eq. (17). The integral $A(|N|) = \int_{\rho_0}^{\rho_1} \rho f_N(\rho) \, d\rho$ is considered in Section 4.

Our numerical data show that $\rho_0 \approx 2|N| - 3.6$, $\rho_1 \approx 2|N| + 3.6$ for $|N| \geq 40$. Such simple formulas suggest that there exists a simple asymptotic solution of Eqs. (22) at large values of $|N|$. It turns out that indeed, this is the case. Let us write Eqs. (22) in the form

$$\int_{\rho_0}^{\rho_1} \rho J_{|N|}(\rho) + iY_{|N|}(\rho) \, d\rho = 0,$$  \hfill (23)

and replace the Bessel functions by their asymptotic forms appropriate for our case in which $\rho_0, \rho_1$ linearly increase with $|N|$:

$$\left(J_{|N|} + iY_{|N|}\right) \left(\frac{|N|}{\cos \beta}\right) \approx \sqrt{\frac{2}{\pi |N| \tan \beta}} \left[e^{i(|N|\tan \beta - |N|\beta - \pi/4)} + O(N^{-1})\right],$$  \hfill (24)

where $|N| \to \infty$ and $\beta$ is fixed [3]. Next, we change the integration variable $\rho$ in (23) to $\beta$ by the substitution $\rho = |N|/\cos \beta$. Then, $\rho_0 = |N|/\cos \beta_0$, $\rho_1 = |N|/\cos \beta_1$. The numerical results suggest that $\beta_0 = \pi/3 - \delta/|N|$, $\beta_1 = \pi/3 + \delta/|N|$. The integration over $\beta$ in the interval $[\beta_0, \beta_1]$ gives the equation for $\delta$, namely $\exp(3i\delta) = -1 + O(1/|N|)$. Thus, $\delta = \pi/3 + O(1/|N|)$, and

$$\rho_0 = 2|N| - \frac{2\pi}{\sqrt{3}} + O(1/|N|), \quad \rho_1 = 2|N| + \frac{2\pi}{\sqrt{3}} + O(1/|N|),$$

in agreement with the numerical results ($2\pi/\sqrt{3} = 3.6275\ldots$). Formulas (21), (24) give in the large $|N|$ limit

$$f_N(\rho) \approx \frac{8}{3} \sin^2[\sqrt{3}(\rho - \rho_0)/4] + O(|N|^{-1}).$$
4 Physical characteristics of the spinning Q-balls

The basic physical characteristics of the spinning Q-balls are calculated from formulas (2-4) in which we insert formula (14):

\[ E = \frac{\pi \lambda^2}{2 \omega^4} \int_{\rho_0}^{\rho_1} d\rho \rho \left[ (\partial_\rho f)^2 + f^2 + \frac{N^2}{\rho^2} f^2 + 2f \right], \tag{25} \]

\[ Q = -\frac{\pi \lambda^2}{2 \omega^5} \int_{\rho_0}^{\rho_1} d\rho \rho f^2, \quad M_z = -NQ \tag{26} \]

(\(\rho_0 = 0\) if \(|N| = 1\)). We see that the angular momentum \(M_z\) is quantized. This has been observed earlier in other models too \[4\].

Formula (25) for the energy can be simplified with the help of two identities valid for our nonnegative solutions \(f(\rho)\) of Eq. (17):

\[ \int_{\rho_0}^{\rho_1} d\rho \rho \left[ (\partial_\rho f)^2 + \frac{N^2}{\rho^2} f^2 \right] = \int_{\rho_0}^{\rho_1} d\rho \rho (f^2 - f), \quad \int_{\rho_0}^{\rho_1} d\rho \rho f^2 = 2 \int_{\rho_0}^{\rho_1} d\rho \rho f. \]

They are obtained by multiplying Eq. (17) by \(\rho f\) or \(\rho^2 f'\), respectively, and integrating by parts. The integral in \(E\) is equal to \(5 \int_{\rho_0}^{\rho_1} d\rho \rho f_N\), and therefore

\[ E = \frac{5\pi \lambda^2}{2 \omega^4} \int_{\rho_0}^{\rho_1} d\rho \rho f_N = -\frac{5}{2} \omega Q. \tag{27} \]

Thus, we need to calculate just one integral \(A(|N|) = \int_{\rho_0}^{\rho_1} d\rho \rho f_N\). It depends only on \(|N|\). Examples of its numerical values are given in Table 1. Calculation based on the asymptotic formula (24) gives

\[ A(|N|) = \frac{32\pi}{3 \sqrt{3}} |N| (1 + \mathcal{O}(|N|^{-1})). \tag{28} \]

It agrees very well with the numerical results already for \(|N| \geq 40\).

The two parameters \(\omega\) and \(N\) that our solutions contain can be related to the basic observables, namely

\[ N = -\frac{M_z}{Q}, \quad \omega = -\frac{2E}{5Q}. \]

Furthermore, the first part of formula (27) yields a relation between the observables, which in the limit of large \(|N|\) has the form

\[ E \approx 5 \left( \frac{\pi^2}{3 \sqrt{3}} \right)^{1/5} \lambda^{2/5} (|M_z||Q|^3)^{1/5} \tag{29} \]
(note the absolute values — $M_z$ and $Q$ can have both signs).

The energy (29) can be regarded as the rest mass of the spinning Q-ball. Moving Q-balls are obtained by applying Lorentz boosts.

5 Remarks

1. The field $\psi$ of our Q-ball solutions reaches exactly the vacuum value $\psi = 0$ at the radius $r_1 = \rho_1/|\omega|$, i.e., at $r_1 = 2(3\sqrt{3}/\pi^2)^{1/5} \lambda^{-2/5} |M_z|^{1/5} |Q|^{-3/5}$ for large $|M_z|$. If $|N| \geq 2$ we also have the inner radius $\rho_0 > 0$ at which too $\psi$ reaches the vacuum value exactly. The approach to the vacuum value is parabolic. For example, formula (21) gives $f_N(\rho) \simeq (\rho - \rho_1)^2/2 + O(\rho - \rho_1)^3$ for $\rho \to \rho_1 -$. Such behavior of the field is typical for the models with V-shaped self interactions [6]. It should be mentioned that similar behavior is observed also in models with a nonstandard kinetic term (so called K-fields) [9].

2. Above we have discussed the simplest axially symmetric spinning Q-balls. One may also consider more general configurations. First, because the field $\psi$ of the Q-balls reaches the vacuum value at the radii $\rho_0, \rho_1$ exactly, one can trivially put arbitrary number of such Q-balls on the $(x^1, x^2)$ plane, provided they do not overlap. In particular, one can have Q-balls in the form of concentric rings, each one with its own values of $Q$ and $M_z$. Second, we expect that there exist radially excited versions of the Q-ball with fixed values of $Q$ and $M_z$. While the profile function $F(r)$ discussed in Section 3 is non negative, in the case of the radially excited Q-balls it will have alternating sign before it parabolically reaches the vacuum value $F = 0$. In the case of non spinning Q-balls this possibility has been shown to exist [5]. Third, one may expect that there exist spinning Q-balls which are not axially symmetric.

3. One can easily check that our spinning Q-balls are stable with respect to radial shrinking or expanding. Nevertheless, we do not expect that they are absolutely stable because they are excited states of non spinning Q-balls. As such, when slightly perturbed they probably will decay into simpler objects, like smaller Q-balls (spinning or not), and will emit a packet of radiation. The dynamics of such processes probably deserves a separate numerical and analytical investigation.

6 Acknowledgement

This work is supported in part by the project SPB nr. 189/6.PRUE/2007/7.
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