Bijections on $r$-Shi and $r$-Catalan Arrangements

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Abstract
Associated with the $r$-Shi arrangement and $r$-Catalan arrangement in $\mathbb{R}^n$, we introduce a cubic matrix for each region to establish two bijections in a uniform way. Firstly, the positions of minimal positive entries in column slices of the cubic matrix will give a bijection from regions of the $r$-Shi arrangement to $O$-rooted labeled $r$-trees. Secondly, the numbers of positive entries in column slices of the cubic matrix will give a bijection from regions of the $r$-Catalan arrangement to pairings of permutation and $r$-Dyck path. Moreover, the numbers of positive entries in row slices of the cubic matrix will recover the Pak-Stanley labeling, a celebrated bijection from regions of the $r$-Shi arrangement to $r$-parking functions.

Keywords: cubic matrix, Shi arrangement, Catalan arrangement, Pak-Stanley labeling

1 Concepts and Backgrounds

This paper aims to establish two bijections: from regions of the $r$-Shi arrangement to $O$-rooted labeled $r$-trees, and from regions of the $r$-Catalan arrangement to pairings of permutation and $r$-Dyck path. To this end, we introduce a cubic matrix for each region to read the combinatorial information from the region.

A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a vector space $V$, see [35,47]. When $V$ is a real space, the set $V \setminus \cup_{H \in \mathcal{A}} H$ consists of finitely many connected components, called regions of $\mathcal{A}$. Denote by $\mathcal{R}(\mathcal{A})$ the set of regions of $\mathcal{A}$. For any positive integers $r$ and $n$, the $r$-Shi arrangement $S^r_n$ in $\mathbb{R}^n$ consists of the following hyperplanes

$$S^r_n : x_i - x_j = -r + 1, -r + 2, \ldots, 0, 1, \ldots, r, \ 1 \leq i < j \leq n.$$
The case of \( r = 1 \) is the classical **Shi arrangement** \( S_n \) introduced by Shi [44] in 1986. Shi further obtained the number of regions of \( S'_n \).

**Theorem 1.1.** [45] For any positive integers \( r \) and \( n \), the number of regions of \( S'_n \) is

\[
|\mathcal{R}(S'_n)| = (rn + 1)^{n-1}.
\]

Let \( O = \{o_1, \ldots, o_r\} \) and \( V = \{v_1, \ldots, v_n\} \) be two disjoint sets of labeled vertices. First introduced by Harary and Palmer [18] in 1968, an **O-rooted labeled** \( r \)-**tree** \( T \) on \( O \cup V \) is a graph having the property: there is a valid rearrangement \( \nu = (v_{i_1}, \ldots, v_{i_n}) \) of vertices \( v_1, \ldots, v_n \), such that each \( v_{i_j} \) with \( j \in [n] \) is adjacent to exactly \( r \) vertices in \( \{o_1, \ldots, o_r, v_{i_1}, \ldots, v_{i_{j-1}}\} \) and, moreover, these \( r \) vertices are themselves mutually adjacent in \( T \). Section 3 will be devoted to some characterizations of the \( O \)-rooted labeled \( r \)-trees. Denote by \( T^n_r \) the set of all \( O \)-rooted labeled \( r \)-trees. In the case of \( r = 1 \), write \( T_n = T^n_1 \), whose members are called **O-rooted labeled trees**. The size of \( T^n_r \) has been counted by Foata [10], Beineke and Pippert [45], Gainer-Dewar and Gessel [16] etc., which extends the Cayley formula \( |T_n| = (n + 1)^{n-1} \) of [7].

**Theorem 1.2.** [45,10] For any positive integers \( r \) and \( n \), the cardinality of \( T^n_r \) is

\[
|T^n_r| = (rn + 1)^{n-1}.
\]

Closely related to \( O \)-rooted labeled \( r \)-trees and \( r \)-Shi arrangement, the **\( r \)-parking function of length** \( n \) is a sequence \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \) such that the monotonic rearrangement \( a_1 \leq a_2 \leq \cdots \leq a_n \) of the numbers \( \alpha_1, \ldots, \alpha_n \) satisfies \( a_i \leq r(i - 1) \). Denote by \( \mathcal{P}^r_n \) the set of all \( r \)-parking functions of length \( n \). In the case of \( r = 1 \), write \( \mathcal{P}_n = \mathcal{P}^1_n \) whose members are called **parking functions of length** \( n \). Explored by Pitman and Stanley [39], Yan [51] etc., the cardinality of \( \mathcal{P}^r_n \) is exactly the same as \( |\mathcal{R}(S'_n)| \) and \( |T^n_r| \), namely,

\[
|\mathcal{P}^r_n| = (rn + 1)^{n-1}.
\]

Naturally we may ask if there are some bijections among \( \mathcal{R}(S'_n) \), \( T^n_r \), and \( \mathcal{P}^r_n \). A celebrated bijection \( \mathcal{R}(S'_n) \to \mathcal{P}^r_n \) (abbreviation for ‘from \( \mathcal{R}(S'_n) \) to \( T^n_r \)’) is the Pak-Stanley labeling which was first suggested by I. Pak in the case of \( r = 1 \), and extended to general \( r \) by R. P. Stanley [48,49]. Later, relevant to the Pak-Stanley labeling, many results on bijections \( \mathcal{R}(S'_n) \to \mathcal{P}^r_n \) have been obtained, see [2,3,8,31,40,47–49] etc.. For the bijection \( \mathcal{P}^r_n \to T^n_r \), currently we just know that it can be established by a composition of three other bijections given in [36] by I. Pak and A. Postnikov. In the case of \( r = 1 \), bijections \( \mathcal{P}_n \to T_n \) have been well studied since 1968, see [11,13,22,25,41,43] etc.. To the best of our knowledge, no explicit bijection \( \mathcal{R}(S'_n) \to T^n_r \) has been established, which is exactly our motivation of this paper. Our first main result is to establish a bijection \( \mathcal{R}(S'_n) \to T^n_r \), see Theorem 2.2. To this end, we will introduce a cubic matrix for \( r \)-Shi arrangement, which will also let us define the Pak-Stanley labeling in an easy way, see Theorem 2.3.

Surprisingly, the cubic matrix method can be applied to the **\( r \)-Catalan arrangement** \( \mathcal{C}^r_n \) in \( \mathbb{R}^n \), the collection of hyperplanes

\[
\mathcal{C}^r_n : \quad x_i - x_j = 0, \pm 1, \ldots, \pm r, \quad \text{for } 1 \leq i < j \leq n.
\]

When \( r = 1 \), denote \( \mathcal{C}_n = \mathcal{C}^1_n \), called the **Catalan arrangement**. The number of regions of \( \mathcal{C}^r_n \) was first obtained by Athanasiadis [1] in 2004.
Theorem 1.3. [1] For any positive integers $r$ and $n$, the number of regions of $\mathcal{C}_n^r$ is

$$|\mathcal{R}(\mathcal{C}_n^r)| = n! C(n, r) = \frac{n!}{rn+1} \binom{rn+n}{n}.$$ 

In Theorem 1.3, the number $C(n, r) = \frac{1}{rn+1} \binom{rn+n}{n}$ is called the **Fuss-Catalan number** or **Raney number**, which counts the number of the $r$-Dyck paths of length $n$. As written in [23], the Fuss-Catalan number was first studied by Fuss [15] in 1791, forty-seven years before Catalan investigated the parentheses problem, see [17] for more results on the Catalan number. The paper [20] presented several combinatorial structures which are counted by Fuss-Catalan numbers. In the case of $r = 1$, $C(n, r) = C_n = \frac{1}{n+1} (2n)_r$ is the **Catalan number**, see [50] for a complete investigation on the Catalan number. Dyck path has many generalizations that have been widely studied in the past, see [6,9,14,21,26,27,29,30,42]. As a generalization of Dyck path, a $r$-Dyck path of length $n$ is a lattice path in the $x$-$y$ plane moving from $(0,0)$ to $(n, rn)$ with steps $(1,0)$ and $(0,1)$ and never going above the line $y = rx$. Denote by $D_n^r$ the collection of all $r$-Dyck paths of length $n$ and $\mathcal{D}_n = D_1^n$ the set of all Dyck paths of length $n$. In 1989, Krattenthaler [24] obtained the number of $r$-Dyck paths of length $n$.

**Theorem 1.4.** [24] For any positive integers $r$ and $n$, the cardinality of $D_n^r$ is

$$|\mathcal{D}_n^r| = C(n, r).$$

As our second main result, we will establish a bijection $\mathcal{R}(\mathcal{C}_n^r) \to \mathcal{S}_n \times D_n^r$ in Theorem 2.6 via the cubic matrix defined for the $r$-Catalan arrangement, which will extend the bijection defined in [47] p. 69.

## 2 Main Results

Our first main result is a bijection $\mathcal{R}(\mathcal{S}_n^r) \to \mathcal{T}_n^r$, which will be stated in Section 2.1 and proved in Section 4. The second main result is a bijection $\mathcal{R}(\mathcal{C}_n^r) \to \mathcal{S}_n \times D_n^r$ and will be given in Section 2.2.

### 2.1 Bijection $\mathcal{R}(\mathcal{S}_n^r) \to \mathcal{T}_n^r$

By introducing a cubic matrix for $r$-Shi arrangement, in this section we establish a bijection $\mathcal{R}(\mathcal{S}_n^r) \to \mathcal{T}_n^r$ and present a straightforward way to view the Pak-Stanley labelling. Given a region $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ and a representative $x = (x_1, x_2, \ldots, x_n) \in \Delta$, define the **cubic matrix** $C_x = (c_{ijk}(x)) \in \mathbb{R}^{n \times n \times r}$ to be

$$c_{ijk}(x) = \begin{cases} 
  x_i - x_j - k, & \text{if } i < j; \\
  0, & \text{if } i = j; \\
  x_i - x_j - k + 1, & \text{if } i > j,
\end{cases}$$

which is an $r$-tuple of square matrices as the index $k$ running from 1 to $r$. For any $i, j \in [n]$, let

$$\text{row}_i(C_x) = (c_{ijk}(x))_{j \in [n], k \in [r]} \quad \text{and} \quad \text{col}_j(C_x) = (c_{ijk}(x))_{i \in [n], k \in [r]}.$$
called the \emph{i-th row slice} and \emph{j-th column slice} of \( C_x \) respectively. Note that each hyperplane \( H \in S_n^r \) is exactly defined by the equation \( H : c_{ijk}(x) = 0 \) for some \( i, j \) and \( k \), and all points of \( \Delta \) lie in the same side of \( H \) since \( \Delta \cap H = \emptyset \). It follows that \( c_{ijk}(x) \) has the same sign for all \( x \in \Delta \), namely, \( \text{Sgn}(c_{ijk}(x)) \) is independent of the choice of representatives \( x \in \Delta \) and can be denoted by
\[
\text{Sgn}_{ijk}(\Delta) = \text{Sgn}(c_{ijk}(x)).
\]
Then \( \text{Sgn}(\Delta) = (\text{Sgn}_{ijk}(\Delta)) \) automatically defines a bijection
\[
\text{Sgn} : \mathcal{R}(S_n^r) \rightarrow \{\text{Sgn}(\Delta) \mid \Delta \in \mathcal{R}(S_n^r)\}.
\]
The symbol \( x \) is understand as either a point of \( \mathbb{R}^n \) or indeterminate depending on its meaning in the context.

**Definition 2.1.** Let \( O = \{o_1, \ldots, o_r\} \) and \( V = \{v_1, \ldots, v_n\} \) be two disjoint sets of labeled vertices. Given a region \( \Delta \in \mathcal{R}(S_n^r) \) and \( x \in \Delta \), for any \( j \in [n] \), let \( f(v_j) = (f_1(v_j), \ldots, f_r(v_j)) \in (O \cup V)^r \) be defined recursively as follows,

\( \text{i} \) if all entries of \( \text{col}_j(C_x) \) are nonpositive, let \( p_j = 0 \) and
\[
f(v_j) = (o_1, o_2, \ldots, o_r),
\]
\( \text{ii} \) otherwise, \( p_j \neq 0 \) and \( \text{col}_j(C_x) \) has a unique minimal positive entry at \( (p_j, q_j) \), let
\[
f(v_j) = (f_1(v_{p_j}), \ldots, f_{q_j-1}(v_{p_j}), f_{q_j+1}(v_{p_j}), \ldots, f_r(v_{p_j}), v_{p_j}),
\]
and let the map \( F : V \rightarrow (O \cup V)^r \) with
\[
F(v_j) = \{f_i(v_j) \mid i \in [r]\}.
\]
Define the graph \( T_x \) on the vertex set \( O \cup V \) such that the vertex \( v_j \) and vertices in \( F(v_j) \) form an \((r + 1)\)-clique for all \( j \in [n] \).

Below is the first main result of this paper, whose proof is highly nontrivial and will be given in Section 4.

**Theorem 2.2.** With the same notations as Definition 2.1, the following map is a bijection,
\[
\Psi_n^r : \mathcal{R}(S_n^r) \rightarrow \mathcal{T}_n^r, \quad \Psi_n^r(\Delta) = T_x \quad \text{for any} \ x \in \Delta.
\]

In the case of \( r = 1 \), the statements of Definition 2.1 and Theorem 2.2 become quiet simple, see Corollary 2.3

**Corollary 2.3.** Let \( O = \{o_1\} \) and \( V = \{v_1, \ldots, v_n\} \) be two disjoint sets of labeled vertices. Given a region \( \Delta \) of \( S_n \), for any \( x \in \Delta \), define an \( n \times n \) matrix \( A_x = (a_{ij}(x)) \) with
\[
a_{ij}(x) = \begin{cases}
x_i - x_j - 1, & \text{if} \ i < j; \\
0, & \text{if} \ i = j; \\
x_i - x_j, & \text{if} \ i > j,
\end{cases}
\]
and a graph \( T_x \) on \( O \cup V \) such that for each \( j \in [n] \), \( v_j \) is adjacent to \( v_{p_j} \), where \( p_j \) is defined as follows,
if column $j$ of $A_x$ has no positive entry, assume $p_j = 0$ and $v_0 = o_1$;

(ii) otherwise, column $j$ of $A_x$ has a unique minimal positive entry at row $p_j$.

Then $T_x$ is an $O$-rooted labeled tree and independent of the choice of representatives $x \in \Delta$. Moreover, the map $\Psi_n : \mathcal{R}(S_n) \to \mathcal{T}_n$ with $\Psi_n(\Delta) = T_x$ is a bijection.

In 1998, a celebrated bijection $\mathcal{R}(S_n^r) \to \mathcal{P}_n^r$ was obtained by Stanley \cite{49} and called the Pak-Stanley labeling, which is defined recursively as follows. Start with the base region $\Delta_0 \in \mathcal{R}(S_n^r)$ with

$$\Delta_0 : x_1 > x_2 > \cdots > x_n > x_1 - 1,$$

whose labeling is assumed to be $\lambda(\Delta_0) = (0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^n$. Suppose $\Delta \in \mathcal{R}(S_n^r)$ has been labeled by $\lambda(\Delta) \in \mathbb{Z}_{\geq 0}^n$, and an unlabeled region $\Delta' \in \mathcal{R}(S_n^r)$ is separated from $\Delta$ by a unique hyperplane $H : c_{ijk}(x) = 0$. Then define the region $\Delta'$ to be labeled by $\lambda(\Delta') = \lambda(\Delta) + e_i$.

Using the cubic matrix $C_x$, Theorem 2.1 of \cite{49} can be restated as follows.

**Theorem 2.4.** \cite{49} Given a region $\Delta$ of $S_n^r$ and $x \in \Delta$, for any $i \in [n]$, let

$$\lambda_i(\Delta) = \text{the number of positive signs of } \text{Sgn(row}_i(C_x)) \text{.}$$

The following map is a bijection

$$\lambda : \mathcal{R}(S_n^r) \to \mathcal{P}_n^r, \quad \Delta \mapsto \lambda(\Delta) = (\lambda_1(\Delta), \ldots, \lambda_n(\Delta)).$$

**Proof.** Note that the base region is

$$\Delta_0 = \{y \in \mathbb{R}^n \mid c_{ijk}(y) < 0, i, j \in [n], k \in [r]\}.$$

If the region $\Delta$ is separated from $\Delta_0$ by the hyperplane $H : c_{ijk}(y) = 0$, then $x \in \Delta$ implies $c_{ijk}(x) > 0$. From the definition of the Pak-Stanley labeling, it is easily seen that $\lambda_i(\Delta)$ is the number of the hyperplanes $H : c_{ijk}(y) = 0$ separating $\Delta$ from $\Delta_0$. Namely, $\lambda_i(\Delta)$ is the number of positive entries in the $i$-th row slice of $C_x$. \hfill $\square$

**Remark 2.5.** Theoretically, the compositions of our bijection $(\Psi_n^r)^{-1} : \mathcal{T}_n^r \to \mathcal{R}(S_n^r)$ in Theorem 2.2 and the Pak-Stanley labeling $\lambda : \mathcal{R}(S_n^r) \to \mathcal{P}_n^r$ in Theorem 2.4 will produce a bijection $\mathcal{T}_n^r \to \mathcal{P}_n^r$, while it seems to be highly complicated and difficult to be stated explicitly.

### 2.2 Bijection $\mathcal{R}(C_n^r) \to \mathfrak{S}_n \times \mathcal{D}_n^r$

In this section, we will establish a bijection $\mathcal{R}(C_n^r) \to \mathfrak{S}_n \times \mathcal{D}_n^r$. Similar as \cite{47} p. 68], the permutation group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permuting coordinates, i.e., if $\pi \in \mathfrak{S}_n$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

Given a region $\Delta \in \mathcal{R}(C_n^r)$ and $x \in \Delta$, there is a unique permutation $\pi_{\Delta} \in \mathfrak{S}_n$, independent of the choice of $x \in \Delta$, such that

$$x_{\pi_{\Delta}(1)} > \cdots > x_{\pi_{\Delta}(n)}.$$

Note that $\mathcal{R}(C_n^r)$ is $\mathfrak{S}_n$-invariant, i.e., for any $\pi \in \mathfrak{S}_n$ and $\Delta \in \mathcal{R}(C_n^r)$, we have

$$\pi(\Delta) = \{\pi(x) \mid x \in \Delta\} \in \mathcal{R}(C_n^r).$$
For $\pi \in S_n$, denote by
\[ \mathcal{R}_\pi(C^n_r) = \{ \Delta \in \mathcal{R}(C^n_r) \mid \pi_\Delta = \pi \}. \]
In particular, let
\[ \mathcal{R}_1(C^n_r) = \{ \Delta \in \mathcal{R}(C^n_r) \mid \pi_\Delta = \text{1 is the identity permutation} \}. \]

It is clear that $\pi$ is a bijection from $\mathcal{R}_\pi(C^n_r)$ to $\mathcal{R}_1(C^n_r)$ and
\[ \mathcal{R}(C^n_r) = \bigsqcup_{\pi \in S_n} \mathcal{R}_\pi(C^n_r). \]

To obtain the bijection $\mathcal{R}(C^n_r) \to S_n \times D^n_r$, it is enough to establish a bijection $\mathcal{R}_1(C^n_r) \to D^n_r$. Given a region $\Delta \in \mathcal{R}_1(C^n_r)$ and a representative $x = (x_1, x_2, \ldots, x_n) \in \Delta$, define the **cubic matrix** $D_x = (d_{ijk}(x)) \in \mathbb{R}^{n \times n \times r}$ to be
\[ d_{ijk}(x) = \begin{cases} x_i - x_j - k, & \text{if } i \neq j; \\ 0, & \text{if } i = j; \end{cases} \]

Similar as before, each hyperplane $H \in C^n_r$ is exactly defined by the equation $H : d_{ijk}(x) = 0$ for some $i, j \in [n]$ with $i \neq j$ and $k \in [r]$. So we still have that $\text{Sgn}(d_{ijk}(x))$ is independent of the choice of representatives $x \in \Delta$.

For any $r$-Dyck path $P \in D^n_r$, if the vertical line $x = i - \frac{1}{2}$ intersects $P$ at the $y$-coordinate $h_i(P)$, the sequence $h(P) = (h_1(P), \ldots, h_n(P)) \in \mathbb{Z}^n$ is nondecreasing and satisfies $0 \leq h_i(P) \leq r(i - 1)$, called the **height sequence** of $P$. Conversely, it is clear that any non-decreasing sequence $h = (h_1, \ldots, h_n)$ with $0 \leq h_i \leq r(i - 1)$ uniquely determines an $r$-Dyck path $P$ of length $n$ such that $h(P) = h$. Indeed, the height sequence of a $r$-Dyck path is also a $r$-parking function. Now we are ready to give the bijection $\mathcal{R}_1(C^n_r) \to D^n_r$. Given any region $\Delta \in \mathcal{R}_1(C^n_r)$ and $x \in \Delta$, let $h(\Delta) = (h_1(\Delta), \ldots, h_n(\Delta))$ be a sequence defined by
\[ h_j(\Delta) = \text{the number of positive signs of Sgn}(\text{col}_j(D_x)), \quad j \in [n]. \quad (5) \]

As we shall see in Theorem 2.6, the sequence $h(\Delta)$ is exactly the height sequence of a $r$-Dyck path of length $n$, say $P_\Delta$, which defines the bijection
\[ \mathcal{R}_1(C^n_r) \to D^n_r, \quad \Delta \mapsto P_\Delta. \quad (6) \]

Now for any region $\Delta \in \mathcal{R}(C^n_r)$, we have $\Delta \in \mathcal{R}_{\pi_\Delta}(C^n_r)$ and $\Delta' = \pi_\Delta(\Delta) \in \mathcal{R}_1(C^n_r)$. By abuse of notations, denote by $P_\Delta$ the corresponding $r$-Dyck path $P_{\Delta'}$ obtained from the above bijection $(6)$, namely $P_\Delta = P_{\pi_\Delta(\Delta)}$ for any $\Delta \in \mathcal{R}(C^n_r)$. Below is our second main result.

**Theorem 2.6.** For any positive integers $r$ and $n$, the following map is a bijection,
\[ \Phi^n_r : \mathcal{R}(C^n_r) \to S_n \times D^n_r, \quad \Phi^n_r(\Delta) = (\pi_\Delta, P_\Delta). \]

**Proof.** Notice from Theorem 1.3 and 1.4 that the both $\mathcal{R}(C^n_r)$ and $S_n \times D^n_r$ have the same cardinality $n!C(n, r)$. By the above arguments, it is enough to show that the map defined in $(6)$ is injective. For any $\Delta \in \mathcal{R}_1(C^n_r)$ and $x = (x_1, \ldots, x_n) \in \Delta$, we have $x_1 > x_2 > \cdots > x_n$. It is easily seen from the definition of the cubic matrix $D_x = (d_{ijk}(x)) \in \mathbb{R}^{n \times n \times r}$ that
\[ (a) \ d_{ijk}(x) > 0 \text{ implies } i < j; \]
(b) if $i < j < j'$, then $d_{ijk}(x) > 0$ implies $d_{ij'k}(x) > 0$ since $d_{ij'k}(x) > d_{ijk}(x)$.

Note from the definition of $h_j(\Delta)$ that for $j \in [n]$,

$$h_j(\Delta) = \# \{(i, k) \in [n] \times [r] \mid d_{ijk}(x) > 0\}.$$  \hfill (7)

The properties (a) and (b) imply $h_j(\Delta) \leq r(j - 1)$ for any $j \in [n]$ and $h_1(\Delta) \leq h_2(\Delta) \leq \cdots \leq h_n(\Delta)$ respectively. So $h(\Delta) = (h_1(\Delta), \ldots, h_n(\Delta))$ is a height sequence of a $r$-Dyck path of length $n$, i.e., the map given in [6] is well-defined. Next we prove the injectivity of the map in [6] by contradiction. Suppose $\Delta$ and $\Omega$ are two distinct regions in $R_1(C_n)$ with $h(\Delta) = h(\Omega)$ and let $x \in \Delta$ and $y \in \Omega$. From $\Delta \neq \Omega$, we have a minimal index $j \in [n]$ such that $h_j(\Delta) = h_j(\Omega)$, from (7) there must exist a pairing $(i', k') \neq (i, k)$ such that

$$d_{i'jk'}(x) = x_i - x_j - k' < 0 \quad \text{and} \quad d_{i'jk'}(y) = y_i - y_j - k' > 0.$$

By property (a), we have $i, i' < j$. If $i = i'$, we have $k' > x_i - x_j > k$ since $d_{ijk}(x) > 0$ and $d_{ij'k'}(x) < 0$ and $k > y_i - y_j > k'$ since $d_{ijk}(y) < 0$ and $d_{ij'k'}(y) > 0$, which is a contradiction. If $i < i'$, we have $k > y_i - y_j > y_{i'} - y_j > k'$ since $d_{ijk}(y) < 0$ and $d_{ij'k'}(y) > 0$. Consider the hyperplane $H : d_{i'j'(k-k')}(z) = 0$. We have

$$d_{i'j'(k-k')}(x) = d_{ijk}(x) - d_{ij'k'}(x) > 0,$$

$$d_{i'j'(k-k')}(y) = d_{ijk}(y) - d_{ij'k'}(y) < 0,$$

which means that the hyperplane $H : d_{i'j'(k-k')}(z) = 0$ separates $\Delta$ from $\Omega$, a contradiction to the minimality of the index $j$. By similar arguments as the case $i < i'$, we can obtain a contradiction for the case $i > i'$. So we can conclude that the map in [6] is injective, which completes the proof. \hfill \Box

It is easily seen that Theorem 2.6 not only extends the bijection $R(C_n) \to \mathcal{S}_n \times D_n$ defined in [17] page 69, but make it more straightforward with the help of the cubic matrix. Below is an example to illustrate the construction of the Dyck path from a region in the case of $r = 1$.

**Example 2.7.** Let $\Delta \in R(C_6)$ be the region

$$\Delta = \left\{ x = (x_1, \ldots, x_6) \in \mathbb{R}^6 \mid \begin{array}{l} \text{if } (i, j) \in \{(1, 2), (1, 3), (2, 3), (3, 4), (5, 6)\}, \text{ then } x_i > x_j \geq x_k, \text{ where } k \in \{6, 5, 4\}, \\
\text{otherwise, } x_i > x_j > x_k, \text{ where } k \in \{4, 5, 6\}. \end{array} \right\}.$$

It is obvious that $\pi_\Delta = 436125 \in \mathcal{S}_6$ and for $1 \leq i < j \leq 6$,

$$\Delta' = \pi_\Delta(\Delta) = \left\{ x = (x_1, \ldots, x_6) \in \mathbb{R}^6 \mid \begin{array}{l} x_i > x_j \geq x_k, \text{ where } k \in \{6, 5, 4\}, \\
\text{otherwise, } x_i > x_j > x_k, \text{ where } k \in \{4, 5, 6\}. \end{array} \right\}.$$

It follows that for any $x \in \Delta'$,

$$\text{Sgn}(c_{ij1}(x)) = \begin{cases} - & \text{if } (i, j) \in \{(1, 2), (1, 3), (2, 3), (3, 4), (5, 6)\}; \\
+ & \text{if } (i, j) \in \{(1, 4), (2, 4), (1, 5), (2, 5), (3, 5), (4, 5), (4, 6), (5, 6)\}. \end{cases}$$

So we have $h(\Delta') = (0, 0, 0, 2, 4, 4)$, which is the height sequence of the Dyck path $P_\Delta = P_{\Delta'}$. Namely, $\Phi^1_n(\Delta) = (\pi_\Delta, P_\Delta)$ with $\pi_\Delta = 436125$ and $P_\Delta$ is the red path of Figure [1].
3 O-Rooted Labeled r-Trees

Preparing for Theorem 2.2, we give some characterizations on O-rooted labeled r-trees in this section. For the structural integrity, below we restate the definition of O-rooted labeled r-trees following from Foata \[10\] in 1971.

**Definition 3.1.** [10] Let \( O = \{o_1, \ldots, o_r\} \) and \( V = \{v_1, \ldots, v_n\} \) be two disjoint sets of labeled vertices. An O-rooted labeled r-tree \( T \) on \( O \cup V \) is a graph having the property: there is a valid rearrangement \( \nu = (v_{i_1}, \ldots, v_{i_n}) \) of vertices \( v_1, \ldots, v_n \), such that each \( v_{i_j} \) with \( j \in [n] \) is adjacent to exactly \( r \) vertices in \( \{o_1, \ldots, o_r, v_{i_1}, \ldots, v_{i_{j-1}}\} \) and, moreover, these \( r \) vertices are themselves mutually adjacent in \( T \). Let

\[
F^\nu_T(v_{i_j}) = \{v \mid v \text{ is adjacent to } v_{i_j} \text{ in } T\} \cap \{o_1, \ldots, o_r, v_{i_1}, \ldots, v_{i_{j-1}}\},
\]

whose members are called **fathers of** \( v_{i_j} \) in \( T \) under \( \nu \).

**Remark 3.2.** Note from the above definition that the father set of \( v_{i_1} \) in \( T \) under \( \nu \) is the root set \( O \), so vertices of \( O \) are mutually adjacent in \( T \). In the case of \( r = 1 \), for any ordinary tree \( T \) on the labeled vertices \( O \cup V \), suppose that \( \nu \) is a rearrangement having the property: \( v_i \) is ahead of \( v_j \) in \( \nu \) whenever \( d_T(v_i, o_1) < d_T(v_j, o_1) \) as distances of two vertices in \( T \). Obviously such \( \nu \) always exists and is valid for defining \( T \) as an O-rooted labeled tree.

To make the above definition more clear, Propositions 3.3-3.5 are characterizations on valid rearrangements and father sets, which might have been obtained by others in the literature but not noticed by us yet. Indeed, the r-trees have been characterized exactly to be the maximal graphs with a given treewidth in [34], and the chordal graphs all of whose maximal cliques are the same size \( r+1 \) and all of whose minimal clique separators are also all the same size \( r \) in [37].

**Proposition 3.3.** Let \( O = \{o_1, \ldots, o_r\} \) and \( V = \{v_1, \ldots, v_n\} \) be two disjoint sets of labeled vertices, and \( T \) an O-rooted labeled r-tree on \( O \cup V \). For each \( i \in [n] \), the father set \( F^\nu_T(v_i) \) is independent of the choice of valid rearrangements \( \nu \) for \( T \), and denoted by \( F_T(v_i) \).

**Proof.** Without loss of generality, we may assume that \( \epsilon = (v_1, \ldots, v_n) \) is a valid rearrangement for \( T \). Given a new valid rearrangement \( \nu \) of vertices \( v_1, \ldots, v_n \) for \( T \), suppose \( s \) is the minimal number such that \( F^\nu_T(v_s) \neq F^\epsilon_T(v_s) \). If \( o_j \in F^\nu_T(v_s) \setminus F^\epsilon_T(v_s) \) for some \( j \in [r] \), then \( o_j \notin F^\epsilon_T(v_s) \).
implies that $v_s$ is not adjacent to $o_j$ in $T$, a contradiction to $o_j \in F_T^s(v_s)$. If $v_t \in F^s_T(v_s) \setminus F^r_T(v_s)$, then we have $t < s$ since $v_t \in F^s_T(v_s)$ and $v_s \notin F^r_T(v_t)$ by the minimality of $s$. Note the fact that $v_s$ is adjacent to $v_t$ in $T$, a contradiction to $v_s \notin F^r_T(v_t)$ and $v_t \notin F^r_T(v_s)$. Hence, $F^r_T(v_i) = F^r_T(v_s)$ for all $i \in [n]$.

**Proposition 3.4.** Let $O = \{o_1, \ldots, o_r\}$ and $V = \{v_1, \ldots, v_n\}$ be two disjoint sets of labeled vertices, and $T$ an $O$-rooted labeled $r$-tree on $O \cup V$. A rearrangement $\nu$ of vertices $v_1, \ldots, v_n$ is valid for $T$ if and only if $v_s$ is ahead of $v_t$ in $\nu$ whenever $v_s \in F_T(v_t)$.

**Proof.** The sufficiency is obvious from the definition of $O$-rooted labeled $r$-tree. To prove the necessity, we may assume that $\nu = (v_1, \ldots, v_n)$ is a valid rearrangement for $T$. Proposition 3.3 implies $F^r_T(v_i) = F^r_T(v_t)$. Now suppose that $\nu$ is a rearrangement such that $v_s$ is ahead of $v_t$ in $\nu$ whenever $v_s \in F^r_T(v_t)$. Let $G^r_T(v_i)$ consist of those vertices $o_1, \ldots, o_r$ who are adjacent to $v_t$ in $T$, and vertices $v_1, \ldots, v_n$ who are adjacent to $v_t$ in $T$ and ahead of $v_s$ in $\nu$. Immediately, we have $F^r_T(v_i) \subseteq G^r_T(v_i)$ for all $i \in [n]$ and $G^r_T(v_t) \cap O = F^r_T(v_t) \cap O$. To obtain the necessity, i.e., $\nu$ is valid for $T$, it is enough to show $F^r_T(v_i) = G^r_T(v_i)$. Suppose $G^r_T(v_i) \neq F^r_T(v_i)$ and $v_s \in G^r_T(v_i) \setminus F^r_T(v_i)$ for some $s, t \in [n]$. By the definition of $G^r_T(v_i)$, $v_s \in G^r_T(v_i)$ implies that $v_s$ is ahead of $v_t$ in $\nu$ and adjacent to $v_t$ in $T$. From the assumption of $\nu$, we have $v_t \notin F^r_T(v_s)$. Note that $v_s$ and $v_t$ are adjacent, a contradiction to $v_t \notin G^r_T(v_s)$ and $v_s \notin F^r_T(v_t)$.

**Proposition 3.5.** Let $O = \{o_1, \ldots, o_r\}$ and $V = \{v_1, \ldots, v_n\}$ be two disjoint sets of labeled vertices, and $F : V \to (O \cup V)$. There is an $O$-rooted labeled $r$-tree $T$ on $O \cup V$ with $F(v_i) = F_T(v_i)$ for all $i \in [n]$ if and only if $F$ satisfies the following properties:

(a) if $v_i \in F(v_{i_2}), \ldots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \ldots, i_j \in [n]$, then $v_{i_j} \notin F(v_{i_1});$

(b) if $F(v_j) \neq O$, then there is a vertex $v_i \in F(v_j)$ such that $|F(v_j) \cap F(v_i)| = r - 1$.

Moreover, both the $r$-tree $T$ and the vertex $v_i$ in (b) are unique.

**Proof.** Let’s prove the second part first. If $T$ is an $O$-rooted labeled $r$-tree on $O \cup V$ with $F(v_i) = F_T(v_i)$, then all vertices of $F(v_i) \cup \{v_1\}$ are mutually adjacent, which exactly form all edges of $T$. So $T$ is uniquely determined by $F(v_i) = F_T(v_i)$. To prove the uniqueness of $v_i$ in (b), note that (a) implies $v_j \notin F(v_j)$ for all $j \in [n]$. Suppose there is another vertex $v_{i'} \in F(v_j)$ with $i' \neq i$ such that $|F(v_j) \cap F(v_{i'})| = r - 1$. Then we have $v_i \in F(v_{i'})$ and $v_{i'} \notin F(v_i)$, a contradiction to (a).

To prove the sufficiency of the first part, we may assume that $\nu = (v_1, \ldots, v_n)$ is a valid rearrangement for $T$. From Proposition 3.4 if $v_j \in F(v_i)$, $v_j$ is ahead of $v_i$ in $\nu$, i.e., $j < i$. So if $v_{i_1} \in F(v_{i_2}), \ldots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \ldots, i_j \in [n]$, then $i_1 < i_j$ which implies $v_{i_j} \notin F(v_{i_1})$ and (a) holds. To prove (b), let $i$ be the largest number with $v_i \in F(v_j)$, $i < j$ obviously. Suppose $v_{i'} \in F(v_j) \setminus F(v_i)$, then $i' < i < j$ and $v_i \notin F(v_{i'})$. Note $v_i, v_{i'} \in F(v_j)$ and all vertices of $F(v_j)$ are mutually adjacent, which is a contradiction to $v_{i'} \notin F(v_i)$ and $v_i \notin F(v_{i'})$. Thus we have $F(v_j) \setminus F(v_i) = \{v_i\}$, i.e., $|F(v_j) \cap F(v_i)| = r - 1$. Moreover, for any $v_{i'} \in F(v_j)$ with $i' < i$, we have at least $v_i, v_{i'} \notin F(v_{i'})$, i.e., $|F(v_j) \cap F(v_{i'})| \leq r - 2$.

To prove the necessity of the first part, from the assumption $v_{i_j} \notin F(v_{i_1})$ whenever $v_{i_1} \in F(v_{i_2}), \ldots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \ldots, i_j \in [n]$, we have $v_{i_1} \notin F(v_{i_2})$ for $1 \leq s < t \leq j$, which implies $v_{i_1} \neq v_{i_2}$, i.e., $i_s \neq i_1$ for $1 \leq s < t \leq j$ since $v_{i_1} \in F(v_{i_2})$ and $v_{i_2} \notin F(v_{i_1})$, and $j \leq n$ consequently. Suppose $v_{i_1} \in F(v_{i_2}), \ldots, v_{i_{k-1}} \in F(v_{i_k})$ for some $i_1, \ldots, i_k \in [n]$, where $k$ is maximal possible. The maximality of $k$ implies $v_{i_k} \notin F(v_i)$ for all $i \in [n]$. Next we use induction on the size of $V$. When $|V| = 1$, note that from (a), $v_1 \in F(v_1)$ produces
$v_1 \notin F(v_1)$, which forces $v_1 \notin F(v_1)$, i.e., $F(v_1) = O$ and the result follows clearly. Let $V' = V \setminus \{v_k\}$ and $F' : V' \rightarrow (O \cup V')$ with $F'(v_i) = F(v_i)$ for all $v_i \in V'$. It is clear that (a) and (b) holds for $F'$. From the induction hypothesis, there is an $O$-rooted labeled $r$-tree $T'$ on $O \cup V'$ such that $F'(v_i) = F'_r(v_i)$ with $v_i \in V'$. Given a valid rearrangement $\nu'$ for $T'$, let $T$ be a graph on the labeled vertex set $O \cup V$ obtained from $T'$ by adding the $r$ edges between $v_{ik}$ and each vertex of $F(v_{ik})$, and let $\nu = (\nu', v_{ik})$. The case of $F(v_{ik}) = O$ is clear. Otherwise, from (b) we have $|F(v_{ik}) \cap F(v_i)| = r - 1$ for some $v_i \in F(v_{ik})$. Note all vertices of $F(v_i) \cup \{v_i\}$ are mutually adjacent in $T'$, which implies that all vertices of $F(v_{ik})$ are also mutually adjacent in $T'$. Consequently, the graph $T$ is an $O$-rooted labeled $r$-tree and $\nu$ is a valid rearrangement for $T$. □

4 Proof of Theorem 2.2

Roughly speaking, in Definition 2.1 the graph $T_x$ is obtained by the following process,

$x \in \Delta \rightarrow p_j, q_j (\text{if } p_j \neq 0) \rightarrow f(v_j) \rightarrow F(v_j) \rightarrow T_x$,

which requires that $p_j, q_j (\text{if } p_j \neq 0), f(v_j), F(v_j)$, and $T_x$ are well-defined for all $j \in [n]$, see Proposition 4.1.

Proposition 4.1. With the same notations as Definition 2.1, the graph $T_x$ is independent of the choice of $x \in \Delta$. Moreover, $T_x$ is an $O$-rooted labeled $r$-tree with $F_{t_x}(v_j) = F(v_j)$ for all $j \in [n]$, namely, the map $\Psi^r_n$ in (4) is well defined.

Proof. Firstly, we will show that $p_j$ and $q_j (\text{if } p_j \neq 0)$ is independent of the choice of $x \in \Delta$. Let $\Delta \in \mathcal{R}(S^i_n)$ and $j \in [n]$. For the case $\beta$ of Definition 2.1 by (2) we have $\text{Sgn}_{ijk}(\Delta) \neq +$ for all $i \in [n]$ and $k \in [r]$, which implies $p_j = 0$ for all $x \in \Delta$. For the case (ii) of Definition 2.1 given any $(i, k) \neq (i', k')$ in $[n] \times [r]$ with $i > i'$, by routine calculations on entries of $C_{x,j}$, we have

$$c_{ijk}(x) - c_{i'jk'}(x) = \begin{cases} c_{i'i'(k-k')}(x), & \text{if } k > k', \quad i > j > i'; \\ -c_{i'i'(k'-k'+1)}(x), & \text{if } k < k', \quad i > j > i'; \\ c_{i'i'(k'-k'+1)}(x), & \text{if } k > k', \quad i > i' > j \text{ or } j > i > i'; \\ -c_{i'i'(k'')} (x), & \text{if } k < k', \quad i > i' > j \text{ or } j > i > i'. \end{cases}$$

For all $x \in \Delta$, we have $\text{Sgn}(c_{ijk}(x) - c_{i'jk'}(x)) = \text{Sgn}(c_{i'i's}(x))$ or $-\text{Sgn}(c_{i'i's}(x))$ for some $s \in [r]$, which means that both $\text{Sgn}(c_{ijk}(x))$ and $\text{Sgn}(c_{ijk}(x) - c_{i'jk'}(x)) \neq 0$ are independent of the choice of $x \in \Delta$. It implies that for all $x \in \Delta$, the $j$-th column slice $\text{col}_j(C_x)$ has a unique minimal positive entry at the same position $(p_j, q_j)$, i.e., $(p_j, q_j)$ is independent of the choice of $x \in \Delta$.

Secondly, we will prove that $f(v_j)$ and $F(v_j)$ is well defined. Let

$$\pi : [n] \rightarrow \{0, 1, \ldots, n\} \quad \text{with } \pi(j) = p_j,$$

$$\pi^{l+1}(j) = \pi(\pi^l(j)) = p_{\pi^l(j)} \text{ for } l \geq 0, \text{ and } \pi^0(j) = j.$$ Note that if $p_j \neq 0$, we have $c_{p_jq_j}(x) > 0$ and

$$c_{p_jq_j}(x) = \begin{cases} x_{p_j} - x_j - q_j, & \text{if } p_j < j; \\ x_{p_j} - x_j - q_j + 1, & \text{if } p_j > j. \end{cases}$$
which implies \( x_{p_j} > x_j \) and \( p_j \neq j \). Namely, we have \( x_{\pi(j)} > x_j \) if \( \pi(j) \neq 0 \). There exists some \( m \in [n] \) such that \( \pi(j), \ldots, \pi^{m-1}(j) \neq 0 \) and \( \pi^m(j) = 0 \), otherwise \( \pi(j), \ldots, \pi^n(j) \neq 0 \) and \( x_j < x_{\pi(j)} < \cdots < x_{\pi^n(j)} \) which is obviously impossible for \( \bm{x} = (x_1, \ldots, x_n) \). Moreover, \( x_j < x_{\pi(j)} < \cdots < x_{\pi^{m-1}(j)} \) implies that \( j, \pi(j), \ldots, \pi^{m-1}(j) \) are mutually distinct. It follows from (i) and (ii) of Definition 2.1 that
\[
f(v_{\pi^{m-2}(j)}) = f(v_{\pi^{m-1}(j)}) = (o_1, \ldots, o_r),
\]
and for all \( l = m-2, \ldots, 1, 0 \), we have
\[
f(v_{\pi^l(j)}) = (f_1(v_{\pi^{l+1}(j)}), \ldots, f_{q_{l}(j)-1}(v_{\pi^{l+1}(j)}), f_{q_{l}(j)}(v_{\pi^{l+1}(j)}), \ldots, f_r(v_{\pi^{l+1}(j)}), v_{\pi^{l+1}(j)}). \tag{9}
\]
Proceeding \( l \) from \( m - 2 \) to \( 0 \) step by step recursively, finally we can obtain \( f(v_j) \), which is well-defined consequently. Moreover, note from the definition that for each \( l = m-2, \ldots, 1, 0 \),
\[
F(v_{\pi^l(j)}) = (F(v_{\pi^{l+1}(j)}) \setminus f_{q_{l}(j)}(v_{\pi^{l+1}(j)})) \cup \{v_{\pi^{l+1}(j)}\},
\]
i.e., \( F(v_{\pi^l(j)}) \) is obtained from \( F(v_{\pi^{l+1}(j)}) \) by removing the vertex \( f_{q_{l}(j)}(v_{\pi^{l+1}(j)}) \) and adding the vertex \( v_{\pi^{l+1}(j)} \). So each \( F(v_{\pi^l(j)}) \) consists of \( r \) members of vertices \( o_1, \ldots, o_r, v_{\pi^{m-1}(j)}, \ldots, v_{\pi^{l+1}(j)} \), which is of size \( r \) since \( j, \pi(j), \ldots, \pi^{m-1}(j) \) are mutually distinct and nonzero. In particular, \( |F(v_j)| = |F(v_{\pi^l(j)})| = r \) and \( F(v_j) \) is well defined.

Finally, it remains to show that there exists uniquely an \( O \)-rooted labeled \( r \)-tree \( T_\bm{x} \) on \( O \cup V \) with \( F_{T_\bm{x}}(v_j) = F(v_j) \) for all \( j \in [n] \). Recall the arguments of the above proof that if \( \pi(j) \neq 0 \), \( F(v_j) \) consists of \( r \) members of vertices \( o_1, \ldots, o_r, v_{\pi^{m-1}(j)}, \ldots, v_{\pi(j)} \), where \( m \) is the smallest integer with \( \pi^m(j) = 0 \) and \( m \geq 2 \). If \( v_i \in F(v_j) \), then \( \pi(j) = p_j \neq 0 \) and
\[
v_i \in \{o_1, \ldots, o_r, v_{\pi^{m-1}(j)}, \ldots, v_{\pi(j)}\},
\]
which follows \( v_i = v_{\pi^l(j)} \), i.e., \( i = \pi^l(j) \) for some positive integer \( l \in [m-1] \). Now suppose \( v_{i_1} \in F(v_{i_2}), \ldots, v_{i_{l-1}} \in F(v_{i_l}) \) for some \( i_1, \ldots, i_l \in [n] \). There exist some positive integers \( l_1, \ldots, l_{l-1} \) such that
\[
i_1 = \pi^{l_1}(i_2), \ldots, i_{l-1} = \pi^{l_{l-1}}(i_l).
\]
We have \( i_l = \pi^l(i_j) \) with \( l = l_1 + \cdots + l_{l-1} \), i.e., \( v_{i_l} = v_{\pi^l(i_j)} \) which implies \( v_{i_l} \notin F(v_{i_1}) \). So the map \( F \) satisfies the property (a) of Proposition 3.3. The property (b) is obvious since \( v_{p_j} \in F(v_j) \) and \( |F(v_j) \cap F(v_{p_j})| = r - 1 \). The proof completes by Proposition 3.3. \qed

**Remark 4.2.** (1) From the above proof, notations of Definition 2.1 can be written more precisely as
\[
p_j = p_j(\Delta), \quad q_j = q_j(\Delta), \quad f = f_\Delta, \quad F = F_\Delta, \text{ and } T_\bm{x} = T_\Delta, \tag{10}
\]
since they are all independent of the choice of \( \bm{x} \in \Delta \). (2) We also have the following observation
\[
V = \{v_1, \ldots, v_n\} \nsubseteq \bigcup_{j=1}^n F_\Delta(v_j), \tag{11}
\]
otherwise, we have \( v_{i_1} \in F_\Delta(v_{i_2}), v_{i_2} \in F_\Delta(v_{i_3}), \ldots \) for an infinite sequence \( i_1, i_2, \ldots \), which is a contradiction since \( i_1 = \pi^{l_1}(i_2), i_2 = \pi^{l_2}(i_3), \ldots \) for some positive integers \( l_1, l_2, \ldots \).
It is easily seen from [11] that for any $i, j, k \in [n]$ and $s, t \in [r]$ with $i > j > k$ and $s + t \leq r$, we have the following facts on linear relations among the entries of the cubic matrix $C_x$,

\begin{align*}
(F1) \quad c_{ijk}(x) + c_{jkt}(x) &= c_{ik(s+t-1)}(x); & (F2) \quad c_{iks}(x) + c_{ktj}(x) &= c_{ij(s+t)}(x); \\
(F3) \quad c_{kis}(x) + c_{ijt}(x) &= c_{kjs(t-1)}(x); & (F4) \quad c_{kjs}(x) + c_{ijt}(x) &= c_{kis(t)}(x); \\
(F5) \quad c_{jk}(x) + c_{ikt}(x) &= c_{jks(t-1)}(x); & (F6) \quad c_{ij}(x) + c_{ikt}(x) &= c_{jk(s+t)}(x).
\end{align*}

Lemma 4.3. If $p_j \neq 0$ and $q_j$ are defined as (ii) of Definition 2.1 then entries of $C_x$ have the following sign relations,

$$\text{Sgn}(c_{ijk}(x)) = \begin{cases} 
\text{Sgn}(c_{ip_j(k-q_j+1)}(x)), & \text{if } q_j \leq k \text{ and } (i, j, p_j) \text{ is even}; \\
-\text{Sgn}(c_{ip_j(q_j-k)}(x)), & \text{if } q_j > k \text{ and } (i, j, p_j) \text{ is even}; \\
-\text{Sgn}(c_{ip_j(q_j-k+1)}(x)), & \text{if } q_j \geq k \text{ and } (i, j, p_j) \text{ is odd}; \\
\text{Sgn}(c_{ip_j(k-q_j)}(x)), & \text{if } q_j < k \text{ and } (i, j, p_j) \text{ is odd},
\end{cases}$$

where $(i, j, p_j)$ is even if $i < j < p_j$, or $p_j < i < j$, or $j < p_j < i$, and odd otherwise.

Proof. We prove the result in the case of $q_j \leq k$ and $i < j < p_j$ whose arguments can be applied to other cases analogously. When $i < j < p_j$, by the fact (F3) we have

$$c_{ip_j, q_j}(x) = c_{ijk}(x) - c_{ip_j(k-q_j+1)}(x),$$

which from the assumption is the unique minimal positive entry in the $j$-th column slice of $C_x$. If $c_{ijk}(x)$ is positive, by the unique minimality of $c_{ip_j, q_j}(x)$ we have $c_{ijk}(x) > c_{ip_j, q_j}(x)$, which implies $c_{ip_j(k-q_j+1)}(x) > 0$. If $c_{ijk}(x)$ is negative, by the positivity of $c_{ip_j, q_j}(x)$ we have $c_{ip_j(k-q_j+1)}(x) < 0$. Namely, $\text{Sgn}(c_{ijk}(x)) = \text{Sgn}(c_{ip_j(k-q_j+1)}(x)).$ \hfill \square

Let $\text{Proj}_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection defined by

$$\text{Proj}_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

It is clear that $\text{Proj}_j(\Delta) \in \mathcal{R}(S_{n-1}^r)$ for any $\Delta \in \mathcal{R}(S_n^r)$. Below is a key lemma to prove the injectivity of the map $\Psi^r$ of (2).

Lemma 4.4. Given $j' \in [n]$, for any $\Delta' \in \mathcal{R}(S_{n-1}^r)$, $i' \in \{0, 1, \ldots, j' - 1, j' + 1, \ldots, n\}$, and $k' \in [r]$ (if $i' \neq 0$), there is at most one region $\Delta \in \mathcal{R}(S_n^r)$ such that $\text{Proj}_{j'}(\Delta) = \Delta'$, $p_{j'}(\Delta) = i'$, $q_{j'}(\Delta) = k'$ (if $i' \neq 0$), and $j' \neq p_j(\Delta)$ for all $j \in [n]$, see (10) for notations $p_j(\Delta)$ and $q_j(\Delta)$.

Proof. We only consider the case of $j' = n$. For general $j'$, the arguments are analogous but more tedious. Suppose two regions $\Delta, \Omega \in \mathcal{R}(S_n^r)$ satisfying that $\text{Proj}_n(\Delta) = \text{Proj}_n(\Omega) = \Delta'$ and

$$p_n(\Delta) = p_n(\Omega) = i', \quad q_n(\Delta) = q_n(\Omega) = k' \text{ (if } i' \neq 0), \quad \text{and } n \neq p_j(\Delta), \quad p_j(\Omega) \text{ for } j \in [n].$$

We will show $\text{Sgn}(\Delta) = \text{Sgn}(\Omega)$, which implies $\Delta = \Omega$ since the map $\text{Sgn} : \mathcal{R}(S_n^r) \to \{\text{Sgn}(\Delta) \mid \Delta \in \mathcal{R}(S_n^r)\}$ is a bijection by (3). Given $z \in \Delta'$, let $x \in \Delta$ and $y \in \Omega$ such that $\text{Proj}_n(x) = \text{Proj}_n(y) = z$. By (2) we have $\text{Sgn}_{ijk}(\Delta) = \text{Sgn}(c_{ijk}(x))$ and $\text{Sgn}_{ijk}(\Omega) = \text{Sgn}(c_{ijk}(y))$. It is enough to show that for all $i, j \in [n]$ and $k \in [r],$

$$\text{Sgn}(c_{ijk}(x)) = \text{Sgn}(c_{ijk}(y)).$$

\hfill (12)
We first claim that for all \( j \in [n] \),
\[
p_j(\Delta) = p_j(\Omega) = p_j \quad \text{and} \quad q_j(\Delta) = q_j(\Omega) = q_j \quad \text{if} \; p_j \neq 0.
\]

Indeed, if \( j = j' = n \), it is obvious from the assumptions. If \( j \neq j' = n \), note \( n \neq p_j(\Delta) \) (\( p_j(\Omega) \) resp.) for all \( j \in [n] \). By the definitions of \( p_j(\Delta) \) and \( q_j(\Delta) \) (\( p_j(\Omega) \) and \( q_j(\Omega) \) resp.), the minimal positive entry of \( \text{col}_j(C_x) \) (\( \text{col}_j(C_y) \) resp.) never appears in the \( n \)-th row slice of \( C_x \) (\( C_y \) resp.). It follows that \( p_j(\Delta) = p_j(\Delta') = p_j(\Omega) \) and \( q_j(\Delta) = q_j(\Delta') = q_j(\Omega) \) for \( j \neq j' \), so the claim holds. Notice that \((12)\) holds if \( p_j = 0 \) since all entries of the \( j \)-th column slice of \( C_x \) are nonpositive, more precisely, \( \text{Sgn}(c_{ijk}(x)) = 0 \) and \( \text{Sgn}(c_{ijk}(x)) = -1 \) if \( i \neq j \). Now we assume \( p_j \neq 0 \) and consider the following cases to prove \((12)\).

1. For \( i, j \in [n-1] \), since \( \text{Proj}_n(x) = \text{Proj}_n(y) = z \), we have \( c_{ijk}(x) = c_{ijk}(y) = c_{ijk}(z) \). Thus \((12)\) holds in this case.

(ii) For \( j = n \) and \( i \in [n] \), note \( p_n(\Delta) = p_n(\Omega) = i' \neq n \) and \( q_n(\Delta) = q_n(\Omega) = k' \) (if \( i' \neq 0 \)). If \( k' \geq k \) and \( i < i' < n \), the 3rd identity of Lemma \(4.3\) implies \( \text{Sgn}(c_{ink}(x)) = -\text{Sgn}(c_{i'i(k'-k+1)}(x)) \) and \( \text{Sgn}(c_{ink}(y)) = -\text{Sgn}(c_{i'i(k'-k+1)}(y)) \). Note from the case of above that \( \text{Sgn}(c_{i'i(k'-k+1)}(x)) = \text{Sgn}(c_{i'i(k'-k+1)}(y)) \). Thus \((12)\) holds in this case. Other cases can be obtained by similar arguments.

(iii) For \( i = n \) and \( j \in [n] \), we have the following four cases.

(C-1). \( q_j \geq k \) and \( p_j < j < i = n \). From the 3rd identity of Lemma \(4.3\) we have
\[
\text{Sgn}(c_{njk}(x)) = -\text{Sgn}(c_{p_jn(q_j-k+1)}(x)) \quad \text{and} \quad \text{Sgn}(c_{njk}(y)) = -\text{Sgn}(c_{p_jn(q_j-k+1)}(y)).
\]

(C-2). \( q_j > k \) and \( j < p_j < i = n \). From the 2nd identity of Lemma \(4.3\) we have
\[
\text{Sgn}(c_{njk}(x)) = -\text{Sgn}(c_{p_jn(q_j-k)}(x)) \quad \text{and} \quad \text{Sgn}(c_{njk}(y)) = -\text{Sgn}(c_{p_jn(q_j-k)}(y)).
\]

(C-3). \( q_j < k \) and \( p_j < j < i = n \). From the 4th identity of Lemma \(4.3\) we have
\[
\text{Sgn}(c_{njk}(x)) = \text{Sgn}(c_{np_j(k-q_j)}(x)) \quad \text{and} \quad \text{Sgn}(c_{njk}(y)) = \text{Sgn}(c_{np_j(k-q_j)}(y)).
\]

(C-4). \( q_j \leq k \) and \( j < p_j < i = n \). From the 1st identity of Lemma \(4.3\) we have
\[
\text{Sgn}(c_{njk}(x)) = \text{Sgn}(c_{np_j(k-q_j+1)}(x)) \quad \text{and} \quad \text{Sgn}(c_{njk}(y)) = \text{Sgn}(c_{np_j(k-q_j+1)}(y)).
\]

It is obvious from (ii) above that \((12)\) holds in cases (C-1) and (C-2). Next we will show \((12)\) holds in (C-3) and (C-4) simultaneously by induction on \( k \). For \( k = 1 \), note that \((12)\) holds if \( p_j < j \) by (C-1), and also holds if \( q_j > k = 1 \) and \( j < p_j \) by (C-2). In particular, we have \( \text{Sgn}(c_{n(n-1)}(x)) = \text{Sgn}(c_{n(n-1)}(y)) \) obviously. The remainder case is \( q_j = 1 \) and \( j < p_j \). From the 1st identity of Lemma \(4.3\) we have \( \text{Sgn}(c_{nj1}(x)) = \text{Sgn}(c_{np_j1}(x)) \). Now consider \( j \) from \( n-2 \) to 1 step by step as follows. For \( j = n-2 \), we have \( p_j = n-1 \) since \( j < p_j \), and \( \text{Sgn}(c_{n(n-2)}(x)) = \text{Sgn}(c_{n(n-2)}(y)) = \text{Sgn}(c_{n(n-1)}(x)) = \text{Sgn}(c_{n(n-1)}(y)) \). For \( j = n-3 \), we have \( p_j = n-1 \) or \( n-2 \) since \( j < p_j \), and \( \text{Sgn}(c_{nj1}(x)) = \text{Sgn}(c_{np_j1}(x)) = \text{Sgn}(c_{nj1}(y)) = \text{Sgn}(c_{np_j1}(y)) \). Continuing above steps, we finally obtain \( \text{Sgn}(c_{nj1}(x)) = \text{Sgn}(c_{nj1}(y)) \) for any \( j \in [n] \), which proves \((12)\) for \( k = 1 \). Now suppose \((12)\) holds in (C-3) and (C-4) for \( 1, \ldots, k-1 \). For general
Let \( k \), since \( k - q_j < k \), it is clear from the induction hypothesis that (12) holds in (C-3). In particular, by (C-1) and (C-3) we have \( \text{Sgn}(c_{n(n-1)k}(x)) = \text{Sgn}(c_{n(n-1)k}(y)) \) for all \( k \in [r] \). For the case (C-4), if \( q_j > 1 \), then \( k - q_j + 1 < k \) and (12) holds in this case by the induction hypothesis. So we only need to consider the case of \( q_j = 1 \) and \( j < p_j \) for (C-4), i.e.,

\[
\text{Sgn}(c_{njk}(x)) = \text{Sgn}(c_{np_jk}(x)) \quad \text{and} \quad \text{Sgn}(c_{njk}(y)) = \text{Sgn}(c_{np_jk}(y)).
\]

Similar as the base case \( k = 1 \), we may consider \( j \) from \( n - 2 \) to 1 step by step, which can prove (12) in this case. E.g. if \( j = n - 2 < p_j \), then \( p_j = n - 1 \) and we have \( \text{Sgn}(c_{n(n-2)k}(x)) = \text{Sgn}(c_{n(n-1)k}(x)) = \text{Sgn}(c_{n(n-1)k}(y)) = \text{Sgn}(c_{n(n-2)k}(y)) \). The proof of (12) in case (iii) completes.

\[
\square
\]

**Proposition 4.5.** The map \( \Psi_n' \) in (4) is injective.

**Proof.** We will use induction on the dimension \( n \geq 2 \). For the induction base \( n = 2 \), it is easy to see that under the map \( \Psi_n' \), all \( 2r+1 \) regions of \( S_n' \) are 1-1 corresponding to \( O \)-rooted labeled \( r \)-trees in \( T_n^r \). Suppose the result holds for \( n - 1 \), i.e., if \( T_n' = \Psi_n'^{r-1}(\Delta') = \Psi_n'^{r-1}(\Omega') = T_n^r \) for any two regions \( \Delta', \Omega' \in R(S_n'^r) \), then we have \( \Delta' = \Omega' \). Now suppose \( \Delta, \Omega \in R(S_n'^r) \) with \( T_\Delta = T_\Omega = T \in T_n^r \). By Proposition 3.3 and Definition 2.1 we have \( F_\Delta(v_j) = F_\Omega(v_j) = F_r(v_j) \) for all \( j \in [n] \). We claim that \( p_j(\Delta) = p_j(\Omega) \) and \( q_j(\Delta) = q_j(\Omega) \) (if \( p_j(\Delta) = p_j(\Omega) \neq 0 \)) for all \( j \in [n] \), whose proof will be given later. From (11), we can take a vertex \( v_{j'} \notin F_\Delta(v_j) = F_\Omega(v_j) \) for all \( j \in [n] \). It is clear that

(a) \( j' \neq p_j(\Delta), j' \neq p_j(\Omega) \) for all \( j \in [n] \).

(b) \( p_{j'}(\Delta) = p_{j'}(\Omega) = i', q_{j'}(\Delta) = q_{j'}(\Omega) = k' \) (if \( i' \neq 0 \)).

Let \( T' \in T_{n-1}^r \) be the \( O \)-rooted labeled \( r \)-tree obtained from \( T \) by removing the vertex \( v_{j'} \) and the edges between \( v_{j'} \) and vertices of \( F_r(v_{j'}) \). Given \( x \in \Delta \) and \( y \in \Omega \), let

\[
\text{Proj}_{j'}(\Delta) = \Delta', \quad \text{Proj}_{j'}(x) = x', \quad \text{Proj}_{j'}(\Omega) = \Omega', \quad \text{Proj}_{j'}(y) = y'.
\]

It follows from \( \text{Proj}_{j'}(x) = x' \) that the cubic matrix \( C_{x'} \in \mathbb{R}^{(n-1) \times (n-1) \times r} \) is obtained from \( C_x \in \mathbb{R}^{n \times n \times r} \) by removing the \( j' \)-th column and row slices from \( C_x \). The above property (a) implies that for each \( j \in [n] \setminus \{j'\} \), the minimal positivity entry \( c_{p_jq_j}(x) \) of \( \text{col}_j(C_x) \) never appears in the \( j' \)-th row slice of \( C_x \). So for \( j \in [n] \setminus \{j'\} \), the minimal positivity entry of \( j \)-th column slice \( \text{col}_j(C_x) \) appears in the same position as \( \text{col}_j(C_{x'}) \), i.e., \( p_j(\Delta) = p_j(\Delta') = p_j \) and \( q_j(\Delta) = q_j(\Delta') = q_j \) (if \( p_j \neq 0 \)). By the definition of \( O \)-rooted labeled \( r \)-tree in Definition 2.1 we have \( T' = T_{x'} = \Psi_{n-1}(\Delta') \), and \( T' = T_{y'} = \Psi_{n-1}(\Omega') \) similarly. From the induction hypothesis, we obtain \( \Delta' = \Omega' \), i.e., \( \text{Proj}_{j'}(\Delta) = \text{Proj}_{j'}(\Omega) = \Delta' \). Combining with the above properties (a) and (b), Lemma 4.3 implies \( \Delta = \Omega \).

To prove the claim under the assumption \( F_\Delta(v_j) = F_\Omega(v_j) = F_r(v_j) \) for all \( j \in [n] \), note

\[
p_j(\Delta) = 0 \iff F_\Delta(v_j) = 0 = F_\Omega(v_j) \iff p_j(\Omega) = 0.
\]

Now for \( p_j(\Delta) \neq 0 \), we have \( p_j(\Omega) \neq 0 \) and from the Definition 2.1

\[
v_{p_j(\Delta)} \in F_\Delta(v_j) = F_\Omega(v_j) \subseteq F_\Omega(v_{p_j(\Omega)}) \cup v_{p_j(\Omega)}.
\]
Suppose \( p_j(\Delta) \neq p_j(\Omega) \). Then we have \( v_{p_j(\Delta)} \in F_\Omega(v_{p_j(\Omega)}) = F_r(v_{p_j(\Omega)}) \), and symmetrically \( v_{p_j(\Omega)} \in F_\Delta(v_{p_j(\Delta)}) = F_r(v_{p_j(\Delta)}) \), which is impossible by Proposition 3.5. Thus \( p_j(\Delta) = p_j(\Omega) \) for all \( j \in [n] \). Define \( \pi : [n] \rightarrow \{0, 1, \ldots, n\} \) with \( \pi(j) = p_j(\Delta) = p_j(\Omega) \) as (8). If \( \pi_j \neq 0 \), we have shown that for some \( m \geq 2 \), we have \( \pi(j), \ldots, \pi^{m-1}(j) \neq 0 \) and \( \pi^m(j) = 0 \). Next we prove \( q_j(\Delta) = q_j(\Omega) \). Let’s start with the convenient notations \( (u; i, u) \) and \( [u; i, u] \) for an \( r \)-tuple \( u = (u_1, \ldots, u_r) \), an element \( u \), and \( i \in [r] \), where

\[
(u; i, u) = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r, u) \quad \text{and} \quad [u; i, u] = \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r, u\}.
\]

It is easy to see that if \( u_1, \ldots, u_r \) and \( u \) are mutually distinct, then the following result holds

\[
[u; i, u] = [u; j, u] \iff i = j \iff (u; i, u) = (u; j, u).
\]

It follows from (ii) of Definition 2.1 that

\[
f_\Delta(v_{\pi^{m-1}(j)}) = O = f_\Omega(v_{\pi^{m-1}(j)}),
\]

For any \( l = m - 2, \ldots, 1, 0 \), it is clear from (9) that

\[
F_\Delta(v_{\pi^l(j)}) = [f_\Delta(v_{\pi^{l+1}(j)}); q_{\pi^l(j)}(\Delta), v_{\pi^{l+1}(j)}],
\]

\[
F_\Omega(v_{\pi^l(j)}) = [f_\Omega(v_{\pi^{l+1}(j)}); q_{\pi^l(j)}(\Omega), v_{\pi^{l+1}(j)}],
\]

and \( F_\Delta(v_{\pi^l(j)}) = F_\Omega(v_{\pi^l(j)}) \). Applying (13) and running \( l \) from \( m - 2 \) to \( 0 \), we finally obtain

\[
q_j(\Delta) = q_j(\Omega) \quad \text{and} \quad f_\Delta(v_j) = f_\Omega(v_j).
\]

**Proof of Theorem 2.2**. We have obtained that the map \( \Psi^r_n : \mathcal{R}(\mathcal{S}^r_n) \rightarrow \mathcal{T}^r_n \) of (4) is well defined by Proposition 4.1 and injective by Proposition 4.5 which is enough to conclude that \( \Psi^r_n \) is a bijection from the fact that both \( \mathcal{R}(\mathcal{S}^r_n) \) and \( \mathcal{T}^r_n \) have the same cardinality by Theorem 1.1 and Theorem 1.2.

**Corollary 4.6**. Given \( j' \in [n] \), for any \( \Delta' \in \mathcal{R}(\mathcal{S}^r_{n-1}) \), \( i' \in \{0, 1, \ldots, j' - 1, j' + 1, \ldots, n\} \) and \( k' \in [r] \) (if \( i' \neq 0 \)), there is a region \( \Delta \in \mathcal{R}(\mathcal{S}^r_n) \) such that \( \text{Proj}_{j'}(\Delta) = \Delta' \), \( p_{j'}(\Delta) = i' \), \( q_{j'}(\Delta) = k' \) (if \( i' \neq 0 \)), and \( j' \neq p_j(\Delta) \) for all \( j \in [n] \).

Note that Corollary 4.6 can be easily obtained from the surjectivity of \( \Psi^r_n \). Recall Lemma 4.4 that the uniqueness of the region \( \Delta \in \mathcal{R}(\mathcal{S}^r_n) \) (if exists) is crucial to guarantee the injectivity of \( \Psi^r_n \) in Proposition 4.5. As a parallel situation, the existence of such region \( \Delta \) in Corollary 4.6 will produce a proof on the surjectivity of \( \Psi^r_n \). However, similar as the Pak-Stanley labeling at the very beginning appeared in [48], we currently have no direct proof on the surjectivity of \( \Psi^r_n \) without using Theorem 1.1 and Theorem 1.2. So it would be of great interest to find a direct proof on Corollary 4.6.

Recall arguments of Proposition 4.5 and Lemma 4.1, which provide an algorithm to construct the region \( \Delta \) of \( r \)-Shi arrangement from an \( O \)-rooted labeled \( r \)-tree \( T \), i.e., the inverse map

\[
(\Psi^r_n)^{-1} : \mathcal{T}^r_n \rightarrow \mathcal{R}(\mathcal{S}^r_n), \quad (\Psi^r_n)^{-1}(T) = \Delta.
\]

As a brief look, below is a small example to illustrate the bijection \( \Psi^r_n \) by constructing the \( O \)-rooted labeled tree from a given region, and its inverse \( (\Psi^r_n)^{-1} \) by constructing the region from a given \( O \)-rooted labeled tree.
Example 4.7. For \( n = 3 \) and \( r = 1 \), Figure 2 describes the complete correspondence between \( \mathcal{R}(S_3) \) and \( T_3 \) under the map \( \Psi_3 \). E.g., let \( \Delta \in \mathcal{R}(S_3) \) be the blue region in Figure 2 defined by
\[
\Delta = \{ 0 < x_1 - x_2 < 1; 0 < x_1 - x_3 < 1; x_2 - x_3 < 0 \},
\]
and \( \mathbf{x} = (0.2, -0.2, 0) \in \Delta \). By Theorem 2.2, we have
\[
A_x = \begin{pmatrix}
0 & -0.6 & -0.8 \\
-0.4 & 0 & -1.2 \\
-0.2 & 0.2 & 0
\end{pmatrix},
\]
and \( p_1(\Delta) = 0, p_2(\Delta) = 3, p_3(\Delta) = 0 \). Namely, in the \( O \)-rooted labeled tree \( T_\mathbf{x} = \Psi_3(\Delta) \), the fathers of \( v_1, v_2 \), and \( v_3 \) are \( o_1, v_3 \), and \( o_1 \) respectively, which exactly determines \( T_\mathbf{x} \) to be the red tree in Figure 2. Conversely, let \( T \in T_3 \) be the green tree in Figure 2 having \( o_1, v_1 \), and \( v_3 \) as the fathers of \( v_1, v_2 \), and \( v_3 \) respectively. If \( \Omega = \Psi_3^{-1}(T) \), it follows from the definition of \( T \) in Theorem 2.2 that \( p_1(\Omega) = 0, p_2(\Omega) = 1, \) and \( p_3(\Omega) = 1 \). Take the leaf \( v_3 \) of \( T \). Let \( T' \) be the tree obtained from \( T \) by removing \( v_3 \) and the edge \( v_3 \sim v_1 \), and \( \Omega' = \text{Proj}_3(\Omega) \). According to the proofs of Lemma 4.4 and Proposition 4.5, we have \( \Psi_3^{-1}(T') = \Omega' \) and
\[
\text{Sgn}(\Omega') = \begin{pmatrix}
0 & + \\
- & 0
\end{pmatrix} \begin{pmatrix}
\text{Sgn}_{11}(\Omega) & \text{Sgn}_{12}(\Omega) \\
\text{Sgn}_{21}(\Omega) & \text{Sgn}_{22}(\Omega)
\end{pmatrix}.
\]
Since \( p_3(\Omega) = 1 \) and \( p_1(\Omega) = 0 \), we have \( \text{Sgn}_{13}(\Omega) = + \) and \( \text{Sgn}_{31}(\Omega) = - \). By Lemma 4.3, we have \( \text{Sgn}_{23}(\Omega) = \text{Sgn}_{21}(\Omega) = - \) and \( \text{Sgn}_{32}(\Omega) = -\text{Sgn}_{13}(\Omega) = - \). The sign matrix \( \text{Sgn}(\Omega) = (\text{Sgn}_{ij}(\Omega))_{3 \times 3} \) exactly determines \( \Omega \) to be the yellow region in Figure 2.

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