POISSON DEGENERATE CENTRAL MOMENTS RELATED TO DEGENERATE DOWLING AND DEGENERATE \( r \)-DOWLING POLYNOMIALS

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ABSTRACT. Degenerate Dowling and degenerate \( r \)-Dowling polynomials were introduced earlier as degenerate versions and further generalizations of Dowling and \( r \)-Dowling polynomials. The aim of this paper is to show their connections with Poisson degenerate central moments for a Poisson random variable with a certain parameter and with Charlier polynomials.

1. INTRODUCTION AND PRELIMINARIES

In recent years, studying various degenerate versions of many special polynomials and numbers, which began with the paper [4] by Carlitz, received regained interests of some mathematicians and many interesting results were discovered (see [12-18] and the references therein). They have been explored by employing several different tools such as combinatorial methods, generating functions, \( p \)-adic analysis, umbral calculus techniques, differential equations, probability theory and analytic number theory.

Degenerate Dowling and degenerate \( r \)-Dowling polynomials were introduced earlier as degenerate versions and further generalizations of Dowling and \( r \)-Dowling polynomials. The aim of this paper is to show their connections with Poisson degenerate central moments for a Poisson random variable with a certain parameter and with Charlier polynomials.

The outline of this paper is as follows. In Section 1, we recall the Stirling numbers of the first and second kinds, Bell polynomials, the degenerate exponential functions, the degenerate Stirling numbers of the first and second kinds, the degenerate Bell polynomials, the Poisson random variable with parameter \( \alpha \), and the Charlier polynomials. In addition, we remind the reader of the Whitney numbers of the first and second kinds, Dowling polynomials, the degenerate Whitney numbers of the second kind, the degenerate Dowling polynomials, the degenerate \( r \)-Whitney numbers of the second kind and the degenerate \( r \)-Dowling polynomials. Section 2 is the main result of this paper. In the following, assume that \( X \) is the Poisson random variable with mean \( \frac{\alpha}{m} \). In Theorem 1, we show that the Poisson degenerate central moment \( E[(mX + 1)_{n, \lambda}] \) is equal to \( D_{m, \lambda}(n, \alpha) \), where \( D_{m, \lambda}(n, x) \) is the degenerate Dowling polynomial. In Theorem 2, we express the same Poisson degenerate central moment in terms of the degenerate Bell polynomials. In Theorem 4, we deduce that \( E[(mX + r)_{n, \lambda}] \) is equal to \( D_{m, \lambda}^{(r)}(n, \alpha) \), where \( D_{m, \lambda}^{(r)}(n, x) \) is the degenerate \( r \)-Dowling polynomial. We express the same in terms of the degenerate Bell polynomials in Corollary 5, and of the degenerate \( r \)-Whitney numbers of the second kind and the Bell polynomials in Theorem 6. Furthermore, it is represented by the Charlier polynomials and the degenerate Stirling numbers of the second kind in Theorems 10 and 11. In the rest of this section, we recall the facts that are needed throughout this paper.

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It is well known that the stirling numbers of the first kind are defined as

\[(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k, \quad (n \geq 0), \quad (\text{see } [1, 2, 3, 6, 7, 21]), \]

where \((x)_0 = 1, \ (x)_n = x(x-1) \cdots (x-n+1), \ (n \geq 1).\)

The Stirling numbers of the second kind are given by

\[x^n = \sum_{k=0}^{n} S_2(n, k)(n)_k, \quad (n \geq 0), \quad (\text{see } [18, 19]).\]

From (1) and (2), we note that

\[
\frac{1}{k!} (\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!},
\]

and

\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see } [5, 6, 17, 19]).
\]

The Bell polynomials are defined by

\[e^x(e^t - 1) = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!}, \quad (\text{see } [5, 6, 19]).\]

Thus, by (5), we get

\[
\phi_n(x) = \sum_{k=0}^{n} S_2(n, k) x^k, \quad (n \geq 0), \quad (\text{see } [4, 16, 19]).
\]

In [12], the degenerate exponentials are defined by

\[e^x e^{(\lambda+1) t} = \sum_{n=0}^{\infty} (x)_{n, \lambda} \frac{t^n}{n!}, \quad e^x(t) = e^x_{\lambda}(t), \quad (0 \neq \lambda \in \mathbb{R}),\]

where

\[(x)_{0, \lambda} = 1, \ (x)_{n, \lambda} = x(x-\lambda)(x-2\lambda) \cdots (x-(n-1)\lambda), \ (n \geq 1), \quad (\text{see } [17]).\]

Let \(\log_{\lambda} t\) be the compositional inverse of \(e^x(t)\) such that \(\log_{\lambda}(e^x(t)) = e^x_{\lambda}(\log_{\lambda} t) = t.\)

Then we have

\[
\log_{\lambda}(1+t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1)_{n, \lambda}}{n!} t^n, \quad (\text{see } [12]).
\]

In view of (1) and (2), the degenerate Stirling numbers of the first kind, \(S_{1, \lambda}(n, k),\) and the degenerate Stirling numbers of the second kind, \(S_{2, \lambda}(n, k),\) are defined by

\[(x)_{n, \lambda} = \sum_{k=0}^{n} S_{1, \lambda}(n, k)(x)_{k, \lambda}, \quad (n \geq 0), \quad (\text{see } [12]).\]

and

\[(x)_{n, \lambda} = \sum_{k=0}^{n} S_{2, \lambda}(n, k)(x)_{k, \lambda}, \quad (n \geq 0) \quad (\text{see } [12]).\]
Note that \( \lim_{\lambda \to 0} \log(1 + t) = \log(1 + t) \), \( \lim_{\lambda \to 0} e_\lambda^x(t) = e^{\alpha t} \), \( \lim_{\lambda \to 0} S_{1,\lambda}(n,k) = S_1(n,k) \), and \( \lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k) \), where \((n,k) \geq 0\).

From (10) and (11), we note that
\[
\frac{1}{k!}(e_\lambda^x(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0),
\]
and
\[
\frac{1}{k!}(\log(1 + t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0),
\]
The degenerate Bell polynomials are defined by
\[
e^{x(e_\lambda^x(t) - 1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see } [12, 15, 16]).
\]
Thus, by (12) and (14), we get
\[
\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k, \quad (n \geq 0), \quad (\text{see } [15, 16, 17]).
\]
A random variable \( X \) is a real valued function defined on a sample space. If \( X \) takes any value in a countable set, then \( X \) is called a discrete random variable. For a discrete random variable \( X \), the probability mass function \( p(a) \) of \( X \) is defined by
\[
p(a) = P(X = a), \quad (\text{see } [10, 11, 20]).
\]
A random variable \( X \) taking on one of the values 0, 1, 2, \ldots is said to be the Poisson random variable with parameter \( \alpha(> 0) \), which is denoted by \( X \sim \text{Poi}(\alpha) \), if the probability mass function of \( X \) is given by
\[
p(i) = P(X = i) = e^{-\alpha} \frac{\alpha^i}{i!}, \quad i = 0, 1, 2, \ldots \quad (\text{see } [10, 11, 20]).
\]
For \( n \geq 1 \), the quantity \( E[X^n] \) of the Poisson random variable \( X \) with parameter \( \alpha(> 0) \), which is called the \( n \)th moment of \( X \), is given by
\[
E[X^n] = \sum_{i=0}^{\infty} i^n p(i) = e^{-\alpha} \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} \frac{i^n}{n!} = \phi_n(\alpha), \quad (\text{see } [10, 11, 20]).
\]
As is well known, the Charlier polynomials \( C_n(x; \alpha) \) are defined by
\[
e^{-\alpha t}(1 + t)^x = \sum_{n=0}^{\infty} C_n(x; \alpha) \frac{t^n}{n!}. \quad (\text{see } [19]),
\]
where \( x, t, \alpha \in \mathbb{R} \).
Thus, by (19), we get
\[
C_n(x; \alpha) = \sum_{i=0}^{n} \left( \sum_{k=0}^{i} \binom{n}{k} (-1)^{i-k} \alpha^{n-k} S_1(k, l) \right) x^l.
\]
A finite lattice \( L \) is geometric if it is a finite semimodular lattice which is also atomic. Dowling constructed an important finite geometric lattice \( Q_n(G) \) out of a finite set of \( n \) elements and a finite
group $G$ of order $m$, called Dowling lattice of rank $n$ over a finite group of order $m$. If $L$ is the Dowling lattice $Q_n(G)$ of rank $n$ over a finite group $G$ of order $m$, then the Whitney numbers of the first kind $V_{Q_n(G)}(n,k)$ and the Whitney numbers of the second kind $W_{Q_n(G)}(n,k)$ are respectively denoted by $V_m(n,k)$ and $W_m(n,k)$. The Whitney numbers $V_m(n,k)$ and $W_m(n,k)$ satisfy the following Stirling number-like relations:

\[(mx+1)^n = \sum_{k=0}^{n} W_m(n,k)m^k(x)_k, \quad (21)\]

\[m^n(x)_n = \sum_{k=0}^{n} V_m(n,k)(mx+1)^k, \quad (n \geq 0), \quad (\text{see } [8, 9, 14]). \quad (22)\]

For $n \geq 0$, Dowling polynomials are given by

\[D_m(n,x) = \sum_{k=0}^{n} W_m(n,k)x^k, \quad (\text{see } [8, 9, 13]). \quad (23)\]

Recently, Kim-Kim considered the degenerate Whitney numbers of the second kind defined by

\[(mx+r)^n = \sum_{k=0}^{n} W_{m,\lambda}(n,k)m^k(x)_k, \quad (n, r \geq 0), \quad (\text{see } [13, 14]). \quad (24)\]

Thus, by (24), we get

\[e_{\lambda}(t)^n \left( \frac{e_{m\lambda}(t)}{m} - 1 \right)^k = \sum_{n=k}^{\infty} W_{m,\lambda}(n,k) \frac{t^n}{n!}. \quad (25)\]

In [14], the degenerate Dowling polynomials are defined by

\[e_{\lambda}(t)e^{\frac{x}{e_{\lambda}(t)}-1} = \sum_{n=0}^{\infty} D_{m,\lambda}(n,x) \frac{t^n}{n!}. \quad (26)\]

By (25) and (26), we get

\[D_{m,\lambda}(n,x) = \sum_{k=0}^{n} W_{m,\lambda}(n,k)x^k, \quad (\text{see } [13, 14]). \quad (27)\]

A further generalization of degenerate Whitney numbers of the second kind, Kim-Kim introduced the degenerate $r$-Whitney numbers of the second kind given by

\[(mx+r)^n = \sum_{k=0}^{n} W_{m,\lambda}^{(r)}(n,k)m^k(x)_k, \quad (n, r \geq 0), \quad (\text{see } [13, 14]). \quad (28)\]

In view of (27), they defined the degenerate $r$-Dowling polynomials given by

\[D_{m,\lambda}^{(r)}(n,x) = \sum_{k=0}^{n} W_{m,\lambda}^{(r)}(n,k)x^k, \quad (n \geq 0), \quad (\text{see } [13, 14]). \quad (29)\]

From (29), we can show that the generating function of the degenerate $r$-Dowling polynomials is given by
\( e_{\lambda}^x(t)e_{\lambda}^{y(\alpha t)-1} = \sum_{n=0}^{\infty} D_{m,\lambda} (m,\alpha) \frac{t^n}{n!}, \) (see [13][14]).

2. Poisson degenerate central moments related to degenerate Dowling and degenerate \( r \)-Dowling polynomials

Let \( X \) be the Poisson random variable with mean \( \frac{\alpha}{m} \). Then we consider the Poisson degenerate central moments given by \( E[(mX + 1)_{n,\lambda}], \quad (n \geq 0) \). We observe that

\[
E[e_{\lambda}^{mX+1}(t)] = \sum_{k=0}^{\infty} e_{\lambda}^{mk+1}(t) p(k)
\]

\[
= e_{\lambda}(t)e^{-\frac{\alpha}{m}} \sum_{k=0}^{\infty} e_{\lambda}^{mk}(t) \frac{1}{k!} \left( \frac{\alpha}{m} \right)^k
\]

\[
= e_{\lambda}(t)e^{-\frac{\alpha}{m}} e_{\lambda}^{e_{\lambda}(t)-1} = e_{\lambda}(t)e_{\lambda}^{e_{\lambda}(t)-1}
\]

\[
= \sum_{n=0}^{\infty} D_{m,\lambda} (n,\alpha) \frac{t^n}{n!}.
\]

On the other hand, by (7), we get

\[
E[e_{\lambda}^{mX+1}(t)] = \sum_{n=0}^{\infty} E[(mX + 1)_{n,\lambda}] \frac{t^n}{n!}.
\]

Therefore, by (31) and (32), we obtain the following theorem.

**Theorem 1.** Let \( X \sim \text{Poi} \left( \frac{\alpha}{m} \right) \). Then we have

\[
E[(mX + 1)_{n,\lambda}] = D_{m,\lambda} (n,\alpha), \quad (n \geq 0).
\]

Note that

\[
E[(mX + 1)^n] = \lim_{\lambda \to 0} E[(mX + 1)_{n,\lambda}] = \lim_{\lambda \to 0} D_{m,\lambda} (n,\alpha) = D_m (n,\alpha), \quad (n \geq 0).
\]

Let \( X \sim \text{Poi} \left( \frac{\alpha}{m} \right) \). It is not difficult to show that

\[
(x + y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda} (y)_{n-k,\lambda}, \quad (n \geq 0).
\]

By (33), we get

\[
E[(mX + 1)_{n,\lambda}] = \sum_{k=0}^{n} \binom{n}{k} (1)_{n-k,\lambda} E[(mX)_{k,\lambda}].
\]

From (17), we have

\[
\sum_{n=0}^{\infty} E[(mX)_{n,\lambda}] \frac{t^n}{n!} = E[e_{\lambda}^{mX}(t)] = e^{-\frac{\alpha}{m}} \sum_{n=0}^{\infty} e_{\lambda}^{mk}(t) \frac{(\alpha/n)^n}{n!}
\]

\[
= e_{\lambda}^{mX} (e_{\lambda}(t)-1) = e_{\lambda}^{mX} (e_{\lambda}^{(m)(t)-1}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\alpha}{m} \right)^m t^n.
\]

Comparing the coefficients on both sides of (35), we have

\[
E[(mX)_{n,\lambda}] = \phi_m \left( \frac{\alpha}{m} \right) m^n, \quad (n \geq 0).
\]
By (15), we get

\begin{equation}
\phi_{n, \frac{\alpha}{m}} \left( \frac{\alpha}{m} \right) = \sum_{k=0}^{n} S_{n, \frac{\alpha}{m}} \left( n, k \right) \left( \frac{\alpha}{m} \right)^{k}.
\end{equation}

Therefore, by (34), (36) and (37), we get

\begin{equation}
E[(mX + 1)_{n, \lambda}] = \sum_{k=0}^{n} \binom{n}{k} (1)_{n-k, \lambda} m^{k} \phi_{k, \frac{\alpha}{m}} \left( \frac{\alpha}{m} \right)
\end{equation}

\begin{equation}
= \sum_{k=0}^{n} \binom{n}{k} (1)_{n-k, \lambda} m^{k} \sum_{j=0}^{k} S_{n, \frac{\alpha}{m}}(k, j) \left( \frac{\alpha}{m} \right)^{j}
\end{equation}

\begin{equation}
= \sum_{j=0}^{n} \alpha^{j} \sum_{k=j}^{n} \binom{n}{k} (1)_{n-k, \lambda} m^{k-j} S_{n, \frac{\alpha}{m}}(k, j).
\end{equation}

Therefore, by (38), we obtain the following theorem.

**Theorem 2.** For \( n \geq 0 \), let \( X \sim Pois(\frac{\alpha}{m}) \). Then we have

\begin{equation}
E[(mX + 1)_{n, \lambda}] = \sum_{k=0}^{n} \binom{n}{k} (1)_{n-k, \lambda} m^{k} \phi_{k, \frac{\alpha}{m}} \left( \frac{\alpha}{m} \right)
\end{equation}

\begin{equation}
= \sum_{j=0}^{n} \alpha^{j} \sum_{k=j}^{n} \binom{n}{k} (1)_{n-k, \lambda} m^{k-j} S_{n, \frac{\alpha}{m}}(k, j).
\end{equation}

When \( m = 1 \), we have

\begin{equation}
\sum_{n=0}^{\infty} D_{1, \lambda}(n, \alpha) \frac{\alpha^{n}}{n!} = \sum_{n=0}^{\infty} E[(X + 1)_{n, \lambda}] \frac{\alpha^{n}}{n!} = E[e^{X+1}(t)]
\end{equation}

\begin{equation}
= e^{\lambda(t)} \sum_{n=0}^{\infty} \frac{e^{\alpha(t)}}{n!} \alpha^{n} = e^{\lambda(t)} e^{\alpha(e_{\lambda}(t))-1}
\end{equation}

\begin{equation}
= e^{\alpha(e_{\lambda}(t))-1} + \frac{d}{d\alpha} e^{\alpha(e_{\lambda}(t))-1} = \sum_{n=0}^{\infty} \left( \phi_{n, \lambda}(\alpha) + \frac{d}{d\alpha} \phi_{n, \lambda}(\alpha) \right) \frac{\alpha^{n}}{n!}.
\end{equation}

Thus, we have

\begin{equation}
\phi_{n, \lambda}(\alpha) + \frac{d}{d\alpha} \phi_{n, \lambda}(\alpha) = D_{1, \lambda}(n, \alpha) = E[(X + 1)_{n, \lambda}], \quad (n \geq 0).
\end{equation}

On the other hand, by (15), we have

\begin{equation}
\frac{d}{d\alpha} \phi_{n, \lambda}(\alpha) = \frac{d}{d\alpha} \sum_{k=0}^{n} S_{2, \lambda}(n, k) \alpha^{k} = \sum_{k=1}^{n} k S_{2, \lambda}(n, k) \alpha^{k-1}
\end{equation}

\begin{equation}
= \sum_{k=0}^{n-1} (k + 1) S_{2, \lambda}(n, k + 1) \alpha^{k}.
\end{equation}

Therefore, by (40), we obtain the following corollary.

**Corollary 3.** For \( n \geq 0 \), let \( X \sim Pois(\alpha) \). Then we have

\begin{equation}
E[(X + 1)_{n, \lambda}] = D_{1, \lambda}(n, \alpha) = \sum_{k=0}^{n} \left( (k + 1) S_{2, \lambda}(n, k + 1) + S_{2, \lambda}(n, k) \right) \alpha^{k}.
\end{equation}
For \( r \geq 0 \), let \( X \) be the Poisson random variable with mean \( \frac{a}{m} \). Then we have

\[
E[e_{\lambda}^{mX+r}(t)] = \sum_{k=0}^{\infty} e_{\lambda}^{mk+r}(t)p(k)
\]

\[
= e_{\lambda}^{r}(t)e^{-\frac{a}{m}} \sum_{k=0}^{\infty} e_{\lambda}^{mk}(t) \left( \frac{\alpha}{m} \right)^{k}
\]

\[
= e_{\lambda}^{r}(t)e^{-\frac{a}{m}} e_{\lambda}^{m}(t) = e_{\lambda}^{r}(t)e_{\lambda}^{m}(t-1) = \sum_{n=0}^{\infty} D_{m,\lambda}^{(r)}(n, \alpha) \frac{t^{n}}{n!}.
\]

The left hand side of (41) is given by

\[
E[e_{\lambda}^{mX+r}(t)] = \sum_{n=0}^{\infty} E[(mX + r)_{n,\lambda}] \frac{t^{n}}{n!}.
\]

Therefore, by (41) and (42), we obtain the following theorem.

**Theorem 4.** For \( m, r \geq 0 \), let \( X \sim \text{Poi}\left(\frac{a}{m}\right) \). Then we have

\[
E[(mX + r)_{n,\lambda}] = D_{m,\lambda}^{(r)}(n, \alpha), \quad (n \geq 0).
\]

Note that

\[
D_{m}^{(r)}(n, \alpha) = \lim_{\lambda \to 0} E[(mX + r)_{n,\lambda}] = E[(mX + r)^{n}].
\]

By (33), we get

\[
E[(mX + r)_{n,\lambda}] = \sum_{k=0}^{n} \binom{n}{k} (r)_{n-k,\lambda} E[(mX)_{k,\lambda}]
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (r)_{n-k,\lambda} m^{k} \phi_{k,\lambda} \left( \frac{\alpha}{m} \right).
\]

Therefore, by (43), we obtain the following corollary.

**Corollary 5.** For \( n, r \geq 0 \), let \( X \sim \text{Poi}\left(\frac{a}{m}\right) \). Then we have

\[
D_{m,\lambda}^{(r)}(n, \alpha) = E[(mX + r)_{n,\lambda}] = \sum_{k=0}^{n} \binom{n}{k} (r)_{n-k,\lambda} m^{k} \phi_{k,\lambda} \left( \frac{\alpha}{m} \right).
\]

When \( r = 1 \), we have

\[
E[(mX + 1)_{n,\lambda}] = D_{m,\lambda}^{(1)}(n, \alpha) = D_{m,\lambda}(n, \alpha).
\]

From (28), we have

\[
D_{m,\lambda}^{(r)}(n, \alpha) = E[(mX + r)_{n,\lambda}] = \sum_{k=0}^{n} W_{m,\lambda}^{(r)}(n,k)m^{k} E[(X)_{k}].
\]

By (3), we get

\[
\sum_{n=0}^{\infty} E[(X)_{n}] \frac{t^{n}}{n!} = E[(1 + r)^{X}] = \sum_{k=0}^{\infty} E[X^{k}] \frac{1}{k!} (\log(1 + t))^{k}
\]

\[
= \sum_{k=0}^{\infty} E[X^{k}] \sum_{n=k}^{\infty} S_{1}(n,k) \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S_{1}(n,k) E[X^{k}] \right) \frac{t^{n}}{n!}.
\]
Since $X \sim \text{Poi}(\frac{\alpha}{m})$, from (17), we have
\[
\sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!} = E[e^{tX}] = e^{\mathbb{E}[e^{tX}]} = e^{\frac{\alpha}{m} (e^t - 1)} = \sum_{k=0}^{\infty} e^{x} \frac{(\frac{\alpha}{m})^k}{k!}.
\]
(46)

Thus, by (46), we get
\[
E[X^k] = \phi_k \left( \frac{\alpha}{m} \right), \quad (k \geq 0).
\]
(47)

From (45) and (47), we have
\[
E[(X)_n] = \sum_{k=0}^{n} S_1(n,k) E[X^k] = \sum_{k=0}^{n} S_1(n,k) \phi_k \left( \frac{\alpha}{m} \right).
\]
(48)

By (44) and (48), we get
\[
D_{m,\lambda}^{(r)}(n, \alpha) = E[(mX + r)_{n,\lambda}] = \sum_{k=0}^{n} W_{m,\lambda}^{(r)}(n,k)m^k E[(X)_k]
\]
(49)

Therefore, by (49), we obtain the following theorem.

**Theorem 6.** For $n \geq 0$, let $X \sim \text{Poi}(\frac{\alpha}{m})$. Then we have
\[
D_{m,\lambda}^{(r)}(n, \alpha) = E[(mX + r)_{n,\lambda}]
\]
\[
= \sum_{j=0}^{n} \left( \sum_{k=j}^{n} W_{m,\lambda}^{(r)}(n,k)m^k S_1(k,j) \right) \phi_j \left( \frac{\alpha}{m} \right).
\]

By Theorem 2, we get
\[
D_{m,\lambda}^{(r)}(n, \alpha) = E[(mX + r)_{n,\lambda}] = \sum_{k=0}^{n} \binom{n}{k} (r - 1)_{n-k,\lambda} E[(mX + 1)_{k,\lambda}]
\]
(50)

On the other hand, by (29), we get
\[
D_{m,\lambda}^{(r)}(n, \alpha) = \sum_{j=0}^{n} \alpha^j W_{m,\lambda}^{(r)}(n, j).
\]
(51)

Therefore, by (50) and (51), we obtain the following theorem.
Theorem 7. For $n, j \geq 0$, we have

$$W_{m,\lambda}^{(r)}(n,j) = \sum_{k=j}^{n} \sum_{l=j}^{k} \binom{n}{k} \binom{k}{l} (r-1)^{n-k,\lambda}(1)_{k-l,\lambda} m^{l-j} S_{2,\lambda}(l,j).$$

From (51) and (43), we note that

$$\sum_{j=0}^{n} \alpha^{j} W_{m,\lambda}^{(r)}(n,j) = D_{m,\lambda}^{(r)}(n,\alpha) = E[(mX + r)_{n,\lambda}]$$

$$= \sum_{j=0}^{n} \binom{n}{j} (r)^{n-j,\lambda} m^{j} \phi_{\lambda}^{j}(\frac{\alpha}{m})$$

$$= \sum_{j=0}^{n} \binom{n}{j} (r)^{n-j,\lambda} m^{j} \sum_{j=0}^{l} \binom{\alpha}{m}^{j} S_{2,\lambda}(l,j)$$

$$= \sum_{j=0}^{n} \alpha^{j} \left( \sum_{j=0}^{n} \binom{n}{j} (r)^{n-j,\lambda} m^{l-j} S_{2,\lambda}(l,j) \right).$$

Therefore, by comparing the coefficients on both sides of (52), we obtain the following theorem.

Theorem 8. For $n, j \geq 0$, we have

$$W_{m,\lambda}^{(r)}(n,j) = \sum_{l=j}^{n} \binom{n}{l} (r)^{n-l,\lambda} m^{l-j} S_{2,\lambda}(l,j).$$

We recall that Charlier polynomials $C_{n}(x;\alpha)$ are given by

$$e^{-\alpha t}(1 + t)^{x} = \sum_{n=0}^{\infty} C_{n}(x;\alpha) \frac{t^{n}}{n!}, \quad (n \geq 0 \text{ and } x, \alpha, t \in \mathbb{R}).$$

Let us take $x = 0$. Then we have

$$e^{-\alpha t} = \sum_{n=0}^{\infty} C_{n}(0;\alpha) \frac{t^{n}}{n!}.$$

Replacing $t$ by $1 - e_{\alpha}^{m}(t)$, we get

$$e^{\alpha(e_{\alpha}^{m}(t)-1)} = \sum_{k=0}^{\infty} C_{k}(0;\alpha)(-1)^{k} \frac{1}{k!} (e_{\alpha}^{m}(t) - 1)^{k}.$$

Thus, by (44) and (55), we get

$$\sum_{n=0}^{\infty} \phi_{n,\lambda}(\alpha) m^{n} \frac{t^{n}}{n!} = e^{\alpha(e_{\alpha}^{m}(mt)-1)} = e^{\alpha(e_{\alpha}^{m}(t)-1)}$$

$$= \sum_{k=0}^{\infty} C_{k}(0;\alpha)(-1)^{k} \frac{1}{k!} (e_{\alpha}^{m}(mt) - 1)^{k}$$

$$= \sum_{k=0}^{\infty} C_{k}(0;\alpha)(-1)^{k} \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) m^{n} \frac{t^{n}}{n!}$$

$$= \sum_{n=0}^{\infty} \left( m^{n} \sum_{k=0}^{n} C_{k}(0;\alpha)(-1)^{k} S_{2,\lambda}(n,k) \right) \frac{t^{n}}{n!}.$$

Therefore, by (56), we obtain the following theorem.
Theorem 9. For $n \geq 0$, let $X \sim \text{Poi}(\alpha)$, Then we have

$$
\phi_{n, \frac{1}{m}}(\alpha) = E[(X)_n] = \sum_{k=0}^{n} C_k(0; \alpha)(-1)^k S_{2, \frac{1}{m}}(n, k).
$$

Let us take $x = 1$ in (53). Then we have

$$
e^{-at} (1 + t) = \sum_{n=0}^{\infty} C_n(1; \alpha) \frac{t^n}{n!}.
$$

Replacing $t$ by $e^m(t) - 1$ and $\alpha$ by $-\frac{\alpha}{m}$, we get

$$
\sum_{n=0}^{\infty} D_{m, \lambda}^{(n)}(n, \alpha) \frac{t^n}{n!} = e^m(t) e^{\bar{m}(t) - 1} = \sum_{k=0}^{\infty} C_k \left(1; -\frac{\alpha}{m}\right) \frac{(e^m(t) - 1)^k}{k!}
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} C_k \left(1; -\frac{\alpha}{m}\right) e^{\bar{m}(n, k)} \right) \frac{t^n}{n!}.
$$

Comparing the coefficients on both sides of (58), we have

$$
D_{m, \lambda}^{(n)}(n, \alpha) = \sum_{k=0}^{n} C_k \left(1; -\frac{\alpha}{m}\right) m^n S_{2, \frac{1}{m}}(n, k), \quad (n \geq 0).
$$

Note that

$$
E[(mX + r)_n] = E[(mX + m + r - m)_n] = \sum_{k=0}^{n} \binom{n}{k} (r - m)_n \lambda E[(mX + m)_k]
= \sum_{k=0}^{n} \binom{n}{k} (r - m)_n \lambda \sum_{j=0}^{k} C_j \left(1; -\frac{\alpha}{m}\right) S_{2, \frac{1}{m}}(k, j) m^k
= \sum_{j=0}^{n} C_j \left(1; -\frac{\alpha}{m}\right) \sum_{k=j}^{n} \binom{n}{k} (r - m)_n \lambda S_{2, \frac{1}{m}}(k, j) m^k.
$$

Therefore, by (59) and (60), we obtain the following theorem.

Theorem 10. For $n \geq 0$, let $X \sim \text{Poi}(\frac{\alpha}{m})$. Then we have

$$
E[(mX + m)_n] = D_{m, \lambda}^{(n)}(n, \alpha) = \sum_{k=0}^{n} C_k \left(1; -\frac{\alpha}{m}\right) m^n S_{2, \frac{1}{m}}(n, k).
$$

Furthermore, we have

$$
D_{m, \lambda}^{(r)}(n, \alpha) = E[(mX + r)_n] = \sum_{j=0}^{n} C_j \left(1; -\frac{\alpha}{m}\right) \sum_{k=j}^{n} \binom{n}{k} (r - m)_n \lambda S_{2, \frac{1}{m}}(k, j) m^k.
$$

Replacing $t$ by $(e^m(t) - 1)$, $\alpha$ by $-\frac{\alpha}{m}$, and $x$ by $\frac{1}{m}$ in (53), we have
\[ e^{\frac{r}{m} \left( e^{\frac{\alpha}{m} (t)} - 1 \right)} = \sum_{k=0}^{\infty} C_k \left( \frac{r}{m} - \frac{\alpha}{m} \right) \frac{1}{k!} \left( e^{\frac{\alpha}{m} (mt)} - 1 \right)^k \]

\[ = \sum_{k=0}^{\infty} C_k \left( \frac{r}{m} - \frac{\alpha}{m} \right) \sum_{n=k}^{\infty} S_{m, \lambda} (n, k) m^n \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( m^n \sum_{k=0}^{n} C_k \left( \frac{r}{m} - \frac{\alpha}{m} \right) S_{m, \lambda} (n, k) \right) \frac{t^n}{n!}. \]

On the other hand, by (61), we get

\[ e^{\frac{r}{m} \left( e^{\frac{\alpha}{m} (t)} - 1 \right)} = \sum_{n=0}^{\infty} D_{m, \lambda} (n, \alpha) \frac{t^n}{n!}. \]

Therefore, by (61) and (62), we obtain the following theorem.

**Theorem 11.** For \( n \geq 0 \), let \( X \sim \text{Poi} \left( \frac{\alpha}{m} \right) \). Then we have

\[ D_{m, \lambda} (n, \alpha) = E[(mX + r)_{n, \lambda}] = m^n \sum_{k=0}^{n} C_k \left( \frac{r}{m} - \frac{\alpha}{m} \right) S_{m, \lambda} (n, k). \]

3. **Conclusion**

In recent years, studying various degenerate versions of many special polynomials and numbers received regained interests of some mathematicians and many interesting results were discovered. Degenerate Dowling and degenerate \( r \)-Dowling polynomials were introduced earlier as degenerate versions and further generalizations of Dowling and \( r \)-Dowling polynomials.

Assume that \( X \) is the Poisson random variable with mean \( \frac{\alpha}{m} \). We showed that the Poisson degenerate central moment \( E[(mX + 1)_{n, \lambda}] \) is equal to \( D_{m, \lambda} (n, \alpha) \) and to an expression involving the degenerate Bell polynomials, respectively in Theorem 1 and Theorem 2. We deduced that \( E[(mX + r)_{n, \lambda}] \) is equal to \( D_{m, \lambda} (n, \alpha) \) in Theorem 4. We expressed the same in terms of the degenerate Bell polynomials in Corollary 5 and of the degenerate \( r \)-Whitney numbers of the second kind and the Bell polynomials in Theorem 6. Furthermore, it is represented by the Charlier polynomials and the degenerate Stirling numbers of the second kind in Theorems 10 and 11.

As one of our future projects, we would like to continue to study degenerate versions of certain special polynomials and numbers and their applications to physics, science and engineering as well as mathematics.

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The authors declare that there is no ethical problem in the production of this paper.
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The authors declare no conflict of interest.

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The authors want to publish this paper in this journal.

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