Asymptotics of a Gauss hypergeometric function with large parameters, III: Application to the Legendre functions of large imaginary order and real degree

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Abstract

We obtain the asymptotic expansion for the Gauss hypergeometric function

$$F(a - \lambda, b + \lambda; c + i\alpha \lambda; z)$$

for $$\lambda \to +\infty$$ with $$a$$, $$b$$ and $$c$$ finite parameters by application of the method of steepest descents. The quantity $$\alpha$$ is real, so that the denominatorial parameter is complex and $$z$$ is a finite complex variable restricted to lie in the sector $$|\arg(1 - z)| < \pi$$. We concentrate on the particular case $$a = 0$$, $$b = c = 1$$, which is associated with the Legendre functions of real degree and imaginary order. The resulting expansions are of Poincaré type and hold in restricted domains of the $$z$$-plane. An expansion is given at the coalescence of two saddle points. Numerical results illustrating the accuracy of the different expansions are given.

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1. Introduction

The Gauss hypergeometric function is defined by

$$F \left( \frac{a, b}{c}; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad (|z| < 1)$$

and elsewhere by analytic continuation, where $$(a)_n = \Gamma(a + n)/\Gamma(a) = a(a + 1)\ldots(a + n - 1)$$ is the Pochhammer symbol or rising factorial. The asymptotic expansion of

$$F \left( \frac{a + \epsilon_1 \lambda, b + \epsilon_2 \lambda}{c + \epsilon_3 \lambda}; z \right)$$

for large values of $$\lambda$$ and fixed complex $$z$$ when the parameters $$\epsilon_j$$ are finite was first considered by Watson [16] in 1918 and recently by the author in [13, 14]; see also [4] for the case of two large parameters. This function may be characterised by the set $$\{\epsilon_1, \epsilon_2, \epsilon_3\}$$, where, by a rescaling of $$\lambda$$ one of the $$\epsilon_j$$ can (if so desired) always be replaced by unity. Watson considered the situation when $$\epsilon_j = 0, \pm 1$$ and examined the cases $$(0, 0, 1)$$, $$(1, -1, 0)$$ and $$(0, -1, 1)$$, together with the additional

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case (1, 2, 0). In [13, 14], the $\epsilon_j$ were taken to be positive constants. It was shown that it is sufficient to consider just three basic types of hypergeometric function corresponding to the parameter sets $(\epsilon, 0, 1)$ (Type A), $(\epsilon, -1, 0)$ (Type B) and $(\epsilon_1, \epsilon_2, 1)$ (Type C), where $\epsilon, \epsilon_1, \epsilon_2 > 0$. An application of the expansion for the case $(\epsilon, \epsilon, 1)$ with $\epsilon > 0$ has arisen in aerodynamics [3, 8]. Expansions of a uniform character when two parameters are large have been given for the case $(1, -1, 0)$ in [7] and $(0, -1, 1), (1, 2, 0)$ in [9].

In this paper, we consider the expansion of the function

$$ F \left( \frac{a - \lambda, b + \lambda}{c + i\alpha \lambda}; z \right) $$

(1.1)

for $\lambda \to +\infty$ with finite $a$, $b$ and $c$. Here the parameter $\alpha \in \mathbb{R}$ is finite and $z$ is a (finite) complex variable restricted to lie in $|\arg(1 - z)| < \pi$. The parameter set in this case is consequently $(-1, 1, i\alpha)$, which is different from the cases previously considered in that the denominatiorial parameter $\epsilon_3 = i\alpha$ is purely imaginary. As in [13, 14], we employ the method of steepest descents applied to a contour integral representation of the function in (1.1) to obtain Poincaré-type expansions. The expansion in the case of the coalescence of two saddle points is also considered.

We concentrate on the situation when the parameters in (1.1) have the values $a = 0$, $b = c = 1$ and consider the function

$$ F_\alpha (\lambda; z) := F \left( \frac{-\lambda, 1 + \lambda}{1 + i\alpha \lambda}; z \right) $$

(1.2)

with $\alpha > 0$. This particular case has arisen in the study of travelling waves in a Toda lattice [19, 17]. This case is also of interest as it is associated with the Legendre functions through the relations [11, p. 353]

$$ P_\nu^{-\mu}(x) = \frac{1}{\Gamma(1 + \mu)} \left( \frac{x - 1}{x + 1} \right)^{\mu/2} F(-\nu, 1 + \nu; 1 + \mu; \frac{1}{2} - \frac{1}{2}x) $$

(1.3)

and

$$ e^{\pi i\mu} Q_\nu^{-\mu}(x) = \frac{\Gamma(1 + \nu - \mu)\Gamma(\mu)}{2\Gamma(1 + \nu + \mu)} \left( \frac{x - 1}{x + 1} \right)^{-\mu/2} F(-\nu, 1 + \nu; 1 - \mu; \frac{1}{2} - \frac{1}{2}x) $$

$$ + \frac{\Gamma(-\mu)}{2} \left( \frac{x - 1}{x + 1} \right)^{\mu/2} F(-\nu, 1 + \nu; 1 + \mu; \frac{1}{2} - \frac{1}{2}x), $$

(1.4)

which define the functions in the complex $x$-plane cut along $(-\infty, 1]$. Thus, as a by-product of our investigation of (1.2) we will obtain the expansions of the Legendre functions $P_\lambda^{\pm i\alpha \lambda}(x)$ and $Q_\lambda^{\pm i\alpha \lambda}(x)$, for large imaginary order and real degree. The expansions for these functions when $x = \sqrt{1 + \alpha^2}$, which corresponds to the above-mentioned coalescence of two saddle points, is also given. We remark that the expansion of $P_\nu^{-\mu}(x)$ and $Q_\nu^{-\mu}(x)$ for $\nu \to +\infty$, $\Re(x) \geq 0$ uniformly valid for $0 \leq \mu/(\nu + \frac{1}{2}) \leq A$, where $A$ is a constant, has been considered by Dunster [6] who employed a differential-equation approach.

We remark that when $z = \frac{1}{2}$ it is possible to give an exact evaluation for $F_\alpha (\lambda; z)$ in the form [11, (15.4.30)]

$$ F_\alpha (\lambda; \frac{1}{2}) = \frac{2^{-i\alpha \lambda} \sqrt{\pi} \Gamma(1 + i\alpha \lambda)}{\Gamma(\frac{1}{2} + \frac{i}{2} \lambda(i\alpha - 1))\Gamma(1 + \frac{i}{2} \lambda(i\alpha + 1))}. $$

(1.5)

2. The expansion of $F_\alpha (\lambda; z)$ for $\lambda \to +\infty$

We take the parameter values $a = 0$, $b = c = 1$ in (1.1) and consider in detail the expansion of $F_\alpha (\lambda; z)$ in (1.2) for $\lambda \to +\infty$; the case of general values of $a$, $b$ and $c$ is considered in Section 2.3. From the series representation of the hypergeometric function it follows that, when $\lambda > 0$ and $\alpha$ is real,

$$ F_{-\alpha} (\lambda; z) = F_\alpha (\lambda; z), $$

(2.1)
where the bar denotes the complex conjugate. It is therefore sufficient to consider $\alpha > 0$ throughout; in addition, we define

$$\theta := \arg z, \quad \phi := \arctan \alpha,$$

where $\theta \in [-\pi, \pi]$ and $\phi \in (0, \frac{\pi}{2})$.

We employ the integral representation [11, p. 388]

$$F \left( \frac{a, b}{c} ; z \right) = \frac{\Gamma(c)\Gamma(1+b-c)}{2\pi i\Gamma(b)} \int_0^{(1+)} \frac{t^{b-1}(t-1)^{c-b-1}}{(1-zt)^a} dt, \quad \Re(b) > 0, \quad \arg(1-z) < \pi$$

where it is supposed that $|\arg(1-z)| < \pi$ and $c-b \neq 1, 2, \ldots$. The integration path is a closed loop that starts from the origin, encircles the point $t = 1$ in the positive sense and returns to the origin without enclosing the point $t = 1/z$. The function $F_\alpha(\lambda; z)$ can then be expressed in the form

$$F_\alpha(\lambda; z) = F \left( \frac{-\lambda, 1+\lambda}{1+i\alpha \lambda} ; z \right) = \frac{G_\alpha(\lambda)}{2\pi i} \int_0^{(1+)} f(t)e^{\lambda \psi(t)} dt,$$

where the phase function $\psi(t)$ and the amplitude function $f(t)$ are

$$\psi(t) = \log t(1-zt) - \beta \log(t-1), \quad f(t) = (t-1)^{-1}, \quad \beta := 1 - i\alpha$$

and

$$G_\alpha(\lambda) := \frac{\Gamma(1+i\alpha \lambda)\Gamma(1+\lambda \beta)}{\Gamma(1+\lambda)} = i\alpha \beta \lambda \frac{\Gamma(i\alpha \lambda)\Gamma(\lambda \beta)}{\Gamma(\lambda)}.$$

The $t$-plane is cut along $(-\infty, 1]$ and also along the ray from the singularity at $t = 1/z$ to infinity in a suitable direction.

The phase function has saddle points where $\psi'(t) = 0$; that is at the points where

$$i\alpha t_s - 1 \quad \frac{z}{1-zt_s} = 0.$$

There are consequently two saddle points, which we label $t_{s1}$ and $t_{s2}$, given by

$$t_{s1}, t_{s2} = \frac{z + \frac{1}{2}i\alpha \mp i(z - z^2 + \frac{1}{2}\alpha^2)^{1/2}}{(1+i\alpha)z}, \quad (2.7)$$

respectively. For sufficiently large $\lambda$, the points $t = 0$ and $t = 1/z$ are zeros of the integrand, so that paths of steepest descent can terminate only at these two points; paths of steepest ascent must terminate at $t = 1$ and at infinity. A typical arrangement of the steepest paths through $t_{s1}$ and $t_{s2}$ is shown in Fig. 1 when $a = 1$ and for different values of $\theta$.

Since $t_{s1}t_{s2} = \{(1+i\alpha)z\}^{-1}$, it follows that

$$\arg t_{s1} + \arg t_{s2} = -(\theta + \phi).$$

Consequently, when $\theta + \phi > 0$ (resp. $< 0$) at least one saddle is situated in the lower (resp. upper) half-plane; when $\theta + \phi = 0$, one saddle is situated in upper half-plane with the other in the lower half-plane. It is to be noted that the saddles coalesce to form a double saddle when $z^2 - z - \frac{1}{4}\alpha^2 = 0$; that is when

$$z = z_d := \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \alpha^2}.$$

The contribution to the integral (2.4) (excluding the pre-factor) from the steepest descent path through the (simple) saddle $t_{sj}$ is given by the formal asymptotic sum [1, p. 265]

$$if(t_{sj})e^{\lambda \psi(t_{sj})+\pi i\gamma_j} \sqrt{\psi''(t_{sj})} \sum_{s=0}^{\infty} \frac{c_s^{(j)}(\frac{1}{2})}{\lambda^{s+\frac{1}{2}}} \quad (j = 1, 2) \quad (2.9)$$

as $\lambda \to +\infty$, where

$$\psi''(t_{sj}) = -\frac{2}{(t_{sj}-1)^2} \left\{ \left( 1 - \frac{1}{t_{sj}} \right) \left( 1 - \frac{1}{t_{sj}} - \beta \right) - \frac{1}{2} \beta(1-\beta) \right\}.$$
Figure 1: Examples of the steepest descent and ascent paths for $|z| = 0.50$ and $\alpha = 1$ when (a) $\theta = 0$, (b) $\theta = \frac{1}{4}\pi$ and (c) $\theta = -\frac{3}{4}\pi$; (d) $|z| = 0.10$, $\theta = 0.46292\pi$ corresponding to a Stokes phenomenon. The saddles are denoted by heavy dots and the arrows indicate the integration path. There is a branch cut along $(-\infty, 1]$ and from the point $1/z$ out to infinity.

The $\gamma_j$ are orientation factors that depend on the direction of integration $\text{arg}(t - t_{sj})$ through the saddle point $t_{sj}$ and have the value either 0 or 1.

2.1 The coefficients $c_s^{(j)}$. The coefficients $c_s^{(j)} \equiv c_s$ (which are functions of $\alpha$ and $z$) for $s \leq 2$ are given explicitly by

$$c_0 = 1, \quad c_1 = \frac{1}{2\psi''} \left\{ 2F_2 - 2\Psi_3F_1 + \frac{5}{6}\Psi_3^2 - \frac{1}{2}\Psi_4 \right\},$$

$$c_2 = \frac{1}{(2\psi'')^2} \left\{ \frac{3}{2}F_4 - \frac{29}{24}\Psi_3F_3 + \frac{1}{4}(\frac{2}{3}\Psi_3^2 - \Psi_4)F_2 - \frac{35}{9}(\Psi_3^3 - \Psi_3\Psi_4 + \frac{6}{35}\Psi_5)F_1 ight.$$ 

$$+ \frac{35}{9}(\frac{11}{24}\Psi_3^4 - \frac{3}{4}(\Psi_3^2 - \frac{1}{6}\Psi_4)\Psi_4 + \frac{1}{5}\Psi_3\Psi_5 - \frac{1}{35}\Psi_6) \right\},$$

(2.10)

where, for brevity, we have defined

$$\Psi_m := \frac{\psi^{(m)}(t)}{\psi''(t)} \quad (m \geq 3), \quad F_m := \frac{f^{(m)}(t)}{f(t)} \quad (m \geq 1)$$

with $\psi(t)$, $f(t)$ and their derivatives being evaluated at $t = t_{s1}$ or $t = t_{s2}$; see, for example, [5, p. 119], [10, p. 127], [12, p. 13].

Alternatively, the $c_s$ can be obtained by an expansion process to yield Wojdylo’s formula [18] given by

$$c_s = \frac{(-)^s 2s \beta_{2s-k}}{\alpha_0^s} \sum_{k=0}^{2s-k} \beta_0 \sum_{j=0}^{k} \frac{(-)^j (s + \frac{1}{2}j)}{j!} B_{kj} \beta_0^j;$$

(2.11)
see also [15, p. 25]. Here \( B_{kj} \equiv B_{kj}(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_{k-j+1}) \) are the partial ordinary Bell polynomials generated by the recursion

\[
B_{kj} = \sum_{r=1}^{k-j+1} \hat{\alpha}_r B_{k-r,j-1}, \quad B_{k0} = \delta_{k0},
\]

where \( \delta_{mn} \) is the Kronecker symbol, and the coefficients \( \hat{\alpha}_r \) and \( \hat{\beta}_r \) appear in the expansions

\[
\psi(t) - \psi(t_s) = \sum_{r=0}^{\infty} \hat{\alpha}_r (t - t_s)^{r+2}, \quad f(t) = \sum_{r=0}^{\infty} \hat{\beta}_r (t - t_s)^r
\]

valid in a neighbourhood of the saddle \( t = t_s \).

### 2.2 The expansion of \( F_\alpha(\lambda; z) \)

It can be seen from Fig. 1 that the steepest ascent path through \( t_{s1} \) crosses the branch cut to pass onto an adjacent Riemann sheet in the \( t \)-plane. When \( \theta \) increases, it is found that in some cases the saddle \( t_{s2} \) (and consequently a portion of the associated steepest descent path) can also pass onto an adjacent sheet. To avoid this difficulty, we make the substitution \( t = e^w \) to find the phase function in (2.5) given by

\[
\psi(w) = (1 - \frac{1}{2} \beta)w + \log(1 - ze^w) - \beta \log 2 \sinh \frac{1}{2}w,
\]

with the image of the saddles given by \( w_j = \log t_{sj} \) \((j = 1, 2)\). The closed circuit surrounding the point \( t = 1 \) in the \( t \)-plane becomes the loop that commences at \(-\infty\), encircles the point \( w = 0 \) in the positive sense and returns to \(-\infty\). Branch cuts are introduced along \((\infty, 1]\) and from the point \( \log 1/z \) out to \( \infty \) parallel to the real \( w \)-axis. This transformation causes all the Riemann sheets in the \( t \)-plane to appear as horizontal strips of width \( 2\pi \) in the \( w \)-plane; the principal sheet corresponds to \(-\pi < \Im(w) \leq \pi \).

Examples of the steepest paths in the \( w \)-plane are illustrated in Fig. 2. In Fig. 2(a) both the saddles \( w_1 \) and \( w_2 \) and their associated steepest descent paths are situated on the principal sheet; in Fig. 2(b) the saddles are again on the principal sheet, but the steepest descent path from \( w_2 \) crosses the line \( \Im(w) = \pi \), which corresponds to passing on to the adjacent sheet in the \( t \)-plane. Fig. 2(c) shows the same situation as Fig. 1(d), namely \(|z| = 0.10, \theta = 0.46292\pi\); for this value of \( \theta \) the saddles \( w_1 \) and \( w_2 \) are connected, with the steepest descent path from \( w_2 \) passing into the strip \( \pi < \Im(w) \leq 3\pi \). The saddle in this strip corresponds to the image of the saddle \( w_1 \) in the principal sheet; the contribution from this image saddle is exponentially smaller (by the factor \( e^{-2\pi\alpha\lambda} \)) than that from \( w_1 \) and so is neglected. Fig. 2(d) shows \(|z| = 0.10, \theta = 0.60\pi \) where the steepest descent path through \( w_2 \) has disconnected from the saddle \( w_2 \) (a Stokes phenomenon) and passes over into the adjacent strip.

Then, from (2.4), we have the expansion

\[
F_\alpha(\lambda; z) \sim G_\alpha(\lambda) \left\{ \frac{e^{\lambda\psi(t_{s1})} f(t_{s1})}{\sqrt{2\pi \psi''(t_{s1})}} \sum_{s=0}^{\infty} \frac{c_s^{(1)}(1/2)^s}{\lambda^{s+1/2}} + \frac{e^{\lambda\psi(t_{s2})} f(t_{s2})}{\sqrt{2\pi \psi''(t_{s2})}} \sum_{s=0}^{\infty} \frac{c_s^{(2)}(1/2)^s}{\lambda^{s+1/2}} \right\}
\]

(2.12)

as \( \lambda \to +\infty \) valid for complex \( z \) in \(|\arg(1 - z)| < \pi \). This expansion will break down in the neighbourhood of the double saddle points \( z = \pm\frac{1}{2} \), and also ceases to be valid in a zone surrounding \( z = 0 \); see below. The orientation factors \( \gamma_1 = \gamma_2 = 0 \) and we note that

\[
e^{\lambda\psi(t_{s1})} = \left\{ \frac{t_{s1}(1 - zt_{s1})}{(t_{s1} - 1)^2} \right\}^\lambda.
\]

If required, an expansion for \( G_\alpha(\lambda) \) in inverse powers of \( \lambda \) is given in the appendix.

The boundary in the upper-half \( z \)-plane on which \( \Re\psi(t_{s1}) = \Re\psi(t_{s2}) \) is shown in Fig. 3 for the particular case of \( \alpha = 1 \). This curve has its endpoints \( A \) and \( B \) at the double saddle points \( z_{d1}^\pm \) in (2.8). Below this curve, and also in \( \Im(z) < 0 \), the contribution from the saddle \( t_{s1} \) dominates that from the saddle \( t_{s2} \); above this curve the saddle \( t_{s2} \) is dominant. In the neighbourhood of the curve both contributions need to be taken into account.

\[\text{For example, this generates the values } B_{41} = \hat{\alpha}_4, B_{42} = \hat{\alpha}_4^2 + 2\hat{\alpha}_1\hat{\alpha}_3, B_{43} = 3\hat{\alpha}_1^2\hat{\alpha}_2 \text{ and } B_{44} = \hat{\alpha}_1^4.\]
The dashed closed curve surrounding \( z = 0 \) (the enclosed domain is denoted by \( D \)) and terminating at \( z_0 \) shows the curve on which \( \Im \psi(t_{s1}) = \Im \psi(t_{s2}) \), where a Stokes phenomenon occurs. As one crosses this loop and passes into its interior the saddle \( t_{s2} \) disconnects from the integration path to leave only the contribution from the saddle \( t_{s1} \); see Fig. 1(d) and Fig. 2(c), (d). As a consequence, the expansion of \( F_\alpha(\lambda; z) \) inside this loop is given by

\[
F_\alpha(\lambda; z) \sim G_\alpha(\lambda) \frac{e^{i\psi(t_{s1})} f(t_{s1})}{\sqrt{2\pi \psi''(t_{s1})}} \sum_{s=0}^{\infty} \frac{C_s^{(1)}(\frac{1}{2})_s}{\lambda^{s+\frac{1}{2}}} \quad (z \in D)
\]  

(2.13)

as \( \lambda \to +\infty \).

In the lower-half \( z \)-plane, the saddle \( t_{s2} \) can pass over onto an adjacent Riemann sheet (in the \( t \)-plane) and great care must be taken to ensure that one uses continuous branches for the functions \( \log t_{s2} \) and \( \log (t_{s2} - 1) \). The curves on which \( t_{s2} \) and \( t_{s2} - 1 \) pass onto adjacent sheets are indicated in Fig. 3 by the dotted curves issuing from \( z = 0 \) and \( z = 1 \), respectively. To the right (resp. left) of the curve issuing from \( z = 0 \) (resp. \( z = 1 \)), \( t_{s2} \) (resp. \( t_{s2} - 1 \)) lies on the principal sheet. It must be emphasised that all the curves in Fig. 3 depend on the value of the parameter \( \alpha > 0 \).

The results of numerical computations carried out with "Mathematica" are presented in Tables 1 and 2. Table 1 shows the absolute relative error\(^2\) in the computation of \( F_\alpha(\lambda; z) \) as a function of the truncation index \( s \) for \( \alpha = 1, \lambda = 80 \) and \( z = 0.50e^{i\theta} \) using the expansion (2.12). We note that when \( \theta = 0 \) the value of \( F_\alpha(\lambda; \frac{1}{2}) \) is given by (1.5). The coefficients \( C_s^{(1,2)} \) were derived from (2.11) and the high-precision evaluation of \( F_\alpha(\lambda; z) \) obtained by the routine Hypergeometric2F1 in "Mathematica." Table 2 shows the absolute relative error as a function of \( \theta \) for different \( |z| \) with the same values of \( \alpha \) and \( \lambda \) and truncation index \( s = 2 \). In this case the coefficients \( C_s^{(1,2)} \) can be

\(^2\)We have adopted the convention in the tables of writing \( x(y) \) for \( x \times 10^y \).
Figure 3: The curve (shown solid) in the $z$-plane on which $\Re \psi(t_{s1}) = \Re \psi(t_{s2})$ when $\alpha = 1$. The endpoints $A$ and $B$ are situated at $z_{A}^\ast = -0.2071$ and $z_{B}^\ast = 1.0271$ corresponding to the double saddle points in the $t$-plane. The dashed loop surrounding $z = 0$ shows the curve on which $\Im \psi(t_{s1}) = \Im \psi(t_{s2})$, where a Stokes phenomenon occurs; the interior of this domain is labelled $\mathcal{D}$. The dotted curves issuing from $z = 0$ and $z = 1$ indicate where $t_{s2}$ and $t_{s2} - 1$, respectively, pass onto an adjacent Riemann sheet in the $t$-plane.

obtained alternatively from (2.10). In the column corresponding to $|z| = 0.06$, all the values of $z$ lie in the domain $\mathcal{D}$ in which only the saddle $t_{s1}$ contributes to $F_{\alpha}(\lambda; z)$. In the column corresponding to $|z| = 0.25$, the error is seen to become relatively large when $\theta = \pi$. This is due to the fact that $z = -0.25$ lies close to the value $z_{zz}^\ast = -0.2071$, which corresponds to the formation of a double saddle point where the expansion (2.12) ceases to be valid.

Table 1: Values of the absolute relative error in the computation of $F_{\alpha}(\lambda; z)$ for different truncation index $s$ in the expansions (2.12) and (2.13) when $\lambda = 80$, $\alpha = 1$ and $z = 0.50e^{i\theta}$.

| $s$ | $\theta = 0$ | $\theta = 0.25\pi$ | $\theta = 0.50\pi$ | $\theta = 0.75\pi$ | $\theta = \pi$ |
|-----|--------------|------------------|------------------|------------------|----------------|
| 0   | 1.042(-08)   | 1.865(-03)       | 3.251(-03)       | 9.873(-04)       | 6.525(-04)     |
| 1   | 5.158(-07)   | 2.927(-06)       | 1.619(-05)       | 1.219(-05)       | 3.182(-05)     |
| 2   | 2.731(-08)   | 3.390(-08)       | 3.139(-07)       | 4.098(-07)       | 1.617(-06)     |
| 3   | 3.050(-11)   | 7.929(-10)       | 7.240(-09)       | 1.910(-08)       | 1.213(-07)     |
| 4   | 2.166(-12)   | 9.013(-12)       | 2.263(-10)       | 1.175(-09)       | 1.195(-08)     |
| 5   | 1.473(-15)   | 2.314(-13)       | 8.728(-12)       | 8.961(-11)       | 1.464(-09)     |

Table 2: Values of the absolute relative error in the computation of $F_{\alpha}(\lambda; z)$ for different $\theta$ and $|z|$ in the expansions (2.12) and (2.13) when $\lambda = 80$, $\alpha = 1$ and truncation index $s = 2$.

| $\theta/\pi$ | $|z| = 0.06$ | $|z| = 0.25$ | $|z| = 0.50$ | $|z| = 0.75$ | $|z| = 1.00$ | $|z| = 1.20$ |
|--------------|------------|------------|------------|------------|------------|------------|
| 0            | 1.340(-08) | 3.566(-08) | 2.731(-08) | 3.566(-08) | 9.869(-08) | 1.265(-08) |
| 0.25         | 4.546(-08) | 7.760(-08) | 3.390(-08) | 4.584(-08) | 9.869(-08) | 1.265(-08) |
| 0.50         | 2.064(-07) | 5.200(-07) | 3.139(-07) | 2.347(-08) | 1.150(-08) | 7.116(-09) |
| 0.75         | 8.228(-07) | 1.159(-05) | 4.098(-07) | 9.420(-08) | 3.681(-08) | 2.153(-08) |
| 1.00         | 1.522(-06) | 7.823(-03) | 1.617(-06) | 1.860(-07) | 5.337(-08) | 2.547(-08) |
| -0.25        | 4.848(-08) | 4.754(-08) | 1.300(-08) | 4.925(-09) | 1.171(-08) | 1.265(-08) |
| -0.50        | 1.597(-07) | 2.028(-07) | 5.965(-08) | 2.347(-08) | 1.150(-08) | 7.116(-09) |
| -0.75        | 6.302(-07) | 3.999(-06) | 4.098(-07) | 9.420(-08) | 4.681(-08) | 2.153(-08) |
Figure 4: An example of the steepest descent and ascent paths when $z = \frac{1}{2}(1 - \sqrt{1 + \alpha^2})$ and $\alpha = 1$. (a) In the $t$-plane with the double saddle at $t_d$. The steepest descent path $SA$ passes over onto the adjacent Riemann sheet and spirals into the origin. (b) The same situation viewed in the $w$-plane where $w_d = \log t_d$. The saddles are denoted by heavy dots and the arrows indicate the integration path. There is a branch cut along $(-\infty, 1]$ and from the point $1/z$ out to infinity (not shown in (a)).

2.3 The expansion in the general case The hypergeometric function in (1.1) has the integral representation from (2.3) given by

$$F\left( a - \lambda, b + \lambda; c + i\alpha \lambda; z \right) = \frac{\Gamma(c + i\alpha \lambda)\Gamma(1 + b - c + \lambda\beta)}{2\pi i \Gamma(b + \lambda)} \int_0^{(1+)} f(t)e^{\lambda\psi(t)} dt,$$

where the amplitude function $f(t)$ is now given by

$$f(t) = \frac{t^{b-1}(t - 1)^{c-b-1}}{(1 - zt)^a}.$$  \hspace{1cm} (2.14)

The phase function $\psi(t)$ is as in (2.5) and consequently the distribution of the saddle points remains the same. It therefore follows that the expansion of $F(a - \lambda, b + \lambda; c + i\alpha \lambda; z)$ for $\lambda \to +\infty$ is given by (2.12) and (2.13) with $f(t)$ replaced by (2.14) and the coefficients $c_{1,2}$ determined from either (2.10) or (2.11). The function $G\alpha(\lambda)$ is replaced by $\frac{\Gamma(c + i\alpha \lambda)\Gamma(1 + b - c + \lambda\beta)}{\Gamma(b + \lambda)}$.

3. The expansion when $z = z_d^-$ for $\lambda \to +\infty$

When $z = z_d^- = \frac{1}{2}(1 - \sqrt{1 + \alpha^2})$, it is seen from (2.7) that the saddles $t_{s1}$ and $t_{s2}$ coalesce to form a double saddle at the point

$$t_d = \frac{z_d^- + \frac{1}{2}i\alpha}{(1 + i\alpha)z_d^-} = \frac{1 - \sqrt{1 + \alpha^2} + i\alpha}{(1 + i\alpha)(1 - \sqrt{1 + \alpha^2})}.$$  \hspace{1cm} (3.1)

In the neighbourhood of the point $z = z_d^-$ the expansions in (2.12) and (2.13) break down. In this section we determine the expansion of $F\alpha(\lambda; z)$ and also that of the general case in (1.1) valid at the coalescence point $z = z_d^-$. The integration path when $z = z_d^-$ is typically as shown in Fig. 4.

3.1 The expansion of $F\alpha(\lambda; z_d^-)$. If we put

$$-u = \psi(t) - \psi(t_d) = A\tau^3 + B\tau^4 + C\tau^5 + D\tau^6 + \ldots, \quad \tau := t - t_d$$

we find the coefficients

$$A = \frac{-(1 + i\alpha)^3}{6\alpha^2 \sqrt{1 + \alpha^2}} \left\{ \alpha^2(-3 + \sqrt{1 + \alpha^2}) + 4(-1 + \sqrt{1 + \alpha^2}) \right\},$$
Inversion yields

\[
B = \frac{(1 + i\alpha)^3(\alpha + 2i)}{8\alpha^3(1 - i\alpha)} \left\{ \alpha^4 - 4\alpha^2(-2 + \sqrt{1 + \alpha^2}) - 8(-1 + \sqrt{1 + \alpha^2}) \right\},
\]
\[
C = \frac{(1 + i\alpha)^4(6 - 5i\alpha - \alpha^2)}{20\alpha^4(1 - i\alpha)\sqrt{1 + \alpha^2}} \left\{ \alpha^4(-5 + \sqrt{1 + \alpha^2}) + 4\alpha^2(-5 + 3\sqrt{1 + \alpha^2}) + 16(-1 + \sqrt{1 + \alpha^2}) \right\},
\]
\[
D = \frac{-i(1 + i\alpha)^4}{24\alpha^5(1 - i\alpha)^2} (8 - 9i\alpha - 3\alpha^2) \left\{ \alpha^6 - 6\alpha^4(-3 + \sqrt{1 + \alpha^2}) - 16\alpha^2(-3 + 2\sqrt{1 + \alpha^2}) - 32(-1 + \sqrt{1 + \alpha^2}) \right\}, \ldots.
\]

Upon differentiation of \(\tau(w)\), we then obtain the expansion

\[
\frac{1}{t - 1} \frac{dT}{dw} = \sum_{m=0}^{\infty} B_m(\alpha) w^{(m-2)/3}
\]

valid in a neighbourhood of \(w = 0\) \((t = t_d)\), where

\[
B_0(\alpha) = \frac{1}{3A^{1/3}T}, \quad B_1(\alpha) = -\frac{1}{9A^{2/3}T^2} \left( 3 + \frac{2BT}{A} \right),
\]
\[
B_2(\alpha) = \frac{1}{3AT^3} \left\{ 1 + \frac{BT}{A} \left( 1 + \frac{BT}{A} \right) - \frac{CT^2}{A} \right\},
\]
\[
B_3(\alpha) = -\frac{1}{243A^{4/3}T^4} \left\{ 81 + 140 \frac{B^3T^3}{A^3} + \frac{126BT^2}{A^2}(B - 2CT) + \frac{108T}{A} (B - CT + DT^2) \right\}, \ldots.
\]

The coefficients \(A, B, C, D\) and the quantity \(T\) are defined above in terms of the parameter \(\alpha\).

Then, from (2.4) and (3.4), we obtain

\[
F_\alpha(\lambda; z_d^-) = \frac{G_\alpha(\lambda)e^{\lambda\psi(t_d)}}{2\pi i} \int_0^\infty e^{-\lambda u} \left\{ \frac{1}{t - 1} \left. \frac{dt}{du} \right|_{ue^{-\pi i}} - \frac{1}{t - 1} \left. \frac{dt}{du} \right|_{ue^{\pi i}} \right\} du
\]
\[
\sim -\frac{G_\alpha(\lambda)}{\pi} e^{\lambda\psi(t_d)} \int_0^\infty e^{-\lambda u} \sum_{m=0}^\infty B_m(\alpha) u^{(m-2)/3} \sin\pi\left(\frac{1}{3}m + \frac{1}{3}\right) du
\]

for \(\lambda \to +\infty\). This therefore produces the expansion

\[
F_\alpha(\lambda; z_d^-) \sim -\frac{G_\alpha(\lambda)}{\pi} e^{\lambda\psi(t_d)} \sum_{m=0}^\infty \frac{B_m(\alpha)\Gamma\left(\frac{1}{3}m + \frac{1}{3}\right)}{\lambda^{(m+1)/3}} \sin\pi\left(\frac{1}{3}m + \frac{1}{3}\right) \quad (\lambda \to +\infty).
\]

Since \(z_d^-\) is real it immediately follows from (2.1) that

\[
F_{-\alpha}(\lambda; z_d^-) = F_\alpha(\lambda; z_d^-).
\]
Due to the complexity of the coefficients it is not practical to present their explicit dependence on the parameter \(\alpha\) for more than the first three terms in the expansion (3.6). If, however, \(\alpha\) is given a numerical value then the inversion process can be carried out with Mathematica to many more terms. In the particular case \(\alpha = 1\), the values of the coefficients \(B_m(\alpha)\) are tabulated in Table 3 for \(m \leq 10\); we observe that the values of \(B_2(\alpha), B_3(\alpha), B_4(\alpha), \ldots\) are not required. Values of \(F_\alpha(\lambda; z^-)\) and its asymptotic estimate from (3.6) with \(m \leq 10\) (sub-optimal truncation) are given in Table 4.

Table 3: Values of the coefficients \(B_m(\alpha)\) to 10dp for \(m \leq 10\) when \(\alpha = 1\).

| \(m\) | \(B_m(\alpha)\) |
|------|------------------|
| 0    | +1.1210852199 + 0.3003938793i |
| 1    | +0.2166214717 + 0.8761432383i |
| 2    | +0.0907067826 + 0.0167746174i |
| 3    | +0.0082200112 + 0.0306774994i |
| 4    | +0.001979294 - 0.007386825i |
| 5    | +0.0001979294 - 0.007386825i |
| 6    | +0.0001979294 - 0.007386825i |
| 7    | -0.0001979294 - 0.007386825i |
| 8    | -0.0001979294 - 0.007386825i |
| 9    | -0.0001979294 - 0.007386825i |
| 10   | -0.0001979294 - 0.007386825i |

A uniform approximation for \(F_\alpha(\lambda; z^-)\) for \(z \simeq z^-\) could be obtained by making the standard cubic transformation (see [11, (2.4.18)]) to \(\psi(t)\) in (2.4). We do not pursue this further here.

### 3.2 The expansion in the general case when \(z = z^-\).

The expansion of the hypergeometric function in (1.1) for general \(a, b\) and \(c\) when \(z = z^-\) follows a similar procedure to that in the specific case of \(F_\alpha(\lambda; z)\). The amplitude function \(f(t)\) is now given by (2.14), which may be expressed in the neighbourhood of the double saddle \(t_d\) as (with \(t = t_d + \tau\))

\[
f(t) = \frac{t^b-1(t-1)^{c-b-1}}{(1-z^-t)^a} = \frac{1}{T} \left(1 + \frac{\tau}{T}\right)^{c-b-1} \left(1 + \frac{\tau}{T} \frac{b-1}{c-b-1}\right)^{1-a},
\]

where \(T\) is defined in (3.3) and

\[
\kappa := \frac{z^-t_d}{1 - z^-t_d} = \frac{1 - \sqrt{1 + \alpha^2 + i\alpha}}{1 + \sqrt{1 + \alpha^2 + i\alpha}}, \quad \hat{T}^{-1} := \frac{t^b_d - (t_d-1)^{c-b-1}}{(1-z^-t_d)^a}.
\]

Expansion about \(\tau = 0\), followed by use of (3.2) to express \(\tau\) in terms of \(w\), produces

\[
\log f(t) = -\log \hat{T} + \sum_{n=1}^{\infty} \gamma_n \left(\frac{w}{A}\right)^{n/3},
\]

where

\[
\gamma_1 = \frac{c - b - 1}{T} \frac{b - 1 - \kappa a}{t_d}, \quad \gamma_2 = -\frac{b - 1 - \kappa^2 a}{2t_d^2} - \frac{c - b - 1}{2T^2} - \frac{\gamma_1 B}{3A^2},
\]

\[
\gamma_3 = -\frac{(B^2 - 3AC)\gamma_1}{9A^2} - \frac{2B\gamma_2}{3A} + \frac{b - 1 + \kappa^3 a}{3t_d^3} + \frac{c - b - 1}{3T^3}, \ldots
\]
Application of Lemma 1 in the appendix therefore shows that

\[ f(t) = \frac{1}{T} \left\{ 1 + \sum_{n=0}^{\infty} D_n \left( \frac{w}{A} \right)^{n/3} \right\}, \]

where

\[ D_1 = \gamma_1, \quad D_2 = \frac{1}{2} \gamma_1^2 + \gamma_2, \quad D_3 = \frac{1}{6} \gamma_1^3 + \gamma_1 \gamma_2 + \gamma_3, \ldots. \]

Then, from (3.2) we obtain the result

\[ f(t) \frac{dt}{dw} = \sum_{m=0}^{\infty} \hat{B}_m(\alpha) w^{(m-2)/3}, \]

valid near \( w = 0 \) \((t = t_d)\), where the first three contributory coefficients are\(^3\)

\[ \hat{B}_0(\alpha) = \frac{1}{3 A^{1/3} T}, \quad \hat{B}_1(\alpha) = \frac{1}{9 A^{2/3} T} \left\{ 3 \left( \frac{c-b-1}{T} + \frac{b-1-\kappa a}{t_d} \right) - \frac{2 B}{A} \right\}, \]

\[ \hat{B}_3(\alpha) = -\frac{1}{243 A^{4/3} T} \left\{ \frac{4}{A^3} (35 B^3 - 63 A B C + 27 A^2 D) - \frac{81}{A^6} (B^2 - A C) D_1 + \frac{54 B D_2}{A} - 81 D_3 \right\}. \]

From this we find the expansion

\[ F \left( \frac{a - \lambda, b + \lambda}{c + i \alpha \lambda}, \frac{z}{T} \right) \sim \frac{\Gamma(c + i \alpha \lambda) \Gamma(1 + b - c + \lambda \beta)}{\Gamma(b + \lambda)} \frac{e^{\lambda \psi(t_d)}}{\pi} \]

\[ \times \sum_{m=0}^{\infty} \hat{B}_m(\alpha) \frac{\Gamma(\frac{3}{4} m + \frac{1}{2})}{\lambda (m+1)^{1/3}} \sin \pi \left( \frac{1}{4} m + \frac{1}{2} \right) \quad (\lambda \to +\infty). \]

The complexity of the higher contributory coefficients \( \hat{B}_m(\alpha) \) \((m \geq 4)\) is such that there appears to be little value in their presentation, although in specific cases with numerical values for \( a, b, c \) and \( \alpha \) it would be quite feasible to continue the inversion process to higher order. It can be verified with some effort that when \( a = 0, b = c = 1 \) we have \( \hat{T} = T \) and the coefficients in (3.8) reduce to those given in (3.5).

4. Application to the expansion of the Legendre functions

From (1.3) and (1.4) we have the Legendre functions of degree \( \lambda \) and order \(-i \alpha \lambda\), where \( \alpha > 0, \lambda > 0 \), given by

\[ P^{\pm i \alpha \lambda}_\lambda(x) = \frac{1}{\Gamma(1 + i \alpha \lambda)} \left( \frac{x - 1}{x + 1} \right)^{i \alpha \lambda/2} F_\alpha(\lambda; \frac{1}{2} - \frac{1}{2} x), \]

\[ e^{-\pi \alpha \lambda} Q^{\pm i \alpha \lambda}_\lambda(x) = \frac{\Gamma(-i \alpha \lambda)}{2} \left( \frac{x - 1}{x + 1} \right)^{i \alpha \lambda/2} F_\alpha(\lambda; \frac{1}{2} - \frac{1}{2} x) \]

\[ + \frac{\Gamma(i \alpha \lambda) \Gamma(1 + \lambda \beta)}{2 \Gamma(1 + \lambda \beta)} \left( \frac{x - 1}{x + 1} \right)^{-i \alpha \lambda/2} F_{-\alpha}(\lambda; \frac{1}{2} - \frac{1}{2} x), \]

where \( F_\alpha(\lambda; z) \) is defined in (1.2) and \( F_{-\alpha}(\lambda; \frac{1}{2} - \frac{1}{2} x) \) is given by the conjugate relation (2.1). These functions are defined\(^4\) in the complex \( x \)-plane cut along \((-\infty, 1]\). The expansions for \( P^{\pm i \alpha \lambda}_\lambda(x) \) and \( Q^{\pm i \alpha \lambda}_\lambda(x) \) then follow from that of \( F_\alpha(\lambda; z) \) given in (2.12) and (2.13).

In the special case \( x = \sqrt{1 + \alpha^2} \), the argument of \( F^{\pm \alpha}_\lambda(\lambda; \frac{1}{2} - \frac{1}{2} x) \) is equal to

\[ z_d^{\pm} = \frac{1}{2}(1 - \sqrt{1 + \alpha^2}) \]

\(^3\)We omit the coefficient \( \hat{B}_2(\alpha) \) as this is not required.

\(^4\)In Mathematica they are obtained numerically by use of the ‘type-3’ Legendre functions.
in (2.8). This corresponds to the coincidence of the two saddle points associated with the integral for \( F_\alpha(\lambda; z) \). From the expansion in (3.6), it therefore follows that
\[
P_\lambda^{-ia\lambda}(\sqrt{1 + \alpha^2}) \sim -\frac{\Gamma(1 + \lambda\beta)}{\Gamma(1 + \lambda)} \left( \frac{\sqrt{1 + \alpha^2} - 1}{\sqrt{1 + \alpha^2} + 1} \right)^{ia\lambda/2} e^{\lambda\psi(t_d)} \frac{S(\lambda; \alpha)}{\pi} \tag{4.3}
\]
and
\[
Q_\lambda^{-ia\lambda}(\sqrt{1 + \alpha^2}) \sim \frac{e^{\pi\alpha\lambda}}{\sinh\pi\alpha\lambda} \left( \frac{\sqrt{1 + \alpha^2} - 1}{\sqrt{1 + \alpha^2} + 1} \right)^{ia\lambda/2} e^{\lambda\psi(t_d)} S(\lambda; \alpha) \tag{4.4}
\]
as \( \lambda \to +\infty \), where
\[
S(\lambda; \alpha) := \sum_{m=0}^{\infty} \frac{B_m(\alpha)\Gamma\left(\frac{1}{3}m + \frac{1}{3}\right)}{\lambda^{(m+1)/3}} \sin\pi\left(\frac{1}{3}m + \frac{1}{3}\right),
\]
t_d is given in (3.1) and \( \psi(t) \) is defined in (2.5).

The expansions for the functions with positive imaginary order then follow from
\[
P_\lambda^{ia\lambda}(\sqrt{1 + \alpha^2}) = \overline{P_\lambda^{-ia\lambda}(\sqrt{1 + \alpha^2})}
\]
and [11, (14.9.14), (14.3.10)]
\[
Q_\lambda^{ia\lambda}(\sqrt{1 + \alpha^2}) = \frac{\Gamma(1 + \lambda\beta)}{\Gamma(1 + \lambda\beta)} e^{-2\pi\alpha\lambda} Q_\lambda^{-ia\lambda}(\sqrt{1 + \alpha^2}).
\]

Appendix: The expansion of \( G_\alpha(\lambda) \)

In this appendix we consider the expansion of \( G_\alpha(\lambda) \) in (2.6) in inverse powers of \( \lambda \). This is given for completeness as the main asymptotic problem under consideration is the large-\( \lambda \) expansion of the integral appearing in (2.4).

It is sufficient to consider \( \alpha > 0 \) since the value of \( G_\alpha(\lambda) \) for \( \alpha < 0 \) is given by its conjugate (when \( \lambda > 0 \)). We use the well-known expansion for \( \log \Gamma(z) \) as \( |z| \to \infty \) is [11, p. 141]
\[
\log \Gamma(z) \sim (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} \quad (|\arg z| < \pi),
\]
where \( B_{2k} \) denote the even-order Bernoulli numbers. In addition, we have the following lemma [2]:

**Lemma 1.** Let \( S(x) = \sum_{n=1}^{\infty} a_n x^{-n} \) as \( x \to \infty \) be a given expansion. Then the composition \( \exp[S(x)] \) has the asymptotic expansion of the following form\(^5\)
\[
\exp[S(x)] \sim \sum_{n=0}^{\infty} b_n x^{-n} \quad (x \to \infty)
\]
where
\[
b_0 = 1, \quad b_n = \frac{1}{n} \sum_{k=1}^{n} ka_kb_{n-k} \quad (n \geq 1).
\]

Then it follows that
\[
\log \left( \frac{G_\alpha(\lambda)}{ia\beta\lambda} \right) = \log \frac{\Gamma(ia\lambda)\Gamma(\lambda\beta)}{G_\alpha(\lambda)}
\]
\[
\sim \frac{1}{2} \log \frac{2\pi}{\lambda} + (ia\lambda - \frac{1}{2}) \log \lambda + (\lambda\beta - \frac{1}{2}) \log (1 - i\alpha) + \sum_{k=1}^{\infty} \frac{A_k}{\lambda^{2k-1}}
\]
\(^5\)The coefficients \( b_n \) can also be expressed in terms of the complete Bell polynomial \( B_n \) in the form \( b_n = B_n(ia_1, 2ia_2, \ldots, nia_n)/n! \).
as $\lambda \to +\infty$, where

$$A_k := \frac{B_{2k}}{2k(2k-1)} \left\{ \frac{1}{(i\alpha)^{2k-1}} + \frac{1}{(1-i\alpha)^{2k-1}} - 1 \right\}.$$

Some straightforward algebra and application of Lemma 1 then produces

$$G_\alpha(\lambda) \sim \sqrt{2\pi\lambda\alpha} \left( 1 + \alpha^2 \right)^{\frac{1}{4}} e^{-\lambda(\frac{1}{2}\pi + \phi)} \exp\left\{ \sum_{k=1}^{\infty} \frac{A_k}{\lambda^{2k-1}} \right\}$$

$$\sim \sqrt{2\pi\lambda\alpha} \left( 1 + \alpha^2 \right)^{\frac{1}{4}} e^{-\lambda(\frac{1}{2}\pi + \phi)} \left\{ 1 + \sum_{k=1}^{\infty} \frac{E_k}{\lambda^k} \right\} \quad (\lambda \to +\infty) \quad (A.1)$$

where

$$\Phi := \lambda\alpha \log\left( \frac{\alpha}{\sqrt{1 + \alpha^2}} \right) - (\lambda + \frac{1}{2})\phi + \frac{1}{4}\pi$$

with $\phi$ defined in (2.2). The first few coefficients $E_k$ are

$$E_1 = A_1, \quad E_2 = \frac{1}{2} A_2^2, \quad E_3 = \frac{1}{6} A_1^3 + A_2, \quad E_4 = \frac{1}{24} A_1^4 + A_1 A_2,$$

$$E_5 = \frac{1}{120} A_1^5 + \frac{1}{2} A_1^2 A_2 + A_3, \quad E_6 = \frac{1}{720} A_1^6 + \frac{1}{6} A_1^3 A_2 + \frac{1}{4} A_2^2 + A_1 A_3, \ldots .$$

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