FROM FLOWS OF $\Lambda$ FLEMING-VIOT PROCESSES TO LOOKDOWN PROCESSES VIA FLOWS OF PARTITIONS

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May 31, 2013

Abstract

The goal of this paper is to unify the stochastic flow of bridges and the lookdown process, which are two approaches to construct the $\Lambda$ Fleming-Viot process along with its genealogy. We first provide a detailed classification of the long-time behaviours of the $\Lambda$ Fleming-Viot. In particular we identify conditions on the measure $\Lambda$ ensuring the existence of a sequence of so-called Eves, generalising the primitive Eve of Bertoin and Le Gall. From a stochastic flow of bridges, when the $\Lambda$ Fleming-Viot admits a sequence of Eves, we then extract a genealogical structure that we call stochastic flow of partitions and which turns out to be a new formulation of the lookdown graph in terms of partitions of integers.

MSC 2010 subject classifications: Primary 60J25; Secondary 60G09, 92D25.

Keywords and phrases: Stochastic flow, Fleming-Viot process, Lookdown process, Partition-valued process, Coalescent, Exchangeable bridge.

1 Introduction

We consider an infinite population where each individual possesses a genetic type, taken to be a point in $[0, 1]$. As time passes the frequencies of the genetic types within the population evolve according to a Markov process $(\rho_t, t \geq 0)$ completely characterised by a finite measure $\Lambda$ on $[0, 1]$ so that the process is usually called the $\Lambda$ Fleming-Viot. It has been introduced by Bertoin and Le Gall in [6], and implicitly by Donnelly and Kurtz in [14]. It takes its values in the set of probability measures on $[0, 1]$, starts from the Lebesgue measure on $[0, 1]$ and evolves through elementary reproduction events that can be informally described as follows. At rate $\nu(du) := u^{-2}\Lambda(du)$ a parent (uniformly chosen among the population) reproduces: a fraction $u$ of individuals dies out and is replaced by individuals with the same type as the parent. It is well-known that the genealogy of the underlying population is given by the so-called $\Lambda$ coalescent [24, 25]. This process takes its values in the set of partitions of integers and can be easily constructed using the consistency of the restrictions of the partitions. To the contrary, there is no obvious way to construct a $\Lambda$ Fleming-Viot. The main objective of this paper is to unify two constructions, namely the lookdown representation of Donnelly and Kurtz [14] and the stochastic flow of bridges of Bertoin and Le Gall [6]. To achieve this goal, we start with a study of the asymptotic properties of the $\Lambda$ Fleming-Viot, which is interesting and useful in its own right. Then we rely on these results to define a lookdown representation pathwise from a flow of bridges.

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1.1 Main results on the asymptotic properties of the \( \Lambda \) Fleming-Viot

As a consequence of the informal definition above, at any given time \( t \geq 0 \) the random measure \( \rho_t \) has an atomic part and a Lebesgue part, namely

\[
\rho_t(dx) = \sum_{i \geq 1} a_i \delta_{x_i}(dx) + \left( 1 - \sum_{i \geq 1} a_i \right) dx
\]

where the \( x_i \)'s lie in \([0, 1]\) and the \( a_i \)'s are non-negative r.v. whose sum is lower than or equal to 1. When this sum is lower than 1 (and so the Lebesgue part of \( \rho_t \) is positive), we say that \( \rho_t \) has dust. Informally it means that at time \( t \), infinitely many individuals do not share their ancestor with anyone else: it should be seen as a residual of the initial condition \( \rho_0(dx) = dx \). Additionally a point \( x \in [0, 1] \) is called ancestral type if there exists \( t > 0 \) such that \( \rho_t(\{x\}) > 0 \). To a given ancestral type \( x \), we associate its emergence time \( \inf \{ t \geq 0 : \rho_t(\{x\}) > 0 \} \) and its extinction time \( \sup \{ t > 0 : \rho_t(\{x\}) > 0 \} \). To avoid trivial reproduction events we assume that \( \Lambda(\{1\}) = 0 \) and we introduce

\[
\Psi(u) := \frac{\Lambda(\{0\})}{2} u^2 + \int_{(0,1)} (e^{-xu} - 1 + xu) \nu(dx), \quad \forall u \geq 0
\]

The behaviour of the \( \Lambda \) Fleming-Viot depends on the intensity of \( \Lambda \) near 0. We propose a classification into four regimes (going from sparse to intense \( \Lambda \)'s), all the stated properties hold almost surely:

1. **Discrete**: \( \int_{[0,1]} \nu(du) < \infty \). For all \( t > 0 \), \( \rho_t \) has dust and finitely many atoms. The ancestral types emerge at strictly positive times and never become extinct. Furthermore, only finitely many ancestral types emerge on every compact interval of time.

2. **Intensive w. dust**: \( \int_{[0,1]} \nu(du) = \infty \) but \( \int_{[0,1]} u \nu(du) < \infty \). For all \( t > 0 \), \( \rho_t \) has dust and infinitely many atoms. The only difference with the previous regime is that on every compact interval of time, infinitely many ancestral types emerge.

3. **Intensive \( \infty \)**: \( \int_{[0,1]} u \nu(du) = \infty \) and \( \int_{\infty} du \Psi(u) = \infty \). For all \( t > 0 \), \( \rho_t \) has no dust and infinitely many atoms. The ancestral types emerge immediately after time 0 and never become extinct.

4. **CDI**: \( \int_{\infty} du \Psi(u) < \infty \). For all \( t > 0 \), \( \rho_t \) has no dust and finitely many atoms. The ancestral types emerge immediately after time 0 and all of them, except one, become extinct in finite time. This phenomenon is usually called Coming Down from Infinity (CDI).

Bertoin and Le Gall proved in [6] that, without condition on the regime of \( \Lambda \), there exists a random point \( e \) uniformly distributed over \([0, 1]\) such that

\[
\rho_t(\{e\}) \overset{t \to \infty}{\longrightarrow} 1
\]

It means that an ancestral type, called the primitive Eve, fixes in the population. Notice that the fixation occurs in finite time iff \( \Lambda \) belongs to regime CDI. Our goal is to further the study of the asymptotic behaviour: does there exist another ancestral type with an overwhelming progeny among the population that does not descend from \( e \)? To formalise (and generalise) this idea we introduce the following definition.

**Definition 1.1** According to the regime of \( \Lambda \), we introduce the Eves as follows:
**Eves - persistent case.** In regimes DISCRETE, INTENSIVE w. DUST and INTENSIVE ∞, we say that the Λ Fleming-Viot admits an infinite sequence of Eves if there exists a collection \((e^i)_{i \geq 1}\) of r.v. such that almost surely for all \(i \geq 1\)

\[ \frac{\rho_t([e^i])}{\rho_t([0,1]\{e^1,\ldots,e^{i-1}\})} \to 1 \quad t \to \infty \]

**Eves - extinction case.** In regime CDI, we say that the Λ Fleming-Viot admits an infinite sequence of Eves if one can order the ancestral types by strictly decreasing extinction times, the sequence is then denoted by \((e^i)_{i \geq 1}\).

Notice that the Λ Fleming-Viot does not necessarily admit an infinite sequence of Eves: the definition relies on a particular asymptotic behaviour (depending on the regime) of the process. We now present results ensuring the existence / non-existence of the sequence of Eves.

**Theorem 1** Suppose that Λ belongs to:

- **Regime** DISCRETE, or
- **Regime** INTENSIVE w. DUST and fulfils

\[ -\int_{(0,1)} u \log u \nu(du) < \infty \]

then the Λ Fleming-Viot does not admit an infinite sequence of Eves.

Let us make some comments on this first result. In regime DISCRETE, we will prove that eventually only the frequency of the primitive Eve makes positive jumps (see Proposition 4.3) so that among the population that does not descend from \(e\) no ancestral type has an overwhelming progeny. In regime INTENSIVE w. DUST our \(u \log u\) condition is verified in particular by any measure Λ whose density is regularly varying at 0 with index \(1 - \alpha, \alpha \in (0,1)\). The proof of the theorem for that regime strongly relies on the lookdown representation: it is based on a fine study of the dust component.

In regime INTENSIVE ∞, the question of existence of an infinite sequence of Eves remains open except for the particular measure \(\Lambda(du) = du\) (this is the intensity measure of the Bolthausen-Sznitman coalescent [11]).

**Proposition 1.2** When \(\Lambda(du) = du\), the Λ Fleming-Viot admits an infinite sequence of Eves.

In regime CDI, the definition of the sequence of Eves requires distinct extinction times for the ancestral types. Therefore we set

\[ E := \{ \text{There exist two ancestral types that become extinct simultaneously} \} \quad (4) \]

On the complement of E, the extinction times are distinct so that the condition of Definition 1.1 is verified. Conversely on E, at least two ancestral types die out simultaneously and Definition 1.1 does not make sense. Therefore the question of existence of the sequence of Eves boils down to determining the probability of E. To the best of our knowledge, the question of simultaneous extinction of ancestral types has never been addressed.

**Theorem 2** Consider regime CDI. The event \(E\) is trivial, that is, \(\mathbb{P}(E) \in \{0,1\}\). If \(\Lambda(\{0\}) > 0\) or \(\Lambda(du) = f(u)du\) where \(f\) is a regularly varying function at \(0^+\) with index \(1 - \alpha\) with \(\alpha \in (1,2)\) then \(\mathbb{P}(E) = 0\) so that the Λ Fleming-Viot admits an infinite sequence of Eves.

Observe that the well-studied class of Beta\((2 - \alpha, \alpha)\) Fleming-Viot with \(\alpha \in (1,2)\) verifies the assumption of the theorem. Notice also that \(\Lambda(du) = \delta_0(du)\) corresponds to the celebrated standard Fleming-Viot process [17] whose genealogy is given by the Kingman coalescent [21]. We refer to Subsection 4.4 for conjectures on the existence of the sequence of Eves in the remaining cases.
1.2 The coupling between the flow of bridges and the lookdown representation

We recall briefly the construction of the $\Lambda$ Fleming-Viot process due to Bertoin and Le Gall [6, 7, 8]. They observed that the distribution function of $p_t$ is equal in law to the composition of two independent copies of the distribution functions of $\rho_s$ and $\rho_{t-s}$, for all $0 < s < t$. Using Kolmogorov’s extension theorem this yields the existence of a consistent collection $(F_{s,t}, -\infty < s \leq t < \infty)$ of distribution functions, called a stochastic flow of bridges, such that for all $s \in \mathbb{R}$ the process $(\rho_{s,t}, t \in [s, \infty))$ is a $\Lambda$ Fleming-Viot (where $\rho_{s,t}$ is the random probability measure with distribution function $F_{s,t}$). A stochastic flow of bridges is then a coupling of an infinity of $\Lambda$ Fleming-Viot processes.

From now on, we consider a measure $\Lambda$ such that the $\Lambda$ Fleming-Viot admits an infinite sequence of Eves. Notice that we do not restrict ourselves to the particular measures exhibited in our results above: we only require Definition 1.1 to be satisfied. From a flow of bridges, we define for all $s \in \mathbb{R}$ almost surely the Eves $(e^i_s)_{i \geq 1}$ of the $\Lambda$ Fleming-Viot process $(\rho_{s,t}, t \in [s, \infty))$. We call $(e^i_s, s \in \mathbb{R})_{i \geq 1}$ the Eves process. It is natural to look at the genealogical relationships between these Eves. For all $s < t$ let $\hat{\Pi}_{s,t}$ be the random partition of integers whose blocks, denoted $(\hat{\Pi}_{s,t}(i))_{i \geq 1}$ in the increasing order of their least element, are defined by

$$\hat{\Pi}_{s,t}(i) := \{ j \in \mathbb{N} : e^j_s \text{ descends from } e^i_s \} = \{ j \in \mathbb{N} : e^j_s \in [F_{s,t}(e^i_s)-, F_{s,t}(e^i_s)] \}$$

In words, we gather in a same block all the Eves at time $t$ who descend from a same Eve at time $s$. We have the following result that relies on the coagulation operator $\text{Coag}$ and the usual distance $d_{\neq}$ (see [4] or Subsection 2.1 for their definitions).

**Theorem 3** The collection of partitions $(\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty)$ enjoys the following properties:

- For every $r < s < t$, $\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})$ a.s. (cocycle property).
- $\hat{\Pi}_{s,t}$ is an exchangeable random partition whose law only depends on $t-s$. Furthermore, for any $s_1 < s_2 < \ldots < s_n$ the partitions $\hat{\Pi}_{s_1,s_2}, \hat{\Pi}_{s_2,s_3}, \ldots, \hat{\Pi}_{s_{n-1},s_n}$ are independent.
- $\hat{\Pi}_{0,0} = 0_{[\infty]} := \{ \{1\}, \{2\}, \ldots \}$ and $\hat{\Pi}_{0,t} \rightarrow 0_{[\infty]}$ in probability as $t \downarrow 0$, for the distance $d_{\neq}$.

Moreover for all $s \in \mathbb{R}$ the process $(\hat{\Pi}_{s-t,s}, t \geq 0)$ is a $\Lambda$ coalescent.

We make use of symbol "hat" in $\hat{\Pi}$ to emphasise the fact that the process goes forward-in-time, by opposition to the usual backward-in-time evolution of partition-valued processes that encode the genealogy. Notice also that $(\hat{\Pi}_{s-t,0}, t \geq 0)$ and $(\hat{\Pi}_{0,t}, t \geq 0)$ do not have the same distribution even if they share the same one-dimensional marginal laws.

We have identified pathwise from the flow of $\Lambda$ Fleming-Viot two objects: the Eves process and the flow of partitions. The next theorem shows that they are sufficient to recover completely the flow of $\Lambda$ Fleming-Viot. Introduce the notation $\varepsilon_s(\hat{\Pi}, (e^i_s)_{i \geq 1})$ to designate the measure-valued process

$$[s, \infty) \ni t \mapsto \sum_{i=1}^{\infty} |\hat{\Pi}_{s,t}(i)| \delta_{e^i_s}(dx) + \left(1 - \sum_{i=1}^{\infty} |\hat{\Pi}_{s,t}(i)|\right) dx$$

(5)

where $|\hat{\Pi}_{s,t}(i)|$ stands for the asymptotic frequency of the $i$-th block of the exchangeable partition $\hat{\Pi}_{s,t}$.

**Theorem 4** Suppose that the $\Lambda$ Fleming-Viot admits an infinite sequence of Eves. The flow of $\Lambda$ Fleming-Viot processes is uniquely decomposed into two random objects: the flow of partitions and the Eves process. More precisely, for all $s \in \mathbb{R}$ almost surely

$$\varepsilon_s(\hat{\Pi}, (e^i_s)_{i \geq 1}) = (\rho_{s,t}, t \in [s, \infty))$$

Furthermore if we are given a $\Lambda$ flow of partitions $\hat{\Pi}'$ and for each $s \in \mathbb{R}$ a sequence of $[0,1]$-valued r.v. $(\lambda_s(i))_{i \geq 1}$ such that almost surely $\varepsilon_s(\hat{\Pi}', (\lambda_s(i))_{i \geq 1}) = (\rho_{s,t}, t \in [s, \infty))$ then
• \((\text{Initial types})\) For all \(s \in \mathbb{R}\), almost surely \(\forall i \geq 1 \chi_s(i) = e_s^i.\)

• \((\text{Genealogy})\) Almost surely, \(\hat{\Pi}' = \hat{\Pi}.\)

This result provides a striking connection with another celebrated construction of the \(\Lambda\) Fleming-Viot: the so-called lookdown representation introduced by Donnelly and Kurtz [14]. Briefly, the lookdown representation consists in a countable particle system initially distributed as a sequence of i.i.d. uniform\([0, 1]\) r.v. and evolving according to a dynamics prescribed by the so-called lookdown graph. At any time, this particle system is exchangeable so that it admits an empirical measure: the collection of empirical measures forms a \(\Lambda\) Fleming-Viot. In Section 3 we will prove a correspondence between lookdown graphs and flows of partitions. The measure-valued process defined in Equation (5) will then appear as the empirical measure of the particle system defined from \(\hat{\Pi}\) (via the correspondence with lookdown graphs) and starting initially from the Eves \(e_s^i, i \geq 1.\) This yields a coupling between a flow of bridges and a lookdown process. Notice that we actually define a flow of lookdown processes since our coupling hold for every \(\Lambda\) Fleming-Viot encoded by the flow of bridges.

**Organisation of the paper** In Section 2, we recall basic definitions on \(\Lambda\) coalescents, stochastic flows of bridges and the lookdown representation. In Section 3, we introduce flows of partitions and make the connection with lookdown graphs. In Section 4, we study the asymptotic behaviour of the \(\Lambda\) Fleming-Viot and prove Theorems 1 and 2. In Section 5, we define the dynamic version of the Eves, that is, the Eves process and we prove Theorem 3. Section 6 is devoted to the proof of Theorem 4. Some technical proofs are postponed to Section 7.

## 2 Preliminaries

### 2.1 \(\Lambda\) coalescents

For all \(n \in \mathbb{N}\cup\{\infty\}\), we denote by \(\mathcal{P}_n\) the set of all partitions of \([n] := \{1, 2, \ldots, n\}\). The set \(\mathcal{P}_\infty\) is equipped with the distance \(d_{\mathcal{P}}\) defined by

\[
\forall \pi, \pi' \in \mathcal{P}_\infty, \quad d_{\mathcal{P}}(\pi, \pi') = 2^{-i} \iff i = \sup\{j \in \mathbb{N} : \pi[j] = \pi'[j]\}
\]

where \(\pi[j]\) is the restriction of \(\pi\) to \([j]\). The metric space \((\mathcal{P}_\infty, d_{\mathcal{P}})\) is compact.

For all \(i \geq 1\), we denote by \(\pi(i)\) the \(i\)-th block of a given partition \(\pi \in \mathcal{P}_\infty\), where the blocks are in the increasing order of their least element. Furthermore, for all \(i \geq 1\), we introduce the asymptotic frequency of the \(i\)-th block of \(\pi\) as

\[
|\pi(i)| := \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} 1_{j \in \pi(i)}
\]

when the limit on the r.h.s. exists.

We define the coagulation operator \(\text{Coag} : \mathcal{P}_\infty \times \mathcal{P}_\infty \to \mathcal{P}_\infty\) as follows. For any elements \(\pi, \pi' \in \mathcal{P}_\infty\), \(\text{Coag}(\pi, \pi')\) is the partition whose blocks are given by

\[
\text{Coag}(\pi, \pi')(i) = \bigcup_{j \in \pi'(i)} \pi(j)
\]

for every \(i \in \mathbb{N}\). This is a Lipschitz-continuous operator and we have

\[
\text{Coag}(\pi, \text{Coag}(\pi', \pi'')) = \text{Coag}(\text{Coag}(\pi, \pi'), \pi'')
\]
for any elements $\pi, \pi', \pi'' \in \mathcal{P}_\infty$, see Section 4.2 in [4] for further details.

Consider a finite measure $\Lambda$ on $[0, 1]$ and let $\nu(du) := u^{-2}\Lambda(du)$. A $\Lambda$ coalescent (see [24, 25]) is a Markov process $(\Pi_t, t \geq 0)$ on $\mathcal{P}_\infty$ started from the partition $0_{[\infty]} := \{1\}$ and such that, for each integer $n \geq 2$, its restriction $(\Pi^{[n]}_t, t \geq 0)$ to $\mathcal{P}_n$ is a continuous time Markov chain that evolves as follows. Fix $t \geq 0$ and $m \in [n]$ and suppose that $\Pi^{[n]}_t$ has $m$ blocks: then any $p$ of them merge at rate

$$\lambda_{m,p} = \int_0^1 u^p(1-u)^{m-p}\nu(du)$$

for all $p \in \{2, \ldots, m\}$. In other words, $\Pi^{[n]}_t$ jumps to $\text{Coag}(\Pi^{[n]}_t, \pi)$ at rate $\lambda_{m,p}$ whenever $\pi \in \mathcal{P}_m$ is composed of $m - p$ singleton blocks and one block with $p$ elements.

From now on, we will systematically assume that $\Lambda(\{1\}) = 0$ to avoid trivial behaviour. Indeed an atom at 1 induces coalescence events merging all the blocks at once. Pitman [24] showed that the number of blocks $#\Pi_t$ is either finite for all time $t > 0$ with probability one or infinite for all time $t > 0$ with probability one. We call the first phenomenon coming down from infinity. Schweinsberg [26] found a necessary and sufficient condition on $\Lambda$ ensuring the coming down from infinity, and later on Bertoin and Le Gall [8] showed that Schweinsberg’s condition is equivalent with $\int_0^\infty du/\Psi(u) < \infty$. The class of measures $\Lambda$ verifying this identity forms regime CDI.

Another important dichotomy concerns the existence of singleton blocks, that is, blocks of $\Pi_t$ with only one element. Pitman [24] showed that the number of singleton blocks in $\Pi_t$ is infinite at all time $t > 0$ with probability one if $\int_{(0,1)} u\nu(du) < \infty$ (DISCRETE and INTENSIVE w. DUST), otherwise (INTENSIVE $\infty$ and CDI) it is null at all time $t > 0$ with probability one. The set of singleton blocks is usually called the dust of the partition.

Finally Freeman [19] showed that in regimes DISCRETE and INTENSIVE w. DUST, in addition to the dust, there is a finite / infinite number of non-singleton blocks according as $\int_{(0,1)} \nu(du)$ is finite / infinite.

Let us provide some important examples. The $\Lambda$ coalescents obtained with

$$\Lambda(du) = \frac{u^{a-1}(1-u)^{b-1}}{\text{Beta}(a,b)}\, du, \quad a, b > 0$$

where Beta$(a,b)$ is Euler Beta function, are called Beta$(a,b)$ coalescents. The particular case where $a = 2 - \alpha, b = \alpha$ with $\alpha \in (0, 2)$ has received much attention those last ten years [2, 3, 9]. The Beta$(2 - \alpha, \alpha)$ coalescent belongs to regime INTENSIVE w. DUST when $\alpha \in (0, 1)$. When $\alpha = 1$, then $\Lambda(du) = du$ and the process is usually called the Bolthausen-Sznitman coalescent [11]: it belongs to regime INTENSIVE $\infty$. When $\alpha \in (1, 2)$, the Beta$(2 - \alpha, \alpha)$ coalescent is in regime CDI. Finally the Kingman coalescent [21] $\Lambda(du) = \delta_0(du)$ is recovered when $\alpha \to 2$.

### 2.2 Stochastic flows of bridges

Informally the $\Lambda$ coalescent merges a fraction $u$ of the current lineages (i.e. blocks) at rate $\nu(du)$. If there were a process describing the corresponding evolution forward-in-time, then at rate $\nu(du)$ a parent would give birth to a fraction $u$ of the total population. To give a precise definition to this forward-in-time process, Bertoin and Le Gall introduced the stochastic flow of bridges.

A bridge is a non-decreasing càdlàg process $F = (F(r), r \in [0, 1])$ with exchangeable increments and such that $F(0) = 0, F(1) = 1$. Kallenberg [20] showed that for any bridge $F$, there exists a sequence of non-negative r.v. $(a_i)_{i \in \mathbb{N}}$ with $a_1 \geq a_2 \geq \ldots \geq 0$, and $\sum_{i=1}^\infty a_i \leq 1$, and a sequence of i.i.d.
uniform \([0, 1]\) r.v. \((U_i)_{i \in \mathbb{N}}\) independent of the sequence \((a_i)_{i \in \mathbb{N}}\) such that a.s. for all \(x \in [0, 1]\),

\[
F(x) = \sum_{i=1}^{\infty} a_i \mathbb{1}_{\{U_i \leq x\}} + \left(1 - \sum_{i=1}^{\infty} a_i\right)x
\]

From any bridge \(F\) and any sequence \((V_p)_{p \geq 1}\) of i.i.d. uniform random variables on \([0, 1]\) independent of \(F\), one can define a random partition \(\pi = \pi(F; (V_p)_{p \geq 1})\) thanks to the following equivalence relation

\[
i \sim j \iff F^{-1}(V_i) = F^{-1}(V_j)
\]

where \(F^{-1}\) denotes the càdlàg inverse of \(F\). As a consequence, there is a one-to-one correspondence between the jumps of \(F\) and the non-singleton blocks of \(\pi\). Moreover \(\pi\) has dust iff \(F\) has a drift component.

In [6] Bertoin and Le Gall defined a consistent collection of bridges in order to obtain, using the above construction, a consistent collection of random partitions.

**Definition 2.1** A flow of bridges is a collection \((F_{s,t}, -\infty < s \leq t < \infty)\) of bridges such that :

- For every \(r < s < t\), \(F_{r,t} = F_{s,t} \circ F_{r,s}\) a.s. (cocycle property).
- The law of \(F_{s,t}\) only depends on \(t - s\). Furthermore, if \(s_1 < s_2 < \ldots < s_n\) the bridges \(F_{s_1,s_2}, F_{s_2,s_3}, \ldots, F_{s_{n-1},s_n}\) are independent.
- \(F_{0,0} = Id\) and \(F_{0,t} \rightarrow Id\) in probability as \(t \downarrow 0\).

Note that symbol \(\circ\) stands for the composition operator of real-valued functions. Given a sequence of i.i.d. uniform \([0, 1]\) r.v. \((V_p)_{p \geq 1}\), independent of a flow of bridges \((F_{s,t}, -\infty < s \leq t < \infty)\), they proved that the process \(t \mapsto \Pi_t := \pi(F_{-t,0}, (V_p)_{p \geq 1})\) is an exchangeable coalescent, see Theorem 1 in [6]. In the particular case of a \(\Lambda\) coalescent, \((F_{s,t}, -\infty < s \leq t < \infty)\) is called a \(\Lambda\) flow of bridges. Hence at any time \(t \geq 0\), the dust and the number of non-singleton blocks of \(\Pi_t\) have the same distribution as the drift and the number of jumps of \(F_{0,t}\). However this property holds only for the one-dimensional marginals: the processes \((F_{0,t}, t \geq 0)\) and \((F_{-t,0}, t \geq 0)\) do not have the same distribution. From the former, Bertoin and Le Gall introduced the \(\Lambda\) Fleming-Viot process. Denote by \(\mathcal{M}_1\) the space of all probability measures on \([0, 1]\), equipped with its weak topology. The \(\Lambda\) Fleming-Viot process is defined as the \(\mathcal{M}_1\)-valued process \((\rho_{0,t}, t \in [0, \infty))\) where

\[
\forall x \in [0, 1], \quad \rho_{0,t}(\{x\}) = F_{0,t}(x)
\]

This Markov process has a Feller semigroup which is characterized by a martingale problem (based on a duality argument with the \(\Lambda\) coalescent) that we do not recall here. Note that, from the stationarity property verified by a flow of bridges, for every \(s \in \mathbb{R}\) the process \((\rho_{s,t}, t \in [s, \infty))\) is also a \(\Lambda\) Fleming-Viot process, where the probability measures \(\rho_{s,t}\) are defined from the bridges \(F_{s,t}\) as above. Since the semigroup is Feller, each process \((\rho_{s,t}, t \in [s, \infty))\) admits a càdlàg modification still denoted \((\rho_{s,t}, t \in [s, \infty))\) for simplicity. The collection of bridges associated to these càdlàg modifications are also still denoted \((F_{s,t}, s \leq t)\).

**Remark 2.2** A stochastic flow of bridges \(F\) is constructed (except in simple cases) via the Kolmogorov extension theorem and, therefore, does not enjoy any regularity property of its trajectories. In particular the cocycle property does not necessarily hold simultaneously for all triplets \(r < s < t\) on a same event of probability one. Consequently the flow of partitions that we will define pathwise from the flow of bridges will suffer from this lack of regularity and will need to be regularised.
2.3 The lookdown process

The lookdown process was introduced by Donnelly and Kurtz in [13, 14] and generalised to the case of \( \Xi \) coalescents in [10]. Let us first define some notation. For each \( n \in \{2, 3, \ldots, \infty\} \), let \( S_n \) be the subset of \( \{0, 1\}^n \) whose elements have at least two coordinates \( 1 \leq i < j \leq n \) equal to 1. For a subset \( p \) of \( \mathbb{R} \times S_\infty \), we denote by \( p|_{[s, t] \times S_n} \) the restriction to \( [s, t] \times S_n \) of the canonical projection of \( p \) on \( \mathbb{R} \times \{0, 1\}^n \).

**Definition 2.3** A deterministic lookdown graph is a countable subset \( p \) of \( \mathbb{R} \times S_\infty \) such that for all \( n \in \mathbb{N} \) and all \( s \leq t \), \( p|_{[s, t] \times S_n} \) has finitely many points.

In the lookdown representation, the population is composed of a countable collection of individuals: at any time each individual is located at a so-called level, taken to be an element of \( \mathbb{N} \). As time passes, the individuals change of levels with the constraint that they can only jump to a higher level. A deterministic lookdown graph should be seen as a collection of elementary reproduction events \( (t, v) \in \mathbb{R} \times S_\infty \), where \( t \) designates the reproduction time and \( v \) determines the levels that participate to this event. More precisely

\[
I_{t,v} := \{ i \geq 1 : v^i = 1 \}
\]

is the set of levels that participate to the reproduction event. If we fix \( s \in \mathbb{R} \) and if we consider a sequence \( (\xi_{s,i}(i))_{i \geq 1} \in [0, 1]^\mathbb{N} \) called initial types, then we define a particle system \( (\xi_{s,t}(1), \xi_{s,t}(2), \ldots), t \geq s \) with values in \([0, 1]^\mathbb{N}\) as follows:

- The initial values are given by \((\xi_{s,i}(i))_{i \geq 1}\).
- At any elementary reproduction event \((t, v) \in p \) with \( t > s \) we have

\[
\begin{align*}
\xi_{s,t}(i) &= \xi_{s,t-}(\min(I_{t,v})) \\
\xi_{s,t}(i) &= \xi_{s,t-}(i - (\#I_{t,v} \cap [i]) - 1) \lor 0
\end{align*}
\]

for all \( i \in I_{t,v} \) and for all \( i \notin I_{t,v} \) (8)

The interpretation is the following. At level \( \min(I_{t,v}) \) is located the parent of the reproduction event. At any level \( i \in I_{t,v} \) a new individual is born with the type \( \xi_{s,t-}(\min(I_{t,v})) \) of the parent. All the individuals alive at time \( t- \) are then redistributed, keeping the same order, on those levels that do not belong to \( I_{t,v} \) (see Figure 1). This particle system is well-defined because of the finiteness of the number of elementary reproduction events falling on any compact interval of time and inducing jumps for the \( n \) first particles (recall our hypothesis on \( p \)).

For every \( t \in [s, \infty) \) we set (under the condition that this object is well-defined)

\[
\Xi_{s,t}(.):= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{\xi_{s,t}(i)}(.)
\]

where the limit is taken in \( \mathcal{M}_1 \). This is the limiting empirical measure of \((\xi_{s,t}(i))_{i \geq 1}\). Of course, this limit does not always exist.

We now explain how one randomises the previous objects so that the collection of limiting empirical measures is almost surely well-defined and forms a \( \Lambda \) Fleming-Viot process. We take \( P \) as a Poisson point process on \( \mathbb{R} \times S_\infty \) with intensity measure \( dt \otimes (\mu_K + \mu_\Lambda) \) where \( \mu_K \) and \( \mu_\Lambda \) are defined by

\[
\mu_\Lambda(.) := \int_{(0,1)} \sigma_u(.) \nu(du) \ , \ \mu_K(.) := \Lambda(0) \sum_{1 \leq i < j} \delta_{s_{i,j}}(.)
\]
to with parameter Poisson point process Definition 2.4 individuals while for all \( \{ i \in \mathbb{N} : i \geq n \} \) admits a process of limiting empirical measures \( \hat{\Pi} \). A key ingredient in the proof of this result is the exchangeability of \( \hat{\Pi} \).

The corresponding flow of partitions, restricted to \( \{ 7 \} \), would be entirely defined by the partitions \( \Pi_{t_1-t_1} = \{ \{ 1 \}, \{ 2, 5, 7 \}, \{ 3 \}, \{ 4, 6 \} \} \), \( \Pi_{t_2-t_2} = \{ \{ 1, 2, 4 \}, \{ 3 \}, \{ 5 \}, \{ 6 \}, \{ 7 \} \} \), \( \Pi_{t_3-t_3} = \{ \{ 1 \}, \{ 2, 5 \}, \{ 3 \}, \{ 4 \}, \{ 6 \}, \{ 7 \} \} \) and so on.

where for all \( u \in (0, 1), \sigma_u(.) \) is the distribution on \( S_{\infty} \) of a sequence of i.i.d. Bernoulli random variables with parameter \( u \), and for all \( 1 \leq i < j \), \( s_{i,j} \) is the element of \( S_{\infty} \) that has only two coordinates equal to \( 1 \): \( i \) and \( j \). The measure \( \mu_\Lambda \) corresponds to reproduction events involving a positive proportion of individuals while \( \mu_K \) corresponds to reproduction events involving only two individuals at once.

**Definition 2.4** A lookdown graph associated with the measure \( \Lambda \) - or \( \Lambda \) lookdown graph in short - is a Poisson point process \( \mathcal{P} \) on \( \mathbb{R} \times S_{\infty} \) with intensity measure \( dt \otimes (\mu_K + \mu_\Lambda) \).

Remark that for \( \mathbb{P} \)-almost all \( \omega \), \( \mathcal{P}(\omega) \) is a deterministic lookdown graph. We fix a time \( s \in \mathbb{R} \) and a sequence of i.i.d. uniform \([0, 1]\) r.v. \( (\xi_{s,i}(i))_{i \geq 1} \). Using Equation (8) with \( p = \mathcal{P}(\omega) \), we get a particle system \( [s, \infty) \ni t \mapsto (\xi_{s,t}(i))_{i \geq 1} \).

**Theorem 5** (Donnelly-Kurtz [14]) With probability one the particle system \( [s, \infty) \ni t \mapsto (\xi_{s,t}(i))_{i \geq 1} \) admits a process of limiting empirical measures \( [s, \infty) \ni t \mapsto \Xi_{s,t} \) which is a càdlàg \( \Lambda \) Fleming-Viot process.

A key ingredient in the proof of this result is the exchangeability of \( (\xi_{s,t}(i))_{i \geq 1} \) at any given time \( t \in [s, \infty) \); it relies on a technical coupling with simple exchangeable models such as the Moran model. In the next section, our definition of the lookdown graph via flows of partitions will make exchangeability immediate.

**Remark 2.5** The lookdown graph can be used to define a collection of lookdown processes indexed by their starting time \( s \in \mathbb{R} \), if we are given for each of them a sequence of initial types. However the corresponding collection of measure-valued processes does not form a flow of \( \Lambda \) Fleming-Viot unless the initial types (taken at different times) are suitably coupled: this is the rôle played by the Eves process as we will see later on.

We end this subsection with a definition that will be useful in the sequel. Consider a lookdown process \( (\xi_t(i), t \in [0, \infty))_{i \geq 1} \) (for simplicity we write \( \xi_t = \xi_{0,t}, \Xi_t = \Xi_{0,t} \)).

**Figure 1**: A lookdown graph. Each arrow corresponds to an elementary reproduction event: the level carrying a dot reproduces on the levels carrying an ending arrow. For example, at time \( t_1 \), level 2 reproduces on levels 5 and 7 while former levels 5, 6 and 7 are pushed up to the next available levels. Note that only finitely many elementary reproduction events affecting the \( n \) first levels occur on any compact interval of time, for each integer \( n \). Furthermore, by tracing back the lineages from a given time, one obtains a \( \Lambda \) coalescent tree.
Definition 2.6 We define \( Y_t(i) \) as the lowest level at time \( t \) that carries the type \( \xi_0(i) \):
\[
Y_t(i) := \inf \{ j \geq 1 : \xi_t(j) = \xi_0(i) \}
\] (10)

3 Flows of partitions

From a stochastic flow of bridges, we will identify in Section 5 the following genealogical structure.

Definition 3.1 A stochastic flow of partitions is a collection of random partitions \( \hat{\Pi} = (\hat{\Pi}_{s,t}, -\infty < s \leq t < \infty) \) that enjoys the following properties:

- For every \( r < s < t \), almost surely \( \hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s}) \) (cocycle property).
- \( \hat{\Pi}_{s,t} \) is an exchangeable random partition whose law only depends on \( t - s \). Furthermore, for any \( s_1 < s_2 < \ldots < s_n \) the partitions \( \hat{\Pi}_{s_1,s_2}, \hat{\Pi}_{s_2,s_3}, \ldots, \hat{\Pi}_{s_{n-1},s_n} \) are independent.
- \( \hat{\Pi}_{0,0} = O[\infty] \) and \( \hat{\Pi}_{0,t} \to O[\infty] \) in probability as \( t \downarrow 0 \), for the distance \( d_P \).

Moreover, when the process \( (\hat{\Pi}_{-t,0}, t \geq 0) \) is a \( \Lambda \) coalescent, we say that \( \hat{\Pi} \) is associated with the measure \( \Lambda \) or, in short, is a \( \Lambda \) flow of partitions.

Remark 3.2 It is worth noting that flows of partitions have been introduced separately by Foucart [18] as a population model for \( \Xi \) Fleming-Viot processes with immigration.

The goal of this section is to show that \( \Lambda \) flows of partitions are equivalent to \( \Lambda \) lookdown graphs. To that end, we first reformulate the deterministic lookdown graph in terms of partitions: we call the corresponding object a deterministic flow of partitions, it enjoys many regularity properties. Second we define, thanks to a Poissonian procedure, a \( \Lambda \) flow of partitions whose trajectories are deterministic flows of partitions so that the equivalence with lookdown graph holds for each trajectory. Third we show that any \( \Lambda \) flow of partitions admits a modification whose trajectories are deterministic flows of partitions so that the equivalence with lookdown graphs holds for the modification.

3.1 Deterministic flows of partitions

Definition 3.3 A deterministic flow of partitions is a collection \((\hat{\pi}_{s,t}, -\infty < s \leq t < \infty)\) of partitions such that

- For all \( r < s < t \), \( \hat{\pi}_{r,t} = \text{Coag}(\hat{\pi}_{s,t}, \hat{\pi}_{r,s}) \) (cocycle property).
- For all \( s \in \mathbb{R} \), \( \lim_{r \uparrow s} \lim_{t \downarrow s} \hat{\pi}_{r,t} =: \hat{\pi}_{s-,s} = O[\infty] \) (left regularity).
- For all \( s \in \mathbb{R} \), \( \lim_{t \downarrow s} \hat{\pi}_{s,t} = \hat{\pi}_{s,s} = O[\infty] \) (right regularity).

Furthermore, if for all \( s \in \mathbb{R} \), \( \hat{\pi}_{s-,s} \) has at most one unique non-singleton block, then we say that \( \hat{\pi} \) is a deterministic flow of partitions without simultaneous mergers.

We begin with an elementary lemma. Recall that \( \pi^{[n]} \) designates the restriction to \([n]\) of a given partition \( \pi \in \mathcal{P}_\infty \).

Lemma 3.4 For all \( s \in \mathbb{R} \) and \( n \in \mathbb{N} \), there exists \( \epsilon > 0 \) such that

\[
\hat{\pi}_{r,t}^{[n]} = O_{[n]}, \quad \forall r, t \text{ s.t. } s \leq r \leq t < s + \epsilon
\]
\[
\hat{\pi}_{r,t}^{[n]} = O_{[n]}, \quad \forall r, t \text{ s.t. } s - \epsilon < r \leq t < s
\]
**Proof** This is a consequence of three facts: the left and right regularities of a deterministic flow of partitions, the cocycle property and the finiteness of the set $\mathcal{P}_n$. □

As a consequence, one can easily deduce that $\hat{\pi}_{s-r,s}$ is well defined for each $s \in \mathbb{R}$ (and thus, the last sentence of Definition 3.3 makes sense). Also for all $s \in \mathbb{R}$, $(\hat{\pi}_{s,t}, t \in [s, \infty))$ is a càdlàg $\mathcal{P}_\infty$-valued function and $(\hat{\pi}_{s-r,s}, r \in [0, \infty))$ is a càdlàg $\mathcal{P}_\infty$-valued function.

The purpose of the next proposition is to show that deterministic flows of partitions are related to deterministic lookdown graphs (see also Figure 1). We introduce, for each $n \in \mathbb{N} \cup \{\infty\}$, $\mathcal{P}_n^*$ as the subset of $\mathcal{P}_n$ whose elements have a unique non-singleton block. In particular, for any subset $I \subset \mathbb{N}$ with at least two elements, we denote by $1_I$ the element of $\mathcal{P}_\infty$ with a unique non-singleton block $I$. The key tool is the map

$$f_n : S_n \longrightarrow \mathcal{P}_n^*$$

that translates an elementary reproduction event in terms of partitions. Plainly, $f_n$ is a bijection from $S_n$ to $\mathcal{P}_n^*$.

**Proposition 3.5** There exists a one-to-one correspondence between the set of deterministic flows of partitions without simultaneous mergers and the set of deterministic lookdown graphs.

**Proof** Consider a deterministic lookdown graph $p$. For each $n \in \mathbb{N}$ and every $s < t$, let $(t_m, v_m)_{1 \leq m \leq q}$ denote the finitely many atoms of $p_{\mathcal{I}[s,t] \times S_n}$ in the increasing order of their time coordinates. We use the map $f_n$ to translate the reproduction events into partitions: $(t_m, f_n(v_m))_{1 \leq m \leq q}$. Subsequently we set

$$\hat{\pi}_{s,t} := \text{Coag}(f_n(v_1), \text{Coag}(f_n(v_{q-1}), \ldots, \text{Coag}(f_n(v_2), f_n(v_1)) \ldots))$$

and $\hat{\pi}_{s,s} := 0_{[n]}$. Obviously, the collection of partitions $(\hat{\pi}_{s,t}, n \in \mathbb{N})$ is compatible and defines by a projective limit a unique partition $\hat{\pi}_{s,t}$ such that its restriction to $[n]$ is $\hat{\pi}_{s,t}^n$ for all $n \in \mathbb{N}$. It is then straightforward to verify that the collection of partitions $(\hat{\pi}_{s,t}, -\infty < s \leq t < \infty)$ is a deterministic flow of partitions without simultaneous mergers.

Conversely consider a deterministic flow of partitions $\hat{\pi}$ without simultaneous mergers. We define the collection of its jumps

$$p := \bigcup_{s : \hat{\pi}_{s-s} \neq 0_{[n]}} \{(s, \hat{\pi}_{s-s}^{-1}(\hat{\pi}_{s-s}))\}$$

which is a subset of $\mathbb{R} \times S_\infty$. Lemma 3.4 ensures that its restriction to $\mathbb{R} \times S_n$ does not accumulate near any point in $\mathbb{R}$. Therefore, we conclude that $p$ is a deterministic lookdown graph.

Finally, using the fact that $f_n$ is a bijection for any $n \in \mathbb{N}$, it is easy to show that the composition of these two procedures is the identity. This entails the one-to-one correspondence. □

The interest of this correspondence is that the flow of partitions entirely and explicitly encodes the genealogical relationships of the lookdown graph. Indeed, consider a deterministic lookdown graph $p$ and let $\hat{\pi}$ be the deterministic flow of partitions associated via the one-to-one correspondence. From $p$ and a given sequence of (distinct) initial types $(\xi_{s,i}(i))_{i \geq 1}$, denote by $(\xi_{s,t}(i), t \in [s, \infty))$ the corresponding lookdown process starting at time $s \in \mathbb{R}$.

**Lemma 3.6** For all $t \in [s, \infty)$ and all $i, j \in \mathbb{N}$

$$\hat{\xi}_{s,t}(j) = \hat{\xi}_{s,s}(i) \iff j \in \hat{\pi}_{s,t}(i)$$

11
Proof Suppose that \( P_{[s,t] \times S} \) has only two points, say \((s_1, v_1)\) and \((s_2, v_2)\). The ancestral lineage of \( \xi_{s,t}(j) \) makes at most two jumps: at time \( s_2 \) from \( \xi_{s,t}(j) \) to \( \xi_{s_2,t}(i_1) \) (for a certain integer \( i_1 \)) and at time \( s_1 \) from \( \xi_{s_1,t}(i_1) \) to \( \xi_{s,s_1}(i) \) (for a certain integer \( i \)). Now observe that the transitions at times \( s_1, s_2 \) once translated in terms of partitions yield that \( j \in \hat{\pi}_{s_2,s_1}(i_1) \) and \( i_1 \in \hat{\pi}_{s_1,s}(i) \). By definition of \( \hat{\pi} \) we have \( \hat{\pi}_j[i] = \hat{\pi}_{s_1,s}(i) = \hat{\pi}_{s_2,t}(j) = 0 \) so that the cocycle property ensures that \( \hat{\pi}_{s,t}(i) = \text{Coag}(\hat{\pi}_{s_2,s_1}, \hat{\pi}_{s_1,s})(i) \). Plainly \( j \) belongs to \( \hat{\pi}_{s,t}(i) \). An easy recursion on the number of elementary reproduction events falling on \((s,t)\) ends the proof.

In words, this means that each block \( \hat{\pi}_{s,t}(i) \) is the set of individuals alive at time \( t \) whose ancestor at time \( s \) had type \( \xi_{s,t}(i) \), for every \( i \geq 1 \). If we suppose that at any time \( t \geq s \) and for any integer \( i \geq 1 \), the block \( \hat{\pi}_{s,t}(i) \) admits an asymptotic frequency denoted by \( |\hat{\pi}_{s,t}(i)| \) then we can introduce the measure-valued process \( \delta_s(\hat{\pi}, (\xi_{s,t}(i)))_{i \geq 1} \) as follows

\[
[s, \infty) \ni t \mapsto \sum_{i \geq 1} [\hat{\pi}_{s,t}(i)] |\delta_{s,t}(i)| (dx) + \left( 1 - \sum_{i \geq 1} |\hat{\pi}_{s,t}(i)| \right) dx \tag{11}
\]

3.2 Poissonian construction of a stochastic flow of partitions

Fix a finite measure \( \Lambda \) on \([0, 1]\) and consider a \( \Lambda \) lookdown graph \( \mathcal{P} \) (recall the definitions of Subsection 2.3). Note that for \( P\)-a.a. \( \omega \in \Omega \), the point collection \( \mathcal{P}(\omega)|_{[s,t] \times S} \) has finitely many points for all \( s < t \) and all \( n \geq 1 \). So without loss of generality, we can assume that it holds for all \( \omega \in \Omega \). Therefore, one can define a deterministic flow of partitions \( \hat{\Pi}^\mathcal{P}(\omega) \) using Proposition 3.5 with \( \mathcal{P}(\omega) \) for each \( \omega \in \Omega \).

**Proposition 3.7** The process \( \hat{\Pi}^\mathcal{P} \) is a \( \Lambda \) flow of partitions.

**Proof** The cocycle property is verified for each trajectory by construction. The independence and continuity properties can be easily obtained. Finally the Poissonian construction of coalescent processes (see [4]) ensures that \( (\hat{\Pi}^\mathcal{P}_{s,0}, t \geq 0) \) is a \( \Lambda \) coalescent.

Observe that the trajectories of the stochastic flow of partitions obtained from this Poisson point process are deterministic flows of partitions. Actually they enjoy several nice regularity properties as the following result shows (the proof is postponed to Section 7).

**Proposition 3.8** Let \( \hat{\Pi} = \hat{\Pi}^\mathcal{P} \). On a same event of probability one, the following holds true:

i) The trajectories of \( \hat{\Pi} \) are deterministic flows of partitions.

ii) (Regularity in frequencies) for every \( s \leq t \) the partitions \( \hat{\Pi}_{s,t}, \hat{\Pi}_{s,t}, \hat{\Pi}_{s,t} \) possess asymptotic frequencies and whenever \( s < t \) we have the following convergences for every integer \( i \geq 1 \)

\[
\lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s,t+\epsilon}(i)| = \lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s+t-\epsilon}(i)| = |\hat{\Pi}_{s,t}(i)|
\]

\[
\lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s,t}(i)| = |\hat{\Pi}_{s,t}(i)|
\]

Fix \( s \in \mathbb{R} \) and let \( (\xi_{s,s}(i))_{i \geq 1} \) be a sequence of i.i.d. uniform[0, 1]. The preceding proposition together with Theorem 5 ensures the following.

**Corollary 3.9** The process \( \delta_s(\hat{\Pi}, (\xi_{s,s}(i)))_{i \geq 1} \) of Equation (11) is well-defined. Additionally it coincides with the process (9) of limiting empirical measures of the lookdown process constructed from \( \mathcal{P} \) and \( (\xi_{s,s}(i))_{i \geq 1} \) from time \( s \) so that it is a càdlàg \( \Lambda \) Fleming-Viot.
Observe that we can define the particle system \((\xi_{s,t}(i))_{i \geq 1}\) at any given time \(t \geq s\) from the partition \(\hat{\Pi}_{s,t}\) and the initial types \((\xi_{s,s}(i))_{i \geq 1}\): the exchangeability of these two objects ensures that \((\xi_{s,t}(i))_{i \geq 1}\) is itself exchangeable (see for instance the proof of Theorem 2.1 in [4]).

### 3.3 Regularisation of a stochastic flow of partitions

We now consider a \(\Lambda\) flow of partitions \(\hat{\Pi}\) in the sense of Definition 3.1. Observe that its trajectories are not necessarily deterministic flows of partitions. Indeed, the cocycle property does not necessarily hold simultaneously for all triplets \(r < s < t\) on a same event of probability one (compare with Remark 2.2). Hence the one-to-one correspondence cannot be directly applied to obtain lookdown graphs. Below we prove the existence of a modification \(\tilde{\Pi}\) whose trajectories are genuine deterministic flows of partitions. The reason that motivates this technical discussion is that we will identify in Section 5 a stochastic flow of partitions embedded into a flow of bridges from which we will construct a lookdown process.

**Proposition 3.10** Let \(\hat{\Pi}\) be a \(\Lambda\) flow of partitions. There exists a collection of partitions \(\tilde{\Pi}\) such that for all \(s \leq t\), almost surely \(\hat{\Pi}_{s,t} = \tilde{\Pi}_{s,t}\) and such that on a same event \(\Omega_{\hat{\Pi}}\) of probability one, all the trajectories \(\tilde{\Pi}(\omega)\) are deterministic flows of partitions.

Our strategy of proof is to restrict to the rational marginals of the flow of partitions \(\hat{\Pi}\), and to prove that they verify the properties of a deterministic flow of partitions up to an event of probability zero. Then, we take right and left limits on this object and prove that we recover a modification of the initial flow. The proof is postponed to Section 7.

For each \(\omega \in \Omega_{\hat{\Pi}}\), let \(P(\omega)\) be the deterministic lookdown graph obtained from \(\tilde{\Pi}(\omega)\) by applying Proposition 3.5. On the complementary event, set any arbitrary values to \(P\). It is plain that \(P\) is a \(\Lambda\) lookdown graph.

**Remark 3.11** From a stochastic flow of partitions, we have been able to define a regularised modification. Note that this operation does not seem possible for a stochastic flow of bridges. Indeed, a key argument in our proof relies on the continuity of the coagulation operator whereas this property does not hold with the composition operator for bridges.

### 4 The \(\Lambda\) Fleming-Viot and its ancestral types

Let \((\rho_t, t \geq 0)\) be a \(\Lambda\) Fleming-Viot process assumed to be càdlàg (this Markov process enjoys the Feller property) and starting from the Lebesgue measure on \([0, 1]\). From the specific form taken by a bridge, for any given time \(t \geq 0\) the measure \(\rho_t\) possesses an atomic part - the jumps of the bridge - and a Lebesgue part - the drift of the bridge so that we recover (1). We will say that \(\rho_t\) has dust if its Lebesgue part is strictly positive. Moreover a point \(x \in [0, 1]\) is called ancestral type if there exists \(t > 0\) such that \(\rho_t(\{x\}) > 0\). We stress that the collection of ancestral types is at most countable. Indeed, in the lookdown representation, there is a countable number of initial types. Since the atomic support of the \(\Lambda\) Fleming-Viot process is included into the set of the initial types of its lookdown representation, the claim follows.

We now rely on the classification into four regimes presented in the introduction. We stress that the asserted facts can be easily obtained thanks to the duality with \(\Lambda\) coalescents, we also mention that this classification into four regimes for the \(\Lambda\) coalescent originally appears in a paper of Freeman [19]. The goal of this section is to prove the results exposed in the introduction on the existence / non-existence of the sequence of Eves of Definition 1.1. But before we achieve this program, we provide several properties of this sequence (when it exists) in order to give more intuition on the Eves. First we observe
that $e^i$ is always well-defined and coincides with the primitive Eve $e$ introduced by Bertoin and Le Gall in [6]. Second we stress that the Eves should be thought of as the ancestral types ordered by importance (in terms of frequency) among the population. In the following proposition, we establish a connection with the lookdown representation.

**Proposition 4.1** Let $\Pi$ be a $\Lambda$ flow of partitions and $(\xi_0(i))_{i \geq 1}$ be i.i.d. uniform $[0, 1]$ r.v. so that the lookdown process $\xi_0(\Pi, (\xi_0(i))_{i \geq 1})$ is a $\Lambda$ Fleming-Viot. Suppose that the $\Lambda$ Fleming-Viot admits an infinite sequence of Eves, then almost surely for each $i \geq 1$, $\xi_0(i) = e^i$.

**Proof** Eves - extinction case. It suffices to remark that both $(e^i_0)_{i \geq 1}$ and $(\xi_0(i))_{i \geq 1}$ are ordered by decreasing extinction times. Indeed in the lookdown representation, the extinction time of the initial type $\xi_0(i)$ corresponds to the hitting time of $+\infty$ by the line $X(i)$ defined in (2.6). Obviously this extinction time is infinite when $i = 1$ and decreases with $i$.

Eves - persistent case. Define $\rho_t(t \in [0, \infty)) := \xi_0(\Pi, (\xi_0(i))_{i \geq 1})$. For every $t \geq 0$, let $(\rho_t(i))_{i \geq 1}$ be the sequence $(\rho_t(\{e^i\}))_{i \geq 1}$ reordered by decreasing values. From the definition of the Eves, we deduce that for every $i \geq 1$

$$P\left(\forall j \in [i], \rho_t(j) = \rho_t(\{e^j\}) > 0\right) \xrightarrow{t \to \infty} 1$$

(12)

Besides, the very definition of the empirical measure of the lookdown representation ensures that for every $i \geq 1$, $\rho_t(\xi_0(j)) = \Pi_{0,t}(i)$ for all $t \geq 0$. Since $\Pi_{0,t}$ is exchangeable, we deduce that the sequence $(\rho_t(\{\xi_0(j)\}))_{i \geq 1}$ is a size-biased reordering of the mass of $\rho_t$ (see Section 2.1.3 in [4], notice that when $\rho_t$ has dust, the size-biased reordering assigns a probability equal to the dust coefficient to choose the mass 0). Henceforth for every $i \geq 1$

$$P\left(\forall j \in [i], \rho_t(\{\xi_0(j)\}) = \rho_t(j) > 0\right) = \mathbb{E}\left[\frac{\rho_t(1)}{1 - \rho_t(1)} \times \cdots \times \frac{\rho_t(i)}{1 - \sum_{j=1}^{i-1} \rho_t(j)}\right]$$

(13)

Combining (12), (13) and the definition of the Eves, we get for every $i \geq 1$

$$P\left(\forall j \in [i], \rho_t(\{\xi_0(j)\}) = \rho_t(\{e^j\})\right) \xrightarrow{t \to \infty} 1$$

So that for every $i \geq 1$

$$P\left(\rho_t(\{\xi_0(i)\}) = \rho_t(\{e^i\})\right) \xrightarrow{t \to \infty} 1$$

This last identity together with the definition of the Eves imply that $\xi_0(i) = e^i$ for every $i \geq 1$. 

Observe that even when the $\Lambda$ Fleming-Viot does not admit an infinite sequence of Eves, the first initial type $\xi_0(1)$ is always equal to $e$, the primitive Eve of Bertoin and Le Gall. We now determine the distribution of the sequence $(e^i)_{i \geq 1}$. In addition, we prove that this sequence of Eves is independent of its process of frequencies. This result allows one to decompose a $\Lambda$ Fleming-Viot process into two independent objects: the Eves and its process of frequencies.

**Proposition 4.2** Consider a measure $\Lambda$ such that the $\Lambda$ Fleming-Viot admits an infinite sequence of Eves. Then $(e^i)_{i \geq 1}$ are i.i.d. uniform $[0, 1]$ independent of the sequence of processes $(\rho_t(\{e^i\}), t \in [0, \infty))_{i \geq 1}$.

**Proof** Denote by $\Phi$ the measurable map that associates to a $\Lambda$ Fleming-Viot process its sequence of Eves. In particular, we have $\Phi(\rho) = (e^i)_{i \geq 1}$. Now consider a sequence $(\xi_0(i))_{i \geq 1}$ of i.i.d. uniform $[0, 1]$ r.v., and an independent $\Lambda$ flow of partitions $\Pi$. Denote by $(\xi_t, t \geq 0) = \xi_0(\Pi, (\xi_0(i))_{i \geq 1})$ the $\Lambda$ Fleming-Viot defined from these objects (recall Corollary 3.9). Hence we can define its sequence of Eves $\Phi(\xi)$. From Proposition 4.1, we deduce that a.s.

$$\Phi(\xi) = (\xi_0(i))_{i \geq 1}$$
Therefore, using the fact that \((\rho_t, t \geq 0) \overset{(d)}{=} (\Xi_t, t \geq 0)\), we deduce that

\[
(\rho, \Phi(\rho)) \overset{(d)}{=} (\Xi, \Phi(\Xi))
\]  

(14)

This implies that \((e_0^i)_{i \geq 1} \overset{(d)}{=} (\xi_0(i))_{i \geq 1}\), and thus, it is a sequence of i.i.d. uniform \([0,1]\) r.v. Moreover, note that the asymptotic frequencies \((\xi_t(\{\xi_0(i)\}), t \geq 0)_{i \geq 1}\) only depend on the flow of partitions \(\Pi\), thus they are independent of the initial types \((\xi_0(i))_{i \geq 1}\). Using a similar identity in law as in Equation (14), we deduce that the sequence \((e_0^i)_{i \geq 1}\) is independent of \((\rho_t(\{e_0^i\}), t \in [0,\infty))_{i \geq 1}\).

We now examine each regime regarding the existence of this sequence of Eves.

4.1 Regimes Discrete and Intensive w. Dust

Proposition 4.3 Consider regime DISCRETE. There exists a random time \(T > 0\) after which the process \(t \mapsto \rho_t(\{e\})\) makes only positive jumps. In other terms, eventually the primitive Eve is the parent of all the elementary reproduction events.

Proof Define the Markov process \(X_t := \rho_t([0,1]\{e\}), t \geq 0\). We know that \(X_t \rightarrow 0\) as \(t \rightarrow \infty\) almost surely. This process evolves at the jump times of a Poisson point process on \([0,\infty) \times [0,1]\] with intensity \(dt \otimes \nu(du)\). Since we are in regime DISCRETE, these atoms are finitely many on any compact interval of time almost surely. We denote by \((t_i, u_i)_{i \in \mathbb{N}}\) these atoms in the increasing order of their time coordinate. Subsequently we write \(X_i := X_{t_i}, i \geq 1\) for simplicity. Now introduce the following random variables: \(\tau_0 := \inf\{i \geq 0 : X_i < 1/2\}\) and for each \(n \geq 0\), \(r_n := \inf\{i > \tau_n : X_i - X_{i-1} > 0\}\) and \(\tau_{n+1} := \inf\{i > r_n : X_i < 1/2\}\). In words, \(\tau_0\) is the first time the process \(X\) hits \((0, 1/2)\) and \(r_0\) is the first time after \(\tau_0\) the process \(X\) makes a positive jump. Recursively \(\tau_{n+1}\) is the first time after \(r_n\) the process \(X\) hits again \((0, 1/2)\), and \(r_{n+1}\) is the time of the next positive jump. Set \(F_k := \mathcal{F}(X_i, 0 \leq i \leq k)\). The proof of the proposition boils down to showing that eventually the sequence \((r_n)_{n \geq 1}\) equals \(+\infty\) almost surely. To that end, we first observe that for all \(n \geq 0\) almost surely

\[
\mathbb{P}(r_{n+1} = \infty | F_{r_{n+1}})1_{\{r_n < \infty\}} = \mathbb{P}(X \text{ makes only negative jumps after time } \tau_{n+1} | X_{\tau_{n+1}})1_{\{r_n < \infty\}} \\
\geq \mathbb{P}(X \text{ makes only negative jumps } | X_0 = 1/2)1_{\{r_n < \infty\}}
\]

Indeed, the probability that \(X\) makes only negative jumps decreases with its initial value. Consequently for all \(n \geq 0\)

\[
\mathbb{P}(r_n < \infty) \leq (1 - \mathbb{P}(X \text{ makes only negative jumps } | X_0 = 1/2))^{n+1}
\]

Hence it is sufficient to show that \(\mathbb{P}(X \text{ makes only negative jumps } | X_0 = 1/2)\) is strictly positive. We proceed as follows. First the probability that \(X\) makes a negative jump at time \(i \geq 1\), conditionally given \(X_{i-1}\) and \(u_i\), is equal to

\[
u_i + (1 - u_i)(1 - X_{i-1}) \geq 1 - X_{i-1}
\]

This can be easily derived from the lookdown representation: either the parent is the level 1 (this occurs with probability \(u_i\)), or it is a level above. By exchangeability this level above has a probability \(1 - X_{i-1}\) to belong to the progeny of \(e\). The asserted formula follows. Second, the transition at a negative jump is given by

\[
X_i = (1 - u_i)X_{i-1}
\]
Consequently the probability that $X$, starting from $1/2$, makes only negative jumps is equal to

$$
\mathbb{E}\left[\prod_{i \geq 1} \left(u_i + (1 - u_i)(1 - \frac{1}{2} \prod_{j \leq i - 1} (1 - u_j))\right)\right] \geq \mathbb{E}\left[\prod_{i \geq 1} \left(1 - \frac{1}{2} \prod_{j \leq i - 1} (1 - u_j)\right)\right] \\
\geq \mathbb{E}\left[\exp\left(\sum_{i \geq 1} \log \left(1 - \frac{1}{2} \prod_{j \leq i - 1} (1 - u_j)\right)\right)\right]
$$

Let us now prove that the (negative) r.v. inside the exponential is finite almost surely. Its expectation is given by

$$
\mathbb{E}\left[\sum_{i \geq 1} \log \left(1 - \frac{1}{2} \prod_{j \leq i - 1} (1 - u_j)\right)\right] \geq \mathbb{E}\left[-\frac{c}{2} \sum_{i \geq 1} \prod_{j \leq i - 1} (1 - u_j)\right] \\
\geq -\frac{c}{2} \sum_{i \geq 1} \mathbb{E}\left[\prod_{j \leq i - 1} (1 - u_j)\right]
$$

where $c$ is a positive constant such that $\log(1 - y) \geq -cy$ for all $y \in [0, 1/2]$. Since the $u_i$’s are i.i.d. with distribution $\nu(du)$ (normalised by its mass), we get

$$
\mathbb{E}\left[\prod_{j \leq i - 1} (1 - u_j)\right] = \left(\frac{\int_{(0,1)}(1-u)\nu(du)}{\int_{(0,1)}\nu(du)}\right)^{i-1}
$$

This ensures that the series at the second line of (15) is finite, which in turn implies the finiteness of the negative r.v. inside the exponential above.

Henceforth the probability that $X$ makes only negative jumps when starting from any value $x \in [0, 1/2]$ is greater than a strictly positive constant uniformly in $x$. This concludes the proof. 

**Proposition 4.4** Consider the lookdown representation in regime INTENSIVE w. DUST, and assume that

$$
-\int_{(0,1)} u \log u \nu(du) < \infty
$$

Then almost surely infinitely many initial types never get a positive frequency.

**Proof** Let us prove that almost surely there exists an integer $n \geq 2$ such that the line $Y(n)$ (recall Definition 2.6) is never chosen as the parent of any elementary reproduction event. This will imply the almost sure existence of one initial type that never gets a positive frequency. Then it will be easy to extend our proof to show the almost sure existence of $k$ lines fulfilling this property, for any $k \geq 1$, which is exactly the statement of the proposition. The proof is divided into four steps. The main idea is to show that any line $Y(n)$ is stochastically greater than a line $\mathcal{Y}$ whose probability of never being chosen as the parent of any elementary reproduction event is strictly positive.

**Step 1.** Observe that conditional on the occurrence of a reproduction event $(t, u) \in \mathbb{R}_+ \times (0, 1)$, the probability that level $n$ is the parent is equal to $u(1 - u)^{n-1}$. This probability is of course decreasing with $n$, thus it confirms the intuition that the higher the line $Y(n)$ is, the higher its probability of never being chosen is.

**Step 2.** Let $u^*$ be a real value in $(0, 1)$ such that $\Lambda(u^*, 1) > 0$. There exist $a \in (1, \infty)$, $\epsilon > 0$ and $n_0 \in \mathbb{N}$ verifying

$$
0 < 1 - a^{-1}\left(1 - \epsilon - \frac{2}{n_0}\right) < u^* , \quad a(4\epsilon^2n_0)^{-1} < 1 \quad , \quad an_0 \geq n_0 + 1
$$
To simplify notation we set $u' := 1 - a^{-1} \left( 1 - \epsilon - \frac{2}{n_0} \right)$. We stress that the rate at which $Y(n)$ jumps from $n$ to a level above $\lfloor an \rfloor$ is greater than a constant $r > 0$ uniformly in $n \geq n_0$. To see this, consider only the elementary reproduction events $(t, u)$ that involve levels 1 and 2. Denote by $Q(u, n)$ the number of levels among $3, 4, \ldots, \lfloor an \rfloor - 1$ that do not participate to the event $(t, u)$. This is a binomial r.v. with $\lfloor an \rfloor - 3$ trials and probability of success $1 - u$. The line located at $n$ at time $t$− jumps to a level above (or equal to) $\lfloor an \rfloor$ at time $t$ if and only if the number of levels between 1 and $\lfloor an \rfloor - 1$ that do not participate to $(t, u)$ is lower than (or equal to) $n - 2$ (see Equation (8)). A simple calculation then yields

$$P(Q(u, n) > n - 2) \leq \frac{\mathbb{V}(Q(u, n))}{(n - 2 - \mathbb{E}(Q(u, n)))^2} \leq \frac{a}{4e^2 n}, \quad \forall n \geq n_0, \forall u \in (u', 1)$$

Consequently for all $n \geq n_0$, the rate at which $Y(n)$ jumps from $n$ to a level above (or equal to) $\lfloor an \rfloor$ is greater than

$$\int_{(u', 1)} u^2 P(Q(u, n) \leq n - 2) \nu(du) \geq \left( 1 - \frac{a}{4e^2 n_0} \right) \Lambda((u', 1)) =: r > 0$$

This shows that the line $Y(n)$ is stochastically above the line $\mathcal{Y}$ that starts at $n_0$, and only evolves at the jump times $t_k$, $k \geq 1$ of a Poisson process with intensity $r dt$. At these jump times, we have the following deterministic jumps $Y_{t_{k+1}} = \lfloor a Y_{t_k} \rfloor$ so that we deduce the existence of a real value $\hat{a} > 1$ such that for all $k \geq 0$, $Y_{t_k} \geq \hat{a}^k$ (implicitly $t_0 := 0$). Notice finally that $\mathcal{Y}$ is not a "real" line of the lookdown graph: its jump times do not coincide with elementary reproduction events.

*Step 3.* Fix $n \geq 2$, define $b(n) := \int_{(0, 1)} u(1 - u)^{n-1} \nu(du)$ and observe that

$$b(n) \leq \int_{(0, n^{\frac{1}{2}})} u \nu(du) + (1 - n^{\frac{1}{2}})^{n-1} \int_{(0, 1)} u \nu(du)$$

(16)

For all $k \geq 0$, conditional on $t_{k+1} - t_k$ the probability that $\mathcal{Y}$ is the parent of no elementary reproduction event on $[t_k, t_{k+1}]$ is equal to

$$\exp \left( - (t_{k+1} - t_k) b(Y_k) \right)$$

Therefore, the probability that $\mathcal{Y}$ is never chosen as the parent is given by

$$q = \prod_{k=0}^{\infty} \mathbb{E}\left[ \exp \left( - (t_{k+1} - t_k) b(Y_k) \right) \right] = \exp \left( \sum_{k=0}^{\infty} \log \left( \frac{r}{r + b(Y_k)} \right) \right)$$

Then observe that $q$ is strictly positive iff $\sum_{k=0}^{\infty} b(Y_k) < \infty$. Recall the bound (16) obtained for $b(n)$, that $b(n)$ is decreasing with $n$ and that $Y_{t_k} \geq \hat{a}^k$ for all $k \geq 0$. It is easy to deduce that $q$ is strictly positive if

$$\sum_{k=0}^{\infty} \int_{(0, \hat{a}^{-\frac{1}{2}})} u \nu(du) < \infty$$

Then observe that this last quantity is equal to

$$\sum_{k=0}^{\infty} (k + 1) \int_{(\hat{a}^{-\frac{k+1}{2}}, \hat{a}^{-\frac{1}{2}})} u \nu(du) = - \frac{2}{\log \hat{a}} \sum_{k=0}^{\infty} \log \left( \hat{a}^{-\frac{k+1}{2}} \right) \int_{(\hat{a}^{-\frac{k+1}{2}}, \hat{a}^{-\frac{1}{2}})} u \nu(du)$$

which is finite since $- \int_{(0, 1)} u \log u \nu(du) < \infty$.

*Step 4.* We start at time 0 with the line $Y(n_0)$. Either it is never chosen as the parent and we are done (this
occurs with probability greater than \( q \), or at a certain random time \( \tau_1 \) it is the parent of an elementary reproduction event. We know that there exists a level \( n' > n_0 \) such that \( Y(n') \) has never been chosen as the parent on \([0, \tau_1] \) (indeed the proportion of dust in \( \rho_t \) is strictly positive). Call \( n_1 := Y_{\tau_1}(n') > n_0 \).

Using the Markov property at time \( \tau_1 \), we consider the lookdown graph shifted by \( \tau_1 \) and the line \( Y(n_1) \) in this shifted lookdown graph. This line has a probability to be never chosen as the parent greater than \( q \). We repeat recursively this procedure: it stops after a finite number of steps almost surely.

\[\blacksquare\]

**Proof of Theorem 1.** In regime \textsc{Discrete}, after time \( T \) the frequency of the ancestral types different from \( e \) make only negative jumps. Therefore for any pair \( x \neq y \) of ancestral types different from \( e \), the ratio \( \rho_t(\{x\})/\rho_t(\{y\}) \) is constant after time \( T \). Consequently there does not exist a sequence of Eves. In regime \textsc{Intensive} \textsc{Dust} under the \( u \log u \) condition, almost surely there exists a level \( n \) in the lookdown representation which never reproduces. Consequently \( \Xi_i(\{\xi_0(n)\}) = 0 \) for all \( t \geq 0 \). Recall from Proposition 4.1 that the Eves, if they exist, are the initial types of the lookdown representation. Obviously level \( n \) cannot be an Eve, and therefore there does not exist an infinite sequence of Eves.

\[\blacksquare\]

### 4.2 Regime \textsc{Intensive} \textsc{Infinite} and the Bolthausen-Sznitman case

When \( \Lambda \) is the Lebesgue measure on \([0, 1] \), one obtains the celebrated Bolthausen-Sznitman coalescent \([11]\). Its \( \Lambda \) Fleming-Viot counterpart belongs to regime \textsc{Intensive} \textsc{Infinite}. The proof of Proposition 1.2 relies strongly on the connection with measure-valued branching processes obtained in \([5, 9]\). We refer to \([12, 16, 23]\) for further details on measure-valued branching processes.

**Proof of Proposition 1.2.** One can construct \( \rho \) by rescaling a measure-valued branching process \((m_t, t \geq 0)\) associated with the Neveu branching mechanism \( \Psi(u) = u \log u \) as follows:

\[
\rho_t(dx) := \frac{m_t(dx)}{m_t([0,1])}, \quad \forall t \geq 0
\]

This result was initially stated for the Bolthausen-Sznitman coalescent by Bertoin and Le Gall in \([5]\), and later on by Birkner et al. for the forward-in-time process in \([9]\). As a consequence of this connection, we deduce that there exists a r.v. \( e \) such that almost surely

\[
\frac{m_t(\{e\})}{m_t([0,1])} \to 1
\]

The branching property ensures that the restrictions of \( m \) to any two disjoint subintervals of \([0, 1]\) are independent. For each integer \( n \geq 1 \), we divide \([0, 1]\) into dyadic subintervals of the form

\[
[0, 2^{-n}), [2^{-n}, 2 \times 2^{-n}), \ldots, [1 - 2^{-n}, 1]
\]

and we consider the corresponding restrictions of \( m \). Obviously, for each subinterval \([i - 1)2^{-n}, i 2^{-n})\) there exists \( e(i, n) \) such that

\[
\frac{m_t(\{e(i, n)\})}{m_t(\{[i - 1)2^{-n}, i 2^{-n})\}) \to 1
\]

Necessarily \( m \) restricted to the union of two subintervals indexed by \( i \neq j \in [2^n] \) admits either \( e(i, n) \) or \( e(j, n) \) as an Eve and therefore

\[
\lim_{t \to \infty} \frac{m_t(\{e(i, n)\})}{m_t(\{e(j, n)\})} \in \{0, +\infty\}
\]

Hence one can order the \( e(i, n), i \in [2^n] \) by asymptotic sizes. Using the consistency of the restrictions when \( n \) varies, one gets the existence of a sequence \( \{e'(t)\}_{t \geq 1} \) fulfilling the formula of the statement.

\[\blacksquare\]

It is rather unfortunate that this simple argument does not apply to other measures in regime \textsc{Intensive} \textsc{Infinite}.

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4.3 Regime CDI and simultaneous extinction of types

We now focus on the coming down from infinity case, and consider the event defined in (4). The objective of what follows is to prove Theorem 2. This is done thanks to Lemma 4.5 and Proposition 4.6.

Lemma 4.5 Consider regime CDI. The event $E$ is trivial, that is, $\mathbb{P}(E) \in \{0, 1\}$.

Proof Consider a $\Lambda$ flow of partitions $\Pi$, let $(\xi_0(i))_{i \geq 1}$ be an independent sequence of i.i.d. uniform $[0, 1]$ r.v. and set $(\rho_t, t \geq 0) := \delta_{0}(\Pi, (\xi_0(i))_{i \geq 1})$. We know that $\rho$ is a $\Lambda$ Fleming-Viot process from Corollary 3.9. We stress that $E$ is independent of the initial types $(\xi_0(i))_{i \geq 1}$ and only depends on the flow of partitions. Thus, we introduce the filtration $\mathcal{F}$ as follows.

$$\forall t \geq 0, \; \mathcal{F}_t := \sigma\{\Pi_{r,s}, 0 \leq r \leq s \leq t\}$$

One easily remarks that $\mathcal{F}_{0+}$ is a trivial $\sigma$-field under $\mathbb{P}$. Set $d^i := \inf\{t \geq 0 : Y_t(i) = \infty\}$, that is, the death time of the $i$-th initial type in the lookdown representation (see Definition 2.6), which is a stopping time of the filtration $\mathcal{F}$. Since $d^i \downarrow 0$ almost surely as $i \to \infty$, we deduce that $\bigcap_{i \geq 1} \mathcal{F}_{d^i} = \mathcal{F}_{0+}$. For all $i \geq 1$, we define the following event

$$E_i := \{\text{There exist two ancestral types that become extinct simultaneously before time } d^i\}$$

and $E_\infty := \bigcap_{i \geq 1} E_i$. Clearly, $E_\infty \in \mathcal{F}_{0+}$ so it has probability 0 or 1 under $\mathbb{P}$.

Case 1 : $\mathbb{P}(E_\infty) = 1$ Since $E_\infty \subset E$, we deduce that $\mathbb{P}(E) = 1$.

Case 2 : $\mathbb{P}(E_\infty) = 0$ Suppose there exists $n \geq 1$ such that $\mathbb{P}(E_n) > 0$. It implies that there exists $i \geq n$ and $p > 0$ such that

$$\mathbb{P}\{(d^i = d^{i+1})\} = p$$

For all $k \geq i$, let

$$\tau_k := \inf\{t \geq 0 : Y_t(i) \geq k\}$$

which is a stopping-time of the filtration $\mathcal{F}$. Remark that

$$\{d^i = d^{i+1}\} = \{Y(i) \text{ and } Y(i + 1) \text{ reach } \infty \text{ simultaneously}\}$$

By applying the Markov property at time $\tau_k$ (and the fact that the distribution of the lookdown graph is invariant by shift in time), we deduce that

$$\mathbb{P}\{(d^i = d^{i+1})\} = \mathbb{P}\{(d^{Y_{\tau_k}(i)} = d^{Y_{\tau_k}(i+1)})\} \leq \mathbb{P}(E_k)$$

Hence, for all $k \geq i$, $\mathbb{P}(E_k) \geq p$. Since the events $E_k, k \geq i$ are nested, we deduce that $\mathbb{P}(E_\infty) \geq p$, which contradicts our assumption. This implies that for all $i \geq 1$

$$\mathbb{P}\{(d^i = d^{i+1})\} = 0$$

which in turn implies that $\mathbb{P}(E) = 0$.

This ends the proof. \hfill \blacksquare

For an important subclass of measures $\Lambda$ in regime CDI we are able to compute $\mathbb{P}(E)$. Recall that we call regularly varying at 0+ with index $\alpha$ any map $f : [0, 1] \to [0, \infty)$ such that $\forall a > 0, f(ax)/f(x) \to a^\alpha$ as $x \downarrow 0$. 

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Proposition 4.6 Suppose that either $\Lambda(du) = f(u)du$ where $f$ is a regularly varying function at $0+$ with index $1 - \alpha$ where $\alpha$ lies in $(1, 2)$, or $\Lambda(\{0\}) > 0$. Then $P(E) = 0$.

Proof Recall the function $\Psi$ from Equation (2). We introduce the map $t \mapsto v(t)$ as the unique solution of
\[
\int_{v(t)}^\infty \frac{du}{\Psi(u)} = t, \quad \forall t > 0
\]
This definition makes sense since we are in regime CDI. Proposition 15 in [1] ensures that for all $\epsilon \in (0, 1)$ we have
\[
P\left(\lim_{t \to 0} \frac{\# \rho_t}{\Psi(\epsilon t)} \geq \frac{1}{1 + \epsilon} : \lim_{t \to 0} \frac{\Psi(\epsilon t)}{\Psi(1 + \epsilon t)} \leq \frac{1}{1 - \epsilon}\right) = 1
\] (17)
where $\# \rho_t$ designates the number of atoms of $\rho_t$. Assume that $\Lambda(du) = f(u)du$ where $f$ is regularly varying at $0+$ with index $1 - \alpha$ (where $\alpha \in (1, 2)$) or that $\Lambda(\{0\}) > 0$ (in which case $\alpha$ is taken equal to 2). Then $\Psi$ is regularly varying at $+\infty$ with index $\alpha$ and $v$ is itself regularly varying at $0+$ with index $-1/(\alpha - 1)$ (see Subsection 7.3 for a proof of this fact). Consequently we have for all $\epsilon \in (0, 1)$
\[
v\left(1 + \frac{1 + \epsilon}{1 - \epsilon}\right) \sim v\left(1 + \frac{1 - \epsilon}{1 + \epsilon}\right) \sim v(t)
\]
Together with (17) this yields
\[
P\left(\lim_{t \to 0} \frac{\# \rho_t}{v(t)} = 1\right) = 1
\]
This forces the jumps of $t \mapsto \# \rho_t$ to be small near $0+$, More precisely, for all $c > 0$
\[
P\left(\lim_{t \to 0} \frac{\# \rho_{t+} - \# \rho_t}{\# \rho_t} < c\right) = 1
\] (18)
We now suppose that $P(E) = 1$ and exhibit a contradiction with the preceding equation using the look-down construction of $\rho$. We use the notation of Subsection 2.3: recall the r.v. $\xi_t, \Xi_t$ and $Y_t(i)$ therein. From $P(E) = 1$, we deduce that the processes $Y(2)$ and $Y(3)$ have a probability $\eta > 0$ to reach $+\infty$ simultaneously. We introduce for every $n \geq 1$ the stopping time
\[
\tau_n := \inf \left\{ t \geq 0 : \Xi_t([0, 1]\{\xi_0(1)\}) \leq \frac{1}{n^2}\right\}
\]
which is finite almost surely since the primitive Eve $e = \xi_0(1)$ fixes. Our strategy is to show that $Y_{\tau_n}(3) - Y_{\tau_n}(2)$ becomes large as $n$ goes to infinity. To that end, we introduce $Z_{\tau_n}$ as the first level above $Y_{\tau_n}(2)$ which is not of type $\xi_0(1)$. Necessarily $Z_{\tau_n}$ is either of type $\xi_0(2)$ or is equal to $Y_{\tau_n}(3)$. In both cases it is located at a level lower than or equal to $Y_{\tau_n}(3)$. Proposition 3.1 in [14] ensures that the sequence $(\xi_{\tau'(i)})_{i \geq 1}$ is exchangeable with empirical measure $\Xi_{\tau'}$ as soon as $\tau'$ is a stopping time in the filtration of $\Xi$. This result can be easily extended to show that $(\xi_{\tau'(i)})_{i \geq 2}$ is exchangeable with empirical measure $\Xi_{\tau'}$, whenever $\tau'$ is a stopping time in the filtration of the pair $(\Xi, \Xi(\{\xi_0(1)\}))$. We deduce that the sequence $(\xi_{\tau_{\tau_n}(i)})_{i \geq 2}$ is exchangeable with empirical measure $\Xi_{\tau_n}$ and, in particular, the law of $(Y_{\tau_n}(2), Z_{\tau_n})$ is characterized as follows. Let $G$ and $G'$ be two r.v. distributed, conditionally given $p = \Xi_{\tau_n}([0, 1]\{\xi_0(1)\})$, as two independent geometric random variables with parameter $p$ that is
\[
\forall k \geq 1, P(G = k \mid p) = (1 - p)^{k-1}p
\]
Then \((Y_{\tau_n}(2), Z_{\tau_n})\) has the same law as \((1 + G, 1 + G + G')\). A simple calculation yields for all \(c > 0\)
\[
\mathbb{P}\left(\frac{G'}{1 + G} \geq c; G \geq n\right) = \mathbb{E} \left[ \sum_{k=n}^{\infty} \sum_{l=c(k+1)}^{\infty} p^2 (1-p)^{k+l-2} \right]
\]
\[
= \mathbb{E} \left[ \frac{p(1-p)^{c-2+n(1+c)}}{1 - (1-p)^{1+c}} \right]
\]
Recall that almost surely \(p \leq 1/n^2\). From the dominated convergence theorem, we get
\[
\mathbb{P}\left(\frac{G'}{1 + G} \geq c; G \geq n\right) \xrightarrow{n \to \infty} \frac{1}{1 + c}
\]
and therefore
\[
\mathbb{P}\left(\frac{Y_{\tau_n}(3) - Y_{\tau_n}(2)}{Y_{\tau_n}(2)} \geq c; Y_{\tau_n}(2) \geq n\right) \geq \mathbb{P}\left(\frac{Z_{\tau_n} - Y_{\tau_n}(2)}{Y_{\tau_n}(2)} \geq c; Y_{\tau_n}(2) \geq n\right) \xrightarrow{n \to \infty} \frac{1}{1 + c} \tag{19}
\]
Take \(c > 0\) such that \((1 + c)^{-1} + \eta > 1\) and call \(A_n\) the event where:
- \(Y(2)\) and \(Y(3)\) reach \(+\infty\) simultaneously.
- \(\frac{Y_{\tau_n}(3) - Y_{\tau_n}(2)}{Y_{\tau_n}(2)} \geq c\).
- \(Y_{\tau_n}(2) \geq n\).

Using (19), we deduce that for \(q = ((1 + c)^{-1} + \eta - 1)/2\) there exists \(n_0 \in \mathbb{N}\) such that for any \(n \geq n_0\) we have \(\mathbb{P}(A_n) > q\). From the Markov property at time \(\tau_n\) and the invariance of the lookdown graph by shift in time, this implies that with probability at least \(q\) there exists \(N \geq n\) such that \(Y((1 + c)N)\) and \(Y(N)\) reach \(+\infty\) simultaneously. Recall that the simultaneous blow-up of \(Y((1 + c)N)\) and \(Y(N)\) is equivalent with a jump of \#\(\Xi\) from (at least) \((1 + c)N\) to (at most) \(N - 1\). Since \(n \geq n_0\) can be chosen arbitrarily large, we find a contradiction with (18). \(\blacksquare\)

### 4.4 Open questions

Consider regime \textsc{Intensive w. Dust} and recall the function \(\Psi\) from (2). In [15], it is shown that the measure-valued branching process with branching mechanism \(\Psi\) (we refer to [16, 23] for a definition of this object) has a residual dust component when \(t\) tends to infinity iff \(-\int_{(0,1)} u \log u \nu(du) < \infty\). However when this \(u \log u\) condition is not fulfilled the frequencies in the population when \(t\) goes to infinity are of comparable order and the measure-valued branching process does not admit an infinite sequence of Eves. As branching processes and \(\Lambda\) Fleming-Viot processes present many similarities [2, 8, 9], it is natural to expect the following behaviour.

**Conjecture 4.7** In regime \textsc{Intensive w. Dust} when \(-\int_{(0,1)} u \log u \nu(du) = \infty\), every initial type of the lookdown representation gets a positive frequency at a certain time. However there does not exist an infinite sequence of Eves.

In regime \textsc{Intensive \(\infty\)}, if we could prove that \(Y(n+1)\) goes to infinity much faster than \(Y(n)\) then we could deduce the existence of a sequence of Eves. But the main difficulty lies in finding a precise upper bound for \(Y(n)\).

**Conjecture 4.8** In regime \textsc{Intensive \(\infty\)}, the \(\Lambda\) Fleming-Viot admits an infinite sequence of Eves without further condition on \(\Lambda\).
Finally in regime CDI, our proof relies strongly on the regular variation of the measure $\Lambda$. However, the similarity between $\Lambda$ Fleming-Viot and branching processes (for which the extinction times of two independent copies are distinct almost surely) suggests the following.

**Conjecture 4.9** In regime CDI, without further condition on $\Lambda$ we have $\mathbb{P}(E) = 0$ and thus, the $\Lambda$ Fleming-Viot always admits an infinite sequence of Eves.

## 5 Eves process and stochastic flow of partitions

In this section, we consider a $\Lambda$ flow of bridges $(F_{s,t}, -\infty < s \leq t < \infty)$ where $\Lambda$ either belongs to the Eves - persistent case or to the Eves - extinction case: we only assume that Definition 1.1 is verified and do not restrict ourselves to the particular measures $\Lambda$ exhibited in Theorem 2 and Proposition 1.2. Recall that for every $s \in \mathbb{R}$, $(\rho_{s,t}, t \in [s, \infty))$ is a càdlàg $\Lambda$ Fleming-Viot process (we use a modification of the initial flow of bridges, as explained in Subsection 2.2).

### 5.1 The Eves process

First, let us recall the dynamics of the primitive Eve in the flow of bridges $F$. For every $s \in \mathbb{R}$, $e^1_s$ is the r.v. in $[0, 1]$ such that

$$
\rho_{s,t}(\{e^1_s\}) \xrightarrow{t \to \infty} 1
$$

For each $s \in \mathbb{R}$, the definition actually makes sense on an event of probability 1. On the complementary event, set $e^1_s := 0$. Using the cocycle property, one gets $e^1_s \in [F_{0,s}(e^1_0) - , F_{0,s}(e^1_s)]$. This implies the following result.

**Proposition 5.1** (Bertoin-Le Gall [6]) For all $s > 0$, almost surely $e^1_0 = F_{0,s}^{-1}(e^1_s)$.

Fix a time $s \in \mathbb{R}$, and define the sequence $(e^1_j)_{j \geq 1}$ as the Eves of the $\Lambda$ Fleming-Viot $(\rho_{s,t}, t \in [s, \infty))$. Remark that this definition holds on an event $\Omega_s$ of probability 1. On the event $\Omega \setminus \Omega_s$ of zero probability, we set any arbitrary values to the sequence $(e^1_j)_{j \geq 1}$. Observe that the sequence $(e^1_j)_{j \geq 1}$ is independent of the past of the flow of bridges until time $s$. Indeed, this sequence only depends on bridges of the form $F_{s,t}, t \geq s$, and thanks to the independence property of the increments of a flow of bridges, we obtain the claimed independence. In the next subsection, we provide a generalisation of Proposition 5.1.

### 5.2 Key property and the flow of partitions

Fix a time $s > 0$. Our next proposition provides the relation between the sequences $(e^1_j)_{j \geq 1}$ and $(e^0_j)_{j \geq 1}$, it relies on a key property due to Bertoin and Le Gall (see Lemma 2 in [6]) which we now recall. Consider a sequence of i.i.d. uniform$[0,1]$ r.v. $(V_i)_{i \geq 1}$ and an independent bridge $B$, and recall the partition $\pi = \pi(B, (V_i)_{i \geq 1})$ as introduced in Subsection 2.2. Denote by $(A_j)_{j \geq 1}$ the blocks of $\pi$ ordered by their smallest element (those blocks are in finite number if $B$ has a finite number of jumps and no drift). For each $j \geq 1$, define $V'_j := B^{-1}(V_i)$ where $i$ is an arbitrary element of $A_j$. If there is a finite number of blocks in $\pi$, complete the sequence with independent uniform$[0, 1]$ random variables. The key property yields that the $(V'_j)_{j \geq 1}$ are i.i.d uniform$[0, 1]$ r.v. independent of $\pi$.

**Proposition 5.2** Introduce $\pi = \pi(F_{0,s}, (e^1_s)_{s \geq 1})$. The sequence $(e^0_j)_{j \geq 1}$ is independent of $\pi$ and almost surely for every block $A_j$ of $\pi$ and any arbitrary element $i \in A_j$, we have

$$
e^0_j = F_{0,s}^{-1}(e^1_{i,s})$$
We turn to the second assertion. We stress that for every follows.

This is a consequence of the cocycle property. Indeed

as the pre-images of the Eves at time 0 from this result. The corresponding partition (restricted to [4]) is \( \hat{\Pi}_{0,s} = \{ \{1, 3\}, \{2\}, \{4\} \} \).

The following technical proof is illustrated by Figure 2.

**Proof** There exists an event of probability 1 on which the definitions of the Eves at times 0 and \( s \) hold and we have \( F_{0,t} = F_{s,t} \circ F_{0,s} \) for all \( t \in [s, \infty) \cap \mathbb{Q} \). We work on this event until the end of the proof. Recall that \( (e_s^j)_{j \geq 0} \) is a sequence of i.i.d uniform \([0, 1]\) r.v. independent of the bridge \( F_{0,s} \). Those r.v. play the rôle of the \( (V_i)_{i \geq 1} \) in the key property presented above.

Let \( K \) be the random number of blocks of \( \pi \). This number is finite in the *Eves - extinction case* while it is infinite in the *Eves - persistent case*, almost surely. Denote by \( (A_j)_{1 \leq j \leq K} \) the blocks in the increasing order of their least element. Then, we can define a sequence of random variables \( V_j := F_{0,s}^{-1}(e_s^j) \) where \( i_j := \min(A_j) \), for all \( j \in [K] \). When \( K \) is finite, we set \( V_j := e_s^j \) for all \( j > K \). The key property ensures that the \( (V_j)_{j \in [K]} \) are independent of the partition \( \pi \). To prove the proposition, it remains to show that:

(i) \( (e_s^j)_{j > K} \) are i.i.d. uniform \([0, 1]\), independent of \( (e_s^j)_{j \in [K]} \) and of \( \pi \).

(ii) \( e_s^j = V_j \) for all \( j \in [K] \) a.s.

We start with the first assertion. In the *Eves - persistent case*, this assertion is trivial since \( K = \infty \) a.s. We consider the *Eves - extinction case*. Observe that \( K \) only depends on the values of the sequence \( (\rho_{0,s}(\{e_s^0\}))_{i \geq 1} \). We then deduce from Proposition 4.2 that \( (e_s^j)_{j > K} \) are i.i.d. uniform \([0, 1]\), independent of \( (e_s^j)_{j \in [K]} \). To prove the independence from \( \pi \), notice that \( \pi \) is a r.v. defined from the sequence \( (e_s^j)_{j \geq 1} \) and the bridge \( F_{0,s} \). The former only depends on the future of the flow after time \( s \), and the latter is obtained from \( (e_s^j)_{j \in [K]} \) and \( (\rho_{0,s}(\{e_s^0\}))_{i \in [K]} \): they are both independent of \( (e_s^j)_{j > K} \). Assertion (i) follows.

We turn to the second assertion. We stress that for every \( j \in [K] \), for all \( t \in [s, \infty) \cap \mathbb{Q} \) (see Figure 2)

\[
\rho_{s,t}(\{e_s^j\}) \leq \rho_{0,t}(\{V_j\}) \leq \rho_{s,t}([0, 1]\setminus\{e_s^1, \ldots, e_s^{j-1}\})
\]

This is a consequence of the cocycle property. Indeed \( V_j \) is the type at time 0 from which descends \( e_s^j \) but none of \( \{e_s^1, \ldots, e_s^{j-1}\} \). When considering the progenies of these individuals at time \( t \geq s \), we see

![Figure 2: An illustration of Proposition 5.2 in the Eves - extinction case. On the left, we recover the Eves at time 0 as the pre-images of the Eves at time s through the bridge F_{0,s}. On the right, the genealogical structure arising from this result. The corresponding partition (restricted to [4]) is \( \hat{\Pi}_{0,s} = \{ \{1, 3\}, \{2\}, \{4\} \} \).](image)
that the progeny of $e_i^j$ is included into that of $V_j^i$, and therefore into the progeny of $[0,1]\setminus \{e_s^1, \ldots, e_s^{i_j-1}\}$. The inequalities (20) follow. To finish the proof, we treat separately the two cases.

**Eves - extinction case.** The processes $t \mapsto \rho_{s,t}([0,1]\setminus \{e_s^1, \ldots, e_s^{i_j-1}\})$ and $t \mapsto \rho_{s,t}(\{e_s^j\})$ reach 0 at the same time. From (20) we conclude that $t \mapsto \rho_{0,t}(\{V_j^i\})$ reaches 0 also at that time. Hence, the collection $(V_j^i)_{j \in [K]}$ is ordered by decreasing extinction times. Finally observe that the $(e_0^j)_{j \in [K]}$ are a reordering of the $(V_j^i)_{j \in [K]}$ (indeed they are both the jump locations of $F_{0,s}$), we conclude that $V_j^i = e_0^j$ for all $j \geq 1$.

**Eves - persistent case.** From Proposition 5.1, we have $e_0^1 = V_1^i$. The very definition of the Eve property implies that for all $j \geq 1$

$$\lim_{t \to \infty} \frac{\rho_{s,t}(\{e_s^j\})}{\rho_{s,t}(\{e_s^1, \ldots, e_s^{i_j-1}\})} = 1$$

Together with (20) we get

$$\lim_{t \to \infty} \frac{\rho_{0,t}(\{V_j^i\})}{\rho_{0,t}(\{0,1\}\setminus \{V_1^i, \ldots, V_{i_j-1}\})} \geq \lim_{t \to \infty} \frac{\rho_{s,t}(\{e_s^j\})}{\rho_{s,t}(\{0,1\}\setminus \{e_s^1, \ldots, e_s^{i_j-1}\})} = 1$$

By uniqueness of the Eves we get $V_j^i = e_0^j$ for all $j \geq 1$.

This ends the proof of the proposition. □

**Remark 5.3** In the Eves - extinction case, the number of blocks $K$ of the partition $\pi(F_{0,s}, (e_i^j)_{i \geq 1})$ is finite a.s. The descendants of individuals $(e_0^j)_{j > K}$ have become extinct by time $s$. In the genealogical interpretation, $(e_0^j)_{1 \leq j \leq K}$ are the ancestors of the $K$ oldest families in the population alive at time $s$.

The preceding proposition motivates the introduction of the following random partitions

$$\hat{\Pi}_{s,t} := \pi(F_{s,t}, (e_i^j)_{i \geq 1}), \quad \forall s \leq t$$

(21)

These partitions provide the genealogical relationships between the Eves as explained in the introduction of this paper.

**Proof of Theorem 3.** The first requirement of Definition 3.1 is a consequence of Proposition 5.2. Indeed, the partition $\hat{\Pi}_{r,t}$ is obtained by applying the key property with the sequence $(e_i^j)_{i \geq 1}$ and the bridge $F_{r,t} = F_{s,t} \circ F_{r,s}$ a.s. Then, remark that $i$ and $j$ are in a same block of $\hat{\Pi}_{r,t}$ iff $F_{r,t}^{-1}(e_i^j) = F_{r,t}^{-1}(e_i^j)$; Call $k, k'$ the integers such that $F_{s,t}^{-1}(e_i^j) = e_k^s$ and $F_{s,t}^{-1}(e_i^j) = e_k'^s$; they are the indices of the blocks of $\hat{\Pi}_{s,t}$ containing $i$ and $j$ respectively. Subsequently $i$ and $j$ are in a same block of $\hat{\Pi}_{r,t}$ iff $k$ and $k'$ are in a same block of $\hat{\Pi}_{r,s}$. The cocycle property follows.

Let us prove the independence of the increments in the case $n = 2$, the general case is obtained by an easy induction. Fix $r < s < t$. Thanks to Proposition 5.2, we know that the sequence $(e_i^j)_{i \geq 1}$ is independent of the partition $\hat{\Pi}_{s,t}$. Additionally, the bridge $F_{r,s}$ is independent of the partition $\hat{\Pi}_{s,t}$ thanks to the independence property of the increments of a flow of bridges and the definition of the latter partition. Given that the partition $\hat{\Pi}_{r,s}$ depends only on the sequence $(e_i^j)_{i \geq 1}$ and the bridge $F_{r,s}$, we deduce the independence between $\hat{\Pi}_{r,s}$ and $\hat{\Pi}_{s,t}$. Furthermore, the fact that the distribution of $\hat{\Pi}_{s,t}$ only depends on $s - t$ is an immediate consequence of the stationarity of flows of bridges.

The convergence in probability of $\hat{\Pi}_{0,t} \to 0_{[\infty]}$ for the distance $d_\rho$ is a consequence of the next lemma.

Finally, since the flow of bridges $F$ is associated with the measure $\Lambda$, we deduce that $(\hat{\Pi}_{-t,0}, t \geq 0)$ is a $\Lambda$ coalescent using the result of Bertoin and Le Gall recalled in Subsection 2.2. □
Lemma 5.4 Consider a collection of bridges \((B_t)_{t \geq 0}\) and an independent sequence of i.i.d. uniform\([0, 1]\) random variables \((V_i)_{i \geq 1}\). The following conditions are equivalent

a) The exchangeable partition \(\pi(B_t, (V_i)_{i \geq 1})\) converges in probability to \(\varnothing_{[\infty]}\) for the distance \(d_{\mathcal{P}}\) as \(t \downarrow 0\).

b) The bridge \(B_t\) converges in probability to \(Id\) in the sense of Skorohod’s topology as \(t \downarrow 0\).

We postpone the proof of this lemma to Section 7.

6 Proof of Theorem 4

We have defined a stochastic flow of partitions \(\tilde{\Pi}\) pathwise from the flow of bridges \(F\). As \(F\) may have many irregularities, so may \(\tilde{\Pi}\). However, using the regularisation procedure of Subsection 3.3, one obtains a modification of the original flow \(\tilde{\Pi}\), still denoted \(\tilde{\Pi}\) for simplicity, such that its trajectories are deterministic flows of partitions, almost surely. Then, the collection of the jumps of \(\tilde{\Pi}\) defines a \(\Lambda\) lookdown graph as proved in Proposition 3.5. For all \(s \in \mathbb{R}\), set \((\Xi_{s,t}, t \in [s, \infty)) := \delta'_s(\tilde{\Pi}, (e'_i)_{i \geq 1})\). We prove Theorem 4 in two steps.

Proof of Theorem 4-Decomposition. Fix \(s \in \mathbb{R}\). Remark that both \((\Xi_{s,t}, t \in [s, \infty))\) and \((\rho_{s,t}, t \in [s, \infty))\) are càdlàg processes. Therefore, to prove that \(\Xi_{s,t} = \rho_{s,t}\), it is sufficient to prove that for every \(t \in [s, \infty)\), we have a.s. \(\Xi_{s,t} = \rho_{s,t}\). Consider a time \(t \in [s, \infty)\). From the definition of \(\Xi_{s,t}\) and thanks to Proposition 5.2, we have almost surely

\[
\rho_{s,t}(dx) = \sum_{i=1}^{\infty} |\Pi_{s,t}(i)| \delta_{e'_i}(dx) + \left(1 - \sum_{i=1}^{\infty} |\Pi_{s,t}(i)| \right) dx = \Xi_{s,t}(dx)
\]

The equality of the measure-valued processes follows. This ensures the decomposition statement of the theorem.

We now focus on the uniqueness statement of the theorem. Let \(\Pi'\) be a \(\Lambda\) flow of partitions. We assume that the trajectories of \(\Pi'\) are deterministic flows of partitions, otherwise the lookdown construction could not be applied to this object. But from the regularisation results of Subsection 3.3, this assumption can be made without loss of generality. For every \(s \in \mathbb{R}\), consider a sequence \((X_s(i))_{i \geq 1}\) of r.v. taking values in \([0, 1]\). We denote by \((X_{s,t}, t \in [s, \infty)) := \delta_s(\Pi', (X_s(i))_{i \geq 1})\) the process of limiting empirical measures of the corresponding lookdown process. We suppose that for every \(s \in \mathbb{R}\), almost surely \((X_{s,t}, t \in [s, \infty)) = (\rho_{s,t}, t \in [s, \infty))\).

Proof of Theorem 4-Uniqueness. Observe that the proof of Proposition 4.1 still holds even if we do not suppose the initial types of the lookdown process to be i.i.d. uniform\([0, 1]\) but we only impose the process of empirical measures to be a \(\Lambda\) Fleming-Viot. Therefore for each \(s \in \mathbb{R}\), we have almost surely for all \(i \geq 1, X_s(i) = e'_s\). It remains to prove that \(\Pi' = \tilde{\Pi}\) a.s.

There exists an event \(\Omega^\ast\) of probability 1 such that on this event, for all rational values \(s \leq t\) and all integer \(i \geq 1\) the following holds true

\[
\rho_{s,t}(\{e'_s\}) = |\Pi_{s,t}(i)| = |\Pi'_{s,t}(i)|
\]

(22)

Some properties will hold both for \(\Pi'\) and \(\tilde{\Pi}\), thus we will use the notation \(\tilde{\Pi}^\ast\) to designate indifferently any of them. The proof of the uniqueness statement of the theorem relies on the following result.
Claim The knowledge of \( (|\hat{\Pi}^{\infty}_{s,t}(i)|, s \leq t \in \mathbb{Q}^2, s \leq t < \infty) \) is sufficient to recover the flow of partitions \( (\hat{\Pi}^{\infty}_{s,t}, -\infty < s \leq t < \infty) \).

Applying this to both flows \( \hat{\Pi} \) and \( \hat{\Pi}' \) together with (22), we obtain that \( \hat{\Pi} = \hat{\Pi}' \) almost surely.

It remains to prove the claim. Proposition 3.8 implies that the asymptotic frequencies at any time can be obtained as right and left limits of those at rational times. Moreover the following lemma shows that the asymptotic frequencies are sufficient to recover all the jumps \( \hat{\Pi}^{\infty}_{s-,s} \), which themselves completely characterise the flow of partitions.

Lemma 6.1 Let \( I \) be a subset of \( \mathbb{N} \). The following assertions are equivalent

1) \( \hat{\Pi}^{\infty}_{s-,s} \) has a unique non-singleton block \( I \).

\[
\forall t \in (s, \infty), \: |\hat{\Pi}^{\infty}_{s,t}(i)| = \sum_{j \in I} |\hat{\Pi}^{\infty}_{s,t}(j)| \quad \text{if} \: i = \min(I)
\]

2) \( \forall t \in (s, \infty), \: |\hat{\Pi}^{\infty}_{s-,t}(i)| = |\hat{\Pi}^{\infty}_{s,t}(j)| \quad \text{if} \begin{cases} i \neq \min(I) \\ i = j - (\# \{I \cap [j]\} - 1) \lor 0 \end{cases} \)

Proof Suppose \( i \). Since \( \hat{\Pi}^{\infty}_{s-,t} = \mathrm{Coag}(\hat{\Pi}^{\infty}_{s,t}, \hat{\Pi}^{\infty}_{s-,s}) \), the very definition of the coagulation operator implies \( i \).

Suppose \( ii \). Since the trajectories of \( \hat{\Pi}^{\infty} \) are deterministic flows of partitions without simultaneous mergers, we know that \( \hat{\Pi}^{\infty}_{s-,s} \) is a partition with at most one non-singleton block. Observe that for all \( \omega \in \Omega^* \), the processes \( t \mapsto |\hat{\Pi}^{\infty}_{s-,t}(i)|(\omega) \) are distinct: either they reach 0 at distinct times (Eves - extinction case) or their asymptotic behaviours differ (Eves - persistent case). The same holds for the processes \( t \mapsto |\hat{\Pi}^{\infty}_{s,t}(j)|(\omega) \). Since \( \hat{\Pi}^{\infty}_{s-,t} = \mathrm{Coag}(\hat{\Pi}^{\infty}_{s,t}, \hat{\Pi}^{\infty}_{s-,s}) \), the equations of \( ii \) imply that the partition \( \hat{\Pi}^{\infty}_{s-,s} \) has a unique non-singleton block \( I \).

A remark on the lookdown ordering Observe that the lookdown construction relies on the future of the underlying measure-valued process: indeed the ordering of the initial types depends on the asymptotic behaviour of their progenies. This idea is made explicit by our definition of the infinite sequence of Eves, and can be originally found in the proof of Theorem 1.1 in [14] (where Donnelly and Kurtz constructed a random permutation between the Moran model and their lookdown process). As a consequence of this observation the filtrations of the lookdown process and the flow of bridges differ.

7 Appendix

7.1 Proof of Proposition 3.8

Without loss of generality, we can assume that for every \( \omega \in \Omega \) the Poisson point process \( \mathcal{P} \) has finitely many atoms on \( [s, t] \times \mathcal{P}^n \) for all \( s < t \) and \( n \geq 1 \). Restricting the flow \( \hat{\Pi} \) to \( [n] \), it is a simple matter to check that the trajectories \( \omega \mapsto \hat{\Pi}(\omega) \) are deterministic flows of partitions. The difficulty of the proof lies now in showing the regularity of the asymptotic frequencies. One can adapt Lemmas 3.4 and 3.5 in [14] to deduce that with probability one for all rational values \( s \) and all integer \( i \geq 1 \) the processes

\[
t \mapsto |\hat{\Pi}_{s,t}(i)|
\]

are well-defined and càdlàg. It remains to show that the same holds when \( s \) is irrational, and to prove that the frequencies are regular when \( s \) or \( t \) varies. We need to distinguish three cases.
1- CDI. The process $r \mapsto \hat{\Pi}_{t-r,t}$ starts with infinitely many blocks, and immediately after time 0, comes down from infinity. Note that this property holds a priori on an event of probability one that depends on $t$, but the cocycle property allows to assert that the coming down from infinity holds for all $t$ simultaneously on a same event of probability one. After time 0, $r \mapsto \hat{\Pi}_{t-r,t}$ evolves at discrete times since on any compact interval of time there are only finitely many elementary reproduction events involving at least two levels among the $n$ first, for every $n \geq 1$. Thus for $\mathbb{P}$-almost all $\omega$ and all $s < t$, there exists a rational value $q(\omega) = q \in (s, t)$ such that $\hat{\Pi}_{s,t} = \hat{\Pi}_{q,t}$ and the existence of asymptotic frequencies follows from the rational case. The limit

$$\lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s+\epsilon,t}(i)| = |\hat{\Pi}_{s,t}(i)|, \forall i \in \mathbb{N}$$

is then obvious. Similarly, there exists a rational value $p(\omega) = p < s$ such that $\hat{\Pi}_{p,t} = \hat{\Pi}_{s-t,t}$. It implies the existence of asymptotic frequencies for the latter along with the limit

$$\lim_{\epsilon \downarrow 0} |\hat{\Pi}_{s-\epsilon,t}(i)| = |\hat{\Pi}_{s,t}(i)|, \forall i \in \mathbb{N}$$

The same kind of arguments apply to show the regularity when $t$ varies.

2- DISCRETE and INTENSIVE w. DUST. On any compact interval of time, only a finite number of elementary reproduction events hit any given level (recall that $\int_{(0,1)} \nu(du) < \infty$), therefore $r \mapsto \hat{\Pi}_{t-r,t}(i)$ evolves at discrete times for any given $i \geq 1$. The arguments of the previous regime can therefore be applied by considering $\hat{\Pi}_{s,t}(i)$ instead of $\hat{\Pi}_{s,t}$.

3- INTENSIVE $\infty$. In this regime, all the partitions have infinitely many blocks and no dust. We start with the existence of asymptotic frequencies for $\hat{\Pi}_{s,t}$ when $s$ is not rational. Fix $i \geq 1$ and a rational value $p \in (s, t)$. For every $n \geq i$, we set

$$\eta(n) := 1 - \sum_{l=1}^{n} |\hat{\Pi}_{p,t}(l)|$$

Since the partitions have no dust, $\eta(n) \to 0$ as $n \to \infty$. Let us denote by $j_1, j_2, \ldots$ the elements of block $\hat{\Pi}_{s,p}(i)$. From the cocycle property, we have:

$$\sum_{j_l \leq n} |\hat{\Pi}_{p,t}(j_l)| \leq \lim_{m \to \infty} \frac{\#\hat{\Pi}_{s,t}(i) \cap [m]}{m} \leq \lim_{m \to \infty} \frac{\#\hat{\Pi}_{s,t}(i) \cap [m]}{m} \leq \sum_{j_l \leq n} |\hat{\Pi}_{p,t}(j_l)| + \eta(n) \quad (23)$$

Letting $n$ go to infinity ensures the existence of $|\hat{\Pi}_{s,t}(i)|$. The same reasoning applies to $\hat{\Pi}_{s-t,t}$ and $\hat{\Pi}_{s-t,t}$. We now prove the regularity properties. Since $p$ is rational, we know that there exists $\epsilon_0(\omega) = \epsilon_0 > 0$ small enough such that

$$|\hat{\Pi}_{p,t+\epsilon}(j) - |\hat{\Pi}_{p,t}(j)|| \leq \frac{\eta(n)}{n}, \forall \epsilon < \epsilon_0, \forall j \leq n$$

Therefore

$$1 - \sum_{l=1}^{n} |\hat{\Pi}_{p,t+\epsilon}(l)| \leq 2\eta(n), \forall \epsilon < \epsilon_0$$

Combined with Equation (23), this ensures the convergence when $t + \epsilon$ goes to $t$. We get similarly the convergence when $t - \epsilon$ goes to $t$. To prove the convergences when $s - \epsilon$ goes to $s$ and $s + \epsilon$ goes to $s$, one remarks that there exists $\epsilon_0(\omega) = \epsilon_0 > 0$ such that no elementary reproduction events affecting at least two levels among $[n]$ fall in $(s - \epsilon_0, s)$ nor in $(s, s + \epsilon_0)$. Hence $\hat{\Pi}_{s-\epsilon,p}(i) \cap [n]$ and $\hat{\Pi}_{s+\epsilon,p}(i) \cap [n]$ do not vary whenever $\epsilon$ is in $(0, \epsilon_0)$. Similar arguments as above end the proof. ■
7.2 Proof of Proposition 3.10

The proof of the proposition relies on three lemmas.

**Lemma 7.1** Consider the restriction \((\hat{\Pi}_{s,t}, s \leq t \in \mathbb{Q})\) of the flow of partitions to its rational marginals. Then there exists an event \(\Omega_{\hat{\Pi}}\) of probability 1 on which:

\[
\forall r < s < t \in \mathbb{Q}, \quad \hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s})
\]

and for every \(s \in \mathbb{R}\) and \(n \in \mathbb{N}\), there exists \(\epsilon(\omega) > 0\) such that

\[
\hat{\Pi}_{p,q}^{[n]} = 0_{[n]}, \quad \forall p, q \in [s, s + \epsilon) \cap \mathbb{Q} \quad \text{and} \quad \hat{\Pi}_{p,q}^{[n]} = 0_{[n]}, \quad \forall p, q \in (s - \epsilon, s) \cap \mathbb{Q}.
\]

**Proof** First, one has

\[
\mathbb{P}\left(\hat{\Pi}_{r,t} = \text{Coag}(\hat{\Pi}_{s,t}, \hat{\Pi}_{r,s}), \forall r < s < t \in \mathbb{Q}^3\right) = 1
\]  

(24)

Second, we only prove that the trajectories, when \(s\) and \(t\) are restricted to \((0, 1)\), are deterministic flows of partitions with probability one since the general case will follow by taking a countable intersection of events of probability one. Observe that almost surely for every integer \(n \geq 1\), the sequence

\[
\left(\#\{i \in [2^n] : \hat{\Pi}_{p,q}^{[n]}(i-1)2^{n-i}2^{-m} \neq 0_{[n]}\}\right)_{m \geq 1}
\]

is bounded. This is trivially the case for a stochastic flow of partitions \(\Pi^{P}\) defined from a \(\Lambda\) lookdown graph \(\mathcal{P}\). Using Equation (24) and the fact that the finite-dimensional distributions of \(\Pi\) and \(\Pi^{P}\) are equal, we deduce that almost surely the sequence is bounded. Therefore, we deduce that nowhere the elementary reproduction events accumulate and the left and right regularity properties of the statement follow.

We now define for every \(s \leq t \in \mathbb{R}\) the partition \(\tilde{\hat{\Pi}}_{s,t}\) on the event \(\Omega_{\hat{\Pi}}\) as follows.

**Lemma 7.2** On the event \(\Omega_{\hat{\Pi}}\) the following random partition is well-defined.

\[
\tilde{\hat{\Pi}}_{s,t} := \begin{cases} 
\hat{\Pi}_{s,t} & \text{if } s, t \in \mathbb{Q}, s < t \\
0_{[\infty]} & \text{if } s = t \\
\lim_{v \downarrow t, v \in \mathbb{Q}} \hat{\Pi}_{s,v} & \text{if } s \in \mathbb{Q}, t \notin \mathbb{Q} \\
\lim_{r \downarrow s, r \in \mathbb{Q}} \hat{\Pi}_{r,t} & \text{if } s \notin \mathbb{Q}, t \in \mathbb{Q} \\
\text{Coag}(\hat{\Pi}_{q,t}, \tilde{\hat{\Pi}}_{s,q}) & \text{for any arbitrary rational } q \in (s, t) \text{ if } s, t \notin \mathbb{Q}
\end{cases}
\]  

(25)

Furthermore, for every \(r < s < t\), \(\tilde{\hat{\Pi}}_{r,t} = \text{Coag}(\tilde{\hat{\Pi}}_{s,t}, \tilde{\hat{\Pi}}_{r,s})\).

**Proof** We work on the event \(\Omega_{\hat{\Pi}}\) throughout this proof. Recall the cocycle property, the left and right regularity properties verified by the flow restricted to its rational marginals. Fix \(s \in \mathbb{Q}\) and let us prove the existence of a limit for \(\hat{\Pi}_{s,v}\) when \(v\) is rational and goes to a given irrational value \(t \in (s, \infty)\). Fix \(n \geq 1\). There exists \(\epsilon = \epsilon(\omega) > 0\) such that for all rational values \(p, q\) in \((t, t+\epsilon)\), \(\hat{\Pi}_{p,q}^{[n]} = 0_{[n]}\). Combined with the cocycle property on rational marginals, this ensures that \(v \mapsto \Pi_{s,v}^{[n]}\) is constant whenever \(v \in (t, t+\epsilon) \cap \mathbb{Q}\). The existence of the limit follows. A similar argument shows the existence of a limit for \(\hat{\Pi}_{v,t}\) when \(v\) is rational and goes to an irrational value \(s\) and \(t\) is a given rational value.
Fix $r < s < t$. If all three are rational, the corresponding cocycle property holds since we are on $\Omega_{\Pi}$. Now suppose that either $s$ is rational or both $r$ and $t$ are rational, then we stress that the corresponding cocycle property still holds. Indeed, take a limiting sequence of rational values for which the cocycle holds and then use the continuity of the coagulation operator (see Subsection 2.1).

Finally, suppose that $s, t \notin \mathbb{Q}$. To verify that our definition of \( \hat{\Pi}_{s,t} \) makes sense, we need to show that $\text{Coag}(\hat{\Pi}_{q,t}, \hat{\Pi}_{s,q})$ does not depend on the value $q \in (s, t) \cap \mathbb{Q}$. Consider two such values $q, q' \in (s, t) \cap \mathbb{Q}$, suppose that $q < q'$ and use Equation (7) to obtain

\[
\text{Coag} \left( \hat{\Pi}_{q', t}, \hat{\Pi}_{s, q'} \right) = \text{Coag} \left( \hat{\Pi}_{q', t}, \text{Coag}(\hat{\Pi}_{q, q'}, \hat{\Pi}_{s, q}) \right)
= \text{Coag} \left( \text{Coag}(\hat{\Pi}_{q', t}, \hat{\Pi}_{q, q'}), \hat{\Pi}_{s, q} \right) = \text{Coag} \left( \hat{\Pi}_{q, t}, \hat{\Pi}_{s, q} \right)
\]

Thus, the definition of $\hat{\Pi}_{s,t}$ does not depend on $q \in (s, t)$.

Finally, consider three irrational $r < s < t$, and two rational values $q, q'$ such that $q \in (r, s)$ and $q' \in (s, t)$.

\[
\text{Coag} \left( \hat{\Pi}_{s,t}, \hat{\Pi}_{r,s} \right) = \text{Coag} \left( \text{Coag}(\hat{\Pi}_{q', t}, \hat{\Pi}_{s, q'}), \text{Coag}(\hat{\Pi}_{q, s}, \hat{\Pi}_{r, q}) \right)
= \text{Coag} \left( \hat{\Pi}_{q', t}, \text{Coag}(\hat{\Pi}_{q, s}, \hat{\Pi}_{r, q}) \right) = \hat{\Pi}_{r,t}
\]

This concludes the proof. \( \blacksquare \)

On the complement of $\Omega_{\Pi}$, set any arbitrary value to $\hat{\Pi}_{s,t}$.

**Lemma 7.3** The collection of partitions $\hat{\Pi}$ is a modification of $\hat{\Pi}$, that is, for every $s \leq t$, a.s. $\hat{\Pi}_{s,t} = \hat{\Pi}_{s,t}$. Furthermore, for each $\omega \in \Omega_{\Pi}$, $\hat{\Pi}(\omega)$ is a deterministic flow of partitions.

**Proof** By definition, for every rational numbers $s \leq t$, $\hat{\Pi}_{s,t} = \hat{\Pi}_{s,t}$ on the event $\Omega_{\Pi}$, so it holds a.s. Suppose that $s \in \mathbb{Q}$. Observe that $(\hat{\Pi}_{s,t}, t \in [s, \infty))$ is a càdlàg process, whose rational marginals coincide almost surely to those of any càdlàg modification of $(\hat{\Pi}_{s,t}, t \in [s, \infty))$ (it can be easily checked that this process has a Feller semigroup), hence $(\hat{\Pi}_{s,t}, t \in [s, \infty))$ is almost surely equal to any given càdlàg modification. This yields for every $t \in (s, \infty)$, a.s. $\hat{\Pi}_{s,t} = \hat{\Pi}_{s,t}$.

Now suppose $s$ irrational, take $t > s$ and fix $n \in \mathbb{N}$. We have for all $q \in (s, t) \cap \mathbb{Q}$

\[
\mathbb{P} \left( \hat{\Pi}^{[n]}_{s,t} = \hat{\Pi}^{[n]}_{s,t} \right) \geq \mathbb{P} \left( \hat{\Pi}^{[n]}_{q,t} = \hat{\Pi}^{[n]}_{q,t} ; \hat{\Pi}^{[n]}_{s,t} = \text{Coag}(\hat{\Pi}^{[n]}_{q,t}, \hat{\Pi}^{[n]}_{s,q}) ; \hat{\Pi}^{[n]}_{s,q} = 0_{[n]} \right)
\]

As $q \downarrow s$, $\mathbb{P}(\hat{\Pi}^{[n]}_{q,t} = 0_{[n]}) \rightarrow 1$ by definition of a stochastic flow of partitions. The cocycle property of a stochastic flow of partitions together with the almost sure identity $\hat{\Pi}_{q,t} = \hat{\Pi}_{q,t}$, we have already proved, ensures that the probability of the event on the r.h.s. tends to 1 as $q \downarrow s$. Thus $\hat{\Pi}_{s,t} = \hat{\Pi}_{s,t}$ almost surely.

Finally, when $t = s$ we know that $\hat{\Pi}_{s,s} = 0_{[\infty]}$ almost surely by definition. Therefore, $\hat{\Pi}$ is a modification of $\hat{\Pi}$.

We need to verify that for all $\omega \in \Omega_{\Pi}$, $\hat{\Pi}(\omega)$ is a deterministic flow of partitions. The cocycle property was proved in the preceding lemma. Let us show the right regularity. Fix $s \in \mathbb{R}$ and $n \in \mathbb{N}$. Recall that there exists $\epsilon = \epsilon(\omega) > 0$ such that for all rational $p < q \in (s, s + \epsilon)$, $\hat{\Pi}^{[n]}_{p,q} = 0_{[n]}$. Letting $p \downarrow s$, we get $\hat{\Pi}^{[n]}_{q,s} = 0_{[n]}$ for all $q \in (s, s + \epsilon) \cap \mathbb{Q}$. Similarly for all $r \in (s, s + \epsilon)$, we have $\hat{\Pi}^{[n]}_{q,r} = 0_{[n]}$ as soon as $q \in (s, r)$. Using the fact that $\hat{\Pi}^{[n]}_{q,r} = \text{Coag}(\hat{\Pi}^{[n]}_{q,r}, \hat{\Pi}^{[n]}_{s,q})$ we get that $\hat{\Pi}^{[n]}_{s,r} = 0_{[n]}$ for all $r \in (s, s + \epsilon)$. This in turn implies that $\hat{\Pi}^{[n]}_{s,r} \rightarrow \hat{\Pi}^{[n]}_{s,s}$ as $r \downarrow s$ and the right regularity is proved. The left regularity is obtained similarly. \( \blacksquare \)
7.3 Calculations on regular variation

We fix $\alpha \in (1, 2)$ and assume that $\Lambda(du) = f(u)du$ with $f(u) = u^{1-\alpha}L(u)$ for all $u \in (0, 1)$ and $L$ is slowly varying at $0^+$ (see [22]). Fix $\lambda \in (0, \infty)$. For all $u > 0$ we have

$$\frac{\Psi(\lambda u)}{\lambda^\alpha} = \Psi(u) + \int_0^\lambda (e^{-xu} - 1 + xu)x^{-1-\alpha}\left(\frac{\lambda}{x} - L(x)\right)dx - \int_1^\lambda (e^{-xu} - 1 + xu)x^{-1-\alpha}L(x)dx$$

Since $L$ is slowly varying at $0^+$, the ratio $L(x)/\lambda L(x)$ goes to $1$ as $x$ tends to $0^+$. This, together with the fact that $\Psi(u)/u \to \infty$ as $u \to \infty$, ensures that

$$\Psi(\lambda u) \sim_{u \to \infty} \lambda^\alpha \Psi(u)$$

Let us now prove that the map $\lambda \mapsto \Psi(\lambda u)/\lambda^\alpha$ is itself regularly varying at $0^+$ with index $-1/(\alpha - 1)$. We have

$$\frac{\Psi(\lambda u)}{\lambda^\alpha} = \int_0^t \frac{\Psi(v(s)) - \Psi(s)\Psi'(v(s))}{\Psi(v(s))^2} v'(s)ds \sim_{t \to 0^+} (\alpha - 1)t$$

In the above calculations, we use the identity $v'(s) = -\Psi(v(s))$ and the regular variation of $\Psi$ at $\infty$ that ensures the convergence $\Psi(u)/\Psi(u) \to \alpha$ as $u \to \infty$ (see Theorem 2 in [22]). Therefore we have proved that

$$\frac{t v'(t)}{v(t)} \sim_{t \to 0^-} \frac{-1}{\alpha - 1}$$

This identity implies, thanks to Theorem 2 in [22], that $v$ is regularly varying at $0^+$ with index $-1/(\alpha - 1)$.

Assume now that $\Lambda(du) = c\delta_0(du) + f(u)du$ where $c > 0$ and $f$ is any positive and measurable function such that $\int_{(0,1)} f(u)du < \infty$. It is simple to show that $\Psi(u) \sim c u^2/2$ as $u \to \infty$. The calculations on $v$ above then applies with $\alpha = 2$. We deduce that $v$ is regularly varying at $0^+$ with index $-1$.

7.4 Proof of Lemma 5.4

Suppose a). Then we know that $B_t \overset{(d)}{\to} Id$ as $t \downarrow 0$ from the continuity Lemma 1 in [6]. Since the limit is deterministic, the convergence in probability in the Skorohod’s metric space follows.

Suppose b). Fix $n \geq 1$ and $\epsilon > 0$, we will prove there exists $t_0 > 0$ such that for all $t \in (0, t_0)$

$$\mathbb{P}(d_{\mathcal{P}}(\pi(B_t), \pi(1d)) < 2^{-n}) > 1 - 2\epsilon$$

There exists $p \in \mathbb{N}$ such that

$$\mathbb{P}\left(\exists i, j \text{ s.t. } 1 \leq i < j \leq n \text{ and } |V_i - V_j| < \frac{2}{p}\right) < \epsilon \quad (26)$$

Moreover, there exists $t_0 > 0$ such that for all $t \in (0, t_0)$

$$\mathbb{P}\left(\bigcap_{0 \leq k \leq p} \left\{B_t\left\lceil \frac{k}{p}\right\rceil \in \left\{\frac{k}{3p}, \frac{k}{p} - \frac{1}{3p}, \frac{k}{p} + \frac{1}{3p}\right\}\right\} \right) > 1 - \epsilon \quad (27)$$

The monotonicity of $B_t^{-1}$, and the two previous equations ensure that

$$\mathbb{P}(d_{\mathcal{P}}(\pi(B_t), \pi(1d)) < 2^{-n}) = \mathbb{P}\left(\{B_t^{-1}(V_1), \ldots, B_t^{-1}(V_n)\} \text{ are all distinct}\right) > 1 - 2\epsilon$$

Since $n$ and $\epsilon$ were arbitrary, we get a).
Acknowledgements  This is part of my PhD thesis. I would like to thank my supervisors, Julien Berestycki and Amaury Lambert, for fruitful discussions throughout this work.

References

[1] J. Berestycki, N. Berestycki, and V. Limic, A small-time coupling between Lambda-coalescents and branching processes, to appear in Ann. Appl. Probab., (2013).

[2] J. Berestycki, N. Berestycki, and J. Schweinsberg, Beta-coalescents and continuous stable random trees, Ann. Probab., 35 (2007), pp. 1835–1887.

[3] J. Berestycki, N. Berestycki, and J. Schweinsberg, Small-time behavior of beta coalescents, Annales de l’Institut Henri Poincare (B) Probability and Statistics, 44 (2008), pp. 214 – 238.

[4] J. Bertoin, Random fragmentation and coagulation processes, vol. 102 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2006.

[5] J. Bertoin and J.-F. Le Gall, The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes, Probability Theory and Related Fields, 117 (2000), pp. 249–266.

[6] ———, Stochastic flows associated to coalescent processes, Probability Theory and Related Fields, 126 (2003), pp. 261–288.

[7] ———, Stochastic flows associated to coalescent processes II: Stochastic differential equations, Annales de l’Institut Henri Poincare (B) Probability and Statistics, 41 (2005), pp. 307 – 333.

[8] ———, Stochastic flows associated to coalescent processes III: Limit theorems, Illinois J. Math., 50 (2006), pp. 147–181.

[9] M. Birkner, J. Blath, M. Capaldo, A. M. Etheridge, M. Möhle, J. Schweinsberg, and A. Wakolbinger, Alpha-stable branching and beta-coalescents, Electronic Journal of Probability, 10 (2005), pp. 303–325.

[10] M. Birkner, J. Blath, M. Möhle, M. Steinruecken, and J. Tams, A modified lookdown construction for the Xi-Fleming-Viot process with mutation and populations with recurrent bottlenecks, Alea, 6 (2009), pp. 25–61.

[11] E. Bolthausen and A.-S. Sznitman, On Ruelle’s probability cascades and an abstract cavity method, Comm. Math. Phys., 197 (1998), pp. 247–276.

[12] D. A. Dawson, Measure-valued Markov processes, vol. 1541 of Lecture Notes in Math., Springer, Berlin, 1993.

[13] P. Donnelly and T. G. Kurtz, A countable representation of the Fleming-Viot measure-valued diffusion, Ann. Probab., 24 (1996), pp. 698–742.

[14] ———, Particle representations for measure-valued population models, Ann. Probab., 27 (1999), pp. 166–205.

[15] T. Duquesne and C. Labbé, On the Eve property for continuous-state branching processes, in preparation, (2013).
[16] A. M. Etheridge, *An introduction to superprocesses*, vol. 20 of University Lecture Series, American Mathematical Society, Providence, RI, 2000.

[17] W. H. Fleming and M. Viot, *Some measure-valued Markov processes in population genetics theory*, Indiana Univ. Math. J., 28 (1979), pp. 817–843.

[18] C. Foucart, *Generalized Fleming-Viot processes with immigration via stochastic flows of partitions*, Alea, 9 (2012), pp. 451–472.

[19] N. Freeman, *The number of non-singleton blocks in Lambda-coalescents with dust*, arXiv:1111.1660, (2011).

[20] O. Kalenberg, *Canonical representations and convergence criteria for processes with interchangeable increments*, Probability Theory and Related Fields, 27 (1973), pp. 23–36. 10.1007/BF00736005.

[21] J. F. C. Kingman, *The coalescent*, Stochastic Processes and their Applications, 13 (1982), pp. 235 – 248.

[22] J. Lamperti, *An occupation time theorem for a class of stochastic processes*, Trans. Amer. Math. Soc., 88 (1958), pp. 380–387.

[23] J.-F. Le Gall, *Spatial branching processes, random snakes and partial differential equations*, Birkhäuser Basel, 1999.

[24] J. Pitman, *Coalescents with multiple collisions*, Ann. Probab., 27 (1999), pp. 1870–1902.

[25] S. Sagitov, *The general coalescent with asynchronous mergers of ancestral lines*, J.Appl.Prob., 36 (1999), pp. 1116–1125.

[26] J. Schweinsberg, *A necessary and sufficient condition for the Lambda-coalescent to come down from infinity*, Electronic Communications in Probability, 5 (2000), pp. 1–11.