Some results on Arithmetico-Geometrico topological indices
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Abstract
Chemical graph theory plays an indispensable role in QSAR(Quantitative structure activity relationship) and QSPS(Qualitative structure property relationship) research. The molecular physico-chemical properties can be elaborated using the data encoded in their specific chemical (molecular) graphs. Topological indices of molecular descriptions are numerical values associated with a graph that throws light on its topology and are graph invariants. A large number of different invariants have been employed with various degrees of success in quantitative and qualitative structure property research. This paper intends to study AG indices of some standard graphs.

Keywords
Arithmetico-Geometrico index, topological indices, wheel graph, gear graph, sunflower graph, fan graph.

AMS Subject Classification
05C07, 05C76, 92E10.

1 Introduction

The numerical values attached with molecular graphs called topological indices play a significant role in chemical studies, analysis and research. A topological index called Weiner index [5] was developed to calculate the boiling point of paraffins, in 1947. A popular index called Zagreb index [3] was defined by Gutman and Trinajstic in 1972. Thereafter many indices are defined namely [1][2] Randic index, topological index etc. In 2016 [4] V.S. Shigehalli and Rachanna Kanavur introduced arithmetic-geometric indices. In this paper we obtain explicit formula for AG indices of standard class of graphs.

Definition 1.1. Arithmetico-Geometrico topological index for a non-empty graph G is defined and denoted as

\[ AG(G) = \sum_{xy \in E(G)} \frac{dx + dy}{2\sqrt{dx \cdot dy}} \]

where \( dx \) and \( dy \) denote the degrees of the vertices of the edge \( xy \)

2. AG indices of some family of graphs

Theorem 2.1. The AG topological index of path graph

\[ AG(P_n) = \begin{cases} 
1 & \text{for } n = 2 \\
\frac{1}{\sqrt{2}} + n - 3 & \text{for } n \geq 3
\end{cases} \]

Proof. For \( n = 2 \), there exists a single edge having two pendant vertices.

So, \[ AG(P_2) = \sum_{xy \in E(P_2)} \frac{dx + dy}{2\sqrt{dx \cdot dy}} = \frac{2}{2} = 1 \]

For \( n = 3 \), there are two edges each having ends of degree 1 and 2.
Theorem 2.3. \( \text{AG}(P_3) = \sum_{xy \in E(P_3)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} \)

\[
\begin{align*}
   & = \frac{1 + 2}{2 \sqrt{1.2}} + \frac{2 + 1}{2 \sqrt{2.1}} \\
   & = \frac{2.3}{2 \sqrt{2}} + 1 \\
   & = \frac{3}{\sqrt{2}} + (4 - 3)
\end{align*}
\]

For \( n = 4 \), there are two pendent edges each of which have end vertices of degree 1 and 2 and one edge with end vertices of degree 2 each. So,

\[
\begin{align*}
   \text{AG}(P_4) & = \sum_{xy \in E(P_4)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} \\
   & = \frac{1 + 2}{2 \sqrt{1.2}} + \frac{2 + 2}{2 \sqrt{2.2}} + \frac{2 + 1}{2 \sqrt{2.1}} \\
   & = \frac{2.3}{2 \sqrt{2}} + n - 3 \\
   & = \frac{3}{\sqrt{2}} + (n - 3)
\end{align*}
\]

Theorem 2.2. \( \text{AG}(C_n) = n \)

Proof. For \( n = 3 \), there are 3 vertices each of which has degree 2. So \( \text{AG}(C_3) = \sum_{xy \in E(C_3)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} = 3 \cdot \frac{3 + 2}{2 \sqrt{2}} = 3.1 = 3 \)

Each graph \( C_n \) has \( n \) edges with each of its vertices with degree 2 and so \( \text{AG}(C_n) = \sum_{xy \in E(C_n)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} = n \cdot \frac{3 + 2}{2 \sqrt{2}} = n \)

Theorem 2.3. \( \text{AG}(K_n) = \binom{n}{2} \)

Proof. For the complete graph \( K_n \), there are \( \frac{n(n-1)}{2} \) edges and each vertex is of degree \( n - 1 \). So,

\[
\begin{align*}
   \text{AG}(K_n) & = \sum_{xy \in E(K_n)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} \\
   & = \sum_{uv \in E(K_n)} \frac{n - 1 + n - 1}{2 \sqrt{(n - 1)(n - 1)}} \\
   & = \sum_{xy \in E(K_n)} 1 = n(n - 1) = \binom{n}{2}
\end{align*}
\]

\[ \square \]

Theorem 2.4. \( \text{AG}(K_{1,n}) = \frac{(n + 1)\sqrt{n}}{2} \)

Proof. For the star graph \( K_{1,n} \), all the \( n \) edges have one end vertex of degree 1 and other of degree \( n \). So,

\[
\begin{align*}
   \text{AG}(K_{1,n}) & = \sum_{xy \in E(K_{1,n})} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} \\
   & = \sum_{u \in E(K_{1,n})} \frac{1 + n}{2 \sqrt{1.n}} \\
   & = \frac{1 + n}{2 \sqrt{1.n}} \sum_{xy \in E(K_{1,n})} 1 \\
   & = \frac{1 + n}{2 \sqrt{1.n}} \frac{(1 + n)\sqrt{n}}{2}
\end{align*}
\]

\[ \square \]

Definition 2.5. The wheel \( W_n \) \( n \geq 3 \) is a graph constructed by attaching a mid-vertex \( v \) with all the vertices of an \( n \)-cycle. The mid-vertex has degree \( n \) and all the other vertices are of degree 3. Also it has \( n + 1 \) vertices and \( 2n \) edges.

Theorem 2.6. For a wheel with order \( n + 1 \),

\[
\text{AG}(W_n) = n(1 + \frac{3 + n}{2\sqrt{3.n}})
\]

Proof. The edges of \( W_n \) can be partitioned into two sets \( E_1 \) and \( E_2 \) as follows.

\[
\begin{align*}
   E_1 & = E(C_n) \in E(W_n) \\
   E_2 & = E(W_n - E(C_n))
\end{align*}
\]

\[
\begin{align*}
   \text{AG}(W_n) & = \sum_{xy \in E(W_n)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} \\
   & = \sum_{xy \in E(E_1)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E(E_2)} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} \\
   & = \sum_{xy \in E(E_1)} \frac{3 + 3}{2 \sqrt{3.n}} + \sum_{xy \in E(E_2)} \frac{3 + n}{2 \sqrt{3.n}} \\
   & = \sum_{xy \in E(E_1)} 1 + \frac{3 + n}{2 \sqrt{3.n}} \sum_{xy \in E(E_2)} 1 \\
   & = n + \frac{3 + n}{2 \sqrt{3.n}} = n(1 + \frac{3 + n}{2\sqrt{3.n}})
\end{align*}
\]

\[ \square \]
Corollary 2.7. For a wheel $W_{2n}$ with order $2n + 1$
$AG(W_{2n}) = 2n(1 + \frac{3 + 2n}{2\sqrt{6n}})$

Theorem 2.8. For an $r$-regular graph of order $n$,
$AG(G) = \frac{nr}{2}$

Proof. For an $r$-regular graph of order $n$, there are $\frac{nr}{2}$ edges. So $AG(G) = \sum_{xy \in E(G)} \frac{1}{\sqrt{2}} = \frac{nr}{2}$.

Corollary 2.9. $AG(K_n) = \binom{n}{2}$

Note that $K_n$ is an $(n - 1)$regular graph of order $n$.

Corollary 2.10. For the Peterson graph $PG$, $AG(PG) = 15$.

Definition 2.11. A gear graph $G_n$ of order $2n + 1$ is a wheel graph where a vertex is appended between each pair of adjacent vertices of the outer cycle. Note that it has 3n edges.

Theorem 2.12. $AG(G_n) = \frac{n}{2\sqrt{3}}(\frac{3 + n}{\sqrt{n}} + 5\sqrt{2})$ where $G_n$ is the gear graph of order $2n + 1$.

Proof. The edges of $G_n$ can be partitioned as follows,
$E_1 = \{xy|\text{deg}(x) = n\ and\ \text{deg}(y) = 3\}$
$E_2 = \{xy|\text{deg}(x) = 2\ and\ \text{deg}(y) = 3\}$

$AG(G_n) = \sum_{xy \in E_1} \frac{dx + dy}{\sqrt{2}\sqrt{dx\cdot dy}} + \sum_{xy \in E_2} \frac{dx + dy}{\sqrt{2}\sqrt{dx\cdot dy}}$
$= \sum_{xy \in E_1} \frac{n + 3}{2\sqrt{3n}} + \sum_{xy \in E_2} \frac{2 + 3}{2\sqrt{2\cdot 3n}}$
$= \frac{n + 3}{2\sqrt{3n}} \sum_{xy \in E_1} 1 + \frac{5}{2\sqrt{6}} \sum_{xy \in E_2} 1$
$= \frac{n + 3}{2\sqrt{3n}} n + \frac{5}{2\sqrt{6}} 2n = \frac{n + 3}{2\sqrt{3n}} + 5\sqrt{2}$

Definition 2.13. The sunflower graph $SF_n$ is a modified wheel with an additional $n$ vertices $w_0, w_1, w_2, \ldots, w_{n-1}$ where $w_i$ is joined to vertices of the edge $e_{i+2}$ for $i = 0, 1, 2, \ldots, n - 1$ where $i + 1$ is taken modulo $n$. The mid-vertex has degree $n$, the cycle vertices have degree 5 and each $w_i$ with degree 2.

Theorem 2.14. For the sunflower graph $SF_n$ of order $2n + 1$,$AG(SF_n) = n(\frac{5 + n}{2\sqrt{3}} + 1 + \frac{5}{\sqrt{3n}})$

Proof. The edges of $SF_n$ can be partitioned into $E_1, E_2, E_3$ as follows.
$E_1 = \{xy|\text{deg}(x) = n\ and\ \text{deg}(y) = 5\}$
$E_2 = \{xy|\text{deg}(x) = 5\ and\ \text{deg}(y) = 5\}$
$E_3 = \{xy|\text{deg}(x) = 5\ and\ \text{deg}(y) = 2\}$

$AG(SF_n) = \sum_{xy \in E_1} \frac{dx + dy}{\sqrt{2}\sqrt{dx\cdot dy}} + \sum_{xy \in E_2} \frac{dx + dy}{\sqrt{2}\sqrt{dx\cdot dy}}$
$+ \sum_{xy \in E_3} \frac{dx + dy}{\sqrt{2}\sqrt{dx\cdot dy}}$
$= \sum_{xy \in E_1} \frac{n + 5}{2\sqrt{5n}} + \sum_{xy \in E_2} \frac{5 + 5}{2\sqrt{5\cdot 3n}} + \sum_{xy \in E_3} \frac{5 + 2}{2\sqrt{5\cdot 2\cdot 3n}}$
$= \frac{n + 5}{2\sqrt{5n}} \sum_{xy \in E_1} 1 + \frac{7}{2\sqrt{10}} \sum_{xy \in E_3} 1$
$= \frac{n + 5}{2\sqrt{5n}} n + \frac{7}{2\sqrt{10}} 2n$
$= n(\frac{n + 5}{2\sqrt{5n}} + 1 + \frac{7}{2\sqrt{10}})$

3. AG index of Graph operations.

Definition 3.1. The sum $G_1 + G_2$ of graphs $G_1$ and $G_2$ of order $n_1$ and $n_2$ and disjoint vertex set $V(G_1)$ and $V(G_2)$ respectively consists of $G_1 \cup G_2$ and edges joining a vertex
of $G_1$ and a vertex of $G_2$ so that $|V(G_1 + G_2)| = |V(G_1)| + |V(G_2)| = n_1 + n_2$ and degree of each vertex $v$ belonging to $G_1$ in $G_1 + G_2$ is $\deg(v) + n_2$ and degree of each vertex $v$ belonging to $G_2$ in $G_1 + G_2$ is $\deg(v) + n_1$. Also $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$

**Definition 3.2.** Fan graph $F_{m,n}$ is a join of $P_n$ and the complement of the graph $K_m$. That is, $K_m + P_n$ is called a fan graph and it has $m + n$ vertices and $mn + n - 1$ edges.

**Theorem 3.3.**

$$AG(K_m + P_n) = \frac{m(m + n + 1)}{\sqrt{n(m + 1)}} + \frac{(n - 2)(m + n + 2)}{\sqrt{n(m + 2)}} + \frac{2m + 3}{\sqrt{(m + 1)(m + 2)}} + |n - 3| \text{ for } n \geq 2$$

**Proof.** The edge set of $K_m + P_n$ can be partitioned into four sets as follows.

$E_1 = \{E(K_m + P_n)\}$ each edge with terminal vertices of degree $m$ and $m + 1$

$E_2 = \{E(K_m + P_n)\}$ each edge with terminal vertices of degree $n$ and $m + 2$

$E_3 = \{E(K_m + P_n)\}$ each edge with terminal vertices of degree $m + 1$ and $m + 2$

$E_4 = \{E(K_m + P_n)\}$ each edge with terminal vertices of degree $m + 2$

Now $E_1$ contains $2m$ edges, $E_2$ contains $(2n - 2)$ edges, $E_3$ contains $(n - 2)m$ edges and $E_4$ contains $(n - 3)$ edges.

$$AG(K_m + P_n) = \sum_{xy \in E_1} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E_2} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E_3} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E_4} \frac{dx + dy}{2 \sqrt{dx \cdot dy}}$$

$$= \sum_{xy \in E_1} \frac{n + m + 1}{2 \sqrt{n(m + 1)}} + \sum_{xy \in E_2} \frac{n + m + 2}{2 \sqrt{n(m + 2)}} + \sum_{xy \in E_3} \frac{m + 1 + m + 2}{2 \sqrt{(m + 1)(m + 2)}} + \sum_{xy \in E_4} \frac{2(m + 2)}{2 \sqrt{(m + 2)(m + 2)}}$$

$$= m - n + m + 1 \sqrt{2 \sqrt{n(m + 1)}} + (n - 2) \frac{n + m + 2}{\sqrt{n(m + 2)}} + \frac{2m + 3}{\sqrt{(m + 1)(m + 2)}} + n - 3$$

**Corollary 3.4.**

$$AG(K_m + P_3) = \frac{m(m + 4)}{\sqrt{3(m + 1)}} + \frac{(m + 5)}{\sqrt{3(m + 2)}} + \frac{2m + 3}{\sqrt{(m + 1)(m + 2)}}$$

**Corollary 3.5.**

$$AG(K_2 + P_3) = \frac{m(m + 4)}{\sqrt{3(m + 1)}} + \frac{(m + 5)}{\sqrt{3(m + 2)}} + \frac{2m + 3}{\sqrt{(m + 1)(m + 2)}}$$

$K_2 + P_3$ is the join of complement of $K_2$ and $P_3$. It has $5$ vertices and $8$ edges. The vertex set are of $2$ types $E_1$ and $E_2$ and $E_1$ has $4$ edges with end vertices of degree $3$ and $3$ and $E_2$ also has $4$ edges with end vertices of degree $3$ and $4$.

$$AG(K_2 + P_3) = \frac{2(2 + 4)}{\sqrt{3(2 + 1)}} + \frac{(2 + 5)}{\sqrt{3(2 + 2)}} + \frac{2 + 3}{\sqrt{(2 + 1)(2 + 2)}}$$

$$= 4 + \frac{7}{2 \sqrt{3}} + \frac{7}{2 \sqrt{3}} = 4 + \frac{7}{\sqrt{3}}$$

**Definition 3.6.** A unicyclic graph which has a unique subgraph isomorphic to a cycle. A vertex on the cycle is called cyclic vertex. Here we consider unicyclic graph with cycle $C_n$ associated with a star $K_{1,n}$ to each cyclic vertex. A pendant on a unicyclic graph is a path of length one with exactly one vertex on the cycle. The non-cyclic vertex of a pendant is called pendant vertex. For every $m, n \in N$ with $n \geq 3$, there is a graph obtained by appending $m$ pendants to each cycle vertex of $C_n$. That is, attaching a copy of $K_{1,m}$ at its vertex of degree $m$ to each cycle vertex. We denote it by the symbol $C_n \odot K_{1,m}$.

**Theorem 3.7.** $AG(C_n \odot K_{1,m}) = n[1 + \frac{m(m + 3)}{2 \sqrt{m + 2}}]$

**Proof.** We have

$$|E(C_n \odot K_{1,m})| = n + mn = n(m + 1) = |V(C_n)| |V(K_{1,m})|$$

$$AG(C_n \odot K_{1,m}) = \sum_{xy \in E(C_n \odot K_{1,m})} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E_k} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E_1} \frac{dx + dy}{2 \sqrt{dx \cdot dy}} + \sum_{xy \in E_2} \frac{dx + dy}{2 \sqrt{dx \cdot dy}}$$

$$= \sum_{xy \in E(C_k)} \frac{m + 2 + m + 2}{2 \sqrt{(m + 2)(m + 2)}} + \sum_{xy \in E_1} \frac{m + 2 + m + 1}{2 \sqrt{(m + 2)(m + 2)}}$$

$$= n + \frac{m + 3}{2 \sqrt{(m + 2)}} mn = n(1 + \frac{m + 3}{2 \sqrt{(m + 2)}})$$
Definition 3.8. A friendship graph $f_n$ is a collection of $n$ triangles with a common vertex, i.e., $f_n = K_1 + nK_2$. It can be obtained from a wheel $W_{2n}$ with a single cycle $C_{2n}$ by eliminating alternate edges of the cycle. Let $v_c$ denote the central vertex, then $d(v_c) = 2n$. Note that $f_n$ has $2n + 1$ vertices and $3n$ edges.

Theorem 3.9. For a friendship graph $f_n$ of order $2n + 1$, $AG(f_n) = n(1 + \frac{2(n+1)}{\sqrt{n}})$

Proof. The edges of $f_n$ can be partitioned into two types as follows.

$E_1 = \{xy|\text{deg}(x) = \text{deg}(y) = 2\}$

$E_2 = \{xy|\text{deg}(x) = 2 \text{ and } \text{deg}(y) = 2n\}$

\[
AG(f_n) = \sum_{xy \in E_1} \frac{dx + dy}{2\sqrt{dx \cdot dy}} + \sum_{xy \in E_2} \frac{dx + dy}{2\sqrt{dx \cdot dy}}
\]

\[
= \sum_{xy \in E_1} \frac{2 + 2}{2\sqrt{2.2}} + \sum_{xy \in E_2} \frac{2n + 2}{2\sqrt{2.2n}}
\]

\[
= \sum_{xy \in E_1} 1 + \sum_{xy \in E_2} 1
\]

\[
= n(1 + \frac{2(n+1)}{\sqrt{n}})
\]

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