Generalization of the Thomas-Wigner Rotation to Uniformly Accelerating Boosts

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ABSTRACT

In the current paper we present a generalization of the transforms from the frame co-moving with an accelerated particle for uniformly accelerated motion into an inertial frame of reference. The motivation is that the real life applications include accelerating and rotating frames with arbitrary orientations more often than the idealized case of inertial frames; our daily experiments happen in Earth-bound laboratories. We use the transforms in order to generalize the Thomas-Wigner rotation to the case of uniformly accelerated boosts.

Keywords: general coordinate transformations, uniform acceleration, Thomas-Wigner rotation.

I. INTRODUCTION

Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2][13], [15]. There is great interest in producing a general solution that deals with arbitrary orientation of the uniform acceleration. The main idea of this paper is to generate a standard blueprint for a general solution. The blueprint relies on transforming the problem geometrically in the “canonical reference frame” of [1], followed by the application of the physical transforms derived for such “canonical” orientations [1][7] and ending with the application of the inverse geometric transformation:

\[ \text{Geometry Transform} \rightarrow \text{Physics Transform} \rightarrow \text{Inverse Geometry Transform} \]  

(1)

We conclude our paper with a practical application of deriving the formula of the Thomas-Wigner rotation due to the composition of accelerated boosts.

II. DYNAMICS IN UNIFORMLY ACCELERATED FRAMES

It is well known that the composition of two non-collinear Lorentz boosts results in a Lorentz transformation that is the composition of a boost and a rotation. This rotation is called Thomas–Wigner rotation. The rotation was discovered by Llewellyn Thomas in 1926 [19] and derived formally by Wigner in 1939 [20]. In this paper we extend the formalism to the case of two successive boosts due to two arbitrary-direction constant accelerations. Consider the case of a particle moving in an arbitrary plane with the normal given by the constant acceleration \( \mathbf{g} = (g_x, g_y, g_z) \) (see Fig. 1).

According to Moller [1], the simpler case when \( \mathbf{g} \) is aligned with the x-axis produces the transformation between the accelerating frame \( S'(r) \) attached to the particle and an inertial frame \( S \):

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
ct'
\end{bmatrix} = \text{Phy_rect} \cdot \begin{bmatrix}
x \\
y \\
z \\
ct
\end{bmatrix}
\]

(2)

where:

\[
\text{Phy_rect} = \begin{bmatrix}
\cosh \frac{\mathbf{g}t}{c} & 0 & \sinh \frac{\mathbf{g}t}{c} \\
0 & 1 & 0 \\
0 & 0 & \cosh \frac{\mathbf{g}t}{c}
\end{bmatrix}
\]

(3)

In the following section, we generalize Moller’s derivation for the arbitrary case \( \mathbf{g} = (g_x, g_y, g_z) \) for obtaining the general four-space coordinate transformations that take us from \( S'(r) \) into \( S \). Expressed in polar coordinates, the acceleration has the form:

\[
\begin{align*}
g_x &= g \cos \theta \cos \phi \\
g_y &= g \sin \theta \cos \phi \\
g_z &= g \sin \phi \\
\phi &= \arcsin \frac{g_y}{g} \\
\theta &= \arctan \frac{g_z}{g_x}
\end{align*}
\]

(4)
The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the “canonical case”, followed by an application of the transformation from the uniformly rotating frame into the inertial frame, ending with the inverse of the first transformation, as shown below:

$$\begin{align*}
\begin{pmatrix}
 x \\
y \\
z
\end{pmatrix}
&=(R_{r}^{-1} \cdot \text{ Phy_rect } \cdot R_{r})
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} \\
&=A
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}
\begin{pmatrix}
 x \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
 ct \\
c t'
\end{pmatrix}
\end{align*}$$

(5)

$$R_{r} = \text{Rot}(e_{z})_{90}^{-\phi} \cdot \text{Rot}(e_{y})_{0} \cdot \text{Rot}_{y}$$

(6)

$$\text{Rot}(e_{z})_{90}^{-\phi} \cdot \text{Rot}_{y}$$ aligns \( g \) with \( e_{x} \), so the first step rotates \( g \) by \( 90^{\circ} - \phi \) around the axis the vector cross-product \( g \times e_{y} = -\frac{g_{x}}{g_{z}} e_{z} + \frac{g_{z}}{g_{x}} e_{x} \), such \( g \) gets aligned with the y-axis (see Fig. 1). For this purpose, we will introduce the pair \( (a,b) = (-\frac{g_{z}}{g_{x}}, \frac{g_{x}}{g_{z}}) \). The following expressions hold [14]:

$$\begin{align*}
\text{Rot}_{y} &=
\begin{bmatrix}
 b & 0 & -a & 0 \\
0 & 1 & 0 & 0 \\
a & 0 & b & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}$$

(7)

$$\begin{align*}
\text{Rot}(e_{z})_{90}^{-\phi} &=
\begin{bmatrix}
 \cos(90^{\circ} - \phi) & -\sin(90^{\circ} - \phi) & 0 \\
0 & 1 & 0 & 0 \\
\sin(90^{\circ} - \phi) & 0 & \cos(90^{\circ} - \phi) & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}$$

(8)

The second step is another rotation around the z-axis by \(-90^{\circ}\) that aligns \( g \) with \( e_{x} \):

$$\begin{align*}
\text{Rot}(e_{x})_{90}^{\phi} &=
\begin{bmatrix}
 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}$$

(9)

Putting it all together:

$$R_{r} = \text{Rot}(e_{z})_{90}^{-\phi} \cdot \text{Rot}(e_{y})_{0} \cdot \text{Rot}_{y}$$

(10)

A = \text{phy_rect} \cdot R_{r}

(11)

Let us consider a second boost given by the constant acceleration \( g' \). Expressed in polar coordinates, the acceleration has the form:

$$\begin{align*}
\phi' &= \arcsin \frac{g_{x}'}{g'} \\
\theta' &= \arctan \frac{g_{z}'}{g_{x}'}
\end{align*}$$

(12)

We now introduce the pair \( (a',b') = (-\frac{g_{z}'}{g_{x}'}, \frac{g_{x}'}{g_{z}'} ) \). The following expressions hold [14]:

$$\begin{align*}
\text{Rot}_{y}' &=
\begin{bmatrix}
 b' & 0 & -a' & 0 \\
0 & 1 & 0 & 0 \\
a' & 0 & b' & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}$$

(13)

$$\begin{align*}
\text{Rot}(e_{y})_{\text{phy_rect}} &=
\begin{bmatrix}
 \sin \phi' & 0 & -\cos \phi' & 0 \\
0 & 1 & 0 & 0 \\
\cos \phi' & 0 & \sin \phi' & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}$$

(14)

$$\begin{align*}
\text{phy_rect}' &=
\begin{bmatrix}
 \cosh \frac{g_{z}'}{c} & 0 & 0 & \sinh \frac{g_{z}'}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \frac{g_{z}'}{c} & 0 & 0 & \cosh \frac{g_{z}'}{c}
\end{bmatrix}
\end{align*}$$

(15)
\[
R_{r} = \text{Rot}(e_{r}) \cdot \text{Rot}(e_{r}) \cdot \text{Rot}(e_{r}) \cdot \text{Rot}(e_{r})
\]

\[
A' = R_{r}^{-1} \cdot \text{Phy}_{\text{rect}} \cdot R_{r} \cdot A \cdot R_{r}^{-1} \cdot \text{Phy}_{\text{rect}} \cdot R_{r}
\]

The boost in the arbitrary direction of acceleration \( g \) followed by the boost in the arbitrary direction of acceleration \( g' \) is therefore completely described by the transformation matrix:

\[
B = A' A = R_{r}^{-1} \cdot \text{Phy}_{\text{rect}} \cdot R_{r} \cdot R_{r}^{-1} \cdot \text{Phy}_{\text{rect}} \cdot R_{r}
\]

Formula (18) represents the generalization of the Thomas-Wigner rotation for the case of uniformly accelerated boosts. While Thomas rotation is a kinematic effect, the effect presented in this paper is a dynamic effect, the rotation is due to the changes in the direction of the acceleration. If the accelerations \( g, g' \) are collinear, then:

\[
R_{r}^{-1} = R_{r}
\]

\[
B = R_{r}^{-1} \cdot \text{Phy}_{\text{rect}} \cdot \text{Phy}_{\text{rect}} \cdot R_{r}
\]

\[
\text{Phy}_{\text{rect}} \cdot \text{Phy}_{\text{rect}} =
\begin{bmatrix}
\cosh(g + g') & 0 & 0 & \sinh(g + g')
\end{bmatrix}
\]

Expression (21) serves as a quick sanity check for the formalism as we can see that the accelerations add algebraically.

III. CONCLUSIONS

We derived the Thomas-Wigner rotation that develops due to the composition of accelerating boosts in arbitrary directions. The solution is of great interest for real life applications, because our Earth-bound laboratories are inertial only in approximation; in real life, the laboratories rotate, and they are accelerated. We produced the generalized solutions for the arbitrary case.

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