Bribery Can Get Harder in Structured Multiwinner Approval Election

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Abstract

We study the complexity of constructive bribery in the context of structured multiwinner approval elections. Given such an election, we ask whether a certain candidate can join the winning committee by adding, deleting, or swapping approvals, where each such action comes at a cost and we are limited by a budget. We assume our elections to either have the candidate interval or the voter interval property, and we require the property to hold also after the bribery. While structured elections usually make manipulative attacks significantly easier, our work also shows examples of the opposite behavior. We conclude by presenting preliminary insights regarding the destructive variant of our problem.

Keywords: bribery, structured domain, approval elections, complexity reversal

1 Introduction

We study the complexity of bribery under the multiwinner approval rule, in the case where the voters’ preferences are structured. Specifically, we use the bribery model of Faliszewski et al. [2017b], where one can either add, delete, or swap approvals, and we consider candidate interval and voter interval preferences [Elkind and Lackner, 2015].

In multiwinner elections, the voters express their preferences over the available candidates and use this information to select a winning committee (i.e., a fixed-size subset of candidates). We focus on one of the simplest and most common scenarios, where each voter specifies the approved candidates, and those with the highest numbers of approvals form the committee. Such elections are used, e.g., to choose city councils, boards of trustees, or to shortlist job candidates. Naturally, there are many other rules and scenarios, but they do not appear in practice as often as this simplest one. For more details on multiwinner voting, we point the readers to the overviews of Faliszewski et al. [2017a] and Lackner and Skowron [2021].

In our scenario, we are given an election, including the contents of all the votes, and, depending on the variant, we can either add, delete, or swap approvals, but each such action comes at a cost. Our goal is to find a cheapest set of actions that ensure that a given candidate joins the winning committee. Such problems, where we modify the votes to ensure a certain outcome, are known under the umbrella name of bribery, and were first studied by Faliszewski et al. [2009], whereas our specific variant is due to Faliszewski et al. [2017b]. Historically, bribery problems indeed aimed to model vote buying, but currently...
more benign interpretations prevail. For example, Faliszewski et al. [2017b] suggest using the cost of bribery as a measure of a candidate’s success: A candidate who did not win, but can be put into the committee at a low cost, certainly did better than one whose bribery is expensive. In particular, since our problem is used for post-election analysis, it is natural to assume that we know all the votes. For other similar interpretations, we point, e.g., to the works of Xia [2012], Shiryaev et al. [2013], Bredereck et al. [2021], Boehmer et al. [2021], or Baumeister and Hogrebe [2021]. Faliszewski and Rothe [2015] give a more general overview of bribery problems.

We assume that our elections either satisfy the candidate interval (CI) or the voter interval (VI) property [Elkind and Lackner, 2015], which correspond to the classic notions of single-peakedness [Black, 1958] and single-crossingness [Mirrlees, 1971, Roberts, 1977] from the world of ordinal elections. Briefly put, the CI property means that the candidates are ordered and each voter approves some interval of them, whereas the VI property requires that the voters are ordered and each candidate is approved by some interval of voters. For example, the CI assumption can be used to model political elections, where the candidates appear on the left-to-right spectrum of opinions and the voters approve those, whose opinions are close enough to their own. Importantly, we require our elections to have the CI/VI property also after the bribery; this approach is standard in bribery problems with structured elections [Brandt et al., 2015, Menon and Larson, 2016, Elkind et al., 2020], as well as in other problems related to manipulating election results [Walsh, 2007, Faliszewski et al., 2011, Fitzsimmons and Hemaspaandra, 2015] (these references are examples only).

**Example 1.** Let us consider a hotel in a holiday resort. The hotel has its base staff, but each month it also hires some additional help. For the coming month, the expectation is to hire extra staff for \( k \) days. Naturally, they would be hired for the days when the hotel is most busy (the decision to request additional help is made a day ahead, based on the observed load; the \( k \) days do not need to be consecutive). Since hotel bookings are typically made in advance, one knows which days are expected to be most busy. However, some people will extend their stays, some will leave early, and some will have to shift their stays. Thus the hotel managers would like to know which days are likely to become the busiest ones after such changes: Then they could inform the extra staff as to when they are expected to be needed, and what changes in this preliminary schedule might happen. Our bribery problem (for the CI setting) captures exactly the problem that the managers want to solve: The days are the candidates, \( k \) is the committee size, and the bookings are the approval votes (note that each booking must regard a consecutive set of days). Prices of adding, deleting, and moving approvals correspond to the likelihood that a particular change actually happens (the managers usually know which changes are more or less likely). Since the bookings must be consecutive, the election has to have the CI property also after the bribery. The managers can solve such bribery problem for each of the days and see which ones can most easily be among the \( k \) busiest ones.

Note that the value of \( k \) can be estimated from previous experience, be limited by the budget for the salaries for the additional workers, depend on the predicted workload, be subject to the availability of the additional workers, and the like. The managers may even want to solve several instances of the problem, with different values of \( k \).

**Example 2.** For the VI setting, let us consider a related scenario. There is a team of archaeologists who booked a set of excavation sites, each for some consecutive number of days (they work on several sites in parallel). The team may want to add some extra staff to those sites that require most working days. However, as in the previous example, the bookings might get extended or shortened. The team’s manager may use bribery to evaluate how likely it is that each of the sites becomes one of the most work-demanding ones. In this case, the days are the voters, and the sites are the candidates.

There are two main reasons why structured elections are studied. Foremost, as in the above examples, sometimes they simply capture the exact problem at hand. Second, many
problems that are intractable in general, become polynomial-time solvable if the elections are structured. Indeed, this is the case for many NP-hard winner-determination problems [Betzler et al., 2013; Elkind and Lackner, 2015; Peters and Lackner, 2020] and for various problems where the goal is to make some candidate a winner [Faliszewski et al., 2011; Magiera and Faliszewski, 2017], including some bribery problems [Brandt et al., 2015; Elkind et al., 2020]. There are also some problems that stay intractable even for structured elections [Schlotter et al., 2017; Yang, 2017] as well as examples of complexity reversals, where assuming structured preferences turns a polynomial-time solvable problem into an intractable one. However, such reversals are rare and, to the best of our knowledge, so far were only observed by Menon and Larson [2016], for the case of weighted elections with three candidates (but see also the work of Fitzsimmons and Hemaspaandra [2015], who find complexity reversals that stem from replacing total ordinal votes with ones that include ties).

Our Contribution. We provide an almost complete picture of the complexity of bribery by either adding, deleting, or swapping approvals under the multiwinner approval voting rule, for the case of CI and VI elections, assuming either that each bribery action has identical unit price or that they can be priced individually (see Table 1). By comparing our results to those for the unrestricted setting, provided by Faliszewski et al. [2017b], we find that any combination of tractability and intractability in the structured and unrestricted setting is possible. For example:

1. Bribery by adding approvals is solvable in polynomial time irrespective if the elections are unrestricted or have the CI or VI properties.
2. Bribery by deleting approvals (where each deleting action is individually priced) is solvable in polynomial time in the unrestricted setting, but becomes NP-hard for CI elections (for VI ones it is still in P).
3. Bribery by swapping approvals only to the designated candidate (with individually priced actions) is NP-hard in the unrestricted setting, but becomes polynomial-time solvable both for CI and VI elections.
4. Bribery by swapping approvals (where each action is individually priced and we are not required to swap approvals to the designated candidate only) is NP-hard in each of the considered settings.

We largely focus on the constructive setting, where the goal is to ensure that some candidate belongs to at least one winning committee (indeed, all the results above are for this setting). However, we also give a glimpse of what happens in the destructive setting, where we want to ensure that a given candidate does not belong to any winning committees. In a certain sense, in this case we also observe a form of “reversal.” Typically, destructive variants of bribery (and related) problems are at least as easy to work with as their constructive counterparts, and lead to more positive results. In our case—albeit this is mostly an intuitive feeling—the situation for CI elections is the opposite. Obtaining the results for the destructive setting seems more challenging and leads to less satisfying theorem statements (e.g., we need less appealing prices) than in the constructive setting.

Possibility of Complexity Reversals. So far, most of the problems studied for structured elections were subproblems of the unrestricted ones. For example, a winner determination algorithm that works for all elections, clearly also works for the structured ones and complexity reversal is impossible. The case of bribery is different because, by assuming structured elections, not only do we restrict the set of possible inputs, but also we constrain the possible actions. Yet, scenarios where bribery is tractable are rare, and only a handful of papers considered bribery in structured domains (we mention those of Brandt et al. [2015],

These references are not complete and are meant as examples.
Unrestricted
Candidate Interval (CI)
Voter Interval (VI)
(prices)
(unit) (any)

AddApprovals
P
P
Faliszewski et al. [2017b]
Thm. 2
Thm. 2

DelApprovals
P
P
Faliszewski et al. [2017b]

SwapApprovals
P
NP-com.
Faliszewski et al. [2017b]

SwapApprovals
P
NP-com.
Faliszewski et al. [2017b]

Table 1: Our results for the CI and VI domains, together with those of Faliszewski et al. [2017b] for the unrestricted setting. SWAP_APPROVALS_TO_P refers to the problem where each action has to move an approval to the preferred candidate.

Fitzsimmons and Hemaspaandra [2015], Menon and Larson [2016], Elkind et al. [2020]), so opportunities for observing complexity reversals were, so far, very limited. We show several such reversals, obtained for very natural settings.

2 Preliminaries

For a positive integer \( t \), we write \([t]\) to mean the set \( \{1, \ldots, t\} \). By writing \([t]_0\) we mean the set \( \{t\} \cup \{0\} \).

Approval Elections. An approval election \( E = (C, V) \) consists of a set of candidates \( C = \{c_1, \ldots, c_m\} \) and a collection of voters \( V = \{v_1, \ldots, v_n\} \). Each voter \( v_i \in V \) has an approval ballot (or, equivalently, an approval set) which contains the candidates that \( v_i \) approves. We write \( v_i \) to refer both to the voter and to his or her approval ballot; the meaning will always be clear from the context.

A multiwinner voting rule is a function \( f \) that given an election \( E = (C, V) \) and a committee size \( k \in [|C|] \) outputs a nonempty family of winning committees (where each committee is a size-\( k \) subset of \( C \)). We disregard the issue of tie-breaking and assume all winning committees to be equally worthy, i.e., we adopt the nonunique winner model.

Given an election \( E = (C, V) \), we let the approval score of a candidate \( c \in C \) be the number of voters that approve \( c \), and we denote it as \( \text{score}_E(c) \). The approval score of a committee \( S \subseteq C \) is \( \text{score}_E(S) = \sum_{c \in S} \text{score}_E(c) \). Given an election \( E \) and a committee size \( k \), the multiwinner approval voting rule, denoted AV, outputs all size-\( k \) committees with the highest approval score. Occasionally we also consider the single-winner approval rule, which is defined in the same way as its multiwinner variant, except that the committee size is fixed to be one. For simplicity, in this case we assume that the rule returns a set of tied winners (rather than a set of tied size-1 winning committees).

Structured Elections. We focus on elections where the approval ballots satisfy either the candidate interval (CI) or the voter interval (VI) properties Elkind and Lackner [2015]:

1. An election has the CI property (is a CI election) if there is an ordering of the candidates (called the societal axis) such that each approval ballot forms an interval with respect to this ordering.

2. An election has the VI property (is a VI election) if there is an ordering of the voters such that each candidate is approved by an interval of the voters (for this ordering).

Given a CI election, we say that the voters have CI ballots or, equivalently, CI preferences; we use analogous conventions for the VI case. As observed by Elkind and Lackner [2015], there are polynomial-time algorithms that test if a given election is CI or VI and, if so, provide
appropriate orders of the candidates or voters; these algorithms are based on solving the consecutive ones problem [Booth and Lueker 1976].

**Notation for CI Elections.** Let us consider a candidate set $C = \{c_1, \ldots, c_m\}$ and a societal axis $\succ = c_1c_2\cdots c_m$. Given two candidates $c_i, c_j$, where $i \leq j$, we write $[c_i, c_j]$ to denote the approval set \{c_i, c_{i+1}, \ldots, c_j\}.

**Bribery Problems.** We focus on the variants of bribery in multiwinner approval elections defined by Faliszewski et al. [2017b]. Let $f$ be a multiwinner voting rule and let $\text{Op}$ be one of $\text{AddApprovals}$, $\text{DelApprovals}$, and $\text{SwapApprovals}$ operations (in our case $f$ will either be AV or its single-winner variant). In the $f$-$\text{Op}$-$\text{Bribery}$ problem we are given an election $E = (C, V)$, a committee size $k$, a preferred candidate $p$, and a nonnegative integer $B$ (the budget). We ask if it is possible to perform at most $B$ unit operations of type $\text{Op}$, so that $p$ belongs to at least one winning committee (this is the constructive variant of the problem; requiring only minor adaptions in proofs, presented in Appendix A all our results for this variant also hold if we require that $p$ belongs to all winning committees):

1. For $\text{AddApprovals}$, a unit operation adds a given candidate to a given voter’s ballot.
2. For $\text{DelApprovals}$, a unit operation removes a given candidate from a given voter’s ballot.
3. For $\text{SwapApprovals}$, a unit operation replaces a given candidate with another one in a given voter’s ballot.

Like Faliszewski et al. [2017b], we also study the variants of $\text{AddApprovals}$ and $\text{SwapApprovals}$ problems where each unit operation must involve the preferred candidate.

We are also interested in the priced variants of the above problems, where each unit operation comes at a cost that may depend both on the voter and the particular affected candidates; we ask if we can achieve our goal by performing operations of total cost at most $B$. We distinguish the priced variants by putting a dollar sign in front of the operation type. For example, $\$\text{AddApprovals}$ means a variant where adding each candidate to each approval ballot has an individual cost.

**Bribery in Structured Elections.** We focus on the bribery problems where the elections have either the CI or the VI property. For example, in the AV-$\$\text{AddApprovals}$-$\text{CI}$-$\text{Bribery}$ problem the input election has the CI property (under a given societal axis) and we ask if it is possible to add approvals with up to a given cost so that (a) the resulting election has the CI property for the same societal axis, and (b) the preferred candidate belongs to at least one winning committee. The VI variants are defined analogously (in particular, the voters’ order witnessing the VI property is given and the election must still have the VI property with respect to this order after the bribery). The convention that the election must have the same structural property before and after the bribery, and the fact that the order witnessing this property is part of the input, is standard in the literature; see, e.g., the works of Faliszewski et al. [2011], Brandt et al. [2015], Menon and Larson [2016], and Elkind et al. [2020]. Further, it also follows naturally from some applications (as in the scenarios from Examples 1 and 2).

**Computational Problems.** For a graph $G$, by $V(G)$ we mean its set of vertices and by $E(G)$ we mean its set of edges. A graph is cubic if each of its vertices is connected to exactly three other ones. Our NP-hardness proofs rely on reductions from variants of the INDEPENDENT SET and X3C problems, both known to be NP-complete [Garey and Johnson 1979, Gonzalez 1985].

**Definition 1.** In the Cubic INDEPENDENT SET problem we are given a cubic graph $G$ and an integer $h$; we ask if $G$ has an independent set of size $h$ (i.e., a set of $h$ vertices such that no two of them are connected).
Definition 2. In the Restricted Exact Cover by 3-Sets problem (RX3C) we are given a universe $X$ of 3n elements and a family $S$ of 3n size-3 subsets of $X$. Each element from $X$ appears in exactly three sets from $S$. We ask if it is possible to choose $n$ sets from $S$ whose union is $X$.

3 Adding Approvals

For the case of adding approvals, all our bribery problems (priced and unpriced, both for the CI and VI domains) remain solvable in polynomial time. Yet, compared to the unrestricted setting, our algorithms require more care. For example, in the unrestricted case it suffices to simply add approvals for the preferred candidate [Faliszewski et al., 2017b] (choosing the voters where they are added in the order of increasing costs for the priced variant); a similar approach works for the VI case, but with a different ordering of the voters.

Theorem 1. AV-$\text{AddApprovals-VI-Bribery} \in P$.

Proof. Consider an input with election $E = (C, V)$, committee size $k$, preferred candidate $p$, and budget $B$. Without loss of generality, we assume that $V = \{v_1, \ldots, v_n\}$ and the order witnessing the VI property is $v_1 \triangleright v_2 \triangleright \cdots \triangleright v_n$. We note that it is neither beneficial nor necessary to ever add approvals for candidates other than $p$. Let $v_i, v_{i+1}, \ldots, v_j$ be the interval of voters that approves $p$, and let $s$ be the lowest number of additional approvals that $p$ needs to obtain to become a member of some winning committee (note that $s$ is easily computable in polynomial time). Our algorithm proceeds as follows: We consider all nonnegative numbers $s_\ell$ and $s_r$ such that (a) $s = s_\ell + s_r$, (b) $i - s_\ell \geq 1$, and (c) $j + s_r \leq n$, and for each of them we compute the cost of adding an approval for $p$ to voters $v_{i-s_\ell}, \ldots, v_{i-1}$ and $v_{j+1}, \ldots, v_{j+s_r}$. We choose the pair that generates lowest cost and we accept if this cost is at most $B$. Otherwise we reject.

The polynomial running time follows directly. Correctness is guaranteed by the fact that we need to maintain the VI property and that it suffices to add approvals for $p$ only. □

The CI case introduces a different complication. Now, adding an approval for the preferred candidate in a given vote also requires adding approvals for all those between him or her and the original approval set. Thus, in addition to bounding the bribery’s cost, we also need to track the candidates whose scores increase.

Theorem 2. AV-$\text{AddApprovals-CI-Bribery} \in P$.

Proof. Our input consists of an election $E = (C, V)$, committee size $k$, preferred candidate $p \in C$, budget $B$, and the information about the costs of all the possible operations (i.e., for each voter and each candidate that he or she does not approve, we have the price for adding this candidate to the voter’s ballot). Without loss of generality, we assume that $C = \{\ell_m', \ldots, \ell_1, p, r_1, \ldots, r_m'\}$, $V = \{v_1, \ldots, v_n\}$, each voter approves at least one candidate and the election is CI with respect to the order:

$$\triangleright = \ell_m' \cdots \ell_2 \ell_1 \ p \ r_1 \ r_2 \cdots r_m'.$$

We start with a few observations. First, if a voter already approves $p$ then there is no point in adding any approvals to his or her ballot. Second, if no voter does not approve $p$, then we should either add any approvals to his or her ballot, or add exactly those approvals that are necessary to ensure that $p$ gets one. For example, if some voter has approval ballot $\{r_3, r_4, r_5\}$ then we may either choose to leave it intact or to extend it to $\{p, r_1, r_2, r_3, r_4, r_5\}$. We let $L = \{\ell_m', \ldots, \ell_1\}$ and $R = \{r_1, \ldots, r_m'\}$, and we partition the voters into three groups, $V_L$, $V'_L$, and $V''_L$, as follows:

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2Without this assumption we could still make our algorithm work. We would guess the number of voters who do not approve any candidates to approve $p$ alone (we would choose these voters to minimize the cost of adding these approvals). Then we would continue as in the proof, but knowing that none of the voters in the group can be bribed further.
1. $V_p$ contains all the voters who approve $p$,

2. $V_L$ contains the voters who approve members of $L$ only,

3. $V_R$ contains the voters who approve members of $R$ only.

Our algorithm proceeds as follows (by guessing we mean iteratively trying all possibilities; Steps 3 and 4 will be described later):

1. Guess the numbers $x_L$ and $x_R$ of voters from $V_L$ and $V_R$ whose approval ballots will be extended to approve $p$.

2. Guess the numbers $t_L$ and $t_R$ of candidates from $L$ and $R$ that will end up with higher approval scores than $p$ (we must have $t_L + t_R < k$ for $p$ to join a winning committee).

3. Compute the lowest cost of extending exactly $x_L$ votes from $V_L$ to approve $p$, such that at most $t_L$ candidates from $L$ end up with more than $\text{score}_E(p) + x_L + x_R$ points (i.e., with score higher than $p$); denote this cost as $B_L$.

4. Repeat the above step for the $x_R$ voters from $V_R$, with at most $t_R$ candidates obtaining more than $\text{score}_E(p) + x_L + x_R$ points; denote the cost of this operation as $B_R$.

5. If $B_L + B_R \leq B$ then accept (reject if no choice of $x_L$, $x_R$, $t_L$, and $t_R$ leads to acceptance).

One can verify that this algorithm is correct (assuming we know how to perform Steps 2 and 3).

Next we describe how to perform Step 3 in polynomial time (Step 4 is handled analogously). We will need some additional notation. For each $i \in [m']$, let $V_L(i)$ consist exactly of those voters from $V_L$ whose approval ballots include candidate $\ell_i$ but do not include $\ell_{i-1}$ (in other words, voters in $V_L(i)$ have approval ballots of the form $[\ell_j, \ell_i]$, where $j \geq i$). Further, for each $i \in [m']$ and each $e \in [[V_L(i)]]_0$ let $\text{cost}(i, e)$ be the lowest cost of extending $e$ votes from $V_L(i)$ to approve $p$ (and, as a consequence, to also approve candidates $\ell_{i-1}, \ldots, \ell_1$).

If $V_L(i)$ contains fewer than $e$ voters then $\text{cost}(i, e) = +\infty$. For each $e \in [x_L]_0$, we define $S(e) = \text{score}_E(p) + e + x_R$. Finally, for each $i \in [m']$, $e \in [x_L]_0$, and $t \in [t_L]_0$ we define function $f(i, e, t)$ so that:

$$f(i, e, t) = \text{the lowest cost of extending exactly } e \text{ votes from } V_L(1) \cup \cdots \cup V_L(i) \text{ (to approve } p) \text{ such that at most } t \text{ candidates among } \ell_1, \ldots, \ell_i \text{ end up with more than } S(e) \text{ points (function } f \text{ takes value } +\infty \text{ if doing so is impossible).}$$

Our goal in Step 3 of the main algorithm is to compute $f(m', x_L, t_L)$, which we do via dynamic programming. To this end, we observe that the following recursive equation holds (let $\chi(i, e)$ be 1 if $\text{score}_E(\ell_i) > S(e)$ and let $\chi(i, e)$ be 0 otherwise; we explain the idea of the equation below):

$$f(i, e, t) = \min_{e' \in [e]_0} \left( \text{cost}(i, e') + f(i-1, e-e', t-\chi(i, e)) \right).$$

The intuition behind this equation is as follows. We consider each possible number $e' \in [e]_0$ of votes from $V_L(i)$ that can be extended to approve $p$. The lowest cost of extending the votes of $e'$ voters from $V_L(i)$ is, by definition, $\text{cost}(i, e')$. Next, we still need to extend $e-e'$ votes from $V_L(i-1), \ldots, V_L(1)$ and, while doing so, we need to ensure that at most $t$ candidates end up with at most $S(e)$ points. Candidate $\ell_i$ cannot get any additional approvals from voters $V_L(i-1), \ldots, V_L(1)$, so he or she exceeds this value exactly if $\text{score}_E(\ell_i) > S(e)$ or, equivalently, if $\chi(i, e) = 1$. This means that we have to ensure that at most $t - \chi(i, e)$ candidates among $\ell_{i-1}, \ldots, \ell_1$ end up with at most $S(e)$ points. However, since we extend $e'$ votes from $V_L(i)$, we know that candidates $\ell_{i-1}, \ldots, \ell_1$ certainly obtain $e'$ additional points (as compared to the input election). Thus we need to ensure that at most $t - \chi(i, e)$ of them end up with score at most $S(e-e')$ after extending the votes from $V_L(1) \cup \cdots \cup V_L(i-1)$. 

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This is ensured by the \( f(i - 1, e - e', t - \chi(i, e)) \) component in the equation (which also provides the lowest cost of the respective operations).

Using the above formula, the fact that \( f(1, e, t) \) can be computed easily for all values of \( e \) and \( t \), and standard dynamic programming techniques, we can compute \( f(m', x_t, t_t) \) in polynomial time. This suffices for completing Step 3 of the main algorithm and we handle Step 4 analogously. Since all the steps of can be performed in polynomial time, the proof is complete.

Both above theorems also apply to the cases where we can only add approvals for the preferred candidate. The algorithm from Theorem 1 is designed to do just that, and for the algorithm from Theorem 2 we can set the price of adding other approvals to be \( +\infty \).

4 Deleting Approvals

The case of deleting approvals is more intriguing. Roughly speaking, in the unrestricted setting it suffices to delete approvals from sufficiently many candidates that have higher scores than \( p \), for whom doing so is least expensive [Faliszewski et al., 2017b]. The same general strategy works for the VI case because we still can delete approvals for different candidates independently.

**Theorem 3.** AV-\( \$\text{DelApprovals-VI-Bribery} \in \text{P} \).

**Proof.** Let our input consist of an election \( E = (C, V) \), preferred candidate \( p \in C \), committee size \( k \), and budget \( B \). We assume that \( V = \{v_1, \ldots, v_n\} \) and the election is VI with respect to ordering the voters by their indices. Let \( s = \text{score}_E(p) \) be the score of \( p \) prior to any bribery. We refer to the candidates with score greater than \( s \) as superior. Since it is impossible to increase the score of \( p \) by deleting approvals, we need to ensure that the number of superior candidates drops to at most \( k - 1 \).

For each superior candidate \( c \), we compute the lowest cost for reducing his or her score to exactly \( s \). Specifically, for each such candidate \( c \) we act as follows. Let \( t = \text{score}_E(c) - s \) be the number of \( c \)'s approvals that we need to delete and let \( v_a, v_a+1, \ldots, v_b \) be the interval of voters that approve \( c \). For each \( i \in [t] \) we compute the cost of deleting \( c \)'s approvals among the first \( i \) and the last \( t - i \) voters in the interval (these are the only operations that achieve our goal and maintain the VI property of the election); we store the lowest of these costs as “the cost of \( c \).”

Let \( S \) be the number of superior candidates (prior to any bribery). We choose \( S - (k - 1) \) of them with the lowest costs. If the sum of these costs is at most \( B \) then we accept and, otherwise, we reject.

For the CI case, our problem turns out to be NP-complete. Intuitively, the reason for this is that in the CI domain deleting an approval for a given candidate requires either deleting all the approvals to the left or all the approvals to the right on the societal axis. Indeed, our main trick is to introduce approvals that must be deleted (at zero cost), but doing so requires choosing whether to delete their left or their right neighbors (at nonzero cost). This result is our first example of a complexity reversal.

**Theorem 4.** AV-\( \$\text{DelApprovals-CI-Bribery} \) is NP-complete.

**Proof.** We give a reduction from RX3C. Let \( I = (X, \mathcal{S}) \) be the input instance, where \( X = \{x_1, \ldots, x_{3n}\} \) is the universe and \( \mathcal{S} = \{S_1, \ldots, S_{3n}\} \) is a family of size-3 subsets of \( X \). By definition, each element of \( X \) belongs to exactly three sets from \( \mathcal{S} \). We form an instance of AV-\( \$\text{DelApprovals-CI-Bribery} \) as follows.

We have the preferred candidate \( p \), for each universe element \( x_i \in X \) we have corresponding universe candidate \( x_i \), for each set \( S_j \in \mathcal{S} \) we have set candidate \( s_j \), and we have set \( D \) of \( 2n \) dummy candidates (where each individual one is denoted by \( \diamond \) ). Let \( C \) be the
set of just-described $8n + 1$ candidates and let $S = \{s_1, \ldots, s_{3n}\}$ contain the set candidates. We fix the societal axis to be:

$$\triangledown = s_1 \cdots s_{3n} \triangledown \cdots \triangledown x_1 \cdots x_{3n} \triangledown$$

Next, we form the voter collection $V$:

1. For each candidate in $S \cup D \cup \{p\}$, we have two voters that approve exactly this candidate. We refer to them as the fixed voters and we set the price for deleting their approvals to be $+\infty$. We refer to their approvals as fixed.

2. For each set $S_j = \{x_a, x_b, x_c\}$, we form three solution voters, $v(s_j, x_a)$, $v(s_j, x_b)$, and $v(s_j, x_c)$, with approval sets $[s_j, x_a]$, $[s_j, x_b]$, and $[s_j, x_c]$, respectively. For a solution voter $v(s_i, x_d)$, we refer to the approvals that $s_i$ and $x_d$ receive as exterior, and to all the other ones as interior. The cost for deleting each exterior approval is one, whereas the cost for deleting the interior approvals is zero. Altogether, there are $9n$ solution voters.

To finish the construction, we set the committee size $k = n + 1$ and the budget $B = 9n$. Below, we list the approval scores prior to any bribery (later we will see that in successful bribes one always deletes all the interior approvals):

1. $p$ has 2 fixed approvals,
2. each universe candidate has 3 exterior approvals (plus some number of interior ones),
3. each set candidate has 3 exterior approvals and 2 fixed ones (plus some number of interior ones), and
4. each dummy candidate has 2 fixed approvals (and $9n$ interior ones).

We claim that there is a bribery of cost at most $B$ that ensures that $p$ belongs to some winning committee if and only if $I$ is a yes-instance of RX3C. For the first direction, let us assume that $I$ is a yes-instance and let $T$ be a size-$n$ subset of $S$ such that $\bigcup_{S_i \in T} S_i = X$ (i.e., $T$ is the desired exact cover). We perform the following bribery: First, for each solution voter we delete all his or her interior approvals. Next, to maintain the CI property (and to lower the scores of some candidates), for each solution voter we delete one exterior approval. Specifically, for each set $S_j = \{x_a, x_b, x_c\}$, if $S_j$ belongs to the cover (i.e., if $S_j \in T$) then we delete the approvals for $x_a$, $x_b$, and $x_c$ in $v(s_j, x_a)$, $v(s_j, x_b)$, and $v(s_j, x_c)$, respectively; otherwise, i.e., if $S_j \notin T$, we delete the approvals for $s_j$ in these votes. As a consequence, all the universe candidates end up with two exterior approvals each, the $n$ set candidates corresponding to the cover end up with three approvals each (two fixed ones and one exterior), the $2n$ remaining set candidates and all the dummy candidates end up with two fixed approvals each. Since $p$ has two approvals, the committee size is $n + 1$, and only $n$ candidates have score higher than $p$, $p$ belongs to some winning committee (and the cost of the bribery is $B$).

For the other direction, let us assume that there is a bribery with cost at most $B$ that ensures that $p$ belongs to some winning committee. It must be the case that this bribery deletes exactly one exterior approval from each solution voter. Otherwise, since there are $9n$ solution voters and the budget is also $9n$, some solution voter would keep both his or her exterior approvals, as well as all the interior ones. This means that after the bribery there would be at least $2n$ dummy candidates with at least three points each. Then, $p$ would not belong to any winning committee. Thus, each solution voter deletes exactly one exterior approval, and we may assume that he or she also deletes all the interior ones (this comes at zero cost and does not decrease the score of $p$).
By the above discussion, we know that all the dummy candidates end up with two fixed approvals, i.e., with the same score as $p$. Thus, for $p$ to belong to some winning committee, at least $5n$ candidates among the set and universe ones also must end up with at most two approvals (at most $n$ candidates can have score higher than $p$). Let $x$ be the number of set candidates whose approval score drops to at most two, and let $y$ be the number of such universe candidates. We have that:

$$0 \leq x \leq 3n, \quad 0 \leq y \leq 3n, \quad \text{and} \quad x + y \geq 5n. \quad (1)$$

Prior to the bribery, each set candidate has five non-interior approvals (including three exterior approvals) so bringing his or her score to at most two costs three units of budget. Doing the same for a universe candidate costs only one unit of budget, as universe candidates originally have only three non-interior approvals. Since our total budget is $9n$, we have:

$$3x + y \leq 9n. \quad (2)$$

Together, inequalities (1) and (2) imply that $x = 2n$ and $y = 3n$. That is, for each universe candidate $x_i$ there is a solution voter $v(s_j, x_d)$ who is bribed to delete the approval for $x_d$ (and, as a consequence of our previous discussion, who is not bribed to delete the approval for $s_j$). We call such solution voters active and we define a family:

$$\mathcal{T} = \{S_j \mid s_j \text{ is approved by some active solution voter}\}.$$

We claim that $\mathcal{T}$ is an exact cover for the RX3C instance $I$. Indeed, by definition of active solution voters we have that $\bigcup_{S_i \in \mathcal{T}} S_i = X$. Further, it must be the case that $|\mathcal{T}| = n$. This follows from the observation that if some solution voter is active then his or her corresponding set candidate $s_j$ has at least three approvals after the bribery (each set candidate receives exterior approvals from exactly three solution voters and these approvals must be deleted if the candidate is to end up with score two; this is possible only if all the three solution voters are not active). Since exactly $2n$ set candidates must have their scores reduced to two, it must be that $3n - |\mathcal{T}| = 2n$, so $|\mathcal{T}| = n$. This completes the proof.

The above proof strongly relies on using 0/1/+∞ prices. The case of unit prices remains open and we believe that resolving it might be quite challenging.

5 Swapping Approvals

In some sense, bribery by swapping approvals is our most interesting scenario because there are cases where a given problem has the same complexity both in the unrestricted setting and for some structured domain (and this happens both for tractability and NP-completeness), as well as cases where the unrestricted variant is tractable but the structured one is not or the other way round.

5.1 Approval Swaps to the Preferred Candidate

Let us first consider a variant of AV-SwapApprovals-Bribery where each unit operation moves an approval from some candidate to the preferred one. We call operations of this form SwapApprovals to $p$. In the unrestricted setting, this problem is in P for unit prices but is NP-complete if the prices are arbitrary. For the CI and VI domains, the problem can be solved in polynomial time for both types of prices. While for the CI domain this is not so surprising—indeed, in this case possible unit operations are very limited—the VI case requires quite some care.

Theorem 5. AV-$\$$SwapApprovals to p-CI-Bribery $\in$ P.
Proof. Consider an input with CI election $E = (C, V)$, committee size $k$, preferred candidate $p$, and budget $B$. W.l.o.g., we assume that $C = \{m', \ldots, l_1, p, r_1, \ldots, r_{m''}\}$, $V = \{v_1, \ldots, v_n\}$, and the societal axis is:

$$\succ = \ell_{m'} \cdots \ell_{l_1} p r_1 r_2 \cdots r_{m''}.$$  

Since unit operations must move approvals to $p$, for each voter $v_i$ exactly one of the following holds:

1. There is $t \in [m']$ such that $v_i$ has approval set $[\ell_t, \ell_1]$ and the only possible operation is to move an approval from $\ell_t$ to $p$ at some given cost.

2. There is $t \in [m'']$ such that $v_i$ has approval set $[r_1, r_1]$ and the only possible operation is to move an approval from $r_t$ to $p$ at some given cost.

3. It is illegal to move any approvals for this voter.

For each candidate $c \in C \setminus \{p\}$ and each integer $x$, we let $f(c, x)$ be the lowest cost of moving $x$ approvals from $c$ to $p$ (we assume that $f(c, x) = +\infty$ if doing so is impossible). By the above discussion, we can compute the values of $f$ in polynomial time.

Our algorithm proceeds as follows. First, we guess score $y \in [n]$ that we expect $p$ to end up with. Second, we let $S$ be the set of candidates that in the input election have score higher than $y$. For each candidate $c \in S$ we define his or her cost to be $f(c, \text{score}_E(c) - y)$, i.e., the lowest cost of moving approvals from $c$ to $p$ so that $c$ ends up with score $y$. Then we let $S'$ be a set of $|S| - (k - 1)$ members of $S$ with the lowest costs (if $|S| \leq k - 1$ then $S'$ is an empty set). For each $c \in S'$, we perform the approval moves implied by $f(c, \text{score}_E(c) - y)$. Finally, we ensure that $p$ has $y$ approvals by performing sufficiently many of the cheapest still-not-performed unit operations (we reject for this value of $y$ if not enough operations remained). If the total cost of all performed unit operations is at most $B$, we accept (indeed, we have just found a bribery that ensures that there are at most $k - 1$ candidates with score higher than $p$ and whose cost is not too high). Otherwise, we reject for this value of $y$. If there is no $y$ for which we accept, we reject.

Our algorithm for the VI case is based on dynamic programming (expressed as searching for a shortest path in a certain graph) and relies on the fact that due to the VI property we avoid performing the same unit operations twice.

Theorem 6. AV-$\$SwapApprovals to p-VI-Bribery $\in P$.

Proof. Consider an instance of our problem with an election $E = (C, V)$, committee size $k$, preferred candidate $p$, and budget $B$. Without loss of generality, we assume that $V = \{v_1, \ldots, v_n\}$ and that the election is VI with respect to the order $v_1 \succ v_2 \succ \cdots \succ v_n$. We also assume that $p$ has at least one approval (if it were not the case, we could try all possible single-approval swaps to $p$ to try all ways in which this assumption can be satisfied; if $p$ is not already a member of some winning committee then we know that $p$ needs to receive at least one approval, so this procedure is correct). On a high level, our algorithm proceeds as follows: We try all pairs of integers $\alpha$ and $\beta$ such that $1 \leq \alpha \leq \beta \leq n$ and, for each of them, we check if there is a bribery with cost at most $B$ that ensures that the preferred candidate is (a) approved exactly by voters $v_\alpha, \ldots, v_\beta$, and (b) belongs to some winning committee. If such a bribery exists then we accept and, otherwise, we reject. Below we describe the algorithm that finds the cheapest successful bribery for a given pair $\alpha, \beta$ (if one exists).

Let $\alpha$ and $\beta$ be fixed. Further, let $x, y$ be two integers such that in the original election $p$ is approved exactly by voters $v_x, v_{x+1}, \ldots, v_y$. Naturally, we require that $\alpha \leq x \leq y \leq \beta$; if this condition is not met then we drop this $\alpha$ and $\beta$. We let $s = \beta - \alpha + 1$ be the score that $p$ is to have after the bribery. We say that a candidate $c \in C \setminus \{p\}$ is dangerous if his or her score in the original election is above $s$. Otherwise, we say that this candidate is safe. Let $D$ be the number of dangerous candidates. For $p$ to become a member of some winning
committee, we need to ensure that after the bribery at most \( k - 1 \) dangerous candidates still have more than \( s \) points (each safe candidate certainly has at most \( s \) points).

To do so, we analyze a certain digraph. For each pair of integers \( a \) and \( b \) such that \( \alpha \leq a \leq b \leq \beta \) and each integer \( d \in [\lceil C \rceil - 1]_0 \) we form a node \((a, b, d)\), corresponding to the fact that there is a bribery after which \( p \) is approved exactly by voters \( v_a, v_{a+1}, \ldots, v_b \) and exactly \( d \) dangerous candidates have scores above \( s \). Given two nodes \( u' = (a', b', d') \) and \( u'' = (a'', b'', d'') \), such that \( a' \geq a'', \ b' \leq b'', \) and \( d'' \in \{d', d' - 1\} \), there is a directed edge from \( u' \) to \( u'' \) with weight cost\((u', u'')\) exactly if there is a candidate \( c \) such that after bribing voters \( v_{a''}, v_{a''+1}, \ldots, v_{b''-1} \) and \( v_{b''+1}, \ldots, v_{b''} \) to move an approval from \( c \) to \( p \) it holds that:

1. voters approving \( c \) still form an interval,
2. if \( c \) is a dangerous candidate and his or her score drops to at most \( s \), then \( d'' = d' - 1 \), and, otherwise, \( d'' = d' \), and
3. the cost of this bribery is exactly cost\((u', u'')\).

One can verify that for each node \( u = (a, b, d) \) the weight of the shortest path from \((x, y, D)\) to \( u \) is exactly the price of the lowest-cost bribery that ensures that \( p \) is approved by voters \( v_a, \ldots, v_b \) and exactly \( d \) dangerous candidates have scores above \( s \) (the VI property ensures that no approval is ever moved twice). Thus it suffices to find a node \((\alpha, \beta, K)\) such that \( K < k \), for which the shortest path from \((x, y, D)\) is at most \( B \). Doing so is possible in polynomial time using, e.g., the classic algorithm of Dijkstra.

\[ \square \]

### 5.2 Arbitrary Swaps

Next, we consider the full variant of bribery by swapping approvals. For the unrestricted domain, the problem is NP-complete for general prices, but admits a polynomial-time algorithm for unit ones \[{Faliszewski et al., 2017}\]. For the CI domain, NP-completeness holds even for the latter.

**Remark 1.** The model of unit prices, applied directly to the case of SwapApprovals-CIBribery, is somewhat unintuitive. For example, consider societal axis \( c_1 \triangleright c_2 \triangleright \cdots \triangleright c_{10} \) and an approval set \( \{c_3, c_5\} \). The costs of swap operations that transform it into, respectively, \( \{c_4, c_6\}, \{c_5, c_7\}, \) and \( \{c_6, c_8\} \) are 1, 2, and 3, as one would naturally expect. Yet, the cost of transforming it into, e.g., \( \{c_8, c_{10}\} \) would also be 3 (move an approval from \( c_3 \) to \( c_8 \), from \( c_4 \) to \( c_9 \), and from \( c_5 \) to \( c_{10} \)), which is not intuitive. Instead, it would be natural to define this cost to be 5 (move the interval by 5 positions to the right). Our proof of Theorem 7 works without change for both these interpretations of unit prices.

**Theorem 7.** AV-SwapApprovals-CIBribery is NP-complete.

**Proof.** We give a reduction from Cubic Independent Set. Let \( G \) be our input graph, where \( V(G) = \{c_1, \ldots, c_n\} \) and \( E(G) = \{e_1, \ldots, e_L\} \), and let \( h \) be the size of the desired independent set. We construct the corresponding AV-SwapApprovals-CIBribery instance as follows.

Let \( B = 3h \) be our budget and let \( t = B + 1 \) be a certain parameter (which we interpret as “more than the budget”). We form the candidate set \( C = V(G) \cup \{p\} \cup F \cup D \), where \( p \) is the preferred candidate, \( F \) is a set of \( t(n+1) \) filler candidates, and \( D \) is a set of \( t \) dummy candidates. Altogether, there are \( t(n+2) + n + 1 \) candidates. We denote individual filler candidates by \( \bigodot \) and individual dummy candidates by \( \bullet \); we fix the societal axis to be:

\[
\triangleright = \bigodot \bigodot \bigodot c_1 \bigodot \bigodot \bigodot c_2 \bigodot \cdots \bigodot \bigodot c_{n-1} \bigodot \bigodot \bigodot c_n \bigodot \cdots \bigodot \bigodot \bullet \bigodot \bigodot \bigodot p
\]

\( t(n+2) + n + 1 \)
For each positive integer \( i \) and each candidate \( c \), we write \( \text{prec}_i(c) \) to mean the \( i \)-th candidate preceding \( c \) in \( \succ \). Similarly, we write \( \text{succ}_i(c) \) to denote the \( i \)-th candidate after \( c \). We introduce the following voters:

1. For each edge \( e_i = \{c_a, c_b\} \) we add an edge voter \( v_{a,b} \) with approval set \([c_a, c_b] \). For each vertex \( c_i \in V(G) \), we write \( V(c_i) \) to denote the set of the three edge voters corresponding to the edges incident to \( c_i \).

2. Recall that \( L = |E(G)| \). For each vertex candidate \( c_i \in V(G) \), we add sufficiently many voters with approval set \([\text{prec}_i(c_i), \text{succ}_i(c_i)] \), so that, together with the score from the edge voters, \( c_i \) ends up with \( L \) approvals.

3. We add \( L - 3 \) voters that approve \( p \).

4. For each group of \( t \) consecutive filler candidates, we add \( L + 4t \) filler voters, each approving all the candidates in the group.

Altogether, \( p \) has score \( L - 3 \), all vertex candidates have score \( L \), the filler candidates have at least \( L + 4t \) approvals each, and the dummy candidates have score 0. We set the committee size to be \( k = t(n + 1) + (n - h) + 1 \). Prior to any bribery, each winning committee consists of \( t(n + 1) \) filler candidates and \( (n - h) + 1 \) vertex ones (chosen arbitrarily). This completes our construction.

Let us assume that \( G \) has a size-\( h \) independent set and denote it with \( S \). For each \( c_i \in S \) and each edge \( e_i = \{c_i, c_j\} \), we bribe edge voter \( v_{i,j} \) to move an approval from \( c_i \) to a filler candidate right next to \( c_j \). This is possible for each of the three edges incident to \( c_i \) because \( S \) is an independent set. As a consequence, each vertex from \( S \) ends up with \( L - 3 \) approvals. Thus only \( n - h \) vertex candidates have score higher than \( p \) and, so, there is a winning committee that includes \( p \).

For the other direction, let us assume that it is possible to ensure that \( p \) belongs to some winning committee via a bribery of cost at most \( B \). Let us consider the election after some such bribery was executed. First, we note that all the filler candidates still have scores higher than \( L + 3t \) (this is so because decreasing a candidate’s score always has at least unit cost and \( B < t \)). Similarly, \( p \) still has score \( L - 3 \) because increasing his or her score, even by one, costs at least \( t \) (indeed, \( p \) is separated from the other candidates by \( t \) dummy candidates). Since \( p \) belongs to some winning committee, this means that at least \( h \) vertex voters must have ended up with score at most \( L - 3 \). In fact, since our budget is \( B = 3h \), a simple counting argument shows that exactly \( h \) of them have score exactly \( L - 3 \), and all the other ones still have score \( L \). Let \( S \) be the set of vertex candidates with score \( L - 3 \). The only way to decrease the score of a vertex candidate \( c_i \) from \( L \) to \( L - 3 \) by spending three units of the budget is to bribe each of the three edge voters from \( V(c_i) \) to move an approval from \( c_i \) to a filler candidate. However, if we bribe some edge voter \( v_{i,j} \) to move an approval from \( c_i \) to a filler candidate, then we cannot bribe that same voter to also move an approval away from \( c_j \) (this would either cost more than \( t \) units of budget or would break the CI condition). Thus it must be the case that the candidates in \( S \) correspond to a size-\( h \) independent set for \( G \).

For the VI domain, the complexity of our problem for unit prices remains open, but for arbitrary prices we show that it is NP-complete. Our proof works even for the single-winner setting. In the unrestricted domain, the single-winner variant is in P \cite{fuliszewski2008approximating}.

**Theorem 8.** AV-$S$SWAPAPPROVALS-VI-Bribery is NP-complete, even for the single-winner case (i.e., for committees of size one).

**Proof.** We give a reduction from RX3C. Let \( I = (X, S) \) be an instance of RX3C, where \( X = \{x_1, \ldots, x_m\} \) is a universe and \( S = \{S_1, \ldots, S_m\} \) is a family of size-3 subsets of \( X \) (recall that each element from \( X \) belongs to exactly three sets from \( S \)). We form a
single-winner approval election with $7n + 1$ voters $V = \{v_0, v_1, \ldots, v_{7n}\}$ and the following candidates:

1. We have the preferred candidate $p$ and the (to be defeated) current winner $d$.
2. For each set $S_i \in S$ we have candidates $s_i, s'_i, s''_i$.

The approvals for these candidates, and the costs of moving them, are as follows (if we do not explicitly list the cost of moving some approval from a given candidate to another, then it is $+\infty$, i.e., this swap is impossible; the construction is illustrated in Figure 1):

1. Candidate $p$ is approved by $4n$ voters, $v_{3n+1}, \ldots, v_{7n}$.
2. Candidate $d$ is approved by $7n$ voters, $v_1, \ldots, v_{7n}$. For each set $S_i = \{x_a, x_b, x_c\}$, where $a < b < c$, the cost of moving $v_a$’s approval from $d$ to $s_i$ is 1, and the costs of moving $v_b$’s and $v_c$’s approvals from $d$ to $s_i$ is 0.
3. For each set $S_i = \{x_a, x_b, x_c\}$, where $a < b < c$, we have the following approvals. Candidate $s_i$ is approved by voter $v_{a-1}$, candidate $s'_i$ is approved by voters $v_{a+1}, \ldots, v_{b-1}$, and candidate $s''_i$ is approved by voters $v_{b+1}, \ldots, v_{c-1}$. The cost of moving the approvals from $s'_i$ or from $s''_i$ to $s_i$ is 0.

One can verify that this election has the VI property for the natural order of the voters (i.e., for $v_0 \succ \cdots \succ v_{7n}$). Candidate $d$ has $7n$ approvals, $p$ has $4n$ approvals, and every other candidate has at most $3n + 1$ approvals. We claim that it is possible to ensure that $p$ becomes a winner of this election by approval-moves of cost at most $B = n$ (such that the election still has the VI property after these moves) if and only if $I$ is a yes-instance of RX3C.

For the first direction, let us assume that $I$ is a yes-instance and that $R \subseteq \{3n\}$ is a size-$n$ set such that $\bigcup_{i \in R} S_i = X$ (naturally, for each $t, \ell \in R$, sets $S_t$ and $S_\ell$ are disjoint). It is possible to ensure that $p$ becomes a winner by making, for each $S_i = \{x_a, x_b, x_c\}$ such that $i \in R$ and $a < b < c$, the following swaps:

1. For each $j \in \{a, b, c\}$, we move $v_j$’s approval from $d$ to $s_i$ (the cost of moving $v_a$’s approval is 1, the two other moves have cost 0).
2. For each $j \in \{a+1, \ldots, b-1\}$, we move $v_j$’s approval from $s'_i$ to $s_i$ (at cost 0).
3. For each $j \in \{b+1, \ldots, c-1\}$, we move $v_j$’s approval from $s''_i$ to $s_i$ (at cost 0).
In total, these moves cost $n$ and, since $R$ corresponds to a cover of $X$, we have that: (a) $p$ is approved by $4n$ voters, (b) $d$ is approved by $4n$ voters, and (c) every other candidate is approved by at most $3n + 1$ voters. Consequently, $p$ is among tied winners of this election.

For the other direction, let us assume that there is a sequence of approval moves that costs at most $n$ and ensures that $p$ is a winner. Since all the moves of approvals from and to $p$ have cost $+\infty$, this means that every candidate ends up with at most $4n$ points. Thus $d$ loses at least $3n$ approvals. No matter what swaps we do, for each $i \in [3n]$ each of $s_1, s'_1$ and $s''_1$ ends up with at most $3n + 1$ approvals so we do not need to count their scores carefully (but we do need to take the VI condition into account for these candidates).

Candidate $d$ can lose approvals only due to voters $v_1, \ldots, v_{3n}$ moving them to candidates in $\{s_1, \ldots, s_{3n}\}$. Let us consider some candidate $s_i$ such that some voter $v_j$ moves an approval from $d$ to $s_i$, and let $a < b < c$ be such that $S_i = \{x_a, x_b, x_c\}$. Due to the costs of moving approvals, it must be that $j \in \{a, b, c\}$. In fact, we claim that all three voters $v_a, v_b, v_c$ move approvals to $s_i$, voters $v_{a+1}, \ldots, v_{b-1}$ move approvals from $s_i$ to $s_j$, and voters $v_{b+1}, \ldots, v_{c-1}$ move approvals from $s''_i$ to $s_j$. This is so, because if those voters $v_a, \ldots, v_j$ would not move their approvals, then—due to the fact that $s_i$ is approved by voter $v_{a-1}$ (and this approval cannot move given our budget)—the approvals for $s_i$ would not satisfy the VI property. Further, voters $v_{j+1}, \ldots, v_c$ also need to move their approvals due to a counting argument: The cost of moving $v_a$’s approval from $d$ to $s_i$ is 1. If we did not move $v_c$’s approvals from $d$ to $s_i$, then it would mean that (globally in our bribery) the average cost of moving an approval from $d$ to some candidate in $\{s_1, \ldots, s_{3n}\}$ would be higher than $1/3$. But since our budget is $n$ and we need to move $3n$ approvals from $d$ to these candidates, this is impossible.

Let $R = \{i \in [3n] \mid$ some voter moves an approval from candidate $d$ to $s_i\}$. By the preceding paragraph, $R$ contains $n$ elements and for each two $i, j \in R$ it must be that sets $S_i$ and $S_j$ are disjoint. Hence, $I$ is a yes-instance.

\section{Destructive Bribery}

We conclude by considering destructive variants of our problems, where the goal is to ensure that a given candidate, often denoted $d$, does not belong to any winning committee (our proofs easily transfer to an alternative model, where one requires only that $d$ does not belong to at least one winning committee; see Appendix \ref{appendix:counterexample}). We use the same bribery actions, except that now we also consider a variant of swapping approvals where we can only move approvals away from $d$.

The destructive variant has been studied for the unrestricted setting by Yang \cite{Yang2020}, for the unpriced cases of adding and deleting approvals. Thus we first establish its complexity also for the priced cases and for swapping approvals. The complexity stays the same as for the constructive variants (the theorem below includes the results of Yang \cite{Yang2020} as special cases).

\begin{theorem}
For unit prices, all destructive variants of our bribery problems for the unrestricted setting are in \#P. For arbitrary prices, the cases of adding and deleting approvals are in \#P, but destructive variants AV-$\$\text{SwapApprovalsAwayFromD-Bribery}$ and AV-$\$\text{SwapApprovals-Bribery}$ are NP-complete.
\end{theorem}

\begin{proof}
\textbf{Case of destructive AV-SwapApprovals-Bribery:} Let our input consist of an election $E = (C, V)$, committee size $k$, candidate $d \in C$, and budget $B$. We can assume that the solution does not contain any transitive swaps, i.e., for $\{c_1, c_2, c_3\} \subseteq C$ a swap of an approval from $c_1$ to $c_2$ and from $c_2$ to $c_3$ (because we could move from $c_1$ to $c_3$ directly). Let us consider some optimal solution. Assume that all swaps from $d$ were already executed and candidate $d$ is left with $t$ approvals. Now we treat $t$ as a fixed score of $d$ and based on the scores prior to making the remaining moves, we group all the candidates into three sets:

1. $L$ — candidates with score lower than $t$.
2. \( T \) – candidates with score equal to \( t \).

3. \( G \) – candidates with score greater than \( t \).

Now let us consider the remaining approval moves in the solution. If there is any swap that moves an approval from candidate \( x \in L \), or to some candidate \( x \in L \), then we can replace it with a swap of an approval from \( d \) to \( x \). Such a move is possible because candidate \( x \) has score lower than \( d \), therefore a vote with an approval for \( d \) and without an approval for \( x \) must exist. If there were a move where \( x \) was getting an approval, then this solution clearly remains correct. If \( x \) were giving an approval to some other candidate \( y \), then by lowering score of \( d \) we achieve that distance of scores of \( d \) and \( y \) remain the same.

If there is any swap that moves an approval from candidate \( x \in T \) to candidate \( y \in T \), then there is a vote with approval for \( x \) and without approval for \( y \). If this vote has an approval for \( d \), then we can make the swap from \( d \) instead of from \( x \). If this vote does not have an approval for \( d \), then there must be a vote where \( d \) is approved but \( x \) is not and we make the swap from \( d \) to \( x \) within that vote, achieving the same effect.

If we implement all the above reasoning, we can be sure that there are left only swaps between candidates from the sets \( T \) and \( G \). By performing these swaps, some candidates from \( G \) may move to sets \( L \) or \( T \), and some candidates from \( T \) may move to sets \( L \) or \( G \). However, if there is a single vote where approval for \( d \) may be moved to any other candidate, then all candidates from \( T \) are moved to \( G \) and all candidates from \( G \) remain in \( G \). This is clearly a move which yields a solution at least as good as the initial one. If it is impossible to make a swap from \( d \), it means that all voters who approve \( d \) also approve all other candidates and all other candidates belong to either \( T \) or \( G \). To solve this case, repeatedly for each candidate from \( G \) who has at least 2 approvals of advantage over \( d \) we move an approval to some candidate \( x \in T \), moving \( x \) to \( G \) (this is correct because all candidates in \( T \) are approved by exactly the same voters as \( d \)).

The above reasoning shows that there is an optimal solution that consists only of two certain types of swaps, which leads to the following algorithm (by guessing \( t \) we mean trying it for all possible values):

1. We guess the number of approvals \( t \) that \( d \) ends up with.

2. In a loop, until \( d \) has exactly \( t \) approvals, among all candidates who have less than \( t + 1 \) approvals find the candidate \( x \) that has the highest score and transfer an approval from \( d \) to \( x \). If there is such a candidate \( x \), there exists a vote where we can move an approval from \( d \) to \( x \), because \( d \) has at least \( t + 1 \) approvals and \( x \) has fewer.

3. If at this point \( d \) does not belong to any winning committee then we accept. Otherwise, either we guessed \( t \) incorrectly or it is a situation where all the voters that approve \( d \) also approve every other candidate.

4. If we have not accepted yet, in a loop until the number of candidates with more than \( t \) approvals is equal to \( k \), find a candidate who has at least \( t + 2 \) approvals and move one of these approvals to some other candidate who has exactly \( t \) approvals (if such a candidate does not exist then reject for this value of \( t \)).

5. If the number of swaps created during the algorithm exceeds \( B \), then we reject (for this value of \( t \)) and accept in the opposite case.

**Case of destructive AV-SwapApprovalsAwayFromD-Bribery:** Let our input consist of an election \( E = (C,V) \), committee size \( k \), candidate \( d \in C \), and budget \( B \). As swaps from \( d \) are the only available operation, our goal is to maximize the number of candidates with score greater than that of \( d \) (without exceeding the budget). Let us assume that the solution exists and leaves \( d \) with exactly \( t \) approvals. There is no point in swapping approvals to candidates who, from the beginning, have more than \( t \) approvals, and there is no point in giving any candidate approvals so that he or she has more than \( t + 1 \) of them.
Hence, given \( t \), a simple greedy algorithm can decide whether the solution exists. It processes the candidates who have scores lower than \( t + 1 \) in descending order of their scores and moves approvals from \( d \) to them until either we exceed the budget or \( d \) ends up with exactly \( t \) approvals. At this point, we check if \( d \) is in some winning committee. If not, we accept. Otherwise, we reject.

**Case of destructive AV-$\text{AddApprovals}$-Bribery and AV-$\text{DeleteApprovals}$-Bribery:** For the case of adding approvals, for each candidate we can independently compute a function saying how much it costs to increase his or her score to a given value (if a candidate already has score higher than \( d \), then this value is equal to 0). Then we select \( k \) candidates for whom getting the score just above that of \( d \) is cheapest. If the sum of their costs is at most equal to the budget, we accept. Otherwise, we reject.

For AV-$\text{DelApprovals}$-Bribery, it never makes sense to delete approvals from candidates other than \( d \). Thus, we keep on deleting \( d \)'s cheapest approvals until he or she is not among the winners (we accept if doing so is within the budget and we reject otherwise).

**Case of destructive AV-$\text{SwapApprovalsAwayFromD}$-Bribery and destructive AV-$\text{SwapApprovals}$-Bribery:**

The reduction for destructive AV-$\text{SwapApprovalsAwayFromD}$-CI-Bribery and destructive AV-$\text{SwapApprovals}$-CI-Bribery from Theorem 11 (described later) can also be applied to this case.

This follows from the fact that the candidate interval property does not constrain swapping approvals in elections, in which each voter supports only one candidate. The reduction from Theorem 11 uses only such voters. Hence, the result therein holds also for destructive AV-$\text{SwapAwayFromDApprovals}$-Bribery and destructive AV-$\text{SwapApprovals}$-Bribery.

For the VI case, we also obtain (or, fail to obtain) almost the same results as in the constructive case (for AV-$\text{SwapApprovals}$-VI-Bribery we use the same proof as in the constructive case, except \( d \) is the distinguished candidate and \( p \) has one extra approval).

**Theorem 10.** Destructive variants of AV-$\text{AddApprovals}$-VI-Bribery and AV-$\text{DeleteApprovals}$-VI-Bribery are in P. Destructive variant of AV-$\text{SwapApprovals}$-VI-Bribery is NP-complete.

**Proof.**

**Case of destructive AV-$\text{AddApprovals}$-VI-Bribery and AV-$\text{DeleteApprovals}$-VI-Bribery:** For the case of adding approvals, for each candidate we can independently compute a function saying how much it costs to increase his or her score to a given value (if a candidate already has score higher than \( d \), then this value is equal to 0): This is just slightly more involved than the unrestricted case. For each candidate \( c \) we check for each pair of integers \( \{x, y\} \) with \( s(c) + x + y > s(d) \) how much it costs to add \( x \) approvals on top of the top-most original approval and to add \( y \) approvals below the bottom-most original approval. (This way, we check every possibility that keeps the interval of approvals intact.) Then we select \( k \) candidates for whom getting the score above that of \( d \) is cheapest. If the sum of their costs is at most equal to the budget, we accept. Otherwise, we reject.

For AV-$\text{DelApprovals}$-Bribery, it never makes sense to delete approvals from candidates other than \( d \). Thus, we check for each pair of integers \( \{x, y\} \) with \( x + y < s(d) \) if deleting the \( x \) top-most and the \( y \) bottom-most approvals of \( d \) is possible with the given budget. We reject if this is not the case for any integer pair.

**Case of destructive AV-$\text{SwapApprovals}$-VI-Bribery:** See proof of Theorem 8.

The case of CI preferences appears to be the most challenging one. Not only do we obtain fewer results than in the constructive setting, but those that we do obtain are less satisfying. Let us illustrate this with AV-$\text{AddApprovals}$-CI-Bribery. We show that
the problem is NP-complete, but to do so, we use a somewhat unappealing trick. Namely, we include some voters who initially do not approve any candidates and we set their price functions so that we can choose one out of four candidates, possibly located far apart in the societal axis, to whom these voters add an approval. Since we did not put extra conditions on the price functions, this is formally correct, but is intuitively unappealing. We also need similar tricks in two other NP-completeness proofs in this section. For example, for the case of swapping approvals away from $d$ we use voters that approve only a single candidate, so we can move this approval arbitrarily (up to constraints implemented with the price function).

Interestingly, if we required each voter to approve at least two candidates, the problem would be in P.

**Theorem 11.** Destructive variants of AV-$\$\text{AddApprovals-CI-Bribery}$, AV-$\$\text{SwapApprovalsAwayFromD-CI-Bribery}$, and AV-$\$\text{SwapApprovals-CI-Bribery}$ are NP-complete. Destructive variant of AV-$\$\text{SwapApprovalsAwayFromD-CI-Bribery}$ is in P.

**Proof.** Case of destructive AV-$\$\text{AddApprovals-CI-Bribery}$: We give a reduction from RX3C. Our input consists of a set $X = \{x_1, \ldots, x_{3n}\}$ and a family of sets $S = \{S_1, \ldots, S_{3n}\}$. We form an election with candidate set $X \cup S \cup \{d\} \cup \{a_1, \ldots, a_{3n}\} \cup \{b_1, \ldots, b_{3n}\}$, where we want $d$ not to be a part of any winning committee. The committee size is $k = 5n$. The societal axis is $x_1 \triangleright a_1 \triangleright \cdots \triangleright x_{3n} \triangleright a_{3n} \triangleright S_1 \triangleright b_1 \triangleright \cdots \triangleright S_{3n} \triangleright b_{3n} \triangleright d$. We have voters so that, initially:

1. $d$ has 3 approvals,
2. each $S_i$ has 1 approval,
3. each $x_j$ has 3 approvals.
4. all the $a_i$ and $b_t$ candidates have zero approvals.

The voters that implement these scores have such high prices, so we cannot add approvals to them. Further, for each set $S_i = \{x_i, x_j, x_k\}$ we form three solution voters. Each of these three voters has an empty approval set and adding each approval costs $+\infty$, except for adding approvals for $S_i, x_i, x_j,$ and $x_k,$ which have unit cost. The budget is $9n$. Prior to any bribery, $d$ is in some winning committees and we need to get $5n$ candidates to have score above 3 to prevent this.

If there is an exact cover of $X$ with $n$ sets from $S$, then we can add approvals so that for each set from the cover, the three corresponding solution voters approve its members (this costs $3n$ and ensures that each candidate from $X$ has 4 approvals). For the $2n$ remaining sets (not forming the cover), each solution voter adds an approval for its corresponding set (this costs $6n$ and ensures that $2n$ set candidates have 4 approvals each). Consequently, there are $5n$ candidates with score higher than $d$ and, so, $d$ does not belong to any winning committee.

Next, assume that there is a bribery after which $d$ does not belong to any winning committee. After performing this bribery, there are $x$ set candidates that have at least 4 approvals each, and $y$ candidates from $X$ that have at least 4 approvals each. Adding these approvals has cost:

$$3x + y \leq 9n$$

This bribery ensures that there are at least $5n$ candidates with score higher than that of $d$, so we know that $x + y \geq 5n$. Together, this implies that $x \leq 2n$, so $y \geq 3n$, which means that $y = 3n$ and, consequently, $x = 2n$. So there are $2n$ sets that get (at least) 4 approvals each. But for these sets, we cannot give approvals to their members of $X$. So the remaining $n$ sets must form a cover.

**Case of destructive AV-$\$\text{SwapApprovalsAwayFromD-CI-Bribery}$ and destructive AV-$\$\text{SwapApprovals-CI-Bribery}$:**
We reduce from Independent Set on regular graphs, where given a (simple) \( \Delta \)-regular graph \( G = (U, E) \) and a positive integer \( h \), we ask whether there is a set of at least \( h \) vertices, in which no two vertices share an edge; the sought set is called an independent set. For some \( u \in U \), let \( E(u) = \{ e : u \in e \} \) be the set of edges incident to \( u \), that is, the edges that have an endpoint in \( u \). In particular, for each vertex \( u \) of some \( \Delta \)-regular graph, it holds that \( |E(u)| = \Delta \). In our proof, it is convenient to define an independent set on \( \Delta \)-regular graphs as a set \( S \) of vertices of the graph that have pairwise disjoint sets of incident edges, that is, \([\bigcup_{u \in S} E(u)] = \Delta |S|\).

Consider an instance \( I = (G, h) \) of Independent Set. Let \( G = (U, E) \) be a \( \Delta \)-regular graph with the set \( U = \{ u_1, u_2, \ldots, u_n \} \) of vertices and the set \( E = \{ e_1, e_2, \ldots, e_m \} \) of edges.

We construct an approval election with committee size \( k = h \) and the candidate set \( C \). Starting with an empty set \( C \), for each vertex \( u \in U \), we add a candidate \( c(u) \) and for each edge \( e \in E \) we add a candidate \( c(e) \). Finally, we add a despised candidate \( d \) obtaining, in total, \( |C| = n + m + 1 \) candidates. The collection of voters \( V \) is build up by two groups of voters. The first group consists of (\( \Delta - 1 \)) voters approving \( d \); these voters cannot be bribed. In the second group, for each edge \( e = \{ u, u' \} \in E \), there is a voter \( v(e) \) that approves \( d \) whose approval can be swapped to support one of \( c(u), c(u') \), and \( c(e) \) at cost 0; any other swap is impossible. Last but not least, we set the budget \( B \) of our new instance to be 0. Accordingly, we implement the forbidden swaps that we described above by giving them cost 1. This concludes the construction of the instance \( I' \) of destructive AV-$\$SWAPAWAYFROM$DAPPROVALS- CI-Bribery that we reduce instance \( I \) to.

To prove the reduction's correctness, let us first assume that \( I \) admits an independent set \( S \) of size at least \( h \). Without loss of generality, we assume that \( S = \{ u_1, u_2, \ldots, u_h \} \) (indeed, each subset of an independent set is an independent set itself and the vertex names can be relabeled). We construct a solution to \( I' \) by swapping the votes as follows. Consider a vertex \( u \in S \). For each incident edge \( e \in E(u) \) of \( u \), we make \( v(e) \) support \( c(u) \) instead of \( d \); by the construction, it comes at cost zero. For every remaining edge, say \( e \in E \setminus \bigcup_{u \in S} E(u) \), we swap the approval of the corresponding vote \( v(e) \) away from \( d \) and give it to \( c(e) \); this action also costs zero budget units. As a result, \( d \) gets \( \Delta - 1 \) approvals, candidates \( c(u_1) \) to \( c(u_h) \) get \( \Delta \) approvals each, and the remaining candidates get either one or zero approvals. Hence, the winning committee of size \( h \) does not contain \( d \) as it exactly consists of candidates \( c(u_1) \) to \( c(u_h) \) and the budget spent is 0.

To prove the opposite direction, assume that there is a bribery action leading to an election \( E' \) in which all winning committees of size \( h \) do not contain \( d \). We claim that each of these committees consists of candidates corresponding to the vertices that form an independent set of size \( h \) in \( G \). However, for ease of presentation and without loss of generality, we fix one such committee \( S \). Observe that the minimal score of \( d \) is \( \Delta - 1 \) as there are exactly \( \Delta - 1 \) voters that cannot be bribed that approve \( d \). Hence, there are at least \( h \) candidates with having at least \( \Delta \) approvals in \( E' \). Note that, by construction, no candidate \( c(e) \) corresponding to an edge \( e \in E \) has a score higher than one in \( E \). Let us fix some \( u \in U \). We now show that candidate \( c(u) \) corresponding to \( u \in U \) cannot obtain more than \( \Delta \) approvals in \( E \). Indeed, \( c(u) \) can only get approvals by bribing voters corresponding to the edges incident to \( u \). Since \( u \) belongs to a \( \Delta \)-regular input graph, it has exactly \( \Delta \) incident edges. So, there is at least \( h \) candidates (corresponding to vertices of \( G \)) with exactly \( \Delta \) approvals in \( E \). From the discussion above it is also clear that to make \( u \) get \( \Delta \) approvals, one needs to bribe all voters corresponding to the incident edges of \( u \) and, obviously, each of these voters contribute to the score of exactly one candidate. Hence, to make all candidates of the winning committee \( S \) have score \( \Delta \), it must hold that for each candidate \( c(u) \in S \), representing vertex \( u \), there is a disjoint set of \( \Delta \) voters, representing edges incident to \( u \), that has to be bribed. So the set of vertices corresponding to members of \( S \) must be an independent set.

This reduction clearly works in polynomial time and forms a parameterized reduction with respect to the value of the committee size, which concludes the proof. Since instance \( I' \)
contains votes that initially approve only one candidate, the argument is also correct for the destructive AV-SwapApprovals-CI-Bribery problem.

**Case of destructive AV-SwapApprovalsAwayFromD-CI-Bribery:** We solve the problem via dynamic programming. The crucial observations are as follows. First, each voter who approves \( d \) and some other candidate(s) either to the left, or to the right can move the approval to just one candidate at unit price. For each candidate \( c_i \), let \( Y_i \) denote the number of voters that approve \( d \) and every candidate between \( d \) and \( c_i \) in the societal order. Second, for voters who only approve \( d \) we can move the approval to any other candidate at unit cost. Let \( X \) denote the total number of such voters.

First, we guess the final score \( s^\ast \) of \( d \) after bribery: it is at least \( s(d) - B \) and at most \( s(d) - 1 \). (Note that we need to swap away \( s(d) - s^\ast < B \) approvals from \( d \).) We have the following binary table \( T[i, k', B', x] \in \{0, 1\} \), where \( T[i, k', B', x] = 1 \) when it is possible move approvals from \( d \) to the first \( i \) candidates in the societal order, while ensuring that \( k' \) of them have score larger than \( s^\ast \), exactly \( B' \) approvals where swapped away from \( d \), and in total \( x \) of these approvals have been swapped by voters only approving \( d \). We initialize the table by setting first setting \( T[1, *, *, *, *) := 0 \) and then updating: (a) \( T[1, 0, B''', x'] := 1 \) if \( s(c_1) + B'' < s^\ast \), \( x' \leq X \), \( x' \leq B'' \), and \( B'' - x' \leq Y_1 \), (b) \( T[1, 1, B'', x'] := 1 \) if \( s(c_1) + B'' > s^\ast \), \( x' \leq X \), \( x' \leq B'' \), and \( B'' - x' \leq Y_1 \). Cases under (a) capture all possibilities to move approvals from \( d \) to the leftmost candidate while keeping its score to at most \( s^\ast \) while cases under (b) capture all possibilities to move approvals from \( d \) to the leftmost candidate while ensuring its score to be larger than the score \( s^\ast \) of \( d \). Entries for \( i > 1 \) (with \( i \) increasing from 2 to \( m \)) are computed as follows. Set \( T[i, *, *, *, *) := 0 \) and then update: (a) \( T[i, k', B'', x'] := 1 \) if there are some \( B^* \) and \( x^\ast \) with \( 0 \leq B^* < B'' \) and \( 0 \leq x^\ast \leq x' \) such that \( s(c_i) + B^* \leq s^\ast \), \( x' \leq X \), \( x' \leq B'' \), \( B'' - x^\ast \leq Y_i \), and \( T[i-1, k', B'' - B^*, x' - x^\ast] = 1 \). (b) \( T[i, k', B'', x'] := 1 \) if there are some \( B^* \) and \( x^\ast \) with \( 0 \leq B^* < B'' \) and \( 0 \leq x^\ast \leq x' \) such that \( s(c_i) + B^* > s^\ast \), \( x' \leq X \), \( x' \leq B'' \), \( B'' - x^\ast \leq Y_i \), and \( T[i-1, k' - 1, B'' - B^*, x' - x^\ast] = 1 \). Similar to the initialization, cases under (a) capture all possibilities to move approvals from \( d \) to the first \( i \) candidates while keeping the score of \( c_i \) to at most \( s^\ast \) while cases under (b) capture all possibilities to move approvals from \( d \) to the first \( i \) candidate while ensuring that the score of \( c_i \) is than the score \( s^\ast \) of \( d \).

Finally, we have if yes-instance if there is some entry \( T[m, k^\ast, s(d) - s^\ast, x^\ast] = 1 \) with \( x^\ast < X \) and \( k^\ast \geq k \).

7 Summary

We have studied bribery in multiwinner approval elections, for the case of candidate interval (CI) and voter interval (VI) preferences. Depending on the setting, our problem can either be easier, harder, or equally difficult as in the unrestricted domain. It would be interesting to extend our work by considering different voting rules (in particular, the Approval-Based Chamberlin–Courant rule [Chamberlin and Courant 1983; Procaccia et al. 2008; Betzler et al. 2013]) and by seeking parameterized complexity results.

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Supplementary Material

A Adapting Proofs to Other Winning Models

In the constructive case, we will adapt our results to the case in which the preferred candidate is in all committees. In the destructive case, after the adaptation, the despised candidate is considered to lose the election if there is at least one winning committee without this candidate.

A.1 Constructive

Theorem 1: Replace “some” by “all” in the definition of \( s \).

Theorem 2: Replace “end up with highe” by “end at with at least as high” in the definition of \( t_\ell \) and \( t_r \), and adjust the definition Step 3 accordingly.

Theorem 3: Definition of “superior candidate” must be adjusted to contain also candidates with score equals to the score of \( p \).

Theorem 4: Add one more fixed approval to \( p \).

Theorem 5: The definition of \( \text{S} \) needs to be adjusted to contain also candidates with the same score as \( p \). The rest remains the same.

Theorem 6: The definition of dangerous candidates needs to be adjusted so that it also contains those with same score as \( p \). Again, the rest can be kept. The rest remains the same.

Theorem 7: Add another voter that only approves \( p \). This way, the \( \Rightarrow \) direction carries over “as is.” In the \( \Leftarrow \) direction we have to observe that bribing this new voter brings nothing good. Indeed, it is the case since when we do so, then we do not have enough budget to do other changes that should decrease the scores of other candidates to be lower than the score of \( p \).

Theorem 8: We add one more voter approving only \( p \) such that the voter cannot be bribed.

A.2 Desctructive

Theorem 9: In the algorithmic results, modify defining groups of candidates to reflect that now it is enough that we have \( k \) candidates (other than \( d \)) with score at least \( t \) instead of at least \( t + 1 \). The reduction is the same as that in Theorem 11.

Theorem 10: For the algorithmic result, we simply need to select \( k \) candidates for which achieving the score of \( d \) is the cheapest. The reduction is the same as that in

Theorem 11: In the case of adding approvals with arbitrary prices, candidate \( d \) should have score 4 in point (1) of the construction. The rest remains the same (up to some fluffy text). In the cases of swapping approvals away from \( d \) with arbitrary prices assuming CI and swapping approvals in the CI domain with arbitrary prices: One should have \( \Delta \) voters supporting \( d \) (that cannot be bribed) instead of \( \Delta - 1 \) of them. In the last case, swapping away from \( d \) for unit prices in CI, the \( k' \) in the definition of the recurrent function needs to be changed to mean the number of candidates with score at least \( s^* \) (some details of the computation of the function have to be adjusted).