Li-Yau Multiplier Set and Optimal Li-Yau Gradient Estimate on Hyperbolic Spaces

Chengjie Yu¹ · Feifei Zhao²

Received: 6 November 2019 / Accepted: 15 October 2020 / Published online: 19 November 2020
© Springer Nature B.V. 2020

Abstract
In this paper, motivated by finding sharp Li-Yau-type gradient estimate for positive solution of heat equations on complete Riemannian manifolds with negative Ricci curvature lower bound, we first introduce the notion of Li-Yau multiplier set and show that it can be computed by heat kernel of the manifold. Then, an optimal Li-Yau-type gradient estimate is obtained on hyperbolic spaces by using recurrence relations of heat kernels on hyperbolic spaces. Finally, as an application, we obtain sharp Harnack inequalities on hyperbolic spaces.

Keywords Heat equation · Li-Yau-type gradient estimate · Heat kernel

Mathematics Subject Classification (2010) Primary 35K05 · Secondary 53C44

1 Introduction
The Li-Yau [18] gradient estimate:
$$\| \nabla \log u \|^2 - \alpha (\log u)_t \leq \frac{n \alpha^2}{2t} + \frac{n \alpha^2 k}{2(\alpha - 1)}$$ (1.1)
for positive solution $u$ of the heat equation on complete Riemannian manifolds with $\text{Ric} \geq -k$ and $k$ a nonnegative constant is of fundamental importance in geometric analysis. Here $\alpha$ is any constant greater than 1.

On complete Riemannian manifolds with nonnegative Ricci curvature, by letting $\alpha \to 1^+$ in Eq. 1.1, one has
$$\| \nabla \log u \|^2 - (\log u)_t \leq \frac{n}{2t}.$$ (1.2)
This estimate is sharp where the equality can be achieved by the fundamental solution of $\mathbb{R}^n$. However, Eq. 1.1 is not sharp when $k > 0$. Finding sharp Li-Yau-type gradient estimate for $k > 0$ is still an unsolved problem (see [8, P. 393]). This is the motivation of this paper. We will assume that $k > 0$ without further indications in the rest of this paper.

Li-Yau-type gradient estimates are important since they give Harnack inequalities immediately by taking integration on geodesics. Many authors have obtained Li-Yau-type gradient estimates in various forms or various settings. For example, in [14], Hamilton obtained a Li-Yau-type gradient estimate in matrix form, and in [15], Hamilton obtained a Li-Yau-type gradient estimate in matrix form for Ricci flow. Hamilton’s works were extended to the Kähler category by Cao-Ni [5] and Cao [4], and further extended to $(p, p)$-forms on Kähler manifolds by Ni and Niu [19]. Recently, in [30], the authors extended the Li-Yau-type gradient estimate to metric measure spaces, and in [7, 28, 29], the authors obtained Li-Yau-type gradient estimates under integral curvature assumptions. Some other Li-Yau-type gradient estimates can be found in [1–3, 6, 9, 16, 20, 22–24]. Here, we only mention some of them that are more related to the topic of this paper and compare them.

A slight improvement of Eq. 1.1:
\[
\|\nabla \log u\|^2 - \alpha (\log u)_t \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 k}{4(\alpha - 1)}
\] (1.3)
was given by Davies [10]. We will call this the Li-Yau-Davies estimate in the rest of this paper.

In [14], Hamilton obtained
\[
\|\nabla \log u\|^2 - e^{2k t} (\log u)_t \leq e^{4k t} \frac{n}{2t}.
\] (1.4)
This estimate is sharp in leading term as $t \to 0^+$ comparing to the Li-Yau-Davies estimate (1.3).

In [1], Bakry and Qian obtained
\[
\|\nabla \log u\|^2 - \left(1 + \frac{2}{3} k t\right)(\log u)_t \leq \frac{n}{2t} + \frac{nk}{2}\left(1 + \frac{1}{3} kt\right).
\] (1.5)
This estimate is also sharp in leading term as $t \to 0^+$. This estimate was also obtained by Li and Xu [17] by a different method.

In [17], Li and Xu obtained
\[
\|\nabla \log u\|^2 - \left(1 + \frac{\sinh(k t) \cosh(k t) - k t}{\sinh^2(k t)}\right)(\log u)_t \leq \frac{nk}{2} \left[\coth(k t) + 1\right].
\] (1.6)
It is not hard to see that Eq. 1.6 is also sharp in leading term as $t \to 0^+$. Moreover, as pointing out in [25], the asymptotic behavior of Eq. 1.6 as $t \to \infty$ is the same as Eq. 1.3 with $\alpha = 2$. The estimates (1.5) and (1.6) are extended to a general form in [21]. Li-Xu’s estimate (1.6) was also obtained by Bakry et al. in [3] by a different method.

One should note that, in the Li-Yau-Davies estimate (1.3), one can choose different $\alpha > 1$ for different given $t$. So, the Li-Yau-Davies estimate also produces gradient estimates with time-dependent parameters. The key feature of the estimates (1.4), Eq. 1.5 is that they are both sharp in leading term as $t \to 0^+$, while the key feature of Li-Xu’s estimate (1.6) is that it is not only sharp in leading term as $t \to 0^+$, but also has a good asymptotic behavior as $t \to +\infty$ which is the same as the Eq. 1.3 for $\alpha = 2$.

For purpose of comparison, we rewrite a Li-Yau-type gradient estimate in the following form:
\[
\beta \|\nabla \log u\|^2 - (\log u)_t \leq \gamma.
\] (1.7)
For example, for the Li-Yau-Davies estimate (1.3),

$$\gamma_{LYD}(\beta, t) = \frac{n}{2t} \left( \frac{1}{\beta} + \frac{kt}{2(1-\beta)} \right)$$  \hspace{1cm} (1.8)

and $\beta \in (0, 1)$. For Hamilton’s estimate (1.4),

$$\beta = \beta_H := e^{-2kt}$$  \hspace{1cm} (1.9)

and

$$\gamma = \gamma_H := \frac{n}{2t} e^{2kt}. \hspace{1cm} (1.10)$$

For Bakry-Qian’s estimate (1.5),

$$\beta = \beta_{BQ} := \frac{1}{1 + \frac{2}{3}kt} \hspace{1cm} (1.11)$$

and

$$\gamma = \gamma_{BQ} := \frac{n}{2t} \cdot \frac{1 + kt + \frac{1}{3}(kt)^2}{1 + \frac{2}{3}kt}. \hspace{1cm} (1.12)$$

For Li-Xu’s estimate (1.6),

$$\beta = \beta_{LX} := \frac{1}{1 + \frac{\sinh(kt)}{\cosh(kt) - kt}} \hspace{1cm} (1.13)$$

and

$$\gamma = \gamma_{LX} := \frac{n}{2t} \cdot \frac{kt[\coth(kt) + 1]}{1 + \frac{\sinh(kt)}{\cosh(kt) - kt}}. \hspace{1cm} (1.14)$$

For a fixed time $t$, a Li-Yau-type gradient estimate

$$\beta_1 \| \nabla \log u \|^2 - (\log u)_t \leq \gamma_1 \hspace{1cm} (1.15)$$

is better than

$$\beta_2 \| \nabla \log u \|^2 - (\log u)_t \leq \gamma_2 \hspace{1cm} (1.16)$$

if $\beta_1 \geq \beta_2$ and $\gamma_1 \leq \gamma_2$. For example, for a fixed time $t \gt 0$, $\gamma_{LYD}(\beta, t)$ achieves its minimum

$$\gamma_m(t) = \frac{n}{2t} \left( 1 + \sqrt{\frac{kt}{2}} \right)^2 \hspace{1cm} (1.17)$$

at

$$\beta_m(t) = \frac{1}{1 + \sqrt{\frac{kt}{2}}}. \hspace{1cm} (1.18)$$

Therefore, for each time $t \gt 0$, the Li-Yau-Davies estimate (1.3) for $\beta = \beta_m(t)$ is better than the Li-Yau-Davies estimate for $\beta < \beta_m(t)$.

In the Li-Yau-Davies estimate, let $\beta = \beta_H := e^{-2kt}$. Then,

$$\gamma_{LYD}(\beta_H(t), t) = \frac{n}{2t} e^{2kt} + \frac{nk}{4(1 - e^{-2kt})} > \gamma_H(t).$$  \hspace{1cm} (1.19)

Comparing this to Hamilton’s estimate (1.4), it seems that Hamilton’s estimate is better than the Li-Yau-Davies estimate (1.3) for $\beta = \beta_H(t)$ for all time. However, note that $\beta_H(t) < \beta_m(t)$ and

$$\gamma_m(t) < \gamma_H(t) \hspace{1cm} (1.20)$$

when $t$ is large enough. So, when $t$ is large enough, the Li-Yau-Davies estimate (1.3) is better than eq1.4. More precisely, let $t_H > 0$ be the intersection point of $\gamma_m(t)$ and $\gamma_H(t)$. 

\section*{Springer}
Then, when \( t \geq t_H \), since \( \beta_m(t) \geq \beta_H(t) \) and \( \gamma_H(t) \geq \gamma_m(t) \), Li-Yau-Davies estimate (1.3) is better than Hamilton’s estimate (1.4), and when \( t < t_H \), since \( \gamma_m(t) > \gamma_H(t) \), Hamilton’s estimate (1.4) is better than the Li-Yau-Davies estimate (1.3) with \( \beta \leq \beta_H(t) \). However, for \( t < t_H \), Hamilton’s estimate (1.4) is not better than the Li-Yau-Davies estimate (1.3) with \( \beta \in (\beta_H(t), 1) \) although it is sharp in leading term as \( t \to 0^+ \).

For Bakry-Qian’s estimate (1.5), since \( \gamma_{BQ}(t) < \gamma_m(t) \) for any \( t > 0 \), Bakry-Qian’s estimate (1.5) is better than the Li-Yau-Davies estimate (1.3) with \( \beta \leq \beta_{BQ}(t) \) for any \( t > 0 \). Moreover, since \( \beta_{BQ}(t) > \beta_H(t) \) and \( \gamma_{BQ}(t) < \gamma_H(t) \) for any \( t > 0 \), Bakry-Qian’s estimate (1.5) is better than Hamilton’s estimate (1.4).

For Li-Xu’s estimate (1.6), let \( t_{LX} \) be the intersection point of \( \gamma_{LX}(t) \) and \( \gamma_m(t) \). Then, when \( t \leq t_{LX} \), Li-Xu’s estimate is better than the Li-Yau-Davies estimate for \( \beta \leq \beta_{LX}(t) \) since \( \gamma_m(t) \geq \gamma_{LX}(t) \) for \( t \leq t_{LX} \). When \( t > t_{LX} \), \( \beta_{LX}(t) < \beta_{LX}(t) \) (see Proposition 5.2 in the Appendix) and

\[
\gamma_{LYD}(\beta_{LX}(t), t) > \gamma_{LX}(t) > \gamma_{LX}(t). \tag{1.21}
\]

Let \( \beta_-(t) < \beta_m(t) < \beta_+(t) \) be such that

\[
\gamma_{LYD}(\beta'_-(t), t) = \gamma_{LYD}(\beta'_+(t), t) = \gamma_{LX}(t). \tag{1.21}
\]

Then, Li-Xu’s estimate (1.6) is better than the Li-Yau-Davies estimate (1.3) for \( \beta \in (0, \beta_{-}(t)] \cup [\beta_{+}(t), \beta_{LX}(t)) \) at each time \( t > t_{LX} \).

For the comparison of Li-Xu’s estimate (1.6) and Bakry-Qian’s estimate (1.5), although one has \( \beta_{BQ}(t) < \beta_{LX}(t) \) for any \( t > 0 \), one cannot conclude that Li-Xu’s estimate (1.6) is better than Bakry-Qian’s estimate (1.5) when \( t \) is large enough, since

\[
\lim_{t \to \infty} \gamma_{BQ}(t) = \frac{nk}{4} < \frac{nk}{2} = \lim_{t \to \infty} \gamma_{LX}(t). \tag{1.22}
\]

Moreover, since \( \beta_H(t) < \beta_{LX}(t) \) and \( \gamma_H(t) > \gamma_{LX}(t) \) for any \( t > 0 \) (see Proposition 5.1 in the Appendix), Li-Xu’s estimate (1.6) is better than Hamilton’s estimate (1.4).

In summary, Hamilton’s estimate (1.4), Bakry-Qian’s estimate (1.5) and Li-Xu’s estimate (1.6) are better than the Li-Yau-Davies estimate (1.3) only for certain range of \( t \) and certain range of \( \beta \). Bakry-Qian’s (1.5) and Li-Xu’s (1.6) estimates are both better than Hamilton’s estimate (1.4).

Motivated by the comparisons above, we introduce the following notion of Li-Yau multiplier set. Let \((M^n, g)\) be a complete Riemannian manifold and

\[
\mathcal{P}(M, g) = \{ u \in C^\infty(\mathbb{R}^+ \times M) \mid u > 0 \text{ and } u_t - \Delta_g u = 0 \}. \tag{1.23}
\]

For \( x \in M \) and \( u \in \mathcal{P}(M, g) \), define the Li-Yau multiplier set of \( u \) at \( x \) as

\[
S(u, x) = \{(t, \beta, \gamma) \in \mathbb{R}^+ \times [0, +\infty) \times \mathbb{R} \mid [\beta]\nabla \log u]^2 - (\log u)_t(t, x) \leq \gamma \}. \tag{1.24}
\]

and the Li-Yau multiplier set of \( u \) at \( x \) and time \( t \) as

\[
S_t(u, x) = \{(\beta, \gamma) \in [0, +\infty) \times \mathbb{R} \mid (t, \beta, \gamma) \in S(u, x) \}. \tag{1.25}
\]

Moreover, we define the Li-Yau multiplier set of \((M, g)\) at \( x \) as

\[
S(M, g, x) = \bigcap_{u \in \mathcal{P}(M, g)} S(u, x), \tag{1.26}
\]

the Li-Yau multiplier set of \((M, g)\) at \( x \) and time \( t \) as

\[
S_t(M, g, x) = \bigcap_{u \in \mathcal{P}(M, g)} S_t(u, x), \tag{1.27}
\]
the Li-Yau multiplier set of \((M, g)\) as
\[
S(M, g) = \bigcap_{x \in M} S(M, g, x), \tag{1.28}
\]
and the Li-Yau multiplier set of \((M, g)\) at time \(t\) as
\[
S_t(M, g) = \bigcap_{x \in M} S_t(M, g, x). \tag{1.29}
\]
A key observation of the Li-Yau multiplier set is as follows:

**Theorem 1.1** Let \((M^n, g)\) be a complete Riemannian manifold with Ricci curvature bounded from below and \(H(t, x, y)\) be its heat kernel. Then,
\[
S(M, g, x) = \bigcap_{y \in M} S(H(\cdot, \cdot, y), x). \tag{1.30}
\]
In other words, \((t, \beta, \gamma) \in S(M, g, x)\) if and only if
\[
\left[ \beta \| \nabla_x \log H \|^2 - (\log H)_t \right] (t, x, y) \leq \gamma \tag{1.31}
\]
for any \(y \in M\). As a consequence,
\[
S(M, g) = \bigcap_{x, y \in M} S(H(\cdot, \cdot, y), x). \tag{1.32}
\]
In other words, \((t, \beta, \gamma) \in S(M, g)\) if and only if
\[
\left[ \beta \| \nabla_x \log H \|^2 - (\log H)_t \right] (t, x, y) \leq \gamma \tag{1.33}
\]
for any \(x, y \in M\).

This observation tells us that to check if a triple \((t, \beta, \gamma)\) belongs to \(S(M, g)\), one only need to check if
\[
\beta \| \nabla \log u \|^2 - (\log u)_t \leq \gamma \tag{1.34}
\]
is true for the heat kernel. This highly simplifies the computation of \(S(M, g)\).

By using the expression
\[
H(t, x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{\|x - y\|^2}{4t}} \tag{1.35}
\]
of heat kernel on the Euclidean space \(\mathbb{R}^n\) with standard metric \(g_E\), it is not hard to see that
\[
S(\mathbb{R}^n, g_E) = \left\{ (t, \beta, \gamma) \mid t > 0, \beta \in [0, 1], \text{ and } \gamma \geq \frac{n}{2t} \right\}. \tag{1.36}
\]
The Li-Yau estimate (1.2) is equivalent to that
\[
S(M^n, g) \supset S(\mathbb{R}^n, g_E) \tag{1.37}
\]
for any complete Riemannian manifold \((M^n, g)\) with nonnegative Ricci curvature. Motivated by this, one may reformulate the problem of finding sharp Li-Yau-type gradient estimate as follows: Let \(M^n_\kappa\) be the space form of dimension \(n\) with constant sectional curvature \(\kappa\) and with standard metric \(g_\kappa\). Do we have
\[
S(M^n, g) \supset S(M^n_\kappa, g_\kappa) \tag{1.38}
\]
for any complete Riemannian manifold \((M^n, g)\) with Ricci curvature not less than \((n - 1)\kappa\)? Laying aside the problem, one still has another problem of finding \(S(M^n_\kappa, g_\kappa)\) for \(\kappa \neq 0\).
Motivated by the work [11] of Davies and Mandouvalos, we will discuss this problem with \( \kappa = -1 \) in this paper.

More precisely, let \( K_n(t, r(x, y)) \) be the heat kernel of the \( n \)-dimensional hyperbolic space, then by using the recurrence relation:

\[
K_{n+2} = -\frac{e^{-nt}}{2\pi \sinh r} \frac{\partial K_n}{\partial r},
\]

we are able to show the following Li-Yau-type gradient estimate on hyperbolic space of odd dimension.

**Theorem 1.2** Let \((M^n, g)\) be an odd dimensional complete Riemannian manifold with constant sectional curvature \(-1\). Then

\[
\left( t, \beta, \frac{n}{2t} + \frac{(n-1)^2}{4(1-\beta)} \right) \in S(M, g),
\]

for any \( t > 0 \) and \( \beta \in [0, 1) \). In other words,

\[
\beta \| \nabla \log u \|^2 - (\log u)_t \leq \frac{n}{2t} + \frac{(n-1)^2}{4(1-\beta)}
\]

for any \( \beta \in [0, 1) \) and any \( u \in \mathcal{P}(M, g) \).

This estimate is sharp in leading term as \( t \to 0^+ \) and sharp as \( t \to \infty \) for hyperbolic spaces.

For hyperbolic space of even dimension, by using the recurrence relation:

\[
K_n(t, r) = \sqrt{2} e^{(n-1)r/4} \int_r^\infty K_{n+1}(t, \rho) \sinh \rho (\cosh \rho - \cosh r)^{1/2} d\rho,
\]

we are only able to obtain a weaker conclusion:

**Theorem 1.3** Let \((M^n, g)\) be an even dimensional complete Riemannian manifold with constant sectional curvature \(-1\). Then

\[
\left( t, \beta, \frac{n+1}{2t} + \frac{(n-1)^2}{4(1-\beta)} \right) \in S(M, g),
\]

for any \( t > 0 \) and \( \beta \in [0, 1) \). In other words,

\[
\beta \| \nabla \log u \|^2 - (\log u)_t \leq \frac{n+1}{2t} + \frac{(n-1)^2}{4(1-\beta)}
\]

for any \( \beta \in [0, 1) \) and any \( u \in \mathcal{P}(M, g) \).

This estimate is better than the Li-Yau-Davies estimate for large \( t \), although it is not sharp in leading term as \( t \to 0^+ \). In fact, for the hyperbolic plane \( \mathbb{H}^2 \) (see Proposition 5.3 in the Appendix),

\[
(\log K_2)_r(t, 0) > \frac{1}{t} + \frac{1}{4}.
\]

So, we can not expect that the same conclusion as in Theorem 1.2 holds on the hyperbolic plane.

It is not hard to see that \( S_{t_1}(M, g) \subset S_{t_2}(M, g) \) for any \( 0 < t_1 < t_2 \) (see (6) of Proposition 2.2). So, we define the Li-Yau multiplier set at time infinity as

\[
S_\infty(M, g) = \bigcup_{t \geq 0} S_t(M, g).
\]
By Eq. 1.36, it is clear that
\[ S_\infty(\mathbb{R}^n, g_E) = [0, 1] \times \mathbb{R}_+. \] (1.45)
A direct corollary of Theorem 1.2 and Theorem 1.3 is as follows.

**Corollary 1.1** Let \((M^n, g)\) be a complete Riemannian manifold with constant sectional curvature \(-1\). Then,
\[ S_\infty(M, g) \supset \{(\beta, \gamma) \mid \beta \in [0, 1) \text{ and } \gamma > (n - 1)^2 4(1 - \beta)\}. \]

Finally, by a standard argument as in [18], we have the following sharp Harnack inequality.

**Theorem 1.4**

1. Let \((M^n, g)\) be an odd dimensional complete Riemannian manifold with constant sectional curvature \(-1\) and \(u \in \mathcal{P}(M, g)\). Then, for any \(x_1, x_2 \in M\) and \(0 < t_1 < t_2\),
\[ u(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^\frac{n}{2} \exp\left(\frac{r^2(x_1, x_2)}{4(t_2 - t_1)} + \frac{(n - 1)^2}{4}(t_2 - t_1) + \frac{n - 1}{2}r(x_1, x_2)\right) u(x_2, t_2). \]

2. Let \((M^n, g)\) be an even dimensional complete Riemannian manifold with constant sectional curvature \(-1\) and \(u \in \mathcal{P}(M, g)\). Then, for any \(x_1, x_2 \in M\) and \(0 < t_1 < t_2\),
\[ u(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^\frac{n+1}{2} \exp\left(\frac{r^2(x_1, x_2)}{4(t_2 - t_1)} + \frac{(n - 1)^2}{4}(t_2 - t_1) + \frac{n - 1}{2}r(x_1, x_2)\right) u(x_2, t_2). \]

Here \(r(x_1, x_2)\) means the distance between \(x_1\) and \(x_2\).

The recurrence relations (1.39) and (1.41) were proved in [11] by using Selberg’s transform. For a simple direct proof and some similar recurrence relations on spheres, see [26].

The key to prove Theorem 1.2 and Theorem 1.3 is the following estimates of heat kernels on hyperbolic spaces.

**Theorem 1.5** Let \(K_n(t, r(x, y))\) be the heat kernel on the hyperbolic space \(\mathbb{H}^n\) and suppose that
\[ K_n(t, r) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{(n-1)^2}{4}t - \frac{r^2}{4\pi \alpha_n(t, r)}}. \] (1.46)

Then

1. \((\alpha_n)_t \geq 0\) when \(n\) is odd;
2. \((\frac{1}{2} \alpha_n)_t \geq 0\) when \(n\) is even;
3. \(0 \leq -(\log \alpha_n)_t \leq \frac{n-1}{2}.

This result was also used to derive a sharp Li-Yau type gradient estimate for general hyperbolic manifolds in [27].

The reason that Theorem 1.2 does not hold for the hyperbolic plane is that (1) of Theorem 1.5 is not true when \(n = 2\). The difference of the odd dimensional case and the even dimensional case (Theorem 1.2 and Theorem 1.3) is mainly due to that we can only have (2) instead of (1) in Theorem 1.5 when \(n\) is even. There is a possibility that (1) of Theorem 1.5 is also true when \(n\) is an even number greater than 2.
The proof of Theorem 1.5 is by a combination of the argument in [11] and some Gronwall-type differential inequalities derived from the recurrence relations of $\alpha_n$.

The rest of this paper is organized as follows. In Section 2, we introduce some elementary properties of Li-Yau multiplier sets and prove Theorem 1.1. In Section 3, we prove Theorem 1.5. In Section 4, we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4. In Section 5, the Appendix, we give the calculations for comparison of Li-Xu’s estimate (1.6) with Hamilton’s estimate (1.4) and for comparison of Li-Xu’s estimate (1.6) with the Li-Yau-Davies estimate (1.3), and show (1.43).

2 Li-Yau Multiplier Set

In this section, we give some simple properties of Li-Yau multiplier sets and prove Theorem 1.1.

First of all, we have the following elementary properties:

**Proposition 2.1** Let $(M^n, g)$ be a complete Riemannian manifold and $u, v \in \mathcal{P}(M, g)$. Then,

(1) $S(u, x)$ is a closed subset of $(0, +\infty) \times [0, \infty) \times \mathbb{R}$.
(2) $S_t(u, x)$ is convex.
(3) if $(\beta, \gamma) \in S_t(u, x)$, then $[0, \beta] \times [\gamma, +\infty) \subset S_t(u, x)$.
(4) $S(u + v, x) \supset (S(u, x) \cap S(v, x))$.

**Proof** The properties (1)–(3) are straightforward from definition. We only need to prove (4).

Let $(t, \beta, \gamma) \in S(u, x) \cap S(v, x)$. Then, at $(t, x)$,

$$
\beta \|\nabla \log(u + v)\|_t^2 - (\log(u + v))_t = (\beta \|\nabla \log u\|_t^2 - (\log u)_t) u^2 + (\beta \|\nabla \log v\|_t^2 - (\log v)_t) v^2 + 2\beta \langle \nabla u, \nabla v\rangle - (u_t v - u v_t)
\leq \frac{\gamma u^2 + \gamma v^2 + (2\beta \langle \nabla \log u, \nabla \log v\rangle - (\log u)_t - (\log v)_t) uv}{(u + v)^2}.
\leq \frac{\gamma u^2 + \gamma v^2 + (\beta \|\nabla \log u\|_t^2 - (\log u)_t + \beta \|\nabla \log v\|_t^2 - (\log v)_t) uv}{(u + v)^2}.
\leq \gamma.
$$

So, $(t, \beta, \gamma) \in S(u + v, x)$.

We also have the following elementary properties for the Li-Yau multiplier set of a complete Riemannian manifold.

**Proposition 2.2** Let $(M^n, g)$ be a complete Riemannian manifold. Then,

(1) $S(M, g, x)$ and $S(M, g)$ are closed in $(0, \infty) \times [0, \infty) \times \mathbb{R}$.
(2) $S(M, g, x) \cap S(M, g)$ are convex.
(3) If $(\beta, \gamma) \in S_t(M, g, x)$, then $[0, \beta] \times [\gamma, +\infty) \subset S_t(M, g, x)$. As a consequence, if $(\beta, \gamma) \in S_t(M, g)$, then $[0, \beta] \times [\gamma, +\infty) \subset S_t(M, g)$.
(4) Let $\varphi$ be an isometric transformation of $(M, g)$, then $S(M, g, x) = S(M, g, \varphi(x))$. As a consequence, if $(M, g)$ is homogeneous, then $S(M, g) = S(M, g, x)$ for any $x \in M$. 

\(\copyright\) Springer
(5) Let \((N, h)\) be another complete Riemannian manifold, and \(\varphi : M \to N\) be a local isometry. Then \(S(M, g, x) \subset S(N, h, \varphi(x))\) for any \(x \in M\). As a consequence, \(S(M, g) \subset S(N, h)\).

(6) \(S_{t_1}(M, g, x) \subset S_{t_2}(M, g, x)\) when \(0 < t_1 < t_2\), for any \(x \in M\). As a consequence, \(S_{t_1}(M, g) \subset S_{t_2}(M, g)\) when \(0 < t_1 < t_2\).

(7) \(\bigcap_{\tau > t} S_{\tau}(M, g, x) = S_t(M, g, x)\) for any \(x \in M\) and \(t > 0\). As a consequence, \(S_t(M, g) = \bigcap_{\tau > t} S_{\tau}(M, g)\).

(8) Let \(\lambda\) be a positive constant. Then, \((t, \beta, \gamma) \in S(M, \lambda^2 g, x)\) if and only if \((\lambda^{-2}t, \beta, \lambda^2 \gamma) \in S(M, g, x)\). As a consequence, \((t, \beta, \gamma) \in S(M, \lambda^2 g)\) if and only if \((\lambda^{-2}t, \beta, \lambda^2 \gamma) \in S(M, g)\).

(9) Let \((N, h)\) be another complete Riemannian manifold. Then, \(S(M \times N, g \times h, (x, y)) \subset S(M, g, x)\) for any \(x \in M\) and \(y \in N\). As a consequence, \(S(M \times N, g \times h) \subset S(M, g)\).

**Proof** The statements (1)–(3) are clearly true by definition. We only give the proofs of (4)–(9).

(4) For any \((t, \beta, \gamma) \in S(M, g, x)\) and \(u \in \mathcal{P}(M, g)\), since \(\varphi\) is an isometry, \(\varphi^*u \in \mathcal{P}(M, g)\).

\[
\begin{align*}
[\beta \|\nabla \log u\|^2 - (\log u)_t](t, \varphi(x)) &= [\beta \|\nabla \log \varphi^*u\|^2 - (\log \varphi^*u)_t](t, x) \\
&\leq \gamma.
\end{align*}
\]

This means that \(S(M, g, x) \subset S(M, g, \varphi(x))\). By applying this to \(\varphi^{-1}\), we obtain the conclusion.

(5) The proof is the same as that of (4). Because a local isometry of complete Riemannian manifolds must be surjective, we obtain the consequence that \(S(M, g) \subset S(N, h)\).

(6) Let \((\beta, \gamma) \in S_{t_1}(M, g, x)\) and \(u \in \mathcal{P}(M, g)\). Let \(v(t, x) = u(t + t_2 - t_1, x)\). It is clear that \(v \in \mathcal{P}(M, g)\). So

\[
\begin{align*}
[\beta \|\nabla \log u\|^2 - (\log u)_t](t_2, x) &= [\beta \|\nabla \log v\|^2 - (\log v)_t](t_1, x) \\
&\leq \gamma.
\end{align*}
\]

which means that \((\beta, \gamma) \in S_{t_2}(M, g, x)\). So, \(S_{t_1}(M, g) \subset S_{t_2}(M, g)\).

(7) Let \((\beta, \gamma) \in \bigcap_{\tau > t} S_{\tau}(M, g, x)\). Then, for any \(u \in \mathcal{P}(M, g)\),

\[
[\beta \|\nabla \log u\|^2 - (\log u)_t](\tau, x) \leq \gamma
\]

for any \(\tau > t\). Setting \(\tau \to t^+\), we have

\[
[\beta \|\nabla \log u\|^2 - (\log u)_t](t, x) \leq \gamma.
\]

So \((\beta, \gamma) \in S_t(M, g, x)\). Combining this and (6), we get the conclusion.

(8) Let \((t, \beta, \gamma) \in S(M, \lambda^2 g, x)\). For any \(u \in \mathcal{P}(M, g)\), let \(v(t, x) = u(\lambda^{-2}t, x)\). Then, \(v \in \mathcal{P}(M, \lambda^2 g)\). So,

\[
[\beta \|\nabla \lambda^{-2} g \log v\|^2 - (\log v)_t](t, \lambda x) \leq \lambda \gamma.
\]

This implies that

\[
[\beta \|\nabla \log u\|^2 - (\log u)_t](\lambda^{-2}t, x) \leq \lambda^2 \gamma.
\]
(9) For any \((t, \beta, \gamma) \in S(M \times N, g \times h, (x, y))\) and \(u \in \mathcal{P}(M, g, x)\), let \(v(t, x, y) = u(t, x)\), then \(v \in \mathcal{P}(M \times N, g \times h)\). So

\[
[\beta \| \nabla \log u \|^2 - (\log u)_t](t, x) = [\beta \| \nabla \log v \|^2 - (\log v)_t](t, x, y) \leq \gamma,
\]

and hence \((t, \beta, \gamma) \in S(M, g)\).

Next, we come to prove Theorem 1.1.

**Proof of Theorem 1.1** Since \(H(\cdot, \cdot, y) \in \mathcal{P}(M, g)\) for any \(y \in M\),

\[
S(M, g, x) \subset \bigcap_{y \in M} S(H(\cdot, \cdot, y), x).
\]  

(2.8)

Conversely, we only need to show that

\[
\bigcap_{y \in M} S(H(\cdot, \cdot, y), x) \subset S(u, x)
\]

for any \(u \in \mathcal{P}(M, g)\).

We first show that Eq. 2.9 is true for any \(u \in \mathcal{P}(M, g)\) that is smooth up to \(t = 0\) and \(u(0, x) = f(x)\) is of compact support. In this case, by the uniqueness of positive solution in [12] (see also [18]),

\[
u(t, x) = \int_M H(t, x, y) f(y) dy.
\]  

(2.10)

So, for any \((t, \beta, \gamma) \in \bigcap_{y \in M} S(H(\cdot, \cdot, y), x)\), we have that, at \((t, x)\),

\[
\beta \| \nabla \log u \|^2 - (\log u)_t
\]

\[
= \left( \int_M H(t, x, y) f(y) dy \right)^{-2} \left( \beta \int_M \int_M \langle \nabla_x H(t, x, y), \nabla_x H(t, x, z) \rangle f(y) f(z) dy dz - \int_M \int_M H_t(t, x, y) H(t, x, z) f(y) f(z) dy dz \right)
\]

\[
= \frac{1}{2} \left( \int_M H(t, x, y) f(y) dy \right)^{-2} \left( 2 \beta \int_M \int_M \langle \nabla_x H(t, x, y), \nabla_x H(t, x, z) \rangle f(y) f(z) dy dz - \int_M \int_M (H(t, x, y) H_t(t, x, z) + H(t, x, y) H_t(t, x, z)) f(y) f(z) dy dz \right)
\]

\[
= \frac{1}{2} \left( \int_M H(t, x, y) f(y) dy \right)^{-2} \times
\]

\[
\left( 2 \beta \int_M \int_M \langle \nabla_x \log H(t, x, y), \nabla_x \log H(t, x, z) \rangle H(t, x, y) H(t, x, z) f(y) f(z) dy dz - \int_M \int_M (\log H(t, x, y))_t + (\log H(t, x, z))_t \rangle H(t, x, y) H(t, x, z) f(y) f(z) dy dz \right)
\]
\[\begin{aligned}
&\leq \frac{1}{2} \left( \int_M H(t, x, y) f(y) dy \right)^{-2} \times \\
&\left( \beta \int_M \int_M (\|\nabla_x \log H(t, x, y)\|^2 + \|\nabla_x \log H(t, x, z)\|^2) H(t, x, y) H(t, x, z) f(y) f(z) dy dz \\
&- \int_M \int_M (\log H(t, x, y))_t + (\log H(t, x, z))_t) H(t, x, y) H(t, x, z) f(y) f(z) dy dz \right) \\
&\leq \gamma.
\end{aligned}\]

Next, we come to show that Eq. 2.9 is true for any \( u \in \mathcal{P}(M, g) \) that is smooth up to \( t = 0 \). By the uniqueness of positive solution by Donnelly [12] again, we have the same expression of \( u \) as in Eq. 2.10 with \( f(x) = u(0, x) \) not necessarily of compact support. Let

\[ \Omega_1 \subset \subset \Omega_2 \subset \subset \cdots \subset \subset \Omega_k \subset \subset \Omega_{k+1} \subset \subset \cdots \]

be an exhaustion of \( M \) and \( \eta_k \) be a smooth function on \( M \) satisfying: (1) \( 0 \leq \eta_k \leq 1 \); (2) \( \text{supp} \eta_k \subset \subset \Omega_{k+1} \) and (3) \( \eta_k|_{\Omega_k} \equiv 1 \) for \( k = 1, 2, \cdots \). Moreover, let

\[ u_k(t, x) = \int_M H(t, x, y) \eta_k(y) f(y) dV(y). \] (2.11)

Then \( u_k \) tends to \( u \) smoothly (see [12, 13]). By the last case, we have

\[ [\beta \|\nabla \log u_k\|^2 - (\log u_k)_t](t, x) \leq \gamma \] (2.12)

for any \( (t, \beta, \gamma) \in \bigcap_{y \in M} S(H(\cdot, \cdot, y), x) \). By letting \( k \to \infty \), we get Eq. 2.9.

Finally, we come the show that Eq. 2.9 is true for any \( u \in \mathcal{P}(M, g) \). For any \( \epsilon > 0 \), let \( u_\epsilon(t, x) = u(t + \epsilon, x) \). Then, by the last case,

\[ [\beta \|\nabla \log u_\epsilon\|^2 - (\log u_\epsilon)_t](t, x) \leq \gamma \] (2.13)

for any \( (t, \beta, \gamma) \in \bigcap_{y \in M} S(H(\cdot, \cdot, y), x) \). By letting \( \epsilon \to 0^+ \), we get Eq. 2.9. \( \square \)

Similarly as in the fundamental work of Li-Yau [18], one has the following relation of Li-Yau multiplier set and Harnack inequality for positive solution of heat equation by the same argument as in [18] (see also [17]).

**Proposition 2.3** Let \( (M^n, g) \) be a complete Riemannian manifold and let \( (t, \beta(t), \gamma(t)) \) with \( t \in (a, b] \) be a curve in \( S(M, g) \) where \( 0 \leq a < b \). Then, for any \( u \in \mathcal{P}(M, g) \), \( a < t_1 < t_2 \leq b \) and \( x_1, x_2 \in M \),

\[ u(x_1, t_1) \leq u(x_2, t_2) \exp \left( \frac{r^2(x_1, x_2)}{4(t_2 - t_1)^2} \int_{t_1}^{t_2} \frac{1}{\beta(t)} dt + \int_{t_1}^{t_2} \gamma(t) dt \right). \] (2.14)

### 3 Analysis of Heat Kernels on Hyperbolic Spaces

In this section, we prove Theorem 1.5 by a combination of the argument of Davies-Mandouvalos [11] and some Gronwall-type differential inequalities derived from the recurrence relations of heat kernels.

The same as in [11], we write the heat kernel \( K_n \) of \( \mathbb{H}^n \) as

\[ K_n(t, r) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{(n-1)r^2}{4} t - \frac{r^2}{4t} a_n(t, r)}. \] (3.1)
Then, by Eq. 1.39, $\alpha_n(t, r)$ satisfies the following recursive identity:

$$\alpha_n = \frac{r}{\sinh r} \alpha_{n-2} - \frac{2t}{\sinh r} \frac{\partial \alpha_{n-2}}{\partial r}. \quad (3.2)$$

Moreover, $\alpha_1 = 1$ and $\alpha_3 = \frac{r}{\sinh r}$. Let $f_1 = \frac{r}{\sinh r}$ and $\sigma = \cosh r$. Then, by Eq. 3.2,

$$\alpha_n = f_1 \alpha_{n-2} - 2t \frac{\partial \alpha_{n-2}}{\partial \sigma}. \quad (3.3)$$

As mentioned in [11], by induction, it is not hard to see that

$$\alpha_{2m+1} = \sum_{i=0}^{m-1} i^i P_{m,i}(f_1, f_2, \cdots, f_m) \quad (3.4)$$

with $P_{m,0}(T_1, T_2, \cdots, T_m) = T_1^m$ and $P_{m,m}(T_1, T_2, \cdots, T_m) = 2^{m-1} T_m$, where

$$f_{m+1} = -\left( \frac{df_m}{d\sigma} \right) = -\frac{1}{\sinh r} \frac{df_m}{dr}$$

for $m = 1, 2, \cdots$. Here $P_{m,i}(T_1, T_2, \cdots, T_m)$’s are polynomials with nonnegative coefficients. As mentioned in [11], by that $\alpha_{2m+1}$ is positive, $f_m$ is decreasing and positive. By making more detailed analysis on $\alpha_{2m+1}$ and $f_m$, we have the following conclusion which will be used later.

**Proposition 3.1** Let $q_m = \frac{f_{m+1}}{f_m}$ for $m = 1, 2, \cdots$. Then

1. $q_m(0) = \frac{m^2}{2^{m+1}}$ for $m = 1, 2, \cdots$;
2. $\lim_{r \to \infty} q_m(r) \cosh r = m$ for $m = 1, 2, \cdots$;
3. $q_m(r) \cosh r \leq m$ and as a consequence $0 \leq -(\log q_m)_r \leq m$ for $m = 1, 2, \cdots$;
4. $0 \leq \cosh r (q_{m+1}(r) - q_m(r)) \leq 1$ and as a consequence, $0 \leq -((\log q_m)_r \leq 1$ for $m = 1, 2, \cdots$;
5. $P_{m,i}(T_1, T_2, \cdots, T_m)$ is a homogenous polynomial of degree $m - i$ for $m = 1, 2, \cdots$ and $i = 0, 1, \cdots, m - 1$;
6. $P_{m,i}(T_1, T_2, \cdots, T_m)$ is a weighted homogenous polynomial of degree $m$ with nonnegative coefficients for $m = 1, 2, \cdots$ and $i = 0, 1, \cdots, m - 1$, when counting the degree of $T_j$ as $j$ for $j = 1, 2, \cdots, m$.

**Proof** By taking derivative to $f_1 \sinh r = r$ with respect to $r$, one has

$$\sigma f_1 - (\sigma^2 - 1) f_2 = 1. \quad (3.5)$$

Taking $m^{th}$ derivative to the last equality with respect to $\sigma$, we have

$$m^2 f_m - (2m + 1) \sigma f_{m+1} + (\sigma^2 - 1) f_{m+2} = 0 \quad (3.6)$$

for any $m = 1, 2, \cdots$.

1. Let $\sigma = 1$, i.e. $r = 0$ in Eq. 3.6. We have

$$m^2 f_m(0) = (2m + 1) f_{m+1}(0). \quad (3.7)$$

So $q_m(0) = \frac{m^2}{2^{m+1}}$.

2. By the expression of $f_1$, it is clear that

$$\lim_{r \to \infty} \frac{\cosh r}{r} f_1 = 1. \quad (3.8)$$
Then, by Eq. 3.5,
\[ \lim_{r \to \infty} \frac{\cosh^2 r}{r} f_2 = 1. \] (3.9)

By Eq. 3.6 and induction, \( \lim_{r \to \infty} \frac{\cosh^n r f_m}{r} = a_m \) exists. Moreover,
\[ m^2 a_m - (2m + 1)a_{m+1} + a_{m+2} = 0 \] (3.10)
with \( a_1 = a_2 = 1 \). By Eq. 3.10,
\[ m(m a_m - a_{m+1}) = (m + 1)a_{m+1} - a_{m+2}. \] (3.11)
So, we have \( m a_m - a_{m+1} = 0 \) for any \( m = 1, 2, \ldots \). This implies that
\[ \lim_{r \to \infty} q_m(r) \cosh r = m. \] (3.12)

(3) By Eq. 3.6,
\[ m(m f_m - \sigma f_{m+1}) - \sigma ((m + 1) f_{m+1} - \sigma f_{m+2}) = f_{m+2} > 0. \] (3.13)
Moreover
\[ (m f_m - \sigma f_{m+1})_\sigma = -((m + 1) f_{m+1} - \sigma f_{m+2}). \] (3.14)
Substituting this into the last inequality, we have
\[ m(m f_m - \sigma f_{m+1}) + \sigma (m f_m - \sigma f_{m+1})_\sigma > 0. \] (3.15)
This implies that
\[ \left( \sigma^m (m f_m - \sigma f_{m+1}) \right)_\sigma > 0. \] (3.16)
Therefore, by (1),
\[ \sigma^m (m f_m - \sigma f_{m+1}) \geq (m f_m - \sigma f_{m+1})|_{\sigma=1} = f_m(0)(m-q_m(0)) = \frac{m^2 + m}{2m + 1} f_m(0) > 0. \] (3.17)
Furthermore,
\[ (f_m)_r + m f_m = -\sinh r_{f_{m+1}} + m f_m \geq -\cosh r_{f_{m+1}} + m f_m = f_m(m - \cosh r q_m) \geq 0. \] (3.18)
So, \( 0 \leq (- \log f_m)_r \leq m \) by that \( f_m \) is decreasing.

(4) By Eq. 3.6,
\[ m^2 - (2m + 1)\sigma q_m + (\sigma^2 - 1)q_m q_{m+1} = 0 \] (3.19)
and
\[ (m + 1)^2 - (2m + 3)\sigma q_{m+1} + (\sigma^2 - 1)q_{m+1} q_{m+2} = 0. \] (3.20)
Taking subtraction of the last two equalities, we have
\[ q_{m+1}(q_m - q_{m+2}) = \frac{2m+1-2\sigma q_m}{\sigma^2-1} + \frac{(2m+3)\sigma}{\sigma^2-1}(q_m - q_{m+1}) \]
\[ > \frac{(2m+3)\sigma}{\sigma^2-1}(q_m - q_{m+1}) \] (3.21)
where we have used (3) in the last inequality. Then,
\[ (q_m - q_{m+1}) = q_m^2 - q_{m+1}^2 - q_{m+1}(q_m - q_{m+2}) \]
\[ < \left( q_m + q_{m+1} - \frac{(2m+3)\sigma}{\sigma^2-1} \right)(q_m - q_{m+1}). \] (3.22)
So,
\[ \left( (\sigma^2 - 1)^{\frac{2m+1}{2}} e^{-Q_m - Q_{m+1} (q_m - q_{m+1})} \right)_\sigma < 0 \] (3.23)
where \( Q_m = \int q_m d\sigma \). This implies that \( q_m - q_{m+1} \leq 0 \).
Furthermore, taking substraction of Eqs. 3.20 and 3.19, one has
\[
(2m + 1)(1 + \sigma(q_m - q_{m+1})) = \frac{\sigma^2 - 1}{\sigma}q_{m+1}\left(\frac{2\sigma^2}{\sigma^2 - 1} + \sigma(q_m - q_{m+2})\right) \geq \frac{\sigma^2 - 1}{\sigma}q_{m+1}[(1 + \sigma(q_m - q_{m+1})) + (1 + \sigma(q_{m+1} - q_{m+2}))]. \tag{3.24}
\]

So
\[
q_{m+1}[1 + \sigma(q_{m+1} - q_{m+2})] \leq \left(\frac{2m + 1}{\sigma^2 - 1} - q_{m+1}\right)(1 + \sigma(q_m - q_{m+1})). \tag{3.25}
\]

Then,
\[
[1 + \sigma(q_m - q_{m+1})]\sigma = [\sinh(\sigma^2 q_m - q_{m+1}) + 1]q_m \geq [\cosh(\sigma^2 q_m - q_{m+1}) + 1]q_m \geq 0. \tag{3.26}
\]

Similarly as before, this implies that
\[
1 + \sigma(q_m - q_{m+1}) \geq 0. \tag{3.27}
\]

Moreover, note that \((q_m)_{\sigma} = q_m(q_m - q_{m+1}) \leq 0\). So, \(q_m\) is decreasing. Furthermore,
\[
(q_m)_{r} + q_m = \sinh r(q_m)_{\sigma} + q_m = [\sinh r(q_m - q_{m+1}) + 1]q_m \geq [\cosh r(q_m - q_{m+1}) + 1]q_m \geq 0.
\]

So \(0 \leq -(\log q_m)_r \leq 1\).

(5) By Eqs. 3.3 and 3.4,
\[
P_{m+1,i} = T_1P_{m,i} + 2\sum_{j=1}^{m} \frac{\partial P_{m,i-1}}{\partial T_j}T_{j+1} \tag{3.29}
\]

for \(i = 0, 1, \cdots, m\). Here, we take \(P_{m,-1} = P_{m,m} = 0\). The conclusion follows by induction on \(m\) and Eq. 3.29.

(6) The conclusion follows by induction on \(m\) and Eq. 3.29.

Next, we come to prove Theorem 1.5.

Proof of Theorem 1.5 (1) By the expression (3.4) of \(\alpha_{2m+1}\), it is clear that \(\alpha_{2m+1}\) is increasing on \(t\).

(2) By Eq. 1.41,
\[
t^{\frac{1}{2}}\alpha_{2m}(t, r) = \frac{\sqrt{2}}{(4\pi)^{\frac{1}{2}}} \int_{r}^{\infty} \frac{\alpha_{2m+1}(t, s)e^{-\frac{r^2-s^2}{4t}}}{\sqrt{\cosh s - \cosh r}} \sinh s ds. \tag{3.30}
\]

Because \(\alpha_{2m+1}\) and \(e^{-\frac{r^2-s^2}{4t}}\) are both increasing for \(s \geq r\) on \(t\), \(t^{\frac{1}{2}}\alpha_{2m}\) is increasing on \(t\).
(3) We first prove the odd dimensional case. Since \( f_m \) is decreasing for \( m = 1, 2, \cdots \), by Eq. 3.4, \( \alpha_{2m+1} \) is decreasing with respect to \( r \). So \(- (\log \alpha_{2m+1})_r \geq 0 \). On the other hand, by (6) of Proposition 3.1, suppose that
\[
P_m,i(f_1, f_2, \cdots , f_m) = \sum_{j_1, j_2, \cdots , j_m \geq 0 \atop j_1 + 2j_2 + \cdots + mj_m = m} a_{m,j_1,j_2,\cdots,j_m} f_1^{j_1} f_2^{j_2} \cdots f_m^{j_m} \tag{3.31}
\]
with \( a_{m,j_1,j_2,\cdots,j_m} \geq 0 \). Then
\[
(\alpha_{2m+1})_r + m\alpha_{2m+1}
= \sum_{i=0}^{m-1} t^i[(P_m,i(f_1, f_2, \cdots , f_m))_r + m P_m,i(f_1, f_2, \cdots , f_m)]
= \sum_{i=0}^{m-1} t^i \sum_{j_1, j_2, \cdots , j_m \geq 0 \atop j_1 + 2j_2 + \cdots + mj_m = m} a_{m,j_1,j_2,\cdots,j_m} [(f_1^{j_1} f_2^{j_2} \cdots f_m^{j_m})_m + m f_1^{j_1} f_2^{j_2} \cdots f_m^{j_m}]
\geq 0 \tag{3.32}
\]
by (3) of Proposition 3.1. So \(- (\log \alpha_{2m+1})_r \leq m \).

Next, we come to show the even dimensional case. By Eq. 1.41,
\[
\alpha_{2m}(t, r) = \frac{\sqrt{2}}{(4\pi t)^{\frac{m}{2}}} \int_r^\infty \frac{\alpha_{2m+1}(t, s) e^{-\frac{x^2}{4t^2}} \sinh s}{\sqrt{\cosh s - \cosh r}} ds
= \frac{1}{(8\pi t)^{\frac{m}{2}}} \int_0^\infty \frac{\alpha_{2m+1}(t, s) e^{\frac{s^2}{2f_1(s)}}}{\sqrt{\cosh s - \cosh r}} dx \tag{3.33}
\]
where \( s = \sqrt{x + r^2} \). By that \( q_m \) is decreasing and the expression (3.4), \( \alpha_{2m+1}(t, s)/f_1(s) \) is decreasing on \( r \). Combining this with Lemma 3.1, we know that \( \alpha_{2m}(t, r) \) is decreasing with respect to \( r \).

Furthermore, by (4) of Proposition 3.1,
\[
0 \leq - \left( \frac{f_m}{f_1} \right)_r \leq m - 1 \tag{3.34}
\]
for \( m = 1, 2, \cdots \). From this and using the same argument as in the proof of the odd dimensional case, we have
\[
- \left[ (\log(\alpha_{2m+1}(t, s)/f_1(s)))_r \right]_s \leq m - 1. \tag{3.35}
\]
Therefore,
\[
- \left[ (\log(\alpha_{2m+1}(t, s)/f_1(s)))_r \right]_r = - \left[ (\log(\alpha_{2m+1}(t, s)/f_1(s)))_r \right]_s \leq m - 1. \tag{3.36}
\]
Then, by Lemma 3.1, we know that
\[
- \left[ (\log(\alpha_{2m+1}(t, s) (\cosh s - \cosh r)^{-\frac{1}{2}}/f_1(s)))_r \right]_r \leq m - \frac{1}{2}. \tag{3.37}
\]
From this and the expression (3.33) of \( \alpha_{2m} \), we get the conclusion. □

In the proof of (3) in Theorem 1.5, we need the following lemma.
Lemma 3.1 For any positive constant $a$,

$$0 \leq \left[ \log \left( \cosh \sqrt{a^2 + r^2} - \cosh r \right) \right]_r \leq 1.$$ 

Proof Note that

$$(\cosh \sqrt{a^2 + r^2} - \cosh r)_r = r \left( \frac{\sinh \sqrt{a^2 + r^2}}{\sqrt{a^2 + r^2}} - \frac{\sinh r}{r} \right) \geq 0 \quad (3.38)$$

by that $\frac{\sinh x}{x}$ is an increasing function. So,

$$\left[ \log \left( \cosh \sqrt{a^2 + r^2} - \cosh r \right) \right]_r \geq 0. \quad (3.39)$$

Moreover, to show that

$$\left[ \log \left( \cosh \sqrt{a^2 + r^2} - \cosh r \right) \right]_r \leq 1, \quad (3.40)$$

it is equivalent to show that

$$F(\rho, r) := \rho (\cosh \rho - \cosh r) - (r \sinh \rho - \rho \sinh r) \geq 0 \quad (3.41)$$

where $\rho = \sqrt{a^2 + r^2} \geq r$. Note that

$$F_\rho = \rho \sinh \rho + \cosh \rho - r \cosh \rho - \cosh r + \sinh r \quad (3.42)$$

and

$$F_{\rho \rho} = 2 \sinh \rho + \rho \cosh \rho - r \sinh \rho > 0 \quad (3.43)$$

for $\rho \geq r$. Hence

$$F_\rho(\rho, r) \geq F_\rho(r, r) = (1 + r) \sinh r - r \cosh r. \quad (3.44)$$

Moreover

$$[F_\rho(r, r)]_r = r \cosh r + \sinh r - r \sinh r \geq 0. \quad (3.45)$$

So,

$$F_\rho(\rho, r) \geq F_\rho(r, r) \geq F_\rho(0, 0) = 0. \quad (3.46)$$

Hence

$$F(\rho, r) \geq F(r, r) = 0. \quad (3.47)$$

This completes the proof of the lemma. \qed

4 Optimal Li-Yau Gradient Estimate on Hyperbolic Spaces

In this section, we come to prove Theorem 1.2, Theorem 1.3 and Theorem 1.4. We first prove Theorem 1.2.

Proof of Theorem 1.2 By (5) of Proposition 2.2, we only need to prove the theorem for hyperbolic spaces. By Eq. 3.1,

$$\log K_n = -\frac{n}{2} \log(4\pi t) - \frac{(n-1)^2}{4} t - \frac{r^2}{4t} + \log \alpha_n. \quad (4.1)$$
Then, by (1) and (3) of Theorem 1.5,
\[ \beta \| \nabla \log K_n \|^2 - (\log K_n)_t, \]
\[ = \beta \left( -\frac{r}{2t} + \log(\alpha_n)_r \right)^2 + \frac{n}{2t} + \frac{(n-1)^2}{4} - \frac{r^2}{4t^2} - (\log \alpha_n)_t \]
\[ = -(1 - \beta) \left( \frac{r}{2t} + \frac{\beta}{1 - \beta} (\log \alpha_n)_r \right)^2 + \frac{\beta}{1 - \beta} (\log \alpha_n)_r^2 + \frac{(n-1)^2}{4} + \frac{n}{2t} - (\log \alpha_n)_t, \]
\[ \leq \frac{n}{2t} + \frac{(n-1)^2}{4(1 - \beta)}. \] (4.2)

Next, we come to prove Theorem 1.3.

**Proof of Theorem 1.3** By (5) of Proposition 2.2, we only need to prove the theorem for hyperbolic spaces. By Eq. 3.1,
\[ \log K_n = -\frac{n+1}{2} \log(4\pi t) - \frac{(n-1)^2}{4} - \frac{r^2}{4t} + \log((4\pi t)^{1/2} \alpha_n). \] (4.3)
By (2) and (3) of Theorem 1.5,
\[ \beta \| \nabla \log K_n \|^2 - (\log K_n)_t, \]
\[ = \beta \left( -\frac{r}{2t} + \log((4\pi t)^{1/2} \alpha_n)_r \right)^2 + \frac{n+1}{2t} + \frac{(n-1)^2}{4} - \frac{r^2}{4t^2} - (\log(4\pi t)^{1/2} \alpha_n)_t \]
\[ \leq -(1 - \beta) \left( \frac{r}{2t} + \frac{\beta}{1 - \beta} (\log \alpha_n)_r \right)^2 + \frac{\beta}{1 - \beta} (\log \alpha_n)_r^2 + \frac{(n-1)^2}{4} + \frac{n+1}{2t} \]
\[ \leq \frac{n+1}{2t} + \frac{(n-1)^2}{4(1 - \beta)}. \] (4.4)

By applying Proposition 2.3, we are able to prove Theorem 1.4.

**Proof of Theorem 1.4** We only need to prove the odd dimensional case. The proof of the even dimensional case is similar.

By Proposition 2.3 and Theorem 1.2, we have
\[ u(x_1, t_1) \leq \left( \frac{t_2}{t_1} \right)^{\frac{n}{2}} \exp \left( \frac{r^2(x_1, x_2)}{4\beta(t_2 - t_1)} + \frac{(n-1)^2(t_2 - t_1)}{4(1 - \beta)} \right) u(x_2, t_2) \] (4.5)
for any constant \( \beta \in (0, 1) \). Let \( \beta = \frac{1}{1 + \frac{(n-1)^2(t_2 - t_1)}{r^2(x_1, x_2)}}, \) which is the minimum point of \( \frac{r^2(x_1, x_2)}{4\beta(t_2 - t_1)} + \frac{(n-1)^2(t_2 - t_1)}{4(1 - \beta)} \). We get the conclusion.

**Appendix**

In this appendix, we give the details in the comparisons of Li-Xu’s estimate (1.6) with the Li-Yau-Davies estimate (1.3) and with Hamilton’s estimate (1.4) respectively which are not that obvious comparing to the other comparisons of estimates in Section 1. Moreover, we will show that \( \alpha_2 \) is deceasing with respect to \( t \).
We first compare Li-Xu’s estimate (1.6) and Hamilton’s estimate (1.4).

**Proposition 5.1** For any \( x \geq 0 \),

\[
\frac{x(\coth x + 1)}{1 + \frac{\sinh x \cosh x - x}{\sinh^2 x}} \leq e^{2x}
\]

and

\[
1 + \frac{\sinh x \cosh x - x}{\sinh^2 x} \leq e^{2x}.
\]

As a consequence, Li-Xu’s estimate (1.6) is better than Hamilton’s estimate (1.4) by letting \( x = kt \).

**Proof** Let \( \beta = \frac{1}{1 + \frac{\sinh x \cosh x - x}{\sinh^2 x}} \). Then, it is clear that

\[
\frac{x(\coth x + 1)}{1 + \frac{\sinh x \cosh x - x}{\sinh^2 x}} = x \left( \frac{1}{\beta} + \frac{x}{\sinh^2 x} \right) \beta
\]

\[
= x + \frac{x^2}{\sinh^2 x} \beta
\]

\[
\leq 1 + x \leq e^{2x}.
\]

(5.1)

For the other inequality, it is equivalent to

\[
f(x) = (e^{2x} - 1) \sinh^2 x - \cosh x \sinh x + x \geq 0.
\]

(5.2)

Note that

\[
f'(x) = 2(e^{2x} - 1) \sinh^2 x + 2(e^{2x} - 1) \sinh x \cosh x \geq 0.
\]

(5.3)

So

\[
f(x) \geq f(0) = 0.
\]

(5.4)

Next, we come to compare Li-Xu’s estimate (1.6) and the Li-Yau-Davies estimate (1.3).

**Proposition 5.2** Let \( \beta = \frac{1}{1 + \frac{\sinh x \cosh x - x}{\sinh^2 x}} \). Then,

1. \( x(\coth x + 1)\beta < \frac{1}{\beta} + \frac{x}{2(1 - \beta)} \) for any \( x > 0 \). As a consequence, \( \gamma_{LYD}(\beta_{LX}(t), t) > \gamma_{LX}(t) \) for any \( t > 0 \) by setting \( x = kt \).

2. The graphs of the functions \( (1 + \sqrt{\frac{x}{2}})^2 \) and \( x(\coth x + 1)\beta \) intersect at only one point \( x_{LX} \geq 8 \). As a consequence, the graphs of the functions \( \gamma_m(t) \) and \( \gamma_{LX}(t) \) intersect at only one point \( t_{LX} = \frac{x_{LX}}{k} \).

3. When \( x \geq x_{LX}, \beta > \frac{1}{1 + \sqrt{\frac{x}{2}}}. \) As a consequence, \( \beta_m(t) < \beta_{LX}(t) \) for \( t \geq t_{LX} \).

**Proof** (1) It is not hard to see that \( \frac{1}{2} \leq \beta \leq 1 \). Then, the same as in the proof of the last proposition, we have

\[
x(\coth x + 1)\beta \leq 1 + x < \frac{1}{\beta} + \frac{x}{2(1 - \beta)}.
\]

(5.5)
(2) The same as in the proof of the last proposition, we have
\[ x \leq x(\coth x + 1) \leq x + 1. \] (5.6)

The graphs of \((1 + \sqrt{x/2})^2\) and \(x + 1\) intersect at \(x = 8\) while the graphs of \((1 + \sqrt{x/2})^2\) and \(x\) intersect at \(x = 6 + 4\sqrt{2}\). Hence, the graphs of \((1 + \sqrt{x/2})^2\) and \(x(\coth x + 1)\) must intersect at some \(x \in [8, 6 + 4\sqrt{2}]\). Moreover, note that
\[
\left[ \left(1 + \sqrt{\frac{x}{2}}\right)^2 \right]_x \leq \frac{3}{4} \quad (5.7)
\]
when \(x \geq 8\) while
\[
[x(\coth x + 1)\beta]_x \geq 1 - \frac{(2x^2 + 2x^3) \cosh x}{\sinh^3 x} \geq 1 - \frac{4x + 4x^2}{\sinh x} > \frac{3}{4} \quad (5.8)
\]
when \(x \geq 8\). Here we have used that \(0 < \beta \leq 1, \frac{x}{\sinh x} \leq 1\) and \(\cosh x \leq 2 \sinh x\) for \(x \geq 8\). This gives us the conclusion.

(3) When \(x \geq x_{LX} \geq 8\),
\[
\sqrt{\frac{x}{2}} \sinh^2 x - \cosh x \sinh x + x \geq 2 \sinh^2 x - \cosh x \sinh x + x > 0, \quad (5.9)
\]
since \(2 \sinh x \geq \cosh x\) when \(x \geq 8\). This gives us the conclusion.

Finally, we come to show that \(\alpha_2\) is decreasing with respect to \(t\).

**Proposition 5.3** For the hyperbolic plane \(\mathbb{H}^2\), \((\log \alpha_2)_t < 0\) and hence
\[
- (\log K_2)_t (t, 0) > \frac{1}{t} + \frac{1}{4}, \quad (5.10)
\]

**Proof** By Eq. 3.33,
\[
\alpha_2 = \frac{\sqrt{2}e^{\frac{x^2}{2}}}{\sqrt{4\pi t}} \int_r^\infty \frac{se^{-\frac{s^2}{2}}}{\sqrt{\cosh s - \cosh r}} \, ds
= \frac{1}{2\sqrt{2\pi}} \int_0^\infty \frac{e^{-\frac{x}{t}}}{\sqrt{\cosh(\sqrt{t/\pi x}^2) - \cosh r}} \, dx \quad (5.11)
\]
where \( x = \frac{s^2 - r^2}{t} \). Note that
\[
\frac{\cosh(\sqrt{tx + r^2}) - \cosh r}{t} = \frac{1}{t} \sum_{i=1}^{\infty} \frac{(tx + r^2)^i - r^{2i}}{(2i)!}
\]
\[
= a_0(x, r) + a_1(x, r)t + a_2(x, r)t^2 + \cdots \tag{5.12}
\]
with \( a_0(x, r), a_1(x, r), a_2(x, r), \cdots \) all positive numbers when \( x > 0 \), hence it is strictly increasing. So, \( (\alpha_2)_t < 0 \). Moreover, by Eq. 3.1, we have
\[
-(\log K_2)_t(t, 0) = \frac{1}{t} + \frac{1}{4} - (\log \alpha_2)_t(t, 0) > \frac{1}{t} + \frac{1}{4}. \tag{5.13}
\]

References

1. Bakry, D., Qian, Z.M.: Harnack inequalities on a manifold with positive or negative Ricci curvature. Rev. Mat. Iberoam. 15(1), 143–179 (1999)
2. Bakry, D., Ledoux, M.: A logarithmic Sobolev form of the Li-Yau parabolic inequality. Rev. Mat. Iberoam. 22(2), 683–702 (2006)
3. Bakry, D., Bolley, F., Gentil, I.: The Li-Yau inequality and applications under a curvature-dimension condition. Ann. Inst. Fourier (Grenoble) 67(1), 397–421 (2017)
4. Cao, H.-D.: On Harnack’s inequalities for the Kähler-Ricci flow. Invent. Math. 109(2), 247–263 (1992)
5. Cao, H.-D., Ni, L.: Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds. Math. Ann. 331(4), 795–807 (2005)
6. Cao, X., Fayyazuddin, L.B., Liu, B.: Differential Harnack estimates for a nonlinear heat equation. J. Funct. Anal. 10, 2312–2330 (2013)
7. Carron G.: Geometric inequalities for manifolds with Ricci curvature in the Kato class. arXiv:1612.03027
8. Chow, B., Chu, S.-C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci flow: techniques and applications. Part I. Geometric aspects. Math Surveys Monogr., vol. 135. American Mathematical Society, Providence (2007)
9. Chow, B., Hamilton, R.S.: Constrained and linear Harnack inequalities for parabolic equations. Invent. Math. 129(2), 213–238 (1997)
10. Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge (1990). x+197 pp. ISBN: 0-521-40997-7
11. Davies, E.B., Mandouvalos, N.: Heat kernel bounds on hyperbolic space and Kleinian groups. Proc. Lond. Math. Soc. (3) 57(1), 182–208 (1988)
12. Donnelly, H.: Uniqueness of positive solutions of the heat equation. Proc. Am. Math. Soc. 99(2), 353–356 (1987)
13. Dodziuk, J.: Maximum principle for parabolic inequalities and the heat flow on open manifolds. Indiana Univ. Math. J. 32(5), 703–716 (1983)
14. Hamilton, R.S.: A matrix Harnack estimate for the heat equation. Commun. Anal. Geom. 1(1), 113–126 (1993)
15. Hamilton, R.S.: The Harnack estimate for the Ricci flow. J. Differ. Geom. 37(1), 225–243 (1993)
16. Lee, P.W.Y.: Generalized Li-Yau estimates and Huisken’s monotonicity formula. ESAIM Control Optim. Calc Var. 23(3), 827–850 (2017)
17. Li, J., Xu, X.: Differential Harnack inequalities on Riemannian manifolds I: linear heat equation. Adv. Math. 226(5), 4456–4491 (2011)
18. Li, P., Yau, S.T.: On the parabolic kernel of the Schrodinger operator. Acta Math. 156(3–4), 153–201 (1986)
19. Ni, L., Niu, Y.: Sharp differential estimates of Li-Yau-Hamilton type for positive (p,p)-forms on Kähler manifolds. Comm. Pure Appl. Math. 64(7), 920–974 (2011)
20. Perelman G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:0211159
21. Qian, B.: Remarks on differential Harnack inequalities. J. Math. Anal. Appl. 409(1), 556–566 (2014)
22. Ren, X.-A., Yao, S., Shen, L.-J., Zhang, G.-Y.: Constrained matrix Li-Yau-Hamilton estimates on Kähler manifolds. Math. Ann. 3–4, 927–941 (2015)
23. Yau, S.T.: On the Harnack inequalities of partial differential equations. Commun. Anal. Geom. 2(3), 431–450 (1994)
24. Yau, S.T.: Harnack inequality for non-self-adjoint evolution equations. Math. Res. Lett. 2(4), 387–399 (1995)
25. Yu, C., Zhao, F.: A note on Li-Yau type gradient estimate. Acta Math. Sci. 39(4), 1185–1194 (2019)
26. Yu, C., Zhao, F.: Heat kernel recurrence for space forms and applications. arXiv:1807.05708, submitted.
27. Yu, C., Zhao, F.: Sharp Li-Yau-type gradient estimates on hyperbolic spaces. J. Geom. Anal. 30(1), 54–68 (2020)
28. Zhang, Q.S., Zhu, M.: Li-Yau gradient bounds under nearly optimal curvature conditions. J. Funct. Anal. 275(2), 478–515 (2018)
29. Zhang, Q.S., Zhu, M.: Li-Yau gradient bound for collapsing manifolds under integral curvature condition. Proc. Amer. Math. Soc. 145(7), 3117–3126 (2017)
30. Zhang, H.-C., Zhu, X.-P.: Local Li-Yau’s estimates on RCD*(K,N) metric measure spaces. Calc. Var. Partial Differ. Equ. 55(4). Paper No. 93, 30 pp (2016)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.