The process $e^+e^- \to \eta(\eta')2\pi$ in the extended NJL model

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Process of electron-positron annihilation into $\eta(\eta')2\pi$ is described within the extended NJL model in the energy range up to about 2 GeV. Contributions of intermediate vector mesons $\rho(770)$ and $\rho(1450)$ are taken into account. Results for the $\eta2\pi$ channel are found to be in a reasonable agreement with experimental data. Predictions for production of $\eta'2\pi$ are given. The corresponding estimations for decays $\tau \to \eta(\eta')2\nu\tau$ are given in Appendix.

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I. INTRODUCTION

The description of hadronic interactions at low energies is problematic. Indeed, perturbative QCD hardly works at energies below 2 GeV. So in this domain various phenomenological models are used, most of them are based on the chiral symmetry of strong interactions. One of the most popular models of such a kind is the Nambu–Jona-Lasinio (NJL) one [1–8]. This model describes spectra of light mesons in the ground states and their interactions using a rather small number of parameters.

Recently in the framework of the extended NJL model, a series of processes of meson production in electron-positron annihilation was described [9, 10]. In the corresponding calculations, we took into account contributions of intermediate vector mesons $\rho(770)$, $\omega(782)$, $\phi(1020)$ and the radial excited states $\rho(1450)$, $\omega(1420)$. The radial excited states are treated with the help of the extended NJL model suggested in papers [14–18]. It was demonstrated that the extended NJL model provides a reasonably good description of a wide class of strong interaction processes at energies up to about 1.5 GeV.

In the present paper we finalize a series of studies by consideration of the reaction $e^+e^- \to \eta(\eta')2\pi$. Both $\rho(770)$ and $\rho(1450)$ intermediate states are taken into account. The applicability of our calculation is limited to the domain of the center-of-mass energies up to about 2 GeV.

The process $e^+e^- \to \eta2\pi$ has been studied experimentally at several facilities: DM1 [19], DM2 [20], ND [21, 22], CMD-2 [23], and BaBar [24]. From the theoretical point of view they were also discussed within several different phenomenological approaches [23, 22, 20]. Comparison of our results with the ones presented in those papers will be given in Conclusions. Below we present the corresponding description in the framework of the extended NJL model and give a comparison with experimental data.

II. THE EXTENDED NJL MODEL

The Lagrangian of quark-meson interactions in the extended NJL model was given in Refs. [12, 13, 15, 16, 18]. After bosonization and diagonalization of the free field Lagrangian, the relevant quark-meson interactions take the form

$$\Delta L^\text{int}_2 = \bar{q}(k')\left(L_I + L_\gamma + L_\pi + L_\eta\right)q(k),$$

$$L_I = i\dot{\hat{A}} - m,$$

$$L_\gamma = \frac{e}{2}\left(\tau_3 + \frac{1}{3}\right)\hat{A},$$

$$L_\pi = A_\rho\tau_3\dot{\rho}(p) - A_{\rho'}\tau_3\dot{\rho'}(p),$$

$$L_\eta = i\tau_5\bar{\eta}\eta,$$

where $\bar{q} = (\bar{u}, \bar{d})$ with $u$ and $d$ quark fields; $m = \text{diag}(m_u, m_d)$, $m_u = m_d = 280$ MeV are the constituent quark masses; $e$ is the electron charge; $\hat{A}$ is the photon field; $\rho$, $\omega$ ($\rho'$, $\omega'$), $\pi$ ($\pi'$), $\eta$, $\eta'$ ($\tilde{\eta}$, $\tilde{\eta}'$) are meson fields (hats over $\eta$ and $\eta'$ mean exited states); $\tau^\pm = (\tau_1 \mp i\tau_2)/\sqrt{2}$, $\tau_{1,2,3}$ are Pauli matrices; and $I$ is

1 Let note that for $\eta$ and $\eta'$ mesons we will use only part of Lagrangian contains interactions with $u$ and $d$ quarks, the full Lagrangian can be found in [18, 27].
the unit matrix. Quantities $A_i$ read

\[
A_\rho = g_{\rho_1} \frac{\sin(\beta + \beta_0)}{\sin(2\beta_0)} + g_{\rho_2} f(k^+^2) \frac{\sin(\beta - \beta_0)}{\sin(2\beta_0)},
\]
\[
A_{\rho'} = g_{\rho_1} \frac{\cos(\beta + \beta_0)}{\sin(2\beta_0)} + g_{\rho_2} f(k^+^2) \frac{\cos(\beta - \beta_0)}{\sin(2\beta_0)},
\]
\[
A_\pi = g_{\pi_1} \frac{\sin(\alpha + \alpha_0)}{\sin(2\alpha_0)} + g_{\pi_2} f(k^+^2) \frac{\sin(\alpha - \alpha_0)}{\sin(2\alpha_0)},
\]
\[
A_{\pi'} = g_{\pi_1} \frac{\cos(\alpha + \alpha_0)}{\sin(2\alpha_0)} + g_{\pi_2} f(k^+^2) \frac{\cos(\alpha - \alpha_0)}{\sin(2\alpha_0)},
\]
\[
A_\eta = g_{\pi_1} \varphi_1^\eta + g_{\pi_2} \varphi_2^\eta f(k^+^2).
\]

Radially-excited states are described in the extended NJL model using the following form factor in the quark-meson interaction:

\[
f(k^+^2) = (1 - |k^+|^2) \Theta(A_3^2 - |k^+|^2),
\]
\[
k^+ = k - \frac{(kp)\rho}{p^2}, \quad d = 1.788 \text{ GeV}^{-2},
\]

where $k$ and $p$ are the quark and meson momenta, respectively; $A_3 = 1.03$ GeV is the cut-off parameter. The coupling constants are defined in the extended NJL model by the integrals containing given form-factors

\[
g_{\pi_1} = \left( \frac{I_0^{(1)}(0)}{Z} \right)^{-1/2} = 3.01,
\]
\[
g_{\pi_2} = \left( \frac{I_2^{(1)}(2)}{Z} \right)^{-1/2} = 4.03,
\]
\[
g_{\rho_1} = \left( \frac{2}{3} \frac{I_0^{(0)}(0)}{Z} \right)^{-1/2} = 6.14,
\]
\[
g_{\rho_2} = \left( \frac{2}{3} \frac{I_2^{(0)}(2)}{Z} \right)^{-1/2} = 9.87,
\]

where $Z$ factor appeared after taking into account pseudoscalar - axial-vector transitions, $Z \approx 1.2$. Note that $g_{\pi_1} \approx m_\pi/F_\pi$, where $F_\pi \approx 93$ MeV is the pion decay constant. The quark loop integrals are defined as:

\[
I_m^{(n)} = -i N_c \int \frac{d^4k}{(2\pi)^4} \frac{(f(k^+^2))^n}{(m^2_a - k^2)^m} \Theta(A_3^2 - k^2),
\]

where $N_c = 3$ is the number of colors. The mixing angles for pseudoscalar and vector mesons are: $\alpha_0 = 58.39^\circ$, $\alpha = 58.70^\circ$, $\beta_0 = 61.44^\circ$, $\beta = 79.85^\circ$. One can find the definition of mixing angles for $\pi$ and $\rho$ mesons in [15][16]. The mixing coefficients for the isoscalar pseudoscalar meson states given in Table 1 were derived in [17][18][27].

\[
\begin{array}{cccccc}
\varphi_1^\eta & \varphi_2^\eta & \varphi_1^{\eta'} & \varphi_2^{\eta'} \\
0.71 & 0.11 & 0.62 & -0.87 & -0.32 & 0.56 \\
0.11 & 0.11 & -0.87 & -0.48 & -0.32 & 0.56 \\
\end{array}
\]

Table I: Mixing coefficients for isoscalar pseudoscalar meson states ($\eta = \eta, \eta', \tilde{\eta}, \tilde{\eta}'$).

For the simplification of the presentation we define:

\[
\varphi_\eta = \frac{1}{\sin(2\alpha_0)} \begin{pmatrix} \sin(\alpha + \alpha_0) \\ \sin(\alpha - \alpha_0) \end{pmatrix},
\]
\[
\varphi_\rho = \frac{1}{\sin(2\beta_0)} \begin{pmatrix} \sin(\beta + \beta_0) \\ \sin(\beta - \beta_0) \end{pmatrix},
\]
\[
\varphi_\rho' = -\frac{1}{\sin(2\beta_0)} \begin{pmatrix} \cos(\beta + \beta_0) \\ \cos(\beta - \beta_0) \end{pmatrix}.
\]

III. PROCESS AMPLITUDES AND CROSS SECTION

The total amplitude of the given process has the form

\[
T = -\frac{4\pi\alpha}{q^2} e^\gamma \mu e H_\mu,
\]

where $q = p_{\pi^+} + p_{\pi^-}$ in the center-of-mass system. The hadronic part of the amplitude takes the form

\[
H_\mu = V_\mu \left( T_\gamma(q^2, s) + \sum_{V=\rho,\rho'} T_V(q^2, s) \right),
\]

where $V_\mu = p_\pi^+ \rho, p_\pi^- \epsilon_{\mu\alpha\beta\gamma}$. Electron-positron annihilation with $\eta\pi\pi$ production is described by Feynman diagrams with virtual photons, Fig. 1 and with intermediate vector $\rho(770)$ and $\rho(1450)$ mesons, Fig. 2. In the following calculations we took into account the ground state $\rho(770)$ and the first radially-excited state $\rho(1450)$

\[
T_\gamma(q^2, s) = \sum_{i=1}^{2} g_{\pi_i} \varphi_\eta^i \left( T_\gamma^{(i-1)}(s) + T_\gamma^{(i-1)}(s) \right),
\]

\[
T_V(q^2, s) = \frac{(C_V/g_V)q^2}{m_V^2 - q^2 - i\sqrt{q^2}|V(q^2)|} \times \sum_{i=1}^{2} \sum_{j=1}^{2} g_{\pi_i} \varphi_\eta^i g_{\pi_j} \varphi_\eta^j \left( T_V^{(i+j-2)}(s) + T_V^{(i+j-2)}(s) \right).
\]

As was shown in previous calculations (see for example [12][13]), vector meson dominance could be directly

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1. Hereafter $\eta = \eta'$.
The vertices \( \gamma \eta \pi \pi \) and \( V \eta \pi \pi \) contain the sum of two terms

\[
T_\Delta^{(n)}(s) = -24 F_\pi g_\pi^3 f_4^{(n)},
\]

\[
T_\Delta^{(n)}(s) = 16 F_\pi g_\pi \sum_{V=\rho,\rho'} \frac{g_{V\rightarrow \pi \pi}}{m_V^2 - s - i\sqrt{2} \Gamma_V(s)} \sum_{i=1}^2 g_{\rho_i}^{V} f_3^{(n+i-1)} \approx 16 F_\pi g_\pi \frac{g_{\rho\rightarrow \pi \pi}}{m_\rho^2 - s - i\sqrt{2} \Gamma_\rho(s)} \sum_{i=1}^2 g_{\rho_i}^{\rho} f_3^{(n+i-1)}.
\]

The term \( T_\Delta^{(n)}(s) \) corresponds to the contribution of quark box-diagram, see Fig. 3. The term \( T_\Delta^{(n)}(s) \) comes from two triangle quark loops connected by a virtual vector meson, see Fig. 4. We neglected the contribution of the \( (1450) \) to the \( T_\Delta^{(n)}(s) \) term, since it is very much suppressed with respect to the \( (770) \) contribution by kinematics and also due to the small partial decay width of \( (1450) \rightarrow 2\pi \) see [16].

Figure 1: The Feynman diagram with an intermediate photon. The dashed circle represents the sum of two sub-diagrams given in Figs. 3 and 4.

Figure 2: The Feynman diagram with intermediate \( 770 \) and \( 1450 \) vector mesons. The dashed circle represents the sum of two sub-diagrams given in Figs. 3 and 4.

Since \( g_{\pi_1}^{V_1} \gg g_{\pi_2}^{V_2} \approx 0 \), we replaced all terms contain vertices with \( \pi \) mesons (see discussion in Refs. [12]).

The second triangle diagram describing decay \( V \rightarrow \pi \pi \) was computed in the framework of the extended NJL model in Refs. [12, 16], it gives:

\[
g_{V\rightarrow \pi \pi} \approx g_{\rho_1}^{V_1} f_3^{(1)} + g_{\rho_2}^{V_2} f_3^{(1)} \frac{I_2^{(1)}}{I_2^{(0)} I_2^{(2)}}.
\]

The transitions of a photon into the vector mesons \( (\rho,\rho') \) denoted by the terms

\[
C_{\gamma V} = \varphi_1^{V_1} + \varphi_2^{V_2} \frac{I_2^{(1)}}{\sqrt{I_2^{(0)} I_2^{(2)}}}.
\]
We chose the fixed width for $\rho(770)$ and the running one $^{10,12}$ for $\rho(1450)$:

\[
\begin{align*}
\Gamma_\rho(s) &= \Gamma_\rho, \\
\Gamma_{\rho'}(s) &= \Theta(2m_\pi - \sqrt{s})\Gamma_{\rho' \to 2\pi} \\
&+ \Theta(\sqrt{s} - 2m_\pi)\left(\Gamma_{\rho' \to 2\pi} + \Gamma_{\rho' \to \omega\pi} \frac{\sqrt{s} - 2m_\pi}{m_\omega - m_\pi}\right)\Theta(m_\omega + m_\pi - \sqrt{s}) \\
&+ \Theta(m_{\rho'} - \sqrt{s})\Theta(\sqrt{s} - m_\omega - m_\pi)\left(\Gamma_{\rho' \to 2\pi} + (\Gamma_{\rho'} - \Gamma_{\rho' \to 2\pi} - \Gamma_{\rho' \to \omega\pi}) \frac{\sqrt{s} - m_\omega - m_\pi}{m_{\rho'} - m_\omega - m_\pi}\right) \\
&+ \Theta(\sqrt{s} - m_{\rho'})\Gamma_{\rho'}(m_{\rho'}^2),
\end{align*}
\]

where $\Gamma_\rho = 147.8$ MeV and $\Gamma_{\rho'}(m_{\rho'}^2) = 400$ MeV are taken from PDG $^{28}$. The values $\Gamma(\rho' \to 2\pi) = 22$ MeV and $\Gamma(\rho' \to \omega\pi^0) = 75$ MeV were calculated in $^{16}$.

The total cross-section takes the form

\[
\sigma(q^2) = \frac{\alpha^2}{192\pi q^4} \int_{s_-}^{s_+} ds \int_{t_-}^{t_+} dt |T(q, s, t)|^2,
\]

where variables are defined as $s = (p_\eta + p_{\pi^+})^2$, $t = (p_\eta + p_{\pi^-})^2$, and the limits are

\[
\begin{align*}
t_\pm &= \frac{1}{4s} \left[ q^2 + m_\eta^2 - 2m_\pi^2 \right] \left[ q^2 + m_\eta^2 - 2m_\pi^2 \right] - \\
&- \left[ \lambda^{1/2}(q^2, s, m_\pi^2) \pm \lambda^{1/2}(m_\eta^2, m_\pi^2, s) \right]^2, \\
s_- &= (m_\eta + m_\pi)^2, \\
s_+ &= (\sqrt{q^2} - m_\pi)^2, \\
\lambda(a, b, c) &= (a - b - c)^2 - 4bc.
\end{align*}
\]

The masses of all particles were taken from PDG $^{28}$: $m_{\pi^\pm} = 139.57$ MeV, $m_\eta = 547.86$ MeV, $m_{\rho'} = 957.78$ MeV, $m_\rho = 775.49$ MeV, $m_{\rho'} = 1465$ MeV. One can see the final results in Figs. 4, 6.

\section*{IV. DISCUSSION AND CONCLUSIONS}

The presented calculation shows that the extended NJL model allows to describe the energy dependence of the total cross section of $e^+e^- \to \eta 2\pi$ in a satisfactory agreement with experimental data in the energy region up to 2 GeV. This allows to expect that our predictions for the channel $e^+e^- \to \eta 2\pi$ are also reasonable.

One of the first attempts to provide a theoretical interpretation of the experimental data for $e^+e^- \to \eta 2\pi$ was presented in paper $^{23}$. The vector meson dominance model was used taking into account intermediate vector mesons $\rho(770)$, $\rho(1450)$, and $\rho(1700)$. A number of free parameters was fitted from the experimental data on the same process. Only one structure which corresponds in our case to the $T_{\Delta}(s)$ in the amplitude of the process was considered.

In paper $^{23}$ a resonance chiral theory was used. This model contains a very large number of free parameters.

However, the contributions of intermediate radial-excited $\rho$ mesons were not taken into account there, while the one due to $\rho(1450)$ meson is certainly important in the description of this process in the energy region under consideration. An advanced application of the same model was presented in Ref. $^{26}$, where intermediate vector mesons $\rho(770)$, $\rho(1450)$, and $\rho(1700)$ were included by means of introduction of additional free parameters. Note that the contribution of $\rho(1700)$ was found to be not very important numerically.

The main difference of our results from the previous ones is that we work in the extended NJL model where all the parameters have been fixed from the beginning. Moreover, for the width of $\rho(1450)$ we take the PDG value 400 MeV, while in the alternative approaches considerably lower values were used, namely: 211 MeV in $^{22}$ and 238 MeV in $^{20}$. As the result our model has a certain predictive power, contrary to the alternative approaches. We would like to underline that the extended NJL model was extensively tested by description of a large series of different processes with strong, weak, and electromagnetic interactions of mesons $^{9,13,17,20,32}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Comparison of the extended NJL model predictions with the BaBar experiment data $^{24}$ for $e^+e^- \to \eta 2\pi$ process.}
\end{figure}
Figure 6: Prediction of the extended NJL model for $e^+e^- \rightarrow \eta' 2\pi$ process.

APPENDIX. THE DECAYS $\tau \rightarrow \eta(\eta') 2\pi\nu$

To proceed the calculation of decay $\tau \rightarrow \eta(\eta') 2\pi\nu$ we complement the Lagrangian with terms contain $W^-$ boson and charged $\rho$ and $\rho'$ mesons

$$\Delta L_2 = \frac{g_{EW}}{\sqrt{2}} \bar{W} + A_\rho \tau_\pm \rho(p) - A_{\rho'} \tau_\pm \rho'(p), \quad (18)$$

where $\tau_\pm = (\tau_1 \mp i\tau_2)/\sqrt{2}$, and $g_{EW}$ is electroweak coupling constant.

Table II: Branching ratios for the processes $\tau \rightarrow \eta(\eta') 2\pi\nu$

| Process | Full amplitude | Only $\rho(770)$ | PDG [28] |
|---------|----------------|--------------------|-----------|
| $B(\tau \rightarrow \eta 2\pi\nu) \cdot 10^3$ | 1.46 | 1.01 | $1.39 \pm 0.10$ |
| $B(\tau \rightarrow \eta' 2\pi\nu) \cdot 10^3$ | 0.09 | 0.12 | < 1.2 |

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