Dyck Paths Categories And Its Relationships With Cluster Algebras

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Abstract

Dyck paths categories are introduced as a combinatorial model of the category of representations of quivers of Dynkin type $A_n$. In particular, it is proved that there is a bijection between some Dyck paths and perfect matchings of some snake graphs. The approach allows us to give formulas for cluster variables in cluster algebras of Dynkin type $A_n$ in terms of Dyck paths.

Keywords and phrases: Auslander-Reiten quiver, cluster algebras, Dyck paths, perfect matchings, quiver representation, snake graph.

Mathematics Subject Classification 2010: 16G20; 16G30; 16G60.

1 Introduction

In the last few years, researches regarding connections between cluster algebras and different fields of mathematics have been growing. For instance, relationships between cluster algebras, quiver representations, combinatorics and number theory have been reported by Fomin et al., Shiffler et al., K. Baur et al., Assem et al. amongst a great number of mathematicians [1, 3, 6–8, 10–12].

Perhaps the Catalan combinatorics (which consists of all the enumeration problems whose solutions are Catalan numbers) is the most appropriated environment for the investigation of cluster algebras of Dynkin type $A_n$. Among all these kinds of problems, it is possible to prove (for example) that the Catalan numbers count [10]:

1. The number of plane binary trees with $n + 1$ endpoints (or $2n + 1$) vertices,
2. The number of ways to parenthesize a string of length $n + 1$ subject to a non associative binary operation,
3. The number of paths $P$ in the $(x, y)$-plane from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that never pass below the $x$-axis. Such paths are called Dyck paths,
4. The number of triangulations of an $(n + 3)$ polygon,
5. The number of clusters of a cluster algebra of Dynkin type $A_n$.
Regarding integer friezes, we point out that Propp in [10] reminds that Conway and Coxeter completely classified the frieze patterns whose entries are positive integers, and showed that these frieze patterns constitute a manifestation of the Catalan numbers. Specifically, that there is a natural association between positive integer frieze patterns and triangulations of regular polygons with labelled vertices. According to Baur and Marsh [3], a connection between cluster algebras and frieze patterns was established by Caldero and Chapoton [5], which showed that frieze patterns can be obtained from cluster algebras of Dynkin type $A_n$.

Another example of the use of the Catalan combinatorics as a tool to describe the structure of cluster algebras, was given by Schiffler et al. [6, 7, 14], who found out formulas for cluster variables based on its relations with some triangulated surfaces and perfect matchings of snake graphs. They also proved that there is a way of obtaining the number of perfect matchings of a given snake graph by associating a suitable continued fraction defined by the sign function of the graph.

Given a non-negative integer $n$ and a triangulation $T$ of a regular polygon with $(n+3)$ vertices. Caldero, Chapoton and Schiffler [4] gave a realization of the category $C_T$ of representations of a quiver $Q_C$ associated to a cluster $C$ of a cluster algebra in terms of the diagonals of the $(n+3)$ polygon. They proved that there is a categorical equivalence between the categories $C_T$ and $\text{Mod} Q_T$, where $C_T$ is the category whose objects are positive integral linear combinations of positive roots (i.e., diagonals that do not belong to the triangulation $T$), whereas $\text{Mod} Q_T$ denotes the category of modules over the quiver $Q_T$ with triangular relations induced by the triangulation $T$.

Follow the ideas of Caldero, Chapoton and Schiffler, in this paper, a combinatorial model of the category of representations of Dynkin quivers with relations is developed by using Dyck paths. To do that, Dyck paths categories are introduced and it is proved that these categories are equivalent to categories of representations of Dynkin quivers of type $A_n$. This approach allows us to realize perfect matchings of snake graphs as objects of suitable Dyck paths categories, and with this machinery a formula for cluster variables based on Dyck paths is obtained.

This paper is distributed as follows; In Section 2, notation and basic definitions to be used throughout the paper are introduced. In Section 3, we define Dyck paths categories and some of its main categorical properties are given in Section 4. In section 5, relationships between objects of the categories of Dyck paths, perfect matchings and cluster algebras are given.

2 Preliminaries

2.1 Cluster Algebras

Fomin and Zelevinsky introduced the term cluster algebra in [10], as a subalgebra of a field of rational functions generated by a set of $n$ cluster variables [8, 11, 12]. The cluster algebras are in connection with different topics in mathematics, as algebraic combinatorics, Lie theory, discrete dynamical systems, tropical geometry, and others. Afterwards, Fomin, Schiffler et al introduced cluster algebras associated to surfaces [7, 3, 14].
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For the sake of clarity, we remind here the definition given by Fomin et al. of a cluster algebra.

Let $T_n$ the $n$-regular tree whose edges are labeled by the numbers $1, \ldots, n$, so that the $n$-edges incident to each vertex receive different labels. The symbol $t^k \rightarrow t'$ is used to denote that vertices $t, t' \in T_n$ are joined by an edge labeled by $k$.

If $\mathcal{F}$ is a field isomorphic to the field of rational functions over $\mathbb{C}$ (alternatively over $\mathbb{Q}$) in $m$ independent variables, then a labeled seed of geometric type over $\mathcal{F}$ is a pair $(\tilde{x}, \tilde{B})$ where:

1. $\tilde{x} = (x_1, x_2, \ldots, x_m)$ is an $m$-tuple of elements of $\mathcal{F}$ forming a free generating set; that is; $x_1, x_2, \ldots, x_m$ are algebraically independent, and $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$,
2. $\tilde{B} = (b_{ij})$ is an $m \times n$ extended skew-symmetrizable integer matrix. $\tilde{B}$ is said to be the extended exchange matrix of the seed. Its top $n \times n$ submatrix $B$ is the exchange matrix.

Let $(\tilde{X}, \tilde{B})$ be a labeled seed as above. Take an index $k \in \{1, 2, \ldots, n\}$. The seed mutation in direction $k$ transforms $(\tilde{x}, \tilde{B})$ into the new labeled seed $\mu_k(\tilde{x}, \tilde{B}) = \tilde{x}', \tilde{B}'$ defined as follows:

$$
\tilde{B}' = \mu_k(\tilde{B}) = (b'_{ij}),
$$

where

$$
b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } i = k \text{ or } j = k, \\
b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0, \\
b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0, \\
b_{ij}, & \text{otherwise.}
\end{cases}
$$

The extended cluster $\tilde{x}' = (x'_1, \ldots, x'_m)$ is given by the identifications $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange rule.

$$
x_kx'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.
$$

A seed pattern is defined by assigning a labeled seed $(\tilde{x}(t), \tilde{B}(t))$ to every vertex, $t \in T_n$, so that the seeds assigned to the end points of any edge $t^k \rightarrow t'$ are obtained from each other by the seed mutation in direction $k$. A seed pattern is uniquely determined by one of its seeds.

Let $(\tilde{x}(t), \tilde{B}(t))_{t \in T_n}$ be a seed pattern as above, and let $X = \bigcup_{t \in T_n} x(t)$ be the set of all cluster variables appearing in its seeds. We let the ground ring be $R = \mathbb{C}[x_{n+1}, \ldots, x_m]$ the polynomial ring generated by the frozen variables.

The cluster algebra $A$ (of geometric type over $R$) associated with the given seed pattern is the $R$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables $A = R[X]$. 
2.1.1 Cluster Algebras From Quivers

For quivers, cluster algebras are defined as follows:

Fix an integer \( n \geq 1 \). In this case, a seed \((Q,u)\) consists of a finite quiver \( Q \) without loops or 2-cycles with vertex set \( \{1, \ldots, n\} \), whereas \( u \) is a free-generating set \( \{u_1, \ldots, u_n\} \) of the field \( \mathbb{Q}(x_1, \ldots, x_n) \).

Let \((Q,u)\) be a seed and \( k \) a vertex of \( Q \). The mutation \( \mu_k(Q,u) \) of \((Q,u)\) at \( k \) is the seed \((Q',u')\), where:

(a) \( Q' \) is obtained from \( Q \) as follows:

1. reverse all arrows incident with \( k \),
2. for all vertices \( i \neq j \) distinct from \( k \), modify the number of arrows between \( i \) and \( j \), in such a way that a system of arrows of the form \( (i \rightarrow j, k \rightarrow i, j \rightarrow k) \) is transformed into the system \( (i \rightarrow j, k \rightarrow i, j \rightarrow k) \).
3. If there are no arrows from \( i \) with target \( k \), the product is taken over the empty set and equals 1.

(b) \( u' \) is obtained from \( u \) by replacing the element \( u_k \) with

\[
u_k = \frac{1}{u_k} \prod_{\text{arrows } i \rightarrow k} u_i + \prod_{\text{arrows } k \rightarrow j} u_j.
\]

If there are no arrows from \( i \) with target \( k \), the product is taken over the empty set and equals 1. It is not hard to see that \( \mu_k(\mu_k(Q,u)) = (Q,u) \). In this case the matrix mutation \( B' \) has the form

\[
b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{else}, \end{cases}
\]

where \([x]_+ = \max(x,0)\). Thus, if \( Q \) is a finite quiver without loops or 2-cycles with vertex set \( \{1, \ldots, n\} \), the following interpretations have place:

1. the clusters with respect to \( Q \) are the sets \( u \) appearing in seeds, \((Q,u)\) obtained from a initial seed \((Q,x)\) by iterated mutation,
2. the cluster variables for \( Q \) are the elements of all clusters,
3. the cluster algebra \( A(Q) \) is the \( \mathbb{Q} \)-subalgebra of the field \( \mathbb{Q}(x_1, \ldots, x_n) \) generated by all the cluster variables.

As example, the cluster variables associated to the quiver \( Q = 1 \rightarrow 2 \) are:

\[\{x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1}{x_1x_2}, \frac{1+x_1}{x_2}\}.$$
2.1.2 Cluster Algebras From Surfaces

Let $S$ be a connected oriented 2−dimensional Riemann surface with nonempty boundary, and let $M$ be a nonempty finite subset of the boundary of $S$, such that each boundary component of $S$ contains at least one point of $M$. The elements of $M$ are called marked points. The pair $(S,M)$ is called a bordered surface with marked points. Marked points in the interior of $S$ are called punctures (For technical reasons, we require that $(S,M)$ is not a disk with 1, 2 or 3 marked points) [7].

An arc $\gamma$ in $(S,M)$ is a curve in $S$, considered up to isotopy, such that:

(i) the endpoints of $\gamma$ are in $M$,
(ii) $\gamma$ does not cross itself, except that its endpoints, may coincide,
(iii) except for the endpoints, $\gamma$ is disjoint from the boundary of $S$,
(iv) $\gamma$ does not cut out a monogon or a bigon.

Curves that connect two marked points and lie entirely on the boundary of $S$ without passing through a third marked point are boundary segments. Note that boundary segments are not arcs. For any two arcs $\gamma, \gamma'$ in $S$, let $e(\gamma, \gamma')$ be the minimal number of crossings of arcs $\alpha$ and $\alpha'$, where $\alpha$ and $\alpha'$ range over all arcs isotopic to $\gamma$ and $\gamma'$, respectively. We say that arcs $\gamma$ and $\gamma'$ are compatible if $e(\gamma, \gamma') = 0$.

An ideal triangulation is a maximal collection of pairwise compatible arcs (together with all boundary segments). Triangulations are connected to each other by sequences of flips. Each flip replaces a single arc $\gamma$ in a triangulation $T$ by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in $T$, forms a new triangulation.

According to Schiffler and Canakci [7], Fomin, Shapiro and Thurston [10] associated a cluster algebra $A(S,M)$ to any bordered surface with marked points $(S,M)$, and the cluster variables of $A(S,M)$ are in bijection with the (tagged) arcs of $(S,M)$.

The following theorem regarding relationships between cluster algebras and surface triangulations was obtained Fomin, Shapiro, and Thurston [8, 9].

Theorem 1. [11] Fix a bordered surface $(S,M)$ and let $A$ be the cluster algebra associated to the signed adjacency matrix of a tagged triangulation. Then the (unlabeled) seed $\Sigma_T$ of $A$ are in bijection with tagged triangulations $T$ of $(S,M)$, and the cluster variables are in bijection with the tagged arcs of $(S,M)$ (so we can denote each by $x_\gamma$, where $\gamma$ is a tagged arc). Moreover, each seed in $A$ is uniquely determined by its cluster. Furthermore, if a tagged triangulation $T'$ is obtained from another tagged triangulation $T$ by flipping a tagged arc $\gamma \in T$ and obtaining $\gamma'$, then $\Sigma_{T'}$ is obtained from $\Sigma_T$ by the seed mutation replacing $x_\gamma$ by $x_{\gamma'}$.

2.2 Snake Graphs and Cluster Variables

In this section, we recall the definition of a snake graph, the number of perfect matchings associated to these graphs, and the way that these concepts can be used to find out a formula for the cluster variables of a cluster algebra associated to a surface [6, 7, 14].

A tile $G$ is a square of fixed side-length in the plane whose sides are parallel or orthogonal to the fixed basis.
We consider a tile $G$ as a graph with four vertices and four edges in the obvious way.

A snake graph $G$ is a connected graph consisting of a finite sequence of tiles $G_1, \ldots, G_d$ with $d \geq 1$, such that for each $i = 1, \ldots, d - 1$:

(i) $G_i$ and $G_{i+1}$ share exactly one edge $e_i$ and this edge is either the north edge of $G_i$ and the south edge of $G_{i+1}$ or the east edge of $G_i$ and the west edge of $G_{i+1}$.

(ii) $G_i$ and $G_j$ have on edge in common whenever $|i - j| \geq 2$.

(iii) $G_i$ and $G_j$ are disjoint whenever $|i - j| \geq 3$.

For notation, $G[i, i+t] = (G_i, \ldots, G_{i+t})$ is the subgraph of $G = (G_1, \ldots, G_n)$, the $d - 1$ edges $e_1, \ldots, e_{d-1}$ which are contained in two tiles are called interior edges of $G$ and the other edges are called boundary edges. A perfect matching $P$ of a graph $G$ is a subset of the set of edges of $G$ such that each vertex of $G$ is incident to exactly one edge in $P$. We let Match($G$) denote the set of all perfect matchings of the graph $G$.

Let $T$ be a triangulation of a surface $(S, M)$ and let $\gamma$ be an arc in $(S, M)$ which is not in $T$. Choose an orientation on $\gamma$, let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. Denote by $s = \tau_0, p_1, p_2, \ldots, p_{d+1} = t$ the ordered points of intersection of $\gamma$ and $T$. For $j = 1, 2, \ldots, d$, let $\tau_j$ be the arc of $T$ containing $p_j$, and let $\Delta_j$ be the two triangles in $T$ on either side of $\tau_j$. Then, for $j = 1, \ldots, d - 1$, arcs $\tau_j$ and $\tau_{j+1}$ form two sides of the triangle $\Delta_j$ in $T$ and we define $e_j$ to be the third arc in this triangle.

Let $G_j$ be the quadrilateral in $T$ that contains $\tau_j$ as a diagonal (a tile) whose edges are arcs in $T$, thus, they are labeled edges. Define a sign function $f$ of the edges $e_1, \ldots, e_d$ by

$$f(e_j) = \begin{cases} +1, & \text{if } e_j \text{ lies on the right of } \gamma \text{ when passing through } \Delta_j, \\ -1, & \text{otherwise.} \end{cases} \tag{4}$$

The labeled snake graph $G_\gamma = (G_1, \ldots, G_d)$ with tiles $G_i$ and sign function $f$ is called the snake graph associated to the arc $\gamma$. Each edge $e$ of $G_\gamma$ is labeled by an arc $\tau(e)$ of the triangulation $T$. Such an arc defines the weight $x(e)$ of the edge $e$ as the cluster variable associated to the arc $\tau(e)$. Thus $x(e) = x_{\tau(e)}$.

In [14] Musiker, Schiffler, and Williams showed a combinatorial formula for cluster variables of a cluster algebra of surface type $A(S, M)$ with principal coefficients $\Sigma_T = (x_T, y_T, B_T)$. In such a case, if $\gamma$ is a generalized arc in a triangulation $T$ which has no self-folded triangles, and $G_\gamma$ is its snake graph. Then the corresponding cluster variable $x_\gamma$ is given by the identity

$$x_\gamma = \frac{1}{\text{cross}(\gamma, T)} \sum_{P \in \text{Match}(G_\gamma)} x(P). \tag{5}$$
where the sum runs over all perfect matchings of \( G \), the summand \( x(P) = \prod_{e \in P} x(e) \) is the weight of the perfect matching \( P \), and \( \text{cross}(T, \gamma) = \prod_{d=1}^{d} x_{\tau_{ij}} \) is the product of all initial cluster variables whose arcs cross \( \gamma \).

A relationship between cluster variables and continued fractions is described by Schiffler and Canakci in [7], who claimed that, the numerator of a continued fraction is equal to the number of perfect matchings of the corresponding abstract snake graph, and that it can therefore be interpreted as the number of terms in the numerator of the Laurent expansion of an associated cluster variable. Thus, the Laurent polynomials of the cluster variable can be recovered from the continued fraction.

### 2.3 Category of Diagonals

In 2006 [4], Caldero, Chapoton, and Schiffler introduced the category of diagonals of a polygon with \( n + 3 \) sides associated to a triangulation \( T \); in this case, the diagonals are called roots which can be classified as negative or positive, negative roots are those roots belonging to the triangulation \( T \).

The combinatorial \( \mathbb{C} \)-linear additive category \( C_T \) is described as follows. The objects are positive integral linear combinations of positive roots, and the space of morphisms from a positive root \( \alpha \) to a positive root \( \alpha' \) is a quotient of the vector space over \( \mathbb{C} \) spanned by pivoting paths from \( \alpha \) to \( \alpha' \). The subspace which defines the quotient is spanned by the so-called mesh relations. For any couple \( \alpha, \alpha' \) of positive roots such that \( \alpha \) is related to \( \alpha' \) by two consecutive pivoting elementary moves with distinct pivots, the mesh relations are given by the identity \( P_{v_1'}P_{v_2'} = P_{v_1'}P_{v_2} \), where \( v_1, v_2 \) (resp. \( v_1', v_2' \)) are the vertices of \( \alpha \) (resp. \( \alpha' \)) such that \( P_{v_1'}P_{v_2} = \alpha' \).

Let \( T \) be a triangulation, then one can define a planar tree \( t_T \) as follows. Its vertices are the triangles of \( T \) and the edges connect adjacent triangles. In the same way, we can define a graph \( Q_T \) whose vertices are the inner edges of \( T \) and are related to each other by an edge, if they bound the same triangle. An orientation can be defined by using graph \( Q_T \), in such a way that a vertex \( i \) connects a vertex \( j \) (denoted \( i \to j \)), if \( -\alpha_j \) can be obtained from the diagonal \( -\alpha_i \) by rotating anticlockwise about their common vertex.

The triangulation \( T \) defines a \( \mathbb{C} \)-linear abelian category \( \text{Mod} \ Q_T \), that is, the category of modules over the quiver \( Q_T \), such that in any triangle, the composition of two successive maps is zero. These relations are named triangle relations.

The following result regarding the category of diagonals was given by Caldero, Chapoton, and Schiffler in [4].

**Theorem 2.** If \( T \) is a triangulation of a polygon with \( n + 3 \) sides then there is a categorical equivalence between the category of diagonals \( C_T \) and the category of modules over the quiver \( Q_T \).

### 3 Dyck Paths Category

In this section, we introduce the category of Dyck paths of length \( 2n \).
3.1 Elementary Shifts

A Dyck path is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(n,n)$ with steps $(1,0)$ and $(0,1)$ such that the path never passes below the line $y = x$. The number of Dyck paths of length $2n$ is equal to the $n$th Catalan number [16]. Henceforth, Dyck words as defined in the following Remark 3 are used to denote Dyck paths.

**Remark 3.** The set of Dyck words is the set of words $w \in X^* = \{U, D\}^*$ characterized by the following two conditions [2]:

- for any left factor $u$ of $w$, $|u|_U \geq |u|_D$,
- $|w|_U = |w|_D$.

where $|w|_a$ is the number of occurrences of the letter $a \in X$ in the word $w$ and the word $u$ is a left factor of the word $w = uv$.

Let $\mathcal{D}_{2n}$ be the set of all Dyck paths of length $2n$, let $UWD = Uw_1 \ldots w_{n-1}D$ be a Dyck path in $\mathcal{D}_{2n}$ with $A = \{UD, DU, UU, DD\}$ the set of choices in $W$.

The support of $UWD$ (denoted by $\text{Supp} UWD \subseteq \{1, 2, \ldots, n-1\} = n-1$) is a set of indices such that

$$\text{Supp} UWD = \{q \in n-1 \mid w_q = UD \text{ or } w_q = UU, 1 \leq q \leq n-1\}.$$ 

A map $f : A \rightarrow A$ such that for any $w \in A$, it holds that $f(w) = f(ab) = w^{-1} = ba$, $a, b \in \{U, D\}$ is said to be a shift. An unitary shift is a map $f_1 : \mathcal{D}_{2n} \rightarrow \mathcal{D}_{2n}$ such that

$$f_1(Uw_1 \ldots w_{i-1}w_iw_{i+1} \ldots w_{n-1}D) = Uw_1 \ldots w_{i-1}f(w_i)w_{i+1} \ldots w_{n-1}D.$$ 

We will denote a unitary shift by a vector of maps from $\mathcal{D}_{2n}$ to itself of the form $(1_1, \ldots, 1_{i-1}, 1_i, 1_{i+1}, \ldots, 1_{n-1})$, where $1_k$ is the identity map associated to the $i$-th coordinate.

An elementary shift is a composition of unitary shifts. A shift path of length $m$ $UWD \rightarrow UW_1D \rightarrow \cdots \rightarrow UW_mD \rightarrow UVD$ from $UWD$ to $UVD$ is a composition of elementary shifts. The set of all Dyck paths in a shift path between $UWD$ and $UVD$ will be denoted by $J$. For notation, we introduce the identity shift as the elementary shift $(1_1, \ldots, 1_{n-1})$.

**Irreversibility condition.** Suppose that a map $R : \mathcal{D}_{2n} \rightarrow \mathcal{D}_{2n}$ is defined by the application of successive elementary shifts to a given Dyck path. Then $R$ is said to be an irreversible relation over $\mathcal{D}_{2n}$ if and only if elementary shifts transforming Dyck paths (from one to the other) are not reversible. In other words, if an elementary shift $F = f_{p_1} \circ \cdots \circ f_{p_q}$ transforms a Dyck path $UWD$ into a Dyck path $UVD$ then there is not an elementary shift $F = f_{q_1} \circ \cdots \circ f_{q_p}$ transforming $UVD$ into $UWD$, for some $p, q \in \mathbb{Z}^+$. 

**Shift Relation.** If there exist two paths $G \circ F$ and $G' \circ F'$ of irreversible relations (of length 2) transforming a Dyck path $UWD$ into the Dyck path $UVD$ over $R$ in the following form:
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Category of Dyck paths of length $2n$. As for the case of diagonals $\mathcal{D}_n$, we can also define a $k$-linear additive category $(\mathcal{D}_{2n}, R)$ based on Dyck paths, in this case, objects are $k$-linear combinations of Dyck paths in $\mathcal{D}_{2n}$ with space of morphisms from a Dyck path $UWD$ to a Dyck path $UVD$ over $R$ being the set

$$\text{Hom}(\mathcal{D}_{2n}, R)(UWD, UVD) = \langle \{g \mid g \text{ is a shift path over } R \} \rangle / \langle \sim_R \rangle.$$ 

The vector space $\text{Hom}(\mathcal{D}_{2n}, R)(UWD, UVD)$ is non-zero if and only if there are shift paths from $UWD$ to $UVD$ and

$$\bigcap_{i \in J} \text{Supp} \ UW'^i D \neq \emptyset,$$

for each shift path, with $UWD$ and $UV D$ in $\mathcal{D}_{2n}$.

Figure 1 shows the elementary shifts over $(\mathcal{D}_6, R)$ associated to an irreversible relation $R$ defined over the set of all Dyck paths of length 6. And such that,

$$R(UWD) = \begin{cases} f_1(UWD), & \text{if } w_1 = UD, \\ f_2(UWD), & \text{if } w_2 = UD. \end{cases}$$

Figure 1: Elementary shifts in $(\mathcal{D}_6, R)$.

3.2 Relations of Type $R_{j_1, \ldots, j_m}^{i_1, \ldots, i_k}$

If $n = \{1, 2, \ldots, n\}$ is an $n$-point chain then $\mathcal{C}_{\{1, n\}}$ stands for all admissible subchains $\mathcal{C}$ of $n$ with $\min \mathcal{C} = 1$ and $\max \mathcal{C} = n$. For instance, $\{1, 6, 8\}$ and $\{1, 3, 8\}$ are three-point subchains contained in $\mathcal{C}_{\{1, 8\}}$. The admissible subchain $\mathcal{C} = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq n$ must satisfy the following constraints for $1 \leq r \leq k$ and $1 \leq s \leq m$:
• If $i_1 = 1$ and $k = m$ then $i_1 < j_1 < \cdots < i_k < j_m = n$.

• If $i_1 = 1$ and $k = m + 1$ then $i_1 < j_1 < \cdots < i_k < j_m < i_k = n$.

• If $j_1 = 1$ and $k = m$ then $j_1 < i_1 < \cdots < j_m < i_k = n$.

• If $j_1 = 1$ and $m = k + 1$ then $j_1 < i_1 < \cdots < j_m < i_k < j_m = n$.

Let $\sigma : \{i_1, j_1\} \to \{0, 1\}$ be a map such that $\sigma(i_1) = 1$ and $\sigma(j_1) = 0$. For $a \in \{i_1, j_1\}$, we assume $i_r(i_{r+\sigma(a)}) \in \{i_1, \ldots, i_k\}$ and $j_{r+1-\sigma(a)}(j_r) \in \{j_1, \ldots, j_m\}$. The orientation between the interval $[i_r, i_{r+1-\sigma(a)}]((j_r, j_{r+1}))$ over $\mathbb{Z}^+$ is from left to right, denoted $[a,b]$ (right to left, denoted $[a,b]$). The following words are defined by using intervals:

- $w_t = \min\{w_s | w_t \leq w_{j_{r+1-\sigma(a)}}, w_s = UD\}$ (if $w_t = \max\{w_s | w_t \leq w_{j_{r+1-\sigma(a)}}, w_s = UD\}$)
- $w_p = \min\{w_s | w_t \leq w_{j_{r+1-\sigma(a)}}, w_s = DU\}$ (if $w_p = \max\{w_s | w_t \leq w_{j_{r+1-\sigma(a)}}, w_s = DU\}$).

We introduce the following elementary shifts:

**ES1.** If $w_s = UD$ for all $w_s \in [i_r, j_{r+1-\sigma(a)}]((j_r, i_{r+\sigma(a)}))$,

\[
\frac{\{j_{r-\sigma(a)}; i_r, j_{r+1-\sigma(a)}; i_{r+1}\}}{([i_{r+\sigma(a)}-1]; i_r, j_{r+\sigma(a)}; i_{r+1})} \frac{([i_r, j_{r+1-\sigma(a)}; j_r, i_{r+\sigma(a)}])}{([i_r, j_{r+1-\sigma(a)}; j_r, i_{r+\sigma(a)}, j_{r+1}]},
\]

then

\[
g(UWD) = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UWD),
\]

if there exists $s \in \mathbb{Z}^+$ such that $j_{r-\sigma(a)} \leq s \leq i_r$, $|s - j_r| > 1$, $w_x = UD$ if $s \leq x \leq i_r$ over $[j_{r-\sigma(a)}, i_r]$ and

\[
w_y = \begin{cases} UD, & \text{if } y = j_{r+1-\sigma(a)}; \\ DU, & \text{otherwise}, \end{cases}
\]

over $[j_{r+1-\sigma(a)}, i_{r+1}]$ for $j_{r+1-\sigma(a)} \neq n - 1$ or the first condition over $[j_{r-\sigma(a)}, i_r]$ for $j_{r+1-\sigma(a)} = n - 1$.

\[
g(UWD) = f_{i_{r+\sigma(a)}} \circ \cdots \circ f_{i_r}(UWD),
\]

if there exists $s \in \mathbb{Z}^+$ such that $i_{r+\sigma(a)} \leq s \leq j_{r+1}$, $|s - i_{r+\sigma(a)}| > 1$, $w_x = UD$ if $i_{r+\sigma(a)} \leq x \leq s$ over $[i_{r+\sigma(a)}, j_{r+1}]$ and

\[
w_y = \begin{cases} UD, & \text{if } y = j_r; \\ DU, & \text{otherwise}, \end{cases}
\]

over $[i_{r+\sigma(a)}, j_r]$ for $j_r \neq 1$ or the first condition over $[i_{r+\sigma(a)}, j_{r+1}]$ for $j_r = 1$, with $i_r \neq 1 (i_{r+\sigma(a)} \neq n - 1)$.
ES2. If \( t = 1 \) or \( n - 1 \) then \( g(UWD) = f_t(UWD) \). Then there are not elementary shifts associated to elements of a subchain different from 1 and \( n - 1 \).

ES3. If \( w_{ij} < w_{ik} < w_{ij+1} \) then \( g(UWD) = f_t(UWD) \).

ES4. If \( w_p = w_{ij+1} \) then

\[
g(UWD) = \begin{cases} f_{ij+1} \circ \cdots \circ f_{ij+1} & \text{if } j_{ij+1} \neq n - 1, \\ f_{ij+1} & \text{if } j_{ij+1} = n - 1. \end{cases}
\]

ES5. If \( w_t < w_p < w_{ij+1} \) then \( g(UWD) = f_p(UWD) \).

For a given subchain \( C = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq \mathbb{N}-1 \), two Dyck paths \( D \) and \( D' \) of length \( 2n \) are said to be related by a relation of type \( R_{j_1 \cdots j_m}^{i_1 \cdots i_k} \) if there is an elementary shift \( ES_i \), \( 1 \leq i \leq 5 \) which transforms either \( D \) into \( D' \) or \( D' \) into \( D \).

**Proposition 4.** The relation \( R_{j_1 \cdots j_m}^{i_1 \cdots i_k} \) is irreversible.

**Proof.** Suppose that there is an elementary shift \( f_{i_1} \circ \cdots \circ f_{i_k} \) from a Dyck path \( UVD \) to a Dyck path \( UV'D \) and that there is an elementary shift \( f_{r_1} \circ \cdots \circ f_{r_l} \) from \( UVD \) to a \( UWD \), then we have five cases:

(i) If \( f_{r_1} \circ \cdots \circ f_{r_l} \) arises from \( ES_1 \) over \( [i_r, j_{r+1} \cdot \sigma(a)] \). Shifts \( ES_2, ES_3 \) and \( ES_5 \) allow to conclude that from \( UVD \) to a \( UWD \), \( f_t = f_{j_{r+1} \cdot \sigma(a)} \circ \cdots \circ f_{i_r} \) or \( f_p = f_{j_{r+1} \cdot \sigma(a)} \circ \cdots \circ f_{i_r} \) and this is a contradiction. If \( ES_1 \) is an elementary shift from \( UVD \) to \( UWD \), then two cases arise: If \( j_{r+1} \cdot \sigma(a) \neq n - 1 \), then

\[
U_v \cdots v_{j_r(a)} \circ \cdots \circ v_{i_r-1} \circ v_{i_r} \cdots v_{j_{r+1} \cdot \sigma(a) + 1} \circ v_{j_{r+1} + 1} \cdots v_n \cdot D,
\]

it turns out that \( f_{j_{r+1} \cdot \sigma(a)} \circ \cdots \circ f_{i_r} \) has the form

\[
U_w \cdots w_{j_r(a)} \circ \cdots \circ w_{i_r-1} \circ w_{i_r} \cdots w_{j_{r+1} \cdot \sigma(a) + 1} \circ w_{j_{r+1} + 1} \cdots w_n \cdot D,
\]

which is a contradiction. If \( j_{r+1} \cdot \sigma(a) = n - 1 \), \( UVD \) is equal to

\[
U_v \cdots v_{j_r(a)} \circ \cdots \circ v_{i_r-1} \circ v_{i_r} \cdots v_{j_{r+1} \cdot \sigma(a) + 1} \circ D,
\]

and \( f_{j_{r+1} \cdot \sigma(a)} \circ \cdots \circ f_{i_r} \) has the shape

\[
U_w \cdots w_{j_r(a)} \circ \cdots \circ w_{i_r-1} \circ w_{i_r} \cdots w_{j_{r+1} \cdot \sigma(a) + 1} \circ D,
\]

again a contradiction. We also get a contradiction if an elementary shift is done by using \( ES_4 \) from \( UVD \) to a \( UWD \), indeed, in these cases it holds that, if \( j_{r+1} \cdot \sigma(a) = n - 1 \), there are \( t \) and \( p \) such that \( p = j_{r+1} \cdot \sigma(a) < t \leq j_{r+1} \) and \( UVD \) is equal to
and $f_{r+1-\sigma(a)} \circ \cdots \circ f_r(UVD)$ is
\[
Uw_1 \ldots w_{i_r-1} w_{i_r} \ldots w_{j_r+1-\sigma(a)} w_{j_r+1-\sigma(a)+1} \ldots w_1 w_{r+1} \ldots w_{n-1} D,
\]

If $v_{j_r+1-\sigma(a)} = n-1$ $f_{r+1-\sigma(a)} = f_{r+1-\sigma(a)} \circ \cdots \circ f_r$, but this is a contradiction.

(ii) If $f_{r_1} \circ \cdots \circ f_{r_2}$ arises from ES2 over $[i_1, j_1]$ then we cannot use elementary shifts defined in cases ES1, ES4, ES5 or ES3, provided that, $i_1 \neq 1$, $t \neq p$ or $1 < t < j_1$. Therefore, ES2 guarantees the existence of a walk from $UVD$ to $UWD$ such that;
\[
Uv_1 \ldots v_{j_1} \ldots v_{n-1} D,
\]

and $f_1(UWD)$ has the form
\[
Uw_1 \ldots w_{j_1} \ldots w_{n-1} D,
\]

which is a contradiction (if $t = n - 1$, the proof is dual).

(iii) If $f_{r_1} \circ \cdots \circ f_{r_2}$ arises from ES3 over $[i_2, j_r+1-\sigma(a)]$, provided that, $i_r < t < p < j_r+1-\sigma(a)$, we conclude that it is not possible to use ES1, ES2, ES4 nor ES5. In the case of ES3 from $UVD$ to a $UWD$, $UVD$ equals
\[
Uv_1 \ldots v_{i_r} \ldots v_{i_r-1} v_t \ldots v_{j_r+1-\sigma(a)} \ldots v_{n-1} D,
\]

and $f_t(UVD)$ has the shape
\[
Uw_1 \ldots w_{i_r} \ldots w_{t+1} w_{t+1} \ldots w_{j_r+1-\sigma(a)} \ldots w_{n-1} D,
\]

but this is a contradiction.

(iv) If $f_{r_1} \circ \cdots \circ f_{r_2}$ arises from ES4 over $[i_2, j_r+1-\sigma(a)]$, provided that $t < p$, we do not use ES2, ES3 nor ES5. If $j+1-\sigma(a) = n-1$, we cannot use ES1. If $j+1-\sigma(a) \neq n-1$ we can use ES1 from $UVD$ to a $UWD$ (Note that, it is not necessary with $v_{m} = UD$ for all $s \in [j+1-\sigma(a) + 1, j+1] \ UVD$ is equal to
\[
Uv_1 \ldots v_{i_r} \ldots v_{i_r-1} v_t \ldots v_{p-1} v_p v_{j_r+1-\sigma(a)} +1 \ldots v_{r+1} v_{r+1} +1 \ldots v_{n-1} D,
\]

it turns out that $g(UVD)$ has the form
\[
Uw_1 \ldots w_{i_r} \ldots w_{i_r-1} w_{i_r} \ldots w_{p-1} w_p w_{j_r+1-\sigma(a)} +1 \ldots w_{r+1} w_{r+1} +1 \ldots w_{n-1} D,
\]

which is a contradiction. Using ES5 from $UVD$ to $UWD$, if $j^+1-\sigma(a) \neq n-1$, $UVD$ is equal to
\[
Uv_1 \ldots v_{i_r} \ldots v_{p-1} v_p v_{j_r+1-\sigma(a)} +1 \ldots v_{r+1} v_{r+1} +1 \ldots v_{n-1} D,
\]
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and $UWD$ has the shape

\[ Uw_1 \ldots w_{t-1} w_t \ldots w_{r-1} w_r D, \]

again a contradiction. If $j_{r+1-\sigma(a)} = n - 1$, $UVD$ is equal to

\[ Uv_1 \ldots v_{t-1} v_t \ldots v_{n-1} D, \]

it turns out that $UWD$ has the shape

\[ Uw_1 \ldots w_{t-1} w_t \ldots w_{r-1} w_r D, \]

this is a contradiction.

(v) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES5 over $[i_r, j_{r+1-\sigma(a)}]$. Then we cannot use ES1, ES2, ES3 nor ES4, because $f_p \neq f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{l_r}$ and $t < p$. Using ES5 from $UVD$ to a $UWD$, we observe that $UVD$ is equal to

\[ Uv_1 \ldots v_{t-1} v_t \ldots v_{p-1} v_p \ldots v_{j_{r+1-\sigma(a)}} \ldots v_{n-1} D, \]

and $f_p(UWD)$ has the form

\[ Uw_1 \ldots w_{t-1} w_t \ldots w_{p-1} w_p \ldots w_{j_{r+1-\sigma(a)}} \ldots w_{n-1} D, \]

again this is a contradiction.

Taking into account that if $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES5 over $[i_r, j_{r+1-\sigma(a)}]$ then same arguments as described above applied dually allow to conclude the proposition. We are done.  

\[ \square \]

3.3 $A_{n-1}$-Dyck Paths Categories

For $n \geq 2$ fixed, the $A_{n-1}$-Dyck paths category is a category of Dyck paths $(\mathcal{D}_{2n}, R)$ where $R$ is a relation of type $R_{i_1 \ldots i_k}^{j_1 \ldots j_m}$ as described before. As an example we let $(\mathcal{D}_3, R_1^3)$ denote the $A_3$-Dyck paths category with the admissible subchain $1 < 3$.

Figure 2 shows all the elementary shifts of $(\mathcal{D}_3, R_1^3)$.

We let $S$ denote the set of all Dyck paths with exactly $n-1$ peaks. The following propositions and lemmas describe some properties of the set $S$.

**Proposition 5.** Let $UWD$ be a Dyck path of length $2n$, then $UWD \in S$ if and only if there is a unique sequence $w_1w_{i+1} \ldots w_{r-1}w_r$ such that

\[ w_i = \begin{cases} UD, & \text{if } l \leq i \leq r; \\ DU, & \text{otherwise}. \end{cases} \]
Proof. Firstly, let $\delta$ be a map $\delta : \{,\} \to \{U,D\}$ where left bracket is associated to the letter $U$ and right bracket is associated to the letter $D$, suppose $UWD \in S$, then there exist bracket-subchains such that $UWD$ can be written in the following form
\[
\left( \begin{array}{c}
1 \\
2 \\
l - 2 \\
l - 1 \\
l \\
l + 1 \\
r - 2 \\
r - 1 \\
r \\
r + 1 \\
n - 2 \\
n - 1 \\
\end{array} \right),
\]
therefore $w_i = UD$ if $l \leq i \leq r$ and $w_i = DU$. On the other hand, suppose $UWD$ has a unique subsequence $w_lw_{l+1}\ldots w_{r-1}w_r$ that satisfies identity (10), then if we apply $\delta^{-1}$ to $UWD$, the sequence
\[
\left( \begin{array}{c}
1 \\
2 \\
l - 2 \\
l - 1 \\
l \\
l + 1 \\
r - 2 \\
r - 1 \\
r \\
r + 1 \\
n - 2 \\
n - 1 \\
\end{array} \right),
\]
is obtained, therefore $UWD \in S$. We are done.

Lemma 6. Let $UWD$ be a Dyck path in $S$, and integers $r,l$ defined as in Proposition 5 with $|r-l| > 0$, then there exists an elementary shift from $UWD$ to another Dyck path with exactly $n - 1$ peaks.

Proof Let $UWD$ be a Dyck path in $S$, let $l$ and $r$ be positive integers such that $w_m = UD$ for $l \leq m \leq r$. Let $l \in [i,r_{i+1-\sigma(a)}]$, we have the following cases:
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(1) If \( l = i_r = 1 \), then
\[
g(UWD) = Uf(w_1)w_2\ldots w_rw_{r+1}\ldots w_{n-1}D \in S.
\]

(2) If \( l = i_r \neq 1 \), then there is \( p = l - 1 \) over \([j_{r-\sigma(a)}, i_r]\) such that
\[
g(UWD) = Uw_1\ldots f(w_p)w_{p+1}\ldots w_m\ldots w_{n-1}D \in S.
\]

(3) If \( i_r < l < j_{r+1-\sigma(a)} \), then
\[
g(UWD) = Uw_1\ldots f(w_j)w_{j+1}\ldots w_{r+1}\ldots w_{n-1}D \in S.
\]

(4) If \( l = j_{r+1-\sigma(a)} \) and \(|l - r| > 0 \), then \( r \in [i_{r+1}, j_{r+1-\sigma(a)}] \) with \(|r_1 - r| > 0 \) and the following cases hold:

(4.1) If \( i_{r+1} \leq r < j_{r+1-\sigma(a)} \), there is \( p = r + 1 \) such that, if \( p \neq j_{r+1-\sigma(a)} \) then
\[
g(UWD) = w_1\ldots w_2\ldots w_r f(w_p)\ldots w_{n-1}D \in S,
\]
if \( p = j_{r+1-\sigma(a)} = n - 1 \), then
\[
g(UWD) = Uw_1\ldots w_r f(w_p)D \in S,
\]
or if \( p = j_{r+1-\sigma(a)} \neq n - 1 \) then
\[
g(UWD) = Uw_1\ldots w_r f(w_p)\ldots f(w_{r+1})\ldots w_{n-1}D \in S.
\]

(4.2) If \( r = j_{r+1-\sigma(a)} \)
\[
g(UWD) = Uw_1\ldots w_1\ldots w_{r-1} f(w_{r-1})\ldots f(w_r)\ldots D \in S.
\]

(4.3) Now, if \(|r_1 - r| > 1 \) or \( r_1 = r + 1 \) and \( r > i_{r+1} + 2 \) then
\[
g(UWD) = Uw_1\ldots f(w_j)\ldots f(w_{r+1})w_{r+1+1}\ldots w_r D \in S.
\]

For \( r \in [j_{r+1-\sigma(a)}, i_{r+1}] \) with \(|r_1 - r| \geq 0 \) we have that:

(4.4) If \( s = t = i_{r+1} = n - 1 \), then
\[
g(UWD) = Uw_1 \ldots w_1 \ldots w_{r+1}f(w_r)D \in S.
\]

On the other hand, if \( s = t = i_{r+1} \neq n - 1 \), then there is \( p \in [i_{r+1}, j_{r+1+2-\sigma(a)}] \) satisfying first condition of (4.1). Thus, if \( j_{r+1-\sigma(a)} < s < i_{r+1} \), it holds that
\[
g(UWD) = Uw_1\ldots w_{p-1} f(w_r)\ldots w_{n-1}D \in S.
\]
(4.5) If \( s = j_{r_1 + 1 - \sigma(a)} \) then \(|r_1 - r| > 0\) (If \(|r_1 - r| = 0\), \(|l - f| = 0\) which is a contradiction)

\[
g(UWD) = Uw_1 \ldots w_l \ldots w_{r_1 - 1} f(w_{r_1}) \ldots f(w_s) w_{r + 1} \ldots w_{n - 1}D \in S. \]

(4.6) Now, suppose that in \( UWD \) \(|r_1 - r| > 0\), then it satisfies the first condition in (4.3).

In case that \( l \in [j_r, i_r + \sigma(a)] \), we have the following cases:

(5) If \( j_r < l \leq i_r + \sigma(a) \), then there exists \( p = l + 1 \) such that, if \( p \neq j_r \), then

\[
g(UWD) = Uw_1 \ldots w_{j_r} \ldots f(w_p) w_{r} \ldots w_{n - 1}D \in S. \]

Note that, if \( p = j_r = 1 \) then

\[
g(UWD) = U f(w_p) w_{r} \ldots w_{n - 1}D \in S, \]

or if \( p = j_r \neq 1 \), then

\[
g(UWD) = Uw_1 \ldots f(w_{i_{r-1} + \sigma(a)}) \ldots f(w_p) w_{r} \ldots w_{n - 1}D \in S. \]

(6) If \( l = j_r \) and \(|l - r| > 0\), then \( r \in [j_{r_1}, i_{r_1 + \sigma(a)}] \) with \(|r_1 - r| \geq 0\), then the following cases hold:

(6.1) If \( j_{r_1} + 2 \leq r \leq i_{r_1 + \sigma(a)} \), then there exists \( p \) satisfying (4.4).

(6.2) If \( j_{r_1} \leq r < j_{r_1} + 2 \), then \(|r_1 - r| > 0\) and if \( r = j_{r_1 + 1} \) satisfies (6.1), or if \( r = j_{r_1} \) then \( UWD \) satisfies (4.5).

(6.3) Now, if \(|r_1 - r| > 0\) then

\[
g(UWD) = U \ldots f(w_i) \ldots f(w_{i_r + \sigma(a)}) w_{i_r + \sigma(a) + 1} \ldots w_s \ldots D \in S, \]

or \( r \in [i_{r_1 + \sigma(a)}, j_{r_1 + 1}] \) with \(|r_1 - 1| \geq 0\) satisfies conditions (4.1), (4.2) and (4.3) for \( i_{r_1 + \sigma(a)} \leq r \leq j_{r_1 + 1} \).

Same arguments are used for the cases \( r \in [i_r, j_{r+1 - \sigma(a)}]([j_r, i_{r+1 + \sigma(a)}]) \) to conclude the lemma. We are done.

\[ \square \]

Lemma 7. Suppose that \( UWD \) is a Dyck path in \( S \) and that integers \( l \) and \( r \) as defined in Proposition 4 are such that \( l = r \), then the following statements hold:

(a) If \( l \notin \{j_s\} \) then there is an elementary shift to a Dyck path with exactly \( n - 1 \) peaks.

(b) If \( l \in \{j_s\} \) then there is an elementary shift from a Dyck path with exactly \( n - 1 \) peaks to \( UWD \).

Proof. Let \( UWD \) be a Dyck path in \( S \), and positive integers \( l \) and \( r \) with \( l = r \).
(a) Suppose \( l \notin \{ j_s \} \) and \( l \in [i_r, j_{r+1} - \sigma(a)] \). If \( i_r \leq l < j_{r+1} - \sigma(a) \), then \( UWD \) satisfies (4.1) and (4.2) of Lemma 6. In particular, if \( l = i_r \) there is \( p' = l - 1 \) in \( [j_{r-\sigma(a)}, i_r] \) that satisfies the first condition of (5) of Lemma 6. The case \( l \in [j_r, i_{r+\sigma(a)}] \) is dual.

(b) Suppose \( l = j_{r+1} - \sigma(a) \), we have the following cases:

(i) If \( |i_r - j_{r+1} - \sigma(a)| = 1 \) (or \( |i_r + 1 - j_{r+1} - \sigma(a)| = 1 \) and \( i_r = 1 \) (or \( i_{r+1} = n - 1 \)), then there is \( UVD \) which is equal to

\[
\begin{align*}
U & \ w_1 \ w_2 \ldots D \in S \ (\text{or } U \ w_1 w_2 \ldots D \in S), \\
& \ U \ w_1 \ w_2 \ldots D \in S \ (\text{or } U \ w_1 w_2 \ldots D \in S),
\end{align*}
\]

and

\[
\begin{align*}
U f(w_1)w_2 \ldots D = UWD \ (\text{or } U \ w_1 f(w_2) \ldots D = UWD)
\end{align*}
\]

(ii) If \( |i_r - j_{r+1} - \sigma(a)| = 1 \) (or \( |i_r + 1 - j_{r+1} - \sigma(a)| = 1 \) and \( i_r \neq 1 \) (or \( i_{r+1} \neq n - 1 \)) then there is \( l' = j_{r-\sigma(a)} \) and \( r' = j_{r+1} - \sigma(a) \) (or \( l' = j_{r+1} - \sigma(a) \) and \( r' = j_{r-2} - \sigma(a) \)) such that \( UVD \) is equal to

\[
\begin{align*}
U & \ w_1 \ w_2 \ldots D \ (\text{or } U \ w_1 w_2 \ldots D \in S), \\
& \ U \ w_1 \ w_2 \ldots D \in S \ (\text{or } U \ w_1 w_2 \ldots D \in S),
\end{align*}
\]

and

\[
\begin{align*}
U f(w_1) \ldots f(w_{r'}-1)w_r \ldots D = UWD \ (\text{or } U \ w_1 f(w_2) \ldots f(w_{r'}) \ldots D = UWD).
\end{align*}
\]

(iii) If \( |i_r - j_{r+1} - \sigma(a)| > 1 \) (or \( |i_r + 1 - j_{r+1} - \sigma(a)| > 1 \)) then there is \( UVD \) which is equal to

\[
\begin{align*}
U & \ w_{l-1} \ w_l \ldots D \ (\text{or } U \ w_l w_{l+1} \ldots D \in S), \\
& \ U \ w_{l-1} \ w_l \ldots D \in S \ (\text{or } U \ w_l w_{l+1} \ldots D \in S),
\end{align*}
\]

and

\[
\begin{align*}
U & \ f(w_{l-1})w_l \ldots D = UWD \ (\text{or } U \ w_l f(w_{l+1}) \ldots D = UWD).
\end{align*}
\]

Similar arguments dually applied can be used to obtain the lemma in the case \( l = j_r \).

We are done.

\[\square\]

**Remark 8.** Note that, in general there is an elementary leftshift and an elementary rightshift over \( S \) and these elementary shifts are disjoint, i.e. if \( f_{p_1} \circ \cdots \circ f_{p_q} \) and \( f_{q_1'} \circ \cdots \circ f_{q_{p'}} \) are elementary left and right shifts, respectively. Then

\[
\{p_1, \ldots, p_q\} \cap \{q'_1, \ldots, q'_{p'}\} = \emptyset,
\]

these elementary shifts are unique according to Lemma 6 and Lemma 7. If \( F^p = f_{p_1} \circ \cdots \circ f_{p_q} \) is an elementary leftshift (rightshift) we write \( F^p_l \) (\( F^p_r \)).

**Proposition 9.** Let \( \mathcal{C} = \{i_1, \ldots, i_k, j_i, \ldots, j_m\} \) be an admissible subchain, then all Dyck paths of \( S \) constitute a connected quiver \( Q \).
Proof. It suffices to prove that $Q$ is connected, to do that, consider Dyck paths $UWD$ and $UVD$ of $S$. Then if there is a shift path between $UWD$ and $UV$ they are connected. Otherwise, Lemmas 6 and 7 allow to define a Dyck path $U WD$ and a shift path $F^{(1)} = f_{p_1}^{(1)} \circ \cdots \circ f_{p_1}^{(1)}$ with $f_{p_1}^{(1)} = f_{m_1}^{(1)} \circ \cdots \circ f_{m_{1+\kappa}}^{(1)}$ such that

$$U WD \xrightarrow{F^{(1)}} \cdots \xrightarrow{F^{(1)}} U WD,$$

and if there is a shift path from $UV D$ to a $U WD$ then they are connected. If there is not a shift path from $UV D$ to $U WD$, then there is a Dyck path $U WD$ and a shift path $F^{(2)} = f_{p_2}^{(2)} \circ \cdots \circ f_{p_2}^{(2)}$ with $f_{p_2}^{(2)} = f_{m_2}^{(2)} \circ \cdots \circ f_{m_{2+\kappa}}^{(2)}$ such that

$$U WD \xrightarrow{F^{(2)}} \cdots \xrightarrow{F^{(2)}} U WD,$$

again, if there is a shift path from $U WD$ to a $UV D$ then they are connected. Since $S$ is finite, the procedure ends in such a way that $U WD$ and $UV D$ are connected and with this argument we are done. $\Box$

Henceforth, let $\mathcal{E}_{\mathcal{D}_n}$ denote the subcategory of $(\mathcal{D}_{2n}, R_{11, \ldots, 11, j_m, \ldots, j_m})$ whose objects are $k$-linear combinations of Dyck paths of $S$. Lemma 10 and Proposition 11 give some properties of the Hom-spaces of this category.

Lemma 10. Let $U WD$, $U WD$, $U WD$ and $U WD$ be Dyck paths in $\mathcal{E}_{\mathcal{D}_n}$ and let

$$F = F_1 \circ F_2$$

(resp. $F_1 \circ F_2$) be a shift path $U WD \xrightarrow{F_1} U WD \xrightarrow{F_2} U WD$ (resp. $U WD \xrightarrow{F_2} U WD \xrightarrow{F_1} U WD$), if there is other shift path $G = G_1 \circ G_2$ such that

$$U WD \xrightarrow{G_2} U WD \xrightarrow{G_2} U WD$$

with $U WD \neq U WD$ then $G_2 = F_1$ and $G_1 = F_2$ (resp. $G_2 = F_2$ and $G_1 = F_1$).

Proof. Let $F = F_1 \circ F_2$ be a shift path such that

$$U \ldots w_{l_1} \ldots w_{l_2} \ldots D \xrightarrow{F_2} U \ldots w_{l_2} \ldots w_{l_3} \ldots D \xrightarrow{F_1} U \ldots v_{r_3} \ldots v_{r_3} \ldots D,$$

with $l_1 = l_2$ and $r_2 = r_3$ and suppose that there is other shift path $G = G_1 \circ G_2$ such that

$$U \ldots w_{l_1} \ldots w_{l_2} \ldots D \xrightarrow{G_2} U \ldots w_{l_2} \ldots w_{l_3} \ldots D \xrightarrow{G_1} U \ldots v_{r_3} \ldots v_{r_3} \ldots D,$$

with $U WD \neq U WD$. Given the elementary rightshift $F_2$, then since $G \neq F_2$, it holds that $U WD$ satisfies the conditions of $U WD$ in order to apply the same elementary leftshift $F_2$, i.e., $F_2 = G_2$ and $l_3 = l_4$. Since $r_1 = r_4$, $U WD$ and $U WD$ satisfy the conditions to apply the same elementary rightshift, i.e., $F_1 = G_1$. Case $F_1 \circ F_2$ is obtained via a dual argument. $\Box$

Proposition 11. If $\text{Hom}_{\mathcal{E}_{\mathcal{D}_n}}(U WD, U WD) \neq 0$ then $\dim_k \text{Hom}_{\mathcal{E}_{\mathcal{D}_n}}(U WD, U WD) = 1$.

Proof. Suppose that $\text{Hom}_{\mathcal{E}_{\mathcal{D}_n}}(U WD, U WD) \neq 0$, then there is a shift path $F$ of the form

$$U WD \xrightarrow{F_0} \cdots \xrightarrow{F_{x_1-1}} U W^{x_1-1} D \xrightarrow{F_{x_1-1}} U W^{x_1-1} D \xrightarrow{F_{x_1}} U W^{x_1} D \xrightarrow{F_{x_1+1}} U W^{x_1+1} D \xrightarrow{F_{x_1+1}} \cdots \xrightarrow{F_{x_n}} U W D,$$
with $x_i \in \{l, r\}$ and for some $m \in \mathbb{Z}^+$. Now, for each pair $F^i_x \circ F^{i+1}_x$ with $x_{i-1} = l$ and $x_i = r$ ($x_{i-1} = r$ and $x_i = l$) that satisfies conditions described in Lemma 10, there is another shift path $F'$ of the form

$$UWD \xrightarrow{F^0 \circ \ldots \circ F^i} UW^{i-1}D \xrightarrow{F^{i+1}_{x_i-1}} UW' \xrightarrow{F^{i+1}_l} UWD,$$

transforming $UWD$ and $UVD$. Thus $F \sim_{R_{i_1 \ldots i_m}^U}$ $F'$.

\[\square\]

4 A Categorical Equivalence

In this section, we establish an equivalence between the category $\mathcal{C}_{2n}$ and the category of representations of a quiver of Dynkin type $A_n$.

4.1 The $\Theta$ Functor

Given an admissible subchain $\mathcal{C} = \{j_1, \ldots, j_m, i_1, \ldots, i_k\}$, $\mathcal{C}_{2n}$ the subcategory of $(\mathcal{D}_{2n}, R_{i_1 \ldots i_m}^U)$ and $Q$ a quiver of type $A_{n-1}$ with $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_m\}$ being the sets of sinks and sources, respectively. Then the $k$-linear additive functor $\Theta : \mathcal{C}_{2n} \rightarrow \text{rep} \ Q$ is defined in such a way that, for an object $UWD \in \mathcal{C}_{2n}$, it holds that,

$$\Theta(UWD) = (\Theta(w_i), \varphi(\Theta(w_i, w_{i+1})))$$

where

$$\Theta(w_i) = \begin{cases} k, & \text{if } w_i = UD, \\ 0, & \text{if } w_i = DU. \end{cases} \quad (11)$$

If $w_i, w_{i+1} \in [i_r, i_{r+1} - \sigma(a)] \cup \{j_r, i_{r+1} - \sigma(a)\}$ then $s(\Theta(w_i, w_{i+1})) = i + 1$, is the starting point of the corresponding arrow, whereas $t(\Theta(w_i, w_{i+1})) = i$ is the ending vertex of the corresponding arrow $s(\Theta(w_i, w_{i+1})) = i$, $t(\Theta(w_i, w_{i+1})) = i + 1$ and,

$$\varphi(w_i, w_{i+1}) : \Theta(w_i) = (\varphi(\Theta(w_i, w_{i+1}))) \rightarrow \Theta(w_i(\Theta(w_i, w_{i+1}))).$$

$$\varphi(w_i, w_{i+1}) = \begin{cases} 1_k, & \text{if } w_i = UD = w_{i+1}, \\ 0, & \text{if } w_i = DU \text{ or } w_{i+1} = DU. \end{cases} \quad (12)$$

Functor $\Theta$ acts on morphisms as follows;

Let

$$f_{q_2} \circ \ldots \circ f_{q_1} = (1_1, \ldots, 1_{q_1-1}, f_{q_1}, \ldots, f_{q_2}, 1_{q_2+1}, \ldots, 1_{n-1}),$$

be a elementary shift between $UWD$ and $UVD$, then:

$$\Theta((1_1, \ldots, 1_{q_1-1}, f_{q_1}, \ldots, f_{q_2}, 1_{q_2+1}, \ldots, 1_{n-1})),
(\Theta(1)_1), \ldots, \Theta(1_{q_1-1}), \Theta(f_{q_1}), \ldots, \Theta(f_{q_2}), \Theta(1_{q_2+1}), \ldots, \Theta(1_{n-1})),
$$

where $\Theta(f_m) = 0$ and,

$$\Theta(1_{m_1}) = \begin{cases} 1_k, & \text{if } w_{m_1} = UD = v_{m_1}, \\ 0, & \text{otherwise}, \end{cases} \quad (13)$$

for $1 \leq m_1 \leq q_1 - 1$, $q_1 \leq m \leq q_2$ and $q_2 + 1 \leq m_1 \leq n - 1$. 

Dyck Paths...
\[\begin{align*}
\text{Remark 12.} & \quad \text{Note that, it is easy to see that } \Theta \text{ is an additive covariant functor.} \\
\text{Lemma 13.} & \quad \text{Let } UWD \text{ and } UVD \text{ be Dyck paths of } \mathbb{C}_x. \text{ If } \text{Hom}_{\mathbb{C}_x}(UWD, UVD) \neq 0 \text{ then } \text{Hom}_{\mathbb{C}_x}(\Theta(UWD), \Theta(UVD)) \neq 0. \\
\text{Proof.} & \quad \text{Suppose } \text{Hom}_{\mathbb{C}_x}(UWD, UVD) \neq 0, \text{ and let } F \text{ be a shift path } UW^a D \rightarrow UW^b D \rightarrow UW^m D \text{ from } UWD = UW^0 D \text{ to } UVD = UW^m D \text{ for some } m \in \mathbb{Z}^+, \text{ then there exist } q_1 \text{ and } q_2 \text{ such that} \\
& \quad \quad \{q_1, q_1 + 1, \ldots, q_2 - 1, q_2\} = \bigcap_{i \in J} \text{Supp } UW^i D,
\end{align*}\]

applying \(\Theta\) we obtain the following diagram:

\[\begin{align*}
\text{Diagram 1.}
\end{align*}\]

where \(c_{q_1} - 1, a_{q_1 - 1}^i, a_{q_2}^i, d_{q_2 + 1}^i \in \{0, k\}, \text{ squares in the diagram are commutative between } q_1 \text{ and } q_2 \text{ (independently of the chosen orientation). For the sub-shift path } F(x,y) \text{ to } F \text{ with } 0 \leq x \leq y \leq m - 1 \text{ there exist positive integers } q_1^{(x,y)} \text{ and } q_2^{(x,y)} \text{ such that} \\
S^{(x,y)} = \{q_1^{(x,y)}(x,y) + 1, \ldots, q_2^{(x,y)}(x,y) - 1, q_2^{(x,y)}(x,y)\} = \bigcap_{i \in J(x,y)} \text{Supp } UW^i D,
\]

and for the diagrams

\[\begin{align*}
\text{Diagram 2.} & \quad \text{Diagram 3.}
\end{align*}\]

we have the following cases:
(1) If \( q_1^{(x,y)} \in [i_r, j_{r+1} - \sigma(a)] \) (\( i_r < q_1^{(x,y)} \leq j_{r+1} - \sigma(a) \)) four cases must be considered.

(1.1) If \( \Theta(w^y_{q_1^{(x,y)} - 1}) = k \) and \( \Theta(w^y_{q_1^{(x,y)} - 1}) = k \), \( q_1^{(x,y)} \) belong to \( D \), which is a contradiction.

(1.2) If \( \Theta(w^y_{q_1^{(x,y)} - 1}) = k \) and \( \Theta(w^y_{q_1^{(x,y)} - 1}) = 0 \), then the Diagram 2 commutes.

(1.3) If \( \Theta(w^y_{q_1^{(x,y)} - 1}) = 0 \) and \( \Theta(w^y_{q_1^{(x,y)} - 1}) = k \), then there is an elementary shift \( f_{q_1^{(x,y)} - 1} \) on the interval and this is again a contradiction.

(1.4) If \( \Theta(w^y_{q_1^{(x,y)} - 1}) = 0 \) and \( \Theta(w^y_{q_1^{(x,y)} - 1}) = 0 \), then the Diagram 2 commutes.

(2) If \( q_2^{(x,y)} \in [j_{r+1} - \sigma(a), i_{r+1}] \) (\( j_{r+1} - \sigma(a) < q_2^{(x,y)} \leq i_{r+1} \)), the conditions (1.1)-(1.4) are satisfied on the interval.

(2.1) If \( \Theta(w^y_{q_2^{(x,y)} - 1}) = k \) and \( \Theta(w^y_{q_2^{(x,y)} - 1}) = 0 \), then they satisfy condition (1.3).

(2.2) If \( \Theta(w^y_{q_2^{(x,y)} - 1}) = 0 \) and \( \Theta(w^y_{q_2^{(x,y)} - 1}) = k \), then they satisfy condition (1.2).

(3) Case \( q_2^{(x,y)} \in [i_r, j_{r+1} - \sigma(a)] \) is similar to case (2) for the Diagram 3.

(4) Case \( q_2^{(x,y)} \in [j_{r+1} - \sigma(a), i_{r+1}] \) is similar to case (1) for the Diagram 3.

Therefore, the Diagram 1 commutes. Since the cases over \( [j_r, i_{r+1}] \) can be showed by using dual arguments. We are done.

**Lemma 14.** Functor \( \Theta \) is faithful and full.

**Proof.** Let \( \phi \) be the map

\[
\phi : \text{Hom}_{\mathcal{E}_{2n}}(UWD, UVD) \to \text{Hom}_{\text{rep } Q}(\Theta(UWD), \Theta(UVD)),
\]

such that \( \phi(\lambda F) = \lambda \Theta(F) \) with \( F = (1_1, \ldots, 1_{q_1}, 1_{q_2}, \ldots, 1_{n-1}) \), for some \( 1 \leq q_1, q_2 \leq n-1 \) and \( \lambda \in k \). Note, \( \phi \) is well defined and Lemma 13 allows us to observe that the image of a non-zero morphism in \( \mathcal{E}_{2n} \) is a non-zero morphism in \( \text{rep } Q \). Thus, \( \phi \) is surjective and injective.

**Theorem 15.** Functor \( \Theta \) is a categorical equivalence between the categories \( \mathcal{E}_{2n} \) and \( \text{rep } Q \).

**Proof.** Lemma 14 implies that functor \( \Theta \) is faithful and full. Now, let \( (M_i, \varphi_a) \in Q_{0, \alpha} \in Q_1 \) be an indecomposable representation in \( \text{rep } Q \) of the form

\[
0 \quad \ldots \quad k \quad 1 \quad k \quad 1 \quad \ldots \quad 1 \quad k \quad 1 \quad k \quad \ldots \quad 0
\]

with \( \{i_1, \ldots, i_k\} \) and \( \{j_1, \ldots, j_m\} \) the sets of sinks and sources respectively. Let \( \varphi_1 : \{0, k\} \to \{DU, UD\} \) be a map such that \( \varphi_1(k) = UD \) and \( \varphi_1(0) = DU \). Define the Dyck path \( UWD \) such that

\[
UWD = U w_1 \ldots w_{q_1 - 1} w_{q_1} \ldots w_{q_2} w_{q_2 + 1} \ldots w_{n-1} D.
\]

Proposition 15 allows us to observe that \( UWD \) has \( n-1 \) peaks over \( \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \) and \( \Theta(UWD) = (M_i, \varphi_a) \in Q_{0, \alpha} \in Q_1 \). Thus, \( \Theta \) is essentially surjective.

\[\square\]
Corollary 16. There exists a bijection \( \varphi \) between the set of representatives of indecomposable representations of rep \( Q \) (denoted \( \text{Ind}(\text{rep} \ Q) \)) and the set of Dyck paths of length \( 2n \) with exactly \( n - 1 \) peaks.

Proof. The Narayana number with exactly \( n - 1 \) peaks over all Dyck paths of length 2n is the triangular number \( T_{n-1} = \frac{(n-1)n}{2} \), which is equal to the number of indecomposable representations of rep \( Q \), then we define \( \varphi : S \to \text{Ind} (\text{rep} \ Q) \) such that \( \varphi(UWD) = \Theta(UWD) \).

Corollary 17. The category \( \mathcal{C}_{2n} \) is an abelian category.

4.2 Properties of the Category \( \mathcal{C}_{2n} \)

In this section, we introduce some properties of \( \mathcal{C}_{2n} \) regarding simple, projective and injective indecomposable objects, also we construct the Auslander-Reiten quiver for algebras of Dynkin type \( A_{n-1} \). Some conditions for morphisms between objects of the category are introduced as well.

Theorem 18. Let \( \mathcal{C} = \{ j_1, \ldots, j_m, i_1, \ldots, i_k \} \) be an admissible subchain, and let \( \mathcal{C}_{2n} \) be the corresponding category, then

(i) Indecomposable simple objects of \( \mathcal{C}_{2n} \) are objects of the form

\[
S(x) = US(w_1^x) \ldots S(w_n^x)D
\]

where

\[
S(w_y^x) = \begin{cases} UD, & \text{if } x = y, \\ DU, & \text{otherwise.} \end{cases}
\]

(ii) Indecomposable projective objects of \( \mathcal{C}_{2n} \) have the form \( P(x) = UP(w_1^x) \ldots P(w_n^x)D \)

where

\[
P(w_y^x) = \begin{cases} UD, & \text{if } x, y \in [i_r, j_{r+1-\sigma(a)}] \text{ or } y \in [i_r, j_{r+\sigma(a)}] \text{ and } x \leq y \text{ (} x \leq y), \\ DU, & \text{otherwise.} \end{cases}
\]

(iii) Indecomposable injective objects of \( \mathcal{C}_{2n} \) have the form \( I(i) = UI(w_1^i) \ldots I(w_n^i)D \)

where

\[
I(w_y^x) = \begin{cases} UD, & \text{if } x, y \in [i_r, j_{r+1-\sigma(a)}] \text{ or } j_{r+\sigma(a)} \text{ and } y \leq x \text{ (} y \leq x), \\ DU, & \text{otherwise.} \end{cases}
\]

Proof. (i) Let \( S(x) = (S(x)_y, \varphi_y) \) be an indecomposable simple object of rep \( Q \) such that \( S(x)_y = k \) if \( x = y \) and \( S(x)_y = 0 \) if \( x \neq y \). Functor \( \Theta \) allows us to observe that, there is \( UWD \in \mathcal{C}_{2n} \) satisfying the required conditions.

(ii) Let \( P(x) = (P(x)_y, \varphi_y) \) be an indecomposable projective object of rep \( Q \), if \( P(x)_y = k \) then there is a path from \( x \) to \( y \), as well as, a source \( i_r, j_{r+1-\sigma(a)} \) \( (j_r) \) and a sink \( i_r, i_{r+\sigma(a)} \) \( (j_r) \) such that \( i_r \leq y \leq x \leq j_{r+1-\sigma(a)} \) \( (j_r \leq x \leq y \leq i_{r+\sigma(a)}) \). Thus, not a path between \( x \) and \( y \), then functor \( \Theta \) determines an object \( UWD \) of \( \mathcal{C}_{2n} \) with \( i_1, \ldots, i_k, j_1, \ldots, j_m \) being an admissible subchain satisfying the required conditions. Case (iii) follows by dually applying the arguments used in the case (ii). \( \square \)
Corollary 19. The indecomposable simple objects of $C_{2n}$ have exactly a subsequence $UUDD$.

Proof. Let $S(x)$ be an indecomposable simple object of $C_{2n}$, then the identity

$$S(x) = U \ldots S(w_{x-1}^x)S(w_x^x)S(w_{x+1}^x) \ldots D = U \ldots DU U DU \ldots DU \ldots D$$

has place as a consequence of Theorem 18. □

Remark 20. The Auslander-Reiten translate can be obtained by using the Coxeter transformation and the dimension vector associated to the support of a Dyck path in $C_{2n}$.

Morphisms in $C_{2n}$ also have the following properties.

Let $UWD$ be a Dyck path of $C_{2n}$, then

$$\begin{align*}
\rho_{UWD} &= w_t \text{ and } b_{UWD} = \max \{w_s \mid w_{i_r} \leq w_s \leq w_{j_r+1-\sigma(a)}, w_s = UD\} \text{ over } [i_r, j_r+1-\sigma(a)], \\
\rho^{UWD} &= \min \{w_s \mid w_{j_r} \leq w_s \leq w_{i_r+\sigma(a)}, w_s = UD\} \text{ and } b^{UWD} = w_t \text{ over } [j_r, i_r+\sigma(a)].
\end{align*}$$

Theorem 21. The vector space $\text{Hom}_{C_{2n}}(UWD, UVD) \neq 0$ if and only if

(i) $\text{Supp}(UWD) \cap \text{Supp}(UVD) \neq \emptyset$,

(ii) $\rho_{UWD} \leq \rho_{UVD}$ and $b_{UWD} \leq b_{UVD}$ over $[i_r, j_r+1-\sigma(a)]$,

(iii) $\rho^{UWD} \geq \rho^{UVD}$ and $b^{UWD} \geq b^{UVD}$ over $[j_r, i_r+\sigma(a)]$.

for all $[i_r, j_r+1-\sigma(a)], [j_r, i_r+\sigma(a)]$ such that $i_r \leq q \leq j_r+1-\sigma(a)$ and $j_r \leq q \leq i_r+\sigma(a)$ with $q \in \text{Supp}(UWD) \cap \text{Supp}(UVD)$.

Proof. The result follows as a consequence of the definition of the functor $\Theta$ and the construction of Lemma 3.1. □

Figure 3 describes a quiver $Q$ of type $A_5$ and the Auslander-Reiten quiver of $\text{rep} Q$.

4.3 A Relationship with Some Nakayama Algebras

In [13] Markovitz, Rubey and Stump presented a connection between the Auslander-Reiten quiver of Nakayama algebras and Dyck paths. In such a work for a Nakayama algebra $A$, they associated the vector space dimension of the indecomposable projective modules $e_i A$ to a Dyck path, this vector is called the Kupisch series. If we take a Nakayama algebra $A = kQ/I$, with $I = \langle x_1 x_4, x_1 x_2 x_3 \rangle$, then the Kupisch series of $kQ/I$ is $[3, 3, 2, 2, 1]$, and the Auslander-Reiten quiver of $kQ/I$ has the shape described in Figure 5.

Let $C_{2(n+1)}$ be the category with the admissible subchain $1 < n$, $j_1 = 1$ and $i_1 = n$, and let $D_i$ be the sets.
\[ Q = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array} \]

Figure 3: Quiver \( Q \) and the Auslander-Reiten quiver of rep \( Q \).

Figure 4: Quiver \( Q \) of type \( A_5 \).

Figure 5: Dyck path associated to \( kQ/I \).

\[
D_1 = \{ X \in Ob(\mathcal{C}_{2(n+1)}) \mid w_1 = UD \},
\]
\[
D_i = \{ X \in Ob(\mathcal{C}_{2(n+1)}) \mid w_m = DU, \ 1 \leq m \leq i - 1 \},
\]
for $1 < i \leq n$. Then, we take the subset $D_{i, j} \subseteq D_i$,
\[
D_{i, j} = \{ Y \in D_i \mid i \leq r_Y \leq m(i, j_i) + i - 1 \},
\]
such that the vector $v = (n - (m(i, j_i) + i - 1))_{i=1}^n$ constitutes an integer partition with $n$ parts. Now, let $\mathfrak{N}_v$ be the full subcategory of $\mathcal{C}_{2(n+1)}$ whose objects are $k$–linear combinations of the Dyck paths in the following set
\[
\mathcal{L} = \bigcup_{i=1}^n D_{i, j_i},
\]
and morphisms defined by the category $\mathcal{C}_{2(n+1)}$.

We assume the following numbering and orientation for a quiver $Q$ associated to a Nakayama algebra

\[
\begin{array}{cccccc}
\circ & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} & \circ \\
1 & 2 & \cdots & n-2 & n-1 & n
\end{array}
\]

Figure 6: Quiver $Q$ of type $A_n$.

The functor $\Theta'$ between the category $\mathfrak{N}_v$ and the category of representations of $kQ/I$ where $kQ/I$ is a Nakayama algebra with Kupisch series $[m(1, j_1), \ldots, m(n, j_n)]$ is defined in such a way that, $\Theta'(UWD) = \Theta(UWD)$ and $\Theta'(F) = \Theta(F)$ for $UWD \in \mathcal{L}$ and $F$ being an elementary shift in $\mathfrak{N}_v$.

**Corollary 22.** The functor $\Theta'$ is an equivalence of categories.

**Proof.** It is a direct consequence of Theorem 15. \qed

As an example, Figure 7 shows the Auslander-Reiten quiver of the Nakayama algebra $A = kQ/I$ associated to the quiver $Q$ shown in Figure 4 with $I = \langle x_3x_4, x_1x_2x_3 \rangle$.

\[
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

Figure 7: Auslander-Reiten quiver of rep $kQ/I$. 
5 Cluster Variables Associated to Dyck Paths

In this section, we construct an alphabet associated to Dyck paths. And it is given a formula for cluster variables of cluster algebras associated to Dynkin diagrams of type $\tilde{A}_n$.

5.1 An Alphabet for Dyck Paths

For $n > 2$, let $U^1_i = u_1 \ldots u_{2n}$ and $U^2_i = u'_1 \ldots u'_{2n}$ be Dyck paths in $D_{2n}$ with the following form:

$$u_j = \begin{cases} U, & \text{if } 1 \leq j \leq i + 1 \text{ or } j = 2(i + 1) + k \leq 2n, \\ D, & \text{if } i + 2 \leq j \leq 2(i + 1) \text{ or } j = 2(i + 1 + k) \leq 2n, \end{cases} \quad (20)$$

and

$$u'_j = \begin{cases} U, & \text{if } 2i < j \leq i + n \text{ or } j = 1 + 2k \leq 2i, \\ D, & \text{if } i + n < j \leq 2n \text{ or } j = 2k \leq 2n, \end{cases} \quad (21)$$

for $k > 0$ and $i \leq n - 2$. The alphabet $H_n$ is the union of the set $\{U^r_i \mid r = 1, 2 \text{ and } 1 \leq i \leq n - 2\}$ and the Dyck path with exactly one peak in $D_{2n}$ (denoted by $E_n$). Figure 8 shows the alphabet $H_3$.

Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $n-1$. We fix two different relations of concatenation $f_1$ and $f_2$ over $H_n$ such that

$$f_1(V_i) = \begin{cases} E_n, & \text{if } V_i = E_n \text{ or } V_i = U^1_1, \\ U^i_{2i+1}, & \text{if } V_i = E_n \text{ or } V_i = U^1_i, \\ U^i_{i+1}, & \text{if } V_i = U^2_i, \end{cases} \quad (22)$$

and

$$f_2(V_i) = \begin{cases} E_n, & \text{if } V_i = U^2_1, \\ U^i_{i+1}, & \text{if } V_i = E_n \text{ or } V_i = U^1_i, \\ U^i_{2i+1}, & \text{if } V_i = U^2_i. \end{cases} \quad (23)$$

Then, we take the set of words $V = V_1 \ldots V_{n-1}$ in $H_n^*$ such that

$$V_i = \begin{cases} f_1(V_{i-1}), & \text{if } i \notin \mathcal{C}, \\ f_2(V_{i-1}), & \text{if } i \in \mathcal{C} - \{1, n - 1\}, \end{cases} \quad (24)$$

for $1 < i \leq n - 2$, $n \geq 4$. This set is denoted by $X_{\mathcal{C}}$, in particular case $X_{(1,2)} = H_3$.\[\]
5.2 Dyck Words and Perfect Matchings

Let $G = (G_1, \ldots, G_{n-1})$ be a snake graph, then we can associate to $G$ an admissible subchain $C$ of $n-1$ in the following way:

If $G_{i-1}, G_i$ and $G_{i+1}$ denote tiles of the following snake graph

$$
\begin{array}{c|c|c}
G_{i-1} & G_i & G_{i+1} \\
\hline
\end{array}
$$

then, $i \in C$ for $1 < i < n - 1$. For example, for the snake graph $G$ shown in Figure 9

$$
\begin{array}{c|c}
G_1 & G_5 \\
\hline
G_3 \\
\hline
G_1 & G_2 \\
\end{array}
$$

Figure 9: Snake graph $G$.

it holds that the corresponding admissible subchain is given by the identity $\{1, 3, 5\} = \{i_1, j_1, i_2\} = \{j_1, i_1, j_2\}$. By notation, $G$ can be written as $G_C$.

The following result establishes a relationship between the alphabet $X_C$ and perfect matchings of snake graphs.

**Lemma 23.** Let $C = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $n-1$. Then, there is a bijective correspondence between the set $X_C$ and the perfect matchings of $G_C$.

**Proof.** Let $C$ be an admissible subchain of $n-1$, $X_C$ be a set of words, and $G_C$ be a snake graph associated to $C$. Assume a numbering over the edges of $G_C$ in the following way:

For boundary edges of $G_i$, we have the following four possibilities

$$
\begin{array}{c|c|c}
G_{i-1} & G_i & G_{i+1} \\
\hline
U_1^{i-1} & U_2^{i-1} \\
\hline
\end{array}
$$

$$
\begin{array}{c|c|c}
G_{i-1} & G_i & G_{i+1} \\
\hline
U_1^{i-1} & U_2^{i-1} \\
\hline
\end{array}
$$

$$
\begin{array}{c|c|c}
G_{i+1} & G_i & G_{i-1} \\
\hline
U_1^{i} & U_2^{i} \\
\hline
\end{array}
$$

$$
\begin{array}{c|c|c}
G_{i+1} & G_i & G_{i-1} \\
\hline
U_1^{i} & U_2^{i} \\
\hline
\end{array}
$$
with \(1 < i < n - 1\) (labeling is given by recurrence). The other edges are labeled with the letter \(E_n\). Now, a perfect matching \(P\) of \(G_c\) can be written as a vector \(v = (v_1, \ldots, v_n)\), where each \(v_i\) corresponds to an edge of \(G_c\) (this vector is unique up to permutation). Define a map \(f : X_c \rightarrow \text{Match}(G_c)\) such that \(f(V_1 \ldots V_{n-2}) = (E_n, V_1, \ldots, V_{n-2}, E_n)\). Firstly, we will prove that \(f\) is well defined by induction over \(n\). To start note that for \(n = 3\), we have the following three cases:

(I) If \(V_1 = E_3\), it turns out that \(f(V_1) = (E_3, E_3, E_3)\), which is given by

\[
\begin{array}{|c|c|c|}
\hline
G_1 & G_2 \\
\hline
\end{array}
\]

(II) If \(V_1 = U_1^1\), it holds that \(f(U_1^1) = (E_3, U_1^1, E_3)\), which is equal to

\[
\begin{array}{|c|c|c|}
\hline
G_1 & G_2 \\
\hline
\end{array}
\]

(III) If \(V_1 = U_2^1\), then \(f(U_2^1) = (E_3, U_2^1, E_3)\), which is of the form

\[
\begin{array}{|c|c|c|}
\hline
G_1 & G_2 \\
\hline
\end{array}
\]

Suppose that the result holds for \(n = k\). Let \(n = k + 1\), by hypothesis \((E_{k+1}, V_1, \ldots, V_k)\) are disjoint sets containing all the previous tiles in \(G_c\), then there are two possibilities for \(k\).

(I) for \(k \in \mathbb{C} \setminus \{1, k + 1\}\), we have the following conditions:

(1.1) If \(V_{k-1} = E_{k+1}\), then \(f(V_1 \ldots E_{k+1} E_{k+1}) = (E_{k+1}, V_1, \ldots, E_{k+1}, E_{k+1})\) and \(f(V_1 \ldots E_{k+1} U_1^{k}) = (E_{k+1}, V_1, \ldots, E_{k+1}, U_1^{k}, E_{k+1})\), which are given by

\[
\begin{array}{|c|c|c|}
\hline
G_{k-1} & G_k & G_{k+1} \\
\hline
\end{array}
\]  
\text{and}  
\begin{array}{|c|c|c|}
\hline
G_{k-1} & G_k & G_{k+1} \\
\hline
\end{array}
\]

(1.2) If \(V_{k-1} = U_1^{k-1}\), then \(f(V_1 \ldots U_1^{k-1} E_{k+1}) = (E_{k+1}, V_1, \ldots, U_1^{k-1}, E_{k+1}, E_{k+1})\) and \(f(V_1 \ldots U_1^{k-1} U_2^{k}) = (E_{k+1}, V_1, \ldots, U_1^{k-1}, U_2^{k}, E_{k+1})\), which are equal to

\[
\begin{array}{|c|c|c|}
\hline
G_{k-2} & G_k & G_{k+1} \\
\hline
\end{array}
\]  
\text{and}  
\begin{array}{|c|c|c|}
\hline
G_{k-2} & G_k & G_{k+1} \\
\hline
\end{array}
\]

(1.3) If \(V_{k-1} = U_2^{k-1}\), then \(f(V_1 \ldots U_2^{k-1} U_1^{k}) = (E_{k+1}, V_1, \ldots, U_2^{k-1}, U_1^{k}, E_{k+1})\) which is of the form

\[
\begin{array}{|c|c|c|}
\hline
G_{k-1} & G_k & G_{k+1} \\
\hline
\end{array}
\]

(II) for \(k \notin \mathbb{C}\), there are the following cases:
(2.1) If \( V_{k-1} = E_{k+1} \), then \( f(V_1 \ldots E_{k+1}U_1^k) = (E_{k+1}, V_1, \ldots, E_{k+1}, U_1^k, E_{k+1}) \), which is given by

\[
\begin{array}{c}
G_{i+1} \\
\downarrow \\
G_{i-1} \\
\downarrow \\
G_i
\end{array}
\]

(2.2) If \( V_{k-1} = U_{1}^{k-1} \), then \( f(V_1 \ldots U_{1}^{k-1}U_1^k) = (E_{k+1}, V_1, \ldots, U_{1}^{k-1}, U_1^k, E_{k+1}) \), which is equal to

\[
\begin{array}{c}
G_{i+1} \\
\downarrow \\
G_{i-1} \\
\downarrow \\
G_i
\end{array}
\]

(2.3) If \( V_{k-1} = U_{2}^{k-1} \), then \( f(V_1 \ldots U_{2}^{k-1}E_{k+1}) = (E_{k+1}, V_1, \ldots, U_{2}^{k-1}, E_{k+1}, E_{k+1}) \) and \( f(V_1 \ldots U_{2}^{k-1}U_{2}^k) = (E_{k+1}, V_1, \ldots, U_{2}^{k-1}, U_{2}^k, E_{k+1}) \), which are of the form

\[
\begin{array}{c}
G_{i+1} \\
\downarrow \\
G_{i-1} \\
\downarrow \\
G_i
\end{array}
\quad \text{and} \quad
\begin{array}{c}
G_{i+1} \\
\downarrow \\
G_{i-1} \\
\downarrow \\
G_i
\end{array}
\]

Dual arguments prove the result for the other labelings. We also note that by definition map \( f \) is injective and surjective.

\[\square\]

**Remark 24.** Each perfect matching of \( G_{\mathcal{C}} \) is in correspondence with just only one object of the \( \mathcal{A}_{n-1} \)-Dyck paths category associated to the admissible subchain \( \mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\} \).

For each Dyck path \( Y = y_1 \ldots y_{2n} \) with \( n - 1 \) peaks, we construct a family of words \( Y \cap X_\mathcal{C} \in H_n^* \) such that;

\[ Y \cap X_\mathcal{C} = \{ Y \cap V^z \mid V^z \in X_\mathcal{C} \} \quad \text{ (25)} \]

where

\[ Y \cap V^z = \begin{cases} V^z & \text{if there exists } j \text{ such that } y_j = v_j^z \text{ for } 1 < j < 2n, \\ E_n & \text{otherwise}, \end{cases} \quad \text{ (26)} \]

with \( V^z = v_1^z \ldots v_{2n}^z \) in \( X_\mathcal{C} \). For the set \( Y \cap X_\mathcal{C} \), it can be defined a relation \( \sim \) such that

\[ Y \cap V^{z_1} \sim Y \cap V^{z_2} \text{ if and only if } Y \cap V^{z_1} \text{ and } Y \cap V^{z_2} \text{ are the same word.} \quad \text{ (27)} \]

In this case, \( \sim \) is an equivalence relation and \( (Y \cap X_\mathcal{C}) / \sim \) is denoted by \( [Y \cap X_\mathcal{C}] \). Also, we remind that a Dyck path \( Y \) can be written as the word \( UW \text{D} = Uw_1 \ldots w_{n-1} \text{D} \), where \( y_1 = U, y_{2n} = \text{D} \) and, \( w_i = y_{2i}, y_{2i+1} \).
Lemma 25. Let \( C = \{i_1, \ldots, i_k, j_1, \ldots, j_m\} \) be an admissible subchain of \( n-1 \) and let \( Y \) a Dyck path of length \( 2n \) with exactly \( n - 1 \) peaks. Then, there is a bijective correspondence between the set \( [Y \cap X_C] \) and the set of perfect matchings of the snake graph belonging to \( G_C \) and induced by the words \( w_t = UD \) in \( Y \).

Proof. Let \( C \) be an admissible subchain of \( n-1 \) and \( Y = UWD \) a Dyck path in \( S \), then by Proposition 5 there are \( l, r \in \mathbb{Z}_{>0} \) with \( 1 \leq l \leq r \leq n - 1 \) such that \( w_t = UD \) for \( l \leq t \leq r \) and \( w_t = DU \) otherwise. Now, let \( G_{t[l, r]} = G[l, r] \) be a snake graph belonging to \( G_C \) induced by \( Y \).

Define a map \( g : [Y \cap X_C] \to \text{Match}(G_{t[l, r]}) \) such that:

(I) If \( 1 < l \leq r < n - 1 \), then \( g([Y \cap V^i]) = g(E_n \ldots E_{i-1}V^i_{r-1} \ldots V^i_1E_n \ldots E_n) = (V^i_{r-1}, \ldots, V^i_1) \).

(II) If \( l = 1 \) and \( 1 \leq l \leq r \leq n - 1 \), then \( g([Y \cap V^i]) = g(V^i_n \ldots V^i_1E_n \ldots E_n) = (E_n, V^i_1, \ldots, V^i_n) \).

(III) If \( r = n - 1 \) and \( 1 \leq l \leq r = n - 1 \), then \( g([Y \cap V^i]) = g(E_n \ldots E_{n-1}V^i_{n-2} \ldots V^i_1E_n) = (V^i_{n-2}, \ldots, V^i_1, V^i_n, E_n) \).

(IV) If \( l = 1 \) and \( r = n - 1 \), then \( g = f \).

Since in the four cases \( g \) is a restriction of \( f \). It follows that \( g \) is a bijection as a consequence of Lemma 23.

5.3 Cluster Variables Formula Based on Dyck Paths Categories

In this section, Dyck paths categories are used to give a formula for cluster variables of cluster algebras of Dynkin type \( A_n \), to do that, we use the category of Dyck paths associated to an admissible subchain. We also present a connection between cluster variables of algebras of type \( A_n-1 \) and Dyck paths with \( n - 1 \) peaks.

Let \( C = \{i_1, \ldots, i_k, j_1, \ldots, j_m\} \) be an admissible subchain of \( n-1 \) and let \( Y = UWD \) be a Dyck path in \( S \), then we define the monomials

\[
\eta_Y = \prod_{UD = w_t \in Y} x_t, \quad (28)
\]

and

\[
X_Y = \prod_{m \in M_Y} x_m, \quad (29)
\]

with \( M_Y \) being the set of indices \( m \) such that

\[
m = \begin{cases} 
  i + 1, & \text{if } U^i_1 \in V, \\
  i, & \text{if } U^i_2 \in V, \\
  0, & \text{if } E_n \in V.
\end{cases} \quad (30)
\]

\( V \in [Y \cap X_C] \). For this case \( x_0 = 1 \).

The following theorem gives the cluster variable associated to a Dyck path in the set \( S \) and its connection with cluster algebras of type \( A_{n-1} \).
Theorem 26. Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $\mathcal{C}_n$, $Y = UWD$ a Dyck path with $n-1$ peaks and $M$ the set of all cluster variables of a cluster algebra of type $A_n$ with $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_m\}$ the sets of sinks and sources, respectively. Then:

(i) The cluster variable associated to $Y$ in the category $\mathcal{C}_{2n}$ is given by

$$X_Y = (\eta_Y)^{-1} \left( \sum_{V \in [Y \cap \mathcal{C}_e]} X_V \right).$$

(ii) There exists a bijective correspondence between Dyck paths with $n-1$ peaks and the set $M \setminus x_0$ with $x_0$ the initial seed.

Proof. Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $\mathcal{C}_n$, and let $T_\mathcal{C}$ be the triangulation of the polygon with $n+2$ vertices given by $\mathcal{C}$

Let $\alpha_{l,r}$ be a diagonal that is not in $T_\mathcal{C}$ that cuts the diagonals $\alpha_1, \ldots, \alpha_{r} \in T_\mathcal{C}$. We define a functor $\chi : C_{T_\mathcal{C}} \rightarrow \mathcal{C}_{2n}$ such that $\chi(\alpha_{l,r}) = UW_{l,r}D$, where

$$w_{ij} = \begin{cases} UD, & \text{if } l \leq j \leq r, \\ DU, & \text{otherwise}, \end{cases}$$

and for any pivoting elementary move $E : \alpha_{l,r} \rightarrow \alpha'_{l',r'}$. $\chi(E)$ is the elementary shift $F = f_{l_1} \circ \cdots \circ f_{l_k}$ from $UW_{l,r}D$ to $UW_{l',r'}D$. Theorems 2 and 15 allow us to establish the following sequence of equivalences:

$$C_{T_\mathcal{C}} \simeq \text{Mod } Q_{T_\mathcal{C}} \simeq \mathcal{C}_{2n},$$

therefore $\chi$ is a categorical equivalence. Thus,

(i) Functor $\chi$ and Lemma 20 allow to establish that $x_\gamma = X_Y$.

(ii) The map $\psi : S \rightarrow M \setminus x_0$ such that $\psi(Y) = X_Y$ is a bijection as a consequence of Theorem 11 and the definition of functor $\chi$. We are done. \qed

For instance, let $\mathcal{C} = \{j_1 = 1, i_1 = 2, j_2 = 4\}$ be an admissible subchain of $\mathcal{C}_4$, the set $X_\mathcal{C}$ is in correspondence with the objects of $\mathcal{C}_{10}$ shown in Figure 11.

Then, for $Y = UDUUDDUDDUD$, we define the set $Y \cap X_\mathcal{C}$ such that

$$[Y \cap X_\mathcal{C}] = \{E_5E_5U_1^3, E_5U_2^2E_5, U_2^2U_1^2U_1^3\}.$$
Thus, identities (28), (29) and (30) define the polynomials

\[ \eta_Y = x_2 x_3, \quad X_{E_5 E_5 U_1^3} = x_0 x_0 x_4, \quad X_{E_5 U_2^3 E_5} = x_0 x_2 x_0, \quad X_{U_1 U_2 U_3} = x_3 x_1 x_4, \]

therefore, the cluster variable associated to the Dyck path \( Y \) is given by the expression

\[ X_Y = \frac{x_4 + x_2 + x_3 x_1 x_4}{x_2 x_3}. \]

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