Rational hyperbolic discounting

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Abstract

This paper shows that the $q$-exponential function rationally evaluate the time discounting. When we consider two processes of wealth accumulation with different frequencies, then the discount rate and the relative frequency between them are essentials to choose the best process. In this context, the exponential discounting is a particular case, where one of the processes has a much higher frequency related to the other. In addition, one can note that some behaviors observed empirically in decision makers, such as subadditivity, magnitude effect, and preference reversal, are consistent with processes which have a low relative frequency.

Keywords: hyperbolic discounting, $q$-exponential, discounted utility model

1. Introduction

The expected utility theory has axioms that define a rational decision maker [1], but experimental observations showed behaviors that violated them [2]. One of the most well documented empirical observation of theses behaviors is the hyperbolic discounting [3]. When mathematical functions are explicitly fitted to experiment data, then a hyperbolic shape fits the data better than the exponential form [4, 5, 6, 7]. However, the $q$-exponential function

$$e_q^{-\rho n} \equiv [1 - (1 - q)\rho n]^{\frac{1}{1-q}}$$

allows greater flexibility of fit for hyperbolic, exponential, and quasi-hyperbolic discounting [8, 9, 10, 11].

Although hyperbolic discounting provide a convenient way to model behavior, such design is not theoretically explained. So, in this paper is used the proposed method in [12] to explore it from a rational perspective, where
the averaging wealth growth is evaluated over time without the need for a utility function, only a dynamic is specified to characterize the growth.

To compute the time average, it should be considered that the market has various strategies and processes interacting [13]. In this context, it is shown here that the $q$-exponential discounting is a rational solution to the intertemporal choice problem, when two processes have different outcomes and time probabilities. More specifically, consider two processes which allow “receiving $m$” with probability $s_m$ and “receiving $M$” with probability $s_M$, where $M > m$ and both rewards are received after a brief period (a day or an hour). Then, the delaying of the amount $M$ in a longer time horizon ($n$ brief periods later) takes us to the discount function $e^{-\rho n q}$, where $\rho$ is the discount rate and $\frac{1}{q-1} = \frac{s_m}{s_M}$ is a relative frequency between these processes.

The relative frequency $s_m/s_M$ usually tends to infinity when, in a short period, receiving a large reward $M$ is much more unlikely than a small reward $m$ ($s_M \ll s_m$). The reason for this is the existence of a consensus in the market that large amounts are more arduous than small amounts. For example, consider the possibilities to earn 1 dollar and 1 million dollars selling candies tomorrow, where the probabilities of success are $s_m \approx 1$ and $s_M \approx 0$, respectively. In this case, $s_m/s_M \to \infty$ and, consequently, $q$ tends to 1 by right side to result in the exponential discount function.

In contrast, a subadditive discounting arises when the big reward is as likely as the small reward or even more likely than it. For example, consider that yesterday you won a lottery prize and heard this good news today. Then, earning 10 dollars selling candies and receiving the 1 million dollars prize have great chances in the near future. Thus, the relative frequency between these processes $s_m/s_M$ is reduced ($q > 1$) and the function $e^{-\rho m}$ discounts slowly over time. Consequently, if discounts are calculated at a frequency greater than the relative frequency between the processes, then, due to subadditivity, short delays are more discounted than long delays, and small rewards are more discounted than large rewards. In addition, in some cases, where frequencies and rewards are different, the delaying of both rewards can reverse the performance of the wealth accumulation process, justifying the preference reversal behavior.

2. Time preference problem

Time preference is the valuation placed on receiving a good on a short date compared with receiving it on a farther date. A typical situation is
choosing to receive a monetary amount \( m \) after a brief period (a day or an hour) or to receive \( M > m \) in a distant time (after some months or years).

Formally, given two hypotheses about the receipt of amounts in different instants in the future, \( \Theta_m = \text{"I receive } m \text{"} \) and \( \Theta_M = \text{"I receive } M \text{"} \), then only one of them must be chosen. The hypothesis \( \Theta_m \) represents the receipt of the amount \( m \) in short period, \( t_m = t_0 + \delta t \), and \( \Theta_M \) represents the receipt of the amount \( M \) in longer time horizons, \( t_M = t_0 + \Delta t \).

We should point out that each hypothesis has an impact on the individual's wealth. If an individual has a wealth \( W_0 \) at present, then the proposition \( \Theta_M \) has the change factor \( 1 + X_M \), while \( \Theta_m \) has the change factor \( 1 + X_m \), where each rate of change factor is calculated as follows:

\[
X_m = \frac{m}{W_0}, \quad (1)
\]
\[
X_M = \frac{M}{W_0}. \quad (2)
\]

Moreover, hypotheses \( \theta_m \) and \( \theta_M \) can be proposed with more modest change factor, \( 1 + x_m < 1 + X_m \) and \( 1 + x_M < 1 + X_M \). In these cases, we can express them by \( \theta_m = \text{"I receive an amount less than } m \text{"} \) and \( \theta_M = \text{"I receive an amount less than } M \text{"} \).

3. Similar statements in the future

A hypothesis is a proposition (or a group of propositions) provisionally anticipated as an explanation of facts, behaviors, or natural phenomena that must be later verified by deduction or experience. Formally, they should always be spoken or written in the present tense because they are referring to the research being conducted. However, in everyday language, hypotheses are guesses for decision-making before the facts are verified. They express the performance of stochastic processes with some certain sense.

The sense of certainty that we have about a hypothesis becomes evident when we express it in the form of future statement. Given that a future sentence indicates an event occurrence after the moment of speech, then it can emphasize the fact realization with an intuitive implicit probability. For instance, consider the following future statements for \( \Theta_M \) and \( \theta_M \):

\( F\Theta_M = \text{"It will at some time be the case that I receive } M \text{";} \)
\( G\theta_M = \text{"It will always be the case that I receive an amount less than } M \text{";} \)
The statement $G\theta_M$ expresses certainty with the modifier "always", while $F\Theta_M$ expresses uncertainty about the instant that the action occurs through the modifier "some time". In temporal logic $F\Theta_M$ is equivalent to saying which there is an instant $t$ in the future where $\Theta_M$ is true, i.e., $\exists t$ such that $(\text{now} < t) \land \Theta_M(t)$, where $\Theta_M(t)$ means that $\Theta_M$ is true at instant $t$. Meanwhile, $G\theta_M$ affirms that $\theta_M$ is always true in the future, $\theta_M(t) \forall t > \text{now}$. [4]

About the sense of certainty, $F$ is a weak operator because $\Theta$ is true only once in the future, while $G$ is a strong operator because it is true in all future periods. If $\Theta_M$ does not always come true, then we can investigate its sense of certainty through the affirmative:

$$GF\Theta_M = \text{“I will frequently receive M”}.$$ 

The future tense $GF\Theta_M$ suggest that the proposition $\Theta_M$ may alternate its logical value several times in the future, but there will always uncertainty about the logic value in any future instant. The adverb "frequently" estimates the frequency at which it is true.

Can two hypotheses with different probabilities and rewards perform similar when repeated indefinitely? The frequencies in which the propositions $\Theta_M$ and $\theta_M$ represent gain are different. In addition, the changes proposed by them are different too. After all, the affirmative $GF\Theta_M$ communicates that the individual will frequently receive $M$ (change factor $1 + X_M$). On the other hand, $G\theta_M$ proposes a small change factor, $1 + x_M$, but continuously over time. Thus, in order to solve this problem, let us consider $\tau(t)$ as the total time where $\Theta_M$ is true and $t$ as the total time where $\theta_M$ is true. Then, the same performance is achieved over time when $(1 + x_M)^t = (1 + X_M)^{\tau(t)}$. If $t$ is also the number of periods under observation, then $\Theta_M$ is true with a frequency given by

$$\lim_{t \to \infty} \frac{\tau(t)}{t} = p. \quad (3)$$

Therefore, the relation between time averages of change factors, $1 + x_M = (1 + X_M)^p$, indicates that the sentences $GF\Theta_M$ and $G\theta_M$ have similar goals in the long run, although $\Theta_M$ and $\theta_M$ have different outcomes and frequencies. This similarity is denoted in this work by

$$GF\Theta_M \sim G\theta_M. \quad (4)$$
We can have a stationary probability when \( t \) is big enough, but the individuals do not have as calculate this probability in practice. In addition, the chances of achieving outcomes in market processes change over time. For example, the probability of receiving a premium changes abruptly after the draw. An alternative to solve this conceptual problem is to replace the probability \( p \) by the sense of certainty \( S_M \), which is an imprecise suggestion (intuition) for the time probability. Figure 1 presents some adverbs of frequency that can suggest the sense of certainty in future tenses.

\[
\begin{array}{ccc}
G & \text{always} & s = 1 \\
 & \text{usually} & s > 1/2 \\
 & \text{frequently} & s > 1/2 \\
 & \text{often} & s = 1/2 \\
GF & \text{sometimes} & s = 1/2 \\
 & \text{occasionally} & s > 1/2 \\
 & \text{rarely} & s < 1/2 \\
 & \text{seldom} & s < 1/2 \\
 & \text{hardly ever} & s = 0 \\
\end{array}
\]

Figure 1: Adverbs of frequency that can suggest the senses of certainty, \( s \). The quantifier “always” indicates certainty, while “never” indicates impossibility. The other adverbs of frequency express uncertainty about the future, \( 0 < s < 1 \).

The axiomatic system of temporal logic proposes that \( G\theta_M \Rightarrow GN\theta_M \), where \( N\theta_M \) stands for \( \theta_M \) is true at the next instant. Therefore, the similarity \( GF\Theta_M \sim G\theta_M \) can be written by

\[
F\Theta_M \sim N\theta_M, \quad (5)
\]

when

\[
1 + x_M \approx (1 + X_M)^{S_M}. \quad (6)
\]

Thus, the statement “It will at some time be the case that I receive \( M \)”, which have a change factor \( 1 + X_M \), is similar to the statement “It will be case that I receive an amount less than \( M \) at the next moment”, which has a change factor \( (1 + X_M)^{S_M} \). Analogously, we have \( F\Theta_m \sim N\theta_m \), if

\[
(1 + x_m) \approx (1 + X_m)^{s_m}, \quad (7)
\]

where \( s_m \) is the sense of certainty concerning the receipt of \( m \) on a short date.
The above procedure, which leads to the equations 6 and 7, is analogous to linguistic meiosis, where the meaning of something is reduced to simultaneously increase something else in its place. In the above mentioned case, proposing $N\theta_M$ (receive an amount less than $M$ at the next moment) suggest greater certainty, because it makes the process more feasible. This procedure is consistent with argumentative suggestions of the economist Eugen von Böhm-Bawerk, where individuals suffer from a systematic tendency to underestimate future wants [15, 3].

4. Hyperbolic discounting

Now suppose, without loss of generality, that $M$ is large enough so that the individual prefers to receive it in the distant future. If we consider $n = \Delta t/\delta t$ periods, where $n$ is the number of attempts to perform $1 + x_m$ until $M$’s receipt date, then we have

$$(1 + x_M) > (1 + x_m)^n. \quad (8)$$

Substitute the expressions 6 and 7 in above expression. Next, substitute the expressions 1 and 2 to obtain

$$\left(1 + \frac{M}{W_0}\right)^{S_M} > \left(1 + \frac{m}{W_0}\right)^{n s_m}. \quad (9)$$

In this case, the rational judgment of the “or” operation between change goals is indicated by the maximum rate of change factor,

$$x_M = \max \left\{x_M, (1 + x_m)^n - 1\right\}$$

$$= \left(1 + \frac{M}{W_0}\right)^{S_M} - 1. \quad (10)$$

In general the time preference solution is presented through a discount function. In order to use this strategy, it is necessary to develop the same form on both sides of the inequality 9. For this, there is a value $\kappa$ such that $\kappa S_M > m$, where we can write

$$\left(1 + \frac{M}{W_0}\right)^{S_M} = \left(1 + \frac{\kappa S_M}{W_0}\right)^{n s_m} > \left(1 + \frac{m}{W_0}\right)^{n s_m}. \quad (11)$$
The discount function undoes the proposed future change in $\Theta_M$, that is,

$$
\frac{1}{\left(1 + \frac{M}{W_0}\right)} = \left(1 + \frac{\kappa S_M}{W_0}\right)^{-\frac{s_m}{s_M} n}.
$$

If $S_M$ is the sense of certainty to receive $M$ on a farther date and $s_M$ is the sense of certainty concerning the receipt of $M$ on a short date (same date to receive $m$), then multiplicative growth provides $S_M = ns_M$ that result in

$$
\frac{1}{\left(1 + \frac{M}{W_0}\right)} = \left(1 + \frac{\kappa S_M}{W_0}n\right)^{-\frac{s_m}{s_M}}.
$$

Thus, it is possible to know the value $M$ to be discounted in any $n$ periods.

Equation 13 describes the hyperbolic discounting $(1 - h \rho)^{\frac{1}{\hat{h}}}$ [8], or $q$-exponential discounting [10], when we reparametrize it by doing

$$
\frac{1}{\hat{h}} = -\frac{s_m}{S_M} n = -\frac{s_m}{s_M},
$$

$$
\rho = \frac{\kappa s_m}{W_0}.
$$

The discount rate $\rho$ is influenced by individual states of scarcity and abundance of goods. For instance, let us consider an individual called Bob. If an object is scarce for him (small $W_0$), then he places a higher preference (great $\rho$). Analogously, if $W_0$ represents an abundance state for him, he has a lower preference (small $\rho$). Therefore, the parameter $W_0$ may cause great variability in experiments to find $\rho$ because the wealth distribution follows the power law [16, 17, 18, 19]. In other words, the wealth $W_0$ can vary abruptly from one individual to another and, consequently, $\rho$ varies too. In Thaler’s experiment, he reported the median responses because there was wide variation among subjects [20]. Moreover, in [3] was presented a tremendous variability in the estimates of literature.

The ratio $h = -s_M / s_m$ influences the shape of the curve over time, where $-1/h = 1/(q - 1)$ is the relative frequency between the wealth accumulation processes. In order to understand this relation, we must remember that the sense of certainty is an approximation (or intuition) of time probabilities, which are occurrence intuitive frequencies of an event in a hypothesis. Therefore, the ratio $s_m / s_M$ communicates how feasible in time the hypothesis $\Theta_m$
is in relation to $\Theta_M$. For example, $e^{-\rho n}$ is equal to the exponential function $e^{-\rho n}$ when $h \to 0^-$. This means that the objective $M$ is very unlikely in a short period ($s_M \ll s_m$), but the process to reach it with a lower amount has high occurrence frequency, $s_m/s_M \to \infty$. Thus, $M$ can only be achieved in the long run with persistent attempts to receive on average $\kappa s_m$ in short periods (See the dashed black curve in Figure 2).

On the other hand, would it be impossible for a large reward to be more likely than a small reward? Such opportunities are rare, but when they arise, they can abruptly change the prospect of wealth growth. For example, consider that yesterday you won a lottery prize and heard this good news today. So, earning $300 selling candy and receiving $1 million has a great chance of success tomorrow. Hence, receiving $1 million has low frequency relative to usual market processes, such as selling candies, $h < 0$. Therefore, in order to achieve a goal after a delay, it will be necessary to compensate the low frequency of occurrences with large rewards in the short term.

The behavior of compensate the low relative frequency with larger rewards can be observed in Figure 2, where we have different iterations of processes to reach the discount 0.5429 after 120 periods. Since the exponential discount

![Figure 2: Discount function $e^{-\rho n}$ versus number of delayed periods: the dashed black curve is the exponential discounting for $\rho = 0.00509$; the o-blue curve is quasi-hyperbolic discounting for $\rho = 5$ and $q = 16$; and the *-red and cyan curves are hyperbolic discounting for $\rho_1 = 0.0099$ and $q_1 = 3$, and $\rho_2 = 0.007$ and $q_2 = 2$, respectively. All these discount functions are approximately equal to 0.5429 in $n = 120$.](image-url)
consists of a process with high relative frequency, then it is possible to reach
the goal with small short-term rewards (low discount rate, \( \rho = 0.00509 \)).
However, other market processes need greater rewards in the short run to
reach the goal when the relative frequency falls. The most extreme case is
the quasi-hyperbolic discount (blue \( \circ \)-curve), which requires a large short-
term reward (\( \rho = 5 \)), but it has the weakest discount after 120 periods, due
to its low relative frequency, \( q = 16 \).

5. Discussion

If the relative frequency is low, then the \( q \)-exponential discounting has
the subadditivity as a mandatory property in the time preference for positive
rewards. The reason for this is \( 1 - q = h = -\frac{\Delta t}{t_m} \) always negative. On this
account, if \( h < 0 \) (\( q > 1 \)), then we have

\[
 e^{-x} e^{-y} = e^{-x-y+(1-q)xy} < e^{-x-y} \quad \forall \ x, y > 0.
\]

This property can be found in experiments of time preference. Daniel Read
pointed out that a common evidence in time preference is the “subadditive
discounting”, in others words, the discounting over a delay is greater when
it is divided into subintervals than when it is left undivided \[21\].

The \( q \)-exponential discounting has decreasing rate over time when a re-
ward is discounted through calculus of continuous compound interest. In
essence, we are compounding continuously when the frequency is increased
to its limit. Hence, if the continuously compounded interest rate is computed
over a process of low relative frequency, \(-\frac{1}{n} \ln e^{-\rho n} \), then this rate decreases
when \( n \) increases. In order to verify this phenomenon as a consequence of
subadditivity, we can develop

\[
 (e_q^{-\rho})^n \leq e_q^{-\rho n} \quad \Rightarrow \quad -\frac{1}{n} \ln (e_q^{-\rho})^n \geq -\frac{1}{n} \ln e_q^{-\rho n}.
\]

If \( h = 1 - q \) does not tend to zero from the left side, then the average discount
rate over shorter time are observed higher than the average discount rate over
longer time horizons, i.e., the average monotonically calculates the expression

\[
 \langle - \ln e_q^{-\rho} \rangle > \left( -\frac{1}{n} \ln e_q^{-\rho n} \right).
\]

This means that if we discount processes with low relative frequency, as if
they have high relative frequency, then the experimental result may draw an
graphic time effect, where the discount rate decreases with the time period to be waited.

In intertemporal choice experiments, a similar “time effect” can be observed as behavior. For example, it was asked to respondents in [20], how much money they would require to make waiting one month, one year and ten years just as attractive as getting the prize of $250 now. The median responses (US $300, US $400 and US $1000) had an continuously compounded annual discount rate of 219% over the one month, 120% over the one year and 19% over the ten years. Other experiments presented a similar pattern as discussed in [3].

In a second behavior, referred to as “magnitude effect”, individuals discount small values more than large values. Here, this effect may also be a consequence of subadditivity. The reason for this is because the magnitude and time effects are mathematically similar in the \( q \)-exponential discounting. More specifically, the discount rate is \( \rho = \kappa s_m / W_0 \) and \( \kappa \) is growing for values of \( M \) (see equation 11). In order to understand the similarity, note that the function \( e^{-\rho n q} \) varies with \( n \rho \). If we fix the value \( n \), for example \( n = 1 \), and we vary only rate \( \rho = r \rho_0 \), where \( \rho_0 \) is constant and \( r > 1 \) is a multiplier which is growing for values of \( M \), then we have the function \( e^{-r \rho_0} \) analogous to \( e^{-\rho n} \). Therefore, the magnitude effect made by \( r \), similarly to the time effect, results in

\[
\langle -\ln e^{-\rho_0} \rangle \geq \left\langle -\frac{1}{r} \ln e^{-r \rho_0} \right\rangle.
\]

The magnitude effect is observed in several intertemporal choice experiments. For example, in Thaler’s investigation [20], the respondents preferred, on average, $30 in 3 months rather than $15 now, $300 in 3 months rather than $250 now, and $3500 in 3 months rather than $3000 now, where the continuously compounded discount rates are 277%, 73% and 62%, respectively. Other experiments have found similar results as discussed in [3].

Another observed behavior in experiments is the “preference reversal”. Initially, when individuals are asked to choose between one apple today and two apples tomorrow, then they may be tempted to prefer only one apple today. However, when the same options are long delayed, for example, choosing between one apple in one year and two apples in one year plus one day, then to add one day to receive two apples becomes acceptable [20].

When we evaluate hypotheses as processes, the delay of rewards changes its performance because the frequency at which it is true is altered. For
example, consider two market processes, where the hypotheses $\theta = \text{“I receive } M\text{”}$ and $\Theta = \text{“I receive } 2M\text{”}$ are evaluated over time. The future tense $F\theta(1) = \text{“It will be case that I receive } M\text{ today”}$ is easier than $F\Theta(1) = \text{“It will be case that I receive } 2M\text{ today”}$. Therefore, if we assume that $F\theta(1)$ is a certainty event, $F\theta(1) = N\theta$, then it is reasonable to assume that there is some uncertainty in $F\Theta(1)$. However, $\theta$ and $\Theta$ are different processes. Thus, if we assume that the uncertainty in $\Theta$ is low, so that receiving $2M$ today is preferable, then

$$
\left(1 + \frac{M}{W_0}\right) < \left(1 + \frac{2M}{W_0}\right)^s, \text{ where } 0 < s < 1.
$$

Now, consider delaying $2M$ until tomorrow. When the waiting time doubles to receive $2M$, $F\Theta(2) = \text{“It will be case that I receive } 2M\text{ tomorrow”}$, then the frequency at which the hypothesis $\Theta$ is true is halved. Therefore, even if $s = 1$, receiving $M$ today is preferable because

$$
\left(1 + \frac{M}{W_0}\right) > \left(1 + \frac{2M}{W_0}\right)^{\frac{s}{2}}.
$$

For this reason, delaying a reward worsens the hypothesis. So, what happens to performing the hypotheses, if we delay the rewards for different dates? An example is considering the future tenses $F\theta(n) = \text{“It will be case that I receive } M\text{ in } n\text{ days”}$ and $F\Theta(n + 1) = \text{“It will be case that I receive } 2M\text{ in } n + 1\text{ days”}$. Since the waiting time to receive $M$ is shorter, then we can realize that the number of trials to receive $M$ will be greater in the future ($(n+1)/n$ trials to receive $M$ for each trial to receive $2M$). Thus, the rational choice between $\theta$ and $\Theta$ depends on the following result:

$$
\max \left\{\left(1 + \frac{M}{W_0}\right)^{\frac{n+1}{n}}, \left(1 + \frac{2M}{W_0}\right)^s\right\}.
$$

When $n = 1$, then it will be preferable to receive the reward $M$ (see equation 19). Depending on the value $s$, this can also happen to other small values of $n$, for example, $n$ equals 2 or 3. However, when $n$ is large enough, the relation $(n+1)/n$ tends to 1 and makes $2M$ a preferable reward (see equation 18). Thus, the performances between the hypotheses reverse when the rewards are shifted to a longer time horizon. Experimental results with similar behaviors can be observed in humans [22, 23, 24, 25] and pigeons [26, 27].
The nonextensive statistical economics, based on Tsallis entropy [28], provides description of asymptotic power laws [29]. In $q$-exponential discounting, to deduce $q = 1 + \frac{s_M}{s_m}$ considering multiplicative dynamic for rational decision-makers, without any consideration of nonlinear time-perception, directly shows as nonergodicity causes nonextensivity. Therefore, the relative frequency between market processes, besides the discount rates, is relevant to understand physical and psychophysical aspects of wealth growth.

6. Conclusion

The problem of choice between two processes with different outcomes and frequencies is a reality within the complexity of the market. In this context, this paper shows that the $q$-exponential discounting is a rational solution for this problem, where the discount rate $\rho$ and the relative frequencies between two processes, $\frac{s_m}{s_M} = \frac{1}{q-1}$, are essential parameters to calculate the discounting.

In usual situations in the market, large amounts are more arduous than small amounts. Consequently, the relative frequency between two processes is usually very high, because, in a short period, receiving a large reward $M$ is much more unlikely than a small reward $m$ ($s_M \ll s_m$). For this reason, the continuous compound interest, based on the exponential discounting, is commonly used as an approximation.

On the other hand, a subadditive discounting may arise when the big reward is as likely as the small reward or even more likely than it. On this account, the relative frequency between processes ($q > 1$) is reduced and the function $e^{-\rho n}$ discounts slowly over time. Hence, if discounts are calculated at a frequency greater than the relative frequency between the processes, then mathematical aspects of nonextensivity are manifested, resulting in short delays more discounted than long delays, and small rewards more discounted than large rewards. In addition, in some cases, where frequencies and rewards are different, the delaying of both rewards can reverse the performance of the wealth accumulation process, justifying the preference reversal behavior.

The multiplicative dynamic, which result in nonergodicity of wealth stochastic growth processes, exhibit aspects of nonextensive statistical mechanics. Assuming rational decision makers, without any consideration of nonlinear time-perception, it is possible to deduce a $q$-exponential function to discount future amounts. Moreover, it is possible to show that some behaviors of hyperbolic discounting are consistent with nonextensive aspects of wealth
growth. Therefore, behavioral finances should take into account the relative frequency between market processes, besides the discount rates, because these factors are relevant in rational decision-making.

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