Infinite Abelian Subalgebras
in Quantum W-Algebras:
An Elementary Proof

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Abstract

An elementary proof is given for the existence of infinite dimensional abelian subalgebras in quantum W-algebras. In suitable realizations these subalgebras define the conserved charges of various quantum integrable systems. We consider all principle W-algebras associated with the simple Lie algebras. The proof is based on the more general result that for a class of vertex operators the quantum operators are related to their classical counterparts by an equivalence transformation.
1. Introduction

For each simple Lie algebra $g$ there is an associated principle $W$-algebra $W(g)$, defined in its free field realization through the intersection of the kernels of screening operators. On the classical level the structure of these $W$-algebras can be investigated by various techniques – most of which, however, fail to have counterparts in the quantum theory. In particular, the mere construction of a quantum $W$-algebra requires novel techniques\[6, 4, 2\]. Conceptually, their existence can be understood by treating the $W$-algebra as a vertex operator algebra whose state space appears as the only non-vanishing cohomology class of a Fock space resolution. Either directly \[4\] (for $sl(n)$) or with reference to the classical limit \[2\] this proves the existence of a quantum $W$-algebra $W(g)$ associated with each simple finite dimensional Lie algebra $g$.

An important feature of these quantum $W(g)$-algebras is the conjectured existence of infinite dimensional abelian subalgebras $I(\hat{g})$ labeled by an affine Lie algebra $\hat{g}$ associated with $g$. For $\hat{g} = \hat{sl}(2)$ this conjecture originally arose in A.B. Zamolodchikov’s program of perturbing a conformal field theory to obtain integrable massive quantum field theories; the generators of the abelian subalgebra playing the role of the conserved charges, which enforce the factorized scattering theory. In suitable realizations the generators of $I(\hat{g})$ can be constructed in terms of rank($g$) scalar fields, which may be free or interacting. In the latter case the generators play the role of the conserved charges in various quantum integrable systems (affine Toda, mKDV, KdV). The conserved charges can be characterized as integrals over differential polynomials in the scalar fields which lie in the kernel of some operator $ad Q_0$ – the conjecture being that this kernel is infinite dimensional.

The proof of this conjecture turned out to be surprisingly difficult. Again a simple transcription of classical techniques fails. In \[3\] a constructive proof was given for the low members of the $A_r^{(1)}$ series. Conceptually one expects that a suitable extension of the cohomological construction of quantum $W$-algebras would also allow one to characterize their infinite abelian subalgebras. The correct complex to achieve this was identified by B. Feigin and E. Frenkel \[1\] as a spectral sequence of Fock spaces whose cohomologies are isomorphic to the exterior algebra $\Lambda^*(\hat{a}^*)$ of the dual space to the principle commutative subalgebra $\hat{a}$ of $\hat{n}_+$. The proof is then based on the stability of the Euler characteristic of the complex under deformations of a parameter. Because of the alternating sum, however, the argument goes through only for affine Lie algebras whose exponents and Coxeter number are odd and even, respectively\[10\]. This excludes $A_n^{(1)}$, $n > 1$, $D_n^{(1)}$, $n > 1$, $E_6^{(1)}$ and $E_7^{(1)}$. Below we will give a proof by elementary techniques which is valid for all affine Lie algebras. In particular this covers the missing cases. The proof rests on the more general result that the operator $Q_0$ is related to its classical counterpart $Q_0^{(0)}$ by an equivalence transformation.
2. $W(g)$-modules and their classical limit

2.1. Let $g$ be a simple finite dimensional Lie algebra of rank $r$, and consider $r$ copies of an infinite dimensional Heisenberg algebra

$$\left[a^a_m, a^b_n\right] = \beta^2 m \delta_{m+n,0} a^{ab}, \quad 1 \leq a, b \leq r, \ m, n \in \mathbb{Z},$$

where $\beta$ is a formal parameter with inverse $\beta^{-1}$ generating the field $\mathbb{C}[\beta]$. It is convenient to use the realization $a^a_m = m x^a_m, \ a^a_m = \beta^2 a_m^{\partial x^a}/a^a_m, \ m > 0$. Introduce Fock spaces labeled by elements $\lambda$ in the dual $h^*$ of the Cartan subalgebra of $g$ by $\pi_0 = C[x^a_1, x^a_2, \ldots], \ \pi_\lambda = \pi_0 \otimes C v_\lambda$ and define a shift operator $T_\lambda : \pi_\lambda^0 \rightarrow \pi_{\lambda+1}, \ P \otimes v_\lambda \rightarrow P \otimes v_{\lambda+1}$. We shall need also Fock spaces $\pi_\lambda$ considered as modules over $\mathbb{C}[\beta^2]$. As linear spaces set $\pi_\lambda = \pi_0 \otimes \mathbb{C}[\beta]$. Let $C : \pi_\lambda \rightarrow \pi_\lambda^0$ denote the projection onto the $\beta$-independent (‘classical’) part.

The Fock spaces $\pi_\lambda$ carry two natural graduations: The principal graduation $\deg x^a_n = n$, deg $v_\lambda = 0 = \deg \beta$ and the power graduation power $x^a_n = 1$, power $v_\lambda = 0 = \text{power } \beta$. Denote by $(\pi_\lambda)_{N,m}$ the subspace of degree $N$ and power $m$.

The central objects in the following are operators of the form

$$Q_\lambda : \pi_0 \rightarrow \pi_\lambda, \quad \lambda \in h^*, \quad (1)$$

defined by $Q_\lambda = T_\lambda (V_\lambda)_1$, where

$$\beta^2 (V_\lambda)_k = \sum_{m \geq 0} S_{m-k} [\{ \lambda \cdot a_{-n}/n \}] S_m [\{-\lambda \cdot a_n/n\}].$$

$S_m[\{x_n\}], \ m \in \mathbb{Z}$ ($S_m = 0, m < 0$) are the elementary Schur polynomials in $x_n$, $n > 0$. It is often convenient to decompose these operators w.r.t. the powers of the annihilation operators appearing. A term of power $N$ in $\lambda \cdot a_m, \ m > 0$ corresponds to an $N$-fold contraction; $N = 1$ is the classical operator, $N = 2$ is the first quantum correction etc.. Explicitly,

$$Q_\lambda = Q_\lambda^0 + \beta^2 Q_\lambda^1 + \ldots$$

$$Q_\lambda^0 = T_\lambda \sum_{m \geq 1} S_{m-1}[\lambda \cdot x_n] \frac{1}{m} \partial^m_\lambda;$$

$$Q_\lambda^1 = T_\lambda \sum_{m \geq 2} S_{m-1}[\lambda \cdot x_n] \left( \sum_{k \neq l, k+l=m} \frac{1}{kl} \partial^A_\lambda \partial^A_\lambda + \frac{1}{2} \sum_{k=m} \frac{1}{k^2} (\partial^A_\lambda)^2 \right),$$

etc. ,

$$\text{(2)}$$

where $\partial^A_\lambda = \sum_a \lambda^a \partial^{\lambda}/\partial x^a$. In particular, the operators $Q_i := Q_{-\alpha_i}, \ i = 1, \ldots, r$ associated with minus the simple roots of $g$ are called screening operators. In this case we set $\pi_i =$
On the r.h.s. \( : \) denotes normal ordering in the oscillators \( \{ x_n \} \). The Fock space representations \( \pi_\lambda \) can be given the structure of a vertex operator algebra (VOA) or meromorphic conformal field theory by augmenting a Virasoro element and a realization of the derivative operator \[. \] The Virasoro element is given by

\[
L_{-2} = \frac{1}{2} x_1 \cdot x_1 + 2(\beta^2 \rho - \rho^\vee) \cdot x_2 ,
\]

where \( \rho, \rho^\vee \) are the Weyl vector of \( g \) and its dual. The derivative operator is realized \( \beta \)-independently as

\[
L_{-1} = \sum_{j=1}^{r} \sum_{m=0}^{r} (m+1) x_m \partial_{m+1}^{(j)} - x_0 \partial_0^{(j)} ,
\]

where \( x_0 \) is defined by \( [x_0, T_\lambda] = (\lambda_j, \lambda)T_\lambda \) and \( \lambda_j \) is the \( j \)-th fundamental weight of \( g \). From the recursion relation for the Schur polynomials one checks \( [L_{-1}, Q_i] = 0, \ 1 \leq i \leq r \).

In the VOA one has a vector-operator correspondence, which associates a unique sequence \( \pi_\lambda \) defined in terms of formal series ('fields') \( \pi_\lambda \). For the Fock spaces two types of composition maps (operator product expansion and normal ordering) are defined. For the Fock spaces \( \pi_\lambda \) the correspondence takes the form

\[
\pi_0 \ni P = (m_1! x_m^{a_1}) \ldots (m_n! x_m^{a_n}) \quad \rightarrow \quad P(z) =: i \partial^{m_1} \phi^{a_1}(z) \ldots i \partial^{m_n} \phi^{a_n}(z) : ,
\]

\[
\pi_\lambda \ni P \otimes v_\lambda \quad \rightarrow \quad : P(z) e^{i\lambda \cdot \phi(z)} :.
\]

On the r.h.s. \( : \) : denotes normal ordering in the oscillators \( \{ a_n^\alpha \} \) and \( \phi^\alpha(z) \) are (free massless) fields with operator product \( \phi^\alpha(z) \phi^\beta(w) = -\beta^2 \delta^{\alpha\beta} \ln(z - w) \). In particular, we denote by \( \mathcal{F}_\lambda \) the space of linear bounded operators \( \pi_\lambda \rightarrow \pi_{\lambda+\lambda'} \) which are obtained as formal residues \( \oint P e^{i\lambda \cdot \phi} \) of the fields corresponding to elements \( P \otimes v_\lambda \) in \( \pi_\lambda \). Then \( \oint : \pi_\lambda/L_{-1} \pi_\lambda \rightarrow \mathcal{F}_\lambda \) provides an isomorphism of linear spaces.

2.2. Next we recall some facts from the cohomological construction of \( W \)-algebras. Consider first the case where \( \beta^2 \) takes numerical values. For all positive irrational values of \( \beta^2 \), both the \( W \)-algebra itself and its irreducible highest weight representations can be described as the only non-vanishing cohomology class of a Fock space resolution. The resolution is of a form similar to the Bernstein-Gelfand-Gelfand (BGG) resolution known for the irreducible \( g \)-modules. The Fock spaces involved are parametrized by elements \( \lambda \in h^* \) lying in an orbit of the Weyl group \( W \) of \( g \). Let \( \alpha_i^\vee \in h, \ 1 \leq i \leq r \) be the simple co-roots and \( \langle \cdot , \cdot \rangle \) the duality pairing. The Weyl group acts by \( r_i * \lambda = r_i(\lambda + \rho) - \rho = \lambda - \langle \lambda + \rho, \alpha_i^\vee \rangle \alpha_i \) on \( h^* \), where \( r_i \) are the fundamental reflections. In the present context
it suffices to consider the Weyl group orbits of dominant integral weights $\Lambda \in P_+$ of $g$.
The resolution is of the form\cite{[1], [2]}

$$0 \to \pi_A(0) \xrightarrow{d_A(0)} \pi_A(1) \xrightarrow{d_A(1)} \ldots \xrightarrow{d_A(|\Delta_+| - 1)} \pi_A(|\Delta_+|) \to 0,$$

where $\pi_A(k) = \bigoplus_{w \in W \mid (w) = k} \pi_{w \ast \Lambda}$ and $\Delta_+$ is the set of positive roots. The differentials $d_A(k), \ 0 \leq k \leq |\Delta_+| - 1$ are certain composite operators built from the $Q_i$'s. We will denote the composition map by $[ \ ]$. The lowest differential $d_A(0)$ is the direct sum of $[Q_i]^{l_i + 1}: \pi_A \to \pi_{r_i \ast \Lambda}$, where $l_i = (\alpha_i, \Lambda), \ 1 \leq i \leq r$. These are linear operators, so that their kernels are graded subspaces of $\pi_A$. For $\Lambda \in P_+$ consider

$$H_A(g) = \bigcap_{i=1}^r \ker ([Q_i]^{l_i + 1}: \pi_A \to \pi_{r_i \ast \Lambda}). \quad (3)$$

This space appears as the lowest, and only non-vanishing cohomology class of the resolution, $H_A(g) = \ker d_A(0)$. Moreover it can be shown to form an irreducible $W(g)$-module. Applying the Euler-Poincare’ principle to the resolution one finds for the character

$$ch_A(g) = \text{Tr}_{H_A(g)}[t^{L_0}] = \frac{t^{h_\Lambda} \prod_{\alpha \in \Delta_+} (1 - t^{(\Lambda + \rho, \alpha)})}{\prod_{n \geq 1} (1 - t^n)}, \quad (4)$$

where $h_\Lambda = \frac{\beta^2}{2}(\Lambda, \Lambda + 2(\beta^2 \rho - \rho'))$ is defined through $L_0 v_\Lambda = h_\Lambda v_\Lambda$.

The relations (3) and (4) hold for all positive irrational values of $\beta^2$. In the present context we wish to treat $\beta$ as a formal variable not taking numerical values. In this case one encounters the problem that for any $P \in \bigcap_{i=1}^r \ker [Q_i]^{l_i + 1}$ also $\beta^{2n} P, \ n > 0$ are elements of $\bigcap_{i=1}^r \ker [Q_i]^{l_i + 1}$ and as elements of the $\mathbb{C}[\beta^2]$-module $\pi_A$ they are formally inequivalent. There are several ways to avoid this overcounting; the one which is technically most useful refers to the classical limit of the above construction. Define $H_A^{(0)}(g) \subset \pi_A^{(0)}$ by

$$H_A^{(0)}(g) = \bigcap_{i=1}^r \ker ([Q_i]^{l_i + 1}: \pi_A^{(0)} \to \pi_{r_i \ast \Lambda}^{(0)}). \quad (5)$$

Also the space $H_A^{(0)}(g)$ appears as the only non-vanishing cohomology class of a BGG-type resolution, which it can be obtained as the $\beta \to 0$ limit of the ‘quantum’ BGG resolution defining $H_A(g)$ in (3). In particular, the composition map for the operators $Q_i^{(0)}$ (for simplicity also denoted by $[ \ ]$) arises as the $\beta \to 0$ limit of the composition map for the $Q_i$’s. Notice that all the quantities in (5) are $\beta$-independent and no ambiguity arises. Set
now  \( \tilde{\pi} = \mathcal{H}^{(0)}_{\Lambda} \oplus \beta^2 \pi_{\Lambda} \), where \( \pi_{\Lambda} = \pi^{(0)}_{\Lambda} \otimes \mathbb{C}[\beta^2] \). In particular \( \tilde{\pi}_{\Lambda} \) is no longer a module over \( \mathbb{C}[\beta^2] \) and no overcounting occurs if one considers

\[
\mathcal{H}_{\Lambda}(g) = \bigcap_{i=1}^r \ker \left( [Q_i]^{i+1} : \tilde{\pi}_{\Lambda} \rightarrow \tilde{\pi}_{r_i \Lambda} \right) \tag{6}
\]

(where \( \tilde{\pi}_{r_i \Lambda} \) could also be replaced with \( \beta^2 \pi_{r_i \Lambda} \)). Generally one can define the BGG-type resolution as before with the Fock spaces \( \pi_{\Lambda \Lambda} \) replaced by \( \tilde{\pi}_{\Lambda \Lambda} \). Again the space \( \mathcal{H}_{\Lambda}(g) \) appears as the only non-vanishing cohomology class of the complex and is an irreducible \( W(\Lambda) \)-module. In particular the character formula (4) is preserved, but now with \( \beta^2 \) considered as a formal variable not taking numerical values. The relation between \( \mathcal{H}_{\Lambda}(g) \subset \tilde{\pi}_{\Lambda} \) and its classical counterpart \( \mathcal{H}_{\Lambda}^{(0)}(g) \subset \pi_{\Lambda}^{(0)} \) can be summarized as follows.

For \( \Lambda \in P_+ \):

i. The restriction \( C_{\Lambda} \) of \( C \) to \( \mathcal{H}_{\Lambda}(g) \) defines as isomorphism of graded linear spaces

\[ C_{\Lambda} : \mathcal{H}_{\Lambda}(g) \rightarrow \mathcal{H}_{\Lambda}^{(0)}(g). \]

ii. Let \( P^{(0)} \in \mathcal{H}_{\Lambda}^{(0)} \) be of degree \( N \) and power \( m \). Then \( P = C_{\Lambda}^{-1}(P^{(0)}) \) is of the form

\[ P = P^{(0)} + \beta^2 P^{(1)} + \ldots + \beta^{2(m-1)} P^{(m-1)}, \tag{7} \]

where \( P^{(i)} \) is of power \( p \leq m - i \) in \( \{ x_i^j \} \). The term of leading power in \( P^{(0)} \) (and hence \( P \)) is a a Weyl invariant polynomial, where the Weyl group acts by

\[ wx_i^n = (w_\alpha_i) \cdot x_n \text{ and } w(x_i^1 \ldots x_i^k) = (w x_i^1 \ldots w x_i^k), \quad w v_{\Lambda} = v_{\Lambda} \text{ on } \pi^{(0)}_{\Lambda}. \]

iii. If \( g \) is simply laced the elements \( P \in \mathcal{H}_{\Lambda}(g) \) are of the form

\[ P = \sum_i c_i(\beta^2) X_i, \tag{8} \]

where \( X_i \in \pi^{(0)}_{\Lambda} \) are monomials in \( \{ x_i^j \} \). The coefficients \( c_i(\beta^2) \) are polynomials in \( \beta^2 \) of the form \( c_i(\beta^2) = c_i + o(\beta^2) \), with \( c_i \neq 0 \).

Let us verify the claims i.–iii. consecutively. i. Being the lowest cohomology class of a BGG-type resolution, the character \( ch_{\Lambda}^{(0)}(g) \) of \( \mathcal{H}_{\Lambda}(g) \) can again be obtained from the Euler-Poincare’ principle. The result just differs by a factor from (4): \( ch_{\Lambda}(g) = t^{h_{\Lambda}} ch_{\Lambda}^{(0)}(g) \). This implies that \( \mathcal{H}_{\Lambda}(g) \) and \( \mathcal{H}_{\Lambda}^{(0)}(g) \) are isomorphic as graded linear spaces. It remains to show that the isomorphism is given by the mapping \( C_{\Lambda} \). Since by construction \( C_{\Lambda} \mathcal{H}_{\Lambda}(g) \subset \mathcal{H}_{\Lambda}^{(0)}(g) \) it suffices to show that \( C_{\Lambda} \) is injective i.e. that some \( P \in \mathcal{H}_{\Lambda}(g) \) is uniquely
determined by its classical part $C_{\Lambda}(P)$. From the form of $Q_i$ in eqn. (2) it is clear that $P \in \mathcal{H}_{\Lambda}(g)$ has a series expansion of the form (7) with $P^{(0)} = C_{\Lambda}(P)$. The uniqueness of this expansion follows from the fact that $P^{(0)}$ is assumed to be $\beta$-independent i.e. ad-hoc $\beta$-dependent linear combinations of elements in $\cap_{i=1}^r \text{Ker} \ Q_i^{(0)}$ are excluded by defining it to be a subspace of $\tilde{\pi}_{\alpha}^{(0)}$. Suppose two states $P$ and $\tilde{P}$ in $\mathcal{H}(g)$ to be given with the same classical part $P^{(0)} = C(P) = C(\tilde{P}) \in \mathcal{H}_{\Lambda}^{(0)}(g)$. Then their first quantum corrections $P^{(1)}$ and $\tilde{P}^{(1)}$ have to differ by an element of $\cap_{i=1}^r \text{Ker} \ Q_i^{(0)}$. If non-zero, this element would give rise to a $\beta$-dependent contribution to the classical part of at least one of the states $P$ or $\tilde{P}$ in conflict with the assumption. Hence $P^{(1)} = \tilde{P}^{(1)}$. By induction one concludes $P^{(k)} = \tilde{P}^{(k)}$, $1 \leq k \leq N - 1$. Thus $P = \tilde{P}$, and $C_{\Lambda}$ is injective.

\textit{ii.} For the first part of \textit{ii.} it remains to show that the corrections $P^{(i)}$ in (7) are of power less or equal to $m - i$ in $\{x_m^i\}$. To check this, decompose $\mathcal{H}_{\Lambda} := \mathcal{H}_{\Lambda}(g)$ into the space of non-derivative states $\tilde{\mathcal{H}}_{\Lambda} = \mathcal{H}_{\Lambda}/L_{-1}\mathcal{H}_{\Lambda}$ and its orthogonal complement. If $P_k$, $k \in I$ is a basis of $\tilde{\mathcal{H}}_{\Lambda}$, the states $L_{-1}P_k$, $k \in I$ form a basis of the orthogonal complement. Thus, it suffices to show the statement for $\tilde{\mathcal{H}}_{\Lambda}$. Let $(\tilde{\pi}_{\alpha})_{N,p}$ be the subspace of $\pi_{\Lambda}/L_{-1}\pi_{\lambda}$ of degree $N$ and power $p$. One then checks that $Q_{\Lambda}^{(k)}$ maps $(\tilde{\pi}_{\alpha})_{N,m}$ to $(\tilde{\pi}_{\alpha})_{N-1,m-k-1}$ (see the proof of eqn. (17) below). This implies (7) for $\tilde{\mathcal{H}}_{N}$.

To verify the second part of the claim \textit{ii.} it is convenient to make use of the fact that there is a bijection from a space $\Omega_{\Lambda}$ of $g$-singlets in level 1 $g$-modules to $\mathcal{H}_{\Lambda}$. On $\Omega_{\Lambda}$ one can introduce a double graduation (power and degree) s.t. the bijection preserves both gradings. Explicitly the image of some $P \in \Omega_{\Lambda}$ is obtained by projecting its $\phi\beta\gamma$ (Wakimoto-type) free field realization onto the $\beta\gamma$-independent part\[3\]. These images are elements of $\mathcal{H}_{\Lambda}$, and by construction the term of leading power is a Weyl invariant polynomial.

\textit{iii.} Let $g^\vee$ be the Lie algebra dual to $g$ (i.e. the Cartan matrix of $g^\vee$ is the transpose of that of $g$) and let $\beta$ momentarily assume numerical values. Denote by $\tilde{Q}_i$ and $\tilde{Q}_i^{\vee}$ the screening operators obtained by replacing $x_n^i = -\alpha_i \cdot x_n$ with $\tilde{x}_n^i = -\beta\alpha_i \cdot x_n$ and $(\tilde{x}_n^i)^\vee = \beta^{-1}\alpha_i \cdot x_n$, respectively (identifying $h$ with $h^*$. For $\beta^2$ irrational there exists a class of irreducible $W(g)$-modules $\tilde{\mathcal{H}}_{\Lambda^+,\Lambda^-}$ labeled by $\Lambda_+ \in P_+$, $\Lambda_- \in P_+^\vee$. These can be described either as $\cap_{i=1}^r[\tilde{Q}_i]^{(\alpha_i,\Lambda_+)+1}$ or as $\cap_{i=1}^r[\tilde{Q}_i^{\vee}]^{(\alpha_i,\Lambda_-)+1}$ on $\pi_{\lambda}$, $\lambda = \beta\Lambda_+ - \beta^{-1}\Lambda_-$. In other words, there exists an isomorphism of $W(g)$-modules

$$\tilde{\mathcal{H}}_{\Lambda^+,\Lambda^-}_{\beta} = \tilde{\mathcal{H}}_{\Lambda^+,\Lambda^-}_{1/\beta},$$

(9)

where $\Lambda^\vee$ is defined by $(\Lambda, \alpha_i) = (\Lambda^\vee, \alpha_i^\vee)$. For $\Lambda_+ = \Lambda_- = 0$ this was shown in \[3\] and by analysing the Kac determinant it can be extended to $\Lambda_+ \in P_+$, $\Lambda_- \in P_+^\vee$. It is known that the singular vectors in the Verma module $M(\lambda)$ can in the free field realization be constructed from $W(g)$-intertwiners $\pi_{\lambda^\vee} \rightarrow \pi_{\lambda}$. For weights of the form
\( \lambda = \beta \Lambda_+ - \beta^{-1} \Lambda_- \), where \( \Lambda_+ \in P_+ \), \( \Lambda_- \in P_+^\vee \) the Verma module contains a singular vector at degree \( (\Lambda_+ + \rho, \alpha)^\vee (\Lambda_- + \rho^\vee, \alpha_i) \) for each \( 1 \leq i \leq r \). The degree of these (and hence all other) singular vectors is left unchanged by the duality transformation \( g \rightarrow g^\vee, \beta \rightarrow -\beta^{-1} \) if \( \Lambda_\pm \rightarrow \Lambda_\pm^\vee \).

The case at hand corresponds to \( (\Lambda_+, \Lambda_-) = (\Lambda, 0) \). Applied to simply-laced Lie algebras \( g \) the duality (9) implies \( \overline{H}_{A,0}|_\beta = \overline{H}_{0,A}|_{-1/\beta} \). If \( \beta^2 \) is irrational also \( 1/\beta^2 \) is irrational, so that \( c h \overline{H}_{A,0} = c h \overline{H}_{0,A} \) (even for the same \( \beta \)). Thus, in fact \( \overline{H}_{A,0} = \overline{H}_{0,A} \). If we now restore \( \beta \) to be a formal variable this forces the elements \( \tilde{P}[\{\tilde{x}_n^i\}] \) of \( \overline{H}_{A,0} = \overline{H}_{0,A} \) to be invariant under the substitution \( \beta \rightarrow -1/\beta \) i.e. to be functions of \( \tilde{B} = \beta - 1/\beta \) only. If \( \tilde{P} \) is of leading power \( m \) in \( \{\tilde{x}_n^i\} \), one returns to the original normalizations via \( P[\{x_n^i\}] = \beta^m \tilde{P}[\{\tilde{x}_n^i/\beta\}] \). This implies that elements \( P \) of \( H_{A}(g) \), rather then being polynomials in \( \beta^2 \), in fact are polynomials in \( B = 1 - \beta^2 \). When rewritten in the form (7) the elements \( P \in H_{A}(g) \) are seen to be of the form (8).

Remark: Of particular interest is the space \( H(g) := H_0(g) \), which encodes information about the operator product expansion of the \( W(g) \)-algebra. By taking \( H(g) \) to be the pre-Hilbert space of a VOA one can show that the quantum \( W(g) \)-algebra exists if and only if \( c h_0(g) \) coincides with the character of the free polynomial algebra generated by elements \( W_{n_i}^i, 1 \leq i \leq r, n_i \geq -(e_i + 1) \), where \( e_i \) are the exponents of \( g \). Explicitly, the condition reads

\[
ch_0(g) \equiv \prod_{1 \leq i \leq r, n_i > e_i} (1 - q^{n_i})^{-1}. \tag{10}
\]

In order to prove the existence of a quantum \( W(g) \)-algebra one thus has to establish (10) from the defining relation (3). In [4, 2] this was achieved by extending \( H_{A}(g) \) to a BGG-type resolution. It then follows that \( H(g) \) contains a set of fundamental vectors \( W_{-e_i}^i, 1 \leq i \leq r \) from which one can build states of the form

\[
W_{-m_1}^{i_1} \cdots W_{-m_n}^{i_n} \otimes v_0, \quad i_j \geq i_{j+1}, \quad m_j \geq 0, \quad m_j \geq m_{j+1} \text{ if } i_j = i_{j+1}. \tag{11}
\]

The validity of the condition (10) implies in particular that these states are linearly independent and exhaust all solutions to \( Q_i P = 0, 1 \leq i \leq r \) at a given degree.

Consider now the spaces \( H_{A}(g) \) modulo \( L_{-1} \)-exact pieces. Set \( H_{\Lambda} := H_{A}(g) := H_{A}(g)/L_{-1}H_{A}(g) \), which inherits the deg \( x_n^i = n \) grading from \( H_{A} \). Let \( (\overline{H}_{A})_N \) denote the subspace of degree \( N \). As before the classical counterparts will carry a superscript ‘(0)’. Let \( \overline{C}_{\Lambda} \) denote the restriction of \( C \) to \( \overline{H}_{A} \). From the previous results it follows that this is a bijection of graded vector spaces

\[
\overline{C}_{\Lambda} : \overline{H}_{A} \rightarrow \overline{H}_{A}^{(0)}, \quad \Lambda \in P_+. \tag{12}
\]
3. Existence of infinite abelian subalgebras

3.1. The commutation relations of generic elements of $W(g)$ can be written in an $su(1,1)$-covariant form, in which all the specific information is encoded in the structure constants (the latter being polynomial or algebraic functions of the central charge). Given this form of the commutation relations one can investigate the conditions under which $W(g)$ possesses abelian subalgebras. It turns out that the existence of an abelian subalgebra amounts to the existence of a preferred basis in which a certain subset of the structure constants vanishes\[5\]. The existence of an infinite dimensional abelian subalgebra in $W(g)$ therefore is an important structural result, somewhat analogous to the existence of a Cartan subalgebra for ordinary Lie algebras.

A first understanding of this result can be obtained from the classical theory. The projection $C$ onto the $\beta$-independent part defines a contraction of the $W(g)$-algebra, which coincides with the classical $W$-algebra $W^{(0)}(g)$. These classical algebras $W^{(0)}(g)$ are known to appear as symmetry algebras in various classical integrable systems (mKdV, KdV, affine Toda). Their conserved charges can be expressed in terms of the generators of the classical $W$-algebra and generate an infinite dimensional abelian subalgebra thereof. This result can be derived from any of the various approaches to classical integrable systems. The formulation which is most useful for the transition to the quantum theory is the following.

Let $\alpha_0$ denote the additional simple root associated with the Dynkin diagram of an affine Lie algebra $\hat{g}$ and set $F := F_{\alpha=0}$, $Q_0 := Q_{-\alpha_0}$, $F_i := F_{-\alpha_i}$, $0 \leq i \leq r$. Since $\alpha_0$ enters only through inner products with elements of $h$ we can identify $\alpha_0$ with its projection $-\theta$ on $h$ (For $\hat{g} = A^{(2)}_r$ this differs by a factor 2 from the standard definition). Note that $\theta \in P_+$, so that the results of section 2 apply to $H_\theta$ and $\hat{H}_\theta$. The classical counterparts of these objects will again carry a superscript ‘(0)’. On the classical level one can show that

$$I^{(0)}(\hat{g}) := \cap_{i=0}^{r} \text{Ker} \left( Q^{(0)}_i : F^{(0)} \to F^{(0)}_i \right)$$

defines an abelian Poisson subalgebra of the classical $W$-algebra $W^{(0)}(g)$ which is linearly generated by elements $I^{(0)}_N = \not\exists J^{(0)}_N$, with $J^{(0)}_N$ of degree $N$, $N - 1 \in E$. The elements $J^{(0)}_N$ are called the classical conserved densities at degree $N$. By definition they are taken to be $\beta$-independent. The index set $E = E(\hat{g})$ are the exponents of $\hat{g}$, an infinite subset of $\mathbb{N}$ uniquely associated to $\hat{g}$. For later use note the following properties of the conserved densities.

- Nonvanishing classical conserved densities $J^{(0)}_N$ exist only for degree $N - 1 \in E$. 

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- The number of linearly independent conserved densities at degree $N$ equals the multiplicity $\#(N - 1)$ of the exponent.

- Their linear hull does not contain elements of leading power less then $N$ in $\{x_n^i\}$.

This follows from the results of Drinfel’d and Sokolov[[9]]. The first two facts are stated explicitly (remarks to prop. 6.6) and the last one can be extracted from the proof of prop. 6.2. We note that only algebras of type $D_2^{(1)}$, $n > 1$ have exponents of multiplicity greater than 1 (the exponents $2n - 1 \mod (4n - 2)$ appear with multiplicity two). In all other cases the conserved densities at some degree $N$, $N - 1 \in E$ are unique. The aim of this paper is to show that analogous results hold in the quantum theory.

**Theorem:**

$$I(\hat{g}) := \bigcap_{i=0}^r \text{Ker } (Q_i : \mathcal{F} \rightarrow \mathcal{F}_i)$$

defines an abelian subalgebra of the quantum $W(g)$-algebra. $I(\hat{g})$ is linearly generated by elements $I_N = \oint J_N$, $N - 1 \in E$ (including multiplicities).

3.2. The problem consists in proving the existence of the elements $J_N$. The remaining properties then follow easily. One checks that $[Q_0, L_{-1}] = 0$ i.e. $L_{-1} \in I(\hat{g})$. If we anticipate that $Q_0^{(0)}$ maps $\hat{\mathcal{H}}_0^{(0)}$ to $\hat{\mathcal{H}}_\theta^{(0)}$ (c.f. the paragraph following eqn. (13)) this means that $Q_0$ maps $\tilde{\pi}_0/L_{-1}\tilde{\pi}_0$ to $\tilde{\pi}_0/L_{-1}\tilde{\pi}_0$. In view of (6) the intersection of the kernels $\bigcap_{i=0}^r \text{Ker } Q_i$ on $\tilde{\pi}_0/L_{-1}\tilde{\pi}_0$ defines a subspace of $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0(g)$. All the elements of $I(\hat{g})$ can thus be expressed in terms of $W$-generators and since for $I_N$, $I_M \in I(\hat{g})$ also $[I_N, I_M] \in I(\hat{g})$ one concludes that $I(\hat{g})$ defines a subalgebra of $W(g)$[[6]]. We wish to show that it is in fact an abelian subalgebra. To see this, recall that the quantum conserved densities $J_N$ are elements of $\hat{\mathcal{H}}_N$. One may thus make use of the bijection (12). If for some $N - 1 \in E$ there exists a quantum conserved density $J_N$ at all, its classical part $\hat{C}J_N$ has to be a classical conserved density. Conversely, given a classical conserved density $J_N^{(0)} \in \hat{\mathcal{H}}_N^{(0)}$ there is only one possible candidate for a quantum conserved density namely $\hat{C}^{-1}(J_N^{(0)})$. In particular it follows that the quantum conserved density $J_N$ associated with $J_N^{(0)}$, if any, must be of leading power $N$ in $\{x_n^i\}$. This forces $I(\hat{g})$ to be abelian: If $J_N(z) = \sum_n (J_N)_nz^{-n-N}$ denotes a conserved density of degree $N$, the conserved charge is the mode of degree $-N + 1$, i.e. $I_N = \oint dz J_N(z) = (J_N)_{-N+1}$. In this situation the general form of the commutator in a VOA reduces to

*It is only for this conclusion that the construction via vertex operators i.e. the condition (13) below is essential. If the elements $\oint \hat{C}^{-1} J_N^{(0)}$ could independently be shown to form a subalgebra, the result on the existence of an infinite abelian subalgebra would follow already at this point.*

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\begin{equation}
[(J_M)_{-M+1}, (J_N)_{-N+1}] = \sum_{K:M+N-K=1} C_{MN}^K (P_K)_{M+N-2}, \quad N-1, M-1 \in E,
\end{equation}

where \( C_{MN}^K \) are the structure constants coupling to the fields \( P_K(z) \) in the VOA which have degrees \( K \) satisfying \( M + N - K = 1 \). Since \( I(\hat{g}) \) is a subalgebra of \( W(g) \) the r.h.s. must again be an element of \( I(\hat{g}) \). In particular every field \( P_K(z) \) for which the structure constant \( C_{MN}^K \) were nonvanishing would have to be of leading power \( M + N - 1 \) in \( \partial \phi^a \). But the terms of power \( M + N - 1 \) on the r.h.s., if any, are the same in the classical and the quantum case. In the classical case they are known to be absent, leaving a r.h.s. which is of leading power less than \( M + N - 1 \) and which thus cannot be a conserved charge. Hence the structure constant \( C_{MN}^K \) has to vanish. In particular this shows that the existence of an abelian subalgebra in \( W(g) \) is equivalent to the vanishing of the set of structure constants \( C_{MN}^K \) with \( K \) running over all fields in the VOA of degree \( K = M + N - 1 \).

To prove the theorem it therefore suffices to show that for every classical conserved density \( J_N^{(0)} \) the unique candidate for its quantum counterpart \( \hat{C}^{-1}J_N^{(0)} \) is annihilated by \( Q_0 \) (modulo \( L_{-1} \)-exact pieces) i.e.

\begin{equation}
Q_0 [\hat{C}^{-1}J_N^{(0)}] = L_{-1} (S \otimes v_\theta), \quad (13)
\end{equation}

for some \( S \in \pi_0 \). The essential point in proving (13) will be that the r.h.s. in fact is an element of \( \hat{H}_\theta \) on which the mapping \( \hat{C}_\theta \) exists and is invertible. For this one has to show that \( Q_0 \) is a well-defined mapping from \( H \) to \( \hat{H}_\theta \) \((*)\). Since \( Q_0 \) commutes with \( L_{-1} \) it then follows that \( Q_0 \) induces a mapping between the equivalence classes modulo total derivatives in \( H \) and \( \hat{H}_\theta \), respectively. In particular, these results will hold for the restrictions to the classical part.

To check \((*)\) recall that both \( H \) and \( \hat{H}_\theta \) are irreducible \( W(g) \)-modules. Thus in effect one has to construct a free field realization of an intertwiner connecting both \( W(g) \)-modules. This can be done via the BGG-type resolution by constructing a sequence of intertwiners \( V_{ww'}^{(i)} : \pi_{w*0} \rightarrow \pi_{w'*\theta} \) connecting the Fock spaces in both resolutions which are labeled by Weyl group elements \( w, w' \) of the same length. These intertwiners have to satisfy a number of consistency conditions which allows one to determine them explicitly\[8\]. The result is that the whole sequence of intertwiners \( V^{(i)} = \{ V_{ww'}, l(w) = l(w') = i \}, \quad 0 \leq i \leq |\Delta_+| \) is already determined by \( V^{(0)} \) and \( V^{(1)} \). Explicitly the latter are given by

\begin{align*}
V^{(0)} &= \{ Q_0 \}, \\
V^{(1)} &= \{ Q_{ij} = \delta_{ij}[Q_0 Q_j^{(\theta,\alpha_j)}] \}.
\end{align*}

These intertwiners make the diagram in Fig. 1 commutative. In particular it follows that elements in \( H \) are mapped into elements of \( \hat{H}_\theta \).
3.3. It remains to prove (13). First observe the identity

$$Q_0^{(1)} C = C_\theta Q_0 \, . \quad (14)$$

To check (14) let $P \in \mathcal{H}$ be given. Then $Q_0 P$ is an element of $\mathcal{H}_\theta \subset \bar{\pi}_\theta$ and for later use we write it in the form

$$Q_0 P = R \otimes v_\theta + L_{-1}(S \otimes v_\theta) \, ,$$

for some $R, S \in \pi_0$. Similarly for $P^{(0)} = CP$ one has

$$Q_0^{(0)} P^{(0)} = R^{(0)} \otimes v_\theta + L_{-1}(S^{(0)} \otimes v_\theta) \, ,$$

for some $R^{(0)}, S^{(0)} \in \pi_0^{(0)}$. But since $Q_0 P = Q_0^{(0)} P^{(0)} + o(\beta^2)$ and $C_\theta$ commutes with $L_{-1}$ it follows that $C_\theta (R \otimes v_\theta) = R^{(0)} \otimes v_\theta$, $C_\theta (S \otimes v_\theta) = S^{(0)} \otimes v_\theta$ and thus $C_\theta Q_0 P = Q_0^{(0)} C P$. This holds for all $P \in \mathcal{H}$ and implies (14). Moreover since both $C_\theta$ and $C$ commute with $L_{-1}$ eqn. (14) projects to an identity $\hat{C}_\theta Q_0 = Q_0^{(0)} \hat{C}$ on $\hat{\mathcal{H}}$, expressing the commutativity of the diagram in Fig. 2. But since both $\hat{C}_\theta$ and $\hat{C}$ are invertible this means that $Q_0$ and $Q_0^{(0)}$ are related by an equivalence transformation

$$Q_0^{(0)} = \hat{C}_\theta Q_0 \hat{C}^{-1} \quad (15)$$

and have the same rank (on all $\hat{\mathcal{H}}_N$, $N \geq 0$). In particular eqn. (15) implies that the kernel of $Q_0^{(0)}$ on $\hat{\mathcal{H}}_N^{(0)}$ and the kernel of $Q_0$ on $\hat{\mathcal{H}}_N$ have the same dimension and hence guarantees that every classical conserved density can be deformed into a quantum conserved density. Explicitly, if $J_N^{(0)} \in \text{Ker} Q_0^{(0)}$ on $\hat{\mathcal{H}}_N^{(0)}$ the unique candidate $\hat{C}^{-1} J_N^{(0)} \in \hat{\mathcal{H}}_N$ for its quantum counterpart will always solve (13): $Q_0 \hat{C}^{-1} J_N^{(0)} = \hat{C}_\theta^{-1} Q_0^{(0)} J_N^{(0)} = 0 \in \hat{\mathcal{H}}_\theta$, which is what we wanted to show.
4. Construction principle

4.1. In the classical theory there exist several fairly efficient methods to compute the conserved densities $J_N^{(0)}$ explicitly. So far none of these methods could be transferred to the quantum theory. The results (13), (15) show that the quantization of a classical conserved density $J_N^{(0)}$ can in principle be done by applying the mapping $\hat{C}^{-1}_N$ (i.e. the restriction of $\hat{C}^{-1}$ to the required degree $N$). Unfortunately no efficient (possibly recursive) method to compute $\hat{C}^{-1}_N$ is available. (Part of the problem being that it is not a homomorphism w.r.t. normal ordering i.e. $\hat{C}(PQ:) \neq \hat{C}(P:)\hat{C}(Q:)$, in general.) In this section we will describe a reasonably efficient method to compute the conserved densities explicitly.

We will introduce two different types of basis $\mathcal{A}$ and $\mathcal{B}$ on $\hat{\mathcal{H}}_N$ which are characterized by certain extremal properties. A basis of type $\mathcal{B}$ is adapted to the operator $Q_0$ and gives an alternative understanding of the result (15). In particular the conserved densities are part of a basis $\mathcal{B}$. In order to construct these basis vectors we exploit the interplay between a basis of type $\mathcal{B}$ and some more easily accessible type of basis $\mathcal{A}$, whose elements have a maximal staggering in their leading powers.

A basis $\mathcal{A}_N = \bigcup_{m=2}^{N} A_{N,m}$ of $\hat{\mathcal{H}}_N$ is called of type $\mathcal{A}$ if

a) $A_{N,m}$ contains the maximal number of linearly independent elements $P \in \mathcal{H}_N$ which are exactly of leading power $m$ in $\{x_n^i\}$.

b) The linear hull $L(A_{N,m})$ does not contain elements of leading power less than $m$ in $\{x_n^i\}$. 

---

Fig.2: Commutative diagram for rank $Q_0^{(0)} = \text{rank } Q_0$. 

\[
\begin{array}{ccc}
\hat{\mathcal{H}} & \xrightarrow{\hat{C}} & \hat{\mathcal{H}}^{(0)} \\
\downarrow Q_0 \quad \quad & & \quad \downarrow Q_0^{(0)} \\
\hat{\mathcal{H}}_{\theta} & \xrightarrow{\hat{C}_\theta} & \hat{\mathcal{H}}^{(0)}_{\theta}
\end{array}
\]
Remark: Clearly a basis of type $A$ in $\mathcal{H}_N$ will induce a basis on $\hat{\mathcal{H}}_N$ with the same properties a) and b). For simplicity we will keep the notation $A_N = \bigcup_{m=2}^{N} A_{N,m}$ for such basis on $\hat{\mathcal{H}}_N$. No confusion will be possible as in this subsection $A_N$ will always refer to a basis on $\mathcal{H}_N$, while from subsection 4.2 on $A_N$ will always refer to a basis on $\hat{\mathcal{H}}_N$.

Condition b) implies that $L(A_{N,m}) \cap \sum_{p<m} L(A_{N,p}) = \{0\}$. Clearly $L(A_{N,m})$ has trivial intersection also with $\sum_{p>m} L(A_{N,p})$. This means that $\mathcal{H}_N$ is the direct sum of the linear hulls

$$\mathcal{H}_N = L(A_{N,N}) \oplus \ldots \oplus L(A_{N,2})$$

thereby introducing a grading on $\mathcal{H}_N$. Of course a basis of this type is not unique – each element of $A_{N,m}$ can be modified by adding an element of $\bigoplus_{p \leq m-1} L(A_{N,p})$ – but the dimensions of the subspaces $L(A_{N,m})$ are unique. The calculation of the associated ‘power characters’ is an open problem. Notice also that $A_{N,1}$ satisfying condition a) of the definition would be empty, which is why $m = 1$ has been omitted in $A_N$. For $N > 2$ also $A_{N,2}$ is empty.

One can fix the ambiguity in the definition of a basis of type $A$ in the following way. Recall from eqn. (7) that for any $P \in \mathcal{H}_N$ the term of leading power is a Weyl invariant polynomial. From the character formula of the spaces $\mathcal{H}_N$, $\mathcal{H}^{(0)}$ and $\Omega_0$ (defined in the proof of eqn. (7)) it follows that $P$, its classical part $P^{(0)}$, and the Weyl invariant polynomial $P_0$ of leading power (c.f. eqn. (18) below) all are in 1-1 correspondence

$$P \overset{1-1}{\leftrightarrow} P^{(0)} \overset{1-1}{\leftrightarrow} P_0.$$ \hspace{1cm} (16)

In particular for $P \in L(A_{N,m})$ the part $P_0$ is of power $m$ and adding elements of $\bigoplus_{p \leq m-1} L(A_{N,p})$ to it will affect only Weyl invariant terms of power less than $m$. Because of the bijection (16) this means that one can always find a basis $A'_N = \bigcup_{m=2}^{N} A'_{N,m}$ of type $A$ s.t. for all $P \in L(A'_{N,m})$ the Weyl invariant polynomial $P_0$ of leading power (c.f. eqn. (18) below) all are in 1-1 correspondence

$$A''_N = \bigcup_{m=2}^{N} A''_{N,m}$$

The basis $A'_N$ of $\mathcal{H}_N$ can be described explicitly. We claim that $A''_{N,m}$ is just given by the set of states (11) of degree $N = \sum_i (m_i + e_i + 1)$ and leading power $m = \sum_i (e_i + 1)$. Denote the latter set of states by $A''_{N,m}$. We wish to show that $A''_{N,m}$ has all the properties characterizing $A'_{N,m}$. Clearly $A''_{N,m}$ contains only linearly independent elements of $\mathcal{H}_N$ which are exactly of leading power $m$. Further $\bigcup_{m=2}^{N} A''_{N,m}$ is a basis of $\mathcal{H}_N$ so that no further linearly independent elements with that property exist. Thus $A''_{N,m}$ satisfies condition a) in the definition of a basis of type $A$. To check condition b) one makes use of the bijections (16). For the Weyl invariant polynomials $P_0$ condition b) is manifestly satisfied and thus holds also for the states $P \in A'''_{N,m}$ associated with them. Hence the
basis $\bigcup_{m=2}^{N} A''_{N,m}$ is of type $A$. It remains to check that the only Weyl invariant terms in some $P \in A'_{N,m}$ are those of power $m$. But this follows because in the generators $W_{-m}^i$, $m \geq e_i + 1$ only the term of power $e_i + 1$ is Weyl invariant. We conclude that the basis $\bigcup_{m=2}^{N} A'_{N,m}$ and $\bigcup_{m=2}^{N} A''_{N,m}$ coincide.

In particular, this yields an explicit construction for a basis of type $A$ in $\mathcal{H}_N$. As remarked before such a basis will induce a basis on $\hat{\mathcal{H}}_N$ with the same properties $a)$ and $b)$. From now on only basis of type $A$ on $\hat{\mathcal{H}}_N$ will be needed and for simplicity we will keep the notation $A_N = \bigcup_{m=2}^{N} A_{N,m}$ for such basis.

4.2. The basis $A_N$ on $\hat{\mathcal{H}}_N$ are defined without reference to the operator $Q_0$. Next we introduce a type of basis $B_N$ adapted to the operator $Q_0$. In particular the conserved densities are part of the basis. In order to define this type of basis some preparations are needed. Let $(\hat{\mathcal{H}}_{\Lambda})_{N,m}$ denote the subspace of $\hat{\mathcal{H}}_\Lambda(g)$ containing the elements of degree $N$ and leading power $m$. Recall the shorthand $\hat{\mathcal{H}}_\Lambda = \hat{\mathcal{H}}_0(g)$. We claim that $Q_0 : \hat{\mathcal{H}}_{N,m} \longrightarrow (\hat{\mathcal{H}}_{\theta})_{N-1,m-2}$ is well defined. (This holds only on the equivalence classes modulo total derivatives. When acting on $\mathcal{H}_N$ the powers get mixed in a complicated fashion.)

The fact that $Q_0$ maps $\hat{\mathcal{H}}$ to $\hat{\mathcal{H}}_{\theta}$ has already been seen in section 3.2. It remains to show that $Q_0$ reduces the degree and the power by 1 and 2 units, respectively. We first show that for generic elements $\lambda \in h^*$ the operator $Q_\lambda$ acts as

$$Q_\lambda : (\hat{\pi})_{N,m} \longrightarrow \bigoplus_{p \leq m-1} (\hat{\pi}_\lambda)_{N-1,p},$$

where $(\hat{\pi}_\lambda)_{N,p}$ is the subspace of $\pi_\Lambda/L-1\pi_\Lambda$ of degree $N$ and power $p$. For the particular choice $\lambda = \theta$ we then show that when acting on $\hat{\mathcal{H}}_{N,m}$ the value $m - 1$ on the r.h.s. can in fact be replaced by $m - 2$.

To check (18) it is convenient to make use of the vector-operator correspondence in the VOA. The action of the operator $Q_\lambda$ on differential polynomials in the fields $i\partial^m \phi^a(z)$ associated with the states $m! x^a_m$ is then induced by the operator product expansion with the normal ordered exponential $\beta^{-2} : \exp(i\lambda \cdot \phi(w)) :$. Consider first a single contraction contribution:

$$\beta^{-2} : i\partial_z^{(n)} \phi^a P(z) : e^{i\lambda \cdot \phi(w)} : \sim \frac{\lambda^a P(z)}{(z-w)^n} : e^{i\lambda \cdot \phi(w)} : + (I)$$

$$\sim \frac{\lambda^a \partial_w^{(n-1)} P(w)}{z-w} : e^{i\lambda \cdot \phi(w)} : + (I) + (II).$$

Here $(I)$ denotes terms coming from the contraction with $P(z)$ and $(II)$ denotes terms of the form $\frac{\cdot \cdot \cdot (w)}{(z-w)^n}$, $n \geq 2$. Clearly, upon integration $\int_z dw$ the terms of type $(II)$ give rise to
total derivative terms and the remainder has both the power and the degree reduced by one unit. By induction on the power of \( P(z) \) one finds \( Q^{(0)}_\lambda : (\tilde{\pi})_{N,m} \to (\tilde{\pi})_{N-1,m-1} \) for the single contractions. Similarly one checks for the higher contractions \( Q^{(k)}_\lambda : (\tilde{\pi})_{N,m} \to (\tilde{\pi})_{N-1,m-k-1} \). In particular this implies (17).

Now specialize to \( \lambda = \theta \). Recall from eqn. (7) that for \( P \in \mathcal{H} \) the term of leading power is a Weyl invariant polynomial. For some \( P \in \mathcal{H}_N \) of leading power \( m \) this means

\[
P = P_0 + P_1(\beta^2),
\]

where \( P_0 \in \pi^{(0)}_0 \) is a Weyl invariant polynomial of power \( m \) in \( \{x_n^i\} \) and \( P_1(\beta^2) \in \pi_0 \) is of power less or equal to \( m - 1 \). Consider now the action of \( Q_0 \) on some \( P \in \hat{\mathcal{H}}_N \). One can consider \( P \) also as an element of \( \mathcal{H}_N \) with the total derivative term fixed by the condition \( L_1P = 0 \). Then \( Q_0P = Q_0P_0 + Q_0P_1 \). But on a Weyl invariant polynomial the action of \( T_\theta^{-1}Q_0 \) and \( T_\alpha^{-1}Q_1 \), \( 1 \leq i \leq r \) coincide, so that \( Q_0P_0 = T_\theta^{-1}Q_0P_0 = T_\theta^{-1}Q_0(P - P_1) = -T_\theta^{-1}Q_0P_1 \). Inserting into (*) this becomes \( Q_0P = (Q_0 - T_\theta^{-1}Q_0)P_1 \). Since \( P_1 \) is an element of \( L(A_{N,m-1}) \), the r.h.s. by (18) lies in \( \bigoplus_{p \leq m-2} \tilde{(\pi_\theta)}_{N-1,p} \). This completes the proof of eqn. (17).

We shall need also the classical counterpart of eqn. (17) stating that \( Q^{(0)}_0 \) maps \( \hat{\mathcal{H}}_{N,m} \) to \( (\hat{\mathcal{H}}^{(0)}_\theta)_{N-1,m-2} \). Indeed eqn. (17) holds also classically. To check this note that equation (18) in particular holds for the single contraction (classical) contributions. In addition equation (19) is preserved with \( P_1(\beta^2) \) replaced by \( P^{(0)}_1 = \tilde{C}P_1 \), so that also the second part of the proof goes through. Eqn. (17) and its classical counterpart can now be used to define a special class of basis on \( \hat{\mathcal{H}}_N \) and \( \hat{\mathcal{H}}^{(0)}_N \), respectively.

A basis \( \mathcal{B}_N \) of \( \hat{\mathcal{H}}_N \) is called of type \( \mathcal{B} \) or extremal w.r.t. \( Q_0 \) if \( \mathcal{B}_N = \bigcup_{m=2}^{N} B_{N,m}, B_{N,m} \subset \hat{\mathcal{H}}_N \) and

a) \( B_{N,m} \) contains the maximal number of linearly independent elements \( P \in \hat{\mathcal{H}}_N \) for which \( Q_0P \) is of the form

\[
Q_0P = R \otimes v_\theta + L_{-1}(S \otimes v_\theta),
\]

where \( R \in \pi_0 \) is of exactly leading power \( m - 2 \) in \( \{x_n^i\} \).

b) The linear hull \( L(B_{N,m}) \) does not contain elements satisfying (20) with \( R \) of power less than \( m - 2 \).

From condition b) one infers that \( L(B_{N,m}) \cap \sum_{p \neq m} L(B_{N,p}) = \{0\} \), so that \( \hat{\mathcal{H}}_N \) is the direct sum of the linear hulls

\[
\hat{\mathcal{H}}_N = L(B_{N,N}) \oplus \ldots \oplus L(B_{N,2}).
\]
Of course a basis of this type is not unique – each element of $B_{N,m}$ can be modified by adding an element of $\bigoplus_{p \leq m-1} L(B_{N,p})$ – but the dimensions of the subspaces $L(B_{N,m})$ are unique. The space $L(B_{N,2})$ is uniquely defined however and is just the space of quantum conserved densities at degree $N$.

Similarly one can define a basis of $\bar{H}_N^{(0)}$ which is extremal w.r.t. $Q_0^{(0)}$. Let $\mathcal{B}_N^{(0)} = \bigcup_{m=2}^{N} B_{N,m}^{(0)}$ denote such a basis. One would expect that $\hat{C}^{-1} B_{N,m}^{(0)}$ then satisfies the defining properties of $B_{N,m}$ for some quantum basis $\hat{B}_N = \bigcup_{m=2}^{N} B_{N,m}$. But this does not follow from the definition and is just the point in question. In fact it is not hard to see that

$$\dim L(B_{N,m}^{(0)}) = \dim L(B_{N,m}), \quad 2 \leq m \leq N.$$  \hspace{1cm} (21)

implies eqn (15). In particular, once (21) has been established, one recovers the existence of the quantum conserved densities for $m=2$

$$\dim L(B_{N,2}) = \dim L(B_{N,2}^{(0)}) = \begin{cases} \#(N-1), & N - 1 \in E \\ 0, & \text{otherwise}, \end{cases}$$

where $\#(N-1)$ is the multiplicity of the exponent $N - 1 \in E$.

4.3. In fact eqn. (21) is stronger than eqn. (15). We shall see below that eqn. (21) implies (15) for any $g$; but (21) follows from (15) only when $g$ is simply-laced. In particular, we claim (21) only for simply-laced Lie algebras.

Let us first check the implication (21) ⇒ (15): Let $B_{N,m}^{(0)} = \{P_1^{(0)}, \ldots, P_k^{(0)}\}$ and $B_{N,m} = \{P_1, \ldots, P_k\}$ be an enumeration of basis vectors. W.l.o.g one may assume that $P_i^{(0)} = \hat{C} P_i$. As in the derivation of eqn. (14) one checks that $\hat{C}_g Q_0 P_i = Q_0^{(0)} \hat{C} P_i$, $1 \leq i \leq k$. This can be done for any $2 \leq m \leq N$ and thus for a basis of $\mathcal{H}_N^{(0)}$. One concludes $\hat{C}_g Q_0 = Q_0^{(0)} \hat{C}$ and hence (15).

Conversely assume now (15) to be given. For $P^{(0)} \in L(B_{N,m}^{(0)})$ one has

$$Q_0[\hat{C}_g^{-1} P^{(0)}] = Q_0^{(0)} P^{(0)} = Q_0^{(0)} P^{(0)} + o(\beta^2).$$

In particular for $m = 2$ the argument $Q_0^{(0)} P^{(0)}$ vanishes (mod $L_{-1}$-exact pieces) and since $\hat{C}_g^{-1}$ is injective also the $o(\beta^2)$ terms have to vanish – which is just a restatement of the conclusion following eqn. (15). For $m > 2$ however $Q_0^{(0)} P^{(0)}$ is of leading power $m - 2$ and in order to confirm $\hat{C}^{-1} P^{(0)} \in L(B_{N,m})$ one has to show that also the quantum corrections are of power $p \leq m - 2$. This does not follow from (15) and requires some additional input. It turns out that the result iii. of section 2.2 provides this extra input but limits the conclusion to simply-laced Lie algebras $g$. We shall apply 2.2.iii. to $\hat{H}_g$
and may rephrase its content as follows: For any \( R \in \hat{H}_\theta \) the set of Fock monomials \( \{ X_i \in \pi_\theta^{(0)} \} \) appearing in the expansion (8) is the same as the one appearing in the expansion of its classical counterpart \( R^{(0)} \in \hat{H}_0 \). Given (*) one may easily check the two inclusions \( \hat{C}^{-1} L(B_{N,m}^{(0)}) \subset L(B_{N,m}) \) and \( L(B_{N,m}) \subset \hat{C}^{-1} L(B_{N,m}^{(0)}) \) separately: Applied to \( R = Q_0[\hat{C}^{-1} P^{(0)}] \in \hat{H}_\theta \), \( P^{(0)} \in L(B_{N,m}^{(0)}) \) one concludes from (*) that both its classical and its quantum terms are of power \( p \leq m - 2 \). Further, by construction of the classical part, no linear combination of elements in \( \hat{C}^{-1} L(B_{N,m}^{(0)}) \) has a rest of leading power less than \( m - 2 \) (modulo \( L^{-1}\)-exact pieces) when acted upon with \( Q_0 \). Hence \( \hat{C}^{-1} L(B_{N,m}^{(0)}) \subset L(B_{N,m}) \). Conversely let \( P \in L(B_{N,m}) \) be given. From (*) one concludes first that its classical part \( P^{(0)} = \hat{C} P \) will have property (20) in the definition of \( B_{N,m} \) and further that the set \( \hat{C} B_{N,m} \) will satisfy condition \( b \) in the definition whenever \( B_{N,m} \) does. Hence \( L(B_{N,m}) \subset \hat{C}^{-1} L(B_{N,m}^{(0)}) \). This verifies the implication (15) \( \Rightarrow \) (21) for \( g \) simply-laced.

4.4. In this section we introduce a system of linear spaces displaying the relation between a basis of type \( A \) and one of type \( B \) on \( \hat{H}_N \). The result, eqn. (29), can be used to compute the quantum conserved densities. The relation between the basis of type \( A \) and \( B \) is encoded in eqn. (17). On the one hand it implies that every element of \( A_{N,m} \) has property (20) in the definition of \( B_{N,m} \). There may however exist linear combinations of elements in \( A_{N,m} \) which lie in one of the lower spaces \( L(B_{N,p}) ; p \leq m - 1 \), i.e.

\[
L(A_{N,m}) \subset \bigoplus_{2 \leq p \leq m} L(B_{N,p}) .
\]  

(22)

On the other hand \( B_{N,m} \) can contain elements of leading power \( m \leq p \leq N \) only. This is because elements of leading power \( p < m \) would by eqn. (17) rest \( R \) (modulo \( L^{-1}\)-exact pieces) which are of power \( m - 3 \) or less – in conflict with condition \( a \) in the definition of \( B_{N,m} \). Thus for \( 2 \leq m \leq N \)

\[
L(B_{N,m}) \subset \bigoplus_{m \leq p \leq N} L(A_{N,p}) .
\]  

(23.m)

For a given basis \( B_N \) this condition can be exploited recursively. For \( m = N \) it reads \( L(B_{N,N}) \subset L(A_{N,N}) \), which is to say that there exists a (non-unique) subspace \( W_N \subset L(A_{N,N}) \) s.t.

\[
W_N \oplus L(B_{N,N}) = L(A_{N,N}) .
\]  

(24)

One can use the ambiguity in the choice of \( W_N \) to achieve that \( Q_0 W_N \) has a rest of power \( p \leq N - 3 \) (modulo \( L^{-1}\)-exact pieces). By definition of \( B_{N,N} \) there exists a basis \( \overline{A}_{N,N} \)
of $L(A_{N,N})$ s.t. $A_{N,N} = B_{N,N} \cup V_N$, where $Q_0 V_N$ has a rest of power $p \leq N - 3$ modulo $L_{-1}$-exact pieces. Thus

$$L(V_N) \oplus L(B_{N,N}) = L(A_{N,N}) .$$

(25)

In the next step insert (25) into the condition (23.11)

$$L(B_{N,N-1}) \subset L(A_{N,N-1}) \oplus L(A_{N,N})$$

$$= L(A_{N,N-1}) \oplus L(B_{N,N}) \oplus L(V_N) .$$

By construction of $V_N$ and eqn. (17) this implies that in fact

$$L(B_{N,N-1}) \subset L(A_{N,N-1}) \oplus L(V_N) .$$

From here one concludes that there exists a (non-unique) subspace $W_{N-1} \subset L(A_{N,N-1}) \oplus L(V_N)$ s.t.

$$W_{N-1} \oplus L(B_{N,N-1}) = L(A_{N,N-1}) \oplus L(V_N) .$$

Again we exploit the ambiguity by chosing a basis $A_{N,N-1} \cup V_N$ of $L(A_{N,N-1}) \oplus L(V_N)$ s.t. $A_{N,N-1} \cup V_N = B_{N,N-1} \cup V_{N-1}$, where $Q_0 V_{N-1}$ has a rest of power $p \leq N - 4$ modulo $L_{-1}$-exact pieces. Thus

$$L(V_{N-1}) \oplus L(B_{N,N-1}) = L(A_{N,N-1}) \oplus L(V_N) .$$

(26)

By induction on $m$ one can now infer the existence of subspaces $W_m \subset L(A_{N,m}) \oplus L(V_{m+1})$ s.t. for $3 \leq m \leq N$

$$W_m \oplus L(B_{N,m}) = L(A_{N,m}) \oplus L(V_{m+1}) .$$

(27.m)

and $Q_0 V_{m+1}$ has a rest of power $p \leq m - 2$ modulo $L_{-1}$-exact pieces. We include (24) in the chain of equations (27.m) by defining $V_{N+1} = \emptyset$. For $m < N$ the equation (27.m) is obtained from the condition (23.m) applied to the basis

$$A_{N}(m + 1) := \left( \bigcup_{p=m+1}^N A_{N,m} \right) \cup \left( \bigcup_{p=2}^m A_{N,m} \right) .$$

By inserting consecutively the equations (28.p) below, $p = N, \ldots, m + 1$ and simplifying the resulting inclusions one arrives at $L(B_{N,m}) \subset L(A_{N,m}) \oplus L(V_{m+1})$, from which one concludes (27.m). The induction step is driven by the choice of a basis $A_{N,m} \cup V_{m+1}$ in $L(A_{N,m}) \oplus L(V_{m+1})$ s.t. $A_{N,m} \cup V_{m+1} = B_{N,m} \cup V_m$, where $Q_0 V_m$ and $Q_0 V_{m+1}$ have a rest of power $p \leq m - 3$ and $p \leq m - 2$, respectively, modulo $L_{-1}$-exact pieces. The equation (27.m) becomes

$$L(V_{m}) \oplus L(B_{N,m}) = L(A_{N,m}) \oplus L(V_{m+1}) ,$$

(28.m)
for $3 \leq m \leq N$. In particular, the basis $\mathcal{A}_N(m+1)$ has been transformed into $\mathcal{A}_N(m)$, completing the induction step. The last equation (28.3) reads $L(V_3) \oplus L(B_{N,3}) = L(\bar{A}_{N,3}) \oplus L(\bar{V}_4)$, where by construction $Q_0V_3$ and $Q_0\bar{V}_4$ have rests of power $p = 0$ and $p \leq 1$, respectively. But this means that $L(V_3)$ actually is the space of conserved densities $L(B_{N,2})$ and we may assume that $V_3 = B_{N,2}$. One can also check that the subsequent equation (22.8) would just reproduce this feature: For $N > 2$ both $V_2$ and $\bar{A}_{N,2}$ are empty, so that (22.8) collapses to $L(B_{N,2}) = L(V_3)$. In summary we have shown:

For a given basis $\mathcal{B}_N$, $N > 2$ there exists a basis $\mathcal{A}_N := \mathcal{A}_N(2) = \bigcup_{m=2}^{N} \mathcal{A}_{N,m}$ and vector spaces $L(V_m)$, $L(\bar{V}_m)$, $3 \leq m \leq N$ s.t.

- $\dim L(V_m) = \dim L(\bar{V}_m)$.
- $L(V_m)$ and $L(\bar{V}_m)$ contain elements of leading power $p \geq m$ only.
- $Q_0V_m$ and $Q_0\bar{V}_m$ have rests of power $p \leq m - 3$ modulo $L_{-1}$-exact pieces.

In particular $L(V_3) = L(\bar{V}_3) = L(B_{N,2})$. The relations among these spaces are summarized in the tableau

$$
\begin{align*}
L(V_N) \oplus L(B_{N,N}) &= L(\bar{A}_{N,N}) \\
L(V_m) \oplus L(B_{N,m}) &= L(\bar{A}_{N,m}) \oplus L(\bar{V}_{m+1}), & 4 \leq m \leq N - 1 \\
L(B_{N,2}) \oplus L(B_{N,3}) &= L(\bar{A}_{N,3}) \oplus L(\bar{V}_4).
\end{align*}
$$

(29)

These features follow directly from the construction outlined. In particular, the equation $\dim L(V_m) = \dim L(\bar{V}_m)$ follows from $\dim L(A_{N,m}) = \dim L(\bar{A}_{N,m})$ and guarantees that the dimensions on both sides of (29) add up correctly: $\sum_{m=1}^{N} \dim L(B_{N,m}) = \sum_{m=1}^{N} \dim L(A_{N,m})$, where both sides equal $\dim \hat{\mathcal{H}}_N$. We repeat that all of the spaces $L(V_m)$, $L(\bar{V}_m)$, $L(\bar{A}_{N,m-1})$, $4 \leq m \leq N$ depend on the initial choice of basis $\mathcal{B}_N$ and only their dimensions are uniquely defined. The exceptional cases are $L(V_3) = L(\bar{V}_3) = L(B_{N,2})$ where already the vector spaces (not only their dimensions) do not depend on the choice of basis. In addition we have

$$
\dim L(V_N) \geq \dim L(V_3) = \dim L(B_{N,2}).
$$

(30)

To see this, recall that quantum conserved densities have to be of leading power $N$ in $\{x_i^n\}$. Clearly $L(B_{N,N})$ is useless for the construction of conserved densities, so that $\dim L(V_N) \geq \dim L(V_3)$. 

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4.5. Now consider the relation between the tableau (29) and its classical counterpart, where the sets \(A_{N,m}, B_{N,m}, V_m, \nabla_m\) are replaced by \(A_{N,m}^{(0)}, B_{N,m}^{(0)}, V_m^{(0)}, \nabla_m^{(0)}\), respectively. We will denote this tableau by (29)(0). The sets \(B_{N,m}^{(0)}, 2 \leq m \leq N\) have already been defined. Let \(A_{N}^{(0)} = \bigcup_{2 \leq m \leq N} A_{N,m}^{(0)}\) denote the classical counterpart of \(A_N\) i.e. a basis of \(\hat{\mathcal{H}}_N^{(0)}\) having properties a) and b) in the definition. Since eqn. (17) is preserved in the classical case, the basis \(A_{N}^{(0)}\) and \(B_{N}^{(0)}\) are related by a chain of equations (23.m),(0), \(2 \leq m \leq N\), replacing (23.m). Given these, the derivation of the classical counterpart of the tableau (29) can be done as before. In particular, this defines the subspaces \(L(A_N^{(0)})\) and \(L(B_N^{(0)})\). The form of the classical tableau (29)(0) will be the same as (29), but in principle the dimensions of some of the linear spaces might be different. However, this is not the case.

Since the subspaces \(L(A_{N,m}^{(0)})\) are defined by means of properties of the leading power terms only (which are left invariant by \(\hat{\mathcal{C}}\)), they manifestly have the same dimensions as their classical counterparts

\[
\dim L(A_{N,m}^{(0)}) = \dim L(A_{N,m}) , \quad 2 \leq m \leq N ,
\]

and one may assume w.l.o.g. that \(A_{N,m}^{(0)} = \hat{\mathcal{C}} A_{N,m}\). For other subspaces \(U \subset \hat{\mathcal{H}}_N\), which are defined through conditions that affect also subleading powers, a relation of the form (31) is not manifest – which is just why the equation (21) required proof. Given that both \(L(A_{N,m})\) and \(L(B_{N,m})\) have the same dimensions as their classical counterparts, the same follows for the remaining spaces \(L(V_m), L(\nabla_m), 3 \leq m \leq N\). This is easily verified by induction from top to bottom and right to left in the tableau (29).

4.6. The tableau (29) gives a means to construct the quantum conserved densities. The method is to calculate the sets \(V_N, \nabla_N, V_{N-1}, \nabla_{N-1}, \ldots, V_3, \nabla_3\) recursively. In the final step one obtains a basis \(V_3 = \nabla_3 = B_{N,2}\) for the space of quantum conserved densities at degree \(N\). Moreover, from the previous results, all of the states invoked are in 1-1 correspondence to their classical counterparts. The calculation in the classical and the quantum case therefore is principally the same – just the coefficients in the linear combinations of the basis vectors in \(\mathcal{A}_N\) get deformed.

In detail, suppose a basis of type \(\mathcal{A}\) to be given. For example one may take the set of states (11) of degree \(N\). Consider now the top row of (29). From (30) one sees that a necessary condition for a conserved density to exist at degree \(N\) is that \(V_N\) is nonempty. For \(N-1 \in E\) the elements of \(V_N\) therefore are the initial material for the construction of the conserved densities. In the second row of (29) one adds the basis elements of \(L(A_{N,N-1})\) and eliminates the space of linear combinations \(L(B_{N,N-1})\) which is certainly useless for the construction of the conserved densities. The remainder \(V_{N-2}\) is used as an
input for the next row etc.. The result \( \dim L(V_N) \geq \dim L(B_{N,2}) = \#(N - 1) \) means that for \( N - 1 \in E \) this process iterates and in the last step yields a basis \( V_3 = B_{N,2} \) for the space of conserved densities.

Examples for such calculations for the low rank \( sl_n \)-cases can be found in [3]. The above scheme (from a slightly different viewpoint) was referred to as ‘elimination algorithm’. In these cases it appeared that in fact all the spaces \( L(V_m) \) are non-trivial and have the same dimension \( \dim L(V_m) = \#(N - 1) \), \( 2 \leq m \leq N \). We expect this to be correct in general.

5. Conclusions

The classical vertex operator \( Q_0^{(0)} \) and the quantum vertex operator \( Q_0 \) were shown to be related by an equivalence transformation. This implies that a basis \( B_N \) containing the conserved densities as some of the basis vectors can smoothly be deformed from the classical to the quantum theory. This class of basis can be characterized by certain extremal properties which give rise to a construction scheme for both the classical and the quantum conserved densities. The results yield a novel understanding of the role of the quantization process in \( W \)-algebras. A better control of the transformation (15) would give a means to quantise many concepts of the classical integrable hierarchies in a systematic fashion.

Acknowledgements: I wish to thank E. Frenkel for stimulating discussions and for contributing to the clarity of the argument.

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